arihant

Skills in Mathematics for

JEE Main & Advanced

Algebra

With Sessionwise Theory & Exercises

Dr. SK Goyal

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ARIHANT PRAKASHAN (Series), MEERUT

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PREFACE

"THE ALGEBRAIC SUM OF ALL THE TRANSFORMATIONS OCCURRING IN A CYCLICAL PROCESS CAN ONLY BE POSITIVE, OR, AS AN EXTREME CASE EQUAL TO NOTHING"

MEANS IF YOU CONTINUOUSLY PUT YOUR EFFORTS ON AN ASPECT YOU HAVE VERY GOOD CHANCE OF POSITIVE OUTCOME i.e. SUCCESS

It is a matter of great pride and honour for me to have received such an overwhelming response to the previous editions of this book from the readers. In a way, this has inspired me to revise this book thoroughly as per the changed pattern of JEE Main & Advanced. I have tried to make the contents more relevant as per the needs of students, many topics have been re-written, a lot of new problems of new types have been added in etcetc. All possible efforts are made to remove all the printing errors that had crept in previous editions. The book is now in such a shape that the students would feel at ease while going through the problems, which will in turn clear their concepts too.

A Summary of changes that have been made in Revised & Enlarged Edition

- Theory has been completely updated so as to accommodate all the changes made in JEE Syllabus & Pattern in recent years.
- The most important point about this new edition is, now the whole text matter of each chapter has been divided into small sessions with exercise in each session. In this way the reader will be able to go through the whole chapter in a systematic way.
- Just after completion of theory, Solved Examples of all JEE types have been given, providing the students a complete understanding of all the formats of JEE questions & the level of difficulty of questions generally asked in JEE.
- Along with exercises given with each session, a complete cumulative exercises have been given at the end of each chapter so as to give the students complete practice for JEE along with the assessment of knowledge that they have gained with the study of the chapter.
- Last 13 Years questions asked in JEE Main & Adv, IIT-JEE & AIEEE have been covered in all the chapters.

However I have made the best efforts and put my all Algebra teaching experience in revising this book. Still I am looking forward to get the valuable suggestions and criticism from my own fraternity i.e. the fraternity of JEE teachers.

I would also like to motivate the students to send their suggestions or the changes that they want to be incorporated in this book.

All the suggestions given by you all will be kept in prime focus at the time of next revision of the book.

Dr. SK Goyal

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SYLLABUS

JEE MAIN

Unit I Sets, Relations and Functions
Sets and their representation, Union, intersection and complement of sets and their algebraic properties, Power set, Relation, Types of relations, equivalence relations, functions, one-one, into and onto functions, composition of functions.

Unit II Complex Numbers
Complex numbers as ordered pairs of reals, Representation of complex numbers in the form a+ib and their representation in a plane, Argand diagram, algebra of complex numbers, modulus and argument (or amplitude) of a complex number, square root of a complex number, triangle inequality.

Unit III Matrices and Determinants
Matrices, algebra of matrices, types of matrices, determinants and matrices of order two and three. Properties of determinants, evaluation of deter-minants, area of triangles using determinants. Adjoint and evaluation of inverse of a square matrix using determinants and elementary transformations, Test of consistency and solution of simultaneous linear equations in two or three variables using determinants and matrices.

Unit IV Permutations and Combinations
Fundamental principle of counting, permutation as an arrangement and combination as selection, Meaning of P(n,r) and C (n,r), simple applications.

Unit V Mathematical Induction
Principle of Mathematical Induction and its simple applications.

Unit VI Binomial Theorem and its Simple Applications
Binomial theorem for a positive integral index, general term and middle term, properties of Binomial coefficients and simple applications.

Unit VII Sequences and Series
Arithmetic and Geometric progressions, insertion of arithmetic, geometric means between two given numbers. Relation between AM and GM Sum upto n terms of special series: $\sum n$, $\sum n^2$, $\sum n^3$. Arithmetico - Geometric progression.

Unit VIII Probability
Probability of an event, addition and multiplication theorems of probability, Baye's theorem, probability distribution of a random variate, Bernoulli and Binomial distribution.

JEE ADVANCED

Algebra
Algebra of complex numbers, addition, multiplication, conjugation, polar representation, properties of modulus and principal argument, triangle inequality, cube roots of unity, geometric interpretations.

Quadratic equations with real coefficients, relations between roots and coefficients, formation of quadratic equations with given roots, symmetric functions of roots.

Arithmetic, geometric and harmonic progressions, arithmetic, geometric and harmonic means, sums of finite arithmetic and geometric progressions, infinite geometric series, sums of squares and cubes of the first n natural numbers.

Logarithms and their Properties
Permutations and combinations, Binomial theorem for a positive integral index, properties of binomial coefficients.

Matrices as a rectangular array of real numbers, equality of matrices, addition, multiplication by a scalar and product of matrices, transpose of a matrix, determinant of a square matrix of order up to three, inverse of a square matrix of order up to three, properties of these matrix operations, diagonal, symmetric and skew-symmetric matrices and their properties, solutions of simultaneous linear equations in two or three variables.

Addition and multiplication rules of probability, conditional probability, independence of events, computation of probability of events using permutations and combinations.

CHAPTER

01

Complex Numbers

Learning Part

Session 1

- Integral Powers of Iota (i)
- Switch System Theory

Session 2

- Definition of Complex Number
- Conjugate Complex Numbers
- Representation of a Complex Number in Various Forms

Session 3

- amp (z) – amp $(-z) = \pm \pi$, According as amp (z) is Positive or Negative
- Square Root of a Complex Number
- Solution of Complex Equations
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- Cube Roots of Unity

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Practice Part

- JEE Type Examples
- Chapter Exercises

Arihant on Your Mobile !
Exercises with the symbol can be practised on your mobile. See inside cover page to activate for free.

The square of any real number, whether positive, negative or zero, is always non-negative i.e. $x^2 \geq 0$ for all $x \in R$. Therefore, there will be no real value of x, which when squared, will give a negative number.

Thus, the equation $x^2 + 1 = 0$ is not satisfied for any real value of x. 'Euler' was the first Mathematician to introduce the symbol i (read 'Iota') for the square root of -1 with the property $i^2 = -1$. The theory of complex number was later on developed by Gauss and Hamilton. According to Hamilton, ''Imaginary number is that number whose square is a negative number ''. Hence, the equation $x^2 + 1 = 0$

$$\Rightarrow \qquad x^2 = -1$$

or $$x = \pm\sqrt{-1}$$

(in the sense of arithmetic, $\sqrt{-1}$ has no meaning).

Symbolically, $\sqrt{-1}$ is denoted by i (the first letter of the word 'Imaginary ').

$\therefore$ Solutions of $x^2 + 1 = 0$ are $x = \pm i$.

Also, i is the unit of complex number, since i is present in every complex number. Generally, if a is positive quantity, then

$$\begin{aligned}\sqrt{-a} \times \sqrt{-a} &= \sqrt{(-1) \times a} \times \sqrt{(-1) \times a} \\ &= \sqrt{-1} \times \sqrt{a} \times \sqrt{-1} \times \sqrt{a} \\ &= i\sqrt{a} \times i\sqrt{a} \\ &= i^2 a = -a\end{aligned}$$

Remark

$\sqrt{-a} = i\sqrt{a}$, where a is positive quantity. Keeping this result in mind, the following computation is correct

$$\sqrt{-a}\,\sqrt{-b} = i\sqrt{a} \cdot i\sqrt{b} = i^2\sqrt{ab} = -\sqrt{ab}$$

where, a and b are positive real numbers.

But the computation, $\sqrt{-a}\,\sqrt{-b} = \sqrt{(-a)(-b)} = \sqrt{|a|\,|b|}$ is wrong. Because the property, $\sqrt{a}\,\sqrt{b} = \sqrt{ab}$ is valid only when atleast one of a and b is non-negative.

If a and b are both negative, then $\sqrt{a}\,\sqrt{b} = -\sqrt{|a||b|}$.

Example 1. Is the following computation correct? If not, give the correct computation.

$$\sqrt{-2}\,\sqrt{-3} = \sqrt{(-2)(-3)} = \sqrt{6}$$

Sol. No,

If a and b are both negative real numbers, then $\sqrt{a}\sqrt{b} = -\sqrt{ab}$

Here, $a = -2$ and $b = -3$.

$\therefore$ $$\sqrt{-2}\,\sqrt{-3} = -\sqrt{(-2)(-3)} = -\sqrt{6}$$

Example 2. A student writes the formula $\sqrt{ab} = \sqrt{a}\,\sqrt{b}$. Then, he substitutes $a = -1$ and $b = -1$ and finds $1 = -1$. Explain, where he is wrong.

Sol. Since, a and b are both negative, therefore $\sqrt{ab} \neq \sqrt{a}\,\sqrt{b}$.

Infact a and b are both negative, then we have $\sqrt{a}\sqrt{b} = -\sqrt{ab}$.

Example 3. Explain the fallacy

$$-1 = i \times i = \sqrt{-1} \times \sqrt{-1} = \sqrt{(-1) \times (-1)} = \sqrt{1} = 1.$$

Sol. If a and b are both negative, then

$$\sqrt{a}\,\sqrt{b} = -\sqrt{|a|\,|b|}$$

$\therefore$ $$\sqrt{-1} \times \sqrt{-1} = -\sqrt{|-1|\,|-1|} = -1$$

Session 1

Integral Powers of Iota (i), Switch System Theory

Integral Powers of Iota (i)

(i) If the index of i is whole number, then

$$i^0 = 1, i^1 = i, i^2 = (\sqrt{-1})^2 = -1,$$

$$i^3 = i \cdot i^2 = -i, i^4 = (i^2)^2 = (-1)^2 = 1$$

To find the value of i^n $(n > 4)$ First divide n by 4. Let q be the quotient and r be the remainder.

i.e. $$4)\,n\,(q$$
$$\underline{-4q}$$
$$r$$

$$\Rightarrow \qquad n = 4q + r$$

When, $0 \leq r \leq 3$

$$\therefore \quad i^n = i^{4q+r} = (i^4)^q\,(i)^r = (1)^q \cdot (i)^r = i^r$$

In general, $i^{4n} = 1$, $i^{4n+1} = i$, $i^{4n+2} = -1$, $i^{4n+3} = -i$ for any whole number n.

(ii) If the index of i is a negative integer, then

$$i^{-1} = \frac{1}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i, i^{-2} = \frac{1}{i^2} = -1,$$

$$i^{-3} = \frac{1}{i^3} = \frac{i}{i^4} = i, i^{-4} = \frac{1}{i^4} = \frac{1}{1} = 1, \text{ etc.}$$

Example 4. Evaluate.

(i) i^{1998}

(ii) i^{-9999}

(iii) $(-\sqrt{-1})^{4n+3}$, $n \in N$

Sol. (i) 1998 leaves remainder 2, when it is divided by 4.

i.e. $4\overline{)1998}\,(499$
$\underline{1996}$
2

$\therefore \quad i^{1998} = i^2 = -1$

Aliter

$$i^{1998} = \frac{i^{2000}}{i^2} = \frac{1}{-1} = -1$$

(ii) 9999 leaves remainder 3, when it is divided by 4.

i.e. $4\overline{)9999}\,(2499$
$\underline{9996}$
3

$\therefore \quad i^{-9999} = \frac{1}{i^{9999}} = \frac{1}{i^3} = \frac{i}{i^4} = \frac{i}{1} = i$

Aliter

$$i^{-9999} = \frac{1}{i^{9999}} = \frac{i}{i^{10000}} = \frac{i}{1} = i$$

(iii) $4n+3$ leaves remainder 3, when it is divided by 4.

i.e., $4\overline{)4n+3}\,(n$
$\underline{4n}$
3

$\therefore \quad i^{4n+3} = i^3 = -i$

Now, $(-\sqrt{-1})^{4n+3} = (-i)^{4n+3} = -(i)^{4n+3}$
$= -(-i)$
$= i$

Aliter $(-\sqrt{-1})^{4n+3} = (-i)^{4n+3} = -i^{4n+3}$
$= -(i^4)^n \cdot i^3$
$= -(1)^n(-i) = i$

Example 5. Find the value of $1+i^2+i^4+i^6+\ldots+i^{2n}$, where $i=\sqrt{-1}$ and $n \in N$.

Sol. $\because \quad 1+i^2+i^4+i^6+\ldots+i^{2n} = 1-1+1-1+\ldots+(-1)^n$

***Case* I** If n is odd, then

$$1+i^2+i^4+i^6+\ldots+i^{2n} = 1-1+1-1+\ldots+1-1=0$$

***Case* II** If n is even, then

$$1+i^2+i^4+i^6+\ldots+i^{2n} = 1-1+1-1+\ldots+1=1$$

Example 6. If $a=\frac{1+i}{\sqrt{2}}$, where $i=\sqrt{-1}$, then find the value of a^{1929}.

Sol. $\because \quad a^2 = \left(\frac{1+i}{\sqrt{2}}\right)^2 = \left(\frac{1+i^2+2i}{2}\right)$

$$= \left(\frac{1-1+2i}{2}\right) = i$$

$\therefore \quad a^{1929} = a \cdot a^{1928} = a \cdot (a^2)^{964} = a\,(i)^{964}$
$= a\,(i)^{4\times 241} = a \cdot (i^4)^{241} = a$

Example 7. Dividing $f(z)$ by $z-i$, where $i=\sqrt{-1}$, we obtain the remainder i and dividing it by $z+i$, we get the remainder $1+i$. Find the remainder upon the division of $f(z)$ by z^2+1.

Sol. $z-i=0 \Rightarrow z=i$

Remainder, when $f(z)$ is divided by $(z-i)=i$

i.e. $f(i)=i$ … (i)

and remainder, when $f(z)$ is divided by $(z+1)=1+i$

i.e. $f(-i)=1+i \quad [\because z+i=0 \Rightarrow z=-i]$ … (ii)

Since, z^2+1 is a quadratic expression, therefore remainder when $f(z)$ is divided by z^2+1, will be in general a linear expression. Let $g(z)$ be the quotient and $az+b$ (where a and b are complex numbers) be the remainder, when $f(z)$ is divided by z^2+1.

Then, $f(z) = (z^2+1)\,g(z) + az + b$ … (iii)

$\therefore \quad f(i) = (i^2+1)\,g(i) + ai + b = ai + b$

or $ai+b=i$ [from Eq. (i)] … (iv)

and $f(-i) = (i^2+1)\,g(-i) - ai + b = -ai + b$

or $-ai+b = 1+i$ [from Eq. (ii)] …(v)

From Eqs. (iv) and (v), we get

$$b = \frac{1}{2} + i \quad \text{and} \quad a = \frac{i}{2}$$

Hence, required remainder $= az + b$

$$= \frac{1}{2}iz + \frac{1}{2} + i$$

The Sum of Four Consecutive Powers of *i* (Iota) is Zero

If $n \in I$ and $i=\sqrt{-1}$, then

$i^n + i^{n+1} + i^{n+2} + i^{n+3} = i^n(1+i+i^2+i^3)$

$= i^n(1+i-1-i) = 0$

Remark

1. $\sum_{r=p}^{m} f(r) = \sum_{r=1}^{m-p+1} f(r+p-1)$
2. $\sum_{r=-p}^{m} f(r) = \sum_{r=1}^{m+p+1} f(r-p-1)$

Example 8. ***Find the value of*** $\sum_{n=1}^{13}(i^n + i^{n+1})$ (where, $i = \sqrt{-1}$)

Sol. $\because \sum_{n=1}^{13}(i^n + i^{n+1}) = \sum_{n=1}^{13} i^n + \sum_{n=1}^{13} i^{n+1} = (i+0) + (i^2+0)$

$= i - 1$ [$\because \sum_{n=2}^{13} i^n = 0$ and $\sum_{n=2}^{13} i^{n+1} = 0$ (three sets of four consecutive powers of i)]

Example 9. ***Find the value of*** $\sum_{n=0}^{100} i^{n!}$ (where, $i = \sqrt{-1}$).

Sol. $n!$ is divisible by 4, $\forall\ n \geq 4$.

$$\therefore \sum_{n=4}^{100} i^{n!} = \sum_{n=1}^{97} i^{(n+3)!}$$

$$= i^0 + i^0 + i^0 + \ldots 97 \text{ times } = 97 \qquad \ldots(i)$$

$$\therefore \sum_{n=0}^{100} i^{n!} = \sum_{n=0}^{3} i^{n!} + \sum_{n=4}^{100} i^{n!}$$

$$= i^{0!} + i^{1!} + i^{2!} + i^{3!} + 97 \qquad \text{[from Eq. (i)]}$$

$$= i^1 + i^1 + i^2 + i^6 + 97 = i + i - 1 - 1 + 97$$

$$= 95 + 2i$$

Example 10. ***Find the value of*** $\sum_{r=1}^{4n+7} i^r$ (where, $i = \sqrt{-1}$).

Sol. $\sum_{r=1}^{4n+7} i^r = i^1 + i^2 + i^3 + \sum_{r=4}^{4n+7} i^r = i - 1 - i + \sum_{r=1}^{4n+4} i^{r+3}$

$= -1 + 0$ [$(n+1)$ sets of four consecutive powers of i]

$= -1$

Example 11. Show that the polynomial $x^{4p} + x^{4q+1} + x^{4r+2} + x^{4s+3}$ is divisible by $x^3 + x^2 + x + 1$, where $p, q, r, s \in N$.

Sol. Let $f(x) = x^{4p} + x^{4q+1} + x^{4r+2} + x^{4s+3}$

and $x^3 + x^2 + x + 1 = (x^2+1)(x+1)$

$= (x+i)(x-i)(x+1)$,

where $i = \sqrt{-1}$

Now, $f(i) = i^{4p} + i^{4q+1} + i^{4r+2} + i^{4s+3} = 1 + i + i^2 + i^3 = 0$

[sum of four consecutive powers of i is zero]

$$f(-i) = (-i)^{4p} + (-i)^{4q+1} + (-i)^{4r+2} + (-i)^{4s+3}$$

$$= 1 + (-i)^1 + (-i)^2 + (-i)^3 = 1 - i - 1 + i = 0$$

and $f(-1) = (-1)^{4p} + (-1)^{4q+1} + (-1)^{4r+2} + (-1)^{4s+3}$

$= 1 - 1 + 1 - 1 = 0$

Hence, by division theorem, $f(x)$ is divisible by $x^3 + x^2 + x + 1$.

Switch System Theory

(Finding Digit in the Unit's Place)

We can determine the digit in the unit's place in a^b, where $a, b \in N$. If last digit of a are 0, 1, 5 and 6, then digits in the unit's place of a^b are 0, 1, 5 and 6 respectively, for all $b \in N$.

Powers of 2

$2^1, 2^2, 2^3, 2^4, 2^5, 2^6, 2^7, 2^8, 2^9, \ldots$ the digits in unit's place of different powers of 2 are as follows :

2, 4, 8, 6, 2, 4, 8, 6, 2,... (period being 4)
↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑
①②③⓪①②③⓪① ... (switch number)

(The remainder when b is divided by 4, can be 1 or 2 or 3 or 0). Then, press the switch number and then we get the digit in unit's place of a^b (just above the switch number) i.e. 'press the number and get the answer'.

Example 12. ***What is the digit in the unit's place of*** $(5172)^{11327}$?

Sol. Here, last digit of a is 2.

The remainder when 11327 is divided by 4, is 3. Then, press switch number 3 and then we get 8.

Hence, the digit in the unit's place of $(5172)^{11327}$ is 8.

Powers of 3

$3^1, 3^2, 3^3, 3^4, 3^5, 3^6, 3^7, 3^8, \ldots$ the digits in unit's place of different powers of 3 are as follows:

3, 9, 7, 1, 3, 9, 7, 1,... (period being 4)
↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑
①②③⓪①②③⓪ ... (switch number)

The remainder when b is divided by 4, can be 1 or 2 or 3 or 0. Now, press the switch number and get the unit's place digit (just above).

Example 13. ***What is the digit in the unit's place of***

$(143)^{86}$?

Sol. Here, last digit of a is 3.

The remainder when 86 is divided by 4, is 2.

Then, press switch number 2 and then we get 9.

Hence, the digit in the unit's place of $(143)^{86}$ is 9.

Powers of 4

$4^1, 4^2, 4^3, 4^4, 4^5, \ldots$ the digits in unit's place of different powers of 4 are as follows:

4, 6, 4, 6, 4, … (period being 2)
↑ ↑ ↑ ↑ ↑
① ⓪ ① ⓪ ① … (switch number)

The remainder when b is divided by 2, can be 1 or 0. Now, press the switch number and get the unit's place digit (just above the switch number).

Example 14. What is the digit in unit's place of $(1354)^{22222}$?

Sol. Here, last digit of a is 4.

The remainder when 22222 is divided by 2, is 0. Then, press switch number 0 and then we get 6.

Hence, the digit in the unit's place of $(1354)^{22222}$ is 6.

Powers of 7

$7^1, 7^2, 7^3, 7^4, 7^5, 7^6, 7^7, 7^8, \ldots$ the digits in unit's place of different powers of 7 are as follows:

7, 9, 3, 1, 7, 9, 3, 1, … (period being 4)
↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑
① ② ③ ⓪ ① ② ③ ⓪ … (switch number)

(The remainder when b is divided by 4, can be 1 or 2 or 3 or 0). Now, press the switch number and get the unit's place digit (just above).

Example 15. What is the digit in the unit's place of $(13057)^{941120579}$?

Sol. Here, last digit of a is 7.

The remainder when 941120579 is divided by 4, is 3. Then, press switch number 3 and then we get 3.

Hence, the digit in the unit's place of $(13057)^{941120579}$ is 3.

Powers of 8

$8^1, 8^2, 8^3, 8^4, 8^5, 8^6, 8^7, 8^8, \ldots$ the digits in unit's place of different powers of 8 are as follows:

8, 4, 2, 6, 8, 4, 2, 6, … (period being 4)
↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑
① ② ③ ⓪ ① ② ③ ⓪ … (switch number)

The remainder when b is divided by 4, can be 1 or 2 or 3 or 0.

Now, press the switch number and get the unit's place digit (just above the switch number).

Example 16. What is the digit in the unit's place of $(1008)^{786}$?

Sol. Here, last digit of a is 8.

The remainder when 786 is divided by 4, is 2. Then, press switch number 2 and then we get 4.

Hence, the digit in the unit's place of $(1008)^{786}$ is 4.

Powers of 9

$9^1, 9^2, 9^3, 9^4, 9^5, \ldots$ the digits in unit's place of different powers of 9 are as follows:

9, 1, 9, 1, 9, … (period being 2)
↑ ↑ ↑ ↑ ↑
① ⓪ ① ⓪ ① … (switch number)

The remainder when b is divided by 2, can be 1 or 0.

Now, press the switch number and get the unit's place digit (just above the switch number).

Example 17. What is the digit in the unit's place of $(2419)^{111213}$?

Sol. Here, last digit of a is 9.

The remainder when 111213 is divided by 2, is 1. Then, press switch number 1 and then we get 9.

Hence, the digit in the unit's place of $(2419)^{111213}$ is 9.

Exercise for Session 1

1 If $(1+i)^{2n}+(1-i)^{2n}=-2^{n+1}$ (where, $i=\sqrt{-1}$) for all those n, which are

(a) even
(b) odd
(c) multiple of 3
(d) None of these

2 If $i=\sqrt{-1}$, the number of values of i^n+i^{-n} for different $n \in I$ is

(a) 1
(b) 2
(c) 3
(d) 4

3 If $a>0$ and $b<0$, then $\sqrt{a}\ \sqrt{b}$ is equal to (where, $i=\sqrt{-1}$)

(a) $-\sqrt{a\cdot|b|}$
(b) $\sqrt{a\cdot|b|}\,i$
(c) $\sqrt{a\cdot|b|}$
(d) None of these

4 Consider the following statements.

$S_1: -6=2i\times 3i=\sqrt{(-4)}\times\sqrt{(-9)}$ (where, $i=\sqrt{-1}$)
$S_2: \sqrt{(-4)}\times\sqrt{(-9)}=\sqrt{(-4)\times(-9)}$
$S_3: \sqrt{(-4)\times(-9)}=\sqrt{36}$
$S_4: \sqrt{36}=6$

Of these statements, the incorrect one is

(a) S_1 only
(b) S_2 only
(c) S_3 only
(d) None of these

5 The value of $\sum_{n=0}^{50} i^{(2n+1)!}$ (where, $i=\sqrt{-1}$) is

(a) i
(b) $47-i$
(c) $48+i$
(d) 0

6 The value of $\sum_{r=-3}^{1003} i^r$ (where $i=\sqrt{-1}$) is

(a) 1
(b) -1
(c) i
(d) $-i$

7 The digit in the unit's place of $(153)^{98}$ is

(a) 1
(b) 3
(c) 7
(d) 9

8 The digit in the unit's place of $(141414)^{12121}$ is

(a) 4
(b) 6
(c) 3
(d) 1

Session 2

Definition of Complex Number, Conjugate Complex Numbers, Representation of a Complex Number in Various Forms

Definition of Complex Number

A number of the form $a + ib$, where $a, b \in R$ and $i = \sqrt{-1}$, is called a **complex number.** It is denoted by z i.e. $z = a + ib$.

A complex number may also be defined as an ordered pair of real numbers; and may be denoted by the symbol (a, b). If we write $z = (a, b)$, then a is called the real part and b is the imaginary part of the complex number z and may be denoted by $\text{Re}(z)$ and $\text{Im}(z)$, respectively i.e., $a = \text{Re}(z)$ and $b = \text{Im}(z)$.

Two complex numbers are said to be equal, if and only if their real parts and imaginary parts are separately equal.

Thus, $a + ib = c + id$

$\Leftrightarrow$ $a = c$ and $b = d$

where, $a, b, c, d \in R$ and $i = \sqrt{-1}$.

i.e. $z_1 = z_2$

$\Leftrightarrow$ $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$

Important Properties of Complex Numbers

1. The complex numbers do not possess the property of order, i.e., $(a + ib) >$ or $< (c + id)$ is not defined. *For example*, $9 + 6i > 3 + 2i$ makes no sense.
2. A real number a can be written as $a + i \cdot 0$. Therefore, every real number can be considered as a complex number, whose imaginary part is zero. Thus, the set of real numbers (R) is a proper subset of the complex numbers (C) i.e. $R \subset C$. Hence, the complex number system is $N \subset W \subset I \subset Q \subset R \subset C$
3. A complex number z is said to be purely real, if $\text{Im}(z) = 0$; and is said to be purely imaginary, if $\text{Re}(z) = 0$. The complex number $0 = 0 + i \cdot 0$ is both purely real and purely imaginary.
4. In real number system, $a^2 + b^2 = 0 \Rightarrow a = 0 = b$.
   But if z_1 and z_2 are complex numbers, then $z_1^2 + z_2^2 = 0$ does not imply $z_1 = z_2 = 0$.
   For example, $z_1 = 1 + i$ and $z_2 = 1 - i$
   Here, $z_1 \neq 0, z_2 \neq 0$
   But $z_1^2 + z_2^2 = (1 + i)^2 + (1 - i)^2 = 1 + i^2 + 2i + 1 + i^2 - 2i$
   $= 2 + 2i^2 = 2 - 2 = 0$
   However, if product of two complex numbers is zero, then atleast one of them must be zero, same as in case of real numbers.
   If $z_1 z_2 = 0$, then $z_1 = 0, z_2 \neq 0$ or $z_1 \neq 0, z_2 = 0$ or $z_1 = 0, z_2 = 0$

Algebraic Operations on Complex Numbers

Let two complex numbers be $z_1 = a + ib$ and $z_2 = c + id$, where $a, b, c, d \in R$ and $i = \sqrt{-1}$.

1. **Addition** $z_1 + z_2 = (a + ib) + (c + id)$
   $= (a + c) + i(b + d)$
2. **Subtraction** $z_1 - z_2 = (a + ib) - (c + id)$
   $= (a - c) + i(b - d)$
3. **Multiplication** $z_1 \cdot z_2 = (a + ib) \cdot (c + id)$
   $= ac + iad + ibc + i^2 bd$
   $= ac + i(ad + bc) - bd$
   $= (ac - bd) + i(ad + bc)$
4. **Division** $\dfrac{z_1}{z_2} = \dfrac{(a + ib)}{(c + id)} \cdot \dfrac{(c - id)}{(c - id)}$

   [multiplying numerator and denominator by $c - id$ where atleast one of c and d is non-zero]

$$= \frac{ac - iad + ibc - i^2 bd}{(c)^2 - (id)^2} = \frac{ac + i(bc - ad) + bd}{c^2 - i^2 d^2}$$

$$= \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2} = \frac{(ac + bd)}{(c^2 + d^2)} + i\frac{(bc - ad)}{(c^2 + d^2)}$$

Remark

$\dfrac{1 + i}{1 - i} = i$ and $\dfrac{1 - i}{1 + i} = -i$, where $i = \sqrt{-1}$.

Properties of Algebraic Operations on Complex Numbers

Let z_1, z_2 and z_3 be any three complex numbers. Then, their algebraic operations satisfy the following properties :

Properties of Addition of Complex Numbers

(i) **Closure law** $z_1 + z_2$ is a complex number.

(ii) **Commutative law** $z_1 + z_2 = z_2 + z_1$

(iii) **Associative law** $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

(iv) **Additive identity** $z+0=z=0+z$, then 0 is called the additive identity.

(v) **Additive inverse** $-z$ is called the additive inverse of z, i.e. $z+(-z)=0$.

Properties of Multiplication of Complex Numbers

(i) **Closure law** $z_1 \cdot z_2$ is a complex number.

(ii) **Commutative law** $z_1 \cdot z_2 = z_2 \cdot z_1$

(iii) **Associative law** $(z_1 \cdot z_2)\, z_3 = z_1\,(z_2 \cdot z_3)$

(iv) **Multiplicative identity** $z \cdot 1 = z = 1 \cdot z$, then 1 is called the multiplicative identity.

(v) **Multiplicative inverse** If z is a non-zero complex number, then $\frac{1}{z}$ is called the multiplicative inverse of z i.e. z. $\frac{1}{z}=1=\frac{1}{z}\cdot z$

(vi) **Multiplication is distributive with respect to addition** $z_1\,(z_2+z_3)=z_1\,z_2+z_1\,z_3$

Conjugate Complex Numbers

The complex numbers $z=(a,b)=a+ib$ and $\bar{z}=(a,-b)=a-ib$, where a and b are real numbers, $i=\sqrt{-1}$ and $b\neq 0$, are said to be complex conjugate of each other (here, the complex conjugate is obtained by just changing the sign of i).

Note that, sum $=(a+ib)+(a-ib)=2a$, which is real.

And product $=(a+ib)(a-ib)=a^2-(ib)^2$

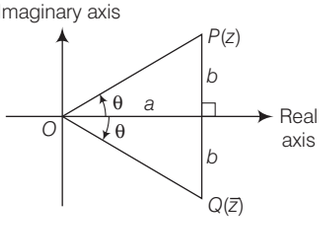
$$=a^2-i^2\,b^2=a^2-(-1)\,b^2$$

$$=a^2+b^2, \text{ which is real.}$$

Geometrically, $\bar{z}$ is the mirror image of z along real axis on argand plane.

Remark

Let $z=-a-ib, a>0, b>0=(-a,-b)$ (III quadrant)



Then, $\bar{z}=-a+ib=(-a,b)$ (II quadrant). Now,

(i) If z lies in I quadrant, then $\bar{z}$ lies in IV quadrant and *vice-versa.*

(ii) If z lies in II quadrant, then $\bar{z}$ lies in III quadrant and *vice-versa.*

Properties of Conjugate Complex Numbers

Let z, z_1 and z_2 be complex numbers. Then,

(i) $\overline{(\bar{z})}=z$

(ii) $z+\bar{z}=2\,\mathrm{Re}\,(z)$

(iii) $z-\bar{z}=2\,\mathrm{Im}\,(z)$

(iv) $z+\bar{z}=0 \;\Rightarrow\; z=-\bar{z} \;\Rightarrow\; z$ is purely imaginary.

(v) $z-\bar{z}=0 \;\Rightarrow\; z=\bar{z} \;\Rightarrow\; z$ is purely real.

(vi) $\overline{z_1 \pm z_2}=\bar{z}_1 \pm \bar{z}_2$ Ingeneral,

$$\overline{z_1 \pm z_2 \pm z_3 \pm \ldots \pm z_n}=\overline{z_1} \pm \overline{z_2} \pm \overline{z_3} \pm \ldots \pm \overline{z_n}$$

(vii) $\overline{z_1 \cdot z_2}=\overline{z_1}\cdot\overline{z_2}$

In general, $\overline{z_1 \cdot z_2 \cdot z_3 \ldots z_n}=\overline{z_1}\cdot\overline{z_2}\cdot\overline{z_3}\ldots\overline{z_n}$

(viii) $\left(\overline{\frac{z_1}{z_2}}\right)=\frac{\bar{z}_1}{\bar{z}_2}, z_2\neq 0$

(ix) $\overline{z^n}=(\bar{z})^n$

(x) $z_1\,\overline{z_2}+\overline{z_1}\,z_2=2\,\mathrm{Re}\,(z_1\,\overline{z_2})=2\,\mathrm{Re}\,(\overline{z_1}\,z_2)$

(xi) If $z=f(z_1,z_2)$, then $\bar{z}=f(\overline{z_1},\overline{z_2})$

Example 18. If $\frac{x-3}{3+i}+\frac{y-3}{3-i}=i$, where $x, y\in R$ and $i=\sqrt{-1}$, find the values of x and y.

Sol. $\because$ $\frac{x-3}{3+i}+\frac{y-3}{3-i}=i$

$\Rightarrow$ $(x-3)(3-i)+(y-3)(3+i)=i(3+i)(3-i)$

$\Rightarrow (3x-xi-9+3i)+(3y+yi-9-3i)=10i$

$\Rightarrow$ $(3x+3y-18)+i(y-x)=10i$

On comparing real and imaginary parts, we get

$$3x+3y-18=0$$

$\Rightarrow$ $x+y=6$ ...(i)

and $y-x=10$ ...(ii)

On solving Eqs. (i) and (ii), we get

$$x=-2,\ y=8$$

Example 19. ***If*** $(a+ib)^5=p+iq$, ***where*** $i=\sqrt{-1}$, ***prove that*** $(b+ia)^5=q+ip$.

Sol. $\because$ $(a+ib)^5=p+iq$

$\therefore$ $\overline{(a+ib)^5}=\overline{p+iq} \;\Rightarrow\; (a-ib)^5=(p-iq)$

$\Rightarrow$ $(-i^2a-ib)^5=(-i^2p-iq)$ $[\because i^2=-1]$

$\Rightarrow$ $(-i)^5(b+ia)^5=(-i)(q+ip)$

$\Rightarrow$ $(-i)(b+ia)^5=(-i)(q+ip)$

$\therefore$ $(b+ia)^5=(q+ip)$

Example 20. Find the least positive integral value of n, for which $\left(\frac{1-i}{1+i}\right)^n$, where $i=\sqrt{-1}$, is purely imaginary with positive imaginary part.

Sol. $\left(\frac{1-i}{1+i}\right)^n=\left(\frac{1-i}{1+i}\times\frac{1-i}{1-i}\right)^n=\left(\frac{1+i^2-2i}{2}\right)^n=\left(\frac{1-1-2i}{2}\right)^n$

$=(-i)^n=$ Imaginary

$\Rightarrow$ $n=1,3,5,\ldots$ for positive imaginary part $n=3$.

Example 21. If the multiplicative inverse of a complex number is $(\sqrt{3}+4i)/19$, where $i=\sqrt{-1}$, find complex number.

Sol. Let z be the complex number.

Then, $z\cdot\left(\frac{\sqrt{3}+4i}{19}\right)=1$

or $z=\frac{19}{(\sqrt{3}+4i)}\times\frac{(\sqrt{3}-4i)}{(\sqrt{3}-4i)}$

$=\frac{19(\sqrt{3}-4i)}{19}=(\sqrt{3}-4i)$

Example 22. ***Find real*** θ***, such that*** $\frac{3+2i\sin\theta}{1-2i\sin\theta}$***, where*** $i=\sqrt{-1}$***, is***

(i) purely real. (ii) purely imaginary.

Sol. Let $z=\frac{3+2i\sin\theta}{1-2i\sin\theta}$

On multiplying numerator and denominator by conjugate of denominator,

$$z=\frac{(3+2i\sin\theta)(1+2i\sin\theta)}{(1-2i\sin\theta)(1+2i\sin\theta)}=\frac{(3-4\sin^2\theta)+8i\sin\theta}{(1+4\sin^2\theta)}$$

$$=\frac{(3-4\sin^2\theta)}{(1+4\sin^2\theta)}+i\frac{(8\sin\theta)}{(1+4\sin^2\theta)}$$

(i) For purely real, $\text{Im}(z)=0$

$\Rightarrow$ $\frac{8\sin\theta}{1+4\sin^2\theta}=0$ or $\sin\theta=0$

$\therefore$ $\theta=n\pi, n\in I$

(ii) For purely imaginary, $\text{Re}(z)=0$

$\Rightarrow$ $\frac{(3-4\sin^2\theta)}{(1+4\sin^2\theta)}=0$ or $3-4\sin^2\theta=0$

or $\sin^2\theta=\frac{3}{4}=\left(\frac{\sqrt{3}}{2}\right)^2=\left(\sin\frac{\pi}{3}\right)^2$

$\therefore$ $\theta=n\pi\pm\frac{\pi}{3}, n\in I$

Example 23. Find real values of x and y for which the complex numbers $-3+ix^2y$ and x^2+y+4i, where $i=\sqrt{-1}$, are conjugate to each other.

Sol. Given, $-3+ix^2y=x^2+y+4i$

$\Rightarrow$ $-3-ix^2y=x^2+y+4i$

On comparing real and imaginary parts, we get

$x^2+y=-3$ ...(i)

and $-x^2y=4$ ...(ii)

From Eq. (ii), we get $x^2=-\frac{4}{y}$

Then, $-\frac{4}{y}+y=-3$ $\left[\text{putting } x^2=-\frac{4}{y}\text{ in Eq. (i)}\right]$

$y^2+3y-4=0\Rightarrow(y+4)(y-1)=0$

$\therefore$ $y=-4,1$

For $y=-4, x^2=1\Rightarrow x=\pm1$

For $y=1, x^2=-4$ [impossible]

$\therefore$ $x=\pm1, y=-4$

Example 24. If $x=-5+2\sqrt{-4}$, find the value of

$$x^4+9x^3+35x^2-x+4.$$

Sol. Since, $x=-5+2\sqrt{-4}\Rightarrow x+5=4i$

$\Rightarrow$ $(x+5)^2=(4i)^2\Rightarrow x^2+10x+25=-16$

$\therefore$ $x^2+10x+41=0$ ...(i)

Now,

$$\begin{array}{r|l} x^2+10x+41 & x^4+9x^3+35x^2-x+4 \;\big|\; x^2-x+4 \\ & \underline{x^4+10x^3+41x^2} \\ & -x^3-6x^2-x+4 \\ & \underline{-x^3-10x^2-41x} \\ & 4x^2+40x+4 \\ & \underline{4x^2+40x+164} \\ & -160 \end{array}$$

$\therefore$ $x^4+9x^3+35x^2-x+4$

$=(x^2+10x+41)(x^2-x+4)-160$

$=0-160=-160$ [from Eq. (i)]

Example 25. Let z be a complex number satisfying the equation $z^2-(3+i)z+\lambda+2i=0$, where $\lambda\in R$ and $i=\sqrt{-1}$. Suppose the equation has a real root, find the non-real root.

Sol. Let α be the real root. Then,

$\alpha^2-(3+i)\alpha+\lambda+2i=0$

$\Rightarrow \quad (\alpha^2 - 3\alpha + \lambda) + i(2 - \alpha) = 0$

On comparing real and imaginary parts, we get

$$\alpha^2 - 3\alpha + \lambda = 0 \quad \text{...(i)}$$

$$\Rightarrow \quad 2 - \alpha = 0 \quad \text{...(ii)}$$

From Eq. (ii), $\alpha = 2$

Let other root be β.

Then, $\alpha + \beta = 3 + i \quad \Rightarrow \quad 2 + \beta = 3 + i$

$\therefore \quad \beta = 1 + i$

Hence, the non-real root is $1 + i$.

Representation of a Complex Number in Various Forms

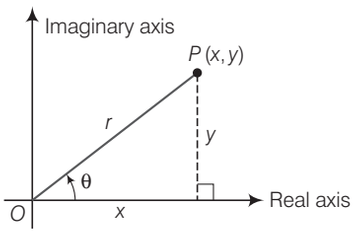
Cartesian Form

(Geometrical Representation)

Every complex number $z = x + iy$, where $x, y \in R$ and $i = \sqrt{-1}$, can be represented by a point in the cartesian plane known as complex plane (Argand plane) by the ordered pair (x, y).

Modulus and Argument of a Complex Number

Let $z = x + iy = (x, y)$ for all $x, y \in R$ and $i = \sqrt{-1}$.



The length OP is called modulus of the complex number z denoted by $|z|$,

i.e. $OP = r = |z| = \sqrt{(x^2 + y^2)}$

and if $(x, y) \neq (0, 0)$, then θ is called the argument or amplitude of z,

i.e. $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ [angle made by OP with positive X-axis]

or $\arg(z) = \tan^{-1}(y / x)$

Also, argument of a complex number is not unique, since if θ is a value of the argument, so also is $2n\pi + \theta$, where $n \in I$. But usually, we take only that value for which $0 \leq \theta < 2\pi$. Any two arguments of a complex number differ by $2n\pi$.

Argument of z will be θ, $\pi - \theta$, $\pi + \theta$ and $2\pi - \theta$ according as the point z lies in I, II, III and IV quadrants respectively, where $\theta = \tan^{-1}\left|\frac{y}{x}\right|$.

Example 26. Find the arguments of $z_1 = 5 + 5i$, $z_2 = -4 + 4i$, $z_3 = -3 - 3i$ and $z_4 = 2 - 2i$, where $i = \sqrt{-1}$.

Sol. Since, z_1, z_2, z_3 and z_4 lies in I, II, III and IV quadrants respectively. The arguments are given by

$$\arg(z_1) = \tan^{-1}\left|\frac{5}{5}\right| = \tan^{-1}1 = \pi/4$$

$$\arg(z_2) = \pi - \tan^{-1}\left|\frac{4}{-4}\right| = \pi - \tan^{-1}1 = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

$$\arg(z_3) = \pi + \tan^{-1}\left|\frac{-3}{-3}\right| = \pi + \tan^{-1}1 = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$$

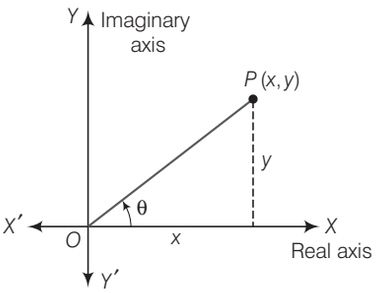
and $\arg(z_4) = 2\pi - \tan^{-1}\left|\frac{-2}{2}\right|$

$$= 2\pi - \tan^{-1}1 = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$$

Principal Value of the Argument

The value θ of the argument which satisfies the inequality $-\pi < \theta \leq \pi$ is called the **principal value** of the argument.

If $z = x + iy = (x, y)$, $\forall\ x, y \in R$ and $i = \sqrt{-1}$, then $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$ always gives the principal value. It depends on the quadrant in which the point (x, y) lies.



(i) $(x, y) \in$ first quadrant $x > 0, y > 0$.

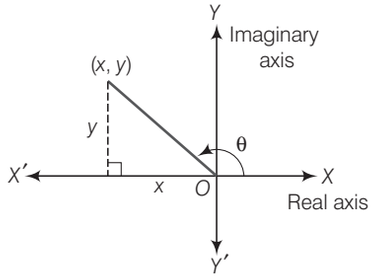
The principal value of $\arg(z) = \theta = \tan^{-1}\left(\frac{y}{x}\right)$

It is an acute angle and positive.

(ii) $(x, y) \in$ second quadrant $x < 0, y > 0$.

The principal value of $\arg(z) = \theta$

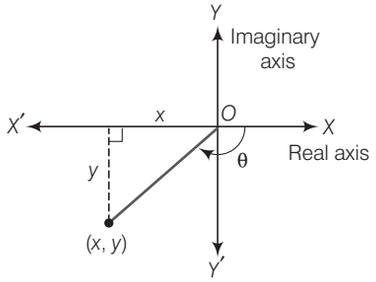
$$= \pi - \tan^{-1}\left(\frac{y}{|x|}\right)$$



It is an obtuse angle and positive.

(iii) $(x, y) \in$ third quadrant $x < 0, y < 0$.

The principal value of $\arg(z) = \theta = -\pi + \tan^{-1}\left(\frac{y}{x}\right)$

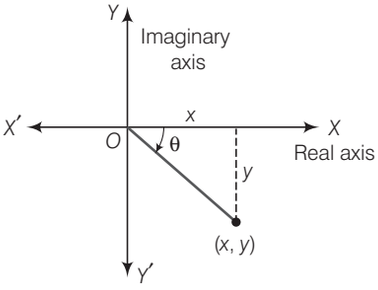


It is an obtuse angle and negative.

(iv) $(x, y) \in$ fourth quadrant $x > 0, y < 0$.

The principal value of $\arg(z) = \theta$

$$= -\tan^{-1}\left(\frac{|y|}{x}\right)$$



It is an acute angle and negative.

Example 27. Find the principal values of the arguments of $z_1 = 2 + 2i$, $z_2 = -3 + 3i$, $z_3 = -4 - 4i$ and $z_4 = 5 - 5i$, where $i = \sqrt{-1}$.

Sol. Since, z_1, z_2, z_3 and z_4 lies in I, II, III and IV quadrants respectively. The principal values of the arguments are given by

$$\tan^{-1}\left(\frac{2}{2}\right),\ \pi - \tan^{-1}\left(\frac{3}{|-3|}\right), -\pi + \tan^{-1}\left(\frac{-4}{-4}\right), -\tan^{-1}\left(\frac{|-5|}{5}\right)$$

or $\tan^{-1} 1, \pi - \tan^{-1} 1, -\pi + \tan^{-1} 1, -\tan^{-1} 1$

or $\frac{\pi}{4}, \pi - \frac{\pi}{4}, -\pi + \frac{\pi}{4}, -\frac{\pi}{4}$ or $\frac{\pi}{4}, \frac{3\pi}{4}, -\frac{3\pi}{4}, -\frac{\pi}{4}$

Hence, the principal values of the arguments of z_1, z_2, z_3 and z_4 are $\frac{\pi}{4}, \frac{3\pi}{4}, -\frac{3\pi}{4}, -\frac{\pi}{4}$, respectively.

Remark

1. Unless otherwise stated, amp z implies principal value of the argument.
2. Argument of the complex number 0 is not defined.
3. If $z_1 = z_2 \Leftrightarrow |z_1| = |z_2|$ and $\arg(z_1) = \arg(z_2)$.
4. If $\arg(z) = \pi/2$ or $-\pi/2$, z is purely imaginary.
5. If $\arg(z) = 0$ or π, z is purely real.

Example 28. Find the argument and the principal value of the argument of the complex number $z = \frac{2+i}{4i + (1+i)^2}$, where $i = \sqrt{-1}$.

Sol. Since, $z = \frac{2+i}{4i + (1+i)^2} = \frac{2+i}{4i + 1 + i^2 + 2i} = \frac{2+i}{6i} = \frac{1}{6} - \frac{1}{3}i$

$\therefore$ z lies in IV quadrant.

Here, $\theta = \tan^{-1}\left|\frac{-\frac{1}{3}}{\frac{1}{6}}\right| = \tan^{-1} 2$

$\therefore$ $\arg(z) = 2\pi - \theta = 2\pi - \tan^{-1} 2$

Hence, principal value of $\arg(z) = -\theta = -\tan^{-1} 2$.

Properties of Modulus

(i) $|z| \geq 0 \Rightarrow |z| = 0$, iff $z = 0$ and $|z| > 0$, iff $z \neq 0$

(ii) $-|z| \leq \text{Re}(z) \leq |z|$ and $-|z| \leq \text{Im}(z) \leq |z|$

(iii) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$

(iv) $z\bar{z} = |z|^2$

(v) $|z_1 z_2| = |z_1||z_2|$

In general, $|z_1 z_2 z_3 \ldots z_n| = |z_1||z_2||z_3| \ldots |z_n|$

(vi) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} (z_2 \neq 0)$

(vii) $|z_1 \pm z_2| \leq |z_1| + |z_2|$

In general, $|z_1 \pm z_2 \pm z_3 \pm \ldots \pm z_n| \leq |z_1| + |z_2| + |z_3| + \ldots + |z_n|$

(viii) $|z_1 \pm z_2| \geq ||z_1| - |z_2||$

(ix) $|z^n| = |z|^n$

(x) $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$

Thus, $|z_1|+|z_2|$ is the greatest possible value of $|z_1+z_2|$ and $||z_1|-|z_2||$ is the least possible value of $|z_1+z_2|$.

(xi) $|z_1 \pm z_2|^2 = (z_1 \pm z_2)(\bar{z}_1 \pm \bar{z}_2) = |z_1|^2 + |z_2|^2 \pm (z_1\bar{z}_2 + \bar{z}_1 z_2)$

or $|z_1|^2 + |z_2|^2 \pm 2\,\mathrm{Re}(z_1\bar{z}_2)$

(xii) $z_1\bar{z}_2 + \bar{z}_1 z_2 = 2|z_1||z_2|\cos(\theta_1 - \theta_2)$, where $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$

(xiii) $|z_1+z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \dfrac{z_1}{z_2}$ is purely imaginary.

(xiv) $|z_1+z_2|^2 + |z_1-z_2|^2 = 2\{|z_1|^2 + |z_2|^2\}$

(xv) $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2+b^2)(|z_1|^2 + |z_2|^2)$, where $a, b \in R$

(xvi) Unimodular i.e., unit modulus

If z is unimodular, then $|z| = 1$. In case of unimodular, let $z = \cos\theta + i\sin\theta, \theta \in R$ and $i = \sqrt{-1}$.

Remark

1. If $f(z)$ is unimodular, then $|f(z)| = 1$ and let $f(z) = \cos\theta + i\sin\theta$, $\theta \in R$ and $i = \sqrt{-1}$.
2. $\dfrac{z}{|z|}$ is always a unimodular complex number, if $z \neq 0$.

(xvii) The multiplicative inverse of a non-zero complex number z is same as its reciprocal and is given by $\dfrac{1}{z} = \dfrac{\bar{z}}{z\bar{z}} = \dfrac{\bar{z}}{|z|^2}$.

Example 29. If $\theta_i \in [0, \pi/6]$, $i = 1, 2, 3, 4, 5$ and $\sin\theta_1 z^4 + \sin\theta_2 z^3 + \sin\theta_3 z^2 + \sin\theta_4 z + \sin\theta_5 = 2$, show that $\dfrac{3}{4} < |z| < 1$.

Sol. Given that,

$\sin\theta_1 z^4 + \sin\theta_2 z^3 + \sin\theta_3 z^2 + \sin\theta_4 z + \sin\theta_5 = 2$

or $2 = |\sin\theta_1 z^4 + \sin\theta_2 z^3 + \sin\theta_3 z^2 + \sin\theta_4 z + \sin\theta_5|$

$2 \le |\sin\theta_1 z^4| + |\sin\theta_2 z^3| + |\sin\theta_3 z^2| + |\sin\theta_4 z| + |\sin\theta_5|$ [by property (vii)]

$\Rightarrow 2 \le |\sin\theta_1||z^4| + |\sin\theta_2||z^3| + |\sin\theta_3||z^2| + |\sin\theta_4||z| + |\sin\theta_5|$ [by property (v)]

$\Rightarrow 2 \le |\sin\theta_1||z|^4 + |\sin\theta_2||z|^3 + |\sin\theta_3||z|^2 + |\sin\theta_4||z| + |\sin\theta_5|$ [by property (ix)] ...(i)

But given, $\theta_i \in [0, \pi/6]$

$\therefore \quad \sin\theta_i \in \left[0, \dfrac{1}{2}\right]$,

i.e. $0 \le \sin\theta_i \le \dfrac{1}{2}$

$\therefore$ Inequality Eq. (i) becomes,

$2 \le \dfrac{1}{2}|z|^4 + \dfrac{1}{2}|z|^3 + \dfrac{1}{2}|z|^2 + \dfrac{1}{2}|z| + \dfrac{1}{2}$

$\Rightarrow \quad 3 \le |z|^4 + |z|^3 + |z|^2 + |z|$

$\Rightarrow \quad 3 \le |z| + |z|^2 + |z|^3 + |z|^4 < |z| + |z|^2 + |z|^3 + |z|^4 + \ldots + \infty$

$\Rightarrow \quad 3 < |z| + |z|^2 + |z|^3 + |z|^4 + \ldots + \infty$

$\Rightarrow \quad 3 < \dfrac{|z|}{1-|z|}$ [here, $|z| < 1$]

$\Rightarrow \quad 3 - 3|z| < |z| \Rightarrow 3 < 4|z|$

$\therefore \quad |z| > \dfrac{3}{4}$

Hence, $\dfrac{3}{4} < |z| < 1$ [$\because |z| < 1$]

Example 30. ***If*** $|z - 2 + i| \le 2$***, find the greatest and least values of*** $|z|$ ***, where*** $i = \sqrt{-1}$.

Sol. Given that, $|z - 2 + i| \le 2$ ...(i)

$\because \quad |z - 2 + i| \ge ||z| - |2 - i||$ [by property (x)]

$\therefore \quad |z - 2 + i| \ge ||z| - \sqrt{5}|$ ...(ii)

From Eqs. (i) and (ii), we get

$||z| - \sqrt{5}| \le |z - 2 + i| \le 2$

$\therefore \quad ||z| - \sqrt{5}| \le 2$

$\Rightarrow \quad -2 \le |z| - \sqrt{5} \le 2$

$\Rightarrow \quad \sqrt{5} - 2 \le |z| \le \sqrt{5} + 2$

Hence, greatest value of $|z|$ is $\sqrt{5} + 2$ and least value of $|z|$ is $\sqrt{5} - 2$.

Example 31. ***If*** z ***is any complex number such that*** $|z + 4| \le 3$***, find the greatest value of*** $|z + 1|$.

Sol. $\because |z + 1| = |(z + 4) - 3|$

$= |(z + 4) + (-3)| \le |z + 4| + |-3|$

$= |z + 4| + 3$

$\le 3 + 3 = 6$ [$\because |z + 4| \le 3$]

$\therefore \quad |z + 1| \le 6$

Hence, the greatest value of $|z + 1|$ is 6.

Example 32. If $|z_1|=1, |z_2|=2, |z_3|=3$ and $|9z_1z_2+4z_3z_1+z_2z_3|=6$, find the value of $|z_1+z_2+z_3|$.

Sol. $\because \quad |z_1|=1 \Rightarrow |z_1|^2=1$

$\Rightarrow \quad z_1\bar{z}_1=1 \Rightarrow \dfrac{1}{z_1}=\bar{z}_1$

$|z_2|=2 \Rightarrow |z_2|^2=4 \Rightarrow z_2\bar{z}_2=4$

$\Rightarrow \quad \dfrac{4}{z_2}=\bar{z}_2$ and $|z_3|=3 \Rightarrow |z_3|^2=9$

$\Rightarrow \quad z_3\bar{z}_3=9 \Rightarrow \dfrac{9}{z_3}=\bar{z}_3$

and given $|9z_1z_2+4z_3z_1+z_2z_3|=6$

$\Rightarrow \quad |z_1z_2z_3|\left|\dfrac{9}{z_3}+\dfrac{4}{z_2}+\dfrac{1}{z_1}\right|=6$

$\Rightarrow \quad |z_1||z_2||z_3||\bar{z}_3+\bar{z}_2+\bar{z}_1|=6$

$\left[\because \dfrac{1}{z_1}=\bar{z}_1, \dfrac{4}{z_2}=\bar{z}_2 \text{ and } \dfrac{9}{z_3}=\bar{z}_3\right]$

$\Rightarrow \quad 1\cdot 2\cdot 3\left|\overline{z_1+z_2+z_3}\right|=6$

$\therefore \quad |z_1+z_2+z_3|=1 \qquad [\because |\bar{z}|=|z|]$

Example 33. ***Prove that***

$$|z_1|+|z_2|=\left|\frac{1}{2}(z_1+z_2)+\sqrt{z_1z_2}\right|+\left|\frac{1}{2}(z_1+z_2)-\sqrt{z_1z_2}\right|.$$

Sol. RHS $=\left|\dfrac{1}{2}(z_1+z_2)+\sqrt{z_1z_2}\right|+\left|\dfrac{1}{2}(z_1+z_2)-\sqrt{z_1z_2}\right|$

$=\left|\dfrac{z_1+z_2+2\sqrt{z_1z_2}}{2}\right|+\left|\dfrac{z_1+z_2-2\sqrt{z_1z_2}}{2}\right|$

$=\dfrac{1}{2}\{|\sqrt{z_1}+\sqrt{z_2}|^2+|\sqrt{z_1}-\sqrt{z_2}|^2\}$

$=\dfrac{1}{2}.2\{|\sqrt{z_1}|^2+|\sqrt{z_2}|^2\}$ [by property (xiv)]

$=|z_1|+|z_2|=$ LHS

Example 34. z_1 and z_2 are two complex numbers, such that $\dfrac{z_1-2z_2}{2-z_1\cdot\overline{z_2}}$ is unimodular, while z_2 is not unimodular. Find $|z_1|$.

Sol. Here, $\left|\dfrac{z_1-2z_2}{2-z_1\bar{z}_2}\right|=1$

$\Rightarrow \quad \left|\dfrac{z_1-2z_2}{2-z_1\bar{z}_2}\right|=1$ [by property (vi)]

$\Rightarrow \quad |z_1-2z_2|=|2-z_1\overline{z_2}|$

$\Rightarrow \quad |z_1-2z_2|^2=|2-z_1\overline{z_2}|^2$

$\Rightarrow \quad (z_1-2z_2)(\overline{z_1-2z_2})=(2-z_1\overline{z_2})(\overline{2-z_1\bar{z}_2})$

[by property (iv)]

$\Rightarrow \quad (z_1-2z_2)(\bar{z}_1-2\bar{z}_2)=(2-z_1\overline{z_2})(2-\bar{z}_1z_2)$

$\Rightarrow \quad z_1\overline{z_1}-2z_1\overline{z_2}-2z_2\overline{z_1}+4z_2\overline{z_2}$

$=4-2\overline{z_1}z_2-2z_1\overline{z_2}+z_1\overline{z_1}z_2\overline{z_2}$

$\Rightarrow \quad |z_1|^2+4|z_2|^2=4+|z_1|^2|z_2|^2$

$\Rightarrow \quad |z_1|^2-|z_1|^2\cdot|z_2|^2+4|z_2|^2-4=0$

$\Rightarrow \quad (|z_1|^2-4)(1-|z_2|^2)=0$

But $|z_2|\neq 1$ [given]

$\therefore \quad |z_1|^2=4$

Hence, $|z_1|=2$

Properties of Arguments

(i) $\arg(z_1z_2)=\arg(z_1)+\arg(z_2)+2k\pi, k\in I$

In general, $\arg(z_1z_2z_3\ldots z_n)$

$=\arg(z_1)+\arg(z_2)+\arg(z_3)+\ldots+\arg(z_n)+2k\pi, k\in I.$

(ii) $\arg\left(\dfrac{z_1}{z_2}\right)=\arg(z_1)-\arg(z_2)+2k\pi, k\in I$

(iii) $\arg\left(\dfrac{z}{\bar{z}}\right)=2\arg(z)+2k\pi, k\in I$

(iv) $\arg(z^n)=n.\arg(z)+2k\pi, k\in I$, where proper value of k must be chosen, so that RHS lies in $(-\pi,\pi]$.

(v) If $\arg\left(\dfrac{z_2}{z_1}\right)=\theta$, then $\arg\left(\dfrac{z_1}{z_2}\right)=2n\pi-\theta$, where $n\in I$.

(vi) $\arg(\bar{z})=-\arg(z)$

Example 35. If $\arg(z_1)=\dfrac{17\pi}{18}$ and $\arg(z_2)=\dfrac{7\pi}{18}$, find the principal argument of z_1z_2 and (z_1/z_2).

Sol. $\arg(z_1z_2)=\arg(z_1)+\arg(z_2)+2k\pi$

$=\dfrac{17\pi}{18}+\dfrac{7\pi}{18}+2k\pi$

$=\dfrac{4\pi}{3}+2k\pi$

$=\dfrac{4\pi}{3}-2\pi=-\dfrac{2\pi}{3}$ [for $k=-1$]

and $\arg\left(\dfrac{z_1}{z_2}\right)=\arg(z_1)-\arg(z_2)+2k\pi$

$=\dfrac{17\pi}{18}-\dfrac{7\pi}{18}+2k\pi=\dfrac{10\pi}{18}+2k\pi$

$=\dfrac{5\pi}{9}+0=\dfrac{5\pi}{9}$ [for $k=0$]

Example 36. If z_1 and z_2 are conjugate to each other, find the principal argument of $(-z_1 z_2)$.

Sol. $\because z_1$ and z_2 are conjugate to each other i.e., $z_2 = \bar{z_1}$, therefore, $z_1 z_2 = z_1 \bar{z_1} = |z_1|^2$

$\therefore \arg(-z_1 z_2) = \arg(-|z_1|^2) = \arg$ [negative real number]

$= \pi$

Example 37. Let z be any non-zero complex number, then find the value of $\arg(z) + \arg(\bar{z})$.

Sol. $\arg(z) + \arg(\bar{z}) = \arg(z\bar{z})$

$= \arg(|z|^2) = \arg$ [positive real number]

$= 0$

(a) Mixed Properties of Modulus and Arguments

(i) $|z_1 + z_2| = |z_1| + |z_2| \Leftrightarrow \arg(z_1) = \arg(z_2)$

(ii) $|z_1 + z_2| = |z_1| - |z_2| \Leftrightarrow \arg(z_1) - \arg(z_2) = \pi$

Proof (i) Let $\arg(z_1) = \theta$ and $\arg(z_2) = \phi$

$\therefore \quad |z_1 + z_2| = |z_1| + |z_2|$

On squaring both sides, we get

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$$

$$\Rightarrow \quad |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \cos(\theta - \phi)$$

$$= |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$$

$\Rightarrow \quad \cos(\theta - \phi) = 1$

$\therefore \quad \theta - \phi = 0$ or $\theta = \phi$

$\therefore \quad \arg(z_1) = \arg(z_2)$

(ii) $\because \quad |z_1 + z_2| = |z_1| - |z_2|$

On squaring both sides, we get

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|$$

$$\Rightarrow \quad |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \cos(\theta - \phi)$$

$$= |z_1|^2 + |z_2|^2 - 2|z_1||z_2|$$

$\Rightarrow \quad \cos(\theta - \phi) = -1$

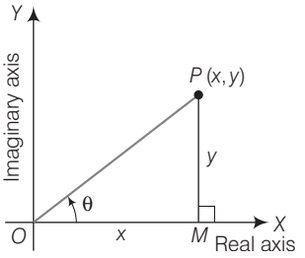
$\therefore \quad \theta - \phi = \pi$ or $\arg(z_1) - \arg(z_2) = \pi$

Remark

1. $|z_1 - z_2| = |z_1| + |z_2| \Leftrightarrow \arg(z_1) = \arg(z_2)$
2. $|z_1 - z_2| = |z_1| - |z_2| \Leftrightarrow \arg(z_1) - \arg(z_2) = \pi$
3. $|z_1 - z_2| = |z_1 + z_2| \Leftrightarrow \arg(z_1) - \arg(z_2) = \pm\frac{\pi}{2}$, $\bar{z}_1 z_2$ and $\frac{z_1}{z_2}$ are purely imaginary.

(b) Trigonometric or Polar or Modulus Argument Form of a Complex Number

Let $z = x + iy$, where $x, y \in R$ and $i = \sqrt{-1}$, z is represented by $P(x, y)$ in the argand plane.



By geometrical representation,

$$OP = \sqrt{(x^2 + y^2)} = |z|$$

$$\angle POM = \theta = \arg(z)$$

In ΔOPM, $x = OP\cos(\angle POM) = |z|\cos(\arg z)$

and $y = OP\sin(\angle POM) = |z|\sin(\arg z)$

$\because \quad z = x + iy$

$\therefore \quad z = |z|(\cos(\arg z) + i\sin(\arg z))$

or $z = r(\cos\theta + i\sin\theta)$

$\bar{z} = r(\cos\theta - i\sin\theta)$

where, $r = |z|$ and $\theta =$ principal value of $\arg(z)$.

Remark

1. $\cos\theta + i\sin\theta$ is also written as $CiS\ \theta$.
2. **Remember**

$1 = \cos 0 + i\sin 0 \quad \Rightarrow \quad -1 = \cos\pi + i\sin\pi$

$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \quad \Rightarrow \quad -i = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$

Example 38. ***Write the polar form of*** $-\frac{1}{2} - \frac{i\sqrt{3}}{2}$ (where, $i = \sqrt{-1}$).

Sol. Let $z = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$. Since, $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ lies in III quadrant.

$\therefore$ Principal value of $\arg(z) = -\pi + \tan^{-1}\left|\frac{-\sqrt{3}/2}{-1/2}\right|$

$= -\pi + \tan^{-1}\sqrt{3} = -\pi + \frac{\pi}{3} = -\frac{2\pi}{3}$

and $|z| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\left(\frac{1}{4} + \frac{3}{4}\right)} = \sqrt{1} = 1$

$\therefore$ Polar form of $z = |z|[\cos(\arg z) + i\sin(\arg z)]$

i.e. $-\frac{1}{2} - \frac{i\sqrt{3}}{2} = \left[\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right]$

(c) Euler's Form

If $\theta \in R$ and $i = \sqrt{-1}$, then $e^{i\theta} = \cos\theta + i\sin\theta$ is known as Euler's identity.

Now, $e^{-i\theta} = \cos\theta - i\sin\theta$

Let $z = e^{i\theta}$

$\therefore \quad |z| = 1$ and $\arg(z) = \theta$

Also, $e^{i\theta} + e^{-i\theta} = 2\cos\theta$ and $e^{i\theta} - e^{-i\theta}\ 2i\sin\theta$

and if $\theta, \phi \in R$ and $i = \sqrt{-1}$, then

(i) $e^{i\theta} + e^{i\phi} = e^{i\left(\frac{\theta+\phi}{2}\right)} \cdot 2\cos\left(\frac{\theta-\phi}{2}\right)$

$\therefore \quad |e^{i\theta} + e^{i\phi}| = 2\left|\cos\left(\frac{\theta-\phi}{2}\right)\right|$

and $\arg(e^{i\theta} + e^{i\phi}) = \left(\frac{\theta+\phi}{2}\right)$

(ii) $e^{i\theta} - e^{i\phi} = e^{i\left(\frac{\theta+\phi}{2}\right)} \cdot 2i\sin\left(\frac{\theta-\phi}{2}\right)$

$\therefore \quad |e^{i\theta} - e^{i\phi}| = 2\left|\sin\left(\frac{\theta-\phi}{2}\right)\right|$

and $\arg(e^{i\theta} - e^{i\phi}) = \frac{\theta+\phi}{2} + \frac{\pi}{2}$ $[\because i = e^{i\pi/2}]$

Remark

1. $e^{i\theta} + 1 = e^{i\theta/2} \cdot 2\cos(\theta/2)$ (Remember)
2. $e^{i\theta} - 1 = e^{i\theta/2} \cdot 2i\sin(\theta/2)$ (Remember)
3. $\frac{e^{i\theta}-1}{e^{i\theta}+1} = i\tan(\theta/2)$ (Remember)
4. If $z = re^{i\theta}$; $|z| = r$, then $\arg(z) = \theta$, $\bar{z} = re^{-i\theta}$
5. If $|z - z_0| = 1$, then $z - z_0 = e^{i\theta}$

Example 39. ***Given that*** $|z - 1| = 1$, ***where*** z ***is a point on the argand plane, show that*** $\frac{z-2}{z} = i\tan(\arg z)$, where $i = \sqrt{-1}$.

Sol. Given, $|z - 1| = 1$

$\therefore \quad z - 1 = e^{i\theta} \Rightarrow z = e^{i\theta} + 1 = e^{i\theta/2} \cdot 2\cos(\theta/2)$

$\therefore \quad \arg(z) = \theta/2$ ...(i)

$$\text{LHS} = \frac{z-2}{z} = \frac{1 + e^{i\theta} - 2}{1 + e^{i\theta}} = \frac{e^{i\theta}-1}{e^{i\theta}+1} = i\tan(\theta/2)$$

$= i\tan(\arg z) = \text{RHS}$ [from Eq. (i)]

Example 40. Let z be a non-real complex number lying on $|z| = 1$, prove that $z = \frac{1 + i\tan\left(\frac{\arg(z)}{2}\right)}{1 - i\tan\left(\frac{\arg(z)}{2}\right)}$ (where, $i = \sqrt{-1}$).

Sol. Given, $|z| = 1$

$\therefore \quad z = e^{i\theta}$ ...(i)

$\Rightarrow \quad \arg(z) = \theta$ ...(ii)

$$\text{RHS} = \frac{1 + i\tan\left(\frac{\arg(z)}{2}\right)}{1 - i\tan\left(\frac{\arg(z)}{2}\right)} = \frac{1 + i\tan(\theta/2)}{1 - i\tan(\theta/2)}$$ [from Eq. (ii)]

$$= \frac{\cos\theta/2 + i\sin\theta/2}{\cos\theta/2 - i\sin\theta/2} = \frac{e^{i\theta/2}}{e^{-i\theta/2}}$$

$= e^{i\theta} = z = \text{LHS}$ [from Eq. (i)]

Example 41. ***Prove that*** $\tan\left(i\ln\left(\frac{a-ib}{a+ib}\right)\right) = \frac{2ab}{a^2 - b^2}$ (where $a, b \in R^+$ and $i = \sqrt{-1}$).

Sol. $\because \quad \left|\frac{a-ib}{a+ib}\right| = \frac{|a-ib|}{|a+ib|} = 1$ $[\because |\bar{z}| = |z|]$

Let $\frac{a-ib}{a+ib} = e^{i\theta}$ ...(i)

By componendo and dividendo , we get

$$\frac{(a-ib)-(a+ib)}{(a-ib)+(a+ib)} = \frac{e^{i\theta}-1}{e^{i\theta}+1} - \frac{b}{a}i = i\tan(\theta/2)$$

or $\tan\left(\frac{\theta}{2}\right) = -\frac{b}{a}$ ...(ii)

$\therefore \quad \text{LHS} = \tan\left(i\ln\left(\frac{a-ib}{a+ib}\right)\right)$

$= \tan(i\ln(e^{i\theta}))$ [from Eq. (i)]

$= \tan(i \cdot i\theta) = -\tan\theta$

$$= -\frac{2\tan\theta/2}{1 - \tan^2\theta/2}$$

$$= -\frac{2(-b/a)}{1 - (-b/a)^2}$$ [from Eq. (ii)]

$$= \frac{2ab}{a^2 - b^2} = \text{RHS}$$

Applications of Euler's Form

If $x, y, \theta \in R$ and $i = \sqrt{-1}$, then

let $z = x + iy$ [cartesian form]

$= |z|(\cos\theta + i\sin\theta)$ [polar form]

$= |z|e^{i\theta}$ [Euler's form]

(i) Product of Two Complex Numbers

Let two complex numbers be

$z_1 = |z_1|e^{i\theta_1}$ and $z_2 = |z_2|e^{i\theta_2}$,

where $\theta_1, \theta_2 \in R$ and $i = \sqrt{-1}$

$$\therefore \quad z_1 \cdot z_2 = |z_1| e^{i\theta_1} \cdot |z_2| e^{i\theta_2} = |z_1||z_2| e^{i(\theta_1+\theta_2)}$$
$$= |z_1||z_2|(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$$

Thus, $|z_1 z_2| = |z_1||z_2|$

and $\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$

(ii) Division of Two Complex Numbers

Let two complex numbers be

$$z_1 = |z_1| e^{i\theta_1} \quad \text{and} \quad z_2 = |z_2| e^{i\theta_2},$$

where $\theta_1, \theta_2 \in R$ and $i = \sqrt{-1}$

$$\therefore \quad \frac{z_1}{z_2} = \frac{|z_1| e^{i\theta_1}}{|z_2| e^{i\theta_2}} = \frac{|z_1|}{|z_2|} e^{i(\theta_1-\theta_2)}$$
$$= \frac{|z_1|}{|z_2|}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2))$$

Thus, $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, (z_2 \neq 0)$

and $\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2)$

(iii) Logarithm of a Complex Number

$$\log_e(z) = \log_e(|z| e^{i\theta}) = \log_e|z| + \log_e(e^{i\theta})$$
$$= \log_e|z| + i\theta = \log_e|z| + i\arg(z)$$

So, the general value of $\log_e(z)$

$$= \log_e(z) + 2n\pi i \, (-\pi < \arg z < \pi).$$

Example 42. If m and x are two real numbers and $i = \sqrt{-1}$, prove that $e^{2mi\cot^{-1}x}\left(\frac{xi+1}{xi-1}\right)^m = 1$.

Sol. Let $\cot^{-1}x = \theta$, then $\cot\theta = x$

$$\therefore \text{LHS} = e^{2mi\cot^{-1}x}\left(\frac{xi+1}{xi-1}\right)^m = e^{2mi\theta}\left(\frac{i\cot\theta+1}{i\cot\theta-1}\right)^m$$
$$= e^{2mi\theta}\left(\frac{i(\cot\theta - i)}{i(\cot\theta + i)}\right)^m = e^{2mi\theta}\left(\frac{\cos\theta - i\sin\theta}{\cos\theta + i\sin\theta}\right)^m$$
$$= e^{2mi\theta}\cdot\left(\frac{e^{-i\theta}}{e^{i\theta}}\right)^m = e^{2mi\theta}\cdot(e^{-2i\theta})^m$$
$$= e^{2mi\theta}\cdot e^{-2mi\theta} = e^0 = 1 = \text{RHS}$$

Example 43. If z and w are two non-zero complex numbers such that $|z| = |w|$ and $\arg(z) + \arg(w) = \pi$, prove that $z = -\overline{w}$.

Sol. Let $\arg(w) = \theta$, then $\arg(z) = \pi - \theta$

$$\therefore \quad z = |z|(\cos(\arg z) + i\sin(\arg z))$$
$$= |z|(\cos(\pi-\theta) + i\sin(\pi-\theta))$$
$$= |z|(-\cos\theta + i\sin\theta) = -|z|(\cos\theta - i\sin\theta)$$
$$= -|w|(\cos(\arg w) - i\sin(\arg w))$$
$$= -|w|(\cos(-\arg w) + i\sin(-\arg w))$$
$$= -|\overline{w}|(\cos(\arg\overline{w}) + i\sin(\arg\overline{w})) = -\overline{w}$$

Example 44. Express $(1+i)^{-i}$, (where, $i = \sqrt{-1}$) in the form $A + iB$.

Sol. Let $A + iB = (1+i)^{-i}$

On taking logarithm both sides, we get

$$\log_e(A+iB) = -i\log_e(1+i)$$
$$= -i\log_e\left(\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\right)$$
$$= -i\log_e\left(\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)$$
$$= -i\log_e(\sqrt{2}\, e^{i\pi/4}) = -i(\log_e\sqrt{2} + \log_e e^{i\pi/4})$$
$$= -i\left(\frac{1}{2}\log_e 2 + \frac{i\pi}{4}\right) = -\frac{i}{2}\log_e 2 + \frac{\pi}{4}$$

$$\therefore A + iB = e^{-\frac{i}{2}\log_e 2 + \frac{\pi}{4}} = e^{\pi/4}\cdot e^{i\log_e 2^{-1/2}}$$
$$= e^{\pi/4}\cdot(\cos(\log_e 2^{-1/2}) + i\sin(\log_e 2^{-1/2}))$$
$$= e^{\pi/4}\cdot\cos\left(\log_e\left(\frac{1}{\sqrt{2}}\right)\right) + i\, e^{\pi/4}\sin\left(\log_e\left(\frac{1}{\sqrt{2}}\right)\right)$$

Example 45. If $\sin(\log_e i^i) = a + ib$, where $i = \sqrt{-1}$, find a and b, hence and find $\cos(\log_e i^i)$.

Sol. $a + ib = \sin(\log_e i^i) = \sin(i\log_e i)$
$$= \sin(i(\log_e|i| + i\arg i))$$
$$= \sin(i(\log_e 1 + (i\pi/2)))$$
$$= \sin(i(0 + (i\pi/2))) = \sin(-\pi/2) = -1$$

$\therefore \quad a = -1, b = 0$

Now, $\cos(\log_e i^i) = \sqrt{1 - \sin^2(\log_e i^i)}$
$$= \sqrt{1-(-1)^2} = \sqrt{(1-1)} = 0$$

Aliter

$\because \quad i^i = (e^{i\pi/2})^i = e^{-\pi/2}$

$\therefore \quad \sin(\log_e i^i) = \sin(\log_e e^{-\pi/2}) = \sin\left(-\frac{\pi}{2}\log_e e\right)$
$$= \sin(-\pi/2) = -1 = a + ib \qquad \text{[given]}$$

$\therefore \quad a = -1, b = 0$

and $\cos(\log_e i^i) = \cos(\log_e e^{-\pi/2})$
$$= \cos\left(-\frac{\pi}{2}\log_e e\right) = \cos\left(-\frac{\pi}{2}\right) = 0$$

Example 46. ***Find the general value of*** $\log_2(5i)$, ***where*** $i = \sqrt{-1}$.

Sol. $\log_2 5i = \frac{\log_e 5i}{\log_e 2} = \frac{1}{\log_e 2}\{\log_e|5i| + i\arg(5i) + 2n\pi i\}$

$$= \frac{1}{\log_e 2}\left\{\log_e 5 + \frac{i\pi}{2} + 2n\pi i\right\}, n \in I$$

Exercise for Session 2

1 If $\frac{1-ix}{1+ix} = a - ib$ and $a^2 + b^2 = 1$, where $a, b \in R$ and $i = \sqrt{-1}$, then x is equal to

(a) $\frac{2a}{(1+a)^2 + b^2}$ (b) $\frac{2b}{(1+a)^2 + b^2}$ (c) $\frac{2a}{(1+b)^2 + a^2}$ (d) $\frac{2b}{(1+b)^2 + a^2}$

2 The least positive integer n for which $\left(\frac{1+i}{1-i}\right)^n = \frac{2}{\pi}\left(\sec^{-1}\frac{1}{x} + \sin^{-1}x\right)(\text{where}, x \neq 0, -1 \leq x \leq 1 \text{and } i = \sqrt{-1})$, is

(a) 2 (b) 4 (c) 6 (d) 8

3 If $z = (3 + 4i)^6 + (3 - 4i)^6$, where $i = \sqrt{-1}$, then $\text{Im}(z)$ equals to

(a) -6 (b) 0 (c) 6 (d) None of these

4 If $(x + iy)^{1/3} = a + ib$, where $i = \sqrt{-1}$, then $\left(\frac{x}{a} + \frac{y}{b}\right)$ is equal to

(a) $4a^2b^2$ (b) $4(a^2 - b^2)$ (c) $4a^2 - b^2$ (d) $a^2 + b^2$

5 If $\frac{3}{2 + \cos\theta + i\sin\theta} = a + ib$, where $i = \sqrt{-1}$ and $a^2 + b^2 = \lambda a - 3$, the value of λ is

(a) 3 (b) 4 (c) 5 (d) 6

6 If $\frac{z-1}{z+1}$ is purely imaginary, then $|z|$ is equal to

(a) $\frac{1}{2}$ (b) 1 (c) $\sqrt{2}$ (d) 2

7 The complex numbers $\sin x + i\cos 2x$ and $\cos x - i\sin 2x$, where $i = \sqrt{-1}$, are conjugate to each other, for

(a) $x = n\pi, n \in I$ (b) $x = 0$ (c) $x = \left(n + \frac{1}{2}\right), n \in I$ (d) no value of x

8 If α and β are two different complex numbers with $|\beta| = 1$, then $\left|\frac{\beta - \alpha}{1 - \bar{\alpha}\beta}\right|$ is equal to

(a) 0 (b) $\frac{1}{2}$ (c) 1 (d) 2

9 If $x = 3 + 4i\,(\text{where}, i = \sqrt{-1})$, the value of $x^4 - 12x^3 + 70x^2 - 204x + 225$, is

(a) -45 (b) 0 (c) 35 (d) 15

10 If $|z_1 - 1| \leq 1, |z_2 - 2| \leq 2, |z_3 - 3| \leq 3$, the greatest value of $|z_1 + z_2 + z_3|$ is

(a) 6 (b) 12 (c) 17 (d) 23

11 The principal value of $\arg(z)$, where $z = \left(1 + \cos\frac{8\pi}{5}\right) + i\sin\frac{8\pi}{5}(\text{where}, i = \sqrt{-1})$ is given by

(a) $-\frac{\pi}{5}$ (b) $-\frac{4\pi}{5}$ (c) $\frac{\pi}{5}$ (d) $\frac{4\pi}{5}$

12 If $|z_1| = 2, |z_2| = 3, |z_3| = 4$ and $|z_1 + z_2 + z_3| = 5$, then $|4z_2z_3 + 9z_3z_1 + 16z_1z_2|$ is

(a) 24 (b) 60 (c) 120 (d) 240

13 If $|z - i| \leq 5$ and $z_1 = 5 + 3i\,(\text{where}, i = \sqrt{-1})$, the greatest and least values of $|iz + z_1|$ are

(a) 7 and 3 (b) 9 and 1 (c) 10 and 0 (d) None of these

14 If z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers, then $\arg\left(\frac{z_1}{z_4}\right) + \arg\left(\frac{z_2}{z_3}\right)$ equals to

(a) 0 (b) $\frac{\pi}{2}$ (c) π (d) $\frac{3\pi}{2}$

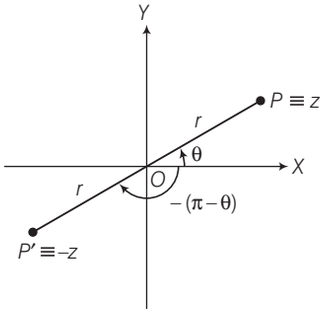
Session 3

amp(z) − amp ($-z$) = ± π; According as amp (z) is Positive or Negative, Square Root of a Complex Number, Solution of Complex Equations, De-Moivre's Theorem, Cube Roots of Unity

amp (z) − amp $(-z)$ = ± π, According as amp (z) is Positive or Negative

***Case* I amp (z) is positive.**

If amp $(z) = \theta$, we have



$$\text{amp}(-z) = -(\angle P'OX) = -(\pi - \theta)$$

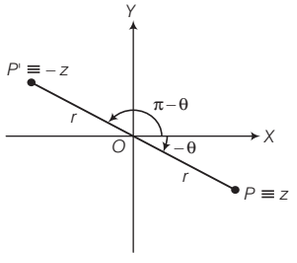
$\therefore$ **amp (z) − amp ($-z$) = π** [here, $OP = OP'$]

***Case* II amp (z) is negative.**

If $\text{amp}(z) = -\theta$

We have, $\text{amp}(-z) = \angle P'OX = \pi - \theta$

$\therefore$ **amp (z) − amp ($-z$) = − π** [here, $OP = OP'$]



Example 47. If $|z_1| = |z_2|$ and $\arg(z_1/z_2) = \pi$, then find the value of $z_1 + z_2$.

Sol. $\because \arg\left(\dfrac{z_1}{z_2}\right) = \pi$

$\Rightarrow$ $\arg(z_1) - \arg(z_2) = \pi$ ...(i)

$\because$ $z_1 = |z_1|(\cos(\arg z_1) + i \sin(\arg z_1))$ ...(ii)

and $z_2 = |z_2|(\cos(\arg z_2) + i \sin(\arg z_2))$ ...(iii)

From Eq. (ii), we get

$$z_1 = |z_2|(\cos(\pi + \arg(z_2)) + i \sin(\pi + \arg(z_2)))$$

[from Eq. (i) and $|z_1| = |z_2|$]

$$= |z_2|(-\cos(\arg z_2) - i \sin(\arg z_2)) = -z_2$$

[from Eq. (iii)]

$\therefore z_1 + z_2 = 0$

Example 48. Let z and w be two non-zero complex numbers, such that $|z| = |w|$ and $\text{amp}(z) + \text{amp}(w) = \pi$, then find the relation between z and w.

Sol. Given, $\text{amp}(z) + \text{amp}(w) = \pi$

$\Rightarrow$ $\text{amp}(z) - \text{amp}(\overline{w}) = \pi$

Here, $|z| = |\overline{w}| = |w|$ [given $|z| = |w|$]

and $\text{amp}(z) > 0$

Then, $z + \overline{w} = 0$

Square Root of a Complex Number

Let $z = x + iy$,

where $x, y \in R$ and $i = \sqrt{-1}$.

Suppose $\sqrt{(x+iy)} = a + ib$ ...(i)

On squaring both sides, we get

$$(x + iy) = (a^2 - b^2) + 2iab$$

On comparing the real and imaginary parts, we get

$$a^2 - b^2 = x \quad \text{...(ii)}$$

and $2ab = y$ ...(iii)

$\therefore$ $a^2 + b^2 = \sqrt{(a^2 - b^2)^2 + 4a^2b^2} = \sqrt{(x^2 + y^2)}$

$$a^2 + b^2 = |z| \quad \text{...(iv)}$$

From Eqs. (ii) and (iv), we get

$$a = \pm\sqrt{\left(\frac{|z| + x}{2}\right)}, \; b = \pm\sqrt{\left(\frac{|z| - x}{2}\right)}$$

or $$a = \pm\sqrt{\left(\frac{|z| + \text{Re}(z)}{2}\right)}, \; b = \pm\sqrt{\left(\frac{|z| - \text{Re}(z)}{2}\right)}$$

Now, from Eq. (i), the required square roots,

$$\text{i.e. } \sqrt{z} = \begin{cases} \pm\left(\sqrt{\dfrac{|z|+\text{Re}(z)}{2}} + i\sqrt{\dfrac{|z|-\text{Re}(z)}{2}}\right), \textbf{ if Im } (z) > 0 \\ \pm\left(\sqrt{\dfrac{|z|+\text{Re}(z)}{2}} - i\sqrt{\dfrac{|z|-\text{Re}(z)}{2}}\right), \textbf{ if Im } (z) < 0 \end{cases}$$

Aliter

If $\sqrt{(x+iy)}$, where $x, y \in R$ and $i = \sqrt{-1}$, then

(i) If y is not even, then multiply and divide in y by 2, then $\sqrt{(x+iy)}$ convert in

$$\sqrt{x+y\sqrt{-1}} = \sqrt{\left(x+2\sqrt{-\frac{y^2}{4}}\right)}.$$

(ii) Factorise: $-\dfrac{y^2}{4}$ say $\alpha, \beta\ (\alpha < \beta)$.

Take that possible factor which satisfy

$$x = (\alpha i)^2 + \beta^2, \text{ if } x > 0 \text{ or } x = \alpha^2 + (i\beta)^2, \text{ if } x < 0$$

(iii) Finally, write $x + iy = (\alpha i)^2 + \beta^2 + 2i\alpha\beta$

or $\alpha^2 + (i\beta)^2 + 2i\alpha\beta$

and take their square root.

$$\text{(iv) } \sqrt{(x+iy)} = \begin{cases} \pm(\alpha i + \beta) \\ \text{or } \pm(\alpha + i\beta) \end{cases} \text{ and } \sqrt{(x-iy)} = \begin{cases} \pm(\beta - i\alpha) \\ \text{or } \pm(\alpha - i\beta) \end{cases}$$

Remark

1. The square root of i is $\pm\left(\dfrac{1+i}{\sqrt{2}}\right)$, where $i = \sqrt{-1}$.
2. The square root of $(-i)$ is $\left(\dfrac{1-i}{\sqrt{2}}\right)$.

Example 49. Find the square roots of the following

(i) $4 + 3i$ (ii) $-5 + 12i$

(iii) $-8 - 15i$ (iv) $7 - 24i$ (where, $i = \sqrt{-1}$)

Sol. (i) Let $z = 4 + 3i$

$\therefore \quad |z| = 5, \text{Re}(z) = 4, \text{Im}(z) = 3 > 0$

$$\because \quad \sqrt{z} = \pm\left(\sqrt{\frac{|z|+\text{Re}(z)}{2}} + i\sqrt{\frac{|z|-\text{Re}(z)}{2}}\right)$$

$$\therefore \sqrt{(4+3i)} = \pm\left(\sqrt{\left(\frac{5+4}{2}\right)} + i\sqrt{\left(\frac{5-4}{2}\right)}\right) = \pm\left(\frac{3+i}{\sqrt{2}}\right)$$

Aliter

$$\sqrt{(4+3i)} = \sqrt{4+3\sqrt{-1}} = \sqrt{4+2\sqrt{\left(-\frac{9}{4}\right)}}$$

$$= \sqrt{\frac{9}{2} - \frac{1}{2} + 2\sqrt{\left(-\frac{9}{4}\right)}}$$

$$= \sqrt{\left\{\left(\frac{3}{\sqrt{2}}\right)^2 + \left(\frac{i}{\sqrt{2}}\right)^2 + 2\cdot\frac{3}{\sqrt{2}}\cdot\frac{i}{\sqrt{2}}\right\}}$$

$$= \sqrt{\left(\frac{3+i}{\sqrt{2}}\right)^2} = \pm\left(\frac{3+i}{\sqrt{2}}\right)$$

(ii) Let $z = -5 + 12i$

$\therefore \quad |z| = 13, \text{Re}(z) = -5, \text{Im}(z) = 12 > 0$

$$\because \quad \sqrt{z} = \pm\left(\sqrt{\frac{|z|+\text{Re}(z)}{2}} + i\sqrt{\frac{|z|-\text{Re}(z)}{2}}\right)$$

$$\therefore \sqrt{(-5+12i)} = \pm\left(\sqrt{\left(\frac{13-5}{2}\right)} + i\sqrt{\left(\frac{13+5}{2}\right)}\right)$$

$$= \pm(2+3i)$$

Aliter

$$\sqrt{(-5+12i)} = \sqrt{(-5+12\sqrt{-1})}$$

$$= \sqrt{(-5+2\sqrt{(-36)})}$$

$$= \sqrt{(-5+2\sqrt{(-9\times 4)})}$$

$$= \sqrt{(-9+4+2\sqrt{(-9\times 4)})}$$

$$= \sqrt{(3i)^2 + 2^2 + 2\cdot 3i\cdot 2}$$

$$= \sqrt{(2+3i)^2} = \pm(2+3i)$$

(iii) Let $z = -8 - 15i$

$\therefore \quad |z| = 17, \text{Re}(z) = -8, \text{Im}(z) = -15 < 0$

$$\therefore \sqrt{(-8-15i)} = \pm\left(\sqrt{\left(\frac{17-8}{2}\right)} - i\sqrt{\left(\frac{17+8}{2}\right)}\right)$$

$$= \pm\left(\frac{3-5i}{\sqrt{2}}\right)$$

Aliter $\sqrt{(-8-15i)} = \sqrt{(-8-15\sqrt{-1})}$

$$= \sqrt{\left(-8-2\sqrt{\left(-\frac{225}{4}\right)}\right)} = \sqrt{\left(-8-2\sqrt{\left(-\frac{25}{2}\times\frac{9}{2}\right)}\right)}$$

$$= \sqrt{\left(\frac{9}{2} - \frac{25}{2} - 2\sqrt{\left(-\frac{25}{2}\times\frac{9}{2}\right)}\right)}$$

$$= \sqrt{\left(\frac{3}{\sqrt{2}}\right)^2 + \left(\frac{5i}{\sqrt{2}}\right)^2 - 2\cdot\frac{3}{\sqrt{2}}\cdot\frac{5i}{\sqrt{2}}}$$

$$= \sqrt{\left(\frac{3-5i}{\sqrt{2}}\right)^2} = \pm\left(\frac{3-5i}{\sqrt{2}}\right)$$

(iv) Let $z = 7 - 24i$

$\therefore \left| z \right| = 25, \text{Re}(z) = 7, \text{Im}(z) = -24 < 0$

$\because \sqrt{z} = \pm\left(\sqrt{\frac{|z| + \text{Re}(z)}{2}} - i\sqrt{\frac{|z| - \text{Re}(z)}{2}}\right)$

$\therefore \sqrt{(7-24i)} = \pm\left(\sqrt{\left(\frac{25+7}{2}\right)} - i\sqrt{\left(\frac{25-7}{2}\right)}\right)$

$= \pm(4 - 3i)$

Aliter

$\sqrt{(7-24i)} = \sqrt{(7-24\sqrt{-1})} = \sqrt{7 - 2\sqrt{(-144)}}$

$= \sqrt{7 - 2\sqrt{(16 \times -9)}}$

$= \sqrt{16 - 9 - 2\sqrt{(16 \times -9)}}$

$= \sqrt{(4)^2 + (3i)^2 - 2 \cdot 4 \cdot 3i}$

$= \sqrt{(4-3i)^2} = \pm(4-3i)$

Example 50. ***Find the square root of*** $x + \sqrt{(-x^4 - x^2 - 1)}$.

Sol. Let $z = x + \sqrt{(-x^4 - x^2 - 1)}$

$= x + i\sqrt{(x^4 + x^2 + 1)}$ $[\because \sqrt{-1} = i]$

$\therefore \left| z \right| = \sqrt{x^2 + (x^4 + x^2 + 1)}$

$= \sqrt{(x^4 + 2x^2 + 1)} = \sqrt{(x^2+1)^2}$

$\therefore \left| z \right| = (x^2 + 1)$

$\text{Re}(z) = x$

$\text{Im}(z) = \sqrt{(x^4 + x^2 + 1)} > 0$

$\because \sqrt{z} = \pm\left(\sqrt{\frac{|z| + \text{Re}(z)}{2}} + i\sqrt{\frac{|z| - \text{Re}(z)}{2}}\right)$

$\therefore \sqrt{x + \sqrt{(-x^4 - x^2 - 1)}}$

$= \pm\left(\sqrt{\left(\frac{x^2 + 1 + x}{2}\right)} + i\sqrt{\left(\frac{x^2 + 1 - x}{2}\right)}\right)$

Aliter

$\sqrt{x + \sqrt{(-x^4 - x^2 - 1)}} = \sqrt{x + 2\sqrt{\left(\frac{-x^4 - x^2 - 1}{4}\right)}}$

$= \sqrt{x + 2\sqrt{\left(\frac{-(x^2+x+1)(x^2-x+1)}{4}\right)}}$

$= \sqrt{x + 2\sqrt{\left[\left(\frac{x^2+x+1}{2}\right) \times -\left(\frac{x^2-x+1}{2}\right)\right]}}$

$= \sqrt{\left(\frac{x^2+x+1}{2}\right) - \left(\frac{x^2-x+1}{2}\right) + 2\sqrt{\left[\left(\frac{x^2+x+1}{2}\right) \times -\left(\frac{x^2-x+1}{2}\right)\right]}}$

$= \sqrt{\left\{\left(\sqrt{\left(\frac{x^2+x+1}{2}\right)}\right)^2 + \left(i\sqrt{\left(\frac{x^2-x+1}{2}\right)}\right)^2 + 2\sqrt{\left(\frac{x^2+x+1}{2}\right)} \cdot i\sqrt{\left(\frac{x^2-x+1}{2}\right)}\right\}}$

$= \sqrt{\left(\sqrt{\left(\frac{x^2+x+1}{2}\right)} + i\sqrt{\left(\frac{x^2-x+1}{2}\right)}\right)^2}$

$= \pm\left(\sqrt{\left(\frac{x^2+x+1}{2}\right)} + i\sqrt{\left(\frac{x^2-x+1}{2}\right)}\right)$

Solution of Complex Equations

Putting $z = x + iy$, where $x, y \in R$ and $i = \sqrt{-1}$ in the given equation and equating the real and imaginary parts, we get x and y, then required solution is $z = x + iy$.

Example 51. Solve the equation $z^2 + \left| z \right| = 0$.

Sol. Let $z = x + iy$, where $x, y \in R$ and $i = \sqrt{-1}$ ...(i)

$\Rightarrow z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$

and $\left| z \right| = \sqrt{(x^2 + y^2)}$

Then, given equation reduces to

$x^2 - y^2 + 2ixy + \sqrt{(x^2 + y^2)} = 0$

On comparing the real and imaginary parts, we get

$x^2 - y^2 + \sqrt{(x^2 + y^2)} = 0$ ...(ii)

and $2xy = 0$ ...(iii)

From Eq. (iii), let $x = 0$ and from Eq. (ii),

$-y^2 + \sqrt{y^2} = 0$

$\Rightarrow -\left| y \right|^2 + \left| y \right| = 0$

$\therefore \left| y \right| = 0, 1$

$\Rightarrow y = 0, \pm 1$

From Eq. (iii), let $y = 0$ and from Eq. (ii),

$x^2 + \sqrt{x^2} = 0$

$\Rightarrow x^2 + \left| x \right| = 0$

$\Rightarrow \left| x \right|^2 + \left| x \right| = 0 \Rightarrow x = 0$

$\therefore$ $x + iy$ are $0 + 0 \cdot i$, $0 + i$, $0 - i$

i.e. $z = 0, i, -i$ are the solutions of the given equation.

Example 52. Find the number of solutions of the equation $z^2 + |z|^2 = 0$.

Sol. $\because \quad z^2 + |z|^2 = 0 \quad \text{or} \quad z^2 + z\bar{z} = 0$

$\Rightarrow \quad z(z + \bar{z}) = 0$

$\therefore \quad z = 0 \quad \ldots(i)$

and $\quad z + \bar{z} = 0 \Rightarrow 2\text{Re}(z) = 0$

$\therefore \quad \text{Re}(z) = 0$

If $\quad z = x + iy \quad [\because x = \text{Re}(z)]$

$= 0 + iy, y \in R$

and $\quad i = \sqrt{-1} \quad \ldots(ii)$

On combining Eqs. (i) and (ii), then we can say that the given equation has infinite solutions.

Example 53. Find all complex numbers satisfying the equation $2|z|^2 + z^2 - 5 + i\sqrt{3} = 0$, where $i = \sqrt{-1}$.

Sol. Let $z = x + iy$, where $x, y \in R$ and $i = \sqrt{-1}$

$\Rightarrow \quad z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$

and $\quad |z| = \sqrt{(x^2 + y^2)}$

Then, given equation reduces to

$2(x^2 + y^2) + x^2 - y^2 + 2ixy - 5 + i\sqrt{3} = 0$

$\Rightarrow \quad (3x^2 + y^2 - 5) + i(2xy + \sqrt{3}) = 0 = 0 + i \cdot 0$

On comparing the real and imaginary parts, we get

$3x^2 + y^2 - 5 = 0 \quad \ldots(i)$

and $\quad 2xy + \sqrt{3} = 0 \quad \ldots(ii)$

On substituting the value of x from Eq. (ii) in Eq. (i), we get

$$3\left(-\frac{\sqrt{3}}{2y}\right)^2 + y^2 - 5 = 0$$

$$\Rightarrow \quad \frac{9}{4y^2} + y^2 = 5$$

or $\quad 4y^4 - 20y^2 + 9 = 0$

$\Rightarrow \quad (2y^2 - 9)(2y^2 - 1) = 0$

$\therefore \quad y^2 = \frac{9}{2}, y^2 = \frac{1}{2} \text{ or } y = \pm\frac{3}{\sqrt{2}}, y = \pm\frac{1}{\sqrt{2}}$

or $\quad y = -\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

From Eq. (ii), we get

$$x = \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2}}$$

$\therefore \quad z = x + iy$

$$= \frac{1}{\sqrt{6}} - \frac{3i}{\sqrt{2}}, -\frac{1}{\sqrt{6}} + \frac{3i}{\sqrt{2}}, \sqrt{\frac{3}{2}} - \frac{i}{\sqrt{2}}, -\sqrt{\frac{3}{2}} + \frac{i}{\sqrt{2}}$$

are the solutions of the given equation.

De-Moivre's Theorem

Statements

(i) If $\theta_1, \theta_2, \theta_3, \ldots, \theta_n \in R$ and $i = \sqrt{-1}$, then

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)\ldots(\cos\theta_n + i\sin\theta_n)$$
$$= \cos(\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n)$$

(ii) If $\theta \in R, n \in I$ (set of integers) and $i = \sqrt{-1}$, then

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

(iii) If $\theta \in R$, $n \in Q$ (set of rational numbers) and $i = \sqrt{-1}$, then $\cos n\theta + i\sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$.

Proof

(i) By Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$

$$\text{LHS} = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)\ldots(\cos\theta_n + i\sin\theta_n)$$
$$= e^{i\theta_1} \cdot e^{i\theta_2} \cdot e^{i\theta_3} \ldots e^{i\theta_n} = e^{i(\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n)}$$
$$= \cos(\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n) = \text{RHS}$$

(ii) If $\theta_1 = \theta_2 = \theta_3 = \ldots = \theta_n = \theta$, then from the above result (i), $(\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)\ldots$ upto n factors

$$= \cos(\theta + \theta + \theta + \ldots \text{ upto } n \text{ times}) + i\sin(\theta + \theta + \theta + \ldots \text{ upto } n \text{ times})$$

i.e., $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

(iii) Let $n = \frac{p}{q}$, where $p, q \in I$ and $q \neq 0$, from above result (ii),

we have $\left(\cos\left(\frac{p}{q}\theta\right) + i\sin\left(\frac{p}{q}\theta\right)\right)^q$

$$= \cos\left(\left(\frac{p}{q}\theta\right)q\right) + i\sin\left(\left(\frac{p}{q}\theta\right)q\right) = \cos p\theta + i\sin p\theta$$

$\Rightarrow \cos\left(\frac{p\theta}{q}\right) + i\sin\left(\frac{p\theta}{q}\right)$ is one of the values of $(\cos p\theta + i\sin p\theta)^{1/q}$

$\Rightarrow \cos\left(\frac{p\theta}{q}\right) + i\sin\left(\frac{p\theta}{q}\right)$ is one of the values of $[(\cos\theta + i\sin\theta)^p]^{1/q}$

$\Rightarrow \cos\left(\frac{p\theta}{q}\right) + i\sin\left(\frac{p\theta}{q}\right)$ is one of the values of $(\cos\theta + i\sin\theta)^{p/q}$

Other Forms of De-Moivre's Theorem

1. $(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta, \forall\ n \in I$

   Proof $(\cos\theta - i\sin\theta)^n = (\cos(-\theta) + i\sin(-\theta))^n$
   $= \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta$

2. $(\sin\theta + i\cos\theta)^n = (i)^n(\cos n\theta - i\sin n\theta), \forall\ n \in I$

   Proof $(\sin\theta + i\cos\theta)^n = (i(\cos\theta - i\sin\theta))^n$
   $= i^n(\cos\theta - i\sin\theta)^n = (i)^n(\cos n\theta - i\sin n\theta)$
   [from remark (1)]

3. $(\sin\theta - i\cos\theta)^n = (-i)^n(\cos n\theta + i\sin n\theta), \forall\ n \in I$

   Proof $(\sin\theta - i\cos\theta)^n = (-i(\cos\theta + i\sin\theta))^n$
   $= (-i)^n(\cos\theta + i\sin\theta)^n$
   $= (-i)^n(\cos n\theta + i\sin n\theta)$

4. $(\cos\theta + i\sin\phi)^n \neq \cos n\theta + i\sin n\phi, \forall\ n \in I$
   [here, $\theta \neq \phi$. ∴ De-Moivre's theorem is not applicable]

5. $\frac{1}{\cos\theta + i\sin\theta} = (\cos\theta + i\sin\theta)^{-1}$
   $= \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$

Example 54. If $z_r = \cos\left(\frac{\pi}{3^r}\right) + i\sin\left(\frac{\pi}{3^r}\right)$, where $i = \sqrt{-1}$, prove that $z_1 z_2 z_3 \ldots$ upto infinity $= i$.

Sol. We have, $z_r = \cos\left(\frac{\pi}{3^r}\right) + i\sin\left(\frac{\pi}{3^r}\right)$

$$\therefore z_1 z_2 z_3 \ldots\infty = \cos\left(\frac{\pi}{3} + \frac{\pi}{3^2} + \frac{\pi}{3^3} + \ldots + \infty\right) + i\sin\left(\frac{\pi}{3} + \frac{\pi}{3^2} + \frac{\pi}{3^3} + \ldots + \infty\right)$$

$$= \cos\left(\frac{\frac{\pi}{3}}{1 - \frac{1}{3}}\right) + i\sin\left(\frac{\frac{\pi}{3}}{1 - \frac{1}{3}}\right) = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$$

$$= 0 + i \cdot 1 = i$$

Example 55. Express $\frac{(\cos\theta + i\sin\theta)^4}{(\sin\theta + i\cos\theta)^5}$ in $a + ib$ form, where $i = \sqrt{-1}$.

Sol. $\because\ (\sin\theta + i\cos\theta)^5 = (i)^5(\cos\theta - i\sin\theta)^5$
$= i(\cos\theta + i\sin\theta)^{-5}$

$$\therefore \frac{(\cos\theta + i\sin\theta)^4}{(\sin\theta + i\cos\theta)^5} = \frac{(\cos\theta + i\sin\theta)^4}{i(\cos\theta + i\sin\theta)^{-5}} = \frac{(\cos\theta + i\sin\theta)^9}{i}$$

$$= \frac{\cos 9\theta + i\sin 9\theta}{i} = -i\cos 9\theta + \sin 9\theta$$

$$= \sin 9\theta - i\cos 9\theta$$

To Find the Roots of $(a + ib)^{p/q}$, where $a, b \in R$; $p, q \in I, q \neq 0$ and $i = \sqrt{-1}$

Let $a + ib = r(\cos\theta + i\sin\theta)$ [polar form]

$$\therefore (a + ib)^{p/q} = \{r(\cos(2n\pi + \theta) + i\sin(2n\pi + \theta))\}^{p/q},\ n \in I$$

$$= r^{p/q}(\cos(2n\pi + \theta) + i\sin(2n\pi + \theta))^{p/q}$$

$$= r^{p/q}\left(\cos\left(\frac{p}{q}(2n\pi + \theta)\right) + i\sin\left(\frac{p}{q}(2n\pi + \theta)\right)\right),$$

where, $n = 0, 1, 2, 3, \ldots, q - 1$

Example 56. Find all roots of $x^5 - 1 = 0$.

Sol. $\because\ x^5 - 1 = 0 \Rightarrow x^5 = 1$

$$\therefore\ x = (1)^{1/5} = (\cos 0 + i\sin 0)^{1/5},$$

where $i = \sqrt{-1}$

$$= [\cos(2n\pi + 0) + i\sin(2n\pi + 0)]^{1/5}$$

$$= \cos\left(\frac{2n\pi}{5}\right) + i\sin\left(\frac{2n\pi}{5}\right),$$

where, $n = 0, 1, 2, 3, 4$

∴ Roots are

$$1, \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right), \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right),$$

$$\cos\left(\frac{6\pi}{5}\right) + i\sin\left(\frac{6\pi}{5}\right), \cos\left(\frac{8\pi}{5}\right) + i\sin\left(\frac{8\pi}{5}\right)$$

Now, $\cos\left(\frac{6\pi}{5}\right) + i\sin\left(\frac{6\pi}{5}\right)$

$$= \cos\left(2\pi - \frac{4\pi}{5}\right) + i\sin\left(2\pi - \frac{4\pi}{5}\right)$$

$$= \cos\left(\frac{4\pi}{5}\right) - i\sin\left(\frac{4\pi}{5}\right)$$

and $\cos\left(\frac{8\pi}{5}\right) + i\sin\left(\frac{8\pi}{5}\right)$

$$= \cos\left(2\pi - \frac{2\pi}{5}\right) + i\sin\left(2\pi - \frac{2\pi}{5}\right)$$

$$= \cos\left(\frac{2\pi}{5}\right) - i\sin\left(\frac{2\pi}{5}\right)$$

Hence, roots are $1, \cos\left(\frac{2\pi}{5}\right) \pm i\sin\left(\frac{2\pi}{5}\right)$

and $\cos\left(\frac{4\pi}{5}\right) \pm i\sin\left(\frac{4\pi}{5}\right)$.

Remark

Five roots are $1, z_1, z_2, \bar{z}_1, \bar{z}_2$ (one real, two complex and two conjugate of complex roots).

Example 57. Find all roots of the equation $x^6 - x^5 + x^4 - x^3 + x^2 - x + 1 = 0$.

Sol. $\because \quad 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 = 0$

$$\Rightarrow \quad 1 \cdot \frac{[1-(-x)^7]}{1-(-x)} = 0, 1 + x \neq 0$$

or $\quad 1 + x^7 = 0, \; x \neq -1 \quad \text{or} \quad x^7 = -1$

$$\therefore \quad x = (-1)^{1/7} = (\cos\pi + i\sin\pi)^{1/7}, \; i = \sqrt{-1}$$

$$= [\cos(2n+1)\pi + i\sin(2n+1)\pi]^{1/7}$$

$$= \cos\left(\frac{(2n+1)\pi}{7}\right) + i\sin\left(\frac{(2n+1)\pi}{7}\right)$$

for $n = 0, 1, 2, 4, 5, 6$.

Remark

$\because$ For $n = 3$, $x = -1$ but here $x \neq -1$

$\therefore \quad n \neq 3$

Cube Roots of Unity

Let $z = (1)^{1/3} \Rightarrow z^3 = 1 \Rightarrow z^3 - 1 = 0$

$\Rightarrow (z-1)(z^2 + z + 1) = 0 \Rightarrow z - 1 = 0 \text{ or } z^2 + z + 1 = 0$

$$\therefore \quad z = 1 \text{ or } z = \frac{-1 \pm \sqrt{(1-4)}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

Therefore, $\boldsymbol{z = 1, \frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}}$, where $i = \sqrt{-1}$.

If second root is represented by ω (omega), third root will be ω^2.

$\therefore$ Cube roots of unity are $1, \omega, \omega^2$ and ω, ω^2 are called non-real complex cube roots of unity.

Remark

1. $\overline{\omega} = \omega^2, (\overline{\omega})^2 = \omega$ 2. $\sqrt{\omega} = \pm\,\omega^2, \sqrt{\omega^2} = \pm\,\omega$
3. $|\omega| = |\omega^2| = 1$

Aliter

Let $\quad z = (1)^{1/3} = (\cos 0 + i\sin 0)^{1/3}, \; i = \sqrt{-1}$

$$= [\cos(2n\pi + 0) + i\sin(2n\pi + 0)]^{1/3}$$

$$= \cos\left(\frac{2n\pi}{3}\right) + i\sin\left(\frac{2n\pi}{3}\right), \text{ where, } n = 0, 1, 2$$

Therefore, roots are

$$1, \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right), \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)$$

or $\quad 1, e^{2\pi i/3}, e^{4\pi i/3}$

If second root is represented by ω, then third root will be ω^2 or if third root is represented by ω, then second root will be ω^2.

Properties of Cube Roots of Unity

(i) $1 + \omega + \omega^2 = 0$ and $\omega^3 = 1$

(ii) **To find the value of $\omega^n (n > 3)$.**

First divide n by 3. Let q be the quotient and r be the remainder.

$$3)\, n\,(q$$
$$\underline{-3q}$$
$$r$$

i.e. $\quad n = 3q + r$, where $0 \leq r \leq 2$

$$\therefore \quad \omega^n = \omega^{3q+r} = (\omega^3)^q \cdot \omega^r = \omega^r$$

In general, $\omega^{3n} = 1, \omega^{3n+1} = \omega, \omega^{3n+2} = \omega^2$

(iii) $1 + \omega^r + \omega^{2r} = \begin{cases} 3, \text{ when } n \text{ is a multiple of } 3 \\ 0, \text{ when } n \text{ is not a multiple of } 3 \end{cases}$

(iv) Cube roots of -1 are $-1, -\omega$ and $-\omega^2$.

(v) $a + b\omega + c\omega^2 = 0 \Rightarrow a = b = c$, if $a, b, c \in R$.

(vi) If a, b, c are non-zero numbers such that $a + b + c = 0 = a^2 + b^2 + c^2$, then $a : b : c = 1 : \omega : \omega^2$.

(vii) A complex number $a + ib$ (where $i = \sqrt{-1}$), for which $|a : b| = 1 : \sqrt{3}$ or $\sqrt{3} : 1$ can always be expressed in terms of ω or ω^2.

For example,

(a) $1 + i\sqrt{3} = -(-1 - i\sqrt{3}) \qquad [\because |1 : \sqrt{3}| = 1 : \sqrt{3}]$

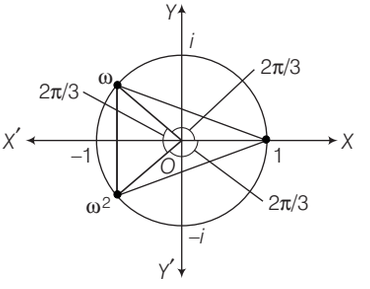
$$= -2\left(\frac{-1 - i\sqrt{3}}{2}\right) = -2\omega^2$$

(b) $\sqrt{3} + i = \frac{i(\sqrt{3} + i)}{i} = \frac{(-1 + i\sqrt{3})}{i}$

$$= \left(\frac{-1 + i\sqrt{3}}{2}\right)\left(\frac{2}{i}\right) \qquad [\because |\sqrt{3} : 1| = \sqrt{3} : 1]$$

$$= \frac{2\omega}{i} = -2i\omega$$

(viii) The cube roots of unity when represented on complex plane lie on vertices of an equilateral triangle inscribed in a unit circle, having centre at origin. One vertex being on positive real axis.



Important Relations in Terms of Cube Root of Unity

(i) $a^2 + ab + b^2 = (a - b\omega)(a - b\omega^2)$

(ii) $a^2 - ab + b^2 = (a + b\omega)(a + b\omega^2)$

(iii) $a^3 + b^3 = (a + b)(a + b\omega)(a + b\omega^2)$

(iv) $a^3 - b^3 = (a - b)(a - b\omega)(a - b\omega^2)$

(v) $a^2 + b^2 + c^2 - ab - bc - ca$
$= (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

(vi) $a^3 + b^3 + c^3 - 3abc$
$= (a + b + c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

Example 58. If ω is a non-real complex cube root of unity, find the values of the following.

(i) ω^{1999}

(ii) ω^{-998}

(iii) $\left(\frac{-1 + i\sqrt{3}}{2}\right)^{3n+2}$, $n \in N$ and $i = \sqrt{-1}$

(iv) $(1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8)$ ... upto $2n$ factors

(v) $\left(\frac{\alpha + \beta\omega + \gamma\omega^2 + \delta\omega^2}{\beta + \alpha\omega^2 + \gamma\omega + \delta\omega}\right)$, where $\alpha, \beta, \gamma, \delta \in R$

(vi) $1 \cdot (2 - \omega)(2 - \omega^2) + 2 \cdot (3 - \omega)(3 - \omega^2) + 3 \cdot (4 - \omega)(4 - \omega^2) + \ldots + \ldots + (n - 1) \cdot (n - \omega)(n - \omega^2)$

Sol. (i) $\omega^{1999} = \omega^{3 \times 666 + 1} = \omega$

(ii) $\omega^{-998} = \frac{1}{\omega^{998}} = \frac{\omega}{\omega^{999}} = \omega$

(iii) $\left(\frac{-1 + i\sqrt{3}}{2}\right)^{3n+2} = \omega^{3n+2} = \omega^{3n} \cdot \omega^2 = (\omega^3)^n \cdot \omega^2$
$= (1)^n \cdot \omega^2 = \omega^2$

(iv) $(1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8)$ ... upto $2n$ factors
$= (1 + \omega)(1 + \omega^2)(1 + \omega)(1 + \omega^2)$ ... upto $2n$ factors
$= (-\omega^2)(-\omega)(-\omega^2)(-\omega)$ ... upto $2n$ factors
$= (\omega^3)(\omega^3)$... upto n factors $= 1 \cdot 1 \cdot 1 \cdot$ ... upto n factors
$= (1)^n = 1$

(v) $\left(\frac{\alpha + \beta\omega + \gamma\omega^2 + \delta\omega^2}{\beta + \alpha\omega^2 + \gamma\omega + \delta\omega}\right) = \frac{\omega(\alpha + \beta\omega + \gamma\omega^2 + \delta\omega^2)}{(\beta\omega + \alpha\omega^3 + \gamma\omega^2 + \delta\omega^2)}$
$= \frac{\omega(\alpha + \beta\omega + \gamma\omega^2 + \delta\omega^2)}{(\beta\omega + \alpha + \gamma\omega^2 + \delta\omega^2)} = \omega$

(vi) $\Sigma(n - 1)(n - \omega)(n - \omega^2) = \Sigma(n^3 - 1) = \Sigma n^3 - \Sigma 1$
$= \left\{\frac{n(n + 1)}{2}\right\}^2 - n$

Example 59. If α, β and γ are the roots of $x^3 - 3x^2 + 3x + 7 = 0$, find the value of $\frac{\alpha - 1}{\beta - 1} + \frac{\beta - 1}{\gamma - 1} + \frac{\gamma - 1}{\alpha - 1}$.

Sol. We have, $x^3 - 3x^2 + 3x + 7 = 0$

$\Rightarrow (x - 1)^3 + 8 = 0$

$\Rightarrow (x - 1)^3 + 2^3 = 0$

$\Rightarrow (x - 1 + 2)(x - 1 + 2\omega)(x - 1 + 2\omega^2) = 0$

$\Rightarrow (x + 1)(x - 1 + 2\omega)(x - 1 + 2\omega^2) = 0$

$\therefore x = -1, 1 - 2\omega, 1 - 2\omega^2$

$\Rightarrow \alpha = -1, \beta = 1 - 2\omega, \gamma = 1 - 2\omega^2$

Then, $\frac{\alpha - 1}{\beta - 1} + \frac{\beta - 1}{\gamma - 1} + \frac{\gamma - 1}{\alpha - 1} = \frac{-2}{-2\omega} + \frac{-2\omega}{-2\omega^2} + \frac{-2\omega^2}{-2}$

$= \frac{1}{\omega} + \frac{1}{\omega} + \omega^2 = \omega^2 + \omega^2 + \omega^2 = 3\omega^2$

Example 60. If $z = \frac{\sqrt{3} + i}{2}$, where $i = \sqrt{-1}$, find the value of $(z^{101} + i^{103})^{105}$.

Sol. $\because z = \frac{\sqrt{3} + i}{2} = \frac{1}{i}\left(\frac{i\sqrt{3} + i^2}{2}\right)$ $[\because i^2 = -1]$

$= -i\left(\frac{-1 + i\sqrt{3}}{2}\right) = -i\omega$

$\therefore z^{101} = (-i\omega)^{101} = -i^{101} \cdot \omega^{101} = -i\omega^2$ and $i^{103} = i^3 = -i$

Then, $z^{101} + i^{103} = -i\omega^2 - i = -i(\omega^2 + 1)$

$= -i(-\omega) = i\omega$

Hence, $(z^{101} + i^{103})^{105} = (i\omega)^{105} = i^{105} \cdot \omega^{105} = i \cdot 1 = i$

Example 61. If $\left(\frac{3}{2} + \frac{i\sqrt{3}}{2}\right)^{50} = 3^{25}(x - iy)$, where $x, y \in R$ and $i = \sqrt{-1}$, find the ordered pair of (x, y).

Sol. $\because \frac{3}{2} + \frac{i\sqrt{3}}{2} = \sqrt{3}\left(\frac{\sqrt{3} + i}{2}\right) = \frac{\sqrt{3}}{i}\left(\frac{i\sqrt{3} + i^2}{2}\right)$

$= -i\sqrt{3}\left(\frac{-1 + i\sqrt{3}}{2}\right) = -i\sqrt{3}\,\omega$

$\therefore \left(\frac{3}{2} + \frac{i\sqrt{3}}{2}\right)^{50} = (-i\sqrt{3}\,\omega)^{50} = i^{50} \cdot 3^{25} \cdot \omega^{50}$

$= -1 \cdot 3^{25} \cdot \omega^2 = -3^{25} \cdot \left(\frac{-1 - i\sqrt{3}}{2}\right)$

$$= 3^{25}\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right) = 3^{25}(x - iy) \qquad \text{[given]}$$

$\therefore \quad x = \frac{1}{2}, y = -\frac{\sqrt{3}}{2}$

$\Rightarrow$ Ordered pair is $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

Example 62. If the polynomial $7x^3 + ax + b$ is divisible by $x^2 - x + 1$, find the value of $2a + b$.

Sol. Let $f(x) = 7x^3 + ax + b$

and $x^2 - x + 1 = (x + \omega)(x + \omega^2)$

$\because$ $f(x)$ is divisible by $x^2 - x + 1$

Then, $f(-\omega) = 0$ and $f(-\omega^2) = 0$

$\Rightarrow \quad -7\omega^3 - a\omega + b = 0$ and $-7\omega^6 - a\omega^2 + b = 0$

or $\quad -7 - a\omega + b = 0$

and $\quad -7 - a\omega^2 + b = 0$

On adding, we get

$$-14 - a(\omega + \omega^2) + 2b = 0$$

or $\quad -14 + a + 2b = 0$ or $a + 2b = 14$ ...(i)

and on subtracting, we get

$$-a(\omega - \omega^2) = 0$$

$\Rightarrow \quad a = 0 \qquad [\because \omega - \omega^2 \neq 0]$

From Eq. (i), we get $b = 7$

$\therefore \quad 2a + b = 7$

Exercise for Session 3

1 The real part of $(1 - i)^{-i}$, where $i = \sqrt{-1}$ is

(a) $e^{-\pi/4}\cos\left(\frac{1}{2}\log_e 2\right)$ (b) $-e^{-\pi/4}\sin\left(\frac{1}{2}\log_e 2\right)$

(c) $e^{\pi/4}\cos\left(\frac{1}{2}\log_e 2\right)$ (d) $e^{-\pi/4}\sin\left(\frac{1}{2}\log_e 2\right)$

2 The amplitude of $e^{e^{-i\theta}}$, where $\theta \in R$ and $i = \sqrt{-1}$ is

(a) $\sin\theta$ (b) $-\sin\theta$

(c) $e^{\cos\theta}$ (d) $e^{\sin\theta}$

3 If $z = i\log_e(2 - \sqrt{3})$, where $i = \sqrt{-1}$, then the $\cos z$ is equal to

(a) i (b) $2i$ (c) 1 (d) 2

4 If $z = i^{i^i}$, where $i = \sqrt{-1}$, then $|z|$ is equal to

(a) 1 (b) $e^{-\pi/2}$ (c) $e^{-\pi}$ (d) e^{π}

5 $\sqrt{(-8 - 6i)}$ is equal to (where, $i = \sqrt{-1}$)

(a) $1 \pm 3i$ (b) $\pm(1 - 3i)$ (c) $\pm(1 + 3i)$ (d) $\pm(3 - i)$

6 $\frac{\sqrt{(5 + 12i)} + \sqrt{(5 - 12i)}}{\sqrt{(5 + 12i)} - \sqrt{(5 - 12i)}}$ is equal to (where, $i = \sqrt{-1}$)

(a) $-\frac{3}{2}i$ (b) $\frac{3}{4}i$ (c) $-\frac{3}{4}i$ (d) $-\frac{3}{2}$

7 If $0 < \text{amp}(z) < \pi$, then $\text{amp}(z) - \text{amp}(-z)$ is equal to

(a) 0 (b) $2\,\text{amp}(z)$ (c) π (d) $-\pi$

8 If $|z_1| = |z_2|$ and $\text{amp}(z_1) + \text{amp}(z_2) = 0$, then

(a) $z_1 = z_2$ (b) $\bar{z}_1 = z_2$ (c) $z_1 + z_2 = 0$ (d) $\bar{z}_1 = \bar{z}_2$

9 The solution of the equation $|z| - z = 1 + 2i$, where $i = \sqrt{-1}$, is

(a) $2 - \frac{3}{2}i$ (b) $\frac{3}{2} + 2i$ (c) $\frac{3}{2} - 2i$ (d) $-2 + \frac{3}{2}i$

10 The number of solutions of the equation $z^2 + \bar{z} = 0$, is

(a) 1 (b) 2
(c) 3 (d) 4

11 If $z_r = \cos\left(\frac{r\alpha}{n^2}\right) + i \sin\left(\frac{r\alpha}{n^2}\right)$, where $r = 1, 2, 3, \ldots, n$ and $i = \sqrt{-1}$, then $\lim_{n \to \infty} z_1 z_2 z_3 \ldots z_n$ is equal to

(a) $e^{i\alpha}$ (b) $e^{-i\alpha/2}$
(c) $e^{i\alpha/2}$ (d) $\sqrt[3]{e^{i\alpha}}$

12 If $\theta \in R$ and $i = \sqrt{-1}$, then $\left(\frac{1 + \sin\theta + i\cos\theta}{1 + \sin\theta - i\cos\theta}\right)^n$ is equal to

(a) $\cos\left(\frac{n\pi}{2} - n\theta\right) + i \sin\left(\frac{n\pi}{2} - n\theta\right)$ (b) $\cos\left(\frac{n\pi}{2} + n\theta\right) + i \sin\left(\frac{n\pi}{2} + n\theta\right)$
(c) $\sin\left(\frac{n\pi}{2} - n\theta\right) + i \cos\left(\frac{n\pi}{2} - n\theta\right)$ (d) $\cos\left(n\left(\frac{\pi}{2} + 2\theta\right)\right) + i \sin\left(n\left(\frac{\pi}{2} + 2\theta\right)\right)$

13 If $i z^4 + 1 = 0$, where $i = \sqrt{-1}$, then z can take the value

(a) $\frac{1+i}{\sqrt{2}}$ (b) $\cos\left(\frac{\pi}{8}\right) + i \sin\left(\frac{\pi}{8}\right)$
(c) $\frac{1}{4i}$ (d) i

14 If $\omega (\neq 1)$ is a cube root of unity, then $(1 - \omega + \omega^2)(1 - \omega^2 + \omega^4)(1 - \omega^4 + \omega^8) \ldots \text{upto } 2n$ factors, is

(a) 2^n (b) 2^{2n}
(c) 0 (d) 1

15 If α, β and γ are the cube roots of p $(p < 0)$, then for any x, y and z, $\frac{x\alpha + y\beta + z\gamma}{x\beta + y\gamma + z\alpha}$ is equal to

(a) $\frac{1}{2}(-1 - i\sqrt{3}), i = \sqrt{-1}$ (b) $\frac{1}{2}(1 + i\sqrt{3}), i = \sqrt{-1}$
(c) $\frac{1}{2}(1 - i\sqrt{3}), i = \sqrt{-1}$ (d) None of these

Session 4

*n*th Root of Unity, Vector Representation of Complex Numbers, Geometrical Representation of Algebraic Operation on Complex Numbers, Rotation Theorem (Coni Method), Shifting the Origin in Case of Complex Numbers, Inverse Points, Dot and Cross Product, Use of Complex Numbers in Coordinate Geometry

n th Root of Unity

Let x be the nth root of unity, then

$$x = (1)^{1/n} = (\cos 0 + i \sin 0)^{1/n}$$

$$= (\cos(2k\pi + 0) + i \sin(2k\pi + 0)^{1/n}$$

[where k is an integer]

$$= (\cos 2k\pi + i \sin 2k\pi)^{1/n}$$

$$\therefore \quad x = \cos\left(\frac{2k\pi}{n}\right) + i \sin\left(\frac{2k\pi}{n}\right)$$

where, $k = 0, 1, 2, 3, \ldots, n-1$

Let $\alpha = \cos\frac{2\pi}{n} + i \sin\frac{2\pi}{n}$, the n, nth roots of unity are α^k $(k = 0, 1, 2, 3, \ldots, n-1)$ i.e, the n, nth roots of unity are $1, \alpha, \alpha^2, \alpha^3, \ldots, \alpha^{n-1}$ which are in GP with common ratio $= e^{2\pi i/n}$.

(a) **Sum of *n*, *n*th roots of unity**

$$1 + \alpha + \alpha^2 + \alpha^3 + \ldots + \alpha^{n-1} = \frac{1 \cdot (1 - \alpha^n)}{(1 - \alpha)}$$

$$= \frac{1 - (\cos 2\pi + i \sin 2\pi)}{1 - \alpha}$$

$$= \frac{1 - (1 + 0)}{1 - \alpha} = 0$$

Remark

$1 + \alpha + \alpha^2 + \alpha^3 + \ldots + \alpha^{n-1} = 0$ is the basic concept to be understood.

(b) **Product of *n*, *n*th roots of unity**

$$1 \times \alpha \times \alpha^2 \times \alpha^3 \times \ldots \times \alpha^{n-1} = \alpha^{1+2+3+\ldots+(n-1)}$$

$$= \alpha^{\frac{(n-1)n}{2}} = \left(\cos\frac{2\pi}{n} + i \sin\frac{2\pi}{n}\right)^{\frac{(n-1)n}{2}}$$

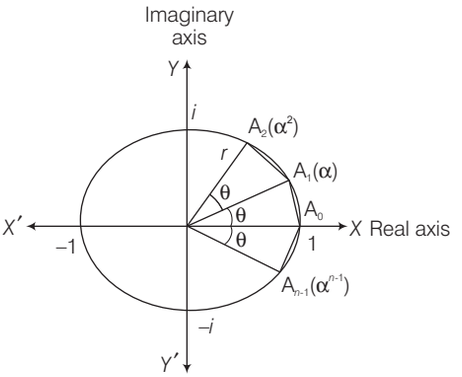
$$= \cos(n-1)\pi + i \sin(n-1)\pi$$

$$= (\cos\pi + i \sin\pi)^{n-1} = (-1)^{n-1}$$

Remark

$1 \cdot \alpha \cdot \alpha^2 \cdot \alpha^3 \ldots \alpha^{n-1} = (-1)^{n-1}$ is the basic concept to be understood.

(c) If α is an imaginary nth root of unity, then other roots are given by $\alpha^2, \alpha^3, \alpha^4, \ldots, \alpha^n$.



(d) $\because \quad 1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1} = 0$

$$\Rightarrow \quad \sum_{k=0}^{n-1} \alpha^k = 0$$

$$\text{or} \quad \sum_{k=0}^{n-1} \cos\left(\frac{2\pi k}{n}\right) + i \sum_{k=0}^{n-1} \sin\left(\frac{2\pi k}{n}\right) = 0$$

$$\Rightarrow \quad \sum_{k=0}^{n-1} \cos\left(\frac{2\pi k}{n}\right) = 0$$

$$\text{and} \quad \sum_{k=0}^{n-1} \sin\left(\frac{2\pi k}{n}\right) = 0$$

These roots are located at the vertices of a regular plane polygon of n sides inscribed in a unit circle having centre at origin, one vertex being on positive real axis.

(e) $x^n - 1 = (x-1)(x-\alpha)(x-\alpha^2)\ldots(x-\alpha^{n-1})$.

Important Benefits

1. If $1, \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1}$ are the n, nth root of unity, then
$(1)^p + (\alpha_1)^p + (\alpha_2)^p + \ldots + (\alpha_{n-1})^p$
$= \begin{cases} 0, \text{ if } p \text{ is not an integral multiple of } n \\ n, \text{ if } p \text{ is an integral multiple of } n \end{cases}$
2. $(1+\alpha_1)(1+\alpha_2)\ldots(1+\alpha_{n-1}) = \begin{cases} 0, \text{ if } n \text{ is even} \\ 1, \text{ if } n \text{ is odd} \end{cases}$
3. $(1-\alpha_1)(1-\alpha_2)\ldots(1-\alpha_{n-1}) = n$
4. $z^n - 1 = (z-1)(z+1)\prod_{r=1}^{(n-2)/2}\left(z^2 - 2z\cos\frac{2r\pi}{n} + 1\right)$, if '$n$' is even.
5. $z^n + 1 = \prod_{r=0}^{(n-2)/2}\left(z^2 - 2z\cos\left(\frac{(2r+1)\pi}{n}\right) + 1\right)$, if n is even.
6. $z^n + 1 = (z+1)\prod_{r=0}^{(n-3)/2}\left(z^2 - 2z\cos\left(\frac{(2r+1)\pi}{n}\right) + 1\right)$, if '$n$' is odd.

The Sum of the Following Series Should be Remembered

(i) $\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos n\theta$

$$= \frac{\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \cdot \cos\left[\left(\frac{n+1}{2}\right)\theta\right]$$

(ii) $\sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta$

$$= \frac{\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \cdot \sin\left[\left(\frac{n+1}{2}\right)\theta\right]$$

Proof

(i) $\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos n\theta$

$= \text{Re}\{e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \ldots + e^{ni\theta}\}$, where $i = \sqrt{-1}$

$$= \text{Re}\left\{\frac{e^{i\theta}\{(e^{i\theta})^n - 1\}}{e^{i\theta} - 1}\right\} = \text{Re}\left\{\frac{e^{i\theta} \cdot e^{ni\theta/2} \cdot 2i\sin\left(\frac{n\theta}{2}\right)}{e^{i\theta/2} \cdot 2i\sin(\theta/2)}\right\}$$

$$= \text{Re}\left\{\frac{\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \cdot e^{\left(\frac{n+1}{2}\right)i\theta}\right\} = \frac{\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \cdot \cos\left[\left(\frac{n+1}{2}\right)\theta\right]$$

(ii) $\sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta$

$= \text{Im}\{e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \ldots + e^{ni\theta}\}$, where $i = \sqrt{-1}$

$$= \text{Im}\left\{\frac{e^{i\theta}\{(e^{i\theta})^n - 1\}}{e^{i\theta} - 1}\right\} = \text{Im}\left\{\frac{e^{i\theta} \cdot e^{\frac{ni\theta}{2}} \cdot 2i\sin\left(\frac{n\theta}{2}\right)}{e^{i\theta/2} \cdot 2i\sin\left(\frac{\theta}{2}\right)}\right\}$$

$$= \text{Im}\left\{\frac{\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \cdot e^{\left(\frac{n+1}{2}\right)i\theta}\right\} = \frac{\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \cdot \sin\left[\left(\frac{n+1}{2}\right)\theta\right]$$

Remark

For $\theta = \frac{2\pi}{n}$, we get

1. $1 + \cos\left(\frac{2\pi}{n}\right) + \cos\left(\frac{4\pi}{n}\right) + \cos\left(\frac{6\pi}{n}\right) + \ldots + \cos\left(\frac{(2n-2)\pi}{n}\right) = 0$

2. $\sin\left(\frac{2\pi}{n}\right) + \sin\left(\frac{4\pi}{n}\right) + \sin\left(\frac{6\pi}{n}\right) + \ldots + \sin\left(\frac{(2n-2)\pi}{n}\right) = 0$

Example 63. If $1, \omega, \omega^2, \ldots, \omega^{n-1}$ are n, nth roots of unity, find the value of $(9-\omega)(9-\omega^2)\ldots(9-\omega^{n-1})$.

Sol. Let $x = (1)^{1/n} \Rightarrow x^n - 1 = 0$

has n roots $1, \omega, \omega^2, \ldots, \omega^{n-1}$

$\therefore \quad x^n - 1 = (x-1)(x-\omega)(x-\omega^2)\ldots(x-\omega^{n-1})$

On putting $x = 9$ in both sides, we get

$$\frac{9^n - 1}{9-1} = (9-\omega)(9-\omega^2)(9-\omega^3)\ldots(9-\omega^{n-1})$$

or $\quad (9-\omega)(9-\omega^2)\ldots(9-\omega^{n-1}) = \frac{9^n - 1}{8}$

Remark

$$\frac{x^n - 1}{x-1} = (x-\omega)(x-\omega^2)\ldots(x-\omega^{n-1})$$

$$\therefore \lim_{x\to 1}\frac{x^n - 1}{x-1} = \lim_{x\to 1}(x-\omega)(x-\omega^2)\ldots(x-\omega^{n-1})$$

$$\Rightarrow \quad n = (1-\omega)(1-\omega^2)\ldots(1-\omega^{n-1})$$

Example 64. ***If*** $a = \cos\left(\frac{2\pi}{7}\right) + i\sin\left(\frac{2\pi}{7}\right)$, ***where*** $i = \sqrt{-1}$, ***find the quadratic equation whose roots are*** $\alpha = a + a^2 + a^4$ ***and*** $\beta = a^3 + a^5 + a^6$.

Sol. $\because \quad a = \cos\left(\frac{2\pi}{7}\right) + i\sin\left(\frac{2\pi}{7}\right)$

$\therefore \quad a^7 = \cos 2\pi + i\sin 2\pi = 1 + 0 = 1$

or $\quad a = (1)^{1/7}$

$\therefore \quad 1, a, a^2, a^3, a^4, a^5, a^6$ are 7, 7 th roots of unity.

$\therefore \quad 1 + a + a^2 + a^3 + a^4 + a^5 + a^6 = 0 \qquad \ldots(i)$

$\Rightarrow \quad (a + a^2 + a^4) + (a^3 + a^5 + a^6) = -1$ or $\alpha + \beta = -1$

and $\alpha\beta = (a + a^2 + a^4)(a^3 + a^5 + a^6)$

$= a^4 + a^6 + a^7 + a^5 + a^7 + a^8 + a^7 + a^9 + a^{10}$

$= a^4 + a^6 + 1 + a^5 + 1 + a + 1 + a^2 + a^3 \quad [\because a^7 = 1]$

$= (1 + a + a^2 + a^3 + a^4 + a^5 + a^6) + 2$

$= 0 + 2 \quad$ [from Eq. (i)]

$= 2$

Therefore, the required equation is

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$ or $x^2 + x + 2 = 0$

Example 65. Find the value of $\sum_{k=1}^{10}\left[\sin\left(\frac{2\pi k}{11}\right) - i\cos\left(\frac{2\pi k}{11}\right)\right]$, where $i = \sqrt{-1}$.

Sol. $\sum_{k=1}^{10}\left[\sin\left(\frac{2\pi k}{11}\right) - i\cos\left(\frac{2\pi k}{11}\right)\right]$

$= -i\sum_{k=1}^{10}\left[\cos\left(\frac{2\pi k}{11}\right) + i\sin\left(\frac{2\pi k}{11}\right)\right]$

$= -i\left\{\sum_{k=0}^{10}\left[\cos\left(\frac{2\pi k}{11}\right) + i\sin\left(\frac{2\pi k}{11}\right)\right] - 1\right\}$

$= -i(0 - 1) \quad$ [sum of 11, 11th roots of unity]

$= i$

Example 66. ***If $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are the n, nth roots of the unity, then find the value of*** $\sum_{i=0}^{n-1}\frac{\alpha_i}{2 - \alpha_i}$.

Sol. Let $x = (1)^{1/n} \Rightarrow x^n = 1 \quad \therefore \quad x^n - 1 = 0$

or $x^n - 1 = (x - \alpha_0)(x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_{n-1})$

$= \prod_{i=0}^{n-1}(x - \alpha_i)$

On taking logarithm both sides, we get

$\log_e(x^n - 1) = \sum_{i=0}^{n-1}\log_e(x - \alpha_i)$

On differentiating both sides w.r.t. x, we get

$\frac{nx^{n-1}}{x^n - 1} = \sum_{i=0}^{n-1}\left(\frac{1}{x - \alpha_i}\right)$

On putting $x = 2$, we get

$\frac{n(2)^{n-1}}{2^n - 1} = \sum_{i=0}^{n-1}\frac{1}{(2 - \alpha_i)}$ ...(i)

Now, $\sum_{i=0}^{n-1}\frac{\alpha_i}{(2 - \alpha_i)} = \sum_{i=0}^{n-1}\left(-1 + \frac{2}{2 - \alpha_i}\right)$

$= -\sum_{i=0}^{n-1}1 + 2\sum_{i=0}^{n-1}\frac{1}{(2 - \alpha_i)} = -(n) + \frac{2 \cdot n \cdot 2^{n-1}}{2^n - 1}$ [from Eq. (i)]

$= -n + \frac{n \cdot 2^n}{2^n - 1} = \frac{n}{2^n - 1}$

Example 67. If $n \geq 3$ and $1, \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1}$ are the n, nth roots of unity, then find the value of $\sum\sum_{1 \leq i < j \leq n-1}\alpha_i\,\alpha_j$.

Sol. Let $x = (1)^{1/n}$

$\therefore \quad x^n = 1 \quad$ or $\quad x^n - 1 = 0$

$\therefore \quad 1 + \alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_{n-1} = 0$

or $\quad \alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_{n-1} = -1$

On squaring both sides, we get

$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \ldots + \alpha^2_{n-1} + 2(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \ldots + \alpha_1\alpha_{n-1} + \alpha_2\alpha_3 + \ldots + \alpha_2\alpha_{n-1} + \ldots + \alpha_{n-2}\alpha_{n-1}) = 1$

or $1^2 + (\alpha_1)^2 + (\alpha_2)^2 + (\alpha_3)^2 + \ldots + (\alpha_{n-1})^2 + 2\sum\sum_{1 \leq i < j \leq n-1}\alpha_i\alpha_j = 1 + 1^2$

$0 + 2\sum\sum_{1 \leq i < j \leq n-1}\alpha_i\alpha_j = 2$

[here, p is not a multiple of n]

$\therefore \quad \sum\sum_{1 \leq i < j \leq n-1}\alpha_i\alpha_j = 1$

Aliter

$\because \quad x^n - 1 = (x - 1)(x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_{n-1})$

On comparing the coefficient of x^{n-2} both sides, we get

$0 = \sum\sum_{0 \leq i < j \leq n-1}\alpha_i\,\alpha_j + \alpha_1 + \alpha_2 + \ldots + \alpha_{n-1}$

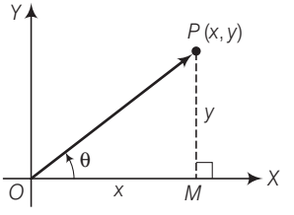
$0 = \sum\sum_{1 \leq i < j \leq n-1}\alpha_i\,\alpha_j - 1$

$[\because 1 + \alpha_1 + \alpha_2 + \ldots + \alpha_{n-1} = 0]$

$\therefore \quad \sum\sum_{1 \leq i < j \leq n-1}\alpha_i\,\alpha_j = 1$

Vector Representation of Complex Numbers

If P is the point (x, y) on the argand plane corresponding to the complex number $z = x + iy$, where $x, y \in R$ and $i = \sqrt{-1}$.



Then, $\overrightarrow{\mathbf{OP}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} \Rightarrow \left|\overrightarrow{\mathbf{OP}}\right| = \sqrt{(x^2 + y^2)} = |z|$

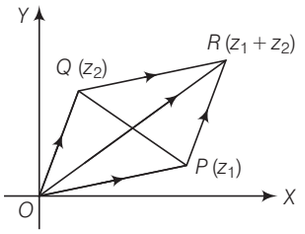
and $\arg(z) =$ direction of the vector $\overrightarrow{\mathbf{OP}} = \tan^{-1}(y/x) = \theta$

Therefore, complex number z can also be represented by $\overrightarrow{\mathbf{OP}}$.

Geometrical Representation of Algebraic Operation on Complex Numbers

(a) Sum

Let the complex numbers $z_1 = x_1 + iy_1 = (x_1, y_1)$ and $z_2 = x_2 + iy_2 = (x_2, y_2)$ be represented by the points P and Q on the argand plane.



Complete the parallelogram $OPRQ$. Then, the mid-points of PQ and OR are the same. The mid-point of

$PQ = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Hence, $R = (x_1 + x_2, y_1 + y_2)$

Therefore, complex number z can also be represented by

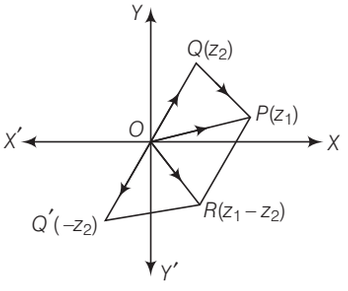
$$\overrightarrow{\mathbf{OR}} = (x_1 + x_2) + i(y_1 + y_2) = (x_1 + iy_1) + (x_2 + iy_2)$$
$$= z_1 + z_2 = (x_1, y_1) + (x_2, y_2)$$

In vector notation, we have

$$z_1 + z_2 = \overrightarrow{\mathbf{OP}} + \overrightarrow{\mathbf{OQ}} = \overrightarrow{\mathbf{OP}} + \overrightarrow{\mathbf{PR}} = \overrightarrow{\mathbf{OR}}$$

(b) Difference

We first represent $-z_2$ by Q', so that QQ' is bisected at O. Complete the parallelogram $OPRQ'$. Then, the point R represents the difference $z_1 - z_2$.



We see that $ORPQ$ is a parallelogram, so that $\overrightarrow{\mathbf{OR}} = \overrightarrow{\mathbf{QP}}$

We have in vectorial notation,

$$z_1 - z_2 = \overrightarrow{\mathbf{OP}} - \overrightarrow{\mathbf{OQ}} = \overrightarrow{\mathbf{OP}} + \overrightarrow{\mathbf{QO}}$$
$$= \overrightarrow{\mathbf{OP}} + \overrightarrow{\mathbf{PR}} = \overrightarrow{\mathbf{OR}} = \overrightarrow{\mathbf{QP}}$$

(c) Product

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1) = r_1 e^{i\theta_1}$

$\therefore \quad |z_1| = r_1$ and $\arg(z_1) = \theta_1$

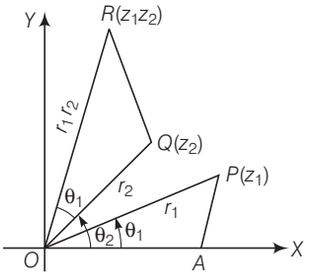
and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2) = r_2 e^{i\theta_2}$

$\therefore \quad |z_2| = r_2$ and $\arg(z_2) = \theta_2$

Then, $z_1 z_2 = r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$= r_1 r_2 \{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$

$\therefore \quad |z_1 z_2| = r_1 r_2$ and $\arg(z_1 z_2) = \theta_1 + \theta_2$



Let P and Q represent the complex numbers z_1 and z_2, respectively.

$\therefore \quad OP = r_1, OQ = r_2$

$\angle POX = \theta_1$ and $\angle QOX = \theta_2$

Take a point A on the real axis OX, such that $OA = 1$ unit. Complete the $\angle OPA$

Now, taking OQ as the base, construct a ΔOQR similar to ΔOPA, so that $\frac{OR}{OQ} = \frac{OP}{OA}$

i.e. $OR = OP \cdot OQ = r_1 r_2$ [since, $OA = 1$ unit]

and $\angle ROX = \angle ROQ + \angle QOX = \theta_1 + \theta_2$

Hence, R is the point representing product of complex numbers z_1 and z_2.

Remark

1. Multiplication by i
   Since, $z = r(\cos\theta + i\sin\theta)$ and $i = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$
   $\therefore \quad iz = r\left[\cos\left(\frac{\pi}{2} + \theta\right) + i\sin\left(\frac{\pi}{2} + \theta\right)\right]$
   Hence, multiplication of z with i, then vector for z rotates a right angle in the positive sense.
2. Thus, to multiply a vector by (-1) is to turn it through two right angles.
3. Thus, to multiply a vector by $(\cos\theta + i\sin\theta)$ is to turn it through the angle θ in the positive sense.

(d) Division

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1) = r_1 e^{i\theta_1}$

$\therefore \quad |z_1| = r_1$ and $\arg(z_1) = \theta_1$

and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2) = r_2 e^{i\theta_2}$

$\therefore \quad |z_2| = r_2$ and $\arg(z_2) = \theta_2$

Then, $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} \cdot \dfrac{(\cos\theta_1 + i\sin\theta_1)}{(\cos\theta_2 + i\sin\theta_2)} \qquad [z_2 \neq 0, r_2 \neq 0]$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

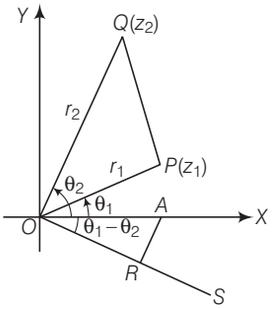
$\therefore \quad \left|\dfrac{z_1}{z_2}\right| = \dfrac{r_1}{r_2}, \arg\left(\dfrac{z_1}{z_2}\right) = \theta_1 - \theta_2$

Let P and Q represent the complex numbers z_1 and z_2, respectively.

$\therefore \quad OP = r_1, OQ = r_2, \ \angle POX = \theta_1$ and $\angle QOX = \theta_2$

Let OS be new position of OP, take a point A on the real axis OX, such that $OA = 1$ unit and through A draw a line making with OA an angle equal to the $\angle OQP$ and meeting OS in R.

Then, R represented by (z_1/z_2).



Now, in similar ΔOPQ and ΔOAR.

$$\frac{OR}{OA} = \frac{OP}{OQ} \Rightarrow OR = \frac{r_1}{r_2}$$

since $OA = 1$ and $\angle AOR = \angle POR - \angle POX = \theta_2 - \theta_1$

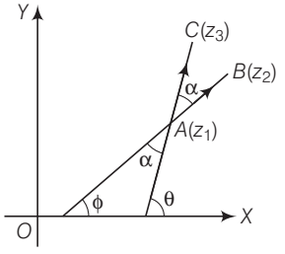
Hence, the vectorial angle of R is $-(\theta_2 - \theta_1)$ i.e., $\theta_1 - \theta_2$.

Remark

If θ_1 and θ_2 are the principal values of z_1 and z_2, then $\theta_1 + \theta_2$ and $\theta_1 - \theta_2$ are not necessarily the principal value of arg (z_1z_2) and arg (z_1 / z_2).

Rotation Theorem (Coni Method)

Let z_1, z_2 and z_3 be the affixes of three points A, B and C respectively taken on argand plane.



Then, we have $\overrightarrow{\mathbf{AC}} = z_3 - z_1$ and $\overrightarrow{\mathbf{AB}} = z_2 - z_1$

and let $\arg \overrightarrow{\mathbf{AC}} = \arg(z_3 - z_1) = \theta$

and $\arg \overrightarrow{\mathbf{AB}} = \arg(z_2 - z_1) = \phi$

Let $\angle CAB = \alpha$

$$\angle CAB = \alpha = \theta - \phi = \arg \overrightarrow{\mathbf{AC}} - \arg \overrightarrow{\mathbf{AB}}$$
$$= \arg(z_3 - z_1) - \arg(z_2 - z_1)$$
$$= \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$$

or angle between AC and AB

$$= \arg\left(\frac{\text{affix of } C - \text{affix of } A}{\text{affix of } B - \text{affix of } A}\right)$$

For any complex number z, we have

$$z = |z| e^{i(\arg z)}$$

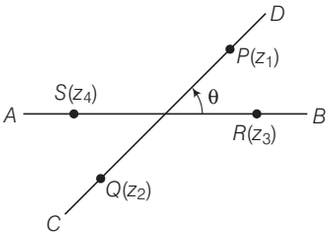
Similarly, $\left(\dfrac{z_3 - z_1}{z_2 - z_1}\right) = \left|\dfrac{z_3 - z_1}{z_2 - z_1}\right| e^{i\left[\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)\right]}$

or $$\frac{z_3 - z_1}{z_2 - z_1} = \frac{|z_3 - z_1|}{|z_2 - z_1|} e^{i(\angle CAB)} = \frac{AC}{AB} e^{i\alpha}$$

Remark

1. Here, only principal values of the arguments are considered.
2. $\arg\left(\dfrac{z_1 - z_2}{z_3 - z_4}\right) = \theta$, if AB coincides with CD, then

   $\arg\left(\dfrac{z_1 - z_2}{z_3 - z_4}\right) = 0$ or π, so that $\dfrac{z_1 - z_2}{z_3 - z_4}$ is real. It follows that if $\dfrac{z_1 - z_2}{z_3 - z_4}$ is real, then the points A, B, C, D are collinear.



3. If AB is perpendicular to CD, then

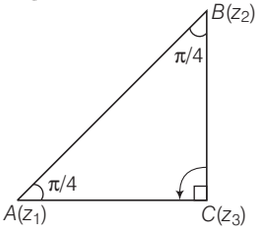
   $\arg\left(\dfrac{z_1 - z_2}{z_3 - z_4}\right) = \pm\dfrac{\pi}{2}$, so $\dfrac{z_1 - z_2}{z_3 - z_4}$ is purely imaginary.

4. It follows that, if $z_1 - z_2 = \pm k(z_3 - z_4)$, where k is purely imaginary number, then AB and CD are perpendicular to each other.

Example 68. ***Complex numbers*** z_1, z_2 ***and*** z_3 ***are the vertices*** A, B, C ***respectively of an isosceles right angled triangle with right angle at*** C. ***Show that*** $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$.

Sol. Since, $\angle ACB = 90°$ and $AC = BC$, then by Coni method

$$\frac{z_1 - z_3}{z_2 - z_3} = \frac{AC}{BC} e^{i\pi/2} = i$$



$$\Rightarrow \quad z_1 - z_3 = i(z_2 - z_3)$$

On squaring both sides, we get

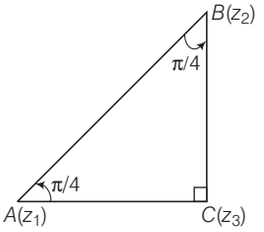
$$(z_1 - z_3)^2 = -(z_2 - z_3)^2$$

$$\Rightarrow \quad z_1^2 + z_3^2 - 2z_1z_3 = -(z_2^2 + z_3^2 - 2z_2z_3)$$

$$\Rightarrow \quad z_1^2 + z_2^2 - 2z_1z_2 = 2(z_1z_3 - z_1z_2 - z_3^2 + z_2z_3)$$

Therefore, $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$

Aliter $CA = CB = \frac{1}{\sqrt{2}} BA$



$\because \quad \angle BAC = (\pi/4)$

$$\therefore \quad \frac{z_2 - z_1}{z_3 - z_1} = \frac{BA}{CA} e^{(i\pi/4)}$$

or $$\frac{z_1 - z_2}{z_1 - z_3} = \sqrt{2}\, e^{(i\pi/4)} \qquad ...(i)$$

and $\angle CBA = (\pi/4)$

$$\therefore \quad \frac{z_3 - z_2}{z_1 - z_2} = \frac{CB}{AB} e^{(i\pi/4)} \text{ or } \frac{z_3 - z_2}{z_1 - z_2} = \frac{1}{\sqrt{2}} e^{(i\pi/4)} \qquad ...(ii)$$

On dividing Eq. (i) by Eq. (ii), we get

$$(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$$

Example 69. Complex numbers z_1, z_2, z_3 are the vertices of A, B, C respectively of an equilateral triangle. Show that

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1.$$

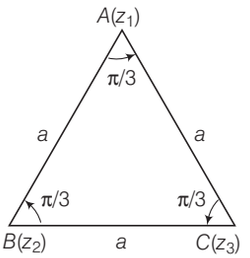
Sol. Let $AB = BC = CA = a$

$\because \quad \angle ABC = \frac{\pi}{3}$

From Coni method, $\frac{z_1 - z_2}{z_3 - z_2} = \frac{a}{a} e^{i\pi/3}$ ...(i)

and $\angle BAC = \frac{\pi}{3}$

From Coni method, $\frac{z_3 - z_1}{z_2 - z_1} = \frac{a}{a} e^{i\pi/3}$ ...(ii)



From Eqs. (i) and (ii), we get $\frac{z_1 - z_2}{z_3 - z_2} = \frac{z_3 - z_1}{z_2 - z_1}$

$$\Rightarrow \quad (z_1 - z_2)(z_2 - z_1) = (z_3 - z_1)(z_3 - z_2)$$

$$\Rightarrow \quad z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

Remark

Triangle with vertices z_1, z_2, z_3, then

(i) $(z_1 - z_2)^2 + (z_2 - z_3)^2 + (z_3 - z_1)^2 = 0$

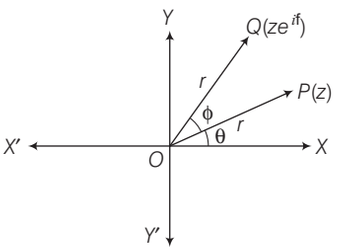
(ii) $(z_1 - z_2)^2 = (z_2 - z_3)(z_3 - z_1)$

(iii) $\sum (z_1 - z_2)(z_2 - z_3) = 0$ (iv) $\sum \frac{1}{(z_1 - z_2)} = 0$

Complex Number as a Rotating Arrow in the Argand Plane

Let $$z = r(\cos\theta + i\sin\theta) = re^{i\theta} \qquad ...(i)$$

be a complex number representing a point P in the argand plane.



Then, $OP = |z| = r$ and $\angle POX = \theta$

Now, consider complex number $z_1 = ze^{i\phi}$

or $$z_1 = re^{i\theta} \cdot e^{i\phi} = re^{i(\theta + \phi)} \qquad \text{[from Eq. (i)]}$$

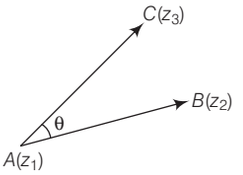
Clearly, the complex number z_1 represents a point Q in the argand plane, when $OQ = r$ and $\angle QOX = \theta + \phi$

Clearly, multiplication of z with $e^{i\phi}$ rotates the vector $\overrightarrow{\mathbf{OP}}$ through angle ϕ in anti-clockwise sense. Similarly, multiplication of z with $e^{-i\phi}$ will rotate the vector $\overrightarrow{\mathbf{OP}}$ in clockwise sense.

Remark

1. If z_1, z_2 and z_3 are the affixes of the three points A, B and C, such that $AC = AB$ and $\angle CAB = \theta$. Therefore,

$$\overrightarrow{AB} = z_2 - z_1 \text{ , } \overrightarrow{AC} = z_3 - z_1.$$



Then, $\overrightarrow{AC}$ will be obtained by rotating $\overrightarrow{AB}$ through an angle θ in anti-clockwise sense and therefore,

$$\overrightarrow{AC} = \overrightarrow{AB}\, e^{i\theta}$$

or $$(z_3 - z_1) = (z_2 - z_1)\, e^{i\theta} \text{ or } \frac{z_3 - z_1}{z_2 - z_1} = e^{i\theta}$$

2. If A, B and C are three points in argand plane, such that $AC = AB$ and $\angle CAB = \theta$, then use the rotation about A to find $e^{i\theta}$, but if $AC \neq AB$, then use Coni method.

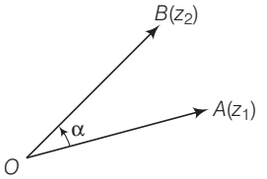
Example 70. Let z_1 and z_2 be roots of the equation $z^2 + pz + q = 0$, where the coefficients p and q may be complex numbers. Let A and B represent z_1 and z_2 in the complex plane. If $\angle AOB = \alpha \neq 0$ and $OA = OB$, where O is the origin, prove that $p^2 = 4q\cos^2(\alpha/2)$.

Sol. Clearly, $\overrightarrow{OB}$ is obtained by rotating $\overrightarrow{OA}$ through angle α.

$$\therefore \quad \overrightarrow{OB} = \overrightarrow{OA}\, e^{i\alpha}$$

$$\Rightarrow \quad z_2 = z_1\, e^{i\alpha}$$

$$\Rightarrow \quad \frac{z_2}{z_1} = e^{i\alpha} \qquad \ldots(i)$$



or $$\frac{z_2}{z_1} + 1 = (e^{i\alpha} + 1)$$

$$\Rightarrow \quad \frac{(z_1 + z_2)}{z_1} = e^{i\alpha/2} \cdot 2\cos(\alpha/2)$$

On squaring both sides, we get

$$\frac{(z_1 + z_2)^2}{z_1^2} = e^{i\alpha} \cdot (4\cos^2 \alpha/2)$$

$$\Rightarrow \frac{(z_1 + z_2)^2}{z_1^2} = \frac{z_2}{z_1} \cdot (4\cos^2 \alpha/2) \qquad \text{[from Eq. (i)]}$$

$$(z_1 + z_2)^2 = 4\, z_1 z_2 \cos^2(\alpha/2)$$

$$(-p)^2 = 4\, q \cos^2(\alpha/2)$$

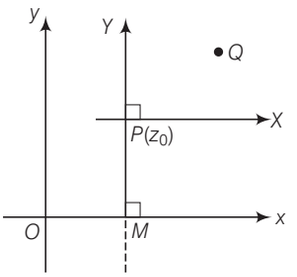
$$\left[\because z_1 \text{ and } z_2 \text{ are the roots of } z^2 + pz + q = 0 \;\; \therefore z_1 + z_2 = -p \text{ and } z_1 z_2 = q\right]$$

or $$p^2 = 4\, q \cos^2(\alpha/2)$$

Shifting the Origin in Case of Complex Numbers

Let O be the origin and P be a point with affix z_0. Let a point Q has affix z with respect to the coordinate system passing through O. When origin is shifted to the point P (z_0), then the new affix Z of the point Q with respect to new origin P is given by $Z = z - z_0$.

i.e., to shift the origin at z_0, we should replace z by $Z + z_0$.



Example 71. ***If z_1, z_2 and z_3 are the vertices of an equilateral triangle with z_0 as its circumcentre, then changing origin to z_0, show that $Z_1^2 + Z_2^2 + Z_3^2 = 0$, where Z_1, Z_2, Z_3 are new complex numbers of the vertices.***

Sol. In an equilateral triangle, the circumcentre and the centroid are the same point.

So, $$z_0 = \frac{z_1 + z_2 + z_3}{3}$$

$$\therefore \quad z_1 + z_2 + z_3 = 3z_0 \qquad \ldots(i)$$

To shift the origin at z_0, we have to replace z_1, z_2, z_3 and z_0 by $Z_1 + z_0, Z_2 + z_0, Z_3 + z_0$ and $0 + z_0$.

Then, Eq. (i) becomes

$$(Z_1 + z_0) + (Z_2 + z_0) + (Z_3 + z_0) = 3(0 + z_0)$$

$$\Rightarrow \quad Z_1 + Z_2 + Z_3 = 0$$

On squaring, we get

$$Z_1^2 + Z_2^2 + Z_3^2 + 2(Z_1Z_2 + Z_2Z_3 + Z_3Z_1) = 0 \qquad \ldots(ii)$$

But triangle with vertices Z_1, Z_2 and Z_3 is equilateral, then

$$Z_1^2 + Z_2^2 + Z_3^2 = Z_1Z_2 + Z_2Z_3 + Z_3Z_1 \qquad \ldots(iii)$$

From Eqs. (ii) and (iii), we get

$$3(Z_1^2 + Z_2^2 + Z_3^2) = 0$$

Therefore, $$Z_1^2 + Z_2^2 + Z_3^2 = 0$$

Inverse Points

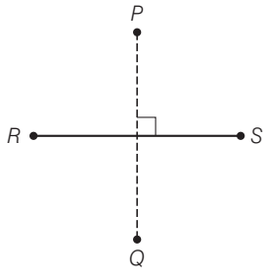
(a) **Inverse points with respect to a line** Two points P and Q are said to be the inverse points with respect to the line RS. If Q is the image of P in RS, i.e., if the line RS is the right bisector of PQ.

Example 72. Show that z_1, z_2 are the inverse points with respect to the line $z\bar{a} + a\bar{z} = b$, if $z_1\bar{a} + a\bar{z}_2 = b$.

Sol. Let RS be the line represented by the equation,

$$z\bar{a} + a\bar{z} = b \qquad \ldots(i)$$

Let P and Q are the inverse points with respect to the line RS. The point Q is the reflection (inverse) of the point P in the line RS, if the line RS is the right bisector of PQ. Take any point z in the line RS, then lines joining z to P and z to Q are equal.

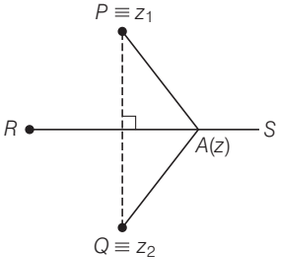


i.e., $\quad |z - z_1| = |z - z_2| \text{ or } |z - z_1|^2 = |z - z_2|^2$

i.e., $(z - z_1)(\bar{z} - \bar{z}_1) = (z - z_2)(\bar{z} - \bar{z}_2)$

$\Rightarrow z(\bar{z}_2 - \bar{z}_1) + \bar{z}(z_2 - z_1) + (z_1\bar{z}_1 - z_2\bar{z}_2) = 0 \qquad \ldots(ii)$

Hence, Eqs. (i) and (ii) are identical, therefore, comparing coefficients, we get



$$\frac{\bar{a}}{\bar{z}_2 - \bar{z}_1} = \frac{a}{z_2 - z_1} = \frac{-b}{z_1\bar{z}_1 - z_2\bar{z}_2}$$

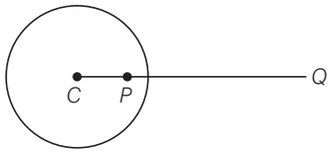
So that, $\quad \dfrac{z_1\bar{a}}{z_1(\bar{z}_2 - \bar{z}_1)} = \dfrac{a\bar{z}_2}{\bar{z}_2(z_2 - z_1)}$

$$= \frac{-b}{z_1\bar{z}_1 - z_2\bar{z}_2} = \frac{z_1\bar{a} + a\bar{z}_2 - b}{0}$$

[by ratio and proportion rule]

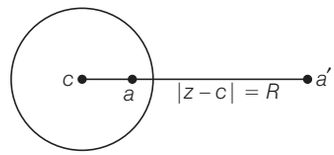
$$z_1\bar{a} + a\bar{z}_2 - b = 0 \quad \text{or} \quad z_1\bar{a} + a\bar{z}_2 = b$$

(b) **Inverse points with respect to a circle** If C is the centre of the circle and P, Q are the inverse points with respect to the circle, then three points C, P, Q are collinear and also $CP \cdot CQ = r^2$, where r is the radius of the circle.



Example 73. Show that inverse of a point a with respect to the circle $|z - c| = R$ (a and c are complex numbers, centre c and radius R) is the point $c + \dfrac{R^2}{\bar{a} - \bar{c}}$.

Sol. Let a' be the inverse point of a with respect to the circle $|z - c| = R$, then by definition,



The points c, a, a' are collinear.

We have, $\arg(a' - c) = \arg(a - c)$

$$= -\arg(\bar{a} - \bar{c}) \qquad [\because \arg\bar{z} = -\arg z]$$

$\Rightarrow \arg(a' - c) + \arg(\bar{a} - \bar{c}) = 0$

$\Rightarrow \arg\{(a' - c)(\bar{a} - \bar{c})\} = 0$

$\therefore (a' - c)(\bar{a} - \bar{c})$ is purely real and positive.

By definition, $|a' - c||a - c| = R^2 \qquad [\because CP \cdot CQ = r^2]$

$\Rightarrow |a' - c||\bar{a} - \bar{c}| = R^2 \qquad [\because |z| = |\bar{z}|]$

$\Rightarrow |(a' - c)(\bar{a} - \bar{c})| = R^2$

$\Rightarrow (a' - c)(\bar{a} - \bar{c}) = R^2$

$[\because (a' - c)(\bar{a} - \bar{c})$ is purely real and positive$]$

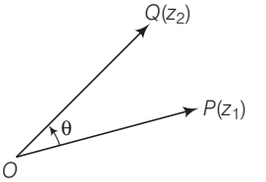
$$\Rightarrow a' - c = \frac{R^2}{\bar{a} - \bar{c}} \Rightarrow a' = c + \frac{R^2}{\bar{a} - \bar{c}}$$

Dot and Cross Product

Let $z_1 = x_1 + iy_1 \equiv (x_1, y_1)$ and $z_2 = x_2 + iy_2 \equiv (x_2, y_2)$, where $x_1, y_1, x_2, y_2 \in R$ and $i = \sqrt{-1}$, be two complex numbers.

If $\angle POQ = \theta$, then from Coni method,

$$\frac{z_2 - 0}{z_1 - 0} = \frac{|z_2|}{|z_1|} e^{i\theta}$$



$$\Rightarrow \frac{z_2\bar{z}_1}{z_1\bar{z}_1} = \frac{|z_2|}{|z_1|} e^{i\theta}$$

$$\Rightarrow \frac{z_2\bar{z}_1}{|z_1|^2} = \frac{|z_2|}{|z_1|} e^{i\theta}$$

$$z_2\bar{z}_1 = |z_1||z_2| e^{i\theta}$$

$$z_2\bar{z}_1 = |z_1||z_2|(\cos\theta + i\sin\theta)$$

$$\Rightarrow \quad \text{Re}(z_2\bar{z}_1) = |z_1||z_2|\cos\theta \qquad \ldots(i)$$

$$\text{and} \quad \text{Im}(z_2\bar{z}_1) = |z_1||z_2|\sin\theta \qquad \ldots(ii)$$

The dot product z_1 and z_2 is defined by,

$$z_1 \cdot z_2 = |z_1||z_2|\cos\theta$$

$$= \text{Re}(\bar{z}_1 z_2) = x_1x_2 + y_1y_2 \qquad \text{[from Eq. (i)]}$$

and cross product of z_1 and z_2 is defined by

$$z_1 \times z_2 = |z_1||z_2|\sin\theta$$
$$= \text{Im}(\bar{z}_1 z_2) = x_1 y_2 - x_2 y_1 \qquad \text{[from Eq. (ii)]}$$

Hence, $z_1 \cdot z_2 = x_1 x_2 + y_1 y_2 = \text{Re}(\bar{z}_1 z_2)$
and $z_1 \times z_2 = x_1 y_2 - x_2 y_1 = \text{Im}(\bar{z}_1 z_2)$

Results for Dot and Cross Products of Complex Number

1. If z_1 and z_2 are perpendicular, then $z_1 \cdot z_2 = 0$
2. If z_1 and z_2 are parallel, then $z_1 \times z_2 = 0$
3. Projection of z_1 on $z_2 = (z_1 \cdot z_2)/|z_2|$
4. Projection of z_2 on $z_1 = (z_1 \cdot z_2)/|z_1|$
5. Area of triangle, if two sides represented by z_1 and z_2 is $\frac{1}{2}|z_1 \times z_2|$.
6. Area of a parallelogram having sides z_1 and z_2 is $|z_1 \times z_2|$.
7. Area of parallelogram, if diagonals represented by z_1 and z_2 is $\frac{1}{2}|z_1 \times z_2|$.

Example 74. If $z_1 = 2 + 5i, z_2 = 3 - i$, where $i = \sqrt{-1}$, find

(i) $z_1 \cdot z_2$ (ii) $z_1 \times z_2$
(iii) $z_2 \cdot z_1$ (iv) $z_2 \times z_1$
(v) acute angle between z_1 and z_2.
(vi) projection of z_1 on z_2.

Sol. (i) $z_1 \cdot z_2 = x_1 x_2 + y_1 y_2 = (2)(3) + (5)(-1) = 1$

(ii) $z_1 \times z_2 = x_1 y_2 - x_2 y_1 = (2)(-1) - (3)(5) = -17$

(iii) $z_2 \cdot z_1 = x_1 x_2 + y_1 y_2 = (2)(3) + (5)(-1) = 1$

(iv) $z_2 \times z_1 = x_2 y_1 - x_1 y_2 = (3)(5) - (2)(-1) = 17$

(v) Let angle between z_1 and z_2 be θ, then

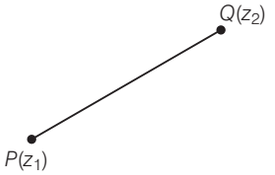
$$z_1 \cdot z_2 = |z_1||z_2|\cos\theta$$
$$\Rightarrow \quad 1 = \sqrt{(4+25)}\sqrt{(9+1)}\cos\theta$$
$$\therefore \quad \cos\theta = \frac{1}{\sqrt{290}} \quad \therefore \quad \theta = \cos^{-1}\left(\frac{1}{\sqrt{290}}\right)$$

(vi) Projection of z_1 on $z_2 = \dfrac{z_1 \cdot z_2}{|z_2|} = \dfrac{1}{\sqrt{(9+1)}} = \dfrac{1}{\sqrt{10}}$

Use of Complex Numbers in Coordinate Geometry

(a) Distance Formula

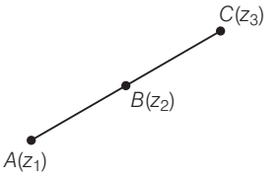
The distance between two points $P(z_1)$ and $Q(z_2)$ is given by



$$PQ = |z_2 - z_1| = |\text{affix of } Q - \text{affix of } P|$$

Remark

1. The distance of a point z from origin, $|z - 0| = |z|$
2. Three points $A(z_1)$, $B(z_2)$ and $C(z_3)$ are collinear, then $AB + BC = AC$



i.e. $|z_1 - z_2| + |z_2 - z_3| = |z_1 - z_3|$.

Example 75. Show that the points representing the complex numbers $(3+2i), (2-i)$ and $-7i$, where $i = \sqrt{-1}$, are collinear.

Sol. Let $z_1 = 3 + 2i, z_2 = 2 - i$ and $z_3 = -7i$.

Then, $|z_1 - z_2| = |1 + 3i| = \sqrt{10}, |z_2 - z_3| = |2 + 6i|$
$= \sqrt{40} = 2\sqrt{10}$

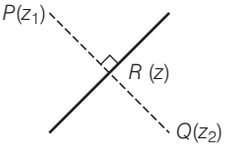
and $|z_1 - z_3| = |3 + 9i| = \sqrt{90} = 3\sqrt{10}$

$\therefore \quad |z_1 - z_2| + |z_2 - z_3| = |z_1 - z_3|$

Hence, the points $(3 + 2i), (2 - i)$ and $-7i$ are collinear.

(b) Equation of the Perpendicular Bisector

If $P(z_1)$ and $Q(z_2)$ are two fixed points and $R(z)$ is moving point, such that it is always at equal distance from $P(z_1)$ and $Q(z_2)$.



i.e. $PR = QR$

or $|z - z_1| = |z - z_2|$

or $z(\bar{z}_1 - \bar{z}_2) + \bar{z}(z_1 - z_2) = z_1\bar{z}_1 - z_2\bar{z}_2$

or $z(\bar{z}_1 - \bar{z}_2) + \bar{z}(z_1 - z_2) = |z_1|^2 - |z_2|^2$

Hence, z lies on the perpendicular bisectors of z_1 and z_2.

Example 76. Find the perpendicular bisector of $3 + 4i$ and $-5 + 6i$, where $i = \sqrt{-1}$.

Sol. Let $z_1 = 3 + 4i$ and $z_2 = -5 + 6i$

If z is moving point, such that it is always equal distance from z_1 and z_2.

i.e. $|z - z_1| = |z - z_2|$

or $z(\bar{z}_1 - \bar{z}_2) + \bar{z}(z_1 - z_2) = |z_1|^2 - |z_2|^2$

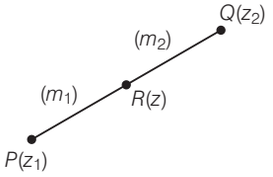
$\Rightarrow z((3 - 4i) - (-5 - 6i)) + \bar{z}((3 + 4i) - (-5 + 6i)) = 25 - 61$

Hence, $(8 + 2i)z + (8 - 2i)\bar{z} + 36 = 0$

which is required perpendicular bisector.

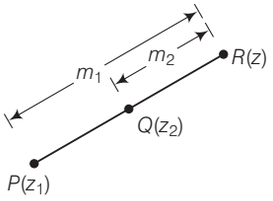
(c) Section Formula

If $R(z)$ divides the joining of $P(z_1)$ and $Q(z_2)$ in the ratio $m_1 : m_2$ $(m_1, m_2 > 0)$.



(i) If $R(z)$ divides the segment PQ internally in the ratio of $m_1 : m_2$, then $z = \dfrac{m_1 z_2 + m_2 z_1}{m_1 + m_2}$

(ii) If $R(z)$ divides the segment PQ externally in the ratio of $m_1 : m_2$, then $z = \dfrac{m_1 z_2 - m_2 z_1}{m_1 - m_2}$

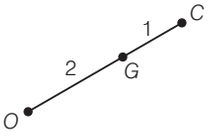


Remark

1. If $R(z)$ is the mid-point of PQ, then affix of R is $\dfrac{z_1 + z_2}{2}$.
2. If z_1, z_2 and z_3 are affixes of the vertices of a triangle, then affix of its centroid is $\dfrac{z_1 + z_2 + z_3}{3}$.
3. In acute angle triangle, orthocentre (O), nine point centre (N), centroid (G) and circumcentre (C) are collinear and $\dfrac{OG}{GC} = \dfrac{2}{1}$, $\dfrac{ON}{NG} = \dfrac{1}{1}$.
4. If z_1, z_2, z_3 and z_4 are the affixes of the vertices of a parallelogram taken in order, then $z_1 + z_3 = z_2 + z_4$.

Example 77. If z_1, z_2 and z_3 are the affixes of the vertices of a triangle having its circumcentre at the origin. If z is the affix of its orthocentre, prove that $z_1 + z_2 + z_3 - z = 0$.

Sol. We know that orthocentre O, centroid G and circumcentre C of a triangle are collinear, such that G divides OC in the ratio $2 : 1$. Since, affix of G is $\dfrac{z_1 + z_2 + z_3}{3}$ and C is the origin. Therefore, by section formula, we get



$$\Rightarrow \quad \frac{z_1 + z_2 + z_3}{3} = \frac{2 \cdot 0 + 1 \cdot z}{2 + 1}$$

$$\Rightarrow \quad z_1 + z_2 + z_3 = z$$

Therefore, $z_1 + z_2 + z_3 - z = 0$

Example 78. Let z_1, z_2 and z_3 be three complex numbers and $a, b, c \in R$, such that $a + b + c = 0$ and $az_1 + bz_2 + cz_3 = 0$, then show that z_1, z_2 and z_3 are collinear.

Sol. Given, $a + b + c = 0$ ...(i)

and $az_1 + bz_2 + cz_3 = 0$ ...(ii)

$\Rightarrow \quad az_1 + bz_2 - (a + b) z_3 = 0$ [from Eq. (i)]

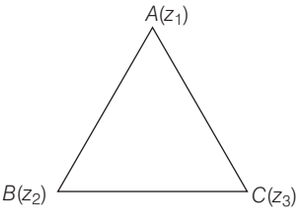
or $z_3 = \dfrac{az_1 + bz_2}{a + b}$

It follows that z_3 divides the line segment joining z_1 and z_2 internally in the ratio $b : a$. (If a, b are of same sign and opposite sign, then externally.)

Hence, z_1, z_2 and z_3 are collinear.

(d) Area of Triangle

If z_1, z_2 and z_3 are the affixes of the vertices of a triangle, then its area $= \dfrac{1}{4} \left| \begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} \right|$



Remark

The area of the triangle with vertices z, ωz and $z + \omega z$ is $\dfrac{\sqrt{3}}{4} |z|^2$, where ω is the cube root of unity.

Example 79. Show that the area of the triangle on the argand plane formed by the complex numbers z, iz and $z + iz$ is $\dfrac{1}{2} |z|^2$, where $i = \sqrt{-1}$.

Sol. Required area $= \dfrac{1}{4} \left| \begin{vmatrix} z & \bar{z} & 1 \\ iz & \overline{iz} & 1 \\ z + iz & \overline{z + iz} & 1 \end{vmatrix} \right|$

$$= \frac{1}{4} \left| \begin{vmatrix} z & \bar{z} & 1 \\ iz & -i\bar{z} & 1 \\ z + iz & \bar{z} + i\bar{z} & 1 \end{vmatrix} \right|$$

$$= \frac{1}{4}\left|\begin{vmatrix} z & \bar{z} & 1 \\ iz & -i\bar{z} & 1 \\ z+iz & \bar{z}-i\bar{z} & 1 \end{vmatrix}\right|$$

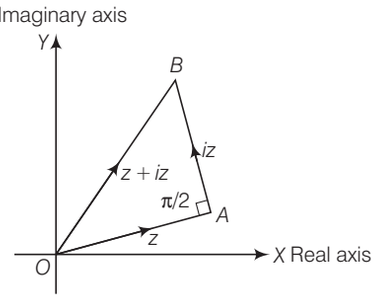
On applying $R_3 \to R_3 - (R_1 + R_2)$, we get

$$\text{Area} = \frac{1}{4}\left|\begin{vmatrix} z & \bar{z} & 1 \\ iz & -i\bar{z} & 1 \\ 0 & 0 & -1 \end{vmatrix}\right| = \frac{1}{4}\left|(-1)(-iz\bar{z} - iz\bar{z})\right|$$

$$= \frac{1}{4}\left|2iz\bar{z}\right| = \frac{1}{2}|i|\left|z\bar{z}\right| = \frac{1}{2}|z|^2$$

Aliter

We have, $iz = z(\cos(\pi/2) + i\sin(\pi/2)) = ze^{(i\pi/2)}$ iz is the vector obtained by rotating vector z in anti-clockwise direction through $(\pi/2)$. Therefore, $OA \perp AB$,



Now, area of $\Delta OAB = \frac{1}{2} OA \times AB = \frac{1}{2}|z||iz|$

$$= \frac{1}{2}|z||i||z| = \frac{1}{2}|z|^2$$

(e) Equation of a Straight Line

(i) Parametric form

Equation of the straight line joining the points having affixes z_1 and z_2 is

$$z = tz_1 + (1-t)z_2, \text{ where } t \in R \sim \{0\}$$

Proof

$$\because \quad z = tz_1 + (1-t)z_2 = \frac{tz_1 + (1-t)z_2}{t + (1-t)}$$

Hence, z divides the line joining z_1 and z_2 in the ratio $1-t : t$. Thus, the points z_1, z_2, z are collinear.

(ii) Non-parametric form

Equation of the straight line joining the points having affixes z_1 and z_2 is

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

or $z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + z_1\bar{z}_2 - \bar{z}_1 z_2 = 0$

Proof Equation of the straight line joining points having affixes z_1 and z_2 is

$$z = tz_1 + (1-t)z_2, \text{ where } t \in R \sim \{0\}$$

$$\Rightarrow \quad z - z_2 = t(z_1 - z_2) \qquad \ldots(i)$$

and $$\overline{z - z_2} = t(\overline{z_1 - z_2})$$

or $$\bar{z} - \bar{z}_2 = t(\bar{z}_1 - \bar{z}_2) \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$\frac{z - z_2}{\bar{z} - \bar{z}_2} = \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} \Rightarrow \frac{z - z_2}{z_1 - z_2} = \frac{\bar{z} - \bar{z}_2}{\bar{z}_1 - \bar{z}_2}$$

$$\Rightarrow \quad \begin{vmatrix} z - z_2 & \bar{z} - \bar{z}_2 & 0 \\ z_1 - z_2 & \bar{z}_1 - \bar{z}_2 & 0 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

Now, applying $R_1 \to R_1 + R_3$ and $R_2 \to R_2 + R_3$, we get

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

or $z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + z_1\bar{z}_2 - \bar{z}_1 z_2 = 0$

Aliter

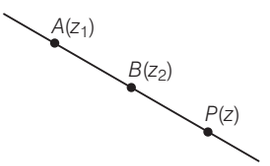
Let $P(z)$ be an arbitrary point on the line, which pass through $A(z_1)$ and $B(z_2)$.

$$\therefore \quad \angle BAP = 0 \text{ or } \pi$$

$$\therefore \quad \arg\left(\frac{z - z_1}{z_2 - z_1}\right) = 0 \text{ or } \pi \quad \text{[by rotation theorem]}$$

$$\Rightarrow \frac{z - z_1}{z_2 - z_1} \text{ is purely real.}$$

$$\therefore \quad \left(\frac{z - z_1}{z_2 - z_1}\right) = \left(\frac{\overline{z - z_1}}{\overline{z_2 - z_1}}\right) \Rightarrow \frac{z - z_1}{z_2 - z_1} = \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1}$$



$$z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + z_1\bar{z}_2 - \bar{z}_1 z_2 = 0$$

or $$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

Remark

If z_1, z_2 and z_3 are collinear, $\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix} = 0$

or $$\sum \overline{z_1}(z_2 - z_3) = 0.$$

(iii) General form The general equation of a straight line is of the form $\bar{a}z + a\bar{z} + b = 0$, where a is a complex number and b is a real number.

Sol. The equation of a straight line passing through points having affixes z_1 and z_2 is given by

$$z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + z_1\bar{z}_2 - \bar{z}_1 z_2 = 0 \quad \ldots(i)$$

On multiplying Eq. (i) by i (where, $i = \sqrt{-1}$), we get

$$zi(\bar{z}_1 - \bar{z}_2) - \bar{z}\, i(z_1 - z_2) + i(z_1\bar{z}_2 - \bar{z}_1 z_2) = 0$$

$$\Rightarrow \quad \bar{z}\{-i(z_1 - z_2)\} + z\{i(\bar{z}_1 - \bar{z}_2)\} + i(z_1\bar{z}_2 - \bar{z}_1 z_2) = 0$$

$$\Rightarrow \bar{z}\{-i(z_1 - z_2)\} + z\overline{\{-i(z_1 - z_2)\}} + \{i(2i\,\mathrm{Im}(z_1\bar{z}_2))\} = 0$$

$$\Rightarrow \bar{z}\{-i(z_1 - z_2)\} + z\overline{\{-i(z_1 - z_2)\}} + \{(-2\mathrm{Im}(z_1\bar{z}_2)\} = 0$$

$$\Rightarrow \quad \bar{z}a + z\bar{a} + b = 0,$$

where, $a = -i(z_1 - z_2), b = -2\mathrm{Im}(z_1\bar{z}_2)$

Hence, the general equation of a straight line is of the form $\bar{a}z + a\bar{z} + b = 0$,

where a is complex number and b is a real number.

(iv) Slope of the line $\bar{a}z + a\bar{z} + b = 0$

Let $A(z_1)$ and $B(z_2)$ be two points on the line $\bar{a}z + a\bar{z} + b = 0$, then

$$\bar{a}z_1 + a\bar{z}_1 + b = 0$$

and $$\bar{a}z_2 + a\bar{z}_2 + b = 0$$

$$\therefore \quad \bar{a}(z_1 - z_2) + a(\bar{z}_1 - \bar{z}_2) = 0$$

$$\Rightarrow \quad \frac{z_1 - z_2}{\overline{z_1 - z_2}} = -\frac{a}{\bar{a}} \qquad [\text{Remember}]$$

$$\text{Complex slope of } AB = -\frac{a}{\bar{a}} = -\frac{\text{coefficient of } \bar{z}}{\text{coefficient of } z}$$

Thus, the complex slope of the line $\bar{a}z + a\bar{z} + b = 0$ is $-\frac{a}{\bar{a}}$.

Remark

The real slope of the line $\bar{a}z + a\bar{z} + b = 0$ is $-\frac{\mathrm{Re}(a)}{\mathrm{Im}(a)}$, i.e. $-\frac{\mathrm{Re}(\text{coefficient of } \bar{z})}{\mathrm{Im}(\text{coefficient of } \bar{z})}$.

Important Theorem

If α_1 and α_2 are the complex slopes of two lines on the argand plane, then prove that the lines are

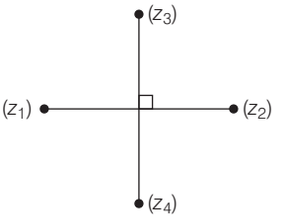
(i) perpendicular, if $\alpha_1 + \alpha_2 = 0$.

(ii) parallel, if $\alpha_1 = \alpha_2$.

Proof Let z_1 and z_2 be the affixes of two points on one line with complex slope α_1 and z_3 and z_4 be the affixes of two points another line with complex slope α_2. Then,

$$\alpha_1 = \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} \quad \text{and} \quad \alpha_2 = \frac{z_3 - z_4}{\bar{z}_3 - \bar{z}_4} \quad \ldots(i)$$

(i) If the lines are perpendicular, then



$$\frac{(z_1 - z_2)}{|z_1 - z_2|} = \frac{(z_3 - z_4)}{|z_3 - z_4|} e^{i\pi/2}$$

$$\Rightarrow \quad \frac{(z_1 - z_2)^2}{|z_1 - z_2|^2} = \frac{(z_3 - z_4)^2}{|z_3 - z_4|^2} e^{i\pi}$$

$$\Rightarrow \quad \frac{(z_1 - z_2)^2}{(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)} = \frac{(z_3 - z_4)^2}{(z_3 - z_4)(\bar{z}_3 - \bar{z}_4)} e^{i\pi}$$

$$\Rightarrow \quad \frac{(z_1 - z_2)}{(\bar{z}_1 - \bar{z}_2)} = \frac{(z_3 - z_4)}{(\bar{z}_3 - \bar{z}_4)}(-1)$$

$$\Rightarrow \quad \alpha_1 = -\alpha_2 \qquad [\text{from Eq. (i)}]$$

$$\therefore \quad \alpha_1 + \alpha_2 = 0$$

(ii) If the lines are parallel, then

$$\frac{z_1 - z_2}{|z_1 - z_2|} = \frac{z_3 - z_4}{|z_3 - z_4|} e^{0}$$

$$\Rightarrow \quad \frac{(z_1 - z_2)^2}{|z_1 - z_2|^2} = \frac{(z_3 - z_4)^2}{|z_3 - z_4|^2}$$

$$\Rightarrow \quad \frac{(z_1 - z_2)^2}{(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)} = \frac{(z_3 - z_4)^2}{(z_3 - z_4)(\bar{z}_3 - \bar{z}_4)}$$

$$\Rightarrow \quad \frac{(z_1 - z_2)}{(\bar{z}_1 - \bar{z}_2)} = \frac{(z_3 - z_4)}{(\bar{z}_3 - \bar{z}_4)}$$

$$\Rightarrow \quad \alpha_1 = \alpha_2$$

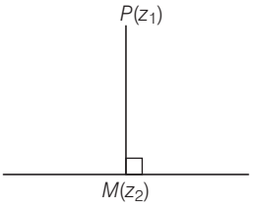
Remark

1. The equation of a line parallel to the line $\bar{a}z + a\bar{z} + b = 0$ is $\bar{a}z + a\bar{z} + \lambda = 0$, where $\lambda \in R$.
2. The equation of a line perpendicular to the line $\bar{a}z + a\bar{z} + b = 0$ is $\bar{a}z - a\bar{z} + i\lambda = 0$ where, $\lambda \in R$ and $i = \sqrt{-1}$

(v) Length of perpendicular from a given point on a given line

The length of perpendicular from a point $P(z_1)$ to the line $\bar{a}z + a\bar{z} + b = 0$ is given by $\frac{|\bar{a}z_1 + a\bar{z}_1 + b|}{2|a|}$.

Proof Let PM be perpendicular from P on the line $\bar{a}\,z + a\,\bar{z} + b = 0$ and let the affix of M be z_2, then



$$PM = |\,z_1 - z_2\,|$$
$$\bar{a}\,z + a\,\bar{z} + b = 0$$

and $M(z_2)$ lies on $\bar{a}\,z + a\,\bar{z} + b = 0$, then

$$\bar{a}\,z_2 + a\,\bar{z}_2 + b = 0 \qquad \ldots(i)$$

Since, PM perpendicular to the line $(\bar{a}\,z + a\,\bar{z} + b = 0)$.

Therefore, $\dfrac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} + \left(-\dfrac{a}{\bar{a}}\right) = 0$

$\Rightarrow \quad \bar{a}\,z_1 - \bar{a}\,z_2 - a\,\bar{z}_1 + a\,\bar{z}_2 = 0$

$\Rightarrow \quad \bar{a}\,z_1 + a\,\bar{z}_1 + b = 2\,a\,\bar{z}_1 + \bar{a}\,z_2 - a\,\bar{z}_2 + b$

$= 2\,a\,\bar{z}_1 - a\bar{z}_2 + (\bar{a}z_2 + b)$

$= 2\,a\,\bar{z}_1 - a\,\bar{z}_2 - a\,\bar{z}_2 \qquad [\because \bar{a}\,z_2 + b = -\,a\,\bar{z}_2]$

$= 2\,a\,(\bar{z}_1 - \bar{z}_2)$

or $|\,\bar{a}\,z_1 + a\,\bar{z}_1 + b\,| = 2\,|\,a\,|\,|\,\bar{z}_1 - \bar{z}_2\,|$

$= 2\,|\,a\,|\,|\,z_1 - z_2\,| \quad [\because |\bar{z}| = |z|]$

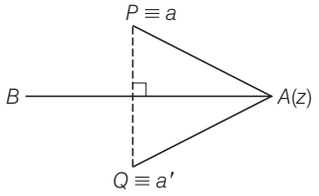
$= 2\,|\,a\,|\,PM$

$\therefore \qquad PM = \dfrac{|\,\bar{a}\,z_1 + a\,\bar{z}_1 + b\,|}{2\,|\,a\,|}$

Example 80. Show that the point a' is the reflection of the point a in the line $z\bar{b} + \bar{z}b + c = 0$, if $a'\bar{b} + \bar{a}b + c = 0$.

Sol. Since, a' is the reflection of point a through the line.

So, the mid-point of PQ



i.e., $\dfrac{a + a'}{2}$ lies on $z\bar{b} + \bar{z}b + c = 0$

or $\bar{b}\left(\dfrac{a + a'}{2}\right) + b\left(\dfrac{\bar{a} + \bar{a'}}{2}\right) + c = 0$

$\Rightarrow \quad \bar{b}\,(a + a') + b\,(\bar{a} + \bar{a'}) + 2c = 0 \qquad \ldots(i)$

Since, $PQ \perp AB$. Therefore,

Complex slope of PQ + Complex slope of $AB = 0$

$\Rightarrow \quad \dfrac{a - a'}{\bar{a} - \bar{a}'} + \left(-\dfrac{b}{\bar{b}}\right) = 0$

$\Rightarrow \quad \bar{b}\,(a - a') - b\,(\bar{a} - \bar{a}') = 0 \qquad \ldots(ii)$

On subtracting Eq. (ii) from Eq. (i), we get

$$a'\bar{b} + \bar{a}b + c = 0$$

Aliter

Equation of perpendicular bisector of PQ is

$z\,(\bar{a}' - \bar{a}\,) + \bar{z}\,(a' - a) - a'\bar{a}' + a\bar{a} = 0 \qquad \ldots(i)$

and given line $\qquad z\bar{b} + \bar{z}\,b + c = 0 \qquad \ldots(ii)$

Since, Eqs. (i) and (ii) are identical, we have

$$\frac{\bar{a}' - \bar{a}}{\bar{b}} = \frac{a' - a}{b} = \frac{a\bar{a} - a'\bar{a}'}{c} = k \qquad [\text{say}]$$

$\because \qquad a' - \bar{a} = \bar{b}\,k,\ a' - a = bk$

and $\qquad a\bar{a} - a'\bar{a}' = ck$

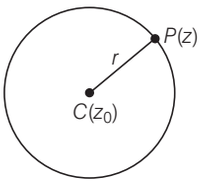
Now, $\qquad a'\bar{b} + \bar{a}b = \left\{a'\left(\dfrac{\bar{a}' - \bar{a}}{k}\right) + \bar{a}\left(\dfrac{a' - a}{k}\right)\right\}$

$$= \frac{1}{k}\,\{a'\bar{a}' - a\,\bar{a}\} = \frac{1}{k}\,(-\,ck) = -\,c$$

Hence, $a'\bar{b} + \bar{a}b + c = 0$

(f) Circle

The equation of a circle whose centre is at point affix z_0 and radius r, is $|\,z - z_0\,| = r$.



Remark

1. If the centre of the circle is at origin and radius r, then its equation is $|\,z\,| = r$.
2. $|\,z - z_0\,| < r$ represents interior of a circle $|\,z - z_0\,| = r$ and $|\,z - z_0\,| > r$ represent exterior of the circle $|\,z - z_0\,| = r$.
3. $r < |\,z - z_0\,| < R$, this region is known as **annulus**.

(i) General Equation of a Circle

The general equation of the circle is

$$z\,\bar{z} + \bar{a}\,z + a\,\bar{z} + b = 0,$$

where a is a complex number and $b \in R$, having centre at $(-a)$ and radius $= \sqrt{|\,a\,|^2 - b}$.

Proof The equation of circle having centre at z_0 and radius r is

$$|\,z - z_0\,| = r$$

$$\Rightarrow \quad |z - z_0|^2 = r^2$$

$$\Rightarrow \quad (z - z_0)(\bar{z} - \bar{z}_0) = r^2$$

$$\Rightarrow \quad z\bar{z} - z\bar{z}_0 - z_0\bar{z} + z_0\bar{z}_0 = r^2$$

$$\Rightarrow \quad z\bar{z} + (-\bar{z}_0)z + (-z_0)\bar{z} + |z_0|^2 - r^2 = 0$$

$$\Rightarrow \quad z\bar{z} + \bar{a}z + a\bar{z} + b = 0$$

where, $a = -z_0$ and $b = |z_0|^2 - r^2$

$$\Rightarrow \quad z\bar{z} + \bar{a}z + a\bar{z} + b = 0$$

where, $b \in R$ represents a circle having centre at $(-a)$ and radius $= \sqrt{|z_0|^2 - b} = \sqrt{|a|^2 - b}$.

Remark

Rule to find the centre and radius of a circle whose equation is given

1. Make the coefficient of $z\bar{z}$ equal to 1 and right hand side equal to zero.
2. The centre of circle will be $= (-a) =$ (– coefficient of $\bar{z}$).
3. Radius $= \sqrt{(|a|^2 - \text{constant term})}$

Example 81. Find the centre and radius of the circle $2z\bar{z} + (3 - i)z + (3 + i)\bar{z} - 7 = 0$, where $i = \sqrt{-1}$.

Sol. The given equation can be written as

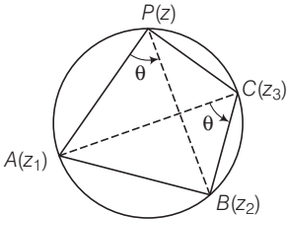
$$z\bar{z} + \left(\frac{\overline{3+i}}{2}\right)z + \left(\frac{3+i}{2}\right)\bar{z} - \frac{7}{2} = 0$$

So, it represent a circle with centre at $-\left(\frac{3+i}{2}\right)$ and radius

$$= \sqrt{\left(\left|-\left(\frac{3+i}{2}\right)\right|^2 + \frac{7}{2}\right)} = \sqrt{\left(\frac{9}{4} + \frac{1}{4} + \frac{7}{2}\right)} = \sqrt{6}$$

(ii) Equation of Circle Through Three Non-Collinear Points

Let $A(z_1), B(z_2), C(z_3)$ be three points on the circle and $P(z)$ be any point on the circle, then



$$\angle ACB = \angle APB$$

Using Coni method,

in ΔACB, $$\frac{z_2 - z_3}{z_1 - z_3} = \frac{BC}{CA}e^{i\theta} \quad \ldots(i)$$

in ΔAPB, $$\frac{z_2 - z}{z_1 - z} = \frac{BP}{AP}e^{i\theta} \quad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$\frac{(z - z_1)(z_2 - z_3)}{(z - z_2)(z_1 - z_3)} = \text{Real} \quad \ldots(iii)$$

Remark

If four points z_1, z_2, z_3, z_4 are concyclic, then $\frac{(z_4 - z_1)(z_2 - z_3)}{(z_4 - z_2)(z_1 - z_3)} =$ real [replacing z by z_4 in Eq. (iii)]

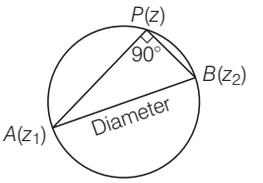
or $$\arg\left[\frac{(z_2 - z_3)(z_4 - z_1)}{(z_1 - z_3)(z_4 - z_2)}\right] = \pi, 0.$$

(iii) Equation of Circle in Diametric Form

If end points of diameter represented by $A(z_1)$ and $B(z_2)$ and $P(z)$ is any point on circle.

$\therefore \quad \angle APB = 90°$

$\therefore$ Complex slope of PA + Complex slope of $PB = 0$



$$\Rightarrow \quad \left(\frac{z - z_1}{\bar{z} - \bar{z}_1}\right) + \left(\frac{z - z_2}{\bar{z} - \bar{z}_2}\right) = 0$$

Hence, $(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$ which is required equation of circle in diametric form.

(iv) Other Forms of Circle

(a) Equation of all circles which are orthogonal to $|z - z_1| = r_1$ and $|z - z_2| = r_2$.

Let the circle be $|z - \alpha| = r$ cut given circles orthogonally.

$\therefore \quad r^2 + r_1^2 = |\alpha - z_1|^2 \quad \ldots(i)$

and $\quad r^2 + r_2^2 = |\alpha - z_2|^2 \quad \ldots(ii)$

On solving,

$$r_2^2 - r_1^2 = \alpha(\bar{z}_1 - \bar{z}_2) + \bar{\alpha}(z_1 - z_2) + |z_2|^2 - |z_1|^2$$

and let $\alpha = a + ib$, $i = \sqrt{-1}$, $a, b \in R$

(b) **Apollonius circle** $\left|\frac{z - z_1}{z - z_2}\right| = k \neq 1$

It is the circle with join of z_3 and z_4 as a diameter, where $z_3 = \frac{z_1 + kz_2}{1 + k}$, $z_4 = \frac{z_1 - kz_2}{1 - k}$

for $k = 1$, the circle reduces to the straight line which is perpendicular bisector of the line segment from z_1 to z_2.

(c) **Circular arc** $\arg\left(\frac{z-z_1}{z-z_2}\right)=\alpha$

This is an arc of a circle in which the chord joining z_1 and z_2 subtends angle α at any point on the arc.

If $\alpha=\pm\frac{\pi}{2}$, then locus of $z$is a circle with the join of z_1 and z_2 as diameter. If $\alpha=0$ or π, then locus is a straight line through the points z_1 and z_2.

(d) The equation $|z-z_1|^2+|z-z_2|^2=k$, will represent a circle, if $k\geq\frac{1}{2}|z_1-z_2|^2$.

Example 82. ***Find all circles which are orthogonal to*** $|z|=1$ ***and*** $|z-1|=4$.

Sol. Let $|z-\alpha|=k$ ...(i)

(where, $\alpha=a+ib$ and $a, b, k\in R$ and $i=\sqrt{-1}$) be a circle which cuts the circles

$|z|=1$ ...(ii)

and $|z-1|=4$ ...(iii)

Orthogonally, then using the property that the sum of squares of their radii is equal to square of distance between centres. Thus, the circle (i) will cut the circles (ii) and (iii) orthogonally, if

$$k^2+1=|\alpha-0|^2=\alpha\bar{\alpha}$$

and $$k^2+16=|\alpha-1|^2=(\alpha-1)(\bar{\alpha}-1)$$

$$=\alpha\bar{\alpha}-(\alpha+\bar{\alpha})+1$$

$\therefore\ 1-(\alpha+\bar{\alpha})-15=0\Rightarrow\alpha+\bar{\alpha}=-14$

$\therefore\quad 2a=-14\Rightarrow a=-7$

$\Rightarrow\quad \alpha=a+ib=-7+ib$

Also, $k^2=|\alpha|^2-1=(-7)^2+b^2-1=b^2+48$

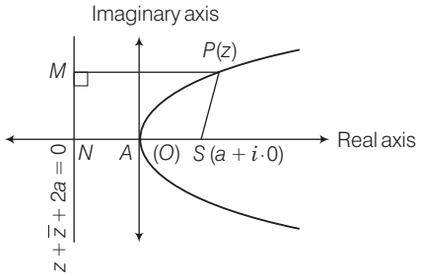
$\Rightarrow\quad k=\sqrt{(b^2+48)}$

Therefore, required family of circles is given by

$$|z+7-ib|=\sqrt{(48+b^2)}.$$

(g) Equation of Parabola

Now, for parabola



$SP=PM$

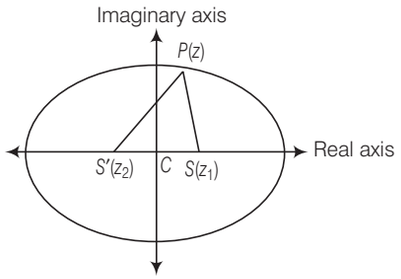
$$|z-a|=\frac{|z+\bar{z}+2a|}{2}$$

or $\quad z\bar{z}-4a(z+\bar{z})=\frac{1}{2}\{z^2+(\bar{z})^2\}$

where, $a\in R$ (focus), directrix is $z+\bar{z}+2a=0$.

(h) Equation of Ellipse

For ellipse



$$SP+S'P=2a$$

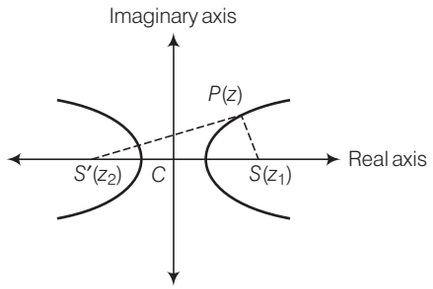
$\Rightarrow\quad |z-z_1|+|z-z_2|=2a$

where, $2a>|z_1-z_2|$ [since, eccentricity <1]

Then, point z describes an ellipse having foci at z_1 and z_2 and $a\in R^+$.

(i) Equation of Hyperbola

For hyperbola



$$SP-S'P=2a\Rightarrow|z-z_1|-|z-z_2|=2a$$

where, $\quad 2a<|z_1-z_2|$ [since, eccentricity >1]

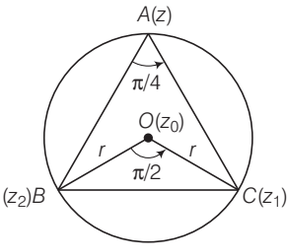
Then, point z describes a hyperbola having foci at z_1 and z_2 and $a\in R^+$.

Examples on Geometry

Example 83. Let $z_1=10+6i, z_2=4+6i$, where $i=\sqrt{-1}$. If z is a complex number, such that the argument of $(z-z_1)/(z-z_2)$ is $\pi/4$, then prove that $|z-7-9i|=3\sqrt{2}$.

Sol. $\because\quad \arg\left(\frac{z-z_1}{z-z_2}\right)=\frac{\pi}{4}$

It is clear that z, z_1, z_2 are non-collinear points. Always a circle passes through z, z_1 and z_2. Let z_0 be the centre of the circle.



On applying rotation theorem in ΔBOC,

$$\frac{z_1 - z_0}{z_2 - z_0} = \frac{OC}{OB} e^{(i\pi/2)} = i \qquad [\because OC = OB]$$

$$\Rightarrow \quad (z_1 - z_0) = i(z_2 - z_0)$$

$$\Rightarrow \quad 10 + 6i - z_0 = i(4 + 6i - z_0)$$

$$\Rightarrow \quad 16 + 2i = (1 - i) z_0$$

or
$$z_0 = \frac{(16 + 2i)}{(1 - i)} \cdot \frac{(1 + i)}{(1 + i)}$$

$$= \frac{16 + 16i + 2i + 2i^2}{2}$$

$$= \frac{14 + 18i}{2} = 7 + 9i$$

and radius, $r = OC = |z_0 - z_1| = |7 + 9i - 10 - 6i|$

$$= |-3 + 3i|$$

$$= \sqrt{(9 + 9)} = 3\sqrt{2}$$

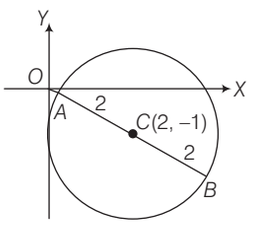
Hence, required equation is

$$|z - z_0| = r$$

$$\Rightarrow \quad |z - 7 - 9i| = 3\sqrt{2}$$

Example 84. If $|z - 2 + i| \le 2$, where $i = \sqrt{-1}$, then find the greatest and least value of $|z|$.

Sol. $\because$ Radius = 2 units



i.e., $AC = CB = 2$ units

$\therefore$ Least value of $|z| = OA = OC - AC = \sqrt{5} - 2$

and greatest value of $|z| = OB = OC + CB = \sqrt{5} + 2$

Hence, greatest value of $|z|$ is $\sqrt{5} + 2$ and least value of $|z|$ is $\sqrt{5} - 2$.

Example 85. ***In the argand plane, the vector $z = 4 - 3i$, where $i = \sqrt{-1}$, is turned in the clockwise sense through 180° and stretched three times. Then, find the complex number represented by the new vector.***

Sol. $\because \quad z = 4 - 3i \quad \Rightarrow \quad |z| = \sqrt{(4)^2 + (-3)^2} = 5$

Let z_1 be the new vector obtained by rotating z in the clockwise sense through 180°, therefore

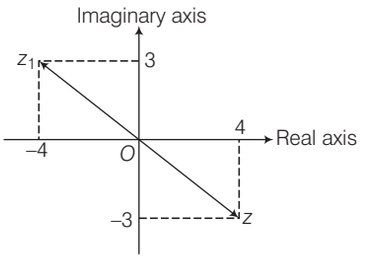
$$z_1 = z\, e^{-i\pi} = -z = -(4 - 3i) = -4 + 3i.$$

The unit vector in the direction of z_1 is $-\frac{4}{5} + \frac{3}{5}i$.

Therefore, required vector $= 3|z|\left(-\frac{4}{5} + \frac{3}{5}i\right)$

$$= 15\left(-\frac{4}{5} + \frac{3}{5}i\right) = -12 + 9i$$

Aliter



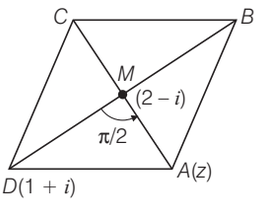
Here, $z_1 = -4 + 3i$

Hence, $3z_1 = -12 + 9i$

Example 86. $ABCD$ is a rhombus. Its diagonals AC and BD intersect at the point M and satisfy $BD = 2AC$. If the points D and M represents the complex numbers $1 + i$ and $2 - i$, where $i = \sqrt{-1}$, respectively, find A.

Sol. Let $A \equiv z$

$\because \quad BD = 2AC$ or $DM = 2AM$



Now, in ΔDMA,

Applying Coni method, we have

$$\frac{z - (2 - i)}{(1 + i) - (2 - i)} = \frac{AM}{DM} e^{i\pi/2} = \frac{1}{2} i$$

$$\Rightarrow \quad z - 2 + i = \frac{i}{2}(-1 + 2i) = -\frac{i}{2} - 1 \text{ or } z = 1 - \frac{3}{2} i$$

$$\therefore \quad A \equiv 1 - \frac{3}{2} i \text{ or } 3 - \frac{i}{2}$$

[if positions of A and C interchange]

If $\left| z \pm \frac{b}{z} \right| = a$, then the greatest and least values of $|z|$ are $\frac{a + \sqrt{(a^2 + 4|b|)}}{2}$ and $\frac{-a + \sqrt{(a^2 + 4|b|)}}{2}$, respectively.

Proof $\left| z \pm \frac{b}{z} \right| \geq \left| |z| - \left| \frac{b}{z} \right| \right|$

$\Rightarrow \quad a \geq \left| |z| - \frac{|b|}{|z|} \right|$

or $\quad -a \leq |z| - \frac{|b|}{|z|} \leq a$

Now, $\quad |z| - \frac{|b|}{|z|} \leq a$

$\Rightarrow \quad |z|^2 - a|z| - |b| \leq 0$

$\therefore \quad \frac{a - \sqrt{(a^2 + 4|b|)}}{2} \leq |z| \leq \frac{a + \sqrt{(a^2 + 4|b|)}}{2}$

or $\quad 0 \leq |z| \leq \frac{a + \sqrt{(a^2 + 4|b|)}}{2}$ ...(i)

and $\quad |z| - \frac{|b|}{|z|} \geq -a \Rightarrow |z|^2 + a|z| - |b| \geq 0$

$\therefore \quad |z| \geq \frac{-a + \sqrt{(a^2 + 4|b|)}}{2}$ ...(ii)

From Eqs. (i) and (ii), we get

$$\frac{-a + \sqrt{(a^2 + 4|b|)}}{2} \leq |z| \leq \frac{a + \sqrt{(a^2 + 4|b|)}}{2}$$

Hence, the greatest value of $|z|$ is $\frac{a + \sqrt{(a^2 + 4|b|)}}{2}$

and the least value of $|z|$ is $\frac{-a + \sqrt{(a^2 + 4|b|)}}{2}$.

Corollary For $b = 1$, $\left| z \pm \frac{1}{z} \right| = a$

Then, $\frac{-a + \sqrt{(a^2 + 4)}}{2} \leq |z| \leq \frac{a + \sqrt{(a^2 + 4)}}{2}$

Example 87. Find the maximum and minimum values of $|z|$ satisfying $\left| z + \frac{1}{z} \right| = 2$.

Sol. Here, $b = 1$ and $a = 2$

$\therefore$ Maximum and minimum values of $|z|$ are $\frac{2 + \sqrt{(4 + 4)}}{2}$ and $\frac{-2 + \sqrt{(4 + 4)}}{2}$ i.e., $1 + \sqrt{2}$ and $-1 + \sqrt{2}$, respectively.

Example 88. If $\left| z + \frac{4}{z} \right| = 2$, find the maximum and minimum values of $|z|$.

Sol. Here, $b = 4$ and $a = 2$.

$\therefore$ Maximum and minimum values of $|z|$ are $\frac{2 + \sqrt{(4 + 16)}}{2}$ and $\frac{-2 + \sqrt{(4 + 16)}}{2}$

i.e. $1 + \sqrt{5}$ and $-1 + \sqrt{5}$, respectively.

Example 89. If $|z| \geq 3$, then determine the least value of $\left| z + \frac{1}{z} \right|$.

Sol. $\because \quad \left| z + \frac{1}{z} \right| \geq \left| |z| - \left| \frac{1}{z} \right| \right| = \left| |z| - \frac{1}{|z|} \right|$ ...(i)

$\because \quad |z| \geq 3 \Rightarrow \frac{1}{|z|} \leq \frac{1}{3}$ or $-\frac{1}{|z|} \geq -\frac{1}{3}$

$\therefore \quad |z| - \frac{1}{|z|} \geq 3 - \frac{1}{3} = \frac{8}{3} \Rightarrow |z| - \frac{1}{|z|} \geq \frac{8}{3}$

or $\quad \left| |z| - \frac{1}{|z|} \right| \geq \frac{8}{3}$ ...(ii)

From Eqs. (i) and (ii), we get

$$\left| z + \frac{1}{z} \right| \geq \frac{8}{3}$$

$\therefore$ Least value of $\left| z + \frac{1}{z} \right|$ is $\frac{8}{3}$.

Exercise for Session 4

1 If z_1, z_2, z_3 and z_4 are the roots of the equation $z^4 = 1$, the value of $\sum_{i=1}^{4} z_i^3$ is

(a) 0 (b) 1 (c) $i, i = \sqrt{-1}$ (d) $1+i, i = \sqrt{-1}$

2 If $z_1, z_2, z_3, \ldots, z_n$ are n, nth roots of unity, then for $k = 1, 2, 3, \ldots, n$

(a) $|z_k| = k|z_{k+1}|$ (b) $|z_{k+1}| = k|z_k|$

(c) $|z_{k+1}| = |z_k| + |z_{k-1}|$ (d) $|z_k| = |z_{k+1}|$

3 If $1, \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1}$ are n, nth roots of unity, then $(1-\alpha_1)(1-\alpha_2)(1-\alpha_3)\ldots(1-\alpha_{n-1})$ equals to

(a) 0 (b) 1 (c) n (d) n^2

4 The value of $\sum_{k=1}^{6} \left(\sin\left(\frac{2\pi k}{7}\right) - i\cos\left(\frac{2\pi k}{7}\right)\right)$, where $i = \sqrt{-1}$, is

(a) -1 (b) 0 (c) $-i$ (d) i

5 If $\alpha \neq 1$ is any nth root of unity, then $S = 1 + 3\alpha + 5\alpha^2 + \ldots$ upto n terms is equal to

(a) $\frac{2n}{1-\alpha}$ (b) $-\frac{2n}{1-\alpha}$ (c) $\frac{n}{1-\alpha}$ (d) $-\frac{n}{1-\alpha}$

6 If a and b are real numbers between 0 and 1, such that the points $z_1 = a + i, z_2 = 1 + bi$ and $z_3 = 0$ form an equilateral triangle, then

(a) $a = b = 2 + \sqrt{3}$ (b) $a = b = 2 - \sqrt{3}$

(c) $a = 2 - \sqrt{3}, b = 2 + \sqrt{3}$ (d) None of these

7 If $|z| = 2$, the points representing the complex numbers $-1 + 5z$ will lie on

(a) a circle (b) a straight line (c) a parabola (d) an ellipse

8 If $|z-2|/|z-3| = 2$ represents a circle, then its radius is equal to

(a) 1 (b) $\frac{1}{3}$ (c) $\frac{3}{4}$ (d) $\frac{2}{3}$

9 If centre of a regular hexagon is at origin and one of the vertex on argand diagram is $1 + 2i$, where $i = \sqrt{-1}$, its perimeter is

(a) $2\sqrt{5}$ (b) $6\sqrt{2}$ (c) $4\sqrt{5}$ (d) $6\sqrt{5}$

10 If z is a complex number in the argand plane, the equation $|z-2| + |z+2| = 8$ represents

(a) a parabola (b) an ellipse (c) a hyperbola (d) a circle

11 If $|z-2-3i| + |z+2-6i| = 4$, where $i = \sqrt{-1}$, then locus of $P(z)$ is

(a) an ellipse (b) ϕ

(c) line segment of points $2+3i$ and $-2+6i$ (d) None of these

12 Locus of the point z satisfying the equation $|iz-1| + |z-1| = 2$, is (where, $i = \sqrt{-1}$)

(a) a straight line (b) a circle (c) an ellipse (d) a pair of straight lines

13 If z, iz and $z + iz$ are the vertices of a triangle whose area is 2 units, the value of $|z|$ is

(a) 1 (b) 2 (c) 4 (d) 8

14 If $\left|z - \frac{4}{z}\right| = 2$, the greatest value of $|z|$ is

(a) $\sqrt{5} - 1$ (b) $\sqrt{3} + 1$ (c) $\sqrt{5} + 1$ (d) $\sqrt{3} - 1$

Shortcuts and Important Results to Remember

1 $\big|\,|z_1|-|z_2|\,\big| \le |z_1+z_2| \le |z_1|+|z_2|$

Thus, $|z_1|+|z_2|$ is the greatest possible value of $|z_1+z_2|$ and $\big|\,|z_1|-|z_2|\,\big|$ is the least possible value of $|z_1+z_2|$.

If $\left|z \pm \frac{b}{z}\right| = a$, then the greatest and least values of $|z|$ are $\frac{a+\sqrt{(a^2+4|b|)}}{2}$ and $\frac{-a+\sqrt{(a^2+4|b|)}}{2}$, respectively.

$$\left|z_1+\sqrt{(z_1^2-z_2^2)}\right| + \left|z_2-\sqrt{(z_1^2-z_2^2)}\right| = |z_1+z_2|+|z_1-z_2|$$

$|z_1+z_2| = |z_1|+|z_2| \Leftrightarrow \arg(z_1) = \arg(z_2)$

i.e. z_1 and z_2 are parallel.

$|z_1+z_2| = |z_1|-|z_2| \Leftrightarrow \arg(z_1) - \arg(z_2) = \pi$

6 $|z_1+z_2| = |z_1-z_2| \Leftrightarrow \arg(z_1) - \arg(z_2) = \pm\pi/2$

7 If $|z_1| = |z_2|$ and $\arg(z_1) + \arg(z_2) = 0$, then z_1 and z_2 are conjugate complex numbers of each other.

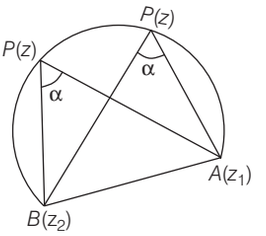
The equation $|z-z_1|^2 + |z-z_2|^2 = k,\ k \in R$ will represent a circle with centre at $\frac{1}{2}(z_1+z_2)$ and radius is $\frac{1}{2}\sqrt{2k-|z_1-z_2|^2}$ provided $k \ge \frac{1}{2}|z_1-z_2|^2$.

9 Area of triangle whose vertices are z, iz and $z+iz$, where $i = \sqrt{-1}$, is $\frac{1}{2}|z|^2$.

10 Area of triangle whose vertices are z, ωz and $z+\omega z$ is $\frac{\sqrt{3}}{4}|z|^2$, where ω is cube root of unity.

11 $\arg(z) - \arg(-z) = \pi$ or $-\pi$ according as $\arg(z)$ is positive or negative.

12 If $\arg\left(\frac{z-z_1}{z-z_2}\right) = \alpha$ (fixed), then the locus of z is a segment of circle.



If $\arg\left(\frac{z-z_1}{z-z_2}\right) = \pm\pi/2$, the locus of z is a circle with z_1 and z_2 as the vertices of diameter.

14 If $\arg\left(\frac{z-z_1}{z-z_2}\right) = 0$ or π, the locus of z is a straight line passing through z_1 and z_2.

15 If three complex numbers are in AP, they lie on a straight line in the complex plane.

16 If three points z_1, z_2, z_3 connected by relation $a z_1 + b z_2 + c z_3 = 0$, where $a+b+c = 0$, the three points are collinear.

17 If z_1, z_2, z_3 are vertices of a triangle, its centroid

$$z_0 = \frac{z_1+z_2+z_3}{3},\ \text{circumcentre } z_1 = \frac{\sum |z_1|^2 (z_2-z_3)}{\sum \bar{z}_1 (z_2-z_3)},$$

$$\text{orthocentre } z = \frac{\sum \bar{z}_1(\bar{z}_2-\bar{z}_3) + \sum |z_1|^2 (z_2-z_3)}{\sum (z_1\bar{z}_2 - \bar{z}_1 z_2)}$$

$$\text{and its area} = \frac{1}{4}\left|\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}\right|.$$

18 If $|z_1| = n_1, |z_2| = n_2, |z_3| = n_3, \ldots, |z_m| = n_m$,

then $\left|\frac{n_1^2}{z_1} + \frac{n_2^2}{z_2} + \frac{n_3^2}{z_3} + \ldots + \frac{n_m^2}{z_m}\right| = |z_1+z_2+z_3+\ldots+z_m|$.

JEE Type Solved Examples :
Single Option Correct Type Questions

■ This section contains **10 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

● ***Ex. 1*** *If z_1 and z_2 be the n th root of unity which subtend right angled at the origin. Then, n must be of the form*

(a) $4k+1$ (b) $4k+2$ (c) $4k+3$ (d) $4k$

Sol. (*d*) The nth roots of unity is given by

$$\cos\left(\frac{2r\pi}{n}\right)+i\sin\left(\frac{2r\pi}{n}\right)=e^{2r\pi i/n},$$

where $r=0, 1, 2, \ldots, (n-1)$

So, let $z_1=e^{2\pi r_1 i/n}$ and $z_2=e^{2\pi r_2 i/n}$, where $0\le r_1, r_2<n$, $r_1\ne r_2$.

It is given that the line segment joining the points having affixes z_1 and z_2, subtends a right angled at the origin. Therefore,

$$\arg\left(\frac{z_1}{z_2}\right)=\pm\frac{\pi}{2}$$

$$\Rightarrow \quad \frac{2\pi r_1}{n}-\frac{2\pi r_2}{n}=\pm\frac{\pi}{2}$$

$$\Rightarrow \quad n=\pm 4(r_1-r_2)$$

$$\therefore \quad n=4k, \text{ where } k=\pm(r_1-r_2)$$

● ***Ex. 2*** *If $|z|=1$ and $\omega=\dfrac{z-1}{z+1}$ (where $z\ne -1$), then $\mathrm{Re}(\omega)$ is*

(a) 0 (b) $-\dfrac{1}{|z+1|^2}$

(c) $\left|\dfrac{z}{z+1}\right|\cdot\dfrac{1}{|z+1|^2}$ (d) $\dfrac{\sqrt{2}}{|z+1|^2}$

Sol. (*a*) We have, $|z|=1$.

Let $z=e^{i\theta}$, where $\theta\in R$ and $i=\sqrt{-1}$.

Then, $\omega=\dfrac{z-1}{z+1}=\dfrac{e^{i\theta}-1}{e^{i\theta}+1}=i\tan\left(\dfrac{\theta}{2}\right)$

$\therefore \quad \mathrm{Re}(\omega)=0$

● ***Ex. 3*** *If a, b, c, a_1, b_1 and c_1 are non-zero complex numbers satisfying $\dfrac{a}{a_1}+\dfrac{b}{b_1}+\dfrac{c}{c_1}=1+i$ and $\dfrac{a_1}{a}+\dfrac{b_1}{b}+\dfrac{c_1}{c}=0$, where $i=\sqrt{-1}$, the value of $\dfrac{a^2}{a_1^2}+\dfrac{b^2}{b_1^2}+\dfrac{c^2}{c_1^2}$ is*

(a) $2i$ (b) $2+2i$ (c) 2 (d) None of these

Sol. (*a*) We have, $\dfrac{a}{a_1}+\dfrac{b}{b_1}+\dfrac{c}{c_1}=1+i$

On squaring both sides, we get

$$\frac{a^2}{a_1^2}+\frac{b^2}{b_1^2}+\frac{c^2}{c_1^2}+2\left(\frac{ab}{a_1b_1}+\frac{bc}{b_1c_1}+\frac{ca}{c_1a_1}\right)=1+i^2+2i$$

$$\Rightarrow \quad \frac{a^2}{a_1^2}+\frac{b^2}{b_1^2}+\frac{c^2}{c_1^2}+2\frac{abc}{a_1b_1c_1}\left(\frac{c_1}{c}+\frac{a_1}{a}+\frac{b_1}{b}\right)=1-1+2i$$

$$\Rightarrow \quad \frac{a^2}{a_1^2}+\frac{b^2}{b_1^2}+\frac{c^2}{c_1^2}+0=2i$$

$$\therefore \quad \frac{a^2}{a_1^2}+\frac{b^2}{b_1^2}+\frac{c^2}{c_1^2}=2i$$

● ***Ex. 4*** *Let z and ω be complex numbers. If $\mathrm{Re}(z)=|z-2|$, $\mathrm{Re}(\omega)=|\omega-2|$ and $\arg(z-\omega)=\dfrac{\pi}{3}$, the value of $\mathrm{Im}(z+\omega)$ is*

(a) $\dfrac{1}{\sqrt{3}}$ (b) $\dfrac{2}{\sqrt{3}}$ (c) $\sqrt{3}$ (d) $\dfrac{4}{\sqrt{3}}$

Sol. (*d*) Let $z=x+iy,\ x, y\in R$ and $i=\sqrt{-1}$

$\because \quad \mathrm{Re}(z)=|z-2|$

$\Rightarrow \quad x=\sqrt{(x-2)^2+y^2}$

$\Rightarrow \quad y^2=4(x-1)$

$\therefore\ z=1+t^2+2ti$, parametric form and let $\omega=p+iq$

Similarly, $\omega=1+s^2+2si$

$\therefore \quad z-\omega=(t^2-s^2)+2i(t-s)$

$\Rightarrow \quad \arg(z-\omega)=\dfrac{\pi}{3}$ [given]

$\therefore \quad \tan^{-1}\left(\dfrac{2}{t+s}\right)=\dfrac{\pi}{3} \ \Rightarrow\ \dfrac{2}{t+s}=\sqrt{3}$

$\Rightarrow \quad (t+s)=\dfrac{2}{\sqrt{3}}$

Now, $z+\omega=2+t^2+s^2+2i(t+s)$

$\therefore \quad \mathrm{Im}(z+\omega)=2(t+s)=\dfrac{4}{\sqrt{3}}$

● ***Ex. 5*** *The mirror image of the curve $\arg\left(\dfrac{z-3}{z-i}\right)=\dfrac{\pi}{6}$, $i=\sqrt{-1}$ in the real axis, is*

(a) $\arg\left(\frac{z+3}{z+i}\right)=\frac{\pi}{6}$ (b) $\arg\left(\frac{z-3}{z+i}\right)=\frac{\pi}{6}$

(c) $\arg\left(\frac{z+i}{z+3}\right)=\frac{\pi}{6}$ (d) $\arg\left(\frac{z+i}{z-3}\right)=\frac{\pi}{6}$

Sol. (*d*) $\because$ The image of z in the real axis is $\bar{z}$.

The image is given by

$$\arg\left(\frac{\bar{z}-3}{\bar{z}-i}\right)=\frac{\pi}{6}$$

$$\Rightarrow \quad -\arg\left(\frac{z-3}{z+i}\right)=\frac{\pi}{6} \qquad [\because \arg(\bar{z})=-\arg(z)]$$

$$\Rightarrow \quad \arg\left(\frac{z+i}{z-3}\right)=\frac{\pi}{6} \qquad \left[\because \arg\left(\frac{z_1}{z_2}\right)=-\arg\left(\frac{z_2}{z_1}\right)\right]$$

● ***Ex. 6*** *The mirror image of the curve* $\arg\left(\frac{z+i}{z-1}\right)=\frac{\pi}{4}$, $i=\sqrt{-1}$ *in the line* $x-y=0$, *is*

(a) $\arg\left(\frac{z+i}{z+1}\right)=\frac{\pi}{4}$ (b) $\arg\left(\frac{z+1}{z-i}\right)=\frac{\pi}{4}$

(c) $\arg\left(\frac{z-i}{z+1}\right)=\frac{\pi}{4}$ (d) $\arg\left(\frac{z+i}{z-1}\right)=\frac{\pi}{4}$

Sol. (*c*) $\because$ The image of z in the line $y=x$ is $i\bar{z}$.

$\therefore$ The image of the given curve is

$$\arg\left(\frac{i\bar{z}+i}{i\bar{z}-1}\right)=\frac{\pi}{4}$$

$$\Rightarrow \quad \arg\left(\frac{\bar{z}+1}{\bar{z}+i}\right)=\frac{\pi}{4}$$

$$\Rightarrow \quad -\arg\left(\frac{z+1}{z-i}\right)=\frac{\pi}{4} \qquad [\because \arg(\bar{z})=-\arg(z)]$$

$$\Rightarrow \quad \arg\left(\frac{z-i}{z+1}\right)=\frac{\pi}{4} \left[\because \arg\left(\frac{z_1}{z_2}\right)=-\arg\left(\frac{z_2}{z_1}\right)\right]$$

● ***Ex. 7*** *If* $z+\frac{1}{z}=1$ *and* $a=z^{2017}+\frac{1}{z^{2017}}$ *and* b *is the last digit of the number* $2^{2^n}-1$„ *when the integer* $n>1$, *the value of* a^2+b^2 *is*

(a) 23 (b) 24 (c) 26 (d) 27

Sol. (c) $\because z+\frac{1}{z}=1 \Rightarrow z^2-z+1=0$

$$\therefore \quad z=\frac{-(-1)\pm\sqrt{(1-4)}}{2}=-\omega,-\omega^2$$

[ω is cube root of unity]

and $z^{2017}=(-\omega)^{2017}=-\omega$,

$z^{2017}=(-\omega^2)^{2017}=-\omega^2$

$$\therefore \quad a=z^{2017}+\frac{1}{z^{2017}}=-\left(\omega+\frac{1}{\omega}\right)=-(\omega+\omega^2)=1$$

and $2^{2^n}=2^{4\cdot 2^{n-4}}=16^{2^{n-4}}$ has last digit 6.

$\therefore \quad b=6-1=5$

Hence, $a^2+b^2=1^2+5^2=26$

● ***Ex. 8*** *If* ω *is complex cube root of unity and a, b, c are such that* $\frac{1}{a+\omega}+\frac{1}{b+\omega}+\frac{1}{c+\omega}=2\omega^2$ *and* $\frac{1}{a+\omega^2}+\frac{1}{b+\omega^2}+\frac{1}{c+\omega^2}=2\omega$, *then the value of* $\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}$ *is equal to*

(a) −2 (b) −1 (c) 1 (d) 2

Sol. (*d*) $\because \quad \frac{1}{a+\omega}+\frac{1}{b+\omega}+\frac{1}{c+\omega}=2\omega^2=\frac{2}{\omega}$

and $\frac{1}{a+\omega^2}+\frac{1}{b+\omega^2}+\frac{1}{c+\omega^2}=2\omega=\frac{2}{\omega^2}$

It is clear that, ω and ω^2 are the roots of the equation

$$\frac{1}{a+x}+\frac{1}{b+x}+\frac{1}{c+x}=\frac{2}{x} \qquad \ldots(i)$$

$$\Rightarrow \quad x\sum(b+x)(c+x)=2(a+x)(b+x)(c+x)$$

$$\Rightarrow \quad x^3-(ab+bc+ca)x-2abc=0$$

$\because$ Coefficient of $x^2=0$, the sum of roots $=0$

$$\Rightarrow \quad \alpha+\omega+\omega^2=0 \Rightarrow \alpha-1=0$$

$$\therefore \quad \alpha=1$$

$\therefore$ Third root is 1.

From Eq. (i), we get

$$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}=2$$

● ***Ex. 9*** *If* a, b *and* c *are distinct integers and* $\omega(\neq 1)$ *is a cube root of unity, then the minimum value of* $|a+b\omega+c\omega^2|+|a+b\omega^2+c\omega|$, *is*

(a) $2\sqrt{3}$ (b) 3 (c) $4\sqrt{2}$ (d) 2

Sol. (*a*) Let $z=a+b\omega+c\omega^2$. Then,

$$|z|^2=z\bar{z}=(a+b\omega+c\omega^2)(a+b\bar{\omega}+c\overline{\omega^2})$$

$$=(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$$

$$=a^2+b^2+c^2-ab-bc-ca$$

$$=\frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]$$

$$\Rightarrow \quad |z|^2\geq\frac{1}{2}\times 6=3 \qquad \left[\begin{array}{l}\because\ a\neq b\neq c\\ \therefore\ |a-b|\geq 1, |b-c|\geq 1\\ \text{and } |a-c|\geq 2\end{array}\right]$$

$\therefore \quad |z| \geq \sqrt{3}$

$\therefore \quad |a + b\omega + c\omega^2| + |a + b\omega^2 + c\omega|$

$= |a + b\omega + c\omega^2| + |a + b\overline{\omega^2} + c\overline{\omega}|$

$= |a + b\omega + c\omega^2| + |a + b\omega + c\omega^2|$

$= 2|a + b\omega + c\omega^2| = 2|z| \geq 2\sqrt{3}$

Hence, the minimum value of $|a + b\omega + c\omega^2| + |a + b\omega^2 + c\omega|$ is $2\sqrt{3}$.

● ***Ex. 10*** *If* $|z - 2i| \leq \sqrt{2}$, *where* $i = \sqrt{-1}$, *then the maximum value of* $|3 - i(z - 1)|$, *is*

(a) $\sqrt{2}$ (b) $2\sqrt{2}$ (c) $2 + \sqrt{2}$ (d) $3 + 2\sqrt{2}$

Sol. (*b*) $\because \quad |3 - i(z - 1)| = |-i(z - 1 - 3i)| = |-i||z - 1 - 3i|$

$= |z - 1 - 3i|$ ...(i)

and $|z - 1 - 3i| = |(z - 2i) + (-1 - i)| \leq |z - 2i| + |-1 - i|$ $[\because |z_1 + z_2| \leq |z_1| + |z_2|]$

$\therefore \quad |z - 1 - 3i| \leq |z - 2i| + \sqrt{2}$

$\Rightarrow \quad |z - 1 - 3i| \leq \sqrt{2} + \sqrt{2}$ $[\because |z - 2i| \leq \sqrt{2}]$

$\Rightarrow \quad |z - 1 - 3i| \leq 2\sqrt{2}$

From Eq. (i), we get

$|3 - i(z - 1)| \leq 2\sqrt{2}$

Hence, the maximum value of $|3 - i(z - 1)|$ is $2\sqrt{2}$.

JEE Type Solved Examples :
More than One Correct Option Type Questions

■ This section contains **5 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of **which more than one** may be correct.

● ***Ex. 11*** *If* $z_1 = a + ib$ *and* $z_2 = c + id$ *are complex numbers such that* $|z_1| = |z_2| = 1$ *and* $\text{Re}(z_1\bar{z}_2) = 0$, *where* $i = \sqrt{-1}$, *then the complex numbers* $\omega_1 = a + ic$ *and* $\omega_2 = b + id$ *satisfy*

(a) $|\omega_1| = 1$ (b) $|\omega_2| = 1$

(c) $\text{Re}(\omega_1\bar{\omega}_2) = 0$ (d) None of these

Sol. (*a, b, c*)

$\because \quad |z_1| = 1, |z_2| = 1 \Rightarrow z_1 = CiS\,\theta, z_2 = CiS\,\phi$

$\text{Re}(z_1\bar{z}_2) = \text{Re}(CiS(\theta - \phi)) = 0$ [given]

$\Rightarrow \quad \cos(\theta - \phi) = 0 \Rightarrow \theta - \phi = \frac{\pi}{2} \Rightarrow \phi = \theta - \frac{\pi}{2}$

and $a = \cos\theta, b = \sin\theta, c = \cos\phi, d = \sin\phi$

$\therefore \quad \omega_1 = a + ic = \cos\theta + i\cos\phi = \cos\theta + i\sin\theta$ $\left[\because \phi = \theta - \frac{\pi}{2}\right]$

and $\omega_2 = b + id = \sin\theta + i\sin\phi = \sin\theta - i\cos\theta$ $\left[\because \phi = \theta - \frac{\pi}{2}\right]$

$\therefore \quad |\omega_1| = 1, |\omega_2| = 1$

and $\text{Re}(\omega_1\bar{\omega}_2) = \text{Re}(\cos\theta + i\sin\theta)(\sin\theta + i\cos\theta) = 0$

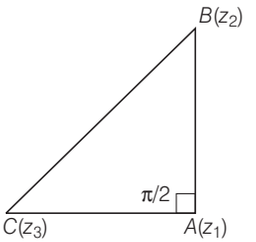
● ***Ex. 12*** *The complex numbers* z_1, z_2, z_3 *satisfying* $(z_2 - z_3) = (1 + i)(z_1 - z_3)$, *where* $i = \sqrt{-1}$, *are vertices of a triangle which is*

(a) equilateral (b) isosceles

(c) right angled (d) scalene

Sol. (*b,c*)

$\because \quad (z_2 - z_3) = (1 + i)(z_1 - z_3)$



$\Rightarrow \quad (z_2 - z_1) = i(z_1 - z_3)$

$\Rightarrow \quad (z_2 - z_1) = -i(z_3 - z_1)$

or $\quad (z_3 - z_1) = i(z_2 - z_1)$

$\Rightarrow \quad \frac{z_3 - z_1}{z_2 - z_1} = e^{i\pi/2}$

AC is obtained by rotating AB about A through $\frac{\pi}{2}$ anti-clockwise.

$\therefore \quad AB = AC, \angle CAB = \frac{\pi}{2}$

Hence, z_1, z_2, z_3 form isosceles right angled triangle.

● ***Ex. 13*** *If* z *satisfies the inequality* $|z - 1| < |z + 1|$, *then one has*

(a) $|z - 2 - i| < |z + 2 - i|, i = \sqrt{-1}$

(b) $|\arg(z + i)| < \frac{\pi}{2}, i = \sqrt{-1}$

(c) $\text{Re}(z) < 0$

(d) $\text{Im}(i\bar{z}) > 0, i = \sqrt{-1}$

Sol. (*a, b, d*)

On putting $z = x + iy$ in the given relation, we have

$(x - 1)^2 + y^2 < (x + 1)^2 + y^2$

$\Rightarrow \quad x > 0$

i.e. $\text{Re}(z) > 0$ ...(i)

and on putting $z = x + iy$ in alternate (a), then

$$(x-2)^2 + (y-1)^2 < (x+2)^2 + (y-1)^2$$

$\Rightarrow$ $x > 0$ [from Eq. (i)]

which is true.

$\therefore$ Real part of $(z + i) = x > 0$,

then $\arg(z + i)$ lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$

and hence $|\arg(z+i)| < \frac{\pi}{2}$

and $\text{Im}(i\bar{z}) = \text{Im}(i(x - iy)) = \text{Im}(y + ix)$

$= x > 0$ [from Eq. (i)]

which is true.

● ***Ex. 14*** *The equation* $z^2 - i|z-1|^2 = 0$, *where* $i = \sqrt{-1}$, *has*

(a) no real root

(b) no purely imaginary root

(c) all roots inside $|z| = 1$

(d) atleast two roots

Sol. (*a, b, c*)

On putting $z = x + iy$, we have

$$(x+iy)^2 - i|x+iy-1|^2 = 0$$

$\Rightarrow$ $x^2 - y^2 + 2ixy - i((x-1)^2 + y^2) = 0$

On comparing real and imaginary parts, we get

$$x^2 - y^2 = 0 \quad \text{and} \quad 2xy = (x-1)^2 + y^2$$

Case **I** When $y = x$, then $2xy = (x-1)^2 + y^2$ reduces to

$$2x^2 = (x-1)^2 + x^2$$

$\Rightarrow$ $0 = -2x + 1$

$\therefore$ $x = \frac{1}{2} = y$

$\Rightarrow$ $z = x + iy = \frac{1+i}{2}$ ...(i)

Case **II** When $y = -x$, then $2xy = (x-1)^2 + y^2$ reduces to

$-2x^2 = (x-1)^2 + x^2$

$\Rightarrow (x-1)^2 + 3x^2 = 0$ which is not possible. ...(ii)

From Eqs. (i) and (ii), we get $z = \frac{1+i}{2}$

i.e., no real and no purely imaginary roots

and $|z| = \frac{1}{\sqrt{2}} < 1$

● ***Ex. 15*** *Let* z_1 *and* z_2 *be two complex numbers represented by points on circles* $|z| = 1$ *and* $|z| = 2$ *respectively, then*

(a) max.$|2z_1 + z_2| = 4$ (b) min.$|z_1 - z_2| = 1$

(c) $\left|z_2 + \frac{1}{z_1}\right| \le 3$ (d) $\left|z_1 + \frac{2}{z_2}\right| \le 2$

Sol. (*a, b, c, d*)

$\because$ $|z_1| = 1$ and $|z_2| = 2$

$\therefore$ $|2z_1 + z_2| \le |2z_1| + |z_2| = 2|z_1| + |z_2|$

$\Rightarrow$ $|2z_1 + z_2| \le 2|z_1| + |z_2| = 2 + 2 = 4$

$\therefore$ $|2z_1 + z_2| \le 4$

$\Rightarrow$ max.$|2z_1 + z_2| = 4$ [alternate (a)]

and $|z_1 - z_2| \ge ||z_1| - |z_2|| = |1 - 2| = 1$

$\Rightarrow$ $|z_1 - z_2| \ge 1$

$\therefore$ min.$|z_1 - z_2| = 1$ [alternate (b)]

$$\left|z_2 + \frac{1}{z_1}\right| \le |z_2| + \left|\frac{1}{z_1}\right| = |z_2| + \frac{1}{|z_1|} = 2 + \frac{1}{1} = 3$$

$\therefore$ $\left|z_2 + \frac{1}{z_1}\right| \le 3$ [alternate (c)]

and $\left|z_1 + \frac{2}{z_2}\right| \le |z_1| + \left|\frac{2}{z_2}\right| = |z_1| + \frac{2}{|z_2|} = 1 + \frac{2}{2} = 2$

$\therefore$ $\left|z_1 + \frac{2}{z_2}\right| \le 2$ [alternate (d)]

JEE Type Solved Examples : Passage Based Questions

■ This section contains **2 solved passages** based upon each of the passage **3 multiple choice** examples have to be answered. Each of these examples has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I (Ex. Nos. 16 to 18)

Consider a quadratic equation $az^2 + bz + c = 0$, where a, b and c are complex numbers.

16. The condition that the equation has one purely imaginary root, is

(a) $(a\bar{b} - \bar{a}b)(b\bar{c} + \bar{b}c) + (c\bar{a} - \bar{c}a)^2 = 0$

(b) $(a\bar{b} + \bar{a}b)(b\bar{c} + \bar{b}c) + (c\bar{a} - \bar{c}a)^2 = 0$

(c) $(a\bar{b} - \bar{a}b)(b\bar{c} - \bar{b}c) + (c\bar{a} + \bar{c}a)^2 = 0$

(d) $(a\bar{b} + \bar{a}b)(b\bar{c} - \bar{b}c) + (c\bar{a} - \bar{c}a)^2 = 0$

Sol. (*b*) $\because$ $az^2 + bz + c = 0$ ...(i)

$\therefore$ $\overline{az^2 + bz + c} = \bar{0}$

$\Rightarrow$ $\bar{a}(\bar{z})^2 + \bar{b}\bar{z} + \bar{c} = 0$

For purely imaginary root,

$$\bar{z} = -z$$

Then, $\bar{a}z^2 - \bar{b}z + \bar{c} = 0$ ...(ii)

From Eqs. (i) and (ii), we get

$$\frac{z^2}{b\bar{c}+\bar{b}c}=\frac{z}{c\bar{a}-\bar{c}a}=\frac{1}{-a\bar{b}-\bar{a}b}$$

$$\Rightarrow \quad z=\frac{b\bar{c}+\bar{b}c}{c\bar{a}-\bar{c}a}=\frac{c\bar{a}-\bar{c}a}{-a\bar{b}-\bar{a}b}$$

$$\therefore \quad (a\bar{b}+\bar{a}b)(b\bar{c}+\bar{b}c)+(c\bar{a}-\bar{c}a)^2=0$$

17. The condition that the equation has one purely real root, is

(a) $(a\bar{b}+\bar{a}b)(b\bar{c}-\bar{b}c)=(c\bar{a}+\bar{c}a)^2$

(b) $(a\bar{b}-\bar{a}b)(b\bar{c}+\bar{b}c)=(c\bar{a}+\bar{c}a)^2$

(c) $(a\bar{b}-\bar{a}b)(b\bar{c}-\bar{b}c)=(c\bar{a}-\bar{c}a)^2$

(d) $(a\bar{b}-\bar{a}b)(b\bar{c}-\bar{b}c)=(c\bar{a}+\bar{c}a)^2$

Sol. (c) $\because \quad az^2+bz+c=0$ ...(i)

$$\Rightarrow \quad \overline{az^2+bz+c}=\bar{0}$$

$$\Rightarrow \quad \bar{a}(\bar{z})^2+\bar{b}\bar{z}+\bar{c}=0$$

For purely real root, $\bar{z}=z$

Then, $\quad \bar{a}z^2+\bar{b}z+\bar{c}=0$ ...(ii)

From Eqs. (i) and (ii), we get

$$\frac{z^2}{b\bar{c}-\bar{b}c}=\frac{z}{c\bar{a}-\bar{c}a}=\frac{1}{a\bar{b}-\bar{a}b}$$

$$\Rightarrow \quad z=\frac{b\bar{c}-\bar{b}c}{c\bar{a}-\bar{c}a}=\frac{c\bar{a}-\bar{c}a}{a\bar{b}-\bar{a}b}$$

$$\Rightarrow \quad (a\bar{b}-\bar{a}b)(b\bar{c}-\bar{b}c)=(c\bar{a}-\bar{c}a)^2$$

18. The condition that the equation has two purely imaginary roots, is

(a) $\frac{a}{\bar{a}}=\frac{b}{\bar{b}}=\frac{c}{\bar{c}}$ (b) $-\frac{a}{\bar{a}}=\frac{b}{\bar{b}}=\frac{c}{\bar{c}}$

(c) $\frac{a}{\bar{a}}=\frac{b}{\bar{b}}=-\frac{c}{\bar{c}}$ (d) $\frac{a}{\bar{a}}=-\frac{b}{\bar{b}}=\frac{c}{\bar{c}}$

Sol. (d) $\because \quad az^2+bz+c=0$ ...(i)

$$\therefore \quad \overline{az^2+bz+c}=\bar{0}$$

$$\Rightarrow \quad \bar{a}(\bar{z})^2+\bar{b}\bar{z}+\bar{c}=0$$

Since, both roots are purely imaginary.

$$\therefore \quad \bar{z}=-z$$

Then, $\quad \bar{a}z^2-\bar{b}z+\bar{c}=0$ ...(ii)

Hence, Eqs. (i) and (ii) are identical.

$$\therefore \quad \frac{a}{\bar{a}}=-\frac{b}{\bar{b}}=\frac{c}{\bar{c}}$$

Passage II (Ex. Nos. 19 to 21)

Let P be a point denoting a complex number z on the complex plane.

i.e. $\quad z=\text{Re}(z)+i\,\text{Im}(z)$, where $i=\sqrt{-1}$

If $\quad \text{Re}(z)=x$ and $\text{Im}(z)=y$, then $z=x+iy$

19. If P moves such that

$$|\text{Re}(z)|+|\text{Im}(z)|=a\,(a\in R^+)$$

The locus of P is

(a) a parallelogram which is not a rhombus

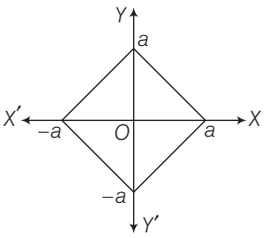
(b) a rhombus which is not a square

(c) a rectangle which is not a square

(d) a square

Sol. (d) $\because \quad |\text{Re}(z)|+|\text{Im}(z)|=a$

$$\Rightarrow \quad |x|+|y|=a$$



$\therefore$ Locus of P is a square.

20. The area of the circle inscribed in the region denoted by $|\text{Re}(z)|+|\text{Im}(z)|=10$ equals to

(a) 50 π sq units (b) 100 π sq units

(c) 55 sq units (d) 110 sq units

Sol. (a) From above, $a=10$

Diameter of circle = Distance between sides of square

= Length of side of square $=a\sqrt{2}=10\sqrt{2}$

or $\quad 2r=10\sqrt{2}$

$$\Rightarrow \quad r=5\sqrt{2}$$

$\therefore$ Area of a circle $=\pi r^2=50\pi$ sq units

21. Number of integral solutions satisfying the inequality $|\text{Re}(z)|+|\text{Im}(z)|<21$, is

(a) 841 (b) 839

(c) 840 (d) 842

Sol. (c) $\because \quad |x|+|y|<21 \;\Rightarrow\; 0\le|x|+|y|\le 20$

If $\quad x>0, y>0, 0\le x+y\le 20$

Number of solutions $= {}^{21}C_2=\frac{21\cdot 20}{2}=210$

$\therefore$ Total integral solutions $=4\times 210=840$

JEE Type Solved Examples :
Single Integer Answer Type Questions

- This section contains **2 examples.** The answer to each example is a **single digit integer** ranging from **0** to **9** (both inclusive).

● ***Ex. 22*** *If* $z_1, z_2 \in C$, $z_1^2 + z_2^2 \in R$, $z_1(z_1^2 - 3z_2^2) = 2$ *and* $z_2(3z_1^2 - z_2^2) = 11$, *the value of* $z_1^2 + z_2^2$ *is*

Sol. **(5)** We have, $z_1(z_1^2 - 3z_2^2) = 2$ ...(i)

and $z_2(3z_1^2 - z_2^2) = 11$ ...(ii)

multiplying Eq. (ii) by $i(\sqrt{-1})$ and then adding in Eq. (i), we get

$$z_1^3 - 3z_1z_2^2 + i(3z_1^2z_2 - z_2^3) = 2 + 11i$$

$$\Rightarrow (z_1 + iz_2)^3 = 2 + 11i \quad ...(iii)$$

Again, multiplying Eq. (ii) by $(-i)$ and then adding in Eq. (i), we get

$$z_1^3 - 3z_1z_2^2 - i(3z_1^2z_2 - z_2^3) = 2 - 11i$$

$$\Rightarrow (z_1 - iz_2)^3 = 2 - 11i \quad ...(iv)$$

Now, on multiplying Eqs. (iii) and (iv), we get

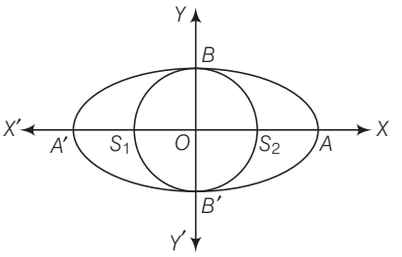
$$(z_1^2 + z_2^2)^3 = 4 + 121 = 125 = 5^3$$

$$\therefore z_1^2 + z_2^2 = 5$$

● ***Ex. 23*** *The number of solutions of the equations* $|z - (4 + 8i)| = \sqrt{10}$ *and* $|z - (3 + 5i)| + |z - (5 + 11i)| = 4\sqrt{5}$, *where* $i = \sqrt{-1}$.

Sol. **(2)** Here, $|z - (4 + 8i)| = \sqrt{10}$

represents a circle with centre (4, 8) and radius $\sqrt{10}$.



Also, $|z - (3 + 5i)| + |z - (5 + 11i)| = 4\sqrt{5}$

represents an ellipse.

$\therefore \quad |(3 + 5i) - (5 + 11i)| = \sqrt{4 + 36} = \sqrt{40} < 4\sqrt{5}$

with foci $S_1(3, 5)$ and $S_2(5, 11)$.

Distance between foci $= S_1S_2 = \sqrt{40} = 2\sqrt{10}$ = Diameter of circle

i.e., $2ae = 2\sqrt{10}$

$\Rightarrow \quad ae = \sqrt{10}$ and $2a = 4\sqrt{5} \Rightarrow a = 2\sqrt{5}$

$\therefore \quad e = \dfrac{ae}{a} = \dfrac{1}{\sqrt{2}}$

Now, $b = a\sqrt{(1 - e^2)} = 2\sqrt{5}\sqrt{\left(1 - \dfrac{1}{2}\right)} = \sqrt{10}$ = Radius of circle

$\therefore$ Centre of the ellipse = Mid-point of S_1 and S_2

$$= \frac{3 + 5i + 5 + 11i}{2} = 4 + 8i \text{ i.e., } (4, 8)$$

which coincides with the centre of the circle and length of minor-axis is equal to the radius of the circle. Hence, there are only (2) two solutions of the given equations.

JEE Type Solved Examples :
Matching Type Questions

- This section contains **2 examples.** Examples 24 and 25 have three statements (A, B and C) given in Column I and four statements (p, q, r and s) in **Column II.** Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II.**

● ***Ex. 24***

	Column I		Column II
(A)	If λ and μ are the greatest and least values of $\|z - 1\|$, if $\|z + 2 + i\| \le 1$, where $i = \sqrt{-1}$, then	(p)	$\lambda + \mu$ = rational
(B)	If λ and μ are the greatest and least values of $\|z - 2\|$, if $\|z + i\| \le 1$, where $i = \sqrt{-1}$, then	(q)	$\lambda + \mu$ = irrational
(C)	If λ and μ are the greatest and least values of $\|z + 2i\|$, if $1 \le \|z - 1\| \le 3$, where $i = \sqrt{-1}$, then	(r)	$\lambda - \mu$ = rational
		(s)	$\lambda - \mu$ = irrational

Sol. (A) → (q, r); (B) → (q, r); (C) → (p, s)

(A) $\because \quad |z+2+i| \le 1$

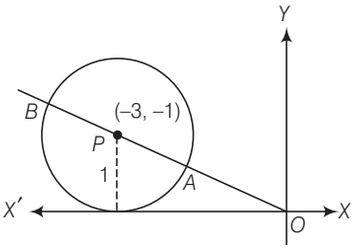
$\Rightarrow \quad |(z-1)+(3+i)| \le 1$

$\Rightarrow \quad |\omega+(3+i)| \le 1$ where, $\omega = z-1$

From the figure, the greatest value of

$|z-1| = |\omega| = |\omega - 0| = OB = OP + PB = \sqrt{10}+1$

$\therefore \quad \lambda = \sqrt{10}+1$



and the least value of $|z-1| = |\omega|$

$= |\omega - 0| = OP - AP = \sqrt{10} - 1$

$\therefore \quad \mu = \sqrt{10} - 1$

$\Rightarrow \quad \lambda + \mu = (\sqrt{10}+1) + (\sqrt{10}-1) = 2\sqrt{10}$ = irrational

and $\quad \lambda - \mu = (\sqrt{10}+1) - (\sqrt{10}-1) = 2$ = rational

Aliter

$\because \quad |z+2+i| \le 1$

$\Rightarrow \quad |(z-1)+(3+i)| \le 1$

$\Rightarrow \quad |\omega+(3+i)| \le 1 \qquad ...(i)$

where, $\omega = z-1$

$\therefore \quad |\omega+(3+i)| \ge ||\omega| - |3+i||$

or $\quad |\omega+(3+i)| \ge ||\omega| - \sqrt{10}| \qquad ...(ii)$

From Eqs. (i) and (ii), we get

$||\omega| - \sqrt{10}| \le |\omega+3+i| \le 1$

$\Rightarrow \quad ||\omega| - \sqrt{10}| \le 1$

or $\quad -1 \le |\omega| - \sqrt{10} \le 1$

or $\quad \sqrt{10} - 1 \le |\omega| \le \sqrt{10} + 1$

$\therefore \quad \lambda = \sqrt{10}+1$ and $\mu = \sqrt{10}-1$

$\Rightarrow \quad \lambda + \mu = 2\sqrt{10}$ = irrational

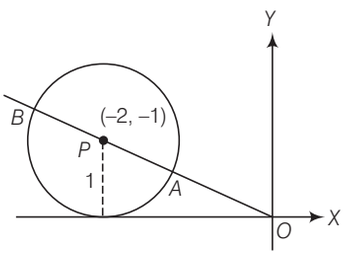
and $\lambda - \mu = 2$ = rational

(B) $\because \quad |z+i| \le 1$

$\Rightarrow \quad |(z-2)+(2+i)| \le 1$

$\Rightarrow \quad |\omega+(2+i)| \le 1$

where, $\quad \omega = z-2$



From the figure, the greatest value of $|z-2| = |\omega|$

$= |\omega - 0| = OB$

$= OP + PB = \sqrt{5}+1$

$\therefore \quad \lambda = \sqrt{5}+1$

and the least value of $|z-2| = |\omega|$

$= |\omega - 0| = OA = OP - AP = \sqrt{5} - 1$

$\therefore \quad \mu = \sqrt{5} - 1$

$\Rightarrow \quad \lambda + \mu = (\sqrt{5}+1) + (\sqrt{5}-1) = 2\sqrt{5}$ = irrational

and $\quad \lambda - \mu = (\sqrt{5}+1) - (\sqrt{5}-1) = 2$ = rational

Aliter

$\because \quad |z+i| \le 1$

$\Rightarrow \quad |(z-2)+(2+i)| \le 1$

$\Rightarrow \quad |\omega+(2+i)| \le 1$

where, $\quad \omega = z-2 \qquad ...(i)$

$\because \quad |\omega+(2+i)| \ge ||\omega| - |2+i||$

or $\quad |\omega+(2+i)| \ge ||\omega| - \sqrt{5}| \qquad ...(ii)$

From Eqs. (i) and (ii), we get

$||\omega| - \sqrt{5}| \le |\omega+2+i| \le 1$

$\Rightarrow \quad ||\omega| - \sqrt{5}| \le 1$

or $\quad -1 \le |\omega| - \sqrt{5} \le 1$ or $\sqrt{5} - 1 \le |\omega| \le \sqrt{5}+1$

$\therefore \quad \lambda = \sqrt{5}+1$ and $\mu = \sqrt{5}-1$

$\Rightarrow \quad \lambda + \mu = 2\sqrt{5}$ = irrational

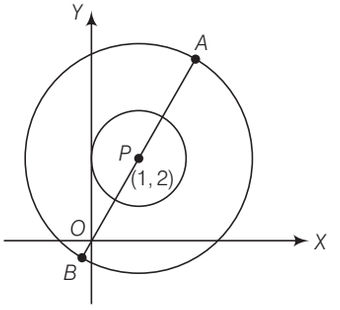
and $\quad \lambda - \mu = 2$ = rational

(C) $\because \quad 1 \le |z-1| \le 3$

$\Rightarrow \quad 1 \le |(z+2i)-(1+2i)| \le 3$

$\Rightarrow \quad 1 \le |\omega-(1+2i)| \le 3 \qquad ...(i)$

where, $\quad \omega = z+2i$



From the figure, the greatest value of $|z+2i| = |\omega|$

$= |\omega - 0| = OA = OP + PA = \sqrt{5}+3$

$\therefore \quad \lambda = 3 + \sqrt{5}$

and the least value of $|z+2i| = |\omega|$

$= |\omega - 0| = OB = PB - OP = 3 - \sqrt{5}$

$\therefore \quad \mu = 3 - \sqrt{5}$

$\Rightarrow \quad \lambda + \mu = (3+\sqrt{5}) + (3-\sqrt{5}) = 6$ = rational

and $\quad \lambda - \mu = (3+\sqrt{5}) - (3-\sqrt{5}) = 2\sqrt{5}$ = irrational

● ***Ex. 25***

Column I		Column II	
(A)	If $\sqrt{(3-4i)}+\sqrt{(-3-4i)}=z$, the principal value of arg (z) can be (where $i=\sqrt{-1}$)	(p)	0
(B)	If $\sqrt{(5+12i)}+\sqrt{(-5+12i)}=z$, the principal value of arg (z) can be (where $i=\sqrt{-1}$)	(q)	$\pm\frac{\pi}{4}$
(C)	If $\sqrt{(-15+8i)}+\sqrt{(-15-8i)}=z$, the principal value of arg (z) can be (where $i=\sqrt{-1}$)	(r)	$\pm\frac{\pi}{2}$
		(s)	$\pm\frac{3\pi}{4}$

Sol. (A) → (q, s); (B) → (q, s); (C) → (p, r)

$$\because\ \sqrt{z}=\pm\left(\sqrt{\frac{|z|+\mathrm{Re}\,(z)}{2}}+i\sqrt{\frac{|z|-\mathrm{Re}(z)}{2}}\right),\ \mathrm{Im}\,(z)>0$$

$$=\pm\left(\sqrt{\frac{|z|+\mathrm{Re}(z)}{2}}-i\sqrt{\frac{|z|-\mathrm{Re}\,(z)}{2}}\right),\ \mathrm{Im}\,(z)<0$$

(A) $\sqrt{(3-4i)}=\pm\left(\sqrt{\frac{5+3}{2}}-i\sqrt{\frac{5-3}{2}}\right)=\pm(2-i)$

$$\sqrt{(-3-4i)}=\pm\left(\sqrt{\frac{5-3}{2}}-i\sqrt{\frac{5+3}{2}}\right)=\pm(1-2i)$$

$$\because\quad z=\sqrt{(3-4i)}+\sqrt{(-3-4i)}$$

$$\therefore\quad z=\pm(2-i)\pm(1-2i)$$

$$=3-3i,\ 1+i,\ -1-i,\ -3+3i$$

$$\therefore\ \text{Principal values of arg}\,(z)=-\frac{\pi}{4},\frac{\pi}{4},-\frac{3\pi}{4},\frac{3\pi}{4}$$

(B) $\sqrt{(5+12i)}=\pm\left(\sqrt{\frac{13+5}{2}}+i\sqrt{\frac{13-5}{2}}\right)$

$$=\pm(3+2i)$$

$$\sqrt{(-5+12i)}=\pm\left(\sqrt{\frac{13-5}{2}}+i\sqrt{\frac{13+5}{2}}\right)$$

$$=\pm(2+3i)$$

$$\because\quad z=\sqrt{(5+12i)}+\sqrt{(-5+12i)}$$

$$=\pm(3+2i)\pm(2+3i)$$

$$=5+5i,\ 1-i,\ -1+i,\ -5-5i$$

$$\therefore\ \text{Principal values of arg}\,(z)=\frac{\pi}{4},-\frac{\pi}{4},\frac{3\pi}{4},-\frac{3\pi}{4}$$

(C) $\sqrt{-15+8i}=\pm\left(\sqrt{\frac{17-15}{2}}+i\sqrt{\frac{17+15}{2}}\right)$

$$=\pm(1+4i)$$

and $\sqrt{-15-8i}=\sqrt{\overline{-15+8i}}=\pm\overline{(1+4i)}$

$$=\pm(1-4i)$$

$$\because\quad z=\sqrt{(-15+8i)}+\sqrt{(-15-8i)}$$

$$=\pm(1+4i)\pm(1-4i)=2,8i,-8i,-2$$

$$\therefore\ \text{Principal values of arg}\,(z)=0,\frac{\pi}{2},-\frac{\pi}{2},\pi.$$

JEE Type Solved Examples :
Statement I and II Type Questions

■ **Directions** Example numbers 26 and 27 are Assertion-Reason type examples. Each of these examples contains two statements:
Statement-1 (Assertion) and **Statement-2** (Reason)
Each of these examples also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement-1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

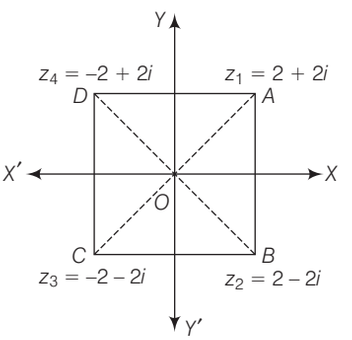
● ***Ex. 26*** *Consider four complex numbers* $z_1=2+2i$, $z_2=2-2i$, $z_3=-2-2i$ *and* $z_4=-2+2i$, *where* $i=\sqrt{-1}$.

Statement-1 z_1, z_2, z_3 *and* z_4 *constitute the vertices of a square on the complex plane because*
Statement-2 *The non-zero complex numbers* $z, \bar{z}, -z, -\bar{z}$ *always constitute the vertices of a square.*

Sol. (*c*) **Statement-1**

$$AB=|z_1-z_2|=|4i|=4,$$
$$BC=|z_2-z_3|=|4|=4,$$
$$CD=|z_3-z_4|=|-4i|=4$$
$$DA=|z_4-z_1|=|-4|=4$$
$$AC=|z_1-z_3|=|4+4i|=4\sqrt{2}$$

and $BD=|z_2-z_4|=|4-4i|=4\sqrt{2}$



It is clear that, $AB = BC = CD = DA$ and $AC = BD$

Hence, z_1, z_2, z_3 and z_4 are the vertices of a square.

$\therefore$ Statement-1 is true.

Statement-2 If $z = a + ib$

If $a \neq b$

Then, $AB = |z - \bar{z}| = |(a + ib) - (a - ib)| = 2|b|$

$BC = |\bar{z} - (-z)| = |\bar{z} + z| = |a - ib + a + ib| = 2|a|$

$\therefore \quad AB \neq BC$

Statement-2 is false.

● ***Ex. 27*** *Consider z_1 and z_2 are two complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$*

Statement-1 *$amp(z_1) - amp(z_2) = 0$*

Statement-2 *The complex numbers z_1 and z_2 are collinear with origin.*

Sol. (b) **Statement-1**

$\because \quad |z_1 + z_2| = |z_1| + |z_2|$ ...(i)

If $\text{amp}(z_1) = \theta_1$ and $\text{amp}(z_2) = \theta_2$, then

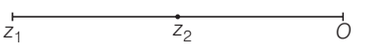
$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \cos(\theta_1 - \theta_2)$$

$$\Rightarrow \quad (|z_1| + |z_2|)^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \cos(\theta_1 - \theta_2)$$

[from Eq. (i)]

$$\Rightarrow \quad |z_1|^2 + |z_2|^2 + 2|z_1||z_2| = |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \cos(\theta_1 - \theta_2)$$

$$\therefore \quad \cos(\theta_1 - \theta_2) = 1$$



$$\Rightarrow \quad \theta_1 - \theta_2 = 0$$

or $\quad \text{amp}(z_1) - \text{amp}(z_2) = 0$

$\therefore$ Statement-1 is true.

Statement-2 Since, z_1, z_2 and O (origin) are collinear, then

$$\text{amp}\left(\frac{O - z_1}{O - z_2}\right) = 0$$

$$\Rightarrow \quad \text{amp}\left(\frac{z_1}{z_2}\right) = 0$$

$$\Rightarrow \quad \text{amp}(z_1) - \text{amp}(z_2) = 0$$

$\therefore$ Statement-2 is true, which is not a correct explanation of Statement-1.

Subjective Type Examples

■ In this section, there are **24 subjective** solved examples.

● ***Ex. 28*** *If $|z - i\,\text{Re}(z)| = |z - \text{Im}(z)|$, then prove that z lies on the bisectors of the quadrants, where $i = \sqrt{-1}$.*

Sol. Let $z = x + iy$, where $x, y \in R$ and $i = \sqrt{-1}$

$$\therefore \quad \text{Re}(z) = x \text{ and } \text{Im}(z) = y$$

Then, $|z - i\,\text{Re}(z)| = |z - \text{Im}(z)|$

$$\Rightarrow \quad |x + iy - ix| = |x + iy - y|$$

$$\Rightarrow \quad |x - i(x - y)| = |(x - y) + iy|$$

$$\Rightarrow \quad \sqrt{x^2 + (x - y)^2} = \sqrt{(x - y)^2 + y^2}$$

$$\Rightarrow \quad x^2 + (x - y)^2 = (x - y)^2 + y^2$$

$$\Rightarrow \quad x^2 = y^2 \text{ or } y = \pm x$$

Hence, z lies on the bisectors of the quadrants.

● ***Ex. 29*** *Find the greatest and the least values of $|z_1 + z_2|$, if $z_1 = 24 + 7i$ and $|z_2| = 6$, where $i = \sqrt{-1}$.*

Sol. $\because \quad z_1 = 24 + 7i$

$$\therefore \quad |z_1| = \sqrt{(24)^2 + (7)^2} = 25$$

$$||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$$

$$\Rightarrow \quad |25 - 6| \leq |z_1 + z_2| \leq 25 + 6$$

or $\quad 19 \leq |z_1 + z_2| \leq 31$

Hence, the least value of $|z_1 + z_2|$ is 19 and the greatest value of $|z_1 + z_2|$ is 31.

● ***Ex. 30*** *Let S denotes the real part of the complex number $z = \dfrac{5 + 2i}{2 - 5i} + \dfrac{20 + 5i}{7 + 6i} + 3i$, where $i = \sqrt{-1}$, K denotes the sum of the imaginary parts of the roots of the equation $z^2 - 8(1 - i)z + 63 - 16i = 0$ and G denotes the value of $\sum_{r=4}^{2012} i^r$, where $i = \sqrt{-1}$, find the value of $S - K + G$.*

Sol. For S,

$$z = \frac{(5 + 2i)}{(2 - 5i)} + \frac{(20 + 5i)}{(7 + 6i)} + 3i$$

$$= \frac{(5 + 2i)(2 + 5i)}{29} + \frac{(20 + 5i)(7 - 6i)}{85} + 3i$$

$$= \frac{0 + 29i}{29} + \frac{170 - 85i}{85} + 3i$$

$$= i + 2 - i + 3i = 2 + 3i$$

$\therefore \text{Re}(z) = 2 \qquad \therefore \quad S = 2$

For K,

Put $z = x + iy$ in the given equation, then

$$(x + iy)^2 - 8(1 - i)(x + iy) + 63 - 16i = 0$$

On comparing the real and imaginary parts, we get

$$x^2 - y^2 - 8(x + y) + 63 = 0 \quad \ldots(i)$$

and $$xy + 4(x - y) = 8 \quad \ldots(ii)$$

On substituting the value of x from Eq. (ii) in Eq. (i), we get

$$y^4 + 16y^3 + \ldots = 0$$

$$\therefore \quad K = -16$$

For G, $$G = \sum_{r=4}^{2012} i^r = \sum_{r=1}^{2009} i^{r+3} = i^{1+3} + 0 = 1$$

$$\therefore \quad S - K + G = 2 - (-16) + 1 = 19$$

● ***Ex. 31*** *If* $|z - 1| = 1$, *where* z *is a point on the argand plane, show that* $\frac{z-2}{z} = i \tan(\arg z)$, *where* $i = \sqrt{-1}$.

Sol. Given, $|z - 1| = 1 \Rightarrow |z - 1|^2 = 1$

$$\Rightarrow \quad (z-1)(\bar{z}-1) = 1 \Rightarrow z\bar{z} - z - \bar{z} = 0$$

$$\Rightarrow \quad (z + \bar{z}) = z\bar{z} \Rightarrow \frac{z}{\bar{z}} + 1 = z$$

$$\Rightarrow \quad \frac{z}{\bar{z}} = z - 1 \quad \ldots(i)$$

Now, RHS $= i \tan(\arg z) = i\left(\frac{\text{Im}(z)}{\text{Re}(z)}\right)$

$$= i\left\{\frac{\frac{z-\bar{z}}{2i}}{\frac{z+\bar{z}}{2}}\right\} = i\left(\frac{z-\bar{z}}{i(z+\bar{z})}\right)$$

$$= \frac{z-\bar{z}}{z+\bar{z}} = \frac{\frac{z}{\bar{z}} - 1}{\frac{z}{\bar{z}} + 1} = \frac{(z-1)-1}{(z-1)+1} = \frac{z-2}{z} \text{ [from Eq. (i)]}$$

= LHS

Aliter

We have, $|z - 1| = 1$ i.e. $(z - 1)$ is unimodular, so we can take

$$z - 1 = \cos\theta + i\sin\theta$$

$$\therefore \quad z - 2 = -1 + \cos\theta + i\sin\theta$$

$$= -2\sin^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= 2i^2\sin^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

or $$z - 2 = 2i\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) \quad \ldots(i)$$

and $$z = 1 + \cos\theta + i\sin\theta$$

$$= 2\cos^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$z = 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) \quad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$\frac{z-2}{z} = i\tan\frac{\theta}{2}$$

Therefore, $\frac{z-2}{z} = i\tan(\arg z)$ $[\because \arg(z) = \theta/2$ from Eq. (ii)]

● ***Ex. 32*** *If* $\arg(z^{1/3}) = \frac{1}{2}\arg(z^2 + \bar{z}\, z^{1/3})$, *find the value of* $|z|$.

Sol. We have, $\arg(z^{1/3}) = \frac{1}{2}\arg(z^2 + \bar{z}\, z^{1/3})$

$$\Rightarrow \quad 2\arg(z^{1/3}) = \arg(z^2 + \bar{z}\, z^{1/3})$$

$$\Rightarrow \quad \arg(z^{2/3}) = \arg(z^2 + \bar{z}\, z^{1/3}) \quad \text{[by property]}$$

$$\Rightarrow \quad \arg(z^2 + \bar{z}\, z^{1/3}) - \arg(z^{2/3}) = 0$$

$$\Rightarrow \quad \arg\left(\frac{z^2 + \bar{z}\, z^{1/3}}{z^{2/3}}\right) = 0 \quad \text{[by property]}$$

$$\Rightarrow \quad \arg\left(z^{4/3} + \frac{\bar{z}}{z^{1/3}}\right) = 0$$

$$\Rightarrow \quad z^{4/3} + \frac{\bar{z}}{z^{1/3}} \text{ is purely real.}$$

$$\Rightarrow \quad \text{Im}\left(z^{4/3} + \frac{\bar{z}}{z^{1/3}}\right) = 0$$

$$\Rightarrow \quad \frac{\left(z^{4/3} + \frac{\bar{z}}{z^{1/3}}\right) - \overline{\left(z^{4/3} + \frac{\bar{z}}{z^{1/3}}\right)}}{2i} = 0$$

$$\Rightarrow \quad z^{4/3} + \frac{\bar{z}}{z^{1/3}} = (\bar{z})^{4/3} + \frac{(\bar{z})}{(\bar{z})^{1/3}}$$

$$\Rightarrow \quad z^{4/3} + \frac{(\bar{z})(\bar{z})^{1/3}}{|z|^{2/3}} = (\bar{z})^{4/3} + \frac{z(z)^{1/3}}{|z|^{2/3}}$$

$$[\because z^{1/3}(\bar{z})^{1/3} = (z\bar{z})^{1/3} = |z|^{2/3}]$$

$$\Rightarrow z^{4/3} - (\bar{z})^{4/3} - \frac{1}{|z|^{2/3}}((z)^{4/3} - (\bar{z})^{4/3}) = 0$$

$$\Rightarrow \quad \{z^{4/3} - (\bar{z})^{4/3}\}\left[1 - \frac{1}{|z|^{2/3}}\right] = 0$$

$$\therefore \quad |z|^{2/3} = 1 \quad [\because z \neq \bar{z}]$$

Therefore, $$|z| = 1$$

● ***Ex. 33*** *C is the complex numbers* $f : C \to R$ *is defined by* $f(z) = |z^3 - z + 2|$. *Find the maximum value of* $f(z)$, *if* $|z| = 1$.

Sol. $\because \quad |z| = 1$

$$\therefore \quad z = e^{i\theta}$$

$$\therefore \quad f(e^{i\theta}) = |e^{3i\theta} - e^{i\theta} + 2| = |e^{2i\theta}(e^{i\theta} - e^{-i\theta}) + 2|$$

$$= |e^{2i\theta} \cdot 2i\sin\theta + 2|$$

$$= |(\cos 2\theta + i\sin 2\theta) \cdot 2i\sin\theta + 2|$$

$$= |(2 - 2\sin 2\theta\sin\theta) + 2i\sin\theta\cos 2\theta|$$

$$= 2|(1 - \sin 2\theta\sin\theta) + i\sin\theta\cos 2\theta|$$

$$= 2\sqrt{(1 - \sin 2\theta\sin\theta)^2 + (\sin\theta\cos 2\theta)^2}$$

$$= 2\sqrt{(1+\sin^2\theta - 2\sin\theta\sin 2\theta)}$$
$$= 2\sqrt{1+\sin^2\theta\,(1-4\cos\theta)}$$
$$= 2\sqrt{1+(1-\cos\theta)(1+\cos\theta)(1-4\cos\theta)}$$

For maximum value, $\cos\theta = -\frac{1}{2}$ $\left[\because \cos\theta \neq -1, 1, \frac{1}{4}\right]$

$\therefore$ Maximum value of $f(z) = 2\cdot\sqrt{1+\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)(3)} = \sqrt{13}$

● ***Ex. 34*** *Prove that the complex numbers z_1 and z_2 and the origin form an isosceles triangle with vertical angle $\frac{2\pi}{3}$, if $z_1^2 + z_2^2 + z_1 z_2 = 0$.*

Sol. Given, $z_1^2 + z_2^2 + z_1 z_2 = 0$

$\Rightarrow$ $(z_1 - \omega z_2)(z_1 - \omega^2 z_2) = 0$

$\Rightarrow$ $z_1 = \omega z_2$ or $z_1 = \omega^2 z_2$

In the first case, $|z_1| = |\omega z_2| \Rightarrow |z_1| = |\omega||z_2| \Rightarrow |z_1| = |z_2|$

Hence, two sides equal

$\text{amp}(z_1) = \text{amp}(\omega)\, z_2 = \text{amp}(\omega) + \text{amp}(z_2)$

$\Rightarrow$ $\text{amp}(z_1) = \frac{2\pi}{3} + \text{amp}(z_2)$

$\Rightarrow$ $\text{amp}(z_1) - \text{amp}(z_2) = \frac{2\pi}{3}$

So, the angle between two sides is $\frac{2\pi}{3}$.

Similarly, the other case

I. Aliter

Given, $z_1^2 + z_2^2 + z_1 z_2 = 0$

$\Rightarrow$ $(z_1 - \omega z_2)(z_1 - \omega^2 z_2) = 0$

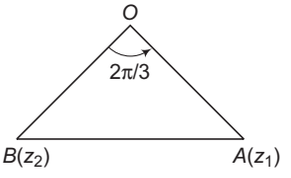
$\Rightarrow$ $z_1 = \omega z_2$ or $z_1 = \omega^2 z_2$

In the first case, $z_1 = \omega z_2$

$\Rightarrow$ $(z_1 - 0) = (z_2 - 0)\, e^{2\pi i/3}$ $\left[\because \omega = e^{\frac{2\pi i}{3}}\right]$

$\Rightarrow$ $\overrightarrow{\mathbf{OA}} = \overrightarrow{\mathbf{OB}}\, e^{2\pi i/3}$

i.e., $\overrightarrow{\mathbf{OA}}$ is obtained by rotating $\overrightarrow{\mathbf{OB}}$ through angle of $\frac{2\pi}{3}$.



$\Rightarrow$ $OA = OB$ and $\angle AOB = \frac{2\pi}{3}$

Thus, triangle formed by z_1, z_2 and origin is isosceles with vertical angle $\frac{2\pi}{3}$.

II. Aliter

Here, $OA = OB$ ...(i)

From Rotation theorem,

$$\frac{z_1 - 0}{z_2 - 0} = \frac{OA}{OB}\, e^{2\pi i/3}$$

$\Rightarrow$ $\frac{z_1}{z_2} = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ [from Eq. (i)]

$\Rightarrow$ $\frac{z_1}{z_2} = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$

$\Rightarrow$ $\left(\frac{z_1}{z_2} + \frac{1}{2}\right) = \frac{i\sqrt{3}}{2}$ ...(ii)

On squaring both sides in Eq. (ii), we get

$\Rightarrow$ $\frac{z_1^2}{z_2^2} + \frac{1}{4} + \frac{z_1}{z_2} = -\frac{3}{4}$

$\Rightarrow$ $\frac{z_1^2}{z_2^2} + \frac{z_1}{z_2} + 1 = 0$

$\Rightarrow$ $z_1^2 + z_1 z_2 + z_2^2 = 0$

$\therefore$ $z_1^2 + z_2^2 + z_1 z_2 = 0$

● ***Ex. 35*** *If $\alpha = e^{2\pi i/7}$, where $i = \sqrt{-1}$ and $f(x) = A_0 + \sum_{k=1}^{20} A_k x^k$, then find the value of $f(x) + f(\alpha x) + f(\alpha^2 x) + \ldots + f(\alpha^6 x)$ independent of α.*

Sol. $\because$ $\alpha = e^{2\pi i/7}$

$\therefore$ $\alpha^7 = e^{2\pi i} = \cos 2\pi + i\sin 2\pi = 1 + 0 = 1$ or $\alpha = (1)^{1/7}$

$\therefore$ $1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6$ are the seven, 7th roots of unity.

$\because f(x) = A_0 + \sum_{k=1}^{20} A_k x^k = \sum_{k=0}^{20} A_k x^k$

Now, $f(x) + f(\alpha x) + f(\alpha^2 x) + \ldots + f(\alpha^6 x)$

$$= \sum_{k=0}^{20} A_k x^k [(1)^k + (\alpha)^k + (\alpha^2)^k + \ldots + (\alpha^6)^k]$$
$$= A_0 x^0 (7) + A_7 x^7 (7) + A_{14} x^{14} (7)$$
$$= 7(A_0 + A_7 x^7 + A_{14} x^{14})$$

$\left[\because (1)^k + (\alpha)^k + (\alpha^2)^k + \ldots + (\alpha^6)^k\right.$

$$= \begin{cases} 7, k \text{ is multiple of } 7 \\ 0, k \text{ is not multiple of } 7 \end{cases}\Bigg]$$

● ***Ex. 36*** *Show that all the roots of the equation $a_1 z^3 + a_2 z^2 + a_3 z + a_4 = 3$, (where $|a_i| \leq 1, i = 1, 2, 3, 4$) lie outside the circle with centre at origin and radius $2/3$.*

Sol. Given that, $a_1 z^3 + a_2 z^2 + a_3 z + a_4 = 3$

We have, $|3| = |a_1z^3 + a_2z^2 + a_3z + a_4|$

$3 \le |a_1z^3| + |a_2z^2| + |a_3z| + |a_4|$

$\Rightarrow \quad 3 \le |a_1||z^3| + |a_2||z^2| + |a_3||z| + |a_4|$

$\Rightarrow \quad 3 \le |a_1||z|^3 + |a_2||z|^2 + |a_3||z| + |a_4|$

$\Rightarrow \quad 3 \le |z|^3 + |z|^2 + |z| + 1 \qquad [\because |a_i| \le 1]$

$\Rightarrow \quad 3 \le 1 + |z| + |z|^2 + |z|^3 < 1 + |z| + |z|^2 + |z|^3 + \ldots$

$\Rightarrow \quad 3 < 1 + |z| + |z|^2 + |z|^3 + \ldots$

$$= \frac{1}{1-|z|} \Rightarrow 3 < \frac{1}{1-|z|}$$

$\Rightarrow \quad 1 - |z| < \frac{1}{3} \Rightarrow \frac{2}{3} - |z| < 0$

$\Rightarrow \quad |z| > 2/3$ or $|z-0| > 2/3$

Hence, all the roots lie in the exterior of circle, $|z-0| = 2/3$.

● ***Ex. 37*** *If A, B and C represent the complex numbers z_1, z_2 and z_3 respectively on the complex plane and the angles at B and C are each equal to $\frac{1}{2}(\pi - \alpha)$, then prove that $(z_2 - z_3)^2 = 4(z_3 - z_1)(z_1 - z_2)\sin^2\left(\frac{\alpha}{2}\right)$.*

Sol. It is given that,

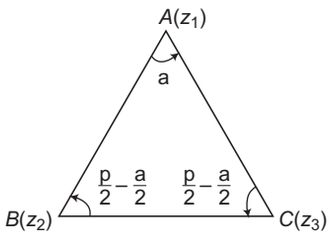
$$\angle ABC = \angle ACB = \frac{\pi}{2} - \frac{\alpha}{2}$$

$\Rightarrow \quad \angle A = \alpha$

$\therefore \quad AC = AB$ ...(i)

So, ΔABC is an isosceles triangle.

Considering rotation of AB about A through angle α, we get



$\Rightarrow \quad \frac{z_3 - z_1}{z_2 - z_1} = e^{i\alpha}$

$\Rightarrow \quad \frac{z_3 - z_1}{z_2 - z_1} = \cos\alpha + i\sin\alpha$ ...(ii)

$\frac{z_3 - z_1}{z_2 - z_1} - 1 = \cos\alpha - 1 + i\sin\alpha$

$\Rightarrow \quad \frac{z_3 - z_2}{z_2 - z_1} = -2\sin^2\frac{\alpha}{2} + 2i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$

or $\quad \frac{z_3 - z_2}{z_2 - z_1} = 2i\sin\frac{\alpha}{2}\left(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right)$ ...(iii)

On squaring both sides, we get

$$\frac{(z_3 - z_2)^2}{(z_2 - z_1)^2} = -4\sin^2\frac{\alpha}{2}\left(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right)^2$$

$$= -4\sin^2\frac{\alpha}{2}(\cos\alpha + i\sin\alpha)$$

[from De-Moivre's theorem]

$$\frac{(z_3 - z_2)^2}{(z_2 - z_1)^2} = -4\sin^2\frac{\alpha}{2}\left(\frac{z_3 - z_1}{z_2 - z_1}\right) \quad \text{[from Eq. (ii)]}$$

Therefore, $(z_2 - z_3)^2 = 4\sin^2(\alpha/2)(z_3 - z_1)(z_1 - z_2)$

Aliter

$\because \quad \angle ABC = \left(\frac{\pi}{2} - \frac{\alpha}{2}\right)$

From Coni method, we have

$$\frac{z_1 - z_2}{z_3 - z_2} = \frac{AB}{BC}e^{i\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)} \qquad \ldots(i)$$

and $\quad \angle ACB = \left(\frac{\pi}{2} - \frac{\alpha}{2}\right)$

From Coni method, we have

$$\frac{z_2 - z_3}{z_1 - z_3} = \frac{BC}{AC}e^{i\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)} \qquad \ldots(ii)$$

On dividing Eq. (ii) by Eq. (i), we get

$$\frac{(z_2 - z_3)^2}{(z_3 - z_1)(z_1 - z_2)} = \frac{(BC)^2}{AB \cdot AC} = \left(\frac{BC}{AB}\right)^2 \qquad [\because AB = AC]$$

$$= \left(\frac{\sin\alpha}{\sin\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)}\right)^2 \qquad \text{[using sine rule]}$$

$$= \left(\frac{2\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)}{\cos\left(\frac{\alpha}{2}\right)}\right)^2 = 4\sin^2\left(\frac{\alpha}{2}\right)$$

Therefore, $(z_2 - z_3)^2 = 4(z_3 - z_1)(z_1 - z_2)\sin^2\alpha/2$

● ***Ex. 38*** *If z_1 and z_2 are two complex numbers such that $\left|\frac{z_1 - z_2}{z_1 + z_2}\right| = 1$, then prove that $\frac{iz_1}{z_2} = k$, where k is a real number. Find the angle between the lines from the origin to the points $z_1 + z_2$ and $z_1 - z_2$ in terms of k.*

Sol. (i) Given, $\left|\frac{z_1 - z_2}{z_1 + z_2}\right| = 1$

$$\Rightarrow \quad \left|\frac{\frac{z_1}{z_2} - 1}{\frac{z_1}{z_2} + 1}\right| = 1$$

$$\Rightarrow \quad \left|\frac{z_1}{z_2} - 1\right| = \left|\frac{z_1}{z_2} + 1\right| \qquad \ldots(i)$$

On squaring Eq. (i) both sides, we have

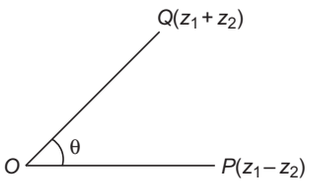
$$\left|\frac{z_1}{z_2}\right|^2 + 1 - 2\,\text{Re}\left(\frac{z_1}{z_2}\right) = \left|\frac{z_1}{z_2}\right|^2 + 1 + 2\,\text{Re}\left(\frac{z_1}{z_2}\right)$$

$$\Rightarrow \quad 4\,\text{Re}\left(\frac{z_1}{z_2}\right) = 0$$

$\Rightarrow \frac{z_1}{z_2}$ is purely imaginary number

$\Rightarrow \frac{z_1}{z_2}$ can be written as $i\frac{z_1}{z_2} = k$, where k is a real number.

(ii) Let θ be the angle between $z_1 - z_2$ and $z_1 + z_2$, then



$$\theta = \arg\left(\frac{z_1+z_2}{z_1-z_2}\right) = \arg\left(\frac{\frac{z_1}{z_2}+1}{\frac{z_1}{z_2}-1}\right) = \arg\left(\frac{-ik+1}{-ik-1}\right)$$

$$= \arg\left(\frac{-1+ik}{1+ik}\right) = \arg\left(\frac{k^2-1+2ik}{k^2+1}\right)$$

Therefore, $\theta = \tan^{-1}\left(\frac{2k}{k^2-1}\right)$

Aliter (i) Given, $\left|\frac{z_1-z_2}{z_1+z_2}\right| = 1$ ...(i)

Let $\frac{z_1-z_2}{z_1+z_2} = \frac{\cos\alpha + i\sin\alpha}{1}$

$\Rightarrow \quad \frac{(z_1-z_2)+(z_1+z_2)}{(z_1+z_2)-(z_1-z_2)} = \frac{1+\cos\alpha+i\sin\alpha}{1-\cos\alpha-i\sin\alpha}$ [by componendo and dividendo]

$$\Rightarrow \quad \frac{z_1}{z_2} = \frac{2\cos^2\left(\frac{\alpha}{2}\right) + 2i\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)}{2\sin^2\left(\frac{\alpha}{2}\right) - 2i\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)}$$

$$\Rightarrow \quad = \frac{2\cos\left(\frac{\alpha}{2}\right)\left(\cos\left(\frac{\alpha}{2}\right) + i\sin\left(\frac{\alpha}{2}\right)\right)}{-2i\sin\left(\frac{\alpha}{2}\right)\left(\cos\left(\frac{\alpha}{2}\right) + i\sin\left(\frac{\alpha}{2}\right)\right)}$$

$$\Rightarrow \quad \frac{z_1}{z_2} = -\frac{\cot\left(\frac{\alpha}{2}\right)}{i}$$

$$\Rightarrow \quad \frac{iz_1}{z_2} = -\cot\left(\frac{\alpha}{2}\right) = k\,(\text{say}) = \text{real}$$

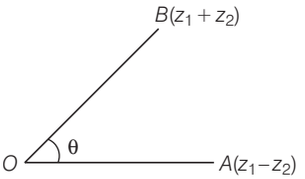
Hence, $\frac{iz_1}{z_2} = k$ ...(ii)

(ii) Now, let the angle between OB and OA be θ, then from Coni method,

$$\frac{z_1+z_2-0}{z_1-z_2-0} = \frac{OB}{OA}\,e^{i\theta} = \left|\frac{z_1+z_2}{z_1-z_2}\right| e^{i\theta}$$

$$\Rightarrow \quad \left(\frac{z_1+z_2}{z_1-z_2}\right) = e^{i\theta} \qquad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad \left(\frac{\frac{z_1}{z_2}+1}{\frac{z_1}{z_2}-1}\right) = e^{i\theta} \Rightarrow \frac{-ki+1}{-ki-1} = e^{i\theta} \qquad \text{[from Eq. (ii)]}$$



$$\Rightarrow \quad \frac{-1+ki}{1+ki} = e^{i\theta}$$

$$\Rightarrow \quad \frac{(-1+ki)(1-ki)}{(1+ki)(1-ki)} = e^{i\theta}$$

$$\Rightarrow \quad \frac{(k^2-1)}{(k^2+1)} + \frac{2ki}{1+k^2} = e^{i\theta}$$

$$\therefore \quad \text{Re}\,(e^{i\theta}) = \cos\theta = \frac{k^2-1}{k^2+1}$$

and $\text{Im}(e^{i\theta}) = \sin\theta = \frac{2k}{k^2+1}$

$$\therefore \quad \tan\theta = \frac{2k}{k^2-1}$$

Therefore, $\theta = \tan^{-1}\left(\frac{2k}{k^2-1}\right)$

● ***Ex. 39*** *If $z = x + iy$ is a complex number with rationals x and y and $|z| = 1$, then show that $|z^{2n} - 1|$ is a rational number for every $n \in N$.*

Sol. Since, $|z| = 1$, where z is unimodular

$\therefore \quad z = \cos\theta + i\sin\theta$

As x and y are rational, $\cos\theta, \sin\theta$ are rationals

$$\therefore \quad |z^{2n} - 1| = \left|z^n\left(z^n - \frac{1}{z^n}\right)\right| = |z|^n\,|z^n - z^{-n}|$$

$$= 1\,|2i\sin n\theta|$$

$$= 2\,|\sin n\theta|$$

Since, $\sin n\theta$ is rational, therefore $|z^{2n} - 1|$ is a rational number.

● **Ex. 40** *If a is a complex number such that $|a|=1$, then find the value of a, so that equation $az^2+z+1=0$ has one purely imaginary root.*

Sol. We have, $az^2+z+1=0$ ...(i)

On taking conjugate both sides, we get

$$\overline{az^2+z+1}=\bar{0}$$

$$\Rightarrow \quad \bar{a}(\bar{z})^2+\bar{z}+1=0$$

$$\Rightarrow \quad \bar{a}(-z)^2+(-z)+1=0$$

[since, z is purely imaginary, $\bar{z}=-z$]

or $\bar{a}z^2-z+1=0$ ...(ii)

Eliminating z from Eqs. (i) and (ii) by cross-multiplication rule, we get

$$(\bar{a}-a)^2+2(a+\bar{a})=0$$

On dividing each by 4, we get

$$\left(\frac{\bar{a}-a}{2}\right)^2+\left(\frac{a+\bar{a}}{2}\right)=0$$

$$\Rightarrow \quad -\left(\frac{a-\bar{a}}{2i}\right)^2+\left(\frac{a+\bar{a}}{2}\right)=0$$

or $-(\text{Im}(a))^2+\text{Re}(a)=0$ ...(iii)

Given, $|a|=1$

Let $a=\cos\alpha+i\sin\alpha$

$\therefore$ $\text{Re}(a)=\cos\alpha$, $\text{Im}(a)=\sin\alpha$

Then, from Eq. (iii), we get

$$-\sin^2\alpha+\cos\alpha=0 \quad \text{or} \quad \cos^2\alpha+\cos\alpha-1=0$$

$$\therefore \quad \cos\alpha=\frac{-1\pm\sqrt{1+4}}{2}$$

Only feasible value of $\cos\alpha=\dfrac{\sqrt{5}-1}{2}$

Hence, $a=\cos\alpha+i\sin\alpha$

where, $\alpha=\cos^{-1}\left(\dfrac{\sqrt{5}-1}{2}\right)$

● **Ex. 41** *If $n\in N>1$, find the sum of real parts of the roots of the equation $z^n=(z+1)^n$.*

Sol. The equation $z^n=(z+1)^n$ will have exactly $n-1$ roots.

We have,

$$\left(\frac{z+1}{z}\right)^n=1 \Rightarrow \left|\left(\frac{z+1}{z}\right)^n\right|=|1|$$

$$\Rightarrow \quad \frac{|z+1|}{|z|}=1$$

$$\Rightarrow \quad |z+1|=|z|$$

$$\Rightarrow \quad |z-(-1)|=|z-0|$$

Therefore, z lies on the right bisector of the segment connecting the points $0+i\cdot 0$ and $-1+0\cdot i$. Thus, $\text{Re}(z)=-1/2$. Hence, roots are collinear and will have their real parts equal to $-1/2$. Hence, sum of the real parts of roots is $\left(-\dfrac{1}{2}(n-1)\right)$.

Aliter

$\because$ $z^n=(z+1)^n$

$$\Rightarrow \quad \left(\frac{z+1}{z}\right)^n=1 \quad \text{or} \quad \frac{z+1}{z}=(1)^{1/n}$$

$$=(\cos 0+i\sin 0)^{1/n}$$

$$=(\cos 2r\pi+i\sin 2r\pi)^{1/n}$$

$$\Rightarrow \quad 1+\frac{1}{z}=\cos\left(\frac{2r\pi}{n}\right)+i\sin\left(\frac{2r\pi}{n}\right)=e^{2r\pi i/n}$$

or $\dfrac{1}{z}=(e^{2r\pi i/n}-1)=e^{\frac{r\pi i}{n}}\cdot 2i\sin\left(\dfrac{\pi r}{n}\right)$

or $z=-\left(\dfrac{1}{2}\right)i\cdot\dfrac{1}{\sin\left(\frac{\pi r}{n}\right)}\cdot e^{-\frac{\pi r i}{n}}$

$$=-\left(\frac{i}{2}\right)\cdot\frac{\left(\cos\frac{\pi r}{n}-i\sin\frac{\pi r}{n}\right)}{\sin\frac{\pi r}{n}}$$

$\therefore$ $\text{Re}(z)=-\dfrac{1}{2}$ [here $r\neq 0$]

where, $r=1,2,3,\ldots,n-1$

Sum of real parts of $z=-\dfrac{1}{2}-\dfrac{1}{2}-\dfrac{1}{2}-\ldots-(n-1)$ times

$$=-\frac{1}{2}(n-1).$$

● **Ex. 42** *Prove that the angle between the line $\bar{a}z+a\bar{z}=0$ and its reflection in the real axis is*

$$\theta=\tan^{-1}\left\{\frac{2\,\text{Re}(a)\,\text{Im}(a)}{\{\text{Im}(a)\}^2-\{\text{Re}(a)\}^2}\right\}.$$

Sol. Let $z=x+iy$, then equation $\bar{a}z+a\bar{z}=0$ can be written as

$$(\bar{a}+a)x+i(\bar{a}-a)y=0$$

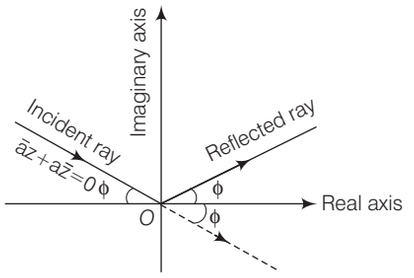
$$\Rightarrow \quad \left(\frac{a+\bar{a}}{2}\right)x+\left(\frac{a-\bar{a}}{2i}\right)y=0$$

$$\Rightarrow \quad \{\text{Re}(a)\}x+\{\text{Im}(a)\}y=0$$

$\therefore$ Slope of the given line $(m)=-\dfrac{\{\text{Re}(a)\}}{\{\text{Im}(a)\}}$

Then, $\tan(180^\circ-\phi)=-\dfrac{\{\text{Re}(a)\}}{\{\text{Im}(a)\}}\Rightarrow\tan\phi=\dfrac{\{\text{Re}(a)\}}{\{\text{Im}(a)\}}$

Hence, angle between the given line and its reflection in real axis



$$= 2\phi = \tan^{-1}\{\tan 2\phi\} = \tan^{-1}\left\{\frac{2\tan\phi}{1-\tan^2\phi}\right\}$$

$$= \tan^{-1}\left\{\frac{2\dfrac{\{\text{Re}(a)\}}{\{\text{Im}(a)\}}}{1-\dfrac{\{\text{Re}(a)\}^2}{\{\text{Im}(a)\}^2}}\right\} = \tan^{-1}\left\{\frac{2\,\text{Re}(a)\,\text{Im}(a)}{\{\text{Im}(a)\}^2-\{\text{Re}(a)\}^2}\right\}$$

● ***Ex. 43*** *Among the complex numbers z which satisfies $|z-25i|\le 15$, find the complex numbers z having*

(i) least positive argument.

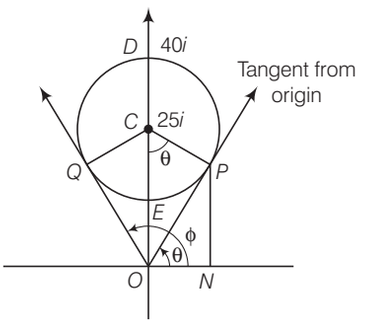
(ii) maximum positive argument.

(iii) least modulus.

(iv) maximum modulus.

Sol. The complex numbers z satisfying the condition

$$|z-25i|\le 15 \qquad \ldots(i)$$



are represented by the points inside and on the circle of radius 15 and centre at the point $C(0, 25)$.

The complex numbers having least positive argument and maximum positive arguments in this region are the points of contact of tangents drawn from origin to the circle.

Here, θ = Least positive argument

and ϕ = Maximum positive argument

$\therefore$ In ΔOCP, $OP = \sqrt{(OC)^2-(CP)^2} = \sqrt{(25)^2-(15)^2} = 20$

and $$\sin\theta = \frac{OP}{OC} = \frac{20}{25} = \frac{4}{5}$$

$$\therefore \quad \tan\theta = \frac{4}{3} \Rightarrow \theta = \tan^{-1}\left(\frac{4}{3}\right)$$

Thus, complex number at P has modulus 20 and argument $\theta = \tan^{-1}\left(\frac{4}{3}\right)$

$$\therefore \quad z_P = 20(\cos\theta + i\sin\theta) = 20\left(\frac{3}{5}+i\frac{4}{5}\right)$$

$$\therefore \quad z_P = 12+16i$$

Similarly, $z_Q = -12+16i$

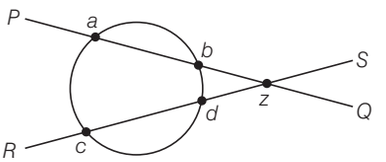
From the figure, E is the point with least modulus and D is the point with maximum modulus.

Hence, $$z_E = \overrightarrow{\mathbf{OE}} = \overrightarrow{\mathbf{OC}} - \overrightarrow{\mathbf{EC}} = 25i - 15i = 10i$$

and $$z_D = \overrightarrow{\mathbf{OD}} = \overrightarrow{\mathbf{OC}} + \overrightarrow{\mathbf{CD}} = 25i + 15i = 40i$$

● ***Ex. 44*** *Two different non-parallel lines meet the circle $|z| = r$ in the points a, b and c, d, respectively. Prove that these lines meet in the point z given by $z = \dfrac{a^{-1}+b^{-1}-c^{-1}-d^{-1}}{a^{-1}b^{-1}-c^{-1}d^{-1}}$, where a, b, c, d are complex constants.*

Sol. Let two non-parallel straight lines PQ, RS meet the circle $|z| = r$ in the points a, b and c, d, then $|a| = r, |b| = r, |c| = r$ and $|d| = r$



or $$|a|^2 = |b|^2 = |c|^2 = |d|^2 = r^2$$

$$\Rightarrow \quad a\bar{a} = b\bar{b} = c\bar{c} = d\bar{d} = r^2$$

$$\therefore \quad \bar{a} = \frac{r^2}{a}, \bar{b} = \frac{r^2}{b}, \bar{c} = \frac{r^2}{c} \text{ and } \bar{d} = \frac{r^2}{d}$$

For line PQ, points a, b and z are collinear, then

$$\begin{vmatrix} z & \bar{z} & 1 \\ a & \bar{a} & 1 \\ b & \bar{b} & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad z(\bar{a}-\bar{b}) - \bar{z}(a-b) + (a\bar{b}-\bar{a}b) = 0$$

$$\Rightarrow \quad z\left(\frac{r^2}{a}-\frac{r^2}{b}\right) - \bar{z}(a-b) + \left(\frac{ar^2}{b}-\frac{br^2}{a}\right) = 0$$

On dividing both sides by $(b-a)$, we get

$$\frac{r^2}{ab}z + \bar{z} - \frac{r^2}{ab}(a+b) = 0$$

$$\Rightarrow \quad \frac{z}{ab} + \frac{\bar{z}}{r^2} - \frac{(a+b)}{ab} = 0 \qquad \ldots(i)$$

Similarly, for line RS, we get

$$\frac{z}{cd} + \frac{\bar{z}}{r^2} - \frac{(c+d)}{cd} = 0 \qquad \ldots(ii)$$

On subtracting Eq. (ii) from Eq. (i), we get

$$z\left(\frac{1}{ab}-\frac{1}{cd}\right) - \frac{(a+b)}{ab} + \frac{(c+d)}{cd} = 0$$

$$\Rightarrow \quad z(a^{-1}b^{-1}-c^{-1}d^{-1}) = a^{-1}+b^{-1}-c^{-1}-d^{-1}$$

Therefore, $$z = \frac{a^{-1}+b^{-1}-c^{-1}-d^{-1}}{a^{-1}b^{-1}-c^{-1}d^{-1}}$$

● ***Ex. 45*** *If n is an odd integer but not a multiple of 3, then prove that* $xy(x+y)(x^2+y^2+xy)$ *is a factor of* $(x+y)^n - x^n - y^n$.

Sol. We have, $xy(x+y)(x^2+y^2+xy) = xy(x+y)(x-\omega y)(x-\omega^2 y)$

and let $f(x, y) = (x+y)^n - x^n - y^n$ ...(i)

On putting $x = 0$ in Eq. (i), we get

$$f(0, y) = y^n - 0 - y^n = 0$$

$\therefore$ $x - 0$ is a factor of Eq. (i).

On putting $y = 0$ in Eq. (i), we get

$$f(x, 0) = x^n - x^n = 0$$

$\therefore$ $y - 0$ is a factor of Eq. (i).

On putting $x = -y$ in Eq. (i), we get

$$f(-y, y) = (-y+y)^n - (-y)^n - y^n$$
$$= 0 - (-y)^n - y^n = y^n - y^n = 0 \quad [\text{because } n \text{ is odd}]$$

$\therefore$ $x + y$ is a factor of Eq. (i).

On putting $x = \omega y$ in Eq. (i), we get

$$f(\omega y, y) = (\omega y + y)^n - (\omega y)^n - y^n$$
$$= y^n [(\omega+1)^n - \omega^n - 1]$$
$$= y^n [(-\omega^2)^n - \omega^n - 1)] \quad [\because 1 + \omega + \omega^2 = 0]$$
$$= -y^n \{\omega^{2n} + \omega^n + 1\} \quad [\text{because } n \text{ is odd}]$$

Since, n is odd but not a multiple of 3, then $n = 3k + 1$ or $n = 3k + 2$, where k is an integer.

$\therefore$ $\omega^{2n} + \omega^n + 1 = 0$ [in both cases] ...(ii)

$\therefore$ $f(\omega y, y) = 0$

$\therefore$ $x - \omega y$ is a factor of Eq. (i).

On putting $x = \omega^2 y$ in Eq. (i), we get

$$f(\omega^2 y, y) = (\omega^2 y + y)^n - (\omega^2 y)^n - y^n$$
$$= y^n \{(\omega^2+1)^n - \omega^{2n} - 1\}$$
$$= y^n \{(-\omega)^n - \omega^{2n} - 1\}$$
$$= -y^n \{\omega^n + \omega^{2n} + 1\} \quad [\text{because, } n \text{ is odd}]$$
$$= 0 \quad [\text{from Eq. (ii)}]$$

$\therefore$ $x - \omega^2 y$ is a factor of Eq. (i).

Combining all the factors, we get

$$(x-0)(y-0)(x+y)(x-\omega y)(x-\omega^2 y)$$

Therefore, $xy(x+y)(x^2+xy+y^2)$ is a factor of $f(x, y) = (x+y)^n - x^n - y^n$.

● ***Ex. 46*** *Interpret the following equations geometrically on the Argand plane.*

(i) $|z-1| + |z+1| = 4$ *(ii)* $\arg(z+i) - \arg(z-i) = \dfrac{\pi}{2}$

(iii) $1 < |z-2-3i| < 4$ *(iv)* $\dfrac{\pi}{4} < \arg(z) < \dfrac{\pi}{3}$

(v) $\log_{\cos \pi/3} \left\{\dfrac{|z-1|+4}{3|z-1|-2}\right\} > 1$

Sol. (i) Since, $|z-1| + |z+1| = 4$

i.e., (distance of z from the point $1 + 0 \cdot i$) + (distance of z from the point $-1 + 0 \cdot i$) = 4 (constant)

i.e., The sum of the distances of z from two fixed points $1 + 0 \cdot i$ and $-1 + 0 \cdot i$ is constant, which is the definition of an ellipse.

Therefore, locus of z satisfying the given condition will be an ellipse with foci at $1 + 0 \cdot i$ and $-1 + 0 \cdot i$ and centre at origin.

(ii) Given that,

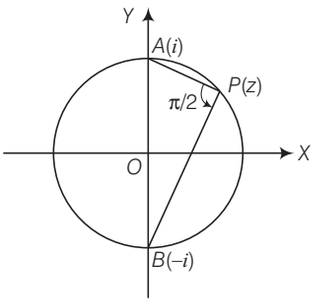
$$\arg(z+i) - \arg(z-i) = \frac{\pi}{2}$$

or $$\arg\left(\frac{z+i}{z-i}\right) = \frac{\pi}{2} \quad \text{...(i)}$$

Let the points A and B have affixes i and $-i$ and the point P has affix z. Then, Eq. (i) can be written as

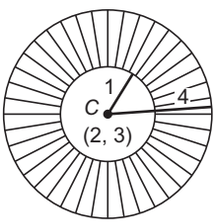
$$\angle BPA = \frac{\pi}{2} \qquad \left[\because \angle BPA = \arg\left(\frac{z+i}{z-i}\right)\right]$$

Thus, locus of $P(z)$ is such that the angle subtended at P by the line joining points A and B is $\dfrac{\pi}{2}$. This is the definition of a circle with diameter AB.



Therefore, locus of point z is a circle with diameter AB and centre at origin with radius 1.

(iii) We have, $1 < |z-2-3i| < 4$ represents a circle with centre at $(2, 3)$ and radius $r \in (1, 4)$.



Since, $|z-2-3i| > 1$ represents the region in the plane outside the circle.

$\therefore$ $|z-2-3i| = 1$ ...(i)

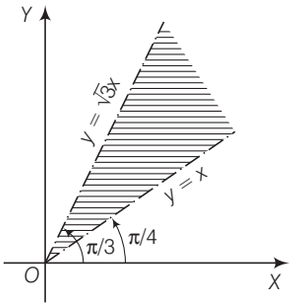
and $|z-2-3i| < 4$ represents the region inside circle.

$\therefore$ $|z-2-3i| = 4$ ...(ii)

Hence, $1 < |z-2-3i| < 4$ represent the angular space between the two circles (i) and (ii).

(iv) We have, $\frac{\pi}{4} < \arg(z) < \frac{\pi}{3}$

Let $z = x + iy \Rightarrow \arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$



The given inequality can be written as

$$\frac{\pi}{4} < \tan^{-1}\left(\frac{y}{x}\right) < \frac{\pi}{3}$$

$$\Rightarrow \quad \tan\frac{\pi}{4} < \frac{y}{x} < \tan\frac{\pi}{3}$$

$$\Rightarrow \quad 1 < \frac{y}{x} < \sqrt{3}$$

$$\Rightarrow \quad x < y < \sqrt{3}\,x$$

This inequality represents the region between the lines

$$y = x \text{ and } y = \sqrt{3}x$$

(v) We have, $\log_{\cos \pi/3}\left\{\frac{|z-1|+4}{3|z-1|-2}\right\} > 1$

or $\log_{1/2}\left\{\frac{|z-1|+4}{3|z-1|-2}\right\} > 1$

$$\Rightarrow \quad \frac{|z-1|+4}{3|z-1|-2} < \frac{1}{2}$$

or $2|z-1|+8 < 3|z-1|-2$

$$\Rightarrow \quad |z-1| > 10$$

Hence, the inequality represents exterior of a circle of radius 10 with centre at (1, 0).

● ***Ex. 47*** *Show that the triangles whose vertices are* z_1, z_2, z_3 *and* z_1', z_2', z_3' *are equilateral, if*

$$(z_1 - z_2)(z_1' - z_2') = (z_2 - z_3)(z_2' - z_3') = (z_3 - z_1)(z_3' - z_1')$$

Sol. From the first two relations, we have

$$\frac{z_1 - z_2}{z_2' - z_3'} = \frac{z_2 - z_3}{z_1' - z_2'} = \frac{(z_1 - z_2) + (z_2 - z_3)}{(z_2' - z_3') + (z_1' - z_2')} = \frac{z_1 - z_3}{z_1' - z_3'}$$

$$\therefore \quad \frac{z_1 - z_2}{z_2' - z_3'} = \frac{z_1 - z_3}{z_1' - z_3'} \qquad \ldots(i)$$

Also, from the last two relations

$$(z_2 - z_3)(z_2' - z_3') = (z_3 - z_1)(z_3' - z_1') \qquad \ldots(ii)$$

On multiplying Eqs. (i) and (ii), we get

$$(z_1 - z_2)(z_2 - z_3) = (z_1 - z_3)^2$$

or $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$

Hence, the triangle whose vertices are z_1, z_2 and z_3 is equilateral.

Similarly, it can be shown that the triangle whose vertices are z_1', z_2' and z_3' is also equilateral.

● ***Ex. 48*** *Show that the triangle whose vertices are* z_1, z_2, z_3 *and* z_1', z_2', z_3' *are directly similar, if*

$$\begin{vmatrix} z_1 & z_1' & 1 \\ z_2 & z_2' & 1 \\ z_3 & z_3' & 1 \end{vmatrix} = 0.$$

Sol. Let A, B, C be the points of affix z_1, z_2, z_3 and A', B', C' be the points of affix z_1', z_2', z_3'.

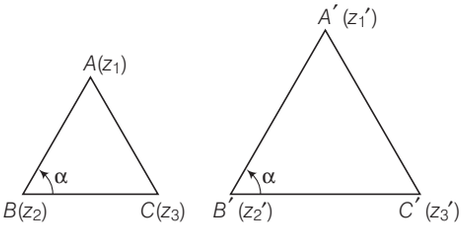
Since, the triangles ABC and $A'B'C'$ are similar, if

$\overrightarrow{\mathbf{BC}} = \lambda\,\overrightarrow{\mathbf{B'C'}}$

i.e., $(z_3 - z_2) = \lambda(z_3' - z_2')$ ...(i)

and $\overrightarrow{\mathbf{CA}} = \lambda\,\overrightarrow{\mathbf{C'A'}}$

i.e., $(z_1 - z_3) = \lambda(z_1' - z_3')$ ...(ii)



On dividing Eq. (i) by Eq. (ii), we get

$$\frac{z_3 - z_2}{z_1 - z_3} = \frac{z_3' - z_2'}{z_1' - z_3'}$$

$$\Rightarrow \quad z_3(z_1' - z_3') - z_2(z_1' - z_3') = z_1(z_3' - z_2') - z_3(z_3' - z_2')$$

$$\Rightarrow \quad z_1(z_2' - z_3') - z_2(z_1' - z_3') + z_3(z_1' - z_2') = 0$$

Hence, $\begin{vmatrix} z_1 & z_1' & 1 \\ z_2 & z_2' & 1 \\ z_3 & z_3' & 1 \end{vmatrix} = 0$

Aliter

Since, ΔABC and $\Delta A'B'C'$ are similar.

If $\frac{AB}{A'B'} = \frac{BC}{B'C'}$ and $\angle ABC = \angle A'B'C' = \alpha$ [say]

Then, from Coni method in ΔABC and $\Delta A'B'C'$, we have

$$\frac{z_1 - z_2}{z_3 - z_2} = \frac{AB}{BC}\,e^{i\alpha} \qquad \ldots(i)$$

and $$\frac{z_1' - z_2'}{z_3' - z_2'} = \frac{A'B'}{B'C'}\,e^{i\alpha} \qquad \ldots(ii)$$

Since, $\frac{AB}{A'B'} = \frac{BC}{B'C'} \therefore \frac{AB}{BC} = \frac{A'B'}{B'C'}$

From Eqs. (i) and (ii), we get $\frac{z_1 - z_2}{z_3 - z_2} = \frac{z_1' - z_2'}{z_3' - z_2'}$,

On simplifying as in Ist method, we get $\begin{vmatrix} z_1 & z_1' & 1 \\ z_2 & z_2' & 1 \\ z_3 & z_3' & 1 \end{vmatrix} = 0$

● ***Ex. 49*** *If ω is the nth root of unity and z_1, z_2 are any two complex numbers, then prove that*
$\sum_{k=0}^{n-1} \left| z_1 + \omega^k z_2 \right|^2 = n\{|z_1|^2 + |z_2|^2\}$, *where* $n \in N$.

Sol. If $1, \omega, \omega^2, \omega^3, ..., \omega^{n-1}$ are the n, nth roots of unity, then

$$\sum_{k=0}^{n-1} \omega^k = 0 \text{ and } \sum_{k=0}^{n-1} (\bar{\omega})^k = 0 \quad ...(i)$$

$$\text{LHS} = \sum_{k=0}^{n-1} \left| z_1 + \omega^k z_2 \right|^2$$

$$= \sum_{k=0}^{n-1} (z_1 + \omega^k z_2)(\bar{z}_1 + (\bar{\omega})^k \bar{z}_2)$$

$$= \sum_{k=0}^{n-1} \{z_1\bar{z}_1 + z_1\bar{z}_2 (\bar{\omega})^k + \bar{z}_1 z_2 \omega^k + z_2\bar{z}_2 (\omega^k)(\bar{\omega})^k\}$$

$$= \sum_{k=0}^{n-1} |z_1|^2 + \sum_{k=0}^{n-1} z_1\bar{z}_2 (\bar{\omega})^k + \sum_{k=0}^{n-1} \bar{z}_1 z_2 \omega^k + \sum_{k=0}^{n-1} |z_2|^2$$

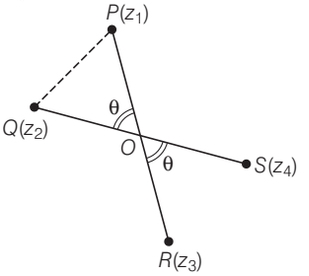
$$= |z_1|^2 \sum_{k=0}^{n-1} 1 + z_1\bar{z}_2 \sum_{k=0}^{n-1} (\bar{\omega})^k + \bar{z}_1 z_2 \sum_{k=0}^{n-1} (\omega)^k + |z_2|^2 \sum_{k=0}^{n-1} 1$$

$= n|z_1|^2 + 0 + 0 + n|z|^2$ [from Eq. (i)]

$= n\{|z_1|^2 + |z_2|^2\} = \text{RHS}$

● ***Ex. 50*** *Let $\sum_{i=1}^{4} a_i = 0$ and $\sum_{j=1}^{4} a_j z_j = 0$, then prove that z_1, z_2, z_3 and z_4 are concyclic, if $a_1 a_2 |z_1 - z_2|^2 = a_3 a_4 |z_3 - z_4|^2$.*

Sol. $\because \sum_{i=1}^{4} a_i = 0$



$\therefore a_1 + a_2 + a_3 + a_4 = 0$

$\Rightarrow (a_1 + a_3) = -(a_2 + a_4)$ ...(i)

and $\sum_{j=1}^{4} a_j z_j = 0$

$\therefore a_1 z_1 + a_2 z_2 + a_3 z_3 + a_4 z_4 = 0$

$\Rightarrow (a_1 z_1 + a_3 z_3) = -(a_2 z_2 + a_4 z_4)$ ...(ii)

On dividing Eq. (ii) by Eq. (i), we get

$$\frac{a_1 z_1 + a_3 z_3}{a_1 + a_3} = \frac{a_2 z_2 + a_4 z_4}{a_2 + a_4} \quad ...(iii)$$

Eq. (iii) implies that point O divides PR in the ratio $a_3 : a_1$ and O divides QS in the ratio $a_4 : a_2$.

Let $OR = a_1 k, OP = a_3 k, OQ = a_4 l, OS = a_2 l$

Now, in ΔOPQ,

$(PQ)^2 = (OP)^2 + (OQ)^2 - 2(OP)(OQ)\cos\theta$

$\Rightarrow |z_1 - z_2|^2 = a_3^2 k^2 + a_4^2 l^2 - 2a_3 a_4 lk \cos\theta$

$\therefore a_1 a_2 |z_1 - z_2|^2 = a_1 a_2 a_3^2 k^2 + a_1 a_2 a_4^2 l^2 - 2a_1 a_2 a_3 a_4 lk \cos\theta$

Similarly, $a_3 a_4 |z_3 - z_4|^2 = a_3 a_4 a_1^2 k^2 + a_3 a_4 a_2^2 l^2 - 2a_1 a_2 a_3 a_4 lk \cos\theta$

From given condition, $a_1 a_2 |z_1 - z_2|^2 = a_3 a_4 |z_3 - z_4|^2$

$\therefore a_1 a_2 a_3^2 k^2 + a_1 a_2 a_4^2 l^2 = a_3 a_4 a_1^2 k^2 + a_3 a_4 a_2^2 l^2$

$\Rightarrow k^2 a_3 a_1 (a_2 a_3 - a_1 a_4) = l^2 a_2 a_4 (a_2 a_3 - a_1 a_4)$

$\Rightarrow (a_1 k)(a_3 k) = (a_2 l)(a_4 l)$

$\Rightarrow OP \cdot OR = OQ \cdot OS$

So, P, Q, R and S are concyclic.

● ***Ex. 51*** *If α and β are the roots of $z + \frac{1}{z} = 2(\cos\theta + i\sin\theta)$, where $0 < \theta < \pi$ and $i = \sqrt{-1}$, show that $|\alpha - i| = |\beta - i|$.*

Sol. Since, $z + \frac{1}{z} = 2(\cos\theta + i\sin\theta)$

$$\therefore z + \frac{1}{z} = 2e^{i\theta} \Rightarrow z^2 - 2e^{i\theta} z + 1 = 0$$

$$\Rightarrow z = \frac{2e^{i\theta} \pm \sqrt{(4e^{2i\theta} - 4)}}{2}$$

$$\Rightarrow z = e^{i\theta} \pm \sqrt{(e^{2i\theta} - 1)} \Rightarrow z = e^{i\theta} \pm \sqrt{e^{i\theta} \cdot 2i\sin\theta}$$

$$\Rightarrow z - i = e^{i\theta} - i \pm \sqrt{e^{i\theta} \cdot 2i\sin\theta}$$

$$= (e^{i\theta} - e^{i\pi/2}) \pm \sqrt{e^{i(\theta+\pi/2)} \cdot 2\sin\theta}$$

$$= e^{i\left(\frac{\theta}{2} + \frac{\pi}{4}\right)} \cdot 2i\sin\left(\frac{\theta}{2} - \frac{\pi}{4}\right) \pm e^{i(\theta/2 + \pi/4)} \cdot \sqrt{2\sin\theta}$$

$$= e^{i(\theta/2 + \pi/4)} \left\{2i\sin\left(\frac{\theta}{2} - \frac{\pi}{4}\right) \pm \sqrt{2\sin\theta}\right\}$$

$$\therefore |z - i| = 1 \cdot \sqrt{4\sin^2\left(\frac{\theta}{2} - \frac{\pi}{4}\right) + 2\sin\theta}$$

$$= \sqrt{2\left(1 - \cos\left(\theta - \frac{\pi}{2}\right)\right) + 2\sin\theta}$$

$$= \sqrt{2 - 2\sin\theta + 2\sin\theta} = \sqrt{2}$$

$\Rightarrow |\alpha - i| = |\beta - i| = \sqrt{2}$ [here, α, β are two values of $z - i$]

Complex Numbers Exercise 1 :
Single Option Correct Type Questions

- This section contains **30 multiple choice questions.** Each question has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct

1. If $\cos(1-i) = a + ib$, where $a, b \in R$ and $i = \sqrt{-1}$, then

(a) $a = \frac{1}{2}\left(e - \frac{1}{e}\right)\cos 1,\ b = \frac{1}{2}\left(e + \frac{1}{e}\right)\sin 1$

(b) $a = \frac{1}{2}\left(e + \frac{1}{e}\right)\cos 1,\ b = \frac{1}{2}\left(e - \frac{1}{e}\right)\sin 1$

(c) $a = \frac{1}{2}\left(e + \frac{1}{e}\right)\cos 1,\ b = \frac{1}{2}\left(e + \frac{1}{e}\right)\sin 1$

(d) $a = \frac{1}{2}\left(e - \frac{1}{e}\right)\cos 1,\ b = \frac{1}{2}\left(e - \frac{1}{e}\right)\sin 1$

2. Number of roots of the equation $z^{10} - z^5 - 992 = 0$, where real parts are negative, is

(a) 3 (b) 4 (c) 5 (d) 6

3. If z and $\bar{z}$ represent adjacent vertices of a regular polygon of n sides with centre at origin and if $\frac{\text{Im}(z)}{\text{Re}(z)} = \sqrt{2} - 1$, the value of n is equal to

(a) 2 (b) 4 (c) 6 (d) 8

4. If $\prod_{p=1}^{r} e^{ip\theta} = 1$, where $\prod$ denotes the continued product and $i = \sqrt{-1}$, the most general value of θ is

(a) $\frac{2n\pi}{r(r-1)}, n \in I$ (b) $\frac{2n\pi}{r(r+1)}, n \in I$

(c) $\frac{4n\pi}{r(r-1)}, n \in I$ (d) $\frac{4n\pi}{r(r+1)}, n \in I$

(where, n is an integer)

5. If $(3+i)(z+\bar{z}) - (2+i)(z-\bar{z}) + 14i = 0$, where $i = \sqrt{-1}$, then $z\bar{z}$ is equal to

(a) 10 (b) 8
(c) -9 (d) -10

6. The centre of a square $ABCD$ is at $z = 0$, A is z_1. Then, the centroid of ΔABC is

(a) $z_1(\cos\pi \pm i\sin\pi)$ (b) $\frac{z_1}{3}(\cos\pi \pm i\sin\pi)$

(c) $z_1\left(\cos\frac{\pi}{2} \pm i\sin\frac{\pi}{2}\right)$ (d) $\frac{z_1}{3}\left(\cos\frac{\pi}{2} \pm i\sin\frac{\pi}{2}\right)$

(where, $i = \sqrt{-1}$)

7. If $z = \frac{\sqrt{3} - i}{2}$, where $i = \sqrt{-1}$, then $(i^{101} + z^{101})^{103}$ equals to

(a) iz (b) z
(c) $\bar{z}$ (d) None of these

8. Let a and b be two fixed non-zero complex numbers and z is a variable complex number. If the lines $a\bar{z} + \bar{a}z + 1 = 0$ and $b\bar{z} + \bar{b}z - 1 = 0$ are mutually perpendicular, then

(a) $ab + \bar{a}\bar{b} = 0$ (b) $ab - \bar{a}\bar{b} = 0$
(c) $\bar{a}b - a\bar{b} = 0$ (d) $a\bar{b} + \bar{a}b = 0$

9. If $\alpha = \cos\left(\frac{8\pi}{11}\right) + i\sin\left(\frac{8\pi}{11}\right)$, where $i = \sqrt{-1}$, then $\text{Re}(\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5)$ is

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) 0 (d) None of these

10. The set of points in an Argand diagram which satisfy both $|z| \le 4$ and $0 \le \arg(z) \le \frac{\pi}{3}$, is

(a) a circle and a line (b) a radius of a circle
(c) a sector of a circle (d) an infinite part line

11. If $f(x) = g(x^3) + xh(x^3)$ is divisible by $x^2 + x + 1$, then

(a) $g(x)$ is divisible by $(x-1)$ but not $h(x)$
(b) $h(x)$ is divisible by $(x-1)$ but not $g(x)$
(c) both $g(x)$ and $h(x)$ are divisible by $(x-1)$
(d) None of the above

12. If the points represented by complex numbers $z_1 = a + ib, z_2 = c + id$ and $z_1 - z_2$ are collinear, where $i = \sqrt{-1}$, then

(a) $ad + bc = 0$ (b) $ad - bc = 0$
(c) $ab + cd = 0$ (d) $ab - cd = 0$

13. Let C denotes the set of complex numbers and R is the set of real numbers. If the function $f : C \to R$ is defined by $f(z) = |z|$, then

(a) f is injective but not surjective
(b) f is surjective but not injective
(c) f is neither injective nor surjective
(d) f is both injective and surjective

14. Let α and β be two distinct complex numbers, such that $|\alpha| = |\beta|$. If real part of α is positive and imaginary part of β is negative, then the complex number $(\alpha + \beta)/(\alpha - \beta)$ may be

(a) zero (b) real and negative
(c) real and positive (d) purely imaginary

15. The complex number z, satisfies the condition $\left|z - \frac{25}{z}\right| = 24$. The maximum distance from the origin of coordinates to the point z, is

(a) 25 (b) 30
(c) 32 (d) None of these

16. The points A, B and C represent the complex numbers $z_1, z_2, (1-i)z_1 + iz_2$ respectively, on the complex plane (where, $i = \sqrt{-1}$). The ΔABC, is

(a) isosceles but not right angled
(b) right angled but not isosceles
(c) isosceles and right angled
(d) None of the above

17. The system of equations $|z+1-i| = \sqrt{2}$ and $|z| = 3$ has (where, $i = \sqrt{-1}$)

(a) no solution (b) one solution
(c) two solutions (d) None of these

18. Dividing $f(z)$ by $z - i$, we obtain the remainder $1 - i$ and dividing it by $z + i$, we get the remainder $1 + i$. Then, the remainder upon the division of $f(z)$ by $z^2 + 1$, is

(a) $i + z$ (b) $1 + z$
(c) $1 - z$ (d) None of these

19. The centre of the circle represented by $|z+1| = 2|z-1|$ on the complex plane, is

(a) 0 (b) $\frac{5}{3}$
(c) $\frac{1}{3}$ (d) None of these

20. If $x = 9^{1/3} \cdot 9^{1/9} \cdot 9^{1/27} \ldots \infty$, $y = 4^{1/3} \cdot 4^{-1/9} \cdot 4^{1/27} \ldots \infty$ and $z = \sum_{r=1}^{\infty} (1+i)^{-r}$, where $i = \sqrt{-1}$, then $\arg(x + yz)$ is equal to

(a) 0 (b) $-\tan^{-1}\left(\frac{\sqrt{2}}{3}\right)$
(c) $-\tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$ (d) $\pi - \tan^{-1}\left(\frac{\sqrt{2}}{3}\right)$

21. If centre of a regular hexagon is at origin and one of the vertices on Argand diagram is $1 + 2i$, where $i = \sqrt{-1}$, then its perimeter is

(a) $2\sqrt{5}$ (b) $4\sqrt{5}$
(c) $6\sqrt{5}$ (d) $8\sqrt{5}$

22. Let $|z_r - r| \le r, \forall\ r = 1, 2, 3, \ldots, n$, then $\left|\sum_{r=1}^{n} z_r\right|$ is less than

(a) n (b) $2n$
(c) $n(n+1)$ (d) $\frac{n(n+1)}{2}$

23. If $\arg\left(\frac{z_1 - \frac{z}{|z|}}{\frac{z}{|z|}}\right) = \frac{\pi}{2}$ and $\left|\frac{z}{|z|} - z_1\right| = 3$, then $|z_1|$ equals to

(a) $\sqrt{3}$ (b) $2\sqrt{2}$ (c) $\sqrt{10}$ (d) $\sqrt{26}$

24. If $|z - 2 - i| = |z|\left|\sin\left(\frac{\pi}{4} - \arg z\right)\right|$, where $i = \sqrt{-1}$, then locus of z, is

(a) a pair of straight lines (b) circle
(c) parabola (d) ellipse

25. If $1, z_1, z_2, z_3, \ldots, z_{n-1}$ are the n, nth roots of unity, then the value of $\sum_{r=1}^{n-1} \frac{1}{(3 - z_r)}$, is

(a) $\frac{n \cdot 3^{n-1}}{3^n - 1} + \frac{1}{2}$ (b) $\frac{n \cdot 3^{n-1}}{3^n - 1} - 1$
(c) $\frac{n \cdot 3^{n-1}}{3^n - 1} + 1$ (d) None of these

26. If $z = (3 + 7i)(\lambda + i\mu)$, when $\lambda, \mu \in I \sim \{0\}$ and $i = \sqrt{-1}$, is purely imaginary then minimum value of $|z|^2$ is

(a) 0 (b) 58
(c) $\frac{3364}{3}$ (d) 3364

27. Given, $z = f(x) + ig(x)$, where $i = \sqrt{-1}$ and $f, g : (0, 1) \to (0, 1)$ are real-valued functions, which of the following hold good?

(a) $z = \frac{1}{1 - ix} + i\left(\frac{1}{1 + ix}\right)$ (b) $z = \frac{1}{1 + ix} + i\left(\frac{1}{1 - ix}\right)$
(c) $z = \frac{1}{1 + ix} + i\left(\frac{1}{1 + ix}\right)$ (d) $z = \frac{1}{1 - ix} + i\left(\frac{1}{1 - ix}\right)$

28. If $z^3 + (3 + 2i)z + (-1 + ia) = 0$, where $i = \sqrt{-1}$, has one real root, the value of a lies in the interval ($a \in R$)

(a) $(-2, -1)$ (b) $(-1, 0)$
(c) $(0, 1)$ (d) $(1, 2)$

29. If m and n are the smallest positive integers satisfying the relation $\left(2\,CiS\,\frac{\pi}{6}\right)^m = \left(4\,CiS\,\frac{\pi}{4}\right)^n$, where $i = \sqrt{-1}$, $(m + n)$ equals to

(a) 60 (b) 72 (c) 96 (d) 120

30. Number of imaginary complex numbers satisfying the equation, $z^2 = \bar{z} \cdot 2^{1-|z|}$ is

(a) 0 (b) 1 (c) 2 (d) 3

Complex Numbers Exercise 2 :
More than One Option Correct Type Questions

■ This section contains **15 multiple choice questions**. Each question has four choices (a), (b), (c) and (d) out of which **MORE THAN ONE** may be correct.

31. If $\frac{z+1}{z+i}$ is a purely imaginary number (where $i=\sqrt{-1}$), then z lies on a
(a) straight line
(b) circle
(c) circle with radius $=\frac{1}{\sqrt{2}}$
(d) circle passing through the origin

32. If z satisfies $|z-1|<|z+3|$, then $\omega=2z+3-i$ (where, $i=\sqrt{-1}$) satisfies
(a) $|\omega-5-i|<|\omega+3+i|$ (b) $|\omega-5|<|\omega+3|$
(c) $\text{Im}(i\omega)>1$ (d) $|\arg(\omega-1)|<\frac{\pi}{2}$

33. If the complex number is $(1+ri)^3=\lambda(1+i)$, when $i=\sqrt{-1}$, for some real λ, the value of r can be
(a) $\cos\frac{\pi}{5}$ (b) $\text{cosec}\frac{3\pi}{2}$
(c) $\cot\frac{\pi}{12}$ (d) $\tan\frac{\pi}{12}$

34. If $z\in C$, which of the following relation(s) represents a circle on an Argand diagram?
(a) $|z-1|+|z+1|=3$ (b) $|z-3|=2$
(c) $|z-2+i|=\frac{7}{3}$ (d) $(z-3+i)(\bar{z}-3-i)=5$
(where, $i=\sqrt{-1}$)

35. If $1, z_1, z_2, z_3, \ldots, z_{n-1}$ be the n, nth roots of unity and ω be a non-real complex cube root of unity, then $\prod_{r=1}^{n-1}(\omega-z_r)$ can be equal to
(a) $1+\omega$ (b) -1
(c) 0 (d) 1

36. If z is a complex number which simultaneously satisfies the equations
$3|z-12|=5|z-8i|$ and $|z-4|=|z-8|$, where $i=\sqrt{-1}$, then $\text{Im}(z)$ can be
(a) 8 (b) 17
(c) 7 (d) 15

37. If $P(z_1)$, $Q(z_2)$, $R(z_3)$ and $S(z_4)$ are four complex numbers representing the vertices of a rhombus taken in order on the complex plane, which one of the following is hold good?
(a) $\frac{z_1-z_4}{z_2-z_3}$ is purely real
(b) $\frac{z_1-z_3}{z_2-z_4}$ is purely imaginary
(c) $|z_1-z_3|\neq|z_2-z_4|$
(d) $\text{amp}\left(\frac{z_1-z_4}{z_2-z_4}\right)\neq\text{amp}\left(\frac{z_2-z_4}{z_3-z_4}\right)$

38. If $|z-3|=\min\{|z-1|,|z-5|\}$, then $\text{Re}(z)$ is equal to
(a) 2 (b) 2.5 (c) 3.5 (d) 4

39. If $\arg(z+a)=\frac{\pi}{6}$ and $\arg(z-a)=\frac{2\pi}{3}$ $(a\in R^+)$, then
(a) $|z|=a$ (b) $|z|=2a$
(c) $\arg(z)=\frac{\pi}{3}$ (d) $\arg(z)=\frac{\pi}{2}$

40. If $z=x+iy$, where $i=\sqrt{-1}$, then the equation $\left|\frac{(2z-i)}{(z+i)}\right|=m$ represents a circle, then m can be
(a) $\frac{1}{2}$ (b) 1 (c) 2 (d) $\in(3, 2\sqrt{3})$

41. Equation of tangent drawn to circle $|z|=r$ at the point $A(z_0)$, is
(a) $\text{Re}\left(\frac{z}{z_0}\right)=1$ (b) $\text{Im}\left(\frac{z}{z_0}\right)=1$
(c) $\text{Im}\left(\frac{z_0}{z}\right)=1$ (d) $z\bar{z}_0+z_0\bar{z}=2r^2$

42. z_1 and z_2 are the roots of the equation $z^2-az+b=0$, where $|z_1|=|z_2|=1$ and a, b are non-zero complex numbers, then
(a) $|a|\leq 1$ (b) $|a|\leq 2$
(c) $\arg(a)=\arg(b^2)$ (d) $\arg(a^2)=\arg(b)$

43. If α is a complex constant, such that $\alpha z^2+z+\bar{\alpha}=0$ has a real root, then
(a) $\alpha+\bar{\alpha}=1$
(b) $\alpha+\bar{\alpha}=0$
(c) $\alpha+\bar{\alpha}=-1$
(d) the absolute value of real root is 1

44. If the equation $z^3+(3+i)z^2-3z-(m+i)=0$, where $i=\sqrt{-1}$ and $m\in R$, has atleast one real root, value of m is
(a) 1 (b) 2 (c) 3 (d) 5

45. If $z^3+(3+2i)z+(-1+ia)=0$, where $i=\sqrt{-1}$, has one real root, the value of a lies in the interval $(a\in R)$
(a) $(-2, 1)$ (b) $(-1, 0)$ (c) $(0, 1)$ (d) $(-2, 3)$

Complex Numbers Exercise 3 : Passage Based Questions

- This section contains **4 passages**. Based upon each of the passage **3 multiple choice questions** have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I

(Q. Nos. 46 to 48)

$\arg(\bar{z}) + \arg(-z) = \begin{cases} \pi, & \text{if } \arg(z) < 0 \\ -\pi, & \text{if } \arg(z) > 0 \end{cases}$, where $-\pi < \arg(z) \le \pi$.

46. If $\arg(z) > 0$, then $\arg(-z) - \arg(z)$ is equal to

(a) $-\pi$ (b) $-\frac{\pi}{2}$

(c) $\frac{\pi}{2}$ (d) π

47. Let z_1 and z_2 be two non-zero complex numbers, such that $|z_1| = |z_2|$ and $\arg(z_1 z_2) = \pi$, then z_1 is equal to

(a) z_2 (b) $\bar{z}_2$

(c) $-z_2$ (d) $-\bar{z}_2$

48. If $\arg(4z_1) - \arg(5z_2) = \pi$, then $\left|\frac{z_1}{z_2}\right|$ is equal to

(a) 1 (b) 1.25

(c) 1.50 (d) 2.50

Passage II

(Q. Nos. 49 to 51)

Sum of four consecutive powers of i (iota) is zero.

i.e., $i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0, \forall\, n \in I$.

49. If $\sum_{n=1}^{25} i^{n!} = a + ib$, where $i = \sqrt{-1}$, then $a - b$, is

(a) prime number

(b) even number

(c) composite number

(d) perfect number

50. If $\sum_{r=-2}^{95} i^r + \sum_{r=0}^{50} i^{r!} = a + ib$, where $i = \sqrt{-1}$, the unit place digit of $a^{2011} + b^{2012}$, is

(a) 2 (b) 3

(c) 5 (d) 6

51. If $\sum_{r=4}^{100} i^{r!} + \prod_{r=1}^{101} i^r = a + ib$, where $i = \sqrt{-1}$, then $a + 75b$, is

(a) 11 (b) 22

(c) 33 (d) 44

Passage III

(Q. Nos. 52 to 54)

For any two complex numbers z_1 and z_2,

$|z_1 - z_2| \ge \begin{cases} |z_1| - |z_2| \\ |z_2| - |z_1| \end{cases}$

and equality holds iff origin z_1 and z_2 are collinear and z_1, z_2 lie on the same side of the origin.

52. If $\left|z - \frac{1}{z}\right| = 2$ and sum of greatest and least values of $|z|$ is λ, then λ^2, is

(a) 2 (b) 4 (c) 6 (d) 8

53. If $\left|z + \frac{2}{z}\right| = 4$ and sum of greatest and least values of $|z|$ is λ, then λ^2, is

(a) 12 (b) 18 (c) 24 (d) 30

54. If $\left|z - \frac{3}{z}\right| = 6$ and sum of greatest and least values of $|z|$ is 2λ, then λ^2, is

(a) 12 (b) 18 (c) 24 (d) 30

Passage IV

(Q. Nos. 55 to 57)

Consider the two complex numbers z and w, such that $w = \frac{z-1}{z+2} = a + ib$, where $a, b \in R$ and $i = \sqrt{-1}$.

55. If $z = C\, iS\, \theta$, which of the following does hold good?

(a) $\sin\theta = \frac{9b}{1-4a}$

(b) $\cos\theta = \frac{1-5a}{1+4a}$

(c) $(1+5a)^2 + (3b)^2 = (1-4a)^2$

(d) All of these

56. Which of the following is the value of $-\frac{b}{a}$, whenever it exists?

(a) $3\tan\left(\frac{\theta}{2}\right)$ (b) $\frac{1}{3}\tan\left(\frac{\theta}{2}\right)$

(c) $-\frac{1}{3}\cot\theta$ (d) $3\cot\frac{\theta}{2}$

57. Which of the following equals to $|z|$?

(a) $|w|$ (b) $(a+1)^2 + b^2$

(c) $a^2 + (b+2)^2$ (d) $(a+1)^2 + (b+1)^2$

Complex Numbers Exercise 4 : Single Integer Answer Type Questions

■ This section contains **10 questions**. The answer to each question is a **single digit integer**, ranging from 0 to 9 (both inclusive).

58. The number of values of z (real or complex) simultaneously satisfying the system of equations

$$1+z+z^2+z^3+\ldots+z^{17}=0$$

and $1+z+z^2+z^3+\ldots+z^{13}=0$ is

59. Number of complex numbers z satisfying $z^3=\bar{z}$ is

60. Let $z=9+ai$, where $i=\sqrt{-1}$ and a be non-zero real.

If $\text{Im}(z^2)=\text{Im}(z^3)$, sum of the digits of a^2 is

61. Number of complex numbers z, such that $|z|=1$ and $\left|\frac{z}{\bar{z}}+\frac{\bar{z}}{z}\right|=1$ is

62. If $x=a+ib$, where $a, b\in R$ and $i=\sqrt{-1}$ and $x^2=3+4i$, $x^3=2+11i$, the value of $(a+b)$ is

63. If $z=\frac{\pi}{4}(1+i)^4\left(\frac{1-\sqrt{\pi}\,i}{\sqrt{\pi}+i}+\frac{\sqrt{\pi}-i}{1+\sqrt{\pi}\,i}\right)$, where $i=\sqrt{-1}$, then $\left(\frac{|z|}{\text{amp}(z)}\right)$ equals to

64. Suppose A is a complex number and $n\in N$, such that $A^n=(A+1)^n=1$, then the least value of n is

65. Let $z_r; r=1,2,3,\ldots,50$ be the roots of the equation $\sum_{r=0}^{50}(z)^r=0$. If $\sum_{r=1}^{50}\frac{1}{(z_r-1)}=-5\lambda$, then λ equals to

66. If $P=\sum_{p=1}^{32}(3p+2)\left(\sum_{q=1}^{10}\left(\sin\frac{2q\pi}{11}-i\cos\frac{2q\pi}{11}\right)\right)^p$, where $i=\sqrt{-1}$ and if $(1+i)P=n(n!), n\in N$, then the value of n is

67. The least positive integer n for which $\left(\frac{1+i}{1-i}\right)^n=\frac{2}{\pi}\sin^{-1}\left(\frac{1+x^2}{2x}\right)$, where $x>0$ and $i=\sqrt{-1}$ is

Complex Numbers Exercise 5 : Matching Type Questions

■ This section contains **4 questions**. Questions 68 and 69 have three statements (A, B and C) given in **Column I** and four statements (p, q, r and s) in **Column II** and questions 70 and 71 have four statements (A, B, C and D) given in **Column I** and five statements (p, q, r, s and t) in **Column II**. Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II**.

68.

	Column I		Column II
(A)	If $\left\|z-\frac{1}{z}\right\|=2$ and if greatest and least values of $\|z\|$ are G and L respectively, then $G-L$, is	(p)	natural number
(B)	If $\left\|z+\frac{2}{z}\right\|=4$ and if greatest and least values of $\|z\|$ are G and L respectively, then $G-L$, is	(q)	prime number
(C)	If $\left\|z-\frac{3}{z}\right\|=6$ and if greatest and least values of $\|z\|$ are G and L respectively, then $G-L$, is	(r)	composite number
		(s)	perfect number

69.

	Column I		Column II
(A)	If $\sqrt{(6+8i)}+\sqrt{(-6+8i)}=z_1, z_2, z_3, z_4$ (where $i=\sqrt{-1}$), then $\|z_1\|^2+\|z_2\|^2\|+\|z_3\|^2+\|z_4\|^2$ is divisible by	(p)	7
(B)	If $\sqrt{(5-12i)}+\sqrt{(-5-12i)}=z_1, z_2, z_3, z_4$ (where $i=\sqrt{-1}$), then $\|z_1\|^2+\|z_2\|^2+\|z_3\|^2+\|z_4\|^2$ is divisible by	(q)	8
(C)	If $\sqrt{(8+15i)}+\sqrt{(-8-15i)}=z_1, z_2, z_3, z_4$(where $i=\sqrt{-1}$), then $\|z_1\|^2+\|z_2\|^2+\|z_3\|^2+\|z_4\|^2$ is divisible by	(r)	13
		(s)	17

70.

	Column I		Column II
(A)	If λ and μ are the unit's place digits of $(143)^{861}$ and $(5273)^{1358}$ respectively, then λ + μ is divisible by	(p)	2
(B)	If λ and μ are the unit's place digits of $(212)^{7820}$ and $(1322)^{1594}$ respectively, then λ + μ is divisible by	(q)	3
(C)	If λ and μ are the unit's place digits of $(136)^{786}$ and $(7138)^{13491}$ respectively, then λ + μ is divisible by	(r)	4
		(s)	5
		(t)	6

71.

	Column I		Column II
(A)	If $\left\|z-\frac{6}{z}\right\|=5$ and maximum and minimum values of $\|z\|$ are λ and μ respectively, then	(p)	$\lambda^{\mu}+\mu^{\lambda}=8$
(B)	If $\left\|z-\frac{7}{z}\right\|=6$ and maximum and minimum values of $\|z\|$ are λ and μ respectively, then	(q)	$\lambda^{\mu}-\mu^{\lambda}=7$
(C)	If $\left\|z-\frac{8}{z}\right\|=7$ and maximum and minimum values of $\|z\|$ are λ and μ respectively, then	(r)	$\lambda^{\mu}+\mu^{\lambda}=7$
		(s)	$\lambda^{\mu}-\mu^{\lambda}=6$
		(t)	$\lambda^{\mu}+\mu^{\lambda}=9$

Complex Numbers Exercise 6 : Statement I and II Type Questions

■ **Directions** (Q. Nos. 72 to 78) are Assertion-Reason type questions. Each of these questions contains two statements:
Statement-1 (Assertion) and **Statement-2** (Reason)
Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

72. **Statement-1** $3+7i>2+4i$, where $i=\sqrt{-1}$.

Statement-2 $3>2$ and $7>4$

73. **Statement-1** $(\cos\theta+i\sin\phi)^3=\cos 3\theta+i\sin 3\phi$, $i=\sqrt{-1}$

Statement-2 $\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^2=i$

74. **Statement-1** Let z_1, z_2 and z_3 be three complex numbers, such that $|3z_1+1|=|3z_2+1|=|3z_3+1|$ and $1+z_1+z_2+z_3=0$, then z_1, z_2, z_3 will represent vertices of an equilateral triangle on the complex plane.

Statement-2 z_1, z_2 and z_3 represent vertices of an equilateral triangle, if $z_1^2+z_2^2+z_3^2+z_1z_2+z_2z_3+z_3z_1=0$.

75. **Statement-1** Locus of z satisfying the equation $|z-1|+|z-8|=5$ is an ellipse.

Statement-2 Sum of focal distances of any point on ellipse is constant for an ellipse.

76. Let z_1, z_2 and z_3 be three complex numbers in AP.

Statement-1 Points representing z_1, z_2 and z_3 are collinear.

Statement-2 Three numbers a, b and c are in AP, if $b-a=c-b$.

77. **Statement-1** If the principal argument of a complex number z is θ, the principal argument of z^2 is 2θ.

Statement-2 $\arg(z^2)=2\arg(z)$

78. Consider the curves on the Argand plane as

$$C_1:\arg(z)=\frac{\pi}{4},$$

$$C_2:\arg(z)=\frac{3\pi}{4}$$

and $C_3:\arg(z-5-5i)=\pi$, where $i=\sqrt{-1}$.

Statement-1 Area of the region bounded by the curves C_1, C_2 and C_3 is $\frac{25}{2}$.

Statement–2 The boundaries of C_1, C_2 and C_3 constitute a right isosceles triangle.

Complex Numbers Exercise 7 :
Subjective Type Questions

- In this section, there are **24 subjective** questions.

79. If z_1, z_2 and z_3 are three complex numbers, then prove that $z_1 \operatorname{Im}(\bar{z}_2 z_3) + z_2 \operatorname{Im}(\bar{z}_3 z_1) + z_3 \operatorname{Im}(\bar{z}_1 z_2) = 0$.

80. The roots z_1, z_2 and z_3 of the equation $x^3 + 3ax^2 + 3bx + c = 0$ in which a, b and c are complex numbers, correspond to the points A, B, C on the Gaussian plane. Find the centroid of the ΔABC and show that it will be equilateral, if $a^2 = b$.

81. If $1, \alpha_1, \alpha_2, \alpha_3$ and α_4 are the roots of $x^5 - 1 = 0$, then prove that
$\frac{\omega - \alpha_1}{\omega^2 - \alpha_1} \cdot \frac{\omega - \alpha_2}{\omega^2 - \alpha_2} \cdot \frac{\omega - \alpha_3}{\omega^2 - \alpha_3} \cdot \frac{\omega - \alpha_4}{\omega^2 - \alpha_4} = \omega$, where ω is a non-real complex root of unity.

82. If z_1 and z_2 both satisfy the relation $z + \bar{z} = 2|z - 1|$ and $\arg(z_1 - z_2) = \frac{\pi}{4}$, find the imaginary part of $(z_1 + z_2)$.

83. If $ax + cy + bz = X$, $cx + by + az = Y$, $bx + ay + cz = Z$, show that
(i) $(a^2 + b^2 + c^2 - bc - ca - ab)(x^2 + y^2 + z^2 - yz - zx - xy) = X^2 + Y^2 + Z^2 - YZ - ZX - XY$
(ii) $(a^3 + b^3 + c^3 - 3abc)(x^3 + y^3 + z^3 - 3xyz) = X^3 + Y^3 + Z^3 - 3XYZ$.

84. For every real number $c \geq 0$, find all complex numbers z which satisfy the equation $|z|^2 - 2iz + 2c(1 + i) = 0$, where $i = \sqrt{-1}$.

85. Find the equations of two lines making an angle of 45° with the line $(2 - i)z + (2 + i)\bar{z} + 3 = 0$, where $i = \sqrt{-1}$ and passing through $(-1, 4)$.

86. For $n \geq 2$, show that
$\left[1 + \left(\frac{1+i}{2}\right)\right]\left[1 + \left(\frac{1+i}{2}\right)^2\right]\left[1 + \left(\frac{1+i}{2}\right)^{2^2}\right] \ldots \left[1 + \left(\frac{1+i}{2}\right)^{2^n}\right] = (1 + i)\left(1 - \frac{1}{2^{2^n}}\right)$, where $i = \sqrt{-1}$.

87. Find the point of intersection of the curves $\arg(z - 3i) = 3\pi/4$ and $\arg(2z + 1 - 2i) = \frac{\pi}{4}$, where $i = \sqrt{-1}$.

88. Show that if a and b are real, then the principal value of $\arg(a)$ is 0 or π, according as a is positive or negative and that of b is $\frac{\pi}{2}$ or $-\frac{\pi}{2}$, according as b is positive or negative.

89. Two different non-parallel lines meet the circle $|z| = r$. One of them at points a and b and the other which is tangent to the circle at c. Show that the point of intersection of two lines is $\frac{2c^{-1} - a^{-1} - b^{-1}}{c^{-2} - a^{-1}b^{-1}}$.

90. A, B and C are the points representing the complex numbers z_1, z_2 and z_3 respectively, on the complex plane and the circumcentre of ΔABC lies at the origin. If the altitude of the triangle through the vertex A meets the circumcircle again at P, prove that P represents the complex number $\left(-\frac{z_2 z_3}{z_1}\right)$.

91. If $|z| \leq 1$ and $|\omega| \leq 1$, show that $|z - \omega|^2 \leq (|z| - |\omega|)^2 + \{\arg(z) - \arg(\omega)\}^2$.

92. Let z, z_0 be two complex numbers. It is given that $|z| = 1$ and the numbers $z, z_0, z\bar{z}_0, 1$ and 0 are represented in an Argand diagram by the points P, P_0, Q, A and the origin, respectively. Show that ΔPOP_0 and ΔAOQ are congruent. Hence, or otherwise, prove that $|z - z_0| = |z\bar{z}_0 - 1|$.

93. Suppose the points $z_1, z_2, \ldots, z_n$ $(z_i \neq 0)$ all lie on one side of a line drawn through the origin of the complex planes. Prove that the same is true of the points $\frac{1}{z_1}, \frac{1}{z_2}, \ldots, \frac{1}{z_n}$. Moreover, show that
$z_1 + z_2 + \ldots + z_n \neq 0$ and $\frac{1}{z_1} + \frac{1}{z_2} + \ldots + \frac{1}{z_n} \neq 0$.

94. If a, b and c are complex numbers and z satisfies $az^2 + bz + c = 0$, prove that $|a||b| = \sqrt{a(\bar{b})^2 c}$ and $|a| = |c| \Leftrightarrow |z| = 1$.

95. Let z_1, z_2 and z_3 be three non-zero complex numbers and $z_1 \neq z_2$. If $\begin{vmatrix} |z_1| & |z_2| & |z_3| \\ |z_2| & |z_3| & |z_1| \\ |z_3| & |z_1| & |z_2| \end{vmatrix} = 0$, prove that
(i) z_1, z_2, z_3 lie on a circle with the centre at origin.
(ii) $\arg\left(\frac{z_3}{z_2}\right) = \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)^2$.

96. Prove that, if z_1 and z_2 are two complex numbers and $c > 0$, then $|z_1 + z_2|^2 \le (1+c)|z_1|^2 + \left(1+\frac{1}{c}\right)|z_2|^2$.

97. Find the circumcentre of the triangle whose vertices are given by the complex numbers z_1, z_2 and z_3.

98. Find the orthocentre of the triangle whose vertices are given by the complex numbers z_1, z_2 and z_3.

99. Prove that the roots of the equation $8x^3 - 4x^2 - 4x + 1 = 0$ are $\cos\frac{\pi}{7}, \cos\frac{3\pi}{7}$ and $\cos\frac{5\pi}{7}$.

Hence, obtain the equations whose roots are

(i) $\sec^2\frac{\pi}{7}, \sec^2\frac{3\pi}{7}, \sec^2\frac{5\pi}{7}$

(ii) $\tan^2\frac{\pi}{7}, \tan^2\frac{3\pi}{7}, \tan^2\frac{5\pi}{7}$

(iii) Evaluate $\sec\frac{\pi}{7} + \sec\frac{3\pi}{7} + \sec\frac{5\pi}{7}$

100. Solve the equation $z^7 + 1 = 0$ and deduce that

(i) $\cos\frac{\pi}{7}\cos\frac{3\pi}{7}\cos\frac{5\pi}{7} = -\frac{1}{8}$

(ii) $\cos\frac{\pi}{14}\cos\frac{3\pi}{14}\cos\frac{5\pi}{14} = \frac{\sqrt{7}}{8}$

(iii) $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14} = \frac{1}{8}$

(iv) $\tan\frac{\pi}{14}\tan\frac{3\pi}{14}\tan\frac{5\pi}{14} = \frac{1}{\sqrt{7}}$

Also, show that

$$(1+y)^7 + (1-y)^7 = 14\left(y^2 + \tan^2\frac{\pi}{14}\right)\left(y^2 + \tan^2\frac{3\pi}{14}\right)\left(y^2 + \tan^2\frac{5\pi}{14}\right)$$

and then deduce that

$$\tan^2\left(\frac{\pi}{14}\right) + \tan^2\left(\frac{3\pi}{14}\right) + \tan^2\left(\frac{5\pi}{14}\right) = 5$$

101. If the complex number z is to satisfy $|z| = 3, |z - \{a(1+i) - i\}| \le 3$ and $|z + 2a - (a+1)i| > 3$, where $i = \sqrt{-1}$ simultaneously for atleast one z, then find all $a \in R$.

102. Write equations whose roots are equal to numbers

(i) $\sin^2\frac{\pi}{2n+1}, \sin^2\frac{2\pi}{2n+1}, \sin^2\frac{3\pi}{2n+1}, \ldots, \sin^2\frac{n\pi}{2n+1}$.

(ii) $\cot^2\frac{\pi}{2n+1}, \cot^2\frac{2\pi}{2n+1}, \cot^2\frac{3\pi}{2n+1}, \ldots, \cot^2\frac{n\pi}{2n+1}$.

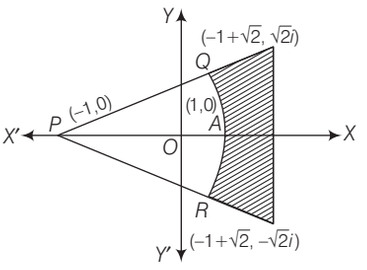
Complex Numbers Exercise 8 : Questions Asked in Previous 13 Years' Exams

- This section contains questions asked in **IIT-JEE, AIEEE, JEE Main & JEE Advanced** from year **2005** to year **2017**.

103. If ω is a cube root of unity but not equal to 1, then minimum value of $|a + b\omega + c\omega^2|$, (where a, b and c are integers but not all equal), is **[IIT-JEE 2005, 3M]**

(a) 0 (b) $\frac{\sqrt{3}}{2}$ (c) 1 (d) 2

104. PQ and PR are two infinite rays. QAR is an arc. Point lying in the shaded region excluding the boundary satisfies **[IIT-JEE 2005, 3M]**



(a) $|z-1| > 2; |\arg(z-1)| < \frac{\pi}{4}$

(b) $|z-1| > 2; |\arg(z-1)| < \frac{\pi}{2}$

(c) $|z+1| > 2; |\arg(z+1)| < \frac{\pi}{4}$

(d) $|z+1| > 2; |\arg(z+1)| < \frac{\pi}{3}$

105. If one of the vertices of the square circumscribing the circle $|z-1| = \sqrt{2}$ is $2 + \sqrt{3}i$, where $i = \sqrt{-1}$. Find the other vertices of the square. **[IIT-JEE 2005, 4M]**

106. If z_1 and z_2 are two non-zero complex numbers, such that $|z_1 + z_2| = |z_1| + |z_2|$, then $\arg(z_1) - \arg(z_2)$ is equal to **[AIEEE 2005, 3M]**

(a) $-\pi$ (b) $-\pi/2$

(c) $\pi/2$ (d) 0

107. If $1, \omega, \omega^2$ are the cube roots of unity, then the roots of the equation $(x-1)^3 + 8 = 0$ are **[AIEEE 2005, 3M]**

(a) $-1, 1+2\omega, 1+2\omega^2$ (b) $-1, 1-2\omega, 1-2\omega^2$

(c) $-1, -1, -1$ (d) None of these

108. If $\omega = \dfrac{z}{z - \dfrac{1}{3}i}$ and $|\omega| = 1$, where $i = \sqrt{-1}$, then z lies on **[AIEEE 2005, 3M]**

(a) a straight line (b) a parabola
(c) an ellipse (d) a circle

109. If $\omega = \alpha + i\beta$, where $\beta \neq 0$, $i = \sqrt{-1}$ and $z \neq 1$, satisfies the condition that $\left(\dfrac{\omega - \bar{\omega}z}{1 - z}\right)$ is purely real, the set of values of z is **[IIT-JEE 2006, 3M]**

(a) $\{z : |z| = 1\}$ (b) $\{z : z = \bar{z}\}$
(c) $\{z : z \neq 1\}$ (d) $\{z : |z| = 1, z \neq 1\}$

110. The value of $\sum_{k=1}^{10}\left(\sin\dfrac{2k\pi}{11} + i\cos\dfrac{2k\pi}{11}\right)$ (where $i = \sqrt{-1}$) is **[AIEEE 2006, 3M]**

(a) i (b) 1
(c) -1 (d) $-i$

111. If $z^2 + z + 1 = 0$, where z is a complex number, the value of $\left(z + \dfrac{1}{z}\right)^2 + \left(z^2 + \dfrac{1}{z^2}\right)^2 + \left(z^3 + \dfrac{1}{z^3}\right)^2 + \ldots + \left(z^6 + \dfrac{1}{z^6}\right)^2$ is **[AIEEE 2006, 6M]**

(a) 18 (b) 54
(c) 6 (d) 12

112. A man walks a distance of 3 units from the origin towards the North-East (N 45° E) direction. From there, he walks a distance of 4 units towards the North-West (N 45° W) direction to reach a point P. Then, the position of P in the Argand plane, is **[IIT-JEE 2007, 3M]**

(a) $3e^{i\pi/4} + 4i$ (b) $(3 - 4i)\, e^{i\pi/4}$
(c) $(4 + 3i)\, e^{i\pi/4}$ (d) $(3 + 4i)\, e^{i\pi/4}$
(where $i = \sqrt{-1}$)

113. If $|z| = 1$ and $z \neq \pm 1$, then all the values of $\dfrac{z}{1 - z^2}$ lie on **[IIT-JEE 2007, 3M]**

(a) a line not passing through the origin
(b) $|z| = \sqrt{2}$
(c) the X-axis
(d) the Y-axis

114. If $|z + 4| \leq 3$, the maximum value of $|z + 1|$ is **[AIEEE 2007, 3M]**

(a) 4 (b) 10
(c) 6 (d) 0

Passage (Q. Nos. 115 to 117)

Let A, B and C be three sets of complex numbers as defined below:

$A = \{z : \text{Im}(z) \geq 1\}$

$B = \{z : |z - 2 - i| = 3\}$

$C = \{z : \text{Re}((1 - i)\, z) = \sqrt{2}\}$, where $i = \sqrt{-1}$ **[IIT-JEE 2008, 4+4+4M]**

115. The number of elements in the set $A \cap B \cap C$, is

(a) 0 (b) 1
(c) 2 (d) ∞

116. Let z be any point in $A \cap B \cap C$. Then, $|z + 1 - i|^2 + |z - 5 - i|^2$ lies between

(a) 25 and 29 (b) 30 and 34
(c) 35 and 39 (d) 40 and 44

117. Let z be any point in $A \cap B \cap C$ and ω be any point satisfying $|\omega - 2 - i| < 3$. Then, $|z| - |\omega| + 3$ lies between

(a) -6 and 3 (b) -3 and 6
(c) -6 and 6 (d) -3 and 9

118. A particle P starts from the point $z_0 = 1 + 2i$, $i = \sqrt{-1}$. It moves first horizontally away from origin by 5 units and then vertically away from origin by 3 units to reach a point z_1. From z_1, the particle moves $\sqrt{2}$ units in the direction of the vector $\hat{\mathbf{i}} + \hat{\mathbf{j}}$ and then it moves through an angle $\dfrac{\pi}{2}$ in anti-clockwise direction on a circle with centre at origin, to reach a point z_2, then the point z_2 is given by **[IIT-JEE 2008, 3M]**

(a) $6 + 7i$ (b) $-7 + 6i$
(c) $7 + 6i$ (d) $-6 + 7i$

119. If the conjugate of a complex numbers is $\dfrac{1}{i - 1}$, where $i = \sqrt{-1}$. Then, the complex number is **[AIEEE 2008, 3M]**

(a) $\dfrac{-1}{i - 1}$ (b) $\dfrac{1}{i + 1}$
(c) $\dfrac{-1}{i + 1}$ (d) $\dfrac{1}{i - 1}$

120. Let $z = x + iy$ be a complex number, where x and y are integers and $i = \sqrt{-1}$. Then, the area of the rectangle whose vertices are the roots of the equation $z\bar{z}^3 + \bar{z}z^3 = 350$, is **[IIT-JEE 2009, 3M]**

(a) 48 (b) 32
(c) 40 (d) 80

121. Let $z = \cos\theta + i\sin\theta$, where $i = \sqrt{-1}$. Then the value of $\sum_{m=1}^{15} \text{Im}(z^{2m-1})$ at $\theta = 2°$ is **[IIT-JEE 2009, 3M]**

(a) $\dfrac{1}{\sin 2°}$ (b) $\dfrac{1}{3\sin 2°}$
(c) $\dfrac{1}{2\sin 2°}$ (d) $\dfrac{1}{4\sin 2°}$

122. If $\left|z - \dfrac{4}{z}\right| = 2$, the maximum value of $|z|$ is equal to **[AIEEE 2009, 4M]**

(a) $2 + \sqrt{2}$ (b) $\sqrt{3} + 1$
(c) $\sqrt{5} + 1$ (d) 2

123. Let z_1 and z_2 be two distinct complex numbers and $z = (1-t)z_1 + iz_2$, for some real number t with $0 < t < 1$ and $i = \sqrt{-1}$. If $\arg(w)$ denotes the principal argument of a non-zero complex number w, then **[IIT-JEE 2010, 3M]**

(a) $|z - z_1| + |z - z_2| = |z_1 - z_2|$

(b) $\arg(z - z_1) = \arg(z - z_2)$

(c) $\begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$

(d) $\arg(z - z_1) = \arg(z_2 - z_1)$

124. Let ω be the complex number $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$, where $i = \sqrt{-1}$, then the number of distinct complex numbers z satisfying $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$, is equal to **[IIT-JEE 2010, 3M]**

(a) 0 (b) 1

(c) 2 (d) 3

125. Match the statements in **Column I** with those in **Column II**.

[**Note** Here, z takes values in the complex plane and Im (z) and Re (z) denote respectively, the imaginary part and the real part of z.] **[IIT- JEE 2010, 8M]**

	Column I		**Column II**
(A)	The set of points z satisfying $\|z - i\|z\|\| = \|z + i\|z\|\|$, where $i = \sqrt{-1}$, is contained in or equal to	(p)	an ellipse with eccentricity 4/5
(B)	The set of points z satisfying $\|z+4\| + \|z-4\| = 10$ is contained in or equal to	(q)	the set of points z satisfying Im $(z) = 0$
(C)	If $\|w\| = 2$, the set of points $z = w - \frac{1}{w}$ is contained in or equal to	(r)	the set of points z satisfying $\|\text{Im}(z)\| \le 1$
(D)	If $\|w\| = 1$, the set of points $z = w + \frac{1}{w}$ is contained in or equal to	(s)	the set of points satisfying $\|\text{Re}(z)\| \le 2$
		(t)	the set of points z satisfying $\|z\| \le 3$

126. If α and β are the roots of the equation $x^2 - x + 1 = 0$, $\alpha^{2009} + \beta^{2009}$ is equal to **[AIEEE 2010, 4M]**

(a) – 1 (b) 1

(c) 2 (d) – 2

127. The number of complex numbers z, such that $|z-1| = |z+1| = |z-i|$, where $i = \sqrt{-1}$, equals to **[AIEEE 2010, 4M]**

(a) 1 (b) 2

(c) ∞ (d) 0

128. If z is any complex number satisfying $|z - 3 - 2i| \le 2$, where $i = \sqrt{-1}$, then the minimum value of $|2z - 6 + 5i|$, is **[IIT-JEE 2011, 4M]**

129. The set $\left\{\text{Re}\left(\frac{2iz}{1-z^2}\right) : z \text{ is a complex number } |z| = 1, z \ne \pm 1\right\}$ is **[IIT-JEE 2011, 2M]**

(a) $(-\infty, -1] \cap [1, \infty)$ (b) $(-\infty, 0) \cup (0, \infty)$

(c) $(-\infty, -1) \cup (1, \infty)$ (d) $[2, \infty)$

130. The maximum value of $\left|\arg\left(\frac{1}{1-z}\right)\right|$ for $|z| = 1, z \ne 1$, is given by **[IIT-JEE 2011, 2M]**

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\frac{2\pi}{3}$

131. Let $w = e^{i\pi/3}$, where $i = \sqrt{-1}$ and a, b, c, x, y and z be non-zero complex numbers such that

$$a + b + c = x$$
$$a + bw + cw^2 = y$$
$$a + bw^2 + cw = z.$$

The value of $\frac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2}$, is **[IIT-JEE 2011, 4M]**

132. Let α and β be real and z be a complex number. If $z^2 + \alpha z + \beta = 0$ has two distinct roots on the line Re $(z) = 1$, then it is necessary that **[AIEEE 2011, 4M]**

(a) $\beta \in (-1, 0)$ (b) $|\beta| = 1$

(c) $\beta \in (1, \infty)$ (d) $\beta \in (0, 1)$

133. If $\omega (\ne 1)$ is a cube root of unity and $(1+\omega)^7 = A + B\omega$, then (A, B) equals to **[AIEEE 2011, 4M]**

(a) (1, 1) (b) (1, 0)

(c) (–1, 1) (d) (0, 1)

134. Let z be a complex number such that the imaginary part of z is non-zero and $a = z^2 + z + 1$ is real. Then, a **cannot** take the value **[IIT-JEE 2012, 3M]**

(a) –1 (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{3}{4}$

135. If $z \ne 1$ and $\frac{z^2}{z-1}$ is real, the point represented by the complex number z lies **[AIEEE 2012, 4M]**

(a) on a circle with centre at the origin

(b) either on the real axis or on a circle not passing through the origin

(c) on the imaginary axis

(d) either on the real axis or on a circle passing through the origin

136. If z is a complex number of unit modulus and argument θ, then $\arg\left(\frac{1+z}{1+\bar{z}}\right)$ equals to **[JEE Main 2013, 4M]**

(a) $\frac{\pi}{2}-\theta$ (b) θ

(c) $\pi-\theta$ (d) $-\theta$

137. Let complex numbers α and $\frac{1}{\bar{\alpha}}$ lie on circles $(x-x_0)^2+(y-y_0)^2=r^2$ and $(x-x_0)^2+(y-y_0)^2=4r^2$, respectively. If $z_0=x_0+iy_0$ satisfies the equation $2|z_0|^2=r^2+2$, then $|\alpha|$ equals to **[JEE Advanced 2013, 2M]**

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$ (c) $\frac{1}{\sqrt{7}}$ (d) $\frac{1}{3}$

138. Let $w=\frac{\sqrt{3}+i}{2}$ and $P=\{w^n : n=1,2,3,\ldots\}$. Further, $H_1=\left\{z\in C:\text{Re}(z)>\frac{1}{2}\right\}$ and $H_2=\left\{z\in C:\text{Re}(z)<\left(-\frac{1}{2}\right)\right\}$, where C is the set of all complex numbers. If $z_1\in P\cap H_1$, $z_2\in P\cap H_2$ and O represents the origin, then $\angle z_1Oz_2$ equals to **[JEE Advanced 2013, 3M]**

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{6}$

(c) $\frac{2\pi}{3}$ (d) $\frac{5\pi}{6}$

Passage (Q. Nos. 139 to 140)

Let $S=S_1\cap S_2\cap S_3$, where $S_1=\{z\in C:|z|<4\}$, $S_2=\left\{z\in C:\text{Im}\left[\frac{z-1+\sqrt{3}i}{1-\sqrt{3}i}\right]>0\right\}$ and $S_3=\{z\in C:\text{Re}\ z>0\}$. **[JEE Advanced 2013, 3+3M]**

139. $\min_{z\in S}|1-3i-z|$ equals to

(a) $\frac{2-\sqrt{3}}{2}$ (b) $\frac{2+\sqrt{3}}{2}$

(c) $\frac{3-\sqrt{3}}{2}$ (d) $\frac{3+\sqrt{3}}{2}$

140. Area of S equals to

(a) $\frac{10\pi}{3}$ (b) $\frac{20\pi}{3}$

(c) $\frac{16\pi}{3}$ (d) $\frac{32\pi}{3}$

141. If z is a complex number such that $|z|\geq 2$, then the minimum value of $\left|z+\left(\frac{1}{2}\right)\right|$, is **[JEE Main 2014, 4M]**

(a) is strictly greater than $\frac{5}{2}$

(b) is equal to $\frac{5}{2}$

(c) is strictly greater than $\frac{3}{2}$ but less than $\frac{5}{2}$

(d) lies in the interval (1, 2)

142. Let $z_k=\cos\left(\frac{2k\pi}{10}\right)+i\sin\left(\frac{2k\pi}{10}\right)$; $k=1, 2,\ldots, 9$. Then, match the column.

	Column I		Column II
(A)	For each z_k there exists a z_j such that $z_k\cdot z_j=1$	(1)	True
(B)	There exists a $k\in\{1, 2, \ldots, 9\}$ such that $z_1\cdot z=z_k$ has no solution z in the set of complex numbers	(2)	False
(C)	$\frac{\lvert1-z_1\rvert\lvert1-z_2\rvert\ldots\lvert1-z_9\rvert}{10}$ equals to	(3)	1
(D)	$1-\sum_{k=1}^{9}\cos\left(\frac{2k\pi}{10}\right)$ equals to	(4)	2

[JEE Advanced 2014, 3M]

Codes

	A	B	C	D		A	B	C	D
(a)	1	2	4	3	(b)	2	1	3	4
(c)	1	2	3	4	(d)	2	1	4	3

143. A complex number z is said to be unimodular if $|z|=1$. Suppose z_1 and z_2 are complex numbers such that $\frac{z_1-2z_2}{2-z_1\bar{z}_2}$ is unimodular and z_2 is not unimodular. Then the point z_1 lies on a **[JEE Main 2015, 4M]**

(a) circle of radius z

(b) circle of radius $\sqrt{2}$

(c) straight line parallel to X-axis

(d) straight line parallel to Y-axis

144. Let $\omega\neq 1$ be a complex cube root of unity. If $(3-3\omega+2\omega^2)^{4n+3}+(2+3\omega-3\omega^2)^{4n+3}+(-3+2\omega+3\omega^2)^{4n+3}=0$, then possible value(s) of n is (are) **[JEE Advanced 2015, 2M]**

(a) 1 (b) 2

(c) 3 (d) 4

145. For any integer k, let $\alpha_k=\cos\left(\frac{k\pi}{7}\right)+i\sin\left(\frac{k\pi}{7}\right)$, where $i=\sqrt{-1}$. The value of the expression $\frac{\sum_{k=1}^{12}|\alpha_{k+1}-\alpha_k|}{\sum_{k=1}^{3}|\alpha_{4k-1}-\alpha_{4k-2}|}$ is **[JEE Advanced 2015, 4M]**

146. A value of θ for which $\frac{2+3i\sin\theta}{1-2i\sin\theta}$ is purely imaginary, is **[JEE Main 2016, 4M]**

(a) $\frac{\pi}{6}$ (b) $\sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$

(c) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (d) $\frac{\pi}{3}$

147. Let $a, b \in R$ and $a^2 + b^2 \neq 0$.

Suppose $S = \left\{z \in C : z = \frac{1}{a+ibt}, t \in R, t \neq 0\right\}$, where $i = \sqrt{-1}$. If $z = x + iy$ and $z \in S$, then (x, y) lies on **[JEE Advanced 2016 4M]**

(a) the circle with radius $\frac{1}{2a}$ and centre $\left(\frac{1}{2a}, 0\right)$ for $a > 0, b \neq 0$

(b) the circle with radius $-\frac{1}{2a}$ and centre $\left(-\frac{1}{2a}, 0\right)$ for $a < 0, b \neq 0$

(c) the X-axis for $a \neq 0, b = 0$

(d) the Y-axis for $a = 0, b \neq 0$

148. Let ω be a complex number such that $2\omega + 1 = z$, when $z = \sqrt{-3}$ if $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2 - 1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix} = 3k$, then k is equal to **[JEE Main 2017, 4M]**

(a) 1 (b) $-z$ (c) z (d) -1

Answers

Exercise for Session 1

1. (d) **2.** (c) **3.** (b) **4.** (b) **5.** (c) **6.** (b)
7. (d) **8.** (a)

Exercise for Session 2

1. (b) **2.** (b) **3.** (b) **4.** (b) **5.** (b) **6.** (b)
7. (d) **8.** (c) **9.** (b) **10.** (b) **11.** (a) **12.** (c)
13. (c) **14.** (a)

Exercise for Session 3

1. (a) **2.** (b) **3.** (d) **4.** (a) **5.** (b) **6.** (a)
7. (c) **8.** (b) **9.** (c) **10.** (d) **11.** (c) **12.** (a)
13. (b) **14.** (b) **15.** (a)

Exercise for Session 4

1. (a) **2.** (d) **3.** (c) **4.** (d) **5.** (b) **6.** (b)
7. (a) **8.** (d) **9.** (d) **10.** (b) **11.** (b) **12.** (a)
13. (b) **14.** (c)

Chapter Exercises

1. (b) **2.** (c) **3.** (d) **4.** (d) **5.** (a) **6.** (d)
7. (b) **8.** (d) **9.** (b) **10.** (c) **11.** (c) **12.** (b)
13. (c) **14.** (d) **15.** (a) **16.** (c) **17.** (a) **18.** (c)
19. (b) **20.** (b) **21.** (c) **22.** (c) **23.** (c) **24.** (c)
25. (d) **26.** (d) **27.** (b) **28.** (b) **29.** (b) **30.** (c)
31. (b,c,d) **32.** (b,c,d) **33.** (b,c,d) **34.** (b,c,d) **35.** (a,c,d) **36.** (a,b)
37. (a,b,c) **38.** (a,d) **39.** (a,c) **40.** (a,b,d) **41.** (a,d)
42. (b,d) **43.** (a,c,d) **44.** (a,d) **45.** (a,b,d)
46. (a) **47.** (d) **48.** (b) **49.** (a) **50.** (c) **51.** (b)
52. (d) **53.** (c) **54.** (a) **55.** (c) **56.** (d) **57.** (b)
58. (1) **59.** (5) **60.** (9) **61.** (8) **62.** (3) **63.** (4)
64. (6) **65.** (5) **66.** (4) **67.** (4)
68. A $\rightarrow$ (p, q); B $\rightarrow$ (p, r); C $\rightarrow$ (p, r, s)
69. A $\rightarrow$ (q); B $\rightarrow$ (q, r); C $\rightarrow$ (q, s)
70. A $\rightarrow$ (p, q, r, t); B $\rightarrow$ (p, s); C $\rightarrow$ (p, r)
71. A $\rightarrow$ (r); B $\rightarrow$ (p, s); C $\rightarrow$ (q, t)
72. (d) **73.** (d) **74.** (c) **75.** (d) **76.** (a) **77.** (d)
78. (d)
82. 2
84. $z = c + i(-1 \pm \sqrt{(1 - c^2 - 2c)})$ for $0 \leq c \leq \sqrt{2} - 1$ and no solution for $c > \sqrt{2} - 1$
85. $(1 - 3i)z + (1 + 3i)\bar{z} - 22 = 0$ and $(3 + i)z + (3 - i)\bar{z} + 14 = 0$
87. No solution **97.** $\frac{\sum |z_1|^2 (z_2 - z_3)}{\sum \overline{z_1}(z_2 - z_3)}$
98. $\frac{\sum z_1^2(\bar{z}_2 - \bar{z}_3) + \sum |z_1|^2 (z_2 - z_3)}{\sum (z_1\overline{z_2} - z_2\overline{z_1})}$
99. (i) $x^3 - 24x^2 + 80x - 64 = 0$
(ii) $x^3 - 21x^2 + 35x - 7 = 0$
(iii) 4
100. Roots of $z^7 + 1 = 0$ are $-1, \alpha, \alpha^3, \alpha^5, \bar{\alpha}, \bar{\alpha}^3, \bar{\alpha}^5$, where $\alpha = \cos\frac{\pi}{7} + i\sin\frac{\pi}{7}$
101. $a \in \left(\frac{1-\sqrt{71}}{2}, \frac{-1-4\sqrt{11}}{5}\right) \cup \left(\frac{-1+4\sqrt{11}}{5}, \frac{1+\sqrt{71}}{2}\right)$
102. (i) ${}^{2n+1}C_1(1-x)^n - {}^{2n+1}C_3(1-x)^{n-1}x + \ldots + (-1)^n x^n = 0$
(ii) ${}^{2n+1}C_1 x^n - {}^{2n+1}C_3 x^{n-1} + {}^{2n+1}C_5 x^{n-2} - \ldots = 0$
103. (c) **104.** (c) **105.** $(1 - \sqrt{3}) + i$, $-i\sqrt{3}$, $(\sqrt{3} + 1) - i$ **106.** (d)
107. (b) **108.** (a) **109.** (d) **110.** (d) **111.** (d) **112.** (d)
113. (d) **114.** (c) **115.** (b) **116.** (c) **117.** (d) **118.** (d)
119. (c) **120.** (a) **121.** (d) **122.** (c) **123.** (a, c, d) **124.** (b)
125. A $\rightarrow$ (q, r); B $\rightarrow$ (p); C $\rightarrow$ (p, s); D $\rightarrow$ (q, r, s, t) **126.** (b) **127.** (a)
128. (5) **129.** (a) **130.** (c) **131.** (3) **132.** (c) **133.** (a)
134. (d) **135.** (d) **136.** (b) **137.** (c) **138.** (c) **139.** (c)
140. (b) **141.** (d) **142.** (c) **143.** (a) **144.** (a, b, d)
145. (4) **146.** (c) **147.** (a,c,d) **148.** (b)

Solutions

1. We have,

$$a + ib = \cos(1 - i) = \cos 1 \cos i + \sin 1 \sin i$$

$$= \cos 1 \cosh 1 + \sin 1\, i \sinh 1$$

$$[\because \cos i = \cosh 1, \sin i \cdot 1 = i \sinh 1]$$

$$= \cos 1 \left(\frac{e + e^{-1}}{2}\right) + i \sin 1 \left(\frac{e - e^{-1}}{2}\right)$$

$$= \frac{1}{2}\left(e + \frac{1}{e}\right)\cos 1 + i \cdot \frac{1}{2}\left(e - \frac{1}{e}\right)\sin 1$$

$\therefore \quad a = \frac{1}{2}\left(e + \frac{1}{e}\right)\cos 1$

and $\quad b = \frac{1}{2}\left(e - \frac{1}{e}\right)\sin 1$

2. Given that, $z^{10} - z^5 - 992 = 0$

Let $\quad t = z^5$

$\Rightarrow \quad t^2 - t - 992 = 0$

$\Rightarrow \quad t = \frac{1 \pm \sqrt{1 + 3968}}{2} = \frac{1 \pm 63}{2} = 32, -31$

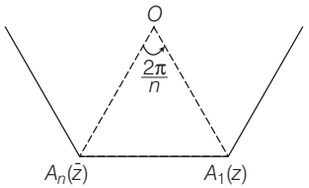
$\therefore \quad z^5 = 32$

and $\quad z^5 = -31$

But the real part is negative, therefore $z^5 = 32$ does not hold.

$\therefore$ Number of solutions is 5.

3. From Coni method,



$$\frac{z - 0}{\bar{z} - 0} = e^{2\pi i/n} \text{ or } \frac{z}{\bar{z}} = e^{2\pi i/n} \quad \ldots (i)$$

But given $\quad \frac{\text{Im}(z)}{\text{Re}(z)} = \sqrt{2} - 1$

$$\Rightarrow \quad \frac{\frac{z - \bar{z}}{2i}}{\frac{z + \bar{z}}{2}} = \sqrt{2} - 1 \quad \Rightarrow \quad \frac{1}{i}\left(\frac{\frac{z}{\bar{z}} - 1}{\frac{z}{\bar{z}} + 1}\right) = \sqrt{2} - 1$$

$$\Rightarrow \quad \left(\frac{e^{2\pi i/n} - 1}{e^{2\pi i/n} + 1}\right) = i(\sqrt{2} - 1) \qquad [\text{from Eq. (i)}]$$

$$\Rightarrow \quad i \tan\left(\frac{\pi}{n}\right) = i(\sqrt{2} - 1)$$

$$\Rightarrow \quad \tan\left(\frac{\pi}{n}\right) = \tan\left(\frac{\pi}{8}\right)$$

$\therefore \quad n = 8$

4. We have, $\quad \prod_{p=1}^{r} e^{ip\theta} = 1$

$$\Rightarrow \quad e^{i\theta} \cdot e^{2i\theta} \cdot e^{3i\theta} \ldots e^{ri\theta} = 1$$

$$\Rightarrow \quad e^{i\theta(1 + 2 + 3 + \ldots + r)} = 1 \quad \Rightarrow \quad e^{i\theta\left(\frac{r(r+1)}{2}\right)} = 1$$

or $\cos\left\{\frac{r(r+1)}{2}\theta\right\} + i \sin\left\{\frac{r(r+1)}{2}\theta\right\} = 1 + i \cdot 0$

On comparing, we get

$$\cos\left\{\frac{r(r+1)}{2}\theta\right\} = 1 \text{ and } \sin\left\{\frac{r(r+1)}{2}\theta\right\} = 0$$

$$\Rightarrow \quad \frac{r(r+1)}{2}\theta = 2m\pi \text{ and } \frac{r(r+1)}{2}\theta = m_1\pi$$

$$\Rightarrow \quad \theta = \frac{4m\pi}{r(r+1)} \text{ and } \theta = \frac{2m_1\pi}{r(r+1)}$$

where, $m, m_1 \in I$

Hence, $\theta = \frac{4n\pi}{r(r+1)}, n \in I.$

5. Let $z = x + iy$, then

$(3 + i)(z + \bar{z}) - (2 + i)(z - \bar{z}) + 14i = 0$ reduces to

$$(3 + i)\,2x - (2 + i)(2iy) + 14i = 0$$

$$\Rightarrow \quad 6x + 2y + i(2x - 4y + 14) = 0$$

On comparing real and imaginary parts, we get

$$6x + 2y = 0$$

$$\Rightarrow \quad 3x + y = 0 \qquad \ldots(i)$$

and $\quad 2x - 4y + 14 = 0$

$$\Rightarrow \quad x - 2y + 7 = 0 \qquad \ldots(ii)$$

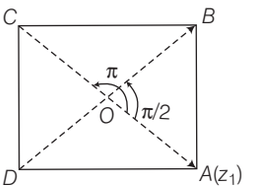
On solving Eqs. (i) and (ii), we get

$$x = -1 \quad \text{and} \quad y = 3$$

$\therefore \quad z = -1 + 3i$

$\therefore \quad z\bar{z} = |z|^2 = |-1 + 3i|^2 = (-1)^2 + (3)^2 = 10$

6. Since, affix of A is z_1.



$\therefore \overrightarrow{\mathbf{OA}} = z_1$ and $\overrightarrow{\mathbf{OB}}$ and $\overrightarrow{\mathbf{OC}}$ are obtained by rotating $\overrightarrow{\mathbf{OA}}$ through $\frac{\pi}{2}$ and π. Therefore, $\overrightarrow{\mathbf{OB}} = iz_1$ and $\overrightarrow{\mathbf{OC}} = -z_1$.

Hence, centroid of $\Delta ABC = \frac{z_1 + iz_1 + (-z_1)}{3}$

$$= \frac{i}{3}z_1 = \frac{z_1}{3}\left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right)$$

If A, B and C are taken in clockwise, then centroid of ΔABC

$$= \frac{1}{3}z_1\left(\cos\frac{\pi}{2} - i \sin\frac{\pi}{2}\right)$$

$\therefore \quad$ Centroid of $\Delta ABC = \frac{z_1}{3}\left(\cos\frac{\pi}{2} \pm i \sin\frac{\pi}{2}\right)$

7. Given that, $z = \dfrac{\sqrt{3} - i}{2} = i\left(\dfrac{-1 - i\sqrt{3}}{2}\right) = i\omega^2$

$\therefore \quad z^{101} = (i\omega^2)^{101} = i^{101}\,\omega^{202} = i\omega$

Now, $i^{101} + z^{101} = i + i\omega = i(-\omega^2)$

$\therefore \quad (i^{101} + z^{101})^{103} = -\,i^{103}\,\omega^{206} = -\,i^3\,\omega^2 = i\omega^2 = z$

8. The complex slope of the line $a\bar{z} + \bar{a}z + 1 = 0$ is $\alpha = -\dfrac{a}{\bar{a}}$

and the complex slope of the line $b\bar{z} + \bar{b}z - 1 = 0$ is $\beta = -\dfrac{b}{\bar{b}}$

Since, both lines are mutually perpendicular, then

$\therefore \quad \alpha + \beta = 0$

$\Rightarrow \quad -\dfrac{a}{\bar{a}} - \dfrac{b}{\bar{b}} = 0$

$\Rightarrow \quad a\bar{b} + \bar{a}b = 0$

9. We have, $\alpha = \cos\left(\dfrac{8\pi}{11}\right) + i\sin\left(\dfrac{8\pi}{11}\right)$

Now, $\text{Re}\,(\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5)$

$= \dfrac{\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \bar{\alpha} + \bar{\alpha}^2 + \bar{\alpha}^3 + \bar{\alpha}^4 + \bar{\alpha}^5}{2}$

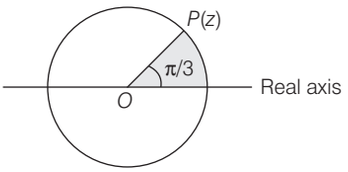
$= \dfrac{-1 + (1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \bar{\alpha} + \bar{\alpha}^2 + \bar{\alpha}^3 + \bar{\alpha}^4 + \bar{\alpha}^5}{2}$

$= \dfrac{-1 + 0}{2}$ [sum of 11, 11th roots of unity]

$= -\dfrac{1}{2}$

10. $|z| \le 4$ ...(i)

and $0 \le \arg(z) \le \dfrac{\pi}{3}$ ...(ii)



which implies the set of points in an argand plane, is a sector of a circle.

11. Since, $x^2 + x + 1 = (x - \omega)(x - \omega^2)$, where ω is the cube root of unity and $f(x) = g(x^3) + x\,h(x^3)$ is divisible by $x^2 + x + 1$. Therefore, ω and ω^2 are the roots of $f(x) = 0$.

$\Rightarrow \quad f(\omega) = 0$ and $f(\omega^2) = 0$

$\Rightarrow \quad g(\omega^3) + \omega\,h(\omega^3) = 0$

and $\quad g((\omega^2)^3) + \omega^2 h(\omega^2)^3 = 0$

$\Rightarrow \quad g(1) + \omega\,h(1) = 0$

and $\quad g(1) + \omega^2 h(1) = 0$

$\Rightarrow \quad g(1) = h(1) = 0$

Hence, $g(x)$ and $h(x)$ both are divisible by $(x - 1)$.

12. Since, z_1, z_2 and $z_1 - z_2$ are collinear.

$$\therefore \quad \begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_1 - z_2 & \bar{z}_1 - \bar{z}_2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_1 - z_2 & \bar{z}_1 - \bar{z}_2 & 1 \end{vmatrix} = 0$$

Applying $R_3 \to R_3 - R_1 + R_2$, then $\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ & & \vdots \\ z_2 & \bar{z}_2 & 1 \\ & & \vdots \\ 0\ldots & 0 & \ldots 1 \end{vmatrix} = 0$

Expand w.r.t. R_3, then

$z_1\bar{z}_2 - \bar{z}_1 z_2 = 0$

$\Rightarrow \quad z_1\bar{z}_2 - \overline{(z_1\bar{z}_2)} = 0$

$\Rightarrow \quad \text{Im}\,(z_1\bar{z}_2) = 0$

$\Rightarrow \quad \text{Im}\,((a + ib)\,\overline{(c + id)}) = 0$

$\Rightarrow \quad \text{Im}\,((a + ib)(c - id)) = 0$

$\Rightarrow \quad bc - ad = 0 \Rightarrow ad - bc = 0$

13. Let $z = a + ib$

$\therefore\ f(a + ib) = \sqrt{(a^2 + b^2)}$

$\Rightarrow \quad f(z) = f(\bar{z}) = f(-z) = f(-\bar{z}) = \sqrt{(a^2 + b^2)}$

$\therefore$ f is not injective (i.e., it is many-one).

but $|z| > 0$ i.e. $f(z) > 0 \Rightarrow f(z) \in R^+$ (Range)

$\Rightarrow R^+ \subset R$

$\therefore$ f is not surjective (i.e., into).

Hence, f is neither injective nor surjective.

14. Let $\alpha = re^{i\theta}$, $\beta = re^{i\phi}$ [$\because |\alpha| = |\beta|$, given]

where, $\theta \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ and $\phi \in (-\pi, 0)$

$$\therefore \quad \frac{\alpha + \beta}{\alpha - \beta} = \frac{re^{i\theta} + re^{i\phi}}{re^{i\theta} - re^{i\phi}} = \frac{e^{i\left(\frac{\theta + \phi}{2}\right)} \cdot 2\cos\left(\frac{\theta - \phi}{2}\right)}{e^{i\left(\frac{\theta + \phi}{2}\right)} \cdot 2i\sin\left(\frac{\theta - \phi}{2}\right)}$$

$$= -\,i\cot\left(\frac{\theta - \phi}{2}\right) = \text{Purely imaginary}$$

15. We have, $|z| = \left|z + \dfrac{25}{z} - \dfrac{25}{z}\right| \le \left|z + \dfrac{25}{z}\right| + \left|\dfrac{25}{-z}\right|$

$\Rightarrow \quad |z| \le 24 + \dfrac{25}{|z|}$

$\Rightarrow \quad |z|^2 - 24|z| - 25 \le 0 \Rightarrow (|z| - 25)(|z| + 1) \le 0$

$\therefore \quad |z| - 25 \le 0$ [$\because |z| + 1 > 0$]

$\Rightarrow \quad |z| \le 25$ or $|z - 0| \le 25$

Hence, the maximum distance from the origin of coordinates to the point z is 25.

16. $\because \quad A \equiv z_1,\ B \equiv z_2,\ C \equiv (1 - i)\,z_1 + iz_2$

$\therefore \quad AB = |z_1 - z_2|$

$BC = |z_2 - (1 - i)\,z_1 - iz_2| = |(1 - i)(z_2 - z_1)|$

$= \sqrt{2}\,|z_1 - z_2|$

and $\quad CA = |(1 - i)\,z_1 + iz_2 - z_1| = |-i(z_1 - z_2)|$

$= |-i|\,|z_1 - z_2| = |z_1 - z_2|$

It is clear that, $AB = CA$ and $(AB)^2 + (CA)^2 = (BC)^2$

$\therefore$ ΔABC is isosceles and right angled.

17. Centre and radius of circle $|z|=3$ ...(i)

are $C_1 \equiv 0,\ r_1 = 3$

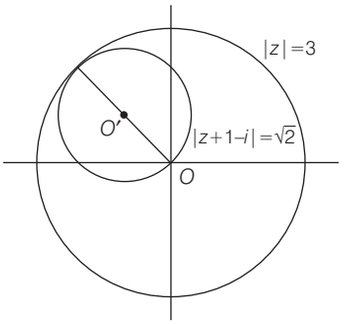
and centre and radius of circle

$$|z+1-i| = \sqrt{2} \quad \text{...(ii)}$$

and $C_2 = -1+i,\ r_2 = \sqrt{2}$

$\because$ $|C_1C_2| = |-1+i| = \sqrt{2}$

and $|C_1C_2| < r_1 - r_2$



Hence, circle (ii) completely inside circle (i)

$\therefore$ Number of solutions = 0

18. We have, $f(z) = g(z)(z^2+1) + h(z)$

where, degree of $h(z)$ < degree of (z^2+1)

$\Rightarrow$ $h(z) = az + b\ ;\ a, b \in C$

$\therefore$ $f(z) = g(z)(z^2+1) + az + b\ ;\ a, b \in C$

$\Rightarrow$ $f(z) = g(z)(z-i)(z+i) + az + b\ ;\ a, b \in C$ ...(i)

Now, $f(i) = 1-i$ [given]

$\Rightarrow$ $ai + b = 1 - i$ [from Eq. (i)] ...(ii)

and $f(-i) = 1+i$ [given]

$\Rightarrow$ $a(-i) + b = 1 + i$ [from Eq. (i)] ...(iii)

On solving Eqs. (ii) and (iii) for a and b, we get

$$a = -1 \text{ and } b = 1$$

$\therefore$ Required remainder, $h(z) = az + b = -z + 1 = 1 - z$

19. We have, $|z+1| = 2|z-1|$

Put $z = x + iy$, we get

$$(x+1)^2 + y^2 = 4[(x-1)^2 + y^2]$$

$\Rightarrow$ $3x^2 + 3y^2 - 10x + 3 = 0$

$\Rightarrow$ $x^2 + y^2 - \frac{10}{3}x + 1 = 0$ ...(i)

On comparing Eq. (i) with the standard equation

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$\Rightarrow$ $g = -\frac{10}{6} = -\frac{5}{3}$ and $f = 0$

$\therefore$ Required centre of circle $\equiv (-g, -f) \equiv \left(\frac{5}{3}, 0\right)$

i.e. $\frac{5}{3} + 0 \cdot i = \frac{5}{3}$

20. $\because$ $x = 9^{1/3} \cdot 9^{1/9} \cdot 9^{1/27} \ldots \infty$

$$= 9^{1/3 + 1/9 + 1/27 + \ldots \infty} = 9^{\frac{1/3}{1-1/3}} = 9^{1/2} = 3$$

$$y = 4^{1/3} \cdot 4^{-1/9} \cdot 4^{1/27} \ldots \infty = 4^{1/3 - 1/9 + 1/27 \ldots \infty}$$

$$= 4^{\frac{1/3}{1+1/3}} = 4^{1/4} = \sqrt{2}$$

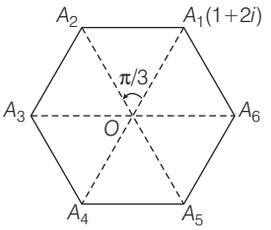
and $z = \sum_{r=1}^{\infty} (1+i)^{-r} = \frac{1}{(1+i)} + \frac{1}{(1+i)^2} + \frac{1}{(1+i)^3} + \ldots \infty$

$$= \frac{\frac{1}{(1+i)}}{1 - \frac{1}{(1+i)}} = \frac{1}{i} = -i$$

Now, $x + yz = 3 - i\sqrt{2}$

$\therefore$ $\arg(x + yz) = \arg(3 - i\sqrt{2}) = -\tan^{-1}\left(\frac{\sqrt{2}}{3}\right)$

21. $\because$ $A_1 \equiv 1 + 2i$



$\therefore$ $A_2 = (1+2i)\, e^{i\pi/3}$

$$= (1+2i)\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) = \frac{1}{2} + \frac{i\sqrt{3}}{2} + i - \sqrt{3}$$

$$= \left(\frac{1}{2} - \sqrt{3}\right) + i\left(\frac{\sqrt{3}}{2} + 1\right)$$

$$\therefore |A_1A_2| = \left|1 + 2i - \left(\frac{1}{2} - \sqrt{3}\right) - i\left(\frac{\sqrt{3}}{2} + 1\right)\right|$$

$$= \left|\frac{1}{2} + \sqrt{3} + i\left(1 - \frac{\sqrt{3}}{2}\right)\right|$$

$$= \sqrt{\left(\frac{1}{2} + \sqrt{3}\right)^2 + \left(1 - \frac{\sqrt{3}}{2}\right)^2} = \sqrt{5}$$

$\therefore$ Perimeter $= 6|A_1A_2| = 6\sqrt{5}$

22. We have,

$$\left|\sum_{r=1}^{n} z_r\right| = \left|\sum_{r=1}^{n} (z_r - r) + r\right| \le \sum_{r=1}^{n} (|z_r - r| + |r|)$$

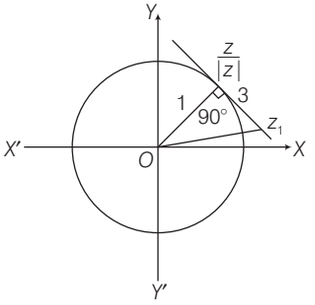
$$= \sum_{r=1}^{n} |z_r - r| + \sum_{r=1}^{n} |r| \le \sum_{r=1}^{n} r + \sum_{r=1}^{n} |r|$$

$$= \frac{n(n+1)}{2} + \frac{n(n+1)}{2} = n(n+1)$$

$\therefore$ $\left|\sum_{r=1}^{n} z_r\right| \le n(n+1)$

23. We have, $\arg\left(\frac{z_1 - \frac{z}{|z|}}{\frac{z}{|z|}}\right) = \frac{\pi}{2}$ and $\left|\frac{z}{|z|} - z_1\right| = 3$

which implies the following diagram



$$\Rightarrow \quad \left|\frac{z}{|z|} - z_1\right| = 3 \Rightarrow |z_1| = \sqrt{9+1} = \sqrt{10}$$

24. Let $z = x + iy = r(\cos\theta + i\sin\theta)$

$\therefore\ |z| = r, \arg(z) = \theta$

Given, $|z - 2 - i| = |z|\left|\sin\left(\frac{\pi}{4} - \arg(z)\right)\right|$

$$\Rightarrow |x + iy - 2 - i| = r\left|\sin\left(\frac{\pi}{4} - \theta\right)\right|$$

$$\Rightarrow |(x-2) + i(y-1)| = r\left|\frac{1}{\sqrt{2}}(\cos\theta - \sin\theta)\right|$$

$$\Rightarrow \sqrt{(x-2)^2 + (y-1)^2} = \frac{1}{\sqrt{2}}|x - y|$$

On squaring both sides, we get

$$2(x^2 + y^2 - 4x - 2y + 5) = x^2 + y^2 - 2xy$$

$$\Rightarrow \quad (x+y)^2 = 2(4x + 2y - 5)$$

which is a parabola.

25. Since, $1, z_1, z_2, z_3, \ldots, z_{n-1}$ are the n, nth roots of unity.

$$\therefore \quad (z^n - 1) = (z-1)(z-z_1)(z-z_2)(z-z_3)\ldots(z-z_{n-1})$$

$$= (z-1)\prod_{r=1}^{n-1}(z - z_r)$$

Taking log on both sides, we get

$$\log_e(z^n - 1) = \log_e(z-1) + \sum_{r=1}^{n-1}\log_e(z - z_r)$$

On differentiating both sides w.r.t.z, we get

$$\frac{nz^{n-1}}{(z^n - 1)} - \frac{1}{(z-1)} = \sum_{r=1}^{n-1}\frac{1}{(z - z_r)}$$

Putting $z = 3$, we get

$$\sum_{r=1}^{n-1}\frac{1}{(3 - z_r)} = \frac{n \cdot 3^{n-1}}{(3^n - 1)} - \frac{1}{2}$$

26. We have,

$$z = (3 + 7i)(\lambda + i\mu)$$

$$= (3\lambda - 7\mu) + i(7\lambda + 3\mu)$$

Since, z is purely imaginary.

$$\therefore \quad 3\lambda - 7\mu = 0$$

$$\Rightarrow \quad \frac{\lambda}{\mu} = \frac{7}{3}$$

$$\because \quad \lambda, \mu \in I - \{0\}$$

For minimum value $\lambda = 7, \mu = 3$

$$\therefore \quad |z|^2 = |(3 + 7i)(\lambda + i\mu)|^2$$

$$= |3 + 7i|^2 |\lambda + i\mu|^2 = 58(\lambda^2 + \mu^2)$$

$$= 58(7^2 + 3^2) = (58)^2 = 3364$$

27. We have,

$$z = f(x) + i\,g(x)$$

where, $i = \sqrt{-1}$ and $f, g : (0,1) \to (0,1)$ are real-valued functions.

(a) $z = \frac{1}{1 - ix} + i\left(\frac{1}{1 + ix}\right)$

$$= \frac{1 + ix}{1 + x^2} + \frac{x + i}{1 + x^2} = \frac{1 + x}{1 + x^2} + i\frac{(1 + x)}{1 + x^2}$$

$$\Rightarrow f(x) = \frac{1 + x}{1 + x^2} \text{ and } g(x) = \frac{1 + x}{1 + x^2}$$

But for $x = 0.5$, $f(0.5) > 1$ and $g(0.5) > 1$, which is out of range.

Hence, (a) is not a correct option.

(b) $z = \frac{1}{1 + ix} + i\left(\frac{1}{1 - ix}\right)$

$$= \frac{1 - ix}{1 + x^2} + \frac{(i - x)}{1 + x^2} = \left(\frac{1 - x}{1 + x^2}\right) + i\left(\frac{1 - x}{1 + x^2}\right)$$

$$\Rightarrow \quad f(x) = \frac{1 - x}{1 + x^2} \text{ and } g(x) = \frac{1 - x}{1 + x^2}$$

Clearly, $f(x), g(x) \in (0,1)$, if $x \in (0,1)$

Hence, (b) is the correct option.

(c) $z = \frac{1 - ix}{1 + x^2} + \frac{i(1 - ix)}{1 + x^2} = \frac{(1 + x)}{(1 + x^2)} + \frac{i(1 - x)}{(1 + x^2)}$

Hence, (c) is not a correct option.

(d) $z = \frac{1}{1 - ix} + i\left(\frac{1}{1 - ix}\right) = \frac{1 + ix}{1 + x^2} + \frac{i(1 + ix)}{(1 + x^2)}$

$$= \frac{(1 - x)}{(1 + x^2)} + \frac{i(1 + x)}{(1 + x^2)}$$

Hence, (d) is not a correct option.

28. Let $z = \alpha$ be a real roots of equation.

$$z^3 + (3 + 2i)z + (-1 + ia) = 0$$

$$\Rightarrow \quad \alpha^3 + (3 + 2i)\alpha + (-1 + ia) = 0$$

$$\Rightarrow \quad (\alpha^3 + 3\alpha - 1) + i(a + 2\alpha) = 0$$

On comparing the real and imaginary parts, we get

$$\alpha^3 + 3\alpha - 1 = 0 \text{ and } a + 2\alpha = 0$$

$$\Rightarrow \quad \alpha = -\frac{a}{2}$$

$$\Rightarrow \quad -\frac{a^3}{8} - \frac{3a}{2} - 1 = 0$$

$$\Rightarrow \quad a^3 + 12a + 8 = 0$$

Let $f(a) = a^3 + 12a + 8$

$\Rightarrow$ $f(-1) < 0$ and $f(0) > 0$

$\therefore$ $a \in (-1, 0)$

29. $CiS\,\frac{\pi}{6} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$

$$= \left(\frac{\sqrt{3}+i}{2}\right) = \frac{1}{i}\left(\frac{-1+i\sqrt{3}}{2}\right) = \frac{\omega}{i} = -i\omega$$

$$\therefore \left(2\,CiS\,\frac{\pi}{6}\right)^m = (-2i\omega)^m = ((-2i\omega)^3)^{m/3} = (8i)^{m/3}$$

$$\text{and} \left(4\,CiS\,\frac{\pi}{4}\right)^n = \left(4\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)^n = (2\sqrt{2}\,(1+i)^n$$

$$= (8(1+i)^2)^{n/2} = (16i)^{n/2}$$

Thus, $(8i)^{m/3} = (16i)^{n/2}$

which is satisfy only when $m = 48$ and $n = 24$

$\therefore$ $m + n = 72$

30. We have, $z^2 = \bar{z} \cdot 2^{1-|z|}$

Taking modulus on both sides, we get

$$|z|^2 = |z| \cdot 2^{1-|z|}$$

$\Rightarrow$ $|z|\,(|z| - 2^{1-|z|}) = 0$ ...(i)

and $\arg(z^2) = \arg(\bar{z} \cdot 2^{1-|z|})$

$\Rightarrow$ $2\arg(z) = \arg(\bar{z}) = -\arg(z)$

$\Rightarrow$ $3\arg(z) = 0$

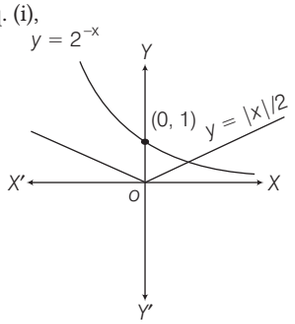
$\therefore$ $\arg(z) = 0$

Then, $y = 0$ $[\because z = x + iy]$

From Eq. (i), $|z| = 0 \Rightarrow x = 0$ $[\because y = 0]$

One solution is $z = 0 + i \cdot 0 = 0$.

Also, from Eq. (i),



$$|z| = 2^{1-|z|} \Rightarrow |x| = 2^{1-x}$$

$\Rightarrow$ $\frac{|x|}{2} = 2^{-x} = y$ (say)

Hence, total number of solutions = 2

31. $\because$ $\frac{z+1}{z+i}$ is a purely imaginary number.

$$\therefore \left(\overline{\frac{z+1}{z+i}}\right) = -\left(\frac{z+1}{z+i}\right) \Rightarrow \frac{\bar{z}+1}{\bar{z}-i} = -\left(\frac{z+1}{z+i}\right)$$

$\Rightarrow$ $(\bar{z}+1)(z+i) + (\bar{z}-i)(z+1) = 0$

$\Rightarrow$ $2z\bar{z} + \bar{z}(1+i) + z(1-i) = 0$

$\Rightarrow$ $z\bar{z} + \left(\frac{1+i}{2}\right)\bar{z} + \left(\frac{1-i}{2}\right)z = 0$

which is a circle and passing through the origin

and radius $= \sqrt{\left|\frac{1+i}{2}\right|^2 - 0} = \left|\frac{1+i}{2}\right| = \frac{1}{\sqrt{2}}$

32. Given, $|z-1| < |z+3|$

$\Rightarrow$ $|z-1|^2 < |z+3|^2$

$\Rightarrow$ $|z|^2 + 1 - 2\,\text{Re}(z) < |z|^2 + 9 + 2\,\text{Re}(3z)$

$\Rightarrow$ $2\,\text{Re}(4z) > -8$

$\Rightarrow$ $\text{Re}(4z) > -4$

$\Rightarrow$ $\frac{4z + 4\bar{z}}{2} > -4$

$\therefore$ $z + \bar{z} > -2$

and $\omega = 2z + 3 - i$

$\therefore$ $\omega + \bar{\omega} = 2z + 3 - i + 2\bar{z} + 3 + i$

$= 2(z + \bar{z}) + 6 > -4 + 6$

$\Rightarrow$ $\omega + \bar{\omega} > 2$

Option (a) $|\omega - 5 - i| < |\omega + 3 + i|$

$\Rightarrow$ $|2z + 3 - i - 5 - i| < |2z + 3 - i + 3 + i|$

$\Rightarrow$ $|2z - 2 - 2i| < |2z + 6|$

$\Rightarrow$ $|z - 1 - i| < |z + 3|$

which is false.

Option (b) $|\omega - 5| < |\omega + 3|$

$\Rightarrow$ $|2z + 3 - i - 5| < |2z + 3 - i + 3|$

$\Rightarrow$ $|2z - 2 - i| < |2z + 6 - i|$

$\Rightarrow$ $\left|z - 1 - \frac{i}{2}\right| < \left|z + 3 - \frac{i}{2}\right|$

$\Rightarrow$ $|z - 1| < |z + 3|$

which is true.

Option (c) $\text{Im}(i\omega) > 1$

$\Rightarrow$ $\frac{i\omega - i\bar{\omega}}{2i} > 1$

$\Rightarrow$ $\frac{i\omega + i\bar{\omega}}{2i} > 1$

$\Rightarrow$ $\omega + \bar{\omega} > 2$

which is true.

Option (d) $|\arg(\omega - 1)| < \frac{\pi}{2}$

$\Rightarrow$ $|\arg(2z + 3 - i - 1)| < \frac{\pi}{2}$

$\Rightarrow$ $|\arg(2z + 2 - i)| < \frac{\pi}{2}$

$\Rightarrow$ $\left|\tan^{-1}\left(\frac{\text{Im}(2z+2-i)}{\text{Re}(2z+2-i)}\right)\right| < \frac{\pi}{2}$

$\therefore$ $\text{Re}(2z + 2 - i) > 0$

$\Rightarrow$ $\frac{(2z + 2 - i) + (2\bar{z} + 2 + i)}{2} > 0$

$\Rightarrow$ $z + \bar{z} + 2 > 0$

$\Rightarrow$ $z + \bar{z} > -2$

which is true.

33. $\therefore$ $(1 + ri)^3 = \lambda(1 + i)$

$\Rightarrow$ $1 + (ri)^3 + 3(1)^2 ri + 3(1)(ri)^2 = \lambda(1 + i)$

$$\Rightarrow \quad 1 - r^3 i + 3ri - 3r^2 = \lambda + i\lambda$$

On comparing real and imaginary parts, we get

$$1 - 3r^2 = \lambda$$

and $$-r^3 + 3r = \lambda$$

Then, $$-r^3 + 3r = 1 - 3r^2$$

$$\Rightarrow \quad r^3 - 3r^2 - 3r + 1 = 0$$

$$\Rightarrow \quad (r^3 + 1) - 3r(r + 1) = 0$$

$$\Rightarrow \quad (r + 1)(r^2 - r + 1 - 3r) = 0$$

$$\Rightarrow \quad (r + 1)(r^2 - 4r + 1) = 0$$

$$\therefore \quad r = -1, 2 \pm \sqrt{3}$$

$$\Rightarrow \quad r = \text{cosec}\ \frac{3\pi}{2}, \tan\frac{\pi}{12}, \cot\frac{\pi}{12}$$

34. **Option** (a) $|z - 1| + |z + 1| = 3$

Here, $|1 - (-1)| < 3$

i.e. $2 < 3$, which is an ellipse.

Option (b) $|z - 3| = 2$

It is a circle with centre 3 and radius 2.

Option (c) $|z - 2 + i| = \frac{7}{3}$

It is a circle with centre $(2 - i)$ and radius $\frac{7}{3}$.

Option (d) $(z - 3 + i)(\bar{z} - 3 - i) = 5$

$$\Rightarrow \quad (z - 3 + i)\overline{(z - 3 + i)} = 5$$

$$\Rightarrow \quad |z - 3 + i|^2 = 5$$

$$\Rightarrow \quad |z - 3 + i| = \sqrt{5}$$

It is a circle with centre at $(3 - i)$ and radius $\sqrt{5}$.

35. Since, $1, z_1, z_2, z_3, \ldots, z_{n-1}$ are the n, nth roots of unity.

Therefore,

$$z^n - 1 = (z - 1)(z - z_1)(z - z_2)\ldots(z - z_{n-1})$$

$$\Rightarrow \quad \frac{z^n - 1}{z - 1} = (z - z_1)(z - z_2)\ldots(z - z_{n-1})$$

$$= \prod_{r=1}^{n-1}(z - z_r)$$

Now, putting $z = \omega$, we get

$$\prod_{r=1}^{n-1}(\omega - z_r) = \frac{\omega^n - 1}{\omega - 1}$$

$$= \begin{cases} 0, & \text{if } n = 3r, r \in Z \\ 1, & \text{if } n = 3r + 1, r \in Z \\ 1 + \omega, & \text{if } n = 3r + 2, r \in Z \end{cases}$$

36. $\because \quad 3|z - 12| = 5|z - 8i|$

$$\therefore \ 9|z - 12|^2 = 25|z - 8i|^2$$

$$\Rightarrow \quad 9(z - 12)(\bar{z} - 12) = 25(z - 8i)(\bar{z} + 8i)$$

$$\Rightarrow \quad 9(z\bar{z} - 12(z + \bar{z}) + 144) = 25(z\bar{z} + 8i(z - \bar{z}) + 64)$$

$$\Rightarrow \quad 16z\bar{z} + 108(z + \bar{z}) + 200(z - \bar{z})i + 304 = 0$$

$$\Rightarrow \quad 16(x^2 + y^2) + 216x - 400y + 304 = 0$$

$$\Rightarrow \quad 2(x^2 + y^2) + 27x - 50y + 38 = 0 \qquad \ldots(i)$$

and $$|z - 4| = |z - 8| \ \Rightarrow |z - 4|^2 = |z - 8|^2$$

$$\Rightarrow \quad |z|^2 + 16 - 2\,\text{Re}\,(4z) = |z|^2 + 64 - 2\,\text{Re}(8z)$$

$$\Rightarrow \quad 8\,\text{Re}\,(z) = 48$$

$$\therefore \quad \text{Re}\,(z) = 6$$

$$\Rightarrow \quad x = 6 \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$2(36 + y^2) + 162 - 50y + 38 = 0$$

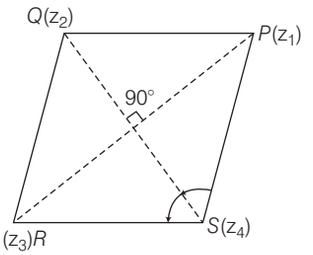
$$\Rightarrow \quad y^2 - 25y + 136 = 0$$

$$\Rightarrow \quad (y - 17)(y - 8) = 0$$

$$\Rightarrow \quad y = 17, 8$$

$$\therefore \quad \text{Im}\,(z) = 17, 8$$

37.



Option (a) $\because \ PS \| QR$

$$\therefore \quad \arg\left(\frac{z_1 - z_4}{z_2 - z_3}\right) = 0$$

$$\Rightarrow \quad \frac{z_1 - z_4}{z_2 - z_3} \text{ is purely real.}$$

Option (b) $\because$ Diagonals of rhombus are perpendicular.

Then, $\arg\left(\frac{z_1 - z_3}{z_2 - z_4}\right) = \frac{\pi}{2}$

$$\Rightarrow \quad \frac{z_1 - z_3}{z_2 - z_4} \text{ is purely imaginary.}$$

Option (c) $\because \quad PR \neq QS$

$$\therefore \quad |z_1 - z_3| \neq |z_2 - z_4|$$

Option (d) $\because \quad \angle QSP = \angle RSQ$

$$\therefore \quad \text{amp}\left(\frac{z_2 - z_4}{z_1 - z_4}\right) = \text{amp}\left(\frac{z_3 - z_4}{z_2 - z_4}\right)$$

$$\Rightarrow \quad -\text{amp}\left(\frac{z_1 - z_4}{z_2 - z_4}\right) = -\text{amp}\left(\frac{z_2 - z_4}{z_3 - z_4}\right)$$

$$\Rightarrow \quad \text{amp}\left(\frac{z_1 - z_4}{z_2 - z_4}\right) = \text{amp}\left(\frac{z_2 - z_4}{z_3 - z_4}\right)$$

38. $\because \ |z - 3| = \min\{|z - 1|, |z - 5|\}$

Case I If $|z - 3| = |z - 1|$

On squaring both sides, we get

$$|z - 3|^2 = |z - 1|^2$$

$$\Rightarrow \quad |z|^2 + 9 - 2\,\text{Re}\,(3z) = |z|^2 + 1 - 2\,\text{Re}\,(z)$$

$$\Rightarrow \quad 4\,\text{Re}\,(z) = 8$$

$$\Rightarrow \quad \text{Re}\,(z) = 2$$

Case II If $|z-3|=|z-5|$

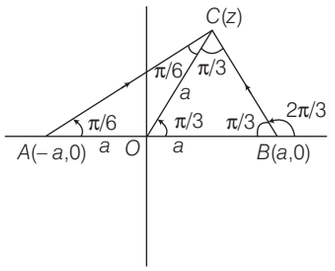
On squaring both sides, we get

$$|z-3|^2=|z-5|^2$$

$$\Rightarrow \quad |z|^2+9-2\,\text{Re}\,(3z)=|z|^2+25-2\,\text{Re}\,(5z)$$

$$\Rightarrow \quad 4\,\text{Re}\,(z)=16 \Rightarrow \text{Re}\,(z)=4$$

39.



From figure, it is clear that z lies on the point of intersection of the rays from A and B.

$\because \angle ACB=90°$ and OBC is an equilateral triangle.

Hence, $OC=a$

$$\Rightarrow \quad |z-0|=a \text{ or } |z|=a$$

and $\arg(z)=\arg(z-0)=\dfrac{\pi}{3}$

40. $\because$ $\left|\dfrac{2z-i}{z+i}\right|=m$

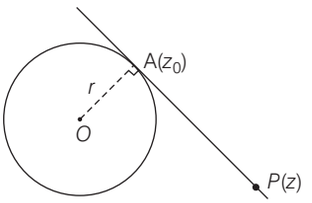
$$\Rightarrow \quad \left|\frac{z-i/2}{z+i}\right|=\frac{m}{2}$$

For circle, $\dfrac{m}{2}\neq 1$

$$\Rightarrow \quad m\neq 2 \text{ and } m>0$$

41. $\therefore$ $A(z_0)$ lie on $|z|=r$

$$\Rightarrow \quad |z_0|=r \Rightarrow |z_0|^2=r^2 \Rightarrow z_0\bar{z}_0=r^2$$



Let $P(z)$ be any point on tangent, then

$$\therefore \quad \angle PAO=\frac{\pi}{2}$$

Complex slope of AP + Complex slope of $OA=0$

$$\Rightarrow \quad \frac{z-z_0}{\bar{z}-\bar{z}_0}+\frac{z_0-0}{\bar{z}_0-0}=0$$

$$\Rightarrow \quad z\bar{z}_0+z_0\bar{z}=2z_0\bar{z}_0$$

$$\Rightarrow \quad z\bar{z}_0+z_0\bar{z}=2r^2$$

$$\Rightarrow \quad z\bar{z}_0=\bar{z}z_0$$

Also, $\dfrac{z\bar{z}_0}{r^2}+\dfrac{z_0\bar{z}}{r^2}=2$

$$\Rightarrow \quad \frac{z\bar{z}_0}{z_0\bar{z}_0}+\frac{z_0\bar{z}}{z_0\bar{z}_0}=2$$

$$\Rightarrow \quad \frac{\dfrac{z}{z_0}+\left(\dfrac{\bar{z}}{z_0}\right)}{2}=1$$

$$\therefore \quad \text{Re}\left(\frac{z}{z_0}\right)=1$$

42. $\because$ $z_1+z_2=a, z_1z_2=b$

and given $|z_1|=|z_2|=1$

Let $z_1=e^{i\theta}$ and $z_2=e^{i\phi}$

$\because$ $|a|=|z_1+z_2|\leq|z_1|+|z_2|=1+1=2$

$\therefore$ $|a|\leq 2$

Also, $\arg(a)=\arg(z_1+z_2)=\arg(e^{i\theta}+e^{i\phi})=\dfrac{\theta+\phi}{2}$

and $\arg(b)=\arg(z_1z_2)=\arg(e^{i\theta+\phi})=\theta+\phi$

$\therefore$ $2\arg(a)=\arg(b) \Rightarrow \arg(a^2)=\arg(b)$

43. $\because$ $\alpha z^2+z+\bar{\alpha}=0$ ...(i)

Then, $\overline{\alpha z^2+z+\bar{\alpha}=0}$

$$\Rightarrow \quad \bar{\alpha}(\bar{z})^2+\bar{z}+\alpha=0$$

$$\Rightarrow \quad \bar{\alpha}z^2+z+\alpha=0 \qquad [\because \bar{z}=z] \ldots(ii)$$

On subtracting Eq. (ii) from Eq. (i), we get

$$(\alpha-\bar{\alpha})z^2-(\alpha-\bar{\alpha})=0$$

$$\Rightarrow \quad \alpha-\bar{\alpha}=0 \text{ and } z^2=1$$

$$\therefore \quad \alpha=\bar{\alpha} \text{ and } z=\pm 1$$

Put $z=\pm 1$ in Eq. (i), we get

$$\alpha+\bar{\alpha}=\pm 1$$

and absolute value of real root = 1

i.e., $|z|=|\pm 1|=1$

44. Let $z=\alpha$ be a real root of equation

$$z^3+(3+i)z^2-3z-(m+i)=0$$

$$\Rightarrow \quad \alpha^3+(3+i)\alpha^2-3\alpha-(m+i)=0$$

$$\Rightarrow \quad (\alpha^3+3\alpha^2-3\alpha-m)+i(\alpha^2-1)=0$$

On comparing real and imaginary parts, we get

$$\alpha^3+3\alpha^2-3\alpha-m=0$$

and $\alpha^2-1=0 \Rightarrow \alpha=\pm 1$

For $\alpha=1$, we get

$$1+3-3-m=0 \Rightarrow m=1$$

For $\alpha=-1$, we get

$$-1+3+3-m=0 \Rightarrow m=5$$

45. Let $z=\alpha$ be a real root of equation

$$z^3+(3+2i)z+(-1+ia)=0$$

$$\Rightarrow \quad \alpha^3+(3+2i)\alpha+(-1+ia)=0$$

$$\Rightarrow \quad (\alpha^3+3\alpha-1)+i(a+2\alpha)=0$$

On comparing real and imaginary parts, we get

$$\alpha^3+3\alpha-1=0$$

and $a+2\alpha=0$

$\Rightarrow \quad \alpha = -\frac{a}{2}$

$\Rightarrow \quad -\frac{a^3}{8} - \frac{3a}{2} - 1 = 0 \Rightarrow a^3 + 12a + 8 = 0$

Let $f(a) = a^3 + 12a + 8$

$\therefore \quad f(-1) < 0,\ f(0) > 0,\ f(-2) < 0$

$f(1) > 0$ and $f(3) > 0$

$\Rightarrow \quad a \in (-2,1)$ or $a \in (-1,0)$ or $a \in (-2,3)$

Sol. (Q. Nos. 46 to 48)

46. $\because \ \arg(z) > 0$

$\therefore \quad \arg(\bar{z}) + \arg(-z) = -\pi$

$\Rightarrow \quad -\arg(z) + \arg(-z) = -\pi$

$\Rightarrow \quad \arg(-z) - \arg(z) = -\pi$

47. $\because \ \arg(z_1 z_2) = \pi$

$\Rightarrow \quad \arg(z_1) + \arg(z_2) = \pi$

$\Rightarrow \quad \arg(z_1) - \arg(\bar{z}_2) = \pi$

Given, $|z_1| = |z_2|$

$\therefore \quad |z_1| = |\bar{z}_2| = |z_2|$

Then, $z_1 + \bar{z}_2 = 0$

$\Rightarrow \quad z_1 = -\bar{z}_2$

48. $\arg(4z_1) - \arg(5z_2) = \pi$

is possible only when $|4z_1| = |5z_2|$

$\Rightarrow \quad \left|\frac{z_1}{z_2}\right| = \frac{5}{4} = 1.25$

and also $4z_1 + 5z_2 = 0$

$\Rightarrow \quad \frac{z_1}{z_2} = -\frac{5}{4}$

$\therefore \quad \left|\frac{z_1}{z_2}\right| = \frac{5}{4} = 1.25$

Sol. (Q. Nos. 49 to 51)

49. $\because \ n!$ is divisible by 4, $\forall\ n \geq 4$.

$\therefore \quad \sum_{n=4}^{25} i^{n!} = \sum_{n=1}^{22} i^{(n+3)!}$

$= i^0 + i^0 + i^0 + \ldots (22 \text{ times}) = 22 \qquad \ldots(i)$

$\therefore \quad \sum_{n=1}^{25} i^{n!} = i^{1!} + i^{2!} + i^{3!} + \sum_{n=4}^{25} i^{n!}$

$= i + i^2 + i^6 + 22 \qquad$ [from Eq. (i)]

$= i - 1 - 1 + 22 = 20 + i$

$\therefore \quad a = 20,\ b = 1$

$\therefore \quad a - b = 20 - 1 = 19$

which is a prime number.

50. $\therefore \ \sum_{r=-2}^{95} i^r + \sum_{r=0}^{50} i^{r!}$

$= \sum_{r=1}^{98} i^{r-3} + \left(i^{0!} + i^{1!} + i^{2!} + i^{3!} + \sum_{r=4}^{50} i^{r!} \right)$

$= (i^{-2} + i^{-1} + 0) + \left(i^1 + i^1 + i^2 + i^6 + \sum_{r=1}^{47} i^{(r+3)!} \right)$

$= (-1 - i) + (i + i - 1 - 1$

$+ (i^0 + i^0 + i^0 + \ldots 47 \text{ times}))$

$= (-1 - i) + (2i - 2 + 47)$

$= 44 + i = a + ib \qquad$ [given]

$\therefore \quad a = 44,\ b = 1$

Unit place digit of $a^{2011} = (44)^{2011}$

$= (44)\left((44)^2\right)^{1005} = (44)(1936)^{1005}$

= (Unit place of 44)

$\times$ (Unit place digit of $(1936)^{1005}$)

= Unit place of $(4 \times 6) = 4$

and unit place digit of $b^{2012} = (1)^{2012} = 1$

Hence, the unit place digit of $a^{2011} + b^{2012} = 4 + 1 = 5$.

51. $\because \ \sum_{r=4}^{100} i^{r!} + \prod_{r=1}^{101} i^r$

$= \sum_{r=1}^{97} i^{(r+3)!} + i^1 \cdot i^2 \cdot i^3 \ldots i^{101}$

$= (i^0 + i^0 + i^0 + \ldots 97 \text{ times}) + i^{1+2+3+\ldots+101}$

$= 97 + i^{5151} = 97 + i^3 = 97 - i$

$\therefore \quad a = 97$ and $b = -1$

Hence, $a + 75b = 97 - 75 = 22$

Sol. (Q. Nos. 52 to 54)

If $\left|z \pm \frac{a}{z}\right| = b$, where $a, b > 0$

$\therefore \quad \left|z \pm \frac{a}{z}\right| \leq |z| + \frac{a}{|z|}$

$\Rightarrow \quad b \leq |z| + \frac{a}{|z|}$

$\Rightarrow \quad |z|^2 - b|z| + a \geq 0$

$\therefore \quad |z| \leq \frac{b - \sqrt{b^2 - 4a}}{2}$

and $|z| \geq \frac{b + \sqrt{b^2 - 4a}}{2} \qquad \ldots(i)$

Also, $\left|z \pm \frac{a}{z}\right| \geq \left||z| - \frac{a}{|z|}\right|$

$\Rightarrow \quad b \geq \left||z| - \frac{a}{|z|}\right|$

$\Rightarrow \quad -b \leq |z| - \frac{a}{|z|} \leq b$

$\Rightarrow \quad -b|z| \leq |z|^2 - a \leq b|z|$

***Case* I** $-b|z| \le |z|^2 - a$

$\Rightarrow \quad |z|^2 + b|z| - a \ge 0$

$\therefore \qquad |z| \le \dfrac{-b - \sqrt{b^2 + 4a}}{2}$

and $\qquad |z| \ge \dfrac{-b + \sqrt{b^2 - 4a}}{2}$

***Case* II** $|z|^2 - a \le b|z|$

$\Rightarrow \quad |z|^2 - b|z| - a \le 0$

$\therefore \quad \dfrac{b - \sqrt{b^2 + 4a}}{2} \le |z| \le \dfrac{b + \sqrt{b^2 + 4a}}{2}$

From Case I and Case II, we get

$$\frac{-b + \sqrt{b^2 + 4a}}{2} \le |z| \le \frac{b + \sqrt{b^2 + 4a}}{2} \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{-b + \sqrt{b^2 + 4a}}{2} \le |z| \le \frac{b + \sqrt{(b^2 + 4a)}}{2}$$

$\therefore$ The greatest value of $|z|$ is $\dfrac{b + \sqrt{b^2 + 4a}}{2}$

and the least value of $|z|$ is $\dfrac{-b + \sqrt{b^2 + 4a}}{2}$.

52. Here, $a = 1$ and $b = 2$

$\lambda =$ Sum of the greatest and least values of $|z|$

$= \sqrt{b^2 + 4a} = \sqrt{4 + 4} = \sqrt{8}$

$\therefore \quad \lambda^2 = 8$

53. Here, $a = 2$ and $b = 4$

$\lambda =$ Sum of the greatest and least value of $|z|$.

$= \sqrt{b^2 + 4a} = \sqrt{16 + 8} = \sqrt{24}$

$\therefore \qquad \lambda^2 = 24$

54. Here, $a = 3$ and $b = 6$

$\lambda =$ Sum of the greatest and least value of $|z|$

$= \sqrt{b^2 + 4a} = \sqrt{36 + 12} = \sqrt{48} = 4\sqrt{3}$

$\Rightarrow \quad \lambda = 2\sqrt{3}$

$\therefore \quad \lambda^2 = 12$

Sol. (Q. Nos. 55 to 57)

$\because \quad W = \dfrac{z - 1}{z + 2} = a + ib$

55. $\because \quad z = CiS\,\theta = e^{i\theta}$

$\therefore \quad \dfrac{e^{i\theta} - 1}{e^{i\theta} + 2} = a + ib$

$\Rightarrow (\cos\theta + i\sin\theta - 1) = (a + ib)(\cos\theta + i\sin\theta + 2)$

On comparing real and imaginary parts, we get

$$\cos\theta - 1 = a\cos\theta + 2a - b\sin\theta$$

$\Rightarrow \quad (1 - a)\cos\theta + b\sin\theta = 2a + 1 \qquad \text{...(i)}$

and $\sin\theta = a\sin\theta + b\cos\theta + 2b$

$$(1 - a)\sin\theta - b\cos\theta = 2b \qquad \text{...(ii)}$$

On squaring and adding Eqs. (i) and (ii), we get

$$(1 - a)^2 + b^2 = (2a + 1)^2 + (2b)^2$$

$\Rightarrow \qquad 3a^2 + 3b^2 + 6a = 0$

$\Rightarrow \qquad a^2 + b^2 + 2a = 0$

From option (c),

$$(1 + 5a)^2 + (3b)^2 = (1 - 4a)^2$$

$\Rightarrow \qquad 9a^2 + 9b^2 + 18a = 0$

$\therefore \qquad a^2 + b^2 + 2a = 0$

56. From Eq. (i), we get

$$(1 - a)\left(\frac{1 - \tan^2\theta/2}{1 + \tan^2\theta/2}\right) + b\left(\frac{2\tan\theta/2}{1 + \tan^2\theta/2}\right) = 2a + 1$$

$$\Rightarrow \quad (1 - a) - (1 - a)\tan^2\frac{\theta}{2} + 2b\tan\frac{\theta}{2} = (2a + 1) + (2a + 1)\tan^2\frac{\theta}{2}$$

$$\Rightarrow \quad (2 + a)\tan^2\frac{\theta}{2} - 2b\tan\frac{\theta}{2} + 3a = 0$$

$$\therefore \qquad \tan\frac{\theta}{2} = \frac{2b \pm \sqrt{4b^2 - 12a(2 + a)}}{2(2 + a)}$$

$$= \frac{2b \pm \sqrt{4b^2 - 12(-b^2)}}{-2b^2/a} \qquad [\because a^2 + b^2 + 2a = 0]$$

$$= \frac{(2b \pm 4b)\,a}{-2b^2} = \frac{6ba}{-2b^2} \text{ or } \frac{-2ab}{-2b^2} = -\frac{3a}{b} \text{ or } \frac{a}{b}$$

$$\therefore \qquad \cot\frac{\theta}{2} = -\frac{b}{3a} \text{ or } \frac{b}{a} \quad \text{or} \quad -\frac{b}{a} = 3\cot\frac{\theta}{2} \text{ or } -\cot\frac{\theta}{2}$$

57. $\therefore \quad a^2 + b^2 + 2a = 0 \Rightarrow (a + 1)^2 + b^2 = 1$

Now, $|z| = 1 = (a + 1)^2 + b^2$

58. $\because \qquad 1 + z + z^2 + z^3 + \ldots + z^{17} = 0$

$$\therefore \qquad \frac{1 \cdot (1 - z^{18})}{(1 - z)} = 0$$

$$\Rightarrow \qquad 1 - z^{18} = 0, 1 - z \ne 0$$

$$\therefore \qquad z^{18} = 1, z \ne 1 \qquad \text{...(i)}$$

and $\quad 1 + z + z^2 + z^3 + \ldots + z^{13} = 0$

$$\therefore \qquad \frac{1 \cdot (1 - z^{14})}{(1 - z)} = 0$$

$$\Rightarrow \qquad 1 - z^{14} = 0, 1 - z \ne 0$$

$$\therefore \qquad z^{14} = 1, z \ne 1 \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$z^{14} \cdot z^4 = 1 \Rightarrow 1 \cdot z^4 = 1$$

$$\therefore \qquad z^4 = 1$$

Then, $\qquad z = 1, -1, i, -i$

$\because \qquad z \ne 1$

$\because \qquad z = -1, i, -i$

Hence, only $z = -1$ satisfy both Eqs. (i) and (ii).

$\therefore$ Number of values of z is 1.

59. We have, $z^3 = \bar{z}$ ...(i)

$\Rightarrow$ $|z|^3 = |\bar{z}| = |z|$

$\Rightarrow$ $|z|(|z|^2 - 1) = 0$

$\Rightarrow$ $|z| = 0$ and $|z|^2 = 1$

Now, $|z|^2 = 1$

$\Rightarrow$ $z\bar{z} = 1 \Rightarrow \bar{z} = \dfrac{1}{z}$

On putting this value in Eq. (i), we get

$$z^3 = \frac{1}{z}$$

$\Rightarrow$ $z^4 = 1$ ...(ii)

Clearly, Eq. (ii) has 4 solutions.

Therefore, the required number of solutions is 5.

60. We have, $z = 9 + ai$

$\Rightarrow$ $z^2 = (81 - a^2) + 18ai$

$z^3 = (729 - 27a^2) + (243a - a^3)\, i$

According to the question, we have

$\text{Im}(z^2) = \text{Im}(z^3)$

$\Rightarrow$ $18a = 243a - a^3 \Rightarrow a(a^2 - 225) = 0$

$\Rightarrow$ $a = 0$ or $a^2 = 225$

But $a \neq 0$

$\therefore$ $a^2 = 225$

$\therefore$ The sum of digits of $a^2 = 2 + 2 + 5 = 9$

61. Let $z = x + iy$

$\because$ $|z| = 1$ ...(i)

$\therefore$ $x^2 + y^2 = 1$

and $\left|\dfrac{z}{\bar{z}} + \dfrac{\bar{z}}{z}\right| = 1$

$\Rightarrow$ $\left|\dfrac{x + iy}{x - iy} + \dfrac{x - iy}{x + iy}\right| = 1$

$\Rightarrow$ $\left|\dfrac{(x + iy)^2 + (x - iy)^2}{x^2 + y^2}\right| = 1$

$\Rightarrow$ $\left|\dfrac{2(x^2 - y^2)}{1}\right| = 1$ [from Eq. (i)]

$\Rightarrow$ $x^2 - y^2 = \pm\dfrac{1}{2}$ ...(ii)

From Eqs. (i) and (ii), we get

$$2x^2 = 1 \pm \frac{1}{2} = \frac{1}{2}, \frac{3}{2}$$

$\Rightarrow$ $x^2 = \dfrac{1}{4}, \dfrac{3}{4} \Rightarrow x = \pm\dfrac{1}{2}, \pm\dfrac{\sqrt{3}}{2}$

For $x = \dfrac{1}{2}, y = \pm\dfrac{\sqrt{3}}{2}$ [from Eq. (i)]

For $x = -\dfrac{1}{2}, y = \pm\dfrac{\sqrt{3}}{2}$ [from Eq. (i)]

For $x = \dfrac{\sqrt{3}}{2}, y = \pm\dfrac{1}{2}$ [from Eq. (i)]

For $x = -\dfrac{\sqrt{3}}{2}, y = \pm\dfrac{1}{2}$ [from Eq. (i)]

$\therefore$ Solutions are $\dfrac{1}{2} \pm \dfrac{i\sqrt{3}}{2}, -\dfrac{1}{2} \pm \dfrac{i\sqrt{3}}{2}, \dfrac{\sqrt{3}}{2} \pm \dfrac{i}{2}, -\dfrac{\sqrt{3}}{2} \pm \dfrac{i}{2}$

Hence, number of solutions is 8.

62. We have, $x = a + ib$

$\Rightarrow$ $x^2 = (a^2 - b^2) + 2iab = 3 + 4i$ [given]

$\therefore$ $a^2 - b^2 = 3$ and $ab = 2$ ...(i)

and $x^3 = x \cdot x^2 = (a + ib)[(a^2 - b^2) + 2iab]$

$= (a^3 - ab^2 - 2ab^2) + i[2a^2b + b(a^2 - b^2)]$

$= (a^3 - 3ab^2) + i(3a^2b - b^3) = 2 + 11i$ [given]

$\therefore$ $a^3 - 3ab^2 = 2$

and $3a^2b - b^3 = 11$ ...(ii)

From Eq. (i), we get

$$a^2 + b^2 = \sqrt{(a^2 - b^2)^2 + 4a^2b^2} = 5$$

Then, $2a^2 = 8, 2b^2 = 2$

$\therefore$ $a^2 = 4, b^2 = 1$

$\Rightarrow$ $a = 2, b = 1$

and $a = -2, b = -1$ [$\because ab = 2$]

Finally, $a = 2, b = 1$ satisfies Eq. (ii).

Hence, $a + b = 2 + 1 = 3$

63. $\because$ $(1 + i)^4 = [(1 + i)^2]^2$

$= (1 + i^2 + 2i)^2 = (1 - 1 + 2i)^2$

$= 4i^2 = -4$ ...(i)

and $\dfrac{1 - \sqrt{\pi}\, i}{\sqrt{\pi} + i} + \dfrac{\sqrt{\pi} - i}{1 + \sqrt{\pi}\, i}$

$= \dfrac{(1 - \sqrt{\pi}\, i)(\sqrt{\pi} - i)}{\pi + 1} + \dfrac{(\sqrt{\pi} - i)(1 - \sqrt{\pi}\, i)}{1 + \pi}$

$= \dfrac{\sqrt{\pi} - i - \pi i - \sqrt{\pi} + \sqrt{\pi} - \pi i - i - \sqrt{\pi}}{\pi + 1}$

$= \dfrac{-2\pi i - 2i}{\pi + 1} = -2i$ ...(ii)

Given, $z = \dfrac{\pi}{4}(1 + i)^4 \left(\dfrac{1 - \sqrt{\pi}\, i}{\sqrt{\pi} + i} + \dfrac{\sqrt{\pi} - i}{1 + \sqrt{\pi}\, i}\right)$

$= \dfrac{\pi}{4}(-4)(-2i) = 2\pi i$ [from Eqs. (i) and (ii)]

Now, $\left(\dfrac{|z|}{\text{amp}\ (z)}\right) = \dfrac{2\pi}{\pi/2} = 4$

64. $\because$ $A^n = 1$

$\Rightarrow$ $A = (1)^{1/n} = e^{2\pi r i/n}, r = 0, 1, 2, \ldots, n - 1$

$\therefore$ $A = 1, e^{2\pi i/n}, e^{4\pi i/n}, e^{6\pi i/n}, \ldots, e^{2\pi(n-1)i/n}$

and $(A + 1)^n = 1 \Rightarrow A + 1 = (1)^{1/n} = e^{2\pi pi/n}$

$$\Rightarrow \quad A = e^{2\pi pi/n} - 1 = e^{p\pi i/n} \cdot 2i \sin\left(\frac{\pi p}{n}\right),$$

$$p = 0, 1, 2, \ldots, n-1$$

$$\therefore \quad A = 0, e^{\pi i/n} \cdot 2i \sin\left(\frac{\pi}{n}\right), e^{2\pi i/n}\, 2i \sin\left(\frac{2\pi}{n}\right), \ldots,$$

$$e^{\pi i\,(n-1)/n} \cdot 2i \sin\left(\frac{\pi\,(n-1)}{n}\right)$$

For $n = 6$,

$$e^{4\pi i/n} = e^{4\pi i/6} = e^{2\pi i/3}$$

$$= \cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$

and $e^{\pi i/n} \cdot 2i \sin\left(\frac{\pi}{n}\right) = e^{\pi i/6} \cdot 2i \sin\left(\frac{\pi}{6}\right)$

$$= \left(\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right) \cdot i$$

$$= \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) i = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$

Hence, the least value of n is 6.

65. Given, $z_1, z_2, z_3, \ldots, z_{50}$ are the roots of the equation

$$\sum_{r=0}^{50} (z)^r = 0, \text{ then}$$

$$\sum_{r=0}^{50} (z)^r = (z - z_1)(z - z_2)(z - z_3)\ldots(z - z_{50}) = \prod_{r=1}^{50} (z - z_r)$$

Taking log on both sides on base e, we get

$$\log_e\left(\sum_{r=0}^{50} (z)^r\right) = \sum_{r=1}^{50} \log_e (z - z_r)$$

On differentiating both sides w.r.t. z, we get

$$\frac{\sum_{r=0}^{50} r(z)^{r-1}}{\sum_{r=0}^{50} (z)^r} = \sum_{r=1}^{50} \frac{1}{(z - z_r)}$$

On putting $z = 1$ in both sides, we get

$$\frac{\sum_{r=0}^{50} r}{\sum_{r=0}^{50} 1} = \sum_{r=1}^{50} \frac{1}{(1 - z_r)}$$

$$\Rightarrow \quad \frac{(1 + 2 + 3 + \ldots + 50)}{51} = -\sum_{r=1}^{50} \frac{1}{(z_r - 1)}$$

$$= -(-5\lambda) \qquad \text{[given]}$$

$$\Rightarrow \quad \frac{\frac{50}{2} \times 51}{51} = 5\lambda$$

$$\Rightarrow \quad \lambda = 5$$

66. $\because \sum_{q=1}^{10}\left(\sin\frac{2q\pi}{11} - i\cos\frac{2q\pi}{11}\right)$

$$= -i \sum_{q=1}^{10}\left(\cos\frac{2q\pi}{11} + i\sin\frac{2q\pi}{11}\right)$$

$$= -i\left\{\sum_{q=0}^{10}\left(\cos\frac{2q\pi}{11} + i\sin\frac{2q\pi}{11}\right) - 1\right\}$$

$$= -i\,\{(\text{sum of 11, 11th roots of unity}) - 1\}$$

$$= -i\,(0 - 1) = i$$

$$\because \quad P = \sum_{p=1}^{32} (3p + 2)\left(\sum_{q=1}^{10}\left(\sin\frac{2q\pi}{11} - i\cos\frac{2q\pi}{11}\right)\right)^p$$

$$= \sum_{p=1}^{32} (3p + 2)(i)^p$$

$$= 3\sum_{p=1}^{32} p(i)^p + 2\sum_{p=1}^{32} (i)^p$$

$$= 3\sum_{p=1}^{32} p(i)^p + 0 = 3S \text{ (say)}$$

where, $S = \sum_{p=1}^{32} p(i)^p$

$$S = 1 \cdot i + 2 \cdot i^2 + 3 \cdot i^3 + \ldots + 31 \cdot i^{31} + 32 \cdot i^{32}$$

$$iS = 1 \cdot i^2 + 2 \cdot i^3 + \ldots + 31 \cdot i^{32} + 32i^{33}$$

$$(1 - i)\,S = (i + i^2 + i^3 + \ldots + i^{32}) - 32i^{33}$$

$$= (0) - 32i$$

$$\therefore \quad S = -\frac{32i \cdot (1 + i)}{(1 - i) \cdot (1 + i)}$$

$$= -16\,(i - 1) = 16\,(1 - i)$$

$$\therefore \quad P = 3S = 48\,(1 - i)$$

Given, $(1 + i)\,P = n(n!) \Rightarrow (1 + i) \cdot 48\,(1 - i) = n(n!)$

$$\Rightarrow \quad 96 = n(n!) \Rightarrow 4(4!) = n(n!)$$

$$\therefore \quad n = 4$$

67. $\because \quad \frac{1 + i}{1 - i} = \frac{(1 + i)^2}{(1 - i)(1 + i)} = \frac{1 + i^2 + 2i}{2} = i$

Given, $\left(\frac{1 + i}{1 - i}\right)^n = \frac{2}{\pi}\sin^{-1}\left(\frac{1 + x^2}{2x}\right)$

$$\Rightarrow \quad i^n = \frac{2}{\pi}\sin^{-1}\left(\frac{1 + x^2}{2x}\right)$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{1 + x^2}{2x}\right) = \frac{\pi}{2}(i)^n$$

$$\Rightarrow \quad \frac{1 + x^2}{2x} = \sin\left(\frac{\pi}{2}(i)^n\right) \qquad \ldots(i)$$

Now, $\text{AM} \geq \text{GM}$

$$\frac{x + \frac{1}{x}}{2} \geq 1 \Rightarrow \frac{x^2 + 1}{2x} \geq 1$$

$$\Rightarrow \quad \sin\left(\frac{\pi}{2}(i)^n\right) \geq 1 \qquad [\because -1 \leq \sin\theta \leq 1]$$

$$\therefore \quad \sin\left(\frac{\pi}{2}(i)^n\right) = 1$$

$$\Rightarrow \quad n = 4, 8, 12, 16, \ldots$$

$\therefore$ Least positive integer, $n = 4$

68. (A) $\rightarrow (p,q)$, (B) $\rightarrow (p,r)$,(C) $\rightarrow (p,r,s)$

If $\left|z \pm \frac{a}{z}\right| = b$, where $a > 0$ and $b > 0$, then

$$\frac{-b+\sqrt{b^2+4a}}{2} \le |z| \le \frac{b+\sqrt{b^2+4a}}{2}$$

(A) Here, $a = 1$ and $b = 2$

Then, $-1+\sqrt{2} \le |z| \le 1+\sqrt{2}$

$\therefore \quad G = 1+\sqrt{2}$

and $\quad L = -1+\sqrt{2}$

$\Rightarrow \quad G - L = 2$ [natural number and prime number]

(B) Here, $a = 2$ and $b = 4$

Then, $-2+\sqrt{6} \le |z| \le 2+\sqrt{6}$

$\therefore \quad G = 2+\sqrt{6}$

and $\quad L = -2+\sqrt{6}$

$\Rightarrow \quad G - L = 4$ [natural number and composite number]

(C) Here, $a = 3$ and $b = 6$

Then, $-3+2\sqrt{3} \le |z| \le 3+2\sqrt{3}$

$\therefore \quad G = 3+2\sqrt{3}$

and $\quad L = -3+2\sqrt{3}$

$\Rightarrow \quad G - L = 6$

[natural number, composite number and perfect number]

69. (A) $\rightarrow$ (q), B $\rightarrow$ (q, r), C $\rightarrow$ (q, s)

We know that,

$$\sqrt{z} = \pm\left(\sqrt{\frac{|z|-\mathrm{Re}(z)}{2}} + i\sqrt{\frac{|z|-\mathrm{Re}(z)}{2}}\right)$$

If $\mathrm{Im}(z) > 0 = \pm\left(\sqrt{\frac{|z|+\mathrm{Re}(z)}{2}} - i\sqrt{\frac{|z|-\mathrm{Re}(z)}{2}}\right)$

If $\mathrm{Im}(z) < 0$

(A) $\sqrt{6+8i} = \pm\left(\sqrt{\frac{10+6}{2}} + i\sqrt{\frac{10-6}{2}}\right)$

$= \pm(2\sqrt{2} + i\sqrt{2})$

$= \pm\sqrt{2}(2+i)$

and $\sqrt{-6+8i} = \pm\left(\sqrt{\frac{10-6}{2}} + i\sqrt{\frac{10+6}{2}}\right)$

$= \pm(\sqrt{2} + i2\sqrt{2}) = \pm\sqrt{2}(1+2i)$

$\therefore \quad z = \sqrt{6+8i} + \sqrt{-6+8i}$

$= \pm\sqrt{2}(2+i) \pm \sqrt{2}(1+2i)$

$= 3\sqrt{2}(1+i), \sqrt{2}(1-i), -3\sqrt{2}(1+i), \sqrt{2}(-1+i)$

$\therefore \quad z_1 = 3\sqrt{2}(1+i), z_2 = \sqrt{2}(1-i),$

$z_3 = -3\sqrt{2}(1+i)$

and $\quad z_4 = \sqrt{2}(-1+i)$

$\therefore \ |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2$

$= 36 + 4 + 36 + 4 = 80$ which is divisible by 8.

(B) $\sqrt{5-12i} = \pm\left(\sqrt{\frac{13+5}{2}} - i\sqrt{\frac{13-5}{2}}\right) = \pm(3-2i)$

and $\sqrt{-5-12i} = \pm\left(\sqrt{\frac{13-5}{2}} - i\sqrt{\frac{13+5}{2}}\right) = \pm(2-3i)$

$\therefore \quad z = \sqrt{5-12i} + \sqrt{-5-12i} = \pm(3-2i) \pm (2-3i)$

$= 5-5i, -1-i, -5+5i, 1+i$

$\therefore \quad z_1 = 5-5i, \ z_2 = -1-i,$

$z_3 = -5+5i$ and $z_4 = 1+i$

$\therefore \ |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 = 50 + 2 + 50 + 2$

$= 104 = 8 \times 13$

(C) $\sqrt{8+15i} = \pm\left(\sqrt{\frac{17+8}{2}} + i\sqrt{\frac{17-8}{2}}\right)$

$= \pm\left(\frac{5}{\sqrt{2}} + \frac{3i}{\sqrt{2}}\right) = \pm\frac{1}{\sqrt{2}}(5+3i)$

and $\sqrt{-8-15i} = \pm\left(\sqrt{\frac{17-8}{2}} - i\sqrt{\frac{17+8}{2}}\right)$

$= \pm\left(\frac{3}{\sqrt{2}} - \frac{5}{\sqrt{2}}i\right) = \pm\frac{1}{\sqrt{2}}(3-5i)$

$\therefore \quad z = \sqrt{8+15i} + \sqrt{-8-15i}$

$= \pm\frac{1}{\sqrt{2}}(5+3i) \pm \frac{1}{\sqrt{2}}(3-5i)$

$z = \frac{1}{\sqrt{2}}(8-2i), \frac{1}{\sqrt{2}}(-2-8i),$

$\frac{1}{\sqrt{2}}(-8+2i), \frac{1}{\sqrt{2}}(2+8i)$

$\therefore \quad z_1 = \sqrt{2}(4-i), \ z_2 = \sqrt{2}(-1-4i)$

$z_3 = \sqrt{2}(-4+i)$ and $z_4 = \sqrt{2}(1+4i)$

$\therefore \ |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 = 34 + 34 + 34 + 34$

$= 136 = 17 \times 8$

70. (A) $\rightarrow$ (p, q, r, t);(B) $\rightarrow$ (p,s);(C) $\rightarrow$ (p, r)

(A) Here, the last digit of 143 is 3. The remainder when 861 is divided by 4 is 1. Then, press switch number 1 and we get 3. Hence, the digit in the units place of $(143)^{861}$ is 3.

$\therefore \quad \lambda = 3$

Next, the last digit of 5273 is 3. The remainder when 1358 is divided by 4 is 2. Then, press switch number 2 and we get 9. Hence, the digit in the units place of $(5273)^{1358}$ is 9.

$\therefore \quad \mu = 9$

Hence, $\quad \lambda + \mu = 3 + 9 = 12$

which is divisible by 2, 3, 4 and 6.

(B) Here, the last digit of 212 is 2. The remainder when 7820 is divided by 4 is 0. Then, press switch number 0 and we get 6. Hence, the digit in the unit's place of $(212)^{7820}$ is 6.

$\therefore \quad \lambda = 6$

Next, the last digit of 1322 is 2. The remainder when 1594 is divided by 4 is 2. Then, press switch number 2 and we get 4.

Hence, the digit in the unit's place of $(1322)^{1594}$ is 4.

$\therefore \quad \mu = 4$

Hence, $\lambda + \mu = 6 + 4 = 10$, which is divisible by 2 and 5.

(C) Here, the last digit of 136 is 6. Therefore, the unit's place of $(136)^{786}$ is 6.

$\therefore \quad \lambda = 6$

Next, the last digit of 7138 is 8. The remainder when 13491 is divided by 4 is 3. Then, press switch number 3 and we get 2. Hence, unit's place of $(7138)^{13491}$ is 2.

$\therefore \quad \mu = 2$

Hence, $\lambda + \mu = 6 + 2 = 8$

which is divisible by 2 and 4.

71. (A) $\rightarrow$ (r); (B) $\rightarrow$ (p,s); (C) $\rightarrow$ (q, t)

If $\left|z - \frac{a}{z}\right| = b$, where $a > 0$ and $b > 0$, then

$$\frac{-b + \sqrt{b^2 + 4a}}{2} \le |z| \le \frac{b + \sqrt{b^2 + 4a}}{2}$$

$$\therefore \quad \lambda = \frac{b + \sqrt{b^2 + 4a}}{2} \text{ and } \mu = \frac{-b + \sqrt{b^2 + 4a}}{2}$$

(A) Here, $a = 6$ and $b = 5$

$\therefore \quad \lambda = 6$ and $\mu = 1$

$\Rightarrow \quad \lambda^{\mu} + \mu^{\lambda} = 6^1 + 1^6 = 7$

and $\quad \lambda^{\mu} - \mu^{\lambda} = 6^1 - 1^6 = 5$

(B) Here, $a = 7$ and $b = 6$

$\therefore \quad \lambda = 7$ and $\mu = 1$

$\therefore \quad \lambda^{\mu} + \mu^{\lambda} = 7^1 + 1^7 = 8$

and $\quad \lambda^{\mu} - \mu^{\lambda} = 7^1 - 1^7 = 6$

(C) Here, $a = 8$ and $b = 7$

$\therefore \quad \lambda = 8$ and $\mu = 1$

$\Rightarrow \quad \lambda^{\mu} + \mu^{\lambda} = 8^1 + 1^8 = 9$

and $\quad \lambda^{\mu} - \mu^{\lambda} = 8^1 - 1^8 = 7$

72. Statement-1 is false because $3 + 7i > 2 + 4i$ is meaningless in the set of complex number as set of complex number does not hold ordering. But Statement-2 is true.

73. Statement-1 is false as

$$(\cos\theta + i \sin\phi)^n \neq \cos n\theta + i \sin n\phi$$

Now, $\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right)^2 = \cos\frac{\pi}{2} + i \sin\frac{\pi}{2}$

$= i$ [by De-Moivre's theorem]

$\therefore$ Statement-2 is true.

74. We have,

$$|3z_1 + 1| = |3z_2 + 1| = |3z_3 + 1|$$

$\therefore z_1, z_2$ and z_3 are equidistant from $\left(-\frac{1}{3}, 0\right)$ and circumcentre of triangle is $\left(-\frac{1}{3}, 0\right)$.

Also, $\quad 1 + z_1 + z_2 + z_3 = 0$

$$\Rightarrow \quad \frac{1 + z_1 + z_2 + z_3}{3} = 0$$

$$\Rightarrow \quad \frac{z_1 + z_2 + z_3}{3} = -\frac{1}{3}$$

$\therefore$ Centroid of the triangle is $\left(-\frac{1}{3}, 0\right)$.

So, the circumcentre and centroid of the triangle coincide. Hence, required triangle is an equilateral triangle.

Therefore, Statement-1 is true. Also, z_1, z_2 and z_3 represent vertices of an equilateral triangle, if $z_1^2 + z_2^2 + z_3^2 - (z_1z_2 + z_2z_3 + z_3z_1) = 0$.

Therefore, Statement-2 is false.

75. We have,

$$|z - 1| + |z - 8| = 5 \quad \ldots(i)$$

Here, $\quad z_1 = 1, z_2 = 8$ and $2a = 5$

Now, $\quad |z_1 - z_2| = |1 - 8| = |-7| = 7$

$\therefore \quad 2a = 5 < 7$

Therefore, locus of Eq. (i) does not represent an ellipse. Hence, Statement-1 is false. Statement-2 is true by the property of ellipse.

76. Since, z_1, z_2 and z_3 are in AP.

$\therefore \quad 2z_2 = z_1 + z_3$

$$\Rightarrow \quad z_2 = \frac{z_1 + z_3}{2}$$

It is clear that, z_2 is the mid-point of z_1 and z_3.

$\therefore z_1, z_2$ and z_3 are collinear.

Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation of Statement-1.

77. Principal argument of a complex number depend upon quadrant and principal argument lies in $(-\pi, \pi]$.

Hence, Statement-1 is always not true and Statement-2 is obviously true.

78. We have, $\quad C_1 : \arg(z) = \frac{\pi}{4}$

$$\Rightarrow \quad \tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4} \quad [\text{let } z = x + iy]$$

$$\Rightarrow \quad \frac{y}{x} = \tan\frac{\pi}{4} = 1$$

$\Rightarrow \quad y = x$

$\therefore \quad C_1 : y = x \quad \ldots(i)$

$C_2 : \arg(z) = \frac{3\pi}{4}$

$$\Rightarrow \quad \tan^{-1}\left(\frac{y}{x}\right) = \frac{3\pi}{4} \quad [\text{let } z = x + iy]$$

$$\Rightarrow \quad \frac{y}{x} = \tan\frac{3\pi}{4} = -1$$

$\Rightarrow \quad y = -x$

$\therefore \quad C_2 : y = -x \quad \ldots(ii)$

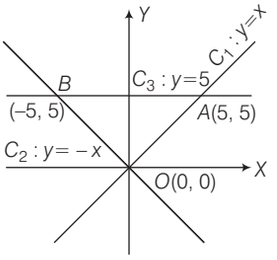
and $C_3 : \arg(z - 5 - 5i) = \pi$

$$\Rightarrow \quad \tan^{-1}\left(\frac{y - 5}{x - 5}\right) = \pi \quad [\text{let } z = x + iy]$$

$$\Rightarrow \quad \frac{y-5}{x-5} = \tan\pi = 0 \quad \Rightarrow y = 5$$

$$\therefore \quad C_3 : y = 5 \qquad \ldots(iii)$$

We get the following figure.



$\therefore$ Area of the region bounded by C_1, C_2 and C_3

$$= \frac{1}{2}\begin{Vmatrix} 5-0 & 5-0 \\ -5-0 & 5-0 \end{Vmatrix} = 25$$

$\therefore$ Statement-1 is false.

Now, $OA = 5\sqrt{2}$, $OB = 5\sqrt{2}$ and $AB = 10$

$\because$ $(OA)^2 + (OB)^2 = (AB)^2$ and $OA = OB$

Therefore, the boundary of C_1, C_2 and C_3 constitutes right isosceles triangle.

Hence, Statement-2 is true.

79. Since, $\text{Im}(\bar{z}_2 z_3) = \dfrac{\bar{z}_2 z_3 - (\overline{\bar{z}_2 z_3})}{2i} = \dfrac{1}{2i}\{\bar{z}_2 z_3 - z_2 \bar{z}_3\}$

$$z_1 \,\text{Im}(\bar{z}_2 z_3) = \frac{1}{2i}\{z_1 \bar{z}_2 z_3 - z_1 z_2 \bar{z}_3\} \qquad \ldots(i)$$

Similarly, $z_2 \,\text{Im}(\bar{z}_3 z_1) = \dfrac{1}{2i}\{z_2 \bar{z}_3 z_1 - z_2 \bar{z}_1 z_3\}$ ...(ii)

and $z_3 \,\text{Im}(\bar{z}_1 z_2) = \dfrac{1}{2i}\{z_3 \bar{z}_1 z_2 - z_3 z_1 \bar{z}_2\}$ ...(iii)

On adding Eqs. (i), (ii) and (iii), we get

$$z_1 \,\text{Im}(\bar{z}_2 z_3) + z_2 \,\text{Im}(\bar{z}_3 z_1) + z_3 \,\text{Im}(\bar{z}_1 z_2) = 0$$

Therefore, this is proved.

80. Since, z_1, z_2 and z_3 are the roots of

$$x^3 + 3ax^2 + 3bx + c = 0,$$

we get $\quad z_1 + z_2 + z_3 = -3a$

$$\Rightarrow \quad \frac{z_1 + z_2 + z_3}{3} = -a$$

and $\quad z_1 z_2 + z_2 z_3 + z_3 z_1 = 3b$

Hence, the centroid of the ΔABC is the point of affix $(-a)$.

Now, the triangle will be equilateral, if

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$$

$$\Rightarrow \quad (z_1 + z_2 + z_3)^2 = 3(z_1 z_2 + z_2 z_3 + z_3 z_1)$$

$$\Rightarrow \quad (-3a)^2 = 3(3b)$$

Therefore, the condition is $a^2 = b$.

81. $\because$ $x^5 - 1 = 0$ has roots $1, \alpha_1, \alpha_2, \alpha_3, \alpha_4$.

$$\therefore \quad (x^5 - 1) = (x-1)(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)$$

$$\Rightarrow \quad \frac{x^5 - 1}{x-1} = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4) \qquad \ldots(i)$$

On putting $x = \omega$ in Eq. (i), we get

$$\frac{\omega^5 - 1}{\omega - 1} = (\omega - \alpha_1)(\omega - \alpha_2)(\omega - \alpha_3)(\omega - \alpha_4)$$

$$\Rightarrow \quad \frac{\omega^2 - 1}{\omega - 1} = (\omega - \alpha_1)(\omega - \alpha_2)(\omega - \alpha_3)(\omega - \alpha_4) \qquad \ldots(ii)$$

and putting $x = \omega^2$ in Eq. (i), we get

$$\frac{\omega^{10} - 1}{\omega^2 - 1} = (\omega^2 - \alpha_1)(\omega^2 - \alpha_2)(\omega^2 - \alpha_3)(\omega^2 - \alpha_4)$$

$$\Rightarrow \quad \frac{\omega - 1}{\omega^2 - 1} = (\omega^2 - \alpha_1)(\omega^2 - \alpha_2)(\omega^2 - \alpha_3)(\omega^2 - \alpha_4) \qquad \ldots(iii)$$

On dividing Eq. (ii) by Eq. (iii), we get

$$\frac{\omega - \alpha_1}{\omega^2 - \alpha_1} \cdot \frac{\omega - \alpha_2}{\omega^2 - \alpha_2} \cdot \frac{\omega - \alpha_3}{\omega^2 - \alpha_3} \cdot \frac{\omega - \alpha_4}{\omega^2 - \alpha_4} = \frac{(\omega^2 - 1)^2}{(\omega - 1)^2}$$

$$= \frac{\omega^4 + 1 - 2\omega^2}{\omega^2 + 1 - 2\omega} = \frac{\omega + 1 - 2\omega^2}{\omega^2 + 1 - 2\omega}$$

$$= \frac{-\omega^2 - 2\omega^2}{-\omega - 2\omega} = \frac{-3\omega^2}{-3\omega} = \omega$$

82. Let $z = x + iy$, then $\dfrac{z + \bar{z}}{2} = x$

$\therefore$ From given relation, we get

$$\Rightarrow \quad x = |\, x + iy - 1 \,|$$

$$\Rightarrow \quad x = |\, (x-1) + iy \,|$$

$$\Rightarrow \quad x^2 = (x-1)^2 + y^2 \quad \Rightarrow 2x = 1 + y^2$$

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

Then, $\quad 2x_1 = 1 + y_1^2$ ...(i)

and $\quad 2x_2 = 1 + y_2^2$ ...(ii)

On subtracting Eq. (ii) from Eq. (i), we get

$$2(x_1 - x_2) = y_1^2 - y_2^2$$

$$2(x_1 - x_2) = (y_1 + y_2)(y_1 - y_2) \qquad \ldots(iii)$$

But, given that $\arg(z_1 - z_2) = \pi/4$

Then, $\quad \tan^{-1}\left(\dfrac{y_1 - y_2}{x_1 - x_2}\right) = \dfrac{\pi}{4} \;\Rightarrow\; \dfrac{y_1 - y_2}{x_1 - x_2} = 1$

$$\therefore \quad y_1 - y_2 = x_1 - x_2 \qquad \ldots(iv)$$

From Eqs. (iii) and (iv), we get

$$y_1 + y_2 = 2 \qquad [\because y_1 - y_2 \neq 0]$$

$$\therefore \quad \text{Im}(z_1 + z_2) = 2$$

Hence, the imaginary part $(z_1 + z_2)$ is 2.

83. (i) LHS $= (a^2 + b^2 + c^2 - bc - ca - ab)$

$$(x^2 + y^2 + z^2 - yz - zx - xy)$$

$$= (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$$

$$(x + y\omega + z\omega^2)(x + y\omega^2 + z\omega)$$

$$= \{(a + b\omega + c\omega^2)(x + y\omega + z\omega^2)\}$$

$$\{(a + b\omega^2 + c\omega)(x + y\omega^2 + z\omega)\}$$

$$= \{ax + cy + bz + \omega(bx + ay + cz)$$

$$+ \omega^2(cx + by + az)\} \times \{ax + cy + bz + \omega^2$$

$$(bx + ay + cz) + \omega(cx + by + az)\}$$

$$= (X + \omega Z + \omega^2 Y)(X + \omega^2 Z + \omega Y)$$

$$= \text{RHS}$$

(ii) $\text{LHS} = (a^3 + b^3 + c^3 - 3abc)(x^3 + y^3 + z^3 - 3xyz)$

$= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) \times (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

$= (a + b + c)(x + y + z)(a^2 + b^2 + c^2 - ab - bc - ca) \times (x^2 + y^2 + z^2 - xy - yz - zx)$ [using (i) part]

$= (ax + ay + az + bx + by + bz + cx + cy + cz)(X^2 + Y^2 + Z^2 - YZ - ZX - XY)$

$= \{(ax + cy + bz) + (cx + by + az) + (bx + ay + cz)\}(X^2 + Y^2 + Z^2 - YZ - ZX - XY)$

$= (X + Y + Z)(X^2 + Y^2 + Z^2 - YZ - ZX - XY)$

$= X^3 + Y^3 + Z^3 - 3XYZ = \text{RHS}$

84. Let $z = x + iy$

$\therefore \quad |z|^2 = x^2 + y^2$

$\therefore \quad x^2 + y^2 - 2i(x + iy) + 2c(1 + i) = 0$

$(x^2 + y^2 + 2y + 2c) + i(-2x + 2c) = 0$

On comparing the real and imaginary parts, we get

$$x^2 + y^2 + 2y + 2c = 0 \quad \ldots(i)$$

and $$-2x + 2c = 0 \quad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$y^2 + 2y + c^2 + 2c = 0$$

$$\Rightarrow \quad y = \frac{-2 \pm \sqrt{4 - 4(c^2 + 2c)}}{2} = -1 \pm \sqrt{(1 - c^2 - 2c)}$$

$\because$ x and y are real.

$\therefore \quad 1 - c^2 - 2c \geq 0$ or $c^2 + 2c + 1 \leq 2$

$(c + 1)^2 \leq (\sqrt{2})^2 \Rightarrow -\sqrt{2} - 1 \leq c \leq \sqrt{2} - 1$

$\therefore \quad 0 \leq c \leq \sqrt{2} - 1 \quad [\because \text{given } c \geq 0]$

Hence, the solution is $z = x + iy = c + i(-1 \pm \sqrt{1 - c^2 - 2c})$

for $0 \leq c \leq \sqrt{2} - 1$

and $z = x + iy \equiv$ no solution for $c > \sqrt{2} - 1$

85. Let $z = x + iy$

$\therefore \quad \text{Re}(z) = x = \dfrac{z + \bar{z}}{2} \quad \ldots(i)$

and $\text{Im}(z) = y = \dfrac{z - \bar{z}}{2i} \quad \ldots(ii)$

The equation $(2 - i)z + (2 + i)\bar{z} + 3 = 0$ can be written as

$$2(z + \bar{z}) - i(z - \bar{z}) + 3 = 0$$

or $$4x + 2y + 3 = 0$$

$\therefore$ Slope of the given line, $m = -2$

Let slope of the required line be m_1, then

$$\tan 45^\circ = \left|\frac{m_1 - m}{1 + m_1 m}\right| \Rightarrow 1 = \left|\frac{m_1 + 2}{1 - 2m_1}\right| \Rightarrow \pm 1 = \frac{m_1 + 2}{1 - 2m_1}$$

$\therefore \quad m_1 = -\dfrac{1}{3}, 3$

$\therefore$ Equation of straight lines through $(-1, 4)$ and having slopes $-\dfrac{1}{3}$ and 3 are $y - 4 = -\dfrac{1}{3}(x + 1)$ and $y - 4 = 3(x + 1)$

$\Rightarrow \quad x + 3y - 11 = 0$ and $3x - y + 7 = 0$

Using Eqs. (i) and (ii), then equations of lines are

$$\frac{z + \bar{z}}{2} + \frac{3(z - \bar{z})}{2i} - 11 = 0$$

and $$\frac{3(z + \bar{z})}{2} - \frac{(z - \bar{z})}{2i} + 7 = 0$$

i.e., $(1 - 3i)z + (1 + 3i)\bar{z} - 22 = 0$

and $(3 + i)z + (3 - i)\bar{z} + 14 = 0$

86. Putting $\dfrac{1 + i}{2} = x$ in LHS, we get

$$\text{LHS} = (1 + x)(1 + x^2)(1 + x^{2^2}) \ldots (1 + x^{2^n})$$

$$= \frac{(1 - x)(1 + x)(1 + x^2)(1 + x^{2^2}) \ldots (1 + x^{2^n})}{(1 - x)}$$

$$= \frac{(1 - x^2)(1 + x^2)(1 + x^{2^2}) \ldots (1 + x^{2^n})}{(1 - x)}$$

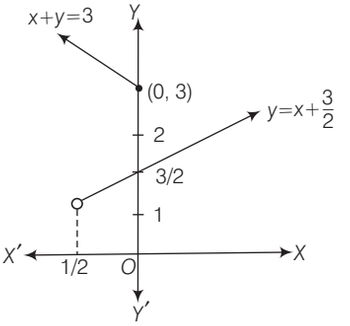
$$= \frac{(1 - x^{2^2})(1 + x^{2^2}) \ldots (1 + x^{2^n})}{(1 - x)}$$

$$= \frac{(1 - x^{2^n})(1 + x^{2^n})}{(1 - x)} = \frac{1 - (x^2)^{2^n}}{(1 - x)}$$

$$= \frac{1 - \left(\frac{i}{2}\right)^{2^n}}{1 - \left(\frac{1 + i}{2}\right)} \qquad \left[\because x = \frac{1 + i}{2}\right]$$

$$= \frac{1 - \frac{1}{2^{2^n}}(1)}{\left(\frac{1 - i}{2}\right)} \cdot \frac{(1 + i)}{(1 + i)} = (1 + i)\left(1 - \frac{1}{2^{2^n}}\right) = \text{RHS}$$

87. Since, $\arg(z - 3i) = 3\pi/4$ is a ray which is start from $3i$ and makes an angle $3\pi/4$ with positive real axis as shown in the figure.



$\therefore$ Equation of ray in cartesian form is

$$y - 3 = \tan(3\pi/4)(x - 0)$$

or $y - 3 = -x$ or $x + y = 3$

and $\arg(2z + 1 - 2i) = \pi/4$

$$\Rightarrow \quad \arg\left(2\left(z + \frac{1}{2} - i\right)\right) = \pi/4$$

or $$\arg(2) + \arg\left(z + \frac{1}{2} - i\right) = \pi/4$$

or $$0 + \arg\left(z + \frac{1}{2} - i\right) = \pi/4$$

or $\arg\left(z-\left(-\frac{1}{2}+i\right)\right)=\pi/4$

which is a ray that start from point $-\frac{1}{2}+i$ and makes an angle $\pi/4$ with positive real axis as shown in the figure.

∴Equation of ray in cartesian form is

$$y-1=1\,[x-(-1/2)] \Rightarrow y=x+3/2$$

From the figure, it is clear that the system of equations has no solution.

88. Let $a=r\cos\alpha$ and $0=r\sin\alpha$ ...(i)

So that, $a^2+0^2=r^2$

$\therefore$ $r=|a|$

Then, $a=|a|\cos\alpha$ [from Eq. (i)]

$\therefore$ $\cos\alpha=\pm 1$

Then, $\cos\alpha=1$ or -1 according as a is + ve or − ve and $\sin\alpha=0$.

Hence, $\alpha=0$ or π according as a is + ve and − ve.

Again, let $0=r_1\cos\beta$ or $b=r_1\sin\beta$ ...(ii)

So that, $0^2+b^2=r_1^2$

$\therefore$ $r_1=|b|$

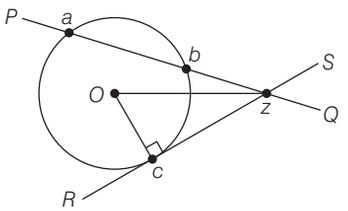
From Eq. (ii), we get $b=|b|\sin\beta$

$\therefore$ $\sin\beta=\pm 1$

Then, $\sin\beta=1$ or -1 according as b is + ve or − ve and $\cos\beta=0$.

Hence, $\beta=\frac{\pi}{2}$ or $-\frac{\pi}{2}$ according as b is +ve or −ve.

89. Let two non-parallel straight lines PQ, RS meet the circle $|z|=r$ in the points a, b and c.



Then, $|a|=r, |b|=r$ and $|c|=r$ or $|a|^2=|b|^2=|c|^2=r^2$

$\therefore$ $a\bar{a}=b\bar{b}=c\bar{c}=r^2$,

then $\bar{a}=\frac{r^2}{a}$, $\bar{b}=\frac{r^2}{b}$ and $\bar{c}=\frac{r^2}{c}$

Points a, b and z are collinear, then $\begin{vmatrix} z & \bar{z} & 1 \\ a & \bar{a} & 1 \\ b & \bar{b} & 1 \end{vmatrix}=0$

$\therefore$ $z(\bar{a}-\bar{b})-\bar{z}(a-b)+a\bar{b}-\bar{a}b=0$

$\Rightarrow$ $z\left(\frac{r^2}{a}-\frac{r^2}{b}\right)-\bar{z}(a-b)+\frac{r^2a}{b}-\frac{r^2b}{a}=0$

On dividing both sides by $r^2(b-a)$, we get

$$\frac{z}{ab}+\frac{\bar{z}}{r^2}=a^{-1}+b^{-1} \quad ...(i)$$

For RS, replace $a=b=c$ in Eq. (i), then

$$\frac{z}{c^2}+\frac{\bar{z}}{r^2}=2c^{-1} \quad ...(ii)$$

On subtracting Eq. (i) from Eq. (ii), we get

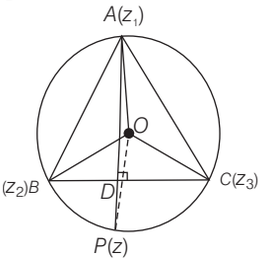
$$z(c^{-2}-a^{-1}b^{-1})=2c^{-1}-a^{-1}-b^{-1}$$

Hence, $z=\frac{2c^{-1}-a^{-1}-b^{-1}}{c^{-2}-a^{-1}b^{-1}}$

which is a required point.

90. $\because AD\perp BC$

$\therefore$ AP is also perpendicular to BC.



Then, $\arg\left(\frac{z_1-z}{z_3-z_2}\right)=\frac{\pi}{2}$

$\therefore$ $\mathrm{Re}\left(\frac{z_1-z}{z_3-z_2}\right)=0$

$\Rightarrow$ $\frac{\frac{z_1-z}{z_3-z_2}+\frac{\bar{z}_1-\bar{z}}{\bar{z}_3-\bar{z}_2}}{2}=0$

$\Rightarrow$ $\frac{z_1-z}{z_3-z_2}+\frac{\bar{z}_1-\bar{z}}{\bar{z}_3-\bar{z}_2}=0$ ...(i)

But O is the circumcentre of ΔABC, then

$$OP=OA=OB=OC$$

$$|z|=|z_1|=|z_2|=|z_3|$$

On squaring the above relation, we get

$$|z|^2=|z_1|^2=|z_2|^2=|z_3|^2$$

$\Rightarrow$ $z\bar{z}=z_1\bar{z}_1=z_2\bar{z}_2=z_3\bar{z}_3$

From first two relations $\frac{\bar{z}_1}{\bar{z}}=\frac{z}{z_1}$ ...(ii)

From first and third relation $\frac{\bar{z}_2}{\bar{z}}=\frac{z}{z_2}$ ...(iii)

and from first and fourth relation $\frac{\bar{z}_3}{\bar{z}}=\frac{z}{z_3}$ ...(iv)

From Eq. (i), we get $\frac{z_1-z}{z_3-z_2}+\frac{\frac{\bar{z}_1}{\bar{z}}-1}{\frac{\bar{z}_3}{\bar{z}}-\frac{\bar{z}_2}{\bar{z}}}=0$ ...(v)

From Eqs. (ii), (iii), (iv) and (v), we get

$$\frac{z_1-z}{z_3-z_2}+\frac{\frac{z}{z_1}-1}{\frac{z}{z_3}-\frac{z}{z_2}}=0$$

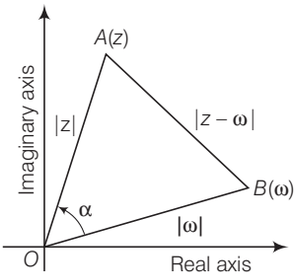
$\Rightarrow$ $\left(\frac{z_1-z}{z_3-z_2}\right)\left\{1+\frac{z_2z_3}{zz_1}\right\}=0 \quad \left[\because \frac{z_1-z}{z_3-z_2}\neq 0\right]$

$\Rightarrow$ $1+\frac{z_2z_3}{zz_1}=0 \Rightarrow z=-\frac{z_2z_3}{z_1}$

91. From the figure,

$$\alpha = (\arg(z) - \arg(\omega)) \quad \ldots(i)$$

and for every $\alpha, \sin^2\frac{\alpha}{2} \le \left(\frac{\alpha}{2}\right)^2$ ...(ii)



In ΔOAB, from cosine rule

$$(AB)^2 = (OA)^2 + (OB)^2 - 2OA \cdot OB \cos\alpha$$

$$\Rightarrow |z-\omega|^2 = |z|^2 + |\omega|^2 - 2|z||\omega|\cos\alpha$$

$$\Rightarrow |z-\omega|^2 = (|z| - |\omega|)^2 + 2|z||\omega|(1-\cos\alpha)$$

$$\Rightarrow |z-\omega|^2 = (|z| - |\omega|)^2 + 4|z||\omega|\sin^2\frac{\alpha}{2}$$

$$\Rightarrow |z-\omega|^2 \le (|z| - |\omega|)^2 + 4|z||\omega|\left(\frac{\alpha}{2}\right)^2 \quad \text{[from Eq. (ii)]}$$

$$\Rightarrow |z-\omega|^2 \le (|z| - |\omega|)^2 + \alpha^2 \quad [\because |z| \le 1, |\omega| \le 1]$$

$$|z-\omega|^2 \le (|z| - |\omega|)^2 + (\arg(z) - \arg(\omega))^2 \quad \text{[from Eq. (i)]}$$

I. Aliter

Let $z = r(\cos\theta + i\sin\theta)$ and $\omega = r_1(\cos\theta_1 + i\sin\theta_1)$,

then $|z| = r$ and $|\omega| = r_1$

Also, $\arg(z) = \theta$ and $\arg(\omega) = \theta_1$

and $r \le 1$ and $r_1 \le 1$ $[\because \text{given } |z| \le 1, |\omega| \le 1]$

We have, $z - \omega = (r\cos\theta - r_1\cos\theta_1) + i(r\sin\theta - r_1\sin\theta_1)$

$$\therefore \quad |z-\omega|^2 = (r\cos\theta - r_1\cos\theta_1)^2 + (r\sin\theta - r_1\sin\theta_1)^2$$

$$\Rightarrow \quad |z-\omega|^2 = r^2 + r_1^2 - 2rr_1\cos(\theta-\theta_1)$$

$$= (r-r_1)^2 + 2rr_1 - 2rr_1\cos(\theta-\theta_1)$$

$$= (r-r_1)^2 + 2rr_1(1-\cos(\theta-\theta_1))$$

$$= (r-r_1)^2 + 4rr_1\sin^2\left(\frac{\theta-\theta_1}{2}\right)$$

$$\le (r-r_1)^2 + 4rr_1\left(\frac{\theta-\theta_1}{2}\right)^2 \quad [\because |\sin\theta| \le |\theta|]$$

$$= (r-r_1)^2 + rr_1(\theta-\theta_1)^2$$

$$\le (r-r_1)^2 + (\theta-\theta_1)^2 \quad [\because r, r_1 \le 1]$$

$$\Rightarrow \quad |z-\omega|^2 \le (|z| - |\omega|)^2 + (\arg z - \arg\omega)^2$$

II. Aliter

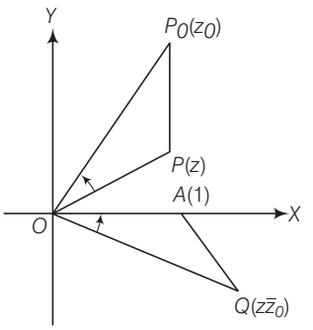
Let $z = r\cos\theta$

and $\omega = r_1\cos\theta_1$

$$\therefore \quad r^2 + r_1^2 - 2rr_1\cos(\theta-\theta_1) \le r^2 + r_1^2 - 2rr_1 + (\theta-\theta_1)^2$$

$$\Rightarrow rr_1\sin^2\left(\frac{\theta-\theta_1}{2}\right) \le \left(\frac{\theta-\theta_1}{2}\right)^2 \quad \left[\begin{matrix}\because r, r_1 \le 1 \\ \text{and } \sin^2 x \le x^2\end{matrix}\right]$$

92. Given, $OA = 1$ and $|z| = 1$



$$\therefore \quad OP = |z-0| = |z| = 1$$

$$\therefore \quad OP = OA$$

$$OP_0 = |z_0 - 0| = |z_0|$$

and $OQ = |z\bar{z}_0 - 0| = |z\bar{z}_0| = |z||\bar{z}_0| = 1|\bar{z}_0| = |z_0|$

$$\therefore \quad OP_0 = OQ$$

Also, $\angle P_0OP = \arg\left(\frac{z_0-0}{z-0}\right) = \arg\left(\frac{z_0}{z}\right) = \arg\left(\frac{\bar{z}\,z_0}{z\,\bar{z}}\right)$

$$= \arg\left(\frac{\bar{z}\,z_0}{|z|^2}\right) = \arg\left(\frac{\bar{z}\,z_0}{1}\right) = -\arg(\overline{\bar{z}\,z_0})$$

$$= -\arg(z\,\bar{z}_0) = \arg\left(\frac{1}{z\,\bar{z}_0}\right)$$

$$= \arg\left(\frac{1-0}{z\,\bar{z}_0 - 0}\right) = \angle AOQ$$

Thus, the triangles POP_0 and AOQ are congruent.

$$\therefore \quad PP_0 = AQ$$

$$|z - z_0| = |z\,\bar{z}_0 - 1|$$

93. Let the equation of line passing through the origin be

$$\bar{a}z + a\bar{z} = 0 \quad \ldots(i)$$

According to the question, $z_1, z_2, \ldots, z_n$ all lie on one side of line (i)

$$\therefore \quad \bar{a}z_i + a\bar{z}_i > 0 \text{ or } < 0 \text{ for all } i = 1, 2, 3, \ldots, n \quad \ldots(ii)$$

$$\Rightarrow \quad \bar{a}\sum_{i=1}^{n} z_i + a\sum_{i=1}^{n}\bar{z}_i > 0 \text{ or } < 0 \quad \ldots(iii)$$

$$\Rightarrow \quad \sum_{i=1}^{n} z_i \ne 0 \quad \left\{\text{If } \sum_{i=1}^{n} z_i = 0, \text{ then } \sum_{i=1}^{n}\bar{z}_i = 0, \text{ hence } \bar{a}\sum_{i=1}^{n} z_i + a\sum_{i=1}^{n}\bar{z}_i = 0\right\}$$

From Eq. (ii), we get

$$\bar{a}z_i + a\bar{z}_i > 0 \text{ or } < 0 \text{ for all } i = 1, 2, 3, \ldots, n$$

$$\Rightarrow \quad \frac{\bar{a}z_i\bar{z}_i}{\bar{z}_i} + \frac{a\bar{z}_iz_i}{z_i} > 0 \text{ or } < 0$$

$$\Rightarrow \quad |z_i|^2\left\{\frac{\bar{a}}{\bar{z}_i} + \frac{a}{z_i}\right\} > 0 \text{ or } < 0$$

$$\Rightarrow \quad \frac{\bar{a}}{\bar{z}_i} + \frac{a}{z_i} > 0 \text{ or } < 0 \text{ for all } i = 1, 2, 3, \ldots, n$$

$$\Rightarrow \quad \frac{1}{z_1}, \frac{1}{z_2}, \ldots, \frac{1}{z_n} \text{ lie on one side of the line } \bar{a}z + a\bar{z} = 0$$

or $\quad \bar{a}\sum_{i=1}^{n}\frac{1}{\bar{z}_i}+a\sum_{i=1}^{n}\frac{1}{z_i}>0 \text{ or } <0$

Therefore, $\sum_{i=1}^{n}\frac{1}{z_i}\neq 0\left\{\text{If } \sum_{i=1}^{n}\frac{1}{z_i}=0, \text{ then } \sum_{i=1}^{n}\frac{1}{\bar{z}_i}=0\right.$

$\Rightarrow \quad \left.\bar{a}\sum_{i=1}^{n}\frac{1}{\bar{z}_i}+a\sum_{i=1}^{n}\frac{1}{z_i}=0\right\}$

94. Given, $|a||b|=\sqrt{a\bar{b}^2c}$; $|a|=|c|$; $az^2+bz+c=0$, then we have to prove that $|z|=1$

On squaring, we get

$|a|^2|b|^2=a\,\bar{b}^2c$ and $|a|^2=|c|^2$

$\Rightarrow \; a\,\bar{a}\,b\,\bar{b}=a\,\bar{b}^2c \quad$ and $\quad a\,\bar{a}=c\,\bar{c}$

$\Rightarrow \quad \bar{a}\,b=\bar{b}\,c \quad$ and $\quad a\,\bar{a}=c\,\bar{c}$ ...(i)

If z_1 and z_2 are the roots of $az^2+bz+c=0$

Then, $\bar{z}_1$ and $\bar{z}_2$ are the roots of $\bar{a}\,(\bar{z})^2+\bar{b}\bar{z}+\bar{c}=0$ ...(A)

$$\left.\begin{aligned}\therefore \quad & z_1+z_2=-\frac{b}{a}, z_1z_2=\frac{c}{a}\\ \text{and} \quad & \bar{z}_1+\bar{z}_2=-\frac{\bar{b}}{\bar{a}}, \bar{z}_1\bar{z}_2=\frac{\bar{c}}{\bar{a}}\end{aligned}\right\} \quad ...(ii)$$

$$\because \quad \frac{1}{z_1}+\frac{1}{z_2}=\frac{z_1+z_2}{z_1\,z_2}=\frac{-b/a}{c/a}=-\frac{b}{c}=-\frac{\bar{b}}{\bar{a}}=\overline{z_1}+\overline{z_2}$$

[from Eqs. (i) and (ii)]

and $\frac{1}{\bar{z}_1}+\frac{1}{\bar{z}_2}=\frac{\bar{z}_1+\bar{z}_2}{\bar{z}_1\,\bar{z}_2}=\frac{-\bar{b}/\bar{a}}{\bar{c}/\bar{a}}$

$$=-\frac{\bar{b}}{\bar{c}}=-\frac{\bar{a}\,bc}{ca\,\bar{a}}=-\frac{b}{a}=z_1+z_2$$

[from Eqs. (i) and (ii)]

Now, it is clear that $z_1=\frac{1}{\bar{z}_1}$ and $z_2=\frac{1}{\bar{z}_2}$

Then, $|z_1|^2=1$ and $|z_2|^2=1$

Hence, $\quad |z|=1$

Conversely For $az^2+bz+c=0$, we have to prove

$|z|=1\Rightarrow|a||b|=\sqrt{a\,\bar{b}^2c}$

and $\quad |a|=|c|$

$|z|=1 \;\Rightarrow\; |z|^2=1 \;\Rightarrow\; z\,\bar{z}=1\Rightarrow z=\frac{1}{\bar{z}}$

From Eq. (A), we get

$$\bar{a}\left(\frac{1}{z}\right)^2+\bar{b}\left(\frac{1}{z}\right)+\bar{c}=0 \text{ or } \bar{c}z^2+\bar{b}z+\bar{a}=0$$

Also, $az^2+bz+c=0$, on comparing

$$\frac{\bar{c}}{a}=\frac{\bar{b}}{b}=\frac{\bar{a}}{c}$$

$\therefore \quad a\,\bar{a}=c\,\bar{c}$ and $\bar{a}\,b=\bar{b}\,cs$

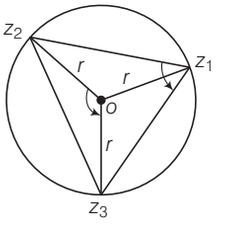
$\Rightarrow \; |a|=|c|$ and $|a||b|=\sqrt{a\,\bar{b}^2c}$

95. (i) Let $z_1=r_1(\cos\alpha+i\sin\alpha)$,

$z_2=r_2(\cos\beta+i\sin\beta)$ and $z_3=r_3(\cos\gamma+i\sin\gamma)$

$\therefore \; |z_1|=r_1, |z_2|=r_2, |z_3|=r_3$

and $\arg(z_1)=\alpha, \arg(z_2)=\beta, \arg(z_3)=\gamma$



From the given condition,

$$\begin{vmatrix} r_1 & r_2 & r_3\\ r_2 & r_3 & r_1\\ r_3 & r_1 & r_2\end{vmatrix}=0$$

$$\Rightarrow \quad r_1^3+r_2^3+r_3^3-3r_1r_2r_3=0$$

$$\Rightarrow \frac{1}{2}(r_1+r_2+r_3)\{(r_1-r_2)^2+(r_2-r_3)^2+(r_3-r_1)^2\}=0$$

Since, $\quad r_1+r_2+r_3\neq 0,$

then $\quad (r_1-r_2)^2+(r_2-r_3)^2+(r_3-r_1)^2=0$

It is possible only when

$$r_1-r_2=r_2-r_3=r_3-r_1=0$$

$$\therefore \quad r_1=r_2=r_3$$

and $\quad |z_1|=|z_2|=|z_3|=r$ [say]

Hence, z_1, z_2, z_3 lie on a circle with the centre at the origin.

(ii) Again, in Δoz_2z_3 by Coni method

$$\arg\left(\frac{z_3-0}{z_2-0}\right)=\angle z_2oz_3\Rightarrow \arg\left(\frac{z_3}{z_2}\right)=\angle z_2oz_3 \quad ...(i)$$

In $\Delta z_2z_1z_3$ by Coni method

$$\arg\left(\frac{z_3-z_1}{z_2-z_1}\right)=\angle z_2\,z_1\,z_3=\frac{1}{2}\angle z_2oz_3 \text{[property of circle]}$$

$$=\frac{1}{2}\arg\left(\frac{z_3}{z_1}\right) \text{ [from Eq. (i)]}$$

$$\therefore \quad \arg\left(\frac{z_3}{z_1}\right)=2\arg\left(\frac{z_3-z_1}{z_2-z_1}\right)$$

Hence, $\arg\left(\frac{z_3}{z_1}\right)=\arg\left(\frac{z_3-z_1}{z_2-z_1}\right)^2$

96. We know that,

$$\text{Re}\,(z_1\bar{z}_2)\le|z_1\bar{z}_2|$$

$$\therefore \; |z_1|^2+|z_2|^2+2\,\text{Re}\,(z_1\bar{z}_2)\le|z_1|^2+|z_2|^2+2|z_1\bar{z}_2|$$

$$\Rightarrow \quad |z_1+z_2|^2\le|z_1|^2+|z_2|^2+2|z_1||z_2| \quad ...(i)$$

Also, AM ≥ GM

$$\therefore \quad \frac{(\sqrt{c}\;|z_1|)^2+\left(\frac{1}{\sqrt{c}}|z_2|\right)^2}{2}\ge\left\{\sqrt{c}\cdot|z_1|^2\cdot\frac{1}{\sqrt{c}}|z_2|^2\right\}^{1/2} \quad [\because c>0]$$

$$\Rightarrow \; c|z_1|^2+\frac{1}{c}|z_2|^2\ge 2|z_1||z_2|$$

$$\therefore \; |z_1|^2+|z_2|^2+2|z_1||z_2|\le|z_1|^2+|z_2|^2+c|z_1|^2+\frac{1}{c}|z_2|^2$$

$$\Rightarrow |z_1|^2+|z_2|^2+2|z_1||z_2|\le(1+c)|z_1|^2+(1+c^{-1})(|z_2|^2) \quad ...(ii)$$

From Eqs. (i) and (ii), we get

$$|z_1 + z_2|^2 \le (1 + c)|z_1|^2 + (1 + c^{-1})|z_2|^2$$

Aliter

Here, $(1 + c)|z_1|^2 + (1 + c^{-1})|z_2|^2 - |z_1 + z_2|^2$

$= (1 + c)\, z_1\bar{z}_1 + \left(1 + \frac{1}{c}\right) z_2\bar{z}_2 - (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$

$= (1 + c)\, z_1\bar{z}_1 + \left(1 + \frac{1}{c}\right) z_2\bar{z}_2 - z_1\bar{z}_1 - z_1\bar{z}_2 - z_2\bar{z}_1 - z_2\bar{z}_2$

$= c z_1\bar{z}_1 + \frac{1}{c} z_2\bar{z}_2 - z_1\bar{z}_2 - z_2\bar{z}_1$

$= \frac{1}{c}\{c^2 z_1\bar{z}_1 + z_2\bar{z}_2 - c z_1\bar{z}_2 - c z_2\bar{z}_1\}$

$= \frac{1}{c}\{c z_1(c\bar{z}_1 - \bar{z}_2) - z_2(c\bar{z}_1 - \bar{z}_2)\}$

$= \frac{1}{c}(c z_1 - z_2)(c\bar{z}_1 - \bar{z}_2) = \frac{1}{c}(c z_1 - z_2)\overline{(c z_1 - z_2)}$

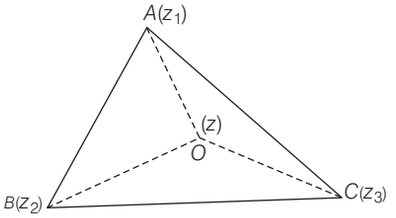
$= \frac{1}{c}|c z_1 - z_2|^2 \ge 0$ as $c > 0$

$\therefore \quad (1 + c)|z_1|^2 + \left(1 + \frac{1}{c}\right)|z_2|^2 - |z_1 + z_2|^2 \ge 0$

Hence, $|z_1 + z_2|^2 \le (1 + c)|z_1|^2 + \left(1 + \frac{1}{c}\right)|z_2|^2$

97. If z be the complex number corresponding to the circumcentre O, then we have

$$OA = OB = OC$$



$\Rightarrow \quad |z - z_1| = |z - z_2| = |z - z_3|$

$\Rightarrow \quad |z - z_1|^2 = |z - z_2|^2 = |z - z_3|^2$

$\Rightarrow \quad (z - z_1)(\bar{z} - \bar{z}_1) = (z - z_2)(\bar{z} - \bar{z}_2)$

$= (z - z_3)(\bar{z} - \bar{z}_3)$ ...(i)

From first two members of Eq. (i), we get

$$\bar{z}(z_2 - z_1) = \bar{z}_1(z - z_1) - \bar{z}_2(z - z_2) \quad ...(ii)$$

and from last two members of Eq. (i), we get

$$\bar{z}(z_3 - z_2) = \bar{z}_2(z - z_2) - \bar{z}_3(z - z_3) \quad ...(iii)$$

Eliminating $\bar{z}$ from Eqs. (ii) and (iii), we get

$(z_2 - z_1)[\bar{z}_2(z - z_2) - \bar{z}_3(z - z_3)] = (z_3 - z_2)$

$[\bar{z}_1(z - z_1) - \bar{z}_2(z - z_2)]$

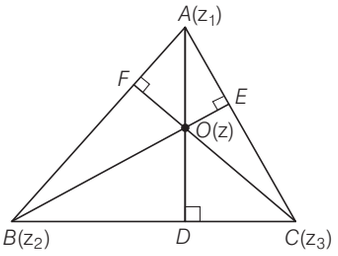
or $z[z_2(z_2 - z_1) - \bar{z}_3(z_2 - z_1) - \bar{z}_1(z_3 - z_2) + \bar{z}_2(z_3 - z_2)]$

$= z_2\bar{z}_2(z_2 - z_1) - z_3\bar{z}_3(z_2 - z_1) - z_1\bar{z}_1(z_3 - z_2) + z_2\bar{z}_2(z_3 - z_2)$

or $z\sum \bar{z}_1(z_2 - z_3) = \sum z_1\bar{z}_1(z_2 - z_3)$

or $$z = \frac{\sum |z_1|^2(z_2 - z_3)}{\sum \bar{z}_1(z_2 - z_3)}$$

98. Let z be the complex number corresponding to the orthocentre O, since $AD \perp BC$, we get



$$\arg\left(\frac{z - z_1}{z_2 - z_3}\right) = \frac{\pi}{2}$$

i.e. $\frac{z - z_1}{z_2 - z_3}$ is purely imaginary.

i.e. $\text{Re}\left(\frac{z - z_1}{z_2 - z_3}\right) = 0$ or $\frac{z - z_1}{z_2 - z_3} + \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_3} = 0$ ...(i)

Similarly, $\frac{z - z_2}{z_3 - z_1} + \frac{\bar{z} - \bar{z}_2}{\bar{z}_3 - \bar{z}_1} = 0$ $[\because BE \perp CA]$ ...(ii)

From Eq. (i), we get

$$\bar{z} = \bar{z}_1 - \frac{(z - z_2)(\bar{z}_2 - \bar{z}_3)}{(z_2 - z_3)} \quad ...(iii)$$

From Eq. (ii), we get

$$\bar{z} = \bar{z}_2 - \frac{(z - z_2)(\bar{z}_3 - \bar{z}_1)}{(z_3 - z_1)} \quad ...(iv)$$

Eliminating $\bar{z}$ from Eqs. (iii) and (iv), we get

$$\bar{z}_1 - \bar{z}_2 = \frac{(z - z_1)}{(z_2 - z_3)}(\bar{z}_2 - \bar{z}_3) - \frac{(z - z_2)(\bar{z}_3 - \bar{z}_1)}{(z_3 - z_1)}$$

or $(z - z_1)(\bar{z}_2 - \bar{z}_3)(z_3 - z_1) - (z - z_2)(\bar{z}_3 - \bar{z}_1)(z_2 - z_3)$

$= (\bar{z}_1 - \bar{z}_2)(z_2 - z_3)(z_3 - z_1)$

or $z\{(\bar{z}_2 - \bar{z}_3)(z_3 - z_1) - (\bar{z}_3 - \bar{z}_1)(z_2 - z_3)\}$

$= (\bar{z}_1 - \bar{z}_2)(z_2 - z_3)(z_3 - z_1) + z_1(\bar{z}_2 - \bar{z}_3)(z_3 - z_1)$

$- z_2(\bar{z}_3 - \bar{z}_1)(z_2 - z_3)$

$\Rightarrow z[\bar{z}_2 z_3 - \bar{z}_2 z_1 - z_3\bar{z}_3 + \bar{z}_3 z_1 - \bar{z}_3 z_2 + z_3\bar{z}_3 + \bar{z}_1 z_2 - \bar{z}_1 z_3]$

$= (\bar{z}_1 - \bar{z}_2)\{z_2 z_3 - z_2 z_1 - z_3^2 + z_3 z_1\}$

$+ (\bar{z}_2 - \bar{z}_3)(z_3 z_1 - z_1^2) + (\bar{z}_3 - \bar{z}_1)(z_2 z_3 - z_2^2)$

$= -\{z_1^2(\bar{z}_2 - \bar{z}_3) + z_2^2(\bar{z}_3 - \bar{z}_1) + z_3^2(\bar{z}_1 - \bar{z}_2)\}$

$+ \{\bar{z}_1 z_2 z_3 - z_2 z_1\bar{z}_1 + z_3 z_1\bar{z}_1 + \bar{z}_2 z_1 z_3$

$- z_1 z_3\bar{z}_3 + z_2 z_3\bar{z}_3 - \bar{z}_1 z_2 z_3\} - z\sum(z_1\bar{z}_2 - z_2\bar{z}_1)$

$= -\sum z_1^2(\bar{z}_2 - \bar{z}_3) - \Sigma z_1\bar{z}_1(z_2 - z_3)$

Hence, $$z = \frac{\sum z_1^2(\bar{z}_2 - \bar{z}_3) + \sum |z_1|^2(z_2 - z_3)}{\sum(z_1\bar{z}_2 - z_2\bar{z}_1)}$$

99. Let $\theta = \frac{1}{7}(2n + 1)\pi$, where $n = 0, 1, 2, 3, \ldots, 6$

$\therefore \quad 7\theta = (2n + 1)\pi$ or $4\theta = (2n + 1)\pi - 3\theta$

or $\cos 4\theta = -\cos 3\theta$

or $2\cos^2 2\theta - 1 = -(4\cos^3\theta - 3\cos\theta)$

or $2(2\cos^2\theta-1)^2-1=-(4\cos^3\theta-3\cos\theta)$

or $8\cos^4\theta+4\cos^3\theta-8\cos^2\theta-3\cos\theta+1=0$

Now, if $\cos\theta=x$, then we have

$8x^4+4x^3-8x^2-3x+1=0$

or $(x+1)(8x^3-4x^2-4x+1)=0$

$x+1\neq 0$ $[\because \theta\neq\pi]$

$\therefore$ $8x^3-4x^2-4x+1=0$ ...(i)

Hence, the roots of this equation are

$$\cos\frac{\pi}{7},\ \cos\frac{3\pi}{7},\ \cos\frac{5\pi}{7}.$$

[since $\cos\frac{9\pi}{7}=\cos\frac{5\pi}{7}$, $\cos\frac{11\pi}{7}$

$=\cos\frac{3\pi}{7}$, $\cos\frac{13\pi}{7}=\cos\frac{\pi}{7}$ and Eq. (i) is cubic]

(i) On putting $\frac{1}{x^2}=y$ or $x=\frac{1}{\sqrt{y}}$ in Eq. (i), then Eq. (i) becomes

$\Rightarrow$ $\frac{8}{y\sqrt{y}}-\frac{4}{y}-\frac{4}{\sqrt{y}}+1=0$

$\Rightarrow$ $\left(1-\frac{4}{y}\right)^2=\left[\frac{4}{\sqrt{y}}\left(1-\frac{2}{y}\right)\right]^2$

or $1+\frac{16}{y^2}-\frac{8}{y}=\frac{16}{y}\left(1+\frac{4}{y^2}-\frac{4}{y}\right)$

or $y^3-24y^2+80y-64=0$ ...(ii)

where $y=\frac{1}{x^2}=\frac{1}{\cos^2\theta}=\sec^2\theta$

Thus, the roots of $x^3-24x^2+80x-61=0$

are $\sec^2\frac{\pi}{7},\sec^2\frac{3\pi}{7},\sec^2\frac{5\pi}{7}$

(ii) Again, putting $y=1+z$ i.e. $z=y-1$

$=\sec^2\theta-1=\tan^2\theta$, Eq. (ii) reduces to

$(1+z)^3-24(1+z)^2+80(1+z)-64=0$

or $z^3-21z^2+35z-7=0$ ...(iii)

Hence, $\tan^2\frac{\pi}{7},\tan^2\frac{3\pi}{7},\tan^2\frac{5\pi}{7}$ are the roots of

$$x^3-21x^2+35x-7=0$$

(iii) Putting $x=\frac{1}{u}$ in Eq. (i), then Eq. (i) reduces to

$u^3-4u^2-4u+8=0$ whose roots are

$$\sec\frac{\pi}{7},\sec\frac{3\pi}{7},\sec\frac{5\pi}{7}.$$

Therefore, sum of the roots is

$$\sec\frac{\pi}{7}+\sec\frac{3\pi}{7}+\sec\frac{5\pi}{7}=4$$

100. Let roots of $z^7+1=0$ are $-1,\alpha,\alpha^3,\alpha^5,\bar{\alpha},\bar{\alpha}^3,\bar{\alpha}^5$,

where $\alpha=\cos\frac{\pi}{7}+i\sin\frac{\pi}{7}$

$\therefore$ $(z^7+1)=(z+1)(z-\alpha)(z-\bar{\alpha})(z-\alpha^3)$

$(z-\bar{\alpha}^3)(z-\alpha^5)(z-\bar{\alpha}^5)$

$\Rightarrow\frac{(z^7+1)}{(z+1)}=(z-\alpha)(z-\bar{\alpha})(z-\alpha^3)(z-\bar{\alpha}^3)(z-\alpha^5)(z-\bar{\alpha}^5)$

$\Rightarrow$ $z^6-z^5+z^4-z^3+z^2-z+1$

$=\left(z^2+1-2z\cos\frac{\pi}{7}\right)\left(z^2+1-2z\cos\frac{3\pi}{7}\right)$

$\left(z^2+1-2z\cos\frac{5\pi}{7}\right)$ ...(A)

Dividing by z^3 on both sides, we get

$\left(z^3+\frac{1}{z^3}\right)-\left(z^2+\frac{1}{z^2}\right)+\left(z+\frac{1}{z}\right)-1$

$=\left(z+\frac{1}{z}-2\cos\frac{\pi}{7}\right)\left(z+\frac{1}{z}-2\cos\frac{3\pi}{7}\right)\left(z+\frac{1}{z}-2\cos\frac{5\pi}{7}\right)$

On putting $z+\frac{1}{z}=2x$, we get

$(8x^3-6x)-(4x^2-2)+2x-1$

$=8\left(x-\cos\frac{\pi}{7}\right)\left(x-\cos\frac{3\pi}{7}\right)\left(x-\cos\frac{5\pi}{7}\right)$

or $8x^3-4x^2-4x+1=8\left(x-\cos\frac{\pi}{7}\right)$

$\left(x-\cos\frac{3\pi}{7}\right)\left(x-\cos\frac{5\pi}{7}\right)$ ...(i)

So, $8x^3-4x^2-4x+1=0$ and this equation has roots

$\cos\frac{\pi}{7},\cos\frac{3\pi}{7},\cos\frac{5\pi}{7}$

$\therefore$ $\cos\frac{\pi}{7}\cos\frac{3\pi}{7}\cos\frac{5\pi}{7}=-\frac{\text{Constant term}}{\text{Coefficient of } x^3}$

$\cos\frac{\pi}{7}\cos\frac{3\pi}{7}\cos\frac{5\pi}{7}=-\frac{1}{8}$ [proved (i) part]

On putting $x=1$ in Eq. (i), we get

$1=8\left(1-\cos\frac{\pi}{7}\right)\left(1-\cos\frac{3\pi}{7}\right)\left(1-\cos\frac{5\pi}{7}\right)$

or $1=8\left(8\sin^2\frac{\pi}{14}\sin^2\frac{3\pi}{14}\sin^2\frac{5\pi}{14}\right)$

Since, $\sin\theta>0$ for $0<\theta<\pi/2$, we get

$\therefore$ $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}=\frac{1}{8}$ ...(ii) [proved (iii) part]

Again, putting $x=-1$ in Eq. (i), we get

$-7=-8\left(1+\cos\frac{\pi}{7}\right)\left(1+\cos\frac{3\pi}{7}\right)\left(1+\cos\frac{5\pi}{7}\right)$

$7=8\left(8\cos^2\frac{\pi}{14}\cos^2\frac{3\pi}{14}\cos^2\frac{5\pi}{14}\right)$

Since, $\cos\theta>0$ for $0<\theta<\pi/2$, we get

$\cos\frac{\pi}{14}\cos\frac{3\pi}{14}\cos\frac{5\pi}{14}=\frac{\sqrt{7}}{8}$ ...(iii) [proved (ii) part]

On dividing Eq. (ii) by Eq. (iii), we get

$\tan\frac{\pi}{14}\tan\frac{3\pi}{14}\tan\frac{5\pi}{14}=\frac{1}{\sqrt{7}}$ [proved (iv) part]

On putting $z = \dfrac{(1+y)}{(1-y)}$ in Eq. (A), we get

$$\frac{(1+y)^7 + (1-y)^7}{2(1-y)^6} = \frac{2^6 \cos^2\frac{\pi}{14}\cos^2\frac{3\pi}{14}\cos^2\frac{5\pi}{14}}{(1-y)^6}$$

$$\left(y^2 + \tan^2\frac{\pi}{14}\right)\left(y^2 + \tan^2\frac{3\pi}{14}\right)\left(y^2 + \tan^2\frac{5\pi}{14}\right)$$

$$\therefore \quad (1+y)^7 + (1-y)^7 = 2^7 \cdot \frac{7}{64}\left(y^2 + \tan^2\frac{\pi}{14}\right)$$

$$\left(y^2 + \tan^2\frac{3\pi}{14}\right)\left(y^2 + \tan^2\frac{5\pi}{14}\right)$$

Using result (ii), we get

$$(1+y)^7 + (1-y)^7 = 14\left(y^2 + \tan^2\frac{\pi}{14}\right)$$

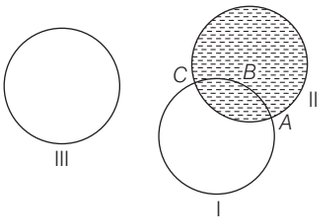
$$\left(y^2 + \tan^2\frac{3\pi}{14}\right)\left(y^2 + \tan^2\frac{5\pi}{14}\right)$$

Equating the coefficient of y^4 on both sides, we get

$${}^7C_4 + {}^7C_4 = 14\left[\tan^2\frac{\pi}{14} + \tan^2\frac{3\pi}{14} + \tan^2\frac{5\pi}{14}\right]$$

Therefore, $\tan^2\dfrac{\pi}{14} + \tan^2\dfrac{3\pi}{14} + \tan^2\dfrac{5\pi}{14} = 5$

101. Equation $|z| = 3$ represents boundary of a circle and equation $|z - \{a(1+i) - i\}| \le 3$ represents the interior and the boundary of a circle and equation $|z + 2a - (a+1)i| > 3$ represents the exterior of a circle. Then, any point which satisfies all the three conditions will lie on first circle, on or inside the second circle and outside the third circle.



For the existence of such a point first two circles must cut or atleast touch each other and first and third circles must not intersect each other. The arcABC of first circle lying inside the second but outside the third circle, represents all such possible points.

Let $z = x + iy$, then equation of circles are

$$x^2 + y^2 = 9 \qquad \ldots(i)$$

$$(x-a)^2 + (y-a+1)^2 = 9 \qquad \ldots(ii)$$

and $\quad (x+2a)^2 + (y-a-1)^2 = 9 \qquad \ldots(iii)$

Circles (i) and (ii) should cut or touch, then distance between their centres $\le$ sum of their radii

$$\Rightarrow \quad \sqrt{(a-0)^2 + (a-1-0)^2} \le 3 + 3$$

$$\Rightarrow \quad a^2 + (a-1)^2 \le 36$$

$$\Rightarrow \quad 2a^2 - 2a - 35 \le 0$$

$$\Rightarrow \quad a^2 - a - \frac{35}{2} \le 0 \text{ or } \left(a - \frac{1+\sqrt{71}}{2}\right)\left(a - \frac{1-\sqrt{71}}{2}\right) \le 0$$

$$\therefore \quad \frac{1-\sqrt{71}}{2} \le a \le \frac{1+\sqrt{71}}{2} \qquad \ldots(iv)$$

Again, circles (i) and (iii) should not cut or touch, then distance between their centres > sum of their radii

$$\sqrt{(-2a-0)^2 + (a+1-0)^2} > 3 + 3$$

or $$\sqrt{5a^2 + 2a + 1} > 6$$

$$\Rightarrow \quad 5a^2 + 2a + 1 > 36$$

or $$5a^2 + 2a - 35 > 0$$

$$\Rightarrow \quad a^2 + \frac{2a}{5} - 7 > 0$$

or $$\left(a - \frac{-1-4\sqrt{11}}{5}\right)\left(a - \frac{-1+4\sqrt{11}}{5}\right) > 0$$

$$\therefore \quad a \in \left(-\infty, \frac{-1-4\sqrt{11}}{5}\right) \cup \left(\frac{-1+4\sqrt{11}}{5}, \infty\right) \qquad \ldots(v)$$

Hence, the common values of a satisfying Eqs. (iv) and (v) are

$$a \in \left(\frac{1-\sqrt{71}}{2}, \frac{-1-4\sqrt{11}}{5}\right) \cup \left(\frac{-1+4\sqrt{11}}{5}, \frac{1+\sqrt{71}}{2}\right)$$

102. (i) From De-moivre's theorem, we know that

$$\sin(2n+1)\alpha = {}^{2n+1}C_1(1-\sin^2\alpha)^n \sin\alpha - {}^{2n+1}C_3(1-\sin^2\alpha)^{n-1}\sin^3\alpha + \ldots + (-1)^n \sin^{2n+1}\alpha$$

It follows that the numbers

$$\sin\frac{\pi}{2n+1}, \sin\frac{2\pi}{2n+1}, \ldots, \sin\frac{n\pi}{2n+1}$$

are the roots of the equation.

$${}^{2n+1}C_1(1-x^2)^n x - {}^{2n+1}C_3(1-x^2)^{n-1}x^3 + \ldots + (-1)^n x^{2n+1} = 0 \text{ of the } (2n+1) \text{ th degree}$$

Consequently, the numbers

$\sin^2\dfrac{\pi}{2n+1}, \sin^2\dfrac{2\pi}{2n+1}, \ldots, \sin^2\dfrac{n\pi}{2n+1}$ are the roots of the equation

${}^{2n+1}C_1(1-x)^n - {}^{2n+1}C_3(1-x)^{n-1}x + \ldots + (-1)^n x^n = 0$ of the nth degree

(ii) From De-moivre's theorem, we know that

$$\sin(2n+1)\alpha = {}^{2n+1}C_1(\cos\alpha)^{2n}\sin\alpha - {}^{2n+1}C_3(\cos\alpha)^{2n-2}\sin^3\alpha + \ldots + (-1)^n \sin^{2n+1}\alpha$$

or $\sin(2n+1)\alpha = \sin^{2n+1}\alpha \{{}^{2n+1}C_1 \cot^{2n}\alpha - {}^{2n+1}C_3 \cot^{2n-2}\alpha + {}^{2n+1}C_5 \cot^{2n-4}\alpha - \ldots\}$

It follows that $\alpha = \dfrac{\pi}{2n+1}, \dfrac{2\pi}{2n+1}, \dfrac{3\pi}{2n+1}, \ldots, \dfrac{n\pi}{2n+1}$

Therefore, equality holds

${}^{2n+1}C_1 \cot^{2n}\alpha - {}^{2n+1}C_3 \cot^{2n-2}\alpha + {}^{2n+1}C_5 \cot^{2n-4}\alpha - \ldots = 0$

It follows that the numbers

$\cot^2\dfrac{\pi}{2n+1}, \cot^2\dfrac{2\pi}{2n+1}, \ldots, \cot^2\dfrac{n\pi}{2n+1}$ are the roots of the equation

$${}^{2n+1}C_1 x^n - {}^{2n+1}C_3 x^{n-1} + {}^{2n+1}C_5 x^{n-2} - \ldots = 0$$

of the nth degree.

103. Let $y = |a + b\omega + c\omega^2|$. For y to be minimum, y^2 must be minimum.

$\therefore \quad y^2 = |a + b\omega + c\omega^2|^2 = (a + b\omega + c\omega^2)\overline{(a + b\omega + c\omega^2)}$

$= (a + b\omega + c\omega^2)(a + b\bar{\omega} + c\bar{\omega}^2)$

$y^2 = (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)3$

$= (a^2 + b^2 + c^2 - ab - bc - ca)$

$= \frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2]$

Since, a, b and c are not equal at a time, so minimum value of y^2 occurs when any two are same and third is differ by 1.

$\Rightarrow$ Minimum of $y = 1$ (as a, b, c are integers)

104. Equation of ray PQ is $\arg(z+1) = \frac{\pi}{4}$

Equation of ray PR is $\arg(z+1) = -\frac{\pi}{4}$

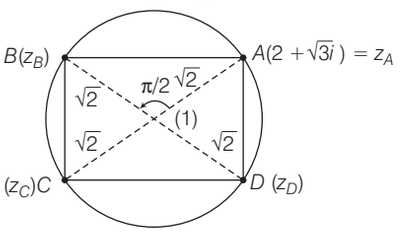
Shaded region is $-\frac{\pi}{4} < \arg(z+1) < \frac{\pi}{4} \Rightarrow |\arg(z+1)| < \frac{\pi}{4}$

$\therefore \quad |PQ| = \sqrt{(\sqrt{2})^2 + (\sqrt{2})^2} = 2$

So, arc QAR is of a circle of radius 2 units with centre at $P(-1, 0)$. All the points in the shaded region are exterior to this circle $|z+1| = 2$.

i.e. $|z+1| > 2$ and $|\arg(z+1)| < \frac{\pi}{4}$.

105. In ΔAOB from Coni method, $\frac{z_B - 1}{z_A - 1} = e^{i\pi/2} = i$



$z_B - 1 = (z_A - 1)\, i$

$\therefore \quad z_B = 1 + (2 + \sqrt{3}i - 1)\, i = 1 + (1 + i\sqrt{3})\, i$

$= 1 + i - \sqrt{3} = 1 - \sqrt{3} + i$

$z_C = 2 - z_A = 2 - (2 + \sqrt{3}i) = -\sqrt{3}\, i$

and $\quad z_D = 2 - z_B = 2 - (1 - \sqrt{3} + i) = 1 + \sqrt{3} - i$

Hence, other vertices are $(1 - \sqrt{3}) + i, -\sqrt{3}i, (1 + \sqrt{3}) - i$.

106. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\therefore |z_1 + z_2| = [(r_1\cos\theta_1 + r_2\cos\theta_2)^2 + (r_1\sin\theta_1 + r_2\sin\theta_2)^2]^{1/2}$

$= [|r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2)]^{1/2} = [(r_1 + r_2)^2]^{1/2}$

$\therefore |z_1 + z_2| = |z_1| + |z_2|$

Therefore, $\cos(\theta_1 - \theta_2) = 1$

$\Rightarrow \quad \theta_1 - \theta_2 = 0$

$\Rightarrow \quad \theta_1 = \theta_2$

Thus, $\arg(z_1) - \arg(z_2) = 0$

107. $(x-1)^3 = -8 \Rightarrow x - 1 = (-8)^{1/3}$

$\Rightarrow \quad x - 1 = -2, -2\omega, -2\omega^2$

$\Rightarrow \quad x = -1, 1 - 2\omega, 1 - 2\omega^2$

108. $\left|\frac{z}{z - \frac{i}{3}}\right| = 1 \Rightarrow |z| = \left|z - \frac{i}{3}\right|$

Clearly, locus of z is perpendicular bisector of line joining points having complex number $0 + i\ 0$ and $0 + \frac{i}{3}$.

Hence, z lies on a straight line.

109. Given, $\left(\frac{\omega - \bar{\omega}z}{1 - z}\right)$ is purely real $\Rightarrow z \neq 1$

$\therefore \quad \left(\frac{\omega - \bar{\omega}z}{1 - z}\right) = \overline{\left(\frac{\omega - \bar{\omega}z}{1 - z}\right)} = \frac{\bar{\omega} - \omega\bar{z}}{1 - \bar{z}}$

$\Rightarrow \quad (\bar{\omega} - \bar{\omega}z)(1 - \bar{z}) = (1 - z)(\bar{\omega} - \omega\bar{z})$

$\Rightarrow \quad (z\bar{z} - 1)(\omega - \bar{\omega}) = 0$

$\Rightarrow \quad (|z|^2 - 1)(2i\beta) = 0 \qquad [\because \omega = \alpha + i\beta]$

$\therefore \quad |z|^2 - 1 = 0$

$\Rightarrow \quad |z| = 1 \text{ and } z \neq 1 \qquad [\because \beta \neq 0]$

110. $\sum_{k=1}^{10} \sin\left(\frac{2k\pi}{11}\right) + i\cos\left(\frac{2k\pi}{11}\right)$

$= i\sum_{k=1}^{10}\left\{\cos\left(\frac{2k\pi}{11}\right) - i\sin\left(\frac{2k\pi}{11}\right)\right\} = i\sum_{k=1}^{10} e^{\frac{-2k\pi i}{11}}$

$= i\left(\sum_{k=0}^{10} e^{\frac{-2k\pi i}{11}} - 1\right) = i(0 - 1)$ [$\because$ sum of 11, 11th roots of unity $= 0$]

$= -i$

111. $\because \ z^2 + z + 1 = 0$

$\therefore \quad z = \omega, \omega^2$

$\because \quad z + \frac{1}{z} = \omega + \frac{1}{\omega} = \omega + \omega^2 = -1$

$\Rightarrow \quad z^2 + \frac{1}{z^2} = \omega^2 + \frac{1}{\omega^2} = \omega^2 + \omega = -1$

$\therefore \quad z^3 + \frac{1}{z^3} = \omega^3 + \frac{1}{\omega^3} = 1 + 1 = 2$

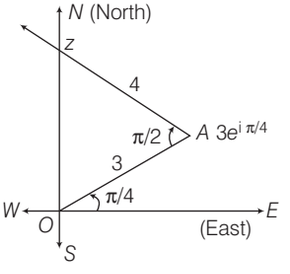
$z^4 + \frac{1}{z^4} = \omega^4 + \frac{1}{\omega^4} = \omega + \frac{1}{\omega} = -1$

$z^5 + \frac{1}{z^5} = \omega^5 + \frac{1}{\omega^5} = \omega^2 + \omega = -1$

and $\quad z^6 + \frac{1}{z^6} = \omega^6 + \frac{1}{\omega^6} = 2$

$\therefore$ Required sum $= (-1)^2 + (-1)^2 + (2)^2 + (-1)^2 + (-1)^2 + (2)^2 = 12$

112. Let $OA = 3$, so that the complex number associated with A is $3e^{i\pi/4}$. If z is the complex number associated with P, then



$$\frac{z-3e^{i\pi/4}}{0-3e^{i\pi/4}}=\frac{4}{3}e^{-\pi/2}=-\frac{4i}{3}$$

$$\Rightarrow \quad 3z-9e^{i\pi/4}=12ie^{i\pi/4} \Rightarrow z=(3+4i)e^{i\pi/4}$$

113. Let $z=\cos\theta+i\sin\theta$

$$\Rightarrow \frac{z}{1-z^2}=\frac{\cos\theta+i\sin\theta}{1-(\cos 2\theta+i\sin 2\theta)}$$

$$=\frac{\cos\theta+i\sin\theta}{2\sin^2\theta-2i\sin\theta\cos\theta}$$

$$=\frac{\cos\theta+i\sin\theta}{-2i\sin\theta(\cos\theta+i\sin\theta)}=\frac{i}{2\sin\theta}$$

Hence, $\frac{z}{1-z^2}$ lies on the imaginary axis i.e. $x=0$ or on Y-axis.

Aliter

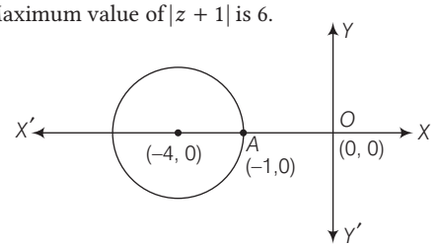
Let $E=\frac{z}{1-z^2}=\frac{z}{z\bar{z}-z^2}=\frac{1}{\bar{z}-z}=-\frac{1}{z-\bar{z}}=-\frac{1}{\left(\frac{z-\bar{z}}{2i}\right)2i}$

$=\frac{i}{2\,\text{Im}\,|z|}$ which is imaginary.

114. $|z+4|\le 3$

$\Rightarrow z$ lies inside or on the circle of radius 3 and centre at $(-4, 0)$.

$\therefore$ Maximum value of $|z+1|$ is 6.



115. Let A = set of points on and above the line $y=1$ in the argand plane.

B = set of points on the circle $(x-2)^2+(y-1)^2=3^2$

$C=\text{Re}\,(1-i)z=\text{Re}\,[(1-i)(x+iy)]=x+y$

$\Rightarrow \quad x+y=\sqrt{2}$

Hence, $(A\cap B\cap C)$ has only one point of intersection.

116. The points $(-1+i)$ and $(5+i)$ are the extremities of diameter of the given circle.

Hence, $|z+1-i|^2+|z-5-i|^2=36$

117. $\because |z-w|\le\big||z|-|w|\big|$

and $|z-w|$ = distance between z and w

z is fixed, hence distance between z and w would be maximum for diametrically opposite points.

$\Rightarrow \quad |z-w|<6 \Rightarrow ||z|-|w||<6$

$\Rightarrow -6<|z|-|w|<6 \Rightarrow -3<|z|-|w|+3<9$

118. $\because \quad z_0=1+2i$

$\therefore \; z_1=6+5i \Rightarrow z_2=-6+7i$

119. Put $(-i)$ in place of i.

Hence, $\frac{-1}{i+1}$

120. $\because \; z\bar{z}(\bar{z}^2+z^2)=350$

Put $z=x+iy$

$\Rightarrow \quad (x^2+y^2)\cdot 2(x^2-y^2)=350$

$\Rightarrow \quad (x^2+y^2)(x^2-y^2)=175=25\times 7$

$\Rightarrow \quad x^2+y^2=25,\; x^2-y^2=7$

$\Rightarrow \quad x^2=16,\; y^2=9$

$\therefore \quad x=\pm 4,\; y=\pm 3;\; x, y\in I$

Area of rectangle $=8\times 6=48$ sq units

121. $\sum_{m=1}^{15}\text{Im}\,(z^{2m-1})=\sum_{m=1}^{15}\text{Im}\,[e^{(2m-1)i\theta}]=\sum_{m=1}^{15}\sin(2m-1)\theta$

$$=\sum_{m=1}^{15}\frac{2\sin(2m-1)\theta\sin\theta}{2\sin\theta}$$

$$=\sum_{m=1}^{15}\frac{\cos(2m-2)\theta-\cos 2m\theta}{2\sin\theta}$$

$$=\frac{\cos 0^\circ-\cos 30\theta}{2\sin\theta}=\frac{1-\cos 60^\circ}{2\sin 2^\circ} \qquad (\because \theta=2^\circ)$$

$$=\frac{1-\frac{1}{2}}{2\sin 2^\circ}=\frac{1}{4\sin 2^\circ}$$

122. $\left|z-\frac{4}{z}\right|\ge\left||z|-\frac{4}{|z|}\right| \Rightarrow 2\ge\left||z|-\frac{4}{|z|}\right|$

$\Rightarrow \quad -2\le|z|-\frac{4}{|z|}\le 2 \Rightarrow -2|z|\le|z|^2-4\le 2|z|$

$\Rightarrow \quad |z|^2+2|z|-4\ge 0$

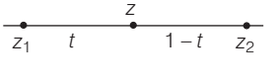
and $\quad 1^2-2|z|-4\le 0$

$\Rightarrow \quad (|z|+1)^2\ge 5$ and $(|z|-1)^2\le 5$

$-\sqrt{5}\le|z|-1\le\sqrt{5}$ and $|z|+1\ge\sqrt{5}$

$\Rightarrow \quad \sqrt{5}-1\le|z|\le\sqrt{5}+1$

123. As $z=(1-t)z_1+tz_2$



$\Rightarrow z_1, z$ and z_2 are collinear.

Thus, options (a) and (d) are correct.

Also, $\frac{z-z_1}{z_2-z_1}=\frac{\bar{z}-\bar{z}_1}{\bar{z}_2-\bar{z}_1}$

Hence, option (c) is correct.

124. $\omega=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$

ω is one of the cube root of unity.

$$\therefore \quad \begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix}=0$$

Applying $R_1\to R_1+R_2+R_3$, we get

$$\begin{vmatrix} z & z & z \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix}=0 \qquad [\because 1+\omega+\omega^2=0]$$

Now, applying $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$, we get

$$\begin{vmatrix} z & 0 & 0 \\ \omega & z+\omega^2-\omega & 1-\omega \\ \omega^2 & 1-w^2 & z+\omega-\omega^2 \end{vmatrix} = 0$$

$\Rightarrow z[(z+\omega^2-\omega)(z+\omega-\omega^2)-(1-\omega)(1-\omega^2)]=0$

$\Rightarrow z[z^2-(\omega^2-\omega)^2-(1-\omega^2-\omega+\omega^3)]=0$

$\Rightarrow z[z^2-(\omega^4+\omega^2-2\omega^3)-1+\omega^2+\omega-\omega^3]=0$

$\Rightarrow z^3=0$

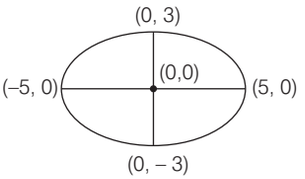
$\therefore z=0$

125. $|z-i|\,z\,||=|z+i|\,z\,||$

(A) Putting $z=x+iy$, we get $y\sqrt{x^2+y^2}=0$

i.e. $\text{Im}(z)=0$

(B) $2ae=8, 2a=10 \Rightarrow 10e=8 \Rightarrow e=\frac{4}{5}$



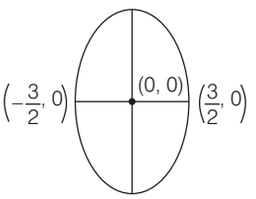
$\therefore \quad b^2=25\left(1-\frac{16}{25}\right)=9$

$\Rightarrow \quad \frac{x^2}{25}+\frac{y^2}{9}=1$

(C) $z=2(\cos\theta+i\sin\theta)-\frac{1}{2(\cos\theta+i\sin\theta)}$

$=2(\cos\theta+i\sin\theta)-\frac{1}{2}(\cos\theta-i\sin\theta)$

$z=\frac{3}{2}\cos\theta+\frac{5}{2}i\sin\theta$



Let $z=x+iy$, then

$x=\frac{3}{2}\cos\theta$ and $y=\frac{5}{2}\sin\theta$

$\Rightarrow \quad \left(\frac{2x}{3}\right)^2+\left(\frac{2y}{5}\right)^2=1$

$\Rightarrow \quad \frac{4x^2}{9}+\frac{4y^2}{25}=1$

$\Rightarrow \quad \frac{x^2}{9/4}+\frac{y^2}{25/4}=1$

$\Rightarrow \quad \frac{9}{4}=\frac{25}{4}(1-e^2)$

$\therefore \quad e^2=1-\frac{9}{25}=\frac{16}{25} \Rightarrow e=\frac{4}{5}$

(D) Let $w=\cos\theta+i\sin\theta$, then

$z=x+iy=w+\frac{1}{w}$

$\Rightarrow \quad x+iy=2\cos\theta$

$\therefore \quad x=2\cos\theta$ and $y=0$

126. $\because \quad x^2-x+1=0$

$\therefore \quad x=\frac{1\pm\sqrt{(1-4)}}{2}=\frac{1\pm i\sqrt{3}}{2}$

$=\frac{1+i\sqrt{3}}{2}$ and $\frac{1-i\sqrt{3}}{2}$

$\therefore \quad x=-\omega^2, -\omega$

$\therefore \quad \alpha=-\omega^2, \beta=-\omega$

$\Rightarrow \quad \alpha^{2009}+\beta^{2009}=-\omega^{4018}-\omega^{2009}$

$=-\omega-\omega^2=-(\omega+\omega^2)$

$=-(-1)=1$

127. $|z-1|=|z+1|=|z-i|$

$\Rightarrow \quad |z-1|^2=|z+1|^2=|z-i|^2$

$\Rightarrow (z-1)(\bar{z}-1)=(z+1)(\bar{z}+1)=(z-i)(\bar{z}+i)$

$\Rightarrow z\bar{z}-z-\bar{z}+1=z\bar{z}+z+\bar{z}+1=z\bar{z}+iz-i\bar{z}+1$

$\Rightarrow \quad -z-\bar{z}=z+\bar{z}=i(z-\bar{z})$

From first two relations,

$2(z+\bar{z})=0 \Rightarrow \text{Re}(z)=0$ ...(i)

From last two relations,

$z+\bar{z}=i(z-\bar{z}) \Rightarrow 2\,\text{Re}(z)=-2\,\text{Im}(z)$

From Eq. (i), $\text{Im}(z)=0$

$\therefore \quad z=\text{Re}(z)+i\,\text{Im}(z)=0+i\cdot 0=0$

Hence, number of solutions is one.

128. We have, $|z-3-2i|\le 2$

$\Rightarrow \quad |2z-6-4i|\le 4$ ...(i)

Now, $|2z-6-4i|=|(2z-6+5i)-9i|$

$\ge ||2z-6+5i|-9|$ ...(ii)

From Eqs. (i) and (ii), we get

$|2z-6+5i|-9|\le 4$

$\Rightarrow -4\le|2z-6+5i|-9\le 4$

$\Rightarrow \quad 5\le|2z-6+5i|\le 13$

Hence, the minimum value of $|2z-6+5i|$ is 5.

129. $\because \quad |z|=1 \quad \therefore \; z=e^{i\theta}$

$\therefore \quad \text{Re}\left(\frac{2iz}{1-z^2}\right)=\text{Re}\left(\frac{2ie^{i\theta}}{1-e^{2i\theta}}\right)=\text{Re}\left(\frac{2i}{e^{-i\theta}-e^{i\theta}}\right)$

$=\text{Re}\left(\frac{2i}{-2i\sin\theta}\right)=\text{Re}\left(-\frac{1}{\sin\theta}\right)$

$=-\frac{1}{\sin\theta}=-\,\text{cosec}\,\theta$

$\because \quad \text{cosec}\,\theta\le -1 \Rightarrow \text{cosec}\,\theta\ge 1$

$\Rightarrow -\text{cosec}\,\theta\ge 1 \Rightarrow -\text{cosec}\,\theta\le -1$

$\Rightarrow -\text{cosec}\,\theta\in(-\infty,-1]\cap[1,\infty)$

$\therefore \quad \text{Re}\left(\frac{2iz}{1-z^2}\right)\in(-\infty,-1]\cap[1,\infty)$

130. $\because\ |z| = 1$. Let $z = e^{i\theta}$

$\therefore \quad z - 1 = e^{i\theta} - 1 = e^{i\theta/2} \cdot 2i \sin(\theta/2)$

$\Rightarrow \quad \dfrac{1}{z-1} = \dfrac{1}{2ie^{i\theta/2} \cdot \sin(\theta/2)} = -\dfrac{ie^{-i\theta/2}}{2\sin(\theta/2)}$

$\Rightarrow \quad \dfrac{1}{1-z} = \dfrac{i \cdot e^{-i\theta/2}}{2\sin(\theta/2)} \quad \therefore \ \arg\left(\dfrac{1}{1-z}\right) = \left(\dfrac{\pi}{2} - \dfrac{\theta}{2}\right)$

$\Rightarrow \quad \left|\arg\left(\dfrac{1}{1-z}\right)\right| = \left|\dfrac{\pi}{2} - \dfrac{\theta}{2}\right|$

$\therefore$ Maximum value of $\left|\arg\left(\dfrac{1}{1-z}\right)\right| = \dfrac{\pi}{2}$

131. $\because\ |x|^2 = x\bar{x} = (a+b+c)(\bar{a}+\bar{b}+\bar{c})$

$= (a+b+c)(\bar{a}+\bar{b}+\bar{c})$

$= |a|^2 + |b|^2 + |c|^2 + a\bar{b} + \bar{a}b + b\bar{c} + \bar{b}c + c\bar{a} + \bar{c}a \quad \ldots(i)$

$|y|^2 = y\bar{y} = (a + b\omega + c\omega^2)(\bar{a} + \bar{b}\bar{\omega} + \bar{c}\bar{\omega}^2)$

$= (a + b\omega + c\omega^2)(\bar{a} + \bar{b}\bar{\omega} + \bar{c}\bar{\omega}^2)$

$= (a + b\omega + c\omega^2)(\bar{a} + \bar{b}\omega^2 + \bar{c}\omega)$

$= |a|^2 + |b|^2 + |c|^2 + a\bar{b}\omega^2 + \bar{a}b\omega$

$+ b\bar{c}\omega^2 + \bar{b}c\omega + c\bar{a}\omega^2 + \bar{c}a\omega \quad \ldots(ii)$

and $|z|^2 = z\bar{z} = (a + b\omega^2 + c\omega)\overline{(a + b\omega^2 + c\omega)}$

$= (a + b\omega^2 + c\omega)(\bar{a} + \bar{b}\bar{\omega}^2 + \bar{c}\bar{\omega})$

$= (a + b\omega^2 + c\omega)(\bar{a} + \bar{b}\omega + \bar{c}\omega^2)$

$= |a|^2 + |b|^2 + |c|^2 + a\bar{b}\omega + \bar{a}b\omega^2$

$+ b\bar{c}w + \bar{b}c\omega^2 + c\bar{a}\omega + \bar{c}a\omega^2 \quad \ldots(iii)$

On adding Eqs. (i), (ii) and (iii), we get

$|x|^2 + |y|^2 + |z|^2 = 3(|a|^2 + |b|^2 + |c|^2)$

$+ 0 + 0 + 0 + 0 + 0 + 0 \ (\because 1 + \omega + \omega^2 = 0)$

$\therefore \quad \dfrac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2} = 3$

132. $\because \quad \mathrm{Re}(z) = 1 \quad \therefore \ \dfrac{z + \bar{z}}{2} = 1 \ \Rightarrow \ z + \bar{z} = 2$

Since, $\alpha, \beta \in R$

$\therefore$ The complex roots are conjugate to each other, if z_1, z_2 are two distinct roots, then $z_1 = \bar{z}_2$ or $\bar{z}_1 = z_2$

$\therefore$ Product of the roots $= z_1 z_2 = \beta$

$\Rightarrow \quad z_1\bar{z}_1 = \beta$

$\therefore \quad \beta = |z_1|^2 = [\mathrm{Re}(z_1)]^2 + \mathrm{Im}\,|z_1|^2$

$= 1 + \mathrm{Im}\,|z_1|^2 > 1$

$[\because$ roots are distinct $\therefore \mathrm{Im}(z_1) \neq 0]$

$\therefore \quad \beta > 1 \quad \text{or} \quad \beta \in (1, \infty)$

133. $\because\ (1+\omega)^7 = (-\omega^2)^7 = -\omega^{14} = -\omega^2 = 1 + \omega$

Given, $(1+\omega)^7 = A + B\omega \ \Rightarrow \ 1 + \omega = A + B\omega$

On comparing, we get $A = 1, B = 1$

$\therefore \quad (A, B) = (1, 1)$

134. Given, $z^2 + z + 1 = a \ \Rightarrow \ z^2 + z + 1 - a = 0$

$\therefore \quad z = \dfrac{-1 \pm \sqrt{(4a-3)}}{2}$

Hence, $a \neq \dfrac{3}{4}$ [for $a = 3/4$, z will be purely real]

135. Let $z = x + iy$, then

$\dfrac{z^2}{z-1} = \dfrac{(x+iy)^2}{(x+iy-1)} = \dfrac{(x^2 - y^2 + 2ix)}{(x-1+iy)}$

$= \dfrac{(x^2 - y^2 + 2ixy)(x-1-iy)}{(x-1+iy)(x-1-iy)}$

$= \dfrac{(x-1)(x^2-y^2) + 2xy^2 + i[2xy(x-1) - y(x^2-y^2)]}{(x-1)^2 + y^2}$

Now, $\mathrm{Im}\left(\dfrac{z^2}{z-1}\right) = 0$

$\Rightarrow \quad 2xy(x-1) - y(x^2 - y^2) = 0$

$\Rightarrow \quad y(2x^2 - 2x - x^2 + y^2) = 0$

$\Rightarrow \quad y(x^2 + y^2 - 2x) = 0$

$\Rightarrow \quad y = 0 \ \text{ or } \ x^2 + y^2 - 2x = 0$

Hence, z lies on the real axis or on a circle passing through the origin.

136. Given, $|z| = 1$ and $\arg(z) = \theta \quad \ldots(i)$

$\Rightarrow \quad |z|^2 = 1 \ \Rightarrow \ z\bar{z} = 1$

$\Rightarrow \quad \bar{z} = \dfrac{1}{z} \quad \ldots(ii)$

$\therefore \quad \arg\left(\dfrac{1+z}{1+\bar{z}}\right) = \arg\left(\dfrac{1+z}{1+1/z}\right)$ [from Eq. (ii)]

$= \arg(z) = \theta$ [from Eq. (i)]

Aliter I

Given, $|z| = 1$ and $\arg(z) = \theta$

$\Rightarrow \quad z = e^{i\theta}$

$\therefore \quad \arg\left(\dfrac{1+z}{1+\bar{z}}\right) = \arg\left(\dfrac{1+e^{i\theta}}{1+e^{-i\theta}}\right) = \arg(e^{i\theta}) = \arg(z) = \theta$

Aliter II Given, $|z| = 1$ and $\arg(z) = \theta$

Let $z = \omega$ (cube root of unity)

$\therefore \quad \arg\left(\dfrac{1+z}{1+\bar{z}}\right) = \arg\left(\dfrac{1+\omega}{1+\bar{\omega}}\right) = \arg\left(\dfrac{1+\omega}{1+\omega^2}\right) \quad (\because \bar{\omega} = \omega^2)$

$= \arg\left(\dfrac{-\omega^2}{-\omega}\right) \quad (\because 1 + \omega + \omega^2 = 0)$

$= \arg(\omega) = \arg(z) = \theta$

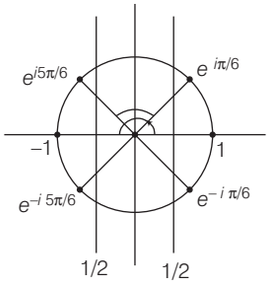
137. $z_0 = 2\alpha - \dfrac{1}{\bar{\alpha}}$

$\because \quad 2|z_0|^2 = r^2 + 2$

$\therefore \quad 2\left|2\alpha - \dfrac{1}{\bar{\alpha}}\right|^2 = r^2 + 2 \Rightarrow 2\left|2\alpha - \dfrac{1}{\bar{\alpha}}\right|^2 = \left|\alpha - \dfrac{1}{\bar{\alpha}}\right|^2 + 2$

$\Rightarrow 7|\alpha|^2 + \dfrac{1}{|\alpha|^2} - 8 = 0 \ \Rightarrow \ |\alpha|^2 = 1 \text{ or } \dfrac{1}{7} \ \Rightarrow \ |\alpha| = 1 \text{ or } \dfrac{1}{\sqrt{7}}$

138.



$$\omega = \frac{\sqrt{3}+i}{2} = e^{i\pi/6}, \quad P = e^{in\pi/6}$$

As $z_1 \in P \cap H_1 \Rightarrow z_1 = 1, e^{i\pi/6}, e^{-i\pi/6}$

As $z_2 \in P \cap H_2 \Rightarrow z_2 = -1, e^{i5\pi/6}, e^{-i5\pi/6}$

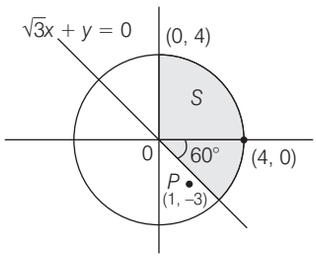
$\angle z_1 O z_2 = 2\pi/3$, where $z_1 = e^{i\pi/6}, z_2 = e^{i5\pi/6}$

Sol. (Q. Nos. 139-140)

Let $z = x + iy, S_1 : x^2 + y^2 < 16$

Now, $\text{Im}\left[\dfrac{(x-1)+i(y+\sqrt{3})}{1-\sqrt{3}i}\right] > 0$

$\Rightarrow \qquad S_2 : \sqrt{3}x + y > 0 \Rightarrow S_3 : x > 0$



139. $\min|1 - 3i - z| = \min|z - 1 + 3i|$

= perpendicular distance of the point $(1, -3)$ from the straight line $\sqrt{3}x + y = 0 = \left|\dfrac{\sqrt{3}-3}{2}\right| = \dfrac{3-\sqrt{3}}{2}$

140. Area of $S = \left(\dfrac{1}{4}\right)\pi \times 4^2 + \left(\dfrac{1}{6}\right)\pi \times 4^2 = \dfrac{20\pi}{3}$

141. Since, $|z| \geq 2$ is the region lying on or outside circle centered at $(0, 0)$ and radius 2. Therefore, $|z + (1/2)|$ is the distance of z from $(-1/2, 0)$, which lies inside the circle.

Hence, minimum value of $|z + (1/2)|$

= distance of $(-1/2, 0)$ from $(-2, 0)$

$$= \sqrt{\left(-\frac{1}{2}+2\right)^2 + |0-0|^2} = 3/2$$

Aliter

$\because \quad |z + (1/2)| \geq \left||z| - \dfrac{1}{2}\right| \geq \left|2 - \dfrac{1}{2}\right| \qquad [\because |z| \geq 2]$

$\therefore \qquad |z + (1/2)| \geq 3/2$

142. Clearly, $z_k^{10} = 1, \forall\ k$, where $z_k \neq 1$

(A) $z_k \cdot z_j = e^{i(2\pi/10)(k+j)} = 1$, if $(k + j)$ is multiple of 10

i.e. possible for each k.

(B) $z_1 \cdot z = z_k$ is clearly incorrect.

(C) Expression $= \dfrac{\left|\lim\limits_{z\to 1}\dfrac{z^{10}-1}{z-1}\right|}{10} = 1$

(D) $1 + \Sigma z_k = 0 \Rightarrow 1 + \sum\limits_{k=1}^{9} \cos\left(\dfrac{2k\pi}{10}\right) = 0$

$\therefore$ Expression $= 2$

143. $\because \qquad \left|\dfrac{z_1 - 2z_2}{2 - z_1\bar{z}_2}\right| = 1$

$\Rightarrow |z_1 - 2z_2|^2 = |2 - z_1\bar{z}_2|^2$

$\Rightarrow (z_1 - 2z_2)\overline{(z_1 - 2z_2)} = (2 - z_1\bar{z}_2)\overline{(2 - z_1\bar{z}_2)}$

$\Rightarrow (z_1 - 2z_2)(\bar{z}_1 - 2\bar{z}_2) = (2 - z_1\bar{z}_2)(2 - \bar{z}_1 z_2)$

$\Rightarrow z_1\bar{z}_1 - 2z_1\bar{z}_2 - 2\bar{z}_1 z_2 + 4z_2\bar{z}_2 = 4 - 2\bar{z}_1 z_2 - 2z_1\bar{z}_2 + z_1\bar{z}_1 z_2\bar{z}_2$

$\Rightarrow |z_1|^2 + 4|z_2|^2 + 4 + |z_1|^2|z_2|^2$

$\Rightarrow \quad (|z_1|^2 - 4)(1 - |z_2|^2) = 0$

$\because \qquad |z_2| \neq 1$

$\therefore \qquad |z_1|^2 = 4 \quad \text{or} \quad |z_1| = 2$

$\Rightarrow$ Point z_1 lies on circle of radius 2.

144. Let $a = 3, b = -3, c = 2$, then

$(a + b\omega + c\omega^2)^{4n+3} + (c + a\omega + b\omega^2)^{4n+3} + (b + c\omega + a\omega^2)^{4n+3} = 0$

$$\Rightarrow (a + b\omega + c\omega^2)^{4n+3}\left\{1 + \left(\frac{c + a\omega + b\omega^2)^{4n+3}}{a + b\omega + c\omega^2}\right)^{4n+3} + \left(\frac{b + c\omega + a\omega^2)}{a + b\omega + c\omega^2}\right)^{4n+3}\right\} = 0$$

$\Rightarrow \quad (a + b\omega + c\omega^2)^{4n+3}(1 + \omega^{4n+3} + (\omega^2)^{4n+3}) = 0$

$\Rightarrow \quad 4n + 3$ should be an integer other than multiple of 3.

$\therefore \quad n = 1, 2, 4, 5$

145. $\because \quad \alpha_k = \cos\left(\dfrac{k\pi}{7}\right) + i\sin\left(\dfrac{k\pi}{7}\right) = e^{i\pi k/7}$

$\therefore \quad \alpha_{k+1} - \alpha k = e^{i\pi(k+1)/7} - e^{i\pi k/7} = e^{i\pi k/7}(e^{i\pi/7} - 1)$

$$= e^{i\pi k/7} \cdot e^{i\pi/14} \cdot 2i\sin\left(\frac{\pi}{14}\right)$$

$$\Rightarrow |\alpha_{k+1} - \alpha_k| = 2\sin\left(\frac{\pi}{14}\right)$$

$$\sum_{k=1}^{12} |\alpha_{k+1} - \alpha_k| = 12 \times 2\sin\left(\frac{\pi}{14}\right) = 24\sin\left(\frac{\pi}{14}\right)$$

and $\alpha_{4k-1} - \alpha_{4k-2} = e^{i\pi(4k-1)/7} - e^{i\pi(4k-2)/7} = e^{i\pi(4k-2)/7}(e^{i\pi/7} - 1)$

$$= e^{i\pi(4k-2)/7} \cdot e^{i\pi/14} \cdot 2i\sin\left(\frac{\pi}{14}\right)$$

$$\Rightarrow \quad |\alpha_{4k-1} - \alpha_{4k-2}| = 2\sin\left(\frac{\pi}{14}\right)$$

$$\therefore \quad \sum_{k=1}^{3} |\alpha_{4k-1} - \alpha_{4k-2}| = 3 \times 2\sin\left(\frac{\pi}{14}\right) = 6\sin\left(\frac{\pi}{14}\right)$$

Hence, $$\frac{\sum_{k=1}^{12}|\alpha_{k+1}-\alpha_k|}{\sum_{k=1}^{3}|\alpha_{4k-1}-\alpha_{4k-2}|}=4$$

146. Let $z=\dfrac{2+3i\sin\theta}{1-2i\sin\theta}$

$\because$ z is purely imaginary

$\therefore$ $\bar{z}=-z$

$\Rightarrow$ $$\left(\frac{2+3i\sin\theta}{1-2i\sin\theta}\right)=-\left(\frac{2+3i\sin\theta}{1-2i\sin\theta}\right)$$

$\Rightarrow$ $$\left(\frac{2-3i\sin\theta}{1+2i\sin\theta}\right)=-\left(\frac{2+3i\sin\theta}{1-2i\sin\theta}\right)$$

$\Rightarrow$ $(2-3i\sin\theta)(1-2i\sin\theta)+(1+2i\sin\theta)(2+3i\sin\theta)=0$

$\Rightarrow$ $4-12\sin^2\theta=0$ or $\sin^2\theta=\dfrac{1}{3}$

$\therefore$ $\theta=\sin^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$

147. $\because$ $$x+iy=\frac{1}{a+ibt}$$

$\Rightarrow$ $$x+iy=\frac{a-ibt}{a^2+b^2t^2}$$

$\Rightarrow$ $$x=\frac{a}{(a^2+b^2t^2)},\ y=-\frac{bt}{(a^2+b^2t^2)}$$

or $$x^2+y^2=\frac{1}{a^2+b^2t^2}=\frac{x}{a}$$

or $$x^2+y^2-\frac{x}{a}=0$$

$\therefore$ Locus of z is a circle with centre $\left(\dfrac{1}{2a},0\right)$

and radius $=\dfrac{1}{2a}$, $a>0$.

Also for $b=0$, $a\neq 0$, we get $y=0$.

$\therefore$ locus is X-axis and for $a=0$, $b\neq 0$, we get $x=0$

$\therefore$ locus is Y-axis.

148. Let $\Delta=\begin{vmatrix}1 & 1 & 1\\ 1 & -\omega^2-1 & \omega^2\\ 1 & \omega^2 & \omega^7\end{vmatrix}=\begin{vmatrix}1 & 1 & 1\\ 1 & \omega & \omega^2\\ 1 & \omega^2 & \omega\end{vmatrix}$

$(\because 1+\omega+\omega^2=0$ and $\omega^3=1)$

Applying $C_1\to C_1+C_2+C_3$, then we get

$$\Delta=\begin{vmatrix}3 & \cdots & 1 & \cdots & 1\\ \vdots & & & & \\ 0 & & \omega & & \omega^2\\ \vdots & & & & \\ 0 & & \omega^2 & & \omega\end{vmatrix}\qquad(\because\ 1+\omega+\omega^2=0)$$

$=3(\omega^2-\omega^4)$

$=3(-1-\omega-\omega)$ $(\because\ \omega^3=1$ and $1+\omega+\omega^2=0)$

$=-3(1+2\omega)$

$=-3z=3k$ (given) $(\because\ 1+2\omega=z)$

$\therefore$ $k=-z$

CHAPTER

02

Theory of Equations

Learning Part

Session 1

- Polynomial in One Variable
- Linear Equation
- Standard Quadratic Equation
- Identity
- Quadratic Equations

Session 2

- Transformation of Quadratic Equations
- Condition for Common Roots

Session 3

- Quadratic Expression
- Wavy Curve Method
- Condition for Resolution into Linear Factors
- Location of Roots (Interval in which Roots Lie)

Session 4

- Equations of Higher Degree
- Rational Algebraic Inequalities
- Roots of Equation with the Help of Graphs

Session 5

- Irrational Equations
- Irrational Inequations
- Exponential Equations
- Exponential Inequations
- Logarithmic Equations
- Logarithmic Inequations

Practice Part

- JEE Type Examples
- Chapter Exercises

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Session 1

Polynomial in One Variable, Identity, Linear Equation, Quadratic Equations, Standard Quadratic Equation

Polynomial in One Variable

An algebraic expression containing many terms of the form cx^n, n being a non-negative integer is called a polynomial,

i.e., $f(x) = a_0 \cdot x^n + a_1 \cdot x^{n-1} + a_2 \cdot x^{n-2} + \ldots + a_{n-1} \cdot x + a_n$,

where x is a variable, $a_0, a_1, a_2, \ldots, a_n$ are constants and $a_0 \neq 0$.

1. Real Polynomial

Let $a_0, a_1, a_2, \ldots, a_n$ be real numbers and x is a real variable. Then,

$f(x) = a_0 \cdot x^n + a_1 \cdot x^{n-1} + a_2 \cdot x^{n-2} + \ldots + a_{n-1} \cdot x + a_n$

is called a real polynomial of real variable (x) with real coefficients.

For example, $5x^3 - 3x^2 + 7x - 4, x^2 - 3x + 1$, etc., are real polynomials.

2. Complex Polynomial

Let $a_0, a_1, a_2, \ldots, a_n$ are complex numbers and x is a varying complex number.

Then $f(x) = a_0 \cdot x^n + a_1 \cdot x^{n-1} + a_2 \cdot x^{n-2} + \ldots + a_{n-1} \cdot x + a_n$

is called a complex polynomial or a polynomial of complex variable with complex coefficients.

For example, $x^3 - 7ix^2 + (3-2i)x + 13, 3x^2 - (2+3i)x + 5i$, etc. (where $i = \sqrt{-1}$) are complex polynomials.

3. Rational Expression or Rational Function

An expression of the form $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials in x, is called a rational expression. As a particular case when $Q(x)$ is a non-zero constant, $\frac{P(x)}{Q(x)}$ reduces to a polynomial.

Thus, every polynomial is a rational expression but a rational expression may or may not be a polynomial.

For example,

(i) $x^2 - 7x + 8$ (ii) $\frac{2}{x-3}$

(iii) $\frac{x^3 - 6x^2 + 11x - 6}{(x-4)}$ (iv) $x + \frac{3}{x}$ or $\frac{x^2+3}{x}$

4. Degree of Polynomial

The highest power of variable (x) present in the polynomial is called the degree of the polynomial.

For example, $f(x) = a_0 \cdot x^n + a_1 \cdot x^{n-1} + a_2 \cdot x^{n-2} + \ldots + a_{n-1} \cdot x + a_n$ is a polynomial in x of degree n.

Remark

A polynomial of degree one is generally called a linear polynomial. Polynomials of degree 2, 3, 4 and 5 are known as quadratic, cubic, biquadratic and pentic polynomials, respectively.

5. Polynomial Equation

If $f(x)$ is a polynomial, real or complex, then $f(x) = 0$ is called a polynomial equation.

(i) A polynomial equation has atleast one root.

(ii) A polynomial equation of degree n has n roots.

Remarks

1. A polynomial equation of degree one is called a **linear equation** i.e. $ax + b = 0$, where $a, b \in C$, set of all complex numbers and $a \neq 0$.
2. A polynomial equation of degree two is called a **quadratic equation** i.e., $ax^2 + bx + c$, where $a, b, c \in C$ and $a \neq 0$.
3. A polynomial equation of degree three is called a **cubic equation** i.e., $ax^3 + bx^2 + cx + d = 0$, where $a, b, c, d \in C$ and $a \neq 0$.
4. A polynomial equation of degree four is called a **biquadratic equation** i.e., $ax^4 + bx^3 + cx^2 + dx + e = 0$, where $a, b, c, d, e \in C$ and $a \neq 0$.
5. A polynomial equation of degree five is called a **pentic equation** i.e., $ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$, where $a, b, c, d, e, f \in C$ and $a \neq 0$.

6. Roots of an Equation

The values of the variable for which an equation is satisfied are called the roots of the equation.

If $x=\alpha$ is a root of the equation $f(x)=0$, then $f(\alpha)=0$.

Remark

The real roots of an equation $f(x)=0$ are the values of x, where the curve $y=f(x)$ crosses X-axis.

7. Solution Set

The set of all roots of an equation in a given domain is called the solution set of the equation.

For example, The roots of the equation $x^3-2x^2-5x+6=0$ are $1,-2,3$, the solution set is $\{1,-2,3\}$.

Remark

Solve or solving an equation means finding its solution set or obtaining all its roots.

Identity

If two expressions are equal for all values of x, then the statement of equality between the two expressions is called an identity.

For example, $(x+1)^2=x^2+2x+1$ is an identity in x.

or

If $f(x)=0$ is satisfied by every value of x in the domain of $f(x)$, then it is called an identity.

For example, $f(x)=(x+1)^2-(x^2+2x+1)=0$ is an identity in the domain C, as it is satisfied by every complex number.

or

If $f(x)=a_0\cdot x^n+a_1\cdot x^{n-1}+a_2\cdot x^{n-2}+\ldots+a_{n-1}\cdot x+a_n=0$ have more than n distinct roots, it is an identity, then

$$a_0=a_1=a_2=\ldots=a_{n-1}=a_n=0$$

For example, If $ax^2+bx+c=0$ is satisfied by more than two values of x, then $a=b=c=0$.

or

In an identity in x coefficients of similar powers of x on the two sides are equal.

For example, If $ax^4+bx^3+cx^2+dx+e=5x^4-3x^3+4x^2-7x-9$ be an identity in x, then

$a=5, b=-3, c=4, d=-7, e=-9$.

Thus, an identity in x satisfied by all values of x, where as an equation in x is satisfied by some particular values of x.

Example 1. ***If equation*** $(\lambda^2-5\lambda+6)x^2+(\lambda^2-3\lambda+2)x+(\lambda^2-4)=0$ ***is satisfied by more than two values of*** x***, find the parameter*** λ***.***

Sol. If an equation of degree two is satisfied by more than two values of unknown, then it must be an identity. Then, we must have

$$\lambda^2-5\lambda+6=0, \lambda^2-3\lambda+2=0, \lambda^2-4=0$$

$\Rightarrow \quad \lambda=2,3$ and $\lambda=2,1$ and $\lambda=2,-2$

Common value of λ which satisfies each condition is $\lambda=2$.

Example 2. ***Show that***

$$\frac{(x+b)(x+c)}{(b-a)(c-a)}+\frac{(x+c)(x+a)}{(c-b)(a-b)}+\frac{(x+a)(x+b)}{(a-c)(b-c)}=1$$

is an identity.

Sol. Given relation is

$$\frac{(x+b)(x+c)}{(b-a)(c-a)}+\frac{(x+c)(x+a)}{(c-b)(a-b)}+\frac{(x+a)(x+b)}{(a-c)(b-c)}=1 \quad \ldots(i)$$

When $x=-a$, then LHS of Eq. (i) $=\dfrac{(b-a)(c-a)}{(b-a)(c-a)}=1$

= RHS of Eq. (i)

When $x=-b$, then LHS of Eq. (i)

$=\dfrac{(c-b)(a-b)}{(c-b)(a-b)}=1=$ RHS of Eq. (i)

and when $x=-c$, then LHS of Eq. (i) $=\dfrac{(a-c)(b-c)}{(a-c)(b-c)}=1$

= RHS of Eq. (i).

Thus, highest power of x occurring in relation of Eq. (i) is 2 and this relation is satisfied by three distinct values of $x(=-a,-b,-c)$. Therefore, it cannot be an equation and hence it is an identity.

Example 3. ***Show that*** $x^2-3|x|+2=0$ ***is an equation.***

Sol. Put $x=0$ in $x^2-3|x|+2=0$

$\Rightarrow \quad 0^2-3|0|+2=2\neq 0$

Since, the relation $x^2-3|x|+2=0$ is not satisfied by $x=0$.

Hence, it is an equation.

Linear Equation

An equation of the form

$$ax+b=0 \quad \ldots(i)$$

where $a, b\in R$ and $a\neq 0$, is a linear equation.

Eq. (i) has an unique root equal to $-\dfrac{b}{a}$.

Example 4. ***Solve the equation*** $\frac{x}{2}+\frac{(3x-1)}{6}=1-\frac{x}{2}$

Sol. We have, $\frac{x}{2}+\frac{(3x-1)}{6}=1-\frac{x}{2}$

or $\frac{x}{2}+\frac{x}{2}+\frac{x}{2}=1+\frac{1}{6}$

or $\frac{3x}{2}=\frac{7}{6}$

or $x=\frac{7}{9}$

Example 5. ***Solve the equation*** $(a-3)x+5=a+2$.

Sol. ***Case*** **I** For $a\neq 3$, this equation is linear, then

$(a-3)\,x=(a-3)$

$\therefore \quad x=\frac{(a-3)}{(a-3)}=1$

Case **II** For $a=3$,

$0\cdot x+5=3+2$

$\Rightarrow \quad 5=5$

Therefore, any real number is its solution.

Quadratic Equations

An equation in which the highest power of the unknown quantity is 2, is called a quadratic equation.

Quadratic equations are of two types :

1. Purely Quadratic Equation

A quadratic equation in which the term containing the first degree of the unknown quantity is absent, is called a purely quadratic equation.

i.e., $ax^2+c=0$,

where $a, c\in C$ and $a\neq 0$.

2. Adfected Quadratic Equation

A quadratic equation in which it contains the terms of first as well as second degrees of the unknown quantity, is called an adfected (or complete) quadratic equation.

i.e., $ax^2+bx+c=0$,

where $a, b, c\in C$ and $a\neq 0, b\neq 0$.

Standard Quadratic Equation

An equation of the form

$$ax^2+bx+c=0 \qquad \text{...(i)}$$

where $a, b, c\in C$ and $a\neq 0$, is called a standard quadratic equation.

The numbers a, b, c are called the coefficients of this equation.

A root of the quadratic Eq. (i) is a complex number α, such that $a\alpha^2+b\alpha+c=0$. Recall that $D=b^2-4ac$ is the discriminant of the Eq. (i) and its roots are given by the following formula.

$$x=\frac{-b\pm\sqrt{D}}{2a} \quad \text{[Shridharacharya method]}$$

Nature of Roots

1. If $a, b, c\in R$ and $a\neq 0$, then

(i) If $D<0$, then Eq. (i) has non-real complex roots.

(ii) If $D>0$, then Eq. (i) has real and distinct roots, namely

$$x_1=\frac{-b+\sqrt{D}}{2a}, x_2=\frac{-b-\sqrt{D}}{2a} \text{ and then}$$

$$ax^2+bx+c=a(x-x_1)(x-x_2). \qquad \text{...(ii)}$$

(iii) If $D=0$, then Eq. (i) has real and equal roots, then

$x_1=x_2=-\frac{b}{2a}$ and then

$$ax^2+bx+c=a(x-x_1)^2. \qquad \text{...(iii)}$$

To represent the quadratic ax^2+bx+c in form Eqs. (ii) or (iii), is to expand it into linear factors.

(iv) If $D\geq 0$, then Eq. (i) has real roots.

(v) If D_1 and D_2 be the discriminants of two quadratic equations, then

(a) If $D_1+D_2\geq 0$, then

- atleast one of D_1 and $D_2\geq 0$.
- if $D_1<0$, then $D_2>0$ and if $D_1>0$, then $D_2<0$.

(b) If $D_1+D_2<0$, then

- atleast one of D_1 and $D_2<0$.
- If $D_1<0$, then $D_2>0$ and if $D_1>0$, then $D_2<0$.

2. If $a, b, c\in Q$ and D is a perfect square of a rational number, the roots are rational and in case it is not a perfect square, the roots are irrational.

3. If $a, b, c\in R$ and $p+iq$ is one root of Eq. (i) $(q\neq 0)$, then the other must be the conjugate $(p-iq)$ and *vice-versa* (where, $p, q\in R$ and $i=\sqrt{-1}$).

4. If $a, b, c\in Q$ and $p+\sqrt{q}$ is one root of Eq. (i), then the other must be the conjugate $p-\sqrt{q}$ and *vice-versa* (where, p is a rational and $\sqrt{q}$ is a surd).

5. If $a=1$ and $b, c\in I$ and the roots of Eq. (i) are rational numbers, these roots must be integers.

6. If $a+b+c=0$ and a, b, c are rational, 1 is a root of the Eq. (i) and roots of the Eq. (i) are rational.

7. $a^2+b^2+c^2-ab-bc-ca=\frac{1}{2}$

$\{(a-b)^2+(b-c)^2+(c-a)^2\}$

$=-\{(a-b)(b-c)+(b-c)(c-a)+(c-a)(a-b)\}$

Example 6. Find all values of the parameter a for which the quadratic equation

$$(a+1)x^2+2(a+1)x+a-2=0$$

(i) has two distinct roots.

(ii) has no roots.

(iii) has two equal roots.

Sol. By the hypothesis, this equation is quadratic and therefore $a \neq -1$ and the discriminant of this equation,

$$D=4(a+1)^2-4(a+1)(a-2)$$
$$=4(a+1)(a+1-a+2)$$
$$=12(a+1)$$

(i) For $a>(-1)$, then $D>0$, this equation has two distinct roots.

(ii) For $a<(-1)$, then $D<0$, this equation has no roots.

(iii) This equation cannot have two equal roots. Since, $D=0$ only for $a=-1$ and this contradicts the hypothesis.

Example 7. ***Solve for*** x,
$(5+2\sqrt{6})^{x^2-3}+(5-2\sqrt{6})^{x^2-3}=10$.

Sol. $\because$ $(5+2\sqrt{6})(5-2\sqrt{6})=1$

$\therefore$ $(5-2\sqrt{6})=\frac{1}{(5+2\sqrt{6})}$

$\therefore$ $(5+2\sqrt{6})^{x^2-3}+(5-2\sqrt{6})^{x^2-3}=10$

reduces to $(5+2\sqrt{6})^{x^2-3}+\left(\frac{1}{5+2\sqrt{6}}\right)^{x^2-3}=10$

Put $(5+2\sqrt{6})^{x^2-3}=t$, then $t+\frac{1}{t}=10$

$\Rightarrow$ $t^2-10t+1=0$

or $t=\frac{10\pm\sqrt{(100-4)}}{2}=(5\pm2\sqrt{6})$

$\Rightarrow$ $(5+2\sqrt{6})^{x^2-3}=(5\pm2\sqrt{6})=(5+2\sqrt{6})^{\pm1}$

$\therefore$ $x^2-3=\pm1$

$\Rightarrow$ $x^2-3=1$ or $x^2-3=-1$

$\Rightarrow$ $x^2=4$ or $x^2=2$

Hence, $x=\pm2, \pm\sqrt{2}$

Example 8. Show that if p, q, r and s are real numbers and $pr=2(q+s)$, then atleast one of the equations $x^2+px+q=0$ and $x^2+rx+s=0$ has real roots.

Sol. Let D_1 and D_2 be the discriminants of the given equations $x^2+px+q=0$ and $x^2+rx+s=0$, respectively.

Now, $D_1+D_2=p^2-4q+r^2-4s=p^2+r^2-4(q+s)$

$=p^2+r^2-2pr$ [given, $pr=2(q+s)$]

$=(p-r)^2\geq0$ [$\because$ p and q are real]

or $D_1+D_2\geq0$

Hence, atleast one of the equations $x^2+px+q=0$ and $x^2+rx+s=0$ has real roots.

Example 9. ***If*** α, β ***are the roots of the equation*** $(x-a)(x-b)=c, c\neq0$. ***Find the roots of the equation*** $(x-\alpha)(x-\beta)+c=0$.

Sol. Since, α, β are the roots of

$$(x-a)(x-b)=c$$

or $(x-a)(x-b)-c=0$,

Then $(x-a)(x-b)-c=(x-\alpha)(x-\beta)$

$\Rightarrow$ $(x-\alpha)(x-\beta)+c=(x-a)(x-b)$

Hence, roots of $(x-\alpha)(x-\beta)+c=0$ are a, b.

Example 10. ***Find all roots of the equation*** $x^4+2x^3-16x^2-22x+7=0$, ***if one root is*** $2+\sqrt{3}$.

Sol. All coefficients are real, irrational roots will occur in conjugate pairs.

Hence, another root is $2-\sqrt{3}$.

$\therefore$ Product of these roots $=(x-2-\sqrt{3})(x-2+\sqrt{3})$

$=(x-2)^2-3=x^2-4x+1$.

On dividing $x^4+2x^3-16x^2-22x+7$ by x^2-4x+1, then the other quadratic factor is x^2+6x+7.

Then, the given equation reduce in the form

$$(x^2-4x+1)(x^2+6x+7)=0$$

$\therefore$ $x^2+6x+7=0$

Then, $x=\frac{-6\pm\sqrt{36-28}}{2}=-3\pm\sqrt{2}$

Hence, the other roots are $2-\sqrt{3}, -3\pm\sqrt{2}$.

Relation between Roots and Coefficients

1. **Relation between roots and coefficients of quadratic equation** If roots of the equation $ax^2+bx+c=0\,(a\neq0)$ be real and distinct and $\alpha<\beta$, then $\alpha=\frac{-b+\sqrt{D}}{2a}, \beta=\frac{-b-\sqrt{D}}{2a}$.

(i) Sum of roots

$$= S = \alpha + \beta = -\frac{b}{a} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}.$$

(ii) Product of roots

$$= P = \alpha\beta = \frac{c}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}.$$

(iii) Difference of roots

$$= D' = \alpha - \beta = \frac{\sqrt{D}}{a} = \frac{\sqrt{\text{Discriminant}}}{\text{Coefficient of } x^2}.$$

2. **Formation of an equation with given roots**
A quadratic equation whose roots are α and β, is given by $(x-\alpha)(x-\beta)=0$ or $x^2-(\alpha+\beta)x+\alpha\beta=0$
i.e. x^2 − (Sum of roots) x + Product of roots = 0

$$\therefore \qquad x^2 - Sx + P = 0.$$

3. **Symmetric function of roots** A function of α and β is said to be symmetric function, if it remains unchanged, when α and β are interchanged.

For example, $\alpha^3+3\alpha^2\beta+3\alpha\beta^2+\beta^3$ is a symmetric function of α and β, whereas $\alpha^3-\beta^3+5\alpha\beta$ is not a symmetric function of α and β. In order to find the value of a symmetric function in terms of $\alpha+\beta, \alpha\beta$ and $\alpha-\beta$ and also in terms of a, b and c.

(i) $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

$$=\left(-\frac{b}{a}\right)^2-2\left(\frac{c}{a}\right)=\frac{b^2-2ac}{a^2}.$$

(ii) $\alpha^2-\beta^2=(\alpha+\beta)(\alpha-\beta)$

$$=\left(-\frac{b}{a}\right)\left(\frac{\sqrt{D}}{a}\right)=-\frac{b\sqrt{D}}{a^2}.$$

(iii) $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$

$$=\left(-\frac{b}{a}\right)^3-3\left(\frac{c}{a}\right)\left(-\frac{b}{a}\right)=-\left(\frac{b^3-3abc}{a^3}\right).$$

(iv) $\alpha^3-\beta^3=(\alpha-\beta)^3+3\alpha\beta(\alpha-\beta)$

$$=\left(\frac{\sqrt{D}}{a}\right)^3+3\left(\frac{c}{a}\right)\left(\frac{\sqrt{D}}{a}\right)=\frac{\sqrt{D}\,(D+3ac)}{a^3}.$$

(v) $\alpha^4+\beta^4=(\alpha^2+\beta^2)^2-2\alpha^2\beta^2$

$$=\left(\frac{b^2-2ac}{a^2}\right)^2-2\left(\frac{c}{a}\right)^2=\frac{b^4+2a^2c^2-4acb^2}{a^4}.$$

(vi) $\alpha^4-\beta^4=(\alpha^2+\beta^2)(\alpha^2-\beta^2)$

$$=-\frac{b\sqrt{D}\,(b^2-2ac)}{a^4}.$$

(vii) $\alpha^5+\beta^5=(\alpha^2+\beta^2)(\alpha^3+\beta^3)-\alpha^2\beta^2(\alpha+\beta)$

$$=\left(\frac{b^2-2ac}{a^2}\right)\left(-\frac{(b^3-3abc)}{a^3}\right)-\frac{c^2}{a^2}\left(-\frac{b}{a}\right)$$

$$=\frac{-(b^5-5ab^3c+5a^2bc^2)}{a^5}.$$

(viii) $\alpha^5-\beta^5=(\alpha^2+\beta^2)(\alpha^3-\beta^3)+\alpha^2\beta^2(\alpha-\beta)$

$$=\left(\frac{b^2-2ac}{a^2}\right)\left(\frac{\sqrt{D}\,(D+3ac)}{a^3}\right)+\left(\frac{c}{a}\right)^2\left(\frac{\sqrt{D}}{a}\right)$$

$$=\frac{\sqrt{D}\,(b^4-3acb^2+3a^2c^2)}{a^5}.$$

Example 11. If one root of the equation $x^2-ix-(1+i)=0, (i=\sqrt{-1})$ is $1+i$, find the other root.

Sol. All coefficients of the given equation are not real, then other root $\neq 1-i$.

Let other root be α, then sum of roots = i

i.e. $\qquad 1+i+\alpha=i \;\Rightarrow\; \alpha=(-1)$

Hence, the other root is (-1).

Example 12. If one root of the equation $x^2-\sqrt{5}\,x-19=0$ is $\frac{9+\sqrt{5}}{2}$, then find the other root.

Sol. All coefficients of the given equation are not rational, then other root $\neq \frac{9-\sqrt{5}}{2}$.

Let other root be α, sum of roots = $\sqrt{5}$

$$\Rightarrow \qquad \frac{9+\sqrt{5}}{2}+\alpha=\sqrt{5} \;\Rightarrow\; \alpha=\frac{-9+\sqrt{5}}{2}$$

Hence, other root is $\frac{-9+\sqrt{5}}{2}$.

Example 13. If the difference between the corresponding roots of the equations $x^2+ax+b=0$ and $x^2+bx+a=0\,(a\neq b)$ is the same, find the value of $a+b$.

Sol. Let α, β be the roots of $x^2+ax+b=0$ and γ, δ be the roots of $x^2+bx+a=0$, then given

$$\alpha-\beta=\gamma-\delta$$

$$\Rightarrow \qquad \frac{\sqrt{a^2-4b}}{1}=\frac{\sqrt{b^2-4a}}{1} \qquad \left[\because \alpha-\beta=\frac{\sqrt{D}}{a}\right]$$

$$\Rightarrow \qquad a^2-4b=b^2-4a$$

$$\Rightarrow \quad (a^2-b^2)+4(a-b)=0 \;\Rightarrow\; (a-b)(a+b+4)=0$$

$$\because \qquad a-b\neq 0$$

$$\therefore \qquad a+b+4=0 \quad \text{or} \quad a+b=-4.$$

Example 14. ***If*** $a+b+c=0$ ***and*** a,b,c ***are rational. Prove that the roots of the equation***

$$(b+c-a)x^2+(c+a-b)x+(a+b-c)=0$$

are rational.

Sol. Given equation is

$$(b+c-a)x^2+(c+a-b)x+(a+b-c)=0 \quad \text{...(i)}$$

$\because (b+c-a)+(c+a-b)+(a+b-c)=a+b+c=0$

$\therefore\ x=1$ is a root of Eq. (i), let other root of Eq. (i) is α, then

$$\text{Product of roots} = \frac{a+b-c}{b+c-a}$$

$$\Rightarrow \quad 1\times\alpha=\frac{-c-c}{-a-a} \qquad [\because a+b+c=0]$$

$$\therefore \quad \alpha=\frac{c}{a} \qquad [\text{rational}]$$

Hence, both roots of Eq. (i) are rational.

Aliter

Let $b+c-a=A, c+a-b=B, a+b-c=C$

Then, $A+B+C=0 \quad [\because a+b+c=0]$ ...(ii)

Now, Eq. (i) becomes

$$Ax^2+Bx+C=0 \quad \text{...(iii)}$$

Discriminant of Eq. (iii),

$$D=B^2-4AC$$
$$=(-C-A)^2-4AC \qquad [\because A+B+C=0]$$
$$=(C+A)^2-4AC$$
$$=(C-A)^2=(2a-2c)^2$$
$$=4(a-c)^2=\text{A perfect square}$$

Hence, roots of Eq. (i) are rational.

Example 15. If the roots of equation $a(b-c)x^2+b(c-a)x+c(a-b)=0$ be equal, prove that a,b,c are in HP.

Sol. Given equation is

$$a(b-c)x^2+b(c-a)x+c(a-b)=0 \quad \text{...(i)}$$

Here, coefficient of x^2 + coefficient of x + constant term = 0

i.e., $a(b-c)+b(c-a)+c(a-b)=0$

Then, 1 is a root of Eq. (i).

Since, its roots are equal.

Therefore, its other root will be also equal to 1.

$$\text{Then, product of roots} = 1\times 1=\frac{c(a-b)}{a(b-c)}$$

$$\Rightarrow \quad ab-ac=ca-bc$$

$$\therefore \quad b=\frac{2ac}{a+c}$$

Hence, a, b and c are in HP.

Example 16. ***If*** α ***is a root of*** $4x^2+2x-1=0$. ***Prove that*** $4\alpha^3-3\alpha$ ***is the other root.***

Sol. Let other root is β,

$$\text{then} \quad \alpha+\beta=-\frac{2}{4}=-\frac{1}{2} \text{ or } \beta=-\frac{1}{2}-\alpha \quad \text{...(i)}$$

and so $4\alpha^2+2\alpha-1=0$, because α is a root of $4x^2+2x-1=0$.

Now, $\beta=4\alpha^3-3\alpha=\alpha(4\alpha^2-3)$

$$=\alpha(1-2\alpha-3) \qquad [\because 4\alpha^2+2\alpha-1=0]$$
$$=-2\alpha^2-2\alpha$$
$$=-\frac{1}{2}(4\alpha^2)-2\alpha$$
$$=-\frac{1}{2}(1-2\alpha)-2\alpha \qquad [\because 4\alpha^2+2\alpha-1=0]$$
$$=-\frac{1}{2}-\alpha=\beta \qquad [\text{from Eq. (i)}]$$

Hence, $4\alpha^3-3\alpha$ is the other root.

Example 17. If α,β are the roots of the equation $\lambda(x^2-x)+x+5=0$. If λ_1 and λ_2 are two values of λ for which the roots α,β are related by $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{4}{5}$, find the value of $\frac{\lambda_1}{\lambda_2}+\frac{\lambda_2}{\lambda_1}$.

Sol. The given equation can be written as

$$\lambda x^2-(\lambda-1)x+5=0$$

$\because \alpha,\beta$ are the roots of this equation.

$$\therefore \quad \alpha+\beta=\frac{\lambda-1}{\lambda} \text{ and } \alpha\beta=\frac{5}{\lambda}$$

But, given $\quad \frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{4}{5}$

$$\Rightarrow \quad \frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{4}{5}$$

$$\Rightarrow \quad \frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}=\frac{4}{5} \quad \Rightarrow \quad \frac{\frac{(\lambda-1)^2}{\lambda^2}-\frac{10}{\lambda}}{\frac{5}{\lambda}}=\frac{4}{5}$$

$$\Rightarrow \quad \frac{(\lambda-1)^2-10\lambda}{5\lambda}=\frac{4}{5} \quad \Rightarrow \quad \lambda^2-12\lambda+1=4\lambda$$

$$\Rightarrow \quad \lambda^2-16\lambda+1=0$$

It is a quadratic in λ, let roots be λ_1 and λ_2, then

$$\lambda_1+\lambda_2=16 \text{ and } \lambda_1\lambda_2=1$$

$$\therefore \quad \frac{\lambda_1}{\lambda_2}+\frac{\lambda_2}{\lambda_1}=\frac{\lambda_1^2+\lambda_2^2}{\lambda_1\lambda_2}=\frac{(\lambda_1+\lambda_2)^2-2\lambda_1\lambda_2}{\lambda_1\lambda_2}$$

$$=\frac{(16)^2-2(1)}{1}=254$$

Example 18. If α, β are the roots of the equation $x^2 - px + q = 0$, find the quadratic equation the roots of which are $(\alpha^2 - \beta^2)(\alpha^3 - \beta^3)$ and $\alpha^3\beta^2 + \alpha^2\beta^3$.

Sol. Since, α, β are the roots of $x^2 - px + q = 0$.

$\therefore \quad \alpha + \beta = p, \alpha\beta = q$

$\Rightarrow \quad \alpha - \beta = \sqrt{(p^2 - 4q)}$

Now, $(\alpha^2 - \beta^2)(\alpha^3 - \beta^3)$

$= (\alpha + \beta)(\alpha - \beta)(\alpha - \beta)(\alpha^2 + \alpha\beta + \beta^2)$

$= (\alpha + \beta)(\alpha - \beta)^2\{(\alpha + \beta)^2 - \alpha\beta\}$

$= p(p^2 - 4q)(p^2 - q)$

and $\alpha^3\beta^2 + \alpha^2\beta^3 = \alpha^2\beta^2(\alpha + \beta) = pq^2$

$S = \text{Sum of roots} = p(p^2 - 4q)(p^2 - q) + pq^2$

$= p(p^4 - 5p^2q + 5q^2)$

$P = \text{Product of roots} = p^2q^2(p^2 - 4q)(p^2 - q)$

$\therefore$ Required equation is $x^2 - Sx + P = 0$

i.e. $x^2 - p(p^4 - 5p^2q + 5q^2)x + p^2q^2(p^2 - 4q)(p^2 - q) = 0$

Exercise for Session 1

1. If $(a^2 - 1)x^2 + (a - 1)x + a^2 - 4a + 3 = 0$ be an identity in x, then the value of a is/are

(a) -1 (b) 1 (c) 3 (d) $-1, 1, 3$

2. The roots of the equation $x^2 + 2\sqrt{3}x + 3 = 0$ are

(a) real and unequal (b) rational and equal

(c) irrational and equal (d) irrational and unequal

3. If $a, b, c \in Q$, then roots of the equation $(b + c - 2a)x^2 + (c + a - 2b)x + (a + b - 2c) = 0$ are

(a) rational (b) non-real (c) irrational (d) equal

4. If $P(x) = ax^2 + bx + c$ and $Q(x) = -ax^2 + dx + c$, where $ac \neq 0$, then $P(x)\,Q(x) = 0$ has atleast

(a) four real roots (b) two real roots

(c) four imaginary roots (d) None of these

5. If roots of the equation $(q - r)x^2 + (r - p)x + (p - q) = 0$ are equal, then p, q, r are in

(a) AP (b) GP (c) HP (d) AGP

6. If one root of the quadratic equation $ix^2 - 2(i + 1)x + (2 - i) = 0, i = \sqrt{-1}$ is $2 - i$, the other root is

(a) $-i$ (b) i (c) $2 + i$ (d $2 - i$

7. If the difference of the roots of $x^2 - \lambda x + 8 = 0$ be 2, the value of λ is

(a) ± 2 (b) ± 4 (c) ± 6 (d) ± 8

8. If $3p^2 = 5p + 2$ and $3q^2 = 5q + 2$ where $p \neq q, pq$ is equal to

(a) $\frac{2}{3}$ (b) $-\frac{2}{3}$ (c) $\frac{3}{2}$ (d) $-\frac{3}{2}$

9. If α, β are the roots of the quadratic equation $x^2 + bx - c = 0$, the equation whose roots are b and c, is

(a) $x^2 + \alpha x - \beta = 0$ (b) $x^2 - [(\alpha + \beta) + \alpha\beta]x - \alpha\beta(\alpha + \beta) = 0$

(c) $x^2 + [(\alpha + \beta) + \alpha\beta]x + \alpha\beta(\alpha + \beta) = 0$ (d) $x^2 + [(\alpha + \beta) + \alpha\beta]x - \alpha\beta(\alpha + \beta) = 0$

10. Let $p, q \in \{1, 2, 3, 4\}$. The number of equations of the form $px^2 + qx + 1 = 0$ having real roots, is

(a) 15 (b) 9 (c) 8 (d) 7

11. If α and β are the roots of the equation $ax^2 + bx + c = 0$ ($a \neq 0, a, b, c$ being different), then $(1 + \alpha + \alpha^2)(1 + \beta + \beta^2)$ is equal to

(a) zero (b) positive (c) negative (d) None of these

Session 2

Transformation of Quadratic Equations, Condition for Common Roots

Transformation of Quadratic Equations

Let α, β be the roots of the equation $ax^2 + bx + c = 0$, then the equation

(i) whose roots are $\alpha + k, \beta + k$, is

$a(x-k)^2 + b(x-k) + c = 0$ [replace x by $(x-k)$]

(ii) whose roots are $\alpha - k, \beta - k$, is

$a(x+k)^2 + b(x+k) + c = 0$ [replace x by $(x+k)$]

(iii) whose roots are $\alpha k, \beta k$, is

$ax^2 + kbx + k^2 c = 0$ $\left[\text{replace } x \text{ by } \left(\frac{x}{k}\right)\right]$

(iv) whose roots are $\frac{\alpha}{k}, \frac{\beta}{k}$, is

$ak^2 x^2 + bkx + c = 0$ [replace x by xk]

(v) whose roots are $-\alpha, -\beta$, is

$ax^2 - bx + c = 0$ [replace x by $(-x)$]

(vi) whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}$, is

$cx^2 + bx + a = 0$ $\left[\text{replace } x \text{ by } \left(\frac{1}{x}\right)\right]$

(vii) whose roots are $-\frac{1}{\alpha}, -\frac{1}{\beta}$, is

$cx^2 - bx + a = 0$ $\left[\text{replace } x \text{ by } \left(-\frac{1}{x}\right)\right]$

(viii) whose roots are $\frac{k}{\alpha}, \frac{k}{\beta}$, is

$cx^2 + kbx + k^2 a = 0$ $\left[\text{replace } x \text{ by } \left(\frac{k}{x}\right)\right]$

(ix) whose roots are $p\alpha + q, p\beta + q$, is

$a\left(\frac{x-q}{p}\right)^2 + b\left(\frac{x-q}{p}\right) + c = 0$ $\left[\text{replace } x \text{ by } \left(\frac{x-q}{p}\right)\right]$

(x) whose roots are $\alpha^n, \beta^n, n \in N$, is

$a(x^{1/n})^2 + b(x^{1/n}) + c = 0$ [replace x by $(x^{1/n})$]

(xi) whose roots are $\alpha^{1/n}, \beta^{1/n}, n \in N$ is

$a(x^n)^2 + b(x^n) + c = 0$ [replace x by (x^n)]

Example 19. If α, β be the roots of the equation $x^2 - px + q = 0$, then find the equation whose roots are $\frac{q}{p-\alpha}$ and $\frac{q}{p-\beta}$.

Sol. Let $\frac{q}{p-\alpha} = x \Rightarrow \alpha = p - \frac{q}{x}$

So, we replacing x by $p - \frac{q}{x}$ in the given equation, we get

$$\left(p - \frac{q}{x}\right)^2 - p\left(p - \frac{q}{x}\right) + q = 0$$

$$\Rightarrow \quad p^2 + \frac{q^2}{x^2} - \frac{2pq}{x} - p^2 + \frac{pq}{x} + q = 0$$

$$\Rightarrow \quad q - \frac{pq}{x} + \frac{q^2}{x^2} = 0$$

or $qx^2 - pqx + q^2 = 0$ or $x^2 - px + q = 0$

is the required equation whose roots are $\frac{q}{p-\alpha}$ and $\frac{q}{p-\beta}$.

Example 20. If α and β are the roots of $ax^2 + bx + c = 0$, then find the roots of the equation $ax^2 - bx(x-1) + c(x-1)^2 = 0$.

Sol. $\because \; ax^2 - bx(x-1) + c(x-1)^2 = 0$ ...(i)

$$\Rightarrow \quad a\left(\frac{x}{x-1}\right)^2 - b\left(\frac{x}{x-1}\right) + c = 0$$

or $$a\left(\frac{x}{1-x}\right)^2 + b\left(\frac{x}{1-x}\right) + c = 0$$

Now, α, β are the roots of $ax^2 + bx + c = 0$.

Then, $\alpha = \frac{x}{1-x}$ and $\beta = \frac{x}{1-x}$

$$\Rightarrow \quad x = \frac{\alpha}{\alpha+1} \text{ and } x = \frac{\beta}{\beta+1}$$

Hence, $\frac{\alpha}{\alpha+1}, \frac{\beta}{\beta+1}$ are the roots of the Eq. (i).

Example 21. If α, β be the roots of the equation $3x^2 + 2x + 1 = 0$, then find value of $\left(\frac{1-\alpha}{1+\alpha}\right)^3 + \left(\frac{1-\beta}{1+\beta}\right)^3$.

Sol. Let $\frac{1-\alpha}{1+\alpha} = x \Rightarrow \alpha = \frac{1-x}{1+x}$

So, replacing x by $\frac{1-x}{1+x}$ in the given equation, we get

$$3\left(\frac{1-x}{1+x}\right)^2+2\left(\frac{1-x}{1+x}\right)+1=0 \Rightarrow x^2-2x+3=0 \quad ...(i)$$

It is clear that $\frac{1-\alpha}{1+\alpha}$ and $\frac{1-\beta}{1+\beta}$ are the roots of Eq. (i).

$$\therefore \quad \left(\frac{1-\alpha}{1+\alpha}\right)+\left(\frac{1-\beta}{1+\beta}\right)=2 \quad ...(ii)$$

and $$\left(\frac{1-\alpha}{1+\alpha}\right)\left(\frac{1-\beta}{1+\beta}\right)=3 \quad ...(iii)$$

$$\therefore \quad \left(\frac{1-\alpha}{1+\alpha}\right)^3+\left(\frac{1-\beta}{1+\beta}\right)^3=\left(\frac{1-\alpha}{1+\alpha}+\frac{1-\beta}{1+\beta}\right)^3-3$$

$$\left(\frac{1-\alpha}{1+\alpha}\right)\left(\frac{1-\beta}{1+\beta}\right)\left(\frac{1-\alpha}{1+\alpha}+\frac{1-\beta}{1+\beta}\right)=2^3-3\cdot3\cdot2=8-18=-10$$

Roots Under Special Cases

Consider the quadratic equation $ax^2+bx+c=0$ ...(i)

where $a, b, c \in R$ and $a \neq 0$. Then, the following hold good :

(i) If roots of Eq. (i) are equal in magnitude but opposite in sign, then sum of roots is zero as well as $D>0$, i.e. $b=0$ and $D>0$.

(ii) If roots of Eq. (i) are reciprocal to each other, then product of roots is 1 as well as $D \geq 0$ i.e., $a=c$ and $D \geq 0$.

(iii) If roots of Eq. (i) are of opposite signs, then product of roots <0 as well as $D>0$ i.e., $a>0, c<0$ and $D>0$ or $a<0, c>0$ and $D>0$.

(iv) If both roots of Eq. (i) are positive, then sum and product of roots >0 as well as $D \geq 0$ i.e., $a>0, b<0, c>0$ and $D \geq 0$ or $a<0, b>0, c<0$ and $D \geq 0$.

(v) If both roots of Eq. (i) are negative, then sum of roots <0, product of roots >0 as well as $D \geq 0$ i.e., $a>0, b>0, c>0$ and $D \geq 0$ or $a<0, b<0, c<0$ and $D \geq 0$.

(vi) If atleast one root of Eq. (i) is positive, then either one root is positive or both roots are positive i.e., point (iii) $\cup$ (iv).

(vii) If atleast one root of Eq. (i) is negative, then either one root is negative or both roots are negative i.e., point (iii) $\cup$ (v).

(viii) If greater root in magnitude of Eq. (i) is positive, then sign of b = sign of $c \neq$ sign of a.

(ix) If greater root in magnitude of Eq. (i) is negative, then sign of $a=$ sign of $b \neq$ sign of c.

(x) If both roots of Eq. (i) are zero, then $b=c=0$.

(xi) If roots of Eq. (i) are 0 and $\left(-\frac{b}{a}\right)$, then $c=0$.

(xii) If roots of Eq. (i) are 1 and $\frac{c}{a}$, then $a+b+c=0$.

Example 22. ***For what values of*** m, ***the equation*** $x^2+2(m-1)x+m+5=0$ ***has*** $(m \in R)$

(i) roots are equal in magnitude but opposite in sign?

(ii) roots are reciprocals to each other?

(iii) roots are opposite in sign?

(iv) both roots are positive?

(v) both roots are negative?

(vi) atleast one root is positive?

(vii) atleast one root is negative?

Sol. Here, $a=1, b=2(m-1)$ and $c=m+5$

$\therefore \quad D=b^2-4ac=4(m-1)^2-4(m+5)$
$=4(m^2-3m-4)$

$\therefore \quad D=4(m-4)(m+1)$ and here $a=1>0$

(i) $b=0$ and $D>0$

$\Rightarrow 2(m-1)=0$ and $4(m-4)(m+1)>0$

$\Rightarrow \quad m=1$ and $m \in (-\infty,-1) \cup (4,\infty)$

$\therefore \quad m \in \phi$ [null set]

(ii) $a=c$ and $D \geq 0$

$\Rightarrow \quad 1=m+5$ and $4(m-4)(m+1) \geq 0$

$\Rightarrow \quad m=-4$ and $m \in (-\infty,-1] \cup [4,\infty)$

$\therefore \quad m=-4$

(iii) $a>0, c<0$ and $D>0$

$\Rightarrow \quad 1>0, m+5<0$ and $4(m-4)(m+1)>0$

$\Rightarrow \quad m<-5$ and $m \in (-\infty,-1) \cup (4,\infty)$

$\therefore \quad m \in (-\infty,-5)$

(iv) $a>0, b<0, c>0$ and $D \geq 0$

$\Rightarrow \quad 1>0, 2(m-1)<0, m+5>0$

and $\quad 4(m-4)(m+1) \geq 0$

$\Rightarrow \quad m<1, m>-5$ and $m \in (-\infty,-1] \cup [4,\infty)$

$\Rightarrow \quad m \in (-5,-1]$

(v) $a>0, b>0, c>0$ and $D \geq 0$

$\Rightarrow \quad 1>0, 2(m-1)>0, m+5>0$

and $\quad 4(m-4)(m+1) \geq 0$

$\Rightarrow \quad m>1, m>-5$ and $m \in (-\infty,-1] \cup [4,\infty)$

$\therefore \quad m \in [4,\infty)$

(vi) Either one root is positive or both roots are positive

i.e., (c) $\cup$ (d)

$\Rightarrow \quad m \in (-\infty,-5) \cup (-5,-1]$

(vii) Either one root is negative or both roots are negative

i.e., (c) $\cup$ (e)

$\Rightarrow \quad m \in (-\infty,-5) \cup [4,\infty)$

Condition for Common Roots

1. Only One Root is Common

Consider two quadratic equations

$$ax^2 + bx + c = 0 \text{ and } a'x^2 + b'x + c' = 0$$

[where $a, a' \neq 0$ and $ab' - a'b \neq 0$]

Let α be a common root, then

$$a\alpha^2 + b\alpha + c = 0 \text{ and } \alpha'\alpha^2 + b'\alpha + c' = 0.$$

On solving these two equations by cross-multiplication, we have

$$\frac{\alpha^2}{bc' - b'c} = \frac{\alpha}{ca' - c'a} = \frac{1}{ab' - a'b}$$

From first two relations, we get

$$\alpha = \frac{bc' - b'c}{ca' - c'a} \quad \ldots(i)$$

and from last two relations, we get

$$\alpha = \frac{ca' - c'a}{ab' - a'b} \quad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$\frac{bc' - b'c}{ca' - c'a} = \frac{ca' - c'a}{ab' - a'b}$$

$$\Rightarrow \quad (ab' - a'b)(bc' - b'c) = (ca' - c'a)^2$$

or $$\begin{vmatrix} a & b \\ a' & b' \end{vmatrix} \times \begin{vmatrix} b & c \\ b' & c' \end{vmatrix} = \begin{vmatrix} c & a \\ c' & a' \end{vmatrix}^2 \quad \text{[remember]}$$

This is the required condition for one root of two quadratic equations to be common.

2. Both Roots are Common

Let α, β be the common roots of the equations $ax^2 + bx + c = 0$ and $a'x^2 + b'x + c' = 0$, then

$$\alpha + \beta = -\frac{b}{a} = -\frac{b'}{a'} \Rightarrow \frac{a}{a'} = \frac{b}{b'} \quad \ldots(iii)$$

and $$\alpha\beta = \frac{c}{a} = \frac{c'}{a'} \Rightarrow \frac{a}{a'} = \frac{c}{c'} \quad \ldots(iv)$$

From Eqs. (iii) and (iv), we get $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$

This is the required condition for both roots of two quadratic equations to be identical.

Remark

To find the common root between the two equations, make the same coefficient of x^2 in both equations and then subtract of the two equations.

Example 23. Find the value of λ, so that the equations $x^2 - x - 12 = 0$ and $\lambda x^2 + 10x + 3 = 0$ may have one root in common. Also, find the common root.

Sol. $\because \quad x^2 - x - 12 = 0$

$\Rightarrow \quad (x - 4)(x + 3) = 0$

$\therefore \quad x = 4, -3$

If $x = 4$ is a common root, then

$$\lambda(4)^2 + 10(4) + 3 = 0$$

$\therefore \quad \lambda = -\frac{43}{16}$

and if $x = -3$ is a common root, then

$$\lambda(-3)^2 + 10(-3) + 3 = 0$$

$\therefore \quad \lambda = 3$

Hence, for $\lambda = -\frac{43}{16}$, common root is $x = 4$

and for $\lambda = 3$, common root is $x = -3$.

Example 24. ***If equations*** $ax^2 + bx + c = 0$, ***(where*** $a, b, c \in R$ ***and*** $a \neq 0$***) and*** $x^2 + 2x + 3 = 0$ ***have a common root, then show that*** $a : b : c = 1 : 2 : 3$.

Sol. Given equations are

$$ax^2 + bx + c = 0 \quad \ldots(i)$$

and $$x^2 + 2x + 3 = 0 \quad \ldots(ii)$$

Clearly, roots of Eq. (ii) are imaginary, since Eqs. (i) and (ii) have a common root. Therefore, common root must be imaginary and hence both roots will be common.

Therefore, Eqs. (i) and (ii) are identical.

$\therefore \quad \frac{a}{1} = \frac{b}{2} = \frac{c}{3}$ or $a : b : c = 1 : 2 : 3$

Example 25. ***If*** a, b, c ***are in GP, show that the equations*** $ax^2 + 2bx + c = 0$ ***and*** $dx^2 + 2ex + f = 0$ ***have a common root, if*** $\frac{a}{d}, \frac{b}{e}, \frac{c}{f}$ ***are in HP.***

Sol. Given equations are

$$ax^2 + 2bx + c = 0 \quad \ldots(i)$$

and $$dx^2 + 2ex + f = 0 \quad \ldots(ii)$$

Since, a, b, c are in GP.

$\therefore \quad b^2 = ac$ or $b = \sqrt{ac}$

From Eq. (i), $ax^2 + 2\sqrt{ac}\,x + c = 0$

or $(\sqrt{a}\,x + \sqrt{c})^2 = 0$ or $x = -\frac{\sqrt{c}}{\sqrt{a}}$

$\because$ Given Eqs. (i) and (ii) have a common root.

Hence, $x = -\frac{\sqrt{c}}{\sqrt{a}}$ also satisfied Eq. (ii), then

$$d\left(\frac{c}{a}\right) - 2e\frac{\sqrt{c}}{\sqrt{a}} + f = 0 \qquad \text{or} \qquad \frac{d}{a} + \frac{f}{c} = \frac{2e}{b}$$

$$\Rightarrow \quad \frac{d}{a} - \frac{2e}{\sqrt{ac}} + \frac{f}{c} = 0 \qquad \therefore \ \frac{d}{a}, \frac{e}{b}, \frac{f}{c} \text{ are in AP.}$$

$$\text{or} \quad \frac{d}{a} - \frac{2e}{b} + \frac{f}{c} = 0 \qquad [\because b = \sqrt{ac}] \qquad \text{Hence, } \frac{a}{d}, \frac{b}{e}, \frac{c}{f} \text{ are in HP.}$$

Exercise for Session 2

1. If α and β are the roots of the equation $2x^2 - 3x + 4 = 0$, then the equation whose roots are α^2 and β^2, is

(a) $4x^2 + 7x + 16 = 0$ (b) $4x^2 + 7x + 6 = 0$ (c) $4x^2 + 7x + 1 = 0$ (d) $4x^2 - 7x + 16 = 0$

2. If α, β are the roots of $x^2 - 3x + 1 = 0$, then the equation whose roots are $\left(\frac{1}{\alpha - 2}, \frac{1}{\beta - 2}\right)$, is

(a) $x^2 + x - 1 = 0$ (b) $x^2 + x + 1 = 0$ (c) $x^2 - x - 1 = 0$ (d) None of these

3. The equation formed by decreasing each root of $ax^2 + bx + c = 0$ by 1 is $2x^2 + 8x + 2 = 0$, then

(a) $a = -b$ (b) $b = -c$ (c) $c = -a$ (d) $b = a + c$

4. If the roots of equation $\frac{x^2 - bx}{ax - c} = \frac{m - 1}{m + 1}$ are equal but opposite in sign, then the value of m will be

(a) $\frac{a - b}{a + b}$ (b) $\frac{b - a}{a + b}$ (c) $\frac{a + b}{a - b}$ (d) $\frac{b + a}{b - a}$

5. If $x^2 + px + q = 0$ is the quadratic equation whose roots are $a - 2$ and $b - 2$, where a and b are the roots of $x^2 - 3x + 1 = 0$, then

(a) $p = 1, q = 5$ (b) $p = 1, q = -5$ (c) $p = -1, q = 1$ (d) None of these

6. If both roots of the equation $x^2 - (m - 3)x + m = 0$ $(m \in R)$ are positive, then

(a) $m \in (3, \infty)$ (b) $m \in (-\infty, 1]$ (c) $m \in [9, \infty)$ (d) $m \in (1, 3)$

7. If the equation $(1 + m)x^2 - 2(1 + 3m)x + (1 + 8m) = 0$, where $m \in R \sim \{-1\}$, has atleast one root is negative, then

(a) $m \in (-\infty, -1)$ (b) $m \in \left(-\frac{1}{8}, \infty\right)$ (c) $m \in \left(-1, -\frac{1}{8}\right)$ (d) $m \in R$

8. If both the roots of $\lambda(6x^2 + 3) + rx + 2x^2 - 1 = 0$ and $6\lambda(2x^2 + 1) + px + 4x^2 - 2 = 0$ are common, then $2r - p$ is equal to

(a) -1 (b) 0 (c) 1 (d) 2

9. If $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a common root $a \neq 0$, then $\frac{a^3 + b^3 + c^3}{abc}$ is equal to

(a) 1 (b) 2 (c) 3 (d) None of these

10. If $a(p + q)^2 + 2bpq + c = 0$ and $a(p + r)^2 + 2bpr + c = 0$, then qr is equal to

(a) $p^2 + \frac{c}{a}$ (b) $p^2 + \frac{a}{c}$ (c) $p^2 + \frac{a}{b}$ (d) $p^2 + \frac{b}{a}$

Session 3

Quadratic Expression, Wavy Curve Method, Condition for Resolution into Linear Factors, Location of Roots

Quadratic Expression

An expression of the form $ax^2 + bx + c$, where $a, b, c \in R$ and $a \neq 0$ is called a quadratic expression in x. So, in general quadratic expression is represented by

$$f(x) = ax^2 + bx + c \text{ or } y = ax^2 + bx + c.$$

Graph of a Quadratic Expression

We have, $y = ax^2 + bx + c = f(x), \quad [a \neq 0]$

$$\Rightarrow \quad y = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a^2}\right]$$

or $$\left(y + \frac{D}{4a}\right) = a\left(x + \frac{b}{2a}\right)^2$$

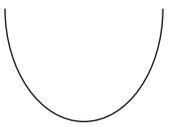
Now, let $y + \frac{D}{4a} = Y$ and $x + \frac{b}{2a} = X$

$$\therefore \quad Y = aX^2$$

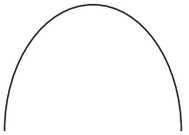
$$\Rightarrow \quad X^2 = \frac{Y}{a}$$

1. The shape of the curve $y = f(x)$ is parabolic.
2. The axis of parabola is $X = 0$ or $x + \frac{b}{2a} = 0$

   or $x = -\frac{b}{2a}$ i.e. parallel to Y-axis.
3. (i) If $\frac{1}{a} > 0 \Rightarrow a > 0$, the parabola open upwards.



(ii) If $\frac{1}{a} < 0 \Rightarrow a < 0$, the parabola open downwards.



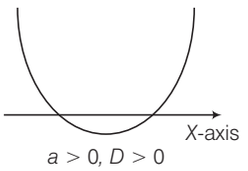
4. **Intersection with axes**

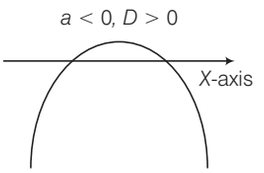
(i) **Intersection with *X*-axis**

For X-axis, $y = 0$.

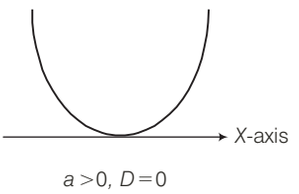
$$\therefore \quad ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{D}}{2a}$$

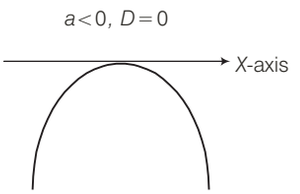
For $D > 0$, parabola cuts X-axis in two real and distinct points



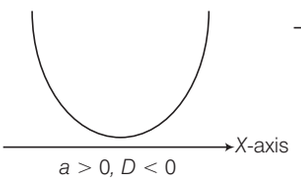


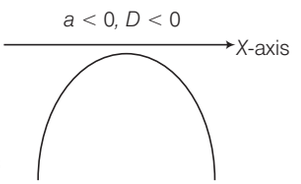
For $D = 0$, parabola touches X-axis in one point i.e., $x = -\frac{b}{2a}$.





For $D < 0$, parabola does not cut X-axis i.e., imaginary values of x.





(ii) **Intersection with *Y*-axis**

For Y-axis, $x = 0$.

$$\therefore \quad y = c$$

5. **Greatest and least values of *f(x)***

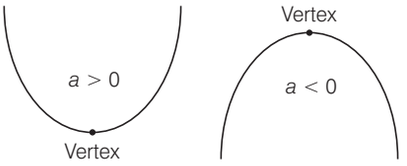
Vertex of the parabola $X^2 = \frac{1}{a}Y$ is

$$X = 0, Y = 0$$

$$\Rightarrow \quad x + \frac{b}{2a} = 0, y + \frac{D}{4a} = 0$$

or $$x = -\frac{b}{2a}, y = -\frac{D}{4a}$$

Hence, vertex of $y = ax^2 + bx + c$ is $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$.



For $a > 0$, $f(x)$ has least value at $x = -\frac{b}{2a}$.

This least value is given by $f\left(-\frac{b}{2a}\right) = -\frac{D}{4a}$

or $y_{\text{least}} = -\frac{D}{4a}$.

$\therefore$ Range of $y = ax^2 + bx + c$ is $\left(-\frac{D}{4a}, \infty\right)$.

For $a < 0$, $f(x)$ has greatest value at $x = -\frac{b}{2a}$.

This greatest value is given by $f\left(-\frac{b}{2a}\right) = -\frac{D}{4a}$

or $y_{\text{greatest}} = -\frac{D}{4a}$

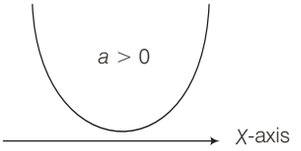
$\therefore$ Range of $y = ax^2 + bx + c$ is $\left(-\infty, -\frac{D}{4a}\right)$.

Sign of Quadratic Expression

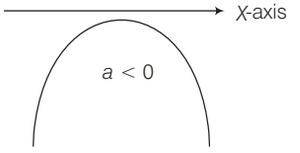
Let $f(x) = ax^2 + bx + c$ or $y = ax^2 + bx + c$,

where $a, b, c \in R$ and $a \neq 0$, for some values of x, $f(x)$ may be positive, negative or zero. This gives the following cases :

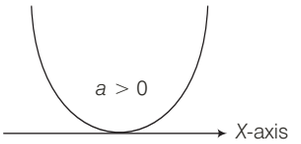
1. **$a > 0$ and $D < 0$.**
   So, $f(x) > 0$ for all $x \in R$,
   i.e. $f(x)$ is positive for all real values of x.



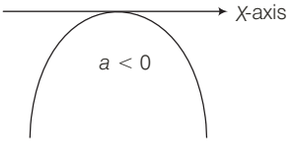
2. **$a < 0$ and $D < 0$**. So, $f(x) < 0$ for all $x \in R$,
   i.e. $f(x)$ is negative for all real values of x.



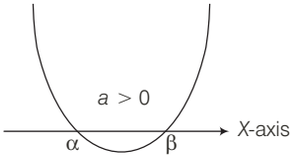
3. **$a > 0$ and $D = 0$**. So, $f(x) \geq 0$ for all $x \in R$,
   i.e. $f(x)$ is positive for all real values of x except at vertex, where $f(x) = 0$.



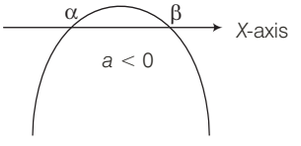
4. **$a < 0$ and $D = 0$**. So, $f(x) \leq 0$ for all $x \in R$,
   i.e. $f(x)$ is negative for all real values of x except at vertex, where $f(x) = 0$.



5. **$a > 0$ and $D > 0$.**
   Let $f(x) = 0$ have two real roots α and β $(\alpha < \beta)$, then $f(x) > 0$ for all $x \in (-\infty, \alpha) \cup (\beta, \infty)$ and $f(x) < 0$ for all $x \in (\alpha, \beta)$.



6. **$a < 0$ and $D > 0$**
   Let $f(x) = 0$ have two real roots α and β $(\alpha < \beta)$,
   then $f(x) < 0$ for all $x \in (-\infty, \alpha) \cup (\beta, \infty)$
   and $f(x) > 0$ for all $x \in (\alpha, \beta)$.



Wavy Curve Method
(Generalised Method of Intervals)

Wave Curve Method is used for solving inequalities of the form

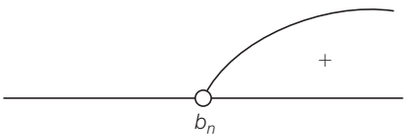
$$f(x) = \frac{(x-a_1)^{k_1}(x-a_2)^{k_2}\ldots(x-a_m)^{k_m}}{(x-b_1)^{p_1}(x-b_2)^{p_2}\ldots(x-b_n)^{p_n}} > 0$$

$(< 0, \geq 0 \text{ or } \leq 0)$,

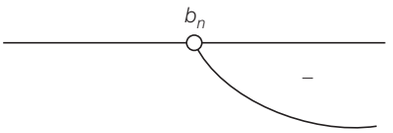
where, $k_1, k_2, \ldots, k_m, p_1, p_2, \ldots, p_n$ are natural numbers and such that $a_i \neq b_j$, where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

We use the following methods:

1. Solve $(x-a_1)^{k_1}(x-a_2)^{k_2}\ldots(x-a_m)^{k_m}=0$ and $(x-b_1)^{p_1}(x-a_2)^{p_2}\ldots(x-b_n)^{p_n}=0$, then we get
   $x=a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n$ [critical points]
2. Assume $a_1<a_2<\ldots<a_m<b_1<b_2<\ldots<b_n$
   Plot them on the real line. Arrange inked (black) circles (●) and un-inked (white) circles (○), such that
   $a_1\ a_2\ldots a_m \quad b_1\ b_2\ldots b_n$
   If $f(x)>0$ ○ ○ ... ○ ○ ○ ... ○
   $f(x)<0$ ○ ○ ... ○ ○ ○ ... ○
   $f(x)\geq 0$ ● ● ... ● ○ ○ ... ○
   $f(x)\leq 0$ ● ● ... ● ○ ○ ... ○
3. Obviously, b_n is the greatest root. If in all brackets before x positive sign and expression has also positive sign, then wave start from right to left, beginning above the number line, i.e.
   $+\dfrac{(x-a_1)^{k_1}(x-a_2)^{k_2}\ldots(x-a_m)^{k_m}}{(x-b_1)^{p_1}(x-b_2)^{p_2}\ldots(x-b_n)^{p_n}}$, then



   and if in all brackets before x positive sign and expression has negative sign, then wave start from right to left, beginning below the number line, i.e.
   $-\dfrac{(x-a_1)^{k_1}(x-a_2)^{k_2}\ldots(x-a_m)^{k_m}}{(x-b_1)^{p_1}(x-b_2)^{p_2}\ldots(x-b_n)^{p_n}}$, then



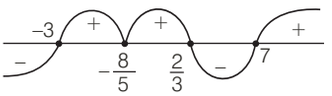
4. If roots occur even times, then sign remain same from right to left side of the roots and if roots occur odd times, then sign will change from right to left through the roots of
   $x=a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n$.
5. The solution of $f(x)>0$ or $f(x)\geq 0$ is the union of all intervals in which we have put the plus sign and the solution of $f(x)<0$ or $f(x)\leq 0$ is the union of all intervals in which we have put the minus sign.

Important Results

1. The point where denominator is zero or function approaches infinity, will never be included in the answer.
2. For $x^2<a^2$ or $|x|<a \Leftrightarrow -a<x<a$
   i.e., $x\in(-a, a)$
3. For $0<x^2<a^2$ or $0<|x|<a$
   $\Leftrightarrow -a<x<a\sim\{0\}$
   i.e., $x\in(-a, a)\sim\{0\}$
4. For $x^2\geq a^2$ or $|x|\geq a \Leftrightarrow x\leq -a$ or $x\geq a$
   i.e., $x\in(-\infty, -a]\cup[a, \infty)$
5. For $x^2>a^2$ or $|x|>a \Leftrightarrow x<-a$ or $x>a$
   i.e., $x\in(-\infty, -a)\cup(a, \infty)$
6. For $a^2\leq x^2\leq b^2$ or $a\leq|x|\leq b$
   $\Leftrightarrow a\leq x\leq b$ or $-b\leq x\leq -a$
   i.e., $x\in[-b, -a]\cup[a, b]$
7. For $a^2<x^2\leq b^2$ or $a<|x|\leq b$
   $\Leftrightarrow a<x\leq b$ or $-b\leq x<-a$
   i.e., $x\in[-b, -a)\cup(a, b]$
8. For $a^2\leq x^2<b^2$ or $a\leq|x|<b$
   $\Leftrightarrow a\leq x<b$ or $-b<x\leq -a$
   i.e., $x\in(-b, -a]\cup[a, b)$
9. For $a^2<x^2<b^2$ or $a<|x|<b$
   $\Leftrightarrow a<x<b$ or $-b<x<-a$
   i.e., $x\in(-b, -a)\cup(a, b)$
10. For $(x-a)(x-b)<0$ and $a<b$, then $a<x<b$
    i.e., $x\in(a, b)$
11. If $(x-a)(x-b)\leq 0$ and $a<b$,
    then $a\leq x\leq b, x\in[a, b]$
12. If $(x-a)(x-b)>0$ and $a<b$, then $x<a$ or $x>b$
    i.e., $x\in(-\infty, a)\cup(b, \infty)$
13. If $(x-a)(x-b)\geq 0$ and $a<b$,
    then $x\leq a$ or $x\geq b$
    i.e., $x\in(-\infty, a]\cup[b, \infty)$

Example 26. Solve the inequality $(x+3)(3x-2)^5(7-x)^3(5x+8)^2\geq 0$.

Sol. We have, $(x+3)(3x-2)^5(7-x)^3(5x+8)^2\geq 0$
$\Rightarrow -(x+3)(3x-2)^5(x-7)^3(5x+8)^2\geq 0$
$\Rightarrow (x+3)(3x-2)^5(x-7)^3(5x+8)^2\leq 0$
[take before x, + ve sign in all brackets]



The critical points are $(-3), \left(-\dfrac{8}{5}\right), \dfrac{2}{3}, 7$.

Hence, $x\in(-\infty, -3]\cup\left[\dfrac{2}{3}, 7\right]\cup\left\{-\dfrac{8}{5}\right\}$.

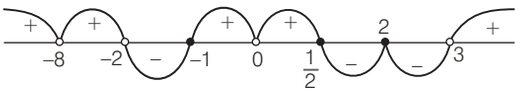
Example 27. ***Solve the inequality***

$$\frac{(x-2)^{10000}(x+1)^{253}\left(x-\frac{1}{2}\right)^{971}(x+8)^4}{x^{500}(x-3)^{75}(x+2)^{93}} \geq 0$$

Sol. We have, $\frac{(x-2)^{10000}(x+1)^{253}\left(x-\frac{1}{2}\right)^{971}(x+8)^4}{x^{500}(x-3)^{75}(x+2)^{93}} \geq 0$

The critical points are $(-8), (-2), (-1), 0, \frac{1}{2}, 2, 3.$

$[\because x \neq -2, 0, 3]$



Hence, $x \in (-\infty, -8] \cup [-8, -2) \cup [-1, 0) \cup \left(0, \frac{1}{2}\right] \cup (3, \infty)$

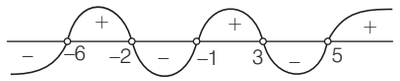
or $\quad x \in (-\infty, -2) \cup [-1, 0) \cup \left(0, \frac{1}{2}\right] \cup (3, \infty)$

Example 28. ***Let*** $f(x) = \frac{(x-3)(x+2)(x+6)}{(x+1)(x-5)}$.

Find intervals, where $f(x)$ ***is positive or negative.***

Sol. We have, $\quad f(x) = \frac{(x-3)(x+2)(x+6)}{(x+1)(x-5)}$

The critical points are $(-6), (-2), (-1), 3, 5$



For $f(x) > 0, \forall\, x \in (-6, -2) \cup (-1, 3) \cup (5, \infty)$

For $f(x) < 0, \forall\, x \in (-\infty, -6) \cup (-2, -1) \cup (3, 5)$

Example 29. Find the set of all x for which

$$\frac{2x}{(2x^2+5x+2)} > \frac{1}{(x+1)}.$$

Sol. We have, $\quad \frac{2x}{(2x^2+5x+2)} > \frac{1}{(x+1)}$

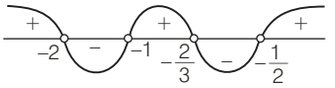
$$\Rightarrow \quad \frac{2x}{(x+2)(2x+1)} - \frac{1}{(x+1)} > 0$$

$$\Rightarrow \quad \frac{(2x^2+2x)-(2x^2+5x+2)}{(x+2)(x+1)(2x+1)} > 0$$

$$\Rightarrow \quad -\frac{(3x+2)}{(x+2)(x+1)(2x+1)} > 0$$

or $\quad \frac{(3x+2)}{(x+2)(x+1)(2x+1)} < 0$

The critical points are $(-2), (-1), \left(-\frac{2}{3}\right), \left(-\frac{1}{2}\right).$



Hence, $x \in (-2, -1) \cup \left(-\frac{2}{3}, -\frac{1}{2}\right).$

Example 30. For $x \in R$, prove that the given expression $\frac{x^2+34x-71}{x^2+2x-7}$ cannot lie between 5 and 9.

Sol. Let $\frac{x^2+34x-71}{x^2+2x-7} = y$

$$\Rightarrow \quad x^2(y-1) + (2y-34)x + 71 - 7y = 0$$

For real values of x, discriminant ≥ 0

$$\therefore \quad (2y-34)^2 - 4(y-1)(71-7y) \geq 0$$

$$\Rightarrow \quad 8y^2 - 112y + 360 \geq 0$$

$$\Rightarrow \quad y^2 - 14y + 45 \geq 0$$

$$\Rightarrow \quad (y-9)(y-5) \geq 0$$

$$\Rightarrow \quad y \in (-\infty, 5] \cup [9, \infty)$$

Hence, y can never lie between 5 and 9.

Example 31. ***For what values of the parameter*** k ***in the inequality*** $\left|\frac{x^2+kx+1}{x^2+x+1}\right| < 3$***, satisfied for all real values of*** x ***?***

Sol. We have, $\quad \left|\frac{x^2+kx+1}{x^2+x+1}\right| < 3$

$$\Rightarrow \quad -3 < \frac{x^2+kx+1}{x^2+x+1} < 3$$

Since, $\quad x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4} > 0$

$$\therefore \quad -3(x^2+x+1) < x^2+kx+1 < 3(x^2+x+1)$$

$$\therefore \quad 4x^2 + (k+3)x + 4 > 0 \qquad \ldots(i)$$

and $\quad 2x^2 - (k-3)x + 2 > 0 \qquad \ldots(ii)$

$\because \quad 4 > 0$ and $2 > 0$

The inequality (i) will be valid, if

$$(k+3)^2 - 4 \cdot 4 \cdot 4 < 0 \Rightarrow (k+3)^2 < 64$$

or $\quad -8 < k+3 < 8$

or $\quad -11 < k < 5 \qquad \ldots(iii)$

and the inequality (ii) will be valid, if

$$(k-3)^2 - 4 \cdot 2 \cdot 2 < 0 \text{ or } (k-3)^2 < 16$$

or $\quad -4 < k-3 < 4$

or $\quad -1 < k < 7 \qquad \ldots(iv)$

The conditions (iii) and (iv) will hold simultaneously, if

$$-1 < k < 5$$

Condition for Resolution into Linear Factors

The quadratic function

$$f(x,y)=ax^2+2hxy+by^2+2gx+2fy+c$$

may be resolved into two linear factors, iff

$$\Delta=abc+2fgh-af^2-bg^2-ch^2=0$$

i.e., $$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$$

Example 32. ***Find the value of*** m ***for which the expression*** $12x^2-10xy+2y^2+11x-5y+m$ ***can be resolved into two rational linear factors.***

Sol. Comparing the given expression with

$ax^2+2hxy+by^2+2gx+2fy+c$, we have

$$a=12,\ h=-5,\ b=2,\ g=\frac{11}{2},\ f=\left(-\frac{5}{2}\right),\ c=m$$

The given expression will have two linear factors, if and only if

$$abc+2fgh-af^2-bg^2-ch^2=0$$

or $$(12)(2)(m)+2\left(-\frac{5}{2}\right)\left(\frac{11}{2}\right)(-5)-(12)\left(-\frac{5}{2}\right)^2-(2)\left(\frac{11}{2}\right)^2-(m)(-5)^2=0$$

$$\Rightarrow \quad 24m+\frac{275}{2}-75-\frac{121}{2}-25m=0 \text{ or } m=2$$

Example 33. If the expression $ax^2+by^2+cz^2+2ayz+2bzx+2cxy$ can be resolved into two rational factors, prove that

$$a^3+b^3+c^3=3abc.$$

Sol. Given expression is

$$ax^2+by^2+cz^2+2ayz+2bzx+2cxy \qquad ...(i)$$

$$=z^2\left[a\left(\frac{x}{z}\right)^2+b\left(\frac{y}{z}\right)^2+c+2a\left(\frac{y}{z}\right)+2b\left(\frac{x}{z}\right)+2c\left(\frac{x}{z}\right)\left(\frac{y}{z}\right)\right]$$

$$=z^2\left[aX^2+bY^2+c+2aY+2bX+2cXY\right]$$

where, $\frac{x}{z}=X$ and $\frac{y}{z}=Y$

Expression (i) will have two rational linear factors in x, y and z, if expression

$aX^2+bY^2+2cXY+2bX+2aY+c$ will have two linear factors, if

$$abc+2abc-aa^2-bb^2-cc^2=0$$

or $$a^3+b^3+c^3=3abc$$

Example 34. Find the linear factors of $x^2-5xy+4y^2+x+2y-2$.

Sol. Given expression is

$$x^2-5xy+4y^2+x+2y-2 \qquad ...(i)$$

Its corresponding equation is

$$x^2-5xy+4y^2+x+2y-2=0$$

or $$x^2-x(5y-1)+4y^2+2y-2=0$$

$$\therefore \quad x=\frac{(5y-1)\pm\sqrt{(5y-1)^2-4\cdot1\cdot(4y^2+2y-2)}}{2}$$

$$=\frac{(5y-1)\pm\sqrt{(9y^2-18y+9)}}{2}$$

$$=\frac{(5y-1)\pm\sqrt{(3y-3)^2}}{2}$$

$$=\frac{(5y-1)\pm(3y-3)}{2}=4y-2,\ y+1$$

$\therefore$ The required linear factors are $(x-4y+2)$ and $(x-y-1)$.

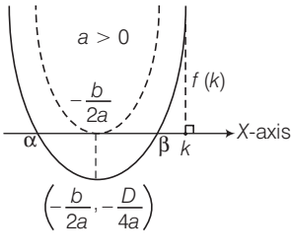
Location of Roots

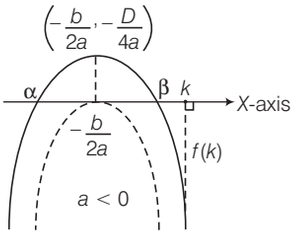
(Interval in which Roots Lie)

Let $f(x)=ax^2+bx+c,\ a,b,c\in R, a\neq0$ and α,β be the roots of $f(x)=0$. Suppose $k,k_1,k_2\in R$ and $k_1<k_2$. Then, the following hold good :

1. Conditions for Number k

(If both the roots of $f(x)=0$ are less than k)





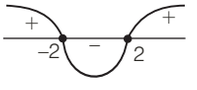
(i) $D\geq0$ (roots may be equal)

(ii) $af(k)>0$

(iii) $k>-\frac{b}{2a}$, where $\alpha\leq\beta$.

Example 35. Find the values of m, for which both roots of equation $x^2-mx+1=0$ are less than unity.

Sol. Let $f(x)=x^2-mx+1$, as both roots of $f(x)=0$ are less than 1, we can take $D\geq0$, $af(1)>0$ and $-\frac{b}{2a}<1$.



(i) **Consider $D \geq 0$** $(-m)^2 - 4 \cdot 1 \cdot 1 \geq 0$

$\Rightarrow \quad (m+2)(m-2) \geq 0$

$\Rightarrow \quad m \in (-\infty, -2] \cup [2, \infty)$ ...(i)

(ii) **Consider $af(1) > 0$** $1(1 - m + 1) > 0$

$\Rightarrow \quad m - 2 < 0 \Rightarrow m < 2$

$\Rightarrow \quad m \in (-\infty, 2)$ ...(ii)

(iii) **Consider** $\left(-\dfrac{b}{2a} < 1\right)$

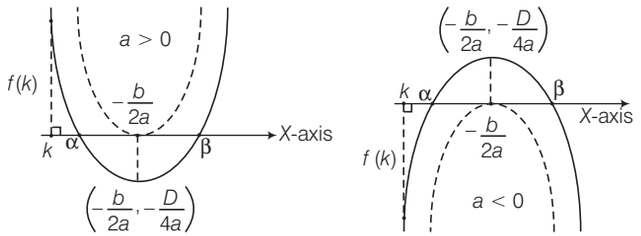
$$\frac{m}{2} < 1 \Rightarrow m < 2$$

$\Rightarrow \quad m \in (-\infty, 2)$ ...(iii)

Hence, the values of m satisfying Eqs. (i), (ii) and (iii) at the same time are $m \in (-\infty, -2]$.

2. Conditions for a Number k

If both the roots of $f(x) = 0$ are greater than k



(i) $D \geq 0$ (roots may be equal)

(ii) $af(k) > 0$

(iii) $k < -\dfrac{b}{2a}$, where $\alpha \leq \beta$.

Example 36. For what values of $m \in R$, both roots of the equation $x^2 - 6mx + 9m^2 - 2m - 2 = 0$ exceed 3?

Sol. Let $f(x) = x^2 - 6mx + 9m^2 - 2m + 2$

As both roots of $f(x) = 0$ are greater than 3, we can take $D \geq 0$, $af(3) > 0$ and $-\dfrac{b}{2a} > 3$.

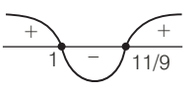
(i) **Consider $D \geq 0$**

$(-6m)^2 - 4 \cdot 1(9m^2 - 2m + 2) \geq 0 \Rightarrow 8m - 8 \geq 0$

$\therefore \quad m \geq 1$ or $m \in [1, \infty)$ ...(i)

(ii) **Consider $af(3) \geq 0$**

$1 \cdot (9 - 18m + 9m^2 - 2m + 2) > 0$



$\Rightarrow \quad 9m^2 - 20m + 11 > 0$

$\Rightarrow \quad (9m - 11)(m - 1) > 0$

$\Rightarrow \quad \left(m - \dfrac{11}{9}\right)(m - 1) > 0$

$\Rightarrow \quad m \in (-\infty, 1) \cup \left(\dfrac{11}{9}, \infty\right)$ ...(ii)

(iii) **Consider** $\left(-\dfrac{b}{2a} > 3\right)$

$$\frac{6m}{2} > 3$$

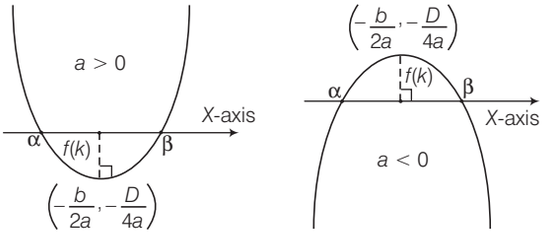
$\Rightarrow \quad m > 1$

$\Rightarrow \quad m \in (1, \infty)$ ...(iii)

Hence, the values of m satisfying Eqs. (i), (ii) and (iii) at the same time are $m \in \left(\dfrac{11}{9}, \infty\right)$.

3. Conditions for a Number k

If k lies between the roots of $f(x) = 0$



(i) $D > 0$ (ii) $af(k) < 0$, where $\alpha < \beta$

Example 37. ***Find all values of p, so that 6 lies between roots of the equation $x^2 + 2(p-3)x + 9 = 0$.***

Sol. Let $f(x) = x^2 + 2(p-3)x + 9$, as 6 lies between the roots of $f(x) = 0$, we can take $D > 0$ and $af(6) < 0$

(i) **Consider $D > 0$**

$\{2(p-3)\}^2 - 4 \cdot 1 \cdot 9 > 0$

$\Rightarrow \quad (p-3)^2 - 9 > 0$

$\Rightarrow \quad p(p-6) > 0$

$\Rightarrow \quad p \in (-\infty, 0) \cup (6, \infty)$ ...(i)

(ii) **Consider $af(6) < 0$**

$1 \cdot \{36 + 12(p-3) + 9\} < 0$

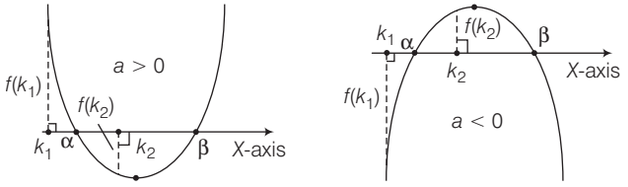
$\Rightarrow \quad 12p + 9 < 0 \Rightarrow p + \dfrac{3}{4} < 0$

$\Rightarrow \quad p \in \left(-\infty, -\dfrac{3}{4}\right)$ ...(ii)

Hence, the values of p satisfying Eqs. (i) and (ii) at the same time are $p \in \left(-\infty, -\dfrac{3}{4}\right)$.

4. Conditions for Numbers k_1 and k_2

If exactly one root of $f(x)=0$ lies in the interval (k_1, k_2)



(i) $D>0$

(ii) $f(k_1)\,f(k_2)<0$, where $\alpha<\beta$.

Example 38. Find the values of m, for which exactly one root of the equation $x^2-2mx+m^2-1=0$ lies in the interval $(-2,4)$.

Sol. Let $f(x)=x^2-2mx+m^2-1$, as exactly one root of $f(x)=0$ lies in the interval $(-2, 4)$, we can take $D>0$ and $f(-2)\,f(4)<0$.

(i) **Consider $D>0$**

$$(-2m)^2-4\cdot1(m^2-1)>0 \implies 4>0$$

$$\therefore \quad m\in R \qquad \ldots(i)$$

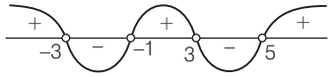
(ii) **Consider $f(-2)\,f(4)<0$**

$$(4+4m+m^2-1)(16-8m+m^2-1)<0$$

$$\Rightarrow \quad (m^2+4m+3)(m^2-8m+15)<0$$

$$\Rightarrow \quad (m+1)(m+3)(m-3)(m-5)<0$$

$$\Rightarrow \quad (m+3)(m+1)(m-3)(m-5)<0$$

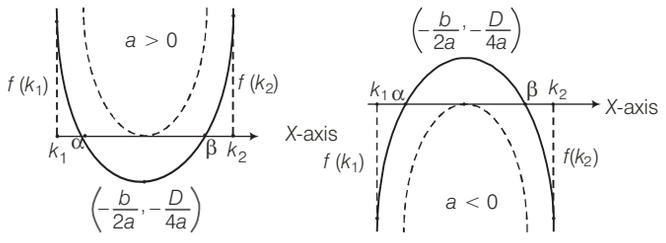


$$\therefore \quad m\in(-3,-1)\cup(3,5) \qquad \ldots(ii)$$

Hence, the values of m satisfying Eqs. (i) and (ii) at the same time are $m\in(-3,-1)\cup(3,5)$.

5. Conditions for Numbers k_1 and k_2

(If both roots $f(x)=0$ are confined between k_1 and k_2)



(i) $D\geq0$ (roots may be equal)

(ii) $af(k_1)>0$

(iii) $af(k_2)>0$

(iv) $k_1<-\dfrac{b}{2a}<k_2$, where $\alpha\leq\beta$ and $k_1<k_2$.

Example 39. Find all values of a for which the equation $4x^2-2x+a=0$ has two roots lie in the interval $(-1,1)$.

Sol. Let $f(x)=4x^2-2x+a$ as both roots of the equation, $f(x)=0$ are lie between $(-1, 1)$, we can take $D\geq0$, $af(-1)>0$, $af(1)>0$ and $-1<\dfrac{1}{4}<1$.

(i) **Consider $D\geq0$**

$$(-2)^2-4\cdot4\cdot a\geq0 \implies a\leq\frac{1}{4} \qquad \ldots(i)$$

(ii) **Consider $a\,f(-1)>0$**

$$4(4+2+a)>0$$

$$\Rightarrow \quad a>-6 \implies a\in(-6,\infty) \qquad \ldots(ii)$$

(iii) **Consider $a\,f(1)>0$**

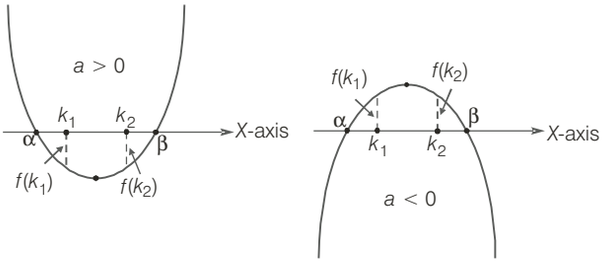
$$4(4-2+a)>0 \implies a>-2$$

$$\Rightarrow \quad a\in(-2,\infty) \qquad \ldots(iii)$$

Hence, the values of a satisfying Eqs. (i), (ii) and (iii) at the same time are $a\in\left(-2,\dfrac{1}{4}\right]$.

6. Conditions for Numbers k_1 and k_2

(If k_1 and k_2 lie between the roots of $f(x)=0$)



(i) $D>0$

(ii) $af(k_1)<0$

(iii) $af(k_2)<0$, where $\alpha<\beta$.

Example 40. ***Find the values of a for which one root of equation $(a-5)x^2-2ax+a-4=0$ is smaller than 1 and the other greater than 2.***

Sol. The given equation can be written as

$$x^2-\left(\frac{2a}{a-5}\right)x+\left(\frac{a-4}{a-5}\right)=0,\ a\neq5.$$

Now, let $f(x)=x^2-\left(\dfrac{2a}{a-5}\right)x+\left(\dfrac{a-4}{a-5}\right)$

As 1 and 2 lie between the roots of $f(x)=0$, we can take $D>0$, $1\cdot f(1)<0$ and $1\cdot f(2)<0$.

(i) **Consider $D > 0$**

$$\left(-\left(\frac{2a}{a-5}\right)\right)^2 - 4\cdot 1\cdot\left(\frac{a-4}{a-5}\right) > 0$$

$$\Rightarrow \quad \frac{36\left(a-\frac{20}{9}\right)}{(a-5)^2} > 0 \qquad [\because a \neq 5]$$

or $$a > \frac{20}{9} \qquad ...(i)$$

(ii) **Consider $1\cdot f(1) < 0$**

$$1^2 - \left(\frac{2a}{a-5}\right) + \left(\frac{a-4}{a-5}\right) < 0 \Rightarrow \frac{9}{(a-5)} > 0 \text{ or } a > 5 \quad ...(ii)$$

(iii) **Consider $1\cdot f(2) < 0$**

$$4 - \frac{4a}{(a-5)} + \left(\frac{a-4}{a-5}\right) < 0$$

$$\Rightarrow \quad \frac{(4a-20-4a+a-4)}{(a-5)} < 0 \quad \Rightarrow \quad \frac{(a-24)}{(a-5)} < 0$$

or $$5 < a < 24 \qquad ...(iii)$$

Hence, the values of a satisfying Eqs. (i), (ii) and (iii) at the same time are $a \in (5, 24)$.

Example 41. ***Let*** $x^2 - (m-3)x + m = 0\ (m \in R)$ ***be a quadratic equation. Find the value of*** m ***for which***

(i) both the roots are smaller than 2.

(ii) both the roots are greater than 2.

(iii) one root is smaller than 2 and the other root is greater than 2.

(iv) exactly one root lies in the interval (1, 2).

(v) both the roots lie in the interval (1, 2).

(vi) one root is greater than 2 and the other root is smaller than 1.

(vii) atleast one root lie in the interval (1, 2).

(viii) atleast one root is greater than 2.

Sol. Let $f(x) = x^2 - (m-3)x + m$

Here, $a = 1, b = -(m-3), c = m$

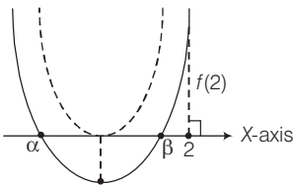
and $D = b^2 - 4ac = (m-3)^2 - 4m$

$= m^2 - 10m + 9 = (m-1)(m-9)$

and x-coordinate of vertex $= -\frac{b}{2a} = \frac{(m-3)}{2}$

(i) **Both the roots are smaller than 2**

$D \geq 0$



i.e., $(m-1)(m-9) \geq 0$

$\therefore \quad m \in (-\infty, 1] \cup [9, \infty) \qquad ...(i)$

$f(2) > 0$

i.e., $4 - 2(m-3) + m > 0$

$\Rightarrow \quad m < 10$

$\therefore \quad m \in (-\infty, 10) \qquad ...(ii)$

and x-coordinate of vertex < 2

i.e., $\frac{(m-3)}{2} < 2 \Rightarrow m < 7$

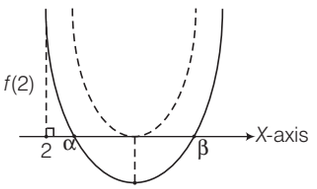
$\therefore \quad m \in (-\infty, 7) \qquad ...(iii)$

On combining Eqs. (i), (ii) and (iii), we get

$m \in (-\infty, 1]$

(ii) **Both the roots are greater than 2**

$D \geq 0$



i.e. $(m-1)(m-9) \geq 0$

$\therefore \quad m \in (-\infty, 1] \in [9, \infty) \qquad ...(i)$

$f(2) > 0$

i.e. $4 - 2(m-3) + m > 0$

$\Rightarrow \quad m < 10$

$\therefore \quad m \in (-\infty, 10) \qquad ...(ii)$

and x-coordinate of vertex > 2

i.e., $\frac{(m-3)}{2} > 2 \Rightarrow m > 7$

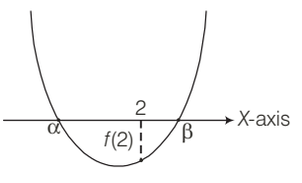
$\therefore \quad m \in (7, \infty) \qquad ...(iii)$

On combining Eqs. (i), (ii) and (iii), we get

$m \in [9, 10)$

(iii) **One root is smaller than 2 and the other root is greater than 2**

$D > 0$



i.e., $(m-1)(m-9) > 0$

$\therefore \quad m \in (-\infty, 1) \cup (9, \infty) \qquad ...(i)$

$f(2) < 0$

i.e. $4 - 2(m-3) + m < 0$

$\therefore \quad m > 10$

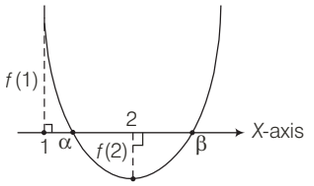
$\therefore \quad m \in (10, \infty)$ ...(ii)

On combining Eqs. (i) and (ii), we get

$m \in (10, \infty)$.

(iv) **Exactly one root lies in the interval (1, 2)**

$\boldsymbol{D > 0}$



i.e., $(m-1)(m-9) > 0$

$\therefore \quad m \in (-\infty, 1) \cup (9, \infty)$ ...(i)

$\boldsymbol{f(1)\, f(2) < 0}$

$(1-(m-3)+m)\,(4-2(m-3)+m) < 0$

$\Rightarrow \quad 4(-m+10) < 0$

$\Rightarrow \quad m - 10 > 0 \Rightarrow m > 10$

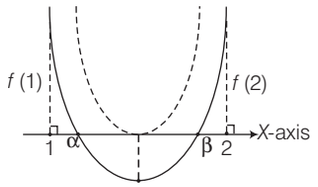
$\therefore \quad m \in (10, \infty)$ ...(ii)

On combining Eqs. (i) and (ii), we get

$m \in (10, \infty)$

(v) **Both the roots lie in the interval (1, 2)**

$\boldsymbol{D \geq 0}$



i.e., $(m-1)(m-9) \geq 0$

$\therefore \quad m \in (-\infty, 1] \cup [9, \infty)$ ...(i)

$\boldsymbol{f(1) > 0}$

i.e., $(1-(m-3)+m) > 0 \Rightarrow 4 > 0$

$\therefore \quad m \in R$ ...(ii)

$\boldsymbol{f(2) > 0}$

i.e., $4 - 2(m-3) + m > 0 \Rightarrow m < 10$

$\therefore \quad m \in (-\infty, 10)$ ...(iii)

$1 <$ x-coordinate of vertex < 2

i.e., $1 < \dfrac{(m-3)}{2} < 2$

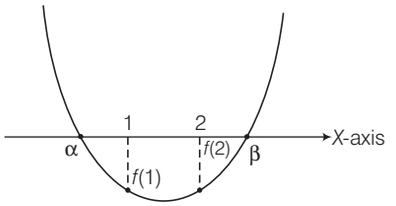
$\Rightarrow \quad 2 < m - 3 < 4$ or $5 < m < 7$

$\therefore \quad m \in (5, 7)$ ...(iv)

On combining Eqs. (i), (ii), (iii) and (iv), we get

$m \in \phi$

(vi) **One root is greater than 2 and the other root is smaller than 1** $\boldsymbol{D > 0}$



i.e., $(m-1)(m-9) > 0$

$\therefore \quad m \in (-\infty, 1) \cup (9, \infty)$ ...(i)

$\boldsymbol{f(1) < 0}$

i.e., $4 < 0$, which is not possible.

Thus, no such 'm' exists.

(vii) **At least one root lie in the interval (1, 2)**

***Case* I** Exactly one root lies in (1, 2)

$m \in (10, \infty)$ [from (iv) part]

***Case* II** Both roots lie in the interval (1, 2).

$m \in \phi$ [from (v) part]

Hence, at least one root lie in the interval (1, 2)

$m \in (10, \infty) \cup \phi$ or $m \in (10, \infty)$

(viii) **Atleast one root is greater than 2**

***Case* I** One root is smaller than 2 and the other root is greater than 2.

Then, $m \in (10, \infty)$ [from (iii) part]

***Case* II** Both the roots are greater than 2, then $m \in [9, 10)$.

Hence, atleast one root is greater than 2.

$\therefore \quad m \in (10, \infty) \cup [9, 10)$ or $m \in [9, 10) \cup (10, \infty)$

Exercise for Session 3

1. If x is real, the maximum and minimum values of expression $\frac{x^2+14x+9}{x^2+2x+3}$ will be

(a) 4, – 5 (b) 5, – 4 (c) – 4, 5 (d) – 4, – 5

2. If x is real, the expression $\frac{x+2}{(2x^2+3x+6)}$ takes all values in the interval

(a) $\left(\frac{1}{13},\frac{1}{3}\right)$ (b) $\left[-\frac{1}{13},\frac{1}{3}\right]$ (c) $\left(-\frac{1}{3},\frac{1}{13}\right)$ (d) None of these

3. If x be real, then the minimum value of $x^2-8x+17$, is

(a) – 1 (b) 0 (c) 1 (d) 2

4. If the expression $\left(mx-1+\frac{1}{x}\right)$ is non-negative for all positive real x, the minimum value of m must be

(a) $-\frac{1}{2}$ (b) 0

(c) $\frac{1}{4}$ (d) $\frac{1}{2}$

5. If the inequality $\frac{mx^2+3x+4}{x^2+2x+2}<5$ is satisfied for all $x \in R$ then

(a) $1<m<5$ (b) $-1<m<5$

(c) $1<m<6$ (d) $m<\frac{71}{24}$

6. The largest negative integer which satisfies $\frac{(x^2-1)}{(x-2)(x-3)}>0$, is

(a) – 4 (b) – 3

(c) – 2 (d) – 1

7. If the expression $2x^2+mxy+3y^2-5y-2$ can be resolved into two rational factors, the value of $|m|$ is

(a) 3 (b) 5

(c) 7 (d) 9

8. If $c>0$ and $4a+c<2b$, then $ax^2-bx+c=0$ has a root in the interval

(a) (0, 2) (b) (2, 4)

(c) (0, 1) (d) (– 2, 0)

9. If the roots of the equation $x^2-2ax+a^2+a-3=0$ are less than 3 then

(a) $a<2$ (b) $2 \le a \le 3$

(c) $3<a \le 4$ (d) $a>4$

10. The set of values of a for which the inequation $x^2+ax+a^2+6a<0$ is satisfied for all $x \in (1,2)$ lies in the interval

(a) (1, 2) (b) [1, 2]

(c) [– 7, 4] (d) None of these

Session 4

Equations of Higher Degree, Rational Algebraic Inequalities, Roots of Equation with the Help of Graphs,

Equations of Higher Degree

The equation $a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n = 0$, where $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are constants but $a_0 \neq 0$, is a polynomial equation of degree n. It has n and only n roots.

Let $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1}, \alpha_n$ be n roots, then

- $\Sigma \alpha_1 = \alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_{n-1} + \alpha_n = (-1)^1 \dfrac{a_1}{a_0}$

  [sum of all roots]
- $\Sigma \alpha_1 \alpha_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \ldots + \alpha_1\alpha_n + \alpha_2\alpha_3 + \ldots + \alpha_2\alpha_n + \ldots + \alpha_{n-1}\alpha_n$

  $= (-1)^2 \dfrac{a_2}{a_0}$ [sum of products taken two at a time]
- $\Sigma \alpha_1 \alpha_2 \alpha_3 = (-1)^3 \dfrac{a_3}{a_0}$

  [sum of products taken three at a time]
- $\alpha_1 \alpha_2 \alpha_3 \ldots \alpha_n = (-1)^n \dfrac{a_n}{a_0}$ **[product of all roots]**

In general, $\Sigma \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_p = (-1)^p \dfrac{a_p}{a_0}$

Remark

1. A polynomial equation of degree n has n roots (real or imaginary).
2. If all the coefficients, i.e., $a_0, a_1, a_2, \ldots, a_n$ are real, then the imaginary roots occur in pairs, i.e. number of imaginary roots is always even.
3. If the degree of a polynomial equation is odd, then atleast one of the roots will be real.
4. $(x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \ldots (x - \alpha_n)$
   $= x^n + (-1)^1 \Sigma \alpha_1 \cdot x^{n-1} + (-1)^2 \Sigma \alpha_1 \alpha_2 \cdot x^{n-2} + \ldots + (-1)^n \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_n$

In Particular

(i) For $n = 3$, if α, β, γ are the roots of the equation $ax^3 + bx^2 + cx + d = 0$, where a, b, c, d are constants and $a \neq 0$, then $\Sigma\alpha = \alpha + \beta + \gamma = (-1)^1 \dfrac{b}{a} = -\dfrac{b}{a}$,

$$\Sigma\alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha = (-1)^2 \frac{c}{a} = \frac{c}{a}$$

and $$\alpha\beta\gamma = (-1)^3 \frac{d}{a} = -\frac{d}{a}$$

or $ax^3 + bx^2 + cx + d = a(x - \alpha)(x - \beta)(x - \gamma)$

$= a(x^3 - \Sigma\alpha \cdot x^2 + \Sigma\alpha\beta \cdot x - \alpha\beta\gamma)]$

(ii) For $n = 4$, if $\alpha, \beta, \gamma, \delta$ are the roots of the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, where a, b, c, d, e are constants and $a \neq 0$, then

$$\Sigma\alpha = \alpha + \beta + \gamma + \delta = (-1)^1 \frac{b}{a} = -\frac{b}{a},$$

$$\Sigma\alpha\beta = (\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = (-1)^2 \frac{c}{a} = \frac{c}{a},$$

$$\Sigma\alpha\beta\gamma = \alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = (-1)^3 \frac{d}{a} = -\frac{d}{a}$$

and $$\alpha\beta\gamma\delta = (-1)^4 \frac{e}{a} = \frac{e}{a}$$

or $ax^4 + bx^3 + cx^2 + dx + e = a(x - \alpha)$

$(x - \beta)(x - \gamma)(x - \delta)$

$= a(x^4 - \Sigma\alpha \cdot x^3 + \Sigma\alpha\beta \cdot x^2 - \Sigma\alpha\beta\gamma \cdot x + \alpha\beta\gamma\delta)$

Example 42. Find the conditions, if roots of the equation $x^3 - px^2 + qx - r = 0$ are in

(i) AP (ii) GP

(iii) HP

Sol. (i) Let roots of the given equation are

$A - D, A, A + D$, then

$$A - D + A + A + D = p \Rightarrow A = \frac{p}{3}$$

Now, A is the roots of the given equation, then it must be satisfy

$$A^3 - pA^2 + qA - r = 0$$

$$\Rightarrow \left(\frac{p}{3}\right)^3 - p\left(\frac{p}{3}\right)^2 + q\left(\frac{p}{3}\right) - r = 0$$

$$\Rightarrow p^3 - 3p^3 + 9qp - 27r = 0$$

or $$2p^3 - 9pq + 27r = 0,$$

which is the required condition.

(ii) Let roots of the given equation are $\frac{A}{R}, A, AR$, then

$$\frac{A}{R}\cdot A\cdot AR=(-1)^3\cdot\left(-\frac{r}{1}\right)=r$$

$$\Rightarrow \quad A^3=r$$

$$\Rightarrow \quad A=r^{\frac{1}{3}}$$

Now, A is the roots of the given equation, then

$$A^3-pA^2+qA-r=0$$

$$\Rightarrow \quad r-p(r)^{2/3}=q(r)^{1/3}-r=0$$

or $$p(r)^{2/3}=q(r)^{1/3}$$

or $$p^3r^2=q^3r$$

or $$p^3r=q^3$$

which is the required condition.

(iii) Given equation is

$$x^3-px^2+qx-r=0 \qquad ...(i)$$

On replacing x by $\frac{1}{x}$ in Eq. (i), then

$$\left(\frac{1}{x}\right)^3-p\left(\frac{1}{x}\right)^2+q\left(\frac{1}{x}\right)-r=0$$

$$\Rightarrow \quad rx^3-qx^2+px-1=0 \qquad ...(ii)$$

Now, roots of Eq. (ii) are in AP.

Let roots of Eq. (ii) are $A-P, A, A+P$, then

$$A-P+A+A+P=\frac{q}{r} \quad \text{or} \quad A=\frac{q}{3r}$$

$\because A$ is a root of Eq. (ii), then

$$rA^3-qA^2+pA-1=0$$

$$\Rightarrow \quad r\left(\frac{q}{3r}\right)^3-q\left(\frac{q}{3r}\right)^2+p\left(\frac{q}{3r}\right)-1=0$$

$$\Rightarrow \quad q^3-3q^3+9pqr-27r^2=0$$

$$\Rightarrow \quad 2q^3-9pqr+27r^2=0,$$

which is the required condition.

Example 43. Solve $6x^3-11x^2+6x-1=0$, if roots of the equation are in HP.

Sol. Put $x=\frac{1}{y}$ in the given equation, then

$$\frac{6}{y^3}-\frac{11}{y^2}+\frac{6}{y}-1=0$$

$$\Rightarrow \quad y^3-6y^2+11y-6=0 \qquad ...(i)$$

Now, roots of Eq. (i) are in AP.

Let the roots be $\alpha-\beta, \alpha, \alpha+\beta$.

Then, sum of roots $=\alpha-\beta+\alpha+\alpha+\beta=6$

$$\Rightarrow \quad 3\alpha=6$$

$$\therefore \quad \alpha=2$$

Product of roots $=(\alpha-\beta)\cdot\alpha\cdot(\alpha+\beta)=6$

$$\Rightarrow \quad (2-\beta)2(2+\beta)=6 \Rightarrow 4-\beta^2=3$$

$$\therefore \quad \beta=\pm 1$$

$\therefore$ Roots of Eqs. (i) are 1, 2, 3 or 3, 2, 1.

Hence, roots of the given equation are $1, \frac{1}{2}, \frac{1}{3}$ or $\frac{1}{3}, \frac{1}{2}, 1$.

Example 44. If α, β, γ are the roots of the equation $x^3-px^2+qx-r=0$, find

(i) $\sum\alpha^2$. (ii) $\sum\alpha^2\beta$. (iii) $\sum\alpha^3$.

Sol. Since, α, β, γ are the roots of $x^3-px^2+qx-r=0$.

$$\therefore \quad \sum\alpha=p, \sum\alpha\beta=q \text{ and } \alpha\beta\gamma=r$$

(i) $\because \sum\alpha\cdot\sum\alpha=p\cdot p$

$$\Rightarrow \quad (\alpha+\beta+\gamma)(\alpha+\beta+\gamma)=p^2$$

$$\Rightarrow \alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\beta\gamma+\gamma\alpha)=p^2$$

or $$\sum\alpha^2+2\sum\alpha\beta=p^2$$

or $$\sum\alpha^2=p^2-2q$$

(ii) $\because \sum\alpha\cdot\sum\alpha\beta=p\cdot q$

$$\Rightarrow \quad (\alpha+\beta+\gamma)\cdot(\alpha\beta+\beta\gamma+\gamma\alpha)=pq$$

$$\Rightarrow \alpha^2\beta+\alpha\beta\gamma+\alpha^2\gamma+\beta^2\alpha+\beta^2\gamma+\alpha\beta\gamma+\gamma^2\beta+\gamma^2\alpha=pq$$

$$\Rightarrow (\alpha^2\beta+\alpha^2\alpha+\beta^2\gamma+\beta^2\gamma+\gamma^2\alpha+\gamma^2\beta)+3\alpha\beta\gamma=pq$$

or $$\sum\alpha^2\beta+3r=pq$$

or $$\sum\alpha^2\beta=pq-3r$$

(iii) $\because \sum\alpha^2\cdot\sum\alpha=(p^2-2q)\cdot p$ [from result (i)]

$$\Rightarrow \quad (\alpha^2+\beta^2+\gamma^2)(\alpha+\beta+\gamma)=p^3-2pq$$

$$\Rightarrow \quad \alpha^3+\beta^3+\gamma^3+(\alpha^2\beta+\alpha^2\gamma+\beta^2\alpha+\beta^2\gamma+\gamma^2\alpha+\gamma^2\beta)=p^3-2pq$$

$$\Rightarrow \quad \sum\alpha^3+\sum\alpha^2\beta=p^3-2pq$$

$$\Rightarrow \quad \sum\alpha^3+pq-3r=p^3-2pq \qquad \text{[from result (ii)]}$$

or $$\sum\alpha^3=p^3-3pq+3r$$

Example 45. If α, β, γ are the roots of the cubic equation $x^3+qx+r=0$, then find the equation whose roots are $(\alpha-\beta)^2, (\beta-\gamma)^2, (\gamma-\alpha)^2$.

Sol. $\because \alpha, \beta, \gamma$ are the roots of the cubic equation

$$x^3+qx+r=0 \qquad ...(i)$$

Then, $$\sum\alpha=0, \sum\alpha\beta=q, \alpha\beta\gamma=-r \qquad ...(ii)$$

If y is a root of the required equation, then

$$y=(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta$$

$$=(\alpha+\beta+\gamma-\gamma)^2-\frac{4\alpha\beta\gamma}{\gamma}$$

$$= (0-\gamma)^2 + \frac{4r}{\gamma} \qquad \text{[from Eq. (ii)]}$$

$$\Rightarrow \quad y = \gamma^2 + \frac{4r}{\gamma}$$

[replacing γ by x which is a root of Eq. (i)]

$$\therefore \quad y = x^2 + \frac{4r}{x}$$

or $$x^3 - yx + 4r = 0 \qquad \ldots(iii)$$

The required equation is obtained by eliminating x between Eqs. (i) and (iii).

Now, subtracting Eq. (iii) from Eq. (i), we get

$$(q+y)\,x - 3r = 0$$

or $$x = \frac{3r}{q+y}$$

On substituting the value of x in Eq. (i), we get

$$\left(\frac{3r}{q+y}\right)^3 + q\left(\frac{3r}{q+y}\right) + r = 0$$

Thus, $$y^3 + 6qy^2 + 9q^2y + (4q^3 + 27r^2) = 0$$

which is the required equation.

Remark

$\Sigma(\alpha-\beta)^2 = -6q, \qquad \Pi(\alpha-\beta)^2 = -(4q^3 + 27r^2)$

Some Results on Roots of a Polynomial Equation

1. **Remainder Theorem** If a polynomial $f(x)$ is divided by a linear function $x-\lambda$, then the remainder is $f(\lambda)$,

   i.e. Dividend = Divisor × Quotient + Remainder

   Let $Q(x)$ be the quotient and R be the remainder, thus

   $$f(x) = (x-\lambda)\,Q(x) + R$$

   $$\Rightarrow \quad f(\lambda) = (\lambda-\lambda)\,Q(\lambda) + R = 0 + R = R$$

Example 46. If the expression $2x^3 + 3px^2 - 4x + p$ has a remainder of 5 when divided by $x+2$, find the value of p.

Sol. Let $$f(x) = 2x^3 + 3px^2 - 4x + p$$

$$\because \quad f(x) = (x+2)Q(x) + 5$$

$$\Rightarrow \quad f(-2) = 5$$

$$\Rightarrow \quad 2(-2)^3 + 3p(-2)^2 - 4(-2) + p = 5 \text{ or } 13p = 13$$

$$\therefore \quad p = 1$$

2. **Factor Theorem** Factor theorem is a special case of Remainder theorem.

   Let $f(x) = (x-\lambda)\,Q(x) + R = (x-\lambda)\,Q(x) + f(\lambda)$

   If $f(\lambda) = 0$, $f(x) = (x-\lambda)\,Q(x)$, therefore $f(x)$ is exactly divisible by $x-\lambda$.

   or

   If λ is a root of the equation $f(x) = 0$, then $f(x)$ is exactly divisible by $(x-\lambda)$ and conversely, if $f(x)$ is exactly divisible by $(x-\lambda)$, then λ is a root of the equation $f(x) = 0$ and the remainder obtained is $f(\lambda)$.

Example 47. ***If $x^2 + ax + 1$ is a factor of $ax^3 + bx + c$, find the conditions.***

Sol. $\because \; ax^3 + bx + c = (x^2 + ax + 1)Q(x)$

Let $$Q(x) = Ax + B,$$

then $$ax^3 + bx + c = (x^2 + ax + 1)(Ax + B)$$

On comparing coefficients of x^3, x^2, x and constant on both sides, we get

$$a = A, \qquad \ldots(i)$$

$$0 = B + aA, \qquad \ldots(ii)$$

$$b = aB + A, \qquad \ldots(iii)$$

and $$c = B \qquad \ldots(iv)$$

From Eqs. (i) and (iv), we get

$$A = a \text{ and } B = c$$

From Eqs. (ii) and (iii), $a^2 + c = 0$ and $b = ac + a$ are the required conditions.

Example 48. A certain polynomial $f(x)$, $x \in R$, when divided by $x-a$, $x-b$ and $x-c$ leaves remainders a, b and c, respectively. Then, find the remainder when $f(x)$ is divided by $(x-a)(x-b)(x-c)$, where a, b, c are distinct.

Sol. By Remainder theorem $f(a) = a$, $f(b) = b$ and $f(c) = c$

Let the quotient be $Q(x)$ and remainder is $R(x)$.

$$\therefore \quad f(x) = (x-a)(x-b)(x-c)Q(x) + R(x)$$

$$\therefore \quad f(a) = 0 + R(a) \Rightarrow R(a) = a$$

$$f(b) = 0 + R(b) \Rightarrow R(b) = b \text{ and } f(c) = 0 + R(c)$$

$$\Rightarrow \quad R(c) = c$$

So, the equation $R(x) - x = 0$ has three roots a, b and c. But its degree is atmost two. So, $R(x) - x$ must be zero polynomial (or identity).

Hence, $R(x) = x$.

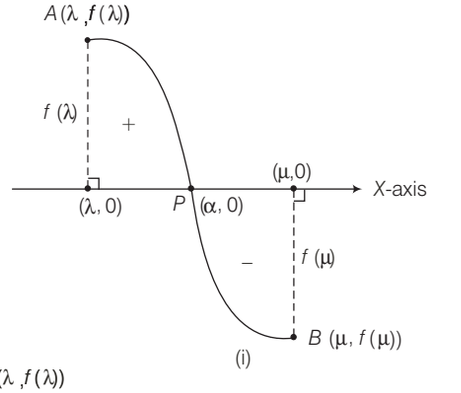
3. Every equation of an odd degree has atleast one real root, whose sign is opposite to that of its last term, provided that the coefficient of the first term is positive.
4. Every equation of an even degree has atleast two real roots, one positive and one negative, whose last term is negative, provided that the coefficient of the first term is positive.
5. If an equation has no odd powers of x, then all roots of the equation are complex provided all the coefficients of the equation have positive sign.

6. If $x = \alpha$ is root repeated m times in $f(x) = 0$ ($f(x) = 0$ is an nth degree equation in x), then $f(x) = (x - \alpha)^m g(x)$
where, $g(x)$ is a polynomial of degree $(n - m)$ and the root $x = \alpha$ is repeated $(m - 1)$ time in $f'(x) = 0$, $(m - 2)$ times in $f''(x) = 0, \ldots, (m - (m - 1))$ times in $f^{m-1}(x) = 0$.

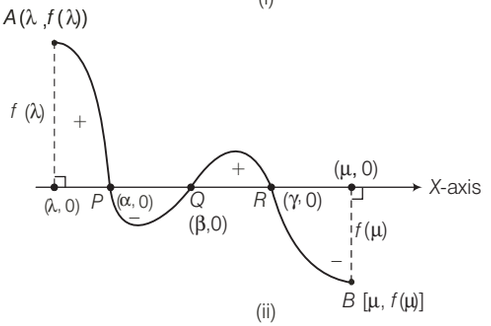
7. Let $f(x) = 0$ be a polynomial equation and λ, μ are two real numbers.
Then, $f(x) = 0$ will have atleast one real root or an odd number of roots between λ and μ, if $f(\lambda)$ and $f(\mu)$ are of opposite signs.
But if $f(\lambda)$ and $f(\mu)$ are of same signs, then either $f(x) = 0$ has no real roots or an even number of roots between λ and μ.

Illustration by Graphs

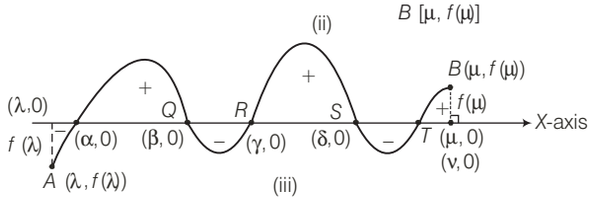
Since, $f(x)$ be a polynomial in x, then graph of $y = f(x)$ will be continuous in every interval.



(i)



(ii)



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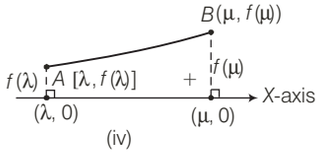

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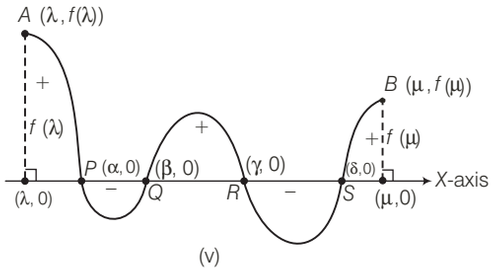

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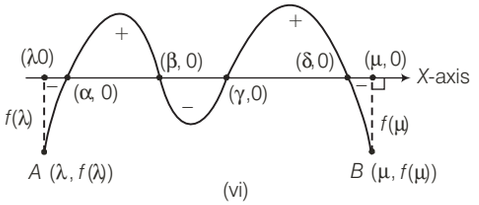


(vi)

(a) In figure (i), (ii) and (iii), $f(\lambda)$ and $f(\mu)$ have opposite signs and equation $f(x) = 0$, has one, three, five roots between λ and μ, respectively.

(b) In figure (iv), (v) and (vi), $f(\lambda)$ and $f(\mu)$ have same signs and equation $f(x) = 0$, has no, four and four roots between λ and μ, respectively.

Example 49. If a, b, c are real numbers, $a \neq 0$. If α is root of $a^2x^2 + bx + c = 0$, β is a root of $a^2x^2 - bx - c = 0$ and $0 < \alpha < \beta$, show that the equation $a^2x^2 + 2bx + 2c = 0$ has a root γ that always satisfies $\alpha < \gamma < \beta$.

Sol. Since, α is a root of $a^2x^2 + bx + c = 0$.

Then, $a^2\alpha^2 + b\alpha + c = 0$ ...(i)

and β is a root of $a^2x^2 - bx - c = 0$,

then $a^2\beta^2 - b\beta - c = 0$ ...(ii)

Let $f(x) = a^2x^2 + 2bx + 2c$

$\therefore$ $f(\alpha) = a^2\alpha^2 + 2b\alpha + 2c = a^2\alpha^2 - 2a^2\alpha^2$ [from Eq. (i)]

$= -a^2\alpha^2$

$\Rightarrow$ $f(\alpha) < 0$ and $f(\beta) = a^2\beta^2 + 2b\beta + 2c$

$= a^2\beta^2 + 2a^2\beta^2$ [from Eq. (ii)]

$= 3a^2\beta^2$

$\Rightarrow$ $f(\beta) > 0$

Since, $f(\alpha)$ and $f(\beta)$ are of opposite signs, then it is clear that a root γ of the equation $f(x) = 0$ lies between α and β.

Hence, $\alpha < \gamma < \beta$ [$\because \alpha < \beta$]

Example 50. If $a < b < c < d$, then show that $(x - a)(x - c) + 3(x - b)(x - d) = 0$ has real and distinct roots.

Sol. Let $f(x) = (x - a)(x - c) + 3(x - b)(x - d)$

Then, $f(a) = 0 + 3(a-b)(a-d) > 0 \quad [\because a-b<0, a-d<0]$
and $f(b) = (b-a)(b-c) + 0 < 0 \quad [\because b-a>0, b-c<0]$
Thus, one root will lie between a and b.
and $f(c) = 0 + 3(c-b)(c-d) < 0 \quad [\because c-b>0, c-d<0]$
and $f(d) = (d-a)(d-c) + 0 > 0 \quad [\because d-a>0, d-c>0]$
Thus, one root will lie between c and d. Hence, roots of equation are real and distinct.

8. Let $f(x)=0$ be a polynomial equation then

(a) the number of positive roots of a polynomial equation $f(x)=0$ (arranged in decreasing order of the degree) cannot exceed the number of changes of signs in $f(x)=0$ as we move from left to right.
For example, Consider the equation
$$2x^3 - x^2 - x + 1 = 0.$$
The number of changes of signs from left to right is 2 (+ to –, then – to +). Then, number of positive roots cannot exceed 2.

(b) The number of negative roots of a polynomial equation $f(x)=0$ cannot exceed the number of changes of signs in $f(-x)$.
For example, Consider the equation
$$5x^4 + 3x^3 - 2x^2 + 5x - 8 = 0$$
Let $f(x) = 4x^4 + 3x^3 - 2x^2 + 5x - 8$
$\therefore \quad f(-x) = 5x^4 - 3x^3 - 2x^2 - 5x - 8$
The number of changes of signs from left to right is (+ to –). Then number of negative roots cannot exceed 1.

(c) If equation $f(x)=0$ have atmost r positive roots and atmost t negative roots, then equation $f(x)=0$ will have atmost $(r+t)$ real roots, i.e. it will have atleast $n-(r+t)$ imaginary roots, where n is the degree of polynomial.
For example, Consider the equation
$$5x^6 - 8x^3 + 3x^5 + 5x^2 + 8 = 0$$
The given equation can be written as
$$5x^6 + 3x^5 - 8x^3 + 5x^2 + 8 = 0$$
Let $f(x) = 5x^6 + 3x^5 - 8x^3 + 5x^2 + 8$
Here, $f(x)$ has two changes in signs.
So, $f(x)$ has atmost two positive real roots
and $f(-x) = 5x^6 - 3x^5 + 8x^3 + 5x^2 + 8$
Here, $f(-x)$ has two changes in signs.
So, $f(x)$ has atmost two negative real roots.
and $x=0$ cannot be root of $f(x)=0$.
Hence, $f(x)=0$ has atmost four real roots, therefore atleast two imaginary roots.

9. **Rolle's Theorem** If $f(x)$ is continuous function in the interval $[a,b]$ and differentiable in interval (a,b) and $f(a)=f(b)$, then equation $f'(x)=0$ will have atleast one root between a and b. Since, every polynomial $f(x)$ is always continuous and differentiable in every interval. Therefore, Rolle's theorem is always applicable to polynomial function in every interval $[a,b]$ if $f(a)=f(b)$.

Example 51. If $2a+3b+6c=0$; $a,b,c \in R$, then show that the equation $ax^2+bx+c=0$ has atleast one root between 0 and 1.

Sol. Given, $2a+3b+6c=0$

$$\Rightarrow \quad \frac{a}{3}+\frac{b}{2}+c=0 \qquad \text{...(i)}$$

Let $f'(x) = ax^2 + bx + c,$

Then, $f(x) = \frac{ax^3}{3} + \frac{bx^2}{2} + cx + d$

Now, $f(0) = d$ and $f(1) = \frac{a}{3} + \frac{b}{2} + c + d$
$= 0 + d \quad$ [from Eq. (i)]

Since, $f(x)$ is a polynomial of three degree, then $f(x)$ is continuous and differentiable everywhere and $f(0)=f(1)$, then by Rolle's theorem $f'(x)=0$ i.e., $ax^2+bx+c=0$ has atleast one real root between 0 and 1.

Reciprocal Equation of the Standard Form can be Reduced to an Equation of Half Its Dimensions

Let the equation be
$$ax^{2m} + bx^{2m-1} + cx^{2m-2} + \ldots + kx^m + \ldots + cx^2 + bx + a = 0$$
On dividing by x^m, then
$$ax^m + bx^{m-1} + cx^{m-2} + \ldots + k + \ldots + \frac{c}{x^{m-2}} + \frac{b}{x^{m-1}} + \frac{a}{x^m} = 0$$
On rearranging the terms, we have
$$a\left(x^m + \frac{1}{x^m}\right) + b\left(x^{m-1} + \frac{1}{x^{m-1}}\right) + c\left(x^{m-2} + \frac{1}{x^{m-2}}\right) + \ldots + k = 0$$
Now, $x^{p+1} + \frac{1}{x^{p+1}} = \left(x^p + \frac{1}{x^p}\right)\left(x + \frac{1}{x}\right) - \left(x^{p-1} + \frac{1}{x^{p-1}}\right)$

Hence, writing z for $x+\frac{1}{x}$ and given to p succession the values $1, 2, 3, \ldots$, we obtain

$$x^2+\frac{1}{x^2}=z^2-2$$

$$x^3+\frac{1}{x^3}=z(z^2-2)-z=z^3-3z$$

$$x^4+\frac{1}{x^4}=z(z^3-3z)-(z^2-2)=z^4-4z^2+2$$

and so on and **generally $x^m+\frac{1}{x^m}$ is of m dimensions in z and therefore the equation in z is of m dimensions**.

Example 52. Solve the equation $2x^4+x^3-11x^2+x+2=0$.

Sol. Since, $x=0$ is not a solution of the given equation.

On dividing by x^2 in both sides of the given equation, we get

$$2\left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)-11=0 \qquad \ldots(i)$$

Put $x+\frac{1}{x}=y$ in Eq. (i), then Eq. (i) reduce in the form

$$2(y^2-2)+y-11=0$$

$$\Rightarrow \quad 2y^2+y-15=0$$

$$\therefore \quad y_1=-3 \text{ and } y_2=\frac{5}{2}$$

Consequently, the original equation is equivalent to the collection of equations

$$\begin{cases} x+\frac{1}{x}=-3 \\ x+\frac{1}{x}=\frac{5}{2} \end{cases},$$

we find that, $x_1=\frac{-3-\sqrt{5}}{2}$, $x_2=\frac{-3+\sqrt{5}}{2}$, $x_3=\frac{1}{2}$, $x_4=2$

Equations which can be Reduced to Linear, Quadratic and Biquadratic Equations

***Type* I An equation of the form**

$$(x-a)(x-b)(x-c)(x-d)=A$$

where, $a<b<c<d$, $b-a=d-c$, can be solved by a change of variable.

i.e. $$y=\frac{(x-a)+(x-b)+(x-c)+(x-d)}{4}$$

$$y=x-\frac{(a+b+c+d)}{4}$$

Example 53. Solve the equation $(12x-1)(6x-1)(4x-1)(3x-1)=5$.

Sol. The given equation can be written as

$$\left(x-\frac{1}{12}\right)\left(x-\frac{1}{6}\right)\left(x-\frac{1}{4}\right)\left(x-\frac{1}{3}\right)=\frac{5}{12\cdot 6\cdot 4\cdot 3} \qquad \ldots(i)$$

Since, $\frac{1}{12}<\frac{1}{6}<\frac{1}{4}<\frac{1}{3}$ and $\frac{1}{6}-\frac{1}{12}=\frac{1}{3}-\frac{1}{4}$

We can introduced a new variable,

$$y=\frac{1}{4}\left[\left(x-\frac{1}{12}\right)+\left(x-\frac{1}{6}\right)+\left(x-\frac{1}{4}\right)+\left(x-\frac{1}{3}\right)\right]$$

$$y=x-\frac{5}{24}$$

On substituting $x=y+\frac{5}{24}$ in Eq. (i), we get

$$\left(y+\frac{3}{24}\right)\left(y+\frac{1}{24}\right)\left(y-\frac{1}{24}\right)\left(y-\frac{3}{24}\right)=\frac{5}{12\cdot 6\cdot 4\cdot 3}$$

$$\Rightarrow \quad \left[y^2-\left(\frac{1}{24}\right)^2\right]\left[y^2-\left(\frac{3}{24}\right)^2\right]=\frac{5}{12\cdot 6\cdot 4\cdot 3}$$

Hence, we find that

$$y^2=\frac{49}{24^2}$$

i.e. $$y_1=\frac{7}{24} \text{ and } y_2=-\frac{7}{24}$$

Hence, the corresponding roots of the original equation are $-\frac{1}{12}$ and $\frac{1}{2}$.

***Type* II** An equation of the form

$$(x-a)(x-b)(x-c)(x-d)=Ax^2$$

where, $ab=cd$ can be reduced to a collection of two quadratic equations by a change of variable $y=x+\frac{ab}{x}$.

Example 54. Solve the equation $(x+2)(x+3)(x+8)(x+12)=4x^2$.

Sol. Since, $(-2)(-12)=(-3)(-8)$, so we can write given equation as

$$(x+2)(x+12)(x+3)(x+8)=4x^2$$

$$\Rightarrow \quad (x^2+14x+24)(x^2+11x+24)=4x^2 \qquad \ldots(i)$$

Now, $x=0$ is not a root of given equation.

On dividing by x^2 in both sides of Eq. (i), we get

$$\left(x+\frac{24}{x}+14\right)\left(x+\frac{24}{x}+11\right)=4 \qquad \ldots(ii)$$

Put $x+\frac{24}{x}=y$, then Eq. (ii) can be reduced in the form

$$(y+14)(y+11)=4 \quad \text{or} \quad y^2+25y+150=0$$

$$\therefore \quad y_1=-15 \text{ and } y_2=-10$$

Thus, the original equation is equivalent to the collection of equations

$$\left[\begin{array}{l} x+\dfrac{24}{x}=-15, \\ x+\dfrac{24}{x}=-10, \end{array}\right.$$

i.e. $$\left[\begin{array}{l} x^2+15x+24=0 \\ x^2+10x+24=0 \end{array}\right.$$

On solving these collection, we get

$$x_1=\frac{-15-\sqrt{129}}{2},\ x_2=\frac{-15+\sqrt{129}}{2},\ x_3=-6,\ x_4=-4$$

***Type* III An equation of the form $(x-a)^4+(x-b)^4=A$**

can also be solved by a change of variable, i.e. making a substitution $y=\dfrac{(x-a)+(x-b)}{2}$.

Example 55. *Solve the equation* $(6-x)^4+(8-x)^4=16$.

Sol. After a change of variable,

$$y=\frac{(6-x)+(8-x)}{2}$$

$$\therefore \quad y=7-x \quad \text{or} \quad x=7-y$$

Now, put $x=7-y$ in given equation, we get

$$(y-1)^4+(y+1)^4=16$$

$$\Rightarrow \quad y^4+6y^2-7=0$$

$$\Rightarrow \quad (y^2+7)(y^2-1)=0$$

$$y^2+7\neq 0$$

[y gives imaginary values]

$$\therefore \quad y^2-1=0$$

Then, $y_1=-1$ and $y_2=1$

Thus, $x_1=8$ and $x_2=6$ are the roots of the given equation.

Rational Algebraic Inequalities

Consider the following types of rational algebraic inequalities

$$\frac{P(x)}{Q(x)}>0,\ \frac{P(x)}{Q(x)}<0,$$

$$\frac{P(x)}{Q(x)}\geq 0,\ \frac{P(x)}{Q(x)}\leq 0$$

If $P(x)$ and $Q(x)$ can be resolved in linear factors, then use *Wavy curve method*, otherwise we use the following statements for solving inequalities of this kind.

(1) $\dfrac{P(x)}{Q(x)}>0\Rightarrow\left\{P(x)\,Q(x)>0\Rightarrow\begin{cases}P(x)>0, Q(x)>0\\ \text{or}\\ P(x)<0, Q(x)<0\end{cases}\right.$

(2) $\dfrac{P(x)}{Q(x)}<0\Rightarrow\left\{P(x)\,Q(x)<0\Rightarrow\begin{cases}P(x)>0, Q(x)<0\\ \text{or}\\ P(x)<0, Q(x)>0\end{cases}\right.$

(3) $\dfrac{P(x)}{Q(x)}\geq 0\Rightarrow\begin{cases}P(x)\,Q(x)\geq 0\\ Q(x)\neq 0\end{cases}\Rightarrow\begin{cases}P(x)\geq 0, Q(x)>0\\ \text{or}\\ P(x)\leq 0, Q(x)<0\end{cases}$

(4) $\dfrac{P(x)}{Q(x)}\leq 0\Rightarrow\begin{cases}P(x)\,Q(x)\leq 0\\ Q(x)\neq 0\end{cases}\Rightarrow\begin{cases}P(x)\geq 0, Q(x)<0\\ \text{or}\\ P(x)\leq 0, Q(x)>0\end{cases}$

Example 56. *Find all values of a for which the set of all solutions of the system*

$$\begin{cases}\dfrac{x^2+ax-2}{x^2-x+1}<2\\ \dfrac{x^2+ax-2}{x^2-x+1}>-3\end{cases}$$

is the entire number line.

Sol. The system is equivalent to

$$\begin{cases}\dfrac{x^2-(a+2)x+4}{x^2-x+1}>0\\ \dfrac{4x^2+(a-3)x+1}{x^2-x+1}>0\end{cases}$$

Since, $x^2-x+1=\left(x-\dfrac{1}{2}\right)^2+\dfrac{3}{4}>0$, this system is

equivalent to $\begin{cases}x^2-(a+2)x+4>0\\ 4x^2+(a-3)x+1>0\end{cases}$

Hence, the discriminants of the both equations of this system are negative.

i.e., $$\begin{cases}(a+2)^2-16<0\\ (a-3)^2-16<0\end{cases}\Rightarrow\ (a+6)(a-2)<0$$

i.e., $x\in(-6,2)$ ...(i)

$\Rightarrow$ $(a+1)(a-7)<0$

i.e. $x\in(-1,7)$ ...(ii)

Hence, from Eqs. (i) and (ii), we get

$$x\in(-1,2)$$

Equations Containing Absolute Values

By definition, $|x| = x$, if $x \geq 0$ $|x| = -x$, if $x < 0$

Example 57. ***Solve the equation*** $x^2 - 5|x| + 6 = 0$.

Sol. The given equation is equivalent to the collection of systems

$$\begin{cases} x^2 - 5x + 6 = 0, \text{ if } x \geq 0 \\ x^2 + 5x + 6 = 0, \text{ if } x < 0 \end{cases} \Rightarrow \begin{cases} (x-2)(x-3) = 0, \text{ if } x \geq 0 \\ (x+2)(x+3) = 0, \text{ if } x < 0 \end{cases}$$

Hence, the solutions of the given equation are

$$x_1 = 2,\ x_2 = 3,\ x_3 = -2,\ x_4 = -3$$

Example 58. ***Solve the equation***

$$\left| \frac{x^2 - 8x + 12}{x^2 - 10x + 21} \right| = -\frac{x^2 - 8x + 12}{x^2 - 10x + 21}.$$

Sol. This equation has the form $|f(x)| = -f(x)$

when, $f(x) = \dfrac{x^2 - 8x + 12}{x^2 - 10x + 21}$

such an equation is equivalent to the collection of systems

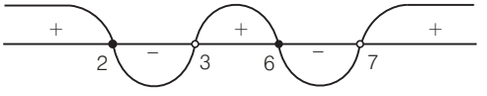
$$\begin{cases} f(x) = -f(x), \text{ if } f(x) \geq 0 \\ f(x) = f(x), \quad \text{if } f(x) < 0 \end{cases}$$

The first system is equivalent to $f(x) = 0$ and the second system is equivalent to $f(x) < 0$ the combining both systems, we get

$$f(x) \leq 0$$

$$\therefore \quad \frac{x^2 - 8x + 12}{x^2 - 10x + 21} \leq 0$$

$$\Rightarrow \quad \frac{(x-2)(x-6)}{(x-3)(x-7)} \leq 0$$



Hence, by *Wavy curve method,*

$$x \in [2, 3) \cup [6, 7)$$

Example 59. Solve the equation $|x - |4 - x|| - 2x = 4$.

Sol. This equation is equivalent to the collection of systems

$$\begin{cases} |x - (4 - x)| - 2x = 4, \text{ if } 4 - x \geq 0 \\ |x + (4 - x)| - 2x = 4, \text{ if } 4 - x < 0 \end{cases}$$

$$\Rightarrow \quad \begin{cases} |2x - 4| - 2x = 4, & \text{if } x \leq 4 \\ 4 - 2x = 4, & \text{if } x > 4 \end{cases} \qquad \ldots(\text{i})$$

The second system of this collection

gives $x = 0$

but $x > 4$

Hence, second system has no solution.

The first system of collection Eq. (i) is equivalent to the system of collection

$$\begin{cases} 2x - 4 - 2x = 4, \text{ if } 2x \geq 4 \\ -2x + 4 - 2x = 4, \text{ if } 2x < 4 \end{cases}$$

$$\Rightarrow \quad \begin{cases} -4 = 4, \text{ if } x \geq 2 \\ -4x = 0, \text{ if } x < 2 \end{cases}$$

The first system is failed and second system gives $x = 0$.

Hence, $x = 0$ is unique solution of the given equation.

Important Forms Containing Absolute Values

***Form* 1 The equation of the form**

$$|f(x) + g(x)| = |f(x)| + |g(x)|$$

is equivalent of the system

$$f(x)\, g(x) \geq 0.$$

Example 60. Solve the equation

$$\left| \frac{x}{x-1} \right| + |x| = \frac{x^2}{|x-1|}.$$

Sol. Let $f(x) = \dfrac{x}{x-1}$ and $g(x) = x$,

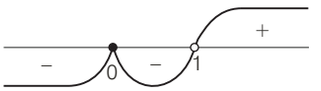
Then, $\quad f(x) + g(x) = \dfrac{x}{x-1} + x = \dfrac{x^2}{x-1}$

$\therefore$ The given equation can be reduced in the form

$$|f(x)| + |g(x)| = |f(x) + g(x)|$$

Hence, $\quad f(x) \cdot g(x) \geq 0$

$$\Rightarrow \quad \frac{x^2}{x-1} \geq 0$$



From *Wavy curve method,* $x \in (1, \infty) \cup \{0\}$.

***Form* 2 The equation of the form**

$$|f_1(x)| + |f_2(x)| + \ldots + |f_n(x)| = g(x) \qquad \ldots(\text{i})$$

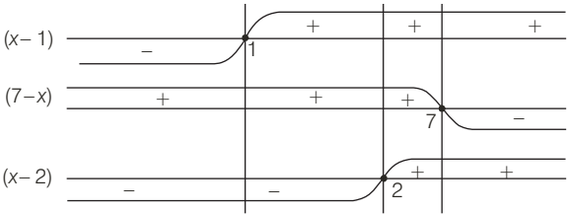
where, $f_1(x), f_2(x), \ldots, f_n(x), g(x)$ are functions of x and $g(x)$ may be constant.

Equations of this form solved by the **method of intervals**. We first find all critical points of $f_1(x), f_2(x), \ldots, f_n(x)$, if coefficient of x is positive, then graph start with positive sign (+) and if coefficient of x is negative, then graph start with negative sign (–). Then, using the definition of the absolute value, we pass from Eq. (i) to a collection of systems which do not contain the absolute value symbols.

Example 61. ***Solve the equation*** $|x-1|+|7-x|+2|x-2|=4$.

Sol. Here, critical points are 1, 2, 7 using the method of intervals, we find intervals when the expressions $x-1, 7-x$ and $x-2$ are of constant signs.

i.e. $x<1, 1<x<2, 2<x<7, x>7$



Thus, the given equation is equivalent to the collection of four systems,

$$\left[\begin{array}{l}\begin{cases}x<1\\-(x-1)+(7-x)-2(x-2)=4\end{cases}\\ \begin{cases}1\le x<2\\(x-1)+(7-x)-2(x-2)=4\end{cases}\\ \begin{cases}2\le x<7\\(x-1)+(7-x)+2(x-2)=4\end{cases}\\ \begin{cases}x\ge 7\\(x-1)-(7-x)+2(x-2)=4\end{cases}\end{array}\right. \Rightarrow \left[\begin{array}{l}\begin{cases}x<1\\x=2\end{cases}\\ \begin{cases}1\le x<2\\x=3\end{cases}\\ \begin{cases}2\le x<7\\x=1\end{cases}\\ \begin{cases}x\ge 7\\x=4\end{cases}\end{array}\right.$$

From the collection of four systems, the given equation has no solution.

Inequations Containing Absolute Values

By definition, $|x|<a \Rightarrow -a<x<a\,(a>0)$

$|x|\le a \Rightarrow -a\le x\le a$

$|x|>a \Rightarrow x<-a$ and $x>a$

and $|x|\ge a \Rightarrow x\le -a$ and $x\ge a$.

Example 62. ***Solve the inequation*** $\left|1-\dfrac{|x|}{1+|x|}\right|\ge\dfrac{1}{2}$.

Sol. The given inequation is equivalent to the collection of systems

$$\begin{cases}\left|1-\dfrac{x}{1+x}\right|\ge\dfrac{1}{2}, \text{ if } x\ge 0\\ \left|1+\dfrac{x}{1-x}\right|\ge\dfrac{1}{2}, \text{ if } x<0\end{cases}\Rightarrow\begin{cases}\dfrac{1}{|1+x|}\ge\dfrac{1}{2}, \text{ if } x\ge 0\\ \dfrac{1}{|1-x|}\ge\dfrac{1}{2}, \text{ if } x<0\end{cases}$$

$$\Rightarrow\begin{cases}\dfrac{1}{1+x}\ge\dfrac{1}{2}, \text{ if } x\ge 0\\ \dfrac{1}{1-x}\ge\dfrac{1}{2}, \text{ if } x<0\end{cases}\Rightarrow\begin{cases}\dfrac{1-x}{1+x}\ge 0, \text{ if } x\ge 0\\ \dfrac{1+x}{1-x}\ge 0, \text{ if } x<0\end{cases}$$

$$\Rightarrow\begin{cases}\dfrac{x-1}{x+1}\le 0, \text{ if } x\ge 0\\ \dfrac{x+1}{x-1}\le 0, \text{ if } x<0\end{cases}$$

For $\dfrac{x-1}{x+1}\le 0$, if $x\ge 0$

$\therefore \quad 0\le x\le 1$ ...(i)

For $\dfrac{x+1}{x-1}\le 0$, if $x<0$

$\therefore \quad -1\le x<0$ ...(ii)

Hence, from Eqs. (i) and (ii), the solution of the given equation is $x\in[-1,1]$.

Aliter

$$\left|1-\frac{|x|}{1+|x|}\right|\ge\frac{1}{2}\Rightarrow\left|\frac{1}{1+|x|}\right|\ge\frac{1}{2}$$

$$\Rightarrow \quad \frac{1}{1+|x|}\ge\frac{1}{2}\Rightarrow 1+|x|\le 2 \text{ or } |x|\le 1$$

$\therefore \quad -1\le x\le 1$ or $x\Rightarrow[-1,1]$

Equations Involving Greatest Integer, Least Integer and Fractional Part

1. Greatest Integer

$[x]$ denotes the greatest integer less than or equal to x i.e., $[x]\le x$. It is also known as **floor** of x.

Thus, $[3.5779]=3, [0.89]=0, [3]=3$

$[-8.7285]=-9$

$[-0.6]=-1$

$[-7]=-7$

In general, if n is an integer and x is any real number between n and $n+1$.

i.e. $n\le x<n+1$, then $[x]=n$

Properties of Greatest Integer

(i) $[x\pm n]=[x]\pm n, n\in I$

(ii) $[-x]=-[x], x\in I$

(iii) $[-x]=-1-[x], x\notin I$

(iv) $[x]-[-x]=2n$, if $x=n, n\in I$

(v) $[x]-[-x]=2n+1$, if $x=n+\{x\}, n\in I$ and $0<\{x\}<1$

(vi) $[x]\ge n\Rightarrow x\ge n, n\in I$

(vii) $[x]>n\Rightarrow x\ge n+1, n\in I$

(viii) $[x]\le n\Rightarrow x<n+1, n\in I$

(ix) $[x]<n\Rightarrow x<n, n\in I$

(x) $n_2\le[x]\le n_1\Rightarrow n_2\le x<n_1+1, n_1, n_2\in I$

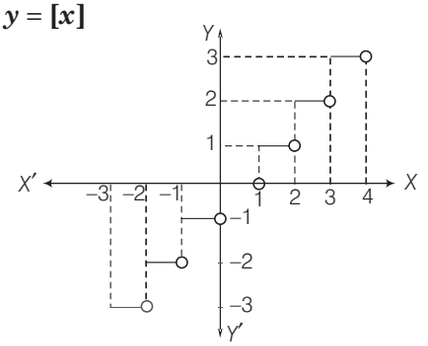
(xi) $[x+y]\ge[x]+[y]$

(xii) $\left[\frac{[x]}{n}\right]=\left[\frac{x}{n}\right], n\in N$

(xiii) $\left[\frac{n+1}{2}\right]+\left[\frac{n+2}{4}\right]+\left[\frac{n+4}{8}\right]+\left[\frac{n+8}{16}\right]+\ldots=n, n\in N$

(xiv) $[x]+\left[x+\frac{1}{n}\right]+\left[x+\frac{2}{n}\right]+\ldots+\left[x+\frac{n-1}{n}\right]=[nx]$, $n\in N$

Graph of $y=[x]$



Remark

Domain and Range of $[x]$ are R and I, respectively.

Example 63. If $[x]$ denotes the integral part of x for real x, then find the value of

$$\left[\frac{1}{4}\right]+\left[\frac{1}{4}+\frac{1}{200}\right]+\left[\frac{1}{4}+\frac{1}{100}\right]+\left[\frac{1}{4}+\frac{3}{200}\right]+\ldots+\left[\frac{1}{4}+\frac{199}{200}\right].$$

Sol. The given expression can be written as

$$\left[\frac{1}{4}\right]+\left[\frac{1}{4}+\frac{1}{200}\right]+\left[\frac{1}{4}+\frac{2}{200}\right]+\left[\frac{1}{4}+\frac{3}{200}\right]+\ldots+\left[\frac{1}{4}+\frac{199}{200}\right]$$

$$=\left[200\cdot\frac{1}{4}\right]=[50]=50 \qquad \text{[from property (xiv)]}$$

Example 64. ***Let $[a]$ denotes the larger integer not exceeding the real number a. If x and y satisfy the equations $y=2[x]+3$ and $y=3[x-2]$ simulaneously, determine $[x+y]$.***

Sol. We have, $y=2[x]+3=3[x-2]$ ...(i)

$\Rightarrow$ $2[x]+3=3([x]-2)$ [from property (i)]

$\Rightarrow$ $2[x]+3=3[x]-6$

$\Rightarrow$ $[x]=9$

From Eq. (i), $y=2\times 9+3=21$

$\therefore$ $[x+y]=[x+21]=[x]+21=9+21=30$

Hence, the value of $[x+y]$ is 30.

2. Least Integer

(x) or $\lceil x\rceil$ denotes the least integer greater than or equal to x i.e., $(x)\geq x$ or $\lceil x\rceil \geq x$. It is also known as **ceilling** of x.

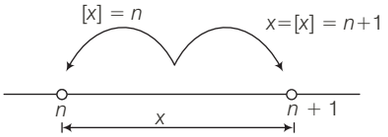
Thus, $(3.578)=4, (0.87)=1,$

$(4)=4$

$\lceil -8.239\rceil=-8, \lceil -0.7\rceil=0$

In general, if n is an integer and x is any real number between n and $n+1$

i.e., $n<x\leq n+1$, then $(x)=n+1$



Relation between Greatest Integer and Least Integer

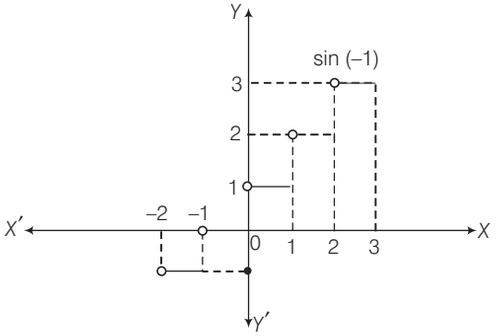
$$(x)=\begin{cases}[x], & x\in I\\ [x]+1, & x\notin I\end{cases}$$

i.e. If $x\in I$, then $x=[x]=(x)$. [remember]

Remark

If $(x)=n$, then $(n-1)<x\leq n$

Graph of $y=(x)=\lceil x\rceil$



Remark

Domain and Range of (x) are R and $[x]+1$, respectively.

Example 65. ***If $[x]$ and (x) are the integral part of x and nearest integer to x, then solve $(x)[x]=1$.***

Sol. ***Case* I** If $x\in I$, then $x=[x]=(x)$

$\therefore$ Given equation convert in $x^2=1$.

$\therefore$ $x=(\pm 1)$

***Case* II** If $x\notin I$, then $(x)=[x]+1$

$\therefore$ Given equation convert in

$$([x]+1)[x]=1 \Rightarrow [x]^2+[x]-1=0$$

or $$[x]=\frac{-1\pm\sqrt{5}}{2} \quad \text{[impossible]}$$

Then, final answer is $x=\pm 1$.

Example 66. Find the solution set of $(x)^2+(x+1)^2=25$, where (x) is the least integer greater than or equal to x.

Sol. ***Case* I** If $x\in I$, then $x=(x)=[x]$

Then, $(x)^2+(x+1)^2=25$ reduces to

$$x^2+\overline{x+1}^2=25 \Rightarrow 2x^2+2x-24=0$$

$$\Rightarrow x^2+x-12=0 \Rightarrow (x+4)(x-3)=0$$

$$\therefore x=-4,3 \quad \ldots(i)$$

***Case* II** If $x\notin I$, then $(x)=[x]+1$

Then, $(x)^2+(x+1)^2=25$ reduces to

$$\{[x]+1\}^2+\{[x+1]+1\}^2=25$$

$$\Rightarrow \{[x]+1\}^2+\{[x]+2\}^2=25$$

$$\Rightarrow 2[x]^2+6[x]-20=0$$

$$\Rightarrow [x]^2+3[x]-10=0$$

$$\Rightarrow \{[x]+5\}\{[x]-2\}=0$$

$$\therefore [x]=-5 \text{ and } [x]=2$$

$$\Rightarrow x\in[-5,-4)\cup[2,3)$$

$\because x\notin I$,

$$\therefore x\in(-5,-4)\cup(2,3) \quad \ldots(ii)$$

On combining Eqs. (i) and (ii), we get

$$x\in(-5,-4]\cup(2,3]$$

3. Fractional Part

$\{x\}$ denotes the fractional part of x, i.e. $0\le\{x\}<1$.

Thus, $\{2\cdot 7\}=0.7, \{5\}=0, \{-3.72\}=0.28$

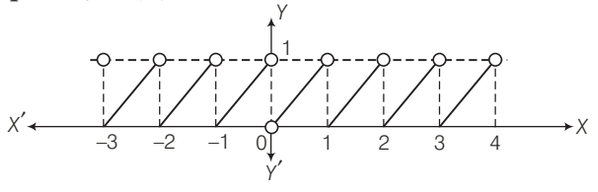
If x is a real number, then $x=[x]+\{x\}$

i.e., $x=n+f$, where $n\in I$ and $0\le f<1$

Properties of Fractional Part of x

(i) $\{x\pm n\}=\{x\}, n\in I$ (ii) If $0\le x<1$, then $\{x\}=x$

Graph of $y=\{x\}$



Remark

1. For proper fraction $0<\{x\}<1$.
2. Domain and range of $\{x\}$ are R and $[0,1)$, respectively.
3. $\{-5.238\}=\{-5-0.238\}=\{-5-1+1-0.238\}$
$=\{-6+0.762\}=\{\bar{6}.762\}=0.762$

Example 67. If $\{x\}$ and $[x]$ represent fractional and integral part of x respectively, find the value of $[x]+\sum_{r=1}^{2000}\frac{\{x+r\}}{2000}$.

Sol. $[x]+\sum_{r=1}^{2000}\frac{\{x+r\}}{2000}=[x]+\sum_{r=1}^{2000}\frac{\{x\}}{2000}$ [from property (i)]

$$=[x]+\frac{\{x\}}{2000}\sum_{r=1}^{2000}1=[x]+\frac{\{x\}}{2000}\times 2000=[x]+\{x\}=x$$

Example 68. ***If $\{x\}$ and $[x]$ represent fractional and integral part of x respectively, then solve the equation*** $x-1=(x-[x])(x-\{x\})$.

Sol. $\because$ $x=[x]+\{x\}, 0\le\{x\}<1$

Thus, given equation reduces to

$$[x]+\{x\}-1=\{x\}[x]$$

$$\Rightarrow \{x\}[x]-[x]-\{x\}+1=0$$

$$\Rightarrow ([x]-1)(\{x\}-1)=0$$

Now, $\{x\}-1\neq 0$ $[\because 0\le\{x\}<1]$

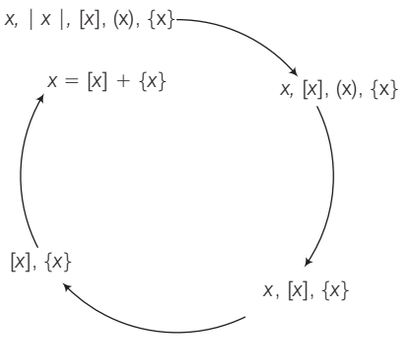
$$\therefore [x]-1=0$$

$$\Rightarrow [x]=1$$

$$\therefore x\in[1,2)$$

Problem Solving Cycle

If a problem has $x, |x|, [x], (x), \{x\}$, then first solve $|x|$, then problem convert in $x, [x], (x), \{x\}$.



Secondly, solve $(x)=\begin{cases}[x], & x\in I\\ [x]+1, & x\notin I\end{cases}$

Then, problem convert in $x, [x], \{x\}$.

Now, put $x=[x]+\{x\}$

Then, problem convert in $[x]$ and $\{x\}$. ...(i)

Since, $0\le\{x\}<1$, then we get $[x]$

From Eq. (i), we get $\{x\}$

Hence, final solution is $x=[x]+\{x\}$.

Example 69. Let $\{x\}$ and $[x]$ denotes the fractional and integral parts of a real number x, respectively. Solve $4\{x\} = x + [x]$.

Sol. $\because \quad x = [x] + \{x\}$ ...(i)

Then, given equation reduces to

$$4\{x\} = [x] + \{x\} + [x]$$

$$\Rightarrow \quad \{x\} = \frac{2}{3}[x] \quad \text{...(ii)}$$

$$\because \quad 0 \le \{x\} < 1 \Rightarrow 0 \le \frac{2}{3}[x] < 1 \text{ or } 0 \le [x] < \frac{3}{2}$$

$$\therefore \quad [x] = 0, 1$$

From Eq. (ii), $\{x\} = 0, \frac{2}{3}$

From Eq. (i), $x = 0, 1 + \frac{2}{3}$ i.e., $x = 0, \frac{5}{3}$

Example 70. Let $\{x\}$ and $[x]$ denotes the fractional and integral part of a real number (x), respectively. Solve $|2x - 1| = 3[x] + 2\{x\}$.

Sol. ***Case* I** $2x - 1 \ge 0$ or $x \ge \frac{1}{2}$

Then, given equation convert to

$$2x - 1 = 3[x] + 2\{x\} \quad \text{...(i)}$$

$$\because \quad x = [x] + \{x\} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$2([x] + \{x\}) - 1 = 3[x] + 2\{x\}$$

$$\therefore \quad [x] = -1$$

$$\therefore \quad -1 \le x < 0 \qquad \left[\because x \ge \frac{1}{2}\right]$$

No solution

***Case* II** $2x - 1 < 0$ or $x < \frac{1}{2}$

Then, given equation reduces to

$$1 - 2x = 3[x] + 2\{x\} \quad \text{...(iii)}$$

$$\because \quad x = [x] + \{x\} \quad \text{...(iv)}$$

From Eqs. (iii) and (iv), we get

$$1 - 2([x] + \{x\}) = 3[x] + 2\{x\}$$

$$\Rightarrow \quad 1 - 5[x] = 4\{x\}$$

$$\therefore \quad \{x\} = \frac{1 - 5[x]}{4} \quad \text{...(v)}$$

Now, $0 \le \{x\} < 1$

$$\Rightarrow \quad 0 \le \frac{1 - 5[x]}{4} < 1$$

$$\Rightarrow \quad 0 \le 1 - 5[x] < 4$$

$$\Rightarrow \quad 0 \ge -1 + 5[x] > -4$$

$$\Rightarrow \quad 1 \ge 5[x] > -3 \text{ or } -\frac{3}{5} < [x] \le \frac{1}{5}$$

$$\therefore \quad [x] = 0$$

From Eq. (v), $\{x\} = \frac{1}{4}$

$$\therefore \quad x = 0 + \frac{1}{4} = \frac{1}{4}$$

Example 71. Solve the equation

$$(x)^2 = [x]^2 + 2x$$

where, $[x]$ and (x) are integers just less than or equal to x and just greater than or equal to x, respectively.

Sol. ***Case* I** If $x \in I$ then

$$x = [x] = (x)$$

The given equation reduces to

$$x^2 = x^2 + 2x$$

$$\Rightarrow \quad 2x = 0 \text{ or } x = 0 \quad \text{...(i)}$$

***Case* II** If $x \notin I$, then $(x) = [x] + 1$

The given equation reduces to

$$([x] + 1)^2 = [x]^2 + 2x$$

$$\Rightarrow \quad 1 = 2(x - [x]) \text{ or } \{x\} = \frac{1}{2}$$

$$\therefore \quad x = [x] + \frac{1}{2} = n + \frac{1}{2},\ n \in I \quad \text{...(ii)}$$

Hence, the solution of the original equation is $x = 0, n + \frac{1}{2}$, $n \in I$.

Example 72. Solve the system of equations in x, y and z satisfying the following equations:

$$x + [y] + \{z\} = 3 \cdot 1$$

$$\{x\} + y + [z] = 4 \cdot 3$$

$$[x] + \{y\} + z = 5 \cdot 4$$

where, $[\cdot]$ and $\{\cdot\}$ denotes the greatest integer and fractional parts, respectively.

Sol. $\because [x] + \{x\} = x, \ [y] + \{y\} = y$ and $[z] + \{z\} = z$,

On adding all the three equations, we get

$$2(x + y + z) = 12.8$$

$$\Rightarrow \quad x + y + z = 6.4 \quad \text{...(i)}$$

Now, adding first two equations, we get

$$x + y + z + [y] + \{x\} = 7.4$$

$$\Rightarrow \quad 6.4 + [y] + \{x\} = 7.4 \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad [y] + \{x\} = 1$$

$$\therefore \quad [y] = 1 \text{ and } \{x\} = 0 \quad \text{...(ii)}$$

On adding last two equations, we get

$$x + y + z + \{y\} + [z] = 9.7$$

$$\{y\} + [z] = 3.3 \quad \text{[from Eq. (ii)]}$$

$$\therefore \quad [z] = 3 \text{ and } \{y\} = 0.3 \quad \text{...(iii)}$$

On adding first and last equations, we get

$$x + y + z + [x] + \{z\} = 8.5$$

$$\Rightarrow \quad [x] + \{z\} = 2.1 \quad \text{[from Eq. (i)]}$$

$$\therefore \quad [x] = 2, \ \{z\} = 0.1 \quad \text{...(iv)}$$

From Eqs. (i), (ii) and (iii), we get

$$x = [x] + \{x\} = 2 + 0 = 2$$

$$y = [y] + \{y\} = 1 + 0.3 = 1.3$$

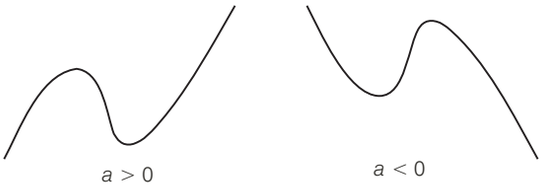
and $\quad z = [z] + \{z\} = 3 + 0.1 = 3.1$

Roots of Equation with the Help of Graphs

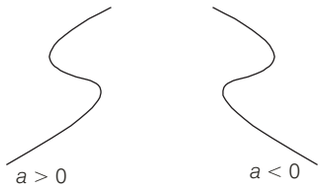
Here, we will discuss some examples to find the roots of equations with the help of graphs.

Important Graphs

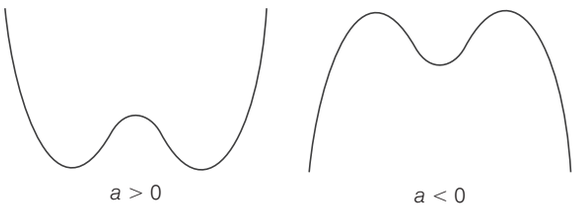
1. $y = ax^3 + bx^2 + cx + d$



2. $x = ay^3 + by^2 + cy + d$



3. $y = ax^4 + bx^3 + cx^2 + dx + e$



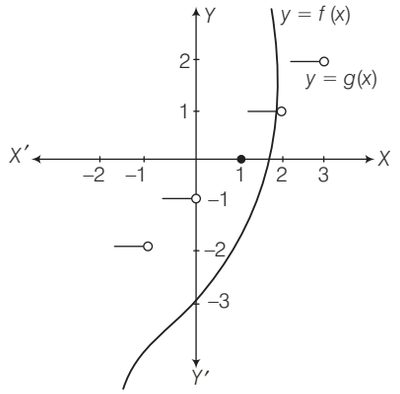
Example 73. Solve the equation $x^3 - [x] = 3$, where $[x]$ denotes the greatest integer less than or equal to x.

Sol. We have, $x^3 - [x] = 3$

$\Rightarrow \quad x^3 - 3 = [x]$

Let $f(x) = x^3 - 3$ and $g(x) = [x]$.

It is clear from the graphs, the point of intersection of two curves $y = f(x)$ and $y = g(x)$ lies between $(1, 0)$ and $(2, 0)$.



$\therefore \quad 1 < x < 2$

We have, $f(x) = x^3 - 3$ and $g(x) = 1$

or $\quad x^3 - 3 = 1 \Rightarrow x^3 = 4$

$\therefore \quad x = (4)^{1/3}$

Hence, $x = 4^{1/3}$ is the solution of the equation $x^3 - [x] = 3$.

Aliter

$\because \quad x = [x] + f, 0 \le f < 1,$

Then, given equation reduces to

$$x^3 - (x - f) = 3 \Rightarrow x^3 - x = 3 - f$$

Hence, it follows that

$$2 < x^3 - x \le 3$$

$$\Rightarrow \quad 2 < x(x+1)(x-1) \le 3$$

Further for $x \ge 2$, we have $x(x+1)(x-1) \ge 6 > 3$

For $x < -1$, we have $x(x+1)(x-1) < 0 < 2$

For $x = -1$, we have $x(x+1)(x-1) = 0 < 2$

For $-1 < x \le 0$, we have $x(x+1)(x-1) \le -x < 1$

and for $0 < x \le 1$, we have $x(x+1)(x-1) < x < x^3 \le 1$

Therefore, x must be $1 < x < 2$

$\therefore \quad [x] = 1$

Now, the original equation can be written as

$$x^3 - 1 = 3 \Rightarrow x^3 = 4$$

Hence, $x = 4^{1/3}$ is the solution of the given equation.

Example 74. Solve the equation $x^3 - 3x - a = 0$ for different values of a.

Sol. We have, $x^3 - 3x - a = 0 \Rightarrow x^3 - 3x = a$

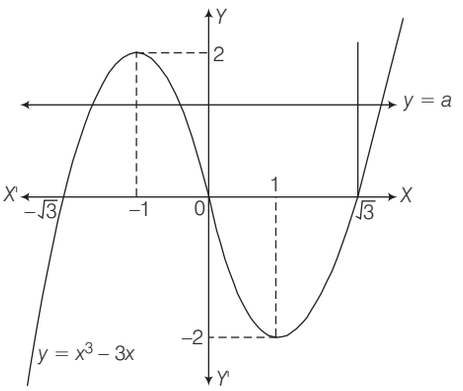
Let $f(x) = x^3 - 3x$ and $g(x) = a$

$\because \quad f'(x) = 0$

$\Rightarrow \quad 3x^2 - 3 = 0$

$\Rightarrow \quad x = -1, 1$

$f''(x) = 6x$



$\therefore \quad f''(-1) = -6 < 0$ and $f''(1) = 6 > 0$

$\therefore f(x)$ local maximum at $x = (-1)$ and local minimum at $x = 1$ and $f(-1) = 2$ and $f(1) = -2$ and $y = g(x) = a$ is a straight line parallel to X-axis.

Following cases arise

***Case* I** When $a > 2$,

In this case $y = f(x)$ and $y = g(x)$ intersects at only one point, so $x^3 - 3x - a = 0$ has only one real root.

***Case* II** When $a = 2$,

In this case $y = f(x)$ and $y = g(x)$ intersects at two points, so $x^3 - 3x - a = 0$ has three real roots, two are equal and one different.

***Case* III** When $-2 < a < 2$,

In this case $y = f(x)$ and $y = g(x)$ intersects at three points, so $x^3 - 3x - a = 0$ has three distinct real roots.

***Case* IV** When $a = -2$,

In this case $y = f(x)$ and $y = g(x)$ touch at one point and intersect at other point, so $x^3 - 3x - a = 0$ has three real roots, two are equal and one different.

***Case* V** When $a < -2$,

In this case $y = f(x)$ and $y = g(x)$ intersects at only one point, so $x^3 - 3x - a = 0$ has only one real root.

Example 75. Show that the equation $x^3 + 2x^2 + x + 5 = 0$ has only one real root, such that $[\alpha] = -3$, where $[x]$ denotes the integral point of x.

Sol. We have, $x^3 + 2x^2 + x + 5 = 0$

$\Rightarrow \quad x^3 + 2x^2 + x = -5$

Let $f(x) = x^3 + 2x^2 + x$ and $g(x) = -5$

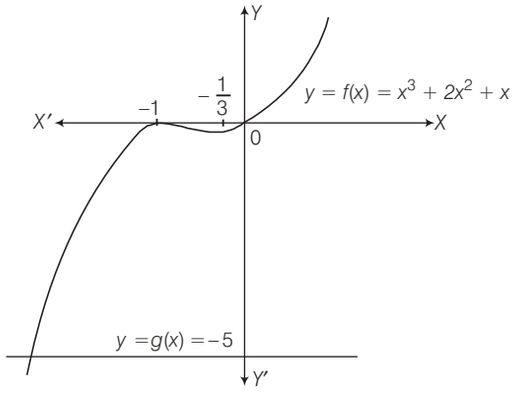
$\because \quad f'(x) = 0 \Rightarrow 3x^2 + 4x + 1 = 0$

$\Rightarrow \quad x = -1, -\frac{1}{3}$ and $f''(x) = 6x + 4$

$\therefore \quad f''(-1) = -2 < 0$ and $f''\left(-\frac{1}{3}\right) = -2 + 4 = 2 > 0$

$\therefore f(x)$ local maximum at $x = -1$ and local minimum at $x = -\frac{1}{3}$

and $f(-1) = 0, f\left(-\frac{1}{3}\right) = -\frac{4}{27}$



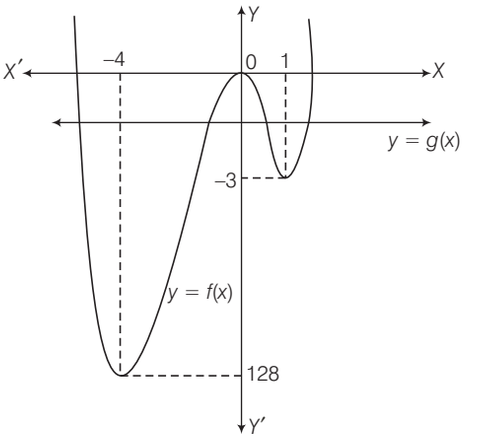
and $f(-2) = -2$ and $f(-3) = -12$

Therefore, x must lie between (-3) and (-2).

i.e. $-3 < \alpha < -2 \Rightarrow [\alpha] = -3$

Example 76. Find all values of the parameter k for which all the roots of the equation $x^4 + 4x^3 - 8x^2 + k = 0$ are real.

Sol. We have, $x^4 + 4x^3 - 8x^2 + k = 0$



$\Rightarrow \quad x^4 + 4x^3 - 8x^2 = -k$

Let $f(x) = x^4 + 4x^3 - 8x^2$ and $g(x) = -k$

$\because \quad f'(x) = 0$

$\Rightarrow \quad 4x^3 + 12x^2 - 16x = 0 \Rightarrow x = -4, 0, 1$

and $f''(x) = 12x^2 + 24x - 16$

$\therefore \quad f''(-4) = 80, \quad f''(0) = -16, f''(1) = 20$

$\therefore f(x)$ has local minimum at $x = -4$ and $x = 1$ and local maximum at $x = 0$

and $f(-4) = -128, f(0) = 0, f(1) = -3$.

Following cases arise

***Case* I** When $-k > 0$ i.e., $k < 0$

In this case $y = x^4 + 4x^3 - 8x^2$ and $y = (-k)$ intersect at two points, so $x^4 + 4x^3 - 8x^2 + k = 0$ has two real roots.

***Case* II** When $-k = 0$ and $-k = -3$, i.e. $k = 0, 3$

In this case $y = x^4 + 4x^3 - 8x^2$ and $y = -k$ intersect at four points, so $x^4 + 4x^3 - 8x^2 + k = 0$ has two distinct real roots and two equal roots.

***Case* III** When $-3 < -k < 0$, i.e. $0 < k < 3$

In this case $y = x^4 + 4x^3 - 8x^2$ and $y = -k$ intersect at four distinct points, so $x^4 + 4x^3 - 8x^2 + k = 0$ has four distinct real roots.

***Case* IV** When $-128 < -k < -3$, i.e. $3 < k < 128$

In this case $y = x^4 + 4x^3 - 8x^2$ and $y = -k$ intersect at two distinct points, so $x^4 + 4x^3 - 8x^2 + k = 0$ has two distinct real roots.

***Case* V** When $-k = -128$ i.e., $k = 128$

In this case $y = x^4 + 4x^3 - 8x^2$ and $y = -k$ touch at one point, so $x^4 + 4x^3 - 8x^2 + k = 0$ has two real and equal roots.

***Case* VI** When $-k < -128$, i.e. $k > 128$

In this case $y = x^4 + 4x^3 - 8x^2$ and $y = -k$ do not intersect, so there is no real root.

Example 77. Let $-1 \le p \le 1$, show that the equation $4x^3 - 3x - p = 0$ has a unique root in the interval $\left[\frac{1}{2}, 1\right]$ and identify it.

Sol. We have, $4x^3 - 3x - p = 0$

$\Rightarrow \quad 4x^3 - 3x = p$

Let $f(x) = 4x^3 - 3x$ and $g(x) = p$

$\therefore \quad f'(x) = 0$

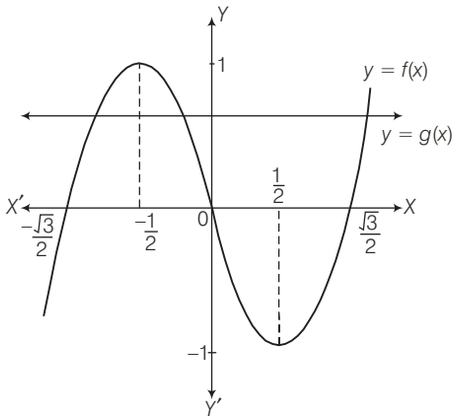
$\Rightarrow \quad 12x^2 - 3 = 0$

$\Rightarrow \quad x = -\frac{1}{2}, -\frac{1}{2}$ and $f''(x) = 24x$

$\therefore \quad f''\left(-\frac{1}{2}\right) = -12 < 0$ and $f''\left(\frac{1}{2}\right) = 12 > 0$

$\therefore f(x)$ has local maximum at $\left(x = -\frac{1}{2}\right)$ and local minimum at $\left(x = \frac{1}{2}\right)$.

Also, $f\left(-\frac{1}{2}\right) = -\frac{4}{8} + \frac{3}{2} = 1$ and $f\left(\frac{1}{2}\right) = \frac{4}{8} - \frac{3}{2} = -1$



We observe that, the line $y = g(x) = p$, where $-1 \le p \le 1$ intersect the curve $y = f(x)$ exactly at point $\alpha \in \left[\frac{1}{2}, 1\right]$.

Hence, $4x^3 - 3x - p = 0$ has exactly one root in the interval $\left[\frac{1}{2}, 1\right]$.

Now, we have to find the value of root α.

Let $\alpha = \cos\theta$, then $4\cos^3\theta - 3\cos\theta - p = 0$

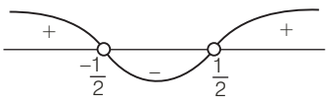
$\Rightarrow \quad \cos 3\theta = p \Rightarrow 3\theta = \cos^{-1}(p)$ or $\theta = \frac{1}{3}\cos^{-1}(p)$

$\therefore \quad \alpha = \cos\theta = \cos\left\{\frac{1}{3}\cos^{-1}(p)\right\}$

Aliter

Let $\phi(x) = 4x^3 - 3x - p$

$\therefore \quad \phi'(x) = 12x^2 - 3 = 12\left(x + \frac{1}{2}\right)\left(x - \frac{1}{2}\right)$



Clearly, $\phi'(x) > 0$ for $x \in \left[\frac{1}{2}, 1\right]$.

Hence, $\phi(x)$ can have atmost one root in $\left[\frac{1}{2}, 1\right]$.

Also, $\phi\left(\frac{1}{2}\right) = -1 - p$ and $\phi(1) = 1 - p$

$\therefore \quad \phi\left(\frac{1}{2}\right)\phi(1) = -(1 - p^2) = (p^2 - 1) \le 0 \qquad [\because -1 \le p \le 1]$

Since, $\phi(x)$ being a polynomial, continuous on $\left[\frac{1}{2}, 1\right]$ and $\phi\left(\frac{1}{2}\right)\phi(1) \le 0$. Therefore, by intermediate value theorem $\phi(x)$ has atleast one root in $\left[\frac{1}{2}, 1\right]$.

Hence, $\phi(x)$ has exactly one root in $\left[\frac{1}{2}, 1\right]$.

Exercise for Session 4

1. If α, β, γ are the roots of $x^3 - x^2 - 1 = 0$, the value of $\sum\left(\frac{1+\alpha}{1-\alpha}\right)$, is equal to

(a) -7 (b) -6
(c) -5 (d) -4

2. If r, s, t are the roots of the equation $8x^3 + 1001x + 2008 = 0$. The value of $(r+s)^3 + (s+t)^3 + (t+r)^3$ is

(a) 751 (b) 752
(c) 753 (d) 754

3. If $\alpha, \beta, \gamma, \delta$ are the roots of equation $x^4 + 4x^3 - 6x^2 + 7x - 9 = 0$, the value of $\prod(1+\alpha^2)$ is

(a) 9 (b) 11
(c) 13 (d) 15

4. If a, b, c, d are four consecutive terms of an increasing AP, the roots of the equation $(x-a)(x-c) + 2(x-b)(x-d) = 0$ are

(a) non-real complex (b) real and equal
(c) integers (d) real and distinct

5. If $x^2 + px + 1$ is a factor of the expression $ax^3 + bx + c$ then

(a) $a^2 - c^2 = ab$ (b) $a^2 + c^2 = -ab$
(c) $a^2 - c^2 = -ab$ (d) None of these

6. The number of real roots of the equation $x^2 - 3|x| + 2 = 0$ is

(a) 1 (b) 2
(c) 3 (d) 4

7. Let $a \neq 0$ and $p(x)$ be a polynomial of degree greater than 2, if $p(x)$ leaves remainder a and $(-a)$ when divided respectively by $x + a$ and $x - a$, the remainder when $p(x)$ is divided by $x^2 - a^2$, is

(a) $2x$ (b) $-2x$
(c) x (d) $-x$

8. The product of all the solutions of the equation $(x-2)^2 - 3|x-2| + 2 = 0$ is

(a) 2 (b) -4
(c) 0 (d) None of these

9. If $0 < x < 1000$ and $\left[\frac{x}{2}\right] + \left[\frac{x}{3}\right] + \left[\frac{x}{5}\right] = \frac{31}{30}x$, where $[x]$ is the greatest integer less than or equal to x, the number of possible values of x is

(a) 32 (b) 33
(c) 34 (d) None of these

10. If $[x]$ is the greatest integer less than or equal to x and (x) be the least integer greater than or equal to x and $[x]^2 + (x)^2 > 25$ then x belongs to

(a) $[3, 4]$ (b) $(-\infty, -4]$
(c) $[4, \infty)$ (d) $(-\infty, -4] \cup [4, \infty)$

Session 5

Irrational Equations, Irrational Inequations, Exponential Equations, Exponential Inequations, Logarithmic Equations, Logarithmic Inequations

Irrational Equations

Here, we consider equations of the type which contain the unknown under the radical sign and the value under the radical sign is known as radicand.

- If roots are all even (i.e. $\sqrt{x}$, $\sqrt[4]{x}, \sqrt[6]{x},\ldots$, etc) of an equation are arithmetic. In other words, if the radicand is negative (i.e. $x<0$), then the root is imaginary, if the radicand is zero, then the root is also zero and if the radicand is positive, then the value of the root is also positive.
- If roots are all odd (i.e. $\sqrt[3]{x}, \sqrt[5]{x}, \sqrt[7]{x},\ldots$ etc) of an equation, then it is defined for all real values of the radicand. If the radicand is negative, then the root is negative, if the radicand is zero, then the root is zero and if the radicand is positive, then the root is positive.

Some Standard Formulae to Solve Irrational Equations

If f and g be functions of $x, k \in N$. Then,

1. $\sqrt[2k]{f}\,\sqrt[2k]{g} = \sqrt[2k]{fg}, f \geq 0, g \geq 0$

2. $\sqrt[2k]{f} \,/\, \sqrt[2k]{g} = \sqrt[2k]{(f/g)}, f \geq 0, g > 0$

3. $|f|\sqrt[2k]{g} = \sqrt[2k]{(f^{2k} g)}, g \geq 0$

4. $\sqrt[2k]{(f/g)} = \sqrt[2k]{|f|} \,/\, \sqrt[2k]{|g|}, fg \geq 0, g \neq 0$

5. $\sqrt[2k]{fg} = \sqrt[2k]{|f|}\,\sqrt[2k]{g}, fg \geq 0$

Example 78. Prove that the following equations has no solutions.

(i) $\sqrt{(2x+7)} + \sqrt{(x+4)} = 0$ (ii) $\sqrt{(x-4)} = -5$

(iii) $\sqrt{(6-x)} - \sqrt{(x-8)} = 2$ (iv) $\sqrt{-2-x} = \sqrt[5]{(x-7)}$

(v) $\sqrt{x} + \sqrt{(x+16)} = 3$ (vi) $7\sqrt{x} + 8\sqrt{-x} + \dfrac{15}{x^3} = 98$

(vii) $\sqrt{(x-3)} - \sqrt{x+9} = \sqrt{(x-1)}$

Sol. (i) We have, $\sqrt{(2x+7)} + \sqrt{(x+4)} = 0$

This equation is defined for $2x + 7 \geq 0$

and $x + 4 \geq 0 \Rightarrow \begin{cases} x \geq -\dfrac{7}{2} \\ x \geq -4 \end{cases}$

$\therefore \quad x \geq -\dfrac{7}{2}$

For $x \geq -\dfrac{7}{2}$, the left hand side of the original equation is positive, but right hand side is zero. Therefore, the equation has no roots.

(ii) We have, $\sqrt{(x-4)} = -5$

The equation is defined for $x - 4 \geq 0$

$\therefore \quad x \geq 4$

For $x \geq 4$, the left hand side of the original equation is positive, but right hand side is negative.

Therefore, the equation has no roots.

(iii) We have, $\sqrt{(6-x)} - \sqrt{x-8} = 2$

The equation is defined for

$6 - x \geq 0$ and $x - 8 \geq 0$

$\therefore \quad \begin{cases} x \leq 6 \\ x \geq 8 \end{cases}$

Consequently, there is no x for which both expressions would have sense. Therefore, the equation has no roots.

(iv) We have, $\sqrt{(-2-x)} = \sqrt[5]{(x-7)}$

This equation is defined for

$-2 - x \geq 0 \quad \Rightarrow \quad x \leq -2$

For $x \leq -2$ the left hand side is positive, but right hand side is negative.

Therefore, the equation has no roots.

(v) We have, $\sqrt{x} + \sqrt{(x+16)} = 3$

The equation is defined for

$x \geq 0$ and $x + 16 \geq 0 \Rightarrow \begin{cases} x \geq 0 \\ x \geq -16 \end{cases}$

Hence, $x \geq 0$

For $x \geq 0$ the left hand side ≥ 4, but right hand side is 3. Therefore, the equation has no roots.

(vi) We have, $7\sqrt{x} + 8\sqrt{-x} + \dfrac{15}{x^3} = 98$

For $x < 0$, the expression $7\sqrt{x}$ is meaningless,

For $x > 0$, the expression $8\sqrt{-x}$ is meaningless

and for $x = 0$, the expression $\frac{15}{x^3}$ is meaningless.

Consequently, the left hand side of the original equation is meaningless for any $x \in R$. Therefore, the equation has no roots.

(vii) We have, $\sqrt{(x-3)} - \sqrt{(x+9)} = \sqrt{x-1}$

This equation is defined for

$$\begin{cases} x-3 \geq 0 \\ x+9 \geq 0 \\ x-1 \geq 0 \end{cases} \Rightarrow \begin{cases} x \geq 3 \\ x \geq -9 \\ x \geq 1 \end{cases}$$

Hence, $x \geq 3$

For $x \geq 3$, $\sqrt{x-3} < \sqrt{x+9}$ i.e. $\sqrt{(x-3)} - \sqrt{(x+9)} < 0$

Hence, for $x \geq 3$, the left hand side of the original equation is negative and right hand side is positive. Therefore, the equation has no roots.

Some Standard Forms to Solve Irrational Equations

***Form* 1 An equation of the form**

$f^{2n}(x) = g^{2n}(x), n \in N$ is equivalent to $f(x) = g(x)$.

Then, find the roots of this equation. If root of this equation satisfies the original equation, then its root of the original equation, otherwise, we say that this root is its **extraneous root**.

Remark

Squaring an Equation May Give Extraneous Roots

Squaring should be avoided as for as possible. If squaring is necessary, then the roots found after squaring must be checked whether they satisfy the original equation or not. If some values of x which do not satisfy the original equation. These values of x are called extraneous roots and are rejected.

Example 79. Solve the equation $\sqrt{x} = x - 2$.

Sol. We have, $\sqrt{x} = x - 2$

On squaring both sides, we obtain

$$x = (x-2)^2$$

$\Rightarrow \quad x^2 - 5x + 4 = 0 \Rightarrow (x-1)(x-4) = 0$

$\therefore \quad x_1 = 1$ and $x_2 = 4$

Hence, $x_1 = 4$ satisfies the original equation, but $x_2 = 1$ does not satisfy the original equation.

$\therefore \quad x_2 = 1$ is the extraneous root.

Example 80. Solve the equation

$$3\sqrt{(x+3)} - \sqrt{(x-2)} = 7.$$

Sol. We have, $3\sqrt{(x+3)} - \sqrt{x-2} = 7$

$\Rightarrow \quad 3\sqrt{(x+3)} = 7 + \sqrt{(x-2)}$

On squaring both sides of the equation, we obtain

$$9x + 27 = 49 + x - 2 + 14\sqrt{x-2}$$

$\Rightarrow \quad 8x - 20 = 14\sqrt{(x-2)}$

$$(4x - 10) = 7\sqrt{x-2}$$

Again, squaring both sides, we obtain

$$16x^2 + 100 - 80x = 49x - 98$$

$\Rightarrow \quad 16x^2 - 129x + 198 = 0$

$\Rightarrow \quad (x-6)\left(x - \frac{33}{16}\right) = 0$

$$x_1 = 6 \text{ and } x_2 = \frac{33}{16}$$

Hence, $x_1 = 6$ satisfies the original equation, but $x_2 = \frac{33}{16}$ does not satisfy the original equation.

$\therefore \; x_2 = \frac{33}{16}$ is the extraneous root.

***Form* 2 An equation in the form**

$\sqrt[2n]{f(x)} = g(x), n \in N$

is equivalent to the system $\begin{cases} g(x) \geq 0 \\ f(x) = g^{2n}(x) \end{cases}$

Example 81. ***Solve the equation***

$$\sqrt{(6 - 4x - x^2)} = x + 4.$$

Sol. We have, $\sqrt{(6-4x-x^2)} = x + 4$

This equation is equivalent to the system

$$\begin{cases} x + 4 \geq 0 \\ 6 - 4x - x^2 = (x+4)^2 \end{cases}$$

$$\Rightarrow \quad \begin{cases} x \geq -4 \\ x^2 + 6x + 5 = 0 \end{cases}$$

On solving the equation $x^2 + 6x + 5 = 0$

We find that, $x_1 = (-1)$ and $x_2 = (-5)$ only $x_1 = (-1)$ satisfies the condition $x \geq -4$.

Consequently, the number -1 is the only solution of the given equation.

***Form* 3 An equation in the form**

$$\sqrt[3]{f(x)} + \sqrt[3]{g(x)} = h(x) \qquad \ldots(i)$$

where $f(x)$, $g(x)$ are the functions of x, but $h(x)$ is a function of x or constant, can be solved as follows cubing both sides of the equation, we obtain

$$f(x) + g(x) + 3\sqrt[3]{f(x)\,g(x)}\,(\sqrt[3]{f(x)} + \sqrt[3]{g(x)}) = h^3(x)$$

$$\Rightarrow \quad f(x) + g(x) + 3\sqrt[3]{f(x)\,g(x)}\,(h(x)) = h^3(x) \qquad \text{[from Eq. (i)]}$$

We find its roots and then substituting, then into the original equation, we choose those which are the roots of the original equation.

Example 82. ***Solve the equation***

$$\sqrt[3]{(2x-1)}+\sqrt[3]{(x-1)}=1.$$

Sol. We have, $\sqrt[3]{(2x-1)}+\sqrt[3]{(x-1)}=1$ ...(i)

Cubing both sides of Eq. (i), we obtain

$$2x-1+x-1+3\cdot\sqrt[3]{(2x-1)(x-1)}$$
$$(\sqrt[3]{(2x-1)}+\sqrt[3]{(x-1)})=1$$
$$\Rightarrow \quad 3x-2+3\cdot\sqrt[3]{(2x^2-3x+1)}\,(1)=1 \qquad \text{[from Eq. (i)]}$$
$$\Rightarrow \quad 3\cdot\sqrt[3]{(2x^2-3x+1)}=3-3x$$
$$\Rightarrow \quad \sqrt[3]{(2x^2-3x+1)}=(1-x)$$

Again cubing both sides, we obtain

$$2x^2-3x+1=(1-x)^3$$
$$\Rightarrow \quad (2x-1)(x-1)=(1-x)^3$$
$$\Rightarrow \quad (2x-1)(x-1)=-(x-1)^3$$
$$\Rightarrow \quad (x-1)\{2x-1+(x-1)^2\}=0$$
$$\Rightarrow \quad (x-1)(x^2)=0$$
$$\therefore \quad x_1=0 \text{ and } x_2=1$$

$\because x_1=0$ is not satisfies the Eq. (i), then $x_1=0$ is an extraneous root of the Eq. (i), thus $x_2=1$ is the only root of the original equation.

***Form* 4 An equation of the form**

$$\sqrt[n]{a-f(x)}+\sqrt[n]{b+f(x)}=g(x).$$

Let $u=\sqrt[n]{a-f(x)}, v=\sqrt[n]{b+f(x)}$

Then, the given equation reduces to the solution of the system of algebraic equations.

$$\begin{cases} u+v=g(x) \\ u^n+v^n=a+b \end{cases}$$

Example 83. Solve the equation

$$\sqrt{(2x^2+5x-2)}-\sqrt{2x^2+5x-9}=1.$$

Sol. Let $u=\sqrt{(2x^2+5x-2)}$

and $v=\sqrt{(2x^2+5x-9)}$

$\therefore \quad u^2=2x^2+5x-2$

and $v^2=2x^2+5x-9$

Then, the given equation reduces to the solution of the system of algebraic equations.

$$u-v=1$$
$$u^2-v^2=7$$
$$\Rightarrow \quad (u+v)(u-v)=7$$
$$\Rightarrow \quad u+v=7 \qquad [\because u-v=1]$$

We get, $u=4, v=3$

$\therefore \quad \sqrt{2x^2+5x-2}=4$

$\therefore \quad 2x^2+5x-18=0$

$\therefore \quad x_1=2 \text{ and } x_2=-9/2$

Both roots satisfies the original equation.

Hence, $x_1=2$ and $x_2=-9/2$ are the roots of the original equation.

Irrational Inequations

We consider, here inequations which contain the unknown under the radical sign.

Some Standard Forms to Solve Irrational Inequations

***Form* 1** An inequation of the form

$$\sqrt[2n]{f(x)}<\sqrt[2n]{g(x)}, n\in N$$

is equivalent to the system $\begin{cases} f(x)\geq 0 \\ g(x)>f(x) \end{cases}$

and inequation of the form $\sqrt[2n+1]{f(x)}<\sqrt[2n+1]{g(x)}, n\in N$

is equivalent to the inequation $f(x)<g(x)$.

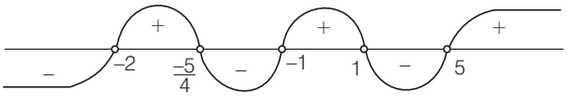
Example 84. Solve the inequation

$$\sqrt[5]{\left[\frac{3}{x+1}+\frac{7}{x+2}\right]}<\sqrt[5]{\frac{6}{x-1}}.$$

Sol. The given inequation is equivalent to

$$\frac{3}{x+1}+\frac{7}{x+2}<\frac{6}{x-1}$$
$$\Rightarrow \quad \frac{4x^2-15x-25}{(x+1)(x+2)(x-1)}<0$$
$$\Rightarrow \quad \frac{(x+5/4)(x-5)}{(x+1)(x+2)(x-1)}<0$$

From Wavy Curve Method :



$$x\in(-\infty,-2)\cup\left(-\frac{5}{4},1\right)\cup(1,5)$$

***Form* 2 An inequation of the form**

$$\sqrt[2n]{f(x)}<g(x), n\in N.$$

is equivalent to the system $\begin{cases} f(x)\geq 0 \\ g(x)>0 \\ f(x)<g^{2n}(x), \end{cases}$

and inequation of the form $\sqrt[2n+1]{f(x)} < g(x), n \in N$

is equivalent to the inequation $f(x) < g^{2n+1}(x)$.

Example 85. Solve the inequation $\sqrt{(x+14)} < (x+2)$.

Sol. We have, $\sqrt{(x+14)} < (x+2)$

This inequation is equivalent to the system

$$\begin{cases} x+14 \geq 0 \\ x+2 > 0 \\ x+14 < (x+2)^2 \end{cases} \Rightarrow \begin{cases} x \geq -14 \\ x > -2 \\ x^2+3x-10 > 0 \end{cases}$$

$$\Rightarrow \quad \begin{cases} x \geq -14 \\ x > -2 \\ (x+5)(x-2) > 0 \end{cases} \Rightarrow \begin{cases} x \geq -14 \\ x > -2 \\ x < -5 \text{ and } x > 2 \end{cases}$$

On combining all three inequation of the system, we get

$$x > 2, \text{ i.e. } x \in (2, \infty)$$

***Form* 3** An inequation of the form

$$\sqrt[2n]{f(x)} > g(x), n \in N$$

is equivalent to the collection of two systems of inequations

i.e. $\begin{cases} g(x) \geq 0 \\ f(x) > g^{2n}(x) \end{cases}$ and $\begin{cases} g(x) < 0 \\ f(x) \geq 0 \end{cases}$

and inequation of the form $\sqrt[2n+1]{f(x)} > g(x), n \in N$

is equivalent to the inequation $f(x) > g^{2n+1}(x)$.

Example 86. ***Solve the inequation***

$\sqrt{(-x^2+4x-3)} > 6-2x$.

Sol. We have, $\sqrt{(-x^2+4x-3)} > 6-2x$

This inequation is equivalent to the collection of two systems, of inequations

i.e. $\begin{cases} 6-2x \geq 0 \\ -x^2+4x-3 > (6-2x)^2 \end{cases}$ and $\begin{cases} 6-2x < 0 \\ -x^2+4x-3 \geq 0 \end{cases}$

$$\Rightarrow \quad \begin{cases} x \leq 3 \\ (x-3)(5x-13) < 0 \end{cases} \text{ and } \begin{cases} x > 3 \\ (x-1)(x-3) \leq 0 \end{cases}$$

$$\Rightarrow \quad \begin{cases} x \leq 3 \\ \frac{13}{5} < x < 3 \end{cases} \text{ and } \begin{cases} x > 3 \\ 1 \leq x < 3 \end{cases}$$

The second system has no solution and the first system has solution in the interval $\left(\frac{13}{5} < x < 3\right)$.

Hence, $x \in \left(\frac{13}{5}, 3\right)$ is the set of solution of the original inequation.

Exponential Equations

If we have an equation of the form $a^x = b\,(a > 0)$, then

(i) $x \in \phi$, if $b \leq 0$

(ii) $x = \log_a b$, if $b > 0, a \neq 1$

(iii) $x \in \phi$, if $a = 1, b \neq 1$

(iv) $x \in R$, if $a = 1, b = 1$ (since, $1^x = 1 \Rightarrow 1 = 1, x \in R$)

Example 87. ***Solve the equation***

$\sqrt{(6-x)}\,(3^{x^2-7.2x+3.9} - 9\sqrt{3}) = 0$.

Sol. We have,

$$\sqrt{(6-x)}\,(3^{x^2-7.2x+3.9} - 9\sqrt{3}) = 0$$

This equation is defined for

$$6 - x \geq 0 \text{ i.e., } x \leq 6 \qquad \ldots(i)$$

This equation is equivalent to the collection of equations

$\sqrt{6-x} = 0$ and $3^{x^2-7.2x+3.9} - 9\sqrt{3} = 0$

$\therefore \qquad x_1 = 6$ and $3^{x^2-7.2x+3.9} = 3^{2.5}$

then $\qquad x^2 - 7.2x + 3.9 = 2.5$

$$x^2 - 7.2x + 1.4 = 0$$

We find that, $\qquad x_2 = \frac{1}{5}$ and $x_3 = 7$

Hence, solution of the original equation are [which satisfies Eq. (i)]

$$x_1 = 6, x_2 = \frac{1}{5}.$$

Some Standard Forms to Solve Exponential Equations

***Form* 1 An equation in the form $a^{f(x)} = 1, a > 0, a \neq 1$**

is equivalent to the equation $f(x) = 0$

Example 88. ***Solve the equation*** $5^{x^2+3x+2} = 1$.

Sol. This equation is equivalent to

$$x^2 + 3x + 2 = 0$$

$$\Rightarrow \qquad (x+1)(x+2) = 0$$

$\therefore x_1 = -1, x_2 = -2$ consequently, this equation has two roots $x_1 = -1$ and $x_2 = -2$.

***Form* 2** An equation in the form

$$f(a^x) = 0$$

is equivalent to the equation $f(t) = 0$, where $t = a^x$.

If $t_1, t_2, t_3, \ldots, t_k$ are the roots of $f(t) = 0$, then

$$a^x = t_1, a^x = t_2, a^x = t_3, \ldots, a^x = t_k$$

Example 89. Solve the equation $5^{2x} - 24 \cdot 5^x - 25 = 0$.

Sol. Let $5^x = t$, then the given equation can reduce in the form

$$t^2 - 24t - 25 = 0$$

$$\Rightarrow \quad (t-25)(t+1) = 0 \Rightarrow t \neq -1,$$

$$\therefore \quad t = 25,$$

then $5^x = 25 = 5^2$, then $x = 2$

Hence, $x_1 = 2$ is only one root of the original equation.

***Form* 3** An equation of the form

$$\alpha a^{f(x)} + \beta b^{f(x)} + \gamma c^{f(x)} = 0,$$

where $\alpha, \beta, \gamma \in R$ and $\alpha, \beta, \gamma \neq 0$ and the bases satisfy the condition $b^2 = ac$ is equivalent to the equation

$$\alpha t^2 + \beta t + \gamma = 0, \text{ where } t = (a/b)^{f(x)}$$

If roots of this equation are t_1 and t_2, then

$$(a/b)^{f(x)} = t_1 \text{ and } (a/b)^{f(x)} = t_2$$

Example 90. Solve the equation $64 \cdot 9^x - 84 \cdot 12^x + 27 \cdot 16^x = 0$.

Sol. Here, $9 \times 16 = (12)^2$.

Then, we divide its both sides by 12^x and obtain

$$\Rightarrow \quad 64 \cdot \left(\frac{3}{4}\right)^x - 84 + 27 \cdot \left(\frac{4}{3}\right)^x = 0 \qquad \ldots(i)$$

Let $\left(\frac{3}{4}\right)^x = t$, then Eq. (i) reduce in the form

$$64t^2 - 84t + 27 = 0$$

$$\therefore \quad t_1 = \frac{3}{4} \text{ and } t_2 = \frac{9}{16}$$

then, $\left(\frac{3}{4}\right)^x = \left(\frac{3}{4}\right)^1$ and $\left(\frac{3}{4}\right)^x = \left(\frac{3}{4}\right)^2$

$$\therefore \quad x_1 = 1 \text{ and } x_2 = 2$$

Hence, roots of the original equation are $x_1 = 1$ and $x_2 = 2$.

***Form* 4 An equation in the form**

$$\alpha \cdot a^{f(x)} + \beta \cdot b^{f(x)} + c = 0,$$

where $\alpha, \beta, c \in R$ and $\alpha, \beta, c \neq 0$ and $ab = 1$ (a and b are inverse positive numbers) is equivalent to the equation

$$\alpha t^2 + ct + \beta = 0, \text{ where } t = a^{f(x)}.$$

If roots of this equation are t_1 and t_2, then $a^{f(x)} = t_1$ and $a^{f(x)} = t_2$.

Example 91. ***Solve the equation***

$15 \cdot 2^{x+1} + 15 \cdot 2^{2-x} = 135$.

Sol. This equation rewrite in the form

$$30.2^x + \frac{60}{2^x} = 135$$

Let $t = 2^x$,

Then, $30t^2 - 135t + 60 = 0$

$$\Rightarrow \quad 6t^2 - 27t + 12 = 0$$

$$\Rightarrow \quad 6t^2 - 24t - 3t + 12 = 0$$

$$\Rightarrow \quad (t-4)(6t-3) = 0$$

Then, $t_1 = 4$ and $t_2 = \frac{1}{2}$

Thus, given equation is equivalent to

$$2^x = 4 \text{ and } 2^x = \frac{1}{2}$$

Then, $x_1 = 2$ and $x_2 = -1$

Hence, roots of the original equation are $x_1 = 2$ and $x_2 = -1$.

***Form* 5 An equation of the form $a^{f(x)} + b^{f(x)} = c$,**

where $a, b, c \in R$ and a, b, c satisfies the condition $a^2 + b^2 = c$, then solution of this equation is $f(x) = 2$ and no other solution of this equation.

Example 92. Solve the equation $3^{x-4} + 5^{x-4} = 34$.

Sol. Here, $3^2 + 5^2 = 34$, then given equation has a solution $x - 4 = 2$.

$\therefore$ $x_1 = 6$ is a root of the original equation.

***Form* 6 An equation of the form $\{f(x)\}^{g(x)}$ is equivalent to the equation**

$$\{f(x)\}^{g(x)} = 10^{g(x) \log f(x)},$$

where $f(x) > 0$.

Example 93. ***Solve the equation*** $5^x \sqrt[x]{8^{x-1}} = 500$.

Sol. We have, $5^x \sqrt[x]{8^{x-1}} = 5^3 \cdot 2^2$

$$\Rightarrow \quad 5^x \cdot 8^{\left(\frac{x-1}{x}\right)} = 5^3 \cdot 2^2$$

$$\Rightarrow \quad 5^x \cdot 2^{\frac{3x-3}{x}} = 5^3 \cdot 2^2$$

$$\Rightarrow \quad 5^{x-3} \cdot 2^{\left(\frac{x-3}{x}\right)} = 1$$

$$\Rightarrow \quad (5.2^{1/x})^{(x-3)} = 1$$

is equivalent to the equation

$$10^{(x-3)\log(5 \cdot 2^{1/x})} = 1$$

$$\Rightarrow \quad (x-3)\log(5 \cdot 2^{1/x}) = 0$$

Thus, original equation is equivalent to the collection of equations

$$x - 3 = 0, \ \log(5 \cdot 2^{1/x}) = 0$$

$$\therefore \quad x_1 = 3, 5 \cdot 2^{1/x} = 1 \Rightarrow 2^{1/x} = \left(\frac{1}{5}\right)$$

$$\therefore \quad x_2 = -\log_5 2$$

Hence, roots of the original equation are $x_1 = 3$ and $x_2 = -\log_5 2$.

Exponential Inequations

When we solve exponential inequation $a^{f(x)} > b\,(a>0)$, we have

(i) $x \in D_f$, if $b \le 0$

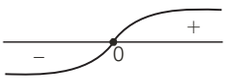
(ii) If $b>0$, then we have $f(x) > \log_a b$, if $a>1$ and $f(x) < \log_a b$, if $0<a<1$ for $a=1$, then $b<1$.

Remark

The inequation $a^{f(x)} \le b$ has no solution for $b \le 0$, $a>0$, $a \ne 1$.

Example 94. ***Solve the inequation*** $3^{x+2} > \left(\frac{1}{9}\right)^{1/x}$.

Sol. We have, $3^{x+2} > (3^{-2})^{1/x} \Rightarrow 3^{x+2} > 3^{-2/x}$



Here, base $3>1$

$$\Rightarrow \quad x+2 > -\frac{2}{x} \quad \Rightarrow \quad \frac{x^2+2x+2}{x} > 0$$

$$\Rightarrow \quad \frac{(x+1)^2+1}{x} > 0 \quad \Rightarrow \quad x>0$$

$$\therefore \quad x \in (0, \infty)$$

Some Standard Forms to Solve Exponential Inequations

Form 1 An inequation of the form

$$f(a^x) \ge 0 \quad \text{or} \quad f(a^x) \le 0$$

is equivalent to the system of collection

$$\begin{cases} t>0, & \text{where } t = a^x \\ f(t) \ge 0 \quad \text{or} \quad f(t) \le 0 \end{cases}$$

Example 95. ***Solve the inequation*** $4^{x+1} - 16^x < 2\log_4 8$.

Sol. Let $4^x = t$, then given inequation reduce in the form

$$4t - t^2 > 2 \cdot \frac{3}{2}$$

$$\Rightarrow \quad t^2 - 4t + 3 < 0 \Rightarrow (t-1)(t-3) < 0$$

$$\Rightarrow \quad 1 < t < 3 \qquad [\because t>0]$$

$$\Rightarrow \quad 1 < 4^x < 3$$

$$\Rightarrow \quad 0 < x < \log_4 3$$

$$\therefore \quad x \in (0, \log_4 3)$$

Form 2 An inequation of the form

$$\alpha a^{f(x)} + \beta b^{f(x)} + \gamma c^{f(x)} \ge 0$$

or $$\alpha a^{f(x)} + \beta b^{f(x)} + \gamma c^{f(x)} \le 0$$

where $\alpha, \beta, \gamma \in R$ and $\alpha, \beta, \gamma \ne 0$ and the bases satisfy the condition $b^2 = ac$ is equivalent to the inequation

$$\alpha t^2 + \beta t + \gamma \ge 0 \quad \text{or} \quad \alpha t^2 + \beta t + \gamma \le 0,$$

where $t = (a/b)^{f(x)}$.

Form 3 An inequation of the form

$$\alpha a^{f(x)} + \beta b^{f(x)} + \gamma \ge 0$$

or $$\alpha a^{f(x)} + \beta b^{f(x)} + \gamma \le 0$$

where $\alpha, \beta, \gamma \in R$ and $\alpha, \beta, \gamma \ne 0$ and $ab = 1$(a and b are inverse (+ve) numbers) is equivalent to the inequation

$$\alpha t^2 + \beta t + \gamma \ge 0 \quad \text{or} \quad \alpha t^2 + \beta t + \gamma \le 0$$

where $t = a^{f(x)}$

Form 4 If an inequation of the exponential form reduces to the solution of homogeneous algebraic inequation, i.e.

$$a_0 f^n(x) + a_1 f^{n-1}(x)\, g(x) + a_2 f^{n-2}(x) g^2(x) + \ldots + a_{n-1} f(x)\, g^{n-1}(x) + a_n\, g^n(x) \ge 0,$$

where $a_0, a_1, a_2, \ldots, a_n$ are constants ($a_0 \ne 0$) and $f(x)$ and $g(x)$ are functions of x.

Example 96. Solve the inequation $2^{2x^2-10x+3} + 6^{x^2-5x+1} \ge 3^{2x^2-10x+3}$.

Sol. The given inequation is equivalent to

$$8 \cdot 2^{2(x^2-5x)} + 6 \cdot 2^{x^2-5x} \cdot 3^{x^2-5x} - 27 \cdot 3^{2(x^2-5x)} \ge 0$$

Let $2^{x^2-5x} = f(x)$ and $3^{x^2-5x} = g(x)$,

then $8 \cdot f^2(x) + 6 f(x) \cdot g(x) - 27 g^2(x) \ge 0$

On dividing in each by $g^2(x)$ $[\because g(x) > 0]$

Then, $8\left(\frac{f(x)}{g(x)}\right)^2 + 6\left(\frac{f(x)}{g(x)}\right) - 27 \ge 0$

and let $\frac{f(x)}{g(x)} = t$ $[\because t>0]$

then $8t^2 + 6t - 27 \ge 0$

$$\Rightarrow \quad \left(t - \frac{3}{2}\right)(t + 9/4) \ge 0$$

$$\Rightarrow \quad t \ge 3/2 \text{ and } t \le -9/4$$

The second inequation has no root. $[\because t>0]$

From the first inequation, $t > 3/2$

$$\left(\frac{2}{3}\right)^{x^2-5x} \ge \left(\frac{2}{3}\right)^{-1} \qquad \left[\because \frac{2}{3} < 1\right]$$

$$\Rightarrow \quad x^2 - 5x \le -1 \Rightarrow x^2 - 5x + 1 \le 0$$

$$\therefore \quad \frac{5-\sqrt{21}}{2} \le x \le \frac{5+\sqrt{21}}{2}$$

Hence, $x \in \left[\frac{5-\sqrt{21}}{2}, \frac{5+\sqrt{21}}{2}\right]$.

Logarithmic Equations

If we have an equation of the form

$$\log_a f(x) = b,\ (a > 0),\ a \neq 1$$

is equivalent to the equation

$$f(x) = a^b \quad (f(x) > 0).$$

Example 97. Solve the equation $\log_3(5 + 4\log_3(x-1)) = 2$.

Sol. We have, $\log_3(5 + 4\log_3(x-1)) = 2$

is equivalent to the equation (here, base $\neq 1, > 0$).

$\therefore \quad 5 + 4\log_3(x-1) = 3^2$

$\Rightarrow \quad \log_3(x-1) = 1 \Rightarrow x - 1 = 3^1$

$\therefore \quad x = 4$

Hence, $x_1 = 4$ is the solution of the original equation.

Some Standard Formulae to Solve Logarithmic Equations

f and g are some functions and $a > 0, a \neq 1$, then, if $f > 0, g > 0$, we have

(i) $\log_a(fg) = \log_a f + \log_a g$

(ii) $\log_a(f/g) = \log_a f - \log_a g$

(iii) $\log_a f^{2\alpha} = 2\alpha \log_a |f|$ (iv) $\log_{a^\beta} f^\alpha = \dfrac{\alpha}{\beta}\log_a f$

(v) $f^{\log_a g} = g^{\log_a f}$ (vi) $a^{\log_a f} = f$

Example 98. Solve the equation $2x^{\log_4 3} + 3^{\log_4 x} = 27$.

Sol. The domain of the admissible values of the equation is $x > 0$. The given equation is equivalent to

$2.3^{\log_4 x} + 3^{\log_4 x} = 27$ [from above result (v)]

$\Rightarrow \quad 3.3^{\log_4 x} = 27$

$\Rightarrow \quad 3^{\log_4 x} = 9$

$\Rightarrow \quad 3^{\log_4 x} = 3^2$

$\Rightarrow \quad \log_4 x = 2$

$\Rightarrow \quad x_1 = 4^2 = 16$ is its only root.

Some Standard Forms to Solve Logarithmic Equations

Form 1 An equation of the form $\log_x a = b, a > 0$ has

(i) Only root $x = a^{1/b}$, if $a \neq 1$ and $b = 0$.

(ii) Any positive root different from unity, if $a = 1$ and $b = 0$.

(iii) No roots, if $a = 1, b \neq 0$.

(iv) No roots, if $a \neq 1, b = 0$.

Example 99. Solve the equation $\log_{(\log_5 x)} 5 = 2$.

Sol. We have, $\log_{(\log_5 x)} 5 = 2$

Base of logarithm > 0 and $\neq 1$.

$\therefore \quad \log_5 x > 0$ and $\log_5 x \neq 1$

$\Rightarrow \quad x > 1$ and $x \neq 5$

$\therefore$ The original equation is equivalent to

$$\log_5 x = 5^{1/2} = \sqrt{5}$$

$\therefore \quad x_1 = 5^{\sqrt{5}}$

Hence, $5^{\sqrt{5}}$ is the only root of the original equation.

Form 2 Equations of the form

(i) $f(\log_a x) = 0, a > 0, a \neq 1$ and

(ii) $g(\log_x A) = 0, A > 0$

Then, Eq. (i) is equivalent to

$f(t) = 0$, where $t = \log_a x$

If $t_1, t_2, t_3, \ldots, t_k$ are the roots of $f(t) = 0$, then

$\log_a x = t_1, \log_a x = t_2, \ldots, \log_a x = t_k$

and Eq. (ii) is equivalent to $f(y) = 0$, where $y = \log_x A$.

If $y_1, y_2, y_3, \ldots, y_k$ are the roots of $f(y) = 0$, then

$\log_x A = y_1, \log_x A = y_2, \ldots, \log_x A = y_k$

Example 100. Solve the equation

$$\frac{1 - 2(\log x^2)^2}{\log x - 2(\log x)^2} = 1.$$

Sol. The given equation can rewrite in the form

$$\frac{1 - 2(2\log x)^2}{\log x - 2(\log x)^2} = 1$$

$$\Rightarrow \quad \frac{1 - 8(\log x)^2}{\log x - 2(\log x)^2} - 1 = 0$$

Let $\log x = t$,

then $\dfrac{1 - 8t^2}{t - 2t^2} - 1 = 0 \Rightarrow \dfrac{1 - 8t^2 - t + 2t^2}{t - 2t^2} = 0$

$$\Rightarrow \quad \frac{1 - t - 6t^2}{(t - 2t^2)} = 0 \Rightarrow \frac{(1 + 2t)(1 - 3t)}{t(1 - 2t)} = 0$$

$$\Rightarrow \quad \begin{cases} t = -\dfrac{1}{2} \\ t = \dfrac{1}{3} \end{cases} \Rightarrow \begin{cases} \log x = -\dfrac{1}{2} \\ \log x = \dfrac{1}{3} \end{cases} \Rightarrow \begin{cases} x_1 = 10^{-1/2} \\ x_2 = 10^{1/3} \end{cases}$$

Hence, $x_1 = \dfrac{1}{\sqrt{10}}$ and $x_2 = \sqrt[3]{10}$ are the roots of the original equation.

Example 101. ***Solve the equation***

$$\log_x^3 10 - 6\log_x^2 10 + 11\log_x 10 - 6 = 0.$$

Sol. Put $\log_x 10 = t$ in the given equation, we get

$$t^3 - 6t^2 + 11t - 6 = 0 \Rightarrow (t-1)(t-2)(t-3) = 0,$$

then $\begin{cases} t = 1 \\ t = 2 \\ t = 3 \end{cases}$

It follows that

$$\begin{cases} \log_x 10 = 1 \\ \log_x 10 = 2 \\ \log_x 10 = 3 \end{cases} \Rightarrow \begin{cases} x = 10 \\ x^2 = 10 \\ x^3 = 10 \end{cases} \Rightarrow \begin{cases} x = 10 \\ x = \sqrt{10} \\ x = \sqrt[3]{10} \end{cases} \quad [\because x > 0 \text{ and} \neq 1]$$

$[\because x > 0 \text{ and } \neq 1]$

$\therefore$ $x_1 = 10$, $x_2 = \sqrt{10}$ and $x_3 = \sqrt[3]{10}$ are the roots of the original equation.

Form 3 Equations of the form

(i) $\log_a f(x) = \log_a g(x), a > 0, a \neq 1$ is equivalent to two ways.

Method I $\begin{cases} g(x) > 0 \\ f(x) = g(x) \end{cases}$

Method II $\begin{cases} f(x) > 0 \\ f(x) = g(x) \end{cases}$

(ii) $\log_{f(x)} A = \log_{g(x)} A, A > 0$ is equivalent to two ways.

Method I $\begin{cases} g(x) > 0 \\ g(x) \neq 1 \\ f(x) = g(x) \end{cases}$

Method II $\begin{cases} f(x) > 0 \\ f(x) \neq 1 \\ f(x) = g(x) \end{cases}$

Example 102. ***Solve the equation***

$$\log_{1/3}\left[2\left(\frac{1}{2}\right)^x - 1\right] = \log_{1/3}\left[\left(\frac{1}{4}\right)^x - 4\right].$$

Sol. The given equation is equivalent to

$$\begin{cases} 2\left(\frac{1}{2}\right)^x - 1 > 0 \\ 2\left(\frac{1}{2}\right)^x - 1 = \left(\frac{1}{4}\right)^x - 4 \end{cases}$$

$$\Rightarrow \begin{cases} \left(\frac{1}{2}\right)^x > \frac{1}{2} \\ \left(\frac{1}{2}\right)^{2x} - 2\left(\frac{1}{2}\right)^x - 3 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x < 1 \\ \left[\left(\frac{1}{2}\right)^x - 3\right]\left[\left(\frac{1}{2}\right)^x + 1\right] = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x < 1 \\ \left(\frac{1}{2}\right)^x = 3, \left(\frac{1}{2}\right)^x + 1 \neq 0 \end{cases} \Rightarrow \begin{cases} x < 1 \\ x = (-\log_2 3) \end{cases}$$

Hence, $x_1 = -\log_2 3$ is the root of the original equation.

Example 103. Solve the equation $\log_{\left(\frac{2+x}{10}\right)} 7 = \log_{\left(\frac{2}{x+1}\right)} 7$.

Sol. The given equation is equivalent to

$$\begin{cases} \frac{2}{x+1} > 0 \\ \frac{2}{x+1} \neq 1 \\ \frac{2+x}{10} = \frac{2}{x+1} \end{cases} \Rightarrow \begin{cases} x + 1 > 0 \\ x \neq 1 \\ x = -6, 3 \end{cases}$$

$\therefore$ $x_1 = 3$ is root of the original equation.

Form 4 Equations of the form

(i) $\log_{f(x)} g(x) = \log_{f(x)} h(x)$ is equivalent to two ways.

Method I $\begin{cases} g(x) > 0 \\ f(x) > 0 \\ f(x) \neq 1 \\ g(x) = h(x) \end{cases}$ Method II $\begin{cases} h(x) > 0 \\ f(x) > 0 \\ f(x) \neq 1 \\ g(x) = h(x) \end{cases}$

(ii) $\log_{g(x)} f(x) = \log_{h(x)} f(x)$ is equivalent to two ways.

Method I $\begin{cases} f(x) > 0 \\ g(x) > 0 \\ g(x) \neq 1 \\ g(x) = h(x) \end{cases}$

Method II $\begin{cases} f(x) > 0 \\ h(x) > 0 \\ h(x) \neq 1 \\ g(x) = h(x) \end{cases}$

Example 104. Solve the equation $\log_{(x^2-1)}(x^3 + 6) = \log_{(x^2-1)}(2x^2 + 5x)$.

Sol. This equation is equivalent to the system

$$\begin{cases} 2x^2 + 5x > 0 \\ x^2 - 1 > 0 \\ x^2 - 1 \neq 1 \\ x^3 + 6 = 2x^2 + 5x \end{cases} \Rightarrow \begin{cases} x < -\frac{5}{2} \text{ and } x > 0 \\ x < -1 \text{ and } x > 1 \\ x \neq \pm\sqrt{2} \\ x = -2, 1, 3 \end{cases}$$

Hence, $x_1 = 3$ is only root of the original equation.

Example 105. Solve the equation $\log_{(x^3+6)}(x^2-1)=\log_{(2x^2+5x)}(x^2-1)$.

Sol. This equation is equivalent to

$$\begin{cases} x^2-1>0 \\ 2x^2+5x>0 \\ 2x^2+5x\neq 1 \\ x^3+6=2x^2+5x \end{cases}$$

$$\Rightarrow \quad \begin{cases} x<-1 \text{ and } x>1 \\ x<-\dfrac{5}{2} \text{ and } x>0 \\ x\neq \dfrac{-5\pm\sqrt{33}}{4} \\ x=-2,1,3 \end{cases}$$

Hence, $x_1=3$ is only root of the original equation.

***Form* 5 An equation of the form** $\log_{h(x)}(\log_{g(x)} f(x))=0$ is equivalent to the system

$$\begin{cases} h(x)>0 \\ h(x)\neq 1 \\ g(x)>0 \\ g(x)\neq 1 \\ f(x)=g(x) \end{cases}$$

Example 106. ***Solve the equation*** $\log_{x^2-6x+8}[\log_{2x^2-2x+8}(x^2+5x)]=0$.

Sol. This equation is equivalent to the system

$$\begin{cases} x^2-6x+8>0 \\ x^2-6x+8\neq 1 \\ 2x^2-2x-8>0 \\ 2x^2-2x-8\neq 1 \\ x^2+5x=2x^2-2x-8 \end{cases}$$

Solve the equations of this system

$$\Rightarrow \quad \begin{cases} x<2 \text{ and } x>4 \\ x\neq 3\pm\sqrt{2} \\ x<\dfrac{1-\sqrt{17}}{2} \text{ and } x>\dfrac{1+\sqrt{17}}{2} \\ x\neq \dfrac{1\pm\sqrt{19}}{2} \\ x=-1,8 \end{cases}$$

$x=-1$, does not satisfy the third relation of this system. Hence, $x_1=8$ is only root of the original equation.

***Form* 6 An equation of the form** $2m\log_a f(x)=\log_a g(x), a>0, a\neq 1, m\in N$ is equivalent to the system

$$\begin{cases} f(x)>0 \\ f^{2m}(x)=g(x) \end{cases}$$

Example 107. ***Solve the equation*** $2\log 2x=\log(7x-2-2x^2)$.

Sol. This equation is equivalent to the system

$$\begin{cases} 2x>0 \\ (2x)^2=7x-2-2x^2 \end{cases}$$

$$\Rightarrow \quad \begin{cases} x>0 \\ 6x^2-7x+2=0 \end{cases}$$

$$\Rightarrow \quad \begin{cases} x>0 \\ (x-1/2)(x-2/3)=0 \end{cases}$$

$$\Rightarrow \quad \begin{cases} x=1/2 \\ x=2/3 \end{cases}$$

Hence, $x_1=1/2$ and $x_2=2/3$ are the roots of the original equation.

***Form* 7 An equation of the form**

$$(2m+1)\log_a f(x)=\log_a g(x), a>0, a\neq 1, m\in N$$

is equivalent to the system $\begin{cases} g(x)>0 \\ f^{2m+1}(x)=g(x) \end{cases}$.

Example 108. ***Solve the equation*** $\log(3x^2+x-2)=3\log(3x-2)$.

Sol. This equation is equivalent to the system

$$\begin{cases} 3x^2+x-2>0 \\ 3x^2+x-2=(3x-2)^3 \end{cases}$$

$$\Rightarrow \quad \begin{cases} (x-2/3)(x-2)>0 \\ (x-2/3)(9x^2-13x+3)=0 \end{cases}$$

$$\Rightarrow \quad \begin{cases} x<2/3 \text{ and } x>2 \\ x=\dfrac{2}{3},\ x=\dfrac{13\pm\sqrt{61}}{18} \end{cases}$$

Original equation has the only root $x_1=\dfrac{13-\sqrt{61}}{18}$.

***Form* 8 An equation of the form**

$$\log_a f(x)+\log_a g(x)=\log_a m(x), a>0, a\neq 1$$

is equivalent to the system

$$\begin{cases} f(x)>0 \\ g(x)>0 \\ f(x)\,g(x)=m(x) \end{cases}$$

Example 109. ***Solve the equation*** $2\log_3 x+\log_3(x^2-3)=\log_3 0.5+5^{\log_5(\log_3 8)}$

Sol. This equation can be written as

$$\log_3 x^2+\log_3(x^2-3)=\log_3 0.5+\log_3 8$$

$$\log_3 x^2+\log_3(x^2-3)=\log_3(4)$$

This is equivalent to the system

$$\begin{cases} x^2 > 0 \\ x^2 - 3 > 0 \\ x^2(x^2-3) = 4 \end{cases} \Rightarrow \begin{cases} x < 0 \text{ and } x > 0 \\ x < -\sqrt{3} \text{ and } x > \sqrt{3} \\ (x^2-4)(x^2+1) = 0 \end{cases}$$

$$\Rightarrow \quad x^2 - 4 = 0 \quad \therefore \quad x = \pm 2, \text{ but } x > 0$$

Consequently, $x_1 = 2$ is only root of the original equation.

***Form* 9 An equation of the form**

$\log_a f(x) - \log_a g(x) = \log_a h(x) - \log_a t(x)$, $a > 0$, $a \neq 1$

is equivalent to the equation

$\log_a f(x) + \log_a t(x) = \log_a g(x) + \log_a h(x)$,

which is equivalent to the system

$$\begin{cases} f(x) > 0 \\ t(x) > 0 \\ g(x) > 0 \\ h(x) > 0 \\ f(x) \cdot t(x) = g(x) \cdot h(x) \end{cases}$$

Example 110. Solve the equation

$$\log_2(3-x) - \log_2\left(\frac{\sin\frac{3\pi}{4}}{5-x}\right) = \frac{1}{2} + \log_2(x+7).$$

Sol. This equation is equivalent to

$$\log_2(3-x) = \log_2\left(\frac{\sin\frac{3\pi}{4}}{5-x}\right) + \frac{1}{2}\log_2 2 + \log_2(x+7)$$

$$\Rightarrow \log_2(3-x) = \log_2\left(\frac{1}{\sqrt{2}(5-x)}\right) + \log_2\sqrt{2} + \log_2(x+7)$$

which is equivalent to the system

$$\begin{cases} 3-x > 0 \\ \dfrac{1}{\sqrt{2}(5-x)} > 0 \\ x+7 > 0 \\ (3-x) = \dfrac{\sqrt{2}(x+7)}{\sqrt{2}(5-x)} \end{cases}$$

$$\Rightarrow \quad \begin{cases} x < 3 \\ x < 5 \\ x > -7 \\ (x-1)(x-8) = 0 \end{cases}$$

Hence, $x_1 = 1$ is only root of the original equation.

Logarithmic Inequations

When we solve logarithmic inequations

(i) $\begin{cases} \log_a f(x) > \log_a g(x) \\ a > 1 \end{cases}$

$$\Rightarrow \begin{cases} g(x) > 0 \\ a > 1 \\ f(x) > g(x) \end{cases}$$

(ii) $\begin{cases} \log_a f(x) > \log_a g(x) \\ 0 < a < 1 \end{cases}$

$$\Rightarrow \begin{cases} f(x) > 0 \\ 0 < a < 1 \\ f(x) < g(x) \end{cases}$$

Example 111. Solve the inequation

$$\log_{2x+3} x^2 < \log_{2x+3}(2x+3).$$

Sol. This inequation is equivalent to the collection of the systems

$$\begin{bmatrix} \begin{cases} 2x+3 > 1 \\ x^2 < 2x+3 \\ 0 < 2x+3 < 1 \\ x^2 > 2x+3 \end{cases} \Rightarrow \begin{bmatrix} \begin{cases} x > -1 \\ (x-3)(x+1) < 0 \end{cases} \\ \begin{cases} -\frac{3}{2} < x < -1 \\ (x-3)(x+1) > 0 \end{cases} \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} \begin{cases} x > -1 \\ -1 < x < 3 \end{cases} & \Rightarrow -1 < x < 3 \\ \begin{cases} -\frac{3}{2} < x < -1 \\ x < -1 \text{ and } x > 3 \end{cases} & \Rightarrow -\frac{3}{2} < x < -1 \end{bmatrix}$$

Hence, the solution of the original inequation is

$$x \in \left(-\frac{3}{2}, -1\right) \cup (-1, 3).$$

Canonical Logarithmic Inequalities

1. $\begin{cases} \log_a x > 0 \\ a > 1 \end{cases} \Rightarrow \begin{cases} x > 1 \\ a > 1 \end{cases}$

2. $\begin{cases} \log_a x > 0 \\ 0 < a < 1 \end{cases} \Rightarrow \begin{cases} 0 < x < 1 \\ 0 < a < 1 \end{cases}$

3. $\begin{cases} \log_a x < 0 \\ a > 1 \end{cases} \Rightarrow \begin{cases} 0 < x < 1 \\ a > 1 \end{cases}$

4. $\begin{cases} \log_a x < 0 \\ 0 < a < 1 \end{cases} \Rightarrow \begin{cases} x > 1 \\ 0 < a < 1 \end{cases}$

Some Standard Forms to Solve Logarithmic Inequations

***Form* 1 Inequations of the form**

	Forms		Collection of systems
(a)	$\log_{g(x)} f(x) > 0$	$\Leftrightarrow$	$\begin{cases} f(x) > 1, \\ g(x) > 1, \end{cases} \begin{cases} 0 < f(x) < 1 \\ 0 < g(x) < 1 \end{cases}$
(b)	$\log_{g(x)} f(x) \geq 0$	$\Leftrightarrow$	$\begin{cases} f(x) \geq 1, \\ g(x) > 1, \end{cases} \begin{cases} 0 < f(x) \leq 1 \\ 0 < g(x) < 1 \end{cases}$
(c)	$\log_{g(x)} f(x) < 0$	$\Leftrightarrow$	$\begin{cases} f(x) > 1, \\ 0 < g(x) < 1, \end{cases} \begin{cases} 0 < f(x) < 1 \\ g(x) > 1 \end{cases}$
(d)	$\log_{g(x)} f(x) \leq 0$	$\Leftrightarrow$	$\begin{cases} f(x) \geq 1, \\ 0 < g(x) < 1, \end{cases} \begin{cases} 0 < f(x) \leq 1 \\ g(x) > 1 \end{cases}$

Example 112. ***Solve the inequation***

$$\log_{\left(\frac{x^2-12x+30}{10}\right)}\left(\log_2 \frac{2x}{5}\right) > 0.$$

Sol. This inequation is equivalent to the collection of two systems

$$\begin{cases} \dfrac{x^2-12x+30}{10} > 1, \\ \log_2\left(\dfrac{2x}{5}\right) > 1, \end{cases}$$

$$\begin{cases} 0 < \dfrac{x^2-12x+30}{10} < 1 \\ 0 < \log_2\left(\dfrac{2x}{5}\right) < 1 \end{cases}$$

On solving the first system, we have

$$\Rightarrow \quad \begin{cases} x^2-12x+20 > 0 \\ \dfrac{2x}{5} > 2 \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} (x-10)(x-2) > 0 \\ x > 5 \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} x < 2 \text{ and } x > 10 \\ x > 5 \end{cases}$$

Therefore, the system has solution $x > 10$.

On solving the second system, we have

$$\Rightarrow \quad \begin{cases} 0 < x^2-12x+30 < 10 \\ 1 < \dfrac{2x}{5} < 2 \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} x^2-12x+30 > 0 \text{ and } x^2-12x+20 < 0 \\ 5/2 < x < 5 \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} x < 6-\sqrt{6} \text{ and } x > 6+\sqrt{6} \text{ and } 2 < x < 10 \\ 0 < x < 5 \end{cases}$$

Therefore, the system has solution $2 < x < 6-\sqrt{6}$ combining both systems, then solution of the original inequations is

$x \in (2, 6-\sqrt{6}) \cup (10, \infty)$.

***Form* 2** Inequations of the form

	Forms		Collection of systems
(a)	$\log_{\phi(x)} f(x) > \log_{\phi(x)} g(x)$	$\Leftrightarrow$	$\begin{cases} f(x) > g(x), \\ g(x) > 0, \\ \phi(x) > 1, \end{cases} \begin{cases} f(x) < g(x) \\ f(x) > 0 \\ 0 < \phi(x) < 1 \end{cases}$
(b)	$\log_{\phi(x)} f(x) \geq \log_{\phi(x)} g(x)$	$\Leftrightarrow$	$\begin{cases} f(x) \geq g(x), \\ g(x) > 0, \\ \phi(x) > 1, \end{cases} \begin{cases} f(x) \leq g(x) \\ f(x) > 0 \\ 0 < \phi(x) < 1 \end{cases}$
(c)	$\log_{\phi(x)} f(x) < \log_{\phi(x)} g(x)$	$\Leftrightarrow$	$\begin{cases} f(x) < g(x), \\ f(x) > 0, \\ \phi(x) > 1, \end{cases} \begin{cases} f(x) > g(x) \\ g(x) > 0 \\ 0 < \phi(x) < 1 \end{cases}$
(d)	$\log_{\phi(x)} f(x) \leq \log_{\phi(x)} g(x)$	$\Leftrightarrow$	$\begin{cases} f(x) \leq g(x), \\ f(x) > 0, \\ \phi(x) > 1, \end{cases} \begin{cases} f(x) \geq g(x) \\ g(x) > 0 \\ 0 < \phi(x) < 1 \end{cases}$

Example 113. Solve the inequation $\log_{(x-3)}(2(x^2-10x+24)) \geq \log_{(x-3)}(x^2-9)$.

Sol. This inequation is equivalent to the collection of systems

$$\begin{cases} 2(x^2-10x+24) \geq x^2-9, \\ x^2-9 > 0, \\ x-3 > 1, \end{cases}$$

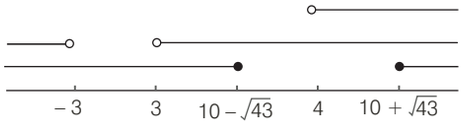
$$\begin{cases} 2(x^2-10x+24) \le x^2-9 \\ 2(x^2-10x+24) > 0 \\ 0 < x-3 < 1 \end{cases}$$

On solving the first system, we have

$$\begin{cases} x^2-20x+57 \ge 0, \\ (x+3)(x-3) > 0, \\ x > 4, \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} x \in (-\infty, 10-\sqrt{43}] \cup [10+\sqrt{43}, \infty) \\ x \in (-\infty, -3) \cup (3, \infty) \\ x \in (4, \infty) \end{cases}$$

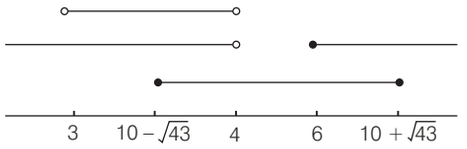
Therefore, the system has solution



$$x \ge 10+\sqrt{43}$$

i.e. $x \in [10+\sqrt{43}, \infty)$

On solving the second system, we have



$$\begin{cases} x^2-20x+57 \le 0, \\ (x-6)(x-4) > 0, \\ 3 < x < 4, \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} x \in [10-\sqrt{43}, 10+\sqrt{43}] \\ x \in (-\infty, 4) \cup (6, \infty) \\ x \in (3, 4) \end{cases}$$

Therefore, the system has solution

$$10-\sqrt{43} \le x < 4,$$

i.e., $x \in [10-\sqrt{43}, 4)$

On combining the both systems, the solution of the original inequation is

$$x \in [10-\sqrt{43}, 4) \cup [10+\sqrt{43}, \infty).$$

Exercise for Session 5

1. The equation $\sqrt{(x+1)} - \sqrt{(x-1)} = \sqrt{(4x-1)}$ has

(a) no solution (b) one solution (c) two solutions (d) more than two solutions

2. The number of real solutions of $\sqrt{(x^2-4x+3)} + \sqrt{(x^2-9)} = \sqrt{(4x^2-14x+6)}$ is

(a) one (b) two (c) three (d) None of these

3. The number of real solutions of $\sqrt{(3x^2-7x-30)} - \sqrt{(2x^2-7x-5)} = x-5$ is

(a) one (b) two (c) three (d) None of these

4. The number of integral values of x satisfying $\sqrt{(-x^2+10x-16)} < x-2$ is

(a) 0 (b) 1 (c) 2 (d) 3

5. The number of real solutions of the equation $\left(\frac{9}{10}\right)^x = -3+x-x^2$ is

(a) 2 (b) 1 (c) 0 (d) None of these

6. The set of all x satisfying $3^{2x} - 3^x - 6 > 0$ is given by

(a) $0 < x < 1$ (b) $x > 1$ (c) $x > 3^{-2}$ (d) None of these

7. The number of real solutions of the equation $2^{x/2} + (\sqrt{2}+1)^x = (3+2\sqrt{2})^{x/2}$ is

(a) one (b) two (c) four (d) infinite

8. The sum of the values of x satisfying the equation $(31+8\sqrt{15})^{x^2-3} + 1 = (32+8\sqrt{15})^{x^2-3}$ is

(a) 3 (b) 0 (c) 2 (d) None of these

9. The number of real solutions of the equation $\log_{0.5} x = |x|$ is

(a) 0 (b) 1 (c) 2 (d) None of these

10. The inequality $(x-1)\ln(2-x) < 0$ holds, if x satisfies

(a) $1 < x < 2$ (b) $x > 0$ (c) $0 < x < 1$ (d) None of these

Shortcuts and Important Results to Remember

1 '0' is neither positive nor negative even integer, '2' is the only even prime number and all other prime numbers are odd, '1' (i.e. unity) is neither a composite nor a prime number and 1, –1 are two units in the set of integers.

2 (i) If $a > 0, b > 0$ and $a < b \Rightarrow a^2 < b^2$

(ii) If $a < 0, b < 0$ and $a < b \Rightarrow a^2 > b^2$

(iii) If $a_1, a_2, a_3, \ldots, a_n \in R$

and $a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2 = 0$

$\Rightarrow \; a_1 = a_2 = a_3 = \ldots = a_n = 0$

3 (i) $\text{Max}(a, b) = \frac{1}{2}(|a + b| + |a - b|)$

(ii) $\text{Min}(a, b) = \frac{1}{2}(|a + b| - |a - b|)$

4 If the equation $f(x) = 0$ has two real roots α and β, then $f'(x) = 0$ will have a real root lying between α and β.

5 If two quadratic equations $P(x) = 0$ and $Q(x) = 0$ have an irrational common root, both roots will be common.

6 In the equation $ax^2 + bx + c = 0\ [a, b, c \in R]$, if $a + b + c = 0$, the roots are $1, \frac{c}{a}$ and if $a - b + c = 0$, the roots are -1 and $\frac{c}{a}$.

7 The condition that the roots of $ax^2 + bx + c = 0$ may be in the ratio $p : q$, is

$pq\,b^2 = ac\,(p + q)^2$ (here, $\alpha : \beta = p : q$)

i.e., $$\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} = \pm\sqrt{\frac{b^2}{ac}}$$

(i) If one root of $ax^2 + bx + c = 0$ is n times that of the other, then $nb^2 = ac\,(n + 1)^2$, here $\alpha : \beta = n : 1$.

(ii) If one root of $ax^2 + bx + c = 0$ is double of the other here $n = 2$, then $2b^2 = 9ac$.

8 If one root of $ax^2 + bx + c = 0$ is nth power of the other, then $(a^n c)^{\frac{1}{n+1}} + (ac^n)^{\frac{1}{n+1}} = -b$.

9 If one root of $ax^2 + bx + c = 0$ is square of the other, then $a^2c + ac^2 + b^3 = 3abc$.

10 If the ratio of the roots of the equation $ax^2 + bx + c = 0$ is equal to the ratio of the roots of $Ax^2 + Bx + C = 0$ and $a \neq 0, A \neq 0$, then $\frac{b^2}{ac} = \frac{B^2}{AC}$.

11 If sum of the roots is equal to sum of their squares then $2ac = ab + b^2$.

12 If sum of roots of $ax^2 + bx + c = 0$ is equal to the sum of their reciprocals, then

$2a^2c = ab^2 + bc^2$, i.e. ab^2, bc^2, ca^2 are in AP

or $\frac{2a}{b} = \frac{b}{c} + \frac{c}{a}$ i.e. $\frac{c}{a}, \frac{a}{b}, \frac{b}{c}$ are in AP.

13 Given, $y = ax^2 + bx + c$

(i) If $a > 0$, $y_{\min} = \frac{4ac - b^2}{4a}$

(ii) If $a < 0$, $y_{\max} = \frac{4ac - b^2}{4a}$

14 If α, β are the roots of $ax^2 + bx + c = 0$ and $S_n = \alpha^n + \beta^n$, then $aS_{n+1} + bS_n + c\,S_{n-1} = 0$.

15 If D_1 and D_2 are discriminants of two quadratics $P(x) = 0$ and $Q(x) = 0$, then

(i) If $D_1D_2 < 0$, then the equation $P(x) \cdot Q(x) = 0$ will have two real roots.

(ii) If $D_1D_2 > 0$, then the equation $P(x) \cdot Q(x) = 0$ has either four real roots or no real root.

(iii) If $D_1D_2 = 0$, then the equation $P(x) \cdot Q(x) = 0$ will have

(a) two equal roots and two distinct roots such that $D_1 > 0$ and $D_2 = 0$ or $D_1 = 0$ and $D_2 > 0$.

(b) only one real solution such that $D_1 < 0$ and $D_2 = 0$ or $D_1 = 0$ and $D_2 < 0$.

16 If $a > 0$ and $x = \sqrt{a + \sqrt{a + \sqrt{a + \ldots + \infty}}}$, then $x = \frac{1 + \sqrt{(4a + 1)}}{2}$.

17 If $a_1, a_2, a_3, \ldots, a_n$ are positive real numbers, then least value of $(a_1 + a_2 + a_3 + \ldots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots + \frac{1}{a_n}\right)$ is n^2.

(i) Least value of $(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = 3^2 = 9$

(ii) Least value of

$$(a + b + c + d)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right) = 4^2 = 16$$

18 **Law of Proportions** If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots$, then each of these ratios is also equal to

(i) $\frac{a + c + e + \ldots}{b + d + f + \ldots}$

(ii) $\left(\frac{pa^n + qc^n + re^n + \ldots}{pb^n + qd^n + rf^n + \ldots}\right)^{1/n}$ (where, $p, q, r, \ldots, n \in R$)

(iii) $\frac{\sqrt{ac}}{\sqrt{bd}} = \frac{\sqrt[n]{(ace \ldots)}}{\sqrt[n]{(bdf \ldots)}}$

19 **Lagrange's Mean Value Theorem** Let $f(x)$ be a function defined on $[a, b]$ such that

contd...

(i) $f(x)$ is continuous on $[a, b]$ and

(ii) $f(x)$ is derivable on (a, b), then $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

20 Lagrange's Identity If $a_1, a_2, a_3, b_1, b_2, b_3 \in R$, then

$(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2 b_2 + a_3b_3)^2$

$= (a_1b_2 - a_2 b_1)^2 + (a_2 b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2$

or $(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2 b_2 + a_3b_3)^2$

$$= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2 + \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}^2$$

Remark

If $(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) \leq (a_1b_1 + a_2b_2 + a_3b_3)^2$,

then $$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}.$$

21 Horner's Method of Synthetic, Division When, we divide a polynomial of degree ≥ 1 by a linear monic polynomial, the quotient and remainder can be found by this method. Consider

$f(x) = a_0 x^n + a_1x^{n-1} + a_2 x^{n-2} + \ldots + a_n$

where $a_0 \neq 0$ and $a_0, a_1, a_2, \ldots, a_n \in R$.

Let $g(x) = (x - \alpha)$ be a linear monic polynomial $\alpha \in R$.

When $g(x) \mid f(x)$; we can find quotient and remainder as follows :

α	a_0	a_1	a_2	$\ldots$	a_n
	0	αa_0	$b_1\alpha$		αb_{n-1}
		a_1	a_2		$a_n + \alpha b_{n-1} = 0$
	a_0	$+ \alpha a_0$	$+ b_1\alpha$		
	$= b_0$	$= b_1$	$= b_2$		

$\therefore \quad f(x) = (x - \alpha)(b_0 x^{n-1} + b_1 x^{n-2} + b_2 x^{n-3} + \ldots + b_{n-1})$

e.g. Find all roots of $x^3 - 6x^2 + 11x - 6 = 0$.

$\because (x - 1)$ is a factor of $x^3 - 6x^2 + 11x - 6$, then

$x = 1$	1	–6	11	–6
	0	1	–5	6
	1	–5	6	0

$\therefore \quad x^3 - 6x^2 + 11x - 6 = (x - 1)(x^2 - 5x + 6)$

$= (x - 1)(x - 2)(x - 3)$

Hence, roots of $x^3 - 6x^2 + 11x - 6 = 0$ are 1, 2 and 3.

JEE Type Solved Examples :
Single Option Correct Type Questions

▪ This section contains **10 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

● ***Ex. 1*** *If* α *and* $\beta(\alpha < \beta)$*, are the roots of the equation* $x^2 + bx + c = 0$*, where* $c < 0 < b$*, then*

(a) $0 < \alpha < \beta$ (b) $\alpha < 0 < \beta < |\alpha|$

(c) $\alpha < \beta < 0$ (d) $\alpha < 0 < |\alpha| < \beta$

Sol. (*b*) $\because$ $\alpha + \beta = -b,\ \alpha\beta = c$ ...(i)

$\because$ $c < 0 \Rightarrow \alpha\beta < 0$

Let $\alpha < 0, \beta > 0$

$\therefore$ $|\alpha| = -\alpha$ and $\alpha < 0 < \beta$ [$\because \alpha < \beta$] ...(ii)

From Eq. (i), we get $-|\alpha| + \beta < 0$

$\Rightarrow$ $\beta < |\alpha|$ ...(iii)

From Eqs. (ii) and (iii), we get

$\alpha < 0 < \beta < |\alpha|$

● ***Ex. 2*** *Let* α, β *be the roots of the equation* $x^2 - x + p = 0$ *and* γ, δ *be the roots of the equation* $x^2 - 4x + q = 0$*. If* α, β, γ *and* δ *are in GP, the integral values of p and q respectively, are*

(a) $-2, -32$ (b) $-2, 3$

(c) $-6, 3$ (d) $-6, -32$

Sol. (*a*) Let r be the common ratio of the GP, then

$\beta = \alpha r, \gamma = \alpha r^2$ and $\delta = \alpha r^3$

$\therefore$ $\alpha + \beta = 1 \Rightarrow \alpha + \alpha r = 1$

or $\alpha(1 + r) = 1$ ...(i)

and $\alpha\beta = p \Rightarrow \alpha(\alpha r) = p$

or $\alpha^2 r = p$ ...(ii)

and $\gamma + \delta = 4 \Rightarrow \alpha r^2 + \alpha r^3 = 4$

or $\alpha r^2(1 + r) = 4$ ...(iii)

and $\gamma\delta = q$

$\Rightarrow$ $(\alpha r^2)(\alpha r^3) = q$

or $\alpha^2 r^5 = q$ ...(iv)

On dividing Eq. (iii) by Eq. (i), we get

$r^2 = 4 \Rightarrow r = -2, 2$

If we take $r = 2$, then α is not integer, so we take $r = -2$.

On substituting $r = -2$ in Eq. (i), we get $\alpha = -1$

Now, from Eqs. (ii) and (iv), we get

$p = \alpha^2 r = (-1)^2(-2) = -2$

and $q = \alpha^2 r^5 = (-1)^2(-2)^5 = -32$

Hence, $(p, q) = (-2, -32)$

● ***Ex. 3*** *Let* $f(x) = \int_1^x \sqrt{(2 - t^2)}\, dt$*, the real roots of the equation* $x^2 - f'(x) = 0$ *are*

(a) ± 1 (b) $\pm \frac{1}{\sqrt{2}}$

(c) $\pm \frac{1}{2}$ (d) 0 and 1

Sol. (*a*) We have, $f(x) = \int_1^x \sqrt{(2 - t^2)}\, dt$

$\Rightarrow$ $f'(x) = \sqrt{(2 - x^2)}$

$\therefore$ $x^2 - f'(x) = 0$

$\Rightarrow$ $x^2 - \sqrt{(2 - x^2)} = 0 \Rightarrow x^4 + x^2 - 2 = 0$

$\Rightarrow$ $x^2 = 1, -2$

$\Rightarrow$ $x = \pm 1$ [only for real value of x]

● ***Ex. 4*** *If* $x^2 + 3x + 5 = 0$ *and* $ax^2 + bx + c = 0$ *have a common root and a, b,* $c \in N$*, the minimum value of* $a + b + c$ *is*

(a) 3 (b) 9

(c) 6 (d) 12

Sol. (*b*) $\because$ Roots of the equation $x^2 + 3x + 5 = 0$ are non-real.

Thus, given equations will have two common roots.

$\Rightarrow$ $\frac{a}{1} = \frac{b}{3} = \frac{c}{5} = \lambda$ [say]

$\therefore$ $a + b + c = 9\lambda$

Thus, minimum value of $a + b + c = 9$ [$\because a, b, c \in N$]

● ***Ex. 5*** *If* $x_1, x_2, x_3, \ldots, x_n$ *are the roots of the equation* $x^n + ax + b = 0$*, the value of* $(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)\ldots(x_1 - x_n)$ *is*

(a) $nx_1 + b$

(b) $n(x_1)^{n-1}$

(c) $n(x_1)^{n-1} + a$

(d) $n(x_1)^{n-1} + b$

Sol. (*c*) $\because x^n + ax + b = (x - x_1)(x - x_2)(x - x_3)\ldots(x - x_n)$

$\Rightarrow (x - x_2)(x - x_3)\ldots(x - x_n) = \frac{x^n + ax + b}{x - x_1}$

On taking $\lim_{x \to x_1}$ both sides, we get

$$(x_1 - x_2)(x_1 - x_3)\ldots(x_1 - x_n) = \lim_{x \to x_1} \frac{x^n + ax + b}{x - x_1} \left[\frac{0}{0}\text{form}\right]$$

$$= \lim_{x \to x_1} \frac{nx^{n-1} + a}{1} = n(x_1)^{n-1} + a$$

● **Ex. 6** *If α, β are the roots of the equation $ax^2 + bx + c = 0$ and $A_n = \alpha^n + \beta^n$, then $a A_{n+2} + bA_{n+1} + cA_n$ is equal to*

(a) 0 (b) 1 (c) $a + b + c$ (d) abc

Sol. (*a*) $\because \alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

$$\therefore \quad A_{n+2} = \alpha^{n+2} + \beta^{n+2}$$
$$= (\alpha + \beta)(\alpha^{n+1} + \beta^{n+1}) - \alpha\beta^{n+1} - \beta\alpha^{n+1}$$
$$= (\alpha + \beta)(\alpha^{n+1} + \beta^{n+1}) - \alpha\beta(\alpha^n + \beta^n)$$
$$= -\frac{b}{a}A_{n+1} - \frac{c}{a}A_n$$
$$\Rightarrow \quad aA_{n+2} + bA_{n+1} + cA_n = 0$$

● **Ex. 7** *If x and y are positive integers such that $xy + x + y = 71$, $x^2y + xy^2 = 880$, then $x^2 + y^2$ is equal to*

(a) 125 (b) 137 (c) 146 (d) 152

Sol. (*c*) $\because \; xy + x + y = 71 \;\Rightarrow\; xy + (x + y) = 71$

and $x^2y + xy^2 = 880 \;\Rightarrow\; xy(x + y) = 880$

$\Rightarrow xy$ and $(x + y)$ are the roots of the quadratic equation.

$$t^2 - 71t + 880 = 0$$
$$\Rightarrow \quad (t - 55)(t - 16) = 0$$
$$\Rightarrow \quad t = 55, 16$$
$$\therefore \quad x + y = 16 \text{ and } xy = 55$$

So, $x^2 + y^2 = (x + y)^2 - 2xy = (16)^2 - 110 = 146$

● **Ex. 8** *If α, β are the roots of the equation $x^2 - 3x + 5 = 0$ and γ, δ are the roots of the equation $x^2 + 5x - 3 = 0$, then the equation whose roots are $\alpha\gamma + \beta\delta$ and $\alpha\delta + \beta\gamma$, is*

(a) $x^2 - 15x - 158 = 0$ (b) $x^2 + 15x - 158 = 0$

(c) $x^2 - 15x + 158 = 0$ (d) $x^2 + 15x + 158 = 0$

Sol. (*d*) $\because \; \alpha + \beta = 3, \alpha\beta = 5, \gamma + \delta = (-5), \gamma\delta = (-3)$

Sum of roots $= (\alpha\gamma + \beta\delta) + (\alpha\delta + \beta\gamma)$
$$= (\alpha + \beta)(\gamma + \delta) = 3 \times (-5) = (-15)$$

Product of roots $= (\alpha\gamma + \beta\delta)(\alpha\delta + \beta\gamma)$
$$= \alpha^2\gamma\delta + \alpha\beta\gamma^2 + \beta\alpha\delta^2 + \beta^2\gamma\delta$$
$$= \gamma\delta(\alpha^2 + \beta^2) + \alpha\beta(\gamma^2 + \delta^2)$$
$$= -3(\alpha^2 + \beta^2) + 5(\gamma^2 + \delta^2)$$
$$= -3[(\alpha + \beta)^2 - 2\alpha\beta] + 5[(\gamma + \delta)^2 - 2\gamma\delta]$$
$$= -3[9 - 10] + 5[25 + 6] = 158$$

$\therefore$ Required equation is $x^2 + 15x + 158 = 0$.

● **Ex. 9** *The number of roots of the equation $\frac{1}{x} + \frac{1}{\sqrt{(1 - x^2)}} = \frac{35}{12}$ is*

(a) 0 (b) 1

(c) 2 (d) 3

Sol. (*d*) Let $\frac{1}{x} = u$ and $\frac{1}{\sqrt{(1 - x^2)}} = v$, then

$$u + v = \frac{35}{12} \text{ and } u^2 + v^2 = u^2v^2$$
$$\Rightarrow \quad (u + v)^2 = \left(\frac{35}{12}\right)^2$$
$$\Rightarrow \quad u^2 + v^2 + 2uv = \left(\frac{35}{12}\right)^2$$
$$\Rightarrow \quad u^2v^2 + 2uv = \left(\frac{35}{12}\right)^2 \qquad [\because u^2 + v^2 = u^2v^2]$$
$$\Rightarrow \quad u^2v^2 + 2uv - \left(\frac{35}{12}\right)^2 = 0$$
$$\Rightarrow \quad \left(uv + \frac{49}{12}\right)\left(uv - \frac{25}{12}\right) = 0$$
$$\Rightarrow \quad uv = -\frac{49}{12}, \; uv = \frac{25}{12}$$

***Case* I** If $uv = -\frac{49}{12}$, then

$$\frac{1}{x} \cdot \frac{1}{\sqrt{(1 - x^2)}} = -\frac{49}{12} \qquad [\text{here } x < 0]$$
$$\Rightarrow \quad x^4 - x^2 + \frac{(12)^2}{(49)^2} = 0$$
$$\Rightarrow \quad x = -\frac{(5 + \sqrt{73})}{14}$$

***Case* II** If $uv = \frac{25}{12}$, then

$$\frac{1}{x} \cdot \frac{1}{\sqrt{(1 - x^2)}} = \frac{25}{12} \qquad [\text{here } x > 0]$$
$$\Rightarrow \quad x^4 - x^2 + \frac{(12)^2}{(25)^2} = 0$$
$$\Rightarrow \quad \left(x^2 - \frac{9}{25}\right)\left(x^2 - \frac{16}{25}\right) = 0 \;\Rightarrow\; x = \frac{3}{5}, \frac{4}{5}$$

On combining both cases,

$$x = -\frac{(5 + \sqrt{73})}{14}, \frac{3}{5}, \frac{4}{5}$$

Hence, number of roots = 3

● **Ex. 10** *The sum of the roots of the equation $2^{33x-2} + 2^{11x+2} = 2^{22x+1} + 1$ is*

(a) $\frac{1}{11}$ (b) $\frac{2}{11}$ (c) $\frac{3}{11}$ (d) $\frac{4}{11}$

Sol. (*b*) Let $2^{11x} = t$, given equation reduces to

$$\frac{t^3}{4} + 4t = 2t^2 + 1$$
$$\Rightarrow \quad t^3 - 8t^2 + 16t - 4 = 0 \;\Rightarrow\; t_1 \cdot t_2 \cdot t_3 = 4$$
$$\Rightarrow \quad 2^{11x_1} \cdot 2^{11x_2} \cdot 2^{11x_3} = 4 \;\Rightarrow\; 2^{11(x_1 + x_2 + x_3)} = 2^2$$
$$\Rightarrow \quad 11(x_1 + x_2 + x_3) = 2$$
$$\therefore \quad x_1 + x_2 + x_3 = \frac{2}{11}$$

JEE Type Solved Examples :
More than One Correct Option Type Questions

- This section contains **5 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of **which more than one** may be correct.

● ***Ex. 11*** *For the equation* $2x^2 - 6\sqrt{2}x - 1 = 0$

(a) roots are rational

(b) roots are irrational

(c) if one root is $(p + \sqrt{q})$, the other is $(-p + \sqrt{q})$

(d) if one root is $(p + \sqrt{q})$, the other is $(p - \sqrt{q})$

Sol. (b,c) As the coefficients are not rational, irrational roots need not appear in conjugate pair.

Here, $\alpha + \beta = 3\sqrt{2}$ and $\alpha\beta = -\frac{1}{2}$

Let $\alpha = p + \sqrt{q}$, then prove that other root $\beta = -p + \sqrt{q}$.

● ***Ex. 12*** *Given that* α, γ *are roots of the equation* $Ax^2 - 4x + 1 = 0$ *and* β, δ *the roots of the equation* $Bx^2 - 6x + 1 = 0$, *such that* α, β, γ *and* δ *are in HP then*

(a) $A = 3$ (b) $A = 4$ (c) $B = 2$ (d) $B = 8$

Sol. (a,d) Since, α, β, γ and δ are in HP, hence $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ and $\frac{1}{\delta}$ are in AP and they may be taken as $a - 3d$, $a - d$, $a + d$ and $a + 3d$. Replacing x by $\frac{1}{x}$, we get the equation whose roots are $a - 3d$, $a + d$ is $x^2 - 4x + A = 0$ and equation whose roots are $a - d$, $a + 3d$ is $x^2 - 6x + B = 0$, then

$$(a - 3d) + (a + d) = 4 \Rightarrow 2(a - d) = 4$$

and $$(a - d) + (a + 3d) = 6 \Rightarrow 2(a + d) = 6$$

$$\therefore \quad a = \frac{5}{2} \text{ and } d = \frac{1}{2}$$

Now, $$A = (a - 3d)(a + d) = \left(\frac{5}{2} - \frac{3}{2}\right)\left(\frac{5}{2} + \frac{1}{2}\right) = 3$$

and $$B = (a - d)(a + 3d) = \left(\frac{5}{2} - \frac{1}{2}\right)\left(\frac{5}{2} + \frac{3}{2}\right) = 8$$

● ***Ex. 13*** *If* $|ax^2 + bx + c| \le 1$ *for all x in [0, 1], then*

(a) $|a| \le 8$ (b) $|b| > 8$

(c) $|c| \le 1$ (d) $|a| + |b| + |c| \le 17$

Sol. (a,c,d) On putting $x = 0, 1$ and $\frac{1}{2}$, we get

$$|c| \le 1 \quad \ldots(i)$$

$$|a + b + c| \le 1 \quad \ldots(ii)$$

and $$|a + 2b + 4c| \le 4 \quad \ldots(iii)$$

From Eqs. (i), (ii) and (iii), we get

$$|b| \le 8 \text{ and } |a| \le 8$$

$$\Rightarrow \quad |a| + |b| + |c| \le 17$$

● ***Ex. 14*** *If* $\cos^4\theta + p$, $\sin^4\theta + p$ *are the roots of the equation* $x^2 + a(2x + 1) = 0$ *and* $\cos^2\theta + q$, $\sin^2\theta + q$ *are the roots of the equation* $x^2 + 4x + 2 = 0$ *then a is equal to*

(a) -2 (b) -1 (c) 1 (d) 2

Sol. (b,d)

$$\because \quad \cos^4\theta - \sin^4\theta = \cos 2\theta$$

$$\Rightarrow \quad \cos^4\theta - \sin^4\theta = \cos^2\theta - \sin^2\theta$$

$$\Rightarrow \quad (\cos^4\theta + p) - (\sin^4\theta + p) = (\cos^2\theta + q) - (\sin^2\theta + q)$$

$$\Rightarrow \quad \frac{\sqrt{4a^2 - 4a}}{1} = \frac{\sqrt{16 - 8}}{1} \quad \left[\because \alpha - \beta = \frac{\sqrt{D}}{a}\right]$$

$$\Rightarrow \quad 4a^2 - 4a = 8 \text{ or } a^2 - a - 2 = 0$$

or $$(a - 2)(a + 1) = 0 \text{ or } a = 2, -1$$

● ***Ex. 15*** *If* α, β, γ *are the roots of* $x^3 - x^2 + ax + b = 0$ *and* β, γ, δ *are the roots of* $x^3 - 4x^2 + mx + n = 0$. *If* α, β, γ *and* δ *are in AP with common difference d then*

(a) $a = m$ (b) $a = m - 5$

(c) $n = b - a - 2$ (d) $b = m + n - 3$

Sol. (b,c,d)

$\because$ a, β, γ, δ are in AP with common difference d, then

$$\beta = \alpha + d, \ \gamma = \alpha + 2d \text{ and } \delta = \alpha + 3d \quad \ldots(i)$$

Given, a, β, γ are the roots of $x^3 - x^2 + ax + b = 0$, then

$$\alpha + \beta + \gamma = 1 \quad \ldots(ii)$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = a \quad \ldots(iii)$$

$$\alpha\beta\gamma = -b \quad \ldots(iv)$$

Also, β, γ, δ are the roots of $x^3 - 4x^2 + mx + n = 0$, then

$$\beta + \gamma + \delta = 4 \quad \ldots(v)$$

$$\beta\gamma + \gamma\delta + \delta\beta = m \quad \ldots(vi)$$

$$\beta\gamma\delta = -n \quad \ldots(vii)$$

From Eqs. (i) and (ii), we get

$$3\alpha + 3d = 1 \quad \ldots(viii)$$

and from Eqs. (i) and (v), we get

$$3\alpha + 6d = 4 \quad \ldots(ix)$$

From Eqs. (viii) and (ix), we get

$$d = 1, \alpha = -\frac{2}{3}$$

Now, from Eq. (i), we get

$$\beta = \frac{1}{3}, \gamma = \frac{4}{3} \text{ and } \delta = \frac{7}{3}$$

From Eqs. (iii), (iv), (vi) and (vii), we get

$$a = -\frac{2}{3}, b = \frac{8}{27}, m = \frac{13}{3}, n = -\frac{28}{27}$$

$$\therefore \quad a = m - 5, n = b - a - 2 \text{ and } b = m + n - 3$$

JEE Type Solved Examples :
Passage Based Questions

- This section contains **2 solved passages** based upon each of the passage **3 multiple choice** examples have to be answered. Each of these examples has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I
(Ex. Nos. 16 to 18)

If G and L are the greatest and least values of the expression $\frac{x^2-x+1}{x^2+x+1}, x \in R$ respectively, then

16. The least value of $G^5 + L^5$ is

(a) 0 (b) 2 (c) 16 (d) 32

Sol. (*b*) Let $y = \frac{x^2-x+1}{x^2+x+1}$

$\Rightarrow \quad x^2y + xy + y = x^2 - x + 1$

$\Rightarrow \quad (y-1)x^2 + (y+1)x + y - 1 = 0 \qquad [\because x \in R]$

$\therefore \quad (y+1)^2 - 4\cdot(y-1)(y-1) \geq 0 \qquad [\because b^2 - 4ac \geq 0]$

$\Rightarrow \quad (y+1)^2 - (2y-2)^2 \geq 0$

$\Rightarrow \quad (3y-1)(y-3) \leq 0$

$\therefore \quad \frac{1}{3} \leq y \leq 3 \Rightarrow G = 3$ and $L = \frac{1}{3} \quad \therefore \; GL = 1$

$$\frac{G^5 + L^5}{2} \geq (GL)^{1/5} = (1)^{1/5} = 1$$

$\Rightarrow \quad \frac{G^5 + L^5}{2} \geq 1$ or $G^5 + L^5 \geq 2$

$\therefore$ Minimum value of $G^5 + L^5$ is 2.

17. G and L are the roots of the equation

(a) $3x^2 - 10x + 3 = 0$ (b) $4x^2 - 17x + 4 = 0$

(c) $x^2 - 7x + 10 = 0$ (d) $x^2 - 5x + 6 = 0$

Sol. (*a*) Equation whose roots are G and L, is

$$x^2 - (G+L)x + GL = 0$$

$\Rightarrow \quad x^2 - \frac{10}{3}x + 1 = 0$ or $3x^2 - 10x + 3 = 0$

18. If $L < \lambda < G$ and $\lambda \in N$, the sum of all values of λ is

(a) 2 (b) 3 (c) 4 (d) 5

Sol. (*b*) $\because L < \lambda < G \Rightarrow \frac{1}{3} < \lambda < 3 \quad \therefore \; \lambda = 1, 2$

Sum of values of $\lambda = 1 + 2 = 3$

Passage II
(Ex. Nos. 19 to 21)

Let a, b, c and d are real numbers in GP. Suppose u, v, w satisfy the system of equations $u + 2v + 3w = 6, 4u + 5v + 6w = 12$ and $6u + 9v = 4$. Further, consider the expressions

$$f(x) = \left(\frac{1}{u} + \frac{1}{v} + \frac{1}{w}\right)x^2 + [(b-c)^2 + (c-a)^2 + (d-b)^2]$$

$x + u + v + w = 0$ *and* $g(x) = 20x^2 + 10(a-d)^2 x - 9 = 0$.

19. $(b-c)^2 + (c-a)^2 + (d-b)^2$ is equal to

(a) $a - d$ (b) $(a-d)^2$ (c) $a^2 - d^2$ (d) $(a+d)^2$

Sol. (*b*) Let $b = ar, c = ar^2$ and $d = ar^3$

Now, $(b-c)^2 + (c-a)^2 + (d-b)^2$

$= (ar - ar^2)^2 + (ar^2 - a)^2 + (ar^3 - ar)^2$

$= a^2r^2(1-r)^2 + a^2(r^2-1)^2 + a^2r^2(r^2-1)^2$

$= a^2(1-r)^2\{r^2 + (r+1)^2 + r^2(r+1)^2\}$

$= a^2(1-r)^2(r^4 + 2r^3 + 3r^2 + 2r + 1)$

$= a^2(1-r)^2(1 + r + r^2)^2 = a^2(1-r^3)^2$

$= (a - ar^3)^2 = (a-d)^2$

20. $(u + v + w)$ is equal to

(a) 2 (b) $\frac{1}{2}$ (c) 20 (d) $\frac{1}{20}$

Sol. (*a*) Now, $u + 2v + 3w = 6$ ...(i)

$4u + 5v + 6w = 12$ ...(ii)

and $6u + 9v = 4$ ...(iii)

From Eqs. (i) and (ii), we get

$2u + v = 0$ ...(iv)

Solving Eqs. (iii) and (iv), we get

$u = -\frac{1}{3}, v = \frac{2}{3}$

Now, from Eq. (i), we get $w = \frac{5}{3}$

$\therefore \quad v + u + w = -\frac{1}{3} + \frac{2}{3} + \frac{5}{3} = 2$

21. If roots of $f(x) = 0$ be α, β, the roots of $g(x) = 0$ will be

(a) α, β (b) $-\alpha, -\beta$ (c) $\frac{1}{\alpha}, \frac{1}{\beta}$ (d) $-\frac{1}{\alpha}, -\frac{1}{\beta}$

Sol. (*c*) Now, $f(x) = \left(\frac{1}{u} + \frac{1}{v} + \frac{1}{w}\right)x^2 + [(b-c)^2 + (c-a)^2 + (d-b)^2]x + u + v + w = 0$

$\Rightarrow \quad f(x) = -\frac{9}{10}x^2 + (a-d)^2 x + 2 = 0$

$\Rightarrow \quad f(x) = -9x^2 + 10(a-d)^2 x + 20 = 0$ ...(v)

Given, roots of $f(x) = 0$ are α and β.

Now, replace x by $\frac{1}{x}$ in Eq. (v), then

$$\frac{-9}{x^2} + \frac{10(a-d)^2}{x} + 20 = 0$$

$\Rightarrow \quad 20x^2 + 10(a-d)^2 x - 9 = 0$

$g(x) = 0$

$\therefore \quad$ Roots of $g(x) = 0$ are $\frac{1}{\alpha}, \frac{1}{\beta}$.

JEE Type Solved Examples :
Single Integer Answer Type Questions

■ This section contains **2 examples.** The answer to each example is a **single digit integer** ranging from **0** to **9** (both inclusive).

● ***Ex. 22*** *If the roots of the equation* $10x^3 - cx^2 - 54x - 27 = 0$ *are in harmonic progression, the value of c is*

Sol. (9) Given, roots of the equation

$$10x^3 - cx^2 - 54x - 27 = 0 \text{ are in HP.} \quad \ldots(i)$$

Now, replacing x by $\frac{1}{x}$ in Eq. (i), we get

$$27x^3 + 54x^2 + cx - 10 = 0 \quad \ldots(ii)$$

Hence, the roots of Eq. (ii) are in AP.

Let $a - d$, a and $a + d$ are the roots of Eq. (ii).

Then, $$a - d + a + a + d = -\frac{54}{27}$$

$$\Rightarrow \quad a = -\frac{2}{3} \quad \ldots(iii)$$

Since, a is a root of Eq. (ii), then

$$27a^3 + 54a^2 + ca - 10 = 0$$

$$\Rightarrow \quad 27\left(-\frac{8}{27}\right) + 54\left(\frac{4}{9}\right) + c\left(-\frac{2}{3}\right) - 10 = 0 \quad \text{[from Eq. (iii)]}$$

$$\Rightarrow \quad 6 - \frac{2c}{3} = 0 \quad \text{or} \quad c = 9$$

● ***Ex. 23*** *If a root of the equation* $n^2 \sin^2 x - 2\sin x - (2n+1) = 0$ *lies in* $[0, \pi/2]$, *the minimum positive integer value of n is*

Sol. (3) $\because \; n^2 \sin^2 x - 2\sin x - (2n+1) = 0$

$$\Rightarrow \quad \sin x = \frac{2 \pm \sqrt{4 + 4n^2(2n+1)}}{2n^2}$$

[by Shridharacharya method]

$$= \frac{1 \pm \sqrt{(2n^3 + n^2 + 1)}}{n^2}$$

$$\because \quad 0 \le \sin x \le 1 \quad [\because x \in [0, \pi/2]]$$

$$\Rightarrow \quad 0 \le \frac{1 + \sqrt{(2n^3 + n^2 + 1)}}{n^2} \le 1$$

$$\Rightarrow \quad 0 \le 1 + \sqrt{(2n^3 + n^2 + 1)} \le n^2$$

$$\Rightarrow \sqrt{(2n^3 + n^2 + 1)} \le (n^2 - 1) \quad [\because n > 1]$$

On squaring both sides, we get

$$2n^3 + n^2 + 1 \le n^4 - 2n^2 + 1$$

$$\Rightarrow \quad n^4 - 2n^3 - 3n^2 \ge 0$$

$$\Rightarrow \quad n^2 - 2n - 3 \ge 0 \quad \Rightarrow \quad (n-3)(n+1) \ge 0$$

$$\Rightarrow \quad n \ge 3$$

$$\therefore \quad n = 3, 4, 5, \ldots$$

Hence, the minimum positive integer value of n is 3.

JEE Type Solved Examples :
Matching Type Questions

■ This section contains **2 examples.** Examples 24 and 25 have three statements (A, B and C) given in Column I and four statements (p, q, r and s) in **Column II.** Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II.**

● ***Ex. 24*** *Column* I *contains rational algebraic expressions and Column* II *contains possible integers which lie in their range. Match the entries of Column* I *with one or more entries of the elements of Column* II.

Column I		Column II	
(A)	$y = \frac{x^2 - 2x + 9}{x^2 + 2x + 9}, x \in R$	(p)	1
(B)	$y = \frac{x^2 - 3x - 2}{2x - 3}, x \in R$	(q)	3
(C)	$y = \frac{2x^2 - 2x + 4}{x^2 - 4x + 3}, x \in R$	(r)	−4
		(s)	−9

Sol. (A) → (p); (B) → (p, q, r, s); (C) → (p, q, s)

(A) $y = \frac{x^2 - 2x + 9}{x^2 + 2x + 9} \quad \Rightarrow \; x^2y + 2xy + 9y = x^2 - 2x + 9$

$$\Rightarrow \quad (y-1)x^2 + 2x(y+1) + 9(y-1) = 0$$

$$\because \quad x \in R$$

$$\therefore \quad 4(y+1)^2 - 4 \cdot 9 \cdot (y-1)^2 \ge 0$$

$$\Rightarrow \quad (y+1)^2 - (3y-3)^2 \ge 0$$

$$\Rightarrow \quad (4y-2)(-2y+4) \ge 0$$

$$\Rightarrow \quad (2y-1)(y-2) \le 0$$

$$\therefore \quad \frac{1}{2} \le y \le 2 \quad \Rightarrow \quad y = 1, 2 \text{ (p)}$$

(B) $\because \quad y = \frac{x^2 - 3x - 2}{2x - 3} \quad \Rightarrow \; 2xy - 3y = x^2 - 3x - 2$

$$\Rightarrow \quad x^2 - x(3 + 2y) + 3y - 2 = 0 \quad \because \quad x \in R$$

$$\therefore \quad (3 + 2y)^2 - 4 \cdot 1 \cdot (3y - 2) \ge 0$$

$$\Rightarrow \quad 4y^2 + 17 \ge 0$$

$$\therefore \quad y \in R \text{ (p, q, r, s)}$$

(C) $\because \quad y = \dfrac{2x^2 - 2x + 4}{x^2 - 4x + 3}$

$\Rightarrow \quad x^2y - 4xy + 3y = 2x^2 - 2x + 4$

$\Rightarrow \quad x^2(y-2) + 2x(1-2y) + 3y - 4 = 0$

$\because \quad x \in R$

$\therefore \quad 4(1-2y)^2 - 4(y-2)(3y-4) \geq 0$

$\Rightarrow \quad (4y^2 - 4y + 1) - (3y^2 - 10y + 8) \geq 0$

$\Rightarrow \quad y^2 + 6y - 7 \geq 0$

$\Rightarrow \quad (y+7)(y-1) \geq 0$

$\therefore \quad y \leq -7 \text{ or } y \geq 1 (p, q, s)$

● ***Ex. 25*** *Entries of Column* I *are to be matched with one or more entries of Column* II.

	Column I		Column II
(A)	If $a + b + 2c = 0$ but $c \neq 0$, then $ax^2 + bx + c = 0$ has	(p)	atleast one root in $(-2, 0)$
(B)	If $a, b, c \in R$ such that $2a - 3b + 6c = 0$, then equation has	(q)	atleast one root in $(-1, 0)$
(C)	Let a, b, c be non-zero real numbers such that	(r)	atleast one root in $(-1, 1)$
	$\int_0^1 (1 + \cos^8 x)(ax^2 + bx + c)dx = \int_0^2 (1 + \cos^8 x)(ax^2 + bx + c)dx$, the equation $ax^2 + bx + c = 0$ has	(s)	atleast one root in $(0, 1)$
		(t)	atleast one root in $(0, 2)$

Sol. (A) $\rightarrow$ (r, s, t); (B) $\rightarrow$ (p, q, r); (C) $\rightarrow$ (r, s, t)

(A) Let $f(x) = ax^2 + bx + c$

Then, $f(1) = a + b + c = -c \quad [\because a + b + 2c = 0]$

and $f(0) = c$

$\therefore \quad f(0)\, f(1) = -c^2 < 0 \quad [\because c \neq 0]$

$\therefore$ Equation $f(x) = 0$ has a root in $(0, 1)$.

$\therefore \quad f(x)$ has a root in $(0, 2)$ as well as in $(-1, 1)$ (r)

(B) Let $f'(x) = ax^2 + bx + c$

$\therefore \quad f(x) = \dfrac{ax^3}{3} + \dfrac{bx^2}{2} + cx + d$

$\therefore \quad f(0) = d$

and $f(-1) = -\dfrac{a}{3} + \dfrac{b}{2} + c + d = -\left(\dfrac{2a - 3b + 6c}{6}\right) + d$

$= 0 + d = d \quad [\because 2a - 3b + 6c = 0]$

Hence, $f(0) = f(-1)$

Hence, $f'(x) = 0$ has atleast one root in $(-1, 0)$ (q)

$\therefore \quad f(x) = 0$ has a root in $(-2, 0)$ (p) as well as $(-1, 1)$ (r)

(C) Let $f(x) = \int (1 + \cos^8 x)(ax^2 + bx + c)dx$

Given, $f(1) - f(0) = f(2) - f(0)$

$\Rightarrow \quad f(1) = f(2)$

$\Rightarrow \quad f'(x) = 0$ has atleast one root in $(0, 1)$.

$\Rightarrow (1 + \cos^8 x)(ax^2 + bx + c) = 0$ has atleast one root in $(0, 1)$.

$\Rightarrow \quad ax^2 + bx + c = 0$ has atleast one root in $(0, 1)$ (s)

$\therefore \quad ax^2 + bx + c = 0$ has a root in $(0, 2)$ (t) as well as in $(-1, 1)$ (r)

JEE Type Solved Examples :
Statement I and II Type Questions

■ **Directions** Example numbers 26 and 27 are Assertion-Reason type examples. Each of these examples contains two statements:
Statement-1 (Assertion) and **Statement-2** (Reason)
Each of these examples also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1

(c) Statement-1 is true, Statement-2 is false

(d) Statement-1 is false, Statement-2 is true

● ***Ex. 26*** **Statement 1** *Roots of* $x^2 - 2\sqrt{3}x - 46 = 0$ *are rational.*

Statement 2 *Discriminant of* $x^2 - 2\sqrt{3}x - 46 = 0$ *is a perfect square.*

Sol. (*d*) In $ax^2 + bx + c = 0, a, b, c \in Q$

[here Q is the set of rational number]

If $D > 0$ and is a perfect square, then roots are real, distinct and rational.

But, here $2\sqrt{3} \notin Q$

$\therefore$ Roots are not rational.

Here, roots are $\dfrac{2\sqrt{3} \pm \sqrt{(12 + 184)}}{2}$

i.e. $\sqrt{3} \pm 7$. [irrational]

But $D = 12 + 184 = 196 = (14)^2$

$\therefore$ Statement-1 is false and Statement-2 is true.

● ***Ex. 27*** **Statement 1** *The equation* $a^x + b^x + c^x - d^x = 0$ *has only one real root, if* $a > b > c > d$.

Statement 2 *If* $f(x)$ *is either strictly increasing or decreasing function, then* $f(x) = 0$ *has only one real root.*

Sol. (*c*) $\because \quad a^x + b^x + c^x - d^x = 0$

$\Rightarrow \quad a^x + b^x + c^x = d^x$

Let $f(x)=\left(\frac{a}{d}\right)^x+\left(\frac{b}{d}\right)^x+\left(\frac{c}{d}\right)^x-1$

$\therefore \quad f'(x)=\left(\frac{a}{d}\right)^x\ln\left(\frac{a}{d}\right)+\left(\frac{b}{d}\right)^x\ln\left(\frac{b}{d}\right)+\left(\frac{c}{d}\right)^x\ln\left(\frac{c}{d}\right)>0$

and $f(0)=2$

$\therefore$ $f(x)$ is increasing function and $\lim_{x\to-\infty} f(x)=-1$

$\Rightarrow$ $f(x)$ has only one real root.

But Statement-2 is false.

For example, $f(x)=e^x$ is increasing but $f(x)=0$ has no solution.

Subjective Type Examples

■ In this section, there are **24 subjective** solved examples.

● ***Ex. 28*** *If* α,β *are roots of the equation* $x^2-p(x+1)-c=0$, *show that* $(\alpha+1)(\beta+1)=1-c$. *Hence, prove that* $\frac{\alpha^2+2\alpha+1}{\alpha^2+2\alpha+c}+\frac{\beta^2+2\beta+1}{\beta^2+2\beta+c}=1$.

Sol. Since, α and β are the roots of the equation,

$$x^2-px-p-c=0$$

$\therefore \quad \alpha+\beta=p$

and $\alpha\beta=-p-c$

Now, $(\alpha+1)(\beta+1)=\alpha\beta+\alpha+\beta+1$

$$=-p-c+p+1=1-c$$

Hence, $(\alpha+1)(\beta+1)=1-c$ ...(i)

Second Part $\text{LHS}=\frac{\alpha^2+2\alpha+1}{\alpha^2+2\alpha+c}+\frac{\beta^2+2\beta+1}{\beta^2+2\beta+c}$

$$=\frac{(\alpha+1)^2}{(\alpha+1)^2-(1-c)}+\frac{(\beta+1)^2}{(\beta+1)^2-(1-c)}$$

$$=\frac{(\alpha+1)^2}{(\alpha+1)^2-(\alpha+1)(\beta+1)}+\frac{(\beta+1)^2}{(\beta+1)^2-(\alpha+1)(\beta+1)}$$ [from Eq. (i)]

$$=\frac{\alpha+1}{\alpha-\beta}+\frac{\beta+1}{\beta-\alpha}=\frac{\alpha-\beta}{\alpha-\beta}=1=\text{RHS}$$

Hence, RHS = LHS

● ***Ex. 29*** *Solve the equation* $x^2+px+45=0$. *It is given that the squared difference of its roots is equal to 144.*

Sol. Let α,β be the roots of the equation $x^2+px+45=0$ and given that

$$(\alpha-\beta)^2=144$$

$\Rightarrow \quad p^2-4\cdot1\cdot45=144 \qquad \left[\because \alpha-\beta=\frac{\sqrt{D}}{a}\right]$

$\Rightarrow \quad p^2=324$

$\therefore \quad p=(\pm 18)$

On substituting $p=18$ in the given equation, we obtain

$$x^2+18x+45=0$$

$\Rightarrow \quad (x+3)(x+15)=0$

$\Rightarrow \quad x=-3,5$

and substituting $p=-18$ in the given equation, we obtain

$$x^2-18x+45=0$$

$$(x-3)(x-15)=0$$

$\Rightarrow \quad x=3,15$

Hence, the roots of the given equation are $(-3),(-15),3$ and 15.

● ***Ex. 30*** *If the roots of the equation* $ax^2+bx+c=0\,(a\neq0)$ *be* α *and* β *and those of the equation* $Ax^2+Bx+C=0$ $(A\neq0)$ *be* $\alpha+k$ *and* $\beta+k$. *Prove that*

$$\frac{b^2-4ac}{B^2-4AC}=\left(\frac{a}{A}\right)^2.$$

Sol. $\because \alpha-\beta=(\alpha+k)-(\beta+k)$

$\Rightarrow \quad \frac{\sqrt{b^2-4ac}}{a}=\frac{\sqrt{(B^2-4AC)}}{A} \qquad \left[\because \alpha-\beta=\frac{\sqrt{D}}{a}\right]$

$\Rightarrow \quad \sqrt{\left(\frac{b^2-4ac}{B^2-4AC}\right)}=\left(\frac{a}{A}\right)$

On squaring both sides, then we get

$$\frac{b^2-4ac}{B^2-4AC}=\left(\frac{a}{A}\right)^2$$

● ***Ex. 31*** *Let* a, b *and* c *be real numbers such that* $a+2b+c=4$. *Find the maximum value of* $(ab+bc+ca)$.

Sol. Given, $a+2b+c=4$

$\Rightarrow \quad a=4-2b-c$

Let $y=ab+bc+ca=a(b+c)+bc$

$$=(4-2b-c)(b+c)+bc$$

$$=-2b^2+4b-2bc+4c-c^2$$

$\Rightarrow \quad 2b^2+2(c-2)b-4c+c^2+y=0$

Since, $b\in R$, so

$$4(c-2)^2-4\times2\times(-4c+c^2+y)\geq0$$

$\Rightarrow \quad (c-2)^2 + 8c - 2c^2 - 2y \geq 0$

$\Rightarrow \quad c^2 - 4c + 2y - 4 \leq 0$

Since, $c \in R$, so $16 - 4(2y-4) \geq 0 \Rightarrow y \leq 4$

Hence, maximum value of $ab + bc + ca$ is 4.

Aliter

$\because \quad \text{AM} \geq \text{GM}$

$\Rightarrow \quad \frac{(a+b)+(b+c)}{2} \geq \sqrt{(a+b)(b+c)}$

$\Rightarrow \quad 2 \geq \sqrt{(ab+bc+ca+b^2)} \qquad [\because a+2b+c=4]$

$\Rightarrow \quad ab+bc+ca \leq 4-b^2$

$\therefore$ Maximum value of $(ab+bc+ca)$ is 4.

● ***Ex. 32*** *Find a quadratic equation whose roots x_1 and x_2 satisfy the condition*
$x_1^2 + x_2^2 = 5, 3(x_1^5 + x_2^5) = 11(x_1^3 + x_2^3)$ *(assume that x_1, x_2 are real).*

Sol. We have, $3(x_1^5 + x_2^5) = 11(x_1^3 + x_2^3)$

$\Rightarrow \quad \frac{x_1^5 + x_2^5}{x_1^3 + x_2^3} = \frac{11}{3}$

$\Rightarrow \quad \frac{(x_1^2 + x_2^2)(x_1^3 + x_2^3) - x_1^2 x_2^2 (x_1 + x_2)}{(x_1^3 + x_2^3)} = \frac{11}{3}$

$\Rightarrow \quad (x_1^2 + x_2^2) - \frac{x_1^2 x_2^2 (x_1 + x_2)}{(x_1 + x_2)(x_1^2 + x_2^2 - x_1 x_2)} = \frac{11}{3}$

$[\because x_1^2 + x_2^2 = 5]$

$\Rightarrow \quad 5 - \frac{x_1^2 x_2^2}{5 - x_1 x_2} = \frac{11}{3}$

$\Rightarrow \quad \frac{4}{3} = \frac{x_1^2 x_2^2}{5 - x_1 x_2}$

$\Rightarrow \quad 3x_1^2 x_2^2 + 4x_1 x_2 - 20 = 0$

$\Rightarrow 3x_1^2 x_2^2 + 10x_1 x_2 - 6x_1 x_2 - 20 = 0$

$\Rightarrow \quad (x_1 x_2 - 2)(3x_1 x_2 + 10) = 0$

$\therefore \quad x_1 x_2 = 2, \left(-\frac{10}{3}\right)$

We have, $(x_1 + x_2)^2 = x_1^2 + x_2^2 + 2x_1 x_2 = 5 + 2x_1 x_2$

$\therefore \quad (x_1 + x_2)^2 = 5 + 4 = 9 \qquad [\text{if } x_1 x_2 = 2]$

$\therefore \quad x_1 + x_2 = \pm 3$

and $(x_1 + x_2)^2 = 5 + 2\left(-\frac{10}{3}\right) = -\frac{5}{3}\left[\text{if } x_1 x_2 = -\frac{10}{3}\right]$

which is not possible, since x_1, x_2 are real.

Thus, required quadratic equations are $x^2 \pm 3x + 2 = 0$.

● ***Ex. 33*** *If each pair of the three equations*
$x^2 + ax + b = 0, x^2 + cx + d = 0$ and $x^2 + ex + f = 0$ *has exactly one root in common, then show that*
$(a + c + e)^2 = 4(ac + ce + ea - b - d - f)$.

Sol. Given equations are

$x^2 + ax + b = 0 \qquad ...(i)$

$x^2 + cx + d = 0 \qquad ...(ii)$

$x^2 + ex + f = 0 \qquad ...(iii)$

Let α, β be the roots of Eq. (i), β, γ be the roots of Eq. (ii) and γ, δ be the roots of Eq. (iii), then

$\alpha + \beta = -a, \alpha\beta = b \qquad ...(iv)$

$\beta + \gamma = -c, \beta\gamma = d \qquad ...(v)$

$\gamma + \alpha = -e, \gamma\alpha = f \qquad ...(vi)$

$\therefore$ LHS $= (a + c + e)^2 = (-\alpha - \beta - \beta - \gamma - \gamma - \alpha)^2$

[from Eqs. (iv), (v) and (vi)]

$= 4(\alpha + \beta + \gamma)^2 \qquad ...(vii)$

RHS $= 4(ac + ce + ea - b - d - f)$

$= 4\{(\alpha + \beta)(\beta + \gamma) + (\beta + \gamma)(\gamma + \alpha) + (\gamma + \alpha)(\alpha + \beta) - \alpha\beta - \beta\gamma - \gamma\alpha)\}$

[from Eqs. (iv), (v) and (vi)]

$= 4(\alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\beta\gamma + 2\gamma\alpha)$

$= 4(\alpha + \beta + \gamma)^2 \qquad ...(viii)$

From Eqs. (vii) and (viii), then we get

$(a + c + e)^2 = 4(ac + ce + ea - b - d - f)$

● ***Ex. 34*** *If α, β are the roots of the equation $x^2 + px + q = 0$ and γ, δ are the roots of the equation $x^2 + rx + s = 0$, evaluate $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)$ in terms of p, q, r and s. Deduce the condition that the equations have a common root.*

Sol. $\because \alpha, \beta$ are the roots of the equation

$x^2 + px + q = 0$

$\therefore \quad \alpha + \beta = -p, \alpha\beta = q \qquad ...(i)$

and γ, δ are the roots of the equation $x^2 + rx + s = 0$

$\therefore \quad \gamma + \delta = -r, \gamma\delta = s \qquad ...(ii)$

Now, $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)$

$= [\alpha^2 - \alpha(\gamma + \delta) + \gamma\delta][\beta^2 - \beta(\gamma + \delta) + \gamma\delta]$

$= (\alpha^2 + r\alpha + s)(\beta^2 + r\beta + s) \qquad [\text{from Eq. (ii)}]$

$= \alpha^2\beta^2 + r\alpha\beta(\alpha + \beta) + r^2\alpha\beta + s(\alpha^2 + \beta^2) + sr(\alpha + \beta) + s^2$

$= \alpha^2\beta^2 + r\alpha\beta(\alpha + \beta) + r^2\alpha\beta + s[(\alpha + \beta)^2 - 2\alpha\beta] + sr(\alpha + \beta) + s^2$

$= q^2 - pqr + r^2 q + s(p^2 - 2q) + sr(-p) + s^2$

$= (q - s)^2 - rpq + r^2 q + sp^2 - prs$

$= (q - s)^2 - rq(p - r) + sp(p - r)$

$= (q - s)^2 + (p - r)(sp - rq) \qquad ...(iii)$

For a common root (let $\alpha = \gamma$ or $\beta = \delta$),

then $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta) = 0 \qquad ...(iv)$

From Eqs. (iii) and (iv), we get

$$(q-s)^2+(p-r)(sp-rq)=0$$

$\Rightarrow (q-s)^2=(p-r)(rq-sp)$, which is the required condition .

● **Ex. 35** *Find all integral values of a for which the quadratic Expresion $(x-a)(x-10)+1$ can be factored as a product $(x+\alpha)(x+\beta)$ of two factors and $\alpha,\beta\in I$.*

Sol. We have, $(x-a)(x-10)+1=(x+\alpha)(x+\beta)$

On putting $x=-\alpha$ in both sides, we get

$$(-\alpha-a)(-\alpha-10)+1=0$$

$$\therefore \quad (\alpha+a)(\alpha+10)=-1$$

$\alpha+a$ and $\alpha+10$ are integers. $[\because a,\alpha\in I]$

$\therefore \quad \alpha+a=-1$ and $\alpha+10=1$

or $\quad \alpha+a=1$ and $\alpha+10=-1$

(i) If $\alpha+10=1$

$\therefore \quad \alpha=-9$, then $a=8$

Similarly, $\beta=-9$

Here, $(x-8)(x-10)+1=(x-9)^2$

(ii) If $\alpha+10=-1$

$\therefore \quad \alpha=-11$, then $a=12$

Similarly, $\beta=12$

Here, $(x-12)(x-10)+1=(x-11)^2$

Hence, $a=8,12$

● **Ex. 36** *Solve the equation*

$$\sqrt{x+3-4\sqrt{(x-1)}}+\sqrt{x+8-6\sqrt{(x-1)}}=1.$$

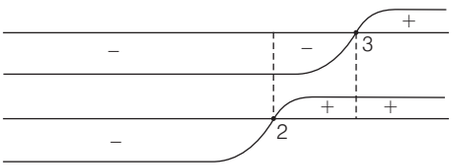
Sol. Let $\sqrt{(x-1)}=t$

We have, $x=t^2+1, t\geq 0$

The given equation reduce in the form

$$\sqrt{(t^2+4-4t)}+\sqrt{(t^2+9-6t)}=1$$

$$\Rightarrow \quad |t-2|+|t-3|=1$$



$$\therefore \quad 2\leq t\leq 3$$

$$\Rightarrow \quad 4\leq t^2\leq 9$$

$$\Rightarrow \quad 4\leq x-1\leq 9$$

$$\Rightarrow \quad 5\leq x\leq 10$$

$\therefore$ Solution of the original equation is $x\in[5,10]$.

● **Ex. 37** *Solve for 'x'*

$1!+2!+3!+\ldots+(x-1)!+x!=k^2$ *and* $k\in I$.

Sol. For $x<4$, the given equation has the only solutions $x=1, k=\pm 1$ and $x=3, k=\pm 3$. Now, let us prove that there are no solutions for $x\geq 4$. The expressions

$$\left.\begin{aligned} 1!+2!+3!+4! &= 33\\ 1!+2!+3!+4!+5! &= 153\\ 1!+2!+3!+4!+5!+6! &= 873\\ 1!+2!+3!+4!+5!+6!+7! &= 5913\end{aligned}\right\}$$ ends with the digit 3.

Now, for $x\geq 4$ the last digit of the sum $1!+2!+\ldots+x!$ is equal to 3 and therefore, this sum cannot be equal to a square of a whole number k (because a square of a whole number cannot end with 3).

● **Ex. 38** *Find the real roots of the equation*

$$\underbrace{\sqrt{x+2\sqrt{x+2\sqrt{x+\ldots+2\sqrt{x+2\sqrt{3x}}}}}}_{n \text{ radical signs}}=x$$

Sol. Rewrite the given equation

$$\sqrt{x+2\sqrt{x+2\sqrt{x+\ldots+2\sqrt{x+2\sqrt{x+2x}}}}}=x \qquad \ldots(i)$$

On replacing the last letter x on the LHS of Eq. (i) by the value of x expressed by Eq. (i), we get

$$x=\underbrace{\sqrt{x+2\sqrt{x+2\sqrt{x+\ldots+\sqrt{x+2x}}}}}_{2n \text{ radical signs}}$$

Further, let us replace the last letter x by the same expression again and again yields.

$$\therefore \quad x=\underbrace{\sqrt{x+2\sqrt{x+2\sqrt{x+\ldots+2\sqrt{x+2x}}}}}_{3n \text{ radical signs}}$$

$$=\underbrace{\sqrt{x+2\sqrt{x+2\sqrt{x+\ldots+2\sqrt{x+2x}}}}}_{4n \text{ radical signs}}=\ldots$$

We can write,

$$x=\sqrt{x+2\sqrt{x+2\sqrt{x+\ldots}}}$$

$$=\underbrace{\lim_{N\to\infty}\sqrt{x+2\sqrt{x+2\sqrt{x+\ldots+2\sqrt{x+2x}}}}}_{N \text{ radical signs}}$$

If follows that

$$x=\sqrt{x+2\sqrt{x+2\sqrt{x+\ldots}}}$$

$$=\sqrt{x+2(\sqrt{x+2\sqrt{x+\ldots}})}=\sqrt{(x+2x)}$$

Hence, $x^2=x+2x$

$\Rightarrow \quad x^2-3x=0$

$\therefore \quad x=0,3$

● **Ex. 39** *Solve the inequation,* $(x^2 + x + 1)^x < 1$.

Sol. Taking logarithm both sides on base 10,

then $x \log (x^2 + x + 1) < 0$

which is equivalent to the collection of systems

$$\left[\begin{cases} x > 0, \\ \log (x^2 + x + 1) < 0, \end{cases} \begin{cases} x < 0, \\ \log (x^2 + x + 1) > 0, \end{cases}\right. \Rightarrow \left[\begin{cases} x > 0, \\ x^2 + x + 1 < 1, \end{cases} \begin{cases} x < 0, \\ x^2 + x + 1 > 1, \end{cases}\right.$$

$$\Rightarrow \left[\begin{cases} x > 0, \\ x(x+1) < 0, \end{cases} \begin{cases} x < 0, \\ x(x+1) > 0 \end{cases}\right. \Rightarrow \left[\begin{cases} x > 0, \\ -1 < x < 0 \end{cases} \begin{cases} x < 0, \\ x > 0 \text{ and } x < (-1) \end{cases}\right.$$

$$\Rightarrow \begin{cases} x \in \phi, \\ x < (-1) \end{cases}$$

Consequently, the interval $x \in (-\infty, -1)$ is the set of all solutions of the original inequation.

Remark

When the inequation is in power, then it is better to take log.

● **Ex. 40** *Solve the equation*

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}} \quad \cfrac{}{1 + \cfrac{1}{x}} = x$$

When in expression on left hand side the sign of a fraction is repeated n times.

Sol. Given equation is

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}} \quad \cfrac{}{1 + \cfrac{1}{x}} = x$$

Let us replace x on the LHS of the given equation by the expression of x. This result in an equation of the same form, which however involves $2n$ fraction lines. Continuing this process on the basis of this transformation, we can write

$$x = 1 + \lim_{n \to \infty} 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}} \quad \cfrac{}{1 + \cfrac{1}{x}} \qquad [n \text{ fractions}]$$

$$\Rightarrow \quad x = 1 + \frac{1}{x} \Rightarrow x^2 - x - 1 = 0$$

$$\therefore \quad x = \frac{1 \pm \sqrt{5}}{2}$$

$$\therefore \quad x_1 = \frac{1 + \sqrt{5}}{2}, x_2 = \frac{1 - \sqrt{5}}{2}$$

satisfy the given equation and this equation has no other roots.

● **Ex. 41** *Solve the system of equations*

$$\begin{cases} |x-1| + |y-2| = 1 \\ y = 2 - |x-1| \end{cases}.$$

Sol. On substituting $|x - 1| = 2 - y$ from second equation in first equation of this system, we get

$$2 - y + |y - 2| = 1$$

Now, consider the following cases

If $y \geq 2$,

then $2 - y + y - 2 = 1 \Rightarrow 0 = 1$

No value of y for $y \geq 2$.

If $y < 2$,

then $2 - y + 2 - y = 1 \Leftrightarrow y = \frac{3}{2}$, which is true.

From the second equation of this system,

$$\frac{3}{2} = 2 - |x - 1|$$

$$\Rightarrow \quad |x - 1| = \frac{1}{2} \Rightarrow x - 1 = \pm \frac{1}{2}$$

$$\Rightarrow \quad x = 1 \pm \frac{1}{2} \Rightarrow x = \frac{1}{2}, \frac{3}{2}$$

Consequently, the set of all solutions of the original system is the set of pairs (x, y), where $x = \frac{1}{2}, \frac{3}{2}$ and $y = \frac{3}{2}$.

● **Ex. 42** *Let* a, b, c *be real and* $ax^2 + bx + c = 0$ *has two real roots* α *and* β, *where* $\alpha < -1$ *and* $\beta > 1$, *then show that* $1 + \frac{c}{a} + \left|\frac{b}{a}\right| < 0$.

Sol. Since, $\alpha < -1$ and $\beta > 1$

$$\alpha + \lambda = -1 \text{ and } \beta = 1 + \mu \qquad [\lambda, \mu > 0]$$

Now, $1 + \frac{c}{a} + \left|\frac{b}{a}\right| = 1 + \alpha\beta + |\alpha + \beta|$

$$= 1 + (-1 - \lambda)(1 + \mu) + |-1 - \lambda + 1 + \mu|$$

$$= 1 - 1 - \mu - \lambda - \lambda\mu + |\mu - \lambda|$$

$$= -\mu - \lambda - \lambda\mu + \mu - \lambda \qquad [\text{if } \mu > \lambda]$$

and $\quad = -\mu - \lambda - \lambda\mu + \lambda - \mu \qquad [\text{if } \lambda > \mu]$

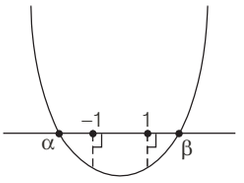
$$\therefore \quad 1 + \frac{c}{a} + \left|\frac{b}{a}\right| = -2\lambda - \lambda\mu \text{ or } -2\mu - \lambda\mu$$

On both cases, $1 + \frac{c}{a} + \left|\frac{b}{a}\right| < 0 \qquad [\because \lambda, \mu > 0]$

Aliter

$\because \quad ax^2 + bx + c = 0, a \neq 0$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$



Let $f(x) = x^2 + \frac{b}{a}x + \frac{c}{a}$

$f(-1) < 0$ and $f(1) < 0$

$\Rightarrow 1 - \frac{b}{a} + \frac{c}{a} < 0$ and $1 + \frac{b}{a} + \frac{c}{a} < 0$

Then, $1 + \left|\frac{b}{a}\right| + \frac{c}{a} < 0$

● ***Ex. 43*** *Solve the equation* $x\left(\frac{3-x}{x+1}\right)\left(x + \frac{3-x}{x+1}\right) = 2.$

Sol. Hence, $x + 1 \neq 0$

and let $x\left(\frac{3-x}{x+1}\right) = u$ and $x + \frac{3-x}{x+1} = v$

$\therefore \quad uv = 2 \quad ...(i)$

and $u + v = x\left(\frac{3-x}{x+1}\right) + x + \left(\frac{3-x}{x+1}\right)$

$$= (x+1)\left(\frac{3-x}{x+1}\right) + x = 3 - x + x = 3$$

$\therefore \quad u + v = 3$ and $uv = 2$

Then, $u = 2, v = 1$ or $u = 1, v = 2$

Given equation is equivalent to the collection

$$\therefore \quad \begin{cases} x\left(\frac{3-x}{x+1}\right) = 2 \\ x + \frac{3-x}{x+1} = 1 \end{cases} \text{ or } \begin{cases} x\left(\frac{3-x}{x+1}\right) = 1 \\ x + \frac{3-x}{x+1} = 2 \end{cases}$$

$$\Rightarrow \quad \begin{cases} x^2 - x + 2 = 0 \\ x^2 - x + 2 = 0 \end{cases} \text{ or } \begin{cases} x^2 - 2x + 1 = 0 \\ x^2 - 2x + 1 = 0 \end{cases}$$

$$\Rightarrow \quad \begin{cases} x^2 - x + 2 = 0 \\ x^2 - 2x + 1 = 0 \end{cases} \Rightarrow \begin{cases} \left(x - \frac{1}{2}\right)^2 + \frac{7}{4} \neq 0 \\ (x-1)^2 = 0 \end{cases}$$

$\therefore \quad (x-1)^2 = 0$

$\Rightarrow x = 1$ is a unique solution of the original equation.

● ***Ex. 44*** *Show that for any real numbers* $a_3, a_4, a_5, \ldots, a_{85}$, *the roots of the equation*

$a_{85}x^{85} + a_{84}x^{84} + \ldots + a_3x^3 + 3x^2 + 2x + 1 = 0$ *are not real.*

Sol. Let $P(x) = a_{85}x^{85} + a_{84}x^{84}$

$$+ \ldots + a_3x^3 + 3x^2 + 2x + 1 = 0 \quad ...(i)$$

Since, $P(0) = 1$, then 0 is not a root of Eq. (i).

Let $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_{85}$ be the complex roots of Eq. (i).

Then, the $\beta_i\left(\text{let}\frac{1}{\alpha_i}\right)$ the complex roots of the polynomial

$$Q(y) = y^{85} + 2y^{84} + 3y^{83} + a_3y^{82} + \ldots + a_{85}$$

It follows that

$$\sum_{i=1}^{85}\beta_i = -2 \text{ and } \sum_{1 \le i < j \le 85}\sum \beta_i\beta_j = 3$$

Then, $\sum_{i=1}^{85}\beta_i^2 = \left(\sum_{i=1}^{85}\beta_i\right)^2 - 2\sum_{1 \le i < j \le 85}\sum \beta_i\beta_j$

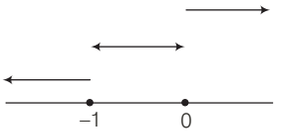
$$= 4 - 6 = -2 < 0$$

Thus, the β_i's is not all real and then α_i's are not all real.

● ***Ex. 45*** *Solve the equation*

$$2^{|x+1|} - 2^x = |2^x - 1| + 1.$$

Sol. Find the critical points :



$x + 1 = 0, 2^x - 1 = 0$

$\therefore \quad x = -1, x = 0$

Now, consider the following cases :

$\boldsymbol{x < -1}$

$$2^{-(x+1)} - 2^x = -(2^x - 1) + 1$$

$$\Rightarrow \quad 2^{-(x+1)} = 2$$

$$\therefore \quad -(x+1) = 1$$

$$\therefore \quad x = -2 \quad ...(i)$$

$\boldsymbol{-1 \le x < 0}$

$$2^{x+1} - 2^x = -(2^x - 1) + 1$$

$$\Rightarrow \quad 2^{x+1} = 2$$

$$\therefore \quad x + 1 = 1$$

$$\therefore \quad x = 0$$

$$x \neq 0 \quad [\because -1 \le x < 0]$$

$\boldsymbol{x \ge 0}$

$$2^{x+1} - 2^x = 2^x - 1 + 1$$

$$\Rightarrow \quad 2^{x+1} = 2 \cdot 2^x$$

$$\Rightarrow \quad 2^{x+1} = 2^{x+1}$$

which is true for $x \ge 0.$ ...(ii)

Now, combining all cases, we have the final solution as

$$x \in [0, \infty) \cup \{-2\}$$

● ***Ex. 46*** *Solve the inequation*

$$-|y| + x - \sqrt{(x^2 + y^2 - 1)} \ge 1.$$

Sol. We have, $-|y| + x - \sqrt{(x^2 + y^2 - 1)} \ge 1$

$$\Rightarrow \quad x - |y| \ge 1 + \sqrt{(x^2 + y^2 - 1)}$$

if $x \ge |y|,$

then squaring both sides,

$$x^2 + y^2 - 2x|y| \ge 1 + x^2 + y^2 - 1 + 2\sqrt{(x^2 + y^2 - 1)}$$

$$\Rightarrow \quad -x|y| \ge \sqrt{(x^2 + y^2 - 1)} \quad ...(i)$$

Since, $x \ge |y| \ge 0$ ...(ii)

Then, LHS of Eq. (i) is non-positive and RHS of Eq. (ii) is non-negative. Therefore, the system is satisfied only, when both sides are zero.

$\therefore$ The inequality Eq. (i) is equivalent to the system.

$$\begin{cases} x|y| = 0 \\ x^2 + y^2 - 1 = 0 \end{cases}$$

The Eq.(i) gives $x = 0$ or $y = 0$. If $x = 0$, then we find $y = \pm 1$ from Eq. (ii) but $x \geq |y|$ which is impossible.

If $y = 0$, then from Eq. (ii), we find

$$x^2 = 1$$

$$\therefore \quad x = 1, -1$$

Taking $x = 1$ $[\because x \geq |y|]$

$\therefore$ The pair $(1, 0)$ satisfies the given inequation. Hence, $(1, 0)$ is the solution of the original inequation.

● ***Ex. 47*** *If $a_1, a_2, a_3, \ldots, a_n (n \geq 2)$ are real and $(n-1)a_1^2 - 2na_2 < 0$, prove that atleast two roots of the equation $x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n = 0$ are imaginary.*

Sol. Let $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are the roots of the given equation.

Then, $\sum\alpha_1 = \alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_n = -a_1$

and $\sum\alpha_1\alpha_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \ldots + \alpha_{n-1}\alpha_n = a_2$

Now, $(n-1)a_1^2 - 2na_2 = (n-1)(\sum\alpha_1)^2 - 2n\sum\alpha_1\alpha_2$

$$= n\{(\sum\alpha_1)^2 - 2\sum\alpha_1\alpha_2\} - (\sum\alpha_1)^2$$

$$= n\sum\alpha_1^2 - (\sum a_1)^2$$

$$= \sum_{1 \leq i < j \leq n} \sum (\alpha_i - \alpha_j)^2$$

But given that $(n-1)a_1^2 - 2na_2 < 0$

$$\Rightarrow \quad \sum_{1 \leq i < j \leq n} \cdot \sum (\alpha_i - \alpha_j)^2 < 0$$

which is true only, when atleast two roots are imaginary.

● ***Ex. 48*** *Solve the inequation $|a^{2x} + a^{x+2} - 1| \geq 1$ for all values of $a\,(a > 0, a \neq 1)$.*

Sol. Using $a^x = t$,

the given inequation can be written in the form

$$|t^2 + a^2 t - 1| \geq 1 \quad \ldots(i)$$

$\because$ $a > 0$ and $a \neq 1$, then $a^x > 0$

$$\therefore \quad t > 0 \quad \ldots(ii)$$

Inequation (i) write in the forms,

$t^2 + a^2 t - 1 \geq 1$ and $t^2 + a^2 t - 1 \leq -1$

$$\therefore \quad t \leq \frac{-a^2 - \sqrt{a^4 + 8}}{2}, t \geq \frac{-a^2 + \sqrt{(a^4 + 8)}}{2}$$

and $-a^2 \leq t \leq 0$

But $t > 0$ [from Eq. (ii)]

$$\therefore \quad t \geq \frac{-a^2 + \sqrt{(a^4 + 8)}}{2}$$

$$\therefore \quad a^x \geq \frac{-a^2 + \sqrt{(a^4 + 8)}}{2}$$

For $0 < a < 1$,

$$x \leq \log_a\left(\frac{-a^2 + \sqrt{(a^4 + 8)}}{2}\right)$$

$$\therefore \quad x \in \left[-\infty, \log_a\left(\frac{-a^2 + \sqrt{(a^4 + 8)}}{2}\right)\right]$$

and for $a > 1$, $x \geq \log_a\left(\frac{-a^2 + \sqrt{(a^4 + 8)}}{2}\right)$

$$\therefore \quad x \in \left(\log_a\left(\frac{-a^2 + \sqrt{(a^4 + 8)}}{2}\right), \infty\right)$$

● ***Ex. 49*** *Solve the inequation*

$$\log_{|x|}(\sqrt{(9 - x^2)} - x - 1) \geq 1.$$

Sol. We rewrite the given inequation in the form,

$$\log_{|x|}(\sqrt{(9 - x^2)} - x - 1) \geq \log_{|x|}(|x|)$$

This inequation is equivalent to the collection of systems.

$$\begin{cases} \sqrt{(9 - x^2)} - x - 1 \geq |x|, \text{ if } |x| > 1 \\ \sqrt{(9 - x^2)} - x - 1 \leq |x| \text{ if } \quad 0 < |x| < 1 \end{cases}$$

and $\begin{bmatrix} \begin{cases} \text{For } x > 1 \\ \sqrt{(9 - x^2)} - x - 1 \geq x \end{cases} \\ \begin{cases} \text{For } x < -1 \\ \sqrt{(9 - x^2)} - x - 1 \geq -x \end{cases} \\ \begin{cases} \text{For } 0 < x < 1 \\ \sqrt{(9 - x^2)} - x - 1 \leq x \end{cases} \\ \begin{cases} \text{For } -1 < x < 0 \\ \sqrt{(9 - x^2)} - x - 1 \leq -x \end{cases} \end{bmatrix} \Rightarrow \begin{bmatrix} \begin{cases} \text{For } x > 1 \\ \sqrt{(9 - x^2)} \geq 2x + 1 \end{cases} \\ \begin{cases} \text{For } x < -1 \\ \sqrt{(9 - x^2)} \geq 1 \end{cases} \\ \begin{cases} \text{For } 0 < x < 1 \\ \sqrt{(9 - x^2)} \leq 2x + 1 \end{cases} \\ \begin{cases} \text{For } -1 < x < 0 \\ \sqrt{(9 - x^2)} \leq 1 \end{cases} \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} \begin{cases} \text{For } x > 1 \\ -\frac{2}{5}(\sqrt{11} + 1) \leq x \leq \frac{2}{5}(\sqrt{11} - 1) \end{cases} \\ \begin{cases} \text{For } x < -1 \\ -2\sqrt{2} \leq x \leq 2\sqrt{2} \end{cases} \\ \begin{cases} \text{For } 0 < x < 1 \\ x \leq -\frac{2}{5}(\sqrt{11} + 1) \text{ and } x \geq \frac{2}{5}(\sqrt{11} - 1) \end{cases} \\ \begin{cases} \text{For } -1 < x < 0 \\ x \leq -2\sqrt{2} \text{ and } x \geq 2\sqrt{2} \end{cases} \end{bmatrix}$$

$$\Rightarrow \quad \left[\begin{array}{l} \begin{cases} x \in \phi \\ -2\sqrt{2} \le x < -1 \end{cases} \\ \begin{cases} \frac{2}{5}(\sqrt{11}-1) \le x < 1 \\ x \in \phi \end{cases} \end{array}\right.$$

Hence, the original inequation consists of the intervals

$$-2\sqrt{2} \le x < -1 \text{ and } \frac{2}{5}(\sqrt{11}-1) \le x < 1.$$

Hence, $x \in [-2\sqrt{2}, -1) \cup \left[\frac{2}{5}(\sqrt{11}-1), 1\right)$

● ***Ex. 50*** *Find all values of 'a' for which the equation* $4^x - a2^x - a + 3 = 0$ has atleast one solution.

Sol. Putting $2^x = t > 0$, then the original equation reduced in the form

$$t^2 - at - a + 3 = 0 \qquad \ldots(i)$$

that the quadratic Eq. (i) should have atleast one positive root $(t > 0)$, then

Discriminant, $D = (-a)^2 - 4 \cdot 1 \cdot (-a+3) \ge 0$

$$\Rightarrow \quad a^2 + 4a - 12 \ge 0$$

$$\Rightarrow \quad (a+6)(a-2) \ge 0$$

$$\therefore \quad a \in (-\infty, -6] \cup [2, \infty)$$

If roots of Eq. (i) are t_1 and t_2, then

$$\begin{cases} t_1 + t_2 = a \\ t_1 t_2 = 3 - a \end{cases}$$

For $a \in (-\infty, -6]$

$t_1 + t_2 < 0$ and $t_1 t_2 > 0$. Therefore, both roots are negative and consequently, the original equation has no solutions.

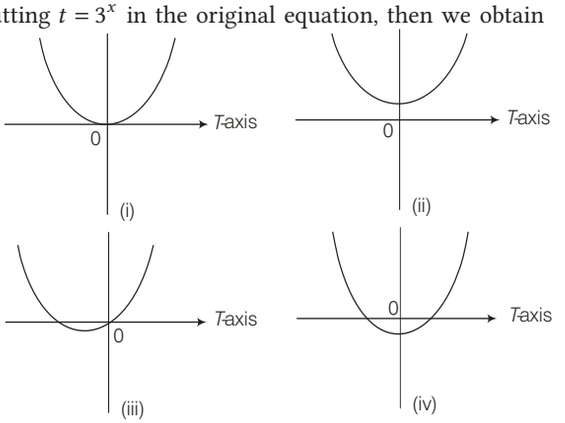
For $a \in [2, \infty)$

$t_1 + t_2 > 0$ and $t_1 t_2 \gtrless 0$, consequently, atleast one of the roots t_1 or t_2, is greater than zero.

Thus, for $a \in [2, \infty)$, the given equation has atleast one solution.

● ***Ex. 51*** *Find all the values of the parameter a for which the inequality* $a9^x + 4(a-1)3^x + a > 1$, *is satisfied for all real values of x .*

Sol. Putting $t = 3^x$ in the original equation, then we obtain



$$at^2 + 4(a-1)t + a > 1$$

$$\Rightarrow \quad at^2 + 4(a-1)t + (a-1) > 0 \qquad [t > 0, \because 3^x > 0]$$

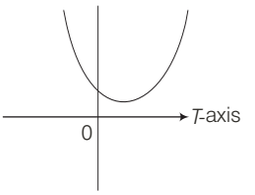
This is possible in two cases. First the parabola $f(t) = at^2 + 4(a-1)t + (a-1)$ opens upwards, with its vertex (turning point) lying in the non-positive part of the T-axis, as shown in the following four figures.

$\therefore \quad a > 0$ and sum of roots ≤ 0

$$\Rightarrow \quad -\frac{4(a-1)}{2a} \le 0 \text{ and } f(0) \ge 0$$

$$\therefore \quad a > 0, a - 1 \ge 0 \text{ and } a - 1 \ge 0$$

Hence, $a \ge 1$



Second the parabola $f(t)$ opens upward, with its vertex lying in positive direction of t, then

$$a > 0, -\frac{4(a-1)}{2a} > 0 \text{ and } D \le 0$$

$$\Rightarrow \quad a > 0, (a-1) < 0$$

and $\quad 16(a-1)^2 - 4(a-1)a \le 0$

$$\Rightarrow \quad a > 0, a < 1$$

and $\quad 4(a-1)(3a-4) \le 0$

$$\Rightarrow \quad a > 0, a < 1 \text{ and } 1 \le a \le \frac{4}{3}$$

These inequalities cannot have simultaneously.

Hence, $a \ge 1$ from Eq. (i).

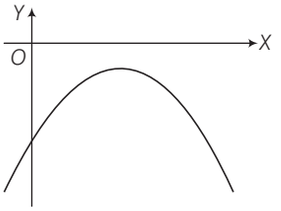
Theory of Equations Exercise 1 :
Single Option Correct Type Questions

- This section contains **30 multiple choice questions.** Each question has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct

1. If a, b, c are real and $a \neq b$, the roots of the equation $2(a-b)x^2 - 11(a+b+c)x - 3(a-b) = 0$ are

(a) real and equal (b) real and unequal
(c) purely imaginary (d) None of these

2. The graph of a quadratic polynomial $y = ax^2 + bx + c; a, b, c \in R$ is as shown.



Which one of the following is not correct?

(a) $b^2 - 4ac < 0$ (b) $\frac{c}{a} < 0$
(c) c is negative
(d) Abscissa corresponding to the vertex is $\left(-\frac{b}{2a}\right)$

3. There is only one real value of *'a'* for which the quadratic equation $ax^2 + (a+3)x + a - 3 = 0$ has two positive integral solutions. The product of these two solutions is

(a) 9 (b) 8 (c) 6 (d) 12

4. If for all real values of a one root of the equation $x^2 - 3ax + f(a) = 0$ is double of the other, $f(x)$ is equal to

(a) $2x$ (b) x^2 (c) $2x^2$ (d) $2\sqrt{x}$

5. A quadratic equation the product of whose roots x_1 and x_2 is equal to 4 and satisfying the relation $\frac{x_1}{x_1 - 1} + \frac{x_2}{x_2 - 1} = 2$, is

(a) $x^2 - 2x + 4 = 0$ (b) $x^2 - 4x + 4 = 0$
(c) $x^2 + 2x + 4 = 0$ (d) $x^2 + 4x + 4 = 0$

6. If both roots of the quadratic equation $x^2 - 2ax + a^2 - 1 = 0$ lie in $(-2, 2)$, which one of the following can be $[a]$? (where $[\cdot]$ denotes the greatest integer function)

(a) -1 (b) 1 (c) 2 (d) 3

7. If $(-2, 7)$ is the highest point on the graph of $y = -2x^2 - 4ax + \lambda$, then λ equals

(a) 31 (b) 11 (c) -1 (d) $-\frac{1}{3}$

8. If the roots of the quadratic equation $(4p - p^2 - 5)x^2 - (2p-1)x + 3p = 0$ lie on either side of unity, the number of integral values of p is

(a) 1 (b) 2 (c) 3 (d) 4

9. Solution set of the equation $3^{2x^2} - 2 \cdot 3^{x^2+x+6} + 3^{2(x+6)} = 0$ is

(a) $\{-3, 2\}$ (b) $\{6, -1\}$ (c) $\{-2, 3\}$ (d) $\{1, -6\}$

10. Consider two quadratic expressions $f(x) = ax^2 + bx + c$ and $g(x) = ax^2 + px + q$ $(a, b, c, p, q \in R, b \neq p)$ such that their discriminants are equal. If $f(x) = g(x)$ has a root $x = \alpha$, then

(a) α will be AM of the roots of $f(x) = 0$ and $g(x) = 0$
(b) α will be AM of the roots of $f(x) = 0$
(c) α will be AM of the roots of $f(x) = 0$ or $g(x) = 0$
(d) α will be AM of the roots of $g(x) = 0$

11. If x_1 and x_2 are the arithmetic and harmonic means of the roots of the equation $ax^2 + bx + c = 0$, the quadratic equation whose roots are x_1 and x_2, is

(a) $abx^2 + (b^2 + ac)x + bc = 0$
(b) $2abx^2 + (b^2 + 4ac)x + 2bc = 0$
(c) $2abx^2 + (b^2 + ac)x + bc = 0$
(d) None of the above

12. $f(x)$ is a cubic polynomial $x^3 + ax^2 + bx + c$ such that $f(x) = 0$ has three distinct integral roots and $f(g(x)) = 0$ does not have real roots, where $g(x) = x^2 + 2x - 5$, the minimum value of $a + b + c$ is

(a) 504 (b) 532 (c) 719 (d) 764

13. The value of the positive integer n for which the quadratic equation $\sum_{k=1}^{n}(x+k-1)(x+k) = 10n$ has solutions α and $\alpha + 1$ for some α, is

(a) 7 (b) 11 (c) 17 (d) 25

14. If one root of the equation $x^2 - \lambda x + 12 = 0$ is even prime, while $x^2 + \lambda x + \mu = 0$ has equal roots, then μ is

(a) 8 (b) 16 (c) 24 (d) 32

15. Number of real roots of the equation $\sqrt{x} + \sqrt{x - \sqrt{(1-x)}} = 1$ is

(a) 0 (b) 1 (c) 2 (d) 3

16. The value of $\sqrt{7 + \sqrt{7 - \sqrt{7 + \sqrt{7 - \ldots}}}}$ upto ∞ is

(a) 5 (b) 4
(c) 3 (d) 2

17. For any real x, the expression $2(k-x)\left[x+\sqrt{x^2+k^2}\right]$ cannot exceed
(a) k^2 (b) $2k^2$
(c) $3k^2$ (d) None of these

18. Given that, for all $x \in R$, the expression $\frac{x^2-2x+4}{x^2+2x+4}$ lies between $\frac{1}{3}$ and 3, the values between which the expression $\frac{9\cdot 3^{2x}+6\cdot 3^x+4}{9\cdot 3^{2x}-6\cdot 3^x+4}$ lies, are
(a) -3 and 1 (b) $\frac{3}{2}$ and 2
(c) -1 and 1 (d) 0 and 2

19. Let α, β, γ be the roots of the equation $(x-a)(x-b)(x-c)=d, d\neq 0$, the roots of the equation $(x-\alpha)(x-\beta)(x-\gamma)+d=0$ are
(a) a, b, d (b) b, c, d
(c) a, b, c (d) $a+d, b+d, c+d$

20. If one root of the equation $ix^2-2(1+i)x+2-i=0$ is $(3-i)$, where $i=\sqrt{-1}$, the other root is
(a) $3+i$ (b) $3+\sqrt{-1}$
(c) $-1+i$ (d) $-1-i$

21. The number of solutions of $|[x]-2x|=4$, where $[x]$ denotes the greatest integer $\leq x$ is
(a) infinite (b) 4 (c) 3 (d) 2

22. If x^2+x+1 is a factor of ax^3+bx^2+cx+d, the real root of $ax^3+bx^2+cx+d=0$ is
(a) $-\frac{d}{a}$ (b) $\frac{d}{a}$ (c) $\frac{a}{d}$ (d) None of these

23. The value of x which satisfy the equation $\sqrt{(5x^2-8x+3)}-\sqrt{(5x^2-9x+4)}=\sqrt{(2x^2-2x)}-\sqrt{(2x^2-3x+1)}$, is
(a) 3 (b) 2
(c) 1 (d) 0

24. The roots of the equation
$$(a+\sqrt{b})^{x^2-15}+(a-\sqrt{b})^{x^2-15}=2a,$$
where $a^2-b=1$, are
(a) $\pm 2, \pm\sqrt{3}$ (b) $\pm 4, \pm\sqrt{14}$
(c) $\pm 3, \pm\sqrt{5}$ (d) $\pm 6, \pm\sqrt{20}$

25. The number of pairs (x, y) which will satisfy the equation $x^2-xy+y^2=4(x+y-4)$, is
(a) 1 (b) 2
(c) 4 (d) None of these

26. The number of positive integral solutions of $x^4-y^4=3789108$ is
(a) 0 (b) 1 (c) 2 (d) 4

27. The value of 'a' for which the equation $x^3+ax+1=0$ and $x^4+ax^2+1=0$, have a common root, is
(a) $a=2$ (b) $a=-2$
(c) $a=0$ (d) None of these

28. The necessary and sufficient condition for the equation $(1-a^2)x^2+2ax-1=0$ to have roots lying in the interval $(0, 1)$, is
(a) $a>0$ (b) $a<0$
(c) $a>2$ (d) None of these

29. Solution set of $x-\sqrt{1-|x|}<0$, is
(a) $\left[-1, \frac{-1+\sqrt{5}}{2}\right)$ (b) $[-1, 1]$
(c) $\left[-1, \frac{-1+\sqrt{5}}{2}\right]$ (d) $\left(-1, \frac{-1+\sqrt{5}}{2}\right)$

30. If the quadratic equations $ax^2+2cx+b=0$ and $ax^2+2bx+c=0 (b\neq c)$ have a common root, $a+4b+4c$, is equal to
(a) -2 (b) -1
(c) 0 (d) 1

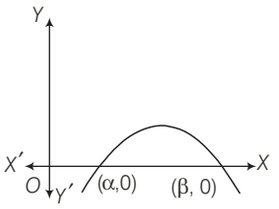
Theory of Equations Exercise 2 : More than One Correct Option Type Questions

- This section contains **15 multiple choice questions**. Each question has four choices (a), (b), (c) and (d) out of which **MORE THAN ONE** may be correct.

31. If $0<a<b<c$ and the roots α, β of the equation $ax^2+bx+c=0$ are non-real complex numbers, then
(a) $|\alpha|=|\beta|$ (b) $|\alpha|>1$
(c) $|\beta|<1$ (d) None of these

32. If A, G and H are the arithmetic mean, geometric mean and harmonic mean between unequal positive integers. Then, the equation $Ax^2-|G|x-H=0$ has
(a) both roots are fractions
(b) atleast one root which is negative fraction
(c) exactly one positive root
(d) atleast one root which is an integer

33. The adjoining graph of $y = ax^2 + bx + c$ shows that



(a) $a < 0$
(b) $b^2 < 4ac$
(c) $c > 0$
(d) a and b are of opposite signs

34. If the equation $ax^2 + bx + c = 0\,(a > 0)$ has two roots α and β such that $\alpha < -2$ and $\beta > 2$, then
(a) $b^2 - 4ac > 0$ (b) $c < 0$
(c) $a + |b| + c < 0$ (d) $4a + 2|b| + c < 0$

35. If $b^2 \geq 4ac$ for the equation $ax^4 + bx^2 + c = 0$, then all the roots of the equation will be real, if
(a) $b > 0, a < 0, c > 0$ (b) $b < 0, a > 0, c > 0$
(c) $b > 0, a > 0, c > 0$ (d) $b > 0, a < 0, c < 0$

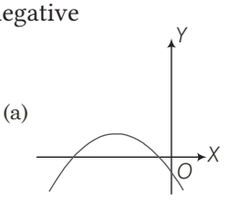
36. If roots of the equation $x^3 + bx^2 + cx - 1 = 0$ from an increasing GP, then
(a) $b + c = 0$
(b) $b \in (-\infty, -3)$
(c) one of the roots is 1
(d) one root is smaller than one and one root is more than one

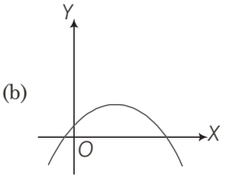
37. Let $f(x) = ax^2 + bx + c$, where $a, b, c \in R$, $a \neq 0$. Suppose $|f(x)| \leq 1, \forall\ x \in [0, 1]$, then
(a) $|a| \leq 8$ (b) $|b| \leq 8$
(c) $|c| \leq 1$ (d) $|a| + |b| + |c| \leq 17$

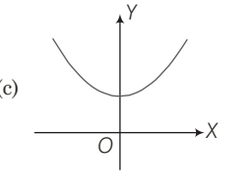
38. $\cos\alpha$ is a root of the equation $25x^2 + 5x - 12 = 0$, $-1 < x < 0$, the value of $\sin 2\alpha$ is
(a) $\frac{24}{25}$ (b) $-\frac{12}{25}$
(c) $-\frac{24}{25}$ (d) $\frac{20}{25}$

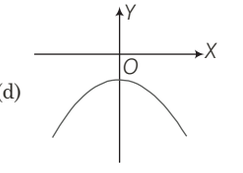
39. If $a, b, c \in R\,(a \neq 0)$ and $a + 2b + 4c = 0$, then equation $ax^2 + bx + c = 0$ has
(a) atleast one positive root
(b) atleast one non-integral root
(c) both integral roots
(d) no irrational root

40. For which of the following graphs of the quadratic expression $f(x) = ax^2 + bx + c$, the product of abc is negative

(a)


(b)


(c)


(d)


41. If $a, b \in R$ and $ax^2 + bx + 6 = 0, a \neq 0$ does not have two distinct real roots, the
(a) minimum possible value of $3a + b$ is -2
(b) minimum possible value of $3a + b$ is 2
(c) minimum possible value of $6a + b$ is -1
(d) minimum possible value of $6a + b$ is 1

42. If $x^3 + 3x^2 - 9x + \lambda$ is of the form $(x - \alpha)^2(x - \beta)$, then λ is equal to
(a) 27 (b) -27
(c) 5 (d) -5

43. If $ax^2 + (b - c)x + a - b - c = 0$ has unequal real roots for all $c \in R$, then
(a) $b < 0 < a$ (b) $a < 0 < b$
(c) $b < a < 0$ (d) $b > a > 0$

44. If the equation whose roots are the squares of the roots of the cubic $x^3 - ax^2 + bx - 1 = 0$ is identical with the given cubic equation, then
(a) $a = b = 0$
(b) $a = 0, b = 3$
(c) $a = b = 3$
(d) a, b are roots of $x^2 + x + 2 = 0$

45. If the equation $ax^2 + bx + c = 0\,(a > 0)$ has two real roots α and β such that $\alpha < -2$ and $\beta > 2$, which of the following statements is/are true?
(a) $4a - 2|b| + c < 0$
(b) $9a - 3|b| + c < 0$
(c) $a - |b| + c < 0$
(d) $c < 0, b^2 - 4ac > 0$

Theory of Equations Exercise 3 : Passage Based Questions

- This section contains **6 passages**. Based upon each of the passage **3 multiple choice questions** have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I

(Q. Nos. 46 to 48)

If G and L are the greatest and least values of the expression $\frac{2x^2-3x+2}{2x^2+3x+2}$, $x \in R$ *respectively.*

46. The least value of $G^{100} + L^{100}$ is

(a) 2^{100} (b) 3^{100} (c) 7^{100} (d) None of these

47. G and L are the roots of the equation

(a) $5x^2 - 26x + 5 = 0$ (b) $7x^2 - 50x + 7 = 0$

(c) $9x^2 - 82x + 9 = 0$ (d) $11x^2 - 122x + 11 = 0$

48. If $L^2 < \lambda < G^2$, $\lambda \in N$, the sum of all values of λ is

(a) 1035 (b) 1081 (c) 1225 (d) 1176

Passage II

(Q. Nos. 49 to 51)

If roots of the equation $x^4 - 12x^3 + cx^2 + dx + 81 = 0$ *are positive.*

49. The value of c is

(a) −27 (b) 27 (c) −54 (d) 54

50. The value of d is

(a) −27 (b) −54 (c) −81 (d) −108

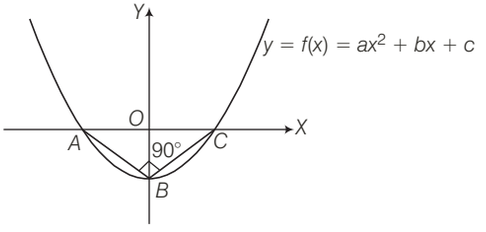
51. Root of the equation $2cx + d = 0$, is

(a) −1 (b) $-\frac{1}{2}$ (c) 1 (d) $\frac{1}{2}$

Passage II

(Q. Nos. 52 to 54)

In the given figure vertices of ΔABC *lie on* $y = f(x) = ax^2 + bx + c$. *The* ΔABC *is right angled isosceles triangle whose hypotenuse* $AC = 4\sqrt{2}$ *units.*



52. $y = f(x)$ is given by

(a) $y = x^2 - 8$ (b) $y = \frac{x^2}{2\sqrt{2}} - 2\sqrt{2}$

(c) $y = x^2 - 4$ (d) $y = \frac{x^2}{2} - \sqrt{2}$

53. Minimum value of $y = f(x)$ is

(a) $-4\sqrt{2}$ (b) $-2\sqrt{2}$

(c) 0 (d) $2\sqrt{2}$

54. Number of integral value of λ for which $\frac{\lambda}{2}$ lies between the roots of $f(x) = 0$, is

(a) 9 (b) 10 (c) 11 (d) 12

Passage III

(Q. Nos. 55 to 57)

Let $f(x) = x^2 + bx + c$ *and* $g(x) = x^2 + b_1 x + c_1$.

Let the real roots of $f(x) = 0$ *be* α, β *and real roots of* $g(x) = 0$ *be* $\alpha + k, \beta + k$ *for same constant k. The least value of* $f(x)$ *is* $-\frac{1}{4}$ *and least value of* $g(x)$ *occurs at* $x = \frac{7}{2}$.

55. The value of b_1 is

(a) −8 (b) −7 (c) −6 (d) 5

56. The least value of $g(x)$ is

(a) −1 (b) $-\frac{1}{2}$ (c) $-\frac{1}{3}$ (d) $-\frac{1}{4}$

57. The roots of $f(x) = 0$ are

(a) 3, 4 (b) −3, 4

(c) −3, − 4 (d) 3, − 4

Passage IV

(Q. Nos. 58 to 60)

If $ax^2 - bx + c = 0$ *have two distinct roots lying in the interval* $(0, 1)$; $a, b, c \in N$.

58. The least value of a is

(a) 3 (b) 4

(c) 5 (d) 6

59. The least value of b is

(a) 5 (b) 6

(c) 7 (d) 8

60. The least value of $\log_5 abc$ is

(a) 1 (b) 2

(c) 3 (d) 4

Passage V

(Q. Nos. 61 to 63)

If $2x^3 + ax^2 + bx + 4 = 0$ (*a and b are positive real numbers) has three real roots.*

61. The minimum value of a^3 is

(a) 108 (b) 216
(c) 432 (d) 864

62. The minimum value of b^3 is

(a) 432 (b) 864
(c) 1728 (d) None of these

63. The minimum value of $(a+b)^3$ is

(a) 1728 (b) 3456
(c) 6912 (d) 864

Passage VI

(Q. Nos. 64 to 66)

If $\alpha, \beta, \gamma, \delta$ *are the roots of the equation* $x^4 + Ax^3 + Bx^2 + Cx + D = 0$ *such that* $\alpha\beta = \gamma\delta = k$ *and* A, B, C, D *are the roots of* $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$ *such that* $A + B = 0$.

64. The value of $\frac{C}{A}$ is

(a) $-\frac{k}{2}$ (b) $-k$ (c) $\frac{k}{2}$ (d) k

65. The value of $(\alpha+\beta)(\gamma+\delta)$ in terms of B and k is

(a) $B-2k$ (b) $B-k$ (c) $B+k$ (d) $B+2k$

66. The correct statement is

(a) $C^2 = AD$ (b) $C^2 = A^2D$ (c) $C^2 = AD^2$ (d) $C^2 = (AD)^2$

Theory of Equations Exercise 4 : Single Integer Answer Type Questions

- This section contains **10 questions**. The answer to each question is a **single digit integer**, ranging from 0 to 9 (both inclusive).

67. The sum of all the real roots of the equation $|x-2|^2 + |x-2| - 2 = 0$ is

68. The harmonic mean of the roots of the equation $(5+\sqrt{2})x^2 - (4+\sqrt{5})x + 8 + 2\sqrt{5} = 0$ is

69. If product of the real roots of the equation, $x^2 - ax + 30 = 2\sqrt{(x^2 - ax + 45)}$, $a > 0$,

is λ and minimum value of sum of roots of the equation is μ. The value of (μ) (where $(\cdot)$ denotes the least integer function) is

70. The minimum value of $\dfrac{\left(x+\frac{1}{x}\right)^6 - \left(x^6 + \frac{1}{x^6}\right) - 2}{\left(x+\frac{1}{x}\right)^3 + x^3 + \frac{1}{x^3}}$ is

(for $x > 0$)

71. Let a, b, c, d are distinct real numbers and a, b are the roots of the quadratic equation $x^2 - 2cx - 5d = 0$. If c and d are the roots of the quadratic equation $x^2 - 2ax - 5b = 0$, the sum of the digits of numerical values of $a+b+c+d$ is

72. If the maximum and minimum values of $y = \dfrac{x^2 - 3x + c}{x^2 + 3x + c}$ are 7 and $\frac{1}{7}$ respectively, the value of c is

73. Number of solutions of the equation $\sqrt{x^2} - \sqrt{(x-1)^2} + \sqrt{(x-2)^2} = \sqrt{5}$ is

74. If α and β are the complex roots of the equation $(1+i)x^2 + (1-i)x - 2i = 0$, where $i = \sqrt{-1}$, the value of $|\alpha - \beta|^2$ is

75. If α, β be the roots of the equation $4x^2 - 16x + c = 0$, $c \in R$ such that $1 < \alpha < 2$ and $2 < \beta < 3$, then the number of integral values of c, are

76. Let r, s and t be the roots of the equation $8x^3 + 1001x + 2008 = 0$ and if $99\lambda = (r+s)^3 + (s+t)^3 + (t+r)^3$, the value of $[\lambda]$ is (where $[\cdot]$ denotes the greatest integer function)

Theory of Equations Exercise 5 : Matching Type Questions

- This section contains **4 questions**. Questions 78 and 80 have three statements (A, B and C) given in **Column I** and four statements (p, q, r and s) in **Column II** and questions 77 and 79 have three statements (A, B and C) given in **Column I** and five statements (p, q, r, s and t) in **Column II**. Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II**.

77. Column I contains rational algebraic expressions and Column II contains possible integers which lie in their range. Match the entries of Column I with one or more entries of the elements of Column II.

Column I		Column II	
(A)	$y=\dfrac{x^2-2x+4}{x^2+2x+4}, x\in R$	(p)	-2
(B)	$y=\dfrac{2x^2+4x+1}{x^2+4x+2}, x\in R$	(q)	-1
(C)	$y=\dfrac{x^2-3x+4}{x-3}, x\in R$	(r)	2
		(s)	3
		(t)	8

78.

Column I		Column II	
(A)	If a, b, c, d are four non-zero real numbers such that $(d+a-b)^2+(d+b-c)^2=0$ and the roots of the equation $a(b-c)x^2+b(c-a)x+c(a-b)=0$ are real and equal, then	(p)	$a+b+c=0$
(B)	If the equation $ax^2+bx+c=0$ and $x^3-3x^2+3x-1=0$ have a common real root, then	(q)	a, b, c are in AP
(C)	Let a, b, c be positive real numbers such that the expression $bx^2+(\sqrt{(a+c)^2+4b^2})x+(a+c)$ is non-negative, $\forall\, x\in R$, then	(r)	a, b, c are in GP
		(s)	a, b, c are in HP

79. Column I contains rational algebraic expressions and Column II contains possible integers of a.

Column I		Column II	
(A)	$y=\dfrac{ax^2+3x-4}{3x-4x^2+a}, x\in R$ and $y\in R$	(p)	0
(B)	$y=\dfrac{ax^2+x-2}{a+x-2x^2}, x\in R$ and $y\in R$	(q)	1
(C)	$y=\dfrac{x^2+2x+a}{x^2+4x+3a}, x\in R$ and $y\in R$	(r)	3
		(s)	5
		(t)	7

80.

Column I		Column II	
(A)	The equation $x^3-6x^2+9x+\lambda=0$ have exactly one root is (1, 3), then $\lvert[\lambda+1]\rvert$ is (where $[\cdot]$ denotes the greatest integer function)	(p)	0
(B)	If $-3<\dfrac{x^2-\lambda x-2}{x^2+x+1}<2, \forall\, x\in R$, then $\lvert[\lambda]\rvert$ is (where $[\cdot]$ denotes the greatest integer function)	(q)	1
(C)	If $x^2+\lambda x+1=0$ and $(b-c)x^2+(c-a)x+(a-b)=0$ have both the roots common, then $\lvert[\lambda-1]\rvert$, (where $[\cdot]$ denotes the greatest integer function)	(r)	2
		(s)	3

Theory of Equations Exercise 6 :
Statement I and II Type Questions

- **Directions** (Q. Nos. 81 to 87) are Assertion-Reason type questions. Each of these questions contains two statements:
  Statement-1 (Assertion) and **Statement-2** (Reason)
  Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below.
  (a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
  (b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
  (c) Statement1 is true, Statement-2 is false
  (d) Statement-1 is false, Statement-2 is true

81. **Statement-1** If the equation $(4p-3)x^2 + (4q-3)x + r = 0$ is satisfied by $x = a$, $x = b$ and $x = c$ (where a, b, c are distinct), then $p = q = \frac{3}{4}$ and $r = 0$.

Statement-2 If the quadratic equation $ax^2 + bx + c = 0$ has three distinct roots, then a, b and c are must be zero.

82. **Statement-1** The equation $x^2 + (2m+1)x + (2n+1) = 0$, where $m, n \in I$, cannot have any rational roots.

Statement-2 The quantity $(2m+1)^2 - 4(2n+1)$, where $m, n \in I$, can never be perfect square.

83. **Statement-1** In the equation $ax^2 + 3x + 5 = 0$, if one root is reciprocal of the other, then a is equal to 5.

Statement-2 Product of the roots is 1.

84. **Statement-1** If one root of $Ax^3 + Bx^2 + Cx + D = 0$, $A \neq 0$, is the arithmetic mean of the other two roots, then the relation $2B^3 + k_1ABC + k_2A^2D = 0$ holds good and then $(k_2 - k_1)$ is a perfect square.

Statement-2 If a, b, c are in AP, then b is the arithmetic mean of a and c.

85. **Statement-1** If x, y, z be real variables satisfying $x + y + z = 6$ and $xy + yz + zx = 8$, the range of variables x, y and z are identical.

Statement-2 $x + y + z = 6$ and $xy + yz + zx = 8$ remains same, if x, y, z interchange their positions.

86. **Statement-1** $ax^3 + bx + c = 0$, where $a, b, c \in R$ cannot have 3 non-negative real roots.

Statement-2 Sum of roots is equal to zero.

87. **Statement-1** The quadratic polynomial $y = ax^2 + bx + c$ ($a \neq 0$ and $a, b, c \in R$) is symmetric about the line $2ax + b = 0$.

Statement-2 Parabola is symmetric about its axis of symmetry.

Theory of Equations Exercise 7 :
Subjective Type Questions

- In this section, there are **24 subjective** questions.

88. For what values of m, the equation $(1+m)x^2 - 2(1+3m)x + (1+8m) = 0$ has $(m \in R)$
(i) both roots are imaginary?
(ii) both roots are equal?
(iii) both roots are real and distinct?
(iv) both roots are positive?
(v) both roots are negative?
(vi) roots are opposite in sign?
(vii) roots are equal in magnitude but opposite in sign?
(viii) atleast one root is positive?
(ix) atleast one root is negative?
(x) roots are in the ratio 2 : 3?

89. For what values of m, then equation $2x^2 - 2(2m+1)x + m(m+1) = 0$ has $(m \in R)$
(i) both roots are smaller tha 2?
(ii) both roots are greater than 2?
(iii) both roots lie in the interval (2, 3)?
(iv) exactly one root lie in the interval (2, 3)?
(v) one root is smaller than 1 and the other root is greater than 1?
(vi) one root is greater than 3 and the other root is smaller than 2?
(vii) atleast one root lies in the interval (2, 3)?
(viii) atleast one root is greater than 2?
(ix) atleast one root is smaller than 2?
(x) roots α and β, such that both 2 and 3 lie between α and β?

90. If r is the ratio of the roots of the equation $ax^2 + bx + c = 0$, show that $\frac{(r+1)^2}{r} = \frac{b^2}{ac}$.

91. If the roots of the equation $\frac{1}{x+p} + \frac{1}{x+q} = \frac{1}{r}$ are equal in magnitude but opposite in sign, show that $p + q = 2r$ and that the product of the roots is equal to $\left(-\frac{p^2+q^2}{2}\right)$.

92. If one root of the quadratic equation $ax^2 + bx + c = 0$ is equal to the nth power of the other, then show that $(ac^n)^{\frac{1}{n+1}} + (a^n c)^{\frac{1}{n+1}} + b = 0$.

93. If α, β are the roots of the equation $ax^2 + bx + c = 0$ and γ, δ those of equation $lx^2 + mx + n = 0$, then find the equation whose roots are $\alpha\gamma + \beta\delta$ and $\alpha\delta + \beta\gamma$.

94. Show that the roots of the equation
$$(a^2 - bc)x^2 + 2(b^2 - ac)x + c^2 - ab = 0$$
are equal, if either $b = 0$ or $a^3 + b^3 + c^3 - 3abc = 0$.

95. If the equation $x^2 - px + q = 0$ and $x^2 - ax + b = 0$ have a common root and the other root of the second equation is the reciprocal of the other root of the first, then prove that $(q-b)^2 = bq(p-a)^2$.

96. If the equation $x^2 - 2px + q = 0$ has two equal roots, then the equation $(1+y)x^2 - 2(p+y)x + (q+y) = 0$ will have its roots real and distinct only, when y is negative and p is not unity.

97. Solve the equation $x^{\log_x (x+3)^2} = 16$.

98. Solve the equation
$$(2+\sqrt{3})^{x^2-2x+1} + (2-\sqrt{3})^{x^2-2x-1} = \frac{101}{10(2-\sqrt{3})}.$$

99. Solve the equation $x^2 + \left(\frac{x}{x-1}\right)^2 = 8$.

100. Solve the equation
$$\sqrt{(x+8) + 2\sqrt{(x+7)}} + \sqrt{(x+1) - \sqrt{(x+7)}} = 4.$$

101. Find all values of a for which the inequation $4^{x^2} + 2(2a+1)2^{x^2} + 4a^2 - 3 > 0$ is satisfied for any x.

102. Solve the inequation $\log_{x^2+2x-3}\left(\frac{|x+4| - |x|}{x-1}\right) > 0$.

103. Solve the system $|x^2 - 2x| + y = 1, x^2 + |y| = 1$.

104. If α, β, γ are the roots of the cubic $x^3 - px^2 + qx - r = 0$. Find the equations whose roots are

(i) $\beta\gamma + \frac{1}{\alpha}, \gamma\alpha + \frac{1}{\beta}, \alpha\beta + \frac{1}{\gamma}$

(ii) $(\beta + \gamma - \alpha), (\gamma + \alpha - \beta), (\alpha + \beta - \gamma)$

Also, find the value of $(\beta + \gamma - \alpha)(\gamma + \alpha - \beta)(\alpha + \beta - \gamma)$.

105. If $A_1, A_2, A_3, \ldots, A_n, a_1, a_2, a_3, \ldots, a_n, a, b, c \in R$, show that the roots of the equation
$$\frac{A_1^2}{x-a_1} + \frac{A_2^2}{x-a_2} + \frac{A_3^2}{x-a_3} + \ldots + \frac{A_n^2}{x-a_n} = ab^2 + c^2 x + ac$$ are real.

106. For what values of the parameter a the equation $x^4 + 2ax^3 + x^2 + 2ax + 1 = 0$ has atleast two distinct negative roots?

107. If $[x]$ is the integral part of a real number x. Then solve $[2x] - [x+1] = 2x$.

108. Prove that for any value of a, the inequation $(a^2+3)x^2 + (a+2)x - 6 < 0$ is true for atleast one negative x.

109. How many real solutions of the equation $6x^2 - 77[x] + 147 = 0$, where $[x]$ is the integral part of x?

110. If α, β are the roots of the equation $x^2 - 2x - a^2 + 1 = 0$ and γ, δ are the roots of the equation $x^2 - 2(a+1)x + a(a-1) = 0$, such that $\alpha, \beta \in (\gamma, \delta)$, find the value of '$a$'.

111. If the equation $x^4 + px^3 + qx^2 + rx + 5 = 0$ has four positive real roots, find the minimum value of pr.

Theory of Equations Exercise 8 : Questions Asked in Previous 13 Years' Exam

- This section contains questions asked in **IIT-JEE, AIEEE, JEE Main & JEE Advanced** from year **2005** to year **2017**.

112. If α, β are the roots of $ax^2 + bx + c = 0$ and $\alpha + \beta$, $\alpha^2 + \beta^2, \alpha^3 + \beta^3$ are in GP, where $\Delta = b^2 - 4ac$, then **[IIT-JEE 2005, 3M]**

(a) $\Delta \neq 0$ (b) $b\Delta = 0$ (c) $cb \neq 0$ (d) $c\Delta = 0$

113. If S is a set of $P(x)$ is polynomial of degree ≤ 2 such that $P(0) = 0, P(1) = 1, P'(x) > 0, \forall x \in (0,1)$, then **[IIT-JEE 2005, 3M]**

(a) $S = 0$

(b) $S = ax + (1-a)x^2, \forall\ a \in (0, \infty)$

(c) $S = ax + (1-a)x^2, \forall\ a \in R$

(d) $S = ax + (1-a)x^2, \forall\ a \in (0, 2)$

114. If the roots of $x^2 - bx + c = 0$ are two consecutive integers, then $b^2 - 4c$ is **[AIEEE 2005, 3M]**

(a) 1 (b) 2
(c) 3 (d) 4

115. If the equation $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x = 0$, $a_1 \neq 0$, $n \geq 2$, has a positive root $x = \alpha$, then the equation $na_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \ldots + a_1 = 0$ has a positive root, which is **[AIEEE 2005, 3M]**

(a) greater than or equal to α
(b) equal to α
(c) greater than α
(d) smaller than α

116. If both the roots of the quadratic equation

$$x^2 - 2kx + k^2 + k - 5 = 0$$

are less than 5, k lies in the interval **[AIEEE 2005, 3M]**

(a) $(-\infty, 4)$ (b) $[4, 5]$
(c) $(5, 6)$ (d) $(6, \infty)$

117. Let a and b be the roots of equation $x^2 - 10cx - 11d = 0$ and those of $x^2 - 10ax - 11b = 0$ are c and d, the value of $a + b + c + d$, when $a \neq b \neq c \neq d$, is **IIT-JEE 2006, 6M]**

118. Let a, b, c be the sides of a triangle. No two of them are equal and $\lambda \in R$. If the roots of the equation $x^2 + 2(a + b + c)x + 3\lambda(ab + bc + ca) = 0$ are real, then **[IIT-JEE 2006, 3M]**

(a) $\lambda < \frac{4}{3}$ (b) $\lambda < \frac{5}{3}$
(c) $\lambda \in \left(\frac{1}{3}, \frac{5}{3}\right)$ (d) $\lambda \in \left(\frac{4}{3}, \frac{5}{3}\right)$

119. All the values of m for which both roots of the equation $x^2 - 2mx + m^2 - 1 = 0$ are greater than -2 but less than 4, lie in the interval **[AIEEE 2006, 3M]**

(a) $-2 < m < 0$ (b) $m > 3$
(c) $-1 < m < 3$ (d) $1 < m < 4$

120. If the roots of the quadratic equation $x^2 + px + q = 0$ are $\tan 30°$ and $\tan 15°$, respectively, the value of $2 + q - p$ is **[AIEEE 2006, 3M]**

(a) 2 (b) 3
(c) 0 (d) 1

121. Let α, β be the roots of the equation $x^2 - px + r = 0$ and $\frac{\alpha}{2}, 2\beta$ be the roots of the equation $x^2 - qx + r = 0$. The value of r is **[IIT-JEE 2007, 3M]**

(a) $\frac{2}{9}(p - q)(2q - p)$ (b) $\frac{2}{9}(q - p)(2p - q)$
(c) $\frac{2}{9}(q - 2p)(2q - p)$ (d) $\frac{2}{9}(2p - q)(2q - p)$

122. If the difference between the roots of the equation $x^2 + ax + 1 = 0$ is less than $\sqrt{5}$, the set of possible values of a, is **[AIEEE 2007, 3M]**

(a) $(-3, 3)$ (b) $(-3, \infty)$
(c) $(3, \infty)$ (d) $(-\infty, -3)$

123. Let a, b, c, p, q be real numbers. Suppose α, β are roots of the equation $x^2 + 2px + q = 0$ and $\alpha, \frac{1}{\beta}$ are the roots of the equation $ax^2 + 2bx + c = 0$, where $\beta^2 \notin \{-1, 0, 1\}$.

Statement-1 $(p^2 - q)(b^2 - ac) \geq 0$ and

Statement-2 $b \neq pa$ or $c \neq qa$ **[IIT-JEE 2008, 3M]**

(a) Statement-1 is true, Statement-2, is true; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement-1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

124. The quadratic equation $x^2 - 6x + a = 0$ and $x^2 - cx + 6 = 0$ have one root in common. The other roots of the first and second equations are integers in the ratio 4 : 3. The common root is **[AIEEE 2008, 3M]**

(a) 4 (b) 3 (c) 2 (d) 1

125. How many real solutions does the equation $x^7 + 14x^5 + 16x^3 + 30x - 560 = 0$ have? **[AIEEE 2008, 3M]**

(a) 1 (b) 3 (c) 5 (d) 7

126. Suppose the cubic $x^3 - px + q = 0$ has three distinct real roots, where $p > 0$ and $q < 0$. Which one of the following holds? **[AIEEE 2008, 3M]**

(a) The cubic has minima at $\left(-\sqrt{\frac{p}{3}}\right)$ and maxima at $\sqrt{\frac{p}{3}}$
(b) The cubic has minima at both $\sqrt{\frac{p}{3}}$ and $\left(-\sqrt{\frac{p}{3}}\right)$
(c) The cubic has maxima at both $\sqrt{\frac{p}{3}}$ and $\left(-\sqrt{\frac{p}{3}}\right)$
(d) The cubic has minima at $\sqrt{\frac{p}{3}}$ and maxima at $\left(-\sqrt{\frac{p}{3}}\right)$

127. The smallest value of k, for which both roots of the equation $x^2 - 8kx + 16(k^2 - k + 1) = 0$ are real, distinct and have value at least 4, is **[IIT-JEE 2009, 4M]**

(a) 6 (b) 4 (c) 2 (d) 0

128. If the roots of the equation $bx^2 + cx + a = 0$ be imaginary, then for all real values of x, the expression $3b^2x^2 + 6bcx + 2c^2$, is **[AIEEE 2009, 4M]**

(a) less than $(-4ab)$ (b) greater than $4ab$
(c) less than $4ab$ (d) greater than $(-4ab)$

129. Let p and q be real numbers such that $p \neq 0, p^3 \neq -q$. If α and β are non-zero complex numbers satisfying $\alpha + \beta = -p$ and $\alpha^3 + \beta^3 = q$, a quadratic equation having $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ as its roots, is **[IIT-JEE 2010, 3M]**

(a) $(p^3 + q)x^2 - (p^3 + 2q)x + (p^3 + q) = 0$
(b) $(p^3 + q)x^2 - (p^3 - 2q)x + (p^3 + q) = 0$
(c) $(p^3 - q)x^2 - (5p^3 - 2q)x + (p^3 - q) = 0$
(d) $(p^3 - q)x^2 - (5p^3 + 2q)x + (p^3 - q) = 0$

130. Consider the polynomial $f(x) = 1 + 2x + 3x^2 + 4x^3$. Let s be the sum of all distinct real roots of $f(x)$ and let $t = |s|$, real number s lies in the interval **[IIT-JEE 2010, 3M]**

(a) $\left(-\frac{1}{4}, 0\right)$ (b) $\left(-11, \frac{3}{4}\right)$ (c) $\left(-\frac{3}{4}, -\frac{1}{2}\right)$ (d) $\left(0, \frac{1}{4}\right)$

131. Let α and β be the roots of $x^2 - 6x - 2 = 0$, with $\alpha > \beta$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$, the value of $\frac{a_{10} - 2a_8}{2a_9}$ is **[IIT-JEE 2011, 3 and JEE Main 2015,4M]**

(a) 1 (b) 2 (c) 3 (d) 4

132. A value of b for which the equations

$$x^2 + bx - 1 = 0 \quad x^2 + x + b = 0$$

have one root in common, is **[IIT-JEE 2011, 3M]**

(a) $-\sqrt{2}$ (b) $-i\sqrt{3}, i = \sqrt{-1}$
(c) $i\sqrt{5}, i = \sqrt{-1}$ (d) $\sqrt{2}$

133. The number of distinct real roots of $x^4 - 4x^3 + 12x^2 + x - 1 = 0$ is **[IIT-JEE 2011, 4M]**

134. Let for $a \neq a_1 \neq 0$, $f(x) = ax^2 + bx + c$, $g(x) = a_1x^2 + b_1x + c_1$ and $p(x) = f(x) - g(x)$. If $p(x) = 0$ only for $x = (-1)$ and $p(-2) = 2$, the value of $p(2)$ is **[AIEEE 2011, 4M]**

(a) 18 (b) 3 (c) 9 (d) 6

135. Sachin and Rahul attempted to solve a quadratic equation. Sachin made a mistake in writing down the constant term and ended up in roots $(4, 3)$. Rahul made a mistake in writing down coefficient of x to get roots $(3, 2)$. The correct roots of equation are **[AIEEE 2011, 4M]**

(a) $-4, -3$ (b) $6, 1$ (c) $4, 3$ (d) $-6, -1$

136. Let $\alpha(a)$ and $\beta(a)$ be the roots of the equation

$$(\sqrt[3]{(1+a)} - 1)x^2 + (\sqrt{(1+a)} - 1)x + (\sqrt[6]{(1+a)} - 1) = 0,$$

where $a > -1$, then $\lim_{a \to 0^+} \alpha(a)$ and $\lim_{a \to 0^+} \beta(a)$, are **[IIT-JEE 2012, 3M]**

(a) $\left(-\frac{5}{2}\right)$ and 1 (b) $\left(-\frac{1}{2}\right)$ and (-1)
(c) $\left(-\frac{7}{2}\right)$ and 2 (d) $\left(-\frac{9}{2}\right)$ and 3

137. The equation $e^{\sin x} - e^{-\sin x} - 4 = 0$ has **[AIEEE 2012, 4M]**

(a) exactly one real root
(b) exactly four real roots
(c) infinite number of real roots
(d) no real roots

138. If the equations $x^2 + 2x + 3 = 0$ and $ax^2 + bx + c = 0$, $a, b, c \in R$ have a common root, then $a : b : c$ is **[JEE Main 2013, 4M]**

(a) 3 : 2 : 1 (b) 1 : 3 : 2 (c) 3 : 1 : 2 (d) 1 : 2 : 3

139. If $a \in R$ and the equation $-3(x - [x])^2 + 2(x - [x]) + a^2 = 0$ (where $[\cdot]$ denotes the greatest integer function) has no integral solution, then all possible values of a lie in the interval **[JEE Main 2014, 4M]**

(a) $(-2, -1)$
(b) $(-\infty, -2) \cup (2, \infty)$
(c) $(-1, 0) \cup (0, 1)$
(d) $(1,2)$

140. Let α, β be the roots of the equation $px^2 + qx + r = 0$, $p \neq 0$. If p, q, r are in AP and $\frac{1}{\alpha} + \frac{1}{\beta} = 4$, the value of $|\alpha - \beta|$, is **[JEE Main 2014, 4M]**

(a) $\frac{\sqrt{34}}{9}$ (b) $\frac{2\sqrt{13}}{9}$
(c) $\frac{\sqrt{61}}{9}$ (d) $\frac{2\sqrt{17}}{9}$

141. Let $a \in R$ and let $f : R \to R$ be given by $f(x) = x^5 - 5x + a$. Then, **[JEE Advanced 2014, 3M]**

(a) $f(x)$ has three real roots, if $a > 4$
(b) $f(x)$ has only one real root, if $a > 4$
(c) $f(x)$ has three real roots, if $a < -4$
(d) $f(x)$ has three real roots, if $-4 < a < 4$

142. The quadratic equation $p(x) = 0$ with real coefficients has purely imaginary roots. Then, $p(p(x)) = 0$ has **[JEE Advanced 2014, 3M]**

(a) only purely imaginary roots
(b) all real roots
(c) two real and two purely imaginary roots
(d) neither real nor purely imaginary roots

143. Let S be the set of all non-zero real numbers α such that the quadratic equation $\alpha x^2 - x + \alpha = 0$ has two distinct real roots x_1 and x_2 satisfying the inequality $|x_1 - x_2| < 1$.

Which of the following intervals is (are) a subset(s) of S? **[JEE Advanced 2015, 4M]**

(a) $\left(-\frac{1}{2}, -\frac{1}{\sqrt{5}}\right)$ (b) $\left(-\frac{1}{\sqrt{5}}, 0\right)$
(c) $\left(0, \frac{1}{\sqrt{5}}\right)$ (d) $\left(\frac{1}{\sqrt{5}}, \frac{1}{2}\right)$

144. The sum of all real values of x satisfying the equation $(x^2 - 5x + 5)^{x^2 + 4x - 60} = 1$ is **[JEE Main 2016, 4M]**

(a) 6 (b) 5 (c) 3 (d) −4

145. Let $-\frac{\pi}{6} < \theta < -\frac{\pi}{12}$. Suppose α_1 and β_1 are the roots of equation $x^2 - 2x\sec\theta + 1 = 0$ and α_2 and β_2 are the roots of the equation $x^2 + 2x\tan\theta - 1 = 0$. If $\alpha_1 > \beta_1$ and $\alpha_2 > \beta_2$, then $\alpha_1 + \beta_2$ equals **[JEE Advanced 2016, 3M]**

(a) $2(\sec\theta - \tan\theta)$ (b) $2\sec\theta$

(c) $-2\tan\theta$ (d) 0

146. If for a positive integer n, the quadratic equation $x(x+1) + (x+1)(x+2) \ldots + (x + \overline{n-1})(x+n) = 10n$ has two consecutive integral solutions, then n is equal to **[JEE Main 2017, 4M]**

(a) 11 (b) 12

(c) 9 (d) 10

Answers

Exercise for Session 1

1.(b) **2.** (c) **3.** (a) **4.** (b) **5.** (a) **6.** (a)
7.(c) **8.** (b) **9.** (c) **10.** (d) **11.** (b)

Exercise for Session 2

1.(a) **2.** (c) **3.** (b) **4.** (a) **5.**(d) **6.** (c)
7. (c) **8.** (b) **9.** (c) **10.** (a)

Exercise for Session 3

1.(a) **2.** (b) **3.** (c) **4.** (c) **5.**(d) **6.** (c)
7.(c) **8.** (a) **9.** (a) **10.** (d)

Exercise for Session 4

1.(c) **2.** (c) **3.** (c) **4.** (d) **5.**(a) **6.** (d)
7.(d) **8.** (c) **9.** (b) **10.** (d)

Exercise for Session 5

1.(a) **2.** (a) **3.** (b) **4.** (c) **5.**(c) **6.** (b)
7.(a) **8.** (b) **9.** (b) **10.** (d)

Chapter Exercises

1.(b) **2.** (b) **3.** (b) **4.** (c) **5.** (a) **6.** (a)
7.(c) **8.** (b) **9.** (c) **10.** (a) **11.** (b) **12.** (c)
13.(b) **14.** (b) **15.** (b) **16.** (c) **17.** (b) **18.** (b)
19.(c) **20.** (d) **21.** (b) **22.** (a) **23.** (c) **24.** (b)
25.(a) **26.** (a) **27.** (b) **28.** (c) **29.** (a) **30.** (c)
31.(a,b) **32.**(b,c) **33.**(a,d) **34.**(a,b,c,d) **35.** (b,d) **36.** (a,b,c,d)
37.(a, b,c,d) **38.** (a,c) **39.** (a,b) **40.** (a,b,c,d) **41.** (a,c) **42.** (b,c)
43.(c,d) **44.**(a,c,d) **45.**(a,c,d)
46.(d) **47.** (b) **48.** (d) **49.**(d) **50.**(d) **51.** (c)
52.(b) **53.** (b) **54.** (c) **55.** (b) **56.** (d) **57.** (a)
58.(c) **59.** (a) **60.** (b) **61.** (c) **62.** (b) **63.** (c)
64.(d) **65.** (a) **66.** (b) **67.** (4) **68.** (4) **69.** (9)
70. (6) **71.** (3) **72.** (4) **73.** (2) **74.** (5) **75.**(3)
76.(7) **77.** (A) $\to$ (r,s), (B) $\to$ (p,q,r,s,t), (C) $\to$ (p,q,t)
78. (A) $\to$ (q,r,s), (B) $\to$ (p), (C) $\to$ (q)
79. (A) $\to$ (q,r,s,t), (B) $\to$ (q,r), (C) $\to$ (p,q)
80.(A) $\to$ (p,q,r,s), (B) $\to$ (p,q), (C) $\to$ (s) **81.**(d) **82.** (a)
83. (a) **84.** (a) **85.** (a) **86.** (d) **87.**(a)

88. (i) $m \in (0, 3)$ (ii) $m = 0, 3$

(iii) $m \in (-\infty, 0) \cup (3, \infty)$ (iv) $m \in (-\infty, -1) \cup [3, \infty)$

(v) $m \in \phi$ (vi) $m \in (-1, -1/8)$

(vii) $m = -1/3$ (viii) $m \in (-\infty, -1) \cup (-1, -1/8) \cup [3, \infty)$

(ix) $m \in (-1, -1/8)$ (x) $m = \frac{81 \pm \sqrt{6625}}{32}$

89. (i) $m \in \left(-\infty, \frac{7-\sqrt{33}}{2}\right)$ (ii) $m \in \left(\frac{7+\sqrt{33}}{2}, \infty\right)$ (iii) $m \in \phi$

(iv) $m \in \left(\frac{7-\sqrt{33}}{2}, \frac{11-\sqrt{73}}{2}\right) \cup \left(\frac{7+\sqrt{33}}{2}, \frac{11+\sqrt{73}}{2}\right)$

(v) $m \in (0, 3)$ (vi) $m \in \left(\frac{7-\sqrt{33}}{2}, \frac{7+\sqrt{33}}{2}\right)$

(vii) $m \in \left(\frac{7-\sqrt{33}}{2}, \frac{11-\sqrt{73}}{2}\right) \cup \left(\frac{7+\sqrt{33}}{2}, \frac{11+\sqrt{73}}{2}\right)$

(viii) $m \in \left(\frac{7-\sqrt{33}}{2}, \frac{7+\sqrt{33}}{2}\right) \cup \left(\frac{7+\sqrt{33}}{2}, \infty\right)$

(ix) $m \in \left(-\infty, \frac{7-\sqrt{33}}{2}\right) \cup \left(\frac{7-\sqrt{33}}{2}, \frac{7+\sqrt{33}}{2}\right)$

(x) $m \in \left(\frac{11-\sqrt{73}}{2}, \frac{7+\sqrt{33}}{2}\right)$

93. $a^2l^2x^2 - ablmx + (b^2 - 2ac)ln + (m^2 - 2ln)ac = 0$

97. $x \in \phi$

98. $x_1 = 1 + \sqrt{1 + \log_{2+\sqrt{3}} 10}$, $x_2 = 1 - \sqrt{1 + \log_{2+\sqrt{3}} 10}$

99. $x_1 = 2$, $x_2 = -1 + \sqrt{3}$ and $x_3 = -1 - \sqrt{3}$

100. $x_1 = 2$

101. $a \in (-\infty, -1) \cup \left(\frac{\sqrt{3}}{2}, \infty\right)$

102. $x \in (-1 - \sqrt{5}, -3) \cup (\sqrt{5} - 1, 5)$

103. The pairs (0, 1), (1, 0), $\left(\frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)$ are solutions of the original system of equations.

104. (i) $ry^3 - q(r+1)y^2 + p(r+1)^2 y - (r+1)^3 = 0$

(ii) $y^3 - py^2 + (4q - p^2)y + (8r - 4pq + p^3) = 0$ and $4pq - p^3 - 8r$

106. $a \in \left(\frac{3}{4}, \infty\right)$ **107.** $x_1 = -1, x_2 = -1/2$ **109.** Four

110. $a \in \left(-\frac{1}{4}, 1\right)$ **111.** 80 **112.** (d) **113.** (d) **114.** (a)
115. (d) **116.** (a) **117.** 1210 **118.**(a) **119.** (c) **120.** (b)
121. (d) **122.** (a) **123.** (b) **124.** (c) **125.** (a) **126.** (d)
127. (c) **128.** (d) **129.** (b) **130.** (c) **131.** (c) **132.** (b)
133.(2) **134.** (a) **135.** (b) **136.** (b) **137.** (d) **138.** (d)
139. (c) **140.** (b) **141.**(b,d) **142.** (d) **143.** (a, d)
144. (c) **145.** (c) **146.** (a)

Solutions

1. We have,

$$2(a-b)\,x^2 - 11(a+b+c)\,x - 3(a-b) = 0$$

$\therefore \quad D = \{-11(a+b+c)\}^2 - 4\cdot 2(a-b)\cdot(-3)(a-b)$

$$= 121(a+b+c)^2 + 24(a-b)^2 > 0$$

Therefore, the roots are real and unequal.

2. Here, $a < 0$

Cut-off Y-axis, $x = 0$

$\Rightarrow \quad y = c < 0$ [from graph]

$\therefore \quad c < 0$

x-coordinate of vertex > 0

$\Rightarrow \quad -\frac{b}{2a} > 0$

$\Rightarrow \quad \frac{b}{a} < 0$

But $a < 0$

$\therefore \quad b > 0$

and y-coordinate of vertex < 0

$\Rightarrow \quad -\frac{D}{4a} < 0 \Rightarrow \frac{D}{4a} > 0$

$\therefore \quad D < 0 \quad [\because a < 0]$

i.e. $b^2 - 4ac < 0$

$\therefore \quad \frac{c}{a} > 0 \quad [\because c < 0, a < 0]$

3. Sum of the roots $= -\frac{(a+3)}{a} = I^+$ [let]

$\therefore \quad a = \left(-\frac{3}{I^+ + 1}\right)$ ...(i)

Product of the roots $= \alpha\beta = \frac{a-3}{a} = I^+ + 2$ ...(ii)

and $D = (a+3)^2 - 4a(a+3)$

$$= \frac{9}{(I^+ + 1)^2}\{(I^+ - 2)^2 - 12\}$$ [from Eq. (i)]

D must be perfect square, then $I^+ = 6$

From Eq. (ii),

Product of the roots $= I^+ + 2 = 6 + 2 = 8$

4. Let α be one root of

$$x^2 - 3ax + f(a) = 0$$

$\Rightarrow \quad \alpha + 2\alpha = 3a \Rightarrow 3\alpha = 3a$

$\Rightarrow \quad \alpha = a$ ...(i)

and $\alpha \cdot 2\alpha = f(a)$

$\Rightarrow \quad f(a) = 2\alpha^2 = 2a^2$ [using Eq. (i)]

$\Rightarrow \quad f(x) = 2x^2$

5. $\because \quad x_1 x_2 = 4$ ...(i)

and $\frac{x_1}{x_1 - 1} + \frac{x_2}{x_2 - 1} = 2$

$\Rightarrow \quad 2x_1x_2 - x_1 - x_2 = 2(x_1x_2 - x_1 - x_2 + 1)$

$\Rightarrow \quad 8 - x_1 - x_2 = 2(4 - x_1 - x_2 + 1)$ [from Eq. (i)]

or $x_1 + x_2 = 2$ ...(ii)

From Eqs. (i) and (ii), required equation is

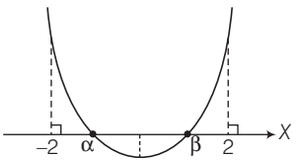
$$x^2 - (x_1 + x_2)\,x + x_1x_2 = 0$$

or $x^2 - 2x + 4 = 0$

6. Let $f(x) = x^2 - 2ax + a^2 - 1$

Now, four cases arise:

***Case* I** $D \geq 0$



$\Rightarrow \quad (-2a)^2 - 4\cdot 1(a^2 - 1) \geq 0$

$\Rightarrow \quad 4 \geq 0$

$\therefore \quad a \in R$

***Case* II** $f(-2) > 0$

$\Rightarrow \quad 4 + 4a + a^2 - 1 > 0$

$\Rightarrow \quad a^2 + 4a + 3 > 0$

$\Rightarrow \quad (a+1)(a+3) > 0$

$\therefore \quad a \in (-\infty, -3) \cup (-1, \infty)$

***Case* III** $f(2) > 0$

$\Rightarrow \quad 4 - 4a + a^2 - 1 > 0$

$\Rightarrow \quad a^2 - 4a + 3 > 0$

$\Rightarrow \quad (a-1)(a-3) > 0$

$\therefore \quad a \in (-\infty, 1) \cup (3, \infty)$

***Case* IV** $-2 <$ x-coordinate of vertex < 2

$\Rightarrow \quad -2 < 2a < 2$

$\therefore \quad a \in (-1, 1)$

Combining all cases, we get $a \in (-1, 1)$

Hence, $[a] = -1, 0$

7. We have, $-\left(\frac{-4a}{2(-2)}\right) = -2$

$\Rightarrow \quad a = 2$

$\therefore \quad y = -2x^2 - 8x + \lambda$ ...(i)

Since, Eq. (i) passes through points $(-2, 7)$

$\therefore \quad 7 = -2(-2)^2 - 8(-2) + \lambda$

$\Rightarrow \quad 7 = -8 + 16 + \lambda$

$\therefore \quad \lambda = -1$

8. Since, the coefficient of $n^2 = (4p - p^2 - 5) < 0$

Therefore, the graph is open downward.

According to the question, 1 must lie between the roots.

Hence, $f(1) > 0$

$\Rightarrow 4p - p^2 - 5 - 2p + 1 + 3p > 0$

$\Rightarrow -p^2 + 5p - 4 > 0$

$\Rightarrow p^2 - 5p + 4 < 0$

$\Rightarrow (p-4)(p-1) < 0$

$\Rightarrow 1 < p < 4$

$\therefore p = 2, 3$

Hence, number of integral values of p is 2.

9. We have, $3^{2x^2} - 2 \cdot 3^{x^2 + x + 6} + 3^{2(x+6)} = 0$

$\Rightarrow (3^{x^2} - 3^{x+6})^2 = 0$

$\Rightarrow 3^{x^2} - 3^{x+6} = 0$

$\Rightarrow 3^{x^2} = 3^{x+6} \Rightarrow x^2 = x + 6$

$\Rightarrow x^2 - x - 6 = 0$

$\Rightarrow (x-3)(x+2) = 0$

$\therefore x = \{-2, 3\}$

10. Given, $b^2 - 4ac = p^2 - 4aq$ ...(i)

and $f(x) = g(x)$

$\Rightarrow ax^2 + bx + c = ax^2 + px + q$

$\Rightarrow (b-p)x = q - c$

$\therefore x = \dfrac{q-c}{b-p} = \alpha$ [given] ...(ii)

From Eq. (i), we get

$(b+p)(b-p) + 4a(q-c) = 0$

$\Rightarrow (b+p)(b-p) + 4a\alpha(b-p) = 0$ [from Eq. (ii)]

or $\alpha = -\dfrac{(b+p)}{4a}$ [$\because b \neq p$]

$= \dfrac{\left(-\dfrac{b}{a}\right) + \left(-\dfrac{p}{a}\right)}{4}$

$= \dfrac{\left[\begin{array}{l}\text{Sum of the roots of } (f(x) = 0) \\ + \text{ Sum of the roots of } (g(x) = 0)\end{array}\right]}{4}$

$=$ AM of the roots of $f(x) = 0$

and $g(x) = 0$

11. Let α and β be the roots of $ax^2 + bx + c = 0$.

$\therefore x_1 = \dfrac{\alpha + \beta}{2} = -\dfrac{b}{2a}$

and $x_2 = \dfrac{2\alpha\beta}{\alpha + \beta} = \dfrac{2 \cdot \dfrac{c}{a}}{-\dfrac{b}{a}} = -\dfrac{2c}{b}$

$\therefore$ The required equation is

$$x^2 - \left[\left(-\frac{b}{2a}\right) + \left(-\frac{2c}{b}\right)\right]x + \frac{2bc}{2ab} = 0$$

i.e. $2abx^2 + (b^2 + 4ac)x + 2bc = 0$

12. Let α_1, α_2 and α_3 be the roots of $f(x) = 0$, such that

$$\alpha_1 < \alpha_2 < \alpha_3$$

and $g(x)$ can take all values from $[-6, \infty)$.

$\because g(x) = (x+1)^2 - 6 \geq -6$

$\therefore \alpha_3 \leq -7, \alpha_2 \leq -8, \alpha_1 \leq -9$

$\therefore a + b + c \geq 719$

$\therefore$ Minimum value of $a + b + c$ is 719.

$\because \alpha_1 + \alpha_2 + \alpha_3 = -a$

$\Rightarrow -a \leq -24$

$\Rightarrow a \geq 24$

$\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 = b$

$\Rightarrow b \geq 191$

and $\alpha_1\alpha_2\alpha_3 = -c$

$\Rightarrow -c \leq -504$

$\Rightarrow c \geq 504$

$\therefore a + b + c \geq 719$

Hence, minimum value of $a + b + c$ is 719.

13. $\because \sum_{k=1}^{n} (x + k - 1)(x + k) = 10n$

$\Rightarrow \sum_{k=1}^{n} x^2 + x(2k-1) + (k-1)k = 10n$

$\Rightarrow nx^2 + x(1 + 3 + 5 + \ldots + (2n-1))$

$+ (0 + 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + (n-1)n) = 10n$

$\Rightarrow nx^2 + x \cdot \dfrac{n}{2}(1 + 2n - 1)$

$+ \left(\dfrac{n(n+1)(2n+1)}{6} - \dfrac{n(n+1)}{2}\right) = 10n$

$\Rightarrow nx^2 + n^2x + \dfrac{n(n^2-1)}{3} = 10n$

$\Rightarrow x^2 + nx + \dfrac{(n^2 - 31)}{3} = 0$ [dividing by n]

$\because (\alpha + 1) - \alpha = \dfrac{\sqrt{D}}{1}$

$1 = \sqrt{D}$

$\Rightarrow D = 1$

$\Rightarrow n^2 - 4 \cdot 1 \cdot \dfrac{(n^2 - 31)}{3} = 1$

$\Rightarrow 3n^2 - 4n^2 + 124 = 3$

$\Rightarrow n^2 = 121$

$\therefore n = 11$

14. Since, 2 is only even prime.

Therefore, we have

$2^2 + \lambda \cdot 2 + 12 = 0$

$\Rightarrow \lambda = 8$

$\therefore x^2 + \lambda x + \mu = 0$

$\Rightarrow x^2 + 8x + \mu = 0$ ...(i)

But Eq. (i) has equal roots.

$\therefore D = 0$

$\Rightarrow 8^2 - 4 \cdot 1 \cdot \mu = 0$

$\Rightarrow \mu = 16$

15. We have, $\sqrt{x}+\sqrt{x-\sqrt{(1-x)}}=1$

$\Rightarrow \quad \sqrt{x-\sqrt{1-x}}=1-\sqrt{x}$

On squaring both sides, we get

$$x-\sqrt{1-x}=1+x-2\sqrt{x}$$

$\Rightarrow \quad -\sqrt{1-x}=1-2\sqrt{x}$

Again, squaring on both sides, we get

$$1-x=1+4x-4\sqrt{x}$$

$$4\sqrt{x}=5x$$

$\Rightarrow \quad \sqrt{x}=\frac{4}{5}$ [on squaring both sides]

$\Rightarrow \quad x=\frac{16}{25}$

Hence, the number of real solutions is 1.

16. Let $x=\sqrt{7+\sqrt{7-\sqrt{7+\sqrt{7-\ldots\infty}}}}$

$\Rightarrow \quad x=\sqrt{7+\sqrt{7-x}}$ [on squaring both sides]

$\Rightarrow \quad x^2-7=\sqrt{7-x}$

$\Rightarrow \quad (x^2-7)^2=7-x$ [again, squaring on both sides]

$\Rightarrow \quad x^4-14x^2+x+42=0$

$\Rightarrow \quad (x-3)(x^3+3x^2-5x-14)=0$

$\Rightarrow \quad (x-3)(x+2)(x^2+x-7)=0$

$\Rightarrow \quad x=3,-2,\frac{-1\pm\sqrt{29}}{2}$

$\therefore \quad x=3$ [$\because x>\sqrt{7}$]

17. Let $y=2(k-x)(x+\sqrt{(x^2+k^2)})$

$\Rightarrow \quad y-2(k-x)x=2(k-x)\sqrt{(x^2+k^2)}$

On squaring both sides, we get

$\Rightarrow \quad y^2+4(k-x)^2x^2-4xy(k-x)=4(k-x)^2(x^2+k^2)$

$\Rightarrow \quad y^2-4xy(k-x)=4(k-x)^2k^2$

$\Rightarrow \quad 4(k^2-y)x^2-4(2k^3-ky)x-y^2+4k^4=0$

Since, x is real.

$\therefore \quad D\geq 0$

$\Rightarrow \quad 16(2k^3-ky)^2-4\cdot 4(k^2-y)(4k^4-y^2)\geq 0$

[using, $b^2-4ac\geq 0$]

$\Rightarrow \quad 4k^6+k^2y^2-4k^4y-(-k^2y^2+4k^6+y^3-4yk^4)\geq 0$

$\Rightarrow \quad 2k^2y^2-y^3\geq 0$

$\Rightarrow \quad y^2(y-2k^2)\leq 0$

$\therefore \quad y\leq 2k^2$

18. We have, $\frac{1}{3}<\frac{x^2-2x+4}{x^2+2x+4}<3,\forall\ x\in R$

$$\frac{1}{3}<\frac{x^2+2x+4}{x^2-2x+4}<3,\forall\ x\in R$$

Let $y=\frac{9\cdot 3^{2x}+6\cdot 3^x+4}{9\cdot 3^{2x}-6\cdot 3^x+4}=\frac{(3^{x+1})^2+2\cdot 3^{x+1}+4}{(3^{x+1})^2-2\cdot 3^{x+1}+4}$

$=\frac{t^2+2t+4}{t^2-2t+4}$, where $t=3^{x+1}$

$\Rightarrow \quad (y-1)t^2-2(y+1)t+4(y-1)=0$

By the given condition, for every $t\in R$,

$$\frac{1}{3}<y<3 \qquad \ldots(i)$$

But $\quad t=3^{x+1}>0$

We have, product of the roots $=4>0$, which is true.

And sum of the roots $=\frac{2(y+1)}{(y-1)}>0$

$\Rightarrow \quad \frac{y+1}{y-1}>0$

$\therefore \quad y\in(-\infty,-1)\cup(1,\infty) \qquad \ldots(ii)$

From Eqs. (i) and (ii), we get

$$1<y<3$$

19. Since α, β and γ are the roots of

$$(x-a)(x-b)(x-c)=d$$

$\Rightarrow \quad (x-a)(x-b)(x-c)-d=(x-\alpha)(x-\beta)(x-\gamma)$

$\Rightarrow \quad (x-\alpha)(x-\beta)(x-\gamma)+d=(x-a)(x-b)(x-c)$

$\Rightarrow$ a, b and c are the roots of

$(x-\alpha)(x-\beta)(x-\gamma)+d=0$

20. Since, all the coefficients of given equation are not real.

Therefore, other root $\neq 3+i$.

Let other root be α.

Then, sum of the roots $=\frac{2(1+i)}{i}$

$\Rightarrow \quad \alpha+3-i=\frac{2(1+i)}{i}$

$\Rightarrow \quad \alpha+3-i=2-2i$

$\therefore \quad \alpha=-1-i$

21. We have, $|[x]-2x|=4$

$\Rightarrow \quad |[x]-2([x]+\{x\})|=4$

$\Rightarrow \quad |[x]+2\{x\}|=4$

which is possible only when

$$2\{x\}=0,1$$

If $\{x\}=0$, then $[x]=\pm 4$ and then $x=-4, 4$ and if $\{x\}=\frac{1}{2}$, then

$$[x]+1=\pm 4$$

$\Rightarrow \quad [x]=3,-5$

$\therefore \quad x=3+\frac{1}{2}$ and $-5+\frac{1}{2}$

$\Rightarrow \quad x=\frac{7}{2},-\frac{9}{2} \Rightarrow x=-4,-\frac{9}{2},\frac{7}{2},4$

22. We know that, x^2+x+1 is a factor of ax^3+bx^2+cx+d.

Hence, roots of $x^2+x+1=0$ are also roots of $ax^3+bx^2+cx+d=0$. Since, ω and ω^2

$\left(\text{where } \omega = -\frac{1}{2} + \frac{3i}{2}\right)$ are two complex roots of $x^2 + x + 1 = 0$.

Therefore, ω and ω^2 are two complex roots of $ax^3 + bx^2 + cx + d = 0$.

We know that, a cubic equation has atleast one real root. Let real root be α. Then,

$$\alpha \cdot \omega \cdot \omega^2 = -\frac{d}{a} \Rightarrow \alpha = -\frac{d}{a}$$

23. We have, $\sqrt{(5x^2 - 8x + 3)} - \sqrt{(5x^2 - 9x + 4)}$

$$= \sqrt{(2x^2 - 2x)} - \sqrt{(2x^2 - 3x + 1)}$$

$$\Rightarrow \sqrt{(5x-3)(x-1)} - \sqrt{(5x-4)(x-1)} = \sqrt{2x(x-1)} - \sqrt{(2x-1)(x-1)}$$

$$\Rightarrow \sqrt{x-1}\,(\sqrt{5x-3} - \sqrt{5x-4}) = \sqrt{x-1}\,(\sqrt{2x} - \sqrt{2x-1})$$

$$\Rightarrow \sqrt{x-1} = 0$$

$$\Rightarrow x = 1$$

24. We have, $(a + \sqrt{b})(a - \sqrt{b}) = a^2 - b = 1$ [given]

$$\therefore (a + \sqrt{b})^{x^2 - 15} + (a - \sqrt{b})^{x^2 - 15} = 2a$$

$$\Rightarrow (a + \sqrt{b})^{x^2 - 15} + \frac{1}{(a + \sqrt{b})^{x^2 - 15}} = 2a$$

Let $y = (a + \sqrt{b})^{x^2 - 15}$

$$\Rightarrow y + \frac{1}{y} = 2a \Rightarrow y^2 - 2ay + 1 = 0$$

$$\Rightarrow y = \frac{2a \pm \sqrt{4a^2 - 4}}{2} = a \pm \sqrt{a^2 - 1}$$

$$\therefore y = a \pm \sqrt{b} = (a + \sqrt{b})^{\pm 1} \quad [\because a^2 - b = 1]$$

$$\Rightarrow (a + \sqrt{b})^{x^2 - 15} = (a + \sqrt{b})^{\pm 1}$$

$$\therefore x^2 - 15 = \pm 1$$

$$\Rightarrow x^2 = 15 \pm 1 \Rightarrow x^2 = 16, 14$$

$$\Rightarrow x = \pm 4, \pm \sqrt{14}$$

25. We have, $x^2 - xy + y^2 = 4(x + y - 4)$

$$\Rightarrow x^2 - x(y + 4) + y^2 - 4y + 16 = 0$$

$$\because x \in R$$

$$\therefore (-(y + 4))^2 - 4 \cdot 1 \cdot (y^2 - 4y + 16) \geq 0$$

[using, $b^2 - 4ac \geq 0$]

$$\Rightarrow y^2 + 8y + 16 - 4y^2 + 16y - 64 \geq 0$$

$$\Rightarrow 3y^2 - 24y + 48 \leq 0$$

$$\Rightarrow y^2 - 8y + 16 \leq 0 \Rightarrow (y - 4)^2 \leq 0$$

$$\therefore (y - 4)^2 = 0$$

$$\therefore y = 4$$

Then, $x^2 - 4x + 16 = 4(x + 4 - 4)$

$$x^2 - 8x + 16 = 0$$

$$(x - 4)^2 = 0$$

$$x = 4$$

Number of pairs is 1 i.e., (4, 4).

26. Since, 3789108 is an even integer. Therefore, $x^4 - y^4$ is also an even integer. So, either both x and y are even integers or both of them are odd integers.

Now, $x^4 - y^4 = (x - y)(x + y)(x^2 + y^2)$

$\Rightarrow x - y, x + y, x^2 + y^2$ must be even integers.

Therefore, $(x - y)(x + y)(x^2 + y^2)$ must be divisible by 8. But 3789108 is not divisible by 8. Hence, the given equation has no solution.

$\therefore$ Number of solutions = 0

27. We have, $x^3 + ax + 1 = 0$

or $x^4 + ax^2 + x = 0$ ...(i)

and $x^4 + ax^2 + 1 = 0$ ...(ii)

From Eqs. (i) and (ii), we get

$$x - 1 = 0$$

$$\Rightarrow x = 1$$

which is a common root.

$$\therefore 1 + a + 1 = 0$$

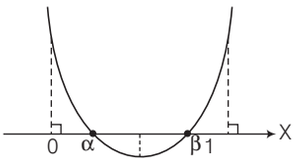
$$\Rightarrow a = -2$$

28. $\because (1 - a^2)x^2 + 2ax - 1 = 0$

$$1 - a^2 \neq 0$$

$$x^2 + \left(\frac{2a}{1 - a^2}\right)x - \left(\frac{1}{1 - a^2}\right) = 0$$

Let $f(x) = x^2 + \left(\frac{2a}{1 - a^2}\right)x - \left(\frac{1}{(1 - a^2)}\right)$



The following cases arise:

***Case* I** $D \geq 0$

$$\left(\frac{2a}{1 - a^2}\right)^2 - 4 \cdot 1 \cdot \left(\frac{-1}{1 - a^2}\right) \geq 0$$

$$\Rightarrow \frac{4a^2}{(1 - a^2)^2} + \frac{4}{(1 - a^2)} \geq 0$$

$$\Rightarrow \frac{4a^2 + 4 - 4a^2}{(1 - a^2)^2} \geq 0$$

$$\Rightarrow \frac{4}{(1 - a^2)^2} \geq 0 \quad \text{[always true]}$$

***Case* II** $f(0) > 0$

$$\Rightarrow \frac{-1}{(1 - a^2)} > 0 \Rightarrow \frac{1}{1 - a^2} < 0$$

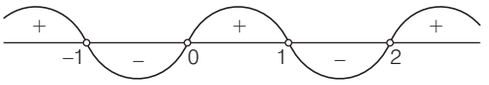
$$\Rightarrow 1 - a^2 < 0$$

$$\therefore a \in (-\infty, -1) \cup (1, \infty)$$

***Case* III** $f(1) > 0$

$$\Rightarrow 1 + \frac{2a}{(1 - a^2)} - \frac{1}{(1 - a^2)} > 0$$

$$\Rightarrow \quad \frac{1-a^2+2a-1}{(1-a^2)} > 0 \;\Rightarrow\; \frac{a^2-2a}{1-a^2} < 0$$



$$\Rightarrow \quad \frac{a(a-2)}{(a+1)(a-1)} > 0$$

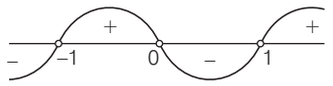
$$\therefore \quad a \in (-\infty, -1) \cup (0, 1) \cup (2, \infty)$$

***Case* IV** $0 <$ x-coordinate of vertex < 1

$$\Rightarrow \quad 0 < -\frac{2a}{2(1-a^2)} < 1 \;\Rightarrow\; 0 < \frac{a}{a^2-1} < 1$$

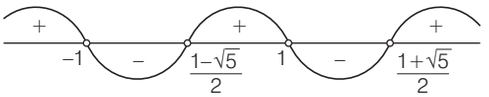
$$\Rightarrow \quad 0 < \frac{a}{(a+1)(a-1)} \text{ and } 1 - \frac{a}{a^2-1} > 0$$

$$\Rightarrow \quad \frac{a}{(a+1)(a-1)} > 0$$



$$\Rightarrow \quad a \in (-1, 0) \cup (1, \infty)$$

and $$\frac{\left(a - \frac{1+\sqrt{5}}{2}\right)\left(a - \frac{1-\sqrt{5}}{2}\right)}{(a+1)(a-1)} > 0$$



and $$a \in (-\infty, -1) \cup \left(\frac{1-\sqrt{5}}{2}, 1\right) \cup \left(\frac{1+\sqrt{5}}{2}, \infty\right)$$

$$\therefore \quad a \in \left(\frac{1-\sqrt{5}}{2}, 0\right) \cup \left(\frac{1+\sqrt{5}}{2}, \infty\right)$$

Combining all cases, we get

$$a > 2$$

29. We have, $x - \sqrt{1-|x|} < 0$ ...(i)

which is defined only when

$$1 - |x| \geq 0$$

$$\Rightarrow \quad |x| \leq 1$$

$$\Rightarrow \quad x \in [-1, 1]$$

Now, from Eq. (i), we get

$$x < \sqrt{1-|x|}$$

***Case* I** If $x \geq 0$, i.e., $0 \leq x \leq 1$

$$x - \sqrt{(1-|x|)} < 0$$

$$\Rightarrow \quad x < \sqrt{(1-x)}$$

On squaring both sides, we get

$$x^2 + x - 1 < 0$$

$$\Rightarrow \quad \frac{-1-\sqrt{5}}{2} < x < \frac{-1+\sqrt{5}}{2}$$

But $$x \geq 0$$

$$\therefore \quad x \in \left[0, \frac{-1+\sqrt{5}}{2}\right)$$

***Case* II** If $x < 0$, i.e., $-1 \leq x < 0$

$$x - \sqrt{(1+x)} < 0$$

$$\Rightarrow \quad x < \sqrt{1+x} \qquad \text{[always true]}$$

$$x \in [-1, 0)$$

Combining both cases, we get

$$x \in \left[-1, \frac{-1+\sqrt{5}}{2}\right)$$

30. We have, $(a \cdot 2b - 2c \cdot a)(2c \cdot c - b \cdot 2b) = (ba - ca)^2$

$$\Rightarrow \quad 2a(b-c) \cdot 2(c^2 - b^2) = a^2(b-c)^2$$

$$\Rightarrow \quad 4a(c-b)(c+b) = a^2(b-c) \qquad [\because b \neq c]$$

$$\Rightarrow \quad 4a(c+b) = -a^2$$

$$\Rightarrow \quad a + 4b + 4c = 0$$

31. $0 < a < b < c, \alpha + \beta = \left(-\frac{b}{a}\right)$ and $\alpha\beta = \frac{c}{a}$

For non-real complex roots,

$$b^2 - 4ac < 0$$

$$\Rightarrow \quad \frac{b^2}{a^2} - \frac{4c}{a} < 0$$

$$\Rightarrow \quad (\alpha+\beta)^2 - 4\alpha\beta < 0$$

$$\Rightarrow \quad (\alpha-\beta)^2 < 0$$

$$\because \quad 0 < a < b < c$$

$\therefore$ Roots are conjugate, then $|\alpha| = |\beta|$

But $$\alpha\beta = \frac{c}{a}$$

$$|\alpha\beta| = \left|\frac{c}{a}\right| > 1 \qquad \left[\because a < c, \therefore \frac{c}{a} > 1\right]$$

$$\Rightarrow \quad |\alpha|\,|\beta| > 1$$

$$\Rightarrow \quad |\alpha|^2 > 1 \text{ or } |\alpha| > 1$$

32. Given equation is

$$Ax^2 - |G|\,x - H = 0 \qquad \text{...(i)}$$

$\therefore$ Discriminant $= (-|G|)^2 - 4A(-H)$

$$= G^2 + 4AH$$

$$= G^2 + 4G^2 \qquad [\because G^2 = AH]$$

$$= 5G^2 > 0$$

$\therefore$ Roots of Eq. (i) are real and distinct.

$\because$ $A = \frac{a+b}{2} > 0, G = \sqrt{ab} > 0, H = \frac{2ab}{a+b} > 0$

[$\because a$ and b are two unequal positive integers]

Let α and β be the roots of Eq. (i). Then,

$$\alpha + \beta = \frac{|G|}{A} > 0$$

and $$\alpha\beta = -\frac{H}{A} < 0$$

and $$\alpha - \beta = \frac{\sqrt{D}}{A} = \frac{G\sqrt{5}}{A} > 0$$

$\therefore \quad \alpha = \dfrac{|G| + G\sqrt{5}}{2A} > 0$

and $\quad \beta = \dfrac{|G| - G\sqrt{5}}{2A} < 0$

Exactly one positive root and atleast one root which is negative fraction.

33. It is clear from graph that the equation $y = ax^2 + bx + c = 0$ has two real and distinct roots. Therefore,

$$b^2 - 4ac > 0 \quad \ldots(i)$$

$\because$ Parabola open downwards.

$\therefore \quad a < 0$

and $y = ax^2 + bx + c$ cuts-off Y-axis at, $x = 0$.

$\therefore \quad y = c < 0$

$\Rightarrow \quad c < 0$

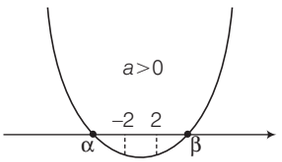
and x-coordinate of vertex > 0

$\Rightarrow \quad -\dfrac{b}{2a} > 0 \Rightarrow \dfrac{b}{a} < 0$

$\Rightarrow \quad b > 0 \quad [\because a < 0]$

It is clear that a and b are of opposite signs.

34. Let $y = ax^2 + bx + c$



Consider the following cases:

***Case* I** $D > 0$

$\Rightarrow \quad b^2 - 4ac > 0$

***Case* II** $af(-2) < 0$

$\Rightarrow \quad a(4a - 2b + c) < 0$

$\Rightarrow \quad 4a - 2b + c < 0$

***Case* III** $af(2) > 0$

$\Rightarrow \quad a(4a + 2b + c) > 0$

$\Rightarrow \quad 4a + 2b + c > 0$

Combining *Case* II and *Case* III, we get

$$4a + 2|b| + c < 0$$

Also, at $x = 0$, $\quad y < 0 \Rightarrow c < 0$

Also, since for $-2 < x < 2$,

$$y < 0$$

$\Rightarrow \quad ax^2 + bx + c < 0$

For $x = 1$, $\quad a + b + c < 0 \quad \ldots(i)$

and for $x = -1$, $\quad a - b + c < 0 \quad \ldots(ii)$

Combining Eqs. (i) and (ii), we get

$$a + |b| + c < 0$$

35. Put $x^2 = y$.

Then, the given equation can be written as

$$f(y) = ay^2 + by + c = 0 \quad \ldots(i)$$

The given equation will have four real roots, i.e. Eq. (i) has two non-negative roots.

Then, $\quad -\dfrac{b}{a} \geq 0$

$$af(0) \geq 0$$

and $\quad b^2 - 4ac \geq 0 \quad$ [given]

$\Rightarrow \quad \dfrac{b}{a} \leq 0$

$$ac \geq 0$$

$\Rightarrow \quad a > 0, b < 0, c > 0$

or $\quad a < 0, b > 0, c < 0$

36. Let the roots be $\dfrac{a}{r}$, a and ar, where $a > 0, r > 1$

$\therefore$ Product of the roots $= 1$

$\Rightarrow \quad \dfrac{a}{r} \cdot a \cdot ar = 1$

$\Rightarrow \quad a^3 = 1$

$\therefore \quad a = 1 \quad$ [one root is 1]

Now, roots are $\dfrac{1}{r}$, 1 and r. Then,

$$\frac{1}{r} + 1 + r = -b$$

$\Rightarrow \quad \dfrac{1}{r} + r = -b - 1 \quad \ldots(i)$

$\because \quad r + \dfrac{1}{r} > 2$

$\Rightarrow \quad -b - 1 > 2$

$\Rightarrow \quad b < -3 \quad$ [from Eq. (i)]

or $\quad b \in (-\infty, -3)$

Also, $\quad \dfrac{1}{r} \cdot 1 + 1 \cdot r + r \cdot \dfrac{1}{r} = c$

$\Rightarrow \quad \dfrac{1}{r} + r + 1 = c = -b \quad$ [from Eq. (i)]

$\therefore \quad b + c = 0$

Now, first root $= \dfrac{1}{r} < 1 \quad [\because$ one root is smaller than one$]$

Second root $= 1$

Third root $= r > 1 \quad [\because$ one root is greater than one$]$

37. We have, $\quad f(x) = ax^2 + bx + c$

$a, b, c \in R \quad [\because a \neq 0]$

On putting $x = 0, 1, \dfrac{1}{2}$, we get

$$|c| \leq 1$$

$$|a + b + c| \leq 1$$

and $\quad \left|\dfrac{1}{4}a + \dfrac{1}{2}b + c\right| \leq 1$

$\Rightarrow \quad -1 \leq c \leq 1,$

$$-1 \leq a + b + c \leq 1$$

and $\quad -4 \leq a + 2b + 4c \leq 4$

$\Rightarrow \quad -4 \leq 4a + 4b + 4c \leq 4$

and $\quad -4 \leq -a - 2b - 4c \leq 4$

On adding, we get

$$-8 \le 3a + 2b \le 8$$

Also, $-8 \le a + 2b \le 8$

$\therefore$ $-16 \le 2a \le 16$

$\Rightarrow$ $|a| \le 8$

$\because$ $-1 \le -c \le 1, -8 \le -a \le 8$

We get, $-16 \le 2b \le 16$

$\Rightarrow$ $|b| \le 8$

$\therefore$ $|a| + |b| + |c| \le 17$

38. $\because$ $x = \dfrac{-5 \pm \sqrt{25 + 1200}}{50} = \dfrac{-5 \pm 35}{50} = \dfrac{30}{50}, \dfrac{-40}{50}$

or $\cos\alpha = \dfrac{3}{5}, \dfrac{-4}{5}$

But $-1 < x < 0$

$\therefore$ $\cos\alpha = -\dfrac{4}{5}$ [lies in II and III quadrants]

$\therefore$ $\sin\alpha = \dfrac{3}{5}$ [lies in II quadrant]

$\therefore$ $\sin\alpha = -\dfrac{3}{5}$ [lies in III quadrant]

$\therefore$ $\sin 2\alpha = 2 \cdot \sin\alpha \cdot \cos\alpha = -\dfrac{24}{25}$ [lies in II quadrant]

$\therefore$ $\sin 2\alpha = 2 \cdot \sin\alpha \cdot \cos\alpha = \dfrac{24}{25}$ [lies in III quadrant]

39. $\because a + 2b + 4c = 0$

$\therefore$ $a\left(\dfrac{1}{2}\right)^2 + b\left(\dfrac{1}{2}\right) + c = 0$

It is clear that one root is $\dfrac{1}{2}$.

Let other root be α. Then, $\alpha + \dfrac{1}{2} = -\dfrac{b}{a}$

$\Rightarrow$ $\alpha = -\dfrac{1}{2} - \dfrac{b}{a}$

which depends upon a and b.

40. $\because$ Cut-off Y-axis, put $x = 0$, i.e. $f(0) = c$

Option (a) $a < 0, c < 0, -\dfrac{b}{2a} < 0$

or $a < 0, c < 0, b < 0$

$\therefore$ $abc < 0$

Option (b) $a < 0, c > 0, -\dfrac{b}{2a} > 0$

or $a < 0, c > 0, b > 0$

$\therefore$ $abc < 0$

Option (c) $a > 0, c > 0, -\dfrac{b}{2a} > 0$

or $a > 0, c > 0, b < 0$

$\therefore$ $abc < 0$

Option (d) $a < 0, c < 0, -\dfrac{b}{2a} < 0$

or $a < 0, c < 0, b < 0$

$\therefore$ $abc < 0$

41. Here, $D \le 0$

and $f(x) \ge 0, \forall\ x \in R$

$\therefore$ $f(3) \ge 0$

$\Rightarrow$ $9a + 3b + 6 \ge 0$

or $3a + b \ge -2$

$\Rightarrow$ Minimum value of $3a + b$ is -2.

and $f(6) \ge 0$

$\Rightarrow$ $36a + 6b + 6 \ge 0$

$\Rightarrow$ $6a + b \ge -1$

$\Rightarrow$ Minimum value of $6a + b$ is -1.

42. Since, $f(x) = x^3 + 3x^2 - 9x + \lambda = (x - \alpha)^2(x - \beta)$

$\therefore$ α is a double root.

$\therefore f'(x) = 0$ has also one root α.

i.e. $3x^2 + 6x - 9 = 0$ has one root α.

$\therefore$ $x^2 + 2x - 3 = 0$ or $(x + 3)(x - 1) = 0$

has the root α which can either -3 or 1.

If $\alpha = 1$, then $f(1) = 0$ gives $\lambda - 5 = 0 \Rightarrow \lambda = 5$.

If $\alpha = -3$, then $f(-3) = 0$ gives

$-27 + 27 + 27 + \lambda = 0$

$\Rightarrow$ $\lambda = -27$

43. We have, $D = (b - c)^2 - 4a(a - b - c) > 0$

$\Rightarrow$ $b^2 + c^2 - 2bc - 4a^2 + 4ab + 4ac > 0$

$\Rightarrow$ $c^2 + (4a - 2b)c - 4a^2 + 4ab + b^2 > 0, \forall c \in R$

Since, $c \in R$, so we have

$(4a - 2b)^2 - 4(-4a^2 + 4ab + b^2) < 0$

$\Rightarrow$ $4a^2 - 4ab + b^2 + 4a^2 - 4ab - b^2 < 0$

$\Rightarrow$ $a(a - b) < 0$

If $a > 0$, then $a - b < 0$

i.e. $0 < a < b$

or $b > a > 0$

If $a < 0$, then $a - b > 0$

i.e. $0 > a > b$

or $b < a < 0$

44. We have, $x^3 - ax^2 + bx - 1 = 0$ ...(i)

Then, $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$= a^2 - 2b$

$\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = (\alpha\beta + \beta\gamma + \gamma\alpha)^2$

$- 2\alpha\beta\gamma(\alpha + \beta + \gamma) = b^2 - 2a$

and $\alpha^2\beta^2\gamma^2 = 1$

Therefore, the equation whose roots are α^2, β^2 and γ^2, is

$x^3 - (a^2 - 2b)x^2 + (b^2 - 2a)x - 1 = 0$ ...(ii)

Since, Eqs. (i) and (ii) are indentical, therefore

$a^2 - 2b = a$ and $b^2 - 2a = b$

Eliminating b, we have

$$\frac{(a^2-a)^2}{4}-2a=\frac{a^2-a}{2}$$

$\Rightarrow$ $a\{a(a-1)^2-8-2(a-1)\}=0$

$\Rightarrow$ $a(a^3-2a^2-a-6)=0$

$\Rightarrow$ $a(a-3)(a^2+a+2)=0$

$\Rightarrow$ $a=0$ or $a=3$ or $a^2+a+2=0$

$\Rightarrow$ $b=0$ or $b=3$

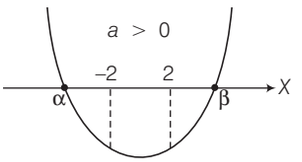
or $b^2+b+2=0$

$\therefore$ $a=b=0$

or $a=b=3$

or a and b are roots of $x^2+x+2=0$.

45. Here, $D>0$



$$b^2-4ac>0$$

or $b^2>4ac$ ...(i)

and $f(0)<0$

$\Rightarrow$ $c<0$ ...(ii)

$f(1)<0$

$\Rightarrow$ $a+b+c<0$ ...(iii)

$f(-1)<0$

$\Rightarrow$ $a-b+c<0$ ...(iv)

$f(2)<0$

$\Rightarrow$ $4a+2b+c<0$ ...(v)

$f(-2)<0$

$\Rightarrow$ $4a-2b+c<0$ ...(vi)

From Eqs. (i) and (ii), we get

$$c<0,\ b^2-4ac>0$$

From Eqs. (iii) and (iv), we get

$$a-|b|+c<0$$

and from Eqs. (v) and (vi), we get

$$4a-2|b|+c<0$$

Solutions (Q. Nos. 46 to 48)

Let $y=\dfrac{2x^2-3x+2}{2x^2+3x+2}$

$\Rightarrow$ $2x^2y+3xy+2y=2x^2-3x+2$

$\Rightarrow 2(y-1)x^2+3(y+1)x+2(y-1)=0$

As $x\in R$

$\therefore$ $D\geq 0$

$\Rightarrow$ $9(y+1)^2-4\cdot 2(y-1)\cdot 2(y-1)\geq 0$

$\Rightarrow$ $9(y+1)^2-16(y-1)^2\geq 0$

$\Rightarrow$ $(3y+3)^2-(4y-4)^2\geq 0$

$\Rightarrow$ $(7y-1)(7-y)\geq 0$

$\Rightarrow$ $(7y-1)(y-7)\leq 0$

$\therefore$ $\dfrac{1}{7}\leq y\leq 7$

$\therefore$ $G=7$ and $L=\dfrac{1}{7}$

$\therefore$ $GL=1$

Now, $\dfrac{G^{100}+L^{100}}{2}\geq (GL)^{100}$ $\Rightarrow$ $\dfrac{G^{100}+L^{100}}{2}\geq 1$

$\Rightarrow$ $G^{100}+L^{100}\geq 2$

46. Least value of $G^{100}+L^{100}$ is 2.

47. The quadratic equation having roots G and L, is

$$x^2-(G+L)x+GL=0$$

$\Rightarrow$ $x^2-\dfrac{50}{7}x+1=0$

$\Rightarrow$ $7x^2-50x+7=0$

48. We have, $L^2<\lambda<G^2$

$\Rightarrow$ $\left(\dfrac{1}{7}\right)^2<\lambda<7^2$

$\Rightarrow$ $\dfrac{1}{49}<\lambda<49$

$\Rightarrow$ $\lambda=1,2,3,\ldots,48$ as $\lambda\in N$

$\therefore$ Sum of all values of $\lambda=1+2+3+\ldots+48=\dfrac{48\times 49}{2}=1176$

Solutions (Q. Nos. 49 to 51)

Let roots be $\alpha,\beta,\gamma,\delta>0$.

$\therefore$ $\alpha+\beta+\gamma+\delta=12$

$(\alpha+\beta)(\gamma+\delta)+\alpha\beta+\gamma\delta=c$

$\alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta)=-d$

$\alpha\beta\gamma\delta=81$

$\because$ $\text{AM}=\dfrac{\alpha+\beta+\gamma+\delta}{4}=3$

and $\text{GM}=(\alpha\beta\gamma\delta)^{1/4}=(81)^{1/4}=3$

$\therefore$ $\text{AM}=\text{GM}$

$\Rightarrow$ $\alpha=\beta=\gamma=\delta=3$

49. $c=(\alpha+\beta)(\gamma+\delta)+\alpha\beta+\gamma\delta$

$=(3+3)(3+3)+3\cdot 3+3\cdot 3=36+18=54$

50. $\because \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta)=-d$

$\therefore$ $d=-\{3\cdot 3\cdot(3+3)+3\cdot 3\cdot(3+3)\}=-108$

51. Required root $=-\dfrac{d}{2c}=-\dfrac{(-108)}{2\times 54}=1$

Solutions (Q. Nos. 52 to 54)

Given that, $AC=4\sqrt{2}$ units

$\therefore$ $AB=BC=\dfrac{AC}{\sqrt{2}}=4$ units

and $OB=\sqrt{(BC)^2-(OC)^2}$

$=\sqrt{(4)^2-(2\sqrt{2})^2}$ $\left[\because OC=\dfrac{AC}{2}\right]$

$=2\sqrt{2}$ units

$\therefore$ Vertices are $A \equiv (-2\sqrt{2}, 0)$,

$$B \equiv (0, -2\sqrt{2})$$

and $$C \equiv (2\sqrt{2}, 0)$$

52. Since, $y = f(x) = ax^2 + bx + c$ passes through A, B and C, then

$$0 = 8a - 2\sqrt{2}b + c - 2\sqrt{2} = c$$

and $$0 = 8a + 2\sqrt{2}b + c$$

We get, $b = 0, a = \dfrac{1}{2\sqrt{2}}$ and $c = -2\sqrt{2}$

$$\therefore \quad y = f(x) = \frac{x^2}{2\sqrt{2}} - 2\sqrt{2}$$

53. Minimum value of $y = \dfrac{x^2}{2\sqrt{2}} - 2\sqrt{2}$ is at $x = 0$.

$$\therefore \quad (y)_{\min} = -2\sqrt{2}$$

54. $f(x) = 0$

$$\Rightarrow \quad \frac{x^2}{2\sqrt{2}} - 2\sqrt{2} = 0 \Rightarrow x = \pm 2\sqrt{2}$$

Given, $$-2\sqrt{2} < \frac{\lambda}{2} < 2\sqrt{2}$$

or $$-4\sqrt{2} < \lambda < 4\sqrt{2}$$

$\therefore$ Initial values of λ are

$$-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5.$$

$\therefore$ Number of integral values is 11.

Solutions. (Q. Nos. 55 to 57)

We have, $$(\alpha - \beta) = (\alpha + k) - (\beta + k)$$

$$\Rightarrow \quad \frac{\sqrt{b^2 - 4c}}{1} = \frac{\sqrt{b_1^2 - 4c_1}}{1}$$

$$\Rightarrow \quad b^2 - 4c = b_1^2 - 4c_1 \qquad \ldots(i)$$

Given, least value of $f(x) = -\dfrac{1}{4} - \dfrac{(b^2 - 4c)}{4 \times 1} = -\dfrac{1}{4}$

$$\Rightarrow \quad b^2 - 4c = 1$$

$$\therefore \quad b^2 - 4c = 1 = b_1^2 - 4c_1 \qquad [\text{from Eq. (i)}] \ldots(ii)$$

Also, given least value of $g(x)$ occurs at $x = \dfrac{7}{2}$.

$$\therefore \quad -\frac{b_1}{2 \times 1} = \frac{7}{2}$$

$$\therefore \quad b_1 = -7$$

55. $b_1 = -7$

56. Least value of $g(x) = -\dfrac{b_1^2 - 4c_1}{4 \times 1} = -\dfrac{1}{4}$ [from Eq. (ii)]

57. $\because g(x) = 0$

$$\therefore \quad x^2 + b_1 x + c_1 = 0$$

$$\Rightarrow \quad x = \frac{-b_1 \pm \sqrt{b_1^2 - 4c_1}}{2}$$

$$= \frac{7 \pm 1}{2} = 3, 4$$

$\therefore$ Roots of $g(x) = 0$ are 3, 4.

Solutions (Q. Nos. 58 to 60)

Let $f(x) = ax^2 - bx + c$ has two distinct roots α and β. Then, $f(x) = a(x - \alpha)(x - \beta)$. Since, $f(0)$ and $f(1)$ are of same sign.

Therefore, $$c(a - b + c) > 0$$

$$\Rightarrow \quad c(a - b + c) \geq 1$$

$$\therefore \quad a^2 \alpha\beta (1 - \alpha)(1 - \beta) \geq 1$$

But $$\alpha(1 - \alpha) = \frac{1}{4} - \left(\frac{1}{2} - \alpha\right)^2 \leq \frac{1}{4}$$

$$\therefore \quad a^2 \alpha\beta (1 - \alpha)(1 - \beta) < \frac{a^2}{16}$$

$$\Rightarrow \quad \frac{a^2}{16} > 1 \Rightarrow a > 4 \qquad [\because \alpha \neq \beta]$$

$$\Rightarrow \quad a \geq 5 \text{ as } a \in I$$

Also, $$b^2 - 4ac \geq 0$$

$$\Rightarrow \quad b^2 \geq 4ac \geq 20$$

$$\Rightarrow \quad b \geq 5$$

Next, $a \geq 5, b \geq 5$, we get $c \geq 1$

$$\therefore \quad abc \geq 25$$

$$\therefore \quad \log_5 abc \geq \log_5 25 = 2$$

58. Least value of a is 5.

59. Least value of b is 5.

60. Least value of $\log_b abc$ is 2.

Solutions. (Q. Nos. 61 to 63)

Let α, β and γ be the roots of $2x^3 + ax^2 + bx + 4 = 0$.

$$\therefore \quad \alpha + \beta + \gamma = -\frac{a}{2}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{b}{2} \text{ and } \alpha\beta\gamma = -2$$

61. $\because$ AM $\geq$ GM

$$\therefore \quad \frac{(-\alpha) + (-\beta) + (-\gamma)}{3} \geq \{(-\alpha)(-\beta)(-\gamma)\}^{1/3}$$

$$\Rightarrow \quad \frac{\frac{a}{2}}{3} \geq (2)^{1/3}$$

$$\therefore \quad a \geq 6(2)^{1/3} \qquad \ldots(i)$$

or $$a^3 \geq 432$$

Hence, minimum value of a^3 is 432.

62. $\because$ AM $\geq$ GM

$$\therefore \quad \frac{(-\alpha)(-\beta) + (-\beta)(-\gamma) + (-\gamma)(-\alpha)}{3}$$

$$\geq \{(-\alpha)(-\beta)(-\beta)(-\gamma)(-\gamma)(-\alpha)\}^{1/3}$$

$$\Rightarrow \quad \frac{b/2}{3} \geq (4)^{1/3}$$

$$\Rightarrow \quad b \geq 6(4)^{1/3} \qquad \ldots(ii)$$

or $$b^3 \geq 864$$

Hence, minimum value of b^3 is 864.

63. From Eqs. (i) and (ii), we get

$$ab \geq 6(2)^{1/3} \cdot 6(4)^{1/3}$$

$\Rightarrow \quad ab \geq 36 \times 2$

$\because \quad \frac{a+b}{2} \geq \sqrt{ab} \geq 6\sqrt{2} \Rightarrow \frac{a+b}{2} \geq 6\sqrt{2}$

$\therefore \quad a + b \geq 12\sqrt{2}$

or $\quad (a+b)^3 \geq 3456\sqrt{2}$

Hence, minimum value of $(a+b)^3$ is $3456\sqrt{2}$.

Solutions (Q. Nos. 64 to 66)

$\therefore \quad \alpha + \beta + \gamma + \delta = -A$ ...(i)

$(\alpha+\beta)(\gamma+\delta) + \alpha\beta + \gamma\delta = B$ ...(ii)

$\alpha\beta(\gamma+\delta) + \gamma\delta(\alpha+\beta) = -C$ ...(iii)

and $\quad \alpha\beta\gamma\delta = D$ ...(iv)

64. $\because \quad \frac{C}{A} = \frac{\alpha\beta(\gamma+\delta) + \gamma\delta(\alpha+\beta)}{\alpha+\beta+\gamma+\delta}$

$= \frac{k(\gamma+\delta) + k(\alpha+\beta)}{\alpha+\beta+\gamma+\delta} \quad [\because \alpha\beta = \gamma\delta = k]$

$= k$ ...(v)

65. From Eq. (ii), we get

$(\alpha+\beta)(\gamma+\delta) = B - (\alpha\beta + \gamma\delta) = B - 2k \quad [\because \alpha\beta = \gamma\delta = k]$

66. From Eq. (iv), we get

$\alpha\beta\gamma\delta = D$

$\Rightarrow \quad k \cdot k = D \quad [\because \alpha\beta = \gamma\delta = k]$

$\Rightarrow \quad \left(\frac{C}{A}\right)^2 = D \quad$ [from Eq. (v)]

$\therefore \quad C^2 = A^2 D$

67. The given equation is $|x-2|^2 + |x-2| - 2 = 0$.

There are two cases:

***Case* I** If $x \geq 2$, then $(x-2)^2 + x - 2 - 2 = 0$

$\Rightarrow \quad x^2 - 3x = 0$

$\Rightarrow \quad x(x-3) = 0$

$\Rightarrow \quad x = 0, 3$

Here, 0 is not possible.

$\therefore \quad x = 3$

***Case* II** If $x < 2$, then

$(x-2)^2 - x + 2 - 2 = 0$

$\Rightarrow \quad x^2 - 5x + 4 = 0$

$\Rightarrow \quad (x-1)(x-4) = 0$

$\Rightarrow \quad x = 1, 4$

Here, 4 is not possible.

$\therefore \quad x = 1$

$\therefore$ The sum of roots $= 1 + 3 = 4$

Aliter

Let $|x-2| = y$.

Then, we get $\quad y^2 + y - 2 = 0$

$\Rightarrow \quad (y-1)(y+2) = 0 \Rightarrow y = 1, -2$

But -2 is not possible.

Hence, $\quad |x-2| = 1 \Rightarrow x = 1, 3$

$\therefore$ Sum of the roots $= 1 + 3 = 4$

68. We have,

$(5+\sqrt{2})x^2 - (4+\sqrt{5})x + 8 + 2\sqrt{5} = 0$

$\therefore$ Sum of the roots $= \frac{4+\sqrt{5}}{5+\sqrt{2}}$

and product of the roots $= \frac{8+2\sqrt{5}}{5+\sqrt{2}}$

$\therefore$ The harmonic mean of the roots

$= \frac{2 \times \text{Product of the roots}}{\text{Sum of the roots}} = \frac{2 \times (8+2\sqrt{5})}{(4+\sqrt{5})} = 4$

69. Let $x^2 - ax + 30 = y$

$\therefore \quad y = 2\sqrt{y+15}$ ...(i)

$\Rightarrow \quad y^2 - 4y - 60 = 0$

$\Rightarrow \quad (y-10)(y+6) = 0$

$\therefore \quad y = 10, -6$

$\Rightarrow \quad y = 10, y \neq -6 \quad [\because y > 0]$

Now, $\quad x^2 - ax + 30 = 10$

$\Rightarrow \quad x^2 - ax + 20 = 0$

Given, $\quad \alpha\beta = \lambda = 20$

$\therefore \quad \frac{\alpha+\beta}{2} \geq \sqrt{\alpha\beta} = \sqrt{20}$

$\Rightarrow \quad \alpha + \beta \geq 2\sqrt{20}$

or $\quad \mu = 4\sqrt{5}$

$\therefore$ Minimum value of μ is $4\sqrt{5}$.

i.e., $\quad \mu = 4\sqrt{5} = 8.9 \Rightarrow (\mu) = 9$

70. $\because N^r = \left(x + \frac{1}{x}\right)^6 - \left(x^6 + \frac{1}{x^6}\right) - 2$

$= \left(x + \frac{1}{x}\right)^6 - \left(x^3 + \frac{1}{x^3}\right)^2 = \left(\left(x + \frac{1}{x}\right)^3 + \left(x^3 + \frac{1}{x^3}\right)\right)$

$\left(\left(x + \frac{1}{x}\right)^3 - \left(x^3 + \frac{1}{x^3}\right)\right)$

$= D^r \cdot \left(3\left(x + \frac{1}{x}\right)\right)$

$\therefore \quad \frac{N^r}{D^r} = 3\left(x + \frac{1}{x}\right) \geq 6$

Hence, minimum value of $\frac{N^r}{D^r}$ is 6.

71. $a + b = 2c$ ...(i)

$ab = -5d$ ...(ii)

$c + d = 2a$ ...(iii)

$cd = -5b$...(iv)

From Eqs. (i) and (iii), we get

$a + b + c + d = 2(a + c)$

$\therefore \quad a + c = b + d$ ...(v)

From Eqs. (i) and (iii), we get

$$b - d = 3(c - a) \quad \ldots(vi)$$

Also, a is a root of $x^2 - 2cx - 5d = 0$

$$\therefore \quad a^2 - 2ac - 5d = 0 \quad \ldots(vii)$$

And c is a root of

$$c^2 - 2ac - 5b = 0 \quad \ldots(viii)$$

From Eqs. (vii) and (viii), we get

$$a^2 - c^2 - 5(d - b) = 0$$

$$\Rightarrow \quad (a + c)(a - c) + 5(b - d) = 0$$

$$\Rightarrow \quad (a + c)(a - c) + 15(c - a) = 0 \quad \text{[from Eq. (vi)]}$$

$$\Rightarrow \quad (a - c)(a + c - 15) = 0$$

$$\therefore \quad a + c = 15, a - c \neq 0$$

From Eq. (v), we get $b + d = 15$

$$\therefore \quad a + b + c + d = a + c + b + d = 15 + 15 = 30$$

$\Rightarrow$ Sum of digits of $a + b + c + d = 3 + 0 = 3$

72. $\because \quad y = \dfrac{x^2 - 3x + c}{x^2 + 3x + c}$

$$\Rightarrow \quad x^2(y - 1) + 3x(y + 1) + c(y - 1) = 0$$

$$\because \quad x \in R$$

$$\therefore \quad 9(y + 1)^2 - 4c(y - 1)^2 \geq 0$$

$$(2\sqrt{c}y - 2\sqrt{c})^2 - (3y + 3)^2 \leq 0$$

$$\Rightarrow \{(2\sqrt{c} + 3)y - (2\sqrt{c} - 3)\}\{(2\sqrt{c} - 3)y - (2\sqrt{c} + 3)\} \leq 0$$

or $$\frac{2\sqrt{c} - 3}{2\sqrt{c} + 3} \leq y \leq \frac{2\sqrt{c} + 3}{2\sqrt{c} - 3}$$

But given, $$\frac{2\sqrt{c} + 3}{2\sqrt{c} - 3} = 7$$

$$\Rightarrow \quad 2\sqrt{c} + 3 = 14\sqrt{c} - 21$$

or $$12\sqrt{c} = 24 \text{ or } \sqrt{c} = 2$$

$$\therefore \quad c = 4$$

73. We have, $\sqrt{x^2} - \sqrt{(x-1)^2} + \sqrt{(x-2)^2} = \sqrt{5}$

$$\Rightarrow \quad |x| - |x - 1| + |x - 2| = \sqrt{5}$$

***Case* I** If $x < 0$, then

$$-x + (x - 1) - (x - 2) = \sqrt{5}$$

$$x = 1 - \sqrt{5}$$

***Case* II** If $0 \leq x < 1$, then

$$x + (x - 1) - (x - 2) = \sqrt{5}$$

$$\Rightarrow \quad x = \sqrt{5} - 1, \text{ which is not possible.}$$

***Case* III** If $1 \leq x < 2$, then

$$x - (x - 1) - (x - 2) = \sqrt{5}$$

$$\Rightarrow \quad x = 3 - \sqrt{5}, \text{ which is not possible.}$$

***Case* IV** If $x > 2$, then

$$x - (x - 1) + (x - 2) = \sqrt{5}$$

$$\Rightarrow \quad x = 1 + \sqrt{5}$$

Hence, number of solutions is 2.

74. $\because \quad (1 + i)x^2 + (1 - i)x - 2i = 0$

$$\Rightarrow \quad x^2 + \frac{(1 - i)}{(1 + i)}x - \frac{2i}{(1 + i)} = 0$$

$$\Rightarrow \quad x^2 - ix - (1 + i) = 0$$

$$\therefore \quad \alpha + \beta = i, \text{ and } \alpha\beta = -(1 + i)$$

$$\therefore \quad \alpha - \beta = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta} = \sqrt{i^2 + 4(1 + i)} = \sqrt{(3 + 4i)}$$

$$|\alpha - \beta| = \sqrt{\sqrt{9 + 16}} = \sqrt{5}$$

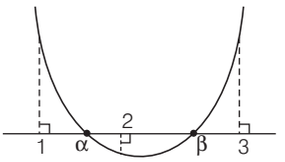
$$\therefore \quad |\alpha - \beta|^2 = 5$$

75. $\because 4x^2 - 16x + c = 0$

$$\Rightarrow \quad x^2 - 4x + \frac{c}{4} = 0$$

Let $$f(x) = x^2 - 4x + \frac{c}{4}$$

Then, the following cases arises:



***Case* I** $D > 0$

$$\Rightarrow \quad 16 - c > 0$$

$$\therefore \quad c < 16$$

***Case* II** $f(1) > 0$

$$\Rightarrow \quad 1 - 4 + \frac{c}{4} > 0$$

$$\Rightarrow \quad \frac{c}{4} > 3$$

$$\therefore \quad c > 12$$

***Case* III** $f(2) < 0$

$$\Rightarrow \quad 4 - 8 + \frac{c}{4} < 0$$

$$\Rightarrow \quad \frac{c}{4} < 4$$

$$\therefore \quad c < 16$$

***Case* IV** $f(3) > 0$

$$\Rightarrow \quad 9 - 12 + \frac{c}{4} > 0$$

$$\Rightarrow \quad \frac{c}{4} > 3$$

$$\Rightarrow \quad c > 12$$

Combining all cases, we get

$$12 < c < 16$$

Thus, integral values of c are 13, 14 and 15.

Hence, number of integral values of c is 3.

76. We have, $$r + s + t = 0 \quad \ldots(i)$$

$$rs + st + tr = \frac{1001}{8} \quad \ldots(ii)$$

and $$rst = -\frac{2008}{8} = -251 \quad \ldots(iii)$$

Now, $(r+s)^3+(s+t)^3+(t+r)^3=(-t)^3+(-r)^3+(-s)^3$

$[\because r+s+t=0]$

$=-(t^3+r^3+s^3)=-3rst \quad [\because r+s+t=0]$

$=-3(-251)=753$

Now, $99\lambda=(r+s)^3+(s+t)^3+(t+r)^3=753$

$\therefore \quad \lambda=\frac{753}{99}=7.6$

$\therefore \quad [\lambda]=7$

77. A → (r,s); B → (p, q, r, s, t); C → (p, q, t)

(A) We have, $y=\frac{x^2-2x+4}{x^2+2x+4}$

$\Rightarrow \quad x^2(y-1)+2(y+1)x+4(y-1)=0$

As $x \in R$, we get

$D \geq 0$

$\Rightarrow \quad 4(y+1)^2-16(y-1)^2 \geq 0$

$\Rightarrow \quad 3y^2-10y+3 \leq 0$

$\Rightarrow \quad (y-3)(3y-1) \leq 0$

$\Rightarrow \quad \frac{1}{3} \leq y \leq 3$

(B) We have, $y=\frac{2x^2+4x+1}{x^2+4x+2}$

$\Rightarrow \quad x^2(y-2)+4(y-1)x+2y-1=0$

As $x \in R$, we get

$D \geq 0$

$\Rightarrow \quad 16(y-1)^2-4(y-2)(2y-1) \geq 0$

$\Rightarrow \quad 4(y-1)^2-(y-2)(2y-1) \geq 0$

$\Rightarrow \quad 2y^2-3y+2 \geq 0$

$\Rightarrow \quad y^2-\frac{3}{2}y+1 \geq 0$

$\Rightarrow \quad \left(y-\frac{3}{4}\right)^2+\frac{7}{16} \geq 0$

$\therefore \quad y \in R$

(C) We have, $y=\frac{x^2-3x+4}{x-3}$

$\Rightarrow \quad x^2-(3+y)x+3y+4=0$

As $x \in R$, we get

$D \geq 0 \Rightarrow (3+y)^2-4(3y+4) \geq 0$

$\Rightarrow \quad y^2-6y-7 \geq 0 \Rightarrow (y+1)(y-7) \geq 0$

$\Rightarrow \quad y \in (-\infty,-1] \cup [7,\infty)$

78. A → (q,r,s); B → (p); C → (q)

(A) $\because (d+a-b)^2+(d+b-c)^2=0$

which is possible only when

$d+a-b=0, d+b-c=0$

$\Rightarrow \quad b-a=c-b$

$\Rightarrow \quad 2b=a+c$

$\therefore$ a, b and c are in AP. ...(i)

$\because \quad a(b-c)+b(c-a)+c(a-b)=0$

$\therefore \quad x=1$ is a root of

$a(b-c)x^2+b(c-a)x+c(a-b)=0$ ...(ii)

Given, roots [Eq. (ii)] are equal.

$\therefore \quad 1\times 1=\frac{c(a-b)}{a(b-c)}$

$\Rightarrow \quad a(b-c)=c(a-b)$

or $\quad b=\frac{2ac}{a+c}$

$\therefore$ a, b and c are in HP. ...(iii)

From Eqs. (i) and (ii), we get

$a=b=c$

$\therefore$ a, b and c are in AP, GP and HP.

(B) $\because \quad x^3-3x^2+3x-1=0$

$\Rightarrow \quad (x-1)^3=0$

$\therefore \quad x=1,1,1$

$\Rightarrow$ Common root, $x=1$

$\therefore \quad a(1)^2+b(1)+c=0$

$\Rightarrow \quad a+b+c=0$

(C) Given, $\quad bx^2+(\sqrt{(a+c)^2+4b^2})\,x+(a+c) \geq 0$

$\therefore \quad D \leq 0$

$\Rightarrow \quad (a+c)^2+4b^2-4b(a+c) \leq 0$

$\Rightarrow \quad (a+c-2b)^2 \leq 0$

or $\quad (a+c-2b)^2=0$

$\therefore \quad a+c=2b$

Hence a, b and c are in AP.

79. A → (q,r,s,t); B → (q,r); C → (p,q)

(A) We have, $y=\frac{ax^2+3x-4}{3x-4x^2+a}$

$\Rightarrow \quad x^2(a+4y)+3(1-y)x-(ay+4)=0$

As $x \in R$, we get

$D \geq 0$

$\Rightarrow \quad 9(1-y)^2+4(a+4y)(ay+4) \geq 0$

$\Rightarrow (9+16a)y^2+(4a^2+46)y+(9+16a) \geq 0, \forall\, y \in R$

$\Rightarrow$ If $9+16a>0$, then $D \leq 0$

Now, $D \leq 0$

$\Rightarrow \quad (4a^2+46)^2-4(9+16a)^2 \leq 0$

$\Rightarrow \quad 4[(2a^2+23)^2-(9+16a)^2] \leq 0$

$\Rightarrow \quad [(2a^2+23)+(9+16a)][(2a^2+23)-(9+16a)] \leq 0$

$\Rightarrow \quad (2a^2+16a+32)(2a^2-16a+14) \leq 0$

$\Rightarrow \quad 4(a+4)^2(a^2-8a+7) \leq 0$

$\Rightarrow \quad a^2-8a+7 \leq 0$

$\Rightarrow \quad (a-1)(a-7) \leq 0$

$\Rightarrow \quad 1 \leq a \leq 7$

$\therefore \quad 9+16a>0$ and $1 \leq a \leq 7$

$\Rightarrow \quad 1 \leq a \leq 7$

(B) We have, $y = \dfrac{ax^2 + x - 2}{a + x - 2x^2}$

$\Rightarrow \quad x^2(a + 2y) + x(1 - y) - (2 + ay) = 0$

As $x \in R$, we get

$D \geq 0$

$\Rightarrow \quad (1 - y)^2 + 4(2 + ay)(a + 2y) \geq 0$

$\Rightarrow \quad (1 + 8a)y^2 + (4a^2 + 14)y + (1 + 8a) \geq 0$

$\Rightarrow$ If $1 + 8a > 0$, then $D \leq 0$

$\Rightarrow \quad (4a^2 + 14)^2 - 4(1 + 8a)^2 \leq 0$

$\Rightarrow \quad 4[(2a^2 + 7)^2 - (1 + 8a)^2] \leq 0$

$\Rightarrow \quad [(2a^2 + 7) + (1 + 8a)][(2a^2 + 7) - (1 + 8a)] \leq 0$

$\Rightarrow \quad (2a^2 + 8a + 8)(2a^2 - 8a + 6) \leq 0$

$\Rightarrow \quad 4(a + 2)^2(a^2 - 4a + 3) \leq 0$

$\Rightarrow \quad a^2 - 4a + 3 \leq 0$

$\Rightarrow \quad (a - 1)(a - 3) \leq 0$

$\Rightarrow \quad 1 \leq a \leq 3$

Thus, $1 + 8a > 0$ and $1 \leq a \leq 3$

$\Rightarrow \quad 1 \leq a \leq 3$

(C) We have, $y = \dfrac{x^2 + 2x + a}{x^2 + 4x + 3a}$

$\Rightarrow \quad x^2(y - 1) + 2(2y - 1)x + a(3y - 1) = 0$

As $x \in R$, we get

$D \geq 0$

$\Rightarrow \quad 4(2y - 1)^2 - 4(y - 1)a(3y - 1) \geq 0$

$\Rightarrow \quad (4 - 3a)y^2 - (4 - 4a)y + (1 - a) \geq 0$

$\Rightarrow$ If $4 - 3a > 0$, then $D \leq 0$

$\Rightarrow \quad (4 - 4a)^2 - 4(4 - 3a)(1 - a) \leq 0$

$\Rightarrow \quad 4(2 - 2a)^2 - 4(4 - 3a)(1 - a) \leq 0$

$\Rightarrow \quad 4 + 4a^2 - 8a - (4 - 7a + 3a^2) \leq 0$

$\Rightarrow \quad a^2 - a \leq 0$

$\Rightarrow \quad a(a - 1) \leq 0$

$\Rightarrow \quad 0 \leq a \leq 1$

80. $A \rightarrow (p,q,r,s); B \rightarrow (p,q); C \rightarrow (s)$

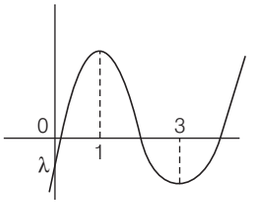
(A) Let $y = f(x) = x^3 - 6x^2 + 9x + \lambda$

$f'(x) = 3x^2 - 12x + 9 = 0$

$\therefore \quad x = 1, 3$

$f''(x) = 6x - 12$

$f''(1) < 0 \quad$ and $\quad f''(3) > 0$



Also, $f(0) < 0 \Rightarrow \lambda < 0$ ...(i)

$f(1) > 0$

$\Rightarrow \quad 1 - 6 + 9 + \lambda > 0$

$\Rightarrow \quad \lambda > -4$ ...(ii)

and $\quad f(3) < 0$

$\Rightarrow \quad 27 - 54 + 27 + \lambda < 0$

$\Rightarrow \quad \lambda < 0$ ...(iii)

From Eqs. (i), (ii) and (iii), we get

$-4 < \lambda < 0$

$\Rightarrow \quad -3 < \lambda + 1 < 1$

$\therefore \quad [\lambda + 1] = -3, -2, -1, 0$

$\therefore \quad |[\lambda + 1]| = 3, 2, 1, 0$

(B) $\because \quad x^2 + x + 1 > 0, \forall\ x \in R$

Given, $\quad -3 < \dfrac{x^2 - \lambda x - 2}{x^2 + x + 1} < 2$

$\Rightarrow \quad -3x^2 - 3x - 3 < x^2 - \lambda x - 2 < 2x^2 + 2x + 2$

$\Rightarrow \quad 4x^2 - (\lambda - 3)x + 1 > 0$

and $\quad x^2 + (\lambda + 2)x + 4 > 0$

$\therefore \quad (\lambda - 3)^2 - 4 \cdot 4 \cdot 1 < 0$

and $\quad (\lambda + 2)^2 - 4 \cdot 1 \cdot 4 < 0$

$\Rightarrow \quad (\lambda - 3)^2 - 4^2 < 0$

and $\quad (\lambda + 2)^2 - 4^2 < 0$

$\Rightarrow \quad -4 < \lambda - 3 < 4$

and $\quad -4 < \lambda + 2 < 4$

or $\quad -1 < \lambda < 7$

and $\quad -6 < \lambda < 2$

We get, $\quad -1 < \lambda < 2$

$\therefore \quad [\lambda] = -1, 0, 1$

$\Rightarrow \quad |[\lambda]| = 0, 1$

(C) $\because \quad (b - c) + (c - a) + (a - b) = 0$

$\therefore\ x = 1$ is a root of

$(b - c)x^2 + (c - a)x + (a - b) = 0$

Also, $x = 1$ satisfies

$x^2 + \lambda x + 1 = 0$

$\Rightarrow \quad 1 + \lambda + 1 = 0$

$\therefore \quad \lambda = -2$

Now, $\quad \lambda - 1 = -3$

$[\lambda - 1] = -3$

$\Rightarrow \quad |[\lambda - 1]| = 3$

81. If quadratic equation $ax^2 + bx + c = 0$ is satisfied by more than two values of x, then it must be an identity.

Therefore, $\quad a = b = c = 0$

$\therefore$ Statement-2 is true.

But in Statement-1,

$4p - 3 = 4q - 3 = r = 0$

Then, $\quad p = q = \dfrac{3}{4}, r = 0$

which is false.

Since, at one value of p or q or r, all coefficients at a time $\neq 0$.

$\therefore$ Statement-1 is false.

82. We have, $x^2 + (2m+1)\,x + (2n+1) = 0$ ...(i)

$m, n \in I$

$\therefore \quad D = b^2 - 4ac$

$= (2m+1)^2 - 4(2n+1)$

is never be a perfect square.

Therefore, the roots of Eq. (i) can never be integers. Hence, the roots of Eq. (i) cannot have any rational root as $a = 1, b, c \in I$.

Hence, both statements are true and Statements –2 is a correct explanation of Statement-1.

83. Let α be one root of equation $ax^2 + 3x + 5 = 0$. Therefore,

$$\alpha \cdot \frac{1}{\alpha} = \frac{5}{a}$$

$$\Rightarrow \quad 1 = \frac{5}{a}$$

$$\Rightarrow \quad a = 5$$

Hence, both the statements are true and Statement-2 is the correct explanation of Statement-1.

84. Let roots of $Ax^3 + Bx^2 + Cx + D = 0$ ...(i)

are $\alpha - \beta, \alpha, \alpha + \beta$ (in AP).

Then, $(\alpha - \beta) + \alpha + (\alpha + \beta) = -\frac{B}{A}$

$\Rightarrow \quad \alpha = -\frac{B}{3A}$, which is a root of Eq. (i).

Then, $A\alpha^3 + B\alpha^2 + C\alpha + D = 0$

$$\Rightarrow \quad A\left(-\frac{B}{3A}\right)^3 + B\left(-\frac{B}{3A}\right)^2 + C\left(-\frac{B}{3A}\right) + D = 0$$

$$\Rightarrow \quad -\frac{B^3}{27A^2} + \frac{B^3}{9A^2} - \frac{BC}{3A} + D = 0$$

$$\Rightarrow \quad 2B^3 - 9ABC + 27A^2D = 0$$

Now, comparing with $2B^3 + k_1ABC + k_2A^2D = 0$, we get

$k_1 = -9, k_2 = 27$

$\therefore \quad k_2 - k_1 = 27 - (-9) = 36 = 6^2$

Hence, both statements are true and Statement-2 is a correct explanation of Statement-1.

85. $\because x, y, z \in R$

$x + y + z = 6$ ...(i)

and $xy + yz + zx = 8$ ...(ii)

$\Rightarrow \quad xy + (x+y)\{6 - (x+y)\} = 8$ [from Eq. (i)]

$\Rightarrow \quad xy + 6x + 6y - (x^2 + 2xy + y^2) = 8$

or $y^2 + (x-6)\,y + x^2 - 6x + 8 = 0$

$\therefore \quad (x-6)^2 - 4 \cdot 1 \cdot (x^2 - 6x + 8) \geq 0, \forall\, y \in R$

$\Rightarrow \quad -3x^2 + 12x + 4 \geq 0 \quad \text{or} \quad 3x^2 - 12x - 4 \leq 0$

or $2 - \frac{4}{\sqrt{3}} \leq x \leq 2 + \frac{4}{\sqrt{3}}$

or $x \in \left[2 - \frac{4}{\sqrt{3}}, 2 + \frac{4}{\sqrt{3}}\right]$

Similarly, $y \in \left[2 - \frac{4}{\sqrt{3}}, 2 + \frac{4}{\sqrt{3}}\right]$

and $z \in \left[2 - \frac{4}{\sqrt{3}}, 2 + \frac{4}{\sqrt{3}}\right]$

Since, Eqs. (i) and (ii) remains same, if x, y, z interchange their positions.

Hence, both statements are true and Statement-2 is a correct explanation of Statement-1.

86. Let $y = ax^3 + bx + c$

$\therefore \quad \frac{dy}{dx} = 3ax^2 + b$

For maximum or minimum $\frac{dy}{dx} = 0$, we get

$$x = \pm\sqrt{-\frac{b}{3a}}$$

***Case* I** If $a > 0, b > 0$, then $\frac{dy}{dx} > 0$

In this case, function is increasing, so it has exactly one root

***Case* II** If $a < 0, b < 0$, then $\frac{dy}{dx} < 0$

In this case, function is decreasing, so it has exactly one root.

***Case* III** $a > 0, b < 0$ or $a < 0, b > 0$, then $y = ax^3 + bx + c$ is maximum at one point and minimum at other point.

Hence, all roots can never be non-negative.

$\therefore$Statement-1 is false. But

$$\text{Sum of roots} = -\frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^3} = 0$$

i.e., Statement-2 is true.

87. Statement-2 is obviously true.

But $y = ax^2 + bx + c$

$$y = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

$$= a\left\{\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a^2}\right\} \qquad [\text{where, } D = b^2 - 4ac]$$

$$\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \frac{1}{a}\left(y + \frac{D}{4a}\right)$$

Let $x + \frac{b}{2a} = X$ and $y + \frac{D}{4a} = Y$.

$\therefore \quad X^2 = \frac{1}{a}Y$

Equation of axis, $X = 0$ i.e. $x + \frac{b}{2a} = 0$

or $2ax + b = 0$

Hence, $y = ax^2 + bx + c$ is symmetric about the line $2ax + b = 0$.

$\therefore$ Both statements are true and Statement-2 is a correct explanation of Statement-1.

88. $\because (1+m)\,x^2 - 2(1+3m)x + (1+8m) = 0$

$\therefore \quad D = 4(1+3m)^2 - 4(1+m)(1+8m) = 4m(m-3)$

(i) Both roots are imaginary.

$\therefore \quad D < 0$

$\Rightarrow \quad 4m(m-3) < 0$

$\Rightarrow \quad 0 < m < 3$

or $\quad m \in (0, 3)$

(ii) Both roots are equal.

$\therefore \quad D = 0$

$\Rightarrow \quad 4m(m-3) = 0$

$\Rightarrow \quad m = 0, 3$

(iii) Both roots are real and distinct.

$\therefore \quad D > 0$

$\Rightarrow \quad 4m(m-3) > 0$

$\Rightarrow \quad m < 0$ or $m > 3$

$\therefore \quad m \in (-\infty, 0) \cup (3, \infty)$

(iv) Both roots are positive.

***Case* I** Sum of the roots > 0

$\Rightarrow \quad \dfrac{2(1+3m)}{(1+m)} > 0$

$\Rightarrow \quad m \in (-\infty, -1) \cup \left(-\dfrac{1}{3}, \infty\right)$

***Case* II** Product of the roots > 0

$\Rightarrow \quad \dfrac{(1+8m)}{(1+m)} > 0$

$m \in (-\infty, -1) \cup \left(-\dfrac{1}{8}, \infty\right)$

***Case* III** $\quad D \geq 0$

$\Rightarrow \quad 4m(m-3) \geq 0$

$m \in (-\infty, 0] \cup [3, \infty)$

Combining all Cases, we get

$m \in (-\infty, -1) \cup [3, \infty)$

(v) Both roots are negative.

Consider the following cases:

***Case* I** Sum of the roots $< 0 \Rightarrow \dfrac{2(1+3m)}{(1+m)} < 0$

$\Rightarrow \quad m \in \left(-1, -\dfrac{1}{3}\right)$

***Case* II** Product of the roots $> 0 \quad \Rightarrow \dfrac{(1+8m)}{(1+m)} > 0$

$\Rightarrow \quad m \in (-\infty, 1) \cup \left(-\dfrac{1}{8}, \infty\right)$

***Case* III** $D \geq 0$

$4m(m-3) \geq 0 \quad \Rightarrow \; m \in (-\infty, 0] \cup [3, \infty)$

Combining all cases, we get

$m \in \phi$

(vi) Roots are opposite in sign, then

***Case* I** Consider the following cases:

Product of the roots < 0

$\Rightarrow \quad \dfrac{(1+8m)}{(1+m)} < 0$

$m \in \left(-1, -\dfrac{1}{8}\right)$

***Case* II** $\quad D > 0 \;\Rightarrow\; 4m(m-3) > 0$

$\Rightarrow \quad m \in (-\infty, 0) \cup (3, \infty)$

Combining all cases, we get

$m \in \left(-1, -\dfrac{1}{8}\right)$

(vii) Roots are equal in magnitude but opposite in sign, then Consider the following cases:

***Case* I** Sum of the roots $= 0$

$\Rightarrow \quad \dfrac{2(1+3m)}{(1+m)} = 0$

$\Rightarrow \quad m = -\dfrac{1}{3}, \; m \neq 1$

***Case* II** $\quad D > 0 \;\Rightarrow\; 4m(m-3) > 0$

$\Rightarrow \quad m \in (-\infty, 0) \cup (3, \infty)$

Combining all cases, we get

$m = -\dfrac{1}{3}$

(viii) Atleast one root is positive, then either one root is positive or both roots are positive.

i.e. $\quad (d) \cup (f)$

or $\quad m \in (-\infty, -1) \cup \left(-1, -\dfrac{1}{8}\right) \cup [3, \infty)$

(ix) Atleast one root is negative, then either one root is negative or both roots are negative.

i.e. $\quad (e) \cup (f) \quad$ or $\quad m \in \left(-1, -\dfrac{1}{8}\right)$

(x) Let roots are 2α are 3α. Then,

Consider the following cases:

***Case* I** Sum of the roots $= 2\alpha + 3\alpha = \dfrac{2(1+3m)}{(1+m)}$

$\Rightarrow \quad \alpha = \dfrac{2(1+3m)}{5(1+m)}$

***Case* II** Product of the roots $= 2\alpha \cdot 3\alpha = \dfrac{(1+8m)}{(1+m)}$

$\Rightarrow \quad 6\alpha^2 = \dfrac{(1+8m)}{(1+m)}$

From Eqs. (i) and (ii), we get

$6\left\{\dfrac{2(1+3m)}{5(1+m)}\right\}^2 = \dfrac{(1+8m)}{(1+m)}$

$\Rightarrow \quad 24(1+3m)^2 = 25(1+8m)(1+m)$

$\Rightarrow \quad 24(9m^2 + 6m + 1) = 25(8m^2 + 9m + 1)$

$16m^2 - 81m - 1 = 0$

or $\quad m = \dfrac{81 \pm \sqrt{(-81)^2 + 64}}{32}$

$\Rightarrow \quad m = \dfrac{81 \pm \sqrt{6625}}{32}$

89. $\because 2x^2 - 2(2m+1)x + m(m+1) = 0 \qquad [\because m \in R]$

$\therefore \quad D = [-2(2m+1)]^2 - 8m(m+1) \qquad [D = b^2 - 4ac]$

$= 4\{(2m+1)^2 - 2m(m+1)\}$

$= 4(2m^2 + 2m + 1)$

$$= 8\left(m^2 + m + \frac{1}{2}\right) = 8\left\{\left(m + \frac{1}{2}\right)^2 + \frac{1}{4}\right\} > 0$$

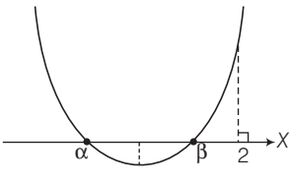
or $D > 0, \forall\ m \in R$ ...(i)

x -coordinate of vertex $= -\dfrac{b}{2a} = \dfrac{2(2m+1)}{4} = \left(m + \dfrac{1}{2}\right)$ ...(ii)

and let

$$f(x) = x^2 - (2m+1)x + \frac{1}{2}m(m+1) \quad ...(iii)$$

(i) Both roots are smaller than 2.



Consider the following cases:

***Case* I** $D \geq 0$

$\therefore$ $m \in R$ [from Eq. (i)]

***Case* II** x -coordinate of vertex < 2.

$\Rightarrow$ $m + \dfrac{1}{2} < 2$ [from Eq. (ii)]

or $m < \dfrac{3}{2}$

***Case* III** $f(2) > 0$

$\Rightarrow$ $4 - (2m+1)\,2 + \dfrac{1}{2}m(m+1) > 0$

$\Rightarrow$ $m^2 - 7m + 4 > 0$

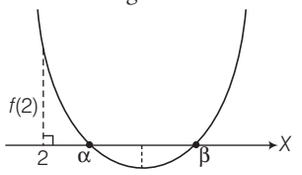
$\therefore$ $m \in \left(-\infty, \dfrac{7-\sqrt{33}}{2}\right) \cup \left(\dfrac{7+\sqrt{33}}{2}, \infty\right)$

Combining all cases, we get

$$m \in \left(-\infty, \frac{7-\sqrt{33}}{2}\right)$$

(ii) Both roots are greater than 2.

Consider the following cases:



***Case* I** $D \geq 0$

$\therefore$ $m \in R$ [from Eq. (i)]

***Case* II** x -coordinate of vertex > 2

$\Rightarrow$ $m + \dfrac{1}{2} > 2$ [from Eq. (ii)]

$\therefore$ $m > \dfrac{3}{2}$

***Case* III** $f(2) > 0$

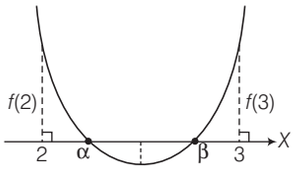
$m \in \left(-\infty, \dfrac{7-\sqrt{33}}{2}\right) \cup \left(7 + \dfrac{\sqrt{33}}{2}, \infty\right)$ [from part (a)]

Combining all cases, we get

$$m \in \left(\frac{7+\sqrt{33}}{2}, \infty\right)$$

(iii) Both roots lie in the interval (2, 3).

Consider the following cases:



***Case* I** $D \geq 0$

$\therefore$ $m \in R$ [from Eq. (i)]

***Case* II** $f(2) > 0$

$\therefore$ $m \in \left(-\infty, \dfrac{7-\sqrt{33}}{2}\right) \cup \left(\dfrac{7+\sqrt{33}}{2}, \infty\right)$ [from part (a)]

***Case* III** $f(3) > 0$

$\Rightarrow$ $9 - 3(2m+1) + \dfrac{1}{2}m(m+1) > 0$

or $m^2 - 11m + 12 > 0$

$\therefore$ $m \in \left(-\infty, \dfrac{11-\sqrt{73}}{2}\right) \cup \left(\dfrac{11+\sqrt{73}}{2}, \infty\right)$

***Case* IV** $2 < x$ -coordinate of vertex < 3

$\Rightarrow$ $2 < m + \dfrac{1}{2} < 3$

or $\dfrac{3}{2} < m < \dfrac{5}{2}$ or $m \in \left(\dfrac{3}{2}, \dfrac{5}{2}\right)$

Combining all cases, we get

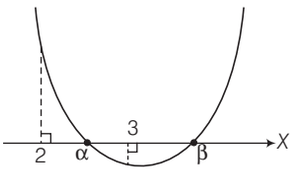
$$m \in \phi$$

(iv) Exactly one root lie in the interval (2,3) .

Consider the following cases:

***Case* I** $D > 0$

$\therefore$ $m \in R$ [from Eq. (i)]



***Case* II** $f(2)\ f(3) < 0$

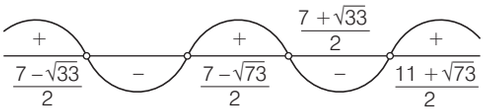
$$\left(4 - 2(2m+1) + \frac{1}{2}m(m+1)\right)$$

$$\left(9 - 3(2m+1) + \frac{1}{2}m(m+1)\right) < 0$$

$\Rightarrow$ $(m^2 - 7m + 4)(m^2 - 11m + 12) < 0$

$\Rightarrow$ $\left(m - \dfrac{7-\sqrt{33}}{2}\right)\left(m - \dfrac{7+\sqrt{33}}{2}\right)$

$$\left(m - \frac{11-\sqrt{73}}{2}\right)\left(m - \frac{11+\sqrt{73}}{2}\right) < 0$$



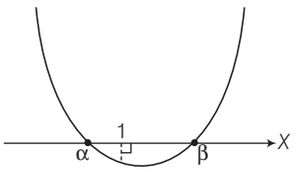
$$\therefore\ m \in \left(\frac{7-\sqrt{33}}{2}, \frac{11-\sqrt{73}}{2}\right) \cup \left(\frac{7+\sqrt{33}}{2}, \frac{11+\sqrt{73}}{2}\right)$$

Combining all cases, we get

$$m \in \left(\frac{7-\sqrt{33}}{2}, \frac{11-\sqrt{73}}{2}\right) \cup \left(\frac{7+\sqrt{33}}{2}, \frac{11+\sqrt{73}}{2}\right)$$

(v) One root is smaller than 1 and the other root is greater than 1.

Consider the following cases:



***Case* I** $D > 0$

$\therefore \quad m \in R$ [from Eq. (i)]

***Case* II** $f(1) < 0$

$\Rightarrow \quad 1-(2m+1)+\frac{1}{2}m(m+1) < 0$ [from Eq. (iii)]

$\Rightarrow \quad m^2 - 3m < 0$

$\Rightarrow \quad m(m-3) < 0$

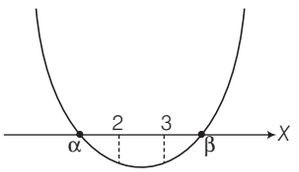
$\therefore \quad m \in (0, 3)$

Combining both cases, we get

$$m \in (0, 3)$$

(vi) One root is greater than 3 and the other root is smaller than 2.

Consider the following cases:



***Case* I** $D > 0$

$\therefore \quad m \in R$ [from Eq. (i)]

***Case* II** $f(2) < 0$

$\Rightarrow \quad m^2 - 7m + 4 < 0$

$$\therefore \quad \frac{7-\sqrt{33}}{2} < m < \frac{7+\sqrt{33}}{2}$$

$$\therefore \quad m \in \left(\frac{7-\sqrt{33}}{2}, \frac{7+\sqrt{33}}{2}\right)$$

***Case* III** $f(3) < 0$

$\Rightarrow \quad m^2 - 11m + 12 < 0$

$$\therefore \quad \frac{11-\sqrt{73}}{2} < m < \frac{11+\sqrt{73}}{2}$$

$$\therefore \quad m \in \left(\frac{11-\sqrt{73}}{2}, \frac{11+\sqrt{73}}{2}\right)$$

Combining all cases, we get

$$m \in \left(\frac{7-\sqrt{33}}{2}, \frac{7+\sqrt{33}}{2}\right)$$

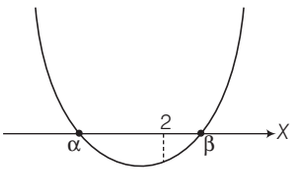
(vii) Atleast one root lies in the interval (2, 3).

i.e. $(d) \cup (c)$

$$\therefore\ m \in \left(\frac{7-\sqrt{33}}{2}, \frac{11-\sqrt{73}}{2}\right) \cup \left(\frac{7+\sqrt{33}}{2}, \frac{11+\sqrt{73}}{2}\right)$$

(viii) Atleast one root is greater than 2.

i.e. (Exactly one root is greater than 2) ∪ (Both roots are greater than 2)



or (Exactly one root is greater than 2) $\cup$ (b) ...(I)

Consider the following cases:

***Case* I** $D > 0$

$\therefore \quad m \in R$ [from Eq. (i)]

***Case* II** $f(2) < 0$

$\Rightarrow \quad m^2 - 7m + 4 < 0$

$$\therefore \quad m \in \left(\frac{7-\sqrt{33}}{2}, \frac{7+\sqrt{33}}{2}\right)$$

Combining both cases, we get

$$m \in \left(\frac{7-\sqrt{33}}{2}, \frac{7+\sqrt{33}}{2}\right) \quad \text{...(II)}$$

Finally from Eqs. (I) and (II), we get

$$m \in \left(\frac{7-\sqrt{33}}{2}, \frac{7+\sqrt{33}}{2}\right) \cup \left(\frac{7+\sqrt{33}}{2}, \infty\right)$$

(ix) Atleast one root is smaller than 2.

i.e. (Exactly one root is smaller than 2) ∪ (Both roots are smaller than 2)

or (h) (II) ∪ (a)

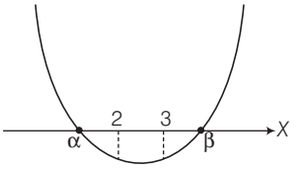
We get, $m \in \left(-\infty, \frac{7-\sqrt{33}}{2}\right) \cup \left(\frac{7-\sqrt{33}}{2}, \frac{7+\sqrt{33}}{2}\right)$

(x) Both 2 and 3 lie between α and β.

Consider the following cases:

***Case* I** $D > 0$

$\therefore \quad m \in R$ [from Eq. (i)]



***Case* II** $f(2) < 0$

$\Rightarrow \quad m^2 - 7m + 4 < 0$

$\therefore \quad m \in \left(\frac{7-\sqrt{33}}{2}, \frac{7+\sqrt{33}}{2}\right)$

***Case* III** $f(3) < 0$

$\Rightarrow \quad m^2 - 11m + 12 < 0$

$\therefore \quad m \in \left(\frac{11-\sqrt{73}}{2}, \frac{11+\sqrt{73}}{2}\right)$

Combining all cases, we get

$$m \in \left(\frac{11-\sqrt{73}}{2}, \frac{7+\sqrt{33}}{2}\right)$$

90. $\because \quad \frac{\alpha}{\beta} = r$

$\Rightarrow \quad \frac{\alpha+\beta}{\alpha-\beta} = \frac{r+1}{r-1}$

[using componendo and dividendo method]

$\Rightarrow \quad \frac{-b/a}{\frac{\sqrt{D}}{a}} = \frac{r+1}{r-1} \Rightarrow b(1-r) = (1+r)\sqrt{D}$

On squaring both sides, we get

$\Rightarrow \quad b^2(1-r)^2 = (1+r)^2(b^2 - 4ac)$

or $\quad (1+r)^2 \cdot 4ac = b^2(4r)$ or $\frac{(1+r)^2}{r} = \frac{b^2}{ac}$

91. We have, $\quad \frac{1}{x+p} + \frac{1}{x+q} = \frac{1}{r}$

$\Rightarrow \quad \frac{(x+q)+(x+p)}{x^2+(p+q)x+pq} = \frac{1}{r}$

$\Rightarrow \quad x^2 + (p+q-2r)x + pq - (p+q)r = 0$

Now, since the roots are equal in magnitudes, but opposite in sign. Therefore,

Sum of the roots = 0

$\Rightarrow \quad p + q - 2r = 0$

$\Rightarrow \quad p + q = 2r \quad \ldots(i)$

and product of the roots $= pq - (p+q)r$

$= pq - (p+q)\left(\frac{p+q}{2}\right)$ [from Eq. (i)]

$= \frac{2pq - p^2 - q^2 - 2pq}{2}$

$= -\frac{p^2+q^2}{2}$

92. Let α be one root of the equation $ax^2 + bx + c = 0$.

Then, other root be α^n.

$\therefore \quad \alpha + \alpha^n = -\frac{b}{a} \quad \ldots(i)$

and $\quad \alpha \cdot \alpha^n = \frac{c}{a}$

$\Rightarrow \quad \alpha^{n+1} = \frac{c}{a}$

$\Rightarrow \quad \alpha = \left(\frac{c}{a}\right)^{\frac{1}{n+1}}$

$\therefore$ From Eq. (i), we get

$$\left(\frac{c}{a}\right)^{\frac{1}{n+1}} + \left(\frac{c}{a}\right)^{\frac{n}{n+1}} = -\frac{b}{a}$$

$\Rightarrow \quad (c)^{\frac{1}{n+1}} \cdot a^{-\frac{1}{n+1}+1} + (c^n)^{\frac{1}{n+1}} \cdot a^{-\frac{n}{n+1}+1} + b = 0$

$\Rightarrow \quad c^{\frac{1}{n+1}} \cdot a^{\frac{n}{n+1}} + (c^n)^{\frac{1}{n+1}} \cdot a^{\frac{1}{n+1}} + b = 0$

$\Rightarrow \quad (a^n c)^{\frac{1}{n+1}} + (c^n a)^{\frac{1}{n+1}} + b = 0$

93. We have, $\quad \alpha + \beta = -\frac{b}{a}$

$$\alpha\beta = \frac{c}{a} \Rightarrow \gamma + \delta = -\frac{m}{l} \text{ and } \gamma\delta = \frac{n}{l}$$

Now, sum of the roots

$= (\alpha\gamma + \beta\delta) + (\alpha\delta + \beta\gamma) = (\alpha+\beta)\gamma + (\alpha+\beta)\delta$

$= (\alpha+\beta)(\gamma+\delta)$

$= \left(-\frac{b}{a}\right)\left(-\frac{m}{l}\right) = \frac{mb}{al}$

and product of the roots

$= (\alpha\gamma + \beta\delta)(\alpha\delta + \beta\gamma)$

$= (\alpha^2 + \beta^2)\gamma\delta + \alpha\beta(\gamma^2 + \delta^2)$

$= \{(\alpha+\beta)^2 - 2\alpha\beta\}\gamma\delta + \alpha\beta\{(\gamma+\delta)^2 - 2\gamma\delta\}$

$= \left\{\left(-\frac{b}{a}\right)^2 - \frac{2c}{a}\right\}\frac{n}{l} + \frac{c}{a}\left\{\left(-\frac{m}{l}\right)^2 - \frac{2n}{l}\right\}$

$= \left\{\frac{b^2-2ac}{a^2}\right\}\frac{n}{l} + \frac{c}{a}\left\{\frac{m^2-2nl}{l^2}\right\} = \frac{(b^2-2ac)ln + (m^2-2nl)ac}{a^2l^2}$

$\therefore$ Required equation is

$$x^2 - \left(\frac{mb}{al}\right)x + \frac{(b^2-2ac)ln + (m^2-2nl)ac}{a^2l^2} = 0$$

$\Rightarrow \quad a^2l^2x^2 - mbalx + (b^2-2ac)ln + (m^2-2nl)ac = 0$

94. Since, the roots are equal.

$\therefore \quad D = 0$

$\Rightarrow \quad 4(b^2-ac)^2 - 4(a^2-bc)(c^2-ab) = 0$

$\Rightarrow \quad (b^2-ac)^2 - (a^2-bc)(c^2-ab) = 0$

$\Rightarrow \quad b(a^3 + b^3 + c^3 - 3abc) = 0$

$\Rightarrow \quad b = 0$ or $a^3 + b^3 + c^3 - 3abc = 0$

95. Let α and β be the roots of $x^2 - px + q = 0$. Then,

$$\alpha + \beta = p \quad \ldots(i)$$

$$\alpha\beta = q \quad \ldots(ii)$$

And α and $\frac{1}{\beta}$ be the roots of $x^2 - ax + b = 0$. Then,

$$\alpha + \frac{1}{\beta} = a \quad \ldots(iii)$$

$$\frac{\alpha}{\beta} = b \quad \ldots(iv)$$

Now, $\text{LHS} = (q-b)^2$

$= \left(\alpha\beta - \frac{\alpha}{\beta}\right)^2$ [from Eqs. (ii) and (iv)]

$= \alpha^2\left(\beta - \frac{1}{\beta}\right)^2 = \alpha^2\left[(\alpha+\beta) - \left(\alpha + \frac{1}{\beta}\right)\right]^2$

$= \alpha^2(p-a)^2$ [from Eqs. (i) and (iii)]

$= \alpha\beta \cdot \frac{\alpha}{\beta}(p-a)^2$

$= bq\,(p-a)^2$ [from Eqs. (ii) and (iv)]

$= \text{RHS}$

96. Since, roots of $x^2 - 2px + q = 0$ are equal.

$\therefore \quad D = 0$

i.e., $(-2p)^2 - 4q = 0$ or $p^2 = q$ ...(i)

Now, $(1+y)\,x^2 - 2(p+y)\,x + (q+y) = 0$

$\therefore$ Discriminant $= 4(p+y)^2 - 4(1+y)(q+y)$

$= 4(p^2 + 2py + y^2 - q - y - qy - y^2)$

$= 4[(2p - q - 1)\,y + p^2 - q]$

$= 4[(2p - p^2 - 1)\,y + 0]$ [from Eq. (i)]

$= -4(p-1)^2 y$

> 0 [$\because y < 0$ and $p \neq 1$]

Hence, roots of $(1+y)\,x^2 - 2(p+y)\,x + (q+y) = 0$ are real and distinct.

97. $x^{\log_x (x+3)^2} = 16$ ...(i)

Equation is defined, when

$x > 0,\ x \neq 1,\ x \neq -3,$

Then, $(x+3)^2 = 4^2$ [by property]

$\Rightarrow \quad x + 3 = \pm 4$

$\therefore \quad x = 1$ and $x = -7$

But $x \neq 1,\ x \neq -7$

i.e. no solution.

$\therefore \quad x \in \phi$

98. $\because (2+\sqrt{3})^{x^2-2x+1} + (2-\sqrt{3})^{x^2-2x-1} = \frac{101}{10(2-\sqrt{3})}$

$\Rightarrow \quad (2+\sqrt{3})^{x^2-2x}\cdot(2+\sqrt{3})(2-\sqrt{3}) + (2-\sqrt{3})^{x^2-2x-1}\cdot(2-\sqrt{3}) = \frac{101}{10}$

$\Rightarrow \quad (2+\sqrt{3})^{x^2-2x} + (2-\sqrt{3})^{x^2-2x} = \frac{101}{10}$

or $(2+\sqrt{3})^{x^2-2x} + \frac{1}{(2+\sqrt{3})^{x^2-2x}} = \frac{101}{10}$ ...(i)

$\left[\because 2-\sqrt{3} = \frac{1}{2+\sqrt{3}}\right]$

Let $(2+\sqrt{3})^{x^2-2x} = \lambda$, then Eq. (i) reduces to

$\lambda + \frac{1}{\lambda} = \frac{101}{10}$

$\Rightarrow \quad 10\lambda^2 - 101\lambda + 10 = 0$

or $(\lambda - 10)(10\lambda - 1) = 0$

$\therefore \quad \lambda = 10, \frac{1}{10}$

$\Rightarrow \quad (2+\sqrt{3})^{x^2-2x} = 10, 10^{-1}$

$\Rightarrow \quad x^2 - 2x = \log_{2+\sqrt{3}} 10, -\log_{2+\sqrt{3}} 10$

$\Rightarrow \quad (x-1)^2 = 1 + \log_{2+\sqrt{3}} 10, 1 - \log_{2+\sqrt{3}} 10$

$\therefore \quad (x-1)^2 = 1 + \log_{2+\sqrt{3}} 10$

[$\because (x-1)^2 \neq 1 - \log_{2+\sqrt{3}} 10$]

$\Rightarrow \quad x = 1 \pm \sqrt{(1 + \log_{2+\sqrt{3}} 10)}$

$\Rightarrow \quad x_1 = 1 + \sqrt{(1 + \log_{2+\sqrt{3}} 10)}$

$x_2 = 1 - \sqrt{(1 + \log_{2+\sqrt{3}} 10)}$

99. We have, $x^2 + \left(\frac{x}{x-1}\right)^2 = 8$

$\Rightarrow \quad \left(x + \frac{x}{x-1}\right)^2 - 2\cdot x \cdot \frac{x}{(x-1)} = 8$

$\Rightarrow \quad \left(\frac{x^2}{x-1}\right)^2 - 2\left(\frac{x^2}{x-1}\right) - 8 = 0$ ...(i)

Let $y = \frac{x^2}{x-1}$. Then, Eq. (i) reduces to

$y^2 - 2y - 8 = 0$

$\Rightarrow \quad (y-4)(y+2) = 0$

$\therefore \quad y = 4, -2$

If $y = 4$, then $4 = \frac{x^2}{x-1}$

or $x^2 - 4x + 4 = 0$

or $(x-2)^2 = 0$

or $x = 2$

$\therefore \quad x_1 = 2$

and if $y = -2$, then $-2 = \frac{x^2}{x-1}$

or $x^2 + 2x - 2 = 0$

$\therefore \quad x = \frac{-2 \pm \sqrt{(4+8)}}{2}$

$\Rightarrow \quad x = -1 \pm \sqrt{3}$

$\therefore \quad x_2 = -1 + \sqrt{3},\ x_3 = -1 - \sqrt{3}$

100. We have, $\sqrt{x + 8 + 2\sqrt{(x+7)}} + \sqrt{(x+1) - \sqrt{(x+7)}} = 4$ ...(i)

Let $\sqrt{(x+7)} = \lambda$ ...(ii)

or $x = \lambda^2 - 7$

Then, Eq. (i) reduces to

$\sqrt{(\lambda^2 - 7 + 8 + 2\lambda)} + \sqrt{(\lambda^2 - 7 + 1 - \lambda)} = 4$

$\Rightarrow \quad (\lambda + 1) + \sqrt{(\lambda^2 - \lambda - 6)} = 4$

or $\sqrt{(\lambda^2 - \lambda - 6)} = 3 - \lambda$

On squaring both sides, we get

$$\lambda^2 - \lambda - 6 = 9 + \lambda^2 - 6\lambda$$

$\Rightarrow \quad 5\lambda = 15$

$\therefore \quad \lambda = 3$

$\Rightarrow \quad \sqrt{(x+7)} = 3 \qquad$ [from Eq. (ii)]

or $\quad x + 7 = 9$

$\therefore \quad x = 2$

and $x = 2$ satisfies Eq. (i).

Hence, $x_1 = 2$

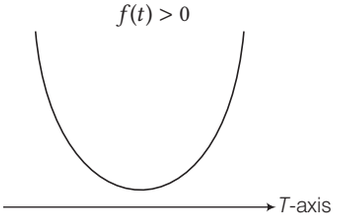
101. We have, $\quad 4^{x^2} + 2(2a+1)2^{x^2} + 4a^2 - 3 > 0 \qquad ...(i)$

Putting $t = 2^{x^2}$ in the Eq. (i), we get

$$t^2 + 2(2a+1)t + 4a^2 - 3 > 0$$

Let $\quad f(t) = t^2 + 2(2a+1)t + 4a^2 - 3 \qquad [\because t > 0, \therefore 2^{x^2} > 0]$

$\because \quad f(t) > 0$



Consider the following cases:

***Case* I** Sum of the roots > 0

$$-2\frac{(2a+1)}{1} > 0$$

$\therefore \quad a \in \left(-\infty, -\frac{1}{2}\right)$

***Case* II** Product of the roots > 0

$\Rightarrow \quad \frac{4a^2 - 3}{1} > 0$

or $\quad a^2 > \frac{3}{4}$

or $\quad a \in \left(-\infty, -\frac{\sqrt{3}}{2}\right) \cup \left(\frac{\sqrt{3}}{2}, \infty\right)$

***Case* III** $\quad D < 0$

$\Rightarrow \quad 4(2a+1)^2 - 4 \cdot 1 \cdot (4a^2 - 3) < 0$

$\Rightarrow \quad 4a + 4 < 0$

$\therefore \quad a < -1$

or $\quad a \in (-\infty, -1)$

Combining all cases, we get

$$a \in (-\infty, -1) \cup \left(\frac{\sqrt{3}}{2}, \infty\right)$$

102. We have, $\quad \log_{x^2+2x-3}\left(\frac{|x+4| - |x|}{x-1}\right) > 0$

The given inequation is valid for

$$\frac{|x+4| - |x|}{(x-1)} > 0$$

and $\quad x^2 + 2x - 3 > 0, \neq 1 \qquad ...(i)$

Now, consider the following cases:

***Case* I** If $0 < x^2 + 2x - 3 < 1$

$\Rightarrow \quad 4 < x^2 + 2x + 1 < 5$

$\Rightarrow \quad 4 < (x+1)^2 < 5$

$\Rightarrow \quad -\sqrt{5} < (x+1) < -2 \quad$ or $\quad 2 < x + 1 < \sqrt{5}$

$\Rightarrow \quad -\sqrt{5} - 1 < x < -3$ or $1 < x < \sqrt{5} - 1$

$\therefore \quad x \in (-\sqrt{5} - 1, -3) \cup (1, \sqrt{5} - 1) \qquad ...(ii)$

Then, $\quad \frac{|x+4| - |x|}{(x-1)} < 1$

Now, $x < -4$, then $\frac{-(x+4) + x}{(x-1)} < 1$

$\Rightarrow \quad 1 + \frac{4}{x-1} > 0$

$\Rightarrow \quad \frac{(x+3)}{(x-1)} > 0$

$\therefore \quad x \in (-\infty, -3) \cup (1, \infty)$

$\Rightarrow \quad x \in (-\infty, -4) \qquad [\because x < -4] \ ...(iii)$

$-4 \le x < 0$, then $\frac{x + 4 + x}{(x-1)} - 1 < 0$

$\Rightarrow \quad \frac{(x+5)}{(x-1)} < 0$

$\therefore \quad x \in (-5, 1)$

$\Rightarrow \quad x \in [-4, 0) \qquad [\because -4 \le x < 0] \ ...(iv)$

and $\quad x \ge 0$, then $\frac{(x+4) - x}{(x-1)} < 1$

$\Rightarrow \quad 1 - \frac{4}{x-1} > 0$

$\Rightarrow \quad \frac{(x-5)}{(x-1)} > 0$

$\therefore \quad x \in (-\infty, 1) \cup (5, \infty)$

$\Rightarrow \quad x \in [0, 1) \cup (5, \infty) \qquad [\because x \ge 0] \ ...(v)$

From Eqs. (iii), (iv) and (v), we get

$$x \in (-\infty, 1) \cup (5, \infty) \qquad ...(vi)$$

Now, common values in Eqs. (ii) and (iv) is

$$x \in (-\sqrt{5} - 1, -3) \qquad ...(vii)$$

***Case* II** If $\quad x^2 + 2x - 3 > 1$

$\Rightarrow \quad x^2 + 2x + 1 > 5 \ \Rightarrow \ (x+1)^2 > 5$

$\Rightarrow \quad x + 1 < -\sqrt{5}$

or $\quad x + 1 > \sqrt{5}$

$\therefore \quad x \in (-\infty, -1 - \sqrt{5}) \cup (\sqrt{5} - 1, \infty) \qquad ...(viii)$

Then, $\quad \frac{|x+4| - |x|}{(x-1)} > 1$

Now, $\quad x < -4$, then $\frac{-4}{x-1} > 1$

$\Rightarrow \quad 1 + \frac{4}{x-1} < 0$

$\Rightarrow \quad \frac{x+3}{x-1} < 0$

$\therefore \quad x \in (-3, 1)$

which is false. $[\because x < -4]$

$-4 \le x < 0$, then $\dfrac{2x+4}{(x-1)} - 1 > 0$

$\Rightarrow \quad \dfrac{(x+5)}{(x-1)} > 0$

$\therefore \quad x \in (-\infty, -5) \cup (1, \infty)$

which is false. $[\because -4 \le x < 0]$

and $x \ge 0$, then $\dfrac{4}{x-1} > 1$

$\Rightarrow \quad 1 - \dfrac{4}{x-1} < 0$

$\Rightarrow \quad \dfrac{x-5}{x-1} < 0$

$\therefore \quad x \in (1, 5)$ ...(ix)

which is false. $[\because x \ge 0]$

Now, common values in Eq. (viii) and (ix) is

$\therefore \quad x \in (\sqrt{5} - 1, 5)$ ...(x)

Combining Eqs. (viii) and (x), we get

$$x \in (-\sqrt{5} - 1, -3) \cup (\sqrt{5} - 1, 5)$$

103. Let $y \ge 0$, then $|y| = y$

and then given system reduces to

$$|x^2 - 2x| + y = 1 \quad ...(i)$$

and $$x^2 + y = 1 \quad ...(ii)$$

From Eqs. (i) and (ii), we get

$$x^2 = |x^2 - 2x|$$

$\Rightarrow \quad x^2 = |x|\,|x-2|$

Now, $x < 0, \quad 0 \le x < 2, \quad x \ge 2$

$x^2 = x(x-2), \quad x^2 = -x(x-2)$

$x^2 = x(x-2)$

$\therefore \quad x = 0$

$\Rightarrow \quad x(x + x - 2) = 0$

$\therefore \quad x = 0$

fail $\therefore x = 0, 1$ fail

$\Rightarrow \quad x = 0, 1$, then $y = 1, 0$

$\therefore$ Solutions are (0, 1) and (1, 0).

If $y < 0$ then $|y| = -y$ and then given system reduces to

$$|x^2 - 2x| + y = 1 \quad ...(iii)$$

and $$x^2 - y = 1 \quad ...(iv)$$

From Eqs. (iii) and (iv), we get

$$|x^2 - 2x| + x^2 = 2$$

$\Rightarrow \quad |x|\,|x-2| + x^2 = 2$

Now, $x < 0, \quad 0 \le x < 2, \quad x \ge 2$

$x(x-2) + x^2 = 2$

$-x(x-2) + x^2 = 2$

$x(x-2) + x^2 = 2$

$\Rightarrow \quad 2x^2 - 2x - 2 = 0 \quad \Rightarrow 2x = 2$

$\Rightarrow \quad x^2 - x - 1 = 0$

$\Rightarrow \quad x^2 - x - 1 = 0$

$\therefore \quad x = 1$

$\therefore \quad x = \dfrac{1 \pm \sqrt{5}}{2}$

$\therefore \quad x = \dfrac{1 \pm \sqrt{5}}{2}$ fail

$\Rightarrow \quad x = \dfrac{1 - \sqrt{5}}{2}$ $[\because x < 0]$

$\Rightarrow \quad x = \dfrac{1-\sqrt{5}}{2}, 1$, then $y = \dfrac{1-\sqrt{5}}{2}, 0$

$\therefore$ Solutions are $\left(\dfrac{1-\sqrt{5}}{2}, \dfrac{1-\sqrt{5}}{2}\right)$ and (1, 0).

Hence, all pairs (0, 1), (1, 0) and $\left(\dfrac{1-\sqrt{5}}{2}, \dfrac{1-\sqrt{5}}{2}\right)$ are solutions of the original system of equations.

104. Given, α, β and γ are the roots of the cubic equation

$$x^3 - px^2 + qx - r = 0 \quad ...(i)$$

$\therefore \quad \alpha + \beta + \gamma = p, \ \alpha\beta + \beta\gamma + \gamma\alpha = q, \ \alpha\beta\gamma = r$

(i) Let $y = \beta\gamma + \dfrac{1}{\alpha}$

$\Rightarrow \quad y = \dfrac{\alpha\beta\gamma + 1}{\alpha} = \dfrac{r+1}{\alpha}$

$\therefore \quad \alpha = \dfrac{r+1}{y}$

From Eq. (i), we get

$$\alpha^3 - p\alpha^2 + q\alpha - r = 0$$

$\Rightarrow \quad \dfrac{(r+1)^3}{y^3} - \dfrac{p(r+1)^2}{y^2} + \dfrac{q(r+1)}{y} - r = 0$

or $\quad ry^3 - q(r+1)y^2 + p(r+1)^2 y - (r+1)^3 = 0$

(ii) Let $y = \beta + \gamma - \alpha = (\alpha + \beta + \gamma) - 2\alpha = p - 2\alpha$

$\therefore \quad \alpha = \dfrac{p-y}{2}$

From Eq. (i), we get

$$\alpha^3 - p\alpha^2 + q\alpha - r = 0$$

$\Rightarrow \quad \dfrac{(p-y)^3}{8} - \dfrac{p(p-y)^2}{4} + \dfrac{q(p-y)}{2} - r = 0$

or $\quad y^3 - py^2 + (4q - p^2)y + (8r - 4pq + p^3) = 0$

Also product of roots $= -(8r - 4pq + p^3)$

105. Assume $\alpha + i\beta$ is a complex root of the given equation, then conjugate of this root, i.e. $\alpha - i\beta$ is also root of this equation. On putting $x = \alpha + i\beta$ and $x = \alpha - i\beta$ in the given equation, we get

$$\frac{A_1^2}{\alpha + i\beta - a_1} + \frac{A_2^2}{\alpha + i\beta - a_2} + \frac{A_3^2}{\alpha + i\beta - a_3} + \ldots + \frac{A_n^2}{\alpha + i\beta - a_n}$$

$$= ab^2 + c^2(\alpha + i\beta) + ac \quad ...(i)$$

and $$\frac{A_1^2}{\alpha - i\beta - a_1} + \frac{A_2^2}{\alpha - i\beta - a_2} + \frac{A_3^2}{\alpha - i\beta - a_3} + \ldots + \frac{A_n^2}{\alpha - i\beta - a_n}$$

$$= ab^2 + c^2(\alpha - i\beta) + ac \quad ...(ii)$$

On subtracting Eq. (i) from Eq. (ii), we get

$$2i\beta\left[\frac{A_1^2}{(\alpha-a_1)^2+\beta^2}+\frac{A_2^2}{(\alpha-a_2)^2+\beta^2}+\frac{A_3^2}{(\alpha-a_3)^2+\beta^2}+\ldots+\frac{A_n^2}{(\alpha-a_n)^2+\beta^2}+c^2\right]=0$$

The expression in bracket $\neq 0$

$\therefore \quad 2i\beta = 0 \Rightarrow \beta = 0$

Hence, all roots of the given equation are real.

106. Given equation is

$$x^4 + 2ax^3 + x^2 + 2ax + 1 = 0 \quad \ldots(i)$$

On dividing by x^2, we get

$$x^2 + 2ax + 1 + \frac{2a}{x} + \frac{1}{x^2} = 0$$

$$\Rightarrow \left(x^2+\frac{1}{x^2}\right)+2a\left(x+\frac{1}{x}\right)+1=0$$

$$\Rightarrow \left(x+\frac{1}{x}\right)^2-2+2a\left(x+\frac{1}{x}\right)+1=0$$

or $$\left(x+\frac{1}{x}\right)^2+2a\left(x+\frac{1}{x}\right)-1=0$$

or $y^2 + 2ay - 1 = 0$, where $y = x + \frac{1}{x}$

$$\therefore \quad y = \frac{-2a \pm \sqrt{(4a^2+4)}}{2} = -a \pm \sqrt{(a^2+1)}$$

Taking '+' sign, we get

$$y = -a + \sqrt{(a^2+1)}$$

$$\Rightarrow \quad x + \frac{1}{x} = -a + \sqrt{(a^2+1)}$$

or $$x^2 + (a - \sqrt{(a^2+1)})\,x + 1 = 0 \quad \ldots(ii)$$

Taking '–' sign, we get $y = -a - \sqrt{(a^2+1)}$

$$\Rightarrow \quad x + \frac{1}{x} = -a - \sqrt{(a^2+1)}$$

or $$x^2 + (a + \sqrt{(a^2+1)})\,x + 1 = 0 \quad \ldots(iii)$$

Let α, β be the roots of Eq. (ii) and γ, δ be the roots of Eq. (iii).

Then, $$\alpha + \beta = \sqrt{(a^2+1)} - a$$

and $$\alpha\beta = 1$$

and $$\gamma + \delta = -\sqrt{(a^2+1)} - a$$

and $$\gamma\delta = 1$$

Clearly, $\alpha + \beta > 0$ and $\alpha\beta > 0$

$\therefore$ Either α, β will be imaginary or both real and positive according to the Eq. (i) has atleast two distinct negative roots. Therefore, both γ and δ must be negative. Therefore,

(i) $\gamma\delta > 0$, which is true as $\gamma\delta = 1$.

(ii) $$\gamma + \delta < 0$$

$$\Rightarrow \quad -(a + \sqrt{(a^2+1)}) < 0$$

$$\Rightarrow \quad a + \sqrt{(a^2+1)} > 0, \text{ which is true for all } a.$$

$$\therefore \quad a \in R$$

(iii) $$D > 0$$

$$\therefore \quad (a + \sqrt{(a^2+1)})^2 - 4 > 0$$

$$\Rightarrow \quad (a + \sqrt{(a^2+1)} + 2)\,(a + \sqrt{(a^2+1)} - 2) > 0$$

$\because \quad a + \sqrt{(a^2+1)} + 2 > 0$

$\therefore \quad a + \sqrt{(a^2+1)} - 2 > 0$

$$\Rightarrow \quad \sqrt{(a^2+1)} > 2 - a$$

$$\begin{cases} a \geq 2 \\ \text{or } a^2 + 1 > (2-a)^2, \text{ if } a < 2 \end{cases}$$

$$\Rightarrow \quad \begin{cases} a \geq 2 \\ \text{or } a > \frac{3}{4}, \text{ if } a < 2 \end{cases}$$

$$\Rightarrow \quad \begin{cases} a \geq 2 \\ \text{or } \frac{3}{4} < a < 2 \end{cases}$$

Hence, $$\frac{3}{4} < a < \infty \text{ or } \quad a \in \left(\frac{3}{4}, \infty\right)$$

107. We have, $[2x] - [x+1] = 2x$

Since, $\text{LHS} = \text{Integer}$

$\therefore \quad \text{RHS} = 2x = \text{Integer}$

$\Rightarrow \quad [2x] = 2x$

Now, $-[x+1] = 0$

$\Rightarrow \quad [x+1] = 0$

or $0 \leq x + 1 < 1$

or $-1 \leq x < 0$

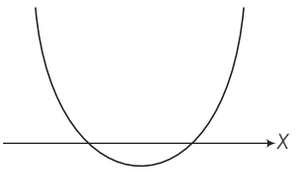
or $-2 \leq 2x < 0$

$\therefore \quad 2x = -2, -1$

or $x = -1, -\frac{1}{2}$

or $x_1 = -1,\ x_2 = -\frac{1}{2}$

108. We have, $(a^2+3)x^2 + (a+2)x - 6 < 0$



Let $f(x) = (a^2+3)\,x^2 + (a+2)\,x - 6$

$\because \quad (a^2+3) > 0$ and $f(x) < 0$

$\therefore \quad D > 0$

$\Rightarrow (a+2)^2 + 24\,(a^2+3) > 0$ is true for all $a \in R$.

109. We have, $6x^2 - 77[x] + 147 = 0$

$$\Rightarrow \quad \frac{6x^2+147}{77} = [x]$$

$$\Rightarrow \quad (0.078)\,x^2 = [x] - 1.9$$

$\because \quad (0.078)\,x^2 > 0 \Rightarrow x^2 > 0$

$\therefore \quad [x] - 1.9 > 0$

or $[x] > 1.9$

$\therefore$ $[x] = 2, 3, 4, 5, \ldots$

If $[x] = 2$, i.e. $2 \le x < 3$

Then, $x^2 = \dfrac{2 - 1.9}{0.078} = 1.28$

$\therefore$ $x = 1.13$ [fail]

If $[x] = 3$, i.e. $3 \le x < 4$

Then, $x^2 = \dfrac{3 - 1.9}{0.078} = 14.1$

$\therefore$ $x = 3.75$ [true]

If $[x] = 4$, i.e. $4 \le x < 5$

Then, $x^2 = \dfrac{4 - 1.9}{0.078} = 26.9$

$\therefore$ $x = 5.18$ [fail]

If $[x] = 5$, i.e. $5 \le x < 6$

Then, $x^2 = \dfrac{5 - 1.9}{0.078} = 39.7$

$\therefore$ $x = 6.3$ [fail]

If $[x] = 6$, i.e. $6 \le x < 7$

Then, $x^2 = \dfrac{6 - 1.9}{0.078} = \dfrac{4.1}{0.078} = 52.56$

$\therefore$ $x = 7.25$ [fail]

If $[x] = 7$, i.e. $7 \le x < 8$

Then, $x^2 = \dfrac{7 - 1.9}{0.078} = \dfrac{5.1}{0.078} = 65.38$

$\therefore$ $x = 8.08$ [fail]

If $[x] = 8$, i.e. $8 \le x < 9$

Then, $x^2 = \dfrac{8 - 1.9}{0.078} = \dfrac{6.1}{0.078} = 78.2$

$\therefore$ $x = 8.8$ [true]

If $[x] = 9$, i.e. $9 \le x < 10$

Then, $x^2 = \dfrac{9 - 1.9}{0.078} = \dfrac{7.1}{0.078} = 91.03$

$\therefore$ $x = 9.5$ [true]

If $[x] = 10$, i.e. $10 \le x < 11$

Then, $x^2 = \dfrac{10 - 1.9}{0.078} = \dfrac{8.1}{0.078} = 103.8$

$\therefore$ $x = 10.2$ [true]

If $[x] = 11$, i.e. $11 \le x < 12$

Then, $x^2 = \dfrac{11 - 1.9}{0.078}$

$= \dfrac{9.1}{0.078} = 116.7$

$\therefore$ $x = 10.8$ [fail]

Other values are fail.

Hence, number of solutions is four.

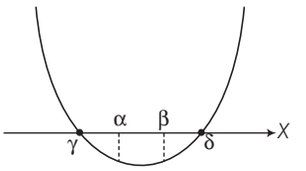
110. Since, the given equation is

$$x^2 - 2x - a^2 + 1 = 0$$

$\Rightarrow$ $(x - 1)^2 = a^2$

$\therefore$ $x - 1 \ne a$ or $x = 1 \pm a$

$\Rightarrow$ $\alpha = 1 + a$ and $\beta = 1 - a$



Let $f(x) = x^2 - 2(a + 1)\, x + a\,(a - 1)$, thus the following conditions hold good:

Consider the following cases:

***Case* I** $D > 0$

$\Rightarrow$ $4\,(a + 1)^2 - 4a\,(a - 1) > 0$

$\Rightarrow$ $3a + 1 > 0$

$\therefore$ $a > -\dfrac{1}{3}$

***Case* II** $f(\alpha) < 0$

$\Rightarrow$ $f(1 + a) < 0$

$\Rightarrow$ $(1 + a)^2 - 2\,(1 + a)\,(1 + a) + a\,(a - 1) < 0$

$\Rightarrow$ $-(1 + a)^2 + a\,(a - 1) < 0$

$\Rightarrow$ $-3a - 1 < 0$

$\Rightarrow$ $a > -\dfrac{1}{3}$

***Case* III** $f(s) = 0$

$\Rightarrow$ $f(1 - a) < 0$

$\Rightarrow$ $(1 - a)^2 - 2\,(a + 1)\,(1 - a) + a\,(a - 1) < 0$

$\Rightarrow$ $(4a + 1)\,(a - 1) < 0$

$\therefore$ $-\dfrac{1}{4} < a < 1$

Combining all cases we get

$$a \in \left(-\frac{1}{4}, 1\right)$$

111. $pr = (-p)\,(-r)$

$= (\alpha + \beta + \gamma + \delta)(\alpha\beta\gamma + \alpha\beta\delta + \gamma\delta\alpha + \gamma\delta\beta)$

$= \alpha^2\beta\gamma + \alpha^2\beta\delta + \alpha^2\gamma\delta + \alpha\beta\gamma\delta + \beta^2\gamma\alpha$

$+ \beta^2\alpha\delta + \alpha\beta\gamma\delta + \beta^2\gamma\delta + \gamma^2\alpha\beta + \alpha\beta\gamma\delta$

$+ \gamma^2\delta\alpha + \gamma^2\delta\beta + \alpha\beta\gamma\delta + \alpha\beta\delta^2 + \gamma\alpha\delta^2 + \gamma\beta\delta^2$

$\because$ AM $\ge$ GM

$\Rightarrow$ $\dfrac{pr}{16} \ge (\alpha^{16}\beta^{16}\gamma^{16}\delta^{16})^{1/6} = \alpha\beta\gamma\delta = 5$

$\Rightarrow$ $\dfrac{pr}{16} \ge 5$

or $pr \ge 80$

$\therefore$ Minimum value of pr is 80.

112. $(\alpha^2 + \beta^2)^2 = (\alpha + \beta)\,(\alpha^3 + \beta^3)$

$\Rightarrow$ $\{(\alpha + \beta)^2 - 2\alpha\beta\}^2 = (\alpha + \beta)\,\{(\alpha + \beta)^3 - 3\alpha\beta\,(\alpha + \beta)\}$

$\Rightarrow$ $\left(\dfrac{b^2}{a^2} - \dfrac{2c}{a}\right)^2 = \left(-\dfrac{b}{a}\right)\left(\dfrac{-b^3}{a^3} + \dfrac{3bc}{a^2}\right)$

$\Rightarrow$ $\left(\dfrac{b^2 - 2ac}{a^2}\right)^2 = \left(\dfrac{-b}{a}\right)\left(\dfrac{-b^3 + 3abc}{a^3}\right)$

$\Rightarrow \quad 4a^2c^2 = acb^2$

$\Rightarrow \quad ac\,(b^2 - 4ac) = 0$

As $\quad a \neq 0$

$\Rightarrow \quad c\Delta = 0$

113. Let $P(x) = bx^2 + ax + c$

As $\quad P(0) = 0$

$\Rightarrow \quad c = 0$

As $\quad P(1) = 1$

$\Rightarrow \quad a + b = 1$

$P(x) = ax + (1 - a)\,x^2$

Now, $\quad P'(x) = a + 2(1 - a)\,x$

As $\quad P'(x) > 0$ for $x \in (0, 1)$

Only option (d) satisfies above condition.

114. Let the roots are α and $\alpha + 1$, where $\alpha \in I$.

Then, sum of the roots $= 2\alpha + 1 = b$

Product of the roots $= \alpha(\alpha + 1) = c$

Now, $\quad b^2 - 4c = (2\alpha + 1)^2 - 4\alpha(\alpha + 1)$

$= 4\alpha^2 + 1 + 4\alpha - 4\alpha^2 - 4\alpha = 1$

$\therefore \quad b^2 - 4c = 1$

115. Let $\quad f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x,$

$f(0) = 0; f(\alpha) = 0$

$\Rightarrow \quad f'(x) = 0$ has atleast one root between $(0, \alpha)$.

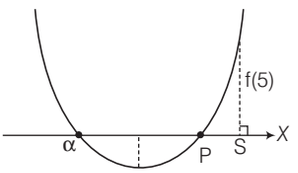
i.e. Equation

$na_n x^{n-1} + (n-1)\,a_{n-1}x^{n-2} + \ldots + a_1 = 0$

has a positive root smaller than α.

116. Let $f(x) = x^2 - 2kx + k^2 + k - 5$

Consider the following cases:



***Case* I** $\quad D \geq 0$

$\Rightarrow \quad 4k^2 - 4.1(k^2 + k - 5) \geq 0$

$\Rightarrow \quad -4(k - 5) \geq 0$

$\Rightarrow \quad k - 5 \leq 0$

$\Rightarrow \quad k \leq 5 \quad \text{or} \quad k \in (-\infty, 5]$

***Case* II** x-Coordinate of vertex $x < 5$

$\Rightarrow \quad \frac{2k}{2} < 5$

$\Rightarrow \quad k < 5$ or $k \in (-\infty, 5)$

***Case* III** $\quad f(5) > 0$

$\Rightarrow \quad 25 - 10k + k^2 + k - 5 > 0$

$\Rightarrow \quad k^2 - 9k + 20 > 0$

$\Rightarrow \quad (k - 4)(k - 5) > 0$ or $k \in (-\infty, 4) \cup (5, \infty)$

Combining all cases, we get

$k \in (-\infty, 4)$

117. We have, $\quad a + b = 10c,\ ab = -11d$

and $\quad c + d = 10a,\ cd = -11b$

$\therefore \quad a + b + c + d = 10(a + c)$

and $\quad abcd = 121\ bd$

$\Rightarrow \quad b + d = 9(a + c)$

and $\quad ac = 121$

Next, $\quad a^2 - 10ac - 11d = 0$

and $\quad c^2 - 10ac - 11b = 0$

$\Rightarrow \quad a^2 + c^2 - 20\,ac - 11(b + d) = 0$

$\Rightarrow \quad (a + c)^2 - 22 \times 121 - 99(a + c) = 0$

$\Rightarrow \quad a + c = 121$ or -22

If $a + c = -22 \Rightarrow a = c$, rejecting these values, we have

$a + c = 121$

$\therefore \quad a + b + c + d = 10(a + c) = 1210$

118. $\quad D \geq 0$

$4(a + b + c)^2 - 12\lambda\,(ab + bc + ca) \geq 0$

$(a^2 + b^2 + c^2) - (3\lambda - 2)(ab + bc + ca) \geq 0$

$\therefore \quad (3\lambda - 2) \leq \frac{(a^2 + b^2 + c^2)}{(ab + bc + ca)}$

Since, $\quad |a - b| < c$

$\Rightarrow \quad a^2 + b^2 - 2ab < c^2 \qquad \text{(i)}$

$|b - c| < a$

$\Rightarrow \quad b^2 + c^2 - 2bc < a^2 \qquad \ldots\text{(ii)}$

$|c - a| < b$

$\Rightarrow \quad c^2 + a^2 - 2ca < b^2 \qquad \ldots\text{(iii)}$

From Eqs. (i), (ii) and (iii), we get

$\frac{a^2 + b^2 + c^2}{ab + bc + ca} < 2 \qquad \ldots\text{(iv)}$

From Eqs. (i) and (iv), we get

$3\lambda - 2 < 2 \ \Rightarrow \ \lambda < \frac{4}{3}$

119. $\because \quad x^2 - 2mx + m^2 - 1 = 0$

$\Rightarrow \quad (x - m)^2 = 1$

$\therefore \quad x - m = \pm 1 \ \text{ or } \ x = m - 1,\ m + 1$

According to the question,

$m - 1 > -2,\ m + 1 > -2$

$\Rightarrow \quad m > -1,\ m > -3$

Then, $\quad m > -1 \qquad \ldots\text{(i)}$

and $\quad m - 1 < 4,\ m + 1 < 4$

$\Rightarrow \quad m < 5,\ m < 3$ and $m < 3 \qquad \ldots\text{(ii)}$

From Eqs. (i) and (ii), we get $\ -1 < m < 3$

120. $x^2 + px + q = 0$

Sum of the roots $= \tan 30° + \tan 15° = -p$

Product of the roots $= \tan 30° \cdot \tan 15° = q$

$\tan 45° = \tan(30° + 15°) \ = \frac{\tan 30° + \tan 15°}{1 - \tan 30° \cdot \tan 15°}$

$$\Rightarrow \quad 1 = \frac{-p}{1-q} \Rightarrow -p = 1-q$$

$$\Rightarrow \quad q - p = 1$$

$$\therefore \quad 2 + q - p = 3$$

121. The equation $x^2 - px + r = 0$ has roots (α, β) and the equation $x^2 - qx + r$ has roots $\left(\frac{\alpha}{2}, 2\beta\right)$.

$$\Rightarrow \quad r = \alpha\beta \text{ and } \alpha + \beta = p \text{ and } \frac{\alpha}{2} + 2\beta = q$$

$$\Rightarrow \quad \beta = \frac{2q-p}{3} \text{ and } \alpha = \frac{2(2p-q)}{3}$$

$$\Rightarrow \quad \alpha\beta = r = \frac{2}{9}(2q-p)(2p-q)$$

122. $\alpha + \beta = -a$

$$|\alpha - \beta| < \sqrt{5} \Rightarrow (\alpha-\beta)^2 < 5$$

$$\Rightarrow \quad a^2 - 4 < 5 \Rightarrow a \in (-3, 3)$$

123. Suppose roots are imaginary, then $\beta = \overline{\alpha}$

and $\frac{1}{\beta} = \overline{\alpha}$

$$\Rightarrow \quad \beta = -\frac{1}{\beta} \qquad \text{[not possible]}$$

$\Rightarrow$ Roots are real $\Rightarrow (p^2 - q)(b^2 - ac) \geq 0$

$\Rightarrow$ Statement –1 is true.

$$-\frac{2b}{a} = \alpha + \frac{1}{\beta}$$

and $\frac{\alpha}{\beta} = \frac{c}{a}, \alpha + \beta = -2p, \alpha\beta = q$

If $\beta = 1$, then $\alpha = q$

$$\Rightarrow \quad c = qa \qquad \text{[not possible]}$$

Also, $\alpha + 1 = \frac{-2b}{a}$

$$\Rightarrow \quad -2p = \frac{-2b}{a}$$

$$\Rightarrow \quad b = ap \qquad \text{[not possible]}$$

$\Rightarrow$ Statement –2 is true but it is not the correct explanation of Statement-1.

124. Let $\alpha, 4\beta$ be roots of $x^2 - 6x + a = 0$ and $\alpha, 3\beta$ be the roots of $x^2 - cx + 6 = 0$.

Then, $\alpha + 4\beta = 6$ and $4\alpha\beta = a$ ...(i)

$\alpha + 3\beta = c$ and $3\alpha\beta = 6$ ...(ii)

From Eqs. (i) and (ii), we get

$$a = 8, \alpha\beta = 2$$

Now, first equation becomes

$$x^2 - 6x + 8 = 0$$

$$\Rightarrow \quad x = 2, 4$$

If $\alpha = 2, 4\beta = 4$, then $3\beta = 3$

If $\alpha = 4, 4\beta = 2$, then $3\beta = \frac{3}{2}$ [non-integer]

$\therefore$ Common root is $x = 2$.

125. Let $f(x) = x^7 + 14x^5 + 16x^3 + 30x - 560$

$$\therefore \quad f'(x) = 7x^6 + 70x^4 + 48x^2 + 30 > 0, \forall\ x \in R$$

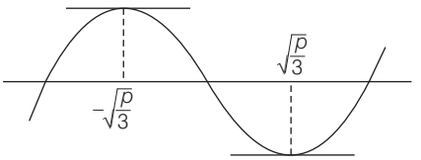
$\Rightarrow f(x)$ is an increasing function, for all $x \in R$

Hence, number of real solutions is 1.

126. Let $f(x) = x^3 - px + q$

$$\therefore \quad f'(x) = 3x^2 - p$$

$$\Rightarrow \quad f''(x) = 6x$$



For maxima or minima, $f'(x) = 0$

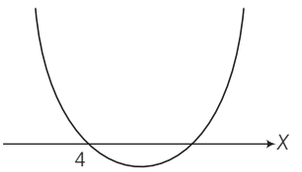
$$\therefore \quad x = \pm\sqrt{\frac{p}{3}}$$

$$\Rightarrow \quad f''\left(\sqrt{\frac{p}{3}}\right) = 6\sqrt{\left(\frac{p}{3}\right)} > 0$$

and $f''\left(-\sqrt{\frac{p}{3}}\right) = -6\sqrt{\frac{p}{3}} < 0$

Hence, given cubic minima at $x = \sqrt{\frac{p}{3}}$ and maxima at $x = -\sqrt{\frac{p}{3}}$.

127. Let $f(x) = x^2 - 8kx + 16(k^2 - k + 1)$



$$\therefore \quad D > 0$$

$$\Rightarrow \quad 64k^2 - 4 \cdot 16(k^2 - k + 1) > 0$$

$$\Rightarrow \quad k > 1 \qquad \text{...(i)}$$

$$\Rightarrow \quad \frac{-b}{2a} > 4 \Rightarrow \frac{8k}{2} > 4$$

$$\Rightarrow \quad k > 1 \qquad \text{...(ii)}$$

and $f(4) \geq 0$

$$\Rightarrow \quad 16 - 32k + 16(k^2 - k + 1) \geq 0$$

$$\Rightarrow \quad k^2 - 3k + 2 \geq 0$$

$$\Rightarrow \quad (k-1)(k-2) \geq 0$$

$$\Rightarrow \quad k \leq 1 \text{ or } k \geq 2 \qquad \text{...(iii)}$$

From Eqs. (i), (ii) and (iii), we get

$$k \geq 2$$

$$k_{\min} = 2$$

128. Since, roots of $bx^2 + cx + a = 0$ are imaginary.

$$\therefore \quad c^2 - 4ab < 0$$

$$\Rightarrow \quad -c^2 > -4ab \qquad \text{...(i)}$$

Let $f(x) = 3b^2x^2 + 6bcx + 2c^2$

Since, $3b^2 > 0$

and $D = (6bc)^2 - 4(3b^2)(2c^2) = 12b^2c^2$

$\therefore$ Minimum value of $f(x) = -\frac{D}{4a} = -\frac{12b^2c^2}{4(3b^2)} = -c^2 > -4ab$

129. $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta}$ ...(i)

and given, $\alpha^3 + \beta^3 = q, \alpha + \beta = -p$

$\Rightarrow$ $(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta) = q$

$\Rightarrow$ $-p^3 + 3p\alpha\beta = q$

or $\alpha\beta = \frac{q+p^3}{3p}$

$\therefore$ From Eq. (i), we get

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{p^2 - \frac{2(q+p^3)}{3p}}{\frac{(q+p^3)}{3p}} = \frac{p^3 - 2q}{(q+p^3)}$$

and product of the roots $= \frac{\alpha}{\beta}\cdot\frac{\beta}{\alpha} = 1$

$\therefore$ Required equation is $x^2 - \left(\frac{p^3-2q}{q+p^3}\right)x + 1 = 0$

or $(q+p^3)x^2 - (p^3 - 2q)x + (q+p^3) = 0$

130. Since, $f'(x) = 12x^2 + 6x + 2$

Here, $D = 6^2 - 4\cdot 12\cdot 2 = 36 - 96 = -60 < 0$

$\therefore$ $f'(x) > 0, \forall\ x \in R$

$\Rightarrow$ Only one real root for $f(x) = 0$

Also, $f(0) = 1, f(-1) = -2$

$\Rightarrow$ Root must lie in $(-1, 0)$.

Taking average of 0 and (-1), $f\left(-\frac{1}{2}\right) = \frac{1}{4}$

$\Rightarrow$ Root must lie in $\left(-1, -\frac{1}{2}\right)$.

Similarly, $f\left(-\frac{3}{4}\right) = -\frac{1}{2}$

$\Rightarrow$ Root must lie in $\left(-\frac{3}{4}, -\frac{1}{2}\right)$.

131. $\because \alpha^2 - 6\alpha - 2 = 0 \Rightarrow \alpha^2 - 2 = 6\alpha$ ...(i)

and $\beta^2 - 6\beta - 2 = 0 \Rightarrow \beta^2 - 2 = 6\beta$ ...(ii)

$\therefore$ $\frac{a_{10} - 2a_8}{2a_9} = \frac{(\alpha^{10} - \beta^{10}) - 2(\alpha^8 - \beta^8)}{2(\alpha^9 - \beta^9)}$

$= \frac{\alpha^8(\alpha^2 - 2) - \beta^8(\beta^2 - 2)}{2(\alpha^9 - \beta^9)}$

$= \frac{\alpha^8\cdot 6\alpha - \beta^8\cdot 6\beta}{2(\alpha^9 - \beta^9)}$ [from Eqs. (i) and (ii)]

$= \frac{6(\alpha^9 - \beta^9)}{2(\alpha^9 - \beta^9)} = 3$

132. Let α be the common root.

Then, $\alpha^2 + b\alpha - 1 = 0$ and $\alpha^2 + \alpha + b = 0$

$\Rightarrow$ $\begin{vmatrix} 1 & b \\ 1 & 1 \end{vmatrix} \times \begin{vmatrix} b & -1 \\ 1 & b \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ b & 1 \end{vmatrix}^2$

$\Rightarrow$ $(1-b)(b^2+1) = (-1-b)^2$

$\Rightarrow$ $b^3 + 3b = 0$

$\therefore$ $b = 0, i\sqrt{3}, -i\sqrt{3}$, where $i = \sqrt{-1}$.

133. Let $f(x) = x^4 - 4x^3 + 12x^2 + x - 1$

$\therefore$ $f'(x) = 4x^3 - 12x^2 + 24x + 1$

$\Rightarrow$ $f''(x) = 12x^2 - 24x + 24$

$= 12(x^2 - 2x + 2)$

$= 12[(x-1)^2 + 1] > 0$

i.e. $f''(x)$ has no real roots.

Hence, $f(x)$ has maximum two distinct real roots, where $f(0) = -1$.

134. Given, $p(x) = f(x) - g(x)$

$\Rightarrow$ $p(x) = (a - a_1)x^2 + (b - b_1)x + (c - c_1)$

It is clear that $p(x) = 0$ has both equal roots -1, then

$$-1 - 1 = -\frac{(b - b_1)}{(a - a_1)}$$

and $-1 \times -1 = \frac{c - c_1}{a - a_1}$

$\Rightarrow$ $b - b_1 = 2(a - a_1)$ and $c - c_1 = (a - a_1)$ ...(i)

Also given, $p(-2) = 2$

$\Rightarrow$ $4(a - a_1) - 2(b - b_1) + (c - c_1) = 2$ ...(ii)

From Eqs. (i) and (ii), we get

$4(a - a_1) - 4(a - a_1) + (a - a_1) = 2$

$\therefore$ $(a - a_1) = 2$ ...(iii)

$\Rightarrow$ $b - b_1 = 4$ and $c - c_1 = 2$ [from Eq. (i)] ...(iv)

Now, $p(2) = 4(a - a_1) + 2(b - b_1) + (c - c_1)$

$= 8 + 8 + 2 = 18$ [from Eqs. (iii) and (iv)]

135. Let the quadratic equation be

$$ax^2 + bx + c = 0$$

Sachin made a mistake in writing down constant term.

$\therefore$ Sum of the roots is correct.

i.e. $\alpha + \beta = 7$

Rahul made a mistake in writing down coefficient of x.

$\therefore$ Product of the roots is correct.

i.e. $\alpha\beta = 6$

$\Rightarrow$ Correct quadratic equation is

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$\Rightarrow$ $x^2 - 7x + 6 = 0$

$\Rightarrow$ $(x-6)(x-1) = 0 \Rightarrow x = 6, 1$

Hence, correct roots are 1 and 6.

136. Let $a + 1 = h^6$

$\therefore$ $(h^2 - 1)x^2 + (h^3 - 1)x + (h - 1) = 0$

$$\Rightarrow \quad \left(\frac{h^2-1}{h-1}\right)x^2+\left(\frac{h^3-1}{h-1}\right)x+1=0$$

As $a \to 0$, then $h \to 1$

$$\lim_{h\to 1}\left(\frac{h^2-1}{h-1}\right)x^2+\lim_{h\to 1}\left(\frac{h^3-1}{h-1}\right)x+1=0$$

$$\Rightarrow \quad 2x^2+3x+1=0$$

$$\Rightarrow \quad 2x^2+2x+x+1=0$$

$$\Rightarrow \quad (2x+1)(x+1)=0$$

$$\therefore \quad x=-1 \text{ and } x=-\frac{1}{2}$$

137. Let $e^{\sin x}=t$ ...(i)

Then, the given equation can be written as

$$t-\frac{1}{t}-4=0 \Rightarrow t^2-4t-1=0$$

$$\therefore \quad t=\frac{4\pm\sqrt{(16+4)}}{2}$$

$$\Rightarrow \quad e^{\sin x}=(2+\sqrt{5}) \qquad [\because e^{\sin x}>0, \therefore \text{taking + ve sign}]$$

$$\Rightarrow \quad \sin x=\log_e(2+\sqrt{5}) \qquad ...(ii)$$

$$\because \quad (2+\sqrt{5})>e \qquad [\because e=2.71828\ldots]$$

$$\Rightarrow \quad \log_e(2+\sqrt{5})>1 \qquad ...(iii)$$

From Eqs. (ii) and (iii), we get

$$\sin x>1 \qquad \text{[which is impossible]}$$

Hence, no real root exists.

138. Given equations are

$$ax^2+bx+c=0 \qquad ...(i)$$

and $$x^2+2x+3=0 \qquad ...(ii)$$

Clearly, roots of Eq. (ii) are imaginary, since Eqs. (i) and (ii) have a common root, therefore common root must be imaginary and hence both roots will be common. Therefore, Eqs. (i) and (ii) are identical.

$$\therefore \quad \frac{a}{1}=\frac{b}{2}=\frac{c}{3} \text{ or } a:b:c=1:2:3$$

139. $\because x-[x]=\{x\}$ [fractional part of x]

For no integral solution, $\{x\} \neq 0$

$$\therefore \quad a \neq 0 \qquad ...(i)$$

The given equation can be written as

$$3\{x\}^2-2\{x\}-a^2=0$$

$$\Rightarrow \quad \{x\}=\frac{2\pm\sqrt{(4+12a^2)}}{6}=\frac{1+\sqrt{(1+3a^2)}}{3} \qquad [\because 0<\{x\}<1]$$

$$\Rightarrow \quad 0<\frac{1+\sqrt{(1+3a^2)}}{3}<1 \Rightarrow \sqrt{(1+3a^2)}<2$$

$$\Rightarrow \quad a^2<1 \Rightarrow -1<a<1 \qquad ...(ii)$$

From Eqs. (i) and (ii), we get

$$a \in (-1, 0) \cup (0, 1)$$

140. $\because \frac{1}{\alpha}+\frac{1}{\beta}=4 \Rightarrow \frac{\alpha+\beta}{\alpha\beta}=4$

$$\Rightarrow \quad \frac{-\frac{q}{p}}{\frac{r}{p}}=4$$

$$\Rightarrow \quad q=-4r \qquad ... (i)$$

Also, given p, q, r are in AP.

$$\therefore \quad 2q=p+r$$

$$\Rightarrow \quad p=-9r \qquad \text{[from Eq. (i)] ...(ii)}$$

Now, $|\alpha-\beta|=\frac{\sqrt{D}}{|a|} \quad \left[\because \text{for } ax^2+bx+c=0, \alpha-\beta=\frac{\sqrt{D}}{a}\right]$

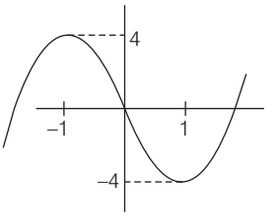
$$=\frac{\sqrt{(q^2-4pr)}}{|p|}$$

$$=\frac{\sqrt{(16r^2+36r^2)}}{9|r|}=\frac{\sqrt{52}\,|r|}{9|r|} \qquad \text{[from Eqs. (i) and (ii)]}$$

$$=\frac{2\sqrt{13}}{9}$$

141. $f(x)=x^5-5x$ and $g(x)=-a$

$$\therefore \quad f'(x)=5x^4-5$$



$$=5(x^2+1)(x-1)(x+1)$$

Clearly, $f(x)=g(x)$ has one real root, if $a>4$ and three real roots, if $|a|<4$.

142. Since, $b=0$ for $p(x)=ax^2+bx+c$, as roots are pure imaginary.

$\Rightarrow x=\pm\sqrt{\frac{(-c\pm i\sqrt{c})}{a}}$, which are clearly neither pure real nor pure imaginary, as $c \neq 0$.

143. $\because \alpha x^2-x+\alpha=0$ has distinct real roots.

$$\therefore \quad D>0$$

$$\Rightarrow \quad 1-4\alpha^2>0 \Rightarrow \alpha \in \left(-\frac{1}{2},\frac{1}{2}\right) \qquad ...(i)$$

Also, $$|x_1-x_2|<1 \Rightarrow |x_1-x_2|^2<1$$

$$\Rightarrow \quad \frac{D}{a^2}<1 \Rightarrow \frac{1-4\alpha^2}{\alpha^2}<1 \Rightarrow \alpha^2>\frac{1}{5}$$

$$\Rightarrow \quad \alpha \in \left(-\infty,-\frac{1}{\sqrt{5}}\right)\cup\left(\frac{1}{\sqrt{5}},\infty\right) \qquad ...(ii)$$

From Eqs. (i) and (ii), we get

$$S=\left(-\frac{1}{2},-\frac{1}{\sqrt{5}}\right)\cup\left(\frac{1}{\sqrt{5}},\frac{1}{2}\right)$$

144. $(x^2-5x+5)^{x^2+4x-60}=1$

Case I

$x^2-5x+5=1$ and $x^2+4x-60$ can be any real number

$\Rightarrow \quad x=1,4$

Case II

$x^2-5x+5=-1$ and $x^2+4x-60$ has to be an even number

$\Rightarrow \quad x=2,3$

For $x=3$, $x^2+4x-60$ is odd, $\therefore\ x\neq 3$

Hence, $x=2$

Case III x^2-5x+5 can be any real number and

$$x^2+4x-60=0$$

$\Rightarrow \quad x=-10,6$

$\Rightarrow$ Sum of all values of $x=1+4+2-10+6=3$

145. $\because\ x^2-2x\sec\theta\, a+1=0 \ \Rightarrow\ x=\sec\theta\pm\tan\theta$

and $$-\frac{\pi}{6}<\theta<-\frac{\pi}{12}$$

$\Rightarrow$ $$\sec\left(-\frac{\pi}{6}\right)>\sec\theta>\sec\left(-\frac{\pi}{12}\right)$$

or $$\sec\left(\frac{\pi}{6}\right)>\sec\theta>\sec\left(\frac{\pi}{12}\right)$$

and $$\tan\left(-\frac{\pi}{6}\right)<\tan\theta<\tan\left(-\frac{\pi}{12}\right)$$

$\Rightarrow$ $$-\tan\left(\frac{\pi}{6}\right)<\tan\theta<-\tan\left(\frac{\pi}{12}\right)$$

or $$\tan\left(\frac{\pi}{6}\right)>-\tan\theta>\tan\left(\frac{\pi}{12}\right)$$

$\because\ \alpha_1,\beta_1$ are roots of $x^2-2x\sec\theta+1=0$ and $\alpha_1>\beta_1$

$\therefore\ \alpha_1=\sec\theta-\tan\theta$ and $\beta_1=\sec\theta+\tan\theta$

$\Rightarrow\ \alpha_2,\beta_2$ are roots of $x^2+2x\tan\theta-1=0$

and $\alpha_2>\beta_2$

$\therefore\ \alpha_2=-\tan\theta+\sec\theta$

and $\beta_2=-\tan\theta-\sec\theta$

Hence, $\alpha_1+\beta_2=-2\tan\theta$

146. $\because\ x(x+1)+(x+1)(x+2)+....+(x+\overline{n-1})(x+n)=10n$

$\Rightarrow\ nx^2+x(1+3+5+\ldots+(2n-1))+(1\,.\,2+2\,.\,3$

$+\ldots+(n-1)\,.\,n)=10n$

or $nx^2+n^2x+\frac{1}{3}(n-1)\,n\,(n+1)=10n$

or $3x^2+3nx+(n^2-1)=30$ $\quad(\because n\neq 0)$

or $3x^2+3nx+(n^2-31)=0$

$\because\ |\alpha-\beta|=1$

or $(\alpha-\beta)^2=1$

or $\frac{D}{a^2}=1$

or $D=a^2$

or $9n^2-12\,.\,(n^2-31)=9$

or $n^2=121$

$\therefore\ n=11$

CHAPTER

03

Sequences and Series

Learning Part

Session 1

- Sequence
- Series
- Progression

Session 2

- Arithmetic Progression

Session 3

- Geometric Sequence or Geometric Progression

Session 4

- Harmonic Sequence or Harmonic Progression

Session 5

- Mean

Session 6

- Arithmetico-Geometric Series (AGS)
- Sigma (Σ) Notation
- Natural Numbers

Session 7

- Application to Problems of Maxima and Minima

Practice Part

- JEE Type Examples
- Chapter Exercises

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The word "Sequence" in Mathematics has same meaning as in ordinary English. A collection of objects listed in a sequence means it has identified first member, second member, third member and so on. The most common examples are depreciate values of certain commodity like car, machinery and amount deposits in the bank for a number of years.

Session 1

Sequence, Series, Progression

Sequence

A succession of numbers arranged in a definite order or arrangement according to some well-defined law is called a sequence.

Or

A sequence is a function of natural numbers (N) with codomain is the set of real numbers (R) [complex numbers (C)]. If range is subset of real numbers (complex numbers), it is called a real sequence (complex sequence).

Or

A mapping $f : N \to C$, then $f(n) = t_n, n \in N$ is called a sequence to be denoted it by
$\{f(1), f(2), f(3), \ldots\} = \{t_1, t_2, t_3, \ldots\} = \{t_n\}$.

The nth term of a sequence is denoted by
$T_n, t_n, a_n, a(n), u_n$, etc.

Remark

The sequence $a_1, a_2, a_3, \ldots$ is generally written as $\{a_n\}$.

For example ,

(i) 1, 3, 5, 7, ... is a sequence, because each term (except first) is obtained by adding 2 to the previous term and $T_n = 2n - 1, n \in N$.

Or

If $T_1 = 1, T_{n+1} = T_{n+2}, n \geq 1$

(ii) 1, 2, 3, 5, 8, 13, ... is a sequence, because each term (except first two) is obtained by taking the sum of preceding two terms.

Or

If $T_1 = 1, T_2 = 2, T_{n+2} = T_n + T_{n+1}, n \geq 1$

(iii) 2, 3, 5, 7, 11, 13, 17, 19, ... is a sequence.

Here, we cannot express $T_n, n \in N$ by an algebraic formula.

Recursive Formula

A formula to determine the other terms of the sequence in terms of its preceding terms is known as recursive formula.

For example,

If $T_1 = 1$ and $T_{n+1} = 6\,T_n, n \in N$.

Then, $T_2 = 6\,T_1 = 6 \cdot 1 = 6$

$T_3 = 6\,T_2 = 6 \cdot 6 = 36$

$T_4 = 6\,T_3 = 6 \cdot 36 = 216 \ldots$

Then, sequence is 1, 6, 36, 216,...

Types of Sequences

There are two types of sequences

1. Finite Sequence

A sequence is said to be finite sequence, if it has finite number of terms. A finite sequence is described by $a_1, a_2, a_3, \ldots, a_n$ or $T_1, T_2, T_3, \ldots, T_n$, where $n \in N$.

For example

(i) 3, 5, 7, 9, ..., 37

(ii) 2, 6, 18, 54, ..., 4374

2. Infinite Sequence

A sequence is said to be an infinite sequence, if it has infinite number of terms. An infinite sequence is described by $a_1, a_2, a_3, \ldots$ or $T_1, T_2, T_3, \ldots$

For example,

(i) $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$

(ii) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

Series

In a sequence, the sum of the directed terms is called a series.

For example, If 1, 4, 7, 10, 13, 16,... is a sequence, then its sum i.e., $1+4+7+10+13+16+\ldots$ is a series.

In general, if $T_1, T_2, T_3, \ldots, T_n, \ldots$ denote a sequence, then the symbolic expression $T_1 + T_2 + T_3 + \ldots + T_n + \ldots$ is called a series associated with the given sequence.

Each member of the series is called its term.

In a series $T_1 + T_2 + T_3 + \ldots + T_r + \ldots$, the sum of first n terms is denoted by S_n. Thus,

$$S_n = T_1 + T_2 + T_3 + \ldots + T_n = \sum_{r=1}^{n} T_r = \sum T_n$$

If S_n denotes the sum of n terms of a sequence.

Then, $S_n - S_{n-1} = (T_1 + T_2 + T_3 + \ldots + T_n) - (T_1 + T_2 + \ldots + T_{n-1}) = T_n$

Thus, $T_n = S_n - S_{n-1}$

Types of Series

There are two types of series

1. Finite Series

A series having finite number of terms is called a finite series.

For example,

(i) $3+5+7+9+\ldots+21$

(ii) $2+6+18+54+\ldots+4374$

2. Infinite Series

A series having an infinite number of terms is called an infinite series.

For example,

(i) $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots$

(ii) $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$

Progression

If the terms of a sequence can be described by an explicit formula, then the sequence is called a progression.

Or

A sequence is said to be progression, if its terms increases (respectively decreases) numerically.

For example, The following sequences are progression :

(i) $1, 3, 5, 7, \ldots$ (ii) $\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \frac{1}{54}, \ldots$

(iii) $1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \ldots$ (iv) $1, 8, 27, 256, \ldots$

(v) $8, -4, 2, -1, \frac{1}{2}, \ldots$

The sequences (iii) and (v) are progressions, because

$$|1| > \left|-\frac{1}{3}\right| > \left|\frac{1}{9}\right| > \left|-\frac{1}{27}\right| > \ldots$$

i.e. $$1 > \frac{1}{3} > \frac{1}{9} > \frac{1}{27} > \ldots$$

and $$|8| > |-4| > |2| > |-1| > \left|\frac{1}{2}\right| > \ldots$$

i.e. $$8 > 4 > 2 > 1 > \frac{1}{2} > \ldots$$

Remark

All the definitions and formulae are valid for complex numbers in the theory of progressions but it should be assumed (if not otherwise stated) that the terms of the progressions are real numbers.

Example 1. ***If*** $f : N \to R$, ***where*** $f(n) = a_n = \frac{n}{(2n+1)^2}$, ***write the sequence in ordered pair form.***

Sol. Here, $a_n = \frac{n}{(2n+1)^2}$

On putting $n = 1, 2, 3, 4, \ldots$ successively, we get

$$a_1 = \frac{1}{(2\cdot 1+1)^2} = \frac{1}{9}, \quad a_2 = \frac{2}{(2\cdot 2+1)^2} = \frac{2}{25}$$

$$a_3 = \frac{3}{(2\cdot 3+1)^2} = \frac{3}{49}, \quad a_4 = \frac{4}{(2\cdot 4+1)^2} = \frac{4}{81}$$

$$\vdots \qquad \vdots \qquad \vdots$$

Hence, we obtain the sequence $\frac{1}{9}, \frac{2}{25}, \frac{3}{49}, \frac{4}{81}, \ldots$

Now, the sequence in ordered pair form is

$$\left\{\left(1, \frac{1}{9}\right), \left(2, \frac{2}{25}\right), \left(3, \frac{3}{49}\right), \left(4, \frac{4}{81}\right), \ldots\right\}$$

Example 2. ***The Fibonacci sequence is defined by*** $a_1 = 1 = a_2, a_n = a_{n-1} + a_{n-2}, n > 2$. ***Find*** $\frac{a_{n+1}}{a_n}$ ***for*** $n = 1, 2, 3, 4, 5$.

Sol. $\because \quad a_1 = 1 = a_2$

$\therefore \quad a_3 = a_2 + a_1 = 1 + 1 = 2,$

$a_4 = a_3 + a_2 = 2 + 1 = 3$

$a_5 = a_4 + a_3 = 3 + 2 = 5$

and $\quad a_6 = a_5 + a_4 = 5 + 3 = 8$

$\therefore \quad \frac{a_2}{a_1} = 1, \frac{a_3}{a_2} = \frac{2}{1} = 2, \frac{a_4}{a_3} = \frac{3}{2}, \frac{a_5}{a_4} = \frac{5}{3}$ and $\frac{a_6}{a_5} = \frac{8}{5}$

Example 3. ***If the sum of*** n ***terms of a series is*** $2n^2 + 5n$ ***for all values of*** n, ***find its 7th term.***

Sol. Given, $S_n = 2n^2 + 5n$

$\Rightarrow \quad S_{n-1} = 2(n-1)^2 + 5(n-1) = 2n^2 + n - 3$

$\therefore \quad T_n = S_n - S_{n-1} = (2n^2 + 5n) - (2n^2 + n - 3) = 4n + 3$

Hence, $T_7 = 4 \times 7 + 3 = 31$

Example 4.

(i) Write $\sum_{r=1}^{n}(r^2 + 2)$ in expanded form.

(ii) Write the series $\frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \ldots + \frac{n}{n+2}$ in sigma form.

Sol. (i) On putting $r = 1, 2, 3, 4, \ldots, n$ in $(r^2 + 2)$,

we get 3, 6, 11, 18, ... , $(n^2 + 2)$

Hence, $\sum_{r=1}^{n}(r^2 + 2) = 3 + 6 + 11 + 18 + \ldots + (n^2 + 2)$

(ii) The rth term of series $= \frac{r}{r+2}$.

Hence, the given series can be written as

$$\frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \ldots + \frac{n}{n+2} = \sum_{r=1}^{n}\left(\frac{r}{r+2}\right)$$

Exercise for Session 1

1 First term of a sequence is 1 and the $(n+1)$ th term is obtained by adding $(n+1)$ to the nth term for all natural numbers n, the 6th term of the sequence is

(a) 7 (b) 13

(c) 21 (d) 27

2. The first three terms of a sequence are 3, 3, 6 and each term after the second is the sum of two terms preceding it, the 8th term of the sequence is

(a) 15 (b) 24

(c) 39 (d) 63

3. If $a_n = \sin\left(\frac{n\pi}{6}\right)$, the value of $\sum_{n=1}^{6} a_n^2$ is

(a) 2 (b) 3

(c) 4 (d) 7

4. If for a sequence $\{a_n\}$, $S_n = 2n^2 + 9n$, where S_n is the sum of n terms, the value of a_{20} is

(a) 65 (b) 75

(c) 87 (d) 97

5. If $a_1 = 2, a_2 = 3 + a_1$ and $a_n = 2a_{n-1} + 5$ for $n > 1$, the value of $\sum_{r=2}^{5} a_r$ is

(a) 130 (b) 160

(c) 190 (d) 220

Session 2

Arithmetic Progression (AP)

Types of Progression

Progressions are various types but in this chapter we will studying only three special types of progressions which are following :

1. Arithmetic Progression (AP)
2. Geometric Progression (GP)
3. Harmonic Progression (HP)

Arithmetic Progression (AP)

An arithmetic progression is a sequence in which the difference between any term and its just preceding term (i.e., term before it) is constant throughout. This constant is called the common difference (abbreviated as CD) and is generally denoted by 'd'.

Or

An arithmetic progression is a sequence whose terms increase or decrease by a fixed number. This fixed number is called the common difference of the AP.

A finite or infinite sequence $\{t_1, t_2, t_3, \ldots, t_n\}$

or $\{t_1, t_2, t_3, \ldots\}$ is said to be an arithmetic progression (AP), if $t_k - t_{k-1} = d$, a constant independent of k, for $k = 2, 3, 4, \ldots, n$ or $k = 2, 3, 4, \ldots$ as the case may be :
The constant d is called the common difference of the AP.

i.e. $$d = t_2 - t_1 = t_3 - t_2 = \ldots = t_n - t_{n-1}$$

Remarks

1. If a be the first term and d be the common difference, then AP can be written as
   $$a, a + d, a + 2d, \ldots, a + (n-1)d, \ldots, \forall\, n \in N.$$
2. If we add the common difference to any term of AP, we get the next following term and if we subtract it from any term, we get the preceding term.
3. The common difference of an AP may be **positive**, **zero**, **negative** or **imaginary.**
4. **Constant AP** common difference of an AP is equal to zero.
5. **Increasing AP** common difference of an AP is greater than zero.
6. **Decreasing AP** common difference of an AP is less than zero.
7. **Imaginary AP** common difference of an AP is imaginary.

Algorithm to determine whether a sequence is an AP or not

***Step* I** Obtain t_n (the nth term of the sequence).

***Step* II** Replace n by $n-1$ in t_n to get t_{n-1}.

***Step* III** Calculate $t_n - t_{n-1}$.

If $t_n - t_{n-1}$ is independent of n, the given sequence is an AP otherwise it is not an AP.

Example 5.

(i) 1, 3, 5, 7, ... (ii) $\pi, \pi + e^{\pi}, \pi + 2e^{\pi}, \ldots$

(iii) $a, a-b, a-2b, a-3b, \ldots$

Sol. (i) Here, 2nd term − 1st term = 3rd term − 2nd term = ...
$\Rightarrow$ $3 - 1 = 5 - 3 = \ldots = 2$, which is a common difference.

(ii) Here, 2nd term − 1st term = 3rd term − 2nd term = ...
$\Rightarrow$ $(\pi + e^{\pi}) - \pi = (\pi + 2e^{\pi}) - (\pi + e^{\pi}) = \ldots$
$= e^{\pi}$, which is a common difference.

(iii) Here, 2nd term − 1st term = 3rd term − 2nd term = ...
$\Rightarrow (a-b) - a = (a-2b) - (a-b) = \ldots$
$= -b$, which is a common difference.

Example 6. Show that the sequence $< t_n >$ defined by $t_n = 5n + 4$ is an AP, also find its common difference.

Sol. We have, $t_n = 5n + 4$

On replacing n by $(n-1)$, we get

$$t_{n-1} = 5(n-1) + 4$$

$\Rightarrow$ $$t_{n-1} = 5n - 1$$

$\therefore$ $$t_n - t_{n-1} = (5n + 4) - (5n - 1) = 5$$

Clearly, $t_n - t_{n-1}$ is independent of n and is equal to 5. So, the given sequence is an AP with common difference 5.

Example 7. Show that the sequence $< t_n >$ defined by $t_n = 3n^2 + 2$ is not an AP.

Sol. We have, $t_n = 3n^2 + 2$

On replacing n by $(n-1)$, we get

$$t_{n-1} = 3(n-1)^2 + 2$$

$\Rightarrow$ $$t_{n-1} = 3n^2 - 6n + 5$$

$\therefore$ $$t_n - t_{n-1} = (3n^2 + 2) - (3n^2 - 6n + 5)$$
$$= 6n - 3$$

Clearly, $t_n - t_{n-1}$ is not independent of n and therefore it is not constant. So, the given sequence is not an AP.

Remark

If the nth term of a sequence is an expression of first degree in n. *For example,* $t_n = An + B$, where A, B are constants, then that sequence will be in AP for $t_n - t_{n-1} = (An + B) - [A(n-1) + B]$ $= A[n - (n-1)] = A =$ constant = Common difference or coefficient of n in t_n Students are advised to consider the above point as a behaviour of standard result.

General Term of an AP

Let 'a' be the first term, 'd' be the common difference and 'l' be the last term of an AP having 'n' terms, where $n \in N$.

Then, AP can be written as $a, a+d, a+2d, \ldots, l-2d, l-d, l$

(i) nth Term of an AP from Beginning

1st term from beginning $= t_1 = a = a + (1-1)\,d$

2nd term from beginning $= t_2 = a + d = a + (2-1)\,d$

3rd term from beginning $= t_3 = a + 2d = a + (3-1)\,d$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

n th term from beginning $= t_n = a + (n-1)\,d, \forall\, n \in N$

Hence, n th term of an AP from beginning

$= t_n = a + (n-1)\,d = l$ [last term]

(ii) nth Term of an AP from End

1st term from end $= t'_1 = l = l - (1-1)\,d$

2nd term from end $= t'_2 = l - d = l - (2-1)\,d$

3rd term from end $= t'_3 = l - 2d = l - (3-1)\,d$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

nth term from end $= t'_n = l - (n-1)\,d, \forall\, n \in N$

Hence, n th term of an AP from end

$t'_n = l - (n-1)\,d = a$ [first term]

Now, it is clear that

$t_n + t'_n = a + (n-1)\,d + l - (n-1)\,d = a + l$

or $t_n + t'_n = a + l$

i.e. In a finite AP, the sum of the terms equidistant from the beginning and end is always same and is equal to the sum of first and last term.

Remark

1. nth term is also called the general term.
2. If last term of AP is t_n and common difference be d, then terms of AP from end are $t_n, t_n - d, t_n - 2d, \ldots$
3. If in a sequence, the terms an alternatively positive and negative, then it cannot be an AP.
4. Common difference of AP $= \dfrac{l-a}{n+1}$, where, a = first term of AP, l = last term of AP and n = number of terms of AP.
5. If t_n, t_{n+1}, t_{n+2} are three consecutive terms of an AP, then $2t_{n+1} = t_n + t_{n+2}$. In particular, if a, b and c are in AP, then $2b = a + c$.

Example 8. Find first negative term of the sequence

$$20, 19\frac{1}{4}, 18\frac{1}{2}, 17\frac{3}{4}, \ldots$$

Sol. The given sequence is an AP in which first term, $a = 20$ and common difference, $d = -\dfrac{3}{4}$. Let the nth term of the given AP be the first negative term. Then,

$$t_n < 0 \;\Rightarrow\; a + (n-1)\,d < 0$$

$$\Rightarrow \quad 20 + (n-1)\left(-\frac{3}{4}\right) < 0$$

$$\Rightarrow \quad 80 - 3n + 3 < 0$$

$$\Rightarrow \quad n > \frac{83}{3} \text{ or } n > 27\frac{2}{3}$$

$$\Rightarrow \quad n = 28$$

Thus, 28th term of the given sequence is the first negative term.

Example 9. If the mth term of an AP is $\dfrac{1}{n}$ and the nth term is $\dfrac{1}{m}$, then find mnth term of an AP.

Sol. If A and B are constants, then r th term of AP is

$$t_r = Ar + B$$

Given, $t_m = \dfrac{1}{n} \Rightarrow Am + B = \dfrac{1}{n}$ ...(i)

and $t_n = \dfrac{1}{m} \Rightarrow An + B = \dfrac{1}{m}$ ...(ii)

From Eqs. (i) and (ii), we get $A = \dfrac{1}{mn}$ and $B = 0$

mn th term $= t_{mn} = Amn + B = \dfrac{1}{mn} \cdot mn + 0 = 1$

Hence, mn th term of the given AP is 1.

Example 10. If $|x-1|$, 3 and $|x-3|$ are first three terms of an increasing AP, then find the 6th term of on AP.

Sol. ***Case*** **I** For $x < 1$,

$$|x-1| = -(x-1)$$

and $|x-3| = -(x-3)$

$\therefore 1 - x$, 3 and $3 - x$ are in AP.

$$\Rightarrow \quad 6 = 1 - x + 3 - x$$

$$\Rightarrow \quad x = -1$$

Then, first three terms are 2, 3, 4,

which is an increasing AP.

$\therefore$ 6th term is 7. [$\because d = 1$]

Case **II** For $1 < x < 3$,

$$|x-1| = x - 1$$

and $|x-3| = -(x-3) = 3 - x$

$\therefore x - 1$, 3 and $3 - x$ are in AP.

$$\Rightarrow \quad 6 = x - 1 + 3 - x$$

$$\Rightarrow \quad 6 = 2 \quad \text{[impossible]}$$

***Case* III** For $x > 3$, $|x-1| = x-1$ and $|x-3| = x-3$

$\therefore\ x-1, 3$ and $x-3$ are in AP.

$\Rightarrow \quad 6 = x-1+x-3 \Rightarrow x = 5$

Then, first three terms are 4, 3, 2, which is a decreasing AP.

Example 11. In the sequence 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, ..., where n consecutive terms have the value n, find the 150th term of the sequence.

Sol. Let the 150th term $= n$

Then, $1+2+3+\ldots+(n-1) < 150 < 1+2+3+\ldots+n$

$$\Rightarrow \quad \frac{(n-1)\,n}{2} < 150 < \frac{n\,(n+1)}{2}$$

$$\Rightarrow \quad n\,(n-1) < 300 < n\,(n+1)$$

Taking first two members

$$n\,(n-1) < 300 \ \Rightarrow\ n^2 - n - 300 < 0$$

$$\Rightarrow \quad \left(n-\frac{1}{2}\right)^2 < 300 + \frac{1}{4}$$

$$\Rightarrow \quad 0 < n < \frac{1}{2} + \frac{\sqrt{1201}}{2}$$

$$\Rightarrow \quad 0 < n < 17.8 \qquad \ldots(i)$$

and taking last two members,

$$n\,(n+1) > 300$$

$$\Rightarrow \quad \left(n+\frac{1}{2}\right)^2 > 300 + \frac{1}{4}$$

$$\therefore \quad n > -\frac{1}{2} + \frac{\sqrt{1201}}{2}$$

$$\Rightarrow \quad n > 16.8 \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$16.8 < n < 17.8$$

$$\Rightarrow \quad n = 17$$

Example 12. ***If*** a_1, a_2, a_3, a_4 ***and*** a_5 ***are in AP with common difference*** $\neq 0$***, find the value of*** $\sum_{i=1}^{5} a_i$ ***when*** $a_3 = 2$.

Sol. $\because a_1, a_2, a_3, a_4$ and a_5 are in AP, we have

$$a_1 + a_5 = a_2 + a_4 = a_3 + a_3 \qquad [\because t_n + t'_n = a + l]$$

$$a_1 + a_5 = a_2 + a_4 = 4 \qquad [\because a_3 = 2]$$

$$a_1 + a_2 + a_3 + a_4 + a_5 = 4 + 2 + 4 = 10$$

$$\Rightarrow \quad \sum_{i=1}^{5} a_i = 10$$

Sum of a Stated Number of Terms of an Arithmetic Series

More than 200 yr ago, a class of German School Children was asked to find the sum of all integers from 1 to 100 inclusive. One boy in the class, an eight year old named **Carl Fredrick Gauss** (1777-1855) who later established his reputation as one of the greatest Mathematicians announced the answer almost at once. The teacher overawed at this asked Gauss to explain how he got this answer. Gauss explained that he had added these numbers in pairs as follows

$$(1+100), (2+99), (3+98), \ldots$$

There are $\frac{100}{2} = 50$ pairs. The answer can be obtained by multiplying 101 by 50 to get 5050.

Sum of n Terms of an AP

Let 'a' be the first term, 'd' the common difference, 'l' be the last term of an AP having n terms and S_n be the sum of n terms, then

$$S_n = a + (a+d) + (a+2d) + \ldots + (l-2d) + (l-d) + l \qquad \ldots(i)$$

Reversing the right hand terms

$$S_n = l + (l-d) + (l-2d) + \ldots + (a+2d) + (a+d) + a \qquad \ldots(ii)$$

On adding Eqs. (i) and (ii), we get

$$2S_n = (a+l) + (a+l) + (a+l) + \ldots + (a+l) + (a+l) + (a+l)$$

$$= (a+l) + (a+l) + \ldots \text{ upto } n \text{ terms} = n(a+l)$$

$$\therefore \quad \boldsymbol{S_n = \frac{n}{2}(a+l)} \qquad \ldots(iii)$$

Now, if we substitute the value of l *viz.*, $l = a + (n-1)\,d$, in this formula, we get

$$S_n = \frac{n}{2}[a + a + (n-1)\,d] = \frac{n}{2}[2a + (n-1)d]$$

$$\therefore \quad S_n = \frac{n}{2}[2a + (n-1)\,d]$$

If we substitute the value of a *viz.,*

$$l = a + (n-1)\,d$$

or $\quad a = l - (n-1)\,d$ in Eq. (iii), then

$$S_n = \frac{n}{2}[2l - (n-1)\,d]$$

If we substitute the value of $a + l$ *viz.,*

$$t_n + t'_n = a + l \text{ in Eq. (iii), then}$$

$$\boldsymbol{S_n = \frac{n}{2}(t_n + t'_n)}$$

***Corollary* I** Sum of first n natural numbers

i.e. $1+2+3+4+\ldots+n$

Here, $\quad a = 1$ and $d = 1$

$$\therefore \quad S = \frac{n}{2}[2\cdot 1 + (n-1)\cdot 1]$$

$$= \frac{n\,(n+1)}{2}$$

***Corollary* II** Sum of first n odd natural numbers

i.e., $1+3+5+\ldots$

Here, $a=1$

and $d=2$

$\therefore \quad S=\frac{n}{2}[2\cdot 1+(n-1)\cdot 2]=n^2$

***Corollary* III** If sum of first n terms is S_n, then sum of next m terms is $S_{m+n}-S_n$.

Important Results with Proof

1. If S_n, t_n and d are sum of n terms, nth term and common difference of an AP respectively, then

$$d=t_n-t_{n-1} \qquad [n\geq 2]$$

$$t_n=S_n-S_{n-1} \qquad [n\geq 2]$$

$$d=S_n-2S_{n-1}+S_{n-2} \qquad [n\geq 3]$$

Proof

$\because \quad S_n=t_1+t_2+t_3+\ldots+t_{n-1}+t_n$

$\Rightarrow \quad S_n=S_{n-1}+t_n$

$\therefore \quad t_n=S_n-S_{n-1}$

but $\quad d=t_n-t_{n-1}$

$$=(S_n-S_{n-1})-(S_{n-1}-S_{n-2})$$

$\therefore \quad d=S_n-2S_{n-1}+S_{n-2}$

2. A sequence is an AP if and only if the sum of its n terms is of the form An^2+Bn, where A and B are constants independent of n.

In this case, the nth term and common difference of the AP are $A(2n-1)+B$ and $2A$, respectively.

Proof As $S_n=An^2+Bn$

$\therefore \quad S_{n-1}=A(n-1)^2+B(n-1)$

$\therefore \quad t_n=S_n-S_{n-1}$

$$=(An^2+Bn)-[A(n-1)^2+B(n-1)]$$

$$=A[n^2-(n-1)^2]+B$$

$$t_n=A(2n-1)+B$$

$\Rightarrow \quad t_{n-1}=A[2(n-1)-1]+B$

$$=A(2n-3)+B$$

Now, $t_n-t_{n-1}=[A(2n-1)+B]-[A(2n-3)+B]$

$$=2A \qquad [\text{a constant}]$$

Hence, the sequence is an AP.

Conversely, consider an AP with first term a and common difference d.

Sum of first n terms $=\frac{n}{2}[2a+(n-1)d]$

$$=\frac{dn^2}{2}+\left(a-\frac{d}{2}\right)n=An^2+Bn,$$

where, $A=\frac{d}{2}, B=a-\frac{d}{2}$

Hence, $S_n=An^2+Bn$, where A and B are constants independent of n.

Hence, the converse is true.

Corollary $\because \quad S_n=An^2+Bn$

$\therefore \quad t_n=A(2n-1)+B$

$t_n=A$ (replacing n^2 by $2n-1$) + coefficient of n

and $\quad d=2A$

i.e. $\quad d=2 \qquad [\text{coefficient of } n^2]$

	S_n	t_n	d
1.	$5n^2+3n$	$5(2n-1)+3=10n-2$	10
2.	$-7n^2+2n$	$-7(2n-1)+2$ $=-14n+9$	-14
3.	$-9n^2-4n$	$-9(2n-1)-4$ $=-18n+5$	-18
4.	$4n^2-n$	$4(2n-1)-1=8n-5$	8

3. If $S_n=an^2+bn+c$, where S_n denotes the sum of n terms of a series, then whole series is not an AP. It is AP from the second term onwards.

Proof As $S_n=an^2+bn+c$ for $n\geq 1$, we get

$$S_{n-1}=a(n-1)^2+b(n-1)+c \text{ for } n\geq 2$$

Now, $\quad t_n=S_n-S_{n-1}$

$\Rightarrow \quad t_n=a(2n-1)+b, n\geq 2$

$\therefore \quad t_{n-1}=a[2(n-1)-1]+b, n\geq 3$

$\Rightarrow \quad t_{n-1}=a(2n-3)+b, n\geq 3$

$\therefore \quad t_n-t_{n-1}=2a=\text{constant}, n\geq 3$

$\therefore \quad t_3-t_2=t_4-t_3=t_5-t_4=\ldots$

But $\quad t_2-t_1=(S_2-S_1)-S_1=S_2-2S_1$

$$=(4a+2b+c)-2(a+b+c)$$

$$=(2a-c) \qquad [\because S_1=t_1]$$

$\therefore \quad t_2-t_1\neq t_3-t_2$

Hence, the whole series is not an AP. It is AP from the second term onwards.

Ratio of Sums is Given

1. If ratio of the sums of m and n terms of an AP is given by

$$\frac{S_m}{S_n} = \frac{Am^2 + Bm}{An^2 + Bn}$$

where A, B are constants and $A \neq 0$.

$\therefore \quad S_m = (Am^2 + Bm)\,k,$

$S_n = (An^2 + Bn)\,k$

$\Rightarrow \quad t_m = S_m - S_{m-1} = [A(2m-1) + B]k$

$t_n = S_n - S_{n-1} = [A(2n-1) + B]k$

$\therefore \quad \frac{t_m}{t_n} = \frac{A(2m-1) + B}{A(2n-1) + B}$

Example 13. The ratio of sums of m and n terms of an AP is $m^2 : n^2$. The ratio of the mth and nth terms is

(a) $(2m+1):(2n-1)$ (b) $m:n$

(c) $(2m-1):(2n-1)$ (d) None of these

Sol. (c) Here, $\frac{S_m}{S_n} = \frac{m^2}{n^2}$ $[\because A = 1, B = 0]$

$\therefore \quad \frac{t_m}{t_n} = \frac{(2m-1)}{(2n-1)}$

$\Rightarrow \quad t_m : t_n = (2m-1):(2n-1)$

2. If ratio of the sums of n terms of two AP's is given by

$$\frac{S_n}{S'_n} = \frac{An + B}{Cn + D}$$

where, A, B, C, D are constants and $A, C \neq 0$

$\therefore \quad S_n = n(An + B)\,k,\ S'_n = n(Cn + D)\,k$

$\Rightarrow \quad t_n = [A(2n-1) + B]k,\ t'_n = [C(2n-1) + D]k$

$\Rightarrow \quad d = t_n - t_{n-1} = 2A,\ d' = t'_n - t'_{n-1} = 2C$

$\therefore \quad \frac{t_n}{t'_n} = \frac{A(2n-1) + B}{C(2n-1) + D}$ and $\frac{d}{d'} = \frac{A}{C}$

Note If $A = 0, C = 0$

Then, $\frac{S_n}{S'_n} = \frac{B}{D} \Rightarrow \frac{t_n}{t'_n} = \frac{B}{D}$ and $\frac{d}{d'} = \frac{0}{0}$ = not defined

Remark

If $\frac{t_n}{t'_n} = \frac{an + b}{cn + d}$

where, a, b, c, d are constants and $a, c \neq 0$, then

$$\frac{S_n}{S'_n} = \frac{a\left(\frac{n+1}{2}\right) + b}{c\left(\frac{n+1}{2}\right) + d}$$

Example 14. The sums of n terms of two arithmetic progressions are in the ratio $(7n+1):(4n+17)$. Find the ratio of their nth terms and also common differences.

Sol. Given, $S_n : S'_n = (7n+1):(4n+17)$

Here, $A = 7, B = 1, C = 4$ and $D = 17$

$\therefore \quad \frac{t_n}{t'_n} = \frac{7(2n-1)+1}{4(2n-1)+17} = \frac{14n-6}{8n+13}$

and $\frac{d}{d'} = \frac{A}{C} = \frac{7}{4}$

Hence, $t_n : t'_n = (14n-6):(8n+13)$ and $d : d' = 7:4$

Example 15. The sums of n terms of two AP's are in the ratio $(3n-13):(5n+21)$. Find the ratio of their 24th terms.

Sol. Given, $S_n : S'_n = (3n-13):(5n+21)$

Here, $A = 3, B = -13, C = 5$ and $D = 21$

$\therefore \quad \frac{t_{24}}{t'_{24}} = \frac{3(2\times 24 - 1) - 13}{5(2\times 24 - 1) + 21} = \frac{128}{256} = \frac{1}{2}$

$\therefore \quad t_{24} : t'_{24} = 1:2$

Example 16. How many terms of the series $20 + 19\frac{1}{3} + 18\frac{2}{3} + \ldots$ must be taken to make 300? Explain the double answer.

Sol. Here, given series is an AP with first term $a = 20$ and the common difference, $d = -\frac{2}{3}$.

Let the sum of n terms of the series be 300.

Then, $S_n = \frac{n}{2}\{2a + (n-1)\,d\}$

$\Rightarrow \quad 300 = \frac{n}{2}\left\{2\times 20 + (n-1)\left(-\frac{2}{3}\right)\right\}$

$\Rightarrow \quad 300 = \frac{n}{3}\{60 - n + 1\}$

$\Rightarrow \quad n^2 - 61n + 900 = 0$

$\Rightarrow \quad (n-25)(n-36) = 0$

$\Rightarrow \quad n = 25 \text{ or } n = 36$

$\therefore$ Sum of 25 terms = Sum of 36 terms = 300

Explanation of double answer

Here, the common difference is negative, therefore terms go on diminishing and $t_{31} = 20 + (31-1)\left(\frac{-2}{3}\right) = 0$ i.e., 31st term becomes zero. All terms after 31st term are negative. These negative terms $(t_{32}, t_{33}, t_{34}, t_{35}, t_{36})$ when added to positive terms $(t_{26}, t_{27}, t_{28}, t_{29}, t_{30})$, they cancel out each other i.e., sum of terms from 26th to 36th terms is zero. Hence, the sum of 25 terms as well as that of 36 terms is 300.

Example 17. Find the arithmetic progression consisting of 10 terms, if the sum of the terms occupying the even places is equal to 15 and the sum of those occupying the odd places is equal to $12\frac{1}{2}$.

Sol. Let the successive terms of an AP be $t_1, t_2, t_3, ..., t_9, t_{10}$.

By hypothesis,

$$t_2 + t_4 + t_6 + t_8 + t_{10} = 15$$

$$\Rightarrow \quad \frac{5}{2}(t_2 + t_{10}) = 15$$

$$\Rightarrow \quad t_2 + t_{10} = 6$$

$$\Rightarrow \quad (a + d) + (a + 9d) = 6$$

$$\Rightarrow \quad 2a + 10d = 6 \qquad ...(i)$$

and $\quad t_1 + t_3 + t_5 + t_7 + t_9 = 12\frac{1}{2}$

$$\Rightarrow \quad \frac{5}{2}(t_1 + t_9) = \frac{25}{2}$$

$$\Rightarrow \quad t_1 + t_9 = 5$$

$$\Rightarrow \quad a + a + 8d = 5$$

$$\Rightarrow \quad 2a + 8d = 5 \qquad ...(ii)$$

From Eqs. (i) and (ii), we get $d = \frac{1}{2}$ and $a = \frac{1}{2}$

Hence, the AP is $\frac{1}{2}, 1, 1\frac{1}{2}, 2, 2\frac{1}{2}, \ldots$

Example 18. ***If N, the set of natural numbers is partitioned into groups $S_1 = \{1\}, S_2 = \{2, 3\}$, $S_3 = \{4, 5, 6\}, \ldots$, find the sum of the numbers in S_{50}.***

Sol. The number of terms in the groups are 1, 2, 3, ...

$\because$ The number of terms in the 50th group = 50

$\therefore$ The last term of 1st group = 1

The last term of 2nd group = 3 = 1 + 2

The last term of 3rd group = 6 = 1 + 2 + 3

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

The last term of 49th group = 1 + 2 + 3 + ... + 49

$\therefore$ First term of 50th group = 1 + (1 + 2 + 3 + ... + 49)

$$= 1 + \frac{49}{2}(1 + 49) = 1226$$

$$\therefore \quad S_{50} = \frac{50}{2}\{2 \times 1226 + (50 - 1) \times 1\}$$

$$= 25 \times 2501 = 62525$$

Example 19. Find the sum of first 24 terms of on AP $t_1, t_2, t_3, \ldots$, if it is known that

$$t_1 + t_5 + t_{10} + t_{15} + t_{20} + t_{24} = 225.$$

Sol. We know that, in an AP the sums of the terms equidistant from the beginning and end is always same and is equal to the sum of first and last term.

Then, $t_1 + t_{24} = t_5 + t_{20} = t_{10} + t_{15}$

but given

$$t_1 + t_5 + t_{10} + t_{15} + t_{20} + t_{24} = 225$$

$$\Rightarrow \quad (t_1 + t_{24}) + (t_5 + t_{20}) + (t_{10} + t_{15}) = 225$$

$$\Rightarrow \quad 3(t_1 + t_{24}) = 225$$

$$\Rightarrow \quad t_1 + t_{24} = 75$$

$$\therefore \quad S_{24} = \frac{24}{2}(t_1 + t_{24}) = 12 \times 75 = 900$$

Example 20. If $(1+3+5+\ldots+p)+(1+3+5+\ldots+q) = (1+3+5+\ldots+r)$, where each set of parentheses contains the sum of consecutive odd integers as shown, then find the smallest possible value of $p+q+r$ (where, $p > 6$).

Sol. We know that, $1 + 3 + 5 + \ldots + (2n - 1) = n^2$

Thus, the given equation can be written as

$$\left(\frac{p+1}{2}\right)^2 + \left(\frac{q+1}{2}\right)^2 = \left(\frac{r+1}{2}\right)^2$$

$$\Rightarrow \quad (p + 1)^2 + (q + 1)^2 = (r + 1)^2$$

Therefore, $(p + 1, q + 1, r + 1)$ form a Pythagorean triplet as $p > 6 \Rightarrow p + 1 > 7$

The first Pythagorean triplet containing a number > 7 is (6, 8, 10).

$$\Rightarrow \quad p + 1 = 8, q + 1 = 6, r + 1 = 10$$

$$\Rightarrow \quad p + q + r = 21$$

Properties of Arithmetic Progression

1. If $a_1, a_2, a_3, \ldots$ are in AP with common difference d, then $a_1 \pm k, a_2 \pm k, a_3 \pm k, \ldots$ are also in AP with common difference d.
2. If $a_1, a_2, a_3, \ldots$ are in AP with common difference d, then $a_1k, a_2k, a_3k, \ldots$ and $\frac{a_1}{k}, \frac{a_2}{k}, \frac{a_3}{k}, \ldots$ are also in AP $(k \neq 0)$ with common differences kd and $\frac{d}{k}$, respectively.
3. If $a_1, a_2, a_3, ..$ and $b_1, b_2, b_3, \ldots$ are two AP's with common differences d_1 and d_2, respectively. Then, $a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3, \ldots$ are also in AP with common difference $(d_1 \pm d_2)$.
4. If $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are two AP's with common differences d_1 and d_2 respectively, then $a_1b_1, a_2b_2, a_3b_3, \ldots$ and $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots$ are not in AP.
5. If $a_1, a_2, a_3, \ldots, a_n$ are in AP, then

$$a_r = \frac{a_{r-k} + a_{r+k}}{2}, \forall\, k, 0 \leq k \leq n - r$$

6. If three numbers in AP whose sum is given are to be taken as $\alpha-\beta, \alpha, \alpha+\beta$ and if five numbers in AP whose sum is given, are to be taken as $\alpha-2\beta, \alpha-\beta, \alpha, \alpha+\beta, \alpha+2\beta$, etc.

In general, If $(2r+1)$ numbers in AP whose sum is given, are to be taken as $(r \in N)$.

$\alpha-r\beta, \alpha-(r-1)\beta, \ldots, \alpha-\beta, \alpha, \alpha+\beta, \ldots,$
$\alpha+(r-1)\beta, \alpha+r\beta$

Remark

1. Sum of three numbers = 3α
Sum of five numbers = 5α
⋮ ⋮ ⋮ ⋮ ⋮
Sum of $(2r+1)$ numbers = $(2r+1)\alpha$

2. From given conditions, find two equations in α and β and then solve them. Now, the numbers in AP can be obtained.

7. If four numbers in AP whose sum is given, are to be taken as
$\alpha-3\beta, \alpha-\beta, \alpha+\beta, \alpha+3\beta$ and if six numbers in AP, whose sum is given are to be taken as $\alpha-5\beta, \alpha-3\beta, \alpha-\beta, \alpha+\beta, \alpha+3\beta, \alpha+5\beta$, etc.

In general If $2r$ numbers in AP whose sum is given, are to be taken as $(r \in N)$.

$\alpha-(2r-1)\beta, \alpha-(2r-3)\beta, \ldots, \alpha-3\beta, \alpha-\beta,$
$\alpha+\beta, \alpha+3\beta, \ldots, \alpha+(2r-3)\beta, \alpha+(2r-1)\beta$

Remark

1. Sum of four numbers = 4α
Sum of six numbers = 6α
⋮ ⋮ ⋮ ⋮ ⋮
Sum of $2r$ numbers = $2r\alpha$

2. From given conditions, find two conditions in α and β and then solve them. Now, the numbers in AP can be obtained.

Example 21. If $S_1, S_2, S_3, \ldots, S_p$ are the sums of n terms of p AP's whose first terms are $1, 2, 3, \ldots, p$ and common differences are $1, 2, 3, \ldots, (2p-1)$ respectively, show that $S_1+S_2+S_3+\ldots.+S_p=\frac{1}{2}np(np+1)$.

Sol. $\because$ $1, 2, 3, \ldots, p$ are in AP.

Then, $2\cdot1, 2\cdot2, 2\cdot3, \ldots, 2p$ are also in AP. ...(i)
[multiplying 2 to each term]

and $1, 3, 5, \ldots, (2p-1)$ are in AP.

Then, $(n-1)\cdot1, (n-1)\cdot3, (n-1)\cdot5, \ldots, (n-1)(2p-1)$ are also in AP. ...(ii)
[multiplying $(n-1)$ to each term]

From Eqs. (i) and (ii), we get

$2\cdot1+(n-1)\cdot1, 2\cdot2+(n-1)\cdot3, 2\cdot3+(n-1)\cdot5, \ldots,$
$2p+(n-1)(2p-1)$ are also in AP. ...(iii)
[adding corresponding terms of Eqs. (i) and (ii)]

From Eq. (iii),

$\frac{n}{2}\{2\cdot1+(n-1)\cdot1\}, \frac{n}{2}\{2\cdot2+(n-1)\cdot3\}, \frac{n}{2}\{2\cdot3+(n-1)\cdot5\}, \ldots,$
$\frac{n}{2}\{2p+(n-1)(2p-1)\}$ are also in AP
[multiplying $\frac{n}{2}$ to each term]

i.e. $S_1, S_2, S_3, \ldots, S_p$ are in AP.

$\therefore\ S_1+S_2+S_3+\ldots+S_p=\frac{p}{2}\{S_1+S_p\}$

$=\frac{p}{2}\left\{\frac{n}{2}[2\cdot1+(n-1)\cdot1]+\frac{n}{2}[2\cdot p+(n-1)(2p-1)]\right\}$

$=\frac{np}{4}\{2+(n-1)+2p+(n-1)(2p-1)\}$

$=\frac{np}{4}(2np+2)=\frac{1}{2}np(np+1)$

Aliter

Here, $S_1=1+2+3+\ldots$ upto n terms $=\frac{n(n+1)}{2}$

$S_2=2+5+8+\ldots$ upto n terms $=\frac{n}{2}[2\cdot2+(n-1)3]$

$=\frac{n(3n+1)}{2}$

Similarly, $S_3=3+8+13+\ldots$ upto n terms $=\frac{n(5n+1)}{2}$, etc.

Now, $S_1+S_2+S_3+\ldots+S_p$

$=\frac{n(n+1)}{2}+\frac{n(3n+1)}{2}+\frac{n(5n+1)}{2}+\ldots$ upto p terms

$=\frac{n}{2}[(n+3n+5n+\ldots \text{upto } p \text{ terms})$
$+(1+1+1+\ldots \text{upto } p \text{ terms})]$

$=\frac{n}{2}\left[\frac{p}{2}(2n+(p-1)2n)+p\right]$

$=\frac{np}{2}[n+n(p-1)+1]=\frac{1}{2}np(np+1)$

Example 22. Let α and β be roots of the equation $x^2-2x+A=0$ and let γ and δ be the roots of the equation $x^2-18x+B=0$. If $\alpha<\beta<\gamma<\delta$ are in arithmetic progression, then find the values of A and B.

Sol. $\because$ $\alpha, \beta, \gamma, \delta$ are in AP.

Let $\beta=\alpha+d, \gamma=\alpha+2d, \delta=\alpha+3d, d>0$
[here, sum of $\alpha, \beta, \gamma, \delta$ is not given]

Given, $\alpha+\beta=2, \alpha\beta=A$

$\Rightarrow$ $2\alpha+d=2, \alpha\beta=A$ ...(i)

and $\gamma+\delta=18, \gamma\delta=B$

$\Rightarrow$ $2\alpha+5d=18, \gamma\delta=B$ ...(ii)

From Eqs. (i) and (ii), we get

$$d = 4, \alpha = -1$$

$$\therefore \quad \beta = 3, \gamma = 7, \delta = 11$$

$$\Rightarrow \quad A = \alpha\beta = (-1)(3) = -3$$

and $\quad B = \gamma\delta = (7)(11) = 77$

Example 23. The digits of a positive integer having three digits are in AP and their sum is 15. The number obtained by reversing the digits is 594 less than the original number. Find the number.

Sol. Let the digit in the unit's place be $a - d$, digit in the ten's place be a and the digit in the hundred's place be $a + d$.

Sum of digits $= a - d + a + a + d = 15$ [given]

$$\Rightarrow \quad 3a = 15$$

$$\therefore \quad a = 5 \qquad \ldots(i)$$

$\therefore$ Original number $= (a - d) + 10a + 100(a + d)$

$$= 111a + 99d = 555 + 99d$$

and number formed by reversing the digits

$$= (a + d) + 10a + 100(a - d)$$

$$= 111a - 99d = 555 - 99d$$

Given, $(555 + 99d) - (555 - 99d) = 594 \Rightarrow 198d = 594$

$$\therefore \quad d = 3$$

Hence, original number $= 555 + 99 \times 3 = 852$

Example 24. If three positive real numbers are in AP such that $abc = 4$, then find the minimum value of b.

Sol. $\because a, b, c$ are in AP.

Let $\quad a = A - D, b = A, c = A + D$

Then, $\quad a = b - D, c = b + D$

Now, $\quad abc = 4$

$$(b - D)\, b\, (b + D) = 4$$

$$\Rightarrow \quad b(b^2 - D^2) = 4$$

$$\Rightarrow \quad b^2 - D^2 < b^2$$

$$\Rightarrow \quad b(b^2 - D^2) < b^3 \Rightarrow 4 < b^3$$

$$\therefore \quad b > (4)^{1/3} \text{ or } b > (2)^{2/3}$$

Hence, the minimum value of b is $(2)^{2/3}$.

Example 25. If a, b, c, d are distinct integers form an increasing AP such that $d = a^2 + b^2 + c^2$, then find the value of $a + b + c + d$.

Sol. Here, sum of numbers i.e., $a + b + c + d$ is not given.

Let $\quad b = a + D, c = a + 2D, d = a + 3D, \forall\ D \in N$

According to hypothesis,

$$a + 3D = a^2 + (a + D)^2 + (a + 2D)^2$$

$$\Rightarrow \quad 5D^2 + 3(2a - 1)\, D + 3a^2 - a = 0 \qquad \ldots(i)$$

$$\therefore \quad D = \frac{-3(2a-1) \pm \sqrt{9(2a-1)^2 - 20(3a^2 - a)}}{10}$$

$$= \frac{-3(2a-1) \pm \sqrt{(-24a^2 - 16a + 9)}}{10}$$

Now, $\quad -24a^2 - 16a + 9 \geq 0$

$$\Rightarrow \quad 24a^2 + 16a - 9 \leq 0$$

$$\Rightarrow \quad -\frac{1}{3} - \frac{\sqrt{70}}{3} \leq a \leq -\frac{1}{3} + \frac{\sqrt{70}}{12}$$

$$\Rightarrow \quad a = -1, 0 \qquad [\because a \in I]$$

When $a = 0$ from Eq. (i), $D = 0, \frac{3}{5}$ (not possible $\because D \in N$) and for $a = -1$

From Eq. (i), $\quad D = 1, \frac{4}{5}$

$$\therefore \quad D = 1 \qquad [\because D \in N]$$

$$\therefore \quad a = -1, b = 0, c = 1, d = 2$$

Then, $\quad a + b + c + d = -1 + 0 + 1 + 2 = 2$

Exercise for Session 2

1. If nth term of the series $25 + 29 + 33 + 37 + \ldots$ and $3 + 4 + 6 + 9 + 13 + \ldots$ are equal, then n equals

(a) 11 (b) 12 (c) 13 (d) 14

2. The rth term of the series $2\frac{1}{2} + 1\frac{7}{13} + 1\frac{1}{9} + \frac{20}{23} + \ldots$ is

(a) $\frac{20}{5r + 3}$ (b) $\frac{20}{5r - 3}$ (c) $20\,(5r + 3)$ (d) $\frac{20}{5r^2 + 3}$

3. In a certain AP, 5 times the 5th term is equal to 8 times the 8th term, its 13th term is

(a) 0 (b) -1 (c) -12 (d) -13

4. If the 9th term of an AP is zero, the ratio of its 29th and 19th terms is

(a) 1 : 2 (b) 2 : 1 (c) 1 : 3 (d) 3 : 1

5. If the pth, qth and rth terms of an AP are a, b and c respectively, the value of $a\,(q - r) + b\,(r - p) + c\,(p - q)$ is

(a) 1 (b) -1 (c) 0 (d) $\frac{1}{2}$

6. The 6th term of an AP is equal to 2, the value of the common difference of the AP which makes the product $a_1a_4a_5$ least is given by

(a) $\frac{8}{5}$ (b) $\frac{5}{4}$ (c) $\frac{2}{3}$ (d) $\frac{1}{3}$

7. The sum of first $2n$ terms of an AP is α and the sum of next n terms is β, its common difference is

(a) $\frac{\alpha - 2\beta}{3n^2}$ (b) $\frac{2\beta - \alpha}{3n^2}$ (c) $\frac{\alpha - 2\beta}{3n}$ (d) $\frac{2\beta - \alpha}{3n}$

8. The sum of three numbers in AP is -3 and their product is 8, then sum of squares of the numbers is

(a) 9 (b) 10 (c) 12 (d) 21

9. Let S_n denote the sum of n terms of an AP, if $S_{2n} = 3S_n$, then the ratio $\frac{S_{3n}}{S_n}$ is equal to

(a) 9 (b) 6 (c) 16 (d) 12

10. The sum of the products of the ten numbers $\pm 1, \pm 2, \pm 3, \pm 4, \pm 5$ taking two at a time, is

(a) -65 (b) 165 (c) -55 (d) 95

11. If $a_1, a_2, a_3, \ldots, a_n$ are in AP, where $a_i > 0$ for all i, the value of

$$\frac{1}{\sqrt{a_1} + \sqrt{a_2}} + \frac{1}{\sqrt{a_2} + \sqrt{a_3}} + \ldots + \frac{1}{\sqrt{a_{n-1}} + \sqrt{a_n}} \text{ is}$$

(a) $\frac{1}{\sqrt{a_1} + \sqrt{a_n}}$ (b) $\frac{1}{\sqrt{a_1} - \sqrt{a_n}}$ (c) $\frac{n}{\sqrt{a_1} - \sqrt{a_n}}$ (d) $\frac{(n - 1)}{\sqrt{a_1} + \sqrt{a_n}}$

Session 3

Geometric Sequence or Geometric Progression (GP)

Geometric Sequence or Geometric Progression (GP)

A geometric progression is a sequence, if the ratio of any term and its just preceding term is constant throughout. This constant quantity is called the common ratio and is generally denoted by ' r'.

Or

A geometric progression (GP) is a sequence of numbers, whose first term is non-zero and each of the term is obtained by multiplying its just preceding term by a constant quantity. This constant quantity is called common ratio of the GP.

Let $t_1, t_2, t_3, \ldots, t_n; t_1, t_2, t_3, \ldots$ be respectively a finite or an infinite sequence. Assume that none of t_n''s is 0 and that $\frac{t_k}{t_{k-1}} = r$, a constant (i.e., independent of k).

For $k = 2, 3, 4, \ldots, n$ or $k = 2, 3, 4, \ldots$ as the case may be. We then call $\{t_k\}_{k=1}^{n}$ or $\{t_k\}_{k=1}^{\infty}$ as the case may be a geometric progression (GP). The constant ratio r is called the common ratio (CR) of the GP.

i.e., $r = \frac{t_2}{t_1} = \frac{t_3}{t_2} = \ldots = \frac{t_n}{t_{n-1}}$

If $t_1 = a$ is the first term of a GP, then

$t_2 = ar,\ t_3 = t_2\, r = ar^2,\ t_4 = t_3\, r = ar^3, \ldots,$

$t_n = t_{n-1}\, r = ar^{n-1}$

It follows that, given that first term a and the common ratio r, the GP can be rewritten as

$a, ar, ar^2, \ldots, ar^{n-1}$ (standard GP) or $a, ar, ar^2, \ldots, ar^{n-1}, \ldots$ (standard GP)

according as it is finite or infinite.

Important Results

1. In a GP, neither $a = 0$ nor $r = 0$.
2. In a GP, no term can be equal to '0'.
3. If in a GP, the terms are alternatively positive and negative, then its common ratio is always negative.
4. If we multiply the common ratio with any term of GP, we get the next following term and if we divide any term by the common ratio, we get the preceding term.
5. The common ratio of GP may be positive, negative or imaginary.
6. If common ratio of GP is equal to unity, then GP is known as **Constant GP**.
7. If common ratio of GP is imaginary or real, then GP is known as **Imaginary GP.**
8. **Increasing and Decreasing GP**
   For a GP to be increasing or decreasing $r > 0$. If $r < 0$, terms of GP are alternatively positive and negative and so neither increasing nor decreasing.

a	$a > 0$	$a > 0$	$a < 0$	$a < 0$
r	$0 < r < 1$	$r > 1$	$r > 1$	$0 < r < 1$
Result	Decreasing	Increasing	Decreasing	Increasing

Example 26.

(i) 1, 2, 4, 8, 16, ... (ii) 9, 3, 1, $\frac{1}{3}, \frac{1}{9}$, ...

(iii) $-2, -6, -18, \ldots$ (iv) $-8, -4, -2, -1, -\frac{1}{2}, \ldots$

(v) $5, -10, 20, \ldots$ (vi) 5, 5, 5, 5,...

(vii) $1, 1+i, 2i, -2+2i, \ldots; i = \sqrt{-1}$

Sol. (i) Here, $a = 1$

and $r = \frac{2}{1} = \frac{4}{2} = \frac{8}{4} = \frac{16}{8} = \ldots = 2$ i.e. $a = 1$ and $r = 2$

Increasing GP $(a > 0, r > 1)$

(ii) Here, $a = 9$

and $r = \frac{3}{9} = \frac{1}{3} = \frac{\frac{1}{3}}{1} = \frac{\frac{1}{9}}{\frac{1}{3}} = \ldots = \frac{1}{3}$ i.e. $a = 9, r = \frac{1}{3}$

Decreasing GP $(a > 0, 0 < r < 1)$

(iii) Here, $a = -2$

and $r = \frac{-6}{-2} = \frac{-18}{-6} = \ldots = 3$

i.e. $a = -2, r = 3$

Decreasing GP $(a < 0, r > 1)$

(iv) Here, $a = -8$

and $r = \frac{-4}{-8} = \frac{-2}{-4} = \frac{-1}{-2} = \frac{-\frac{1}{2}}{-1} = \ldots. = \frac{1}{2}$

i.e. $a = -8, r = \frac{1}{2}$

Increasing GP $(a < 0, 0 < r < 1)$

(v) Here, $a = 5$

and $r = \frac{-10}{5} = \frac{20}{-10} = \ldots = -2$ i.e., $a = 5, r = -2$

Neither increasing nor decreasing $(r < 0)$

(vi) Here, $a = 5$

and $r = \frac{5}{5} = \frac{5}{5} = \frac{5}{5} = \ldots = 1$ i.e., $a = 5, r = 1$

Constant GP $(r = 1)$

(vii) Here, $a = 1$

and $r = \frac{1+i}{1} = \frac{2i}{1+i} = \frac{-2+2i}{2i} = \ldots$

$= (1+i) = \frac{2i(1-i)}{(1+i)(1-i)} = \frac{(-1+i)i}{i^2} = \ldots$

$= (1+i) = (i+1) = (1+i) = \ldots$

i.e., $a = 1, r = 1 + i$

Imaginary GP $(r =$ imaginary$)$

Example 27. Show that the sequence $< t_n >$ defined by $t_n = \frac{2^{2n-1}}{3}$ for all values of $n \in N$ is a GP. Also, find its common ratio.

Sol. We have, $t_n = \frac{2^{2n-1}}{3}$

On replacing n by $n - 1$, we get

$$t_{n-1} = \frac{2^{2n-3}}{3} \Rightarrow \frac{t_n}{t_{n-1}} = \frac{\frac{2^{2n-1}}{3}}{\frac{2^{2n-3}}{3}} = 2^2 = 4$$

Clearly, $\frac{t_n}{t_{n-1}}$ is independent of n and is equal to 4. So, the given sequence is a GP with common ratio 4.

Example 28. ***Show that the sequence*** $< t_n >$ ***defined by*** $t_n = 2 \cdot 3^n + 1$ ***is not a GP.***

Sol. We have, $t_n = 2 \cdot 3^n + 1$

On replacing n by $(n - 1)$ in t_n, we get

$$t_{n-1} = 2 \cdot 3^{n-1} + 1$$

$$\Rightarrow \quad t_{n-1} = \frac{(2 \cdot 3^n + 3)}{3}$$

$$\therefore \quad \frac{t_n}{t_{n-1}} = \frac{(2 \cdot 3^n + 1)}{\frac{(2 \cdot 3^n + 3)}{3}} = \frac{3(2 \cdot 3^n + 1)}{(2 \cdot 3^n + 3)}$$

Clearly, $\frac{t_n}{t_{n-1}}$ is not independent of n and is therefore not constant. So, the given sequence is not a GP.

General Term of a GP

Let 'a' be the first term, 'r' be the common ratio and 'l' be the last term of a GP having 'n' terms. Then, GP can be written as $a, ar, ar^2, \ldots, \frac{l}{r^2}, \frac{l}{r}, l$

(i) nth Term of a GP from Beginning

1st term from beginning $= t_1 = a = ar^{1-1}$

2nd term from beginning $= t_2 = ar = ar^{2-1}$

3rd term from beginning $= t_3 = ar^2 = ar^{3-1}$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

nth term from beginning $= t_n = ar^{n-1}, \forall\, n \in N$

Hence, n th term of a GP from beginning

$$t_n = ar^{n-1} = l \qquad \text{[last term]}$$

(ii) nth Term of a GP from End

1st term from end $= t'_1 = l = \frac{l}{r^{1-1}}$

2nd term from end $= t'_2 = \frac{l}{r} = \frac{l}{r^{2-1}}$

3rd term from end $= t'_3 = \frac{l}{r^2} = \frac{l}{r^{3-1}}$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

nth term from end $= t'_n = \frac{l}{r^{n-1}}, \forall\, n \in N$

Hence, nth term of a GP from end

$$= t'_n = \frac{l}{r^{n-1}} = a \qquad \text{[first term]}$$

Now, it is clear that $t_k \times t'_k = ar^{k-1} \times \frac{l}{r^{k-1}} = a \times l$

or $\quad t_k \times t'_k = a \times l, \forall\, 1 \le k \le n$

i.e. in a finite GP of n terms, the product of the k th term from the beginning and the k th term form the end is independent of k and equals the product of the first and last terms.

Remark

1. nth term is also called the general term.
2. If last term of GP be t_n and CR is r, then terms of GP from end are $t_n, \frac{t_n}{r}, \frac{t_n}{r^2}, \ldots$
3. If in a GP, the terms are alternatively positive and negative, then its common ratio is always negative.

4. If a and l represent first and last term of a GP respectively, then common ratio of GP $= r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}}$

5. If t_n, t_{n+1}, t_{n+2} are three consecutive terms of a GP, then
$$\frac{t_{n+1}}{t_n} = \frac{t_{n+2}}{t_{n+1}} \Rightarrow t_{n+1}^2 = t_n\, t_{n+2}$$
In particular, if a, b, c are in GP, then $\frac{b}{a} = \frac{c}{b} \Rightarrow b^2 = ac$
On squaring, $\frac{b^2}{a^2} = \frac{c^2}{b^2}$
Hence, a^2, b^2, c^2 are also in GP.

Example 29. If first term of a GP is a, third term is b and $(n+1)$th term is c. The $(2n+1)$th term of a GP is

(a) $c\sqrt{\frac{b}{a}}$ (b) $\frac{bc}{a}$ (c) abc (d) $\frac{c^2}{a}$

Sol. Let common ratio $= r$

$\therefore \quad b = ar^2 \Rightarrow r = \sqrt{\frac{b}{a}}$

Also, $\quad c = ar^n \Rightarrow r^n = \frac{c}{a}$

$\therefore \quad t_{2n+1} = ar^{2n} = a(r^n)^2 = a\left(\frac{c}{a}\right)^2 = \frac{c^2}{a}$

Hence, (d) is the correct answer.

Example 30. The $(m+n)$th and $(m-n)$th terms of a GP are p and q, respectively. Then, the mth term of the GP is

(a) $p\left(\frac{q}{p}\right)^{\frac{m}{2n}}$ (b) $\sqrt{pq}$

(c) $\sqrt{\frac{p}{q}}$ (d) None of these

Sol. Let a be the first term and r be the common ratio, then
$$t_{m+n} = p \Rightarrow ar^{m+n-1} = p \quad ...(i)$$
$$t_{m-n} = q \Rightarrow ar^{m-n-1} = q \quad ...(ii)$$
From Eqs. (i) and (ii), we get
$$ar^{m+n-1} \times ar^{m-n-1} = p \times q$$
$$\Rightarrow \quad a^2 r^{2m-2} = pq \Rightarrow ar^{m-1} = \sqrt{pq}$$
$$\Rightarrow \quad t_m = \sqrt{pq}$$
Hence, (b) is the correct answer.

Example 31. If $\sin\theta, \sqrt{2}(\sin\theta + 1), 6\sin\theta + 6$ are in GP, then the fifth term is

(a) 81 (b) $81\sqrt{2}$ (c) 162 (d) $162\sqrt{2}$

Sol. $[\sqrt{2}(\sin\theta + 1)]^2 = \sin\theta\,(6\sin\theta + 6)$
$\Rightarrow [(\sin\theta + 1)\,2(\sin\theta + 1) - 6\sin\theta] = 0$

We get, $\quad \sin\theta = -1, \frac{1}{2}$

$\therefore \quad \sin\theta = \frac{1}{2}$ [$\sin\theta = -1$ is not possible]

then first term $= a = \sin\theta = \frac{1}{2}$ and common ratio
$$= r = \frac{\sqrt{2}\left(\frac{1}{2} + 1\right)}{\left(\frac{1}{2}\right)} = 3\sqrt{2}$$
$\therefore \quad t_5 = ar^4 = \frac{1}{2}(3\sqrt{2})^4 = 162$

Hence, (c) is the correct answer.

Example 32. The 1025th term in the sequence 1, 22, 4444, 88888888, ... is

(a) 2^9 (b) 2^{10}
(c) 2^{11} (d) 2^{12}

Sol. The number of digits in each term of the sequence are 1, 2, 4, 8, ... which are in GP. Let 1025th term is 2^n.
Then,
$$1 + 2 + 4 + 8 + ... + 2^{n-1} < 1025 \le 1 + 2 + 4 + 8 + ... + 2^n$$
$$\Rightarrow \quad \frac{(2-1)(1 + 2 + 2^2 + 2^3 + ... + 2^{n-1})}{(2-1)} < 1025$$
$$\le \frac{(2-1)(1 + 2 + 2^2 + 2^3 + ... + 2^n)}{(2-1)}$$
$$\Rightarrow \quad 2^n - 1 < 1025 \le 2^{n+1} - 1 \Rightarrow 2^n < 1026 \le 2^{n+1} \quad ...(i)$$
or $\quad 2^{n+1} \ge 1026 > 1024$
$$\Rightarrow \quad 2^{n+1} > 2^{10} \Rightarrow n + 1 > 10$$
$$\therefore \quad n > 9 \quad \therefore n = 10$$
[which is always satisfy Eq. (i)]

Hence, (b) is the correct answer.

Example 33. ***If a, b, c are real numbers such that $3(a^2 + b^2 + c^2 + 1) = 2(a + b + c + ab + bc + ca)$, then a, b, c are in***

(a) AP only (b) GP only
(c) GP and AP (d) None of these

Sol. Given, $3(a^2 + b^2 + c^2 + 1) = 2(a + b + c + ab + bc + ca)$
$$\Rightarrow \quad 2(a^2 + b^2 + c^2 - ab - bc - ca) + (a^2 + b^2 + c^2 - 2a - 2b - 2c + 3)$$
$$\Rightarrow \quad \{(a-b)^2 + (b-c)^2 + (c-a)^2\} + \{(a-1)^2 + (b-1)^2 + (c-1)^2\} = 0$$
$\Rightarrow a - b = b - c = c - a = 0$ and $a - 1 = b - 1 = c - 1 = 0$
$$\Rightarrow \quad a = b = c = 1$$
$\Rightarrow a, b, c$ are in GP and AP.
Hence, (c) is the correct answer.

Sum of a Stated Number of Terms of a Geometric Series

The game of chess was invented by **Grand Vizier Sissa Ben Dhair** for the Indian king **Shirham.** Pleased with the game, the king asked the **Vizier** what he would like as reward. The **Vizier** asked for one grain of wheat to be placed on the first square of the chess, two grains on the second, four grains on the third and so on (each time doubling the number of grains). The king was surprised of the request and told the **vizier** that he was fool to ask for so little.

The inventor of chess was no fool. He told the king ''What I have asked for is more wheat that you have in the entire kingdom, in fact it is more than there is in the whole world'' He was right. There are 64 squares on a chess board and on the nth square he was asking for 2^{n-1} grains, if you add the numbers

i.e., $$S = 1 + 2 + 2^2 + 2^3 + \ldots + 2^{62} + 2^{63} \quad \ldots(i)$$

On multiplying both sides by 2, then

$$2S = 2 + 2^2 + 2^3 + 2^4 + \ldots + 2^{63} + 2^{64} \quad \ldots(ii)$$

On subtracting Eq. (i) from Eq. (ii), we get

$S = 2^{64} - 1 = 18, 446, 744, 073, 709, 551, 615$ grains i.e., represent more wheat that has been produced on the Earth.

Sum of n Terms of a GP

Let a be the first term, r be the common ratio, l be the last term of a GP having n terms and S_n the sum of n terms, then

$$S_n = a + ar + ar^2 + \ldots + \frac{l}{r^2} + \frac{l}{r} + l \quad \ldots(i)$$

On multiplying both sides by r (the common ratio)

$$r\,S_n = ar + ar^2 + ar^3 + \ldots + \frac{l}{r} + l + lr \quad \ldots(ii)$$

On subtracting Eq. (ii) from Eq. (i), we have

$$S_n - r\,S_n = a - lr \text{ or } S_n(1 - r) = a - lr$$

$$\therefore \quad \boldsymbol{S_n = \frac{a - lr}{1 - r}}, \text{ when } r < 1$$

$$\boldsymbol{S_n = \frac{lr - a}{r - 1}}, \text{ when } r > 1$$

Now, $$l = t_n = ar^{n-1}$$

Then, above formula can be written as

$$\boldsymbol{S_n = \frac{a(1 - r^n)}{(1 - r)}} \text{ when } \boldsymbol{r < 1}, \boldsymbol{S_n = \frac{a(r^n - 1)}{(r - 1)}},$$

when $\boldsymbol{r > 1}$

If $r = 1$, the above formulae cannot be used. But, then the GP reduces to $a, a, a, \ldots$

$\therefore \quad S_n = a + a + a + \ldots n$ times $= na$

Sum to Infinity of a GP, when the Numerical Value of the Common Ratio is Less than Unity, i.e. It is a Proper Fraction

If a be the first term, r be the common ratio of a GP, then

$$S_n = \frac{a(1 - r^n)}{(1 - r)} = \frac{a}{(1 - r)} - \frac{ar^n}{(1 - r)}$$

Let $-1 < r < 1$ i.e. $|r| < 1$, then $\lim_{n \to \infty} r^n \to 0$

Let S_∞ denote the sum to infinity of the GP, then

$$S_\infty = \frac{a}{(1 - r)},$$

where $\quad -1 < r < 1$

Recurring Decimal

Recurring decimal is a very good example of an infinite geometric series and its value can be obtained by means of infinite geometric series as follows

$0.3\dot{2}\dot{7} = 0.327272727\ldots$ to infinity

$$= 0.3 + 0.027 + 0.00027 + 0.0000027 + \ldots \text{ upto infinity}$$

$$= \frac{3}{10} + \frac{27}{10^3} + \frac{27}{10^5} + \frac{27}{10^7} + \ldots \text{ upto infinity}$$

$$= \frac{3}{10} + \frac{27}{10^3}\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + \ldots \text{ upto infinity}\right)$$

$$= \frac{3}{10} + \frac{27}{10^3}\left(\frac{1}{1 - \frac{1}{10^2}}\right)$$

$$= \frac{3}{10} + \frac{27}{990} = \frac{297 + 27}{990}$$

$$= \frac{324}{990} \qquad \text{[rational number]}$$

Aliter (Best method)

Let P denotes the figure which do not recur and suppose them p in number, Q denotes the recurring period consisting of q figures. Let R denotes the value of the recurring decimal.

Then, $$R = 0 \cdot PQQQ \ldots$$

$\therefore \quad 10^p \times R = P \cdot QQQ \ldots$

and $$10^{p+q} \times R = PQ \cdot QQQ \ldots$$

$\therefore$ Therefore, by subtraction $R = \dfrac{PQ - P}{(10^{p+q} - 10^p)}$.

***Corollary* I** If $R = 0 \cdot QQQ \ldots$

Then, $R = \dfrac{Q}{10^q - 1}$ (when Q denote the recurring period consisting of q figures)

For example, If $R = 0.\dot{3}$, then $R = \dfrac{3}{10^1 - 1} = \dfrac{1}{3}$

***Corollary* II** The value of recurring decimal is always rational number.

Example 34. Find the value of $0.32\dot{5}\dot{8}$.

Sol. Let $R = 0.3258$

$\Rightarrow$ $R = 0.3258585858 \ldots$ ...(i)

Here, number of figures which are not recurring is 2 and number of figures which are recurring is also 2.

Then, $100R = 32.58585858 \ldots$ ...(ii)

and $10000R = 3258.58585858 \ldots$ ...(iii)

On subtracting Eq. (ii) from Eq. (iii), we get

$$9900R = 3226$$

$$\therefore \quad R = \frac{3226}{9900}$$

Hence, $R = \dfrac{1613}{4950}$

Shortcut Methods for Recurring Decimals

1. The numerator of the vulgar fraction is obtained by subtracting the non-recurring figure from the given figure.
2. The denominator consists of as many 9's as there are recurring figure and as many zero as there are non-recurring figure.

   For example,

   (i) $0.36\dot{5}\dot{4} = \dfrac{3654 - 36}{9900} = \dfrac{3618}{9900}$

   (ii) $1.3\dot{2}\dot{7} = 1 + 0.3\dot{2}\dot{7} = 1 + \dfrac{327 - 3}{990} = \dfrac{1314}{990}$

   (iii) $0.\dot{3} = \dfrac{3 - 0}{9} = \dfrac{1}{3}$

Example 35. ***Find the sum upto n terms of the series $a + aa + aaa + aaaa + \ldots$, $\forall\ a \in N$ and $1 \le a \le 9$.***

Sol. Let $S = a + aa + aaa + aaaa + \ldots$ upto n terms

$= a(1 + 11 + 111 + 1111 + \ldots \text{ upto } n \text{ terms})$

$= \dfrac{a}{9}(9 + 99 + 999 + 9999 + \ldots \text{ upto } n \text{ terms})$

$= \dfrac{a}{9}\{(10^1 - 1) + (10^2 - 1) + (10^3 - 1) + (10^4 - 1) + \ldots$ upto n terms$\}$

$= \dfrac{a}{9}\{(10 + 10^2 + 10^3 + \ldots \text{ upto } n \text{ terms}) - (1 + 1 + 1 + \ldots n \text{ times})\}$

$= \dfrac{a}{9}\left\{\dfrac{10(10^n - 1)}{10 - 1} - n\right\} = \dfrac{a}{9}\left\{\dfrac{10}{9}(10^n - 1) - n\right\}$ [Remember]

In Particular

(i) For $a = 1, 1 + 11 + 111 + \ldots = \dfrac{1}{9}\left\{\dfrac{10}{9}(10^n - 1) - n\right\}$

(ii) For $a = 2, 2 + 22 + 222 + \ldots = \dfrac{2}{9}\left\{\dfrac{10}{9}(10^n - 1) - n\right\}$

(iii) For $a = 3, 3 + 33 + 333 + \ldots = \dfrac{3}{9}\left\{\dfrac{10}{9}(10^n - 1) - n\right\}$

(iv) For $a = 4, 4 + 44 + 444 + \ldots = \dfrac{4}{9}\left\{\dfrac{10}{9}(10^n - 1) - n\right\}$

(v) For $a = 5, 5 + 55 + 555 + \ldots = \dfrac{5}{9}\left\{\dfrac{10}{9}(10^n - 1) - n\right\}$

(vi) For $a = 6, 6 + 66 + 666 + \ldots = \dfrac{6}{9}\left\{\dfrac{10}{9}(10^n - 1) - n\right\}$

(vii) For $a = 7, 7 + 77 + 777 + \ldots = \dfrac{7}{9}\left\{\dfrac{10}{9}(10^n - 1) - n\right\}$

(viii) For $a = 8, 8 + 88 + 888 + \ldots = \dfrac{8}{9}\left\{\dfrac{10}{9}(10^n - 1) - n\right\}$

(ix) For $a = 9, 9 + 99 + 999 + \ldots = \dfrac{9}{9}\left\{\dfrac{10}{9}(10^n - 1) - n\right\}$

Example 36. Find the sum upto n terms of the series $0.b + 0.bb + 0.bbb + 0.bbbb + \ldots, \forall\ b \in N$ and $1 \le b \le 9$.

Sol. Let $S = 0.b + 0.bb + 0.bbb + 0.bbbb + \ldots$ upto n terms

$= b(0.1 + 0.11 + 0.111 + 0.1111 + \ldots \text{ upto } n \text{ terms})$

$= \dfrac{b}{9}(0.9 + 0.99 + 0.999 + 0.9999 + \ldots. \text{ upto } n \text{ terms})$

$= \dfrac{b}{9}\{(1 - 0.1) + (1 - 0.01) + (1 - 0.001) + (1 - 0.0001) + \ldots$ upto n terms$\}$

$= \dfrac{b}{9}\{(1 + 1 + 1 + 1 + \ldots \text{ upto } n \text{ times}) - (0.1 + 0.01 + 0.001 + 0.0001 + \ldots \text{ upto } n \text{ terms})\}$

$= \dfrac{b}{9}\left\{n - \left(\dfrac{1}{10} + \dfrac{1}{10^2} + \dfrac{1}{10^3} + \dfrac{1}{10^4} + \ldots \text{ upto } n \text{ terms}\right)\right\}$

$= \dfrac{b}{9}\left\{n - \dfrac{\dfrac{1}{10}\left(1 - \left(\dfrac{1}{10}\right)^n\right)}{1 - \dfrac{1}{10}}\right\} = \dfrac{b}{9}\left\{n - \dfrac{1}{9}\left[1 - \left(\dfrac{1}{10}\right)^n\right]\right\}$ [Remember]

In Particular

(i) For $b = 1$,

$$0.1 + 0.11 + 0.111 + \ldots = \frac{1}{9}\left\{n - \frac{1}{9}\left[1 - \left(\frac{1}{10}\right)^n\right]\right\}$$

(ii) For $b = 2$,

$$0.2 + 0.22 + 0.222 + \ldots = \frac{2}{9}\left\{n - \frac{1}{9}\left[1 - \left(\frac{1}{10}\right)^n\right]\right\}$$

(iii) For $b = 3$,

$$0.3 + 0.33 + 0.333 + \ldots = \frac{3}{9}\left\{n - \frac{1}{9}\left[1 - \left(\frac{1}{10}\right)^n\right]\right\}$$

(iv) For $b = 4$,

$$0.4 + 0.44 + 0.444 + \ldots = \frac{4}{9}\left\{n - \frac{1}{9}\left[1 - \left(\frac{1}{10}\right)^n\right]\right\}$$

(v) For $b = 5$,

$$0.5 + 0.55 + 0.555 + \ldots = \frac{5}{9}\left\{n - \frac{1}{9}\left[1 - \left(\frac{1}{10}\right)^n\right]\right\}$$

(vi) For $b = 6$,

$$0.6 + 0.66 + 0.666 + \ldots = \frac{6}{9}\left\{n - \frac{1}{9}\left[1 - \left(\frac{1}{10}\right)^n\right]\right\}$$

(vii) For $b = 7$,

$$0.7 + 0.77 + 0.777 + \ldots = \frac{7}{9}\left\{n - \frac{1}{9}\left[1 - \left(\frac{1}{10}\right)^n\right]\right\}$$

(viii) For $b = 8$,

$$0.8 + 0.88 + 0.888 + \ldots = \frac{8}{9}\left\{n - \frac{1}{9}\left[1 - \left(\frac{1}{10}\right)^n\right]\right\}$$

(ix) For $b = 9$,

$$0.9 + 0.99 + 0.999 + \ldots = \frac{9}{9}\left\{n - \frac{1}{9}\left[1 - \left(\frac{1}{10}\right)^n\right]\right\}$$

Example 37. If N, the set of natural numbers is partitioned into groups $S_1 = \{1\}, S_2 = \{2, 3\}$, $S_3 = \{4, 5, 6, 7\}$, $S_4 = \{8, 9, 10, 11, 12, 13, 14, 15\}, \ldots$, then find the sum of the numbers in S_{50}.

Sol. The number of terms in the groups are $1, 2, 2^2, 2^3, \ldots$

$\therefore$ The number of terms in the 50th group $= 2^{50-1} = 2^{49}$

$\because$ The first term of 1st group $= 1 = 2^0 = 2^{1-1}$

The first term of 2nd group $= 2 = 2^1 = 2^{2-1}$

The first term of 3rd group $= 4 = 2^2 = 2^{3-1}$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

The first term of 50th group $= 2^{50-1} = 2^{49}$

$$\therefore \quad S_{50} = \frac{2^{49}}{2}\{2 \times 2^{49} + (2^{49} - 1) \times 1\}$$
$$= 2^{48}(2^{50} + 2^{49} - 1)$$
$$= 2^{48}[2^{49}(2 + 1) - 1] = 2^{48}(3 \cdot 2^{49} - 1)$$

Example 38. ***If*** $S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}}$, ***then calculate the least value of*** n ***such that*** $2 - S_n < \frac{1}{100}$.

Sol. Given, $S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}} = \frac{1 \cdot \left[1 - \left(\frac{1}{2}\right)^n\right]}{\left(1 - \frac{1}{2}\right)}$

$$\Rightarrow \quad S_n = 2 - \frac{1}{2^{n-1}}$$

$$\Rightarrow \quad 2 - S_n = \frac{1}{2^{n-1}} < \frac{1}{100} \qquad \left[\because 2 - S_n < \frac{1}{100}\right]$$

$$\Rightarrow \quad 2^{n-1} > 100 > 2^6$$

$$\Rightarrow \quad 2^{n-1} > 2^6$$

$$\therefore \quad n - 1 > 6 \Rightarrow n > 7$$

Hence, the least value of n is 8.

Example 39. If $x = 1 + a + a^2 + a^3 + \ldots + \infty$ and $y = 1 + b + b^2 + b^3 + \ldots + \infty$ show that $1 + ab + a^2b^2 + a^3b^3 + \ldots + \infty = \frac{xy}{x + y - 1}$, where $0 < a < 1$ and $0 < b < 1$.

Sol. Given, $x = 1 + a + a^2 + a^3 + \ldots + \infty = \frac{1}{1 - a}$

$$\Rightarrow \quad x - ax = 1$$

$$\therefore \quad a = \left(\frac{x - 1}{x}\right) \qquad \ldots(i)$$

and $\quad y = 1 + b + b^2 + b^3 + \ldots + \infty$

Similarly, $\quad b = \left(\frac{y - 1}{y}\right) \qquad \ldots(ii)$

Since, $\quad 0 < a < 1, 0 < b < 1$

$$\therefore \quad 0 < ab < 1$$

Now, $1 + ab + a^2b^2 + a^3b^3 + \ldots + \infty = \frac{1}{1 - ab}$

$$= \frac{1}{1 - \left(\frac{x - 1}{x}\right)\left(\frac{y - 1}{y}\right)} \qquad \text{[from Eqs. (i) and (ii)]}$$

$$= \frac{xy}{xy - xy + x + y - 1}$$

Hence, $1 + ab + a^2b^2 + a^3b^3 + \ldots + \infty = \frac{xy}{x + y - 1}$

Properties of Geometric Progression

1. If $a_1, a_2, a_3, \ldots$ are in GP with common ratio r, then $a_1k, a_2k, a_3k, \ldots$ and $\frac{a_1}{k}, \frac{a_2}{k}, \frac{a_3}{k}, \ldots$ are also in GP $(k \neq 0)$ with common ratio r.
2. If $a_1, a_2, a_3, \ldots$ are in GP with common ratio r, then $a_1 \pm k, a_2 \pm k, a_3 \pm k, \ldots$. are not in GP $(k \neq 0)$.
3. If $a_1, a_2, a_3, \ldots$ are in GP with common ratio r, then
   (i) $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots$ are also in GP with common ratio $\frac{1}{r}$.
   (ii) $a_1^n, a_2^n, a_3^n, \ldots$ are also in GP with common ratio r^n and $n \in Q$.
   (iii) $\log a_1, \log a_2, \log a_3, \ldots$ are in AP $(a_i > 0, \forall\, i)$ In this case, the converse also holds good.
4. If $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are two GP's with common ratios r_1 and r_2, respectively. Then,
   (i) $a_1b_1, a_2b_2, a_3b_3, \ldots$ and $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots$ are also in GP with common ratios r_1r_2 and $\frac{r_1}{r_2}$, respectively.
   (ii) $a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3, \ldots$ are not in GP.
5. If $a_1, a_2, a_3, \ldots, a_{n-2}, a_{n-1}, a_n$ are in GP. Then,
   (i) $a_1a_n = a_2a_{n-1} = a_3a_{n-2} = \ldots$
   (ii) $a_r = \sqrt{a_{r-k}\ a_{r+k}}, \forall\, k, 0 \leq k \leq n - r$
   (iii) $\frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \ldots = \frac{a_n}{a_{n-1}}$
   $\Rightarrow \quad a_2^2 = a_3a_1, a_3^2 = a_2a_4, \ldots$
   Also, $a_2 = a_1r, a_3 = a_1r^2,$
   $a_4 = a_1r^3, \ldots, a_n = a_1r^{n-1}$
   where, r is the common ratio of GP.
6. If three numbers in GP whose product is given are to be taken as $\frac{a}{r}, a, ar$ and if five numbers in GP whose product is given are to be taken as $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$, etc.

   In general If $(2m+1)$ numbers in GP whose product is given are to be taken as $(m \in N)$

   $$\frac{a}{r^m}, \frac{a}{r^{m-1}}, \ldots, \frac{a}{r}, a, ar, \ldots, ar^{m-1}, ar^m$$

Remark

1. Product of three numbers = a^3
   Product of five numbers = a^5
   ⋮ ⋮ ⋮ ⋮ ⋮
   Product of $(2m + 1)$ numbers = a^{2m+1}
2. From given conditions, find two equations in a and r and then solve them. Now, the numbers in GP can be obtained.

7. If four numbers in GP whose product is given are to be taken as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$ and if six numbers in GP whose product is given are to be taken as $\frac{a}{r^5}, \frac{a}{r^3}, \frac{a}{r}, ar, ar^3, ar^5$, etc.

   In general If $(2m)$ numbers in GP whose product is given are to be taken as $(m \in N)$

   $$\frac{a}{r^{2m-1}}, \frac{a}{r^{2m-3}}, \ldots, \frac{a}{r^3}, \frac{a}{r}, ar, ar^3, \ldots, ar^{2m-3}, ar^{2m-1}$$

Remark

1. Product of four numbers = a^4
   Product of six numbers = a^6
   ⋮ ⋮ ⋮ ⋮ ⋮
   Product of $(2m)$ numbers = a^{2m}
2. From given conditions, find two equations in a and r and then solve them. Now, the numbers in GP can be obtained.

Example 40. If $S_1, S_2, S_3, \ldots, S_p$ are the sum of infinite geometric series whose first terms are 1, 2, 3, ..., p and whose common ratios are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{p+1}$ respectively, prove that

$$S_1 + S_2 + S_3 + \ldots + S_p = \frac{p(p+3)}{2}.$$

Sol. $\because \quad S_p = \dfrac{p}{1 - \dfrac{1}{p+1}} = (p+1)$

$\therefore \quad S_1 = 2, S_2 = 3, S_3 = 4, \ldots$

$\therefore \quad \text{LHS} = S_1 + S_2 + S_3 + \ldots + S_p$

$$= 2 + 3 + 4 + \ldots + (p+1) = \frac{p}{2}(2 + p + 1)$$

$$= \frac{p(p+3)}{2} = \text{RHS}$$

Example 41. ***Let x_1 and x_2 be the roots of the equation $x^2 - 3x + A = 0$ and let x_3 and x_4 be the roots of the equation $x^2 - 12x + B = 0$. It is known that the numbers x_1, x_2, x_3, x_4 (in that order) form an increasing GP. Find A and B.***

Sol. $\because x_1, x_2, x_3, x_4$ are in GP.

Let $x_2 = x_1 r, x_3 = x_1 r^2, x_4 = x_1 r^3$

[here, product of x_1, x_2, x_3, x_4 are not given]

Given, $x_1 + x_2 = 3, x_1 x_2 = A$

$\Rightarrow \quad x_1(1 + r) = 3, x_1^2 r = A$ ...(i)

and $x_3 + x_4 = 12, x_3 x_4 = B$

$\Rightarrow x_1 r^2 (1 + r) = 12, x_1^2 r^5 = B$ ...(ii)

From Eqs. (i) and (ii),

$r^2 = 4 \Rightarrow r = 2$ [for increasing GP]

From Eq. (i), $x_1 = 1$

Now, $A = x_1^2 r = 1^2 \cdot 2 = 2$ [from Eq. (i)]

and $B = x_1^2 r^5 = 1^2 \cdot 2^5 = 32$ [from Eq. (ii)]

Example 42. ***Suppose a, b, c are in AP and a^2, b^2, c^2 are in GP, if $a > b > c$ and $a + b + c = \frac{3}{2}$, then find the values of a and c.***

Sol. Since, a, b, c are in AP and sum of a, b, c is given.

Let $a = b - D, c = b + D$ $[D < 0] [\because a > b > c]$

and given $a + b + c = \frac{3}{2}$

$\Rightarrow b - D + b + b + D = \frac{3}{2}$

$\therefore \quad b = \frac{1}{2}$

Then, $a = \frac{1}{2} - D$ and $c = \frac{1}{2} + D$

Also, given a^2, b^2, c^2 are in GP, then $(b^2)^2 = a^2 c^2$

$\Rightarrow \quad \pm b^2 = ac \Rightarrow \pm \frac{1}{4} = \frac{1}{4} - D^2$

$\Rightarrow \quad D^2 = \frac{1}{4} \pm \frac{1}{4} = \frac{1}{2}$ $[\because D \neq 0]$

$\therefore \quad D = \pm \frac{1}{\sqrt{2}} \Rightarrow D = -\frac{1}{\sqrt{2}}$ $\left[\because D < 0\right]$

Hence, $a = \frac{1}{2} + \frac{1}{\sqrt{2}}$ and $c = \frac{1}{2} - \frac{1}{\sqrt{2}}$

Example 43. If the continued product of three numbers in GP is 216 and the sum of their products in pairs is 156, then find the sum of three numbers.

Sol. Here, product of numbers in GP is given.

$\therefore$ Let the three numbers be $\frac{a}{r}, a, ar$.

Then, $\frac{a}{r} \cdot a \cdot ar = 216$

$\Rightarrow \quad a^3 = 216$

$\therefore \quad a = 6$

Sum of the products in pairs = 156

$\Rightarrow \quad \frac{a}{r} \cdot a + a \cdot ar + ar \cdot \frac{a}{r} = 156$

$\Rightarrow \quad a^2 \left(\frac{1}{r} + r + 1\right) = 156 \Rightarrow 36 \left(\frac{1 + r^2 + r}{r}\right) = 156$

$\Rightarrow \quad 3 \left(\frac{1 + r + r^2}{r}\right) = 13 \Rightarrow 3r^2 - 10r + 3 = 0$

$\Rightarrow \quad (3r - 1)(r - 3) = 0 \Rightarrow r = \frac{1}{3}$ or $r = 3$

Putting the values of a and r, the required numbers are 18, 6, 2 or 2, 6, 18. Hence, the sum of numbers is 26.

Example 44. Find a three-digit number whose consecutive digits form a GP. If we subtract 792 from this number, we get a number consisting of the same digits written in the reverse order. Now, if we increase the second digit of the required number by 2, then the resulting digits will form an AP.

Sol. Let the three digits be a, ar, ar^2, then according to hypothesis

$100a + 10ar + ar^2 - 792 = 100ar^2 + 10ar + a$

$\Rightarrow \quad 99a(1 - r^2) = 792$

$\Rightarrow \quad a(1 + r)(1 - r) = 8$ ...(i)

and $a, ar + 2, ar^2$ are in AP.

Then, $2(ar + 2) = a + ar^2$

$\Rightarrow \quad a(r^2 - 2r + 1) = 4 \Rightarrow a(r - 1)^2 = 4$ ...(ii)

On dividing Eq. (i) by Eq. (ii), we get

$\frac{r + 1}{r - 1} = -2 \Rightarrow r = \frac{1}{3}$

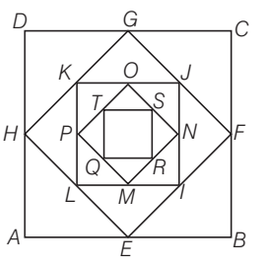
From Eq. (ii), $a = 9$

Thus, digits are 9, 3, 1 and so the required number is 931.

Examples on Application of Progression in Geometrical Figures

Example 45. ***A square is drawn by joining the mid-points of the sides of a given square. A third square is drawn inside the second square in the same way and this process continues indefinitely. If a side of the first square is 16 cm, then determine the sum of the areas of all the squares.***

Sol. Let a be the side length of square, then



$AB = BC = CD = DA = a$

$\because E, F, G, H$ are the mid-points of AB, BC, CD and DA, respectively.

$$\therefore \quad EF = FG = GH = HE = \frac{a}{\sqrt{2}}$$

and I, J, K, L are the mid-points of EF, FG, GH and HE, respectively.

$$\therefore \quad IJ = JK = KL = LI = \frac{a}{2}$$

Similarly, $MN = NO = OP = PM = \frac{a}{2\sqrt{2}}$ and

$$QR = RS = ST = TQ = \frac{a}{4}, \ldots$$

S = Sum of areas

$$= ABCD + EFGH + IJKL + MNOP + QRST + \ldots$$

$$= a^2 + \left(\frac{a}{\sqrt{2}}\right)^2 + \left(\frac{a}{2}\right)^2 + \left(\frac{a}{2\sqrt{2}}\right)^2 + \ldots$$

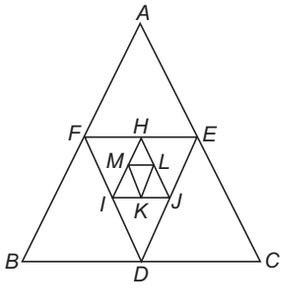
$$= a^2\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\right)$$

$$= a^2\left(\frac{1}{1-\frac{1}{2}}\right) = 2a^2 = 2(16)^2 \qquad [\because a = 16 \text{ cm}]$$

$$= 512 \text{ sq cm}$$

Example 46. One side of an equilateral triangle is 24 cm. The mid-points of its sides are joined to form another triangle whose mid-points, in turn, are joined to form still another triangle. This process continues, indefinitely. Find the sum of the perimeters of all the triangles.

Sol. Let a be the side length of equilateral triangle, then $AB = BC = CA = a$



$\because D, E, F$ are the mid-points of BC, CA and AB, respectively.

$$\therefore \quad EF = FD = DE = \frac{a}{2}$$

and H, I, J are the mid-points of EF, FD and DE, respectively.

$$\therefore \quad IJ = JH = HI = \frac{a}{4}$$

Similarly, $KL = ML = KM = \frac{a}{8}, \ldots$

P = Sum of perimeters $= 3\left(a + \frac{a}{2} + \frac{a}{4} + \frac{a}{8} + \ldots\right)$

$$= 3\left(\frac{a}{1-\frac{1}{2}}\right) = 6a = 6 \times 24 = 144 \text{ cm} \qquad [\because a = 24 \text{ cm}]$$

Example 47. ***Let $S_1, S_2, \ldots$ be squares such that for each $n \geq 1$, the length of a side of S_n equals the length of a diagonal of S_{n+1}. If the length of a side of S_1 is 10 cm and the area of S_n less than 1 sq cm. Then, find the value of n.***

Sol. We have, length of a side of

S_n = length of diagonal of S_{n+1}

$\Rightarrow$ Length of a side of $S_n = \sqrt{2}$ (length of a side of S_{n+1})

$\Rightarrow$ $\frac{\text{Length of a side of } S_{n+1}}{\text{Length of a side of } S_n} = \frac{1}{\sqrt{2}}$, for all $n \geq 1$

$\Rightarrow$ Sides of $S_1, S_2, S_3, \ldots$ form a GP with common ratio $\frac{1}{\sqrt{2}}$ and first term 10.

$$\therefore \quad \text{Side of } S_n = 10\left(\frac{1}{\sqrt{2}}\right)^{n-1} = \frac{10}{2^{\frac{(n-1)}{2}}}$$

$$\Rightarrow \text{ Area of } S_n = (\text{Side})^2 = \frac{100}{2^{n-1}}$$

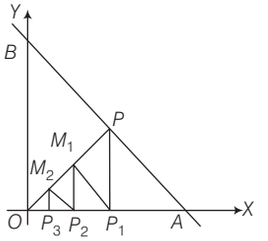
Now, given area of $S_n < 1$

$$\Rightarrow \quad \frac{100}{2^{n-1}} < 1 \Rightarrow 2^{n-1} > 100 > 2^6$$

$$\Rightarrow \quad 2^{n-1} > 2^6 \Rightarrow n - 1 > 6$$

$$\therefore \quad n > 7 \text{ or } n \geq 8$$

Example 48. The line $x + y = 1$ meets X-axis at A and Y-axis at B, P is the mid-point of AB, P_1 is the foot of perpendicular from P to OA, M_1 is that of P_1 from OP; P_2 is that of M_1 from OA, M_2 is that of P_2 from OP; P_3 is that of M_2 from OA and so on. If P_n denotes the nth foot of the perpendicular on OA, then find OP_n.



Sol. We have,

$$(OM_{n-1})^2 = (OP_n)^2 + (P_n M_{n-1})^2$$

$$= (OP_n)^2 + (OP_n)^2 = 2(OP_n)^2 = 2\alpha_n^2 \text{ [say]}$$

Also, $(OP_{n-1})^2 = (OM_{n-1})^2 + (P_{n-1}M_{n-1})^2$

$$\Rightarrow \quad \alpha_{n-1}^2 = 2\alpha_n^2 + \frac{1}{2}\alpha_{n-1}^2 \Rightarrow \alpha_n^2 = \frac{1}{4}\alpha_{n-1}^2$$

$$\Rightarrow \quad \alpha_n = \frac{1}{2}\alpha_{n-1}$$

$$\Rightarrow \quad OP_n = \alpha_n = \frac{1}{2}\alpha_{n-1} = \frac{1}{2^2}\alpha_{n-2} = \ldots = \frac{1}{2^n}$$

$$\therefore \quad OP_n = \left(\frac{1}{2}\right)^n$$

Use of GP in Solving Practical Problems

In this part, we will see how the formulae relating to GP can be made use of in solving practical problems.

Example 49. Dipesh writes letters to four of his friends. He asks each of them to copy the letter and mail to four different persons with the request that they continue the chain similarly. Assuming that the chain is not broken and that it costs 25 paise to mail one letter, find the total money spent on postage till the 8th set of letters is mailed.

Sol. Number of letters in the 1st set = 4 (These are letters sent by Dipesh)

Number of letters in the 2nd set = 4 + 4 + 4 + 4 = 16

Number of letters in the 3rd set

$$= 4 + 4 + 4 + \ldots + 16 \text{ terms} = 64$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

The number of letters sent in the 1st set, 2nd set, 3rd set, ... are respectively 4, 16, 64, ... which is a GP with $a = 4$,

$$r = \frac{16}{4} = \frac{64}{16} = 4$$

$\therefore$ Total number of letters in all the first 8 sets

$$= \frac{4(4^8 - 1)}{4 - 1} = 87380$$

$\therefore$ Total money spent on letters $= 87380 \times \frac{25}{100} = ₹21845$

Example 50. An insect starts from a point and travels in a straight path 1 mm in the first second and half of the distance covered in the previous second in the succeeding second. In how much time would it reach a point 3 mm away from its starting point.

Sol. Distance covered by the insect in the 1st second = 1 mm

Distance covered by it in the 2nd second $= 1 \times \frac{1}{2} = \frac{1}{2}$ mm

Distance covered by it in the 3rd second $= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ mm

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

The distance covered by the insect in 1st second, 2nd second, 3rd second, ... are respectively $1, \frac{1}{2}, \frac{1}{4}, \ldots$, which are in GP with $a = 1, r = \frac{1}{2}$. Let time taken by the insect in covering 3 mm be n seconds.

$$\therefore 1 + \frac{1}{2} + \frac{1}{4} + \ldots + n \text{ terms} = 3$$

$$\Rightarrow \quad \frac{1 \cdot \left[1 - \left(\frac{1}{2}\right)^n\right]}{1 - \frac{1}{2}} = 3$$

$$\Rightarrow \quad 1 - \left(\frac{1}{2}\right)^n = \frac{3}{2}$$

$$\Rightarrow \quad \left(\frac{1}{2}\right)^n = -\frac{1}{2}$$

$$\Rightarrow \quad 2^n = -2$$

which is impossible because $2^n > 0$

$\therefore$ Our supposition is wrong.

$\therefore$ There is no $n \in N$, for which the insect could never 3 mm in n seconds.

Hence, it will never to able to cover 3 mm.

Remark

The maximum distance that the insect could cover is 2 mm.

i.e., $$1 + \frac{1}{2} + \frac{1}{4} + \ldots = \frac{1}{1 - \frac{1}{2}} = 2$$

Example 51. The pollution in a normal atmosphere is less than 0.01%. Due to leakage of a gas from a factory, the pollution is increased to 20%. If every day 80% of the pollution is neutralised, in how many days the atmosphere will be normal?

Sol. Let the pollution on 1st day = 20

The pollution on 2nd day $= 20 \times 20\% = 20(0.20)$

The pollution on 3rd day $= 20(0.20)^2$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

Let in n days the atmosphere will be normal

$$\therefore \quad 20(0.20)^{n-1} < 0.01$$

$$\Rightarrow \quad \left(\frac{2}{10}\right)^{n-1} < \frac{1}{2000}$$

Taking logarithm on base 10, we get

$$(n-1)(\log 2 - \log 10) < \log 1 - \log 2000$$

$$\Rightarrow \quad (n-1)(0.3010 - 1) < 0 - (0.3010 + 3)$$

$$\Rightarrow \quad n - 1 > \frac{3.3010}{0.6990}$$

$$\Rightarrow \quad n > 5.722$$

Hence, the atmosphere will be normal in 6 days.

Exercise for Session 3

1. The fourth, seventh and the last term of a GP are 10, 80 and 2560, respectively. The first term and number of terms in GP are

(a) $\frac{4}{5}$, 12 (b) $\frac{4}{5}$, 10 (c) $\frac{5}{4}$, 12 (d) $\frac{5}{4}$, 10

2. If the first and the nth terms of a GP are a and b respectively and if P is the product of the first n terms, then P^2 is equal to

(a) ab (b) $(ab)^{n/2}$ (c) $(ab)^n$ (d) None of these

3. If $a_1, a_2, a_3, (a_1 > 0)$ are three successive terms of a GP with common ratio r, the value of r for which $a_3 > 4a_2 - 3a_1$ holds is given by

(a) $1 < r < 3$ (b) $-3 < r < -1$ (c) $r < 1$ or $r > 3$ (d) None of these

4. If $x, 2x + 2, 3x + 3$ are in GP, the fourth term is

(a) 27 (b) -27 (c) 13.5 (d) -13.5

5. In a sequence of 21 terms the first 11 terms are in AP with common difference 2 and the last 11 terms are in GP with common ratio 2, if the middle term of the AP is equal to the middle term of GP, the middle term of the entire sequence is

(a) $-\frac{10}{31}$ (b) $\frac{10}{31}$ (c) $-\frac{32}{31}$ (d) $\frac{32}{31}$

6. Three distinct numbers x, y, z form a GP in that order and the numbers $7x + 5y, 7y + 5z, 7z + 5x$ form an AP in that order. The common ratio of GP is

(a) -4 (b) -2 (c) 10 (d) 18

7. The sum to n terms of the series $11 + 103 + 1005 + \ldots$ is

(a) $\frac{1}{9}(10^n - 1) + n^2$ (b) $\frac{1}{9}(10^n - 1) + 2n$ (c) $\frac{10}{9}(10^n - 1) + n^2$ (d) $\frac{10}{9}(10^n - 1) + 2n$

8. In an increasing GP, the sum of the first and last term is 66, the product of the second and the last but one is 128 and the sum of the sum of the terms is 126, then the number of terms in the series is

(a) 6 (b) 8 (c) 10 (d) 12

9. If S_1, S_2, S_3 be respectively the sum of $n, 2n$ and $3n$ terms of a GP, then $\frac{S_1(S_3 - S_2)}{(S_2 - S_1)^2}$ is equal to

(a) 1 (b) 2 (c) 3 (d) 4

10. If $|a| < 1$ and $|b| < 1$, then the sum of the series $1 + (1 + a)b + (1 + a + a^2)b^2 + (1 + a + a^2 + a^3)b^3 + \ldots$ is

(a) $\frac{1}{(1-a)(1-b)}$ (b) $\frac{1}{(1-a)(1-ab)}$ (c) $\frac{1}{(1-b)(1-ab)}$ (d) $\frac{1}{(1-a)(1-b)(1-ab)}$

11. If the sides of a triangle are in GP and its larger angle is twice the smallest, then the common ratio r satisfies the inequality

(a) $0 < r < \sqrt{2}$ (b) $1 < r < \sqrt{2}$ (c) $1 < r < 2$ (d) $r > \sqrt{2}$

12. If $ax^3 + bx^2 + cx + d$ is divisible by $ax^2 + c$, then a, b, c, d are in

(a) AP (b) GP (c) HP (d) None of these

13. If $(r)_n$ denotes the number $r\,r\,r \ldots$ (n digits), where $r = 1, 2, 3, \ldots, 9$ and $a = (6)_n, b = (8)_n, c = (4)_{2n}$, then

(a) $a^2 + b + c = 0$ (b) $a^2 + b - c = 0$ (c) $a^2 + b - 2c = 0$ (d) $a^2 + b - 9c = 0$

14. $0.4\dot{2}\dot{7}$ represents the rational number

(a) $\frac{47}{99}$ (b) $\frac{47}{110}$ (c) $\frac{47}{999}$ (d) $\frac{49}{99}$

15. If the product of three numbers in GP be 216 and their sum is 19, then the numbers are

(a) 4, 6, 9 (b) 4, 7, 8 (c) 3, 7, 9 (d) None of these

Session 4

Harmonic Sequence or Harmonic Progression (HP)

Harmonic Sequence or Harmonic Progression (HP)

A Harmonic Progression (HP) is a sequence, if the reciprocals of its terms are in Arithmetic Progression (AP)

i.e., $t_1, t_2, t_3, \ldots$ is HP if and only if $\frac{1}{t_1}, \frac{1}{t_2}, \frac{1}{t_3}, \ldots$ is an AP.

For example, The sequence

(i) $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \ldots$ (ii) $2, \frac{5}{2}, \frac{10}{3}, \ldots$

(iii) $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \ldots$ are HP's.

Remark

1. No term of HP can be zero.
2. The most general or standard HP is
$$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \frac{1}{a+3d}, \ldots.$$

Example 52. ***If*** a, b, c ***are in HP, then show that*** $\frac{a-b}{b-c} = \frac{a}{c}$.

Sol. Since, a, b, c are in HP, therefore

$\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP

i.e. $\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$

or $\frac{a-b}{ab} = \frac{b-c}{bc}$ or $\frac{a-b}{b-c} = \frac{a}{c}$

Remark

A HP may also be defined as a series in which every three consecutive terms (say I, II, III) satisfy $\frac{\text{I}-\text{II}}{\text{II}-\text{III}} = \frac{\text{I}}{\text{III}}$ this relation.

Example 53. Find the first term of a HP whose second term is $\frac{5}{4}$ and the third term is $\frac{1}{2}$.

Sol. Let a be the first term. Then, $a, \frac{5}{4}, \frac{1}{2}$ are in HP.

Then, $\frac{a - \frac{5}{4}}{\frac{5}{4} - \frac{1}{2}} = \frac{a}{\frac{1}{2}}$ [from above note]

$\Rightarrow \frac{4a-5}{5-2} = 2a$

$\Rightarrow 4a - 5 = 6a$ or $2a = -5$

$\therefore a = -\frac{5}{2}$

(i) nth Term of HP from Beginning

Let a be the first term, d be the common difference of an AP. Then, nth term of an AP from beginning $= a + (n-1)\,d$

Hence, the nth term of HP from beginning

$$= \frac{1}{a + (n-1)\,d}, \forall\, n \in N$$

(ii) nth Term of HP from End

Let l be the last term, d be the common difference of an AP. Then,

nth term of an AP from end $= l - (n-1)\,d$

Hence, the nth term of HP from end $= \frac{1}{l - (n-1)\,d}, \forall\, n \in N$

Remark

1. $\frac{1}{n\text{th term of HP from beginning}} + \frac{1}{n\text{th term of HP from end}}$
$= a + l = \frac{1}{\text{first term of HP}} + \frac{1}{\text{last term of HP}}$
2. There is no general formula for the sum of any number of quantities in HP are generally solved by inverting the terms and making use of the corresponding AP.

Example 54. ***If*** $\frac{1}{a} + \frac{1}{c} + \frac{1}{a-b} + \frac{1}{c-b} = 0$, ***then prove that*** a, b, c ***are in HP, unless*** $b = a + c$.

Sol. We have, $\frac{1}{a} + \frac{1}{c} + \frac{1}{a-b} + \frac{1}{c-b} = 0$

$\Rightarrow \left(\frac{1}{a} + \frac{1}{c-b}\right) + \left(\frac{1}{c} + \frac{1}{a-b}\right) = 0$

$\Rightarrow \frac{(c-b+a)}{a\,(c-b)} + \frac{(a-b+c)}{c\,(a-b)} = 0$

$\Rightarrow (a+c-b)\left[\frac{1}{a(c-b)} + \frac{1}{c(a-b)}\right] = 0$

$\Rightarrow (a+c-b)\,[2ac - b\,(a+c)] = 0$

If $a + c - b \neq 0$, then $2ac - b(a + c) = 0$

or $b = \dfrac{2ac}{a + c}$

Therefore, a, b, c are in HP and if $2ac - b(a + c) \neq 0$, then $a + c - b = 0$ i.e., $b = a + c$.

Example 55. If $a_1, a_2, a_3, \ldots, a_n$ are in HP, then prove that $a_1a_2 + a_2a_3 + a_3a_4 + \ldots + a_{n-1}a_n = (n-1)\, a_1a_n$

Sol. Given, $a_1, a_2, a_3, \ldots, a_n$ are in HP.

$\therefore$ $\dfrac{1}{a_1}, \dfrac{1}{a_2}, \dfrac{1}{a_3}, \ldots, \dfrac{1}{a_n}$ are in AP.

Let D be the common difference of the AP, then

$$\frac{1}{a_2} - \frac{1}{a_1} = \frac{1}{a_3} - \frac{1}{a_2} = \frac{1}{a_4} - \frac{1}{a_3} = \ldots = \frac{1}{a_n} - \frac{1}{a_{n-1}} = D$$

$$\Rightarrow \quad \frac{a_1 - a_2}{a_1a_2} = \frac{a_2 - a_3}{a_2a_3} = \frac{a_3 - a_4}{a_3a_4} = \ldots = \frac{a_{n-1} - a_n}{a_{n-1}a_n} = D$$

$$\Rightarrow \quad a_1a_2 = \frac{a_1 - a_2}{D}, a_2a_3 = \frac{a_2 - a_3}{D}, a_3a_4 = \frac{a_3 - a_4}{D}, \ldots, a_{n-1}a_n = \frac{a_{n-1} - a_n}{D}$$

On adding all such expressions, we get

$$a_1a_2 + a_2a_3 + a_3a_4 + \ldots + a_{n-1}a_n = \frac{a_1 - a_n}{D} = \frac{a_1a_n}{D}\left(\frac{1}{a_n} - \frac{1}{a_1}\right)$$

$$= \frac{a_1a_n}{D}\left[\frac{1}{a_1} + (n-1)\, D - \frac{1}{a_1}\right] = (n-1)\, a_1a_n$$

Hence, $a_1a_2 + a_2a_3 + a_3a_4 + \ldots + a_{n-1}a_n = (n-1)\, a_1a_n$

Remark

In particular case,

1. when $n = 4$ $a_1a_2 + a_2a_3 + a_3a_4 = 3a_1a_4$
2. when $n = 6$ $a_1a_2 + a_2a_3 + a_3a_4 + a_4a_5 + a_5a_6 = 5\, a_1a_6$

Example 56. ***The sum of three numbers in HP is 37 and the sum of their reciprocals is $\frac{1}{4}$. Find the numbers.***

Sol. Three numbers in HP can be taken as $\dfrac{1}{a-d}, \dfrac{1}{a}, \dfrac{1}{a+d}$.

Then, $\dfrac{1}{a-d} + \dfrac{1}{a} + \dfrac{1}{a+d} = 37$ ...(i)

and $a - d + a + a + d = \dfrac{1}{4}$

$\therefore$ $a = \dfrac{1}{12}$

From Eq. (i), $\dfrac{12}{1 - 12d} + 12 + \dfrac{12}{1 + 12d} = 37$

$$\Rightarrow \quad \frac{12}{1 - 12d} + \frac{12}{1 + 12d} = 25 \Rightarrow \frac{24}{1 - 144d^2} = 25$$

$$\Rightarrow \quad 1 - 144d^2 = \frac{24}{25} \text{ or } d^2 = \frac{1}{25 \times 144}$$

$\therefore$ $d = \pm \dfrac{1}{60}$

$\therefore$ $a - d, a, a + d$ are $\dfrac{1}{15}, \dfrac{1}{12}, \dfrac{1}{10}$ or $\dfrac{1}{10}, \dfrac{1}{12}, \dfrac{1}{15}$.

Hence, three numbers in HP are 15, 12, 10 or 10, 12, 15.

Example 57. ***If pth, qth and rth terms of a HP be respectively a, b and c, then prove that $(q - r)\, bc + (r - p)\, ca + (p - q)\, ab = 0$.***

Sol. Let A and D be the first term and common difference of the corresponding AP. Now, a, b, c are respectively the p th, q th and r th terms of HP.

$\therefore \dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}$ will be respectively the p th, q th and r th terms of the corresponding AP.

$$\Rightarrow \quad \frac{1}{a} = A + (p-1)\, D \quad \ldots(i)$$

$$\frac{1}{b} = A + (q-1)\, D \quad \ldots(ii)$$

$$\frac{1}{c} = A + (r-1)\, D \quad \ldots(iii)$$

On subtracting Eq. (iii) from Eq. (ii), we get

$$\frac{1}{b} - \frac{1}{c} = (q - r)\, D \Rightarrow bc\,(q - r) = \frac{(c - b)}{D} = -\frac{(b - c)}{D}$$

So, LHS $= (q - r)\, bc + (r - p)\, ca + (p - q)\, ab$

$$= -\frac{1}{D}\{b - c + c - a + a - b\} = 0 = \text{RHS}$$

Theorem Relating to the Three Series

If a, b, c are three consecutive terms of a series, then

if $\dfrac{a-b}{b-c} = \dfrac{a}{a}$, then a, b, c are in AP.

if $\dfrac{a-b}{b-c} = \dfrac{a}{b}$, then a, b, c are in GP and if $\dfrac{a-b}{b-c} = \dfrac{a}{c}$, then a, b, c are in HP.

Mixed Examples on AP, GP and HP

Example 58. ***If a, b, c are in AP and a^2, b^2, c^2 be in HP. Then, prove that $-\dfrac{a}{2}, b, c$ are in GP or else $a = b = c$.***

Sol. Given, a, b, c are in AP.

$$\therefore \quad b = \frac{a+c}{2} \quad \ldots(i)$$

and a^2, b^2, c^2 are in HP.

$$\therefore \quad b^2 = \frac{2a^2c^2}{a^2+c^2} \quad \ldots(ii)$$

From Eq. (ii) $b^2\{(a+c)^2 - 2ac\} = 2a^2c^2$

$$\Rightarrow \quad b^2\{(2b)^2 - 2ac\} = 2a^2c^2 \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad 2b^4 - acb^2 - a^2c^2 = 0$$

$$\Rightarrow \quad (2b^2 + ac)(b^2 - ac) = 0$$

$$\Rightarrow \quad 2b^2 + ac = 0 \text{ or } b^2 - ac = 0$$

If $2b^2 + ac = 0$, then $b^2 = -\frac{1}{2}ac$ or $-\frac{a}{2}$, b, c are in GP

and if $b^2 - ac = 0 \Rightarrow a, b, c$ are in GP.

But given, a, b, c are in AP.

Which is possible only when $a = b = c$

Example 59. ***If*** a, b, c ***are in HP,*** b, c, d ***are in GP and*** c, d, e ***are in AP, then show that*** $e = \dfrac{ab^2}{(2a-b)^2}$.

Sol. Given, a, b, c are in HP.

$$\therefore \quad b = \frac{2ac}{a+c} \text{ or } c = \frac{ab}{2a-b} \quad \ldots(i)$$

Given, b, c, d are in GP.

$$\therefore \quad c^2 = bd \quad \ldots(ii)$$

and given, c, d, e are in AP.

$$\therefore \quad d = \frac{c+e}{2}$$

$$\Rightarrow \quad e = 2d - c$$

$$e = \left(\frac{2c^2}{b} - c\right) \quad \text{[from Eq. (ii)]} \ldots(iii)$$

From Eqs. (i) and (iii), $e = \dfrac{2}{b}\left(\dfrac{ab}{2a-b}\right)^2 - \left(\dfrac{ab}{2a-b}\right)$

$$= \frac{ab}{(2a-b)^2}\{2a - (2a-b)\}$$

$$= \frac{ab^2}{(2a-b)^2}$$

Example 60. If a, b, c, d and e be five real numbers such that a, b, c are in AP; b, c, d are in GP; c, d, e are in HP. If $a = 2$ and $e = 18$, then find all possible values of b, c and d.

Sol. Given, a, b, c are in AP,

$$\therefore \quad b = \frac{a+c}{2} \quad \ldots(i)$$

b, c, d are in GP,

$$\therefore \quad c^2 = bd \quad \ldots(ii)$$

and c, d, e are in HP.

$$\therefore \quad d = \frac{2ce}{c+e} \quad \ldots(iii)$$

Now, substituting the values of b and d in Eq. (ii), then

$$c^2 = \left(\frac{a+c}{2}\right)\left(\frac{2ce}{c+e}\right)$$

$$\Rightarrow \quad c(c+e) = e(a+c)$$

$$\Rightarrow \quad c^2 = ae \quad \ldots(iv)$$

Given, $a = 2, e = 18$

From Eq. (iv), $c^2 = (2)(18) = 36$

$$\therefore \quad c = \pm 6$$

From Eq. (i), $b = \dfrac{2 \pm 6}{2} = 4, -2$

and from Eq. (ii), $d = \dfrac{c^2}{b} = \dfrac{36}{b} = \dfrac{36}{4} \text{ or } \dfrac{36}{-2}$

$$\therefore \quad d = 9 \text{ or } -18$$

Hence, $c = 6, b = 4, d = 9$ or $c = -6, b = -2, d = -18$

Example 61. If three positive numbers a, b and c are in AP, GP and HP as well, then find their values.

Sol. Since a, b, c are in AP, GP and HP as well

$$\therefore \quad b = \frac{a+c}{2} \quad \ldots(i)$$

$$b^2 = ac \quad \ldots(ii)$$

and $$b = \frac{2ac}{a+c} \quad \ldots(iii)$$

From Eqs. (i) and (ii), we have

$$\left(\frac{a+c}{2}\right)^2 = ac$$

or $$(a+c)^2 = 4ac$$

or $$(a+c)^2 - 4ac = 0$$

or $$(a-c)^2 = 0$$

$$\therefore \quad a = c \quad \ldots(iv)$$

On putting $c = a$ in Eq. (i), we get $b = \dfrac{a+a}{2} = a$ ...(v)

From Eqs. (iv) and (v), $a = b = c$, thus the three numbers will be equal.

Remark

1. If three positive numbers are in any two of AP, GP and HP, then it will be also in third.
2. Thus, if three positive numbers are in any two of AP, GP and HP, then they will be in the third progression and the numbers will be equal.

Exercise for Session 4

1. If a, b, c are in AP and b, c, d be in HP, then

(a) $ab = cd$ (b) $ad = bc$ (c) $ac = bd$ (d) $abcd = 1$

2. If a, b, c are in AP, then $\frac{a}{bc}, \frac{1}{c}, \frac{2}{b}$ are in

(a) AP (b) GP (c) HP (d) None of these

3. If a, b, c are in AP and a, b, d are in GP, then $a, a-b, d-c$ will be in

(a) AP (b) GP (c) HP (d) None of these

4. If $x, 1, z$ are in AP and $x, 2, z$ are in GP, then $x, 4, z$ will be in

(a) AP (b) GP (c) HP (d) None of these

5. If a, b, c are in GP, $a-b, c-a, b-c$ are in HP, then $a + 4b + c$ is equal to

(a) 0 (b) 1 (c) – 1 (d) None of these

6. If $(m+1)\text{th}, (n+1)\text{th}$ and $(r+1)\text{th}$ terms of an AP are in GP and m, n, r are in HP, then the value of the ratio of the common difference to the first term of the AP is

(a) $-\frac{2}{n}$ (b) $\frac{2}{n}$ (c) $-\frac{n}{2}$ (d) $\frac{n}{2}$

7. If a, b, c are in AP and a^2, b^2, c^2 are in HP, then

(a) $a = b = c$ (b) $2b = 3a + c$ (c) $b^2 = \sqrt{\left(\frac{ac}{8}\right)}$ (d) None of these

8. If a, b, c are in HP, then $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in

(a) AP (b) GP (c) HP (d) None of these

9. If $\frac{x+y}{2}, y, \frac{y+z}{2}$ are in HP, then x, y, z are in

(a) AP (b) GP (c) HP (d) None of these

10. If $\frac{a+b}{1-ab}, b, \frac{b+c}{1-bc}$ are in AP, then $a, \frac{1}{b}, c$ are in

(a) AP (b) GP (c) HP (d) None of these

Session 5

Mean

Mean

Arithmetic Mean

If three terms are in AP, then the middle term is called the **Arithmetic Mean** (or shortly written as AM) between the other two, so if a, b, c are in AP, then b is the AM of a and c.

(i) Single AM of n Positive Numbers

Let n positive numbers be $a_1, a_2, a_3, \ldots, a_n$ and A be the AM of these numbers, then

$$A = \frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}$$

In particular Let a and b be two given numbers and A be the AM between them, then a, A, b are in AP.

$$\therefore \quad A = \frac{a+b}{2}$$

Remark

1. AM of $2a, 3b, 5c$ is $\frac{2a + 3b + 5c}{3}$.
2. AM of $a_1, a_2, a_3, \ldots, a_{n-1}, 2a_n$ is $\frac{a_1 + a_2 + a_3 + \ldots + a_{n-1} + 2a_n}{n}$.

(ii) Insert n-Arithmetic Mean Between Two Numbers

Let a and b be two given numbers and $A_1, A_2, A_3, \ldots, A_n$ are AM's between them.

Then, $a, A_1, A_2, A_3, \ldots, A_n, b$ will be in AP.

Now, $b = (n+2)$ th term $= a + (n+2-1)\,d$

$$\therefore \quad d = \left(\frac{b-a}{n+1}\right)$$

[Remember] [where, d = common difference] ...(i)

$$\therefore \quad A_1 = a + d,\ A_2 = a + 2d,\ \ldots,\ A_n = a + nd$$

$$\Rightarrow \quad A_1 = a + \left(\frac{b-a}{n+1}\right),\ A_2 = a + 2\left(\frac{b-a}{n+1}\right),\ \ldots,\ A_n$$

$$= a + n\left(\frac{b-a}{n+1}\right)$$

Corollary **I The sum of n AM's between two given quantities is equal to n times the AM between them.**

Let two numbers be a and b and $A_1, A_2, A_3, \ldots, A_n$ are n AM's between them.

Then, $a, A_1, A_2, A_3, \ldots, A_n, b$ will be in AP.

$\therefore$ Sum of n AM's between a and b

$= A_1 + A_2 + A_3 + \ldots + A_n$

$= \frac{n}{2}(A_1 + A_n)$ [$\because A_1, A_2, A_3, \ldots, A_n$ are in AP]

$= \frac{n}{2}(a + d + a + nd) = \frac{n}{2}[2a + (n+1)\,d]$

$= \frac{n}{2}(2a + b - a)$ [from Eq. (i)]

$= n\left(\frac{a+b}{2}\right) = n$ [AM between a and b]

[Remember]

Aliter $A_1 + A_2 + A_3 + \ldots + A_n$

$= (a + A_1 + A_2 + A_3 + \ldots + A_n + b) - (a + b)$

$= \frac{(n+2)}{2}(a+b) - (a+b) = n\left(\frac{a+b}{2}\right)$

$= n$ [AM of a and b]

Aliter

[This method is applicable only when n is even]

$A_1 + A_2 + A_3 + \ldots + A_{n-2} + A_{n-1} + A_n$

$= (A_1 + A_n) + (A_2 + A_{n-1}) + (A_3 + A_{n-2}) + \ldots$

upto $\frac{n}{2}$ terms

$= (a+b) + (a+b) + (a+b) + \ldots$ upto $\frac{n}{2}$ times

[$\because T_n + T'_n = a + l$]

$= \frac{n}{2}(a+b) = n\left(\frac{a+b}{2}\right) = n$ [AM of a and b]

Corollary **II The sum of m AM's between any two numbers is to the sum of n AM's between them as $m : n$.**

Let two numbers be a and b.

$\therefore$ Sum of m AM's between a and $b = m$ [AM of a and b] ...(i)

Similarly, sum of n AM's between a and $b = n$ [AM of a and b] ...(ii)

$$\therefore \quad \frac{\text{Sum of } m \text{ AM's}}{\text{Sum of } n \text{ AM's}} = \frac{m(\text{AM of } a \text{ and } b)}{n(\text{AM of } a \text{ and } b)} = \frac{m}{n}$$

Example 62. ***If*** a, b, c ***are in AP and*** p ***is the AM between*** a ***and*** b ***and*** q ***is the AM between*** b ***and*** c, ***then show that*** b ***is the AM between*** p ***and*** q.

Sol. $\because$ a, b, c are in AP.

$$\therefore \quad 2b = a + c \qquad \ldots(i)$$

$\because$ p is the AM between a and b.

$$\therefore \quad p = \frac{a+b}{2} \qquad \ldots(ii)$$

$\because$ q is the AM between b and c.

$$\therefore \quad q = \frac{b+c}{2} \qquad \ldots(iii)$$

On adding Eqs. (ii) and (iii), then

$$p + q = \frac{a+b}{2} + \frac{b+c}{2} = \frac{a+c+2b}{2} = \frac{2b+2b}{2} \quad \text{[using Eq. (i)]}$$

$$\therefore \quad p + q = 2b \text{ or } b = \frac{p+q}{2}$$

Hence, b is the AM between p and q.

Example 63. Find n, so that $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}(a \neq b)$ be the AM between a and b.

Sol. $\because$ $$\frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \frac{a+b}{2}$$

$$\Rightarrow \quad \frac{b^{n+1}\left[\left(\frac{a}{b}\right)^{n+1}+1\right]}{b^n\left[\left(\frac{a}{b}\right)^n+1\right]} = \frac{b}{2}\left[\left(\frac{a}{b}\right)+1\right]$$

$$\Rightarrow \quad 2\left[\left(\frac{a}{b}\right)^{n+1}+1\right] = \left[\left(\frac{a}{b}\right)^n+1\right]\left(\frac{a}{b}+1\right)$$

Let $$\frac{a}{b} = \lambda$$

$$\therefore \quad 2\lambda^{n+1} + 2 = (\lambda^n + 1)(\lambda + 1)$$

$$\Rightarrow \quad 2\lambda^{n+1} + 2 = \lambda^{n+1} + \lambda^n + \lambda + 1$$

$$\Rightarrow \quad \lambda^{n+1} - \lambda^n - \lambda + 1 = 0 \Rightarrow (\lambda^n - 1)(\lambda - 1) = 0$$

$$\lambda - 1 \neq 0 \qquad [\because a \neq b]$$

$$\therefore \quad \lambda^n - 1 = 0 \Rightarrow \lambda^n = 1 = \lambda^0$$

$$\Rightarrow \quad n = 0$$

Example 64. There are n AM's between 3 and 54 such that 8th mean is to $(n-2)$ th mean as 3 to 5. Find n.

Sol. Let $A_1, A_2, A_3, \ldots, A_n$ be n AM's between 3 and 54.

If d be the common difference, then

$$d = \frac{54-3}{n+1} = \frac{51}{n+1} \qquad \ldots(i)$$

According to the example,

$$\frac{A_8}{A_{n-2}} = \frac{3}{5}$$

$$\Rightarrow \quad 5(3+8d) = 3[3+(n-2)d] \Rightarrow 6 = d(3n-46)$$

$$\Rightarrow \quad 6 = (3n-46)\frac{51}{(n+1)} \qquad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad 6n + 6 = 153n - 2346 \Rightarrow 147n = 2352$$

$$\therefore \quad n = 16$$

Example 65. If 11 AM's are inserted between 28 and 10, then find the three middle terms in the series.

Sol. Let $A_1, A_2, A_3, \ldots, A_{11}$ be 11 AM's between 28 and 10.

If d be the common difference, then

$$d = \frac{10-28}{12} = -\frac{3}{2}$$

Total means = 11 (odd)

$$\therefore \quad \text{Middle mean} = \left(\frac{11+1}{2}\right)\text{th} = 6\text{th} = A_6$$

Then, three middle terms are A_5, A_6 and A_7.

$$\therefore \quad A_5 = 28 + 5d = 28 - \frac{15}{2} = \frac{41}{2}$$

$$A_6 = 28 + 6d = 28 - 9 = 19$$

and $$A_7 = 28 + 7d = 28 - \frac{21}{2} = \frac{35}{2}$$

Example 66. If a, b, c are in AP, then show that $a^2(b+c) + b^2(c+a) + c^2(a+b) = \frac{2}{9}(a+b+c)^3$.

Sol. $\because$ a, b, c are in AP.

$$\therefore \quad b = \frac{a+c}{2} \text{ i.e., } 2b = a + c \qquad \ldots(i)$$

$$\text{LHS} = a^2(b+c) + b^2(c+a) + c^2(a+b)$$

$$= (a^2b + a^2c) + b^2(2b) + (c^2a + c^2b)$$

$$= b(a^2 + c^2) + ac(a+c) + 2b^3$$

$$= b[(a+c)^2 - 2ac] + ac(2b) + 2b^3$$

$$= b(a+c)^2 + 2b^3 = b(2b)^2 + 2b^3 = 6b^3$$

$$\text{RHS} = \frac{2}{9}(a+b+c)^3 = \frac{2}{9}(2b+b)^3$$

$$= \frac{2}{9} \times 27b^3 = 6b^3$$

Hence, LHS = RHS

Geometric Mean

If three terms are in GP, then the middle term is called the **Geometric Mean** (or shortly written as GM) between the other two, so if a, b, c are in GP, then b is the GM of a and c.

(i) Single GM of n Positive Numbers

Let n positive numbers be $a_1, a_2, a_3, \ldots, a_n$ and G be the GM of these numbers, then $G = (a_1 a_2 a_3 \ldots a_n)^{1/n}$

In particular Let a and b be two numbers and G be the GM between them, then a, G, b are in GP.
Hence, $G = \sqrt{ab}$; $a > 0, b > 0$

Remark

1. If $a < 0$, $b < 0$, then $G = -\sqrt{ab}$
2. If $a < 0$, $b > 0$ or $a > 0$, $b < 0$, then GM between a and b does not exist.

Example

(i) The GM between 4 and 9 is given by

$$G = \sqrt{4 \times 9} = 6$$

(ii) The GM between -4 and -9 is given by

$$G = \sqrt{-4 \times -9} = -6$$

(iii) The GM between -4 and 9 or 4 and -9 does not exist.

i.e. $\sqrt{(-4) \times 9} = \sqrt{-1}\,\sqrt{36} = 6i$

and $\sqrt{4 \times (-9)} = \sqrt{-1}\,\sqrt{36} = 6i$

(ii) Insert n-Geometric Mean Between Two Numbers

Let a and b be two given numbers and $G_1, G_2, G_3, \ldots, G_n$ are n GM's between them.

Then, $a, G_1, G_2, G_3, \ldots, G_n, b$ will be in GP.

Now, $b = (n+2)$ th term $= ar^{n+2-1}$

$$\therefore \quad r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}} \text{ [where } r = \text{common ratio] [Remember]} \quad \ldots(i)$$

$$\therefore \quad G_1 = ar, G_2 = ar^2, \ldots, G_n = ar^n$$

$$\Rightarrow \quad G_1 = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}, G_2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}, \ldots, \; G_n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

Corollary **The product of n geometric means between a and b is equal to the nth power of the geometric mean between a and b.**

Let two numbers be a and b and $G_1, G_2, G_3, \ldots, G_n$ are n GM's between them.

Then, $a, G_1, G_2, G_3, \ldots, G_n, b$ will be in GP.

$\therefore$ Product of n GM's between a and b

$$= G_1 G_2 G_3 \ldots G_n = (ar)(ar^2)(ar^3)\ldots(ar^n)$$

$$= a^{1+1+1+\ldots+1} \cdot r^{1+2+3+\ldots+n}$$

$$= a^n \cdot r^{\frac{n(n+1)}{2}} = a^n \cdot \left[\left(\frac{b}{a}\right)^{\frac{1}{n+1}}\right]^{\frac{n(n+1)}{2}} \quad \text{[from Eq. (i)]}$$

$$= a^n \left(\frac{b}{a}\right)^{\frac{n}{2}} = a^{n/2} b^{n/2} = (\sqrt{ab})^n$$

$$= [\text{GM of } a \text{ and } b]^n \quad \text{[Remember]}$$

Aliter [This method is applicable only when n is even]

$$G_1 G_2 G_3 \ldots G_{n-2}\, G_{n-1}\, G_n = (G_1\, G_n)(G_2 G_{n-1})(G_3 G_{n-2}) \ldots \frac{n}{2} \text{ factors}$$

$$= (ab)(ab)(ab)\ldots\frac{n}{2} \text{ factors} \quad [\because T_n \times T_n' = a \times l]$$

$$= (ab)^{n/2} = (\sqrt{ab})^n = [\text{GM of } a \text{ and } b]^n$$

Example 67. ***If a be one AM and G_1 and G_2 be two geometric means between b and c, then prove that $G_1^3 + G_2^3 = 2abc$.***

Sol. Given, a = AM between b and c

$$\Rightarrow \quad a = \frac{b+c}{2} \Rightarrow 2a = b + c \quad \ldots(i)$$

Again, b, G_1, G_2, c are in GP.

$$\therefore \quad \frac{G_1}{b} = \frac{G_2}{G_1} = \frac{c}{G_2} \Rightarrow b = \frac{G_1^2}{G_2}, c = \frac{G_2^2}{G_1}$$

and $G_1 G_2 = bc$ ...(ii)

From Eqs. (i) and (ii),

$$2a = \frac{G_1^2}{G_2} + \frac{G_2^2}{G_1} = \frac{G_1^3 + G_2^3}{G_1 G_2} = \frac{G_1^3 + G_2^3}{bc} \quad [\because G_1 G_2 = bc]$$

$$\Rightarrow \quad G_1^3 + G_2^3 = 2abc$$

Example 68. ***If one geometric mean G and two arithmetic means p and q be inserted between two quantities, then show that $G^2 = (2p - q)(2q - p)$.***

Sol. Let the two quantities be a and b, then

$$G^2 = ab \quad \ldots(i)$$

Again, a, p, q, b are in AP.

$$\therefore \quad p - a = q - p = b - q$$

$$\Rightarrow \quad a = 2p - q$$

$$b = 2q - p \quad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$G^2 = (2p - q)(2q - p)$$

Example 69. ***Find n, so that $\frac{a^{n+1} + b^{n+1}}{a^n + b^n} (a \neq b)$ be the GM between a and b.***

Sol. $\because \quad \frac{a^{n+1} + b^{n+1}}{a^n + b^n} = \sqrt{ab}$

$$\Rightarrow \quad \frac{b^{n+1}\left[\left(\frac{a}{b}\right)^{n+1}+1\right]}{b^{n}\left[\left(\frac{a}{b}\right)^{n}+1\right]}=b\sqrt{\frac{a}{b}} \quad \Rightarrow \quad \frac{\left(\frac{a}{b}\right)^{n+1}+1}{\left(\frac{a}{b}\right)^{n}+1}=\left(\frac{a}{b}\right)^{\frac{1}{2}}$$

Let $\frac{a}{b}=\lambda$

$$\Rightarrow \quad \frac{\lambda^{n+1}+1}{\lambda^{n}+1}=\lambda^{\frac{1}{2}} \Rightarrow \lambda^{n+1}+1=\lambda^{n+\frac{1}{2}}+\lambda^{\frac{1}{2}}$$

$$\Rightarrow \quad \lambda^{n+\frac{1}{2}}(\lambda^{\frac{1}{2}}-1)-(\lambda^{\frac{1}{2}}-1)=0$$

$$\Rightarrow \quad (\lambda^{\frac{1}{2}}-1)(\lambda^{n+\frac{1}{2}}-1)=0$$

$$\Rightarrow \quad \lambda^{\frac{1}{2}}-1\neq 0 \qquad [\because a\neq b]$$

$$\therefore \quad \lambda^{n+\frac{1}{2}}-1=0$$

$$\Rightarrow \quad \lambda^{n+\frac{1}{2}}=1=\lambda^{0}$$

$$\Rightarrow \quad n+\frac{1}{2}=0 \text{ or } n=-\frac{1}{2}$$

Example 70. Insert five geometric means between $\frac{1}{3}$ and 9 and verify that their product is the fifth power of the geometric mean between $\frac{1}{3}$ and 9.

Sol. Let G_1, G_2, G_3, G_4, G_5 be 5 GM's between $\frac{1}{3}$ and 9.

Then, $\frac{1}{3}, G_1, G_2, G_3, G_4, G_5, 9$ are in GP.

Here, r = common ratio $=\left(\frac{9}{\frac{1}{3}}\right)^{1/6}=3^{\frac{1}{2}}=\sqrt{3}$

$$\therefore \quad G_1=ar=\frac{1}{3}\cdot\sqrt{3}=\frac{1}{\sqrt{3}}$$

$$G_2=ar^2=\frac{1}{3}\cdot 3=1$$

$$G_3=ar^3=\frac{1}{3}\cdot 3\sqrt{3}=\sqrt{3}$$

$$G_4=ar^4=\frac{1}{3}\cdot 9=3$$

$$G_5=ar^5=\frac{1}{3}\cdot 9\sqrt{3}=3\sqrt{3}$$

Now, Product $= G_1\times G_2\times G_3\times G_4\times G_5$

$$=\frac{1}{\sqrt{3}}\times 1\times\sqrt{3}\times 3\times 3\sqrt{3}=9\sqrt{3}=(3)^{\frac{5}{2}}=\left(\sqrt{\frac{1}{3}\times 9}\right)^{5}$$

$$=\left[\text{GM of } \frac{1}{3} \text{ and } 9^{5}\right]$$

An Important Theorem

Let a and b be two real, positive and unequal numbers and A, G are arithmetic and geometric means between them, then

(i) a and b are the roots of the equation

$$x^2-2Ax+G^2=0 \qquad \text{[Remember]}$$

(ii) a and b are given by $A\pm\sqrt{(A+G)(A-G)}$ [Remember]

(iii) $A>G$ [Remember]

Proof $\because A$ is the AM between a and b, then

$$A=\frac{a+b}{2} \Rightarrow a+b=2A \qquad \ldots(i)$$

and G is the GM between a and b, then

$$G=\sqrt{ab} \Rightarrow ab=G^2 \qquad \ldots(ii)$$

$\therefore a$ and b are the roots of the equation, then

x^2- (sum of roots) $x+$ product of roots $=0$

$$\Rightarrow \quad x^2-(a+b)x+ab=0$$

i.e. $x^2-2Ax+G^2=0$ is the required equation.

$$\Rightarrow \quad x=\frac{2A\pm\sqrt{(-2A)^2-4\cdot 1\cdot G^2}}{2\cdot 1}=A\pm\sqrt{(A^2-G^2)}$$

$$\therefore \quad x=A\pm\sqrt{(A+G)(A-G)}$$

Now, for real, positive and unequal numbers of a and b,

$$(A+G)(A-G)>0 \Rightarrow (A-G)>0$$

$$\therefore \quad A>G$$

Remark

1. If a and b are real and positive numbers, then $A\geq G$
2. If $a_1, a_2, a_3, \ldots, a_n$ are n positive numbers, then AM $\geq$ GM i.e.,
$$\frac{a_1+a_2+a_3+\ldots+a_n}{n}\geq (a_1a_2a_3\ldots a_n)^{1/n}$$
3. (i) If $a>0, b>0$ or $a<0, b<0$ and $\lambda_1>0, \lambda_2>0$, then
$\lambda_1\frac{a}{b}+\lambda_2\frac{b}{a}\geq 2\sqrt{\lambda_1\lambda_2}$
if $\frac{a}{b}=x>0$ and $\lambda_1=\lambda_2=1$, then $x+\frac{1}{x}\geq 2$
(ii) If $a>0, b<0$ or $a<0, b>0$ and $\lambda_1>0, \lambda_2>0$, then
$\lambda_1\frac{a}{b}+\lambda_2\frac{b}{a}\leq -2\sqrt{\lambda_1\lambda_2}$
if $\frac{a}{b}=x<0$ and $\lambda_1>0, \lambda_2>0$ then, $x+\frac{1}{x}\leq -2$

Example 71. AM between two numbers whose sum is 100 is to the GM as 5 : 4, find the numbers.

Sol. Let the numbers be a and b.

Then, $a+b=100$

or $2A=100$

$\Rightarrow \quad A = 50$ ...(i)

and given, $\frac{A}{G} = \frac{5}{4} \Rightarrow \frac{50}{G} = \frac{5}{4}$ [from Eq. (i)]

$\therefore \quad G = 40$ ...(ii)

From important theorem $a, b = A \pm \sqrt{(A+G)(A-G)}$

$= 50 \pm \sqrt{(50+40)(50-40)}$

$= 50 \pm 30 = 80, 20$

$\therefore \quad a = 80, b = 20$

or $\quad a = 20, b = 80$

Example 72. If $a_1, a_2, \ldots, a_n$ are positive real numbers whose product is a fixed number c, then find the minimum value of $a_1 + a_2 + \ldots + a_{n-1} + 3a_n$.

Sol. $\because$ AM $\geq$ GM

$$\therefore \frac{a_1 + a_2 + \ldots + a_{n-1} + 3a_n}{n} \geq (a_1 a_2 \ldots a_{n-1} 3a_n)^{1/n} = (3c)^{1/n}$$

$$\Rightarrow a_1 + a_2 + \ldots + a_{n-1} + 3a_n \geq n\,(3c)^{1/n}$$

Hence, the minimum value of $a_1 + a_2 + \ldots + a_{n-1} + 3a_n$ is $n\,(3c)^{1/n}$.

Harmonic Mean

If three terms are in HP, then the middle term is called the **Harmonic Mean** (or shortly written as HM) between the other two, so if a, b, c are in HP, then b is the HM of a and c.

(i) Single HM of n Positive Numbers

Let n positive numbers be $a_1, a_2, a_3, \ldots, a_n$ and H be the HM of these numbers, then

$$H = \frac{n}{\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots + \frac{1}{a_n}\right)}$$

In particular Let a and b be two given numbers and H be the HM between them a, H, b are in HP.

Hence, $H = \frac{2}{\frac{1}{a} + \frac{1}{b}}$ i.e., $H = \frac{2ab}{(a+b)}$

Remark

HM of a, b, c is $\frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}$ or $\frac{3abc}{ab + bc + ca}$.

Caution The AM between two numbers a and b is $\frac{a+b}{2}$. It does not follow that HM between the same numbers is $\frac{2}{a+b}$. The HM is the reciprocals of $\frac{\frac{1}{a} + \frac{1}{b}}{2}$ i.e., $\frac{2ab}{(a+b)}$.

(ii) Insert n-Harmonic Mean Between Two Numbers

Let a and b be two given numbers and $H_1, H_2, H_3, \ldots, H_n$ are n HM's between them.

Then, $a, H_1, H_2, H_3, \ldots, H_n, b$ will be in HP, if D be the common difference of the corresponding AP.

$\therefore b = (n+2)$ th term of HP.

$$= \frac{1}{(n+2)\text{th term of corresponding AP}}$$

$$= \frac{1}{\frac{1}{a} + (n+2-1)\,D}$$

$$\Rightarrow \quad D = \frac{\frac{1}{b} - \frac{1}{a}}{(n+1)} \qquad \text{[Remember]}$$

$$\therefore \quad \frac{1}{H_1} = \frac{1}{a} + D,\ \frac{1}{H_2} = \frac{1}{a} + 2D, \ldots, \frac{1}{H_n} = \frac{1}{a} + nD$$

$$\Rightarrow \quad \frac{1}{H_1} = \frac{1}{a} + \frac{(a-b)}{ab\,(n+1)}, \frac{1}{H_2} = \frac{1}{a} + \frac{2\,(a-b)}{ab\,(n+1)}, \ldots, \frac{1}{H_n}$$

$$= \frac{1}{a} + \frac{n\,(a-b)}{ab\,(n+1)}$$

Corollary **The sum of reciprocals of n harmonic means between two given numbers is n times the reciprocal of single HM between them.**

Let two numbers be a and b and $H_1, H_2, H_3, \ldots, H_n$ are n HM's between them. Then, $a, H_1, H_2, H_3, \ldots, H_n, b$ will be in HP.

$$\therefore \quad \frac{1}{H_1} + \frac{1}{H_2} + \frac{1}{H_3} + \ldots + \frac{1}{H_n} = \frac{n}{2}\left(\frac{1}{H_1} + \frac{1}{H_n}\right) \qquad \left[\because S_n = \frac{n}{2}(a+l)\right]$$

$$= \frac{n}{2}\left(\frac{1}{a} + D + \frac{1}{b} - D\right) = \frac{n}{2}\left(\frac{1}{a} + \frac{1}{b}\right)$$

$$= \frac{n}{\left(\frac{2}{\frac{1}{a} + \frac{1}{b}}\right)} = \frac{n}{[\text{HM of } a \text{ and } b]}$$

Aliter [This method is applicable only when n is even]

$$\frac{1}{H_1} + \frac{1}{H_2} + \frac{1}{H_3} + \ldots + \frac{1}{H_{n-2}} + \frac{1}{H_{n-1}} + \frac{1}{H_n}$$

$$= \left(\frac{1}{H_1} + \frac{1}{H_n}\right) + \left(\frac{1}{H_2} + \frac{1}{H_{n-1}}\right) + \left(\frac{1}{H_3} + \frac{1}{H_{n-2}}\right) + \ldots \text{upto } \frac{n}{2} \text{ terms}$$

$$=\left(\frac{1}{a}+D+\frac{1}{b}-D\right)+\left(\frac{1}{a}+2D+\frac{1}{b}-2D\right)$$
$$+\left(\frac{1}{a}+3D+\frac{1}{b}-3D\right)+\ldots\text{upto }\frac{n}{2}\text{ terms}$$
$$=\left(\frac{1}{a}+\frac{1}{b}\right)+\left(\frac{1}{a}+\frac{1}{b}\right)+\left(\frac{1}{a}+\frac{1}{b}\right)+\ldots\text{ upto }\frac{n}{2}\text{ terms}$$
$$=\frac{n}{2}\left(\frac{1}{a}+\frac{1}{b}\right)=\frac{n}{\left(\frac{2}{\frac{1}{a}+\frac{1}{b}}\right)}=\frac{n}{(\text{HM of } a \text{ and } b)}$$

Example 73. If H be the harmonic mean between x and y, then show that $\frac{H+x}{H-x}+\frac{H+y}{H-y}=2$

Sol. We have, $H=\frac{2xy}{x+y}$

$\therefore \quad \frac{H}{x}=\frac{2y}{x+y}$ and $\frac{H}{y}=\frac{2x}{x+y}$

By componendo and dividendo, we have

$$\frac{H+x}{H-x}=\frac{2y+x+y}{2y-x-y}=\frac{x+3y}{y-x}$$

and $$\frac{H+y}{H-y}=\frac{2x+x+y}{2x-x-y}=\frac{3x+y}{x-y}$$

$$\therefore \quad \frac{H+x}{H-x}+\frac{H+y}{H-y}=\frac{x+3y}{y-x}+\frac{3x+y}{x-y}$$
$$=\frac{x+3y-3x-y}{y-x}=\frac{2(y-x)}{(y-x)}=2$$

Aliter $\frac{H+x}{H-x}+\frac{H+y}{H-y}=2$

$$\Rightarrow\left(\frac{H+x}{H-x}-1\right)=\left(1-\frac{H+y}{H-y}\right)\Rightarrow\frac{2x}{H-x}=\frac{-2y}{H-y}$$

i.e. $Hx-xy=-Hy+xy \Rightarrow H(x+y)=2xy$

i.e. $H=\frac{2xy}{(x+y)}$

which is true as, x, H, y are in HP. Hence, the required result.

Example 74. ***If*** $a_1, a_2, a_3, \ldots, a_{10}$ ***be in AP and*** $h_1, h_2, h_3, \ldots, h_{10}$ ***be in HP. If*** $a_1=h_1=2$ ***and*** $a_{10}=h_{10}=3$***, then find the value of*** $a_4 h_7$.

Sol. $\because$ $a_1, a_2, a_3, \ldots, a_{10}$ are in AP.

If d be the common difference, then

$$d=\frac{a_{10}-a_1}{9}=\frac{3-2}{9}=\frac{1}{9}$$

$$\therefore \quad a_4=a_1+3d=2+\frac{3}{9}=2+\frac{1}{3}=\frac{7}{3} \qquad \ldots(\text{i})$$

and given $h_1, h_2, h_3, \ldots, h_{10}$ are in HP.

If D be common difference of corresponding AP.

Then, $$D=\frac{\frac{1}{h_{10}}-\frac{1}{h_1}}{9}=\frac{\frac{1}{3}-\frac{1}{2}}{9}=-\frac{1}{54}$$

$$\therefore \quad \frac{1}{h_7}=\frac{1}{h_1}+6D=\frac{1}{2}-\frac{6}{54}=\frac{1}{2}-\frac{1}{9}=\frac{7}{18}\Rightarrow h_7=\frac{18}{7}$$

Hence, $$a_4\cdot h_7=\frac{7}{3}\times\frac{18}{7}=6$$

Example 75. Find n, so that $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}(a\neq b)$ be HM between a and b.

Sol. $\because$ $$\frac{a^{n+1}+b^{n+1}}{a^n+b^n}=\frac{2ab}{a+b}$$

$$\Rightarrow \quad \frac{b^{n+1}\left[\left(\frac{a}{b}\right)^{n+1}+1\right]}{b^n\left[\left(\frac{a}{b}\right)^n+1\right]}=\frac{b^2\left[2\left(\frac{a}{b}\right)\right]}{b\left(\frac{a}{b}+1\right)}$$

$$\Rightarrow \quad \frac{\left(\frac{a}{b}\right)^{n+1}+1}{\left(\frac{a}{b}\right)^n+1}=\frac{2\left(\frac{a}{b}\right)}{\left(\frac{a}{b}\right)+1}$$

Let $$\frac{a}{b}=\lambda$$

Then, $$\frac{\lambda^{n+1}+1}{\lambda^n+1}=\frac{2\lambda}{\lambda+1}$$

$$\Rightarrow \quad (\lambda+1)(\lambda^{n+1}+1)=2\lambda(\lambda^n+1)$$
$$\Rightarrow \quad \lambda^{n+2}+\lambda+\lambda^{n+1}+1=2\lambda^{n+1}+2\lambda$$
$$\Rightarrow \quad \lambda^{n+2}-\lambda^{n+1}-\lambda+1=0$$
$$\Rightarrow \quad \lambda^{n+1}(\lambda-1)-1(\lambda-1)=0$$
$$\Rightarrow \quad (\lambda-1)(\lambda^{n+1}-1)=0$$
$$\Rightarrow \quad \lambda-1\neq 0 \qquad [\because a\neq b]$$
$$\therefore \quad \lambda^{n+1}-1=0$$
$$\Rightarrow \quad \lambda^{n+1}=1=\lambda^0$$
$$\Rightarrow \quad n+1=0 \text{ or } n=-1$$

Example 76. Insert 6 harmonic means between 3 and $\frac{6}{23}$.

Sol. Let $H_1, H_2, H_3, H_4, H_5, H_6$ be 6 HM's between 3 and $\frac{6}{23}$.

Then, $3, H_1, H_2, H_3, H_4, H_5, H_6, \frac{6}{23}$ are in HP.

$$\Rightarrow \quad \frac{1}{3}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \frac{1}{H_4}, \frac{1}{H_5}, \frac{1}{H_6}, \frac{23}{6} \text{ are in AP.}$$

Let common difference of this AP be D.

$$\therefore \quad D = \frac{\frac{23}{6} - \frac{1}{3}}{7} = \frac{(23-2)}{7\times 6} = \frac{21}{7\times 6} = \frac{1}{2}$$

$$\therefore \quad \frac{1}{H_1} = \frac{1}{3} + D = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

$$\Rightarrow \quad H_1 = \frac{6}{5} = 1\frac{1}{5}$$

$$\frac{1}{H_2} = \frac{1}{3} + 2D = \frac{1}{3} + 1 = \frac{4}{3} \Rightarrow H_2 = \frac{3}{4}$$

$$\frac{1}{H_3} = \frac{1}{3} + 3D = \frac{1}{3} + \frac{3}{2} = \frac{11}{6} \Rightarrow H_3 = \frac{6}{11}$$

$$\frac{1}{H_4} = \frac{1}{3} + 4D = \frac{1}{3} + 2 = \frac{7}{3} \Rightarrow H_4 = \frac{3}{7}$$

$$\frac{1}{H_5} = \frac{1}{3} + 5D = \frac{1}{3} + \frac{5}{2} = \frac{17}{6} \Rightarrow H_5 = \frac{6}{17}$$

and $$\frac{1}{H_6} = \frac{1}{3} + 6D = \frac{1}{3} + 3 = \frac{10}{3} \Rightarrow H_6 = \frac{3}{10}$$

$\therefore$ HM's are $1\frac{1}{5}, \frac{3}{4}, \frac{6}{11}, \frac{3}{7}, \frac{6}{17}, \frac{3}{10}$.

Important Theorem 1

Let a and b be two real, positive and unequal numbers and A, G and H are arithmetic, geometric and harmonic means respectively between them, then

(i) A, G, H form a GP i.e., $G^2 = AH$ [Remember]

(ii) $A > G > H$ [Remember]

Proof

(i) $\because$ $$A = \frac{a+b}{2}, G = \sqrt{ab} \text{ and } H = \frac{2ab}{a+b}$$

Now, $$AH = \left(\frac{a+b}{2}\right)\left(\frac{2ab}{a+b}\right) = ab = G^2$$

Therefore, $G^2 = AH$ i.e. A, G, H are in GP.

Remark

The result $AH = G^2$ will be true for n numbers, if they are in GP.

(ii) $\because$ $A > G$ [from important theorem of GM] ...(i)

or $$\frac{A}{G} > 1$$

$$\Rightarrow \quad \frac{G}{H} > 1 \qquad \left[\because \frac{A}{G} = \frac{G}{H} \Rightarrow G^2 = AH\right]$$

$$\Rightarrow \quad G > H \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$A > G > H$$

Remark

If $a_1, a_2, a_3, \ldots, a_n$ are n positive numbers, then AM ≥ GM ≥ HM i.e.,

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \geq (a_1 a_2 \ldots a_n)^{1/n} \geq \frac{n}{\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right)}$$

Sign of equality (AH = GM = HM) holds when numbers are equal i.e., $a_1 = a_2 = \ldots = a_n$.

Important Theorem 2

If A, G, H are arithmetic, geometric and harmonic means of three given numbers a, b and c, then the equation having a, b, c as its roots is

$$\boldsymbol{x^3 - 3Ax^2 + \frac{3G^3}{H}x - G^3 = 0} \quad \text{[Remember]}$$

Proof $\because A = \text{AM of } a, b, c = \frac{a+b+c}{3}$

i.e., $$a + b + c = 3A \ldots \text{(i)}$$

$$G = \text{GM of } a, b, c = (abc)^{1/3}$$

i.e. $$abc = G^3 \qquad \text{...(ii)}$$

and $$H = \text{HM of } a, b, c$$

$$= \frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} = \frac{3abc}{ab + bc + ca} = \frac{3G^3}{ab + bc + ca}$$

[from Eq. (ii)]

i.e. $$ab + bc + ca = \frac{3G^3}{H} \qquad \text{...(iii)}$$

$\therefore a, b, c$ are the roots of the equation

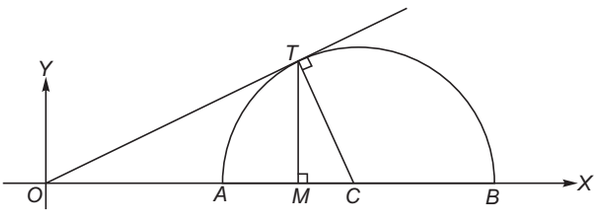
$$x^3 - (a+b+c)\,x^2 + (ab + bc + ca)\,x - abc = 0$$

i.e., $$x^3 - 3Ax^2 + \frac{3G^3}{H}x - G^3 = 0$$

[from Eqs. (i), (ii) and (iii)]

Geometrical Proof of $A > G > H$

Let $OA = a$ unit and $OB = b$ unit and AB be a diameter of semi-circle. Draw tangent OT to the circle and TM perpendicular to AB.



Let C be the centre of the semi-circle.

$\because$ $$\frac{OA + OB}{2} = \frac{(OC - AC) + (OC + CB)}{2}$$

$$= \frac{2\,OC}{2} = OC \qquad [\because AC = CB = \text{radius of circle}]$$

$\therefore \quad OC = \dfrac{a+b}{2}$ [i.e. OC = arithmetic mean]

$\Rightarrow \quad A = \dfrac{a+b}{2}$

Now, from geometry

$$(OT)^2 = OA \times OB = ab = G^2$$

$\therefore \quad OT = G$, the geometric mean

Now, from similar ΔOCT and ΔOMT, we have

$$\frac{OM}{OT} = \frac{OT}{OC} \text{ or } OM = \frac{(OT)^2}{OC} = \frac{ab}{\frac{a+b}{2}} = \frac{2ab}{a+b}$$

$\therefore \quad OM = H$, the harmonic mean

Also, it is clear from the figure, that

$$OC > OT > OM \text{ i.e. } A > G > H$$

Example 77. If $A^x = G^y = H^z$, where A, G, H are AM, GM and HM between two given quantities, then prove that x, y, z are in HP.

Sol. Let $A^x = G^y = H^z = k$

Then, $A = k^{1/x}, G = k^{1/y}, H = k^{1/z}$

$\because \quad G^2 = AH \Rightarrow (k^{1/y})^2 = k^{1/x} \cdot k^{1/z}$

$\Rightarrow \quad k^{2/y} = k^{1/x + 1/z} \Rightarrow \dfrac{2}{y} = \dfrac{1}{x} + \dfrac{1}{z} \Rightarrow \dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}$ are in AP.

Hence, x, y, z are in HP.

Example 78. The harmonic mean of two numbers is 4, their arithmetic mean A and geometric mean G satisfy the relation $2A + G^2 = 27$. Find the numbers.

Sol. Let the numbers be a and b.

Given, $H = 4$

$\because \quad G^2 = AH = 4A$ ...(i)

and given $2A + G^2 = 27$

$\Rightarrow \quad 2A + 4A = 27$ [from Eq. (i)]

$\therefore \quad A = \dfrac{9}{2}$

From Eq. (i), $G^2 = 4 \times \dfrac{9}{2} = 18$

Now, from important theorem of GM

$$a, b = A \pm \sqrt{(A^2 - G^2)} = \frac{9}{2} \pm \sqrt{\left(\frac{81}{4} - 18\right)}$$

$$= \frac{9}{2} \pm \frac{3}{2} = 6, 3 \text{ or } 3, 6$$

Example 79. If the geometric mean is $\dfrac{1}{n}$ times the harmonic mean between two numbers, then show that the ratio of the two numbers is

$$1 + \sqrt{(1 - n^2)} : 1 - \sqrt{(1 - n^2)}.$$

Sol. Let the two numbers be a and b.

Given, $G = \dfrac{1}{n} H$ ...(i)

Now, $G^2 = AH$

$\Rightarrow \quad \dfrac{H^2}{n^2} = AH$ [from Eq. (i)]

$\therefore \quad A = \dfrac{H}{n^2}$ ...(ii)

Now, from important theorem of GM

$$a, b = A \pm \sqrt{(A^2 - G^2)} = \frac{H}{n^2} \pm \sqrt{\left(\frac{H^2}{n^4} - \frac{H^2}{n^2}\right)}$$

$$= \frac{H}{n^2}[1 \pm \sqrt{(1 - n^2)}]$$

$$\therefore \quad \frac{a}{b} = \frac{\frac{H}{n^2}[1 + \sqrt{(1 - n^2)}]}{\frac{H}{n^2}[1 - \sqrt{(1 - n^2)}]}$$

$\therefore \quad a : b = 1 + \sqrt{(1 - n^2)} : 1 - \sqrt{(1 - n^2)}$

Example 80. If three positive unequal quantities a, b, c be in HP, then prove that $a^n + c^n > 2b^n, n \in N$

Sol. $\because \quad G > H$

$\therefore \quad \sqrt{ac} > b$

$\Rightarrow \quad (ac)^{\frac{n}{2}} > b^n$ or $a^{\frac{n}{2}} c^{\frac{n}{2}} > b^n$ ...(i)

Also, $(a^{\frac{n}{2}} - c^{\frac{n}{2}})^2 > 0 \Rightarrow a^n + c^n - 2a^{\frac{n}{2}} c^{\frac{n}{2}} > 0$

$\Rightarrow \quad a^n + c^n > 2\, a^{\frac{n}{2}} c^{\frac{n}{2}} > 2b^n$ [from Eq. (i)]

$\therefore \quad a^n + c^n > 2b^n$

Example 81.

(i) If a, b, c, d be four distinct positive quantities in AP, then

(a) $bc > ad$

(b) $c^{-1}d^{-1} + a^{-1}b^{-1} > 2(b^{-1}d^{-1} + a^{-1}c^{-1} - a^{-1}d^{-1})$

(ii) If a, b, c, d be four distinct positive quantities in GP, then

(a) $a + d > b + c$

(b) $c^{-1}d^{-1} + a^{-1}b^{-1} > 2(b^{-1}d^{-1} + a^{-1}c^{-1} - a^{-1}d^{-1})$

(iii) If a, b, c, d be four distinct positive quantities in HP, then

(a) $a + d > b + c$ (b) $ad > bc$

Sol. (i) $\because$ a, b, c, d are in AP.

(a) Applying AM > GM

For first three members, $b > \sqrt{ac}$

$\Rightarrow \quad b^2 > ac$ ...(i)

and for last three members, $c > \sqrt{bd}$

$\Rightarrow \quad c^2 > bd \quad \ldots(ii)$

From Eqs. (i) and (ii), we get

$$b^2c^2 > (ac)(bd)$$

Hence, $bc > ad$

(b) Applying AM > HM

For first three members,

$$b > \frac{2ac}{a+c}$$

$\Rightarrow \quad ab + bc > 2ac \quad \ldots(iii)$

For last three members, $c > \frac{2bd}{b+d}$

$$bc + cd > 2bd \quad \ldots(iv)$$

From Eqs. (iii) and (iv), we get

$$ab + bc + bc + cd > 2ac + 2bd$$

or $\quad ab + cd > 2(ac + bd - bc)$

Dividing in each term by $abcd$, we get

$c^{-1}d^{-1} + a^{-1}b^{-1} > 2(b^{-1}d^{-1} + a^{-1}c^{-1} - a^{-1}d^{-1})$

(ii) $\because$ a, b, c, d are in GP.

(a) Applying AM > GM

For first three members,

$$\frac{a+c}{2} > b$$

$\Rightarrow \quad a + c > 2b \quad \ldots(v)$

For last three members, $\frac{b+d}{2} > c$

$\Rightarrow \quad b + d > 2c \quad \ldots(vi)$

From Eqs. (v) and (vi), we get

$a + c + b + d > 2b + 2c$ or $a + d > b + c$

(b) Applying GM > HM

For first three members, $b > \frac{2ac}{a+c}$

$\Rightarrow \quad ab + bc > 2ac \quad \ldots(vii)$

For last three members, $c > \frac{2bd}{b+d}$

$\Rightarrow \quad bc + cd > 2bd \quad \ldots(viii)$

From Eqs. (vii) and (viii), we get

$$ab + bc + bc + cd > 2ac + 2bd$$

or $\quad ab + cd > 2(ac + bd - bc)$

Dividing in each term by $abcd$, we get

$c^{-1}d^{-1} + a^{-1}b^{-1} > 2(b^{-1}d^{-1} + a^{-1}c^{-1} - a^{-1}d^{-1})$

(iii) $\because$ a, b, c, d are in HP.

(a) Applying AM > HM

For first three members,

$$\frac{a+c}{2} > b$$

$\Rightarrow \quad a + c > 2b \quad \ldots(ix)$

For last three members, $\frac{b+d}{2} > c$

$\Rightarrow \quad b + d > 2c \quad \ldots(x)$

From Eqs. (ix) and (x), we get

$$a + c + b + d > 2b + 2c$$

or $\quad a + d > b + c$

(b) Applying GM > HM

For first three members, $\sqrt{ac} > b$

$\Rightarrow \quad ac > b^2 \quad \ldots(xi)$

For last three members,

$$\sqrt{bd} > c$$

$\Rightarrow \quad bd > c^2 \quad \ldots(xii)$

From Eqs. (xi) and (xii), we get

$$(ac)(bd) > b^2c^2$$

or $\quad ad > bc$

Exercise for Session 5

1. If the AM of two positive numbers a and b $(a > b)$ is twice of their GM, then $a : b$ is

(a) $2 + \sqrt{3} : 2 - \sqrt{3}$ (b) $7 + 4\sqrt{3} : 7 - 4\sqrt{3}$

(c) $2 : 7 + 4\sqrt{3}$ (d) $2 : \sqrt{3}$

2. If $A_1, A_2; G_1, G_2$ and H_1, H_2 are two arithmetic, geometric and harmonic means, respectively between two quantities a and b, then which of the following is not the value of ab is?

(a) A_1H_2 (b) A_2H_1

(c) G_1G_2 (d) None of these

3. The GM between -9 and -16, is

(a) 12 (b) -12

(c) -13 (d) None of these

4. Let $n \in N, n > 25$. If $A, G \text{ and } H$ denote the arithmetic mean, geometric mean and harmonic mean of 25 and n. Then, the least value of n for which $A, G, H \in \{25, 26, \ldots, n\}$, is

(a) 49 (b) 81

(c) 169 (d) 225

5. If 9 harmonic means be inserted between 2 and 3, then the value of $A + \frac{6}{H} + 5$ (where A is any of the AM's and H is the corresponding HM), is

(a) 8 (b) 9

(c) 10 (d) None of these

6. If $H_1, H_2, \ldots, H_n$ be n harmonic means between a and b, then $\frac{H_1 + a}{H_1 - a} + \frac{H_n + b}{H_n - b}$ is

(a) n (b) $n + 1$

(c) $2n$ (d) $2n - 2$

7. The AM of two given positive numbers is 2. If the larger number is increased by 1, the GM of the numbers becomes equal to the AM to the given numbers. Then, the HM of the given numbers is

(a) $\frac{3}{2}$ (b) $\frac{2}{3}$

(c) $\frac{1}{2}$ (d) 2

8. If $a, a_1, a_2, a_3, \ldots, a_{2n}, b$ are in AP and $a, b_1, b_2, b_3, \ldots, b_{2n}, b$ are in GP and h is the HM of a and b, then $\frac{a_1 + a_{2n}}{b_1 b_{2n}} + \frac{a_2 + a_{2n-1}}{b_2 b_{2n-1}} + \ldots + \frac{a_n + a_{n+1}}{b_n b_{n+1}}$ is equal to

(a) $\frac{2n}{h}$ (b) $2nh$

(c) nh (d) $\frac{n}{h}$

Session 6

Arithmetico-Geometric Series (AGS), Sigma (Σ) Notation, Natural Numbers

Arithmetico-Geometric Series (AGS)

Definition

A series formed by multiplying the corresponding terms of an AP and a GP is called **Arithmetico - Geometric Series** (or shortly written as AGS)

For example, $1+4+7+10+\ldots$ is an AP and $1+x+x^2+x^3+\ldots$ is a GP.

Multiplying together the corresponding terms of these series, we get

$1+4x+7x^2+10x^3+\ldots$ which is an

Arithmetico-Geometric Series.

Again, $a+(a+d)+(a+2d)+\ldots+[a+(n-1)d]$ is a typical AP

and $1+r+r^2+\ldots+r^{n-1}$ is a typical GP.

Multiplying together the corresponding terms of these series, we get

$$a+(a+d)\,r+(a+2d)\,r^2+\ldots+[a+(n-1)d]\,r^{n-1}$$

which is called a standard Arithmetico-Geometric series.

Sum of n Terms of an Arithmetico-Geometric Series

Let the series be $a+(a+d)\,r+(a+2d)\,r^2+\ldots$
$+[a+(n-1)d]\,r^{n-1}$

Let S_n denotes the sum to n terms, then

$$S_n=a+(a+d)\,r+(a+2d)\,r^2+\ldots+[a+(n-2)\,d]\,r^{n-2}+[a+(n-1)\,d]\,r^{n-1} \quad \ldots(\text{i})$$

Multiplying both sides of Eq. (i) by r, we get

$$rS_n=ar+(a+d)\,r^2+(a+2d)\,r^3+\ldots+[a+(n-2)\,d]\,r^{n-1}+[a+(n-1)\,d]\,r^n \quad \ldots(\text{ii})$$

Subtracting Eq. (ii) from Eq. (i), we get

$$(1-r)\,S_n=a+(dr+dr^2+\ldots+dr^{n-1})-[a+(n-1)\,d]\,r^n$$

$$=a+\frac{dr(1-r^{n-1})}{(1-r)}-[a+(n-1)\,d]\,r^n$$

$$\therefore \quad \boldsymbol{S_n=\frac{a}{(1-r)}+\frac{dr\,(1-r^{n-1})}{(1-r)^2}-\frac{[a+(n-1)\,d]r^n}{(1-r)}} \quad \ldots(\text{iii})$$

Remark

The above result (iii) is not used as standard formula in any examination. You should follow all steps as shown above.

To Deduce the Sum upto Infinity from the Sum upto *n* Terms of an Arithmetico - Geometric Series, when $|r|<1$

From Eq. (iii), we have

$$S_n=\frac{a}{1-r}+\frac{dr}{(1-r)^2}-\frac{dr^n}{(1-r)^2}-\frac{[a+(n-1)\,d]r^n}{(1-r)}$$

If $|r|<1$, when $n\to\infty, r^n\to 0$

and $\frac{dr^n}{(1-r)^2}$ and $\frac{[a+(n-1)\,d]r^n}{(1-r)}$ both $\to 0$

$$\therefore \quad S_\infty=\frac{a}{(1-r)}+\frac{dr}{(1-r)^2}$$

Independent method Let S_∞ denotes the sum to infinity, then

$$S_\infty=a+(a+d)\,r+(a+2d)\,r^2+(a+3d)\,r^3+\ldots \text{ upto } \infty \quad \ldots(\text{iv})$$

Multiplying both sides of Eq. (iv) by r, we get

$$rS_\infty=ar+(a+d)\,r^2+(a+2d)\,r^3+\ldots \text{ upto } \infty \quad \ldots(\text{v})$$

Subtracting Eq. (v) from Eq. (iv), we get

$$(1-r)\,S_\infty=a+(dr+dr^2+dr^3+\ldots \text{ upto } \infty)$$

$$=a+\frac{dr}{(1-r)}$$

$$\therefore \quad S_\infty=\frac{a}{(1-r)}+\frac{dr}{(1-r)^2}$$

Example 82. Find the sum of the series
$1+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+\ldots$
(i) to n terms. (ii) to infinity.

Sol. The given series can be written as

$$1+4\left(\frac{1}{5}\right)+7\left(\frac{1}{5}\right)^2+10\left(\frac{1}{5}\right)^3+\ldots$$

The series is an Arithmetico-Geometric series, since each term is formed by multiplying corresponding terms of the series 1, 4, 7, ... which are in AP and

$1, \frac{1}{5}, \frac{1}{5^2}, \ldots$ which are in GP.

$$\therefore \quad T_n = [n \text{ th term of } 1, 4, 7, \ldots] \left[n \text{ th term of } 1, \frac{1}{5}, \left(\frac{1}{5}\right)^2, \ldots \right]$$

$$= [1+(n-1)3] \times 1 \cdot \left(\frac{1}{5}\right)^{n-1} = (3n-2)\left(\frac{1}{5}\right)^{n-1}$$

$$\therefore \; T_{n-1} = (3n-5)\left(\frac{1}{5}\right)^{n-2}$$

(i) Let sum of n terms of the series is denoted by S_n.

Then, $S_n = 1+4\left(\frac{1}{5}\right)+7\left(\frac{1}{5}\right)^2+\ldots$

$$+(3n-5)\left(\frac{1}{5}\right)^{n-2}+(3n-2)\left(\frac{1}{5}\right)^{n-1} \quad \ldots(i)$$

Multiplying both the sides of Eq. (i) by $\frac{1}{5}$, we get

$$\therefore \quad \frac{1}{5}S_n = \frac{1}{5}+4\left(\frac{1}{5}\right)^2+7\left(\frac{1}{5}\right)^3+\ldots+(3n-5)\left(\frac{1}{5}\right)^{n-1}$$

$$+(3n-2)\left(\frac{1}{5}\right)^n \quad \ldots(ii)$$

Subtracting Eq. (ii) from Eq. (i), we get

$$\left(1-\frac{1}{5}\right)S_n = 1+3\left[\frac{1}{5}+\left(\frac{1}{5}\right)^2+\left(\frac{1}{5}\right)^3+\ldots+\left(\frac{1}{5}\right)^{n-1}\right]$$

$$-(3n-2)\left(\frac{1}{5}\right)^n$$

$$\text{or} \quad \frac{4}{5}S_n = 1+3\left[\left(\frac{1}{5}\right)+\left(\frac{1}{5}\right)^2+\left(\frac{1}{5}\right)^3+\ldots+(n-1) \text{ terms}\right]$$

$$-(3n-2)\left(\frac{1}{5}\right)^n$$

$$= 1+3\left\{\frac{\frac{1}{5}\left[1-\left(\frac{1}{5}\right)^{n-1}\right]}{1-\frac{1}{5}}\right\}-(3n-2)\left(\frac{1}{5}\right)^n$$

$$= 1+\frac{3}{4}\left\{1-\left(\frac{1}{5}\right)^{n-1}\right\}-(3n-2)\left(\frac{1}{5}\right)^n$$

$$\therefore \quad S_n = \frac{5}{4}+\frac{15}{16}\left[1-\left(\frac{1}{5}\right)^{n-1}\right]-\frac{(3n-2)}{4}\left(\frac{1}{5}\right)^{n-1}$$

$$= \frac{35}{16}-\frac{(12n+7)}{16}\left(\frac{1}{5}\right)^{n-1}$$

(ii) $S_\infty = 1+4\left(\frac{1}{5}\right)+7\left(\frac{1}{5}\right)^2+10\left(\frac{1}{5}\right)^3+\ldots \text{ upto } \infty \quad \ldots(iii)$

Multiplying both sides of Eq. (i) by $\frac{1}{5}$, we get

$$\frac{1}{5}S_\infty = \left(\frac{1}{5}\right)+4\left(\frac{1}{5}\right)^2+7\left(\frac{1}{5}\right)^3+\ldots \text{ upto } \infty \quad \ldots(iv)$$

Subtracting Eq. (iv) from Eq. (iii), we get

$$\left(1-\frac{1}{5}\right)S_\infty = 1+3\left[\left(\frac{1}{5}\right)+\left(\frac{1}{5}\right)^2+\left(\frac{1}{5}\right)^3+\ldots \text{upto } \infty\right]$$

$$= 1+3\left(\frac{\frac{1}{5}}{1-\frac{1}{5}}\right) = 1+\frac{3}{4}$$

$$\Rightarrow \quad \frac{4}{5}S_\infty = \frac{7}{4}$$

$$\therefore \quad S_\infty = \frac{35}{16}$$

Example 83. If the sum to infinity of the series $1+4x+7x^2+10x^3+\ldots$ is $\frac{35}{16}$, find x.

Sol. Let $S_\infty = 1+4x+7x^2+10x^3+\ldots \text{ upto } \infty \quad \ldots(i)$

Multiplying both sides of Eq. (i) by x we get

$$x\,S_\infty = x+4x^2+7x^3+10x^4+\ldots \text{ upto } \infty \quad \ldots(ii)$$

Subtracting Eq. (ii) from Eq. (i), we get

$$(1-x)\,S_\infty = 1+3x+3x^2+3x^3+\ldots \text{ upto } \infty$$

$$= 1+3(x+x^2+x^3+\ldots\text{upto } \infty) = 1+3\left(\frac{x}{1-x}\right) = \frac{(1+2x)}{(1-x)}$$

$$\therefore \quad S_\infty = \frac{(1+2x)}{(1-x)^2} = \frac{35}{16} \qquad \left[\because S_\infty = \frac{35}{16}\right]$$

$$\Rightarrow \quad 16+32x = 35-70x+35x^2$$

$$\Rightarrow \quad 35x^2-102x+19 = 0$$

$$\Rightarrow \quad (7x-19)(5x-1) = 0$$

$$x \neq \frac{19}{7}$$

[$\because$ for infinity series common ratio $-1 < x < 1$]

Hence, $x = \frac{1}{5}$

Example 84. ***Find the sum of the series*** $1+2^2x+3^2x^2+4^2x^3+\ldots$ ***up to*** ∞, $|x|<1$.

Sol. Here, the numbers $1^2, 2^2, 3^2, 4^2, \ldots$ i.e. 1, 4, 9, 16, … are not in AP but 1, $4-1=3$, $9-4=5$, $16-9=7$, … are in AP.

Let $S_\infty = 1+2^2x+3^2x^2+4^2x^3+\ldots$ upto ∞

$$= 1+4x+9x^2+16x^3+\ldots \text{ upto } \infty \quad \ldots(i)$$

Multiplying both sides of Eq. (i) by x, we get

$$xS_\infty = x+4x^2+9x^3+16x^4+\ldots \text{ upto } \infty \quad \ldots(ii)$$

Subtracting Eq. (ii) from Eq. (i), we get

$$(1-x)S_\infty = 1+3x+5x^2+7x^3+\ldots \text{ upto } \infty \quad \ldots(iii)$$

Again, multiplying both sides of Eq. (iii) by x, we get

$$x(1-x)S_\infty = x+3x^2+5x^3+7x^4+\ldots \text{ upto } \infty \quad \ldots(iv)$$

Subtracting Eq. (iv) from Eq. (iii), we get

$$(1-x)(1-x)S_\infty = 1+2x+2x^2+2x^3+\ldots \text{ upto } \infty$$

$$= 1+2(x+x^2+x^3+\ldots \text{ upto } \infty)$$

$$= 1+2\left(\frac{x}{1-x}\right)=\frac{(1+x)}{(1-x)}$$

$$\therefore \quad S_\infty = \frac{(1+x)}{(1-x)^3}$$

Sigma (Σ) Notation

Σ is a letter of greek alphabets and it is called 'sigma'. The symbol sigma (Σ) represents the sum of similar terms. Usually sum of n terms of any series is represented by placing Σ the nth term of the series. But if we have to find the sum of k terms of a series whose nth term is u_n, this will be represented by $\sum_{n=1}^{k} u_n$.

For example, $\sum_{n=1}^{n=9} n$, i.e. $\sum_{1}^{9} n$ only means the sum of n similar terms when n varies from 1 to 9.

Thus, $\sum_{1}^{9} n = 1+2+3+4+5+6+7+8+9$

Remark

Shortly Σ is written in place of $\sum_{1}^{n}$.

Properties of Sigma Notation

1. $\sum_{r=1}^{n} T_r = T_1+T_2+T_3+\ldots+T_n$, when T_n is the general term of the series.

2. $\sum_{r=1}^{n}(T_r \pm T_r') = \sum_{r=1}^{n} T_r \pm \sum_{r=1}^{n} T_r'$

[sigma operator is distributive over addition and subtraction]

3. $\sum_{r=1}^{n} T_r T_r' \neq \left(\sum_{r=1}^{n} T_r\right)\left(\sum_{r=1}^{n} T_r'\right)$

[sigma operator is not distributive over multiplication]

4. $\sum_{r=1}^{n}\left(\frac{T_r}{T_r'}\right) \neq \frac{\left(\sum_{r=1}^{n} T_r\right)}{\left(\sum_{r=1}^{n} T_r'\right)}$

[sigma operator is not distributive over division]

5. $\sum_{r=1}^{n} a T_r = a\sum_{r=1}^{n} T_r$ [where a is constant]

6. $\sum_{j=1}^{n}\sum_{i=1}^{n} T_i T_j = \left(\sum_{i=1}^{n} T_i\right)\left(\sum_{j=1}^{n} T_j\right)$

[where i and j are independent]

Examples on Sigma Notation

(i) $\sum_{i=1}^{m} a = a+a+a+\ldots$ upto m times $= am$

(ii) $\sum a = a+a+a+\ldots$ upto n times $= an$

i.e. $\sum 5 = 5n$, $\sum 3 = 3n$

(iii) $\sum_{i=1}^{5}(i^2-3i) = \sum_{i=1}^{5} i^2 - 3\sum_{i=1}^{5} i$

$$= (1^2+2^2+3^2+4^2+5^2)-3(1+2+3+4+5)$$

$$= 55-45 = 10$$

(iv) $\sum_{r=1}^{3}\left(\frac{r+1}{2r+4}\right) = \left(\frac{1+1}{2\cdot 1+4}\right)+\left(\frac{2+1}{2\cdot 2+4}\right)+\left(\frac{3+1}{2\cdot 3+4}\right)$

$$= \frac{2}{6}+\frac{3}{8}+\frac{4}{10} = \frac{40+45+48}{120} = \frac{133}{120} = 1\frac{13}{120}$$

Important Theorems on Σ (Sigma) Operator

Theorem 1 $\sum_{r=1}^{n} f(r+1)-f(r) = f(n+1)-f(1)$

Theorem 2

$$\sum_{r=1}^{n} f(r+2)-f(r) = f(n+2)+f(n+1)-f(2)-f(1)$$

Proof (Theorem 1) $\sum_{r=1}^{n} f(r+1) - f(r)$

$$= [f(2) - f(1)] + [f(3) - f(2)] + [f(4) - f(3)] + \ldots + [f(n+1) - f(n)]$$
$$= f(n+1) - f(1)$$

Proof (Theorem 2)

$$\sum_{r=1}^{n} f(r+2) - f(r) = \sum_{r=1}^{n} [f(r+2) - f(r+1)] + [f(r+1) - f(r)]$$
$$= \sum_{r=1}^{n} f(r+2) - f(r+1) + \sum_{r=1}^{n} f(r+1) - f(r)$$
$$= [f(n+2) - f(2)] + [f(n+1) - f(1)] \quad \text{[from Theorem 1]}$$
$$= f(n+2) + f(n+1) - f(2) - f(1)$$

Remark

1. $\sum_{r=1}^{n} f(r+k) - f(r) = \sum_{m=1}^{k} f(n+m) - \sum_{m=1}^{k} f(m), \forall\ k \in N$
2. $\sum_{r=1}^{n} f(2r+1) - f(2r-1) = f(2n+1) - f(1)$
3. $\sum_{r=1}^{n} f(2r) - f(2r-2) = f(2n) - f(0)$

Natural Numbers

The positive integers 1, 2, 3, ... are called natural numbers. These form an AP with first term and common difference, each equal to unity.

(i) Sum of the First *n* Natural Numbers

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} = \Sigma n$$

$$\Rightarrow \quad \Sigma n = \frac{n(n+1)}{2} \quad \text{[Remember]}$$

(ii) Sum of the First *n* Odd Natural Numbers

$$1 + 3 + 5 + \ldots \text{ upto } n \text{ terms} = \frac{n}{2}[2 \cdot 1 + (n-1) \cdot 2] = n^2$$

$$\Rightarrow \quad \sum (2n-1) = n^2 \quad \text{[Remember]}$$

(iii) Sum of the Squares of the First *n* Natural Numbers

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \sum n^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof We know that, $r^3 - (r-1)^3 = 3r^2 - 3r + 1$

Taking $\sum_{r=1}^{n}$ on both sides, we get

$$\sum_{r=1}^{n} r^3 - (r-1)^3 = 3\sum_{r=1}^{n} r^2 - 3\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$$

$$\Rightarrow \quad n^3 - 0^3 = 3\sum n^2 - 3\sum n + n \quad \ldots\text{(i)}$$

[from important Theorem 1]

Substituting the value of $\sum n$ in Eq. (i), we get

$$\Rightarrow \quad n^3 = 3\sum n^2 - \frac{3 \cdot n(n+1)}{2} + n$$

$$\Rightarrow \quad 3\sum n^2 = n^3 + \frac{3n(n+1)}{2} - n = \frac{n}{2}(2n^2 + 3n + 1)$$
$$= \frac{n(n+1)(2n+1)}{2}$$

$$\Rightarrow \quad \sum n^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{[Remember]}$$

Independent Proof We know that,

$$(2r+1)^3 - (2r-1)^3 = 24r^2 + 2$$

Taking $\sum_{r=1}^{n}$ on both sides, we get

$$\sum_{r=1}^{n} (2r+1)^3 - (2r-1)^3 = \sum_{r=1}^{n} (24r^2 + 2)$$

$$\Rightarrow \quad (2n+1)^3 - 1^3 = 24\sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} 1$$

[from points to consider-2]

$$\Rightarrow \quad (2n+1)^3 - 1 = 24\sum n^2 + 2n$$

$$\Rightarrow \quad (2n+1)^3 - (2n+1) = 24\sum n^2$$

$$\Rightarrow \quad (2n+1)[(2n+1)^2 - 1] = 24\sum n^2$$

$$\Rightarrow \quad (2n+1)(2n+1+1)(2n+1-1) = 24\sum n^2$$

$$\Rightarrow \quad \sum n^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{[Remember]}$$

(iv) Sum of the Cubes of the First *n* Natural Numbers

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \sum n^3 = \left(\sum n\right)^2 = \left\{\frac{n(n+1)}{2}\right\}^2$$

Proof We know that,

$$r^4 - (r-1)^4 = 4r^3 - 6r^2 + 4r - 1$$

Taking $\sum_{r=1}^{n}$ on both sides, we get

$$\sum_{r=1}^{n} r^4 - (r-1)^4 = 4\sum_{r=1}^{n} r^3 - 6\sum_{r=1}^{n} r^2 + 4\sum_{r=1}^{n} r - \sum_{r=1}^{n} 1$$

$$\Rightarrow \quad n^4 - 0^4 = 4\sum n^3 - 6\sum n^2 + 4\sum n - n \qquad \text{...(i)}$$

[from important theorem 1]

Substituting the values of $\sum n^2$ and $\sum n$ in Eq. (i), we get

$$\Rightarrow \quad n^4 = 4\sum n^3 - \frac{6n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2} - n$$

$$\Rightarrow 4\sum n^3 = n^4 + n(n+1)(2n+1) - 2n(n+1) + n$$

$$= n[n^3 + (n+1)(2n+1) - 2(n+1) + 1]$$

$$= n(n^3 + 2n^2 + n)$$

$$= n^2(n+1)^2$$

$$\therefore \quad \sum n^3 = \left\{\frac{n(n+1)}{2}\right\}^2 = (\sum n)^2 \qquad \text{[Remember]}$$

Independent Proof We know that,

$$r^2(r+1)^2 - r^2(r-1)^2 = 4r^3$$

Taking $\sum_{r=1}^{n}$ on both sides, we get

$$\sum_{r=1}^{n} r^2(r+1)^2 - r^2(r-1)^2 = 4\sum_{r=1}^{n} r^3$$

$$\Rightarrow \quad n^2(n+1)^2 - 1^2 \cdot 0^2 = 4\sum n^3$$

[from important Theorem 1]

$$\Rightarrow \quad \sum n^3 = \left\{\frac{n(n+1)}{2}\right\}^2 = (\sum n)^2 \qquad \text{[Remember]}$$

Corollary $1^3 + 2^3 + 3^3 + \ldots + n^3 = (1 + 2 + 3 + \ldots + n)^2$

(v) Sum of the Powers Four of the First *n* Natural Numbers

$$1^4 + 2^4 + 3^4 + \ldots + n^4 = \sum n^4$$

$$= \frac{n(n+1)(2n+1)(3n^2 + 3n - 1)}{30}$$

Proof We know that,

$$r^5 - (r-1)^5 = 5r^4 - 10r^3 + 10r^2 - 5r + 1$$

Taking $\sum_{r=1}^{n}$ on both sides, we get

$$\sum_{r=1}^{n} r^5 - (r-1)^5 = 5\sum_{r=1}^{n} r^4 - 10\sum_{r=1}^{n} r^3 + 10\sum_{r=1}^{n} r^2 - 5\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$$

$$\Rightarrow \quad n^5 - 0^5 = 5\sum n^4 - 10\sum n^3 + 10\sum n^2 - 5\sum n + n \qquad \text{...(i)}$$

[from important Theorem 1]

Substituting the values of $\sum n$, $\sum n^2$, $\sum n^3$ in Eq. (i), we get

$$\Rightarrow \quad n^5 = 5\sum n^4 - \frac{10n^2(n+1)^2}{4} + \frac{10n(n+1)(2n+1)}{6} - \frac{5n(n+1)}{2} + n$$

$$\therefore \quad 5\sum n^4 = n\left\{n^4 + \frac{5n(n+1)^2}{2} - \frac{5(n+1)(2n+1)}{3} + \frac{5(n+1)}{2} - 1\right\}$$

$$= \frac{n}{6}\{6n^4 + 15n(n^2 + 2n + 1) - 10(2n^2 + 3n + 1) + 15n + 15 - 6\}$$

$$\Rightarrow \quad \sum n^4 = \frac{n}{30}(6n^4 + 15n^3 + 10n^2 - 1)$$

$$= \frac{n(n+1)(2n+1)(3n^2 + 3n - 1)}{30}$$

Remark

If nth term of a sequence is given by $T_n = an^3 + bn^2 + cn + d$, where a, b, c, d are constants.

Then, sum of n terms, $S_n = \Sigma T_n = a\Sigma n^3 + b\,\Sigma n^2 + c\,\Sigma n + d\,\Sigma 1$

This can be evaluated using the above results.

Example 85. *Find the sum to n terms of the series* $1^2 + 3^2 + 5^2 + \ldots$ *upto n terms.*

Sol. Let T_n be the nth term of this series, then

$$T_n = [1 + (n-1)2]^2 = (2n-1)^2 = 4n^2 - 4n + 1$$

$\therefore$ Sum of n terms $S_n = \Sigma T_n = 4\Sigma n^2 - 4\Sigma n + \Sigma 1$

$$= \frac{4n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n$$

$$= \frac{n}{3}(4n^2 + 6n + 2 - 6n - 6 + 3)$$

$$= \frac{n(4n^2 - 1)}{3}$$

Example 86. *Find the sum to n terms of the series* $1 \cdot 2^2 + 2 \cdot 3^2 + 3 \cdot 4^2 + \ldots$.

Sol. Let T_n be the nth term of this series, then

$T_n = (n$th term of 1, 2, 3, ...) (nth term of $2^2, 3^2, 4^2, \ldots$)

$$= n(n+1)^2 = n^3 + 2n^2 + n$$

$\therefore$ Sum of n terms $S_n = \Sigma T_n$

$$2 = \Sigma n^3 + 2\Sigma n^2 + \Sigma n$$

$$= \left\{\frac{n(n+1)}{2}\right\}^2 + 2\left\{\frac{n(n+1)(2n+1)}{6}\right\} + \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} + \frac{2(2n+1)}{3} + 1\right\}$$

$$= \frac{n(n+1)}{12}(3n^2 + 3n + 8n + 4 + 6)$$

$$= \frac{n(n+1)(3n^2 + 11n + 10)}{12} = \frac{n(n+1)(n+2)(3n+5)}{12}$$

Example 87. ***Find the sum of*** n ***terms of the series whose*** n***th terms is (i)*** $n(n-1)(n+1)$ ***(ii)*** $n^2 + 3^n$.

Sol. (i) We have, $T_n = n(n-1)(n+1) = n^3 - n$

$\therefore$ Sum of n terms $S_n = \Sigma T_n = \Sigma n^3 - \Sigma n$

$$= \left\{\frac{n(n+1)}{2}\right\}^2 - \left\{\frac{n(n+1)}{2}\right\}$$

$$= \frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} - 1\right\}$$

$$= \frac{n(n+1)(n-1)(n+2)}{4}$$

(ii) We have, $T_n = n^2 + 3^n$

$\therefore$ Sum of n terms $S_n = \Sigma T_n = \Sigma n^2 + \Sigma 3^n$

$$= \Sigma n^2 + (3^1 + 3^2 + 3^3 + \ldots + 3^n)$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{3(3^n - 1)}{(3-1)}$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{3}{2}(3^n - 1)$$

Example 88. ***Find the sum of the series***

$$\frac{1^3}{1} + \frac{1^3 + 2^3}{1+3} + \frac{1^3 + 2^3 + 3^3}{1+3+5} + \ldots \text{ upto } n \text{ terms.}$$

Sol. Let T_n be the nth term of the given series. Then,

$$T_n = \frac{(1^3 + 2^3 + 3^3 + \ldots + n^3)}{(1 + 3 + 5 + \ldots + (2n-1))} = \frac{\left\{\frac{n(n+1)}{2}\right\}^2}{\frac{n}{2}(1 + 2n - 1)}$$

$$= \frac{(n+1)^2}{4} = \frac{1}{4}(n^2 + 2n + 1)$$

Let S_n denotes the sum of n terms of the given series. Then,

$$S_n = \Sigma T_n = \frac{1}{4}\Sigma(n^2 + 2n + 1)$$

$$= \frac{1}{4}(\Sigma n^2 + 2\Sigma n + \Sigma 1)$$

$$= \frac{1}{4}\left\{\frac{n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2} + n\right\}$$

$$= \frac{n}{24}\{2n^2 + 3n + 1 + 6n + 6 + 6\}$$

Hence, $S_n = \dfrac{n(2n^2 + 9n + 13)}{24}$

Example 89. Show that

$$\frac{1\cdot 2^2 + 2\cdot 3^2 + \ldots + n\cdot(n+1)^2}{1^2\cdot 2 + 2^2\cdot 3 + \ldots + n^2\cdot(n+1)} = \frac{3n+5}{3n+1}.$$

Sol. Let T_n and T_n' be the nth terms of the series in numerator and denominator of LHS. Then,

$\therefore \quad T_n = n(n+1)^2$ and $T_n' = n^2(n+1)$

$\therefore \quad \text{LHS} = \dfrac{\Sigma T_n}{\Sigma T_n'} = \dfrac{\Sigma n(n+1)^2}{\Sigma n^2(n+1)} = \dfrac{\Sigma(n^3 + 2n^2 + n)}{\Sigma(n^3 + n^2)}$

$$= \frac{\Sigma n^3 + 2\Sigma n^2 + \Sigma n}{\Sigma n^3 + \Sigma n^2}$$

$$= \frac{\left\{\frac{n(n+1)}{2}\right\}^2 + 2\left\{\frac{n(n+1)(2n+1)}{6}\right\} + \left\{\frac{n(n+1)}{2}\right\}}{\left\{\frac{n(n+1)}{2}\right\}^2 + \left\{\frac{n(n+1)(2n+1)}{6}\right\}}$$

$$= \frac{\frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} + \frac{2(2n+1)}{3} + 1\right\}}{\frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} + \frac{(2n+1)}{3}\right\}}$$

$$= \frac{\frac{1}{6}(3n^2 + 3n + 8n + 4 + 6)}{\frac{1}{6}(3n^2 + 3n + 4n + 2)}$$

$$= \frac{(3n^2 + 11n + 10)}{(3n^2 + 7n + 2)} = \frac{(3n+5)(n+2)}{(3n+1)(n+2)} = \frac{(3n+5)}{(3n+1)} = \text{RHS}$$

Example 90. Find the sum of the series $1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + 3\cdot 4\cdot 5 + \ldots$ upto n terms.

Sol. Here, $T_n = \{n\text{th term of } 1, 2, 3, \ldots\} \times \{n\text{th term of } 2, 3, 4, \ldots\} \times \{n\text{th term of } 3, 4, 5, \ldots\}$

$\therefore \quad T_n = n(n+1)(n+2) = n^3 + 3n^2 + 2n$

$\therefore \quad S_n = \text{Sum of } n \text{ terms of the series}$

$$= \Sigma T_n = \Sigma n^3 + 3\Sigma n^2 + 2\Sigma n$$

$$= \left\{\frac{n(n+1)}{2}\right\}^2 + 3\left\{\frac{n(n+1)(2n+1)}{6}\right\} + 2\left\{\frac{n(n+1)}{2}\right\}$$

$$= \frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} + (2n+1) + 2\right\}$$

$$= \frac{n(n+1)}{4}(n^2 + n + 4n + 2 + 4)$$

$$= \frac{n(n+1)(n+2)(n+3)}{4}$$

Example 91. Find sum to n terms of the series $1 + (2+3) + (4+5+6) + \ldots$.

Sol. Now, number of terms in first bracket is 1, in the second bracket is 2, in the third bracket is 3, etc. Therefore, the number of terms in the nth bracket will be n.

Let the sum of the given series of n terms = S

$\therefore$ Number of terms in $S = 1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$

Also, the first term of S is 1 and common difference is also 1.

$$\therefore \quad S = \frac{\left\{\dfrac{n(n+1)}{2}\right\}}{2}\left[2\cdot 1 + \left(\frac{n(n+1)}{2} - 1\right)\cdot 1\right]$$

$$= \frac{n(n+1)}{8}(4 + n^2 + n - 2)$$

$$= \frac{n(n+1)(n^2+n+2)}{8}$$

Example 92. *Find the sum of the series*

$1\cdot n + 2\cdot(n-1) + 3\cdot(n-2) + 4\cdot(n-3) + \ldots + (n-1)\cdot 2 + n\cdot 1$

also, find the coefficient of x^{n-1} in the expansion of $(1 + 2x + 3x^2 + \ldots + nx^{n-1})^2$.

Sol. The rth term of the given series is

$$T_r = r\cdot(n - r + 1) = (n+1)\,r - r^2$$

$\therefore$ Sum of the series

$$S_n = \sum_{r=1}^{n} T_r = (n+1)\sum_{r=1}^{n} r - \sum_{r=1}^{n} r^2 = (n+1)\,\Sigma n - \Sigma n^2$$

$$= (n+1)\frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n(n+1)}{6}(3n + 3 - 2n - 1) = \frac{n(n+1)(n+2)}{6}$$

Now,

$$(1 + 2x + 3x^2 + \ldots + nx^{n-1})^2 = (1 + 2x + 3x^2 + \ldots + nx^{n-1}) \times (1 + 2x + 3x^2 + \ldots + nx^{n-1})$$

$\therefore$ Coefficient of x^{n-1} in $(1 + 2x + 3x^2 + \ldots + nx^{n-1})^2$

$$= 1\cdot n + 2\cdot(n-1) + 3\cdot(n-2) + \ldots + n\cdot 1$$

$$= S_n = \frac{n(n+1)(n+2)}{6}$$

Method of Differences

If the differences of the successive terms of a series are in AP or GP, we can find the nth term of the series by the following steps.

***Step* I** Denote the nth term and the sum of the series upto n terms of the series by T_n and S_n, respectively.

***Step* II** Rewrite the given series with each term shifted by one place to the right.

***Step* III** Then, subtract the second expression of S_n from the first expression to obtain T_n.

Example 93. Find the nth term and sum of n terms of the series, $1 + 5 + 12 + 22 + 35 + \ldots$.

Sol. The sequence of differences between successive terms is 4, 7, 10, 13,... . Clearly, it is an AP with common difference 3. So, let the nth term of the given series be T_n and sum of n terms be S_n.

Then, $S_n = 1 + 5 + 12 + 22 + 35 + \ldots + T_{n-1} + T_n$ ...(i)

$S_n = 1 + 5 + 12 + 22 + \ldots + T_{n-1} + T_n$ ...(ii)

Subtracting Eq. (ii) from Eq. (i), we get

$$0 = 1 + 4 + 7 + 10 + 13 + \ldots + (T_n - T_{n-1}) - T_n$$

$$\Rightarrow \quad T_n = 1 + 4 + 7 + 10 + 13 + \ldots n \text{ terms}$$

$$= \frac{n}{2}\{2\cdot 1 + (n-1)\,3\} = \frac{1}{2}(3n^2 - n)$$

Hence, $T_n = \dfrac{3}{2}n^2 - \dfrac{1}{2}n$

$\therefore$ Sum of n terms $S_n = \Sigma T_n = \dfrac{3}{2}\Sigma n^2 - \dfrac{1}{2}\Sigma n$

$$= \frac{3}{2}\left(\frac{n(n+1)(2n+1)}{6}\right) - \frac{1}{2}\left(\frac{n(n+1)}{2}\right)$$

$$= \frac{n(n+1)}{4}(2n + 1 - 1)$$

$$= \frac{1}{2}n^2(n+1) = \frac{1}{2}(n^3 + n^2)$$

Example 94. *Find the nth term and sum of n terms of the series,* $1 + 3 + 7 + 15 + 31 + \ldots$.

Sol. The sequence of differences between successive terms is 2, 4, 8, 16, ... Clearly, it is a GP with common ratio 2. So, let the nth term and sum of the series upto n terms of the series be T_n and S_n, respectively. Then,

$S_n = 1 + 3 + 7 + 15 + 31 + \ldots + T_{n-1} + T_n$ ...(i)

$S_n = 1 + 3 + 7 + 15 + \ldots + T_{n-1} + T_n$ ...(ii)

Subtracting Eq. (ii) from Eq. (i), we get

$$0 = 1 + 2 + 4 + 8 + 16 + \ldots + (T_n - T_{n-1}) - T_n$$

$$\Rightarrow \quad T_n = 1 + 2 + 4 + 8 + 16 + \ldots \text{ upto } n \text{ terms}$$

$$= \frac{1\cdot(2^n - 1)}{2 - 1}$$

Hence, $T_n = (2^n - 1)$

$\therefore$ Sum of n terms $S_n = \Sigma T_n = \Sigma(2^n - 1) = \Sigma 2^n - \Sigma 1$

$$= (2 + 2^2 + 2^3 + \ldots + 2^n) - n$$

$$= \frac{2\cdot(2^n - 1)}{(2 - 1)} - n = 2^{n+1} - 2 - n$$

Example 95. *Find the nth term of the series* $1 + 4 + 10 + 20 + 35 + \ldots$

Sol. The sequence of first consecutive differences is 3, 6, 10, 15, ... and second consecutive differences is 3, 4, 5, ... Clearly, it is an AP with common difference 1. So, let the nth term and sum of the series upto n terms of the series be T_n and S_n, respectively.

Then,

$S_n = 1 + 4 + 10 + 20 + 35 + ... + T_{n-1} + T_n$ ...(i)

$S_n = \quad 1 + 4 + 10 + 20 + ... + T_{n-1} + T_n$ ...(ii)

Subtracting Eq. (ii) from Eq. (i), we get

$$0 = 1 + 3 + 6 + 10 + 15 + ... + (T_n - T_{n-1}) - T_n$$

$\Rightarrow \quad T_n = 1 + 3 + 6 + 10 + 15 + ...$ upto n terms

or $\quad T_n = 1 + 3 + 6 + 10 + 15 + ... + t_{n-1} + t_n$ ...(iii)

$T_n = 1 + 3 + 6 + 10 + ... + t_{n-1} + t_n$ ...(iv)

Now, subtracting Eq. (iv) from Eq. (iii), we get

$$0 = 1 + 2 + 3 + 4 + 5 + ... + (t_n - t_{n-1}) - t_n$$

or $\quad t_n = 1 + 2 + 3 + 4 + 5 + ...$ upto n terms

$$= \Sigma n = \frac{n(n+1)}{2}$$

$\therefore \quad T_n = \Sigma t_n = \frac{1}{2}(\Sigma n^2 + \Sigma n)$

$$= \frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right)$$

$$= \frac{1}{2} \cdot \frac{n(n+1)}{6}(2n+1+3) = \frac{1}{6}n(n+1)(n+2)$$

Example 96. *Find the nth term of the series* $1+5+18+58+179+\ldots$

Sol. The sequence of first consecutive differences is 4, 13, 40, 121, ... and second consecutive differences is 9, 27, 81, ... Clearly, it is a GP with common ratio 3. So, let the nth term and sum of the series upto n terms of the series be T_n and S_n, respectively. Then,

$S_n = 1 + 5 + 18 + 58 + 179 + ... + T_{n-1} + T_n$ ...(i)

$S_n = 1 + 5 + 18 + 58 + ... + T_{n-1} + T_n$ ...(ii)

Subtracting Eq. (ii) from Eq. (i), we get

$$0 = 1 + 4 + 13 + 40 + 121 + ... + (T_n - T_{n-1}) - T_n$$

$\Rightarrow \quad T_n = 1 + 4 + 13 + 40 + 121 + ...$ upto n terms

or $\quad T_n = 1 + 4 + 13 + 40 + 121 + ... + t_{n-1} + t_n$ ...(iii)

$T_n = \quad 1 + 4 + 13 + 40 + ... + t_{n-1} + t_n$ ...(iv)

Now, subtracting Eq. (iv) from Eq. (iii), we get

$$0 = 1 + 3 + 9 + 27 + 81 + ... + (t_n - t_{n-1}) - t_n$$

or $\quad t_n = 1 + 3 + 9 + 27 + 81 + ...$ upto n terms

$$= \frac{1 \cdot (3^n - 1)}{(3-1)} = \frac{1}{2}(3^n - 1)$$

$\therefore \quad T_n = \Sigma t_n = \frac{1}{2}(\Sigma 3^n - \Sigma 1)$

$$= \frac{1}{2}\{(3 + 3^2 + 3^3 + ... + 3^n) - n\}$$

$$= \frac{1}{2}\left\{\frac{3(3^n - 1)}{(3-1)} - n\right\}$$

$$= \frac{3}{4}(3^n - 1) - \frac{1}{2}n$$

Method of Differences (Shortcut) to find nth term of a Series

The nth term of the series can be written directly on the basis of successively differences, we use the following steps to find the nth term T_n of the given sequence.

***Step* I** If the first consecutive differences of the given sequence are in AP, then take

$T_n = a(n-1)(n-2) + b(n-1) + c$, where a, b, c are constants. Determine a, b, c by putting $n = 1, 2, 3$ and putting the values of T_1, T_2, T_3.

***Step* II** If the first consecutive differences of the given sequence are in GP, then take

$T_n = ar^{n-1} + bn + c$, where a, b, c are constants and r is the common ratio of GP. Determine a, b, c by putting $n = 1, 2, 3$ and putting the values of T_1, T_2, T_3.

***Step* III** If the differences of the differences computed in Step I are in AP, then take

$T_n = a(n-1)(n-2)(n-3) + b(n-1)(n-2) + c(n-1) + d$, where a, b, c, d are constants.Determine by putting $n = 1, 2, 3, 4$ and putting the values of T_1, T_2, T_3, T_4.

***Step* IV** If the differences of the differences computed in Step I are in GP with common ratio r, then take

$T_n = ar^{n-1} + bn^2 + cn + d$, where a, b, c, d are constants. Determine by putting $n = 1, 2, 3, 4$ and putting the values of T_1, T_2, T_3, T_4.

Example 97. Find the nth term and sum of n terms of the series $2+4+7+11+16+\ldots$.

Sol. The sequence of first consecutive differences is 2, 3, 4, 5, ... Clearly, it is an AP.

Then, nth term of the given series be

$T_n = a(n-1)(n-2) + b(n-1) + c$ ...(i)

Putting $n = 1, 2, 3$, we get

$$2 = c \Rightarrow 4 = b + c \Rightarrow 7 = 2a + 2b + c$$

After solving, we get $a = \frac{1}{2}, b = 2, c = 2$

Putting the values of a, b, c in Eq. (i), we get

$$T_n = \frac{1}{2}(n-1)(n-2) + 2(n-1) + 2 = \frac{1}{2}(n^2 + n + 2)$$

Hence, sum of series $S_n = \Sigma T_n = \frac{1}{2}(\Sigma n^2 + \Sigma n + 2\Sigma 1)$

$$= \frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} + 2n\right)$$

$$= \frac{1}{6}n(n^2 + 3n + 8)$$

Example 98. ***Find the nth term and sum of n terms of the series*** $5+7+13+31+85+\ldots$.

Sol. The sequence of first consecutive differences is 2, 6, 18, 54, ... Clearly, it is a GP with common ratio 3. Then, nth term of the given series be

$$T_n = a(3)^{n-1} + bn + c \quad \ldots(i)$$

Putting $n = 1, 2, 3$, we get

$$5 = a + b + c \quad \ldots(ii)$$
$$7 = 3a + 2b + c \quad \ldots(iii)$$
$$13 = 9a + 3b + c \quad \ldots(iv)$$

Solving these equations, we get

$$a = 1, b = 0, c = 4$$

Putting the values of a, b, c in Eq. (i), we get

$$T_n = 3^{n-1} + 4$$

Hence, sum of the series

$$S_n = \Sigma T_n = \Sigma(3^{n-1} + 4) = \Sigma(3^{n-1}) + 4\Sigma 1$$
$$= (1 + 3 + 3^2 + \ldots + 3^{n-1}) + 4n$$
$$= 1 \cdot \frac{(3^n - 1)}{(3 - 1)} + 4n = \frac{1}{2}(3^n + 8n - 1)$$

Example 99. ***Find the nth term of the series*** $1+2+5+12+25+46+\ldots$.

Sol. The sequence of first consecutive differences is 1, 3, 7, 13, 21, ... The sequence of the second consecutive differences is 2, 4, 6, 8, ... Clearly, it is an AP. Then, nth term of the given series be

$$T_n = a(n-1)(n-2)(n-3) + b(n-1)(n-2) + c(n-1) + d \quad \ldots(i)$$

Putting $n = 1, 2, 3, 4$, we get

$$1 = d \quad \ldots(ii)$$
$$2 = c + d \quad \ldots(iii)$$
$$5 = 2b + 2c + d \quad \ldots(iv)$$
$$12 = 6a + 6b + 3c + d \quad \ldots(v)$$

After, solving these equations, we get

$$a = \frac{1}{3}, b = 1, c = 1, d = 1$$

Putting the values of a, b, c, d in Eq. (i), we get

$$T_n = \frac{1}{3}(n^3 - 6n^2 + 11n - 6) + (n^2 - 3n + 2) + (n - 1) + 1$$
$$= \frac{1}{3}(n^3 - 3n^2 + 5n) = \frac{n}{3}(n^2 - 3n + 5)$$

Example 100. Find the nth term of the series $2+5+12+31+86+\ldots$.

Sol. The sequence of first consecutive differences is 3, 7, 19, 55, ... The sequence of the second consecutive differences is 4, 12, 36, ... Clearly, it is a GP with common ratio 3. Then, nth term of the given series be

$$T_n = a(3)^{n-1} + bn^2 + cn + d \quad \ldots(i)$$

Putting $n = 1, 2, 3, 4$, we get

$$2 = a + b + c + d \quad \ldots(ii)$$
$$5 = 3a + 4b + 2c + d \quad \ldots(iii)$$
$$12 = 9a + 9b + 3c + d \quad \ldots(iv)$$
$$31 = 27a + 16b + 4c + d \quad \ldots(v)$$

After, solving these equations, we get

$$a = 1, b = 0, c = 1, d = 0$$

Putting the values of a, b, c, d in Eq. (i), we get

$$T_n = 3^{n-1} + n$$

Method of Differences

(Maha Shortcut)

To find $t_1 + t_2 + t_3 + \ldots + t_{n-1} + t_n$

Let $S_n = t_1 + t_2 + t_3 + \ldots + t_{n-1} + t_n$

Then, $\Delta t_1, \Delta t_2, \Delta t_3, \ldots, \Delta t_{n-1}$ [1st order differences]

$\Delta^2 t_1, \Delta^2 t_2, \Delta^2 t_3, \ldots, \Delta^2 t_{n-1}$ [2nd order differences]

$\vdots \quad \vdots \quad \vdots$

$$\therefore\ t_n = {}^{n-1}C_0\, t_1 + {}^{n-1}C_1\, \Delta t_1 + {}^{n-1}C_2\, \Delta^2 t_1 + \ldots + {}^{n-1}C_{r-1}\, \Delta^{r-1} t_1$$

and $S_n = {}^{n}C_1\, t_1 + {}^{n}C_2\, \Delta t_1 + {}^{n}C_3\, \Delta^2 t_1 + \ldots + {}^{n}C_r\, \Delta^r t_1$

where, $\Delta t_1 = t_2 - t_1, \Delta t_2 = t_3 - t_2$, etc.

$\Delta^2 t_1 = \Delta t_2 - \Delta t_1, \Delta^3 t_1 = \Delta^2 t_2 - \Delta^2 t_1$, etc.

Example 101. Find the n th term and sum to n terms of the series $12+40+90+168+280+432+\ldots$.

Sol. Let $S_n = 12 + 40 + 90 + 168 + 280 + 432 + \ldots$, then

1st order differences are 28, 50, 78, 112, 152, ... (i.e. $\Delta t_1, \Delta t_2, \Delta t_3, \ldots$)

and 2nd order differences are

22, 28, 34, 40, ... (i.e. $\Delta^2 t_1, \Delta^2 t_2, \Delta^2 t_3, \ldots$)

and 3rd order differences are

6, 6, 6, 6,... (i.e. $\Delta^3 t_1, \Delta^3 t_2, \Delta^3 t_3, \ldots$)

and 4th order differences are

0, 0, 0, 0, ... (i.e. $\Delta^4 t_1, \Delta^4 t_2, \Delta^4 t_3, \ldots$)

$$\therefore\quad t_n = 12 \cdot {}^{n-1}C_0 + 28 \cdot {}^{n-1}C_1 + 22 \cdot {}^{n-1}C_2 + 6 \cdot {}^{n-1}C_3$$
$$= 12 + 28(n-1) + \frac{22(n-1)(n-2)}{2} + \frac{6(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3}$$
$$= n^3 + 5n^2 + 6n$$

and $S_n = 12 \cdot {}^{n}C_1 + 28 \cdot {}^{n}C_2 + 22 \cdot {}^{n}C_3 + 6 \cdot {}^{n}C_4$

$$= 12n + \frac{28n(n-1)}{2} + \frac{22n(n-1)(n-2)}{1\cdot 2\cdot 3} + \frac{6\cdot n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}$$

$$= \frac{n}{12}(n+1)(3n^2+23n+46)$$

V_n Method

To find the sum of the series of the forms

I. $a_1a_2 \ldots a_r + a_2a_3 \ldots a_{r+1} + \ldots + a_na_{n+1} \ldots a_{n+r-1}$

II. $\frac{1}{a_1a_2 \ldots a_r} + \frac{1}{a_2a_3 \ldots a_{r+1}} + \ldots + \frac{1}{a_na_{n+1} \ldots a_{n+r-1}}$

where, $a_1, a_2, a_3, \ldots, a_n, \ldots$ are in AP.

***Solution of form* I** Let S_n be the sum and T_n be the nth term of the series, then

$$S_n = a_1a_2 \ldots a_r + a_2a_3 \ldots a_{r+1} + \ldots + a_na_{n+1} + \ldots + a_{n+r-1}$$

$$\therefore \quad T_n = a_na_{n+1}a_{n+2} \ldots a_{n+r-2}\, a_{n+r-1} \quad \ldots(\text{i})$$

Let $V_n = a_na_{n+1}a_{n+2} \ldots a_{n+r-2}\, a_{n+r-1}\, a_{n+r}$

[taking one extra factor in T_n for V_n]

$$\therefore \quad V_{n-1} = a_{n-1}a_n\, a_{n+1} \ldots a_{n+r-3}\, a_{n+r-2}\, a_{n+r-1}$$

$$\Rightarrow \quad V_n - V_{n-1} = a_na_{n+1}a_{n+2} \ldots a_{n+r-1}(a_{n+r} - a_{n-1}) = T_n(a_{n+r} - a_{n-1}) \quad [\text{from Eq. (i)}] \ldots(\text{ii})$$

Let d be the common difference of AP, then

$$a_n = a_1 + (n-1)d$$

Then, from Eq. (ii)

$$V_n - V_{n-1} = T_n[\{a_1 + (n+r-1)d\} - \{a_1 + (n-2)d\}] = (r+1)d\,T_n$$

$$\Rightarrow \quad T_n = \frac{1}{(r+1)d}(V_n - V_{n-1})$$

$$\therefore \quad S_n = \Sigma T_n = \sum_{n=1}^{n} T_n = \frac{1}{(r+1)d}\sum_{n=1}^{n}(V_n - V_{n-1})$$

$$= \frac{1}{(r+1)d}(V_n - V_0)$$

[from important Theorem 1 of Σ]

$$= \frac{1}{(r+1)(a_2 - a_1)}(a_na_{n+1} \ldots a_{n+r} - a_0a_1 \ldots a_r)$$

***Corollary* I** If $a_1, a_2, a_3, \ldots, a_n, \ldots$ are in AP, then

(i) For $r = 2$, $a_1a_2 + a_2a_3 + \ldots + a_na_{n+1} = \frac{1}{3(a_2 - a_1)}(a_na_{n+1}a_{n+2} - a_0a_1a_2)$

(ii) For $r = 3$,

$$a_1a_2a_3 + a_2a_3a_4 + \ldots + a_na_{n+1}a_{n+2} = \frac{1}{4(a_2 - a_1)}(a_na_{n+1}a_{n+2}a_{n+3} - a_0a_1a_2a_3)$$

***Corollary* II**

(i) $1\cdot 2 + 2\cdot 3 + \ldots + n(n+1) = \frac{1}{3}\{n(n+1)(n+2) - 0\cdot 1\cdot 2\} = \frac{n(n+1)(n+2)}{3}$

(ii) $1\cdot 3\cdot 5\cdot 7 + 3\cdot 5\cdot 7\cdot 9 + \ldots + (2n-1)\cdot(2n+1)(2n+5)\cdot(2n+3)\cdot$

$$= \frac{1}{11}\{(2n-1)(2n+1)(2n+3)(2n+5)(2n+7) - (-1)\cdot 1\cdot 3\cdot 5\cdot 7\}$$

$$= \frac{1}{11}\{(2n-1)(2n+1)(2n+3)(2n+5)(2n+7) + 105\}$$

***Solution of form* II** Let d be the common difference of AP, then $a_n = a_1 + (n-1)d$

Let sum of the series and nth term are denoted by S_n and T_n, respectively. Then,

$$S_n = \frac{1}{a_1a_2 \ldots a_r} + \frac{1}{a_2a_3 \ldots a_{r+1}} + \ldots + \frac{1}{a_na_{n+1} \ldots a_{n+r-1}}$$

$$\therefore \quad T_n = \frac{1}{a_na_{n+1}a_{n+2} \ldots a_{n+r-2}\, a_{n+r-1}} \quad \ldots(\text{i})$$

Let $V_n = \frac{1}{a_{n+1}\, a_{n+2} \ldots a_{n+r-2}\, a_{n+r-1}} \quad \ldots(\text{ii})$

[leaving first factor from denominator of T_n]

So, $V_{n-1} = \frac{1}{a_na_{n+1} \ldots a_{n+r-3}\, a_{n+r-2}}$

$$\Rightarrow \quad V_n - V_{n-1} = \frac{1}{a_{n+1}\, a_{n+2} \ldots a_{n+r-2}\, a_{n+r-1}} - \frac{1}{a_na_{n+1} \ldots a_{n+r-3}\, a_{n+r-2}}$$

$$= \frac{a_n - a_{n+r-1}}{a_na_{n+1}\, a_{n+2} - a_{n+r-2}\, a_{n+r-1}}$$

$$= T_n(a_n - a_{n+r-1}) \quad [\text{from Eq. (i)}]$$

$$= T_n[\{a_1 + (n-1)d\} - \{a_1 + (n+r-2)d\}]$$

$$= d(1-r)\,T_n$$

$$\therefore \quad T_n = \frac{(V_n - V_{n-1})}{d(1-r)}$$

$$\therefore \quad S_n = \Sigma T_n = \sum_{n=1}^{n}\frac{(V_n - V_{n-1})}{d(1-r)} = \frac{1}{d(1-r)}(V_n - V_0)$$

[from important Theorem 1 of Σ]

$$= \frac{1}{(a_2 - a_1)(1-r)}\left\{\frac{1}{a_{n+1}\, a_{n+2} \ldots a_{n+r-2}\, a_{n+r-1}} - \frac{1}{a_1\, a_2 \ldots a_{r-2}\, a_{r-1}}\right\}$$

Hence, the sum of n terms is $S_n = \frac{1}{(r-1)(a_2 - a_1)}$

$$\left\{\frac{1}{a_1 a_2 \ldots a_{r-1}} - \frac{1}{a_{n+1}\, a_{n+2} \ldots a_{n+r-1}}\right\}$$

***Corollary* I** If $a_1, a_2, a_3, \ldots, a_n, \ldots$ are in AP, then

(i) For $r = 2$,

$$\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \ldots + \frac{1}{a_n a_{n+1}} = \frac{1}{(a_2 - a_1)}$$

$$\left\{\frac{1}{a_1} - \frac{1}{a_{n+1}}\right\} = \frac{1}{d}\left(\frac{a_{n+1} - a_1}{a_1\, a_{n+1}}\right)$$

$$= \frac{1}{d}\left(\frac{a_1 + nd - a_1}{a_1 a_{n+1}}\right) = \frac{n}{a_1\, a_{n+1}}$$

(ii) For $r = 3$, $\frac{1}{a_1 a_2 a_3} + \frac{1}{a_2 a_3 a_4} + \ldots + \frac{1}{a_n\, a_{n+1}\, a_{n+2}}$

$$= \frac{1}{2(a_2 - a_1)}\left\{\frac{1}{a_1 a_2} - \frac{1}{a_{n+1}\, a_{n+2}}\right\}$$

(iii) For $r = 4$,

$$\frac{1}{a_1 a_2 a_3 a_4} + \frac{1}{a_2 a_3 a_4 a_5} + \ldots + \frac{1}{a_n a_{n+1} a_{n+2} a_{n+3}}$$

$$= \frac{1}{3(a_2 - a_1)}\left\{\frac{1}{a_1 a_2 a_3} - \frac{1}{a_{n+1}\, a_{n+2}\, a_{n+3}}\right\}$$

***Corollary* II**

(i) $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

(ii) $\frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \frac{1}{3\cdot 4\cdot 5} + \ldots + \frac{1}{n(n+1)(n+2)}$

$$= \frac{1}{2}\left\{\frac{1}{1\cdot 2} - \frac{1}{(n+1)(n+2)}\right\} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$$

(iii) $\frac{1}{1\cdot 3\cdot 5\cdot 7} + \frac{1}{3\cdot 5\cdot 7\cdot 9}$

$$+ \ldots + \frac{1}{(2n-1)(2n+1)(2n+3)(2n+5)}$$

$$= \frac{1}{6}\left\{\frac{1}{1\cdot 3\cdot 5} - \frac{1}{(2n+1)(2n+3)(2n+5)}\right\}$$

$$= \frac{1}{90} - \frac{1}{6(2n+1)(2n+3)(2n+5)}$$

Example 102. Find the sum upto n terms of the series $1\cdot 4\cdot 7\cdot 10 + 4\cdot 7\cdot 10\cdot 13 + 7\cdot 10\cdot 13\cdot 16 + \ldots$

Sol. Let T_n be the nth term of the given series.

$\therefore T_n = (n\text{th term of } 1, 4, 7, \ldots)(n\text{th term of } 4, 7, 10, \ldots)$

$(n\text{th term of } 7, 10, 13, \ldots)(n\text{th term of } 10, 13, 16, \ldots)$

$$T_n = (3n-2)(3n+1)(3n+4)(3n+7) \quad \ldots(\text{i})$$

$$\therefore V_n = (3n-2)(3n+1)(3n+4)(3n+7)(3n+10)$$

$$V_{n-1} = (3n-5)(3n-2)(3n+1)(3n+4)(3n+7)$$

$$\Rightarrow \quad V_n = (3n+10)\, T_n \quad [\text{from Eq. (i)}]$$

and $\quad V_{n-1} = (3n-5)\, T_n$

$$\therefore \quad V_n - V_{n-1} = 15\, T_n$$

$$\therefore \quad T_n = \frac{1}{15}(V_n - V_{n-1}))$$

$$\therefore \quad S_n = \Sigma T_n = \sum_{n=1}^{n} \frac{1}{15}(V_n - V_{n-1})$$

$$= \frac{1}{15}(V_n - V_0)$$

[from important Theorem 1 of Σ]

$$= \frac{1}{15}\{(3n-2)(3n+1)(3n+4)(3n+7)(3n+10) - (-2)(1)(4)(7)(10)\}$$

$$= \frac{1}{15}\{(3n-2)(3n+1)(3n+4)(3n+7)(3n+10) + 560\}$$

Shortcut Method

$$S_n = \frac{1}{(\text{last factor of III term} - \text{first factor of I term})}$$

(Taking one extra factor in T_n in last − Taking one extra factor in I term in start)

$$= \frac{1}{(16-1)}\{(3n-2)(3n+1)(3n+4)(3n+7)(3n+10) - (-2)\cdot 1\cdot 4\cdot 7\cdot 10\}$$

$$= \frac{1}{15}\{(3n-2)(3n+1)(3n+4)(3n+7)(3n+10) + 560\}$$

Example 103. Find the sum to n terms of the series $\frac{1}{1\cdot 3\cdot 5\cdot 7\cdot 9} + \frac{1}{3\cdot 5\cdot 7\cdot 9\cdot 11} + \frac{1}{5\cdot 7\cdot 9\cdot 11\cdot 13} + \ldots$.
Also, find the sum to infinity terms.

Sol. Let T_n be the nth term of the given series.

Then, $\quad T_n = \frac{1}{(2n-1)(2n+1)(2n+3)(2n+5)(2n+7)} \quad \ldots(\text{i})$

$$\therefore \quad V_n = \frac{1}{(2n+1)(2n+3)(2n+5)(2n+7)}$$

[leaving first factor from denominator of T_n]

$$V_{n-1} = \frac{1}{(2n-1)(2n+1)(2n+3)(2n+5)}$$

$$\Rightarrow \quad V_n - V_{n-1} = \frac{1}{(2n+1)(2n+3)(2n+5)(2n+7)} - \frac{1}{(2n-1)(2n+1)(2n+3)(2n+5)}$$

$$= \frac{(2n-1) - (2n+7)}{(2n-1)(2n+1)(2n+3)(2n+5)(2n+7)}$$

$$= -8\, T_n \quad [\text{from Eq. (i)}]$$

$$\therefore \quad T_n = -\frac{1}{8}(V_n - V_{n-1})$$

$$\therefore \quad S_n = \Sigma T_n = \sum_{n=1}^{n} T_n = -\frac{1}{8}\sum_{n=1}^{n}(V_n - V_{n-1}) = -\frac{1}{8}(V_n - V_0)$$

[from Important Theorem 1 of Σ]

$$= \frac{1}{8}(V_0 - V_n)$$

$$= \frac{1}{8}\left\{\frac{1}{1\cdot3\cdot5\cdot7} - \frac{1}{(2n+1)(2n+3)(2n+5)(2n+7)}\right\}$$

$$= \frac{1}{840} - \frac{1}{8(2n+1)(2n+3)(2n+5)(2n+7)}$$

and $S_\infty = \frac{1}{840} - \frac{1}{\infty} = \frac{1}{840} - 0 = \frac{1}{840}$

Shortcut Method

$$\frac{1}{1\cdot3\cdot5\cdot7\cdot9} + \frac{1}{3\cdot5\cdot7\cdot9\cdot11} + \frac{1}{5\cdot7\cdot9\cdot11\cdot13} + \ldots + \frac{1}{(2n-1)(2n+1)(2n+3)(2n+5)(2n+7)} \quad \ldots(i)$$

Now, in each term in denominator

$$9-1 = 11-3 = 13-5 = \ldots = (2n+7)-(2n-1) = 8$$

Then, Eq. (i) can be written as

$$= \frac{1}{8}\left\{\frac{9-1}{1\cdot3\cdot5\cdot7\cdot9} + \frac{11-3}{3\cdot5\cdot7\cdot9\cdot11} + \frac{13-5}{5\cdot7\cdot9\cdot11\cdot13} + \ldots + \frac{(2n+7)-(2n-1)}{(2n-1)(2n+1)(2n+3)(2n+5)(2n+7)}\right\}$$

$$= \frac{1}{8}\left\{\frac{1}{1\cdot3\cdot5\cdot7} - \frac{1}{3\cdot5\cdot7\cdot9} + \frac{1}{3\cdot5\cdot7\cdot9} - \frac{1}{5\cdot7\cdot9\cdot11} + \frac{1}{5\cdot7\cdot9\cdot11} - \frac{1}{7\cdot9\cdot11\cdot13} + \ldots + \frac{1}{(2n-1)(2n+1)(2n+3)(2n+5)} - \frac{1}{(2n+1)(2n+3)(2n+5)(2n+7)}\right\}$$

$$= \frac{1}{8}\left\{\frac{1}{1\cdot3\cdot5\cdot7} - \frac{1}{(2n+1)(2n+3)(2n+5)(2n+7)}\right\}$$

[middle terms are cancelled out]

$$= \frac{1}{840} - \frac{1}{8(2n+1)(2n+3)(2n+5)(2n+7)} = S_n \quad \text{[say]}$$

$\therefore$ Sum to infinity terms $= S_\infty = \frac{1}{840} - 0 = \frac{1}{840}$

Maha Shortcut Method

Taking $\frac{1}{8}$ outside the bracket

$\left(\text{i.e. } \frac{1}{9-1} = \frac{1}{11-3} = \frac{1}{13-5} = \ldots\right)$ and in bracket leaving last factor of denominator of first term – leaving first factor of denominator of last term

i.e., $S_n = \frac{1}{8}\left(\frac{1}{1\cdot3\cdot5\cdot7} - \frac{1}{(2n+1)(2n+3)(2n+5)(2n+7)}\right)$

$$\therefore \quad S_\infty = \frac{1}{8}\left(\frac{1}{1\cdot3\cdot5\cdot7} - 0\right) = \frac{1}{840}$$

Example 104. ***If*** $\sum_{r=1}^{n} T_r = \frac{n(n+1)(n+2)(n+3)}{12}$, ***where*** T_r ***denotes the*** r***th term of the series. Find*** $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{T_r}$.

Sol. We have, $T_n = \sum_{r=1}^{n} T_r - \sum_{r=1}^{n-1} T_r$

$$= \frac{n(n+1)(n+2)(n+3)}{12} - \frac{(n-1)n(n+1)(n+2)}{12}$$

$$= \frac{n(n+1)(n+2)}{12}[(n+3)-(n-1)]$$

$$= \frac{n(n+1)(n+2)}{3} \quad \frac{1}{T_n} = \frac{3}{n(n+1)(n+2)}$$

$$\therefore \quad \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{T_r} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{3}{r(r+1)(r+2)}$$

$$= 3\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}$$

$$= 3\lim_{n\to\infty}\left(\frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + \frac{1}{3\cdot4\cdot5} + \ldots + \frac{1}{n(n+1)(n+2)}\right)$$

Maha Shortcut Method

$$= 3\lim_{n\to\infty}\frac{1}{2}\left(\frac{1}{1\cdot2} - \frac{1}{(n+1)(n+2)}\right)$$

$$= \frac{3}{2}\left(\frac{1}{2} - 0\right) = \frac{3}{4}$$

Exercise for Session 6

1. The sum of the first n terms of the series $\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+\ldots$ is

(a) $2^n - n - 1$ (b) $1 - 2^{-n}$ (c) $n + 2^{-n} - 1$ (d) $2^n - 1$

2. $2^{1/4} \cdot 4^{1/8} \cdot 8^{1/16} \cdot 16^{1/32} \ldots$ is equal to

(a) 1 (b) $\frac{3}{2}$ (c) 2 (d) $\frac{5}{2}$

3. $1 + 3 + 7 + 15 + 31 + \ldots$ upto n terms equals

(a) $2^{n+1} - n$ (b) $2^{n+1} - n - 2$ (c) $2^n - n - 2$ (d) None of these

4. 99th term of the series $2 + 7 + 14 + 23 + 34 + \ldots$ is

(a) 9998 (b) 9999 (c) 10000 (d) 100000

5. The sum of the series $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \ldots$ upto n terms is

(a) $n\,(n+1)\,(n+2)$ (b) $(n+1)\,(n+2)\,(n+3)$

(c) $\frac{1}{4}n\,(n+1)\,(n+2)\,(n+3)$ (d) $\frac{1}{4}\,(n+1)\,(n+2)\,(n+3)$

6. $\frac{1}{1 \cdot 2}+\frac{1}{2 \cdot 3}+\frac{1}{3 \cdot 4}+\ldots+\frac{1}{n\,(n+1)}$ equals

(a) $\frac{1}{n\,(n+1)}$ (b) $\frac{n}{n+1}$

(c) $\frac{2n}{n+1}$ (d) $\frac{2}{n\,(n+1)}$

7. Sum of the n terms of the series $\frac{3}{1^2}+\frac{5}{1^2+2^2}+\frac{7}{1^2+2^2+3^3}+\ldots$ is

(a) $\frac{2n}{n+1}$ (b) $\frac{4n}{n+1}$

(c) $\frac{6n}{n+1}$ (d) $\frac{9n}{n+1}$

8. If $t_n = \frac{1}{4}(n+2)(n+3)$ for $n = 1, 2, 3, \ldots$, then $\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}+\ldots+\frac{1}{t_{2003}}$ equals

(a) $\frac{4006}{3006}$ (b) $\frac{4003}{3007}$

(c) $\frac{4006}{3008}$ (d) $\frac{4006}{3009}$

9. The value of $\frac{1}{(1+a)(2+a)}+\frac{1}{(2+a)(3+a)}+\frac{1}{(3+a)(4+a)}+\ldots$ upto ∞ is

(where, a is constant)

(a) $\frac{1}{1+a}$ (b) $\frac{2}{1+a}$

(c) ∞ (d) None of these

10. If $f(x)$ is a function satisfying $f(x+y) = f(x)\,f(y)$ for all $x, y \in N$ such that $f(1) = 3$ and $\sum_{x=1}^{n} f(x) = 120$. Then, the value of n is

(a) 4 (b) 5 (c) 6 (d) None of these

Session 7

Application to Problems of Maxima and Minima (Without Calculus)

Application to Problems of Maxima and Minima (Without Calculus)

Suppose that $a_1, a_2, a_3, ..., a_n$ are n positive variables and k is constant, then

(i) If $a_1 + a_2 + a_3 + ... + a_n = k$ (constant), the value of $a_1 a_2 a_3 \ldots a_n$ is greatest when $a_1 = a_2 = a_3 = ... = a_n$, so that the greatest value of $a_1 a_2 a_3 \ldots a_n$ is $\left(\frac{k}{n}\right)^n$.

Proof $\because$ AM $\geq$ GM

$$\therefore \quad \frac{a_1 + a_2 + a_3 + \ldots + a_n}{n} \geq (a_1 a_2 a_3 \ldots a_n)^{1/n}$$

$$\Rightarrow \quad \frac{k}{n} \geq (a_1\ a_2\ a_3 \ldots a_n)^{1/n}$$

$$\text{or} \quad (a_1 a_2 a_3 \ldots a_n) \leq \left(\frac{k}{n}\right)^n$$

Here, $a_1 = a_2 = a_3 = \ldots = a_n$

$\therefore$ Greatest value of $a_1 a_2 a_3 \ldots a_n$ is $\left(\frac{k}{n}\right)^n$.

Example 105. Find the greatest value of xyz for positive values of x, y, z subject to the condition $yz + zx + xy = 12$.

Sol. Given, $yz + zx + xy = 12$ (constant), the value of $(yz)(zx)(xy)$ is greatest when $yz = zx = xy$

Here, $n = 3$ and $k = 12$

Hence, greatest value of $(yz)(zx)(xy)$ is $\left(\frac{12}{3}\right)^3$ i.e. 64.

$\therefore$ Greatest value of $x^2y^2z^2$ is 64.

Thus, greatest value of xyz is 8.

Aliter

Given $yz + zx + xy = 12$, the greatest value of $(yz)(zx)(xy)$ is greatest when

$$yz = zx = xy = c \qquad \text{[say]}$$

Since, $yz + zx + xy = 12$

$\therefore \quad c + c + c = 12$

$\Rightarrow \quad 3c = 12$ or $c = 4$

$\therefore \quad yz = zx = xy = 4$

Hence, greatest value of $(yz)(zx)(xy)$ is $4 \cdot 4 \cdot 4$

i.e., greatest value of $x^2y^2z^2$ is 64.

Hence, greatest value of xyz is 8.

Example 106. Find the greatest value of x^3y^4, if $2x + 3y = 7$ and $x \geq 0, y \geq 0$.

Sol. To find the greatest value of x^3y^4 or $(x)(x)(x)(y)(y)(y)(y)$

Here, x repeats 3 times and y repeats 4 times.

Given, $2x + 3y = 7$,

then multiplying and dividing coefficients of x and y by 3 and 4, respectively.

Rewrite $3\left(\frac{2x}{3}\right) + 4\left(\frac{3y}{4}\right) = 7$

$$\text{or} \underset{1}{\left(\frac{2x}{3}\right)} + \underset{2}{\left(\frac{2x}{3}\right)} + \underset{3}{\left(\frac{2x}{3}\right)} + \underset{4}{\left(\frac{3y}{4}\right)} + \underset{5}{\left(\frac{3y}{4}\right)} + \underset{6}{\left(\frac{3y}{4}\right)} + \underset{7}{\left(\frac{3y}{4}\right)} = 7$$

Here, $k = 7$ and $n = 7$

Hence, greatest value of

$$\left(\frac{2x}{3}\right)\left(\frac{2x}{3}\right)\left(\frac{2x}{3}\right)\left(\frac{3y}{4}\right)\left(\frac{3y}{4}\right)\left(\frac{3y}{4}\right)\left(\frac{3y}{4}\right) \text{ is } \left(\frac{7}{7}\right)^7.$$

or greatest value of $\frac{2^3 \cdot 3^4}{3^3 \cdot 4^4} x^3y^4$ is 1.

Thus, greatest value of x^3y^4 is $\frac{32}{3}$.

(ii) If $a_1 a_2 a_3 \ldots a_n = k$ (constant), the value of $a_1 + a_2 + a_3 + \ldots + a_n$ is least when $a_1 = a_2 = a_3 = \ldots = a_n$, so that the least of $a_1 + a_2 + a_3 + \ldots + a_n$ is $n(k)^{1/n}$.

Proof $\because$ AM $\geq$ GM

$$\therefore \frac{a_1 + a_2 + a_3 + \ldots + a_n}{n} \geq (a_1 a_2 a_3 \ldots a_n)^{1/n} = (k)^{1/n}$$

$$\Rightarrow \quad \frac{a_1 + a_2 + a_3 + \ldots + a_n}{n} \geq (k)^{1/n}$$

$$\text{or} \quad a_1 + a_2 + a_3 + \ldots + a_n \geq n(k)^{1/n}$$

Here, $a_1 = a_2 = a_3 = \ldots = a_n$

$\therefore$ Least value of $a_1 + a_2 + a_3 + \ldots + a_n$ is $n(k)^{1/n}$

Example 107. ***Find the least value of*** $3x+4y$ ***for positive values of*** x ***and*** y***, subject to the condition*** $x^2y^3=6$**.**

Sol. Given, $x^2y^3=6$

or $(x)(x)(y)(y)(y)=6$

Here, x repeats 2 times and y repeats 3 times

$$\therefore \quad 3x+4y=2\left(\frac{3x}{2}\right)+3\left(\frac{4y}{3}\right)$$

$$=\underset{1}{\left(\frac{3x}{2}\right)}+\underset{2}{\left(\frac{3x}{2}\right)}+\underset{3}{\left(\frac{4y}{3}\right)}+\underset{4}{\left(\frac{4y}{3}\right)}+\underset{5}{\left(\frac{4y}{3}\right)}$$

multiplying and dividing coefficient of x and y by 2 and 3 respectively and write $x^2y^3=6$

$$\Rightarrow\left(\frac{3x}{2}\right)\left(\frac{3x}{2}\right)\left(\frac{4y}{3}\right)\left(\frac{4y}{3}\right)\left(\frac{4y}{3}\right)=\frac{3^2}{2^2}\times\frac{4^3}{3^3}\times 6=32$$

Here, $n=5$ and $k=32$

Hence, least value of $\frac{3x}{2}+\frac{3x}{2}+\frac{4y}{3}+\frac{4y}{3}+\frac{4y}{3}$

$$=5(32)^{1/5}=10$$

i.e. least value of $3x+4y=10$

Example 108. Find the minimum value of $bcx+cay+abz$, when $xyz=abc$.

Sol. To find the minimum value of

$$bcx+cay+abz,$$

write, $xyz=abc$

or $(bcx)(cay)(abz)=a^3b^3c^3=k$ [constant]

Here, $n=3$

Hence, minimum value of $bcx+cay+abz=n(k)^{1/n}$

$$=3(a^3b^3c^3)^{1/3}=3abc$$

An Important Result

If $a_i>0,\ i=1,2,3,\ldots,n$ which are not identical, then

(i) $\dfrac{a_1^m+a_2^m+\ldots+a_n^m}{n}>\left(\dfrac{a_1+a_2+\ldots+a_n}{n}\right)^m$; If $m<0$ or $m>1$

(ii) $\dfrac{a_1^m+a_2^m+\ldots+a_n^m}{n}<\left(\dfrac{a_1+a_2+\ldots+a_n}{n}\right)^m$;

If $0<m<1$

Remark

If $a_1=a_2=\ldots.=a_n$, then use equal sign in inequalities.

Example 109. If a,b,c be positive real numbers, prove that $\dfrac{a}{b+c}+\dfrac{b}{c+a}+\dfrac{c}{a+b}\geq\dfrac{3}{2}$.

Sol. Arithmetic mean of (-1) th powers

$\geq(-1)$ th power of arithmetic mean

$$\frac{\left(\frac{b+c}{a+b+c}\right)^{-1}+\left(\frac{c+a}{a+b+c}\right)^{-1}+\left(\frac{a+b}{a+b+c}\right)^{-1}}{3}$$

$$\geq\left(\frac{\frac{b+c}{a+b+c}+\frac{c+a}{a+b+c}+\frac{a+b}{a+b+c}}{3}\right)^{-1}$$

$$\Rightarrow\quad \frac{\frac{a+b+c}{b+c}+\frac{a+b+c}{c+a}+\frac{a+b+c}{a+b}}{3}\geq\left(\frac{2}{3}\right)^{-1}$$

$$\Rightarrow\quad \frac{a}{b+c}+1+\frac{b}{c+a}+1+\frac{c}{a+b}+1\geq\frac{9}{2}$$

$$\Rightarrow\quad \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{9}{2}-3$$

or $\dfrac{a}{b+c}+\dfrac{b}{c+a}+\dfrac{c}{a+b}\geq\dfrac{3}{2}$

Example 110. ***If*** a ***and*** b ***are positive and*** $a+b=1$***, show that*** $\left(a+\dfrac{1}{a}\right)^2+\left(b+\dfrac{1}{b}\right)^2>\dfrac{25}{2}$**.**

Sol. Since, AM of 2nd powers > 2nd power of AM

$$\therefore\quad \frac{\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2}{2}>\left(\frac{a+\frac{1}{a}+b+\frac{1}{b}}{2}\right)^2$$

$$=\frac{1}{4}(a+b+a^{-1}+b^{-1})^2=\frac{1}{4}(1+a^{-1}+b^{-1})^2 \qquad [\because a+b=1]$$

$$\therefore\quad \left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2>\frac{1}{2}(1+a^{-1}+b^{-1})^2 \qquad \ldots(\text{i})$$

Again, $\dfrac{a^{-1}+b^{-1}}{2}>\left(\dfrac{a+b}{2}\right)^{-1}=\left(\dfrac{1}{2}\right)^{-1}=2$

or $\dfrac{a^{-1}+b^{-1}}{2}>2$

$$\Rightarrow\quad a^{-1}+b^{-1}>4$$

$$\therefore\quad (1+a^{-1}+b^{-1})>5 \text{ or } (1+a^{-1}+b^{-1})^2>25$$

$$\Rightarrow\quad \frac{1}{2}(1+a^{-1}+b^{-1})^2>\frac{25}{2} \qquad \ldots(\text{ii})$$

From Eqs. (i) and (ii), we get

$$\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2>\frac{25}{2}$$

Exercise for Session 7

1. The minimum value of $4^x + 4^{2-x}$, $x \in R$ is

(a) 0 (b) 2
(c) 4 (d) 8

2. If $0 < \theta < \pi$, then the minimum value of $\sin^3\theta + \text{cosec}^3\theta + 2$, is

(a) 0 (b) 2
(c) 4 (d) 8

3. If a, b, c and d are four real numbers of the same sign, then the value of $\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}$ lies in the interval

(a) $[2, \infty)$ (b) $[3, \infty)$
(c) $(4, \infty)$ (d) $[4, \infty)$

4. If $0 < x < \frac{\pi}{2}$, then the minimum value of $2(\sin x + \cos x + \text{cosec}\, 2x)^3$ is

(a) 27 (b) 13.5
(c) 6.75 (d) 0

5. If $a + b + c = 3$ and $a > 0, b > 0, c > 0$, then the greatest value of $a^2b^3c^2$ is

(a) $\frac{3^4 \cdot 2^{10}}{7^7}$ (b) $\frac{3^{10} \cdot 2^4}{7^7}$
(c) $\frac{3^2 \cdot 2^{12}}{7^7}$ (d) $\frac{3^{12} \cdot 2^2}{7^7}$

6. If $x + y + z = a$ and the minimum value of $\frac{a}{x} + \frac{a}{y} + \frac{a}{z}$ is 81^{λ}, then the value of λ is

(a) $\frac{1}{2}$ (b) 1
(c) $\frac{1}{4}$ (d) 2

7. a, b, c are three positive numbers and abc^2 has the greatest value $\frac{1}{64}$, then

(a) $a = b = \frac{1}{2}, c = \frac{1}{4}$ (b) $a = b = c = \frac{1}{3}$
(c) $a = b = \frac{1}{4}, c = \frac{1}{2}$ (d) $a = b = c = \frac{1}{4}$

Shortcuts and Important Results to Remember

1 If $T_n = An + B$, i.e. nth term of an AP is a linear expression in n, where A, B are constants, then coefficient of n i.e., A is the common difference.

2 If $S_n = Cn^2 + Dn$ is the sum of n terms of an AP, where C and D are constants, then common difference of AP is $2C$ i.e., 2 times the coefficient of n^2.

3 (i) $d = T_n - T_{n-1}\ [n \geq 2]$ (ii) $T_n = S_n - S_{n-1}\ [n \geq 2]$
(iii) $d = S_n - 2S_{n-1} + S_{n-2}\ [n \geq 3]$

4 If for two different AP's

$$\frac{S_n}{S_n'} = \frac{An^2 + Bn}{Cn^2 + Cn} \quad \text{or} \quad \frac{An + B}{Cn + D}$$

Then, $\dfrac{T_n}{T_n'} = \dfrac{A(2n-1) + B}{C(2n-1) + D}$

5 If for two different AP's

$$\frac{T_n}{T_n'} = \frac{An + B}{Cn + D}, \text{ then } \frac{S_n}{S_n'} = \frac{A\left(\frac{n+1}{2}\right) + B}{C\left(\frac{n+1}{2}\right) + D}$$

6 If $T_p = q$ and $T_q = p$, then $T_{p+q} = 0, T_r = p + q - r$

7 If $pT_p = qT_q$ of an AP, then $T_{p+q} = 0$

8 If $S_p = S_q$ for an AP, then $S_{p+q} = 0$

9 If $S_p = q$ and $S_q = p$ of an AP, then $S_{p+q} = -(p+q)$

10 If $T_p = P$ and $T_q = Q$ for a GP, then $T_n = \left[\dfrac{P^{n-q}}{Q^{n-p}}\right]^{1/(p-q)}$

11 If $T_{m+n} = p, T_{m-n} = q$ for a GP, then

$$T_m = \sqrt{pq}, \quad T_n = p\left(\frac{q}{p}\right)^{m/2n}$$

12 If $T_m = n, T_n = m$ for a HP, then

$$T_{m+n} = \frac{mn}{(m+n)}, T_{mn} = 1, T_p = \frac{mn}{p}$$

13 If $T_p = qr, T_q = pr$ for a HP, then $T_r = pq$

14 No term of HP can be zero and there is no formula to find S_n for HP.

15 a, b, c are in AP, GP or HP as $\dfrac{a-b}{b-c} = \dfrac{a}{a}$ or $\dfrac{a}{b}$ or $\dfrac{a}{c}$.

16 If A, G, H be AM, GM and HM between a and b, then

$$\frac{a^{n+1} + b^{n+1}}{a^n + b^n} = \begin{cases} A, & \text{when } n = 0 \\ G, & \text{when } n = -\dfrac{1}{2} \\ H, & \text{when } n = -1 \end{cases}$$

17 If A and G are the AM and GM between two numbers a, b, then a, b are given by $A \pm \sqrt{(A+G)(A-G)}$

18 If a, b, c are in GP, then $a + b, 2b, b + c$ are in HP.

19 If a, b, c are in AP, then $\lambda^a, \lambda^b, \lambda^c$ are in GP, where $\lambda > 0, \lambda \neq 1$.

20 If $-1 < r < 1$, then GP is said to be convergent, if $r < -1$ or $r > 1$, then GP is said to be divergent and if $r = -1$, then series is oscillating.

21 If a, b, c, d are in GP, then
$(a \pm b)^n, (b \pm c)^n, (c \pm d)^n$ are in GP, $\forall\ n \in I$

22 If a, b, c are in AP as well as in GP, then $a = b = c$.

23 The equations $a_1x + a_2y = a_3$, $a_4x + a_5y = a_6$ has a unique solution, if $a_1, a_2, a_3, a_4, a_5, a_6$ are in AP and common difference $\neq 0$.

24 For n positive quantities $a_1, a_2, a_3, \ldots, a_n$

$$\text{AM} \geq \text{GM} \geq \text{HM}$$

sign of equality (AM = GM = HM) holds when quantities are equal

i.e. $a_1 = a_2 = a_3 = \ldots = a_n$.

25 For two positive numbers a and b (AM) (HM) = $(\text{GM})^2$, the result will be true for n numbers, if they are in GP.

26 If odd numbers of (say $2n + 1$) AM's, GM's and HM's be inserted between two numbers, then their middle means [i.e., $(n+1)$ th mean] are in GP.

27 If a^2, b^2, c^2 are in AP.

$\Rightarrow$ $\dfrac{1}{b+c}, \dfrac{1}{c+a}, \dfrac{1}{a+b}$ are in AP.

28 Coefficient of x^{n-1} and x^{n-2} in
$(x - a_1)(x - a_2)(x - a_3)\ldots(x - a_n)$
are $-(a_1 + a_2 + a_3 + \ldots + a_n)$ and Σa_1a_2, respectively
where, $\Sigma a_1a_2 = \dfrac{(\Sigma a_1)^2 - \Sigma a_1^2}{2}$.

29 $1 + 3 + 5 + \ldots$ upto n terms $= n^2$

30 $2 + 6 + 12 + 20 + \ldots$ upto n terms $= \dfrac{n(n+1)(n+2)}{3}$

31 $1 + 3 + 7 + 13 + \ldots$ upto n terms $= \dfrac{n(n^2+2)}{3}$

32 $1 + 5 + 14 + 30 + \ldots$ upto n terms $= \dfrac{n(n+1)^2(n+2)}{12}$

33 If $a_1, a_2, a_3, \ldots, a_n$ are the non-zero terms of a non-constant AP, then

$$\frac{1}{a_1a_2} + \frac{1}{a_2a_3} + \frac{1}{a_3a_4} + \ldots + \frac{1}{a_{n-1}a_n} = \frac{(n-1)}{a_1a_n}$$

JEE Type Solved Examples :
Single Option Correct Type Questions

■ This section contains **10 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

● ***Ex. 1*** *If $b-c$, $2b-\lambda$, $b-a$ are in HP, then $a-\frac{\lambda}{2}$, $b-\frac{\lambda}{2}$, $c-\frac{\lambda}{2}$ are is*

(a) AP (b) GP
(c) HP (d) None of these

Sol. (b) $(2b-\lambda)=\frac{2(b-c)(b-a)}{(b-c)+(b-a)}$

$\Rightarrow \quad (2b-\lambda)=(2b-(a+c))=2[b^2-(a+c)b+ac]$

$\Rightarrow \quad 2b^2-2b\lambda+\lambda(a+c)-2ac=0$

$\Rightarrow \quad b^2-b\lambda+\frac{\lambda}{2}(a+c)-ac=0$

$\Rightarrow \left(b-\frac{\lambda}{2}\right)^2-\frac{\lambda^2}{4}+\frac{\lambda}{2}(a+c)-ac=0$

$\Rightarrow \quad \left(b-\frac{\lambda}{2}\right)^2=\frac{\lambda^2}{4}-\frac{\lambda}{2}(a+c)+ac$

$\Rightarrow \quad \left(b-\frac{\lambda}{2}\right)^2=\left(a-\frac{\lambda}{2}\right)\left(c-\frac{\lambda}{2}\right)$

Hence, $a-\frac{\lambda}{2}$, $b-\frac{\lambda}{2}$, $c-\frac{\lambda}{2}$ are in GP.

● ***Ex. 2*** *Let $a_1, a_2, a_3, \ldots, a_{10}$ are in GP with $a_{51}=25$ and $\sum_{i=1}^{101}a_i=125$, then the value of $\sum_{i=1}^{101}\left(\frac{1}{a_i}\right)$ equals*

(a) 5 (b) $\frac{1}{5}$ (c) $\frac{1}{25}$ (d) $\frac{1}{125}$

Sol. (b) Let 1st term be a and common ratio be r, then

$$\sum_{i=1}^{101}\frac{1}{a_i}=125$$

$\Rightarrow \quad (a_1+a_1r+a_1r^2+\ldots+a_1r^{100})=125$

$\Rightarrow \quad \frac{a_1(1-r^{101})}{(1-r)}=125 \qquad [\text{let } 0<r<1] \ldots(\text{i})$

$\therefore \quad \sum_{i=1}^{101}\frac{1}{a_i}=\frac{1}{a_i}+\frac{1}{a_ir}+\frac{1}{a_ir^2}+\ldots+\frac{1}{a_ir^{100}}=\frac{\frac{1}{a_i}\left[\left(\frac{1}{r}\right)^{101}-1\right]}{\left(\frac{1}{r}-1\right)}$

$\left[\text{here}\frac{1}{r}>1\right]$

$=\frac{(1-r^{101})}{a_1r^{100}(1-r)}=\frac{1}{a_1r^{100}}\times\frac{125}{a_1}$ [from Eq. (i)]

$=\frac{125}{(a_1r^{50})^2}=\frac{125}{(a_{51})^2}=\frac{125}{(25)^2}=\frac{1}{5}$

● ***Ex. 3*** *If $x=111\ldots1$ (20 digits), $y=333\ldots3$ (10 digits) and $z=222\ldots2$ (10 digits), then $\frac{x-y^2}{z}$ equals*

(a) $\frac{1}{2}$ (b) 1 (c) 2 (d) 4

Sol. (b) $\because x=\frac{1}{9}(999\ldots9)=\frac{1}{9}(10^{20}-1)$,

$y=\frac{1}{3}(999\ldots9)=\frac{1}{3}(10^{10}-1)$

and $z=\frac{2}{9}(999\ldots9)=\frac{2}{9}(10^{10}-1)]$

$\therefore \quad \frac{x-y^2}{z}=\frac{\frac{1}{9}(10^{20}-1)-\frac{1}{9}(10^{10}-1)^2}{\frac{2}{9}(10^{10}-1)}$

$=\frac{10^{10}+1-(10^{10}-1)}{2}=1$

● ***Ex. 4*** *Consider the sequence 1, 2, 2, 3, 3, 3, ..., where n occurs n times. The number that occurs as 2011th terms is*

(a) 61 (b) 62
(c) 63 (d) 64

Sol. (c) The last 4 occurs as $1+2+3+4=10$th term. The last n occurs as $\left(\frac{n(n+1)}{2}\right)^{\text{th}}$ term, the last 62 occurs as $\left(\frac{62\times63}{2}\right)^{\text{th}}=1953$rd term and the last 63 occurs as $\left(\frac{63\times64}{2}\right)^{\text{th}}=2016$th term.

$\therefore$ 63 occurs from 1954th term to 2016th term.

Hence, $(2011)^{\text{th}}$ term is 63.

● ***Ex. 5*** *Let $S=\sum_{r=1}^{117}\frac{1}{2[\sqrt{r}]+1}$, when $[\cdot]$ denotes the greatest integer function and if $S=\frac{p}{q}$, when p and q are co-primes, the value of $p+q$ is*

(a) 20 (b) 76 (c) 19 (d) 69

Sol. (b) $\because \quad S=\sum_{r=1}^{117}\frac{1}{2[\sqrt{r}]+1}$

$$=\frac{3}{2\cdot1+1}+\frac{5}{2\cdot2+1}+\frac{7}{2\cdot3+1}+\ldots+\frac{19}{2\cdot9+1}+\frac{18}{2\cdot10+1}$$

$$=9+\frac{18}{21}=9+\frac{6}{7}=\frac{69}{7}$$

$\therefore \quad p=69$ and $q=7 \Rightarrow p+q=69+7=76$

● **Ex. 6** *If a, b, c are non-zero real numbers, then the minimum value of the expression* $\frac{(a^8+4a^4+1)(b^4+3b^2+1)(c^2+2c+2)}{a^4b^2}$ *equals*

(a) 12 (b) 24 (c) 30 (d) 60

Sol. (c) Let $P=\frac{(a^8+4a^4+1)(b^4+3b^2+1)(c^2+2c+2)}{a^4b^2}$

$$=\left(a^4+4+\frac{1}{a^4}\right)\left(b^2+3+\frac{1}{b^2}\right)\{(c+1)^2+1)\}$$

$\because \quad =a^4+4+\frac{1}{a^4}\geq 6, b^2+3+\frac{1}{b^2}\geq 5$ and $(c+1)^2+1\geq 1$

$\left[\because x+\frac{1}{x}\geq 2 \text{ for } x>0\right]$

$\therefore \quad P\geq 6\cdot5\cdot1=30 \Rightarrow P\geq 30$

Hence, the required minimum value is 30.

● **Ex. 7** *If the sum of m consecutive odd integers is m^4, then the first integer is*

(a) m^3+m+1 (b) m^3+m-1

(c) m^3-m-1 (d) m^3-m+1

Sol. (d) Let $2a+1, 2a+3, 2a+5, \ldots$ be the AP, then

$$m^4=(2a+1)+(2a+3)+(2a+5)+\ldots \text{ upto } m \text{ terms}$$

$$=\frac{m}{2}\{2(2a+1)+(m-1)\cdot2\}=m(2a+1+m-1)$$

$\Rightarrow \quad m^3=(2a+1)+m-1$

$\therefore \quad 2a+1=m^3-m+1$

● **Ex. 8** *The value of* $\sum_{r=1}^{\infty}\frac{(4r+5)5^{-r}}{r(5r+5)}$ *is*

(a) $\frac{1}{5}$ (b) $\frac{2}{5}$ (c) $\frac{1}{25}$ (d) $\frac{2}{125}$

Sol. (a) $\sum_{r=1}^{\infty}\frac{(4r+5)5^{-r}}{r(5r+5)}=\lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{(5r+5)-r}{r(5r+5)}\right)\cdot\frac{1}{5^r}$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{5r+5}\right)\frac{1}{5^r}$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{1}{r\cdot5^{-r}}-\frac{1}{(r+1)5^{r+1}}\right)$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{1}{5}-\frac{1}{(n+1)5^{n+1}}\right)=\frac{1}{5}-0=\frac{1}{5}$$

● **Ex. 9** *Let λ be the greatest integer for which* $5p^2-16, 2p\lambda, \lambda^2$ *are distinct consecutive terms of an AP, where* $p\in R$. *If the common difference of the AP is* $\left(\frac{m}{n}\right)$, $m,n\in N$ *and m, n are relative prime, the value of* $m+n$ *is*

(a) 133 (b) 138 (c) 143 (d) 148

Sol. (c) $\because 5p^2-16, 2p\lambda, \lambda^2$ are in AP, then

$$4p\lambda=5p^2-16+\lambda^2$$

$\Rightarrow \quad 5p^2-4p\lambda+\lambda^2-16=0 \quad \ldots(i)$

$\therefore \quad B-4AC\geq 0 \quad [\because p\in R]$

$\Rightarrow \quad 16\lambda^2-4\cdot5\cdot(\lambda^2-16)\geq 0$

$\Rightarrow \quad -\lambda^2+80\geq 0$ or $\lambda^2\geq 80$

$\Rightarrow \quad -\sqrt{80}\leq\lambda\leq\sqrt{80}$

$\therefore \quad \lambda=8$ [greatest integer]

From Eq. (i), $5p^2-32p+48=0$

$\Rightarrow \quad (p-4)(5p-12)=0$

$\therefore \quad p=4, p=\frac{12}{5}$

$\Rightarrow \quad p=\frac{12}{5}, p\neq 4$

[for $p=4$ all terms are equal]

Now, common difference $=\lambda^2-2p\lambda$

$$=64-16\times\frac{12}{5}=64\left(1-\frac{3}{5}\right)=\frac{128}{5}=\frac{m}{n} \quad \text{[given]}$$

$\therefore \quad m=128$ and $n=5$

Hence, $m+n=143$

● **Ex. 10** *If* $2\lambda, \lambda$ *and* $[\lambda^2-14], \lambda\in R-\{0\}$ *and [.] denotes the greatest integer function are the first three terms of a GP in order, then the 51th term of the sequence,* $1, 3\lambda, 6\lambda, 10\lambda, \ldots$ *is*

(a) 5104 (b) 5304

(c) 5504 (d) 5704

Sol. (b) $\because 2\lambda, \lambda, [\lambda^2-14]$ are in GP, then

$$\lambda^2=2\lambda[\lambda^2-14]$$

$\Rightarrow \quad \frac{\lambda}{2}=[\lambda^2-14]$

$\therefore \lambda$ must be an even integer

Hence, $\lambda=4$

Now, required sequence 1,12, 24, 40, ...

or $1, 4(1+2), 4(1+2+3), 4(1+2+3+4), \ldots$

$\therefore \quad 51\text{th term}=4(1+2+3+\ldots+51)$

$$=4\cdot\frac{51}{2}(1+51)=4\cdot51\cdot26=5304$$

JEE Type Solved Examples :
More than One Correct Option Type Questions

■ This section contains **5 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of **which more than one** may be correct.

● ***Ex. 11*** *The first three terms of a sequence are* $3, -1, -1$. *The next terms are*

(a) 2 (b) 3 (c) $-\frac{5}{27}$ (d) $-\frac{5}{9}$

Sol. (b, d) The given sequence is not an AP or GP or HP. It is an AGP, $3, (3+d)r, (3+2d)r^2, \ldots$

$\Rightarrow \quad (3+d)r = -1, (3+2d)r^2 = -1$

Eliminating r, we get $(3+d)^2 = -(3+2d)$

$\Rightarrow \quad d^2 + 8d + 12 = 0 \Rightarrow d = -2, -6,$

then $r = -1, \frac{1}{3}$

$\therefore$ Next term is $(3+3d)r^3 = 3, -\frac{5}{9}$

● ***Ex. 12*** *There are two numbers a and b whose product is* 192 *and the quotient of AM by HM of their greatest common divisor and least common multiple is* $\frac{169}{48}$. *The smaller of a and b is*

(a) 2 (b) 4 (c) 6 (d) 12

Sol. (b, d) If $G = \text{GED}$ of a and b, $L = \text{LCM}$ of a and b, we have $GL = ab = 192$ ...(i)

$$\frac{\text{AM}}{\text{HM}} \text{ of } G \text{ and } L \text{ is } \left(\frac{G+L}{2}\right)\left(\frac{G+L}{2GL}\right) = \frac{169}{48}$$

$\Rightarrow \quad (G+L)^2 = \frac{169}{12}GL = \frac{169}{12} \times 192 = 13^2 \cdot 4^2$

$\Rightarrow \quad G + L = 52$ but $GL = 192$

$\Rightarrow \quad G = 4, L = 48 \Rightarrow a = 4, b = 48$ or $a = 12, b = 16$

● ***Ex. 13*** *Consider a series* $\frac{1}{2} + \frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \frac{5}{2^5} + \ldots + \frac{\lambda n}{2^n}$. *If* S_n *denotes its sum to n terms, then* S_n *cannot be*

(a) 2 (b) 3 (c) 4 (d) 5

Sol. (a, b, c, d)

$$\because \quad Sn = \frac{1}{2} + \frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \frac{5}{2^5} + \ldots + \frac{\lambda_n}{2^n}$$

$$= \frac{3}{4} + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \frac{5}{2^5} + \ldots + \frac{\lambda_n}{2^n}\right)$$

$$+ \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2^2} + \frac{2}{2^3} + \ldots + \frac{\lambda_n}{2^n}\right) - \frac{1}{4} - \frac{\lambda_n}{2^{n+2}} - \frac{\lambda_n}{2^{n+1}}$$

$$\Rightarrow \quad S_n = \frac{3}{4} + \frac{1}{4}S_n + \frac{1}{2}S_n - \frac{1}{4} - \frac{\lambda_n}{2^{n+2}} - \frac{\lambda_n}{2^{n+1}}$$

$$\Rightarrow \frac{1}{4}S_n = \frac{1}{2} - \frac{\lambda_n}{2^{n+2}} - \frac{\lambda_n}{2^{n+1}} \Rightarrow S_n = 2 - \frac{\lambda_n}{2^{n+1}} - \frac{\lambda_n}{2^{n-1}} < 2$$

● ***Ex. 14*** *If* $S_r = \sqrt{r + \sqrt{r + \sqrt{r + \sqrt{\ldots \infty}}}}$, $r > 0$ *then which the following is/are correct.*

(a) S_r, S_6, S_{12}, S_{20} are in AP

(b) S_4, S_9, S_{16} are irrational

(c) $(2S_{4-1})^2, (2S_{5-1})^2 (2S_{6-1})^2$ are in AP

(d) S_2, S_{12}, S_{56} are in GP

Sol. (a, b, c, d)

$$\because S_r = \sqrt{r + \sqrt{r + \sqrt{+\sqrt{\ldots \infty}}}} = \sqrt{r + S_r}$$

$$\Rightarrow \quad S_r^2 - S_r - r = 0$$

$$\therefore \quad S_r = \frac{1 + \sqrt{(1+4r)}}{2} \qquad [\because r > 0]$$

Alternate (a) S_2, S_6, S_{12}, S_{20} i.e., 2, 3, 4, 5 are in AP.

Alternate (b) S_4, S_9, S_{16} i.e., $\frac{1+\sqrt{17}}{2}, \frac{1+\sqrt{37}}{2}, \frac{1+\sqrt{65}}{2}$ are irrationals.

Alternate (c) $(2S_{4-1})^2, (2S_{5-1})^2, (2S_{6-1})^2$ i.e., 17, 21, 25 are in AP

Alternate (d) S_2, S_{12}, S_{56} i.e., 2, 4, 8 are in GP.

● ***Ex. 15*** *If* $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ *are in AP and a, b, −2c are in GP, where a, b, c are non-zero, then*

(a) $a^3 + b^3 + c^3 = 3abc$ (b) $-2a, b, -2c$ are in AP

(c) $-2a, b, -2c$ are in GP (d) $a^2, b^2, 4c^2$ are in GP

Sol. (a, b, d)

$\because \quad \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP $\Rightarrow$ a, b, c are in HP

$$\therefore \quad b = \frac{2ab}{a+c} \qquad \ldots(i)$$

and $a, b, -2c$ are in GP, then $b^2 = -2ac$ ...(ii)

From Eqs. (i) and (ii), we get

$$b = \frac{-b^2}{a+c} \Rightarrow a + b + c = 0 \qquad [\because b \neq 0]$$

$\therefore \quad a^3 + b^3 + c^3 = 3abc$ and $a, b, -2c$ are in GP

$\Rightarrow \quad a^2, b^2, 4c^2$ are also in GP and $a + b + c = 0$

$\Rightarrow \quad 2b = -2a - 2c$

$\therefore \quad -2a, b, -2c$ are in AP.

JEE Type Solved Examples : Passage Based Questions

- This section contains **3 solved passages** based upon each of the passage **3 multiple choice** examples have to be answered. Each of these examples has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I

(Ex. Nos. 16 to 18)

Consider a sequence whose sum to n terms is given by the quadratic function $S_n = 3n^2 + 5n$.

16. *The nature of the given series is*

(a) AP (b) GP (c) HP (d) AGP

Sol. (*a*) $\because \quad S_n = 3n^2 + 5n$

$\therefore \quad T_n = S_n - S_{n-1}$

$= (3n^2 + 5n) - [3(n-1)^2 + 5(n-1)]$

$= 3(2n-1) + 5 = 6n + 2$

The *n*th term is a linear function in *n*. Hence, sequence must be an AP.

17. *For the given sequence, the number 5456 is the*

(a) 153 th term (b) 932 th term

(c) 707 th term (d) 909 th term

Sol. (*d*) Given, $T_n = 5456$

$\Rightarrow \quad 6n + 2 = 5456 \Rightarrow 6n = 5454$

$\therefore \quad n = 909$

$\therefore$ The number 5456 is the 909 th term.

18. *Sum of the squares of the first 3 terms of the given series is*

(a) 1100 (b) 660 (c) 799 (d) 1000

Sol. (*b*) $T_1^2 + T_2^2 + T_3^2 = 8^2 + 14^2 + 20^2 = 64 + 196 + 400 = 660$

Passage II

(Ex. Nos. 19 to 21)

Let r be the number of identical terms in the two AP's. Form the sequence of identical terms, it will be an AP, then the rth term of this AP make $t_r \le$ *the smaller of the last term of the two AP's.*

19. *The number of terms common to two AP's* 3, 7, 11, ..., 407 *and* 2, 9, 16, ..., 709 *is*

(a) 14 (b) 21 (c) 28 (d) 35

Sol. (*a*) Sequence 3, 7, 11, ... , 407 has common difference = 4 and sequence 2, 9, 16, ... , 709 has common difference = 7.

Hence, the sequence with common terms has common difference LCM of 4 and 7 which is 28.

The first common term is 23.

Hence, the sequence is 23, 51, 79, ... , 387 which has 14 terms.

Aliter By inspection, first common term to both the series is 23, second common term is 51, third common term is 79 and so on. These numbers form an AP 23, 51, 79, ...

Since, $T_{14} = 23 + 13(28) = 387 < 407$

and $T_{15} = 23 + 14(28) = 415 > 407$

Hence, number of common terms = 14

20. *The 10th common term between the series* $3 + 7 + 11 + \ldots$ *and* $1 + 6 + 11 + \ldots$ *is*

(a) 189 (b) 191 (c) 211 (d) 213

Sol. (b) Series 3 + 7 + 11 + ... has common difference = 4 and series 1 + 6 + 11 + ... has common difference = 5

Hence, the series with common terms has common difference LCM of 4 and 5 which is 20.

The first common terms is 11.

Hence, the series is 11 + 31 + 51 + 71 + ...

$\therefore \quad t_{10} = 11 + (10-1)(20) = 191$

Aliter t_n for $3 + 7 + 11 + \ldots = 3 + (n-1)(4) = 4n - 1$

and t_m for $1 + 6 + 11 + \ldots = 1 + (m-1)(5) = 5m - 4$

For a common term, $4n - 1 = 5m - 4$ i.e., $4n = 5m - 3$

For $m = 3$, $n = 3$ gives the first common term i.e., 11.

For $m = 7$, $n = 8$ gives the second common term i.e., 31.

For $m = 11$, $n = 13$ gives the third common term i.e., 51.

Hence, the common term series is 11 + 31 + 51 + ...

$\therefore \quad t_{10} = 11 + (10-1)\,20 = 191$

21. *The value of largest term common to the sequences* 1, 11, 21, 31, ... *upto* 100 *terms and* 31, 36, 41, 46, ... *upto* 100 *terms, is*

(a) 281 (b) 381 (c) 471 (d) 521

Sol. (*d*) Sequence 1, 11, 21, 31, ... has common difference = 10 and sequence 31, 36, 41, 46, ... has common difference = 5. Hence, the sequence with common terms has common difference LCM of 10 and 5 which is 10.

The first common term is 31.

Hence, the sequence is 31, 41, 51, 61, 71, ...(i)

Now, t_{100} of first sequence $= 1 + (100-1)\,10 = 991$

and t_{100} of second sequence $= 31 + (100-1)\,5 = 526$

Value of largest common term < 526

$\therefore t_n$ of Eq. (i) is $31 + (n-1)\,10 = 10n + 21$

$t_{50} = 10 \times 50 + 21 = 521$

is the value of largest common term.

Aliter Let *m*th term of the first sequence be equal to the *n*th term of the second sequence, then

$1 + (m-1)\,10 = 31 + (n-1)\,5$

$\Rightarrow \quad 10m - 9 = 5n + 26 \Rightarrow 10m - 35 = 5n$

$\Rightarrow \quad 2m - 7 = n \le 100 \Rightarrow 2m \le 107$

$\Rightarrow \quad m \le 53\frac{1}{2}$

$\therefore$ Largest value of $m = 53$

$\therefore$ Value of largest term $= 1 + (53 - 1)\,10 = 521$

Passage III

(Ex. Nos. 22 to 24)

We are giving the concept of arithmetic mean of mth power. Let $a_1, a_2, a_3, \ldots, a_n$ be n positive real numbers (not all equal) and m be a real number. Then,

$$\frac{a_1^m + a_2^m + a_3^m + \ldots + a_n^m}{n} > \left(\frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}\right)^m,$$

if $m \in R \sim [0, 1]$

However, if $m \in (0, 1)$, then

$$\frac{a_1^m + a_2^m + a_3^m + \ldots + a_n^m}{n} < \left(\frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}\right)^m$$

Obviously, if $m = \{0, 1\}$, then

$$\frac{a_1^m + a_2^m + a_3^m + \ldots + a_n^m}{n} = \left(\frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}\right)^m.$$

22. *If $x > 0, y > 0, z > 0$ and $x + y + z = 1$, the minimum value of* $\frac{x}{2-x} + \frac{y}{2-y} + \frac{z}{2-z}$, is

(a) 0.2 (b) 0.4
(c) 0.6 (d) 0.8

Sol. (*c*) Since, AM of (-1) th powers $\ge (-1)$ th powers of AM

$$\therefore \frac{(2-x)^{-1} + (2-y)^{-1} + (2-z)^{-1}}{3} \ge \left(\frac{2 - x + 2 - y + 2 - z}{3}\right)^{-1}$$

$$= \left[\frac{6 - (x + y + z)}{3}\right]^{-1} = \left(\frac{6-1}{3}\right)^{-1} = \frac{3}{5} \qquad [\because x + y + z = 1]$$

$$\Rightarrow \quad \frac{(2-x)^{-1} + (2-y)^{-1} + (2-z)^{-1}}{3} \ge \frac{3}{5}$$

or $$\frac{1}{3}\left[\frac{1}{2-x} + \frac{1}{2-y} + \frac{1}{2-z}\right] \ge \frac{3}{5}$$

$$\Rightarrow \quad \frac{1}{2-x} + \frac{1}{2-y} + \frac{1}{2-z} \ge \frac{9}{5}$$

or $$\frac{2}{2-x} + \frac{2}{2-y} + \frac{2}{2-z} \ge \frac{18}{5}$$

or $$1 + \frac{x}{2-x} + 1 + \frac{y}{2-y} + 1 + \frac{z}{2-z} \ge \frac{18}{5}$$

or $$\frac{x}{2-x} + \frac{y}{2-y} + \frac{z}{2-z} \ge \frac{18}{5} - 3$$

Hence, $$\frac{x}{2-x} + \frac{y}{2-y} + \frac{z}{2-z} \ge \frac{3}{5} = 0.6$$

$\Rightarrow \quad \frac{x}{2-x} + \frac{y}{2-y} + \frac{z}{2-z} \ge 0.6$

Thus, minimum value of $\frac{x}{2-x} + \frac{y}{2-y} + \frac{z}{2-z}$ is 0.6.

23. *If $\sum_{i=1}^{n} a_i^2 = \lambda, \forall\, a_i \ge 0$ and if greatest and least values of $\left(\sum_{i=1}^{n} a_i\right)^2$ are λ_1 and λ_2 respectively, then $(\lambda_1 - \lambda_2)$ is*

(a) $n\lambda$ (b) $(n-1)\lambda$
(c) $(n+2)\lambda$ (d) $(n+1)\lambda$

Sol. (*b*) $\because$ AM of 2nd powers $\ge$ 2nd power of AM

$$\therefore \frac{a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2}{n} \ge \left(\frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}\right)^2$$

$$\Rightarrow \quad \frac{\lambda}{n} \ge \left(\frac{\sum_{i=1}^{n} a_i}{n}\right)^2 \qquad \therefore \left(\sum_{i=1}^{n} a_i\right)^2 \le n\,\lambda \qquad \ldots(i)$$

Also, $(a_1 + a_2 + a_3 + \ldots + a_n)^2 = a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2 + 2\sum a_1 a_2$

$$= \lambda + 2\sum a_1 a_2 \ge \lambda$$

$$\therefore \quad \left(\sum_{i=1}^{n} a_i\right)^2 \ge \lambda \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$\lambda \le \left(\sum_{i=1}^{n} a_i\right)^2 \le n\,\lambda$$

$\therefore \quad \lambda_1 = n\,\lambda$ and $\lambda_2 = \lambda$

Then, $\lambda_1 - \lambda_2 = (n-1)\,\lambda$

24. *If sum of the mth powers of first n odd numbers is λ, $\forall\, m > 1$, then*

(a) $\lambda < n^m$ (b) $\lambda > n^m$ (c) $\lambda < n^{m+1}$ (d) $\lambda > n^{m+1}$

Sol. (*d*) $\because m > 1$

$$\therefore \quad \frac{1^m + 3^m + 5^m + \ldots + (2n-1)^m}{n} > \left(\frac{1 + 3 + 5 + \ldots + (2n-1)}{n}\right)^m$$

$$= \left(\frac{\frac{n}{2}(1 + 2n - 1)}{n}\right)^m = n^m$$

$\therefore 1^m + 3^m + 5^m + \ldots + (2n-1)^m > n^{m+1}$

Hence, $\lambda > n^{m+1}$

JEE Type Solved Examples :
Single Integer Answer Type Questions

▪ This section contains **2 examples.** The answer to each example is a **single digit integer** ranging from **0** to **9** (both inclusive).

● ***Ex. 25*** *A sequence of positive terms $A_1, A_2, A_3, \ldots, A_n$ satisfies the relation $A_{n+1} = \frac{3(1+A_n)}{(3+A_n)}$. Least integral value of A_1 for which the sequence is decreasing can be*

Sol. (2) $\because \quad A_{n+1} = \frac{3(1+A_n)}{(3+A_n)}$. For $n = 1,\ A_2 = \frac{3(1+A_1)}{(3+A_1)}$

For $n = 2,\ A_3 = \frac{3(1+A_2)}{(3+A_2)}$

$$= \frac{3\left(1+\frac{3(1+A_1)}{(3+A_1)}\right)}{3+\frac{3(1+A_1)}{(3+A_1)}} = \frac{6+4A_1}{4+2A_1} = \frac{3+2A_1}{2+A_1}$$

$\because$ Given, sequence can be written as

$$A_1, \frac{3(1+A_1)}{(3+A_1)}, \frac{(3+2A_1)}{(2+A_1)}, \ldots$$

Given, $A_1 > 0$ and sequence is decreasing, then

$$A_1 > \frac{3(1+A_1)}{(3+A_1)}, \frac{3(1+A_1)}{(3+A_1)} > \frac{(3+2A_1)}{(2+A_1)}$$

$\Rightarrow \quad A_1^2 > 3 \quad$ or $\quad A_1 > \sqrt{3}$

$\therefore \quad A_1 = 2 \qquad$ [least integral value of A_1]

● ***Ex. 26*** *When the ninth term of an AP is divided by its second term we get 5 as the quotient, when the thirteenth term is divided by sixth term the quotient is 2 and the remainder is 5, then the second term is*

Sol. (7) Let a be the first term and d be the common difference, then $T_9 = 5\,T_2$

$\Rightarrow \quad (a+8d) = 5(a+d)$

$\therefore \quad 4a = 3d \qquad \ldots(i)$

and $\quad T_{13} = T_6 \times 2 + 5$

$\Rightarrow \quad a + 12d = 2(a+5d) + 5$

$\Rightarrow \quad 2d = a + 5 \qquad \ldots(ii)$

From Eqs. (i) and (ii), we get

$a = 3 \quad$ and $\quad d = 4$

$\therefore \quad T_2 = a + d = 7$

JEE Type Solved Examples :
Matching Type Questions

▪ This section contains **2 examples.** Examples 27 has three statements (A, B and C) given in **Column I** and four statements (p, q, r and s) in **Column II** example 28 has four statements (A, B, C and D) given in **Column I** and five statements (p, q, r, s and t) in **Column II**. Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II.**

● ***Ex. 27***

	Column I		Column II
(A)	If $a_1, a_2, a_3, \ldots$ are in AP and $a_1 + a_6 + a_{10} + a_{21} + a_{25} + a_{30} = 120$, then $\sum_{i=1}^{30} a_i$ is	(p)	400
(B)	If $a_1, a_2, a_3, \ldots$ are in AP and $a_1 + a_5 + a_9 + a_{13} + a_{17} + a_{21} + a_{25} = 112$, then $\sum_{i=1}^{25} a_i$ is	(q)	600
(C)	If $a_1, a_2, a_3, \ldots$ are in AP and $a_1 + a_4 + a_7 + a_{10} + a_{13} + a_{16} = 375$, then $\sum_{i=1}^{16} a_i$ is	(r)	800
		(s)	1000

Sol. (A) $\rightarrow$ (q); (B) $\rightarrow$ (p); (C) $\rightarrow$ (s)

(A) $\because a_1, a_2, a_3, \ldots$ are in AP.

$\therefore \quad a_1 + a_{30} = a_6 + a_{25} = a_{10} + a_{21} = \lambda \qquad$ [say]

$\because \quad a_1 + a_6 + a_{10} + a_{21} + a_{25} + a_{30} = 120$

$\therefore \quad 3\lambda = 120$

$\Rightarrow \quad \lambda = 40$

Then, $\quad \sum_{i=1}^{30} a_i = \frac{30}{2}(a_1 + a_{30}) = 15 \times \lambda = 15 \times 40 = 600$

(B) $\because a_1, a_2, a_3, \ldots$ are in AP.

$\therefore \quad a_1 + a_{25} = a_5 + a_{21}$

$= a_9 + a_{17} = a_{13} + a_{13} = \lambda \qquad$ [say]

$\because \quad a_1 + a_5 + a_9 + a_{13} + a_{17} + a_{21} + a_{25} = 112$

$\therefore \quad 3\lambda + \frac{\lambda}{2} = 112$

$\Rightarrow \quad \frac{7\lambda}{2} = 112$

$\Rightarrow \quad \lambda = 32$

Then, $\quad \sum_{i=1}^{25} a_i = \frac{25}{2}(a_1 + a_{25}) = \frac{25}{2} \times 32 = 400$

(C) $\because a_1, a_2, a_3, \ldots$ are in AP.

$\therefore \quad a_1 + a_{16} = a_4 + a_{13} = a_7 + a_{10} = \lambda$ [say]

$\because \quad a_1 + a_4 + a_7 + a_{10} + a_{13} + a_{16} = 375$

$\therefore \quad 3\lambda = 375 \quad \therefore \quad \lambda = 125$

Then, $\sum_{i=1}^{16} a_i = \frac{16}{2}(a_1 + a_{16})$

$= 8 \times \lambda = 8 \times 125 = 1000$

Ex. 28

	Column I		Column II
(A)	If $a > 0, b > 0, c > 0$ and the minimum value of $a(b^2 + c^2) + b(c^2 + a^2) + c(a^2 + b^2)$ is λabc, then λ is	(p)	2
(B)	If a, b, c are positive, $a + b + c = 1$ and the minimum value of $\left(\frac{1}{a} - 1\right)\left(\frac{1}{b} - 1\right)\left(\frac{1}{c} - 1\right)$ is λ, then λ is	(q)	4
(C)	If $a > 0, b > 0, c > 0, s = a + b + c$ and the minimum value of $\frac{2s}{s-a} + \frac{2s}{s-b} + \frac{2s}{s-c}$ is $(\lambda - 1)$, then λ is	(r)	6
(D)	If $a > 0, b > 0, c > 0, a, b, c$ are in GP and the the minimum value of $\left(\frac{a}{b}\right)^{\lambda} + \left(\frac{c}{b}\right)^{\lambda}$ is 2, then λ is	(s)	8
		(t)	10

Sol. (A) → (r); (B) → (s); (C) → (t); (D) → (p, q, r, s, t)

(A) $\because$ AM ≥ GM

$\therefore \quad \frac{ab^2 + ac^2 + bc^2 + ba^2 + ca^2 + cb^2}{6}$

$\geq (ab^2 \cdot ac^2 \cdot bc^2 \cdot ba^2 \cdot ca^2 \cdot cb^2)^{1/6} = abc$

$\therefore \quad a(b^2 + c^2) + b(c^2 + a^2) + c(a^2 + b^2) \geq 6abc$

$\Rightarrow \quad \lambda = 6$

(B) $\because$ AM ≥ GM

For b, c, we get $\frac{(b + c)}{2} \geq \sqrt{bc}$

$\Rightarrow \quad (b + c) \geq 2\sqrt{bc}$ ...(i)

For c, a, we get

$\frac{(c + a)}{2} \geq \sqrt{ca}$

$\Rightarrow \quad (c + a) \geq 2\sqrt{ca}$ ...(ii)

and for a, b, we get

$\frac{(a + b)}{2} \geq \sqrt{ab}$

$\Rightarrow \quad (a + b) \geq 2\sqrt{ab}$ ...(iii)

On multiplying Eqs. (i), (ii) and (iii), we get

$(b + c)(c + a)(a + b) \geq 8abc$

$\Rightarrow \quad (1 - a)(1 - b)(1 - c) \geq 8abc \quad [\because a + b + c = 1]$

$\Rightarrow \quad \left(\frac{1}{a} - 1\right)\left(\frac{1}{b} - 1\right)\left(\frac{1}{c} - 1\right) \geq 8$

$\therefore \quad \lambda = 8$

(C) $\because$ AM ≥ HM

$\frac{(s-a) + (s-b) + (s-c)}{3} \geq \frac{3}{\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c}}$

$\Rightarrow \quad \frac{3s - (a+b+c)}{3} \geq \frac{3}{\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c}}$

$\Rightarrow \quad \frac{3s - s}{3} \geq \frac{3}{\left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c}\right)}$

$\Rightarrow \frac{2s}{s-a} + \frac{2s}{s-b} + \frac{2s}{s-c} \geq 9$

Here, $\lambda - 1 = 9$

$\therefore \quad \lambda = 10$

(D) If a, b, c are in GP.

Then, $a^{\lambda}, b^{\lambda}, c^{\lambda}$ are also in GP.

Then, AM ≥ GM

$\frac{a^{\lambda} + c^{\lambda}}{2} \geq b^{\lambda}$

$\Rightarrow \quad a^{\lambda} + c^{\lambda} \geq 2b^{\lambda}$

$\Rightarrow \quad \left(\frac{a}{b}\right)^{\lambda} + \left(\frac{c}{b}\right)^{\lambda} \geq 2$

$\therefore \quad \lambda \in R$

Hence, $\lambda = 2, 4, 6, 8, 10$

JEE Type Solved Examples :
Statement I and II Type Questions

■ **Directions** Example numbers 29 to 32 are Assertion-Reason type examples. Each of these examples contains two statements:
Statement-1 (Assertion) and **Statement-2** (Reason)
Each of these examples also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below.
(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement-1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

● ***Ex. 29.*** **Statement 1** *The sum of first n terms of the series* $1^2 - 2^2 + 3^2 - 4^2 + 5^2 - \ldots$ *can be* $= \pm \frac{n(n+1)}{2}$.

Statement 2 *Sum of first n natural numbers is* $\frac{n(n+1)}{2}$.

Sol. (a) Clearly, nth term of the given series is negative or positive according as n is even or odd, respectively.

Case **I** When n is even, in this case the given series is

$1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \ldots + (n-1)^2 - n^2$

$= (1^2 - 2^2) + (3^2 - 4^2) + (5^2 - 6^2) + \ldots + [(n-1)^2 - n^2]$

$= (1-2)(1+2) + (3-4)(3+4) + (5-6)(5+6) + \ldots + (n-1-n)(n-1+n)$

$= -(1 + 2 + 3 + 4 + 5 + 6 + \ldots + (n-1) + n) = \frac{n(n+1)}{2}$

Case **II** When n is odd, in this case the given series is

$1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \ldots + (n-2)^2 - (n-1)^2 + n^2$

$= (1^2 - 2^2) + (3^2 - 4^2) + (5^2 - 6^2) + \ldots + [(n-2)^2 - (n-1)^2] + n^2$

$= (1-2)(1+2) + (3-4)(3+4) + (5-6)(5+6) + \ldots + [(n-2) - (n-1)][(n-2) + (n-1)] + n^2$

$= -[1 + 2 + 3 + 4 + 5 + 6 + \ldots + (n-2) + (n-1)] + n^2$

$= -\frac{(n-1)(n-1+1)}{2} + n^2 = \frac{n(n+1)}{2}$

It is clear that Statement-1 is true, Statement-2 is true and Statement-2 is correct explanation for Statement-1.

● ***Ex. 30*** **Statement 1** *If a, b, c are three positive numbers in GP, then* $\left(\frac{a+b+c}{3}\right)\left(\frac{3abc}{ab+bc+ca}\right) = (abc)^{2/3}$.

Statement-2 *(AM) (HM)=(GM)^2 is true for positive numbers.*

Sol. (c) If a, b be two real, positive and unequal numbers, then

$$\text{AM} = \frac{a+b}{2},\ \text{GM} = \sqrt{ab} \text{ and } \text{HM} = \frac{2ab}{a+b}$$

$\therefore$ AM) (HM) = (GM)^2

This result will be true for n numbers, if they are in GP.
Hence, Statement-1 is true and Statement-2 is false.

● ***Ex. 31*** *Consider an AP with a as the first term and d is the common difference such that S_n denotes the sum to n terms and a_n denotes the nth term of the AP. Given that for some $m, n \in N$, $\frac{S_m}{S_n} = \frac{m^2}{n^2}$ $(m \neq n)$.*

Statement 1 $d = 2a$ *because*

Statement 2 $\frac{a_m}{a_n} = \frac{2m+1}{2n+1}$

Sol. (c) $\because$ $\frac{S_m}{S_n} = \frac{m^2}{n^2}$

Let $S_m = m^2 k,\ S_n = n^2 k$

$\therefore$ $a_m = S_m - S_{m-1} = m^2 k - (m-1)^2 k$

$\Rightarrow$ $a_m = (2m-1)k$

Similarly, $a_n = (2n-1)k$ $\therefore$ $\frac{a_m}{a_n} = \frac{2m-1}{2n-1}$

Statement-2 is false.

Also, $\because$ $a_1 = k, a_2 = 3k, a_3 = 5k, \ldots$

Given, $a_1 = a = k$

$\therefore$ $a_1 = a, a_2 = 3a, a_3 = 5a, \ldots$

$\therefore$ Common difference $d = a_2 - a_1 = a_3 - a_2 = \ldots$

$\Rightarrow$ $d = 2a$

$\therefore$ Statement-1 is true.

● ***Ex. 32*** **Statement-1** 1, 2, 4, 8, ... *is a GP,* 4, 8, 16, 32, ... *is a GP and* $1+4, 2+8, 4+16, 8+32, \ldots$ *is also a GP.*

Statement-2 *Let general term of a GP with common ratio r be T_{k+1} and general term of another GP with common ratio r be T'_{k+1}, then the series whose general term $T''_{k+1} = T_{k+1} + T'_{k+1}$ is also a GP with common ratio r.*

Sol. (a) 1, 2, 4, 8, ...

Common ratio $r = 2$

$\therefore$ $T_{k+1} = 1 \cdot (2)^{k+1-1} = 2^k$

and 4, 8, 16, 32, ...

Common ratio, $r = 2$

$\therefore$ $T'_{k+1} = 4 \cdot (2)^{k+1-1} = 4 \cdot 2^k$

Then, $T_{k+1} + T'_{k+1} = 5 \cdot 2^k = T''_{k+1}$

Common ratio of $T''_{k+1} = \frac{5 \cdot 2^k}{5 \cdot 2^{k-1}} = 2$, which is true.

Hence, Statement-1 and Statement-2 both are true and Statement-2 is the correct explanation of Statement-1.

Subjective Type Examples

■ In this section, there are **24 subjective** solved examples.

● ***Ex. 33*** *In a set of four numbers, the first three are in GP and the last three are in AP with a common difference of 6. If the first number is same as the fourth, then find the four numbers.*

Sol. Let the last three numbers in AP, be $a, a+6, a+12$.
[$\because$ 6 is the common difference]

If first number is b, then four numbers are
$b, a, a+6, a+12$

But given, $b = a+12$

$\therefore$ Four numbers are $a+12, a, a+6, a+12$ ...(i)

Since, first three numbers are in GP.

Then, $a^2 = (a+12)(a+6)$

$\Rightarrow$ $a^2 = a^2 + 18a + 72$

$\Rightarrow$ $18a + 72 = 0$

$\therefore$ $a = -4$ [from Eq. (i)]

Hence, four numbers are $8, -4, 2, 8$.

● ***Ex. 34*** *Find the natural number a for which* $\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$*, where the function* f *satisfies* $f(x+y) = f(x)\,f(y)$ *for all natural numbers* x, y *and further* $f(1) = 2$.

Sol. Given, $f(x+y) = f(x)\,f(y)$ ...(i)

and $f(1) = 2$ ...(ii)

On putting $x = y = 1$ in Eq. (i), we get

$f(1+1) = f(1)\,f(1) = 2 \cdot 2$

$\therefore$ $f(2) = 2^2$ ...(iii)

Now, on putting $x = 1, y = 2$ in Eq. (i), we get

$f(1+2) = f(1)\,f(2) = 2 \cdot 2^2$ [from Eqs. (ii) and (iii)]

$\therefore$ $f(3) = 2^3$

On putting $x = y = 2$ in Eq. (i), we get

$f(2+2) = f(2)\,f(2) = 2^2 \cdot 2^2$ [from Eq. (iii)]

$\therefore$ $f(4) = 2^4$

$\vdots \quad \vdots \quad \vdots$

Similarly, $f(\lambda) = 2^{\lambda}, \lambda \in N$

$\therefore$ $f(a+k) = 2^{a+k}, a+k \in N$

$\because$ $\sum_{k=1}^{n} f(a+k) = 16(2^n - 1) \Rightarrow \sum_{k=1}^{n} 2^{a+k} = 16(2^n - 1)$

$\Rightarrow$ $2^a \sum_{k=1}^{n} 2^k = 16(2^n - 1)$

$\Rightarrow$ $2^a (2^1 + 2^2 + 2^3 + \ldots + 2^n) = 16(2^n - 1)$

$\Rightarrow$ $2^a \cdot \frac{2(2^n - 1)}{(2-1)} = 16(2^n - 1)$

$\Rightarrow$ $2^{a+1} = 16 = 2^4$

$\Rightarrow$ $a + 1 = 4$

$\therefore$ $a = 3$

● ***Ex. 35*** *If n is a root of* $x^2(1-ac) - x(a^2 + c^2) - (1+ac) = 0$ *and if n harmonic means are inserted between a and c, find the difference between the first and the last means.*

Sol. Let $H_1, H_2, H_3, \ldots, H_n$, are n harmonic means, then $a, H_1, H_2, H_3, \ldots, H_n, b$ are in HP.

$\therefore \frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \ldots, \frac{1}{H_n}, \frac{1}{b}$ are in AP.

If d be the common difference, then $\frac{1}{c} = \frac{1}{a} + (n+2-1)\,d$

$\therefore$ $d = \frac{(a-c)}{ac(n+1)}$ ...(i)

$\Rightarrow$ $\frac{1}{h_1} = \frac{1}{a} + d$ and $\frac{1}{h_n} = \frac{1}{c} - d$

$\therefore$ $h_1 - h_n = \frac{a}{1+ad} - \frac{c}{1-cd} = \frac{a}{1 + \frac{(a-c)}{c(n+1)}} - \frac{c}{1 - \frac{(a-c)}{a(n+1)}}$

$= \frac{ac(n+1)}{cn+a} - \frac{ac(n+1)}{an+c} = ac(n+1)\left(\frac{1}{cn+a} - \frac{1}{an+c}\right)$

$= ac(n+1)\left(\frac{an + c - cn - a}{acn^2 + (a^2 + c^2)n + ac}\right)$

$= \frac{ac(a-c)(n^2-1)}{acn^2 + (a^2 + c^2)n + ac}$ ...(ii)

But given n is a root of

$x^2(1-ac) - x(a^2 + c^2) - (1+ac) = 0.$

Then, $n^2(1-ac) - n(a^2 + c^2) - (1+ac) = 0$

or $acn^2 + (a^2 + c^2)n + ac = n^2 - 1,$

then from Eq. (ii), $h_1 - h_n = \frac{ac(a-c)(n^2-1)}{(n^2-1)} = ac(a-c)$

● ***Ex. 36*** *A number consists of three-digits which are in GP the sum of the right hand and left hand digits exceeds twice the middle digit by 1 and the sum of the left hand and middle digits is two third of the sum of the middle and right hand digits. Find the number.*

Sol. Let the three digits be a, ar and ar^2, then number is

$$100a + 10ar + ar^2 \quad \ldots(i)$$

Given, $a + ar^2 = 2ar + 1$

or $a(r^2 - 2r + 1) = 1$

or $a(r-1)^2 = 1 \quad \ldots(ii)$

Also, given $a + ar = \frac{2}{3}(ar + ar^2) \implies 3 + 3r = 2r + 2r^2$

or $2r^2 - r - 3 = 0$ or $(r+1)(2r-3) = 0$

$\therefore \quad r = -1, \frac{3}{2}$

For $r = -1, \quad a = \frac{1}{(r-1)^2} = \frac{1}{4} \notin I$

$\therefore \quad r \neq -1$

For $r = \frac{3}{2}, a = \frac{1}{\left(\frac{3}{2} - 1\right)^2} = 4$ [from Eq. (ii)]

From Eq. (i), number is $400 + 10 \cdot 4 \cdot \frac{3}{2} + 4 \cdot \frac{9}{4} = 469$

● ***Ex. 37*** *Find the value of the expression* $\sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=1}^{j} 1$.

Sol. $\sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=1}^{j} 1 = \sum_{i=1}^{n} \sum_{j=1}^{i} j = \sum_{i=1}^{n} \frac{i(i+1)}{2}$

$$= \frac{1}{2}\left[\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i\right] = \frac{1}{2}\left[\sum n^2 + \sum n\right]$$

$$= \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right]$$

$$= \frac{n(n+1)}{12}[2n + 1 + 3] = \frac{n(n+1)(n+2)}{6}$$

● ***Ex. 38*** *Three numbers are in GP whose sum is* 70. *If the extremes be each multiplied by* 4 *and the mean by* 5, *then they will be in AP. Find the numbers.*

Sol. Let the three numbers in GP be $\frac{a}{r}, a, ar$.

Given, $\frac{a}{r} + a + ar = 70 \quad \ldots(i)$

and $\frac{4a}{r}, 5a, 4ar$ are in AP.

$\therefore \quad 10a = \frac{4a}{r} + 4ar$ or $\frac{10a}{4} = \frac{a}{r} + ar$

or $\frac{5a}{2} = 70 - a$ [from Eq. (i)]

or $5a = 140 - 2a$ or $7a = 140$

$\therefore \quad a = 20$

From Eq. (i), we get

$$\frac{20}{r} + 20 + 20r = 70$$

or $\frac{20}{r} + 20r = 50$

or $2 + 2r^2 = 5r$ or $2r^2 - 5r + 2 = 0$

or $(r-2)(2r-1) = 0 \quad \therefore \quad r = 2$ or $\frac{1}{2}$

Hence, the three numbers are 10, 20, 40 or 40, 20, 10.

● ***Ex. 39*** *If the sum of m terms of an AP is equal to the sum of either the next n terms or the next p terms, then prove that* $(m+n)\left(\frac{1}{m} - \frac{1}{p}\right) = (m+p)\left(\frac{1}{m} - \frac{1}{n}\right)$.

Sol. Let the AP be $a, a + d, a + 2d, \ldots$

Given, $T_1 + T_2 + \ldots + T_m = T_{m+1} + T_{m+2} + \ldots + T_{m+n} \quad \ldots(i)$

On adding $T_1 + T_2 + \ldots + T_m$ both sides in Eq. (i), we get

$2(T_1 + T_2 + \ldots + T_m) = T_1 + T_2 + \ldots + T_m + T_{m+1} + \ldots + T_{m+n}$

$\implies \quad 2S_m = S_{m+n}$

$\therefore \quad 2 \cdot \frac{m}{2}[2a + (m-1)d] = \frac{m+n}{2}[2a + (m+n-1)d]$

Let $2a + (m-1)d = x$

$\implies \quad mx = \frac{m+n}{2}\{x + nd\}$

$\implies \quad (m-n)x = (m+n)nd \quad \ldots(ii)$

Again, $T_1 + T_2 + \ldots + T_m = T_{m+1} + T_{m+2} + \ldots + T_{m+p}$

Similarly, $(m-p)x = (m+p)pd \quad \ldots(iii)$

On dividing Eq. (ii) by Eq. (iii), we get

$$\frac{m-n}{m-p} = \frac{(m+n)n}{(m+p)p}$$

$\implies \quad (m-n)(m+p)p = (m-p)(m+n)n$

On dividing both sides by mnp, we get

$$(m+p)\left(\frac{1}{n} - \frac{1}{m}\right) = (m+n)\left(\frac{1}{p} - \frac{1}{m}\right)$$

Hence, $(m+n)\left(\frac{1}{m} - \frac{1}{p}\right) = (m+p)\left(\frac{1}{m} - \frac{1}{n}\right)$

● ***Ex. 40*** *Find the sum of the products of every pair of the first n natural numbers.*

Sol. We find that

$$S = 1\cdot2 + 1\cdot3 + 1\cdot4 + \ldots + 2\cdot3 + 2\cdot4 + \ldots + 3\cdot4 + 3\cdot5 + \ldots + (n-1)\cdot n \quad \ldots(i)$$

$\because \quad [1 + 2 + 3 + \ldots + (n-1) + n]^2 = 1^2 + 2^2 + 3^2 + \ldots + (n-1)^2 + n^2 + 2[1\cdot2 + 1\cdot3 + 1\cdot4 + \ldots + 2\cdot3 + 2\cdot4 + \ldots + 3\cdot4 + 3\cdot5 + \ldots + (n-1)\cdot n]$

$(\sum n)^2 = \sum n^2 + 2S$ [from Eq. (i)]

$$\Rightarrow \quad S = \frac{(\sum n)^2 - \sum n^2}{2}$$

$$= \frac{\left\{\frac{n(n+1)}{2}\right\}^2 - \frac{n(n+1)(2n+1)}{6}}{2}$$

$$= \frac{\frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6}}{2}$$

$$= \frac{n(n+1)}{24}[3n(n+1) - 2(2n+1)]$$

$$= \frac{n(n+1)(3n^2 - n - 2)}{24}$$

Hence, $S = \frac{(n-1)n(n+1)(3n+2)}{24}$

● ***Ex. 41*** *If* $I_n = \int_0^{\pi/4} \tan^n x\,dx$, *show that* $\frac{1}{I_2 + I_4}, \frac{1}{I_3 + I_5}, \frac{1}{I_4 + I_6}, \frac{1}{I_5 + I_7}, \ldots$ *form an AP. Find its common difference.*

Sol. We have,

$$I_n + I_{n+2} = \int_0^{\pi/4} (\tan^n x + \tan^{n+2} x)\,dx$$

$$= \int_0^{\pi/4} \tan^n x\,(1 + \tan^2 x)\,dx$$

$$= \int_0^{\pi/4} \tan^n x \cdot \sec^2 x\,dx = \left[\frac{\tan^{n+1} x}{n+1}\right]_0^{\pi/4} = \frac{1}{n+1}$$

Hence, $\frac{1}{I_n + I_{n+2}} = n + 1$

On putting $n = 2, 3, 4, 5, \ldots$

$\therefore \quad \frac{1}{I_2 + I_4} = 3, \frac{1}{I_3 + I_5} = 4, \frac{1}{I_4 + I_6} = 5, \frac{1}{I_5 + I_7} = 6, \ldots$

Hence, $\frac{1}{I_2 + I_4}, \frac{1}{I_3 + I_5}, \frac{1}{I_4 + I_6}, \frac{1}{I_5 + I_7}, \ldots$ are in AP with common difference 1.

● ***Ex. 42*** *If the sum of the terms of an infinitely decreasing GP is equal to the greatest value of the function* $f(x) = x^3 + 3x - 9$ *on the interval* $[-5, 3]$ *and the difference between the first and second terms is* $f'(0)$, *then show that the common ratio of the progression is* $\frac{2}{3}$.

Sol. Given, $f(x) = x^3 + 3x - 9$

$\therefore \quad f'(x) = 3x^2 + 3$

Hence, $f'(x) > 0$ in $[-5, 3]$, then $f(x)$ is an increasing function on $[-5, 3]$ and therefore, $f(x)$ will have greatest value at $x = 3$.

Thus, greatest value of $f(x)$ is

$$f(x) = 3^3 + 3 \cdot 3 - 9 = 27$$

Let $a, ar, ar^2, \ldots$ be a GP with common ratio $|r| < 1$ [$\because$ given infinitely GP]

and also given $S_\infty = 27$

so, $\frac{a}{1-r} = 27$ ...(i)

and $a - ar = f'(0)$

$\Rightarrow \quad a(1-r) = f'(0) = 3$ [$\because f'(0) = 3$]

$\therefore \quad a(1-r) = 3$ ...(ii)

From Eqs. (i) and (ii), we get

$$(1-r)^2 = \frac{1}{9} \Rightarrow 1 - r = \pm\frac{1}{3}$$

$\therefore \quad r = 1 \pm \frac{1}{3}$

So, $r = \frac{4}{3}, \frac{2}{3} \Rightarrow r \neq \frac{4}{3}$ [$\because |r| < 1$]

Hence, $r = \frac{2}{3}$

● ***Ex. 43*** *Solve the following equations for x and y*

$$\log_{10} x + \frac{1}{2}\log_{10} x + \frac{1}{4}\log_{10} x + \ldots = y$$

and $\frac{1 + 3 + 5 + \ldots + (2y-1)}{4 + 7 + 10 + \ldots + (3y+1)} = \frac{20}{7\log_{10} x}$.

Sol. From the first equation

$$\log_{10} x \left\{1 + \frac{1}{2} + \frac{1}{4} + \ldots + \infty\right\} = y$$

$$\Rightarrow \quad \log_{10} x \left\{\frac{1}{1 - \frac{1}{2}}\right\} = y$$

$$\Rightarrow \quad 2\log_{10} x = y \quad \ldots(i)$$

From the second equation

$$\frac{1 + 3 + 5 + \ldots + (2y-1)}{4 + 7 + 10 + \ldots + (3y+1)} = \frac{20}{7\log_{10} x}$$

$$\Rightarrow \quad \frac{\frac{y}{2}(1 + 2y - 1)}{\frac{y}{2}(4 + 3y + 1)} = \frac{20}{7\log_{10} x}$$

$$\Rightarrow \quad \frac{2y}{3y+5} = \frac{20}{7\log_{10} x}$$

$$\Rightarrow \quad 7y(2\log_{10} x) = 60y + 100$$

$$\Rightarrow \quad 7y(y) = 60y + 100 \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad 7y^2 - 60y - 100 = 0$$

$$\therefore \quad (y - 10)(7y + 10) = 0$$

$$\therefore \quad y = 10, y \neq \frac{-10}{7}$$

[because y being the number of terms in series $\Rightarrow y \in N$]

From Eq. (i), we have

$$2\log_{10} x = 10 \Rightarrow \log_{10} x = 5$$

$$\therefore \quad x = 10^5$$

Hence, required solution is $x = 10^5, y = 10$

● ***Ex. 44*** *If* $0 < x < \frac{\pi}{2}$, $\exp[(\sin^2 x + \sin^4 x + \sin^6 x + \ldots + \infty)\log_e 2]$ *satisfies the quadratic equation* $x^2 - 9x + 8 = 0$, *find the value of* $\frac{\sin x - \cos x}{\sin x + \cos x}$.

Sol. $0 < x < \frac{\pi}{2}$

$\therefore \quad 0 < \sin^2 x < 1$

Then, $\sin^2 x + \sin^4 x + \sin^6 x + \ldots + \infty$

$$= \frac{\sin^2 x}{1 - \sin^2 x} = \tan^2 x$$

$\therefore \exp[(\sin^2 x + \sin^4 x + \sin^6 x + \ldots + \infty)\log_e 2]$

$$= \exp(\tan^2 x \cdot \log_e 2) = \exp(\log_e 2^{\tan^2 x})$$

$$= e^{\log_e 2^{\tan^2 x}} = 2^{\tan^2 x}$$

Let $y = 2^{\tan^2 x}$

Because y satisfies the quadratic equation.

Then, $y^2 - 9y + 8 = 0$

So, $y = 1, 8$

if $y = 1 = 2^{\tan^2 x}$

$\Rightarrow \quad 2^{\tan^2 x} = 2^0$

$\Rightarrow \quad \tan^2 x = 0$

$\therefore \quad x = 0$ [impossible] [$\because x > 0$]

Now, if $y = 8 = 2^{\tan^2 x}$

$\Rightarrow \quad 2^{\tan^2 x} = 2^3$

$\Rightarrow \quad \tan^2 x = 3$

$\therefore \quad \tan x = \sqrt{3}$

$$\therefore \quad \frac{\sin x - \cos x}{\sin x + \cos x} = \frac{\tan x - 1}{\tan x + 1} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \times \frac{\sqrt{3} - 1}{\sqrt{3} - 1}$$

$$= \frac{(\sqrt{3} - 1)^2}{3 - 1} = \frac{3 + 1 - 2\sqrt{3}}{2}$$

Hence, $\frac{\sin x - \cos x}{\sin x + \cos x} = 2 - \sqrt{3}$

● ***Ex. 45*** *The natural numbers are arranged in the form given below*

1

2 3

4 5 6 7

8 9 10 11 12 13 14 15

.....

.....

The rth group containing 2^{r-1} numbers. Prove that sum of the numbers in the nth group is $2^{n-2}[2^n + 2^{n-1} - 1]$.

Sol. Let 1st term of the r th group be T_r and the 1st terms of successive rows are 1, 2, 4, 8, ..., respectively.

$$T_r = 1 \cdot 2^{r-1} = 2^{r-1}$$

Hence, the sum of the numbers in the r th group is

$$= \frac{2^{r-1}}{2}\{2 \cdot 2^{r-1} + (2^{r-1} - 1) \cdot 1\}$$

[$\because$ number of terms in rth group is 2^{r-1}]

$$= 2^{r-2}\{2^r + 2^{r-1} - 1\}$$

Hence, sum of the numbers in the nth group is $2^{n-2}[2^n + 2^{n-1} - 1]$.

● ***Ex. 46*** *If* a, b, c *are in HP, then prove that* $\frac{a+b}{2a-b} + \frac{c+b}{2c-b} > 4$.

Sol. Since, a, b, c are in HP.

$$\therefore \quad \frac{2}{b} = \frac{1}{a} + \frac{1}{c} \qquad \ldots(i)$$

and let $P = \frac{a+b}{2a-b} + \frac{c+b}{2c-b}$

$$= \frac{a + \frac{2ac}{a+c}}{2a - \frac{2ac}{a+c}} + \frac{c + \frac{2ac}{a+c}}{2c - \frac{2ac}{a+c}} \qquad \text{[from Eq. (i)]}$$

$$= \frac{a+3c}{2a} + \frac{3a+c}{2c} = 1 + \frac{3}{2}\left(\frac{c}{a} + \frac{a}{c}\right) \qquad \ldots(ii)$$

$\because \quad$ AM > GM [$\because a \neq c$]

$$\therefore \quad \left(\frac{c}{a} + \frac{a}{c}\right) > 2$$

$$\Rightarrow \quad \frac{3}{2}\left(\frac{c}{a} + \frac{a}{c}\right) > 3$$

or $\quad 1 + \frac{3}{2}\left(\frac{c}{a} + \frac{a}{c}\right) > 1 + 3$ or $P > 4$

Hence, $\frac{a+b}{2a-b} + \frac{c+b}{2c-b} > 4$

● ***Ex. 47*** *Find the sum of n terms of the series* $\frac{1}{1+1^2+1^4} + \frac{2}{1+2^2+2^4} + \frac{3}{1+3^2+3^4} + \ldots$.

Sol. The n th term of the given series is $T_n = \frac{n}{(1+n^2+n^4)}$

$$\therefore \text{Sum of } n \text{ terms} = S_n = \sum T_n = \sum \frac{n}{(1+n^2+n^4)}$$

$$= \sum \frac{n}{(1+n+n^2)(1-n+n^2)}$$

$$= \frac{1}{2}\sum\left(\frac{1}{1-n+n^2} - \frac{1}{1+n+n^2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{1-1+1} - \frac{1}{1+n+n^2}\right) \quad \text{[by property]}$$

$$= \frac{(n+n^2)}{2(1+n+n^2)} = \frac{n(n+1)}{2(n^2+n+1)}$$

● ***Ex. 48*** *The value of xyz is 55 or $\frac{343}{55}$ according as the series a, x, y, z, b is an AP or HP. Find the values of a and b given that they are positive integers.*

Sol. If a, x, y, z, b are in AP.

Then, b = Fifth term $= a + (5-1)d$

where, d is common difference]

$$\therefore \quad d = \frac{b-a}{4}$$

$$\therefore \quad x \cdot y \cdot z = (a+d)(a+2d)(a+3d) = 55 \quad \text{[given]}$$

$$\Rightarrow \quad \left(\frac{b+3a}{4}\right)\left(\frac{2a+2b}{4}\right)\left(\frac{a+3b}{4}\right) = 55$$

$$\Rightarrow \quad (a+3b)(a+b)(3a+b) = 55 \times 32 \quad \ldots(i)$$

If they are in HP.

The common difference of the associated AP is $\frac{1}{4}\left(\frac{1}{b} - \frac{1}{a}\right)$.

i.e. $\frac{(a-b)}{4ab}$

$$\therefore \quad \frac{1}{x} = \frac{1}{a} + \frac{(a-b)}{4ab}$$

$$\Rightarrow \quad x = \frac{4ab}{a+3b}$$

$$\therefore \quad \frac{1}{y} = \frac{1}{a} + \frac{2(a-b)}{4ab}$$

$$\Rightarrow \quad y = \frac{4ab}{2a+2b} = \frac{2ab}{a+b}$$

and $\quad \frac{1}{z} = \frac{1}{a} + \frac{3(a-b)}{4ab}$

$$\Rightarrow \quad z = \frac{4ab}{3a+b}$$

$$\therefore \quad xyz = \frac{4ab}{(a+3b)} \cdot \frac{2ab}{(a+b)} \cdot \frac{4ab}{(3a+b)} = 343 \quad \text{[given]}$$

$$\Rightarrow \quad \frac{32\,a^3b^3}{55 \times 32} = \frac{343}{55} \quad \text{[from Eq. (i)]}$$

or $\quad a^3b^3 = 343$

$$\Rightarrow \quad ab = 7$$

Hence, $\quad a = 7, b = 1$

or $\quad a = 1, b = 7$

● ***Ex. 49*** *Find the sum of the first n terms of the series*

$1^3 + 3\cdot 2^2 + 3^3 + 3\cdot 4^2 + 5^3 + 3\cdot 6^2 + \ldots$

If (i) n is even, (ii) n is odd.

Sol. ***Case* I** If n is even.

Let $\quad n = 2m$

$$\therefore \quad S = 1^3 + 3\cdot 2^2 + 3^3 + 3\cdot 4^2 + 5^3 + 3\cdot 6^2 + \ldots + (2m-1)^3 + 3(2m)^2$$

$$= \{1^3 + 3^3 + 5^3 + \ldots + (2m-1)^3\} + 3\{2^2 + 4^2 + 6^2 + \ldots + (2m)^2\}$$

$$= \sum_{r=1}^{m}(2r-1)^3 + 3\cdot 4\sum_{r=1}^{m} r^2$$

$$= \sum_{r=1}^{m}\{8r^3 - 12r^2 + 6r - 1\} + 12\sum_{r=1}^{m} r^2$$

$$= 8\sum_{r=1}^{m} r^3 - 12\sum_{r=1}^{m} r^2 + 6\sum_{r=1}^{m} r - \sum_{r=1}^{m} 1 + 12\sum_{r=1}^{m} r^2$$

$$= 8\sum_{r=1}^{m} r^3 + 6\sum_{r=1}^{m} r - \sum_{r=1}^{m} 1$$

$$= 8\cdot\frac{m^2(m+1)^2}{4} + 6\frac{m(m+1)}{2} - m$$

$$= 2m^2(m+1)^2 + 3m(m+1) - m$$

$$= m[2m^3 + 4m^2 + 5m + 2]$$

$$= \frac{n}{2}\left[2\left(\frac{n}{2}\right)^3 + 4\left(\frac{n}{2}\right)^2 + 5\left(\frac{n}{2}\right) + 2\right] \quad \left[\because m = \frac{n}{2}\right]$$

Hence, $S = \frac{n}{8}(n^3 + 4n^2 + 10n + 8)$ ...(i)

***Case* II** If n is odd.

Then, $(n+1)$ is even in the case

Sum of first n terms = Sum of first $(n+1)$ terms $-$ $(n+1)$ th term

$$= \frac{(n+1)}{8}[(n+1)^3 + 4(n+1)^2 + 10(n+1) + 8] - 3(n+1)^2$$

$$= \frac{1}{8}(n+1)[n^3 + 3n^2 + 3n + 1 + 4n^2 + 8n + 4 + 10n + 10 + 8 - 24n - 24]$$

Hence, $S = \frac{1}{8}(n+1)[n^3 + 7n^2 - 3n - 1]$

● ***Ex. 50*** *Find out the largest term of the sequence*

$\frac{1}{503}, \frac{4}{524}, \frac{9}{581}, \frac{16}{692}, \ldots$.

Sol. General term can be written as $T_n = \frac{n^2}{500 + 3n^3}$

Let $U_n = \frac{1}{T_n} = \frac{500}{n^2} + 3n$

Then, $\frac{dU_n}{dn} = -\frac{1000}{n} + 3$

and $\frac{d^2U_n}{dn^2} = \frac{3000}{n^4}$

For maxima or minima of U_n, we have

$$\frac{dU_n}{dn} = 0 \Rightarrow n^3 = \frac{1000}{3}$$

$$\Rightarrow \quad n = \left(\frac{1000}{3}\right)^{1/3} \text{ (not an integer) and } 6 < \left(\frac{1000}{3}\right)^{1/3} < 7$$

But n is an integer, therefore for the maxima or minima of U_n we will take n as the nearest integer to $\left(\frac{1000}{3}\right)^{1/3}$.

Since, $\left(\frac{1000}{3}\right)^{1/3}$ is more close to 7 than to 6. Thus, we take $n = 7$.

Further $\frac{d^2U^n}{dn^2} = +ve$, then U_n will be minimum and therefore, T_n will be maximum for $n = 7$.

Hence, T_7 is largest term. So, largest term in the given sequence is $\frac{49}{1529}$.

● ***Ex. 51*** *If* $f(r) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{r}$ *and* $f(0) = 0$, *find* $\sum_{r=1}^{n} (2r+1)\, f(r)$.

Sol. Since, $\sum_{r=1}^{n} (2r+1)\, f(r)$

$$= \sum_{r=1}^{n} (r^2 + 2r + 1 - r^2)\, f(r) = \sum_{r=1}^{n} \{(r+1)^2 - r^2\}\, f(r)$$

$$= \sum_{r=1}^{n} \{(r+1)^2 f(r) - (r+1)^2 f(r+1) + (r+1)^2 f(r+1) - r^2 f(r)\}$$

$$= \sum_{r=1}^{n} (r+1)^2 \{f(r) - f(r+1)\} + \sum_{r=1}^{n} \{(r+1)^2 f(r+1) - r^2 f(r)\}$$

$$= -\sum_{r=1}^{n} \frac{(r+1)^2}{(r+1)} + \sum_{r=1}^{n-1} (r+1)^2 f(r+1) + (n+1)^2 f(n+1) - \sum_{r=1}^{n} r^2 f(r) \qquad \left[\because f(r+1) - f(r) = \frac{1}{r+1}\right]$$

$$= -\sum_{r=1}^{n} (r+1) + \{2^2 f(2) + 3^2 f(3) + \ldots + n^2 f(n)\} + (n+1)^2 f(n+1) - \{1^2 f(1) + 2^2 f(2) + 3^2 f(3) + \ldots + n^2 f(n)\}$$

$$= -\sum_{r=1}^{n} r - \sum_{r=1}^{n} 1 + (n+1)^2 f(n+1) - 1^2 f(1)$$

$$= -\frac{n(n+1)}{2} - n + (n+1)^2 f(n+1) - f(1)$$

$$= (n+1)^2 f(n+1) - \frac{n(n+3)}{2} - 1 \qquad [\because f(1) = 1]$$

$$= (n+1)^2 f(n+1) - \frac{(n^2 + 3n + 2)}{2}$$

Hence, this is the required result.

● ***Ex. 52*** *If the equation* $x^4 - 4x^3 + ax^2 + bx + 1 = 0$ *has four positive roots, find the values of a and b.*

Sol. Let x_1, x_2, x_3, x_4 are the roots of the equation

$$x^4 - 4x^3 + ax^2 + bx + 1 = 0 \qquad \ldots(i)$$

$\therefore\ x_1 + x_2 + x_3 + x_4 = 4$ and $x_1 x_2 x_3 x_4 = 1$

$\because \quad \text{AM} = \frac{x_1 + x_2 + x_3 + x_4}{4} = \frac{4}{4} = 1$

and $\text{GM} = (x_1 x_2 x_3 x_4)^{1/4} = (1)^{1/4} = 1$

i.e., AM = GM

which is true only when $x_1 = x_2 = x_3 = x_4 = 1$

Hence, given equation has all roots identical, equal to 1 i.e., equation have form

$$(x-1)^4 = 0$$

$$\Rightarrow \quad x^4 - 4x^3 + 6x^2 - 4x + 1 = 0 \qquad \ldots(ii)$$

On comparing Eqs. (i) and (ii), we get

$$a = 6, b = -4$$

● ***Ex. 53*** *Evaluate* $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 n}{3^m (n \cdot 3^m + m \cdot 3^n)}$.

Sol. Let $S = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 n}{3^m (n \cdot 3^m + m \cdot 3^n)}$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\left(\frac{3^m}{m}\right)\left(\frac{3^m}{m} + \frac{3^n}{n}\right)}$$

Now, let $a_m = \frac{3^m}{m}$ and $a_n = \frac{3^n}{n}$

Then, $S = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{a_m (a_m + a_n)} \qquad \ldots(i)$

By interchanging m and n, then

$$S = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{a_n (a_n + a_m)} \qquad \ldots(ii)$$

On adding Eqs. (i) and (ii), we get

$$2S = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{a_m a_n} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{mn}{3^m 3^n}$$

$$= \left(\sum_{n=1}^{\infty} \frac{n}{3^n}\right)^2 = \left[1\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right)^2 + 3\left(\frac{1}{3}\right)^3 + \ldots\right]^2$$

$$= (S')^2 \qquad \ldots(\text{iii})$$

where, $S' = 1\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right)^2 + 3\left(\frac{1}{3}\right)^3 + \ldots + \infty$

$$\frac{1}{3}S' = \quad 1\left(\frac{1}{3}\right)^2 + 2\left(\frac{1}{3}\right)^3 \quad + \ldots + \infty$$

$$\frac{2}{3}S' = \quad \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 \quad + \ldots + \infty$$

$$= \frac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{1}{2}$$

$$\therefore \qquad S' = \frac{3}{4}$$

From Eq. (iii), we get $2S = \left(\frac{3}{4}\right)^2$

$$\therefore \qquad S = \frac{9}{32}$$

● ***Ex. 54*** *Find the value of* $\sum_{\substack{i=0 \\ (i \neq j \neq k)}}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{3^i\, 3^j\, 3^k}$

Sol. Let $S = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{3^i\, 3^j\, 3^k} \qquad [i \neq j \neq k]$

We will first of all find the sum without any restriction on i, j, k.

Let $\quad S_1 = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{3^i\, 3^j\, 3^k} = \left(\sum_{i=0}^{\infty} \frac{1}{3^i}\right)^3$

$$= \left(\frac{3}{2}\right)^3 = \frac{27}{8}$$

***Case* I** If $i = j = k$

Let $\quad S_2 = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{3^i\, 3^j\, 3^k}$

$$= \sum_{i=0}^{\infty} \frac{1}{3^{3i}} = 1 + \frac{1}{3^3} + \frac{1}{3^6} + \ldots = \frac{1}{1 - \frac{1}{3^3}} = \frac{27}{26}$$

***Case* II** If $i = j \neq k$

Let $\quad S_3 = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{3^i\, 3^j\, 3^k} = \left(\sum_{i=0}^{\infty} \frac{1}{3^{2i}}\right)\left(\sum_{k=0}^{\infty} \frac{1}{3^k}\right)$

$[\because k \neq i]$

$$= \sum_{i=0}^{\infty} \frac{1}{3^{2i}}\left(\frac{3}{2} - \frac{1}{3^i}\right) = \sum_{i=0}^{\infty} \frac{3}{2} \cdot \frac{1}{3^{2i}} - \sum_{i=0}^{\infty} \frac{1}{3^{3i}}$$

$$= \frac{3}{2} \cdot \frac{9}{8} - \frac{27}{26} = \frac{135}{208}$$

Hence required sum, $S = S_1 - S_2 - 3S_3$

$$= \frac{27}{8} - \frac{27}{26} - 3\left(\frac{135}{208}\right) = \frac{27 \times 26 - 27 \times 8 - 3 \times 135}{208} = \frac{81}{208}$$

● ***Ex. 55*** *Let* $S_n, n = 1, 2, 3, \ldots$ *be the sum of infinite geometric series, whose first term is n and the common ratio is* $\frac{1}{n+1}$.

Evaluate

$$\lim_{n \to \infty} \frac{S_1 S_n + S_2\, S_{n-1} + S_3 S_{n-2} + \ldots + S_n S_1}{S_1^2 + S_2^2 + \ldots + S_n^2}.$$

Sol. $\because \qquad S_n = \frac{n}{1 - \frac{1}{n+1}} \quad \Rightarrow \quad S_n = n + 1$

$$\therefore\ S_1 S_n + S_2 S_{n-1} + S_3 S_{n-2} + \ldots + S_n S_1$$

$$= \sum_{r=1}^{n} S_r S_{n-r+1} = \sum_{r=1}^{n} (r+1)(n-r+2)$$

$$= \sum_{r=1}^{n} [(n+1)r - r^2 + (n+2)]$$

$$= (n+1)\sum_{r=1}^{n} r - \sum_{r=1}^{n} r^2 + (n+2)\sum_{r=1}^{n} 1$$

$$= (n+1)\sum n - \sum n^2 + (n+2) \cdot n$$

$$= \frac{(n+1)\, n\, (n+1)}{2} - \frac{n\,(n+1)(2n+1)}{6} + (n+2)\, n$$

$$= \frac{n}{6}(n^2 + 9n + 14) \qquad \ldots(\text{i})$$

and $S_1^2 + S_2^2 + \ldots + S_n^2 = \sum_{r=1}^{n} S_r^2 = \sum_{r=1}^{n} (r+1)^2 = \sum_{r=0}^{n} (r+1)^2 - 1^2$

$$= \frac{(n+1)(n+2)(2n+3)}{6} - 1$$

$$= \frac{n}{6}(2n^2 + 9n + 13) \qquad \ldots(\text{ii})$$

From Eqs. (i) and (ii), we get

$$\lim_{n \to \infty} \frac{S_1 S_n + S_2\, S_{n-1} + S_3 S_{n-2} + \ldots + S_n S_1}{S_1^2 + S_2^2 + \ldots + S_n^2}$$

$$= \lim_{n \to \infty} \frac{\frac{n}{6}(n^2 + 9n + 14)}{\frac{n}{6}(2n^2 + 9n + 13)} = \lim_{n \to \infty} \frac{\left(1 + \frac{9}{n} + \frac{14}{n^2}\right)}{\left(2 + \frac{9}{n} + \frac{13}{n^2}\right)}$$

$$= \frac{1 + 0 + 0}{2 + 0 + 0} = \frac{1}{2}$$

● ***Ex. 56*** *The nth term of a series is given by* $t_n = \dfrac{n^5 + n^3}{n^4 + n^2 + 1}$ *and if sum of its n terms can be expressed as* $S_n = a_n^2 + a + \dfrac{1}{b_n^2 + b}$, *where* a_n *and* b_n *are the nth terms of some arithmetic progressions and a, b are some constants, prove that* $\dfrac{b_n}{a_n}$ *is a constant.*

Sol. Since, $t_n = \dfrac{n^5 + n^3}{n^4 + n^2 + 1}$

$$= n - \frac{n}{n^4 + n^2 + 1}$$

$$= n + \frac{1}{2(n^2 + n + 1)} - \frac{1}{2(n^2 - n + 1)}$$

Sum of n terms $S_n = \sum t_n$

$$= \sum n + \frac{1}{2}\left\{\sum\left(\frac{1}{n^2 + n + 1} - \frac{1}{n^2 - n + 1}\right)\right\}$$

$$= \frac{n(n+1)}{2} + \frac{1}{2}\left(\frac{1}{n^2 + n + 1} - 1\right) \quad [\text{by property}]$$

$$= \frac{n^2}{2} + \frac{n}{2} - \frac{1}{2} + \frac{1}{2n^2 + 2n + 2}$$

$$= \left(\frac{n}{\sqrt{2}} + \frac{1}{2\sqrt{2}}\right)^2 - \frac{1}{8} - \frac{1}{2} + \frac{1}{\left(n\sqrt{2} + \frac{1}{\sqrt{2}}\right)^2 + \frac{3}{2}}$$

$$= \left(\frac{n}{\sqrt{2}} + \frac{1}{2\sqrt{2}}\right)^2 - \frac{5}{8} + \frac{1}{\left(n\sqrt{2} + \frac{1}{\sqrt{2}}\right)^2 + \frac{3}{2}}$$

but given, $S_n = a_n^2 + a + \dfrac{1}{b_n^2 + b}$

On comparing, we get

$$a_n = \frac{n}{\sqrt{2}} + \frac{1}{2\sqrt{2}},\ a = -\frac{5}{8},\ b_n = \left(n\sqrt{2} + \frac{1}{\sqrt{2}}\right),\ b = \frac{3}{2}$$

$\therefore$ $\dfrac{b_n}{a_n} = 2$, which is constant.

Sequences and Series Exercise 1 :
Single Option Correct Type Questions

■ This section contains **30 multiple choice questions.** Each question has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct

1. If the numbers x, y, z are in HP, then $\frac{\sqrt{yz}}{\sqrt{y}+\sqrt{z}}, \frac{\sqrt{xz}}{\sqrt{x}+\sqrt{z}}, \frac{\sqrt{xy}}{\sqrt{x}+\sqrt{y}}$ are in

(a) AP (b) GP
(c) HP (d) None of these

2. If $a_1, a_2, \ldots$ are in HP and $f_k = \sum_{r=1}^{n} a_r - a_k$, then $2^{\alpha_1}, 2^{\alpha_2}, 2^{\alpha_3}, 2^{\alpha_4}, \ldots$ are in

$\left\{\text{where}\,\alpha_1 = \frac{a_1}{f_1}, \alpha_2 = \frac{a_2}{f_2}, \alpha_3 = \frac{a_3}{f_3}, \ldots\right\}$

(a) AP (b) GP
(c) HP (d) None of these

3. ABC is a right angled triangle in which $\angle B = 90°$ and $BC = a$. If n points $L_1, L_2, \ldots, L_n$ on AB are such that AB is divided in $n+1$ equal parts and $L_1M_1, L_2M_2, \ldots, L_nM_n$ are line segments parallel to BC and $M_1, M_2, \ldots, M_n$ are on AC, the sum of the lengths of $L_1M_1, L_2M_2, \ldots, L_nM_n$ is

(a) $\frac{a(n+1)}{2}$ (b) $\frac{a(n-1)}{2}$
(c) $\frac{an}{2}$
(d) impossible to find from the given data

4. Let S_n $(1 \le n \le 9)$ denotes the sum of n terms of the series $1 + 22 + 333 + \ldots + \underbrace{999\ldots9}_{9 \text{ times}}$, then for $2 \le n \le 9$

(a) $S_n - S_{n-1} = \frac{1}{9}(10^n - n^2 + n)$

(b) $S_n = \frac{1}{9}(10^n - n^2 + 2n - 2)$

(c) $9(S_n - S_{n-1}) = n(10^n - 1)$

(d) None of the above

5. If a, b, c are in GP, then the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root, if $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in

(a) AP (b) GP
(c) HP (d) None of these

6. Sum of the first n terms of the series $\frac{1}{2} + \frac{3}{4} + \frac{7}{8} + \frac{15}{16} + \ldots$ is equal to

(a) $2^n - n - 1$ (b) $1 - 2^{-n}$
(c) $n + 2^{-n} - 1$ (d) $2^n - 1$

7. If in a ΔPQR, $\sin P, \sin Q, \sin R$ are in AP, then

(a) the altitudes are in AP (b) the altitudes are in HP
(c) the medians are in GP (d) the medians are in AP

8. Let $a_1, a_2, \ldots, a_{10}$ be in AP and $h_1, h_2, \ldots, h_{10}$ be in HP. If $a_1 = h_1 = 2$ and $a_{10} = h_{10} = 3$, then $a_4 h_7$ is

(a) 2 (b) 3
(c) 5 (d) 6

9. If $I_n = \int_0^{\pi} \frac{1 - \sin 2nx}{1 - \cos 2x}\,dx$, then $I_1, I_2, I_3, \ldots$ are in

(a) AP (b) GP
(c) HP (d) None of these

10. If $a(b-c)x^2 + b(c-a)xy + c(a-b)y^2$ is a perfect square, the quantities a, b, c are in

(a) AP (b) GP
(c) HP (d) None of these

11. The sum to infinity of the series, $1 + 2\left(1 - \frac{1}{n}\right) + 3\left(1 - \frac{1}{n}\right)^2 + \ldots$ is

(a) n^2 (b) $n(n+1)$
(c) $n\left(1 + \frac{1}{n}\right)^2$ (d) None of these

12. If $\log_3 2, \log_3(2^x - 5)$ and $\log_3\left(2^x - \frac{7}{2}\right)$ are in AP, x is equal to

(a) 2 (b) 3
(c) 4 (d) 2, 3

13. Let a, b, c be three positive prime numbers. The progression in which $\sqrt{a}, \sqrt{b}, \sqrt{c}$ can be three terms (not necessarily consecutive), is

(a) AP (b) GP
(c) HP (d) None of these

14. If n is an odd integer greater than or equal to 1, the value of $n^3 - (n-1)^3 + (n-2)^3 - \ldots + (-1)^{n-1} 1^3$ is

(a) $\frac{(n+1)^2(2n-1)}{4}$ (b) $\frac{(n-1)^2(2n-1)}{4}$
(c) $\frac{(n+1)^2(2n+1)}{4}$ (d) None of these

15. If the sides of a right angled triangle form an AP, the sines of the acute angles are

(a) $\frac{3}{5}, \frac{4}{5}$ (b) $\sqrt{3}, \frac{1}{3}$

(c) $\sqrt{\frac{\sqrt{5}-1}{2}}, \sqrt{\frac{\sqrt{5}+1}{2}}$ (d) $\frac{\sqrt{3}}{2}, \frac{1}{2}$

16. The sixth term of an AP is equal to 2. The value of the common difference of the AP which makes the product $a_1\ a_4\ a_5$ least, is given by

(a) $\frac{8}{5}$ (b) $\frac{5}{4}$

(c) $\frac{2}{3}$ (d) None of these

17. If the arithmetic progression whose common difference is non-zero, the sum of first $3n$ terms is equal to the sum of the next n terms. The ratio of the sum of the first $2n$ terms to the next $2n$ terms is

(a) $\frac{1}{5}$ (b) $\frac{2}{3}$

(c) $\frac{3}{4}$ (d) None of these

18. The coefficient of x^{n-2} in the polynomial $(x-1)(x-2)(x-3)\ldots(x-n)$, is

(a) $\frac{n(n^2+2)(3n+1)}{24}$

(b) $\frac{n(n^2-1)(3n+2)}{24}$

(c) $\frac{n(n^2+1)(3n+4)}{24}$

(d) None of the above

19. Consider the pattern shown below:

Row 1 1

Row 2 3 5

Row 3 7 9 11

Row 4 13 15 17, 19, etc.

The number at the end of row 60 is

(a) 3659 (b) 3519

(c) 3681 (d) 3731

20. Let a_n be the nth term of an AP. If $\sum_{r=1}^{100} a_{2r} = \alpha$ and $\sum_{r=1}^{100} a_{2r-1} = \beta$, the common difference of the AP is

(a) $\alpha - \beta$ (b) $\beta - \alpha$

(c) $\frac{\alpha-\beta}{2}$ (d) None of these

21. If a_1, a_2, a_3, a_4, a_5 are in HP, then $a_1a_2 + a_2\ a_3 + a_3a_4 + a_4a_5$ is equal to

(a) $2\ a_1a_5$ (b) $3a_1a_5$

(c) $4a_1a_5$ (d) $6a_1a_5$

22. If a, b, c and d are four positive real numbers such that $abcd = 1$, the minimum value of $(1+a)(1+b)(1+c)(1+d)$ is

(a) 1 (b) 4

(c) 16 (d) 64

23. If a, b, c are in AP and $(a+2b-c)(2b+c-a)(c+a-b) = \lambda\ abc$, then λ is

(a) 1 (b) 2

(c) 4 (d) None of these

24. If $a_1, a_2, a_3, \ldots$ are in GP with first term a and common ratio r, then

$\frac{a_1a_2}{a_1^2 - a_2^2} + \frac{a_2a_3}{a_2^2 - a_3^2} + \frac{a_3a_4}{a_3^2 - a_4^2} + \ldots + \frac{a_{n-1}\ a_n}{a_{n-1}^2 - a_n^2}$ is equal to

(a) $\frac{nr}{1-r^2}$ (b) $\frac{(n-1)r}{1-r^2}$ (c) $\frac{nr}{1-r}$ (d) $\frac{(n-1)r}{1-r}$

25. The sum of the first ten terms of an AP is four times the sum of the first five terms, the ratio of the first term to the common difference is

(a) $\frac{1}{2}$ (b) 2 (c) $\frac{1}{4}$ (d) 4

26. If $\cos(x-y)$, $\cos x$ and $\cos(x+y)$ are in HP, the $\cos x \sec\left(\frac{y}{2}\right)$ is equal to

(a) $\pm\sqrt{2}$ (b) $\frac{1}{\sqrt{2}}$

(c) $-\frac{1}{\sqrt{2}}$ (d) None of these

27. If 11 AM's are inserted between 28 and 10, the number of integral AM's is

(a) 5 (b) 6

(c) 7 (d) 8

28. If x, y, z are in GP $(x, y, z > 1)$, then $\frac{1}{2x+\ln x}, \frac{1}{4x+\ln y}, \frac{1}{6x+\ln z}$ are in

(a) AP (b) GP

(c) HP (d) None of these

29. The minimum value of the quantity $\frac{(a^2+3a+1)(b^2+3b+1)(c^2+3c+1)}{abc}$, where $a, b, c \in R^+$, is

(a) $\frac{11^3}{2^3}$ (b) 125

(c) 25 (d) 27

30. Let $a_1, a_2, \ldots$ be in AP and $q_1, q_2, \ldots$ be in GP. If $a_1 = q_1 = 2$ and $a_{10} = q_{10} = 3$, then

(a) $a_7\ q_{19}$ is not an integer (b) $a_{19}\ q_7$ is an integer

(c) $a_7\ q_{19} = a_{19}\ q_{10}$ (d) None of these

Sequences and Series Exercise 2 :
More than One Correct Option Type Questions

■ This section contains **15 multiple choice questions**. Each question has four choices (a), (b), (c) and (d) out of which **MORE THAN ONE** may be correct.

31. If $a(n) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{2^n - 1}$, then

(a) $a(100) < 100$ (b) $a(100) > 100$
(c) $a(200) > 100$ (d) $a(200) < 100$

32. If the first and $(2n-1)$ th term of an AP, GP and HP are equal and their nth terms are a, b and c respectively, then

(a) $a = b = c$ (b) $a \geq b \geq c$
(c) $a + c = b$ (d) $ac - b^2 = 0$

33. For $0 < \phi < \frac{\pi}{2}$, if $x = \sum_{n=0}^{\infty} \cos^{2n}\phi$, $y = \sum_{n=0}^{\infty} \sin^{2n}\phi$ and $z = \sum_{n=0}^{\infty} \cos^{2n}\phi \sin^{2n}\phi$, then

(a) $xyz = xz + y$ (b) $xyz = xy + z$
(c) $xyz = x + y + z$ (d) $xyz = yz + x$

34. If a, b, c are in AP and a^2, b^2, c^2 are in HP, then which of the following could hold true?

(a) $-\frac{a}{2}, b, c$ are in GP (b) $a = b = c$
(c) a^3, b^3, c^3 are in GP (d) None of these

35. The next term of the GP $x, x^2 + 2, x^3 + 10$ is

(a) 0 (b) 6 (c) $\frac{729}{16}$ (d) 54

36. If the sum of n consecutive odd numbers is $25^2 - 11^2$, then

(a) $n = 14$ (b) $n = 16$
(c) first odd number is 23 (d) last odd number is 49

37. The GM of two positive numbers is 6. Their AM is A and HM is H satisfy the equation $90A + 5H = 918$, then A may be equal to

(a) $\frac{1}{5}$ (b) 5 (c) $\frac{5}{2}$ (d) 10

38. If the sum to n terms of the series $\frac{1}{1\cdot3\cdot5\cdot7} + \frac{1}{3\cdot5\cdot7\cdot9} + \frac{1}{5\cdot7\cdot9\cdot11} + \ldots$ is $\frac{1}{90} - \frac{\lambda}{f(n)}$, then

(a) $f(0) = 15$ (b) $f(1) = 105$
(c) $f(\lambda) = \frac{640}{27}$ (d) $\lambda = \frac{1}{3}$

39. For the series,

$$S = 1 + \frac{1}{(1+3)}(1+2)^2 + \frac{1}{(1+3+5)}(1+2+3)^2 + \frac{1}{(1+3+5+7)}(1+2+3+4)^2 + \ldots$$

(a) 7th term is 16 (b) 7th term is 18
(c) sum of first 10 terms is $\frac{505}{4}$ (d) sum of first 10 terms is $\frac{405}{4}$

40. Let $E = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$, then

(a) $E < 3$ (b) $E > \frac{3}{2}$ (c) $E < 2$ (d) $E > 2$

41. Let S_n $(n \geq 1)$ be a sequence of sets defined by

$S_1 = \{0\}, S_2 = \left\{\frac{3}{2}, \frac{5}{2}\right\}, S_3 = \left\{\frac{8}{3}, \frac{11}{3}, \frac{14}{3}\right\}$,

$S_4 = \left\{\frac{15}{4}, \frac{19}{4}, \frac{23}{4}, \frac{27}{4}\right\}$, ..., then

(a) third element in S_{20} is $\frac{439}{20}$
(b) third element in S_{20} is $\frac{431}{20}$
(c) sum of the elements in S_{20} is 589
(d) sum of the elements in S_{20} is 609

42. Which of the following sequences are unbounded?

(a) $\left(1 + \frac{1}{n}\right)^n$ (b) $\left(\frac{2n+1}{n+2}\right)$ (c) $\left(1 + \frac{1}{n}\right)^{n^2}$ (d) $\tan n$

43. Let a sequence $\{a_n\}$ be defined by

$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{3n}$, then

(a) $a_2 = \frac{11}{12}$ (b) $a_2 = \frac{19}{20}$
(c) $a_{n+1} - a_n = \frac{(9n+5)}{(3n+1)(3n+2)(3n+3)}$
(d) $a_{n+1} - a_n = \frac{-2}{3(n+1)}$

44. Let $S_n(x) = \left(x^{n-1} + \frac{1}{x^{n-1}}\right) + 2\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + \ldots + (n-1)\left(x + \frac{1}{x}\right) + n$, then

(a) $S_1(x) = 1$ (b) $S_1(x) = x + \frac{1}{x}$
(c) $S_{100}(x) = \frac{1}{x^{99}}\left(\frac{x^{100}-1}{x-1}\right)^2$ (d) $S_{100}(x) = \frac{1}{x^{100}}\left(\frac{x^{100}-1}{x-1}\right)^2$

45. All the terms of an AP are natural numbers and the sum of the first 20 terms is greater than 1072 and less than 1162. If the sixth term is 32, then

(a) first term is 7 (b) first term is 12
(c) common difference is 4 (d) common difference is 5

Sequences and Series Exercise 3 : Passage Based Questions

- This section contains **8 passages**. Based upon each of the passage **3 multiple choice questions** have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I

(Q. Nos. 46 to 48)

S_n be the sum of n terms of the series $\frac{8}{5}+\frac{16}{65}+\frac{24}{325}+\ldots$

46. The value of $\lim_{n\to\infty} S_n$ is

(a) 0 (b) $\frac{1}{2}$ (c) 2 (d) 4

47. The seventh term of the series is

(a) $\frac{56}{2505}$ (b) $\frac{56}{6505}$ (c) $\frac{56}{5185}$ (d) $\frac{56}{9605}$

48. The value of S_8, is

(a) $\frac{288}{145}$ (b) $\frac{1088}{545}$ (c) $\frac{81}{41}$ (d) $\frac{107}{245}$

Passage II

(Q. Nos. 49 to 51)

Two consecutive numbers from 1, 2, 3, ..., n are removed. The arithmetic mean of the remaining numbers is $\frac{105}{4}$.

49. The value of n lies in

(a) (41, 51) (b) (52, 62) (c) (63, 73) (d) (74, 84)

50. The removed numbers

(a) are less than 10 (b) lies between 10 to 30
(c) lies between 30 to 70 (d) greater than 70

51. Sum of all numbers is

(a) less than 1000 (b) lies between 1200 to 1500
(c) greater than 1500 (d) None of these

Passage III

(Q. Nos. 52 to 54)

There are two sets A and B each of which consists of three numbers in AP whose sum is 15 and where D and d are the common differences such that $D = 1 + d, d > 0$. If $p = 7(q - p)$, where p and q are the product of the numbers respectively in the two series.

52. The value of p is

(a) 105 (b) 140 (c) 175 (d) 210

53. The value of q is

(a) 200 (b) 160 (c) 120 (d) 80

54. The value of $7D + 8d$ is

(a) 37 (b) 22 (c) 67 (d) 52

Passage IV

(Q. Nos. 55 to 57)

There are two sets A and B each of which consists of three numbers in GP whose product is 64 and R and r are the common ratios such that $R = r + 2$. If $\frac{p}{q}=\frac{3}{2}$, where p and q are sum of numbers taken two at a time respectively in the two sets.

55. The value of p is

(a) 66 (b) 72 (c) 78 (d) 84

56. The value of q is

(a) 54 (b) 56 (c) 58 (d) 60

57. The value of $r^R + R^r$ is

(a) 5392 (b) 368 (c) 32 (d) 4

Passage V

(Q. Nos. 58 to 60)

The numbers 1, 3, 6, 10, 15, 21, 28, ... are called triangular numbers. Let t_n denotes the nth triangular number such that $t_n = t_{n-1} + n, \forall\, n \geq 2$.

58. The value of t_{50} is

(a) 1075 (b) 1175 (c) 1275 (d) 1375

59. The number of positive integers lying between t_{100} and t_{101} are

(a) 99 (b) 100 (c) 101 (d) 102

60. If $(m+1)$ is the nth triangular number, then $(n-m)$ is

(a) $1+\sqrt{(m^2+2m)}$ (b) $1+\sqrt{(m^2+2)}$
(c) $1+\sqrt{(m^2+m)}$ (d) None of these

Passage VI

(Q. Nos. 61 to 63)

Let $A_1, A_2, A_3, \ldots, A_m$ be arithmetic means between -3 and 828 and $G_1, G_2, G_3, \ldots, G_n$ be geometric means between 1 and 2187. Product of geometric means is 3^{35} and sum of arithmetic means is 14025.

61. The value of n is

(a) 45 (b) 30 (c) 25 (d) 10

62. The value of m is

(a) 17 (b) 34 (c) 51 (d) 68

63. The value of $G_1 + G_2 + G_3 + \ldots + G_n$ is

(a) 2044 (b) 1022
(c) 511 (d) None of these

Passage VII

(Q. Nos. 64 to 66)

Suppose α, β *are roots of* $ax^2 + bx + c = 0$ *and* γ, δ *are roots of* $Ax^2 + Bx + C = 0$.

64. If $\alpha, \beta, \gamma, \delta$ are in AP, then common difference of AP is

(a) $\frac{1}{4}\left(\frac{b}{a} - \frac{B}{A}\right)$ (b) $\frac{1}{3}\left(\frac{b}{a} - \frac{B}{A}\right)$

(c) $\frac{1}{2}\left(\frac{c}{a} - \frac{B}{A}\right)$ (d) $\frac{1}{3}\left(\frac{c}{a} - \frac{C}{A}\right)$

65. If a, b, c are in GP as well as $\alpha, \beta, \gamma, \delta$ are in GP, then A, B, C are in

(a) AP only (b) GP only

(c) AP and GP (d) None of these

66. If $\alpha, \beta, \gamma, \delta$ are in GP, then common ratio of GP is

(a) $\sqrt{\left(\frac{bA}{aB}\right)}$ (b) $\sqrt{\left(\frac{aB}{bA}\right)}$

(c) $\sqrt{\left(\frac{bC}{cB}\right)}$ (d) $\sqrt{\left(\frac{cB}{bC}\right)}$

Passage VIII (Q. Nos. 67 to 69)

Suppose p *is the first of* n $(n > 1)$ *arithmetic means between two positive numbers* a *and* b *and* q *the first of* n *harmonic means between the same two numbers.*

67. The value of p is

(a) $\frac{na + b}{n + 1}$ (b) $\frac{nb + a}{n + 1}$ (c) $\frac{na - b}{n + 1}$ (d) $\frac{nb - a}{n + 1}$

68. The value of q is

(a) $\frac{(n-1)\,ab}{nb + a}$ (b) $\frac{(n+1)\,ab}{nb + a}$ (c) $\frac{(n+1)\,ab}{na + b}$ (d) $\frac{(n-1)\,ab}{na + b}$

69. Final conclusion is

(a) q lies between p and $\left(\frac{n+1}{n-1}\right)^2 p$

(b) q lies between p and $\left(\frac{n+1}{n-1}\right)p$

(c) q does not lie between p and $\left(\frac{n+1}{n-1}\right)^2 p$

(d) q does not lie between p and $\left(\frac{n+1}{n-1}\right)p$

Sequences and Series Exercise 4 : Single Integer Answer Type Questions

- This section contains **10 questions**. The answer to each question is a **single digit integer**, ranging from 0 to 9 (both inclusive).

70. Let a, b, c, d be positive real numbers with $a < b < c < d$. Given that a, b, c, d are the first four terms of an AP and a, b, d are in GP. The value of $\frac{ad}{bc}$ is $\frac{p}{q}$, where p and q are prime numbers, then the value of q is

71. If the coefficient of x in the expansion of $\prod_{r=1}^{110}(1 + rx)$ is $\lambda\,(1+10)\,(1+10+10^2)$, then the value of λ is

72. A 3-digit palindrome is a 3-digit number (not starting with zero) which reads the same backwards as forwards *For example,* 242. The sum of all even 3-digit palindromes is $2^{n_1} \cdot 3^{n_2} \cdot 5^{n_3} \cdot 7^{n_4} \cdot 11^{n_5}$, alue of $n_1 + n_2 + n_3 + n_4 + n_5$ is

73. If n is a positive integer satisfying the equation $2 + (6 \cdot 2^2 - 4 \cdot 2) + (6 \cdot 3^2 - 4 \cdot 3) + \ldots + (6 \cdot n^2 - 4 \cdot n) = 140$, then the value of n is

74. Let $S(x) = 1 + x - x^2 - x^3 + x^4 + x^5 - x^6 - x^7 + \ldots + \infty$, where $0 < x < 1$. If $S(x) = \frac{\sqrt{2}+1}{2}$, then the value of $(x + 1)^2$ is

75. The sequence $a_1, a_2, a_3, \ldots$ is a geometric sequence with common ratio r. The sequence $b_1, b_2, b_3, \ldots$ is also a geometric sequence. If $b_1 = 1$, $b_2 = \sqrt[4]{7} - \sqrt[4]{28} + 1$, $a_1 = \sqrt[4]{28}$ and $\sum_{n=1}^{\infty} \frac{1}{a_n} = \sum_{n=1}^{\infty} b_n$, then the value of $(1 + r^2 + r^4)$ is

76. Let (a_1, b_1) and (a_2, b_2) are the pair of real numbers such that 10, a, b, ab constitute an arithmetic progression. Then, the value of $\left(\frac{2a_1a_2 + b_1b_2}{10}\right)$ is

77. If one root of $Ax^3 + Bx^2 + Cx + D = 0$, $A \neq 0$, is the arithmetic mean of the other two roots, then the relation $2B^3 + \lambda ABC + \mu A^2 D = 0$ holds good. Then, the value of $2\lambda + \mu$ is

78. If $|x| > 1$, then sum of the series $\frac{1}{1+x} + \frac{2}{1+x^2} + \frac{2^2}{1+x^4} + \frac{2^3}{1+x^8} + \ldots$ upto ∞ is $\frac{1}{x - \lambda}$, then the value of λ is

79. Three non-zero real numbers form an AP and the squares of these numbers taken in same order form a GP. If the possible common ratios are $(3 \pm \sqrt{k})$ where $k \in N$, then the value of $\left[\frac{k}{8} - \frac{8}{k}\right]$ is (where [ ] denotes the greatest integer function).

Sequences and Series Exercise 5 : Matching Type Questions

■ This section contains **4 questions**. Questions 80, 81 and 82 have three statements (A, B and C) and question 83 has four statements (A, B, C and D) given in **Column I** and questions 80 and 81 have four statements (p, q, r and s), question 82 has five statements (p, q, r, s and t) and question 83 has three statements (p, q and r) in **Column II**, respectively. Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II**.

80.

	Column I		Column II
(A)	a, b, c, d are in AP, then	(p)	$a+d>b+c$
(B)	a, b, c, d are in GP, then	(q)	$ad>bc$
(C)	a, b, c, d are in HP, then	(r)	$\frac{1}{a}+\frac{1}{d}>\frac{1}{b}+\frac{1}{c}$
		(s)	$ad<bc$

81.

	Column I		Column II
(A)	For an AP $a_1, a_2, a_3, \ldots, a_n, \ldots$; $a_1=\frac{5}{2}$; $a_{10}=16$. If $a_1+a_2+\ldots+a_n=110$, then 'n' equals	(p)	9
(B)	The interior angles of a convex non-equiangular polygon of 9 sides are in AP. The least positive integer that limits the upper value of the common difference between the measures of the angles in degrees is	(q)	10
(C)	For an increasing GP, $a_1, a_2, a_3, \ldots, a_n, \ldots$; $a_6=4\,a_4$; $a_9-a_7=192$, if $a_4+a_5+a_6+\ldots+a_n=1016$, then n equals	(r)	11
		(s)	12

82.

	Column I		Column II
(A)	If $a_1, a_2, a_3, \ldots$ are in AP and $a_1+a_4+a_7+a_{14}+a_{17}+a_{20}=165$, $\alpha=a_2+a_6+a_{15}+a_{19}$ and $\beta=2(a_9+a_{12})-(a_3+a_{18})$, then	(p)	$\alpha=2\beta$
(B)	If $a_1, a_2, a_3, \ldots$ are in AP and $a_1+a_5+a_{10}+a_{15}+a_{20}+a_{24}=195$, $\alpha=a_2+a_7+a_{18}+a_{23}$ and $\beta=2(a_3+a_{22})-(a_8+a_{17})$, then	(q)	$\alpha+2\beta=260$
(C)	If $a_1, a_2, a_3, \ldots$ are in AP and $a_1+a_7+a_{10}+a_{21}+a_{24}+a_{30}=225$, $\alpha=a_2+a_7+a_{24}+a_{29}$ and $\beta=2(a_{10}+a_{21})-(a_3+a_{28})$, then	(r)	$\alpha+2\beta=220$
		(s)	$\alpha-\beta=5\lambda, \lambda\in I$
		(t)	$\alpha+\beta=15\mu, \mu\in I$

83.

	Column I		Column II
(A)	If $4a^2+9b^2+16c^2=2(3ab+6bc+4ca)$, where a, b, c are non-zero numbers, then a, b, c are in	(p)	AP
(B)	If $17a^2+13b^2+5c^2=(3ab+15bc+5ca)$, where a, b, c are non-zero numbers, then a, b, c are in	(q)	GP
(C)	If $a^2+9b^2+25c^2=abc\left(\frac{15}{a}+\frac{5}{b}+\frac{3}{c}\right)$, where a, b, c are non-zero numbers, then a, b, c are in	(r)	HP
(D)	If $(a^2+b^2+c^2)p^2-2p(ab+bc+ca)+(a^2+b^2+c^2)\le 0$, where a, b, c, p are non-zero numbers, then a, b, c are in		

Sequences and Series Exercise 6 : Statement I and II Type Questions

■ **Directions** (Q. Nos. 84 to 90) are Assertion-Reason type questions. Each of these questions contains two statements:
Statement-1 (Assertion) and **Statement-2** (Reason)
Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

84. **Statement 1** 4, 8, 16 are in GP and 12, 16, 24 are in HP.
Statement 2 If middle term is added in three consecutive terms of a GP, resultant will be in HP.

85. **Statement 1** If the nth term of a series is $2n^3 + 3n^2 - 4$, then the second order differences must be an AP.
Statement 2 If nth term of a series is a polynomial of degree m, then mth order differences of series are constant.

86. **Statement 1** The sum of the products of numbers $\pm a_1, \pm a_2, \pm a_3, \ldots, \pm a_n$ taken two at a time is $-\sum_{i=1}^{n} a_i^2$.
Statement 2 The sum of products of numbers $a_1, a_2, a_3, \ldots, a_n$ taken two at a time is denoted by $\sum_{1 \le i < j \le n} a_i a_j$.

87. **Statement 1** $a + b + c = 18\,(a, b, c > 0)$, then the maximum value of abc is 216.
Statement 2 Maximum value occurs when $a = b = c$

88. **Statement 1** If $4a^2 + 9b^2 + 16c^2 = 2(3ab + 6bc + 4ca)$, where a, b, c are non-zero real numbers, then a, b, c are in GP.
Statement-2 If $(a_1 - a_2)^2 + (a_2 - a_3)^2 + (a_3 - a_1)^2 = 0$, then $a_1 = a_2 = a_3, \forall\ a_1, a_2, a_3 \in R$.

89. **Statement 1** If a and b be two positive numbers, where $a > b$ and $4 \times$ GM $= 5 \times$ HM for the numbers. Then, $a = 4b$.
Statement 2 (AM) (HM) $=$ (GM)^2 is true for positive numbers.

90. **Statement1** The difference between the sum of the first 100 even natural numbers and the sum of the first 100 odd natural numbers is 100.
Statement 2 The difference between the sum of the first n even natural numbers and sum of the first n odd natural numbers is n.

Sequences and Series Exercise 7 : Subjective Type Questions

■ In this section, there are **24 subjective** questions.

91. The p th, $(2p)$ th and $(4p)$ th terms of an AP, are in GP, then find the common ratio of GP.

92. Find the sum of n terms of the series $(a + b) + (a^2 + ab + b^2) + (a^3 + a^2 b + ab^2 + b^3) + \ldots$, where $a \neq 1, b \neq 1$ and $a \neq b$.

93. The sequence of odd natural numbers is divided into groups 1; 3, 5; 7, 9, 11; ... and so on. Show that the sum of the numbers in nth group is n^3.

94. Let a, b, c are respectively the sums of the first n terms, the next n terms and the next n terms of a GP. Show that a, b, c are in GP.

95. If the first four terms of an arithmetic sequence are $a, 2a, b$ and $(a - 6 - b)$ for some numbers a and b, find the sum of the first 100 terms of the sequence.

96. If $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$ upto $\infty = \frac{\pi^2}{6}$, find
(i) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$ upto ∞
(ii) $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$ upto ∞

97. If the arithmetic mean of $a_1, a_2, a_3, \ldots, a_n$ is a and $b_1, b_2, b_3, \ldots, b_n$ have the arithmetic mean b and $a_i + b_i = 1$ for $i = 1, 2, 3, \ldots, n$, prove that

$$\sum_{i=1}^{n} (a_i - a)^2 + \sum_{i=1}^{n} a_i b_i = nab$$

98. If $a_1, a_2, a_3, \ldots$ is an arithmetic progression with common difference 1 and $a_1 + a_2 + a_3 + \ldots + a_{98} = 137$, then find the value of $a_2 + a_4 + a_6 + \ldots + a_{98}$.

99. If $t_1 = 1, t_r - t_{r-1} = 2^{r-1}, r \ge 2$, find $\sum_{r=1}^{n} t_r$.

100. Prove that $I_1, I_2, I_3, \ldots$ form an AP, if

(i) $I_n = \int_0^{\pi} \frac{\sin 2nx}{\sin x}\, dx$ (ii) $I_n = \int_0^{\pi} \left(\frac{\sin nx}{\sin x}\right)^2 dx$

101. Consider the sequence $S = 7 + 13 + 21 + 31 + \ldots + T_n$, find the value of T_{70}.

102. Find value of $\left(x + \frac{1}{x}\right)^3 + \left(x^2 + \frac{1}{x^2}\right)^3 + \ldots + \left(x^n + \frac{1}{x^n}\right)^3$.

103. If a_m be the mth term of an AP, show that

$$a_1^2 - a_2^2 + a_3^2 - a_4^2 + \ldots + a_{2n-1}^2 - a_{2n}^2 = \frac{n}{(2n-1)}(a_1^2 - a_{2n}^2).$$

104. If three unequal numbers are in HP and their squares are in AP, show that they are in the ratio
$1 + \sqrt{3} : -2 : 1 - \sqrt{3}$ or $1 - \sqrt{3} : -2 : 1 + \sqrt{3}$.

105. If $a_1, a_2, a_3, \ldots, a_n$ are in AP with $a_1 = 0$, prove that

$$\frac{a_3}{a_2} + \frac{a_4}{a_3} + \ldots + \frac{a_n}{a_{n-1}} - a_2\left(\frac{1}{a_2} + \frac{1}{a_3} + \ldots + \frac{1}{a_{n-2}}\right) = \frac{a_{n-1}}{a_2} + \frac{a_2}{a_{n-1}}$$

106. Balls are arranged in rows to form an equilateral triangle. The first row consists of one ball, the second row of two balls and so on. If 669 more balls are added, then all the balls can be arranged in the shape of a square and each of the sides, then contains 8 balls less than each side of the triangle. Determine the initial number of balls.

107. If $\theta_1, \theta_2, \theta_3, \ldots, \theta_n$ are in AP whose common difference is d, then show that

$$\sin d \{\sec\theta_1 \sec\theta_2 + \sec\theta_2 \sec\theta_3 + \ldots + \sec\theta_{n-1} \sec\theta_n\} = \tan\theta_n - \tan\theta_1.$$

108. Show that,

$$(1 + 5^{-1})(1 + 5^{-2})(1 + 5^{-4})(1 + 5^{-8}) \ldots (1 + 5^{-2^n}) = \frac{5}{4}(1 - 5^{-2^{n+1}})$$

109. Evaluate $S = \sum_{n=0}^{n} \frac{2^n}{(a^{2^n} + 1)}$ (where $a > 1$).

110. Find the sum to infinite terms of the series

$$\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{2}{9}\right) + \ldots + \tan^{-1}\left(\frac{2^{n-1}}{1 + 2^{2n-1}}\right) + \ldots$$

111. Find the sum to n terms, whose nth term is
$\tan[\alpha + (n-1)\beta]\tan(\alpha + n\beta)$.

112. If $\sum_{r=1}^{n} T_r = \frac{n}{8}(n+1)(n+2)(n+3)$, find $\sum_{r=1}^{n} \frac{1}{T_r}$.

113. If S_1, S_2, S_3 denote the sum of n terms of 3 arithmetic series whose first terms are unity and their common difference are in HP, prove that

$$n = \frac{2S_3S_1 - S_1S_2 - S_2S_3}{S_1 - 2S_2 + S_3}.$$

114. Three friends whose ages form a GP divide a certain sum of money in proportion to their ages. If they do that three years later, when the youngest is half the age of the oldest, then he will receive ₹ 105 more than he gets now and the middle friend will get ₹ 15 more than he gets now. Find the ages of the friends.

Sequences and Series Exercise 8 : Questions Asked in Previous 13 Year's Exam

- This section contains questions asked in **IIT-JEE, AIEEE, JEE Main & JEE Advanced** from year **2005** to year **2017**.

115. If a, b, c are in AP and $|a|, |b|, |c| < 1$ and

$$x = 1 + a + a^2 + \ldots + \infty$$
$$y = 1 + b + b^2 + \ldots + \infty$$
$$z = 1 + c + c^2 + \ldots + \infty$$

Then, x, y, z will be in **[AIEEE 2005, 3M]**

(a) AP (b) GP (c) HP

(d) None of these

116. If $a_n = \frac{3}{4} - \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 - \ldots + (-1)^{n-1}\left(\frac{3}{4}\right)^n$ and $b_n = 1 - a_n$, then find the least natural number n_0 such that $b_n > a_n, \forall\, n \geq n_0$. **[IIT-JEE 2006, 6M]**

117. Let $a_1, a_2, a_3, \ldots$ be terms are in AP, if $\frac{a_1 + a_2 + \ldots + a_p}{a_1 + a_2 + \ldots + a_q} = \frac{p^2}{q^2}$, $p \neq q$, then $\frac{a_6}{a_{21}}$ equals **[AIEEE 2006, 4.5M]**

(a) $\frac{41}{11}$ (b) $\frac{7}{2}$

(c) $\frac{2}{7}$ (d) $\frac{11}{41}$

118. If $a_1, a_2, \ldots, a_n$ are in HP, then the expression $a_1a_2 + a_2a_3 + \ldots + a_{n-1}a_n$ is equal to **[AIEEE 2006, 6M]**

(a) $n(a_1 - a_n)$

(b) $(n-1)(a_1 - a_n)$

(c) na_1a_n

(d) $(n-1)a_1a_n$

119. Let V_r denotes the sum of the first r terms of an arithmetic progression whose first term is r and the common difference is $(2r-1)$. Let $T_r = V_{r+1} - V_r - 2$ and $Q_r = T_{r+1} - T_r$ for $r = 1, 2, \ldots$ **[IIT-JEE 2007, 4+4+4M]**

(i) The sum $V_1 + V_2 + \ldots + V_n$ is

(a) $\frac{1}{12} n(n+1)(3n^2 - n + 1)$

(b) $\frac{1}{12} n(n+1)(3n^2 + n + 2)$

(c) $\frac{1}{2} n(2n^2 - n + 1)$

(d) $\frac{1}{3}(2n^3 - 2n + 3)$

(ii) T_r is always

(a) an odd number (b) an even number

(c) a prime number (d) a composite number

(iii) Which one of the following is a correct statement?

(a) $Q_1, Q_2, Q_3, \ldots$ are in AP with common difference 5

(b) $Q_1, Q_2, Q_3, \ldots$ are in AP with common difference 6

(c) $Q_1, Q_2, Q_3, \ldots$ are in AP with common difference 11

(d) $Q_1 = Q_2 = Q_3 = \ldots$

120. Let A_1, G_1, H_1 denote the arithmetic, geometric and harmonic means respectively, of two distinct positive numbers. For $n \geq 2$, let A_{n-1}, G_{n-1} and H_{n-1} has arithmetic, geometric and harmonic means as A_n, G_n, H_n, respectively. **[IIT-JEE 2007, 4+4+4M]**

(i) Which one of the following statement is correct?

(a) $G_1 > G_2 > G_3 > \ldots$ (b) $G_1 < G_2 < G_3 < \ldots$

(c) $G_1 = G_2 = G_3 = \ldots$

(d) $G_1 < G_3 < G_5 < \ldots$ and $G_2 > G_4 > G_6 > \ldots$

(ii) Which of the following statement is correct?

(a) $A_1 > A_2 > A_3 > \ldots$

(b) $A_1 < A_2 < A_3 < \ldots$

(c) $A_1 > A_3 > A_5 > \ldots$ and $A_2 < A_4 < A_6 < \ldots$

(d) $A_1 < A_3 < A_5 < \ldots$ and $A_2 > A_4 > A_6 > \ldots$

(iii) Which of the following statement is correct?

(a) $H_1 > H_2 > H_3 > \ldots$

(b) $H_1 < H_2 < H_3 < \ldots$

(c) $H_1 > H_3 > H_5 > \ldots$ and $H_2 < H_4 < H_6 < \ldots$

(d) $H_1 < H_3 < H_5 < \ldots$ and $H_2 > H_4 > H_6 > \ldots$

121. If a geometric progression consisting of positive terms, each term equals the sum of the next two terms, then the common ratio of this progression equals **[AIEEE 2007, 3M]**

(a) $\frac{1}{2}(1 - \sqrt{5})$ (b) $\frac{1}{2}\sqrt{5}$

(c) $\sqrt{5}$ (d) $\frac{1}{2}(\sqrt{5} - 1)$

122. Suppose four distinct positive numbers a_1, a_2, a_3, a_4 are in GP. Let $b_1 = a_1, b_2 = b_1 + a_2, b_3 = b_2 + a_3$ and $b_4 = b_3 + a_4$.

Statement 1 The numbers b_1, b_2, b_3, b_4 are neither in AP nor in GP.

Statement 2 The numbers b_1, b_2, b_3, b_4 are in HP. **[IIT-JEE 2008, 3M]**

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1

(c) Statement-1 is true, Statement-2 is false

(d) Statement-1 is false, Statement-2 is true

123. The first two terms of a geometric progression add upto 12 the sum of the third and the fourth terms is 48, if the terms of the geometric progression are alternately positive and negative, then the first term is **[AIEEE 2008, 3M]**

(a) -12 (b) 12 (c) 4 (d) -4

124. If the sum of first n terms of an AP is cn^2, then the sum of squares of these n terms is **[IIT-JEE 2009, 3M]**

(a) $\frac{n(4n^2 - 1)c^2}{6}$ (b) $\frac{n(4n^2 + 1)c^2}{3}$

(c) $\frac{n(4n^2 - 1)c^2}{3}$ (d) $\frac{n(4n^2 + 1)c^2}{6}$

125. The sum to infinity of the series $1 + \frac{2}{3} + \frac{6}{3^2} + \frac{10}{3^3} + \frac{14}{3^4} + \ldots$ is **[AIEEE 2009, 4M]**

(a) 6 (b) 2 (c) 3 (d) 4

126. Let $S_k, k = 1, 2, \ldots, 100$, denote the sum of the infinite geometric series whose first term is $\frac{k-1}{k!}$ and common ratio is $\frac{1}{k}$. Then, the value of $\frac{100^2}{100!} + \sum_{k=2}^{100} |(k^2 - 3k + 1) S_k|$ is **[IIT-JEE 2010, 3M]**

127. Let $a_1, a_2, a_3, \ldots, a_{11}$ be real numbers satisfying $a_1 = 15, 27 - 2a_2 > 0$ and $a_k = 2a_{k-1} - a_{k-2}$ for $k = 3, 4, \ldots, 11$. If $\frac{a_1^2 + a_2^2 + \ldots + a_{11}^2}{11} = 90$, then the value of $\frac{a_1 + a_2 + \ldots + a_{11}}{11}$ is equal to **[IIT-JEE 2010, 3M]**

128. A person is to count 4500 currency notes. Let a_n denotes the number of notes he counts in the nth minute. If $a_1 = a_2 = \ldots = a_{10} = 150$ and $a_{10}, a_{11}, \ldots$ are in AP with common difference -2, then the time taken by him to count all notes is **[AIEEE 2010, 8M]**

(a) 34 min (b) 125 min

(c) 135 min (d) 24 min

129. The minimum value of the sum of real numbers $a^{-5}, a^{-4}, 3a^{-3}, 1, a^8$ and a^{10} with $a > 0$ is **[IIT-JEE 2011, 4M]**

130. A man saves ₹ 200 in each of the first three months of his service. In each of the subsequent months his saving increases by ₹ 40 more than the saving of immediately previous month. His total saving from the start of service will be ₹ 11040 after **[AIEEE 2011, 4M (Paper I)]**

(a) 19 months (b) 20 months
(c) 21 months (d) 18 months

131. Let a_n be the nth term of an AP, if $\sum_{r=1}^{100} a_{2r} = \alpha$ and $\sum_{r=1}^{100} a_{2r-1} = \beta$, then the common difference of the AP is **[AIEEE 2011, 4M (Paper II)]**

(a) $\frac{\alpha - \beta}{200}$ (b) $\alpha - \beta$
(c) $\frac{\alpha - \beta}{100}$ (d) $\beta - \alpha$

132. If a_1, a_2, a_3,...be in harmonic progression with $a_1 = 5$ and $a_{20} = 25$. The least positive integer n for which $a_n < 0$ is **[IIT-JEE 2012, 3M]**

(a) 22 (b) 23
(c) 24 (d) 25

133. **Statement 1** The sum of the series $1 + (1 + 2 + 4) + (4 + 6 + 9) + (9 + 12 + 16) + \ldots + (361 + 380 + 400)$ is 8000.

Statement 2 $\sum_{k=1}^{n} (k^3 - (k-1)^3) = n^3$ for any natural number n. **[AIEEE 2012, 4M]**

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement-1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

134. If 100 times the 100th term of an AP with non-zero common difference equals the 50 times its 50th term, then the 150th term of this AP is **[AIEEE 2012, 4M]**

(a) 150 times its 50th term (b) 150
(c) zero (d) −150

135. If x, y, z are in AP and $\tan^{-1} x$, $\tan^{-1} y$, $\tan^{-1} z$ are also in AP, then **[JEE Main 2013, 4M]**

(a) $2x = 3y = 6z$ (b) $6x = 3y = 2z$
(c) $6x = 4y = 3z$ (d) $x = y = z$

136. The sum of first 20 terms of the sequence 0.7, 0.77, 0.777, ..., is **[JEE Main 2013, 4M]**

(a) $\frac{7}{9}(99 - 10^{-20})$ (b) $\frac{7}{81}(179 + 10^{-20})$
(c) $\frac{7}{9}(99 + 10^{-20})$ (d) $\frac{7}{81}(179 - 10^{-20})$

137. Let $S_n = \sum_{k=1}^{4n} (-1)^{\frac{k(k+1)}{2}} \cdot k^2$, then S_n can take value(s) **[JEE Advanced 2013, 4M]**

(a) 1056 (b) 1088 (c) 1120 (d) 1332

138. A pack contains n cards numbered from 1 to n. Two consecutive numbered cards are removed from the pack and the sum of the numbers on the remaining cards is 1224. If the smaller of the numbers on the removed cards is k, then $k - 20$ is equal to **[JEE Advanced 2013, 4M]**

139. If $(10)^9 + 2(11)^1(10)^8 + 3(11)^2(10)^7 + \ldots + (10)(11)^9 = k(10)^9$, then k is equal to **[JEE Main 2014, 4M]**

(a) 100 (b) 110 (c) $\frac{121}{10}$ (d) $\frac{441}{100}$

140. Three positive numbers form an increasing GP. If the middle terms in this GP is doubled, the new numbers are in AP. Then, the common ratio of the GP is **[JEE Main 2014, 4M]**

(a) $2 - \sqrt{3}$ (b) $2 + \sqrt{3}$ (c) $\sqrt{2} + \sqrt{3}$ (d) $3 + \sqrt{2}$

141. Let a, b, c be positive integers such that $\frac{b}{a}$ is an integer. If a, b, c are in geometric progression and the arithmetic mean of a, b, c is $b + 2$, the value of $\frac{a^2 + a - 14}{a + 1}$ is **[JEE Advanced 2014, 3M]**

142. The sum of first 9 terms of the series $\frac{1^3}{1} + \frac{1^3 + 2^3}{1 + 3} + \frac{1^3 + 2^3 + 3^3}{1 + 3 + 5} + \ldots$ is **[JEE Main 2015, 4M]**

(a) 192 (b) 71 (c) 96 (d) 142

143. If m is the AM of two distinct real numbers l and n $(l, n > 1)$ and G_1, G_2 and G_3 are three geometric means between l and n, then $G_1^4 + 2G_2^4 + G_3^4$ equals **[JEE Main 2015, 4M]**

(a) $4l^2m^2n^2$ (b) $4l^2mn$
(c) $4lm^2n$ (d) $4lmn^2$

144. Suppose that all the terms of an arithmetic progression (AP) are natural numbers. If the ratio of the sum of the first seven terms to the sum of the first eleven terms is 6 : 11 and the seventh term lies between 130 and 140, then the common difference of this AP is **[JEE Main 2015, 4M]**

145. If the 2nd, 5th and 9th terms of a non-eustant AP are in GP, then the common ratio of this GP is **[JEE Main 2016, 4M]**

(a) 1 (b) $\frac{7}{4}$ (c) $\frac{8}{5}$ (d) $\frac{4}{3}$

146. If the sum of the first ten terms of the series $\left(1\frac{3}{5}\right)^2 + \left(2\frac{2}{5}\right)^2 + \left(3\frac{1}{5}\right)^2 + 4^2 + \left(4\frac{4}{5}\right)^2 + \ldots$ is $\frac{16}{5}$ m, then m equal to **[JEE Main 2016, 4M]**

(a) 100 (b) 99 (c) 102 (d) 101

147. Let $b_i > 1$ for $i = 1, 2, \ldots, 101$. Suppose $\log_e b_1, \log_e b_2, \log_e b_3, \ldots, \log_e b_{101}$ are in Arithmetic Progression (AP) with the common difference $\log_e 2$. Suppose $a_1, a_2, a_3, \ldots, a_{101}$ are in AP. Such that, $a_1 = b_1$ and $a_{51} = b_{51}$. If $t = b_1 + b_2 + \ldots + b_{51}$ and $s = a_1 + a_2 + \ldots + a_{51}$, then **[JEE Advanced 2016, 3M]**

(a) $s > t$ and $a_{101} > b_{101}$ (b) $s > t$ and $a_{101} < b_{101}$
(c) $s < t$ and $a_{101} > b_{101}$ (d) $s < t$ and $a_{101} < b_{101}$

148. For any three positive real numbers a, b and c, $9(25a^2 + b^2) + 25(c^2 - 3ac) = 15b(3a + c)$. Then **[JEE Main 2017, 4M]**

(a) a, b and c are in GP
(b) b, c and a are in GP
(c) b, c and a are in AP
(d) a, b and c are in AP

Answers

Exercise for Session 1

1. (c) **2.** (d) **3.** (b) **4.** (c) **5.** (a)

Exercise for Session 2

1. (b) **2.** (a) **3.** (a) **4.** (b) **5.** (c) **6.** (c)

Exercise for Session 3

1. (b) **2.** (d) **3.** (b) **4.** (c) **5.** (d)

Exercise for Session 4

1. (c) **2.** (c) **3.** (c) **4.** (d) **5.** (a) **6.** (a)

Exercise for Session 5

1. (c) **2.** (a) **3.** (a) **4.** (c) **5.** (b) **6.** (b)
7. (b) **8.** (b) **9.** (a) **10.** (b)

Exercise for Session 6

1. (b) **2.** (d) **3.** (b) **4.** (c) **5.** (a) **6.** (a)
7. (a) **8.** (c) **9.** (b) **10.** (c)

Exercise for Session 7

1. (a) **2.** (d) **3.** (b) **4.** (d) **5.** (c) **6.** (c)
7. (a) **8.** (a)

Exercise for Session 8

1. (c) **2.** (c) **3.** (b) **4.** (a) **5.** (c) **6.** (b)
7. (c) **8.** (d) **9.** (a) **10.** (a)

Exercise for Session 9

1. (d) **2.** (c) **3.** (d) **4.** (a) **5.** (b) **6.** (a)
7. (c)

Chapter Exercises

1. (a) **2.** (d) **3.** (c) **4.** (c) **5.** (a) **6.** (c)
7. (b) **8.** (d) **9.** (a) **10.** (c) **11.** (a) **12.** (b)
13. (d) **14.** (a) **15.** (a) **16.** (c) **17.** (a) **18.** (b)
19. (a) **20.** (d) **21.** (c) **22.** (c) **23.** (c) **24.** (b)
25. (a) **26.** (a) **27.** (a) **28.** (c) **29.** (b) **30.** (c)
31. (a,c) **32.** (b,d) **33.** (b,c) **34.** (a,b) **35.** (c,d) **36.** (a,c,d)
37. (a,d) **38.** (a,b,c) **39.** (a,c) **40.** (b,c) **41.** (a,c) **42.** (c,d)
43. (b,c) **44.** (a,c) **45.** (a,d)
46. (c) **47.** (d) **48.** (a) **49.** (a) **50.** (a) **51.** (b)
52. (a) **53.** (c) **54.** (b) **55.** (d) **56.** (b) **57.** (c)
58. (c) **59.** (b) **60.** (d) **61.** (d) **62.** (b) **63.** (d)
64. (a) **65.** (b) **66.** (b) **67.** (a) **68.** (b) **69.** (c)
70. (3) **71.** (5) **72.** (8) **73.** (4) **74.** (2) **75.** (7)
76. (3) **77.** (9) **78.** (1) **79.** (0)
80. (A) $\to$ (r, s); (B) $\to$ (p, r); (C) $\to$ (p, q)
81. (A) $\to$ (r); (B) $\to$ (p); (C) $\to$ (q)
82. (A) $\to$ (p,r,s,t); (B) $\to$ (p,q,s,t); (C) $\to$ (p,s,t)
83. (A) $\to$ (r); (B) $\to$ (p); (C) $\to$ (r); (D) $\to$ (q)
84. (a) **85.** (a) **86.** (b) **87.** (a) **88.** (d) **89.** (c)
90. (a)

91. 2.

92. $\dfrac{1}{(a-b)}\left\{\dfrac{a^2(1-a^n)}{(1-a)} - \dfrac{b^2(1-b^n)}{(1-b)}\right\}$

95. -5050 **96.** (i) $\dfrac{\pi^2}{8}$ (ii) $\dfrac{\pi^2}{12}$ **98.** 93

99. $2^{n+1} - n - 2$ **101.** 5113

102. $\dfrac{x^3(1-x^{3n})}{1-x^3} + \dfrac{(1-x^{3n})}{x^{3n}(1-x^3)} + \dfrac{3x(1-x^n)}{(1-x)} + \dfrac{3(1-x^n)}{x^n(1-x)}$

106. 1540 **109.** $\dfrac{1}{a-1}$

110. $\dfrac{\pi}{4}$ **111.** $\dfrac{\dfrac{\sin n\beta}{\cos(\alpha + n\beta)\cos\alpha} - n\tan\beta}{\tan\beta}$

112. $\dfrac{n(n+3)}{2(n+1)(n+2)}$ **114.** 12, 18, 27 **115.** (c)

116. (7) **117.** (d) **118.** (d) **119.** (i) (b), (ii) (d), (iii) (b)
120. (i) (c), (ii) (a), (iii) (b) **121.** (d) **122.** (c) **123.** (a) **124.** (c)
125. (c) **126.** (3) **127.** (0) **128.** (a) **129.** (8) **130.** (c)
131. (c) **132.** (d) **133.** (a) **134.** (c) **135.** (d) **136.** (b)
137. (a,d) **138.** (5) **139.** (a) **140.** (b) **141.** (4) **142.** (c)
143. (c) **144.** (a) **145.** (d) **146.** (d) **147.** (b) **148.** (c)

Solutions

1. $\because$ x, y, z are in HP.

$\therefore$ $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in AP.

$\therefore$ $$\frac{1}{x} - \frac{1}{y} = \frac{1}{y} - \frac{1}{z} \quad \text{...(i)}$$

$$\frac{\sqrt{yz}}{\sqrt{y} + \sqrt{z}} = \frac{1}{\frac{1}{\sqrt{y}} + \frac{1}{\sqrt{z}}} = a \quad \text{[say]}$$

$$\frac{\sqrt{zx}}{\sqrt{x} + \sqrt{z}} = \frac{1}{\frac{1}{\sqrt{z}} + \frac{1}{\sqrt{x}}} = b \quad \text{[say]}$$

and $$\frac{\sqrt{xy}}{\sqrt{x} + \sqrt{y}} = \frac{1}{\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}}} = c \quad \text{[say]}$$

$\because$ $$\frac{a-b}{b-c} = \frac{\frac{1}{\frac{1}{\sqrt{y}} + \frac{1}{\sqrt{z}}} - \frac{1}{\frac{1}{\sqrt{z}} + \frac{1}{\sqrt{x}}}}{\frac{1}{\frac{1}{\sqrt{z}} + \frac{1}{\sqrt{x}}} - \frac{1}{\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}}}} = \frac{\frac{1}{x} - \frac{1}{y}}{\frac{1}{y} - \frac{1}{z}} = \frac{a}{a}$$

[from Eq. (i)]

Hence, $\frac{\sqrt{yz}}{\sqrt{y} + \sqrt{z}}, \frac{\sqrt{zx}}{\sqrt{z} + \sqrt{x}}, \frac{\sqrt{xy}}{\sqrt{x} + \sqrt{y}}$ are in AP.

2. $\because$ $a_1, a_2, a_3, \ldots$ are in HP.

$\Rightarrow$ $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots$ are in AP. ...(i)

$\because$ $$f_k = \sum_{r=1}^{n} a_r - a_k$$

$\Rightarrow$ $$a_k + f_k = \sum_{r=1}^{n} a_r = \lambda \quad \text{[say]}$$

$\Rightarrow$ $$a_1 + f_1 = a_2 + f_2 = a_3 + f_3 = \ldots = \lambda$$

From Eq. (i), $\frac{\lambda}{a_1}, \frac{\lambda}{a_2}, \frac{\lambda}{a_3}, \ldots$ are also in AP.

$\Rightarrow$ $\frac{a_1 + f_1}{a_1}, \frac{a_2 + f_2}{a_2}, \frac{a_3 + f_3}{a_3}, \ldots$ are also in AP.

Subtracting from each term by 1, we get

$\frac{f_1}{a_1}, \frac{f_2}{a_2}, \frac{f_3}{a_3}, \ldots$ are also in AP.

$\Rightarrow$ $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \frac{1}{\alpha_3}, \ldots$ are in AP.

$\therefore$ $\alpha_1, \alpha_2, \alpha_3, \ldots$ are in HP.

$\therefore$ $2^{\alpha_1}, 2^{\alpha_2}, 2^{\alpha_3}, \ldots$ are not in AP/GP/HP.

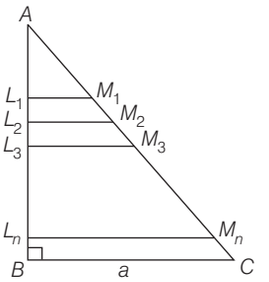
3. $\because$ $$\frac{AL_1}{AB} = \frac{L_1M_1}{BC} \Rightarrow L_1M_1 = \frac{a}{n+1}$$

Similarly, $L_2M_2 = \frac{2a}{n+1}$

$L_3M_3 = \frac{3a}{n+1}$

$\vdots \quad \vdots \quad \vdots$

$L_nM_n = \frac{na}{n+1}$



$$L_1M_1 + L_2M_2 + \ldots + L_nM_n$$
$$= \frac{a}{(n+1)}(1 + 2 + 3 + \ldots + n)$$
$$= \frac{a}{(n+1)} \cdot \frac{n(n+1)}{2} = \frac{na}{2}$$

4. $\because$ $S_n = 1 + 22 + 333 + \ldots + \underbrace{nnnn\ldots n}_{n \text{ terms}}$

$\therefore$ $$S_n - S_{n-1} = \underbrace{nnnn\ldots n}_{n \text{ times}} = \underbrace{n(111\ldots1)}_{n \text{ times}}$$

$$= n(10^{n-1} + 10^{n-2} + \ldots + 10 + 1) = \frac{n(10^n - 1)}{10 - 1}$$

$\therefore$ $9(S_n - S_{n-1}) = n(10^n - 1)$

5. Given that a, b, c are in GP.

Then, $b^2 = ac$ ...(i)

and equations $ax^2 + 2bx + c = 0$

and $dx^2 + 2ex + f = 0$ have a common root. ...(A)

Now, $ax^2 + 2bx + c = 0$

$\Rightarrow$ $ax^2 + 2\sqrt{ac}\,x + c = 0$ [by Eq. (i)]

$\Rightarrow$ $(\sqrt{a}x + \sqrt{c})^2 = 0 \Rightarrow \sqrt{a}x + \sqrt{c} = 0$

$\Rightarrow$ $x = -\frac{\sqrt{c}}{\sqrt{a}}$ [repeated]

By the condition (A), $\left(-\frac{\sqrt{c}}{\sqrt{a}}\right)$ be the root of $dx^2 - 2ex + f = 0$

So, it satisfy the equation

$$d\left(-\sqrt{\frac{c}{a}}\right)^2 + 2e\left(-\sqrt{\frac{c}{a}}\right) + f = 0$$

$\Rightarrow$ $\frac{dc}{a} - 2e\frac{\sqrt{c}}{\sqrt{a}} + f = 0 \Rightarrow \frac{d}{a} - \frac{2e}{\sqrt{ac}} + \frac{f}{c} = 0$

$\Rightarrow$ $\frac{d}{a} - \frac{2e}{b} + \frac{f}{c} = 0 \Rightarrow \frac{d}{a} + \frac{f}{c} = 2\left(\frac{e}{b}\right)$

So, $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in AP.

6. $\because$ $S_n = \frac{1}{2} + \frac{3}{4} + \frac{7}{8} + \frac{15}{16} + \ldots n$ up to terms

$$= \left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{4}\right) + \left(1 - \frac{1}{8}\right) + \ldots n \text{ up to terms}$$
$$= \left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{2^2}\right) + \left(1 - \frac{1}{2^3}\right) + \ldots + \left(1 - \frac{1}{2^n}\right)$$

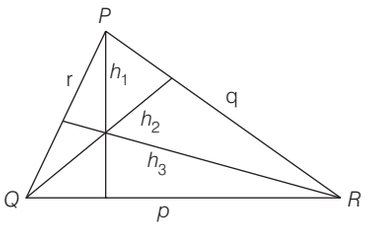
$$= n - \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}}\right)$$

$$= n - \frac{1}{2} + \frac{1\left[1 - \left(\frac{1}{2}\right)^n\right]}{\left(1 - \frac{1}{2}\right)}$$

$$\left[\text{by sum GP}, S_n = \frac{a(1-r^n)}{1-r}, \text{if } 0 < r < 1\right]$$

$$= n - 1 + \frac{1}{2^n} = n - 1 + 2^{-n}$$

7. Let triangle be the area of ΔPQR.



$\therefore \quad \Delta = \frac{1}{2} \times p \times h_1$ [h_1, h_2, h_3 are altitudes]

$\Rightarrow \quad h_1 = \frac{2\Delta}{p}$ ...(i)

Similarly, $h_2 = \frac{2\Delta}{q}$ ...(ii)

and $h_3 = \frac{2\Delta}{r}$ ...(iii)

According to the question, sin P, sin Q, sin R are in AP.

Then, kp, kq, kr are in AP [by sine rule]

$\Rightarrow$ p, q, r are in AP.

$\Rightarrow \quad \frac{2\Delta}{h_1}, \frac{2\Delta}{h_2}, \frac{2\Delta}{h_3}$ are in AP. [by Eqs. (i), (ii) and (iii)]

$\Rightarrow \quad \frac{1}{h_1}, \frac{1}{h_2}, \frac{1}{h_3}$ are in AP.

$\Rightarrow$ h_1, h_2, h_3 are in HP.

$\Rightarrow$ Altitudes are in HP.

8. Given that, $a_1, a_2, \ldots, a_{10}$ be in AP.

Let d be the common difference of AP.

$\therefore \quad d = \frac{a_{10} - a_1}{10 - 1}$

$d = \frac{3-2}{9}$ [given that, $a_1 = h_1 = 2$ and $a_{10} = h_{10} = 3$]

$d = \frac{1}{9}$

$\therefore \quad a_4 = a_1 + 3d = 2 + \frac{3}{9} = 2 + \frac{1}{3} = \frac{7}{3}$

Now, $h_1, h_2, \ldots, h_{10}$ be in HP.

So, common difference of respective AP.

$$D = \frac{\frac{1}{h_{10}} - \frac{1}{h_1}}{10 - 1} = \frac{\frac{1}{3} - \frac{1}{2}}{9} = \frac{-1}{9 \times 6} - \frac{-1}{54}$$

So, $\frac{1}{h_7} = \frac{1}{h_1} + 6D \Rightarrow \frac{1}{\lambda_7} = \frac{1}{2} + 6\left(\frac{-1}{54}\right) = \frac{1}{2} - \frac{1}{9}$

$$\frac{1}{h_7} = \frac{7}{18} \Rightarrow h_7 = \frac{18}{7}$$

So, $a_4 h_7 = \frac{7}{3} \times \frac{18}{7} = 6$

9. $\because \quad I_n = \int_0^{\pi} \frac{1 - \sin 2nx}{1 - \cos 2x}\, dx \Rightarrow I_n = \int_0^{\pi} \frac{1 - \sin 2nx}{2 \sin^2 x}\, dx$

$\Rightarrow \quad I_{n+1} + I_{n-1} - 2I_n$

$$= \frac{1}{2}\int_0^{\pi} \frac{[1 - \sin 2(n+1)x + 1 - \sin 2(n-1)x - 2 + 2\sin 2nx]}{\sin^2 x}\, dx$$

$$= \frac{1}{2}\int_0^{\pi} \frac{[\sin 2nx - \sin 2(n+1)x] + [\sin 2nx - \sin 2(n-1)x]}{\sin^2 x}\, dx$$

$$= \frac{1}{2}\int_0^{\pi} \frac{-2\cos(2n+1)x \sin(x) + 2\cos(2n-1)x \sin x}{\sin^2 x}\, dx$$

$$= \int_0^{\pi} \frac{\sin x[\cos(2n-1)x - \cos(2n+1)x]}{\sin^2 x}\, dx$$

$$= \int_0^{\pi} \frac{2\sin 2nx \sin x}{\sin x}\, dx = 2\int_0^{\pi} \sin 2nx\, dx = \frac{2}{2n}[-\cos 2nx]_0^{\pi}$$

$$= -\frac{1}{n}(1 - 1) = 0$$

$\therefore \quad I_{n+1} + I_{n-1} = 2I_n$

$\Rightarrow$ $I_{n-1} + I_n, I_{n+1}$ are in AP.

$\therefore$ $I_1, I_2, I_3, \ldots$ are in AP.

10. Given that,

$a(b-c)x^2 + b(c-a)xy + c(a-b)y^2$ is perfect square.

$\therefore \quad b^2(c-a)^2 = 4a(b-c) \cdot c(a-b)$

$\Rightarrow \quad b^2(c-a)^2 = 4ac(a-b)(b-c)$

$\Rightarrow \quad [a(b-c) + c(a-b)]^2 = 4ac(a-b)(b-c)$

$[\because a(b-c) + b(c-a) + c(a-b) = 0]$

$\Rightarrow \quad [a(b-c) - c(a-b)]^2 = 0$

$\Rightarrow \quad a(b-c) - c(a-b) = 0$

$\Rightarrow \quad ab - ac - ca + bc = 0 \Rightarrow b(a+c) = 2ac$

$\Rightarrow \quad b = \frac{2ac}{a+c}$

$\Rightarrow$ a, b, c are in HP.

11. Let $S = 1 + 2\left(1 - \frac{1}{n}\right) + 3\left(1 - \frac{1}{n}\right)^2 + \ldots + \infty$

$$\left(1 - \frac{1}{n}\right)S = \left(1 - \frac{1}{n}\right) + 2\left(1 - \frac{1}{n}\right) + \ldots + \infty$$

$$- \quad - \quad -$$

$$S\left(1 - 1 + \frac{1}{n}\right) = 1 + \left(1 - \frac{1}{n}\right) + \left(1 - \frac{1}{n}\right)^2 + \ldots + \infty$$

$$\Rightarrow \quad \frac{S}{n} = \frac{1}{1-\left(1-\frac{1}{n}\right)} \qquad \left[S_\infty = \frac{a}{1-r} \text{ by GP}\right]$$

$$\Rightarrow \quad S = \frac{n}{\frac{1}{n}}$$

$$\Rightarrow \quad S = n^2$$

12. $\because \log_3 2, \log_3 (2^x - 5)$ and $\log_3\left(2^x - \frac{7}{2}\right)$ are in AP. ...(i)

For defined, $2^x - 5 > 0$ and $2^x - \frac{7}{2} > 0$

$\therefore \quad 2^x > 5$ ...(ii)

From Eq. (i), $2, 2^x - 5, 2^x - \frac{7}{2}$ are in GP.

$$\therefore \quad (2^x - 5)^2 = 2 \cdot \left(2^x - \frac{7}{2}\right)$$

$$\Rightarrow \quad 2^{2x} - 12 \cdot 2^x + 32 = 0$$

$$\Rightarrow \quad (2^x - 8)(2^x - 4) = 0$$

$$\therefore \quad 2^x = 8, 4$$

$$\Rightarrow \quad 2^x = 8 = 2^3, 2^x \neq 4 \qquad [\text{fromEq. (ii)}]$$

$$\therefore \quad x = 3$$

13. $\because a, b, c$ are positive prime numbers.

Let $\sqrt{a}, \sqrt{b}, \sqrt{c}$ are 3 terms of AP. [not necessarily consecutive]

Then, $\sqrt{a} = A + (p-1)D$ ...(i)

$\sqrt{b} = A + (q-1)D$ ...(ii)

$\sqrt{c} = A + (r-1)D$ ...(iii)

[A and D be the first term and common difference of AP]

$\sqrt{a} - \sqrt{b} = (p-q)D$ ...(iv)

$\sqrt{b} - \sqrt{c} = (q-r)D$ ...(v)

$\sqrt{c} - \sqrt{a} = (r-p)D$ ...(vi)

On dividing Eq. (iv) by Eq. (v), we get

$$\frac{\sqrt{a} - \sqrt{b}}{\sqrt{b} - \sqrt{c}} = \frac{p-q}{q-r} \quad \text{...(vii)}$$

Since, p, q, r are natural numbers and a, b, c are positive prime numbers, so

Eq. (vii) does not hold.

So, $\sqrt{a}, \sqrt{b}$ and $\sqrt{c}$ cannot be the 3 terms of AP.

[not necessarily consecutive]

Similarly, we can show that $\sqrt{a}, \sqrt{b}, \sqrt{c}$ cannot be any 3 terms of GP and HP. [not necessarily, consecutive]

14. Given that n is an odd integer greater than or equal to 1.

$$S_n = n^3 - (n-1)^3 + (n-2)^3 - \ldots + (-1)^{n-1} 1^3$$

$$= 1^3 - 2^3 + \ldots + (n-2)^3 - (n-1)^3 + n^3$$

[$\because n$ is odd integer, so $(n-1)$ is even integer]

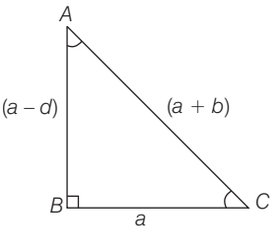
$$= (1^3 + 2^3 + \ldots + n^3) - 2 \cdot 2^3 \left(1^3 + 2^3 + \ldots + \frac{n-1}{2} \text{terms}\right)$$

$$= \left[\frac{n(n+1)}{2}\right]^2 - 16 \cdot \left[\frac{\frac{n-1}{2}\left(\frac{n-1}{2}+1\right)}{2}\right]^2$$

$$= \frac{n^2(n+1)^2}{4} - \frac{4(n-1)^2(n+1)^2}{16} = \frac{(n+1)^2}{4}[n^2 - (n-1)^2]$$

$$= \frac{(n-1)^2}{4} \cdot (2n-1)(1) = \frac{(2n-1)(n+1)^2}{4}$$

15. Let the sides of right angled triangle be $(a-d), a, (a+d)$ $(a > d)$.



By Pythagoras theorem,

$$(a+d)^2 = a^2 + (a-d)^2$$

$$a^2 + d^2 + 2ad = a^2 + a^2 + d^2 - 2ad$$

$$a^2 = 4ad$$

$$a = 4d \qquad [\text{since } a \neq 0] \quad \text{...(i)}$$

According to the question, $\sin A = \frac{a}{a+d} = \frac{4d}{5d} = \frac{4}{5}$

$$\sin C = \frac{a-d}{a+d} = \frac{3d}{5d} = \frac{3}{5}$$

16. $T_6 = 2$

Let d be common difference of AP and a be the first term of AP.

$$T_6 = 2$$

$$\Rightarrow \quad a + 5d = 2 \quad \text{...(i)}$$

Let $A = a_1 a_4 a_5$

$$A = a(a+3d)(a+4d)$$

[using $T_n = a + (n-1)d$ and from Eq. (i) $a = 2 - 5d$]

$$A = (2-5d)(2-2d)(2-d)$$

$$A = 8 - 32d + 34d^2 - 10d^3$$

For max and min values of A, $\frac{dA}{dd} = 0$

$$-30d^2 + 68d - 32 = 0 \Rightarrow 15d^2 - 34d + 16 = 0$$

$$15d^2 - (24d + 10d) + 16 = 0$$

$$15d^2 - 24d - 10d + 16 = 0$$

$$3d(5d-8) - 2(5d-8) = 0$$

$$(5d-8)(3d-2) = 0$$

$$d = \frac{8}{5} \text{ or } d = \frac{2}{3}$$

For $d = \frac{2}{3}$, $\frac{d^2A}{dd^2} > 0$

So, A is least for $d = \frac{2}{3}$.

17. Given, common difference $\neq 0$

$$S_{3n} = S_{4n} - S_{3n}$$

$\Rightarrow$ $2 \cdot S_{3n} = S_{4n}$ [let $S_n = Pn^2 + Qn$]

$\Rightarrow$ $2 \cdot [P(3n)^2 + Q(3n)] = P(4n)^2 + Q(4n)$

$\Rightarrow$ $2Pn^2 + 2Qn = 0$

or $Q = -nP$ ...(i)

$$\therefore \quad \frac{S_{2n}}{S_{4n} - S_{2n}} = \frac{P(2n)^2 + Q(2n)}{[P(4n)^2 + Q(4n)] - [P(2n)^2 + Q(2n)]}$$

$$= \frac{2n(2nP + Q)}{12\,Pn^2 + 2nQ} = \frac{2nP + Q}{6nP + Q}$$

$$= \frac{2nP - nP}{6nP - nP} = \frac{1}{5} \qquad \text{[from Eq. (i)]}$$

18. Let $f(x) = (x-1)(x-2)(x-3)\ldots(x-n)$

$$= x^n - S_1 x^{n-1} + S_2 x^{n-2} - \ldots + (-1)^n (1 \cdot 2 \cdot 3 \ldots n)$$

So, coefficient of x^{n-2} in $f(x) = S_2 = (1 \cdot 2 + 1 \cdot 3 + \ldots)$

= Sum of product of first n natural number taken 2 at time

$$= \frac{1}{2}[(1 + 2 + \ldots + n)^2 - (1^2 + 2^2 + \ldots + n^2)]$$

$$= \frac{1}{2}\left[\left\{\frac{n(n+1)}{2}\right\}^2 - \frac{n(n+1)(2n+1)}{6}\right]$$

$$= \frac{1}{2} \cdot \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} - \frac{2n+1}{3}\right]$$

$$= \frac{1}{2} \cdot \frac{n(n+1)}{2}\left[\frac{3n^2 + 3n - 4n - 2}{6}\right]$$

$$= \frac{n(n+1)(3n^2 - n - 2)}{24} = \frac{n(n+1)(3n+2)(n-1)}{24}$$

$$= \frac{n(n^2 - 1)(3n + 2)}{24}$$

19. If last term of nth row is T_n, then

Let $S = 1 + 5 + 11 + 19 + \ldots + T_n$

$S = \quad 1 + 5 + 11 + \ldots + T_{n-1} + T_n$

$- \quad - \quad - \quad - \quad - \quad -$

$0 = 1 + 4 + 6 + 8 + \ldots + n \text{ terms} - T_n$

$$T_n = 1 + 2(2 + 3 + 4 + \ldots + (n-1) \text{ terms})$$

$$= 1 + 2 \cdot \frac{(n-1)}{2}[2 \cdot 2 + (n-2) \cdot 1]$$

$$= 1 + (n-1)(n+2)$$

$$= 1 + n^2 + n - 2$$

$\Rightarrow$ $T_n = n^2 + n - 1$

$\therefore$ $T_{60} = (60)^2 + 60 - 1 = 3600 + 59 = 3659$

20. Given that, $\sum_{r=1}^{100} a_{2r} = \alpha$

$\Rightarrow$ $a_2 + a_4 + \ldots + a_{200} = \alpha$ ...(i)

and $\sum_{r=1}^{100} a_{2r-1} = \beta$

$\Rightarrow$ $a_1 + a_3 + \ldots + a_{199} = \beta$

On subtracting Eq. (ii) from Eq. (i), we get

$(a_2 - a_1) + (a_4 - a_3) + \ldots + (a_{200} - a_{199}) = \alpha - \beta$

$d + d + \ldots$ up to 100 terms $= \alpha - \beta$

[beacause a_n be the nth term of AP with common difference d]

$$100\,d = \alpha - \beta$$

$$d = \frac{\alpha - \beta}{100}$$

21. Given that, a_1, a_2, a_3, a_4, a_5 are in HP.

$\therefore$ $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \frac{1}{a_4}, \frac{1}{a_5}$ are in AP.

$\Rightarrow$ $\frac{1}{a_2} - \frac{1}{a_1} = \frac{1}{a_3} - \frac{1}{a_2} = \frac{1}{a_4} - \frac{1}{a_3} = \frac{1}{a_5} - \frac{1}{a_4} = d$ [say]

$\therefore$ $a_1 - a_2 = a_1 a_2 d \Rightarrow a_2 - a_3 = a_2 a_3 d$

$a_3 - a_4 = a_3 a_4 d \Rightarrow a_4 - a_5 = a_4 a_5 d$

On adding all, we get

$$a_1a_2 + a_2a_3 + a_3a_4 + a_4a_5 = \frac{a_1 - a_5}{d} = a_1a_5\left(\frac{\frac{1}{a_5} - \frac{1}{a_1}}{d}\right) = 4\,a_1a_5$$

22. $\because (1+a)(1+b)(1+c)(1+d)$

$= 1 + a + b + c + d + ab + ac + ad + bc + bd + cd$

$+ abc + abd + cda + cdb + abcd$ [16 terms]

$\therefore$ AM $\geq$ GM

$$\frac{(1+a)(1+b)(1+c)(1+d)}{16} \geq (a^8b^8c^8d^8)^{1/16}$$

$= (abcd)^{1/2} = (1)^{1/2} = 1$ [$\because abcd = 1$]

$\Rightarrow$ $\frac{(1+a)(1+b)(1+c)(1+d)}{16} \geq 1$

$\Rightarrow$ $(1+a)(1+b)(1+c)(1+d) \geq 16$

$\therefore$ Minimum value of $(1+a)(1+b)(1+c)(1+d)$ is 16.

23. $\because a, b, c$ are in AP.

$\therefore$ $2b = a + c$ ...(i)

Now, $(a + 2b - c)(2b + c - a)(c + a - b)$

$= (a + a + c - c)(a + c + c - a)(2b - b)$ [from Eq. (i)]

$= (2a)(2c)(b) = 4abc$

$\therefore$ $\lambda = 4$

24. $a_1, a_2, \ldots, a_n$ are in GP with first term a and common ratio r.

$$S_n = \underbrace{\frac{a_1a_2}{a_1^2 - a_2^2} + \frac{a_2a_3}{a_2^2 - a_3^2} + \ldots + \frac{a_{n-1}\,a_n}{a_{n-1}^2 - a_n^2}}_{(n-1) \text{ times}} \qquad \ldots(i)$$

$$T_n = \frac{a_{n-1}\,a_n}{a_{n-1}^2 - a_n^2} = \frac{a_{n-1}\,a_n}{(a_{n-1} - a_n)(a_{n-1} + a_n)}$$

$$= \frac{1}{\left(1 - \frac{a_n}{a_{n-1}}\right)\left(1 + \frac{a_{n-1}}{a_n}\right)}$$

$$= \frac{1}{(1-r)\left(1 + \frac{1}{r}\right)} = \frac{r}{(r+1)(1-r)} \qquad \text{[by GP]}$$

$\therefore$ $S_n = \sum_{n=2}^{n} T_n = \sum_{n=2}^{n} \frac{r}{(1-r^2)} = \frac{(n-1)\,r}{(1-r^2)}$

25. According to the question, for AP

$$S_{10} = 4\,S_5$$

$$\frac{10}{2}(2a + 9d) = 4 \cdot \frac{5}{2}(2a + 4d) \quad \left[\text{by } S_n = \frac{n}{2}[2a + (n-1)\,d]\right]$$

$$10a + 45d = 20a + 40d$$

$$\Rightarrow \qquad 10a = 5d \;\Rightarrow\; \frac{a}{d} = \frac{1}{2}$$

26. $\because \cos(x-y), \cos x, \cos(x+y)$ are in HP.

$$\therefore \qquad \cos x = \frac{2\cos(x-y)\cos(x+y)}{\cos(x-y) + \cos(x+y)}$$

$$\Rightarrow \qquad \cos x = \frac{2(\cos^2 x - \sin^2 y)}{2\cos x \cos y}$$

$$\Rightarrow \quad \cos^2 x \cos y = \cos^2 x - \sin^2 y$$

$$\Rightarrow \quad \cos^2 x(1 - \cos y) = \sin^2 y$$

$$= (1 + \cos y)(1 - \cos y)$$

$$\Rightarrow \qquad \cos^2 x = (1 + \cos y) \qquad [\because 1 - \cos y \neq 0]$$

$$\Rightarrow \qquad \cos^2 x = 2\cos^2 \frac{y}{2}$$

$$\Rightarrow \qquad \cos^2 x \sec^2\left(\frac{y}{2}\right) = 2$$

$$\therefore \qquad \cos x \sec\left(\frac{y}{2}\right) = \pm\sqrt{2}$$

27. Let 11 AM's are $A_1, A_2, A_3, \ldots, A_{11}$.

Given, $28, A_1, A_2, A_3, \ldots, A_{11}, 10$ are in AP.

$$\therefore \qquad d = \frac{10 - 28}{12} = -\frac{3}{2}$$

$$\therefore \qquad A_i = 28 + id = 28 - \frac{3}{2}i$$

It is clear that $A_2, A_4, A_6, A_8, A_{10}$ are integral AM's.

Hence, number of integral AM's are 5.

28. $\because x, y, z$ are in GP $\qquad [x, y, z > 1]$

$\therefore \ln x, \ln y, \ln z$ are in AP

and $2x, 4x, 6x$ are also in AP. $\qquad [x > 1]$

By property,

$2x + \ln x, 4x + \ln y, 6x + \ln z$ are also in AP.

$$\therefore \quad \frac{1}{2x + \ln x}, \frac{1}{4x + \ln y}, \frac{1}{6x + \ln z} \text{ are in HP.}$$

29. Let $A = \dfrac{(a^2 + 3a + 1)(b^2 + 3b + 1)(c^2 + 3c + 1)}{abc}$

$$= \left(\frac{a^2 + 3a + 1}{a}\right)\left(\frac{b^2 + 3b + 1}{b}\right)\left(\frac{c^2 + 3c + 1}{c}\right)$$

$$= \left(a + 3 + \frac{1}{a}\right)\left(b + 3 + \frac{1}{b}\right)\left(c + 3 + \frac{1}{c}\right),$$

where $a, b, c \in R^+$.

Applying AM $\geq$ GM on a and $\dfrac{1}{a}$,

$$a + \frac{1}{a} \geq 2 \;\Rightarrow\; a + \frac{1}{b} + 3 \geq 5$$

Similarly, $\quad b + \dfrac{1}{b} \geq 2 \;\Rightarrow\; b + \dfrac{1}{b} + 3 \geq 5$

and $\qquad c + \dfrac{1}{c} \geq 2 \;\Rightarrow\; c + \dfrac{1}{c} + 3 \geq 5$

$$\therefore \quad \left(a + \frac{1}{a} + 3\right)\left(b + \frac{1}{b} + 3\right)\left(c + \frac{1}{c} + 3\right) \geq 125$$

So, $\qquad A \geq 5 \cdot 5 \cdot 5 \;\Rightarrow\; A \geq 125$

Minimum value of A is 125.

30. $a_1, a_2, \ldots$,are in AP and $q_1, q_2, \ldots$ are in GP.

$$a_1 = q_1 = 2 \text{ and } a_{10} = q_{10} = 3$$

Let d be the common diference of AP

i.e., $\qquad d = \dfrac{3-2}{9} = \dfrac{1}{9}$

Then, $\qquad a_7 = a_1 + 6d = 2 + 6d = 2 + 6 \times \dfrac{1}{9} = \dfrac{8}{3}$

$$a_{19} = a_1 + 18d = 2 + 18d$$

$$= 2 + 18 \times \frac{1}{9} = \frac{36}{9} = 4$$

Let r be the common ratio of GP i.e., $r = \left(\dfrac{3}{2}\right)^{1/9}$

Then, $\qquad q_7 = q_1 r^6 = 2r^6$

$$= 2 \cdot \left(\frac{3}{2}\right)^{6 \times \frac{1}{9}} = 2\left(\frac{3}{2}\right)^{2/3}$$

$$q_{10} = q_1 r^9 = 2r^9 = 2 \cdot \left(\frac{3}{2}\right)^{9 \times \frac{1}{9}} = 3$$

$$q_{19} = q_1 \cdot r^{18} = 2 \cdot r^{18}$$

$$= 2 \cdot \left(\frac{3}{2}\right)^{18 \times \frac{1}{9}} = 2\left(\frac{3}{2}\right)^{18/9} = \frac{9}{2}$$

(a) $a_7 q_{19} = \dfrac{8}{3} \times \dfrac{9}{2} = 12$, which is an integer.

(b) $a_{19} q_7 = 4 \times 2 \times \left(\dfrac{3}{2}\right)^{2/3} = 8\left(\dfrac{3}{2}\right)^{2/3}$,

which is not an integer.

(c) $a_7 q_{19} = \dfrac{8}{3} \times \dfrac{9}{2} = 12$; $a_{19} q_{10} = 4 \times 3 = 12$

31. $\because a(n) = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \ldots + \dfrac{1}{2^n - 1}$

$$= 1 + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right)$$

$$+ \left(\frac{1}{8} + \ldots + \frac{1}{15}\right) + \ldots + \frac{1}{2^n - 1}$$

$$= 1 + \left(\frac{1}{2} + \frac{1}{2^2 - 1}\right) + \left(\frac{1}{2^2} + \frac{1}{5} + \frac{1}{6} + \frac{1}{2^3 - 1}\right) + \ldots + \frac{1}{2^n - 1}$$

$$\therefore \qquad a(n) < 1 + 1 + \ldots + n \text{ terms}$$

$$\Rightarrow \qquad a(n) < n$$

$$\Rightarrow \qquad a(100) < 100$$

Also, $a(n) = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \ldots + \frac{1}{2^n - 1}$

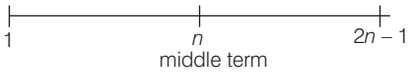
$$= 1 + \frac{1}{2} + \left(\frac{1}{2^1 + 1} + \frac{1}{2^2}\right) + \left(\frac{1}{2^2 + 1} + \frac{1}{6} + \frac{1}{7} + \frac{1}{2^3}\right) + \ldots + \left(\frac{1}{2^{n-1} + 1} + \ldots + \frac{1}{2^n}\right) - \frac{1}{2^n}$$

$$a(n) > 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \ldots + \frac{2^{n-1}}{2^n} - \frac{1}{2^n}$$

$$a(n) > \left(1 - \frac{1}{2^n}\right) + \frac{n}{2} \Rightarrow a(n) > \frac{n}{2}$$

$\therefore\ a(200) > 100$

32.



In a AP of $(2n-1)$ terms, nth term $= a$

In a GP of $(2n-1)$ terms, nth term $= b$

In a HP of $(2n-1)$ terms, nth term $= c$

a, b, c will be arithmetic mean, geometric mean, harmonic mean, respectively.

So, $a \geq b \geq c$ and $b^2 = ac$

33. $\because\ 0 < \phi < \frac{\pi}{2}$

$\therefore\quad 0 < \sin\phi < 1$ and $0 < \cos\phi < 1$

$\therefore\quad x = \sum_{n=0}^{\infty} \cos^{2n}\phi = 1 + \cos^2\phi + \cos^4\phi + \ldots + \infty$

$$= \frac{1}{1 - \cos^2\phi} = \frac{1}{\sin^2\phi}$$

or $\quad \sin^2\phi = \frac{1}{x}$ ...(i)

and $\quad y = \sum_{n=0}^{\infty} \sin^{2n}\phi = 1 + \sin^2\phi + \sin^4\phi + \ldots + \infty$

$$= \frac{1}{1 - \sin^2\phi} = \frac{1}{\cos^2\phi}$$

or $\quad \cos^2\phi = \frac{1}{y}$ ...(ii)

From Eqs. (i) and (ii),

$$\sin^2\phi + \cos^2\phi = \frac{1}{x} + \frac{1}{y}$$

$$1 = \frac{1}{x} + \frac{1}{y}$$

$\therefore\quad xy = x + y$ ...(iii)

and $\quad z = \sum_{n=0}^{\infty} \cos^{2n}\phi \sin^{2n}\phi$

$$= 1 + \cos^2\phi \sin^2\phi + \cos^4\phi \sin^4\phi + \ldots$$

$$= \frac{1}{1 - \sin^2\phi \cos^2\phi} = \frac{1}{1 - \frac{1}{xy}} \text{ [from Eqs. (i) and (ii)]}$$

$\Rightarrow\quad z = \frac{xy}{xy - 1}$

$\Rightarrow\quad xyz = z + xy$

and $\quad xyz = z + x + y$ [from Eq. (iii)]

34. $\because\ a, b, c$ are in AP $\Rightarrow b = \frac{a + c}{2}$ ...(i)

and a^2, b^2, c^2 are in HP.

$\Rightarrow\quad b^2 = \frac{2a^2c^2}{a^2 + c^2}$ ...(ii)

$\Rightarrow\quad b^2\{a^2 + c^2\} = 2a^2c^2$

$\Rightarrow\quad b^2\{(a + c)^2 - 2ac\} = 2a^2c^2$ [from Eq. (i)]

$\Rightarrow\quad b^2(4b^2 - 2ac) = 2a^2c^2$

$\Rightarrow\quad 2b^4 - ac(b^2) - a^2c^2 = 0$

$\Rightarrow\quad (b^2 - ac)(2b^2 + ac) = 0$

If $\quad b^2 - ac = 0$

a, b, c are in GP.

But given a, b, c are in AP.

$\therefore\quad a = b = c$

and if $\quad 2b^2 + ac = 0$

then $\frac{-a}{2}, b, c$ are in GP.

35. According to the question, x, $x^2 + 2$ and $x^3 + 10$ are in GP.

So, $\quad (x^2 + 2)^2 = x(x^3 + 10)$

$\Rightarrow\quad x^4 + 4 + 4x^2 - x^4 - 10x = 0$

$\Rightarrow\quad 4x^2 - 10x + 4 = 0$

$\Rightarrow\quad 2x^2 - 5x + 2 = 0$

$\Rightarrow\quad 2x^2 - 4x - x + 2 = 0$

$\Rightarrow\quad 2x(x - 2) - 1(x - 2) = 0$

$\Rightarrow\quad (x - 2)(2x - 1) = 0$

$\Rightarrow\quad x = 2$ or $x = \frac{1}{2}$

For $x = 2$, first 3 terms are 2, 6, 18.

So, 4th term of GP $= 2 \cdot (3)^3 = 54$

For $x = \frac{1}{2}$, first 3 terms are $\frac{1}{2}, \frac{9}{4}, \frac{81}{8}$.

So, $\quad T_4 = \frac{1}{2}\left(\frac{9}{2}\right)^3 = \frac{1}{2} \times \frac{729}{8} = \frac{729}{16}$

36. Let n consecutive odd numbers be

$$2k + 1, 2k + 3, 2k + 5, \ldots, 2k + 2n - 1$$

According to question, sum of these n numbers

$$= \frac{n}{2}[2k + 1 + 2k + 2n - 1] = n(2k + n)$$

$$= n^2 + 2kn = (n + k)^2 - k^2$$

Given that, $(n + k)^2 - k^2 = 25^2 - 11^2$

$\Rightarrow\quad n + k = 25$ and $k = 11 \quad \Rightarrow \quad n = 14$ and $k = 11$

So, first term $= 2k + 1 = 23$

Last term $= 2k + 2n - 1 = 22 + 28 - 1 = 22 + 27 = 49$

37. $\because G = 6$ and $G^2 = AH$

$\Rightarrow \quad H = \dfrac{36}{A}$

Given, $\quad 90A + 5H = 918$

$\Rightarrow \quad 90A + 5 \times \dfrac{36}{A} = 918 \;\Rightarrow\; 5A + \dfrac{10}{A} = 51$

$\Rightarrow \quad 5A^2 - 51A + 10 = 0 \;\Rightarrow (A - 10)(5A - 1) = 0$

$\therefore \quad A = 10, \dfrac{1}{5}$

38. $\because \quad T_n = \dfrac{1}{(2n-1)(2n+1)(2n+3)(2n+5)}$

$\therefore \quad S_n = \sum_{n=1}^{n} T_n$a

$$S_n = \frac{1}{6}\sum_{n=1}^{n} \frac{(2n+5)-(2n-1)}{(2n-1)(2n+1)(2n+3)(2n+5)}$$

$$= \frac{1}{6}\sum_{n=1}^{n}\left(\frac{1}{(2n-1)(2n+1)(2n+3)} - \frac{1}{(2n+1)(2n+3)(2n+5)}\right)$$

$$= \frac{1}{6}\left(\frac{1}{1\cdot 3\cdot 5} - \frac{1}{(2n+1)(2n+3)(2n+5)}\right)$$

$$= \frac{1}{90} - \frac{\frac{1}{6}}{(2n+1)(2n+3)(2n+5)}$$

$\therefore \quad \lambda = \dfrac{1}{6}$

and $\quad f(n) = (2n+1)(2n+3)(2n+5)$

$\therefore \quad f(0) = 15$

$f(1) = 105$

and $f(\lambda) = f\left(\dfrac{1}{6}\right) = \left(\dfrac{1}{3}+1\right)\left(\dfrac{1}{3}+3\right)\left(\dfrac{1}{3}+5\right) = \dfrac{640}{27}$

39. $\because S = 1 + \dfrac{1}{(1+3)}(1+2)^2 + \dfrac{1}{(1+3+5)}(1+2+3)^2 + \ldots$

$$T_n = \frac{1}{(1+3+5+7+\ldots n \text{ terms})}\cdot(1+2+3+4+\ldots n \text{ terms})^2$$

$$= \frac{1}{\left[\frac{n}{2}[2\cdot 1 + (n-1)\cdot 2]\right]}\cdot\left(\frac{n(n+1)}{2}\right)^2 = \frac{(n+1)^2}{4}$$

(a) $T_7 = \dfrac{(7+1)^2}{4} = \dfrac{64}{4} = 16$

(b) $S_{10} = \sum_{n=1}^{10}\left(\dfrac{n+1}{2}\right)^2 = \dfrac{1}{4}\sum_{n=1}^{10}(n^2+2n+1)$

$$= \frac{1}{4}\left(\sum_{n=1}^{10} n^2 + 2\sum_{n=1}^{10} n + \sum_{n=1}^{10} 1\right)$$

$$= \frac{1}{4}\left(\frac{10\times 11\times 21}{6} + \frac{2\times 10\times 11}{2} + 10\right)$$

$$= \frac{1}{4}(385 + 110 + 10) = \frac{505}{4}$$

40. $\quad E = \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots$

$$E < 1 + \frac{1}{(1)(2)} + \frac{1}{(2)(3)} + \ldots$$

$$E < 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots$$

$$E < 2 \qquad \ldots(i)$$

$$E > 1 + \frac{1}{(2)(3)} + \frac{1}{(3)(4)} + \ldots$$

$$E > 1 + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots$$

$$E > 1 + \frac{1}{2}; \quad E > \frac{3}{2}$$

41. $\because S_1 = \{0\}$

$$S_2 = \left\{\frac{3}{2}, \frac{5}{2}\right\}$$

$$S_3 = \left\{\frac{8}{3}, \frac{11}{3}, \frac{14}{3}\right\}$$

$$S_4 = \left\{\frac{15}{4}, \frac{19}{4}, \frac{23}{4}, \frac{27}{4}\right\}$$

$$\vdots \quad \vdots \quad \vdots$$

Let $\quad S = 3 + 8 + 15 + \ldots + T_{19}$

$$\underline{S = \quad 3 + \quad 8 + \ldots + T_{18} + T_{19}}$$

$$0 = 3 + 5 + 7 + \ldots + 19 \text{ terms} - T_{19}$$

$T_{19} = 3 + 5 + 7 + \ldots + 19$ terms

$\therefore \quad T_{19} = \dfrac{19}{2}(6 + 18\times 2) = \dfrac{19}{2}\times 42 = 399$

$$S_{20} = \left\{\frac{399}{20}, \frac{419}{20}, \frac{439}{20}, \ldots\right\}$$

$\therefore$ Third element of $S_{20} = \dfrac{439}{20}$

Sum of elements of $S_{20} = \dfrac{20}{2}\times\dfrac{1}{2}[2\times 399 + 19\times 20]$

$= 399 + 190 = 589$

42. (a) $\because S = \left(1 + \dfrac{1}{n}\right)^n$

$$S = 1 + {}^nC_1\left(\frac{1}{n}\right) + {}^nC_2\left(\frac{1}{n}\right)^2 + {}^nC_3\left(\frac{1}{n}\right)^3 + \ldots + {}^nC_n\left(\frac{1}{n}\right)^n$$

$$S = 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\left(\frac{1}{n^2}\right) + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n^3}\right) + \ldots + \frac{n(n-1)\ldots 1}{n!}\left(\frac{1}{n^n}\right)$$

$$S = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \ldots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots\left(1 - \frac{n-1}{n}\right)$$

$$S < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}$$

$$S < 1 + 1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \ldots + \frac{1}{1\cdot 2\cdot 3 \ldots n}$$

$$S < 1 + 1 + \frac{1}{2} + \frac{1}{2\cdot 2} + \ldots + \frac{1}{2\cdot 2 \ldots 2}$$

$$S < 1 + \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} \Rightarrow S < 1 + 2\left(1 - \frac{1}{2^n}\right)$$

$$S < 3 - \frac{1}{2^{n-1}} \quad \therefore S < 3, \forall\ n$$

Also, $S = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \ldots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \ldots \left(1 - \frac{n-1}{n}\right)$

$S > 1 + 1;\ S > 2$

$\therefore$ S is bounded.

(b) $\because \quad a_n = \frac{2n+1}{n+2}$

For $n_1 = 1, a_1 = \frac{3}{3} = 1,$

for $n = 2, a_2 = \frac{5}{4} = 1.25$

$\vdots \quad \vdots \quad \vdots$

Now, $a_{n+1} - a_n > 0 \Rightarrow a_{n+1} > a_n$

$\therefore$ a_n represents the increasing sequence

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n+1}{n+2} = \lim_{n\to\infty} \frac{n\left(2 + \frac{1}{n}\right)}{n\left(1 + \frac{2}{n}\right)} = \frac{2}{1} = 2$$

$\therefore$ $\{a_n\}$ is bounded sequence.

(c) $\because \quad a_n = \left(1 + \frac{1}{n}\right)^{n^2}$

For $n = 1, a_1 = 2,$

for $n = 2, a_2 = \left(1 + \frac{1}{2}\right)^4 = \left(\frac{3}{2}\right)^4 = \frac{3^4}{2^4} = \frac{81}{16} = 5.06$ [approximate]

$$\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^{n^2} = e^{\lim_{n\to\infty} \frac{1}{n} \times n^2} = e^{\lim_{n\to\infty} n} = e^{\infty} = \infty$$

$\therefore$ $\{a_n\}$ represents unbounded sequence.

(d) $\because a_n = \tan n$

$$a_n = n + \frac{n^3}{3} + \frac{2}{15}n^5 + \ldots + \infty$$

and we know that $-\infty < \tan n < \infty$

So, $\{a_n\}$ is unbounded sequence.

43. $\because \quad a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{3n}$

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{n+2n}$$

$$a_n = \sum_{\alpha=1}^{2n} \frac{1}{n+\alpha}$$

$$a_2 = \sum_{\alpha=1}^{4} \frac{1}{2+\alpha} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{20 + 15 + 12 + 10}{60}$$

$$= \frac{57}{60} = \frac{19}{20}$$

Now, $a_{n+1} - a_n = \left(\frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{3n+3}\right) - \left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{3n}\right)$

$$= \frac{1}{3n+1} + \frac{1}{3n+2} + \frac{1}{3n+3} - \frac{1}{n+1}$$

$$= \frac{1}{3n+1} + \frac{1}{3n+2} - \frac{2}{3(n+1)}$$

$$= \frac{9n^2 + 15n + 6 + 9n^2 + 12n + 3 - 18n^2 - 18n - 4}{(3n+1)(3n+2)(3n+3)}$$

$$= \frac{9n+5}{(3n+1)(3n+2)(3n+3)}$$

44. $S_n(x) = \left(x^{n-1} + \frac{1}{x^{n-1}}\right) + 2\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + \ldots + (n-1)\left(x + \frac{1}{x}\right) + n$

Let $S' = x^{n-1} + 2x^{n-2} + 3x^{n-3} + \ldots + (n-1)\,x$

$$\frac{S'}{x} = \quad x^{n-2} + 2x^{n-3} + \ldots + (n-2)\,x + (n-1)$$

$$S'\left(1 - \frac{1}{x}\right) = x^{n-1} + x^{n-2} + x^{n-3} + \ldots + x - (n-1)$$

$$S'\frac{(x-1)}{x} = \frac{x\cdot(x^{n-1} - 1)}{(x-1)} - (n-1)$$

$$\Rightarrow \quad S' = \frac{x^2}{(x-1)^2}(x^{n-1} - 1) - \frac{(n-1)\,x}{(x-1)}$$

$$S'' = \frac{1}{x^{n-1}} + \frac{2}{x^{n-2}} + \ldots + \frac{(n-1)}{x}$$

$$\Rightarrow \quad S'' = \frac{1}{x^n}\left[x + 2x^2 + \ldots + (n-1)\,x^{n-1}\right]$$

$$= \frac{1}{x^n}\frac{\left[(n-1)\,x^n - nx^{n-1} + 1\right]}{(x-1)^2}$$ [similarly as above]

$\therefore \quad S_n(x) = S' + S'' + n$

$$\Rightarrow \quad S_n(x) = \frac{1}{x^{(n-1)}}\left(\frac{x^n - 1}{x - 1}\right)^2 \qquad \ldots(i)$$

For $n = 1, S_1(x) = 1$

$$S_{100}(x) = \frac{1}{x^{99}}\left(\frac{x^{100} - 1}{x - 1}\right)^2$$

45. Let the AP start with n and common difference d, then according to question,

$$n + 5d = 32$$

$$n = 32 - 5d \qquad \ldots(i)$$

and $1072 < n + (n + d) + \ldots + (n + 19d) < 1162$

$$1072 < 20n + \frac{19 \times 20}{2} d < 1162$$

$$1072 < 640 - 100d + 190d < 1162$$

$$432 < 90d < 522$$

$$4.8 < d < 5.8$$

Let d is natural number, so $d = 5$

$\therefore$ $n = 32 - 5 \times 5 = 7$

First term is 7 and common difference is 5.

Sol. (Q. Nos. 46 to 48)

Let $S_n = \frac{8}{5} + \frac{16}{65} + \frac{24}{325} + \ldots$

$$T_r = \frac{8r}{4r^4 + 1} = \frac{8r}{(2r^2 + 2r + 1)(2r^2 - 2r + 1)}$$

$$= 2\left[\frac{(2r^2 + 2r + 1) - (2r^2 - 2r + 1)}{(2r^2 + 2r + 1)(2r^2 - 2r + 1)}\right]$$

$$= 2\left[\frac{1}{2r^2 - 2r + 1} - \frac{1}{2r^2 + 2r + 1}\right]$$

46. $\lim\limits_{n \to \infty} S_n = \lim\limits_{n \to \infty} \sum\limits_{r=1}^{n} T_r$

$$= \lim_{n \to \infty} \sum_{r=1}^{n} 2\left(\frac{1}{2r^2 - 2r + 1} - \frac{1}{2r^2 + 2r + 1}\right)$$

$$= 2 \lim_{n \to \infty}\left(1 - \frac{1}{2n^2 + 2n + 1}\right) = 2(1 - 0) = 2$$

47. $T_7 = \frac{8 \times 7}{4 \times 7^4 + 1} = \frac{56}{9605}$

48. $S_8 = \sum\limits_{r=1}^{8} T_r = 2 \sum\limits_{r=1}^{8}\left(\frac{1}{2r^2 - 2r + 1} - \frac{1}{2r^2 + 2r + 1}\right)$

$$= 2\left(1 - \frac{1}{2(8)^2 + 2(8) + 1}\right) = 2\left(1 - \frac{1}{145}\right) = \frac{288}{145}$$

Sol. (Q. Nos. 4-6)

Let p and $(p + 1)$ be removed numbers from 1, 2, 3, ... n, then Sum of the remaining numbers

$$= \frac{n(n+1)}{2} - (2p + 1)$$

From given condition,

$$\frac{105}{4} = \frac{\frac{n(n+1)}{2} - (2p + 1)}{(n - 2)}$$

$\Rightarrow$ $2n^2 - 103n - 8p + 206 = 0$

Since, n and p are integers, so n must be even.

Let $n = 2r$, we get

$$p = \frac{4r^2 + 103(1 - r)}{4}$$

Since, p is an integer, then $(1 - r)$ must be divisible by 4.

Let $r = 1 + 4t$, we get

$$n = 2 + 8t \text{ and } p = 16t^2 - 95t + 1$$

Now, $1 \le p < n$

$\Rightarrow$ $1 \le 16t^2 - 95t + 1 < 8t + 2$

$\Rightarrow$ $t = 6 \Rightarrow n = 50$ and $p = 7$

49. Hence, the value of n lies in (41,51).

50. Hence, removed numbers are 7 and 8.

51. Sum of all numbers $= \frac{50(50 + 1)}{2} = 1275$

Sol. (Q. Nos. 52 to 54)

Let $A = \{A - D, A, A + D\}$; $B = \{a - d, a, a + d\}$

According to the question,

$$A - D + A + A + D = 15$$

$\Rightarrow$ $3A = 15$

$\Rightarrow$ $A = 5$ ...(i)

and $a - d + a + a + d = 15$

$\Rightarrow$ $a = 5$ ...(ii)

and $D = 1 + d$ ...(iii)

$$p = (A - D)\, A\, (A + D)$$

$$p = A(A^2 - D^2)$$ ...(iv)

$$p = 5(25 - D^2)$$ ...(v)

Similarly, $q = 5(25 - d^2)$

Given that, $p = 7(q - p)$

$$8p = 7q$$

From Eqs. (iv) and (v), we get

$$8 \times 5(25 - D^2) = 7 \times 5(25 - d^2)$$

$$200 - 8D^2 = 175 - 7d^2$$

$$25 = 8D^2 - 7d^2$$

$$25 = 8(1 + d)^2 - 7d^2 \quad \text{[from Eq. (iii)]}$$

$$25 = 8 + 8d^2 + 16d - 7d^2$$

$$17 - d^2 - 16d = 0$$

$$d^2 + 16d - 17 = 0$$

$$(d + 17)(d - 1) = 0$$

$$d = -17 \text{ or } d = 1$$

$\Rightarrow$ $d = 1$ $[\because d > 0]$

$\Rightarrow$ $D = 2$

52. $p = 5(25 - D^2) = 5(25 - 4) = 5(21) = 105$

53. $q = 5(25 - d^2) = 5(25 - 1) = 120$

54. $7D + 8d = 14 + 8 = 22$

Sol. (Q. Nos. 55 to 57)

Let $A = \left\{\frac{A}{R}, A, AR\right\}$

$$B = \left\{\frac{a}{r}, a, ar\right\}$$

According to the question, $\frac{A}{R} \cdot A \cdot AR = 64$

$\Rightarrow$ $A^3 = 64 \Rightarrow A = 4$ ...(i)

$$\frac{a}{r} \cdot a \cdot ar = 64 \Rightarrow a^3 = 64 \Rightarrow a = 4$$ ...(ii)

and $R = r + 2$ ...(iii)

$$p = \frac{A}{R}\cdot A + A\cdot AR + AR\cdot\frac{A}{R}$$

$$= \frac{A^2}{R} + A^2R + A^2 = \frac{16}{R} + 16R + 16$$

$$q = \frac{a}{r}\cdot a + a\cdot ar + ar\cdot\frac{a}{r}$$

$$= \frac{a^2}{r} + a^2r + a^2 = \frac{16}{r} + 16r + 16$$

Given that, $\frac{p}{q} = \frac{3}{2}$

So, $$\frac{(16 + 16R^2 + 16R)\,r}{(16 + 16r^2 + 16r)\,R} = \frac{3}{2}$$

$$\frac{(1 + R^2 + R)\,r}{(1 + r^2 + r)\,R} = \frac{3}{2}$$

From Eq. (iii), $R = r + 2$

$\Rightarrow$ $$\frac{(1 + r^2 + 4 + 4r + r + 2)\,r}{(1 + r + r^2)\,(r + 2)} = \frac{3}{2}$$

$\Rightarrow$ $$\frac{r^3 + 5r^2 + 7r}{r^3 + 3r^2 + 3r + 2} = \frac{3}{2}$$

$\Rightarrow$ $r^3 - r^2 - 5r + 6 = 0$

$\Rightarrow$ $(r - 2)(r^2 + r - 3) = 0$

$\Rightarrow$ $r = 2$ or $r = \frac{-1 \pm \sqrt{13}}{2}$

So, $R = 4$

55. $p = 16\left(\frac{1}{R} + R + 1\right) = 16\left(\frac{1}{4} + 4 + 1\right) = \frac{16}{4}(21) = 84$

56. $q = 16\left(\frac{1}{r} + r + 1\right) = 16\left(\frac{1}{2} + 2 + 1\right) = \frac{16}{2}\times 7 = 8\times 7 = 56$

57. $r^R + R^r = (4)^2 + (2)^4 = 16 + 16 = 32$

Sol. (Q. Nos. 58 to 60)

Given sequence, 1, 3, 6, 10, 15, 21, 28, ...

where $t_n = t_{n-1} + n, \forall\ n \geq 2$

So, $t_n = [t_{n-2} + (n-1)] + n$

$= t_{n-3} + (n-2) + (n-1) + n$

$\vdots \quad \vdots \quad \vdots$

$t_n = t_1 + 2 + 3 + ... + (n-1) + n$

$t_n = 1 + 2 + 3 + ... + n$

$t_n = \frac{n(n+1)}{2}$ ...(i)

58. $t_{50} = \frac{50\times 51}{2} = 25\times 51 = 1275$

59. $t_{100} = \frac{100\times 101}{2} = 5050$

$t_{101} = \frac{101\times 102}{2} = 101\times 51 = 5151$

Number of positive integers lying between t_{100} and t_{101}

$= 5151 - 5050 - 1$

$= 101 - 1 = 100$

60. According to the question, $(m + 1)$ is the nth triangular number, then

$$\frac{n(n+1)}{2} = m + 1$$

$$n^2 + n - 2(m + 1) = 0$$

$$n = \frac{-1 \pm \sqrt{1 + 8(m+1)}}{2}$$

$$= \frac{-1 + \sqrt{(8m + 9)}}{2}$$

$\therefore$ $$n - m = \frac{-1 + \sqrt{8m + 9} - 2m}{2}$$

Sol. (Q. Nos. 61 to 63)

$A_1, A_2, A_3, ..., A_m$ are arithmetic means between -3 and 828.

So, $A_1 + A_2 + ... + A_m = m\frac{(a+b)}{2}$

$\Rightarrow$ $A_1 + A_2 + ... + A_m = m\left(\frac{-3 + 288}{2}\right)$

$\Rightarrow$ $14025 = m\left(\frac{825}{2}\right)$

[given that sum of AM's = 14025]

$\Rightarrow$ $m = 17\times 2$

$\therefore$ $m = 34$ ...(i)

Now, $G_1, G_2, ..., G_n$ be the GM's between 1 and 2187.

$\therefore$ $G_1G_2G_3 ... G_n = (ab)^{n/2}$

$\Rightarrow$ $3^{35} = (1\times 2187)^{n/2}$ $\Rightarrow$ $3^{35} = 3^{7n/2}$

So, $35 = \frac{7n}{2}$

$\Rightarrow$ $n = 10$ ...(ii)

61. $n = 10$ [by Eq. (ii)]

62. $m = 34$ [by Eq. (i)]

63. $G_1 + G_2 + ... + G_n = r + r^2 + r^3 + ... + r^n$

$= r + r^2 + r^3 + ... + r^{10} = r\frac{(1 - r^{10})}{1 - r}$

$\left[\because r = \left(\frac{l}{a}\right)^{1/n+1} = \left(\frac{2187}{1}\right)^{1/11} = 3^{7/11}\right]$

$= 3^{7/11}\frac{(1 - 3^{70/11})}{(1 - 3^{7/11})}$

Solution (Q. Nos. 64 to 66)

$\because$ $\alpha + \beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a}, \alpha - \beta = \frac{\sqrt{b^2 - 4ac}}{a}$

and $\gamma + \delta = -\frac{B}{A}, \gamma\delta = \frac{C}{A}, \gamma - \delta = \frac{\sqrt{B^2 - 4AC}}{A}$

64. Since, α, β, γ are in AP.

Let $\beta = \alpha + D, \gamma = \alpha + 2D$ and $\delta = \alpha + 3D$

$\because$ $\alpha + \beta = \frac{-b}{a} \Rightarrow \alpha + \alpha + D = -\frac{b}{a}$

or $2\alpha + D = -\frac{b}{a}$ ...(i)

and $\gamma+\delta=-\frac{B}{A}\Rightarrow 2\alpha+5D=-\frac{B}{A}$ ...(ii)

From Eqs. (i) and (ii), we get

$$4D=\left(-\frac{B}{A}+\frac{b}{a}\right) \text{ or } D=\frac{1}{4}\left(\frac{b}{a}-\frac{B}{A}\right)$$

65. Since, $\alpha, \beta, \gamma, \delta, \ldots$ are in GP.

$\therefore \quad \frac{\beta}{\alpha}=\frac{\gamma}{\beta}=\frac{\delta}{\gamma}$

$\Rightarrow \quad \frac{\beta}{\alpha}=\frac{\delta}{\gamma}\Rightarrow\frac{\alpha}{\gamma}=\frac{\beta}{\delta}$

$\Rightarrow \quad \frac{\alpha+\beta}{\gamma+\delta}=\sqrt{\frac{\alpha\beta}{\gamma\delta}}$

$\Rightarrow \quad \frac{-\frac{b}{a}}{-\frac{B}{A}}=\sqrt{\frac{\frac{c}{a}}{\frac{C}{A}}}\Rightarrow\frac{b^2A^2}{a^2B^2}=\frac{cA}{aC}$

$\Rightarrow \quad \frac{acA^2}{aB^2}=\frac{cA}{C} \Rightarrow B^2=AC$

Hence, A, B, C are in GP.

66. Since, $\alpha, \beta, \gamma, \delta, \ldots$ are in GP.

$\therefore \quad r=\frac{\beta}{\alpha}=\frac{\gamma}{\beta}=\frac{\delta}{\gamma}$

$\Rightarrow \quad \alpha+\beta=\alpha+\alpha r=-\frac{b}{a}$

$\Rightarrow \quad \alpha(1+r)=-\frac{b}{a}$ ...(i)

and $\gamma+\delta=\alpha r^2+\alpha r^3=-\frac{B}{A}$

$\Rightarrow \quad \alpha r^2(1+r)=-\frac{B}{A}$ ...(ii)

From Eqs. (i) and (ii), we get

$$r^2=\frac{Ba}{bA}$$

$\therefore \quad r=\sqrt{\frac{aB}{bA}}$

Sol. (Q. Nos. 67 to 69)

For $n>1$, we have $n+1>n-1$

$\Rightarrow \quad \frac{n+1}{n-1}>1 \Rightarrow p\left(\frac{n+1}{n-1}\right)^2>p \qquad [\because p>0]$ ...(i)

Now, $p=a+d$

Since, $a, p, b,$ are in AP.

And $d=\frac{b-a}{n+1}$

67. $p=a+\frac{(b-a)}{n+1}=\frac{na+b}{n+1}$

68. $\frac{1}{q}=\frac{1}{a}+D=\frac{1}{a}+\frac{\frac{1}{b}-\frac{1}{a}}{n+1}$

$\Rightarrow \quad q=\frac{ab(n+1)}{a+bn}$

69. Again, $\frac{p}{q}=\left(\frac{an+b}{n+1}\right)\times\left[\frac{a+bn}{ab(n+1)}\right]$

$$=\frac{a^2n+abn^2+b^2n+ab}{ab(n+1)^2}$$

$$=\frac{n\left(\frac{a}{b}+\frac{b}{a}\right)+(n^2+1)}{(n+1)^2}$$

$\Rightarrow \quad \frac{p}{q}-1=\frac{n\left(\frac{a}{b}+\frac{b}{a}-2\right)}{(n+1)^2}=\frac{n\left(\frac{\sqrt{a}}{\sqrt{b}}-\frac{\sqrt{b}}{\sqrt{a}}\right)^2}{(n+1)^2}$

So, $\frac{p}{q}-1>0 \Rightarrow \frac{p}{q}>1 \Rightarrow p>q$ ...(iii)

From Eqs. (i) and (ii), we get

$$q<p<\left(\frac{n+1}{n-1}\right)^2 p$$

70. a, b, c, d are positive real numbers with

$a<b<c<d$ ...(A)

According to the question, a, b, c, d are in AP.

$\Rightarrow b=a+\alpha, c=a+2\alpha$ and $d=a+3\alpha$ ...(i)

α be the common difference

and a, b, d are in GP.

$\Rightarrow \quad b^2=ad$ ...(ii)

From Eqs. (i) and (ii), we get

$$(a+\alpha)^2=a(a+3\alpha)$$

$\Rightarrow \quad a^2+\alpha^2+2a\alpha=a^2+3a\alpha$

$\Rightarrow \quad \alpha^2=a\alpha$

$\Rightarrow \quad a(\alpha-a)=0$

$\Rightarrow \quad \alpha=0 \text{ or } \alpha=a$

$a\neq 0$ by (A), so $\alpha=a$

From Eq. (i), $b=2a, c=3a$ and $d=4a$

$$\frac{ad}{bc}=\frac{a\cdot 4a}{2a\cdot 3a}=\frac{2}{3}=\left(\frac{p}{q}\right)$$

where, p and q are prime numbers.

So, $q=3$

71. $\because \sum_{r=1}^{110}(1+rx)=(1+x)(1+2x)(1+3x)\ldots(1+110x)$

$$=1^{110}+(x+2x+3x+\ldots+110x)\,1^{109}+\ldots$$

So, coefficient of x in

$$\sum_{r=1}^{110}(1+rx)=(1+2+3+\ldots+110)=\frac{110\times 111}{2}=55\times 111$$

$$=6105$$

Now, $\lambda(1+10)(1+10+10^2)=\lambda(11)(111)$

$\Rightarrow \quad \lambda(111)(11)=6105 \Rightarrow \lambda=5$

72. Let number of the form palindrome be $\alpha\beta\alpha$.

Now, If $\alpha\beta\alpha$ is even, then α may be 2, 4, 6, 8 and β take values 0, 1, 2, ..., 9 .

So, total number of palindrome (even) $=10\times 4=40$

To find the sum of all even 3 digit plaindrome
So, sum of number start with 2
$= (200 + 2) \times 10 + (0 + 1 + 2 + 3 + ... + 9) \times 10$
$= 2020 + 450 = 2470$
Sum of number start with $4 = (404) \times 10 + 450$
Similarly, sum of number start with $6 = (606) \times 10 + 450$
Similarly, sum of number start with $8 = (808) \times 10 + 450$
$\therefore$ Total sum $= (202 + 404 + 606 + 808) \times 10 + 450 \times 4$
$= 20200 + 1800 = 22000$
$= 2^4 \times 5^3 \times 11$

On comparing $2^4 \times 5^3 \times 11^1$ with
$2^{n_1} \times 3^{n_2} \times 5^{n_3} \times 7^{n_4} \times 11^{n_5}$,
$n_1 = 4, n_2 = 3, n_3 = 0, n_4 = 0, n_5 = 1$
Now, $n_1 + n_2 + n_3 + n_4 + n_5 = 8$

73. $\because 2 + (6 \cdot 2^2 - 4 \cdot 2) + (6 \cdot 3^2 - 4 \cdot 3) + ... + (6 \cdot n^2 - 4 \cdot n) = 140$

$\Rightarrow 2 + 6(2^2 + 3^2 + ... + n^2) - 4 \cdot (2 + 3 + ... + n) = 140$

$\Rightarrow 2 + 6\left(\frac{n(n+1)(2n-1)}{6} - 1\right) - 4\left(\frac{n(n+1)}{2} - 1\right) = 140$

$\Rightarrow 2 + n(n+1)(2n+1) - 6 - 2n(n+1) + 4 = 140$

$\Rightarrow n(n+1)(2n+1) - 2n(n+1) - 140 = 0$

$\Rightarrow 2n^3 + 3n^2 + n - 2n^2 - 2n - 140 = 0$

$\Rightarrow 2n^3 + n^2 - n - 140 = 0$

$\Rightarrow (n-4)(2n^2 + 9n + 35) = 0$

$\Rightarrow n = 4$ or $2n^2 + 9n + 35 = 0$

$\Rightarrow 2n^2 + 9n + 35 = 0$

$\Rightarrow n = \frac{-9 \pm \sqrt{81 - 280}}{4}$

$\therefore n = \frac{9 \pm \sqrt{-199}}{4}$ [complex values]

Only positive integer value of n is 4.

74. $S(x) = 1 + x - x^2 - x^3 + x^4 + x^5 - x^6 - x^7 + ... + \infty$
where $x \in (0, 1)$

$S(x) = (1 + x) - x^2(1 + x) + x^4(1 + x) - x^6(1 + x) + ... + \infty$

$\Rightarrow S(x) = (1 + x)[1 - x^2 + x^4 - x^6 + ... + \infty]$

$\Rightarrow S(x) = (1 + x)\left(\frac{1}{1 + x^2}\right)$ $\left[\because S_\infty = \frac{a}{1 - r} \text{ for GP}\right]$

According to the question, $S(x) = \frac{\sqrt{2} + 1}{2}$

So, $\frac{1 + x}{1 + x^2} = \frac{\sqrt{2} + 1}{2}$

$\Rightarrow 2 + 2x = (\sqrt{2} + 1)x^2 + \sqrt{2} + 1$

$\Rightarrow (\sqrt{2} + 1)x^2 - 2x - 2 + \sqrt{2} + 1 = 0$

$\Rightarrow (\sqrt{2} + 1)x^2 - 2x + \sqrt{2} - 1 = 0$

$\Rightarrow (\sqrt{2} + 1)x^2 - 2x + \frac{1}{\sqrt{2} + 1} = 0$

$\Rightarrow [(\sqrt{2} + 1)x]^2 - 2(\sqrt{2} + 1)x + 1 = 0$

$\Rightarrow [(\sqrt{2} + 1)x - 1]^2 = 0$

$\Rightarrow x = \frac{1}{\sqrt{2} + 1}$ [repeated]

So, $x = \sqrt{2} - 1$

$\therefore (x + 1)^2 = 2$

75. $a_1, a_2, a_3, ...$ are in GP with common ratio r
and $b_1, b_2, b_3, ...$ is also a GP i.e. $b_1 = 1$
$b_2 = \sqrt[4]{7} - \sqrt[4]{28} + 1, a_1 = \sqrt[4]{28}$
and $\sum_{n=1}^{\infty} \frac{1}{a_n} = \sum_{n=1}^{\infty} b_n$

$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + ... + \infty = b_1 + b_2 + b_3 + ... + \infty$

$\Rightarrow \frac{1}{\sqrt[4]{28}} + \frac{1}{\sqrt[4]{28}\, r} + \frac{1}{\sqrt[4]{28}\, r^2} + ... + \infty$
$= 1 + (\sqrt[4]{7} - \sqrt[4]{28} + 1) + (\sqrt[4]{7} - \sqrt[4]{28} + 1)^2 + ... + \infty$

$\Rightarrow \frac{\frac{1}{\sqrt[4]{28}}}{1 - \frac{1}{r}} = \frac{1}{1 - \sqrt[4]{7} + \sqrt[4]{28} - 1}$

$\Rightarrow \frac{r}{(r-1)\sqrt[4]{28}} = \frac{1}{\sqrt[4]{7}(\sqrt[4]{4} - 1)}$

$\Rightarrow \frac{r}{(r-1)} \frac{1}{\sqrt[4]{4}} = \frac{1}{(\sqrt[4]{4} - 1)}$

Let $\sqrt[4]{4} = \alpha$, we get

$\Rightarrow \frac{r}{(r-1)\alpha} = \frac{1}{\alpha - 1}$

$\Rightarrow r\alpha - r = r\alpha - \alpha \Rightarrow r = \alpha$

$\Rightarrow r = \sqrt[4]{4}$

Now, $1 + r^2 + r^4 = 1 + (\sqrt[4]{4})^2 + (\sqrt[4]{4})^4$
$= 1 + 4^{1/2} + 4 = 1 + 2 + 4 = 7$

76. Let $a = 10 + D$ ...(i)
$b = 10 + 2D$ ...(ii)
$ab = 10 + 3D$ ...(iii)

On substituting the values of a and b in Eq. (iii), we get
$(10 + D)(10 + 2D) = (10 + 3D)$

$\Rightarrow 2D^2 + 27D + 90 = 0$

$\therefore D = -6, D = -\frac{15}{2}$

$\therefore a_1 = 10 - 6 = 4,$
$a_2 = 10 - \frac{15}{2} = \frac{5}{2}$

and $b_1 = 10 - 12 = -2, b_2 = 10 - 15 = -5$

Now, $\left(\frac{2a_1a_2 + b_1b_2}{10}\right) = \left(\frac{2 \times 10 + 10}{10}\right) = 3$

77. Given equation, $Ax^3 + Bx^2 + Cx + D = 0$ ...(i)

where, $A \neq 0$

Let roots are α, β, γ, then $\beta = \dfrac{\alpha + \gamma}{2}$ ...(ii)

Given relation, $2B^3 + \lambda\, ABC + \mu\, A^2D = 0$ ...(iii)

From Eq. (i), $\alpha + \beta + \gamma = -\dfrac{\beta}{A}$

$\Rightarrow \quad 3\beta = -\dfrac{B}{A}$ [from Eq. (ii)]

$\Rightarrow \quad \beta = -\dfrac{B}{3A}$

Now, β satisfy Eq. (i), so

$$A\left(\frac{-B}{3A}\right)^3 + B\left(\frac{-B}{3A}\right)^2 + C\left(\frac{-B}{3A}\right) + D = 0$$

$$\Rightarrow \quad \frac{-B^3}{27A^2} + \frac{B^3}{9A^2} - \frac{BC}{3A} + D = 0$$

$$\Rightarrow \quad \frac{2}{27}\frac{B^3}{A^2} - \frac{BC}{3A} + D = 0$$

$$\Rightarrow \quad 2B^3 - 9\, ABC + 27\, DA^2 = 0$$

Compare with Eq. (iii), we get

$$\lambda = -9, \mu = 27$$

$$2\lambda + \mu = -18 + 27 = 9$$

78. Let $P = \lim\limits_{n \to \infty}\left(\dfrac{1}{1+x} + \dfrac{2}{1+x^2} + \dfrac{2^2}{1+x^4} + \ldots \text{ upto } n \text{ terms}\right)$

$$= \lim_{n \to \infty} \sum_{r=0}^{n}\left(\frac{2^r}{1+x^{2^r}} + \frac{2^r}{1-x^{2^r}} - \frac{2^r}{1-x^{2^r}}\right)$$

$$= \lim_{n \to \infty} \sum_{r=0}^{n}\left(\frac{2^{r+1}}{1-x^{2^{r+1}}} - \frac{2^r}{1-x^{2^r}}\right)$$

$$= \lim_{n \to \infty}\left(\frac{2^{n+1}}{1-x^{2^{n+1}}} - \frac{1}{1-x}\right)$$

$$= \lim_{n \to \infty} \frac{\dfrac{2^{n+1}}{x^{2^{n+1}}}}{\dfrac{1}{x^{2^{n+1}}} - 1} - \frac{1}{1-x} = 0 - \frac{1}{1-x}$$

$$= \frac{1}{x-1} = \frac{1}{x-\lambda} \quad \text{[given]}$$

$\therefore \quad \lambda = 1$

79. Let number of AP are $(a-d), a, (a+d)$.

According to the question, $(a-d)^2, a^2, (a+d)^2$ are in GP.

$\therefore \quad (a^2)^2 = (a-d)^2 (a+d)^2$

$\Rightarrow \quad a^4 = (a^2 - d^2)^2$

$\Rightarrow \quad a^4 = a^4 + d^4 - 2a^2d^2$

$\Rightarrow \quad a^2(a^2 - 2d^2) = 0$

$\Rightarrow \quad a \neq 0$, so $a^2 = 2d^2$

$\Rightarrow \quad a = \pm\sqrt{2}d$ ...(i)

Let common ratio of GP is r.

$\therefore \quad r^2 = \dfrac{(a+d)^2}{(a-d)^2}$

$\Rightarrow \quad r^2 = \dfrac{a^2 + d^2 + 2ad}{a^2 + d^2 - 2ad}$

$\Rightarrow \quad r^2 = \dfrac{2d^2 + d^2 + 2\sqrt{2}d^2}{2d^2 + d^2 - 2\sqrt{2}\,d^2}$ [from Eq. (i) for $a = \sqrt{2}\,d$]

$\Rightarrow \quad r^2 = \dfrac{(3+2\sqrt{2})\,d^2}{(3-2\sqrt{2})\,d^2}$

$\Rightarrow \quad r^2 = \dfrac{(3+2\sqrt{2})(3+2\sqrt{2})}{9-8}$

$\Rightarrow \quad r^2 = (3+2\sqrt{2})^2$

$\Rightarrow \quad r^2 = (3+\sqrt{8})^2$

$\therefore \quad r = \pm(3+\sqrt{8})$

$\Rightarrow \quad r = 3 + \sqrt{8}$ [$\because r$ is positive]

Similarly, for $a = -\sqrt{2}\,d$, we get

$$r = \pm(3-\sqrt{8})$$

$\Rightarrow \quad r = (3-\sqrt{8})$ [$\because r$ is positive]

Compare r with $3 \pm \sqrt{k}$, we get

$$k = 8$$

$\therefore \quad \left[\dfrac{k}{8} - \dfrac{8}{k}\right] = \left[\dfrac{8}{8} - \dfrac{8}{8}\right]$

$$= [1-1] = [0] = 0$$

80. **(A)** a, b, c, d are in AP [a, b, c, d are positive real numbers]

By AM > GM, for a, b, c

$$b > \sqrt{ac}$$

$\Rightarrow \quad b^2 > ac$ ...(i)

Now, applying for b, c, d

$$c > \sqrt{bd} \;\Rightarrow\; c^2 > bd \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$b^2c^2 > (ac)(bd) \;\Rightarrow\; bc > ad$$

Again, applying AM > HM for a, b, c

$$b > \frac{2}{\frac{1}{a} + \frac{1}{c}} \Rightarrow \frac{1}{a} + \frac{1}{c} > \frac{2}{b} \quad \text{...(iii)}$$

For last 3 terms b, c, d

$$c > \frac{2}{\frac{1}{b} + \frac{1}{d}} \Rightarrow \frac{1}{b} + \frac{1}{d} > \frac{2}{c} \quad \text{...(iv)}$$

From Eqs. (iii) and (iv), we get

$$\frac{1}{a} + \frac{1}{c} + \frac{1}{b} + \frac{1}{d} > \frac{2}{b} + \frac{2}{c}$$

$\Rightarrow \quad \dfrac{1}{a} + \dfrac{1}{d} > \dfrac{1}{b} + \dfrac{1}{c}$

(B) a, b, c, d are in GP.

For a, b, c applying AM > GM,

$\Rightarrow \quad \dfrac{a+c}{2} > b \;\Rightarrow\; a + c > 2b$ ...(i)

Similarly, for b, c, d

$$b + d > 2c \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$a + b + c + d > 2b + 2c \implies a + d > b + c$$

Now, applying GM > HM for $a, b\ c$

$$b > \frac{2ac}{a + c}$$

$$\Rightarrow \quad \frac{1}{c} + \frac{1}{a} > \frac{2}{c} \quad \text{...(iii)}$$

Similarly, for b, c, d, we get

$$\frac{1}{d} + \frac{1}{b} > \frac{2}{c} \quad \text{...(iv)}$$

On adding Eqs. (iii) and (iv), we get

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} > 2\left(\frac{1}{b} + \frac{1}{c}\right)$$

$$\Rightarrow \quad \frac{1}{a} + \frac{1}{d} > \frac{1}{b} + \frac{1}{c}$$

(C) a, b, c, d are in HP.

Applying AM > HM for a, b, c

$$\frac{a + c}{2} > b$$

$$\Rightarrow \quad a + c > 2b \quad \text{...(i)}$$

Similarly, for last 3 terms b, c, d

$$\Rightarrow \quad b + d > 2c \quad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$a + b + c + d > 2b + 2c$$

$$\Rightarrow \quad a + d > b + c$$

Again, applying GM > HM for a, b, c

$$\sqrt{ac} > b$$

$$\Rightarrow \quad ac > b^2 \quad \text{...(iii)}$$

Similarly, for b, c, d

$$\Rightarrow \quad bd > c^2 \quad \text{...(iv)}$$

On multiplying Eqs. (iii) and (iv), we get

$$abcd > b^2c^2$$

$$ad > bc$$

81. **(A)** $a_1, a_2, a_3, ..., a_n, ...$ are in AP

and $$a_1 = \frac{5}{2},\ a_{10} = 16$$

$$\therefore \quad a_1 + a_2 + ... + a_n = 110$$

$$\Rightarrow \quad \frac{n}{2}(a_1 + a_n) = 110$$

$$\Rightarrow \quad \frac{n}{2}\left[\frac{5}{2} + \frac{5}{2} + (n-1)\,d\right] = 110 \quad \text{...(i)}$$

Now, $$d = \frac{a_{10} - a_1}{10 - 1} = \frac{16 - \frac{5}{2}}{9} = \frac{27}{9 \times 2} = \frac{3}{2} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{n}{2}\left[5 + (n-1)\frac{3}{2}\right] = 110$$

$$\Rightarrow \quad 5n + (n^2 - n)\frac{3}{2} = 220$$

$$\Rightarrow \quad 3n^2 + 7n - 440 = 0$$

$$\Rightarrow \quad 3n^2 + 40n - 33n - 440 = 0$$

$$\Rightarrow \quad n(3n + 40) - 11(3n + 40) = 0$$

$$\Rightarrow \quad (3n + 40)(n - 11) = 0$$

So, $$n = -\frac{40}{3} \text{ or } n = 11$$

$$\therefore \quad n = 11 \qquad [n \in N]$$

(B) Let first angle = a [in degrees]

Common difference = d [in degrees]

Number of sides $n = 9$

$\therefore$ Sum of interior angles $= (n - 2) \times 180°$

$$\Rightarrow \quad \frac{n}{2}[2a + (n-1)\,d] = (n-2) \times 180°$$

$$\Rightarrow \quad \frac{9}{2}(2a + 8d) = 7 \times 180°$$

$$\Rightarrow \quad a + 4d = 140°$$

and largest angle $T_9 = a + 8d < 180°$

$$\Rightarrow \quad 4d < 40$$

$$\Rightarrow \quad d < 10$$

$$\therefore \quad d = 9$$

(C) Given increasing GP,

$$a_1, a_2, ..., a_n, ...$$

where $$a_6 = 4a_4$$

$$a_1 r^5 = 4a_1 r^3 \qquad [r \text{ is the common ratio}]$$

$$\Rightarrow \quad r^2 = 4$$

$$\Rightarrow \quad r = 2 \qquad [\because \text{increasing GP}]$$

and $$a_9 - a_7 = 192$$

$$a_1(r^8 - r^6) = 192$$

$$a_1(256 - 64) = 192$$

$$a_1 = \frac{192}{192}$$

$$a_1 = 1$$

Then, $a_2 = 2$, $a_3 = 4$ and $a_4 + a_5 + ... + a_n = 1016$

$$(a_1 + a_2 + ... + a_n) - (a_1 + a_2 + a_3) = 1016$$

$$\frac{1(2^n - 1)}{2 - 1} = 1016 + 7$$

$$2^n = 1023 + 1 = 1024 = 2^{10}$$

$$\therefore \quad n = 10$$

82. **(A)** $a_1, a_2, ...$ are in AP.

$a_1 + a_4 + a_7 + a_{14} + a_{17} + a_{20} = 165$ [In an AP, sum of the terms equidistant from the 1st and last is equal to sum of 1st and last terms]

$$\Rightarrow \quad 3(a_1 + a_{20}) = 165$$

$$\Rightarrow \quad a_1 + a_1 + 19d = 55$$

d is the common difference of AP.

$$2a_1 + 19d = 55 \quad \text{...(i)}$$

Now, $$\alpha = a_2 + a_6 + a_{15} + a_{19}$$

$$\alpha = 2(a_2 + a_{19})$$

$$\alpha = 2(a_1 + d + a_1 + 18d)$$
$$\alpha = 2(2a_1 + 19d) \quad ...(ii)$$

and $\beta = 2(a_9 + a_{12}) - (a_3 + a_{18})$
$\beta = 2(a_1 + 8d + a_1 + 11d) - (a_1 + 2d + a_1 + 17d)$
$\beta = 2(2a_1 + 19d) - (2a_1 + 19d)$
$\beta = 2a_1 + 19d \quad ...(iii)$

From Eqs. (i) and (iii), we get
$\alpha = 2\beta$

From Eqs. (i), (ii) and (iii), we get
$\alpha + 2\beta = 4(2a_1 + 19d) = 4(55) = 220$ [from Eq.(i)]
$\alpha + \beta = 3(2a_1 + 19d)$
$= 3 \times 55 = 165 = 15 \times 11 = 15\mu$, where $\mu \in I$
$\alpha - \beta = 2a_1 + 19d$
$= 55 = 5 \times 11 = 5\lambda$, where $\lambda \in I$

(B) $a_1, a_2, ...$ are in AP.
$a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 195$
$3(a_1 + a_{24}) = 195$
$\Rightarrow \quad a_1 + a_{24} = 65 \quad ...(i)$
$\Rightarrow \quad 2a_1 + 23d = 65$
Now, $\alpha = a_2 + a_7 + a_{18} + a_{23}$
$= 2(a_2 + a_{23}) = 2(2a_1 + 23d)$
$\alpha = 130$ [from Eq. (i)]
$\beta = 2(a_2 + a_{22}) - (a_8 + a_{17})$
$= 2(2a_1 + 23d) - (2a_1 + 23d)$
$= 130 - 65 = 65$
Then, $\alpha = 2\beta$
$\alpha + 2\beta = 130 + 130 = 260$
$\alpha + \beta = 195 = 15 \times 13 = 15\mu$, where $\mu = 13$
and $\alpha - \beta = 130 - 65 = 65$
$= 5 \times 13 = 5\lambda$, where $\lambda = 13$

(C) $a_1, a_2, ...$ are in AP.
$a_1 + a_7 + a_{10} + a_{21} + a_{24} + a_{30} = 225$
$3(a_1 + a_{30}) = 225$
$2a_1 + 29d = 75 \quad ...(i)$
Now, $\alpha = a_2 + a_7 + a_{24} + a_{29}$
$\alpha = 4a_1 + 58d = 2(2a_1 + 29d)$
$= 2 \times 75 = 150$
$\alpha = 150 \quad ...(ii)$
and $\beta = 2(a_{10} + a_{21}) - (a_3 + a_{28})$
$= 2(2a_1 + 29d) - (2a_1 + 29d) = 150 - 75$
$\beta = 75 \quad ...(iii)$
Then, $\alpha = 2\beta$
$\alpha + 2\beta = 150 + 150 = 300$ and $\alpha - \beta = 150 - 75 = 75$
$= 5 \times 15 = 5\lambda$, where $\lambda = 15$
and $\alpha + \beta = 150 + 75 = 225 = 15 \times 15 = 15\mu$, where $\mu = 15$

83. **(A)** $4a^2 + 9b^2 + 16c^2 = 2(3ab + 6bc + 4ca)$
$(2a)^2 + (3b)^2 + (4c)^2 - (2a)(3b) - (2a)(4c) - (2a)(4c) = 0$
$\frac{1}{2}\{(2a - 3b)^2 + (3b - 4c)^2 + (4c - 2a)^2\} = 0$
$\Rightarrow \quad 2a - 3b = 0$ and $3b - 4c = 0$
and $4c - 2a = 0 \Rightarrow 2a = 3b$
and $3b = 4c$ and $4c = 2a$
$\Rightarrow \quad a = \frac{3}{2}b$ and $b = \frac{4}{3}c$ and $c = \frac{1}{2}a$
$\Rightarrow \quad a = \frac{3}{2}b$ and $b = \frac{4}{3}c$ and $c = \frac{3}{4}b$
So, a, b, c are $\frac{3}{2}b, b, \frac{3}{4}b$
Reciprocal of the terms $\frac{2}{3b}, \frac{1}{b}, \frac{4}{3b}$,
which is in AP.
So, these a, b, c are in HP.

(B) $17a^2 + 13b^2 + 5c^2 = 3ab + 15bc + 5ca$
$\Rightarrow \quad 34a^2 + 26a^2 + 10c^2 - 6ab - 30bc - 10ca = 0$
$\Rightarrow \quad (3a - b)^2 + (5b - 3c)^2 + (c - 5a)^2 = 0$
$\Rightarrow \quad 3a - b = 0$ and $5b - 3c = 0$ and $c - 5a = 0$
$\Rightarrow \quad \frac{a}{1} = \frac{b}{3} = \frac{c}{5} = \lambda$ [say]
$\therefore \quad a = \lambda, b = 3\lambda, c = 5\lambda$
Hence, a, b, c are in AP.

(C) $a^2 + 9b^2 + 25c^2 = abc\left(\frac{15}{a} + \frac{5}{b} + \frac{3}{c}\right)$
$\Rightarrow \quad (a)^2 + (3b)^2 + (5c)^2 - 15bc - 5ac - 3ab = 0$
$\Rightarrow \quad \frac{1}{2}\{(a - 3b)^2 + (3b - 5c)^2 + (5c - a)^2\} = 0$
$\Rightarrow \quad a - 3b = 0$ and $3b - 5c = 0$ and $5c - a = 0$
$\Rightarrow \quad a = 3b$ and $3b = 5c$ and $5c = a$
$\Rightarrow \quad a = 3b$ and $b = \frac{5}{3}c$ and $c = \frac{a}{5}$
$\Rightarrow \quad a = 3b$ and $b = \frac{5}{3}c$ and $c = \frac{3}{5}b$
So, a, b, c are of the form $3b, b, \frac{3b}{5}$.
Reciprocal of $3b, b, \frac{3b}{5}$ are $\frac{1}{3b}, \frac{1}{b}, \frac{5}{3b}$, which are in AP.
$\left[\because \frac{1}{b} - \frac{1}{3b} = \frac{2}{3b}$ and $\frac{5}{3b} - \frac{1}{b} = \frac{2}{3b}\right]$

(D) $(a^2 + b^2 + c^2)p^2 - 2p(ab + bc + ca) + a^2 + b^2 + c^2 \le 0$
$\Rightarrow \quad (a^2p^2 + b^2 - 2abp) + (b^2p^2 + c^2 - 2pbc)$
$+ (c^2p^2 + a^2 - 2acp) \le 0$
$\Rightarrow \quad (ap - b)^2 + (bp - c)^2 + (cp - a)^2 \le 0$
$\Rightarrow \quad (ap - b)^2 + (bp - c)^2 + (cp - a)^2 = 0$
$\Rightarrow \quad ap - b = 0$ and $bp - c = 0$ and $cp - a = 0$
$\Rightarrow \quad p = \frac{b}{a}$ and $p = \frac{c}{b}$ and $p = \frac{a}{c}$
$\Rightarrow \quad \frac{b}{a} = \frac{c}{b} = \frac{a}{c}$
$\Rightarrow a, b, c$ are in GP.

84. If a, b, c are in GP.
Then, $b^2 = ac$
If middle term is added, then $a + b, 2b$ and $c + b$ are in GP.

$$\frac{\text{I}-\text{II}}{\text{II}-\text{III}}=\frac{a+b-2b}{2b-(c+b)} \qquad [\text{here, I}=a+b,\ \text{II}=2b,\ \text{III}=c+b]$$

$$=\frac{a-b}{b-c}=\frac{ab-b^2}{b^2-bc}=\frac{ab-ac}{ac-bc} \qquad [\because b^2=ac]$$

$$=\frac{a(b-c)(a+b)(b+c)}{c(a-b)(a+b)(b+c)}$$

$$=\frac{a(b^2-c^2)(a+b)}{c(a^2-b^2)(b+c)}$$

$$=\frac{a(ac-c^2)(a+b)}{c(a^2-ac)(b+c)};\ \frac{a+b}{b+c}=\frac{\text{I}}{\text{III}}$$

Hence, $a+b, 2b, b+c$ are in HP.

Hence, both statements are true and Statement-2 is correct explanation for Statement-1.

85. $\because T_n=2n^3+3n^2-4$

Sequence is 1, 24, 77, 172, 321, ...
First order difference 23, 53, 95, 149, ...
Second order difference 30, 42, 54, ...
which are in AP.

$\therefore$ **Statenemt-1** is true.

$\because T_n$ is of three degree and third order difference will be constant.

Statement-2 is true, which is correct explanation for Statement-1.

86. **Statement-1** Let S be the required sum of product of numbers.

$$\left(\sum_{i=1}^{n}x_i\right)^2=\sum_{i=1}^{n}x_i^2+2\sum\sum_{1\le i<j\le n}x_i x_j$$

$$\therefore \quad (a_1-a_1+a_2-a_2+\ldots+a_n-a_n)^2=2\sum_{i=1}^{n}a_i^2+2S$$

$$\therefore \qquad S=-\sum_{i=1}^{n}a_i^2$$

$\therefore$ **Statement-1** is true.

Statement-2 is true but not correct explanation for Statement-1.

87. **Statement-1** $a+b+c=18,\ a, b, c>0$

Applying AM $\ge$ GM for a, b, c

$$\frac{a+b+c}{3}\ge\sqrt[3]{abc} \ \Rightarrow\ \sqrt[3]{abc}\le 6\Rightarrow abc\le 216$$

Maximum value of abc is 216 which occurs at $a=b=c$.
Statement-2 is the correct explanation for Statement-1.

88. **Statement-1**

$$4a^2+9b^2+16c^2-2(3ab+6bc+4ca)=0$$

$$\Rightarrow (2a)^2+(3b)^2+(4c)^2-(2a)(3b)-(3b)(4c)-(2a)(4c)=0$$

$$\Rightarrow \frac{1}{2}\{(2a-3b)^2+(3b-4c)^2+(4c-2a)^2\}=0$$

$$\Rightarrow 2a-3b=0 \text{ and } 3b-4c=0 \text{ and } 4c-2a=0$$

$$\Rightarrow \text{and } b=\frac{4c}{3} \text{ and } c=\frac{a}{2}\Rightarrow a=\frac{3b}{2} \text{ and } b=\frac{4c}{3} \text{ and } c=\frac{3b}{4}$$

Then, a, b, c are of the form $\frac{3b}{2}, b, \frac{3b}{4}$, which are in HP.

So, Statement-1 is false.

Statement-2 If

$$(a_1-a_2)^2+(a_2-a_3)^2+(a_3-a_1)^2=0$$

$$\Rightarrow a_1-a_2=0 \text{ and } a_2-a_3=0 \text{ and } a_3-a_1=0$$

$$\Rightarrow a_1=a_2=a_3,\ \forall\ a_1, a_2, a_3\in R$$

So, Statement-2 is true.

89. $\because A=\frac{a+b}{2},\ G=\sqrt{ab}$ and $H=\frac{2ab}{a+b}$

Given, $4G=5H$ ...(i)

and $G^2=AH$

$\therefore$ $H=\frac{G^2}{A}$ ...(ii)

From Eqs. (i) and (ii), we get

$$4G=\frac{5G^2}{A}\Rightarrow 4A=5G$$

$$\Rightarrow 2(a+b)=5\sqrt{ab}$$

$$\Rightarrow 4(a^2+b^2+2ab)=25ab$$

$$\Rightarrow 4a^2-17ab+4b^2=0$$

$$\Rightarrow (a-4b)(4a-b)=0$$

$$a=4b,\ 4a-b\ne 0 \qquad [\because a>b]$$

$\therefore$ Statement-1 is true.

Statement-2 is true only for two numbers, if numbers more than two, then this formula (AM) (HM) = $(\text{GM})^2$ is true, if numbers are in GP.

Statement-2 is false for positive numbers.

90. **Statement-1** Sum of first 100 even natural numbers

$$E_1=2+4+\ldots+200=\frac{2(100\times 101)}{2}=10100$$

Sum of 100 odd natural numbers $=1+3+\ldots+199$

$$O=\frac{100}{2}(1+199)=10000$$

$$\therefore \qquad E-O=100$$

So, Statement-1 is true.

Statement-2 Sum of first n natural even numbers

$$E=2+4+\ldots+2n=\frac{2n(n+1)}{2}=n^2+n$$

Sum of first n odd natural numbers

$$O=1+3+\ldots+(2n-1)$$

$$=\frac{n}{2}[1+2n-1]=n^2$$

So, $E-O=n^2+n-n^2=n$

Statement-2 is true and correct explanation for Statement-1.

91. Let $T_n=An+B$

$\therefore$ $T_p=Ap+B,$

$$T_{2p}=2Ap+B,\ T_{4p}=4Ap+B$$

$\therefore$ T_p, T_{2p}, T_{4p} are in G P.

$$\therefore \qquad (2Ap+B)^2=(Ap+B)(4Ap+B)$$

$$\Rightarrow ABp=0$$

$$\therefore \qquad B=0,\ A\ne 0,\ p\ne 0$$

$\Rightarrow$ Common ratio, $r=\frac{T_{2p}}{T_p}=\frac{2Ap+0}{Ap+0}=2$

92. $a \neq 1, b \neq 0$ and $a \neq b$

Let $S = (a + b) + (a^2 + ab + b^2) + (a^3 + a^2b + ab^2 + b^3) + \ldots + n$ terms

$$= \frac{1}{(a-b)}[(a^2 - b^2) + (a^3 - b^3) + (a^4 - b^4) + \ldots + n \text{ terms}]$$

$$= \frac{1}{(a-b)}[a^2(1 + a + \ldots + n \text{ terms}) - b^2(1 + b + b^2 + \ldots + n \text{ terms})]$$

$$= \frac{1}{(a-b)}\left[a^2 \cdot \frac{1 \cdot (a^n - 1)}{(a-1)} - b^2 \cdot \frac{1 \cdot (b^n - 1)}{(b-1)}\right]$$

$$= \frac{1}{(a-b)}\left[a^2 \frac{(1-a^n)}{(1-a)} - b^2 \frac{(1-b^n)}{(1-b)}\right]$$

93. Sequence of natural number is divided into group.

1, 3, 5, 7, 9, 11, ...

$\therefore$ nth row contains n elements

1st element of nth row $= n^2 - (n-1)$

Least element of nth row $= n^2 + (n-1)$

$\therefore$ Sum of the element in the nth row

$$= \frac{n}{2}(a + l) = \frac{n}{2}[n^2 - (n-1) + n^2 + (n-1)]$$

$$= \frac{n}{2}[n^2 - n + 1 + n^2 + n - 1] = \frac{n}{2}[2n^2] = n^3$$

94. $$a = S_n = \frac{a(r^n - 1)}{r - 1} \quad \ldots(i)$$

$$b = S_{2n} - S_n = \frac{a(r^{2n} - 1)}{(r-1)} - \frac{a(r^n - 1)}{(r-1)} = \frac{a(r^n - 1)}{(r-1)}(r^n) \quad \ldots(ii)$$

$$c = S_{3n} - S_{2n} = \frac{a(r^{3n} - 1)}{(r-1)} - \frac{a(r^{2n} - 1)}{(r-1)}$$

$$= \frac{a(r^n - 1)}{(r-1)}(r^{2n} + r^n + 1 - r^n - 1) = \frac{a(r^n - 1)}{(r-1)} \cdot (r^n)^2 \quad \ldots(iii)$$

From Eqs. (i), (ii) and (iii), $b^2 = ac$, so a, b, c are in GP.

95. First four terms of an AP are $a, 2a, b$ and $(a - 6 - b)$.

So, $2a - a = a - 6 - b - b$

$\Rightarrow$ $a = a - 6 - 2b$

$\Rightarrow$ $-2b = 6 \Rightarrow b = -3$

and $2a - a = b - 2a$

$\Rightarrow$ $b = 3a \Rightarrow a = -1$

$\therefore$ First terms $a = -1$ and $d = a = -1$

$$S_{100} = \frac{100}{2}[2a + (100 - 1)d]$$

$$= 50[-2 + 99(-1)]$$

$$= 50(-2 - 99) = 50(-101) = -5050$$

96. (i) $$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \infty = \frac{\pi^2}{6} \quad \ldots(i)$$

$$\therefore \quad \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots + \infty$$

$$= \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots + \infty\right) - \left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \ldots + \infty\right)$$

$$= \frac{\pi^2}{6} - \frac{1}{4}\frac{\pi^2}{6} = \frac{3}{4} \times \frac{\pi^2}{6} = \frac{\pi^2}{8}$$

(ii) $$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots + \infty = \left(\frac{1}{1^2} - \frac{1}{3^3} + \ldots\right) - \frac{1}{2^2}\left(\frac{1}{1^2} - \frac{1}{2^2} + \ldots + \infty\right)$$

$$= \frac{\pi^2}{8} - \frac{1}{4} \times \frac{\pi^2}{6} = \frac{\pi^2}{12} \quad \text{[by part (i)]}$$

97. $\Sigma a_i b_i = \Sigma a_i (1 - a_i) = na - \Sigma a_i^2$

$$= na - \Sigma(a_i - a + a)^2$$

$$= na - \Sigma[(a_i - a)^2 + a^2 + 2a(a_i - a)]$$

$$= na - \Sigma[(a_i - a)^2 - \Sigma a^2 - 2a\Sigma(a_i - a)$$

$\therefore$ $\Sigma a_i b_i + \Sigma(a_i - a)^2 = na - na^2 - 2a(na - na)$

$$= na(1 - a) = nab \qquad \left[\begin{array}{c}\because \Sigma b_i = \Sigma 1 - \Sigma a_i \\ \therefore nb = n - na \\ \text{or } a + b = 1\end{array}\right]$$

98. $a_1 + a_2 + \ldots + a_{98} = 137$

$$\frac{98}{2}(a_1 + a_{98}) = 137$$

$$a_1 + a_2 + 97 = \frac{137}{49}; \quad 2a_1 + 97 = \frac{137}{49}$$

$$2a_1 = \frac{137}{49} - 97; \quad a_1 = \frac{1}{2}\frac{(137 - 4753}{49}$$

$$a_1 = -\frac{4616}{2 \times 49}; \quad a_1 = \frac{2308}{49} \quad \ldots(i)$$

Now, $a_2 + a_4 + \ldots + a_{98} = (a_1 + 1) + (a_1 + 3) + \ldots + (a_1 + 97)$ $[\because d = 1]$

$$= 49a_1 + (1 + 3 + \ldots + 97)$$

$$= -49 \times \frac{2308}{49} + \frac{49}{2}(1 + 97)$$

$$= -2308 + 49^2$$

$$= -2308 + 2401 = 93$$

99. $t_1 = 1$ and $t_r - t_{r-1} = 2^{r-1}, r \geq 2$

$$t_2 - t_1 = 2$$
$$t_3 - t_2 = 2^2$$
$$t_4 - t_3 = 2^3$$
$$\vdots \quad \vdots \quad \vdots$$
$$t_n - t_{n-1} = 2^{n-1}$$

Addiing columnwise, we get

$$t_n - t_1 = 2 + 2^2 + \ldots + 2^{n-1}$$

$$t_n = 1 + 2 + 2^2 + \ldots + 2^{n-1}$$

$$t_n = \frac{1 \cdot (2^n - 1)}{2 - 1} \Rightarrow t_n = 2^n - 1$$

So, $$\sum_{r=1}^{n} t_r = t_1 + t_2 + \ldots + t_n = (2 - 1) + (2^2 - 1) + \ldots + (2^n - 1)$$

$$= (2 + 2^2 + \ldots + 2^n) - n = \frac{2 \cdot (2^n - 1)}{(2 - 1)} - n = 2^{n+1} - 2 - n$$

$$= 2^{n+1} - n - 2$$

100. (i) $I_n = \int_0^{\pi} \frac{\sin 2nx}{\sin x}\, dx = \int_0^{\pi} \frac{\sin 2n(\pi - x)}{\sin x} dx$

$= \int_0^{\pi} \frac{\sin (2n\pi - 2nx)}{\sin x}\, dx$

$I_n = -\int_0^{\pi} \frac{\sin 2nx}{\sin x}\, dx$

$I_n = -I_n \Rightarrow 2I_n = 0 \Rightarrow I_n = 0$

$\therefore \quad I_1 = I_2 = I_3 = \ldots = 0$

which is a constant series.

$\therefore$ This series is AP with common difference 0 and first term o.

(ii) $I_n = \int_0^{\pi} \frac{\sin^2 nx}{\sin^2 x}\, dx$

Let $f(x) = \frac{\sin^2 nx}{\sin^2 x}$

Hence, $f(\pi - x) = f(x)$

So, $I_n = 2\int_0^{\pi/2} \frac{\sin^2 nx}{\sin^2 x} dx$

Now, $I_{n+1} + I_{n-1} - 2I_n$

$= 2\int_0^{\pi/2} \left\{ \frac{(\sin^2(n+1)\, x - \sin^2 nx)}{\sin^2 x} + \frac{(\sin^2(n-1)x - \sin^2 nx)}{\sin^2 x} \right\} dx$

$= 2\int_0^{\pi/2} \frac{\sin (2n+1)\, x \sin x - \sin (2n-1)\, x \sin x}{\sin^2 x}\, dx$

$= 2\int_0^{\pi/2} \frac{\sin (2n+1)\, x - \sin (2n-1)x}{\sin x}\, dx$

$= 2\int_0^{\pi/2} \frac{2\cos 2nx \sin x}{\sin x} dx$

$= 4\int_0^{\pi/2} \cos 2nx\, dx = \frac{4}{2n}[\sin 2nx]_0^{\pi/2} = \frac{2}{n} \cdot 0 = 0$

$\therefore\ I_{n+1} + I_{n-1} = 2I_n \quad \therefore I_1, I_2, I_3, \ldots$ are in A P.

101. $S = 7 + 13 + 21 + 31 + \ldots + T_n$

$S = \quad 7 + 13 + 21 + \ldots + T_{n-1} + T_n$

$0 = 7 + 6 + 8 + 10 + \ldots + n \text{ terms} - T_n$

$T_n = 7 + 6 + 8 + 10 + \ldots + n \text{ terms}$

$T_n = 7 + \{6 + 8 + 10 + \ldots + (n-1) \text{ terms}\}$

$T_n = 7 + \frac{(n-1)}{2}(12 + (n-2)2)$

$T_n = 7 + \frac{(n-1)}{2}(8 + 2n)$

$T_n = 7 + (n-1)(4 + n)$

$T_{70} = 7 + 69 \times 74 = 7 + 5106 = 5113$

102. $\left(x + \frac{1}{x}\right)^3 + \left(x^2 + \frac{1}{x^2}\right)^3 + \ldots + \left(x^n + \frac{1}{x^n}\right)^3 = \sum_{n=1}^{n} \left(x^n + \frac{1}{x^n}\right)^3$

$= \sum_{n=1}^{n} \left(x^{3n} + \frac{1}{x^{3n}} + 3\left(x^n + \frac{1}{x^n}\right)\right)$

$= \sum_{n=1}^{n} x^{3n} + \sum_{n=1}^{n} \frac{1}{x^{3n}} + 3 \sum_{n=1}^{n} x^n + 3 \sum_{n=1}^{n} \frac{1}{x^n}$

$= \frac{x^3(1 - x^{3n})}{(1 - x^3)} + \frac{(1 - x^{3n})}{x^{3n}(1 - x^3)} + \frac{3x(1 - x^n)}{(1 - x)} + \frac{3(1 - x^n)}{x^n(1 - x)}$

103. Let d be the common difference of AP.

LHS $= a_1^2 - a_2^2 + a_3^2 - a_4^2 + \ldots + a_{2n-1}^2 - a_{2n}^2$

$(a_1 - a_2)(a_1 + a_2) + (a_3 - a_4)(a_3 + a_4) + \ldots + (a_{2n-1} - a_{2n})(a_{2n-1} + a_{2n})$

$= -d\,(a_1 + a_2 + \ldots + a_{2n-1} + a_{2n})$

$= -d\,[(a_1 + a_{2n}) + (a_2 + a_{2n-1}) + \ldots + (a_n + a_{n+1})]$

$= -dn\,(a_1 + a_{2n})$

$= -dn\,\frac{(a_1^2 - a_{2n}^2)}{(a_1 - a_{2n})} = \frac{-dn(a_1^2 - a_{2n}^2)}{(1 - 2n)d}$ $[\because a_{2n} = a_1 + (2n-1)\, d]$

$= \frac{n}{2n-1}(a_1^2 - a_{2n}^2)$

104. Let a, b, c (unequal number) are in HP

$\therefore \quad b = \frac{2ac}{a + c}$

$\Rightarrow \quad \frac{b}{2} = \frac{ac}{a + c} = \lambda$ [say]

$\Rightarrow \quad b = 2\lambda$ and $ac = \lambda(a + c)$ ...(i)

Now, a^2, b^2, c^2 are in AP

So, $b^2 = \frac{a^2 + c^2}{2} \Rightarrow 2b^2 = a^2 + c^2$

$\Rightarrow \quad 2(2\lambda)^2 = (a + c)^2 - 2ac$

$\Rightarrow \quad (a + c)^2 - 2\lambda(a + c) - 8\lambda^2 = 0$

$\Rightarrow \quad (a + c - 4\lambda)(a + c + 2\lambda) = 0$

$\Rightarrow \quad a + c = 4\lambda$ or $a + c = -2\lambda$

***Case* I** If $a + c = 4\lambda$

$\therefore \quad ac = 4\lambda^2$ [from Eq. (i)]

$\Rightarrow \quad (a - c)^2 = (a + c)^2 - 4ac$

$\Rightarrow \quad (a - c)^2 = 16\lambda^2 - 16\lambda^2$

$\Rightarrow \quad (a - c)^2 = 0 \Rightarrow a = c$

Let given that a, b, c are distinct, so $a + c = 4\lambda$ is not valid.

***Case* II** If $a + c = -2\lambda$

$\Rightarrow \quad ac = -2\lambda^2$ [from Eq. (i)]

$\therefore \quad (a - c)^2 = (a + c)^2 - 4ac$

$\Rightarrow \quad (a - c)^2 = 4\lambda^2 + 8\lambda^2 \Rightarrow (a - c) = \pm 2\sqrt{3}\lambda$ ...(ii)

If $a - c = 2\sqrt{3}\lambda$, ...(iii)

then $a + c = 2\lambda$

From Eqs. (ii) and (iii), we get

$a = (\sqrt{3} - 1)\lambda$ and $c = -(1 + \sqrt{3})\lambda$

$\therefore \quad a : b : c = (\sqrt{3} - 1)\lambda : 2\lambda : -(\sqrt{3} + 1)\lambda$

$a : b : c = (\sqrt{3} - 1) : 2 : -(\sqrt{3} + 1)$

$\Rightarrow \quad a : b : c = (1 - \sqrt{3}) : -2 : (\sqrt{3} + 1)$

If $a - c = -2\sqrt{3}\lambda$, ...(iv)

then $a + c = -2\lambda$ ...(v)

From Eqs. (iv) and (v), we get

$$a = -(\sqrt{3}+1)\lambda \text{ and } c = (\sqrt{3}-1)\lambda$$

$\therefore \quad a : b : c = -(\sqrt{3}+1)\,\lambda : 2\lambda : (\sqrt{3}-1)\lambda$

$\Rightarrow \quad a : b : c = (1+\sqrt{3}) : -2 : (1-\sqrt{3})$

105. $a_1, a_2, a, \ldots, a_n$ are in AP with $a_1 = 0$ and common difference d $[d \neq 0]$

$\therefore a_2 = d, a_3 = 2d, \ldots, a_n = (n-1)\,d$

$$\text{LHS} = \frac{a_3}{a_2} + \frac{a_4}{a_3} + \frac{a_5}{a_4} + \ldots + \frac{a_n}{a_{n-1}} - a_2\left(\frac{1}{a_2} + \frac{1}{a_3} + \ldots + \frac{1}{a_{n-2}}\right)$$

$$= \frac{1}{a_2}(a_3 - a_2) + \frac{1}{a_3}(a_4 - a_2) + \ldots + \frac{(a_{n-1} - a_2)}{a_{n-2}} + \frac{a_n}{a_{n-1}}$$

$$= \frac{1}{d}(2d - d) + \frac{1}{2d}(3d - d) + \ldots + \frac{[(n-2)\,d - d]}{(n-3)\,d} + \frac{(n-1)\,d}{(n-2)\,d}$$

$$= [1 + 1 + \ldots + (n-3) \text{ times}] + \frac{n-1}{n-2}$$

$$= (n-3) + \frac{(n-1)}{n-2} = (n-3) + \frac{(n-2)+1}{(n-2)}$$

$$= (n-3) + 1 + \frac{1}{n-2} = n - 2 + \frac{1}{n-2}$$

$$= \frac{a\,(n-2)\,d}{d} + \frac{d}{(n-2)\,d} = \frac{a_{n-1}}{a_2} + \frac{a_2}{a_{n-1}} = \text{RHS}$$

106. Let one side of equilateral triangle contains n balls. Then

Number of balls (initially) $= 1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$

According to the question, $\dfrac{n(n+1)}{2} + 669 = (n-8)^2$

$$\Rightarrow \quad n^2 + n + 1338 = 2n^2 - 32n + 128$$

$$\Rightarrow \quad n^2 - 33n - 1210 = 0$$

$$\Rightarrow \quad (n-55)(n+22) = 0 \;\Rightarrow\; n = 55 \text{ or } n = -22$$

which is not possible

$\therefore \quad n = 55$

So, $\dfrac{n(n+1)}{2} = \dfrac{55 \times 56}{2} = 55 \times 28 = 1540$

107. $\theta_1, \theta_2, \theta_3, \ldots, \theta_n$ are in AP.

So, $\theta_2 - \theta_1 = \theta_3 - \theta_2 = \ldots = \theta_n - \theta_{n-1} = d$

$\therefore$ LHS $= \sin d\,[\sec\theta_1 \sec\theta_2 + \sec\theta_2 \sec\theta_3 + \ldots + \sec\theta_{n-1} \sec\theta_n]$

$$= \sin d\left[\frac{1}{\cos\theta_1 \cos\theta_2} + \frac{1}{\cos\theta_2 \cos\theta_3} + \ldots + \frac{1}{\cos\theta_{n-1} \cos\theta_n}\right]$$

$$= \frac{\sin d}{\cos\theta_1 \cos\theta_2} + \frac{\sin d}{\cos\theta_2 \cos\theta_3} + \ldots + \frac{\sin d}{\cos\theta_{n-1} \cos\theta_n}$$

$$= \frac{\sin(\theta_2 - \theta_1)}{\cos\theta_1 \cos\theta_2} + \frac{\sin(\theta_3 - \theta_2)}{\cos\theta_2 \cos\theta_3} + \ldots + \frac{\sin(\theta_n - \theta_{n-1})}{\cos\theta_{n-1} \cos\theta_n}$$

$$= (\tan\theta_2 - \tan\theta_1) + (\tan\theta_3 - \tan\theta_2) + \ldots + (\tan\theta_n - \tan\theta_{n-1})$$

$$= \tan\theta_n - \tan\theta_1 = \text{RHS}$$

108. LHS $= (1 + 5^{-1})(1 + 5^{-2})(1 + 5^{-4}) \ldots (1 + 5^{-2^n})$

$$= \left(1 + \frac{1}{5}\right)\left(1 + \frac{1}{5^2}\right)\left(1 + \frac{1}{5^4}\right) \ldots \left(1 + \frac{1}{5^{2^n}}\right)$$

$$= \frac{\left(1 - \frac{1}{5}\right)}{\left(1 - \frac{1}{5}\right)}\left(1 + \frac{1}{5}\right)\left(1 + \frac{1}{5^2}\right)\left(1 + \frac{1}{5^4}\right) \ldots \left(1 + \frac{1}{5^{2^n}}\right)$$

$$= \frac{5}{4}\left[\left(1 + \frac{1}{5^2}\right)\left(1 + \frac{1}{5^2}\right)\left(1 + \frac{1}{5^4}\right) \ldots \left(1 + \frac{1}{5^{2^n}}\right)\right]$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$= \frac{5}{4}\left(1 - \frac{1}{5^{2^{n+1}}}\right) = \frac{5}{4}(1 - 5^{-2^{n+1}}) = \text{RHS}$$

109. $S = \displaystyle\sum_{n=0}^{\infty} \frac{2^n}{a^{2^n} + 1}, (a > 1)$

$$S_n = \sum_{n=0}^{n} \frac{2^n}{a^{2^n} + 1}$$

$$= \frac{1}{a+1} + \frac{2}{a^2+1} + \frac{4}{a^4+1} + \frac{8}{a^8+1} + \ldots + \frac{2^n}{a^{2^n}+1}$$

$$= \frac{1}{1+a} + \frac{2}{1+a^2} + \frac{4}{1+a^4} + \frac{8}{1+a^8} + \ldots + \frac{2^n}{1+a^{2^n}}$$

$$= \left(-\frac{1}{1-a} + \frac{1}{1-a}\right) + \frac{1}{1+a} + \frac{2}{1+a^2} + \frac{4}{1+a^4} + \ldots + \frac{2^n}{1+a^{2^n}}$$

$$= \frac{1}{a-1} + \left(\frac{1}{1-a} + \frac{1}{1+a}\right) + \frac{2}{1+a^2} + \frac{4}{1+a^4} + \ldots + \frac{2^n}{1+a^{2^n}}$$

$$= \frac{1}{a-1} + \left(\frac{2}{1-a^2} + \frac{2}{1+a^2}\right) + \frac{4}{1+a^4} + \ldots + \frac{2^n}{1+a^{2^n}}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$S_n = \frac{1}{a-1} + \frac{2^{n+1}}{1 - a^{2^{n+1}}}$$

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1}{a-1} + \frac{2^{n+1}}{1 - a^{2^{n+1}}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{a-1} + \frac{\frac{2^{n+1}}{a^{2^{n+1}}}}{\frac{1}{a^{2^{n+1}}} - 1}\right) = \frac{1}{a-1} + \frac{0}{0-1} = \frac{1}{a-1}$$

110. $T_n = \tan^{-1}\left(\dfrac{2^{n-1}}{1 + 2^{2n-1}}\right) = \tan^{-1}\left(\dfrac{2^{n-1}}{1 + 2^n \cdot 2^{n-1}}\right)$

$$= \tan^{-1}\left(\frac{2^n - 2^{n-1}}{1 + 2^n \cdot 2^{n-1}}\right) = \tan^{-1}2^n - \tan^{-1}2^{n-1}$$

$S_n = T_1 + T_2 + \ldots + T_n$

$= (\tan^{-1}2^1 - \tan^{-1}2^0) + (\tan^{-1}2^2 - \tan^{-1}2^1) + \ldots + (\tan^{-1}2^n - \tan^{-1}2^{n-1})$

$= (\tan^{-1}2^n - \tan^{-1}1)$

$$S_n = \tan^{-1}2^n - \frac{\pi}{4}$$

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\tan^{-1}2^n - \frac{\pi}{4}\right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

111. $T_n = \tan[\alpha + (n-1)\beta]\tan(\alpha + n\beta)$

$\tan\beta = \tan[(\alpha + n\beta) - \{\alpha + (n-1)\beta\}]$

$$\tan\beta = \frac{\tan(\alpha + n\beta) - \tan(\alpha + (n-1)\beta\}]}{1 + \tan(\alpha + n\beta)\tan\{\alpha + (n-1)\beta\}]}$$

$\therefore 1 + T_n = \cot\beta[\tan(\alpha + n\beta) - \tan\{\alpha + (n-1)\beta\}]$

$T_n = \cot\beta[\tan(\alpha + n\beta) - \tan\{\alpha + (n-1)\beta\}] - 1$

For $n = 1$,

$T_1 = \cot\beta[\tan(\alpha + \beta) - \tan\alpha] - 1$

For $n = 2$,

$T_2 = \cot\beta[\tan(\alpha + 2\beta) - \tan(\alpha + \beta)] - 1$

For $n = 3$,

$T_3 = \cot\beta[\tan(\alpha + 3\beta) - \tan(\alpha + 2\beta)] - 1$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

For $n = n$,

$T_n = \cot\beta[\tan(\alpha + n\beta) - \tan(\alpha + (n-1)\beta\}] - 1$

Sum columnwise,

$$S_n = T_1 + T_2 + T_3 + \ldots + T_n = \cot\beta[\tan(\alpha + n\beta) - \tan\alpha] - n$$

$$= \frac{\frac{\sin(\alpha + n\beta)}{\cos(\alpha + n\beta)} + \frac{\sin\alpha}{\cos\alpha}}{\tan\beta} - n = \frac{\frac{\sin(\alpha + n\beta - \alpha)}{\cos\alpha\cos(\alpha + n\beta)}}{\tan\beta} - n$$

$$= \frac{\frac{\sin n\beta}{\cos(\alpha + n\beta)\cos\alpha} - n\tan\beta}{\tan\beta}$$

112. $S_n = \sum_{r=1}^{n} T_r = \frac{n(n+1)(n+2)(n+3)}{8}$

$$T_r = S_r - S_{r-1} = \frac{r(r+1)(r+2)(r+3)}{8} - \frac{(r-1)r(r+1)(r+2)}{8}$$

$$= \frac{r(r+1)(r+2)}{2}$$

$$\frac{1}{T_r} = \frac{2}{r(r+1)(r+2)} = \frac{(r+2)-r}{r(r+1)(r+2)} = \left(\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}\right)$$

$$\sum_{r=1}^{n}\frac{1}{T_r} = \sum_{r=1}^{n}\left(\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}\right)$$

$$= \sum_{r=1}^{n}\left\{\left(\frac{1}{r} - \frac{1}{r+1}\right) - \left(\frac{1}{r+1} - \frac{1}{r+2}\right)\right\}$$

$$= \left(\frac{1}{1} - \frac{1}{n+1}\right) - \left(\frac{1}{2} - \frac{1}{n+2}\right)$$

$$= \frac{1}{2} + \frac{1}{n+2} - \frac{1}{n+1} = \frac{n(n+3)}{2(n+1)(n+2)}$$

113. Let d_1, d_2 and d_3 be the common differences of the 3 arithmetic progressions.

$\therefore \quad S_i = \frac{n}{2}[2 \times a + (n-1)d_i], \forall\ i = 1, 2, 3$

$\Rightarrow \quad S_i = \frac{n}{2}[2 + (n-1)d_i]$

$\Rightarrow \quad S_i = n + \frac{n(n-1)}{2}d_i \ \Rightarrow\ d_i = \frac{2(S_i - n)}{n(n-1)}$

Given that d_1, d_2, d_3 are in HP.

$\therefore \quad \frac{1}{d_1}, \frac{1}{d_2}, \frac{1}{d_3}$ are in AP.

$\therefore \quad \frac{2}{d_2} = \frac{1}{d_1} + \frac{1}{d_3}$

$\Rightarrow \quad \frac{2}{\frac{2(S_2 - n)}{n(n-1)}} = \frac{1}{\frac{2(S_1 - n)}{n(n-1)}} + \frac{1}{\frac{2(S_3 - n)}{n(n-1)}}$

$\Rightarrow \quad \frac{2}{S_2 - n} = \frac{1}{S_1 - n} + \frac{1}{S_3 - n}$

$\Rightarrow \quad \frac{2}{S_2 - n} = \frac{S_3 + S_1 - 2n}{(S_1 - n)(S_3 - n)}$

$\Rightarrow 2[S_1S_3 - (S_1 + S_3)n + n^2] = (S_2 - n)(S_1 + S_3 - 2n)$

$\Rightarrow 2S_1S_3 - 2(S_1 + S_3)n + 2n^2$

$= S_1S_2 + S_2S_3 - 2nS_2 - n(S_1 + S_3) + 2n^2$

$\Rightarrow \ 2S_1S_3 - S_2S_3 - S_1S_2 = n(S_1 + S_3 - 2S_2)$

$\Rightarrow \quad n = \frac{(2S_1S_3 - S_2S_3 - S_1S_2)}{(S_1 - 2S_2 + S_3)}$

114. Let their ages be a, ar, ar^2.

After 3 yr, their ages will be $a + 3$, $ar + 3$, $ar^2 + 3$.

Given, $2(a + 3) = ar^2 + 3$ ...(i)

Let x rupees be the sum of the money divided.

And let $y = a + ar + ar^2$ ...(ii)

Then, $y + 9 = a + 3 + (ar + 3) + (ar^2 + 3)$

We have, $\frac{x(a+3)}{(y+9)} = \frac{xa}{y} + 105$

$\Rightarrow \quad x\left(\frac{a+3}{y+9} - \frac{a}{y}\right) = 105$ ...(iii)

Also, $\frac{x(ar+3)}{(y+9)} = \frac{xar}{y} + 15$

$\Rightarrow \quad x\left[\frac{ar+3}{y+9} - \frac{ar}{y}\right] = 15$ ...(iv)

On dividing Eq. (iii) by Eq. (iv), we get

$$\frac{y(a+3) - a(y+9)}{y(ar+3) - ar(y+9)} = 7 \ \Rightarrow\ \frac{y - 3a}{y - 3ar} = 7$$

$\Rightarrow \quad 6y = 21ar - 3a \ \Rightarrow\ y = \frac{a(7r-1)}{2}$

From Eq. (ii),

$$\frac{a(7r-1)}{2} = a + ar + ar^2$$

$\Rightarrow \quad 5ar = 3a + 2ar^2$ ...(v)

From Eqs. (i) and (ii),

$$a = 12, r = \frac{3}{2}$$

Let ages of these friends are $12, 12 \times \frac{3}{2}, 12 \times \left(\frac{3}{2}\right)^2$ i.e. 12, 18, 27.

115. Clearly, $x = \frac{1}{1-a}$, $y = \frac{1}{1-b}$ and $z = \frac{1}{1-c}$.

Since, a, b, c are in AP.

$\Rightarrow \quad 1 - a, 1 - b, 1 - c$ are also in AP.

$\Rightarrow \quad \frac{1}{1-a}, \frac{1}{1-b}, \frac{1}{1-c}$ are in HP.

$\therefore x, y, z$ are in HP.

116. $\because\ B_n = 1 - A_n > A_n \Rightarrow A_n < \frac{1}{2}$

Now, $A_n = \dfrac{\frac{3}{4}\left(1 - \left(-\frac{3}{4}\right)^n\right)}{1 + \frac{3}{4}} < \frac{1}{2} \Rightarrow \left(-\frac{3}{4}\right)^n > -\frac{1}{6}$

Obviously, it is true for all even values of n.

But for $n = 1, -\frac{3}{4} < -\frac{1}{6}$

$n = 3\left(-\frac{3}{4}\right)^3 = -\frac{27}{24} < -\frac{1}{6}$

$n = 5, \left(-\frac{3}{4}\right)^5 = -\frac{243}{1024} > -\frac{1}{6}$

which is true for $n = 7$ obviously, $n_0 = 7$

Aliter $B_n = 1 - A_n > A_n$

$\Rightarrow \quad A_n < \frac{1}{2} \Rightarrow \frac{3}{4}\dfrac{\left(1 - \left(-\frac{3}{4}\right)^n\right)}{1 + \frac{3}{4}} < \frac{1}{2} \Rightarrow \left(-\frac{3}{4}\right)^n > -\frac{1}{6}$

Obviously, it is true for all even values of n.

But for $n = 1, -\frac{3}{4} < -\frac{1}{6}$

$n = 3, \left(-\frac{3}{4}\right)^3 = -\frac{27}{64} < -\frac{1}{6}$

$n = 5, \left(-\frac{3}{4}\right)^5 = -\frac{243}{1024} < -\frac{1}{6}$

and for $n = 7 \Rightarrow \left(-\frac{3}{4}\right)^7 = -\frac{2187}{12288} > -\frac{1}{6}$

Hence, minimum natural number $n_0 = 7$.

117. $\because\ \dfrac{\frac{p}{2}[2a_1 + (p-1)d]}{\frac{q}{2}[2a_1 + (q-1)d]} = \frac{p^2}{q^2}$

$\Rightarrow \dfrac{2a_1 + (p-1)d}{2a_1 + (q-1)d} = \frac{p}{q} \Rightarrow \dfrac{a_1 + \left(\frac{p-1}{2}\right)d}{a_1 + \left(\frac{q-1}{2}\right)d} = \frac{p}{q}$

For $\frac{a_6}{a_{21}}$, $p = 11$ and $q = 41 \Rightarrow \frac{a_6}{a_{21}} = \frac{11}{41}$

118. $\frac{1}{a_2} - \frac{1}{a_1} = \frac{1}{a_3} - \frac{1}{a_2} = \ldots = \frac{1}{a_n} - \frac{1}{a_{n-1}} = d$ [say]

Then, $a_1a_2 = \dfrac{a_1 - a_2}{d}, a_2a_3 = \dfrac{a_2 - a_3}{d}, \ldots, a_{n-1}a_n = \dfrac{a_{n-1} - a_n}{d}$

$\therefore\ a_1a_2 + a_2a_3 + \ldots + a_{n-1}a_n = \dfrac{a_1 - a_n}{d}$

Also, $\dfrac{1}{a_n} = \dfrac{1}{a_1} + (n-1)d \Rightarrow \dfrac{a_1 - a_n}{d} = (n-1)a_1a_n$

$\therefore\ a_1a_2 + a_2a_3 + \ldots + a_{n-1}a_n = (n-1)a_1a_n$

119. (i) $V_r = \dfrac{r}{2}[(2r + (r-1)(2r-1)] = \frac{1}{2}(2r^3 - r^2 + r)$

$\therefore\ \sum_{r=1}^{n} V_r = \frac{1}{2}\left[2\sum_{r=1}^{n} r^3 - \sum_{r=1}^{n} r^3 + \sum_{r=1}^{n} r\right]$

$= \frac{1}{2}\left[2\left(\frac{n(n+1)}{2}\right)^2 - \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right]$

$= \frac{1}{12}n(n+1)(3n^2 + n + 2)$

(ii) $V_{r+1} - V_r = (r+1)^3 - r^3 - \frac{1}{2}[(r+1)^2 - r^2] + \frac{1}{2}(1)$

$= 3r^2 + 2r + 1$

$\therefore\quad T_r = 3r^2 + 2r - 1$

$= (r+1)(3r-1)$, which is a composite number.

(iii) Since, $T_r = 3r^2 + 2r - 1$

$\therefore\quad T_{r+1} = 3(r+1)^2 + 2(r+1) - 1$

$\therefore\quad Q_r = T_{r+1} - T_r = 3[2r+1] + 2[1]$

$\Rightarrow\quad Q_r = 6r + 5$

$\Rightarrow\quad Q_{r+1} = 6(r+1) + 5$

Common difference $= Q_{r+1} - Q_r = 6$

120. (i) $A_1 = \dfrac{a+b}{2}; G_1 = \sqrt{ab}; H_1 = \dfrac{2ab}{a+b}$

$A_n = \dfrac{A_{n-1} + H_{n-1}}{2}$, $G_n = \sqrt{A_{n-1}H_{n-1}}$ and $H_n = \dfrac{2A_{n-1}H_{n-1}}{A_{n-1} + H_{n-1}}$

Clearly, $G_1 = G_2 = G_3 = \ldots = \sqrt{ab}$

(ii) A_2 is AM of A_1 and H_1 and $A_1 > H_1$

$\Rightarrow\quad A_1 > A_2 > H_1$

A_3 is AM of A_2 and H_2 and $A_2 > H_2$

$\Rightarrow\quad A_2 > A_3 > H_2$

$\therefore\quad A_1 > A_2 > A_3 > \ldots$

(iii) As above $A_1 > H_2 > H_1$, $A_1 > H_3 > H_2$

$\therefore\quad H_1 < H_2 < H_3 < \ldots$

121. Let geometric progression is $a, ar, ar^2, \ldots,$ $[a, r > 0]$

$\because\quad a = ar + ar^2$

$\Rightarrow\quad r^2 + r - 1 = 0 \Rightarrow r = \dfrac{-1 \pm \sqrt{5}}{2}$

$\therefore\quad r = \dfrac{\sqrt{5} - 1}{2}$

122. $b_1 = a_1, b_2 = b_1 + a_2 = a_1 + a_2, b_3 = b_2 + a_3 = a_1 + a_2 + a_3$

and $b_4 = b_3 + a_4 = a_1 + a_2 + a_3 + a_4$

Hence, b_1, b_2, b_3, b_4 are neither in AP nor in GP and nor in HP

123. Let $a, ar, ar^2, \ldots$

$a + ar = 12$ ...(i)

and $ar^2 + ar^3 = 48$ ...(ii)

On dividing Eq. (ii) by Eq. (i), we get

$r^2 = 4$, if $r \neq -1$

$\therefore\quad r = -2$

[$\because$ terms are alternatively positive and negative]

Now, from Eq. (i), $a = -12$

124. $\because\quad S_n = cn^2$

$\therefore\quad t_n = S_n - S_{n-1} = c(2n-1)$

$\Sigma t_n^2 = c^2\Sigma(2n-1)^2$

$= c^2\Sigma(4n^2 - 4n + 1) = c^2\{4\Sigma n^2 - 4\Sigma n + \Sigma 1\}$

$= c^2\left\{\dfrac{4n(n+1)(2n+1)}{6} - \dfrac{4n(n+1)}{2} + n\right\}$

$$= c^2 n\left\{\frac{2}{3}(2n^2+3n+1) - 2n - 2 + 1\right\}$$

$$= \frac{c^2 n}{3}(4n^2-1) = \frac{n(4n^2-1)c^2}{3}$$

125. Let $S = 1 + \frac{2}{3} + \frac{6}{3^2} + \frac{10}{3^3} + \frac{14}{3^4} + \ldots$ ...(i)

$\therefore \frac{1}{3}S = \quad \frac{1}{3} + \frac{2}{3^2} + \frac{6}{3^3} + \frac{10}{3^4} + \ldots$ ...(ii)

On subtracting Eq. (ii) from Eq. (i), we get

$$\frac{2}{3}S = 1 + \frac{1}{3} + \frac{4}{3^2} + \frac{4}{3^3} + \frac{4}{3^4} + \ldots$$

$$= \frac{4}{3} + 4\left(\frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \ldots\right) = \frac{4}{3} + 4\left\{\frac{\frac{1}{3^2}}{1-\frac{1}{3}}\right\} = \frac{4}{3} + \frac{2}{3} = 2$$

$\therefore \quad S = 3$

126. $S_k = \frac{a}{1-r} = \frac{\frac{k-1}{k!}}{1-\frac{1}{k}} = \frac{k}{k!} = \frac{1}{(k-1)!}$

Now, $\sum_{k=2}^{100} |(k^2-3k+1)S_k| = \sum_{k=2}^{100} \left|(k^2-3k+1)\cdot\frac{1}{(k-1)!}\right|$

$$= \sum_{k=2}^{100}\left|\frac{(k-1)}{(k-2)} = \frac{k}{(k-1)!}\right|$$

$$= \left|\frac{1}{0!} - \frac{2}{1!}\right| + \left|\frac{2}{1!} - \frac{3}{2!}\right| + \left|\frac{3}{2!} - \frac{4}{3!}\right| + \ldots + \left|\frac{99}{98!} - \frac{100}{99!}\right|$$

$$= \left(\frac{2}{1!} - \frac{1}{0!}\right) + \left(\frac{2}{1!} - \frac{3}{2!}\right) + \left(\frac{3}{2!} - \frac{4}{3!}\right) + \ldots + \left(\frac{99}{98!} - \frac{100}{99!}\right)$$

$$= 3 - \frac{100}{99!} = 3 - \frac{(100)^2}{100!}$$

$\therefore \frac{(100)^2}{100!} + \sum_{k=2}^{100} |(k^2-3k+1)S_k| = 3$

127. $\because \quad a_k = 2a_{k-1} - a_{k-2}$ or $a_{k-1} = \frac{a_{k-2} + a_k}{2}$

$\therefore \quad a_1, a_2, a_3, \ldots$ are in AP.

$\because \quad \frac{a_1^2 + a_2^2 + a_3^2 + \ldots + a_{11}^2}{11} = 90 \Rightarrow \sum_{k=1}^{11} a_i^2 = 11 \times 90$

$$\Rightarrow \quad \sum_{k=1}^{11}(a_1 + (i-1)d)^2 = 11 \times 90$$

$$\Rightarrow \quad \sum_{k=1}^{11}\{a_1^2 + 2a_1 d(i-1) + d^2(i-1)^2\} = 11 \times 90$$

$$\Rightarrow \quad 11 \times a_1^2 + 2a_1 d(0+1+2+3+\ldots+10) + d^2(0^2+1^2+2^2+\ldots+10^2) = 11 \times 90$$

$$\Rightarrow \quad 11 \times 15^2 + 2 \times 15 \times d \cdot \left(\frac{10 \cdot 11}{2}\right) + d^2 \cdot \left(\frac{10 \cdot 11 \cdot 21}{6}\right) = 11 \times 90$$

$$\Rightarrow \quad 385d^2 + 1650d + 1485 = 0 \qquad [\because a_1 = 15]$$

$$\Rightarrow \quad 7d^2 + 30d + 27 = 0$$

$$\Rightarrow \quad (7d+9)(d+3) = 0$$

$$\therefore \quad d = -3, d \neq -\frac{9}{7} \qquad [\because 27 - 2a_2 > 0]$$

$$\therefore \quad \frac{a_1 + a_2 + a_3 + \ldots + a_{11}}{11} = \frac{\frac{11}{2}\{2a_1 + (11-1)d\}}{11}$$

$$= a_1 + 5d = 15 - 15 = 0$$

128. Till 10th minute, number of counted notes = 1500

$$\therefore 3000 = \frac{n}{2}\{2 \times 148 + (n-1) \times -2\} = n(148 - n + 1)$$

$$\Rightarrow \quad n^2 - 149n + 3000 = 0$$

$$\Rightarrow \quad (n-125)(n-24) = 0$$

$\therefore \quad n = 125, 24$

$n = 125$ is not possible.

$\therefore \quad n = 24$

$\therefore$ Total time = 10 + 24 = 34 min

129. $\because$ AM $\geq$ GM

$$\therefore \quad \frac{a^{-5} + a^{-4} + a^{-3} + a^{-3} + a^{-3} + 1 + a^8 + a^{10}}{8} \geq (a^{-5} \cdot a^{-4} \cdot a^{-3} \cdot a^{-3} \cdot a^{-3} \cdot 1 \cdot a^8 \cdot a^{10})^{1/8} = (1)^{1/8} = 1$$

$$\Rightarrow \quad \frac{a^{-5} + a^{-4} + 3a^{-3} + 1 + a^8 + a^{10}}{8} \geq 1$$

$$\therefore \quad a^{-5} + a^{-4} + 3a^{-3} + 1 + a^8 + a^{10} \geq 8$$

$\Rightarrow$ Required minimum value = 8

130. Let the time taken to save ₹ 11040 be $(n+3)$ months.

For first 3 months, he saves ₹ 200 each month.

$\therefore$ In $(n+3)$ month,

$$3 \times 200 + \frac{n}{2}[2(240) + (n-1) \times 40] = 11040$$

$$\Rightarrow \quad 600 + \frac{n}{2}[40(12 + n - 1)] = 11040$$

$$\Rightarrow \quad 600 + 20n(n+11) = 11040$$

$$\Rightarrow \quad n^2 + 11n - 522 = 0$$

$$\Rightarrow \quad (n-18)(n+29) = 0$$

$\therefore \quad n = 18$, neglecting $n = -29$

$\therefore$ Total time = $n + 3 = 21$ months

131. Given, $a_2 + a_4 + a_6 + \ldots + a_{200} = \alpha$ ...(i)

and $a_1 + a_3 + a_5 + \ldots + a_{199} = \beta$ ...(ii)

On subtracting Eq. (ii) from Eq. (i), we get

$$(a_2 - a_1) + (a_4 - a_3) + (a_6 - a_5) + \ldots + (a_{200} - a_{199}) = \alpha - \beta$$

$$\Rightarrow \quad d + d + d + \ldots + d = \alpha - \beta \Rightarrow 100d = \alpha - \beta$$

$$\therefore \quad d = \frac{(\alpha - \beta)}{100}$$

132. $\because a_1, a_2, a_3, \ldots$ are in HP.

$\therefore \quad \frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots$ are in AP.

Let D be the common difference of this AP, then

$$\frac{1}{a_{20}} = \frac{1}{a_1} + (20-1)D$$

$$\Rightarrow \quad D = \frac{\frac{1}{25} - \frac{1}{5}}{19} = -\frac{4}{25 \times 19}$$

and $\frac{1}{a_n} = \frac{1}{a_1} + (n-1)\,D = \frac{1}{5} - \frac{4(n-1)}{25\times 19} = \left(\frac{95-4n+4}{25\times 19}\right)$

$= \left(\frac{99-4n}{25\times 19}\right) < 0 \qquad [\because a_n < 0]$

$\Rightarrow \quad 99 - 4n < 0 \quad \Rightarrow \quad n > 24.75$

Hence, the least positive integer is $n = 25$.

133. $\because \quad (1) = (1-0)(1^2 + 1\cdot 0 + 0^2) = 1^3 - 0^3$

$(1+2+4) = (2-1)(2^2 + 2\cdot 1 + 1^2) = 2^3 - 1^3$

$(4+6+9) = (3-2)(3^2 + 3\cdot 2 + 2^2) = 3^3 - 2^3$

$\vdots \qquad \vdots \qquad \vdots$

$(361 + 380 + 400) = (20-19)(20^2 + 20\cdot 19 + 19^2) = 20^3 - 19^3$

Required sum

$= (1^3 - 0^3) + (2^3 - 1^3) + (3^3 - 2^3) + \ldots + (20^2 - 19^3) = 20^3 = 8000$

Also, $\sum_{k=1}^{n} k^3 - (k-1)^3 = \sum_{k=1}^{n} \{k-(k-1)\}\{k^2 + k(k-1) + (k-1)^2\}$

$= \sum_{k=1}^{n} (3k^2 - 3k + 1) = 3\,\Sigma n^2 - 3\Sigma n + \Sigma 1$

$= \frac{3n(n+1)(2n+1)}{6} - \frac{3n(n+1)}{2} + n$

$= \frac{n}{2}(2n^2 + 3n + 1 - 3n - 3 + 2) = n^3$

Both statements are correct and Statement-2 is the correct explanation of Statement-1.

134. Let a be the first term and d be the common difference. Then,

$100\,T_{100} = 50\,T_{50}$

$\Rightarrow \quad 100\,(a + 99d) = 50\,(a + 49d)$

$\Rightarrow \quad 2\,(a + 99d) = (a + 49d) \quad \Rightarrow \quad a + 149\,d = 0$

$\therefore \quad T_{150} = 0$

135. $\because$ x, y, z are in AP.

Let $x = y - d,\ z = y + d$ ...(i)

Also, given $\tan^{-1} x, \tan^{-1} y, \tan^{-1} z$ are in AP.

$\therefore \quad 2\tan^{-1} y = \tan^{-1} x + \tan^{-1} z$

$\Rightarrow \quad \tan^{-1}\left(\frac{2y}{1-y^2}\right) = \tan^{-1}\left(\frac{x+z}{1-xz}\right)$

$\Rightarrow \quad \frac{2y}{1-y^2} = \frac{x+z}{1-xz} \Rightarrow \frac{2y}{1-y^2} = \frac{2y}{1-(y^2-d^2)}$

$\Rightarrow \quad y^2 = y^2 - d^2 \qquad$ [from Eq. (i)]

$\therefore \quad d = 0$

From Eq. (i), $x = y$ and $z = y$

$\therefore \quad x = y = z$

Aliter

$\because \quad x, y, z$ are in AP. ...(i)

$\therefore \quad 2y = x + z$ ...(ii)

Also, $\tan^{-1} x, \tan^{-1} y, \tan^{-1} z$ are in AP.

$\therefore \quad 2\tan^{-1} y = \tan^{-1} x + \tan^{-1} z$

$\Rightarrow \quad \tan^{-1}\left(\frac{2y}{1-y^2}\right) = \tan^{-1}\left(\frac{x+z}{1-xz}\right)$

$\Rightarrow \quad \frac{2y}{1-y^2} = \frac{x+z}{1-xz} = \frac{2y}{1-xz} \qquad$ [from Eq. (ii)]

$\Rightarrow \quad y^2 = zx$

$\therefore\ x, y, z$ are in GP. ...(iii)

From Eqs. (i) and (ii) x, y, z are in AP and also in GP, then $x = y = z$.

136. $S = 0.7 + 0.77 + 0.777 + \ldots$ upto 20 terms

$= \frac{7}{9}(0.9 + 0.99 + 0.999 + \ldots \text{ upto 20 terms})$

$= \frac{7}{9}[(1-0.1) + (1-0.01) + (1-0.001) + \ldots \text{ upto 20 terms}]$

$= \frac{7}{9}\left[(1+1+1+\ldots \text{ upto 20 times}) - \left(\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \ldots \text{ upto 20 terms}\right)\right]$

$= \frac{7}{9}\left[20 - \frac{\frac{1}{10}\left(1 - \left(\frac{1}{10}\right)^{20}\right)}{1 - \frac{1}{10}}\right] = \frac{7}{9}\left[\frac{180 - 1 + 10^{-20}}{9}\right]$

$= \frac{7}{81}(179 + 10^{-20})$

137. $S_n = -1^2 - 2^2 + 3^2 + 4^2 - 5^2 - 6^2 + 7^2 + 8^2 - \ldots + (4n-1)^2 + (4n)^2$

$= (3^2 - 1^2) + (4^2 - 2^2) + (7^2 - 5^2) + (8^2 - 6^2) + \ldots + [\{(4n-1)^2 - (4n-3)^2\} + \{(4n)^2 - (4n-2)^2\}]$

$= 4\,[2 + 3 + 6 + 7 + 10 + 11 + \ldots + (4n-2) + (4n-1)]$

$= 8\,\{(1 + 3 + 5 + \ldots + (2n-1)\} + 4\,\{3 + 7 + 11 + \ldots + (4n-1)\}$

$= 16n^2 + 4n = 4n(4n+1),\ n \in N$

Satisfied by (a) and (d), where $n = 8, 9$, respectively.

138. Let two consecutive numbers are k and $k+1$ such that $1 \le k \le n-1$, then

$(1 + 2 + 3 + \ldots + n) - (k + k + 1) = 1224$

$\Rightarrow \quad \frac{n(n+1)}{2} - (2k+1) = 1224 \text{ or } k = \frac{n^2 + n - 2450}{4}$

Now, $\quad 1 \le \frac{n^2 + n - 2450}{4} \le n - 1 \Rightarrow 49 < n < 51$

$\therefore \quad n = 50 \Rightarrow k = 25$

Hence, $\quad k - 20 = 25 - 20 = 5$

139. The given series can be written as

$k = 1 + 2\left(\frac{11}{10}\right) + 3\left(\frac{11}{10}\right)^2 + \ldots + 9\left(\frac{11}{10}\right)^8 + 10\left(\frac{11}{10}\right)^9$ ...(i)

On multiplying both sides by $\left(\frac{11}{10}\right)$, then

$\frac{11k}{10} = \left(\frac{11}{10}\right) + 2\left(\frac{11}{10}\right)^2 + 3\left(\frac{11}{10}\right)^3 + \ldots + 9\left(\frac{11}{10}\right)^9 + 10\left(\frac{11}{10}\right)^{10}$ ...(ii)

Now, on subtracting Eq. (ii) from Eq. (i), then

$-\frac{k}{10} = \underbrace{1 + \left(\frac{11}{10}\right) + \left(\frac{11}{10}\right)^2 + \ldots + \left(\frac{11}{10}\right)^2}_{10 \text{ times}} - 10\left(\frac{11}{10}\right)^{10}$

$= \frac{1\cdot\left\{\left(\frac{11}{10}\right)^{10} - 1\right\}}{\left(\frac{11}{10} - 1\right)} - 10\left(\frac{11}{10}\right)^{10}$

$$\Rightarrow \quad k = -100 \cdot \left\{ \left(\frac{11}{10}\right)^{10} - 1 \right\} + 100 \left(\frac{11}{10}\right)^{10} = 100$$

140. Let a, ar, ar^2 are in GP. $\because$ GP is increasing.

$\therefore \quad r > 1$

New numbers $a, 2ar, ar^2$ are in AP.

$\therefore \quad 4ar = a + ar^2 \Rightarrow r^2 - 4r + 1 = 0$

Hence, $r = \dfrac{4 \pm \sqrt{(16-4)}}{2} = 2 + \sqrt{3}$ $[\because r > 1]$

141. Let $\dfrac{b}{a} = \dfrac{c}{b} = r$

$\therefore \quad b = ar, c = ar^2$

Given, $\dfrac{a+b+c}{3} = b + 2 \Rightarrow 1 + \dfrac{b}{a} + \dfrac{c}{a} = 3\left(\dfrac{b}{a}\right) + \dfrac{6}{a}$

$\Rightarrow \quad 1 + r + r^2 = 3r + \dfrac{6}{a}$

Now, for $a = 6$, only we get $r = 0, 2$ [rational]

So, $r = 2$

$\Rightarrow \quad (a, b, c) = (6, 12, 24)$

$\therefore \quad \dfrac{a^2 + a - 14}{a+1} = \dfrac{36 + 6 - 14}{6+1} = 4$

142. $T_n = \dfrac{1^3 + 2^3 + 3^3 + \ldots + n^3}{1 + 3 + 5 + \ldots + (2n-1)} = \dfrac{\left(\dfrac{n(n+1)}{2}\right)^2}{\dfrac{n(1+2n-1)}{2}} = \dfrac{(n+1)^2}{4}$

$= \dfrac{1}{4}(n^2 + 2n + 1)$

$\therefore S_n = \dfrac{1}{4}(\Sigma n^2 + 2\Sigma n + \Sigma 1) = \dfrac{1}{4}\left[\dfrac{n(n+1)(2n+1)}{6} + \dfrac{2n(n+1)}{2} + n\right]$

$S_9 = \dfrac{1}{4}[285 + 90 + 9] = 96$

143. Given, $m = \dfrac{l+n}{2} \Rightarrow l + n = 2m$ ...(i)

and l, G_1, G_2, G_3, n are in GP.

$\therefore \quad \dfrac{G_1}{l} = \dfrac{G_2}{G_1} = \dfrac{G_3}{G_2} = \dfrac{n}{G_3}$

$\Rightarrow \quad G_1G_3 = \ln, G_1^2 = lG_2, G_2^2 = G_3G_1, G_3^2 = nG_2$ ...(ii)

Now, $G_1^4 + 2G_2^4 + G_3^4 = l^2G_2^2 + 2G_2^4 + n^2G_2^2$

$= G_2^2(l^2 + 2G_2^2 + n^2)$ [from Eq. (ii)]

$= G_3G_1(l^2 + 2G_3G_1 + n^2)$ [from Eq. (ii)]

$= \ln(l^2 + 2\ln + n^2)$ [from Eq. (ii)]

$= \ln(l+n)^2 = \ln(2m)^2$ [from Eq. (i)] $= 4lm^2n$

144. Let first term $= a$ and common difference $= d$

$\because \quad \dfrac{\text{sum of seven terms}}{\text{sum of eleven terms}} = \dfrac{6}{11}$

$\Rightarrow \quad \dfrac{\frac{7}{2}(a_1 + a_7)}{\frac{11}{2}(a_1 + a_{11})} = \dfrac{6}{11} \Rightarrow \dfrac{\frac{7}{2}(2a + 6d)}{\frac{11}{2}(2a + 10d)} = \dfrac{6}{11}$

or $a = 9d$ and $130 < a_7 < 140$

$\Rightarrow \quad 130 < a_1 + 6d < 140 \Rightarrow 130 < 15d < 140$

$\therefore \quad 8\dfrac{2}{3} < d < 9\dfrac{1}{3} \Rightarrow d = 9$ $(\because a, d \in N)$

145. $\because$ $a + d, a + 4d, a + 8d$ are in GP $(d \neq 0)$

$\therefore \quad (a + 4d)^2 = (a + d)(a + 8d)$

$\Rightarrow \quad a = 8d$

$\therefore$ Common ratio $= \dfrac{a + 4d}{a + d} = \dfrac{8d + 4d}{8d + d} = \dfrac{4}{3}$ $(\because a = 8d)$

Aliter

Let the GP be a, ar, ar^2 and terms of AP and $A + d, A + 4d, A + 8d$, then

$$r = \frac{ar^2 - ar}{ar - a} = \frac{(A + 8d) - (A + 4d)}{(A + 4d) - (A + d)} = \frac{4}{3}$$

146. $\left(\dfrac{8}{5}\right)^2 + \left(\dfrac{12}{5}\right)^2 + \left(\dfrac{16}{5}\right)^2 + \left(\dfrac{20}{5}\right)^2 + \ldots + \left(\dfrac{44}{5}\right)^2$

$= \dfrac{16}{25}(2^2 + 3^2 + 4^2 + 5^2 + \ldots 11^2)$

$= \dfrac{16}{25}\left(\dfrac{11 \cdot (11+1) \cdot (22+1)}{6} - 1\right)$

$= \dfrac{16}{25} \times 505 = \dfrac{16}{5} \times 101 = \dfrac{16}{5}$ m (given)

$\therefore m = 101$

147. $\because$ $\log_e b_1, \log_e b_2, \log_e b_3, \ldots, \log_e b_{101}$ are in AP.

$\Rightarrow$ $b_1, b_2, b_3, \ldots, b_{101}$ are in GP with common ratio 2.

$(\because$ common difference $= \log_e 2)$

Also, $a_1, a_2, a_3, \ldots a_{101}$ are in AP.

where, $a_1 = b_1$ and $a_{51} = b_{51}$

$\therefore$ $b_2, b_3, \ldots, b_{50}$ are GM's and $a_2, a_3, \ldots, a_{50}$ are AM's between b_1 and b_{51}.

$\because \quad \text{GM} < \text{AM}$

$\Rightarrow \quad b_2 < a_2, b_3 < a_3, \ldots, b_{50} < a_{50}$

$\therefore \quad b_1 + b_2 + b_3 + \ldots + b_{51} < a_1 + a_2 + a_3 + \ldots + a_{51}$

$\Rightarrow \quad t < s$

Also, $a_1, a_2, a_3, \ldots, a_{101}$ are in AP and $b_1, b_2, b_3, \ldots. b_{101}$ are in GP.

$\because \quad a_1 = b_1$ and $a_{51} = b_{51}$

$\therefore \quad b_{101} > a_{101}$

148. $(15a)^2 + (3b)^2 + (5c)^2 - 45ab - 15bc - 75ac = 0$

$\Rightarrow \quad \dfrac{1}{2}\{(15a - 3b)^2 + (3b - 5c)^2 + (5c - 15a)^2\} = 0$

$\Rightarrow \quad (15a - 3b)^2 + (3b - 5c)^2 + (5c - 15a)^2 = 0$

or $15a - 3b = 0, 3b - 5c = 0, 5c - 15a = 0$

$\because \quad b = 5a, c = 3a$

$\Rightarrow$ $5a, 3a, a$ are in AP i.e. b, c, a are in AP.

CHAPTER

04

Logarithms and Their Properties

Learning Part

Session 1

- Definition
- Characteristic and Mantissa

Session 2

- Principle Properties of Logarithm

Session 3

- Properties of Monotonocity of Logarithm
- Graphs of Logarithmic Functions

Practice Part

- JEE Type Examples
- Chapter Exercises

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The technique of logarithms was introduced by **John Napier** (1550-1617). The logarithm is a form of indices which is used to simplify the algebraic calculations. The operations of multiplication, division of a very large number becomes quite easy and get converted into simple operations of addition and subtraction, respectively. The results obtained are correct upto some decimal places.

Session 1

Definition, Characteristic and Mantissa

Definition

The logarithm of any positive number, whose base is a number (>0) different from 1, is the index or the power to which the base must be raised in order to obtain the given number.

i.e. if $a^x = b$ (where $a > 0, \neq 1$), then x is called the logarithm of b to the base a and we write $\log_a b = x$, clearly $b > 0$. Thus, $\log_a b = x \Leftrightarrow a^x = b, a > 0, a \neq 1$ and $b > 0$.

If $a = 10$, then we write log b rather than $\log_{10} b$. If $a = e$, we write ln b rather than $\log_e b$. Here, *'e'* is called as **Napier's base** and has numerical value equal to 2.7182. Also, $\log_{10} e$ is known as **Napierian constant**.

i.e. $\log_{10} e = 0.4343$

$\therefore$ $\ln b = 2.303 \log_{10} b$

$$\left[\text{since, } \ln b = \log_{10} b \times \log_e 10 = \frac{1}{\log_{10} e} \times \log_{10} b = \frac{1}{0.4343} \log_{10} b = 2.303 \log_{10} b\right]$$

Remember

(i) $\log 2 = \log_{10} 2 = 0.3010$

(ii) $\log 3 = \log_{10} 3 = 0.4771$

(iii) $\ln 2 = 2.303 \log 2 = 0.693$

(iv) $\ln 10 = 2.303$

***Corollary* I** From the definition of the logarithm of the number b to the base a, we have an identity

$$a^{\log_a b} = b, a > 0, a \neq 1 \text{ and } b > 0$$

which is known as the **Fundamental Logarithmic Identity**.

***Corollary* II** The function defined by $f(x) = \log_a x, a > 0, a \neq 1$ is called logarithmic function. Its domain is $(0, \infty)$ and range is R (set of all real numbers).

***Corollary* III** $a^x > 0, \forall\, x \in R$

(i) If $a > 1$, then a^x is monotonically increasing.

For example, $5^{2.7} > 5^{2.5}, 3^{222} > 3^{111}$

(ii) If $0 < a < 1$, then a^x is monotonically decreasing.

For example, $\left(\frac{1}{5}\right)^{2.7} < \left(\frac{1}{5}\right)^{2.5}, (0.7)^{222} < (0.7)^{212}$

***Corollary* IV**

(i) If $a > 1$, then $a^{-\infty} = 0$

$\therefore$ $\log_a 0 = -\infty$ (if $a > 1$)

(ii) If $0 < a < 1$, then $a^{\infty} = 0$

$\therefore$ $\log_a 0 = +\infty$ (if $0 < a < 1$)

***Corollary* V** (i) $\log_a b \to \infty$, if $a > 1, b \to \infty$

(ii) $\log_a b \to -\infty$, if $0 < a < 1, b \to \infty$

Remark

1. 'log' is the abbreviation of the word 'logarithm'.
2. **Common logarithm** (Brigg's logarithms) The base is 10.
3. If $x < 0, a > 0$ and $a \neq 1$, then $\log_a x$ is an imaginary.
4. If $a > 1$, $\log_a x = \begin{cases} +\text{ve}, & x > 1 \\ 0, & x = 1 \\ -\text{ve}, & 0 < x < 1 \end{cases}$

   And if $0 < a < 1, \log_a x = \begin{cases} +\text{ve}, & 0 < x < 1 \\ 0, & x = 1 \\ -\text{ve}, & x > 1 \end{cases}$
5. $\log_a 1 = 0$ $(a > 0, a \neq 1)$

   $\log_a a = 1$ $(a > 0, a \neq 1)$ and $\log_{(1/a)} a = -1$ $(a > 0, a \neq 1)$

Example 1. Find the value of the following :

(i) $\log_9 27$ (ii) $\log_{3\sqrt{2}} 324$

(iii) $\log_{1/9}(27\sqrt{3})$ (iv) $\log_{(5+2\sqrt{6})}(5-2\sqrt{6})$

(v) $\log_{0.2} 0.008$ (vi) $2^{2\log_4 5}$

(vii) $(0.4)^{-\log_{2.5}\left\{\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots\right\}}$ (viii) $(0.05)^{\log_{\sqrt{20}}(0.\bar{3})}$

Sol. (i) Let $x = \log_9 27$

$\Rightarrow \quad 9^x = 27 \Rightarrow 3^{2x} = 3^3 \Rightarrow 2x = 3$

$\therefore \quad x = \frac{3}{2}$

(ii) Let $x = \log_{3\sqrt{2}} 324$

$\Rightarrow (3\sqrt{2})^x = 324 = 2^2 \cdot 3^4 \Rightarrow (3\sqrt{2})^x = (3\sqrt{2})^4$

$\therefore \quad x = 4$

(iii) Let $x = \log_{1/9}(27\sqrt{3})$

$\Rightarrow \quad \left(\frac{1}{9}\right)^x = 27\sqrt{3} \Rightarrow 3^{-2x} = 3^{7/2} \Rightarrow -2x = 7/2$

$\therefore \quad x = -\frac{7}{4}$

(iv) $\because (5+2\sqrt{6})(5-2\sqrt{6}) = 1$

or $5+2\sqrt{6} = \frac{1}{5-2\sqrt{6}}$ ...(i)

Now, let $x = \log_{(5+2\sqrt{6})}(5-2\sqrt{6})$

$= \log_{1/(5-2\sqrt{6})} 5-2\sqrt{6} = -1$ [from Eq. (i)]

(v) Let $x = \log_{0.2} 0.008$

$\Rightarrow \quad (0.2)^x = 0.008 \Rightarrow (0.2)^x = (0.2)^3 \Rightarrow x = 3$

(vi) Let $x = 2^{2\log_4 5} = 4^{\log_4 5} = 5$

(vii) Let $x = (0.4)^{-\log_{2.5}\left\{\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots\right\}}$

$= \left(\frac{4}{10}\right)^{-\log_{2.5}\left\{\frac{\frac{1}{3}}{1-\frac{1}{3}}\right\}} = \left(\frac{2}{5}\right)^{-\log_{2.5}\left(\frac{1}{2}\right)} = \left(\frac{5}{2}\right)^{\log_{5/2}\left(\frac{1}{2}\right)} = \frac{1}{2}$

(viii) Let $x = (0.05)^{\log_{\sqrt{20}}(0.\bar{3})} = (0.05)^{\log_{\sqrt{20}}(\lambda)}$ ...(i)

where, $\lambda = 0.\bar{3}$

Then, $\lambda = 0.33333\ldots$ ...(ii)

$\Rightarrow \quad 10\lambda = 3.33333\ldots$ ...(iii)

On subtracting Eq. (ii) from Eq. (iii), we get

$9\lambda = 3 \Rightarrow \lambda = \frac{1}{3}$

Now, from Eq. (i), $x = (0.05)^{\log_{\sqrt{20}}\left(\frac{1}{3}\right)}$

$= \left(\frac{1}{20}\right)^{\log_{(20)^{1/2}(3)^{-1}}} = \left(\frac{1}{20}\right)^{-\frac{1}{1/2}\log_{20} 3}$

$= 20^{(2\log_{20} 3)} = 20^{\log_{20} 3^2} = 3^2 = 9$

Example 2. ***Find the value of the following:***

(i) $\log_{\tan 45^\circ}\cot 30^\circ$ (ii) $\log_{(\sec^2 60^\circ - \tan^2 60^\circ)}\cos 60^\circ$

(iii) $\log_{(\sin^2 30^\circ + \cos^2 30^\circ)} 1$ (iv) $\log_{30} 1$

Sol. (i) Here, base $= \tan 45^\circ = 1$ tan

$\therefore$ log is not defined.

(ii) Here, base $= \sec^2 60^\circ - \tan^2 60^\circ = 1$

$\therefore$ log is not defined.

(iii) $\because \log_{(\sin^2 30^\circ + \cos^2 30^\circ)} 1 = \log_1 1 \neq 1$

$\because$ Here, base $= 1$

$\therefore$ log is not defined.

(iv) $\log_{30} 1 = 0$

Characteristic and Mantissa

The integral part of a logarithm is called the **characteristic** and the fractional part (decimal part) is called **mantissa**.

i.e., $\log N =$ Integer + Fractional or decimal part (+ve)

$\downarrow \qquad \downarrow$

Characteristic Mantissa

The mantissa of log of a number is always kept positive.

i.e., if $\log 564 = 2.751279$, then 2 is the characteristic and 0.751279 is the mantissa of the given number 564.

And if $\log 0.00895 = -2.0481769$

$= -2 - 0.0481769$

$= (-2-1) + (1 - 0.0481769)$

$= -3 + 0.9518231$

Hence, -3 is the characteristic and 0.9518231 (not 0.0481769) is mantissa of log 0.00895.

In short, $-3 + 0.9518231$ is written as $\bar{3}.9518231$.

Remark

1. If $N > 1$, the characteristic of $\log N$ will be one less than the number of digits in the integral part of N.
   For example, If $\log 235.68 = 2.3723227$
   Here, $N = 235.68$
   $\therefore$ Number of digits in the integral part of $N = 3$
   $\Rightarrow$ Characteristic of $\log 235.68 = N - 1 = 3 - 1 = 2$
2. If $0 < N < 1$, the characteristic of $\log N$ is negative and numerically it is one greater than the number of zeroes immediately after the decimal part in N.
   For example, If $\log 0.0000279 = \bar{5}.4456042$
   Here, four zeroes immediately after the decimal point in the number 0.0000279 is $(\overline{4+1})$, i.e. $\bar{5}$.
3. If the characteristics of $\log N$ be n, then number of digits in N is $(n+1)$ (Here, $N > 1$).
4. If the characteristics of $\log N$ be $-n$, then there exists $(n-1)$ number of zeroes after decimal part of N (here, $0 < N < 1$).

Example 3. ***If*** $\log 2 = 0.301$ ***and*** $\log 3 = 0.477$, ***find the number of digits in*** 6^{20}.

Sol. Let $P = 6^{20} = (2 \times 3)^{20}$

$$\therefore \quad \log P = 20 \log(2 \times 3) = 20\{\log 2 + \log 3\}$$
$$= 20\{0.301 + 0.477\}$$
$$= 20 \times 0.778 = 15.560$$

Since, the characteristic of $\log P$ is 15, therefore the number of digits in P will be 15 + 1, i.e. 16.

Example 4. Find the number of zeroes between the decimal point and first significant digit of $(0.036)^{16}$, where $\log 2 = 0.301$ and $\log 3 = 0.477$.

Sol. Let $P = (0.036)^{16} \Rightarrow \log P = 16 \log (0.036)$

$$= 16 \log\left(\frac{36}{1000}\right) = 16 \log\left(\frac{2^2 \cdot 3^2}{1000}\right)$$
$$= 16\{\log 2^2 + \log 3^2 - \log 10^3\}$$
$$= 16\{2\log 2 + 2\log 3 - 3\}$$
$$= 16\{2 \times 0.301 + 2 \times 0.477 - 3\}$$
$$= 16\{1.556 - 3\} = 24.896 - 48$$
$$= -48 + 24 + 0.896$$
$$= -24 + 0.896 = \overline{24} + 0.896$$

$\therefore$ The required number of zeroes = 24 − 1 = 23.

Exercise for Session 1

1. The value of $\log_{2\sqrt{3}} 1728$ is
   (a) 6 (b) 8
   (c) 3 (d) 5

2. The value of $\log_{(8-3\sqrt{7})}(8 + 3\sqrt{7})$ is
   (a) –2 (b) –1
   (c) 0 (d) Not defined

3. The value of $(0.16)^{\log_{2.5}\left\{\frac{1}{3} + \frac{1}{3^2} + \ldots\right\}}$ is
   (a) 2 (b) 4
   (c) 6 (d) 8

4. If $\log 2 = 0.301$, the number of integers in the expansion of 4^{17} is
   (a) 9 (b) 11
   (c) 13 (d) 15

5. If $\log 2 = 0.301$, then the number of zeroes between the decimal point and the first significant figure of 2^{-34} is
   (a) 9 (b) 10
   (c) 11 (d) 12

Session 2

Principle Properties of Logarithm

Principle Properties of Logarithm

Let m and n be arbitrary positive numbers, $a > 0, a \neq 1, b > 0, b \neq 1$ and α, β be any real numbers, then

(i) $\log_a (mn) = \log_a m + \log_a n$

In general, $\log_a (x_1 x_2 x_3 \ldots x_n) = \log_a x_1 + \log_a x_2 + \log_a x_3 + \ldots + \log_a x_n$ (where, $x_1, x_2, x_3, \ldots, x_n > 0$)

Or

$$\log_a \left(\prod_{i=1}^{n} x_i \right) = \sum_{i=1}^{n} \log_a x_i, \forall x_i > 0$$

where, $i = 1, 2, 3, \ldots, n$.

(ii) $\log_a \left(\frac{m}{n} \right) = \log_a m - \log_a n$

(iii) $\log_a m^{\alpha} = \alpha \log_a m$

(iv) $\log_{a^{\beta}} m = \frac{1}{\beta} \log_a m$ (v) $\log_b m = \frac{\log_a m}{\log_a b}$

Remark

1. $\log_b a \cdot \log_a b = 1 \Leftrightarrow \log_b a = \frac{1}{\log_a b}$
2. $\log_b a \cdot \log_c b \cdot \log_a c = 1$
3. $\log_y x \cdot \log_z y \cdot \log_a z = \log_a x$
4. $e^{\ln a^x} = a^x$

Extra Properties of Logarithm

(i) $a^{\log_b x} = x^{\log_b a}, b \neq 1, a, b, x$ are positive numbers.

(ii) $a^{\log_a x} = x, a > 0, a \neq 1, x > 0$

(iii) $\log_{a^k} x = \frac{1}{k} \log_a x, \ a > 0, a \neq 1, x > 0$

(iv) $\log_a x^{2k} = 2k \log_a |x|, \ a > 0, a \neq 1, k \in I$

(v) $\log_{a^{2k}} x = \frac{1}{2k} \log_{|a|} x, \ x > 0, a > 0, a \neq \pm 1$ and $k \in I \sim \{0\}$

(vi) $\log_{a^{\alpha}} x^{\beta} = \frac{\beta}{\alpha} \log_a x, \ x > 0, a > 0, a \neq 1, \alpha \neq 0$

(vii) $\log_a x^2 \neq 2 \log_a x, \ a > 0, a \neq 1$

Since, domain of $\log_a (x^2)$ is $R \sim \{0\}$ and domain of $\log_a x$ is $(0, \infty)$ are not same.

(viii) $a^{\log_b a} = \sqrt{a}$, if $b = a^2, a > 0, b > 0, b \neq 1$

(ix) $a^{\log_b a} = a^2$, if $b = \sqrt{a}, a > 0, b > 0, b \neq 1$

Example 5. Solve the equation $3 \cdot x^{\log_5 2} + 2^{\log_5 x} = 64$.

Sol. $\because \quad 3 \cdot x^{\log_5 2} + 2^{\log_5 x} = 64$

$\Rightarrow \quad 3 \cdot 2^{\log_5 x} + 2^{\log_5 x} = 64$ [by extra property (i)]

$\Rightarrow \quad 4 \cdot 2^{\log_5 x} = 64$

$\Rightarrow \quad 2^{\log_5 x} = 4^2 = 2^4$

$\therefore \quad \log_5 x = 4$

$\Rightarrow \quad x = 5^4 = 625$

Example 6. If $4^{\log_{16} 4} + 9^{\log_3 9} = 10^{\log_x 83}$, find x.

Sol. $\because \quad 4^{\log_{16} 4} = \sqrt{4} = 2$ [by extra property (ix)]

and $\quad 9^{\log_3 9} = 9^2 = 81$ [by extra property (viii)]

$\therefore \quad 4^{\log_{16} 4} + 9^{\log_3 9} = 2 + 81 = 83 = 10^{\log_x 83}$

$\Rightarrow \quad \log_{10} 83 = \log_x 83$

$\therefore \quad x = 10$

Example 7. Prove that $a^{\sqrt{\log_a b}} - b^{\sqrt{\log_b a}} = 0$.

Sol. $\because \quad a^{\sqrt{(\log_a b)}} = a^{\sqrt{\log_a b} \times \sqrt{\log_a b} \times \sqrt{\log_b a}}$

$= a^{\log_a b \cdot \sqrt{(\log_b a)}}$

$= b^{\sqrt{\log_b a}}$ [by extra property (ii)]

Hence, $a^{\sqrt{(\log_a b)}} - b^{\sqrt{(\log_b a)}} = 0$

Example 8. ***Prove that*** $\frac{\log_2 24}{\log_{96} 2} - \frac{\log_2 192}{\log_{12} 2} = 3$.

Sol. LHS $= \frac{\log_2 24}{\log_{96} 2} - \frac{\log_2 192}{\log_{12} 2}$

$= \log_2 24 \times \log_2 96 - \log_2 192 \times \log_2 12$

Now, let $12 = \lambda$, then

LHS $= \log_2 2\lambda \times \log_2 8\lambda - \log_2 16\lambda \times \log_2 \lambda$

$$= (\log_2 2 + \log_2 \lambda)(\log_2 8 + \log_2 \lambda) - (\log_2 16 + \log_2 \lambda)\log_2 \lambda$$
$$= (\log_2 2 + \log_2 \lambda)(\log_2 2^3 + \log_2 \lambda) - (\log_2 2^4 + \log_2 \lambda)\log_2 \lambda$$
$$= (1 + \log_2 \lambda)(3\log_2 2 + \log_2 \lambda) - (4\log_2 2 + \log_2 \lambda)\log_2 \lambda$$
$$= (1 + \log_2 \lambda)(3 + \log_2 \lambda) - \log_2 \lambda(4 + \log_2 \lambda)$$
$$= 3$$
$$= \text{RHS}$$

Example 9. Solve for a, λ, if $\log_\lambda a \cdot \log_5 \lambda \cdot \log_\lambda 25 = 2$.

Sol. Here, $\lambda > 0, \lambda \neq 1$

We have, $\log_\lambda a \cdot \{\log_5 \lambda \cdot \log_\lambda 25\} = 2$

$\Rightarrow (\log_\lambda a)\{\log_5 25\} = 2$

$\Rightarrow (\log_\lambda a)\{\log_5 5^2\} = 2$

$\Rightarrow (\log_\lambda a)\{2\log_5 5\} = 2$

$\Rightarrow (\log_\lambda a)\{2\} = 2$

$\therefore \log_\lambda (a) = 1$ or $a = \lambda$

Exercise for Session 2

1 If $a = \log_{24} 12, b = \log_{48} 36$ and $c = \log_{36} 24$, $1 + abc$ is equal to

(a) $2ab$ (b) $2bc$ (c) $2ca$ (d) $ba + bc$

2 The value of $\log_4[\log_2\{\log_2(\log_3 81)\}]$ is equal to

(a) -1 (b) 0 (c) 1 (d) 2

3 $\log_2 \log_2 (\underbrace{\sqrt{\sqrt{\ldots\sqrt{2}}}}_{n \text{ times}})$ is equal to

(a) 0 (b) 1 (c) n (d) $-n$

4 If $a = \log_3 5, b = \log_{17} 25$, which one of the following is correct?

(a) $a < b$ (b) $a = b$ (c) $a > b$ (d) None of these

5 The value of $\log_{0.75} \log_2 \sqrt{\sqrt[-2]{(0.125)}}$ is equal to

(a) -1 (b) 0 (c) 1 (d) None of these

Session 3

Properties of Monotonocity of Logarithm, Graphs of Logarithmic Functions

Properties of Monotonocity of Logarithm

1. Constant Base

(i) $\log_a x > \log_a y \Leftrightarrow \begin{cases} x > y > 0, \text{ if } a > 1 \\ 0 < x < y, \text{ if } 0 < a < 1 \end{cases}$

(ii) $\log_a x < \log_a y \Leftrightarrow \begin{cases} 0 < x < y, \text{ if } a > 1 \\ x > y > 0, \text{ if } 0 < a < 1 \end{cases}$

(iii) $\log_a x > p \Leftrightarrow \begin{cases} x > a^p, \text{ if } a > 1 \\ 0 < x < a^p, \text{ if } 0 < a < 1 \end{cases}$

(iv) $\log_a x < p \Leftrightarrow \begin{cases} 0 < x < a^p, \text{ if } a > 1 \\ x > a^p, \text{ if } 0 < a < 1 \end{cases}$

2. Variable Base

(i) $\log_x a$ is defined, if $a > 0, x > 0, x \neq 1$.

(ii) If $a > 1$, then $\log_x a$ is monotonically decreasing in $(0, 1) \cup (1, \infty)$.

(iii) If $0 < a < 1$, then $\log_x a$ is monotonically increasing in $(0, 1) \cup (1, \infty)$.

Very Important Concepts

(i) If $a > 1, p > 1$, then $\log_a p > 0$

(ii) If $0 < a < 1, p > 1$, then $\log_a p < 0$

(iii) If $a > 1, 0 < p < 1$, then $\log_a p < 0$

(iv) If $p > a > 1$, then $\log_a p > 1$

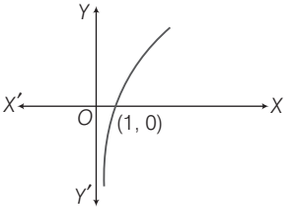
(v) If $a > p > 1$, then $0 < \log_a p < 1$

(vi) If $0 < a < p < 1$, then $0 < \log_a p < 1$

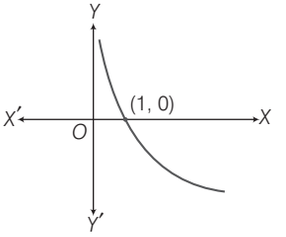
(vii) If $0 < p < a < 1$, then $\log_a p > 1$

Graphs of Logarithmic Functions

1. **Graph of $y = \log_a x$, if $a > 1$ and $x > 0$**



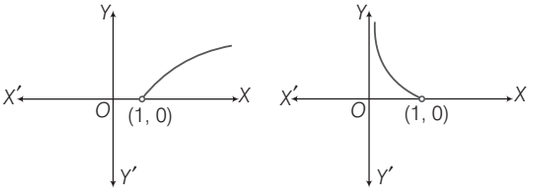
2. **Graph of $y = \log_a x$, if $0 < a < 1$ and $x > 0$**



Remark

1. If the number x and the base 'a' are on the same side of the unity, then the logarithm is positive.

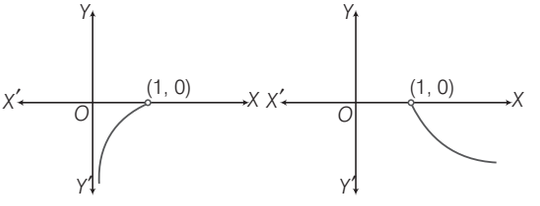
Case I $y = \log_a x, a > 1, x > 1$ ***Case II*** $y = \log_a x, 0 < a < 1, 0 < x < 1$



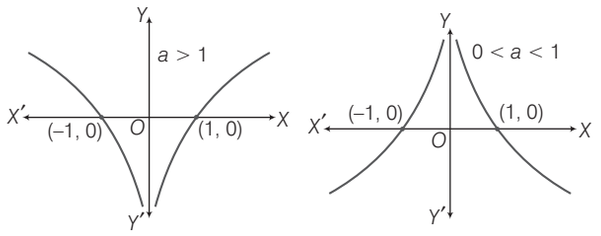
2. If the number x and the base a are on the opposite sides of the unity, then the logarithm is negative.

Case I $y = \log_a x, a > 1, 0 < x < 1$

Case II $y = \log_a x, 0 < a < 1, x > 1$



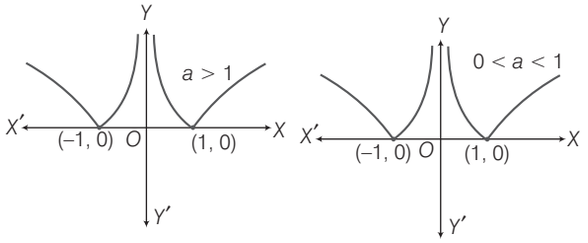
3. Graph of $y = \log_a |x|$



Remark

Graphs are symmetrical about *Y*-axis.

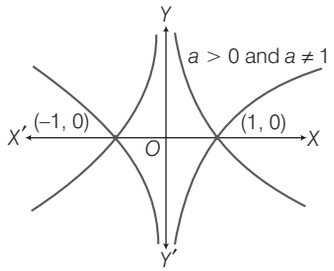
4. Graph of $y = |\log_a |x||$



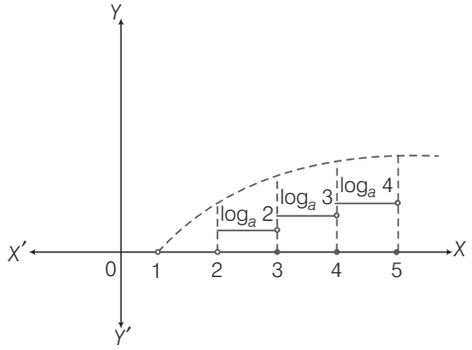
Remark

Graphs are same in both cases i.e., $a > 1$ and $0 < a < 1$.

5. **Graph of $|y| = \log_a |x||$**



6. **Graph of $y = \log_a [x]$, $a > 1$ and $x \geq 1$**
(where $[\cdot]$ denotes the greatest integer function)
Since, when $1 \leq x < 2, [x] = 1 \Rightarrow \log_a [x] = 0$
when $2 \leq x < 3, [x] = 2 \Rightarrow \log_a [x] = \log_a 2$
when $3 \leq x < 4, [x] = 3 \Rightarrow \log_a [x] = \log_a 3$ and so on.



Example 10. Arrange in ascending order $\log_2(x), \log_3(x), \log_e(x), \log_{10}(x)$, if
(i) $x > 1$ (ii) $0 < x < 1$.

Sol. $\because 2 < e < 3 < 10$

(i) For $x > 1$, $\log_x 2 < \log_x e < \log_x 3 < \log_x 10$

$$\Rightarrow \quad \frac{1}{\log_2(x)} < \frac{1}{\log_e(x)} < \frac{1}{\log_3(x)} < \frac{1}{\log_{10}(x)}$$

$$\Rightarrow \quad \log_2(x) > \log_e(x) > \log_3(x) > \log_{10}(x)$$

Hence, ascending order is

$$\log_{10}(x) < \log_3(x) < \log_e(x) < \log_2(x)$$

(ii) For $0 < x < 1$, $\log_x 2 > \log_x e > \log_x 3 > \log_x 10$

$$\Rightarrow \frac{1}{\log_2(x)} > \frac{1}{\log_e(x)} > \frac{1}{\log_3(x)} > \frac{1}{\log_{10}(x)}$$

$$\therefore \quad \log_2(x) < \log_e(x) < \log_3(x) < \log_{10}(x)$$

which is in ascending order.

Example 11. ***If*** $\log 11 = 1.0414$, ***prove that*** $10^{11} > 11^{10}$.

Sol. $\because \quad \log 10^{11} = 11 \log 10 = 11$

and $\log 11^{10} = 10 \log 11 = 10 \times 1.0414 = 10.414$

It is clear that, $11 > 10.414$

$$\Rightarrow \quad \log 10^{11} > \log 11^{10} \qquad [\because \text{here, base} = 10]$$

$$\Rightarrow \quad 10^{11} > 11^{10}$$

Example 12. ***If*** $\log_2(x-2) < \log_4(x-2)$, ***find the interval in which*** x ***lies.***

Sol. Here, $x - 2 > 0$

$$\Rightarrow \quad x > 2 \qquad \ldots(i)$$

and $\log_2(x-2) < \log_{2^2}(x-2) = \frac{1}{2}\log_2(x-2)$

$$\Rightarrow \quad \log_2(x-2) < \frac{1}{2}\log_2(x-2)$$

$$\Rightarrow \quad \frac{1}{2}\log_2(x-2) < 0 \;\Rightarrow\; \log_2(x-2) < 0$$

$$\Rightarrow \quad x - 2 < 2^0 \;\Rightarrow\; x - 2 < 1$$

$$\Rightarrow \quad x < 3 \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$2 < x < 3 \text{ or } x \in (2, 3)$$

Example 13. ***Prove that*** $\log_n(n+1) > \log_{(n+1)}(n+2)$ ***for any natural number*** $n > 1$.

Sol. Since, $\frac{n+1}{n} = 1 + \frac{1}{n} > 1 + \frac{1}{n+1} = \left(\frac{n+2}{n+1}\right)$

For $n > 1$,

$$\log_n\left(\frac{n+1}{n}\right) > \log_{n+1}\left(\frac{n+1}{n}\right) > \log_{n+1}\left(\frac{n+2}{n+1}\right)$$

$\Rightarrow \quad \log_n(n+1) - \log_n n > \log_{(n+1)}(n+2) - \log_{(n+1)}(n+1)$

$\Rightarrow \quad \log_n(n+1) - 1 > \log_{(n+1)}(n+2) - 1$

$\therefore \quad \log_n(n+1) > \log_{(n+1)}(n+2)$ **Hence proved.**

How to Find Minimum Value of

$\lambda_1 \log_a x + \lambda_2 \log_x a, a > 0, x > 0,$
$a \neq 1, x \neq 1$ and $\lambda_1, \lambda_2 \in R_+$

$\because \quad$ AM $\geq$ GM

$\Rightarrow \quad \dfrac{\lambda_1 \log_a x + \lambda_2 \log_x a}{2} \geq \sqrt{(\lambda_1 \log_a x)(\lambda_2 \log_x a)} = \sqrt{\lambda_1 \lambda_2}$

$\Rightarrow \quad \lambda_1 \log_a x + \lambda_2 \log_x a \geq 2\sqrt{\lambda_1 \lambda_2}$

Hence, the minimum value of $\lambda_1 \log_a x + \lambda_2 \log_x a$ is $2\sqrt{\lambda_1 \lambda_2}$.

Example 14. ***Find the least value of the expression*** $2\log_{10} x - \log_x 0.01$, ***where*** $x > 0, x \neq 1$.

Sol. Let $P = 2\log_{10} x - \log_x 0.01 = 2\log_{10} x - \log_x (10^{-2})$

$= 2(\log_{10} x + \log_x 10)$

$\geq 2 \cdot 2 = 4$ [by above article]

$\therefore \quad P \geq 4$

Hence, the least value of P is 4.

Example 15. ***Which is smaller 2 or*** $(\log_\pi 2 + \log_2 \pi)$***?***

Sol. Let $P = \log_\pi 2 + \log_2 \pi > 2$ [by above article] [$\because \pi \neq 2$]

$\therefore \quad P > 2$

$\Rightarrow \quad (\log_\pi 2 + \log_2 \pi) > 2$

Hence, the smaller number is 2.

Exercise for Session 3

1 If $\log_{0.16}(a+1) < \log_{0.4}(a+1)$, then a satisfies

(a) $a > 0$ (b) $0 < a < 1$ (c) $-1 < a < 0$ (d) None of these

2 The value of x satisfying the inequation $x^{\frac{1}{\log_{10} x}} \cdot \log_{10} x < 1$, is

(a) $0 < x < 10$ (b) $0 < x < 10^{10}$ (c) $0 < x < 10^{1/10}$ (d) None of these

3 If $\log_{\text{cosec}\, x} \sin x > 0$, then

(a) $x > 0$ (b) $x < 0$ (c) $-1 < x < 1$ (d) None of these

4 The value of $\log_{10} 3$ lies in the interval

(a) $\left(\frac{2}{5}, \frac{1}{2}\right)$ (b) $\left(0, \frac{1}{2}\right)$ (c) $\left(0, \frac{2}{5}\right)$ (d) None of these

5 The least value of n in order that the sum of first n terms of the infinite series $1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \ldots$, should differ from the sum of the series by less than 10^{-6}, is (given, $\log 2 = 0.30103, \log 3 = 0.47712$)

(a) 14 (b) 27 (c) 53 (d) 57

Shortcuts and Important Results to Remember

1 For a non-negative number 'a' and $n \geq 2, n \in N$, $\sqrt[n]{a} = a^{1/n}$.

2 The number of positive integers having base a and characteristic n is $a^{n+1} - a^n$.

3 Logarithm of zero and negative real number is not defined.

4 $|\log_b a + \log_a b| \geq 2, \forall\, a > 0, a \neq 1, b > 0, b \neq 1$.

5 $\log_2 \log_2 \underbrace{\sqrt{\sqrt{\sqrt{\sqrt{\ldots\sqrt{2}}}}}}_{n \text{ times}} = -n$

6 $a^{\sqrt{\log_a b}} = b^{\sqrt{\log_b a}}$

7 Logarithms to the base 10 are called common logarithms (Brigg's logarithms).

8 If $x = \log_c b + \log_b c$, $y = \log_a c + \log_c a$, $z = \log_a b + \log_b a$, then $x^2 + y^2 + z^2 - 4 = xyz$.

JEE Type Solved Examples :
Single Option Correct Type Questions

■ This section contains **8 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

● ***Ex. 1*** *The expression* $\log_2 5 - \sum_{k=1}^{4} \log_2\left(\sin\left(\frac{k\pi}{5}\right)\right)$ *reduces to* $\frac{p}{q}$*, where p and q are co-prime, the value of* $p^2 + q^2$ *is*

(a) 13 (b) 17 (c) 26 (d) 29

Sol. (*b*) Let $p = \log_2 5 - \sum_{k=1}^{4} \log_2\left(\sin\left(\frac{k\pi}{5}\right)\right)$

$$= \log_2 5 - \left\{\log_2\left(\sin\left(\frac{\pi}{5}\right)\right) + \log_2\left(\sin\left(\frac{2\pi}{5}\right)\right) + \log_2\left(\sin\left(\frac{3\pi}{5}\right)\right) + \log_2\left(\sin\left(\frac{4\pi}{5}\right)\right)\right\}$$

$$= \log_2 5 - \log_2\left\{\sin\frac{\pi}{5}\cdot\sin\frac{2\pi}{5}\cdot\sin\frac{3\pi}{5}\cdot\sin\frac{4\pi}{5}\right\}$$

$$= \log_2 5 - \log_2\left\{\sin^2\left(\frac{\pi}{5}\right)\cdot\sin^2\left(\frac{2\pi}{5}\right)\right\}$$

$$= \log_2 5 - \log_2\left\{\frac{(1-\cos 72^\circ)(1-\cos 144^\circ)}{4}\right\}$$

$$= \log_2 5 - \log_2\left\{\frac{(1-\sin 18^\circ)(1+\cos 36^\circ)}{4}\right\}$$

$$= \log_2 5 - \log_2\left\{\frac{\left(1-\frac{\sqrt{5}-1}{4}\right)\left(1+\frac{\sqrt{5}+1}{4}\right)}{4}\right\}$$

$$= \log_2 5 - \log_2\left\{\frac{(5-\sqrt{5})(5+\sqrt{5})}{64}\right\} = \log_2 5 - \log_2\left(\frac{5}{16}\right)$$

$$= \log_2\left(5\times\frac{16}{5}\right) = \log_2 2^4 = \frac{4}{1} = \frac{p}{q} \quad \text{[given]}$$

$\therefore \quad p = 4, q = 1$

Hence, $p^2 + q^2 = 4^2 + 1^2 = 17$

● ***Ex. 2*** *If* $3 \le a \le 2015, 3 \le b \le 2015$ *such that* $\log_a b + 6\log_b a = 5$*, the number of ordered pairs (a, b) of integers is*

(a) 48 (b) 50 (c) 52 (d) 54

Sol. (*c*) Let $x = \log_a b$ ...(i)

$$\Rightarrow \quad x + \frac{6}{x} = 5 \Rightarrow x^2 - 5x + 6 = 0 \Rightarrow x = 2, 3$$

From Eq. (i), we get $\log_a b = 2, 3$

$\Rightarrow \quad b = a^2$ or a^3

The pairs (*a*, *b*) are

$(3, 3^2), (4, 4^2), (5, 5^2), (6, 6^2), \ldots, (44, 44^2)$ and $(3, 3^3), (4, 4^3), (5, 5^3), \ldots, (12, 12^3)$.

Hence, there are $42 + 10 = 52$ pairs.

● ***Ex. 3*** *The lengths of the sides of a triangle are* $\log_{10} 12$, $\log_{10} 75$ *and* $\log_{10} n$*, where* $n \in N$*. If a and b are the least and greatest values of n respectively, the value of* $b - a$ *is divisible by*

(a) 221 (b) 222 (c) 223 (d) 224

Sol. (*c*) In a triangle,

$$\log_{10} 12 + \log_{10} 75 > \log_{10} n \Rightarrow n < 12 \times 75 = 900$$

$\therefore \quad n < 900$ ...(i)

and $\log_{10} 12 + \log_{10} n > \log_{10} 75$

$\Rightarrow \quad n > \frac{75}{12} = \frac{25}{4}$

$\therefore \quad n > \frac{25}{4}$ ...(ii)

From Eqs. (i) and (ii), we get $\frac{25}{4} < n < 900$

$\therefore \quad n = 7, 8, 9, 10, \ldots, 899$

Hence, $a = 7, b = 899$

$\therefore \quad b - a = 892 = 4 \times 223$

Hence, $b - a$ is divisible by 223.

● ***Ex. 4*** *If* $5\log_{abc}(a^3 + b^3 + c^3) = 3\lambda\left(\frac{1+\log_3(abc)}{\log_3(abc)}\right)$ *and* $(abc)^{a+b+c} = 1$ *and* $\lambda = \frac{m}{n}$*, where m and n are relative primes, the value of* $|m+n| + |m-n|$ *is*

(a) 8 (b) 10 (c) 12 (d) 14

Sol. (*b*) $\because \quad (abc)^{a+b+c} = 1 = (abc)^0$

$\therefore \quad a + b + c = 0 \Rightarrow a^3 + b^3 + c^3 = 3abc$

Now, LHS $= 5\log_{abc}(a^3 + b^3 + c^3) = 5\log_{abc}(3abc)$ ...(i)

and RHS $= 3\lambda\left(\frac{1+\log_3(abc)}{\log_3(abc)}\right) = 3\lambda\left(\frac{\log_3(3abc)}{\log_3(abc)}\right)$

$= 3\lambda\log_{abc}(3abc)$ ...(ii)

From Eqs. (i) and (ii), we get

$5\log_{abc}(3abc) = 3\lambda\log_{abc}(3abc)$

$\therefore \quad \lambda = \frac{5}{3} = \frac{m}{n}$ [given]

$\Rightarrow \quad m = 5, n = 3$

Hence, $|m+n| + |m-n| = 8 + 2 = 10$

● **Ex. 5** *If* $a^{\log_b c} = 3 \cdot 3^{\log_4 3} \cdot 3^{\log_4 3^{\log_4 3}} \cdot 3^{\log_4 3^{\log_4 3^{\log_4 3}}} \ldots \infty$, *where* $a, b, c \in Q$, *the value of abc is*

(a) 9 (b) 12 (c) 16 (d) 20

Sol. (*c*) $a^{\log_b c} = 3^{1+\log_4 3 + (\log_4 3)^2 + (\log_4 3)^3 + \ldots \infty}$

$= 3^{1/(1-\log_4 3)} = 3^{1/\log_4 (4/3)} = 3^{\log_{4/3} 4}$

$\therefore \quad a = 3, b = \frac{4}{3}, c = 4$

Hence, $abc = 3 \cdot \frac{4}{3} \cdot 4 = 16$

● **Ex. 6** *Number of real roots of equation*

$3^{\log_3 (x^2 - 4x + 3)} = (x - 3)$ *is*

(a) 0 (b) 1 (c) 2 (d) infinite

Sol. (*a*) $\because \quad 3^{\log_3 (x^2 - 4x + 3)} = (x - 3) \quad \ldots(i)$

Eq. (i) is defined, if $x^2 - 4x + 3 > 0$

$\Rightarrow \quad (x-1)(x-3) > 0$

$\Rightarrow \quad x < 1$ or $x > 3 \quad \ldots(ii)$

Eq. (i) reduces to $x^2 - 4x + 3 = x - 3 \Rightarrow x^2 - 5x + 6 = 0$

$\therefore \quad x = 2, 3 \quad \ldots(iii)$

From Eqs. (ii) and (iii), use get $x \in \phi$

$\therefore$ Number of real roots = 0

● **Ex. 7** *If* $\log_6 a + \log_6 b + \log_6 c = 6$, *where a, b, c* $\in N$ *and a, b, c are in GP and* $b - a$ *is a square of an integer, then the value of* $a + b - c$ *is*

(a) 21 (b) 15 (c) 9 (d) 3

Sol. (*b*) $\because \quad \log_6 a + \log_6 b + \log_6 c = 6$

$\Rightarrow \quad \log_6 (abc) = 6$

$\Rightarrow \quad abc = 6^6$

$\Rightarrow \quad b^3 = 6^6 \quad [\because b^2 = ac]$

$\Rightarrow \quad b = 36$

Also, $b - a = 36 - a$ is a square for $a = 35, 32, 27, 20, 11$

Now, $\quad c = \frac{b^2}{a} = \frac{36^2}{a}$ is an integer for $a = 27$

$\therefore \quad a = 27, b = 36, c = 48$

Hence, $a + b - c = 27 + 36 - 48 = 15$

● **Ex. 8** *If* $x = \log_{2a}\left(\frac{bcd}{2}\right), y = \log_{3b}\left(\frac{acd}{3}\right),$

$z = \log_{4c}\left(\frac{abd}{4}\right)$ *and* $w = \log_{5d}\left(\frac{abc}{5}\right)$ *and*

$\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} + \frac{1}{w+1} = \log_{abcd} N + 1$, *the value of N is*

(a) 40 (b) 80

(c) 120 (d) 160

Sol. (*c*) $\because \quad x = \log_{2a}\left(\frac{bcd}{2}\right)$

$\Rightarrow \quad x + 1 = \log_{2a}\left(\frac{2abcd}{2}\right) = \log_{2a}(abcd)$

$\therefore \quad \frac{1}{x+1} = \log_{abcd} 2a$

Similarly, $\frac{1}{y+1} = \log_{abcd} 3b, \frac{1}{z+1} = \log_{abcd} 4c$

and $\frac{1}{w+1} = \log_{abcd} 5d$

$\therefore \quad \frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} + \frac{1}{w+1} = \log_{abcd}(2a \cdot 3b \cdot 4c \cdot 5d)$

$= \log_{abcd}(120abcd)$

$= \log_{abcd} 120 + 1$

$= \log_{abcd} N + 1 \quad$ [given]

Hence, $\quad N = 120$

JEE Type Solved Examples :
More than One Correct Option Type Questions

▪ This section contains **4 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of **which more than one** may be correct.

● **Ex. 9** *The equation*

$(\log_{10} x + 2)^3 + (\log_{10} x - 1)^3 = (2\log_{10} x + 1)^3$ *has*

(a) no natural solution (b) two rational solutions

(c) no prime solution (d) one irrational solution

Sol. (*b, c, d*) Let $\log_{10} x + 2 = a$ and $\log_{10} x - 1 = b$

$\therefore \quad a + b = 2\log_{10} x + 1$, then given equation reduces to

$a^3 + b^3 = (a+b)^3$

$\Rightarrow \quad 3ab(a+b) = 0 \Rightarrow a = 0$ or $b = 0$ or $a + b = 0$

$\Rightarrow \quad \log_{10} x + 2 = 0$ or $\log_{10} x - 1 = 0$

or $\quad 2\log_{10} x + 1 = 0$

$\Rightarrow \quad x = 10^{-2}$ or $x = 10$ or $x = 10^{-1/2}$

Hence, $\quad x = \frac{1}{100}$ or $x = 10$ or $x = \frac{1}{\sqrt{10}}$

● **Ex. 10** *The value of* $\frac{\log_5 9 \cdot \log_7 5 \cdot \log_3 7}{\log_3 \sqrt{6}} + \frac{1}{\log_4 \sqrt{6}}$ *is co-prime with*

(a)1 (b) 3 (c) 4 (d) 5

Sol. (*a, b, d*) Let $P = \dfrac{\log_5 9 \cdot \log_7 5 \cdot \log_3 7}{\log_3 \sqrt{6}} + \dfrac{1}{\log_4 \sqrt{6}}$

$$= \frac{\log_3 9}{\log_3 \sqrt{6}} + \log_{\sqrt{6}} 4 = \log_{\sqrt{6}} 9 + \log_{\sqrt{6}} 4$$

$$= \log_{\sqrt{6}}(36) = \log_{\sqrt{6}}(\sqrt{6})^4 = 4 \implies P = 4$$

which is co-prime with 1, 3, 4 and 5.

● ***Ex. 11*** *Which of the following quantities are irrational for the quadratic equation*
$(\log_{10} 8)x^2 - (\log_{10} 5)x = 2(\log_2 10)^{-1} - x$ *?*

(a) Sum of roots (b) Product of roots
(c) Sum of coefficients (d) Discriminant

Sol. (*c, d*) $\because$ $(\log_{10} 8)x^2 - (\log_{10} 5)x = 2(\log_2 10)^{-1} - x$

$\Rightarrow$ $(3\log_{10} 2)x^2 + (1 - \log_{10} 5)x - 2\log_{10} 2 = 0$

$\Rightarrow$ $(3\log_{10} 2)x^2 + (\log_{10} 2)x - 2\log_{10} 2 = 0$

Now, Sum of roots $= -\dfrac{1}{3}$ = Rational

Product of roots $= -\dfrac{2}{3}$ = Rational

Sum of coefficients $= 3\log_{10} 2 + \log_{10} 2 - 2\log_{10} 2$
$= 2\log_{10} 2$ = Irrational

Discriminant $= (\log_{10} 2)^2 + 24(\log_{10} 2)^2$
$= 25\,(\log_{10} 2)^2$ = Irrational

● ***Ex. 12*** *The system of equations*
$\log_{10}(2000xy) - \log_{10} x \cdot \log_{10} y = 4$
$\log_{10}(2yz) - \log_{10} y \cdot \log_{10} z = 1$
and $\log_{10}(zx) - \log_{10} z \cdot \log_{10} x = 0$
has two solutions (x_1, y_1, z_1) *and* (x_2, y_2, z_2)*, then*

(a) $x_1 + x_2 = 101$ (b) $y_1 + y_2 = 25$
(c) $x_1 x_2 = 100$ (d) $z_1 z_2 = 100$

Sol. (*a, b, c, d*) Let $\log_{10} x = a$, $\log_{10} y = b$ and $\log_{10} z = c$
Then, given equations reduces to

$$a + b - ab = 4 - \log_{10} 2000 = \log_{10} 5 \quad \ldots(i)$$

$$b + c - bc = 1 - \log_{10} 2 = \log_{10} 5 \quad \ldots(ii)$$

and $c + a - ca = 0$ ...(iii)

From Eqs. (i) and (ii), we get

$$a + b - ab = b + c - bc$$

$\Rightarrow$ $(c - a) - b(c - a) = 0$

$\Rightarrow$ $(c - a)(1 - b) = 0$

$1 - b \neq 0, c - a = 0 \implies c = a$

From Eq. (iii), we get

$$2a - a^2 = 0 \implies a = 0, 2$$

Then, $c = a \implies c = 0, 2$

and $b = \log_{10} 5, 2 - \log_{10} 5$

$\therefore$ $\log_{10} x = 0, 2 \implies x = 10^0, 10^2$

$\Rightarrow$ $x = 1, 100$

$\Rightarrow$ $x_1 = 1,\ x_2 = 100$

and $\log_{10} y = \log_{10} 5, 2 - \log_{10} 5$
$= \log_{10} 5, \log_{10} 20$

$\Rightarrow$ $y = 5, 20$

$\Rightarrow$ $y_1 = 5, y_2 = 20$

and $\log_{10} z = 0, 2 \implies z = 10^0, 10^2$

$\Rightarrow$ $z = 1, 100$

$\Rightarrow$ $z_1 = 1, z_2 = 100$

Finally, $x_1 + x_2 = 1 + 100 = 101$, $y_1 + y_2 = 5 + 20 = 25$,
$x_1 x_2 = 1 \times 100 = 100$ and $z_1 z_2 = 1 \times 100 = 100$

JEE Type Solved Examples : Passage Based Questions

■ This section contains **2 solved passages** based upon each of the passage **3 multiple choice** examples have to be answered. Each of these examples has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I

(Ex. Nos. 13 to 15)

Suppose that $\log_{10}(x - 2) + \log_{10} y = 0$ and
$\sqrt{x} + \sqrt{(y - 2)} = \sqrt{(x + y)}$.

13. The value of x is
(a) $2 + \sqrt{2}$ (b) $1 + \sqrt{2}$ (c) $2\sqrt{2}$ (d) $4 - \sqrt{2}$

14. The value of y is
(a) 2 (b) $2\sqrt{2}$ (c) $1 + \sqrt{2}$ (d) $2 + 2\sqrt{2}$

15. If $x^{2t^2 - 6} + y^{6 - 2t^2} = 6$, the value of $t_1\, t_2\, t_3\, t_4$ is
(a) 1 (b) 2 (c) 4 (d) 8

Sol. (Ex. Nos. 13-15)

$\because$ $\log_{10}(x - 2) + \log_{10} y = 0$

$\therefore$ $x - 2 > 0, y > 0$

$\Rightarrow$ $x > 2, y > 0$ ...(i)

and $\log_{10}\{(x - 2)y\} = 0$

$\Rightarrow$ $(x - 2)y = 10^0 = 1$

$\therefore$ $(x - 2)y = 1$ ...(ii)

Also, given that $\sqrt{x} + \sqrt{(y - 2)} = \sqrt{(x + y)}$

$\therefore$ $x \geq 0, y - 2 \geq 0, x + y \geq 0$

$\Rightarrow \quad x \geq 0, y \geq 2$ ...(iii)

On squaring both sides, we get

$$x + y - 2 + 2\sqrt{x}\sqrt{(y-2)} = x + y$$

$\Rightarrow \quad \sqrt{x}\sqrt{y-2} = 1$

$\Rightarrow \quad x(y-2) = 1$ ...(iv)

From Eqs. (i) and (iii),we get

$$x > 2, y \geq 2$$

and from Eqs. (ii) and (iv), we get $y = x$

From Eq. (ii), $(x-2)x = 1$

$\Rightarrow \quad x^2 - 2x - 1 = 0$

$\therefore \quad x = \dfrac{2 \pm \sqrt{4+4}}{2}$ [neglect $-$ve sign, since $x > 2$]

13. (b) $x = (\sqrt{2}+1)$.

14. (c) $y = x = \sqrt{1+1}$

15. (d) $\because \quad x^{2t^2-6} + y^{6-2t^2} = 6$

$\Rightarrow \quad x^{2t^2-6} + (x^{-1})^{2t^2-6} = 6$

$\Rightarrow \quad (x^2)^{t^2-3} + (x^{-2})^{t^2-3} = 6$

$\Rightarrow \quad (3+2\sqrt{2})^{t^2-3} + (3-2\sqrt{2})^{t^2-3} = 6$

Now, we get $t^2 - 3 = \pm 1$

$\Rightarrow \quad t^2 = 4, 2$

$\therefore \quad t = \pm 2, \pm\sqrt{2}$

$\therefore \quad t_1 t_2 t_3 t_4 = (2)(-2)(\sqrt{2})(-\sqrt{2}) = 8$

Passage II

(Ex. Nos. 16 to 18)

If $10^{\log_p\{\log_q(\log_r x)\}} = 1$ and $\log_q\{\log_r(\log_p x)\} = 0$.

16. The value of x is

(a) q^r (b) r^q (c) r^p (d) rq

Sol. (b) $\because \quad 10^{\log_p[\log_q(\log_r x)]} = 1 = 10^0$

$\Rightarrow \quad \log_p\{\log_q(\log_r x)\} = 0$

$\Rightarrow \quad \log_q(\log_r x) = 1 \Rightarrow \log_r x = q$

$\Rightarrow \quad x = r^q$ ...(i)

and $\log_q\{\log_r(\log_p x)\} = 0$

$\Rightarrow \quad \log_r(\log_p x) = 1 \Rightarrow \log_p x = r$

$\therefore \quad x = p^r$ ...(ii)

From Eqs. (i) and (ii), we get $x = r^q = p^r$

17. The value of p is

(a) $r^{q/r}$ (b) rq (c) 1 (d) $r^{r/q}$

Sol. (a) $\because \quad r^q = p^r$ ...(iii)

$\Rightarrow \quad p = r^{q/r}$

18. The value of q is

(a) $r^{p/r}$ (b) $p\log_p r$ (c) $r\log_r p$ (d) $r^{r/p}$

Sol. (c) From Eq. (iii),

$$q\log r = r\log p \Rightarrow q = r\left(\frac{\log p}{\log r}\right) = r\log_r p$$

JEE Type Solved Examples :
Single Integer Answer Type Questions

▪ This section contains **2 examples.** The answer to each example is a **single digit integer** ranging from **0** to **9** (both inclusive).

● ***Ex. 19*** *If x_1 and x_2 are the solutions of the equation $x^{\log_{10} x} = 100x$ such that $x_1 > 1$ and $x_2 < 1$, the value of $\dfrac{x_1 x_2}{2}$ is*

Sol. (5) $\because \quad x^{\log_{10} x} = 100x$

Taking logarithm on both sides on base 10, then we get

$$\log_{10} x \cdot \log_{10} x = \log_{10} 100 + \log_{10} x$$

$\Rightarrow \quad (\log_{10} x)^2 - \log_{10} x - 2 = 0$

$\Rightarrow \quad (\log_{10} x - 2)(\log_{10} x + 1) = 0$

$\therefore \quad \log_{10} x = 2, -1 \Rightarrow x = 10^2, 10^{-1}$

$\therefore \quad x_1 = 100, x_2 = \dfrac{1}{10}$

$\therefore \quad \dfrac{x_1 x_2}{2} = 5$

● ***Ex. 20*** *If $(31.6)^a = (0.0000316)^b = 100$, the value of $\dfrac{1}{a} - \dfrac{1}{b}$ is*

Sol. (3) $\because \quad (31.6)^a = (0.0000316)^b = 100$

$\Rightarrow \quad a\log_{10}(31.6) = b\log_{10}(0.0000316) = \log_{10}100$

$\Rightarrow \quad a\log_{10}(31.6) = b\log_{10}(31.6 \times 10^{-6}) = 2$

$\Rightarrow \quad a\log_{10}(31.6) = b\log_{10}(31.6) - 6b = 2$

$\Rightarrow \quad \dfrac{2}{a} = \log_{10}(31.6)$

and $\dfrac{2}{b} = \log_{10}(31.6) - 6$

$\therefore \quad \dfrac{2}{a} - \dfrac{2}{b} = 6$

$\Rightarrow \quad \dfrac{1}{a} - \dfrac{1}{b} = 3$

JEE Type Solved Examples : Matching Type Questions

■ This section contains **2 examples.** Examples 24 and 25 have three statements (A, B and C) given in Column I and four statements (p, q, r and s) in **Column II.** Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II.**

● Ex. 21

	Column I		Column II
(A)	If x_1 and x_2 satisfy the equation $(x+1)^{\log_{10}(x+1)} = 100(x+1)$, then the value of $(x_1+1)(x_2+1)+5$ is	(p)	irrational
(B)	The product of all values of x which make the following statement true $(\log_3 x)(\log_5 9) - \log_x 25 + \log_3 2 = \log_3 54$, is	(q) (r)	rational prime
(C)	If $\log_b a = -3$, $\log_b c = 4$ and if the value of x satisfying the equation $a^{3x} = c^{x-1}$ is expressed in the form p/q, where p and q are relatively prime, then q is	(s)	composite
		(t)	twin prime

Sol. A $\to$ (q, s, t), B $\to$(p), C $\to$ (q, r)

(A) $(x+1)^{\log_{10}(x+1)} = 100(x+1)$

Taking logarithm on both sides on base 10, then we get

$\log_{10}(x+1)\cdot\log_{10}(x+1) = \log_{10}100 + \log_{10}(x+1)$

$\Rightarrow \quad \{\log_{10}(x+1)\}^2 = 2 + \log_{10}(x+1)$

$\Rightarrow \quad \{\log_{10}(x+1)\}^2 - \log_{10}(x+1) - 2 = 0$

$\Rightarrow \quad \{\log_{10}(x+1) - 2\}\{\log_{10}(x+1) + 1\} = 0$

$\therefore \quad \log_{10}(x+1) = 2, -1$

$\Rightarrow \quad (x+1) = 10^2, 10^{-1}$

$\therefore \quad (x_1+1)(x_2+1) = 10^2 \times 10^{-1} = 10$

$\Rightarrow \quad (x_1+1)(x_2+1)+5 = 10+5$

$= 15 = 3\times 5$

(B) $\because (\log_3 x)(\log_5 9) - \log_x 25 + \log_3 2 = \log_3 54$

$\Rightarrow \quad 2\log_5 x - 2\log_x 5 = \log_3 54 - \log_3 2$

$= \log_3(27) = 3$

Let $\log_5 x = \lambda$, then

$$2\lambda - \frac{2}{\lambda} = 3$$

$\Rightarrow \quad 2\lambda^2 - 3\lambda - 2 = 0$

$\Rightarrow \quad 2\lambda^2 - 4\lambda + \lambda - 2 = 0$

$2\lambda(\lambda-2) + 1(\lambda-2) = 0 \Rightarrow \lambda = -\frac{1}{2}, 2$

$\therefore \quad \log_5 x = -\frac{1}{2}, 2$

$\Rightarrow \quad x = 5^{-1/2}, 5^2$ or $x = \frac{1}{\sqrt{5}}, 25$

$\therefore$ Product of the values of $x = \frac{1}{\sqrt{5}} \times 25 = 5\sqrt{5}$

(C) $\because \quad \log_b a = -3$ and $\log_b c = 4$

$\therefore \quad \log_c a = -\frac{3}{4}$ ...(i)

and $a^{3x} = c^{x-1}$

$\Rightarrow \quad 3x\log a = (x-1)\log c$

$\Rightarrow \quad 3x\log_c a = x-1$

$\Rightarrow \quad 3x \times -\frac{3}{4} = x-1$ [from Eq. (i)]

$\Rightarrow \quad -9x = 4x - 4$ or $x = \frac{4}{13}$

$\therefore \quad q = 13$ [prime and rational]

● Ex. 22

	Column I		Column II
(A)	If α and β are the roots of $ax^2 + bx + c = 0$, where $a = 2^{\log_2 3} - 3^{\log_3 2}$, $b = 1 + 2^{\sqrt{\log_2 3}} - 3^{\sqrt{\log_3 2}}$ and $c = \log_2 \log_2 \sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{2}}}}}}$, then HM of α and β is	(p) (q)	divisible by 2 divisible by 4
(B)	The sum of the solutions of the equation	(r)	divisible by 6
	$\lvert x-1\rvert^{\log_2 x^2 - 2\log_x 4} = (x-1)^7$ is	(s)	divisible by 8
(C)	If $5(\log_y x + \log_x y) = 26$, $xy = 64$, then the value of $\lvert x-y\rvert$ is	(t)	divisible by 10

Sol. A $\to$ (p, q, r), B $\to$ (p, r), C $\to$ (p, r, t)

(A) $\because \quad a = 3-2 = 1, b = 1, c = \log_2 \log_2 2^{2^{-6}}$

$= \log_2(2^{-6}) = -6$

The equation reduces to $x^2 + x - 6 = 0$

$\therefore \quad \alpha + \beta = -1, \ \alpha\beta = -6$

$\therefore \quad \text{HM} = \frac{2\alpha\beta}{\alpha+\beta} = \frac{2(-6)}{(-1)} = 12$

(B) Obviously, $x = 2$ is a solution. Since, LHS is positive, $x - 1 > 0$. The equation reduces to

$\log_2 x^2 - 2\log_x 4 = 7$

$\Rightarrow \quad 2\lambda - \frac{4}{\lambda} = 7$, where $\lambda = \log_2 x$

$\Rightarrow \quad 2\lambda^2 - 7\lambda - 4 = 0 \Rightarrow \lambda = 4, -\frac{1}{2}$

$\therefore \quad \log_2 x = 4, -\frac{1}{2} \Rightarrow x = 2^4, 2^{-1/2}$

$\Rightarrow \quad x = 16, \frac{1}{\sqrt{2}}$

$\Rightarrow \quad x = 16, x \neq \frac{1}{\sqrt{2}} \quad [\because x > 1]$

$\therefore$ Solutions are $x = 2, 16$

$\therefore$ Sum of solutions $= 2 + 16 = 18$

(C) If $\alpha = \log x, \beta = \log y$

$\therefore \quad \log_y x + \log_x y = \frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

$\therefore \quad 5(\log_y x + \log_x y) = 26$

$\Rightarrow \quad \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{26}{5}$

Let $\frac{\alpha}{\beta} = \lambda$, then $\lambda + \frac{1}{\lambda} = \frac{26}{5}$

$\Rightarrow \quad 5\lambda^2 - 26\lambda + 5 = 0$

$\Rightarrow \quad 5\lambda^2 - 25\lambda - \lambda + 5 = 0$

$\Rightarrow \quad (\lambda - 5)(5\lambda - 1) = 0$

$\Rightarrow \quad \lambda = 5, \frac{1}{5}$

$\therefore \quad \frac{\alpha}{\beta} = 5, \frac{1}{5} \Rightarrow \frac{\alpha}{\beta} = 5$

$\Rightarrow \quad \alpha = 5\beta$ ...(i)

and $\alpha + \beta = \log x + \log y = \log(xy) = \log(64)$

$\therefore \quad \alpha + \beta = 6\log 2$ ...(ii)

From Eqs. (i) and (ii), we get

$\beta = \log 2$ and $\alpha = 5\log 2$

$\Rightarrow \quad y = 2, x = 32$ or $y = 32, x = 2$

$\therefore \quad |x - y| = 30$

JEE Type Solved Examples :
Statement I and II Type Questions

■ **Directions** Example numbers 23 to 24 are Assertion-Reason type examples. Each of these examples contains two statements:
Statement-1 (Assertion) and **Statement-2** (Reason)
Each of these examples also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement-1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

● ***Ex. 23*** **Statement-1** *If* $N = \left(\frac{1}{0.4}\right)^{20}$, *then N contains 7 digits before decimal.*

Statement-2 *Characteristic of the logarithm of N to the base* 10 *is* 7.

Sol. $(d) \because \quad N = \left(\frac{1}{0.4}\right)^{20} = \left(\frac{10}{2^2}\right)^{20}$

$\Rightarrow \log_{10} N = 20(1 - 2\log_{10} 2) = 20(1 - 2 \times 0.3010)$
$= 20 \times 0.3980 = 7.9660$

Since, characteristic of $\log_{10} N$ is 7, therefore the number of digits in N will be $7 + 1$, i.e. 8.

Hence, Statement-1 is false and Statement-2 is true.

● ***Ex. 24*** **Statement-1** *If* $p, q \in N$ *satisfy the equation* $x^{\sqrt{x}} = (\sqrt{x})^x$ *and* $q > p$, *then q is a perfect number.*

Statement-2 *If a number is equal to the sum of its factor, then number is known as perfect number.*

Sol. $(d) \because \quad x^{\sqrt{x}} = (\sqrt{x})^x$

Taking logarithm on both sides on base e, then

$\ln(x)^{\sqrt{x}} = \ln(\sqrt{x})^x$

$\Rightarrow \quad \sqrt{x} \ln x = x \ln\sqrt{x} \Rightarrow \sqrt{x} \ln x = \frac{x}{2} \ln x$

$\Rightarrow \quad \ln x\left(\sqrt{x} - \frac{x}{2}\right) = 0$

$\Rightarrow \quad \ln x \cdot \sqrt{x} \cdot \left(1 - \frac{\sqrt{x}}{2}\right) = 0$

$\Rightarrow \quad \ln x = 0, \sqrt{x} = 0, 1 - \frac{\sqrt{x}}{2} = 0$

$\therefore \quad x = 1, 0, 4$

$\because \quad x \in N$

$\therefore \quad x = 1, 4 \Rightarrow p = 1$ and $q = 4$

$\because \quad 4 = 1 \times 2 \times 2 \Rightarrow 4 \neq 1 + 2 + 2$

$\therefore q$ is not a perfect number.

Hence, Statement-1 is false and Statement-2 is true.

Subjective Type Questions

■ In this section, there are **21 subjective** solved examples.

● ***Ex. 25*** *Prove that* $\log_3 5$ *is an irrational.*

Sol. Let $\log_3 5$ is rational.

$\therefore \quad \log_3 5 = \frac{p}{q}$, where p and q are co-prime numbers.

$\Rightarrow \quad 5 = 3^{p/q} \Rightarrow 3^p = 5^q$

which is not possible, hence our assumption is wrong.
Hence, $\log_3 5$ is an irrational.

● ***Ex. 26*** *Find the value of the expression*

$$(\log 2)^3 + \log 8 \cdot \log 5 + (\log 5)^3.$$

Sol. $\because \quad \log 2 + \log 5 = \log(2 \cdot 5) = \log 10 = 1$ ...(i)

$\Rightarrow \quad (\log 2 + \log 5)^3 = 1$

$\Rightarrow \quad (\log 2)^3 + (\log 5)^3 + 3\log 2 \log 5(\log 2 + \log 5) = 1^3$

$\Rightarrow \quad (\log 2)^3 + (\log 5)^3 + \log 2^3 \log 5(1) = 1$ [from Eq. (i)]

$\Rightarrow \quad (\log 2)^3 + \log 8 \log 5 + (\log 5)^3 = 1$

● ***Ex. 27*** *If* $\lambda^{\log_3 5} = 81$, *find the value of* $\lambda^{(\log_3 5)^2}$.

Sol. $\because \quad \lambda^{\log_3 5} = 81$

$\therefore (\lambda^{\log_3 5})^{\log_3 5} = (81)^{\log_3 5}$

$\Rightarrow \quad \lambda^{(\log_3 5)^2} = 3^{4\log_3 5} = 3^{\log_3 5^4} = 5^4 = 625$

● ***Ex. 28*** *Find the product of the positive roots of the equation* $\sqrt{(2009)}(x)^{\log_{2009} x} = x^2$.

Sol. Given, $\sqrt{(2009)}(x)^{\log_{2009} x} = x^2$

Taking logarithm both sides on base 2009, then

$$\log_{2009} \sqrt{(2009)} + \log_{2009} x \cdot \log_{2009} x = \log_{2009} x^2$$

$\Rightarrow \quad \frac{1}{2} + (\log_{2009} x)^2 = 2\log_{2009} x$ [for $x > 0$]

$\Rightarrow \quad (\log_{2009} x)^2 - 2\log_{2009} x + \frac{1}{2} = 0$

If roots are x_1 and x_2, then $\log_{2009} x_1 + \log_{2009} x_2 = 2$

$\Rightarrow \quad \log_{2009}(x_1 x_2) = 2$ or $x_1 x_2 = (2009)^2$

● ***Ex. 29*** *Prove that* $\log_7 11$ *is greater than* $\log_8 5$.

Sol. $\because \quad 11 > 5$

$\Rightarrow \quad \log 11 > \log 5$ ...(i)

and $8 > 7$

$\Rightarrow \quad \log 8 > \log 7$ ...(ii)

From Eqs. (i) and (ii), we get

$$\log 11 \cdot \log 8 > \log 7 \cdot \log 5$$

$\Rightarrow \quad \frac{\log 11}{\log 7} > \frac{\log 5}{\log 8} \Rightarrow \log_7 11 > \log_8 5$

● ***Ex. 30*** *Given,* $a^2 + b^2 = c^2$. *Prove that*

$\log_{b+c} a + \log_{c-b} a = 2\log_{c+b} a \cdot \log_{c-b} a, \forall\, a > 0, a \neq 1$

$c - b > 0, c + b > 0$

$c - b \neq 1, c + b \neq 1.$

Sol. LHS $= \log_{b+c} a + \log_{c-b} a$

$$= \frac{1}{\log_a (c+b)} + \frac{1}{\log_a (c-b)}$$

$$= \frac{\log_a (c+b) + \log_a (c-b)}{\log_a (c+b)\log_a (c-b)}$$

$$= \frac{\log_a (c^2 - b^2)}{\log_a (c+b) \cdot \log_a (c-b)}$$

$$= \frac{\log_a a^2}{\log_a (c+b) \cdot \log_a (c-b)} \qquad [\because c^2 - b^2 = a^2]$$

$$= \frac{2\log_a a}{\log_a (c+b) \cdot \log_a (c-b)}$$

$$= \frac{2}{\log_a (c+b) \cdot \log_a (c-b)}$$

$$= 2\log_{c+b} a \cdot \log_{c-b} a = \text{RHS}$$

● ***Ex. 31*** *Let* $a > 0, c > 0, b = \sqrt{ac}$, a, c *and* $ac \neq 1, N > 0$.

Prove that $\frac{\log_a N}{\log_c N} = \frac{\log_a N - \log_b N}{\log_b N - \log_c N}$.

Sol. RHS $= \frac{\log_a N - \log_b N}{\log_b N - \log_c N} = \frac{\frac{1}{\log_N a} - \frac{1}{\log_N b}}{\frac{1}{\log_N b} - \frac{1}{\log_N c}}$

$$= \frac{(\log_N b - \log_N a)}{(\log_N c - \log_N b)} \cdot \frac{\log_N c}{\log_N a}$$

$$= \frac{\log_N \left(\frac{b}{a}\right)}{\log_N \left(\frac{c}{b}\right)} \cdot \frac{\log_a N}{\log_c N} = \frac{\log_a N}{\log_c N} = \text{LHS}$$

$$\left[\because b = \sqrt{ac} \Rightarrow b^2 = ac \Rightarrow \frac{b}{a} = \frac{c}{b}\right]$$

Ex. 32 *If* $a^x = b, b^y = c, c^z = a, x = \log_b a^{k_1}, y = \log_c b^{k_2}$, $z = \log_a c^{k_3}$, *find the minimum value of* $3k_1 + 6k_2 + 12k_3$.

Sol. $\because \quad a = c^z = (b^y)^z \quad [\because c = b^y]$

$= b^{yz} = (a^x)^{yz} = a^{xyz} \quad [\because b = a^x]$

$\therefore \quad xyz = 1$

Also, $xyz = \log_b a^{k_1} \cdot \log_c b^{k_2} \cdot \log_a c^{k_3}$

$= k_1 \cdot k_2 \cdot k_3 \cdot \log_b a \cdot \log_c b \cdot \log_a c$

$1 = k_1 k_2 k_3$

$\because \quad$ AM $\geq$ GM

$\therefore \quad \dfrac{3k_1 + 6k_2 + 12k_3}{3} \geq (3k_1 \cdot 6k_2 \cdot 12k_3)^{1/3}$

$= (3 \cdot 6 \cdot 12 \cdot k_1 k_2 k_3)^{1/3}$

$= (3 \cdot 6 \cdot 12)^{1/3} \quad [\because k_1 k_2 k_3 = 1]$

$= (2^3 \cdot 3^3)^{1/3} = 6$

or $3k_1 + 6k_2 + 12k_3 \geq 18$

$\therefore$ Minimum value of $3k_1 + 6k_2 + 12k_3$ is 18.

Ex. 33 *If* $x = 1 + \log_a bc, y = 1 + \log_b ca, z = 1 + \log_c ab$, *prove that* $xyz = xy + yz + zx$.

Sol. $\because \quad x = 1 + \log_a bc = 1 + \dfrac{\log bc}{\log a} = 1 + \dfrac{\log b + \log c}{\log a}$

$= \dfrac{\log a + \log b + \log c}{\log a}$

or $\dfrac{1}{x} = \dfrac{\log a}{\log a + \log b + \log c}$ ...(i)

Similarly, $\dfrac{1}{y} = \dfrac{\log b}{\log a + \log b + \log c}$ ...(ii)

and $\dfrac{1}{z} = \dfrac{\log c}{\log a + \log b + \log c}$ ...(iii)

On adding Eqs. (i), (ii) and (iii), we get

$\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 1$

or $xyz = xy + yz + zx$

Ex. 34 *If* $\dfrac{\ln a}{(b-c)} = \dfrac{\ln b}{(c-a)} = \dfrac{\ln c}{(a-b)}$, *prove that*

$a^{b+c} \cdot b^{c+a} \cdot c^{a+b} = 1$

Also, prove that $a^{b+c} + b^{c+a} + c^{a+b} \geq 3$.

Sol. Since, $a > 0, b > 0, c > 0$

$\dfrac{\ln a}{(b-c)} = \dfrac{\ln b}{(c-a)} = \dfrac{\ln c}{(a-b)}$

$= \dfrac{(b+c)\ln a + (c+a)\ln b + (a+b)\ln c}{0}$

[using ratio and proportion]

$\therefore \quad (b+c)\ln a + (c+a)\ln b + (a+b)\ln c = 0$

$\Rightarrow \quad \ln a^{b+c} + \ln b^{c+a} + \ln c^{a+b} = 0$

$\Rightarrow \quad \ln \{a^{b+c} \cdot b^{c+a} \cdot c^{a+b}\} = 0$

$\Rightarrow \quad a^{b+c} \cdot b^{c+a} \cdot c^{a+b} = e^0 = 1 \quad$ ...(i)

Again, AM $\geq$ GM

$\Rightarrow \quad \dfrac{a^{b+c} + b^{c+a} + c^{a+b}}{3} \geq (a^{b+c} \cdot b^{c+a} \cdot c^{a+b})^{1/3}$

$= (1)^{1/3} = 1 \quad$ [from Eq. (i)]

or $a^{b+c} + b^{c+a} + c^{a+b} \geq 3$

Ex. 35 *Simplify* $5^{\log_{1/5}(1/2)} + \log_{\sqrt{2}}\left(\dfrac{4}{\sqrt{7}+\sqrt{3}}\right) + \log_{1/2}\left(\dfrac{1}{10+2\sqrt{21}}\right)$.

Sol. $\because \quad 5^{\log_{1/5}\left(\frac{1}{2}\right)} = 5^{\log_5(2)} = 2$

$\log_{\sqrt{2}}\left(\dfrac{4}{\sqrt{7}+\sqrt{3}}\right) = \log_{\sqrt{2}}\left(\dfrac{4(\sqrt{7}-\sqrt{3})}{(\sqrt{7}+\sqrt{3})(\sqrt{7}-\sqrt{3})}\right)$

$= \log_{\sqrt{2}}(\sqrt{7}-\sqrt{3})$

$= \log_{2^{1/2}}(\sqrt{7}-\sqrt{3})^1$

$= \dfrac{1}{1/2}\log_2(\sqrt{7}-\sqrt{3})$

$= \log_2(\sqrt{7}-\sqrt{3})^2 = \log_2(10 - 2\sqrt{21})$

and $\log_{1/2}\left(\dfrac{1}{10+2\sqrt{21}}\right) = \log_2(10 + 2\sqrt{21})$

Hence,

$5^{\log_{1/5}(1/2)} + \log_{\sqrt{2}}\left(\dfrac{4}{\sqrt{7}+\sqrt{3}}\right) + \log_{1/2}\left(\dfrac{1}{10+2\sqrt{21}}\right)$

$= 2 + \log_2(10 - 2\sqrt{21}) + \log_2(10 + 2\sqrt{21})$

$= 2 + \log_2\{(10 - 2\sqrt{21})(10 + 2\sqrt{21})\}$

$= 2 + \log_2(100 - 84) = 2 + \log_2(2)^4 = 2 + 4 = 6$

Ex. 36 *Find the square of the sum of the roots of the equation* $\log_2 x \cdot \log_3 x \cdot \log_5 x = \log_2 x \cdot \log_3 x + \log_3 x \cdot \log_5 x + \log_5 x \cdot \log_2 x$.

Sol. Let $\log_2 x = A$, $\log_3 x = B$ and $\log_5 x = C$, then the given equation can be written as

$ABC = AB + BC + CA = ABC\left(\dfrac{1}{C} + \dfrac{1}{A} + \dfrac{1}{B}\right)$

$\Rightarrow \quad ABC\left(\dfrac{1}{A} + \dfrac{1}{B} + \dfrac{1}{C} - 1\right) = 0$

or $A = 0, B = 0, C = 0, \dfrac{1}{A} + \dfrac{1}{B} + \dfrac{1}{C} - 1 = 0$

$\underbrace{\log_2 x = 0, \log_3 x = 0, \log_5 x = 0}_{x > 0}, \underbrace{\log_x 2 + \log_x 3 + \log_x 5 = 0}_{x > 0,\ x \neq 1}$

or $x = 2^0, x = 3^0, x = 5^0, \log_x(2 \cdot 3 \cdot 5) = 0$

or $x = 1, x = 1, x = 1, x = 30$

$\therefore$ Roots are 1 and 30.

Hence, the required value

$$= (1 + 30)^2 = (31)^2 = 961$$

● **Ex. 37** *Given that* $\log_2 a = \lambda, \log_4 b = \lambda^2$ *and* $\log_{c^2}(8) = \dfrac{2}{\lambda^3 + 1}$, *write* $\log_2\left(\dfrac{a^2 b^5}{c^4}\right)$ *as a function of* 'λ' $(a, b, c > 0, c \neq 1)$.

Sol. $\because \log_2 a = \lambda \Rightarrow a = 2^\lambda$

$\Rightarrow \log_4 b = \lambda^2$

$\Rightarrow b = 4^{\lambda^2} = 2^{2\lambda^2}$

and $\log_{c^2}(8) = \dfrac{2}{\lambda^3 + 1}$

$\Rightarrow \dfrac{3}{2}\log_c 2 = \dfrac{2}{\lambda^3 + 1}$

$\Rightarrow \log_c 2 = \dfrac{4}{3(\lambda^3 + 1)}$

or $\log_2 c = \dfrac{3(\lambda^3 + 1)}{4}$ or $c = 2^{\left\{\frac{3(\lambda^3+1)}{4}\right\}}$

$\therefore \log_2\left(\dfrac{a^2 b^5}{c^4}\right) = \log_2(a^2 b^5 c^{-4})$

$= \log_2\{2^{2\lambda} \cdot 2^{10\lambda^2} \cdot 2^{-3(\lambda^3+1)}\}$

$= \log_2\{2^{2\lambda + 10\lambda^2 - 3(\lambda^3+1)}\}$

$= 2\lambda + 10\lambda^2 - 3(\lambda^3 + 1)$

● **Ex. 38** *Given that* $\log_2 3 = a, \log_3 5 = b, \log_7 2 = c$, *express the logarithm of the number* 63 *to the base* 140 *in terms of* a, b *and* c.

Sol. $\because \log_2 3 = a$ ...(i)

$\Rightarrow b = \log_3 5 = \dfrac{\log_2 5}{\log_2 3} = \dfrac{\log_2 5}{a}$ [from Eq. (i)]

$\therefore \log_2 5 = ab$ ...(ii)

and $\log_7 2 = c$

$\Rightarrow \dfrac{1}{\log_2 7} = c$ or $\log_2 7 = \dfrac{1}{c}$ ...(iii)

Now, $\log_{140} 63 = \dfrac{\log_2 63}{\log_2 140} = \dfrac{\log_2(3^2 \times 7)}{\log_2(2^2 \times 5 \times 7)}$

$= \dfrac{2\log_2 3 + \log_2 7}{2 + \log_2 5 + \log_2 7} = \dfrac{2a + \dfrac{1}{c}}{2 + ab + \dfrac{1}{c}}$

[from Eqs. (i), (ii) and (iii)]

$= \left(\dfrac{2ac + 1}{2c + abc + 1}\right)$

● **Ex. 39** *Show that the sum of the roots of the equation* $x + 1 = 2\log_2(2^x + 3) - 2\log_4(1980 - 2^{-x})$ *is* $\log_2 11$.

Sol. Given,

$x + 1 = 2\log_2(2^x + 3) - 2\log_4(1980 - 2^{-x})$

$= 2\log_2(2^x + 3) - 2\log_{2^2}(1980 - 2^{-x})^1$

$= 2\log_2(2^x + 3) - 2 \cdot \dfrac{1}{2}\log_2(1980 - 2^{-x})$

$= \log_2(2^x + 3)^2 - \log_2(1980 - 2^{-x})$

$= \log_2\left\{\dfrac{(2^x + 3)^2}{1980 - 2^{-x}}\right\}$

or $2^{x+1} = \dfrac{(2^x + 3)^2}{1980 - 2^{-x}}$

$\Rightarrow 1980(2^{x+1}) - 2 = 2^{2x} + 9 + 6 \cdot 2^x$

$\Rightarrow 2^{2x} - 3954 \cdot 2^x + 11 = 0$ ...(i)

If x_1, x_2 are the roots of Eq. (i), then

$2^{x_1} \cdot 2^{x_2} = 11$ or $2^{x_1 + x_2} = 11$

$\Rightarrow x_1 + x_2 = \log_2 11$

● **Ex. 40** *Solve the following equations for* x *and* y

$\log_{100}|x + y| = \dfrac{1}{2}, \log_{10} y - \log_{10}|x| = \log_{100} 4$.

Sol. $\because \log_{100}|x + y| = \dfrac{1}{2}$

$\Rightarrow |x + y| = (100)^{1/2} = 10$

$\Rightarrow |x + y| = 10$ ...(i)

and $\log_{10} y - \log_{10}|x| = \log_{100} 4, y > 0$

$\Rightarrow \log_{10}\left(\dfrac{y}{|x|}\right) = \log_{10^2} 2^2 = \dfrac{2}{2}\log_{10} 2$

$\Rightarrow \log_{10}\left(\dfrac{y}{|x|}\right) = \log_{10} 2 \Rightarrow \dfrac{y}{|x|} = 2$

$\Rightarrow y = 2|x|$ ...(ii)

From Eqs. (i) and (ii),we get

$|x + 2|x|| = 10$ ...(iii)

***Case* I** If $x > 0$, then $|x| = x$

From Eq. (iii),

$|x + 2x| = 10$

$$\Rightarrow \quad 3|x| = 10 \Rightarrow |x| = \frac{10}{3}$$

$$\therefore \quad x = \frac{10}{3}, y = \frac{20}{3} \qquad \text{[from Eq. (ii)]}$$

***Case* II** If $x < 0$, then $|x| = -x$

From Eq. (iii),

$$|x - 2x| = 10$$

$$\Rightarrow \quad |-x| = 10 \Rightarrow |x| = 10$$

$$\therefore \quad -x = 10$$

$$\Rightarrow \quad x = -10$$

From Eq. (ii), $y = 20$

Hence, solutions are $\left\{\frac{10}{3}, \frac{20}{3}\right\}$, $\{-10, 20\}$.

● ***Ex. 41*** *Solve the following equation for x*

$$\frac{6}{5}a^{\log_a x \cdot \log_{10} a \cdot \log_a 5} - 3^{\log_{10}(x/10)} = 9^{\log_{100} x + \log_4 2}.$$

Sol. $\because \ \frac{6}{5}\cdot a^{\log_a x \cdot \log_{10} a \cdot \log_a 5} - 3^{\log_{10}(x/10)} = 9^{\log_{100} x + \log_4 2}$

$$\Rightarrow \quad \frac{6}{5}\cdot x^{\log_{10} 5} - 3^{(\log_{10} x - 1)} = 3^{2\left(\frac{1}{2}\log_{10} x + \frac{1}{2}\right)} \quad \text{[by property]}$$

$$\Rightarrow \quad \frac{6}{5}\cdot 5^{\log_{10} x} - \frac{3^{\log_{10} x}}{3} = 3^{\log_{10} x + 1} \quad \text{[by property]}$$

Let $\log_{10} x = \lambda$, then

$$\Rightarrow \quad \frac{6}{5}\cdot 5^{\lambda} - \frac{3^{\lambda}}{3} = 3\cdot 3^{\lambda}$$

$$\Rightarrow \quad \frac{6}{5}\cdot 5^{\lambda} = 3^{\lambda}\left(\frac{1}{3} + 3\right) = \frac{10}{3}\cdot 3^{\lambda}$$

$$\Rightarrow \quad 5^{\lambda - 2} = 3^{\lambda - 2} \text{ which is possible only, where } \lambda = 2.$$

$$\Rightarrow \quad \log_{10} x = 2$$

$$\therefore \quad x = 10^2 = 100$$

● ***Ex. 42*** *Find the value of x satisfying the equation*

$$|x-1|^{\log_3 x^2 - 2\log_x 9} = (x-1)^7.$$

Sol. The given equation is,

$$|x-1|^{\log_3 x^2 - 2\log_x 9} = (x-1)^7 \qquad \ldots(i)$$

This equation is defined for

$$x^2 > 0, x > 0, x \neq 1 \text{ and } x - 1 \geq 1$$

$$\Rightarrow \quad x \geq 2, \text{ then Eq. (i) reduces to}$$

$$(x-1)^{\log_3 x^2 - 2\log_x 9} = (x-1)^7$$

Taking log on both sides, then

$$(\log_3 x^2 - 2\log_x 9)\log(x-1) = 7\log(x-1)$$

$$\Rightarrow \quad \log(x-1)\{\log_3 x^2 - 2\log_x 9 - 7\} = 0$$

$$\Rightarrow \quad \log(x-1)\left\{2\log_3 x - \frac{4}{\log_3 x} - 7\right\} = 0$$

$$\Rightarrow \quad \log(x-1)\{2(\log_3 x)^2 - 7\log_3 x - 4\} = 0$$

$$\Rightarrow \quad \log(x-1)(\log_3 x - 4)(2\log_3 x + 1) = 0$$

$$\Rightarrow \quad \log(x-1) = 0, \log_3 x = 4, \log_3 x = -\frac{1}{2}$$

$$\Rightarrow \quad x - 1 = (10)^0, x = 3^4, x = 3^{-1/2}$$

$$\Rightarrow \quad x - 1 = 1, x = 81, x = \frac{1}{\sqrt{3}}$$

$$\therefore \quad x = 2, 81 \qquad \left[\because x \geq 2, \therefore x \neq \frac{1}{\sqrt{3}}\right]$$

● ***Ex. 43*** *Find all real numbers x which satisfy the equation* $2\log_2 \log_2 x + \log_{1/2}\log_2(2\sqrt{2}x) = 1$.

Sol. Given,

$$2\log_2 \log_2 x + \log_{1/2}\log_2(2\sqrt{2}x) = 1$$

$$\Rightarrow \quad 2\log_2\log_2 x - \log_2\log_2(2\sqrt{2}x) = 1$$

$$\Rightarrow \quad 2\log_2 \log_2 x - \log_2\{\log_2(2\sqrt{2}) + \log_2 x\} = 1$$

$$\Rightarrow \quad 2\log_2 \log_2 x - \log_2\left\{\frac{3}{2} + \log_2 x\right\} = 1$$

Let $\log_2 x = \lambda$, then $2\log_2 \lambda - \log_2\left(\frac{3}{2} + \lambda\right) = 1$

$$\Rightarrow \quad \log_2 \lambda^2 - \log_2\left(\frac{3}{2} + \lambda\right) = 1$$

$$\Rightarrow \quad \log_2\left\{\frac{\lambda^2}{\frac{3}{2} + \lambda}\right\} = 1 \Rightarrow \frac{\lambda^2}{\frac{3}{2} + \lambda} = 2^1$$

$$\Rightarrow \quad \lambda^2 = 3 + 2\lambda \Rightarrow \lambda^2 - 2\lambda - 3 = 0$$

$$\Rightarrow \quad (\lambda - 3)(\lambda + 1) = 0$$

$$\therefore \quad \lambda = 3, -1$$

or $\log_2 x = 3, -1$

$$\Rightarrow \quad x = 2^3, 2^{-1}$$

$$\Rightarrow \quad x = 8, \frac{1}{2} \qquad \ldots(i)$$

But the given equation is valid only when,

$$x > 0, 2\sqrt{2}x > 0, \log_2 x > 0, \log_2(2\sqrt{2}x) > 0$$

$$\Rightarrow \quad x > 0, x > 0, x > 1, x > \frac{1}{2\sqrt{2}}$$

Hence, $x > 1$

From Eq. (i), the solution of the given equation is $x = 8$.

● ***Ex. 44*** *Solve for x,*

$$\log_{3/4}\log_8(x^2+7) + \log_{1/2}\log_{1/4}(x^2+7)^{-1} = -2.$$

Sol. Given,

$$\log_{3/4}\log_8(x^2+7) + \log_{1/2}\log_{1/4}(x^2+7)^{-1} = -2$$

$$\Rightarrow \log_{3/4}\log_{2^3}(x^2+7)+\log_{2^{-1}}\log_{2^{-2}}(x^2+7)^{-1}=-2$$

$$\Rightarrow \log_{3/4}\left\{\frac{1}{3}\log_2(x^2+7)\right\}-\log_2\left\{\frac{1}{2}\log_2(x^2+7)\right\}=-2$$

Let $\log_2(x^2+7)=6\lambda$ ...(i)

Then, $\log_{3/4}(2\lambda)-\log_2(3\lambda)=-2$

$$\Rightarrow \frac{\log_2(2\lambda)}{\log_2(3/4)}-\log_2(3\lambda)=-2$$

$$\Rightarrow \frac{1+\log_2\lambda}{\log_2 3-\log_2 4}-(\log_2 3+\log_2\lambda)=-2$$

$$\Rightarrow \frac{1+\log_2\lambda}{\log_2 3-2}-(\log_2 3+\log_2\lambda)=-2$$

Again, let $\log_2\lambda=A$ and $\log_2 3=B$, then

$$\frac{1+A}{B-2}-(B+A)=-2$$

$$\Rightarrow 1+A-B^2-AB+2B+2A=-2B+4$$

$$\Rightarrow A(3-B)=B^2-4B+3=(B-1)(B-3)$$

$$\Rightarrow A=-(B-1)$$

$[\because B-3\neq 0, \text{ i.e. } \log_2 3\neq 3]$

$$\Rightarrow A+B=1 \Rightarrow \log_2\lambda+\log_2 3=1$$

$$\Rightarrow \log_2(3\lambda)=1$$

$$\Rightarrow 3\lambda=2$$

$$\Rightarrow 3\cdot\frac{1}{6}\log_2(x^2+7)=2 \text{ [from Eq. (i)]}$$

$$\Rightarrow \log_2(x^2+7)=4$$

$$\Rightarrow x^2+7=2^4=16 \text{ or } x^2=9$$

$$\therefore x=\pm 3$$

● ***Ex. 45*** *Prove that*

$$2^{\left(\sqrt{\log_a\sqrt[4]{ab}+\log_b\sqrt[4]{ab}}-\sqrt{\log_a\sqrt[4]{b/a}+\log_b\sqrt[4]{a/b}}\right)\sqrt{\log_a b}}=\begin{cases}2, b\geq a>1.\\ 2^{\log_a b}, 1<b<a\end{cases}.$$

Sol. Since, $\sqrt{\log_a\sqrt[4]{ab}+\log_b\sqrt[4]{ab}}=\sqrt{\frac{1}{4}\log_a(ab)+\frac{1}{4}\log_b(ab)}$

$$=\sqrt{\frac{1}{4}(1+\log_a b+\log_b a+1)}$$

$$=\sqrt{\left(\frac{\log_a b+\frac{1}{\log_a b}+2}{4}\right)}=\sqrt{\left(\frac{\sqrt{|\log_a b|}+\frac{1}{\sqrt{|\log_a b|}}}{2}\right)^2}$$

$$=\frac{1}{2}\left(\sqrt{|\log_a b|}+\frac{1}{\sqrt{|\log_a b|}}\right)$$

and $\sqrt{\log_a\sqrt[4]{(b/a)}+\log_b\sqrt[4]{(a/b)}}$

$$=\sqrt{\frac{1}{4}\log_a\left(\frac{b}{a}\right)+\frac{1}{4}\log_b\left(\frac{a}{b}\right)}$$

$$=\sqrt{\frac{1}{4}(\log_a b-1+\log_b a-1)}$$

$$=\sqrt{\frac{\log_a b+\frac{1}{\log_a b}-2}{4}}$$

$$=\frac{\left|\sqrt{|\log_a b|}-\frac{1}{\sqrt{|\log_a b|}}\right|}{2}$$

$\therefore \sqrt{\log_a\sqrt[4]{ab}+\log_b\sqrt[4]{ab}}-\sqrt{\log_a\sqrt[4]{b/a}+\log_b\sqrt[4]{(a/b)}}$

P (say)

$$=\frac{1}{2}\left\{\sqrt{|\log_a b|}+\frac{1}{\sqrt{|\log_a b|}}+\left|\sqrt{|\log_a b|}-\frac{1}{\sqrt{|\log_a b|}}\right|\right\}$$

***Case* I** If $b\geq a>1$, then

$$P=\frac{1}{2}\left\{\sqrt{|\log_a b|}+\frac{1}{\sqrt{|\log_a b|}}-\sqrt{|\log_a b|}+\frac{1}{\sqrt{|\log_a b|}}\right\}$$

$$=\frac{1}{\sqrt{\log_a b}}$$

$\therefore 2^{P\sqrt{\log_a b}}=2^1=2$

***Case* II** If $1<b<a$, then

$$P=\frac{1}{2}\left\{\sqrt{|\log_a b|}+\frac{1}{\sqrt{|\log_a b|}}+\sqrt{|\log_a b|}-\frac{1}{\sqrt{|\log_a b|}}\right\}$$

$$=\sqrt{|\log_a b|}$$

$\therefore 2^{P\sqrt{\log_a b}}=2^{\log_a b}$

Logarithms and Their Properties Exercise 1 :
Single Option Correct Type Questions

■ This section contains **20 multiple choice questions.** Each question has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct

1. If $\log_{10} 2 = 0.3010\ldots$, the number of digits in the number 2000^{2000} is
(a) 6601 (b) 6602 (c) 6603 (d) 6604

2. There exist a positive number λ, such that $\log_2 x + \log_4 x + \log_8 x = \log_\lambda x$, for all positive real numbers x.
If $\lambda = \sqrt[b]{a}$, where $a, b \in N$, the smallest possible value of $(a+b)$ is equal to
(a) 12 (b) 63 (c) 65 (d) 75

3. If a, b and c are the three real solutions of the equation

$$x^{\log_{10}^2 x + \log_{10} x^3 + 3} = \frac{2}{\frac{1}{\sqrt{x+1}-1} - \frac{1}{\sqrt{x+1}+1}}$$

where, $a > b > c$, then a, b, c are in
(a) AP (b) GP
(c) HP (d) $a^{-1} + b^{-1} = c^{-1}$

4. If $f(n) = \prod_{i=2}^{n-1} \log_i (i+1)$, the value of $\sum_{k=1}^{100} f(2^k)$ equals
(a) 5010 (b) 5050 (c) 5100 (d) 5049

5. If $\log_3 27 \cdot \log_x 7 = \log_{27} x \cdot \log_7 3$, the least value of x, is
(a) 7^{-3} (b) 3^{-7} (c) 7^3 (d) 3^7

6. If $x = \log_5 (1000)$ and $y = \log_7 (2058)$, then
(a) $x > y$ (b) $x < y$
(c) $x = y$ (d) None of these

7. If $\log_5 120 + (x-3) - 2\log_5 (1 - 5^{x-3}) = -\log_5 (0.2 - 5^{x-4})$, then x is
(a) 1 (b) 2 (c) 3 (d) 4

8. If $x_n > x_{n-1} > \ldots > x_2 > x_1 > 1$, the value of

$$\log_{x_1} \log_{x_2} \log_{x_3} \ldots \log_{x_n} x_n^{\cdot^{\cdot^{\cdot^{x_1}}}}$$ is

(a) 0 (b) 1
(c) 2 (d) undefined

9. If $\frac{x(y+z-x)}{\log x} = \frac{y(z+x-y)}{\log y} = \frac{z(x+y-z)}{\log z}$, then $x^y y^x = z^y y^z$ is equal to
(a) $z^x x^z$ (b) $x^z y^x$ (c) $x^y y^z$ (d) $x^x y^y$

10. If $y = a^{\frac{1}{1-\log_a x}}$ and $z = a^{\frac{1}{1-\log_a y}}$, then x is equal to
(a) $a^{\frac{1}{1+\log_a z}}$ (b) $a^{\frac{1}{2+\log_a z}}$ (c) $a^{\frac{1}{1-\log_a z}}$ (d) $a^{\frac{1}{2-\log_a z}}$

11. If $\log_{0.3}(x-1) < \log_{0.09}(x-1)$, then x lies in the interval
(a) $(-\infty, 1)$
(b) $(1, 2)$
(c) $(2, \infty)$
(d) None of the above

12. The value of $a^x - b^y$ is (where $x = \sqrt{\log_a b}$ and $y = \sqrt{\log_b a}$, $a > 0$, $b > 0$ and $a, b \neq 1$)
(a) 1 (b) 2
(c) 0 (d) -1

13. If $x = 1 + \log_a bc$, $y = 1 + \log_b ca$, $z = 1 + \log_c ab$, then $\frac{xyz}{xy + yz + zx}$ is equal to
(a) 0 (b) 1
(c) -1 (d) 2

14. The value of $a^{\frac{\log_b (\log_b N)}{\log_b a}}$ is
(a) $\log_a N$ (b) $\log_b N$
(c) $\log_N a$ (d) $\log_N b$

15. The value of $49^A + 5^B$, where $A = 1 - \log_7 2$ and $B = -\log_5 4$ is
(a) 10.5 (b) 11.5
(c) 12.5 (d) 13.5

16. The number of real values of the parameter λ for which $(\log_{16} x)^2 - \log_{16} x + \log_{16} \lambda = 0$ with real coefficients will have exactly one solution is
(a) 1 (b) 2
(c) 3 (d) 4

17. The number of roots of the equation $x^{\log_x (x+3)^2} = 16$ is
(a) 1 (b) 0
(c) 2 (d) 4

18. The point on the graph $y = \log_2 \log_6 \{2^{\sqrt{(2x+1)}} + 4\}$, whose y-coordinate is 1 is
(a) (1, 1) (b) (6,1)
(c) (8, 1) (d) (12, 1)

19. Given, $\log 2 = 0.301$ and $\log 3 = 0.477$, then the number of digits before decimal in $3^{12} \times 2^8$ is
(a) 7 (b) 8
(c) 9 (d) 11

20. The number of solution(s) for the equation $2\log_x a + \log_{ax} a + 3\log_{a^2 x} a = 0$, is
(a) one (b) two
(c) three (d) four

Logarithms and Their Properties Exercise 2 : More than One Correct Option Type Questions

■ This section contains **9 multiple choice questions**. Each question has four choices (a), (b), (c) and (d) out of which **MORE THAN ONE** may be correct.

21. If $x^{(\log_2 x)^2 - 6\log_2 x + 11} = 64$, then x is equal to
(a) 2 (b) 4 (c) 6 (d) 8

22. If $\log_\lambda x \cdot \log_5 \lambda = \log_x 5$, $\lambda \neq 1, \lambda > 0$, then x is equal to
(a) λ (b) 5 (c) $\frac{1}{5}$ (d) None of these

23. If $S = \{x : \sqrt{\log_x \sqrt{3x}}$, where $\log_3 x > -1\}$, then
(a) S is a finite set (b) $S \in \phi$
(c) $S \subset (0, \infty)$ (d) S properly contains $\left(\frac{1}{3}, \infty\right)$

24. If x satisfies $\log_2(9^{x-1} + 7) = 2 + \log_2(3^{x-1} + 1)$, then
(a) $x \in Q$
(b) $x \in N$
(c) $x \in \{x \in Q : x < 0\}$
(d) $x \in N_e$ (set of even natural numbers)

25. $\log_p \log_p \underbrace{\sqrt[p]{\sqrt[p]{\sqrt[p]{\ldots\sqrt[p]{p}}}}}_{n \text{ times}}$, $p > 0$ and $p \neq 1$ is equal to
(a) n (b) $-n$
(c) $\frac{1}{n}$ (d) $\log_{1/p}(p^n)$

26. If $\log_a x = \alpha$, $\log_b x = \beta$, $\log_c x = \gamma$ and $\log_d x = \delta$, $x \neq 1$ and $a, b, c, d \neq 0, > 1$, then $\log_{abcd} x$ equals
(a) $\leq \frac{\alpha + \beta + \gamma + \delta}{16}$ (b) $\geq \frac{\alpha + \beta + \gamma + \delta}{16}$
(c) $\frac{1}{\alpha^{-1} + \beta^{-1} + \gamma^{-1} + \delta^{-1}}$ (d) $\frac{1}{\alpha\beta\gamma\delta}$

27. If $\log_{10} 5 = a$ and $\log_{10} 3 = b$, then
(a) $\log_{10} 8 = 3(1 - a)$ (b) $\log_{40} 15 = \frac{(a + b)}{(3 - 2a)}$
(c) $\log_{243} 32 = \left(\frac{1 - a}{b}\right)$ (d) All of these

28. If x is a positive number different from 1, such that $\log_a x$, $\log_b x$ and $\log_c x$ are in AP, then
(a) $\log b = \frac{2(\log a)(\log c)}{(\log a + \log c)}$ (b) $b = \frac{a + c}{2}$
(c) $b = \sqrt{ac}$ (d) $c^2 = (ac)^{\log_a b}$

29. If $|a| < |b|$, $b - a < 1$ and a, b are the real roots of the equation $x^2 - |\alpha| x - |\beta| = 0$, the equation $\log_{|b|}\left|\frac{x}{a}\right| - 1 = 0$ has
(a) one root lying in interval $(-\infty, a)$
(b) one root lying in interval (b, ∞)
(c) one positive root
(d) one negative root

Logarithms and Their Properties Exercise 3 : Passage Based Questions

■ This section contains **4 passages**. Based upon each of the passage **3 multiple choice questions** have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I

(Q. Nos. 30 to 32)

Let $\log_2 N = a_1 + b_1$, $\log_3 N = a_2 + b_2$ *and* $\log_5 N = a_3 + b_3$, *where* $a_1, a_2, a_3 \in I$ *and* $b_1, b_2, b_3 \in [0, 1)$.

30. If $a_1 = 5$ and $a_2 = 3$, the number of integral values of N is
(a) 16 (b) 32 (c) 48 (d) 64

31. If $a_1 = 6$, $a_2 = 4$ and $a_3 = 3$, the largest integral value of N is
(a) 124 (b) 63
(c) 624 (d) 127

32. If $a_1 = 6$, $a_2 = 4$ and $a_3 = 3$, the difference of largest and smallest integral values of N, is
(a) 2 (b) 8
(c) 14 (d) 20

Passage II

(Q. Nos. 33 to 35)

Let 'S' denotes the antilog of 0.5 to the base 256 and 'K' denotes the number of digits in 6^{10} *(given* $\log_{10} 2 = 0.301$, $\log_{10} 3 = 0.477$*) and G denotes the number of positive integers, which have the characteristic 2, when the base of logarithm is 3.*

33. The value of G is
(a) 18 (b) 24 (c) 30 (d) 36

34. The value of KG is
(a) 72 (b) 144 (c) 216 (d) 288

35. The value of SKG is

(a) 1440 (b) 17280
(c) 2016 (d) 2304

Passage III
(Q. Nos. 36 to 38)

Suppose 'U' denotes the number of digits in the number $(60)^{100}$ *and 'M' denotes the number of cyphers after decimal, before a significant figure comes in* $(8)^{-296}$. *If the fraction U/M is expressed as rational number in the lowest term as* p/q *(given* $\log_{10} 2 = 0.301$ *and* $\log_{10} 3 = 0.477$*).*

36. The value of p is

(a) 1 (b) 2 (c) 3 (d) 4

37. The value of q is

(a) 5 (b) 2
(c) 3 (d) 4

38. The equation whose roots are p and q, is

(a) $x^2 - 3x + 2 = 0$ (b) $x^2 - 5x + 6 = 0$
(c) $x^2 - 7x + 12 = 0$ (d) $x^2 - 9x + 20 = 0$

Passage IV (Q. Nos. 39 to 41)

Let G, O, E *and* L *be positive real numbers such that* $\log(G\cdot L) + \log(G\cdot E) = 3, \log(E\cdot L) + \log(E\cdot O) = 4,$ $\log(O\cdot G) + \log(O\cdot L) = 5$ *(base of the log is 10).*

39. If the value of the product $(GOEL)$ is λ, the value of $\sqrt{\log\lambda\sqrt{\log\lambda\sqrt{\log\lambda\ldots}}}$ is

(a) 3 (b) 4
(c) 5 (d) 7

40. If the minimum value of $3G + 2L + 2O + E$ is $2^{\lambda}3^{\mu}5^{\nu}$, where λ, μ and ν are whole numbers, the value of $\sum(\lambda^{\mu} + \mu^{\lambda})$ is

(a) 7 (b) 13
(c) 19 (d) None of these

41. If $\log\left(\frac{G}{O}\right)$ and $\log\left(\frac{O}{E}\right)$ are the roots of the equation

(a) $x^2 + x = 0$ (b) $x^2 - x = 0$
(c) $x^2 - 2x + 3 = 0$ (d) $x^2 - 1 = 0$

Logarithms and Their Properties Exercise 4 : Single Integer Answer Type Questions

- This section contains **10 questions**. The answer to each question is a **single digit integer**, ranging from 0 to 9 (both inclusive).

42. If $x, y \in R^+$ and $\log_{10}(2x) + \log_{10} y = 2$ and $\log_{10} x^2 - \log_{10}(2y) = 4$ and $x + y = \frac{m}{n}$, where m and n are relative prime, the value of $m - 3n^6$ is

43. A line $x = \lambda$ intersects the graph of $y = \log_5 x$ and $y = \log_5(x + 4)$. The distance between the points of intersection is 0.5. Given $\lambda = a + \sqrt{b}$, where a and b are integers, the value of $(a + b)$ is

44. If the left hand side of the equation $a(b-c)x^2 + b(c-a)xy + c(a-b)y^2 = 0$ is a perfect square, the value of $\left\{\frac{\log(a+c) + \log(a-2b+c)}{\log(a-c)}\right\}^2, (a, b, c \in R^+, a > c)$ is

45. Number of integers satisfying the inequality $\left(\frac{1}{3}\right)^{\frac{|x+2|}{2-|x|}} > 9$ is

46. If $x > 2$ is a solution of the equation $|\log_{\sqrt{3}}(x-2)| + |\log_3(x-2)| = 2$, then the value of x is

47. Number of integers satisfying the inequality $\log_2\sqrt{x} - 2\log^2_{1/4} x + 1 > 0$, is

48. The value of $b(>0)$ for which the equation $2\log_{1/25}(bx + 28) = -\log_5(12 - 4x - x^2)$ has coincident roots, is

49. The value of $\frac{2^{\log_{2^{1/4}} 2} - 3^{\log_{27} 125} - 4}{7^{4\log_{49} 2} - 3}$ is

50. If x_1 and x_2 $(x_2 > x_1)$ are the integral solutions of the equation $(\log_5 x)^2 + \log_{5x}\left(\frac{5}{x}\right) = 1$, the value of $|x_2 - 4x_1|$ is

51. If $x = \log_{\lambda} a = \log_a b = \frac{1}{2}\log_b c$ and $\log_{\lambda} c = nx^{n+1}$, the value of n is

Logarithms and Their Properties Exercise 5 : Matching Type Questions

- This section contains **3 questions**. Questions 52 to 54 have four statements (A, B, C and D) given in **Column I** and four statements (p, q, r and s) in **Column II**. Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II**.

52.

	Column I		Column II
(A)	$\frac{\log_3 243}{\log_2 \sqrt{32}}$	(p)	positive integer
(B)	$\frac{2\log 6}{(\log 12 + \log 3)}$	(q)	negative integer
(C)	$\log_{1/3}\left(\frac{1}{9}\right)^{-2}$	(r)	rational but not integer
(D)	$\frac{\log_5 16 - \log_5 4}{\log_5 128}$	(s)	prime

53.

	Column I		Column II
(A)	The expression $\sqrt{\log^2_{0.5} 8}$ has the value equal to	(p)	1
(B)	The value of the expression $(\log_{10} 2)^3 + \log_{10} 8 \cdot \log_{10} 5 + (\log_{10} 5)^3 + 3$, is	(q)	2
(C)	Let $N = \log_2 15 \cdot \log_{1/6} 2 \cdot \log_3\left(\frac{1}{6}\right)$. The value of $[N]$ is (where $[\cdot]$ denotes the greatest integer function)	(r)	3
(D)	If $(52.6)^a = (0.00526)^b = 100$, the value of $\frac{1}{a} - \frac{1}{b}$ is	(s)	4

54.

	Column I		Column II
(A)	If $\log_{1/x}\left\{\frac{2(x-2)}{(x+1)(x-5)}\right\} \geq 1$, then x can belongs to	(p)	$\left(0, \frac{1}{3}\right]$
(B)	If $\log_3 x - \log^2_3 x \leq \frac{3}{2}\log_{(1/2\sqrt{2})} 4$, then x can belongs to	(q)	(1, 2]
(C)	If $\log_{1/2}(4-x) \geq \log_{1/2} 2 - \log_{1/2}(x-1)$, then x belongs to	(r)	[3, 4)
(D)	Let α and β are the roots of the quadratic equation $(\lambda^2 - 3\lambda + 4)x^2 - 4(2\lambda - 1)x + 16 = 0$, if α and β satisfy the condition $\beta > 1 > \alpha$, then p can lie in	(s)	(3, 8)

Logarithms and Their Properties Exercise 6 : Statement I and II Type Questions

- **Directions** Question numbers 55 to 60 are Assertion-Reason type questions. Each of these questions contains two statements:
  Statement-1 (Assertion) and **Statement-2** (Reason) Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below.
  (a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
  (b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
  (c) Statement-1 is true, Statement-2 is false
  (d) Statement-1 is false, Statement-2 is true

55. **Statement-1** $\log_{10} x < \log_3 x < \log_e x < \log_2 x$ $(x > 0, x \neq 1)$.
Statement-2 If $0 < x < 1$, then $\log_x a > \log_x b \Rightarrow 0 < a < b$.

56. **Statement-1** The equation $7^{\log_7 (x^3+1)} - x^2 = 1$ has two distinct real roots.
Statement-2 $a^{\log_a N} = N$, where $a > 0$, $a \neq 1$ and $N > 0$.

57. **Statement-1** $\left(\frac{1}{3}\right)^7 < \left(\frac{1}{3}\right)^4$
$\Rightarrow \quad 7\log\left(\frac{1}{3}\right) < 4\log\left(\frac{1}{3}\right) \Rightarrow 7 < 4$
Statement-2 If $ax < ay$, where $a < 0$, $x, y > 0$, then $x > y$.

58. **Statement-1** The equation $x^{\log_x (1-x)^2} = 9$ has two distinct real solutions.
Statement-2 $a^{\log_a b} = b$, when $a > 0$, $a \neq 1$, $b > 0$.

59. **Statement-1** The equation $(\log x)^2 + \log x^2 - 3 = 0$ has two distinct solutions.
Statement-2 $\log x^2 = 2\log x$.

60. **Statement-1** $\log_x 3 \cdot \log_{x/9} 3 = \log_{81}(3)$ has a solution.
Statement-2 Change of base in logarithms is possible.

Logarithms and Their Properties Exercise 7 : Subjective Type Questions

■ In this section, there are **27 subjective** questions.

61. (i) If $\log_7 12 = a$, $\log_{12} 24 = b$, then find value of $\log_{54} 168$ in terms of a and b.

(ii) If $\log_3 4 = a$, $\log_5 3 = b$, then find the value of $\log_3 10$ in terms of a and b.

62. If $\frac{\ln a}{b-c} = \frac{\ln b}{c-a} = \frac{\ln c}{a-b}$, prove the following.

(i) $abc = 1$

(ii) $a^a \cdot b^b \cdot c^c = 1$

(iii) $a^{b^2+bc+c^2} \cdot b^{c^2+ca+a^2} \cdot c^{a^2+ab+b^2} = 1$

(iv) $a + b + c \geq 3$

(v) $a^a + b^b + c^c \geq 3$

(vi) $a^{b^2+bc+c^2} + b^{c^2+ca+a^2} + c^{a^2+ab+b^2} \geq 3$

63. Prove that $\log_{10} 2$ lies between $\frac{1}{3}$ and $\frac{1}{4}$.

64. If $\log 2 = 0.301$ and $\log 3 = 0.477$, find the number of integers in

(i) 5^{200} (ii) 6^{20}

(iii) the number of zeroes after the decimal is 3^{-500}.

65. If $\log 2 = 0.301$ and $\log 3 = 0.477$, find the value of $\log(3.375)$.

66. Find the least value of $\log_2 x - \log_x (0.125)$ for $x > 1$.

67. Without using the tables, prove that

$$\frac{1}{\log_3 \pi} + \frac{1}{\log_4 \pi} > 2.$$

68. Solve the following equations.

(i) $x^{1+\log_{10} x} = 10x$

(ii) $\log_2(9 + 2^x) = 3$

(iii) $2 \cdot x^{\log_4 3} + 3^{\log_4 x} = 27$

(iv) $\log_4 \log_3 \log_2 x = 0$

(v) $x^{\frac{\log_{10} x + 5}{3}} = 10^{5+\log_{10} x}$

(vi) $\log_3\left(\log_9 x + \frac{1}{2} + 9^x\right) = 2x$

(vii) $4^{\log_{10} x+1} - 6^{\log_{10} x} - 2 \cdot 3^{\log_{10} x^2+2} = 0$

(viii) $\frac{\log_{10}(x-3)}{\log_{10}(x^2-21)} = \frac{1}{2}$

(ix) $x^{\log_2 x + 4} = 32$

(x) $\log_a x = x$, where $a = x^{\log_4 x}$

(xi) $\log_{\sqrt{2}\sin x}(1 + \cos x) = 2$

69. Find a rational number, which is 50 times its own logarithm to the base 10.

70. Find the value of the expression

$$\frac{2}{\log_4 (2000)^6} + \frac{3}{\log_5 (2000)^6}.$$

71. Find the value of x satisfying $\log_a\{1 + \log_b\{1 + \log_c(1 + \log_p x)\}\} = 0$.

72. Find the value of $4^{5\log_{4\sqrt{2}}(3-\sqrt{6}) - 6\log_8(\sqrt{3}-\sqrt{2})}$.

73. Solve the following inequations.

(i) $\log_{(2x+3)} x^2 < 1$

(ii) $\log_{2x}(x^2 - 5x + 6) < 1$

(iii) $\log_2(2 - x) < \log_{1/2}(x + 1)$

(iv) $\log_{x^2}(x + 2) < 1$

(v) $3^{\log_3 \sqrt{(x-1)}} < 3^{\log_3 (x-6)} + 3$

(vi) $\log_{1/2}(3x - 1)^2 < \log_{1/2}(x + 5)^2$

(vii) $\log_{10} x + 2 \leq \log_{10}^2 x$

(viii) $\log_{10}(x^2 - 2x - 2) \leq 0$

(ix) $\log_x\left(2x - \frac{3}{4}\right) > 2$

(x) $\log_{1/3} x < \log_{1/2} x$

(xi) $\log_{2x+3} x^2 < \log_{2x+3}(2x + 3)$

(xii) $\log_2^2 x + 3\log_2 x \geq \frac{5}{2}\log_{4\sqrt{2}} 16$

(xiii) $(x^2 + x + 1)^x < 1$

(xiv) $\log_{(3x^2+1)} 2 < \frac{1}{2}$

(xv) $x^{(\log_{10} x)^2 - 3\log_{10} x + 1} > 1000$

(xvi) $\log_4\{14 + \log_6(x^2 - 64)\} \leq 2$

(xvii) $\log_2(9 - 2^x) \leq 10^{\log_{10}(3-x)}$

(xviii) $\log_a\left(\frac{2x+3}{x}\right) \geq 0$ for

(a) $a > 1$, (b) $0 < a < 1$

(xix) $1 + \log_2(x - 1) \leq \log_{x-1} 4$

(xx) $\log_{5x+4}(x^2) \leq \log_{5x+4}(2x + 3)$

74. Solve $\sqrt{\log_x (ax)^{1/5} + \log_a (ax)^{1/5}} + \sqrt{\log_a\left(\frac{x}{a}\right)^{1/5} + \log_x\left(\frac{a}{x}\right)^{1/5}} = a.$

75. It is known that $x = 9$ is root of the equation,

$\log_\pi(x^2 + 15a^2) - \log_\pi(a - 2) = \log_\pi \frac{8ax}{a-2}$

find the other roots of this equation.

76. Solve $\log_4(\log_3 x) + \log_{1/4}(\log_{1/3} y) = 0$ and $x^2 + y^2 = \frac{17}{4}$.

77. Find the real value(s) of x satisfying the equation $\log_{2x}(4x) + \log_{4x}(16x) = 4$.

78. Find the sum and product of all possible values of x which makes the following statement true

$$\log_6 54 + \log_x 16 = \log_{\sqrt{2}} x - \log_{36}\left(\frac{4}{9}\right).$$

79. Solve the equation

$$\frac{3}{2}\log_4(x+2)^3 + 3 = \log_4(4-x)^3 + \log_4(x+6)^3.$$

80. Solve $\log_2(4^{x+1}+4)\cdot\log_2(4^x+1) = \log_{1/\sqrt{2}}\left(\frac{1}{\sqrt{8}}\right)$.

81. Solve the system of equations $2^{\sqrt{x}+\sqrt{y}} = 256$ and $\log_{10}\sqrt{xy} - \log_{10}\left(\frac{3}{2}\right) = 1$.

82. Solve the system of equations $\log_2 y = \log_4(xy-2)$, $\log_9 x^2 + \log_3(x-y) = 1$.

83. Find the solution set of the inequality

$$2\log_{1/4}(x+5) > \frac{9}{4}\log_{\frac{1}{3\sqrt{3}}}(9) + \log_{\sqrt{(x+5)}}(2).$$

84. Solve $\log_3(\sqrt{x} + |\sqrt{x}-1|) = \log_9(4\sqrt{x} - 3 + 4|\sqrt{x}-1|)$.

85. In the inequality

$$(\log_2 x)^4 - \left(\log_{1/2}\frac{x^5}{4}\right)^2 - 20\log_2 x + 148 < 0$$

holds true in (a, b), where $a, b \in N$. Find the value of $ab(a+b)$.

86. Find the value of x satisfying the equation

$$\sqrt{(\log_3 \sqrt[3]{3x} + \log_x \sqrt[3]{3x})\cdot\log_3 x^3} + \sqrt{\left(\log_3 \sqrt[3]{\left(\frac{x}{3}\right)} + \log_x \sqrt[3]{\left(\frac{3}{x}\right)}\right)\log_3 x^3} = 2.$$

87. If P is the number of natural numbers whose logarithm to the base 10 have the characteristic P and Q is the number of natural numbers reciprocals of whose 3 logarithms to the base 10 have the characteristic $-q$, show that $\log_{10} P - \log_{10} Q = p - q + 1$.

Logarithms and Their Properties Exercise 8 : Questions Asked in Previous 13 Year's Exam

- This section contains questions asked in **IIT-JEE, AIEEE, JEE Main & JEE Advanced** from year **2005** to year **2017**.

88. Let $a = \log_3 \log_3 2$ and an integer k satisfying $1 < 2^{(-k+3^{-a})} < 2$, then k equals to **[IIT-JEE 2008, 1.5M]**

(a) 0 (b) 1
(c) 2 (d) 3

89. Let (x_0, y_0) be solution of the following equations $(2x)^{\ln 2} = (3y)^{\ln 3}$ and $3^{\ln x} = 2^{\ln y}$, then x_0 is **[IIT-JEE 2011, 3M]**

(a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) 6

90. The value of

$$6 + \log_{3/2}\left(\frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\cdots}}}\right) \text{ is}$$

[IIT-JEE 2012, 4M]

91. If $3^x = 4^{x-1}$, then x equals **[JEE Advanced 2013, 3M]**

(a) $\frac{2\log_3 2}{2\log_3 2 - 1}$ (b) $\frac{2}{2 - \log_2 3}$
(c) $\frac{1}{1 - \log_4 3}$ (d) $\frac{2\log_2 3}{2\log_2 3 - 1}$

Answers

Exercise for Session 1

1. (a) **2.** (b) **3.** (b) **4.** (b) **5.** (b)

Exercise for Session 2

1. (b) **2.** (b) **3.** (d) **4.** (c) **5.** (c)

Exercise for Session 3

1. (c) **2.** (c) **3.** (d) **4.** (a) **5.** (c)

Chapter Exercises

1. (c) **2.** (d) **3.** (b) **4.** (b) **5.** (a) **6.** (a)
7. (a) **8.** (b) **9.** (a) **10.** (c) **11.** (c) **12.** (c)
13. (b) **14.** (b) **15.** (c) **16.** (b) **17.** (b) **18.** (d)
19. (c) **20.** (b)
21. (a, b, d) **22.** (b, c) **23.** (c, d) **24.** (a, b) **25.** (b, d)
26. (a, c) **27.** (a, b, c, d) **28.** (a, d) **29.** (c, d)
30. (b) **31.** (d) **32.** (a) **33.** (a) **34.** (b) **35.** (d)
36. (b) **37.** (c) **38.** (b) **39.** (b) **40.** (a) **41.** (d)
42. (9) **43.** (6) **44.** (4) **45.** (3) **46.** (9) **47.** (3)
48. (4) **49.** (7) **50.** (1) **51.** (2)
52. (A) → (p, s), (B) → (p), (C) → (q), (D) → (r)
53. (A) → (r), (B) → (s), (C) → (q), (D) → (q)
54. (A) → (q), (B) → (p), (C) → (q, r), (D) → (s)
55. (d) **56.** (d) **57.** (d) **58.** (d) **59.** (c) **60.** (d)
61. (i) $\frac{ab+1}{a(8-5b)}$ (ii) $\frac{ab+2}{2b}$ **64.** (i) 140 (ii)16 (iii) 238 **65.** 0.528
66. $2\sqrt{3}$ **68.** (i) 10, $\frac{1}{10}$ (ii) $x \in \phi$
(iii) $x = 16$ (iv) $x = 8$ (v) $\{10^{-5}, 10^{3}\}$
(vi) $x = \frac{1}{3}$ (vii) $x = \frac{1}{100}$ (viii) $x = 5$ (ix) $x = 2$ or $\frac{1}{32}$
(x) $x = 2$ (xi) $x = \frac{\pi}{3}$

69. 100 **70.** $\frac{1}{6}$ **71.** 1 **72.** 9
73. (i) $x \in \left(-\frac{3}{2}, 3\right) \cup \{-1, 0\}$ (ii) $x \in \left(0, \frac{1}{2}\right) \cup (1, 2) \cup (3, 6)$
(iii) $x \in \left(-1, \frac{1-\sqrt{5}}{2}\right) \cup \left(\frac{1+\sqrt{5}}{2}, 2\right)$
(iv) $x \in (-2, 1) \cup (2, \infty) \sim \{-1, 0\}$ (v) $x > 6$
(vi) $x \in (-\infty, -5) \cup (-5, -1) \cup (3, \infty)$
(vii) $x \in (0, 10^{-1}] \cup [10^{2}, \infty)$
(viii) $x \in [-1, 1-\sqrt{3}) \cup (1+\sqrt{3}, 3]$
(ix) $x \in \left(\frac{3}{8}, \frac{1}{2}\right) \cup \left(1, \frac{3}{2}\right)$ (x) $x \in (0, 1)$
(xi) $x \in \left(-\frac{3}{2}, -1\right) \cup (-1, 3)$ (xii) $x \in \left(0, \frac{1}{16}\right] \cup [2, \infty)$
(xiii) $x \in (-\infty, -1)$ (xiv) $x \in (-\infty, -1) \cup (1, \infty)$
(xv) $x \in (1000, \infty)$ (xvi) $x \in [-10, -8) \cup (8, 10]$
(xvii) $x \in (-\infty, 0]$
(xviii) (a) $x \in (-\infty, -3] \cup (0, \infty)$ (b) $x \in \left[-3, -\frac{3}{2}\right)$
(xix) $x \in (2, 3]$ (xx) $x \in \left(-\frac{3}{5}, -\frac{3}{2}\right) \cup [-1, 0) \cup (0, 3]$

74. $x = a^{4/5_{a2}}$ **75.** $x = 15$ for $a = 3$
76. $x = 2$ or $\frac{1}{2}$, $y = \frac{1}{2}$ or 2
77. $x = 1, 2^{-3/2}$ **78.** Sum = $\frac{9}{2}$, Product = 2
79. $x = 2$ **80.** $x = 0$ **81.** (9, 25) and (25, 9)
82. $x = 3$, $y = 2$ **83.** $x \in (-5, -4) \cup (-3, -1)$
84. $x = \frac{25}{64}$ **85.** 3456 **86.** $x \in (1, 3]$ **88.** (b)
89. (c) **90.** (4) **91.** (a, b, c)

Solutions

1. $\log_{10}2 = 0.3010$

Let $y = 2000^{2000}$

$$\log_{10} y = 2000 \log_{10} 2000 = 2000 \times (\log_{10} 2 + 3)$$
$$= 2000 \times 3.3010 = 6602$$

So, the number of digits in $2000^{2000} = 6602 + 1 = 6603$.

2. $\because\ \lambda > 0$ and $\lambda \neq 1$ and $x > 0$

$$\log_2 x + \log_4 x + \log_8 x = \log_\lambda x$$

$$\Rightarrow \quad \log_2 x + \frac{1}{2}\log_2 x + \frac{1}{3}\log_2 x = \log_\lambda x$$

$$\Rightarrow \quad \frac{11}{6}\log_2 x = \log_\lambda x$$

$$\Rightarrow \quad \frac{11}{6\log_x 2} = \frac{11}{\log_x \lambda}$$

$$\Rightarrow \quad 11\log_x \lambda - 6\log_x 2 = 0$$

$$\Rightarrow \quad \log_x\left(\frac{\lambda^{11}}{2^6}\right) = 0 \ \Rightarrow\ \frac{\lambda^{11}}{2^6} = 1$$

$$\Rightarrow \quad \lambda^{11} = 2^6 \ \Rightarrow\ \lambda = 2^{6/11}$$

$$\Rightarrow \quad \lambda = (2^6)^{1/11} \qquad \ldots(i)$$

Given that, $\lambda = \sqrt[b]{a}$ and $a, b \in N$

$$\Rightarrow \quad \lambda = a^{\frac{1}{b}} \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$a = 2^6 \quad \text{and} \quad b = 11$$

$$\Rightarrow \quad a + b = 64 + 11 = 75$$

3. $x^{\log_{10}^2 x + \log_{10} x^3 + 3} = \dfrac{2}{\dfrac{1}{\sqrt{x+1}-1} - \dfrac{1}{\sqrt{x+1}+1}}$

Given, a, b and c are real solution Eq. (i) and $a > b > c$ and for Eq. (i) to be defined $x > 0$, $x > -1 \ \Rightarrow\ x > 0$ from Eq. (i),

$$x^{\log_{10}^2 x + 3\log_{10} x + 3} = \frac{2x}{2}$$

On taking logarithm both sides on base 10, then

$$(\log_{10}^2 x + 3\log_{10} x + 3)\log_{10} x = \log_{10} x$$

$$\Rightarrow\ (\log_{10}^2 x + 3\log_{10} x + 2)\log_{10} x = 0$$

$$\Rightarrow (\log_{10} x + 1)(\log_{10} x + 2)\log_{10} x = 0$$

$$\therefore \quad \log_{10} x = -2, -1, 0$$

$$\therefore \quad x = 10^{-2}, 10^{-1}, 10^0$$

$$x = \frac{1}{100}, \frac{1}{10}, 1$$

So, a, b, c can take values $a = 1, b = \dfrac{1}{10}, c = \dfrac{1}{100}$ $\quad(\because a > b > c)$

$$\therefore \quad a, b, c \in \text{GP}$$

4. $f(n) = \prod_{i=2}^{n-1} \dfrac{\log(i+1)}{\log(i)} = \dfrac{\log(n)}{\log(2)} = \log_2 n$

$$\therefore \quad f(2^k) = k$$

Then, $\sum_{k=1}^{100} f(2^k) = \sum_{k=1}^{100} k = \dfrac{100 \cdot (100+1)}{2} = 5050$

5. $\log_3 27 \cdot \log_x 7 = \log_{27} x \cdot \log_7 3$ ... (i)

Eq. (i) valid for $x > 0, x \neq 1$

On solving Eq. (i),

$$\log_3(3^3) \cdot \log_x 7 = \frac{1}{3}\log_3 x \cdot \log_7 3$$

$$\Rightarrow \quad 9 \cdot \log_x 7 = \log_7 x$$

$$\Rightarrow \quad 9 = (\log_7 x)^2$$

$$\Rightarrow \quad \log_7 x = \pm 3$$

$$\Rightarrow \quad x = 7^3 \quad \text{or} \quad x = 7^{-3}$$

Then, the least value of x is $\dfrac{1}{7^3}$ i.e., 7^{-3}.

6. $\because\ x = \log_5(5^3 \times 8) = 3 + \log_5 8$

$$\Rightarrow \quad x - 3 = \log_5 8 \qquad \ldots(i)$$

and $\quad y = \log_7(7^3 \times 6) = 3 + \log_7 6$

$$\Rightarrow \quad y - 3 = \log_7 6 \qquad \ldots(ii)$$

$\because \quad 8 > 6$ and $7 > 5$

$$\Rightarrow \quad \log 8 > \log 6 \text{ and } \log 7 > \log 5$$

or $\quad (\log 8)(\log 7) > (\log 6)(\log 5)$

$$\Rightarrow \quad \log_5 8 > \log_7 6$$

$$\Rightarrow \quad x - 3 > y - 3 \qquad \text{[from Eqs. (i) and (ii)]}$$

$$\therefore \quad x > y$$

7. $\because\ \log_5 120 + (x-3) - 2\log_5(1 - 5^{x-3}) = -\log_5(0.2 - 5^{x-4})$

$$\Rightarrow \quad \log_5(5 \times 24) + (x-3)$$
$$= \log_5(1 - 5^{x-3})^2 - \log_5\left(\frac{1 - 5^{x-3}}{5}\right)$$

$$\Rightarrow \quad 1 + \log_5 24 + (x-3) = \log_5\{5 \cdot (1 - 5^{x-3})\}$$

$$\Rightarrow \quad 1 + \log_5(24 \cdot 5^{x-3}) = 1 + \log_5(1 - 5^{x-3})$$

$$\Rightarrow \quad 24 \cdot 5^{x-3} = 1 - 5^{x-3}$$

$$\Rightarrow \quad 25 \cdot 5^{x-3} = 1$$

$$\Rightarrow \quad 5^{x-1} = 5^0$$

$$\therefore \quad x - 1 = 0 \ \Rightarrow\ x = 1$$

8. Given, $x_n > x_{n-1} > \cdots > x_2 > x_1 > 1$

$$\therefore \quad \log_{x_1} \log_{x_2} \log_{x_3} \cdots \log_{x_n} x_n^{x_{n-1}^{\cdot^{\cdot^{x_1}}}}$$
$$= \log_{x_1} \log_{x_2} \log_{x_3} \cdots \log x_{n-1}^{x_{n-1}^{x_{n-2}^{\cdot^{\cdot^{x_1}}}}}$$
$$= \log_{x_1} x_1 = 1 \qquad (\because \log_a a = 1)$$

9. Let $\dfrac{x(y+z-x)}{\log x} = \dfrac{y(z+x-y)}{\log y} = \dfrac{z(x+y-z)}{\log z} = \dfrac{1}{n}$

Then, $\quad \log x = nx(y+z-x) \qquad \ldots(i)$

$$\log y = ny(z+x-y) \qquad \ldots(ii)$$

and $\quad \log z = nz(x+y-z) \qquad \ldots(iii)$

$\therefore \quad y\log x + x\log y = y\log z + z\log y$

$= z\log x + x\log z$

$\Rightarrow \quad \log(x^y \cdot y^x) = \log(y^z \cdot z^y) = \log(x^z \cdot z^x)$

$\Rightarrow \quad x^y \cdot y^x = y^z \cdot z^y = z^x \cdot x^z$

10. $\because \quad y = a^{\frac{1}{1-\log_a x}}$

$\Rightarrow \quad \log_a y = \dfrac{1}{1-\log_a x}$ ...(i)

and $z = a^{\frac{1}{1-\log_a y}}$

or $\log_a z = \dfrac{1}{1-\log_a y}$ ...(ii)

From Eqs. (i) and (ii), we get

$$\log_a z = \frac{1}{1-\left(\dfrac{1}{1-\log_a x}\right)} = 1 - \frac{1}{\log_a x}$$

$\Rightarrow \quad \dfrac{1}{\log_a x} = (1-\log_a z) \Rightarrow \log_a x = \dfrac{1}{(1-\log_a z)}$

$\therefore \quad x = a^{\frac{1}{1-\log_a z}}$

11. $\log_{0.3}(x-1) < \log_{0.09}(x-1)$ ...(i)

Eq. (i) defined for $x > 1$, ...(ii)

$\Rightarrow \quad \log_{0.3}(x-1) - \log_{(0.3)^2}(x-1) < 0$

$\Rightarrow \quad \log_{0.3}(x-1) - \dfrac{1}{2}\log_{0.3}(x-1) < 0$

$\Rightarrow \quad \dfrac{1}{2}\log_{0.3}(x-1) < 0$

$\Rightarrow \quad \log_{0.3}(x-1) < 0$

$\Rightarrow \quad (x-1) > (0.3)^0$

[$\because$ base of log is lie in (0, 1)]

$\Rightarrow \quad x > 2$ ...(iii)

From Eqs. (ii) and (iii), we get

$x > 2 \Rightarrow x \in (2, \infty)$

12. $\because \quad a^x = a^{\sqrt{\log_a b}}$

$= a^{\sqrt{\log_a b}\cdot\sqrt{\log_a b}\sqrt{\log_b a}} = a^{\log_a b\sqrt{\log_b a} = b\sqrt{\log_b a}} = b^y$

$\therefore \quad a^x - b^y = 0$

13. $\because \quad x = 1 + \log_a bc = \log_a a + \log_a bc = \log_a(abc)$

$\therefore \quad \dfrac{1}{x} = \log_{abc} a$ ...(i)

Similarly, $\dfrac{1}{y} = \log_{abc} b$ ...(ii)

and $\dfrac{1}{z} = \log_{abc} c$ ...(iii)

On adding Eqs. (i), (ii) and (iii), we get

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \log_{abc} abc = 1$$

$$\Rightarrow \frac{xy + yz + zx}{xyz} = 1 \text{ or } \frac{xyz}{xy + yz + zx} = 1$$

14. $a^{\frac{\log_b(\log_b N)}{\log_b a}} = a^{\log_a(\log_b N)} = \log_b N$

15. $49^A + 5^B = ?$

$A = 1 - \log_7 2$

$A = \log_7 7 - \log_7 2$

$A = \log_7 \dfrac{7}{2} \Rightarrow 7^A = \dfrac{7}{2} \Rightarrow 49^A = \dfrac{49}{4}$

and $B = -\log_5 4 = \log_5\left(\dfrac{1}{4}\right) \Rightarrow 5^B = \dfrac{1}{4}$

$\therefore \quad 49^A + 5^B = \dfrac{49}{4} + \dfrac{1}{4} = \dfrac{50}{4} = 12.5$

16. $(\log_{16} x)^2 - \log_{16} x + \log_{16}\lambda = 0$ ...(i)

Eq. (i) defined for $x > 0, \lambda > 0 \left(\log_{16} x - \dfrac{1}{2}\right)^2 - \dfrac{1}{4} + \log_{16}\lambda = 0$

For exactly one solution,

$\log_{16} x - \dfrac{1}{2} = 0$

$\therefore \quad -\dfrac{1}{4} + \log_{16}\lambda = 0 \Rightarrow \log_{16}\lambda = \dfrac{1}{4}$

or $\lambda = (16)^{1/4} = 2$

17. $x^{\log_x (x+3)^2} = 16$ ...(i)

From Eq. (i), $x > 0$ and $x \neq 1$ ... (ii)

By Eq. (i), $(x+3)^2 = 16$

$\Rightarrow \quad x + 3 = \pm 4$

$\Rightarrow \quad x = 1$ or $x = -7$

From Eq. (ii), no values of x satisfy Eq. (i).

$\therefore$ Number of values of x satisfy Eq. (i)

$\therefore$ Number of roots = 0

18. Given, $y = \log_2 \log_6(2^{\sqrt{2x+1}} + 4)$ ...(i)

From Eq. (i) to be defined,

$2x + 1 > 0 \Rightarrow x > -\dfrac{1}{2}$ ... (ii)

We find value of x for which $y = 1$

$\therefore \quad 1 = \log_2 \log_6(2^{\sqrt{2x+1}} + 4)$

$\Rightarrow \quad \log_6(2^{\sqrt{2x+1}} + 4) = 2$

$\Rightarrow \quad 2^{\sqrt{2x+1}} + 4 = 36$

$\Rightarrow \quad 2^{\sqrt{2x+1}} = 32 = 2^5 \Rightarrow \sqrt{2x+1} = 5$

$\Rightarrow \quad 2x + 1 = 25 \Rightarrow x = 12$

So, required point is (12, 1).

19. Given that, $\log 2 = 0.301$

$\log 3 = 0.477$

Let $y = 3^{12} \times 2^8$

$\log y = 12\log 3 + 8\log 2$

$= 12 \times (0.477) + 8(0.301) = 8.132$

So, number of digits before decimal in $3^{12} \times 2^8 = 8 + 1 = 9$

20. Given, equation $2\log_x a + \log_{ax} a + 3\log_{a^2x} a = 0$ ...(i)

$$\Rightarrow \quad \frac{2}{\log_a x} + \frac{1}{1+\log_a x} + \frac{3}{2+\log_a x} = 0 \quad \text{...(ii)}$$

Let $\log_a x = t$

Then, Eq. (ii),

$$\frac{2}{t} + \frac{1}{1+t} + \frac{3}{2+t} = 0 \Rightarrow 6t^2 + 11t + 4 = 0$$

$$\Rightarrow \quad t = -\frac{4}{3} \text{ or } -\frac{1}{2}$$

So, $x = a^{-4/3}$ or $x = a^{-1/2}$

Two value of x possible for which Eq. (i) is defined and satisfy.

21. Decimal on $x > 0$ and $x \neq 1$.

Taking logarithm on both sides on base 2, we get

$$\{(\log_2 x)^2 - 6\log_2 x + 11\}\log_2 x = 6$$

Let $\log_2 x = t$

$$\therefore \quad t^3 - 6t^2 + 11t - 6 = 0$$

$$\Rightarrow \quad (t-1)(t-2)(t-3) = 0 \Rightarrow t = 1, 2, 3$$

$$\Rightarrow \quad \log_2 x = 1, 2, 3$$

$$\Rightarrow \quad x = 2, 2^2, 2^3$$

22. $\log_\lambda x \cdot \log_5 \lambda = \log_x 5$ ...(i)

$$\lambda \neq 1, \lambda > 0 \text{ and } x > 0, \quad x \neq 1$$

$$\Rightarrow \quad \log_5 x = \log_x 5 \quad \Rightarrow \quad (\log_5 x)^2 = 1$$

$$\Rightarrow \quad \log_5 x = \pm 1 \quad \Rightarrow \quad x = 5^1 \text{ and } 5^{-1}$$

$$\therefore \quad x = 5 \text{ and } \frac{1}{5}$$

23. $S = \{x : \sqrt{\log_x \sqrt{3x}} : \log_3 x > -1\}$

$$\log_3 x > -1$$

$$\Rightarrow \quad x > \frac{1}{3} \quad \text{...(i)}$$

Let $y = \sqrt{\log_x \sqrt{3x}}$, $x \neq 1$

To be defined y, $3x > 0 \Rightarrow x > 0$ ...(ii)

and $\log_x \sqrt{3x} \geq 0$ ...(iii)

From Eqs. (i) and (iii),

for $x \in \left(\frac{1}{3}, 1\right) \Rightarrow \sqrt{3}x \leq 1$

$$\Rightarrow \quad 3x \leq 1 \quad \Rightarrow \quad x \leq \frac{1}{3}$$

No solution for this case.

Now, for $x > 1$, from Eq. (iii), $\sqrt{3x} \geq 1 \Rightarrow x \geq \frac{1}{3}$

$$\therefore \quad x > 1$$

24. Given equation,

$$\log_2(9^{x-1} + 7) = 2 + \log_2(3^{x-1} + 1)$$

$$\Rightarrow \quad \log_2 \frac{\{3^{2(x-1)} + 7\}}{3^{(x-1)} + 1} = 2$$

$$\Rightarrow \quad 3^{2(x-1)} + 7 = 4 \cdot \{3^{(x-1)} + 1\}$$

$$\Rightarrow \quad \{3^{(x-1)}\}^2 - 4 \cdot 3^{(x-1)} + 3 = 0$$

$$\Rightarrow \quad (3^{x-1} - 3)(3^{x-1} - 1) = 0$$

$$\Rightarrow \quad x - 1 = 1 \text{ or } x - 1 = 0$$

$$\Rightarrow \quad x = 2 \text{ or } x = 1$$

25. $y = \log_p \log_p (\underbrace{\sqrt[p]{\sqrt[p]{\cdots\sqrt[p]{p}}}}_{n \text{ times}})$ $[p > 0, p \neq 1]$

$$= \log_p \left\{ \log_p (\underbrace{\sqrt[p]{\sqrt[p]{\cdots\sqrt[p]{p}}}}_{(n-1) \text{ times}})^{\frac{1}{p}} \right\} = \log_p \left\{ \frac{1}{p} \log_p (\underbrace{\sqrt[p]{\sqrt[p]{\cdots\sqrt[p]{p}}}}_{(n-1) \text{ times}}) \right\}$$

$$= \log_p \left\{ \frac{1}{p} \cdot \frac{1}{p} \log_p (\underbrace{\sqrt[p]{\sqrt[p]{\cdots\sqrt[p]{p}}}}_{(n-2) \text{ times}}) \right\}$$

$$= \log_p \left(\frac{1}{p^n} \right) = -n, \quad \log_{1/p} p^n = -n$$

26. $\log_a x = \alpha$, $\log_b x = \beta$, $\log_c x = \gamma$, $\log_d x = \delta$

$$\Rightarrow \quad \log_x a = \alpha^{-1} \quad \text{...(i)}$$

$$\Rightarrow \quad \log_x b = \beta^{-1} \quad \text{...(ii)}$$

$$\Rightarrow \quad \log_x c = \gamma^{-1} \quad \text{...(iii)}$$

$$\Rightarrow \quad \log_x d = \delta^{-1} \quad \text{...(iv)}$$

On adding Eqs. (i), (ii), (iii) and (iv) , we get

$$\log_x (abcd) = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} \quad \text{...(v)}$$

$$\therefore \quad \log_{abcd} x = \frac{1}{a^{-1} + \beta^{-1} + \gamma^{-1} + \delta^{-1}}$$

For $\alpha, \beta, \gamma, \delta$

$$\text{AM} \geq \text{HM} \Rightarrow \frac{\alpha + \beta + \gamma + \delta}{4} \geq \frac{4}{\alpha^{-1} + \beta^{-1} + \gamma^{-1} + \delta^{-1}}$$

or $$\frac{1}{\alpha^{-1} + \beta^{-1} + \gamma^{-1} + \delta^{-1}} \leq \frac{\alpha + \beta + \gamma + \delta}{16}$$

or $$\log_{abcd} x \leq \frac{\alpha + \beta + \gamma + \delta}{16} \quad \text{[from Eq. (v)]}$$

27. $\because \quad \log_{10} 5 = a \text{ and } \log_{10} 3 = b$ ...(i)

$$\therefore \quad \log_{10} 2 = \log_{10}\left(\frac{10}{5}\right) = 1 - a \quad \text{...(ii)}$$

Option (a)

$$\therefore \quad \log_{10} 8 = 3\log_{10} 2 = 3(1-a) \quad \text{[from Eq. (ii)]}$$

Option (b) $\log_{40} 15 = \frac{\log_{10} 15}{\log_{10} 40} = \frac{\log_{10}(5 \times 3)}{\log_{10}(2^3 \times 5)}$

$$= \frac{\log_{10} 5 + \log_{10} 3}{\log_{10} 2^3 + \log_{10} 5}$$

$$= \frac{a + b}{3(1-a) + a} = \frac{(a+b)}{(3-2a)}$$

Option (c) $\log_{243} 32 = \log_{3^5} 2^5 = \frac{5}{5}\log_5 2 = \frac{\log_{10} 2}{\log_{10} 2}$

$$= \frac{1-a}{a} \quad \text{[from Eqs. (i) and (ii)]}$$

Hence, all options are correct.

28. $\because x > 0$ and $x \neq 1$

Given, $\log_a x$, $\log_b x$ and $\log_c x$ are in AP.

$$\Rightarrow \quad 2\log_b x = \log_a x + \log_c x$$

$$\Rightarrow \quad \frac{2\log x}{\log b} = \frac{\log x}{\log a} + \frac{\log x}{\log c}$$

$$\Rightarrow \quad \frac{2}{\log b} = \frac{1}{\log a} + \frac{1}{\log c} \qquad \begin{bmatrix} \log x \neq 0 \\ \therefore x \neq 1 \end{bmatrix}$$

$$\Rightarrow \quad \log b = \frac{2(\log a)(\log c)}{(\log a + \log c)}$$

Also, $$\frac{\log b}{\log a} = \frac{2\log c}{\log a + \log c}$$

$$\Rightarrow \quad \log_a b = \frac{\log c^2}{\log(ac)} = \log_{(ac)} c^2$$

$$\therefore \quad c^2 = (ac)^{\log_a b}$$

29. $|a| < |b|$, $b - a < 1$

$$a, b \in x^2 - |a|\, x - |\beta| = 0 \qquad \ldots(i)$$

So, $$\left.\begin{aligned} a + b &= |\alpha| \\ ab &= -|\beta| \end{aligned}\right\} \qquad \ldots(ii)$$

Given equation, $\log_{|b|}\left|\frac{x}{a}\right| - 1 = 0$, $\log_{|b|}\left|\frac{x}{a}\right| = 1$

$$\Rightarrow \quad \left|\frac{x}{a}\right| = |b|^1$$

$$\Rightarrow \quad |x| = |ab|$$

$$\Rightarrow \quad |x| = |\beta| \qquad \text{[from Eq. (ii)]}$$

$$\therefore \quad x = \pm\beta$$

Sol. (Q. Nos. 30 to 32)

$$\because \quad \log_2 N = a_1 + b_1$$

$$\Rightarrow \quad b_1 = \log_2 N - a_1$$

Given, $0 \leq b_1 < 1 \Rightarrow 0 \leq \log_2 N - a_1 < 1$

$$\Rightarrow \quad a_1 \leq \log_2 N < 1 + a_1$$

$$\Rightarrow \quad 2^{a_1} \leq N < 2^{1+a_1} \qquad \ldots(i)$$

Similarly, $3^{a_2} \leq N \;\; 3^{1+a_2} \qquad \ldots(ii)$

and $5^{a_3} \leq N < 5^{1+a_3} \qquad \ldots(iii)$

30. Here, $a_1 = 5$ and $a_2 = 3$, then from Eqs. (i) and (ii),

$$2^5 \leq N < 2^6 \quad \text{and} \quad 3^3 \leq N < 3^4$$

$\therefore$ Common values of N are 32, 33, 34, …, 63

Number of integral values of N are 32.

31. Here, $a_1 = 6$, $a_2 = 4$ and $a_3 = 3$, then from Eqs. (i), (ii) and (iii),

$$2^6 \leq N < 2^7, 3^4 \leq N < 3^5 \quad \text{and} \quad 5^3 \leq N < 5^4$$

$\Rightarrow$ 64, 65, 66, …, 127, 81, 82, 83,…, 242 and 125, 126, …, 624

$\therefore$ Largest common value = 127

32. Here, $a_1 = 6$, $a_2 = 4$ and $a_3 = 3$

From question number 31, we get

64, 65, 66,..., 127; 81, 82, 83, ..., 242 and 125, 126, …,624

$\therefore$ Largest common value = 127

and smallest common value = 125

$\therefore$ Difference = 127 − 125 = 2

Sol. (Q. Nos. 33 to 35)

S = Antilog of (0.5) to the base 256

$$\Rightarrow \quad \log_{256} S = 0.5$$

$$S = (256)^{0.5} = (2^8)^{1/2}$$

$$S = 2^4$$

$$S = 16 \qquad \ldots(i)$$

K = Number of digits in 6^{10}

$$[\because \log_{10} 2 = 0.301,\ \log_{10} 3 = 0.477]$$

Let $\alpha = 6^{10}$

$$\log \alpha = 10\log_{10} 6 = 10(0.301 + 0.477)$$

$$= 10(0.778)$$

$$\log(6^{10}) = 7.78$$

So, $x = 7 + 1$, $x = 8$

Number of positive integers which have characteristic 2, when the base of logarithm is 3

$$= 3^{2+1} - 3^2 = 18$$

$\therefore$ $G = 18$

33. The value of $G = 18$

34. The value of $KG = 8 \times 18 = 144$

35. The value of $SKG = 16 \times 8 \times 18 = 16 \times 144 = 2304$

Sol. (Q. Nos. 36 to 38)

U = Number of digits in $(60)^{100}$

Let $\alpha = (60)^{100}$

$$\log_{10} \alpha = 100\log_{10} 60 = 100(1 + \log_{10} 2 + \log_{10} 3)$$

$$= 100\,(1.778)$$

$$\log_{10} \alpha = 177.8$$

So, $U = 177 + 1 \;\Rightarrow\; U = 178 \qquad \ldots(i)$

M = Number of cyphers after decimal, before a significant figure comes in $(8)^{-296}$

Let $\beta = (8)^{-296}$

$$\log_{10} \beta = (-296)\log_{10} 8 = (-296) \times 3\log_{10} 2$$

$$\log_{10} \beta = (-296) \times 3 \times (0.301)$$

$$= -267.288 = -267 - 0.288$$

$$= -267 - 1 + (1 - 0.288) = -268 + 0.712$$

$$\log_{10} \beta = \overline{268}.712$$

$\therefore$ $M = 268 - 1 = 267$

Now, $$\frac{U}{M} = \frac{178}{267}$$

According to the question,

$$\frac{U}{M} = \frac{2}{3}$$

$$\Rightarrow \quad \frac{U}{M} = \frac{p}{q}$$

So, $p = 2$

and $q = 3$

36. The value of $p = 2$.

37. The value of $q = 3$.

38. The equation whose roots are p and q is $x^2 - 5x + 6 = 0$.

Sol. (Q. Nos. 39 to 41)

According to question, $G, O, E, L > 0$ and are real numbers.

Such that,

$\log_{10}(G \cdot L) + \log_{10}(G \cdot E) = 3 \Rightarrow \log_{10} G^2 LE = 3$

$\Rightarrow \quad G^2 LE = 10^3$ ...(i)

and $\log_{10} E \cdot L + \log_{10} E \cdot O = 4$

$\Rightarrow \quad \log_{10} E^2 \cdot L \cdot O = 4$

$\Rightarrow \quad E^2 \cdot L \cdot O = 10^4$ ...(ii)

and $\log_{10}(O \cdot G) + \log_{10}(O \cdot L) = 5$

$\Rightarrow \quad \log_{10} O^2 GL = 5 \Rightarrow O^2 GL = 10^5$ ...(iii)

From Eqs. (i), (ii) and (iii), we get

$$G^3 O^3 E^3 L^3 = 10^{12}$$

$$GOEL = 10^4 \quad \text{...(iv)}$$

$\Rightarrow \quad \lambda = 10^4$

39. Now, let

$$y = \sqrt{\log\lambda\sqrt{\log\lambda\sqrt{\log\lambda\cdots}}} = (\log\lambda)^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots}$$

$$= (\log\lambda)^{\frac{1/2}{1-1/2}} = (\log\lambda)$$

$$= \log 10^4 = 4\log 10 = 4$$

40. Minimum of $3G + 2L + 2O + E = 2^{\lambda}3^{\mu}5^{\upsilon}$

where $\lambda, \mu, \upsilon \in W$

Apply AM $\geq$ GM for $3G, 2L, 2O, E$

$$\frac{3G + 2L + 2O + E}{8} \geq \sqrt[8]{G^3 \times L^2 \times O^2 \times E}$$

So, $$8 \times \sqrt[8]{G^3 L^2 O^2 E} = 2^{\lambda}3^{\mu}5^{\upsilon} \quad \text{...(v)}$$

(equality hold, if $G = L = O = E$)

From Eqs. (i) and (iii) of Q. 10, we get

$$G^3 L^2 O^2 E = 10^8$$

From Eq. (v), $$8 \times (10^8)^{1/8} = 2^{\lambda}3^{\mu}5^{\upsilon}$$

$$8 \times 10 = 2^{\lambda}3^{\mu}5^{\upsilon}$$

$$2^4 \times 5^1 = 2^{\lambda}3^{\mu}5^{\upsilon}$$

$\therefore \quad \lambda = 4, \upsilon = 1, \mu = 0$

$$\Sigma(\lambda^{\mu} + \mu^{\lambda}) = (4^0 + 0^4) + (0^1 + 1^0) + (1^4 + 4^1)$$

$$= (1 + 0) + (0 + 1) + 1 + 4 = 7$$

41. $\log_{10}\left(\frac{G}{O}\right) + \log_{10}\left(\frac{O}{E}\right) = \log_{10}\left(\frac{G}{E}\right) = \log_{10} 1 = 0$

[divide Eq. (iv) and Eq. (ii) of Q. 39]

$$P = \log_{10}\frac{G}{O} \cdot \log_{10}\frac{O}{E} = \log\left(\frac{1}{10}\right)\log(10) = -1$$

[by dividing Eq. (i) by Eq. (ii) and dividing Eq. (iii) by Eq. (iv) in Q. 10]

$$= x^2 - 0 \cdot x + (-1) = 0 = x^2 - 1$$

42. $\log_{10}(2x) + \log_{10} y = 2 \Rightarrow 2xy = 10^2$ ...(i)

and $\log_{10} x^2 - \log_{10} 2y = 4$

$\Rightarrow \quad \frac{x^2}{2y} = 10^4$ ...(ii)

From Eqs. (i) and (ii), $x^3 = 10^6 \Rightarrow x = 100$

From Eq. (i), $y = \frac{1}{2}$

$\therefore \quad x + y = 100 + \frac{1}{2} = \frac{201}{2} = \frac{m}{n}$ (given)

$\therefore \quad m = 201$ and $n = 2$

$\Rightarrow \quad m - 3n^6 = 201 - 3(2)^6 = 201 - 192 = 9$

43. Solving, $x = \lambda$ and $y = \log_5 x$, we get

$$A \equiv (\lambda, \log_5 \lambda), \lambda > 0$$

and solving $x = \lambda$ and $y = \log_5(x + 4)$, we get

$$B \equiv \{\lambda, \log_5(\lambda + 4)\}, \lambda > -4$$

Given, $AB = 0.5$

$\Rightarrow \quad \log_5(\lambda + 4) - \log_5 \lambda = 0.5$

$\Rightarrow \quad \frac{\lambda + 4}{\lambda} = (5)^{1/2} = \sqrt{5}$

$\Rightarrow \quad \lambda = \frac{4}{\sqrt{5} - 1} = 4\frac{(\sqrt{5} + 1)}{4}$

$= 1 + \sqrt{5} = a + \sqrt{b}$ [given]

$\therefore \quad a = 1$ and $b = 5$

Then, $a + b = 1 + 5 = 6$

44. $\therefore \quad a(b - c)x^2 + b(c - a)xy + c(a - b)y^2 = b, y \neq 0$ ...(i)

$$a(b - c)\left(\frac{x}{y}\right)^2 + b(c - a)\left(\frac{x}{y}\right) + c(a - b) = 0$$

Let $\frac{x}{y} = X$

$\Rightarrow \quad a(b - c)X^2 + b(c - a)X + c(a - b) = 0$

$\because \quad a(b - c) + b(c - a) + c(a - b) = 0$

$\therefore \quad X = 1$

$\because$ Eq. (i) is perfect square.

$\therefore$ Roots are equal.

$\therefore \quad 1 \times 1 = \frac{c(a - b)}{a(b - c)}$

$\Rightarrow \quad b = \frac{2ac}{a + c}$ ...(ii)

Now, $\log(a + c) + \log(a - 2b + c)$

$= \log\{(a + c)^2 - 2b(a + c)\}$

$= \log\{(a + c)^2 - 4ac\}$ [from Eq. (ii)]

$= \log(a - c)^2 = 2\log(a - c)$

$\Rightarrow \quad \frac{\log(a + c) + \log(a - 2b + c)}{\log(a - c)} = 2$

$\therefore \quad \left\{\frac{\log(a + c) + \log(a - 2b + c)}{\log(a - c)}\right\}^2 = 4$

45. According to the question, $x \in I$

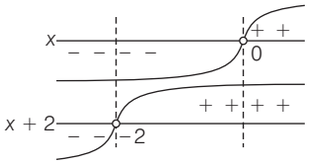
Given equation, $\left(\frac{1}{3}\right)^{\frac{|x+2|}{2-|x|}} > 9 \quad [x \neq \pm 2]$ ...(i)

$\Rightarrow \quad 3^{\frac{|x+2|}{2-|x|}} > 3^2 \Rightarrow -\frac{|x + 2|}{2 - |x|} > 2$

$$\Rightarrow \quad \frac{|x+2|}{|x|-2}-2>0$$

$$\Rightarrow \quad \frac{|x+2|-2|x|+4}{|x|-2}>0 \qquad \ldots(ii)$$

***Case* I** If $x<-2$, $-\dfrac{x-2+2x+4}{-x-2}>0$



$$\frac{x+2}{-(x+2)}>0 \Rightarrow -1>0$$

which is not possible.

***Case* II** $-2<x<0$, then Eq. (ii)

$$\Rightarrow \quad \frac{x+2+2x+4}{-x-2}>0 \Rightarrow \frac{3x+6}{-(x+2)}>0$$

$$\frac{-3(x+2)}{(x+2)}>0-3>0$$

which is not possible.

***Case* III** when $x>0$

From Eq. (ii),

$$\frac{x+2-2x+4}{x-2}>0 \Rightarrow \frac{-x+6}{x-2}>0$$

$$\frac{x-6}{x-2}<0$$

$$2<x<6$$

So, the integer values of $x=3, 4, 5$

So, the number of integer values of x is 3.

46. $x>2$

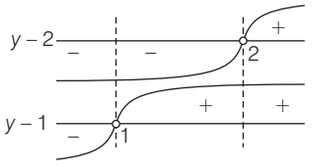
$$|\log_{\sqrt{3}} x-2|+|\log_3 x-2|=2$$

$$|2\log_3 x-2|+|\log_3 x-2|=2$$

$$2|\log_3 x-1|+|\log_3 x-2|=2$$

Let $\log_3 x=y$ ...(i)

Then, Eq. (i) $\Rightarrow 2|y-1|+|y-2|=2$ ...(ii)



***Case* I** $y<1$, then $x<3$

Eq. (ii) becomes $-2y+2-y+2=2$

$$-3y=-2, y=\frac{2}{3}$$

$$\log_3 x=\frac{2}{3} \qquad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad x=3^{2/3}$$

which is less than 2, so not acceptable.

***Case* II** $1<y<2$, then $3<x<9$

From Eq. (ii), $2(y-1)-(y-2)=2$

$$\Rightarrow \quad y=2$$

$$\Rightarrow \quad \log_3 x=2$$

$$\therefore \quad x=3^2=9 \qquad \text{[impossible]}$$

***Case* III** $y\geq 2$, then $x\geq 9$

From Eq. (ii), $2(y-1)+(y-2)=2$

$$\therefore \quad y=2, \log_3 x=2$$

$$\therefore \quad x=9 \qquad \text{[acceptable]}$$

47. Given equation is

$$\log_2\sqrt{x}-2\log_{1/4}^2 x+1>0 \qquad \ldots(i)$$

From Eq. (i), $x>0$

Eq. (i) $\Rightarrow \dfrac{1}{2}\log_2 x-\dfrac{2}{(-2)^2}\log_2^2 x+1>0$

$$\Rightarrow \quad \frac{1}{2}\log_2 x-\frac{1}{2}\log_2^2 x+1>0$$

$$\Rightarrow \quad (\log_2 x)^2-(\log_2 x)-2<0$$

$$\Rightarrow \quad (\log_2 x-2)(\log_2 x+1)<0$$

$$\Rightarrow \quad -1<\log_2 x<2$$

$$\Rightarrow \quad 2^{-1}<x<2^2$$

$$\Rightarrow \quad \frac{1}{2}<x<4$$

$$\Rightarrow \quad x\in I, \text{ so } x=1, 2, 3$$

So, number of integer value of x is 3.

48. Given that, $b>0$

$$2\log_{1/25}(bx+28)=-\log_5(12-4x-x^2) \qquad \ldots(i)$$

$$\frac{2}{(-2)}\log_5(bx+28)=-\log_5(12-4x-x^2)$$

$$\Rightarrow \quad bx+28=12-4x-x^2$$

and $bx+28>0$

and $12-4x-x^2>0$

$$\Rightarrow \quad x^2+(4+b)x+16=0 \qquad \ldots(ii)$$

and $x>\dfrac{-28}{b}$ and $-6<x<2$

Since, Eq. (i) has coincident roots, so discriminant Eq. (ii) is zero.

$$(4+b)^2-64=0$$

$$b+4=\pm 8$$

$$b=4 \text{ or } b=-12$$

Since, $b>0$ so $b=4$

for this value $x>-7$ and $-6<x<2$

49. $\dfrac{2^{\log_{1/4}2}-3^{\log_{27}125}-4}{7^{4\log_{49}2}-3}=\dfrac{2^{4\log_2 2}-3^{\log_{3^3}5^3}-4}{7^{4\log_{7^2}2^1}}$

$$=\frac{2^4-5-4}{7^{2\log_7 2}-3}=\frac{16-9}{2^2-3}=7$$

50. $(\log_5 x)^2+\log_{5x}\left(\dfrac{5}{x}\right)=1, x>0, x\neq\dfrac{1}{5}$

$$\Rightarrow \quad (\log_5 x)^2+\frac{\log_5\left(\frac{5}{x}\right)}{\log_5(5x)}=1 \Rightarrow (\log_5 x)^2+\frac{1-\log_5 x}{1+\log_5 x}=1$$

Let $\log_5 x=t$, then

$$t^2 + \frac{1-t}{1+t} = 1$$

$\Rightarrow \quad t^3 + t^2 - 2t = 0$

$\Rightarrow \quad t(t+2)(t-1) = 0 \quad \Rightarrow \quad t = -2, 0, 1$

$\Rightarrow \quad x = 5^{-2}, 5^0, 5^1$

$\Rightarrow \quad x = \frac{1}{25}, 1, 5$

$x_1, x_2 \in I$

$\therefore \quad x_1 = 1, x_2 = 5$

$\therefore \quad |x_2 - 4x_1| = |5 - 4| = 1$

51. Given, $x = \log_\lambda a = \log_a b = \frac{1}{2}\log_b c$ and $\log_\lambda c = nx^{n+1}$

$x = \log_\lambda a = \log_a b = \log_b \sqrt{c}$ and $\log_\lambda c = nx^{n+1}$ ...(i)

From Eq. (i), $\log_\lambda a \times \log_a b + \log_b \sqrt{c} = x^3$

$\log_\lambda \sqrt{c} = x^3, \frac{1}{2}\log_\lambda c = x^3$

$\log_\lambda c = 2x^3$

Compare with $\log_\lambda c = nx^{n+1}$

$\Rightarrow \quad n = 2$

52. (A) $\frac{\log_3 243}{\log_2 \sqrt{32}} = \frac{\log_3 3^5}{-\frac{1}{2}\log_2 2^5} = \frac{5 \times 2}{5} = 2$ (p,s)

(B) $\frac{2\log 6}{\log 12 + \log 3} = \frac{2\log 6}{\log 36} = \frac{2\log 6}{2\log 6} = 1$ (p)

(C) $\log_{1/3}\left(\frac{1}{9}\right)^{-2} = -\log_3 3^4 = -4$ (q)

(D) $\frac{\log_5 16 - \log_5 4}{\log_5 128} = \frac{\log_5\left(\frac{16}{4}\right)}{\log_5 (2)^7} = \frac{\log_5 (2)^2}{\log_5 (2)^7} = \frac{2}{7}$ (r)

53. (A) $\sqrt{\log^2_{(0.5)^2} 8} = \sqrt{\log^2_{1/2} 8} = \sqrt{(\log_{2^{-1}} 2^3)^2}$

$= \sqrt{\left(\frac{3}{-1}\log_2 2\right)^2} = \sqrt{(-3)^2} = \sqrt{9} = 3$ (r)

(B) $(\log_{10} 2)^3 + \log_{10} 8 \cdot \log_{10} 5 + (\log_{10} 5)^3$

$= (\log_{10} 2)^3 + 3\log_{10} 2 \log_{10} 5 + (\log_{10} 5)^3$

$= (\log_{10} 2)^3 + 3 \cdot \log_{10} 2 \cdot \log_{10} 5 \cdot (\log_{10} 2 + \log_{10} 5) + (\log_{10} 5)^3$

$[\because \log_{10} 2 + \log_{10} 5 = \log_{10} 10 = 1]$

$= (\log_{10} 2 + \log_{10} 5)^3 = (\log_{10} 10)^3 = (1)^3 = 1$

$3 + (\log_{10} 2)^3 + \log_{10} 8 \cdot \log_{10} 5 + (\log_{10} 5)^3$

$= 3 + 1 = 4$ (s)

(C) $N = \log_2 15 \cdot \log_{1/6} 2 \cdot \log_3 \frac{1}{6}$

$= \log_2 15 (-\log_6 2)(-\log_3 6)$

$= \frac{\log 15}{\log 2} \times \frac{\log 2}{\log 6} \times \frac{\log 6}{\log 3} = \log_3 15$

$9 < 15 < 27$

$2 < \log_3 15 < 3$

So, $[N] = 2$ (q)

(D) $(52.6)^a = (0.00526)^b = 100$

$(52.6)^a = 100$ and $(0.00526)^b = 100$

$52.6 = 10^{\frac{2}{a}}$ ...(i)

$(52.6)^b \times 10^{-4b} = 10^2$

$(52.6)^b = 10^{2+4b}$

$\Rightarrow \quad 52.6 = 10^{\left(\frac{2+4b}{b}\right)}$ ...(ii)

From Eqs. (i) and (ii), we get

$10^{\frac{2}{a}} = 10^{\frac{2+4b}{b}}$

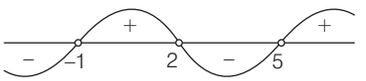
$\Rightarrow \quad \frac{2}{a} = \frac{2}{b} + 4$

$\Rightarrow \quad \frac{1}{a} - \frac{1}{b} = 2$ (q)

54. (A) Given that, $\log_{1/x}\frac{2(x-2)}{(x+1)(x-5)} \geq 1$ ...(i)

for log to be defined $\frac{(x-2)}{(x+1)(x-5)} > 0$,

then $x \in (-1, 2) \cup (5, \infty)$



Let $x > 0$ and $x \neq 1$

So, $x \in (0, 1) \cup (1, 2) \cup (5, \infty)$

***Case* I** $x \in (0, 1)$ ...(ii)

$\frac{1}{x} > 1$

$\therefore$ By Eq. (i), $\log_{\frac{1}{x}}\frac{2(x-2)}{(x+1)(x-5)} \geq 1$

$\Rightarrow \quad \frac{2(x-2)}{(x+1)(x-5)} \geq \frac{1}{x}$

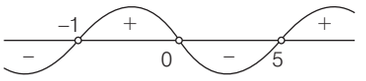
$\Rightarrow \quad \frac{2(x-2)}{(x+1)(x-5)} - \frac{1}{x} \geq 0$

$\Rightarrow \quad \frac{2x(x-2) - (x+1)(x+5)}{x(x+1)(x-5)} \geq 0$

$\Rightarrow \quad \frac{2x^2 - 4x - x^2 + 4x + 5}{x(x+1)(x-5)} \geq 0$

$\Rightarrow \quad \frac{x^2 + 5}{x(x+1)(x-5)} \geq 0$

$\Rightarrow \quad x(x+1)(x-5) > 0$



$\Rightarrow \quad x \in (-1, 0) \cup (5, \infty)$

But by Eq. (ii), $x \in (0, 1)$

So, no solution for this case.

***Case* II** Let $x \in (1, 2) \cup (5, \infty)$ ...(iii)

$$\frac{1}{x} < 1$$

Eq. (i) $\Rightarrow \log_{\frac{1}{x}} \frac{2(x-2)}{(x+1)(x-5)} \geq 1$

$$\frac{2(x-2)}{(x+1)(x-5)} \leq \frac{1}{x}$$

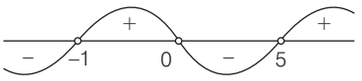
$$\Rightarrow \quad \frac{2(x-2)}{(x+1)(x-5)} - \frac{1}{x} \leq 0$$

$$\Rightarrow \quad \frac{x^2+5}{x(x+1)(x-5)} \leq 0 \qquad \text{[by case I]}$$

$$\Rightarrow \quad x(x+1)(x-5) < 0$$

$$\Rightarrow \quad x \in (-\infty, -1) \cup (0, 5) \qquad \ldots(iv)$$

Eq. (iii), $x \in (1, 2) \cup (5, \infty)$



From Eqs. (iii) and (iv), $x \in (1, 2]$ (q)

(B) $\log_3 x - \log_3^2 x \leq \frac{3}{2} \log_{\frac{1}{2\sqrt{2}}} 4$ ...(i)

defined, when $x > 0$

$$\log_3 x - \log_3^2 x \geq \frac{3}{2} \times \left(\frac{-2}{3}\right) \times 2 \times 1$$

$$\Rightarrow \quad \log_3 x - \log_3^2 x + 2 \leq 0$$

$$\Rightarrow \quad \log_3^2 x - \log_3 x - 2 \geq 0$$

$$\Rightarrow \quad (\log_3 x - 2)(\log_3 x + 1) \geq 0$$

$$\Rightarrow \quad \log_3 x \leq -1$$

or $\log_3 x \geq 2$

$$\Rightarrow \quad x \leq \frac{1}{3} \text{ or } x \geq 9$$

From Eq. (i), $x > 0$

So, $x \in \left(0, \frac{1}{3}\right] \cup [9, \infty)$ (p)

(C) $\log_{\frac{1}{2}} (4 - x) \geq \log_{\frac{1}{2}} 2 - \log_{\frac{1}{2}} (x - 1)$ ...(i)

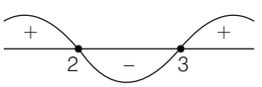
$$\Rightarrow \quad \log_{\frac{1}{2}} \frac{(4-x)(x-1)}{2} \geq 0$$

$$\Rightarrow \quad -\frac{(x-4)(x-1)}{2} \leq 1$$

$$\Rightarrow \quad (x-4)(x-1) \geq -2$$

$$\Rightarrow \quad x^2 - 5x + 4 + 2 \geq 0$$

$$\Rightarrow \quad x^2 - 5x + 6 \geq 0$$



$$(x-3)(x-2) \geq 0$$

$$x \leq 2 \text{ or } x \geq 3 \qquad \ldots(ii)$$

From Eq. (i) to be defined, $4 - x > 0$ and $x - 1 > 0$

$$x < 4 \text{ and } x > 1 \qquad \ldots(iii)$$

From Eqs. (ii) and (iii),

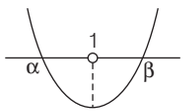
$$x \in (1, 2] \cup [3, 4) \text{ (q, r)}$$

(D) Given equation is

$$(\lambda^2 - 3\lambda + 4)\, x^2 - 4(2\lambda - 1)\, x + 16 = 0 \qquad \ldots(i)$$

$$\lambda^2 - 3\lambda + 4 = \lambda^2 - 3\lambda + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 4 = \left(\lambda - \frac{3}{2}\right)^2 + \frac{7}{4}$$

So, $\lambda^2 - 3\lambda + 4 > 0, \forall\ \lambda \in R$



and $D > 0$

$\Rightarrow$ We get $\lambda > \frac{15}{8}$ ...(ii)

Let $f(x) = (\lambda^2 - 3\lambda + 4)\, x^2 = 4(2\lambda - 1)\, x + 16$

$\therefore \quad f(1) < 0$ by graph of $f(x)$

$$\lambda^2 - 11\lambda + 24 < 0$$

$$(\lambda - 3)(\lambda - 8) < 0$$

$$3 < \lambda < 8 \qquad \ldots(iii)$$

From Eqs. (ii) and (iii), we get

$$3 < \lambda < 8 \ \Rightarrow\ \lambda \in (3, 8) \text{ (s)}$$

55. If $0 < a < b$

Statement-1 If $x > 1$

$$\Rightarrow \quad \log_x a < \log_x b$$

$\therefore$ **Statement-2** If $0 < x < 1$

$$\Rightarrow \quad \log_x a > \log_x b$$

$\therefore$ Statement-2 is true, also

$$10 > 3 > e > 2$$

If $x > 1$,

then $\log_x 10 > \log_x 3 > \log_x e > \log_x 2$

$$\Rightarrow \quad \frac{1}{\log_x 10} < \frac{1}{\log_x 3} < \frac{1}{\log_x e} < \frac{1}{\log_x 2}$$

$$\Rightarrow \quad \log_{10} x < \log_3 x < \log_e x < \log_2 x$$

and for $0 < x < 1$

We get, $\log_{10} x > \log_3 x > \log_e x > \log_2 x$

It is clear that for $x > 0, x \neq 1$

Statement-1 is false.

56. **Statement-1** $7^{\log_7 (x^3 + 1)} - x^2 = 1$ ...(i)

$$\left.\begin{aligned} x^3 + 1 - x^2 &= 1 \\ x^3 - x^2 &= 0 \\ x^2(x-1) &= 0 \end{aligned}\right| \begin{aligned} &\text{for this } x^3 + 1 > 0 \\ &\Rightarrow x^3 > -1 \\ &\Rightarrow x > -1 \end{aligned}$$

$x = 0$ (repeated) or $x = 1$

Thus, Eq. (i) has 2 repeated roots.

$\therefore$ Statement-1 is false.

Statement-2 $a^{\log_a N} = N$, $a > 0$, $a \neq 1$ and $N > 0$

which is true.

57. **Statement-1** $\left(\frac{1}{3}\right)^7 < \left(\frac{1}{3}\right)^4$. Taking log on both sides,

$$\log_e \left(\frac{1}{3}\right)^7 < \log_e \left(\frac{1}{3}\right)^4$$

$$7 \log_e \frac{1}{3} < 4 \log_e \frac{1}{3}$$

Now, $\log_e \frac{1}{3} < 0$ $[\because 2 < e < 3]$

So, $7 > 4$

Statement-1 is false. ...(i)

Statement-2 $ax < ay$

and $a < 0,\ x > 0,\ y > 0$

Eq. (i) divide by a, we get $x > y$

Statement-2 is true.

58. **Statement-1** $x^{\log_x (1-x)^2} = 9$

$$(1-x)^2 = 9 \qquad \begin{cases} \text{Eq. (i) is defined, if} \\ x \neq 1,\ x > 0 \end{cases}$$

$$1 - x = \pm 3$$

$\therefore$ $x = -2$ or 4

$x = 4$ [acceptable]

$\therefore$ Eq. (i) has only one solution.

Statement-1 is false.

Statement-2 $a^{\log_a b} = b$, where $a > 0,\ a \neq 1,\ b > 0$

which is true.

59. **Statement-1** $(\log x)^2 + \log x^2 - 3 = 0$ ...(i)

$\Rightarrow$ $(\log x)^2 + 2 \log x - 3 = 0$

$\Rightarrow$ $(\log x + 3)(\log x - 1) = 0$

$\Rightarrow$ $\log x = -3$ or $\log x = 1$

$\Rightarrow$ $x = 10^{-3}$ or $x = 10$

Eq. (i) is defined for $x > 0$.

So, Eq. (i) has 2 distinct solutions.

Statement-2 $\log x^2 \neq 2 \log x$

$\because$ LHS has domain $x \in R$ and RHS has domain $x \in (0, \infty)$

$\therefore$ Statement-2 is false.

60. **Statement-1**

$$\log_x 3 \cdot \log_{x/9} 3 = \log_{81} 3 \quad \text{...(i)}$$

Eq. (i) holds, if $x > 0,\ x \neq 1,\ x \neq 9$

By Eq. (i), $\frac{1}{\log_3 x} \cdot \frac{1}{(\log_3 x + 2)} = \frac{1}{4}$

$$(\log_3 x)^2 + 2 \log_3 x - 4 = 0$$

$$(\log_3 x)^2 + 2 \log_3 x + 4 = 8$$

$$(\log_3 x + 2)^2 = 8$$

$$\log_3 x + 2 = \pm 2\sqrt{2}$$

$$\log_3 x = 2(-1 \pm \sqrt{2})$$

$\therefore$ $x = 3^{2(-1 \pm \sqrt{2})}$

Two values of x satisfying Eq. (i)

So, Statement-1 is false.

Statement-2 Change of bases in logarithm is possible.

$\therefore$ Statement-2 is true.

61. (i) $\because a = \log_7 12 = \frac{\log 12}{\log 7} = \frac{2 \log 2 + \log 3}{\log 7}$

$$a = \frac{2 + \log_2 3}{\log_2 7} \quad \text{...(i)}$$

and $b = \log_{12} 24 = \frac{\log 24}{\log 12} = \frac{3 \log 2 + \log 3}{2 \log 2 + \log 3}$

$$= \frac{3 + \log_2 3}{2 + \log_2 3} \quad \text{...(ii)}$$

Let $\log_2 3 = \lambda$ and $\log_2 7 = \mu$

From Eq. (i), $a = \frac{2 + \lambda}{\mu}$

and from Eq. (ii), $b = \frac{3 + \lambda}{2 + \lambda}$, we get

$$\lambda = \frac{3 - 2b}{b - 1} \text{ and } \mu = \frac{1}{a(b-1)}$$

$\therefore$ $\log_{54} 168 = \frac{\log 168}{\log 54} = \frac{\log (2^3 \times 3 \times 7)}{\log (3^3 \times 2)}$

$$= \frac{3 \log 2 + \log 3 + \log 7}{3 \log 3 + \log 2}$$

$$= \frac{3 + \log_2 3 + \log_2 7}{3 \log_2 3 + 1} = \frac{3 + \lambda + \mu}{3\lambda + 1}$$

$$= \frac{3 + \frac{3 - 2b}{b-1} + \frac{1}{a(b-1)}}{\frac{3(3-2b)}{b-1} + 1}$$

$$= \frac{(ab + 1)}{a(8 - 5b)}$$

(ii) $\because$ $a = \log_3 4$ and $b = \log_5 3$

$\therefore$ $ab = \log_5 4$ ...(i)

Now, $\log_3 10 = \frac{\log_5 10}{\log_5 3} = \frac{2 \log_5 10}{2 \log_5 3}$

$$= \frac{\log_5 (100)}{2b} = \frac{\log_5 (4 \times 25)}{2b}$$

$$= \frac{\log_5 4 + 2}{2b} = \frac{ab + 2}{2b} \quad \text{[from Eq. (i)]}$$

62. $\because \frac{\ln a}{b-c} = \frac{\ln b}{c-a} = \frac{\ln c}{a-b}$ [by using law of proportion]

(i) $\because$ $\frac{\ln a}{b-c} = \frac{\ln b}{c-a} = \frac{\ln c}{a-b}$

$$= \frac{\ln a + \ln b + \ln c}{b - c + c - a + a - b} = \frac{\ln (abc)}{0}$$

$\Rightarrow \ln (abc) = 0 \Rightarrow abc = 1$

(ii) $\frac{\ln a}{b-c} + \frac{\ln b}{c-a} + \frac{\ln c}{a-b} = \frac{a \ln a + b \ln b + c \ln c}{a(b-c) + b(c-a) + c(a-b)}$

$$= \frac{\ln a^a + \ln b^b + \ln c^c}{0} = \frac{\ln (a^a \cdot b^b \cdot c^c)}{0}$$

$\Rightarrow \ln (a^a b^b c^c) = 0$

$\Rightarrow$ $a^a b^b c^c = 1$

(iii) $\dfrac{\ln a}{b-c}=\dfrac{\ln b}{c-a}=\dfrac{\ln c}{a-b}$

$$=\frac{\begin{matrix}[(b^2+bc+c^2)\ln a+(c^2+ca+a^2)\ln b\\+(a^2+ab+b^2)\ln c]\end{matrix}}{\begin{matrix}[(b^2+bc+c^2)(b-c)+(c^2+ca+a^2)(c-a)\\+(a^2+ab+b^2)(a-b)]\end{matrix}}$$

$$=\frac{\ln a^{b^2+bc+c^2}+\ln b^{c^2+ca+a^2}+\ln c^{a^2+ab+b^2}}{(b^3-c^3)+(c^3-a^3)+(a^3-b^3)}$$

$$=\frac{\ln\left(a^{b^2+bc+c^2}\cdot b^{c^2+ca+a^2}\cdot c^{a^2+ab+b^2}\right)}{0}$$

$\Rightarrow \ln\left(a^{b^2+bc+c^2}\cdot b^{c^2+ca+a^2}\cdot c^{a^2+ab+b^2}\right)=0$

$\therefore \quad a^{b^2+bc+c^2}\cdot b^{c^2+ca+a^2}\cdot c^{a^2+ab+b^2}=1$

(iv) $\because$ AM $\geq$ GM

$\therefore \quad \dfrac{a+b+c}{3}\geq (abc)^{1/3}=(1)^{1/3}=1$ [from Eq. (i)]

$\therefore \quad \dfrac{a+b+c}{3}\geq 1 \Rightarrow a+b+c\geq 3$

(v) $\because$ AM $\geq$ GM

$\Rightarrow \quad \dfrac{a^a+b^b+c^c}{3}\geq (a^a\cdot b^b\cdot c^c)^{1/3}$

$=(1)^{1/3}=1$ [from Eq. (ii)]

$\Rightarrow \quad \dfrac{a^a+b^b+c^c}{3}\geq 1 \Rightarrow a^a+b^b+c^c\geq 3$

(vi) $\because$ AM $\geq$ GM $\quad \dfrac{a^{b^2+bc+c^2}+b^{c^2+ca+a^2}+c^{a^2+ab+b^2}}{3}$

$\geq (a^{b^2+bc+c^2}\cdot b^{c^2+ca+a^2}\cdot c^{a^2+ab+b^2})^{1/3}$

$=(1)^{1/3}$ [from Eq. (iii)]

$=1$

$\Rightarrow \quad \dfrac{a^{b^2+bc+c^2}+b^{c^2+ca+a^2}+c^{a^2+ab+b^2}}{3}\geq 1$

$\Rightarrow \quad a^{b^2+bc+c^2}+b^{c^2+ca+a^2}+c^{a^2+ab+b^2}\geq 3$

63. To prove $\log_{10}2$ lies between $\dfrac{1}{3}$ and $\dfrac{1}{4}$

$2^{12}=4096$

$1000<4096<10000$

$10^3<2^{12}<10^4$

Taking logarithm to the base 10,

$\log_{10}10^3<\log_{10}2^{12}<\log_{10}10^4$

$3<12\log_{10}2<4\Rightarrow \dfrac{1}{4}<\log_{10}2<\dfrac{1}{3}$

64. $\log 2=0.301$

$\log 3=0.477$

(i) Let $\alpha=5^{200}$

$\log\alpha=200\log 5=200(\log 10-\log 2)=200(1-0.301)$

$=200\times 0.699=139.8$

So, number of integers in $5^{200}=139+1=140$.

(ii) $\alpha=6^{20}$

$\therefore \log\alpha=20\log 6=20(\log 2+\log 3)$

$=20(0.310+0.477)$

$=20\times 0.778=15.560$

So, number of integers in $6^{20}=15+1=16$

(iii) Let $\alpha=3^{-500}$

$\log\alpha=-500\log 3=-500\times(0.477)=-238.5$

$=-239+0.5=\overline{239}.5$

So, number of zeroes after the decimal in $3^{-500}=239-1=238$

65. Given that, $\log_{10}2=0.301$

and $\log_{10}3=0.477$

$\log 3.375=\log(3375)-\log 10^3=\log 5^3\times 3^3-3\log 5\times 2$

$=3\log 5+3\log 3-3\log 5-3\log 2$

$=3(0.477)-3(0.301)=3(0.176)$

$=0.528$

66. Let $P=\log_2 x-\log_x(0.125)=\log_2 x-\log_x\left(\dfrac{1}{8}\right)$

$=\log_2 x+3\log_x 2$

$\therefore \quad$ AM $\geq$ GM

$\Rightarrow \quad \dfrac{\log_2 x+3\log_x 2}{2}\geq\sqrt{(\log_2 x)(3\log_x 2)}=\sqrt{3}$

$\therefore \quad \dfrac{P}{2}\geq\sqrt{3}$

$\Rightarrow \quad P\geq 2\sqrt{3}$

$\therefore$ Least value of $\log_2 x-\log_x(0.125)$ is $2\sqrt{3}$.

67. Let $y=\dfrac{1}{\log_3\pi}+\dfrac{1}{\log_4\pi}=\log_\pi 3+\log_\pi 4$

$=\log_\pi 12$

Now, $12>\pi^2$

$\log_\pi 12>\log_\pi\pi^2 \quad \therefore \quad y>2$

68. (i) $\therefore \quad x^{1+\log_{10}x}=10x$ ...(i)

$\Rightarrow \quad x\cdot x^{\log_{10}x}=10x$

$\Rightarrow \quad x[x^{\log_{10}x}-10]=0$

$x\neq 0$, so $x^{\log_{10}x}-10=0$

$\Rightarrow \quad x^{\log_{10}x}=10$

$\Rightarrow \quad \log_{10}x=\log_x 10$

$\Rightarrow \quad (\log_{10}x)^2=1$

$\Rightarrow \quad \log_{10}x=\pm 1$

$\Rightarrow \quad x=10^{\pm 1}$

$\Rightarrow \quad x=10 \text{ or } \dfrac{1}{10}$ [$\because x>0$]

(ii) $\log_2(9+2^x)=3$

$\Rightarrow \quad 9+2^x=8$

$\Rightarrow \quad 2^x=-1$

which is not possible, so $x\in\phi$.

(iii) $2 \cdot x^{\log_4 3} + 3^{\log_4 x} = 27$

$2 \cdot 3^{\log_4 x} + 3^{\log_4 x} = 27 \qquad \left[\because a^{\log_b c} = c^{\log_b a}\right]$

$3^{\log_4 (x+1)} = 3^3$

$\log_4 (x+1) = 3$

$\log_4 x = 2$

$x = 16$

(iv) $\log_4 \log_3 \log_2 x = 0$ ...(i)

Defined for $x > 0$, $\log_2 x > 0$ and $\log_3 \log_2 x > 0$

$\Rightarrow \quad x > 0,\ x > 1,\ x > 3$

$\therefore \quad x > 3$

$\log_3 \log_2 x = 1$

$\log_2 x = 3,\ x = 8$

which satisfy Eq. (i).

(v) $x^{\frac{\log_{10} x + 5}{3}} = 10^{5 + \log_{10} x}$ ...(i)

Defined for $x > 0$

Let $\log_{10} x = y$

$\Rightarrow \quad x = 10^y$

By Eq. (i), $10^{y\left(\frac{y+5}{3}\right)} = 10^{5+y}$

$\Rightarrow \quad y^2 + 5y = 15 + 3y$

$\Rightarrow \quad y^2 + 2y - 15 = 0$

$\Rightarrow \quad (y+5)(y-3) = 0$

$\Rightarrow \quad y = -5 \text{ or } y = 3$

$\Rightarrow \quad x = \frac{1}{10^5} \text{ or } x = 10^3$

$\therefore \quad x = \{10^{-5}, 10^3\}$

(vi) $\log_3 \left(\log_9 x + \frac{1}{2} + 9^x\right) = 2x$

Defined for $x > 0$,

$\log_9 x + \frac{1}{2} + 9^x = 9^x$

$\Rightarrow \quad \log_9 x = -\frac{1}{2} \ \Rightarrow \ x = 9^{-\frac{1}{2}}$

$\Rightarrow \quad x = 3^{-1}$

$\therefore \quad x = \frac{1}{3}$

(vii) $4^{\log_{10} x + 1} - 6^{\log_{10} x} - 2 \cdot 3^{\log_{10} x^2 + 2} = 0$ ...(i)

$\Rightarrow \ 2^{2\log_{10} x + 2} - (2 \times 3)^{\log_{10} x} - 2 \cdot 3^{2\log_{10} x + 2} = 0$

Let $\log_{10} x = \lambda$, then

$2^{2\lambda + 2} - (2 \times 3)^{\lambda} - 2 \cdot 3^{2\lambda + 2} = 0$

$\Rightarrow \quad 2^2 - \left(\frac{3}{2}\right)^{\lambda} - 2 \cdot 3^2 \cdot \left(\frac{3}{2}\right)^{2\lambda} = 0$

Let $\left(\frac{3}{2}\right)^{\lambda} = \mu$

$\therefore \quad 18\mu^2 + \mu - 4 = 0$

$\Rightarrow \quad 18\mu^2 + 9\mu - 8\mu - 4 = 0$

$\Rightarrow \quad 9\mu(2\mu + 1) - 4(2\mu + 1) = 0$

$\therefore \quad \mu = -\frac{1}{2}, \mu = \frac{4}{9}$

$\mu \neq -\frac{1}{2}$

$\therefore \quad \mu = \frac{4}{9}$

$\left(\frac{3}{2}\right)^{\lambda} = \left(\frac{3}{2}\right)^{-2} \Rightarrow \ \lambda = -2$

Hence, $x = 10^{\lambda} = 10^{-2} \ \frac{1}{100}$

(viii) $\frac{\log_{10}(x-3)}{\log_{10}(x^2 - 21)} = \frac{1}{2}$

is defined for $x > 1$ and $x^2 > 21$.

$\therefore \quad x > \sqrt{21}$ ...(i)

$\Rightarrow \quad 2\log_{10}(x-3) = \log_{10}(x^2 - 21)$

$\Rightarrow \quad \log_{10}(x-3)^2 = \log_{10}(x^2 - 21)$

$\Rightarrow \quad (x-3)^2 = x^2 - 21$

$x^2 - 6x + 9 = x^2 - 21$

$\therefore \quad x = 5$

satisfy Eq. (i), hence $x = 5$.

(ix) $x^{\log_2 x + 4} = 32$

Defined for $x > 0$,

$\log_2 x + 4 = \log_x 2^5$

$\log_2 x + 4 = \frac{5}{\log_2 x}$

$(\log_2 x)^2 + 4\log_2 x - 5 = 0$

$(\log_2 x + 5)(\log_2 x - 1) = 0$

$\Rightarrow \quad \log_2 x = -5 \text{ or } \log_2 x = 1$

$\Rightarrow \quad x = 2^{-5} \text{ or } x = 2^1$

$\therefore \quad x = \frac{1}{32} \text{ or } x = 2$

which satisfy Eq. (i).

(x) $\log_a x = x$ ...(i)

and $a = x^{\log_4 x}$ ...(ii)

Defined for $x > 0$

From Eq. (i), $x = a^x$

$a^x = x,\ a = x^{1/x}$

From Eq. (ii), $x^{\frac{1}{x}} = x^{\log_4 x}$

$\Rightarrow \quad \frac{1}{x} = \log_4 x$

$\Rightarrow \quad x = \log_x 4 \ \Rightarrow \ x^x = 4$

$\therefore \quad x = 2$

(xi) $\log_{\sqrt{2}\sin x}(1 + \cos x) = 2$ ...(i)

Defined for $1 + \cos x > 0$, $\sqrt{2}\sin x > 0$

and $\sqrt{2}\sin x \neq 1$, then

$1 + \cos x = 2\sin^2 x$

$1 + \cos x = 2 - 2\cos^2 x$

$$2\cos^2 x + \cos x - 1 = 0$$
$$(2\cos x - 1)(\cos x + 1) = 0$$
$$1 + \cos x \neq 0$$
So, $$\cos x = \frac{1}{2}$$

$x = \frac{\pi}{3}$, Eq. (i) is defined for that value of x.

69. Let rational number be x, then

$$x = 50 \log_{10} x \Rightarrow 2x = 100 \cdot \log_{10} x$$

Taking logarithm to the base 10, then

$$\log_{10} 2 + \log_{10} x = 2 + \log_{10} (\log_{10} x)$$

Let $\log_{10} x = \lambda$

$\therefore \quad \log_{10} 2 + \lambda = 2 + \log_{10} (\lambda)$

$\Rightarrow \quad \log_{10}\left(\frac{\lambda}{2}\right) = \lambda - 2$

which is true for $\lambda = 2$.

$\therefore \quad \log_{10} x = 2$

$\Rightarrow \quad x = 10^2 = 100$

70. Let $y = \frac{2}{\log_4 (2000)^6} + \frac{3}{\log_5 (2000)^6}$

$= 2 \log_{(2000)^6} 4 + 3 \log_{(2000)^6} 5$

$= \log_{(2000)^6} 4^2 + \log_{(2000)^6} 5^3$

$= \log_{(2000)^6} (4^2 \times 5^3)$

$= \frac{1}{6} \log_{2000} 2000 = \frac{1}{6}$

71. $\log_a [1 + \log_b \{1 + \log_c (1 + \log_p x)\}] = 0$ ...(i)

$\Rightarrow \quad 1 + \log_b \{1 + \log_c (1 + \log_p x)\} = 1$

$\Rightarrow \quad \log_b \{1 + \log_c (1 + \log_p x)\} = 0$

$\Rightarrow \quad 1 + \log_c (1 + \log_p x) = 1$

$\Rightarrow \quad \log_c (1 + \log_p x) = 0$

$\Rightarrow \quad 1 + \log_p x = 1$

$\Rightarrow \quad \log_p x = 0$

$\Rightarrow \quad x = p^0$

$\Rightarrow \quad x = 1$

Eq. (i) is satisfied for this value of x.

72. $\because 5 \log_{4\sqrt{2}} (3 - \sqrt{6}) - 6 \log_8 (\sqrt{3} - \sqrt{2})$

$= 5 \log_{2^{5/2}} (3 - \sqrt{6}) - 6 \log_{2^3} (\sqrt{3} - \sqrt{2})$

$= 5 \times \frac{1}{5/2} \log_2 (3 - \sqrt{6}) - 6 \times \frac{1}{3} \log_2 (\sqrt{3} - \sqrt{2})$

$= \log_2 (3 - \sqrt{6})^2 - \log_2 (\sqrt{3} - \sqrt{2})^2$

$= \log_2 \left(\frac{3 - \sqrt{6}}{\sqrt{3} - \sqrt{2}}\right)^2 = \log_2 \left(\frac{\sqrt{3}(\sqrt{3} - \sqrt{2})}{(\sqrt{3} - \sqrt{2})}\right)^2 = \log_2 3$

$\therefore \quad = 4^{5 \log_{4\sqrt{2}} (3 - \sqrt{6}) - 6 \log_8 (\sqrt{3} - \sqrt{2})}$

$= 4^{\log_2 3} = 2^{2 \log_2 3} = 2^{\log_2 9} = 9$

73. (i) $\log_{2x+3} x^2 < 1$ ...(i)

***Case* I** $0 < 2x + 3 < 1$, i.e. $-\frac{3}{2} < x < -1$

Eq. (i), $x^2 > 2x + 3$

$x^2 - 2x - 3 > 0$

$(x - 3)(x + 1) > 0$

$x < -1$ or $x > 3$

$\therefore \quad x \in \left(-\frac{3}{2}, -1\right)$ ...(ii)

***Case* II** $2x + 3 > 1 \Rightarrow x > -1$

Eq. (i), $x^2 < 2x + 3$

$(x - 3)(x + 1) < 0 - 1 < x < 3$

$\Rightarrow \quad x \in (-1, 3]$ ...(iii)

Eq. (i), $x \neq 0$ ...(iv)

Eqs. (ii), (iii) and (iv), $x \in \left(-\frac{3}{2}, 3\right) \cup \{-1, 0\}$

(ii) $\log_{2x} (x^2 - 5x + 6) < 1$ ...(i)

For Eq. (i) to be defined $2x > 0$ and $2x \neq 1$

So, $x > 0$ and $x \neq \frac{1}{2}$

and $x^2 - 5x + 6 > 0 \Rightarrow x < 2$ or $x > 3$ ...(ii)

***Case* I** $0 < 2x < 1 \Rightarrow 0 < x < \frac{1}{2}$ ...(iii)

From Eq. (i), $\log_{2x} (x^2 - 5x + 6) < 1$

$x^2 - 5x + 6 < 2x$

$x^2 - 7x + 5 > 0$

$(x - 6)(x - 1) > 0$

$x < 1$ or $x > 6$ ...(iv)

From Eqs. (iii), (iv) and (ii)

$$x \in \left(0, \frac{1}{2}\right) \quad \text{...(A)}$$

***Case* II** $2x > 1 \Rightarrow x > \frac{1}{2}$ ...(v)

From Eq. (i), $\log_{2x} (x^2 - 5x + 6) < 1$

$\Rightarrow \quad x^2 - 5x + 6 < 2x$

$\Rightarrow \quad x^2 - 7x + 6 < 0$

$\Rightarrow \quad 1 < x < 6$ ...(vi)

From Eqs. (ii), (v) and (vi),

$x \in (1, 2) \cup (3, 6)$ ...(B)

From Eqs. (A) and (B), $x \in \left(0, \frac{1}{2}\right) \cup (1, 2) \cup (3, 6)$

(iii) $\log_2 (2 - x) < \log_{1/2} (x + 1)$ ...(i)

From Eq. (i) to be defined $2 - x > 0 \Rightarrow x < 2$

and $x + 1 > 0 \Rightarrow x > -1$

So, $x \in (-1, 2)$ ...(ii)

Now, from Eq. (i), $\log_2 (2 - x) + \log_2 (x + 1) < 0$

$(2 - x)(x + 1) < 1$

$(x - 2)(x + 1) + 1 > 0$

$x^2 - x - 2 + 1 > 0$

$x^2 - x - 1 > 0$

$$\Rightarrow \quad x < \frac{1-\sqrt{5}}{2}$$

or $$x > \frac{1+\sqrt{5}}{2} \quad \ldots(iii)$$

From Eqs. (ii) and (iii),

$$x \in \left(-1, \frac{1-\sqrt{5}}{2}\right) \cup \left(\frac{1+\sqrt{5}}{2}, 2\right)$$

(iv) $\log_{x^2}(x+2) < 1$ ...(i)

From Eq. (i) to be defined, $x + 2 > 0 \Rightarrow x > -2$ ...(A)

and $x \in R,\ x \neq 0$ and $x \neq 1$

***Case* I** $x \in (-1, 1) \sim \{0\}$ ...(ii)

Eq. (i), $(x+2) > x^2$

$$x^2 - x - 2 < 0$$

$$(x-2)(x+1) < 0$$

$$-1 < x < 2 \quad \ldots(iii)$$

From Eqs. (ii), (iii) and (A),

$$x \in (-1, 0) \cup (0, 1) \quad \ldots(B)$$

***Case* II** $x \in (-\infty, -1) \cup (1, \infty)$ ...(iv)

Eq. (i), $x + 2 < x^2$

$$x^2 - x - 2 > 0$$

$$x < -1 \text{ or } x > 2 \quad \ldots(v)$$

From Eqs. (iv), (v) and (A),

$$x \in (-2, -1) \cup (2, \infty) \quad \ldots(C)$$

From Eqs. (B) and (C),

$$x \in (-2, 1) \cup (2, \infty) \sim \{-1, 0\}$$

(v) $3^{\log_3 \sqrt{x-1}} < 3^{\log_3 (x-6)} + 3$ ...(i)

From Eq. (i) to be defined

$$x - 1 > 0 \implies x > 1 \quad \ldots(ii)$$

and $x - 6 > 0 \implies x > 6$ ...(iii)

From Eqs. (ii) and (iii), $x > 6$ ...(iv)

Eq. (i), $\sqrt{x-1} - (x-6) - 3 < 0$

$$\sqrt{x-1} - x + 3 < 0$$

$$\sqrt{x-1} < (x-3)\ x - 1 < (x-3)^2$$

$$x^2 + 9 - 6x - x + 1 > 0$$

$$x^2 - 7x + 10 > 0$$

$$(x-5)(x-2) > 0$$

$$x < 2 \text{ or } x > 5 \quad \ldots(v)$$

From Eqs. (iv) and (v), $x > 6$

(vi) $\log_{1/2}(3x-1)^2 < \log_{1/2}(x+5)^2$ ...(i)

From Eq. (i) to be defined $x \neq \frac{1}{3},\ x \neq -5$ ...(ii)

Eq. (i), $(3x-1)^2 > (x+5)^2$

$$(3x - 1 - x - 5)(3x - 1 + x + 5) > 0$$

$$(2x-6)(4x+4) > 0$$

$$(x-3)(x+1) > 0$$

$$x < -1 \text{ or } x > 3 \quad \ldots(iii)$$

From Eqs. (ii) and (iii),

$$x \in (-\infty, -5) \cup (-5, -1) \cup (3, \infty)$$

(vii) $\log_{10} x + 2 \leq \log_{10}^2 x$ ...(i)

From Eq. (i), $x > 0$ ...(ii)

$$\log_{10}^2 x + \log_{10} x - 2 \geq 0$$

$$(10_{10}\ x - 2)(\log_{10} x + 1) \geq 0$$

$$\log_{10} x \leq -1 \text{ or } \log_{10} x \geq 2$$

$$x \leq \frac{1}{10} \text{ or } x \geq 100 \quad \ldots(iii)$$

From Eqs. (ii) and (iii),

$$x \in \left(0, \frac{1}{10}\right] \cup [100, \infty)$$

or $$x \in (0, 10^{-1}] \cup [10^2, \infty)$$

(viii) $\log_{10}(x^2 - 2x - 2) \leq 0$ ...(i)

From Eq. (i), $x^2 - 2x - 2 > 0$

$$x^2 - 2x + 1 - 3 > 0$$

$$(x-1)^2 - (\sqrt{3})^2 > 0$$

$$[x - (1+\sqrt{3})][x - (1-\sqrt{3})] > 0$$

$\therefore$ $x \in (-\infty, 1-\sqrt{3}) \cup (1+\sqrt{3}, \infty)$ ...(ii)

$$x^2 - 2x - 2 \leq 1$$

$$x^2 - 2x - 3 \leq 0$$

$$(x-3)(x+1) \leq 0$$

$$-1 \leq x \leq 3 \quad \ldots(iii)$$

From Eqs. (ii) and (iii), we get

$$x \in [-1, 1-\sqrt{3}) \cup (1+\sqrt{3}, 3]$$

(ix) $\log_x \left(2x - \frac{3}{4}\right) > 2$ ...(i)

From Eq. (i) to be defined $x > 0,\ x \neq 1,\ 2x - \frac{3}{4} > 0$

$$x > 0,\ x \neq 1,\ x > \frac{3}{8} \quad \ldots(ii)$$

From Eq. (i), $\log_x \left(2x - \frac{3}{4}\right) > 2$

***Case* I** $0 < x < 1$ ...(iii)

$$2x - \frac{3}{4} < x^2$$

$$8x - 3 - 4x^2 < 0$$

$$4x^2 - 8x + 3 > 0$$

$$4x^2 - 6x - 2x + 3 > 0$$

$$(2x-1)(2x-3) > 0$$

$$x < \frac{1}{2} \text{ or } x > \frac{3}{2} \quad \ldots(iv)$$

From Eqs. (ii), (iii) and (iv),

$$x \in \left(\frac{3}{8}, \frac{1}{2}\right) \quad \ldots(v)$$

***Case* II** $x > 1$ ...(vi)

Eq. (i) $\Rightarrow \quad 2x - \frac{3}{4} > x^2$

$$8x - 3 > 4x^2$$

$$4x^2 - 8x + 3 < 0$$

$$\frac{1}{2} < x < \frac{3}{2} \quad \ldots(vii)$$

From Eqs. (ii), (vi) and (vii), we get

$$x \in \left(1, \frac{3}{2}\right) \quad \ldots(viii)$$

From Eqs. (v) and (viii), we get

$$x \in \left(\frac{3}{8}, \frac{1}{2}\right) \cup \left(1, \frac{3}{2}\right)$$

(x) $\log_{1/3} x < \log_{1/2} x \, (x > 0)$

$$\Rightarrow \quad \log_3 x > \log_2 x$$

$$\Rightarrow \quad \frac{\log x}{\log 3} - \frac{\log x}{\log 2} > 0$$

$$\log x \left(\frac{\log 3 - \log 2}{\log 2 \log 3}\right) < 0$$

$$\Rightarrow \quad \log x < 0 \Rightarrow x < 1$$

So, $x \in (0, 1)$

(xi) $\log_{2x+3} x^2 < \log_{2x+3} (2x + 3)$ ...(i)

From Eq. (i) to be defined,

$$2x + 3 > 0$$

$$x > -\frac{3}{2}$$

$$2x + 3 \neq 1$$

$$x \neq -1$$

$$x \in R - \{0\} \quad \ldots(A)$$

From Eq. (i), $\log_{2x+3} x^2 < 1$ ...(ii)

***Case* I** $0 < 2x + 3 < 1 \Rightarrow -\frac{3}{2} < x < -1$ ...(iii)

From Eq. (ii), $\log_{2x+3} x^2 < 1$

$$\Rightarrow \quad x^2 > 2x + 3 \Rightarrow x^2 - 2x - 3 > 0$$

$$\Rightarrow \quad (x - 3)(x + 1) > 0$$

$$\Rightarrow \quad x < -1 \text{ or } x > 3 \quad \ldots(iv)$$

From Eqs. (A), (iii) and (iv), $x \in \left(-\frac{3}{2}, -1\right)$ ...(B)

***Case* II** If $2x + 3 > 1 \Rightarrow x > -1$ ...(v)

$$\log_{2x+3} x^2 < 1$$

$$x^2 < 2x + 3$$

$$x^2 - 2x - 3 < 0$$

$$(x - 3)(x + 1) < 0$$

$$\Rightarrow \quad -1 < x < 3 \quad \ldots(vi)$$

So, Eqs. (A), (v) and (vi), $x \in (-1, 3)$ ...(C)

From Eqs. (B) and (C),

$$x \in \left(-\frac{3}{2}, -1\right) \cup (-1, 3)$$

(xii) $\log_2^2 x + 3 \log_2 x \geq \frac{5}{2} \log_{4\sqrt{2}} 16$ ...(i)

$$\log_2^2 x + 3 \log_2 x - \frac{5}{2} \times \frac{2}{5} \log_2 16 \geq 0$$

$$\log_2^2 x + 3 \log_2 x - 4 \geq 0$$

$$(\log_2 x + 4)(\log_2 x - 1) \geq 0$$

$$\log_2 x \leq -4 \text{ or } \log_2 x \geq 1 \Rightarrow x \leq \frac{1}{16}$$

or $x \geq 2$ ...(ii)

From Eq. (i), $x > 0$ ...(iii)

From Eqs. (ii) and (iii), $x \in \left(0, \frac{1}{16}\right] \in [2, \infty)$

(xiii) $\because \; (x^2 + x + 1)^x < 1$

Taking logarithm on both sides, then

$$x \log (x^2 + x + 1) < 0$$

$\because \quad x^2 + x + 1 > 0, \forall \; x \in R$

***Case* I** If $x > 0$ ...(i)

Then, $\log (x^2 + x + 1) < 0$

$$\therefore \quad x^2 + x + 1 < 1$$

$$\Rightarrow \quad x(x + 1) < 0$$

$$\Rightarrow \quad -1 < x < 0 \quad \ldots(ii)$$

From Eqs. (i) and (ii), $x \in \phi$

***Case* II** If $x < 0$ ...(iii)

Then, $\log (x^2 + x + 1) > 0$

$$\Rightarrow \quad x^2 + x + 1 > 1$$

$$\Rightarrow \quad x(x + 1) > 0$$

$$\therefore \quad x \in (-\infty, -1) \cup (0, \infty) \quad \ldots(iv)$$

From Eqs. (iii) and (iv), we get

$$x \in (-\infty, -1)$$

(xiv) $\log_{(3x^2+1)} 2 < \frac{1}{2}$

$$2 < (3x^2 + 1)^{1/2}$$

$$(3x^2 + 1 > 1, \forall \; x \in R)$$

$$4 < 3x^2 + 1$$

$$3x^2 > 3$$

$$x^2 > 1$$

$$x < -1 \text{ or } x > 1$$

$$\Rightarrow \quad x \in (-\infty, -1) \cup (1, \infty)$$

(xv) $x^{(\log_{10} x)^2 - 3 \log_{10} x + 1} > 1000$ ...(i)

From Eq. (i) to be defined, $x > 0$ and $x \neq 1$

Let $\log_{10} x = y \Rightarrow x = 10^y$

From Eq. (i), $10^{y(y^2 - 3y + 1)} > 10^3$

$$\Rightarrow \quad y^3 - 3y^2 + y - 3 > 0$$

$$\Rightarrow \quad y^2 (y - 3) + 1 (y - 3) > 0$$

$$\Rightarrow \quad (y - 3)(y^2 + 1) > 0$$

$$\Rightarrow \quad y > 3$$

$$\Rightarrow \quad \log_{10} x > 3$$

$$\Rightarrow \quad x > 1000$$

$$\Rightarrow \quad x \in (1000, \infty)$$

(xvi) $\log_4 \{14 + \log_6 (x^2 - 64)\} \leq 2$ ...(i)

$$14 + \log_6 (x^2 - 64) \leq 16$$

$$\log_6 (x^2 - 64) \leq 2$$

$$x^2 - 64 \le 36$$

$$x^2 \le 100$$

$$-10 \le x \le 10 \quad \ldots(ii)$$

From Eq. (i), $x^2 - 64 > 0$

$$\Rightarrow \quad x < -8 \text{ or } x > 8 \quad \ldots(iii)$$

From Eqs. (ii) and (iii),

$$x \in [-10, -8) \cup (8, 10]$$

(xvii) $\log_2 (9 - 2^x) \le 10^{\log_{10} (3 - x)}$ ...(i)

From Eq. (i) to be defined,

$$9 - 2^x > 0 \Rightarrow 9 > 2^x$$

$$\Rightarrow \quad 2^x < 9 \Rightarrow x < \log_2 9$$

$$3 - x > 0 \Rightarrow x < 3$$

Then, $x < 3$ ...(ii)

From Eq. (i), $\log_2 (9 - 2^x) \le 3 - x$

$$\Rightarrow \quad 9 - 2^x \le 2^{3 - x}$$

$$\Rightarrow \quad 9 - 2^x - 8 \cdot 2^{-x} \le 0$$

$$\Rightarrow \quad (2^x)^2 - 9 2^x + 8 \ge 0$$

$$\Rightarrow \quad (2^x - 8)(2^x - 1) \ge 0$$

$$\Rightarrow \quad 2^x \le 1 \text{ or } 2^x \ge 8$$

$$\Rightarrow \quad x \le 0 \text{ or } x \ge 3 \quad \ldots(iii)$$

From Eqs. (ii) and (iii), $x \le 0 \Rightarrow x \in (-\infty, 0]$

(xviii) $\log_a \left(\dfrac{2x + 3}{x} \right) \ge 0$ ...(i)

From inequation (a), $a > 1$

By Eq. (i), $\dfrac{2x + 3}{x} > 0$

$$\Rightarrow \quad \left[\frac{x - \left(-\frac{3}{2} \right)}{x - 0} \right] > 0$$

$$\Rightarrow \quad x < -\frac{3}{2} \text{ or } x > 0 \quad \ldots(ii)$$

From Eq. (i), $\log_a \left(2 + \dfrac{3}{x} \right) \ge 0$

$$2 + \frac{3}{x} \ge 1$$

$$\frac{3 + x}{x} \ge 0$$

$$\frac{x - (-3)}{x - 0} \ge 0$$

$$x \le -3 \text{ or } x \ge 0 \quad \ldots(iii)$$

From Eqs. (ii) and (iii),

$$x \le -3 \text{ or } x > 0 \quad \ldots(iv)$$

$$\Rightarrow \quad x \in (-\infty, -3] \cup (0, \infty)$$

From inequation in (b), $0 < a < 1$

From Eq. (i), $\dfrac{2x + 3}{x} \le 1$

$$\Rightarrow \quad \frac{x + 3}{x} \le 0$$

$$\Rightarrow \quad -3 \le x \le 0$$

$$\Rightarrow \quad x \in [-3, 0] \quad \ldots(v)$$

From Eqs. (ii) and (v), we get $x \in \left[-3, -\dfrac{3}{2} \right)$

(xix) $1 + \log_2 (x - 1) \le \log_{(x - 1)} 4$ ...(i)

From Eq. (i) to be defined, $x - 1 > 0 \Rightarrow x > 1$

and $x - 1 \ne 1 \Rightarrow x \ne 2$ ...(ii)

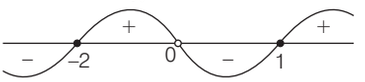
By Eq. (i), $1 + \log_2 (x - 1) \le 2 \log_{(x - 1)} 2$

Let $\log_2 (x - 1) = \lambda$, then

$$1 + \lambda \le \frac{2}{\lambda}$$

$$\Rightarrow \quad \frac{\lambda^2 + \lambda - 2}{\lambda} \le 0$$

$$\Rightarrow \quad \frac{(\lambda + 2)(\lambda - 1)}{\lambda} \le 0$$



$$\Rightarrow \quad \lambda \le -2 \text{ or } 0 < \lambda \le 1$$

$$\Rightarrow \quad \log_2 (x - 1) \le -2$$

or $0 < \log_2 (x - 1) \le 1$

$$\Rightarrow \quad x - 1 \le 2^{-2} \text{ or } 2^0 < x - 1 \le 2^1$$

$$\Rightarrow \quad x \le \frac{5}{4} \text{ or } 2 < x \le 3 \quad \ldots(iii)$$

From Eqs. (ii) and (iii), we get

$$x \in (2, 3]$$

(xx) $\log_{5x + 4} x^2 \le \log_{5x + 4} (2x + 3)$ ...(i)

From Eq. (i) to be defined, $5x + 4 > 0 \Rightarrow x > -\dfrac{4}{5}$

$$5x + 4 \ne 1 \Rightarrow x \ne -\frac{3}{5}$$

$$2x + 3 > 0 \Rightarrow x > -\frac{3}{2}$$

and $x \in (-\infty, \infty) - \{0\}$

$$\Rightarrow \quad x \in \left(-\frac{4}{5}, -\frac{3}{5} \right) \cup \left(-\frac{3}{5}, 0 \right) \cup (0, \infty) \quad \ldots(ii)$$

From Eq. (i), $\log_{5x + 4} x^2 \le \log_{5x + 4} (2x + 3)$

$$\log_{5x + 4} \frac{x^2}{2x + 3} \le 0 \quad \ldots(iii)$$

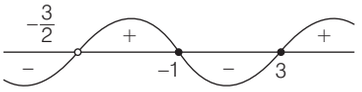
***Case* I** $0 < 5x + 4 < 1$

$$\Rightarrow \quad -\frac{4}{5} < x < -\frac{3}{5} \quad \ldots(iv)$$

From Eq. (iii), $\dfrac{x^2}{2x + 3} \ge 1$

$$\frac{x^2 - 2x - 3}{2x + 3} \ge 0$$

$$\frac{(x-3)(x+1)}{\left[x-\left(-\frac{3}{2}\right)\right]} \geq 0$$



$$x \in \left(-\frac{3}{2}, -1\right] \cup [3, \infty) \quad \ldots(v)$$

From Eqs. (ii), (iv) and (v), $x \in \phi$ ...(vi)

***Case* II** $5x + 4 > 1 \Rightarrow x > -\frac{3}{5}$ ...(vii)

From Eq. (iii), $\frac{x^2}{2x+3} \leq 1$

$$\left[\frac{(x-3)(x+1)}{\left\{x-\left(-\frac{3}{2}\right)\right\}}\right] \leq 0$$

$\Rightarrow x < -\frac{3}{2}$ or $x \in [-1, 3]$ ...(viii)

From Eqs. (ii), (vii) and (viii),

$$x \in \left(-\frac{3}{5}, -\frac{3}{2}\right) \cup [-1, 0) \cup (0, 3] \quad \ldots(ix)$$

From Eqs. (vi) and (ix), we get

$$x \in \left(-\frac{3}{5}, -\frac{3}{2}\right) \cup [-1, 0) \cup (0, 3]$$

74. Given equation is

$$\sqrt{\log_x (ax)^{1/5} + \log_a (ax)^{1/5}} + \sqrt{\log_a \left(\frac{x}{a}\right)^{1/5} + \log_x \left(\frac{a}{x}\right)^{1/5}} = a \quad \ldots(i)$$

$$\frac{1}{\sqrt{5}} \sqrt{1 + \log_x a + 1 + \log_a x} + \frac{1}{\sqrt{5}} \sqrt{\log_a x - 1 + \log_x a - 1} = a$$

$$\sqrt{\log_a x + \frac{1}{\log_a x} + 2} + \sqrt{\log_a x + \frac{1}{\log_a x} - 2} = \sqrt{5}a$$

$$\left|\sqrt{|\log_a x|} + \frac{1}{\sqrt{|\log_a x|}}\right| + \left|\sqrt{|\log_a x|} - \frac{1}{\sqrt{|\log_a x|}}\right| = \sqrt{5}a \quad \ldots(ii)$$

Let $\sqrt{|\log_a x|} = y \quad [y \geq 0]$

$$\left|y + \frac{1}{y}\right| + \left|y - \frac{1}{y}\right| = \sqrt{5}a \quad \ldots(iii)$$

***Case* I** $x \geq a > 1$ Eq. (iii) $\Rightarrow y + \frac{1}{y} + y - \frac{1}{y} = \sqrt{5}a$

$$\Rightarrow 2y = \sqrt{5}a$$

$$2\sqrt{|\log_a x|} = \sqrt{5}a$$

$$\sqrt{|\log_a x|} = \frac{\sqrt{5}}{2}a$$

$$\log_a x = \frac{5}{4}a^2$$

$$x = a^{\frac{5}{4}a^2}$$

***Case* II** $1 < x < a$ ...(v)

By Eq. (iii), $y + \frac{1}{y} - y + \frac{1}{y} = \sqrt{5}a$

$$\frac{2}{y} = \sqrt{5}a$$

$$y = \frac{2}{\sqrt{5}a}$$

$$\sqrt{|\log_a x|} = \frac{2}{\sqrt{5}a}$$

$$\Rightarrow \log_a x = \frac{4}{5a^2}$$

$$x = a^{4/5a^2}$$

75. Given equation,

$$\log_\pi (x^2 + 15a^2) - \log_\pi (a-2) = \log_\pi \frac{8ax}{a-2} \quad \ldots(i)$$

Eq. (i) is defined, if $a - 2 > 0 \Rightarrow a > 2$ ...(ii)

$$\frac{8ax}{a-2} > 0$$

By Eq. (ii), $a > 2$

So, $ax > 0$, then $x > 0$

Eq. (i) for $x = 9, a > 0$

$$\log_\pi \frac{(x^2 + 15a^2)}{(a-2)} = \log_\pi \frac{8ax}{a-2}$$

$$x^2 + 15a^2 = 8ax \quad \ldots(iii)$$

$$(x - 3a)(x - 5a) = 0$$

$\therefore \quad x = 3a$ and $x = 5a$

For $a = 3$, $x = 9$ and $x = 15$

$\Rightarrow x = 15$ for $a = 3$

76. Given that,

$$\log_4 (\log_3 x) + \log_{1/4} (\log_{1/3} y) = 0 \quad \ldots(i)$$

$$\Rightarrow \frac{1}{2}\log_2 \log_3 x - \frac{1}{2}\log_2(-\log_3 y) = 0$$

$$\Rightarrow \frac{1}{2}\left[\log_2 \left(\frac{\log_3 x}{-\log_3 y}\right)\right] = 0$$

$$\Rightarrow -\frac{\log_3 x}{\log_3 y} = 1$$

$$\Rightarrow \log_3 x = -\log_3 y$$

$$\Rightarrow \log_3 x = \log_3 \left(\frac{1}{y}\right)$$

$$\Rightarrow x = \frac{1}{y} \quad \ldots(ii)$$

Also, given that, $x^2 + y^2 = \frac{17}{4}$

$$x^2 + \frac{1}{x^2} = \frac{17}{4}$$

$$\left(x+\frac{1}{x}\right)^2=\frac{17}{4}+2$$

$$x+\frac{1}{x}=\frac{5}{2} \qquad \text{[by Eq. (i) } x>0, y>0\text{]}$$

$$x+\frac{1}{x}=2+\frac{1}{2}$$

$$\therefore \quad x=2 \text{ or } \frac{1}{2}$$

For these values of x, $y=\frac{1}{2}$ or 2 [by Eq. (ii)]

77. $\log_{2x} 4x+\log_{4x} 16x=4$ ...(i)

From Eq. (i) is defined for $x>0$, $x\neq\frac{1}{2}$, $x\neq\frac{1}{4}$ ...(ii)

$$\Rightarrow \quad \frac{\log 4x}{\log 2x}+\frac{\log 16x}{\log 4x}=4$$

$$\Rightarrow \quad \frac{2\log 2+\log x}{\log 2+\log x}+\frac{4\log 2+\log x}{2\log 2+\log x}=4$$

On dividing by log 2, then

$$\frac{2+\log_2 x}{1+\log_2 x}+\frac{4+\log_2 x}{2+\log_2 x}=4$$

Let $\log_2 x=\lambda$, then

$$(2+\lambda)^2+(1+\lambda)(4+\lambda)=4(1+\lambda)(2+\lambda)$$

$$\Rightarrow \quad 2\lambda^2+9\lambda+8=4\lambda^2+12\lambda+8$$

$$\Rightarrow \quad 2\lambda^2+3\lambda=0$$

$$\therefore \quad \lambda=0, \lambda=-\frac{3}{2}$$

$$\Rightarrow \quad \log_2 x=0, \log_2 x=-\frac{3}{2}$$

$$\therefore \quad x=2^0, x=2^{-3/2}$$

or $x=1, x=2^{-3/2}$

78. Given equation,

$$\log_6 54+\log_x 16=\log_{\sqrt{2}} x-\log_{36}\frac{4}{9} \qquad ...(i)$$

Eq. (i) holds, if $x>0$, $x\neq 1$

From Eq. (i),

$$1+\log_6 9+4\log_x 2=2\log_2 x-\log_6\frac{2}{3}$$

$$\Rightarrow \quad 1+\log_6 9+\log_6\frac{2}{3}+4\log_x 2-2\log_2 x=0$$

$$\Rightarrow \quad 2+4\log_x 2-2\log_2 x=0$$

$$\Rightarrow \quad (\log_2 x)^2-\log_2 x-2=0$$

$$\Rightarrow \quad \log_2 x=2 \text{ or } \log_2 x=-1$$

$$\Rightarrow \quad x=4 \text{ or } x=\frac{1}{2}$$

Sum of the values of x satisfy Eq. (i) $=4+\frac{1}{2}=\frac{9}{2}$

Product of the values of x satisfy Eq. (i) $=4\times\frac{1}{2}=2$

79. Let $\frac{3}{2}\log_4(x+2)^3+3=\log_4(4-x)^3+\log_4(x+6)^3$ ...(i)

Eq. (i) holds, if $4-x>0$ and $x+0>0$, $x+2>0$

i.e., $-2<x<4$ ...(ii)

From Eq. (i),

$$\frac{3}{2}\times 2\times\frac{1}{2}\log_2|(x+2)|+3=\frac{1}{2}\times 3\log_2(4-x)+\frac{1}{2}\times 3\log_2(x+6)$$

$$\Rightarrow \quad \log_2(x+2)+2=\log_2(4-x)+\log_2(x+6)$$

$$\Rightarrow \quad \log_2\{4(x+2)\}=\log_2\{(4-x)(x+6)\}$$

$$\Rightarrow \quad 4(x+2)=(4-x)(x+6)$$

$$4x+8=-x^2-2x+24$$

$$x^2+6x-16=0$$

$$(x+8)(x-2)=0$$

$$\therefore \quad x=-8, x=2 \qquad ...(iii)$$

From Eqs. (ii) and (iii), we get $x=2$

80. $\log_2(4^{x+1}+4)\cdot\log_2(4^x+1)=\log_{1/\sqrt{2}}\left(\frac{1}{\sqrt{8}}\right)$ ...(i)

Eq. (i) defined, for $4^x+1>0$ which is true for all $x\in R$.

$$\log_2[4(4^x+1)]\cdot\log_2(4^x+1)=\log_{\sqrt{2}}\sqrt{8}=3$$

$$[2+\log_2(4^x+1)]\log_2(4^x+1)=3$$

Let $\log_2(4^x+1)=y$

$$\therefore \quad (y+2)y=3$$

$$y^2+2y-3=0$$

$$(y+3)(y-1)=0$$

$$y=1 \text{ or } y=-3$$

$$\therefore \quad \log_2(4^x+1)=1 \text{ or } \log_2(4^x+1)=-3$$

$$4^x+1=2 \text{ or } 4^x+1=\frac{1}{8}$$

$$4^x=1 \text{ or } 4^x=\frac{1}{8}-1$$

$x=0$ or $4^x=-\frac{7}{8}$ which is not possible.

$$\therefore \quad x=0$$

81. $2^{\sqrt{x}+\sqrt{y}}=256$

$$\Rightarrow \quad 2^{\sqrt{x}+\sqrt{y}}=2^8$$

$$\Rightarrow \quad \sqrt{x}+\sqrt{y}=8 \qquad ...(i)$$

Also, given that, $\log_{10}\sqrt{xy}-\log_{10}\frac{3}{2}=1$ ...(ii)

which is defined, $xy>0$

So, Eq. (ii) $\Rightarrow \quad \log_{10}\sqrt{xy}=\log_{10}\left(10\times\frac{3}{2}\right)$

$$\Rightarrow \quad \sqrt{xy}=15$$

$$\Rightarrow \quad xy=225 \qquad ...(iii)$$

From Eq. (i), $x+y+2\sqrt{xy}=64$

$$x+y=64-30$$

$$x+y=34$$

From Eq. (iii), $xy=225$

After solving, we get $x=9$ or $x=25$, then $y=25$ or $y=9$

Hence, solutions are (9, 25) and (25, 9).

82. Given, $\log_2 y = \log_4 (xy - 2)$ ...(i)

Eq. (i) defined for $y > 0$ and $xy - 2 > 0$ ...(ii)

$$xy > 0 \quad ...(iii)$$

From Eqs. (ii) and (iii) $\Rightarrow$ $y > 0, x > 0$

By Eq. (i), $y = \sqrt{xy - 2}$

$$y^2 - xy + 2 = 0 \quad ...(iv)$$

$$y(x - y) = 2 \quad ...(v)$$

Also given that,

$$\log_9 x^2 + \log_3 (x - y) = 1 \quad ...(vi)$$

which is defined for $x \in R - \{0\}$ and $x - y > 0$

$\Rightarrow$ $x > y$

By Eq. (vi), $x(x-y) = 3 \Rightarrow x^2 - xy = 3$ ...(vii)

and $x(x-y) = 3$ ...(viii)

Form Eqs. (iv) and (vii), $y^3 + 2 = x^2 - 3$

$$x^2 - y^2 = 5 \quad ...(ix)$$

On dividing Eq. (v) by Eq. (viii),

$$\frac{y}{x} = \frac{2}{3} \Rightarrow y = \frac{2x}{3} \quad ...(x)$$

From Eqs. (ix) and (x),

$x = 3$ and $y = 2$

83. Given that,

$$2 \log_{1/4} (x + 5) > \frac{9}{4} \log_{1/3\sqrt{3}} 9 + \log_{\sqrt{x+5}} 2 \quad ...(i)$$

By Eq. (i), $x + 5 > 0 \Rightarrow x > -5$

$$x + 5 \neq 1 \Rightarrow x \neq -4$$

So, $x \in (-5, -4) \cup (-4, \infty)$ ...(ii)

Now, by Eq. (i)

$$\frac{2}{-2} \log_2 (x + 5) - \frac{9}{4} \times \left(\frac{-2}{3}\right) \log_3 9 - 2 \log_{x+5} 2 > 0$$

$$-\log_2 (x + 5) + 3 - 2 \log_{x+5} 2 > 0$$

$$-\log_2 (x + 5) - \frac{2}{\log_2 (x + 5)} + 3 > 0 \quad ...(iii)$$

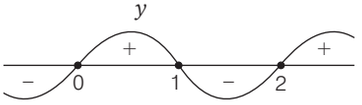
Now, let $\log_2 (x + 5) = y$, then Eq. (iii) becomes

$$-y - \frac{2}{y} + 3 > 0$$

$$\Rightarrow \quad \frac{-y^2 + 3y - 2}{y} > 0$$

$$\Rightarrow \quad \frac{y^2 - 3y + 2}{y} < 0$$

$$\Rightarrow \quad \frac{(y-2)(y-1)}{y} < 0$$



$$\Rightarrow \quad y < 0 \text{ or } 1 < y < 2$$

$$\Rightarrow \quad \log_2 (x + 5) < 0 \text{ or } 1 < \log_2 (x + 5) < 2$$

$$\Rightarrow \quad x + 5 < 1 \text{ or } 2 < x + 5 < 4$$

$$\Rightarrow \quad x < -4 \quad ...(iv)$$

or $-3 < x < -1$ ...(v)

From Eqs. (ii), (iv) and (v),

$$x \in (-5, -4) \cup (-3, -1)$$

84. $\log_3 (\sqrt{x} + |\sqrt{x} - 1|) = \log_9 (4\sqrt{x} - 3 + 4|\sqrt{x} - 1|)$ ...(i)

From Eq. (i) is defined, if $x \geq 0$

then $\log_3 (\sqrt{x} + |\sqrt{x} - 1|) = \log_{3^2} (4\sqrt{x} - 3 + 4|\sqrt{x} - 1|)$

$$\Rightarrow \quad 2(\sqrt{x} + |\sqrt{x} - 1|) = 4\sqrt{x} - 3 + 4|\sqrt{x} - 1|$$

$$\Rightarrow \quad 3 - 2\sqrt{x} = 2|\sqrt{x} - 1|$$

On squaring both sides, then

$$9 + 4x - 12\sqrt{x} = 4x + 4 - 4\sqrt{x}$$

$$\Rightarrow \quad 8\sqrt{x} = 5$$

$$\therefore \quad x = \frac{25}{64}$$

85. $(\log_2 x)^4 - \left(\log_{1/2} \frac{x^5}{4}\right)^2 - 20 \log_2 x + 148 < 0$

From Eq. (i), $x > 0$

$$\Rightarrow \quad (\log_2 x)^4 - (5 \log_2 x - 2)^2 - 20 \log_2 x + 148 < 0 \quad ...(i)$$

$$(\log_2 x)^4 - 25 \log_2^2 x - 4 + 20 \log_x x - 20 \log_2 x + 148 < 0$$

$$(\log_2 x)^4 - 25 \log_2^2 x + 144 < 0$$

$$\{(\log_2 x)^2 - 16\}\{(\log_2 x)^2 - 9\} < 0$$

$$9 < (\log_2 x)^2 < 16$$

$$3 < \log_2 x < 4 \text{ or } -4 < \log_2 x < -3$$

$$8 < x < 16 \quad ...(ii)$$

or $\frac{1}{16} < x < \frac{1}{8}$ ...(iii)

According to the question in Eq. (i) holds, for $x \in (a, b)$ where $a, b \in N$

So, from Eq. (ii), $a = 8$, $b = 16$

$$\therefore \quad ab(a + b) = 8 \times 16(8 + 16)$$

$$= 144 \times 24 = 3456$$

86. $\sqrt{(\log_3 \sqrt[3]{3x}) + (\log_x \sqrt[3]{3x}) \log_3 x^3}$

$$+ \sqrt{\left(\log_3 \sqrt[3]{\frac{x}{3}} + \log_x \sqrt[3]{\frac{3}{x}}\right) \log_3 x^3} = 2 \quad ...(i)$$

Eq. (i) is defined for $x > 0$, $x \neq 1$

From Eq. (i), $\sqrt{\frac{1}{3}(1 + \log_3 x + 1 + \log_x 3)\, 3 \log_3 x}$

$$+ \sqrt{\frac{1}{3}(\log_3 x - 1 + \log_x 3 - 1)\, 3 \log_3 x}$$

$$= 2\sqrt{\left(\log_3 x + \frac{1}{\log_3 x} + 2\right)} + \sqrt{\left(\log_3 x + \frac{1}{\log_3 x} - 2\right)}$$

$$= 2\sqrt{|\log_x 3|}$$

$$\Rightarrow \quad \sqrt{|\log_3 x|} + \frac{1}{\sqrt{|\log_3 x|}} + \left|\sqrt{|\log_3 x|} - \frac{1}{\sqrt{|\log_3 x|}}\right| = 2\sqrt{|\log_x 3|} \quad ...(ii)$$

***Case* I** If $x \geq 3$, $\sqrt{\log_3 x} - \dfrac{1}{\sqrt{\log_3 x}} > 0$

$$2\sqrt{\log_3 x} = 2\sqrt{\log_x 3}$$

$$(\log_3 x)^2 = 1 \Rightarrow x = 3 \text{ or } \frac{1}{3}, \text{ so } x = 3$$

***Case* II** If $1 < x < 3$, $\dfrac{2}{\sqrt{\log_3 x}} = 2\sqrt{\log_x 3}$

$$\Rightarrow \quad \log_3 x \cdot \log_x 3 = 1$$

$$\Rightarrow \quad 1 = 1$$

which is true, for all $x \in (1, 3]$.

So, $x \in (1, 3]$

87. P = Number of natural numbers, whose logarithms to the base 10 have characteristic p.

Let 'x' represent the natural number, i.e.

$$x = \lambda \times 10^p \ [\lambda = \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \ldots]$$

So, P = Number of natural numbers which have $(p + 1)$ digits

$$= 9 \cdot 10^p - 1 + 1 = 9 \cdot 10^p$$

Q = Number of natural numbers which have (q) digits.

$$Q = 9 \cdot 10^{q-1} - 1 + 1 = 9 \cdot 10^{q-1}$$

So, $\log_{10} P - \log_{10} Q = \log_{10}(9 \cdot 10^p) - \log_{10}(9 \cdot 10^{q-1})$

$$= (\log_{10} 9 + p) - (\log_{10} 9 + q - 1)$$

$$= p - q + 1$$

88. $\because \quad a = \log_3 \log_3 2$

$$\Rightarrow \quad 3^a = \log_3 2$$

$$\therefore \quad 3^{-a} = \log_2 3$$

Now, $\quad 1 < 2^{-k + 3^{-a}} < 2^1$

$$\Rightarrow \quad 2^0 < 2^{-k + 3^{-a}} < 2^1$$

$$\therefore \quad 0 < -k + 3^{-a} < 1$$

$$\Rightarrow \quad 0 < -k + \log_2 3 < 1$$

$$\Rightarrow \quad 0 > k - \log_2 3 > -1$$

$$\Rightarrow \quad \log_2 3 - 1 < k < \log_2 3$$

$$\therefore \quad k = 1$$

89. $\because (2x)^{\ln 2} = (3y)^{\ln 3}$

Taking log with base e on both sides, then

$$\ln 2 (\ln 2 + \ln x) = \ln 3 (\ln 3 + \ln y) \quad \ldots(i)$$

and $\quad 3^{\ln x} = 2^{\ln y}$

Taking log with base e on both sides, then

$$\ln x \cdot \ln 3 = \ln y \cdot \ln 2 \quad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$\ln 2 (\ln 2 + \ln x) = \ln 3 \left(\ln 3 + \frac{\ln x \cdot \ln 3}{\ln 2} \right)$$

$$\Rightarrow \quad \ln x \left(\frac{(\ln 3)^2}{\ln 2} - \ln 2 \right) = -((\ln 3)^2 - (\ln 2)^2)$$

$$\therefore \quad \ln x = -\ln 2 = \ln \left(\frac{1}{2} \right)$$

$$\Rightarrow \quad x = \frac{1}{2}$$

$$\therefore \quad x_0 = \frac{1}{2}$$

90. Let $S = \dfrac{1}{3\sqrt{2}} \sqrt{4 - \dfrac{1}{3\sqrt{2}} \sqrt{4 - \dfrac{1}{3\sqrt{2}} \sqrt{4 - \dfrac{1}{3\sqrt{2}} \ldots \infty}}}$

$$\Rightarrow \quad S = \frac{1}{3\sqrt{2}} \sqrt{4 - S}$$

or $\quad (3\sqrt{2}\, S)^2 = 4 - S$

$$\Rightarrow \quad 18S^2 + S - 4 = 0$$

$$\Rightarrow \quad (9S - 4)(2S + 1) = 0$$

$$\therefore \quad 9S - 4 = 0 \qquad [\because 2S + 1 \neq 0]$$

or $\quad S = \dfrac{4}{9} = \left(\dfrac{3}{2} \right)^{-2}$

$$\Rightarrow \quad \log_{3/2} S = -2 \Rightarrow 6 + \log_{3/2} S = 6 - 2 = 4$$

Hence,

$$6 + \log_{3/2} \left(\frac{1}{3\sqrt{2}} \sqrt{4 - \frac{1}{3\sqrt{2}} \sqrt{4 - \frac{1}{3\sqrt{2}} \sqrt{4 - \frac{1}{3\sqrt{2}} \ldots}}} \right) = 4$$

91. $(3/4)^x = 1/4$

Taking log with base 2

$$\Rightarrow \quad x(\log_2 3 - 2) = -2$$

$$\therefore \quad x = \frac{2}{2 - \log_2 3} = \frac{1}{1 - \log_4 3} \Rightarrow \text{(b, c)}$$

and taking log with base 3

$$\Rightarrow \quad x(1 - \log_3 4) = -2 \log_3 2$$

$$\therefore \quad x = \frac{2 \log_3 2}{2 \log_3 2 - 1}$$

CHAPTER

05

Permutations and Combinations

Learning Part

Session 1

- Fundamental Principle of Counting
- Factorial Notation

Session 2

- Divisibility Test
- Principle of Inclusion and Exclusion
- Permutation

Session 3

- Number of Permutations Under Certain Conditions
- Circular Permutations
- Restricted Circular Permutations

Session 4

- Combination
- Restricted Combinations

Session 5

- Combinations from Identical Objects

Session 6

- Arrangement in Groups
- Multinomial Theorem
- Multiplying Synthetically

Session 7

- Rank in a Dictionary
- Gap Method [when particular objects are never together]

Practice Part

- JEE Type Examples
- Chapter Exercises

Arihant on Your Mobile !

Exercises with the symbol can be practised on your mobile. See inside cover page to activate for free.

In everyday life, we need to know about the number of ways of doing certain work from given number of available options. *For example,* Three persons A, B and C are applying for a job in which only one post is vacant. Clearly, vacant post can be filled either by A or B or C i.e., total number of ways doing this work is three.

Again, let two persons A and B are to be seated in a row, then only two possible ways of arrangement is AB or BA. In two arrangements, persons are same but their order is different. Thus, in arranging things, order of things is important.

Session 1

Fundamental Principle of Counting, Factorial Notation

Fundamental Principle of Counting

(i) Multiplication Principle

If an operation can be performed in 'm' different ways, following which a second operation can be performed in 'n' different ways, then the two operations in succession can be performed in $m \times n$ ways. This can be extended to any finite number of operations.

Note For AND → '×' (multiply)

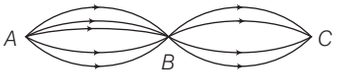
Example 1. A hall has 12 gates. In how many ways can a man enter the hall through one gate and come out through a different gate?

Sol. Since, there are 12 ways of entering into the hall. After entering into the hall, the man come out through a different gate in 11 ways.

Hence, by the fundamental principle of multiplication, total number of ways is $12 \times 11 = 132$ ways.

Example 2. There are three stations A, B and C, five routes for going from station A to station B and four routes for going from station B to station C. Find the number of different ways through which a person can go from A to C via B.

Sol. Since, there are five routes for going from A to B. So, there are four routes for going from B to C.



Hence, by the fundamental principle of multiplication, total number of different ways

$= 5 \times 4$ [i.e., A to B and then B to C]

$= 20$ ways

(ii) Addition Principle

If an operation can be performed in 'm' different ways and another operation, which is independent of the first operation, can be performed in 'n' different ways. Then, either of the two operations can be performed in $(m + n)$ ways. This can be extended to any finite number of mutually exclusive operation.

Note For OR → '+' (Addition)

Example 3. There are 25 students in a class in which 15 boys and 10 girls. The class teacher select either a boy or a girl for monitor of the class. In how many ways the class teacher can make this selection?

Sol. Since, there are 15 ways to select a boy, so there are 10 ways to select a girl.

Hence, by the fundamental principle of addition, either a boy or a girl can be performed in $15 + 10 = 25$ ways.

Example 4. There are 4 students for Physics, 6 students for Chemistry and 7 students for Mathematics gold medal. In how many ways one of these gold medals be awarded?

Sol. The Physics, Chemistry and Mathematics student's gold medal can be awarded in 4, 6 and 7 ways, respectively.

Hence, by the fundamental principle of addition, number ways of awarding one of the three gold medals.

$= 4 + 6 + 7 = 17$ ways.

Factorial Notation

Let n be a positive integer. Then, the continued product of first 'n' natural numbers is called factorial n, to be denoted by $n!$ or n i.e., $n! = n(n-1)(n-2)\ldots 3 \cdot 2 \cdot 1$

Note When n is negative or a fraction, $n!$ is not defined.

Some Important Properties

(i) $n! = n(n-1)! = n(n-1)(n-2)!$

(ii) $0! = 1! = 1$

(iii) $(2n)! = 2^n n![1 \cdot 3 \cdot 5 \ldots (2n-1)]$

(iv) $\dfrac{n!}{r!} = n(n-1)(n-2)\ldots(r+1)$ $[r < n]$

(v) $\dfrac{n!}{(n-r)!} = n(n-1)(n-2)\ldots(n-r+1)$ $[r < n]$

(vi) $\dfrac{1}{n!} + \dfrac{1}{(n+1)!} = \dfrac{\lambda}{(n+2)!}$, then $\lambda = (n+2)^2$

(vii) If $x! = y! \Rightarrow x = y$ or $x = 0, y = 1$
or $x = 1, y = 0$

Example 5. Find n, if $(n+2)! = 60 \times (n-1)!$.

Sol. $\because (n+2)! = (n+2)(n+1)n(n-1)!$

$$\Rightarrow \quad \frac{(n+2)!}{(n-1)!} = (n+2)(n+1)n$$

$$\Rightarrow \quad 60 = (n+2)(n+1)n \qquad \text{[given]}$$

$$\Rightarrow \quad 5 \times 4 \times 3 = (n+2) \times (n+1) \times n$$

$$\therefore \quad n = 3$$

Example 6. Evaluate $\sum_{r=1}^{n} r \times r!$.

Sol. We have, $\sum_{r=1}^{n} r \times r! = \sum_{r=1}^{n} \{(r+1) - 1\} r! = \sum_{r=1}^{n} (r+1)! - r!$

$$= (n+1)! - 1!$$

[put $r = n$ in $(r+1)!$ and $r = 1$ is $r!$]

$$= (n+1)! - 1$$

Example 7. Find the remainder when $\sum_{r=1}^{n} r!$ is divided by 15, if $n \geq 5$.

Sol. Let

$$N = \sum_{r=1}^{n} r! = 1! + 2! + 3! + 4! + 5! + 6! + 7! + \ldots + n!$$

$$= (1! + 2! + 3! + 4!) + (5! + 6! + 7! + \ldots + n!)$$

$$= 33 + (5! + 6! + 7! + \ldots + n!)$$

$$\Rightarrow \quad \frac{N}{15} = \frac{33}{15} + \frac{(5! + 6! + 7! + \ldots + n!)}{15}$$

$$= 2 + \frac{3}{15} + \text{Integer} \qquad [\text{as } 5!, 6!, \ldots \text{ are divisible by } 15]$$

$$= \frac{3}{15} + \text{Integer}$$

Hence, remainder is 3.

Exponent of prime p in $n!$

Exponent of prime p in $n!$ is denoted by $E_p(n!)$, where p is prime number and n is a natural number. The last integer amongst 1, 2, 3, ..., $(n-1)$, n which is divisible by p is $\left[\dfrac{n}{p}\right]p$, where $[\cdot]$ denotes the greatest integer function.

$$\therefore \quad E_p(n!) = E_p(1 \cdot 2 \cdot 3 \ldots (n-1) \cdot n)$$

$$= E_p\left(p \cdot 2p \cdot 3p \ldots (n-1)p \cdot \left[\frac{n}{p}\right] \; p\right)$$

[because the remaining natural numbers from 1 to n are not divisible by p]

$$= \left[\frac{n}{p}\right] + E_p\left(1 \cdot 2 \cdot 3 \ldots \left[\frac{n}{p}\right]\right) \qquad \ldots(\text{i})$$

Now, the last integer amongs 1, 2, 3, ..., $\left[\dfrac{n}{p}\right]$ which is divisible by p is $\left[\dfrac{\frac{n}{p}}{p}\right] = \left[\dfrac{n}{p^2}\right]$. Now, from Eq. (i), we get

$$E_p(n!) = \left[\frac{n}{p}\right] + E_p\left(p, 2p, 3p, \ldots, \left[\frac{n}{p^2}\right]p\right)$$

[because the remaining natural numbers from 1 to $\left[\dfrac{n}{p}\right]$ are not divisible by p]

$$\therefore \quad E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + E_p\left(1 \cdot 2 \cdot 3 \ldots \left[\frac{n}{p^2}\right]\right)$$

Similarly, we get

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \ldots + \left[\frac{n}{p^s}\right]$$

where, s is the largest natural number such that

$$p^s \leq n < p^{s+1}$$

Note Number of zeroes at the end of $n!$ = $E_5(n!)$.

Example 8. ***Find the exponent of*** 3 ***in*** 100! .

Sol. In terms of prime factors 100! can be written as $2^a \cdot 3^b \cdot 5^c \cdot 7^d \ldots$

Now, $b = E_3(100!)$

$$= \left[\frac{100}{3}\right] + \left[\frac{100}{3^4}\right] + \left[\frac{100}{3^3}\right] + \left[\frac{100}{3^4}\right] + \ldots$$

$$= 33 + 11 + 3 + 1 + 0 + \ldots = 48$$

Hence, exponent of 3 is 48.

Aliter

$\because \quad 100! = 1 \times 2 \times 3 \times 4 \times 5 \times \ldots \times 98 \times 99 \times 100$

$= (1 \times 2 \times 4 \times 5 \times 7 \times \ldots \times 98 \times 100)$

$(3 \times 6 \times 9 \times \ldots \times 96 \times 99)$

$= k \times 3^{33}(1 \times 2 \times 3 \times \ldots \times 32 \times 33)$

$= [k(1 \times 2 \times 4 \times 5 \times \ldots \times 31 \times 32)]$

$(3 \times 6 \times 9 \times \ldots \times 30 \times 33)$

$= 3^{33} k_1 \times 3^{11}(1 \times 2 \times 3 \times \ldots \times 10 \times 11)$

$= 3^{44}\ [k_1\ (1 \times 2 \times 4 \times 5 \times \ldots \times 10 \times 11)](3 \times 6 \times 9)$

$= k_2 \times 3^{44} \times 3^4 \times 2 = k_3 \times 3^{48}$

Hence, exponent of 3 in 100! is 48.

Example 9. *Prove that 33! is divisible by 2^{19} and what is the largest integer n such that 33! is divisible by 2^n?*

Sol. In terms of prime factors, 33! can be written as $2^a \cdot 3^b \cdot 5^c \cdot 7^d \cdot \ldots$

Now, $E_2(33!) = \left[\frac{33}{2}\right] + \left[\frac{33}{2^2}\right] + \left[\frac{33}{2^3}\right] + \left[\frac{33}{2^4}\right] + \left[\frac{33}{2^5}\right] + \ldots$

$= 16 + 8 + 4 + 2 + 1 + 0 + \ldots$

$= 31$

Hence, the exponent of 2 in 33! is 31. Now, 33! is divisible by 2^{31} which is also divisible by 2^{19}.

$\therefore$ Largest value of n is 31.

Example 10. *Find the number of zeroes at the end of 100!.*

Sol. In terms of prime factors, 100! can be written as $2^a \cdot 3^b \cdot 5^c \cdot 7^d \ \ldots$

Now, $E_2(100!) = \left[\frac{100}{2}\right] + \left[\frac{100}{2^2}\right] + \left[\frac{100}{2^3}\right] + \left[\frac{100}{2^4}\right] + \left[\frac{100}{2^5}\right] + \left[\frac{100}{2^6}\right]$

$= 50 + 25 + 12 + 6 + 3 + 1 = 97$

and $E_5(100!) = \left[\frac{100}{5}\right] + \left[\frac{100}{5^2}\right]$

$= 20 + 4 = 24$

$\therefore \quad 100! = 2^{97} \cdot 3^b \cdot 5^{24} \cdot 7^d \ \ldots = 2^{73} \cdot 3^b \cdot (2 \times 5)^{24} \cdot 7^d \ \ldots$

$= 2^{73} \cdot 3^b \cdot (10)^{24} \cdot 7^d \ \ldots$

Hence, number of zeroes at the end of 100! is 24.

or Exponent of 10 in 100! = min (97, 24) = 24.

Aliter

Number of zeroes at the end of 100!

$= E_5(100!) = \left[\frac{100}{5}\right] + \left[\frac{100}{5^2}\right] + \ldots$

$= 20 + 4 + 0 + \ldots = 24$

Example 11. For how many positive integral values of n does $n!$ end with precisely 25 zeroes?

Sol. $\because$ Number of zeroes at the end of $n! = 25$ [given]

$\Rightarrow \quad E_5(x!) = 25$

$\Rightarrow \quad \left[\frac{n}{5}\right] + \left[\frac{n}{25}\right] + \left[\frac{n}{125}\right] + \ldots = 25$

It's easy to see that $n = 105$ is the smallest satisfactory value of n. The next four values of n will also work (i.e., $n = 106, 107, 108, 109$). Hence, the answer is 5.

Example 12. Find the exponent of 80 in 180!.

Sol. $\because \ 80 = 2^4 \times 5$

$\therefore \quad E_2(180!) = \left[\frac{180}{2}\right] + \left[\frac{180}{2^2}\right] + \left[\frac{180}{2^3}\right] + \left[\frac{180}{2^4}\right]$

$+ \left[\frac{180}{2^5}\right] + \left[\frac{180}{2^6}\right] + \left[\frac{180}{2^7}\right] + \ldots$

$= 90 + 45 + 22 + 11 + 5 + 2 + 1 = 176$

and $E_5(180!) = \left[\frac{180}{5}\right] + \left[\frac{180}{5^2}\right] + \left[\frac{180}{5^3}\right] + \ldots$

$= 36 + 7 + 1 + 0 + \ldots$

$= 44$

Now, exponent of 16 in 180! is $\left[\frac{176}{4}\right] = 44$, where $[\cdot]$ denotes the greatest integer function. Hence, the exponent of 80 in 180! is 44.

Exercise for Session 1

1. There are three routes: air, rail and road for going from Chennai to Hyderabad. But from Hyderabad to Vikarabad, there are two routes, rail and road. The number of routes from Chennai to Vikarabad via Hyderabad is

(a) 4 (b) 5 (c) 6 (d) 7

2. There are 6 books on Mathematics, 4 books on Physics and 5 books on Chemistry in a book shop. The number of ways can a student purchase either a book on Mathematics or a book on Chemistry, is

(a) 10 (b) 11 (c) 9 (d) 15

3. If a, b and c are three consecutive positive integers such that $a < b < c$ and $\frac{1}{a!} + \frac{1}{b!} = \frac{\lambda}{c!}$, the value of $\sqrt{\lambda}$ is

(a) a (b) b (c) c (d) $a + b + c$

4. If $n!$, $3 \times n!$ and $(n+1)!$ are in GP, then $n!$, $5 \times n!$ and $(n+1)!$ are in

(a) AP (b) GP (c) HP (d) AGP

5. Sum of the series $\sum_{r=1}^{n} (r^2 + 1)\, r!$ is

(a) $(n+1)!$ (b) $(n+2)! - 1$ (c) $n \cdot (n+1)!$ (d) $n \cdot (n+2)!$

6. If $15! = 2^{\alpha} \cdot 3^{\beta} \cdot 5^{\gamma} \cdot 7^{\delta} \cdot 11^{\theta} \cdot 13^{\phi}$, the value of $\alpha - \beta + \gamma - \delta + \theta - \phi$ is

(a) 4 (b) 6 (c) 8 (d) 10

7. The number of naughts standing at the end of 125! is

(a) 29 (b) 30 (c) 31 (d) 32

8. The exponent of 12 in 100! is

(a) 24 (b) 25 (c) 47 (d) 48

9. The number 24! is divisible by

(a) 6^{24} (b) 24^{6} (c) 12^{12} (d) 48^{5}

10. The last non-zero digit in 20! is

(a) 2 (b) 4 (c) 6 (d) 8

11. The number of prime numbers among the numbers $105! + 2, 105! + 3, 105! + 4, \ldots, 105! + 104$ and $105! + 105$ is

(a) 31 (b) 32 (c) 33 (d) None of these

Session 2

Divisibility Test, Principle of Inclusion and Exclusion, Permutation

Divisibility Test

In decimal system all numbers are formed by the digits 0, 1, 2, 3, ..., 9. If $\boldsymbol{a\ b\ c\ d\ e}$ is a five-digit number in decimal system, then we can write that.

$\boldsymbol{a\ b\ c\ d\ e} = 10^4 \cdot a + 10^3 \cdot b + 10^2 \cdot c + 10 \cdot d + e.$

Number $\boldsymbol{a\ b\ c\ d\ e}$ will be divisible

(1) **by 2**, if e is divisible by 2.

(2) **by 4**, if $2d + e$ is divisible by 4.

(3) **by 8**, if $4c + 2d + e$ is divisible by 8.

(4) **by $\boldsymbol{2^t}$**, if number formed by last t digits is divisible by 2^t.

For example, Number 820101280 is divisible by 2^5 because 01280 is divisible by 2^5.

(5) **by 5**, if $e = 0$ or 5.

(6) **by $\boldsymbol{5^t}$**, if number formed by last t digits is divisible by 5^t.

For example, Number 1128375 is divisible by 5^3 because 375 is divisible by 5^3.

(7) **by 3**, if $a + b + c + d + e$(sum of digits) is divisible by 3.

(8) **by 9**, if $a + b + c + d + e$ is divisible by 9.

(9) **by 6**, if $e =$ even and $a + b + c + d + e$ is divisible by 3.

(10) **by 18**, if $e =$ even and $a + b + c + d + e$ is divisible by 9.

(11) **by 7**, if $abcd - 2e$ is divisible by 7.

For example, Number 6552 is divisible by 7 because $655 - 2 \times 2 = 651 = 93 \times 7$ is divisible by 7.

(12) **by 11**, if $\underbrace{a + c + e}_{\text{Sum of digits at odd places}} - \underbrace{b + d}_{\text{Sum of digits at even places}}$ is divisible by 11.

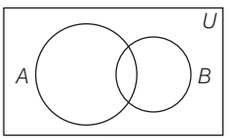
For example, Number 15222163 is divisible by 11 because $(1 + 2 + 2 + 6) - (5 + 2 + 1 + 3) = 0$ is divisible by 11.

(13) **by 13,** if $abcd + 4e$ is divisible by 13.

For example, Number 1638 is divisible by 13 because $163 + 4 \times 8 = 195 = 15 \times 13$ is divisible by 13.

Principle of Inclusion and Exclusion

1. If A and B are finite sets, from the Venn diagram (i), it is clear that

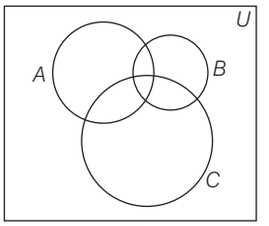


(i)

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

and $n(A' \cap B') = n(U) - n(A \cup B)$

2. If A, B and C are three finite sets, then from the Venn diagram (ii), it is clear that



(ii)

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$$

and $n(A' \cap B' \cap C') = n(U) - n(A \cup B \cup C)$

Note If $A_1, A_2, A_3, \ldots, A_n$ are finite sets, then

$$n(A_1 \cup A_2 \cup \ldots \cup A_n) = \Sigma n(A_i) - \Sigma n(A_i \cap A_j) + \Sigma n(A_i \cap A_j \cap A_k) - \ldots + (-1)^n \Sigma n(A_1 \cap A_2 \cap \ldots \cap A_n)$$

and $n(A'_1 \cap A'_2 \cap \ldots \cap A'_n) = n(U) - n(A_1 \cup A_2 \cup \ldots \cup A_n)$.

Example 13. Find the number of positive integers from 1 to 1000, which are divisible by atleast 2, 3 or 5.

Sol. Let A_k be the set of positive integers from 1 to 1000, which is divisible by k. Obviously, we have to find $n(A_2 \cup A_3 \cup A_5)$. If $[\cdot]$ denotes the greatest integer function, then

$$n(A_2) = \left[\frac{1000}{2}\right] = [500] = 500$$

$$n(A_3) = \left[\frac{1000}{3}\right] = [333.33] = 333$$

$$n(A_5)=\left[\frac{1000}{5}\right]=[200]=200$$

$$n(A_2\cap A_3)=\left[\frac{1000}{6}\right]=[166.67]=166$$

$$n(A_3\cap A_5)=\left[\frac{1000}{15}\right]=[66.67]=66$$

$$n(A_2\cap A_5)=\left[\frac{1000}{10}\right]=[100]=100$$

and $n(A_2\cap A_3\cap A_5)=\left[\frac{1000}{30}\right]=[33.33]=33$

From Principle of Inclusion and Exclusion

$$n(A_2\cup A_3\cup A_5)=n(A_2)+n(A_3)+n(A_5)-n(A_2\cap A_3)-n(A_3\cap A_5)-n(A_2\cap A_5)+n(A_2\cap A_3\cap A_5)$$
$$=500+333+200-166-66-100+33=734$$

Hence, the number of positive integers from 1 to 1000, which are divisible by atleast 2, 3 or 5 is 734.

Note

The number of positive integers from 1 to 1000, which are not divisible by 2, 3 or 5 is $n(A'_2\cap A'_3\cap A'_5)$.

$\therefore n(A'_2\cap A'_3\cap A'_5)=n(U)-n(A_2\cup A_3\cup A_5)$ [here, $n(U)=1000$]
$=1000-734=266$

Permutation

Each of the different arrangements which can be made by taking some or all of a number of things is called a permutation. In permutation, order of the arrangement is important.

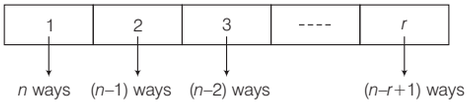
Important Results

1. **The number of permutations of n different things, taking r at a time is denoted by ${}^nP_r$ or $P(n,r)$ or $A(n,r)$, then**

$$^nP_r=n(n-1)(n-2)\ldots(n-r+1)=\frac{n!}{(n-r)!}$$

where, $n\in N$, $r\in W$ and $0\le r\le n$.

Proof LHS = ${}^nP_r$ = Number of ways of filling up r vacant places simultaneously from n different things



$$=n(n-1)(n-2)\ldots(n-r+1)$$
$$=\frac{n(n-1)(n-2)\ldots(n-r+1)\times(n-r)}{(n-r)!}$$
$$=\frac{n!}{(n-r)!}=\text{RHS}$$

Note

(i) The number of permutations of n different things taken all at a time = ${}^nP_n=n!$

(ii) ${}^nP_0=1$, ${}^nP_1=n$ and ${}^nP_{n-1}={}^nP_n=n!$

(iii) ${}^nP_r=n({}^{n-1}P_{r-1})=n(n-1)({}^{n-2}P_{r-2})$
$=n(n-1)(n-2)({}^{n-3}P_{r-3})=\ldots$

(iv) ${}^{n-1}P_r=(n-r)\,{}^{n-1}P_{r-1}$

(v) $\frac{{}^nP_r}{{}^nP_{r-1}}=(n-r+1)$

Example 14. If ${}^{56}P_{r+6}:{}^{54}P_{r+3}=30800:1$, find ${}^rP_2$.

Sol. We have, $\frac{{}^{56}P_{r+6}}{{}^{54}P_{r+3}}=\frac{30800}{1}$

$\Rightarrow (56)(55)\frac{{}^{54}P_{r+4}}{{}^{54}P_{r+3}}=\frac{30800}{1}$ [from note (iii)]

$\Rightarrow \frac{{}^{54}P_{r+4}}{{}^{54}P_{r+3}}=10$

$\Rightarrow 54-(r+4)+1=10$ [from note (v)]

$r=41$

$\therefore {}^rP_2={}^{41}P_2=41\cdot 40=1640$

Example 15. If ${}^{n+5}P_{n+1}=\frac{11(n-1)}{2}\cdot{}^{n+3}P_n$, find n.

Sol. We have, $\frac{{}^{n+5}P_{n+1}}{{}^{n+3}P_n}=\frac{11(n-1)}{2}$

$\Rightarrow \frac{(n+5)(n+4)\cdot{}^{n+3}P_{n-1}}{{}^{n+3}P_n}=\frac{11(n-1)}{2}$ [from note (iii)]

$\frac{(n+5)(n+4)}{(n+3-n+1)}=\frac{11(n-1)}{2}$ [from note (v)]

$\Rightarrow (n+5)(n+4)=22(n-1)$

$\Rightarrow n^2-13n+42=0$

$\Rightarrow (n-6)(n-7)=0$

$\therefore n=6,7$

Example 16. If ${}^{m+n}P_2=90$ and ${}^{m-n}P_2=30$, find the values of m and n.

Sol. $\because {}^{m+n}P_2=90=10\times 9={}^{10}P_2$

$\therefore m+n=10$ ...(i)

and ${}^{m-n}P_2=30=6\times 5={}^6P_2$

$\therefore m-n=6$ ...(ii)

From Eqs. (i) and (ii), we get

$m=8$ and $n=2$.

Example 17. Find the value of r, if

(i) ${}^{11}P_r = 990$

(ii) ${}^{8}P_5 + 5 \cdot {}^{8}P_4 = {}^{9}P_r$

(iii) ${}^{22}P_{r+1} : {}^{20}P_{r+2} = 11 : 52$

Sol. (i) $\because {}^{11}P_r = 990 = 11 \times 10 \times 9 = {}^{11}P_3$

$\therefore \quad r = 3$

(ii) $\because \quad {}^{8}P_5 + 5 \cdot {}^{8}P_4 = {}^{9}P_r$

$\Rightarrow \quad {}^{8}P_4 \left(\frac{{}^{8}P_5}{{}^{8}P_4} + 5 \right) = {}^{9}P_r$

$\Rightarrow \quad {}^{8}P_4 (8 - 5 + 1 + 5) = {}^{9}P_r$ [from note (v)]

$\Rightarrow \quad 9 \cdot {}^{8}P_4 = {}^{9}P_r$

$\Rightarrow \quad {}^{9}P_5 = {}^{9}P_r$ [from note (iii)]

$\therefore \quad r = 5$

(iii) $\because \; {}^{22}P_{r+1} : {}^{20}P_{r+2} = 11 : 52$

$\Rightarrow \quad \frac{{}^{22}P_{r+1}}{{}^{20}P_{r+2}} = \frac{11}{52}$

$\Rightarrow \quad \frac{22 \cdot 21 \cdot {}^{20}P_{r-1}}{(19 - r) \cdot (20 - r) \cdot (21 - r) \cdot {}^{20}P_{r-1}} = \frac{11}{52}$

[from note (iii) and (iv)]

$\Rightarrow \quad (21 - r) \cdot (20 - r) \cdot (19 - r) = 52 \times 2 \times 21$

$= 14 \times 13 \times 12$

$\therefore \quad r = 7$

Example 18. Prove that
${}^{1}P_1 + 2 \cdot {}^{2}P_2 + 3 \cdot {}^{3}P_3 + \ldots + n \cdot {}^{n}P_n = {}^{n+1}P_{n+1} - 1$.

Sol. LHS $= {}^{1}P_1 + 2 \cdot {}^{2}P_2 + 3 \cdot {}^{3}P_3 + \ldots + n \cdot {}^{n}P_n$

$= \sum_{r=1}^{n} r \cdot {}^{r}P_r = \sum_{r=1}^{n} \{(r + 1) - 1\} \cdot {}^{r}P_r$

$= \sum_{r=1}^{n} \{(r + 1) \cdot {}^{r}P_r - {}^{r}P_r)\}$

$= \sum_{r=1}^{n} ({}^{r+1}P_{r+1} - {}^{r}P_r)$ [from note (iii)]

$= {}^{n+1}P_{n+1} - {}^{1}P_1 = {}^{n+1}P_{n+1} - 1$

$=$ RHS

Example 19. Determine the number of permutations of the letters of the word 'SIMPLETON' taken all at a time.

Sol. There are 9 letters in the word 'SIMPLETON' and all the 9 letters are different. Hence, the number of permutations taking all the letters at a time

$= {}^{9}P_9 = 9! = 362880$

Note Total number of letters in English alphabet = 26

(i) Number of vowels = 5
i.e., A, E, I, O, U [strong vowels]

(ii) *W* and *Y* an half vowels. [weak vowels]

(iii) Number of consonants = 21 [except vowels]
i.e., B, C, D, F, G, ..., Y, Z

(iv) Words which contains all vowels are
EDUCATION, EQUATION, ...

(iv) Words which do not contains any vowels are
SKY, FLY, TRY, ...

Example 20. How many different signals can be given using any number of flags from 4 flags of different colours?

Sol. The signals can be made by using one or more flags at a time. Hence, by the fundamental principle of addition, the total number of signals

$= {}^{4}P_1 + {}^{4}P_2 + {}^{4}P_3 + {}^{4}P_4$

$= 4 + (4 \times 3) + (4 \times 3 \times 2) + (4 \times 3 \times 2 \times 1)$

$= 4 + 12 + 24 + 24 = 64$

Example 21. Find the total number of 9-digit numbers which have all different digits.

Sol. Number of digits are 10 (0, 1, 2, 3, 4, 5, 6, 7, 8, 9)

Total number of 9-digit numbers $= {}^{10}P_9$

Out of these, the number of numbers having zero at the first place $= {}^{9}P_8$

Hence, required number of numbers $= {}^{10}P_9 - {}^{9}P_8$

$= 10 \times {}^{9}P_8 - {}^{9}P_8 = 9 \times {}^{9}P_8$

$= 9 \times \frac{9!}{1!} = 9 \times 9!$

Note Total number of n digit numbers $(1 \le n \le 10)$, which have all different digits $= {}^{10}P_n - {}^{9}P_{n-1}$

Example 22. A 5-digit number divisible by 3 is to be formed using the numbers 0, 1, 2, 3, 4 and 5 without repetition. Find total number of ways in which this can be done.

Sol. A number will be divisible by 3, if sum of the digits in number be divisible by 3.

Here, $0 + 1 + 2 + 3 + 4 + 5 = 15$, which is divisible by 3. Therefore, the digit that can be left out, while the sum still is multiple of 3, is either 0 or 3.

If 0 left out

Then, possible numbers $= {}^{5}P_5 = 5! = 120$

If 3 left out

Then, possible numbers $= {}^{5}P_5 - {}^{4}P_4 = 5! - 4! = 120 - 24 = 96$

Hence, required total numbers $= 120 + 96 = 216$

Example 23. A 5-digit number is formed by the digits 1, 2, 3, 4, 5 without repetition. Find the number of the numbers, thus formed divisible by 4.

Sol. Let a 5-digit number be *abcde*.

It will be divisible by 4, if $2d + e$ is divisible by 4.

$\Rightarrow$ $\underbrace{2d + e}_{\text{Even}}$ is divisible by 4 $\quad\therefore$ e must be even.

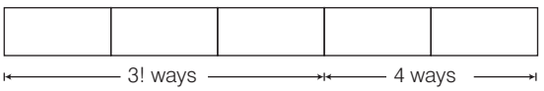
$\Rightarrow$ $\underbrace{2\left(d + \frac{e}{2}\right)}_{\text{Should be even}}$ is divisible by 4

Then, $e = 2, d = 1, 3, 5$ and $e = 4, d = 2$ } Total four cases

$\therefore$ Required number of ways $= 4 \times \underbrace{3!}_{\text{Number of ways}} = 24$

filling *abc* after filling *de*.

Aliter A number will be divisible by 4, if the last two digits of the number is divisible by 4, then for divisible by 4, last two digits 12 or 24 or 32 or 52



Hence, the number formed is divisible by $4 = 3! \times 4 = 24$.

Example 24. ***Find the number of permutations of letters a b c d e f g taken all together if neither 'beg' nor 'cad' pattern appear.***

Sol. The total number of permutations without any restriction is 7!

$$n(U) = 7! = 5040$$

Let $n(A)$ be the number of permutations in which *'beg'* pattern always appears

$$b\ e\ g\ a\ c\ d\ f$$

i.e., $n(A) = 5! = 120$

and let $n(B)$ be the number of permutations in which *'cad'* pattern always appears

$$c\ a\ d\ b\ e\ f\ g$$

i.e., $n(B) = 5! = 120$

Now, $n(A \cap B)$ = Number of permutations in which *'beg'* and *'cad'* pattern appear

$$b\ e\ g\ c\ a\ d\ f$$

i.e., $n(A \cap B) = 3! = 6$

Hence, the number of permutations in which *'beg'* and *'cad'* patterns do not appear is $n(A' \cap B')$.

or $n(A' \cap B') = n(U) - n(A \cup B)$

$= n(U) - [n(A) + n(B) - n(A \cap B)]$

$= 5040 - 120 - 120 + 6 = 4806$

2. The number of permutations of n things taken all at a time, p are alike of one kind, q are alike of second kind and r are alike of a third kind and the rest $n - (p + q + r)$ are all different is $\frac{n!}{p!\,q!\,r!}$

Proof Let the required number of permutations be x. Since, p different things can be arranged among themselves in $p!$ ways, therefore if we replace p identical things by p different things, which are also different from the rest of things, the number of permutations will become $x \times p!$

Again, if we replace q identical things by q different things, the number of permutations will become $(x \times p!) \times q!$

Again, if we replace r identical things by r different things, the number of permutations will become $(x \times p! \times q!) \times r!$. Now, all the n things are different and therefore, number of permutations should be $n!$.

Thus, $x \times p! \times q! \times r! = n!$

$\therefore$ $x = \frac{n!}{p!\,q!\,r!}$

Remark

The above theorem can be extended further i.e. if there are n things taken all at a time, p_1 are alike of one kind, p_2 are alike of second kind, p_3 are alike of third kind, ..., p_r are alike of rth kind such that $p_1 + p_2 + p_3 + \ldots + p_r = n$, the number of permutations of these n things is $\frac{n!}{p_1!\,p_2!\,p_3!\ldots p_r!}$.

Example 25. How many words can be formed with the letters of the word 'ARIHANT' by rearranging them?

Sol. Here, total letters 7, in which 2A's but the rest are different. Hence, the number of words formed $= \frac{7!}{2!} = 2520$

Example 26. Find the number of permutations of the letters of the words 'DADDY DID A DEADLY DEED'.

Sol. Here, total letters 19, in which 9D 's, 3A's, 2Y's, 3E's and rest occur only once.

$\therefore$ Required number of permutations $= \frac{19!}{9! \times 3! \times 2! \times 3!}$

Example 27. How many words can be formed with the letters of the words

(i) HIGH SCHOOL and

(ii) INTERMEDIATE by rearranging them?

Sol. (i) Here, total letters are 10, in which 3H's and 2O's, but the rest are different. Hence, the number of words formed $= \frac{10!}{3!\,2!}$

(ii) Here, total letters are 12, in which 2I's, 2T's and 3E's but the rest are different. Hence, the number of words formed $= \frac{12!}{2!\,2!\,3!}$ **Note** [For Remember]

High School = 10 th class = Total number of letters are 10

Intermediate = 12 th class = Total number of letters are 12

3. The number of permutations of n different things taken r at a time when each thing may be repeated any number of times is n^r.

Proof Since, the number of permutations of n different things taken r at a time = Number of ways in which r blank places can be filled by n different things.

Clearly, the first place can be filled in n ways. Since, each thing may be repeated, the second place can be filled in n ways. Similarly, each of the 3rd, 4th, …, rth place can be filled in n ways.

By multiplication principle, the number of permutations of n different things taken r at a time when each thing may be repeated any number of times

$$= n \times n \times n \times \ldots \times r \text{ factors}$$
$$= n^r$$

Corollary When $r = n$

i.e., the number of permutations of n different things, taken all at a time, when each thing may be repeated any number of times in each arrangements is n^n.

Example 28. ***A child has four pockets and three marbles. In how many ways can the child put the marbles in its pockets?***

Sol. The first marble can be put into the pocket in 4 ways, so can the second. Thus, the number of ways in which the child can put the marbles $= 4 \times 4 \times 4 = 4^3 = 64$ ways

Example 29. There are m men and n monkeys $(n > m)$. If a man have any number of monkeys. In how many ways may every monkey have a master?

Sol. The first monkey can select his master by m ways and after that the second monkey can select his master again by m ways, so can the third and so on.

All monkeys can select master $= m \times m \times m \ldots$ upto n factors $= (m)^n$ ways

Example 30. How many four digit numbers can be formed by using the digits 1, 2, 3, 4, 5, 6, 7, if atleast one digit is repeated ?

Sol. The numbers that can be formed when repetition of digits is allowed are $7^4 = 2401$.

The numbers that can be formed when all the digits are distinct when repetition is not allowed are ${}^7P_4 = 840$.

Therefore, the numbers that can be formed when atleast one digit is repeated $= 7^4 - {}^7P_4$

$$= 2401 - 840 = 1561$$

Example 31. In how many ways can 4 prizes be distributed among 5 students, if no student gets all the prizes?

Sol. The number of ways in which the 4 prizes can be given away to the 5 students, if a student can get any number of prizes $= 5^4 = 625$.

Again, the number of ways in which a student gets all the 4 prizes = 5, since there are 5 students and any one of them may get all the four prizes.

Therefore, the required number of ways in which a student does not get all the prizes $= 625 - 5 = 620$.

Example 32. Find the number of n-digit numbers, which contain the digits 2 and 7, but not the digits 0, 1, 8, 9.

Sol. The total number without any restrictions containing digits 2, 3, 4, 5, 6, 7 is $n(U) = 6^n$.

The total number of numbers that contain 3, 4, 5, 6, 7 is $n(A) = 5^n$.

The total number of numbers that contain 2, 3, 4, 5, 6 is $n(B) = 5^n$.

The total number of numbers that contain 3, 4, 5, 6 is $n(A \cap B) = 4^n$.

The total number of numbers that do not contain digits 2 and 7 is $n(A \cup B)$

i.e., $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$$= 5^n + 5^n - 4^n = 2(5^n) - 4^n$$

Hence, the total number of numbers that contain 2 and 7 is $n(A' \cap B')$

$\therefore \quad n(A' \cap B') = n(U) - n(A \cup B) = 6^n - 2 \cdot (5^n) + 4^n$

Example 33. Show that the total number of permutations of n different things taken not more than r at a time, when each thing may be repeated any number of times is $\dfrac{n(n^r - 1)}{(n-1)}$.

Sol. Given, total different things $= n$

The number of permutations of n things taken one at a time $= {}^nP_1 = n$, now if we taken two at a time (repetition is allowed), then first place can be filled by n ways and second place can again be filled in n ways.

$\therefore$ The number of permutations of n things taken two at a time

$$= {}^nP_1 \times {}^nP_1 = n \times n = n^2$$

Similarly, the number of permutations of n things taken three at a time $= n^3$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

The number of permutations of n things taken r at a time $= n^r$. Hence, the total number of permutations

$$= n + n^2 + n^3 + \ldots + n^r$$
$$= \frac{n(n^r - 1)}{(n-1)} \qquad \text{[sum of } r \text{ terms of a GP]}$$

Exercise for Session 2

1. If ${}^{n}P_5 = 20 \cdot {}^{n}P_3$, then n equals

(a) 4 (b) 8 (c) 6 (d) 7

2. If ${}^{9}P_5 + 5 \cdot {}^{9}P_4 = {}^{n}P_r$, then $n + r$ equals

(a) 13 (b) 14 (c) 15 (d) 16

3. If ${}^{m+n}P_2 = 56$ and ${}^{m-n}P_3 = 24$, then $\frac{{}^{m}P_3}{{}^{n}P_2}$ equals

(a) 20 (b) 40 (c) 60 (d) 80

4. If ${}^{2n+1}P_{n-1} : {}^{2n-1}P_n = 7 : 10$, then ${}^{n}P_3$ equals

(a) 60 (b) 24 (c) 120 (d) 6

5. In a train, five seats are vacant, the number of ways three passengers can sit, is

(a) 10 (b) 20 (c) 30 (d) 60

6. If a denotes the number of permutations of $x + 2$ things taken all at a time, b the number of permutations of x things taken 11 at a time and c the number of permutations of $(x - 11)$ things taken all at a time such that $a = 182\,bc$, the value of x is

(a) 10 (b) 12 (c) 15 (d) 18

7. The number of nine non-zero digits such that all the digits in the first four places are less than the digit in the middle and all the digits in the last four places are greater than that in the middle, is

(a) 48 (b) 7560 (c) 10080 (d) 576

8. Total number of words that can be formed using all letters of the word 'DIPESH' that neither begins with 'I' nor ends with 'D' is equal to

(a) 504 (b) 480 (c) 624 (d) 696

9. The number of all five digit numbers which are divisible by 4 that can be formed from the digits 0, 1, 2, 3, 4 (without repetition), is

(a) 36 (b) 30 (c) 34 (d) None of these

10. The number of words can be formed with the letters of the word 'MATHEMATICS' by rearranging them, is

(a) $\frac{11!}{2!\,2!}$ (b) $\frac{11!}{2!}$ (c) $\frac{11!}{2!\,2!\,2!}$ (d) $11!$

11. Six identical coins are arranged in a row. The number of ways in which the number of tails is equal to the number of heads, is

(a) 9 (b) 20 (c) 40 (d) 120

12. A train time table must be compiled for various days of the week so that two trains twice a day depart for three days, one train daily for two days and three trains once a day for two days. How many different time tables can be compiled?

(a) 140 (b) 210 (c) 133 (d) 72

13. Five persons entered the lift cabin on the ground floor of an 8 floor house. Suppose each of them can leave the cabin independently at any floor beginning with the first. The total number of ways in which each of the five persons can leave the cabin at any one of the floor, is

(a) 5^7 (b) 7^5 (c) 35 (d) 2520

14. Four die are rolled. The number of ways in which atleast one die shows 3, is

(a) 625 (b) 671 (c) 1256 (d) 1296

15. The number of 4-digit numbers that can be made with the digits 1, 2, 3, 4 and 5 in which atleast two digits are identical, is

(a) $4^5 - 5!$ (b) 505 (c) 600 (d) 120

16. There are unlimited number of identical balls of three different colours. How many arrangements of atmost 7 balls in a row can be made by using them?

(a) 2187 (b) 343 (c) 399 (d) 3279

Session 3

Number of Permutations Under Certain Conditions, Circular Permutations, Restricted Circular Permutations

Number of Permutations Under Certain Conditions

(i) Number of permutations of n different things, taken r at a time, when a particular thing is to be always included in each arrangement, is $r \cdot {}^{n-1}P_{r-1}$

Corollary Number of permutations of n different things, taken r at a time, when p particular things is to be always included in each arrangement, is

$$p\,!\,(r-(p-1))^{n-p}P_{r-p}.$$

(ii) Number of permutations of n different things, taken r at a time, when a particular thing is never taken in each arrangement, is

$${}^{n-1}P_r.$$

(iii) Number of permutations of n different things, taken all at a time, when m specified things always come together, is

$$m\,!\times(n-m+1)\,!$$

(iv) Number of permutations of n different things, taken all at a time, when m specified things never come together, is

$$n\,!-m\,!\times(n-m+1)\,!$$

Example 34. How many permutations can be made out of the letters of the word 'TRIANGLE' ? How many of these will begin with T and end with E ?

Sol. The word 'TRIANGLE' has eight different letters, which can be arranged themselves in 8! ways.

$\therefore$ Total number of permutations $= 8\,! = 40320$

Again, when T is fixed at the first place and E at the last place, the remaining six can be arranged themselves in $6\,!$ ways.

$\therefore$ The number of permutations which begin with T and end with E $= 6! = 720.$

Example 35. In how many ways can the letters of the word 'INSURANCE' be arranged, so that the vowels are never separate?

Sol. The word 'INSURANCE' has nine different letters, combine the vowels into one bracket as (IUAE) and treating them as one letter we have six letters viz.

(IUAE) N S R N C and these can be arranged among themselves in $\frac{6\,!}{2\,!}$ ways and four vowels within the bracket can be arranged themselves in $4\,!$ ways.

$\therefore$ Required number of words $= \frac{6\,!}{2\,!}\times 4\,! = 8640$

Example 36. How many words can be formed with the letters of the word 'PATALIPUTRA' without changing the relative positions of vowels and consonants?

Sol. The word 'PATALIPUTRA' has eleven letters, in which 2P's, 3A's, 2T's, 1L, 1U, 1R and 1I. Vowels are AAIUA

These vowels can be arranged themselves in $\frac{5\,!}{3\,!} = 20$ ways.

The consonants are PTLPTR these consonants can be arranged themselves in $\frac{6\,!}{2\,!\,2\,!} = 180$ ways

$\therefore$ Required number of words

$$= 20 \times 180 = 3600 \text{ ways.}$$

Example 37. Find the number of permutations that can be had from the letters of the word 'OMEGA'

(i) O and A occuping end places.

(ii) E being always in the middle.

(iii) Vowels occuping odd places.

(iv) Vowels being never together.

Sol. There are five letters in the word 'OMEGA'.

(i) When O and A occuping end places

i.e., M E G (OA)

the first three letters (M, E, G) can be arranged themselves by $3\,! = 6$ ways and last two letters (O, A) can be arranged themselves by $2\,! = 2$ ways.

$\therefore$ Total number of such words

$$= 6 \times 2 = 12 \text{ ways.}$$

(ii) When E is the fixed in the middle, then there are four places left to be filled by four remaining letters O, M, G and A and this can be done in 4 ! ways.

∴ Total number of such words = 4 ! = 24 ways.

(iii) Three vowels (O, E, A) can be arranged in the odd places in 3 ! ways (1st, 3rd and 5th) and the two consonants (M, G) can be arranged in the even places in 2 ! ways (2nd and 4th)

∴ Total number of such words

$= 3! \times 2! = 12$ ways.

(iv) Total number of words = 5 ! = 120

Combine the vowels into one bracket as (OEA) and treating them as one letter, we have

(OEA), M, G and these can be arranged themselves in 3 ! ways and three vowels with in the bracket can be arranged themselves in 3 ! ways.

∴ Number of ways when vowels come together

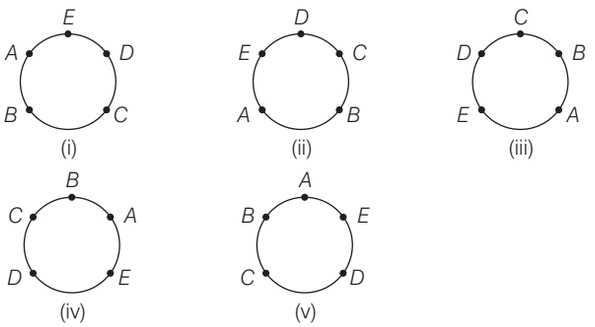
$= 3! \times 3! = 36$ ways.

Hence, number of ways when vowels being never together = 120 − 36 = 84 ways.

Circular Permutations

(i) Arrangements round a circular table

Consider five persons *A*, *B*, *C*, *D* and *E* on the circumference of a circular table in order which has no head now, shifting *A*, *B*, *C*, *D* and E one position in anti-clockwise direction we will get arragements as follows



We see that, if 5 persons are sitting at a round table, they can be shifted five times and five different arrangements. Thus, obtained will be the same, because anti-clockwise order of *A*, *B*, *C*, *D* and *E* does not change.

But if *A*, *B*, *C*, *D* and *E* are sitting in a row and they are shifted in such an order that the last occupies the place of first, then the five arrangements will be different. Thus, if there are 5 things, then for each circular arrangement number of linear arrangements is 5.

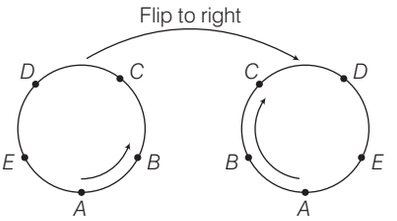
Similarly, if *n* different things are arranged along a circle for each circular arrangement number of linear arrangements is *n*.

Therefore, the number of linear arrangements of *n* different things = *n* × number of circular arrangements of *n* different things

(ii) Arrangements of beads or flowers (all different) around a circular necklace or garland

Consider five beads *A*, *B*, *C*, *D* and *E* in a necklace or five flowers *A*, *B*, *C*, *D* and *E* in a garland, etc. If the necklace or garland on the left is turned over, we obtain the arrangement on the right i.e. anti-clockwise and clockwise order of arrangement is not different we will get arrangements as follows:

We see that arrangements in figures are not different.



Then, the number of circular permutations of *n* different things taken all at a time is $\frac{1}{2}(n-1)!$, if clockwise and anti-clockwise orders are taken as not different.

Example 38. Find the number of ways in which 12 different beads can be arranged to form a necklace.

Sol. 12 different beads can be arranged among themselves in a circular order in $(12-1)! = 11!$ ways. Now, in the case of necklace, there is no distinction between clockwise and anti-clockwise arrangements. So, the required number of arrangements $= \frac{1}{2}(11!)$.

Example 39. Consider 21 different pearls on a necklace. How many ways can the pearls be placed in on this necklace such that 3 specific pearls always remain together?

Sol. After fixing the places of three pearls, treating 3 specific pearls = 1 unit. So, we have now

18 pearls + 1 unit = 19 and the number of arrangement will be $(19-1)! = 18!$

Also, the number of ways of 3 pearls can be arranged between themselves is 3 ! = 6.

Since, there is no distinction between the clockwise and anti-clockwise arrangements.

So, the required number of arrangements $= \frac{1}{2}18! \cdot 6 = 3(18!)$.

Restricted Circular Permutations

***Case* I** If clockwise and anti-clockwise orders are taken as different, then the number of circular permutations of n different things taken r at a time.

$$= \frac{{}^{n}P_r}{r} = \frac{1}{r} \cdot \frac{n!}{(n-r)!}$$

Note For checking correctness of formula, put $r = n$, then we get $(n-1)!$ [result (5) (i)]

Example 40. In how many ways can 24 persons be seated round a table, if there are 13 sets ?

Sol. In case of circular table, the clockwise and anti-clockwise orders are different, the required number of circular permutations $= \frac{{}^{24}P_{13}}{13} = \frac{24!}{13 \times 11!}$.

$\Rightarrow n! = n \times$ number of circular arrangements of n different things

$\Rightarrow$ Number of circular arrangements of n different things

$$= \frac{n!}{n} = (n-1)!$$

Hence, the number of circular permutations of n different things taken all at a time is $(n-1)!$, if clockwise and anti-clockwise orders are taken as different.

Example 41. Find the number of ways in which three Americans, two British, one Chinese, one Dutch and one Egyptian can sit on a round table so that persons of the same nationality are separated.

Sol. The total number of persons without any restrictions is

$$n(U) = (8-1)!$$
$$= 7! = 5040$$

When, three Americans (A_1, A_2, A_3) are sit together,

$$n(A) = 5! \times 3!$$
$$= 720$$

When, two British (B_1, B_2) are sit together

$$n(B) = 6! \times 2!$$
$$= 1440$$

When, three Americans (A_1, A_2, A_3) and two British (B_1, B_2) are sit together $n(A \cap B) = 4! \times 3! \times 2! = 288$

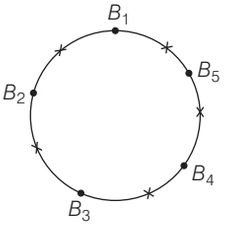
$$\therefore \quad n(A \cup B) = n(A) + nB) - n(A \cap B)$$
$$= 720 + 1440 - 288 = 1872$$

Hence, $n(A \cap B') = n(U) - n(A \cup B)$

$$= 5040 - 1872$$
$$= 3168$$

Example 42. In how many different ways can five boys and five girls form a circle such that the boys and girls alternate?

Sol. After fixing up one boy on the table, the remaining can be arranged in 4 ! ways but boys and girls are to alternate. There will be 5 places, one place each between two boys these five places can be filled by 5 girls in 5 ! ways.



Hence, by the principle of multiplication, the required number of ways $= 4! \times 5! = 2880$.

Example 43. 20 persons were invited to a party. In how many ways can they and the host be seated at a circular table ? In how many of these ways will two particular persons be seated on either side of the host?

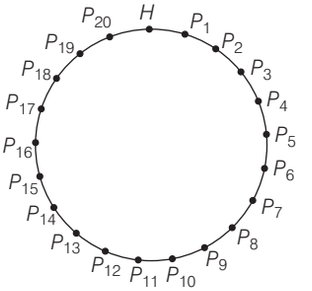
Sol. **I Part** Total persons on the circular table

$$= 20 \text{ guest} + 1 \text{ host} = 21$$

They can be seated in $(21-1)! = 20!$ ways.

II Part After fixing the places of three persons (1 host + 2 persons).

Treating (1 host + 2 persons) = 1 unit, so we have now {(remaining 18 persons + 1 unit) = 19} and the number of arrangement will be $(19-1)! = 18!$ also these two particular persons can be seated on either side of the host in 2 ! ways.



Hence, the number of ways of seating 21 persons on the circular table such that two particular persons be seated on either side of the host $= 18! \times 2! = 2 \times 18!$

***Case* II** If clockwise and anti-clockwise orders are taken as not different, then the number of circular permutations of n different things taken r at a time $= \frac{{}^{n}P_r}{2r} = \frac{1}{2r} \cdot \frac{n!}{(n-r)!}$

Note

For checking correctness of formula put $r = n$, then we get $\frac{(n-1)!}{2}$ [result (5) (ii)]

Example 44. ***How many necklace of 12 beads each can be made from 18 beads of various colours?***

Sol. In the case of necklace, there is no distinction between the clockwise and anti-clockwise arrangements, the required number of circular permutations.

$$= \frac{{}^{18}P_{12}}{2 \times 12} = \frac{18!}{6! \times 24} = \frac{18 \times 17 \times 16 \times 15 \times 14 \times 13!}{6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 24} = \frac{119 \times 13!}{2}$$

Exercise for Session 3

1. How many words can be formed from the letters of the word 'COURTESY' whose first letter is C and the last letter is Y ?

(a) 6! (b) 8! (c) 2(6)! (d) 2(7)!

2. The number of words that can be made by writing down the letters of the word 'CALCULATE' such that each word starts and ends with a consonant, is

(a) $\frac{3}{2}(7)!$ (b) 2(7)! (c) $\frac{5}{2}(7)!$ (d) 3(7)!

3. The number of words can be formed from the letters of the word 'MAXIMUM', if two consonants cannot occur together, is

(a) 4! (b) $3! \times 4!$

(c) 3! (d) $\frac{4!}{3!}$

4. All the letters of the word 'EAMCET' are arranged in all possible ways. The number of such arrangements in which two vowels are not adjacent to each other, is

(a) 54 (b) 72

(c) 114 (d) 360

5. How many words can be made from the letters of the word 'DELHI', if L comes in the middle in every word?

(a) 6 (b) 12 (c) 24 (d) 60

6. In how many ways can 5 boys and 3 girls sit in a row so that no two girls are sit together?

(a) $5! \times 3!$ (b) ${}^4P_3 \times 5!$ (c) ${}^6P_3 \times 5!$ (d) ${}^5P_3 \times 3!$

7. There are n numbered seats around a round table. Total number of ways in which n_1 $(n_1 < n)$ persons can sit around the round table, is equal to

(a) ${}^nC_{n_1}$ (b) ${}^nP_{n_1}$ (c) ${}^nC_{n_1-1}$ (d) ${}^nP_{n_1-1}$

8. In how many ways can 7 men and 7 women can be seated around a round table such that no two women can sit together?

(a) 7! (b) $7! \times 6!$ (c) $(6!)^2$ (d) $(7!)^2$

9. The number of ways that 8 beads of different colours be string as a necklace, is

(a) 2520 (b) 2880 (c) 4320 (d) 5040

10. If 11 members of a committee sit at a round table so that the President and secretary always sit together, then the number of arrangements, is

(a) $9! \times 2$ (b) 10! (c) $10! \times 2$ (d) 11!

11. In how many ways can 15 members of a council sit along a circular table, when the secretary is to sit on one side of the Chairman and the deputy secretary on the other side?

(a) $12! \times 2$ (b) 24 (c) $15! \times 2$ (d) 30

Session 4

Combination, Restricted Combinations

Combination

Each of the different groups or selections which can be made by some or all of a number of given things without reference to the order of the things in each group is called a combination.

Important Result

(1) The number of combinations of n different things taken r at a time is denoted by ${}^nC_r$ or $C(n, r)$ or $\left(\frac{n}{r}\right)$.

Then,

$$
\begin{aligned}
{}^nC_r &= \frac{n!}{r!(n-r)!} \qquad [0 \le r \le n] \\
&= \frac{{}^nP_r}{r!} \\
&= \frac{n(n-1)(n-2)\ldots(n-r+1)}{r(r-1)(r-2)\ldots 2\cdot 1}, \; n \in N \text{ and } r \in W
\end{aligned}
$$

Proof Let the number of combinations of n different things taken r at a time be ${}^nC_r$.

Now, each combination consists of r different things and these r things can be arranged among themselves in $r!$ ways.

Thus, for one combination of r different things, the number of arrangements is $r!$.

Hence, for ${}^nC_r$ combinations, number of arrangements is

$$r! \times {}^nC_r \qquad \ldots(i)$$

But number of permutations of n different things taken r at a time is ${}^nP_r$. $\ldots(ii)$

From Eqs. (i) and (ii), we get

$$r! \times {}^nC_r = {}^nP_r = \frac{n!}{(n-r)!}$$

$$\therefore \quad {}^nC_r = \frac{n!}{r!(n-r)!}, \; r \in W \text{ and } n \in N$$

Note the following facts:

(i) ${}^nC_r$ is a natural number

(ii) ${}^nC_r = 0$, if $r > n$

(iii) ${}^nC_0 = {}^nC_n = 1$, ${}^nC_1 = n$

(iv) ${}^nP_r = {}^nC_r$, if $r = 0$ or 1

(v) ${}^nC_r = {}^nC_{n-r}$, if $r > \frac{n}{2}$

(vi) If ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$

(vii) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$ [Pascal's rule]

(viii) ${}^nC_r = \frac{n}{r} \cdot {}^{n-1}C_{r-1}$

(ix) $n \cdot {}^{n-1}C_{r-1} = (n-r+1) \cdot {}^nC_{r-1}$

(x) $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$

(xi) (a) If n is even, ${}^nC_r$ is greatest for $r = \frac{n}{2}$

(b) If n is odd, ${}^nC_r$ is greatest for $r = \frac{n-1}{2}$ or $\frac{n+1}{2}$

(xii) ${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n$

(xiii) ${}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots = 2^{n-1}$

(xiv) ${}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_n = 2^{2n}$

(xv) ${}^nC_n + {}^{n+1}C_n + {}^{n+2}C_2 + {}^{n+3}C_n + \ldots + {}^{2n-1}C_n = {}^{2n}C_{n+1}$

Example 45. If ${}^{15}C_{3r} = {}^{15}C_{r+3}$, find ${}^rC_2$.

Sol. We have, ${}^{15}C_{3r} = {}^{15}C_{r+3}$

$\Rightarrow \quad 3r = r + 3$

or $\quad 3r + r + 3 = 15$

$\Rightarrow \quad 2r = 3$ or $4r = 12$

$\Rightarrow \quad r = \frac{3}{2}$ or $r = 3$

but $r \in W$, so that $r \neq \frac{3}{2}$

$\therefore \quad r = 3$

Then, ${}^rC_2 = {}^3C_2 = {}^3C_1 = 3$

Example 46. If ${}^{n}C_9 = {}^{n}C_7$, find n.

Sol. We have, ${}^{n}C_9 = {}^{n}C_7 \Rightarrow n = 9 + 7 \quad [\because 9 \neq 7]$

$\therefore \quad n = 16$

Example 47. ***Prove that***

$$\binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2} = \binom{n+2}{r}$$

Sol. $\therefore \binom{n}{r} = {}^{n}C_r$

$$\therefore \text{ LHS} = \binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2}$$

$$= {}^{n}C_r + 2\,{}^{n}C_{r-1} + {}^{n}C_{r-2}$$

$$= ({}^{n}C_r + {}^{n}C_{r-1}) + ({}^{n}C_{r-1} + {}^{n}C_{r-2})$$

$$= {}^{n+1}C_r + {}^{n+1}C_{r-1} = {}^{n+2}C_r$$

$$= \binom{n+2}{r} = \text{RHS}$$

Example 48. If ${}^{2n}C_3 : {}^{n}C_3 = 11:1$, find the value of n.

Sol. We have,

$${}^{2n}C_3 : {}^{n}C_3 = 11:1$$

$$\Rightarrow \quad \frac{{}^{2n}C_3}{{}^{n}C_3} = \frac{11}{1}$$

$$\Rightarrow \quad \frac{\frac{2n(2n-1)(2n-2)}{1\cdot 2\cdot 3}}{\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}} = 11 \;\Rightarrow\; \frac{4(2n-1)}{(n-2)} = 11$$

$$\Rightarrow \quad 8n - 4 = 11n - 22 \quad \Rightarrow 3n = 18$$

$$\therefore \quad n = 6$$

Example 49. If ${}^{n+1}C_{r+1} : {}^{n}C_r : {}^{n-1}C_{r-1} = 11:6:3$, find the values of n and r.

Sol. Here, $\frac{{}^{n+1}C_{r+1}}{{}^{n}C_r} = \frac{11}{6}$

$$\Rightarrow \quad \frac{n+1}{r+1}\cdot\frac{{}^{n}C_r}{{}^{n}C_r} = \frac{11}{6} \qquad \left[\because {}^{n}C_r = \frac{n}{r}\cdot {}^{n-1}C_{r-1}\right]$$

$$\Rightarrow \quad \frac{n+1}{r+1} = \frac{11}{6}$$

$$\Rightarrow \quad 6n + 6 = 11r + 11$$

$$\Rightarrow \quad 6n - 11r = 5 \qquad \ldots(i)$$

and $\frac{{}^{n}C_r}{{}^{n-1}C_{r-1}} = \frac{6}{3}$

$$\Rightarrow \quad \frac{n}{r}\cdot\frac{{}^{n-1}C_{r-1}}{{}^{n-1}C_{r-1}} = \frac{6}{3} \qquad \left[\because {}^{n}C_r = \frac{n}{r}\cdot {}^{n-1}C_{r-1}\right]$$

$$\Rightarrow \quad \frac{n}{r} = 2$$

$$\Rightarrow \quad n = 2r \qquad \ldots(ii)$$

On solving Eqs. (i) and (ii), we get $n = 10$ and $r = 5$.

Example 50. If ${}^{n}C_{r-1} = 36$, ${}^{n}C_r = 84$ and ${}^{n}C_{r+1} = 126$, find r.

Sol. Here, $\frac{{}^{n}C_r}{{}^{n}C_{r-1}} = \frac{84}{36}$

$$\Rightarrow \quad \frac{n-r+1}{r} = \frac{7}{3} \qquad \left[\because \frac{{}^{n}C_r}{{}^{n}C_{r-1}} = \frac{n-r+1}{r}\right]$$

$$\Rightarrow \quad 3n - 3r + 3 = 7r$$

$$\Rightarrow \quad 10r - 3n = 3 \qquad \ldots(i)$$

and $\frac{{}^{n}C_{r+1}}{{}^{n}C_r} = \frac{n-(r+1)+1}{(r+1)} = \frac{126}{84} \quad \left[\because \frac{{}^{n}C_r}{{}^{n}C_{r-1}} = \frac{n-r+1}{r}\right]$

$$\Rightarrow \quad \frac{n-r}{r+1} = \frac{3}{2}$$

$$\Rightarrow \quad 2n - 2r = 3r + 3$$

$$\Rightarrow \quad 5r - 2n = -3$$

or $\quad 10r - 4n = -6 \qquad \ldots(ii)$

On subtracting Eq. (ii) from Eq. (i), we get

$$n = 9$$

From Eq. (i), we get

$$10r - 27 = 3 \;\Rightarrow\; 10r = 30$$

$$\therefore \quad r = 3$$

Example 51. Prove that product of r consecutive positive integers is divisible by $r!$.

Sol. Let r consecutive positive integers be (m), $(m+1), (m+2), \ldots, (m+r-1)$, where $m \in N$.

$\therefore$ Product $= m(m+1)(m+2)\ldots(m+r-1)$

$$= \frac{(m-1)!\, m(m+1)(m+2)\ldots(m+r-1)}{(m-1)!}$$

$$= \frac{(m+r-1)!}{(m-1)!} = \frac{r!\cdot(m+r-1)!}{r!\,(m-1)!}$$

$[\because {}^{m+r-1}C_r$ is natural number$]$

$$= r!\cdot {}^{m+r-1}C_r,$$

which is divisible by $r!$.

Example 52. Evaluate

$${}^{47}C_4 + \sum_{j=0}^{3} {}^{50-j}C_3 + \sum_{k=0}^{5} {}^{56-k}C_{53-k}.$$

Sol. We have, ${}^{47}C_4 + \sum_{j=0}^{3} {}^{50-j}C_3 + \sum_{k=0}^{5} {}^{56-k}C_{53-k}$

$$= {}^{47}C_4 + \sum_{j=0}^{3} {}^{50-j}C_3 + \sum_{k=0}^{5} {}^{56-k}C_3 \quad [\because {}^{n}C_r = {}^{n}C_{n-r}]$$

$$= {}^{47}C_4 + ({}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3 + {}^{47}C_3) + ({}^{56}C_3 + {}^{55}C_3 + {}^{54}C_3 + {}^{53}C_3 + {}^{52}C_3 + {}^{51}C_3)$$

$$= {}^{47}C_4 + {}^{47}C_3 + {}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3 + {}^{52}C_3 + {}^{53}C_3 + {}^{54}C_3 + {}^{55}C_3 + {}^{56}C_3$$

$$= ({}^{47}C_4 + {}^{47}C_3) + {}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3 + {}^{52}C_3 + {}^{53}C_3 + {}^{54}C_3 + {}^{55}C_3 + {}^{56}C_3$$

$$= {}^{48}C_4 + {}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3 + \ldots + {}^{56}C_3$$

$$= {}^{49}C_4 + {}^{49}C_3 + {}^{50}C_3 + \ldots + {}^{56}C_3$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$= {}^{56}C_4 + {}^{56}C_3 = {}^{57}C_4$$

Example 53. Prove that the greatest value of ${}^{2n}C_r$ $(0 \le r \le 2n)$ is ${}^{2n}C_n$ (for $1 \le r \le n$).

Sol. We have, $\dfrac{{}^{2n}C_r}{{}^{2n}C_{r-1}} = \dfrac{2n-r+1}{r} \quad \left[\because \dfrac{{}^{n}C_r}{{}^{n}C_{r-1}} = \dfrac{n-r+1}{r}\right]$

$$= \frac{2(n-r)+(r+1)}{r} = \frac{1+2(n-r)+1}{r} > 1$$

$$\Rightarrow \quad \frac{{}^{2n}C_r}{{}^{2n}C_{r-1}} > 1 \qquad [\text{for } 1 \le r \le n]$$

$$\therefore \quad {}^{2n}C_{r-1} < {}^{2n}C_r$$

On putting $r = 1, 2, 3, \ldots, n$,

then ${}^{2n}C_0 < {}^{2n}C_1$, ${}^{2n}C_1 < {}^{2n}C_2, \ldots, {}^{2n}C_{n-1} < {}^{2n}C_n$

On combining all inequalities, we get

$$\Rightarrow \quad {}^{2n}C_0 < {}^{2n}C_1 < {}^{2n}C_2 < {}^{2n}C_3 < \ldots < {}^{2n}C_{n-1} < {}^{2n}C_n$$

but ${}^{2n}C_r = {}^{2n}C_{2n-r}$, it follows that

$${}^{2n}C_{2n} < {}^{2n}C_{2n-1} < {}^{2n}C_{2n-2} < {}^{2n}C_{2n-3} < \ldots < {}^{2n}C_{n+1} < {}^{2n}C_n$$

Hence, the greatest value of ${}^{2n}C_r$ is ${}^{2n}C_n$.

Example 54. Thirty six games were played in a football tournament with each team playing once against each other. How many teams were there?

Sol. Let the number of teams be n.

Then number of matches to be played is ${}^{n}C_2 = 36$

$$\Rightarrow \quad {}^{n}C_2 = \frac{9 \times 8}{1 \times 2} = {}^{9}C_2$$

$$\Rightarrow \quad n = 9$$

Restricted Combinations

(i) The number of selections (combinations) of r objects out of n different objects, when

(a) k particular things are always included $= {}^{n-k}C_{r-k}$.

(b) k particular things are never included $= {}^{n-k}C_r$.

(ii) The number of combinations of r things out of n different things, such that k particular things are not together in any selection $= {}^{n}C_r - {}^{n-k}C_{r-k}$

(iii) The number of combinations of n different objects taking r at a time when, p particular objects are always included and q particular objects are always excluded $= {}^{n-p-q}C_{r-p}$

Note

(i) The number of selections of r consecutive things out of n things in a row $= n - r + 1$.

(ii) The number of selections of r consecutive things out of n things along a circle $= \begin{cases} n, \text{ if } r < n \\ 1, \text{ if } r = n \end{cases}$.

Example 55. In how many ways can a cricket, eleven players by chosen out of a batch 15 players, if

(i) a particular is always chosen.

(ii) a particular player is never chosen?

Sol. (i) Since, particular player is always chosen. It means that $11 - 1 = 10$ players are selected out of the remaining $15 - 1 = 14$ players.

$\therefore$ Required number of ways $= {}^{14}C_{10} = {}^{14}C_4$

$$= \frac{14 \cdot 13 \cdot 12 \cdot 11}{1 \cdot 2 \cdot 3 \cdot 4} = 1001$$

(ii) Since, particular player is never chosen. It means that 11 players are selected out of the remaining $15 - 1 = 14$ players.

$\therefore$ Required number of ways $= {}^{14}C_{11} = {}^{14}C_3$

$$= \frac{14 \cdot 13 \cdot 12}{1 \cdot 2 \cdot 3} = 364$$

Example 56. How many different selections of 6 books can be made from 11 different books, if

(i) two particular books are always selected.

(ii) two particular books are never selected?

Sol. (i) Since, two particular books are always selected. It means that $6 - 2 = 4$ books are selected out of the remaining $11 - 2 = 9$ books.

$\therefore$ Required number of ways $= {}^{9}C_4 = \dfrac{9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4} = 126.$

(ii) Since, two particular books are never selected. It means that 6 books are selected out of the remaining $11 - 2 = 9$ books.

$\therefore$ Required number of ways $= {}^9C_6$

$$= {}^9C_3 = \frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} = 84.$$

Example 57. A person tries to form as many different parties as he can, out of his 20 friends. Each party should consist of the same number. How many friends should be invited at a time? In how many of these parties would the same friends be found?

Sol. Let the person invite r number of friends at a time. Then, the number of parties are ${}^{20}C_r$, which is maximum, when $r = 10$.

If a particular friend will be found in p parties, then p is the number of combinations out of 20 in which this particular friend must be included. Therefore, we have to select 9 more from 19 remaining friends.

Hence, $p = {}^{19}C_9$

(2) The number of ways (or combinations) of n different things selecting atleast one of them is $2^n - 1$. This can also be stated as the total number of combinations of n different things.

Proof For each things, there are two possibilities, whether it is selected or not selected.

Hence, the total number of ways is given by total possibilities of all the things which is equal to $2 \times 2 \times 2 \times \ldots \times n$ factors $= 2^n$

But, this includes one case in which nothing is selected.

Hence, the total number of ways of selecting one or more of n different things $= 2^n - 1$

Aliter Number of ways of selecting one, two, three, ..., n things from n different things are

$${}^nC_1, {}^nC_2, {}^nC_3, \ldots, {}^nC_n, \text{ respectively.}$$

Hence, the total number of ways or selecting atleast one thing is

$${}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_n$$
$$= ({}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n) - {}^nC_0 = 2^n - 1$$

Example 58. Mohan has 8 friends, in how many ways he invite one or more of them to dinner?

Sol. Mohan select one or more than one of his 8 friends. So, required number of ways

$$= {}^8C_1 + {}^8C_2 + {}^8C_3 + \ldots + {}^8C_8$$
$$= 2^8 - 1 = 255.$$

Example 59. ***A question paper consists of two sections having respectively, 3 and 5 questions. The following note is given on the paper "It is not necessory to attempt all the questions one question from each section is compulsory". In how many ways can a candidate select the questions?***

Sol. Here, we have two sections A and B (say), the section A has 3 questions and section B has 5 questions and one question from each section is compulsory, according to the given direction.

$\therefore$ Number of ways selecting one or more than one question from section A is $2^3 - 1 = 7$

and number of ways selecting one or more than one question from section B is $2^5 - 1 = 31$

Hence, by the principle of multiplication, the required number of ways in which a candidate can select the questions

$$= 7 \times 31 = 217.$$

Example 60. A student is allowed to select atleast one and atmost n books from a collection of $(2n+1)$ books. If the total number of ways in which he can select books is 63, find the value of n.

Sol. Given, student select atleast one and atmost n books from a collection of $(2n + 1)$ books. It means that he select one book or two books or three books or ... or n books. Hence, by the given hypothesis.

$${}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + \ldots + {}^{2n+1}C_n = 63 \quad \ldots(i)$$

Also, the sum of binomial coefficients, is

$${}^{2n+1}C_0 + {}^{2n+1}C_1 + \ldots + {}^{2n+1}C_n + {}^{2n+1}C_{n+1} + \ldots + {}^{2n+1}C_{n+1}$$
$$= (1+1)^{2n+1} = 2^{2n+1}$$

$$\Rightarrow \quad {}^{2n+1}C_0 + 2({}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_n) + {}^{2n+1}C_{2n+1} = 2^{2n+1} \quad [\because {}^nC_r = {}^nC_{n-r}]$$

$$\Rightarrow \quad 1 + 2 \times 63 + 1 = 2^{2n+1} \Rightarrow 128 = 2^{2n+1}$$

$$\Rightarrow \quad 2^7 = 2^{2n+1} \Rightarrow 7 = 2n + 1$$

$$\therefore \quad n = 3$$

Example 61. There are three books of Physics, four of Chemistry and five of Mathematics. How many different collections can be made such that each collection consists of

(i) one book of each subject,

(ii) atleast one book of each subject,

(iii) atleast one book of Mathematics.

Sol. (i) ${}^3C_1 \times {}^4C_1 \times {}^5C_1 = 3 \times 4 \times 5 = 60$

(ii) $(2^3 - 1) \times (2^4 - 1) \times (2^5 - 1) = 7 \times 15 \times 31 = 3255$

(iii) $(2^5 - 1) \times 2^7 = 31 \times 128 = 3968$

Exercise for Session 4

1. If ${}^{43}C_{r-6} = {}^{43}C_{3r+1}$, the value of r is

(a) 6 (b) 8 (c) 10 (d) 12

2. If ${}^{18}C_{15} + 2({}^{18}C_{16}) + {}^{17}C_{16} + 1 = {}^{n}C_3$, the value of n is

(a) 18 (b) 20 (c) 22 (d) 24

3. If ${}^{20}C_{n+2} = {}^{n}C_{16}$, the value of n is

(a) 7 (b) 10 (c) 13 (d) None of these

4. If ${}^{47}C_4 + \sum_{r=1}^{5} {}^{52-r}C_3$ is equal to

(a) ${}^{47}C_6$ (b) ${}^{52}C_5$ (c) ${}^{52}C_4$ (d) None of these

5. If ${}^{n}C_3 + {}^{n}C_4 > {}^{n+1}C_3$ then

(a) $n > 6$ (b) $n < 6$ (c) $n > 7$ (d) $n < 7$

6. The Solution set of ${}^{10}C_{x-1} > 2 . {}^{10}C_x$ is

(a) {1, 2, 3} (b) {4, 5, 6} (c) {8, 9, 10} (d) {9, 10, 11}

7. If ${}^{2n}C_2 : {}^{n}C_2 = 9:2$ and ${}^{n}C_r = 10$, then r is equal to

(a) 2 (b) 3 (c) 4 (d) 5

8. If ${}^{2n}C_3 : {}^{n}C_2 = 44:3$, for which of the following value of r, the value of ${}^{n}C_r$ will be 15.

(a) $r = 3$ (b) $r = 4$ (c) $r = 5$ (d) $r = 6$

9. If ${}^{n}C_r = {}^{n}C_{r-1}$ and ${}^{n}P_r = {}^{n}P_{r+1}$, the value of n is

(a) 2 (b) 3 (c) 4 (d) 5

10. If ${}^{n}P_r = 840$, ${}^{n}C_r = 35$, the value of n is

(a) 1 (b) 3 (c) 5 (d) 7

11. If ${}^{n}P_3 + {}^{n}C_{n-2} = 14n$, the value of n is

(a) 5 (b) 6 (c) 8 (d) 10

12. There are 12 volleyball players in all in a college, out of which a team of 9 players is to be formed. If the captain always remains the same, in how many ways can the team be formed ?

(a) 36 (b) 99 (c) 108 (d) 165

13. In how many ways a team of 11 players can be formed out of 25 players, if 6 out of them are always to be included and 5 are always to be excluded

(a) 2002 (b) 2008 (c) 2020 (d) 8002

14. A man has 10 friends. In how many ways he can invite one or more of them to a party?

(a) 10! (b) 2^{10} (c) 10! – 1 (d) $2^{10} - 1$

15. In an examination, there are three multiple choice questions and each question has four choices. Number of ways in which a student can fail to get all answers correct, is

(a) 11 (b) 12 (c) 27 (d) 63

16. In an election, the number of candidates is 1 greater than the persons to be elected . If a voter can vote in 254 ways, the number of candidates is

(a) 6 (b) 7 (c) 8 (d) 10

17. The number of groups that can be made from 5 different green balls, 4 different blue balls and 3 different red balls, if atleast one green and one blue ball is to be included

(a) 3700 (b) 3720 (c) 4340 (d) None of these

18. A person is permitted to select atleast one and atmost n coins from a collection of $(2n + 1)$ distinct coins. If the total number of ways in which he can select coins is 255, then n equals

(a) 4 (b) 8 (c) 16 (d) 32

Session 5

Combinations from Identical Objects

Combinations from Identical Objects

(i) The number of combinations of n identical objects taking r objects $(r \le n)$ at a time $= 1$.

(ii) The number of combinations of zero or more objects from n identical objects $= n + 1$.

(iii) The total number of combinations of atleast one out of $a_1 + a_2 + a_3 + \ldots + a_n$ objects, where a_1 are alike of one kind, a_2 are alike of second kind, a_3 are alike of third kind, ..., a_n are alike of nth kind

$$= (a_1 + 1)(a_2 + 1)(a_3 + 1)\ldots(a_n + 1) - 1$$

Example 62. How many selections of atleast one red ball from a bag containing 4 red balls and 5 black balls, balls of the same colour being identical?

Sol. Number of selections of atleast one red ball from 4 identical red balls $= 4$

Number of selections of any number of black balls from 5 identical black balls

$$= 5 + 1 = 6$$

$\therefore$ Required number of selections of balls

$$= 4 \times 6 = 24$$

Example 63. There are p copies each of n different books. Find the number of ways in which a non-empty selection can be made from them.

Sol. Since, copies of the same book are identical.

$\therefore$ Number of selections of any number of copies of a book is $p + 1$. Similarly, in the case for each book.

Therefore, total number of selections is $(p + 1)^n$.

But this includes a selection, which is empty i.e., zero copy of each book. Excluding this, the required number of non-empty selections is $(p + 1)^n - 1$.

Example 64. There are 4 oranges, 5 apples and 6 mangoes in a fruit basket and all fruits of the same kind are identical. In how many ways can a person make a selection of fruits from among the fruits in the basket?

Sol. Zero or more oranges can be selected out of 4 identical oranges $= 4 + 1 = 5$ ways.

Similarly, for apples number of selection $= 5 + 1 = 6$ ways and mangoes can be selected in $6 + 1 = 7$ ways.

$\therefore$ The total number of selections when all the three kinds of fruits are selected $= 5 \times 6 \times 7 = 210$

But, in one of these selection number of each kind of fruit is zero and hence this selection must be excluded.

$\therefore$ Required number $= 210 - 1 = 209$

Combinations when both Identical and Distinct Objects are Present

The number of combinations (selections) of one or more objects out of $a_1 + a_2 + a_3 + \ldots + a_n$ objects, where a_1 are alike of one kind, a_2 are alike of second kind, a_3 are alike of third kind, ..., a_n are alike of nth kind and k are distinct.

$$= \{(a_1 + 1)(a_2 + 1)(a_3 + 1)\cdots(a_n + 1)\}$$
$$({}^kC_0 + {}^kC_1 + {}^kC_2 + \ldots + {}^kC_k) - 1$$
$$= (a_1 + 1)(a_2 + 1)(a_3 + 1) + \ldots + (a_n + 1)2^k - 1$$

Example 65. Find the number of ways in which one or more letters can be selected from the letters

AAAAA BBBB CCC DD EFG.

Sol. Here, $5A$'s are alike, 4 B's are alike, 3 C's are alike, $2D$'s are alike and E, F, G are different.

$\therefore$ Total number of combinations

$$= (5 + 1)(4 + 1)(3 + 1)(2 + 1)2^3 - 1$$
$$= 6 \cdot 5 \cdot 4 \cdot 3 \cdot 8 - 1$$
$$= 2879$$

[excluding the case, when no letter is selected]

Explanation Selection from $(AAAAA)$ can be made by 6 ways such include no A, include one A, include two A, include three A, include four A, include five A. Similarly, selections from $(BBBB)$ can be made in 5 ways, selections from (CCC) can be made in 4 ways, selections from (DD) can be made in 3 ways and from E, F, G can be made in $2 \times 2 \times 2$ ways.

Number of Divisors of N

Every natural number N can always be put in the form $N = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3} \ldots p_k^{\alpha_k}$, where $p_1, p_2, p_3, \ldots, p_k$ are distinct primes and $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k \in W$.

(i) The total number of divisors of N including 1 and N
$= (\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1)\ldots(\alpha_k + 1)$

(ii) The total number of divisors of N excluding 1 and N
$= (\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1)\ldots(\alpha_k + 1) - 2$

(iii) The total number of divisors of N excluding either 1 or $N = (\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1)\ldots(\alpha_k + 1) - 1$

(iv) Sum of all divisors $= (p_1^0 + p_1^1 + p_1^2 + p_1^3 + \ldots + p_1^{\infty_1})$
$(p_2^0 + p_2^1 + p_2^2 + p_2^3 + \ldots + p_2^{\alpha_2})\ldots$
$(p_k^0 + p_k^1 + p_k^2 + p_k^3 + \ldots + p_k^{\alpha_k})$
$$= \left(\frac{1 - p_1^{\alpha_1 + 1}}{1 - p_1}\right) \cdot \left(\frac{1 - p_2^{\alpha_2 + 1}}{1 - p_2}\right) \ldots \left(\frac{1 - p_k^{\alpha_k + 1}}{1 - p_k}\right)$$

(v) Sum of proper divisors (excluding 1 and the expression itself)
= Sum of all divisors $- (N + 1)$

(vi) The number of even divisors of N are possible only if $p_1 = 2$, otherwise there is no even divisor.

$\therefore$ Required number of **even** divisors
$$= \alpha_1 (\alpha_2 + 1)(\alpha_3 + 1) + \ldots + (\alpha_k + 1)$$

(vii) The number of **odd** divisors of N

***Case* I** If $p_1 = 2$, the number of odd divisors
$$= (\alpha_2 + 1)(\alpha_3 + 1)\ldots(\alpha_k + 1)$$

***Case* II** If $p_1 \neq 2$, the number of odd divisors
$$= (\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1)\ldots(\alpha_k + 1)$$

(viii) The number of ways in which N can be resolved as a product of two factors
$$= \begin{cases} \frac{1}{2}(\alpha_1 + 1)(\alpha_2 + 1)\ldots(\alpha_k + 1), & \text{if } N \text{ is not a perfect square} \\ \frac{1}{2}\{(\alpha_1 + 1)(\alpha_2 + 1)\ldots(\alpha_k + 1) + 1\}, & \text{if } N \text{ is a perfect square} \end{cases}$$

(ix) The number of ways in which a composite number N can be resolved into two factors which are relatively prime (or coprime) to each other is equal to 2^{n-1}, where n is the number of different factors (or different primes) in N.

Example 66. Find the number of proper factors of the number 38808. Also, find sum of all these divisors.

Sol. The number $38808 = 2^3 \cdot 3^2 \cdot 7^2 \cdot 11$

Hence, the total number of proper factors (excluding 1 and itself i.e., 38808)
$= (3 + 1)(2 + 1)(2 + 1)(1 + 1) - 2$
$= 72 - 2 = 70$

And sum of all these divisors (proper)
$= (2^0 + 2^1 + 2^2 + 2^3)(3^0 + 3^1 + 3^2)$
$(7^0 + 7^1 + 7^2)(11^0 + 11^1) - 1 - 38808$
$= (15)(13)(57)(12) - 38809$
$= 133380 - 38809$
$= 94571$

Example 67. Find the number of even proper divisors of the number 1008.

Sol. $\because 1008 = 2^4 \times 3^2 \times 7^1$

$\therefore$ Required number of even proper divisors
= Total number of selections of atleast one 2 and any number of 3's or 7's.
$= 4 \times (2 + 1) \times (1 + 1) - 1 = 23$

Example 68. Find the number of odd proper divisors of the number 35700. Also, find sum of the odd proper divisors.

Sol. $\because 35700 = 2^2 \times 3^1 \times 5^2 \times 7^1 \times 17^1$

$\therefore$ Required number of odd proper divisors
= Total number of selections of zero 2 and any number of 3's or 5's or 7's or 17's
$= (1 + 1)(2 + 1)(1 + 1)(1 + 1) - 1 = 23$

$\therefore$ The sum of odd proper divisors
$= (3^0 + 3^1)(5^0 + 5^1 + 5^2)(7^0 + 7^1)(17^0 + 7^1) - 1$
$= 4 \times 31 \times 8 \times 18 - 1$
$= 17856 - 1 = 17855$

Example 69. If $N = 10800$, find the

(i) the number of divisors of the form $4m + 2, \forall\, m \in W$.

(ii) the number of divisors which are multiple of 10.

(iii) the number of divisors which are multiple of 15.

Sol. We have, $N = 10800 = 2^4 \times 3^3 \times 5^2$

(i) $\because (4m + 2) = 2(2m + 1)$, in any divisor of the form $4m + 2$, 2 should be exactly 1.
So, the number of divisors of the form
$(4m + 2) = 1 \times (3 + 1) \times (2 + 1) = 1 \times 4 \times 3 = 12$

(ii) $\therefore$ The required number of proper divisors
= Total number of selections of atleast one 2 and one 5 from 2, 2, 2, 2, 3, 3, 3, 5, 5
$= 4 \times (3 + 1) \times 2 = 32$

(iii) ∴ The required number of proper divisors
= Total number of selections of atleast one 3 and one 5 from 2, 2, 2, 2, 3, 3, 3, 5, 5
$= (4+1)\times 3\times 2 = 30$

Example 70. Find the number of divisors of the number $N = 2^3\cdot 3^5\cdot 5^7\cdot 7^9\cdot 9^{11}$, which are perfect square.

Sol. $\because N = 2^3\cdot 3^5\cdot 5^7\cdot 7^9\cdot 9^{11}$

$= 2^3\cdot 3^5\cdot 5^7\cdot 7^9\cdot 3^{22}$

$= 2^3\cdot 3^{27}\cdot 5^7\cdot 7^9$

For perfect square of N, each prime factor must occur even number of times.

2 can be taken in 2 ways (i.e., 2^0 or 2^2)

3 can be taken in 14 ways (i.e., 3^0 or 3^2 or 3^4 or 3^6 or 3^8 or 3^{10} or 3^{12} or 3^{14} or 3^{16} or 3^{18} or 3^{20} or 3^{22} or 3^{24} or 3^{26})

5 can taken in 4 ways (i.e., 5^0 or 5^2 or 5^4 or 5^6)

and 7 can taken in 5 ways
(i.e., 7^0 or 7^2 or 7^4 or 7^4 or 7^6 or 7^8)

Hence, total divisors which are perfect squares
$= 2\times 14\times 4\times 5 = 560$

Example 71. In how many ways the number 10800 can be resolved as a product of two factors?

Sol. Let $N = 10800 = 2^4\times 3^3\times 5^2$

Here, N is not a perfect square [∵ power of 3 is odd]

Hence, the number of ways $= \frac{1}{2}(4+1)(3+1)(2+1) = 30$

Example 72. In how many ways the number 18900 can be split in two factors which are relatively prime (or coprime)?

Sol. Let $N = 18900 = 2^2\cdot 3^3\cdot 5^2\cdot 7^1$

Relatively prime or coprime Two numbers not necessarily prime are said to be relatively prime or coprime, if their HCF (highest common factor) is one as 2, 3, 5, 7 are relatively prime numbers.

∴ $n = 4$ [number of different primes in N]

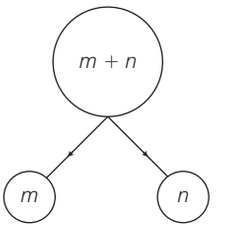
Hence, number of ways in which a composite number N can be resolved into two factors which are relatively prime or coprime $= 2^{4-1} = 2^3 = 8$

Division of Objects Into Groups

(a) Division of Objects Into Groups of Unequal Size

Theorem Number of ways in which $(m+n)$ distinct objects can be divided into two unequal groups containing m and n objects is $\frac{(m+n)!}{m!\,n!}$.

Proof The number of ways in which $(m+n)$ distinct objects are divided into two groups of the size m and n
= The number of ways m objects are selected out of $(m+n)$ objects to form one of the groups, which can be done in ${}^{m+n}C_m$ ways. The other group of n objects is formed by the remaining n objects.



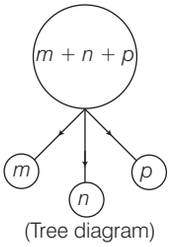
(Tree diagram)

$= {}^{m+n}C_m\cdot {}^{n}C_n = \frac{(m+n)}{m!\,n!}$

Corollary **I** The number of ways to distribute $(m+n)$ distinct objects among 2 persons in the groups containing m and n objects
= (Number of ways to divide) × (Number of groups)

$= \frac{(m+n)}{m!\,n!}\times 2!$

Corollary **II** The number of ways in which $(m+n+p)$ distinct objects can be divided into three unequal groups containing m, n and p objects, is



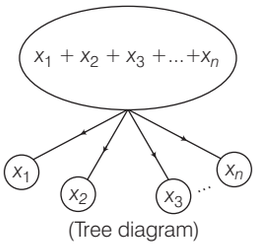
(Tree diagram)

${}^{m+n+p}C_m\cdot {}^{n+p}C_n\cdot {}^{p}C_p = \frac{(m+n+p)!}{m!\,n!\,p!}$

Corollary **III** The number of ways to distribute $(m+n+p)$ distinct objects among 3 persons in the groups containing m, n and p objects
= (Number of ways to divide) × (Number of groups)

$= \frac{(m+n+p)!}{m!\,n!\,p!}\times 3!$

Corollary **IV** The number of ways in which $(x_1+x_2+x_3+\cdots+x_n)$ distinct objects can be divided into n unequal groups containing x_1, $x_2, x_3, \ldots, x_n$ objects, is



(Tree diagram)

$$\frac{(x_1 + x_2 + x_3 + \cdots + x_n)!}{x_1!\, x_2!\, x_3! \cdots x_n!}.$$

***Corollary* V** The number of ways to distribute $(x_1 + x_2 + x_3 + \ldots + x_n)$ distinct objects among n persons in the groups containing $x_1, x_2, \ldots, x_n$ objects

= (Number of ways to divide) × (Number of groups)

$$= \frac{(x_1 + x_2 + x_3 + \ldots + x_n)!}{x_1!\, x_2!\, x_3! \ldots x_n!} \times n!$$

(b) Division of Objects Into Groups of Equal Size

The number of ways in which mn distinct objects can be divided equally into m groups, each containing n objects and

(i) If order of groups is not important is.

$$= \left(\frac{(mn)!}{(n!)^m}\right) \times \frac{1}{m!}.$$

(ii) If order of groups is important is.

$$\left(\frac{(mn)!}{(n!)^m} \times \frac{1}{m!}\right) \times m! = \frac{(mn)!}{(n!)^m}.$$

Note Division of $14n$ objects into 6 groups of $2n, 2n, 2n, 2n, 3n, 3n$, size is $\dfrac{\left(\dfrac{(14n)!}{(2n)!\,(2n)!\,(2n)!\,(2n)!\,(3n)!\,(3n)!}\right)}{4!\,2!} = \dfrac{(14n)}{((2n)!)^4\ ((3n)!)^2} \times \dfrac{1}{4!\,2!}$

Now, the distribution ways of these 6 groups among 6 persons is

$$\frac{(14n)!}{[(2n)!)]^4\ [(3n)!]^2} \times \frac{1}{4!\,2!} \times 6! = \frac{(14n)!}{[(2n)!]^4\ [(3n)!]^2} \times 15$$

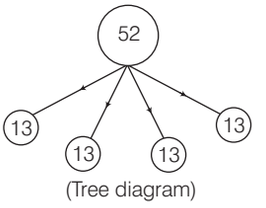
Example 73. *In how many ways can a pack of 52 cards be*

(i) distributed equally among four players in order?

(ii) divided into four groups of 13 cards each?

(iii) divided into four sets, three of them having 17 cards each and fourth just one card?

Sol. (i) Here, order of group is important, then the numbers of ways in which 52 different cards can be divided equally into 4 players is



(Tree diagram)

$$\frac{52!}{4!\,(13!)^4} \times 4! = \frac{52!}{(13!)^4}$$

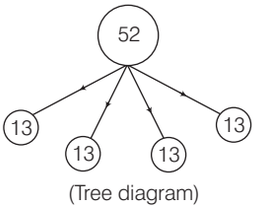
Aliter Each player will get 13 cards. Now, first player can be given 13 cards out of 52 cards in $^{52}C_{13}$ ways. Second player can be given 13 cards out of remaining 39 cards (i.e., $52 - 13 = 39$) in $^{39}C_{13}$ ways. Third player can be given 13 cards out of remaining 26 cards (i.e., $39 - 13 = 26$) in $^{26}C_{13}$ ways and fourth player can be given 13 cards out of remaining 13 cards (i.e., $26 - 13 = 13$) in $^{13}C_{13}$ ways.

Hence, required number of ways

$$= {}^{52}C_{13} \times {}^{39}C_{13} \times {}^{26}C_{13} \times {}^{13}C_{13}$$

$$= \frac{52!}{13!\,39!} \times \frac{39!}{13!\,26!} \times \frac{26!}{13!\,13!} \times 1 = \frac{52!}{(13!)^4}$$

(ii) Here, order of group is not important, then the number of ways in which 52 different cards can be divided equally into 4 groups is



(Tree diagram)

$$\frac{52!}{4!\,(13!)^4}$$

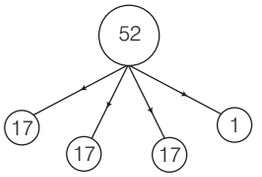
Aliter Each group will get 13 cards. Now, first group can be given 13 cards out of 52 cards in $^{52}C_{13}$ ways. Second group can be given 13 cards out of remaining 39 cards (i.e., $52 - 13 = 39$) in $^{39}C_{13}$ ways. Third group can be given 13 cards out of remaining 26 cards (i.e., $39 - 13 = 26$) in $^{26}C_{13}$ ways and fourth group can be given 13 cards out of remaining 13 cards (i.e., $26 - 13 = 13$) in $^{13}C_{13}$ ways. But the all (four) groups can be interchanged in 4! ways. Hence, the required number of ways

$$= {}^{52}C_{13} \times {}^{39}C_{13} \times {}^{26}C_{13} \times {}^{13}C_{13} \times \frac{1}{4!}$$

$$= \frac{52!}{13!\,39!} \times \frac{39!}{13!\,26!} \times \frac{26!}{13!\,13!} \times 1 \times \frac{1}{4!} = \frac{52!}{(13!)^4\,4!}$$

(iii) First, we divide 52 cards into two sets which contains 1 and 51 cards respectively, is

$$\frac{52!}{1!\,51!}$$



Now, 51 cards can be divided equally in three sets each contains 17 cards (here order of sets is not important) in $\frac{51!}{3!\,(17!)^3}$ ways.

Hence, the required number of ways

$$= \frac{52!}{1!\,51!} \times \frac{51!}{3!(17!)^3}$$

$$= \frac{52!}{1!\,3!(17)^3} = \frac{52!}{(17!)^3\,3!}$$

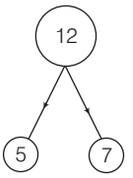
Aliter First set can be given 17 cards out of 52 cards in ${}^{52}C_{17}$. Second set can be given 17 cards out of remaining 35 cards (i.e., $52 - 17 = 35$) in ${}^{35}C_{17}$. Third set can be given 17 cards out of remaining 18 cards (i.e., $35 - 17 = 18$) in ${}^{18}C_{17}$ and fourth set can be given 1 card out of 1 card in ${}^{1}C_{1}$. But the first three sets can interchanged in 3! ways. Hence, the total number of ways for the required distribution

$$= {}^{52}C_{17} \times {}^{35}C_{17} \times {}^{18}C_{17} \times {}^{1}C_{1} \times \frac{1}{3}!$$

$$= \frac{52!}{17!\,35!} \times \frac{35!}{17!\,1!} \times \frac{18!}{17!\,18!} \times 1 \times \frac{1}{3!} = \frac{(52)!}{(17!)^3\,3!}$$

Example 74. In how many ways can 12 different balls be divided between 2 boys, one receiving 5 and the other 7 balls? Also, in how many ways can these 12 balls be divided into groups of 5, 4 and 3 balls, respectively?

Sol. **I Part** Here, order is important, then the number of ways in which 12 different balls can be divided between two boys which contains
5 and 7 balls respectively, is



(Tree diagram)

$$= \frac{12!}{5!\,7!} \times 2! = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7!}{(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)7!} \cdot 2 = 1584$$

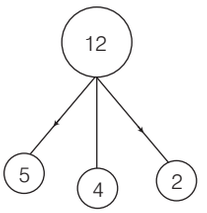
Aliter First boy can be given 5 balls out of 12 balls in ${}^{12}C_{5}$. Second boy can be given 7 balls out of 7 balls (i.e., $12 - 5 = 7$) but there order is important boys interchange by (2 types), then required number of ways

$$= {}^{12}C_{5} \times {}^{7}C_{7} \times 2! = \frac{12!}{5!\,7!} \times 1 \times 2!$$

$$= \frac{12! \times 2}{5! \times 7!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7! \cdot 2}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 7!} = 1584.$$

II Part Here, order is not important, then the number of ways in which 12 different balls can be divided into three groups of 5,4 and 3 balls respectively, is

$$= \frac{12!}{5!\,4!\,3!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5!}{5! \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} = 27720$$



(Tree diagram)

Aliter First group can be given 5 balls out of 12 balls in ${}^{12}C_{5}$ ways. Second group can be given 4 balls out of remaining 7 balls $(12 - 5 = 7)$ in ${}^{7}C_{4}$ and 3 balls can be given out of remaining 3 balls in ${}^{3}C_{3}$.

Hence, the required number of ways (here order of groups are not important)

$$= {}^{12}C_{5} \times {}^{7}C_{4} \times {}^{3}C_{3}$$

$$= \frac{12!}{5!\,7!} \times \frac{7!}{4!\,3!} \times 1$$

$$= \frac{12!}{5!\,4!\,3!}$$

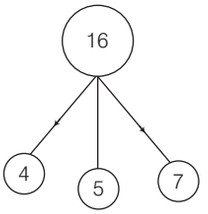
Example 75. In how many ways can 16 different books be distributed among three students A, B, C so that B gets 1 more than A and C gets 2 more than B ?

Sol. Let A gets n books, then B gets $n + 1$ and C gets $n + 3$.

Now, $n + (n + 1) + (n + 3) = 16$

$\Rightarrow \quad 3n = 12$

$\therefore \quad n = 4$



(Tree diagram)

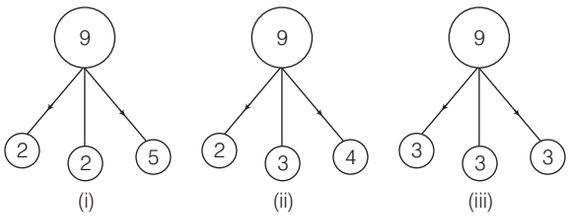
$\Rightarrow$ A, B, C gets 4, 5 and 7 books, respectively.

Hence, the total number of ways for the required distribution

$$= \frac{16!}{4!\,5!\,7!}$$

Example 76. In how many ways can 9 different books be distributed among three students if each receives atleast 2 books?

Sol. If each receives atleast 2 books, then the division as shown by tree diagrams



The number of division ways for tree diagrams (i), (ii) and (iii) are

$$\frac{9!}{(2!)^2\,(5!)} \times \frac{1}{2!}, \frac{9!}{2!\,3!\,4!} \text{ and } \frac{9!}{(3!)^3} \times \frac{1}{3!}, \text{ respectively.}$$

Hence, the total number of ways of distribution of these groups among 3 students is

$$\left[\frac{9!}{(2!)^2\,(5!)} \times \frac{1}{2!} + \frac{9!}{2!\,3!\,4!} + \frac{9!}{(3!)^3} \times \frac{1}{3!}\right] \times 3!$$

$$= [378 + 1260 + 280] \times 6$$

$$= 11508$$

Exercise for Session 5

1. There are 3 oranges, 5 apples and 6 mangoes in a fruit basket (all fruits of same kind are identical). Number of ways in which fruits can be selected from the basket, is

(a) 124 (b) 125 (c) 167 (d) 168

2. In a city no two persons have identical set of teeth and there is no person without a tooth. Also, no person has more than 32 teeth. If we disguard the shape and size of tooth and consider only the positioning of the teeth, the maximum population of the city is

(a) 2^{32} (b) $(32)^2 - 1$ (c) $2^{32} - 1$ (d) 2^{31}

3. If $a_1, a_2, a_3, \ldots, a_{n+1}$ be $(n+1)$ different prime numbers, then the number of different factors (other than 1) of $a_1^m \cdot a_2 \cdot a_3, \ldots, a_{n+1}$, is

(a) $m+1$ (b) $(m+1)2^n$ (c) $m \cdot 2^n + 1$ (d) None of these

4. Number of proper factors of 2400 is equal to

(a) 34 (b) 35
(c) 36 (d) 37

5. The sum of the divisors of $2^5 \cdot 3^4 \cdot 5^2$ is

(a) $3^2 \cdot 7^1 \cdot 11^2$ (b) $3^2 \cdot 7^1 \cdot 11^2 \cdot 31$
(c) $3 \cdot 7 \cdot 11 \cdot 31$ (d) None of these

6. The number of proper divisors of $2^p \cdot 6^q \cdot 21^r$, $\forall\ p, q, r \in N$, is

(a) $(p+q+1)(q+r+1)(r+1)$ (b) $(p+q+1)(q+r+1)(r+1)-2$
(c) $(p+q)(q+r)r-2$ (d) $(p+q)(q+r)r$

7. The number of odd proper divisors of $3^p \cdot 6^q \cdot 15^r$, $\forall\ p, q, r \in N$, is

(a) $(p+1)(q+1)(r+1)-2$ (b) $(p+1)(q+1)(r+1)-1$
(c) $(p+q+r+1)(r+1)-2$ (d) $(p+q+r+1)(r+1)-1$

8. The number of proper divisors of 1800, which are also divisible by 10, is

(a) 18 (b) 27 (c) 34 (d) 43

9. Total number of divisors of 480 that are of the form $4n+2$, $n \geq 0$, is equal to

(a) 2 (b) 3 (c) 4 (d) 5

10. Total number of divisors of $N = 2^5 \cdot 3^4 \cdot 5^{10} \cdot 7^6$ that are of the form $4n+2$, $n \geq 1$, is equal to

(a) 54 (b) 55 (c) 384 (d) 385

11. Total number of divisors of $N = 3^5 \cdot 5^7 \cdot 7^9$ that are of the form $4n + 1, n \geq 0$ is equal to

(a) 15 (b) 30 (c) 120 (d) 240

12. Number of ways in which 12 different books can be distributed equally among 3 persons, is

(a) $\frac{12!}{(4!)^3}$ (b) $\frac{12!}{(3!)^4}$ (c) $\frac{12!}{(4!)^4}$ (d) $\frac{12!}{(3!)^3}$

13. Number of ways in which 12 different things can be distributed in 3 groups, is

(a) $\frac{12!}{(4!)^3}$ (b) $\frac{12!}{3!(4!)^3}$ (c) $\frac{12!}{4!(3!)^3}$ (d) $\frac{12!}{(3!)^4}$

14. Number of ways in which 12 different things can be distributed in 5 sets of 2, 2, 2, 3, 3, things is

(a) $\frac{12!}{(3!)^2(2!)^3}$ (b) $\frac{12!5!}{(3!)^2(2!)^3}$ (c) $\frac{12!}{(3!)^3(2!)^4}$ (d) $\frac{12!5!}{(3!)^2(2!)^4}$

15. Number of ways in which 12 different things can be divided among five persons so that they can get 2, 2, 2, 3, 3 things respectively, is

(a) $\frac{12!}{(3!)^2(2!)^3}$ (b) $\frac{12!5!}{(3!)^2(2!)^3}$ (c) $\frac{12!}{(3!)^2(2!)^4}$ (d) $\frac{12!5!}{(3!)^2(2!)^4}$

16. The total number of ways in which $2n$ persons can be divided into n couples, is

(a) $\frac{2n!}{(n!)^2}$ (b) $\frac{2n!}{(2n!)^n}$ (c) $\frac{2n!}{n!(2n!)^2}$ (d) None of these

17. n different toys have to be distributed among n children. Total number of ways in which these toys can be distributed so that exactly one child gets no toy, is equal to

(a) $n!$ (b) $n! {}^nC_2$ (c) $(n-1)! {}^nC_2$ (d) $n! {}^{n-1}C_2$

18. In how many ways can 8 different books be distributed among 3 students if each receives atleast 2 books?

(a) 490 (b) 980 (c) 2940 (d) 5880

Session 6

Arrangement in Groups, Multinomial Theorem, Multiplying Synthetically

Arrangement in Groups

(a) The number of ways in which n different things can be arranged into r different groups is $r(r+1)(r+2)\ldots(r+n-1)$ or $n!\cdot {}^{n-1}C_{r-1}$ according as blank groups are or are not admissible.

Proof

(i) Let n letters $a_1, a_2, a_3, \ldots, a_n$ be written in a row in any order. All the arrangements of the letters in r, groups, blank groups being admissible, can be obtained thus, place among the letters $(r-1)$ marks of partition and arrange the $(n+r-1)$ things (consisting of letters and marks) in all possible orders. Since, $(r-1)$ of the things are alike, the number of different arrangements is $\frac{(n+r-1)!}{(r-1)!} = r(r+1)(r-2)\ldots(r+n-1)$.

(ii) All the arrangements of the letters in r groups, none of the groups being blank, can be obtained as follows:

(I) Arrange the letters in all possible orders. This can be done in $n!$ ways.

(II) In every such arrangement, place $(r-1)$ marks of partition in $(r-1)$ out of the $(n-1)$ spaces between the letters. This can be done in ${}^{n-1}C_{r-1}$ ways.

Hence, the required number is $n!\cdot {}^{n-1}C_{r-1}$.

Example 77. In how many ways 5 different balls can be arranged into 3 different boxes so that no box remains empty?

Sol. The required number of ways $= 5!\cdot {}^{5-1}C_{3-1} = 5!\cdot {}^{4}C_{2}$

$$= (120)\cdot\left(\frac{4\cdot 3}{1\cdot 2}\right) = 720$$

Aliter

Each box must contain atleast one ball, since no box remains empty. Boxes can have balls in the following systems

Box	I	II	III
Number of balls	1	1	3

Or

Box	I	II	III
Number of balls	1	2	2

All 5 balls can be arranged by 5! ways and boxes can be arranged in each system by $\frac{3!}{2!}$.

Hence, required number of ways $= 5!\times\frac{3!}{2!} + 5!\times\frac{3!}{2!}$

$$= 120\times 3 + 120\times 3 = 720$$

(b) The number of ways in which n different things can be distributed into r different groups is $r^n - {}^{r}C_1(r-1)^n + {}^{r}C_2(r-2)^n - \ldots + (-1)^{r-1}\cdot {}^{r}C_{r-1}$

Or

$$\sum_{p=0}^{r}(-1)^p\cdot {}^{r}C_p\cdot(r-p)^n$$

Or

Coefficient of x^n in $n!(e^x-1)^r$.

Here, blank groups are not allowed.

Proof In any distribution, denote the groups by $g_1, g_2, g_3, \ldots, g_r$ and consider the distributions in which blanks are allowed.

The total number of these is r^n.

The number in which g_1 is blank, is $(r-1)^n$.

Therefore, the number in which g_1 is not blank, is

$$r^n - (r-1)^n$$

of these last, the number in which g_2 is blank, is

$$(r-1)^n - (r-2)^n$$

Therefore, the number in which g_1, g_2 are not blank, is

$$r^n - 2(r-1)^n + (r-2)^n$$

of these last, the number in which g_3 is blank, is

$$(r-1)^n - 2(r-2)^n + (r-3)^n$$

Therefore, the number in which g_1, g_2, g_3 are not blank, is

$$r^n - 3(r-1)^n + 3(r-2)^n - (r-3)^n$$

This process can be continued as far as we like and it is obvious that the coefficients are formed as in a binomial expansion.

Hence, the number of distributions in which no one of x assigned groups is blank, is

$$r^n - {}^{x}C_1(r-1)^n + {}^{x}C_2(r-2)^n - \ldots + (-1)^x(r-x)^n$$

when $x = r$, then

$$r^n - {}^rC_1(r-1)^n + {}^rC_2(r-2)^n - \ldots + (-1)^{r-1} \cdot {}^rC_{r-1} (r-(r-1))^n + (-1)^r \cdot {}^rC_r (r-r)^n$$

Or

$$r^n - {}^rC_1(r-1)^n + {}^rC_2(r-2)^n - \ldots + (-1)^{r-1} \cdot {}^rC_{r-1}$$

Aliter

By Principle of Inclusion and Exclusion

Let A_i denotes the set of distribution of things, if ith group gets nothing. Then, $n(A_i) = (r-1)^n$

[as n things can be distributed among $(r-1)$ groups in $(r-1)^n$ ways]

Then, $n(A_i \cap A_j)$ represents number of distribution ways in which groups i and j get no object. Then,

$$n(A_i \cap A_j) = (r-2)^n$$

Also, $n(A_i \cap A_j \cap A_k) = (r-3)^n$

This process can be continued, then the required number is

$$\begin{aligned} n(A_1' A_2' \cap \ldots \cap A_r') &= n(U) - n(A_1 \cup A_2 \cup \ldots \cup A_r) \\ &= r^n - \Big\{ \sum n(A_i) - \sum n(A_i \cap A_j) + \sum n(A_i \cap A_j \cap A_k) \ldots \\ &\qquad + (-1)^n \sum n(A_1 \cap A_2 \cap \ldots \cap A_r) \Big\} \\ &= r^n - \{ {}^rC_1(r-1)^n - {}^rC_2(r-2)^n + {}^rC_3(r-3)^n - \ldots + (-1)^r \cdot {}^rC_{r-1} \} \\ &= r^n - {}^rC_1(r-1)^n + {}^rC_2(r-2)^n - {}^rC_3(r-3)^n + \ldots + (-1)^{r-1} \cdot {}^rC_{r-1} \cdot 1 \end{aligned}$$

Note Coefficient of x^r in $e^{px} = \frac{p^r}{r!}$.

Example 78. In how many ways 5 different balls can be distributed into 3 boxes so that no box remains empty?

Sol. The required number of ways

$$= 3^5 - {}^3C_1(3-1)^5 + {}^3C_2(3-2)^5 - {}^3C_3(3-3)^5$$

$$= 243 - 96 + 3 - 0 = 150$$

Or

Coefficient of x^5 in $5!(e^x - 1)^3$

$= $ Coefficient of x^5 in $5!(e^{3x} - 3e^{2x} + 3e^x - 1)$

$$= 5!\left(\frac{3^5}{5!} - 3 \times \frac{2^5}{5!} + 3 \times \frac{1}{5!}\right) = 3^5 - 3 \cdot 2^5 + 3 = 243 - 96 + 3 = 150$$

Aliter

Each box must contain atleast one ball, since number box remains empty. Boxes can have balls in the following systems

Box	I	II	III
Number of balls	1	1	3

Or

Box	I	II	III
Number of balls	1	2	2

The number of ways to distribute the balls in I system

$$= {}^5C_1 \times {}^4C_1 \times {}^3C_3$$

$\therefore$ The total number of ways to distribute 1, 1, 3 balls to the boxes

$$= {}^5C_1 \times {}^4C_1 \times {}^3C_3 \times \frac{3!}{2!} = 5 \times 4 \times 1 \times 3 = 60$$

and the number of ways to distribute the balls in II system

$$= {}^5C_1 \times {}^4C_2 \times {}^2C_2$$

$\therefore$ The total number of ways to distribute 1, 2, 2 balls to the boxes

$$= {}^5C_1 \times {}^4C_2 \times {}^2C_2 \times \frac{3!}{2!}$$

$$= 5 \times 6 \times 1 \times 3 = 90$$

$\therefore$ The required number of ways $= 60 + 90 = 150$

Example 79. In how many ways can 5 different books be tied up in three bundles?

Sol. The required number of ways $= \frac{1}{3!}(3^5 - {}^3C_1 \cdot 2^5 + {}^3C_2 \cdot 1^5)$

$$= \frac{150}{6} = 25$$

Example 80. ***If*** $n(A) = 5$ ***and*** $n(B) = 3$***, find number of onto mappings from*** A ***to*** B***.***

Sol. We know that in onto mapping, each image must be assigned atleast one pre-image.

This is equivalent to number of ways in which 5 different balls (pre-images) can be distributed in 3 different boxes (images), if no box remains empty. The total number of onto mappings from A to B

$$= 3^5 - {}^3C_1(3-1)^5 + {}^3C_2(3-2)^5$$

$$= 243 - 96 + 3 = 150$$

(c) The number of ways in which n identical things can be distributed into r different groups is

$${}^{n+r-1}C_{r-1} \text{ or } {}^{n-1}C_{r-1}$$

According, as blank groups are or are not admissible.

Proof

If blank groups are not allowed Any such distribution can be effected as follows: place the n things in a row and put marks of partition in a selection of $(r-1)$ out of the $(n-1)$ spaces between them. This can be done in ${}^{n-1}C_{r-1}$.

If blank groups are allowed The number of distribution is the same as that of $(n+r)$ things of the same sort into r groups with no blank groups. For such a distribution can be effected thus, put one of the

$(n+r)$ things into each of the r groups and distribute the remaining n things into r groups, blank lots being allowed. Hence, the required number is ${}^{n+r-1}C_{r-1}$.

Aliter The number of distribution of n identical things into r different groups is the coefficient of x^n in $(1+x+x^2+\ldots+\infty)^r$ or in $(x+x^2+x^3+\ldots+\infty)^r$ according as blank groups are or are not allowed.

These expressions are respectively equal to $(1-x)^{-r}$ and $x^r(1-x)^{-r}$

Hence, coefficient of x^n in two expressions are ${}^{n+r-1}C_{r-1}$ and ${}^{n-1}C_{r-1}$, respectively.

Example 81. In how many ways 5 identical balls can be distributed into 3 different boxes so that no box remains empty?

Sol. The required number of ways $= {}^{5-1}C_{3-1} = {}^4C_2 = \dfrac{4\cdot 3}{1\cdot 2} = 6$

Aliter Each box must contain atleast one ball, since no box remains empty. Boxes can have balls in the following systems.

Box	I	II	III
Number of balls	1	1	3

Or

Box	I	II	III
Number of balls	1	2	2

Here, balls are identical but boxes are different the number of combinations will be 1 in each systems.

$\therefore$ Required number of ways $= 1\times\dfrac{3!}{2!}+1\times\dfrac{3!}{2!}=3+3=6$

Example 82. Four boys picked up 30 mangoes. In how many ways can they divide them, if all mangoes be identical?

Sol. Clearly, 30 mangoes can be distributed among 4 boys such that each boy can receive any number of mangoes.

Hence, total number of ways $= {}^{30+4-1}C_{4-1}$

$$= {}^{33}C_3 = \frac{33\cdot 32\cdot 31}{1\cdot 2\cdot 3} = 5456$$

Example 83. Find the positive number of solutions of $x+y+z+w=20$ under the following conditions

(i) Zero value of x, y, z and w are included.

(ii) Zero values are excluded.

Sol. (i) Since, $x+y+z+w=20$

Here, $x\ge 0, y\ge 0, z\ge 0, w\ge 0$

The number of Sols of the given equation in this case is same as the number of ways of distributing 20 things among 4 different groups.

Hence, total number of Sols $= {}^{20+4-1}C_{4-1}$

$$= {}^{23}C_3 = \frac{23\cdot 22\cdot 21}{1\cdot 2\cdot 3} = 1771$$

(ii) Since, $x+y+z+w=20$ ...(i)

Here, $x\ge 1, y\ge 1, z\ge 1, w\ge 1$

or $x-1\ge 0, y-1\ge 0, z-1\ge 0, w-1\ge 0$

Let $x_1 = x-1 \Rightarrow x = x_1+1$

$y_1 = y-1 \Rightarrow y = y_1+1$

$z_1 = z-1 \Rightarrow z = z_1+1$

$w_1 = w-1 \Rightarrow w = w_1+1$

Then, from Eq. (i), we get

$$x_1+1+y_1+1+z_1+1+w_1+1=20$$

$$\Rightarrow \quad x_1+y_1+z_1+w_1=16$$

and $x_1\ge 0, y_1\ge 0, z_1\ge 0, w_1\ge 0$

Hence, total number of Solutions $= {}^{16+4-1}C_{4-1}$

$$= {}^{19}C_3 = \frac{19\cdot 18\cdot 17}{1\cdot 2\cdot 3} = 57\cdot 17 = 969$$

Aliter

Part (ii) $\because\ x+y+z+w=20$

$x\ge 1, y\ge 1, z\ge 1, w\ge 1$

Hence, total number of solutions

$$= {}^{20-1}C_{4-1} = {}^{19}C_3 = 969$$

Example 84. How many integral solutions are there to $x+y+z+t=29$, when $x\ge 1$, $y>1$, $z\ge 3$ and $t\ge 0$?

Sol. Since, $x+y+z+t=29$ ...(i)

and x, y, z, t are integers

$\therefore \quad x\ge 1, y\ge 2, z\ge 3, t\ge 0$

$\Rightarrow \quad x-1\ge 0, y-2\ge 0, z-3\ge 0, t\ge 0$

Let $x_1 = x-1, x_2 = y-2, x_3 = z-3$

or $x = x_1+1, y = x_2+2, z = x_3+3$ and then $x_1\ge 0$, $x_2\ge 0$, $x_3\ge 0$, $t\ge 0$

From Eq. (i), we get

$$x_1+1+x_2+2+x_3+3+t=29$$

$$\Rightarrow \quad x_1+x_2+x_3+t=23$$

Hence, total number of solutions $= {}^{23+4-1}C_{4-1}$

$$= {}^{26}C_3 = \frac{26\cdot 25\cdot 24}{1\cdot 2\cdot 3} = 2600$$

Aliter

$\because \quad x+y+z+t=29$ ...(i)

and $x\ge 1, y-1\ge 1, z-2\ge 1, t+1\ge 1$

Let $x_1 = x, y_1 = y-1, z_1 = z-2, t_1 = t+1$

or $x = x_1, y = y_1+1, z = z_1+2, t = t_1-1$

and then $x_1\ge 1, y_1\ge 1, z_1\ge 1, t_1\ge 1$

From Eq. (i), $x_1+y_1+1+z_1+2+t_1-1=29$

$$\Rightarrow \quad x_1+y_1+z_1+t_1=27$$

Hence, total number of solutions $= {}^{27-1}C_{4-1} = {}^{26}C_3$

$$= \frac{26\cdot 25\cdot 24}{1\cdot 2\cdot 3} = 2600$$

Example 85. How many integral Solutions are there to the system of equations $x_1 + x_2 + x_3 + x_4 + x_5 = 20$ and $x_1 + x_2 = 15$, when $x_k \geq 0$?

Sol. We have, $x_1 + x_2 + x_3 + x_4 + x_5 = 20$ ...(i)

and $x_1 + x_2 = 15$ ...(ii)

Then, from Eqs. (i) and (ii), we get two equations

$x_3 + x_4 + x_5 = 5$ ...(iii)

$x_1 + x_2 = 15$ ...(iv)

and given $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, $x_4 \geq 0$ and $x_5 \geq 0$

Then, number of solutions of Eq. (iii)

$$= {}^{5+3-1}C_{3-1} = {}^{7}C_2$$

$$= \frac{7 \cdot 6}{1 \cdot 2} = 21$$

and number of solutions of Eq. (iv)

$$= {}^{15+2-1}C_{2-1} = {}^{16}C_1 = 16$$

Hence, total number of solutions of the given system of equations

$$= 21 \times 16 = 336$$

Example 86. Find the number of non-negative integral solutions of $3x + y + z = 24$.

Sol. We have,

$3x + y + z = 24$ and given $x \geq 0, y \geq 0, z \geq 0$

Let $x = k$

$\therefore$ $y + z = 24 - 3k$ ...(i)

Here, $24 \geq 24 - 3k \geq 0 [\because x \geq 0]$

Hence, $0 \leq k \leq 8$

The total number of integral solutions of Eq. (i) is

$${}^{24-3k+2-1}C_{2-1} = {}^{25-3k}C_1 = 25 - 3k$$

Hence, the total number of Sols of the original equation

$$= \sum_{k=0}^{8}(25 - 3k) = 25\sum_{k=0}^{8}1 - 3\sum_{k=0}^{8}k$$

$$= 25 \cdot 9 - 3 \cdot \frac{8 \cdot 9}{2} = 225 - 108 = 117$$

(d) The number of ways in which n identical things can be distributed into r groups so that no group contains less than l things and more than m things ($l < m$) is coefficient of x^{n-lr} in the expansion of $(1 - x^{m-l+1})^r (1 - x)^{-r}$.

Proof Required number of ways

= Coefficient of x^n in the expansion of

$$(x^l + x^{l+1} + x^{l+2} + \ldots + x^m)^r$$

[$\because$ no group contains less than l things and more than m things, here r groups]

= Coefficient of x^n in the expansion of $x^{lr}(1 + x + x^2 + \ldots + x^{m-l})^r$

= Coefficient of x^{n-lr} in the expansion of $(1 + x + x^2 + \ldots + x^{m-l})^r$

= Coefficient of x^{n-lr} in the expansion of

$$\left(\frac{1 \cdot (1 - x^{m-l+1})}{(1 - x)}\right)^r$$

[sum of $m - l + 1$ terms of GP]

= Coefficient of x^{n-lr} in the expansion of

$$(1 - x^{m-l+1})^r (1 - x)^{-r}$$

Example 87. In how many ways can three persons, each throwing a single dice once, make a sum of 15 ?

Sol. Number on the faces of the dice are 1, 2, 3, 4, 5, 6 (least number 1, greatest number 6)

Here, $l = 1, m = 6, r = 3$ and $n = 15$

$\therefore$ Required number of ways = Coefficient of $x^{15-1\times3}$ in the expansion of $(1 - x^6)^3 (1 - x)^{-3}$

= Coefficient of x^{12} in the expansion of

$$(1 - 3x^6 + 3x^{12})(1 + {}^3C_1 x + {}^4C_2 x^2 + \ldots + {}^8C_6 x^6 + \ldots + {}^{14}C_{12} x^{12} + \ldots)$$

$$= {}^{14}C_{12} - 3 \times {}^8C_6 + 3 = {}^{14}C_2 - 3 \times {}^8C_2 + 3$$

$$= 91 - 84 + 3 = 10$$

Example 88. ***In how many ways in which an examiner can assign 30 marks to*** 8 ***questions, giving not less than*** 2 ***marks to any question.***

Sol. If examiner given marks any seven question 2 (each) marks, then marks on remaining questions given by examiner $= -7 \times 2 + 30 = 16$

If x_i are the marks assigned to ith question, then $x_1 + x_2 + x_3 + \ldots + x_8 = 30$ and $2 \leq x_i \leq 16$ for $i = 1, 2, 3, \ldots, 8$.

Here, $l = 2, m = 16, r = 8$ and $n = 30$

$\therefore$ Required number of ways

= Coefficient of $x^{30-2\times8}$ in the expansion of

$$(1 - x^{16-2+1})^8 (1 - x)^{-8}$$

= Coefficient of x^{14} in the expansion of

$$(1 - x^{15})^8 (1 + {}^8C_1 x + {}^9C_2 x^2 + \ldots + {}^{21}C_{14} x^{14} + \ldots)$$

= Coefficient of x^{14} in the expansion of

$$(1 + {}^8C_1 x + {}^9C_2 x^2 + \ldots + {}^{21}C_{14} x^{14} + \ldots)$$

$$= {}^{21}C_{14} = {}^{21}C_7$$

Note Coefficient of x^r in the expansion of $(1 - x)^{-n}$ is ${}^{n+r-1}C_r$.

(e) If a group has n things in which p are identical, then the number of ways of selecting r things from a group is

$\sum_{r=0}^{r} {}^{n-p}C_r$ **or** $\sum_{r=r-p}^{r} {}^{n-p}C_r$, **according as** $r \le p$ **or** $r \ge p$.

Example 89. A bag has contains 23 balls in which 7 are identical. Then, find the number of ways of selecting 12 balls from bag.

Sol. Here, $n = 23,\ p = 7,\ r = 12\ (r > p)$

$$\therefore \text{ Required number of selections} = \sum_{r=5}^{12} {}^{16}C_r$$

$$= {}^{16}C_5 + {}^{16}C_6 + {}^{16}C_7 + {}^{16}C_8 + {}^{16}C_9 + {}^{16}C_{10} + {}^{16}C_{11} + {}^{16}C_{12}$$

$$= ({}^{16}C_5 + {}^{16}C_6) + ({}^{16}C_7 + {}^{16}C_8) + ({}^{16}C_9 + {}^{16}C_{10}) + ({}^{16}C_{11} + {}^{16}C_{12})$$

$$= {}^{17}C_6 + {}^{17}C_8 + {}^{17}C_{10} + {}^{17}C_{12} \qquad [\because {}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r]$$

$$= {}^{17}C_{11} + {}^{17}C_9 + {}^{17}C_{10} + {}^{17}C_{12} \qquad [\because {}^{n}C_r = {}^{n}C_{n-r}]$$

$$= ({}^{17}C_{11} + {}^{17}C_{12}) + ({}^{17}C_9 + {}^{17}C_{10})$$

$$= {}^{18}C_{12} + {}^{18}C_{10} = {}^{18}C_6 + {}^{18}C_8$$

Derangements Any change in the order of the things in a group is called a derangement.

Or

When 'n' things are to be placed at 'n' specific places but none of them is placed on its specified position, then we say that the 'n' things are deranged.

Or

Assume $a_1, a_2, a_3, \ldots, a_n$ be n distinct things such that their positions are fixed in a row. If we now rearrange $a_1, a_2, a_3, \ldots, a_n$ in such a way that no one occupy its original position, then such an arrangement is called a derangement.

Consider 'n' letters and 'n' corresponding envelops. The number of ways in which letters can be placed in the envelopes (one letter in each envelope) so that no letter is placed in correct envelope is

$$n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}\right)$$

Proof n letters are denoted by 1, 2, 3, …, n. Let A_i denote the set of distribution of letters in envelopes (one letter in each envelop) so that the i th letter is placed in the corresponding envelope, then

$$n(A_i) = 1 \times (n-1)!$$

[$\because$ the remaining $(n-1)$ letters can be placed in $(n-1)$ envelopes is $(n-1)!$]

and $n(A_i \cap A_j) = 1 \times 1 \times (n-2)!$ [$\because$ i and j can be placed in their corresponding envelopes and remaining $(n-2)$ letters can be placed in $(n-2)$ envelopes in $(n-2)!$ way]

Also, $n(A_i \cap A_j \cap A_k) = 1 \times 1 \times 1 \times (n-3)!$

Hence, the required number is

$$n(A_1' \cap A_2' \cap A_3' \cap \ldots \cap A_n')$$

$$= n(U) - n(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n)$$

$$= n! - \Big\{\sum n(A_i) - \sum n(A_i \cap A_j) + \sum n(A_i \cap A_j \cap A_k) - \ldots + (-1)^n \sum n(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n)$$

$$= n! - \{{}^{n}C_1 \times (n-1)! - {}^{n}C_2 \times (n-2)! + {}^{n}C_3 \times (n-3)! - \ldots + (-1)^{n-1} \times {}^{n}C_n \times 1!\}$$

$$= n! - \left\{\frac{n \times (n-1)!}{1!} - \frac{n(n-1)}{2!} \times (n-2)! + \frac{n(n-1)(n-2)}{3!} \times (n-3)! - \ldots + (-1)^{n-1} \times 1\right\}$$

$$= n! - \left\{\frac{n!}{1!} - \frac{n!}{2!} + \frac{n!}{3!} - \ldots + (-1)^n \cdot 1\right\}$$

$$= n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^n}{n!}\right]$$

Maha Short Cut Method

If $\quad D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^n}{n!}\right]$

Then, $\boldsymbol{D_{n+1} = (n+1)D_n + (-1)^{n+1}},\ \forall\ \boldsymbol{x \in N}$

and $\quad \boldsymbol{D_{n+1} = n(D_n + D_{n-1})},\ \forall\ \boldsymbol{x \in N - \{1\}}$

where $\quad D_1 = 0$

For $n = 1$, from result I

$$D_2 = 2D_1 + (-1)^2 = 0 + 1 = 1$$

For $n = 2$, from result I

$$D_3 = 3D_2 + (-1)^3 = 3 \times 1 - 1 = 2$$

For $n = 3$, from result I

$$D_4 = 4D_3 + (-1)^4 = 4 \times 2 + 1 = 9$$

For $n = 4$, from result I

$$D_5 = 5D_4 + (-1)^5 = 5 \times 9 - 1 = 44$$

For $n = 5$, from result I

$$D_6 = 6D_5 + (-1)^6 = 6 \times 44 + 1 = 265$$

Note $D_1 = 0,\ D_2 = 1,\ D_3 = 2,\ D_4 = 9,\ D_5 = 44,\ D_6 = 265$ [Remember]

Remark

If r things goes to wrong place out of n things, then $(n-r)$ things goes to original place (here $r < n$).

If D_n = Number of ways, if all n things goes to wrong places.

and D_r = Number of ways, if r things goes to wrong places.

If r goes to wrong places out of n, then $(n-r)$ goes to correct places.

Then, $D_n = {}^nC_{n-r}\, D_r$

where, $D_r = r!\left(1 - \frac{1}{1!} + \frac{1}{2!}! - \frac{1}{3!} + \ldots + (-1)^r \frac{1}{r!}\right)$

If atleast p things goes to wrong places, then $D_n = \sum_{r=p}^{n} {}^nC_{n-r} \cdot D_r$

Example 90. A person writes letters to six friends and addresses the corresponding envelopes. In how many ways can the letters be placed in the envelopes so that (i) atleast two of them are in the wrong envelopes. (ii) all the letters are in the wrong envelopes.

Sol. (i) The number of ways in which atleast two of them in the wrong envelopes

$$= \sum_{r=2}^{6} {}^6C_{6-r} \cdot D_r$$

$$= {}^6C_4 \times D_2 + {}^6C_3 \times D_3 + {}^6C_2 \times D_4 + {}^6C_1 \times D_5 + {}^6C_0 \times D_6$$

$$= 15D_2 + 20D_3 + 15D_4 + 6D_5 + D_6 \quad \text{[from note]}$$

$$= 15 \times 1 + 20 \times 2 + 15 \times 9 + 6 \times 44 + 265$$

$$= 719$$

(ii) The number of ways in which all letters be placed in wrong envelopes $= D_6 = 265$ [from note]

Aliter

(i) The number of all the possible ways of putting 6 letters into 6 envelopes is $6!$.

Number of ways to place all letters correctly into corresponding envelopes $= 1$

and number of ways to place one letter in the wrong envelope and other 5 letters in the write envelope $= 0$

[$\because$ It is not possible that only one letter goes in the wrong envelope, when if 5 letters goes in the right envelope, then remaining one letter also goes in the write envelope]

Hence, number of ways to place atleast two letters goes in the wrong envelopes

$$= 6! - 0 - 1 = 6! - 1 = 720 - 1 = 719$$

(ii) The number of ways 1 letter in 1 address envelope, so that one letter is in wrong envelope $= 0$ ...(i)

[because it is not possible that only one letter goes in the wrong envelope]

The number of ways to put 2 letters in 2 addressed envelopes so that all are in wrong envelopes

= The number of ways without restriction – The number of ways in which all are in correct envelopes – The number of ways in which 1 letter is in the correct envelope

$$= 2! - 1 - 0 = 2 - 1$$

$$= 1 \quad \text{...(ii) [from Eq. (i)]}$$

The number of ways to put 3 letters in 3 addressed envelopes so that all are in wrong envelopes

= The number of ways without restriction – The number of ways in which all are in correct envelopes – The number of ways in which 1 letter is in the correct envelope – The number of ways in which 2 letter are in correct envelope

$$= 3! - 1 - {}^3C_1 \times 1 - 0 \quad \text{[from Eqs. (i) and (ii)]}$$

$$= 2$$

[${}^3C_1$ means that select one envelope to put the letter correctly]

The number of ways to put 4 letters in 4 addressed envelopes so that all are in wrong envelopes

= The number of ways without restriction – The number of ways in which all are in correct envelopes – The number of ways in which 1 letter is in the correct envelope – The number of ways in which 2 letters are in correct envelopes – The number of ways in which 3 letters are in correct envelopes

$$= 4! - 1 - {}^4C_1 \times 2 - {}^4C_2 \times 1 - {}^4C_3 \times 0$$

[from Eqs. (i), (ii) and (iii)]

$$= 24 - 1 - 8 - 6 - 0 = 9 \quad \text{...(iv)}$$

The number of ways to put 5 letters in 5 addressed envelopes so that all are in wrong envelopes

= The number of ways without restriction – The number of ways in which all are in correct envelopes – The number of ways in which 1 letter is in the correct envelopes – The number of ways in which 2 letters are in correct envelopes – The number of ways in which 3 letters are in correct envelopes – The number of ways in which 4 letters are in correct envelopes

$$= 5! - 1 - {}^5C_1 \times 9 - {}^5C_2 \times 2 - {}^5C_3 \times 1 - {}^5C_4 \times 0$$

[from Eqs. (i), (ii), (iii) and (iv)]

$$= 120 - 1 - 45 - 20 - 10 - 0 = 44$$

The number of ways to put 6 letters in 6 addressed envelopes so that all are in wrong envelopes

= The number of ways without restriction − The number of ways in which all are in correct envelopes − The number of ways in which 1 letter is in the correct envelope − The number of ways in which 2 letters are in correct envelopes − The number of ways in which 3 letters are in correct envelopes − The number of ways in which 4 letters are in correct envelopes − The number of ways in which 5 letters are in correct envelopes.

$$= 6! - 1 - {}^6C_1 \times 44 - {}^6C_2 \times 9 - {}^6C_3 \times 2 - {}^6C_4 \times 1 - {}^6C_5 \times 0$$

[from Eqs. (i), (ii), (iii), (iv) and (v)]

$$= 720 - 1 - 264 - 135 - 40 - 15 = 720 - 455 = 265$$

Multinomial Theorem

(i) If there are l objects of one kind, m objects of second kind, n objects of third kind and so on, then the number of ways of choosing r objects out of these objects (i.e., $l + m + n + \ldots$) is the coefficient of x^r in the expansion of

$$(1 + x + x^2 + x^3 + \ldots + x^l)(1 + x + x^2 + \ldots + x^m)(1 + x + x^2 + \ldots + x^n)$$

Further, if one object of each kind is to be included, then the number of ways of choosing r objects out of these objects (i.e., $l + m + n + \ldots$) is the coefficient of x^r in the expansion of

$$(x + x^2 + x^3 + \ldots + x^l)(x + x^2 + x^3 + \ldots + x^m)(x + x^2 + x^3 + \ldots + x^n)\ldots$$

(ii) If there are l objects of one kind, m objects of second kind, n objects of third kind and so on, then the number of possible arrangements/permutations of r objects out of these objects (i.e., $l + m + n + \ldots$) is the coefficient of x^r in the expansion of

$$r!\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^l}{l!}\right)\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^m}{m!}\right)\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}\right)\ldots.$$

Different Cases of Multinomial Theorem

***Case* I If upper limit of a variable is more than or equal to the sum required, then the upper limit of that variable can be taken as infinite.**

Example 91. In how many ways the sum of upper faces of four distinct die can be five?

Sol. Here, the number of required ways will be equal to the number of solutions of $x_1 + x_2 + x_3 + x_4 = 5$ i.e., $1 \le x_i \le 6$ for $i = 1, 2, 3, 4$.

Since, upper limit is 6, which is greater than required sum, so upper limit taken as infiite. So, number of Sols is equal to coefficient of α^5 in the expansion of $(1 + \alpha + \alpha^2 + \ldots + \infty)^4$

$$= \text{Coefficient of } \alpha^5 \text{ in the expansion of } (1 - \alpha)^{-4}$$

$$= \text{Coefficient of } \alpha^5 \text{ in the expansion of } (1 + {}^4C_1\alpha + {}^5C_2\alpha^2 + \ldots)$$

$$= {}^8C_5 = {}^8C_3 = \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} = 56$$

***Case* II If the upper limit of a variable is less than the sum required and the lower limit of all variables is non-negative, then the upper limit of that variable is that given in the problem.**

Example 92. ***In an examination, the maximum marks each of three papers is*** 50 ***and the maximum mark for the fourth paper is*** 100***. Find the number of ways in which the candidate can score*** 60% ***marks in aggregate.***

Sol. Aggregate of marks $= 50 \times 3 + 100 = 250$

$$\therefore \text{ 60\% of the aggregate} = \frac{60}{100} \times 250 = 150$$

Let the marks scored by the candidate in four papers be x_1, x_2, x_3 and x_4. Here, the number of required ways will be equal to the number of Sols of $x_1 + x_2 + x_3 + x_4 = 150$ i.e., $0 \le x_1, x_2, x_3 \le 50$ and $0 \le x_4 \le 100$.

Since, the upper limit is 100 < required sum (150).

The number of solutions of the equation is equal to coefficient of α^{150} in the expansion of

$$(\alpha^0 + \alpha^1 + \alpha^2 + \ldots + \alpha^{50})^3(\alpha^0 + \alpha^1 + \alpha^2 + \ldots + \alpha^{100})$$

$$= \text{Coefficient of } \alpha^{150} \text{ in the expansion of } (1 - \alpha^{51})^3(1 - \alpha^{10})(1 - \alpha)^{-4}$$

$$= \text{Coefficient of } \alpha^{150} \text{ in the expansion of } (1 - 3\alpha^{51} + 3\alpha^{102})(1 - \alpha^{101})(1 + {}^4C_1\alpha + {}^5C_2\alpha^2 + \ldots + \infty)$$

$$= \text{Coefficient of } \alpha^{150} \text{ in the expansion of } (1 - 3\alpha^{51} - \alpha^{101} + 3\alpha^{102})(1 + {}^4C_1\alpha + {}^5C_2\alpha^2 + \ldots + \infty)$$

$$= {}^{153}C_{150} - 3 \times {}^{102}C_{99} - {}^{52}C_{49} + 3 \times {}^{51}C_{48}$$

$$= {}^{153}C_3 - 3 \times {}^{102}C_3 - {}^{52}C_3 + 3 \times {}^{51}C_3$$

$$= 110556$$

Very Important Trick

On multiplying $p_0 + p_1\alpha + p_2\alpha^2 + p_3\alpha^3 + \ldots + p_n\alpha^n$ by $(1+\alpha)$, we get

$$p_0 + (p_0 + p_1)\alpha + (p_1 + p_2)\alpha^2 + (p_2 + p_3)\alpha^3 + \ldots + (p_{n-2} + p_{n-1})\alpha^{n-1} + (p_{n-1} + p_n)\alpha^n + p_n\alpha^{n+1}$$

i.e., we just add coefficient of α^r with coefficient of α^{r-1} (i.e., previous term) to get coefficient α^r in product.

Now, coefficient of $\alpha^r = p_{r-1} + p_r$

On multiplying $p_0 + p_1\alpha + p_2\alpha^2 + p_3\alpha^3 + \ldots + p_n\alpha^n$ by $(1+\alpha+\alpha^2)$

we get, $p_0 + (p_0 + p_1)\alpha + (p_0 + p_1 + p_2)\alpha^2 + (p_1 + p_2 + p_3)\alpha^3 + (p_2 + p_3 + p_4)\alpha^4 + \ldots$

i.e., to find coefficient of α^r in product and add this with 2 preceding coefficients.

Now, coefficient of $\alpha^r = p_{r-2} + p_{r-1} + p_r$

Similarly, in product of $p_0 + p_1\alpha + p_2\alpha^2 + \ldots$ with $(1+\alpha+\alpha^2+\alpha^3)$, the coefficient of α^r in product will be

$$\underbrace{p_{r-3} + p_{r-2} + p_{r-1}}_{\text{3 preceding coefficients}} + p_r$$

and in product of $p_0 + p_1\alpha + p_2\alpha^2 + \ldots$ with $(1+\alpha+\alpha^2+\alpha^3+\alpha^4)$, the coefficient of α^r in product will be $\underbrace{p_{r-4} + p_{r-3} + p_{r-2} + p_{r-1}}_{\text{4 preceding coefficients}} + p_r$

Finally, in product of $p_0 + p_1\alpha + p_2\alpha^2 + \ldots$ with $(1+\alpha+\alpha^2+\alpha^3+\ldots+\text{upto}\,\infty)$, the coefficient of α^r in product will be $\underbrace{p_0 + p_1 + p_2 + \ldots + p_{r-1}}_{\text{all preceding coefficients}} + p_r$

Example 93. ***Find the coefficient of*** α^6 ***in the product*** $(1+\alpha+\alpha^2)(1+\alpha+\alpha^2)(1+\alpha+\alpha^2+\alpha^3)(1+\alpha)(1+\alpha)(1+\alpha)$.

Sol. The given product can be written as

$$(1+\alpha+\alpha^2)(1+\alpha+\alpha^2)(1+\alpha+\alpha^2+\alpha^3)(1+\alpha)^3$$

$$\text{or } (1+\alpha+\alpha^2)(1+\alpha+\alpha^2)(1+\alpha+\alpha^2+\alpha^3)(1+3\alpha+3\alpha^2+\alpha^3)$$

Multiplying Synthetically

1	α	α^2	α^3	α^4	α^5	α^6	...
1	3	3	1	0	0	0	

... on multiplying by $1+\alpha+\alpha^2+\alpha^3 \rightarrow$ To each coefficient add 3 preceding coefficients

1	4	7	8	7	4	1	...

...on multiplying by $1+\alpha+\alpha^2 \rightarrow$ To each coefficient add 2 preceding coefficients.

1	5	12	19	22	19	12	...

...on multiplying by $1+\alpha+\alpha^2 \rightarrow$ To each coefficient add 2 preceding coefficients.

...	...	...	...	...	...	53	...

Hence, required coefficient is 53.

Example 94. ***Find the number of different selections of*** 5 ***letters which can be made from*** 5*A*'***s,*** 4*B* '***s,*** 3*C*'***s,*** 2*D*'***s and*** 1*E*

Sol. All selections of 5 letters are given by 5th degree terms in

$$(1 + A + A^2 + A^3 + A^4 + A^5)(1 + B + B^2 + B^3 + B^4)$$

$$(1 + C + C^2 + C^3)(1 + D + D^2)(1 + E)$$

$\therefore$ Number of 5 letter selections

= Coefficient of α^5 in $(1+\alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5)$

$$(1+\alpha+\alpha^2+\alpha^3+\alpha^4)(1+\alpha+\alpha^2+\alpha^3)$$

$$(1+\alpha+\alpha^2)(1+\alpha)$$

Multiplying synthetically

1	α	α^2	α^3	α^4	α^5 ...
1	1	1	1	1	1
---------- $\times 1 + \alpha + \alpha + \alpha^2 + \alpha^3 + \alpha^4$					
1	2	3	4	5	5 ...
---------- $\times 1 + \alpha + \alpha^2 + \alpha^3$					
1	3	6	10	14	17 ...
---------- $\times 1 + \alpha + \alpha^2$					
1	4	10	19	30	41 ...
---------- $\times 1 + \alpha$					
1	5	14	29	49	**71**

Hence, required coefficient is 71.

Example 95. Find the number of combinations and permutations of 4 letters taken from the word EXAMINATION.

Sol. There are 11 letters

A, A, N, N, X, M, T, O.

Then, number of combinations

= coefficient of x^4 in $(1 + x + x^2)^3(1 + x)^5$

[$\because$ 2A's, 2I 's, 2N's, 1E, 1X, 1M, 1T and 1O]

= Coefficient of x^4 in $\{(1 + x)^3 + x^6 + 3(1 + x)^2 x^2$

$$+3(1+x)x^4\}(1+x)^5$$

= Coefficient of x^4 in

$$\{(1+x)^8 + x^6(1+x)^5 + 3x^2(1+x)^7 + 3x^4(1+x)^6\}$$

$$= {}^8C_4 + 0 + 3\cdot{}^7C_2 + 3 = \frac{8\cdot7\cdot6\cdot5}{1\cdot2\cdot3\cdot4} + 3\cdot\frac{7\cdot6}{1\cdot2} + 3 = 70 + 63 + 3$$

$$= 136$$

and number of permutations

$$= \text{Coefficient of } x^4 \text{ in } 4!\left(1 + \frac{x}{1!} + \frac{x^2}{2!}\right)^3\left(1 + \frac{x}{1!}\right)^5$$

$$= \text{Coefficient of } x^4 \text{ in } 4!\left(1 + x + \frac{x^2}{2}\right)^3(1+x)^5$$

= Coefficient of x^4 in

$$4!\left\{(1+x)^3 + \frac{x^6}{8} + \frac{3}{2}(1+x)^2x^2 + \frac{3}{4}x^4(1+x)\right\}(1+x)^5$$

= Coefficient of x^4 in

$$4!\left\{(1+x)^8 + \frac{x^6}{8}(1+x)^5 + \frac{3}{2}x^2(1+x)^7 + \frac{3}{4}x^4(1+x)^6\right\}$$

$$= 4!\left\{{}^8C_4 + 0 + \frac{3}{2}\cdot{}^7C_2 + \frac{3}{4}\right\} = 24\left\{\frac{8\cdot7\cdot6\cdot5}{1\cdot2\cdot3\cdot4} + \frac{3}{2}\cdot\frac{7\cdot6}{1\cdot2} + \frac{3}{4}\right\}$$

$$= 8\cdot7\cdot6\cdot5 + 6(3\cdot7\cdot6) + 6\cdot3 = 1680 + 756 + 18 = 2454$$

Aliter There are 11 letters:

A, A, I, I, N, N, E, X, M, T, O

The following cases arise:

***Case* I All letters different** The required number of choosing 4 different letters from 8 different (A, I, N, E, X, M, T, O) types of the letters

$$= {}^8C_4 = \frac{8\cdot7\cdot6\cdot5}{1\cdot2\cdot3\cdot4} = 70$$

and number of permutations $= {}^8P_4 = 8\cdot7\cdot6\cdot5 = 1680$

***Case* II Two alike of one type and two alike of another type** This must be 2A's, 2I's or 2I's, 2N's, or 2N's, 2A's.

∴ Number of selections $= {}^3C_2 = 3$

For example, [for arrangements]

A	A	I	I

and number of permutations $= 3\cdot\frac{4!}{2!\,2!} = 18$

***Case* III Two alike and two different** This must be 2A's or 2I's or 2N's

and for each case 7 different letters.

For example, for 2A's, 7 differents's are I, N, E, X, M, T, O

For example, [for arrangements]

A	X	A	O

∴ Number of selections $= {}^3C_1 \times {}^7C_2 = 3\times\frac{7\times6}{1\times2} = 63$

and number of permutations $= 63\cdot\frac{4!}{2!} = 756$

From Case I, II and III

The required number of combinations $= 70 + 3 + 63 = 136$

and number of permutations $= 1680 + 18 + 756 = 2454$

Note Number of combinations and permutations of 4 letters taken from the word MATHEMATICS are 136 and 2454 respectively, as like of EXAMINATION.

Number of Solutions with the Help of Multinomial Theorem

***Case* I** If the equation

$$\alpha + 2\beta + 3\gamma + \ldots + q\theta = n \qquad \ldots(i)$$

(a) If zero included, the number of solution of Eq. (i)

= Coefficient of x^n in $(1 + x + x^2 + \ldots)$

$$(1 + x^2 + x^4 + \ldots)(1 + x^3 + x^6 + \ldots)\ldots$$

$$(1 + x^q + x^{2q} + \ldots)$$

= Coefficient of x^n in

$$(1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}\ldots(1-x^q)^{-1}$$

(b) If zero excluded, then the number of solutions of Eq. (i)

= Coefficient of x^n in $(x + x^2 + x^3 + \ldots)$

$$(x^2 + x^4 + x^6 + \ldots)(x^3 + x^6 + x^9 + \ldots)$$

$$\ldots(x^q + x^{2q} + \ldots)$$

= Coefficient of x^n in $x^{1+2+3+\ldots+q}(1-x)^{-1}$

$$(1-x^2)^{-1}(1-x^3)^{-1}\ldots(1-x^q)^{-1}$$

= Coefficient of $x^{n-\frac{q(q+1)}{2}}$ in

$$(1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}\ldots(1-x^q)^{-1}$$

Example 96. Find the number of non-negative integral solutions of $x_1 + x_2 + x_3 + 4x_4 = 20$.

Sol. Number of non-negative integral solutions of the given equation

$$= \text{Coefficient of } x^{20} \text{ in } (1-x)^{-1}(1-x)^{-1}(1-x)^{-1}(1-x^4)^{-1}$$

$$= \text{Coefficient of } x^{20} \text{ in } (1-x)^{-3}(1-x^4)^{-1}$$

$$= \text{Coefficient of } x^{20} \text{ in } (1 + {}^3C_1 + {}^4C_2x^2 + {}^5C_3x^3 + {}^6C_4x^4 + \ldots + {}^{10}C_8x^8 + \ldots + {}^{14}C_{12}x^{12} + \ldots + {}^{18}C_{16}x^{16} + \ldots + {}^{22}C_{20}x^{20} + \ldots)(1 + x^4 + x^8 + x^{12} + x^{16} + x^{20} + \ldots)$$

$$= 1 + {}^6C_4 + {}^{10}C_8 + {}^{14}C_{12} + {}^{18}C_{16} + {}^{22}C_{20}$$

$$= 1 + {}^6C_2 + {}^{10}C_2 + {}^{14}C_2 + {}^{18}C_2 + {}^{22}C_2$$

$$= 1 + \left(\frac{6\cdot5}{1\cdot2}\right) + \left(\frac{10\cdot9}{1\cdot2}\right) + \left(\frac{14\cdot13}{1\cdot2}\right) + \left(\frac{18\cdot17}{1\cdot2}\right) + \left(\frac{22\cdot21}{1\cdot2}\right)$$

$$= 1 + 15 + 45 + 91 + 153 + 231 = 536$$

Example 97. Find the number of positive unequal integral solutions of the equation $x + y + z + w = 20$.

Sol. We have, $x + y + z + w = 20$ ...(i)

Assume $x < y < z < w$. Here, $x, y, z, w \geq 1$

Now, let $x = x_1, y - x = x_2, z - y = x_3$ and $w - z = x_4$

$\therefore$ $x = x_1, y = x_1 + x_2, z = x_1 + x_2 + x_3$ and $w = x_1 + x_2 + x_3 + x_4$

From Eq. (i), $4x_1 + 3x_2 + 2x_3 + x_4 = 20$

Then, $x_1, x_2, x_3, x_4 \geq 1$

$\because$ $4x_1 + 3x_2 + 2x_3 + x_4 = 20$ ...(ii)

$\therefore$ Number of positive integral solutions of Eq. (ii)

= Coefficient of x^{20-10} in $(1-x^4)^{-1}(1-x^3)^{-1}(1-x^2)^{-1}(1-x)^{-1}$

= Coefficient of x^{10} in $(1-x^4)^{-1}(1-x^3)^{-1}(1-x^2)^{-1}(1-x)^{-1}$

= Coefficient of x^{10} in $(1 + x^4 + x^8 + x^{12} + \ldots) \times (1 + x^3 + x^6 + x^9 + x^{12} + \ldots) \times (1 + x^2 + x^4 + x^6 + x^8 + x^{10} + \ldots) \times (1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + \ldots)$

= Coefficient of x^{10} in $(1 + x^3 + x^6 + x^9 + x^4 + x^7 + x^{10} + x^8) \times (1 + x^2 + x^4 + x^6 + x^8 + x^{10})(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10})$ [neglecting higher powers]

= Coefficient of x^{10} in $(1 + x^2 + x^4 + x^6 + x^8 + x^{10} + x^3 + x^5 + x^7 + x^9 + x^6 + x^8 + x^{10} + x^9 + x^4 + x^6 + x^8 + x^{10} + x^7 + x^9 + x^{10} + x^8 + x^{10})(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10})$ [neglecting higher powers]

$= 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 23$

But x, y, z and w can be arranged in ${}^4P_4 = 4! = 24$

Hence, required number of Sols $= (23)(24) = 552$

Example 98. In how many ways can 15 identical blankets be distribted among six beggars such that everyone gets atleast one blanket and two particular beggars get equal blankets and another three particular beggars get equal blankets.

Sol. The number of ways of distributing blankets is equal to the number of solutions of the equation $3x + 2y + z = 15$, where $x \geq 1$, $y \geq 1$, $z \geq 1$ which is equal to coefficient of α^{15} in

$(\alpha^3 + \alpha^6 + \alpha^9 + \alpha^{12} + \alpha^{15} + \ldots) \times (\alpha^2 + \alpha^4 + \alpha^6 + \alpha^8 + \alpha^{10} + \alpha^{12} + \alpha^{14} + \ldots) \times (\alpha + \alpha^2 + \alpha^3 + \ldots + \alpha^{15} + \ldots)$

= Coefficient of α^9 in $(1 + \alpha^3 + \alpha^6 + \alpha^9) \times (1 + \alpha^2 + \alpha^4 + \alpha^6 + \alpha^8) \times (1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 + \alpha^7 + \alpha^8 + \alpha^9)$ [neglecting higher powers]

= Coefficient of α^9 in $(1 + \alpha^2 + \alpha^4 + \alpha^6 + \alpha^8 + \alpha^3 + \alpha^5 + \alpha^7 + \alpha^9 + \alpha^6 + \alpha^8 + \alpha^9) \times (1 + \alpha + \alpha^2 \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 + \alpha^7 + \alpha^8 + \alpha^9)$ [neglecting higher powers]

$= 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 12$

***Case* II** If the inequation

$$x_1 + x_2 + x_3 + \ldots + x_m \leq n \quad \text{...(i)}$$

[when the required sum is not fixed]

In this case, we introduce a dummy variable x_{m+1}.

So that,

$$x_1 + x_2 + x_3 + \ldots + x_m + x_{m+1} = n,$$

$$x_{m+1} \geq 0 \quad \text{...(ii)}$$

Here, the number of Sols of Eqs. (i) and (ii) will be same.

Example 99. Find the number of positive integral solutions of the inequation $3x + y + z \leq 30$.

Sol. Let dummy variable w, then

$$3x + y + z + w = 30, w \geq 0 \quad \text{...(i)}$$

Now, let $a = x - 1, b = y - 1, c = z - 1, d = w$, then

$$3a + b + c + d = 25, \text{ where } a, b, c, d \geq 0 \quad \text{...(ii)}$$

$\therefore$ Number of positive integral solutions of Eq. (i)

= Number of non-negative integral solutions of Eq. (ii)

= Coefficient of α^{25} in $(1 + \alpha^3 + \alpha^6 + \ldots)(1 + \alpha + \alpha^2 + \ldots)^3$

= Coefficient of α^{25} in $(1 + \alpha^3 + \alpha^6 + \ldots)(1 - \alpha)^{-3}$

= Coefficient of α^{25} in $(1 + \alpha^3 + \alpha^6 + \ldots)(1 + {}^3C_1\alpha + {}^4C_2\alpha^2 + \ldots)$

$= {}^{27}C_{25} + {}^{24}C_{22} + {}^{21}C_{19} + {}^{18}C_{16} + {}^{15}C_{13} + {}^{12}C_{10} + {}^9C_7 + {}^6C_4 + {}^3C_1$

$= {}^{27}C_2 + {}^{24}C_2 + {}^{21}C_2 + {}^{18}C_2 + {}^{15}C_2 + {}^{12}C_2 + {}^9C_2 + {}^6C_2 + {}^3C_1$

$= 351 + 276 + 210 + 153 + 105 + 66 + 36 + 15 + 3 = 1215$

Aliter

From Eq. (ii), $3a + b + c + d = 25$, where $a, b, c, d \geq 0$

Clearly, $0 \leq a \leq 8$, if $a = k$, then

$$b + c + d = 25 - 3k \quad \ldots(iii)$$

Hence, number of non-negative integral solutions of Eq. (iii) is

$${}^{25-3k+3-1}C_{3-1} = {}^{27-3k}C_2 = \frac{(27-3k)(26-3k)}{2}$$

$$= \frac{3}{2}(3k^2 - 53k + 234)$$

Therefore, required number is

$$\frac{3}{2}\sum_{k=0}^{8}(3k^2 - 53k + 234)$$

$$= \frac{3}{2}\left[3\cdot\left(\frac{8\times 9\times 17}{6}\right) - 53\cdot\left(\frac{8\times 9}{2}\right) + 234\times 9\right] = 1215$$

Example 100. In how many ways can we get a sum of atmost 15 by throwing six distinct dice ?

Sol. Let x_1, x_2 , x_3, x_4, x_5 and x_6 be the number that appears on the six dice.

The number of ways = Number of solutions of the inequation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 15$$

Introducing a dummy variable $x_7 (x_7 \geq 0)$, the inequation becomes an equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 15$$

Here, $1 \leq x_i \leq 6$ for $i = 1, 2, 3, 4, 5, 6$ and $x_7 \geq 0$.

Therefore, number of solutions

= Coefficient of x^{15} in $(x + x^2 + x^3 + x^4 + x^5 + x^6)^6 \times (1 + x + x^2 + \ldots)$

= Coefficient of x^9 in $(1 - x^6)^6 (1 - x)^{-7}$

= Coefficient of x^9 in $(1 - 6x^6)(1 + {}^7C_1 x + {}^8C_2 x^2 + \ldots)$ [neglecting higher powers]

$= {}^{15}C_9 - 6 \times {}^9C_3 = {}^{15}C_6 - 6 \times {}^9C_3$

$= 5005 - 504 = 4501$

***Case* III** If the inequation

$$x_1 + x_2 + x_3 + \ldots + x_n \geq n$$

[when the values of $x_1, x_2, \ldots, x_n$ are restricted]

In this case first find the number of solutions of $x_1 + x_2 + x_3 + \ldots + x_n \leq n - 1$ and then subtract it from the total number of solutions.

Example 101. In how many ways can we get a sum greater than 15 by throwing six distinct dice?

Sol. Let x_1, x_2, x_3, x_4, x_5 and x_6 be the number that appears on the six dice.

The number of ways = Number of solutions of the inequation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 > 15$$

Here, $1 \leq x_i \leq 6$, $i = 1, 2, 3, 4, 5, 6$

Total number of cases $= 6^6 = 2^6 \times 3^6 = 64 \times 729 = 46656$

and number of ways to get the sum less than or equal to 15, which is 4501 [from Example 100]

Hence, the number of ways to get a sum greater than 15 is $46656 - 4501 = 42155$

***Case* IV** If the equation

$$x_1 x_2 x_3 \ldots x_n = 2^{\alpha_1} \cdot 3^{\alpha_2} \cdot 5^{\alpha_3} \ldots$$

where $\alpha_1, \alpha_2, \alpha_3, \ldots$ are natural numbers.

In this case number of positive integral solutions $(x_1, x_2, x_3, \ldots, x_n)$ are

$$({}^{\alpha_1+n-1}C_{n-1})({}^{\alpha_2+n-1}C_{n-1})({}^{\alpha_3+n-1}C_{n-1})\ldots$$

Example 102. Find the total number of positive integral solutions for (x, y, z) such that $xyz = 24$.

Sol. $\because$ $xyz = 24 = 2^3 \times 3^1$

Hence, total number of positive integral solutions

$$= ({}^{3+3-1}C_{3-1})({}^{1+3-1}C_{3-1})$$

$$= {}^5C_2 \times {}^3C_2 = 30$$

Aliter

$\because$ $xyz = 24 = 2^3 \times 3^1$

Now, consider three boxes x, y, z.

3 can be put in any of the three boxes.

Also, 2, 2, 2 can be distributed in the three boxes in ${}^{3+3-1}C_{3-1} = {}^5C_2 = 10$ ways. Hence, the total number of positive integral solutions = the number of distributions which is given by $3 \times 10 = 30$.

Geometrical Problems

(a) If there are n points in a plane out of these points no three are in the same line except m points which are collinear, then

(i) Total number of different lines obtained by joining these n points is ${}^nC_2 - {}^mC_2 + 1$

(ii) Total number of different triangles formed by joining these n points is ${}^nC_3 - {}^mC_3$

(iii) Total number of different quadrilateral formed by joining these n points is

$${}^nC_4 - ({}^mC_3 \cdot {}^nC_1 + {}^mC_4 \cdot {}^nC_0)$$

Example 103. There are 10 points in a plane out of these points no three are in the same straight line except 4 points which are collinear. How many

(i) straight lines (ii) trian-gles

(iii) quadrilateral, by joining them?

Sol. (i) Required number of straight lines

$$= {}^{10}C_2 - {}^4C_2 + 1 = \frac{10\cdot 9}{1\cdot 2} - \frac{4\cdot 3}{1\cdot 2} + 1 = 45 - 6 + 1 = 40$$

(ii) Required number of triangles

$$= {}^{10}C_3 - {}^4C_3 = \frac{10\cdot 9\cdot 8}{1\cdot 2\cdot 3} - {}^4C_1 = 120 - 4 = 116$$

(iii) Required number of quadrilaterals

$$= {}^{10}C_4 - ({}^4C_3 \cdot {}^6C_1 + {}^4C_4 \cdot {}^6C_0)$$

$$= \frac{10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4} - ({}^4C_1 \cdot {}^6C_1 + 1.1)$$

$$= 210 - (4\times 6 + 1) = 210 - 25 = 185$$

(b) If there are n points in a plane out of these points no any three are collinear, then

(i) Total points of intersection of the lines joining these n points $= {}^pC_2$, where $p = {}^nC_2$

(ii) If n points are the vertices of a polygon, then total number of diagonals $= {}^nC_2 - n = \frac{n(n-3)}{2}$

Example 104. How many number of points of intersection of n straight lines, if n satisfies ${}^{n+5}P_{n+1} = \frac{11(n-1)}{2} \times {}^{n+3}P_n$?

Sol. We have, $${}^{n+5}P_{n+1} = \frac{11(n-1)}{2} \times {}^{n+3}P_n$$

$$\Rightarrow \quad \frac{(n+5)!}{4!} = \frac{11(n-1)}{2} \times \frac{(n+3)!}{3!}$$

$$\Rightarrow \quad \frac{(n+5)(n+4)}{4} = \frac{11(n-1)}{2}$$

$$\Rightarrow \quad n^2 - 13n + 42 = 0 \Rightarrow (n-6)(n-7) = 0$$

$$\Rightarrow \quad n = 6 \text{ or } n = 7$$

The number of points of intersection of lines is ${}^6C_2$ or ${}^7C$

$$= 15 \text{ or } 21$$

Example 105. The interior angles of a regular polygon measure 150° each. Then, find the number of diagonals of the polygon.

Sol. Each exterior angle $= 30°$

$$\therefore \text{ Number of sides} = \frac{360°}{30°} = \frac{360 \times \frac{\pi}{180}}{30 \times \frac{\pi}{180}} = 12$$

$$\therefore \text{ Number of diagonals} = {}^{12}C_2 - 12 = 66 - 12 = 54$$

Example 106. In a polygon the number of diagonals is 77. Find the number of sides of the polygon.

Sol. Let number of sides of the polygon $= n$, then ${}^nC_2 - n = 77$

$$\Rightarrow \quad \frac{n(n-1)}{2} - n = 77 \Rightarrow \frac{n(n-3)}{2} = \frac{14\times 11}{2}$$

we get, $n = 14$

(c) n straight lines are drawn in a plane such that no two of them are parallel and no three of them are concurrent. Then, number of parts into which these lines divides the plane is equal to

$$1 + \sum_{k=1}^{n} k, \text{ .e. } \frac{(n^2+n+2)}{2}$$

Example 107. If n lines are drawn in a plane such that no two of them are parallel and no three of them are concurrent, such that these lines divide the plane in 67 parts, then find number of different points at which these lines will cut.

Sol. Given number of straight lines $= n$, then

$$1 + \sum_{k=1}^{n} k = 67 \Rightarrow \frac{n^2+n+2}{2} = 67$$

$$\Rightarrow \quad n^2 + n - 132 = 0 \Rightarrow (n+12)(n-11) = 0$$

$$\therefore \quad n = 11, n \neq -12$$

Hence, required number of points $= {}^nC_2 = {}^{11}C_2 = \frac{11\cdot 10}{2}$

$$= 55$$

(d) If m parallel lines in a plane are intersected by a family of other n parallel lines. Then, total number of parallelograms so formed

$$= {}^mC_2 \cdot {}^nC_2 \text{ i.e., } \frac{mn(m-1)(n-1)}{4}$$

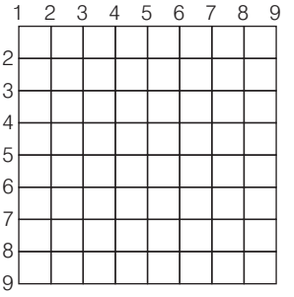
Example 108. Find number of rectangles in a chess board, which are not a square.

Sol. Number of rectangles $= {}^9C_2 \times {}^9C_2 = (36)^2 = 1296$

Number of squares $= 8\times 8 + 7\times 7 + 6\times 6 + \ldots + 1\times 1$

$$= 204$$

$\therefore$ Required number $= 1296 - 204 = 1092$



Square can be formed as follows :

To form the smallest square, select any two consecutive lines from the given (here 9) vertical and horizontal lines. This can be done in 8×8 ways (1-2, 2-3, 3-4, ..., 8-9)

Again, to form the square consists of four small squares, select the lines as follows (1-3, 2-4, 3-5,..., 7-9) from both vertical and horizontal lines, thus 7×7 squares are obtained. Proceed in the same way)

Note If n parallel lines are intersected by another n parallel lines, then number of rhombus $= \sum (n-1)^2 = \dfrac{(n-1)\,n(2n-1)}{6}$

(e) Number of Rectangles and Squares

(i) Number of rectangles of any size in a square of $n \times n$ is $\sum_{r=1}^{n} r^3$ and number of squares of any size is $\sum_{r=1}^{n} r^2$.

(ii) In a rectangle of $n \times p\ (n < p)$ number of rectangles of any size is $\dfrac{np}{4}(n+1)\,(p+1)$ and number of squares of any size is

$$\sum_{r=1}^{n} (n+1-r)\,(p+1-r).$$

Example 109. Find the number of rectangles excluding squares from a rectangle of size 9×6.

Sol. Here, $n = 6$ and $p = 9$

$\therefore$ Number of rectangles excluding square

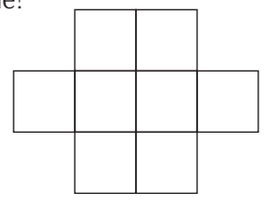
$$= \frac{6 \cdot 9}{4}(6+1)\,(9+1) - \sum_{r=1}^{6} (7-r)\,(10-r)$$

$$= 945 - \sum_{r=1}^{6} (70 - 17r + r^2) = 945 - 154 = 791$$

(f) If there are n rows, first row has α_1 squares, 2nd row has α_2 squares, 3rd row has α_3 squares, ... and nth row has α_n squares. If we have to filled up the squares with β X_s such that each row has atleast one X. The number of ways = Coefficient of x^{β} in

$$({}^{\alpha_1}C_1 x + {}^{\alpha_2}C_2\, x^2 + \ldots + {}^{\alpha_1}C_{\alpha_1} x^{\alpha_1}) \times ({}^{\alpha_2}C_1 x + {}^{\alpha_2}C_2\, x^2 + \ldots + {}^{\alpha_2}C_{\alpha_2} x^{\alpha_2}) \times ({}^{\alpha_3}C_1 x + {}^{\alpha_3}C_2\, x^2 + \ldots + {}^{\alpha_3}C_{\alpha_3} x^{\alpha_3}) \times \ldots \times ({}^{\alpha_n}C_1 x + {}^{\alpha_n}C_2\, x^2 + \ldots + {}^{\alpha_n}C_{\alpha_n} x^{\alpha_n})$$

Example 110. Six X's have to be placed in the squares of the figure below, such that each row contains atleast one X. In how many different ways can this be done?



Sol. The required number of ways

= Coefficient of x^6 in $({}^2C_1 x + {}^2C_2\, x^2)({}^4C_1 x + {}^4C_2\, x^2 + {}^4C_3 x^3 + {}^4C_4 x^4)({}^2C_1 x + {}^2C_2 x^2)$

= Coefficient of x^3 in $(2+x)^2\,(4 + 6x + 4x^2 + x^3)$

= Coefficient of x^3 in $(4 + 4x + x^2)(4 + 6x + 4x^2 + x^3)$

$= 4 + 16 + 6$

$= 26$

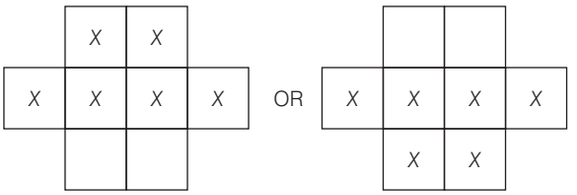
Aliter

In the given figure there are 8 squares and we have to place $6X$'s this can be done in

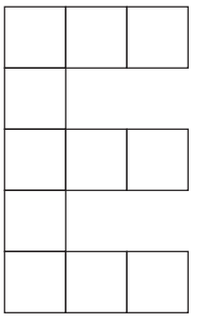
$${}^8C_6 = {}^8C_2 = \frac{8 \cdot 7}{1 \cdot 2} = 28 \text{ ways}$$

But these include the possibility that either headed row or lowest row may not have any X. These two possibilities are to be excluded.

$\therefore$ Required number of ways $= 28 - 2 = 26$



Example 111. In how many ways the letters of the word DIPESH can be placed in the squares of the adjoining figure so that no row remains empty?



Sol. If all letters are same, then number of ways

= Coefficient of x^6 in $({}^3C_1 x + {}^3C_2\, x^2 + {}^3C_3 x^3)^3\,({}^1C_1 x)^2$

= Coefficient of x in $(3 + 3x + x^2)^3$

= Coefficient of x in $(3 + 3x)^3$

[neglecting higher degree term]

$= 27 \times {}^3C_1 = 81$

But in DIPESH all letters are different.

$\therefore$ Required number of ways $= 81 \times 6!$

Exercise for Session 6

1. If number of ways in which 7 different balls can be distributed into 4 different boxes, so that no box remains empty is 100 λ, the value of λ is

(a) 18 (b) 108 (c) 1008 (d) 10008

2. If number of ways in which 7 different balls can be distributed into 4 boxes, so that no box remains empty is 48 λ, the value of λ is

(a) 231 (b) 331 (c) 431 (d) 531

3. If number of ways in which 7 identical balls can be distributed into 4 boxes, so that no box remains empty is 4λ, the value of λ is

(a) 5 (b) 7 (c) 9 (d) 11

4. Number of non-negative integral solutions of the equation $a + b + c = 6$ is

(a) 28 (b) 32 (c) 36 (d) 56

5. Number of integral solutions of $a + b + c = 0$, $a \geq -5$, $b \geq -5$ and $c \geq -5$, is

(a) 272 (b) 136 (c) 240 (d) 120

6. If a, b and c are integers and $a \geq 1, b \geq 2$ and $c \geq 3$. If $a + b + c = 15$, the number of possible solutions of the equation is

(a) 55 (b) 66 (c) 45 (d) None of these

7. Number of integral solutions of $2x + y + z = 10$ ($x \geq 0$, $y \geq 0$, $Z \geq 0$) is

(a) 18 (b) 27 (c) 36 (d) 51

8. A person writes letters to six friends and addresses the corresponding envelopes. Let x be the number of ways so that atleast two of the letters are in wrong envelopes and y be the number of ways so that all the letters are in wrong envelopes. Then, $x - y$ is equal to

(a) 719 (b) 265 (c) 454 (d) None of these

9. A person goes for an examination in which there are four papers with a maximum of m marks from each paper. The number of ways in which one can get $2m$ marks, is

(a) ${}^{2m+3}C_3$ (b) $\left(\frac{1}{3}\right)(m+1)(2m^2+4m+1)$

(c) $\left(\frac{1}{3}\right)(m+1)(2m^2+4m+3)$ (d) None of these

10. The number of selections of four letters from the letters of the word ASSASSINATION, is

(a) 72 (b) 71 (c) 66 (d) 52

11. The number of positive integral solutions of $2x_1 + 3x_2 + 4x_3 + 5x_4 = 25$, is

(a) 20 (b) 22 (c) 23 (d) None of these

12. If a, b, and c are positive integers such that $a + b + c \leq 8$, the number of possible values of the ordered triplet (a, b, c) is

(a) 84 (b) 56 (c) 83 (d) None of these

13. The total number of positive integral solutions of $15 < x_1 + x_2 + x_3 \leq 20$ is equal to

(a) 685 (b) 785 (c) 1125 (d) None of these

14. The total number of integral solutions for (x, y, z) such that $xyz = 24$, is

(a) 36 (b) 90 (c) 120 (d) None of these

15. There are 12 points in a plane in which 6 are collinear. Number of different straight lines that can be drawn by joining them, is

(a) 51 (b) 52 (c) 132 (d) 18

16. 4 points out of 11 points in a plane are collinear. Number of different triangles that can be drawn by joining them, is

(a) 165 (b) 161 (c) 152 (d) 159

17. The number of triangles that can be formed with 10 points as vertices, n of them being collinear, is 110. Then, n is

(a) 3 (b) 4 (c) 5 (d) 6

18. $ABCD$ is a convex quadrilateral. 3, 4, 5 and 6 points are marked on the sides AB, BC, CD and DA, respectively. The number of triangles with vertices on different sides, is

(a) 270 (b) 220 (c) 282 (d) None of these

19. There are 10 points in a plane of which no three points are collinear and 4 points are concyclic. The number of different circles that can be drawn through atleast 3 points of these points, is

(a) 116 (b) 120 (c) 117 (d) None of these

20. 4 points out of 8 points in a plane are collinear. Number of different quadrilateral that can be formed by joining them, is

(a) 56 (b) 60 (c) 76 (d) 53

21. There are $2n$ points in a plane in which m are collinear $(n > m > 4)$. Number of quadrilateral formed by joining these lines

(a) is equal to ${}^{2n}C_4 - {}^{m}C_4$ (b) is greater than ${}^{2n}C_4 - {}^{m}C_4$

(c) is less than ${}^{2n}C_4 - {}^{m}C_4$ (d) None of these

22. In a polygon the number of diagonals is 54. The number of sides of the polygon, is

(a) 10 (b) 12 (c) 9 (d) None of these

23. In a polygon no three diagonals are concurrent. If the total number of points of intersection of diagonals interior to the polygon be 70, then the number of diagonals of the polygon, is

(a) 20 (b) 28 (c) 8 (d) None of these

24. n lines are drawn in a plane such that no two of them are parallel and no three of them are concurrent. The number of different points at which these lines will cut, is

(a) $\sum_{k=1}^{n-1} k$ (b) $n(n-1)$ (c) n^2 (d) None of these

25. Six straight lines are drawn in a plane such that no two lines are parallel and no three lines are concurrent. Then, the number of parts into which these lines divide the plane, is

(a) 15 (b) 22 (c) 29 (d) 36

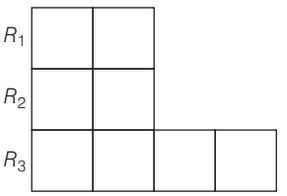
26. A parallelogram is cut by two sets of m lines parallel to its sides. The number of parallelogram thus formed, is

(a) $({}^{m}C_2)^2$ (b) $({}^{m+1}C_2)^2$ (c) $({}^{m+2}C_2)^2$ (d) None of these

27. The number of rectangles excluding squares from a rectangle of size 11×8 is 48λ, then the value of λ is

(a) 13 (b) 23 (c) 43 (d) 53

28. The number of ways the letters of the word PERSON can be placed in the squares of the figure shown so that no row remains empty, is



(a) $24 \times 6!$ (b) $26 \times 6!$ (c) $26 \times 7!$ (d) $27 \times 7!$

Session 7

Rank in a Dictionary, Gap Method

Rank in a Dictionary

The dictionary format means words are arranged in alphabetical order.

Following Examples will help you learn how to find the rank in the dictionary.

Example 112. If the letters of the word are arranged as in dictionary, find the rank of the following words.

(i) RAJU (ii) UMANG

(iii) AIRTEL

Sol. **(i)** In a dictionary, the letters in alphabetical order are A, J, R, U

$\therefore$ The first word is AJRU.

Number of words beginning with A = Number of ways arranging J, R, U = 3 ! = 6

Number of words beginning with J = 3 ! = 6

The next word begin with R and it is RAJU.

$\therefore$ Number of words before RAJU = 12

$\therefore$ Rank of word RAJU = 13

(ii) The letters in alphabetical order are A, G, M, N, U

$\therefore$ The first word is AGMNU

Number of words beginning with A = 4 ! = 24

Number of words beginning with G = 4 ! = 24

Number of words beginning with M = 4 ! = 24

Number of words beginning with N = 4 ! = 24

Number of words beginning with UA = 3 ! = 6

Number of words beginning with UG = 3 ! = 6

Number of words beginning with UMAG = 1 ! = 1

Number of words beginning with UMANG = 1

$\therefore$ Rank of the word

UMANG = 24 + 24 + 24 + 24 + 6 + 6 + 1 + 1 = 110

(iii) The letters in alphabetical order are A, E, I, L, R, T

$\therefore$ The first word is AEILRT

Number of words beginning with AE = 4 ! = 24

Number of words beginning with AIE = 3 ! = 6

Number of words beginning with AIL = 3 ! = 6

Number of words beginning with AIRE = 2 ! = 2

Number of words beginning with AIRL = 2 ! = 2

Number of words beginning with AIRTEL = 1

$\therefore$ Rank of the word AIRTEL= 24 + 6 + 6 + 2 + 2 + 1 = 41

Example 113. If letters of the word are arranged as in dictionary, find the rank of the following words.

(i) INDIA (ii) SURITI (iii) DOCOMO

Sol. (i) The letters in alphabetical order are A, D, I, I, N

$\therefore$ The first word is ADIIN

Number of words beginning with A = $\frac{4!}{2!}$ = 12

Number of words beginning with D = $\frac{4!}{2!}$ = 12

Number of words beginning with IA = 3 ! = 6

Number of words beginning with ID = 3 ! = 6

Number of words beginning with II = 3 ! = 6

Number of words beginning with INA = 2 ! = 2

Number of words beginning with INDA = 1 ! = 1

Number of words beginning with INDIA = 1

$\therefore$ Rank of the word INDIA

= 12 + 12 + 6 + 6 + 6 + 2 + 1 + 1 = 46

(ii) The letters in alphabetical order are I, I, R, S, T, U

$\therefore$ The first word is IIRSTU

Number of words beginning with I = 5 ! = 120

Number of words beginning with R = $\frac{5!}{2!}$ = 60

Number of words beginning with SI = 4 ! = 24

Number of words beginning with SR = $\frac{4!}{2!}$ = 12

Number of words beginning with ST = $\frac{4!}{2!}$ = 12

Number of words beginning with SUI = 3 ! = 6

Number of words beginning with SURII = 1 ! = 1

Number of words beginning with SURITI = 1

$\therefore$ Rank of the word SURITI

= 120 + 60 + 24 + 12 + 12 + 6 + 1 + 1 = 236

(iii) The letters in alphabetical order are C, D, M, O, O, O

$\therefore$ The first word is CDMOOO

Number of words beginning with C = $\frac{5!}{3!}$ = 20

Number of words beginning with DC = $\frac{4!}{3!}$ = 4

Number of words beginning with DM = $\frac{4!}{3!}$ = 4

Number of words beginning with DOCM = $\frac{2!}{2!}$ = 1

Number of words beginning with DOCOMO = 1

$\therefore$ Rank of the word DOCOMO = 20 + 4 + 4 + 1 + 1 = 30

Gap Method

[when particular objects are never together]

Example 114. ***There are 10 candidates for an examination out of which 4 are appearing in Mathematics and remaining 6 are appearing in different subjects. In how many ways can they be seated in a row so that no two Mathematics candidates are together?***

Sol. In this method first arrange the remaining candidates

Here, remaining candidates = 6

$$\times 0 \times 0 \times 0 \times 0 \times 0 \times 0 \times$$

$\times$: Places available for Mathematics candidates

0 : Places for others

Remaining candidates can be arranged in 6 ! ways. There are seven places available for Mathematics candidates so that no two Mathematics candidates are together. Now, four candidates can be placed in these seven places in ${}^{7}P_{4}$ ways.

Hence, the total number of ways $= 6! \times {}^{7}P_{4} = 720 \times 840$

$= 604800$

Example 115. In how many ways can 7 plus (+) and 5 minus (−) signs be arranged in a row so that no two minus (−) signs are together?

Sol. In this method, first arrange the plus (+) signs.

Here, minus (−) signs = 5

$$0 + 0 + 0 + 0 + 0 + 0 + 0 + 0$$

We can put minus (−) sign in any of the 8 places in the above arrangement i.e., we have to select 5 places out of 8 which can be done is ${}^{8}C_{5}$ ways $= {}^{8}C_{3}$ ways $= 56$ ways.

Example 116. Find the number of ways in which 5 girls and 5 boys can be arranged in a row, if no two boys are together.

Sol. In this example, there is no condition for arranging the girls. Now, 5 girls can be arranged in 5 ! ways.

$$\times G \times G \times G \times G \times G \times$$

When girls are arranged, six gaps are generated as shown above with '$\times$'.

Now, boys must occupy the places with '$\times$' marked, so that no two boys are together.

Therefore, five boys can be arranged in these six gaps in ${}^{6}P_{5}$ ways.

Hence, total number of arrangement is $5! \times {}^{6}P_{5}$.

Example 117. Find the number of ways in which 5 girls and 5 boys can be arranged in a row, if boys and girls are alternate.

Sol. First five girls can be arranged in 5 ! ways

i.e., $\times G \times G \times G \times G \times G$ or $G \times G \times G \times G \times G \times$

Now, if girls and boys are alternate, then boys can occupy places with '$\times$' as shows above.

Hence, total number of arrangements is

$$5! \times 5! + 5! \times 5! = 2 \times (5!)^{2}$$

Use of Set Theory

A set is well defined collection of distinct objects.

Subset

If every element of a set A is also an element of a set B, then A is called the subset B, we write

$$A \subset B \Leftrightarrow \{x \in A \Rightarrow x \in B\}$$

Union

The union of two sets A and B is the set of all those elements which are either in A or in B or in both. This set is denoted by $A \cup B$ or $A + B$.

Symbolically, $A \cup B = \{x : x \in A \text{ or } x \in B\}$

Intersection

The intersection of two sets A and B is the set of all elements which are common in A and B. This set is denoted by $A \cap B$ or AB.

Symbolically, $A \cap B = \{x : x \in A \text{ and } x \in B\}$

Example 118. ***A is a set containing n elements. A subset P_1 of A is chosen. The set A is reconstructed by replacing the elements of P_1. Next, a subset P_2 of A is chosen and again the set is reconstructed by replacing the elements of P_2. In this way m (> 1) subsets $P_1, P_2, \ldots, P_m$ of A are chosen. Find the number of ways of choosing $P_1, P_2, \ldots, P_m$, so that***

(i) $P_1 \cap P_2 \cap P_3 \cap \ldots \cap P_m = \phi$

(ii) $P_1 \cup P_2 \cup P_3 \cup \ldots \cup P_m = A$

Sol. Let $A = \{a_1, a_2, a_3, \ldots, a_n\}$

(i) For each a_i $(1 \le i \le n)$, we have either $a \in P_j$ or $a_i \notin P_j$ $(1 \le j \le m)$. i.e., there are 2^m choices in which $a_i (1 \le i \le n)$ may belong to the P_j's.

Out of these, there is only one choice, in which $a_i \in P_j$ for all $j = 1, 2, \ldots, m$ which is not favourable for $P_1 \cap P_2 \cap P_3 \cap \ldots \cap P_m$ to be ϕ. Thus, $a_i \notin P_1 \cap P_2 \cap \ldots \cap P_m$ in $(2^m - 1)$ ways. Since, there are n elements in the set A, the total number of choices is $(2^m - 1)^n$.

(ii) There is exactly one choice, in which, $a_i \in P_j$ for all $j = 1, 2, 3, ..., m$ which is not favourable for $P_1 \cup P_2 \cup P_3 \cup ... \cup P_m$ to be equal to A. Thus, a_i can belong to $P_1 \cup P_2 \cup P_3 \cup ... \cup P_m$ in $(2^m - 1)$ ways.

Since, there are n elements in the set A, the number of ways in which $P_1 \cup P_2 \cup P_3 \cup ... \cup P_m$ can be equal to A is $(2^m - 1)^n$.

Example 119. ***A is a set containing n elements. A subset P of A is chosen. The set A is reconstructed by replacing the elements of P. A subset of A is again chosen. Find the number of ways of choosing P and Q, so that***

(i) $P \cap Q$ contains exactly r elements.

(ii) $P \cap Q$ contains exactly 2 elements.

(iii) $P \cap Q = \phi$

Sol. Let $A = \{a_1, a_2, a_3, ..., a_n\}$

(i) The r elements in P and Q such that $P \cap Q$ can be chosen out of n is ${}^nC_r$ ways a general element of A must satisfy one of the following possibilities [here, general element be $a_i (1 \le i \le n)$]

(i) $a_i \in P$ and $a_i \in Q$

(ii) $a_i \in P$ and $a_i \notin Q$

(iii) $a_i \in P$ and $a_i \in Q$

(iv) $a_i \notin P$ and $a_i \notin Q$

Let $a_1, a_2, ..., a_r \in P \cap Q$

There is only one choice each of them (i.e., (i) choice) and three choices (ii), (iii) and (iv) for each of remaining $(n - r)$ elements.

Hence, number of ways of remaining elements $= 3^{n-r}$

Hence, number of ways in which $P \cap Q$ contains exactly r elements $= {}^nC_r \times 3^{n-r}$

(i) Put $r = 2$, then ${}^nC_2 \times 3^{n-2}$

(iii) Put $r = 0$, then ${}^nC_0 \times 3^n = 3^n$

Sum of digits

(i) The sum of the digits in the unit's place of all numbers formed with the help of $a_1, a_2, ..., a_n$ taken all at a time is $(n-1)!(a_1 + a_2 + ... + a_n)$
(repetition of digits not allowed)

Example 120. Find the sum of the digits in the unit's place of all numbers formed with the help of 3, 4, 5, 6 taken all at a time.

Sol. Sum of the digits in the unit's place
$= (4-1)!(3+4+5+6) = 6 \times 18 = 108$

(ii) The sum of all digit numbers that can be formed using the digits $a_1, a_2, ..., a_n$ (repetition of digits not allowed)

is $= (n-1)!(a_1 + a_2 + ... + a_n)\dfrac{(10^n - 1)}{9}$

Example 121. ***Find the sum of all five digit numbers that can be formed using the digits 1, 2, 3, 4 and 5 (repetition of digits not allowed)***

Sol. Required sum $= (5-1)!(1+2+3+4+5)\left(\dfrac{10^5 - 1}{9}\right)$

$= 24 \cdot 15 \cdot 11111 = 3999960$

Aliter

Since, one of the numbers formed with the 5 digits a, b, c, d and e is $10^4 a + 10^3 b + 10^2 c + 10d + e$;

Hence, $10^4 a$ will occur altogether in 4 ! ways similarly each of $10^4 b, 10^4 c, 10^4 d, 10^4 e$ will occur in 4 ! ways.

Hence, if all the numbers formed with the digits be written one below the other, thus

$$10^4 \cdot a + 10^3 \cdot b + 10^2 \cdot c + 10 \cdot d + e$$

$$10^4 \cdot b + 10^3 \cdot c + 10^2 \cdot d + 10 \cdot e + a$$

$$10^4 \cdot c + 10^3 \cdot d + 10^2 \cdot e + 10 \cdot a + b$$

$$10^4 \cdot d + 10^3 \cdot e + 10^2 \cdot a + 10 \cdot b + c$$

$$10^4 \cdot e + 10^3 \cdot a + 10^2 \cdot b + 10 \cdot c + d$$

Hence, the required sum

$= 4! \times (a + b + c + d + e) \times (10^4 + 10^3 + 10^2 + 10 + 1)$

$= 4! \times (1 + 2 + 3 + 4 + 5)(11111) = 3999960$

Difference between Permutation and Combination

Problems of permutations	Problems of combinations
1. Arrangements	Selections, choose
2. Standing in a line, seated in a row	Distributed group is formed
3. Problems on digits	Committee
4. Problems on letters from a word	Geometrical problems

Exercise for Session 7

1. The letters of the word ‘‘DELHI’’ are arranged in all possible ways as in a dictionary, the rank of the word ‘‘DELHI’’ is
   (a) 4 (b) 5
   (c) 6 (d) 7

2. The letters of the word ‘‘KANPUR’’ are arranged in all possible ways as in a dictionary, the rank of the word ‘‘KANPUR’’ from last is
   (a) 121 (b) 122
   (c) 598 (d) 599

3. The letters of the word ‘‘MUMBAI’’ are arranged in all possible ways as in a dictionary, the rank of the word ‘‘MUMBAI’’ is
   (a) 297 (b) 295
   (c) 299 (d) 301

4. The letters of the word ‘‘CHENNAI’’ are arranged in all possible ways as in a dictionary, then rank of the word ‘‘CHENNAI’’ from last is
   (a) 2016 (b) 2017
   (c) 2018 (d) 2019

5. If all permutations of the letters of the word ‘‘AGAIN’’ are arranged as in a dictionary, then 50th word is
   (a) NAAGI (b) NAGAI
   (c) NAAIG (d) NAIAG

Shortcuts and Important Results to Remember

1 When two dice are thrown, the number of ways of getting a total r (sum of numbers on upper faces), is

(i) $r-1$, if $2 \le r \le 7$

(ii) $13-r$, if $8 \le r \le 12$

2 When three dice are thrown, the number of ways of getting a total r (sum of numbers on upper faces), is

(i) ${}^{r-1}C_2$, if $3 \le r \le 8$

(ii) 25, if $r = 9$

(iii) 27, if $r = 10, 11$

(iv) 25, if $r = 12$

(v) ${}^{20-r}C_2$, if $13 \le r \le 18$

3 The product of k consecutive positive integers is divisible by $k!$.

4 Number of zeroes in $n! = E_5(n!)$

5 n straight lines are drawn in the plane such that no two lines are parallel and no three lines are concurrent. Then, the number of parts into which these lines divides the plane is equal to $\frac{(n^2+n+2)}{2}$.

6 ${}^nC_r$ is divisible by n only, if n is a prime number $(1 \le r \le n-1)$.

7 The number of diagonals in n-gon (n sides closed polygon) is $\frac{n(n-3)}{2}$.

8 In n-gon no three diagonals are concurrent, then the total number of points of intersection of diagonals interior to the polygon is ${}^nC_4$.

9 Consider a polygon of n sides, then number of triangles in which no side is common with that of the polygon are $\frac{1}{6}n(n-4)(n-5)$.

10 If m parallel lines in a plane are intersected by a family of other n parallel lines. The total number of parallelograms so formed $= {}^mC_2 \cdot {}^nC_2 = \frac{mn(m-1)(n-1)}{4}$

11 Highest power of prime p in ${}^nC_r$, since

$$ {}^nC_r = \frac{n!}{r!(n-r)!} $$

If $H_p(n!) = \alpha$,

$H_p(r!) = \beta$

and $H_p\{(n-r)!\} = \gamma$

Then, $H_p({}^nC_r) = \alpha - (\beta + \gamma)$

12 Highest power of prime p in ${}^nP_r$, since

$$ {}^nP_r = \frac{n!}{(n-r)!} $$

If $H_p(n!) = \lambda$, $H_p\{(n-r)!\} = \mu$. Then, $H_p({}^nP_r) = \lambda - \mu$

13 If there are n rows. Ist row has $m_1$squares, IInd row has m_2 squares, IIIrd row has m_3 squares and so on. If we placed λ X's in the squares such that each row contains atleast one X. Then the number of ways = Coefficient of x^λ in

$({}^{m_1}C_1 x + {}^{m_1}C_2 x^2 + \ldots + {}^{m_1}C_{m_1} x^{m_1})$

$\times ({}^{m_2}C_1 x + {}^{m_2}C_2 x^2 + {}^{m_2}C_3 x^3 + \ldots + {}^{m_2}C_{m_2} x^{m_2}) \times$

$({}^{m_3}C_1 x + {}^{m_3}C_2 x^2 + \ldots + {}^{m_3}C_{m_3} x^{m_3}) \times \ldots$

If $\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$, $\forall\, x, y, n \in N$

$\Rightarrow \quad (x-n)(y-n) = n^2$

$\therefore \quad x = n + \lambda,$

$y = n + \frac{n^2}{\lambda},$

where λ is divisor of n^2.

Then, number of integral solutions (x, y) is equal to number of divisors of n^2.

If $n = 3$, $n^2 = 9 = 3^2$, the equation has 3 solutions.

$(x, y) = (4, 12), (6, 6), (12, 4)$

JEE Type Solved Examples :
Single Option Correct Type Questions

■ This section contains **10 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

● ***Ex. 1*** *Number of words of* 4 *letters that can be formed with the letters of the word IIT JEE, is*

(a) 42 (b) 82 (c) 102 (d) 142

Sol. (*c*) There are 6 letters I, I, E, E, T, J

The following cases arise:

***Case* I** All letters are different

$$^4P_4 = 4! = 24$$

***Case* II** Two alike and two different

$$^2C_1 \times {}^3C_2 \times \frac{4!}{2!} = 72$$

***Case* III** Two alike of one kind and two alike of another kind

$$^2C_2 \times \frac{4!}{2!\,2!} = 6$$

Hence, number of words = 24 + 72 + 6 = 102

Aliter

Number of words = Coefficient of x^4 in

$$4!\left(1 + \frac{x}{1!} + \frac{x^2}{2!}\right)^2 (1+x)^2$$

$$= \text{Coefficient of } x^4 \text{ in } 6[(1+x)^2 + 1]^2 (1+x)^2$$

$$= \text{Coefficient of } x^4 \text{ in } 6[(1+x)^6 + 2(1+x)^4 + (1+x)^2]$$

$$= 6[{}^6C_4 + 2 \cdot {}^4C_4 + 0] = 6(15+2) = 102$$

● ***Ex. 2*** *Let* y *be element of the set* $A = \{1, 2, 3, 5, 6, 10, 15, 30\}$ *and* x_1, x_2, x_3 *be integers such that* $x_1 x_2 x_3 = y$*, the number of positive integral solutions of* $x_1 x_2 x_3 = y$*, is*

(a) 27 (b) 64 (c) 81 (d) 256

Sol. (*b*) Number of solutions of the given equations is the same as the number of solutions of the equation

$$x_1\, x_2\, x_3\, x_4 = 30 = 2 \times 3 \times 5$$

Here, x_4 is infact a dummy variable.

If $x_1\, x_2\, x_3 = 15$, then $x_4 = 2$ and if $x_1 x_2\, x_3 = 5$, then $x_4 = 6$, etc.

Thus, $x_1\, x_2\, x_3\, x_4 = 2 \times 3 \times 5$

Each of 2, 3 and 5 will be factor of exactly one of x_1, x_2, x_3, x_4 in 4 ways.

∴ Required number = $4^3 = 64$

● ***Ex. 3*** *The number of positive integer solutions of* $a + b + c = 60$*, where* a *is a factor of* b *and* c*, is*

(a) 184 (b) 200 (c) 144 (d) 270

Sol. (*c*) ∵ a is a factor of b and $c \Rightarrow a$ divides 60

∴ $a = 1, 2, 3, 4, 5, 6, 10, 12, 15, 30$ [∵ $a \neq 60$]

and $b = ma$, $c = na$, when $m, n \geq 1$

∵ $a + b + c = 60$

$\Rightarrow a + ma + na = 60 \Rightarrow m + n = \left(\frac{60}{a} - 1\right)$

∴ Number of solutions = $^{\frac{60}{a}-1-1}C_{2-1} = \left(\frac{60}{a} - 2\right)$

Hence, total number of solutions for all values of a

$$= 58 + 28 + 18 + 13 + 10 + 8 + 4 + 3 + 2 + 0 = 144$$

● ***Ex. 4*** *The number of times the digit* 3 *will be written when listing the integers from* 1 *to* 1000*, is*

(a) 269 (b) 271 (c) 300 (d) 302

Sol. (*c*) Since, 3 does not occur in 1000. So, we have to count the number of times 3 occurs, when we list the integers from 1 to 999.

Any number between 1 and 999 is of the form xyz, where $0 \leq x, y, z \leq 9$.

Let us first count the number in which 3 occurs exactly once. Since, 3 can occur at one place in $^3C_1$ ways, there are $^3C_1 \times 9 \times 9 = 243$ such numbers. Next 3 can occur in exactly two places in $^3C_2 \times 9 = 27$ such numbers. Lastly, 3 can occur in all three digits in one number only. Hence, the number of times, 3 occurs is $1 \times 243 + 2 \times 27 + 3 \times 1 = 300$

● ***Ex. 5*** *Number of points having position vector* $a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$*, where* $a, b, c \in \{1, 2, 3, 4, 5\}$ *such that* $2^a + 3^b + 5^c$ *is divisible by* 4*, is*

(a) 70 (b) 140

(c) 210 (d) 280

Sol. (*a*) ∵ $2^a + 3^b + 5^c = 2^a + (4-1)^b + (4+1)^c$

$$= 2^a + 4k + (-1)^b + (1)^c$$

$$= 2^a + 4k + (-1)^b + 1$$

I. $a = 1$, b = even, c = any number

II. $a \neq 1$, b = odd, c = any number

∴ Required number of ways = $1 \times 2 \times 5 + 4 \times 3 \times 5 = 70$

[∵ even numbers = 2, 4; odd numbers = 1, 3, 5 and any numbers = 1, 2, 3, 4, 5]

● ***Ex. 6*** *Number of positive unequal integral solutions of the equation* $x + y + z = 12$ *is*

(a) 21 (b) 42 (c) 63 (d) 84

Sol. (*b*) We have, $x + y + z = 12$ ...(i)

Assume $x < y < z$. Here, $x, y, z \geq 1$

$\therefore$ Solutions of Eq. (i) are
(1, 2, 9), (1, 3, 8), (1, 4, 7), (1, 5, 6), (2, 3, 7), (2, 4, 6) and (3, 4, 5).

Number of positive integral solutions of Eq. (i) = 7 but x, y, z can be arranged in 3! = 6

Hence, required number of solutions = $7 \times 6 = 42$

Aliter

Let $x = \alpha, y - x = \beta, z - y = \gamma$

$\therefore\ x = \alpha, y = \alpha + \beta, z = \alpha + \beta + \gamma$

From Eq. (i), $3\alpha + 2\beta + \gamma = 12; \alpha, \beta, \gamma \geq 1$

$\therefore$ Number of positive integral solutions of Eq. (i)

= Coefficient of λ^{12} in

$$(\lambda^3 + \lambda^6 + \lambda^9 + \lambda^{12} + \ldots)$$
$$(\lambda^2 + \lambda^4 + \lambda^6 + \lambda^8 + \lambda^{10} + \lambda^{12} + \ldots)$$
$$(\lambda + \lambda^2 + \lambda^3 + \ldots + \lambda^{12})$$

= Coefficient of λ^6 in $(1 + \lambda^3 + \lambda^6)(1 + \lambda^2 + \lambda^4 + \lambda^6)$
$$(1 + \lambda + \lambda^2 + \lambda^3 + \lambda^4 + \lambda^5 + \lambda^6)$$

= Coefficient of λ^6 in $(1 + \lambda^2 + \lambda^4 + \lambda^6 + \lambda^3 + \lambda^5 + \lambda^6)$
$$\times (1 + \lambda + \lambda^2 + \lambda^3 + \lambda^4 + \lambda^5 + \lambda^6)$$

$= 1 + 1 + 1 + 1 + 1 + 1 + 1 = 7$

but x, y, z can be arranged in 3! = 6

Hence, required number of solutions = $7 \times 6 = 42$

● ***Ex. 7*** *12 boys and 2 girls are to be seated in a row such that there are atleast 3 boys between the 2 girls. The number of ways this can be done is $\lambda \times 12!$, the value of λ is*

(a) 55 (b) 110 (c) 20 (d) 45

Sol. (*b*) Let $P =$ Number of ways, 12 boys and 2 girls are seated in a row

$$= 14! = 14 \times 13 \times 12! = 182 \times 12!$$

$P_1 =$ Number of ways, the girls can sit together
$$= (14 - 2 + 1) \times 2! \times 12! = 26 \times 12!$$

$P_2 =$ Number of ways, one boy sits between the girls
$$= (14 - 3 + 1) \times 2! \times 12! = 24 \times 12!$$

$P_3 =$ Number of ways, two boys sit between the girls
$$= (14 - 4 + 1) \times 2! \times 12! = 22 \times 12!$$

$\therefore$ Required number of ways $= (182 - 26 - 24 - 22) \times 12!$
$$= 110 \times 12! = \lambda \times 12! \quad \text{[given]}$$

$\therefore\quad \lambda = 110$

● ***Ex. 8*** *A is a set containing n elements. A subset P of A is chosen. The set A is reconstructed by replacing the elements of P. A subset Q of A is again chosen, the number of ways of choosing so that $(P \cup Q)$ is a proper subset of A, is*

(a) 3^n (b) 4^n (c) $4^n - 2^n$ (d) $4^n - 3^n$

Sol. (*d*) Let $A = \{a_1, a_2, a_3, \ldots, a_n\}$

a general element of A must satisfy one of the following possibilities.

[here, general element be $a_i (1 \leq i \leq n)$]

(i) $a_i \in P, a_i \in Q$ (ii) $a_i \in P, a_i \notin Q$

(iii) $a_i \notin P, a_i \in Q$ (iv) $a_i \notin P, a_i \notin Q$

Therefore, for one element a_i of A, we have four choices (i), (ii), (iii) and (iv).

$\therefore$ Total number of cases for all elements $= 4^n$

and for one element a_i of A, such that $a_i \in P \cup Q$, we have three choices (i), (ii) and (iii).

$\therefore$ Number of cases for all elements belong to $P \cup Q = 3^n$

Hence, number of ways in which atleast one element of A does not belong to

$$P \cup Q = 4^n - 3^n.$$

● ***Ex. 9*** *Let N be a natural number. If its first digit (from the left) is deleted, it gets reduced to $\frac{N}{29}$. The sum of all the digits of N is*

(a) 14 (b) 17
(c) 23 (d) 29

Sol. (*a*) Let $N = a_n\ a_{n-1}\ a_{n-2} \ldots a_3\ a_2\ a_1\ a_0$

$$= a_0 + 10a_1 + 10^2 a_2 + \ldots + 10^{n-1} a_{n-1} + 10^n a_n \quad \ldots(i)$$

Then, $\frac{N}{29} = a_{n-1}\ a_{n-2}\ a_{n-3} \ldots a_3\ a_2\ a_1\ a_0$

$$= a_0 + 10a_1 + 10^2 a_2 + \ldots + 10^{n-2} a_{n-2} + 10^{n-1} a_{n-1}$$

or $\quad N = 29(a_0 + 10a_1 + 10^2 a_2 + \ldots$
$$+ 10^{n-2} a_{n-2} + 10^{n-1} a_{n-1}) \quad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$10^n \cdot a_n = 28(a_0 + 10a_1 + 10^2 a_2 + \ldots + 10^{n-1} a_{n-1})$$

$\Rightarrow$ 28 divides $10^n \cdot a_n \Rightarrow a_n = 7, n \geq 2 \Rightarrow 5^2 = a_0 + 10a_1$

The required N is 725 or 7250 or 72500, etc.

$\therefore$ The sum of the digits is 14.

● ***Ex. 10*** *If the number of ways of selecting n cards out of unlimited number of cards bearing the number 0, 9, 3, so that they cannot be used to write the number 903 is 93, then n is equal to*

(a) 3 (b) 4
(c) 5 (d) 6

Sol. (*c*) We cannot write 903.

If in the selection of n cards, we get either

(9 or 3), (9 or 0), (0 or 3), (only 0), (only 3) or (only 9).

For (9 or 3) can be selected $= 2 \times 2 \times 2 \times \ldots \times n$ factors $= 2^n$

Similarly, (9 or 0) or (0 or 3) can be selected $= 2^n$

In the above selection (only 0) or (only 3) or (only 9) is repeated twice.

$\therefore$ Total ways $= 2^n + 2^n + 2^n - 3 = 93$

$\Rightarrow\quad 3 \cdot 2^n = 96 \Rightarrow\ 2^n = 32 = 2^5$

$\therefore\quad n = 5$

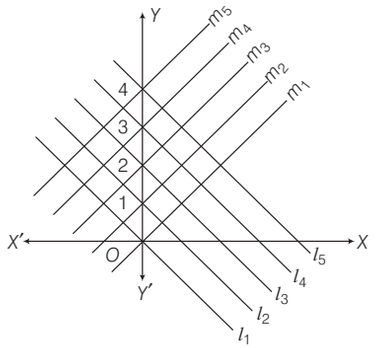
JEE Type Solved Examples :
More than One Correct Option Type Questions

- This section contains **5 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of **which more than one** may be correct.

● ***Ex. 11*** *In a plane, there are two families of lines* $y = x + r$, $y = -x + r$, *where* $r \in \{0, 1, 2, 3, 4\}$. *The number of squares of diagonals of the length 2 formed by the lines is*

(a) 9 (b) 16 (c) $\frac{3}{2} \cdot {}^4C_2$ (d) ${}^5C_2 + {}^3P_2$

Sol. (a, c) There are two sets of five parallel lines at equal distances. Clearly, lines like l_1, l_3, m_1 and m_3 form a square whose diagonal's length is 2.



$\therefore$ The number of required squares $= 3 \times 3 = 9 = \frac{3}{2} \cdot {}^4C_2$

[$\because$ choices are (l_1, l_2), (l_2, l_4) and (l_3, l_5) for one set, etc.]

● ***Ex. 12*** *Number of ways in which three numbers in AP can be selected from* 1, 2, 3, …, n, *is*

(a) $\left(\frac{n-1}{2}\right)^2$, if n is even (b) $\frac{n(n-2)}{4}$, if n is even

(c) $\frac{(n-1)^2}{4}$, if n is odd (d) $\frac{n(n+1)}{2}$, if n is odd

Sol. (b, c) If a, b, c are in AP, then $a + c = 2b$

a and c both are odd or both are even.

***Case* I** If n **is even**

Let $n = 2m$ in which m are even and m are odd numbers.

$\therefore$ Number of ways $= {}^mC_2 + {}^mC_2 = 2 \cdot {}^mC_2 = 2 \cdot \frac{m(m-1)}{2}$

$= \frac{n}{2}\left(\frac{n}{2} - 1\right) = \frac{n(n-2)}{4}$ [$\because n = 2m$]

***Case* II** If n **is odd**

Let $n = 2m + 1$ in which m are even and $m + 1$ are odd numbers.

$\therefore$ Number of ways $= {}^mC_2 + {}^{m+1}C_2$

$= \frac{m(m-1)}{2} + \frac{(m+1)m}{2} = m^2 = \frac{(n-1)^2}{4}$ [$\because n = 2m + 1$]

● ***Ex. 13*** *If n objects are arranged in a row, then number of ways of selecting three of these objects so that no two of them are next to each other, is*

(a) ${}^{n-2}C_3$ (b) ${}^{n-3}C_3 + {}^{n-3}C_2$

(c) $\frac{(n-2)(n-3)(n-4)}{6}$ (d) ${}^nC_2$

Sol. (a, b, c) Let a_0 be the number of objects to the left of the first object chosen, a_1 be the number of objects between the first and the second, a_2 be the number of objects between the second and the third and a_3 be the number of objects to the right of the third object. Then,

$a_0, a_3 \geq 0$ and $a_1, a_2 \geq 1$

also $a_0 + a_1 + a_2 + a_3 = n - 3$

Let $a = a_0 + 1$, $b = a_3 + 1$, then $a \geq 1$, $b \geq 1$ such that $a + a_1 + a_2 + b = n - 1$

The total number of positive integral solutions of this equation is ${}^{n-1-1}C_{4-1} = {}^{n-2}C_3 = {}^{n-3}C_3 + {}^{n-3}C_2$

$= \frac{(n-2)(n-3)(n-4)}{1 \cdot 2 \cdot 3}$

● ***Ex. 14*** *Given that the divisors of* $n = 3^p \cdot 5^q \cdot 7^r$ *are of the form* $4\lambda + 1$, $\lambda \geq 0$. *Then,*

(a) $p + r$ is always even (b) $p + q + r$ is even or odd

(c) q can be any integer (d) if p is even, then r is odd

Sol. (a, b, c)

$\because$ $3^p = (4-1)^p = 4\lambda_1 + (-1)^p$,

$5^q = (4+1)^q = 4\lambda_2 + 1$

and $7^r = (8-1)^r = 8\lambda_3 + (-1)^r$

Hence, both p and r must be odd or both must be even. Thus, $p + r$ is always even. Also, $p + q + r$ can be odd or even.

● ***Ex. 15*** *Number of ways in which* 15 *identical coins can be put into* 6 *different bags*

(a) is coefficient of x^{15} in $x^6(1 + x + x^2 + \ldots \infty)^6$, if no bag remains empty

(b) is coefficient of x^{15} in $(1 - x)^{-6}$

(c) is same as number of the integral solutions of $a + b + c + d + e + f = 15$

(d) is same as number of non-negative integral solutions of $\sum_{i=1}^{6} x_i = 15$

Sol. (a, b, d) Let bags be x_1, x_2, x_3, x_4, x_5 and x_6, then $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 15$.

$\therefore$ For no bags remains empty, number of ways

$=$ Coefficient of x^{15} in $(x^1 + x^2 + x^3 + \ldots \infty)^6$

$=$ Coefficient of x^{15} in $x^6(1 + x + x^2 + \ldots \infty)^6$

$=$ Coefficient of x^9 in $(1 - x)^{-6}$

In option (c), it is not mentioned that solution is positive integral.

JEE Type Solved Examples :
Passage Based Questions

■ This section contains **3 solved passages** based upon each of the passage **3 multiple choice** examples have to be answered. Each of these examples has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I
(Ex. Nos. 16 to 18)

All the letters of the word 'AGAIN' be arranged and the words thus formed are known as 'Simple Words'. Further two new types of words are defined as follows:

(i) ***Smart word:*** *All the letters of the word 'AGAIN' are being used, but vowels can be repeated as many times as we need.*

(ii) ***Dull word:*** *All the letters of the word 'AGAIN' are being used, but consonants can be repeated as many times as we need.*

16. If a vowel appears in between two similar letters, the number of simple words is
(a) 12 (b) 6 (c) 36 (d) 14

17. Number of 7 letter smart words is
(a) 1500 (b) 1050 (c) 1005 (d) 150

18. Number of 7 letter dull words in which no two vowels are together, is
(a) 402 (b) 420 (c) 840 (d) 42

Sol.

16. (*b*)

N	A-I-A	G

$\therefore$ Required number of simple words $= 3! = 6$

17. (*b*)

A	G	A	I	N	A	A
					I	I
					A	I

$\therefore$ Number of 7 letter smart words

$$= \frac{7!}{4!} + \frac{7!}{2!\,3!} + \frac{7!}{3!\,2!} = 210 + 420 + 420 = 1050$$

18. (*b*) Now, 3 vowels A, I, A are to be placed in the five available places.

$$\begin{cases} \times\,N\times\,G\times\,N\times\,N\times \\ \quad \text{OR} \\ \times\,N\times\,G\times\,G\times\,G\times \\ \quad \text{OR} \\ \times\,N\times\,G\times\,G\times\,N\times \end{cases}$$

Hence, required number of ways

$$= {}^5C_3 \times \frac{3!}{2!} \times \left\{\frac{4!}{3!} + \frac{4!}{3!} + \frac{4!}{2!\,2!}\right\}$$

$$= 30(4 + 4 + 6) = 420$$

Passage II (Ex. Nos. 19 to 21)

Consider a polygon of sides 'n' which satisfies the equation $3 \cdot {}^nP_4 = {}^{n-1}P_5$.

19. Rajdhani express travelling from Delhi to Mumbai has n stations enroute. Number of ways in which a train can be stopped at 3 stations if no two of the stopping stations are consecutive, is
(a) 20 (b) 35 (c) 56 (d) 84

20. Number of quadrilaterals that can be formed using the vertices of a polygon of sides 'n' if exactly 1 side of the quadrilateral is common with side of the n-gon, is
(a) 96 (b) 100 (c) 150 (d) 156

21. Number of quadrilaterals that can be made using the vertices of the polygon of sides 'n' if exactly two adjacent sides of the quadrilateral are common to the sides of the n-gon, is
(a) 50 (b) 60 (c) 70 (d) 80

Sol. $\because \quad 3 \cdot {}^nP_4 = {}^{n-1}P_5$

It is clear that $n \geq 6$.

$\therefore \quad 3 \cdot n(n-1)(n-2)(n-3) = (n-1)(n-2)(n-3)(n-4)(n-5)$

$\Rightarrow \quad (n-1)(n-2)(n-3)(n^2 - 12n + 20) = 0$

$\Rightarrow \quad (n-1)(n-2)(n-3)(n-10)(n-2) = 0$

$\therefore \quad n = 10, n \neq 1, 2, 3 \qquad [\because n \geq 6]$

$\Rightarrow \quad n = 10$

19. (*d*) Let a_0 be the number of stations to the left of the station I chosen, a_1 be the number of stations between the station I and station II, a_2 be the number of stations between the station II and station III and a_3 be the number of stations to the right of the third station. Then,

$a_0, a_3 \geq 0$ and $a_1, a_2 \geq 1$

Also, $a_0 + a_1 + a_2 + a_3 = n + 1 - 3$

Let $a = a_0 + 1, b = a_3 + 1$, then $a, b \geq 1$ such that

$a + a_1 + a_2 + b = n$

$\therefore$ Required number of ways $= {}^{n-1}C_{4-1} = {}^9C_3$ [here, $n = 10$]

$= 84$

20. (*c*) Number of quadrilaterals of which exactly one side is the side of the n-gon

$= n \times {}^{n-4}C_2 = 10 \times {}^6C_2 = 150 \qquad [\because n = 10]$

21. (*a*) Number of quadrilaterals of which exactly two adjacent sides of the quadrilateral are common to the sides of the *n*-gon

$$= n \times {}^{n-5}C_1 = n(n-5) = 10 \times 5 \qquad [\because n = 10]$$
$$= 50$$

Passage III
(Ex. Nos. 22 to 23)

Consider the number $N = 2016$.

22. Number of cyphers at the end of ${}^{N}C_{N/2}$ is

(a) 0 (b) 1
(c) 2 (d) 3

23. Sum of all even divisors of the number N is

(a) 6552 (b) 6448
(c) 6048 (d) 5733

Sol.

22. (*c*) $\because \quad {}^{N}C_{N/2} = {}^{2016}C_{1008} = \dfrac{(2016)!}{[(1008)!]^2}$

$$E_5(2016!) = \left[\frac{2016}{5}\right] + \left[\frac{2016}{5^2}\right] + \left[\frac{2016}{5^3}\right] + \left[\frac{2016}{5^4}\right]$$
$$= 403 + 80 + 16 + 3 = 502$$

and $\quad E_5(1008!) = \left[\dfrac{1008}{5}\right] + \left[\dfrac{1008}{5^2}\right] + \left[\dfrac{1008}{5^3}\right] + \left[\dfrac{1008}{5^4}\right]$

$$= 201 + 40 + 8 + 1 = 250$$

Hence, the number of cyphers at the end of ${}^{2016}C_{1008}$

$$= 502 - 250 - 250 = 2$$

23. (*b*) $\because \quad N = 2016 = 2^5 \cdot 3^2 \cdot 7^1$

$\therefore$ Sum of all even divisors of the number N

$$= (2 + 2^2 + 2^3 + 2^4 + 2^5)(1 + 3 + 3^2)(1 + 7^1) = 6448$$

JEE Type Solved Examples :
Single Integer Answer Type Questions

- This section contains **2 examples.** The answer to each example is a **single digit integer** ranging from **0** to **9** (both inclusive).

● ***Ex. 24*** *If* $\binom{18}{r-2} + 2\binom{18}{r-1} + \binom{18}{r} \geq \binom{20}{13}$, *then the number of values of r are*

Sol. (7) We have, $\binom{18}{r-2} + 2\binom{18}{r-1} + \binom{18}{r} \geq \binom{20}{13}$

It means that ${}^{18}C_{r-2} + 2 \cdot {}^{18}C_{r-1} + {}^{18}C_r \geq {}^{20}C_{13}$

$\Rightarrow \quad ({}^{18}C_{r-2} + {}^{18}C_{r-1}) + ({}^{18}C_{r-1} + {}^{18}C_r) \geq {}^{20}C_7$

$\Rightarrow \quad {}^{19}C_{r-1} + {}^{19}C_r \geq {}^{20}C_7$

$\Rightarrow \quad {}^{20}C_r \geq {}^{20}C_7$

or $\quad {}^{20}C_r \geq {}^{20}C_{13}$

$\Rightarrow \quad 7 \leq r \leq 13$

$\therefore \quad r = 7, 8, 9, 10, 11, 12, 13$

Hence, the number of values of r are 7.

● ***Ex. 25*** *If* λ *be the number of* 3*-digit numbers are of the form xyz with* $x < y$, $z < y$ *and* $x \neq 0$, *the value of* $\dfrac{\lambda}{30}$ *is*

Sol. (8) Since, $x \geq 1$, then $y \geq 2$ $\qquad [\because x < y\ ]$

If $y = n$, then x takes values form 1 to $n-1$ and z can take the values from 0 to $n-1$ (i.e., n values).

Thus, for each values of $y (2 \leq y \leq 9)$, x and z take $n(n-1)$ values.

Hence, the 3-digit numbers are of the form xyz

$$= \sum_{n=2}^{9} n(n-1) = \sum_{n=1}^{9} n(n-1) \qquad [\because \text{at } n = 1,\ n(n-1) = 0]$$
$$= \sum_{n=1}^{9} n^2 - \sum_{n=1}^{9} n$$
$$= \frac{9(9+1)(18+1)}{6} - \frac{9(9+1)}{2}$$
$$= 285 - 45$$
$$= 240 = \lambda \qquad [\text{given}]$$

$\therefore \quad \dfrac{\lambda}{30} = 8$

JEE Type Solved Examples :
Matching Type Questions

- This section contains **2 examples.** Examples 26 and 27 have four statements (A, B and C) given in **Column I** and four statements (p, q, r and s) in **Column II.** Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II.**

● *Ex. 26*

	Column I	Column II	
(A)	The sum of the factors of 8! which are odd and are the form $3\lambda + 2$, $\lambda \in N$, is	(p)	384
(B)	The number of divisors of $n = 2^7 \cdot 3^5 \cdot 5^3$ which are the form $4\lambda + 1$, $\lambda \in N$, is	(q)	240
(C)	Total number of divisors of $n = 2^5 \cdot 3^4 \cdot 5^{10} \cdot 7^6$ which are the form $4\lambda + 2$, $\lambda \geq 1$, is	(r)	11
(D)	Total number of divisors of $n = 3^5 \cdot 5^7 \cdot 7^9$ which are the form $4\lambda + 1$, $\lambda \geq 0$, is	(s)	40

Sol. (A) $\rightarrow$ *s*; (B) $\rightarrow$ *r*; (C) $\rightarrow$ *p*; (D) $\rightarrow$ *q*

(A) Here, $8! = 2^7 \cdot 3^2 \cdot 5^1 \cdot 7^1$

So, the factors may be 1, 5, 7, 35 of which 5 and 35 are of the form $3\lambda + 2$.

$\therefore$ Sum is 40.

(B) Number of odd numbers $= (5 + 1)(3 + 1) = 24$

Required number = 12, but 1 is included.

$\therefore$ Required number of numbers $= 12 - 1 = 11$ of the form $4\lambda + 1$.

(C) Here, $4\lambda + 2 = 2(2\lambda + 1)$

$\therefore$ Total divisors $= 1 \cdot 5 \cdot 11 \cdot 7 - 1 = 384$

[$\because$ one is subtracted because there will be case when selected powers of 3, 5 and 7 are zero]

(D) Here, any positive integer power of 5 will be in the form of $4\lambda + 1$ when even powers of 3 and 7 will be in the form of $4\lambda + 1$ and odd powers of 3 and 7 will be in the form of $4\lambda - 1$.

$\therefore$ Required divisors $= 8(3 \cdot 5 + 3 \cdot 5) = 240$

● *Ex. 27*

	Column I	Column II	
(A)	Four dice (six faced) are rolled. The number of possible outcomes in which atleast one die shows 2, is	(p)	210
(B)	Let A be the set of 4-digit numbers $a_1a_2a_3a_4$, where $a_1 > a_2 > a_3 > a_4$. Then, $n(A)$ is equal to	(q)	480
(C)	The total number 3-digit numbers, the sum of whose digits is even, is equal to	(r)	671
(D)	The number of 4-digit numbers that can be formed from the digits 0, 1, 2, 3, 4, 5, 6, 7, so that each number contains digit 1, is	(s)	450

Sol. (A) $\rightarrow$ *r*; (B) $\rightarrow$ *p*; (C) $\rightarrow$ *s*; (D) $\rightarrow$ *q*

(A) The number of possible outcomes with 2 on atleast one die = The total number of outcomes with 2 on atleast one die = (The total number of outcomes) – (The number of outcomes in which 2 does not appear on any dice) $= 6^4 - 5^4 = 1296 - 625 = 671$

(B) Any selection of four digits from the 10 digits 0, 1, 2, 3,..., 9 gives one number. So, the required number of numbers is ${}^{10}C_4$ i.e., 210.

(C) Let the number be $n = pqr$. Since, $p + q + r$ is even, p can be filled in 9 ways and q can be filled in 10 ways.
r can be filled in number of ways depending upon what is the sum of p and q.
If $(p + q)$ is odd, then r can be filled with any one of five odd digits.
If $(p + q)$ is even, then r can be filled with any one of five even digits.
In any case, r can be filled in five ways.
Hence, total number of numbers is $9 \times 10 \times 5 = 450$

(D) After fixing 1 at one position out of 4 places, 3 places can be filled by ${}^7P_3$ ways. But for some numbers whose fourth digit is zero, such type of ways is ${}^6P_2$. Therefore, total number of ways is ${}^7P_3 - {}^6P_2 = 480$

JEE Type Solved Examples :
Statement I and II Type Questions

■ **Directions** Example numbers 28 and 29 are Assertion-Reason type examples. Each of these examples contains two statements:
Statement-1 (Assertion) and **Statement-2** (Reason)
Each of these examples also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below.
(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement-1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

● ***Ex. 28*** **Statement-1** *Number of rectangles on a chessboard is* ${}^{8}C_2 \times {}^{8}C_2$.

Statement-2 *To form a rectangle, we have to select any two of the horizontal lines and any two of the vertical lines.*

Sol. (*d*) In a chessboard, there are 9 horizontal lines and 9 vertical lines.
∴ Number of rectangles of any size are ${}^{9}C_2 \times {}^{9}C_2$.
Hence, Statement-1 is false and Statement-2 is true.

● ***Ex. 29*** **Statement-1** *If* $f : \{a_1, a_2, a_3, a_4, a_5\} \to \{a_1, a_2, a_3, a_4, a_5\}$, *f is onto and* $f(x) \neq x$ *for each* $x \in \{a_1, a_2, a_3, a_4, a_5\}$, *is equal to* 44.

● **Statement-2** *The number of derangement for n objects is* $n!\sum_{r=0}^{n}\frac{(-1)^r}{r!}$.

Sol. (*a*) $\because D_n = n!\sum_{r=0}^{n}\frac{(-1)^r}{r!} = n!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\ldots+\frac{(-1)^n}{n!}\right)$

$$\therefore \quad D_5 = 5!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}\right)$$
$$= 120\left(\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-\frac{1}{120}\right)$$
$$= 6-20+5-1$$
$$= 65-21$$
$$= 44$$

Hence, Statement-1 is true, Statement-2 is true and Statement-2 is a correct explanation for Statement-1.

Subjective Type Examples

■ In this section, there are **17 subjective** solved examples.

● ***Ex. 30*** *Solve the inequality*

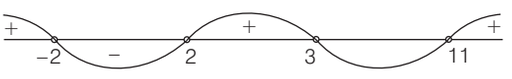
$${}^{x-1}C_4 - {}^{x-1}C_3 - \frac{5}{4}{}^{x-2}A_2 < 0, x \in N.$$

Sol. We have, ${}^{x-1}C_4 - {}^{x-1}C_3 - \frac{5}{4}{}^{x-2}A_2 < 0$

$$\Leftrightarrow \frac{(x-1)(x-2)(x-3)(x-4)}{1\cdot2\cdot3\cdot4} - \frac{(x-1)(x-2)(x-3)}{1\cdot2\cdot3} - \frac{5}{4}\cdot(x-2)(x-3) < 0$$

$$\Leftrightarrow (x-1)(x-2)(x-3)(x-4) - 4(x-1)(x-2)(x-3) - 30(x-2)(x-3) < 0$$

$$\Leftrightarrow (x-2)(x-3)\{(x-1)(x-4) - 4(x-1) - 30\} < 0$$



$$\Leftrightarrow (x-2)(x-3)\{x^2 - 9x - 22\} < 0$$

$$\Leftrightarrow (x-2)(x-3)(x+2)(x-11) < 0$$

From wavy curve method

$$x \in (-2,2) \cup (3,11)$$

but $x \in N$

$$\therefore \quad x = 1, 4, 5, 6, 7, 8, 9, 10 \qquad \ldots(i)$$

From inequality,

$$x-1 \geq 4, x-1 \geq 3, x-2 \geq 2$$

or $x \geq 5, x \geq 4, x \geq 4$

Hence, $x \geq 5$ ...(ii)

From Eqs. (i) and (ii), solutions of the inequality are

$$x = 5, 6, 7, 8, 9, 10.$$

● ***Ex. 31*** *Find the sum of the series*

$$(1^2+1)1! + (2^2+1)2! + (3^2+1)3! + \ldots + (n^2+1)n!.$$

Sol. Let $S_n = (1^2+1)1! + (2^2+1)2! + (3^2+1)3! + \ldots + (n^2+1)n!$

$$\therefore \quad n\text{th term} \quad T_n = (n^2+1)n!$$
$$= \{(n+1)(n+2) - 3(n+1) + 2\}n!$$
$$T_n = (n+2)! - 3(n+1)! + 2n!$$

Putting $n = 1, 2, 3, 4, ..., n$

Then, $T_1 = 3! - 3 \cdot 2! + 2 \cdot 1!$

$T_2 = 4! - 3 \cdot 3! + 2 \cdot 2!$

$T_3 = 5! - 3 \cdot 4! + 2 \cdot 3!$

$T_4 = 6! - 3 \cdot 5! + 2 \cdot 4!$

.....

.....

$T_{n-1} = (n+1)! - 3n! + 2(n-1)!$

$T_n = (n+2)! - 3(n+1)! + 2n!$

$\therefore \quad S_n = T_1 + T_2 + T_3 + ... + T_n$

$= (n+2)! - 2(n+1)!$ [the rest cancel out]

$= (n+2)(n+1)! - 2(n+1)!$

$= (n+1)!(n+2-2)$

$= n(n+1)!$

● ***Ex. 32*** *Find the negative terms of the sequence*

$$x_n = \frac{^{n+4}P_4}{P_{n+2}} - \frac{143}{4P_n}.$$

Sol. We have,

$$x_n = \frac{^{n+4}P_4}{P_{n+2}} - \frac{143}{4P_n}$$

$$\therefore \quad x_n = \frac{(n+4)(n+3)(n+2)(n+1)}{(n+2)!} - \frac{143}{4n!}$$

$$= \frac{(n+4)(n+3)(n+2)(n+1)}{(n+2)(n+1)n!} - \frac{143}{4n!}$$

$$= \frac{(n+4)(n+3)}{n!} - \frac{143}{4n!} = \frac{(4n^2+28n-95)}{4n!}$$

$\because$ x_n is negative

$\therefore \quad \frac{(4n^2+28n-95)}{4n!} < 0$

which is true for $n = 1,2$.

Hence, $x_1 = -\frac{63}{4}$ and $x_2 = -\frac{23}{8}$ are two negative terms.

● ***Ex. 33*** *How many integers between* 1 *and* 1000000 *have the sum of the digits equal to* 18*?*

Sol. Integers between 1 and 1000000 will be 1, 2, 3, 4, 5 or 6 digits and given sum of digits = 18

Thus, we need to obtain the number of solutions of the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 18 \quad ...(i)$$

where, $0 \le x_i \le 9, i = 1, 2, 3, 4, 5, 6$

Therefore, the number of solutions of Eq. (i), will be

= Coefficient of x^{18} in $(x^0 + x^1 + x^2 + x^3 + ... + x^9)^6$

= Coefficient of x^{18} in $\left(\frac{1-x^{10}}{1-x}\right)^6$

= Coefficient of x^{18} in $(1-x^{10})^6(1-x)^{-6}$

= Coefficient of x^{18} in $(1-6x^{10})(1 + {}^6C_1 x + {}^7C_2 x^2 + ... + {}^{13}C_8 x^8 + ... + {}^{23}C_{18} x^{18} + ...)$

$= {}^{23}C_{18} - 6 \cdot {}^{13}C_8 = {}^{23}C_5 - 6 \cdot {}^{13}C_5$

$= \frac{23 \cdot 22 \cdot 21 \cdot 20 \cdot 19}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - 6 \cdot \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$

$= 33649 - 7722 = 25927$

● ***Ex. 34*** *How many different car licence plates can be constructed, if the licences contain three letters of the English alphabet followed by a three digit number,*

(i) if repetition are allowed?

(ii) if repetition are not allowed?

Sol. (i) Total letters = 26 (i.e., $A, B, C, ..., X, Y, Z$)

and total digit number = 10 (i.e., 0, 1, 2, ..., 9)

If three letters on plate is represented by, then first place can be filled = 26

Second place can again be filled = 26

[$\because$ repetition are allowed]

and third place can again be filled = 26

Hence, three letters can be filled = $26 \times 26 \times 26$

$= (26)^3$ ways

and three digit numbers on plate by 999 ways

(i.e., 001, 002, ..., 999)

Hence, by the principle of multiplication, the required number of ways = $(26)^3(999)$ ways

(ii) Here, three letters out of 26 can be filled = $^{26}P_3$

[$\because$ repetition are not allowed]

and three digit can be filled out of 10 = $^{10}P_3$

[$\because$ repetition are not allowed]

Hence, required number of ways = $(^{26}P_3)(^{10}P_3)$ ways.

● ***Ex. 35*** *A man has* 7 *relatives,* 4 *of them are ladies and* 3 *gentlemen, has wife, has also* 7 *relatives,* 3 *of them are ladies and* 4 *are gentlemen. In how many ways can they invite a dinner party of* 3 *ladies and* 3 *gentlemen so that there are* 3 *of them man's relatives and* 3 *of the wife's relatives?*

Sol. The four possible ways of inviting 3 ladies and 3 gentlemen for the party with the help of the following table :

Man's relatives		Wife's relatives		Number of ways
4 Ladies	3 Gentlemen	3 Ladies	4 Gentlemen	
0	3	3	0	${}^4C_0 \times {}^3C_3 \times {}^3C_3 \times {}^4C_0 = 1$
1	2	2	1	${}^4C_1 \times {}^3C_2 \times {}^3C_2 \times {}^4C_1 = 144$
2	1	1	2	${}^4C_2 \times {}^3C_1 \times {}^3C_1 \times {}^4C_2 = 324$
3	0	0	3	${}^4C_3 \times {}^3C_0 \times {}^3C_0 \times {}^4C_3 = 16$

$\therefore$ Required number of ways to invite $= 1 + 144 + 324 + 16$
$= 485$

● **Ex. 36** *A team of ten is to be formed from* 6 *male doctors and* 10 *nurses of whom* 5 *are male and* 5 *are female. In how many ways can this be done, if the team must have atleast* 4 *doctors and atleast 4 nurses with atleast* 2 *male nurses and atleast* 2 *female nurses?*

Sol.

6 Doctors	5 Male nurses	5 Female nurses	Number of ways of selection
4	4	2	${}^6C_4 \times {}^5C_4 \times {}^5C_2 = 750$
4	3	3	${}^6C_4 \times {}^5C_3 \times {}^5C_3 = 1500$
4	2	4	${}^6C_4 \times {}^5C_2 \times {}^5C_4 = 750$
5	3	2	${}^6C_5 \times {}^5C_3 \times {}^5C_2 = 600$
5	2	3	${}^6C_5 \times {}^5C_2 \times {}^5C_3 = 600$
6	2	2	${}^6C_6 \times {}^5C_2 \times {}^5C_2 = 100$
			$\therefore$ Total ways = 4300

● **Ex. 37** *A number of four different digits is formed with the help of the digits* 1,2,3,4,5,6,7 *in all possible ways.*

(i) How many such numbers can be formed?
(ii) How many of these are even?
(iii) How many of these are exactly divisible by 4?
(iv) How many of these are exactly divisible by 25?

Sol. Here total digit = 7 and no two of which are alike

(i) Required number of ways = Taking 4 out of 7
$= {}^7P_4 = 7 \times 6 \times 5 \times 4 = 840$

(ii) For even number last digit must be 2 or 4 or 6. Now the remaining three first places on the left of 4-digit numbers are to be fillled from the remaining 6-digits and this can be done in
${}^6P_3 = 6 \cdot 5 \cdot 4 = 120$ ways
and last digit can be filled in 3 ways.
$\therefore$ By the principle of multiplication, the required number of ways
$= 120 \times 3 = 360$

(iii) For the number exactly divisible by 4, then last two digit must be divisible by 4, the last two digits are *viz.*, 12, 16, 24, 32, 36, 52, 56, 64, 72, 76. Total 10 ways.
Now, the remaining two first places on the left of 4-digit numbers are to be filled from the remaining 5-digits and this can be done in ${}^5P_2 = 20$ ways.
Hence, by the principle of multiplication, the required number of ways
$= 20 \times 10 = 200$

(iv) For the number exactly divisible by 25, then last two digit must be divisible by 25, the last two digits are *viz.,* 25, 75. Total 2 ways.
Now, the remaining two first places on the left of 4-digit number are to be filled from the remaining 5-digits and this can be done in ${}^5P_2 = 20$ ways.
Hence, by the principle of multiplication, the required number of ways
$= 20 \times 2 = 40$

● **Ex. 38** *India and South Africa play One Day International Series until one team wins* 4 *matches. No match ends in a draw. Find in how many ways the series can be won?*

Sol. Taking I for India and S for South Africa. We can arrange I and S to show the wins for India and South Africa, respectively.

For example., $ISSSS$ means first match is won by India which is followed by 4 wins by South Africa. This is one way in which series can be won.

Suppose, South Africa wins the series, then last match is always won by South Africa.

	Wins of I	Wins of S	Number of ways
(i)	0	4	1
(ii)	1	4	$\frac{4!}{3!} = 4$
(iii)	2	4	$\frac{5!}{2!3!} = 10$
(iv)	3	4	$\frac{6!}{3!3!} = 20$

$\therefore$ Total number of ways = 35
In the same number of ways, India can win the series.
$\therefore$ Total number of ways in which the series can be won
$= 35 \times 2 = 70$

● ***Ex. 39*** *Let n and k be positive integers such that* $n \geq \frac{k(k+1)}{2}$. *Find the number of solutions* $(x_1, x_2, \ldots, x_k)$, $x_1 \geq 1, x_2 \geq 2, \ldots, x_k \geq k$ *all integers satisfying* $x_1 + x_2 + \ldots + x_k = n$.

Sol. We have, $x_1 + x_2 + \ldots + x_k = n$ ...(i)

Now, let $y_1 = x_1 - 1,\ y_2 = x_2 - 2, \ldots, y_k = x_k - k$

$\therefore \quad y_1 \geq 0, y_2 \geq 0, \ldots, y_k \geq 0$

On substituting the values $x_1, x_2, \ldots, x_k$ in terms of $y_1, y_2, \ldots, y_k$ in Eq. (i), we get

$$y_1 + 1 + y_2 + 2 + \ldots + y_k + k = n$$

$$\Rightarrow \quad y_1 + y_2 + \ldots + y_k = n - (1 + 2 + 3 + \ldots + k)$$

$$\therefore \quad y_1 + y_2 + \ldots + y_k = n - \frac{k(k+1)}{2} = A \text{ (say)} \qquad \ldots(ii)$$

The number of non-negative integral solutions of the Eq. (ii) is

$$= {}^{k+A-1}C_A = \frac{(k + A - 1)!}{A!(k-1)!}$$

where, $A = n - \frac{k(k+1)}{2}$

● ***Ex. 40*** *Find the number of all whole numbers formed on the screen of a calculator which can be recognised as numbers with (unique) correct digits when they are read inverted. The greatest number formed on its screen is* 999999.

Sol. The number can use digits 0, 1, 2, 5, 6, 8 and 9 because they can be recognised as digits when they are see inverted.

A number can't begin with ,therefore all numbers having at unit's digit should no be counted. (when those numbers will be read inverted they will begin with).

No. of digits	Total numbers
1	7
2	$6^2 = 36$
3	$6 \times 7 \times 6 = 252$
4	$6 \times 7^2 \times 6 = 1764$
5	$6 \times 7^3 \times 6 = 12348$
6	$6 \times 7^4 \times 6 = 86436$
$\therefore$	Total = 100843

● ***Ex. 41*** *How many different numbers which are smaller than* 2×10^8 *and are divisible by* 3, *can be written by means of the digits* 0, 1 *and* 2*?*

Sol. 12, 21 ... 122222222 are form the required numbers we can assume all of them to be nine digit in the form $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9$ and can use 0 for a_1; a_2 and a_0 and a_0, a_1, a_2 and a_3... and so on to get 8-digit, 7-digit, 6-digit numbers etc. a_1 can assume one of the 2 values of 0 or 1. $a_2, a_3, a_4, a_5, a_6, a_7, a_8$ can assume any of 3 values 0, 1, 2.

The number for which $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = a_8 = a_9 = 0$ must be eleminated. The sum of first 8-digits i.e., $a_1 + a_2 + \ldots + a_8$ can be in the form of $3n - 2$ or $3n - 1$ or $3n$.

In each case a_9 can be chosen from 0,1,2 in only 1 way so that the sum of all 9-digits in equal to $3n$.

$\therefore$ Total numbers $= 2 \times 3^7 \times 1 - 1 = 4374 - 1 = 4373$.

● ***Ex. 42*** *There are n straight lines in a plane such that* n_1 *of them are parallel in one direction,* n_2 *are parallel in different direction and so on,* n_k *are parallel in another direction such that* $n_1 + n_2 + \ldots + n_k = n$. *Also, no three of the given lines meet at a point. Prove that the total number of points of intersection is*

$$\frac{1}{2}\left\{n^2 - \sum_{r=1}^{k} n_r^2\right\}.$$

Sol. Total number of points of intersection when no two of n given lines are parallel and no three of them are concurrent, is ${}^nC_2$. But it is given that there are k sets of $n_1, n_2, n_3, \ldots, n_k$ parallel lines such that no line in one set is parallel to line in another set.

Hence, total number of points of intersection

$$= {}^nC_2 - ({}^{n_1}C_2 + {}^{n_2}C_2 + \ldots + {}^{n_k}C_2)$$

$$= \frac{n(n-1)}{2} - \left\{\frac{n_1(n_1-1)}{2} + \frac{n_2(n_2-1)}{2} + \ldots + \frac{n_k(n_k-1)}{2}\right\}$$

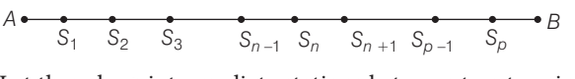
$$= \frac{n(n-1)}{2} - \frac{1}{2}\{(n_1^2 + n_2^2 + \ldots + n_k^2) - (n_1 + n_2 + \ldots + n_k)\}$$

$$= \frac{n(n-1)}{2} - \frac{1}{2}\left\{\sum_{r=1}^{k} n_r^2 - n\right\}$$

$$= \frac{n^2}{2} - \frac{1}{2}\sum_{r=1}^{k} n_r^2 = \frac{1}{2}\left\{n^2 - \sum_{r=1}^{k} n_r^2\right\}$$

● ***Ex. 43*** *There are p intermediate stations on a railway line from one terminus to another. In how many ways a train can stop at* 3 *of these interediate stations, if no two of these stopping stations are to be consecutive?*

Sol.



Let there be p intermediate stations between two terminus stations A and B as shown above.

Number of ways the train can stop in three intermediate stations = ${}^{p}C_3$

These are comprised of two exclusive cases *viz.*

(i) atleast two stations are consecutive.

(ii) now two of which is consecutive.

Now, there are $(p-1)$ pairs of consecutive intermediate stations.

In order to get a station trio in which atleast two stations are consecutive, each pair can be associated with a third station in $(p-2)$ ways. Hence, total number of ways in which 3 stations consisting of atleast two consecutive stations, can be chosen in $(p-1)(p-2)$ ways. Among these, each triplet of consecutive stations occur twice.

For example , the pair (S_n, S_{n-1}) when combined with S_{n+1} and the pair (S_n, S_{n+1}) when combined with S_{n-1} gives the same triplet and is counted twice. So, the number of three consecutive stations trio should be subtracted.

Now, number of these three consecutive stations trio is $(p-2)$.

Hence, the number of ways the triplet of stations consisting of atleast two consecutive stations can be chosen in

$$= \{(p-1)(p-2) - (p-2)\} \text{ ways}$$

$$= (p-2)^2 \text{ ways}$$

Therefore, number of ways the train can stop in three consecutive stations

$$= {}^{p}C_3 - (p-2)^2 = \frac{p(p-1)(p-2)}{1\cdot 2\cdot 3} - (p-2)^2$$

$$= (p-2)\left[\frac{p^2 - p - 6p + 12}{6}\right] = \frac{(p-2)(p^2 - 7p + 12)}{6}$$

$$= \frac{(p-2)(p-3)(p-4)}{1\cdot 2\cdot 3} = {}^{(p-2)}C_3$$

● ***Ex. 44*** *How many different 7-digit numbers are there and sum of whose digits is even?*

Sol. Let us consider 10 successive 7-digit numbers

$a_1a_2a_3a_4a_5a_6\ 0,$

$a_1a_2a_3a_4a_5a_6\ 1,$

$a_1a_2a_3a_4a_5a_6\ 2,$

.....

$a_1a_2a_3a_4a_5a_6\ 9$

where, a_1, a_2, a_3, a_4, a_5 and a_6 are some digits. We see that half of these 10 numbers, i.e. 5 numbers have an even sum of digits.

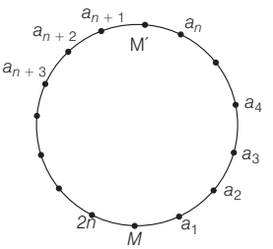
The first digit a_1 can assume 9 different values and each of the digits a_2, a_3, a_4, a_5 and a_6 can assume 10 different values.

The last digit a_7 can assume only 5 different values of which the sum of all digits is even.

$\therefore$ There are $9 \times 10^5 \times 5 = 45 \times 10^5$, 7-digit numbers the sum of whose digits is even.

● ***Ex. 45*** *There are 2n guests at a dinner party. Supposing that the master and mistress of the house have fixed seats opposite one another and that there are two specified guests who must not be placed next to one another. Find the number of ways in which the company can be placed.*

Sol. Let the M and M' represent seats of the master and mistress respectively and let $a_1, a_2, a_3, \ldots, a_{2n}$ represent the $2n$ seats.



Let the guests who must not be placed next to one another be called P and Q.

Now, put P at a_1 and Q at any position, other than a_2, say at a_3, then remaining $(2n-2)$ guests can be arranged in the remaining $(2n-2)$ positions in $(2n-2)!$ ways. Hence, there will be altogether $(2n-2)(2n-2)!$ arrangements of the guests, when P is at a_1.

The same number of arrangements when P is at a_n or a_{n+1} or a_{2n}. Thus, for these positions $(a_1, a_n, a_{n+1}, a_{2n})$ of P, there are altogether $4(2n-2)(2n-2)!$ ways. ...(i)

If P is at a_2, then there are altogether $(2n-3)$ positions for Q. Hence, there will be altogether $(2n-3)(2n-2)!$ arrangements of the guests, when P is at a_2.

The same number of arrangements can be made when P is at any other position excepting the four positions

$$a_1, a_n, a_{n+1}, a_{2n}.$$

Hence, for these $(2n-4)$ positions of P, there will be altogether

$(2n-4)(2n-3)(2n-2)!$ arrangements of the guests ...(ii)

Hence, from Eqs. (i) and (ii), the total number of ways of arranging the guests

$= 4(2n-2)(2n-2)! + (2n-4)(2n-3)(2n-2)!$

$= (4n^2 - 6n + 4)(2n-2)!$

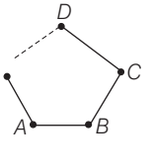
● ***Ex. 46*** *Find the number of triangles whose angular points are at the angular points of a given polygon of n sides, but none of whose sides are the sides of the polygon.*

Sol. A polygon of n sides has n angular points. Number of triangles formed from these n angular points = ${}^{n}C_3$.

These are comprised of two exclusive cases *viz.*

(i) atleast one side of the triangle is a side of the polygon.

(ii) no side of the triangle is a side of the polygon.



Let AB be one side of the polygon. If each angular point of the remaining $(n-2)$ points are joined with A and B, we get a triangle with one side AB.

$\therefore$ Number of triangles of which AB is one side $= (n-2)$

Likewise, number of triangles of which BC is one side $= (n-2)$ and of which atleast one side is the side of the polygon $= n(n-2)$.

Out of these triangle, some are counted twice. For example , the triangle when C is joined with AB is ΔABC, is taken when AB is taken as one side. Again triangle formed when A is joined with BC is counted when BC is taken as one side.

Number of such triangles $= n$

So, the number of triangles of which one side is the side of the triangle

$$= n(n-2) - n \ = n(n-3)$$

Hence, the total number of required triangles

$$= {}^{n}C_3 - n(n-3) = \frac{1}{6}n(n-4)(n-5)$$

● ***Ex. 47*** *Prove that* $(n!)!$ *is divisible by* $(n!)^{(n-1)!}$.

Sol. First we show that the product of p consecutive positive integers is divisible by $p!$. Let the p consecutive integers be $m, m+1, m+2, \ldots, m+p-1$. Then,

$$m(m+1)(m+2)\ldots(m+p-1) = \frac{(m+p-1)!}{(m-1)!}$$

$$= p! \frac{(m+p-1)!}{(m-1)!\,p!}$$

$$= p!\,{}^{m+p-1}C_p$$

Since, ${}^{m+p-1}C_p$ is an integer.

$$\therefore \quad {}^{m+p-1}C_p = \frac{m(m+1)(m+2)\ldots(m+p-1)}{p!}$$

Now, $(n!)!$ is the product of the positive integers from 1 to $n!$. We write the integers from 1 to $n!$ is $(n-1)!$ rows as follows:

1	2	3	...	n
$n+1$	$n+2$	$n+3$	...	$2n$
$2n+1$	$2n+2$	$2n+3$	...	$3n$
:	:	:	:	:
$n!-n+1$	$n!-n+2$	$n!-n+3$	....	$n!$

Each of these $(n-1)!$ rows contain n consecutive positive integers. The product of the consecutive integers in each row is divisible by $n!$. Thus, the product of all the integers from 1 to $n!$ is divisible by $(n!)^{(n-1)!}$.

Permutations and Combinations Exercise 1 :
Single Option Correct Type Questions

■ This section contains **30 multiple choice questions.** Each question has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct

1. A lady gives a dinner party to 5 guests to be selected from nine friends. The number of ways of forming the party of 5, given that two of the friends will not attend the party together, is
(a) 56 (b) 126 (c) 91 (d) None of these

2. If a, b, c and d are odd natural numbers such that $a+b+c+d=20$, the number of values of the ordered quadruplet (a, b, c, d) is
(a) 165 (b) 455 (c) 310 (d) None of these

3. If l = LCM of 8!, 10! and 12! and h = HCF of 8!, 10! and 12!, then $\frac{l}{h}$ is equal to
(a) 132 (b) 11800
(c) 11880 (d) None of these

4. The number of positive integers satisfying the inequality ${}^{n+1}C_{n-2} - {}^{n+1}C_{n-1} \le 100$ is
(a) 9 (b) 8
(c) 5 (d) None of these

5. The number of ways in which a score of 11 can be made from a through by three persons, each throwing a single die once, is
(a) 45 (b) 18 (c) 27 (d) 68

6. The number of positive integers with the property that they can be expressed as the sum of the cubes of 2 positive integers in two different ways is
(a) 1 (b) 100 (c) infinite (d) 0

7. In a plane there are 37 straight lines, of which 13 pass through the point A and 11 pass through the point B. Besides, no three lines pass through one point, no line passes through both points A and B and no two are parallel, the number of intersection points the lines have, is
(a) 535 (b) 601 (c) 728 (d) 963

8. If a denotes the number of permutations of $x+2$ things taken all at a time, b the number of permutations of x things taken 11 at a time and c the number of permutations of $x-11$ things taken all at a time such that $a = 182bc$, the value of x is
(a) 15 (b) 12 (c) 10 (d) 18

9. The number of numbers less than 1000 that can be formed out of the digits 0, 1, 2, 3, 4 and 5, no digit being repeated, is
(a) 130 (b) 131 (c) 156 (d) 158

10. If the permutations of a, b, c, d and e taken all together are written down in alphabetical order as in dictionary and numbered, the rank of the permutation *debac* is
(a) 90 (b) 91 (c) 92 (d) 93

11. On a railway there are 20 stations. The number of different tickets required in order that it may be possible to travel from every station to every station, is
(a) 210 (b) 225 (c) 196 (d) 105

12. A set containing n elements. A subset P of A is chosen. The set A is reconstructed by replacing the element of P. A subset Q of A is again chosen the number of ways of choosing P and Q, so that $P \cap Q = \phi$, is
(a) $2^{2n} - {}^{2n}C_n$ (b) 2^n (c) $2^n - 1$ (d) 3^n

13. The straight lines l_1, l_2, l_3 are parallel and lie in the same plane. A total number of m points on l_1 ; n points on l_2 ; k points on l_3, the maximum number of triangles formed with vertices at these points is
(a) ${}^{m+n+k}C_3$ (b) ${}^{m+n+k}C_3 - {}^mC_3 - {}^nC_3 - {}^kC_3$
(c) ${}^mC_3 + {}^nC_3 + {}^kC_3$ (d) None of these

14. Let A be a set of $n(\ge 3)$ distinct elements. The number of triplets (x, y, z) of the set A in which atleast two coordinates are equal to
(a) ${}^nP_3$ (b) $n^3 - {}^nP_3$
(c) $3n^2 - 2n$ (d) $3n^2(n-1)$

15. The total number of five-digit numbers of different digits in which the digit in the middle is the largest, is
(a) $2^2 \cdot 3^2 \cdot 7^2$ (b) $2^3 \cdot 3 \cdot 7^3$
(c) $2^2 \cdot 3^3 \cdot 7^2$ (d) $2^3 \cdot 3^2 \cdot 7^3$

16. The total number of words that can be formed using all letters of the word 'RITESH' that neither begins with I nor ends with R, is
(a) 504 (b) 480
(c) 600 (d) 720

17. A man has three friends. The number of ways he can invite one friend everyday for dinner on six successive nights, so that no friend is invited more than three times, is
(a) 360 (b) 420 (c) 170 (d) 510

18. The number of three digit numbers of the form xyz such that $x < y, z \le y$ and $x \ne 0$, is
(a) 240 (b) 244 (c) 276 (d) 285

19. The letters of the word 'MEERUT' are arranged in all possible ways as in a dictionary, then the rank of the word 'MEERUT' is
(a) 119 (b) 120 (c) 121 (d) 122

20. The number of ways in which 10 candidates $A_1, A_2, \ldots, A_{10}$ can be ranked so that A_1 is always above A_2, is
(a) 10! (b) $\frac{10!}{2}$
(c) 9! (d) None of these

21. Let A be the set of four digit numbers $a_1a_2\,a_3a_4$, where $a_1 > a_2 > a_3 > a_4$, then $n(A)$ is
(a) 126 (b) 84
(c) 210 (d) None of these

22. The number of distinct rational numbers x such that $0 < x < 1$ and $x = \frac{p}{q}$, where $p, q \in \{1, 2, 3, 4, 5, 6\}$ is
(a) 15 (b) 13 (c) 12 (d) 11

23. The total number of integral solutions of $xyz = 24$ is
(a) 30 (b) 36 (c) 90 (d) 120

24. If $ABCD$ is a convex quadrilateral with 3, 4, 5 and 6 points, marked on sides AB, BC, CD and DA respectively, then the number of triangles with vertices on different sides, is
(a) 220 (b) 270 (c) 282 (d) 342

25. The number of ways can a team of six horses be selected out of a stud of 16, so that there shall always be three out of A, B, C, D, E, F but never AD, BE or CF together, is
(a) 720 (b) 840 (c) 960 (d) 1260

26. The number of polynomials of the form $x^3 + ax^2 + bx + c$ that are divisible by $x^2 + 1$, where a, b, $c \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, is
(a) 10 (b) 15 (c) 5 (d) 8

27. Let $x_1, x_2, x_3, \ldots, x_k$ be the divisors of positive integer 'n' (including 1 and x). If $x_1 + x_2 + \ldots + x_k = 75$, then $\sum_{i=1}^{k} \frac{1}{x_i}$ is equal to
(a) $\frac{k^2}{75}$ (b) $\frac{75}{k}$ (c) $\frac{n^2}{75}$ (d) $\frac{75}{n}$

28. The total number of functions 'f' from the set $\{1, 2, 3\}$ into the set $\{1, 2, 3, 4, 5\}$ such that $f(i) \le f(j), \forall\, i < j$, is
(a) 35 (b) 30 (c) 50 (d) 60

29. Ten persons numbered 1, 2, 3, ..., 10 play a chess tournament, each player playing against every player exactly one game. It is known that no game ends in a draw. Let $w_1, w_2, w_3, \ldots, w_{10}$ be the number of games won by player 1, 2, 3, ..., 10 respectively and $l_1, l_2, l_3, \ldots, l_{10}$ be the number of games lost by the players 1, 2, 3, ..., 10 respectively, then
(a) $\Sigma w_i^2 = 81 - \Sigma l_i^2$ (b) $\Sigma w_i^2 = 81 + \Sigma l_i^2$
(c) $\Sigma w_i^2 = \Sigma l_i^2$ (d) None of these

30. In the next world cup of cricket there will be 12 teams, divided equally in two groups. Teams of each group will play a match against each other. From each group 3 top teams will qualify for the next round. In this round each team will play against others once.
Four top teams of this round will qualify for the semi- final round, where each team will play against the others once. Two top teams of this round will go to the final round, where they will play the best of three matches. The minimum number of matches in the next world cup will be
(a) 54 (b) 53 (c) 38 (d) 37

Permutations and Combinations Exercise 2 : More than One Correct Option Type Questions

- This section contains **10 multiple choice questions**. Each question has four choices (a), (b), (c) and (d) out of which **MORE THAN ONE** may be correct.

31. If $300! = 3^m \times$ an integer, then
(a) $m = 148$ (b) $m = 150$
(c) It is equivalent to number of n in $150! = 2^{n-2} \times$ an integer
(d) $m = {}^{150}C_2$

32. If $102! = 2^{\alpha} \cdot 3^{\beta} \cdot 5^{\gamma} \cdot 7^{\delta} \ldots$, then
(a) $\alpha = 98$ (b) $\beta = 2\gamma + 1$ (c) $\alpha = 2\beta$ (d) $2\gamma = 3\delta$

33. The number of ways of choosing triplet (x, y, z) such that $z \ge \max\{x, y\}$ and $x, y, z \in \{1, 2, 3, \ldots, n, n+1\}$, is
(a) ${}^{n+1}C_3 + {}^{n+2}C_3$ (b) $\frac{n(n+1)(2n+1)}{6}$
(c) $1^2 + 2^2 + 3^2 + \ldots + n^2$ (d) $2({}^{n+2}C_3) - {}^{n+1}C_2$

34. Let n be 4-digit integer in which all the digits are different. If x is the number of odd integers and y is the number of even integers, then
(a) $x < y$ (b) $x > y$
(c) $x + y = 4500$ (d) $|x - y| = 56$

35. Let $S = \{1, 2, 3, \ldots, n\}$. If X denotes the set of all subsets of S containing exactly two elements, then the value of $\sum_{A \in X} (\text{min. } A)$ is given by
(a) ${}^{n+1}C_3$ (b) ${}^{n}C_3$
(c) $\frac{n(n^2-1)}{6}$ (d) $\frac{n(n^2-3n+2)}{6}$

36. Let $p = 2520$, x = number of divisors of p which are multiple of 6, y = number of divisors of p which are multiple of 9, then
(a) $x = 12$ (b) $x = 24$ (c) $y = 12$ (d) $y = 16$

37. If N denotes the number of ways of selecting r objects out of n distinct objects $(r \geq n)$ with unlimited repetition but with each object included atleast once in selection, then N is
(a) ${}^{r-1}C_{r-n}$ (b) ${}^{r-1}C_{n}$
(c) ${}^{r-1}C_{n-1}$ (d) ${}^{r-1}C_{r-n-1}$

38. There are three teams x, $x+1$ and y childrens and total number of childrens in the teams is 24. If two childrens of the same team do not fight,then
(a) maximum number of fights is 190
(b) maximum number of fights is 191
(c) maximum number of fights occur when $x = 7$
(d) maximum number of fights occur when $x = 8$

39. Let N denotes the number of ways in which $3n$ letters can be selected from $2n$ A's, $2n$$B$'s and $2n$$C$'s. Then,
(a) $3 \mid (N-1)$
(b) $n \mid (N-1)$
(c) $(n+1) \mid (N-1)$
(d) $3n(n+1) \mid (N-1)$

40. If $\alpha = x_1x_2x_3$ and $\beta = y_1y_2y_3$ are two 3-digit numbers, then the number of pairs of α and β can be formed so that α can be subtracted from β without borrowing, is
(a) $2! \times 10! \times 10!$ (b) $(45)(55)^2$
(c) $3^2 \cdot 5^3 \cdot 11^2$ (d) 136125

Permutations and Combinations Exercise 3 : Passage Based Questions

- This section contains **5 passages**. Based upon each of the passage **3 multiple choice questions** have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I (Q. Nos. 41 to 43)

Consider the word $W = TERRORIST$.

41. Number of four letter words that can be made using only the letters from the word W, if each word must contains atleast one vowel, is
(a) 588 (b) 504 (c) 294 (d) 600

42. Number of arrangements of the word W, if no two R's are together, is
(a) 11460 (b) 10400
(c) 12600 (d) 9860

43. Number of arrangements of the word W, if R's as well as T's are separated, is
(a) 9860 (b) 1080 (c) 10200 (d) 11400

Passage II (Q. Nos. 44 to 46)

Different words are formed by arranging the letters of the word 'SUCCESS'.

44. The number of words in which C's are together but S's are separated, is
(a) 120 (b) 96
(c) 24 (d) 420

45. The number of words in which no two C's and no two S's are together, is
(a) 120 (b) 96
(c) 24 (d) 180

46. The number of words in which the consonants appear in alphabetic order, is
(a) 42 (b) 40 (c) 420 (d) 480

Passage III (Q. Nos. 47 to 49)

Different words are being formed by arranging the letters of the word 'ARRANGE'.

47. The number of words in which the two R's are not together, is
(a) 1260 (b) 960
(c) 900 (d) 600

48. The number of words in which neither two R's nor two A's come together, is
(a) 1260 (b) 900
(c) 660 (d) 240

49. The rank of the word 'ARRANGE' in the dictionary is
(a) 340 (b) 341 (c) 342 (d) 343

Passage IV (Q. Nos. 50 to 52)

Let $S(n)$ denotes the number of ordered pairs (x, y) satisfying $\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$, $\forall\, x, y, n \in N$.

50. $S(10)$ equals
(a) 3 (b) 6 (c) 9 (d) 12

51. $S(6) + S(7)$ equals
(a) $S(3) + S(4)$ (b) $S(5) + S(6)$ (c) $S(8) + S(9)$ (d) $S(1) + S(11)$

52. $\sum_{r=1}^{10} S(r)$ equals
(a) 47 (b) 48 (c) 49 (d) 50

Passage V (Q. Nos. 53 to 55)

Let $f(n)$ denotes the number of different ways, the positive integer n can be expressed as the sum of the 1's and 2's. For example, $f(4)=5$

i.e., $4=1+1+1+1$

$=1+1+2=1+2+1=2+1+1=2+2$

53. The value of $f\{f(6)\}$ is

(a) 376 (b) 377 (c) 321 (d) 370

54. The number of solutions of the equation $f(n)=n$, where $n \in N$ is

(a) 1 (b) 2 (c) 3 (d) 4

55. In a stage show, $f(4)$ superstars and $f(3)$ junior artists participate. Each one is going to present one item, then the number of ways the sequence of items can be planned, if no two junior artists present their items consecutively, is

(a) 144 (b) 360 (c) 4320 (d) 14400

Permutations and Combinations Exercise 4 : Single Integer Answer Type Questions

- This section contains **10 questions**. The answer to each question is a **single digit integer**, ranging from 0 to 9 (both inclusive).

56. The ten's digit of $1!+2!+3!+\ldots+97!$ is

57. The exponent of 7 in ${}^{100}C_{50}$ is

58. Let P_n denotes the number of ways in which three people can be selected out of n people sitting in a row, in two of them are consecutive. If $P_{n+1}-P_n=15$, the value of n is

59. If the letters of the word are arranged as in a dictionary. m and n are the rank of the words BULBUL and NANNU respectively, then the value of $m-4n$ is

60. An n-digit number is a positive number with exactly n digits. Nine hundred distinct n-digit numbers are to be formed using only the three digits 2, 5 and 7. The smallest value of n for which this is possible, is

61. If a, b, c are three natural numbers in AP such that $a+b+c=21$ and if possible number of ordered triplet (a, b, c) is λ, then the vlaue of $(\lambda-5)$ is

62. If 2λ is the number of ways of selecting 3 member subset of {1, 2, 3, ..., 29}, so that the numbers form of a GP with integer common ratio, then the value of λ is

63. In a certain test, there are n questions. In this test, 2^{n-k} students gave wrong answers to atleast k questions, where $k=1, 2, 3, \ldots, n$. If the total number of wrong answers given is 127, then the value of n is

64. A 7-digit number made up of all distinct digits 8, 7, 6, 4, 2, x and y, is divisible by 3. The, possible number of ordered pair (x, y) is

65. There are five points A, B, C, D and E. No three points are collinear and no four are concyclic. If the line AB intersects of the circles drawn throught the five points. The number of points of intersection on the line apart from A and B is

Permutations and Combinations Exercise 5 : Matching Type Questions

- This section contains **5 questions**. Questions 66 to 70 have four statements (A, B, C and D) given in **Column I** and four statements (p, q, r and s) in **Column II**. Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II**.

66.

	Column I		Column II
(A)	${}^{n+4}C_{n+1}-{}^{n+3}C_n=15(n+2)$, then n equals	(p)	19
(B)	$11\cdot{}^{n}P_4=20\cdot{}^{n-2}P_4$, then n equals	(q)	27
(C)	${}^{2n}C_3=11\cdot{}^{n}C_3$, then n equals	(r)	16
(D)	${}^{n+2}C_8:{}^{n-2}P_4=57:16$, then n equals	(s)	6

67.

	Column I		Column II
(A)	Number of increasing permutations of m symbols are there from the n set numbers $(a_1, a_2, \ldots, a_n)$, where the order among the numbers, is given by $a_1 < a_2 < a_3 < \ldots < a_{m-1} < a_m$ is	(p)	n^m
(B)	There are m men and n monkeys. Number of ways in which every monkey has a master, if a man can have any number of monkeys is	(q)	${}^mC_n$
(C)	Number of ways in which r red balls and $(m-1)$ green balls can be arranged in a line, so that no two red balls are together is (balls of the same colour are alike)	(r)	${}^nC_m$
(D)	Number of ways in which 'm' different toys can be distributed in n children, if every child may received any number of toys is	(s)	m^n

68.

	Column I		Column II
(A)	Number of straight lines joining any two of 10 points of which four point are collinear is	(p)	30
(B)	Maximum number of points of intersection of 10 straight lines in the plane is	(q)	60
(C)	Maximum number of points of intersection of 6 circles in the plane is	(r)	40
(D)	Maximum number of points of intersection of 6 parabolas is	(s)	45

69. Consider a 6×6 chessboard. Then, match the following columns.

	Column I		Column II
(A)	Number of rectangles is	(p)	${}^{10}C_5$
(B)	Number of squares is	(q)	441
(C)	Number of ways three squares can be selected, if they are not in same row or column is	(r)	91
(D)	In how many ways eleven '+' sign can be arranged in the squares, if no row remains empty	(s)	2400

70. 5 balls are to be placed in 3 boxes. Each box can hold all the 5 balls. Number of ways in which the balls can be placed so that no box remain empty, if

	Column I		Column II
(A)	balls are identical but boxes are different	(p)	2
(B)	balls are different but boxes are identical	(q)	25
(C)	balls as well as boxes are identical	(r)	50
(D)	balls as well as boxes are identical but boxes kept in a row	(s)	6

Permutations and Combinations Exercise 6 : Statement I and II Type Questions

■ **Directions** (Q. Nos. 71 to 82) are Assertion-Reason type questions. Each of these questions contains two statements:
Statement-1 (Assertion) and **Statement-2** (Reason)
Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

71. **Statement-1** The smallest positive integer n such that $n!$ can be expressed as a product of n-3 consecutive integers, is 6.
Statement-2 Product of three consecutive integers is divisible by 6.

72. **Statement-1** A number of four different digits is formed with the help of the digits 1, 2, 3, 4, 5, 6, 7 in all possible ways. The number of ways which are exactly divisible by 4 is 200.
Statement-2 A number divisible by 4, if units place digit is also divisible by 4.

73. **Statement-1** The number of divisors of 10! is 280.
Statement-2 $10! = 2^p \cdot 3^q \cdot 5^r \cdot 7^s$, where $p, q, r, s \in N$.

74. **Statement-1** Number of permutations of 'n' dissimilar things taken 'n' at a time is $n!$.
Statement-2 If $n(A) = n(B) = n$, then the total number of functions from A to B are $n!$.

75. **Statement-1** If N the number of positive integral solutions of $x_1\ x_2\ x_3\ x_4 = 770$, then N is divisible by 4 distinct prime numbers.

Statement-2 Prime numbers are 2, 3, 5, 7, 11, 13, ...

76. **Statement-1** The total number of ways in which three distinct numbers in AP, can be selected from the set {1, 2, 3, ..., 21}, is equal to 100.

Statement-2 If a, b, c are in AP, then $a + c = 2b$.

77. **Statement-1** The number of even divisors of the numbers $N = 12600$ is 54.

Statement-2 0, 2, 4, 6, 8, ... are even integers.

78. **Statement-1** A 5-digit number divisible by 3 is to be formed using the digits 0, 1, 2, 3, 4, 5 and 6 without repetition, then the total number of ways this can be done is 216.

Statement-2A number is divisible by 3, if sum of its digits is divisible by 3.

79. **Statement-1** The sum of the digits in the ten's place of all numbers formed with the help of 3, 4, 5, 6 taken all at a time is 108.

Statement-2 The sum of the digits in the ten's places = The sum of the digits is the units's place.

80. **Statement-1** There are $p \geq 8$ points in space no four of which are in the same with exception of $q \geq 3$ points which are in the same plane, then the number of planes each containing three points is ${}^{p}C_3 - {}^{q}C_3$.

Statement-2 3 non-collinear points always determine unique plane.

81. **Statement-1** The highest power of 3 in ${}^{50}C_{10}$ is 4.

Statement-2 If p is any prime number, then power of p in $n!$ is equal to $\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \ldots$, where $[\cdot]$ denotes the greatest integer function.

82. **Statement-1** A convex quindecagon has 90 diagonals.

Statement-2 Number of diagonals in a polygon is ${}^{n}C_2 - n$.

Permutations and Combinations Exercise 7 : Subjective Type Questions

- In this section, there are **17 subjective** questions.

83. Given that ${}^{n}C_{n-r} + 3{}^{n}C_{n-r+1} + 3$.
${}^{n}C_{n-r+2} + {}^{n}C_{n-r+3} = {}^{x}C_r$. Find x

84. Solve the equation $3{}^{x+1}C_2 + P_2 x = 4{}^{x}A_2, x \in N$.

85. How many positive terms are there in the sequence (x_n) if $x_n = \frac{195}{4P_n} - \frac{{}^{n+3}A_3}{P_{n+1}}, n \in N$?

86. Prove that ${}^{n-1}C_3 + {}^{n-1}C_4 > {}^{n}C_3$ if $n > 7$.

87. In how many ways can a mixed doubles game in tennis be arranged from 5 married couples, if no husband and wife play in the same game?

88. In how many ways, we can choose two teams of mixed double for a tennis tournament from four couples such that if any couple participates, then it is in the same team?

89. A family consists of a grandfather, 5 sons and daughters and 8 grand child. They are to be seated in a row for dinner. The grand children wish to occupy the 4 seats at each end and the grandfather refuses to have a grand child on either side of him. In how many ways can the family be made to sit?

90. A tea party is arranged for 16 people along two sides of a large table with 8 chairs on each side. Four men sit on one particular side and two on the other side. In how many ways can they be seated?

91. Every man who has lived on earth has made a certain number of handshakes. Prove that the number of men who have made an odd number of handshakes is even.

92. A train is going from Cambridge to London stops at nine intermediate stations. Six persons enter the train during the journey with six different tickets. How many different sets of tickets they have had?

93. n different things are arranged around a circle. In how many ways can 3 objects be selected when no two of the selected objects are consecutive?

94. A boat 's crew consists of 8 men, 3 of whom can only row on one side and 2 only on the other. Find the number of ways in which the crew can be arranged.

95. In how many different ways can a set A of $3n$ elements be partitioned into 3 subsets of equal number of elements ? (The subsets P, Q, R form a partition if $P \cup Q \cup R = A, P \cap R = \phi, Q \cap R = \phi, R \cap P = \phi$.)

96. A square of n units by n units is divided into n^2 squares each of area 1 sq unit. Find the number of ways in which 4 points (out of $(n+1)^2$ vertices of the squares) can be chosen so that they form the vertices of a square.

97. How many sets of 2 and 3 (different) numbers can be formed by using numbers between 0 and 180 (both including) so that 60 is their average?

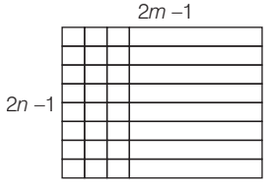
98. There are n straight lines in a plane, no two of which are parallel and no three passes through the same point. Their point of intersection are joined. Show that the number of fresh lines thus introduced, is $\frac{1}{8}n(n-1)(n-2)(n-3)$.

99. 6 balls marked as 1, 2, 3, 4, 5 and 6 are kept in a box. Two players A and B start to take out 1 ball at a time from the box one after another without replacing the ball till the game is over. The number marked on the ball is added each time to the previous sum to get the sum of numbers marked on the balls taken out. If this sum is even, then 1 point is given to the player. The first player to get 2 points is declared winner. At the start of the game, the sum is 0. If A starts to take out the ball, find the number of ways in which the game can be won.

Permutations and Combinations Exercise 8 : Questions Asked in Previous 13 Year's Exam

■ This section contains questions asked in **IIT-JEE, AIEEE, JEE Main & JEE Advanced** from year **2005** to year **2017**.

100. There is a rectangular sheet of dimension $(2m-1)\times(2n-1)$, (where $m>0, n>0$). It has been divided into square of unit area by drawing line perpendicular to the sides. Find the number of rectangles having sides of odd unit length. **[IIT-JEE 2005, 3M]**



(a) $(m+n+1)^2$ (b) $mn(m+1)(n+1)$
(c) 4^{m+n-2} (d) m^2n^2

101. If the letters of the word SACHIN arranged in all possible ways and these words are written out as in dictionary, then the word SACHIN appears at series number **[AIEEE 2005, 3M]**
(a) 603 (b) 602 (c) 601 (d) 600

102. If r, s, t are prime numbers and p, q are the positive integers such that LCM of p, q is $r^2t^4s^2$, then the number of ordered pair (p, q) is **[IIT-JEE 2006, 3M]**
(a) 252 (b) 254 (c) 225 (d) 224

103. At an election, a voter may vote for any number of candidates, not greater than number to be elected. There are 10 candidates and 4 are to be selected. If a voter votes for atleast one candidate, then number of ways in which he can vote, is **[AIEEE 2006, 4.5M]**
(a) 5040 (b) 6210
(c) 385 (d) 1110

104. The letters of the word COCHIN are permuted and all the permutations are arranged in an alphabetical order as in English dictionary. The number of words that appear before the word COHIN is **[IIT-JEE 2007, 3M]**
(a) 360 (b) 192
(c) 96 (d) 48

105. The set $S=\{1,2,3,\ldots,12\}$ to be partitioned into three sets A, B, C of equal size. Thus, $A\cup B\cup C=S$, $A\cap B=B\cap C=A\cap C=\phi$. The number of ways to partition S, is **[AIEEE 2007, 3M]**
(a) $\frac{12!}{3!(4!)^3}$ (b) $\frac{12!}{3!(3!)^4}$
(c) $\frac{12!}{(4!)^3}$ (d) $\frac{12!}{(3!)^4}$

106. Consider all possible permutations of the letters of the word ENDEANOEL. Match the statements/ expressions in **Column I** with the statements/ expressions in **Column II**. **[IIT-JEE 2008, 6M]**

	Column I		Column II
(A)	The number of permutations containing the word ENDEA is	(p)	5!
(B)	The number of permutations in which the letters E occurs in the first and the last positions, is	(q)	$2\times5!$
(C)	The number of permutations in which none of the letters D, L, N occurs in the last five positions, is	(r)	$7\times5!$
(D)	The number of permutations in which the letters A, E, O occur only in odd positions, is	(s)	$21\times5!$

107. How many different words can be formed by Jumbling the letters in the word MISSISSIPPI in which no two S's are adjacent? **[AIEEE 2008, 3M]**
(a) $6\cdot7\cdot{}^8C_4$ (b) $6\cdot8\cdot{}^7C_4$
(c) $7\cdot{}^6C_4\cdot{}^8C_4$ (d) $8\cdot{}^6C_4\cdot{}^7C_4$

108. In a shop, there are five types of ice-creams available. A child buys six ice-creams. **[AIEEE 2008, 3M]**

Statement-1 The number of different ways the child can buy the six ice-creams is ${}^{10}C_4$.

Statement-2 The number of different ways the child can buy six ice-creams is equal to the number of different ways to arranging 6A's and 4B's in a row. **[AIEEE 2008, 3M]**

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement-1 true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

109. The number of 7-digit integers, with sum of the digits equal to10 and formed by using the digits 1, 2 and 3 only, is **[IIT-JEE 2009, 3M]**
(a) 55 (b) 66 (c) 77 (d) 88

110. From 6 different novels and 3 different dictionaries, 4 novels and 1 dictionary are to be selected and arranged in a row on a shelf so that the dictionary is always in the middle. Then, the number of such arrangements is **[AIEEE 2009, 4M]**
(a) atleast 1000
(b) less than 500
(c) atleast 500 but less than 750
(d) atleast 750 but less than 1000

111. There are two urns. Urn A has 3 distinct red balls and urn B has 9 distinct blue balls. From each urn, two balls are taken out at random and then transferred to the other. The number of ways in which this can be done, is **[AIEEE 2010, 4M]**
(a) 36 (b) 66 (c) 108 (d) 3

112. **Statement-1** The number of ways distributing 10 identical balls in 4 distinct boxes such that no box is empty, is $^{9}C_{3}$.

Statement-2 The number of ways of choosing any 3 places from 9 different places is $^{9}C_{3}$ **[AIEEE 2011, 4M]**
(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement-1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

113. There are 10 points in a plane, out of these 6 are collinear. If N is the number of triangles formed by joining these points, then **[AIEEE 2011, 4M]**
(a) $N > 190$ (b) $N \le 100$
(c) $100 < N \le 140$ (d) $140 < N \le 190$

114. The total number of ways in which 5 balls of different colours can be distributed among 3 persons, so that each person gets atleast one ball is **[IIT-JEE 2012, 3M]**
(a) 75 (b) 150 (c) 210 (d) 243

■ **Directions** (Q. Nos. 115 to 116) Let a_n denotes the number of all n-digits positive integer formed by the digits 0, 1 or both such that no consecutive digits in them are 0. Let b_n be the number of such n-digit integers ending with digit 1 and c_n be the number of such n digits integers ending with digit 0.

115. The value of b_6, is **[IIT-JEE 2012, 3+3M]**
(a) 7 (b) 8 (c) 9 (d) 11

116. Which of the following is correct?
(a) $a_{17} = a_{16} + a_{15}$ (b) $c_{17} \ne c_{16} + c_{15}$
(c) $b_{17} \ne b_{16} + c_{16}$ (d) $a_{17} = c_{17} + b_{16}$

117. Assuming the balls to be identical except for difference in colours, the number of ways in which one or more balls can be selected from 10 white, 9 green and 7 black balls, is **[AIEEE 2012, 4M]**
(a) 630 (b) 879 (c) 880 (d) 629

118. Let T_n be the number of all possible triangles formed by joining vertices of an n-sided regular polygon. If $T_{n+1} - T_n = 10$, the value of n is **[JEE Main 2013, 4M]**
(a) 5 (b) 10 (c) 8 (d) 7

119. Consider the set of eight vectors $V = [a\hat{i} + b\hat{j} + c\hat{k} : a, b, c \in \{-1, 1\}]$. Three non-coplanar vectors can be chosen from V in 2^p ways, then p is **[JEE Advanced 2013, 4M]**

120. Let $n_1 < n_2 < n_3 < n_4 < n_5$ be positive integers such that $n_1 + n_2 + n_3 + n_4 + n_5 = 20$, the number of such distinct arrangements $(n_1, n_2, n_3, n_4, n_5)$ is **[JEE Advanced 2014, 3M]**

121. For $n \ge 2$ be an integer. Take n distinct points on a circle and join each pair of points by a line segment. Colour the line segment joining every pair of adjacent points by blue and the rest by red. If the number of red and blue line segments are equal, the value of n is **[JEE Advanced 2014, 3M]**

122. Six cards and six envelopes are numbered $1, 2, 3, 4, 5, 6$ and cards are to be placed in envelopes, so that each envelope contains exactly one card and no card is placed in the envelope bearing the same number and moreover the card numbered 1 is always placed in envelope numbered 2. Then the number of ways it can be done, is **[JEE Advanced 2014, 3M]**
(a) 264 (b) 265
(c) 53 (d) 67

123. The number of integers greater than 6000 that can be formed using the digits 3,5,6,7 and 8 without repetition, is **[JEE Main 2015, 4M]**
(a) 120 (b) 72
(c) 216 (d) 192

124. Let n be the number of ways in which 5 boys and 5 girls can stand in a queue in such away that all the girls stand consecutively in the queue. Let m be the number of ways in which 5 boys and 5 girls can stand in a queue in such a way that exactly four girls stand consecutively in the queue, the value of $\frac{m}{n}$ is **[JEE Advanced 2015, 3M]**

125. If all the words (with or without meaning having five letters, formed using the letters of the word SMALL and arranged as in a dictionary, then the position of the word SMALL is **[JEE Main 2016, 4M]**

(a) 59th (b) 52nd (c) 58th (d) 46th

126. A debate club consists of 6 girls and 4 boys. A team of 4 members is to be selected from this club including the selection of a captain (from among these 4 members) for the team. If the team has to include at most one boy, then the number of ways of selecting the team is **[JEE Advanced 2016, 3M]**

(a) 380 (b) 320
(c) 260 (d) 95

127. A man X has 7 friends, 4 of them are ladies and 3 are men. His wife Y also has 7 friends, 3 of them are ladies and 4 are men. Assume X and Y have no common friends. Then the total number of ways in which X and Y together can throw a party inviting 3 ladies and 3 men, so that 3 friends of each of X and Y are in this party, is **[JEE Main 2017, 4M]**

(a) 484 (b) 485
(c) 468 (d) 469

Answers

Exercise for Session 1

1. (c) **2.** (b) **3.** (c) **4.** (a) **5.** (c) **6.** (b)
7. (c) **8.** (c) **9.** (b) **10.** (b) **11.** (d)

Exercise for Session 2

1. (b) **2.** (c) **3.** (c) **4.** (d) **5.** (d) **6.** (b)
7. (d) **8.** (a) **9.** (b) **10.** (c) **11.** (b) **12.** (b)
13. (b) **14.** (b) **15.** (b) **16.** (d)

Exercise for Session 3

1. (a) **2.** (c) **3.** (a) **4.** (b) **5.** (c) **6.** (c)
7. (b) **8.** (b) **9.** (a) **10.** (a) **11.** (a)

Exercise for Session 4

1. (d) **2.** (b) **3.** (d) **4.** (c) **5.** (a) **6.** (c)
7. (a) **8.** (b) **9.** (b) **10.** (d) **11.** (a) **12.** (d)
13. (a) **14.** (d) **15.** (d) **16.** (c) **17.** (b) **18.** (a)

Exercise for Session 5

1. (c) **2.** (c) **3.** (d) **4.** (a) **5.** (b) **6.** (b)
7. (d) **8.** (a) **9.** (c) **10.** (c) **11.** (d) **12.** (a)
13. (b) **14.** (c) **15.** (d) **16.** (c) **17.** (b) **18.** (c)

Exercise for Session 6

1. (c) **2.** (c) **3.** (a) **4.** (a) **5.** (b) **6.** (a)
7. (c) **8.** (c) **9.** (c) **10.** (a) **11.** (d) **12.** (b)
13. (a) **14.** (c) **15.** (b) **16.** (b) **17.** (c) **18.** (d)
19. (c) **20.** (d) **21.** (c) **22.** (b) **23.** (a) **24.** (a)
25. (b) **26.** (c) **27.** (c) **28.** (b)

Exercise for Session 7

1. (b) **2.** (d) **3.** (a) **4.** (c) **5.** (c)

Chapter Exercises

1. (c) **2.** (a) **3.** (c) **4.** (b) **5.** (c) **6.** (c)
7. (a) **8.** (b) **9.** (b) **10.** (d) **11.** (a) **12.** (d)
13. (b) **14.** (b) **15.** (c) **16.** (a) **17.** (d) **18.** (c)
19. (d) **20.** (b) **21.** (c) **22.** (d) **23.** (d) **24.** (d)
25. (c) **26.** (a) **27.** (d) **28.** (a) **29.** (c) **30.** (b)
31. (a, c) **32.** (a,b,c,d) **33.** (a,b,c,d) **34.** (a,d) **35.** (a, c)
36. (b,d) **37.** (a, c) **38.** (b,c) **39.** (a,b,c,d) **40.** (b,c,d)
41. (a) **42.** (c) **43.** (c) **44.** (c) **45.** (b) **46.** (a)
47. (c) **48.** (c) **49.** (c) **50.** (c) **51.** (c) **52.** (b)
53. (b) **54.** (c) **55.** (d) **56.** (1) **57.** (0) **58.** (8)
59. (3) **60.** (7) **61.** (8) **62.** (6) **63.** (7) **64.** (8)
65. (8) **66.** (A) → (q); (B) → (r); (C) → (s); (D) → (p)
67. (A) → (r); (B) → (s); (C) → (q); (D) → (p)
68. (A) → (r); (B) → (s); (C) → (p); (D) → (q)
69. (A) → (s); (B) → (r); (C) → (s); (D) → (p)
70. (A) → (s); (B) → (q); (C) → (p); (D) → (s)
71. (b) **72.** (c) **73.** (d) **74.** (c) **75.** (d) **76.** (b)
77. (b) **78.** (d) **79.** (a) **80.** (d) **81.** (d) **82.** (a)
83. $x = n + 3$ **84.** $x = 3$ **85.** 4 **87.** 60 **88.** 42
89. 11520 **90.** ${}^{8}P_4 \times {}^{8}P_2 \times 10!$ **92.** ${}^{45}C_6$
93. $\frac{n(n-4)(n-5)}{6}$ **94.** 1728 **95.** $\frac{(3n)!}{6(n!)^3}$ **96.** $\frac{n^2(n+1)}{2}$
97. 4530 **99.** 96 **100.** (d) **101.** (c) **102.** (d)
103. (c) **104.** (c) **105.** (c)
106. (A) → (p); (B) → (s); (C) → (q); (D) → (q)
107. (c) **108.** (a) **109.** (c) **110.** (a) **111.** (c) **112.** (a)
113. (b) **114.** (b) **115.** (b) **116.** (a) **117.** (b) **118.** (a)
119. (5) **120.** (7) **121.** (5) **122.** (c) **123.** (d) **124.** (5)
125. (c) **126.** (a) **127.** (b)

Solutions

1. ${}^{9}C_5 - {}^{9-2}C_{5-2} = {}^{9}C_4 - {}^{7}C_3$

$= 126 - 35 = 91$

2. Let $a = 2x - 1, b = 2y - 1, c = 2z - 1, d = 2w - 1$

where, $x, y, z, w \in N$

Then, $a + b + c + d = 20$

$\Rightarrow \quad x + y + z + w = 12$

$\therefore$ Number of ordered quadruplet $= {}^{12-1}C_{4-1}$

$= {}^{11}C_3 = \dfrac{11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3} = 165$

3. l = LCM of 8!, 10! and 12! = 12!

and h = HCF of 8!, 10! and 12! = 8!

$\therefore \quad \dfrac{l}{h} = \dfrac{12!}{8!} = 12 \cdot 11 \cdot 10 \cdot 9 = 11880$

4. ${}^{n+1}C_{n-2} - {}^{n+1}C_{n-1} \le 100$

$\Rightarrow \quad {}^{n+1}C_3 - {}^{n+1}C_2 \le 100$

$\Rightarrow \quad \dfrac{(n+1)n(n-1)}{6} - \dfrac{(n+1)n}{2} \le 100$

$\Rightarrow \quad n(n+1)(n-4) \le 600$

It is true for $n = 2, 3, 4, 5, 6, 7, 8, 9$

5. Coefficient of x^{11} in $(x + x^2 + x^3 + x^4 + x^5 + x^6)^3$

= Coefficient of x^8 in $(1 + x + x^2 + x^3 + x^4 + x^5)^3$

= Coefficient of x^8 in $(1 - x^6)^3(1 - x)^{-3}$

= Coefficient of x^8 in $(1 - 3x^6)(1 + {}^{3}C_1 x + \ldots)$

$= {}^{10}C_2 - 3 \times {}^{4}C_2 = 45 - 18 = 27$

6. ${}^{n}C_2, n \in N$, infinite numbers.

7. $\because$ 13 lines pass through A and 11 lines pass through B.

$\therefore$ Number of intersection points

$= {}^{37}C_2 - {}^{13}C_2 - {}^{11}C_2 + 2$ [$\because$ two points A and B]

$= 535$

8. $\because \quad a = 182bc$

$\Rightarrow \quad (x+2)! = 182 \times {}^{x}P_{11} \times (x-11)!$

$\Rightarrow \quad (x+2)! = 182 \times x!$

$\Rightarrow \quad (x+2)(x+1) = 14 \times 13$

$\Rightarrow \quad x + 1 = 13$

$\therefore \quad x = 12$

9. $6 + 5 \times 5 + 5 \times 5 \times 4 = 131$

10. The letters in alphabetical order are $abcde$

$a \to 4! = 24$ $\qquad$ $db \to 3! = 6$

$b \to 4! = 24$ $\qquad$ $dc \to 3! = 6$

$c \to 4! = 24$ $\qquad$ $dea \to 2! = 2$

$da \to 3! = 6$ $\qquad$ $debac \to 1$

$\therefore$ The rank of $debac = 24 + 24 + 24 + 6 + 6 + 6 + 2 + 1 = 93$

11. Number of different tickets

$= 20 + 19 + 18 + 17 + \ldots + 3 + 2 + 1 = 210$

12. Let $A = \{a_1, a_2, a_3, \ldots, a_n\}$

(i) $a_i \in P, a_i \in Q$ $\qquad$ (ii) $a_i \in P, a_i \notin Q$

(iii) $a_i \notin P, a_i \in Q$ $\qquad$ (iv) $a_i \notin P, a_i \notin Q$, where $1 \le i \le n$

$\because \quad P \cap Q = \phi$ $\qquad$ [cases in favour 3 i.e., (ii), (iii), (iv)]

$\therefore$ Required number of ways $= 3^n$

13. Total points on all three lines $= m + n + k$

$\therefore$ Maximum number of triangles $= {}^{m+n+k}C_3 - {}^{m}C_3 - {}^{n}C_3 - {}^{k}C_3$

14. Required number of triplets = Total number of triplets without restrictions – Number of triplets with all different coordinates

$= n^3 - {}^{n}P_3$

15. Let middle largest digit be r, then digits available for remaining four places are 0, 1, 2, 3, ..., $r-1$.

Number of ways filling remaining four places

$$= \sum_{r=4}^{9} ({}^{r}P_4 - {}^{r-1}P_3) = \sum_{r=4}^{9} (r-1) \times {}^{r-1}P_3$$

$$= 3 \times {}^{3}P_3 + 4 \times {}^{4}P_3 + 5 \times {}^{5}P_3 + 6 \times {}^{6}P_3 + 7 \times {}^{7}P_3 + 8 \times {}^{8}P_3$$

$= 5292 = 2^2 \cdot 3^3 \cdot 7^2$

16. Required number of words $= 6! - 5! - 5! + 4! = 504$

17. Let x, y, z be the friends and a, b, c denote the case when x is invited a times, y is invited b times and z is invited c times. Now, we have the following possibilities $(a, b, c) = (1, 2, 3)$ or $(2, 2, 2)$ or $(3, 3, 0)$ [grouping of 6 days of week]

Hence, the total number of ways

$$= \frac{6!}{1!\,2!\,3!} \times 3! + \frac{6!}{2!\,2!\,2!} \times \frac{3!}{3!} + \frac{6!}{3!\,3!\,0!} \times \frac{3!}{2!}$$

$= 360 + 90 + 60 = 510$

18. If $y = n$, then x takes values from 1 to $n - 1$ and z can take values from 0 to n (i.e., $(n + 1)$ values). Thus, for each value of y $(2 \le y \le 9)$, x and z take $(n - 1)(n + 1)$ values.

Hence, the 3-digit numbers are of the form xyz

$$= \sum_{x=2}^{9} (n-1)(n+1) = \sum_{x=2}^{9} (n^2 - 1) = 276$$

19. The letters in alphabetical order are EEMRTU

$\text{E} \to 5! = 120$

$\text{MEERTU} \to 1! = 1$

$\text{MEERUT} \to 1$

$\therefore \quad$ Rank of MEERUT $= 120 + 1 + 1 = 122$

20. The candidates can be ranked in 10! ways. In half of these ways, A_1 is above A_2 and in another half, A_2 is above A_1. So, required number of ways $= \dfrac{10!}{2}$.

21. Any selection of four digits from the ten digits 0, 1, 2, 3, ..., 9 gives one number.

So, the required number of numbers $= {}^{10}C_4 = 210$

22. As $0 < x < 1$, we have $p < q$

The number of rational numbers $= 5 + 4 + 3 + 2 + 1 = 15$

When p, q have a common factor, we get some rational numbers which are not different from those already counted.

There are 4 such numbers $\frac{2}{4}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}$.

$\therefore$ The required number of rational numbers $= 15 - 4 = 11$

23. We have,

$$24 = 2 \times 3 \times 4 = 2 \times 2 \times 6$$
$$= 1 \times 6 \times 4 = 1 \times 3 \times 8$$
$$= 1 \times 2 \times 12 = 1 \times 1 \times 24$$

The number of positive integral solutions of $xyz = 24$ is

$$3! + \frac{3!}{2!} + 3! + 3! + 3! + \frac{3!}{2!} = 30$$

and number of integral solutions having two negative factors is ${}^3C_2 \times 30 = 90$.

Hence, number of integral solutions $= 30 + 90 = 120$

24. Total number of triangles = Number of triangles with vertices on sides $[(AB, BC, CD) + (AB, BC, DA) + (AB, CD, DA) + (BC, CD, DA)]$

$= {}^3C_1 \times {}^4C_1 \times {}^5C_1 + {}^3C_1 \times {}^4C_1 \times {}^6C_1 + {}^3C_1 \times {}^5C_1 \times {}^6C_1 + {}^4C_1 \times {}^5C_1 \times {}^6C_1$

$= 60 + 72 + 90 + 120 = 342$

25. $\because$ 16 horses = 10 horses + (A, B, C, D, E, F)

$\therefore$ The number of ways $= {}^{10}C_3 \times$ (Number of ways of choosing out of A, B, C, D, E, F, so that AD, BE and CF are not together)

$= {}^{10}C_3 \times$ (One from each of pairs AD, BE, CF)

$= {}^{10}C_3 \times {}^2C_1 \times {}^2C_1 \times {}^2C_1 = 960$

26. We have, $i^3 + ai^2 + bi + c = 0$

and $(-i)^3 + a(-i)^2 + b(-i) + c = 0$

$\Rightarrow \quad (c - a) + (b - 1)i = 0$

and $\quad (c - a) - i(b - 1) = 0$

$\Rightarrow \quad b = 1, a = c$

Thus, total number of such polynomials $= {}^{10}C_1 = 10$

27. $\sum_{i=1}^{k} \frac{1}{x_i} = \frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_k} = \frac{\sum_{i=1}^{k} x_i}{n} = \frac{75}{n}$

[as LCM of $x_1, x_2, x_3, \ldots, x_k$ is n]

28. Let '1' be associated with 'r', $r \in \{1, 2, 3, 4, 5\}$, then '2' can be associated with $r, r + 1, \ldots, 5$. Let '2' be associated with 'j' then 3 can be associated with $j, j + 1, \ldots, 5$. Thus, required number of functions

$$\sum_{r=1}^{5} \left(\sum_{j=r}^{5} (6 - j) \right) = \sum_{r=1}^{5} \frac{(6 - r)(7 - r)}{2} = 35$$

29. Clearly, each player will play 9 games.

$\therefore$ Total number of games $= {}^{10}C_2 = 45$

Clearly, $w_i + l_i = 9$ and $\Sigma w_i = \Sigma l_i = 45$

$\Rightarrow \quad w_i = 9 - l_i \Rightarrow w_i^2 = 81 - 18l_i + l_i^2$

$\Rightarrow \quad \Sigma w_i^2 = \Sigma 81 - 18 \Sigma l_i + \Sigma l_i^2$

$= 81 \times 10 - 18 \times 45 + \Sigma l_i^2 = \Sigma l_i^2$

30. The number of matches in the first round $= {}^6C_2 + {}^6C_2 = 30$

The number of matches in the next round $= {}^6C_2 = 15$

and the number of matches in the semi-final round $= {}^4C_2 = 6$

$\therefore$ The required number of matches $= 30 + 15 + 6 + 2 = 53$

[$\because$ for 'best of three' atleast two matches are played]

31. $E_3(300!) = \left[\frac{300}{3}\right] + \left[\frac{300}{3^2}\right] + \left[\frac{300}{3^3}\right] + \left[\frac{300}{3^4}\right] + \left[\frac{300}{3^5}\right]$

$= 100 + 33 + 11 + 3 + 1 = 148$

$\therefore \quad m = 148$

and $E_2(150!) = \left[\frac{150}{2}\right] + \left[\frac{150}{2^2}\right] + \left[\frac{150}{2^3}\right] + \ldots + \left[\frac{150}{2^7}\right]$

$= 75 + 37 + 18 + 9 + 4 + 2 + 1 = 146$

$\therefore \quad n - 2 = 146$

$\Rightarrow \quad n = 148$

32. $\because \quad E_2(102!) = 98$, $E_3(102!) = 49$,

$E_5(102!) = 24$ and $E_7(102!) = 16$

$\therefore \ \alpha = 98, \beta = 49, \gamma = 24$ and $\delta = 16$

33. Triplets with

(i) $x = y < z$ (ii) $x < y < z$

(iii) $y < x < z$

can be chosen in ${}^{n+1}C_2, {}^{n+1}C_3, {}^{n+1}C_3$ ways.

$\therefore \quad {}^{n+1}C_2 + {}^{n+1}C_3 + {}^{n+1}C_3 = {}^{n+2}C_3 + {}^{n+1}C_3$

$= 2({}^{n+2}C_3) - {}^{n+1}C_2$

$= \frac{n(n+1)(2n+1)}{6}$

34. **When x is odd**

Unit's place filled by 1, 3, 5, 7, 9.

$\therefore \quad x = 8 \times 8 \times 7 \times 5 = 2240$

When x is even

Unit's place filled by 0, 2, 4, 6, 8.

$\therefore \quad y = 8 \times 8 \times 7 \times 4 + 9 \times 8 \times 7 \times 1 = 2296$

$\Rightarrow \quad x < y$ and $|x - y| = 56$

35. $\sum_{A \in X} \min A$

$$\sum_{r=1}^{n-1} r(n - r) = n \sum_{r=1}^{n-1} r - \sum_{r=1}^{n-1} r^2$$
$$= \frac{n(n-1)n}{2} - \frac{(n-1)n(2n-1)}{6}$$
$$= \frac{(n+1) \cdot n \cdot (n-1)}{1 \cdot 2 \cdot 3} = {}^{n+1}C_3 = \frac{n(n^2 - 1)}{6}$$

36. $\because \ p = 2520 = 2^3 \cdot 3^2 \cdot 5^1 \cdot 7^1$

$= 6 \cdot 2^2 \cdot 3^1 \cdot 5^1 \cdot 7^1 = 9 \cdot 2^3 \cdot 5^1 \cdot 7^1$

$\therefore \quad x = (2 + 1)(1 + 1)(1 + 1)(1 + 1) = 24$

and $\quad y = (3 + 1)(1 + 1)(1 + 1) = 16$.

37. $\because x_1 + x_2 + x_3 + \ldots + x_n = r, \forall x_i \geq 1, (1 \leq i \leq n)$

Total number of such solutions $= {}^{r-1}C_{n-1} = {}^{r-1}C_{r-n}$

38. $\because x + x + 1 + y = 24$

$\Rightarrow \quad y = 23 - 2x$

Let N = Total number of fights subject to the condition that any two children of one team do not fight.

$\therefore \quad N = {}^{24}C_2 - ({}^{x}C_2 + {}^{x+1}C_2 + {}^{y}C_2)$

$= {}^{24}C_2 - ({}^{x}C_2 + {}^{x+1}C_2 + {}^{23-2x}C_2)$

$= 23 - 3x^2 + 45x$

$\therefore \quad \dfrac{dN}{dx} = 0 - 6x + 45$

For maximum or minimum, put $\dfrac{dN}{dx} = 0 \Rightarrow x = 7.5$

$\Rightarrow \quad x = 7 \quad [\because x \in I]$

Now, $\dfrac{d^2N}{dx^2} < 0$

$\therefore$ N will be maximum when $x = 7$

and $N = 23 - 3(7)^2 + 45 \times 7 = 191$

39. $\because x + y + z = 3n$

$\Rightarrow \quad N =$ Coefficient of α^{3n} in $(1 + \alpha + \alpha^2 + \ldots + \alpha^{2n})^3$

$=$ Coefficient of α^{3n} in $(1 - \alpha^{2n+1})^3 (1 - \alpha)^{-3}$

$=$ Coefficient of α^{3n} in $(1 - 3\alpha^{2n+1})\ (1 + {}^3C_1\,\alpha + \ldots)$

$= {}^{3n+2}C_{3n} - 3 \cdot {}^{n+1}C_{n-1}$

$= {}^{3n+2}C_2 - 3 \cdot {}^{n+1}C_2 = 3n^2 + 3n + 1$

$\therefore \quad N - 1 = 3n(n+1)$

40. Since, α can be subtracted from β without borrowing, if $y_i \geq x_i$, for $i = 1, 2, 3$.

Let $x_i = \lambda$

If $i = 1$, then $\lambda = 1, 2, 3, \ldots, 9$

and if $i = 2$ and 3, then $\lambda = 0, 1, 2, 3, \ldots, 9$

Hence, total number of ways of choosing the pair α, β

$$= \left(\sum_{\lambda=1}^{9} (10 - \lambda)\right)\left(\sum_{\lambda=0}^{9} (10 - \lambda)\right)^2 = (45)(55)^2$$

41. There are 9 letters

T, T, R, R, R, E, O, I, S

λ = Number of four lettered words (no restriction)

= Coefficient of x^4 in

$$4!\left(1 + \frac{x}{1!} + \frac{x^2}{2!}\right)\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}\right)(1 + x)^4 = 626$$

μ = Number of four lettered words (no vowels)

= Coefficient of x^4 in

$$4!\left(1 + \frac{x}{1!} + \frac{x^2}{2!}\right)\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}\right)(1 + x) = 38$$

$\therefore$ Required ways (atleast one vowel) $= 626 - 38 = 588$

42. $\times$ T $\times$ E $\times$ O $\times$ I $\times$ S $\times$ T $\times$

There are 7 available places for RRR.

$\therefore$ Required ways $= {}^{7}C_3 \times \dfrac{6!}{2!} = 12600$

43. $\times$ E $\times$ T T $\times$ O $\times$ I $\times$ S $\times$

Number of ways having TT together and RRR separated

$= {}^{6}C_3 \times 5! = 2400$

Hence, number of arrangements of the word W, if R's as well as T's are separated $= 12600 - 2400 = 10200$

44. $\times$ U $\times$ CC $\times$ E $\times$

Hence, required number of ways $= {}^{4}C_3 \times 3! = 24$

45. $\times$ U $\times$ C $\times$ C $\times$ E $\times$

There are five available places for three SSS.

$\therefore$ Total number of ways no two S's together $= {}^{5}C_3 \times \dfrac{4!}{2!} = 120$

Hence, number of words having CC separated and SSS separated $= 120 - 24 = 96$

46. Total number of ways $= \dfrac{7!}{2!3!} = 420$

Consonants in SUCCESS are S, C, C, S, S

Number of ways arranging consonants $= \dfrac{5!}{2!3!} = 10$

Hence, number of words in which consonants appear in alphabetic order $= \dfrac{420}{10} = 42$

47. $\times$A $\times$ A $\times$ N $\times$ G $\times$ E $\times$

Hence, required number of ways $= {}^{6}C_2 \times \dfrac{5!}{2!} = 900$

48. The number of ways in which two A's are together

i.e., $\times$ A A $\times$ N $\times$ G $\times$ E $\times$ is ${}^{5}C_2 \times 4! = 240$

Hence, number of ways in which neither two R's no two A's come together $= 900 - 240 = 660$

49. The letters in alphabetical order are AAEGNRR

$\text{AA} \rightarrow \dfrac{5!}{2!} = 60\text{ARA} \rightarrow 4! = 24$

$\text{AE} \rightarrow \dfrac{5!}{2!} = 60\,\text{ARE} \rightarrow 4! = 24$

$\text{AG} \rightarrow \dfrac{5!}{2!} = 60\text{ARG} \rightarrow 4! = 24$

$\text{AN} \rightarrow \dfrac{5!}{2!} = 60\text{ARN} \rightarrow 4! = 24$

$\text{ARRAE} \rightarrow 2! = 2 \qquad \text{ARRAG} \rightarrow 2! = 2$

$\text{ARRANEG} \rightarrow 1 \qquad \text{ARRANGE} \rightarrow 1$

$\therefore$ Rank in dictionary

$= 60 + 60 + 60 + 60 + 24 + 24 + 24 + 24 + 2 + 2 + 1 + 1 = 342$

■ ***Sol.*** (Q. Nos. 50-52)

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n} \Rightarrow (x - n)(y - n) = n^2$$

$\therefore \quad x = n + \lambda, y = n + \dfrac{n^2}{\lambda}$

where, λ is divisor of n^2.

The number of integral solutions (x, y) is equal to the number of divisors of n^2.

If $n = 3, n^2 = 9 = 3^2$, then the equation has 3 solutions.

$(x, y) = (4, 12), (6, 6), (12, 4)$

50. $\because \quad 10^2 = 2^2 \cdot 5^2$

$\therefore \quad S(10) = 3 \times 3 = 9$

51. $\because \quad 6^2 = 2^2 \cdot 3^2$

$\Rightarrow \quad S(6) = 3 \times 3 = 9$ and $7^2 \Rightarrow S(7) = 3$

$\therefore \quad S(6) + S(7) = 12$

Also, $8^2 = 2^6$

$\Rightarrow \quad S(8) = 7$ and $9^2 = 3^4 \Rightarrow S(9) = 5$

$\therefore \quad S(8) + S(9) = 12$

$\Rightarrow \quad S(6) + S(7) = S(8) + S(9) = 12$

52. $\because \quad 1^2 \to S(1) = 1, 2^2 \to S(2) = 3, 3^2 \to S(3) = 3,$

$4^2 \to 2^4 \to S(4) = 5, 5^2 \to S(5) = 3, S(6) = 9$

$S(7) = 3, S(8) = 7, S(9) = 5$ and $S(10) = 9$ [from above]

$$\therefore \quad \sum_{r=1}^{10} S(r) = S(1) + S(2) + S(3) + S(4) + S(5) + S(6) + S(7) + S(8) + S(9) + S(10)$$

$$= 1 + 3 + 3 + 5 + 3 + 9 + 3 + 7 + 5 + 9 = 48$$

53. $\because \quad f(6) = {}^6C_0 + {}^5C_1 + {}^4C_2 + {}^3C_3 = 13$

$\therefore f\{f(6)\} = f(13) = {}^{13}C_0 + {}^{12}C_1 + {}^{11}C_2 + {}^{10}C_3 + {}^9C_4 + {}^8C_5 + {}^7C_6$

$= 1 + 12 + 55 + 120 + 126 + 56 + 7 = 377$

54. $\because \quad f(1) = {}^1C_0 = 1, f(2) = {}^2C_0 + {}^1C_1 = 2, f(3) = {}^3C_0 + {}^2C_1 = 3,$

$f(4) = 5$ [given]

and $f(5) = {}^5C_0 + {}^4C_1 + {}^3C_2 = 8$

Thus, we say that $f(n) > n$ for $n = 4, 5, 6, \ldots$

Hence, number of solutions for $f(n) = n$ is 3.

55. Number of superstars $= f(4) = 5$

and number of junior artists $= f(3) = 3$

$\times S_1 \times S_2 \times S_3 \times S_4 \times S_5 \times$ [S_i for superstars]

$\therefore$ Required number of ways $= {}^6C_3 \times 5! \times 3! = 14400$

56. For $n \geq 10$, the number of zeros in $n! \geq 2$

$\therefore \quad 1! + 2! + 3! + 4! + \ldots + 97! = \ldots 13$

$\therefore \quad$ Ten's digit $= 1$

57. $\because \quad {}^{100}C_{50} = \dfrac{100!}{(50!)^2}$

$$\therefore \quad E_7(100!) = \left[\frac{100}{7}\right] + \left[\frac{100}{7^2}\right] = 14 + 2 = 16$$

and $\quad E_7(50!) = \left[\dfrac{50}{7}\right] + \left[\dfrac{50}{7^2}\right] = 7 + 1 = 8$

$\therefore \quad E_7$ in $({}^{100}C_{50}) = 16 - 2 \times 8 = 0$

58. ${}^{n-1}C_3 - {}^{n-2}C_3 = 15$

$\Rightarrow \quad n = 8$ [$\because P_n = {}^{n-2}C_3$]

59. For BULBUL, the letters in alphabetical order are BBLLUU

$$\text{BB} \to \frac{4!}{2!2!} = 6 \text{ BULBLU} \to 1$$

$$\text{BL} \to \frac{4!}{2!} = 12 \text{ BULBUL} \to 1$$

$$\text{BU} \to \frac{3!}{2!} = 39$$

$\therefore \quad m = 6 + 12 + 3 + 1 + 1 = 23$

For NANNU

The letters in alphabetical order are ANNNU

$$\text{A} \to \frac{4!}{3!} = 4 \text{ NANNU} \to 1$$

$\therefore \quad n = 4 + 1 = 5$

Hence, $m - 4n = 23 - 20 = 3$

60. Each of the n digits can be anyone of the three 2, 5 or 7.

$\therefore$ The number of n-digit numbers is 3^n.

$\Rightarrow \quad 3^n > 900$

$\Rightarrow \quad n = 7, 8, 9 \ldots$

Hence, smallest value of n is 7.

61. $a + b + c = 21 \Rightarrow 3b = 21 \Rightarrow b = 7$ [$\because a + c = 2b$]

$\Rightarrow \quad a + b + c = 21 \Rightarrow a + c = 14$

$\Rightarrow \quad \lambda = {}^{14-1}C_{2-1} = {}^{13}C_1 = 13$

Hence, $\lambda - 5 = 13 - 5 = 8$

62. $2\lambda =$ Number of selecting 3 member subsets of {1, 2, 3, ..., 29} which are in

GP with common ratio (2 or 3 or 4 or 5).

$$= \left[\frac{29}{2^2}\right] + \left[\frac{29}{3^2}\right] + \left[\frac{29}{4^2}\right] + \left[\frac{29}{5^2}\right]$$

$= 7 + 3 + 1 + 1 = 12$

$\therefore \quad \lambda = 6$

63. The number of students answering exactly $i (1 \leq i \leq n-1)$ questions wrongly is $2^{n-i} - 2^{n-i-1}$. The number of students answering all n questions wrongly is 2^0.

Hence, the total number of wrong answers

$$\sum_{i=1}^{n-1} i(2^{n-i} - 2^{n-i-1} + n(2^0) = 127$$

$$\Rightarrow 2^{n-1} + 2^{n-2} + 2^{n-3} + \ldots + 2^1 + 2^0 = 127$$

$\Rightarrow \quad 2^n - 1 = 127$

$\Rightarrow \quad 2^n = 128 = 2^7$

$\therefore \quad n = 7$

64. The sum of digits is divisible by 3.

i.e., $8 + 7 + 6 + 4 + 2 + x + y$ or $27 + x + y$ is divisible by 3

$\therefore \quad x + y$ must be divisible by 3.

Then, possible ordered pairs are

(0, 3), (3, 0), (1, 5), (5, 1), (0, 9), (9, 0), (3, 9), (9, 3)

$\therefore$ Number of ordered pairs = 8

65. Number of circles through ACD, ACE, ADE intersect the line $AB = 3$ and

Number of circles through BCD, BCE, BDE intersect the line $AB = 3$ and

Number of circles through CDE intersects the line $AB = 2$

Hence, number of points of intersection $= 3 + 3 + 2 = 8$

66. **(A)** ${}^{n+4}C_{n+1} - {}^{n+3}C_n = 15(n+2)$

$\Rightarrow \quad {}^{n+3}C_{n+1} + {}^{n+3}C_n - {}^{n+3}C_n = 15(n+2)$

$\Rightarrow \quad {}^{n+3}C_{n+1} = 15(n+2)$

$\Rightarrow \quad {}^{n+3}C_2 = 15(n+2)$

$\Rightarrow \quad \dfrac{(n+3)}{2} = 15 \Rightarrow n = 27$

(B) $11\cdot {}^{n}P_4 = 20\,.\,{}^{n-2}P_4$

$\Rightarrow 11.n(n-1)(n-2)(n-3) = 20(n-2)(n-3)(n-4)(n-5)$

$\Rightarrow \quad 11n(n-1) = 20(n-4)(n-5) \qquad [\because n \neq 2, 3]$

$\Rightarrow \quad 9n^2 - 169n + 400 = 0$

$\therefore \quad n = 16 \;\Rightarrow\; n \neq \dfrac{25}{9}$

(C) ${}^{2n}C_3 = 11\cdot {}^{n}C_3$

$\Rightarrow \quad \dfrac{2n(2n-1)(2n-2)}{1\cdot 2\cdot 3} = \dfrac{11\cdot n(n-1)(n-2)}{1\cdot 2\cdot 3}$

$\therefore \quad n = 6$

(D) ${}^{n+2}C_8 : {}^{n-2}P_4 = 57:16 \;\Rightarrow\; \dfrac{{}^{n+2}C_8}{{}^{n-2}P_4} = \dfrac{57}{16}$

$\Rightarrow \quad \dfrac{(n+2)}{8}\cdot\dfrac{n+1}{7}\cdot\dfrac{(n)}{6}\cdot\dfrac{(n-1)}{5}\cdot\dfrac{{}^{n-2}C_4}{{}^{n-2}P_4} = \dfrac{57}{16}$

$\Rightarrow \quad (n+2)(n+1)\,n(n-1) = 21\cdot 20\cdot 19\cdot 18$

$\therefore \quad n = 19$

67. **(A)** $\dfrac{{}^{n}P_m}{m!} = {}^{n}C_m$

(B) Required ways $= \underbrace{m\times m\times m\times \ldots \times m}_{n \text{ times}} = m^n$

(C) $\times G \times G \times G \times G \times \ldots \times G \times$

Here, number of gaps for red balls $= m - 1 + 1 = m$

$\therefore$ Required ways $= {}^{m}C_n$

(D) Required ways $= \dfrac{n\times n\times n\times \ldots \times n}{m \text{ times}} = n^m$

68. **(A)** Required lines $= {}^{10}C_2 - {}^{4}C_2 + 1 = 40$

(B) Maximum number of points $= {}^{10}C_2 = 45$

(C) Maximum number of points $= {}^{6}C_2 \times 2 = 30$

(D) Maximum number of points $= {}^{6}C_2 \times 4 = 60$

69. **(A)** Number of rectangles $= {}^{7}C_2 \times {}^{7}C_2 = 441$

[select two vertical and two horizontal lines]

(B) Number of squares $= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = 91$

(C) First square can be selected in 36 ways, second square can be selected in $(36 - 6 - 5) = 25$ ways and third square can be selected in $(25 - 5 - 4) = 16$ ways.

$\therefore$ Required ways $= 36 \times 25 \times 15 = 2400$

(D) $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 11$

where, $a_1, a_2, a_3, a_4, a_5, a_6, \geq 1$

$\therefore$ Required ways $= {}^{11-1}C_{6-1} = {}^{10}C_5$

70. **(A)** $\square\square\square$ or $\square\square\square$ Required ways

$= 1\times 1\times 1\times \dfrac{3!}{2!} + 1\times 1\times 1\times \dfrac{3!}{2!} = 6$

(B) $\square\square\square$ or $\square\square\square$ Required ways

$= \dfrac{{}^{5}C_3 \times {}^{2}C_1 \times {}^{1}C_1}{2} + \dfrac{{}^{5}C_2 \times {}^{3}C_2 \times {}^{1}C_1}{2} = 25$

(C) $\square\square\square$ or $\square\square\square$ Required ways $= 1\times 1\times 1 + 1\times 1\times 1 = 2$

(D) $\square\square\square$ or $\square\square\square$ Required ways $= 3 + 3 = 6$

71. Statement-1 is True

$\because$ $6! = 720 = 8\times 9\times 10$ i.e., Product of $6 - 3 = 3$ consecutive integers and Statement-2 is also true, but Statement-2 is not a correct explanation for Statement-1

72. For the number exactly divisible by 4, then last two digits must be divisible by 4, the last two digits are 12, 16, 24, 32, 36, 52, 56, 64, 72, 76. Total 10 ways.

Now, the remaining two first places on the left of 4 digit numbers are to be filled from remaining 5 digits and this can be done in ${}^{5}P_2 = 20$ ways.

$\therefore$ Required number of ways $= 20\times 10 = 200$

Hence, Statement-1 is true and Statement-2 is false.

73. $\because 10! = 2^8\cdot 3^4\cdot 5^2\cdot 7^1$

$\therefore$ Total number of divisors

$= (8+1)(4+1)(2+1)(1+1) = 270$

Hence, Statement-1 is false and Statement-2 is true.

74. $\because$ Number of permutations of n dissimilar things taken n at a time $= {}^{n}P_n = n!$

$\therefore$ Statement-1 is true and Statement-2 is false.

[$\because$ number of function $= n^n$]

75. $\because x_1\, x_2\, x_3\, x_4 = 2\cdot 5\cdot 7\cdot 11$

Each of 2, 5, 7, 11 can assign in 4 ways.

$\therefore$ Required number of solutions $= 4\times 4\times 4\times 4 = 4^4 = 2^8 = 256$

Hence, Statement-1 is false and Statement-2 is true.

76. $\because a + c = 2b$

i.e., sum of two numbers is even, then both numbers are even or odd. In {1,2,3,4,...,21}, 11 numbers are odd and 10 numbers are even.

Then, total number of ways $= {}^{11}C_2 + {}^{10}C_2 = 55 + 45 = 100$

Hence, both statements are true but Statement-2 is not a correct explanation for Statement-1.

77. $\because$ $N = 12600 = 2^3\cdot 3^2\cdot 5^2\cdot 7^1$

$\therefore$ Number of even divisors $= 3\cdot(2+1)\cdot(2+1)\cdot(1+1) = 54$

Both statements are true but Statement-2 is not a correct explanation for Statement-1.

78. We know that a number is divisible by 3, if the sum of its digits is divisible by 3. Now, out of 0, 1, 2, 3, 4, 5, 6 if we take 1,2,4,5,6 or 1,2,3,4,5 or 0,3,4,5,6 or 0,2,3,4,6 or 0,1,3,5,6 or 0,1,2,4,5 or 0,1,2,3,6

$\therefore$ Total number of ways $= 2\times {}^{5}P_5 + 5\times({}^{5}P_5 - {}^{4}P_4)$

$= 240 + 480$

$= 720$

Statement-1 is false, Statement-2 is true.

79. The sum of the digits in the ten's place

= The sum of the digits in the unit's place

$= (4-1)!(3+4+5+6) = 108$

Both statements are true and Statement-2 is a correct explanation for Statement-1.

80. Number of planes each containing three points

$= {}^{p}C_3 - {}^{q}C_3 + 1$

$\therefore$ Statement-1 is false and Statement-2 is always true.

81. $\because \quad {}^{50}C_{10} = \dfrac{50!}{10!40!}$

$\therefore \quad E_3(50!) = \left[\dfrac{50}{3}\right] + \left[\dfrac{50}{9}\right] + \left[\dfrac{50}{27}\right] + \left[\dfrac{50}{81}\right] + \ldots$

$= 16 + 5 + 1 + 0 + \ldots = 22$

$E_3(40!) = \left[\dfrac{40}{3}\right] + \left[\dfrac{40}{9}\right] + \left[\dfrac{40}{27}\right] + \left[\dfrac{40}{81}\right] + \ldots$

$= 13 + 4 + 1 + 0 = 18$

and $\quad E_3(10!) = \left[\dfrac{10}{3}\right] + \left[\dfrac{10}{9}\right] + \left[\dfrac{10}{27}\right] + \ldots = 3 + 1 + 0 = 4$

Hence, highest power of 3 in ${}^{50}C_{10} = 22 - (18 + 4) = 0$

$\therefore$ Statement-1 is false, Statement-2 is true.

82. Number of diagonals in quindecagon $= {}^{15}C_2 - 15 = 105 - 15 = 90$

Both statements are true and Statement-2 is a correct explanation for Statement-1.

83. We have, ${}^nC_{n-r} + 3{}^nC_{n-r+1} + 3{}^nC_{n-r+2} + {}^nC_{n-r+3} = {}^nC_r$

$\Leftrightarrow ({}^nC_{n-r} + {}^nC_{n-r+1}) + 2({}^nC_{n-r+1} + {}^nC_{n-r+2}) + ({}^nC_{n-r+2} + {}^nC_{n-r+3}) = {}^xC_r$

$\Leftrightarrow \quad {}^{n+1}C_{n-r+1} + 2.{}^{n+1}C_{n-r+2} + {}^{n+1}C_{n-r+3} = {}^xC_r$

$\Leftrightarrow ({}^{n+1}C_{n-r+1} + {}^{n+1}C_{n-r+2}) + ({}^{n+1}C_{n-r+2} + {}^{n+1}C_{n-r+3}) = {}^xC_r$

$\Leftrightarrow \quad {}^{n+2}C_{n-r+2} + {}^{n+2}C_{n-r+3} = {}^xC_r$

$\Leftrightarrow \quad {}^{n+3}C_{n-r+3} = {}^xC_r$

$\Leftrightarrow \quad {}^{n+3}C_r = {}^xC_r \qquad [\because {}^nC_r = {}^nC_{n-r}]$

Hence, $\quad x = n + 3$

84. We have, $\quad 3^{x+1}C_2 + P_2 \cdot x = 4^xA_2$

$\Leftrightarrow \quad \dfrac{3(x+1)x}{1 \cdot 2} + 2!x = 4 \cdot x(x-1)$

$\Leftrightarrow \quad 3x^2 + 3x + 4x = 8x^2 - 8x$

$\Leftrightarrow \quad 5x^2 - 15x = 0$

$\Leftrightarrow \quad 5x(x-3) = 0$

$\therefore \quad x \neq 0 \qquad [\because x \in N]$

Hence, $x = 3$ is the solution of the given equation.

85. We have, $\quad x_n = \dfrac{195}{4P_n} - \dfrac{{}^{n+3}A_3}{P_{n+1}}$

$\therefore \quad x_n = \dfrac{195}{4 \cdot n!} - \dfrac{(n+3)(n+2)(n+1)}{(n+1)!}$

$= -\dfrac{195}{4 \cdot n!} - \dfrac{(n+3)(n+2)}{n!}$

$= \dfrac{195 - 4n^2 - 20n - 24}{4 \cdot n!} = \dfrac{171 - 4n^2 - 20n}{4 \cdot n!}$

$\because \quad x_n$ is positive.

$\therefore \quad \dfrac{171 - 4n^2 - 20n}{4 \cdot n!} > 0$

$\Rightarrow \quad 4n^2 + 20n - 171 < 0$

which is true for $n = 1, 2, 3, 4$

Hence, the given sequence (x_n) has 4 positive terms.

86. We have, $\quad {}^{n-1}C_3 + {}^{n-1}C_4 > {}^nC_3 \qquad [\because {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r]$

$\Leftrightarrow \quad {}^nC_4 > {}^nC_3$

$\Leftrightarrow \quad \dfrac{n!}{4!(n-4)!} > \dfrac{n!}{3!(n-3)!}$

$\Leftrightarrow \quad \dfrac{1}{4(n-4)!} > \dfrac{1}{(n-3)(n-4)!} \qquad [\because m! = m(m-1)!]$

$\Leftrightarrow \quad n - 3 > 4 \Leftrightarrow n > 7$

87. Now, let sides of game be A and B. Given 5 married couples, i.e., 5 husbands and 5 wives. Now 2 husbands for two sides A and B be selected out of $5 = {}^5C_2 = 10$ ways.

After choosing the two husbands their wives are to be excluded (since no husband and wife play in the same game). So, we are to choose 2 wives out of remaining $5 - 2 = 3$ wives i.e., ${}^3C_2 = 3$ ways. Again two wives can interchange their sides A and B in $2! = 2$ ways.

By the principle of multiplication.

The required number of ways $= 10 \times 3 \times 2 = 60$.

88. ***Case* I When no couple is chosen**

We can choose 2 men in ${}^4C_2 = 6$ ways and hence two teams can be formed in $2 \times 6 = 12$ ways.

***Case* II When only one couple is chosen**

A couple can be chosen in ${}^4C_1 = 4$ ways and the other team can be chosen in ${}^3C_1 \times {}^2C_1 = 6$ ways. Hence, two teams can be formed in $4 \times 6 = 24$ ways.

***Case* III When two couples are chosen**

Then team can be chosen in ${}^4C_2 = 6$ ways.

Hence, total ways $= 12 + 24 + 6 = 42$.

89. The total number of seats

$= 1$ grandfather $+ 5$ sons and daughters $+ 8$ grand children $= 14$

The grand children with to occupy the 4 seats on either side of the table $4!$ ways $= 24$ ways

and grandfather can occupy a seat in $(5 - 1)$ ways $= 4$ ways

[Since 4 gaps between 5 sons and daughters]

and the remaining seat can be occupied in $5!$ ways

$= 120$ ways [5 seats for sons and daughters]

Hence, required number of ways, By the principle of multiplication law $= 24 \times 4 \times 120 = 11520$

90. There are 8 chairs on each side of the table. Let sides be represented by A and B. Let four persons sit on side A, then number of ways arranging 4 persons on 8 chairs on side $A = {}^8P_4$ and then two persons sit on side B, then number of ways arranging 2 persons on 8 chairs on side $B = {}^8P_2$ and arranging the remaining 10 persons in remaining 10 chairs in $10!$ ways.

Hence, the total number of ways in which the persons can be arranged $= {}^8P_4 \times {}^8P_2 \times 10! = \dfrac{8! \cdot 8!\, 10!}{4!\, 6!}$

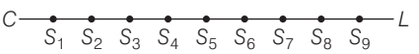
91. The total number of handshake participations by all men what so ever is an even number, which is twice the number of handshakes.

The sum of all participations by men having an even number of handshakes is an even number, which is the sum of several even numbers.

The sum of all participations by men having an odd number of handshakes is an even number, which is an even number minus an even number.

The number of men having an odd number of handshakes must be even for the sum of the odd numbers of their participations be even.

92.



For S_1, 9 different tickets are available, one for each of the remaining 9 stations, similarly at S_2, 8 different tickets are available and so on.

Thus, total number of different tickets

$$= 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 45$$

So, the six different tickets must be any six of these 45 and there are evidently as many different sets of 6 tickets as there are combinations of 45 things taken 6 at a time.

Hence, the required number = ${}^{45}C_6$.

93. Let the object be denoted by $a_1, a_2, a_3, \ldots, a_n$ arranged in a circle, we have to select 3 objects so that no two of them are consecutive. For this, we first find the number of ways in which 2 or 3 objects are consecutive. Now, number of ways in which 2 or 3 objects are consecutive, is obtained as follows with a_1. The number of such triples is

$a_1a_2a_3, a_1a_2a_4, a_1a_2a_5, \ldots, a_1a_2a_{n-1}$.

[Since, we have excluded $a_1a_2a_n$, so it will be repeated again. If we start with a_n, then we shall get triples : $a_na_1a_2, a_na_1a_3$]

So, number of such triples when we start with a_1, is $(n-3)$. Similarly, with $a_2, a_3, a_4, \ldots$, we shall get the numbers of triples that is $(n-3)$.

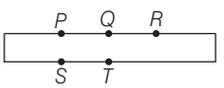
But total number of triples is ${}^nC_3$.

Hence, required number of ways $= {}^nC_3 - n(n-3)$

$$= \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} - n(n-3) = \frac{n}{6}[n^2 - 3n + 2 - 6n + 18]$$

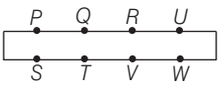
$$= \frac{n}{6}(n^2 - 9n + 20) = \frac{n}{6}(n-4)(n-5)$$

94. Let the men P, Q, R, S, T, U, V, W and suppose P, Q, R remain only on one side and S, T on the other as represented in figure.



Then, since 4 men must row on each side, of the remaining 3, one must be placed on the side of P, Q, R and the other two on the side of S, T and this can evidently be done in 3 ways, for we can place any one of the three on the side of P, Q, R.

Now, 3 ways of distributing the crew let us first consider one, say that in which U is on the side of P, Q, R as shown in figure.



Now, P, Q, R, U can be arranged in 4! ways and S, T, V, W can be arranged in 4! ways. Hence, total number of ways arranging the men = $4! \times 4! = 576$

Hence, the number of ways of arranging the crew

$$= 3 \times 576$$

$$= 1728$$

95. The required number of ways = The number of ways in which $3n$ different things can be divided in 3 equal groups = The number of ways to distribute $3n$ different things equally among three persons $= \frac{3n!}{3!(n!)^3} = \frac{3n!}{6(n!)^3}$

96. Number of squares of area n^2 square units $= 1^2$

Number of squares of area $(n-1)^2$ square units $= 2^2$

Number of squares of area $(n-2)^2$ square units $= 3^2$

.....

Number of squares of area 1^2 square units $= n^2$

Adding gives $N_1 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

When *n* is even

Number of squares of area $\frac{n^2}{2}$ square units $= 1^2$

Number of squares of area $\frac{(n-2)^2}{2}$ square units $= 3^2$

.....

Number of squares of area $\frac{2^2}{2}$ square units $= (n-1)^2$

Adding gives $N_2 = 1^2 + 3^2 + 5^2 + \ldots + (n-1)^2 = \frac{n(n-1)(n+1)}{6}$

When *n* is odd

Number of squares of area $\frac{(n-1)^2}{2}$ square units $= 2^2$

Number of squares of area $\frac{(n-3)^2}{2}$ square units $= 4^2$

Number of squares of area $\frac{(n-5)^2}{2}$ square units $= 6^2$

.....

Number of squares of area $\frac{2^2}{2}$ square units $= (n-1)^2$

Adding gives $N_2 = 2^2 + 4^2 + 6^2 + \ldots + (n-1)^2 = \frac{n(n-1)(n+1)}{6}$

∴Total number of squares formed which can be obtained by taking 4 points out of $(n+1)^2$ points $= N_1 + N_2$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n-1)(n+1)}{6} = \frac{n^2(n+1)}{2}$$

97. **(i) Set of 2 numbers**

Let a and b be 2 numbers $\frac{a+b}{2} = 60 \Rightarrow a + b = 120$

a and b both cannot be equal to or greater than 60

[∵ 60 cannot be used twice]

Let $0 \le a \le 59$ and $61 \le b \le 120$

The total number of ways in which a can be chosen

$$= {}^{60}C_1 = 60$$

The value of b depends on the value of a and there is 1 value of b corresponding to 1 of a.

∴ Total number of sets having 2 numbers = 60

(ii) Set of 3 numbers

Let a, b, c be the three numbers

Then, $\frac{a+b+c}{3} = 60 \Rightarrow a+b+c = 180$

***Case* I** Let $0 \le a \le 59$, $0 \le b \le 59$ and $c \ge 60$

a can be chosen in ${}^{60}C_1 = 60$ ways

b can be chosen in ${}^{59}C_1 = 59$ ways

[$\because$ b cannot use the value of a]

$\therefore$ Number of ways in which a and b can be chosen

$= 60 \times 59 = 3540$

Now, $1 \le a+b \le 117$ and there is only one value of c for 1 value of $a+b$ so that $a+b+c = 180$.

$\therefore$ Number of ways in which a, b, c can be chosen

$= 60 \times 59 = 3540$

***Case* II** $a = 60$

$\therefore$ $b+c = 120$

The number of ways in which b and c can assume values

$= 60$ [from Eq. (i)]

$\therefore$ Number of ways in which a, b, c can be chosen $= 60$

***Case* III** $61 \le a \le 90, 61 \le b \le 90$ and $c \le 60$

a can assume values in ${}^{30}C_1 = 30$ ways

b can assume values in ${}^{29}C_1 = 29$ ways

The value of c depends on the value of a and b

$\therefore$ Number of ways in which a, b, c can be chosen

$= 30 \times 29 = 870$

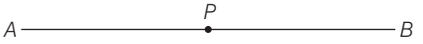
$\therefore$ Total number of ways in which sets of 3 numbers can be chosen

$= 3540 + 60 + 870 = 4470$

$\therefore$ Total number of ways in which sets of 2 and 3 numbers can be chosen

$= 4470 + 60 = 4530$

98. Let AB be any one of n straight lines and suppose it is intersected by some other straight line CD at P.



Then, it is clear that AB contains $(n-1)$ of the points of intersection because it is intersected by the remaining $(n-1)$ $(n-1)$ straight lines in $(n-1)$ different points. So, the aggregate number of points contained in the n straight lines is $n(n-1)$. But in making up this aggregate, each point has evidently been counted twice. For instance, the point P has been counted once among the points situated on AB and again among those on CD.

Hence, the actual number of points $= \frac{n(n-1)}{2}$

Now, we have to find the number of new lines formed by joining these points. The number of new lines passing through P is evidently equal to the number of points lying outside the lines AB and CD for getting a new line joining P with each of these points only.

Since, each of the lines AB and CD contained $(n-2)$ points besides the point P, the number of points situated on AB and CD

$= 2(n-2) - 1$

$= (2n-3)$

$\therefore$ The number of points outside AB and CD

$= \frac{n(n-1)}{2} - (2n-3)$

The number of new lines passing through P and similarly through each other points.

$\therefore$ The aggregate number of new lines passing through the points

$= \frac{n(n-1)}{2}\left\{\frac{n(n-1)}{2} - (2n-3)\right\}$

But in making up this aggregate, every new line is counted twice. For instance, if Q is one of the points outside AB and CD, the line PQ is counted once among the lines passing through P and again among these passing through Q.

Hence, actual number of fresh lines introduced

$= \frac{1}{2}\left[\frac{n(n-1)}{2}\left\{\frac{n(n-1)}{2} - (2n-3)\right\}\right]$

$= \frac{1}{8}n(n-1)(n-2)(n-3)$

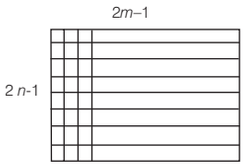
99. Denoting A_1, B_1, A_2 and B_2 for their taking out the ball, a chart is made to denote the winner.

S. No.		A_1	B_1	A_2	B_2	Number of ways
1.	**Points** Number on the ball Sum	1 Even (1 of 3) Even	1 Even (1 of 2) Even	0 Odd (1 of 3) Odd	2 Odd (1 of 2) Even	${}^3C_1 \times {}^2C_1 \times {}^3C_1 \times {}^2C_1 = 36$
2.	**Points** Number on the ball Sum	−1 Odd (1 of 3)Odd	1 Odd (1 of 2) Even	0 Even (1 of 3)Even	2 Even (1 of 2) Even	${}^3C_1 \times {}^2C_1 \times {}^3C_1 \times {}^2C_1 = 36$
3.	**Points** Number on the ball Sum	1 Even (1 of 3) Even	2 Odd (1 of 3) Odd	0 Odd (1 of 3) Even		${}^3C_1 \times {}^3C_1 \times {}^2C_1 = 18$
4.	**Points** Number on the ball Sum	1 Even (1 of 3) Even	1 Even (1 of 2) Even	2 Even (1 of 1)Even		${}^3C_1 \times {}^2C_1 \times {}^1C_1 = 6$

$\therefore$ Total number of ways in which the game can be won when A starts the game $= 36 + 36 + 18 + 6 = 96$

100. Along horizontal side one unit can be taken in $(2m-1)$ ways and 3 unit side can be taken in $(2m-3)$ ways. The number of ways of selecting a side horizontally is

$$(2m-1+2m-3+2m-5+...+3+1)=\frac{m}{2}(2m-1+1)=m^2$$



Similarly, the number of ways along vertical side is

$$(2n-1+2n-3+...+5+3+1)=\frac{n}{2}(2n-1+1)=n^2$$

$\therefore$ Total number of rectangles $=m^2n^2$

101. Words starting with A, C, H, I, N are each equals to 5!

$\therefore$ Total words $=5\times5!=600$

The first word starting with S is SACHIN.

$\therefore$ SACHIN appears in dictionary at serial number 601.

102. Required number of ordered pair (p,q) is

$$(2\times3-1)(2\times5-1)(2\times3-1)-1=224$$

103. ${}^{10}C_1+{}^{10}C_2+{}^{10}C_3+{}^{10}C_4=10+45+120+210=385$

104. In a word COCHIN, the second place can be filled in ${}^4C_1$ ways and the remaining four alphabets can be arranged in 4! ways in four different places. The next 97th word will be COCHIN.

Hence, the number of words that appear before the word COCHIN is 96.

105. 12 different objects are to be divided into 3 groups of equal size, which are named as A, B and C.

Number of ways $={}^{12}C_4\times{}^8C_4\times{}^4C_4=\frac{12!}{(4!)^3}$

106. (A) $\rightarrow$ (p); (B) $\rightarrow$ (s); (C) $\rightarrow$ (q); (D) $\rightarrow$ (q)

(A) ENDEA, N, O, E, L are five different letters, then permutations $=5!$.

(B) If E is in the first and last position, then permutations

$$=\frac{7!}{2!}=\frac{7\times6\times5!}{2}=21\times5!$$

(C) For first four letters $=\frac{4!}{2!}=4\times3=12$ and for last five letters $=\frac{5!}{3!}=\frac{5!}{6}$, then permutations $=12\times\frac{5!}{6}=2\times5!$

(D) For A, E and O $=\frac{5!}{3!}$ and for others $=\frac{4!}{2!}=12$, then permutations $=\frac{5!}{3!}\times12=\frac{5!}{6}\times12=2\times5!$.

107. Other than S seven letters M, I, I, I, P, P, I can be arranged in $\frac{7!}{2!4!}=\frac{7\cdot6\cdot5}{1\cdot2}=7\cdot{}^6C_2=7\cdot{}^6C_4$

Now, four S can be placed in 8 spaces in ${}^8C_4$ ways.

Hence, required number of ways $=7\cdot{}^6C_4\cdot{}^8C_4$

108. $x_1+x_2+x_3+x_4+x_5=6\Rightarrow{}^{6+5-1}C_{5-1}={}^{10}C_4$

109. Coefficient of x^{10} in $(x+x^2+x^3)^7$

$\Rightarrow$ Coefficient of x^3 in $(1+x+x^2)^7$

$\Rightarrow$ Coefficient of x^3 in $\left(\frac{1-x^3}{1-x}\right)^7=(1-x^3)^7(1-x)^{-7}$

$\Rightarrow$ Coefficient of x^3 in $(1-7x^3)(1+{}^7C_1x+{}^8C_2x^2+{}^9C_3x^3+...)$

$$={}^9C_3-7=\frac{9\cdot8\cdot7}{1\cdot2\cdot3}-7=77$$

Aliter

The digits are 1, 1, 1, 1, 1, 2, 3, or 1, 1, 1, 1, 2, 2, 2

Hence, number of seven digit numbers formed $=\frac{7!}{5!}+\frac{7!}{4!3!}$

$$=42+35=77$$

110. 4 novels can be selected from 6 novels in ${}^6C_4$ ways. 1 dictionary can be selected from 3 dictionaries in ${}^3C_1$ ways.

As the dictionary selected is fixed in the middle, the remaining 4 novels can be arranged in 4! ways.

$\therefore$ The required number of ways of arrangement.

$$={}^6C_4\times{}^3C_1\times4!=1080$$

111. Total number of ways $={}^3C_2\times{}^9C_2={}^3C_1\times{}^9C_2=3\times\frac{9\times8}{1\times2}$

$$=3\times9\times4=108$$

112. The number of ways of distributing 10 identical balls in 4 different boxes such that no box is empty $={}^{10-1}C_{4-1}={}^9C_3$

Statement-1 is true.

The number of ways of choosing any 3 places from 9 different places $={}^9C_3$

Statement-2 is true.

Both statements are true but statement-2 is not a correct explanation for statement-1.

Aliter Let a, b, c, d are the balls in four boxes, then $a+b+c+d=10$ and $a\geq1, b\geq1, c\geq1, d\geq1$[$\because$ no box is empty]

$\therefore$Number of solutions $={}^{10-1}C_{4-1}={}^9C_3$

113. Number of triangles $={}^{10}C_3-{}^6C_3$

$$\Rightarrow\quad N=\frac{10\cdot9\cdot8}{1\cdot2\cdot3}-\frac{6\cdot5\cdot4}{1\cdot2\cdot3}\Rightarrow N=120-20\Rightarrow N=100$$

$\therefore\quad N\leq100$

114. $\because$ Each person gets atleast one ball.

$\therefore$ 3 persons can have 5 balls in the following systems

Person	I	II	III
No. of balls	1	1	3

or

Person	I	II	III
No. of balls	1	2	2

The number of ways to distribute the balls in first system

$$={}^5C_1\times{}^4C_1\times{}^3C_3$$

$\therefore$ The total number of ways to distribute 1,1,3 balls to the persons $={}^5C_1\times{}^4C_1\times{}^3C_3\times\frac{3!}{2!}=60$

and the number of ways to distribute the balls in second system $={}^5C_1\times{}^4C_2\times{}^2C_2$

Hence, the total number of ways to distribute 1,2,2 balls to the persons $={}^5C_1\times{}^4C_2\times{}^2C_2\times\frac{3!}{2!}=90$

$\therefore$ The required number of ways $=60+90=150$

Aliter The required number of ways

$$=3^5-{}^3C_1(3-1)^5+{}^3C_2(3-2)^5-{}^3C_3(3-3)^5$$

$$=243-96+3-0=150$$

115. $\because a_n$ = number of all n-digit positive integers formed by the digits 0,1 or both such that no consecutive digits in them are zero.

and b_n = number of such n-digit integers ending with 1

c_n = number of such n-digit integers ending with 0.

Clearly $a_n = b_n + c_n$ [$\because a_n$ can end with 0 or 1]

Also, $b_n = a_{n-1}$ and $c_n = a_{n-2}$

[$\because$ if last digit is 0, second last has to be 1]

$\therefore$ We get $a_n = a_{n-1} + a_{n-2}, n \geq 3$

Also, $a_1 = 1, a_2 = 2$

By the recurring formula $a_3 = a_2 + a_1 = 3$

$a_4 = a_3 + a_2 = 3 + 2 = 5$

$a_5 = a_4 + a_3 = 5 + 3 = 8$

Also, $b_6 = a_5 = 8$

116. By recurring formula, $a_{17} = a_{16} + a_{15}$ is correct.

Also, $C_{17} \neq C_{16} + C_{15} \Rightarrow a_{15} \neq a_{14} + a_{13}$ [$\because C_n = a_{n-2}$]

$\therefore$ Incorrect. Similarly, other parts are also incorrect.

117. Required number of ways

$$= (10+1)(9+1)(7+1) - 1 = 880 - 1 = 879$$

118. $\because T_{n+1} - T_n = 10 \Rightarrow {}^{n+1}C_3 - {}^nC_3 = 10 \Rightarrow {}^nC_2 + {}^nC_3 - {}^nC_3 = 10$

$$\Rightarrow \quad {}^nC_2 = 10 = \frac{20}{2} = \frac{5 \cdot 4}{1 \cdot 2} = {}^5C_2 \Rightarrow n = 5$$

119. Given 8 vectors are $(1,1,1),(-1,1,1),(1,-1,1),(1,1,-1),(-1,-1,1),(1,-1,-1),(-1,1,-1),(-1,-1,-1)$ there are 4 diagonals of a cube. Now, for 3 non-coplanar vectors first we select 3 groups of diagonals and its opposite in ${}^4C_3 = 4$ ways. Then one vector from each group can be selected in $2 \times 2 \times 2 = 8$ ways.

$\therefore$ Total ways $= 4 \times 8 = 32 = 2^5 = 2^p$ (given)

Hence, $p = 5$

120. If n_1, n_2, n_3, n_4 take minimum values 1,2,3,4 respectively, then n_5 will be maximum 10.

$\therefore$ Corresponding to $n_5 = 10$, there is only one solution

$n_1 = 1, n_2 = 2, n_3 = 3, n_4 = 4$

Corresponding to $n_5 = 9$, we can have,

$n_1 = 1, n_2 = 2, n_3 = 3, n_4 = 5$ i.e., one solution

Corresponding to $n_5 = 8$, we can have,

$n_1 = 1, n_2 = 2, n_3 = 3, n_4 = 6$

or $n_1 = 1, n_2 = 2, n_3 = 4, n_4 = 5$ i.e., two solutions

Corresponding to $n_5 = 7$, we can have

$n_1 = 1, n_2 = 2, n_3 = 4, n_4 = 6$

or $n_1 = 1, n_2 = 3, n_3 = 4, n_4 = 5$ i.e., two solutions

Corresponding to $n_5 = 6$, we can have

$n_1 = 2, n_2 = 3, n_3 = 4, n_4 = 5$ i.e., one solution

Thus, there can be 7 solutions.

121. Number of adjacent lines = n

Number of non-adjacent lines $= {}^nC_2 - n = \frac{n(n-3)}{2}$

$\therefore \quad \frac{n(n-3)}{2} = n \Rightarrow \frac{n(n-5)}{2} = 0 \Rightarrow n = 0$ or 5

But $n \geq 2 \Rightarrow n = 5$

122. $\because$ Card numbered 1 is always placed in envelope numbered 2, we can consider two cases.

***Case* I** Card numbered 2 is placed in envelope numbered 1, then it is derangement of 4 objects, which can be done in

$$4!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right) = 9 \text{ ways}$$

***Case* II** Card numbered 2 is not placed in envelope numbered 1, then it is derangement of 5 objects, which can be done in

$$5!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right) = 44 \text{ ways}$$

$\therefore$ Total ways $= 9 + 44 = 53$ ways

123. Four digit numbers can be arranged in $3 \times 4! = 72$ ways and five digit numbers can be arranged in $5! = 120$ ways

$\therefore$ Number of integers $= 72 + 120 = 192$

124. $n = 5! \times 6!$

For m: 5 boys can stand in a row in 5!, creating 6 alternate space for girls. A group of 4 girls can be selected in ${}^5C_4$ ways. A group of 4 and single girl can be arranged at 2 places out of 6 in ${}^6P_2$ ways. Also, 4 girls can arrange themselves in 4! ways.

$\therefore \quad m = 5! \times {}^6P_2 \times {}^5C_4 \times 4! = 5! \times 30 \times 5 \times 4! = 5! \times 6! \times 5$

$\Rightarrow \quad \frac{m}{n} = \frac{5! \times 6! \times 5}{5! \times 6!} = 5$

125. Words starting with A, L, M $= \frac{4!}{2!} + 4! + \frac{4!}{2!} = 48$

Words starting with SA, SL $= \frac{3!}{2!} + 3! = 9$

Rank of the word SMALL $= 48 + 9 + 1 = 58$

126. Either one boy will be selected or no boy will be selected. Also out of four members one captain is to be selected.

$\therefore$ Required number of ways $= ({}^4C_1 \times {}^6C_3 + {}^6C_4) \times {}^4C_1$

$$= (4 \times 20 + 15) \times 4 = 95 \times 4 = 380$$

127.

X		Y		
$4L$	$3M$	$3L$	$4M$	
3	0	0	3	$= {}^4C_3 \times {}^3C_0 \times {}^3C_0 \times {}^4C_3 = 16$
2	1	1	2	$= {}^4C_2 \times {}^3C_1 \times {}^3C_1 \times {}^4C_2 = 324$
1	2	2	1	$= {}^4C_1 \times {}^3C_2 \times {}^3C_2 \times {}^4C_1 = 144$
0	3	3	0	$= {}^4C_0 \times {}^3C_3 \times {}^3C_3 \times {}^4C_0 = 1$

CHAPTER

06

Binomial Theorem

Learning Part

Session 1

- Binomial Theorem for Positive Integral Index
- Pascal's Triangle

Session 2

- General Term
- Middle Terms
- Greatest Term
- Trinomial Expansion

Session 3

- Two Important Theorems
- Divisibility Problems

Session 4

- Use of Complex Numbers in Binomial Theorem
- Multinomial Theorem
- Use of Differentiation
- Use of Integration
- When Each Term is Summation Contains the Product of Two Binomial Coefficients or Square of Binomial Coefficients
- Binomial Inside Binomial
- Sum of the Series

Practice Part

- JEE Type Examples
- Chapter Exercises

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Session 1

Binomial Theorem for Positive Integral Index, Pascal's Triangle

An algebraic expression consisting of two dissimilar terms with positive or negative sign between them is called a binomial expressions.

For example, $x + a, x^2 a - \frac{a}{x}, \frac{p}{x^2} - \frac{q}{x^4}, 5 - x,$

$(x^2 + 1)^{1/3} - \frac{1}{\sqrt{(x^3 + 1)}}$, etc., are called binomial expressions.

Remarks

1. An algebraic expression consisting of three dissimilar terms is called a trinomial. e.g. $a + 2b + c, x - 2y + 3z, 2\alpha - \frac{3}{\beta} + \gamma,$ etc. are called the trinomials.
2. In general, expressions consisting more than two dissimilar terms are known as multinomial expressions.

Binomial Theorem for Positive Integral Index

If $x, a \in C$ and $n \in N$, then

$$(x+a)^n = {}^nC_0\, x^{n-0}\, a^0 + {}^nC_1\, x^{n-1} a^1 + {}^nC_2\, x^{n-2}\, a^2 + \ldots + {}^nC_r\, x^{n-r}\, a^r + \ldots + {}^nC_{n-1}\, x^1\, a^{n-1} + {}^nC_n\, x^0\, a^n \quad \ldots(\text{i})$$

or $$(x+a)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r}\, a^r$$

Hence, ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$ are called binomial coefficients.

Remark

1. In each term, the degree is n and the coefficient of $x^{n-r}\, a^r$ is equal to the number of ways $\underbrace{x, x, x, \ldots, x}_{(n-r) \text{ times}}, \underbrace{a, a, a, \ldots, a}_{r \text{ times}}$ can be arranged, which is given by $\frac{n!}{(n-r)!\, r!} = {}^nC_r$

   For example, $(x + a)^5 = \frac{5!}{5!\,0!} x^5\, a^0 + \frac{5!}{4!\,1!} x^4\, a + \frac{5!}{3!\,2!} x^3\, a^2 + \frac{5!}{2!\,3!} x^2\, a^3 + \frac{5!}{1!\,4!} x\, a^4 + \frac{5!}{0!\,5!} x^0\, a^5$

   $= {}^5C_0\, x^5 + {}^5C_1\, x^4\, a + {}^5C_2\, x^3\, a^2 + {}^5C_3\, x^2\, a^3 + {}^5C_4\, x\, a^4 + {}^5C_5\, a^5$

2. Let $S = (x + a)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r}\, a^r$

   Replacing r by $n - r$, we have

   $$S = (x + a)^n = \sum_{r=0}^{n} {}^nC_{n-r}\, x^{n-(n-r)}\, a^{n-r} = \sum_{r=0}^{n} {}^nC_{n-r}\, x^r\, a^{n-r}$$

   $$= {}^nC_n\, a^n + {}^nC_{n-1}\, a^{n-1}\, x + {}^nC_{n-2}\, a^{n-2}\, x^2 + \ldots + {}^nC_0\, x^n$$

   Thus, replacing r by $n - r$, we are infact writing the binomial expansion in reverse order.

Some Important Points

1. Replacing a by $(-a)$ in Eq. (i), we get

$$(x-a)^n = {}^nC_0\, x^{n-0}\, a^0 - {}^nC_1\, x^{n-1}\, a^1 + {}^nC_2\, x^{n-2}\, a^2 - \ldots + \ldots + (-1)^r\, {}^nC_r\, x^{n-r}\, a^r + \ldots + (-1)^n\, {}^nC_n\, x^0\, a^n \quad \ldots(\text{ii})$$

   or $$(x-a)^n = \sum_{r=0}^{n} (-1)^r\, {}^nC_r\, x^{n-r}\, a^r$$

2. On adding Eqs. (i) and (ii), we get

$$(x+a)^n + (x-a)^n = 2\{{}^nC_0\, x^{n-0}\, a^0 + {}^nC_2\, x^{n-2}\, a^2 + {}^nC_4\, x^{n-4}\, a^4 + \ldots\}$$

   $= 2$ {Sum of terms at odd places}

   The last term is ${}^nC_n\, a^n$ or ${}^nC_{n-1}\, x\, a^{n-1}$, according as n is even or odd, respectively.

3. On subtracting Eq. (ii) from Eq. (i), we get

$$(x+a)^n - (x-a)^n = 2\{{}^nC_1\, x^{n-1}\, a^1 + {}^nC_3\, x^{n-3}\, a^3 + {}^nC_5\, x^{n-5}\, a^5 + \ldots\}$$

   $= 2$ {Sum of terms at even places}

   The last term is ${}^nC_{n-1}\, x\, a^{n-1}$ or ${}^nC_n\, a^n$, according as n is even or odd, respectively.

4. Replacing x by 1 and a by x in Eq. (i), we get

$$(1+x)^n = {}^nC_0\, x^0 + {}^nC_1\, x^1 + {}^nC_2\, x^2 + \ldots + {}^nC_r\, x^r + \ldots + {}^nC_{n-1}\, x^{n-1} + {}^nC_n\, x^n \quad \ldots(\text{iii})$$

   or $$(1+x)^n = \sum_{r=0}^{n} {}^nC_r\, x^r$$

5. Replacing x by $(-x)$ in Eq. (iii), we get

$$(1-x)^n = {}^nC_0\, x^0 - {}^nC_1\, x^1 + {}^nC_2\, x^2 - \ldots + (-1)^r\, {}^nC_r\, x^r + \ldots + {}^nC_n(-1)^n x^n$$

$$\text{or } (1-x)^n = \sum_{r=0}^{n}(-1)^r\, {}^nC_r\, x^r$$

Example 1. Expand $\left(2a-\frac{3}{b}\right)^5$ by binomial theorem.

Sol. Using binomial theorem, we get

$$\left(2a-\frac{3}{b}\right)^5 = {}^5C_0\,(2a)^{5-0}\left(-\frac{3}{b}\right)^0 + {}^5C_1(2a)^{5-1}\left(-\frac{3}{b}\right)^1 + {}^5C_2\,(2a)^{5-2}\left(-\frac{3}{b}\right)^2 + {}^5C_3\,(2a)^{5-3}\left(-\frac{3}{b}\right)^3 + {}^5C_4\,(2a)^{5-4}\left(-\frac{3}{b}\right)^4 + {}^5C_5\,(2a)^{5-5}\left(-\frac{3}{b}\right)^5$$

$$= {}^5C_0\,(2a)^5 - {}^5C_1\,(2a)^4\left(\frac{3}{b}\right) + {}^5C_2\,(2a)^3\left(\frac{3}{b}\right)^2 - {}^5C_3\,(2a)^2\left(\frac{3}{b}\right)^3 + {}^5C_4\,(2a)^1\left(\frac{3}{b}\right)^4 - {}^5C_5\left(\frac{3}{b}\right)^5$$

$$= 32a^5 - \frac{240\,a^4}{b} + \frac{720\,a^3}{b^2} - \frac{1080\,a^2}{b^3} + \frac{810\,a}{b^4} - \frac{243}{b^5}$$

Example 2. ***Simplify***

$$(x+\sqrt{(x^2-1)})^6 + (x-\sqrt{(x^2-1)})^6.$$

Sol. Let $\sqrt{(x^2-1)} = a$

Then, $(x+a)^6 + (x-a)^6 = 2\{{}^6C_0\, x^{6-0}\, a^0 + {}^6C_2\, x^{6-2}\, a^2 + {}^6C_4\, x^{6-4}\, a^4 + {}^6C_6\, x^{6-6}\, a^6\}$

$= 2\{x^6 + 15x^4a^2 + 15x^2a^4 + a^6\}$ [from point (2)]

$= 2\{x^6 + 15x^4\,(x^2-1) + 15x^2\,(x^2-1)^2 + (x^2-1)^3\}$ [$\because a = \sqrt{x^2-1}$]

$= 2(32x^6 - 48x^4 + 18x^2 - 1)$

Example 3. In the expansion of $(x+a)^n$, if sum of odd terms is P and sum of even terms is Q, prove that

(i) $P^2 - Q^2 = (x^2-a^2)^n$

(ii) $4PQ = (x+a)^{2n} - (x-a)^{2n}$

Sol. $\because (x+a)^n = {}^nC_0\, x^{n-0}a^0 + {}^nC_1\, x^{n-1}a^1 + {}^nC_2\, x^{n-2}\, a^2 + {}^nC_3\, x^{n-3}\, a^3 + \ldots + \ldots + {}^nC_n\, x^{n-n}\, a^n$

$= ({}^nC_0\, x^n + {}^nC_2\, x^{n-2}\, a^2 + {}^nC_4\, x^{n-4}\, a^4 + \ldots) + ({}^nC_1\, x^{n-1}\, a^1 + {}^nC_3\, x^{n-3}\, a^3 + {}^nC_5\, x^{n-5}\, a^5 + \ldots)$

$= P + Q$ (given) ...(i)

and $(x-a)^n = {}^nC_0\, x^{n-0}\, a^0 - {}^nC_1\, x^{n-1}\, a^1 + {}^nC_2\, x^{n-2}\, a^2 - {}^nC_3\, x^{n-3}\, a^3 + \ldots + {}^nC_n\, x^{n-n}\, a^n$

$= ({}^nC_0\, x^n + {}^nC_2\, x^{n-2}\, a^2 + {}^nC_4\, x^{n-4}\, a^4 + \ldots) - ({}^nC_1\, x^{n-1}\, a + {}^nC_3\, x^{n-3}\, a^3 + {}^nC_5\, x^{n-5}\, a^5 + \ldots)$

$= P - Q$ (given) ...(ii)

(i) $P^2 - Q^2 = (P+Q)(P-Q)$

$= (x+a)^n \cdot (x-a)^n$

$= (x^2-a^2)^n$ [from Eqs. (i) and (ii)]

(ii) $(x+a)^{2n} - (x-a)^{2n} = [(x+a)^n]^2 - [(x-a)^n]^2$

$= (P+Q)^2 - (P-Q)^2$

$= 4PQ$ [from Eqs. (i) and (ii)]

Example 4. ***Show that*** $(101)^{50} > (100)^{50} + (99)^{50}$.

Sol. Since, $(101)^{50} - (99)^{50} = (100+1)^{50} - (100-1)^{50}$

$= 2\{{}^{50}C_1\,(100)^{49} + {}^{50}C_3\,(100)^{47} + {}^{50}C_5\,(100)^{45} + \ldots\}$

$= 2 \times {}^{50}C_1\,(100)^{49} + 2\{{}^{50}C_3\,(100)^{47} + {}^{50}C_5\,(100)^{45} + \ldots\}$

$= (100)^{50} + (\text{a positive number}) > (100)^{50}$

Hence, $(101)^{50} - (99)^{50} > (100)^{50}$

$\Rightarrow$ $(101)^{50} > (100)^{50} + (99)^{50}$

Example 5. ***If*** $a_n = \sum_{r=0}^{n}\frac{1}{{}^nC_r}$, ***find the value of*** $\sum_{r=0}^{n}\frac{r}{{}^nC_r}$.

Sol. Let $P = \sum_{r=0}^{n}\frac{r}{{}^nC_r}$ ...(i)

Replacing r by $(n-r)$ in Eq. (i), we get

$$P = \sum_{r=0}^{n}\frac{(n-r)}{{}^nC_{n-r}} = \sum_{r=0}^{n}\frac{(n-r)}{{}^nC_r} \quad [\because {}^nC_r = {}^nC_{n-r}] \ldots(ii)$$

On adding Eqs. (i) and (ii), we get

$$2P = \sum_{r=0}^{n}\frac{n}{{}^nC_r} = n\sum_{r=0}^{n}\frac{1}{{}^nC_r} = na_n \quad [\text{given}]$$

$$\therefore \quad P = \frac{n}{2}a_n$$

Hence, $\sum_{r=0}^{n}\frac{r}{{}^nC_r} = \frac{n}{2}a_n$

Properties of Binomial Expansion $(x+a)^n$

(i) This expansion has $(n+1)$ terms.

(ii) Since, ${}^nC_r = {}^nC_{n-r}$, we have

$${}^nC_0 = {}^nC_n = 1$$

$${}^nC_1 = {}^nC_{n-1} = n$$

$${}^nC_2 = {}^nC_{n-2} = \frac{n(n-1)}{2!} \text{ and so on.}$$

(iii) In any term, the suffix of C is equal to the index of a and the index of $x = n - (\text{suffix of } C)$.

(iv) In each term, sum of the indices of x and a is equal to n.

Properties of Binomial Coefficient

(i) ${}^nC_r$ can also be represented by $C(n, r)$ or $\binom{n}{r}$.

(ii) ${}^nC_x = {}^nC_y$, then either $x = y$ or $n = x + y$.

So, ${}^nC_r = {}^nC_{n-r} = \frac{n!}{r!(n-r)!}$

(iii) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(iv) $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$

(v) ${}^nC_r = \frac{n}{r} \cdot {}^{n-1}C_{r-1}$

Pascal's Triangle

Coefficients of binomial expansion can also be easily determined by Pascal's triangle.

$(x+a)^0$ 1

$(x+a)^1$ 1 1

$(x+a)^2$ 1 2 1

$(x+a)^3$ 1 3 3 1

$(x+a)^4$ 1 4 6 4 1

$(x+a)^5$ 1 5 10 10 5 1

Pascal triangle gives the direct binomial coefficients.

For example,

$$(x+a)^4 = 1 \cdot x^4 + 4 \cdot x^3 \cdot a + 6 \cdot x^2 a^2 + 4 \cdot x\, a^3 + 1 \cdot a^4$$

$$= x^4 + 4x^3 a + 6x^2 a^2 + 4x\, a^3 + a^4$$

How to Construct a Pascal's Triangle

Binomial coefficients in the expansion of $(x+a)^3$ are

1 3 3 1

1 (1+3) (3+3) (3+1) 1

Then, 1 4 6 4 1

are the binomial coefficients in the expansion of $(x+a)^4$.

Example 6. ***Find the number of dissimilar terms in the expansion of*** $(1 - 3x + 3x^2 - x^3)^{33}$.

Sol. $(1 - 3x + 3x^2 - x^3)^{33} = [(1-x)^3]^{33} = (1-x)^{99}$

Therefore, number of dissimilar terms in the expansion of $(1 - 3x + 3x^2 - x^3)^3$ is 100.

Example 7. ***Find the value of*** $\sum_{r=1}^{n} \frac{r \cdot {}^nC_r}{{}^nC_{r-1}}$.

Sol. $\because \quad \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$

$$\therefore \quad \frac{r \cdot {}^nC_r}{{}^nC_{r-1}} = (n-r+1)$$

$$\therefore \quad \sum_{r=1}^{n} \frac{r \cdot {}^nC_r}{{}^nC_{r-1}} = \sum_{r=1}^{n} (n-r+1) = \sum_{r=1}^{n} (n+1) - \sum_{r=1}^{n} r$$

$$= (n+1) \sum_{r=1}^{n} 1 - (1 + 2 + 3 + \ldots + n)$$

$$= (n+1) \cdot n - \frac{n(n+1)}{2} = \frac{n(n+1)}{2}$$

Example 8. ***Let*** C_r ***stands for*** ${}^nC_r$, ***prove that***

$$(C_0 + C_1)(C_1 + C_2)(C_2 + C_3) \ldots (C_{n-1} + C_n) = \frac{(n+1)^n}{n!} (C_0 C_1 C_2 \ldots C_{n-1}).$$

Sol. $\text{LHS} = (C_0 + C_1)(C_1 + C_2)(C_2 + C_3) \ldots (C_{n-1} + C_n)$

$$= \prod_{r=1}^{n} (C_{r-1} + C_r) = \prod_{r=1}^{n} ({}^{n+1}C_r) \quad [\because {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r]$$

$$= \prod_{r=1}^{n} \left(\frac{n+1}{r}\right) {}^nC_{r-1} \qquad \left[\because {}^nC_r = \frac{n}{r} \cdot {}^{n-1}C_{r-1}\right]$$

$$= \prod_{r=1}^{n} (n+1) \cdot \prod_{r=1}^{n} \frac{1}{r} \cdot \prod_{r=1}^{n} C_{r-1}$$

$$= (n+1)^n \cdot \frac{1}{n!} \cdot (C_0 C_1 C_2 \ldots C_{n-1})$$

$$= \frac{(n+1)^n}{n!} (C_0\, C_1\, C_2 \ldots C_{n-1}) = \text{RHS}$$

Example 9. ***Find the sum of the series***

$$\sum_{r=0}^{n}(-1)^r\,{}^nC_r\left\{\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\frac{15^r}{2^{4r}}+\text{... upto } m \text{ terms}\right\}.$$

Sol. $\because (1-x)^n=\sum_{r=0}^{n}(-1)^r\,{}^nC_r\,x^r$ ...(i)

Let $P=\sum_{r=0}^{n}(-1)^r\,{}^nC_r\left\{\left(\frac{1}{2}\right)^r+\left(\frac{3}{4}\right)^r+\left(\frac{7}{8}\right)^r+\left(\frac{15}{16}\right)^r+\text{... upto } m \text{ terms}\right\}$

$$=\sum_{r=0}^{n}(-1)^r\,{}^nC_r\cdot\left(\frac{1}{2}\right)^r+\sum_{r=0}^{n}(-1)^r\,{}^nC_r\cdot\left(\frac{3}{4}\right)^r+\sum_{r=0}^{n}(-1)^r\,{}^nC_r\cdot\left(\frac{7}{8}\right)^r+\sum_{r=0}^{n}(-1)^r\,{}^nC_r\cdot\left(\frac{15}{16}\right)^r+\ldots \text{ upto } m \text{ terms}$$

$$=\left(1-\frac{1}{2}\right)^n+\left(1-\frac{3}{4}\right)^n+\left(1-\frac{7}{8}\right)^n+\left(1-\frac{15}{16}\right)^n+\text{... upto } m \text{ terms}$$

$$=\left(\frac{1}{2}\right)^n+\left(\frac{1}{2}\right)^{2n}+\left(\frac{1}{2}\right)^{3n}+\left(\frac{1}{2}\right)^{4n}+\text{... upto } m \text{ terms}$$

$$=\frac{\left(\frac{1}{2}\right)^n\left[1-\left\{\left(\frac{1}{2}\right)^n\right\}^m\right]}{1-\left(\frac{1}{2}\right)^n}$$

$$=\frac{(2^{mn}-1)}{2^{mn}\,(2^n-1)}$$

Exercise for Session 1

1. The value of $\sum_{r=0}^{10} r\cdot{}^{10}C_r\cdot 3^r\cdot(-2)^{10-r}$ is

(a) 10 (b) 20 (c) 30 (d) 300

2. The number of dissimilar terms in the expansion of $\left(x+\frac{1}{x}+x^2+\frac{1}{x^2}\right)^{15}$ are

(a) 61 (b) 121 (c) 255 (d) 16

3. The expansion $\{x+(x^3-1)^{1/2}\}^5+\{x-(x^3-1)^{1/2}\}^5$ is a polynomial of degree

(a) 5 (b) 6 (c) 7 (d) 8

4. $(\sqrt{2}+1)^6-(\sqrt{2}-1)^6$ is equal to

(a) 101 (b) $70\sqrt{2}$ (c) $140\sqrt{2}$ (d) $120\sqrt{2}$

5. The total number of dissimilar terms in the expansion of $(x+a)^{100}+(x-a)^{100}$ after simplification will be

(a) 202 (b) 51
(c) 50 (d) 101

6. The number of non-zero terms in the expansion of $(1+3\sqrt{2}x)^9+(1-3\sqrt{2}x)^9$, is

(a) 0 (b) 5
(c) 9 (d) 10

7. If $(1+x)^n=\sum_{r=0}^{n}C_r\,x^r$, $\left(1+\frac{C_1}{C_0}\right)\left(1+\frac{C_2}{C_1}\right)\ldots\left(1+\frac{C_n}{C_{n-1}}\right)$ is equal to

(a) $\frac{n^{n-1}}{(n-1)!}$ (b) $\frac{(n+1)^{n-1}}{(n-1)!}$

(c) $\frac{(n+1)^n}{n!}$ (d) $\frac{(n+1)^{n+1}}{n!}$

8. If ${}^{n+1}C_{r+1}:{}^nC_r:{}^{n-1}C_{r-1}=11:6:3$, nr is equal to

(a) 20 (b) 30
(c) 0 (d) 50

Session 2

General Term, Middle Terms, Greatest Term, Trinomial Expansion

General Term

The term ${}^nC_r\ x^{n-r}\ a^r$ is the $(r+1)$ th term from beginning in the expansion of $(x+a)^n$. It is usually called the general term and it is denoted by T_{r+1}.
i.e., $\boldsymbol{T_{r+1} = {}^nC_r\ x^{n-r}\ a^r}$

Example 10. Find the 7th term in the expansion of $\left(4x - \frac{1}{2\sqrt{x}}\right)^{13}$.

Sol. Seventh term, $T_7 = T_{6+1} = {}^{13}C_6\,(4x)^{13-6}\left(-\frac{1}{2\sqrt{x}}\right)^6$

$$= {}^{13}C_6 \cdot 4^7 \cdot x^7 \cdot \frac{1}{2^6 \cdot x^3}$$

$$= {}^{13}C_6 \cdot 2^8 \cdot x^4$$

Example 11. ***Find the coefficient of x^8 in the expansion of $\left(x^2 - \frac{1}{x}\right)^{10}$.***

Sol. Here, $T_{r+1} = {}^{10}C_r\,(x^2)^{10-r}\left(-\frac{1}{x}\right)^r$

$$= {}^{10}C_r\ x^{20-2r} \cdot (-1)^r \cdot \frac{1}{x^r}$$

$$= {}^{10}C_r\,(-1)^r \cdot x^{20-3r} \qquad \ldots(i)$$

Now, in order to find out the coefficient of $x^8, 20-3r$ must be 8.

i.e. $20 - 3r = 8$

$\therefore$ $r = 4$

Hence, putting $r = 4$ in Eq. (i), we get

Required coefficient $= (-1)^4 \cdot {}^{10}C_4 = \frac{10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4} = 210$

Example 12. Find

(i) the coefficient of x^7 in the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$.

(ii) the coefficient of x^{-7} in the expansion of $\left(ax - \frac{1}{bx^2}\right)^{11}$.

Also, find the relation between a and b, so that these coefficients are equal.

Sol. (i) Here, $T_{r+1} = {}^{11}C_r\,(ax^2)^{11-r}\left(\frac{1}{bx}\right)^{11}$

$$= {}^{11}C_r \cdot \frac{a^{11-r}}{b^r} \cdot x^{22-3r} \qquad \ldots(i)$$

Now, in order to find out the coefficient of $x^7, 22-3r$ must be 7,

i.e. $22 - 3r = 7$

$\therefore$ $r = 5$

Hence, putting $r = 5$ in Eq. (i), we get

Required coefficient $= {}^{11}C_5 \cdot \frac{a^6}{b^5}$

(ii) Here, $T_{R+1} = {}^{11}C_R\,(ax)^{11-R}\left(-\frac{1}{bx^2}\right)^R$

$$= {}^{11}C_R\,(a)^{11-R}\left(-\frac{1}{b}\right)^R \cdot x^{11-3R}$$

$$= (-1)^R \cdot {}^{11}C_R \cdot \frac{a^{11-R}}{b^R} \cdot x^{11-3R} \qquad \ldots(ii)$$

Now, in order to find out the coefficient of x^{-7}, $11-3R$ must be -7.

i.e., $11 - 3R = -7 \Rightarrow R = 6$. Hence, putting $R = 6$ in Eq. (ii), we get

Required coefficient

$$= (-1)^6 \cdot {}^{11}C_6 \cdot \frac{a^5}{b^6} = {}^{11}C_5 \cdot \frac{a^5}{b^6} \qquad [\because {}^nC_r = {}^nC_{n-r}]$$

Also given, coefficient of x^7 in $\left(ax^2 + \frac{1}{bx}\right)^{11}$ = coefficient of x^{-7} in $\left(ax - \frac{1}{bx^2}\right)^{11}$

$$\Rightarrow {}^{11}C_5 \cdot \frac{a^6}{b^5} = {}^{11}C_5 \cdot \frac{a^5}{b^6} \Rightarrow ab = 1$$

which is the required relation between a and b.

Example 13. ***Find the term independent of x in the expansion of*** $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$.

Sol. Here, $T_{r+1} = {}^9C_r \left(\frac{3}{2}x^2\right)^{9-r} \left(-\frac{1}{3x}\right)^r$

$$= (-1)^r \cdot {}^9C_r \cdot \left(\frac{3}{2}\right)^{9-r} \cdot \left(\frac{1}{3}\right)^r \cdot x^{18-3r} \quad \ldots(\text{i})$$

If this term is independent of x, then the index of x must be zero, i.e., $18 - 3r = 0 \Rightarrow r = 6$

Therefore, $(r+1)$ th term, i.e., 7th term is independent of x and its value by putting $r = 6$ in Eq. (i)

$$= (-1)^6 \cdot {}^9C_6 \cdot \left(\frac{3}{2}\right)^3 \cdot \left(\frac{1}{3}\right)^6 = {}^9C_3 \cdot \frac{1}{2^3 \cdot 3^3}$$

$$= \frac{9 \cdot 8 \cdot 7}{(1 \cdot 2 \cdot 3)\, 2^3 \cdot 3^3} = \frac{7}{18}$$

$(p+1)$ th Term From End in the Expansion of $(x+a)^n$

$(p+1)$ th term from end in the expansion of $(x+a)^n$

$=(p+1)$ th term from beginning in the expansion of $(a+x)^n$

$= {}^nC_p \; a^{n-p} \; x^p$

Example 14. Find the 4th term from the end in the expansion of $\left(\frac{x^3}{2} - \frac{2}{x^2}\right)^7$.

Sol. 4th term from the end in the expansion of $\left(\frac{x^3}{2} - \frac{2}{x^2}\right)^7$

= 4th term from beginning in the expansion of $\left(-\frac{2}{x^2} + \frac{x^3}{2}\right)^7$

$$= {}^7C_3 \left(-\frac{2}{x^2}\right)^{7-3} \left(\frac{x^3}{2}\right)^3 = \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} \cdot \frac{2^4}{x^8} \cdot \frac{x^9}{2^3} = 70x$$

Example 15. Find the $(n+1)$th term from the end in the expansion of $\left(2x - \frac{1}{x}\right)^{3n}$.

Sol. $(n+1)$th term from the end in the expansion of $\left(2x - \frac{1}{x}\right)^{3n}$

$= (n+1)$ th term from beginning in the expansion of $\left(-\frac{1}{x} + 2x\right)^{3n}$

$$= T_{n+1} = {}^{3n}C_n \left(-\frac{1}{x}\right)^{3n-n} (2x)^n = {}^{3n}C_n \cdot 2^n \cdot x^{-n}$$

How to Find Free from Radical Terms or Rational Terms in the Expansion of $(a^{1/p} + b^{1/q})^N, \forall\, a, b \in$ Prime Numbers

First, find $T_{r+1} = {}^NC_r \, (a^{1/p})^{N-r} \, (b^{1/q})^r$

$$\therefore \quad T_{r+1} = {}^NC_r \cdot a^{(N-r)/p} \cdot b^{r/q}$$

By inspection, putting the values of $0 \le r \le N$, when indices of a and b are integers.

Remark

1. If indices of a and b are positive integers.
   Then, free from radical terms = Terms which are integers
   $\therefore$ Number of non-integral terms = Total terms – Number of integral terms
2. If indices of a and b both are not positive integers.
   Then, free from radical terms = Rational terms – Integral terms
3. Number of irrational terms = Total terms – Number of rational terms

Example 16. ***Find the number of terms in the expansion of*** $(\sqrt[4]{9} + \sqrt[6]{8})^{500}$ ***which are integers.***

Sol. Since, $(\sqrt[4]{9} + \sqrt[6]{8})^{500} = (9^{1/4} + 8^{1/6})^{500} = (3^{1/2} + 2^{1/2})^{500}$

[$\because a, b \in$ prime numbers]

$\therefore$ General term, $T_{r+1} = {}^{500}C_r \, (3^{1/2})^{500-r} \cdot (2^{1/2})^r$

$$= {}^{500}C_r \cdot 3^{\frac{500-r}{2}} \cdot 2^{r/2}$$

$$= {}^{500}C_r \cdot 3^{250 - r/2} \cdot 2^{r/2}$$

Now, $0 \le r \le 500$

For $r = 0, 2, 4, 6, 8, \ldots, 500$, indices of 3 and 2 are positive integers.

Hence, number of terms which are integers = 250 + 1 = 251

Example 17. ***Find the sum of all rational terms in the expansion of*** $(3^{1/5} + 2^{1/3})^{15}$.

Sol. The general term in the expansion of $(3^{1/5} + 2^{1/3})^{15}$ is

$$T_{r+1} = {}^{15}C_r \, (3^{1/5})^{15-r} \cdot (2^{1/3})^r$$

$$= {}^{15}C_r \cdot 3^{3 - \frac{r}{5}} \cdot 2^{\frac{r}{3}}$$

Now, $0 \le r \le 15$

For $r = 0, 15$

Rational terms are T_{0+1} and T_{15+1}.

Then, $T_{0+1} = {}^{15}C_0 \cdot 3^3 \cdot 2^0 = 27$

and $T_{15+1} = {}^{15}C_{15} \cdot 3^0 \cdot 2^5 = 32$

$\therefore$ Sum of all rational terms = 27 + 32 = 59

Example 18. Find the number of irrational terms in the expansion of $(\sqrt[8]{5}+\sqrt[6]{2})^{100}$.

Sol. Since, $(\sqrt[8]{5}+\sqrt[6]{2})^{100}=(5^{1/8}+2^{1/6})^{100}$

$\therefore$ General term, $T_{r+1}={}^{100}C_r\,(5^{1/8})^{100-r}\,(2^{1/6})^r$

$={}^{100}C_r\,(5)^{(100-r)/8}\cdot(2)^{r/6}$

As, 2 and 5 are coprime.

$\therefore$ T_{r+1} will be rational, if $(100-r)$ is a multiple of 8 and r is a multiple of 6.

Also, $0\le r\le 100$

$\therefore$ $r=0,6,12,18,\ldots,96$

Now, $100-r=4,10,16,\ldots,100$ ...(i)

and $100-r=0,8,16,24,\ldots,100$ ...(ii)

The common terms in Eqs. (i) and (ii) are 16, 40, 64 and 88.

$\therefore$ $r=84,60,36,12$ gives rational terms.

$\therefore$ The number of irrational terms $=101-4=97$

Problems Regarding Three/Four Consecutive Terms or Coefficients

(i) If consecutive coefficients are given

In this case, divide consecutive coefficients pairwise, we get equations and then solve them.

Example 19. ***Let n be a positive integer. If the coefficients of rth, $(r+1)$th and $(r+2)$th terms in the expansion of $(1+x)^n$ are in AP, then find the relation between n and r.***

Sol. $\because$ $T_r=T_{(r-1)+1}={}^nC_{r-1}\,x^{r-1}$

$T_{r+1}={}^nC_r\,x^r$ and $T_{r+2}=T_{(r+1)+1}={}^nC_{r+1}\,x^{r+1}$

$\therefore$ Coefficients of rth, $(r+1)$th and $(r+2)$th terms in the expansion of $(1+x)^n$ are ${}^nC_{r-1}$, ${}^nC_r$, ${}^nC_{r+1}$.

$\because$ Given, ${}^nC_{r-1}$, ${}^nC_r$, ${}^nC_{r+1}$ are in AP.

and $n\ge r+1$

$\therefore$ $\dfrac{{}^nC_{r-1}}{{}^nC_r},1,\dfrac{{}^nC_{r+1}}{{}^nC_r}$ are also in AP.

$\Rightarrow$ $\dfrac{r}{n-r+1},1,\dfrac{n-r}{r+1}$ are in AP.

$\Rightarrow$ $1-\dfrac{r}{n-r+1}=\dfrac{n-r}{r+1}-1$ $\Rightarrow$ $\dfrac{n-2r+1}{n-r+1}=\dfrac{n-2r-1}{r+1}$

$\Rightarrow$ $nr-2r^2+r+n-2r+1$

$=n^2-2nr-n-nr+2r^2+r+n-2r-1$

$\Rightarrow$ $n^2-4nr+4r^2=n+2$ $\Rightarrow$ $(n-2r)^2=n+2$

***Corollary* I** For $r=2,n=7$ [$\because n\ge 3$]

***Corollary* II** For $r=5,n=7,14$ [$\because n\ge 6$]

Example 20. ***If a,b,c and d are any four consecutive coefficients in the expansion of $(1+x)^n$, then prove that:***

(i) $\dfrac{a}{a+b}+\dfrac{c}{c+d}=\dfrac{2b}{b+c}$.

(ii) $\left(\dfrac{b}{b+c}\right)^2>\dfrac{ac}{(a+b)(c+d)}$, if $x>0$.

Sol. Let a, b, c and d be the coefficients of the r th, $(r+1)$th, $(r+2)$th and $(r+3)$th terms respectively, in the expansion of $(1+x)^n$. Then,

$T_r=T_{r-1+1}={}^nC_{r-1}\,x^{r-1}$

$\therefore$ $a={}^nC_{r-1}$ ...(i)

$\because$ $T_{r+1}={}^nC_r\,x^r$

$\therefore$ $b={}^nC_r$ ...(ii)

$\because$ $T_{r+2}=T_{(r+1)+1}={}^nC_{r+1}\,x^{r+1}$

$\therefore$ $c={}^nC_{r+1}$ ....(iii)

and $T_{r+3}=T_{(r+2)+1}={}^nC_{r+2}\,x^{r+2}$

$\therefore$ $d={}^nC_{r+2}$ ...(iv)

From Eqs. (i) and (ii), we get

$a+b={}^nC_{r-1}+{}^nC_r={}^{n+1}C_r$

$=\dfrac{n+1}{r}\cdot{}^nC_{r-1}=\left(\dfrac{n+1}{r}\right)a$

$\therefore$ $\dfrac{a}{a+b}=\dfrac{r}{n+1}$ ...(v)

From Eqs. (ii) and (iii), we get

$b+c={}^nC_r+{}^nC_{r+1}={}^{n+1}C_{r+1}$

$=\left(\dfrac{n+1}{r+1}\right){}^nC_r=\left(\dfrac{n+1}{r+1}\right)b$

$\therefore$ $\dfrac{b}{b+c}=\dfrac{r+1}{n+1}$ ...(vi)

From Eqs. (iii) and (iv), we get

$c+d={}^nC_{r+1}+{}^nC_{r+2}={}^{n+1}C_{r+2}$

$=\left(\dfrac{n+1}{r+2}\right){}^nC_{r+1}=\left(\dfrac{n+1}{r+2}\right)c$

$\therefore$ $\dfrac{c}{c+d}=\dfrac{r+2}{n+1}$ ...(vii)

From Eqs. (v), (vi) and (vii), we get

$\dfrac{a}{a+b},\dfrac{b}{b+c}$ and $\dfrac{c}{c+d}$ are in AP.

(i) $\dfrac{a}{a+b}+\dfrac{c}{c+d}=2\left(\dfrac{b}{b+c}\right)$

or $\dfrac{a}{a+b}+\dfrac{c}{c+d}=\dfrac{2b}{b+c}$

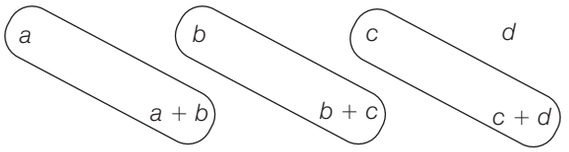
(ii) AM > GM

$$\therefore \quad \left(\frac{b}{b+c}\right) > \sqrt{\left(\frac{a}{a+b}\right)\left(\frac{c}{c+d}\right)}$$

$$\Rightarrow \quad \left(\frac{b}{b+c}\right)^2 > \frac{ac}{(a+b)(c+d)}$$

Remembering Method

$\because$



$\therefore \quad \frac{a}{a+b}, \frac{b}{b+c}$ and $\frac{c}{c+d}$ are in AP.

(ii) If consecutive terms are given

In this case, divide consecutive terms pairwise. i.e., If four consecutive terms are $T_r, T_{r+1}, T_{r+2}, T_{r+3}$. Then, find $\frac{T_{r+1}}{T_r}, \frac{T_{r+2}}{T_{r+1}}, \frac{T_{r+3}}{T_{r+2}} \Rightarrow \lambda_1, \lambda_2, \lambda_3$ (say). Then, divide λ_2 by λ_1 and λ_3 by λ_2 and solve.

Example 21. ***If the 2nd, 3rd and 4th terms in the expansion of*** $(x+y)^n$ ***are 240, 720 and 1080 respectively, find*** x, y and n.

Sol. Given, $T_2 = T_{1+1} = {}^nC_1 \cdot x^{n-1} \cdot y = 240$ ...(i)

$$T_3 = T_{2+1} = {}^nC_2 \cdot x^{n-2} \cdot y^2 = 720 \quad \text{...(ii)}$$

and $T_4 = T_{3+1} = {}^nC_3 \cdot x^{n-3} \cdot y^3 = 1080$ ...(iii)

On dividing Eq. (ii) by Eq. (i), we get

$$\frac{{}^nC_2 \cdot x^{n-2} \cdot y^2}{{}^nC_1 \cdot x^{n-1} \cdot y} = \frac{720}{240}$$

$$\Rightarrow \quad \left(\frac{n-2+1}{2}\right) \cdot \frac{y}{x} = 3 \Rightarrow \frac{y}{x} = \frac{6}{n-1} \quad \text{...(iv)}$$

Also, dividing Eq. (iii) by Eq. (ii), we get

$$\frac{{}^nC_3 \cdot x^{n-3} \cdot y^3}{{}^nC_2 \cdot x^{n-2} \cdot y^2} = \frac{1080}{720}$$

$$\Rightarrow \quad \left(\frac{n-3+1}{3}\right) \cdot \frac{y}{x} = \frac{3}{2} \Rightarrow \frac{y}{x} = \frac{9}{2(n-2)} \quad \text{...(v)}$$

From Eqs. (iv) and (v), we get

$$\frac{6}{n-1} = \frac{9}{2(n-2)}$$

$$\Rightarrow \quad 12n - 24 = 9n - 9$$

$$\Rightarrow \quad 3n = 15$$

$$\therefore \quad n = 5$$

From Eq. (iv), we get $y = \frac{3}{2}x$ ...(vi)

From Eqs. (i) and (vi), we get

$${}^5C_1 \cdot x^4 \cdot y = 240 \quad \Rightarrow 5 \cdot x^4 \cdot \frac{3}{2}x = 240$$

$$\therefore \quad x^5 = 32 = 2^5 \Rightarrow x = 2$$

From Eq. (vi), we get $y = 3$

Hence, $x = 2, y = 3$ and $n = 5$

Middle Terms

The middle term depends upon the value of n.

(i) **When n is even** The total number of terms in the expansion of $(x+a)^n$ is $n+1$ (odd). So, there is only one middle term, i.e., $\left(\frac{n}{2}+1\right)$th term is the middle term. It is given by $T_{n/2+1} = {}^nC_{n/2}\, x^{n/2}\, a^{n/2}$

(ii) **When n is odd** The total number of terms in the expansion of $(x+a)^n$ is $n+1$ (even). So, there are two middle terms, i.e., $\left(\frac{n+1}{2}\right)$th and $\left(\frac{n+3}{2}\right)$th are two middle terms. They are given by

$$T_{\frac{n+1}{2}} = T_{\left(\frac{n-1}{2}\right)+1} = {}^nC_{\frac{n-1}{2}} \cdot x^{\frac{n+1}{2}} \cdot a^{\frac{n-1}{2}}$$

and $T_{\frac{n+3}{2}} = T_{\left(\frac{n+1}{2}\right)+1} = {}^nC_{\frac{n+1}{2}} \cdot x^{\frac{n-1}{2}} \cdot a^{\frac{n+1}{2}}$

Example 22. ***Find the middle term in the expansion of*** $\left(\frac{a}{x}+bx\right)^{12}$.

Sol. The number of terms in the expansion of $\left(\frac{a}{x}+bx\right)^{12}$ is 13 (odd), its middle term is $\left(\frac{12}{2}+1\right)$ th, i.e., 7th term.

$\therefore$ Required term, $T_7 = T_{6+1} = {}^{12}C_6 \left(\frac{a}{x}\right)^6 (bx)^6$

$$= {}^{12}C_6\, a^6\, b^6 = 924\, a^6\, b^6$$

Example 23. Find the middle term in the expansion of $\left(3x - \frac{x^3}{6}\right)^9$.

Sol. The number of terms in the expansion of $\left(3x - \frac{x^3}{6}\right)^9$ is 10 (even). So, there are two middle terms, i.e. $\left(\frac{9+1}{2}\right)$th and $\left(\frac{9+3}{2}\right)$th terms. They are given by T_5 and T_6.

$$\therefore \quad T_5 = T_{4+1} = {}^9C_4\,(3x)^5\left(-\frac{x^3}{6}\right)^4$$

$$= \frac{9\cdot 8\cdot 7\cdot 6}{1\cdot 2\cdot 3\cdot 4}\cdot 3^5\,x^5\cdot\frac{x^{12}}{6^4} = \frac{189}{8}\,x^{17}$$

and $$T_6 = T_{5+1} = {}^9C_5\,(3x)^4\left(-\frac{x^3}{6}\right)^5$$

$$= -\,{}^9C_4\cdot 3^4\cdot x^4\cdot\frac{x^{15}}{6^5}$$

$$= -\frac{9\cdot 8\cdot 7\cdot 6}{1\cdot 2\cdot 3\cdot 4}\cdot 3^4\cdot\frac{x^{19}}{6^5} = -\frac{21}{16}\,x^{19}$$

Example 24. Show that the middle term in the expansion of $(1+x)^{2n}$ is

$$\frac{1\cdot 3\cdot 5\ldots(2n-1)}{n!}\cdot 2^n\,x^n,\ n \text{ being a positive integer.}$$

Sol. The number of terms in the expansion of $(1+x)^{2n}$ is $2n+1$ (odd), its middle term is $(n+1)$th term.

$\therefore$ Required term $= T_{n+1}$

$$= {}^{2n}C_n\,x^n = \frac{2n!}{n!\,n!}\,x^n = \frac{(1\cdot 2\cdot 3\cdot 4\ldots(2n-1)\cdot 2n)}{n!\,n!}\,x^n$$

$$= \frac{\{1\cdot 3\cdot 5\ldots(2n-1)\}\{2\cdot 4\cdot 6\ldots 2n\}}{n!\,n!}\,x^n$$

$$= \frac{\{1\cdot 3\cdot 5\ldots(2n-1)\}\,2^n\,(1\cdot 2\cdot 3\ldots n)}{n!\,n!}\,x^n$$

$$= \frac{\{1\cdot 3\cdot 5\ldots(2n-1)\}\,2^n\,n!}{n!\,n!}\,x^n = \frac{1\cdot 3\cdot 5\ldots(2n-1)}{n!}\,2^n\,x^n$$

Greatest Term

If T_r and T_{r+1} are the rth and $(r+1)$th terms in the expansion of $(x+a)^n$, then

$$\frac{T_{r+1}}{T_r} = \frac{{}^nC_r\cdot x^{n-r}\cdot a^r}{{}^nC_{r-1}\cdot x^{n-r+1}\cdot a^{r-1}} = \left(\frac{n-r+1}{r}\right)\cdot\frac{a}{x}$$

Let numerically, T_{r+1} be the greatest term in the above expansion. Then,

$$T_{r+1}\geq T_r \quad\text{or}\quad \frac{T_{r+1}}{T_r}\geq 1 \;\Rightarrow\; \left(\frac{n-r+1}{r}\right)\left|\frac{a}{x}\right|\geq 1$$

[$\because$ a may be + ve or − ve]

or
$$r \leq \frac{(n+1)}{\left(1+\left|\frac{x}{a}\right|\right)} \qquad \ldots(\text{i})$$

Now, on substituting values of n, x and a in Eq. (i), we get

$$r \leq m+f \text{ or } r\leq m$$

where, $$m\in N \text{ and } 0<f<1$$

In the first case, T_{m+1} is the greatest term, while in the second case, T_m and T_{m+1} are the greatest terms and both are equal (numerically).

Shortcut Method

To find the greatest term (numerically) in the expansion of $(x+a)^n$.

Now, $$(x+a)^n = a^n\left(1+\frac{x}{a}\right)^n$$

Calculate $$m = \frac{\left|\frac{x}{a}\right|(n+1)}{\left(\left|\frac{x}{a}\right|+1\right)}$$

***Case* I** If $m\in$ Integer, then T_m and T_{m+1} are the greatest terms and both are equal (numerically).

***Case* II** If $m\notin$ Integer, then $T_{[m]+1}$ is the greatest term, where $[\cdot]$ denotes the greatest integer function.

Example 25. Find numerically the greatest term in the expansion of $(2+3x)^9$, when $x=3/2$.

Sol. Let T_{r+1} be the greatest term in the expansion of $(2+3x)^9$, we have

$$\frac{T_{r+1}}{T_r} = \left(\frac{9-r+1}{r}\right)\left|\frac{3x}{2}\right| = \left(\frac{10-r}{r}\right)\left|\frac{3}{2}\times\frac{3}{2}\right| = \frac{90-9r}{4r}$$

[$\because$ $x=3/2$]

$$\therefore \quad \frac{T_{r+1}}{T_r}\geq 1$$

$$\Rightarrow \quad \frac{90-9r}{4r}\geq 1 \;\Rightarrow\; 90\geq 13r$$

$$\therefore \quad r\leq \frac{90}{13} = 6\frac{12}{13}$$

or $$r\leq 6\frac{12}{13}$$

$\therefore$ Maximum value of r is 6.

So, greatest term $= T_{6+1} = {}^9C_6\,(2)^{9-6}\,(3x)^6$

$$= {}^9C_3\cdot 2^3\cdot\left(3\times\frac{3}{2}\right)^6$$

$$= \frac{9\cdot 8\cdot 7}{1\cdot 2\cdot 3}\cdot\frac{2^3\cdot 3^{12}}{2^6} = \frac{7\times 3^{13}}{2}$$

Aliter Since, $(2+3x)^9 = 2^9\left(1+\frac{3x}{2}\right)^9$

Now, $m = \dfrac{(9+1)\left|\dfrac{3x}{2}\right|}{\left|\dfrac{3x}{2}\right|+1} = \dfrac{10\times\dfrac{9}{4}}{\dfrac{9}{4}+1}$ $[\because x = 3/2]$

$= \dfrac{90}{13} = 6\dfrac{12}{13} \neq$ Integer

$\therefore$ The greatest term in the expansion is

$T_{[m]+1} = T_{6+1}$ in $(2+3x)^9$

$= {}^9C_6\,(2)^{9-6}\,(3x)^6 = {}^9C_3 \cdot 2^3 \cdot \left(\dfrac{3^2}{2}\right)^6$ $[\because x = 3/2]$

$= \dfrac{9\cdot 8\cdot 7}{1\cdot 2\cdot 3}\cdot\dfrac{3^{12}}{2^3} = \dfrac{7\times 3^{13}}{2}$

Example 26. ***Find numerically the greatest term in the expansion of*** $(3-5x)^{11}$***, when*** $x = \dfrac{1}{5}$.

Sol. Let T_{r+1} be the greatest term in the expansion of $(3-5x)^{11}$, we have

$\dfrac{T_{r+1}}{T_r} = \left(\dfrac{11-r+1}{r}\right)\left|-\dfrac{5x}{3}\right|$

$= \left(\dfrac{12-r}{r}\right)\left|-\dfrac{1}{3}\right| = \dfrac{12-r}{3r}$ $[\because x = 1/5]$

$\therefore \quad \dfrac{T_{r+1}}{T_r} \geq 1 \;\Rightarrow\; \dfrac{12-r}{3r} \geq 1 \;\Rightarrow 12 \geq 4r$

$\therefore \quad r \leq 3 \;\Rightarrow\; r = 2, 3$

So, the greatest terms are T_{2+1} and T_{3+1}.

$\therefore$ Greatest term (when $r = 2$) $= T_{2+1} = {}^{11}C_2\,(3)^9\,(-5x)^2$

$= \dfrac{11\cdot 10}{1\cdot 2}\cdot 3^9\cdot(1)^2 = 55\times 3^9$ $[\because x = 1/5]$

and greatest term (when $r = 3$) $= T_{3+1}$

$= \left|{}^{11}C_3\,(3)^8\,(-5x)^3\right| = \left|{}^{11}C_3\,(3)^8\,(-1)^3\right|$ $[\because x = 1/5]$

$= {}^{11}C_3 \cdot 3^8 = \dfrac{11\cdot 10\cdot 9}{1\cdot 2\cdot 3}\cdot 3^8 = 55\times 3^9$

From above, we say that the values of both greatest terms are equal.

Aliter

Since, $(3-5x)^{11} = 3^{11}\left(1-\dfrac{5x}{3}\right)^{11}$

Now, $m = \dfrac{(11+1)\left|-\dfrac{5x}{3}\right|}{\left|-\dfrac{5x}{3}\right|+1} = \dfrac{12\times\left|-\dfrac{1}{3}\right|}{\left|-\dfrac{1}{3}\right|+1}$ $\left[\because x = \dfrac{1}{5}\right]$

$= \dfrac{4}{\dfrac{1}{3}+1} = 3$

Since, the greatest terms in the expansion are T_3 and T_4.

$\therefore$ Greatest term (when $r = 2$) $= {}^{11}C_2\,(3)^9\,(-5x)^2$

$= {}^{11}C_2\,(3)^9\,(-1)^2$ $\left[\because x = \dfrac{1}{5}\right]$

$= \dfrac{11\cdot 10}{1\cdot 2}\cdot 3^9 = 55\times 3^9$

and greatest term (when $r = 3$) $= \left|{}^{11}C_3\,(3)^8\,(-5x)^3\right|$

$= \left|{}^{11}C_3\,(3)^8\,(-1)^3\right|$ $\left[\because x = \dfrac{1}{5}\right]$

$= \dfrac{11\cdot 10\cdot 9}{1\cdot 2\cdot 3}\cdot 3^8 = 55\times 3^9$

Greatest Coefficient

(i) If n is even, then greatest coefficient is ${}^nC_{n/2}$.

(ii) If n is odd, then greatest coefficients are ${}^nC_{(n-1)/2}$ and ${}^nC_{(n+1)/2}$.

Example 27. Show that, if the greatest term in the expansion of $(1+x)^{2n}$ has also the greatest coefficient, then x lies between $\dfrac{n}{n+1}$ and $\dfrac{n+1}{n}$.

Sol. In the expansion of $(1+x)^{2n}$, the middle term is $\left(\dfrac{2n}{2}+1\right)$th

i.e., $(n+1)$th term, we know that from binomial expansion, middle term has greatest coefficient.

$[\because$ Terms $T_1, T_2, T_3, \ldots, T_n, T_{n+1}, T_{n+2}, \ldots]$

$\therefore \quad T_n < T_{n+1} > T_{n+2}$

$\Rightarrow \quad \dfrac{T_{n+1}}{T_n} = \dfrac{{}^{2n}C_n\cdot x^n}{{}^{2n}C_{n-1}\cdot x^{n-1}} = \dfrac{2n-n+1}{n}\cdot x$

$\Rightarrow \quad \dfrac{T_{n+1}}{T_n} > 1$ or $\dfrac{n+1}{n}\cdot x > 1$

or $\quad x > \dfrac{n}{n+1}$ ...(i)

and $\quad \dfrac{T_{n+2}}{T_{n+1}} = \dfrac{{}^{2n}C_{n+1}\,x^{n+1}}{{}^{2n}C_n\,x^n} = \dfrac{2n-(n+1)+1}{n+1}\cdot x$

$= \dfrac{n}{n+1}\cdot x$

$\Rightarrow \quad \dfrac{T_{n+2}}{T_{n+1}} < 1 \;\Rightarrow\; \dfrac{n}{n+1}\cdot x < 1$ or $x < \dfrac{n+1}{n}$ ...(ii)

From Eqs. (i) and (ii), we get

$$\frac{n}{n+1} < x < \frac{n+1}{n}$$

Corollary For $n = 5$

$$\frac{5}{6} < x < \frac{6}{5}$$

Important Properties of the Binomial Coefficients

In the binomial expansion of $(1+x)^n$. Let us denote the coefficients ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_r, \ldots, {}^nC_n$ by $C_0, C_1, C_2, \ldots, C_r, \ldots, C_n$, respectively.

(i) **The coefficients of the terms equidistant from the beginning and the end are equal**

The $(r+1)$th term from the beginning in the expansion of $(1+x)^n$ is ${}^nC_r\, x^r$.

$\therefore$ The coefficient of the $(r+1)$th term from the beginning is ${}^nC_r$ and the $(r+1)$th term from the end in the expansion of $(1+x)^n = (r+1)$ th term from the beginning in the expansion of $(x+1)^n = {}^nC_r\, x^{n-r}$

$\therefore$ The coefficient of the $(r+1)$th term from the end is ${}^nC_r$.

Hence, the coefficients of $(r+1)$th term from the beginning and the end are equal.

(ii) **The sum of the binomial coefficients in the expansion of $(1+x)^n$**

$\because \quad (1+x)^n = {}^nC_0 + {}^nC_1\, x + {}^nC_2\, x^2 + {}^nC_3\, x^3 + \ldots + {}^nC_n\, x^n$

Putting $x = 1$, we get

$$2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n$$

or $\quad C_0 + C_1 + C_2 + \ldots + C_n = 2^n$

$\therefore$ Sum of binomial coefficients $= 2^n$

(iii) **The sum of the coefficients of the odd terms = The sum of the coefficients of the even terms**

$\because \quad (1+x)^n = {}^nC_0 + {}^nC_1\, x + {}^nC_2\, x^2 + {}^nC_3\, x^3 + \ldots + {}^nC_n\, x^n$

Putting $x = -1$, we get

$$0 = {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + {}^nC_4 - {}^nC_5 + \ldots$$

or $\quad {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots = {}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots$

Since, the sum of all the coefficients is 2^n, therefore each side is equal to $\frac{2^n}{2}$ i.e. 2^{n-1}.

Hence, $C_1 + C_3 + C_5 + \ldots = C_0 + C_2 + C_4 + \ldots = 2^{n-1}$

Remark

1. In the expansion of $(x - 2y + 3z)^n$, putting $x = y = z = 1$, then we get the sum of coefficients $= (1 - 2 + 3)^n = 2^n$.
2. In the expansion of $(1 + x + x^2)^n$, putting $x = 1$, we get the sum of coefficients $= (1 + 1 + 1)^n = 3^n$.

Trinomial Expansion

For $n \in N, (1+x+x^2)^n = \sum_{r=0}^{2n} a_r x^r$

$= a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots + a_{2n} x^{2n}$ ...(i)

There are $(2n+1)$ terms. The middle coefficient is a_n which is also the greatest.

$$a_0 = a_{2n}, a_1 = a_{2n-1}, .., a_r = a_{2n-r}$$

The coefficients of $(1+x+x^2)^n$ for $n = 0, 1, 2, \ldots$ can be arranged in a triangle.

```
                                1
                            1   1   1
                        1   2   3   2   1
                    1   3   6   7   6   3   1
                1   4   10  16  19  16  10  4   1
            1   5   15  30  45  51  45  30  15  5   1
            .   .   .   .   .   .   .   .   .   .   .
            .   .   .   .   .   .   .   .   .   .   .
```

i.e., The rows contains the coefficients for $n = 0, 1, 2, 3, \ldots$.

Each entry other than two entries at the ends is the sum of three entries above it.

$15 = 1 + 4 + 10, 30 = 16 + 10 + 4$, etc.

Putting $x = 1$ and $x = -1$ in Eq. (i), we get

$$a_0 + a_1 + a_2 + a_3 + \ldots + a_{2n} = 3^n$$

[sum of all coefficients] ...(ii)

and $\quad a_0 - a_1 + a_2 - a_3 + \ldots + a_{2n} = 1$ ...(iii)

On adding Eqs. (ii) and (iii), we get

$$a_0 + a_2 + a_4 + \ldots + a_{2n} = \frac{3^n + 1}{2}$$

[sum of coefficients of even powers of x]

On subtracting Eq. (ii) from Eq. (i), we get

$$a_1 + a_3 + a_5 + \ldots + a_{2n-1} = \frac{3^n - 1}{2}$$

[sum of coefficients of odd powers of x]

Putting $x = i(\sqrt{-1})$ in Eq. (i), we get

$$a_0 + a_1\, i + a_2\, i^2 + a_3\, i^3 + a_4\, i^4 + a_5\, i^5 + \ldots + a_{2n}\, i^{2n} = i^n$$

$$\Rightarrow \quad (a_0 - a_2 + a_4 - \ldots) + i(a_1 - a_3 + a_5 - \ldots) = i^n$$

$$(a_0 - a_2 + a_4 - \ldots) + i(a_1 - a_3 + a_5 - \ldots) = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^n = \cos\left(\frac{n\pi}{2}\right) + i\sin\left(\frac{n\pi}{2}\right)$$

On comparing real and imaginary parts, we get

$$a_0 - a_2 + a_4 - \ldots = \cos\left(\frac{n\pi}{2}\right)$$

and $a_1 - a_3 + a_5 - \ldots = \sin\left(\frac{n\pi}{2}\right)$

Putting $x = \omega$ and ω^2 (cube roots of unity) in Eq. (i), we get

$$a_0 + a_1\,\omega + a_2\,\omega^2 + a_3\,\omega^3 + a_4\,\omega^4 + \ldots = 0 \quad \ldots(\text{iv})$$

and $a_0 + a_1\,\omega^2 + a_2\,\omega^4 + a_3\,\omega^6 + a_4\,\omega^8 + \ldots = 0 \quad \ldots(\text{v})$

On adding Eqs. (ii), (iv) and (v) and then dividing by 3, we get

$$a_0 + a_3 + a_6 + \ldots = 3^{n-1}$$

Note

(i) $a_1 + a_4 + a_7 + \ldots = a_2 + a_5 + a_8 + \ldots = 3^{n-1}$

(ii) $a_0 + a_4 + a_8 + \ldots = \frac{1}{4}\left\{3^n + 1 + 2\cos\left(\frac{n\pi}{2}\right)\right\}$

(iii) $a_1 + a_5 + a_9 + \ldots = \frac{1}{4}\left\{3^n - 1 + 2\sin\left(\frac{n\pi}{2}\right)\right\}$

(iv) $a_0 + a_6 + a_{12} + \ldots = \frac{1}{6}\left\{3^n + 1 + 2^{n+1}\cos\left(\frac{n\pi}{3}\right)\right\}$

(v) $\sum_{r=1}^{2n} r \cdot a_r = n \cdot 3^n$ (vi) $\sum_{r=1}^{2n} (-1)^{r-1} \cdot r \cdot a_r = -n$

Example 28. Find the sum of coefficients in the expansion of the binomial $(5p - 4q)^n$, where n is a positive integer.

Sol. Putting $p = q = 1$ in $(5p - 4q)^n$, the required sum of coefficients $= (5 - 4)^n = 1^n = 1$

Example 29. In the expansion of $(3^{-x/4} + 3^{5x/4})^n$, if the sum of binomial coefficients is 64 and the term with the greatest binomial coefficient exceeds the third by $(n - 1)$, find the value of x.

Sol. Given sum of the binomial coefficients in the expansion of $(3^{-x/4} + 3^{5x/4})^n = 64$

Then, putting $3^{-x/4} = 3^{5x/4} = 1$

$\therefore \quad (1 + 1)^n = 64 \Rightarrow 2^n = 2^6$

$\therefore \quad n = 6$

We know that, middle term has the greatest binomial coefficients. Here, $n = 6$

$\therefore$ Middle term $= \left(\frac{n}{2} + 1\right)$th term $= 4$th term $= T_4$

and given that $T_4 = (n - 1) + T_3$

$\Rightarrow \quad T_{3+1} = (6 - 1) + T_{2+1}$

$\Rightarrow \quad {}^6C_3\,(3^{-x/4})^3\,(3^{5x/4})^3 = 5 + {}^6C_2\,(3^{-x/4})^4\,(3^{5x/4})^2$

$\Rightarrow \quad 20 \cdot 3^{3x} = 5 + 15 \cdot 3^{3x/2}$

Let $3^{3x/2} = t$

$\therefore \quad 20\,t^2 = 5 + 15\,t$

$\Rightarrow \quad 4t^2 - 3t - 1 = 0$

$\Rightarrow \quad (4t + 1)(t - 1) = 0$

$\therefore \quad t = 1, \; t \neq -\frac{1}{4} \Rightarrow 3^{3x/2} = 1 = 3^0$

$\therefore \quad \frac{3x}{2} = 0 \text{ or } x = 0$

Example 30. ***Find the values of***

(i) $\frac{1}{(n-1)!} + \frac{1}{(n-3)!\,3!} + \frac{1}{(n-5)!\,5!} + \ldots$

(ii) $\frac{1}{12!} + \frac{1}{10!\,2!} + \frac{1}{8!\,4!} + \ldots + \frac{1}{12!}$

Sol. (i) $\because \; 1! = 1$

$\therefore$ The given series can be written as

$$\frac{1}{(n-1)!\,1!} + \frac{1}{(n-3)!\,3!} + \frac{1}{(n-5)!\,5!} + \ldots \quad \ldots(\text{i})$$

$\because$ Sum of values of each terms in factorial are equal.

i.e. $(n - 1) + 1 = (n - 3) + 3 = (n - 5) + 5 = \ldots = n$

From Eq. (i),

$$\frac{1}{n!}\left[\frac{n!}{(n-1)!\,1!} + \frac{n!}{(n-3)!\,3!} + \frac{n!}{(n-5)!\,5!} + \ldots\right]$$

$$= \frac{1}{n!}({}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots) = \frac{2^{n-1}}{n!}$$

(ii) $\because \; 0! = 1$

$\therefore$ The given series can be written as

$$\frac{1}{12!0!} + \frac{1}{10!\,2!} + \frac{1}{8!\,4!} + \ldots + \frac{1}{0!\,12!} \quad \ldots(\text{ii})$$

$\because$ Sum of values of each terms in factorial are equal

i.e., $12 + 0 = 10 + 2 = 8 + 4 = \ldots = 12$

From Eq. (ii), $\frac{1}{12!}\left[\frac{12!}{12!\,0!} + \frac{12!}{10!\,2!} + \frac{12!}{8!\,4!} + \ldots + \frac{12!}{0!\,12!}\right]$

$$= \frac{1}{12!}({}^{12}C_0 + {}^{12}C_2 + {}^{12}C_4 + \ldots + {}^{12}C_{12}) = \frac{2^{12-1}}{12!} = \frac{2^{11}}{12!}$$

Example 31. Prove that the sum of the coefficients in the expansion of $(1 + x - 3x^2)^{2163}$ is -1.

Sol. Putting $x = 1$ in $(1 + x - 3x^2)^{2163}$, the required sum of coefficients $= (1 + 1 - 3)^{2163} = (-1)^{2163} = -1$

Example 32. If the sum of the coefficients in the expansion of $(\alpha x^2 - 2x + 1)^{35}$ is equal to the sum of the coefficients in the expansion of $(x - \alpha y)^{35}$, find the value of α.

Sol. Given, sum of the coefficients in the expansion of $(\alpha x^2 - 2x + 1)^{35}$

= Sum of the coefficients in the expansion of $(x - \alpha y)^{35}$

Putting $x = y = 1$, we get

$$(\alpha - 1)^{35} = (1 - \alpha)^{35}$$

$$\Rightarrow \quad (\alpha - 1)^{35} = -(\alpha - 1)^{35}$$

$$\Rightarrow \quad 2(\alpha - 1)^{35} = 0$$

$$\Rightarrow \quad \alpha - 1 = 0$$

$$\therefore \quad \alpha = 1$$

Example 33. *If* $(1 + x - 2x^2)^{20} = \sum_{r=0}^{40} a_r x^r$, *then find the value of* $a_1 + a_3 + a_5 + \ldots + a_{39}$.

Sol. $\because$ $(1 + x - 2x^2)^{20} = \sum_{r=0}^{40} a_r x^r$ ...(i)

Putting $x = 1$, we get $0 = \sum_{r=0}^{40} a_r$

or $a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + \ldots + a_{39} + a_{40} = 0$ ...(ii)

Putting $x = -1$ in Eq. (i), we get

$$(-2)^{20} = \sum_{r=0}^{40} (-1)^r a_r$$

or $a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + \ldots - a_{39} + a_{40} = 2^{20}$ ...(iii)

On subtracting Eq. (iii) from Eq. (ii), we get

$$2[a_1 + a_3 + a_5 + \ldots + a_{39}] = -2^{20}$$

or $$a_1 + a_3 + a_5 + \ldots + a_{39} = -2^{19}$$

Corollary On adding Eqs. (ii) and (iii) and then dividing by 2, we get $a_0 + a_2 + a_4 + \ldots + a_{40} = 2^{19}$

Exercise for Session 2

1. If the rth term in the expansion of $(1 + x)^{20}$ has its coefficient equal to that of the $(r + 4)$th term, then r is

(a) 7 (b) 9 (c) 11 (d) 13

2. If the fourth term in the expansion of $\left(px + \frac{1}{x}\right)^n$ is $\frac{5}{2}$, then $n + p$ is equal to

(a) $\frac{9}{2}$ (b) $\frac{11}{2}$ (c) $\frac{13}{2}$ (d) $\frac{15}{2}$

3. If in the expansion of $\left(\sqrt[3]{2} + \frac{1}{\sqrt[3]{3}}\right)^n$, the ratio of 7th term from the beginning to the 7th term from the end is $\frac{1}{6}$, then n is

(a) 3 (b) 5 (c) 7 (d) 9

4. The number of integral terms in the expansion of $(5^{1/2} + 7^{1/8})^{1024}$ is

(a) 128 (b) 129 (c) 130 (d) 131

5. In the expansion of $(7^{1/3} + 11^{1/9})^{6561}$, the number of terms free from radicals is

(a) 715 (b) 725 (c) 730 (d) 750

6. If the coefficients of three consecutive terms in the expansion of $(1 + x)^n$ are 165, 330 and 462 respectively, the value of n is

(a) 7 (b) 9 (c) 11 (d) 13

7. If the coefficients of 5th, 6th and 7th terms in the expansion of $(1 + x)^n$ are in AP, then n is equal to

(a) 7 only (b) 14 only (c) 7 or 14 (d) None of these

8. If the middle term in the expansion of $\left(x^2 + \frac{1}{x}\right)^n$ is $924\, x^6$, the value of n is

(a) 8 (b) 12 (c) 16 (d) 20

9. If the sum of the binomial coefficients in the expansion of $\left(x^2 + \frac{2}{x^3}\right)^n$ is 243, the term independent of x is equal to

(a) 40 (b) 30 (c) 20 (d) 10

10. In the expansion of $(1 + x)(1 + x + x^2)\ldots(1 + x + x^2 + \ldots + x^{2n})$, the sum of the coefficients is

(a) 1 (b) $2n!$ (c) $2n! + 1$ (d) $(2n + 1)!$

Session 3

Two Important Theorems, Divisibility Problems

Two Important Theorems

Theorem 1 If $(\sqrt{P}+Q)^n = I+f$, where I and n are positive integers, n being odd and $0 \le f < 1$, then show that $(I+f)f = k^n$, where $P - Q^2 = k > 0$ and $\sqrt{P} - Q < 1$.

Proof Given, $\sqrt{P} - Q < 1 \quad \therefore \quad 0 < (\sqrt{P} - Q)^n < 1$

Now, let $(\sqrt{P} - Q)^n = f'$, where $0 < f' < 1$

Also $$I + f = (\sqrt{P} + Q)^n \quad ...(i)$$

$$0 \le f < 1 \quad ...(ii)$$

$$f' = (\sqrt{P} - Q)^n \quad ...(iii)$$

and $$0 < f' < 1 \quad ...(iv)$$

On subtracting Eq. (iii) from Eq. (i), we get

$$I + f - f' = (\sqrt{P} + Q)^n - (\sqrt{P} - Q)^n$$

$$= 2[{}^nC_1 (\sqrt{P})^{n-1} \cdot Q + {}^nC_3 (\sqrt{P})^{n-3} \cdot Q^3 + \ldots]$$

$$= 2 \text{ (integer)} = \text{Even integer} \quad ...(v)$$

[Since, n is odd, RHS contains even powers of $\sqrt{P}$, so RHS is an even integer]

$\therefore$ LHS is also an integer.

$\because$ I is an integer.

$\therefore$ $(f - f')$ is also an integer.

$\Rightarrow$ $$f - f' = 0 \quad [\because -1 < (f - f') < 1]$$

or $$f = f'$$

From Eq. (v), I is an even integer and

$$(I+f)f = (I+f)f' = (\sqrt{P}+Q)^n (\sqrt{P}-Q)^n$$

$$= (P - Q^2)^n = k^n$$

Remark

If n is even integer, then $(\sqrt{P} + Q)^n + (\sqrt{P} - Q)^n = I + f + f'$

Since, LHS and I are integers.

$\therefore$ $(f + f')$ is also an integer.

$\Rightarrow$ $f + f' = 1 \quad [\because 0 < (f + f') < 2]$

$\therefore$ $f' = 1 - f$

Hence, $(I + f)(1 - f) = (I + f)f' = (\sqrt{P} + Q)^n (\sqrt{P} - Q)^n$

$= (P - Q^2)^n = k^n$

Theorem 2 If $(P + \sqrt{Q})^n = I + f$, where I and n are positive integers and $0 \le f < 1$, show that $(I + f)(1 - f) = k^n$, where $P^2 - Q = k > 0$ and $P - \sqrt{Q} < 1$.

Proof Given, $P - \sqrt{Q} < 1$

$\therefore$ $0 < (P - \sqrt{Q})^n < 1$

Now, let $(P - \sqrt{Q})^n = f'$, where $0 < f' < 1$

Also, $$I + f = (P + \sqrt{Q})^n \quad ...(i)$$

$$0 \le f < 1 \quad ...(ii)$$

$$f' = (P - \sqrt{Q})^n \quad ...(iii)$$

and $$0 < f' < 1 \quad ...(iv)$$

On adding Eqs. (i) and (iii), we get

$$I + f + f' = (P + \sqrt{Q})^n + (P - \sqrt{Q})^n$$

$$= 2[{}^nC_0 P^n + {}^nC_2 P^{n-2} (\sqrt{Q})^2 + {}^nC_4 P^{n-4} (\sqrt{Q})^4 + \ldots]$$

$$= 2 \text{ (integer)} = \text{Even integer} \quad ...(v)$$

[Since, RHS contains even power of $\sqrt{Q}$, so RHS is an even integer]

$\therefore$ LHS is also an integer.

$\because$ I is an integer.

$\Rightarrow$ $f + f'$ is also an integer.

$\therefore$ $f + f' = 1 \quad [\because 0 < (f + f') < 2]$

or $f' = 1 - f$

From Eq. (v), I = even integer $- 1$ = odd integer and

$$(I + f)(1 - f) = (I + f)f'$$

$$= (P + \sqrt{Q})^n (P - \sqrt{Q})^n = (P^2 - Q)^n = k^n$$

Example 34. Show that the integral part of $(5 + 2\sqrt{6})^n$ is odd, where n is natural number.

Sol. $(5 + 2\sqrt{6})^n$ can be written as $(5 + \sqrt{24})^n$

Now, let $$I + f = (5 + \sqrt{24})^n \quad ...(i)$$

$$0 \le f < 1 \quad ...(ii)$$

and let $$f' = (5 - \sqrt{24})^n \quad ...(iii)$$

$$0 < f' < 1 \quad ...(iv)$$

On adding Eqs. (i) and (iii), we get

$$I + f + f' = (5 + \sqrt{24})^n + (5 - \sqrt{24})^n$$

$$I + 1 = 2p,$$

$\forall$ $p \in N$ = Even integer [from theorem 2]

$\therefore$ $I = 2p - 1$ = Odd integer

Example 35. Show that the integral part of $(5\sqrt{5} + 11)^{2n+1}$ is even, where $n \in N$.

Sol. $(5\sqrt{5} + 11)^{2n+1}$ can be written as $(\sqrt{125} + 11)^{2n+1}$

Now, let $$I + f = (\sqrt{125} + 11)^{2n+1} \quad ...(i)$$

$$0 \le f < 1 \quad ...(ii)$$

and let $$f' = (\sqrt{125} - 11)^{2n+1} \quad ...(iii)$$

$$0 < f' < 1 \quad ...(iv)$$

On subtracting Eq. (iii) from Eq. (i), we get

$$I + f - f' = (\sqrt{125} + 11)^{2n+1} - (\sqrt{125} - 11)^{2n+1}$$

$$I + 0 = 2p, \forall\ p \in N = \text{Even integer}$$

[from theorem 1]

$$\therefore \quad I = 2p = \text{Even integer}$$

Example 36. Let $R = (6\sqrt{6} + 14)^{2n+1}$ and $f = R - [R]$, where $[\cdot]$ denotes the greatest integer function. Find the value of $Rf, n \in N$.

Sol. $(6\sqrt{6} + 14)^{2n+1}$ can be written as $(\sqrt{216} + 14)^{2n+1}$ and given that $f = R - [R]$

and $R = (6\sqrt{6} + 14)^{2n+1} = (\sqrt{216} + 14)^{2n+1}$

$$\therefore \quad [R] + f = (\sqrt{216} + 14)^{2n+1} \quad \ldots(i)$$

$$0 \le f < 1 \quad \ldots(ii)$$

Let $f' = (\sqrt{216} - 14)^{2n+1}$ ...(iii)

$$0 < f' < 1 \quad \ldots(iv)$$

On subtracting Eq. (iii) from Eq. (i), we get

$$[R] + f - f' = (\sqrt{216} + 14)^{2n+1} - (\sqrt{216} - 14)^{2n+1}$$

$[R] + 0 = 2p, \forall\ p \in N = \text{Even integer}$ [from theorem 1]

$$\therefore \quad f - f' = 0 \text{ or } f = f'$$

Now, $Rf = Rf' = (\sqrt{216} + 14)^{2n+1}(\sqrt{216} - 14)^{2n+1}$

$$= (216 - 196)^{2n+1} = (20)^{2n+1}$$

Example 37. If $(7 + 4\sqrt{3})^n = s + t$, where n and s are positive integers and t is a proper fraction, show that $(1 - t)(s + t) = 1$.

Sol. $(7 + 4\sqrt{3})^n$ can be written as $(7 + \sqrt{48})^n$

$$\therefore \quad s + t = (7 + \sqrt{48})^n \quad \ldots(i)$$

$$0 < t < 1 \quad \ldots(ii)$$

Now, let $t' = (7 - \sqrt{48})^n$ ...(iii)

$$0 < t' < 1 \quad \ldots(iv)$$

On adding Eqs. (i) and (iii), we get

$$s + t + t' = (7 + \sqrt{48})^n + (7 - \sqrt{48})^n$$

$s + 1 = 2p, \forall\ p \in N = \text{Even integer}$ [from theorem 2]

$$\therefore \quad t + t' = 1 \text{ or } 1 - t = t'$$

Then, $(1 - t)(s + t) = t'(s + t) = (7 - \sqrt{48})^n (7 + \sqrt{48})^n$

[from Eqs. (i) and (iii)]

$$= (49 - 48)^n = (1)^n = 1$$

Example 38. If $x = (8 + 3\sqrt{7})^n$, where n is a natural number, prove that the integral part of x is an odd integer and also show that $x - x^2 + x[x] = 1$, where $[\cdot]$ denotes the greatest integer function.

Sol. $(8 + 3\sqrt{7})^n$ can be written as $(8 + \sqrt{63})^n$

$$\therefore \quad x = [x] + f$$

or $[x] + f = (8 + \sqrt{63})^n$ ...(i)

$$0 \le f < 1 \quad \ldots(ii)$$

Now, let $f' = (8 - \sqrt{63})^n$ ...(iii)

$$0 < f' < 1 \quad \ldots(iv)$$

On adding Eqs. (i) and (iii), we get

$$[x] + f + f' = (8 + \sqrt{63})^n + (8 - \sqrt{63})^n$$

$[x] + 1 = 2p, \forall\ p \in N = \text{Even integer}$

[from theorem 2]

$$\therefore \quad [x] = 2p - 1 = \text{Odd integer}$$

i.e., Integral part of x = Odd integer

$$\because \quad f + f' = 1 \implies 1 - f = f' \quad \ldots(v)$$

LHS $= x - x^2 + x[x] = x - x(x - [x]) = x - xf$

$[\because x = [x] + f]$

$= x(1 - f) = x f'$ [from Eq.(v)]

$= (8 + \sqrt{63})^n (8 - \sqrt{63})^n$ [from Eqs.(i) and (iii)]

$= (64 - 63)^n = (1)^n = 1 = \text{RHS}$

Remark

Sometimes, students find it difficult to decide whether a problem is on addition or subtraction. Now, if $x = [x] + f$ and $0 < f' < 1$ and if $[x] + f + f' =$ Integer. Then, addition and if $[x] + f - f' =$ Integer, the subtraction and values of $(f + f')$ and $(f - f')$ are 1 and 0, respectively.

Divisibility Problems

Type I

(i) $(x^n - a^n)$ is divisible by $(x - a)$, $\forall\ n \in N$.

(ii) $(x^n + a^n)$ is divisible by $(x + a)$, $\forall\ n \in$ Only odd natural numbers.

Example 39. Show that $1992^{1998} - 1955^{1998} - 1938^{1998} + 1901^{1998}$ is divisible by 1998.

Sol. Here, $n = 1998$ (Even)

$\therefore$ Only result (i) applicable.

Let $P = 1992^{1998} - 1955^{1998} - 1938^{1998} + 1901^{1998}$

$$= (1992^{1998} - 1955^{1998}) - (1938^{1998} - 1901^{1998})$$

divisible by $(1992 - 1955)$ i.e. 37; divisible by $(1938 - 1901)$ i.e. 37

$\therefore$ P is divisible by 37.

Also, $P = (1992^{1998} - 1938^{1998}) - (1955^{1998} - 1901^{1998})$

divisible by $(1992 - 1938)$ i.e., 54; divisible by $(1955 - 1901)$ i.e., 54

$\therefore$ P is also divisible by 54.

Hence, P is divisible by 37×54, i.e., 1998.

Example 40. Prove that $2222^{5555} + 5555^{2222}$ is divisible by 7.

Sol. We have, $2222^{5555} + 5555^{2222}$

$$= (2222^{5555} + 4^{5555}) + (5555^{2222} - 4^{2222}) - (4^{5555} - 4^{2222}) \quad \ldots(i)$$

The number $(2222^{5555} + 4^{5555})$ is divisible by $2222 + 4 = 2226 = 7 \times 318$, which is divisible by 7 and the number $(5555^{2222} - 4^{2222})$ is divisible by

$5555 - 4 = 5551 = 7 \times 793$, which is divisible by 7 and the number $(4^{5555} - 4^{2222}) = 4^{2222}(4^{3333} - 1) = 4^{2222}(64^{1111} - 1^{1111})$ is divisible by $64 - 1 = 63 = 7 \times 9$, which is divisible by 7.

Therefore, each brackets of Eq. (i) are divisible by 7. Hence, $2222^{5555} + 5555^{2222}$ is divisible by 7.

Type II To show that an Expression is Divisible by An Integer

Solution Process

(i) If a, p, n and r are positive integers, first of all write $a^{pn+r} = a^{pn} \cdot a^r = (a^p)^n \cdot a^r$

(ii) If we will show that the given expression is divisible by c. Then, expression $a^p = \{1 + (a^p - 1)\}$, if some power of $(a^p - 1)$ has c as a factor.

or $a^p = \{2 + (a^p - 2)\}$, if some power of $(a^p - 2)$ has c as a factor.

or $a^p = \{3 + (a^p - 3)\}$, if some power of $(a^p - 3)$ has c as a factor.

⋮ ⋮ ⋮ ⋮ ⋮ ⋮

or $a^p = \{k + (a^p - k)\}$, if some power of $(a^p - k)$ has c as a factor.

Example 41. ***If*** n ***is any positive integer, show that*** $2^{3n+3} - 7n - 8$ ***is divisible by*** 49.

Sol. Given expression

$$= 2^{3n+3} - 7n - 8 = 2^{3n} \cdot 2^3 - 7n - 8$$
$$= 8^n \cdot 8 - 7n - 8 = 8(1+7)^n - 7n - 8$$
$$= 8(1 + {}^nC_1 \cdot 7 + {}^nC_2 \cdot 7^2 + \ldots + {}^nC_n \cdot 7^n) - 7n - 8$$
$$= 8 + 56n + 8({}^nC_2 \cdot 7^2 + \ldots + {}^nC_n \cdot 7^n) - 7n - 8$$
$$= 49n + 8({}^nC_2 \cdot 7^2 + \ldots + {}^nC_n \cdot 7^n)$$
$$= 49\{n + 8({}^nC_2 + \ldots + {}^nC_n \cdot 7^{n-2})\}$$

Hence, $2^{3n+3} - 7n - 8$ is divisible by 49.

Example 42. If 10^n divides the number $101^{100} - 1$, find the greatest value of n.

Sol. We have, $101^{100} - 1 = (1 + 100)^{100} - 1$

$$= 1 + {}^{100}C_1 \cdot 100 + {}^{100}C_2 \cdot 100^2 + \ldots + {}^{100}C_{100}\, 100^{100} - 1$$
$$= {}^{100}C_1 \cdot 100 + {}^{100}C_2 \cdot 100^2 + \ldots + {}^{100}C_{100} \cdot 10^{100}$$
$$= (100)(100) + {}^{100}C_2 \cdot 100^2 + \ldots + {}^{100}C_{100} \cdot 100^{100}$$
$$= (100)^2 [1 + {}^{100}C_2 + \ldots + 100^{98}]$$
$$= 100^2\, k, \text{ where } k \text{ is a positive integer}$$

Therefore, $101^{100} - 1$ is divisible by 100^2 i.e., 10^4.

$\therefore \quad n = 4$

How to Find Remainder by Using Binomial Theorem

If a, p, n and r are positive integers, then to find the remainder when a^{pn+r} is divided by b, we adjust power of a to a^{pn+r} which is very close to b, say with difference 1 i.e., $b \pm 1$. Also, the remainder is always positive. When number of the type $5n - 2$ is divided by 5, then we have

$$5)\, 5n - 2\, (n$$
$$-\frac{5n}{-2}$$

We can write $-2 = -2 - 3 + 3 = -5 + 3$

or $$\frac{5n-2}{5} = \frac{5n-5+3}{5} = n - 1 + \frac{3}{5}$$

Hence, the remainder is 3.

Example 43. If 7^{103} is divided by 25, find the remainder.

Soln. We have, $7^{103} = 7 \cdot 7^{102} = 7 \cdot (7^2)^{51} = 7(49)^{51} = 7(50-1)^{51}$

$$= 7[(50)^{51} - {}^{51}C_1(50)^{50} + {}^{51}C_2(50)^{49} - \ldots - 1]$$
$$= 7[(50)^{51} - {}^{51}C_1(50)^{50} + {}^{51}C_2(50)^{49} - \ldots + {}^{51}C_{50}(50)] - 7 - 18 + 18$$
$$= 7[50((50)^{50} - {}^{51}C_1(50)^{49} + {}^{51}C_2(50)^{48} - \ldots + {}^{51}C_{50})] - 25 + 18$$
$$= 7[50k] - 25 + 18, \text{ where } k \text{ is an integer.}$$
$$= 25[14k - 1] + 18 = 25p + 18 \qquad [\text{where } p \text{ is an integer}]$$

Now, $\frac{7^{103}}{25} = p + \frac{18}{25}$. Hence, the remainder is 18.

Example 44. Find the remainder, when $5^{5^{5^{\cdots^5}}}$ (24 times 5) is divided by 24.

Sol. Here, $5^{5^{5^{\cdots^5}}}$ (23 times 5) is an odd natural number.

Let $5^{5^{5^{\cdots^5}}}$ (23 times 5) $= 2m + 1$

Now, let $x = 5^{5^{5^{\cdots^5}}}$ (24 times 5) $= 5^{2m+1} = 5 \cdot 5^{2m}$, where m is a natural number.

$\therefore \quad x = 5 \cdot (5^2)^m = 5(24+1)^m$
$$= 5[{}^mC_0(24)^m + {}^mC_1(24)^{m-1} + \ldots + {}^mC_{m-1}(24) + 1]$$
$$= 5(24k + 1) = 24(5k) + 5$$

$\therefore \quad \frac{x}{24} = 5k + \frac{5}{24}$

Hence, the remainder is 5.

Example 45. If 7 divides $32^{32^{32}}$, then find the remainder.

Solution. We have, $32 = 2^5$

$\therefore \quad 32^{32} = (2^5)^{32} = 2^{160} = (3-1)^{160}$
$$= {}^{160}C_0(3)^{160} - {}^{160}C_1(3)^{159} + \ldots - {}^{160}C_{159}(3) + 1$$
$$= 3(3^{159} - {}^{160}C_1(3)^{158} + \ldots - {}^{160}C_{159}) + 1$$
$$= 3m + 1, \; m \in I^+$$

Now, $32^{32^{32}} = 32^{3m+1} = 2^{5(3m+1)} = 2^{15m+5}$

$= 2^2 \cdot 2^{3(5m+1)} = 4(8)^{5m+1} = 4(7+1)^{5m+1}$

$= 4[{}^{5m+1}C_0(7)^{5m+1} + {}^{5m+1}C_1(7)^{5m} + {}^{5m+1}C_2(7)^{5m-1} + \ldots + {}^{5m+1}C_{5m}(7) + 1]$

$= 4[7({}^{5m+1}C_0(7)^{5m} + {}^{5m+1}C_1(7)^{5m-1} + \ldots + {}^{5m+1}C_{5m}) + 1]$

$= 4[7k+1]$, where k is positive integer $= 28k + 4$

$\therefore \quad \dfrac{32^{32^{32}}}{7} = 4k + \dfrac{4}{7}$

Hence, the remainder is 4.

How to Find Last Digit, Last Two Digits, Last Three Digits, ... and so on.

If a, p, n and r are positive integers, then a^{pn+r} is adjust of the form $(10k \pm 1)^m$, where k and m are positive integers. For last digit, take 10 common. For last two digits, take 100 common, for last three digits, take 1000 common , ... and so on.

i.e. $(10k \pm 1)^m = (10k)^m + {}^mC_1(10k)^{m-1}(\pm 1) + {}^mC_2(10k)^{m-2}(\pm 1)^2 + \ldots + {}^mC_{m-2}(10k)^2(\pm 1)^{m-2} + {}^mC_{m-1}(10k)(\pm 1)^{m-1} + (\pm 1)^m$

For last digit $= 10\lambda + (\pm 1)^m$

For last two digits $= 100\mu + {}^mC_{m-1}(10k)(\pm 1)^{m-1} + (\pm 1)^m$

For last three digits $= 1000\nu + {}^mC_{m-2}(10k)^2(\pm 1)^{m-2} + {}^mC_{m-1}(10k)(\pm 1)^{m-1} + (\pm 1)^m$ and so on where $\lambda, \mu, \nu \in I$.

Example 46. Find the last two digits of 3^{400}.

Sol. We have, $3^{400} = (3^2)^{200} = (9)^{200} = (10-1)^{200}$

$= (10)^{200} - {}^{200}C_1(10)^{199} + {}^{200}C_2(10)^{198} - {}^{200}C_3(10)^{197} + \ldots + {}^{200}C_{198}(10)^2 - {}^{200}C_{199}(10) + 1$

$= 100\mu - {}^{200}C_{199}(10) + 1$, where $\mu \in I$

$= 100\mu - {}^{200}C_1(10) + 1 = 100\mu - 2000 + 1$

$= 100(\mu - 20) + 1 = 100p + 1$, where p is an integer.

Hence, the last two digits of 3^{400} is $00 + 1 = 01$.

Example 47. If the number is 17^{256}, find the

(i) last digit. (ii) last two digits.

(iii) last three digits of 17^{256}.

Sol. Since, $17^{256} = (17^2)^{128} = (289)^{128} = (290-1)^{128}$

$\therefore 17^{256} = {}^{128}C_0(290)^{128} - {}^{128}C_1(290)^{127} + {}^{128}C_2(290)^{126} - {}^{128}C_3(290)^{125} + \ldots - {}^{128}C_{125}(290)^3 + {}^{128}C_{126}(290)^2 - {}^{128}C_{127}(290) + 1$

(i) **For last digit**

$17^{256} = 290[{}^{128}C_0(290)^{127} - {}^{128}C_1(290)^{126} + {}^{128}C_2(290)^{125} - \ldots - {}^{128}C_{127}(1)] + 1$

$= 290(k) + 1$, where k is an integer.

$\therefore$ Last digit $= 0 + 1 = 1$

(ii) **For last two digits,**

$17^{256} = (290)^2[{}^{128}C_0(290)^{126} - {}^{128}C_1(290)^{125} + {}^{128}C_2(290)^{124} - \ldots + {}^{128}C_{126}(1)] - {}^{128}C_{127}(290) + 1$

$= 100m - {}^{128}C_{127}(290) + 1$, where m is an integer.

$= 100m - {}^{128}C_1(290) + 1 = 100m - 128 \times 290 + 1$

$= 100m - 128 \times (300 - 10) + 1$

$= 100(m - 384) + 1281$

$= 100n + 1281$, where n is an integer.

$\therefore$ Last two digits $= 00 + 81 = 81$

(iii) **For last three digits,**

$17^{256} = (290)^3[{}^{128}C_0(290)^{125} - {}^{128}C_1(290)^{124} + {}^{128}C_2(290)^{123} - \ldots - {}^{128}C_{125}(1)] + {}^{128}C_{126}(290)^2 - {}^{128}C_{127}(290) + 1$

$= 1000m + {}^{128}C_{126}(290)^2 - {}^{128}C_{127}(290) + 1$

where, m is an integer

$= 1000m + {}^{128}C_2(290)^2 - {}^{128}C_1(290) + 1$

$= 1000m + \dfrac{(128)(127)}{2}(290)^2 - 128 \times 290 + 1$

$= 1000m + (128)(127)(290)(145) - (128)(290) + 1$

$= 1000m + (128)(290)(127 \times 145 - 1) + 1$

$= 1000m + (128)(290)(18414) + 1$

$= 1000m + 683527680 + 1$

$= 1000m + 683527000 + 680 + 1$

$= 1000(m + 683527) + 681$

$\therefore$ Last three digits $= 000 + 681 = 681$

Two Important Results

(i) $2 \le \left(1 + \dfrac{1}{n}\right)^n < 3, n \ge 1, n \in N$

(ii) If $n > 6$, then $\left(\dfrac{n}{3}\right)^n < n! < \left(\dfrac{n}{2}\right)^n$

Example 48. ***Find the positive integer just greater than*** $(1 + 0.0001)^{10000}$.

Sol. $(1 + 0.0001)^{10000} = \left(1 + \dfrac{1}{10000}\right)^{10000}$

We know that, $2 \le \left(1+\frac{1}{n}\right)^n < 3,\ n \ge 1,\ n \in N$ [Result (i)]

Hence, positive integer just greater than $(1+0.0001)^{10000}$ is 3.

Example 49. Find the greater number is 100^{100} and $(300)!$.

Sol. Using Result (ii), We know that, $\left(\frac{n}{3}\right)^n < n!$

Putting $n = 300$, we get

$$(100)^{300} < (300)! \quad \ldots(i)$$

But $$(100)^{100} < (100)^{300} \quad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$(100)^{100} < (100)^{300} < (300)!$$

$$\Rightarrow \quad (100)^{100} < (300)!$$

Hence, the greater number is $(300)!$.

Example 50. Find the greater number in $300!$ and $\sqrt{300^{300}}$.

Sol. Since, $(100)^{150} > 3^{150}$

$$\Rightarrow \quad (100)^{150} \cdot (100)^{150} > 3^{150} \cdot (100)^{150}$$

$$\Rightarrow \quad (100)^{300} > (300)^{150}$$

or $$(100)^{300} > \sqrt{300^{300}} \quad \ldots(i)$$

Using result (ii), $\left(\frac{n}{3}\right)^n < n!$

Putting $n = 300$, we get $(100)^{300} < 300!$ ...(ii)

From Eqs. (i) and (ii), we get

$$\sqrt{300^{300}} < (100)^{300} < 300!$$

$$\Rightarrow \quad \sqrt{300^{300}} < 300!$$

Hence, the greater number is $300!$.

Exercise for Session 3

1. If $x = (7+4\sqrt{3})^{2n} = [x] + f$, where $n \in N$ and $0 \le f < 1$, then $x(1-f)$ is equal to

(a) 1 (b) 0 (c) -1 (d) even integer

2. If $(5+2\sqrt{6})^n = I + f;\ n, I \in N$ and $0 \le f < 1$, then I equals

(a) $\frac{1}{f} - f$ (b) $\frac{1}{1+f} - f$ (c) $\frac{1}{1-f} - f$ (d) $\frac{1}{1+f} + f$

3. If $n > 0$ is an odd integer and $x = (\sqrt{2}+1)^n, f = x - [x]$, then $\frac{1-f^2}{f}$ is

(a) an irrational number (b) a non-integer rational number (c) an odd number (d) an even number

4. Integral part of $(\sqrt{2}+1)^6$ is

(a) 196 (b) 197 (c) 198 (d) 199

5. $(103)^{86} - (86)^{103}$ is divisible by

(a) 7 (b) 13 (c) 17 (d) 23

6. Fractional part of $\frac{2^{78}}{31}$ is

(a) $\frac{2}{31}$ (b) $\frac{4}{31}$ (c) $\frac{8}{31}$ (d) $\frac{16}{31}$

7. The unit digit of $17^{1983} + 11^{1983} - 7^{1983}$ is

(a) 1 (b) 2 (c) 3 (d) 0

8. The last two digits of the number $(23)^{14}$ are

(a) 01 (b) 03 (c) 09 (d) 27

9. The last four digits of the number 3^{100} are

(a) 2001 (b) 3211 (c) 1231 (d) 0001

10. The remainder when 23^{23} is divided by 53 is

(a) 17 (b) 21 (c) 30 (d) 47

Session 4

Use of Complex Numbers in Binomial Theorem, Multinomial Theorem, Use of Differentiation, Use of Integration, Binomial Inside Binomial, Sum of the Series

Use of Complex Numbers in Binomial Theorem

If $\theta \in R, n \in N$ and $i = \sqrt{-1}$, then

$$(\cos\theta + i\sin\theta)^n = {}^nC_0 (\cos\theta)^{n-0} (i\sin\theta)^0 + {}^nC_1 (\cos\theta)^{n-1} (i\sin\theta)^1 + {}^nC_2 (\cos\theta)^{n-2} (i\sin\theta)^2 + {}^nC_3 (\cos\theta)^{n-3} (i\sin\theta)^3 + \ldots$$

or $\cos n\theta + i\sin n\theta = \cos^n\theta + i\cdot {}^nC_1 (\cos\theta)^{n-1}\sin\theta - {}^nC_2 (\cos\theta)^{n-2}\sin^2\theta - i\cdot {}^nC_3 (\cos\theta)^{n-3}\sin^3\theta + \ldots$

On comparing real and imaginary parts, we get

$$\cos n\theta = \cos^n\theta - {}^nC_2 (\cos\theta)^{n-2}\sin^2\theta - {}^nC_4 (\cos\theta)^{n-4}\sin^4\theta - \ldots$$

and $\sin n\theta = {}^nC_1 (\cos\theta)^{n-1}\sin\theta - {}^nC_3 (\cos\theta)^{n-3}\sin^3\theta + {}^nC_5 (\cos\theta)^{n-5}\sin^5\theta - \ldots$

Example 51. ***If*** $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + \ldots$, find the values of

(i) $C_0 - C_2 + C_4 - C_6 + \ldots$

(ii) $C_1 - C_3 + C_5 - C_7 + \ldots$

(iii) $C_0 + C_3 + C_6 + \ldots$

Sol. $\because (1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + C_5x^5 + \ldots$

Putting $x = i$, where $i = \sqrt{-1}$, then

$$(1+i)^n = C_0 + C_1 i + C_2 i^2 + C_3 i^3 + C_4 i^4 + C_5 i^5 + \ldots$$
$$= (C_0 - C_2 + C_4 - \ldots) + i(C_1 - C_3 + C_5 - \ldots) \quad \ldots(i)$$

Also, $(1+i)^n = \left[\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\right]^n$

$$= 2^{n/2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^n$$
$$= 2^{n/2}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) \quad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$(C_0 - C_2 + C_4 - \ldots) + i(C_1 - C_3 + C_5 - \ldots) = 2^{n/2}\cos\left(\frac{n\pi}{4}\right) + i\cdot 2^{n/2}\sin\left(\frac{n\pi}{4}\right)$$

On comparing real and imaginary parts, we get

$$C_0 - C_2 + C_4 - \ldots = 2^{n/2}\cos\left(\frac{n\pi}{4}\right) \quad \text{[part (i)]}$$

$$C_1 - C_3 + C_5 - \ldots = 2^{n/2}\sin\left(\frac{n\pi}{4}\right) \quad \text{[part (ii)]}$$

We have, $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + C_5x^5 + C_6x^6 + \ldots$

Putting $x = 1, \omega, \omega^2$ (cube roots of unity) and adding, we get

$$3(C_0 + C_3 + C_6 + \ldots) = 2^n + (1+\omega)^n + (1+\omega^2)^n$$
$$= 2^n + (-\omega^2)^n + (-\omega)^n = 2^n + (-1)^n(\omega^{2n} + \omega^n)$$
$$= 2^n + (-1)^n\left\{e^{\frac{4\pi in}{3}} + e^{\frac{2\pi in}{3}}\right\}$$
$$= 2^n + (-1)^n\cdot e^{n\pi i}\cdot 2\cos\left(\frac{n\pi}{3}\right)$$
$$= 2^n + (-1)^n\cdot(-1)^n\cdot 2\cos\left(\frac{n\pi}{3}\right)$$
$$= 2^n + (-1)^{2n}\cdot 2\cos\left(\frac{n\pi}{3}\right) = 2^n + 2\cos\left(\frac{n\pi}{3}\right)$$

$$\therefore\ C_0 + C_3 + C_6 + \ldots = \frac{1}{3}\left\{2^n + 2\cos\left(\frac{n\pi}{3}\right)\right\}$$

Example 52. ***Find the value of*** ${}^{4n}C_0 + {}^{4n}C_4 + {}^{4n}C_8 + \ldots + {}^{4n}C_{4n}$.

Sol. $\because\ 4 - 0 = 8 - 4 = \ldots = 4$

$\therefore$ Four roots of unity $(1)^{1/4}$ are $1, -1, i, -i$, we have

$$(1+x)^{4n} = {}^{4n}C_0 + {}^{4n}C_1x + {}^{4n}C_2x^2 + {}^{4n}C_3x^3 + \ldots$$

Putting $x = 1, -1, i, -i$ and then adding, we get

$$4({}^{4n}C_0 + {}^{4n}C_4 + {}^{4n}C_8 + \ldots) = 2^{4n} + 0 + (1+i)^{4n} + (1-i)^{4n}$$
$$= 2^{4n} + (2i)^{2n} + (-2i)^{2n}$$

$$= 2^{4n} + 2^{2n}(-1)^n + 2^{2n}(-1)^n$$

$$= 2^{4n} + (-1)^n \cdot 2^{2n+1}$$

$$\therefore \quad {}^{4n}C_0 + {}^{4n}C_4 + {}^{4n}C_8 + \ldots = 2^{4n-2} + (-1)^n \cdot 2^{2n-1}$$

Remark

If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n$, then

(i) $C_0 + C_4 + C_8 + C_{12} + \ldots = \frac{1}{2}\left\{2^{n-1} + 2^{n/2}\cos\left(\frac{n\pi}{4}\right)\right\}$

(ii) $C_1 + C_5 + C_9 + C_{13} + \ldots = \frac{1}{2}\left\{2^{n-1} + 2^{n/2}\sin\left(\frac{n\pi}{4}\right)\right\}$

(iii) $C_0 + C_6 + C_{12} + \ldots = \frac{1}{3}\left\{2^{n-1}\cos\left(\frac{n\pi}{4}\right) + 3^{n/2}\cos\left(\frac{n\pi}{6}\right)\right\}$

Multinomial Theorem

If n is a positive integer and $x_1, x_2, x_3, \ldots, x_k \in C$, then

$$(x_1 + x_2 + x_3 + \ldots + x_k)^n = \sum \frac{n!}{(\alpha_1!)(\alpha_2!)(\alpha_3!)\ldots(\alpha_k!)} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \ldots x_k^{\alpha_k}$$

where, $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k$ are all non-negative integers such that $\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_k = n$.

Remark

The coefficient of $x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot x_3^{\alpha_3} \ldots x_k^{\alpha_k}$ in the expansion of $(x_1 + x_2 + x_3 + \ldots + x_k)^n$ is $\sum \frac{n!}{(\alpha_1!)(\alpha_2!)(\alpha_3!)\ldots(\alpha_k!)}$.

In Particular

(i) $(a+b+c)^n = \sum \frac{n!}{(\alpha!)(\beta!)(\gamma!)} a^\alpha b^\beta c^\gamma$ such that $\alpha + \beta + \gamma = n$

(ii) $(a+b+c+d)^n = \sum \frac{n!}{(\alpha!)(\beta!)(\gamma!)(\delta!)} a^\alpha b^\beta c^\gamma d^\delta$ such that $\alpha + \beta + \gamma + \delta = n$

Example 53. ***Find the coefficient of $a^4 b^3 c^2 d$ in the expansion of $(a-b+c-d)^{10}$.***

Sol. The coefficient of $a^4 b^3 c^2 d$ in the expansion of

$$(a-b+c-d)^{10} \text{ is } (-1)^4 \frac{10!}{4!3!2!1!} = 12600$$

[powers of b and d are 3 and 1 $\therefore (-1)^3(-1)$]

Example 54. ***Find the coefficient of $a^3 b^4 c^5$ in the expansion of $(bc+ca+ab)^6$.***

Sol. In this case, write $a^3 b^4 c^5 = (ab)^x (bc)^y (ca)^z$ say

$$\therefore \quad a^3 b^4 c^5 = a^{z+x} \cdot b^{x+y} \cdot c^{y+z}$$

$$\Rightarrow \quad z + x = 3,\ x + y = 4$$

$$y + z = 5$$

On adding all, we get $2(x+y+z) = 12$

$$\therefore \quad x + y + z = 6$$

Then, $x = 1, y = 3, z = 2$

Therefore, the coefficient of $a^3 b^4 c^5$ in the expansion of $(bc+ca+ab)^6$ or the coefficient of $(ab)^1 (bc)^3 (ca)^2$ in the expansion of $(bc+ca+ab)^6$ is $\frac{6!}{1!3!2!}$, i.e. 60.

Aliter

Coefficient of $a^3 b^4 c^5$ in the expansion of $(bc+ca+ab)^6$

= Coefficient of $a^3 b^4 c^5$ in the

expansion of $(abc)^6 \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^6$

= Coefficient of $\left(\frac{1}{a}\right)^3 \left(\frac{1}{b}\right)^2 \left(\frac{1}{c}\right)^1$ in the expansion of

$\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^6$ is $\frac{6!}{3!2!1!} = 60$

Number of Distinct or Dissimilar Terms in the Multinomial Expansion

Statement The number of distinct or dissimilar terms in the multinomial expansion of $(x_1 + x_2 + x_3 + \ldots + x_k)^n$ is ${}^{n+k-1}C_{k-1}$.

Proof We have, $(x_1 + x_2 + x_3 + \ldots + x_k)^n$

$$= \sum \frac{n!}{(\alpha_1!)(\alpha_2!)(\alpha_3!)\ldots(\alpha_k!)} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \ldots x_k^{\alpha_k}$$

where, $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k$ are non-negative integers such that

$$\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_k = n \qquad \ldots(i)$$

Here, the number of terms in the expansion of

$$(x_1 + x_2 + x_3 + \ldots + x_k)^n$$

= The number of non-negative integral solutions of the Eq. (i)

$= {}^{n+k-1}C_{k-1}$

Example 55. ***Find the total number of distinct or dissimilar terms in the expansion of $(x+y+z+w)^n$, $n \in N$.***

Sol. The total number of distinct or dissimilar terms in the expansion of $(x+y+z+w)^n$ is

$$= {}^{n+4-1}C_{4-1} = {}^{n+3}C_3 = \frac{(n+3)(n+2)(n+1)}{1 \cdot 2 \cdot 3}$$

$$= \frac{(n+1)(n+2)(n+3)}{6}$$

I. Aliter

We know that, $(x+y+z+w)^n = \{(x+y)+(z+w)\}^n$

$$= (x+y)^n + {}^nC_1 (x+y)^{n-1}(z+w) + {}^nC_2 (x+y)^{n-2}(z+w)^2 + \ldots + {}^nC_n (z+w)^n$$

$\therefore$ Number of terms in RHS

$= (n+1) + n\cdot 2 + (n-1)\cdot 3 + \ldots + 1\cdot(n+1)$

$$= \sum_{r=0}^{n} (n-r+1)(r+1)$$

$$= \sum_{r=0}^{n} (n+1) + nr - r^2 = (n+1)\sum_{r=0}^{n} 1 + n\sum_{r=0}^{n} r - \sum_{r=0}^{n} r^2$$

$$= (n+1)\cdot(n+1) + n\cdot\frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{(n+1)(n+2)(n+3)}{6}$$

II. Aliter

$$(x+y+z+w)^n = \sum \frac{n!}{n_1!\, n_2!\, n_3!\, n_4!}\, x^{n_1} y^{n_2} z^{n_3} w^{n_4}$$

where, n_1, n_2, n_3, n_4 are non-negative integers subject to the condition $n_1 + n_2 + n_3 + n_4 = n$

Hence, number of the distinct terms

= Coefficient of x^n in $(x^0 + x^1 + x^2 + \ldots + x^n)^4$

$$= \text{Coefficient of } x^n \text{ in } \left(\frac{1-x^{n+1}}{1-x}\right)^4$$

= Coefficient of x^n in $(1-x^{n+1})^4 (1-x)^{-4}$

= Coefficient of x^n in $(1-x)^{-4}$ $\quad [\because x^{n+1} > x^n]$

$$= {}^{n+3}C_n = {}^{n+3}C_3 = \frac{(n+3)(n+2)(n+1)}{6}$$

Greatest Coefficient in Multinomial Expansion

The greatest coefficient in the expansion of $(x_1 + x_2 + x_3 + \ldots + x_k)^n$ is $\dfrac{n!}{(q!)^{k-r}((q+1)!)^r}$, where q is the quotient and r is the remainder when n is divided by k i.e.

$$k)\, n\, (q$$
$$\underline{\quad r}$$

Example 56. ***Find the greatest coefficient in the expansion of*** $(a+b+c+d)^{15}$.

Sol. Here, $n = 15$ and $k = 4$ $\quad [\because a, b, c, d$ are four terms$]$

$$4)\, 15\, (3$$
$$\underline{12}$$
$$3$$

$\therefore \quad q = 3$ and $r = 3$

Hence, greatest coefficient $= \dfrac{15!}{(3!)^1 (4!)^3}$

Coefficient of x^r in Multinomial Expansion

If n is a positive integer and $a_1, a_2, a_3, \ldots, a_k \in C$, then coefficient of x^r in the expansion of $(a_1 + a_2 x + a_3 x^2 + \ldots + a_k x^{k-1})^n$, is

$$\sum \frac{n!}{(\alpha_1!)(\alpha_2!)(\alpha_3!)\ldots(\alpha_k!)}\, a_1^{\alpha_1} a_2^{\alpha 2} a_3^{\alpha_3} \ldots a_k^{\alpha_k}$$

where, $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k$ are non-negative integers such that $\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_k = n$

and $\quad \alpha_2 + 2\alpha_3 + 3\alpha_4 + \ldots + (k-1)\alpha_k = r$

Example 57. Find the coefficient of x^7 in the expansion of $(1+3x-2x^3)^{10}$.

Sol. Coefficient of x^7 in the expansion of $(1+3x-2x^3)^{10}$ is

$$= \sum \frac{10!}{\alpha!\,\beta!\,\gamma!} (1)^\alpha (3)^\beta (-2)^\gamma$$

where, $\alpha + \beta + \gamma = 10$ and $\beta + 3\gamma = 7$

The possible values of α, β and γ are given below

α	β	γ
3	7	0
5	4	1
7	1	2

$\therefore$ Coefficient of x^7

$$= \frac{10!}{3!\,7!\,0!}(1)^3 (3)^7 (-2)^0 + \frac{10!}{5!\,4!\,1!}(1)^5 (3)^4 (-2)^1 + \frac{10!}{7!\,1!\,2!}(1)^7 (3)^1 (-2)^2$$

$$= 262440 - 204120 + 4320 = 62640$$

Use of Differentiation

This method applied only when the numericals occur as the product of the binomial coefficients, if

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n$$

Solution Process

(i) If last term of the series leaving the plus or minus sign is m, then divide m by n. If q is the quotient and r is the remainder.

i.e. $\quad m = nq + r \quad$ or $\quad n)\, m\, (q$

$$\underline{nq}$$
$$r$$

Then, replace x by x^q in the given series and multiplying both sides of the expression by x^r.

(ii) After this, differentiate both sides w.r.t. x and put $x = 1$ or -1 or $i (i = \sqrt{-1})$, etc. According to the given series.

(iii) If product of two numericals (or square of numericals) or three numericals (or cube of numericals), then differentiate twice or thrice.

Example 58. *If* $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$, *prove that* $C_1 + 2C_2 + 3C_3 + \ldots + nC_n = n \cdot 2^{n-1}$.

Sol. Here, last term of $C_1 + 2C_2 + 3C_3 + \ldots + nC_n$ is nC_n i.e., n and last term with positive sign.

Then, $n = n \cdot 1 + 0$ or $n) n (1$

$\underline{0}$

Here, $q = 1$ and $r = 0$

Then, the given series is

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n$$

Differentiating both sides w.r.t. x, we get

$$n(1+x)^{n-1} = 0 + C_1 + 2C_2 x + 3C_3 x^2 + \ldots + nC_n x^{n-1}$$

Putting $x = 1$, we get

$$n \cdot 2^{n-1} = C_1 + 2C_2 + 3C_3 + \ldots + nC_n$$

or $$C_1 + 2C_2 + 3C_3 + \ldots + nC_n = n \cdot 2^{n-1}$$

I. Aliter

$$C_1 + 2C_2 + 3C_3 + \ldots + nC_n$$
$$= n + 2 \cdot \frac{n(n-1)}{1 \cdot 2} + 3 \cdot \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} + \ldots + n \cdot 1$$
$$= n \left\{ 1 + (n-1) + \frac{(n-1)(n-2)}{1 \cdot 2} + \ldots + 1 \right\}$$

Let $n - 1 = N$, then

$$\text{LHS} = (1+N) \left\{ 1 + N + \frac{N(N-1)}{1 \cdot 2} + \ldots + 1 \right\}$$
$$= (1+N) \{ 1 + {}^N C_1 + {}^N C_2 + \ldots + {}^N C_N \}$$
$$= (1+N) 2^N = n \cdot 2^{n-1} = \text{RHS}$$

II. Aliter

$$\text{LHS} = C_1 + 2C_2 + 3C_3 + \ldots + nC_n = \sum_{r=1}^{n} r \cdot {}^n C_r$$
$$= \sum_{r=1}^{n} r \cdot \frac{n}{r} \cdot {}^{n-1} C_{r-1} \qquad \left[\because {}^n C_r = \frac{n}{r} \cdot {}^{n-1} C_{r-1} \right]$$
$$= n \sum_{r=1}^{n} {}^{n-1} C_{r-1}$$
$$= n ({}^{n-1} C_0 + {}^{n-1} C_1 + {}^{n-1} C_2 + \ldots + {}^{n-1} C_{n-1}]$$
$$= n \cdot 2^{n-1} = \text{RHS}$$

Example 59. *If* $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$, prove that $C_0 + 2C_1 + 3C_2 + \ldots + (n+1) C_n = (n+2) 2^{n-1}$.

Sol. Here, last term of $C_0 + 2C_1 + 3C_2 + \ldots + (n+1) C_n$ is $(n+1) C_n$ i.e., $(n+1)$ and last term with positive sign.

and $$n + 1 = n \cdot 1 + 1$$

or $$n) n + 1 (1$$
$$\underline{-n}$$
$$1$$

Here, $q = 1$ and $r = 1$

The given series is

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$$

Now, replacing x by x^1 and multiplying both sides by x, we get

$$x(1+x)^n = C_0 x + C_1 x^2 + C_2 x^3 + \ldots + C_n x^{n+1}$$

Differentiating both sides w.r.t. x, we get

$$x \cdot n (1+x)^{n-1} + (1+x)^n \cdot 1 = C_0 + 2C_1 x + 3C_2 x^2 + \ldots + (n+1) C_n x^n$$

Putting $x = 1$, we get

$$n (2)^{n-1} + 2^n = C_0 + 2C_1 + 3C_2 + \ldots + (n+1) C_n$$

or $$C_0 + 2C_1 + 3C_2 + \ldots + (n+1) C_n = (n+2) 2^{n-1}$$

I. Aliter

$$\text{LHS} = C_0 + 2C_1 + 3C_2 + \ldots + (n+1) C_n$$
$$= C_0 + (1+1) C_1 + (1+2) C_2 + \ldots + (1+n) C_n$$
$$= (C_0 + C_1 + C_2 + \ldots + C_n) + (C_1 + 2C_2 + \ldots + n C_n)$$

[use example 58]

$$= 2^n + n \cdot 2^{n-1} = (n+2) 2^{n-1} = \text{RHS}$$

II. Aliter

$$\text{LHS} = C_0 + 2C_1 + 3C_2 + \ldots + (n+1) C_n$$
$$= \sum_{r=1}^{n+1} r \cdot {}^n C_{r-1} = \sum_{r=1}^{n+1} (r - 1 + 1) \cdot {}^n C_{r-1}$$
$$= \sum_{r=1}^{n+1} (r-1) \cdot {}^n C_{r-1} + {}^n C_{r-1}$$
$$= \sum_{r=1}^{n+1} n \cdot {}^{n-1} C_{r-2} + \sum_{r=1}^{n+1} {}^n C_{r-1}$$

$$\left[\because {}^n C_{r-1} = \frac{n}{r-1} \cdot {}^{n-1} C_{r-2} \right]$$

$$= n (0 + {}^{n-1} C_0 + {}^{n-1} C_1 + {}^{n-1} C_2 + \ldots + {}^{n-1} C_{n-1}) + ({}^n C_0 + {}^n C_1 + {}^n C_2 + \ldots + {}^n C_n)$$
$$= n \cdot 2^{n-1} + 2^n = (n+2) \cdot 2^{n-1} = \text{RHS}$$

Example 60. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$, prove that $C_0 + 3C_1 + 5C_2 + \ldots + (2n+1)C_n = (n+1)2^n$.

Sol. Here, last term of $C_0 + 3C_1 + 5C_2 + \ldots + (2n+1)C_n$ is $(2n+1)C_n$ i.e., $(2n+1)$ and last term with positive sign.

Then, $2n + 1 = n \cdot 2 + 1$

or $n)\,2n + 1(2$

$\underline{-\,2n}$

1

Here, $q = 2$ and $r = 1$

The given series is

$$(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$$

Now, replacing x by x^2, we get

$$(1+x^2)^n = C_0 + C_1x^2 + C_2x^4 + \ldots + C_nx^{2n}$$

On multiplying both sides by x^1, we get

$$x(1+x^2)^n = C_0x + C_1x^3 + C_2x^5 + \ldots + C_nx^{2n+1}$$

On differentiating both sides w.r.t. x, we get

$$x \cdot n(1+x^2)^{n-1} \cdot 2x + (1+x^2)^n \cdot 1 = C_0 + 3C_1x^2 + 5C_2x^4 + \ldots + (2n+1)C_nx^{2n}$$

Putting $x = 1$, we get

$$n \cdot 2^{n-1} \cdot 2 + 2^n = C_0 + 3C_1 + 5C_2 + \ldots + (2n+1)C_n$$

or $C_0 + 3C_1 + 5C_2 + \ldots + (2n+1)C_n = (n+1)2^n$

I. Aliter

$$\begin{aligned}\text{LHS} &= C_0 + 3C_1 + 5C_2 + \ldots + (2n+1)C_n \\ &= C_0 + (1+2)C_1 + (1+4)C_2 + \ldots + (1+2n)C_n \\ &= (C_0 + C_1 + C_2 + \ldots + C_n) + 2(C_1 + 2C_2 + \ldots + nC_n) \\ &= 2^n + 2 \cdot n \cdot 2^{n-1} = 2^n + n \cdot 2^n \qquad \text{[from Illustration 58]} \\ &= (n+1)2^n = \text{RHS}\end{aligned}$$

II. Aliter

$$\begin{aligned}\text{LHS} &= C_0 + 3C_1 + 5C_2 + \ldots + (2n+1)C_n \\ &= \sum_{r=0}^{n} (2r+1)\,{}^nC_r = \sum_{r=0}^{n} 2r \cdot {}^nC_r + \sum_{r=0}^{n} {}^nC_r \\ &= 2\sum_{r=0}^{n} r \cdot {}^nC_r + \sum_{r=0}^{n} {}^nC_r \\ &= 2\sum_{r=0}^{n} r \cdot \frac{n}{r} \cdot {}^{n-1}C_{r-1} + \sum_{r=0}^{n} {}^nC_r \quad \left[\because {}^nC_r = \frac{n}{r} \cdot {}^{n-1}C_{r-1}\right] \\ &= 2n\sum_{r=0}^{n} {}^{n-1}C_{r-1} + \sum_{r=0}^{n} {}^nC_r \\ &= 2n\,(0 + {}^{n-1}C_0 + {}^{n-1}C_1 + {}^{n-1}C_2 + \ldots + {}^{n-1}C_{n-1}) \\ &\qquad + ({}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n) \\ &= 2n \cdot 2^{n-1} + 2^n = (n+1) \cdot 2^n = \text{RHS}\end{aligned}$$

Example 61. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$, prove that $1^2 \cdot C_1 + 2^2 \cdot C_2 + 3^2 \cdot C_3 + \ldots + n^2 \cdot C_n = n(n+1) \cdot 2^{n-2}$.

Sol. Here, last term of $1^2 \cdot C_1 + 2^2 \cdot C_2 + 3^2 \cdot C_3 + \ldots + n^2 \cdot C_n$ is $n^2 \cdot C_n$ i.e., n^2. Linear factors of n^2 are n and n; [start always with greater factor] and last term with positive sign.

and $n = n \cdot 1 + 0$ or $n)\,n(1$

$\underline{-\,n}$

0

Here, $q = 1$ and $r = 0$

Then, the given series is

$$(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_nx^n$$

On differentiating both sides w.r.t. x, we get

$$nx(1+x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + \ldots + nC_nx^{n-1} \qquad \ldots(i)$$

and in last term, numerical is nC_n i.e., n and power of $(1+x)$ is $n-1$.

Then, $n = (n-1) \cdot 1 + 1$ or $n-1)\,n\,(1$

$n-1$

$\underline{-\;+}$

1

Here, $q = 1$ and $r = 1$

Now, multiplying both sides by x in Eq. (i), then

$$nx(1+x)^{n-1} = C_1x + 2C_2x^2 + 3C_3x^3 + \ldots + nC_nx^n$$

Differentiating on both sides w.r.t. x, we get

$$n\{x \cdot (n-1)(1+x)^{n-2} + (1+x)^{n-1} \cdot 1\} = C_1 \cdot 1 + 2^2C_2x + 3^2C_3x^2 + \ldots + n^2C_nx^{n-1}$$

Putting $x = 1$, we get

$$n\{1 \cdot (n-1) \cdot 2^{n-2} + 2^{n-1}\} = 1^2 \cdot C_1 + 2^2 \cdot C_2 + 3^2 \cdot C_3 + \ldots + n^2 \cdot C_n$$

or $1^2 \cdot C_1 + 2^2 \cdot C_2 + 3^2 \cdot C_3 + \ldots + n^2 \cdot C_n = n(n+1)2^{n-2}$

Aliter

$$\begin{aligned}\text{LHS} &= 1^2 \cdot C_1 + 2^2 \cdot C_2 + 3^2 \cdot C_3 + \ldots + n^2 \cdot C_n \\ &= \sum_{r=1}^{n} r^2 \cdot {}^nC_r = \sum_{r=1}^{n} r^2 \cdot \frac{n}{r} \cdot {}^{n-1}C_{r-1} \\ &\qquad \left[\because {}^nC_r = \frac{n}{r} \cdot {}^{n-1}C_{r-1}\right] \\ &= n\sum_{r=1}^{n} r \cdot {}^{n-1}C_{r-1} = n\sum_{r=1}^{n} \{(r-1)+1\} \cdot {}^{n-1}C_{r-1} \\ &= n\sum_{r=1}^{n} (r-1) \cdot {}^{n-1}C_{r-1} + n\sum_{r=1}^{n} {}^{n-1}C_{r-1} \\ &= n\sum_{r=1}^{n} (n-1) \cdot {}^{n-2}C_{r-2} + n\sum_{r=1}^{n} {}^{n-1}C_{r-1}\end{aligned}$$

$$= n(n-1)\sum_{r=1}^{n} {}^{n-2}C_{r-2} + n\sum_{r=1}^{n} {}^{n-1}C_{r-1}$$

$$= n(n-1)(0 + {}^{n-2}C_0 + {}^{n-2}C_1 + {}^{n-2}C_2 + \ldots + {}^{n-2}C_{n-2}) + n({}^{n-1}C_0 + {}^{n-1}C_1 + {}^{n-1}C_2 + \ldots + {}^{n-1}C_{n-1})$$

$$= n(n-1)\cdot 2^{n-2} + n\cdot 2^{n-1} = n(n+1)2^{n-2} = \text{RHS}$$

Example 62. *If* $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$, *prove that* $(1\cdot 2)C_2 + (2\cdot 3)C_3 + \ldots + \{(n-1)\cdot n\}C_n = n(n-1)2^{n-2}$.

Sol. Here, last term of $(1\cdot 2)C_2 + (2\cdot 3)C_3 + \ldots + \{(n-1)\cdot n\}C_n$ is $(n-1)nC_n$ i.e. $(n-1)\,n$

[start with greater factor here greater factor is n] and last term with positive sign, then $n = n\cdot 1 + 0$

or $n)\,n\,(1$

$-\,n$

0

Here, $q = 1$ and $r = 0$

The given series is

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n$$

Differentiating on both sides w.r.t. x, we get

$$n(1+x)^{n-1} = 0 + C_1 + 2C_2 x + 3C_3 x^2 + \ldots + nC_n x^{n-1}$$

Again, differentiating on both sides w.r.t. x, we get

$$n(n-1)(1+x)^{n-2} = 0 + 0 + (1\cdot 2)C_2 + (2\cdot 3)C_3 x + \ldots + \{(n-1)\cdot n\}C_n x^{n-2}$$

Putting $x = 1$, we get

$$n(n-1)(1+1)^{n-2} = (1\cdot 2)C_2 + (2\cdot 3)C_3 + \ldots + \{(n-1)n\}\cdot C_n$$

or $(1\cdot 2)C_2 + (2\cdot 3)C_3 + \ldots + \{(n-1)n\}\cdot C_n = n(n-1)2^{n-2}$

I. Aliter

$$\text{LHS} = (1\cdot 2)C_2 + (2\cdot 3)C_3 + (3\cdot 4)C_4 + \ldots + \{(n-1)n\}\cdot C_n$$

$$= (1\cdot 2)\frac{n(n-1)}{1\cdot 2} + (2\cdot 3)\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} + (3\cdot 4)\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4} + \ldots + (n-1)n\cdot 1$$

$$= n(n-1)\left\{1 + \frac{(n-2)}{1} + \frac{(n-2)(n-3)}{1\cdot 2} + \ldots + 1\right\}$$

Now, in bracket, let $n-2 = N$, then

$$= n(n-1)\left\{1 + \frac{N}{1} + \frac{N(N-1)}{2!} + \ldots + 1\right\}$$

$$= n(n-1)\{{}^{N}C_0 + {}^{N}C_1 + \ldots + {}^{N}C_N\}$$

$$= n(n-1)2^N = n(n-1)2^{n-2} = \text{RHS}$$

II. Aliter

$$\text{LHS} = (1\cdot 2)C_2 + (2\cdot 3)C_3 + \ldots + \{(n-1)\cdot n\}C_n$$

$$= \sum_{r=2}^{n}(r-1)\cdot r\cdot {}^{n}C_r$$

$$= \sum_{r=2}^{n}(r-1)\cdot r\cdot \frac{n}{r}\cdot\frac{n-1}{(r-1)}\cdot {}^{n-2}C_{r-2}$$

$$= (n-1)\,n\sum_{r=2}^{n} {}^{n-2}C_{r-2}$$

$$= (n-1)\,n({}^{n-2}C_0 + {}^{n-2}C_1 + {}^{n-2}C_2 + \ldots + {}^{n-2}C_{n-2})$$

$$= (n-1)\,n\cdot 2^{n-2} = \text{RHS}$$

Example 63. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n$, prove that $C_0 - 2C_1 + 3C_2 - 4C_3 + \ldots + (-1)^n (n+1)C_n = 0$.

Sol. Numerical value of last term of $C_0 - 2C_1 + 3C_2 - 4C_3 + \ldots + (-1)^n (n+1)C_n$ is $(n+1)C_n$ i.e., $(n+1)$, then

$n+1 = n\cdot 1 + 1$ or $n)\,n+1\,(1$

$-\,n$

1

Here, $q = 1$ and $r = 1$

The given series is

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n$$

On multiplying both sides by x, we get

$$x(1+x)^n = C_0 x + C_1 x^2 + C_2 x^3 + C_3 x^4 + \ldots + C_n x^{n+1}$$

On differentiating both sides w.r.t. x, we get

$$x\cdot n(1+x)^{n-1} + (1+x)^n\cdot 1 = C_0 + 2C_1 x + 3C_2 x^2 + 4C_3 x^3 + \ldots + (n+1)C_n x^n$$

Putting $x = -1$, we get

$$0 = C_0 - 2C_1 + 3C_2 - 4C_3 + \ldots + (-1)^n (n+1)C_n$$

or $C_0 - 2C_1 + 3C_2 - 4C_3 + \ldots + (-1)^n (n+1)C_n = 0$

I. Aliter

$$\text{LHS} = C_0 - 2C_1 + 3C_2 - 4C_3 + \ldots + (-1)^n (n+1)C_n$$

$$= C_0 - (C_1 + C_1) + (C_2 + 2C_2) - (C_3 + 3C_3) + \ldots + (-1)^n \{C_n + nC_n\}$$

$$= \{C_0 - C_1 + C_2 - C_3 + \ldots + (-1)^n C_n\} + \{-C_1 + 2C_2 - 3C_3 + \ldots + (-1)^n nC_n\}$$

$$= (1-1)^n + \left\{\begin{matrix} -n + 2\cdot\dfrac{n(n-1)}{1\cdot 2} - 3\dfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3} \\ + \ldots + (-1)^n\cdot n \end{matrix}\right\}$$

$$= 0 + n\left\{-1 + (n-1) - \frac{(n-1)(n-2)}{1\cdot 2} + \ldots + (-1)^n\right\}$$

$$= 0 - n\left\{1 - (n-1) + \frac{(n-1)(n-2)}{1\cdot 2} - \ldots + (-1)^{n-1}\right\}$$

Let in bracket, put $n-1 = N$, we get

$$\text{LHS} = 0 - n\left\{1 - N + \frac{N(N-1)}{1\cdot 2} - \ldots + (-1)^N\right\}$$

$$= 0 - n\{{}^NC_0 - {}^NC_1 + {}^NC_2 - \ldots + (-1)^N\ {}^NC_N\}$$

$$= 0 - n(1-1)^N = 0 - 0 = 0 = \text{RHS}$$

II. Aliter

$$\text{LHS} = C_0 - 2C_1 + 3C_2 - 4C_3 + \ldots + (-1)^n (n+1) C_n$$

$$= \sum_{r=0}^{n} (-1)^r (r+1)\, {}^nC_r = \sum_{r=0}^{n} (-1)^r [r\cdot {}^nC_r + {}^nC_r]$$

$$= \sum_{r=0}^{n} (-1)^r [n\cdot {}^{n-1}C_{r-1} + {}^nC_r] \quad \left[\because {}^nC_r = \frac{n}{r}\cdot {}^{n-1}C_{r-1}\right]$$

$$= n\sum_{r=0}^{n} (-1)^r \cdot {}^{n-1}C_{r-1} + \sum_{r=0}^{n} (-1)^r \cdot {}^nC_r$$

$$= -n\sum_{r=0}^{n} (-1)^{r-1} \cdot {}^{n-1}C_{r-1} + \sum_{r=0}^{n} (-1)^r \cdot {}^nC_r$$

$$= -n(1-1)^{n-1} + (1-1)^n = 0 + 0 = 0 = \text{RHS}$$

Example 64. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n$, prove that $C_1 - 2C_2 + 3C_3 - \ldots + (-1)^{n-1} n C_n = 0$.

Sol. Numerical value of last term of

$C_1 - 2C_2 + 3C_3 - \ldots + (-1)^{n-1} n C_n$ is nC_n i.e., n, then

and $n = n\cdot 1 + 0$ or $n)\,n\,(1$

$-\ n$

1

Here, $q = 1$ and $r = 0$

The given series is

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n$$

On differentiating both sides w.r.t. x, we get

$$n(1+x)^{n-1} = 0 + C_1 + 2C_2 x + 3C_3 x^2 + \ldots + n C_n x^{n-1}$$

Putting $x = -1$, we get

$$0 = C_1 - 2C_2 + 3C_3 - \ldots + (-1)^{n-1} n C_n$$

or $C_1 - 2C_2 + 3C_3 - \ldots + (-1)^{n-1} n C_n = 0$

I. Aliter

$$\text{LHS} = C_1 - 2C_2 + 3C_3 - \ldots + (-1)^{n-1} n\cdot C_n$$

$$= n - 2\cdot\frac{n(n-1)}{1\cdot 2} + 3\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} - \ldots + (-1)^{n-1}\cdot n\cdot 1$$

$$= n\left\{1 - \frac{(n-1)}{1} + \frac{(n-1)(n-2)}{1\cdot 2} - \ldots + (-1)^{n-1}\right\}$$

In bracket, put $n-1 = N$, then

$$\text{LHS} = n\left\{1 - \frac{N}{1} + \frac{N(N-1)}{1\cdot 2} - \ldots + (-1)^N\right\}$$

$$= n\{{}^NC_0 - {}^NC_1 + {}^NC_2 - \ldots + (-1)^N\ {}^NC_N\}$$

$$= n(1-1)^N = 0 = \text{RHS}$$

II. Aliter

$$\text{LHS} = C_1 - 2C_2 + 3C_3 - \ldots + (-1)^{n-1}\cdot n\, C_n$$

$$= \sum_{r=1}^{n} (-1)^{r-1}\cdot r\cdot {}^nC_r$$

$$= \sum_{r=1}^{n} (-1)^{r-1}\cdot n\cdot {}^{n-1}C_{r-1} \quad \left[\because {}^nC_r = \frac{n}{r}\cdot {}^{n-1}C_{r-1}\right]$$

$$= n\sum_{r=1}^{n} (-1)^{r-1}\cdot {}^{n-1}C_{r-1}$$

$$= n(1-1)^{n-1} = 0 = \text{RHS}$$

Example 65. *If* $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n$, *prove that* $C_0 - 3C_1 + 5C_2 - \ldots + (-1)^n (2n+1) C_n = 0$.

Sol. The numerical value of last term of

$C_0 - 3C_1 + 5C_2 - \ldots + (-1)^n (2n+1) C_n$ is $(2n+1) C_n$

i.e. $(2n+1)$

and $2n + 1 = 2\cdot n + 1$ or $n)\,2n+1\,(2$

$\underline{-\ 2n}$

1

Here, $q = 2$ and $r = 1$

The given series is

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n \text{ now,}$$

replacing x by x^2, then we get

$$(1+x^2)^n = C_0 + C_1 x^2 + C_2 x^4 + \ldots + C_n x^{2n}$$

On multiplying both sides by x, we get

$$x(1+x^2)^n = C_0 x + C_1 x^3 + C_2 x^5 + \ldots + C_n x^{2n+1}$$

On differentiating both sides w.r.t. x, we get

$$x\cdot n(1+x^2)^{n-1} 2x + (1+x^2)^n \cdot 1 = C_0 + 3C_1 x^2 + 5C_2 x^4 + \ldots + (2n+1) C_n x^{2n}$$

Putting $x = i$ in both sides, we get

$$0 + 0 = C_0 - 3C_1 + 5C_2 - \ldots + (2n+1)(-1)^n C_n$$

or $C_0 - 3C_1 + 5C_2 - \ldots + (-1)^n (2n+1) C_n = 0$

I. Aliter

$$\text{LHS} = C_0 - 3C_1 + 5C_2 - \ldots + (-1)^n (2n+1) C_n$$

$$= C_0 - (1+2) C_1 + (1+4) C_2 - \ldots + (-1)^n (1+2n) C_n$$

$$= (C_0 - C_1 + C_2 - \ldots + (-1)^n C_n) - 2(C_1 - 2C_2 + \ldots + (-1)^{n-1} n\cdot C_n)$$

$= (1-1)^n - 2 \cdot 0$ [from Example 64]

$= 0 = \text{RHS}$

II. Aliter

$$\text{LHS} = C_0 - 3C_1 + 5C_2 - \ldots + (-1)^n (2n+1) C_n$$

$$= \sum_{r=1}^{n} (-1)^r (2r+1)\, {}^nC_r = \sum_{r=1}^{n} (-1)^r [2r \cdot {}^nC_r + {}^nC_r]$$

$$= 2\sum_{r=1}^{n} n \cdot {}^{n-1}C_{r-1} + \sum_{r=1}^{n} (-1)^r \cdot {}^nC_r$$

$$= 2n(1-1)^{n-1} + (1-1)^n = 0 + 0 = 0 = \text{RHS}$$

Use of Integration

This method is applied only when the numericals occur as the denominator of the binomial coefficient.

Solution Process

If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n$, then integrate both sides between the suitable limits which gives the required series.

1. If the sum contains $C_0, C_1, C_2, \ldots, C_n$ are all positive signs, then integrate between limits 0 to 1.
2. If the sum contains alternate signs (i.e., *+, −),* then integrate between limits −1 to 0.
3. If the sum contains odd coefficients (i.e., $C_0, C_2, C_4, \ldots$), then integrate between −1 to +1.
4. If the sum contains even coefficients (i.e., $C_1, C_3, C_5, \ldots$), then subtracting (2) from (1) and then dividing by 2.
5. If in denominator of binomial coefficient product of two numericals, then integrate two times first times taken limits between 0 to x and second times take suitable limits.

Example 66. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$, prove that

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}.$$

Sol. $\because (1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$ ...(i)

Integrating both sides of Eq. (i) within limits 0 to 1, then we get

$$\int_0^1 (1+x)^n\, dx = \int_0^1 (C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n)\, dx$$

$$\Rightarrow \quad = \left[C_0 x + \frac{C_1 x^2}{2} + \frac{C_2 x^3}{3} + \ldots + \frac{C_n x^{n+1}}{n+1} \right]_0^1$$

$$\Rightarrow \frac{2^{n+1}-1}{n+1} = C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots + \frac{C_n}{n+1}$$

or $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$

I. Aliter

$$\text{LHS} = C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots + \frac{C_n}{n+1}$$

$$= 1 + \frac{n}{1\cdot 2} + \frac{n(n-1)}{1\cdot 2\cdot 3} + \ldots + \frac{1}{n+1}$$

$$= \frac{1}{n+1}\left[(n+1) + \frac{(n+1)n}{1\cdot 2} + \frac{(n+1)\,n\,(n-1)}{1\cdot 2\cdot 3} + \ldots + 1 \right]$$

Put $n+1 = N$, then

$$\text{LHS} = \frac{1}{N}\left[N + \frac{N(N-1)}{2!} + \frac{N(N-1)(N-2)}{3!} + \ldots + 1 \right]$$

$$= \frac{1}{N}[{}^NC_1 + {}^NC_2 + {}^NC_3 + \ldots + {}^NC_N]$$

$$= \frac{1}{N}[(1+1)^N - 1] = \frac{2^N - 1}{N} = \frac{2^{n+1}-1}{n+1} = \text{RHS}$$

II. Aliter

$$\text{LHS} = C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots + \frac{C_n}{n+1} = \sum_{r=0}^{n} \frac{C_r}{r+1}$$

$$= \sum_{r=0}^{n} \frac{{}^nC_r}{(r+1)} = \sum_{r=0}^{n} \frac{{}^{n+1}C_{r+1}}{(n+1)} \left[\because \frac{{}^{n+1}C_{r+1}}{n+1} = \frac{{}^nC_r}{r+1} \right]$$

$$= \frac{1}{(n+1)} \sum_{r=0}^{n} {}^{n+1}C_{r+1}$$

$$= \frac{1}{(n+1)} ({}^{n+1}C_1 + {}^{n+1}C_2 + {}^{n+1}C_3 + \ldots + {}^{n+1}C_{n+1})$$

$$= \frac{1}{n+1}(2^{n+1} - 1) = \frac{2^{n+1}-1}{n+1} = \text{RHS}$$

Example 67. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n$, prove that

$$C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \ldots + (-1)^n \frac{C_n}{n+1} = \frac{1}{n+1}.$$

Sol. $\because (1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$ ...(i)

Integrating on both sides of Eq. (i) within limits − 1 to 0, then we get

$$\int_{-1}^{0} (1+x)^n\, dx = \int_{-1}^{0} (C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n)\, dx$$

$$\Rightarrow \left[\frac{(1+x)^n}{n+1}\right]_{-1}^{0} = \left[C_0 x + \frac{C_1 x^2}{2} + \frac{C_2 x^3}{3} + \ldots + \frac{C_n x^{n+1}}{n+1}\right]_{-1}^{0}$$

$$\Rightarrow \quad \frac{1-0}{n+1} = 0 - \left(-C_0 + \frac{C_1}{2} - \frac{C_2}{3} + \ldots + (-1)^{n+1}\frac{C_n}{n+1}\right)$$

$$\Rightarrow \quad \frac{1}{n+1} = C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \ldots + (-1)^{n+2}\frac{C_n}{n+1}$$

$$\Rightarrow \quad \frac{1}{n+1} = C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \ldots + (-1)^n \frac{C_n}{n+1}$$

$$[\because (-1)^{n+2} = (-1)^n (-1)^2 = (-1)^n]$$

Hence, $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \ldots + (-1)^n \frac{C_n}{n+1} = \frac{1}{n+1}$

I. Aliter

$$\text{LHS} = C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \ldots + (-1)^n \frac{C_n}{n+1}$$

$$= 1 - \frac{n}{2} + \frac{n(n-1)}{1\cdot 2\cdot 3} - \ldots + (-1)^n \frac{1}{n+1} = \frac{1}{(n+1)}$$

$$\left[(n+1) - \frac{(n+1)n}{1\cdot 2} + \frac{(n+1)n(n-1)}{1\cdot 2\cdot 3} - \ldots + (-1)^n\right]$$

Put $n+1 = N$, we get

$$= \frac{1}{N}\left[N - \frac{N(N-1)}{1\cdot 2} + \frac{N(N-1)(N-2)}{1\cdot 2\cdot 3} - \ldots + (-1)^{N-1}\right]$$

$$= \frac{1}{N}[{}^N C_1 - {}^N C_2 + {}^N C_3 - \ldots + (-1)^{N-1}]$$

$$= -\frac{1}{N}[-{}^N C_1 + {}^N C_2 - {}^N C_3 + \ldots + (-1)^N {}^N C_N]$$

$$= -\frac{1}{N}[{}^N C_0 - {}^N C_1 + {}^N C_2 - {}^N C_3 + \ldots + (-1)^N {}^N C_N - {}^N C_0]$$

$$= -\frac{1}{N}[(1-1)^N - {}^N C_0] = -\frac{1}{N}[0-1] = \frac{1}{N}$$

$$= \frac{1}{n+1} = \text{RHS}$$

II. Aliter

$$\text{LHS} = C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \ldots + (-1)^n \frac{C_n}{n+1} = \sum_{r=0}^{n} \frac{(-1)^r \cdot C_r}{r+1}$$

$$= \sum_{r=0}^{n} \frac{(-1)^r \cdot {}^n C_r}{r+1}$$

$$= \sum_{r=0}^{n} (-1)^r \cdot \frac{{}^{n+1}C_{r+1}}{(n+1)} \qquad \left[\because \frac{{}^{n+1}C_{r+1}}{n+1} = \frac{{}^n C_r}{r+1}\right]$$

$$= \frac{1}{(n+1)} \sum_{r=0}^{n} (-1)^r \cdot {}^{n+1}C_{r+1}$$

$$= \frac{1}{(n+1)}({}^{n+1}C_1 - {}^{n+1}C_2 + {}^{n+1}C_3 - \ldots + (-1)^n \cdot {}^{n+1}C_{n+1})$$

$$= \frac{1}{(n+1)}\{{}^{n+1}C_0 - ({}^{n+1}C_0 - {}^{n+1}C_1 + {}^{n+1}C_2 - {}^{n+1}C_3 + \ldots + (-1)^{n+1}\ {}^{n+1}C_{n+1})\}$$

$$= \frac{1}{(n+1)}[1 - (1-1)^{n+1}] = \frac{1}{(n+1)}[1-0] = \frac{1}{n+1} = \text{RHS}$$

Example 68. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n$, prove that $\frac{C_0}{1} + \frac{C_2}{3} + \frac{C_4}{5} + \ldots = \frac{2^n}{n+1}$.

Sol. $\because (1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + \ldots + C_n x^n \quad \ldots(\text{i})$

Integrating on both sides of Eq. (i) within limits -1 to 1, then we get

$$\int_{-1}^{1} (1+x)^n\, dx = \int_{-1}^{1}(C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + \ldots + C_n x^n)\, dx$$

$$= \int_{-1}^{1}(C_0 + C_2 x^2 + C_4 x^4 + \ldots)dx + \int_{-1}^{1}(C_1 x + C_3 x^3 + \ldots)dx$$

$$= 2\int_{0}^{1}(C_0 + C_2 x^2 + C_4 x^4 + \ldots)\, dx + 0$$

[by property of definite integral]

[since, second integral contains odd function]

$$\left[\frac{(1+x)^{n+1}}{n+1}\right]_{-1}^{1} = 2\left[\left(C_0 x + \frac{C_2 x^3}{3} + \frac{C_4 x^5}{5} + \ldots\right)\right]_{0}^{1}$$

$$\Rightarrow \qquad \frac{2^{n+1}}{n+1} = 2\left(C_0 + \frac{C_2}{3} + \frac{C_4}{5} + \ldots\right)$$

or $C_0 + \frac{C_2}{3} + \frac{C_4}{5} + \ldots = \frac{2^n}{n+1}$

I. Aliter

$$\text{LHS} = C_0 + \frac{C_2}{3} + \frac{C_4}{5} + \ldots$$

$$= 1 + \frac{n(n-1)}{1\cdot 2\cdot 3} + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \ldots$$

$$= \frac{1}{(n+1)}\left\{\frac{n+1}{1} + \frac{(n+1)n(n-1)}{1\cdot 2\cdot 3} + \frac{(n+1)n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \ldots\right\}$$

$$= \frac{1}{n+1}\{{}^{n+1}C_1 + {}^{n+1}C_3 + {}^{n+1}C_5 + \ldots\}$$

$$= \frac{1}{(n+1)}[\text{sum of even binomial coefficients of } (1+x)^{n+1}]$$

$$= \frac{2^{n+1-1}}{n+1} = \frac{2^n}{n+1} = \text{RHS}$$

II. Aliter LHS $= \frac{C_0}{1} + \frac{C_2}{3} + \frac{C_4}{5} + \ldots$

***Case* I** If n is odd say $n = 2m+1, \forall\ m \in W$, then

$$\text{LHS} = \sum_{r=0}^{m} \frac{{}^{2m+1}C_{2r}}{2r+1} = \sum_{r=0}^{m} \frac{{}^{2m+2}C_{2r+1}}{(2m+1)} \left[\because \frac{{}^{2m+1}C_{2r}}{2r+1} = \frac{{}^{2m+2}C_{2r+1}}{2m+1}\right]$$

$$= \frac{1}{(2m+1)} \cdot 2^{2m+2-1} = \frac{2^n}{n+1} = \text{RHS} \quad [\because n = 2m+1]$$

***Case* II** If n is even say $n = 2m, \forall\ m \in N$, then

$$\text{LHS} = \sum_{r=0}^{m} \frac{{}^{2m}C_{2r}}{2r+1} = \sum_{r=0}^{m} \frac{{}^{2m+1}C_{2r+1}}{(2m+1)} \left[\because \frac{{}^{2m+1}C_{2r+1}}{2m+1} = \frac{{}^{2m}C_{2r}}{2r+1}\right]$$

$$= \frac{2^{2m+1-1}}{2m+1} = \frac{2^n}{n+1} = \text{RHS} \quad [\because n = 2m]$$

Example 69.

If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n$, prove that $\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \ldots = \frac{2^n - 1}{n+1}$.

Sol. We know that, from Examples (66) and (67)

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \frac{C_3}{4} + \frac{C_4}{5} + \frac{C_5}{6} + \ldots = \frac{2^{n+1} - 1}{n+1} \quad \ldots(\text{i})$$

and $$C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + \frac{C_4}{5} - \frac{C_5}{6} + \ldots = \frac{1}{n+1} \quad \ldots(\text{ii})$$

On subtracting Eq. (ii) from Eq. (i), we get

$$2\left(\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \ldots\right) = \frac{2^{n+1} - 2}{n+1}$$

On dividing each sides by 2, we get

$$\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \ldots = \frac{2^n - 1}{n+1}$$

I. Aliter LHS $= \frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \ldots$

$$= \frac{n}{1 \cdot 2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{n(n-1)(n-2)(n-3)(n-4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \ldots$$

$$= \frac{1}{n+1}\left[\frac{(n+1)n}{1 \cdot 2} + \frac{(n+1)n(n-1)(n-2)}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{(n+1)n(n-1)(n-2)(n-3)(n-4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \ldots\right]$$

Put $n+1 = N$, then

$$\text{LHS} = \frac{1}{N}\left[\frac{N(N-1)}{2!} + \frac{N(N-1)(N-2)(N-3)}{4!} + \frac{N(N-1)(N-2)(N-3)(N-4)(N-5)}{6!} + \ldots\right]$$

$$= \frac{1}{N}[{}^{N}C_2 + {}^{N}C_4 + {}^{N}C_6 + \ldots]$$

$$= \frac{1}{N}[({}^{N}C_0 + {}^{N}C_2 + {}^{N}C_4 + {}^{N}C_6 + \ldots) - {}^{N}C_0]$$

$$= \frac{1}{N}[2^{N-1} - 1] = \frac{2^n - 1}{n+1} = \text{RHS}$$

II. Aliter

$$\text{LHS} = \frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \ldots$$

***Case* I** If n is odd say $n = 2m+1, \forall\ m \in W$, then

$$\text{LHS} = \sum_{r=0}^{m} \frac{{}^{2m+1}C_{2r+1}}{2r+2} = \sum_{r=0}^{m} \frac{{}^{2m+2}C_{2r+2}}{(2m+2)} \left[\because \frac{{}^{2m+2}C_{2r+2}}{2m+2} = \frac{{}^{2m+1}C_{2r+1}}{2r+2}\right]$$

$$= \frac{1}{(2m+2)}({}^{2m+2}C_2 + {}^{2m+2}C_4 + \ldots + {}^{2m+2}C_{2m+2})$$

$$= \frac{1}{(2m+2)} \cdot (2^{2m+2-1} - {}^{2m+1}C_0) = \frac{2^n - 1}{n+1} \quad [\because 2m+1 = n]$$

$= \text{RHS}$

***Case* II** If n is even say $n = 2m, \forall\ m \in N$, then

$$\text{LHS} = \sum_{r=0}^{m-1} \frac{{}^{2m}C_{2r+1}}{(2r+2)} = \sum_{r=0}^{m-1} \frac{{}^{2m+1}C_{2r+2}}{(2m+1)} \left[\because \frac{{}^{2m+1}C_{2r+2}}{2m+1} = \frac{{}^{2m}C_{2r+1}}{2r+2}\right]$$

$$= \frac{1}{(2m+1)} \sum_{r=0}^{m-1} {}^{2m+1}C_{2r+2}$$

$$= \frac{1}{(2m+1)}({}^{2m+1}C_2 + {}^{2m+1}C_4 + {}^{2m+1}C_6 + \ldots + {}^{2m+1}C_{2n})$$

$$= \frac{1}{(2m+1)} \cdot (2^{2m+1-1} - {}^{2m+1}C_0)$$

$$= \frac{2^n - 1}{n+1} = \text{RHS} \quad [\because n = 2m]$$

Example 70.

If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$, ***prove that***

$$3C_0 + 3^2\frac{C_1}{2} + \frac{3^3 C_2}{3} + \frac{3^4 C_3}{4} + \ldots + \frac{3^{n+1} C_n}{n+1} = \frac{4^{n+1} - 1}{n+1}.$$

Sol. $\because \quad (1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_nx^n \quad \ldots(i)$

Integrating on both sides of Eq. (i) within limits 0 to 3, we get

$$\int_0^3 (1+x)^n\,dx = \int_0^3 (C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_nx^n)\,dx$$

$$\Rightarrow \left[\frac{(1+x)^{n+1}}{n+1}\right]_0^3 = \left[C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \frac{C_3x^4}{4} + \ldots + \frac{C_n\,x^{n+1}}{n+1}\right]_0^3$$

$$\Rightarrow \frac{4^{n+1}-1}{n+1} = 3C_0 + \frac{3^2C_1}{2} + \frac{3^3C_2}{3} + \frac{3^4C_3}{4} + \ldots + \frac{3^{n+1}C_n}{n+1}$$

Hence,

$$3C_0 + \frac{3^2C_1}{2} + \frac{3^3C_2}{3} + \frac{3^4C_3}{4} + \ldots + \frac{3^{n+1}C_n}{n+1} = \frac{4^{n+1}-1}{n+1}$$

I. Aliter

$$\text{LHS} = 3C_0 + \frac{3^2C_1}{2} + \frac{3^3C_2}{3} + \frac{3^4C_3}{4} + \ldots + \frac{3^{n+1}C_n}{n+1}$$

$$= 3\cdot 1 + \frac{3^2\cdot n}{2} + \frac{3^3\cdot n(n-1)}{1\cdot 2\cdot 3} + \frac{3^4\cdot n(n-1)(n-2)}{1\cdot 2\cdot 3\cdot 4} + \ldots + \frac{3^{n+1}}{n+1}$$

$$= \frac{1}{(n+1)}\left[3\cdot(n+1) + \frac{3^2(n+1)\,n}{1\cdot 2} + \frac{3^3(n+1)\,n(n-1)}{1\cdot 2\cdot 3} + \frac{3^4(n+1)\,n(n-1)(n-2)}{1\cdot 2\cdot 3\cdot 4} + \ldots + 3^{n+1}\right]$$

Put $n+1 = N$, then

$$\text{LHS} = \frac{1}{N}\left[3N + \frac{3^2N(N-1)}{2!} + \frac{3^3N(N-1)(N-2)}{3!} + \frac{3^4N(N-1)(N-2)(N-3)}{4!} + \ldots + 3^N\right]$$

$$= \frac{1}{N}\left[{}^NC_1(3) + {}^NC_2(3)^2 + {}^NC_3(3)^3 + \ldots + {}^NC_N(3)^N\right]$$

$$= \frac{1}{N}\left[{}^NC_0 + {}^NC_1(3) + {}^NC_2(3)^2 + {}^NC_3(3)^3 + \ldots + {}^NC_N(3)^N - {}^NC_0\right]$$

$$= \frac{1}{N}\{(1+3)^N - 1\} = \frac{4^N-1}{N} = \frac{4^{n+1}-1}{n+1} = \text{RHS}$$

II. Aliter

$$\text{LHS} = 3C_0 + 3^2\frac{C_1}{2} + \frac{3^3C_2}{3} + \frac{3^4C_3}{4} + \ldots + \frac{3^{n+1}C_n}{n+1}$$

$$= \sum_{r=0}^{n}\frac{3^{r+1}\cdot {}^nC_r}{(r+1)} = \sum_{r=0}^{n}\frac{3^{r+1}\cdot {}^{n+1}C_{r+1}}{(n+1)} \qquad \left[\because \frac{{}^{n+1}C_{r+1}}{n+1} = \frac{{}^nC_r}{r+1}\right]$$

$$= \frac{1}{(n+1)}\sum_{r=0}^{n}{}^{n+1}C_{r+1}\cdot 3^{r+1}$$

$$= \frac{1}{(n+1)}({}^{n+1}C_1\cdot 3 + {}^{n+1}C_2\cdot 3^2 + {}^{n+1}C_3\cdot 3^3 + \ldots + {}^{n+1}C_{n+1}\cdot 3^{n+1})$$

$$= \frac{1}{(n+1)}[(1+3)^{n+1} - {}^{n+1}C_0]$$

$$= \frac{4^{n+1}-1}{n+1} = \text{RHS}$$

Example 71. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$, show that $\frac{2^2}{1\cdot 2}C_0 + \frac{2^3}{2\cdot 3}C_1 + \frac{2^4}{3\cdot 4}C_2 + \ldots + \frac{2^{n+2}C_n}{(n+1)(n+2)} = \frac{3^{n+2}-2n-5}{(n+1)(n+2)}$.

Sol. Given,

$$(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_nx^n \quad \ldots(i)$$

Integrating both sides of Eq. (i) within limits 0 to x, we get

$$\int_0^x (1+x)^n\,dx = \int_0^x (C_0 + C_1x + C_2x^2 + \ldots + C_nx^n)\,dx$$

$$\Rightarrow \left[\frac{(1+x)^{n+1}}{n+1}\right]_0^x = \left[C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \ldots + \frac{C_n\,x^{n+1}}{n+1}\right]_0^x$$

$$\Rightarrow \frac{(1+x)^{n+1}-1}{(n+1)} = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \ldots + \frac{C_nx^{n+1}}{n+1} \quad \ldots(ii)$$

Again, integrating both sides of Eq. (ii) within limits 0 to 2, we get

$$\int_0^2 \frac{(1+x)^{n+1}-1}{(n+1)}\,dx = \int_0^2 \left(C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \ldots + \frac{C_nx^{n+1}}{n+1}\right)dx$$

$$\Rightarrow \frac{1}{(n+1)}\left(\frac{(1+x)^{n+2}}{n+2} - x\right)\Bigg]_0^2 = \left[\frac{C_0x^2}{1\cdot 2} + \frac{C_1x^3}{2\cdot 3} + \frac{C_2x^4}{3\cdot 4} + \ldots + \frac{C_n\,x^{n+2}}{(n+1)(n+2)}\right]_0^2$$

$$\Rightarrow \frac{1}{(n+1)}\left\{\frac{3^{n+2}}{n+2} - 2 - \frac{1}{n+2}\right\} = \frac{2^2}{1\cdot 2}C_0 + \frac{2^3}{2\cdot 3}C_1 + \frac{2^4}{3\cdot 4}C_2 + \ldots + \frac{2^{n+2}C_n}{(n+1)(n+2)}$$

Hence, $\frac{2^2}{1\cdot 2}C_0+\frac{2^3}{2\cdot 3}C_1+\frac{2^4}{3\cdot 4}C_2+\ldots+\frac{2^{n+2}C_n}{(n+1)(n+2)}$

$$=\frac{3^{n+2}-2n-5}{(n+1)(n+2)}$$

I. Aliter

$$\text{LHS}=\frac{2^2}{1\cdot 2}C_0+\frac{2^3}{2\cdot 3}C_1+\frac{2^4}{3\cdot 4}C_2+\ldots+\frac{2^{n+2}C_n}{(n+1)(n+2)}$$

$$=\frac{2^2}{1\cdot 2}(1)+\frac{2^3}{2\cdot 3}\cdot n+\frac{2^4}{3\cdot 4}\frac{n(n-1)}{1\cdot 2}+\ldots+\frac{2^{n+2}\cdot 1}{(n+1)(n+2)}$$

$$=\frac{1}{(n+1)(n+2)}\left\{\frac{(n+2)(n+1)}{1\cdot 2}2^2+\frac{(n+2)(n+1)n}{1\cdot 2\cdot 3}2^3+\frac{(n+2)(n+1)n(n-1)}{1\cdot 2\cdot 3\cdot 4}2^4+\ldots+2^{n+2}\right\}$$

Put $n+2=N$, then we get

$$=\frac{1}{N(N-1)}\left\{\frac{N(N-1)}{1\cdot 2}2^2+\frac{N(N-1)(N-2)}{1\cdot 2\cdot 3}2^3+\frac{N(N-1)(N-2)(N-3)}{1\cdot 2\cdot 3\cdot 4}2^4+\ldots+2^N\right\}$$

$$=\frac{1}{N(N-1)}\{{}^NC_2(2)^2+{}^NC_3(2)^3+{}^NC_4(2)^4+\ldots+{}^NC_N(2)^N]$$

$$=\frac{1}{N(N-1)}\{{}^NC_0+{}^NC_1(2)+{}^NC_2(2)^2+{}^NC_3(2)^3+{}^NC_4(2)^4+\ldots+{}^NC_N(2)^N-{}^NC_0-{}^NC_1(2)]$$

$$=\frac{1}{N(N-1)}\{(1+2)^N-1-2N\}$$

$$=\frac{3^{n+2}-1-2(n+2)}{(n+2)(n+1)}=\frac{3^{n+2}-2n-5}{(n+1)(n+2)}=\text{RHS}$$

II. Aliter

$$\text{LHS}=\frac{2^2}{1\cdot 2}\cdot C_0+\frac{2^3}{2\cdot 3}\cdot C_1+\frac{2^4}{3\cdot 4}\cdot C_2+\ldots+\frac{2^{n+2}\cdot C_n}{(n+1)(n+2)}$$

$$=\sum_{r=1}^{n+1}\frac{2^{r+1}}{r(r+1)}\cdot{}^nC_{r-1}$$

$$=\sum_{r=1}^{n+1}\frac{2^{r+1}\cdot{}^{n+2}C_{r+1}}{(n+1)(n+2)}\quad\left[\because\frac{{}^{n+2}C_{r+1}}{(n+1)(n+2)}=\frac{{}^nC_{r-1}}{r(r+1)}\right]$$

$$=\frac{1}{(n+1)(n+2)}\sum_{r=1}^{n+1}{}^{n+2}C_{r+1}\cdot 2^{r+1}$$

$$=\frac{1}{(n+1)(n+2)}[{}^{n+2}C_2\cdot 2^2+{}^{n+2}C_3\cdot 2^3+\ldots+{}^{n+2}C_{n+2}\cdot 2^{n+2}]$$

$$=\frac{1}{(n+1)(n+2)}[(1+2)^{n+2}-{}^{n+2}C_0-{}^{n+2}C_1\cdot 2^1]$$

$$=\frac{(3^{n+2}-2n-5)}{(n+1)(n+2)}=\text{RHS}$$

When Each Term in Summation Contains the Product of Two Binomial Coefficients or Square of Binomial Coefficients

Solution Process

1. **If difference of the lower suffixes of binomial coefficients in each term is same.**

i.e. ${}^nC_0\,{}^nC_2+{}^nC_1\cdot{}^nC_3+{}^nC_2\cdot{}^nC_4+\ldots$

Here, $2-0=3-1=4-2=\ldots=2$

***Case* I If each term of series is positive, then**

$$(1+x)^n=C_0+C_1x+C_2x^2+\ldots+C_nx^n \quad\ldots(i)$$

Interchanging 1 and x, we get

$$(x+1)^n=C_0x^n+C_1x^{n-1}+C_2x^{n-2}+\ldots+C_n \quad\ldots(ii)$$

Then, multiplying Eqs. (i) and (ii) and equate the coefficients of suitable power of x on both sides.

Replacing x by $\frac{1}{x}$ in Eq. (i), then we get

$$\left(1+\frac{1}{x}\right)^n=C_0+\frac{C_1}{x}+\frac{C_2}{x^2}+\ldots+\frac{C_n}{x^n} \quad\ldots(iii)$$

Then, multiplying Eqs. (i) and (iii) and equate the coefficients of suitable power of x on both sides.

Example 72. ***If*** $(1+x)^n=C_0+C_1x+C_2x^2+C_3x^3+\ldots+C_nx^n$**, *prove that***

$$C_0C_r+C_1C_{r+1}+C_2C_{r+2}+\ldots+C_{n-r}C_n=\frac{2n!}{(n-r)!(n+r)!}.$$

Sol. Here, differences of lower suffixes of binomial coefficients in each term is r.

i.e., $r-0=r+1-1=r+2-2=\ldots=n-(n-r)=r$

Given,

$$(1+x)^n=C_0+C_1x+C_2x^2+\ldots+C_{n-r}x^{n-r}+\ldots+C_nx^n \quad\ldots(i)$$

Now,

$$(x+1)^n=C_0x^n+C_1x^{n-1}+C_2x^{n-2}+\ldots+C_rx^{n-r}+C_{r+1}x^{n-r-1}+C_{r+2}x^{n-r-2}+\ldots+C_n \quad\ldots(ii)$$

On multiplying Eqs. (i) and (ii), we get

$$(1+x)^{2n}=(C_0+C_1x+C_2x^2+\ldots+C_{n-r}x^{n-r}+\ldots+C_nx^n)\times(C_0x^n+C_1x^{n-1}+C_2x^{n-2}+\ldots+C_rx^{n-r}+C_{r+1}x^{n-r-1}+C_{r+2}x^{n-r-2}+\ldots+C_n) \quad\ldots(iii)$$

Now, coefficient of x^{n-r} on LHS of Eq. (iii) $= {}^{2n}C_{n-r}$

$$= \frac{2n!}{(n-r)!\,(n+r)!}$$

and coefficient of x^{n-r} on RHS of Eq. (iii)

$$= C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \ldots + C_{n-r}C_n$$

But Eq. (iii) is an identity, therefore coefficient of x^{n-r} in

RHS = coefficient of x^{n-r} in LHS.

$$\Rightarrow C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \ldots + C_{n-r}C_n = \frac{2n!}{(n-r)!\,(n+r)!}$$

Aliter

Given,

$$(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_r\,x^r + C_{r+1}x^{r+1} + C_{r+2}x^{r+2} + \ldots + C_{n-r}x^r + \ldots + C_nx^n \quad \ldots(i)$$

Now, $\left(1+\frac{1}{x}\right)^n = C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \ldots + \frac{C_r}{x^r} + \frac{C_{r+1}}{x^{r+1}} + \frac{C_{r+2}}{x^{r+2}} + \ldots + \frac{C_{n-r}}{x^{n-r}} + \ldots + \frac{C_n}{x^n}$ ...(ii)

On multiplying Eqs. (i) and (ii), we get

$$\frac{(1+x)^{2n}}{x^n} = (C_0 + C_1x + C_2x^2 + \ldots + C_rx^r + C_{r+1}x^{r+1} + C_{r+2}x^{r+2} + \ldots + C_{n-r}x^{n-r} + \ldots + C_nx^n) \times \left(C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \ldots + \frac{C_r}{x^r} + \frac{C_{r+1}}{x^{r+1}} + \frac{C_{r+2}}{x^{r+2}} + \ldots + \frac{C_{n-r}}{x^{n-r}} + \ldots + \frac{C_n}{x^n}\right) \quad \ldots(iii)$$

Now, coefficient of $\frac{1}{x^r}$ in RHS

$$= (C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \ldots + C_{n-r}C_n)$$

$\therefore$ Coefficient of $\frac{1}{x^r}$ in LHS = Coefficient of x^{n-r} in

$$(1+x)^{2n} = {}^{2n}C_{n-r} = \frac{2n!}{(n-r)!\,(n+r)!}$$

But Eq. (iii) is an identity, therefore coefficient of $\frac{1}{x^r}$ in

RHS = coefficient of $\frac{1}{x^r}$ in LHS.

$$\Rightarrow \quad C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \ldots + C_{n-r}C_n = \frac{2n!}{(n-r)!\,(n+r)!}$$

***Corollary* I** For $r = 0$,

$$C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2 = \frac{2n!}{(n!)^2}$$

***Corollary* II** For $r = 1$,

$$C_0C_1 + C_1C_2 + C_2C_3 + \ldots + C_{n-1}C_n = \frac{2n!}{(n-1)!\,(n+1)!}$$

***Corollary* III** For $r = 2$,

$$C_0C_2 + C_1C_3 + C_2C_4 + \ldots + C_{n-2}C_n = \frac{2n!}{(n-2)!\,(n+2)!}$$

Example 73. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$, prove that $C_0^2 + C_1^2 + C_2^2 = \frac{2n!}{n!\,n!} = \frac{1\cdot3\cdot5\ldots(2n-1)}{n!}\cdot2^n$.

Sol. Given, $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$ ...(i)

Now, $(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots + C_n$ ...(ii)

On multiplying Eqs. (i) and (ii), we get

$$(1+x)^{2n} = (C_0 + C_1x + C_2x^2 + \ldots + C_nx^n) \times (C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots + C_n) \quad \ldots(iii)$$

Now, coefficient of x^n in RHS

$$= C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2$$

And coefficient of x^n in LHS $= {}^{2n}C_n = \frac{2n!}{n!\,n!}$

$$= \frac{1\cdot2\cdot3\cdot4\cdot5\ldots(2n-1)\,2n}{n!\,n!} = \frac{1\cdot3\cdot5\ldots(2n-1)\,2^n\,n!}{n!\,n!}$$

But Eq. (iii) is an identity, therefore coefficient of x^n in RHS = coefficient of x^n in LHS.

$$\Rightarrow C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2 = \frac{2n!}{n!\,n!} = \frac{1\cdot3\cdot5\cdot\ldots\cdot(2n-1)}{n!}\cdot2^n$$

Aliter

Given, $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$ ...(i)

Now, $\left(1+\frac{1}{x}\right)^n = C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \ldots + \frac{C_n}{x^n}$ ...(ii)

On multiplying Eqs. (i) and (ii), we get

$$\frac{(1+x)^{2n}}{x^n} = (C_0 + C_1x + C_2x^2 + \ldots + C_nx^n) \times \left(C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \ldots + \frac{C_n}{x^n}\right) \quad \ldots(iii)$$

Now, constant term in RHS $= C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2$

Constant term in LHS = Constant term in $\frac{(1+x)^{2n}}{x^n}$

$$= \text{Coefficient of } x^n \text{ in } (1+x)^{2n} = {}^{2n}C_n = \frac{2n!}{n!\,n!}$$

$$= \frac{n!\,2^n\,[1\cdot3\cdot5\ldots(2n-1)]}{n!\,n!} = \frac{2^n\,[1\cdot3\cdot5\ldots(2n-1)]}{n!}$$

But Eq. (iii) is an identity, therefore the constant term in RHS = constant term in LHS.

$$\Rightarrow {C_0}^2 + {C_1}^2 + {C_2}^2 + \ldots + {C_n}^2 = \frac{2n!}{n!\,n!} = \frac{\{1\cdot 3\cdot 5\ldots(2n-1)\}}{n!}2^n$$

***Case* II If terms of the series alternately positive and negative, then**

$$(1-x)^n = C_0 - C_1x + C_2x^2 - \ldots + (-1)^n C_n x^n \quad \ldots(i)$$

$$\text{and } (x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots + C_n \quad \ldots(ii)$$

Then, multiplying Eqs. (i) and (ii) and equate the coefficient of suitable power of x on both sides.

Or

Replacing x by $\frac{1}{x}$ in Eq. (i), we get

$$\left(1-\frac{1}{x}\right)^n = C_0 - \frac{C_1}{x} + \frac{C_2}{x^2} - \ldots + (-1)^n\frac{C_n}{x^n} \quad \ldots(iii)$$

Then, multiplying Eqs. (i) and (iii) and equate the coefficient of suitable power of x on both sides.

Example 74. *Prove that*

$$({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - \ldots + ({}^{2n}C_{2n})^2 = (-1)^n \cdot {}^{2n}C_n.$$

Sol. Since, $(1-x)^{2n} = {}^{2n}C_0 - {}^{2n}C_1x + {}^{2n}C_2x^2 - \ldots + (-1)^{2n}\cdot {}^{2n}C_{2n}x^{2n}$

$$\text{or } (1-x)^{2n} = {}^{2n}C_0 - {}^{2n}C_1x + {}^{2n}C_2x^2 - \ldots + {}^{2n}C_{2n}x^{2n} \quad \ldots(i)$$

$$\text{and } (x+1)^{2n} = {}^{2n}C_0x^{2n} + {}^{2n}C_1x^{2n-1} + {}^{2n}C_2x^{2n-2} + \ldots + {}^{2n}C_{2n} \quad \ldots(ii)$$

On multiplying Eqs. (i) and (ii), we get

$$(x^2-1)^{2n} = ({}^{2n}C_0 - {}^{2n}C_1x + {}^{2n}C_2x^2 - \ldots + {}^{2n}C_{2n}x^{2n}) \times ({}^{2n}C_0x^{2n} + {}^{2n}C_1x^{2n-1} + {}^{2n}C_2x^{2n-2} + \ldots + {}^{2n}C_{2n}) \quad \ldots(iii)$$

Now, coefficient of x^{2n} in RHS

$$= ({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - \ldots + ({}^{2n}C_{2n})^2$$

Now, LHS can also be written as $(1-x^2)^{2n}$.

$\therefore$ General term in LHS, $T_{r+1} = {}^{2n}C_r(-x^2)^r$

Putting $r = n$, we get $T_{n+1} = (-1)^n \cdot {}^{2n}C_n x^{2n}$

$\Rightarrow$ Coefficient of x^{2n} in LHS $= (-1)^n \cdot {}^{2n}C_n$

But Eq. (iii) is an identity, therefore coefficient of x^{2n} in RHS = coefficient of x^{2n} in LHS

$$\Rightarrow ({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - \ldots + ({}^{2n}C_{2n})^2 = (-1)^n \cdot {}^{2n}C_n$$

Aliter

Since, $(1+x)^{2n} = {}^{2n}C_0 + {}^{2n}C_1x + {}^{2n}C_2x^2 + \ldots + {}^{2n}C_{2n}x^{2n}$ ...(i)

and $\left(1-\frac{1}{x}\right)^{2n} = {}^{2n}C_0 - \frac{{}^{2n}C_1}{x} + \frac{{}^{2n}C_2}{x^2} - \ldots + \frac{{}^{2n}C_{2n}}{x^{2n}}$ ...(ii)

On multiplying Eqs. (i) and (ii), we get

$$\frac{(x^2-1)^{2n}}{x^{2n}} = ({}^{2n}C_0 + {}^{2n}C_1x + {}^{2n}C_2x^2 + \ldots + {}^{2n}C_{2n}x^{2n}) \times ({}^{2n}C_0 - \frac{{}^{2n}C_1}{x} + \frac{{}^{2n}C_2}{x^2} - \ldots + \frac{{}^{2n}C_{2n}}{x^{2n}}) \ldots(iii)$$

Now, constant term in RHS

$$= ({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - \ldots + ({}^{2n}C_{2n})^2$$

Constant term in LHS = Constant term in $\frac{(x^2-1)^{2n}}{x^{2n}}$

= Coefficient of x^{2n} in $(x^2-1)^{2n}$

= Coefficient of x^{2n} in $(1-x^2)^{2n}$

$= {}^{2n}C_n(-1)^n = (-1)^n \cdot {}^{2n}C_n$

But Eq. (iii) is an identity, therefore the constant term in RHS = constant term in LHS.

$$\Rightarrow ({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - \ldots + ({}^{2n}C_{2n})^2 = (-1)^n \cdot {}^{2n}C_n$$

Example 75. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$, prove that $C_0^2 - C_1^2 + C_2^2 - \ldots + (-1)^n \cdot C_n^2 = 0$ or $(-1)^{n/2}\cdot\frac{n!}{(n/2)!\,(n/2)!}$, according as n is odd or even. Also, evaluate ${C_0}^2 + {C_1}^2 + {C_2}^2 - \ldots + (-1)^n \cdot {C_n}^2$ for $n = 10$ and $n = 11$

Sol. Since, $(1-x)^n = C_0 - C_1x + C_2x^2 - \ldots + (-1)^n C_nx^n$ ...(i)

and $(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots + C_n$ ...(ii)

On multiplying Eqs. (i) and (ii), we get

$$(1-x^2)^n = \{C_0 - C_1x + C_2x^2 - \ldots + (-1)^n C_nx^n\} \times (C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots + C_n) \quad \ldots(iii)$$

Now, coefficient of x^n in RHS

$$= C_0^2 - C_1^2 + C_2^2 - \ldots + (-1)^n C_n^2$$

General term in LHS $= T_{r+1} = {}^nC_r(-x^2)^r = {}^nC_r(-1)^r x^{2r}$

Putting $2r = n$, we get $r = n/2$

$\therefore \quad T_{(n/2)+1} = {}^nC_{n/2}(-1)^{n/2}x^n$

$\therefore$ Coefficient of x^n in LHS $= {}^nC_{n/2}(-1)^{n/2}$

$$= (-1)^{n/2}\cdot\frac{n!}{(n/2)!\,(n/2)!}$$

$$= \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^{n/2}\dfrac{n!}{(n/2)!\,(n/2)!}, & \text{if } n \text{ is even} \end{cases} \quad \left[\because \left(\frac{\text{odd}}{2}\right)! = \infty\right]$$

But Eq. (iii) is an identity, therefore coefficient of x^n in RHS = coefficient of x^n in LHS.

$$\Rightarrow \quad C_0^2 - C_1^2 + C_2^2 - \ldots + (-1)^n C_n^2 = \begin{cases} 0 & \text{, if } n \text{ is odd} \\ (-1)^{n/2} \dfrac{n!}{(n/2)!\,(n/2)!}, & \text{if } n \text{ is even} \end{cases}$$

Now, for $n = 10$,

$$C_0^2 - C_1^2 + C_2^2 - \ldots + C_{10}^2 = (-1)^{10/2} \frac{10!}{5!\,5!} = -252 \qquad [\because 10 \text{ is even}]$$

and from $n = 11$,

$$C_0^2 - C_1^2 + C_2^2 - \ldots - C_{11}^2 = 0 \qquad [\because 11 \text{ is odd}]$$

Aliter

Since, $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$ ...(i)

Replacing x by $-\frac{1}{x}$, then we get

$$\left(1 - \frac{1}{x}\right)^n = C_0 - \frac{C_1}{x} + \frac{C_2}{x^2} - \ldots + (-1)^n \frac{C_n}{x^n} \qquad \ldots(ii)$$

On multiplying Eqs. (i) and (ii), we get

$$\frac{(x^2-1)^n}{x^n} = (C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n) \times \left(C_0 - \frac{C_1}{x} + \frac{C_2}{x^2} - \ldots + (-1)^n \frac{C_n}{x^n}\right) \qquad \ldots(iii)$$

Now, constant term in RHS

$$= C_0^1 - C_1^2 + C_2^2 - \ldots + (-1)^n C_n^2$$

$\therefore$ Constant term in LHS

$$= \text{Constant term in } \frac{(x^2-1)^n}{x^n}$$

$$= \text{Coefficient of } x^n \text{ in } (x^2-1)^n$$

$$= \text{Coefficient of } x^n \text{ in } {}^nC_{n/2}\,(x^2)^{n-(n/2)}\,(-1)^{n/2}$$

$$= (-1)^{n/2} \cdot {}^nC_{n/2}$$

$$= (-1)^{n/2} \cdot \frac{n!}{(n/2)!\,(n/2)!}$$

$$= \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^{n/2} \cdot \dfrac{n!}{(n/2)!\,(n/2)!}, & \text{if } n \text{ is even} \end{cases}$$

But Eq. (iii) is an identity, therefore the constant term in RHS = constant term in LHS.

$$\Rightarrow \quad C_0^2 - C_1^2 + C_2^2 - \ldots + (-1)^n C_n^2 = \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^{n/2} \cdot \dfrac{n!}{(n/2)!\,(n/2)!}, & \text{if } n \text{ is even} \end{cases}$$

2. If sum of the lower suffixes of binomial coefficients in each term is same.

i.e., $C_0 C_n + C_1 C_{n-1} + C_2 C_{n-2} + \ldots + C_n C_0$

Here, $0 + n = 1 + (n-1) = 2 + (n-2) = \ldots = n + 0 = n$

***Case* I** If each term of series is positive, then

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n \qquad \ldots(i)$$

and $$(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n \qquad \ldots(ii)$$

Then, multiplying Eqs. (i) and (ii) and equate the coefficient of suitable power of x on both sides.

Example 76. Prove that ${}^{m+n}C_r = {}^mC_r + {}^mC_{r-1}\,{}^nC_1 + {}^mC_{r-2}\,{}^nC_2 + \ldots + {}^nC_r$ if $r < m$, $r < n$ and m, n, r are positive integers.

Sol. Here, sum of lower suffixes of binomial coefficients in each term is r.

i.e. $$r = r - 1 + 1 = r - 2 + 2 = \ldots = r = r$$

Since,

$$(1+x)^m = {}^mC_0 + {}^mC_1 x + \ldots + {}^mC_{r-2} x^{r-2} + {}^mC_{r-1} x^{r-1} + {}^mC_r x^r + \ldots + {}^mC_m x^m \ldots(i)$$

and $$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_r x^r + \ldots + {}^nC_n x^n \ldots(ii)$$

On multiplying Eqs. (i) and (ii), we get

$$(1+x)^{m+n} = ({}^mC_0 + {}^mC_1 x + \ldots + {}^mC_{r-2} x^{r-2} + {}^mC_{r-1} x^{r-1} + {}^mC_r x^r + \ldots + {}^mC_m x^m) \times ({}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_r x^r + \ldots + {}^nC_n x^n) \ldots(iii)$$

Now, coefficient of x^r in RHS

$$= {}^mC_r \cdot {}^nC_0 + {}^mC_{r-1} \cdot {}^nC_1 + {}^mC_{r-2} \cdot {}^nC_2 + \ldots + {}^mC_0 \cdot {}^nC_r$$

$$= {}^mC_r + {}^mC_{r-1} \cdot {}^nC_1 + {}^mC_{r-2} \cdot {}^nC_2 + \ldots + {}^nC_r$$

Coefficient of x^r in LHS $= {}^{m+n}C_r$

But Eq. (iii) is an identity, therefore coefficient of x^r in LHS = coefficient of x^r in RHS.

$$\Rightarrow \quad {}^{m+n}C_r = {}^mC_r + {}^mC_{r-1} \cdot {}^nC_1 + {}^mC_{r-2} \cdot {}^nC_2 + \ldots + {}^nC_r$$

***Case* II** If terms of the series alternately positive and negative, then

$$(1-x)^n = C_0 - C_1 x + C_2 x^2 - \ldots + (-1)^n C_n x^n \ldots(i)$$

and $$(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n \quad \ldots(ii)$$

Then, multiplying Eqs. (i) and (ii) and equate the coefficient of suitable power of x on both sides.

Example 77. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$, prove that $C_0 C_n - C_1 C_{n-1} + C_2 C_{n-2} - \ldots + (-1)^n C_n C_0 = 0$ or $(-1)^{n/2} \dfrac{n!}{(n/2)!\,(n/2)!}$, according as n is odd or even.

Sol. Given, $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_{n-2}x^{n-2} + C_{n-1}x^{n-1} + C_n x^n$ ...(i)

and $(1-x)^n = C_0 - C_1x + C_2x^2 - \ldots + (-1)^n C_n x^n$ ...(ii)

On multiplying Eqs. (i) and (ii), we get

$$(1-x^2)^n = (C_0 + C_1x + C_2x^2 + \ldots + C_{n-2}x^{n-2} + C_{n-1}x^{n-1} + C_n x^n) \times (C_0 - C_1x + C_2x^2 - \ldots + (-1)^n C_n x^n) \quad \ldots(iii)$$

Now, coefficient of x^n in RHS

$$= C_0C_n - C_1C_{n-1} + C_2C_{n-2} - \ldots + (-1)^n C_nC_0$$

Now, general term in LHS,

$$T_{r+1} = {}^nC_r(-x^2)^r = (-1)^r \cdot {}^nC_r\, x^{2r}$$

Putting $2r = n$, we get

$$r = n/2$$

Now, $T_{n/2+1} = (-1)^{n/2} \cdot {}^nC_{n/2}\, x^n$

$\therefore$ Coefficient of x^n in LHS $= (-1)^{n/2} \cdot {}^nC_{n/2}$

$$= (-1)^{n/2} \cdot \frac{n!}{(n/2)!\,(n/2)!}$$

$$= \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^{n/2} \cdot \dfrac{n!}{(n/2)!\,(n/2)!}, & \text{if } n \text{ is even} \end{cases}$$

But Eq. (iii) is an identity, therefore the coefficient of x^n in RHS = coefficient of x^n in LHS.

$$\Rightarrow C_0C_n - C_1C_{n-1} + C_2C_{n-2} - \ldots + (-1)^n C_nC_0$$

$$= \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^{n/2} \cdot \dfrac{n!}{(n/2)!\,(n/2)!}, & \text{if } n \text{ is even} \end{cases}$$

3. If each term is the product of two binomial coefficient divided or multiplied by an integer, then integrating or differentiating by preceeding method. Then, multiplying two series and equate the coefficient of suitable power of x on both sides.

Example 78.

If $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_nx^n$, prove that

$$C_1^2 + 2C_2^2 + 3C_3^2 + \ldots + nC_n^2 = \frac{(2n-1)!}{((n-1)!)^2}.$$

Sol. Given, $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_nx^n$

Differentiating both sides w.r.t. x, we get

$$n(1+x)^{n-1} = 0 + C_1 + 2C_2x + 3C_3x^2 + \ldots + nC_nx^{n-1}$$

$$\Rightarrow n(1+x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + \ldots + nC_nx^{n-1} \quad \ldots(i)$$

and $(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + C_3x^{n-3} + \ldots + C_n$ ...(ii)

On multiplying Eqs. (i) and (ii), then we get

$$n(1+x)^{2n-1} = (C_1 + 2C_2x + 3C_3x^2 + \ldots + nC_nx^{n-1}) \times (C_0x^n + C_1x^{n-1} + C_2x^{n-2} + C_3x^{n-3} + \ldots + C_n) \quad \ldots(iii)$$

Now, coefficient of x^{n-1} on RHS

$$= C_1^2 + 2C_2^2 + 3C_3^2 + \ldots + nC_n^2$$

and coefficient of x^{n-1} on LHS

$$= n \cdot {}^{2n-1}C_{n-1} = n \cdot \frac{(2n-1)!}{(n-1)!\,n!}$$

$$= \frac{(2n-1)!}{(n-1)!\,(n-1)!} = \frac{(2n-1)!}{\{(n-1)!\}^2}$$

But Eq. (iii) is an identity, therefore the coefficient of x^{n-1} in RHS = coefficient of x^{n-1} in LHS.

$$\Rightarrow \quad C_1^2 + 2C_2^2 + 3C_3^2 + \ldots + nC_n^2 = \frac{(2n-1)!}{\{(n-1)!\}!}$$

Example 79.

If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$, prove that $C_0^2 + \frac{C_1^2}{2} + \frac{C_2^2}{3} + \ldots + \frac{C_n^2}{n+1} = \frac{(2n+1)!}{\{(n+1)!\}^2}$.

Sol. Given, $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$

Integrating both sides w.r.t. x within limits 0 to x, then we get

$$\int_0^x (1+x)^n\,dx = \int_0^x (C_0 + C_1x + C_2x^2 + \ldots + C_nx^n)\,dx$$

$$\frac{(1+x)^{n+1}-1}{(1+n)} = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \ldots + \frac{C_nx^{n+1}}{n+1} \quad \ldots(i)$$

and $(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots + C_n$ ...(ii)

Multiplying Eqs. (i) and (ii), we get

$$\frac{1}{(n+1)}\{(1+x)^{2n+1} - (1+x)^n\} = \left(C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \ldots + \frac{C_nx^{n+1}}{n+1}\right) \times (C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots + C_n) \quad \ldots(iii)$$

Now, coefficient of x^{n-1} in RHS of Eq. (iii)

$$= C_0^2 + \frac{C_1^2}{2} + \frac{C_2^2}{3} + \ldots + \frac{C_n^2}{n+1}$$

and coefficient of x^{n+1} in LHS of Eq. (iii)

$$= \frac{1}{(n+1)}\{{}^{2n+1}C_{n+1} - 0\}$$

$$= \frac{1}{(n+1)} \cdot \frac{(2n+1)!}{(n+1)!\,n!}$$

$$= \frac{(2n+1)!}{(n+1)!\,(n+1)!} = \frac{(2n+1)!}{\{(n+1)!\}^2}$$

But Eq. (iii) is an identity, therefore coefficient of x^{n+1} in RHS of Eq. (iii) = coefficient of x^{n+1} in LHS of Eq. (iii).

$$\Rightarrow \quad C_0^2 + \frac{C_1^2}{2} + \frac{C_2^2}{3} + \ldots + \frac{C_n^2}{n+1} = \frac{(2n+1)!}{\{(n+1)!\}^2}$$

Binomial Inside Binomial

The upper suffices of binomial coefficients are different but lower suffices are same.

Example 80. Evaluate $\sum_{r=0}^{n} {}^{n+r}C_n$.

Sol. $\sum_{r=0}^{n} {}^{n+r}C_n = {}^{n}C_n + {}^{n+1}C_n + {}^{n+2}C_n + \ldots + {}^{2n}C_n$

= Coefficient of x^n in

$[(1+x)^n + (1+x)^{n+1} + (1+x)^{n+2} + \ldots + \ldots + (1+x)^{2n}]$

$= \text{Coefficient of } x^n \text{ in } \left[\frac{(1+x)^n[(1+x)^{n+1}-1]}{(1+x)-1}\right]$

$= \text{Coefficient of } x^{n+1} \text{ in } [(1+x)^{2n+1} - (1+x)^n]$

$= {}^{2n+1}C_{n+1} - 0 = {}^{2n+1}C_n$

Example 81. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$, prove that

$C_0 \cdot {}^{2n}C_n - C_1 \cdot {}^{2n-2}C_n + C_2 \cdot {}^{2n-4}C_n - \ldots = 2^n$

Sol. LHS $= C_0 \cdot {}^{2n}C_n - C_1 \cdot {}^{2n-2}C_n + C_2 \cdot {}^{2n-4}C_n - \ldots$

= Coefficient of x^n in

$[C_0(1+x)^{2n} - C_1(1+x)^{2n-2} + C_2(1+x)^{2n-4} - \ldots]$

= Coefficient of x^n in

$[C_0(1+x)^2]^n - C_1[(1+x)^2]^{n-1} + C_2[(1+x)^2]^{n-2} - \ldots]$

= Coefficient of x^n in $[[(1+x)^2 - 1]^n]$

= Coefficient of x^n in $(2x + x^2)^n$

= Constant term in $(2+x)^n = 2^n$ = RHS

Example 82. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n$, prove that

$C_0 \cdot {}^{2n}C_n - C_1 \cdot {}^{2n-1}C_n + C_2 \cdot {}^{2n-2}C_n - C_3 \cdot {}^{2n-3}C_n + \ldots + (-1)^n C_n \cdot {}^{n}C_n = 1$

Sol. LHS $= C_0 \cdot {}^{2n}C_n - C_1 \cdot {}^{2n-1}C_n + C_2 \cdot {}^{2n-2}C_n - C_3 \cdot {}^{2n-3}C_n + \ldots + (-1)^n C_n \cdot {}^{n}C_n$

= Coefficient of x^n in

$[C_0(1+x)^{2n} - C_1(1+x)^{2n-1} + C_2(1+x)^{2n-2} - C_3(1+x)^{2n-3} + \ldots + (-1)^n C_n \cdot (1+x)^n]$

= Coefficient of x^n in

$(1+x)^n [C_0(1+x)^n - C_1(1+x)^{n-1} + C_2(1+x)^{n-2} - C_3(1+x)^{n-3} + \ldots + (-1)^n C_n \cdot 1]$

= Coefficient of x^n in $(1+x)^n [((1+x)-1)^n]$

= Coefficient of x^n in $(1+x)^n \cdot x^n$ d

= Constant term in $(1+x)^n = 1$ = RHS

Sum of the Series

***Case* I When *i* and *j* are independent.**

In this summation, three types of terms occur, when $i < j$, $i = j$ and $i > j$,

i.e., $\sum_{i=0}^{n}\sum_{j=0}^{n} a_i a_j = \sum_{i=0}^{n}\left\{a_i\left(\sum_{j=0}^{n} a_j\right)\right\}$

$$= \sum_{i=0}^{n} a_i \sum_{j=0}^{n} a_j = \left(\sum_{i=0}^{n} a_i\right)^2 \text{ or } \left(\sum_{j=0}^{n} a_j\right)^2$$

***Corollary* I** $\sum_{i=0}^{n}\sum_{j=0}^{n} {}^{n}C_i \, {}^{n}C_j = \left(\sum_{i=0}^{n} {}^{n}C_i\right)^2$

$$= (2^n)^2 = 2^{2n}$$

Example 83. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$, find the values of the following.

(i) $\sum_{i=0}^{n}\sum_{j=0}^{n} (C_i + C_j)$

(ii) $\sum_{i=0}^{n}\sum_{j=0}^{n} (i+j) C_i C_j$

Sol. (i) $\sum_{i=0}^{n}\sum_{j=0}^{n} (C_i + C_j) = \sum_{i=0}^{n}\sum_{j=0}^{n} C_i + \sum_{i=0}^{n}\sum_{j=0}^{n} C_j$

$$= \sum_{j=0}^{n}\left(\sum_{i=0}^{n} C_i\right) + \sum_{i=0}^{n}\left(\sum_{j=0}^{n} C_j\right)$$

$$= \sum_{j=0}^{n}(2^n) + \sum_{i=0}^{n}(2^n) = (n+1)\cdot 2^n + (n+1)\cdot 2^n$$

$$= 2(n+1)2^n = (n+1)2^{n+1}$$

(ii) $\sum_{i=0}^{n}\sum_{j=0}^{n}(i+j)C_i C_j = \sum_{i=0}^{n}\sum_{j=0}^{n} iC_i C_j + \sum_{i=0}^{n}\sum_{j=0}^{n} jC_i C_j$

$$= \sum_{i=0}^{n} i\,C_i\left(\sum_{j=0}^{n} C_j\right) + \sum_{j=0}^{n} j\,C_j\left(\sum_{i=0}^{n} C_i\right)$$

$$= \sum_{i=0}^{n} i\, C_i\, (2^n) + \sum_{j=0}^{n} j\, C_j\, (2^n)$$

$$= 2^n \sum_{i=0}^{n} i\, {}^nC_i + 2^n \sum_{j=0}^{n} j\, {}^nC_j$$

$$= 2^n \sum_{i=0}^{n} i \cdot \frac{n}{i} \cdot {}^{n-1}C_{i-1} + 2^n \sum_{j=0}^{n} j \cdot \frac{n}{j} \cdot {}^{n-1}C_{j-1}$$

$$= n \cdot 2^n \sum_{i=0}^{n} {}^{n-1}C_{i-1} + n \cdot 2^n \sum_{j=0}^{n} {}^{n-1}C_{j-1}$$

$$= n \cdot 2^n \cdot 2^{n-1} + n \cdot 2^n \cdot 2^{n-1}$$

$$= n \cdot 2 \cdot 2^{2n-1} = n \cdot 2^{2n}$$

***Case* II When *i* and *j* are dependent.**

In this summation, when $i < j$ is equal to the sum of the terms when $i > j$, if a_i and a_j are symmetrical. So, in this case

$$\sum_{i=0}^{n} \sum_{j=0}^{n} a_i\, a_j = \sum\sum_{0 \le i < j \le n} a_i\, a_j + \sum\sum_{i=j} a_i a_j + \sum\sum_{0 \le j < i \le n} a_i a_j$$

$$= 2 \sum\sum_{0 \le i < j \le n} a_i\, a_j + \sum\sum_{i=j} a_i\, a_j$$

$$\Rightarrow \quad \sum\sum_{0 \le i < j \le n} a_i\, a_j = \frac{\sum_{i=0}^{n} \sum_{j=0}^{n} a_i\, a_j - \sum\sum_{i=j} a_i\, a_j}{2}$$

When a_i and a_j are not symmetrical, we find the sum by listing all the terms.

***Corollary* I**

$$\sum\sum_{0 \le i < j \le n} {}^nC_i\, {}^nC_j = \frac{\sum_{i=0}^{n} \sum_{j=0}^{n} {}^nC_i\, {}^nC_j - \sum\sum_{i=j} {}^nC_i \cdot {}^nC_j}{2}$$

$$= \frac{(2^n)^2 - \sum_{i=0}^{n} ({}^nC_i)^2}{2} = \frac{2^{2n} - {}^{2n}C_n}{2} = 2^{2n-1} - \frac{2n!}{2(n!)^2}$$

Example 84. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$, find the values of the following.

(i) $\sum\sum_{0 \le i < j \le n} C_i$ (ii) $\sum\sum_{0 \le i < j \le n} j\, C_i$

(iii) $\sum\sum_{i \ne j} C_i\, C_j$ (iv) $\sum\sum_{0 \le i \le j \le n} C_i\, C_j$

(v) $\sum\sum_{0 \le i < j \le n} (C_i \pm C_j)^2$

(vi) $\sum\sum_{0 \le i < j \le n} (i + j)\, C_i\, C_j$

(vii) $\sum\sum_{0 \le i < j \le n} (i \cdot j)\, C_i\, C_j$

Sol. (i) $$\sum\sum_{0 \le i < j \le n} C_i = \frac{\left(\sum_{i=0}^{n} \sum_{j=0}^{n} C_i\right) - \sum\sum_{i=j} C_i}{2}$$

$$= \frac{(n+1) \sum_{i=0}^{n} C_i - \sum_{i=0}^{n} C_i}{2} = n \cdot 2^{n-1}$$

(ii) $$\sum\sum_{0 \le i < j \le n} j\, C_i = \sum_{r=0}^{n-1} {}^nC_r \{(r+1) + (r+2) + (r+3) + \ldots + n\}$$

$$= \sum_{r=0}^{n-1} {}^nC_r \cdot \frac{(n-r)(n+r+1)}{2}$$

$$= \frac{1}{2} \sum_{r=0}^{n-1} {}^nC_r\, (n^2 - r^2 + n - r)$$

$$= \frac{1}{2}(n^2 + n) \sum_{r=0}^{n-1} {}^nC_r - \frac{1}{2} \sum_{r=0}^{n-1} r \cdot {}^nC_r - \frac{1}{2} \sum_{r=0}^{n-1} r^2 \cdot {}^nC_r$$

$$= \frac{1}{2}(n^2 + n)(2^n - 1) - \frac{1}{2} \cdot n \cdot (2^{n-1} - 1) - \frac{1}{2} \cdot n\, [(n-1)(2^{n-2} - 1) + 2^{n-1} - 1]$$

$$= n\,(3n+1) \cdot 2^{n-3}$$

Remark

Here, j and C_i are not symmetrical.

(iii) Here, $i \ne j$ i.e., $i > j$ or $i < j$

But C_i and C_j are symmetrical.

$$\therefore \quad \sum\sum_{i \ne j} C_i\, C_j = 2 \sum\sum_{0 \le i < j \le n} C_i\, C_j$$

$$= 2\left(\frac{2^{2n} - {}^{2n}C_n}{2}\right) \quad \text{[from corollary I]}$$

$$= 2^{2n} - {}^{2n}C_n$$

(iv) $$\sum\sum_{0 \le i \le j \le n} C_i C_j = \sum\sum_{0 \le i < j \le n} C_i C_j + \sum\sum_{i=j} C_i C_j$$

$$= \frac{1}{2}(2^{2n} - {}^{2n}C_n) + {}^{2n}C_n \quad \text{[from corollary I]}$$

$$= \frac{1}{2}(2^{2n} + {}^{2n}C_n)$$

(v) $$\sum\sum_{0 \le i < j \le n} (C_i \pm C_j)^2 = \sum\sum_{0 \le i < j \le n} (C_i^2 + C_j^2 \pm 2\, C_i C_j)$$

$$= \sum\sum_{0 \le i < j \le n} (C_i^2 + C_j^2) \pm 2 \sum\sum_{0 \le i < j \le n} C_i\, C_j$$

$$\because \sum\sum_{0 \le i < j \le n} (C_i^2 + C_j^2)$$

$$= \frac{\sum_{i=0}^{n}\sum_{j=0}^{n}(C_i^2 + C_j^2) - 2\sum_{i=0}^{n} C_i^2}{2}$$

$$= \frac{\sum_{i=0}^{n}\left(\sum_{j=0}^{n} C_i^{\,2} + \sum_{j=0}^{n} C_j^2\right) - 2\cdot {}^{2n}C_n}{2}$$

$$= \frac{\sum_{i=0}^{n}\left((n+1)\,C_i^{\,2} + {}^{2n}C_n\right) - 2\cdot {}^{2n}C_n}{2}$$

$$= \frac{(n+1)\sum_{i=0}^{n} C_i^{\,2} + {}^{2n}C_n \sum_{i=0}^{n} 1 - 2\cdot {}^{2n}C_n}{2}$$

$$= \frac{(n+1)\cdot {}^{2n}C_n + {}^{2n}C_n\cdot(n+1) - 2\cdot {}^{2n}C_n}{2}$$

$$= n\cdot {}^{2n}C_n$$

$$\therefore \quad \sum\sum_{0\le i<j\le n} (C_i \pm C_j)^2 = n\cdot {}^{2n}C_n \pm (2^{2n} - {}^{2n}C_n)$$

[from corollary 1]

$$= (n \mp 1)\,{}^{2n}C_n \pm 2^{2n}; \quad \sum\sum_{0\le i<j\le n} (i+j)C_i C_j$$

Remark

$$\sum\sum_{0\le i<j\le n} (C_i + C_j) = n\cdot 2^n$$

(vi) $\sum\sum_{0\le i<j\le n} (i+j)C_i C_j$

Let $P = \sum\sum_{0\le i<j\le n} (i+j)\,C_i\,C_j$ ...(i)

Replacing i by $n-i$ and j by $n-j$ in Eq. (i), then we get

$$P = \sum\sum_{0\le i<j\le n} (n-i+n-j)\,C_{n-i}\,C_{n-j}$$

[$\because$ sum of binomial expansion does not change if we replace r by $n-r$]

$$P = \sum\sum_{0\le i<j\le n} (2n-i-j)\,C_i\,C_j$$

[$\because {}^nC_r = {}^nC_{n-r}$] ...(ii)

On adding Eqs. (i) and (ii), we get

$$2P = 2n \sum\sum_{0\le i<j\le n} C_i\,C_j$$

or $$P = n\sum\sum_{0\le i<j\le n} C_i\,C_j = \frac{n}{2}(2^{2n} - {}^{2n}C_n)$$

[from corollary I]

(vii) $\sum\sum_{0\le i<j\le n} (i\cdot j)\,C_i\,C_j = \sum\sum_{0\le i<j\le n} (i\,{}^nC_i)(j\cdot {}^nC_j)$

$$= n^2 \sum\sum_{0\le i<j\le n} {}^{n-1}C_{i-1}\,{}^{n-1}C_{j-1}$$

$$= n^2\left[\frac{2^{2(n-1)} - {}^{2n-2}C_{n-1}}{2}\right] \text{[from corollary I]}$$

$$= n^2\left(2^{2n-3} - \frac{1}{2}\cdot {}^{2n-2}C_{n-1}\right)$$

Exercise for Session 4

1. The coefficient of $a^4 b^8 c^9 d^9$ in the expansion of $(abc + abd + acd + bcd)^{10}$ is

(a) $10!$ (b) $\frac{10!}{4!8!9!9!}$ (c) 2520 (d) None of these

2. If $(1+2x+3x^2)^{10} = a_0 + a_1 x + a_2 x^2 + \ldots + a_{20} x^{20}$, then a_1 equals

(a) 210 (b) 20 (c) 10 (d) None of these

3. If $(1+x+x^2+x^3)^5 = a_0 + a_1 x + a_2 x^2 + \ldots + a_{15} x^{15}$, then a_{10} equals

(a) 99 (b) 100 (c) 101 (d) 110

4. Coefficient of x^{15} in $(1+x+x^3+x^4)^n$ is

(a) $\sum_{r=0}^{5} {}^nC_{5-r} \cdot {}^nC_{3r}$ (b) $\sum_{r=0}^{5} {}^nC_{5r}$ (c) $\sum_{r=0}^{5} {}^nC_{2r}$ (d) $\sum_{r=0}^{3} {}^nC_{3-r} \cdot {}^nC_{5r}$

5. The number of terms in the expansion of $\left(x^2 + 1 + \frac{1}{x^2}\right)^n, n \in N$ is

(a) ${}^{n+2}C_2$ (b) ${}^{n+3}C_2$ (c) ${}^{2n+1}C_{2n}$ (d) ${}^{3n+1}C_{3n}$

6. If $(1+x)^{10} = a_0 + a_1 x + a_2 x^2 + \ldots + a_{10} x^{10}$, then $(a_0 - a_2 + a_4 - a_6 + a_8 - a_{10})^2 + (a_1 - a_3 + a_5 - a_7 + a_9)^2$ is equal to

(a) 2^9 (b) 3^9 (c) 2^{10} (d) 3^{10}

7. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n, n$ being even the value of $C_0 + (C_0 + C_1) + (C_0 + C_1 + C_2) + \ldots + (C_0 + C_1 + C_2 + \ldots + C_{n-1})$ is equal to

(a) $n \cdot 2^n$ (b) $n \cdot 2^{n-1}$ (c) $n \cdot 2^{n-2}$ (d) $n \cdot 2^{n-3}$

8. The value of $\frac{C_0}{1\cdot 3} - \frac{C_1}{2\cdot 3} + \frac{C_2}{3\cdot 3} - \frac{C_3}{4\cdot 3} + \ldots + (-1)^n \frac{C_n}{(n+1)\cdot 3}$ is

(a) $\frac{3}{n+1}$ (b) $\frac{n+1}{3}$ (c) $\frac{1}{3(n+1)}$ (d) None of these

9. The value of $\binom{50}{0}\binom{50}{1} + \binom{50}{1}\binom{50}{2} + \ldots + \binom{50}{49}\binom{50}{50}$, where ${}^nC_r = \binom{n}{r}$, is

(a) $\binom{100}{50}$ (b) $\binom{100}{51}$ (c) $\binom{50}{25}$ (d) $\binom{50}{25}^2$

10. If C_r stands for ${}^4C_r$, then $C_0 C_4 - C_1 C_3 + C_2 C_2 - C_3 C_1 + C_4 C_0$ is equal to

(a) C_1 (b) C_2 (c) C_3 (d) C_4

11. The sum $\sum_{r=0}^{n} (r+1)({}^nC_r)^2$ is equal to

(a) $\frac{(n+2)(2n-1)!}{n!(n-1)!}$ (b) $\frac{(n+2)(2n+1)!}{n!(n-1)!}$ (c) $\frac{(n+2)(2n+1)!}{n!(n+1)!}$ (d) $\frac{(n+2)(2n-1)!}{n!(n+1)!}$

12. $\sum_{r=1}^{n} \left(\sum_{p=0}^{r-1} {}^nC_r \, {}^rC_p \, 2^p \right)$ is equal to

(a) $4^n - 3^n + 1$ (b) $4^n - 3^n - 1$ (c) $4^n - 3^n + 2$ (d) $4^n - 3^n$

13. $\left(\sum_{r=0}^{10} {}^{10}C_r \right) \left(\sum_{m=0}^{10} (-1)^m \frac{{}^{10}C_m}{2^m} \right)$ is equal to

(a) 1 (b) 2^5 (c) 2^{10} (d) 2^{20}

14. The value of $\sum\sum\sum\sum_{0 \le i < j < k < l \le n} 2$ is equal to

(a) $2(n+1)^3$ (b) $2 \cdot {}^{n+1}C_4$ (c) $2(n+1)^4$ (d) $2 \cdot {}^{n+2}C_3$

Shortcuts and Important Results to Remember

1 $(r+1)$th term from end in the expansion of $(x+y)^n = (r+1)$th term from beginning in the expansion of $(y+x)^n$.

2 If ${}^nC_{r-1}, {}^nC_r, {}^nC_{r+1}$ are in AP, then $(n-2r)^2 = n+2$ or $r = \frac{1}{2}(n \pm \sqrt{(n+2)})$ for $r = 2,\ n = 7$ and for $r = 5, n = 7, 14$.

3 Four consecutive binomial coefficients can never be in AP.

4 Three consecutive binomial coefficients can never be in GP or HP.

5 If a, b, c, d are four consecutive coefficients in the expansion of $(1+x)^n$, then $\frac{a}{a+b}, \frac{b}{b+c}, \frac{c}{c+d}$ are in AP.

(i) $\frac{a}{a+b} + \frac{c}{c+d} = 2\left(\frac{b}{b+c}\right)$

(ii) $\left(\frac{b}{b+c}\right)^2 > \frac{ac}{(a+b)(c+d)}$

6 If greatest term in $(1+x)^{2n}$ has the greatest coefficient, then $\frac{n}{n+1} < x < \frac{n+1}{n}$.

7 (a) The coefficient of x^{n-1} in the expansion of

$$(x-1)(x-2)(x-3)\ldots(x-n) = -(1+2+3+\ldots+n)$$
$$= -\frac{n(n+1)}{2} = -{}^{n+1}C_2$$

(b) The coefficient of x^{n-1} in the expansion of

$$(x+1)(x+2)(x+3)\ldots(x+n)$$
$$= (1+2+3+\ldots+n) = \frac{n(n+1)}{2} = {}^{n+1}C_2$$

8 The number of terms in the expansion of

$$(x+a)^n + (x-a)^n = \begin{cases} \frac{n+2}{2}, \text{ if } n \text{ is even} \\ \frac{n+1}{2}, \text{ if } n \text{ is odd} \end{cases}$$

9 The number of terms in the expansion of

$$(x+a)^n - (x-a)^n = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{n+1}{2}, & \text{if } n \text{ is odd} \end{cases}$$

10 The number of terms in the expansion of multinomial $(x_1 + x_2 + x_3 + \ldots + x_m)^n$, when $x_1, x_2, x_3, \ldots, x_m \in C$ and $n \in N$, is ${}^{n+m-1}C_{m-1}$.

11 The number of terms in the expansion of $\left(ax^p + \frac{b}{x^p} + c\right)^n$, where $n, p \in N$ and a, b, c are constants, is $2n+1$.

12 If the coefficients of pth and qth terms in the expansion of $(1+x)^n$ are equal, then $p+q = n+2$, where $p, q, n \in N$.

13 If the coefficients of x^r, x^{r+1} in the expansion of $\left(a + \frac{x}{b}\right)^n$ are equal, then $n = (r+1)(ab+1) - 1$, where $n, r \in N$ and a, b are constants.

14 Coefficient of x^m in the expansion of $\left(ax^p + \frac{b}{x^q}\right)^n$ = Coefficient of T_{r+1}, where $r = \frac{np-m}{p+q}$, where $p, q, n \in N$ and a, b are constants.

15 The term independent of x in the expansion of $\left(ax^p + \frac{b}{x^q}\right)^n$ is T_{r+1}, where $r = \frac{np}{p+q}$, where $n,\ p, q \in N$ and a, b are constants.

16 Sum of the coefficients in the expansion of $(ax+by)^n$ is $(a+b)^n$, where $n \in N$ and a, b are constants.

17 If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$ and $p+q = 1$, then

(i) $\sum_{r=0}^{n} r \cdot C_r \cdot p^r \cdot q^{n-r} = np$

(ii) $\sum_{r=0}^{n} r^2 \cdot C_r \cdot p^r \cdot q^{n-r} = n^2p^2 + npq$

18 If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$, then

(i) $\sum_{r=0}^{n} r \cdot C_r = n \cdot 2^{n-1}$ (ii) $\sum_{r=0}^{n} \frac{C_r}{r+1} = \frac{2^{n+1}-1}{n+1}$

(iii) $\sum_{r=0}^{n} r^2 \cdot C_r = n(n+1)2^{n-2}$ (iv) $\sum_{r=0}^{n} (-1)^r \cdot r \cdot C_r = 0$

(v) $\sum_{r=0}^{n} \frac{(-1)^r C_r}{r+1} = \frac{1}{n+1}$

(vi) $\sum_{r=0}^{n} (-1)^r \frac{C_r}{r} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$

(vii) $\sum_{r=0}^{n} (-1)^r \cdot r^2 \cdot C_r = 0$

(viii) $\sum_{r=0}^{n} (-1)^r \cdot (a-r)(b-r)C_r = 0, \forall\ n > 3$

(ix) $\sum_{r=0}^{n} (-1)^r (a-r)(b-r)(c-r)C_r = 0, \forall\ n > 3$

(x) $\sum_{r=0}^{n} (-1)^r (a-r)^3 C_r = 0, \forall\ n > 3$

(xi) $\sum_{r=0}^{n} r(r-1)(r-2)\ldots(r-k+1)C_r x^{r-k} = \frac{d^k}{dx^k}(1+x)^n$

for $k = 2$, $\sum_{r=0}^{n} r(r-1)C_r = \frac{d^2}{dx^2}[(1+x)^n]_{x=1} = n(n-1)2^{n-2}$

and for $k = 3$; $\sum_{r=0}^{n} r(r-1)(r-2)(-1)^{r-3}C_r$

$$= \frac{d^3}{dx^3}[(1+x)^n]_{x=-1} = 0$$

JEE Type Solved Examples :
Single Option Correct Type Questions

■ This section contains **10 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

● ***Ex. 1*** *If* $\binom{2n+1}{0}+\binom{2n+1}{3}+\binom{2n+1}{6}+\ldots=170$, *then n equals*

(a) 2 (b) 4 (c) 6 (d) 8

Sol. (b) $\because\ (1+x)^{2n+1}={}^{2n+1}C_0+{}^{2n+1}C_1x+{}^{2n+1}C_2x^2+{}^{2n+1}C_3x^3+{}^{2n+1}C_4x^4+{}^{2n+1}C_5x^5+{}^{2n+1}C_6x^6+\ldots$

Putting $x=1, \omega, \omega^2$ (where ω is cube root of unity) and adding, we get

$2^{2n+1}+(1+\omega)^{2n+1}+(1+\omega^2)^{2n+1}=3({}^{2n+1}C_0+{}^{2n+1}C_3+{}^{2n+1}C_6+\ldots$

$\Rightarrow\quad 2^{2n+1}-\omega^{2(2n+1)}-\omega^{2n+1}=3({}^{2n+1}C_0+{}^{2n+1}C_3+{}^{2n+1}C_6+\ldots)\ [\because 1+\omega+\omega^2=0]$

$\Rightarrow\quad {}^{2n+1}C_0+{}^{1+2n}C_3+{}^{2n+1}C_6+\ldots=\frac{1}{3}(2^{2n+1}-\omega^{2(2n+1)}-\omega^{2n+1})$

$\Rightarrow\quad \binom{2n+1}{0}+\binom{2n+1}{3}+\binom{2n+1}{6}+\ldots=\frac{1}{3}(2^{2n+1}-\omega^{2(2n+1)}-\omega^{2n+1})$

$\Rightarrow\quad 170=\frac{1}{3}(2^{2n+1}-\omega^{2(2n+1)}-\omega^{2n+1})$

For $n=4, 170=\frac{1}{3}(512-1-1)=170$ $[\because \omega^3=1]$

Hence, $n=4$

● ***Ex. 2*** $({}^mC_0+{}^mC_1-{}^mC_2-{}^mC_3)+({}^mC_4+{}^mC_5-{}^mC_6-{}^mC_7)+\ldots=0$ *if and only if for some positive integer k, m is equal to*

(a) $4k$ (b) $4k+1$ (c) $4k-1$ (d) $4k+2$

Sol. (c) If $\theta\in R$ and $i=\sqrt{-1}$, then $(\cos\theta+i\sin\theta)^n$

$={}^mC_0(\cos\theta)^m+{}^mC_1(\cos\theta)^{m-1}(i\sin\theta)+{}^mC_2(\cos\theta)^{m-2}(i\sin\theta)^2+\ldots+{}^mC_m(i\sin\theta)^m$

$(\cos m\theta+i\sin m\theta)=[{}^mC_0(\cos\theta)^m-{}^mC_2(\cos\theta)^{m-2}\cdot\sin^2\theta+{}^mC_4(\cos\theta)^{m-4}\sin^4\theta-\ldots]+i[{}^mC_1(\cos\theta)^{m-1}\cdot\sin\theta-{}^mC_3(\cos\theta)^{m-3}\sin^3\theta+\ldots]$ [using Demoivre's theorem]

Comparing real and imaginary parts, we get

$\cos m\theta={}^mC_0(\cos\theta)^m-{}^mC_2(\cos\theta)^{m-2}\sin^2\theta+{}^mC_4(\cos\theta)^{m-4}\sin^4\theta-\ldots\ldots(i)$

$\sin m\theta={}^mC_1(\cos\theta)^{m-1}\cdot\sin\theta-{}^mC_3(\cos\theta)^{m-3}\cdot\sin^3\theta+\ldots$ ...(ii)

On adding Eqs. (i) and (ii), we get

$\cos m\theta+\sin m\theta={}^mC_0(\cos\theta)^m+{}^mC_1(\cos\theta)^{m-1}\cdot\sin\theta-{}^mC_2(\cos\theta)^{m-2}\sin^2\theta-{}^mC_3(\cos\theta)^{m-3}\sin^3\theta+{}^mC_4(\cos\theta)^{m-4}\sin^4\theta+\ldots\sin\left(m\theta+\frac{\pi}{4}\right)$

$=(\cos\theta)^m\left\{{}^mC_0+{}^mC_1\tan\theta-{}^mC_2\tan^2\theta-{}^mC_3\tan^3\theta+{}^mC_4\tan^4\theta+{}^mC_5\tan^5\theta-\ldots\right\}$

Putting $\theta=\frac{\pi}{4}$, $\sqrt{2}\sin\left(\frac{(m+1)\pi}{4}\right)=\frac{1}{2^{m/2}}$

$\left\{({}^mC_0+{}^mC_1-{}^mC_2-{}^mC_3)+({}^mC_4+{}^mC_5-{}^mC_6-{}^mC_7)\right.$
$\left.+\ldots+({}^mC_{m-3}+{}^mC_{m-2}-{}^mC_{m-1}-{}^mC_m)\right\}$

$\because ({}^mC_0+{}^mC_1-{}^mC_2-{}^mC_3)+({}^mC_4+{}^mC_5-{}^mC_6-{}^mC_7)+\ldots=0$ [given]

$\therefore\ \sin\left(\frac{(m+1)\pi}{4}\right)=0\ \Rightarrow\ \frac{(m+1)\pi}{4}=k\pi$

or $m=4k-1, \forall k\in I$

● ***Ex. 3*** *If coefficient of* x^n *in the expansion of* $(1+x)^{101}(1-x+x^2)^{100}$ *is non-zero, then n cannot be of the form*

(a) $3\lambda+1$ (b) 3λ (c) $3\lambda+2$ (d) $4\lambda+1$

Sol. (c) $\because (1+x)^{101}(1-x+x^2)^{100}=(1+x)((1+x)(1-x+x^2))^{100}$

$=(1+x)(1+x^3)^{100}$

$=(1+x)(1+{}^{100}C_1x^3+{}^{100}C_2x^6+{}^{100}C_3x^9+\ldots+\ldots+{}^{100}C_{10}x^{300})$

Clearly, in this expression x^3 will present if $n=3\lambda$ or $n=3\lambda+1$. So, n cannot be of the form $3\lambda+2$.

● ***Ex. 4*** *The sum* $\sum_{i=0}^{m}\binom{10}{i}\binom{20}{m-i}$, *(where* $\frac{p}{q}=0$, *if* $p<q$*) is maximum when m is*

(a) 5 (b) 10 (c) 15 (d) 20

Sol. (c) $\sum_{i=0}^{m}\binom{10}{i}\binom{20}{m-i}=\sum_{i=0}^{m}{}^{10}C_i\,{}^{20}C_{m-i}$

$={}^{10}C_0\cdot{}^{20}C_m+{}^{10}C_1\cdot{}^{20}C_{m-1}+{}^{10}C_2\cdot{}^{20}C_{m-2}+\ldots+{}^{10}C_m\cdot{}^{20}C_0$

= Coefficient of x^m in the expansion of product $(1+x)^{10}(1+x)^{20}$

= Coefficient of x^m in the expansion of $(1+x)^{30}={}^{30}C_m$

To get maximum value of the given sum, ${}^{30}C_m$ should be maximum. Which is so, when $m=\frac{30}{2}=15$

● **Ex. 5** *If* $^{n-1}C_r = (k^2-3)\cdot {}^nC_{r+1}$ *then* k *belongs to*

(a) $(-\infty,-2]$ (b) $[2,\infty)$
(c) $[-\sqrt{3},\sqrt{3}]$ (d) $(\sqrt{3},2]$

Sol. (d) $\because {}^{n-1}C_r = (k^2-3)\cdot {}^nC_{r+1}$

$$\Rightarrow \quad k^2-3 = \frac{{}^{n-1}C_r}{{}^nC_{r+1}} = \frac{r+1}{n} \qquad \ldots(i)$$

$$\Rightarrow \quad 0 \le r \le n-1$$

$$\Rightarrow \quad 1 \le r+1 \le n$$

$$\Rightarrow \quad \frac{1}{n} \le \frac{r+1}{n} \le 1$$

$$\Rightarrow \quad \frac{1}{n} \le (k^2-3) \le 1$$

$$\Rightarrow \quad 3+\frac{1}{n} \le k^2 \le 4 \quad \text{or} \quad 3 < k^2 \le 4 \qquad [\text{here, } n \ge 2]$$

$$\therefore \quad k \in [-2,\sqrt{3}) \cup (\sqrt{3},2]$$

● **Ex. 6** *If* $\left(x+\frac{1}{x}+1\right)^6 = a_0 + \left(a_1x+\frac{b_1}{x}\right) + \left(a_2x^2+\frac{b_2}{x^2}\right)+\ldots+\left(a_6x^6+\frac{b_6}{x^6}\right)$,

the value of a_0 *is*

(a) 121 (b) 131
(c) 141 (d) 151

Sol. (c) $\because \left(x+\frac{1}{x}+1\right)^6 = \sum_{r=0}^{6} {}^6C_r\left(x+\frac{1}{x}\right)^r$ for constant term r must be even integer.

$$\therefore a_0 = {}^6C_0 + {}^6C_2 \times {}^2C_1 + {}^6C_4 \times {}^4C_2 + {}^6C_6 \times {}^6C_3$$
$$= 1+30+90+20 = 141$$

● **Ex. 7** *The coefficient of* x^{50} *in the series*

$$\sum_{r=1}^{101} rx^{r-1}(1+x)^{101-r} \text{ is}$$

(a) $^{100}C_{50}$ (b) $^{101}C_{50}$
(c) $^{102}C_{50}$ (d) $^{103}C_{50}$

Sol. (c) Let $S = \sum_{r=1}^{101} rx^{r-1}(1+x)^{101-r}$

$$= (1+x)^{100} + 2x(1+x)^{99} + 3x^2(1+x)^{98} + \ldots + 101x^{100}$$

$$S = (1+x)^{100}\left\{1+2\left(\frac{x}{1+x}\right)+3\left(\frac{x}{1+x}\right)^2+\ldots+101\left(\frac{x}{1+x}\right)^{100}\right\} \qquad \ldots(i)$$

$$\therefore \frac{Sx}{(1+x)} = (1+x)^{100}\left\{\left(\frac{x}{1+x}\right)+2\left(\frac{x}{1+x}\right)^2 + 3\left(\frac{x}{1+x}\right)^3+\ldots+101\left(\frac{x}{1+x}\right)^{101}\right\} \qquad \ldots(ii)$$

On subtracting Eq. (ii) from Eq. (i), then we get

$$\frac{S}{(1+x)} = (1+x)^{100}\left\{1+\left(\frac{x}{1+x}\right)+\left(\frac{x}{1+x}\right)^2 + \ldots+\left(\frac{x}{1+x}\right)^{100} - 101\left(\frac{x}{1+x}\right)^{101}\right\}$$

$$= (1+x)^{100}\left\{\frac{1\cdot\left(1-\left(\frac{x}{1+x}\right)^{101}\right)}{1-\left(\frac{x}{1+x}\right)} - 101\left(\frac{x}{1+x}\right)^{101}\right\}$$

$$\therefore \quad S = (1+x)^{102} - x^{101}(1+x) - 101x^{101}$$

and coefficient of x^{50} in $S = {}^{102}C_{50}$.

● **Ex. 8** *The largest integer* λ *such that* 2^{λ} *divides* $3^{2^n}-1$, $n \in N$ *is*

(a) $n-1$ (b) n (c) $n+1$ (d) $n+2$

Sol. (d) $\because 3^{2^n}-1 = (4-1)^{2^n}-1$

$$= (4^{2^n} - {}^{2^n}C_1\cdot 4^{2^n-1} + {}^{2^n}C_2\cdot 4^{2^n-2} - \ldots - {}^{2^n}C_{2^n-1}\cdot 4 + 1) - 1$$

$$= 4^{2^n} - 2^n\cdot 4^{2^n-1} + \frac{2^n(2^n-1)}{2}\cdot 4^{2^n-2} - \ldots - 2^n\cdot 4$$

$$= 2^{n+2}(2^{2^{n+1}-n-2} - 2^{2^{n+1}-4} + \ldots - 1) = 2^{n+2}(\text{Integer})$$

Hence, $3^{2^n}-1$ is divisible by $2^{n+2}\cdot\lambda = n+2$

● **Ex. 9** The last term in the binomial expansion of $\left(\sqrt[3]{2}-\frac{1}{\sqrt{2}}\right)^n$ is $\left(\frac{1}{3\sqrt[3]{9}}\right)^{\log_3 8}$, the 5th term from beginning is

(a) $^{10}C_6$ (b) $2\,^{10}C_4$
(c) $\frac{1}{2}\cdot{}^{10}C_4$ (d) None of the above

Sol. (a) Since, last term in the expansion of $\left(\sqrt[3]{2}-\frac{1}{\sqrt{2}}\right)^n$

$$= \left(\frac{1}{3\cdot\sqrt[3]{9}}\right)^{\log_3 8} \Rightarrow {}^nC_n\cdot\left(-\frac{1}{\sqrt{2}}\right)^n = \left(\frac{1}{3\cdot\sqrt[3]{9}}\right)^{\log_3 8}$$

$$\Rightarrow \quad (-1)^n\cdot\left(\frac{1}{2}\right)^{n/2} = \left(\frac{1}{3^{5/3}}\right)^{\log_3 8} = (3^{-5/3})^{\log_3 2^3}$$

$$= 3^{-\frac{5}{3}\times 3\times\log_3 2} = 3^{-5\log_3 2} = 3^{\log_3 2^{-5}} = 2^{-5} = \left(\frac{1}{2}\right)^5$$

$$\Rightarrow \quad (-1)^n\cdot\left(\frac{1}{2}\right)^{n/2} = \left(\frac{1}{2}\right)^5 \therefore \quad n = 10$$

Now, 5th term from beginning $= {}^{10}C_4(\sqrt[3]{2})^6\left(-\frac{1}{\sqrt{2}}\right)^4$

$$= {}^{10}C_4\cdot 2^2\cdot\frac{1}{2^2} = {}^{10}C_4 = {}^{10}C_6$$

● ***Ex.* 10** If $f(x)=\sum_{r=1}^{n}\{r^2({}^nC_r - {}^nC_{r-1})+(2r+1){}^nC_r\}$ *and* $f(30)=30(2)^{\lambda}$, *then the value of* λ *is*

(a) 3 (b) 4 (c) 5 (d) 6

Sol. (c) Here, $f(x)=\sum_{r=1}^{n}\{r^2({}^nC_r - {}^nC_{r-1})+(2r+1){}^nC_r\}$

$$=\sum_{r=1}^{n}(r^2+2r+1){}^nC_r - r^2\cdot{}^nC_{r-1}$$

$$=\sum_{r=1}^{n}((r+1)^2\cdot{}^nC_r - r^2\cdot{}^nC_{r-1})$$

$$=(n+1)^2\cdot{}^nC_n - 1^2\cdot{}^nC_0$$

$$=(n+1)^2-1=(n^2+2n)$$

$\therefore$ $f(30)=(30)^2+2(30)=960$

$=30\times32=30(2)^5=30(2)^{\lambda}$ [given]

Hence, $\lambda=5$

JEE Type Solved Examples :
More than One Correct Option Type Questions

■ This section contains **5 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of **which more than one** may be correct.

● ***Ex.* 11** *Let* $a_n=\left(1+\frac{1}{n}\right)^n$. *Then for each* $n\in N$

(a) $a_n\geq 2$ (b) $a_n<3$ (c) $a_n<4$ (d) $a_n<2$

Sol. (a, b, c)

$\because$ $a_n=\left(1+\frac{1}{n}\right)^n={}^nC_0+{}^nC_1\cdot\left(\frac{1}{n}\right)+\sum_{r=2}^{n}{}^nC_r\left(\frac{1}{n}\right)^2$

$$=2+\sum_{r=2}^{n}{}^nC_r\left(\frac{1}{n}\right)^2$$

$\therefore$ $a_n\geq 2$ for all $n\in N$

Also, $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e=2.7182\ldots$

$\therefore$ $a_n<e$

Finally, $2\leq a_n<e$

● ***Ex.* 12** *Let* $S_n(x)=\sum_{k=0}^{n}{}^nC_k\ \sin(kx)\cos((n-k)x)$ *then*

(a) $S_5\left(\frac{\pi}{2}\right)=16$ (b) $S_7\left(\frac{-\pi}{2}\right)=64$

(c) $S_{50}(\pi)=0$ (d) $S_{51}(-\pi)=-2^{50}$

Sol. (a, b, c)

$\because$ $S_n(x)=\sum_{k=0}^{n}{}^nC_k\sin(kx)\cos((n-k)x)$ ...(i)

Replace k by $n-k$ in Eq. (i), then

$$S_n(x)=\sum_{k=0}^{n}{}^nC_{n-k}\sin((n-k)x)\cos(kx)$$

or $S_n(x)=\sum_{k=0}^{n}{}^nC_k\sin((n-k)x)\cos(kx)$ ...(ii)

On adding Eqs. (i) and (ii), we get

$$2S_n(x)=\sum_{k=0}^{n}{}^nC_k\cdot\sin(nx)=2^n\cdot\sin(nx)$$

$\Rightarrow$ $S_n(x)=2^{n-1}\cdot\sin(nx)$

$\therefore$ $S_5\left(\frac{\pi}{2}\right)=2^4\cdot\sin\left(\frac{5\pi}{2}\right)=16,$

$S_7\left(-\frac{\pi}{2}\right)=2^6\cdot\sin\left(-\frac{7\pi}{2}\right)=2^6\times-1\times-1=64$

$S_{50}(\pi)=2^{49}\cdot\sin(50\pi)=0$

and $S_{51}(-\pi)=2^{50}\cdot\sin(-51\pi)=0$

● ***Ex.* 13** *If* $a+b=k$, *when* $a,b>0$ *and* $S(k,n)=\sum_{r=0}^{n}r^2({}^nC_r)a^r\cdot b^{n-r}$, *then*

(a) $S(1,3)=3(3a^2+ab)$ (b) $S(2,4)=16(4a^2+ab)$

(c) $S(3,5)=25(5a^2+ab)$ (d) $S(4,6)=36(6a^2+ab)$

Sol. (a, b)

$\because$ $S(k,n)=\sum_{r=0}^{n}r^2\cdot({}^nC_r)a^r\cdot b^{n-r}$

$$=b^n\sum_{r=0}^{n}r^2\cdot\left(\frac{n}{r}\cdot{}^{n-1}C_{r-1}\right)\cdot\left(\frac{a}{b}\right)^r$$

$$=nb^n\sum_{r=0}^{n}((r-1)+1)^{n-1}C_{r-1}\cdot\left(\frac{a}{b}\right)^r$$

$$=nb^n\sum_{r=0}^{n}((n-1)\cdot{}^{n-2}C_{r-2}+{}^{n-1}C_{r-1})\left(\frac{a}{b}\right)^r$$

$$=nb^n\cdot(n-1)\cdot\left(\frac{a}{b}\right)^2\sum_{r=0}^{n}{}^{n-2}C_{r-2}\left(\frac{a}{b}\right)^{r-2}+nb^n\cdot\left(\frac{a}{b}\right)\sum_{r=0}^{n}{}^{n-1}C_r\left(\frac{a}{b}\right)^{r-1}$$

$$=nb^n\cdot(n-1)\left(\frac{a}{b}\right)^2\left(1+\frac{a}{b}\right)^{n-2}+nb^n\cdot\left(\frac{a}{b}\right)\cdot\left(1+\frac{a}{b}\right)^{n-1}$$

$$=n(n-1)a^2k^{n-2}+nak^{n-1}$$

$$=n^2a^2k^{n-2}+nak^{n-2}(k-a)=n^2a^2k^{n-2}+nabk^{n-2}$$

$\therefore S(1,3)=9a^2+3ab=3(3a^2+ab)$ [$\because$ $a+b=k$]

$S(2,4)=16(4a^2+ab)$

$S(3,5)=135(5a^2+ab)$

$S(4,6)=1536(6a^2+ab)$

● **Ex. 14** *The value of x, for which the ninth term in the expansion of* $\left\{\frac{\sqrt{10}}{(\sqrt{x})^{5\log_{10} x}} + x \cdot x^{\frac{1}{2\log_{10} x}}\right\}^{10}$ *is* 450 *is equal to*

(a) 10 (b) 10^2 (c) $\sqrt{10}$ (d) $10^{-2/5}$

Sol. (b, d) Let $\log_{10} x = \lambda \Rightarrow x = 10^{\lambda}$ ...(i)

Given, $T_9 = 450$

$\Rightarrow {}^{10}C_8 \cdot \left(\frac{\sqrt{10}}{10^{5\frac{\lambda^2}{2}}}\right)^2 \cdot (10^{\lambda} \cdot 10^{1/2})^8 = 450$

$\Rightarrow {}^{10}C_2 \cdot \frac{10}{10^{5\lambda^2}} \cdot 10^{8^{\lambda}} \cdot 10^4 = 450$

$\Rightarrow 10^{8\lambda + 4 - 5\lambda^2} = 1 = 10^0$

$\Rightarrow 8\lambda + 4 - 5\lambda^2 = 0$

$\Rightarrow 5\lambda^2 - 8\lambda - 4 = 0$

$\Rightarrow \lambda = 2, -2/5$

$\Rightarrow x = 10^2, 10^{-2/5}$ [from Eq. (i)]

● **Ex. 15** *For a positive integer n, if the expansion of* $\left(\frac{5}{x^2} + x^4\right)$ *has a term independent of x, then n can be*

(a) 18 (b) 27 (c) 36 (d) 45

Sol. (a, b, c, d) Let $(r+1)$th term of $\left(\frac{5}{x^2} + x^4\right)^n$ be independent of x. We have, $T_{r+1} = {}^nC_r \left(\frac{5}{x^2}\right)^{n-r} (x^4)^r = {}^nC_r \cdot 5^{n-r} \cdot x^{6r-2n}$

For this term to be independent of x,

$6r - 2n = 0$ or $n = 3r$

For $r = 6, 9, 12, 15,$

$n = 18, 27, 36, 45.$

JEE Type Solved Examples : Passage Based Questions

■ This section contains **2 solved passages** based upon each of the passage **3 multiple choice** examples have to be answered. Each of these examples has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I (Ex. Nos. 16 to 18)

Consider $(1 + x + x^2)^n = \sum_{r=0}^{2n} a_r x^r$, *where* $a_0, a_1,$ $a_2, \ldots, a_{2n}$ *are real numbers and n is a positive integer.*

16. The value of $\sum_{r=0}^{n-1} a_{2r}$ is

(a) $\frac{9^n - 2a_{2n} - 1}{4}$ (b) $\frac{9^n - 2a_{2n} + 1}{4}$

(c) $\frac{9^n + 2a_{2n} - 1}{4}$ (d) $\frac{9^n + 2a_{2n} + 1}{4}$

17. The value of $\sum_{r=1}^{n} a_{2r-1}$ is

(a) $\frac{9^n - 1}{2}$ (b) $\frac{9^n - 1}{4}$ (c) $\frac{9^n + 1}{2}$ (d) $\frac{9^n + 1}{4}$

18. The value of a_2 is

(a) ${}^{4n+1}C_2$ (b) ${}^{3n+1}C_2$

(c) ${}^{2n+1}C_2$ (d) ${}^{n+1}C_2$

Sol.

We have, $(1 + x + x^2)^{2n} = \sum_{r=0}^{4n} a_r x^r$ ...(i)

Replacing x by $\frac{1}{x}$ in Eq. (i), we get

$\left(1 + \frac{1}{x} + \frac{1}{x^2}\right)^{2n} = \sum_{r=0}^{4n} a_r \left(\frac{1}{x}\right)^r$

$\Rightarrow (1 + x + x^2)^{2n} = \sum_{r=0}^{4n} a_r x^{4n-r}$ ...(ii)

From Eqs. (i) and (ii), we get $\sum_{r=0}^{4n} a_r x^r = \sum_{r=0}^{4n} a_r x^{4n-r}$

Equating the coefficient of x^{4n-r} on both sides, we get

$a_{4n-r} = a_r$ for $0 \le r \le 4n$

Hence, $a_r = a_{4n-r}$

Putting $x = 1$ in Eq. (i), then

$\sum_{r=0}^{4n} a_r = 3^{2n} = 9^n$ ...(iii)

Putting $x = -1$ in Eq. (i), then $\sum_{r=0}^{4n} (-1)^r a_r = 1$ ...(iv)

16. (b) On adding Eqs. (iii) and (iv), we get

$2(a_0 + a_2 + a_4 + \ldots + a_{2n-2} + a_{2n} + \ldots + a_{4n}) = 9^n + 1$

$\Rightarrow 2[2(a_0 + a_2 + a_4 + \ldots + a_{2n-2}) + a_{2n}) = 9^n + 1$ [$\because a_r = a_{4n-r}$]

$\therefore a_0 + a_2 + a_4 + \ldots + a_{2n-2} = \frac{9^n - 2a_{2n} + 1}{4}$

$\Rightarrow \sum_{r=0}^{n-1} a_{2r} = \frac{9^n - 2a_{2n} + 1}{4}$

17. (b) On subtracting Eq. (iv) from Eq. (iii), we get

$2(a_1 + a_3 + a_5 + \ldots + a_{2n-1} + a_{2n+1} + \ldots + a_{4n-1}) = 9^n - 1$

$\Rightarrow 2[2(a_1 + a_3 + a_5 + \ldots + a_{2n-1}] = 9^n - 1$ [$\because a_r = a_{4n-r}$]

$\therefore\ a_1 + a_3 + a_5 + \ldots + a_{2n-1} = \dfrac{9^n - 1}{4}$

$\Rightarrow \quad \sum_{r=1}^{n} a_{2r-1} = \dfrac{9^n - 1}{4}$

18. (c) $\because a_2$ = Coefficient of x^2 in $(1 + x + x^2)^{2n}$

$$\therefore (1 + x + x^2)^{2n} = \sum_{\alpha+\beta+\gamma=2n} \frac{2n!}{\alpha!\beta!\gamma!}(1)^{\alpha}(x)^{\beta}(x^2)^{\gamma}$$

$$= \sum_{\alpha+\beta+\gamma=2n} \frac{2n!}{\alpha!\beta!\gamma!} x^{\beta+2\gamma}$$

For a_2, $\beta + 2\gamma = 2$

Possible values of α, β, γ are $(2n-2, 2, 0)$ and $(2n-1, 0, 1)$.

$$\therefore \quad a_2 = \frac{2n!}{(2n-2)!\,2!\,0!} + \frac{2n!}{(2n-1)!\,0!\,1!}$$

$$= {}^{2n}C_2 + {}^{2n}C_1 = {}^{2n+1}C_2$$

Passage II

(Ex. Nos. 19 to 21)

Let $\quad S = \sum_{r=1}^{30} \dfrac{{}^{30+r}C_r\,(2r-1)}{{}^{30}C_r\,(30+r)}, K = \sum_{r=0}^{30} ({}^{30}C_r)^2$

and $\quad G = \sum_{r=0}^{60} (-1)^r\,({}^{60}C_r)^2$

19. The value of $(G - S)$ is

(a) 0 (b) 1 (c) 2^{30} (d) 2^{60}

20. The value of $(SK - SG)$ is

(a) 0 (b) 1
(c) 2^{30} (d) 2^{60}

21. The value of $K + G$ is

(a) $2S - 2$ (b) $2S - 1$
(c) $2S + 1$ (d) $2S + 2$

Sol.

$$\because S = \sum_{r=1}^{30} \frac{{}^{30+r}C_r\,(2r-1)}{{}^{30}C_r\,(30+r)} = \sum_{r=1}^{30} \frac{{}^{30+r}C_r}{{}^{30}C_r}\left(1 - \frac{30-r+1}{30+r}\right)$$

$$= \sum_{r=1}^{30}\left[\frac{{}^{30+r}C_r}{{}^{30}C_r} - \frac{{}^{30+r}C_r}{{}^{30}C_r}\cdot\frac{(30-r+1)}{(30+r)}\right]$$

$$= \sum_{r=1}^{30}\left[\frac{{}^{30+r}C_r}{{}^{30}C_r} - \frac{\dfrac{(30+r)}{r}\cdot{}^{29+r}C_{r-1}}{{}^{30}C_r}\cdot\frac{(31-r)}{30+r}\right]$$

$$= \sum_{r=1}^{30}\left[\frac{{}^{30+r}C_r}{{}^{30}C_r} - \frac{{}^{29+r}C_{r-1}}{{}^{30}C_{r-1}}\right] \quad \left[\because \frac{{}^{n}C_r}{{}^{n}C_{r-1}} = \frac{n-r+1}{r}\right]$$

For $n = 30\left(\dfrac{31-r}{r}\cdot{}^{30}C_r = {}^{30}C_{r-1}\right)$

$$= \frac{{}^{30+30}C_{30}}{{}^{30}C_{30}} - \frac{{}^{29+1}C_0}{{}^{30}C_0} = {}^{60}C_{30} - 1$$

$$K = \sum_{r=0}^{30} ({}^{30}C_r)^2 = {}^{60}C_{30} \text{ and } G = \sum_{r=0}^{60} (-1)^r\,({}^{60}C_r)^2$$

$$= ({}^{60}C_0)^2 - ({}^{60}C_1)^2 + ({}^{60}C_2)^2 - \ldots + ({}^{60}C_{60})^2 = {}^{60}C_{30}$$

[$\because n = 60$ is even]

19. (b) $G - S = {}^{60}C_{30} - ({}^{60}C_{30} - 1) = 1$

20. (a) $SK - SG = S(K - G) = S(G - G) = 0$ [$\because K = G$]

21. (d) $K + G = 2\cdot{}^{60}C_{30} = 2(S + 1) = 2S + 2$

JEE Type Solved Examples :
Single Integer Answer Type Questions

- This section contains **2 examples.** The answer to each example is a **single digit integer** ranging from **0** to **9** (both inclusive).

● ***Ex. 22*** *The digit at unit's place in* $2^{9^{100}}$ *is*

Sol. (2) $\because\ 9^{100} = (2\cdot 4 + 1)^{100} = 4n + 1$ [say]

[where n is positive integer]

$$\therefore \quad 2^{9^{100}} = 2^{4n+1} = 2^{4n}\cdot 2 = (16)^n \cdot 2$$

The digit at unit's place in $(16)^n = 6$.

$\therefore$ The digit at unit's place in $(16)^n \cdot 2 = 2$

● ***Ex. 23*** *If* $(1+x)^n = \sum_{r=0}^{n} a_r\,x^r$, $b_r = 1 + \dfrac{a_r}{a_{r-1}}$ *and* $\prod_{r=1}^{n} b_r = \dfrac{(101)^{100}}{100!}$, *then the value of* $\dfrac{n}{20}$ *is*

Sol. (5) Here, $a_r = {}^{n}C_r$

$$\therefore \quad b_r = 1 + \frac{a_r}{a_{r-1}} = 1 + \frac{{}^{n}C_r}{{}^{n}C_{r-1}}$$

$$= 1 + \frac{n-r+1}{r} = \frac{(n+1)}{r}$$

$$\Rightarrow \quad \prod_{r=1}^{n} b_r = \prod_{r=1}^{n} \frac{(n+1)}{r}$$

$$= \frac{(n+1)}{1}\cdot\frac{(n+1)}{2}\cdot\frac{(n+1)}{3}\cdots\frac{(n+1)}{n} = \frac{(n+1)^n}{n!}$$

$$= \frac{(101)^{100}}{100!} \quad \text{[given]}$$

$$\therefore \quad n = 100 \Rightarrow \frac{n}{20} = 5$$

JEE Type Solved Examples : Matching Type Questions

- This section contains **2 examples.** Examples 24 and 25 have three statements (A, B and C) given in Column I and four statements (p, q, r and s) in **Column II.** Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II.**

● *Ex. 24*

	Column I		Column II
(A)	If m and n are the numbers of rational terms in the expansions of $(\sqrt{2}+3^{1/5})^{10}$ and $(\sqrt{3}+5^{1/8})^{256}$ respectively, then	(p)	$n-m=6$
(B)	If m and n are the numbers of irrational terms in the expansions of $(2^{1/2}+3^{1/5})^{40}$ and $(5^{1/10}+2^{1/6})^{100}$ respectively, then	(q)	$m+n=20$
(C)	If m and n are the numbers of rational terms in the expansions of $(1+\sqrt{2}+3^{1/3})^{6}$ and $(1+\sqrt[3]{2}+\sqrt[5]{3})^{15}$ respectively, then	(r)	$n-m=31$
		(s)	$m+n=35$
		(t)	$n-m=39$

Sol. (A) → (r, s); (B) → (t); (C) → (p, q)

(A) $\because\ (\sqrt{2}+3^{1/5})^{10}=(2^{1/2}+3^{1/5})^{10}$

$\therefore\quad T_{r+1}={}^{10}C_r\cdot 2^{\frac{10-r}{2}}\cdot 3^{\frac{r}{5}}$

For rational terms, $r=0, 10$ $\quad[\because 0\le r\le 10]$

$\therefore$ Number of rational terms $=2$

i.e., $\quad m=2$ and $(\sqrt{3}+5^{1/8})^{256}=(3^{1/2}+5^{1/8})^{256}$

$\therefore\ T_{R+1}={}^{256}C_R\cdot 3^{\frac{256-R}{2}}\cdot 5^{R/8}$

For rational terms, $r=0, 8, 16, 24, 32, \ldots, 256$ $[\because 0\le r\le 256]$

$\therefore$ Number of rational terms $=1+32=33$

i.e., $n=33\ \Rightarrow\ m+n=35$ (s) and $n-m=31$

(B) T_{r+1} in $(2^{1/3}+3^{1/5})^{40}={}^{40}C_r\cdot 2^{\frac{40-r}{3}}\cdot 3^{r/5}$

For rational terms, $r=10, 25, 40$ $[\because 0\le r\le 40]$

$\because$ Number of rational terms $=3$

$\therefore$ Number of irrational terms

$=$ Total terms $-$ Number of rational terms

$=41-3=38$ i.e. $m=38$

and T_{R+1} in $(5^{1/10}+2^{1/6})^{100}={}^{100}C_R\cdot 5^{\frac{100-R}{10}}\cdot 2^{R/6}$

rational terms, $R=0, 30, 60, 90$ $\quad[\because 0\le R\le 100]$

$\because$ Number of rational terms $=4$

$\therefore$ Number of irrational terms $=101-4=97$

i.e. $n=97\ \Rightarrow\ m+n=100, n-m=97-38=39$

(C) $\because\ (1+\sqrt{2}+3^{1/3})^6=(1+2^{1/2}+3^{1/3})^6$

$$=\sum_{\alpha+\beta+\gamma=6}\frac{6!}{\alpha!\beta!\gamma!}(1)^{\alpha}(2^{1/2})^{\beta}(3^{1/3})^{\gamma}$$

$$=\sum_{\alpha+\beta+\gamma=6}\frac{6!}{\alpha!\beta!\gamma!}2^{\beta/2}\cdot 3^{\gamma/3}$$

Values of (α,β,γ) for rational terms are $(0,0,6)$, $(1,2,3),(3,0,3),(0,6,0),(2,4,0),(4,2,0),(6,0,0)$.

$\therefore$ Number of rational terms $=7$ i.e., $m=7$

and $(1+\sqrt[3]{2}+\sqrt[5]{3})^{15}=(1+2^{1/3}+3^{1/5})^{15}$

$$=\sum_{\alpha+\beta+\gamma=15}\frac{15!}{\alpha!\beta!\gamma!}(1)^{\alpha}(2^{1/3})^{\beta}(3^{1/5})^{\gamma}$$

$$=\sum_{\alpha+\beta+\gamma=15}\frac{15!}{\alpha!\beta!\gamma!}2^{\beta/3}\cdot 3^{\gamma/5}$$

of (α,β,γ) for rational terms are

$(5,0,10),(2,3,10),(10,0,5),(7,3,5),(4,6,5),(1,9,5),$
$(15,0,0),(12,3,0),(9,6,0),(6,9,0),(3,12,0),(15,0,0)$.

$\therefore$ Number of rational terms $=13$ i.e. $n=13$

Hence, $\quad m+n=20$ and $n-m=6$

● *Ex. 25* *If* $(1+x)^n=\sum_{r=0}^{n}C_r\,x^r$, *match the following.*

	Column I		Column II
(A)	If $S=\sum_{r=0}^{n}\lambda\,C_r$ and values of S are a, b, c for $\lambda=1, r, r^2$ respectively, then	(p)	$a=b+c$
(B)	If $S=\sum_{r=0}^{n}(-1)^r\ \lambda\,C_r$ and values of S are a, b, c for $\lambda=1, r, r^2$ respectively, then	(q)	$a+b=c+2$
(C)	If $S=\sum_{r=0}^{n}\frac{\lambda\,C_r}{(r+1)}$ and values of S are a, b, c for $\lambda=1, r, r^2$ respectively, then	(r)	$a^3+b^3+c^3=3abc$
		(s)	$b^{c-a}+(c-a)^b=1$
		(t)	$a+c=4b$

Sol. (A) → (p, q); (B) → (p, r, t); (C) →(s, t)

(A) For $\lambda=1$, $a=\sum_{r=0}^{n}C_r=2^n$

For $\lambda=r$, $b=\sum_{r=0}^{n}r\,C_r=\sum_{r=0}^{n}r\cdot\frac{n}{r}\cdot{}^{n-1}C_{r-1}$

$$=n\sum_{r=0}^{n}{}^{n-1}C_{r-1}=n\cdot 2^{n-1}$$

and for $\lambda=r^2$, $c=\sum_{r=0}^{n}r^2\,C_r=\sum_{r=0}^{n}r^2\cdot\frac{n}{r}\cdot{}^{n-1}C_{r-1}$

$$=n\sum_{r=0}^{n}r\cdot{}^{n-1}C_{r-1}=n\sum_{r=1}^{n}r\cdot{}^{n-1}C_{r-1}$$

$$= n\left[\sum_{r=1}^{n}\{(r-1)+1\}^{n-1}C_{r-1}\right]$$

$$= n\left[\sum_{r=1}^{n}(r-1)\cdot{}^{n-1}C_{r-1}+\sum_{r=1}^{n}{}^{n-1}C_{r-1}\right]$$

$$= n\left[\sum_{r=1}^{n}(r-1)\frac{(n-1)}{(r-1)}\cdot{}^{n-2}C_{r-2}+2^{n-1}\right]$$

$$= n\left[(n-1)\sum_{r=1}^{n}{}^{n-2}C_{r-2}+2^{n-1}\right]$$

$$= n\left[(n-1)\cdot 2^{n-2}+2^{n-1}\right] = n(n+1)2^{n-2}$$

For $n=1, a=2, b=1, c=1$ and for $n=2, a=4, b=4, c=6$ $\Big\}$ $a=b+c$, $a+b=c+2$

(B) For $\lambda=1$, $a=\sum_{r=0}^{n}(-1)^r\cdot C_r=0$

For $\lambda=r$,

$$b=\sum_{r=0}^{n}(-1)^r\cdot r\cdot C_r=\sum_{r=0}^{n}(-1)^r\cdot r\cdot\frac{n}{r}\cdot{}^{n-1}C_{r-1}$$

$$= n\sum_{r=1}^{n}(-1)^r\cdot{}^{n-1}C_{r-1} = n(1-1)^{n-1}=0$$

and for $\lambda=r^2, c=\sum_{r=0}^{n}(-1)^r\cdot r^2\cdot C_r$

$$=\sum_{r=0}^{n}(-1)^r\cdot r^2\cdot\frac{n}{r}\cdot{}^{n-1}C_{r-1}$$

$$= n\sum_{r=0}^{n}(-1)^r\cdot r\cdot{}^{n-1}C_{r-1}$$

$$= n\sum_{r=0}^{n}(-1)^r\{(r-1)+1\}^{n-1}C_{r-1}$$

$$= n\sum_{r=0}^{n}(-1)^r(r-1)^{n-1}C_{r-1}+n\sum_{r=0}^{n}(-1)^r\cdot{}^{n-1}C_{r-1}$$

$$= 0+0=0$$

$\therefore a=b=c=0 \Rightarrow a=b+c$

$\Rightarrow a^3+b^3+c^3=3abc \Rightarrow a+c=4b$

(C) For $\lambda=1$, $a=\sum_{r=0}^{n}\frac{C_r}{(r+1)}=\frac{1}{(n+1)}\sum_{r=0}^{n}\left(\frac{n+1}{r+1}\right)\cdot{}^nC_r$

$$=\frac{1}{(n+1)}\sum_{r=0}^{n}{}^{n+1}C_{r+1}=\frac{1}{n+1}(2^{n+1}-1)$$

$$=\frac{2^{n+1}-1}{n+1}$$

For $\lambda=r, b=\sum_{r=0}^{n}\frac{r\cdot C_r}{(r+1)}=\sum_{r=0}^{n}\left(1-\frac{1}{r+1}\right)C_r$

$$=2^n-\left(\frac{2^{n+1}-1}{n+1}\right)=\frac{(n-1)2^n+1}{n+1}$$

For $\lambda=r^2, c=\sum_{r=0}^{n}\frac{r^2\cdot C_r}{(r+1)}=\sum_{r=0}^{n}\left((r-1)+\frac{1}{r+1}\right)C_r$

$$=\sum_{r=0}^{n}r\cdot C_r-\sum_{r=0}^{n}C_r+\sum_{r=0}^{n}\frac{C_r}{r+1}$$

$$= n\cdot 2^{n-1}-2^n+\frac{2^{n+1}-1}{n+1}$$

$$=\frac{(n^2-n+2)2^{n-1}-1}{(n+1)}$$

For $n=1, a=\frac{3}{2}, b=\frac{1}{2}, c=\frac{1}{2}$ and for $n=2, a=\frac{7}{3}, b=\frac{5}{3}, c=\frac{7}{3}$; $\Big\}$ $a+c=4b$, $b^{c-a}+(c-a)^b=1$

JEE Type Solved Examples :
Statement I and II Type Questions

■ **Directions** Example numbers 26 and 27 are Assertion-Reason type examples. Each of these examples contains two statements:
Statement-1 (Assertion) and **Statement-2** (Reason)
Each of these examples also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement-1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

● ***Ex. 26*** **Statement-1** (7^9+9^7) *is divisible by 16*

Statement-2 (x^y+y^x) *is divisible by* $(x+y), \forall\, x, y$.

Sol. (c) $7^9+9^7=(8-1)^9+(8+1)^7$

$$=(8^9-{}^9C_1\cdot 8^8+{}^9C_2\cdot 8^7-{}^9C_3\cdot 8^6+\ldots+{}^9C_8\cdot 8-1)$$
$$+(8^7+{}^7C_1\cdot 8^6+{}^7C_2\cdot 8^5+\ldots+{}^7C_6\cdot 8+1)$$
$$=8^9-9\cdot 8^8+8^7\cdot({}^9C_2+1)+8^6(-{}^9C_3+7)$$
$$+8^5({}^9C_4+{}^7C_2)+\ldots+8({}^9C_8+{}^7C_6)$$
$$=64\lambda \qquad [\lambda \text{ is an integer}]$$

$\therefore 7^9+9^7$ is divisible by 16.

$\therefore$ Statement-1 is true. Statement-2 is false.

● ***Ex. 27.*** **Statement-1** *Number of distinct terms in the sum of expansion* $(1+ax)^{10}+(1-ax)^{10}$ *is 22.*

Statement-2 *Number of terms in the expansion of* $(1+x)^n$ *is* $n+1, \forall\, n\in N$.

Sol. (d) $\because (1+ax)^{10}+(1-ax)^{10}=2\{1+{}^{10}C_2(ax)^2+{}^{10}C_4(ax)^4+{}^{10}C_6(ax)^6+{}^{10}C_8(ax)^8+{}^{10}C_{10}(ax)^{10}\}$

$\therefore$ Number of distinct terms = 6

$\Rightarrow$ Statement-1 is false but Statement-2 is obviously true.

Subjective Type Examples

● **Ex. 28** *Find the coefficient independent of x in the expansion of* $(1+x+2x^3)\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$.

Sol. $(r+1)$ th term in the expansion of $\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$

i.e., $$T_{r+1}={}^9C_r\left(\frac{3}{2}x^2\right)^{9-r}\left(-\frac{1}{3x}\right)^r$$

$$={}^9C_r\left(\frac{3}{2}\right)^{9-r}\cdot x^{18-2r}\cdot\left(-\frac{1}{3}\right)^r\cdot x^{-r}$$

$$={}^9C_r\left(\frac{3}{2}\right)^{9-r}\cdot\left(-\frac{1}{3}\right)^r\cdot x^{18-3r}$$

Hence, general term in the expansion of $(1+x+2x^3)\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9={}^9C_r\left(\frac{3}{2}\right)^{9-r}\left(-\frac{1}{3}\right)^r\cdot x^{18-3r}$

$$+{}^9C_r\left(\frac{3}{2}\right)^{9-r}\left(-\frac{1}{3}\right)^r\cdot x^{19-3r}$$

$$+2\,{}^9C_r\left(\frac{3}{2}\right)^{9-r}\left(-\frac{1}{3}\right)^r\cdot x^{21-3r}$$

For independent term, putting $18-3r=0, 19-3r=0$, $21-3r=0$ respectively, we get

$r=6, r=19/3$ [impossible] $r=7$, second term do not given the independent term.

Hence, coefficient independent of x

$$={}^9C_6\cdot\left(\frac{3}{2}\right)^3\cdot\left(-\frac{1}{3}\right)^6+0+2\cdot{}^9C_7\cdot\left(\frac{3}{2}\right)^2\left(-\frac{1}{3}\right)^7$$

$$={}^9C_3\cdot\frac{27}{8.729}-2\cdot{}^9C_2\cdot\frac{9}{4}\cdot\frac{1}{2187}=\frac{7}{18}-\frac{2}{27}=\frac{17}{54}$$

● **Ex. 29** *If* $(1+x)^n=C_0+C_1x+C_2x^2+\ldots+C_nx^n$, *show that* $\sum_{r=0}^{n}C_r^3$ *is equal to the coefficient of* x^ny^n *in the expansion of* $\{(1+x)(1+y)(x+y)\}^n$.

Sol. $(1+x)^n(y+1)^n(x+y)^n=\sum_{r=0}^{n}C_rx^r$

$$\sum_{s=0}^{n}C_sy^{n-s}\sum_{t=0}^{n}C_tx^{n-t}y^t \qquad \ldots(i)$$

Since, C_0^3 is the coefficient of $x^0y^{n-0}x^{n-0}y^0$

i.e., $x^ny^n\ (r=s=t=0)$

Now, C_1^3 is the coefficient of $x^1y^{n-1}x^{n-1}y$

i.e., $x^ny^n\ (r=s=t=1)$

And C_k^3 is the coefficient of $x^ky^{n-k}x^{n-k}y^k$

i.e., $x^ny^n\ (r=s=t=k)$

Hence, the coefficient of x^ny^n in $(1+x)^n(y+1)^n(x+y)^n$

$$=C_0^3+C_1^3+C_2^3+\ldots+C_n^3=\sum_{r=0}^{n}C_r^3$$

● **Ex. 30** *Let* $(1+x^2)^2(1+x)^n=\sum_{k=0}^{n+4}a_kx^k$. *If* a_1, a_2 *and* a_3 *are in AP, find* n.

Sol. We have,

$$(1+x^2)^2(1+x)^n=(1+2x^2+x^4)\times({}^nC_0+{}^nC_1x+{}^nC_2x^2+{}^nC_3x^3+\ldots)$$

$$=a_0+a_1x+a_2x^2+a_3x^3+\ldots \qquad \text{[say]}$$

Now, comparing the coefficients of x, x^2 and x^3, we get

$$a_1={}^nC_1, a_2=2\cdot{}^nC_0+{}^nC_2, a_3=2\cdot{}^nC_1+{}^nC_3 \qquad \ldots(i)$$

In a_1, $n\geq1$, in a_2, $n\geq2$ and in a_3, $n\geq3$

$\therefore$ $n\geq3$ $\ldots(ii)$

From Eq. (i),

$$a_1=n, a_2=2+\frac{n(n-1)}{1\cdot2}=\frac{n^2-n+4}{2}$$

and $$a_3=2n+\frac{n(n-1)(n-2)}{1\cdot2\cdot3}=\frac{n^3-3n^2+14n}{6}$$

Since, a_1, a_2, a_3 are in AP.

Therefore, $2a_2=a_1+a_3$

$$\Rightarrow n^2-n+4=n+\frac{n^3-3n^2+14n}{6}$$

$$\Rightarrow n^3-9n^2+26n-24=0$$

or $(n-2)(n-3)(n-4)=0$

$\therefore$ $n=2,3,4$

Hence, $n=3,4$ [from Eq. (ii)]

● **Ex. 31** *If* $(1-x^3)^n=\sum_{r=0}^{n}a_rx^r(1-x)^{3n-2r}$, *find* a_r, *where* $n\in N$.

Sol. We have, $(1-x^3)^n=\sum_{r=0}^{n}a_rx^r(1-x)^{3n-2r}$

$$\Rightarrow(1-x)^n(1+x+x^2)^n=\sum_{r=0}^{n}\frac{a_r\cdot x^r(1-x)^{3n}}{(1-x)^{2r}}$$

$$\Rightarrow\frac{(1-x)^n(1+x+x^2)^n}{(1-x)^{3n}}=\sum_{r=0}^{n}\frac{a_r\cdot x^r}{(1-x)^{2r}}$$

$$\Rightarrow\left[\frac{1+x+x^2}{(1-x)^2}\right]^n=\sum_{r=0}^{n}a_r\cdot\frac{x^r}{(1-x)^{2r}}$$

$$\Rightarrow \quad \left[\frac{(1-x)^2+3x}{(1-x)^2}\right]^n = \sum_{r=0}^{n} a_r \left[\frac{x}{(1-x)^2}\right]^r$$

$$\Rightarrow \quad \left[1+3\left(\frac{x}{(1-x)^2}\right)\right]^n = \sum_{r=0}^{n} a_r \left[\frac{x}{(1-x)^2}\right]^r \quad \ldots(i)$$

Let $A = \dfrac{x}{(1-x)^2}$

Then, Eq. (i) becomes $(1+3A)^n = \sum_{r=0}^{n} a_r A^r$

On comparing the coefficient of A^r, we get

$${}^nC_r \cdot 3^r = a_r$$

Hence, $a_r = {}^nC_r \cdot 3^r$

● ***Ex. 32*** *If $a_0, a_1, a_2, \ldots, a_{2n}$ are the coefficients in the expansion of $(1+x+x^2)^n$ in ascending powers of x, show that $a_0^2 - a_1^2 - a_2^2 - \ldots + a_{2n}^2 = a_n$.*

Sol. We have, $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + a_{2n}x^{2n}$ ...(i)

Replacing x by $\left(-\dfrac{1}{x}\right)$ in Eq. (i), we get

$$\left(1-\frac{1}{x}+\frac{1}{x^2}\right)^n = a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} - \ldots + \frac{a_{2n}}{x^{2n}} \quad \ldots(ii)$$

On multiplying Eqs. (i) and (ii), we get

$$(1+x+x^2)^n \times \left(1-\frac{1}{x}+\frac{1}{x^2}\right)^n = (a_0 + a_1x + a_2x^2 + \ldots + a_{2n}x^{2n}) \times \left(a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} - \ldots + \frac{a_{2n}}{x^{2n}}\right)$$

$$\Rightarrow \frac{(1+x^2+x^4)^n}{x^{2n}} = (a_0 + a_1x + a_2x^2 + \ldots + a_{2n}x^{2n}) \times \left(a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} - \ldots + \frac{a_{2n}}{x^{2n}}\right) \ldots(iii)$$

Constant term in RHS $= a_0^2 - a_1^2 + a_2^2 - \ldots + a_{2n}^2$

Now, constant term in $\dfrac{(1+x^2+x^4)^n}{x^{2n}}$ = Coefficient of x^{2n} in $(1+x^2+x^4)^n = a_n$ [replacing x by x^2 in Eq. (i)]

But Eq. (iii) is an identity, therefore, the constant term in RHS = constant term in LHS.

$$a_0^2 - a_1^2 + a_2^2 - \ldots + a_{2n}^2 = a_n$$

● ***Ex. 33*** *Show that no three consecutive binomial coefficients can be in (i) GP and (ii) HP.*

Sol. (i) Suppose that the r th, $(r+1)$th and $(r+2)$th coefficients of $(1+x)^n$ are in GP.

i.e., ${}^nC_{r-1}$, ${}^nC_r$, ${}^nC_{r+1}$ are in GP.

Then, $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{{}^nC_{r+1}}{{}^nC_r}$

$$\Rightarrow \quad \frac{n-r+1}{r} = \frac{n-r}{r+1} \quad \left[\because \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}\right]$$

$$\Rightarrow \quad (n-r+1)(r+1) = r(n-r)$$

$$\Rightarrow \quad nr + n - r^2 - r + r + 1 = nr - r^2$$

$$\Rightarrow \quad n + 1 = 0$$

$$\Rightarrow \quad n = -1$$

which is not possible, since n is a positive integer.

(ii) Suppose that rth, $(r+1)$th and $(r+2)$th coefficients of $(1+x)^n$ are in HP,

i.e. ${}^nC_{r-1}$, ${}^nC_r$, ${}^nC_{r+1}$ are in HP.

Then, $\dfrac{2}{{}^nC_r} = \dfrac{1}{{}^nC_{r-1}} + \dfrac{1}{{}^nC_{r+1}}$

$$\Rightarrow \quad 2 = \frac{{}^nC_r}{{}^nC_{r-1}} + \frac{{}^nC_r}{{}^nC_{r+1}} \quad \left[\because \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}\right]$$

$$\Rightarrow \quad 2 = \frac{n-r+1}{r} + \frac{r+1}{n-r}$$

$$\Rightarrow \quad 2r(n-r) = (n-r+1)(n-r) + r(r+1)$$

$$\Rightarrow \quad 2nr - 2r^2 = n^2 - nr - nr + r^2 + n - r + r^2 + r$$

$$\Rightarrow \quad n^2 - 4nr + 4r^2 + n = 0 \quad \Rightarrow \quad (n-2r)^2 + n = 0$$

which is not possible, as $(n-2r)^2 \geq 0$ and n is a positive integer.

● ***Ex. 34*** *Evaluate $\sum_{i=0}^{n}\sum_{j=1}^{n} {}^nC_j \cdot {}^jC_i, i \leq j$.*

Sol. We have, $\sum_{i=0}^{n}\sum_{j=1}^{n} {}^nC_j \cdot {}^jC_i$

$$= {}^nC_1({}^1C_0 + {}^1C_1) + {}^nC_2({}^2C_0 + {}^2C_1 + {}^2C_2) + {}^nC_3({}^3C_0 + {}^3C_1 + {}^3C_2 + {}^3C_3) + {}^nC_4({}^4C_0 + {}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4) + \ldots + {}^nC_n({}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n)$$

$$= {}^nC_1(2) + {}^nC_2(2)^2 + {}^nC_3(2)^3 + \ldots + {}^nC_n(2)^n = (1+2)^n - 1$$

$$= 3^n - 1$$

● ***Ex. 35*** *Find the remainder, when 27^{40} is divided by 12.*

Sol. We have, $27^{40} = (3^3)^{40} = 3^{120} = 3\cdot(3)^{119} = 3\cdot(4-1)^{119}$

$= 3(4n-1)$, where n is some integer

$= 12n - 3 = 12n - 12 + 9 = 12(n-1) + 9$

$= 12m + 9$, where m is some integer.

$$\therefore \quad \frac{27^{40}}{12} = m + \frac{9}{12}$$

Hence, the remainder is 9.

● ***Ex. 36*** *Show that* $[(\sqrt{3}+1)^{2n}]+1$ *is divisible by* 2^{n+1}, $\forall\, n \in N$, *where* $[\cdot]$ *denotes the greatest integer function.*

Sol. Let $x=(\sqrt{3}+1)^{2n}=[x]+f$ ...(i)

where, $0 \le f < 1$

and $(\sqrt{3}-1)^{2n}=f'$ ...(ii)

where, $0 < f' < 1$

On adding Eqs. (i) and (ii), we get

$$[x]+f+f'=(\sqrt{3}+1)^{2n}+(\sqrt{3}-1)^{2n}$$
$$=(4+2\sqrt{3})^{n}+(4-2\sqrt{3})^{n}$$
$$=2^{n}\{(2+\sqrt{3})^{n}+(2-\sqrt{3})^{n}\}$$
$$=2^{n}\cdot 2\{{}^{n}C_0(2)^{n}+{}^{n}C_2(2)^{n-2}(\sqrt{3})^{2}+{}^{n}C_4(2)^{n-4}(\sqrt{3})^{4}+\ldots\}$$

$\therefore\ [x]+f+f'=2^{n+1}k$, where k is an integer. ...(iii)

Hence, $(f+f')$ is an integer.

i.e., $f+f'=1$ $\quad[\because 0<(f+f')<2]$

From Eq. (iii), we get

$$[x]+1=2^{n+1}k$$

$\Rightarrow\ [(\sqrt{3}+1)^{2n}]+1=2^{n+1}k$ [from Eq. (i)]

which shows that $[(\sqrt{3}+1)^{2n}]+1$ divisible by 2^{n+1}, $\forall\, n \in N$.

● ***Ex. 37*** *Find the number of rational terms and also find the sum of rational terms in* $(\sqrt{2}+\sqrt[3]{3}+\sqrt[6]{5})^{10}$.

Sol. We have, $(\sqrt{2}+\sqrt[3]{3}+\sqrt[6]{5})^{10}=(2^{1/2}+3^{1/3}+5^{1/6})^{10}$

$$=\sum_{\alpha+\beta+\gamma=10}\frac{10!}{\alpha!\,\beta!\,\gamma!}2^{\alpha/2}\cdot 3^{\beta/3}\cdot 5^{\gamma/6}$$

For rational terms,

$\alpha=0, 2, 4, 6, 8, 10,\ \beta=0, 3, 6, 9, \gamma=0, 6$

Since, $0 \le \alpha, \beta, \gamma \le 10$.

$\therefore$ Possible triplets are $(4, 0, 6), (4, 6, 0), (10, 0, 0)$.

There exists three rational terms.

$\therefore$ Required sum

$$=\frac{10!}{4!\,0!\,6!}2^{2}\cdot 5+\frac{10!}{4!\,6!\,0!}2^{2}\cdot 3^{2}+\frac{10!}{10!\,0!\,0!}2^{5}$$

$$=4200+7560+32=11792$$

● ***Ex. 38*** *Find the remainder, when* $(1690^{2608}+2608^{1690})$ *is divided by* 7.

Sol. We have, $1690^{2608}+2608^{1690}=(1690^{2608}-3^{2608})+(2608^{1690}-4^{1690})+(3^{2608}+4^{1690})$

The number $(1690^{2608}-3^{2608})$ is divisible by

$1690-3=1687=7\times 241$ which is divisible by 7, the difference $(2608^{1690}-4^{1690})$ is also divisible by 7, since it is divisible by $2608-4=2604=7\times 372$.

As to sum $3^{2608}+4^{1690}$, it can be rewritten as

$$3\cdot(3^{3})^{869}+4\cdot(4^{3})^{563}$$
$$=3(28-1)^{869}+4(63+1)^{563}$$
$$=3(7m-1)+4(7n+1)$$

[where, m and n are some positive integers]

where p is some positive integer.

Hence, the remainder is 1.

● ***Ex. 39*** *If* $C_0, C_1, C_2, \ldots, C_n$ *are the binomial coefficients in the expansion of* $(1+x)^{n}$, *prove that*

$$(C_0+2C_1+C_2)(C_1+2C_2+C_3)\ldots(C_{n-1}+2C_n+C_{n+1})=\frac{(n+2)^{n}}{(n+1)!}\prod_{r=1}^{n}(C_{r-1}+C_r).$$

Sol. LHS $=(C_0+2C_1+C_2)(C_1+2C_2+C_3)\ldots(C_{n-1}+2C_n+C_{n+1})$

$$=\prod_{r=1}^{n}({}^{n}C_{r-1}+2\,{}^{n}C_r+{}^{n}C_{r+1})$$
$$=\prod_{r=1}^{n}\{({}^{n}C_{r-1}+{}^{n}C_r)+({}^{n}C_r+{}^{n}C_{r+1})\}$$
$$=\prod_{r=1}^{n}({}^{n+1}C_r+{}^{n+1}C_{r+1})$$ [by Pascal's rule]

$$=\prod_{r=1}^{n}({}^{n+2}C_{r+1})=\prod_{r=1}^{n}\left(\frac{n+2}{r+1}\right){}^{n+1}C_r\left[\because {}^{n}C_r=\frac{n}{r}\cdot{}^{n-1}C_{r-1}\right]$$
$$=\prod_{r=1}^{n}\left(\frac{n+2}{r+1}\right)({}^{n}C_{r-1}+{}^{n}C_r)=\prod_{r=1}^{n}\left(\frac{n+2}{r+1}\right)\prod_{r=1}^{n}(C_{r-1}+C_r)$$
$$=\frac{(n+2)}{2}\cdot\frac{(n+2)}{3}\cdot\frac{(n+2)}{4}\ldots\frac{(n+2)}{(n+1)}\prod_{r=1}^{n}(C_{r-1}+C_r)$$
$$=\frac{(n+2)^{n}}{(n+1)!}\prod_{r=1}^{n}(C_{r-1}+C_r)=\text{RHS}$$

● ***Ex. 40*** *If* $\sum_{r=0}^{2n}a_r(x-2)^{r}=\sum_{r=0}^{2n}b_r(x-3)^{r}$ *and* $a_k=1$, $\forall\, k \ge n$, *show that* $b_n={}^{2n+1}C_{n+1}$.

Sol. $\because\ \sum_{r=0}^{2n}a_r(x-2)^{r}=\sum_{r=0}^{2n}b_r(x-3)^{r}$

Let $y=x-3 \Rightarrow y+1=x-2$

So, the given expression reduces to

$$\sum_{r=0}^{2n}a_r(1+y)^{r}=\sum_{r=0}^{2n}b_r y^{r}$$

$$\Rightarrow \quad a_0 + a_1(1+y) + a_2(1+y)^2 + \ldots + a_{2n}(1+y)^{2n}$$
$$= b_0 + b_1 y + \ldots + b_{2n} y^{2n}$$

Using $a_k = 1, \forall\, k \geq n$, we get

$$a_0 + a_1(1+y) + a_2(1+y)^2 + \ldots + a_{n-1}(1+y)^{n-1}$$
$$+ (1+y)^n + (1+y)^{n+1} + \ldots + (1+y)^{2n}$$
$$= b_0 + b_1 y + \ldots + b_n y^n + \ldots + b_{2n} y^{2n}$$

On comparing the coefficient of y^n on both sides, we get

$${}^nC_n + {}^{n+1}C_n + {}^{n+2}C_n + \ldots + {}^{2n}C_n = b_n$$

$$\Rightarrow \quad {}^{n+1}C_{n+1} + {}^{n+1}C_n + {}^{n+2}C_n + \ldots + {}^{2n}C_n = b_n$$

$$[\because\ {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r]$$

$$\Rightarrow {}^{n+2}C_{n+1} + {}^{n+2}C_n + \ldots + {}^{2n}C_n = b_n$$

[adding first two terms]

If we combine terms on LHS finally, we get

$${}^{2n+1}C_{n+1} = b_n$$

● ***Ex. 41*** *(i) If n is an odd natural number, prove that*

$$\sum_{r=0}^{n} \frac{(-1)^r}{{}^nC_r} = 0.$$

(ii) If n is an even natural number, find the value of

$$\sum_{r=0}^{n} \frac{(-1)^r}{{}^nC_r}.$$

Sol. (i) We have, $\sum_{r=0}^{n} \frac{(-1)^r}{{}^nC_r} = \sum_{r=0}^{\frac{n+1}{2}} \left[\frac{(-1)^r}{{}^nC_r} + \frac{(-1)^{n-r}}{{}^nC_{n-r}} \right]$

$$= \sum_{r=0}^{\frac{n+1}{2}} (-1)^r \left[\frac{1}{{}^nC_r} + \frac{(-1)^n}{{}^nC_{n-r}} \right] = \sum_{r=0}^{\frac{n+1}{2}} (-1)^r \left[\frac{1}{{}^nC_r} - \frac{1}{{}^nC_r} \right]$$

$= 0$ $[\because n$ is odd and ${}^nC_r = {}^nC_{n-r}]$

(ii) We have,

$$\sum_{r=0}^{n} \frac{(-1)^r}{{}^nC_r} = \sum_{r=0}^{\frac{n}{2}-1} \left[\frac{(-1)^r}{{}^nC_r} + \frac{(-1)^{n-r}}{{}^nC_{n-r}} \right] + \frac{(-1)^{n/2}}{{}^nC_{n/2}}$$

$$= \sum_{r=0}^{\frac{n}{2}-1} (-1)^r \left[\frac{1}{{}^nC_r} + \frac{(-1)^n}{{}^nC_r} \right] + \frac{(-1)^{n/2}}{{}^nC_{n/2}}$$

$$= \sum_{r=0}^{\frac{n}{2}-1} (-1)^r \left[\frac{1}{{}^nC_r} + \frac{1}{{}^nC_r} \right] + \frac{(-1)^{n/2}}{{}^nC_{n/2}}$$

$$= \left[\sum_{r=0}^{\frac{n}{2}-1} (-1)^r \cdot \frac{2}{{}^nC_r} \right] + \frac{(-1)^{n/2}}{{}^nC_{n/2}}$$

● ***Ex. 42*** *If* $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n$, *show that*

$$C_1 - \frac{C_2}{2} + \frac{C_3}{3} - \ldots (-1)^{n-1} \frac{C_n}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}.$$

Sol. We know that,

$$(1-x)^n = C_0 - C_1 x + C_2 x^2 - \ldots + (-1)^n C_n x^n$$

or $C_0 - (1-x)^n = C_1 x - C_2 x^2 + C_3 x^3 - \ldots + (-1)^{n-1} C_n x^n$

$$\Rightarrow 1 - (1-x)^n = C_1 x - C_2 x^2 + C_3 x^3 - \ldots + (-1)^{n-1} C_n x^n$$

Dividing in each side by x, then

$$\frac{1-(1-x)^n}{x} = C_1 - C_2 x + C_3 x^2 - \ldots + (-1)^{n-1} C_n x^{n-1}$$

On integrating within limits 0 to 1, we have

$$\int_0^1 \frac{1-(1-x)^n}{x}\, dx = \int_0^1 (C_1 - C_2 x + C_3 x^2 - \ldots + (-1)^{n-1} C_n x^{n-1})\, dx$$

$$= \left[C_1 x - \frac{C_2 x^2}{2} + \frac{C_3 x^3}{3} - \ldots + (-1)^{n-1} C_n \frac{x^n}{n} \right]_0^1$$

$$\int_0^1 \frac{1-(1-x)^n}{x}\, dx = C_1 - \frac{C_2}{2} + \frac{C_3}{3} - \ldots + \frac{(-1)^{n-1}}{n} C_n$$

Putting $1 - x = t$ in integral,

$$\Rightarrow \quad dx = -dt$$

when $x \to 1, t \to 0$ and when $x \to 0, t \to 1$

$$\therefore \int_0^1 \frac{(1-t^n)}{(1-t)}(-dt) = C_1 - \frac{C_2}{2} + \frac{C_3}{3} - \ldots + (-1)^{n-1} \frac{C_n}{n}$$

$$\Rightarrow \quad \int_0^1 \frac{(1-t^n)}{(1-t)}\, dt = C_1 - \frac{C_2}{2} + \frac{C_3}{3} - \ldots + (-1)^{n-1} \frac{C_n}{n}$$

$$\Rightarrow \int_0^1 (1 + t + t^2 + \ldots + t^{n-1})\, dt = C_1 - \frac{C_2}{2} + \frac{C_3}{3} - \ldots + (-1)^{n-1} \frac{C_n}{n}$$

$$\Rightarrow \left[t + \frac{t^2}{2} + \frac{t^3}{3} + \ldots + \frac{t^n}{n} \right]_0^1 = C_1 - \frac{C_2}{2} + \frac{C_3}{3} - \ldots + (-1)^{n-1} \frac{C_n}{n}$$

$$\Rightarrow 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} = C_1 - \frac{C_2}{2} + \frac{C_3}{3} - \ldots + (-1)^{n-1} \frac{C_n}{n}$$

Hence, $C_1 - \frac{C_2}{2} + \frac{C_3}{3} - \ldots + (-1)^{n-1} \frac{C_n}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$

● ***Ex. 43*** *If* $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n$, *find the sum of the seriesd*

$$\frac{C_0}{2} - \frac{C_1}{6} + \frac{C_2}{10} - \frac{C_3}{14} + \ldots + (-1)^n \frac{C_n}{4n+2}.$$

Sol. Let $S = \frac{C_0}{2} - \frac{C_1}{6} + \frac{C_2}{10} - \frac{C_3}{14} + \ldots + (-1)^n \frac{C_n}{4n+2}$

$$= \frac{1}{2}\left(\frac{C_0}{1} - \frac{C_1}{3} + \frac{C_2}{5} - \frac{C_3}{7} + \ldots + (-1)^n \frac{C_n}{2n+1}\right) \quad \ldots(i)$$

Consider, $(1-x^2)^n = C_0 - C_1 x^2 + C_2 x^4 - C_3 x^6 + \ldots + (-1)^n C_n x^{2n}$

$$\Rightarrow \int_0^1 (1-x^2)^n \, dx = \int_0^1 (C_0 - C_1 x^2 + C_2 x^4 - C_3 x^6 + \ldots + (-1)^n C_n x^{2n}) \, dx$$

$$\Rightarrow \int_0^1 (1-x^2)^n \, dx = \left[C_0 x - \frac{C_1 x^3}{3} + \frac{C_2 x^5}{5} - \frac{C_3 x^7}{7} + \ldots + (-1)^n \frac{C_n x^{2n+1}}{2n+1}\right]$$

$$\Rightarrow \int_0^1 (1-x^2)^n \, dx = C_0 - \frac{C_1}{3} + \frac{C_2}{5} - \frac{C_3}{7} + \ldots + (-1)^n \frac{C_n}{2n+1}$$

From Eq. (i),

$$\int_0^1 (1-x^2)^n \, dx = 2S \quad \text{or} \quad S = \frac{1}{2}\int_0^1 (1-x^2)^n \, dx$$

Put $x = \sin\theta$ i.e., $dx = \cos\theta \, d\theta$

$$S = \frac{1}{2}\int_0^{\pi/2} \cos^{2n+1}\theta \, d\theta$$

By using Walli's formula,

$$S = \frac{1}{2} \cdot \frac{2n(2n-2)(2n-4)\ldots 4 \cdot 2}{(2n+1)(2n-1)(2n-3)\ldots 3 \cdot 1} \cdot 1$$

$$= \frac{1}{2} \cdot \frac{\{2n(2n-2)(2n-4)\ldots 4 \cdot 2)\}^2}{(2n+1)!}$$

$$= \frac{1}{2} \cdot \frac{(2^n n!)^2}{(2n+1)!} = 2^{2n-1} \frac{(n!)^2}{(2n+1)!}$$

● ***Ex. 44*** *If* $(1+x)^n = \sum_{r=0}^{n} C_r x^r$, *then prove that*

$$\sum\sum_{0 \le i < j \le n} \left(\frac{i}{C_i} + \frac{j}{C_j}\right) = \frac{n^2}{2} \sum_{r=0}^{n} \frac{1}{C_r}.$$

Sol. Let $S = \sum\sum_{0 \le i < j \le n} \left(\frac{i}{C_i} + \frac{j}{C_j}\right)$ ...(i)

Replacing i by $n-i$ and j by $n-j$, we get

$$S = \sum\sum_{0 \le i < j \le n} \left(\frac{n-i}{C_{n-i}} + \frac{n-j}{C_{n-j}}\right) = \sum\sum_{0 \le i < j \le n} \left(\frac{n-i}{C_i} + \frac{n-j}{C_j}\right)$$

$[\because C_r = C_{n-r}]$ ...(ii)

On adding Eqs. (i) and (ii), we get

$$2S = n \sum\sum_{0 \le i < j \le n} \left(\frac{1}{C_i} + \frac{1}{C_j}\right)$$

$$\therefore \quad S = \frac{n}{2} \sum\sum_{0 \le i < j \le n} \left(\frac{1}{C_i} + \frac{1}{C_j}\right) = \frac{n}{2}\left(\sum_{r=0}^{n-1} \frac{n-r}{C_r} + \sum_{r=1}^{n} \frac{r}{C_r}\right)$$

$$= \frac{n}{2}\left(\sum_{r=0}^{n} \frac{n-r}{C_r} + \sum_{r=0}^{n} \frac{r}{C_r}\right) = \frac{n}{2}\left(\sum_{r=0}^{n} \frac{n}{C_r}\right) = \frac{n^2}{2} \sum_{r=0}^{n} \frac{1}{C_r}$$

● ***Ex. 45*** *If* $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n$, *show that*

$$\sum_{r=0}^{n} \frac{C_r \, 3^{r+4}}{(r+1)(r+2)(r+3)(r+4)} = \frac{1}{(n+1)(n+2)(n+3)(n+4)}\left(4^{n+4} - \sum_{t=0}^{3} {}^{n+4}C_t \, 3^t\right).$$

Sol. LHS $= \sum_{r=0}^{n} \frac{C_r \cdot 3^{r+4}}{(r+1)(r+2)(r+3)(r+4)}$

$$= \sum_{r=0}^{n} \frac{C_r \cdot 3^{r+4}}{\frac{(r+1)(r+2)(r+3)(r+4)}{4!} 4!}$$

$$= \sum_{r=0}^{n} \frac{C_r \cdot 3^{r+4}}{{}^{r+4}C_4 \cdot 4!} = \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} \cdot \frac{3^{r+4}}{\frac{(r+4)!}{4!r!} \cdot 4!}$$

$$= \sum_{r=0}^{n} \frac{n! \cdot 3^{r+4}}{(n-r)! \cdot (n+4)!}$$

$$= \sum_{r=0}^{n} \frac{n! \cdot 3^{r+4}}{(n-r)! \cdot (r+4)!} \cdot \frac{(n+1)(n+2)(n+3)(n+4)}{(n+1)(n+2)(n+3)(n+4)}$$

$$= \sum_{r=0}^{n} \frac{(n+4)! \, 3^{r+4}}{(n-r)!(r+4)!(n+1)(n+2)(n+3)(n+4)}$$

$$= \frac{1}{(n+1)(n+2)(n+3)(n+4)}\left[\sum_{r=0}^{n} \frac{(n+4)! \cdot 3^{r+4}}{(n-r)! \cdot (r+4)!}\right]$$

$$= \frac{1}{(n+1)(n+2)(n+3)(n+4)}\left[\sum_{r=0}^{n} {}^{n+4}C_{r+4} \, 3^{r+4}\right]$$

$$= \frac{1}{(n+1)(n+2)(n+3)(n+4)}\left\{\sum_{t=4}^{n+4} {}^{n+4}C_t \, 3^t\right\} \quad [\text{put } r+4 = t]$$

$$= \frac{1}{(n+1)(n+2)(n+3)(n+4)}\left\{\sum_{t=0}^{n+4} {}^{n+4}C_t \, 3^t - \sum_{t=0}^{3} {}^{n+4}C_t \, 3^t\right\}$$

$$= \frac{1}{(n+1)(n+2)(n+3)(n+4)}\left\{(1+3)^{n+4} - \sum_{t=0}^{3} {}^{n+4}C_t \, 3^t\right\}$$

$$= \frac{1}{(n+1)(n+2)(n+3)(n+4)}\left\{4^{n+4} - \sum_{t=0}^{3} {}^{n+4}C_t \, 3^t\right\}$$

= RHS

● ***Ex. 46*** *Prove that* $\sum_{k=0}^{9} x^k$ *divides* $\sum_{k=0}^{9} x^{kkkk}$.

Sol. Let $S_1 = \sum_{k=0}^{9} x^{kkkk} = x^0 + x^{1111} + x^{2222} + \ldots + x^{9999}$

and $S_2 = \sum_{k=0}^{9} x^k = x^0 + x^1 + x^2 + \ldots + x^9$

Now, $S_1 - S_2 = \sum_{k=0}^{9} (x^{kkkk} - x^k) = \sum_{k=0}^{9} x^k \left((x^{10})^{kkk} - 1\right)$

$= [(x^{10})^{kkk} - 1] \sum_{k=0}^{9} x^k = \lambda \sum_{k=0}^{9} x^k$

$\Rightarrow \quad S_1 - S_2 = \lambda S_2 \quad \Rightarrow \quad S_1 = (1+\lambda) S_2$

Hence, $\sum_{k=0}^{9} x^{kkkk}$ is divisible by $\sum_{k=0}^{9} x^k$.

● ***Ex. 47*** *Prove that* $\sum_{r=1}^{k} (-3)^{r-1} \cdot {}^{3n}C_{2r-1} = 0$, *where* $k = \frac{3n}{2}$ *and n is an even positive integer.*

Sol. Given, n is an even positive integer.

Let $n = 2m$; $\therefore \; k = 3m, \; m \in N$

$$\text{LHS} = \sum_{r=1}^{k} (-3)^{r-1} \, {}^{3n}C_{2r-1} = \sum_{r=1}^{3m} (-3)^{r-1} \, {}^{6m}C_{2r-1}$$

$$= {}^{6m}C_1 - 3 \cdot {}^{6m}C_3 + 3^2 \cdot {}^{6m}C_5 - \ldots + (-3)^{3m-1} \, {}^{6m}C_{6m-1} \quad \ldots(\text{i})$$

Consider $(1 + i\sqrt{3})^{6m} = {}^{6m}C_0 + {}^{6m}C_1 (i\sqrt{3}) + {}^{6m}C_2 (i\sqrt{3})^2 + {}^{6m}C_3 (i\sqrt{3})^3 + {}^{6m}C_4 (i\sqrt{3})^4 + {}^{6m}C_5 (i\sqrt{3})^5 + \ldots + {}^{6m}C_{6m-1} (i\sqrt{3})^{6m-1} + {}^{6m}C_{6m} (i\sqrt{3})^{6m}$ ...(ii)

Now, $(1 + i\sqrt{3})^{6m} = \left\{(-2)\left(\frac{-1 - i\sqrt{3}}{2}\right)\right\}^{6m} = (-2\omega^2)^{6m}$

$= 2^{6m}$, where ω^2 is cube root of unity.

Then, Eq. (ii) can be written as

$$2^{6m} = \{{}^{6m}C_0 - {}^{6m}C_2 \cdot 3 + {}^{6m}C_4 \cdot 3^2 - \ldots + (-3)^{3m} \cdot {}^{6m}C_{6m}\} + i\sqrt{3}\{{}^{6m}C_1 - {}^{6m}C_3 \cdot 3 + {}^{6m}C_5 \cdot 3^2 - \ldots + (-3)^{3m-1} \cdot {}^{6m}C_{6m-1}\}$$

On comparing the imaginary part on both sides, we get

$$\sqrt{3}({}^{6m}C_1 - 3 \cdot {}^{6m}C_3 + 3^2 \cdot {}^{6m}C_5 - \ldots + (-3)^{3m-1} \cdot {}^{6m}C_{6m-1}) = 0$$

or $${}^{6m}C_1 - 3 \cdot {}^{6m}C_3 + 3^2 \cdot {}^{6m}C_5 - \ldots + (-3)^{3m-1} \cdot {}^{6m}C_{6m-1} = 0$$

$$\Rightarrow \quad \sum_{r=1}^{3m} (-3)^{r-1} \cdot {}^{6m}C_{2r-1} = 0$$

or $\sum_{r=1}^{k} (-3)^{r-1} \cdot {}^{3n}C_{2r-1} = 0$, where $n = 2m$ and $k = 3m$

● ***Ex. 48*** *Prove that*

$${}^nC_3 + {}^nC_7 + {}^nC_{11} + \ldots = \frac{1}{2}\left\{2^{n-1} - 2^{n/2} \sin\frac{n\pi}{4}\right\}$$

Sol. In given series difference in lower suffices is 4.

i.e., $7 - 3 = 11 - 7 = \ldots = 4$

Now, $(1)^{1/4} = (\cos 0 + i \sin 0)^{1/4}$

$= (\cos 2r\pi + i \sin 2r\pi)^{1/4}$

$= \cos\frac{r\pi}{2} + i \sin\frac{r\pi}{2}$, where $r = 0, 1, 2, 3$

Four roots of unity $= 1, i, -1, -i = 1, \alpha, \alpha^2, \alpha^3$ [say]

and $(1 + x)^n = \sum_{r=0}^{n} {}^nC_r \, x^r$

Putting $x = 1, \alpha, \alpha^2, \alpha^3$, we get $2^n = \sum_{r=0}^{n} {}^nC_r$ ...(i)

$(1 + \alpha)^n = \sum_{r=0}^{n} {}^nC_r \alpha^r$ ...(ii)

$(1 + \alpha^2)^n = \sum_{r=0}^{n} {}^nC_r \alpha^{2r}$ ...(iii)

and $(1 + \alpha^3)^n = \sum_{r=0}^{n} {}^nC_r \alpha^{3r}$ ...(iv)

On multiplying Eq. (i) by 1, Eq. (ii) by α, Eq. (iii) by α^2 and Eq. (iv) by α^3 and adding, we get

$$\Rightarrow 2^n + \alpha(1+\alpha)^n + \alpha^2(1+\alpha^2)^n + \alpha^3(1+\alpha^3)^n = \sum_{r=0}^{n} {}^nC_r (1 + \alpha^{r+1} + \alpha^{2r+2} + \alpha^{3r+3}) \quad \ldots(\text{v})$$

For $r = 3, 7, 11, \ldots$ RHS of Eq. (v)

$$= {}^nC_3 (1 + \alpha^4 + \alpha^8 + \alpha^{12}) + {}^nC_7 (1 + \alpha^8 + \alpha^{16} + \alpha^{24}) + {}^nC_{11} (1 + \alpha^{12} + \alpha^{24} + \alpha^{36}) + \ldots$$

$$= 4({}^nC_3 + {}^nC_7 + {}^nC_{11} + \ldots) \qquad [\because \alpha^4 = 1]$$

and LHS of Eq. (v)

$$= 2^n + i(1+i)^n + i^2(1+i^2)^n + i^3(1+i^3)^n$$

$$= 2^n + i(1+i)^n + 0 - i(1-i)^n$$

$$= 2^n + i\{(1+i)^n - (1-i)^n\}$$

Since, $\left[(1+i)^n = \left[\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\right]^n\right]$

$$= 2^n + i2^{n/2} \cdot 2i \sin\frac{n\pi}{4} = 2^{n/2}\left\{\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right\}^n$$

$$= 2^n - 2^{n/2} \cdot 2 \sin\frac{n\pi}{4} = 2^{n/2}\left\{\cos\frac{n\pi}{4} + i \sin\frac{n\pi}{4}\right\}$$

Hence, $4({}^nC_3 + {}^nC_7 + {}^nC_{11} + \ldots) = 2\left(2^{n-1} - 2^{n/2} \sin\frac{n\pi}{4}\right)$

$$\Rightarrow \quad {}^nC_3 + {}^nC_7 + {}^nC_{11} + \ldots = \frac{1}{2}\left(2^{n-1} - 2^{n/2} \sin\frac{n\pi}{4}\right)$$

● **Ex. 49** *Evaluate* $\sum_{i=0}^{n-1}\sum_{j=1+i}^{n+1} {}^nC_i \; {}^{n+1}C_j$.

Sol. Let $P = \sum_{i=0}^{n-1}\sum_{j=1+i}^{n+1} {}^nC_i \; {}^{n+1}C_j$

$$= \sum_{j=1}^{n+1} {}^nC_0 \; {}^{n+1}C_j + \sum_{j=2}^{n+1} {}^nC_1 \; {}^{n+1}C_j + \sum_{j=3}^{n+1} {}^nC_2 \; {}^{n+1}C_j + \ldots + \sum_{j=n}^{n+1} {}^nC_{n-1} \; {}^{n+1}C_j$$

$$= {}^nC_0 \sum_{j=1}^{n+1} {}^{n+1}C_j + {}^nC_1 \sum_{j=2}^{n+1} {}^{n+1}C_j + {}^nC_2 \sum_{j=3}^{n+1} {}^{n+1}C_j + \ldots + {}^nC_{n-1} \sum_{j=n}^{n+1} {}^{n+1}C_j$$

$$= {}^nC_0 ({}^{n+1}C_1 + {}^{n+1}C_2 + {}^{n+1}C_3 + \ldots + {}^{n+1}C_{n+1}) + {}^nC_1 ({}^{n+1}C_2 + {}^{n+1}C_3 + {}^{n+1}C_4 + \ldots + {}^{n+1}C_{n+1}) + {}^nC_2 ({}^{n+1}C_3 + {}^{n+1}C_4 + {}^{n+1}C_5 + \ldots + {}^{n+1}C_{n+1}) + \ldots + {}^nC_{n-1} ({}^{n+1}C_n + {}^{n+1}C_{n+1})$$

$$= {}^{n+1}C_1 \cdot {}^nC_0 + {}^{n+1}C_2 ({}^nC_0 + {}^nC_1) + {}^{n+1}C_3 ({}^nC_0 + {}^nC_1 + {}^nC_2) + \ldots + {}^{n+1}C_{n+1} ({}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_{n-1})$$

$$= ({}^nC_0 + {}^nC_1) \cdot {}^nC_0 + ({}^nC_1 + {}^nC_2)({}^nC_0 + {}^nC_1) + ({}^nC_2 + {}^nC_3)({}^nC_0 + {}^nC_1 + {}^nC_2) + \ldots + ({}^nC_n + {}^nC_{n-1})({}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_{n-1}) + n$$

$$= ({}^nC_0)^2 + ({}^nC_1)^2 + ({}^nC_2)^2 + \ldots + ({}^nC_{n-1})^2 + 2\{{}^nC_0 \cdot {}^nC_1 + {}^nC_0 \cdot {}^nC_2 + {}^nC_0 \cdot {}^nC_3 + \ldots + {}^nC_0 \cdot {}^nC_{n-1} + \ldots + {}^nC_{n-2} \; {}^nC_{n-1}\} + 2^n - 1 + n$$

$$= ({}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_{n-1})^2 + 2^n - 1 + n$$

$$= (2^n - 1)^2 + 2^n - 1 + n = 2^{2n} - 2^n + n$$

● **Ex. 50** *If* $(9 + 4\sqrt{5})^n = I + f$, *n and I being positive integers and f is a proper fraction, show that* $(I - 1)\, f + f^2$ *is an even integer.*

Sol. $(9 + 4\sqrt{5})^n = I + f$ ...(i)

$0 \le f < 1$ ...(ii)

Let $f' = (9 - 4\sqrt{5})^n$ ...(iii)

and $0 < f' < 1$ ...(iv)

From Eqs. (i) and (iii), we get

$$I + f + f' = (9 + 4\sqrt{5})^n + (9 - 4\sqrt{5})^n$$
$$= 2\{9^n + {}^nC_2 \, 9^{n-2} (4\sqrt{5})^2 + \ldots\}$$
$$= 2N, \text{ where } N \text{ is a positive integer.}$$

and from Eqs. (ii) and (iii), we get $0 < f + f' < 2$

Since, $f + f'$ is an integer.

$\therefore \quad f + f' = 1$

Now, $I + 1 = 2N \implies 1 = 2N - 1$ ...(v)

$\because \quad (I + f)(1 - f) = (9 + 4\sqrt{5})^n \; f'$

$$= (9 + 4\sqrt{5})^n (9 - 4\sqrt{5})^n = 1^n = 1$$

$\therefore \quad (I - 1) f + f^2 = I - 1 = 2N - 1 - 1 = 2N - 2$ [from Eq. (v)]

= An even integer

● **Ex. 51** *If* P_r *is the coefficient of* x^r *in the expansion of* $(1+x)^2 \left(1+\frac{x}{2}\right)^2 \left(1+\frac{x}{2^2}\right)^2 \left(1+\frac{x}{2^3}\right)^2 \ldots$, *prove that* $P_r = \frac{2^2}{(2^r - 1)}(P_{r-1} + P_{r-2})$ *and* $P_4 = \frac{1072}{315}$.

Sol. Let $(1+x)^2 \left(1+\frac{x}{2}\right)^2 \left(1+\frac{x}{2^2}\right)^2 \left(1+\frac{x}{2^3}\right)^2 \ldots$,

$$= 1 + P_1 x + P_2 x^2 + P_3 x^3 + P_4 x^4 + \ldots + P_{r-1} x^{r-1} + P_r x^r + \ldots \quad \ldots(i)$$

Replacing x by $\frac{x}{2}$, we get

$$\left(1+\frac{x}{2}\right)^2 \left(1+\frac{x}{2^2}\right)^2 \left(1+\frac{x}{2^3}\right)^2 \left(1+\frac{x}{2^4}\right)^2 \ldots = \left[1 + P_1\left(\frac{x}{2}\right) + P_2\left(\frac{x}{2}\right)^2 + P_3\left(\frac{x}{2}\right)^3 + \ldots\right]$$

On multiplying both sides by $(1+x)^2$, we get

$$(1+x)^2 \left(1+\frac{x}{2}\right)^2 \left(1+\frac{x}{2^2}\right)^2 \left(1+\frac{x}{2^3}\right)^2 \ldots = (1+x)^2 \left[1 + P_1\left(\frac{x}{2}\right) + P_2\left(\frac{x}{2}\right)^2 + P_3\left(\frac{x}{2}\right)^3 + \ldots\right] \quad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$1 + P_1 x + P_2 x^2 + P_3 x^3 + P_4 x^4 + \ldots + P_{r-1} x^{r-1} + P_r x^r + \ldots = (1+x)^2 \left[1 + P_1\left(\frac{x}{2}\right) + P_2\left(\frac{x}{2}\right)^2 + P_3\left(\frac{x}{2}\right)^3 + \ldots\right]$$

On equating coefficient of x^r, we get

$$P_r = P_r\left(\frac{1}{2^r}\right) + 2P_{r-1}\left(\frac{1}{2^{r-1}}\right) + P_{r-2}\left(\frac{1}{2^{r-2}}\right)$$

$$\Rightarrow \quad P_r\left(1 - \frac{1}{2^r}\right) = \frac{1}{2^{r-2}}(P_{r-1} + P_{r-2})$$

$$\Rightarrow \quad P_r = \frac{2^2}{(2^r - 1)}(P_{r-1} + P_{r-2})$$

Now, $P_0 = 1, \; P_1 = 2 + 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots = 4$

$$P_2 = \frac{2^2 (P_1 + P_0)}{2^2 - 1} = \frac{20}{3},$$

$$P_3 = \frac{2^2 (P_2 + P_1)}{2^3 - 1} = \frac{128}{21}$$

and $$P_4 = \frac{2^2 (P_3 + P_2)}{2^4 - 1} = \frac{4\left(\frac{128}{21} + \frac{20}{3}\right)}{15} = \frac{1072}{315}$$

Binomial Theorem Exercise 1 :
Single Option Correct Type Questions

■ This section contains **30 multiple choice questions.** Each question has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct

1. If $\sum_{r=0}^{n}(-1)^r\ {}^nC_r\left[\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\ldots \text{ upto } m \text{ terms}\right]$ $= f(n)\left(1-\frac{1}{2^{mn}}\right)$,

$\int_{-3}^{3} f(x^3 \ln x)\, d(x^3 \ln x)$ is equal to

(a) -6 (b) -3

(c) 3 (d) Cannot be determined

2. The coefficient of $(a^3 \cdot b^6 \cdot c^8 \cdot d^9 \cdot e \cdot f)$ in the expansion of $(a+b+c-d-e-f)^{31}$ is

(a) 123210 (b) 23110

(c) 3110 (d) None of these

3. The sum of rational terms in $(\sqrt{2}+\sqrt{3}+\sqrt[6]{5})^{10}$, is

(a) 12632 (b) 1260

(c) 126 (d) None of these

4. If $(1+x-3x^2)^{2145} = a_0 + a_1 x + a_2 x^2 + \ldots$, then $a_0 - a_1 + a_2 - a_3 + \ldots$ ends with

(a) 1 (b) 3

(c) 7 (d) 9

5. In the expansion of $\left(\sqrt{\frac{q}{p}}+\sqrt[10]{\frac{p^7}{q^3}}\right)^n$, there is a term similar to pq, then that term is equal to

(a) $45pq$ (b) $120\ pq$

(c) $210\ pq$ (d) $252\ pq$

6. Let $(5+2\sqrt{6})^n = I + f$, where $n, I \in N$ and $0 < f < 1$, then the value of $f^2 - f + I \cdot f - I$, is

(a) a natural number (b) a negative integer

(c) a prime number (d) an irrational number

7. If $x + \frac{1}{x} = 1$ and $p = x^{4000} + \frac{1}{x^{4000}}$ and q is the digit at unit place in the number $2^{2^n}+1$, $n \in N$ and $n > 1$, then $p+q$, is

(a) 8 (b) 6

(c) 7 (d) None of these

8. If the number of terms in $\left(x+1+\frac{1}{x}\right)^n$ $(n \in I^+)$ is 401, then n is greater than

(a) 201 (b) 200

(c) 199 (d) None of these

9. $\sum_{r=0}^{n-1}\frac{{}^nC_r}{{}^nC_r + {}^nC_{r+1}}$ is equal to

(a) $\frac{n}{2}$ (b) $\frac{n+1}{2}$

(c) $\frac{n(n+1)}{2}$ (d) $\frac{n(n-1)}{2(n+1)}$

10. The largest term in the expansion of $\left(\frac{b}{2}+\frac{b}{2}\right)^{100}$ is

(a) b^{100} (b) $\left(\frac{b}{2}\right)^{100}$

(c) ${}^{100}C_{50}\left(\frac{b}{2}\right)^{100}$ (d) ${}^{100}C_{50}b^{100}$

11. If the fourth term of $\left(\sqrt{x^{\left(\frac{1}{1+\log x}\right)}}+\sqrt[12]{x}\right)^6$ is equal to 200 and $x > 1$, x is equal to

(a) $10\sqrt{2}$ (b) 10 (c) 10^4 (d) $\frac{10}{\sqrt{2}}$

12. The coefficient of x^m in $(1+x)^m + (1+x)^{m+1} + \ldots + (1+x)^n$, $m \le n$, is

(a) ${}^{n+1}C_{m+1}$ (b) ${}^{n-1}C_{m-1}$

(c) ${}^nC_m$ (d) ${}^nC_{m+1}$

13. The number of values of 'r' satisfying the equation ${}^{39}C_{3r-1} - {}^{39}C_{r^2} = {}^{39}C_{r^2-1} - {}^{39}C_{3r}$ is

(a) 1 (b) 2

(c) 3 (d) 4

14. The sum $S = {}^{20}C_2 + 2\cdot{}^{20}C_3 + 3\cdot{}^{20}C_4 + \ldots + 19\cdot{}^{20}C_{20}$ is equal to

(a) $1+5\cdot 2^{20}$ (b) $1+2^{21}$

(c) $1+9\cdot 2^{20}$ (d) 2^{20}

15. The remainder, if $1+2+2^2+2^3+\ldots+2^{1999}$ is divided by 5, is

(a) 0 (b) 1

(c) 2 (d) 3

16. Coefficient of $\frac{1}{x}$ in the expansion of $(1+x)^n(1+1/x)^n$ is

(a) $\frac{n!}{(n-1)!(n+1)!}$ (b) $\frac{2n!}{(n-1)!(n+1)!}$

(c) $\frac{n!}{(2n-1)!(2n+1)!}$ (d) $\frac{2n!}{(2n-1)!(2n+1)!}$

17. The last two digits of the number 19^{9^4} is
(a) 19 (b) 29
(c) 39 (d) 81

18. If the second term in the expansion of $\left(\sqrt[13]{a}+\frac{a}{\sqrt{a^{-1}}}\right)^n$ is $14a^{5/2}$, the value of $\frac{{}^nC_3}{{}^nC_2}$ is
(a) 4 (b) 3
(c) 12 (d) 6

19. If $6^{83}+8^{83}$ is divided by 49, the remainder is
(a) 0 (b) 14
(c) 35 (d) 42

20. The sum of all the rational terms in the expansion of $(3^{1/4}+4^{1/3})^{12}$ is
(a) 91 (b) 251
(c) 273 (d) 283

21. Last four digits of the number $N=7^{100}-3^{100}$ is
(a) 2000 (b) 4000
(c) 6000 (d) 8000

22. If 5^{99} is divided by 13, the remainder is
(a) 2 (b) 4 (c) 6 (d) 8

23. The value of $\left\{\frac{3^{2003}}{28}\right\}$, where $\{.\}$ denotes the fractional part function is
(a) 17/28 (b) 19/28
(c) 23/28 (d) 5/28

24. The value of $\sum_{r=0}^{20} r\,(20-r)({}^{20}C_r)^2$ is equal to
(a) $400\ {}^{37}C_{20}$ (b) $400\ {}^{40}C_{19}$
(c) $400\ {}^{38}C_{19}$ (d) $400\ {}^{38}C_{20}$

25. If $(3+x^{2008}+x^{2009})^{2010}=a_0+a_1x+a_2x^2+\ldots+a_nx^n$, the value of $a_0-\frac{a_1}{2}-\frac{a_2}{2}+a_3-\frac{a_4}{2}-\frac{a_5}{2}+a_6-\ldots$ is
(a) 1 (b) 2^{2010}
(c) 5^{2010} (d) 3^{2010}

26. The total number of terms which depend on the value of x in the expansion of $\left(x^2-2+\frac{1}{x^2}\right)^n$ is
(a) $2n+1$ (b) $2n$
(c) $n+1$ (d) n

27. The coefficient of x^{10} in the expansion of $(1+x^2-x^3)^8$, is
(a) 420 (b) 476
(c) 532 (d) 588

28. The number of real negative terms in the binomial expansion of $(1+ix)^{4n-2}$, $n\in N, n>0, i=\sqrt{-1}$, is
(a) n (b) $n+1$
(c) $n-1$ (d) $2n$

29. $\sum_{p=1}^{n}\sum_{m=p}^{n}\binom{n}{m}\binom{m}{p}$ is equal to
(a) 3^n (b) 2^n
(c) 3^n+2^n (d) 3^n-2^n

30. The largest real value of x, such that $\sum_{r=0}^{4}\left(\frac{5^{4-r}}{(4-r)!}\right)\left(\frac{x^r}{r!}\right)=\frac{8}{3}$ is
(a) $2\sqrt{2}-5$ (b) $2\sqrt{2}+5$
(c) $-2\sqrt{2}-5$ (d) $-2\sqrt{2}+5$

Binomial Theorem Exercise 2 : More than One Correct Option Type Questions

- This section contains **15 multiple choice questions**. Each question has four choices (a), (b), (c) and (d) out of which **MORE THAN ONE** may be correct.

31. If in the expansion of $(1+x)^m(1-x)^n$, the coefficients of x and x^2 are 3 and -6 respectively, the values of m and n are
(a) 3 (b) 6 (c) 9 (d) 12

32. If the coefficients of rth, $(r+1)$th and $(r+2)$th terms in the expansion of $(1+x)^{14}$ are in AP, then r is /are
(a) 5 (b) 9
(c) 10 (d) 12

33. If n is a positive integer and $(3\sqrt{3}+5)^{2n+1}=\alpha+\beta$, where α is an integer and $0<\beta<1$, then
(a) α is an even integer
(b) $(\alpha+\beta)^2$ is divisible by 2^{2n+1}
(c) the integer just below $(3\sqrt{3}+5)^{2n+1}$ divisible by 3
(d) α is divisible by 10

34. If $(8+3\sqrt{7})^n=P+F$, where P is an integer and F is a proper fraction, then
(a) P is an odd integer (b) P is an even integer
(c) $F(P+F)=1$ (d) $(1-F)(P+F)=1$

35. The value of x for which the 6th term in the expansion of

$$\left\{2^{\log_2\sqrt{(9^{x-1}+7)}}+\frac{1}{2^{\left(\frac{1}{5}\right)\log_2(3^{x-1}+1)}}\right\}^7 \text{ is 84, is}$$

(a) 4 (b) 3
(c) 2 (d) 1

36. Consider the binomial expansion of $\left(\sqrt{x}+\frac{1}{2\cdot\sqrt[4]{x}}\right)^n, n\in N$, where the terms of the expansion are written in decreasing powers of x. If the coefficients of the first three terms form an arithmetic progression, then the statement(s) which hold good is /are

(a) Total number of terms in the expansion of the binomial is 8
(b) Number of terms in the expansion with integral power of x is 3
(c) There is no term in the expansion which is independent of x
(d) Fourth and fifth are the middle terms of the expansion

37. Let $(1+x^2)^2\,(1+x)^n = a_0 + a_1x + a_2x^2 + \ldots$, if a_1, a_2 and a_3 are in AP, the value of n is

(a) 2 (b) 3
(c) 4 (d) 7

38. 10th term of $\left(3-\sqrt{\frac{17}{4}+3\sqrt{2}}\right)^{20}$ is

(a) an irrational number (b) a rational number
(c) a positive integer (d) a negative integer

39. If $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_nx^n$, then

$C_0 - (C_0 + C_1) + (C_0 + C_1 + C_2) - (C_0 + C_1 + C_2 + C_3) + \ldots + (-1)^{n-1}(C_0 + C_1 + C_2 + \ldots + C_{n-1})$, when n is even integer is

(a) a positive value (b) a negative value
(c) divisible by 2^{n-1} (d) divisible by 2^n

40. If $f(n) = \sum_{i=0}^{n}\binom{30}{30-i}\binom{20}{30-i}$, then

(a) maximum value of $f(n)$ is ${}^{50}C_{25}$
(b) $f(0) + f(1) + f(2) + \ldots + f(50) = 2^{50}$
(c) $f(n)$ is always divisible by 50
(d) $f^2(0) + f^2(1) + f^2(2) + \ldots + f^2(50) = {}^{100}C_{50}$

41. Number of values of r satisfying the equation ${}^{69}C_{3r-1} - {}^{69}C_{r^2} = {}^{69}C_{r^2-1} - {}^{69}C_{3r}$ is

(a) 1 (b) 2
(c) 3 (d) 7

42. If the middle term of $\left(x+\frac{1}{x}\sin^{-1}x\right)^8$ is equal to $\frac{630}{16}$, the values of x is/are

(a) $-\frac{\pi}{3}$ (b) $-\frac{\pi}{6}$
(c) $\frac{\pi}{6}$ (d) $\frac{\pi}{3}$

43. If $b^2 < ac$, the sum of the coefficients in the expansion of $(a\alpha^2x^2 + 2b\alpha x + c)^n, (a, b, c, \alpha \in R, n \in N)$, is

(a) + ve, if $a > 0$ (b) + ve, if $c > 0$
(c) − ve, if $a < 0$, n is odd (d) + ve, if $c < 0$, n is even

44. In the expansion of $\left(x^2+1+\frac{1}{x^2}\right)^n, n\in N$, then

(a) number of terms $= 2n + 1$
(b) term independent of $x = 2^{n-1}$
(c) coefficient of $x^{2n-2} = n$
(d) coefficient of $x^2 = n$

45. The coefficient of the $(r+1)$th term of $\left(x+\frac{1}{x}\right)^{20}$, when expanded in the descending powers of x, is equal to the coefficient of the 6th term of $\left(x^2+2+\frac{1}{x^2}\right)^{10}$ when expanded in ascending powers of x. The value of r is

(a) 5 (b) 6
(c) 14 (d) 15

Binomial Theorem Exercise 3 : Passage Based Questions

- This section contains **7 passages**. Based upon each of the passage **3 multiple choice questions** have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I (Q. Nos. 46 to 48)

Consider $(1+x+x^2)^n = \sum_{r=0}^{2n} a_r\, x^r$, *where* $a_0, a_1, a_2, \ldots, a_{2n}$ *are real numbers and n is a positive integer.*

46. The value of $\sum_{r=0}^{n-1} a_r$ is

(a) $\frac{-3^n - a_n}{2}$ (b) $\frac{3^n - a_n}{2}$ (c) $\frac{a_n - 3^n}{2}$ (d) $\frac{3^n + a_n}{2}$

47. If n is even, the value of $\sum_{r=0}^{n/2-1} a_{2r}$ is

(a) $\frac{3^n - 1 + a_n}{2}$ (b) $\frac{3^n - 1 - a_n}{4}$

(c) $\frac{3^n + 1 + a_n}{2}$ (d) $\frac{3^n + 1 - 2a_n}{4}$

48. If n is odd, the value of $\sum_{r=1}^{\frac{n+1}{2}} a_{2r-1}$ is

(a) $\frac{3^n - 1 + 2a_n}{2}$ (b) $\frac{3^n - 1 + 2a_n}{4}$

(c) $\frac{3^n + 1 + 2a_n}{2}$ (d) $\frac{3^n + 1 - 2a_n}{4}$

Passage II (Q. Nos. 49 to 51)

If $(1+x+2x^2)^{20} = a_0 + a_1x + a_2x^2 + \ldots + a_{40}x^{40}$.

49. The value of $a_0 + a_2 + a_4 + \ldots + a_{38}$ is

(a) $2^{19}(2^{19}-1)$ (b) $2^{20}(2^{19}-1)$

(c) $2^{19}(2^{20}-1)$ (d) $2^{20}(2^{20}-1)$

50. The value of $a_1 + a_3 + a_5 + \ldots + a_{37}$ is

(a) $2^{19}(2^{19}-20)$ (b) $2^{19}(2^{20}-21)$

(c) $2^{19}(2^{19}-21)$ (d) $2^{19}(2^{19}-19)$

51. The value of $\frac{a_{39}}{a_{40}}$, is

(a) 2^{20} (b) (c) 10 (d) 1

Passage III (Q. Nos. 52 to 54)

Suppose, m divided by n, then quotient q and remainder r

i.e.

$$n\overline{)m}\,(q$$
$$\underline{\quad}$$
$$r$$

or $m = nq + r$, $\forall\ m, n, q, r \in I$ *and* $n \neq 0$

52. If a is the remainder when 5^{40} is divided by 11 and b is the remainder when 2^{2011} is divided by 17, the value of $a+b$ is

(a) 7 (b) 8

(c) 9 (d) 10

53. If $19^{93} - 13^{99}$ is divided by 162, the remainder is

(a) 8 (b) 4

(c) 1 (d) 0

54. If 13^{99} is divided by 81, the remainder is

(a) 13 (b) 23

(c) 39 (d) 55

Passage IV (Q. Nos. 55 to 57)

Consider the binomial expansion $R = (1+2x)^n = I + f$, *where I is the integral part of R and f is the fractional part of R*, $n \in N$. *Also, the sum of coefficients of R is 2187.*

55. The value of $(n+Rf)$ for $x = \frac{1}{\sqrt{2}}$ is

(a) 7 (b) 8 (c) 9 (d) 10

56. If ith term is the greatest term for $x = 1/3$, then i equals

(a) 4 (b) 5 (c) 6 (d) 7

57. If kth term is having greatest coefficient, the sum of all possible values of k, is

(a) 7 (b) 9

(c) 11 (d) 13

Passage V (Q. Nos. 58 to 60)

If $(x+a_1)(x+a_2)(x+a_3)\ldots(x+a_n) = x^n + S_1x^{n-1} + S_2x^{n-2} + \ldots + S_n$

where, $S_1 = \sum_{i=1}^{n} a_i$, $S_2 = \sum\sum_{1 \le i < j \le n} a_ia_j$, $S_3 = \sum\sum\sum_{1 \le i < j < k \le n} a_ia_ja_k$

and so on.

58. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$, the coefficient of x^n in the expansion of $(x+C_0)(x+3C_1)(x+5C_2)\ldots(x+(2n+1)C_n)$, is

(a) $n \cdot 2^n$ (b) $(n+1)\cdot 2^n$

(c) $n \cdot 2^{n+1}$ (d) $n \cdot 2^n + 1$

59. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$, the coefficient of x^{n-1} in the expansion of $(x+C_0)(x+C_1)(x+C_2)\ldots(x+C_n)$ is

(a) $2^{2n-1} - \frac{1}{2}\,{}^{2n}C_{n-1}$ (b) $2^{2n-1} - \frac{1}{2}\,{}^{2n}C_n$

(c) $2^{2n-1} - \frac{1}{2}\,{}^{2n+1}C_n$ (d) $2^{2n-1} - \frac{1}{2}\,{}^{2n+1}C_{n-1}$

60. Coefficient of x^7 in the expansion of $(1+x)^2(3+x)^3(5+x)^4$ is

(a) 112 (b) 224 (c) 342 (d) 416

Passage VI (Q. Nos. 61 to 63)

Let us consider the binomial expression

$$A=\left(x^2+\frac{3}{x}\right)^m \text{ and } B=\left(\frac{5x}{2}+\frac{x^{-2}}{2}\right)^n$$

Sum of coefficients of expansion of B is 6561. The difference of the coefficient of third term to the second term in the expansion of A is equal to 117.

61. The value of m is

(a) 4 (b) 5 (c) 6 (d) 7

62. If n^m is divided by 7, the remainder is

(a) 1 (b) 2 (c) 3 (d) 5

63. The ratio of the coefficient of second term from the beginning and the end in the expansion of B, is

(a) 125 (b) 625
(c) 3125 (d) 15625

Passage VII (Q. Nos. 64 to 66)

Let us consider the binomial expression $(1+x)^n=\sum_{r=0}^{n} a_r x^r$, *where* a_4, a_5 and a_6 *are in AP,* $(n<10)$. *Consider another binomial expression of* $A=(\sqrt[3]{2}+\sqrt[4]{3})^{13n}$, *the expression of A contains some rational terms* $T_{a_1}, T_{a_2}, T_{a_3}, \ldots, T_{a_m}$ $(a_1<a_2<a_3<\ldots<a_m)$.

64. The value of $\sum_{i=1}^{n} a_i$ is

(a) 63 (b) 127
(c) 255 (d) 511

65. The value of a_m is

(a) 87 (b) 88
(c) 89 (d) 90

66. The common difference of the arithmetic progression $a_1, a_2, a_3, \ldots, a_m$ is

(a) 6 (b) 8
(c) 10 (d) 12

Binomial Theorem Exercise 4 : Single Integer Answer Type Questions

- This section contains **10 questions**. The answer to each question is a **single digit integer**, ranging from 0 to 9 (both inclusive).

67. For integer $n>1$, the digit at unit's place in the number $\sum_{r=0}^{100} r!+2^{2^n}$ is

68. If $(1+x+x^2+x^3)^n=\sum_{r=0}^{3n} a_r x^r$ and $\sum_{r=0}^{3n} a_r=k$ and if $\sum_{r=0}^{3n} r\,a_r=\frac{\lambda n k}{2}$, the value of λ is

69. The number of rational terms in the expansion of $\left(\sqrt[3]{4}+\frac{1}{\sqrt[4]{6}}\right)^{20}$ is

70. If $2^{2006}+2006$ is divided by 7, the remainder is

71. The last two digits of the natural number 19^{9^4} is ab, the value of $b-3a$ is

72. If $\frac{\left[{}^nC_r+4\cdot{}^nC_{r+1}+6\cdot{}^nC_{r+2}+4\cdot{}^nC_{r+3}+{}^nC_{r+4}\right]}{\left[{}^nC_r+3\cdot{}^nC_{r+1}+3\cdot{}^nC_{r+2}+{}^nC_{r+3}\right]}=\frac{n+\lambda}{r+\lambda}$, the value of λ is

73. The value of $99^{50}-99\cdot98^{50}+\frac{99\cdot98}{1\cdot2}(97)^{50}-\ldots+99$ is

74. If the greatest term in the expansion of $(1+x)^{2n}$ has the greatest coefficient if and only if $x\in\left(\frac{10}{11},\frac{11}{10}\right)$ and the fourth term in the expansion of $\left(\lambda x+\frac{1}{x}\right)^m$ is $\frac{n}{4}$, the value of $m\lambda$ is

75. If the value of $(n+2)\cdot{}^nC_0\cdot2^{n+1}-(n+1)\cdot{}^nC_1\cdot2^n+n\cdot{}^nC_2\cdot2^{n-1}-\ldots$ is equal to $k(n+1)$, the value of k is

76. If $(1+x+x^2+\ldots+x^9)^4(x+x^2+x^3+\ldots+x^9)=\sum_{r=1}^{45} a_r x^r$ and the value of $a_2+a_6+a_{10}+\ldots+a_{42}$ is λ, the sum of all digits of λ is

Binomial Theorem Exercise 5 : Matching Type Questions

- This section contains **5 questions**. Questions 77, 78 and 79 have three statements (A, B and C) given in **Column I** and five statements (p, q, r, s and t) in **Column II** and questions 80 and 81 have four statements (A, B, C and D) given in **Column I** and four statements (p, q, r and s) in **Column II**. Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II**.

77.

	Column I		Column II
(A)	If λ and μ are the unit's place digit in m^n and n^m respectively, where m and n are the number of rational and irrational terms in the expansion of $(7^{1/3} + 11^{1/9})^{6561}$ respectively, then	(p)	$\lambda^2 + \mu^2 = 1$
(B)	If λ and μ are the unit's place digit in m^n and n^m respectively, where m and n are the number of terms with integral coefficients and number of terms with non-integral coefficients in the expansion of $(7^{1/3} + 5^{1/2} \cdot x)^{600}$ respectively, then	(q)	$\lambda^\mu + \mu^\lambda = 1$
(C)	If λ and μ are the unit's place digit in m^n and n^m respectively, where m and n are the number of rational and irrational terms in the expansion of $(\sqrt{2} + \sqrt[3]{3} + \sqrt[6]{7})^{10}$ respectively, then	(r)	$\lambda + \mu = 4$
		(s)	$\sqrt{\lambda\sqrt{\lambda\sqrt{\lambda\sqrt{\lambda\ \ldots\infty}}}} = \mu$
		(t)	$\lambda + \mu = \lambda^\mu$

78.

	Column I		Column II
(A)	If $\binom{18}{r-2} + 2\binom{18}{r-1} + \binom{18}{r} \geq \binom{20}{13}$, then the values of r is /are	(p)	5
(B)	The digit in the unit's place of the number $183! + 3^{183}$ is less than	(q)	6
(C)	If the 4th term in the expansion of $\left(ax + \frac{1}{x}\right)^n$ is $5/2$, then na is less than	(r)	7
		(s)	8
		(t)	9

79.

	Column I		Column II
(A)	The sum of binomial coefficients of terms containing power of x more than x^{30} in $(1+x)^{61}$ is divisible by	(p)	2^{57}
(B)	The sum of binomial coefficients of rational terms in the expansion of $(1+\sqrt{3})^{62}$ is divisible by	(q)	2^{58}
(C)	If $\left(x + \frac{1}{x} + x^2 + \frac{1}{x^2}\right)^{31} = a_0 x^{-62} + a_1 x^{-61} + a_2 x^{-60} + \ldots + a_{124} x^{62}$, then $a_1 + a_3 + a_5 + \ldots + a_{123}$ is divisible by	(r)	2^{59}
		(s)	2^{60}
		(t)	2^{61}

80.

	Column I		Column II
(A)	If $11^n + 21^n$ is divisible by 16, then n can be	(p)	4
(B)	The remainder, when 3^{37} is divided by 80, is less than	(q)	5
(C)	In the expansion of $(1+x)^{29}$ coefficient of $(r+1)$th term is equal to that of $(r+k)$th term, then the value of k cannot be	(r)	6
(D)	If the ratio of 2nd and 3rd terms in the expansion of $(a+b)^n$ is equal to ratio of 3rd and 4th terms in the expansion of $(a+b)^{n+3}$, then n is less than	(s)	7

81.

	Column I		Column II
(A)	If number of dissimilar terms in the expansion of $(x + 2y + 3z)^n \quad (n \in N)$ is $an^2 + bn + c$, then	(p)	$a + b + c = 3$
(B)	If number of dissimilar terms in the expansion of $(x + y + z)^{2n+1} - (x + y - z)^{2n+1} \quad (n \in N)$ is $an^2 + bn + c$, then	(q)	$a + b + c = 4$
(C)	If number of dissimilar terms in the expansion of $(x - y + z)^n + (x + y - z)^n \quad (n \in$ is even natural number) is $an^2 + bn + c$, then	(r)	$a + b = 2c$
(D)	If number of dissimilar terms in the expansion of $\left(\frac{x^2 + 1 + x^4}{x^2}\right)^{\Sigma n}$ $(n \in N)$ is $an^2 + bn + c$, then	(s)	$b + c = 8a$

Binomial Theorem Exercise 6 :
Statement I and II Type Questions

■ **Directions** (Q. Nos. 71 to 82) are Assertion-Reason type questions. Each of these questions contains two statements:
Statement-1 (Assertion) and **Statement-2** (Reason) Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below.
(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

82. **Statement-1** Greatest coefficient in the expansion of $(1+3x)^6$ is ${}^6C_3 \cdot 3^3$.

Statement-2 Greatest coefficient in the expansion of $(1+x)^{2n}$ is the middle term.

83. **Statement-1** The term independent of x in the expansion of $\left(x^2+\frac{1}{x^2}+2\right)^{25}$ is ${}^{50}C_{25}$.

Statement-2 In a binomial expansion middle term is independent of x.

84. **Statement-1** In the expansion of $(1+x)^n$, if coefficient of 31st and 32nd terms are equal, then $n=61$.

Statement-2 Middle term in the expansion of $(1+x)^n$, has greatest coefficient.

85. **Statement-1** The number of terms in the expansion of $\left(x+\frac{1}{x}+1\right)^n$ is $(2n+1)$.

Statement-2 The number of terms in the expansion of $(x_1+x_2+x_3+\ldots+x_m)^n$ is ${}^{n+m-1}C_{m-1}$.

86. **Statement-1** 4^{101} when divided by 101 leaves the remainder 4.

Statement-2 (n^p-n) when divided by 'p' leaves remainder zero when $n \geq 2, n \in N$ and p is a prime number.

87. **Statement-1** $11^{25}+12^{25}$ when divided by 23 leaves the remainder zero.

Statement-2 a^n+b^n is always divisible by $(a+b), \forall\, n \in N$.

88. **Statement-1** The maximum value of the term independent of x in the expansion of $(ax^{1/6}+bx^{1/3})^9$ is 84.

Statement-2 $a^2+b=2$.

Binomial Theorem Exercise 7 :
Subjective Type Questions

■ In this section, there are **24 subjective** questions.

89. Determine the value of x in the expression of $(x+x^{\log_{10} x})^5$, if the third term in the expansion is 1000000.

90. Find the value of
$$\frac{18^3+7^3+3\cdot 18\cdot 7\cdot 25}{(3^6+6\cdot 243\cdot 2+15\cdot 81\cdot 4+20\cdot 27\cdot 8+15\cdot 9\cdot 16+6\cdot 3\cdot 32+64)}.$$

91. Simplify $\left(\frac{a+1}{a^{2/3}-a^{1/3}+1}-\frac{a-1}{a-a^{1/2}}\right)^{10}$ into a binomial and determine the terms independent of a.

92. Show that there will be a term independent of x in the expansion of $(x^a+x^{-b})^n$ only, if an is a multiple of $(a+b)$.

93. If a, b and c are the three consecutive coefficients in the expansion of a power of $(1+x)$, prove that the index of the power is $\frac{2ac+b(a+c)}{b^2-ac}$.

94. Find n in the binomial $\left[\sqrt[3]{2}+\frac{1}{\sqrt[3]{3}}\right]^n$, if the ratio of 7th term from beginning to 7th term from the end is $1/6$.

95. If $S_n = {}^nC_0\,{}^nC_1+{}^nC_1\,{}^nC_2+\ldots+{}^nC_{n-1}\,{}^nC_n$ and if $\frac{S_{n+1}}{S_n}=\frac{15}{4}$, find n.

96. If $(1+x)^n = C_0+C_1x+C_2x^2+\ldots+C_nx^n$, prove that $\frac{C_1}{C_0}+2\frac{C_2}{C_1}+3\frac{C_3}{C_2}+\ldots+n\frac{C_n}{C_{n-1}}=\frac{n(n+1)}{2}$

97. Which term in the expansion of $\left[\sqrt[3]{\left(\frac{a}{\sqrt{b}}\right)}+\sqrt{\left(\frac{b}{\sqrt[3]{a}}\right)}\right]^{21}$ contains a and b to one and same power.

98. Find the coefficient of x^r in the expansion of
$$(x+3)^{n-1}+(x+3)^{n-2}(x+2)+(x+3)^{n-3}(x+2)^2+\ldots+(x+2)^{n-1}.$$

99. Prove that, if p is a prime number greater than 2, the difference $[(2+\sqrt{5})^p] - 2^{p+1}$ is divisible by p, where $[.]$ denotes greatest integer.

100. If $((x))$ represents the least integer greater than x, prove that $((\{(\sqrt{3}+1)^{2n}\}))$, $n \in N$ is divisible by 2^{n+1}.

101. Solve the equation
$${}^{11}C_1 x^{10} - {}^{11}C_3 x^8 + {}^{11}C_5 x^6 - {}^{11}C_7 x^4 + {}^{11}C_9 x^2 - {}^{11}C_{11} = 0.$$

102. If $g(x) = \sum_{r=0}^{200} \alpha_r \cdot x^r$ and $f(x) = \sum_{r=10}^{200} \beta_r x^4, \beta_r = 1$ for $r \geq 100$ and $g(x) = f(1+x)$, show that the greatest coefficient in the expansion of $(1+x)^{201}$ is α_{100}.

103. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$, find the value of
$$\sum\sum_{0 \leq i < j \leq n} (i+j)(C_i + C_j + C_i C_j).$$

104. Evaluate $\sum\sum_{0 \leq i \neq j \leq 10} {}^{21}C_i \cdot {}^{21}C_j$.

105. Find the coefficients of x^4 in the expansions of
(i) $(1 + x + x^2 + x^3)^{11}$.
(ii) $(2 - x + 3x^2)^6$.

106. Prove the identity
$$\frac{1}{{}^{2n+1}C_r} + \frac{1}{{}^{2n+1}C_{r+1}} = \frac{2n+2}{2n+1} \cdot \frac{1}{{}^{2n}C_r},$$
use it to prove $\sum_{r=1}^{r=2n-1} \frac{(-1)^{r-1} r}{{}^{2n}C_r} = \frac{n}{n+1}$.

107. Let $a_0, a_1, a_2, \ldots$ are the coefficients in the expansion of $(1 + x + x^2)^n$ arranged order of x. Find the value of $a_r - {}^nC_1 a_{r-1} + {}^nC_r a_{r-2} - \ldots + (-1)^r {}^nC_r a_0$, where r is not divisible by 3.

108. If for z as real or complex.
$(1 + z^2 + z^4)^8 = C_0 + C_1 z^2 + C_2 z^4 + \ldots + C_{16} z^{32}$,
prove that
(i) $C_0 - C_1 + C_2 - C_3 + \ldots + C_{16} = 1$
(ii) $C_0 + C_3 + C_6 + C_9 + C_{12} + C_{15} + (C_2 + C_5 + C_8 + C_{11} + C_{14})\omega + (C_1 + C_4 + C_7 + C_{10} + C_{13} + C_{16})\omega^2 = 0$,
where ω is a cube root of unity.

109. Let $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{2n} x^{2n}$ and $g(x) = b_0 + b_1 x + b_2 x^2 + \ldots + b_{n-1} x^{n-1} + x^n + x^{n+1} + \ldots + x^{2n}$.
If $f(x) = g(x+1)$, find a_n in terms of n.

110. If $a_0, a_1, a_2, \ldots$ are the coefficients in the expansion of $(1 + x + x^2)^n$ in ascending powers of x, prove that
(i) $a_0 a_1 - a_1 a_2 + a_2 a_3 - \ldots = 0$
(ii) $a_0 a_2 - a_1 a_3 + a_2 a_4 - \ldots + a_{2n-2} a_{2n} = a_{n+1}$
(iii) if $E_1 = a_0 + a_3 + a_6 + \ldots; E_2 = a_1 + a_4 + a_7 + \ldots$ and $E_3 = a_2 + a_5 + a_8 + \ldots$, then $E_1 = E_2 = E_3 = 3^{n-1}$

111. Prove that $(n-1)^2 C_1 + (n-3)^2 C_3 + (n-5)^2 C_5 + \ldots = n(n+1)2^{n-3}$, where C_r stands for ${}^nC_r$.

112. Show that $\frac{C_0}{1} - \frac{C_1}{4} + \frac{C_2}{7} - \ldots + (-1)^n \frac{C_n}{3n+1} = \frac{3^n \cdot n!}{1 \cdot 4 \cdot 7 \ldots (3n+1)}$, where C_r stands for ${}^nC_r$.

Binomial Theorem Exercise 8 : Questions Asked in Previous 13 Year's Exams

- This section contains questions asked in **IIT-JEE, AIEEE, JEE Main & JEE Advanced** from year **2005** to year **2017**.

113. The value of $\binom{30}{0}\binom{30}{10} - \binom{30}{1}\binom{30}{11} + \binom{30}{2}\binom{30}{12} + \ldots + \binom{30}{20}\binom{30}{30}$ is **[IIT JEE 2005, 3M]**
(a) ${}^{60}C_{20}$ (b) ${}^{30}C_{10}$ (c) ${}^{60}C_{30}$ (d) ${}^{40}C_{30}$

114. If the coefficients of pth, $(p+1)$th and $(p+2)$th terms in expansion of $(1+x)^n$ are in AP, then **[AIEEE 2005, 3M]**
(a) $n^2 - 2np + 4p^2 = 0$
(b) $n^2 - n(4p+1) + 4p^2 - 2 = 0$
(c) $n^2 - n(4p+1) + 4p^2 = 0$
(d) None of the above

115. If the coefficient of x^7 in $\left(ax^2 + \frac{1}{bx}\right)^{11}$ is equal to the coefficient of x^{-7} in $\left(ax - \frac{1}{bx^2}\right)^{11}$, then ab is equal to **[AIEEE 2005, 3M]**

(a) 1 (b) 1 / 2
(c) 2 (d) 3

116. For natural numbers m and n, if $(1-y)^m (1+y)^n = 1 + a_1 y + a_2 y^2 + \ldots$ and $a_1 = a_2 = 10$, then (m, n) is **[AIEEE 2006, 3M]**

(a) (20, 45) (b) (35, 20)
(c) (45, 35) (d) (35, 45)

117. In the binomial expansion of $(a-b)^n$, $n \geq 5$, the sum of 5th and 6th terms is zero, $\frac{a}{b}$ equals **[AIEEE 2007, 3M]**

(a) $\frac{5}{n-4}$ (b) $\frac{6}{n-5}$
(c) $\frac{n-5}{6}$ (d) $\frac{n-4}{5}$

118. The sum of the series ${}^{20}C_0 - {}^{20}C_1 + {}^{20}C_2 - {}^{20}C_3 + \ldots + {}^{20}C_{10}$ is **[AIEEE 2007, 3M]**

(a) $-{}^{20}C_{10}$ (b) $\frac{1}{2}{}^{20}C_{10}$
(c) 0 (d) ${}^{20}C_{10}$

119. **Statement-1** $\sum_{r=0}^{n} (r+1)\,{}^nC_r = (n+2)\cdot 2^{n-1}$

Statement-2 $\sum_{r=0}^{n} (r+1)\,{}^nC_r\, x^r = (1+x)^n + nx(1+x)^{n-1}$. **[AIEEE 2007]**

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement-1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

120. The remainder left out when $8^{2n} - (62)^{2n+1}$ is divided by 9, is **[AIEEE 2009, 4M]**

(a) 8 (b) 0 (c) 2 (d) 7

121. For $r = 0, 1, 2, \ldots, 10$, let A_r, B_r and C_r denote respectively, the coefficients of x^r in the expansion of $(1+x)^{10}, (1+x)^{20}$ and $(1+x)^{30}$, $\sum_{r=1}^{10} A_r (B_{10} B_r - C_{10} A_r)$ is equal to **[IIT-JEE 2010, 5M]**

(a) $B_{10} - C_{10}$ (b) $A_{10}(B_{10} - C_{10}A_{10})$
(c) 0 (d) $C_{10} - B_{10}$

122. Let $S_1 = \sum_{j=1}^{10} j(j-1)\cdot {}^{10}C_j$, $S_2 = \sum_{j=1}^{10} j \cdot {}^{10}C_j$ and $S_3 = \sum_{j=1}^{10} j^2 \cdot {}^{10}C_j$ **[IIT-JEE 2010]**

Statement-1 $S_3 = 55 \times 2^9$

Statement-2 $S_1 = 90 \times 2^8$ and $S_2 = 10 \times 2^8$

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement-1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

123. The coefficient of x^7 in the expansion of $(1 - x - x^2 + x^3)^6$, is **[AIEEE 2011, 4M]**

(a) – 132 (b) – 144
(c) 132 (d) 144

124. If n is a positive integer, then $(\sqrt{3}+1)^{2n} - (\sqrt{3}-1)^{2n}$ is **[AIEEE 2012, 4M]**

(a) an odd positive integer
(b) an even positive integer
(c) a rational number other than positive integer
(d) an irrational number

125. The term independent of x in the expansion of $\left(\frac{x+1}{x^{2/3} - x^{1/3} + 1} - \frac{x-1}{x - x^{1/2}}\right)^{10}$ is **[JEE Main, 2013, 4M]**

(a) 120 (b) 210
(c) 310 (d) 4

126. The coefficients of three consecutive terms of $(1+x)^{n+5}$ are in the ratio 5 : 10 : 14, the value of n is **[JEE Advanced 2013M]**

127. If the coefficients of x^3 and x^4 in the expansion of $(1 + ax + bx^2)(1-2x)^{18}$ in powers of x are both zero, then (a, b) is equal to **[JEE Main 2014, 3M]**

(a) $\left(14, \frac{272}{3}\right)$ (b) $\left(16, \frac{272}{3}\right)$
(c) $\left(14, \frac{251}{3}\right)$ (d) $\left(16, \frac{251}{3}\right)$

128. Coefficient of x^{11} in the expansion of $(1+x^2)^4 (1+x^3)^7 (1+x^4)^{12}$ is **[JEE Advanced 2014, 3M]**

(a) 1051 (b) 1106 (c) 1113 (d) 1120

129. The sum of coefficients of integral powers of x in the binomial expansion of $(1 - 2\sqrt{x})^{50}$, is **[JEE Main 2015, 4M]**

(a) $\frac{1}{2}(2^{50} + 1)$ (b) $\frac{1}{2}(3^{50} + 1)$
(c) $\frac{1}{2}(3^{50})$ (d) $\frac{1}{2}(3^{50} - 1)$

130. The coefficients of x^9 in the expansion of $(1+x)(1+x^2)(1+x^3)\ldots(1+x^{100})$ is

[JEE Advanced 2015, 4M]

131. If the number of terms in the expansion of $\left(1-\frac{2}{x}+\frac{4}{x^2}\right)^n$, $x \neq 0$ is 28, then the sum of the coefficients of all the terms in this expansion, is **[JEE Main 2016, 4M]**

(a) 243 (b) 729 (c) 64 (d) 2187

132. Let m be the smallest positive integer such that the coefficient of x^2 in the expansion of $(1+x)^2+(1+x)^3+\ldots\ldots+(1+x)^{49}+(1+mx)^{50}$ is $(3n+1)^{51}C_3$ for some positive integer n. Then the value of n is **[JEE Advanced 2016, 3M]**

133. The value of $({}^{21}C_1-{}^{10}C_1)+({}^{21}C_2-{}^{10}C_2)+({}^{21}C_3-{}^{10}C_3)+({}^{21}C_4-{}^{10}C_4)+\ldots\ldots+({}^{21}C_{10}-{}^{10}C_{10})$ is

[JEE Advanced 2017, 4M]

(a) $2^{20}-2^{10}$ (b) $2^{21}-2^{11}$
(c) $2^{21}-2^{10}$ (d) $2^{20}-2^9$

Answers

Exercise for Session 1

1. (c) **2.** (a) **3.** (c) **4.** (c) **5.** (b) **6.** (b)
7. (c) **8.** (d)

Exercise for Session 2

1. (b) **2.** (c) **3.** (d) **4.** (b) **5.** (c) **6.** (c)
7. (c) **8.** (b) **9.** (a) **10.** (d)

Exercise for Session 3

1. (a) **2.** (c) **3.** (d) **4.** (b) **5.** (c) **6.** (c)
7. (a) **8.** (c) **9.** (a) **10.** (c)

Exercise for Session 4

1. (c) **2.** (b) **3.** (c) **4.** (a) **5.** (a) **6.** (c)
7. (b) **8.** (c) **9.** (b) **10.** (b) **11.** (a) **12.** (d)
13. (a) **14.** (b)

Chapter Exercises

1. (d) **2.** (d) **3.** (a) **4.** (b) **5.** (d) **6.** (b)
7. (b) **8.** (c) **9.** (a) **10.** (c) **11.** (b) **12.** (a)
13. (b) **14.** (c) **15.** (a) **16.** (b) **17.** (a) **18.** (a)
19. (c) **20.** (d) **21.** (d) **22.** (d) **23.** (b) **24.** (d)
25. (b) **26.** (b) **27.** (b) **28.** (a) **29.** (d) **30.** (a)
31. (c,d) **32.** (a,b) **33.** (a,d) **34.** (a,d) **35.** (c,d) **36.** (b,c)
37. (b,c) **38.** (a,d) **39.** (b,c) **40.** (a,b,d) **41.** (c,d) **42.** (a,d)
43. (a,b,c,d) **44.** (a,c) **45.** (a,d)
46. (b) **47.** (d) **48.** (b) **49.** (c) **50.** (b) **51.** (c)
52. (c) **53.** (d) **54.** (d) **55.** (b) **56.** (a) **57.** (b)
58. (b) **59.** (b) **60.** (d) **61.** (c) **62.** (a) **63.** (d)
64. (b) **65.** (c) **66.** (d)
67. (0) **68.** (3) **69.** (3) **70.** (8) **71.** (6) **72.** (4)
73. (0) **74.** (3) **75.** (4) **76.** (9)
77. (A) $\rightarrow$ (q, r); (B) $\rightarrow$ (p, q, t); (C) $\rightarrow$ (s)
78. (A) $\rightarrow$ (r, s, t); (B) $\rightarrow$ (s, t); (C) $\rightarrow$ (p, q, r, s, t)
79. (A) $\rightarrow$ (p, q, r, s); (B) $\rightarrow$ (p, q, r, s, t); (C) $\rightarrow$ (p, q, r, s, t)
80. (A) $\rightarrow$ (q, s); (B) $\rightarrow$ (p, q, r, s); (C) $\rightarrow$ (q, s); (D) $\rightarrow$ (r, s)
81. (A) $\rightarrow$ (p, r); (B) $\rightarrow$ (q); (C) $\rightarrow$ (s); (D) $\rightarrow$ (p, r)
82. (d) **83.** (c) **84.** (b) **85.** (b) **86.** (d) **87.** (c)
88. (a)
89. $x = 10$ or $10^{-5/2}$ **90.** 1 **91.** 210 **94.** 9 **95.** 4,2
97. 10 **98.** ${}^nC_r(3^{n-r}-2^{n-r})$ **101.** $x = \cot\left(\frac{r\pi}{11}\right)$, $r = \pm 1, \pm 2, \ldots, \pm 5$
103. $n^2 \cdot 2^n + n\left\{2^{2n-1}-\frac{2n!}{2(n!)^2}\right\}$ **104.** $\frac{1}{2}\left[2^{40}-\frac{42!}{2(21!)^2}\right]$
105. (i) 990 (ii) 3660 **107.** 0 **109.** ${}^{2n+1}C_{n+1}$
113. (b) **114.** (b) **115.** (a) **116.** (d) **117.** (d) **118.** (b)
119. (a) **120.** (c) **121.** (d) **122.** (b) **123.** (b) **124.** (d)
125. (b) **126.** (6) **127.** (b) **128.** (c) **129.** (b) **130.** (8)
131. (b) **132.** (5) **133.** (a)

Solutions

1. $\because \sum_{r=0}^{n}(-1)^r\ {}^nC_r\left[\left(\frac{1}{2}\right)^r+\left(\frac{3}{4}\right)^r+\left(\frac{7}{8}\right)^r+\cdots\text{upto } m \text{ terms}\right]$

$$=\left(1-\frac{1}{2}\right)^n+\left(1-\frac{3}{4}\right)^n+\left(1-\frac{7}{8}\right)^n+\cdots\text{ upto } m \text{ terms}$$

$$=\frac{1}{2^n}+\frac{1}{2^{2n}}+\frac{1}{2^{3n}}+\cdots\text{upto } m \text{ terms}$$

$$=\frac{\frac{1}{2^n}\left[1-\left(\frac{1}{2^n}\right)^m\right]}{\left(1-\frac{1}{2^n}\right)}=\left(\frac{1}{2^n-1}\right)\left(1-\frac{1}{2^{mn}}\right)$$

$\therefore f(n)=\dfrac{1}{2^n-1}$

$\therefore \int_{-3}^{3} f(x^3\ln x)\cdot d(x^3\ln x)$

$$=\int_{-3}^{3}\frac{1}{(2^{x^3\ln x}-1)}\cdot(3x^2\ln x+x^2)dx$$

Since, $\ln x$ cannot be defined for $x<0$.

$\therefore$ Above integral cannot be calculated.

2. Coefficient of $(a^3\cdot b^6\cdot c^8\cdot d^9\cdot e\cdot f)$ in given expansion

$$=(-1)^9\cdot(-1)^1\cdot(-1)^1\cdot\frac{31!}{3!6!8!9!1!1!}$$

3. General term of given expression

$$=\frac{10!}{\alpha!\beta!\gamma!}2^{\alpha/2}\cdot3^{\beta/3}\cdot5^{\gamma/6} \qquad \ldots(i)$$

α, β, γ satisfying two following property

$$0\le\alpha,\beta,\gamma\le10;\ \alpha+\beta+\gamma=10$$

$\therefore$ $\alpha=0,2,4,6,8,10;\ \beta=0,3,6;\ \gamma=0,6$

Hence, possible pairs of $(\alpha,\beta,\gamma)=(4,6,0);(4,0,6);(10,0,0)$

$\therefore$ There exists three rational terms.

So, sum of rational terms

$$=\frac{10!}{4!6!}\cdot2^2\cdot3^2+\frac{10!}{4!6!}2^2\cdot5^1+\frac{10!}{10!}2^5=12632$$

4. We have,

$$(1+x-3x^2)^{2145}=a_0+a_1x+a_2x^2+\cdots$$

On putting $x=-1$, we get

$$a_0-a_1+a_2-\ldots=(-3)^{2145}$$

But we know that,

$$3^1=3,\ 3^2=9,\ 3^3=27,\ 3^4=81$$

$\therefore$ $a_0-a_1+a_2+\ldots=[(-3)^4]^{536}(-3)^1$

$\therefore$ End digit of $(-3)^{2145}$

$=$ End digit of $[(-3)^4]^{536}\times$ End digit of $(-3)^1$

$=1\times3=3$

5. We have,

$$T_{r+1}={}^nC_r\left(\sqrt{\frac{q}{p}}\right)^{n-r}\left(10\sqrt{\frac{p^7}{q^3}}\right)^r={}^nC_r(q)^{\frac{n-r}{2}-\frac{3r}{10}}\times(p)^{\frac{r-n}{2}+\frac{7r}{10}}$$

$$={}^nC_r\cdot q^{\frac{5n-8r}{10}}\cdot p^{\frac{12r-5n}{10}}$$

For coefficient of pq, we put

$$\frac{5n-8r}{10}=1,\ \frac{12r-5n}{10}=1$$

$\Rightarrow$ $5n-8r-10=0,\ 12r-5n-10=0$

$\Rightarrow$ $r=5,\ n=10$

$\therefore$ $T_6={}^{10}C_5\ pq=252\ pq$

6. We have,

$$(5+2\sqrt{6})^n=(5+\sqrt{24})^n$$

Now, let $I+f=(5+\sqrt{24})^n$ ...(i)

$$0\le f<1 \qquad \ldots(ii)$$

and $f'=(5-\sqrt{24})^n$ ...(iii)

$$0<f'<1 \qquad \ldots(iv)$$

On adding Eqs. (i) and (iii), we get

$$I+f+f'=2k \text{ (even integer)}$$

$\Rightarrow$ $f'+f'=1$

$\Rightarrow$ $f'=1-f$

$\therefore$ $f^2-f+If-I=f(f-1)+I(f-1)$

$$=(f-1)(I+f)$$

$$=-(1-f)(I+f)=-f'(I+f)$$

$$=-(5-\sqrt{24})^n(5+\sqrt{24})^n$$

$$=-(25-24)^n=-1$$

$=$ a negative integer

7. Given, $x+\dfrac{1}{x}=1 \Rightarrow x^2-x+1=0$

$\Rightarrow$ $(x+\omega)(x+\omega^2)=0$

$\Rightarrow$ $x=-\omega,-\omega^2$

$\therefore$ $p=(-\omega)^{4000}+\dfrac{1}{(-\omega)^{4000}}=\omega^{4000}+\dfrac{1}{\omega^{4000}}$

$$=\omega+\frac{1}{\omega}=\frac{\omega^2+1}{\omega}=-\frac{\omega}{\omega}=-1$$

Also, for $x=-\omega^2,\ p=-1$

For $n>1, 2^n=4k,\ k\in N$

$\therefore$ $2^{2^n}=2^{4k}=(16)^k=$ last digit number is 6

Now, $q=$ unit digit at unit place in the number $(2^{2^n}+1)$

$$=6+1=7$$

$\therefore$ $p+q=-1+7=6$

8. Now, $\left(x+1+\dfrac{1}{x}\right)^n=\dfrac{(1+x+x^2)^n}{x^n}$

Since, $(1+x+x^2)^n$ is of the form

$a_0+a_1x+a_2x^2+\cdots+a_{2n}x^{2n}$ which contains $2n+1$ terms.

$\therefore$ $2n+1=401 \Rightarrow 2n=400 \Rightarrow n=200$

which is greater than 199.

9. We have, $\sum_{r=0}^{n-1} \frac{{}^nC_r}{{}^nC_r + {}^nC_{r+1}} = \sum_{r=0}^{n-1} \frac{{}^nC_r}{{}^{n+1}C_{r+1}}$

$$= \sum_{r=0}^{n-1} \frac{{}^nC_r}{\frac{n+1}{r+1}\,{}^nC_r} = \sum_{r=0}^{n-1} \frac{r+1}{n+1}$$

$$= \frac{1}{n+1}[1 + 2 + \cdots + n] = \frac{n(n+1)}{2(n+1)} = \frac{n}{2}$$

10. Here, $n = 100$, so the total number of terms is 101.

$\therefore$ Largest term = Middle term = 51th term

$$= {}^{100}C_{50}\left(\frac{b}{2}\right)^{50}\left(\frac{b}{2}\right)^{50} = {}^{100}C_{50}\left(\frac{b}{2}\right)^{100}$$

11. We have, $T_4 = {}^6C_3\left(\sqrt{x^{\left(\frac{1}{1+\log x}\right)}}\right)^3 (x^{1/12})^3 = 200$ [given]

$$\Rightarrow \quad 20\,(x^{\frac{3}{2(1+\log x)}})\,x^{1/4} = 200$$

$$\Rightarrow \quad x^{\frac{3}{2(1+\log x)} + \frac{1}{4}} = 10$$

On taking logarithm on base 10, we get

$$\left[\frac{3}{2(1+\log x)} + \frac{1}{4}\right]\log x = 1$$

$$\Rightarrow \quad \frac{(6+1+\log x)\log x}{4(1+\log x)} = 1$$

$$\Rightarrow \quad (\log x)^2 + 3\log x - 4 = 0$$

$$\Rightarrow \quad (\log x + 4)(\log x - 1) = 0$$

$$\Rightarrow \quad \log x = -4, 1$$

$$\therefore \quad x = 10^{-4}, 10$$

But $x > 1$

$$\therefore \quad x = 10$$

12. $\because (1+x)^m + (1+x)^{m+1} + \cdots + (1+x)^n$

$$= \frac{(1+x)^m\{(1+x)^{n-m+1} - 1\}}{(1+x) - 1} = \frac{(1+x)^{n+1} - (1+x)^m}{x}$$

$\therefore$ Coefficient of x^m in $(1+x)^m + (1+x)^{m+1} + \cdots + (1+x)^n$

or coefficient of x^m in $\frac{(1+x)^{n+1} - (1+x)^m}{x}$

or coefficient of x^{m+1} in $(1+x)^{n+1} - (1+x)^m$

$$= {}^{n+1}C_{m+1} - 0 = {}^{n+1}C_{m+1}$$

13. We have, ${}^{39}C_{3r-1} + {}^{39}C_{3r} = {}^{39}C_{r^2} + {}^{39}C_{r^2-1}$

$$\Rightarrow \quad {}^{40}C_{3r} = {}^{40}C_{r^2}$$

$$\Rightarrow \quad 3r = r^2 \text{ or } 40 - 3r = r^2$$

$$\Rightarrow \quad r = 0, 3 \text{ or } r^2 + 3r - 40 = 0$$

$$\Rightarrow \quad (r+8)(r-5) = 0 \;\Rightarrow\; r = 0, 3, 5, -8$$

But $r = 0, -8$ do not satisfy the given equation

$\therefore \quad r = 3, 5$

14. We have, $(1+x)^{20} = {}^{20}C_0 + {}^{20}C_1\,x + {}^{20}C_2\,x^2 + \cdots + {}^{20}C_{20}\,x^{20}$

On dividing by x, we get

$$\frac{(1+x)^{20}}{x} = \frac{{}^{20}C_0}{x} + {}^{20}C_1 + {}^{20}C_2 x + {}^{20}C_3 x^2 + \cdots + {}^{20}C_{20}\,x^{19}$$

On differentiating w.r.t. x, we get

$$\frac{20(1+x)^{19}\cdot x - (1+x)^{20}}{x^2} = \frac{-{}^{20}C_0}{x^2} + 0 + {}^{20}C_2 + 2\cdot{}^{20}C_3 x + \cdots + 19\cdot{}^{20}C_{20}\,x^{18}$$

On putting $x = 1$, we get

$$20(2)^{19} - (2)^{20} = -\frac{1}{1} + {}^{20}C_2 + 2\cdot{}^{20}C_3 + \cdots + 19\cdot{}^{20}C_{20}$$

$$\therefore \quad {}^{20}C_2 + 2\cdot{}^{20}C_3 + \cdots + 19\cdot{}^{20}C_{20} = 1 + 10\cdot 2^{20} - 2^{20} = 1 + 9\cdot 2^{20}$$

15. We have, $S = \frac{1\,(2^{2000} - 1)}{2 - 1} = 2^{2000} - 1 = (2^2)^{1000} - 1$

$$= (5-1)^{1000} - 1$$

$$= (5^{1000} - {}^{1000}C_1\cdot 5^{999} + {}^{1000}C_2\cdot 5^{998} \cdots + {}^{1000}C_{998}\cdot 5^2 - {}^{1000}C_{999}\cdot 5 + 1) - 1$$

$$= 5(5^{999} - {}^{1000}C_1\cdot 5^{998} + {}^{1000}C_2\cdot 5^{997} - \cdots - {}^{1000}C_{999})$$

$\therefore$ Remainder is 0.

16. Now, $(1+x)^n\left(1 + \frac{1}{x}\right)^n = \frac{(1+x)^{2n}}{x^n}$

$\therefore$ Coefficient of x^{-1} in $(1+x)^n\left(1 + \frac{1}{x}\right)^n$

= Coefficient of x^{n-1} in $(1+x)^{2n} = {}^{2n}C_{n-1} = \frac{(2n)!}{(n-1)!\,(n+1)!}$

17. $\because \quad 19^{9^4} = (20-1)^{9^4} = (20-1)^{6521} = -1 + (6521)\times 20 + \text{multiple of } 100$

$$= -1 + 20 + \text{multiple of } 100$$

$$= 19 + \text{multiple of } 100$$

$\therefore$ Last two digits of the number 19^{9^4} is 19.

18. $T_2 = {}^nC_1\,(\sqrt[13]{a})^{n-1}\left(\frac{a}{\sqrt{a^{-1}}}\right)^1 = 14a^{5/2}$ [given]

$$\Rightarrow \quad n(a)^{\frac{n-1}{13}}\,a^{1+\frac{1}{2}} = 14a^{5/2}$$

$$\Rightarrow \quad na^{\frac{n-1}{13}}\,a^{3/2} = 14a^{5/2}$$

When we put $n = 14$, then it satisfies the above equation

$$\therefore \quad \frac{{}^nC_3}{{}^nC_2} = \frac{{}^{14}C_3}{{}^{14}C_2} = \frac{14-3+1}{3} = 4$$

19. $6^{83} + 8^{83} = (7-1)^{83} + (7+1)^{83}$

$$= 2(7^{83} + {}^{83}C_2\cdot 7^{81} + {}^{83}C_4\cdot 7^{79} + \ldots + {}^{83}C_{80}7^3 + {}^{83}C_{82}7)$$

$$= 2\{49m + {}^{83}C_{82}\cdot 7\}$$

where, m is an integer

$$= 98m + 2\cdot{}^{83}C_1\cdot 7 = 98m + 2\cdot 83\cdot 7$$

$$= 98m + 2(77+6)\cdot 7 = 49(2m+22) + 84$$

$= 49(2m + 22) + 49 + 35$

$= 49(2m + 23) + 35 = 49n + 35$

where n is an integer.

Hence, the remainder is 35.

20. In the expansion of $(3^{1/4} + 2^{2/3})^{12}$, the general term is

$$T_{r+1} = {}^{12}C_r\,(3^{1/4})^{12-r}\,(2^{2/3})^r = {}^{12}C_r\,3^{3-\frac{r}{4}}\,2^{\frac{2r}{3}}$$

Now, $0 \le r \le 12$

$\therefore$ $r = 0, 12$

Rational terms are T_{0+1} and T_{12+1}

Now, $T_1 = {}^{12}C_0 3^3 2^0 = 27$

$\Rightarrow$ $T_{13} = {}^{12}C_{12}3^0 \cdot 2^8 = 256$

$\therefore$ Required sum $= T_1 + T_{13}$

$= 27 + 256 = 283$

21. $N = 7^{100} - 3^{100} = (7^2)^{50} - (3^2)^{50}$

$= (50-1)^{50} - (10-1)^{50}$

$= [(50)^{50} - {}^{50}C_1\,(50)^{49} + {}^{50}C_2(50)^{48} - {}^{50}C_3 (50)^{47} + \cdots + {}^{50}C_{48}\,(50)^2 - {}^{50}C_{49}(50) + 1]$

$- [10^{50} - {}^{50}C_1 \cdot 10^{49} + {}^{50}C_2(10)^{48} - {}^{50}C_3(10)^{47} + \cdots + {}^{50}C_{48}\,(10)^2 - {}^{50}C_{49}(10) + 1]$

$= [10^4 m - {}^{50}C_{47}(50)^3 + {}^{50}C_{48}(50)^2 - {}^{50}C_{49}\,(50) + 1]$

$- [10^4 n - {}^{50}C_{47}(10)^3 + {}^{50}C_{48}(10)^2 - {}^{50}C_{49}(10) + 1]$

when m and n are integers.

$= 10^4 p - {}^{50}C_3[(50)^3 - (10)^3] + {}^{50}C_2[(50)^2 - (10)^2] - {}^{50}C_1[(50) - (10)]$

When p is an integer.

$= 10^4 p - 124 \times 196 \times 10^5 + 294 \times 10^4 - 2000 = 10^4 q - 2000$

When q is an integer.

$= 10^4 q - 10^4 + 10^4 - 2000 = 10^4(q-1) + 8000$

$\therefore$ Last four digits $= 0000 + 8000 = 8000$

22. Let $P = 5^{99} = 5 \times 5^{98} = 5(25)^{49} = 5(26-1)^{49}$

$= 5[{}^{49}C_0(26)^{49} - {}^{49}C_1(26)^{48} + {}^{49}C_2(26)^{47} - \cdots + {}^{49}C_{48}(26) - {}^{49}C_{49} \cdot 1]$

$= 5 \times 26k - 5$, when k is an integer.

$\therefore$ $\dfrac{P}{13} = 10k - \dfrac{5}{13} = 10k - 1 + \dfrac{8}{13}$

Hence, the remainder is 8.

23. Now, $\dfrac{3^{2003}}{28} = \dfrac{3^2 \times 3^{2001}}{28} = \dfrac{9}{28}(3^3)^{667} = \dfrac{9}{28}(28-1)^{667}$

$= \dfrac{9}{28}\{(28)^{667} - {}^{667}C_1(28)^{666} + {}^{667}C_2(28)^{665} - \cdots + {}^{667}C_{666}(28) - 1\}$

$= 9k - \dfrac{9}{28}$, where k is an integer.

$= (9k-1) + \dfrac{19}{98}$

or $\left\{\dfrac{3^{2003}}{28}\right\} = \left\{(9k-1) + \dfrac{19}{28}\right\} = \dfrac{19}{28}$

24. $\sum_{r=0}^{20} r(20-r) \times ({}^{20}C_r)^2$

$= \sum_{r=0}^{20} r \times {}^{20}C_r(20-r) \times {}^{20}C_{20-r} = \sum_{r=0}^{20} 20 \cdot {}^{19}C_{r-1} \times 20 \times {}^{19}C_{19-r}$

$= 400 \sum_{r=0}^{20} {}^{19}C_{r-1} \times {}^{19}C_{19-r}$

$= 400 \times$ Coefficient of x^{18} in $(1+x)^{19}(1+x)^{19}$

$= 400 \times {}^{38}C_{18} = 400 \times {}^{38}C_{20}$

25. Given, $(3 + x^{2008} + x^{2009})^{2010} = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$

On putting $x = \omega$ and ω^2 respectively, we get

$(3 + \omega^{2008} + \omega^{2009})^{2010} = a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 + \cdots$

or $(3 + \omega + \omega^2)^{2010} = a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 + \cdots$

or $2^{2010} = a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 + a_4\omega^4 + a_5\omega^5 + a_6\omega^6 + \cdots$ ...(i)

and $[3 + (\omega^2)^{2008} + (\omega^2)^{2009}]^{2010}$

$= a_0 + a_1\omega^2 + a_2\omega^4 + a_3\omega^6 + a_4\omega^8 + a_5\omega^{10} + a_6\omega^{12} + \cdots$

or $(3 + \omega^2 + \omega)^{2010}$

$= a_0 + a_1\omega^2 + a_2\omega^4 + a_3\omega^6 + a_4\omega^8 + a_5\omega^{10} + a_6\omega^{12} + \cdots$

$\Rightarrow 2^{2010} = a_0 + a_1\omega^2 + a_2\omega^4 + a_3\omega^6 + a_4\omega^8 + a_5\omega^{10} + a_6\omega^{12}$ ...(ii)

On adding Eqs. (i) and (ii), we get

$2 \times 2^{2010} = 2a_0 + a_1(\omega + \omega^2) + a_2(\omega^2 + \omega^4) + a_3(\omega^3 + \omega^6) + a_4(\omega^4 + \omega^8) + a_5(\omega^5 + \omega^{10}) + a_6(\omega^6 + \omega^{12}) + \cdots$

$= 2a_0 - a_1 - a_2 + 2a_3 - a_4 - a_5 + 2a_6 - \cdots$

$\Rightarrow$ $2^{2010} = a_0 - \dfrac{1}{2}a_1 - \dfrac{1}{2}a_2 + a_3 - \dfrac{1}{2}a_4 - \dfrac{1}{2}a_5 + a_6 - \cdots$

26. Now, $\left(x^2 - 2 + \dfrac{1}{x^2}\right)^n = \dfrac{(x^4 - 2x^2 + 1)^n}{x^{2n}} = \dfrac{(x^2-1)^{2n}}{x^{2n}}$

$\therefore$ Total number of terms that are dependent of x is equal to number of terms in the expansion of $(x^2-1)^{2n}$ that have degree of x different from $2n$, which is given by $(2n+1) - 1 = 2n$.

27. Given expansion can be rewritten as $[1 + x^2(1-x)]^8$

$= {}^8C_0 + {}^8C_1x^2(1-x) + {}^8C_2\,x^4(1-x)^2$

$+ {}^8C_3x^6(1-x)^3 + {}^8C_4\,x^8(1-x)^4 + {}^8C_5x^{10}(1-x)^5 + \cdots$

There are only two terms, which we get the coefficient of x^{10}.

$\therefore$ Coefficient of $x^{10} = {}^8C_4[$Coefficient of x^2 in $(1-x)^4] + {}^8C_5\,[$Coefficient of x^0 in $(1-x)^5]$

$= {}^8C_4({}^4C_2) + {}^8C_5\,(1)$

$= ({}^8C_4)({}^4C_2) + {}^8C_3 = (70)(6) + 56 = 476$

28. $(1+ix)^{4n-2} = {}^{4n-2}C_0 + {}^{4n-2}C_1(ix) + {}^{4n-2}C_2(ix)^2 + \cdots + {}^{4n-2}C_{4n-2}\,(ix)^{4n-2}$

Here, we see that Ist negative term is T_3 and the next term is T_7 and the last negative term is T_{4n-1}.

Now, $3, 7, \cdots, 4n-1$

It is an AP.

$\because \quad l = a + (N-1)d$

$\therefore \quad 4n - 1 = 3 + (N-1)4$

$\Rightarrow \quad n - 1 = N - 1 \Rightarrow N = n$

29. $\because \quad \binom{n}{m}\binom{m}{p} = \frac{n!}{m!(n-m)!} \times \frac{m!}{p!(m-p)!}$

$$= \frac{n!}{(n-m)!\, p!(m-p)!} = \binom{n}{p}\binom{n-p}{m-p}$$

$$\therefore \sum_{p=1}^{n}\sum_{m=p}^{n}\binom{n}{m}\binom{m}{p} = \sum_{p=1}^{n}\sum_{m=p}^{n}\binom{n}{p}\binom{n-p}{m-p}$$

$$= \sum_{p=1}^{n}\binom{n}{p}\sum_{m=p}^{n}\binom{n-p}{m-p}$$

$$= \sum_{p=1}^{n}\binom{n}{p}\sum_{t=0}^{n-p}\binom{n-p}{t} \quad [\text{where, } t = m-p]$$

$$= 2^n \sum_{p=1}^{n}\binom{n}{p}2^{n-p}$$

$$= 2^n \sum_{p=1}^{n}\binom{n}{p}\frac{1}{2^p} = 2^n\left[\left(1+\frac{1}{2}\right)^n - 1\right] = 3^n - 2^n$$

30. Given, $\sum_{r=0}^{4}\frac{5^{4-r}}{(4-r)!}\left(\frac{x^r}{r!}\right) = \frac{8}{3}$

$\Rightarrow \quad \frac{(5+x)^4}{4!} = \frac{8}{3}$

$\Rightarrow \quad (5+x)^4 = 64 = (2\sqrt{2})^4 \Rightarrow 5 + x = \pm 2\sqrt{2}$

$\therefore \quad x = 2\sqrt{2} - 5 \text{ or } x = -2\sqrt{2} - 5$

Hence, largest real value of x is $2\sqrt{2} - 5$.

31. We have,

Coefficient of x in $(1+x)^m(1-x)^n = {}^mC_1 - {}^nC_1$

and coefficient of x^2 in $(1+x)^m(1-x)^n = {}^mC_2 - {}^mC_1\,{}^nC_1 + {}^nC_2$

According to the question, ${}^mC_1 - {}^nC_1 = 3$

$\Rightarrow \quad m - n = 3 \quad \ldots(i)$

and $\quad {}^mC_2 - {}^mC_1\,{}^nC_1 + {}^nC_2 = -6$

$\Rightarrow \quad \frac{m(m-1)}{2} - mn + \frac{n(n-1)}{2} = -6$

$\Rightarrow \quad (m-n)^2 - (m+n) = -12$

$\Rightarrow \quad 9 - (m+n) = -12 \quad [\text{from Eq. (i)}]$

or $\quad m + n = 21 \quad \ldots(ii)$

From Eqs. (i) and (ii), we get

$$m = 12 \text{ and } n = 9$$

32. Coefficient of rth, $(r+1)$ th and $(r+2)$th terms in $(1+x)^{14}$ are ${}^{14}C_{r-1}$, ${}^{14}C_r$ and ${}^{14}C_{r+1}$, respectively.

Now, according to the question, $2({}^{14}C_r) = {}^{14}C_{r-1} + {}^{14}C_{r+1}$

On dividing both sides by ${}^{14}C_r$, we get

$$2 = \frac{{}^{14}C_{r-1}}{{}^{14}C_r} + \frac{{}^{14}C_{r+1}}{{}^{14}C_r}$$

$\Rightarrow \quad 2 = \frac{r}{14-r+1} + \frac{14-(r+1)+1}{r+1}$

$\Rightarrow \quad 2 = \frac{r}{15-r} + \frac{14-r}{r+1}$

$\Rightarrow \quad 2(15-r)(r+1) = r(r+1) + (15-r)(14-r)$

$\Rightarrow \quad -2r^2 + 28r + 30 = 2r^2 - 28r + 210$

$\Rightarrow \quad 4r^2 - 56r + 180 = 0 \Rightarrow r^2 - 14r + 45 = 0$

$\Rightarrow \quad (r-9)(r-5) = 0$

$\Rightarrow \quad r = 5, 9$

33. $(3\sqrt{3}+5)^{2n+1} = (\sqrt{27}+5)^{2n+1}$

Now, let $\alpha + \beta = (\sqrt{27}+5)^{2n+1} \quad \ldots(i)$

$0 < \beta < 1 \quad \ldots(ii)$

and let $\quad \beta' = (\sqrt{27}-5)^{2n+1} \quad \ldots(iii)$

$0 < \beta' < 1 \quad \ldots(iv)$

On subtracting Eq. (iii) from Eq. (i), we get

$$\alpha + \beta - \beta' = (\sqrt{27}+5)^{2n+1} - (\sqrt{27}-5)^{2n+1} \quad \ldots(v)$$

$\Rightarrow \quad \alpha + 0 = 2p$ (even integer), $\forall\, p \in N$

$\Rightarrow \quad \alpha = 2p =$ even integer

Also, from Eq. (v), we get

$\alpha = (\sqrt{27}+5)^{2n+1} - (\sqrt{27}-5)^{2n+1}$ divisible by $(\sqrt{27}+5) - (\sqrt{27}-5)$, i.e. divisible by 10.

34. We have, $(8+3\sqrt{7})^n = (8+\sqrt{63})^n$

Now, let $\quad P + F = (8+\sqrt{63})^n \quad \ldots(i)$

$0 < F < 1 \quad \ldots(ii)$

and let $\quad F' = (8-\sqrt{63})^n \quad \ldots(iii)$

$0 < F' < 1 \quad \ldots(iv)$

On adding Eqs. (i) and (iii), we get

$$P + F + F' = (8+\sqrt{63})^n + (8-\sqrt{63})^n \quad \ldots(v)$$

$\Rightarrow \quad P + 1 = 2p$ (even integer), $\forall\, p \in N$

$\Rightarrow \quad P = 2p - 1 =$ odd integer

$\therefore \quad F' = 1 - F$

$\therefore \quad (1-F)(P+F) = F'(P+F) = (8-\sqrt{63})^n(8+\sqrt{63})^n$

$$= (64-63)^n = 1^n = 1$$

35. We have, 6th term in the expansion of

$$\left\{2^{\log_2\sqrt{(9^{x-1}+7)}} + 2^{\frac{1}{(1/5)\log_2(3^{x-1}+1)}}\right\}^7$$

or $\quad \left\{\sqrt{(9^{x-1}+7)} + \frac{1}{(3^{x-1}+1)^{1/5}}\right\}^7$ is $T_6 = T_{5+1}$

$$= {}^7C_5\,(\sqrt{9^{x-1}+7})^2\left\{\frac{1}{(3^{x-1}+1)^{1/5}}\right\}^5$$

$$= {}^7C_2 \cdot \frac{(9^{x-1}+7)}{(3^{x-1}+1)} = 21 \cdot \frac{(9^{x-1}+7)}{(3^{x-1}+1)} = 84 \quad [\text{given}]$$

$\Rightarrow \quad (9^{x-1}+7) = 4(3^{x-1}+1)$

Let $3^{x-1} = \lambda$, then

$$\lambda^2 - 4\lambda + 3 = 0$$

or $\quad (\lambda - 3)(\lambda - 1) = 0$

$\therefore \quad \lambda = 3, 1$

$\Rightarrow \quad 3^{x-1} = 3^1, 3^0$

or $\quad x - 1 = 1, 0$ or $x = 2, 1$

36. $\left(\sqrt{x} + \frac{1}{2\cdot\sqrt[4]{x}}\right)^n$ or $\left(x^{1/2} + \frac{1}{2}x^{-1/4}\right)^n$

$= {}^nC_0 \cdot x^{\frac{n}{2}} + {}^nC_1 \cdot \left(\frac{1}{2}\right)\cdot x^{\frac{2n-3}{4}} + {}^nC_2 \cdot \left(\frac{1}{2}\right)^2 \cdot x^{\frac{n-3}{2}} + \cdots$

According to the question,

$${}^nC_0, {}^nC_1\left(\frac{1}{2}\right), {}^nC_2\left(\frac{1}{2}\right)^2 \text{ are in AP.}$$

$\therefore \quad {}^nC_1 = {}^nC_0 + {}^nC_2\left(\frac{1}{2}\right)^2$

$\Rightarrow \quad n = 1 + \frac{n(n-1)}{4\cdot 2}$

$\Rightarrow \quad n^2 - 9n + 8 = 0$

$\Rightarrow \quad (n-8)(n-1) = 0$

$\therefore \quad n = 8, n \neq 1$

option (a) Number of terms $= 8 + 1 = 9$

option (b) Now, $T_{r+1} = {}^8C_r \cdot x^{4-\frac{r}{2}} \cdot \left(\frac{1}{2}\right)^r \cdot x^{-\frac{r}{4}}$

$\because \quad 0 \le r \le 8$

For integral powers of x, $r = 0, 4, 8$

$\therefore$ Number of terms in the expansion with integral power of x is 3.

option (c) From option (b),

$$T_{r+1} = {}^8C_r \cdot x^{4-\frac{3r}{4}} \cdot \left(\frac{1}{2}\right)^r$$

For independent of x,

$$4 - \frac{3r}{4} = 0$$

$$r = \frac{16}{3} \notin W$$

$\therefore$ No terms in the given expansion which is independent of x.

option (d) Middle term is

$$T_5 = {}^8C_4 \cdot x \cdot \left(\frac{1}{2}\right)^4$$

i.e. only one middle term.

37. We have,

Coefficient of x, x^2 and x^3 in $(1 + x^2)^2 (1 + x)^n$

i.e., values of a_1, a_2 and a_3 in $(1 + 2x^2 + x^4)(1 + x)^n$

$\Rightarrow \quad a_1 = {}^nC_1, a_2 = {}^nC_2 + 2$ and $a_3 = {}^nC_3 + 2\,{}^nC_1$

According to the question,

$$2a_2 = a_1 + a_3$$

$\Rightarrow \quad 2({}^nC_2 + 2) = {}^nC_1 + ({}^nC_3 + 2\,{}^nC_1)$

$\Rightarrow \quad \left[2\frac{n(n-1)}{2}\right] + 4 = 3n + \frac{n(n-1)(n-2)}{6}$

$\Rightarrow \quad n^3 - 9n^2 + 26n - 24 = 0$

$\Rightarrow \quad (n-2)(n^2 - 7n + 12) = 0$

$\Rightarrow \quad (n-2)(n-3)(n-4) = 0$

$\Rightarrow \quad n = 2, 3, 4$

Hence, $n = 3, 4$ $(n \neq 2, \because {}^nC_3$ is not defined)

38. We have, $\frac{17}{4} + 3\sqrt{2} = \frac{1}{4}(9 + 8 + 12\sqrt{2}) = \frac{1}{4}(3 + 2\sqrt{2})^2$

$\Rightarrow \quad 3 - \sqrt{\frac{17}{4} + 3\sqrt{2}} = 3 - \frac{1}{2}(3 + 2\sqrt{2}) = \left(\frac{3}{2} - \sqrt{2}\right)$

$\therefore$ 10th term in $\left(3 - \sqrt{\frac{17}{4} + 3\sqrt{2}}\right)^{20}$ is

$$T_{9+1} = {}^{20}C_9\left(\frac{3}{2}\right)^{20-9}(-\sqrt{2})^9$$

$$= {}^{20}C_9(-1)^9 3^{11}\cdot 2^{-11+\frac{9}{2}}$$

$$= -\,{}^{20}C_9 3^{11} 2^{-\frac{13}{2}}$$

which is a negative and an irrational number.

39. We have, $C_0 - (C_0 + C_1) + (C_0 + C_1 + C_2) - (C_0 + C_1 + C_2 + C_3) + \cdots (-1)^{n-1}(C_0 + C_1 + \cdots + C_{n-1})$

For even integer, take $n = 2m$, we get

$= C_0 - (C_0 + C_1) + (C_0 + C_1 + C_2) - (C_0 + C_1 + C_2 + C_3) + \cdots - (C_0 + C_1 + \cdots + C_{2m-1})$

$= -(C_1 + C_3 + C_5 + \cdots + C_{2m-1})$

$= -(C_1 + C_3 + C_5 + \cdots + C_{n-1}) \quad [\because n = 2m]$

$= -2^{n-1}$

40. We have,

$$f(n) = \sum_{i=0}^{n}\binom{30}{30-i}\binom{20}{n-i} = \sum_{i=0}^{n}\binom{30}{i}\binom{20}{n-i} = {}^{50}C_n$$

$\therefore f(n)$ is greatest, when $n = 25$

$\therefore$ Maximum value of $f(n)$ is ${}^{50}C_{25}$.

Also, $\quad f(0) + f(1) + \cdots + f(50)$

$$= {}^{50}C_0 + {}^{50}C_1 + {}^{50}C_2 + \cdots + {}^{50}C_{50} = 2^{50}$$

Also, ${}^{50}C_n$ is not divisible by 50 for any n as 50 is not a prime number.

$$\sum_{n=0}^{50}(f(n))^2 = ({}^{50}C_0)^2 + ({}^{50}C_1)^2 + \cdots + ({}^{50}C_{50})^2 = {}^{100}C_{50}$$

41. ${}^{69}C_{3r-1} + {}^{69}C_{3r} = {}^{69}C_{r^2-1} + {}^{69}C_{r^2}$

$\Rightarrow \quad {}^{70}C_{3r} = {}^{70}C_{r^2}$

$\Rightarrow \quad r^2 = 3r$ or $70 - 3r = r^2$

$\Rightarrow \quad r = 0, 3$ or $r^3 + 3r - 70 = 0$

$\Rightarrow \quad r = 0, 3$ or $(r + 10)(r - 7) = 0$

$\Rightarrow \quad r = 0, 3, 7, -10$

But $r = 0, -10$ do not satisfies the given equation.
Hence, two values of r satisfies,
i.e. $r = 3, 7$

42. Here, n is even, so middle term is $\left(\frac{8}{2}+1\right)$th, i.e. 5th term.

$$\therefore \quad T_5 = {}^8C_4(x)^4\left(\frac{\sin^{-1}x}{x}\right)^4 = \frac{630}{16} \qquad \text{[given]}$$

$$\Rightarrow \quad 70(\sin^{-1}x)^4 = \frac{630}{10} \Rightarrow (\sin^{-1}x)^4 = \frac{9}{16}$$

$$\Rightarrow \quad (\sin^{-1}x)^2 = \frac{3}{4} \Rightarrow \sin^{-1}x = \pm\frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad x = -\frac{\pi}{3}, \frac{\pi}{3} \qquad \left[\because \sin^{-1}x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right]$$

43. Sum of coefficients $= (a\alpha^2 + 2b\alpha + c)^n$

Let $f(\alpha) = a\alpha^2 + 2b\alpha + c$

Now, $D = 4b^2 - 4ac = 4(b^2 - ac) < 0$

$\therefore$ $f(\alpha) < 0$ or $f(\alpha) > 0, \forall\ \alpha \in R$

If $a > 0$, then $f(\alpha) > 0$

$\Rightarrow (a\alpha^2 + 2b\alpha + c)^n > 0$

If $c > 0$, i.e. $f(0) > 0 \Rightarrow f(\alpha) > 0$

$\Rightarrow (a\alpha^2 + 2b\alpha + c)^n > 0$

If $a < 0$, then $f(\alpha) < 0$

$\Rightarrow (a\alpha^2 + 2b\alpha + c)^n < 0$, if n is odd

If $c < 0$, then $f(0) < 0 \Rightarrow f(\alpha) < 0$

$\Rightarrow (a\alpha^2 + 2b\alpha + c)^n > 0$, if n is even.

44. $\because \left(x^2 + 1 + \frac{1}{x^2}\right)^n = \frac{(1 + x^2 + x^4)^n}{x^{2n}}$

$$= \frac{a_0 + a_1x^2 + a_2x^4 + \ldots + a_{2n}x^{4n}}{x^{2n}}$$

$\therefore$ Number of terms $= 2n + 1$

Term independent of $x = a_n$ = Constant term in $\left(x^2 + 1 + \frac{1}{x^2}\right)^n$

$$= \text{Coefficient of } x^{2n} \text{ in } (1 + x^2 + x^4)^n$$
$$= \text{Coefficient of } x^n \text{ in } (1 + x + x^2)^n$$
$$= \frac{d^n}{dx^n}(1 + x + x^2)^n \neq 2^{n-1}$$

Coefficient of x^{2n-2} in $\left(x^2 + 1 + \frac{1}{x^2}\right)^n$

$$= \text{Coefficient of } x^{4n-2} \text{ in } (1 + x^2 + x^4)^n$$
$$= \text{Coefficient of } x^{2n-1} \text{ in } (1 + x + x^2)^n$$

Now, let $(1 + x + x^2)^n = \lambda_0 + \lambda_1 x + \lambda_2 x^2 + \ldots + \lambda_{2n-1}x^{2\lambda-1} + \lambda_{2n}x^{2n}$

On replacing x by $\frac{1}{x}$, we get

$$\left(1 + \frac{1}{x} + \frac{1}{x^2}\right)^n = \lambda_0 + \frac{\lambda_1}{x} + \frac{\lambda_2}{x^2} + \ldots + \frac{\lambda_{2n-1}}{x^{2n-1}} + \frac{\lambda_{2n}}{x^{2n}}$$

or $(1 + x + x^2)^n = \lambda_{2n} + \lambda_{2n-1}x + \ldots + \lambda_1x^{2n-1} + \lambda_0x^{2n}$

On differentiating both sides w.r.t. x, we get

$$n(1 + x + x^2)^n \cdot (1 + 2x) = \lambda_{2n-1} + \ldots + 2n\lambda_0x^{2n-1}$$

On putting $x = 0$, we get $\lambda_{2n-1} = n$

Hence, coefficient of $x^{2n-2} = n$

and coefficient of x^2 in $\left(x^2 + 1 + \frac{1}{x^2}\right)^n$

$$= \text{Coefficient of } x^{2n+2} \text{ in } (1 + x^2 + x^4)^n$$
$$= \text{Coefficient of } x^{n+1} \text{ in } (1 + x + x^2)^n$$
$$= \frac{d^{n+1}}{dx^{n+1}}(1 + x + x^2)^n \neq n$$

45. Now, $\left(x + \frac{1}{x}\right)^{20} = {}^{20}C_0x^{20} + {}^{20}C_1x^{18} + {}^{20}C_2x^{16} + {}^{20}C_3x^{14} + \ldots + {}^{20}C_9x^2 + {}^{20}C_{10} + {}^{20}C_{11}x^{-2} + \ldots + {}^{20}C_{20}x^{-20}$

$$T_{r+1} = {}^{20}C_r \cdot x^{20-2r} \qquad \ldots(i)$$

and $\left(x^2 + 2 + \frac{1}{x^2}\right)^{10} = \left(\frac{1}{x} + x\right)^{20}$

$$= {}^{20}C_0x^{-20} + {}^{20}C_1x^{-18} + {}^{20}C_2x^{-16} + \ldots + {}^{20}C_{10} + {}^{20}C_{11}x^2 + {}^{20}C_{12}x^4 + \ldots + {}^{20}C_{20}x^{20}$$

$\therefore$ $T_6 = T_{5+1} = {}^{20}C_5x^{-10}$ ...(ii)

According to the question, ${}^{20}C_r = {}^{20}C_5$

$\therefore$ $r = 5$ or $20 = r + 5 \Rightarrow Sr = 5, 15$

■ ***Sol.*** (Q. Nos. 46 to 48)

46. We have, $(1 + x + x^2)^n = \sum_{r=0}^{2n} a_rx^r$ ...(i)

On replacing x by $\frac{1}{x}$, we get

$$\left(1 + \frac{1}{x} + \frac{1}{x^2}\right)^n = \sum_{r=0}^{2n} a_r\left(\frac{1}{x}\right)^r$$

$$\Rightarrow \quad (1 + x + x^2)^n = \sum_{r=0}^{2n} a_rx^{2n-r} \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$\sum_{r=0}^{2n} a_rx^r = \sum_{r=0}^{2n} a_rx^{2n-r}$$

Equating the coefficient of x^{2n-r} on both sides, we get

$$a_{2n-r} = a_r \qquad \ldots(iii)$$
$$0 \leq r \leq 2n$$

On putting $r = 0, 1, 2, 3, \ldots, n-1, n$, we get

$$a_{2n} = a_0$$
$$a_{2n-1} = a_1$$
$$a_{2n-2} = a_2$$
$$a_{2n-3} = a_3$$
$$\vdots \quad \vdots \quad \vdots$$
$$a_{n+1} = a_{n-1}, a_n = a_n$$

Then, $a_0 + a_1 + a_2 + \ldots + a_{n-1}$

$= a_{n+1} + a_{n+2} + \ldots + a_{2n}$ ...(iv)

and on putting $x = 1$ in Eq. (i), we get

$$\sum_{r=0}^{2n} a_r = 3^n$$

$\Rightarrow (a_0 + a_1 + a_2 + \ldots + a_{n-1}) + a_n + (a_{n+1} + a_{n+2} + \ldots + a_{2n}) = 3^n$

From Eq.(iv), we get

$2(a_0 + a_1 + a_2 + \ldots + a_{n-1}) = 3^n - a_n$

or $\sum_{r=0}^{n-1} a_r = \frac{(3^n - a_n)}{2}$

47. On putting $x = 1$ and $x = -1$ in Eq. (i), we get

$3^n = a_0 + a_1 + a_2 + \ldots + a_{2n}$ ...(v)

$1 = a_0 - a_1 + a_2 - a_3 + \ldots + a_{2n}$ ...(vi)

On adding and subtracting Eqs. (v) and (vi), we get

$\frac{3^n + 1}{2} = (a_0 + a_2 + \ldots + a_{2n})$ ...(vii)

$\frac{3^n - 1}{2} = (a_1 + a_3 + a_5 + \ldots + a_{2n-1})$ ...(viii)

Also, $a_r = a_{2n-r}$

Put $r = 0, 2, 4, 6, \ldots, n-1$

$a_0 = a_{2n}, a_2 = a_{2n-2}, a_4 = a_{2n-4}, \ldots$

$a_{n-1} = a_{n+1}$

From Eq. (vii), we get

$\frac{3^n + 1}{2} = 2(a_0 + a_2 + \ldots + a_{n-2}) + a_n$

$\frac{3^n + 1 - 2a_n}{4} = a_0 + a_2 + \ldots + a_{n-1}$

48. From Eq. (viii), we get

$\frac{3^n - 1}{2} = 2(a_1 + a_3 + \ldots + a_n) - a_n$

$\frac{3^n - 1 + 2a_n}{4} = (a_1 + a_3 + \ldots + a_n)$

■ ***Sol.*** (Q. Nos. 49 to 51)

Given, $(1 + x + 2x^2)^{20} = a_0 + a_1 x + a_2 x^2 + \ldots + a_{40} x^{40}$

On putting $x = 1$ and $x = -1$ respectively, we get

$a_0 + a_1 + a_2 + \ldots + a_{40} = 4^{20} = 2^{40}$ ...(i)

and $a_0 - a_1 + a_2 - \ldots + a_{40} = 2^{20}$ ...(ii)

From Eqs. (i) and (ii), we get

$a_0 + a_2 + a_4 + \ldots + a_{38} + a_{40} = 2^{19}(2^{20} + 1)$ ...(iii)

and $a_1 + a_3 + a_5 + \ldots + a_{37} + a_{39} = 2^{19}(2^{20} - 1)$ ...(iv)

Also, replacing x by $\frac{1}{x}$ in given expression, we get

$$\left(1 + \frac{1}{x} + \frac{2}{x^2}\right)^{20} = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \ldots + \frac{a_{38}}{x^{38}} + \frac{a_{39}}{x^{39}} + \frac{a_{40}}{x^{40}}$$

$\Rightarrow (2 + x + x^2)^{20} = a_0 x^{40} + a_1 x^{39} + \ldots + a_{38} x^2 + a_{39} x + a_{40}$ ...(v)

On putting $x = 0$, we get $a_{40} = 2^{20}$ ...(vi)

On differentiating both sides of Eq. (v) w.r.t. x and put $x = 0$, we get

$a_{39} = 20(2)^{19}$ ...(vii)

49. $a_0 + a_2 + a_4 + \ldots + a_{38} + a_{40} = 2^{19}(2^{20} + 1)$ [from Eq. (iii)]

$\Rightarrow a_0 + a_2 + a_4 + \ldots + a_{38} = 2^{19}(2^{20} + 1) - a_{40}$

$= 2^{19}(2^{20} + 1) - 2^{20}$ [from Eq. (vi)]

$= 2^{19}(2^{20} - 1)$

50. $a_1 + a_3 + a_5 + \ldots + a_{37} + a_{39} = 2^{19}(2^{20} - 1)$ [from Eq. (iv)]

$\Rightarrow a_1 + a_3 + a_5 + \ldots + a_{37} = 2^{19}(2^{20} - 1) - a_{39}$

$= 2^{19}(2^{20} - 1) - 20(2)^{19}$ [from Eq. (vii)]

$= 2^{19}(2^{20} - 21)$

51. From Eqs. (vi) and (vii), we get $\frac{a_{39}}{a_{40}} = \frac{20(2)^{19}}{2^{20}} = 10$

■ ***Sol.*** (Q. Nos. 52 to 54)

52. $5^{40} = (5^2)^{20} = (22 + 3)^{20} = 22\lambda + 3^{20}, \lambda \in N$

Also, $3^{20} = (3^2)^{10} = (11 - 2)^{10} = 11\mu + 2^{10}, \mu \in N$

Now, $2^{10} = 1024 = 11 \times 93 + 1$

$\therefore$ Remainder, $a = 1$

Also, $2^{2011} = 2^3(2^4)^{502} = 2^3(17 - 1)^{502}$

$= 8[(17)^{502} - {}^{502}C_1(17)^{501} + \ldots - {}^{502}C_{501}(17) + 1]$

$= 8(17\lambda + 1), \lambda \in N = 8 \times 17\lambda + 8$

$\therefore$ Remainder, $b = 8$

Hence, $a + b = 1 + 8 = 9$

53. $19^{93} - 13^{99} =$ (odd number) $-$ (odd number) $=$ even number

$\therefore$ $19^{93} - 13^{99}$ is divisible by 2.

Now, $19^{93} - 13^{99} = (18 + 1)^{93} - (12 + 1)^{99}$

$= [(18)^{93} + {}^{93}C_1(18)^{92} + {}^{93}C_2(18)^{91} \ldots + \ldots + {}^{93}C_{92}(18) + 1]$

$- [(12)^{99} + {}^{99}C_1(12)^{98} + {}^{99}C_2(12)^{97} + \ldots + {}^{99}C_{98}(12) + 1]$

$= (18)^2\lambda + {}^{93}C_1 \times 18 - (12)^2\mu - {}^{99}C_1(12)$

When λ and μ are integers

$= (18)^2\lambda - (12)^2\mu + 486$

$= 81 \times 4\lambda - 12^2({}^{99}C_2 + 12 \cdot {}^{99}C_3) + 81p + 486$

$= 81$ (integer), where p is an integer.

But 2 and 81 are co-prime.

$\therefore$ $19^{93} - 13^{99}$ is divisible by 162.

54. $13^{99} = (12 + 1)^{99} = (12)^{99} + {}^{99}C_1(12)^{98} + {}^{99}C_2(12)^{97} + \ldots + {}^{99}C_{97}(12)^2 + {}^{99}C_{98}(12) + 1$

$= \{(12)^{99} + {}^{99}C_1(12)^{98} + {}^{99}C_2(12)^{97} + \ldots + {}^{99}C_{97}(12)^2\} + {}^{99}C_1(12) + 1$

$= 81\lambda + 99 \times 12 + 1$, where λ is an integer

$= 81\lambda + 81 \times 14 + 55$

$\therefore$ Remainder $= 55$

■ ***Sol.*** (Q. Nos. 55 to 57)

Here, $(1+2)^n = 2187$

$$3^n = 2187 = 3^7 \Rightarrow n = 7$$

At $x = \frac{1}{\sqrt{2}}$, $R = (\sqrt{2}+1)^7 = I + f$

55. Let $f' = (\sqrt{2}-1)^7$, $0 < f' < 1$

$\therefore \quad Rf' = (\sqrt{2}+1)^7 (\sqrt{2}-1)^7 = (1)^7 = 1$

$\therefore \quad (n + Rf) = 7 + 1 = 8$

56. Here, $m = \left| \frac{(n+1)(x)}{1+x} \right|$

$$= \left| \frac{(7+1)\left(2 \times \frac{1}{3}\right)}{1 + 2 \times \frac{1}{3}} \right| = -\frac{8 \times \frac{2}{3}}{\frac{5}{3}} = \frac{16}{5} = 3.2$$

$T_{[m]+1} = T_{3+1} = T_4$

57. Here, $n = 7$

$\therefore$ Greatest coefficient $= \frac{{}^7C_{7-1}}{2}$ or ${}^7C_3$

and $\frac{{}^7C_{7+1}}{2}$, i.e. ${}^7C_4$

Sum of values of $k = (3+1) + (4+1) = 9$

■ ***Sol.*** (Q. Nos. 58 to 60)

58. $(x + C_0)(x + 3 \cdot C_1)(x + 5 \cdot C_2) + \ldots \{x + (2n+1) \cdot C_n\}$

$= x^{n+1} + x^n \{C_0 + 3 \cdot C_1 + 5 \cdot C_2 + \ldots + (2n+1) \cdot C_n\}$

$\therefore$ Coefficient of $x^n = C_0 + 3 \cdot C_1 + 5 \cdot C_2 + \ldots + (2n+1) \cdot C_n$

$= (C_0 + C_1 + C_2 + \ldots + C_n)$

$\quad + 2\{C_1 + 2 \cdot C_2 + \ldots n \cdot C_n\}$

$$= 2^n + 2\left\{n + 2 \cdot \frac{n(n-1)}{2} + \ldots + n\right\}$$

$$= 2^n + 2n\left\{1 + (n-1) + \frac{(n-1)(n-2)}{1 \cdot 2} + \ldots + 1\right\}$$

$$= 2^n + 2n\{{}^{n-1}C_0 + {}^{n-1}C_1 + {}^{n-1}C_2 + \ldots + {}^{n-1}C_{n-1}\}$$

$$= 2^n + 2n(1+1)^{n-1} = 2^n + n \cdot 2^n = (n+1)2^n$$

59. $(x + C_0)(x + C_1)(x + C_2) + \ldots + (x + C_n)$

$$= x^{n+1} + \left(\sum_{r=0}^{n} C_r\right) x^n + \left(\sum_{0 \le i}\sum_{j \le n} C_i C_j\right) x^{n-1} + \ldots$$

$\therefore$ Coefficient of x^{n-1} in $\sum\sum_{0 \le i \le j \le n} C_i C_j$

$$= \frac{1}{2}\left\{\left(\sum_{r=0}^{n} C_r\right)^2 - \left(\sum_{r=0}^{n} C_r^2\right)\right\} = \frac{1}{2}\{2^{2n} - {}^{2n}C_n\}$$

$$= 2^{2n-1} - \frac{1}{2} \cdot {}^{2n}C_n$$

60. $(x+1)^2 (x+3)^3 (x+5)^4$

$= (x+1)(x+1)(x+3)(x+3)(x+3)(x+5)(x+5)$

$\quad (x+5)(x+5)$

$= x^9 + (1+1+3+3+3+5+5+5+5)\,x^8$

$\quad + (1 \cdot 1 + 1 \cdot 3 + 1 \cdot 3 + 1 \cdot 3 + 1 \cdot 5 + \ldots + 1 \cdot 5 + 3 \cdot 3$

$\quad + 3 \cdot 3 + 3 \cdot 5 + \ldots + 5 \cdot 5)\,x^7 + \ldots$

$\therefore$ Coefficient of $x^7 = (1+1+3+3+3+5+5+5+5)^2$

$$\frac{-(1^2 + 1^2 + 3^2 + 3^2 + 3^2 + 5^2 + 5^2 + 5^2 + 5^2)}{2}$$

$$= \frac{(31)^2 - (129)}{2} = \frac{961 - 129}{2} = 416$$

■ ***Sol.*** (Q. Nos. 61 to 83)

Since, sum of coefficient of B is 6561.

$$\therefore \quad \left(\frac{5}{2} + \frac{1}{2}\right)^n = 6561$$

$$\Rightarrow \quad 3^n = 6561 \Rightarrow 3^n = 3^8$$

$$\therefore \quad n = 8$$

61. Coefficient $(T_3 - T_2) = 117$

$${}^mC_2 3^2 - {}^mC_1 3^1 = 117$$

$$\Rightarrow \quad m = 6$$

62. $n^m = 8^6 = (1+7)^6 = (1+7k)$

Hence, remainder is 1.

63. $$\frac{\text{Coefficient of } T_2 \text{ in } \left(\frac{5x}{2} + \frac{x^{-2}}{2}\right)^8}{\text{Coefficient of } T_2 \text{ in } \left(\frac{x^{-2}}{2} + \frac{5x}{2}\right)^8} = \frac{{}^8C_1 \left(\frac{5}{2}\right)^7 \left(\frac{1}{2}\right)}{{}^8C_1 \left(\frac{1}{2}\right)^7 \left(\frac{5}{2}\right)} = 5^6 = 15625$$

■ ***Sol.*** (Q. Nos. 64 to 66)

$\because a_4, a_5, a_6$ i.e., ${}^nC_4, {}^nC_5, {}^nC_6$ are in AP, then

$$2 \cdot {}^nC_5 = {}^nC_4 + {}^nC_6$$

$$\Rightarrow \quad 2 = \frac{{}^nC_4}{{}^nC_5} + \frac{{}^nC_6}{{}^nC_5} = \frac{5}{n-5+1} + \frac{n-6+1}{6}$$

$$\Rightarrow \quad 2 = \frac{5}{n-4} + \frac{n-5}{6}$$

$$\Rightarrow \quad 12n - 48 = 30 + n^2 - 9n + 20$$

$$\Rightarrow \quad n^2 - 21n + 98 = 0 \Rightarrow n = 7, 14$$

Hence, $n = 7$ $[\because n < 10]$

Also, $A = (\sqrt[3]{2} + \sqrt[4]{3})^{13n} = (2^{1/3} + 3^{1/4})^{91}$

$\therefore \quad T_{r+1} = {}^{91}C_r (2^{1/3})^{91-r} \cdot (3^{1/4})^r$

$$= {}^{91}C_r \cdot 2^{\frac{91-r}{3}} \cdot 3^{r/4} \qquad \ldots(i)$$

64. $\sum_{i=1}^{n} a_i = \sum_{i=1}^{7} a_i = a_1 + a_2 + a_3 + \ldots + a_7$

$$= {}^7C_1 + {}^7C_2 + {}^7C_3 + \ldots + {}^7C_7 = 2^7 - 1 = 127$$

65. From Eq. (i), we get

$$0 \le r \le 91$$

For rational terms, $r = 4, 16, 28, 40, 52, 64, 76, 88$

Rational terms are $T_5, T_{17}, T_{29}, T_{41}, T_{53}, T_{65}, T_{77}, T_{89}$

$\therefore \quad a_m = 89$

66. Also, 5, 17, 29, 41 53,..., 89
are in AP with common difference 12.

67. The unit digit of 2^{2^n} is always 6 for $n > 1$.

Now, $\sum_{r=0}^{100} r! = 0! + 1! + 2! + 3! + 4! + 10(k); k \in N$

$= 1 + 1 + 2 + 6 + 24 + 10\,k = 34 + 10\,k$

$\therefore$ Unit digit of $\sum_{r=0}^{100} r! + 2^{2^n}$

$=$ Unit place of $\sum_{r=0}^{100} r!$ + Unit place of 2^{2^n}

$= 4 + 0 + 6 = 10$, its unit place is 0.

68. Given, $\sum_{r=0}^{3n} a_r x^r = (1 + x + x^2 + x^3)^n$

It is clear that a_r is the coefficient of x^r in the expansion of $(1 + x + x^2 + x^3)^n$.

On replacing x by $\frac{1}{x}$ in the given equation , we get

$$\sum_{r=0}^{3n} a_r \left(\frac{1}{x}\right)^r = \frac{(1 + x + x^2 + x^3)^n}{x^{3n}}$$

Here, a_r represents the coefficient of 3^{3n-r} in $(1 + x + x^2 + x^3)^n$.

Thus, $a_r = a_{3n-r}$ ...(i)

Let $I = \sum_{r=0}^{3n} r \times a_r = \sum_{r=0}^{3n} (3n - r)\, a_{3n-r}$

[replacing r by $3n - r$]

$= \sum_{r=0}^{3n} (3n - r)\, a_r$ [from Eq. (i)]

$= 3n \sum_{r=0}^{3n} - \sum_{r=0}^{3n} r a r$

$\Rightarrow \quad 2I = 3nk \;\Rightarrow\; I = \frac{3nk}{2} \;\therefore\; \lambda = 3$

69. We have, $T_{r+1} = {}^{20}C_r \cdot 4^{\frac{20-r}{3}} \cdot 6^{-r/4}$

$= {}^{20}C_r \cdot 2^{\frac{40-2r}{3}} \cdot 2^{-r/4} \cdot 3^{-r/4}$

$= {}^{20}C_r \cdot 2^{\frac{160-11r}{12}} \cdot 3^{-r/4}$

For rational terms, $\frac{r}{4}$ and $\frac{160 - 11r}{12}$ must be integers and

$0 \le r \le 20$.

$\therefore$ $\frac{r}{4}$ is an integer.

$\Rightarrow \quad r = 0, 4, 8, 12, 16, 20$

Clearly, for $r = 8, 16$ and 20 $\frac{160 - 11r}{12}$ is also an integer.

$\therefore$ The number of rational terms is 3.

70. We have, $2^{2006} = 2^2 (2^3)^{668}$

$= 4 (1 + 7)^{668} = 4 (1 + 7k) = 4 + 28k$

$\therefore\; 2^{2006} + 2006 = 4 + 28k + 7 \times (286) + 4$

Hence, remainder is 8.

71. We have, $19^{9^4} = (20 - 1)^{6561}$

$= (20)^{6561} - {}^{6561}C_1 (20)^{6560} + {}^{6561}C_2 (20)^{6559}$

$- \ldots - {}^{6561}C_{6559} (20)^2 + {}^{6561}C_{6560} (20)^2 - 1$

$= 1000\,k - {}^{6561}C_2 (400) + {}^{6561}C_1 (20) - 1$

where, k is an integer.

$= 1000p + 6561 \times 20 - 1 = 1000p + 131220 - 1$

where, p is an integer.

$= 1000p + 131219$

$\therefore \quad ab = 19$

i.e. $\quad a = 1, b = 9$

Hence, $b - 3a = 9 - 3 = 6$

72. ${}^nC_r + 4 \cdot {}^nC_{r+1} + 6 \cdot {}^nC_{r+2} + 4 \cdot {}^nC_{r+3} + {}^nC_{r+4}$

$= ({}^nC_r + {}^nC_{r+1}) + 3\,({}^nC_{r+1} + {}^nC_{r+2})$

$+ 3\,({}^nC_{r+2} + {}^nC_{r+3}) + ({}^nC_{r+3} + {}^nC_{r+4})$

$= {}^{n+1}C_{r+1} + 3 \cdot {}^{n+1}C_{r+2} + 3 \cdot {}^{n+1}C_{r+3} + {}^{n+1}C_{r+4}$

$= ({}^{n+1}C_{r+1} + {}^{n+1}C_{r+2}) + 2\,({}^{n+1}C_{r+2}$

$+ {}^{n+1}C_{r+3}) + ({}^{n+1}C_{r+3} + {}^{n+1}C_{r+4})$

$= {}^{n+2}C_{r+2} + 2 \cdot {}^{n+2}C_{r+3} + {}^{n+2}C_{r+4}$

$= ({}^{n+2}C_{r+2} + {}^{n+2}C_{r+3}) + ({}^{n+2}C_{r+3} + {}^{n+2}C_{r+4})$

$= {}^{n+3}C_{r+3} + {}^{n+3}C_{r+4}$

$= {}^{n+4}C_{r+4} = \frac{n+4}{r+4} {}^{n+3}C_{r+3}$

Similarly, ${}^nC_r + 3 \cdot {}^nC_{r+1} + 3 \cdot {}^nC_{r+2} + {}^nC_{r+3} = {}^{n+3}C_{r+3}$

$\therefore \quad \frac{n+4}{r+4} = \frac{n+\lambda}{r+\lambda} \Rightarrow\; \lambda = 4$

73. $99^{50} - 99 \cdot 98^{50} + \frac{99 \cdot 98}{1 \cdot 2} (97)^{50} - \ldots + 99$

$= 99^{50} - {}^{99}C_1 (98)^{50} + {}^{99}C_2 (97)^{50} - \ldots + {}^{99}C_{98} \cdot 1$

$= {}^{99}C_0 (99)^{50} - {}^{99}C_1 (99 - 1)^{50} + {}^{99}C_2 (99 - 2)^{50} - \ldots$

$+ {}^{99}C_{98} (99 - 98)^{50} - {}^{99}C_{99} (99 - 99)^{50}$

$= (99)^{50} \{{}^{99}C_0 - {}^{99}C_1 + {}^{99}C_2 - \ldots + {}^{99}C_{98} - {}^{99}C_{99}\}$

$+ {}^{50}C_1 (99)^{49} \{{}^{99}C_1 - 2 \cdot {}^{99}C_2 + 3 \cdot {}^{99}C_3 - \ldots\}$

$= 0 + 0 = 0$

74. Given,

$\because$ Greatest term in the expansion of $(1 + x)^{2n}$ has the greatest coefficient.

$\therefore \quad T_{n+1} = {}^{2n}C_n x^n$ (greatest term)

$\therefore \quad T_n < T_{n+1} > T_{n+2}$

$\Rightarrow \quad {}^{2n}C_{n-1} \cdot x^{n-1} < {}^{2n}C_n \cdot x^n > {}^{2n}C_{n+1} \cdot x^{n+1}$

$\Rightarrow \quad \frac{{}^{2n}C_{n-1}}{{}^{2n}C_n} \cdot \frac{1}{x} < 1 > \frac{{}^{2n}C_{n+1}}{{}^{2n}C_n} \cdot x$

$\Rightarrow \quad \frac{n}{n+1} \cdot \frac{1}{x} < 1 > \frac{n}{n+1}\, x$

$\Rightarrow \quad x > \frac{n}{n+1}$ and $x < \frac{n+1}{n}$

i.e., $x \in \left(\frac{n}{n+1}, \frac{n+1}{n}\right)$

Given, $x \in \left(\frac{10}{11}, \frac{11}{10}\right)$

$\therefore \quad n = 10$

Also, T_4 in $\left(\lambda x + \frac{1}{x}\right)^m = {}^mC_3(\lambda x)^{m-3}\left(\frac{1}{x}\right)^3 = \frac{n}{4}$ [given]

$${}^mC_3 \cdot \lambda^{m-3} \cdot x^{m-6} = \frac{n}{4} \quad \text{[given]}$$

$$= \frac{10}{4} \quad [\because n = 10]$$

$$= \frac{5}{2}$$

Put $m - 6 = 0$, we get

$$m = 6$$

$$\therefore \quad {}^6C_3 \cdot \lambda^3 = \frac{5}{2}$$

$$\Rightarrow \quad \lambda^3 = \frac{5}{2} \times \frac{1}{20} = \frac{1}{8} = \left(\frac{1}{2}\right)^3$$

$$\therefore \quad \lambda = \frac{1}{2}$$

Hence, $m\lambda = 6 \times \frac{1}{2} = 3$

75. We know that, $(x-1)^n = {}^nC_0x^n - {}^nC_1x^{n-1} + {}^nC_2x^{n-2} - \ldots$

$$\therefore \quad x^2(x-1)^n = {}^nC_0x^{n+2} - {}^nC_1x^{n+1} + {}^nC_2x^n - \ldots$$

On differentiating w.r.t. x, we get

$$2x(x-1)^n + x^2 \cdot n(x-1)^{n-1}$$

$$= (n+2)\,{}^nC_0\,x^{n+1} - (n+1)\,{}^nC_1x^n + n \cdot {}^nC_2\,x^{n-1} - \ldots$$

On putting $x = 2$, we get

$$(n+2)\,{}^nC_0\,2^{n+1} - (n+1) \cdot {}^nC_1 \cdot 2^n + n \cdot {}^nC_2 2^{n-1} - \ldots$$

$$= 4 + 4n = 4(1+n)$$

$$\therefore \quad k = 4$$

76. On putting $x = 1, -1, i, -i$ in the given expression, we get

$$10^4 \times 9 = a_1 + a_2 + a_3 + a_4 + \ldots + a_{45} \quad \ldots(i)$$

$$0 = -a_1 + a_2 - a_3 + a_4 + \ldots - a_{45} \quad \ldots(ii)$$

$$(1+i)^4 \cdot i = a_1 i - a_2 - a_3 i + a_4 + \ldots$$

$$\Rightarrow \quad -i(2i)^2 = -a_1 i + a_2 + a_3 i - a_4 - \ldots$$

$$\Rightarrow \quad 4i = -a_1 i + a_2 + a_3 i - a_4 - \ldots \quad \ldots(iii)$$

and $\quad -4i = a_1 i + a_2 - a_3 i - a_4 - \ldots \quad \ldots(iv)$

On adding Eqs. (i), (ii), (iii) and (iv), we get

$$4(a_2 + a_6 + a_{10} + a_{14} + \ldots + a_{42}) = 9 \times 10^4$$

or $\quad a_2 + a_6 + a_{10} + \ldots + a_{42} = 22500 = \lambda$ [given]

$\therefore$ Required sum $= 2 + 2 + 5 + 0 + 0 = 9$

77. **(A)** General term, $T_{r+1} = {}^{6561}C_r\,(7^{1/3})^{6561-r}\,(11^{1/9})^r$

$$= {}^{6561}C_r 7^{\left(\frac{6561-r}{3}\right)} 11^{(r/9)}$$

For rational term, r should be a multiple of 9, i.e., $r = 0, 9, 18, \ldots, 6561$

Total rational terms, $m = 730$

and irrational terms, $n = 6562 - 730 = 5832$

Let $\quad \lambda =$ unit digit of $(730)^{5832} = 0$

and $\quad \mu =$ unit digit of $(5832)^{730}0$.

$=$ unit digit of $(2^5)^{145} \cdot 2^5$

$= (2)(2) = 4$

$\therefore \quad \lambda^\mu + \mu^\lambda = (0)^4 + (4)^0 = 1$

and $\quad \lambda + \mu = 0 + 4 = 4$

(B) General term, $t_{r+1} = {}^{600}C_r\,(7^{1/3})^{600-r}\,(x5^{1/2})^r$

$$= {}^{600}C_r\,(7)^{\frac{600-r}{3}}\,5^{r/2}\,x^r$$

For rational term, r should be multiple of 6.

i.e. $\quad r = 0, 6, \ldots, 600$

$\therefore$ Total rational terms, $m = 101$

and total irrational terms,

$$n = 601 - 101 = 500$$

Let $\quad \lambda =$ unit digit of $(m)^n$

$=$ unit digit of $(101)^{500} = 1$

and $\quad \mu =$ unit digit of $(500)^{101} = 0$

$\therefore \quad \lambda^2 + \mu^2 = (1)^2 + (0)^2 = 1$

$$\lambda^\mu + \mu^\lambda = (1)^0 + (0)^1 = 1$$

and $\quad \lambda + \mu = 1 + 0 = 1 = \lambda^\mu$

(C) $(\sqrt{2} + \sqrt[3]{3} + \sqrt[6]{7})^{10} = \Sigma \frac{10!}{\alpha!\beta!\gamma!}(\sqrt{2})^\alpha\,(\sqrt[3]{3})^\beta\,(\sqrt[6]{7})^\gamma$

$$= \Sigma \frac{10!}{\alpha!\beta!\gamma!} 2^{\alpha/2} \cdot 3^{\beta/3} \cdot 3^{\gamma/6}$$

$\because \quad \alpha + \beta + \gamma = 10$

For rational terms, $\alpha = 0, 2, 4, 6, 8, 10$, $\beta = 0, 3, 6, 9$ and $\gamma = 0, 6$

Possible triplets are $(4, 6, 0), (10, 0, 0), (4, 0, 6)$.

$\therefore$ Total rational terms, $m = 3$

Total irrational term, $n = {}^{10+2}C_2 - 3 = 63$

Let $\quad \lambda =$ unit's place digit of $3^{63} = (3^4)^{15} \cdot 3^3 = 1 \times 27 = 7$

and $\quad \mu =$ unit's place digit of $63^3 =$ unit digit of $3^3 = 7$

Now, $\sqrt{\lambda\sqrt{\lambda\sqrt{\lambda\sqrt{\lambda\sqrt{\lambda \ldots}}}}} \infty = \mu$

$$\Rightarrow \quad \lambda^{\left\{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\right\}} = \mu\lambda^{\frac{1/2}{1-1/2}} = \mu \quad \therefore \quad \lambda = \mu = 7$$

78. **(A)** Given, $\quad \binom{18}{r-2} + 2\binom{18}{r-1} + \binom{18}{r} \geq \binom{20}{13}$

$$\Rightarrow \quad \binom{18}{r-2} + \binom{18}{r-1} + \binom{18}{r-1} + \binom{18}{r} \geq \binom{20}{13}$$

$$\Rightarrow \quad \binom{19}{r-1} + \binom{19}{r} \geq \binom{20}{13}$$

$$\Rightarrow \quad \binom{20}{r} \geq \binom{20}{7} \Rightarrow 7 \leq r \leq 13$$

$$\therefore \quad r = 7, 8, 9, 10, 11, 12, 13$$

(B) The unit digit of 183! is 0.

Now, $3^{183} = (3^4)^{45}(3)^3$

Unit digit of 3^{183} = Unit digit of $(81)^{45} \times$ Unit digit of 27

$= 1 \times 7 = 7$

$\therefore$ Unit digit of $183! + 3^{183} = 0 + 7 = 7$

which is less than 8 and 9.

(C) $T_4 = {}^nC_3 (ax)^{n-3}\left(\frac{1}{x}\right)^3 = \frac{5}{2}$

$\Rightarrow {}^nC_3\, a^{n-3}x^{n-6} = \frac{5}{2} \Rightarrow n = 6$ and $a = \frac{1}{3}$

$\therefore \quad na = 6 \times \frac{1}{3} = 2$

79. **(A)** The coefficient of power of x more than x^{30} in $(1 + x)^{61}$ is

$${}^{61}C_{31} + {}^{61}C_{32} + \ldots + {}^{61}C_{61}$$

We know that, $(1 + 1)^{61} = {}^{61}C_0 + {}^{61}C_1 + {}^{61}C_2 + \ldots + {}^{61}C_{61}$

$\Rightarrow \quad 2^{61} = 2({}^{61}C_{31} + {}^{61}C_{32} + \ldots + {}^{61}C_{61})$

$\Rightarrow \quad {}^{61}C_{31} + {}^{61}C_{32} + \ldots + {}^{61}C_{61} = 2^{60}$

Hence, 2^{60} is divisible by $2^{57}, 2^{58}, 2^{59}, 2^{60}$.

(B) General term is, $T_{r+1} = {}^{62}C_r (\sqrt{3})^r = {}^{62}C_r 3^{r/2}$

For rational term = r should be multiple of 2.

i.e. $\quad r = 0, 2, 4, 6, \ldots, 62$

$\therefore$ Required sum $= T_1 + T_3 + \ldots + T_{63}$

$= {}^{62}C_0 + {}^{62}C_2 + \ldots + {}^{62}C_{62} = 2^{62-1} = 2^{61}$

Hence, 2^{61} is divisible by $2^{57}, 2^{58}, 2^{59}, 2^{60}, 2^{61}$.

(C) Put $x = 1$ and $x = -1$ in given expression, then we get

$$4^{31} = a_0 + a_1 + a_2 + a_3 + \ldots + a_{124} \quad \ldots(i)$$

and $\quad 0 = a_0 - a_1 + a_2 - a_3 + \ldots + a_{124} \quad \ldots(ii)$

On subtracting Eq. (ii) from Eq. (i), we get

$$2^{62} = 2(a_1 + a_3 + \ldots + a_{123})$$

$$\Rightarrow \quad 2^{61} = (a_1 + a_s + \ldots + a_{123})$$

$\therefore\ a_1 + a_3 + \ldots + a_{123}$ is divisible by $2^{57}, 2^{58}, 2^{59}, 2^{60}, 2^{61}$.

80. **(A)** $(11)^n + (21)^n = (16 - 5)^n + (16 + 5)^n$

$= 2[{}^nC_0 (16)^n + {}^nC_2 (16)^{n-2}(5)^2$

$+ {}^nC_4(16)^{n-4}(5)^4 + {}^nC_6(16)^{n-6}(5)^6 + \ldots]$

Hence, given expression is divisible by 16, if

$$n = 1, 3, 5, 7$$

(B) $3^{37} = (3^4)^9 \cdot 3 = (81)^9 \cdot 3 = 3(80 + 1)^9]$

$= 3[{}^9C_1 (80)^8 + {}^9C_2 (80)^7 + \ldots + 1]$

$\therefore$ Remainder of 3^{37} is 3.

(C) $\quad T_{r+1} = T_{r+k} \Rightarrow {}^{29}C_r = {}^{29}C_{r+k-1}$

$\Rightarrow 29 - r = r + k - 1 \Rightarrow 30 = 2r + k \quad [\because r \leq 29]$

For even values of k, i.e., $k = 0, 2, 4, 6, 8, \ldots, 28$,

(D) Given, $\frac{T_2}{T_3} = \frac{T'_3}{T'_4}$

$$\Rightarrow \quad \frac{{}^nC_1 (a)^{n-1}(b)^1}{{}^nC_2 (a)^{n-2}(b)^2} = \frac{{}^{n+3}C_2 (a)^{n+1}(b)^2}{{}^{n+3}C_3 (a)^n (b)^3}$$

$$\Rightarrow \quad \frac{{}^nC_1}{{}^nC_2} \times \frac{a}{b} = \frac{{}^{n+3}C_2}{{}^{n+3}C_3} \cdot \frac{a}{b}$$

$$\Rightarrow \quad \frac{{}^nC_2}{{}^nC_1} = \frac{{}^{n+3}C_3}{{}^{n+3}C_2}$$

$$\Rightarrow \quad \frac{n-2+1}{2} = \frac{n+3-3+1}{3}$$

$$\Rightarrow \quad 3n - 3 = 2n + 2$$

$$\therefore \quad n = 5$$

81. **(A)** Number of dissimilar terms in the expansion of $(x + 2y + 3z)^n\ (n \in N)$

$$= {}^{n+3-1}C_{3-1} = {}^{n+2}C_2 = \frac{1}{2}n^2 + \frac{3}{2}n + 1$$

$$\therefore \quad a = \frac{1}{2}, b = \frac{3}{2}, c = 1$$

Hence, $\quad a + b + c = \frac{1}{2} + \frac{3}{2} + 1 = 3$

and $\quad a + b = \frac{1}{2} + \frac{3}{2} = 2 = 2c$

(B) We have,

$(x + y + z)^{2n+1} = \{(x + y) + z\}^{2n+1}$

$= (x + y)^{2n+1} + {}^{2n+1}C_1 (x + y)^{2n} z$

$+ {}^{2n+1}C_2 (x + y)^{2n-1} z^2 + {}^{2n+1}C_3 (x + y)^{2n-2} z^3 + \ldots$

$+ {}^{2n+1}C_{2n} (x + y) z^{2n} + {}^{2n+1}C_{2n+1} z^{2n+1}$

and $(x + y - z)^{2n+1} = \{(x + y) - z\}^{2n+1}$

$= (x + y)^{2n+1} - {}^{2n+1}C_1 (x + y)^{2n} z + {}^{2n+1}C_2 (x + y)^{2n-1} z^2$

$- {}^{2n+1}C_3 (x + y)^{2n-2} z^3 + \ldots + {}^{2n+1}C_{2n} (x + y) z^{2n} - {}^{2n+1}C_{2n+1} z^{2n+1}$

$\therefore (x + y + z)^{2n+1} - (x + y - z)^{2n+1}$

$= 2\{{}^{2n+1}C_1 (x + y)^{2n} z + {}^{2n+1}C_3 (x + y)^{2n-2} z^3 + \ldots + z^{2n+1}\}$

$\therefore$ The number of dissimilar terms in the expansion of $(x + y + z)^{2n+1} - (x + y - z)^{2n+1}$

$= (2n + 1) + (2n - 1) + (2n - 3) + \ldots 5 + 3 + 1$

$= \frac{(n+1)}{2}(2n + 1 + 1) = (n + 1)^2$

$= n^2 + 2n + 1$

$\therefore \quad a = 1, b = 2, c = 1$

Hence, $\quad a + b + c = 1 + 2 + 1 = 4$

(C) We have, $(x - y + z)^n = \{x - (y - z)\}^n$

$= x^n - {}^nC_1 x^{n-1}(y - z) + {}^nC_2 x^{n-2}(y - z)^2$

$+ {}^nC_3 x^{n-3}(y - z)^3$

$+ \ldots - {}^nC_{n-1} x (y - z)^{n-1} + {}^nC_n (y - z)^n$

and $\quad (x + y - z)^n = (x + y - z)^n$

$= x^n + {}^nC_1 x^{n-1}(y-z) + {}^nC_2 x^{n-2}(y-z)^2 + {}^nC_3 x^{n-3}(y-z)^3$
$+ \ldots + {}^nC_{n-1} x(y-z)^{n-1} + (y-z)^n$
$\therefore (x-y+z)^n + (x+y-z)^n$
$= 2[x^n + {}^nC_2 x^{n-2}(y-z)^2 + {}^nC_4 x^{n-4}(y-z)^n + \ldots + (y-z)^n]$

$\therefore$ The number of dissilmilar terms in the expansion of $(x-y+z)^n + (x+y-z)^n = 1 + 3 + 5 + \ldots + (n+1)$

$$= \frac{\frac{(n+2)}{2}}{2}(1+n+1) = \frac{(n+2)^2}{4} = \frac{1}{4}(n^2+n+1)$$

$\therefore \quad a = \frac{1}{4}, b = 1, c = 1$

Hence, $b + c = 1 + 1 = 2 = 8a$

(D) $\because \left(\frac{x^2 + 1 + x^4}{x^2}\right)^{\Sigma n}$

$$= \frac{a_0 + a_2 x^2 + a_4 x^4 + \ldots + a_{2n(n+1)} x^{2n(n+1)}}{x^{n(n+1)}}$$

$\therefore$ Number of terms $= \frac{1}{2} \cdot 2n(n+1) + 1 = n^2 + n + 1$

$\therefore \quad a = 1, b = 1, c = 1$

Hence, $a + b + c = 1 + 1 + 1 = 3$

and $a + b = 1 + 1 = 2 = 2c$

82. Statement-2 is obviously correct.

Now, we have $(1+3x)^6 = {}^6C_0 + {}^6C_1(3x) + {}^6C_2(3x)^2 + {}^6C_3(3x)^3 + {}^6C_4(3x)^4 + {}^6C_5(3x)^5 + {}^6C_6(3x)^6$

$\therefore$ Greatest coefficient in $(1+3x)^6$ is ${}^6C_6 3^6$.

So, Statement-1 is wrong.

83. We have, $\left(x^2 + \frac{1}{x^2} + 2\right)^{25} = \left(x + \frac{1}{x}\right)^{50}$

$\therefore \quad T_{r+1} = {}^{50}C_r \cdot C^{50-r} \cdot \left(\frac{1}{x}\right)^r = {}^{50}C_r x^{50-2r}$

For independent of x, we put

$50 - 2r = 0 \Rightarrow r = 25$

$\therefore \quad T_{25+1} = {}^{50}C_{25}$

But in binomial expansion of $(x+a)^n$, middle terms is independent of x, iff $x \cdot a = 1$.

84. We have,

Coefficient of 31st term in $(1+x)^n$ = Coefficient of 32nd term in $(1+x)^n$

$\Rightarrow$ Coefficient of T_{30+1} = Coefficient of T_{31+1}

$\Rightarrow \quad {}^nC_{30} = {}^nC_{31} \Rightarrow n = 30 + 31 = 61$

Hence, both statements are correct but Statement-2 is not the correct Explanation of Statement-1.

85. We have, $\left(x + \frac{1}{x} + 1\right)^n = 1 + {}^nC_1\left(x + \frac{1}{x}\right) + {}^nC_2\left(x + \frac{1}{x}\right)^2 + \ldots + {}^nC_n\left(x + \frac{1}{x}\right)^n$

$\therefore$ Number of terms $= 1 + n + n = 2n + 1$

$\therefore$ Both the statements are correct but Statement-2 is not the correct explanation of Statement-1.

86. $4^{101} - 4 = 4(4^{100} - 1) = 4(16^{50} - 1)$
$= 4(16^{25} + 1)(16^{25} - 1)$
$= 4$ (divisible by $16+1$) (divisible by $16-1$)
$=$ divisible by 102

$\therefore 4^{101} - 4$ is divisible by 102.

or if 4^{101} is divisible by 102, then remainder is 4.

$\therefore$ Statement-1 is false but Statement-2 is obviously true.

87. $\because (x^n + a^n)$ is always divisible by $(x+a)$ when n is odd natural number. Therefore, $(11^{25} + 12^{25})$ is divisible by $11 + 12 = 23$.

$\therefore$ Statement-1 is always true but Statement-2 is false.

for n even natural number.

88. $T_{r+1} = {}^9C_r (ax^{1/6})^{9-r} (bx^{-1/3})^r = {}^9C_r \cdot a^{9-r} \cdot b^r \cdot x^{\frac{9-r}{6} - \frac{r}{3}}$

For independent of x, put $\frac{9-r}{6} - \frac{5}{3} = 0$

$\Rightarrow \quad 9 - r - 2r = 0$

$\Rightarrow \quad r = 3$

$\therefore \quad T_{3+1} = {}^9C_3 \cdot a^6 b^3 = 84a^6 b^3$

Now using A M $\geq$ GM

$\Rightarrow \quad \frac{a^2 + b}{2} \geq (a^2 b)^{1/2} \Rightarrow \frac{2}{2} \geq (a^2 b)^{1/2} \qquad [\because a^2 + b = 2]$

$\therefore \quad a^2 b \leq 1 \Rightarrow (a^2 b)^3 \leq 1^3 \Rightarrow 84a^6 b^3 \leq 84$

$\therefore \quad T_4 \leq 84$

Hence, both statements are true and Statement-2 is the correct explanation of Statement-1.

89. We have, $10000 = T_3 = T_{2+1} = {}^5C_2 x^{(5-2)} (x^{\log_{10} x})^2$

$\Rightarrow \quad 100000 = x^3 \cdot x^{2\log_{10} x} = x^{3 + 2\log_{10} x}$

$\Rightarrow 3 + 2\log_{10} x = \log_x 100000 = 5\log_x 10 = \frac{5}{\log_{10} x}$

$\Rightarrow 2(\log_{10} x)^2 + 3\log_{10} x - 5 = 0$

Put $\log_{10} x = y$, we get

$2y^2 + 3y - 5 = 0 \Rightarrow y = -\frac{5}{2}$ or 1

$\therefore \quad \log_{10} x = -\frac{5}{2}$ or 1

$\Rightarrow \quad x = 10$ or $10^{-5/2}$

90. We have, $\frac{18^3 + 7^3 + 3 \cdot 18 \cdot 7 \cdot 25}{\left[\begin{matrix} 3^6 + 6 \cdot 243 \cdot 2 + 15 \cdot 81 \cdot 4 + 20 \cdot 27 \cdot 8 \\ + 15 \cdot 9 \cdot 16 \quad + 6 \cdot 3 \cdot 32 + 64 \end{matrix}\right]}$

$$= \frac{(18+7)^3}{(3+2)^6} = \frac{(25)^3}{(5)^6} = \frac{(25)^3}{(25)^3} = 1$$

91. We have, $\left(\frac{a+1}{a^{2/3} - a^{1/3} + 1} - \frac{a-1}{a - a^{1/2}}\right)^{10}$

$$= \left[\frac{(a^{1/3})^3 + 1^3}{a^{2/3} - a^{1/3} + 1} - \frac{(a^{1/2})^2 - 1^2}{a^{1/2}(a^{1/2} - 1)}\right]^{10}$$

$$= \left((a^{1/3} + 1) - \frac{a^{1/2} + 1}{a^{1/2}} \right)^{10} = (a^{1/3} - a^{-1/2})^{10}$$

Now, $T_{r+1} = {}^{10}C_r \, (a^{1/3})^{10-r} \, (-a^{-1/2})^r$ ...(i)

$$= {}^{10}C_r \, a^{\frac{10-r}{3} - \frac{r}{2}} (-1)^r$$

It will be independent of a, if

$$\frac{10-r}{3} - \frac{r}{2} = 0 \implies 20 - 2r - 3r = 0$$

$$\Rightarrow \quad r = 4$$

Putting $r = 4$ in Eq. (i), we get

$$T_5 = {}^{10}C_4 \, (-1)^4 = {}^{10}C_4 = 210$$

92. The general term in $(x^a + x^{-b})^n$ is

$$T_{r+1} = {}^nC_r \, (x^a)^{n-r} \, (x^{-b})^r = {}^nC_r x^{a(n-r) - br} = {}^nC_r x^{an - (a+b)r}$$

For independent of x, we must have $an - (a + b) = r = 0$

$$\Rightarrow \quad r = \frac{an}{a+b} \implies an = (a+b)\, r, \, r \in N$$

$\Rightarrow$ an is multiple of $(a + b)$.

93. Let n be the index of power in $(1 + x)$. Then, ${}^nC_r = a$ ...(i)

$${}^nC_{r+1} = b \quad \text{...(ii)}$$

and ${}^nC_{r+2} = c$ ...(iii)

From Eqs. (i) and (ii), we get

$$\frac{{}^nC_r}{{}^nC_{r+1}} = \frac{a}{b}$$

$$\Rightarrow \quad \frac{r+1}{n-r} = \frac{a}{b} \implies r = \frac{an-b}{a+b} \quad \text{...(iv)}$$

From Eqs. (ii) and (iii), we get

$$\frac{{}^nC_{r+1}}{{}^nC_{r+2}} = \frac{b}{c}$$

$$\Rightarrow \quad \frac{r+2}{n-r-1} = \frac{b}{c} \implies r = \frac{bn-b-2c}{b+c} \quad \text{...(v)}$$

From Eqs. (iv) and (v), we get

$$\frac{bn-b-2c}{b+c} = \frac{an-b}{a+b}$$

$$\Rightarrow \quad (b^2 - ac)\, n = 2ac + b\,(a+c)$$

$$\Rightarrow \quad n = \frac{2ac + b\,(a+c)}{(b^2 - ac)}$$

94. Given expansion is $\left[\sqrt[3]{2} + \frac{1}{\sqrt[3]{3}} \right]^n$.

$\therefore$ 7th term from the beginning

$$= {}^nC_6 \, (2)^{\frac{n-6}{3}} \cdot (3^{-1/3})^6 = {}^nC_6 \, 2^{\frac{n-6}{3}} \cdot 3^{-2}$$

$\therefore$ Again, 7th term from end in $\left[\sqrt[3]{2} + \frac{1}{\sqrt[3]{3}} \right]^n$

$$= \text{7th term from beginning in } \left[\frac{1}{\sqrt[3]{3}} + \sqrt[3]{3} \right]^n$$

$$= {}^nC_6 (3^{-1/3})^{n-6} (2)^2 = {}^nC_6 \, 3^{\frac{6-n}{3}} \cdot 4$$

According to the question, we have

$$\frac{{}^nC_6 \cdot 2^{\frac{n-6}{3}} \cdot 3^{-2}}{{}^nC_6 \cdot 4 \cdot 3^{\frac{6-n}{3}}} = \frac{1}{6} \implies 2^{\frac{n-12}{3}} \cdot 3^{\frac{n-12}{3}} = \frac{1}{6}$$

$$\Rightarrow \quad (6)^{\frac{n-12}{3}} = 6^{-1} \implies \frac{n-12}{2} = -1, \, n = 9$$

95. We know that, $(1 + x)^n \, (x + 1)^n$

$$= [{}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_n x^n]$$
$$\times [{}^nC_0 x^n + {}^nC_1 x^{n-1} + {}^nC_2 x^{n-2} + \ldots + {}^nC_n]$$

Equating coefficient of x^{n+1} on both sides, we get

$${}^{2n}C_{n+1} = [{}^nC_0 \, {}^nC_1 + {}^nC_1 \, {}^nC_2 + \ldots + {}^nC_{n-1} \, {}^nC_n]$$

$\therefore \quad S_n = {}^{2n}C_{n+1}$

But $\frac{S_{n+1}}{S_n} = \frac{15}{4} \implies \frac{{}^{2n+2}C_{n+2}}{{}^{2n}C_{n+1}} = \frac{15}{4}$

$$\Rightarrow \quad \frac{(2n+2)}{(n+2)} \cdot \frac{{}^{2n+1}C_{n+1}}{{}^{2n}C_{n+1}} = \frac{15}{4}$$

$$\Rightarrow \quad \frac{2\,(n+1)}{(n+2)} \cdot \frac{{}^{2n+1}C_n}{{}^{2n}C_{n-1}} = \frac{15}{4}$$

$$\Rightarrow \quad \frac{2(n+1)}{(n+2)} \cdot \frac{2n+1}{n} = \frac{15}{4}$$

$$\Rightarrow \quad 8\,(2n^2 + 3n + 1) = 15n^2 + 30n$$

$$\Rightarrow \quad n^2 - 6n + 8 = 0$$

$\therefore \quad n = 4, 2$

96. $\frac{C_1}{C_0} + 2 \cdot \frac{C_2}{C_1} + 3 \cdot \frac{C_3}{C_2} + \ldots + n \cdot \frac{C_n}{C_{n-1}}$

$$= \frac{n}{1} + 2 \cdot \frac{n(n-1)}{2n} + 3 \cdot \frac{n(n-1)(n-2)}{3!} \times \frac{2!}{n(n-1)} + \ldots + n \cdot \frac{1}{n}$$

$$= n + (n-1) + (n-2) + \ldots + 1 \, \frac{n(n+1)}{2}$$

97. We have, $\left[\sqrt[3]{\left(\frac{a}{\sqrt{b}} \right)} + \sqrt{\left(\frac{b}{\sqrt[3]{a}} \right)} \right]^{21} = [ab^{-1/2}]^{1/3} + (ba^{-1/3})^{1/2}]^{21}$

Let T_{r+1} contain a and b to one and the same power.

$\therefore \quad T_{r+1} = {}^{21}C_r \, (ab^{-1/2})^{\frac{21-r}{3}} \, (ba^{-1/3})^{r/2}$

$$= {}^{21}C_r \cdot a^{\frac{21-r}{3} - \frac{r}{6}} \cdot b^{\frac{r}{2} - \frac{21-r}{6}}$$

$\because \quad \frac{21-r}{3} - \frac{r}{6} = \frac{r}{2} - \frac{21-r}{6} \quad \therefore \quad \frac{21-r}{2} = \frac{2r}{3}$

$\Rightarrow \quad 63 - 3r = 4r \implies 63 = 7r \implies r = 9$

$\therefore$ Required term $= r + 1 = 10$

98. Given series is a GP.

$$\therefore S = (x+3)^{n-1} \frac{\left[1 - \left(\frac{x+2}{x+3} \right)^n \right]}{1 - \frac{x+2}{x+3}} = (x+3)^n \, \frac{[(x+3)^n - (x+2)^n]}{(x+3)^n}$$

$$= (x+3)^n - (x+2)^n = (3+x)^n - (2+x)^n$$

$\therefore$ Coefficient of x^r in $S = {}^nC_r\, 3^{n-r} - {}^nC_r 2^{n-r}$

$= {}^nC_r\,(3^{n-r} - 2^{n-r})$

99. Let $(2+\sqrt{5})^p$ or $(\sqrt{5}+2)^p = I + f$ ...(i)

$0 \le f < 1$ ...(ii)

$(\sqrt{5}-2)^p = f'$ ...(iii)

and $0 < f' < 1$ ...(iv)

On subtracting Eq. (iii) from Eq. (i), we get

$I + f - f' = (\sqrt{5}+2)^p - (\sqrt{5}-2)^p$

$= 2\{{}^pC_1(\sqrt{5})^{p-1}\cdot 2^1 + {}^pC_3(\sqrt{5})^{p-3}\cdot 2^3 + \ldots + {}^pC_p\,(\sqrt{5})^0 2^p\}$

[$\because$ integer value of $f - f' = 0$]

$\therefore \quad [(\sqrt{5}+2)^p] + 0 - 2^{p+1}$

$= 2\{{}^pC_1\,(\sqrt{5})^{p-1}\cdot 2 + {}^pC_3\,(\sqrt{5})^{p-3}\cdot 2^3 + \ldots + {}^pC_{p-2}(\sqrt{5})^2\cdot 2^{p-2}\}$

$= p\lambda = p$ (integer)

$\therefore \quad [(\sqrt{5}+2)^p] - 2^{p+1}$ is divisible by P.

100. Let $(\sqrt{3}+1)^{2n} = I + f,\ 0 \le f < 1$

and $f' = (\sqrt{3}-1)^{2n}$

$\therefore I + f + f' = (\sqrt{3}+1)^{2n} + (\sqrt{3}-1)^{2n}$

$= [(\sqrt{3}+1)^2]^n + [(\sqrt{3}-1)^2]^n$

$= (4+2\sqrt{3})^n + (4-2\sqrt{3})^n$

$= 2^n\,(2+\sqrt{3})^n + 2^n\,(2-\sqrt{3})^n$

$= 2^n\cdot 2\{{}^nC_0 2^n + {}^nC_2(2^{n-2})\,(3) + {}^nC_4(2^{n-4})\,(3^2) + \ldots\}$

$\therefore \quad I + 1 = 2^{n+1},\ k \in I$ [$\because f + f' = 1$]

or $[(\{(\sqrt{3}+1)^{2n}\})] = 2^{n+1}k$

Hence, $[(\{(\sqrt{3}+1)^{2n}\})],\ n \in N$ is divisible by 2^{n+1}.

101. ${}^{11}C_1\cdot x^{10} - {}^{11}C_3\cdot x^8 + {}^{11}C_5\cdot x^6 - {}^{11}C_7 x^4 + {}^{11}C_9\,x^2 - {}^{11}C_{11} = 0$

Now, $(1+ix)^{11} = {}^{11}C_0 + {}^{11}C_1 ix + {}^{11}C_2\,(ix)^2 + {}^{11}C_3\,(ix)^3 + {}^{11}C_4\,(ix)^4 + {}^{11}C_5\,(ix)^5 + {}^{11}C_6\,(ix)^6 + {}^{11}C_7\,(ix)^7 + \ldots + {}^{11}C_{11}\,(ix)^{11}$

$\Rightarrow \quad (1+ix)^{11} = ({}^{11}C_0 - {}^{11}C_2\,x^2 + {}^{11}C_4\,x^4 - {}^{11}C_6\,x^6 + {}^{11}C_8\,x^8 + {}^{11}C_{10}\,x^{10}) + i\,({}^{11}C_1\,x - {}^{11}C_3\,x^3 + {}^{11}C_5\,x^5 - {}^{11}C_7\,x^7 + {}^{11}C_9\,x^9 - {}^{11}C_{11}\,x^{11})$

Comparing real part on both sides, we get

$(1+ix)^{11} = ({}^{11}C_{11} - {}^{11}C_9\,x^2 + {}^{11}C_7\,x^4 - {}^{11}C_5\,x^6 + {}^{11}C_3\,x^8 - {}^{11}C_1\,x^{10})$ [$\because {}^nC_r = {}^nC_{n-r}$]

$\Rightarrow \text{Re}\,\{(1+ix)^{11}\} = -({}^{11}C_1\,x^{10} - {}^{11}C_3\,x^8 + {}^{11}C_5\,x^6 - {}^{11}C_7\,x^4 + {}^{11}C_9\,x^2 - {}^{11}C_{11})$

$\Rightarrow \quad {}^{11}C_1 x^{10} - {}^{11}C_3\,x^8 + {}^{11}C_5\,x^6 - {}^{11}C_7\,x^4 + {}^{11}C_9\,x^2 - {}^{11}C_{11}$

$= -\,\text{Re}\,\{1+ix\}^{11}\}$

Let $x = \cot\theta = -\text{Re}\{1 - i\cot\theta\}''$

$= -\,\text{Re}\left[\left\{\frac{\sin\theta + i\cos\theta}{\sin\theta}\right\}^{11}\right] = -\,\text{Re}\left\{\frac{(i)^{11}\,(\cos\theta - i\sin\theta)}{\sin\theta}\right\}^{11}$

$= -\text{Re}\left\{\frac{-i\,(\cos 11\theta - i\sin 11\theta)}{\sin^{11}\theta}\right\} = \frac{\sin 11\theta}{\sin^{11}\theta} = 0$ [given]

$\theta \ne 0$

$\therefore \quad \sin 11\theta = r\pi$

or $\theta = \frac{r\pi}{11},\ r = \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$

$\therefore \quad x = \cot\,\theta = \cot\left(\frac{r\pi}{11}\right)$

$r = \pm 1, \pm 2, \ldots, \pm 5$

102. Since, $g(x) = f(1+x)$

$g(x) = \sum_{r=0}^{200} \alpha_r x^r$ and $f(x) = \sum_{r=0}^{200} \beta_r\, x^r$

$\therefore \quad \sum_{r=0}^{200} \alpha_r\, x^r = \sum_{r=0}^{200} \beta_r\,(1+x)^r$

Now, $\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_{200}\,x^{200}$

$= \beta_0 + \beta_1\,(1+x) + \beta_2\,(1+x)^2 + \ldots + \beta_{100}\,(1+x)^{100} + \beta_{101}\,(1+x)^{101} + \ldots + \beta_{200}\,(1+x)^{200}$

$\Rightarrow \quad \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_{200}\,x^{200}$

$= \beta_0 + \beta_1\,(1+x) + \beta_2\,(1+x)^2 + \ldots + (1+x)^{100} + (1+x)^{101} + \ldots + (1+x)^{200}$ [$\because \beta_{100} = \beta_{101} = \ldots = \beta_{200} = 1$]

Equating the coefficient of x^{100}, we get

$\alpha_{100} = {}^{100}C_{100} + {}^{101}C_{100} + {}^{102}C_{100} + \ldots + {}^{200}C_{100}$

$= {}^{101}C_{101} + {}^{101}C_{100} + {}^{102}C_{100} + \ldots + {}^{200}C_{100}$

$= {}^{102}C_{101} + {}^{102}C_{100} + \ldots + {}^{200}C_{100}$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$= {}^{200}C_{101} + {}^{200}C_{100} = {}^{201}C_{101}$ ...(i)

Again, greatest coefficient in the expansion of $(1+x)^{201}$

= Coefficient of middle term

= Coefficient of T_{101} or coefficient of T_{102}

$= {}^{201}C_{100}$ or ${}^{201}C_{101} = {}^{201}C_{100}$ ...(ii)

From Eqs. (i) and (ii), we get

The greatest coefficient in the expansion of $(1+x)^{201}$.

103. $P = \sum\sum_{0 \le i < j \le n} (i+j)\,(C_i + C_j + C_iC_j)$ ...(i)

Replacing i by $n-i$ and j by $n-j$ in Eq. (i), we get

$P = \sum\sum_{0 \le i < j \le n} (n-i+n-j)\,(C_{n-i} + C_{n-j} + C_{n-i}\,C_{n-j})$

$= \sum\sum_{0 \le i < j \le n} (2n-i-j)\,(C_i + C_j + C_iC_j)$ [$\because {}^nC_r = {}^nC_{n-r}$] ...(ii)

On adding Eqs. (i) and (ii), then we get

$2P = 2n \sum\sum_{0 \le i < j \le n} (C_i + C_j + C_iC_j)$

$\therefore \quad P = n \sum\sum_{0 \le i < j \le n} (C_i + C_j) + n \sum\sum_{0 \le i < j \le n} C_iC_j$

$$= n \cdot n\,(C_0 + C_1 + C_2 + \ldots + C_n) + \frac{n}{2}(2^{2n} - {}^{2n}C_n)$$

$$= n^2 \cdot 2^n + n \cdot 2^{2n-1} - \frac{n}{2} \cdot {}^{2n}C_n$$

$$= n^2 \cdot 2^n + n\left\{2^{2n-1} - \frac{2n!}{2(n!)^2}\right\}$$

104. $\sum\limits_{0 \le i \ne j \le 10} \sum {}^{21}C_i \cdot {}^{21}C_j$

$$= \frac{1}{2}\left[\sum_{i=0}^{10}\sum_{j=0}^{10} {}^{21}C_i\,{}^{21}C_j\right] - \sum_{i=0}^{10}({}^{21}C_i)^2$$

$$= \frac{1}{2}\left[\sum_{i=0}^{10} {}^{21}C_i 2^{21-r}\right] - \frac{1}{2}\sum_{i=0}^{10}({}^{21}C_i)^2$$

$$= \frac{2^{20} \cdot 2^{20}}{2} - \frac{{}^{42}C_{21}}{2 \times 2} = \frac{1}{2}\left[2^{40} - \frac{(42)!}{2\,(21!)^2}\right]$$

105. (i) We have, $(1 + x + x^2 + x^3)^{11} = (1 + x)^{11}\,(1 + x^2)^{11}$

$$= (1 + {}^{11}C_1 x + {}^{11}C_2 x^2 + {}^{11}C_3 x^3 + {}^{11}C_4 x^4 + \ldots) \times (1 + {}^{11}C_1 x^2 + {}^{11}C_2 x^4 + \ldots)$$

$\therefore$ Coefficient of $x^4 = {}^{11}C_2 + {}^{11}C_2 \times {}^{11}C_1 + {}^{11}C_4$

$$= 55 + 605 + 330 = 990$$

(ii) $[(2 - x) + 3x^2]^6$

$$= {}^6C_0(2 - x)^6 + {}^6C_1\,(2 - x)^5\,(3x)^2 + {}^6C_2\,(2 - x)^4\,(3x^2)^2 + \ldots$$

$$= {}^6C_0\,[{}^6C_4(2)^2] + {}^6C_1 \times 3\,[{}^5C_2(2)^3] + {}^6C_2 \times 9\,[{}^4C_0(2)^4]$$

[equating coefficient of x^4]

$$= 60 + 1440 + 2160 = 3660$$

106. LHS $= \dfrac{1}{{}^{2n+1}C_r} + \dfrac{1}{{}^{2n+1}C_{r+1}}$

$$= \frac{(2n + 1 - r)!\,r!}{(2n + 1)!} + \frac{(2n - r)!\,(r + 1)!}{(2n + 1)!}$$

$$= (2n - r)!\,(r)!\left[\frac{(2n + 1 - r) + r + 1}{(2n + 1)!}\right]$$

$$= \frac{(2n - r)!\,(r)!(2n + 2)}{(2n + 1)\,(2n)!} = \frac{2n + 2}{2n + 1} \cdot \frac{1}{{}^{2n}C_r}$$

Now, $\sum\limits_{r=1}^{2n-1} \dfrac{(-1)^{r-1} \cdot r}{{}^{2n}C_r} = \sum\limits_{r=1}^{2n-1}(-1)^{r-1}\left(\dfrac{2n+1}{2n+2}\right)\left[\dfrac{1}{{}^{2n+1}C_r} + \dfrac{1}{{}^{2n+1}C_{r+1}}\right] \times r$

$$= \left(\frac{2n+1}{2n+2}\right)\sum_{r=1}^{2n-1}(-1)^{r-1}\left(\frac{1}{{}^{2n+1}C_r} + \frac{1}{{}^{2n+1}C_{r+1}}\right) \times r$$

$$= \left(\frac{2n+1}{2n+2}\right)\left[\left(\frac{1}{{}^{2n+1}C_1} + \frac{1}{{}^{2n+1}C_2}\right) - 2\left(\frac{1}{{}^{2n+1}C_2} + \frac{1}{{}^{2n+1}C_3}\right) + 3\left(\frac{1}{{}^{2n+1}C_3} + \frac{1}{{}^{2n+1}C_4}\right) - 4\left(\frac{1}{{}^{2n+1}C_4} + \frac{1}{{}^{2n+1}C_5}\right) + \ldots + (2n-1)\left(\frac{1}{{}^{2n+1}C_{2n-1}} + \frac{1}{{}^{2n+1}C_{2n}}\right)\right.$$

$$= \left(\frac{2n+1}{2n+2}\right)\left[\frac{1}{{}^{2n+1}C_1} - \frac{1}{{}^{2n+1}C_2} + \frac{1}{{}^{2n+1}C_3} - \frac{1}{{}^{2n+1}C_4} + \ldots + \frac{(2n-1)}{{}^{2n+1}C_{2n}}\right]$$

$$= \left(\frac{2n+1}{2n+2}\right)\left[\frac{1}{{}^{2n+1}C_1} - \frac{1}{{}^{2n+1}C_2} + \frac{1}{{}^{2n+1}C_3} - \frac{1}{{}^{2n+1}C_4} + \ldots - \frac{1}{{}^{2n+1}C_{2n}} + \frac{2n}{{}^{2n+1}C_{2n}}\right]$$

$$= \left(\frac{2n+1}{2n+2}\right)\left[\left(\frac{1}{{}^{2n+1}C_1} - \frac{1}{{}^{2n+1}C_{2n}}\right) - \left(\frac{1}{{}^{2n+1}C_2} - \frac{1}{{}^{2n+1}C_{2n-1}}\right) + \ldots + \frac{2n}{{}^{2n+1}C_{2n}}\right]$$

$$= \left(\frac{2n+1}{2n+2}\right)\left[0 + \frac{2n}{{}^{2n+1}C_{2n}}\right] = \left(\frac{2n+1}{2n+2}\right)\left(\frac{2n}{2n+1}\right)$$

$$= \frac{2n}{2n+2} = \frac{n}{n+1}$$

107. Given, $(1 + x + x^2)^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_{r-2}x^{r-2} + a_{r-1}x^{r-1} + a_r x^r + \ldots + a_{2n}x^{2n}$ ...(i)

and $(1 - x)^n = {}^nC_0 - {}^nC_1 x + {}^nC_2 x^2 - \ldots + (-1)^r\,{}^nC_r x^r + \ldots + (-1)^n\,{}^nC_n$ ...(ii)

On multiplying Eqs. (i) and (ii) and equating coefficient of x^r on both sides, we get

Coefficient of x^r in $(1 - x^3)^n$

$$= {}^nC_0\,a_r - {}^nC_1 a_{r-1} + {}^nC_2 a_{r-2} - \ldots + (-1)^r\,{}^nC_r a_r$$

Since, r is not a multiple of 3, therefore the expression $(1 - x^3)^n$ does not contain x^r in any term.

$\therefore$ Coefficient of x^r in $(1 - x^3)^n = 0$

Hence, $a_r - {}^nC_1\,a_{r-1} + {}^nC_2\,a_{r-2} - \ldots + (-1)^r\,{}^nC_r\,a_0 = 0$

108. Given, $(1 + z^2 + z^4)^8 = C_0 + C_1 z^2 + C_2 z^4 + \ldots + C_{16} z^{32}$

(i) Put $z = i$, we get

$$(1 - 1 + 1)^8 = C_0 - C_1 + C_2 - C_3 + \ldots + C_{16}$$

$$C_0 - C_1 + C_2 - C_3 + \ldots + C_{16} = 1$$

(ii) Put $z = \omega$, we get

$$(1 + \omega^2 + \omega^4)^8 = C_0 + C_1\omega^2 + C_2\omega^4 + C_3\omega^6 + \ldots + C_{16}\omega^{32}$$

$$\Rightarrow (1 + \omega^2 + \omega)^8 = C_0 + C_1\omega^2 + C_2\omega + C_3 + C_4\omega^2 + C_5\omega + \ldots + C_{16}\omega^2$$

$$\Rightarrow 0 = (C_0 + C_3 + C_6 + \ldots + C_{15}) + (C_2 + C_5 + \ldots + C_{14})\,\omega + (C_1 + C_4 + \ldots + C_{16})\,\omega^2$$

109. Given, $f(x) = g(x + 1)$

$$\therefore a_0 x^0 + a_1 x + a_2 x^2 + \ldots + a_{2n}x^{2n} = b_0 + b_1(x+1) + b_2(x+1)^2 + \ldots + b_{n-1}(x+1)^{n-1} + (x+1)^n + (x+1)^{n+1} + (x+1)^{n+2} + \ldots + (x+1)^{2n}$$

Equating coefficient of x^n on both sides, we get

$$a_n = {}^nC_n + {}^{n+1}C_n + \ldots + {}^{2n}C_n$$

$= {}^{n+1}C_{n+1} + {}^{n+1}C_n + \ldots + {}^{2n}C_n \qquad [\because {}^nC_n = {}^{n+1}C_{n+1}]$

$= {}^{n+2}C_{n+1} + \ldots + {}^{2n}C_n = {}^{2n}C_{n+1} + {}^{2n}C_n$

$[\because {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r]$

$= {}^{2n+1}C_{n+1}$

110. Let $(1 + x + x^2)^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_{2n} x^{2n}$ ...(i)

Replacing x by $\left(-\frac{1}{x}\right)$, we get

$$\left(1 - \frac{1}{x} + \frac{1}{x^2}\right)^n = a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} - \ldots + \frac{a_{2n}}{x^{2n}}$$

$\Rightarrow (1 - x + x^2)^n = a_0 x^{2n} - a_1 x^{2n-1} + a_2 x^{2n-2} - \ldots + a_{2n} x^{2n}$ ...(ii)

(i) Multiplying Eqs. (i) and (ii) and equating the coefficient of x^{2n+1}, then we get

Coefficient of x^{2n+1} in $(1 + x^2 + x^4)^n$

$= a_0 a_1 - a_1 a_2 + a_2 a_3 - \ldots$

In RHS, put $x^2 = y$, we get

$a_0 a_1 - a_1 a_2 + a_2 a_3 - \ldots = 0$ (only even powers contains)

(ii) Multiplying Eqs. (i) and (ii) and equating the coefficient of x^{2n+2}

Coefficient of x^{2n+2} in $(1 + x^2 + x^4)^n$

$=$ Coefficient of y^{n+1} in $(1 + y + y^2)^n$

$= a_0 a_2 - a_1 a_3 + a_2 a_4 - \ldots + a_{2n-2} \cdot a_{2n}$

$[\because \text{ put } x^2 = y]$

$= a_{n+1}$

(iii) Put $x = 1$, ω and ω^2 in Eq. (i), we get

$3^n = a_0 + a_1 + a_2 + a_3 + \ldots + a_{2n}$ ...(iii)

$\Rightarrow (1 + \omega + \omega^2)^n = a_0 + a_1 \omega + a_2 \omega^2 + a_3 \omega^3 + \ldots + a_{2n} \omega^{2n}$

$\Rightarrow 0 = a_0 + a_1 \omega + a_2 \omega^2 + a_3 + \ldots$ ...(iv)

and $(1 + \omega^2 + \omega^4)^n = a_0 + a_1 \omega^2 + a_2 \omega^4 + a_3 \omega^6 + \ldots$

$\Rightarrow 0 = a_0 + a_1 \omega^2 + a_2 \omega + a_3 + \ldots$ ...(v)

on adding Eqs. (iii), (iv) and (v), we get

$3^n = 3a_0 + a_1(1 + \omega + \omega^2) + a_2(1 + \omega^2 + \omega) + 3a_3 + \ldots +$

$\Rightarrow 3^n = 3(a_0 + a_3 + a_6 + \ldots) \Rightarrow a_0 + a_3 + a_6 + \ldots = 3^{n-1}$

On multiplying Eqs. (iv) and (v) by ω^2 and ω, respectively and then adding Eqs. (iii), (iv) and (v), we get

$3^n = a_0(1 + \omega^2 + \omega) + a_1(1 + \omega^3 + \omega^3) + a_2(1 + \omega^4 + \omega^3) + a_3(1 + \omega + \omega^2) + \ldots$

$= 3(a_1 + a_4 + \ldots)$

$\Rightarrow a_1 + a_4 + \ldots = 3^{n-1}$

Again, multiplying Eq. (iv) by ω and Eq. (v) by ω^2, respectively and then adding Eqs. (iii), (iv) and (v), we get

$3^n = 3(a_2 + a_5 + a_8 + \ldots)$

$\Rightarrow a_2 + a_5 + a_8 + \ldots = 3^{n-1}$

Hence, $a_0 + a_3 + a_6 + \ldots = a_1 + a_4 + a_7 + \ldots$

$= a_2 + a_5 + a_8 + \ldots = 3^{n-1}$

111. LHS $= (n-1)^2 C_1 + (n-3)^2 C_3 + (n-5)^2 C_5 + \ldots$

$= n^2(C_1 + C_3 + C_5 + \ldots) - 2n(C_1 + 3C_3 + 5C_5 + \ldots) + (1^2 C_1 + 3^2 C_3 + 5^2 C_5 + \ldots)$

$= n^2(2^{n-1})(-2n)\left(\frac{n}{1} \cdot {}^{n-1}C_0 + \frac{n}{3} \cdot 3 \cdot {}^{n-1}C_2 + \frac{n}{5} \cdot 5 \cdot {}^{n-1}C_4 + \ldots\right) + \left(1^2 \cdot \frac{n}{1} \cdot {}^{n-1}C_0 + 3^2 \cdot \frac{n}{3} \cdot {}^{n-1}C_2 + 5^2 \cdot \frac{n}{5} \cdot {}^{n-1}C_4 + \ldots\right)$

$= n^2 \cdot 2^{n-1} - 2n^2({}^{n-1}C_0 + {}^{n-1}C_2 + {}^{n-1}C_4 + \ldots) + n(1 \cdot {}^{n-1}C_0 + 3 \cdot {}^{n-1}C_2 + 5 \cdot {}^{n-1}C_4 + \ldots)$

$= n^2 \cdot 2^{n-1} - 2n^2 \cdot 2^{n-2} + n[{}^{n-1}C_0 + (2+1) \cdot {}^{n-1}C_2 + (4+1) \cdot {}^{n-1}C_4 + \ldots]$

$= n^2 \cdot 2^{n-2}(2-2) + n[({}^{n-1}C_0 + {}^{n-1}C_2 + {}^{n-1}C_4 + \ldots)] + (2 \cdot {}^{n-1}C_2 + 4 \cdot {}^{n-1}C_4 + \ldots)]$

$= 0 + n[2^{n-2} + (n-1)\{{}^{n-2}C_1 + {}^{n-2}C_3 + \ldots\}]$

$= n[2^{n-2} + (n-1) \cdot 2^{n-3}] = n \cdot 2^{n-3}(2 + n - 1)$

$= n(n+1)2^{n-3} = \text{RHS}$

112. $(1 - x^3)^n = C_0 - C_1 x^3 + C_2 x^6 - C_3 x^9 + \ldots + (-1)^n C_n x^{3n}$

Then, $\int_0^1 (1 - x^3)^n \, dx$

$= \int_0^1 (C_0 - C_1 x^3 + C_2 x^6 - \ldots + (-1)^n C_n (x^{3n}) \, dx$

$= \left[\frac{C_0 x}{1} - \frac{C_1 x^4}{4} + \frac{C_2 x^7}{7} - \ldots + (-1)^n \frac{C_n x^{3n+1}}{3n+1}\right]_0^1$

$= \frac{C_0}{1} - \frac{C_1}{4} + \frac{C_2}{7} - \ldots + (-1)^n \frac{C_n}{3n+1}$

Let $I_n = \int_0^1 (1 - x^3)^n \cdot 1 \, dx$

$= [(1 - x^3)^n \cdot x]_0^1 - \int_0^1 n(1 - x^3)^{n-1} \cdot (-3x^2) \cdot x \, dx$

$= 0 - 3n \int_0^1 (1 - x^3)^{n-1}(1 - x^3 - 1) \, dx$

$= -3n(I_n - I_{n-1})$

$\Rightarrow I_n = \frac{3n}{(3n+1)} I_{n-1}; \quad I_{n-1} = \frac{3(n-1)}{(3n-2)} I_{n-2}$

$I_{n-2} = \frac{3(n-2)}{(3n-5)} I_{n-3}$

$\vdots \qquad \vdots \qquad \vdots$

$I_3 = \frac{3 \cdot 3}{10} I_2; \quad I_2 = \frac{3 \cdot 2}{7} I_1; \quad I_1 = \frac{3 \cdot 1}{4} I_0 = \frac{3 \cdot 1}{3}(1)$

$\therefore \quad I_n = \frac{(3 \cdot 1)(3 \cdot 2)(3 \cdot 3) \ldots (3 \cdot n)}{1 \cdot 4 \cdot 7 \cdot 10 \ldots (3n+1)} \cdot 1 = \frac{3^n \cdot n!}{1 \cdot 4 \cdot 7 \cdot 10 \ldots (3n+1)}$

Hence, $\frac{C_0}{1} - \frac{C_1}{4} + \frac{C_2}{7} - \ldots + (-1)^n \frac{C_n}{3n+1} = \frac{3^n \cdot n!}{1 \cdot 4 \cdot 7 \ldots (3n+1)}$

113. $(1-x)^{30} = {}^{30}C_0\,x^0 - {}^{30}C_1\,x^1 + {}^{30}C_2\,x^2 + \ldots$ ...(i)

$(x+1)^{30} = {}^{30}C_0\,x^{30} + {}^{30}C_1\,x^{29} + {}^{30}C_2\,x^{28}$

$+ \ldots + {}^{30}C_{10}\,x^{20} + \ldots + {}^{30}C_{30}\,x^0$ ...(ii)

On multiplying Eqs. (i) and (ii) and equating the coefficient of x^{20} on both sides, we get

required sum = coefficient of x^{20} in $(1-x^2)^{30} = {}^{30}C_{10}$

114. Coefficients of p th, $(p+1)$ th and $(p+2)$ th terms in expansion of $(1+x)^n$ are ${}^nC_{p-1}$, ${}^nC_p$, ${}^nC_{p+1}$.

Then, $2\cdot{}^nC_p = {}^nC_{p-1} + {}^nC_{p+1}$

$$2 = \frac{{}^nC_{p-1}}{{}^nC_p} + \frac{{}^nC_{p+1}}{{}^nC_p}$$

$$2 = \frac{p}{n-p+1} + \frac{n-p}{p+1}$$

$\Rightarrow \quad 2(n-p+1)(p+1) = p(p+1) + (n-p)(n-p+1)$

$\Rightarrow \; n^2 - n(4p+1) + 4p^2 - 2 = 0$

115. In the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$, the general term is

$$T_{r+1} = {}^{11}C_r\,(ax^2)^{11-r}\left(\frac{1}{bx}\right)^r = {}^{11}C_r a^{11-r}\,\frac{1}{b^r}\,x^{22-3r}$$

For x^7, we must have $22-3r=7 \Rightarrow r=5$ and then the

coefficient of $x^7 = {}^{11}C_5 \cdot a^{11-5}\,\frac{1}{b^5} = {}^{11}C_5\,\frac{a^6}{b^5}$

Similarly, in the expansion of $\left(ax - \frac{1}{bx^2}\right)^{11}$, the general term is

$$T_{r+1} = {}^{11}C_r\,(-1)^r\,\frac{a^{11-r}}{b^r}\cdot x^{11-3r}$$

For x^{-7} we must have, $11-3r=-7$

$\Rightarrow \; r=6$ and then coefficient of x^{-7} is ${}^{11}C_6\,\frac{a^5}{b^6} = {}^{11}C_5\,\frac{a^5}{b^6}$

As given, $\quad {}^{11}C_5\,\frac{a^6}{b^5} = {}^{11}C_5\,\frac{a^5}{b^6} \;\Rightarrow ab = 1$

116. $\because\; (1-y)^m(1+y)^n = (1 - {}^mC_1\,y + {}^mC_2\,y^2 \ldots)$

$\times(1 + {}^nC_1\,y + {}^nC_2\,y^2 + \ldots)$

$= 1 + (n-m)\,y + ({}^mC_2 + {}^nC_2 - mn) + \ldots$

Then $\quad a_1 = n - m = 10$ [given]...(i)

and $\quad {}^mC_2 + {}^nC_2 - mn = a_2 = 10$ (given)

$\Rightarrow \quad \frac{m(m-1)}{2} + \frac{n(n-1)}{2} - mn = 10$

$\Rightarrow \quad m^2 + n^2 - m - n - 2mn = 20$

$\Rightarrow \quad (n-m)^2 - (m+n) = 20 \quad [\because n-m=10]$

or $\quad 100 - (m-n = 20$

$\therefore \quad m+n = 80$ ...(ii)

On solving Eqs. (i) and (ii), we get

$n = 45,\ m = 35$

Hence, $\quad (m, n) = (35, 45)$

117. $\because\; T_5 + T_6 = 0 \;\Rightarrow\; \frac{T_6}{T_5} = -1$

$$\Rightarrow \quad \frac{{}^nC_5\,(a)^{n-5}(-b)^5}{{}^nC_4(a)^{n-4}(-b)^4} = -1 \;\Rightarrow\; \left(\frac{n-5+1}{5}\right) = \frac{a}{b}$$

$$\therefore \quad \frac{a}{b} = \frac{n-4}{5}$$

118. ${}^{20}C_0 - {}^{20}C_1 + {}^{20}C_2 - {}^{20}C_3 + \ldots$

${}^{20}C_9 + {}^{20}C_{10} - {}^{20}C_{11} + {}^{20}C_{12} - \ldots + {}^{20}C_{20} = 0$

$\Rightarrow\; 2\{{}^{20}C_0 - {}^{20}C_1 + {}^{20}C_2 - {}^{20}C_3 + \ldots - {}^{20}C_9\} + {}^{20}C_{10} = 0$

$\Rightarrow\; 2\{{}^{20}C_0 - {}^{20}C_1 + {}^{20}C_2 - {}^{20}C_3 + \ldots + {}^{20}C_9 + {}^{20}C_{10}\} = {}^{20}C_{10}$

$\Rightarrow\; {}^{20}C_0 - {}^{20}C_1 + {}^{20}C_2 - {}^{20}C_3 + \ldots - {}^{20}C_9 + {}^{20}C_{10} = \frac{1}{2}\,{}^{20}C_{10}$

119. $\sum_{r=0}^{n}(r+1)\,{}^nC_r = \sum_{r=0}^{n} r\cdot{}^nC_r + \sum_{r=0}^{n}{}^nC_r$

$$= \sum_{r=0}^{n} r\cdot\frac{n}{r}\cdot{}^{n-1}C_{r-1} + \sum_{r=0}^{n}{}^nC_r$$

$$= n\sum_{r=0}^{n}{}^{n-1}C_{r-1} + \sum_{r=0}^{n}{}^nC_r$$

$$= n\cdot 2^{n-1} + 2^n = (n+2)\,2^{n-1}$$

Statement-1 is true.

and $\sum_{r=0}^{n}(r+1)\,{}^nC_r\,x^r = \sum_{r=0}^{n} r\cdot{}^nC_r\cdot x^r + \sum_{r=0}^{n}{}^nC_r\cdot x^r$

$$= \sum_{r=0}^{n} r\cdot\frac{n}{r}\cdot{}^{n-1}C_{r-1}\cdot x^r + \sum_{r=0}^{n}{}^nC_r\cdot x^r$$

$$= n\sum_{r=0}^{n}{}^{n-1}C_{r-1}\cdot x^r + \sum_{r=0}^{n}{}^nC_r\cdot x^r$$

$$= nx(1+x)^{n-1} + (1+x)^n$$

On substituting $x=1$, then we get

$$\sum_{r=0}^{n}(r+1)\,{}^nC_r = n\cdot 2^{n-1} + 2^n = (n+2)\,2^{n-1}$$

Hence, Statement-2 is also true and it is a correct explanation for Statement-1.

120. $8^{2n} - (62)^{2n+1} = (64)^n - (62)^{2n+1}$

$= (63+1)^n - (63-1)^{2n+1}$

$= (1+63)^n + (1-63)^{2n+1}$

$= \{1 + {}^nC_1\cdot 63 + {}^nC_2(63)^2 + \ldots + {}^nC_n(63)^n\}$

$+ \{1 - {}^{2n+1}C_1(63) + {}^{2n+1}C_2(63)^2 \ldots - {}^{2n+1}C_{2n+1}(62)^{2n+1}\}$

$= 2 + 63\{{}^nC_1 + {}^nC_2\cdot 63 + \ldots + {}^nC_n(63)^{n-1} - {}^{2n+1}C_1$

$+ {}^{2n+1}C_2\cdot 63 \ldots - {}^{2n+1}C_{2n+1}(63)^{2n}\}$

$\therefore$ Remainder is 2.

121. $\because \quad A_r = {}^{10}C_r,\ B_r = {}^{20}C_r$ and $C_r = {}^{30}C_r$

$$\therefore \quad \sum_{r=1}^{10} A_r(B_{10}B_r - C_{10}A_r) = \sum_{r=1}^{10}{}^{10}C_r\,({}^{20}C_{10}\cdot{}^{20}C_r - {}^{30}C_{10}\cdot{}^{10}C_r)$$

$$= {}^{20}C_{10}\sum_{r=1}^{10}({}^{10}C_r)({}^{20}C_r) - {}^{30}C_{10}\sum_{r=1}^{10}({}^{10}C_r)^2$$

$$= {}^{20}C_{10}({}^{30}C_{10} - 1) - {}^{30}C_{10}({}^{20}C_{10} - 1) = C_{10} - B_{10}$$

122. Use ${}^nC_r = \frac{n}{r}\cdot{}^{n-1}C_{r-1}$, then $\quad S_1 = \sum_{C=1}^{10} C(C-1)\cdot{}^{10}C_C$

$$= \sum_{C=1}^{10} C(C-1)\cdot\frac{10.9}{C(C-1)}\cdot{}^8C_{C-2} = 90\sum_{C=1}^{10}{}^8C_{C-2} = 90\times 2^8$$

$$S_2 = \sum_{C=1}^{10} C \cdot {}^{10}C_C = \sum_{C=1}^{10} C \cdot \frac{10}{C} \cdot {}^{9}C_{C-1} = 10 \sum_{C=1}^{10} {}^{9}C_{C-1} = 10 \times 2^9$$

and $S_3 = \sum_{C=1}^{10} C^2 \cdot {}^{10}C_C = \sum_{C=1}^{10} C^2 \cdot \frac{10}{C} \cdot {}^{9}C_{C-1}$

$$= 10 \sum_{C=1}^{10} ((C-1)+1) \cdot {}^{9}C_{C-1}$$

$$= 10 \sum_{C=1}^{10} (9 \cdot {}^{8}C_{C-2} + {}^{9}C_{C-1}) = 10(9.2^8 + 2^9) = 55 \times 2^9$$

Both statements are true but Statement-2 is not correct
Explanation for Statement-1.

123. Here, $(1 - x - x^2 + x^3)^6 = (1 - x)^6 (1 - x^2)^6 = (1 - x^2)^6 (1 - x)^6$

$$= (1 - {}^{6}C_1 x^2 + {}^{6}C_2 x^4 - {}^{6}C_3 x^6 + \ldots) \times (1 - {}^{6}C_1 x + {}^{6}C_2 x^2 - {}^{6}C_3 x^3 + {}^{6}C_4 x^4 - {}^{6}C_5 x^5 + {}^{6}C_6 x^6)$$

$\therefore$ Coefficient of x^7 in $(1 - x - x^2 + x^3)^6$

$$= {}^{6}C_1 \times {}^{6}C_5 + ({}^{6}C_2 \times (-{}^{6}C_3) + \{(-{}^{6}C_3) \times (-{}^{6}C_1)\}$$

$$= 36 - 300 + 120 = -144$$

124. $(\sqrt{3}+1)^{2n} - (\sqrt{3}-1)^{2n} = 2\{{}^{2n}C_1(\sqrt{3})^{2n-1} + {}^{2n}C_3 (\sqrt{3})^{2n-3} + \ldots + {}^{2n}C_{2n-1}(\sqrt{3})\}$

$$= 2\sqrt{3}\{{}^{2n}C_1(\sqrt{3})^{2n-2} + {}^{2n}C_3(\sqrt{3})^{2n-4} + \ldots + {}^{2n}C_{2n-1}\}$$

$$= 2\sqrt{3}\{{}^{2n}C_1 (3)^{n-1} + {}^{2n}C_3(3)^{n-2} + \ldots {}^{2n}C_{2n-1}\}$$

$$= \sqrt{3} \times \text{even integer}$$

125. $\because \quad x + 1 = (x^{1/3})^3 + 1^3 = (x^{1/3} + 1)(x^{2/3} - x^{1/3} + 1)$

$$\therefore \quad \frac{x+1}{x^{2/3} - x^{1/3} + 1} = x^{1/3} + 1$$

and $\quad x - 1 = (x^{1/2})^2 - 1^2 = (x^{1/2} + 1)(x^{1/2} - 1)$

Now, $\frac{x-1}{x - x^{1/2}} = \frac{(x-1)}{x^{1/2}(x^{1/2} - 1)} = \frac{x^{1/2} + 1}{x^{1/2}} = 1 + x^{-1/2}$,

then $\left(\frac{x+1}{x^{2/3} - x^{1/3} + 1} - \frac{x-1}{x - x^{1/2}} \right)^{10} = (x^{1/3} - x^{-1/2})^{10}$

$\therefore \quad T_{r+1} = {}^{10}C_r (x^{1/3})^{10-r} (-x^{-1/2})^r$

For independent of x, $\frac{10-r}{3} - \frac{r}{2} = 0 \Rightarrow r = 4$

$$\therefore \quad T_{4+1} = {}^{10}C_4 = \frac{10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4} = 210$$

126. Since, ${}^{n+5}C_r : {}^{n+5}C_{r-1} = 2 \Rightarrow 3r = n + 6$ ...(i)

and ${}^{n+5}C_{r+1} : {}^{n+5}C_r = \frac{7}{5} \Rightarrow 12r = 5n + 18$ ...(ii)

From Eqs. (i) and (ii), we get

$$4(n+6) = 5n + 18 \Rightarrow n = 6$$

127. $(1 + ax + bx^2)(1 - 2x)^{18}$

$$= (1 + ax + bx^2)[1 - {}^{18}C_1(2x) + {}^{18}C_2 (2x)^2 - {}^{18}C_3(2x)^3 + {}^{18}C_4 (2x)^4 - \ldots]$$

According to the question, Coefficient of x^3

$$= -{}^{18}C_3 \cdot 8 + a \cdot {}^{18}C_2 \cdot 2^2 - b \cdot {}^{18}C_1 \cdot 2 = 0$$

$$\Rightarrow 17a - b = \frac{544}{3} \quad \ldots(i)$$

and coefficient of $x^4 = {}^{18}C_4 \cdot 2^4 - {}^{18}C_3 \cdot 2^3 \cdot a + {}^{18}C_2 \cdot 2^2 \cdot b = 0$

$$\Rightarrow \quad 32a - 3b = 240 \quad \ldots(ii)$$

On solving Eqs. (i) and (ii), we get

$$a = 16, \; b = \frac{272}{3}$$

$$\therefore \quad (a, b) = \left(16, \frac{272}{3}\right)$$

128. $\because (1 + x^2)^4 (1 + x^3)^7 (1 + x^4)^{12}$

$$= (1 + {}^{4}C_1 x^2 + {}^{4}C_2 x^4 + {}^{4}C_3 x^6 + {}^{4}C_4 x^8) \times (1 + {}^{7}C_1 x^3 + {}^{7}C_2 x^6 + {}^{7}C_3 x^9 + \ldots) \times (1 + {}^{12}C_1 x^4 + {}^{12}C_2 x^8 + \ldots)$$

Required coefficient

$$= {}^{12}C_2 \cdot {}^{7}C_1 \cdot 1 + {}^{12}C_1 \cdot {}^{7}C_1 \cdot {}^{4}C_2 + {}^{7}C_1 \cdot {}^{4}C_4 + {}^{7}C_3 \cdot {}^{4}C_1$$

$$= 462 + 504 + 7 + 140 = 1113$$

129. $\because \quad T_{r+1} = {}^{50}C_r (-2\sqrt{x})^r = {}^{50}C_r (-2)^r \cdot x^{r/2}$

For integral powers of x, $r = 0, 2, 4, 6, \ldots, 50$

$\therefore$ Required sum $= {}^{50}C_0 + 2^2 \cdot {}^{50}C_2 + 2^4 \cdot {}^{50}C_4 + \ldots + 2^{50} \cdot {}^{50}C_{50}$

$$= \frac{1}{2}[(1+2)^{50} + (1-2)^{50}] = \frac{1}{2}(3^{50} + 1)$$

130. In the Expansion of $(1 + x)(1 + x^2)(1 + x^3)\ldots(1 + x^{100})$.

x^9 can be found in the following ways

$x^9, x^{1+8}, x^{2+7}, x^{3+6}, x^{4+5}, x^{1+2+6}, x^{1+3+5}, x^{2+3+4}$

There are 8 cases

The coenfficient of x^9 in each cases is 1

$\therefore$ Required coefficient = 8

131. Total number of terms $= {}^{n+2}C_2 = 28$

$\Rightarrow (n+2)(n+1) = 56 = (6+2)(6+1)$

$\therefore \quad n = 6$

Sum of coefficients $= (1 - 2 + 4)^n = 3^6 = 729$

[**Note** In the solution it is considered that different terms in the expansion having same powers are not merged, as such it should be a bonus question]

132. Coefficient of x^2 in the expansion

$$= {}^{2}C_2 + {}^{3}C_2 + {}^{4}C_2 + {}^{5}C_2 + \ldots + {}^{49}C_2 + {}^{50}C_2 \cdot m^2$$

$$= {}^{3}C_3 + {}^{3}C_2 + {}^{4}C_2 + {}^{5}C_2 + \ldots + {}^{49}C_2 + {}^{50}C_2 \cdot m^2$$

$$= {}^{4}C_3 + {}^{4}C_2 + {}^{5}C_2 + \ldots + {}^{49}C_2 + {}^{50}C_2 \cdot m^2$$

$$= {}^{50}C_3 + {}^{50}C_2 \cdot m^2 \quad \text{(Applying again and again Pascal's rule)}$$

$$= ({}^{50}C_3 + {}^{50}C_2) + {}^{50}C_2(m^2 - 1)$$

$$= {}^{51}C_3 + {}^{50}C_2(m^2 - 1) = (3n+1)\,{}^{51}C_3 \quad \text{(given)}$$

or $\quad {}^{50}C_2(m^2 - 1) = 3n \cdot {}^{51}C_3$

or $\quad \frac{m^2 - 1}{3n} = \frac{51}{3} = 17 \quad \text{or} \quad \frac{m^2 - 1}{51} = n$

for $m = 16, n = 5$

133. $({}^{21}C_1 + {}^{21}C_2 + {}^{21}C_3 + {}^{21}C_4 + \ldots + {}^{21}C_{10}) - ({}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + \ldots + {}^{10}C_{10})$

$$= \frac{1}{2}(2^{21} - 2) - (2^{10} - 1) = (2^{20} - 1) - (2^{10} - 1)$$

$$= 2^{20} - 2^{10}$$

CHAPTER

07

Determinants

Learning Part

Session 1

- Definition of Determinants
- Expansion of Determinant
- Sarrus Rule for Expansion
- Window Rule for Expansion

Session 2

- Minors and Cofactors
- Use of Determinants in Coordinate Geometry
- Properties of Determinants

Session 3

- Examples on Largest Value of a Third Order Determinant
- Multiplication of Two Determinants of the Same Order
- System of Linear Equations
- Cramer's Rule
- Nature of Solutions of System of Linear Equations
- System of Homogeneous Linear Equations

Session 4

- Differentiation of Determinant
- Integration of a Determinant
- Walli's Formula
- Use of Σ in Determinant

Practice Part

- JEE Type Examples
- Chapter Exercises

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Session 1

Definition of Determinants, Expansion of Determinant, Sarrus Rule for Expansion, Window Rule for Expansion

Determinants were invented independently by **Gabriel Cramer**, whose now well-known rule for solving linear system was published in 1750, although not in present day notation. The now-standard "Vertical line notation", i.e. "| |" was given in 1841 by **Arthur Cayley**. The working knowledge of determinants is a basic necessity for a student. Determinants have wide applications in Engineering, Science, Economics, Social science, etc.

Definition of Determinants

Consider the system of two homogeneous linear equations

$$a_1 x + b_1 y = 0 \quad \text{...(i)}$$
$$a_2 x + b_2 y = 0 \quad \text{...(ii)}$$

in the two variables x and y. From these equations, we obtain

$$-\frac{a_1}{b_1} = \frac{y}{x} = -\frac{a_2}{b_2} \Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2}$$

$$\Rightarrow \quad a_1 b_2 - a_2 b_1 = 0$$

The result $a_1 b_2 - a_2 b_1$ is represented by $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

which is known as determinant of order two. The quantities a_1, b_1, a_2 and b_2 are called constituents or elements of the determinant and $a_1 b_2 - a_2 b_1$ is called its value.

The horizontal lines are called rows and vertical lines are called columns. Here, this determinant consists two rows and two columns.

For example, The value of the determinant

$$\begin{vmatrix} 2 & 3 \\ 4 & -5 \end{vmatrix} = 2 \times (-5) - 3 \times 4 = -10 - 12 = -22$$

Now, let us consider the system of three homogeneous linear equations

$$a_1 x + b_1 y + c_1 z = 0 \quad \text{...(i)}$$
$$a_2 x + b_2 y + c_2 z = 0 \quad \text{...(ii)}$$
$$a_3 x + b_3 y + c_3 z = 0 \quad \text{...(iii)}$$

On solving Eqs. (ii) and (iii) for x, y and z by cross-multiplication, we get

$$\frac{x}{b_2 c_3 - b_3 c_2} = \frac{y}{c_2 a_3 - c_3 a_2} = \frac{z}{a_2 b_3 - a_3 b_2} = k \quad \text{[say]}$$

$$\Rightarrow \quad x = k(b_2 c_3 - b_3 c_2), y = k(c_2 a_3 - c_3 a_2)$$

and $$z = k(a_2 b_3 - a_3 b_2)$$

On putting these values of x, y and z in Eq. (i), we get

$$a_1(b_2 c_3 - b_3 c_2) + b_1(c_2 a_3 - c_3 a_2) + c_1(a_2 b_3 - a_3 b_2) = 0$$

or $$a_1(b_2 c_3 - b_3 c_2) - b_1(c_3 a_2 - c_2 a_3) + c_1(a_2 b_3 - a_3 b_2) = 0 \quad \text{...(iv)}$$

or $$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0 \quad \text{...(v)}$$

Usually this is written as $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

Here, the expression $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ consisting of three rows and three columns, is called determinant of order three.

The quantities $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3$ and c_3 are called constituents or elements of the determinant.

Remark

1. A determinant is generally denoted by D or Δ.
2. A determinant of the nth order consists of n rows and n columns and its expansion contains $n!$ terms.
3. A determinant of nth order consists of n rows and n columns.
   $\therefore$ Number of constituents in determinant = n^2
4. In a determinant the horizontal lines counting from top to bottom 1st, 2nd, 3rd, … respectively, known as rows and denoted by $R_1, R_2, R_3, \ldots$ and vertical lines from left to right 1st, 2nd, 3rd, … respectively, known as columns and denoted by $C_1, C_2, C_3, \ldots$.
5. Shape of every determinant is square.
6. Sign system for order 2, order 3, order 4, … are given by

$$\begin{vmatrix} + & - \\ - & + \end{vmatrix}, \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}, \begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}, \ldots$$

Expansion of Determinant

(i) Expansion of two order

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & \\ & b_2 \end{vmatrix} - \begin{vmatrix} & b_1 \\ a_2 & \end{vmatrix} = a_1 b_2 - b_1 a_2$$

For example, $\begin{vmatrix} 5 & -4 \\ -3 & 2 \end{vmatrix} = \begin{vmatrix} 5 & \\ & 2 \end{vmatrix} - \begin{vmatrix} & -4 \\ -3 & \end{vmatrix}$

$$= 10 - 12 = -2$$

(ii) Expansion of third order

(a) *With respect to first row.*

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1 (b_2 c_3 - b_3 c_2) - b_1 (a_2 c_3 - a_3 c_2) + c_1 (a_2 b_3 - a_3 b_2)$$

(b) *With respect to second column.*

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ & \vdots & \\ a_2 & b_2 & c_2 \\ & \vdots & \\ a_3 & b_3 & c_3 \end{vmatrix} = -b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

$$= -b_1 (a_2 c_3 - a_3 c_2) + b_2 (a_1 c_3 - a_3 c_1) - b_3 (a_1 c_2 - a_2 c_1)$$

Remark

A determinant can be expanded along any of its row or column. Value of the determinant remains same in any of the cases.

Example 1. Find the value of the determinant

$$\begin{vmatrix} 1 & 2 & 4 \\ 3 & 4 & 9 \\ 2 & -1 & 6 \end{vmatrix}$$

Sol. Expanding the determinant along the first row

$$\Delta_1 = 1 \begin{vmatrix} 4 & 9 \\ -1 & 6 \end{vmatrix} - 2 \begin{vmatrix} 3 & 9 \\ 2 & 6 \end{vmatrix} + 4 \begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix}$$

$$= 1(24 + 9) - 2(18 - 18) + 4(-3 - 8)$$

$$= 33 - 0 - 44$$

$$= -11$$

and expanding the determinant along third column

$$\Delta_2 = 4 \begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix} - 9 \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} + 6 \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

$$= 4(-3 - 8) - 9(-1 - 4) + 6(4 - 6)$$

$$= -44 + 45 - 12$$

$$= 1 - 12$$

$$= -11$$

and expanding the determinant along second column

$$\Delta_3 = -2 \begin{vmatrix} 3 & 9 \\ 2 & 6 \end{vmatrix} + 4 \begin{vmatrix} 1 & 4 \\ 2 & 6 \end{vmatrix} - (-1) \begin{vmatrix} 1 & 4 \\ 3 & 9 \end{vmatrix}$$

$$= -2(18 - 18) + 4(6 - 8) + 1(9 - 12)$$

$$= 0 - 8 - 3$$

$$= -11$$

Hence, $\Delta_1 = \Delta_2 = \Delta_3$

Example 2. ***If*** $\Delta = \begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$,

prove that $2 \le \Delta \le 4$.

Sol. Given, $\Delta = \begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$

Expanding along first row, we get

$$\Delta = 1 \begin{vmatrix} 1 & \sin\theta \\ -\sin\theta & 1 \end{vmatrix} - \sin\theta \begin{vmatrix} -\sin\theta & \sin\theta \\ -1 & 1 \end{vmatrix} + 1 \begin{vmatrix} -\sin\theta & 1 \\ -1 & -\sin\theta \end{vmatrix}$$

$$= (1 + \sin^2\theta) - \sin\theta(-\sin\theta + \sin\theta) + (\sin^2\theta + 1)$$

$$= 2(1 + \sin^2\theta)$$

Again, $0 \le \sin^2\theta \le 1$

$\Rightarrow \quad 1 \le (1 + \sin^2\theta) \le 1 + 1$

$\Rightarrow \quad 2 \le 2(1 + \sin^2\theta) \le 4$

$\therefore \quad 2 \le \Delta \le 4$

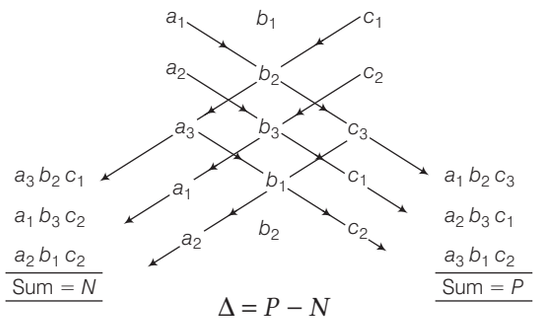
Sarrus Rule for Expansion

Sarrus gave a rule for a determinant of order 3.

Rule Write down the three rows of the Δ and rewrite the first two rows. The three diagonals sloping down to the right given the three terms and the three diagonals sloping down to the left also given the three terms.

If $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Rule

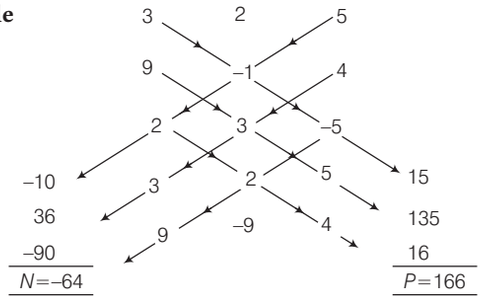


$\therefore \quad \Delta = P - N$

Example 3. ***Expand*** $\begin{vmatrix} 3 & 2 & 5 \\ 9 & -1 & 4 \\ 2 & 3 & -5 \end{vmatrix}$ ***by Sarrus rule.***

Sol. Let $\Delta = \begin{vmatrix} 3 & 2 & 5 \\ 9 & -1 & 4 \\ 2 & 3 & -5 \end{vmatrix}$

Rule

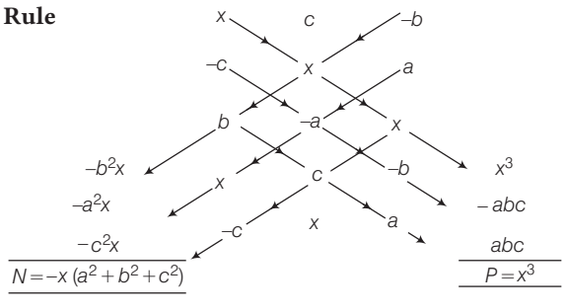


$\therefore \quad \Delta = P - N = 166 - (-64) = 230$

Example 4. ***If*** $a, b, c \in R$***, find the number of real roots of the equation*** $\begin{vmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{vmatrix} = 0$

Sol. Let $\Delta = \begin{vmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{vmatrix}$

Rule



$\therefore \quad \Delta = P - N$

$= x^3 + x(a^2 + b^2 + c^2) = 0 \qquad$ [given]

$\therefore \quad x = 0$ or $x^2 = -(a^2 + b^2 + c^2)$

$\Rightarrow \quad x = 0$ or $x = \pm i\sqrt{(a^2 + b^2 + c^2)}$, where $i = \sqrt{-1}$

Hence, number of real roots is one.

Window Rule for Expansion

Window rule valid only for third order determinant.

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

In this method, rewrite first two elements of second row and third row, then

Rule

$$\begin{matrix} a_1 & b_1 & c_1 & & \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{matrix}$$

Now, taking positive sign with a_1, b_1 and c_1.

$$\Delta = a_1(b_2 c_3 - b_3 c_2) + b_1(c_2 a_3 - c_3 a_2) + c_1(a_2 b_3 - a_3 b_2)$$

Example 5. Expand $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 6 & 2 \\ 5 & 9 & 4 \end{vmatrix}$ by window rule.

Sol. Let $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 6 & 2 \\ 5 & 9 & 4 \end{vmatrix}$

Rule:

$$\begin{matrix} 1 & 2 & 3 & & \\ 4 & 6 & 2 & 4 & 6 \\ 5 & 9 & 4 & 5 & 9 \end{matrix}$$

$\therefore \quad \Delta = 1(24 - 18) + 2(10 - 16) + 3(36 - 30)$

$= 6 - 12 + 18 = 12$

Example 6. Find the value of the determinant $\begin{vmatrix} -1 & 2 & 1 \\ 3 + 2\sqrt{2} & 2 + 2\sqrt{2} & 1 \\ 3 - 2\sqrt{2} & 2 - 2\sqrt{2} & 1 \end{vmatrix}$.

Sol. Let $\Delta = \begin{vmatrix} -1 & 2 & 1 \\ 3 + 2\sqrt{2} & 2 + 2\sqrt{2} & 1 \\ 3 - 2\sqrt{2} & 2 - 2\sqrt{2} & 1 \end{vmatrix}$ and let $2\sqrt{2} = \lambda$,

then $\Delta = \begin{vmatrix} -1 & 2 & 1 \\ 3 + \lambda & 2 + \lambda & 1 \\ 3 - \lambda & 2 - \lambda & 1 \end{vmatrix}$

Rule

$$\begin{matrix} -1 & 2 & 1 & & \\ 3 + \lambda & 2 + \lambda & 1 & 3 + \lambda & 2 + \lambda \\ 3 - \lambda & 2 - \lambda & 1 & 3 - \lambda & 2 - \lambda \end{matrix}$$

Now, $\Delta = -1(2 + \lambda - 2 + \lambda) + 2(3 - \lambda - 3 - \lambda) + 1[(3 + \lambda)(2 - \lambda) - (3 - \lambda)(2 + \lambda)]$

$= -2\lambda - 4\lambda + (-2\lambda) = -8\lambda = -16\sqrt{2}$

$[\because \lambda = 2\sqrt{2}]$

Exercise for Session 1

1 Sum of real roots of the equation $\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$ is

(a) -2 (b) -1 (c) 0 (d) 1

2 If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy, i = \sqrt{-1}$, then

(a) $x = 3, y = 1$ (b) $x = 1, y = 3$
(c) $x = 0, y = 3$ (d) $x = 0, y = 0$

3 If $p\lambda^4 + q\lambda^3 + r\lambda^2 + s\lambda + t = \begin{vmatrix} \lambda^2 + 3\lambda & \lambda - 1 & \lambda + 3 \\ \lambda^2 + 1 & 2 - \lambda & \lambda - 3 \\ \lambda^2 - 3 & \lambda + 4 & 3\lambda \end{vmatrix}$, then t is equal to

(a) 7 (b) 14
(c) 21 (d) 28

4 If one root of the equation $\begin{vmatrix} 7 & 6 & x^2 - 13 \\ 2 & x^2 - 13 & 2 \\ x^2 - 13 & 3 & 7 \end{vmatrix} = 0$ is $x = 2$, the sum of all other five roots is

(a) $2\sqrt{15}$ (b) -2
(c) $\sqrt{20} + \sqrt{15} - 2$ (d) None of these

5 If A, B and C are the angles of a non-right angled ΔABC, the value of $\begin{vmatrix} \tan A & 1 & 1 \\ 1 & \tan B & 1 \\ 1 & 1 & \tan C \end{vmatrix}$ is

(a) 0 (b) 1 (c) 2 (d) 3

6 If $\Delta = \begin{vmatrix} 1 & 3\cos\theta & 1 \\ \sin\theta & 1 & 3\cos\theta \\ 1 & \sin\theta & 1 \end{vmatrix}$, the maximum value of Δ is

(a) -10 (b) $-\sqrt{10}$ (c) $\sqrt{10}$ (d) 10

7 If the value of the determinant $\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix}$ is positive, then $(a, b, c > 0)$

(a) $abc > 1$ (b) $abc > -8$ (c) $abc < -8$ (d) $abc > -2$

Session 2

Minors and Cofactors, Use of Determinants in Coordinate Geometry, Properties of Determinants

Minors and Cofactors

Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{21} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$

be a determinant of order n, $n \geq 2$, then the determinant of order $n-1$ obtained from the determinant Δ after deleting the ith row and jth column is called the **minor of the element a_{ij}** and it is usually denoted by M_{ij}, where $i = 1, 2, 3, \ldots, n$ and $j = 1, 2, 3, \ldots, n$.

If M_{ij} is the minor of the element a_{ij} in the determinant Δ, then $(-1)^{i+j} M_{ij}$ is called the **cofactor of the element a_{ij}**. It is usually denoted by C_{ij}.

Thus, $C_{ij} = (-1)^{i+j} M_{ij}$

$$= \begin{cases} M_{ij}, & \text{if} \quad i+j \text{ is an even integer} \\ -M_{ij}, & \text{if} \quad i+j \text{ is an odd integer} \end{cases}$$

(i) Let $\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, then

$M_{11} = |a_{22}| = a_{22}, M_{12} = |a_{21}| = a_{21},$

$M_{21} = |a_{12}| = a_{12}, M_{22} = |a_{11}| = a_{11}$ and

$C_{11} = (-1)^{1+1} M_{11} = a_{22},$

$C_{12} = (-1)^{1+2} M_{12} = -a_{21},$

$C_{21} = (-1)^{2+1} M_{21} = -a_{12}$

and $C_{22} = (-1)^{2+2} M_{22} = a_{11}$

(ii) Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Determinants of minors and cofactors are

$$\Delta^M = \begin{vmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{vmatrix}, \quad \Delta^c = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & a_{32} & a_{33} \end{vmatrix}$$

where

$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, C_{11} = (-1)^{1+1} M_{11} = M_{11}$

$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, C_{12} = (-1)^{1+2} M_{12} = -M_{12}$

$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}, C_{13} = (-1)^{1+3} M_{13} = M_{13}$

$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, C_{21} = (-1)^{2+1} M_{21} = -M_{21}$

$M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, C_{22} = (-1)^{2+2} M_{22} = M_{22}$

$M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}, C_{23} = (-1)^{2+3} M_{23} = -M_{23}$

$M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, C_{31} = (-1)^{3+1} M_{31} = M_{31}$

$M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, C_{32} = (-1)^{3+2} M_{32} = -M_{32}$

$M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, C_{33} = (-1)^{3+3} M_{33} = M_{33}$

Important Results for Cofactors

1. The sum of products of the elements of any row or column with their corresponding cofactors is equal to the value of the determinant.

i.e., $\Delta = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$

$= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$

$= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} = a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33}$

Now, value of n order determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & a_{nn} \end{vmatrix}$$

$= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \ldots + a_{1n}C_{1n}$

(when expanded along first row)

2. The sum of the product of element of any row (or column) with corresponding cofactors of another row (or column) is equal to zero.

i.e., $a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} = 0$,

$a_{11}C_{13} + a_{21}C_{23} + a_{31}C_{33} = 0$, etc.

3. If the value of a n order determinant is Δ, then the value of the determinant formed by the cofactors of corresponding elements of the given determinant is given by

$$\Delta^c = \Delta^{n-1}$$

i.e., in case of second order determinant

$$\Delta^c = \Delta$$

and third order determinant $\Delta^c = \Delta^2$.

Example 7. ***Find the determinants of minors and cofactors of the determinant*** $\begin{vmatrix} 2 & 3 & 4 \\ 7 & 2 & -5 \\ 8 & -1 & 3 \end{vmatrix}$.

Sol. Here, $M_{11} = \begin{vmatrix} 2 & -5 \\ -1 & 3 \end{vmatrix} = 6 - 5 = 1$

[delete 1st row and 1st column]

$\therefore \quad C_{11} = (-1)^{1+1} M_{11} = M_{11} = 1$

$M_{12} = \begin{vmatrix} 7 & -5 \\ 8 & 3 \end{vmatrix} = 21 + 40 = 61$

[delete 1st row and 2nd column]

$\therefore \quad C_{12} = (-1)^{1+2} M_{12} = -M_{12} = -61$

$M_{13} = \begin{vmatrix} 7 & 2 \\ 8 & -1 \end{vmatrix} = -7 - 16 = -23$

[delete 1st row and 3rd column]

$\therefore \quad C_{13} = (-1)^{1+3} M_{13} = M_{13} = -23$

$M_{21} = \begin{vmatrix} 3 & 4 \\ -1 & 3 \end{vmatrix}$ [delete 2nd row and 1st column]

$= 9 + 4 = 13$

$\therefore \quad C_{21} = (-1)^{2+1} M_{21} = -M_{21} = -13$

$M_{22} = \begin{vmatrix} 2 & 4 \\ 8 & 3 \end{vmatrix}$ [delete 2nd row and 2nd column]

$= 6 - 32 = -26$

$\therefore \quad C_{22} = (-1)^{2+2} M_{22} = M_{22} = -26$

$M_{23} = \begin{vmatrix} 2 & 3 \\ 8 & -1 \end{vmatrix}$ [delete 2nd row and 3rd column]

$= -2 - 24 = -26$

$\therefore \quad C_{23} = (-1)^{2+3} M_{23} = -M_{23} = 26$

$M_{31} = \begin{vmatrix} 3 & 4 \\ 2 & -5 \end{vmatrix}$ [delete 3rd row and 1st column]

$= -15 - 8 = -23$

$\therefore \quad C_{31} = (-1)^{3+1} M_{31} = M_{31} = -23$

$M_{32} = \begin{vmatrix} 2 & 4 \\ 7 & -5 \end{vmatrix}$ [delete 3rd row and 2nd column]

$= -10 - 28 = -38$

$\therefore \quad C_{32} = (-1)^{3+2} M_{32} = -M_{32} = 38$

and $M_{33} = \begin{vmatrix} 2 & 3 \\ 7 & 2 \end{vmatrix} = 4 - 21 = -17$

[delete 3rd row and 3rd column]

$\therefore \quad C_{33} = (-1)^{3+3} M_{33} = M_{33} = -17$

Hence, determinants of minors and cofactors are

$\begin{vmatrix} 1 & 61 & -23 \\ 13 & -26 & -26 \\ -23 & -38 & -17 \end{vmatrix}$ and $\begin{vmatrix} 1 & -61 & -23 \\ -13 & -26 & 26 \\ -23 & 38 & -17 \end{vmatrix}$, respectively.

Goyal's Method for Cofactors
(Direct Method)

This method applied only for third order determinant.

Method If $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

***Step* I** Write down the three rows of the Δ and rewrite first two rows.

i.e.
$$\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix}$$

***Step* II** Alter Step I, rewrite first two columns.

i.e.,
$$\begin{matrix} a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \\ c_1 & c_2 & c_3 & c_1 & c_2 \\ a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \end{matrix}$$

***Step* III** After step II, deleting first row and first column, then we get all cofactors i.e.

$$\begin{matrix} b_2 & b_3 & b_1 & b_2 \\ c_2 & c_3 & c_1 & c_2 \\ a_2 & a_3 & a_1 & a_2 \\ b_2 & b_3 & b_1 & b_2 \end{matrix}$$

or $\Delta^c = \begin{vmatrix} b_2 c_3 - b_3 c_2 & b_3 c_1 - b_1 c_3 & b_1 c_2 - b_2 c_1 \\ c_2 a_3 - c_3 a_2 & c_3 a_1 - c_1 a_3 & c_1 a_2 - c_2 a_1 \\ a_2 b_3 - a_3 b_2 & a_3 b_1 - a_1 b_3 & a_1 b_2 - a_2 b_1 \end{vmatrix}$

Example 8. Find the determinant of cofactors of the determinant $\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix}$ by Direct Method.

Sol. Let $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix}$

***Step* I** Write down the three rows of the Δ and rewrite first two rows.

$$\begin{matrix} & 1 & 2 & 3 \\ & -4 & 3 & 6 \\ \text{i.e.,} & 2 & -7 & 9 \\ & 1 & 2 & 3 \\ & -4 & 3 & 6 \end{matrix}$$

***Step* II** After step I, rewrite first two columns

$$\begin{matrix} & 1 & 2 & 3 & 1 & 2 \\ & -4 & 3 & 6 & -4 & 3 \\ \text{i.e.,} & 2 & -7 & 9 & 2 & -7 \\ & 1 & 2 & 3 & 1 & 2 \\ & -4 & 3 & 6 & -4 & 3 \end{matrix}$$

***Step* III** After Step II, deleting first row and first column, then we get all cofactors i.e.,

$$\begin{matrix} 3 & 6 & -4 & 3 \\ -7 & 9 & 2 & -7 \\ 2 & 3 & 1 & 2 \\ 3 & 6 & -4 & 3 \end{matrix} \quad \text{or} \quad \Delta^c = \begin{vmatrix} 69 & 48 & 22 \\ -39 & 3 & 11 \\ 3 & -18 & 11 \end{vmatrix}$$

Example 9. If the value of a third order determinant is 11, find the value of the square of the determinant formed by the cofactors.

Sol. Here, $n = 3$ and $\Delta = 11$

$$\therefore \quad (\Delta^c)^2 = (\Delta^2)^2 = \Delta^4 = 11^4 = 14641$$

Use of Determinants in Coordinate Geometry

(i) Area of triangle whose vertices are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

(ii) If points $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) and collinear, then

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

(iii) If $a_r x + b_r y + c_r = 0; r = 1, 2, 3$ are the sides of a triangle, then the area of the triangle is given by

$$\Delta = \frac{1}{|2\,C_1 C_2 C_3|}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$$

where C_1, C_2 and C_3 are the cofactors of the elements c_1, c_2 and c_3 respectively, in the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(iv) Equation of straight line passing through two points (x_1, y_1) and (x_2, y_2) is

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

(v) If three lines $a_r x + b_r y + c_r = 0;\ r = 1, 2, 3$ are concurrent, then

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

(vi) If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines, then

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

(vii) Equation of circle through three non-collinear points $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) is given by

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

Some Useful Operations

(i) The interchange of *i*th row and *j*th row is denoted by $R_i \leftrightarrow R_j$. (In case of column $C_i \leftrightarrow C_j$)

(ii) The addition of m times the elements of *j*th row to the corresponding elements of *i*th row is denoted by $R_i \to R_i + mR_j$.
(In case of column $C_i \to C_i + mC_j$)

(iii) The addition of m times the elements of jth row and n times the elements of k th row to the corresponding elements of ith row is denoted by
$R_i \to R_i + mR_j + nR_k$.
(In case of column $C_i \to C_i + mC_j + nC_k$)

Properties of Determinants

We shall establish certain properties of a determinant of the third order but reader should note that these are capable of application to a determinant of any order.

Property I The value of a determinant remains unaltered when rows are changed into corresponding columns and columns are changed into corresponding rows.

Proof Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Expanding the determinant along the first row, then

$$\Delta = a_1(b_2 c_3 - b_3 c_2) - b_1(a_2 c_3 - a_3 c_2) + c_1(a_2 b_3 - a_3 b_2)$$
$$= a_1(b_2 c_3 - b_3 c_2) - a_2(b_1 c_3 - b_3 c_1) + a_3(b_1 c_2 - b_2 c_1)$$
$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$= \Delta'$, where Δ' be the value of the determinant when rows of determinant Δ are changed into corresponding columns.

Property II If any two rows (or two columns) of a determinant are interchanged, then the sign of determinant is changed and the numerical value remains unaltered.

Proof Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Expanding the determinant along the first row, then

$$\Delta = a_1(b_2 c_3 - b_3 c_2) - b_1(a_2 c_3 - a_3 c_2) + c_1(a_2 b_3 - a_3 b_2)$$
$$= -a_2(b_1 c_3 - b_3 c_1) + b_2(a_1 c_3 - a_3 c_1) - c_2(a_1 b_3 - a_3 b_1)$$
$$= -[a_2(b_1 c_3 - b_3 c_1) - b_2(a_1 c_3 - a_3 c_1) + c_2(a_1 b_3 - a_3 b_1)]$$
$$= -\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad [\text{by } R_1 \leftrightarrow R_2]$$

Hence, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Remark

If any row (or column) of a determinant Δ be passed over m rows (or columns), then the resulting determinant $= (-1)^m \Delta$.

Property III If two rows (or columns) of a determinant are identical, then the value of the determinant is zero.

Proof Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = -\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = -\Delta$

$[\text{by } R_1 \leftrightarrow R_3]$

$\Rightarrow \quad 2\Delta = 0$

$\therefore \quad \Delta = 0$

Property IV If the elements of any row (or any column) of a determinant be each multiplied by the same factor k, then the value of the determinant is multiplied by k.

Proof Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 C_{11} + b_1 C_{12} + c_1 C_{13}$

Then, $\begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = ka_1 C_{11} + kb_1 C_{12} + kc_1 C_{13}$

(where C_{11}, C_{12} and C_{13} are the cofactors of a_1, b_1 and c_1 in Δ)

$$= k(a_1 C_{11} + b_1 C_{12} + c_1 C_{13}) = k\Delta$$

Property V If every element of some column (or row) is the sum of two items, then the determinant is equal to the sum of two determinants; one containing one the first term in place of each sum, the other only the second term. The remaining elements of both determinants are the same as in the given determinant i.e.,

$$\begin{vmatrix} a_1 + x & b_1 & c_1 \\ a_2 + y & b_2 & c_2 \\ a_3 + z & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & b_1 & c_1 \\ y & b_2 & c_2 \\ z & b_3 & c_3 \end{vmatrix}$$

Proof Let $\Delta = \begin{vmatrix} a_1 + x & b_1 & c_1 \\ a_2 + y & b_2 & c_2 \\ a_3 + z & b_3 & c_3 \end{vmatrix}$

Expanding the determinant along first column, then

$$\Delta=(a_1+x)\begin{vmatrix}b_2 & c_2\\ b_3 & c_3\end{vmatrix}-(a_2+y)\begin{vmatrix}b_1 & c_1\\ b_3 & c_3\end{vmatrix}+(a_3+z)\begin{vmatrix}b_1 & c_1\\ b_2 & c_2\end{vmatrix}$$

$$=a_1\begin{vmatrix}b_2 & c_2\\ b_3 & c_3\end{vmatrix}-a_2\begin{vmatrix}b_1 & c_1\\ b_3 & c_3\end{vmatrix}+a_3\begin{vmatrix}b_1 & c_1\\ b_2 & c_2\end{vmatrix}+x\begin{vmatrix}b_2 & c_2\\ b_3 & c_3\end{vmatrix}-y\begin{vmatrix}b_1 & c_1\\ b_3 & c_3\end{vmatrix}+z\begin{vmatrix}b_1 & c_1\\ b_2 & c_2\end{vmatrix}$$

$$=\begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}+\begin{vmatrix}x & b_1 & c_1\\ y & b_2 & c_2\\ z & b_3 & c_3\end{vmatrix}$$

Remark

1. $\begin{vmatrix}a_1+b_1+c_1 & d_1+e_1 & f_1\\ a_2+b_2+c_2 & d_2+e_2 & f_2\\ a_3+b_3+c_3 & d_3+e_3 & f_3\end{vmatrix}=\begin{vmatrix}a_1 & d_1 & f_1\\ a_2 & d_2 & f_2\\ a_3 & d_3 & f_3\end{vmatrix}+\begin{vmatrix}a_1 & e_1 & f_1\\ a_2 & e_2 & f_2\\ a_3 & e_3 & f_3\end{vmatrix}+\begin{vmatrix}b_1 & d_1 & f_1\\ b_2 & d_2 & f_2\\ b_3 & d_3 & f_3\end{vmatrix}+\begin{vmatrix}b_1 & e_1 & f_1\\ b_2 & e_2 & f_2\\ b_3 & e_3 & f_3\end{vmatrix}+\begin{vmatrix}c_1 & d_1 & f_1\\ c_2 & d_2 & f_2\\ c_3 & d_3 & f_3\end{vmatrix}+\begin{vmatrix}c_1 & e_1 & f_1\\ c_2 & e_2 & f_2\\ c_3 & e_3 & f_3\end{vmatrix}$

2. If each element of first row of a determinant consists of algebraic sum of p elements, second row consists of algebraic sum of q elements, third row consists of algebraic sum of r elements and so on.

Then, given determinant is equivalent to the sum of $p\times q\times r\times\ldots$ other determinants in each of which the elements consists of single term.

Property VI The value of the determinant does not change, if the elements of any row (or column) are increased or diminished by equimultiples of the corresponding elements of any other row (or column) of the determinant.

i.e., $\begin{vmatrix}a_1+mb_1+nc_1 & b_1 & c_1\\ a_2+mb_2+nc_2 & b_2 & c_2\\ a_3+mb_3+nc_3 & b_3 & c_3\end{vmatrix}=\begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}$

Proof $\begin{vmatrix}a_1+mb_1+nc_1 & b_1 & c_1\\ a_2+mb_2+nc_2 & b_2 & c_2\\ a_3+mb_3+nc_3 & b_3 & c_3\end{vmatrix}=\begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}+m\begin{vmatrix}b_1 & b_1 & c_1\\ b_2 & b_2 & c_2\\ b_3 & b_3 & c_3\end{vmatrix}+n\begin{vmatrix}c_1 & b_1 & c_1\\ c_2 & b_2 & c_2\\ c_3 & b_3 & c_3\end{vmatrix}$

$$=\begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}+m\cdot 0+n\cdot 0=\begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}$$

Property VII If each element on one side or other side or both side of the principal diagonal of determinant is zero, then the value of the determinant is the product of the diagonal element.

i.e., $\begin{vmatrix}a & 0 & 0\\ f & b & 0\\ e & d & c\end{vmatrix}=\begin{vmatrix}a & i & h\\ 0 & b & g\\ 0 & 0 & c\end{vmatrix}=\begin{vmatrix}a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & c\end{vmatrix}=abc$

Proof Let $\Delta=\begin{vmatrix}a\cdots & 0\cdots & 0\\ f & b & 0\\ e & d & c\end{vmatrix}$

Expanding along R_1, we get $a\begin{vmatrix}b & 0\\ d & c\end{vmatrix}=a(bc)=abc$

Property VIII If determinant Δ becomes zero on putting $x=\alpha$, then we say that $(x-\alpha)$ is a factor of Δ.

i.e., if $\Delta=\begin{vmatrix}x & 5 & 2\\ x^2 & 9 & 4\\ x^3 & 16 & 8\end{vmatrix}$

at $x=2, \Delta=0$ [because C_1 and C_3 are identical at $x=2$]

and at $x=0, \Delta=0$ [because all elements of C_1 are zero]

Hence, $(x-0)$ and $(x-2)$ are the factors of Δ.

Remark

1. It should be noted that while applying operations on determinant that atleast one row (or column) must remain unchanged.

2. Maximum number of operations at a time = order – 1

3. It should be noted that, if the row (or column) which is changed by multiplied a non-zero number, then the determinant will be divided by that number.

Examples on Properties

Example 10. Evaluate $\begin{vmatrix}13 & 16 & 19\\ 14 & 17 & 20\\ 15 & 18 & 21\end{vmatrix}$.

Sol. Let $\Delta=\begin{vmatrix}13 & 16 & 19\\ 14 & 17 & 20\\ 15 & 18 & 21\end{vmatrix}$

Applying $R_2\to R_2-R_1$ and $R_3\to R_3-R_1$, then

$$\Delta=\begin{vmatrix}13 & 16 & 19\\ 1 & 1 & 1\\ 2 & 2 & 2\end{vmatrix}=2\begin{vmatrix}13 & 16 & 19\\ 1 & 1 & 1\\ 1 & 1 & 1\end{vmatrix}=0$$

[$\because R_2$ and R_3 are identical]

Example 11. Prove that $\begin{vmatrix}\alpha & \beta & \gamma\\ \theta & \phi & \psi\\ \lambda & \mu & \nu\end{vmatrix} = \begin{vmatrix}\beta & \mu & \phi\\ \alpha & \lambda & \theta\\ \gamma & \nu & \psi\end{vmatrix}$.

Sol. LHS $= \begin{vmatrix}\alpha & \beta & \gamma\\ \theta & \phi & \psi\\ \lambda & \mu & \nu\end{vmatrix} = \begin{vmatrix}\alpha & \theta & \lambda\\ \beta & \phi & \mu\\ \gamma & \psi & \nu\end{vmatrix}$

[interchanging rows and columns]

$= (-1)\begin{vmatrix}\alpha & \lambda & \theta\\ \beta & \mu & \phi\\ \gamma & \nu & \psi\end{vmatrix}$ $[C_2 \leftrightarrow C_3]$

$= (-1)^2\begin{vmatrix}\beta & \mu & \phi\\ \alpha & \lambda & \theta\\ \gamma & \nu & \psi\end{vmatrix}$ $[R_1 \leftrightarrow R_2]$

$= \begin{vmatrix}\beta & \mu & \phi\\ \alpha & \lambda & \theta\\ \gamma & \nu & \psi\end{vmatrix} =$ RHS

Example 12. Use the properties of determinant and without expanding, prove that

$$\begin{vmatrix}b+c & q+r & y+z\\ c+a & r+p & z+x\\ a+b & p+q & x+y\end{vmatrix} = 2\begin{vmatrix}a & p & x\\ b & q & y\\ c & r & z\end{vmatrix}.$$

Sol. Let LHS $= \Delta = \begin{vmatrix}b+c & q+r & y+z\\ c+a & r+p & z+x\\ a+b & p+q & x+y\end{vmatrix}$

Applying $R_1 \to R_1 + R_2 + R_3$, then

$$\Delta = \begin{vmatrix}2(a+b+c) & 2(p+q+r) & 2(x+y+z)\\ c+a & r+p & z+x\\ a+b & p+q & x+y\end{vmatrix}$$

$$= 2\begin{vmatrix}a+b+c & p+q+r & x+y+z\\ c+a & r+p & z+x\\ a+b & p+q & x+y\end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$\Delta = 2\begin{vmatrix}a+b+c & p+q+r & x+y+z\\ -b & -q & -y\\ -c & -r & -z\end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, then

$$\Delta = 2\begin{vmatrix}a & p & x\\ -b & -q & -y\\ -c & -r & -z\end{vmatrix} = 2(-1)(-1)\begin{vmatrix}a & p & x\\ b & q & y\\ c & r & z\end{vmatrix}$$

$$= 2\begin{vmatrix}a & p & x\\ b & q & y\\ c & r & z\end{vmatrix} = \text{RHS}$$

Example 13. Without expanding as far as possible, prove that

$$\begin{vmatrix}1 & 1 & 1\\ x & y & z\\ x^3 & y^3 & z^3\end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z).$$

Sol. Let $\Delta = \begin{vmatrix}1 & 1 & 1\\ x & y & z\\ x^3 & y^3 & z^3\end{vmatrix}$

for $x = y$, $\Delta = 0$ $[\because C_1$ and C_2 are identical$]$

Hence, $(x-y)$ is a factor of Δ. Similarly, $(y-z)$ and $(z-x)$ are factors of Δ. But Δ is a homogeneous expression of the 4 th degree in x, y and z.

There must be one more factor of the 1st degree in x, y and z say $k(x+y+z)$, where k is a constant.

Let $\Delta = k(x-y)(y-z)(z-x)(x+y+z)$

On putting $x = 0$, $y = 1$ and $z = 2$, then

$$\begin{vmatrix}1 & 1 & 1\\ 0 & 1 & 2\\ 0 & 1 & 8\end{vmatrix} = k(0-1)(1-2)(2-0)(0+1+2)$$

$\Rightarrow \quad 1\cdot(8-2) = k(-1)(-1)(2)(3) \quad \therefore \quad k = 1$

$\Rightarrow \quad \Delta = (x-y)(y-z)(z-x)(x+y+z) = \text{RHS}$

Example 14. Solve for x,

$$\begin{vmatrix}4x & 6x+2 & 8x+1\\ 6x+2 & 9x+3 & 12x\\ 8x+1 & 12x & 16x+2\end{vmatrix} = 0.$$

Sol. Applying $C_2 \to C_2 - \frac{3}{2}C_1$ and $C_3 \to C_3 - 2C_1$

Then, $\begin{vmatrix}4x & 2 & 1\\ 6x+2 & 0 & -4\\ 8x+1 & -(3/2) & 0\end{vmatrix} = 0$

$\Rightarrow \quad 4x(0-6) - (6x+2)\left(0+\frac{3}{2}\right) + (8x+1)(-8-0) = 0$

$\Rightarrow \quad -97x - 11 = 0 \quad \Rightarrow \quad x = -\frac{11}{97}$

Example 15. Prove that

$$\begin{vmatrix}a^2+1 & ab & ac\\ ab & b^2+1 & bc\\ ac & bc & c^2+1\end{vmatrix} = 1 + a^2 + b^2 + c^2.$$

Sol. Let LHS $= \Delta = \begin{vmatrix}a^2+1 & ab & ac\\ ab & b^2+1 & bc\\ ac & bc & c^2+1\end{vmatrix}$

On taking common a, b and c from R_1, R_2 and R_3 respectively, then

$$\Delta = abc\begin{vmatrix} \dfrac{a^2+1}{a} & b & c \\ a & \dfrac{b^2+1}{b} & c \\ a & b & \dfrac{c^2+1}{c} \end{vmatrix}$$

Now, multiplying in C_1, C_2 and C_3 by a, b and c respectively, then

$$\Delta = \begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, then

$$\Delta = \begin{vmatrix} 1+a^2+b^2+c^2 & b^2 & c^2 \\ 1+a^2+b^2+c^2 & b^2+1 & c^2 \\ 1+a^2+b^2+c^2 & b^2 & c^2+1 \end{vmatrix}$$

$$= (1+a^2+b^2+c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 1 & b^2+1 & c^2 \\ 1 & b^2 & c^2+1 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$\Delta = (1+a^2+b^2+c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ & \ddots & \\ 0 & 1 & 0 \\ & & \ddots \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (1+a^2+b^2+c^2)\cdot 1\cdot 1\cdot 1 = (1+a^2+b^2+c^2) = \text{RHS}$$

Example 16. If a, b and c are all different and if $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$, prove that $abc = -1$.

Sol. Let $\Delta = \begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$

$$= \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= (-1)\begin{vmatrix} 1 & a^2 & a \\ 1 & b^2 & b \\ 1 & c^2 & c \end{vmatrix} + abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \qquad [\text{by } C_1 \leftrightarrow C_3]$$

$$= (-1)^2\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \qquad [\text{by } C_2 \leftrightarrow C_3]$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}(1+abc)$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}(1+abc)$$

$$= (b-a)(c-a)(1+abc)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$$

Applying $R_3 \to R_3 - R_2$, then

$$= (b-a)(c-a)(1+abc)\begin{vmatrix} 1 & a & a^2 \\ & \ddots & \\ 0 & 1 & b+a \\ & & \ddots \\ 0 & 0 & c-b \end{vmatrix}$$

$$= (b-a)(c-b)(c-a)(1+abc)$$

$$= (a-b)(b-c)(c-a)(1+abc)$$

But given that, $\Delta = 0$

$\therefore \quad (a-b)(b-c)(c-a)(1+abc) = 0$

$\Rightarrow \quad 1+abc = 0$

[since a, b and c are different, so $a \neq b, b \neq c, c \neq a$]

Hence, $abc = -1$

Exercise for Session 2

1 If λ and μ are the cofactors of 3 and -2 respectively, in the determinant $\begin{vmatrix} 1 & 0 & -2 \\ 3 & -1 & 2 \\ 4 & 5 & 6 \end{vmatrix}$, the value of $\lambda + \mu$ is

(a) 5 (b) 7 (c) 9 (d) 11

2 If a, b and c are distinct and $D = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$, then the square of the determinant of its cofactors is divisible by

(a) $(a^2 + b^2 + c^2)^2$ (b) $(ab + bc + ca)^2$ (c) $(a + b + c)^2$ (d) $(a + b + c)^4$

3 An equilateral triangle has each of its sides of length 4 cm. If $(x_r, y_r)(r = 1, 2, 3)$ are its vertices, the value of $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2$ is

(a) 192 (b) 768 (c) 1024 (d) 128

4 If the lines $ax + y + 1 = 0$, $x + by + 1 = 0$ and $x + y + c = 0$ (a, b and c being distinct and different from 1) are concurrent, the value of $\frac{a}{a-1} + \frac{b}{b-1} + \frac{c}{c-1}$ is

(a) 0 (b) 1 (c) 2 (d) 3

5 If $p + q + r = 0 = a + b + c$, the value of the determinant $\begin{vmatrix} pa & qb & rc \\ qc & ra & pb \\ rb & pc & qa \end{vmatrix}$ is

(a) 0 (b) $pa + qb + rc$ (c) 1 (d) None of the above

6 If p, q and r are in AP, the value of determinant $\begin{vmatrix} a^2 + 2^{n+1} + 2p & b^2 + 2^{n+2} + 3q & c^2 + p \\ 2^n + p & 2^{n+1} + q & 2q \\ a^2 + 2^n + p & b^2 + 2^{n+1} + 2q & c^2 - r \end{vmatrix}$ is

(a) 1 (b) 0 (c) $a^2b^2c^2 - 2^n$ (d) $(a^2 + b^2 + c^2) - 2^n q$

7 Let $\{D_1, D_2, D_3, \ldots, D_n\}$ be the set of third order determinants that can be made with the distinct non-zero real numbers $a_1, a_2, \ldots, a_9$. Then,

(a) $\sum_{i=1}^{n} D_i = 1$ (b) $\sum_{i=1}^{n} D_i = 0$ (c) $D_i = D_j, \forall i, j$ (d) None of these

8 If $\begin{vmatrix} x & 3 & 6 \\ 3 & 6 & x \\ 6 & x & 3 \end{vmatrix} = \begin{vmatrix} 2 & x & 7 \\ x & 7 & 2 \\ 7 & 2 & x \end{vmatrix} = \begin{vmatrix} 4 & 5 & x \\ 5 & x & 4 \\ x & 4 & 5 \end{vmatrix} = 0$, then x is equal to

(a) 0 (b) -9 (c) 3 (d) None of these

9 If $a + b + c = 0$, the one root of $\begin{vmatrix} a - x & c & b \\ c & b - x & a \\ b & a & c - x \end{vmatrix} = 0$ is

(a) 1 (b) 2 (c) $a^2 + b^2 + c^2$ (d) 0

10 If $a^2 + b^2 + c^2 = -2$ and $f(x) = \begin{vmatrix} 1 + a^2x & (1 + b^2)x & (1 + c^2)x \\ (1 + a^2)x & 1 + b^2x & (1 + c^2)x \\ (1 + a^2)x & (1 + b^2)x & 1 + c^2x \end{vmatrix}$, the $f(x)$ is a polynomial of degree

(a) 0 (b) 1 (c) 2 (d) 3

11 If a, b, c, d, e and f are in GP, the value of $\begin{vmatrix} a^2 & d^2 & x \\ b^2 & e^2 & y \\ c^2 & f^2 & z \end{vmatrix}$, is

(a) depends on x and y (b) depends on x and z (c) depends on y and z (d) independent of x, y and z

Session 3

Examples on Largest Value of a Third Order Determinant, Multiplication of Two Determinants of the Same Order, System of Linear Equations, Cramer's Rule, Nature of Solutions of System of Linear Equations, System of Homogeneous Linear Equations

Examples on Largest Value of a Third Order Determinant

Example 17. ***Find the largest value of a third order determinant whose elements are 0 or 1.***

Sol. Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$\Delta = a_1(b_2 c_3 - b_3 c_2) - b_1(a_2 c_3 - a_3 c_2) + c_1(a_2 b_3 - a_3 b_2)$$
$$= (a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2) - (b_1 c_3 a_2 + b_2 c_1 a_3 + b_3 c_2 a_1)$$

Since, each element of Δ is either 0 or 1, therefore the value of the Δ cannot exceed 3. But to attain this value, each expression with a positive sign must equal 1, while those with a negative sign must be 0. However, if $a_1 b_2 c_3 = a_2 b_3 c_1 = a_3 b_1 c_2 = 1$, every element of the determinant must be 1, making its value zero. Thus, noting that

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 2$$

The largest value of Δ is 2.

Example 18. Find the largest value of a third order determinant, whose elements are 1 or –1.

Sol. Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$\therefore \Delta = a_1(b_2 c_3 - b_3 c_2) - b_1(a_2 c_3 - a_3 c_2) + c_1(a_2 b_3 - a_3 b_2)$$
$$= (a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2) - (b_1 c_3 a_2 + b_2 c_1 a_3 + b_3 c_2 a_1)$$

Since, each element of Δ is either 1 or –1, therefore the value of the Δ cannot exceed 6. But it can be 6 only if

$$a_1 b_2 c_3 = a_2 b_3 c_1 = a_3 b_1 c_2 = 1 \quad \ldots(i)$$

and $$b_1 c_3 a_2 = b_2 c_1 a_3 = b_3 c_2 a_1 = -1 \quad \ldots(ii)$$

In the first case, the product of the nine elements of the determinant equals 1, while it is –1 in the second case, so the two cannot occur simultaneously i.e., the determinant cannot equal 6. The following determinant satisfies the given conditions and equals the largest value

$$\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = -1(1-1) - 1(-1-1) + 1(1+1) = 4$$

Example 19. Show that the value of a third order determinant whose all elements are 1 or –1 is an even number.

Sol. Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Applying $R_2 \to R_2 - \frac{a_2}{a_1} R_1$ and $R_3 \to R_3 - \frac{a_3}{a_1} R_1$, then

$$\Delta = \begin{vmatrix} a_1 & \cdots & b_1 & \cdots & c_1 \\ \vdots & & & & \\ 0 & & b_2 - \frac{a_2}{a_1} b_1 & & c_2 - \frac{a_2}{a_1} c_1 \\ \vdots & & & & \\ 0 & & b_3 - \frac{a_3}{a_1} b_1 & & c_3 - \frac{a_3}{a_1} c_1 \end{vmatrix}$$

Expanding along C_1, we get

$$\Delta = a_1 \left\{ \left(b_2 - \frac{a_2}{a_1} b_1 \right) \left(c_3 - \frac{a_3}{a_1} c_1 \right) - \left(b_3 - \frac{a_3}{a_1} b_1 \right) \left(c_2 - \frac{a_2}{a_1} c_1 \right) \right\} \quad \ldots(i)$$

Since, $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ are 1 or –1.

$\therefore$ $b_2, \frac{a_2}{a_1} b_1, c_3, \frac{a_3}{a_1} c_1, b_3, \frac{a_3}{a_1} b_1, c_2, \frac{a_2}{a_1} c_1$ are 1 or –1

$\Rightarrow$ $b_2 - \frac{a_2}{a_1} b_1, c_3 - \frac{a_3}{a_1} c_1, b_3 - \frac{a_3}{a_1} b_1, c_2 - \frac{a_2}{a_1} c_1$ are 2, –2 or 0.

$\therefore$ $\left(b_2 - \frac{a_2}{a_1} b_1 \right) \left(c_3 - \frac{a_3}{a_1} c_1 \right)$

and $\left(b_3 - \frac{a_3}{a_1} b_1 \right) \left(c_2 - \frac{a_2}{a_1} c_1 \right)$ are 4, – 4

or 0 = an even number

From Eq. (i), Δ = an even number ($a_1 = 1$ or –1)

Multiplication of Two Determinants of the Same Order

Let the two determinants of third order be

$$\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } \Delta_2 = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

Let Δ be their product.

Method of Multiplication (Row by Row)

Take the first row of Δ_1 and the first row of Δ_2 i.e., a_1, b_1, c_1 and $\alpha_1, \beta_1, \gamma_1$ multiplying the corresponding elements and add. The result is $a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1$ is the first element of first row of Δ.

Now, similar product first row of Δ_1 and second row of Δ_2 gives $a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2$ is the second element of first row of Δ and the product of first row of Δ_1 and third row of Δ_2 gives $a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3$ is the third element of first row of Δ. The second row and third row of Δ is obtained by multiplying second row and third row of Δ_1 with 1st, 2nd, 3rd row of Δ_2 in the above manner.

$$\text{Hence, } \Delta = \Delta_1 \times \Delta_2 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}$$

Multiplication can also be performed row by column or column by row or column by column as required in the problem.

Example 20. Evaluate $\begin{vmatrix} 1 & 2 & 3 \\ -2 & 3 & 2 \\ 3 & 4 & -4 \end{vmatrix} \times \begin{vmatrix} -2 & 1 & 3 \\ 3 & -2 & 1 \\ 2 & 1 & -2 \end{vmatrix}$.

Using the concept of multiplication of determinants.

Sol. Let $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ -2 & 3 & 2 \\ 3 & 4 & -4 \end{vmatrix} \times \begin{vmatrix} -2 & 1 & 3 \\ 3 & -2 & 1 \\ 2 & 1 & -2 \end{vmatrix}$

On multiplying row by row, we get

$$\Delta = \begin{vmatrix} -2+2+9 & 3-4+3 & 2+2-6 \\ 4+3+6 & -6-6+2 & -4+3-4 \\ -6+4-12 & 9-8-4 & 6+4+8 \end{vmatrix}$$

$$= \begin{vmatrix} 9 & 2 & -2 \\ 13 & -10 & -5 \\ -14 & -3 & 18 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_3$ and $C_2 \to C_2 + C_3$, then

$$\Delta = \begin{vmatrix} 7 & 0 & -2 \\ 8 & -15 & -5 \\ 4 & 15 & 18 \end{vmatrix}$$

Applying $R_2 \to R_2 + R_3$, then

$$\Delta = \begin{vmatrix} 7 & 0 & -2 \\ \vdots & \vdots & \\ 12 & 0 & 13 \\ \vdots & \vdots & \\ 4 & \cdots\ 15 & \cdots\ 18 \end{vmatrix}$$

Expanding along C_2, we get

$$-15 \begin{vmatrix} 7 & -2 \\ 12 & 13 \end{vmatrix} = -15(91+24) = -15 \times 115 = -1725$$

Example 21. If $ax_1^2 + by_1^2 + cz_1^2 = ax_2^2 + by_2^2 + cz_2^2 = ax_3^2 + by_3^2 + cz_3^2 = d$,
$ax_2x_3 + by_2y_3 + cz_2z_3 = ax_3x_1 + by_3y_1 + cz_3z_1 = ax_1x_2 + by_1y_2 + cz_1z_2 = f$,
then prove that

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = (d-f)\left\{\frac{(d+2f)}{abc}\right\}^{1/2}$$

Sol. Let LHS $= \Delta = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$

$$\therefore \Delta^2 = \Delta \times \Delta = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \times \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \times \begin{vmatrix} ax_1 & by_1 & cz_1 \\ ax_2 & by_2 & cz_2 \\ ax_3 & by_3 & cz_3 \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} ax_1^2 + by_1^2 + cz_1^2 & ax_1x_2 + by_1y_2 + cz_1z_2 & ax_3x_1 + by_3y_1 + cz_3z_1 \\ ax_1x_2 + by_1y_2 + cz_1z_2 & ax_2^2 + by_2^2 + cz_2^2 & ax_2x_3 + by_2y_3 + cz_2z_3 \\ ax_3x_1 + by_3y_1 + cz_3z_1 & ax_2x_3 + by_2y_3 + cz_2z_3 & ax_3^2 + by_3^2 + cz_3^2 \end{vmatrix}$$ [multiplying row by row]

$$= \frac{1}{abc}\begin{vmatrix} d & f & f \\ f & d & f \\ f & f & d \end{vmatrix}$$ [given]

Applying $C_1 \to C_1 + C_2 + C_3$, then

$$= \frac{1}{abc}\begin{vmatrix} d+2f & f & f \\ d+2f & d & f \\ d+2f & f & d \end{vmatrix} = \frac{(d+2f)}{abc}\begin{vmatrix} 1 & f & f \\ 1 & d & f \\ 1 & f & d \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$= \frac{(d+2f)}{abc}\begin{vmatrix} 1 & f & f \\ & \ddots & \\ 0 & d-f & 0 \\ & \ddots & \\ 0 & 0 & d-f \end{vmatrix} = \frac{(d+2f)}{abc}(d-f)^2$$

$$\therefore \Delta = (d-f)\left\{\frac{d+2f}{abc}\right\}^{1/2} = \text{RHS}$$

An Important Property

If A_1, B_1 and $C_1, \ldots$ are respectively the cofactors of the elements a_1, b_1 and $c_1, \ldots$ of the determinant.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \Delta \neq 0, \text{ then } \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \Delta^2$$

Proof Consider

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1A_1 + b_1B_1 + c_1C_1 & a_1A_2 + b_1B_2 + c_1C_2 & a_1A_3 + b_1B_3 + c_1C_3 \\ a_2A_1 + b_2B_1 + c_2C_1 & a_2A_2 + b_2B_2 + c_2C_2 & a_2A_3 + b_2B_3 + c_2C_3 \\ a_3A_1 + b_3B_1 + c_3C_1 & a_3A_2 + b_3B_2 + c_3C_2 & a_3A_3 + b_3B_3 + c_3C_3 \end{vmatrix}$$

[multiplying row by row]

$$= \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} = \Delta^3 \qquad \left(\text{as } a_iA_j + b_iB_j + c_iC_j = \begin{cases} \Delta, & i = j \\ 0, & i \neq j \end{cases}\right)$$

$$\Rightarrow \Delta \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \Delta^3 \quad \text{or} \quad \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \Delta^2$$

$[\because \Delta \neq 0]$

Note Let $\Delta \neq 0$ and Δ^c denotes the determinant formed by the cofactors of Δ and n is order of determinant, then

$$\Delta^c = \Delta^{n-1}$$

This is known as **power cofactor formula.**

Example 22. Show that

$$\begin{vmatrix} a^2+x^2 & ab-cx & ac+bx \\ ab+cx & b^2+x^2 & bc-ax \\ ac-bx & bc+ax & c^2+x^2 \end{vmatrix} = \begin{vmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{vmatrix}^2.$$

Sol. Let $\Delta = \begin{vmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{vmatrix}$

Cofactors of 1st row of Δ are $x^2 + a^2, cx + ab, ac - bx$, cofactors of 2nd row of Δ are $ab - cx, x^2 + b^2, ax + bc$ and cofactors of 3rd row of Δ are $ac + bx, bc - ax, x^2 + c^2$.

Hence, the determinant of the cofactors of Δ is

$$\Delta^c = \begin{vmatrix} a^2+x^2 & ab+cx & ac-bx \\ ab-cx & b^2+x^2 & bc+ax \\ ac+bx & bc-ax & c^2+x^2 \end{vmatrix}$$

Interchanging rows into columns, we get

$$\Delta^c = \begin{vmatrix} a^2+x^2 & ab-cx & ac+bx \\ ab+cx & b^2+x^2 & bc-ax \\ ac-bx & bc+ax & c^2+x^2 \end{vmatrix} = \begin{vmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{vmatrix}^2 \quad [\because \Delta^c = \Delta^2]$$

Example 23. Prove the following by multiplication of determinants and power cofactor formula

$$\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix}$$

$$= \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$$

Sol. Let $\Delta = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}$. Expanding along R_1, then

$$\Delta = 0 - c(0 - ab) + b(ac - 0) = 2abc$$

$$\therefore \quad \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \Delta^2 = (2abc)^2 = 4a^2b^2c^2 \qquad \ldots(\text{i})$$

Also, $\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} \times \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}$

$$= \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix} \qquad \ldots(\text{ii})$$

[multiplying row by row]

and $\Delta^c = \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = \Delta^{3-1} = \Delta^2$

$$= \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 \quad \ldots(iii)$$

From Eqs. (i), (ii) and (iii), we get

$$\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix}$$

$$= \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$$

Express a Determinant Into Product of Two Determinants

Consider the determinant $\begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 \end{vmatrix}$

Let $\Delta = \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 \end{vmatrix}$

By the property of determinant, Δ can be written as

$$\Delta = \begin{vmatrix} a_1\alpha_1 & a_1\alpha_2 + b_1\beta_2 \\ a_2\alpha_1 & a_2\alpha_2 + b_2\beta_2 \end{vmatrix} + \begin{vmatrix} b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 \\ b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1\alpha_1 & a_1\alpha_2 \\ a_2\alpha_1 & a_2\alpha_2 \end{vmatrix} + \begin{vmatrix} a_1\alpha_1 & b_1\beta_2 \\ a_2\alpha_1 & b_2\beta_2 \end{vmatrix} + \begin{vmatrix} b_1\beta_1 & a_1\alpha_2 \\ b_2\beta_1 & a_2\alpha_2 \end{vmatrix} + \begin{vmatrix} b_1\beta_1 & b_1\beta_2 \\ b_2\beta_1 & b_2\beta_2 \end{vmatrix}$$

$$= \alpha_1\alpha_2 \begin{vmatrix} a_1 & a_1 \\ a_2 & a_2 \end{vmatrix} + \alpha_1\beta_2 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \beta_1\alpha_2 \begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix} + \beta_1\beta_2 \begin{vmatrix} b_1 & b_1 \\ b_2 & b_2 \end{vmatrix}$$

$$= 0 + \alpha_1\beta_2 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} - \beta_1\alpha_2 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + 0$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} (\alpha_1\beta_2 - \alpha_2\beta_1)$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}$$

$$\therefore \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}$$

Example 24. Prove that

$$\begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 & a_1\alpha_3 + b_1\beta_3 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 & a_2\alpha_3 + b_2\beta_3 \\ a_3\alpha_1 + b_3\beta_1 & a_3\alpha_2 + b_3\beta_2 & a_3\alpha_3 + b_3\beta_3 \end{vmatrix} = 0.$$

Sol. LHS $= \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 & a_1\alpha_3 + b_1\beta_3 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 & a_2\alpha_3 + b_2\beta_3 \\ a_3\alpha_1 + b_3\beta_1 & a_3\alpha_2 + b_3\beta_2 & a_3\alpha_3 + b_3\beta_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & 0 \\ \alpha_2 & \beta_2 & 0 \\ \alpha_3 & \beta_3 & 0 \end{vmatrix} \quad \text{[row by row]}$$

$= 0 \times 0 = 0 =$ RHS

Example 25. Prove that

$$\begin{vmatrix} 2 & \alpha+\beta+\gamma+\delta & \alpha\beta+\gamma\delta \\ \alpha+\beta+\gamma+\delta & 2(\alpha+\beta)(\gamma+\delta) & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) \\ \alpha\beta+\gamma\delta & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) & 2\alpha\beta\gamma\delta \end{vmatrix} = 0.$$

Sol. LHS $= \begin{vmatrix} 2 & \alpha+\beta+\gamma+\delta & \alpha\beta+\gamma\delta \\ \alpha+\beta+\gamma+\delta & 2(\alpha+\beta)(\gamma+\delta) & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) \\ \alpha\beta+\gamma\delta & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) & 2\alpha\beta\gamma\delta \end{vmatrix}$

$$= \begin{vmatrix} 1 & 1 & 0 \\ \alpha+\beta & \gamma+\delta & 0 \\ \alpha\beta & \gamma\delta & 0 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 0 \\ \gamma+\delta & \alpha+\beta & 0 \\ \gamma\delta & \alpha\beta & 0 \end{vmatrix} \quad \text{[row by row]}$$

$= 0 \times 0 = 0 =$ RHS

Example 26. ***Prove that***

$$\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix} = 0.$$

Sol. LHS $= \begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix}$

$$= \begin{vmatrix} \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \\ \cos C & \sin C & 0 \end{vmatrix} \times \begin{vmatrix} \cos P & \sin P & 0 \\ \cos Q & \sin Q & 0 \\ \cos R & \sin R & 0 \end{vmatrix} \quad \text{[row by row]}$$

$= 0 \times 0 = 0 =$ RHS

Example 27. ***If*** α, β and γ ***are real numbers, without expanding at any stage, prove that***

$$\begin{vmatrix} 1 & \cos(\beta-\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & 1 & \cos(\gamma-\beta) \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & 1 \end{vmatrix} = 0.$$

Sol. LHS $= \begin{vmatrix} 1 & \cos(\beta-\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & 1 & \cos(\gamma-\beta) \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & 1 \end{vmatrix}$

$= \begin{vmatrix} \cos(\alpha-\alpha) & \cos(\beta-\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & \cos(\beta-\beta) & \cos(\gamma-\beta) \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & \cos(\gamma-\gamma) \end{vmatrix}$

$= \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 0 \end{vmatrix} \times \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 0 \end{vmatrix}$

$= 0 \times 0 = 0 =$ RHS

Example 28. If $a, b, c, x, y, z \in R$, prove that

$$\begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (a-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix} = \begin{vmatrix} (1+ax)^2 & (1+bx)^2 & (1+cx)^2 \\ (1+ay)^2 & (1+by)^2 & (1+cy)^2 \\ (1+az)^2 & (1+bz)^2 & (1+cz)^2 \end{vmatrix}.$$

Sol. LHS $= \begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (a-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix}$

$= \begin{vmatrix} a^2-2ax+x^2 & b^2-2bx+x^2 & c^2-2cx+x^2 \\ a^2-2ay+y^2 & b^2-2by+y^2 & c^2-2cy+y^2 \\ a^2-2az+z^2 & b^2-2bz+z^2 & c^2-2cz+z^2 \end{vmatrix}$

$= \begin{vmatrix} 1 & 2x & x^2 \\ 1 & 2y & y^2 \\ 1 & 2z & z^2 \end{vmatrix} \times \begin{vmatrix} a^2 & -a & 1 \\ b^2 & -b & 1 \\ c^2 & -c & 1 \end{vmatrix}$ [row by row]

$= \begin{vmatrix} 1 & 2x & x^2 \\ 1 & 2y & y^2 \\ 1 & 2z & z^2 \end{vmatrix} \times (-1)(-1) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

[$C_1 \leftrightarrow C_3$ and taking (-1) common from second determinant]

$= \begin{vmatrix} 1 & 2x & x^2 \\ 1 & 2y & y^2 \\ 1 & 2z & z^2 \end{vmatrix} \times \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

$= \begin{vmatrix} 1+2ax+a^2x^2 & 1+2bx+b^2x^2 & 1+2cx+c^2x^2 \\ 1+2ay+a^2y^2 & 1+2by+b^2y^2 & 1+2cy+c^2y^2 \\ 1+2az+a^2z^2 & 1+2bz+b^2z^2 & 1+2cz+c^2z^2 \end{vmatrix}$

[multiplying row by row]

$= \begin{vmatrix} (1+ax)^2 & (1+bx)^2 & (1+cx)^2 \\ (1+ay)^2 & (1+by)^2 & (1+cy)^2 \\ (1+az)^2 & (1+bz)^2 & (1+cz)^2 \end{vmatrix} =$ RHS

System of Linear Equations

(i) Consistent equations *Definite and unique solution* [Intersecting lines]

A system of (linear) equations is said to be consistent, if it has atleast one solution.

For example, System of equations $\left.\begin{matrix} x+y=2 \\ x-y=6 \end{matrix}\right\}$ is consistent because it has a solution $x=4, y=-2$. Here, two lines intersect at one point.

i.e., intersecting lines.

(ii) Inconsistent equations *No solution* [Parallel lines]

A system of (linear) equations is said to be inconsistent, if it has no solution.

Let $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$, then $\frac{a_1}{a_2}=\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$

$\Rightarrow$ Given equations are inconsistent.

For example, System of equations $\left.\begin{matrix} x+y=2 \\ 2x+2y=5 \end{matrix}\right\}$ is inconsistent because it has no solution i.e., there is no value of x and y which satisfy both the equations. Here, the two lines are parallel.

(iii) Dependent equations *Infinite solutions* [Identical lines]

A system of (linear) equations is said to be dependent, if it has infinite solutions.

Let $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$, then $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}\Rightarrow$ Given equations are dependent.

For example, System of equations $\left.\begin{matrix} x+2y=3 \\ 2x+4y=6 \end{matrix}\right\}$ is dependent because it has infinite solutions i.e., there are infinite values of x and y satisfy both the equations. Here, the two lines are identical.

Cramer's Rule

System of linear equations in two variables

Let us consider a system of equations be

$$\left.\begin{aligned} a_1x + b_1y + c_1 = 0 \\ a_2x + b_2y + c_2 = 0 \end{aligned}\right\} \text{ where } \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$$

On solving by cross-multiplication, we get

$$\frac{x}{(b_1c_2 - b_2c_1)} = \frac{y}{(c_1a_2 - c_2a_1)} = \frac{1}{(a_1b_2 - a_2b_1)}$$

or $$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

or $$x = \frac{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, y = \frac{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

System of Linear Equations in Three Variables

Let us consider a system of linear equations be

$$a_1x + b_1y + c_1z = d_1 \quad \ldots(\text{i})$$
$$a_2x + b_2y + c_2z = d_2 \quad \ldots(\text{ii})$$
$$a_3x + b_3y + c_3z = d_3 \quad \ldots(\text{iii})$$

Here, $$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},$$

$$\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ and } \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

If $\Delta \neq 0$, then

$$\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1x + b_1y + c_1z & b_1 & c_1 \\ a_2x + b_2y + c_2z & b_2 & c_2 \\ a_3x + b_3y + c_3z & b_3 & c_3 \end{vmatrix}$$

Applying $C_1 \to C_1 - yC_2 - zC_3$, then

$$\Delta_1 = \begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix} = x \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = x\Delta$$

$\therefore$ $$x = \frac{\Delta_1}{\Delta}, \text{ where } \Delta \neq 0$$

Similarly, $\Delta_2 = y\Delta$ and $\Delta_3 = z\Delta$

$\therefore$ $$y = \frac{\Delta_2}{\Delta} \text{ and } z = \frac{\Delta_3}{\Delta}$$

Thus, $$x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta}, z = \frac{\Delta_3}{\Delta}, \text{ where } \Delta \neq 0 \quad \ldots(\text{iv})$$

The rule given in Eq. (iv) to find the values of x, y and z is called the **CRAMER'S RULE**.

Remark

1. Δ_i is obtained by replacing elements of ith columns by d_1, d_2, d_3, where $i = 1, 2, 3$.
2. Cramer's rule can be used only when $\Delta \neq 0$.

Nature of Solution of System of Linear Equations

Let us consider a system of linear equations be

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Now, there are two cases arise:

***Case* I** If $\Delta \neq 0$

In this case, $x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta}, z = \frac{\Delta_3}{\Delta}$

Then, system will have unique finite solutions and so equations are **consistent**.

***Case* II** If $\Delta = 0$

(a) **When atleast one of $\Delta_1, \Delta_2, \Delta_3$ be non-zero**

(i) Let $\Delta_1 \neq 0$, then from $\Delta_1 = x\Delta$ will not be satisfied for any value of x because $\Delta = 0$ and $\Delta_1 \neq 0$ and hence no value of x is possible.

(ii) Let $\Delta_2 \neq 0$, then from $\Delta_2 = y\Delta$ will not be satisfied for any value of y because $\Delta = 0$ and $\Delta_2 \neq 0$ and hence no value of y is possible.

(iii) Let $\Delta_3 \neq 0$, then from $\Delta_3 = z\Delta$ will not be satisfied for any value of z because $\Delta = 0$ and $\Delta_3 \neq 0$ and hence no value of z is possible.

Thus, if $\Delta = 0$ and any of $\Delta_1, \Delta_2, \Delta_3$ is non-zero. Then, the system has no solution i.e., equations are **inconsistent.**

(b) **When $\Delta_1 = \Delta_2 = \Delta_3 = 0$**

In this case, $\left.\begin{aligned} \Delta_1 = x\Delta \\ \Delta_2 = y\Delta \\ \Delta_3 = z\Delta \end{aligned}\right\}$ will be true for all values of x, y and z.

But, since $a_1x + b_1y + c_1z = d_1$, therefore only two of x, y and z will be independent and third will be dependent on the other two.

Thus, the system will have infinite number of solutions i.e., equations are **consistent**.

Remark

1. If $\Delta \neq 0$, the system will have unique finite solution and so equations are consistent.
2. If $\Delta = 0$ and atleast one of $\Delta_1, \Delta_2, \Delta_3$ be non-zero, then the system has no solution i.e., equations are inconsistent.
3. If $\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$, the equations will have infinite number of solutions i.e. equations are consistent.

Example 29. Solve the following system of equations by Cramer's rule.

$$x + y = 4 \text{ and } 3x - 2y = 9$$

Sol. Here, $\Delta = \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} = -2 - 3 = -5 \neq 0$

$$\Delta_1 = \begin{vmatrix} 4 & 1 \\ 9 & -2 \end{vmatrix} = -8 - 9 = -17$$

and $\Delta_2 = \begin{vmatrix} 1 & 4 \\ 3 & 9 \end{vmatrix} = 9 - 12 = -3$

Then, by Cramer's rule

$$x = \frac{\Delta_1}{\Delta} = \frac{-17}{-5} = \frac{17}{5} \text{ and } y = \frac{\Delta_2}{\Delta} = \frac{-3}{-5} = \frac{3}{5}$$

$\therefore \quad x = \frac{17}{5}, y = \frac{3}{5}$

Example 30. Solve the following system of equations by Cramer's rule.

$$x + y + z = 9$$
$$2x + 5y + 7z = 52$$
$$2x + y - z = 0$$

Sol. Here, $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{vmatrix}$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, then

$$= \begin{vmatrix} 1 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \\ 2 & & 3 & & 5 \\ \vdots & & & & \\ 2 & & -1 & & -3 \end{vmatrix}$$

Expanding along R_1, then

$$\Delta = 1\begin{vmatrix} 3 & 5 \\ -1 & -3 \end{vmatrix} = -9 + 5 = -4 \neq 0, \quad \Delta_1 = \begin{vmatrix} 9 & 1 & 1 \\ 52 & 5 & 7 \\ 0 & 1 & -1 \end{vmatrix}$$

Applying $C_2 \to C_2 + C_3$, then

$$\Delta_1 = \begin{vmatrix} 9 & & 2 & & 1 \\ & & & & \vdots \\ 52 & & 12 & & 7 \\ & & & & \vdots \\ 0 & \cdots & 0 & \cdots & -1 \end{vmatrix}$$

Expanding along R_3, then

$$\Delta_1 = (-1)\begin{vmatrix} 9 & 2 \\ 52 & 12 \end{vmatrix} = -(108 - 104) = -4$$

$$\Rightarrow \qquad \Delta_2 = \begin{vmatrix} 1 & 9 & 1 \\ 2 & 52 & 7 \\ 2 & 0 & -1 \end{vmatrix}$$

Applying $C_1 \to C_1 + 2C_3$, then

$$\Delta_2 = \begin{vmatrix} 3 & & 9 & & 1 \\ & & & & \vdots \\ 16 & & 52 & & 7 \\ & & & & \vdots \\ 0 & \cdots & 0 & \cdots & -1 \end{vmatrix}$$

Expanding along R_3, then

$$\Delta_2 = (-1)\begin{vmatrix} 3 & 9 \\ 16 & 52 \end{vmatrix} = -(156 - 144) = -12 \text{ and } \Delta_3 = \begin{vmatrix} 1 & 1 & 9 \\ 2 & 5 & 52 \\ 2 & 1 & 0 \end{vmatrix}$$

Applying $C_1 \to C_1 - 2C_2$, then

$$\Delta_3 = \begin{vmatrix} -1 & & 1 & & 9 \\ & & \vdots & & \\ -8 & & 5 & & 52 \\ & & \vdots & & \\ 0 & \cdots & 1 & \cdots & 0 \end{vmatrix}$$

Expanding along R_3, then

$$\Delta_3 = (-1)\begin{vmatrix} -1 & 9 \\ -8 & 52 \end{vmatrix}$$
$$= -(-52 + 72) = -20$$

Then, by Cramer's rule

$$x = \frac{\Delta_1}{\Delta} = \frac{-4}{-4} = 1, y = \frac{-12}{-4} = 3$$

and $\qquad z = \frac{\Delta_3}{\Delta} = \frac{-20}{-4} = 5$

$\therefore \qquad x = 1, y = 3, z = 5$

Example 31. ***For what values of*** p ***and*** q***, the system of equations***

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + pz = q \text{ has}$$

(i) unique solution?
(ii) an infinitely many solutions?
(iii) no solution?

Sol. Given equations are

$$x + y + z = 6 \quad \Rightarrow \quad x + 2y + 3z = 10$$
$$x + 2y + pz = q$$

$$\therefore \quad \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & p \end{vmatrix} = (p-3) \Rightarrow \Delta_1 = \begin{vmatrix} 6 & 1 & 1 \\ 10 & 2 & 3 \\ q & 2 & p \end{vmatrix}$$

$$= 6(2p-6) - 1(10p-3q) + (20-2q)$$

$$= 2p + q - 16$$

$$\Delta_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 10 & 3 \\ 1 & q & p \end{vmatrix}$$

$$= 1(10p-3q) - 6(p-3) + 1(q-10) = 4p - 2q + 8$$

$$\text{and } \Delta_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & 2 & 10 \\ 1 & 2 & q \end{vmatrix}$$

$$= 1(2q-20) - 1(q-10) + 6(2-2) = q - 10$$

(i) For unique solution, $A \neq 0 \Rightarrow p \neq 3, q \in R$

(ii) For infinitely many solutions, $\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$

$\therefore \quad p = 3, q = 10$

(iii) For no solution, $\Delta = 0$ and atleast one of $\Delta_1, \Delta_2, \Delta_3$ is non-zero is $p = 3$ and $q \neq 10$.

Condition for Consistency of Three Linear Equations in Two Unknowns

Let us consider a system of linear equations in x and y

$$a_1 x + b_1 y + c_1 = 0 \quad \ldots(\text{i})$$

$$a_2 x + b_2 y + c_2 = 0 \quad \ldots(\text{ii})$$

$$a_3 x + b_3 y + c_3 = 0 \quad \ldots(\text{iii})$$

will be consistent, the values of x and y obtained from any two equations satisfy the third equation.

On solving Eqs. (ii) and (iii) by Cramer's rule, we have

$$\frac{x}{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}} = \frac{y}{\begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}$$

These values of x and y will satisfy Eq. (i), then

$$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + b_1 \begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0$$

$$\Rightarrow \quad a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0$$

$$\therefore \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

which is the required condition.

Remark

For consistency of three linear equations in two knowns, the number of solution is one.

Example 32. Find the value of λ, if the following equations are consistent

$$x + y - 3 = 0$$
$$(1+\lambda)x + (2+\lambda)y - 8 = 0$$
$$x - (1+\lambda)y + (2+\lambda) = 0$$

Sol. The given equations in two unknowns are consistent, then

$$\begin{vmatrix} 1 & 1 & -3 \\ (1+\lambda) & (2+\lambda) & -8 \\ 1 & -(1+\lambda) & (2+\lambda) \end{vmatrix} = 0$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 + 3C_1$, then

$$\begin{vmatrix} 1 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \\ (1+\lambda) & & 1 & & (3\lambda - 5) \\ \vdots & & & & \\ 1 & & -(2+\lambda) & & (5+\lambda) \end{vmatrix} = 0$$

Expanding along R_1, then

$$1 \cdot \begin{vmatrix} 1 & 3\lambda - 5 \\ -(2+\lambda) & (5+\lambda) \end{vmatrix} = 0$$

$$\Rightarrow \quad (5+\lambda) + (2+\lambda)(3\lambda - 5) = 0$$

$$\Rightarrow \quad 3\lambda^2 + 2\lambda - 5 = 0 \quad \text{or} \quad (3\lambda + 5)(\lambda - 1) = 0$$

$$\therefore \quad \lambda = 1, -5/3$$

System of Homogeneous Linear Equations

Let us consider a system of homogeneous linear equations in three unknown x, y and z be

$$a_1 x + b_1 y + c_1 z = 0 \quad \ldots(\text{i})$$

$$a_2 x + b_2 y + c_2 z = 0 \quad \ldots(\text{ii})$$

$$a_3 x + b_3 y + c_3 z = 0 \quad \ldots(\text{iii})$$

Here, $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

***Case* I** If $\Delta \neq 0$, then $x = 0, y = 0, z = 0$ is the only solution of above system. This solution is called a **Trivial solution**.

***Case* II** If $\Delta = 0$, atleast one of x, y and z is non-zero. This solution is called a **Non-trivial solution**.

Explanation From Eqs. (ii) and (iii), we get

$$\frac{x}{(b_2 c_3 - b_3 c_2)} = \frac{y}{(c_2 a_3 - c_3 a_2)} = \frac{z}{(a_2 b_3 - a_3 b_2)}$$

or $$\frac{x}{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}} = \frac{y}{\begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}} = k[\text{say}] \ (\neq 0)$$

$$\therefore \quad x = k\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, y = k\begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} \text{ and } z = k\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

On putting these values of x, y and z in Eq. (i), we get

$$a_1\left\{k\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}\right\} + b_1\left\{k\begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix}\right\} + c_1\left\{k\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}\right\} = 0$$

$$\Rightarrow \quad a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0 \quad [\because k \neq 0]$$

or $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \text{ or } \Delta = 0$$

This is the condition for system have **Non-trivial solution.**

Remark

1. If $\Delta \neq 0$, the given system of equations has **only zero** solution for all its variables, then the given equations are said to have **Trivial solution**.
2. If $\Delta = 0$, the given system of equations has **no solution** or **infinite solutions** for all its variables, then the given equations are said to have **Non-trivial solution**.

Example 33. *Find all values of λ for which the equations*

$$(\lambda - 1)x + (3\lambda + 1)y + 2\lambda z = 0$$
$$(\lambda - 1)x + (4\lambda - 2)y + (\lambda + 3)z = 0$$
$$2x + (3\lambda + 1)y + 3(\lambda - 1)z = 0$$

possess non-trivial solution and find the ratios $x : y : z$, where λ has the smallest of these values.

Sol. The given system of linear equations has non-trivial solution, then we must have

$$\begin{vmatrix} \lambda - 1 & 3\lambda + 1 & 2\lambda \\ \lambda - 1 & 4\lambda - 2 & \lambda + 3 \\ 2 & 3\lambda + 1 & 3(\lambda - 1) \end{vmatrix} = 0$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$\begin{vmatrix} \lambda - 1 & 3\lambda + 1 & 2\lambda \\ 0 & \lambda - 3 & -\lambda + 3 \\ 3 - \lambda & 0 & \lambda - 3 \end{vmatrix} = 0$$

Applying $C_3 \to C_3 + C_2$, then

$$\begin{vmatrix} \lambda - 1 & 3\lambda + 1 & 5\lambda + 1 \\ 0 & \lambda - 3 & 0 \\ 3 - \lambda & 0 & \lambda - 3 \end{vmatrix} = 0$$

Expanding along R_2, we get

$$(\lambda - 3)\begin{vmatrix} \lambda - 1 & 5\lambda + 1 \\ 3 - \lambda & \lambda - 3 \end{vmatrix} = 0$$

$$\Rightarrow \quad (\lambda - 3)[(\lambda - 1)(\lambda - 3) - (3 - \lambda)(5\lambda + 1)] = 0$$

$$\Rightarrow \quad (\lambda - 3)^2 \cdot 6\lambda = 0$$

$$\therefore \quad \lambda = 0, 3$$

Here, smallest value of λ is 0.

$\therefore$ The first two equations can be written as $x - y = 0$ and $x + 2y - 3z = 0$.

Using Cramer's rule, we get

$$\frac{x}{\begin{vmatrix} -1 & 0 \\ 2 & -3 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 0 & 1 \\ -3 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix}}$$

$$\Rightarrow \quad \frac{x}{3} = \frac{y}{3} = \frac{z}{3} \Rightarrow \frac{x}{1} = \frac{y}{1} = \frac{z}{1}$$

$$\therefore \quad x : y : z = 1 : 1 : 1$$

Example 34. *Given, $x = cy + bz, y = az + cx$ and $z = bx + ay$, where x, y and z are not all zero, prove that $a^2 + b^2 + c^2 + 2abc = 1$.*

Sol. The given equation can be rewritten as

$$x - cy - bz = 0$$
$$-cx + y - az = 0$$
$$-bx - ay + z = 0$$

Since, x, y and z are not all zero, the system will have non-trivial solution, if

$$\begin{vmatrix} 1 & -c & -b \\ -c & 1 & -a \\ -b & -a & 1 \end{vmatrix} = 0$$

Applying $C_2 \to C_2 + cC_1$ and $C_3 \to C_3 + bC_1$, then

$$\begin{vmatrix} 1 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \\ -c & & 1 - c^2 & & -a - bc \\ \vdots & & & & \\ -b & & -a - bc & & 1 - b^2 \end{vmatrix} = 0$$

Expanding along R_1, we get

$$1\begin{vmatrix} 1 - c^2 & -a - bc \\ -a - bc & 1 - b^2 \end{vmatrix} = 0$$

$$\Rightarrow \quad (1 - c^2)(1 - b^2) - (a + bc)^2 = 0$$

$$\Rightarrow \quad 1 - b^2 - c^2 + b^2c^2 - a^2 - b^2c^2 - 2abc = 0$$

$$\Rightarrow \quad a^2 + b^2 + c^2 + 2abc = 1$$

Exercise for Session 3

1. Number of second order determinants which have maximum values whose each entry is either –1or 1 is equal to
(a) 2 (b) 4 (c) 6 (d) 8

2. Minimum value of a second order determinant whose each entry is either 1 or 2 is equal to
(a) 0 (b) –1 (c) –2 (d) –3

3. If $l_i^2 + m_i^2 + n_i^2 = 1, (i = 1, 2, 3)$ and $l_i l_j + m_i m_j + n_i n_j = 0, (i \neq j; i, j = 1, 2, 3)$ and $\Delta = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$, then
(a) $|\Delta| = 3$ (b) $|\Delta| = 2$ (c) $|\Delta| = 1$ (d) $|\Delta| = 0$

4. Let $\Delta_0 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and Δ_1 denotes the determinant formed by the cofactors of elements of Δ_0 and Δ_2 denote the determinant formed by the cofactors of Δ_1 and so on. Δ_n denotes the determinant formed by the cofactors of Δ_{n-1}, the determinant value of Δ_n is
(a) Δ_0^{2n} (b) $\Delta_0^{2^n}$ (c) $\Delta_0^{n^2}$ (d) Δ_0^2

5. If $\begin{vmatrix} 1 & x & x^2 \\ x & x^2 & 1 \\ x^2 & 1 & x \end{vmatrix} = 3$, then the value of $\begin{vmatrix} x^3 - 1 & 0 & x - x^4 \\ 0 & x - x^4 & x^3 - 1 \\ x - x^4 & x^3 - 1 & 0 \end{vmatrix}$, is
(a) 6 (b) 9 (c) 18 (d) 27

6. The value of the determinant $\begin{vmatrix} (a_1 - b_1)^2 & (a_1 - b_2)^2 & (a_1 - b_3)^2 & (a_1 - b_4)^2 \\ (a_2 - b_1)^2 & (a_2 - b_2)^2 & (a_2 - b_3)^2 & (a_2 - b_4)^2 \\ (a_3 - b_1)^2 & (a_3 - b_2)^2 & (a_3 - b_3)^2 & (a_3 - b_4)^2 \\ (a_4 - b_1)^2 & (a_4 - b_2)^2 & (a_4 - b_3)^2 & (a_4 - b_4)^2 \end{vmatrix}$, is
(a) depends on $a_i, i = 1, 2, 3, 4$ (b) depends on $b_i, i = 1, 2, 3, 4$ (c) depends on $c_i, i = 1, 2, 3, 4$ (d) 0

7. Value of $\begin{vmatrix} 1 + x_1 & 1 + x_1 x & 1 + x_1 x^2 \\ 1 + x_2 & 1 + x_2 x & 1 + x_2 x^2 \\ 1 + x_3 & 1 + x_3 x & 1 + x_3 x^2 \end{vmatrix}$ depends upon
(a) only x (b) only x_1 (c) only x_2 (d) None of these

8. If the system of linear equations $x + y + z = 6, x + 2y + 3z = 14$ and $2x + 5y + \lambda z = \mu$ $(\lambda, \mu \in R)$ has a unique solution, then
(a) $\lambda \neq 8$ (b) $\lambda = 8$ and $\mu \neq 36$ (c) $\lambda = 8$ and $\mu = 36$ (d) None of these

9. The system of equations $ax - y - z = a - 1, x - ay - z = a - 1, x - y - az = a - 1$ has no solution, if a is
(a) either –2 or 1 (b) –2 (c) 1 (d) not (–2)

10. The system of equations $x + 2y - 4z = 3, 2x - 3y + 2z = 5$ and $x - 12y + 16z = 1$ has
(a) inconsistent solution (b) unique solution (c) infinitely many solutions (d) None of these

11. If $c < 1$ and the system of equations $x + y - 1 = 0, 2x - y - c = 0$ and $-bx + 3by - c = 0$ is consistent, then the possible real values of b are
(a) $b \in \left(-3, \frac{3}{4}\right)$ (b) $b \in \left(-\frac{3}{2}, 4\right)$ (c) $b \in \left(-\frac{3}{4}, 3\right)$ (d) None of these

12. The equations $x + 2y = 3, y - 2x = 1$ and $7x - 6y + a = 0$ are consistent for
(a) $a = 7$ (b) $a = 1$ (c) $a = 11$ (d) None of these

13. Values of k for which the system of equations $x + ky + 3z = 0, kx + 2y + 2z = 0$ and $2x + 3y + 4z = 0$ possesses non-trivial solution
(a) $\left\{2, \frac{5}{4}\right\}$ (b) $\left\{-2, \frac{5}{4}\right\}$ (c) $\left\{2, -\frac{5}{9}\right\}$ (d) $\left\{-2, -\frac{5}{4}\right\}$

Session 4

Differentiation of Determinant, Integration of a Determinant, Walli's Formula, Use of Σ in Determinant

Differentiation of Determinant

Let $\Delta(x)$ be a determinant of order n. If we write $\Delta(x) = [C_1\ C_2\ C_3 \ldots C_n]$, where $C_1, C_2, C_3, \ldots, C_n$ denotes 1st, 2nd, 3rd, …, nth columns respectively, then

$$\begin{aligned}\Delta'(x) &= [C_1'\ C_2\ C_3 \cdots C_n] + [C_1\ C'_2\ C_3 \cdots C_n] \\ &\quad + [C_1\ C_2\ C_3' \cdots C_n] + \cdots + [C_1\ C_2\ C_3 \cdots C_n'] \\ &= \sum [C_1'\ C_2\ C_3 \cdots C_n]\end{aligned}$$

where C_i' denotes the column which contains the derivative of all the functions in the ith column C_i. Also, if

$$\Delta(x) = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ \vdots \\ R_n \end{bmatrix}$$

where $R_1, R_2, R_3, \ldots, R_n$ denote 1st, 2nd, 3rd, …, nth rows respectively, then

$$\Delta'(x) = \begin{bmatrix} R_1' \\ R_2 \\ R_3 \\ \vdots \\ R_n \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2' \\ R_3 \\ \vdots \\ R_n \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2 \\ R_3' \\ \vdots \\ R_n \end{bmatrix} + \cdots + \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ \vdots \\ R_n' \end{bmatrix} = \sum \begin{bmatrix} R_1' \\ R_2 \\ R_3 \\ \vdots \\ R_n \end{bmatrix}$$

where R_i' denotes the row which contains the derivative of all the functions in the ith row R_i.

***Corollary* I** For $n = 2$,

$$\Delta(x) = [C_1\ C_2], \text{ then } \Delta'(x) = [C_1'\ C_2] + [C_1\ C_2']$$

Also, if $\Delta(x) = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$, then $\Delta'(x) = \begin{bmatrix} R_1' \\ R_2 \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2' \end{bmatrix}$

For example, Let $\Delta(x) = \begin{vmatrix} a_1(x) & b_1(x) \\ a_2(x) & b_2(x) \end{vmatrix}$, then

$$\Delta'(x) = \begin{vmatrix} a_1'(x) & b_1'(x) \\ a_2(x) & b_2(x) \end{vmatrix} + \begin{vmatrix} a_1(x) & b_1(x) \\ a_2'(x) & b_2'(x) \end{vmatrix}$$

[derivative according to rowwise]

***Corollary* II For $n = 3$,** $\Delta(x) = [C_1\ C_2\ C_3]$, then

$$\Delta'(x) = [C_1'\ C_2\ C_3] + [C_1\ C_2'\ C_3] + [C_1\ C_2\ C_3']$$

Also, if $\Delta(x) = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}$, then $\Delta'(x) = \begin{bmatrix} R'_1 \\ R_2 \\ R_3 \end{bmatrix} + \begin{bmatrix} R_1 \\ R'_2 \\ R_3 \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2 \\ R'_3 \end{bmatrix}$

For example, Let $\Delta(x) = \begin{vmatrix} a_1(x) & a_2(x) & a_3(x) \\ b_1(x) & b_2(x) & b_3(x) \\ c_1(x) & c_2(x) & c_3(x) \end{vmatrix}$, then

$$\begin{aligned}\Delta'(x) &= \begin{vmatrix} a_1'(x) & a_2'(x) & a_3'(x) \\ b_1(x) & b_2(x) & b_3(x) \\ c_1(x) & c_2(x) & c_3(x) \end{vmatrix} \\ &\quad + \begin{vmatrix} a_1(x) & a_2(x) & a_3(x) \\ b_1'(x) & b_2'(x) & b_3'(x) \\ c_1(x) & c_2(x) & c_3(x) \end{vmatrix} + \begin{vmatrix} a_1(x) & a_2(x) & a_3(x) \\ b_1(x) & b_2(x) & b_3(x) \\ c_1'(x) & c_2'(x) & c_3'(x) \end{vmatrix}\end{aligned}$$

[derivative according to rowwise]

Remark

1. In a third order determinant, if two rows (columns) consist functions of x and third row (column) is constant, let

$$\Delta(x) = \begin{vmatrix} a_1(x) & a_2(x) & a_3(x) \\ b_1(x) & b_2(x) & b_3(x) \\ c_1 & c_2 & c_3 \end{vmatrix}, \text{ then}$$

$$\Delta'(x) = \begin{vmatrix} a_1'(x) & a_2'(x) & a_3'(x) \\ b_1(x) & b_2(x) & b_3(x) \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1(x) & a_2(x) & a_3(x) \\ b_1'(x) & b_2'(x) & b_3'(x) \\ c_1 & c_2 & c_3 \end{vmatrix}$$

2. In a third order determinant, if only one row (column) consists functions of x and other rows (columns) are constant, let

$$\Delta(x) = \begin{vmatrix} a_1(x) & a_2(x) & a_3(x) \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}, \text{ then } \Delta'(x) = \begin{vmatrix} a_1'(x) & a_2'(x) & a_3'(x) \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

and in general

$$\frac{d^n}{dx^n}\{\Delta(x)\} = \begin{vmatrix} \frac{d^n}{dx^n}\{a_1(x)\} & \frac{d^n}{dx^n}\{a_2(x)\} & \frac{d^n}{dx^n}\{a_3(x)\} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Important Derivatives
(Committed to Memory)

If a and b are constants and $n \in N$, then

1. if $y=(ax+b)^n$, then $\dfrac{d^n y}{dx^n}=n!\,a^n$
2. if $y=\sin(ax+b)$, then $\dfrac{d^n y}{dx^n}=\sin\left(\dfrac{n\pi}{2}+ax+b\right)\cdot a^n$
3. if $y=\cos(ax+b)$, then $\dfrac{d^n y}{dx^n}=\cos\left(\dfrac{n\pi}{2}+ax+b\right)\cdot a^n$

Example 35. ***If*** $f(x)=\begin{vmatrix}\sin x & \cos x & \sin x\\ \cos x & -\sin x & \cos x\\ x & 1 & 1\end{vmatrix}$, ***find the value of*** $2^{f'(0)}+\{f'(1)\}^2$.

Sol. $f'(x)=\begin{vmatrix}\cos x & -\sin x & \cos x\\ \cos x & -\sin x & \cos x\\ x & 1 & 1\end{vmatrix}+\begin{vmatrix}\sin x & \cos x & \sin x\\ -\sin x & -\cos x & -\sin x\\ x & 1 & 1\end{vmatrix}$

$+\begin{vmatrix}\sin x & \cos x & \sin x\\ \cos x & -\sin x & \cos x\\ 1 & 0 & 0\end{vmatrix}$ [derivative according to rowwise]

$=0+0+1\begin{vmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{vmatrix}=\cos^2 x+\sin^2 x=1$

$\therefore\ f'(x)=1 \Rightarrow \quad f'(0)=1$ and $f'(1)=1$

$\Rightarrow \quad 2^{f'(0)}+\{f'(1)\}^2=2^1+1^2=3$

Example 36. ***Let*** $f(x)=\begin{vmatrix}\cos x & \sin x & \cos x\\ \cos 2x & \sin 2x & 2\cos 2x\\ \cos 3x & \sin 3x & 3\cos 3x\end{vmatrix}$, ***then find the value of*** $f'\left(\dfrac{\pi}{2}\right)$.

Sol. Given, $f(x)=\begin{vmatrix}\cos x & \sin x & \cos x\\ \cos 2x & \sin 2x & 2\cos 2x\\ \cos 3x & \sin 3x & 3\cos 3x\end{vmatrix}$

$\therefore \quad f'(x)=\begin{vmatrix}-\sin x & \sin x & \cos x\\ -2\sin 2x & \sin 2x & 2\cos 2x\\ -3\sin 3x & \sin 3x & 3\cos 3x\end{vmatrix}$

$+\begin{vmatrix}\cos x & \cos x & \cos x\\ \cos 2x & 2\cos 2x & 2\cos 2x\\ \cos 3x & 3\cos 3x & 3\cos 3x\end{vmatrix}+\begin{vmatrix}\cos x & \sin x & -\sin x\\ \cos 2x & \sin 2x & -4\sin 2x\\ \cos 3x & \sin 3x & -9\sin 3x\end{vmatrix}$

[derivative according to columnwise]

$\Rightarrow f'\left(\dfrac{\pi}{2}\right)=\begin{vmatrix}-1 & 1 & 0\\ 0 & 0 & -2\\ 3 & -1 & 0\end{vmatrix}+0+\begin{vmatrix}0 & 1 & -1\\ -1 & 0 & 0\\ 0 & -1 & 9\end{vmatrix}$

[$\because C_2=C_3$ in second determinant]

$=2(1-3)+1(9-1)=-4+8=4$

Example 37. Let α be a repeated root of a quadratic equation $f(x)=0$ and $A(x)$, $B(x)$ and $C(x)$ be polynomials of degree 3, 4, and 5 respectively, show that $\begin{vmatrix}A(x) & B(x) & C(x)\\ A(\alpha) & B(\alpha) & C(\alpha)\\ A'(\alpha) & B'(\alpha) & C'(\alpha)\end{vmatrix}$ is divisible by $f(x)$, where prime ($'$) denotes the derivatives.

Sol. Since, α is a repeated root of the quadratic equation $f(x)=0$, then $f(x)$ can be written as $f(x)=a(x-\alpha)^2$, where a is some non-zero constant.

Let $g(x)=\begin{vmatrix}A(x) & B(x) & C(x)\\ A(\alpha) & B(\alpha) & C(\alpha)\\ A'(\alpha) & B'(\alpha) & C'(\alpha)\end{vmatrix}$

$g(x)$ is divisible by $f(x)$, if it is divisible by $(x-a)^2$ i.e., $g(\alpha)=0$ and $g'(\alpha)=0$. As $A(x)$, $B(x)$ and $C(x)$ are polynomials of degree 3, 4 and 5, respectively.

$\therefore$ Degree of $g(x)\geq 2$

Now, $g(\alpha)=\begin{vmatrix}A(\alpha) & B(\alpha) & C(\alpha)\\ A(\alpha) & B(\alpha) & C(\alpha)\\ A'(\alpha) & B'(\alpha) & C'(\alpha)\end{vmatrix}=0$

[$\because R_1$ and R_2 are identical]

Also, $g'(x)=\begin{vmatrix}A'(x) & B'(x) & C'(x)\\ A(\alpha) & B(\alpha) & C(\alpha)\\ A'(\alpha) & B'(\alpha) & C'(\alpha)\end{vmatrix}$

$\therefore \quad g'(\alpha)=\begin{vmatrix}A'(\alpha) & B'(\alpha) & C'(\alpha)\\ A(\alpha) & B(\alpha) & C(\alpha)\\ A'(\alpha) & B'(\alpha) & C'(\alpha)\end{vmatrix}=0$

[$\because R_1$ and R_3 are identical]

This implies that $f(x)$ divides $g(x)$.

Example 38. Find the coefficient of x in the determinant $\begin{vmatrix}(1+x)^{a_1b_1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3}\\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3}\\ (1+x)^{a_3b_1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3}\end{vmatrix}$

Sol. We know that, if $f(x)$ be a polynomial, then coefficient of x^n in $f(x)=\dfrac{1}{n!}f^n(0)$.

Let $f(x)=\begin{vmatrix}(1+x)^{a_1b_1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3}\\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3}\\ (1+x)^{a_3b_1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3}\end{vmatrix}$

$\therefore\ f'(x)=\begin{vmatrix}a_1b_1(1+x)^{a_1b_1-1} & a_1b_2(1+x)^{a_1b_2-1} & a_1b_3(1+x)^{a_1b_3-1}\\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3}\\ (1+x)^{a_3b_1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3}\end{vmatrix}$

$$+\begin{vmatrix} (1+x)^{a_1b_1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3} \\ a_2b_1(1+x)^{a_2b_1-1} & a_2b_2(1+x)^{a_2b_2-1} & a_2b_3(1+x)^{a_2b_3-1} \\ (1+x)^{a_3b_1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3} \end{vmatrix}$$

$$+\begin{vmatrix} (1+x)^{a_1b_1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3} \\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3} \\ a_3b_1(1+x)^{a_3b_1-1} & a_3b_2(1+x)^{a_3b_2-1} & a_3b_3(1+x)^{a_3b_3-1} \end{vmatrix}$$

$$\therefore\quad f'(0)=\begin{vmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}+\begin{vmatrix} 1 & 1 & 1 \\ a_2b_1 & a_2b_2 & a_3b_3 \\ 1 & 1 & 1 \end{vmatrix}+\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{vmatrix}$$

$$=0+0+0=0$$

$\therefore$ Coefficient of x in $f(x)=\dfrac{f'(0)}{1!}=0$

Aliter

Let $\begin{vmatrix} (1+x)^{a_1b_1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3} \\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3} \\ (1+x)^{a_3b_1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3} \end{vmatrix}=A+Bx+Cx^2+\cdots$

On differentiating both sides w.r.t. x and then put $x=0$ in both sides, we get

$$B=\begin{vmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}+\begin{vmatrix} 1 & 1 & 1 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ 1 & 1 & 1 \end{vmatrix}+\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{vmatrix}$$

$=0+0+0=0$

Hence, coefficient of x in given determinant is 0.

Example 39. If $\Delta(x)=\begin{vmatrix} \alpha+x & \theta+x & \lambda+x \\ \beta+x & \phi+x & \mu+x \\ \gamma+x & \psi+x & \gamma+x \end{vmatrix}$,

show that $\Delta''(x)=0$ and $\Delta(x)=\Delta(0)+Sx$, where S denotes the sum of all the cofactors of all elements in $\Delta(0)$ and dash denotes the derivative.

Sol. We have, $\Delta'(x)=\begin{vmatrix} 1 & \theta+x & \lambda+x \\ 1 & \phi+x & \mu+x \\ 1 & \psi+x & \nu+x \end{vmatrix}+\begin{vmatrix} \alpha+x & 1 & \lambda+x \\ \beta+x & 1 & \mu+x \\ \gamma+x & 1 & \nu+x \end{vmatrix}$

$$+\begin{vmatrix} \alpha+x & \theta+x & 1 \\ \beta+x & \phi+x & 1 \\ \gamma+x & \psi+x & 1 \end{vmatrix}$$

Applying $C_2\to C_2-xC_1$ and $C_3\to C_3-xC_1$ in first, $C_1\to C_1-xC_2$ and $C_3\to C_3-xC_2$ in second and $C_1\to C_1-xC_3$ and $C_2\to C_2-xC_3$ in third, then

$$\Delta'(x)=\begin{vmatrix} 1 & \theta & \lambda \\ 1 & \phi & \mu \\ 1 & \psi & \nu \end{vmatrix}+\begin{vmatrix} \alpha & 1 & \lambda \\ \beta & 1 & \mu \\ \gamma & 1 & \nu \end{vmatrix}+\begin{vmatrix} \alpha & \theta & 1 \\ \beta & \phi & 1 \\ \gamma & \psi & 1 \end{vmatrix}$$

$$\left\{\text{sum of all cofactors in } \Delta(0), \text{where } \Delta(0)=\begin{vmatrix} \alpha & \theta & \lambda \\ \beta & \phi & \mu \\ \gamma & \psi & \nu \end{vmatrix}\right\}$$

$\therefore\quad \Delta''(x)=0$ [$\because S$ is constant]

Since, $\Delta'(x)=S$

On integrating $\Delta(x)=Sx+C$

$\therefore\quad \Delta(0)=0+C$

Hence, $\Delta(x)=Sx+\Delta(0)$

Example 40. ***If*** $f(x)=\begin{vmatrix} x^n & \sin x & \cos x \\ n! & \sin\left(\dfrac{n\pi}{2}\right) & \cos\left(\dfrac{n\pi}{2}\right) \\ \pi & \pi^2 & \pi^3 \end{vmatrix}$,

then find the value of $\dfrac{d^n}{dx^n}\{f(x)\}$ ***at*** $x=0$, $n\in I$.

Sol. $\dfrac{d^n}{dx^n}\{f(x)\}=\begin{vmatrix} \dfrac{d^n}{dx^n}(x^n) & \dfrac{d^n}{dx^n}(\sin x) & \dfrac{d^n}{dx^n}(\cos x) \\ n! & \sin\left(\dfrac{n\pi}{2}\right) & \cos\left(\dfrac{n\pi}{2}\right) \\ \pi & \pi^2 & \pi^3 \end{vmatrix}$

$$=\begin{vmatrix} n! & \sin\left(\dfrac{n\pi}{2}+x\right) & \cos\left(\dfrac{n\pi}{2}+x\right) \\ n! & \sin\left(\dfrac{n\pi}{2}\right) & \cos\left(\dfrac{n\pi}{2}\right) \\ \pi & \pi^2 & \pi^3 \end{vmatrix}$$

$$\therefore\quad \frac{d^n}{dx^n}\{f(x)\}\text{ at}(x=0)=\begin{vmatrix} n! & \sin\left(\dfrac{n\pi}{2}\right) & \cos\left(\dfrac{n\pi}{2}\right) \\ n! & \sin\left(\dfrac{n\pi}{2}\right) & \cos\left(\dfrac{n\pi}{2}\right) \\ \pi & \pi^2 & \pi^3 \end{vmatrix}$$

$=0$ [$\because R_1$ and R_2 are identical]

Integration of a Determinant

Let $\Delta(x)=\begin{vmatrix} f(x) & g(x) & h(x) \\ p & q & r \\ l & m & n \end{vmatrix}$

where p, q, r, l, m and n are constants, then

$$\int_a^b \Delta(x)\,dx=\begin{vmatrix} \int_a^b f(x)dx & \int_a^b g(x)dx & \int_a^b h(x)\,dx \\ p & q & r \\ l & m & n \end{vmatrix}$$

Remark

If in a determinant, the elements of more than one columns or rows are functions of x, then the integration can be done only after evaluation or expansion of the determinant.

Important Integrals (Committed to Memory)

1. (i) $\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x+\cos^n x}\,dx=\frac{\pi}{4}$

$$=\int_0^{\pi/2} \frac{\cos^n x}{\sin^n x+\cos^n x}\,dx, \forall n\in R$$

(ii) $\int_0^{\pi/2} \frac{\tan^n x}{1+\tan^n x}\,dx=\frac{\pi}{4}=\int_0^{\pi/2} \frac{dx}{1+\tan^n x}, \forall n\in R$

(iii) $\int_0^{\pi/2} \frac{dx}{1+\cot^n x}=\frac{\pi}{4}=\int_0^{\pi/2} \frac{\cot^n x}{1+\cot^n x}\,dx, \forall n\in R$

2. (i) $\int_0^{\pi/2} \ln\sin x\,dx=\int_0^{\pi/2} \ln\cos x\,dx=-\frac{\pi}{2}\ln 2$

or $\frac{\pi}{2}\ln\left(\frac{1}{2}\right)$

(ii) $\int_0^{\pi/2} \ln\tan x\,dx=\int_0^{\pi/2} \ln\cot x\,dx=0$

(iii) $\int_0^{\pi/2} \ln\sec x\,dx=\int_0^{\pi/2} \ln\operatorname{cosec} x\,dx=\frac{\pi}{2}\ln 2$

Walli's Formula

(An easy way to evaluate $\int_0^{\pi/2} \sin^m x\cos^n x\,dx$, where $m, n\in W$) We have, $\int_0^{\pi/2} \sin^m x\cos^n x\,dx$

$$=\frac{\{(m-1)(m-3)\ldots 2 \text{ or } 1\}\,\{(n-1)(n-3)\ldots 2 \text{ or } 1\}}{\{(m+n)(m+n-2)(m+n-4)\ldots 2 \text{ or } 1\}}$$

where, p is $\pi/2$, if m and n are both even, otherwise $p=1$. The last factor in each of three products is either 1 or 2. In case any of m or n is 1, we simply write 1 as the only factor to replace its product. If any of m or n is zero provided, we put 1 as the only factor in its product and we regard 0 as even.

For example,

1. $\int_0^{\pi/2} \sin^6 x\cos^4 x\,dx=\frac{[5\cdot 3\cdot 1][3\cdot 1]}{[10\cdot 8\cdot 6\cdot 4\cdot 2]}\cdot\frac{\pi}{2}=\frac{3\pi}{512}$

2. $\int_0^{\pi/2} \sin^6 x\cos^3 x\,dx=\frac{[5\cdot 3\cdot 1][2]}{[9\cdot 7\cdot 5\cdot 3\cdot 1]}\cdot 1=\frac{2}{63}$

3. $\int_0^{\pi/2} \sin^5 x\cos^7 x\,dx=\frac{[4\cdot 2][6\cdot 4\cdot 2]}{[12\cdot 10\cdot 8\cdot 6\cdot 4\cdot 2]}\cdot 1=\frac{1}{120}$

4. $\int_0^{\pi/2} \sin^8 x\,dx=\frac{[7\cdot 5\cdot 3\cdot 1]}{[8\cdot 6\cdot 4\cdot 2]}\cdot\frac{\pi}{2}=\frac{35\pi}{256}$

5. $\int_0^{\pi/2} \cos^7 x\,dx=\frac{[6\cdot 4\cdot 2]}{[7\cdot 5\cdot 3\cdot 1]}\cdot 1=\frac{16}{35}$

6. $\int_0^{\pi/2} \sin^{10} x\cos x\,dx=\frac{[9\cdot 7\cdot 5\cdot 3\cdot 1][1]}{[11\cdot 9\cdot 7\cdot 5\cdot 3\cdot 1]}\cdot 1=\frac{1}{11}$

Example 41. ***If*** $\Delta(x)=\begin{vmatrix} a & b & c\\ 6 & 4 & 3\\ x & x^2 & x^3\end{vmatrix}$, ***then find*** $\int_0^1 \Delta(x)\,dx$.

Sol. $\int_0^1 \Delta(x)\,dx=\begin{vmatrix} a & b & c\\ 6 & 4 & 3\\ \int_0^1 x\,dx & \int_0^1 x^2dx & \int_0^1 x^3dx\end{vmatrix}$

$$=\begin{vmatrix} a & b & c\\ 6 & 4 & 3\\ \left[\frac{x^2}{2}\right]_0^1 & \left[\frac{x^3}{3}\right]_0^1 & \left[\frac{x^4}{4}\right]_0^1\end{vmatrix}=\begin{vmatrix} a & b & c\\ 6 & 4 & 3\\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4}\end{vmatrix}$$

Applying $R_2\to R_2-12R_3$, then $\int_0^1 \Delta(x)dx=\begin{vmatrix} a & b & c\\ 0 & 0 & 0\\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4}\end{vmatrix}=0$

Example 42. ***If***

$$f(x)=\begin{vmatrix} \sin^5 x & \ln\sin x & \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\\ n & \sum_{k=1}^{n} k & \prod_{k=1}^{n} k\\ \frac{8}{15} & \frac{\pi}{2}\ln\left(\frac{1}{2}\right) & \frac{\pi}{4}\end{vmatrix},$$

then find the value of $\int_0^{\pi/2} f(x)\,dx$.

Sol. $\int_0^{\pi/2} f(x)\,dx$

$$=\begin{vmatrix} \int_0^{\pi/2}\sin^5 x\,dx & \int_0^{\pi/2}\ln\sin x\,dx & \int_0^{\pi/2}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\\ n & \sum_{k=1}^{n} k & \prod_{k=1}^{n} k\\ \frac{8}{15} & \frac{\pi}{2}\ln\left(\frac{1}{2}\right) & \frac{\pi}{4}\end{vmatrix},$$

$$=\begin{vmatrix} \frac{4}{5}\cdot\frac{2}{3} & -\frac{\pi}{2}\ln 2 & \frac{\pi}{4}\\ n & \sum_{k=1}^{n} k & \prod_{k=1}^{n} k\\ \frac{8}{15} & \frac{\pi}{2}\ln\left(\frac{1}{2}\right) & \frac{\pi}{4}\end{vmatrix}$$ [by Walli's formula]

$$=\begin{vmatrix} \frac{8}{15} & \frac{\pi}{2}\ln\left(\frac{1}{2}\right) & \frac{\pi}{4}\\ n & \sum_{k=1}^{n} k & \prod_{k=1}^{n} k\\ \frac{8}{15} & \frac{\pi}{2}\ln\left(\frac{1}{2}\right) & \frac{\pi}{4}\end{vmatrix}=0$$ [since R_1 and R_3 are identical]

Example 43. ***Let*** $f(x)$
$$= \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix},$$ ***then find the value of*** $\int_0^{\pi/2} f(x)\,dx$.

Sol. Applying $C_2 \to C_2 - \cos^2 x\, C_1$, then

$$f(x) = \begin{vmatrix} \sec x & 0 & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x - \cos^4 x & \operatorname{cosec}^2 x \\ 1 & 0 & \cos^2 x \end{vmatrix}$$

[expanding along C_2]

$$= (\cos^2 x - \cos^4 x) \begin{vmatrix} \sec x & \sec^2 x + \cot x \operatorname{cosec} x \\ 1 & \cos^2 x \end{vmatrix}$$

$$= (\cos^2 x - \cos^4 x)(\cos x - \sec^2 x - \cot x \operatorname{cosec} x)$$

$$= \cos^2 x (1 - \cos^2 x)\left(\cos x - \frac{1}{\cos^2 x} - \frac{\cos x}{\sin^2 x}\right)$$

$$= \cos^2 x \sin^2 x \left(\cos x - \frac{1}{\cos^2 x} - \frac{\cos x}{\sin^2 x}\right)$$

$$= \cos^3 x \sin^2 x - \sin^2 x - \cos^3 x$$

$$= -\cos^3 x (1 - \sin^2 x) - \sin^2 x$$

$$f(x) = -\cos^5 x - \sin^2 x$$

$$\therefore \int_0^{\pi/2} f(x)\,dx = -\int_0^{\pi/2} \cos^5 x\,dx - \int_0^{\pi/2} \sin^2 x\,dx$$

$$= -\left(\frac{4}{5}\cdot\frac{2}{3}\cdot 1\right) - \left(\frac{1}{2}\cdot\frac{\pi}{2}\right) = -\left(\frac{8}{15} + \frac{\pi}{4}\right)$$

[by Walli's formula]

Use of Σ in Determinant

If $\Delta(r) = \begin{vmatrix} f(r) & g(r) & h(r) \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix}$

where a, b, c, a_1, b_1 and c_1 are constants, independent of r, then

$$\sum_{r=1}^{n} \Delta(r) = \begin{vmatrix} \sum_{r=1}^{n} f(r) & \sum_{r=1}^{n} g(r) & \sum_{r=1}^{n} h(r) \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix}$$

Remark

If in a determinant, the elements of more than one columns or rows are function of r, then the Σ can be done only after evaluation or expansion of the determinant.

Important Summation

(Committed to Memory)

1. $\sum_{r=1}^{n} r = \Sigma n = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

2. $\sum_{r=1}^{n} r^2 = \Sigma n^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

3. $\sum_{r=1}^{n} r^3 = \Sigma n^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3$
$$= \left[\frac{n(n+1)}{2}\right]^2 = (\Sigma n)^2$$

4. $\sum_{r=1}^{n} a = \Sigma a = \underbrace{a + a + a + \ldots + a}_{n \text{ times}} = an$

5. $\sum_{r=1}^{n} (\lambda - 1)\lambda^{r-1} = \lambda^n - 1, \forall\, \lambda \neq 1$ and $\lambda > 1$

6. $\sum_{r=1}^{n} \sin[\alpha + (r-1)\beta] = \frac{\sin\left\{\alpha + \frac{(n-1)\beta}{2}\right\} \sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$

Particular For $\alpha = \beta = \theta$.

$$\sum_{r=1}^{n} \sin r\theta = \frac{\sin\left\{\left(\frac{n+1}{2}\right)\theta\right\} \cdot \sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$$

7. $\sum_{r=1}^{n} \cos\{\alpha + (r-1)\beta\} = \frac{\cos\left\{\alpha + \frac{(n-1)}{2}\beta\right\} \sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$

Particular For $\alpha = \beta = \theta$.

$$\sum_{r=1}^{n} \cos r\theta = \frac{\cos\left\{\left(\frac{n+1}{2}\right)\theta\right\} \sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$$

8. $\sum_{r=1}^{n} \{f(r+1) - f(r)\} = f(n+1) - f(1)$

Particular $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \sum_{r=1}^{n} \left(\frac{1}{r} - \frac{1}{r+1}\right)$
$$= \frac{1}{1} - \frac{1}{n+1} = \frac{n}{n+1}$$

9. $\sum_{r=1}^{n} {}^nC_r = 2^n$

Remark

Capital pie $\prod$ is not direct applicable in determint i.e.,

$$\prod_{r=1}^{n}\Delta(r) \neq \begin{vmatrix} \prod_{r=1}^{n} f(r) & \prod_{r=1}^{n} g(r) & \prod_{r=1}^{n} h(r) \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix}$$

Explanation $\prod_{r=1}^{n}\Delta(r) = \Delta(1)\times\Delta(2)\times\ldots\times\Delta(n)$

$$= \begin{vmatrix} f(1) & g(1) & h(1) \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix} \times \begin{vmatrix} f(2) & g(2) & h(2) \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix} \times \ldots \times \begin{vmatrix} f(n) & g(n) & h(n) \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix}$$

$$\neq \begin{vmatrix} \prod_{r=1}^{n} f(r) & \prod_{r=1}^{n} g(r) & \prod_{r=1}^{n} h(r) \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix}$$

Example 44. *Let n be a positive integer and* $\Delta_r = \begin{vmatrix} 2r-1 & {}^{n}C_r & 1 \\ n^2-1 & 2^n & n+1 \\ \cos^2 n^2 & \cos^2 n & \cos^2(n+1) \end{vmatrix}$, *prove that* $\sum_{r=0}^{n}\Delta_r = 0$.

Sol. We have, $\Delta_r = \begin{vmatrix} 2r-1 & {}^{n}C_r & 1 \\ n^2-1 & 2^n & n+1 \\ \cos^2(n^2) & \cos^2 n & \cos^2(n+1) \end{vmatrix}$

$$\therefore \quad \sum_{r=0}^{n}\Delta_r = \begin{vmatrix} \sum_{r=0}^{n}(2r-1) & \sum_{r=0}^{n} {}^{n}C_r & \sum_{r=0}^{n} 1 \\ n^2-1 & 2^n & n+1 \\ \cos^2(n^2) & \cos^2 n & \cos^2(n+1) \end{vmatrix}$$

Now, $\sum_{r=0}^{n}(2r-1) = 2\sum_{r=0}^{n} r - \sum_{r=0}^{n} 1$

$$= 2(0+1+2+3+\ldots+n) - \underbrace{(1+1+1+\ldots+1)}_{(n+1)\text{ times}}$$

$$= \frac{2n(n+1)}{2} - (n+1) = (n+1)(n-1) = n^2-1$$

$$\sum_{r=0}^{n}\Delta_r = \begin{vmatrix} n^2-1 & 2^n & n+1 \\ n^2-1 & 2^n & n+1 \\ \cos^2(n^2) & \cos^2 n & \cos^2(n+1) \end{vmatrix} = 0$$

[since R_1 and R_2 are identical]

Example 45. Let n be a positive integer and $\Delta_r = \begin{vmatrix} r^2+r & r+1 & r-2 \\ 2r^2+3r-1 & 3r & 3r-3 \\ r^2+2r+3 & 2r-1 & 2r-1 \end{vmatrix}$ and $\sum_{r=1}^{n}\Delta_r = an^2+bn+c$, find the value of $a+b+c$.

Sol. We have, $\Delta_r = \begin{vmatrix} r^2+1r & r+1 & r-2 \\ 2r^2+3r-1 & 3r & 3r-3 \\ r^2+2r+3 & 2r-1 & 2r-1 \end{vmatrix}$

Applying $R_2 \to R_2 - (R_1 + R_3)$, then

$$\Delta_r = \begin{vmatrix} r^2+r & & r+1 & & r-2 \\ \vdots & & & & \\ -4 & \ldots & 0 & \ldots & 0 \\ \vdots & & & & \\ r^2+2r+3 & & 2r-1 & & 2r-1 \end{vmatrix}$$

Expanding along R_2, we get

$$= 4\begin{vmatrix} r+1 & r-2 \\ 2r-1 & 2r-1 \end{vmatrix}$$

$$= 4[(r+1)(2r-1) - (r-2)(2r-1)]$$

$$= 24r - 12$$

Now, $\sum_{r=1}^{n}\Delta_r = 24\sum_{r=1}^{n} r - 12\sum_{r=1}^{n} 1$

$$= 24\frac{n(n+1)}{2} - 12n = 12n(n+1-1)$$

$$= 12n^2 = an^2 + bn + c \qquad \text{[given]}$$

For $n = 1$, we have

$$a + b + c = 12$$

Exercise for Session 4

1. If $f(x)=\begin{vmatrix} x & x^2 & x^3 \\ 1 & 2 & 3 \\ 0 & 1 & x \end{vmatrix}$, $f'(1)$ is equal to

(a) -1 (b) 0 (c) 1 (d) 2

2. Let $f(x)=\begin{vmatrix} \sec x & x^2 & x \\ 2\sin x & x^3 & 2x^2 \\ \tan 3x & x^2 & x \end{vmatrix}$, $\lim_{x\to 0}\frac{f(x)}{x^4}$ is equal to

(a) 0 (b) -1 (c) 2 (d) 3

3. Let $\begin{vmatrix} x & 2 & x \\ x^2 & x & 6 \\ x & x & 6 \end{vmatrix} = Ax^4 + Bx^3 + Cx^2 + Dx + E$, the value of $5A + 4B + 3C + 2D + E$ is equal to

(a) -16 (b) -11 (c) 0 (d) 16

4. Let $f(x)=\begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$, where p is a constant. Then $\frac{d^3}{dx^3}\{f(x)\}$ at $x=0$ is

(a) p (b) $p+p^2$ (c) $p+p^3$ (d) independent of p

5. If $y=\sin mx$, the value of the determinant $\begin{vmatrix} y & y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & y_8 \end{vmatrix}$, where $y_n=\frac{d^n y}{dx^n}$, is

(a) m^2 (b) m^3 (c) m^9 (d) None of these

6. Let $f(x)=\begin{vmatrix} 2\cos^2 x & \sin 2x & -\sin x \\ \sin 2x & 2\sin^2 x & \cos x \\ \sin x & -\cos x & 0 \end{vmatrix}$, the value of $\int_0^{\pi/2}\{f(x)+f'(x)\}\,dx$, is

(a) $\frac{\pi}{2}$ (b) π (c) $\frac{3\pi}{2}$ (d) 2π

7. If $f(x)=\begin{vmatrix} \cos x & e^{x^2} & 2x\cos^2\left(\frac{x}{2}\right) \\ x^2 & \sec x & \sin x + x^3 \\ 1 & 2 & x+\tan x \end{vmatrix}$, the value of $\int_{-\pi/2}^{\pi/2}(x^2+1)[f(x)+f'(x)]\,dx$, is

(a) -1 (b) 0 (c) 1 (d) 2

8. If $f(x)=\begin{vmatrix} \sin^2 x + \cos^4 x & \ln\cos x & \frac{1}{1+(\tan x)^{\sqrt{2}}} \\ \frac{\pi}{7} & \pi^2 & \pi^4 \\ \frac{1}{16} & -\frac{1}{2}\ln 2 & \frac{1}{4} \end{vmatrix}$, the value of $\int_0^{\pi/2} f(x)\,dx$ is

(a) 2 (b) -1 (c) 0 (d) None of these

9. If $\Delta_k=\begin{vmatrix} 1 & n & n \\ 2k & n^2+n+1 & n^2+n \\ 2k-1 & n^2 & n^2+n+1 \end{vmatrix}$ and $\sum_{k=1}^{n}\Delta_k=56$, then n is equal to

(a) 4 (b) 6 (c) 8 (d) None of these

10. The value of $\sum_{r=2}^{n}(-2)^r\begin{vmatrix} {}^{n-2}C_{r-2} & {}^{n-2}C_{r-1} & {}^{n-2}C_r \\ -3 & 1 & 1 \\ 2 & -1 & 0 \end{vmatrix}$ $(n>2)$ is

(a) $2n-1+(-1)^n$ (b) $2n+1+(-1)^n$ (c) $2n-3+(-1)^n$ (d) None of these

Shortcuts and Important Results to Remember

1 **Symmetric Determinant** The elements situated at equal distance from the diagonal are equal both in magnitude and sign. i.e. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$

2 **Skew-symmetric Determinant** All the diagonal elements are zero and the elements situated at equal distance from the diagonal are equal in magnitude but opposite in sign. The value of skew-symmetric determinant of even order is always a perfect square and that of odd order is always zero i.e. $\begin{vmatrix} 0 & a \\ -a & 0 \end{vmatrix} = a^2$ and $\begin{vmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{vmatrix} = 0$

3 **Circulant Determinant** The elements of the rows (or columns) are in cyclic order. i.e.,

(i) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

(ii) $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$

$= (a-b)(b-c)(c-a)(ab+bc+ca)$

(iii) $\begin{vmatrix} a & bc & abc \\ b & ca & abc \\ c & ab & abc \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = abc(a-b)(b-c)(c-a)$

(iv) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

(v) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc)$

Remark

These results direct applicable in lengthy questions as behaviour of standard results.

4 (i) If $\Delta = 0$, then $\Delta^C = 0$, where Δ^C denotes the determinant of cofactors of elements of Δ.

(ii) If $\Delta \neq 0$, then $\Delta^C = \Delta^{n-1}$, where n is order of Δ.

(iii) Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

The sum of products of the elements of any row or column with the corresponding cofactors is equal to the value of determinant, i.e.

$$a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} = \Delta$$

and sum of products of the elements of any row or column with the cofactors of the corresponding elements of any other row or column is zero, i.e.,

$$a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} = a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33} = 0$$

5 A homogeneous system of equations is never consistent.

6 **Conjugate of a Determinant** If a_i, b_i and $c_i \in C$ $(i = 1, 2, 3)$ and $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, then $\overline{\Delta} = \begin{vmatrix} \overline{a_1} & \overline{b_1} & \overline{c_1} \\ \overline{a_2} & \overline{b_2} & \overline{c_2} \\ \overline{a_3} & \overline{b_3} & \overline{c_3} \end{vmatrix}$

(i) If Δ is purely real, then $\Delta = \overline{\Delta}$

(ii) If Δ is purely imaginary, then $\overline{\Delta} = -\Delta$

7 (i) If $x_1, x_2, x_3, \ldots$ are in AP or $a^{x_1}, a^{x_2}, a^{x_3}, \ldots$ are in GP, then $\begin{vmatrix} x_1 & x_2 & x_3 \\ x_{n+1} & x_{n+2} & x_{n+3} \\ x_{2n+1} & x_{2n+2} & x_{2n+3} \end{vmatrix} = 0$

(ii) If $a_1, a_2, a_3, \ldots$ are in GP and $a_i > 0, i = 1, 2, 3, \ldots$,

then $\begin{vmatrix} \log a_n & \log a_{n+1} & \log \alpha_{n+2} \\ \log a_{n+3} & \log a_{n+4} & \log a_{n+5} \\ \log a_{n+6} & \log a_{n+7} & \log a_{n+8} \end{vmatrix} = 0$

JEE Type Solved Examples :
Single Option Correct Type Questions

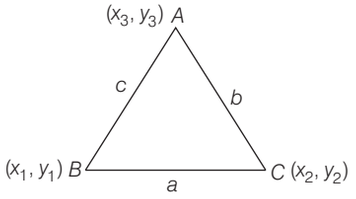
▪ This section contains **10 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

● **Ex. 1** *If* $(x_1-x_2)^2+(y_1-y_2)^2=a^2$, $(x_2-x_3)^2+(y_2-y_3)^2=b^2$ *and* $(x_3-x_1)^2+(y_3-y_1)^2=c^2$ *and* $k\begin{vmatrix} x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}^2 = (a+b+c)(b+c-a)(c+a-b)(a+b-c)$, *the value of k is*

(a) 1 (b) 2 (c) 4 (d) 8

Sol. (*c*) Consider the triangle with vertices $B(x_1, y_1)$, $C(x_2, y_2)$ and $A(x_3, y_3)$ and

$$AB = c,\ BC = a \text{ and } CA = b$$



$$\therefore \quad \text{Area of } \Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix} \quad \ldots(i)$$

Also, area of $\Delta ABC = \sqrt{s(s-a)(s-b)(s-c)}$,

where $2s = a+b+c$ ...(ii)

From Eqs. (i) and (ii), we get

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix} = \sqrt{s(s-a)(s-b)(s-c)}$$

On squaring and simplifying, we get

$$4\begin{vmatrix} x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}^2 = (a+b+c)(b+c-a)(c+a-b)(a+b-c)$$

Hence, the value of k is 4.

● **Ex. 2** *If a, b and c are complex numbers, the determinant* $\Delta = \begin{vmatrix} 0 & -b & -c\\ \bar{b} & 0 & -a\\ \bar{c} & \bar{a} & 0\end{vmatrix}$ *is*

(a) a non-zero real number (b) purely imaginary
(c) 0 (d) None of these

Sol. (*b*) We observe that,

$$\bar{\Delta} = \begin{vmatrix} 0 & -\bar{b} & -\bar{c}\\ b & 0 & -\bar{a}\\ c & a & 0\end{vmatrix}$$

$$\Rightarrow \quad \bar{\Delta} = \begin{vmatrix} 0 & b & c\\ -\bar{b} & 0 & a\\ -\bar{c} & -\bar{a} & 0\end{vmatrix}$$

[interchanging rows and columns]

$$= -\begin{vmatrix} 0 & -b & -c\\ \bar{b} & 0 & -a\\ \bar{c} & \bar{a} & 0\end{vmatrix}$$

[taking (– 1) common from each row]

$$\Rightarrow \quad \bar{\Delta} = -\Delta$$

Hence, Δ is purely imaginary.

● **Ex. 3** *The equation*

$$\begin{vmatrix} (1+x)^2 & (1-x)^2 & -(2+x^2)\\ 2x+1 & 3x & 1-5x\\ x+1 & 2x & 2-3x\end{vmatrix} + \begin{vmatrix} (1+x)^2 & 2x+1 & x+1\\ (1-x)^2 & 3x & 2x\\ 1-2x & 3x-2 & 2x-3\end{vmatrix} = 0 \text{ has}$$

(a) no real solution
(b) 4 real solutions
(c) two real and two non-real solutions
(d) infinite number of solutions, real or non-real

Sol. (*d*) Interchanging rows and columns in first determinant, then

$$\begin{vmatrix} (1+x)^2 & 2x+1 & x+1\\ (1-x)^2 & 3x & 2x\\ -(2+x^2) & 1-5x & 2-3x\end{vmatrix} + \begin{vmatrix} (1+x)^2 & 2x+1 & x+1\\ (1-x)^2 & 3x & 2x\\ 1-2x & 3x-2 & 2x-3\end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} (1+x)^2 & 2x+1 & x+1\\ (1-x)^2 & 3x & 2x\\ -(1+x)^2 & -2x-1 & -x-1\end{vmatrix} = 0$$

Applying $R_3 \to R_3 + R_1$, then

$$\Rightarrow \quad \begin{vmatrix} (1+x)^2 & 2x+1 & x+1 \\ (1-x)^2 & 3x & 2x \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$$\Rightarrow \quad 0 = 0$$

which is true for all values of x.

Hence, given equation has infinite number of solutions, real or non-real.

● ***Ex. 4*** *If X, Y and Z are positive numbers such that Y and Z have respectively 1 and 0 at their unit's place and*

$$\Delta = \begin{vmatrix} X & 4 & 1 \\ Y & 0 & 1 \\ Z & 1 & 0 \end{vmatrix}$$

If $(\Delta + 1)$ is divisible by 10, then X has at its unit's place

(a) 0 (b) 1

(c) 2 (d) 3

Sol. (*c*) Let $X = 10x + \lambda, Y = 10y + 1$ and $Z = 10z$, where $x, y, z \in N$, then

$$\Delta = \begin{vmatrix} X & 4 & 1 \\ Y & 0 & 1 \\ Z & 1 & 0 \end{vmatrix} = \begin{vmatrix} 10x+\lambda & 4 & 1 \\ 10y+1 & 0 & 1 \\ 10z & 1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 10x & 4 & 1 \\ 10y & 0 & 1 \\ 10z & 1 & 0 \end{vmatrix} + \begin{vmatrix} \lambda & 4 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= 10 \begin{vmatrix} x & 4 & 1 \\ y & 0 & 1 \\ z & 1 & 0 \end{vmatrix} + (1 - \lambda)$$

$\Rightarrow \Delta + 1 = 10k + (2 - \lambda)$,

where $k = \begin{vmatrix} x & 4 & 1 \\ y & 0 & 1 \\ z & 1 & 0 \end{vmatrix}$.

It is given that $(\Delta + 1)$ is divisible by 10. Therefore, $2 - \lambda = 0$ i.e., $\lambda = 2$

$\therefore \quad X = 10x + 2$

$\Rightarrow 2$ is at unit's place of X.

● ***Ex. 5*** *The number of distinct values of a 2×2 determinant whose entries are from the set $\{-1, 0, 1\}$, is*

(a) 3 (b) 4 (c) 5 (d) 6

Sol. (*c*) Possible values are $-2, -1, 0, 1, 2$

i.e., $\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1, \begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} = 0, \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1,$

$\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2, \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2$

● ***Ex. 6*** *If* $f(x) = \begin{vmatrix} (1+x)^a & (1+2x)^b & 1 \\ 1 & (1+x)^a & (1+2x)^b \\ (1+2x)^b & 1 & (1+x)^a \end{vmatrix}; a, b$

being positive integers, then

(a) constant term in $f(x)$ is 4

(b) coefficient of x in $f(x)$ is 0

(c) constant term in $f(x)$ is $(a - b)$

(d) constant term in $f(x)$ is $(a + b)$

Sol. (*b*) Let $\begin{vmatrix} (1+x)^a & (1+2x)^b & 1 \\ 1 & (1+x)^a & (1+2x)^b \\ (1+2x)^b & 1 & (1+x)^a \end{vmatrix}$

$$= A + Bx + Cx^2 + \ldots$$

Put $x = 0$, then $A = 0$

On differentiating both sides w.r.t. x and then put $x = 0$

$$\begin{vmatrix} a & 2b & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 0 & a & 2b \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2b & 0 & a \end{vmatrix} = B$$

$\therefore \quad 0 + 0 + 0 = B$

$\Rightarrow \quad B = 0$

Hence, constant term in $f(x)$ is zero and coefficient of x in $f(x)$ is 0.

● ***Ex. 7*** *If* $f_j = \sum_{i=0}^{2} a_{ij} x^i$, $j = 1, 2, 3$ *and* f_j' *and* f_j'' *are denoted by* $\frac{df_j}{dx}$ *and* $\frac{d^2 f_j}{dx^2}$ *respectively, then* $g(x) = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}$ *is*

(a) a constant (b) a linear in x

(c) a quadratic in x (d) a cubic in x

Sol. (*a*) $\because g'(x) = \begin{vmatrix} f_1' & f_2' & f_3' \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix} + \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1'' & f_2'' & f_3'' \\ f_1'' & f_2'' & f_3'' \end{vmatrix} + \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1''' & f_2''' & f_3''' \end{vmatrix}$

$= 0 + 0 + 0$ [$\because f_j$ is a quadratic function]

$\therefore \quad g(x) = c =$ constant

● ***Ex. 8*** *Let* $\Delta_a = \begin{vmatrix} (a-1) & n & 6 \\ (a-1)^2 & 2n^2 & 4n-2 \\ (a-1)^3 & 3n^3 & 3n^2 - 3n \end{vmatrix}$,

the value of $\sum_{a=1}^{n} \Delta_a$ *is*

(a) 0 (b) $\frac{(n-1)n}{2}$

(c) $\frac{(n-1)n^2}{2}$ (d) $\frac{(n-1)n(2n-1)}{3}$

Sol. (a) $\sum_{a=1}^{n}\Delta_a = \begin{vmatrix} \sum_{a=1}^{n}(a-1) & n & 6 \\ \sum_{a=1}^{n}(a-1)^2 & 2n^2 & 4n-2 \\ \sum_{a=1}^{n}(a-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$

$$= \begin{vmatrix} \frac{(n-1)n}{2} & n & 6 \\ \frac{(n-1)n(2n-1)}{6} & 2n^2 & 4n-2 \\ \frac{(n-1)^2n^2}{4} & 3n^3 & 3n^2-3n \end{vmatrix}$$

$$= \frac{(n-1)n^2}{2}\begin{vmatrix} 1 & 1 & 6 \\ \frac{2n-1}{3} & 2n & 4n-2 \\ \frac{(n-1)n}{2} & 3n^2 & 3n^2-3n \end{vmatrix}$$

[taking $\frac{(n-1)n}{2}$ and n common from C_1 and C_2]

Applying $C_3 \to C_3 - 6C_1$, then

$$\sum_{a=1}^{n}\Delta_a = \frac{(n-1)n^2}{2}\begin{vmatrix} 1 & 1 & 0 \\ \frac{2n-1}{3} & 2n & 0 \\ \frac{(n-1)n}{2} & 3n^2 & 0 \end{vmatrix} = 0$$

● ***Ex. 9*** *If* $\Delta(x) = \begin{vmatrix} 1 & \cos x & 1-\cos x \\ 1+\sin x & \cos x & 1+\sin x-\cos x \\ \sin x & \sin x & 1 \end{vmatrix}$, *then* $\int_0^{\pi/2}\Delta(x)\,dx$ *is equal to*

(a) $-\frac{1}{2}$ (b) 0 (c) $\frac{1}{4}$ (d) $\frac{1}{2}$

Sol. (a) Applying $C_3 \to C_3 + C_2 - C_1$, then

$$\Delta(x) = \begin{vmatrix} 1 & \cos x & 0 \\ 1+\sin x & \cos x & 0 \\ \sin x & \sin x & 1 \end{vmatrix}$$

$$= 1(\cos x - \cos x - \cos x \sin x) = -\frac{1}{2}\sin 2x$$

$$\therefore \int_0^{\pi/2}\Delta(x)dx = -\frac{1}{2}\int_0^{\pi/2}\sin 2x\,dx$$

$$= \frac{1}{4}[\cos 2x]_0^{\pi/2} = \frac{1}{4}(-1-1) = -\frac{1}{2}$$

● ***Ex. 10*** *Number of values of a for which the system of equations* $a^2x + (2-a)y = 4 + a^2$ *and* $ax + (2a-1)y = a^5 - 2$ *possess no solution, is*

(a) 0 (b) 1
(c) 2 (d) infinite

Sol. (c) $\because \Delta = \begin{vmatrix} a^2 & 2-a \\ a & 2a-1 \end{vmatrix} = a^2(2a-1) - a(2-a)$

$$= 2a(a+1)(a-1)$$

For no solution, $\Delta = 0$

$\therefore$ $a = -1, 0, 1$

$\Rightarrow$ $\Delta_1 = \begin{vmatrix} 4+a^2 & 2-a \\ a^5-2 & 2a-1 \end{vmatrix}$

Values of Δ_1 at $a = -1, 0, 1$ are $-6, 0, 6$ respectively and

$$\Delta_2 = \begin{vmatrix} a^2 & 4+a^2 \\ a & a^5-2 \end{vmatrix}$$

Values of Δ_2 at $a = -1, 0, 1$ are $2, 0, -6$, respectively.

For no solution,

$\Delta = 0$ and atleast one of Δ_1, Δ_2 is non-zero.

$\therefore$ $a = -1, 1$

JEE Type Solved Examples :
More than One Correct Option Type Questions

■ This section contains **5 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of **which more than one** may be correct.

● ***Ex. 11*** *The determinant* $\begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix}$ *is divisible by*

(a) $a+b+c$ (b) $(a+b)(b+c)(c+a)$
(c) $a^2+b^2+c^2$ (d) $(a-b)(b-c)(c-a)$

Sol. (a,c,d) Applying $C_2 \to C_2 - C_1 - 2C_3$, then

$$\begin{vmatrix} a^2 & -(b^2+c^2) & bc \\ b^2 & -(c^2+a^2) & ca \\ c^2 & -(a^2+b^2) & ab \end{vmatrix} = -\begin{vmatrix} a^2 & b^2+c^2 & bc \\ b^2 & c^2+a^2 & ca \\ c^2 & a^2+b^2 & ab \end{vmatrix}$$

Applying $C_2 \to C_2 + C_1$, then

$$= -\begin{vmatrix} a^2 & a^2+b^2+c^2 & bc \\ b^2 & a^2+b^2+c^2 & ca \\ c^2 & a^2+b^2+c^2 & ab \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$= -\begin{vmatrix} a^2 & \dots & a^2+b^2+c^2 & \dots & bc \\ & & \vdots & & \\ b^2-a^2 & & 0 & & c(a-b) \\ & & \vdots & & \\ c^2-a^2 & & 0 & & -b(c-a) \end{vmatrix}$$

$$= (a^2+b^2+c^2)\begin{vmatrix} -(a+b)(a-b) & c(a-b) \\ (c+a)(c-a) & -b(c-a) \end{vmatrix}$$

$$= (a-b)(c-a)(a^2+b^2+c^2)\begin{vmatrix} -(a+b) & c \\ c+a & -b \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$, then

$$= (a-b)(c-a)(a^2+b^2+c^2)\begin{vmatrix} -(a+b+c) & c \\ (a+b+c) & -b \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2)$$

● ***Ex. 12*** *The value of* θ *lying between* $-\frac{\pi}{4}$ *and* $\frac{\pi}{2}$ *and* $0 \le A \le \frac{\pi}{2}$ *and satisfying the equation*

$$\begin{vmatrix} 1+\sin^2 A & \cos^2 A & 2\sin 4\theta \\ \sin^2 A & 1+\cos^2 A & 2\sin 4\theta \\ \sin^2 A & \cos^2 A & 1+2\sin 4\theta \end{vmatrix} = 0, \text{ are}$$

(a) $A = \frac{\pi}{4}, \theta = -\frac{\pi}{8}$ (b) $A = \frac{3\pi}{8} = \theta$

(c) $A = \frac{\pi}{5}, \theta = -\frac{\pi}{8}$ (d) $A = \frac{\pi}{6}, \theta = \frac{3\pi}{8}$

Sol. (*a, b, c, d*)

$$\because \quad \begin{vmatrix} 1+\sin^2 A & \cos^2 A & 2\sin 4\theta \\ \sin^2 A & 1+\cos^2 A & 2\sin 4\theta \\ \sin^2 A & \cos^2 A & 1+2\sin 4\theta \end{vmatrix} = 0$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$\begin{vmatrix} 1+\sin^2 A & \cos^2 A & 2\sin 4\theta \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 0$$

Applying $C_1 \to C_1 + C_2$, then

$$\begin{vmatrix} 2 & \cos^2 A & 2\sin 4\theta \\ & \vdots & \\ 0 \quad \dots & 1 \quad \dots & 0 \\ & \vdots & \\ -1 & 0 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad 1(2+2\sin 4\theta) = 0$$

$$\therefore \quad \sin 4\theta = -1$$

$$\Rightarrow \quad 4\theta = (2n-1)\frac{\pi}{2} \Rightarrow \theta = (2n-1)\frac{\pi}{8}$$

For $n = 0, 2$, then $\theta = -\frac{\pi}{8}, \frac{3\pi}{8}$ and $A \in R$

● ***Ex. 13*** *The digits A, B, C are such that the three digit numbers A*88, 6*B*8, 86*C are divisible by* 72, *the determinant* $\begin{vmatrix} A & 6 & 8 \\ 8 & B & 6 \\ 8 & 8 & C \end{vmatrix}$ *is divisible by*

(a) 72 (b) 144 (c) 288 (d) 216

Sol. (*a, b, c*)

$\because$ $A88, 6B8, 86C$ are divisible by 72.

$\therefore$ $A88, 6B8, 86C$ are also divisible by 9.

$\Rightarrow \quad A+8+8, 6+B+8, 8+6+C$

are divisible by 9, then $A = 2, B = 4, C = 4$

Let $\Delta = \begin{vmatrix} A & 6 & 8 \\ 8 & B & 6 \\ 8 & 8 & C \end{vmatrix} = \begin{vmatrix} 2 & 6 & 8 \\ 8 & 4 & 6 \\ 8 & 8 & 4 \end{vmatrix}$

$= 2(16-48) - 6(32-48) + 8(64-32) = 288$

Hence, Δ is divisible by 72, 144 and 288.

● ***Ex. 14*** *If p, q, r* and *s are in AP and*

$$f(x) = \begin{vmatrix} p+\sin x & q+\sin x & p-r+\sin x \\ q+\sin x & r+\sin x & -1+\sin x \\ r+\sin x & s+\sin x & s-q+\sin x \end{vmatrix}$$

such that $\int_0^1 f(x)\,dx = -2$, *the common difference of the AP can be*

(a) -1 (b) 1/2 (c) 1 (d) 2

Sol. (a, c)

$$\because f(x) = \frac{1}{2}\begin{vmatrix} p+\sin x & q+\sin x & p-r+\sin x \\ 2q+2\sin x & 2r+2\sin x & -2+2\sin x \\ r+\sin x & s+\sin x & s-q+\sin x \end{vmatrix}$$

Applying $R_2 \to R_2 - (R_1 + R_3)$, then

$$f(x) = \frac{1}{2}\begin{vmatrix} p+\sin x & q+\sin x & p-r+\sin x \\ & & \vdots \\ 0 \quad \dots & 0 \quad \dots & -2 \\ & & \vdots \\ r+\sin x & s+\sin x & s-q+\sin x \end{vmatrix}$$

$[\because 2q = p+r, 2r = q+s \text{ and } p+s = q+r]$

$$= -\frac{(-2)}{2}\begin{vmatrix} p+\sin x & q+\sin x \\ r+\sin x & s+\sin x \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, then

$$f(x) = \begin{vmatrix} p + \sin x & D \\ p + 2D + \sin x & D \end{vmatrix}$$

[where D = common difference]

$$= D[p + \sin x - p - 2D - \sin x] = -2D^2$$

and $$\int_0^1 f(x)dx = -4$$

$$\Rightarrow \quad \int_0^1 (-2D^2)dx = -4 \Rightarrow -2D^2 = -2$$

$$\therefore \quad D^2 = 1 \Rightarrow D = \pm 1$$

● ***Ex. 15*** *If the system of equations $a^2x - by = a^2 - b$ and $bx - b^2y = 2 + 4b$ possess an infinite number of solutions, the possible values of a and b are*

(a) $a = 1, b = -1$ (b) $a = 1, b = -2$

(c) $a = -1, b = -1$ (d) $a = -1, b = -2$

Sol. (a, b, c, d)

Here, $\Delta = \begin{vmatrix} a^2 & -b \\ b & -b^2 \end{vmatrix} = -a^2b^2 + b^2 = -(a^2 - 1)b^2$

If $\Delta = 0$, then $a^2 = 1, b = 0$

Now, $$\Delta_1 = \begin{vmatrix} a^2 - b & -b \\ 2 + 4b & -b^2 \end{vmatrix}$$

For $a^2 = 1, \Delta_1 = \begin{vmatrix} 1 - b & -b \\ 2 + 4b & -b^2 \end{vmatrix} = b(b+1)(b+2)$

and $$\Delta_2 = \begin{vmatrix} a^2 & a^2 - b \\ b & 2 + 4b \end{vmatrix}$$

For $a^2 = 1, \Delta_2 = \begin{vmatrix} 1 & 1 - b \\ b & 2 + 4b \end{vmatrix} = (b+1)(b+2)$

For infinite number of solutions, $\Delta = \Delta_1 = \Delta_2 = 0$

$\therefore \; a^2 = 1, b = -1, -2 \Rightarrow a = \pm 1, b = -1, b = -2$

JEE Type Solved Examples : Passage Based Questions

■ This section contains **2 solved passages** based upon each of the passage **3 multiple choice** examples have to be answered. Each of these examples has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I

(Ex. Nos. 16 to 18)

Consider the system of equations

$$2x + ay + 6z = 8; \quad x + 2y + bz = 5$$

$$x + y + 3z = 4$$

16. The system has unique solution, if

(a) $a = 2, b = 3$ (b) $a = 2, b \neq 3$

(c) $a \neq 2, b = 3$ (d) $a \neq 2, b \neq 3$

17. The system has infinite solutions, if

(a) $a = 2, b \in R$ (b) $a = 3, b \in R$

(c) $a \in R, b = 2$ (d) $a \in R, b = 3$

18. The system has no solution, if

(a) $a = 2, b = 3$ (b) $a = 2, b \neq 3$

(c) $a \neq 2, b = 3$ (d) $a \neq 2, b \neq 3$

Sol. $\Delta = \begin{vmatrix} 2 & a & 6 \\ 1 & 2 & b \\ 1 & 1 & 3 \end{vmatrix} = 2(6 - b) - a(3 - b) + 6(1 - 2)$

$$= ab - 3a - 2b + 6 = (a - 2)(b - 3)$$

$$\Delta_1 = \begin{vmatrix} 8 & a & 6 \\ 5 & 2 & b \\ 4 & 1 & 3 \end{vmatrix} = 8(6 - b) - a(15 - 4b) + 6(5 - 8)$$

$$= 4ab - 15a - 8b + 30 = (a - 2)(4b - 15)$$

$$\Delta_2 = \begin{vmatrix} 2 & 8 & 6 \\ 1 & 5 & b \\ 1 & 4 & 3 \end{vmatrix} = 0 \qquad [\because R_1 = 2R_3]$$

$$\Delta_3 = \begin{vmatrix} 2 & a & 8 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{vmatrix} = 2(8 - 5) - a(4 - 5) + 8(1 - 2)$$

$$= 6 + a - 8 = a - 2$$

16. (*d*) The system has unique solution, if

$$\Delta \neq 0$$

$$\Rightarrow \quad (a - 2)(b - 3) \neq 0$$

$$\Rightarrow \quad a \neq 2, b \neq 3$$

17. (*a*) The system has infinite solution, if

$$\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$$

$$\Rightarrow \quad a - 2 = 0$$

or $$a = 2, b \in R$$

18. (*c*) The system has no solution, if

$\Delta = 0$ and atleast one of Δ_1, Δ_2 and Δ_3 is non-zero.

$$\Rightarrow \quad a \neq 2, b = 3$$

Passage II

(Ex. Nos. 19 to 20)

Let ${}^{x}C_i$, ${}^{x^2}C_i$ *and* ${}^{x^3}C_i\ (i=1,2,3)$ *be Binomial coefficients, where* $x \in N$

and $f(x)=12\begin{vmatrix} {}^{x}C_1 & {}^{x}C_2 & {}^{x}C_3 \\ {}^{x^2}C_1 & {}^{x^2}C_2 & {}^{x^2}C_3 \\ {}^{x^3}C_1 & {}^{x^3}C_2 & {}^{x^3}C_3 \end{vmatrix}$, *then*

19. $f(x)$ is a polynomial of degree

(a) 6 (b) 10

(c) 14 (d) 18

20. If $f(x)=(x-1)^m x^n (x+1)^p$, where $m, n, p \in N$, then the value of $\sum mn$ is

(a) 32 (b) 43

(c) 44 (d) 56

Sol.

$$\because f(x)=12\begin{vmatrix} {}^{x}C_1 & {}^{x}C_2 & {}^{x}C_3 \\ {}^{x^2}C_1 & {}^{x^2}C_2 & {}^{x^2}C_3 \\ {}^{x^3}C_1 & {}^{x^3}C_2 & {}^{x^3}C_3 \end{vmatrix}$$

$$=12\begin{vmatrix} x & \frac{x(x-1)}{2} & \frac{x(x-1)(x-2)}{6} \\ x^2 & \frac{x^2(x^2-1)}{2} & \frac{x^2(x^2-1)(x^2-2)}{6} \\ x^3 & \frac{x^3(x^3-1)}{2} & \frac{x^3(x^3-1)(x^3-2)}{6} \end{vmatrix}$$

Taking x, x^2 and x^3 common from R_1, R_2 and R_3, then

$$f(x)=x\cdot x^2\cdot x^3\begin{vmatrix} 1 & (x-1) & (x-1)(x-2) \\ 1 & (x^2-1) & (x^2-1)(x^2-2) \\ 1 & (x^3-1) & (x^3-1)(x^3-2) \end{vmatrix}$$

$$=x^6(x-1)^2\begin{vmatrix} 1 & 1 & x-2 \\ 1 & x+1 & (x+1)(x^2-2) \\ 1 & x^2+x+1 & (x^2+x+1)(x^3-2) \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$ and $R_3 \to R_3 - R_2$, then

$$f(x)=x^6(x-1)^2\begin{vmatrix} 0 & -x & x(3-x-x^2) \\ \vdots & & \\ 1 & \cdots\ x+1\ \cdots & (x+1)(x^2-2) \\ \vdots & & \\ 0 & x^2 & x^2(x^2+x^3-3) \end{vmatrix}$$

Expanding along C_1, then

$$=-x^6(x-1)^2\begin{vmatrix} -x & x(3-x-x^2) \\ x^2 & x^2(x^2+x^3-3) \end{vmatrix}$$

$$=-x^9(x-1)^2\begin{vmatrix} -1 & 3-x-x^2 \\ 1 & x^2+x^3-3 \end{vmatrix}$$

$$=-x^9(x-1)^2(-x^2-x^3+3-3+x+x^2)$$

$$=x^{10}(x-1)^2(x^2-1)=x^{10}(x-1)^3(x+1)$$

19. (*c*) $f(x)$ is a polynomial of degree 14.

20. (*b*) Here, $m=3, n=10$ and $p=1$

$\therefore \sum mn = mn + np + pm = 30 + 10 + 3 = 43$

JEE Type Solved Examples :
Single Integer Answer Type Questions

- This section contains **2 examples.** The answer to each example is a **single digit integer** ranging from **0** to **9** (both inclusive).

● ***Ex. 21*** *If* $\Delta_r = \begin{vmatrix} r & r-1 \\ r-1 & r \end{vmatrix}$, *where* r *is a natural number, the value of* $\sqrt[10]{\sum_{r=1}^{1024}\Delta_r}$ *is*

Sol. (4) $\because$ $\Delta_r = r^2-(r-1)^2$

$$\therefore \quad \sum_{r=1}^{1024}\Delta_r=(1024)^2-(1-1)^2=(1024)^2=2^{20}$$

$$\Rightarrow \quad \sqrt[10]{\sum_{r=1}^{1024}\Delta_r}=2^2=4$$

● ***Ex. 22*** *If* P, Q *and* R *are the angles of a triangle, the value of* $\begin{vmatrix} \tan P & 1 & 1 \\ 1 & \tan Q & 1 \\ 1 & 1 & \tan R \end{vmatrix}$ *is*

Sol. (2) $\begin{vmatrix} \tan P & 1 & 1 \\ 1 & \tan Q & 1 \\ 1 & 1 & \tan R \end{vmatrix} = \tan P(\tan Q \tan R - 1)$

$$-1(\tan R - 1) + 1(1 - \tan Q)$$

$$= \tan P \tan Q \tan R - (\tan P + \tan Q + \tan R) + 2$$

$$= 0 + 2$$

$[\because$ In ΔPQR, $\tan P + \tan Q + \tan R = \tan P \tan Q \tan R]$

$$= 2$$

JEE Type Solved Examples : Matching Type Questions

■ This section contains 2 eamples. Example 23 have four statements (A, B, C and D) given in **Column I** and four statement (p, q, r and s) in **Column II** and example 24 have three statements (A, B and C) given in **Column I** and four statements (p, q, r and s) in **Column II**. Any given statement in **Column I** can have correct matching with one or more statements(s) given in **Column II.**

● ***Ex. 23*** *Let $f(x)$ denotes the determinant*

$$f(x)=\begin{vmatrix} x^2 & 2x & 1+x^2 \\ x^2+1 & x+1 & 1 \\ x & -1 & x-1 \end{vmatrix}.$$

On expansion $f(x)$ is seen to be a 4th *degree polynomial given by $f(x)=a_0x^4+a_1x^3+a_2x^2+a_3x+a_4$.*
Using differentiation of determinant or otherwise match the entries in Column I *with one or more entries of the elements of Column* II.

Column I		Column II	
(A)	$a_0^2+a_1$ is divisible by	(p)	2
(B)	$a_2^2+a_4$ is divisible by	(q)	3
(C)	$a_0^2+a_2$ is divisible by	(r)	4
(D)	$a_4^2+a_3^2+a_1^2$ is divisible by	(s)	5

Sol. (A) → (p, s); (B) → (p, r); (C) → (p, q); (D) → (q)

$\because$ $$f(x)=\begin{vmatrix} x^2 & 2x & 1+x^2 \\ x^2+1 & x+1 & 1 \\ x & -1 & x-1 \end{vmatrix}$$

Applying $C_1 \to C_1 - C_3$, then

$$f(x)=\begin{vmatrix} -1 & 2x & 1+x^2 \\ x^2 & x+1 & 1 \\ 1 & -1 & x-1 \end{vmatrix}$$

Expanding along R_1, then

$$f(x)=-(x^2-1+1)-2x(x^3-x^2-1)+(1+x^2)(-x^2-x-1)$$

$$=-3x^4+x^3-3x^2+x-1 \quad \ldots(i)$$

According to the question, we get

$$f(x)=a_0x^4+a_1x^3+a_2x^2+a_3x+a_4 \quad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$a_0=-3, a_1=1, a_2=-3, a_3=1, a_4=-1$$

(A) $a_0^2+a_1=(-3)^2+1=9+1=10=2\times5$

(B) $a_2^2+a_4=(-3)^2-1=9-1=8=2\times4$

(C) $a_0^2+a_2=(-3)^2-3=9-3=6=2\times3$

(D) $a_4^2+a_3^2+a_1^2=(-1)^2+(1)^2+(1)^2=1+1+1=3$

● ***Ex. 24*** *Suppose a, b and c are distinct and x, y and z are connected by the system of equations $x+ay+a^2z=a^3$, $x+by+b^2z=b^3$ and $x+cy+c^2z=c^3$.*

Column I		Column II	
(A)	For $x=1, y=2$ and $z=3$, $(a+b+c)^{-(ab+bc+ca)}$ is divisible by	(p)	3
(B)	For $x=4, y=3$ and $z=2$, $(ab+bc+ca)^{abc}$ is divisible by	(q)	6
(C)	For $x=6, y=4$ and $z=2$, $(abc)^{a+b+c}$ is divisible by	(r)	9
		(s)	12

Sol. (A) → (p, r) (B) → (p, r); (C) → (p, q, r, s)

$a\neq b\neq c$

$$\Delta=\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}=(a-b)(b-c)(c-a)$$

$$\Delta_1=\begin{vmatrix} a^3 & a & a^2 \\ b^3 & b & b^2 \\ c^3 & c & c^2 \end{vmatrix}=\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}=abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$=abc(a-b)(b-c)(c-a)$$

$$\Delta_2=\begin{vmatrix} 1 & a^3 & a^2 \\ 1 & b^3 & b^2 \\ 1 & c^3 & c^2 \end{vmatrix}=-\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

$$=-(a-b)(b-c)(c-a)(ab+bc+ca)$$

and $$\Delta_3=\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}=(a-b)(b-c)(c-a)(a+b+c)$$

By Cramer's rule, we get

$$x=\frac{\Delta_1}{\Delta}=abc$$

$$y=\frac{\Delta^2}{\Delta}=-(ab+bc+ca),\ z=\frac{\Delta_3}{\Delta}=a+b+c$$

(A) $(a+b+c)^{-(ab+bc+ca)}=z^y=3^2=9$, which is divisible by 3 and 9.

(B) $(ab+bc+ca)^{abc}=(-y)^x=(-3)^4=81$, which is divisible by 3 and 9.

(C) $(abc)^{a+b+c}=x^3=6^2=36$, which is divisible by 3, 6, 9 and 12.

JEE Type Solved Examples :
Statement I and II Type Questions

■ **Directions** Example numbers 25 and 26 are Assertion-Reason type examples. Each of these examples contains two statements:
Statement-1 (Assertion) and **Statement-2** (Reason)
Each of these examples also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below.
(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement-1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

● ***Ex. 25*** **Statement-1** *Let*

$$\Delta_r = \begin{vmatrix} (r-1) & n! & 6 \\ (r-1)^2 & (n!)^2 & 4n-2 \\ (r-1)^3 & (n!)^3 & 3n^2-2n \end{vmatrix}, \text{ then } \prod_{r=2}^{n+1} \Delta_r = 0.$$

Statement-2 $\prod_{r=2}^{n+1} \Delta_r = \Delta_2 \cdot \Delta_3 \cdot \Delta_4 \ldots \Delta_{n+1}$

Sol. (*d*) $\because \prod_{r=2}^{n+1} \Delta_r = \Delta_2 \cdot \Delta_3 \cdot \Delta_4 \ldots \Delta_{n+1}$

$$= \begin{vmatrix} 1 & n! & 6 \\ 1 & (n!)^2 & 4n-2 \\ 1 & (n!)^3 & 3n^3-2n \end{vmatrix} \times \begin{vmatrix} 2 & n! & 6 \\ 4 & (n!)^2 & 4n-2 \\ 8 & (n!)^3 & 3n^2-2n \end{vmatrix}$$

$$\times \ldots \times \begin{vmatrix} n & n! & 6 \\ n^2 & (n!)^2 & 4n-2 \\ n^3 & (n!)^3 & (3n^2-2n) \end{vmatrix} \neq 0$$

$\therefore$ Statement-1 is false and Statement-2 is true.

● ***Ex. 26*** *Consider the determinant*

$$f(x) = \begin{vmatrix} 0 & x^2-a & x^3-b \\ x^2+a & 0 & x^2+c \\ x^4+b & x-c & 0 \end{vmatrix}.$$

Statement-1 $f(x) = 0$ *has one root* $x = 0$.

Statement-2 *The value of skew-symmetric determinant of odd order is always zero.*

Sol. (a) For $x = 0$, the determinant reduces to the determinant of a skew-symmetric of odd order which is always zero. Hence, $x = 0$ is the solution of given equation $f(x) = 0$.

Subjective Type Examples

■ In this section, there are **20 subjective** solved examples.

● ***Ex. 27*** *A determinant of second order is made with the elements* 0 *and* 1. *Find the number of determinants with non-negative values.*

Sol. The number of determinants that can be made with 0 and 1

$$= 2 \times 2 \times 2 \times 2 = 16$$

and there are only three determinants of second order with negative values

i.e., $\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$

Therefore, number of determinants with non-negative values $= 16 - 3 = 13$

● ***Ex. 28*** *Prove that* $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$

$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$, *hence find the value of the determinant, if* a, b *and* c *are the roots of the equation* $px^3 - qx^2 + rx - s = 0$.

Sol. Let $\Delta = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$

Since, the answer contain abc, then taking a, b and c common from R_1, R_2 and R_3 respectively, then

$$\Delta = abc \begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

But answer also contains $\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$,

then applying $R_1 \to R_1 + R_2 + R_3$

$\therefore$ $\Delta = abc$

$$\begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

Taking $\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ common from R_1, then

$$\Delta = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, then

$$\Delta = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 0 & 1 \\ \frac{1}{b} & 1 & \frac{1}{b} \\ \frac{1}{c} & 0 & \frac{1}{c}+1 \end{vmatrix}$$

Expanding along C_2, then

$$\Delta = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 1 \\ \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

Hence, $\Delta = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

2nd Part $\Delta = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = abc + bc + ca + ab$

$$= \frac{s}{p} + \frac{r}{p} = \left(\frac{s+r}{p}\right)$$

● ***Ex. 29*** *If a, b and c are positive and are the pth, qth and rth terms, respectively of a GP. Show without expanding that*

$$\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = 0.$$

Sol. Let A be the first term and R be the common ratio of GP, then

$$a = p\text{th term} = AR^{p-1}$$
$$b = q\text{th term} = AR^{q-1}$$
$$c = r\text{th term} = AR^{r-1}$$

$\therefore$ $\log a = \log A + (p-1)\log R$,
$\log b = \log A + (q-1)\log R$ and
$\log c = \log A + (r-1)\log R$

$$\therefore \quad \text{LHS} = \begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \log A + (p-1)\log R & p & 1 \\ \log A + (q-1)\log R & q & 1 \\ \log A + (r-1)\log R & r & 1 \end{vmatrix}$$

Applying $C_1 \to C_1 - (\log A)C_3$, then

$$= \begin{vmatrix} (p-1)\log R & p & 1 \\ (q-1)\log R & q & 1 \\ (r-1)\log R & r & 1 \end{vmatrix} = \log R \begin{vmatrix} (p-1) & p & 1 \\ (q-1) & q & 1 \\ (r-1) & r & 1 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_3$, then

$$= \log R \begin{vmatrix} p & p & 1 \\ q & q & 1 \\ r & r & 1 \end{vmatrix} = 0 \text{ [since } C_1 \text{ and } C_2 \text{ are identical]}$$

$$= \text{RHS}$$

● ***Ex. 30*** *Prove that*

$$\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = 4(b+c)(c+a)(a+b).$$

Sol. Let $\Delta = \begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix}$

On putting $a+b=0$, $b=-a$

Then, $\Delta = \begin{vmatrix} -2a & 0 & a+c \\ 0 & 2a & c-a \\ c+a & c-a & -2c \end{vmatrix}$

Expanding along R_1, then

$\Delta = -2a\{-4ac-(c-a)^2\} - 0 + (a+c)\{0-2a(c+a)\}$

$= 2a(c+a)^2 - 2a(c+a)^2 = 0$

Hence, $(a+b)$ is a factor of Δ, similarly $(b+c)$ and $(c+a)$ are the factors of Δ.

On expansion of determinant we can see that each term of the determinant is a homogeneous expression in a, b and c of degree 3 and also RHS is a homogeneous expression of degree 3.

Let $\Delta = k(a+b)(b+c)(c+a)$

or $\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = k(a+b)(b+c)(c+a)$

On putting $a=0$, $b=1$ and $c=2$, we get

$$\begin{vmatrix} 0 & 1 & 2 \\ 1 & -2 & 3 \\ 2 & 3 & -4 \end{vmatrix} = k(0+1)(1+2)(2+0)$$

$\Rightarrow \quad 0-1(-4-6)+2(3+4) = 6k$

$\Rightarrow \quad 24 = 6k$

$\therefore \quad k = 4$

Hence, $\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = 4(a+b)(b+c)(c+a)$

● ***Ex. 31*** *If $bc+qr = ca+rp = ab+pq = -1$, show that* $\begin{vmatrix} ap & bp & cr \\ a & b & c \\ p & q & r \end{vmatrix} = 0.$

Sol. Given equations can be rewritten as

$$bc + qr + 1 = 0 \quad \ldots(i)$$
$$ca + rp + 1 = 0 \quad \ldots(ii)$$
$$ab + pq + 1 = 0 \quad \ldots(iii)$$

On multiplying Eqs. (i), (ii) and (iii) by ap, bq and cr respectively, we get

$$(abc)p + (pqr)a + ap = 0$$
$$(abc)q + (pqr)b + bq = 0$$
$$(abc)r + (pqr)c + cr = 0$$

These equations are consistent, given equations three but abc and pqr are two.

Hence, $$\begin{vmatrix} p & a & ap \\ q & b & bq \\ r & c & cr \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} p & q & r \\ a & b & c \\ ap & bq & cr \end{vmatrix} = 0$$

[interchanging rows into columns]

$$\Rightarrow \quad (-1)\begin{vmatrix} ap & bq & cr \\ a & b & c \\ p & q & r \end{vmatrix} = 0 \qquad [R_1 \leftrightarrow R_3]$$

Hence, $$\begin{vmatrix} ap & bq & cr \\ a & b & c \\ p & q & r \end{vmatrix} = 0$$

● ***Ex. 32*** *If α and β are the roots of the equations $x^2 - 2x + 4 = 0$, find the value of* $\begin{vmatrix} \sum\alpha & \sum\alpha^2 & \sum\alpha^3 \\ \sum\alpha^2 & \sum\alpha^3 & \sum\alpha^4 \\ \sum\alpha^3 & \sum\alpha^4 & \sum\alpha^5 \end{vmatrix}$.

Sol. Given, $x^2 - 2x + 4 = 0$

$$\therefore \quad x = 1 \pm i\sqrt{3}$$
$$\therefore \quad \alpha = 1 + i\sqrt{3}$$
and $$\beta = 1 - i\sqrt{3}$$

$$\Rightarrow \quad \alpha = -2\left(\frac{-1 - i\sqrt{3}}{2}\right) \text{ and } \beta = -2\left(\frac{-1 + i\sqrt{3}}{2}\right)$$

$\alpha = -2\omega^2$ and $\beta = -2\omega$, where ω is the cube root of unity.

$$\sum\alpha = \alpha + \beta = -2(\omega + \omega)^2 = -2(-1) = 2$$
$$\sum\alpha^2 = \alpha^2 + \beta^2 = 4\omega^4 + 4\omega^2 = 4(\omega + \omega)^2 = 4(-1) = -4$$
$$\sum\alpha^3 = \alpha^3 + \beta^3 = -8\omega^6 - 8\omega^3 = -8 - 8 = -16$$
$$\sum\alpha^4 = \alpha^4 + \beta^4 = 16\omega^8 + 16\omega^4 = 16(\omega^2 + \omega)$$
$$= 16(-1) = -16$$

and $$\sum\alpha^5 = \alpha^5 + \beta^5 = -32\omega^{10} - 32\omega^5$$
$$= -32(\omega + \omega^2) = -32(-1) = 32$$

Let $$\Delta = \begin{vmatrix} \sum\alpha & \sum\alpha^2 & \sum\alpha^3 \\ \sum\alpha^2 & \sum\alpha^3 & \sum\alpha^4 \\ \sum\alpha^3 & \sum\alpha^4 & \sum\alpha^5 \end{vmatrix} = \begin{vmatrix} 2 & -4 & -16 \\ -4 & -16 & -16 \\ -16 & -16 & 32 \end{vmatrix}$$

$$= 2(-4)(-16)\begin{vmatrix} 1 & -2 & -8 \\ 1 & 4 & 4 \\ 1 & 1 & -2 \end{vmatrix} = 128\begin{vmatrix} 1 & -2 & -8 \\ 1 & 4 & 4 \\ 1 & 1 & -2 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$\Delta = 128\begin{vmatrix} 1 & -2 & -8 \\ 0 & 6 & 12 \\ 0 & 3 & 6 \end{vmatrix}$$

Expanding along C_1, we get

$$\Delta = 128 \cdot 1 \cdot \begin{vmatrix} 6 & 12 \\ 3 & 6 \end{vmatrix} = 128(36 - 36) = 0$$

● ***Ex. 33*** *If $a^2 + b^2 + c^2 = 1$, prove that*

$$\begin{vmatrix} a^2 + (b^2 + c^2)\cos\phi & ab(1 - \cos\phi) & ac(1 - \cos\phi) \\ ba(1 - \cos\phi) & b^2 + (c^2 + a^2)\cos\phi & bc(1 - \cos\phi) \\ ca(1 - \cos\phi) & cb(1 - \cos\phi) & c^2 + (a^2 + b^2)\cos\phi \end{vmatrix}$$

is independent of a, b and c.

Sol. Let

$$\Delta = \begin{vmatrix} a^2 + (b^2 + c^2)\cos\phi & ab(1 - \cos\phi) & ac(1 - \cos\phi) \\ ba(1 - \cos\phi) & b^2 + (c^2 + a^2)\cos\phi & bc(1 - \cos\phi) \\ ca(1 - \cos\phi) & cb(1 - \cos\phi) & c^2 + (a^2 + b^2)\cos\phi \end{vmatrix}$$

On multiplying C_1, C_2 and C_3 by a, b and c respectively and taking a, b and c common from R_1, R_2 and R_3 respectively, we get

$$\Delta = \frac{abc}{abc}\begin{vmatrix} a^2 + (b^2 + c^2)\cos\phi & b^2(1 - \cos\phi) & c^2(1 - \cos\phi) \\ a^2(1 - \cos\phi) & b^2 + (c^2 + a^2)\cos\phi & c^2(1 - \cos\phi) \\ a^2(1 - \cos\phi) & b^2(1 - \cos\phi) & c^2 + (a^2 + b^2)\cos\phi \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, then

$$\Delta = \begin{vmatrix} a^2+b^2+c^2 & b^2 1-\cos\phi) & c^2(1-\cos\phi) \\ a^2+b^2+c^2 & b^2+(c^2+a^2)\cos\phi & c^2(1-\cos\phi) \\ a^2+b^2+c^2 & b^2(1-\cos\phi) & c^2+(a^2+b^2)\cos\phi \end{vmatrix}$$

Taking $a^2+b^2+c^2$ common from C_1, then

$$\Delta = (a^2+b^2+c^2)\begin{vmatrix} 1 & b^2(1-\cos\phi) & c^2(1-\cos\phi) \\ 1 & b^2+(c^2+a^2)\cos\phi & c^2(1-\cos\phi) \\ 1 & b^2(1-\cos\phi) & c^2+(a^2+b^2)\cos\phi \end{vmatrix}$$

Applying $R_1 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$\Delta = 1\begin{vmatrix} 1 & b^2(1-\cos\phi) & c^2(1-\cos\phi) \\ 0 & (a^2+b^2+c^2)\cos\phi & 0 \\ 0 & 0 & (a^2+b^2+c^2)\cos\phi \end{vmatrix}$$

$$= (a^2+b^2+c^2)^2\cos^2\phi$$

[by property, since all elements zero below leading diagonal]

$$= 1^2 \cdot \cos\phi = \cos^2\phi \qquad [\because a^2+b^2+c^2 = 1]$$

which is independent of a, b and c.

● ***Ex. 34*** *If $a \neq 0$ and $a \neq 1$, show that*

$$\begin{vmatrix} x+1 & x & x \\ x & x+a & x \\ x & x & x+a^2 \end{vmatrix} = a^3\left[1+\frac{x(a^3-1)}{a^2(a-1)}\right].$$

Sol. Let

$$\text{LHS} = \Delta = \begin{vmatrix} x+1 & x & x \\ x & x+a & x \\ x & x & x+a^2 \end{vmatrix} = \begin{vmatrix} x+1 & x & x \\ x+0 & x+a & x \\ x+0 & x & x+a^2 \end{vmatrix}$$

$$= \begin{vmatrix} x & x & x \\ x & x+a & x \\ x & x & x+a^2 \end{vmatrix} + \begin{vmatrix} 1 & x & x \\ 0 & x+a & x \\ 0 & x & x+a^2 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ in first determinant, then

$$\Delta = \begin{vmatrix} x & x & x \\ 0 & a & 0 \\ 0 & 0 & a^2 \end{vmatrix} + \begin{vmatrix} 1 & x & x \\ 0 & x+a & x \\ 0 & x & x+a^2 \end{vmatrix}$$

Expanding first determinant by property, since all elements below leading diagonal are zero and expanding second determinant along C_1, then

$$\Delta = x \cdot a \cdot a^2 + 1 \cdot \begin{vmatrix} x+a & x \\ x & x+a^2 \end{vmatrix}$$

$$= xa^3 + \{(x+a)(x+a^2) - x^2\}$$

$$= xa^3 + (x^2 + a^2x + ax + a^3 - x^2)$$

$$= xa^3 + a^2x + ax + a^3 = a^3 + x(a^3 + a^2 + a)$$

$$= a^3 + \frac{x \cdot a(a^3-1)}{(a-1)} = a^3\left[1 + \frac{x(a^3-1)}{a^2(a-1)}\right] = \text{RHS}$$

● ***Ex. 35*** (i) *Prove that* $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$

$$= \begin{vmatrix} \alpha^2 & \beta^2 & \beta^2 \\ \beta^2 & \alpha^2 & \beta^2 \\ \beta^2 & \beta^2 & \alpha^2 \end{vmatrix},$$

where $\alpha^2 = a^2+b^2+c^2$ *and* $\beta^2 = ab+bc+ca$.

(ii) *Prove that* $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$ *is divisible by* $(a+b+c)^2$. *Find the quotient.*

(iii) *Prove that* $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$

$$= \begin{vmatrix} a^2 & c^2 & 2ac-b^2 \\ 2ab-c^2 & b^2 & a^2 \\ b^2 & 2bc-a^2 & c^2 \end{vmatrix}.$$

(iv) *Prove that* $\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$

$$= (a^3+b^3+c^3-3abc)^2.$$

Sol. (i) Let $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$\therefore$ Determinant of cofactors of Δ is

$$\Delta^c = \begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix} = \Delta^{3-1} = \Delta^2$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 \qquad \text{...(i)}$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= \begin{vmatrix} a^2+b^2+c^2 & ab+bc+ca & ab+bc+ca \\ ab+bc+ca & a^2+b^2+c^2 & ab+bc+ca \\ ab+bc+ca & ab+bc+ca & a^2+b^2+c^2 \end{vmatrix} \quad \text{[row by row]}$$

$$= \begin{vmatrix} \alpha^2 & \beta^2 & \beta^2 \\ \beta^2 & \alpha^2 & \beta^2 \\ \beta^2 & \beta^2 & \alpha^2 \end{vmatrix}$$

Hence, $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix} = \begin{vmatrix} \alpha^2 & \beta^2 & \beta^2 \\ \beta^2 & \alpha^2 & \beta^2 \\ \beta^2 & \beta^2 & \alpha^2 \end{vmatrix}$

(ii) From Eq. (i), we get

$$\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$$

$$= (a^3+b^3+c^3-3abc)^2$$

$$= (a+b+c)^2(a^2+b^2+c^2-ab-bc-ca)^2$$

Therefore, $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$ is divisible by $(a+b+c)^2$.

Hence, the quotient is $(a^2+b^2+c^2-ab-bc-ca)^2$.

(iii) From Eq. (i), we get

$$\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$$

Let $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & -c & b \\ b & -a & c \\ c & -b & a \end{vmatrix}$$

$$= \begin{vmatrix} a^2 & c^2 & 2ac-b^2 \\ 2ab-c^2 & b^2 & a^2 \\ b^2 & 2bc-a^2 & c^2 \end{vmatrix} \quad \text{[row by row]}$$

Hence, $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$

$$= \begin{vmatrix} a^2 & c^2 & 2ac-b^2 \\ 2ab-c^2 & b^2 & a^2 \\ b^2 & 2bc-a^2 & c^2 \end{vmatrix}$$

(iv) LHS $= \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix} \text{ [row by row]}$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$$

$= (a+b+c)^2(a^2+b^2+c^2-ab-bc-ca)^2$ [from Eq. (ii)]

$= [-(a^3+b^3+c^3-3abc)]^2$

$= (a^3+b^3+c^3-3abc)^2 = \text{RHS}$

● ***Ex. 36*** *Let α and β be the roots of the equation $ax^2+bx+c=0$. Let $S_n = \alpha^n + \beta^n$ for $n \geq 1$. Evaluate the determinant* $\begin{vmatrix} 3 & 1+S_1 & 1+S_2 \\ 1+S_1 & 1+S_2 & 1+S_3 \\ 1+S_2 & 1+S_3 & 1+S_4 \end{vmatrix}$.

Sol. Since, α and β are the roots of the equation $ax^2+bx+c=0$.

$$\therefore \ \alpha+\beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a} \text{ and } \alpha-\beta = \frac{\sqrt{D}}{a}$$

Let $\Delta = \begin{vmatrix} 3 & 1+S_1 & 1+S_2 \\ 1+S_1 & 1+S_2 & 1+S_3 \\ 1+S_2 & 1+S_3 & 1+S_4 \end{vmatrix}$

$$= \begin{vmatrix} 3 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} = \Delta_1 \times \Delta_1 \quad \text{[say]}$$

$$\therefore \quad \Delta = \Delta_1^2 \quad \text{...(i)}$$

$$\therefore \quad \Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, then

$$\Delta_1 = \begin{vmatrix} 1 & 0 & 0 \\ 1 & \alpha-1 & \beta-1 \\ 1 & \alpha^2-1 & \beta^2-1 \end{vmatrix}$$

Expanding along R_1, then

$$\Delta_1 = \begin{vmatrix} \alpha - 1 & \beta - 1 \\ \alpha^2 - 1 & \beta^2 - 1 \end{vmatrix} = (\alpha - 1)(\beta - 1) \begin{vmatrix} 1 & 1 \\ \alpha + 1 & \beta + 1 \end{vmatrix}$$

$$= \{\alpha\beta - (\alpha + \beta) + 1\}(\beta - \alpha)$$

$$\therefore \quad \Delta = \Delta_1^2 = [\alpha\beta - (\alpha + \beta) + 1]^2 (\beta - \alpha)^2$$

$$= \left(\frac{c}{a} + \frac{b}{a} + 1\right)^2 \cdot \frac{D}{a^2} = \frac{(a + b + c)^2 (b^2 - 4ac)}{a^4}$$

● ***Ex. 37*** *If A, B and C are the angles of a triangle, show that*

(i) $\begin{vmatrix} \sin 2A & \sin C & \sin B \\ \sin C & \sin 2B & \sin A \\ \sin B & \sin A & \sin 2C \end{vmatrix} = 0.$

(ii) $\begin{vmatrix} -1 + \cos B & \cos C + \cos B & \cos B \\ \cos C + \cos A & -1 + \cos A & \cos A \\ -1 + \cos B & -1 + \cos A & -1 \end{vmatrix} = 0.$

Sol. (i) LHS $= \begin{vmatrix} \sin 2A & \sin C & \sin B \\ \sin C & \sin 2B & \sin A \\ \sin B & \sin A & \sin 2C \end{vmatrix}$

$$= \begin{vmatrix} 2ka\cos A & kc & kb \\ kc & 2kb\cos B & ka \\ kb & ka & 2kc\cos C \end{vmatrix} \text{[from sine rule]}$$

$$= k^3 \begin{vmatrix} 2a\cos A & c & b \\ c & 2b\cos B & a \\ b & a & 2c\cos C \end{vmatrix}$$

$$= k^3 \begin{vmatrix} a\cos A + a\cos A & a\cos B + b\cos A & a\cos C + c\cos A \\ a\cos B + b\cos A & b\cos B + b\cos B & b\cos C + c\cos B \\ a\cos C + c\cos A & c\cos B + b\cos C & c\cos C + c\cos C \end{vmatrix}$$

$$= k^3 \begin{vmatrix} \cos A & a & 0 \\ \cos B & b & 0 \\ \cos C & c & 0 \end{vmatrix} \times \begin{vmatrix} a & \cos A & 0 \\ b & \cos B & 0 \\ c & \cos C & 0 \end{vmatrix} = 0 \times 0 = 0 = \text{RHS}$$

(ii) LHS $= \begin{vmatrix} -1 + \cos B & \cos C + \cos B & \cos B \\ \cos C + \cos A & -1 + \cos A & \cos A \\ -1 + \cos B & -1 + \cos A & -1 \end{vmatrix}$

Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$, then

$$= \begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$$

$$= \frac{1}{a} \begin{vmatrix} -a & \cos C & \cos B \\ a\cos C & -1 & \cos A \\ a\cos B & \cos A & -1 \end{vmatrix}$$

Applying $C_1 \to C_1 + bC_2 + cC_3$, then

$$= \frac{1}{a} \begin{vmatrix} 0 & \cos C & \cos B \\ 0 & -1 & \cos A \\ 0 & \cos A & -1 \end{vmatrix} = 0 = \text{RHS}$$

● ***Ex. 38*** *Without expanding at any stage, evaluate the value of the determinant*

$$\begin{vmatrix} 2 & \tan A\cot B + \cot A\tan B & \tan A\cot C + \cot A\tan C \\ \tan B\cot A + \cot B\tan A & 2 & \tan B\cot C + \cot B\tan C \\ \tan C\cot A + \cot C\tan A & \tan C\cot B + \cot C\tan B & 2 \end{vmatrix}.$$

Sol. The given determinant can be written as the product of two determinants

$$\begin{vmatrix} \tan A & \cot A & 0 \\ \tan B & \cot B & 0 \\ \tan C & \cot C & 0 \end{vmatrix} \times \begin{vmatrix} \cot A & \tan A & 0 \\ \cot B & \tan B & 0 \\ \cot C & \tan C & 0 \end{vmatrix} = 0 \times 0 = 0$$

● ***Ex. 39*** *Suppose that digit numbers $A28, 3B9$ and $62C$, where A, B and C are integers between 0 and 9 are divisible by a fixed integer k, prove that the determinant* $\begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix}$ *is also divisible by k.*

Sol. Given, $A28, 3B9$ and $62C$ are divisible by k, then

$$A28 = 100A + 20 + 8 = n_1 k \quad \ldots(\text{i})$$

$$3B9 = 300 + 10B + 9 = n_2 k \quad \ldots(\text{ii})$$

and $$62C = 600 + 20 + C = n_3 k \quad \ldots(\text{iii})$$

where $n_1, n_2, n_3 \in I$ (integers).

Let $$\Delta = \begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix}$$

Applying $R_2 \to R_2 + 10R_3 + 100R_1$, then

$$A = \begin{vmatrix} A & 3 & 6 \\ 100A + 20 + 8 & 300 + 10B + 9 & 600 + 20 + C \\ 2 & B & 2 \end{vmatrix}$$

$$= \begin{vmatrix} A & 3 & 6 \\ n_1 k & n_2 k & n_3 k \\ 2 & B & 2 \end{vmatrix} \quad \text{[using Eqs. (i), (ii) and (iii)]}$$

$$= k \begin{vmatrix} A & 3 & 6 \\ n_1 & n_2 & n_3 \\ 2 & B & 2 \end{vmatrix}$$

Hence, Δ is divisible by k.

● ***Ex. 40*** *If* $\Delta = \begin{vmatrix} \sin x & \sin(x+h) & \sin(x+2h) \\ \sin(x+2h) & \sin x & \sin(x+h) \\ \sin(x+h) & \sin(x+2h) & \sin x \end{vmatrix}$, *find* $\lim_{h \to 0} \left(\frac{\Delta}{h^2} \right)$.

Sol. Let $a = \sin x$, $b = \sin(x+h)$ and $c = \sin(x+2h)$

$$\Delta = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = (a^3 + b^3 + c^3 - 3abc)$$

$$= \frac{1}{2}(a+b+c)\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right]$$

Now, $a - b = \sin x - \sin(x+h) = -2\cos\left(x + \frac{h}{2}\right)\sin\frac{h}{2}$

$b - c = \sin(x+h) - \sin(x+2h) = -2\cos\left(x + \frac{3h}{2}\right)\sin\frac{h}{2}$

and $c - a = \sin(x+2h) - \sin x = 2\cos(x+h)\sin h$

$$\therefore \quad \frac{\Delta}{h^2} = \frac{1}{2}(a+b+c) \left[\left(\frac{a-b}{h} \right)^2 + \left(\frac{b-c}{h} \right)^2 + \left(\frac{c-a}{h} \right)^2 \right]$$

$$= \frac{1}{2}\left[\sin x + \sin(x+h) + \sin(x+2h)\right] \times \left[\left(\frac{-2\cos\left(x + \frac{h}{2}\right)\sin\frac{h}{2}}{h} \right)^2 + \left(\frac{-2\cos\left(x + \frac{3h}{2}\right)\sin\frac{h}{2}}{h} \right)^2 + \left(\frac{2\cos(x+h)\sin h}{h} \right)^2 \right]$$

$$\therefore \quad \lim_{h \to 0} \frac{\Delta}{h^2} = \frac{1}{2}(3\sin x)(\cos^2 x + \cos^2 x + 4\cos^2 x)$$

$$\therefore \quad = 9 \sin x \cos^2 x$$

● ***Ex. 41*** *If* $f(x) = \begin{vmatrix} x+c_1 & x+a & x+a \\ x+b & x+c_2 & x+a \\ x+b & x+b & x+c_3 \end{vmatrix}$, *show that* $f(x)$ *is linear in x. Hence, deduce that* $f(0) = \frac{bg(a) - ag(b)}{(b-a)}$, *where* $g(x) = (c_1 - x)(c_2 - x)(c_3 - x)$.

Sol. Since, $f(x) = \begin{vmatrix} x+c_1 & x+a & x+a \\ x+b & x+c_2 & x+a \\ x+b & x+b & x+c_3 \end{vmatrix}$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_2$, then

$$f(x) = \begin{vmatrix} x+c_1 & a-c_1 & 0 \\ x+b & c_2-b & a-c_2 \\ x+b & 0 & c_3-b \end{vmatrix}$$

$$f(x) = x \begin{vmatrix} 1 & a-c_1 & 0 \\ 1 & c_2-b & a-c_2 \\ 1 & 0 & c_3-b \end{vmatrix} + \begin{vmatrix} c_1 & a-c_1 & 0 \\ b & c_2-b & a-c_2 \\ b & 0 & c_3-b \end{vmatrix}$$

So, $f(x)$ is linear.

Let $f(x) = Px + Q$

Then, $f(-a) = -aP + Q$, $f(-b) = -bP + Q$

$$f(0) = 0 \cdot P + Q = Q = \frac{bf(-a) - af(-b)}{(b-a)} \quad \text{...(ii)}$$

From Eq. (i), we get

$$f(-a) = \begin{vmatrix} c_1 - a & 0 & 0 \\ b-a & c_2-a & 0 \\ b-a & b-a & c_3-a \end{vmatrix}$$

$$= (c_1 - a)(c_2 - a)(c_3 - a)$$

Similarly, $f(-b) = (c_1 - b)(c_2 - b)(c_3 - b)$

and $g(x) = (c_1 - x)(c_2 - x)(c_3 - x)$

$$g(a) = f(-a)$$

$$\therefore \quad g(b) = f(-b)$$

Now, from Eq. (ii), we get

$$f(0) = \frac{bg(a) - ag(b)}{(b-a)}$$

● ***Ex. 42*** *If* $f(x)$ *is a polynomial of degree* < 3, *prove that*

$$\begin{vmatrix} 1 & a & f(a)/(x-a) \\ 1 & b & f(b)/(x-b) \\ 1 & c & f(c)/(x-c) \end{vmatrix} \div \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \frac{f(x)}{(x-a)(x-b)(x-c)}.$$

Sol. $\frac{f(x)}{(x-a)(x-b)(x-c)} = \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$ [let] ...(i)

On comparing the various powers of x, we get

$$\Rightarrow \quad \begin{cases} A = \frac{f(a)}{(a-b)(a-c)} = -\frac{f(a)}{(a-b)(c-a)} \\ B = \frac{f(b)}{(b-a)(b-c)} = -\frac{f(b)}{(a-b)(b-c)} \\ C = \frac{f(c)}{(c-a)(c-b)} = -\frac{f(c)}{(b-c)(c-a)} \end{cases}$$

Now, from Eq. (i), we get

$$\frac{f(x)}{(x-a)(x-b)(x-c)}$$

$$= \frac{(c-b)\frac{f(a)}{(x-a)} - (c-a)\frac{f(b)}{(x-b)} + (b-a)\frac{f(c)}{(x-c)}}{(a-b)(b-c)(c-a)}$$

$$= \frac{\begin{vmatrix} 1 & a & f(a)/(x-a) \\ 1 & b & f(b)/(x-b) \\ 1 & c & f(c)/(x-c) \end{vmatrix}}{\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}}$$

● ***Ex. 43*** *If* $f(a,b) = \dfrac{f(b)-f(a)}{b-a}$ *and* $f(a,b,c) = \dfrac{f(b,c)-f(a,b)}{(c-a)}$, *prove that*

$$f(a,b,c) = \begin{vmatrix} f(a) & f(b) & f(c) \\ 1 & 1 & 1 \\ a & b & c \end{vmatrix} \div \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}.$$

Sol. LHS $= f(a,b,c) = \dfrac{f(b,c)-f(a,b)}{(c-a)}$

$$= \frac{\frac{f(c)-f(b)}{(c-b)} - \frac{f(b)-f(a)}{(b-a)}}{(c-a)}$$

$$= \frac{(b-a)\{f(c)-f(b)\} - (c-b)\{f(b)-f(a)\}}{(b-a)(c-b)(c-a)}$$

$$= \frac{(f(a)\cdot(c-b) - f(b)\cdot(c-a) + f(c)\cdot(b-a))}{(a-b)(b-c)(c-a)}$$

$$= \frac{f(a)\begin{vmatrix} 1 & 1 \\ b & c \end{vmatrix} - f(b)\begin{vmatrix} 1 & 1 \\ a & c \end{vmatrix} + f(c)\begin{vmatrix} 1 & 1 \\ a & b \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} f(a) & f(b) & f(c) \\ 1 & 1 & 1 \\ a & b & c \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}}$$

$$= \begin{vmatrix} f(a) & f(b) & f(c) \\ 1 & 1 & 1 \\ a & b & c \end{vmatrix} \div \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \text{RHS}$$

● ***Ex. 44*** *Let S be the sum of all possible determinants of order 2 having 0, 1, 2 and 3 as their elements. Find the common root* α *of the equations*

$$x^2 + ax + [m+1] = 0,$$

$$x^2 + bx + [m+4] = 0$$

and $x^2 - cx + [m+15] = 0,$

such that $\alpha > S$, *where* $a+b+c=0$ *and*

$$m = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{2n} \frac{r}{\sqrt{(n^2+r^2)}}$$

and [.] denotes the greatest integer function.

Sol. Let α be a common root of the given equations, then

$$\alpha^2 + a\alpha + [m+1] = 0$$

$$\Rightarrow \quad \alpha^2 + a\alpha + [m] + 1 = 0 \qquad \ldots(i)$$

$$\alpha^2 + b\alpha + [m+4] = 0$$

$$\Rightarrow \quad \alpha^2 + b\alpha + [m] + 4 = 0 \qquad \ldots(ii)$$

and $\quad \alpha^2 - c\alpha + [m+15] = 0$

$$\Rightarrow \quad \alpha^2 - c\alpha + [m] + 15 = 0 \qquad \ldots(iii)$$

On adding Eqs. (i) and (ii) and subtracting Eq. (iii), we get

$$a^2 + (a+b+c)\alpha + [m] - 10 = 0$$

$$\alpha^2 + 0 + [m] - 10 = 0 \qquad [\because a+b+c=0]$$

$$\Rightarrow \quad \alpha^2 + [m] - 10 = 0 \qquad \ldots(iv)$$

Also, $\quad m = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{2n} \frac{r}{\sqrt{n^2+r^2}}$

$$= \lim_{n\to\infty} \sum_{r=1}^{2n} \frac{1}{n} \cdot \frac{r/n}{\sqrt{1+(r/n)^2}} = \int_0^2 \frac{x}{\sqrt{(1+x^2)}}\,dx$$

$$[\sqrt{(1+x^2)}]_0^2 = \sqrt{5} - 1$$

Now, $\quad [m] = [\sqrt{5} - 1] = 1$

From Eq. (iv), we get

$$\alpha^2 + 1 - 10 = 0 \Rightarrow \alpha^2 = 9$$

$$\therefore \quad \alpha = \pm 3$$

Now, number of determinants of order 2 having

$$0, 1, 2, 3 = 4! = 24$$

Let $\Delta_1 = \begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix}$ be one such determinant and their exists another determinant.

Let $\quad \Delta_2 = \begin{vmatrix} a_3 & a_4 \\ a_1 & a_2 \end{vmatrix}$ [obtained on interchanging R_1 and R_2]

such that $\Delta_1 + \Delta_2 = 0$

$\because S$ = Sum of all the 24 determinants $= 0$

Since, $\quad \alpha > S \Rightarrow \alpha > 0$

$$\therefore \quad \alpha = 3$$

● ***Ex. 45*** *If a_1, a_2, a_3 and $b_1, b_2, b_3 \in R$ and are such that $a_i b_j \neq 1$ for $1 \le i, j \le 3$,*

$$\begin{vmatrix} \frac{1-a_1^3b_1^3}{1-a_1b_1} & \frac{1-a_1^3b_2^3}{1-a_1b_2} & \frac{1-a_1^3b_3^3}{1-a_1b_3} \\ \frac{1-a_2^3b_1^3}{1-a_2b_1} & \frac{1-a_2^3b_2^3}{1-a_2b_2} & \frac{1-a_2^3b_3^3}{1-a_2b_3} \\ \frac{1-a_3^3b_1^3}{1-a_3b_1} & \frac{1-a_3^3b_2^3}{1-a_3b_2} & \frac{1-a_3^3b_3^3}{1-a_3b_3} \end{vmatrix} > 0$$ *provided either*

$a_1 < a_2 < a_3$ and $b_1 < b_2 < b_3$ or $a_1 > a_2 > a_3$ and $b_1 > b_2 > b_3$.

Sol. Since, $\dfrac{x^3 - y^3}{x - y} = \dfrac{(x-y)(x^2 + xy + y^2)}{(x-y)} = x^2 + xy + y^2$

Hence, the given determinant becomes

$$\begin{vmatrix} 1 + a_1b_1 + a_1^2b_1^2 & 1 + a_1b_2 + a_1^2b_2^2 & 1 + a_1b_3 + a_1^2b_3^2 \\ 1 + a_2b_1 + a_2^2b_1^2 & 1 + a_2b_2 + a_2^2b_2^2 & 1 + a_2b_3 + a_2^2b_3^2 \\ 1 + a_3b_1 + a_3^2b_1^2 & 1 + a_3b_2 + a_3^2b_2^2 & 1 + a_3b_3 + a_3^2b_3^2 \end{vmatrix} > 0$$

$$\Rightarrow \quad \begin{vmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{vmatrix} \times \begin{vmatrix} 1 & b_1 & a_1^2 \\ 1 & b_2 & b_2^2 \\ 1 & b_3 & b_3^2 \end{vmatrix} > 0$$

$$\Rightarrow \quad (a_1 - a_2)(a_2 - a_3)(a_3 - a_1)(b_1 - b_2)(b_2 - b_3)(b_3 - b_1) > 0$$

$$\left\{ \because \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a) \right\}$$

***Case* I** If $a_1 < a_2 < a_3$ and $b_1 < b_2 < b_3$, then

$$(a_1 - a_2) < 0, (a_2 - a_3) < 0$$

and $(b_1 - b_2) < 0, (b_2 - b_3) < 0$

$(a_1 - a_3) < 0$

and $(b_1 - b_3) < 0$

$\therefore \quad (a_3 - a_1) > 0$

$\therefore \quad (b_3 - b_1) > 0$

Then, $(a_1 - a_2)(a_2 - a_3)(a_3 - a_1) > 0$

and $(b_1 - b_2)(b_2 - b_3)(b_3 - b_1) > 0$

$\therefore \quad (a_1 - a_2)(a_2 - a_3)(a_3 - a_1)(b_1 - b_2)(b_2 - b_3)(b_3 - b_1) > 0$

which is true.

***Case* II** If $a_1 > a_2 > a_3$ and $b_1 > b_2 > b_3$

$\therefore \quad a_1 - a_2 > 0, a_2 - a_3 > 0$

and $b_1 - b_2 > 0, b_2 - b_3 > 0$

$a_1 - a_3 > 0 \Rightarrow a_3 - a_1 < 0$

and $b_1 - b_3 > 0 \Rightarrow b_3 - b_1 < 0$

Hence, $(a_1 - a_2)(a_2 - a_3)(a_3 - a_1) < 0$

and $(b_1 - b_2)(b_2 - b_3)(b_3 - b_1) < 0$

$\therefore \quad (a_1 - a_2)(a_2 - a_3)(a_3 - a_1)(b_1 - b_2)(b_2 - b_3)(b_3 - b_1) > 0$

which is true.

● ***Ex. 46*** *Show that a six-digit number abcdef is divisible by 11, if and only if $ab + cd + ef$ is divisible by 11. Hence or otherwise, find one set of values of two-digit numbers x, y and z, so that the value of the determinant* $\begin{vmatrix} x & 23 & 42 \\ 13 & 37 & y \\ 19 & z & 34 \end{vmatrix}$ *is divisible by 99 (without expanding the determinant).*

Sol. Since, $abcdef = ab0000 + cd00 + ef$

$= (9999 + 1)\,ab + (99 + 1)\,cd + ef$

$= 9999\,ab + 99\,cd + ab + cd + ef$

Given, $abcdef$ is divisible by 11, if and only if $ab + cd + ef$ is divisible by 11. Now, let $x = ab$, $y = cd$ and $z = ef$.
[each being a two-digit number]

Again, let $\Delta = \begin{vmatrix} x & 23 & 42 \\ 13 & 37 & y \\ 19 & z & 34 \end{vmatrix} = \begin{vmatrix} ab & 23 & 42 \\ 13 & 37 & cd \\ 19 & ef & 34 \end{vmatrix}$

Applying $R_1 \to R_1 + 100\,R_2 + 10000\,R_3$, we get

$$\Delta = \begin{vmatrix} 1913ab & ef3723 & 34cd42 \\ 13 & 37 & cd \\ 19 & ef & 34 \end{vmatrix}$$

Now, $1913ab$ is divisible by 11, if and only if
$19 + 13 + ab = 32 + ab$ is divisible by $11 \Rightarrow ab = 01, 12, 23, \ldots$

Again, $1913ab$ is divisible by 9, if and only if

$1 + 9 + 1 + 3 + a + b = 14 + a + b$ is divisibe by 9.

The above two conditions are satisfied for $a = 6$, $b = 7$. Thus, $x = 67$. Similarly, $y = 23$ and $z = 39$.

Determinants Exercise 1 :
Single Option Correct Type Questions

■ This section contains **30 multiple choice questions.** Each question has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct

1. If $f(n) = \alpha^n + \beta^n$ and $\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix}$ $= k(1-\alpha)^2 (1-\beta)^2 (\alpha-\beta)^2$, k^2 d is equal to

(a) 1 (b) -1
(c) $\alpha\beta$ (d) $\alpha\beta\gamma$

2. Let $\Delta(x) = \begin{vmatrix} x+a & x+b & x+a-c \\ x+b & x+c & x-1 \\ x+c & x+d & x-b+d \end{vmatrix}$ and

$\int_0^2 \Delta(x)\,dx = -16$, where a, b, c and d are in AP, then the common difference of the AP is equal to

(a) ± 1 (b) ± 2 (c) ± 3 (d) ± 4

3. If $\Delta(x) = \begin{vmatrix} x & 1+x^2 & x^3 \\ \log_e(1+x^2) & e^x & \sin x \\ \cos x & \tan x & \sin^2 x \end{vmatrix}$, then

(a) $\Delta(x)$ is divisible by x (b) $\Delta(x) = 0$
(c) $\Delta'(x) = 0$ (d) None of these

4. If a, b and c are sides of a triangle and

$\begin{vmatrix} a^2 & b^2 & c^2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix} = 0$, then

(a) ΔABC is an equilateral triangle
(b) ΔABC is a right angled isosceles triangle
(c) ΔABC is an isosceles triangle
(d) None of the above

5. If $\begin{vmatrix} \alpha & x & x & x \\ x & \beta & x & x \\ x & x & \gamma & x \\ x & x & x & \delta \end{vmatrix} = f(x) - xf'(x)$, then $f(x)$ is equal to

(a) $(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$
(b) $(x+\alpha)(x+\beta)(x+\gamma)(x+\delta)$
(c) $2(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$
(d) None of the above

6. If $\begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & a+b & c \end{vmatrix} = 0$, the line $ax + by + c = 0$ passes through the fixed point which is

(a) (1, 2) (b) (1, 1) (c) (– 2, 1) (d) (1, 0)

7. If $f(x) = a + bx + cx^2$ and α, β and γ are the roots of the equation $x^3 = 1$, then $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is equal to

(a) $f(\alpha) + f(\beta) + f(\gamma)$
(b) $f(\alpha) f(\beta) + f(\beta) f(\gamma) + f(\gamma) f(\alpha)$
(c) $f(\alpha) f(\beta) f(\gamma)$
(d) $-f(\alpha) f(\beta) f(\gamma)$

8. When the determinant $\begin{vmatrix} \cos 2x & \sin^2 x & \cos 4x \\ \sin^2 x & \cos 2x & \cos^2 x \\ \cos 4x & \cos^2 x & \cos 2x \end{vmatrix}$ is expanded in powers of $\sin x$, the constant term in that expression is

(a) 1 (b) 0 (c) -1 (d) 2

9. If [ ] denotes the greatest integer less than or equal to the real number under consideration and $-1 \le x < 0, 0 \le y < 1, 1 \le z < 2$, the value of the determinant $\begin{vmatrix} [x]+1 & [y] & [z] \\ [x] & [y]+1 & [z] \\ [x] & [y] & [z]+1 \end{vmatrix}$ is

(a) $[x]$ (b) $[y]$
(c) $[z]$ (d) None of these

10. The determinant $\begin{vmatrix} y^2 & -xy & x^2 \\ a & b & c \\ a' & b' & c' \end{vmatrix}$ is equal to

(a) $\begin{vmatrix} bx+ay & cx+by \\ b'x+a'y & c'x+b'y \end{vmatrix}$ (b) $\begin{vmatrix} a'x+b'y & bx+cy \\ ax+by & b'x+c'y \end{vmatrix}$

(c) $\begin{vmatrix} bx+cy & ax+by \\ b'x+c'y & a'x+b'y \end{vmatrix}$ (d) $\begin{vmatrix} ax+by & bx+cy \\ a'x+b'y & b'x+c'y \end{vmatrix}$

11. If A, B and C are angles of a triangle, the value of $\begin{vmatrix} e^{2iA} & e^{-iC} & e^{-iB} \\ e^{-iC} & e^{2iB} & e^{-iA} \\ e^{-iB} & e^{-iBA} & e^{2iC} \end{vmatrix}$ is (where $i = \sqrt{-1}$)

(a) 1 (b) -1 (c) -2 (d) -4

12. If $\begin{vmatrix} x^n & x^{n+2} & x^{2n} \\ 1 & x^a & a \\ x^{n+5} & x^{a+6} & x^{2n+5} \end{vmatrix} = 0, \forall\ x \in R$, where $n \in N$, the value of a is

(a) n (b) $n-1$
(c) $n+1$ (d) None of these

13. If x, y and z are the integers in AP lying between 1 and 9 and $x\,51, y\,41$ and $z\,31$ are three digits number, the value of $\begin{vmatrix} 5 & 4 & 3 \\ x\,51 & y\,41 & z\,31 \\ x & y & z \end{vmatrix}$ is

(a) $x + y + z$
(b) $x - y + z$
(c) 0
(d) None of the above

14. If $a_1\,b_1\,c_1, a_2\,b_2\,c_2$ and $a_3 b_3 c_3$ are three digit even natural numbers and $\Delta = \begin{vmatrix} c_1 & a_1 & b_1 \\ c_2 & a_2 & b_2 \\ c_3 & a_3 & b_3 \end{vmatrix}$, then Δ is

(a) divisible by 2 but not necessarily by 4
(b) divisible by 4 but not necessarily by 8
(c) divisible by 8
(d) None of the above

15. If a, b and c are sides of ΔABC such that

$$\begin{vmatrix} c & b\cos B + c\beta & a\cos A + b\alpha + c\gamma \\ a & c\cos B + a\beta & b\cos A + c\alpha + a\gamma \\ b & a\cos B + b\beta & c\cos A + a\alpha + b\gamma \end{vmatrix} = 0$$

$\left(\text{where } \alpha, \beta, \gamma \in R^+ \text{ and } \angle A, \angle B, \angle C \neq \dfrac{\pi}{2}\right)$, ΔABC is

(a) an isosceles (b) an equilateral
(c) can't say (d) None of these

16. If x_1, x_2 and y_1, y_2 are the roots of the equations $3x^2 - 18x + 9 = 0$ and $y^2 - 4y + 2 = 0$, the value of the determinant $\begin{vmatrix} x_1x_2 & y_1y_2 & 1 \\ x_1 + x_2 & y_1 + y_2 & 2 \\ \sin(\pi x_1x_2) & \cos(\pi/2 y_1y_2) & 1 \end{vmatrix}$ is

(a) 0 (b) 1
(c) 2 (d) None of these

17. If the value of $\Delta = \begin{vmatrix} {}^{10}C_4 & {}^{10}C_5 & {}^{11}C_m \\ {}^{11}C_6 & {}^{11}C_7 & {}^{12}C_{m+2} \\ {}^{12}C_8 & {}^{12}C_9 & {}^{13}C_{m+4} \end{vmatrix}$ is equal to zero, then m is equal to

(a) 6 (b) 4
(c) 5 (d) None of these

18. The value of the determinant $\begin{vmatrix} 1 & \sin(\alpha - \beta)\theta & \cos(\alpha - \beta)\theta \\ a & \sin\alpha\theta & \cos\alpha\theta \\ a^2 & \sin(\alpha - \beta)\theta & \cos(\alpha - \beta)\theta \end{vmatrix}$ is independent of

(a) α (b) β
(c) θ (d) a

19. If $f(x)$, $g(x)$ and $h(x)$ are polynomials of degree 4 and $\begin{vmatrix} f(x) & g(x) & h(x) \\ a & b & c \\ p & q & r \end{vmatrix} = mx^4 + nx^3 + rx^2 + sx + t$ be an identity in x, then $\begin{vmatrix} f'''(0) - f''(0) & g'''(0) - g''(0) & h'''(0) - h''(0) \\ a & b & c \\ p & q & r \end{vmatrix}$ is equal to

(a) $2(3n + r)$ (b) $3(2n - r)$
(c) $3(2n + r)$ (d) $2(3n - r)$

20. If $f(x) = \begin{vmatrix} \cos(x + \alpha) & \cos(x + \beta) & \cos(x + \gamma) \\ \sin(x + \alpha) & \sin(x + \beta) & \sin(x + \gamma) \\ \sin(\beta - \gamma) & \sin(\gamma - \alpha) & \sin(\alpha - \beta) \end{vmatrix}$, then $f(\theta) - 2f(\phi) + f(\psi)$ is equal to

(a) 0 (b) $\alpha - \beta$
(c) $\alpha + \beta + \gamma$ (d) $\alpha + \beta - \gamma$

21. If $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a - b)(b - c)(c - a)(a + b + c)$, where a, b and c are all different, then the determinant $\begin{vmatrix} 1 & 1 & 1 \\ (x - a)^2 & (x - b)^2 & (x - c)^2 \\ (x - b)(x - c) & (x - c)(x - a) & (x - a)(x - b) \end{vmatrix}$ vanishes when

(a) $a + b + c = 0$ (b) $x = \dfrac{1}{3}(a + b + c)$
(c) $x = \dfrac{1}{2}(a + b + c)$ (d) $x = a + b + c$

22. Let $a, b, c \in R$ such that no two of them are equal and satisfy $\begin{vmatrix} 2a & b & c \\ b & c & 2a \\ c & 2a & b \end{vmatrix} = 0$, the equation $24ax^2 + 4bx + c = 0$ has

(a) atleast one root in $\left[0, \dfrac{1}{2}\right]$
(b) atleast one root in $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$
(c) atleast one root in $[-1, 0]$
(d) atleast two roots in $[0, 2]$

23. The number of positive integral solution of the equation $\begin{vmatrix} x^3 + 1 & x^2y & x^2z \\ xy^2 & y^3 + 1 & y^2z \\ xz^2 & yz^2 & z^3 + 1 \end{vmatrix} = 11$ is

(a) 0 (b) 3 (c) 6 (d) 12

24. If $f(x) = ax^2 + bx + c, a, b, c \in R$ and equation $f(x) - x = 0$ has imaginary roots $\alpha, \beta\ \gamma$ and δ be the roots of $f(f(x)) - x = 0$, then $\begin{vmatrix} 2 & \alpha & \delta \\ \beta & 0 & \alpha \\ \gamma & \beta & 1 \end{vmatrix}$ is

(a) 0 (b) purely real
(c) purely imaginary (d) None of these

25. If the system of equations $2x - y + z = 0$, $x - 2y + z = 0$, $tx - y + 2z = 0$ has infinitely many solutions and $f(x)$ be a continuous function, such that $f(5 + x) + f(x) = 2$, then $\int_0^{-2t} f(x)\,dx$ is equal to

(a) 0 (b) $-2t$ (c) 5 (d) t

26. If $(1 + ax + bx^2)^4 = a_0 + a_1 x + a_2 x^2 + \ldots + a_8 x^8$, where $a, b, a_0, a_1, \ldots, a_8 \in R$ such that $a_0 + a_1 + a_2 \neq 0$ and $\begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_0 \\ a_2 & a_0 & a_1 \end{vmatrix} = 0$, then

(a) $a = \frac{3}{4}, b = \frac{5}{8}$ (b) $a = \frac{1}{4}, b = \frac{5}{32}$
(c) $a = 1, b = \frac{2}{3}$ (d) None of these

27. Given, $f(x) = \log_{10} x$ and $g(x) = e^{\pi i x}$.
If $\phi(x) = \begin{vmatrix} f(x)\cdot g(x) & (f(x))^{g(x)} & 1 \\ f(x^2)\cdot g(x^2) & (f(x^2))^{g(x^2)} & 0 \\ f(x^3)\cdot g(x^3) & (f(x^3))^{g(x^3)} & 1 \end{vmatrix}$, the value of $\phi(10)$, is

(a) 1 (b) 2 (c) 0 (d) None of these

28. The value of the determinant $\begin{vmatrix} 1 & (\alpha^{2x} - \alpha^{-2x})^2 & (\alpha^{2x} + \alpha^{-2x})^2 \\ 1 & (\beta^{2x} - \beta^{-2x})^2 & (\beta^{2x} + \beta^{-2x})^2 \\ 1 & (\gamma^{2x} - \gamma^{-2x})^2 & (\gamma^{2x} + \gamma^{-2x})^2 \end{vmatrix}$, is

(a) 0 (b) $(\alpha\beta\gamma)^{2x}$ (c) $(\alpha\beta\gamma)^{-2x}$ (d) None of these

29. If a, b and c are non-zero real numbers and if the equations $(a-1)x = y + z, (b-1)y = z + x$, $(c-1)z = x + y$ has a non-trivial solution, then $ab + bc + ca$ equals to

(a) $a + b + c$ (b) abc (c) 1 (d) None of these

30. The set of equations $\lambda x - y + (\cos\theta) z = 0$, $3x + y + 2z = 0, (\cos\theta) x + y + 2z = 0, 0 \le \theta \le 2\pi$, has non-trivial solution(s)

(a) for no value of λ and θ
(b) for all value of λ and θ
(c) for all values of λ and only two values of θ
(d) for only one value of λ and all values of θ

Determinants Exercise 2 : More than One Correct Option Type Questions

■ This section contains **15 multiple choice questions**. Each question has four choices (a), (b), (c) and (d) out of which **MORE THAN ONE** may be correct.

31. The determinant $\Delta = \begin{vmatrix} a^2 + x^2 & ab & ac \\ ab & b^2 + x^2 & bc \\ ac & bc & c^2 + x^2 \end{vmatrix}$ is divisible by

(a) x (b) x^2 (c) x^3 (d) x^4

32. The value of the determinant $\begin{vmatrix} \sqrt{6} & 2i & 3 + \sqrt{6} \\ \sqrt{12} & \sqrt{3} + \sqrt{8}i & 3\sqrt{2} + \sqrt{6}\,i \\ \sqrt{18} & \sqrt{2} + \sqrt{12}\,i & \sqrt{27} + 2i \end{vmatrix}$, is (where $i = \sqrt{-1}$)

(a) complex (b) real (c) irrational (d) rational

33. If $D_k = \begin{vmatrix} 2^{k-1} & \frac{1}{k(k+1)} & \sin k\theta \\ x & y & z \\ 2^n - 1 & \frac{n}{n+1} & \frac{\sin\left(\frac{n+1}{2}\right)\theta \sin\frac{n}{2}\theta}{\sin\frac{\theta}{2}} \end{vmatrix}$, then $\sum_{k=1}^{n} D_k$ is equal to

(a) 0 (b) independent of n
(c) independent of θ (d) independent of x, y and z

34. The determinant $\begin{vmatrix} a & b & a\alpha + b \\ b & c & b\alpha + c \\ a\alpha + b & b\alpha + c & 0 \end{vmatrix}$ is equal to zero, if

(a) a, b and c are in AP
(b) a, b, c, are in GP
(c) a, b and c are in HP
(d) $(x - \alpha)$ is a factor of $ax^2 + 2bx + c$

35. Let $f(x) = \begin{vmatrix} 2\cos x & 1 & 0 \\ 1 & 2\cos x & 1 \\ 0 & 1 & 2\cos x \end{vmatrix}$, then

(a) $f\left(\frac{\pi}{3}\right) = -1$

(b) $f'\left(\frac{\pi}{3}\right) = \sqrt{3}$

(c) $\int_0^{\pi} f(x)dx = 0$

(d) $\int_{-\pi}^{\pi} f(x)\,dx = 0$

36. If $\Delta(x)=\begin{vmatrix} x^2-5x+3 & 2x-5 & 3 \\ 3x^2+x+4 & 6x+1 & 9 \\ 7x^2-6x+9 & 14x-6 & 21 \end{vmatrix}$
$= ax^3+bx^2+cx+d$, then
(a) $a=0$ (b) $b=0$ (c) $c=0$ (d) $d=47$

37. If a, b and c are the sides of a triangle and A, B and C are the angles opposite to a, b and c respectively, then
$\Delta=\begin{vmatrix} a^2 & b\sin A & C\sin A \\ b\sin A & 1 & \cos A \\ C\sin A & \cos A & 1 \end{vmatrix}$ is independent of
(a) a (b) b (c) c (d) A, B, C

38. Let $f(a,b)=\begin{vmatrix} a & a^2 & 0 \\ 1 & (2a+b) & (a+b)^2 \\ 0 & 1 & (2a+3b) \end{vmatrix}$, then
(a) $(a+b)$ is a factor of $f(a,b)$
(b) $(a+2b)$ is a factor of $f(a,b)$
(c) $(2a+b)$ is a factor of $f(a,b)$
(d) a is a factor of $f(a,b)$

39. If $f(x)=\begin{vmatrix} \sec^2 x & 1 & 1 \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cot^2 x \end{vmatrix}$, then
(a) $\int_{-\pi/4}^{\pi/4} f(x)\,dx=\frac{1}{16}(3\pi+8)$
(b) $f'\left(\frac{\pi}{2}\right)=0$
(c) maximum value of $f(x)$ is 1
(d) minimum value of $f(x)$ is 0

40. If $\begin{vmatrix} a & a+x^2 & a+x^2+x^4 \\ 2a & 3a+2x^2 & 4a+3x^2+2x^4 \\ 3a & 6a+3x^2 & 10a+6x^2+3x^4 \end{vmatrix}$
$= a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6+a_7x^7$ and
$f(x)=+a_0x^2+a_3x+a_6$, then
(a) $f(x)\geq 0, \forall\ x\in R$ if $a>0$
(b) $f(x)=0$, only if $a=0$
(c) $f(x)=0$, has two equal roots
(d) $f(x)=0$, has more than two root if $a=0$

41. If $\Delta(x)=\begin{vmatrix} 4x-4 & (x-2)^2 & x^3 \\ 8x-4\sqrt{2} & (x-2\sqrt{2})^2 & (x+1)^3 \\ 12x-4\sqrt{3} & (x-2\sqrt{3})^2 & (x-1)^3 \end{vmatrix}$, then
(a) term independent of x in $\Delta(x)=16(5-\sqrt{2}-\sqrt{3})$
(b) coefficient of x in $\Delta(x)=48(1+\sqrt{2}-\sqrt{3})$
(c) coefficient of x in $\Delta(x)=16(5+\sqrt{2}-\sqrt{3})$
(d) coefficient of x in $\Delta(x)$ is divisible by 16

42. If
$f(x)=\begin{vmatrix} 3 & 3x & 3x^2+2a^2 \\ 3x & 3x^2+2a^2 & 3x^3+6a^2x \\ 3x^2+2a^3 & 3x^3+6a^2x & 3x^4+12a^2x^2+2a^4 \end{vmatrix}$,
then
(a) $f'(x)=0$
(b) $y=f(x)$ is a straight line parallel to X-axis
(c) $\int_0^2 f(x)\,dx=32a^4$
(d) None of the above

43. If $a>b>c$ and the system of equations $ax+by+cz=0$, $bx+cy+az=0$, $cx+ay+bz=0$ has a non-trivial solution, then both the roots of the quadratic equation at^2+bt+c, are
(a) real
(b) of opposite sign
(c) positive
(d) complex

44. The values of λ and b for which the equations $x+y+z=3$, $x+3y+2z=6$ and $x+\lambda y+3z=b$ have
(a) a unique solution, if $\lambda\neq 5$, $b\in R$
(b) no solution, if $\lambda\neq 5$, $b=9$
(c) infinite many solution, $\lambda=5$, $b=9$
(d) None of the above

45. Let λ and α be real. Let S denote the set of all values of λ for which the system of linear equations
$\lambda x+(\sin\alpha)y+(\cos\alpha)z=0$
$x+(\cos\alpha)y+(\sin\alpha)z=0$
$-x+(\sin\alpha)y-(\cos\alpha)z=0$
has a non-trivial solution, then S contains
(a) $(-1,1)$ (b) $[-\sqrt{2},-1]$
(c) $[1,\sqrt{2}]$ (d) $(-2,2)$

Determinants Exercise 3 : Passage Based Questions

- This section contains **7 passages**. Based upon each of the passage **3 multiple choice questions** have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I

(Q. Nos. 46 to 48)

Consider the system of equations

$x + y + z = 5;\ x + 2y + 3z = 9;\ x + 3y + \lambda z = \mu$

The system is called smart, brilliant, good and lazy according as it has solution, unique solution, infinitely many solutions and no solution, respectively.

46. The system is smart, if

(a) $\lambda \neq 5$ or $\lambda = 5$ and $\mu = 13$ (b) $\lambda \neq 5$ and $\mu = 13$

(c) $\lambda \neq 5$ and $\mu \neq 13$ (d) $\lambda \neq 5$ or $\lambda = 5$ and $\mu \neq 13$

47. The system is good, if

(a) $\lambda \neq 5$ or $\lambda = 5$ and $\mu = 13$ (b) $\lambda = 5$ and $\mu = 13$

(c) $\lambda = 5$ and $\mu \neq 13$

(d) $\lambda \neq 5$, μ is any real number

48. The system is lazy, if

(a) $\lambda \neq 5$ or $\lambda = 5$ and $\mu = 13$ (b) $\lambda = 5$ and $\mu = 13$

(c) $\lambda = 5$ and $\mu \neq 13$ (d) $\lambda \neq 5$ or $\lambda = 5$ and $\mu \neq 13$

Passage II

(Q. Nos. 49 to 51)

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ *and* $C_{ij} = (-1)^{i+j} M_{ij}$, *where* M_{ij} *is a determinant obtained by deleting ith row and jth column, then* $\begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix} = \Delta^2$.

49. If $\begin{vmatrix} 1 & x & x^2 \\ x & x^2 & 1 \\ x^2 & 1 & x \end{vmatrix} = 5$ and $\Delta = \begin{vmatrix} x^3 - 1 & 0 & x - x^4 \\ 0 & x - x^4 & x^3 - 1 \\ x - x^4 & x^3 - 1 & 0 \end{vmatrix}$, then sum of digits of Δ^2, is

(a) 7 (b) 8 (c) 13 (d) 11

50. Suppose $a, b, c \in R,\ a + b + c > 0,\ A = bc - a^2, B = ca - b^2$ and $C = ab - c^2$ and $\begin{vmatrix} A & B & C \\ B & C & A \\ C & A & B \end{vmatrix} = 49$, then the value of $a^3 + b^3 + c^3 - 3abc$, is

(a) –7 (b) 7 (c) –2401 (d) 2401

51. If $a^3 + b^3 + c^3 - 3abc = -3$ and $A = bc - a^2,\ B = ca - b^2$ and $C = ab - c^2$, then the value of $aA + bB + cC$, is

(a) –3 (b) 3 (c) –9 (d) 9

Passage III

(Q. Nos. 52 to 54)

If α, β, γ *are the roots of* $x^3 + 2x^2 - x - 3 = 0$

52. The value of $\begin{vmatrix} \alpha & \beta & \gamma \\ \gamma & \alpha & \beta \\ \beta & \gamma & \alpha \end{vmatrix}$ is equal to

(a) 14 (b) –2 (c) 10 (d) 14

53. If the absolute value of the expression $\frac{\alpha - 1}{\alpha + 2} + \frac{\beta - 1}{\beta + 2} + \frac{\gamma - 1}{\gamma + 2}$ can be expressed as $\frac{m}{n}$, where m and n are co-prime, the value of $\begin{vmatrix} m & n^2 \\ m - n & m + n \end{vmatrix}$, is

(a) 17 (b) 27 (c) 37 (d) 47

54. If $a = \alpha^2 + \beta^2 + \gamma^2,\ b = \alpha\beta + \beta\gamma + \gamma\alpha$, the value of $\begin{vmatrix} a & b & b \\ b & a & b \\ b & b & a \end{vmatrix}$, is

(a) 14 (b) 49 (c) 98 (d) 196

Passage IV

(Q. Nos. 55 to 57)

Suppose $f(x)$ *is a function satisfying the following conditions:*

(i) $f(0) = 2,\ f(1) = 1$

(ii) $f(x)$ *has a minimum value at* $x = \frac{5}{2}$

(iii) For all $x, f'(x) = \begin{vmatrix} 2ax & 2ax - 1 & 2ax + b + 1 \\ b & b + 1 & -1 \\ 2(ax + b) & 2ax + 2b + 1 & 2ax + b \end{vmatrix}$.

55. The value of $f(2) + f(3)$ is

(a) 1 (b) $\frac{3}{2}$ (c) 2 (d) $\frac{5}{2}$

56. The number of solutions of the equation $f(x) + 1 = 0$ is

(a) 0 (b) 1 (c) 2 (d) infinite

57. Range of $f(x)$ is

(a) $\left(-\infty, \frac{7}{16}\right]$ (b) $\left[\frac{3}{4}, \infty\right)$ (c) $\left[\frac{7}{16}, \infty\right)$ (d) $\left(-\infty, \frac{3}{4}\right]$

Passage V

(Q. Nos. 58 to 60)

$$If \begin{vmatrix} x & e^{x-1} & (x-1)^3 \\ x-\ln x & \cos(x-1) & (x-1)^2 \\ \tan x & \sin^2 x & \cos^2 x \end{vmatrix} = a_0 + a_1(x-1) + a_2(x-1)^2 + \ldots$$

58. The value of $\cos^{-1}(a_1)$, is

(a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) π

59. The value of $\lim_{x \to a_0} (\sin x)^x$ is

(a) 1 (b) e (c) $e-1$ (d) None of these

60. The equation whose roots are a_0 and a_1, is

(a) $x^2 - x = 0$ (b) $x^2 - 2x = 0$
(c) $x^2 - 3x = 0$ (d) None of these

Passage VI

(Q. Nos. 61 to 63)

$$Let\ \Delta = \begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix}\ and\ the\ equation$$

$x^3 - px^2 + qx - r = 0$ *has roots* $a,\ b,\ c$, *where* $a, b, c \notin R^+$.

61. The value of Δ is

(a) $\le 9r^3$ (b) $\ge 27r^2$ (c) $\le 27r^2$ (d) $\ge 81r^3$

62. If a, b, c are in GP, then

(a) $r^3 = p^3 q$ (b) $p^3 = r^3 q$ (c) $p^3 = q^3 r$ (d) $q^3 = p^3 r$

63. If $\Delta = 27$ and $a^2 + b^2 + c^2 = 2$, then the value of $\sum a^2 b$, is

(a) $3(2\sqrt{2} - p)$ (b) $3(2\sqrt{2} - r)$
(c) $3(2\sqrt{2} - q)$ (d) $3(2\sqrt{2} - p - q)$

Passage VII

(Q. Nos. 64 to 66)

$$If\ \Delta n = \begin{vmatrix} a^2+n & ab & ac \\ ab & b^2+n & bc \\ ac & bc & c^2+n \end{vmatrix},\ n \in N\ and\ the\ equation$$

$x^3 - \lambda x^2 + 11x - 6 = 0$ *has roots* $a,\ b,\ c$ *and* a, b, c *are in AP.*

64. The value of $\sum_{r=1}^{7} \Delta_n$ is

(a) $(12)^3$ (b) $(14)^3$ (c) $(26)^3$ (d) $(28)^3$

65. The value of $\frac{\Delta 2_n}{\Delta_n}$ is

(a) < 8 (b) = 8
(c) > 8 (d) None of these

66. The value of $\sum_{r=1}^{30} \left(\frac{27\Delta_r - \Delta 3_r}{27r^2} \right)$ is

(a) 130 (b) 190 (c) 280 (d) 340

Determinants Exercise 4 : Single Integer Answer Type Questions

- This section contains **10 questions**. The answer to each question is a **single digit integer**, ranging from 0 to 9 (both inclusive).

67. If $\begin{vmatrix} 3^2+k & 4^2 & 3^2+3+k \\ 4^2+k & 5^2 & 4^2+4+k \\ 5^2+k & 6^2 & 5^2+5+k \end{vmatrix} = 0$,

the value of $\sqrt{2^k \sqrt{2^k \sqrt{2^k \ldots \infty}}}$ is

68. Let α, β and γ are three distinct roots of

$\begin{vmatrix} x-1 & -6 & 2 \\ -6 & x-2 & -4 \\ 2 & -4 & x-6 \end{vmatrix} = 0$, the value of $\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \right)^{-1}$ is

69. If $\begin{vmatrix} x & e^{x-1} & (x-1)^3 \\ x-\ln x & \cos(x-1) & (x-1)^2 \\ \tan x & \sin^2 x & \cos^2 x \end{vmatrix} = \sum_{r=0}^{n} a_r (x-1)^r$,

the value of $(2^{a_0} + 3^{a_1})^{a_1 + 1}$ is

70. If $\begin{vmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos\gamma \\ \cos\beta & \cos\gamma & 1 \end{vmatrix} = \begin{vmatrix} 0 & \cos\alpha & \cos\beta \\ \cos\alpha & 0 & \cos\gamma \\ \cos\beta & \cos\gamma & 0 \end{vmatrix}$,

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma$ is equal to

71. Let $f(a, b, c) = \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$, the greatest integer $n \in N$ such that $(a+b+c)^{n'}$ divides $f(a, b, c)$ is

72. If $0 \le \theta \le \pi$ and the system of equations

$$x = (\sin\theta)\, y + (\cos\theta)\, z$$
$$y = z + (\cos\theta)\, x$$
$$z = (\sin\theta)\, x + y$$

has a non-trivial solution, then $\frac{8\theta}{\pi}$ is equal to

73. The value of the determinant $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix}$ is

74. If a, b, c and d are the roots of the equation $x^4 + 2x^3 + 4x^2 + 8x + 16 = 0$, the value of the determinant $\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix}$ is

75. If $a \neq 0, b \neq 0, c \neq 0$ and $\begin{vmatrix} 1+a & 1 & 1 \\ 1+b & 1+2b & 1 \\ 1+c & 1+c & 1+3c \end{vmatrix} = 0$, the value of $|a^{-1} + b^{-1} + c^{-1}|$ is equal to

76. If the system of equations
$$ax + hy + g = 0; hx + by + f = 0$$
and $ax^2 + 2hxy + by^2 + 2gx + 2fy + c + \lambda = 0$ has a unique solution and
$\dfrac{abc + 2fgh - af^2 - bg^2 - ch^2}{h^2 - ab} = 8$, the value of λ is

Determinants Exercise 5 : Matching Type Questions

■ This section contains **5 questions**. Questions 77 to 81 have three statements (A, B and C) given in **Column I** and four statements (p, q, r and s) in **Column II**. Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II**.

77.

	Column I		Column II
(A)	If a,b,c are three complex numbers such that $a^2+b^2+c^2=0$ and $\begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix} = \lambda a^2b^2c^2$, then λ is divisible by	(p)	2
(B)	If $a,b,c \in R$ and $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 5a+2b & 7a+5b+2c \\ 3a & 7a+3b & 9a+7b+3c \end{vmatrix} = -1024$, then a is divisible by	(q)	3
(C)	Let $\Delta(x) = \begin{vmatrix} x-1 & 2x^2-5 & x^3-1 \\ 2x^2+5 & 2x+2 & x^3+3 \\ x^3-1 & x+1 & 3x^2-2 \end{vmatrix}$ and $ax+b$ be the remainder, when $\Delta(x)$ is divided by x^2-1, then $4a+2b$ is divisible by	(r)	4
		(s)	5
		(t)	6

78.

	Column I		Column II
(A)	Let $f_1(x) = x + a_1$, $f_2(x) = x^2 + b_1x + b_2$, $x_1 = 2$, $x_2 = 3$ and $x_3 = 5$ and $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ f_1(x_1) & f_1(x_2) & f_1(x_3) \\ f_2(x_1) & f_2(x_2) & f_2(x_3) \end{vmatrix}$, then Δ is	(p)	Even number
(B)	If $\lvert a_1 - b_1 \rvert = 6$ and $f(x) = \begin{vmatrix} 1 & b_1 & a_1 \\ 1 & b_1 & 2a_1 - x \\ 1 & 2b_1 - x & a_1 \end{vmatrix}$, then the minimum value of $f(x)$ is	(q)	Prime number
(C)	If coefficient of x in $f(x) = \begin{vmatrix} x-2 & (x-1)^2 & x^3 \\ x-1 & x^2 & (x+1)^3 \\ x & (x+1)^2 & (x+2)^3 \end{vmatrix}$ is λ, then $\lvert\lambda\rvert$ is	(r)	Odd number
		(s)	Composite number
		(t)	Perfect number

79.

	Column I		Column II	
(A)	If $\begin{vmatrix} x^2+3x & x-1 & x+3 \\ x^2+1 & 2+3x & x-3 \\ x^2-3 & x+4 & 3x \end{vmatrix} = ax^4+bx^3+cx^2+dx+e$, then $e+a$ is divisible by	(p)	2
(B)	If $\begin{vmatrix} x-1 & 5x & 7 \\ x^2-1 & x-1 & 8 \\ 2x & 3x & 0 \end{vmatrix} = ax^3+bx^2+cx+d$, then $(e+a-3)$ is divisible by	(q)	3
(C)	If $\begin{vmatrix} x^3+4x & x+3 & x-2 \\ x-2 & 5x & x-1 \\ x-3 & x+2 & 4x \end{vmatrix} = ax^5+bx^4+cx^3+dx^2+ex+f$, then $(f+e)$ is divisible by	(r)	5
		(s)	6
		(t)	7

80.

	Column I		Column II
(A)	If $a^2+b^2+c^2=1$ and $\Delta = \begin{vmatrix} a^2+(b^2+c^2)d & ab(1-d) & ca(1-d) \\ ab(1-d) & b^2+(c^2+a^2)d & bc(1-d) \\ ca(1-d) & bc(1-d) & c^2+(a+b^2)d \end{vmatrix}$, then Δ is	(p)	independent of a
(B)	If $\Delta = \begin{vmatrix} \frac{1}{c} & \frac{1}{c} & \frac{-(a+b)}{c^2} \\ \frac{-(b+c)}{a^2} & \frac{1}{a} & \frac{1}{a} \\ \frac{-bd(b+c)}{a^2c} & \frac{(ad+2bd+cd)}{ac} & \frac{-(a+b)bd}{ac^2} \end{vmatrix}$, then Δ is	(q)	independent of b
(C)	If $\Delta = \begin{vmatrix} \sin a & \cos a & \sin(a+d) \\ \sin b & \cos b & \sin(b+d) \\ \sin c & \cos c & \sin(c+d) \end{vmatrix}$, then Δ is	(r)	independent of c
		(s)	independent of d
		(t)	zero

81.

	Column I		Column II
(A)	If n be the number of distinct values of 2×2 determinant whose entries are from the set $\{-1,0,1\}$, then $(n-1)^2$ is divisible by	(p)	2
(B)	If n be the number of 2×2 determinants with non-negative values whose entries from the set $\{0,1\}$, then $(n-1)$ is divisible by	(q)	3
(C)	If n be the number of 2×2 determinants with negative values whose entries from the set $\{-1, 1\}$, then $n(n+1)$ is divisible by	(r)	4
		(s)	5
		(t)	6

Determinants Exercise 6 : Statement I and II Type Questions

■ **Directions** (Q. Nos. 82 to 87) are Assertion-Reason type questions. Each of these questions contains two statements:

Statement-1 (Assertion) and **Statement-2** (Reason) Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1

(c) Statement1 is true, Statement-2 is false

(d) Statement-1 is false, Statement-2 is true

82. **Statement-1** If $\Delta(r)=\begin{vmatrix} r & r+1 \\ r+3 & r+4 \end{vmatrix}$ then $\sum_{r=1}^{n}\Delta(r)=-3n$

Statement-2 If $\Delta(r)=\begin{vmatrix} f_1(r) & f_2(r) \\ f_3(r) & f_4(r) \end{vmatrix}$

then $\sum_{r=1}^{n}\Delta(r)=\begin{vmatrix} \sum_{r=1}^{n} f_1(r) & \sum_{r=1}^{n} f_2(r) \\ \sum_{r=1}^{n} f_3(r) & \sum_{r=1}^{n} f_4(r) \end{vmatrix}$

83. Consider the determinant

$$\Delta=\begin{vmatrix} a_1+b_1x^2 & a_1x^2+b_1 & c_1 \\ a_2+b_2x^2 & a_2x^2+b_2 & c_2 \\ a_3+b_3x^2 & a_3x^2+b_3 & c_3 \end{vmatrix}=0,$$

where $a_i, b_i, c_i \in R (i=1,2,3)$ and $x \in R$.

Statement-1 The value of x satisfying $\Delta=0$ are $x=1,-1$

Statement-2 If $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0$, then $\Delta=0$.

84. **Statement-1** The value of determinant

$$\begin{vmatrix} \sin\pi & \cos\left(x+\frac{\pi}{4}\right) & \tan\left(x-\frac{\pi}{4}\right) \\ \sin\left(x-\frac{\pi}{4}\right) & -\cos\left(\frac{\pi}{2}\right) & \ln\left(\frac{x}{y}\right) \\ \cot\left(\frac{\pi}{4}+x\right) & \ln\left(\frac{y}{x}\right) & \tan\pi \end{vmatrix}$$ is zero.

Statement-2 The value of skew-symmetric determinant of odd order equals zero.

85. **Statement-1** $f(x)=\begin{vmatrix} (1+x)^{11} & (1+x)^{12} & (1+x)^{13} \\ (1+x)^{21} & (1+x)^{22} & (1+x)^{23} \\ (1+x)^{31} & (1+x)^{32} & (1+x)^{33} \end{vmatrix}$,

the coefficient of x in $f(x)=0$

Statement-2 If $P(x)=a_0+a_1x+a_2x^2+a_3x^3+\ldots+a_nx^n$, then $a_1=P'(0)$, where dash denotes the differential coefficient.

86. **Statement-1** If system of equations $2x+3y=a$ and $bx+4y=5$ has infinite solution, then $a=\frac{15}{4}, b=\frac{8}{5}$

Statement-2 Straight lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ are parallel, if $\frac{a_1}{a_2}=\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$

87. **Statement-1** The value of the determinant $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{vmatrix}\neq 0$

Statement-2 Neither of two rows or columns of $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{vmatrix}$ is identical.

88. **Statement-1** The digits A, B and C re such that the three digit numbers $A88$, $6B8$, $86C$ are divisible by 72, then the determinant $\begin{vmatrix} A & 6 & 8 \\ 8 & B & 6 \\ 8 & 8 & C \end{vmatrix}$ is divisible by 288.

Statement-2 $A=B=?$

Determinants Exercise 7 : Subjective Type Questions

■ In this section, there are **20 subjective** questions.

89. Prove that $\begin{vmatrix} b+c & c & b \\ c & c+a & a \\ b & a & a+b \end{vmatrix} = 4abc.$

90. Prove that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3.$

91. Find the value of determinant $\begin{vmatrix} \sqrt{13}+\sqrt{3} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{26} & 5 & \sqrt{10} \\ 3+\sqrt{65} & \sqrt{15} & 5 \end{vmatrix}.$

92. Find the value of the determinant $\begin{vmatrix} bc & ca & ab \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$, where a, b and c respectively are the pth, qth and rth terms of a harmonic progression.

93. Without expanding the determinant at any stage, prove that $\begin{vmatrix} -5 & 3+5i & \frac{3}{2}-4i \\ 3-5i & 8 & 4+5i \\ \frac{3}{2}+4i & 4-5i & 9 \end{vmatrix}$ has a purely real value.

94. Prove without expanding that $\begin{vmatrix} ah+bg & g & ab+ch \\ bf+ba & f & hb+bc \\ af+bc & c & bg+fc \end{vmatrix} = a \begin{vmatrix} ah+bg & a & h \\ bf+ba & h & b \\ af+bc & g & f \end{vmatrix}.$

95. If A, B and C are the angles of a triangle and

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix} = 0,$$

then prove that ΔABC must be isosceles.

96. Prove that $\begin{vmatrix} \beta\gamma & \beta\gamma'+\beta'\gamma & \beta'\gamma' \\ \gamma\alpha & \gamma\alpha'+\gamma'\alpha & \gamma'\alpha' \\ \alpha\beta & \alpha\beta'+\alpha'\beta & \alpha'\beta' \end{vmatrix}$
$= (\alpha\beta'-\alpha'\beta)(\beta\gamma'-\beta'\gamma)(\gamma\alpha'-\gamma'\alpha).$

97. If $y = \frac{u}{v}$, where u and v are functions of x, show that

$$v^3 \frac{d^2y}{dx^2} = \begin{vmatrix} u & v & 0 \\ u' & v & v \\ u'' & v'' & 2v' \end{vmatrix}.$$

98. Show that the determinant $\Delta(x)$ is given by $\Delta(x) =$ $\begin{vmatrix} \sin(x+\alpha) & \cos(x+\alpha) & a+x\sin\alpha \\ \sin(x+\beta) & \cos(x+\beta) & b+x\sin\beta \\ \sin(x+\gamma) & \cos(x+\gamma) & c+x\sin\gamma \end{vmatrix}$. is independent of x.

99. Evaluate $\begin{vmatrix} {}^{x}C_1 & {}^{x}C_2 & {}^{x}C_3 \\ {}^{y}C_1 & {}^{y}C_2 & {}^{y}C_3 \\ {}^{z}C_1 & {}^{z}C_2 & {}^{z}C_3 \end{vmatrix}.$

100. (i) Find maximum value of

$$f(x) = \begin{vmatrix} 1+\sin^2 x & \cos^2 x & 4\sin 2x \\ \sin^2 x & 1+\cos^2 x & 4\sin 2x \\ \sin^2 x & \cos^2 x & 1+4\sin 2x \end{vmatrix}.$$

(ii) Let A, B and C be the angles of a triangle, such that $A \geq B \geq C$.

Find the minimum value of Δ, where

$$\Delta = \begin{vmatrix} \sin^2 A & \sin A\cos A & \cos^2 A \\ \sin^2 B & \sin B\cos B & \cos^2 B \\ \sin^2 C & \sin C\cos C & \cos^2 C \end{vmatrix}.$$

101. If $f(x) = \begin{vmatrix} x^2-4x+6 & 2x^2+4x+10 & 3x^2-2x+16 \\ x-2 & 2x+2 & 3x-1 \\ 1 & 2 & 3 \end{vmatrix}$, then find the value of $\int_{-3}^{3} \frac{x^2\sin x}{1+x^6} f(x)dx.$

102. If $Y = sX$ and $Z = tX$ all the variables beings functions of x, then prove that $\begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = X^3 \begin{vmatrix} s_1 & t_1 \\ s_2 & t_2 \end{vmatrix}$, where suffixes denote the order of differentiation with respect to x.

103. If f, g and h are differentiable functions of x and $\Delta = \begin{vmatrix} f & g & h \\ (xf)' & (xg)' & (xh)' \\ (x^2f)'' & (x^2g)'' & (x^2h)'' \end{vmatrix}$, then prove that

$$\Delta' = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3f'')' & (x^3g'')' & (x^3h'')' \end{vmatrix}.$$

104. If $|a_1| > |a_2| + |a_3|$, $|b_2| > |b_1| + |b_3|$ and $|c_3| > |c_1| + |c_2|$, then show that $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \neq 0.$

105. Show that $\begin{vmatrix} (a-a_1)^{-2} & (a-a_1)^{-1} & a_1^{-1} \\ (a-a_2)^{-2} & (a-a_2)^{-1} & a_2^{-1} \\ (a-a_3)^{-2} & (a-a_3)^{-1} & a_3^{-1} \end{vmatrix} = \pm \frac{a^2\,\Pi(a_i - a_j)}{\Pi a_i\,\Pi(a-a_i)^2}$. Write out the terms of the product in the numerator and give the resulting expression its correct sign.

106. Show that in general there are three values of t for which the following system of equations has a non-trivial solution $(a-t)x + by + cz = 0$, $bx + (c-t)y + az = 0$ and $cx + ay + (b-t)z = 0$.

Express the product of these values of t in the form of a determinant.

107. Eliminates

(i) a, b and c

(ii) x, y, z from the equations

$$-a + \frac{by}{z} + \frac{cz}{y} = 0,\quad -b + \frac{cz}{x} + \frac{ax}{z} = 0$$

and $$-c + \frac{ax}{y} + \frac{by}{x} = 0.$$

108. If x, z and y are not all zero and if

$ax + by + cz = 0$, $bx + cy + az = 0$ and $cx + ay + bz = 0$, then prove that $x : y : z = 1 : 1 : 1$ or $1 : \omega : \omega^2$ or $1 : \omega^2 : \omega$

Determinants Exercise 8 : Questions Asked in Previous 13 Year's Exam

- This section contains questions asked in **IIT-JEE, AIEEE, JEE Main & JEE Advanced** from year **2005** to year **2017**.

109. If $a^2 + b^2 + c^2 = -2$ and $f(x) = \begin{vmatrix} 1+a^2x & (1+b^2)x & (1+c^2)x \\ (1+a^2)x & 1+b^2x & (1+c^2)x \\ (1+a^2)x & (1+b^2)x & 1+c^2x \end{vmatrix}$, then $f(x)$ is a polynomial of degree **[AIEEE 2005, 3M]**

(a) 3 (b) 2 (c) 1 (d) 0

110. The system of equations

$$\alpha x + y + z = \alpha - 1,$$
$$x + \alpha y + z = \alpha - 1$$
and $$x + y + \alpha z = \alpha - 1$$

has no solution, if α is **[AIEEE 2005, 3M]**

(a) not -2 (b) 1

(c) -2 (d) Either -2 or 1

111. If $a_1, a_2, a_3, \ldots, a_n, \ldots$ are in GP, then the determinant $\Delta = \begin{vmatrix} \log a_n & \log a_{n+1} & \log a_{n+2} \\ \log a_{n+3} & \log a_{n+4} & \log a_{n+5} \\ \log a_{n+6} & \log a_{n+7} & \log a_{n+8} \end{vmatrix}$ is equal to **[AIEEE 2005, 3M]**

(a) 1 (b) 0 (c) 4 (d) 2

112. If $D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix}$ for $x \neq 0,\ y \neq 0$, then D is **[AIEEE 2007, 3M]**

(a) divisible by neither x nor y

(b) divisible by both x and y

(c) divisible by x but not y

(d) divisible by y but not x

113. Consider the system of equations

$$x - 2y + 3z = -1$$
$$-x + y - 2z = k$$
$$x - 3y + 4z = 1$$

Statement-1 The system of equations has no solutions for $k \neq 3$. **[IIT-JEE 2008, 3M]**

and

Statement-2 The determinant $\begin{vmatrix} 1 & 3 & -1 \\ -1 & -2 & k \\ 1 & 4 & 1 \end{vmatrix} \neq 0$, for $k \neq 3$.

(a) Statement-1 is true, Statement-2 is true and Statement-2 is a correct explanation for Statement-1.

(b) Statement-1 is true, Statement-2 is true and Statement-2 is not a correct explanation for Statement-1.

(c) Statement-1 is true, Statement-2 is false.

(d) Statement-1 is false, Statement-2 is true.

114. Let a, b, c be any real numbers. Suppose that there are real numbers x, y, z not all zero such that $x = cy + bz$, $y = az + cx$ and $z = bx + ay$. Then, $a^2 + b^2 + c^2 + 2abc$ is equal to **[AIEEE 2008, 3M]**

(a) -1 (b) 0 (c) 1 (d) 2

115. Let a, b, c be such that $b(a+c) \neq 0$. If $\begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ (-1)^{n+2}a & (-1)^{n+1}b & (-1)^n c \end{vmatrix} = 0$, then the value of n is **[AIEEE 2009, 4M]**

(a) any integer (b) zero

(c) an even integer (d) any odd integer

116. If $f(\theta)=\begin{vmatrix} 1 & \tan\theta & 1 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$, then the set $\left\{f(\theta):0\le\theta<\frac{\pi}{2}\right\}$ is **[IIT-JEE 2011, 2M]**

(a) $(-\infty,-1)\cup(1,\infty)$ (b) $[2,\infty)$
(c) $(-\infty,0]\cup[2,\infty)$ (d) $(-\infty,-1]\cup[1,\infty)$

117. The number of values of k for which the linear equations

$$4x+ky+2z=0$$
$$kx+4y+z=0$$
$$2x+2y+z=0$$

Possess a non-zero solution is **[AIEEE 2011,4M]**

(a) zero (b) 3 (c) 2 (d) 1

118. If the trivial solution is the only solution of the system of equations

$$x-ky+z=0$$
$$kx+3y-kz=0$$
$$3x+y-z=0$$

Then, the set of values of k is

(a) $\{2,-3\}$ (b) $R-\{2,-3\}$
(c) $R-\{2\}$ (d) $R-\{-3\}$ **[AIEEE 2011, 4M]**

119. The number of values of k for which the system of equations $(k+1)x+8y=4k;\ kx+(k+3)y=3k-1$ has no solution, is

(a) 1 (b) 2
(c) 3 (d) infinite **[JEE Main 2013, 4M]**

120. If $\alpha,\ \beta\neq 0$ and $f(n)=\alpha^n+\beta^n$ and

$$\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix}=k(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2,$$

then k is equal to **[JEE Main 2014, 4M]**

(a) 1 (b) -1
(c) $\alpha\beta$ (d) $1/\alpha\beta$

121. The set of all values of λ for which the system of linear equations

$$2x_1-2x_2+x_3=\lambda x_1$$
$$2x_1-3x_2+2x_3=\lambda x_2$$
$$-x_1+2x_2=\lambda x_3$$

has a non-trivial solution **[JEE Main 2015, 4M]**

(a) contains two elements
(b) contains more than two elements
(c) is an empty set
(d) is a singleton

122. Which of the following values of α satisfy the equation

$$\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2 \\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2 \end{vmatrix}=-648\alpha\ ?$$

[JEE Advanced 2015, 4M]

(a) -4 (b) 9
(c) -9 (d) 4

123. The system of linear equations

$$x+\lambda y-z=0$$
$$\lambda x-y-z=0$$
$$x+y-\lambda z=0$$

has a non-trivial solution for **[JEE Main 2016, 4M]**

(a) exactly one-value of λ
(b) exactly two values of λ
(c) exactly three values of λ
(d) infinitely many values of λ.

124. The total number of distinct $x\in R$ for which

$$\begin{vmatrix} x & x^2 & 1+x^3 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x^2 & 1+27x^3 \end{vmatrix}=10 \text{ is}$$

[JEE Advanced 2016, 3M]

125. Let $a,\lambda,\mu\in R$. Consider the system of linear equations

$$ax+2y=\lambda$$
$$3x-2y=\mu$$

Which of the following statement(s) is (are) correct? **[JEE Advanced 2016, 4M]**

(a) If $a=-3$, then the system has infinitely many solutions for all values of λ and μ
(b) If $a\neq -3$, then the system has a unique solution for all values of λ and μ
(c) If $\lambda+\mu=0$, then the system has infinitely many solutions for $a=-3$
(d) If $\lambda+\mu\neq 0$, then the system has no solution for $a=-3$

126. If S is the set of distinct values of 'b' for which the following system of linear equations

$$x+y+z=1$$
$$x+ay+z=1$$
$$ax+by+z=0$$

has no solution, then S is **[JEE Main 2017, 4M]**

(a) an infinite set
(b) a finite set containing two or more elements
(c) a singleton
(d) an empty set

Answers

Exercise for Session 1

1. (d) 2. (d) 3. (c) 4. (b) 5. (c) 6. (d)
7. (b)

Exercise for Session 2

1. (c) 2. (d) 3. (a) 4. (c) 5. (a) 6. (b)
7. (b) 8. (b) 9. (d) 10. (c) 11. (d)

Exercise for Session 3

1. (b) 2. (c) 3. (c) 4. (b) 5. (b) 6. (d)
7. (d) 8. (a) 9. (b) 10. (c) 11. (c) 12. (a)
13. (a) 14. (a)

Exercise for Session 4

1. (c) 2. (b) 3. (b) 4. (d) 5. (d) 6. (b)
7. (b) 8. (c) 9. (d) 10. (a)

Chapter Exercises

1. (a) 2. (b) 3. (a) 4. (c) 5. (a) 6. (b)
7. (d) 8. (c) 9. (c) 10. (d) 11. (d) 12. (c)
13. (c) 14. (a) 15. (b) 16. (a) 17. (c) 18. (a)
19. (d) 20. (a) 21. (b) 22. (a) 23. (b) 24. (b)
25. (b) 26. (b) 27. (c) 28. (a) 29. (b) 30. (a)
31. (a, b, c, d) 32. (b, d) 33. (a, b, c, d) 34. (b, d)
35. (a, c, d) 36. (a, b, c) 37. (a, b, c, d) 38. (a, b, d)
39. (a, b, c, d) 40. (a, c, d) 41. (a, b) 42. (a, b)
43. (a, b) 44. (a, c) 45. (a, b, c) 46. (a)
47. (b) 48. (c) 49. (c) 50. (b) 51. (b)
52. (c) 53. (c) 54. (d) 55. (a) 56. (a) 57. (c)
58. (c) 59. (a) 60. (d) 61. (b) 62. (d) 63. (b)
64. (b) 65. (a) 66. (c) 67. (2) 68. (9) 69. (2)
70. (1) 71. (3) 72. (6) 73. (1) 74. (8) 75. (3)
76. (8) 77. (A) $\to$ (p, r); (B) $\to$ (p, r); (C) $\to$ (p, q, s, t)
78. (A) $\to$ (p, s, t); (B) $\to$ (r, t); (C) $\to$ (p, q)
79. (A) $\to$ (r); (B) $\to$ (r, t); (C) $\to$ (p, q, s)
80. (A) $\to$ (p, q, r); (B) $\to$ (p, q, r, s, t); (C) $\to$ (p, q, r, s, t)
81. (A) $\to$ (p, r); (B) $\to$ (p, q, r, t); (C) $\to$ (p, r, s)
82. (c) 83. (b) 84. (a) 85. (a) 86. (b) 87. (b)
88. (c) 91. $15\sqrt{2} - 25\sqrt{3}$
92. 0 99. $\frac{1}{12}xyz(x-y)(y-z)(z-x)$
100. (i) 6 (ii) 0 101. 0
105. $-a^2(a_1 - a_2)(a_2 - a_3)(a_3 - a_1)$
106. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$
107. (i) $\frac{yz}{x^2} + \frac{zx}{y^2} + \frac{xy}{z^2} + 1 = 0$ (ii) $a^3 + b^3 + c^3 = 5\,abc$
109. (b) 110. (c) 111. (b) 112. (b) 113. (a) 114. (c)
115. (d) 116. (b) 117. (c) 118. (b) 119. (a) 120. (a)
121. (a) 122. (b, c) 123. (c) 124. (2) 125. (b,c,d)
126. (c)

Solutions

1. $\because f(n)=\alpha^n+\beta^n$

Let $\Delta=\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix}$

$$=\begin{vmatrix} 3 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$$

$$=\begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}\times\begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}=\begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}^2$$

Applying $C_2\to C_2-C_1$ and $C_3\to C_3-C_1$, then

$$\Delta=\begin{vmatrix} 1 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \\ 1 & & \alpha-1 & & \beta-1 \\ \vdots & & & & \\ 1 & & \alpha^2-1 & & \beta^2-1 \end{vmatrix}^2$$

Expanding along R_1, we get

$$\Delta=\begin{vmatrix} \alpha-1 & \beta-1 \\ \alpha^2-1 & \beta^2-1 \end{vmatrix}^2=(\alpha-1)^2(\beta-1)^2\begin{vmatrix} 1 & 1 \\ \alpha+1 & \beta+1 \end{vmatrix}^2$$

$$=(\alpha-1)^2(\beta-1)^2(\beta-\alpha)^2=(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$$

$$=k(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2 \qquad \text{[given]}$$

$\therefore\quad k=1$

2. $\because a, b, c$ and d are in AP. Let D be the common difference,then

$$b=a+D,\ c=a+2D,\ d=a+3D \qquad \text{...(i)}$$

and $\Delta(x)=\begin{vmatrix} x+a & x+b & x+a-c \\ x+b & x+c & x-1 \\ x+c & x+d & x-b+d \end{vmatrix}$

On putting the values of b, c and d from Eq.(i) in $\Delta(x)$, then

$$\Delta(x)=\begin{vmatrix} x+a & x+a+D & x-2D \\ x+a+D & x+a+2D & x-1 \\ x+a+2D & x+a+3D & x+2D \end{vmatrix}$$

Applying $R_2\to R_2-\frac{1}{2}(R_1+R_3)$, then

$$\Delta(x)=\begin{vmatrix} x+a & x+a+D & x-2D \\ & & \vdots \\ 0 & \cdots\quad 0 & \cdots\quad -1 \\ & & \vdots \\ x+a+2D & x+a+3D & x+2D \end{vmatrix}$$

Expanding along R_2, then

$$\Delta(x)=\begin{vmatrix} x+a & x+a+D \\ x+a+2D & x+a+3D \end{vmatrix}$$

Applying $R_2\to R_2-R_1$, then

$$\Delta(x)=\begin{vmatrix} x+a & x+a+D \\ 2D & 2D \end{vmatrix}$$

$$=2D(x+a-x-a-D)=-2D^2$$

Also, $\int_0^2\Delta(x)\,dx=-16$

$\Rightarrow\quad -2D^2(2)=-16$

$\therefore\quad D^2=4$ or $D=\pm 2$

3. Let $\Delta(x)=\begin{vmatrix} x & 1+x^2 & x^3 \\ \log_e(1+x^2) & e^x & \sin x \\ \cos x & \tan x & \sin^2 x \end{vmatrix}$

$$=a+bx+cx^2+\ldots$$

On putting $x=0$, we get

$$\begin{vmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}=a$$

$\therefore\quad 0=a$

or $a=0$, then

$$\Delta(x)=bx+cx^2+\ldots$$

Hence, $\Delta(x)$ is divisible by x.

4. Given, $\begin{vmatrix} a^2 & b^2 & c^2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix}=0$

$$\Rightarrow\quad \begin{vmatrix} a^2 & b^2 & c^2 \\ a^2+2a+1 & b^2+2b+1 & c^2+2c+1 \\ a^2-2a+1 & b^2-2b+1 & c^2-2c+1 \end{vmatrix}=0$$

Applying $R_2\to R_2-R_3$, then

$$\begin{vmatrix} a^2 & b^2 & c^2 \\ 4a & 4b & 4c \\ a^2-2a+1 & b^2-2b+1 & c^2-2c+1 \end{vmatrix}=0$$

Applying $R_3\to R_3-R_1+\frac{1}{2}R_2$, then

$$4\begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}=0$$

$$\Rightarrow\quad -\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}=0 \qquad [\because R_1\to R_3]$$

$$\Rightarrow\quad \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}=0$$

$\Rightarrow\quad (a-b)(b-c)(c-a)=0$

$\therefore\quad a-b=0$ or $b-c=0$ or $c-a=0$

$\Rightarrow\quad a=b$ or $b=c$ or $c=a$

Hence, ΔABC is an isosceles triangle.

5. Let $\Delta = \begin{vmatrix} \alpha & x & x & x \\ x & \beta & x & x \\ x & x & \gamma & x \\ x & x & x & \delta \end{vmatrix}$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$ and $C_4 \to C_4 - C_1$, then

$$\Delta = \begin{vmatrix} \alpha & x-\alpha & x-\alpha & x-\alpha \\ x & \beta - x & 0 & 0 \\ x & 0 & \gamma - x & 0 \\ x & 0 & 0 & \delta - x \end{vmatrix}$$

Expanding along first column, then

$$\Delta = \alpha(\beta - x)(\gamma - x)(\delta - x) - x(x-\alpha)(\gamma - x)(\delta - x) + x(\delta - x)(x-\alpha)(x-\beta) - x(x-\alpha)(\beta - x)(\gamma - x)$$

$$= (x-\alpha)(x-\beta)(x-\gamma)(x-\delta) - x[(x-\alpha)(x-\gamma)(x-\delta) + (x-\beta)(x-\gamma)(x-\delta) + (x-\alpha)(x-\beta)(x-\delta) + (x-\alpha)(x-\beta)(x-\gamma)] \quad \text{[given]}$$

$$= f(x) - x f'(x)$$

$\therefore \quad f(x) = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$

6. Given, $\begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & a+b & c \end{vmatrix} = 0$

$$\Rightarrow \quad \frac{1}{a}\begin{vmatrix} a^2 & b-c & c+b \\ a^2+ac & b & c-a \\ a^2-ab & a+b & c \end{vmatrix} = 0$$

Applying $C_1 \to C_1 + bC_2 + cC_3$, then

$$\Rightarrow \quad \frac{1}{a}\begin{vmatrix} a^2+b^2+c^2 & b-c & c+b \\ a^2+b^2+c^2 & b & c-a \\ a^2+b^2+c^2 & a+b & c \end{vmatrix} = 0$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$\Rightarrow \quad \frac{1}{a}\begin{vmatrix} a^2+b^2+c^2 & \cdots & b-c & \cdots & c-b \\ \vdots & & & & \\ 0 & & c & & -b-a \\ \vdots & & & & \\ 0 & & a+c & & -b \end{vmatrix} = 0$$

Expanding along C_1, then

$$\Rightarrow \quad \frac{(a^2+b^2+c^2)}{a}\begin{vmatrix} c & -b-a \\ a+c & -b \end{vmatrix} = 0$$

$$\Rightarrow \quad \frac{(a^2+b^2+c^2)}{a}[(-bc + (b+a)(a+c)] = 0$$

$$\Rightarrow \quad \frac{(a^2+b^2+c^2)(-bc+ab+bc+a^2+ac)}{a} = 0$$

$$\Rightarrow \quad (a^2+b^2+c^2)(a+b+c) = 0$$

$\because \quad a^2+b^2+c^2 \neq 0$

$\therefore \quad a+b+c = 0$

Therefore, line $ax + by + c = 0$ passes through the fixed point (1, 1).

7. $\because \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3+b^3+c^3-3abc)$

$$= -(a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$$

[where ω is cube roots of unity]

$$= -f(\alpha) f(\beta) f(\gamma) \qquad [\because \alpha = 1, \beta = \omega, \gamma = \omega^2]$$

8. Let $\Delta = \begin{vmatrix} \cos 2x & \sin^2 x & \cos 4x \\ \sin^2 x & \cos 2x & \cos^2 x \\ \cos 4x & \cos^2 x & \cos 2x \end{vmatrix}$

$$= \begin{vmatrix} 1-2\sin^2 x & \sin^2 x & 1-8\sin^2 x(1-\sin^2 x) \\ \sin^2 x & 1-2\sin^2 x & 1-\sin^2 x \\ 1-8\sin^2 x(1-\sin^2 x) & 1-\sin^2 x & 1-2\sin^2 x \end{vmatrix}$$

The required constant term is $\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$

Applying $C_3 \to C_3 - C_1$, then

$$\begin{vmatrix} 1 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & & 1 & & 1 \\ \vdots & & & & \\ 1 & & 1 & & 0 \end{vmatrix} = 1(0-1) = -1$$

9. $\because \quad -1 \le x < 0 \Rightarrow [x] = -1$

$0 \le y < 1 \Rightarrow [y] = 0$

$1 \le z < 2 \Rightarrow [z] = 1$

$$\text{Let } \Delta = \begin{vmatrix} [x]+1 & [y] & [z] \\ [x] & [y]+1 & [z] \\ [x] & [y] & [z]+1 \end{vmatrix} = \begin{vmatrix} 0 & & 0 & & 1 \\ & & \vdots & & \\ -1 & \cdots & 1 & \cdots & 1 \\ & & \vdots & & \\ -1 & & 0 & & 2 \end{vmatrix}$$

Expanding along C_2, then $\Delta = \begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} = 1 = [z]$

10. Let $\Delta = \begin{vmatrix} y^2 & -xy & x^2 \\ a & b & c \\ a' & b' & c' \end{vmatrix} = \frac{1}{xy}\begin{vmatrix} xy^2 & -xy & x^2y \\ ax & b & cy \\ a'x & b' & c'y \end{vmatrix}$

Applying $C_1 \to C_1 + yC_2$ and $C_3 \to C_3 + xC_2$, then

$$\Delta = \frac{1}{xy}\begin{vmatrix} 0 & \cdots & -xy & \cdots & 0 \\ & & \vdots & & \\ ax+by & & b & & bx+cy \\ & & \vdots & & \\ a'x+b'y & & b' & & b'x+c'y \end{vmatrix}$$

Expanding along R_1, then

$$= \frac{1}{xy} \cdot xy \cdot \begin{vmatrix} ax+by & bx+cy \\ a'x+b'y & b'x+c'y \end{vmatrix}$$

$$= \begin{vmatrix} ax+by & bx+cy \\ a'x+b'y & b'x+c'y \end{vmatrix}$$

11. $\because$ In a triangle $A + B + C = \pi$ and $e^{\pi} = \cos\pi + i\sin\pi = -1$

$$e^{i(B+C)} = e^{i(\pi - A)} = e^{i\pi}\cdot e^{iA} = -e^{-iA}$$

$\Rightarrow$ $e^{-i(B+C)} = -e^{iA}$

Similarly, $e^{-i(A+B)} = -e^{iC}$ and $e^{-i(C+A)} = -e^{iB}$

Taking e^{iA}, e^{iB}, e^{iC} common from R_1, R_2 and R_3 respectively, we get

$$\Delta = e^{iA}\cdot e^{iB}\cdot e^{iC}\begin{vmatrix} e^{iA} & e^{-i(A+C)} & e^{-i(A+B)} \\ e^{-i(B+C)} & e^{iB} & e^{-i(A+B)} \\ e^{-i(B+C)} & e^{-i(A+C)} & e^{iC} \end{vmatrix}$$

$$= e^{i\pi} = \begin{vmatrix} e^{iA} & -e^{iB} & -e^{iC} \\ -e^{iA} & e^{iB} & -e^{iC} \\ -e^{iA} & -e^{iB} & e^{iC} \end{vmatrix}$$

Taking e^{iA}, e^{iB}, e^{iC} common from C_1, C_2 and C_3 respectively, we get

$$\Delta = (-1)\,e^{iA}\cdot e^{iB}\cdot e^{iC}\begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix}$$

$$= (-1)\,e^{i\pi}\begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix}$$

$$= (-1)(-1)\begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix}$$

Applying $C_2 \to C_2 + C_1$ and $C_3 \to C_3 + C_1$, then

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ -1 & 0 & -2 \\ -1 & -2 & 0 \end{vmatrix} = 1(0-4) = -4$$

12. Taking x^5 common from R_3, then

$$x^5\begin{vmatrix} x^n & x^{n+2} & x^{2n} \\ 1 & x^a & a \\ x^n & x^{a+1} & x^{2n} \end{vmatrix} = 0,\ \forall\ x \in R$$

$\Rightarrow$ $a + 1 = n + 2 \ \Rightarrow\ a = n + 1$

13. Since, x, y and z are in AP.

$\therefore$ $2y = x + z$ ...(i)

Let $\Delta = \begin{vmatrix} 5 & 4 & 3 \\ x51 & y41 & z31 \\ x & y & z \end{vmatrix}$

$$= \begin{vmatrix} 5 & 4 & 3 \\ 100x + 50 + 1 & 100y + 40 + 1 & 100z + 30 + 1 \\ x & y & z \end{vmatrix}$$

Applying $R_2 \to R_2 - \frac{1}{2}(R_1 + R_3)$, then

$$= \begin{vmatrix} 5 & 0 & 3 \\ 100x + 50 + 1 & 0 & 100z + 30 + 1 \\ x & 0 & z \end{vmatrix}$$ [from Eq. (i)]

$= 0$ [$\because$ all elements of C_2 are zeroes]

14. As $a_1b_1c_1$, $a_2b_2c_2$ and $a_3b_3c_3$ are even natural numbers each of c_1, c_2, c_3 is divisible by 2.

Let $C_i = 2\lambda_i$ for $i = 1, 2, 3$ and $\lambda_i \in N$, then

$$\Delta = \begin{vmatrix} 2\lambda_1 & a_1 & b_1 \\ 2\lambda_2 & a_2 & b_2 \\ 2\lambda_3 & a_3 & b_3 \end{vmatrix} = 2\begin{vmatrix} \lambda_1 & a_1 & b_1 \\ \lambda_2 & a_2 & b_2 \\ \lambda_3 & a_3 & b_3 \end{vmatrix} = 2m$$

where m is some natural number. Thus, Δ is divisible by 2. That Δ may not be divisible by 4 can be seen by taking the three numbers as 112, 122 and 134.

$$\Delta = \begin{vmatrix} 2 & 1 & 1 \\ 2 & 1 & 2 \\ 4 & 1 & 3 \end{vmatrix} = 2(3-2) - 1(6-8) + 1(2-4) = 2$$

which is divisible by 2 but not by 4.

15. Let $\Delta = \begin{vmatrix} c & b\cos B + c\beta & a\cos A + b\alpha + c\gamma \\ a & c\cos B + a\beta & b\cos A + c\alpha + a\gamma \\ b & a\cos B + b\beta & c\cos A + a\alpha + b\gamma \end{vmatrix}$

Applying $C_2 \to C_2 - \beta C_1$ and $C_3 \to C_3 - \gamma C_1$, then

$$\Delta = \begin{vmatrix} c & b\cos B & a\cos A + b\alpha \\ a & c\cos B & b\cos A + c\alpha \\ b & a\cos B & c\cos A + a\alpha \end{vmatrix}$$

Applying $C_3 \to C_3 - \alpha\sec B\, C_2$, then

$$\Delta = \begin{vmatrix} c & b\cos B & a\cos A \\ a & c\cos B & b\cos A \\ b & a\cos B & c\cos A \end{vmatrix} = \cos A\cos B\begin{vmatrix} c & b & a \\ a & c & b \\ b & a & c \end{vmatrix}$$

Applying $C_1 \leftrightarrow C_3$, then $\Delta = -\cos A\cos B\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$$= -\cos A\cos B(a+b+c)\cdot\frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2]$$

Given, $\cos A \neq 0$, $\cos B \neq 0$ and $a + b + c \neq 0$

$\because$ $\Delta = 0$

$\therefore$ $(a-b)^2 + (b-c)^2 + (c-a)^2 = 0$

which is independent, when $a - b = 0$, $b - c = 0$ and $c - a = 0$

i.e., $a = b = c$

Hence, ΔABC is an equilateral.

16. Here, $\left.\begin{matrix} x_1 + x_2 = 6, & x_1x_2 = 3 \\ \text{and}\quad y_1 + y_2 = 4, & y_1y_2 = 2 \end{matrix}\right\}$ ...(i)

Let $\Delta = \begin{vmatrix} x_1x_2 & y_1y_2 & 1 \\ x_1 + x_2 & y_1 + y_2 & 2 \\ \sin(\pi x_1x_2) & \cos\left(\frac{\pi}{2}y_1y_2\right) & 1 \end{vmatrix}$

$$= \begin{vmatrix} 3 & 2 & 1 \\ 6 & 4 & 2 \\ \sin 3\pi & \cos\left(\frac{\pi}{4}\right) & 1 \end{vmatrix}$$ [from Eq. (i)]

Applying $R_2 \to R_2 - 2R_1$, then $\Delta = \begin{vmatrix} 3 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 1 \end{vmatrix} = 0$

17. $\because \Delta = \begin{vmatrix} {}^{10}C_4 & {}^{10}C_5 & {}^{11}C_m \\ {}^{11}C_6 & {}^{11}C_7 & {}^{12}C_{m+2} \\ {}^{12}C_8 & {}^{12}C_9 & {}^{13}C_{m+4} \end{vmatrix}$

Applying $C_2 \to C_2 + C_1$ and use Pascal's rule $({}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r)$, then

$$\Delta = \begin{vmatrix} {}^{10}C_4 & {}^{11}C_5 & {}^{11}C_m \\ {}^{11}C_6 & {}^{12}C_7 & {}^{12}C_{m+2} \\ {}^{12}C_8 & {}^{13}C_9 & {}^{13}C_{m+4} \end{vmatrix} = 0 \qquad \text{[given]}$$

$\therefore \quad m = 5$

18. Let $\Delta = \begin{vmatrix} 1 & \sin(\alpha-\beta)\theta & \cos(\alpha-\beta)\theta \\ a & \sin\alpha\theta & \cos\alpha\theta \\ a^2 & \sin(\alpha-\beta)\theta & \cos(\alpha-\beta)\theta \end{vmatrix}$

Applying $R_1 \to R_1 - R_3$, then

$$\Delta = \begin{vmatrix} 1-a^2 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \\ a & & \sin\alpha\theta & & \cos\alpha\theta \\ \vdots & & & & \\ a^2 & & \sin(\alpha-\beta)\theta & & \cos(\alpha-\beta)\theta \end{vmatrix}$$

Expanding along R_1, then

$$\Delta = (1-a^2)\begin{vmatrix} \sin\alpha\theta & \cos\alpha\theta \\ \sin(\alpha-\beta)\theta & \cos(\alpha-\beta)\theta \end{vmatrix}$$

$$= (1-a^2)\,[\sin\alpha\theta \cdot \cos(\alpha-\beta)\theta - \cos\alpha\theta \cdot \sin(\alpha-\beta)\theta]$$

$$= (1-a^2)\sin(\alpha\theta - \alpha\theta + \beta\theta) = (1-a^2)\sin\beta\theta$$

19. Let $F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ a & b & c \\ p & q & r \end{vmatrix} = mx^4 + nx^3 + rx^2 + sx + t$ ...(i)

On differentiating twice and thrice of Eq. (i) w.r.t.x, then

$$F''(x) = \begin{vmatrix} f''(x) & g''(x) & h''(x) \\ a & b & c \\ p & q & r \end{vmatrix}$$

$$= 12mx^2 + 6nx + 2r \qquad \text{...(ii)}$$

$$F'''(x) = \begin{vmatrix} f'''(x) & g'''(x) & h'''(x) \\ a & b & c \\ p & q & r \end{vmatrix} = 24mx + 6n \qquad \text{...(iii)}$$

On putting $x = 0$ in Eqs. (ii) and (iii), we get

$$\begin{vmatrix} f''(0) & g''(0) & h''(0) \\ a & b & c \\ p & q & r \end{vmatrix} = 2r \qquad \text{...(iv)}$$

and $$\begin{vmatrix} f''(0) & g''(0) & h''(0) \\ a & b & c \\ p & q & r \end{vmatrix} = 6n \qquad \text{...(v)}$$

Now, subtracting Eq. (iv) from Eq. (v), we get

$$\begin{vmatrix} f'''(0) & -f''(0) & g'''(0)-g''(0) & h'''(0) & -h''(0) \\ & a & & b & c \\ & p & & q & r \end{vmatrix}$$

$$= 6n - 2r = 2(3n - r)$$

20. $\because f(x) = \begin{vmatrix} \cos(x+\alpha) & \cos(x+\beta) & \cos(x+\gamma) \\ \sin(x+\alpha) & \sin(x+\beta) & \sin(x+\gamma) \\ \sin(\beta-\gamma) & \sin(\gamma-\alpha) & \sin(\alpha-\beta) \end{vmatrix}$

On differentiating both sides w.r.t. x, then

$$f'(x) = \begin{vmatrix} -\sin(x+\alpha) & -\sin(x+\beta) & -\sin(x+\gamma) \\ \sin(x+\alpha) & \sin(x+\beta) & \sin(x+\gamma) \\ \sin(\beta-\gamma) & \sin(\gamma-\alpha) & \sin(\alpha-\beta) \end{vmatrix}$$

$$+ \begin{vmatrix} \cos(x+\alpha) & \cos(x+\beta) & \cos(x+\gamma) \\ \cos(x+\alpha) & \cos(x+\beta) & \cos(x+\gamma) \\ \sin(\beta-\gamma) & \sin(\gamma-\alpha) & \sin(\alpha-\beta) \end{vmatrix}$$

$$= -\begin{vmatrix} \sin(x+\alpha) & \sin(x+\beta) & \sin(x+\gamma) \\ \sin(x+\alpha) & \sin(x+\beta) & \sin(x+\gamma) \\ \sin(\beta-\gamma) & \sin(\gamma-\alpha) & \sin(\alpha-\beta) \end{vmatrix}$$

$$+ \begin{vmatrix} \cos(x+\alpha) & \cos(x+\beta) & \cos(x+\gamma) \\ \cos(x+\alpha) & \cos(x+\beta) & \cos(x+\gamma) \\ \sin(\beta-\gamma) & \sin(\gamma-\alpha) & \sin(\alpha-\beta) \end{vmatrix}$$

$$= 0 + 0 \qquad [\because R_1 \text{ and } R_2 \text{ are identical}]$$

$$= 0$$

$\therefore f(x) = c$ [constant]

Now, $f(\theta) - 2f(\phi) + f(\psi) = c - 2c + c = 0$

21. Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$

Taking a, b, c common from C_1, C_2, C_3, then $= abc \begin{vmatrix} \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ 1 & 1 & 1 \\ a^2 & b^2 & c^2 \end{vmatrix}$

On multiplying in R_1 by abc, then

$$\Delta = \begin{vmatrix} bc & ca & ab \\ 1 & 1 & 1 \\ a^2 & b^2 & c^2 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 1 \\ bc & ca & ab \\ a^2 & b^2 & c^2 \end{vmatrix} \qquad [R_1 \leftrightarrow R_2]$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} \qquad [R_2 \leftrightarrow R_3]$$

$$= (a-b)(b-c)(c-a)(a+b+c)$$

Now, $D = \begin{vmatrix} 1 & 1 & 1 \\ (x-a)^2 & (x-b)^2 & (x-c)^2 \\ (x-b)(x-c) & (x-c)(x-a) & (x-a)(x-b) \end{vmatrix}$

$$= (b-a)(c-b)(a-c)(3x-a-b-c)$$

Now, given that a, b and c are all different, then $D = 0$

$\therefore \quad 3x - a - b - c = 0$

$\Rightarrow \quad x = \frac{1}{3}(a+b+c)$

22. Given, determinant

$$2a(bc-4a^2)-b(b^2-2ac)+c(2ab-c^2)=0$$

$$\Rightarrow \quad -[(2a)^3+b^3+c^3-3\cdot 2a\cdot b\cdot c]=0$$

$$\Rightarrow \quad \frac{1}{2}(2a+b+c)[(2a-b)^2+(b-c)^2+(c-2a)^2]=0$$

$$\Rightarrow \quad 2a+b+c=0 \qquad \text{...(i) } [\because b\neq c]$$

Let $f(x)=8ax^3+2bx^2+cx+d$

$$\therefore\ f(0)=d \text{ and for } \left(\frac{1}{2}\right)=a+\frac{b}{2}+\frac{c}{2}+d=\frac{2a+b+c}{2}+d$$

$$=\frac{0}{2}+d=d \qquad [\text{from Eq. (i)}]$$

$$\Rightarrow \quad f(0)=f\left(\frac{1}{2}\right)$$

So, $f(x)$ satisfies Rolle's theorem and hence $f'(x)=0$ has atleast one root in $\left[0,\frac{1}{2}\right]$.

23. Given, $\begin{vmatrix} x^3+1 & x^2y & x^2z \\ xy^2 & y^3+1 & y^2z \\ xz^2 & yz^2 & z^3+1 \end{vmatrix}=11$

Taking x, y, z common from C_1, C_2, C_3 respectively, then

$$\Rightarrow \quad xyz\begin{vmatrix} \frac{x^3+1}{x} & x^2 & x^2 \\ y^2 & \frac{y^3+1}{y} & y^2 \\ z^2 & z^2 & \frac{z^3+1}{z} \end{vmatrix}=11$$

On multiplying R_1 by x, R_2 by y and R_3 by z, we get

$$\Rightarrow \quad \begin{vmatrix} x^3+1 & x^3 & x^3 \\ y^3 & y^3+1 & y^3 \\ z^3 & z^3 & z^3+1 \end{vmatrix}=11$$

Applying $R_1\to R_1+R_2+R_3$, then

$$\begin{vmatrix} x^3+y^3+z^3+1 & x^3+y^3+z^3+1 & x^3+y^3+z^3+1 \\ y^3 & y^3+1 & y^3 \\ z^3 & z^3 & z^3+1 \end{vmatrix}=11$$

Applying $C_2\to C_2-C_1$ and $C_3\to C_3-C_1$, then

$$\begin{vmatrix} x^3+y^3+z^3+1 & \ddots & 0 & 0 \\ y^3 & \ddots & 1 & 0 \\ z^3 & & 0 & \ddots\ 1 \end{vmatrix}=11$$

$$\Rightarrow \quad x^3+y^3+z^3+1=11$$

$$\Rightarrow \quad x^3+y^3+z^3=10$$

Therefore, the ordered triplets are (2, 1, 1), (1, 2, 1) and (1, 1, 2).

24. $\because f(x)-x=0$ has imaginary roots.

Then, $f(x)-x>0$ or $f(x)-x<0, \forall\ x\in R$

for $f(x)-x>0, \forall\ x\in R$,

then $f[f(x)]-f(x)>0, \forall\ x\in R$

On adding, we get

$$f[f(x)]-x>0, \forall\ x\in R$$

Similarly, $f[f(x)]-x<0, \forall\ x\in R$

Thus, roots of the equation $f[f(x)]-x=0$ are imaginary

Let $z=\begin{vmatrix} 2 & \alpha & \delta \\ \beta & 0 & \alpha \\ \gamma & \beta & 1 \end{vmatrix}$

Then, $\bar{z}=\begin{vmatrix} 2 & \bar{\alpha} & \bar{\delta} \\ \bar{\beta} & 0 & \bar{\alpha} \\ \bar{\gamma} & \bar{\beta} & 1 \end{vmatrix}=\begin{vmatrix} 2 & \beta & \gamma \\ \alpha & 0 & \beta \\ \delta & \alpha & 1 \end{vmatrix}=\begin{vmatrix} 2 & \alpha & \delta \\ \beta & 0 & \alpha \\ \gamma & \beta & 1 \end{vmatrix}=z$

Hence, z is purely real.

25. For infinitely many solutions

$$\Delta=\Delta_1=\Delta_2=\Delta_3=0$$

$$\Delta=0 \ \Rightarrow\ \begin{vmatrix} 2 & -1 & 1 \\ 1 & -2 & 1 \\ t & -1 & 2 \end{vmatrix}=0 \ \Rightarrow\ t=5$$

For $t=5, \Delta_1=\Delta_2=\Delta_3=0$

Now, $\int_0^{-2t} f(x)\,dx=\int_0^{-10} f(x)\,dx=\int_0^{-5} f(x)\,dx+\int_{-5}^{-10} f(x)\,dx$

$$=\int_{-5}^{-10} f(x+5)\,dx+\int_{-5}^{-10} f(x)\,dx$$

$$=\int_{-5}^{-10}[f(x+5)+f(x)]\,dx$$

$$=\int_{-5}^{-10} 2dx=2(-10+5)$$

$$=-10=-2t$$

26. On putting $x=0$, we get $a_0=1$

On differentiating both sides w.r.t. x and putting $x=0$, we get

$$a_1=4a$$

On differentiating again w.r.t. x and putting $x=0$, we get

$$2a_2=12a^2+8b$$

or $$a_2=6a^2+4b$$

Also, given $\begin{vmatrix} a_1 & a_1 & a_2 \\ a_0 & a_2 & a_0 \\ a_2 & a_0 & a_1 \end{vmatrix}=0$

$$\Rightarrow \quad -(a_0^3+a_1^3+a_2^3-3a_0a_1a_2)=0$$

$$\Rightarrow \quad \frac{1}{2}(a_0+a_1+a_2)[(a_0-a_1)^2+(a_1-a_2)^2+(a_2-a_0)^2]=0$$

$$\because \quad a_0+a_1+a_2\neq 0$$

$$\therefore \quad (a_0-a_1)^2+(a_1-a_2)^2+(a_2-a_0)^2=0$$

$$\Rightarrow \quad a_0-a_1=0, a_1-a_2=0, a_2-a_0=0$$

$$\therefore \quad a_0=a_1=a_2$$

$$\Rightarrow \quad 1=4a=6a^2+4b$$

$$\Rightarrow \quad a=\frac{1}{4} \text{ and } b=\frac{5}{32}$$

27. $\because f(x)=\log_{10}x$ and $g(x)=e^{\pi i x}$

$$\therefore \quad f(10)=\log_{10}10=1$$

and $$g(10)=e^{10\pi i}=(-1)^{10}=1$$

$$f(10^2) = \log_{10} 10^2 = 2$$

and $$g(10^2) = e^{100\pi i} = (-1)^{100} = 1$$

$$f(10^3) = \log_{10} 10^3 = 3$$

and $$g(10^3) = e^{1000\pi i} = (-1)^{1000} = 1$$

Given, $\phi(x) = \begin{vmatrix} f(x)\cdot g(x) & [f(x)]^{g(x)} & 1 \\ f(x^2)\cdot g(x^2) & [f(x^2)]^{g(x^2)} & 0 \\ f(x^3)\cdot g(x^3) & [f(x^3)]^{g(x^3)} & 1 \end{vmatrix}$

$\therefore \quad \phi(10) = \begin{vmatrix} f(10)\cdot g(10) & [f(10)]^{g(10)} & 1 \\ f(10^2)\cdot g(10^2) & [f(10^2)]^{g(10^2)} & 0 \\ f(10^3)\cdot g(10^3) & [f(10^3)]^{g(10^3)} & 1 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 0 \\ 3 & 3 & 1 \end{vmatrix} = 0$$

28. Let $\Delta = \begin{vmatrix} 1 & (\alpha^{2x} - \alpha^{-2x})^2 & (\alpha^{2x} + \alpha^{-2x})^2 \\ 1 & (\beta^{2x} - \beta^{-2x})^2 & (\beta^{2x} + \beta^{-2x})^2 \\ 1 & (\gamma^{2x} - \gamma^{-2x})^2 & (\gamma^{2x} + \gamma^{-2x})^2 \end{vmatrix}$

Applying $C_3 \to C_3 - C_2$, then $\Delta = \begin{vmatrix} 1 & (\alpha^{2x} - \alpha^{-2x})^2 & 4 \\ 1 & (\beta^{2x} - \beta^{-2x})^2 & 4 \\ 1 & (\gamma^{2x} - \gamma^{-2x})^2 & 4 \end{vmatrix} = 0$

29. The given equations can be written as

$$(a-1)\,x - y - z = 0,$$

$$-x + (b-1)\,y - z = 0$$

and $$-x - y + (c-1)\,z = 0$$

For non-trivial solution

$$\begin{vmatrix} a-1 & -1 & -1 \\ -1 & b-1 & -1 \\ -1 & -1 & c-1 \end{vmatrix} = 0$$

Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$, then

$$\begin{vmatrix} a & 0 & -1 \\ 0 & b & -1 \\ -c & -c & c-1 \end{vmatrix} = 0$$

Expanding along R_1, then

$\Rightarrow \quad a(bc - b - c) - 0 - 1(0 + bc) = 0$

$\Rightarrow \quad ab + bc + ca = abc$

30. For non-trivial solution $\begin{vmatrix} \lambda & -1 & \cos\theta \\ 3 & 1 & 2 \\ \cos\theta & 1 & 2 \end{vmatrix} = 0$

Applying $R_3 \to R_3 - R_2$, then

$$\begin{vmatrix} \lambda & -1 & \cos\theta \\ \vdots & & \\ 3 & 1 & 2 \\ \vdots & & \\ \cos\theta - 3 & \cdots\ 0\ \cdots & 0 \end{vmatrix} = 0$$

Expanding along R_3, then

$\Rightarrow \quad (\cos\theta - 3)(-2 - \cos\theta) = 0$

$\Rightarrow \quad (\cos\theta - 3)(2 + \cos\theta) = 0$

$\cos\theta = 3, -2$, where -2 is neglected.

Hence, $\begin{vmatrix} \lambda & -1 & \cos\theta \\ 3 & 1 & 2 \\ \cos\theta & 1 & 2 \end{vmatrix} > 0$ only trivial solution is possible.

31. $\because \Delta = \begin{vmatrix} a^2 + x^2 & ab & ac \\ ab & b^2 + x^2 & bc \\ ac & bc & c^2 + x^2 \end{vmatrix}$

Taking a, b, c common from R_1, R_2, R_3 respectively, then

$$\Delta = abc \begin{vmatrix} \dfrac{a^2 + x^2}{a} & b & c \\ a & \dfrac{b^2 + x^2}{b} & c \\ a & b & \dfrac{c^2 + x^2}{c} \end{vmatrix}$$

On multiplying in C_1, C_2, C_3 by a, b, c respectively, then

$$\Delta = \begin{vmatrix} a^2 + x^2 & b^2 & c^2 \\ a^2 & b^2 + x^2 & c^2 \\ a^2 & b^2 & c^2 + x^2 \end{vmatrix}$$

Now, applying $C_1 \to C_1 + C_2 + C_3$, then

$$\Delta = \begin{vmatrix} x^2 + a^2 + b^2 + c^2 & b^2 & c^2 \\ x^2 + a^2 + b^2 + c^2 & b^2 + x^2 & c^2 \\ x^2 + a^2 + b^2 + c^2 & b^2 & c^2 + x^2 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$\Delta = \begin{vmatrix} x^2 + a^2 + b^2 + c^2 & b^2 & c^2 \\ 0 & \ddots\ x^2 & 0 \\ 0 & 0\ \ddots & x^2 \end{vmatrix}$$

$$= x^4 (x^2 + a^2 + b^2 + c^2)$$

32. Let $\Delta = \begin{vmatrix} \sqrt{6} & 2i & 3 + \sqrt{6} \\ \sqrt{12} & \sqrt{3} + \sqrt{8}\,i & 3\sqrt{2} + \sqrt{6}\,i \\ \sqrt{18} & \sqrt{2} + \sqrt{12}\,i & \sqrt{27} + 2i \end{vmatrix}$

Applying $R_2 \to R_2 - \sqrt{2}\,R_1$ and $R_3 \to R_3 - \sqrt{3}\,R_1$, then

$$\Delta = \begin{vmatrix} \sqrt{6} & \cdots & 2i & \cdots & 3 + \sqrt{6} \\ \vdots & & & & \\ 0 & & \sqrt{3} & & -2\sqrt{3} + 6i \\ \vdots & & & & \\ 0 & & \sqrt{2} & & -3\sqrt{2} + 2i \end{vmatrix}$$

Expanding along C_1, we get

$$= \sqrt{6} \begin{vmatrix} \sqrt{3} & -2\sqrt{3} + \sqrt{6}i \\ \sqrt{2} & -3\sqrt{2} + 2i \end{vmatrix}$$

$$= \sqrt{6}\,[-3\sqrt{6} + 2i\sqrt{3} + 2\sqrt{6} - 2i\sqrt{3}]$$

$$= \sqrt{6}\,(-\sqrt{6}) = -6 \qquad \text{[real and rational]}$$

33. $\sum_{k=1}^{n} 2^{k-1} = 1 + 2 + 2^2 + \ldots + 2^n = 2^n - 1$

$$\sum_{k=1}^{n}\frac{1}{k(k+1)}=\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)$$

$$=\frac{1}{1}-\frac{1}{n+1}=\frac{n}{n+1}$$

and $\sum_{k=1}^{n}\sin k\theta=\dfrac{\sin\left(\frac{n+1}{2}\right)\theta\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$

Given, $D_k=\begin{vmatrix} 2^{k-1} & \frac{1}{k(k+1)} & \sin k\theta \\ x & y & z \\ 2^n-1 & \frac{n}{n+1} & \frac{\sin\left(\frac{n+1}{2}\right)\theta\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \end{vmatrix}$

$$\therefore \sum_{k=1}^{n}D_k=\begin{vmatrix} \sum_{k=1}^{n}2^{k-1} & \sum_{k=1}^{n}\frac{1}{k(k+1)} & \sum_{k=1}^{n}\sin k\theta \\ x & y & z \\ 2^n-1 & \frac{n}{n+1} & \frac{\sin\left(\frac{n+1}{2}\right)\theta\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \end{vmatrix}$$

$$=\begin{vmatrix} 2^n-1 & \frac{n}{n+1} & \frac{\sin\left(\frac{n+1}{2}\right)\theta\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \\ x & y & z \\ 2^n-1 & \frac{n}{n+1} & \frac{\sin\left(\frac{n+1}{2}\right)\theta\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \end{vmatrix}=0$$

34. We have, $\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix}=0$

Applying $C_3\to C_3-\alpha C_1-C_2$, then

$$\begin{vmatrix} a & b & 0 \\ b & c & 0 \\ aa+b & ba+c & -(aa^2+2ba+c) \end{vmatrix}=0$$

Expanding along C_3, we get

$$-(a\alpha^2+2b\alpha+c)(ac-b^2)=0$$

$$\Rightarrow\quad (a\alpha^2+2b\alpha+c)(b^2-ac)=0$$

$$\Rightarrow\quad b^2-ac=0 \text{ and } a\alpha^2+2b\alpha+c=0$$

i.e. a, b and c are in GP and $(x-\alpha)$ is a factor of $ax^2+2bx+c=0$.

35. $\because\quad f(x)=\begin{vmatrix} 2\cos x & 1 & 0 \\ 1 & 2\cos x & 1 \\ 0 & 1 & 2\cos x \end{vmatrix}$

$$=2\cos x(4\cos^2 x-1)-1(2\cos x-0)+0$$

$$=2\cos x(4\cos^2 x-1-1)$$

$$=4\cos x(2\cos^2 x-1)$$

$$=4\cos x\cos 2x$$

$$=2(\cos 3x+\cos x)$$

Option (a)

$$f\left(\frac{\pi}{3}\right)=2\left(\cos\frac{3\pi}{3}+\cos\frac{\pi}{3}\right)=2\left(-1+\frac{1}{2}\right)=-1$$

Option (b)

$$f'(x)=2(-3\sin 3x-\sin x)$$

$$\therefore\quad f'\left(\frac{\pi}{3}\right)=2\left(-3\sin\pi-\sin\frac{\pi}{3}\right)=2\left(0-\frac{\sqrt{3}}{2}\right)=-\sqrt{3}$$

Option (c)

$$\int_0^{\pi}f(x)\,dx=2\int_0^{\pi}(\cos 3x+\cos x)\,dx=2\left[\frac{\sin 3x}{3}+\sin x\right]_0^{\pi}$$

$$=2[(0+0)-(0+0)]=0$$

Option (d)

$$\int_0^{\pi}f(x)\,dx=2\int_{-\pi}^{\pi}(\cos 3x+\cos x)\,dx=4\int_0^{\pi}(\cos 3x+\cos x)\,dx$$

$$=0 \qquad \text{[from option (c)]}$$

36. $\because\ \Delta(x)=\begin{vmatrix} x^2-5x+3 & 2x-5 & 3 \\ 3x^3+x+4 & 6x+1 & 9 \\ 7x^2-6x+9 & 14x-6 & 21 \end{vmatrix}$

Applying $R_2\to R_2-3R_1$ and $R_3\to R_3-7R_1$, then

$$=\begin{vmatrix} x^2-5x+3 & \cdots & 2x-5 & \cdots & 3 \\ & & & & \vdots \\ 16x-5 & & 16 & & 0 \\ & & & & \vdots \\ 29x-12 & & 29 & & 0 \end{vmatrix}$$

Expanding along C_3, we get $=3\begin{vmatrix} 16x-5 & 16 \\ 29x-12 & 29 \end{vmatrix}$

Applying $C_1\to C_1-xC_2$, then

$$\Delta(x)=3\begin{vmatrix} -5 & 16 \\ -12 & 29 \end{vmatrix}=3(-145+192)=3\times 47$$

$$=141=ax^3+bx^2+cx+d \text{ [given]}$$

$$\therefore\quad a=0, b=0, c=0, d=141$$

37. $\because\ \Delta=\begin{vmatrix} a^2 & b\sin A & c\sin A \\ b\sin A & 1 & \cos A \\ c\sin A & \cos A & 1 \end{vmatrix}$

Taking common a from each R_1 and C_1, then

$$\Delta = \begin{vmatrix} 1 & \frac{b\sin A}{a} & \frac{c\sin A}{a} \\ \frac{b\sin A}{a} & 1 & \cos A \\ \frac{c\sin A}{a} & \cos A & 1 \end{vmatrix} = \begin{vmatrix} 1 & \sin B & \sin C \\ \sin B & 1 & \cos A \\ \sin C & \cos A & 1 \end{vmatrix}$$

[by sine rule]

Applying $C_2 \to C_2 - \sin B C_1$ and $C_3 \to C_3 - \sin C C_1$, then

$$\Delta = \begin{vmatrix} 1 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \\ \sin B & & 1-\sin^2 B & & \cos A - \sin B \sin C \\ \vdots & & & & \\ \sin C & & \cos A - \sin B \sin C & & 1-\sin^2 C \end{vmatrix}$$

Expanding along R_1, then

$$\Delta = \begin{vmatrix} \cos^2 B & \cos[\pi - (B+C)] \\ \cos[\pi - (B+C)] - \sin B \sin C & -\sin B \sin C \\ & \cos^2 C \end{vmatrix}$$

$[\because A + B + C = \pi]$

$$= \begin{vmatrix} \cos^2 B & -\cos(B+C) - \sin B \sin C \\ -\cos(B+C) - \sin B \sin C & \cos^2 C \end{vmatrix}$$

$$= \begin{vmatrix} \cos^2 B & -\cos B \cos C \\ -\cos B \cos C & \cos^2 C \end{vmatrix}$$

$$= \cos^2 B \cos^2 C - \cos^2 B \cos^2 C = 0$$

38. $\because f(a,b) = \begin{vmatrix} a & a^2 & 0 \\ 1 & (2a+b) & (a+b)^2 \\ 0 & 1 & (2a+3b) \end{vmatrix}$

Applying $C_2 \to C_2 - aC_1$, then

$$f(a,b) = \begin{vmatrix} a & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \\ 1 & & (a+b) & & (a+b)^2 \\ \vdots & & & & \\ 0 & & 1 & & (2a+3b) \end{vmatrix}$$

Expanding along R_1, then

$$f(a,b) = a\begin{vmatrix} (a+b) & (a+b)^2 \\ 1 & (2a+3b) \end{vmatrix}$$

$$= a(a+b)\begin{vmatrix} 1 & (a+b) \\ 1 & (2a+3b) \end{vmatrix}$$

$$= a(a+b)(2a+3b-a-b)$$

$$= a(a+b)(a+2b)$$

39. $\because$ $f(x) = \begin{vmatrix} \sec^2 x & 1 & 1 \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cot^2 x \end{vmatrix}$

Applying $C_2 \to C_2 - \cos^2 x\, C_1$, then

$$f(x) = \begin{vmatrix} \sec^2 x & 0 & 1 \\ & \vdots & \\ \cos^2 x & \cdots \quad \cos^2 x - \cos^4 x \quad \cdots & \operatorname{cosec}^2 x \\ & \vdots & \\ 1 & 0 & \cot^2 x \end{vmatrix}$$

Expanding along C_2, then

$$f(x) = \sin^2 x \cos^2 x \begin{vmatrix} \sec^2 x & 1 \\ 1 & \cot^2 x \end{vmatrix}$$

$$= \sin^2 x \cos^2 x (\operatorname{cosec}^2 x - 1)$$

$$= \sin^2 x \cos^2 x \cot^2 x = \cos^4 x$$

option (a)

$$\int_{-\pi/4}^{\pi/4} f(x)\,dx = \int_{-\pi/4}^{\pi/4} \cos^4 x\,dx = 2\int_0^{\pi/4} \cos^4 x\,dx$$

$$= 2\int_0^{\pi/4} \left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$= 2 \times \frac{1}{2}\int_0^{\pi/2} \left(\frac{1+\cos x}{2}\right)^2 dx$$

$$= \frac{1}{4}\int_0^{\pi/2} (1 + 2\cos x + \cos^2 x)\,dx$$

$$= \frac{1}{4}\int_0^{\pi/2} 1 \cdot dx + \frac{1}{2}\int_0^{\pi/2} \cos x\,dx + \frac{1}{4}\int_0^{\pi/2} \cos^2 x\,dx$$

$$= \frac{1}{4}\left(\frac{\pi}{2} - 0\right) + \frac{1}{2}(\sin x)_0^{\pi/2} + \frac{1}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= \frac{\pi}{8} + \frac{1}{2}(1-0) + \frac{\pi}{16} = \frac{1}{16}(2\pi + 8 + \pi) = \frac{1}{16}(3\pi + 8)$$

option (b)

$\because$ $f'(x) = 4\cos^3 x \cdot (-\sin x)$

$\therefore$ $f'\left(\frac{\pi}{2}\right) = 0$

option (c) and (d)

$\because$ $0 \le \cos^4 x \le 1$

$\because$ Maximum value of $f(x)$ is 1.

and minimum value of $f(x)$ is 0.

40. Let $\Delta = \begin{vmatrix} a & a+x^2 & a+x^2+x^4 \\ 2a & 3a+2x^2 & 4a+3x^2+2x^4 \\ 3a & 6a+3x^2 & 10a+6x^2+3x^4 \end{vmatrix}$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$, then

$$\Delta = \begin{vmatrix} a & a+x^2 & a+x^2+x^4 \\ 0 & a & 2a+x^2 \\ 0 & 3a & 7a+3x^2 \end{vmatrix}$$

Applying $R_3 \to R_3 - 3R_2$, then

$$\Delta = \begin{vmatrix} a & \ddots & a+x^2 & a+x^2+x^4 \\ 0 & & a & 2a+x^2 \\ 0 & & 0 & \ddots \quad a \end{vmatrix}$$

$$= a^3 = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 \quad \text{[given]}$$

$\therefore \quad a_0 = a^3, a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0, a_6 = 0, a_7 = 0$

and $\quad f(x) = a_0 x^2 + a_3 x + a_6 = a^3 x^2$

option (a) $f(x) \geq 0 \Rightarrow a^3 x^2 \geq 0$

If $a^3 > 0$, then $x^2 \geq 0$

$\therefore \quad a > 0, x \in R$

option (b) If $a = 0$, then $f(x) = 0$

and $\quad$ If $x = 0$, then $f(x) = 0$

$\therefore$ Aliter (b) is fail

option (c) $f(x) = 0$

$\Rightarrow \quad a^3 x^2 = 0$ or $x^2 = 0$

$\therefore \quad x = 0, 0$

option (d) For $a = 0$, $f(x) = 0$ is an identity, then it has more than two roots.

41. Let $\Delta(x) = \begin{vmatrix} 4x-4 & (x-2)^2 & x^3 \\ 8x-4\sqrt{2} & (x-2\sqrt{2})^2 & (x+1)^3 \\ 12x-4\sqrt{3} & (x-2\sqrt{3})^2 & (x-1)^3 \end{vmatrix}$

$= a_0 + a_1 x + a_2 x^2 + \ldots \ldots$(i)

On putting $x = 0$ in Eq. (i), then

$$\begin{vmatrix} -4 & 4 & 0 \\ -4\sqrt{2} & 8 & 1 \\ -4\sqrt{3} & 12 & -1 \end{vmatrix} = a_0$$

or $\quad a_0 = -4(-8-12) - 4(4\sqrt{2} + 4\sqrt{3})$

$= 16(5 - \sqrt{2} - \sqrt{3}) =$ term independent of x in Δ.

Also, on differentiating Eq. (i) w.r.t. x and then put $x = 0$, we get

$$\begin{vmatrix} 4 & -4 & 0 \\ -4\sqrt{2} & 8 & 1 \\ -4\sqrt{3} & 12 & -1 \end{vmatrix} + \begin{vmatrix} -4 & 4 & 0 \\ 8 & -4\sqrt{2} & 3 \\ -4\sqrt{3} & 12 & -1 \end{vmatrix} + \begin{vmatrix} -4 & 4 & 0 \\ -4\sqrt{2} & 8 & 1 \\ 12 & -4\sqrt{3} & 3 \end{vmatrix} = a_1$$

$\therefore \quad a_1 = 4(-8-12) + 4(4\sqrt{2} + 4\sqrt{3})$

$- 4(4\sqrt{2} - 36) - 4(-8 + 12\sqrt{3})$

$- 4(24 + 4\sqrt{3}) - 4(-12\sqrt{2} - 12)$

$= 48 + 48\sqrt{2} - 48\sqrt{3} = 48(1 + \sqrt{2} - \sqrt{3})$

$=$ Coefficient of x in $\Delta(x)$

42. $\because \quad f(x) = \begin{vmatrix} 3 & 3x & 3x^2 + 2a^2 \\ 3x & 3x^2 + 2a^2 & 3x^3 + 6a^2 x \\ 3x^2 + 2a^2 & 3x^3 + 6a^2 x & 3x^4 + 12a^2 x^2 + 2a^4 \end{vmatrix}$

Applying $C_3 \to C_3 - xC_2$ and $C_2 \to C_2 - xC_1$, then

$$f(x) = \begin{vmatrix} 3 & 0 & 2a^2 \\ 3x & 2a^2 & 4a^2 x \\ 3x^2 + 2a^2 & 4a^2 x & 6a^2 x^2 + 2a^4 \end{vmatrix}$$

Applying $C_3 \to C_3 - x C_2$, then

$$= \begin{vmatrix} 3 & 0 & 2a^2 \\ 3x & 2a^2 & 2a^2 x \\ 3x^2 + 2a^2 & 4a^2 x & 2a^2 x^2 + 2a^4 \end{vmatrix}$$

$$= 4a^4 \begin{vmatrix} 3 & 0 & 1 \\ 3x & 1 & x \\ 3x^2 + 2a^2 & 2x & x^2 + a^2 \end{vmatrix}$$

Applying $C_1 \to C_1 - 3C_3$, then

$$f(x) = 4a^4 \begin{vmatrix} 0 & 0 & 1 \\ \vdots & & \\ 0 & 1 & x \\ \vdots & & \\ -a^2 \quad \cdots & 2x \quad \cdots & x^2 + a^2 \end{vmatrix}$$

Expanding along C_1, we get

$= 4a^4 [-a^2 (0 - 1)] = 4a^6$

$\therefore \quad f'(x) = 0$

i.e. $y = f(x)$ is a straight line parallel to X-axis.

43. $\because a > b > c$ and given equations are

$$ax + by + cz = 0,$$
$$bx + cy + az = 0$$

and $\quad cx + ay + bz = 0$

For non-trivial solution

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

$\Rightarrow \quad 3abc - (a^3 + b^3 + c^3) = 0$

$\therefore \quad a + b + c = 0$

If α and β be the roots of $at^2 + bt + c = 0$

$\therefore \quad \alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

and $\quad D = b^2 - 4ac = (-a - c)^2 - 4ac = (a - c)^2 > 0$

For opposite sign $|\alpha - \beta| > 0$

$\Rightarrow \quad (\alpha - \beta)^2 > 0 \Rightarrow (\alpha + \beta)^2 - 4\alpha\beta > 0$

$\Rightarrow \quad \frac{b^2}{a^2} - \frac{4c}{a} > 0 \Rightarrow (-a - c)^2 - 4ac > 0$

$\Rightarrow \quad (a - c)^2 > 0$, true

Hence, the roots are real and have opposite sign.

44. Here, $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & \lambda & 3 \end{vmatrix} = 1(9 - 2\lambda) - 1(3 - 2) + 1(\lambda - 3)$

$= -(\lambda - 5)$

$$\Delta_1 = \begin{vmatrix} 3 & 1 & 1 \\ 6 & 3 & 2 \\ b & \lambda & 3 \end{vmatrix} = 3(9 - 2\lambda) - 1(18 - 2b) + 1(6\lambda - 3b)$$

$= -(b - 9)$

$$\Delta_2 = \begin{vmatrix} 1 & 3 & 1 \\ 1 & 6 & 2 \\ 1 & b & 3 \end{vmatrix} = 1$$

$$(18 - 2b) - 3(3 - 2) + 1(b - 6) = -(b - 9)$$

and $\Delta_3 = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 3 & 6 \\ 1 & \lambda & b \end{vmatrix} = 1$

$$(3b - 6\lambda) - 1(b - 6) + 3(\lambda - 3) = (2b - 3\lambda - 3)$$

Aliter (a) for unique solution $\Delta \neq 0$

i.e. $\lambda \neq 5, b \in R$

Aliter (b) for no solution

$D = 0$ and atleast one of $\Delta_1, \Delta_2, \Delta_3$ is non-zero

$\therefore \quad \lambda = 5, b \neq 9$

Aliter (c) For infinite many solution

$\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$

$\therefore \quad \lambda = 5, b = 9$

45. For non-trivial solutions

$$\begin{vmatrix} \lambda & \sin\alpha & \cos\alpha \\ 1 & \cos\alpha & \sin\alpha \\ -1 & \sin\alpha & -\cos\alpha \end{vmatrix} = 0$$

Expanding along C_1, we get

$\Rightarrow \quad \lambda(-\cos^2\alpha - \sin^2\alpha) - 1(-\sin\alpha\cos\alpha - \sin\alpha\cos\alpha)$

$$-1(\sin^2\alpha - \cos^2\alpha) = 0$$

$\Rightarrow \quad -\lambda + \sin2\alpha + \cos2\alpha = 0$

$\Rightarrow \quad \lambda = (\sin2\alpha + \cos2\alpha)$

$\because \quad -\sqrt{2} \le \sin2\alpha + \cos2\alpha \le \sqrt{2}$

$\therefore \quad -\sqrt{2} \le \lambda \le \sqrt{2}$

$\Rightarrow \quad S = [-\sqrt{2}, \sqrt{2}]$

Sol. (Q. Nos. 46 to 48)

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & \lambda \end{vmatrix} = (\lambda - 5),$$

$$\Delta_1 = \begin{vmatrix} 5 & 1 & 1 \\ 9 & 2 & 3 \\ \mu & 3 & \lambda \end{vmatrix} = (\lambda + \mu - 18),$$

$$\Delta_2 = \begin{vmatrix} 1 & 5 & 1 \\ 1 & 9 & 3 \\ 1 & \mu & \lambda \end{vmatrix} = (4\lambda - 2\mu + 6)$$

and $\Delta_3 = \begin{vmatrix} 1 & 1 & 5 \\ 1 & 2 & 9 \\ 1 & 3 & \mu \end{vmatrix} = (\mu - 13)$

46. The system is smart, if

$\Delta \neq 0 \Rightarrow \lambda \neq 5$

or $\quad \Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$

$\Rightarrow \quad \lambda = 5$ and $\mu = 13$

47. The system is good, if

$\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$

$\Rightarrow \quad \lambda = 5$ and $\mu = 13$

48. The system is lazy, if

$\Delta = 0$ and atleast one of $\Delta_1, \Delta_2, \Delta_3 \neq 0$

$\Rightarrow \quad \lambda = 5$ and $\mu \neq 13$

Sol. (Q. Nos. 49 to 51)

$$\because \quad \begin{vmatrix} bc - a^2 & ca - b^2 & ab - c^2 \\ ca - b^2 & ab - c^2 & bc - a^2 \\ ab - c^2 & bc - a^2 & ca - b^2 \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 \qquad \text{...(i)}$$

For $a = 1, b = x$ and $c = x^2$

$$\begin{vmatrix} x^3 - 1 & 0 & x - x^4 \\ 0 & x - x^4 & x^3 - 1 \\ x - x^4 & x^3 - 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ x & x^2 & 1 \\ x^2 & 1 & x \end{vmatrix}^2$$

$\therefore \quad \Delta = 5^2 = 25$

49. $\because \quad \Delta^2 = (25)^2 = 625$

Sum of digits of $\Delta^2 = 6 + 2 + 5 = 13$

50. From Eq. (i), we get

$$\begin{vmatrix} A & B & C \\ B & C & A \\ C & A & B \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$$

$$\Rightarrow \quad 49 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$$

$$\Rightarrow \quad q \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \pm 7$$

$\Rightarrow \quad -(a^3 + b^3 + c^3 - 3abc) = \pm 7$

$\Rightarrow \quad a^3 + b^3 + c^3 - 3abc = \mp 7$

$\therefore \quad a^3 + b^3 + c^3 - 3abc = 7 \qquad [\because a + b + c > 0]$

51. $\because \quad aA + bB + cC = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc)$

$$= -(-3) = 3$$

Sol. (Q. Nos. 52 to 54)

$\because \quad \alpha + \beta + \gamma = -2, \alpha\beta + \beta\gamma + \gamma\alpha = -1$ and $\alpha\beta\gamma = 3$

52. $\begin{vmatrix} \alpha & \beta & \gamma \\ \gamma & \alpha & \beta \\ \beta & \gamma & \alpha \end{vmatrix} = -\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix} = \alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma$

$$= (\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha)$$

$$= (\alpha + \beta + \gamma)[(\alpha + \beta + \gamma)^2 - 3(\alpha\beta + \beta\gamma + \gamma\alpha)]$$

$$= (-2)(4 - 9) = 10$$

53. Let $x = \dfrac{\alpha - 1}{\alpha + 2} \Rightarrow \alpha = \dfrac{2x + 1}{1 - x}$

$\because \quad \alpha$ is a root of $x^3 + 2x^2 - x - 3 = 0$

$\Rightarrow \quad \alpha^3 + 2\alpha^2 - \alpha - 3 = 0$

$$\Rightarrow \left(\frac{2x+1}{1-x}\right)^3+2\left(\frac{2x+1}{1-x}\right)^2-\left(\frac{2x+1}{1-x}\right)-3=0$$

$$\Rightarrow x^3+6x^2+21x-1=0 \quad ...(i)$$

Hence, $\frac{\alpha-1}{\alpha+2}, \frac{\beta-1}{\beta+2}$ and $\frac{\gamma-1}{\gamma+2}$ are the roots of Eq. (i), then

$$\frac{\alpha-1}{\alpha+2}+\frac{\beta-1}{\beta+2}+\frac{\gamma-1}{\gamma+2}=-6$$

$$\therefore \left|\frac{\alpha-1}{\alpha+2}+\frac{\beta-1}{\beta+1}+\frac{\gamma-1}{\gamma+2}\right|=\frac{6}{1}=\frac{m}{n}$$

$$\Rightarrow m=6 \text{ and } n=1,$$

then $$\begin{vmatrix} m & n^2 \\ m-n & m+n \end{vmatrix}=\begin{vmatrix} 6 & 1 \\ 5 & 7 \end{vmatrix}=42-5=37$$

54. $\because$ $$\begin{vmatrix} a & b & b \\ b & a & b \\ b & b & a \end{vmatrix}=\begin{vmatrix} a & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}^2=(\alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma)^2$$

$$=(\alpha+\beta+\gamma)^2[(\alpha+\beta+\gamma)^2-3(\alpha\beta+\beta\gamma+\gamma\alpha)]^2$$

$$=(-2)^2[(-2)^2+3]^2=4\times 49=196$$

Sol. (Q. Nos. 55 to 57)

$\because$ $$f'(x)=\begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$$

Applying $C_2 \to C_2-C_1$ and $C_3 \to C_3-C_1$, then

$$f'(x)=\begin{vmatrix} 2ax & -1 & b+1 \\ b & 1 & -1-b \\ 2ax+2b & 1 & -b \end{vmatrix}$$

Applying $R_3 \to R_3-R_1$, then

$$f'(x)=\begin{vmatrix} 2ax & -1 & b+1 \\ b & 1 & -1-b \\ 2b & 2 & -2b-1 \end{vmatrix}$$

Applying $R_3 \to R_3-2R_2$, then

$$f'(x)=\begin{vmatrix} 2ax & -1 & b+1 \\ & & \vdots \\ b & 1 & -1-b \\ & & \vdots \\ 0 \quad \cdots & 0 \quad \cdots & 1 \end{vmatrix}$$

$$\Rightarrow f'(x)=(2ax+b)$$

$$\therefore f(x)=ax^2+bx+c$$

$$f(0)=2 \Rightarrow c=2 \quad ...(i)$$

and $$f(1)=1 \Rightarrow a+b+2=1 \Rightarrow a+b=-1 \quad ...(ii)$$

Also, $$f'\left(\frac{5}{2}\right)=0 \Rightarrow 5a+b=0 \quad ...(iii)$$

From Eqs. (ii) and (iii), we get

$$a=\frac{1}{4} \text{ and } b=-\frac{5}{4}$$

$$\therefore f(x)=\frac{x^2}{4}-\frac{5x}{4}+2$$

55. $\because$ $f(2)+f(3)=\left(\frac{4}{4}-\frac{10}{4}+2\right)+\left(\frac{9}{4}-\frac{15}{4}+2\right)=1$

56. $\because$ $f(x)+1=0 \Rightarrow \frac{x^2}{4}-\frac{5x}{4}+3=0$

$\because$ $D=\frac{25}{16}-3=-\frac{23}{16}<0$

$\therefore$ Number of solutions = 0

57. Minimum value of $f(x)=-\frac{D}{4a}=-\frac{-\left(\frac{25}{16}-2\right)}{1}=\frac{7}{16}$

Hence, range of $f(x)$ is $\left[\frac{7}{16}, \infty\right)$

Sol. (Q. Nos. 58 to 60)

Put $x=1$ on both sides, we get

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ \tan 1 & \sin^2 1 & \cos^2 1 \end{vmatrix}=a_0 \Rightarrow 0=a_0$$

we observe that

$$a_1=f'(1)$$

where $$f(x)=\begin{vmatrix} x & e^{x-1} & (x-1)^3 \\ x-\ln x & \cos(x-1) & (x-1)^2 \\ \tan x & \sin^2 x & \cos^2 x \end{vmatrix}$$

$$f'(x)=\begin{vmatrix} 1 & e^{x-1} & 3(x-1)^2 \\ x-\ln x & \cos(x-1) & (x-1)^2 \\ \tan x & \sin^2 x & \cos^2 x \end{vmatrix}$$

$$+\begin{vmatrix} x & e^{x-1} & (x-1)^3 \\ 1-\frac{1}{x} & -\sin(x-1) & 2(x-1) \\ \tan x & \sin^2 x & \cos^2 x \end{vmatrix}$$

$$+\begin{vmatrix} x & e^{x-1} & (x-1)^3 \\ x-\ln x & \cos(x-1) & (x-1)^2 \\ \sec^2 x & \sin 2x & -\sin 2x \end{vmatrix}$$

$$\Rightarrow f'(1)=\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ \tan 1 & \sin^2 1 & \cos^2 1 \end{vmatrix}+\begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ \tan 1 & \sin^2 1 & \cos^2 1 \end{vmatrix}$$

$$+\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ \sec^2 1 & \sin 2 & -\sin 2 \end{vmatrix}$$

$$=0+0+0=0$$

$$\therefore a_1=0$$

58. $\cos^{-1}(a_1)=\cos^{-1}(0)=\frac{\pi}{2}$

59. Let $P=\lim_{x\to a_0}(\sin x)^x=\lim_{x\to 0}(\sin x)^x$

$$\therefore \ln p=\lim_{x\to 0} x \ln \sin x \qquad [\text{form } (0\times\infty)]$$

$$=\lim_{x\to 0}\frac{\ln \sin x}{Yx}=\lim_{x\to 0}\frac{\cot x}{-Yx^2} \qquad [\text{ by } L' \text{ Hospital's Rule}]$$

$$= -\lim_{x\to 0}\frac{x^2}{\tan x} = -1\times 0 = 0$$

$\therefore \quad P = e^0 = 1$

60. Required Equation is

$$(x-a_0)(x-a_1) = 0$$

$\Rightarrow \quad (x-0)(x-0) = 0$

$\Rightarrow \quad x^2 = 0$

Sol. (Q. Nos. 61 to 63)

Multiplying R_1, R_2, R_3 by a, b, c respectively and then taking a, b, c common from C_1, C_2, C_3, we get

$$\Delta = \begin{vmatrix} -bc & ab+ac & ac+ab \\ ab+bc & -ac & bc+ab \\ ac+bc & bc+ac & -ab \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$ and then taking $(ab+bc+ca)$ from C_2 and C_3, we get

$$\Delta = (ab+bc+ca)^2 \begin{vmatrix} -bc & 1 & 1 \\ ab+bc & -1 & 0 \\ ac+bc & 0 & -1 \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we get

$$= (ab+bc+ca)^2 \begin{vmatrix} ab+bc+ca & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \\ ab+bc & & -1 & & 0 \\ \vdots & & & & \\ ac+bc & & 0 & & -1 \end{vmatrix}$$

$$= (ab+bc+ca)^3 \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = (ab+bc+ca)^3$$

Also, a, b and c are the roots of

$$x^3 - px^2 + qx - r = 0$$

$\because \quad a+b+c = p,\ ab+bc+ca = q,\ abc = r$

$\Rightarrow \quad \Delta = q^3 \qquad ...(i)$

61. $\because$ AM $\geq$ GM

$$\Rightarrow \quad \left(\frac{ab+bc+ca}{3}\right) \geq (ab\cdot bc\cdot ca)^{1/3}$$

$$\Rightarrow \quad \frac{q}{3} \geq (r^2)^{1/3} \ \Rightarrow\ q^3 \geq 27r^2$$

or $\quad \Delta \geq 27r^2$ [from Eq. (i)]

62. $\because$ a, b and c are in GP.

$\therefore \quad mb^2 = ac \Rightarrow b^3 = abc = r \ \Rightarrow\ b = r^{1/3}$

and b is a root of $x^3 - px^2 + qx - r = 0$

$\Rightarrow \quad b^3 - pb^2 + qb - r = 0$

$\Rightarrow \quad r - pr^{2/3} + qr^{1/3} - r = 0$

$\Rightarrow \quad p^3r^2 = q^3r$

$\therefore \quad q^3 = p^3r$

63. $\because \quad \Delta = 27 \Rightarrow q^3 = 27$

$\therefore \quad q = 3$

or $\quad ab+bc+ca = 3$ and $a^2+b^2+c^2 = 2$

$\therefore \quad \sum a^2b = a^2b + a^2c + b^2a + b^2c + c^2a + c^2b$

$$= (a+b+c)(ab+bc+ca) - 3abc$$

$$= 3p - 3r$$

$$= 6\sqrt{2} - 3r$$

$[\because (a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)]$

$= 3(2\sqrt{2} - r) \qquad [\because p^2 = 8 \Rightarrow p = 2\sqrt{2}]$

Sol. (Q. Nos. 64 to 66)

Taking a, b, c common from R_1, R_2, R_3 respectively and then multiplying by a, b, c is C_1, C_2, C_3 respectively, we get

$$\Delta_n = \begin{vmatrix} a^2+n & b^2 & c^2 \\ a^2 & b^2+n & c^2 \\ a^2 & b^2 & c^2+n \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, then

$$\Delta_n = \begin{vmatrix} n+a^2+b^2+c^2 & b^2 & c^2 \\ n+a^2+b^2+c^2 & b^2+n & c^2 \\ n+a^2+b^2+c^2 & b^2 & c^2+n \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$\Delta_n = \begin{vmatrix} n+a^2+b^2+c^2 & b^2 & c^2 \\ & \ddots & \\ 0 & n & 0 \\ & & \ddots \\ 0 & 0 & n \end{vmatrix}$$

$\Delta_n = n^3 + n^2(a^2+b^2+c^2) \qquad ...(i)$

Also, $a+b+c = \lambda$

$3b = \lambda \qquad [\because a, b, c \text{ are in AP}]$

$\therefore \quad b = \frac{\lambda}{3}$

Also, b is root of $x^3 - \lambda x^2 + 11x - 6 = 0$

$\Rightarrow \quad b^3 - \lambda b^2 + 11b - 6 = 0$

$\Rightarrow \quad \frac{\lambda^3}{27} - \frac{\lambda^3}{9} + \frac{11\lambda}{3} - 6 = 0$

$\Rightarrow \quad 2\lambda^3 - 99\lambda + 162 = 0$

$\because \quad \lambda = 6$

Then, equation becomes $x^3 - 6x^2 + 11x - 6 = 0$

$\therefore \quad x = 1, 2, 3$

Let $\quad a = 1, b = 2$ and $c = 3$

From Eq. (i), we get

$$\Delta_n = n^3 + 14n^2$$

$$\therefore \quad \sum_{n=1}^{n} \Delta_n = \frac{n(n+1)(3n^2+59n+28)}{12}$$

64. $\displaystyle\sum_{r=1}^{7} \Delta_r = \frac{7\cdot 8(147+59\cdot 7+28)}{12} = (14)^3$

65. $\dfrac{\Delta_{2n}}{\Delta_n} = \dfrac{8(n+7)}{(n+14)} < 8$

$\therefore \quad \dfrac{\Delta_{2n}}{\Delta_n} < 8$

66. $\because \quad \Delta_r = r^3 + 14r^2$

$$\therefore \quad \frac{27\Delta_r - \Delta_{3r}}{27r^2} = \frac{28}{3}$$

$$\Rightarrow \quad \sum_{r=1}^{30}\left(\frac{27\Delta_r - \Delta_{3r}}{27r^2}\right) = \frac{28}{3}\times 30 = 280$$

67. We have, $\begin{vmatrix} 3^2+k & 4^2 & 3^2+3+k \\ 4^2+k & 5^2 & 4^2+4+k \\ 5^2+k & 6^2 & 5^2+5+k \end{vmatrix} = 0$

Applying $C_3 \to C_3 - C_1$, then

$$\begin{vmatrix} 3^2+k & 4^2 & 3 \\ 4^2+k & 5^2 & 4 \\ 5^2+k & 6^2 & 5 \end{vmatrix} = 0$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$\begin{vmatrix} 9+k & 16 & 3 \\ 7 & 9 & 1 \\ 16 & 20 & 2 \end{vmatrix} = 0$$

$$\Rightarrow (9+k)(18-20) - 16(14-16) + 3(140-144) = 0$$

$$\Rightarrow \quad -18 - 2k + 32 - 12 = 0 \quad \Rightarrow 2k = 2$$

$$\therefore \quad k = 1$$

Now, $\sqrt{2^k\sqrt{2^k\sqrt{2^k\ldots}}}\infty = (2^k)^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\infty}$

$$= (2^k)^{\frac{\frac{1}{2}}{1-\frac{1}{2}}} = 2^k = 2^1 = 2$$

68. We have, $\begin{vmatrix} x-1 & -6 & 2 \\ -6 & x-2 & -4 \\ 2 & -4 & x-6 \end{vmatrix} = 0$

Applying $C_2 \to C_2 + 3C_3$, then

$$\begin{vmatrix} x-1 & 0 & 2 \\ -6 & x-14 & -4 \\ 2 & 3x-22 & x-6 \end{vmatrix} = 0$$

Expanding along R_1, then

$$(x-1)\{(x-14)(x-6) + 4(3x-22)\} - 0 + 2\{-18x + 132 - 2x + 2\} = 0$$

$$\Rightarrow (x-1)(x^2 - 8x - 4) + 2(-20x + 160) = 0$$

$$\Rightarrow \quad x^3 - 9x^2 - 36x + 324 = 0$$

$$\Rightarrow \quad (x-9)(x-6)(x+6) = 0$$

$$\therefore \quad x = 9 \text{ or } 6 \text{ or } -6$$

Now, let $\alpha = 9, \beta = 6, \gamma = -6$

$$\therefore \quad \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{1}{9} + \frac{1}{6} - \frac{1}{6} = \frac{1}{9}$$

$$\therefore \quad \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right)^{-1} = 9$$

69. We have, $\begin{vmatrix} x & e^{x-1} & (x-1)^3 \\ x - \ln x & \cos(x-1) & (x-1)^2 \\ \tan x & \sin^2 x & \cos^2 x \end{vmatrix}$

$$= a_0 + a_1(x-1) + a_2(x-1)^2 + \ldots + a_n(x-1)^n \quad \ldots(\text{i})$$

On putting $x = 1$ in Eq. (i), we get

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ \tan 1 & \sin^2 1 & \cos^2 1 \end{vmatrix} = a_0 + 0 + 0 + \ldots$$

$$\therefore \quad a_0 = 0 \qquad [\because R_1 \text{ and } R_2 \text{ an identical}]$$

On differentiating Eq. (i) both sides w.r.t.x, then

$$\begin{vmatrix} 1 & e^{x-1} & 3(x-1)^2 \\ x - \ln x & \cos(x-1) & (x-1)^2 \\ \tan x & \sin^2 x & \cos^2 x \end{vmatrix} + \begin{vmatrix} x & e^{x-1} & (x-1)^3 \\ \left(1-\frac{1}{x}\right) & -\sin(x-1) & 2(x-1) \\ \tan x & \sin^2 x & \cos^2 x \end{vmatrix} + \begin{vmatrix} x & e^{x-1} & (x-1)^3 \\ x - \ln x & \cos(x-1) & (x-1)^2 \\ \sec^2 x & \sin 2x & -\sin 2x \end{vmatrix}$$

$$= 0 + a_1 + 2a_2(x-1) + 3a_3(x-1)^2 + \ldots + na_n(x-1)^{n-1}$$

Now, on putting $x = 1$, we get

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ \tan 1 & \sin^2 1 & \cos^2 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ \tan 1 & \sin^2 1 & \cos^2 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ \sec^2 1 & \sin 2 & -\sin 2 \end{vmatrix}$$

$$= a_1 + 0 + 0 + \ldots + 0$$

$$\therefore a_1 = 0 + 0 + 0 = 0$$

Hence, $(2^{a_0} + 3^{a_1})^{a_1+1} = (2^0 + 3^0)^{0+1} = (1+1)^1 = 2^1 = 2$

70. Given, $\begin{vmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos\gamma \\ \cos\beta & \cos\gamma & 1 \end{vmatrix} = \begin{vmatrix} 0 & \cos\alpha & \cos\beta \\ \cos\alpha & 0 & \cos\gamma \\ \cos\beta & \cos\gamma & 0 \end{vmatrix}$

$$\Rightarrow \quad 1(1-\cos^2\gamma) - \cos\alpha(\cos\alpha - \cos\beta\cos\gamma) + \cos\beta(\cos\gamma\cos\alpha - \cos\beta)$$

$$= 0 - \cos\alpha(0 - \cos\beta\cos\gamma) + \cos\beta(\cos\gamma\cos\alpha - 0)$$

$$\Rightarrow 1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma = 2\cos\alpha\cos\beta\cos\gamma$$

$$\Rightarrow 1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma = 0$$

Hence, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

71. $\because f(a,b,c) = \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, then

$$f(a,b,c) = \begin{vmatrix} (b+c)^2 & a^2-(b+c)^2 & a^2-(b+c)^2 \\ b^2 & (c+a)^2-b^2 & 0 \\ c^2 & 0 & (a+b)^2-c^2 \end{vmatrix}$$

$$= \begin{vmatrix} (b+c)^2 & (a+b+c)(a-b-c) & (a+b+c)(a-b-c) \\ b^2 & (c+a+b)(c+a-b) & 0 \\ c^2 & 0 & (a+b+c)(a+b-c) \end{vmatrix}$$

$$= (a+b+c)^2 \begin{vmatrix} (b+c)^2 & a-b-c & a-b-c \\ b^2 & c+a-b & 0 \\ c^2 & 0 & a+b-c \end{vmatrix}$$

Applying $R_1 \to R_1 - (R_2 + R_3)$, then

$$f(a,b,c) = (a+b+c)^2 \begin{vmatrix} 2bc & -2c & -2b \\ b^2 & c+a-b & 0 \\ c^2 & 0 & a+b-c \end{vmatrix}$$

Applying $C_2 \to C_2 + \frac{1}{b} C_1$ and $C_3 \to C_3 + \frac{1}{c} C_1$, then

$$f(a,b,c) = (a+b+c)^2 \begin{vmatrix} 2bc & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \\ b^2 & & c+a & & \frac{b^2}{c} \\ \vdots & & & & \\ c^2 & & \frac{c^2}{b} & & a+b \end{vmatrix}$$

Expanding along R_1, then

$$f(a,b,c) = (a+b+c)^2 [2bc\{(c+a)(a+b)-bc\}]$$
$$= (a+b+c)^2 \{2bc(ac+bc+a^2+ab-bc)\}$$
$$= 2bc(a+b+c)^2 a(a+b+c)$$
$$= 2abc(a+b+c)^3$$

We get, greatest integer $n \in N$ such that $(a+b+c)^n$ divides $f(a,b,c)$ is 3.

72. The system of equations has a non-trivial solution, then

$$\begin{vmatrix} 1 & -\sin\theta & -\cos\theta \\ -\cos\theta & 1 & -1 \\ -\sin\theta & -1 & 1 \end{vmatrix} = 0$$

Applying $C_3 \to C_3 + C_2$, then

$$\begin{vmatrix} 1 & \cdots & -\sin\theta & \cdots & -\sin\theta - \cos\theta \\ & & & & \vdots \\ -\cos\theta & & 1 & & 0 \\ & & & & \vdots \\ -\sin\theta & & -1 & & 0 \end{vmatrix} = 0$$

Expanding along C_3, then

$$(-\sin\theta - \cos\theta)(\cos\theta + \sin\theta) = 0$$
$$\Rightarrow \quad (\sin\theta + \cos\theta)^2 = 0$$
$$\Rightarrow \quad \sin\theta + \cos\theta = 0$$
$$\Rightarrow \quad \sin\theta = -\cos\theta$$
$$\therefore \quad \tan\theta = -1$$
$$\Rightarrow \quad \theta = \frac{3\pi}{4} \qquad [\because \theta \in [0, \pi]]$$

Hence, $\frac{8\theta}{\pi} = 6$

73. Let $\Delta = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix}$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ and $R_4 \to R_4 - R_1$, then

$$\Delta = \begin{vmatrix} 1 & \cdots & 1 & \cdots & 1 & \cdots & 1 \\ \vdots & & & & & & \\ 0 & & 1 & & 2 & & 3 \\ \vdots & & & & & & \\ 0 & & 2 & & 5 & & 9 \\ \vdots & & & & & & \\ 0 & & 3 & & 9 & & 19 \end{vmatrix}$$

Expanding along C_1, then $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 9 \\ 3 & 9 & 19 \end{vmatrix}$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - R_1$, then $= \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 3 & 10 \end{vmatrix}$

Expanding along C_1, we get $\Delta = 1 \begin{vmatrix} 1 & 3 \\ 3 & 10 \end{vmatrix} = 10 - 9 = 1$

74. Let $\Delta = \begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix}$

Taking a, b, c, d common from R_1, R_2, R_3 and R_4 respectively, then

$$\Delta = abcd \begin{vmatrix} 1+\frac{1}{a} & \frac{1}{a} & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} & \frac{1}{c} \\ \frac{1}{d} & \frac{1}{d} & \frac{1}{d} & 1+\frac{1}{d} \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3 + R_4$ and taking $\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right)$ common, we get

$$\Delta = abcd \left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right) \begin{vmatrix} 1 & 1 & 1 & 1 \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} & \frac{1}{c} \\ \frac{1}{d} & \frac{1}{d} & \frac{1}{d} & 1+\frac{1}{d} \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$ and $C_4 \to C_4 - C_1$, then

$$\Delta = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\begin{vmatrix} 1 & \ddots & 0 & 0 & 0 \\ \frac{1}{b} & & 1 & 0 & 0 \\ & \ddots & & & \\ \frac{1}{c} & & 0 & 1 & 0 \\ & & & \ddots & \\ \frac{1}{d} & & 0 & 0 & 1 \end{vmatrix}$$

$$= abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)1\cdot 1\cdot 1$$

$$= abcd + (bcd + acd + abd + abc) = \sigma_4 + \sigma_3$$

$$= \frac{16}{1} + \left(-\frac{8}{1}\right) = 8$$

75. Given, $\begin{vmatrix} 1+a & 1 & 1 \\ 1+b & 1+2b & 1 \\ 1+c & 1+c & 1+3c \end{vmatrix} = 0$

Taking a, b, c common from R_1, R_2 and R_3 respectively, then

$$abc\begin{vmatrix} 1+\frac{1}{a} & \frac{1}{a} & \frac{1}{a} \\ 1+\frac{1}{b} & 2+\frac{1}{b} & \frac{1}{b} \\ 1+\frac{1}{c} & 1+\frac{1}{c} & 3+\frac{1}{c} \end{vmatrix} = 0$$

Applying $R_1 \to R_1 + R_2 + R_3$ and taking $\left(3+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ common, we get

$$abc\left(3+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 1 & 1 \\ 1+\frac{1}{b} & 2+\frac{1}{b} & \frac{1}{b} \\ 1+\frac{1}{c} & 1+\frac{1}{c} & 3+\frac{1}{c} \end{vmatrix} = 0$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, then

$$abc\left(3+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \\ 1+\frac{1}{2} & & 1 & & -1 \\ \vdots & & & & \\ 1+\frac{1}{c} & & 0 & & 2 \end{vmatrix} = 0$$

Expanding along R_1, we get

$$2\,abc\left(3+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = 0$$

$\because \quad a \neq 0, b \neq 0, c \neq 0$

$\therefore \quad \frac{1}{a}+\frac{1}{b}+\frac{1}{c} = -3$ or $|a^{-1}+b^{-1}+c^{-1}| = 3$

76. Given equations

$$ax + hy + g = 0, \quad \ldots(i)$$

$$hx + by + f = 0 \quad \ldots(ii)$$

and $ax^2 + 2hxy + by^2 + 2gx + 2fy + c + \lambda = 0 \quad \ldots(iii)$

Eq. (iii), can be written as

$$x(ax + hy + g) + y(hx + by + f) + gx + fy + c + \lambda = 0$$

$\Rightarrow \quad x\cdot 0 + y\cdot 0 + gx + fy + c + \lambda = 0$ [from Eqs. (i) and (ii)]

$\Rightarrow \quad gx + fy + c + \lambda = 0 \quad \ldots(iv)$

According to the question Eqs. (i), (ii) and (iii) has unique solution. So, Eqs. (i), (ii) and (iv) has unique solution,

then $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c+\lambda \end{vmatrix} = 0$

$\Rightarrow \quad a(bc + b\lambda - f^2) - h(ch + h\lambda - fg) + g(hf - bg)$

$\Rightarrow \quad (abc + 2fgh - af^2 - bg^2 - ch^2) = \lambda(h^2 - ab)$

or $\quad \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{h^2 - ab} = \lambda$

According to the question, $\lambda = 8$

77. (A) $\to$ (p, r); (B) $\to$ (p, r); (C) $\to$ (p, q, s, t)

(A) Using $a^2 + b^2 + c^2 = 0$, we get

$$\Delta = \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix} = \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$$

$$= abc\begin{vmatrix} -a & a & a \\ b & -b & b \\ c & c & -c \end{vmatrix}$$

[taking a, b, c common from C_1, C_2, C_3 respectively]

Applying $C_2 \to C_2 + C_1$ and $C_3 \to C_3 + C_1$, then

$$\Delta = abc\begin{vmatrix} -a & \ldots & 0 & \ldots & 0 \\ \vdots & & & & \\ b & & 0 & & 2b \\ \vdots & & & & \\ c & & 2c & & 0 \end{vmatrix}$$

$$= (abc)(-a)(-4\,bc) = 4a^2\,b^2\,c^2$$

$\therefore \quad \lambda = 4$

(B) Let $\quad \Delta = \begin{vmatrix} a & a+b & a+b+c \\ 2a & 5a+2b & 7a+5b+2c \\ 3a & 7a+3b & 9a+7b+3c \end{vmatrix}$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$, then

$$\Delta = \begin{vmatrix} a & \cdots & a+b & \cdots & a+b+c \\ \vdots & & & & \\ 0 & & 3a & & 5a+3b \\ \vdots & & & & \\ 0 & & 4a & & 6a+4b \end{vmatrix}$$

$$= a\begin{vmatrix} 3a & 5a+3b \\ 4a & 6a+4b \end{vmatrix}$$

$$= a(18a^2 + 12ab - 20a^2 - 12ab)$$

$$= -2a^3 = -1024 \quad \text{[given]}$$

$\Rightarrow \quad a^3 = 512 = 8^3$

$\therefore \quad a = 8$

(C) Let $\Delta(x)=\begin{vmatrix} x-1 & 2x^2-5 & x^3-1 \\ 2x^2+5 & 2x+2 & x^3+3 \\ x^3-1 & x+1 & 3x^2-2 \end{vmatrix}$ ...(i)

According to the question,

$\Delta(x)=(x^2-1)\,P(x)+ax+b$

$\therefore\ \Delta(1)=a+b$ and $\Delta(-1)=-a+b$ ...(ii)

From Eq. (i), we get

$$\Delta(1)=\begin{vmatrix} 0 & \cdots & 3 & \cdots & 0 \\ & & \vdots & & \\ 7 & & 4 & & 4 \\ & & \vdots & & \\ 0 & & 2 & & 1 \end{vmatrix}=3(7-0)=21$$

$$\text{and } \Delta(-1)=\begin{vmatrix} -2 & \cdots & -3 & \cdots & 2 \\ & & \vdots & & \\ 7 & & 0 & & 2 \\ & & \vdots & & \\ -2 & & 0 & & 1 \end{vmatrix}=3(7+4)=33$$

From Eq. (ii), $a+b=21$ and $-a+b=33$,

we get $a=-6,\ b=27$

$\therefore\quad 4a+2b=-24+54=30$

78. (A) $\rightarrow$ (p, s, t); (B) $\rightarrow$ (r, t); (C) $\rightarrow$ (p, q)

(A) $\because\quad \Delta=\begin{vmatrix} 1 & 1 & 1 \\ f_1(x_1) & f_1(x_2) & f_1(x_3) \\ f_2(x_1) & f_2(x_2) & f_2(x_3) \end{vmatrix}=\begin{vmatrix} 1 & 1 & 1 \\ f_1(2) & f_1(3) & f_1(5) \\ f_2(2) & f_2(3) & f_2(5) \end{vmatrix}$

$$=\begin{vmatrix} 1 & 1 & 1 \\ 2+a_1 & 3+a_1 & 5+a_1 \\ 4+2b_1+b_2 & 9+3b_1+b_2 & 25+5b_1+b_2 \end{vmatrix}$$

Applying $C_2\rightarrow C_2-C_1$ and $C_3\rightarrow C_3-C_1$, then

$$\Delta=\begin{vmatrix} 1 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \\ 2+a_1 & & 1 & & 3 \\ \vdots & & & & \\ 4+2b_1+b_2 & & 5+b_1 & & 21+3b \end{vmatrix}$$

$$=\begin{vmatrix} 1 & 3 \\ 5+b_1 & 21+3b_1 \end{vmatrix}$$

$$=21+3b_1-15-3b_1=6$$

(B) $\because\ f(x)=\begin{vmatrix} 1 & b_1 & a_1 \\ 1 & b_1 & 2a_1-x \\ 1 & 2b_1-x & a_1 \end{vmatrix}$

Applying $R_2\rightarrow R_2-R_1$ and $R_3\rightarrow R_3-R_1$, then

$$f(x)=\begin{vmatrix} 1 & \cdots & b_1 & \cdots & a_1 \\ \vdots & & & & \\ 0 & & 0 & & a_1-x \\ \vdots & & & & \\ 0 & & b_1-x & & 0 \end{vmatrix}$$

$$=-(a_1-x)(b_1-x)=-x^2+(a_1+b_1)x-a_1\,b_1$$

Minimum value of $f(x)=-\dfrac{D}{4a}=-\dfrac{(a_1+b_1)^2-4a_1b_1}{4(-1)}$

$$=\frac{(a_1-b_1)^2}{4}=\frac{36}{4}=9$$

(C) $\because\ f(x)$ is a polynomial of degree atmost 6 in x.

If $f(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6$

$\Rightarrow \lambda=a_1=f'(0)$

$$=\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 8 \end{vmatrix}+\begin{vmatrix} -2 & -2 & 0 \\ -1 & 0 & 1 \\ 0 & 2 & 8 \end{vmatrix}+\begin{vmatrix} -2 & 1 & 0 \\ -1 & 0 & 3 \\ 0 & 1 & 12 \end{vmatrix}$$

$$=-8-12+18=-2$$

$\therefore\qquad |\lambda|=2$

79. (A) $\rightarrow$ (r); (B) $\rightarrow$ (r, t); (C) $\rightarrow$ (p, q, s)

(A) Let $f(x)=\begin{vmatrix} x^2+3x & x-1 & x+3 \\ x^2+1 & 2+3x & x-3 \\ x^2-3 & x+4 & 3x \end{vmatrix}$

$$f(x)=ax^4+bx^3+cx^2+dx+e \qquad ...(i)$$

$$\therefore\quad e=f(0)=\begin{vmatrix} 0 & -1 & 3 \\ 1 & 2 & -3 \\ -3 & 4 & 0 \end{vmatrix}=0+1(0-9)+3(4+6)=21$$

Dividing both sides of Eq. (i) by x^4 i.e., C_1 by x^2, C_2 by x and C_3 by x and then taking $\lim\limits_{x\to\infty}$, we get

$$a=\begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{vmatrix}=1(8)-1(2)+1(-2)=4$$

Hence, $e+a=25$

(B) Let $f(x)=\begin{vmatrix} x-1 & 5x & 7 \\ x^2-1 & x-1 & 8 \\ 2x & 3x & 0 \end{vmatrix}=ax^3+bx^2+cx+d$ ...(i)

$$\therefore\quad c=f'(0)=\begin{vmatrix} 1 & 0 & 7 \\ 0 & -1 & 8 \\ 2 & 0 & 0 \end{vmatrix}+\begin{vmatrix} -1 & 5 & 7 \\ -1 & 1 & 8 \\ 0 & 3 & 0 \end{vmatrix}+\begin{vmatrix} -1 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$$=2(0+7)-3(-8+7)+0=17$$

Dividing both sides of Eq. (i) by x^3 i.e., c_1 by x^2, c_2 by x and taking $\lim\limits_{x\to\infty}$, we get

$$a=\begin{vmatrix} 0 & & 5 & & 7 \\ \vdots & & & & \\ 1 & \cdots & 1 & \cdots & 8 \\ \vdots & & & & \\ 0 & & 3 & & 0 \end{vmatrix}=-1(0-21)=21$$

Hence, $c+a-3=35$

(C) Let $g(x)=\begin{vmatrix} x^3+4x & x+3 & x-2 \\ x-2 & 5x & x-1 \\ x-3 & x+2 & 4x \end{vmatrix}$

$$=ax^5+bx^4+cx^3+dx^2+ex+f$$

$$\therefore \quad f = g(0) = \begin{vmatrix} 0 & 3 & -2 \\ -2 & 0 & -1 \\ -3 & 2 & 0 \end{vmatrix} = 0 - 3(0-3) - 2(-4-0) = 17$$

$$\text{and } e = g'(0) = \begin{vmatrix} 4 & 3 & -2 \\ 1 & 0 & -1 \\ 1 & 2 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & -2 \\ -2 & 5 & -1 \\ -3 & 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 3 & 1 \\ -2 & 0 & 1 \\ -3 & 2 & 4 \end{vmatrix}$$

$$= 1 - 23 + 11 = -11$$

Hence, $f + e = 17 - 11 = 6$

80. (A) $\rightarrow$ (p, q, r); (B) $\rightarrow$ (p, q, r, s, t); (C) $\rightarrow$ (p, q, r, s, t)

(A) Taking common a, b, c from R_1, R_2 and R_3 respectively and then multiplying in C_1, C_2 and C_3 by a, b, c respectively, then

$$\Delta = \begin{vmatrix} a + (b^2 + c^2)d & b^2(1-d) & c^2(1-d) \\ a^2(1-d) & b^2 + (c^2 + a^2)d & c^2(1-d) \\ a^2(1-d) & b^2(1-d) & c^2 + (a^2 + b^2)d \end{vmatrix}$$

Applying $C_1 \rightarrow C_1 + C_2 + C_3$, then

$$\Delta = \begin{vmatrix} 1 & b^2(1-d) & c^2(1-d) \\ 1 & b^2 + (c^2 + a^2)d & c^2(1-d) \\ 1 & b^2(1-d) & c^2 + (a^2 + b^2)d \end{vmatrix}$$

$$[\because a^2 + b^2 + c^2 = 1]$$

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, then

$$\Delta = \begin{vmatrix} 1 & b^2(1-d) & c^2(1-d) \\ 0 & d & 0 \\ 0 & 0 & d \end{vmatrix} = d^2$$

$$[\because a^2 + b^2 + c^2 = 1]$$

(B) Multiplying C_1 by a, C_2 by b and C_3 by c, then

$$\Delta = \frac{1}{abc} \begin{vmatrix} \frac{a}{c} & \frac{b}{c} & -\frac{(a+b)}{c} \\ -\frac{(b+c)}{a} & \frac{b}{a} & \frac{c}{a} \\ -\frac{bd(b+c)}{ac} & \frac{bd(a+2b+c)}{ac} & -\frac{(a+b)bd}{ac} \end{vmatrix}$$

Applying $C_1 \rightarrow C_1 + C_2 + C_3$, then

$$\Delta = \frac{1}{abc} \begin{vmatrix} 0 & \frac{b}{c} & -\frac{(a+b)}{c} \\ 0 & \frac{b}{a} & \frac{c}{a} \\ 0 & \frac{bd(a+2b+c)}{ac} & -\frac{(a+b)bd}{ac} \end{vmatrix} = 0$$

(C) Applying $C_3 \rightarrow C_3 - \cos d\, C_1 - \sin d\, C_2$, then

$$\Delta = \begin{vmatrix} \sin a & \cos a & 0 \\ \sin b & \cos b & 0 \\ \sin c & \cos c & 0 \end{vmatrix} = 0$$

81. (A) $\rightarrow$ (p, r); (B) $\rightarrow$ (p, q, r, t); (C) $\rightarrow$ (p, r, s)

(A) Possible values are -2 , -1, 0, 1, 2 and numbering determinant $= 3^4 = 81$

i.e., $\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1, \begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} = 0, \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1, \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2,$

$\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2 \quad \therefore n = 5 \Rightarrow (n-1)^2 = 16$

(B) There are only three determinants of second order with negative value

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}$$

Number of possible determinants with elements 0 and 1 are $2^4 = 16$. Therefore, number of determinants with non-negative values is 13.

$\therefore \qquad n = 13$

$\Rightarrow \qquad (n-1) = 12$

(C) There are only four determinants of second order with negative value

$$\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}, \begin{vmatrix} -1 & -1 \\ -1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & -1 \\ -1 & -1 \end{vmatrix}$$

$\therefore \qquad n = 4 \Rightarrow n(n+1) = 20$

82. **Statement-1**

$$\Delta(r) = \begin{vmatrix} r & r+1 \\ r+3 & r+4 \end{vmatrix} = r(r+4) - (r+1)(r+3)$$

$$= (r^2 + 4r) - (r^2 + 4r + 3) = -3$$

$$\therefore \quad \sum_{r=1}^{n} \Delta(r) = \sum_{r=1}^{n} (-3)$$

$$= \underbrace{(-3) + (-3) + (-3) + \ldots + (-3)}_{n \text{ times}} = -3n$$

$\Rightarrow$ Statement-1 is true.

Statement-2

$$\Delta(r) = \begin{vmatrix} f_1(r) & f_2(r) \\ f_3(r) & f_4(r) \end{vmatrix} = f_1(r)\, f_4(r) - f_2(r)\, f_3(r)$$

$$\therefore \quad \sum_{r=1}^{n} \Delta(r) = \sum_{r=1}^{n} [f_1(r)\, f_4(r) - f_2(r)\, f_3(r)]$$

$$= \sum_{r=1}^{n} [f_1(r)\, f_4(r)] - \sum_{r=1}^{n} [f_2(r)\, f_3(r)] \qquad \ldots(i)$$

$$\text{and } \begin{vmatrix} \sum_{r=1}^{n} f_1(r) & \sum_{r=1}^{n} f_2(r) \\ \sum_{r=1}^{n} f_3(r) & \sum_{r=1}^{n} f_4(r) \end{vmatrix}$$

$$= \left(\sum_{r=1}^{n} f_1(r)\right)\left(\sum_{r=1}^{n} f_4(r)\right) - \left(\sum_{r=1}^{n} f_2(r)\right)\left(\sum_{r=1}^{n} f_3(r)\right) \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we get $\sum_{r=1}^{n} \Delta(r) \neq \begin{vmatrix} \sum_{r=1}^{n} f_1(r) & \sum_{r=1}^{n} f_2(r) \\ \sum_{r=1}^{n} f_3(r) & \sum_{r=1}^{n} f_4(r) \end{vmatrix}$

$\therefore$ Statement-2 is false.

Hence, Statement-1 is true and Statement-2 is false.

83. $\because \quad \Delta = \begin{vmatrix} a_1 + b_1x^2 & a_1x^2 + b_1 & c_1 \\ a_2 + b_2x^2 & a_2x^2 + b_2 & c_2 \\ a_3 + b_3x^2 & a_3x^2 + b_3 & c_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} 1 & x^2 & 0 \\ x^2 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad \ldots(i)$$

Statement-1 If $\Delta = 0$, then

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} 1 & x^2 & 0 \\ x^2 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} 1 & x^2 & 0 \\ \vdots & & \vdots \\ x^2 & 1 & 0 \\ \vdots & & \vdots \\ 0 \quad \cdots & 0 \quad \cdots & 1 \end{vmatrix} = 0 \Rightarrow 1 - x^4 = 0 \text{ or } x^4 = 1 \qquad [\because x^2 \neq -1]$$

Statement-1 is true

Now, if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$, then

$\Delta = 0$ [from Eq. (i)]

Statement-2 is also true.

Hence, both the statements are true but Statement-2 is not a correct explanation of Statement-1.

84. Statement-2 is always true for Statement-1

$$\cos\left(x + \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{2} - \left(\frac{\pi}{4} - x\right)\right) = \sin\left(\frac{\pi}{4} - x\right)$$

$$= -\sin\left(x - \frac{\pi}{4}\right)$$

$$\cot\left(\frac{\pi}{4} + x\right) = \cot\left(\frac{\pi}{2} - \left(\frac{\pi}{4} - x\right)\right) = \tan\left(\frac{\pi}{4} - x\right)$$

$$= -\tan\left(x - \frac{\pi}{4}\right)$$

Also, $\ln\left(\frac{y}{x}\right) = -\ln\left(\frac{x}{y}\right)$

Therefore, determinant given in Statement-1 is skew-symmetric and hence its value is zero. Hence, both statements are true and Statement-2 is a correct explanation of Statement-1.

85. $\begin{vmatrix} (1+x)^{11} & (1+x)^{12} & (1+x)^{13} \\ (1+x)^{21} & (1+x)^{22} & (1+x)^{23} \\ (1+x)^{31} & (1+x)^{32} & (1+x)^{33} \end{vmatrix} = A_0 + A_1x + A_2x^2 + \ldots \quad [\text{ let}]$

On differentiating both sides w.r.t.x and then put $x = 0$, we get

$$\begin{vmatrix} 11 & 12 & 13 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 21 & 22 & 23 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 31 & 32 & 33 \end{vmatrix} = 0 + A_1 + 0 + 0 + \ldots$$

$$\Rightarrow \qquad 0 + 0 + 0 = A_1 \quad \therefore \quad A_1 = 0$$

$\therefore$ Coefficient of x in $f(x) = 0$

Both statements are true, Statement-2 is a correct explanation of Statement-1.

86. Here, $\Delta = \begin{vmatrix} 2 & 3 \\ b & 4 \end{vmatrix} = 8 - 3b,$

$$\Delta_1 = \begin{vmatrix} a & 3 \\ 5 & 4 \end{vmatrix} = 4a - 15$$

and $\Delta_2 = \begin{vmatrix} 2 & a \\ b & 5 \end{vmatrix} = 10 - ab$

For infinite solutions, $\Delta = \Delta_1 = \Delta_2 = 0$

We get, $a = \frac{15}{4}$ and $b = \frac{8}{3}$

$\therefore$ Statement-1 is true and if lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are parallel, then

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$$

$\therefore$ Statement-2 is true, but in Statement-1

$$\frac{2}{b} = \frac{3}{4} = \frac{a}{5}$$

$$\Rightarrow \qquad \frac{3}{4} = \frac{3}{4} = \frac{3}{4}$$

[both equation are identical]

$\therefore$ Statement-2 is not a correct explanation for Statement-1.

87. $\because \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{vmatrix} = 1(0 - 48) - 2(0 - 42) + 3(32 - 35)$

$$= -48 + 84 - 9$$

$$= 84 - 57 = 27 \neq 0$$

$\therefore$ Statement-1 is true.

Also, in given determinant neither two rows or columns are identical, Statement-2 is true, Statement-2 is not a correct explanation for Statement-1.

88. $\because A88, 6B8, 86C$ are divisible by 72, then $A88 = 72\lambda$, $6B8 = 72\mu$ and $86C = 72\upsilon$, where $\lambda, \mu, \upsilon \in N$.

$$\because \qquad \begin{vmatrix} A & 6 & 8 \\ 8 & B & 6 \\ 8 & 8 & C \end{vmatrix}$$

Applying $R_3 \to R_3 + 10R_2 + 100R_1$, then

$$\begin{vmatrix} A & 6 & 8 \\ 8 & B & 6 \\ 100A + 80 + 8 & 600 + 10B + 8 & 800 + 60 + c \end{vmatrix}$$

$$= \begin{vmatrix} A & 6 & 8 \\ 8 & B & 6 \\ 72\lambda & 72\mu & 72\upsilon \end{vmatrix} = 72 \begin{vmatrix} A & 6 & 8 \\ 8 & B & 6 \\ \lambda & \mu & \upsilon \end{vmatrix} \qquad \ldots(i)$$

Now, $A88$ is also divisible by 9, then

$A + 8 + 8 = A + 16$ is divisible by 9

$\therefore \qquad A = 2$

and $6B8$ is also divisible by 9, then $6 + B + 8 = B + 14$ is divisible by 9

$\therefore \qquad B = 4$

From Eq. (i), we get

$$= 72\begin{vmatrix} 2 & 6 & 8 \\ 8 & 2 & 6 \\ \lambda & \mu & \upsilon \end{vmatrix} = 288\begin{vmatrix} 1 & 3 & 4 \\ 4 & 1 & 3 \\ \lambda & \mu & \upsilon \end{vmatrix} = 288 \qquad \text{[integer]}$$

Statement-1 is true and Statement-2 is false.

89. Let $\Delta = \begin{vmatrix} b+c & c & b \\ c & c+a & a \\ b & a & a+b \end{vmatrix}$

Applying $R_1 \to R_1 - (R_2 + R_3)$, then

$$\therefore \quad \Delta = \begin{vmatrix} 0 & -2a & -2a \\ c & c+a & a \\ b & a & a+b \end{vmatrix}$$

Taking $(-2a)$ common from R_1, then

$$\Delta = (-2a)\begin{vmatrix} 0 & 1 & 1 \\ c & c+a & a \\ b & a & a+b \end{vmatrix}$$

Applying $C_2 \to C_2 - C_3$, then

$$\therefore \quad \Delta = (-2a)\begin{vmatrix} 0 & 0 & 1 \\ c & c & a \\ b & -b & a+b \end{vmatrix}$$

Expanding along R_1, we get

$$\Delta = (-2a)\cdot 1 \cdot \begin{vmatrix} c & c \\ b & -b \end{vmatrix} = (-2a)(-2bc)$$

Hence, $\Delta = 4abc$

90. Let $\Delta = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

Since, the answer is $(a+b+c)^3$, we shall try to get $(a+b+c)$.

Applying $R_1 \to R_1 + R_2 + R_3$, then

$$\Delta = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Taking $(a+b+c)$ common from R_1, we get

$$\Delta = (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Applying $C_1 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$\therefore \quad \Delta = (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -a-b-c & 0 \\ 2c & 0 & -c-a-b \end{vmatrix}$$

[by property, since all elements above leading diagonal are zero]

$$= (a+b+c)\cdot 1 \cdot (-a-b-c)\cdot(-c-a-b)$$

Hence, $\Delta = (a+b+c)^3$

91. Let $\Delta = \begin{vmatrix} \sqrt{13}+\sqrt{3} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{26} & 5 & \sqrt{10} \\ 3+\sqrt{65} & \sqrt{15} & 5 \end{vmatrix}$

$$= \begin{vmatrix} \sqrt{3} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{15} & 5 & \sqrt{10} \\ 3 & \sqrt{15} & 5 \end{vmatrix} + \begin{vmatrix} \sqrt{13} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{26} & 5 & \sqrt{10} \\ \sqrt{65} & \sqrt{15} & 5 \end{vmatrix}$$

Taking common from 1st determinant $\sqrt{3}$, $\sqrt{5}$ and $\sqrt{5}$ from C_1, C_2 and C_3 respectively and taking common from 2nd determinant $\sqrt{13}$, $\sqrt{5}$ and $\sqrt{5}$ from C_1, C_3 and C_3 respectively, we get

$$= \sqrt{3}\times\sqrt{5}\times\sqrt{5}\begin{vmatrix} 1 & 2 & 1 \\ \sqrt{5} & \sqrt{5} & \sqrt{2} \\ \sqrt{3} & \sqrt{3} & \sqrt{5} \end{vmatrix} + \sqrt{13}\times\sqrt{5}\times\sqrt{5}\begin{vmatrix} 1 & 2 & 1 \\ \sqrt{2} & \sqrt{5} & \sqrt{2} \\ \sqrt{5} & \sqrt{3} & \sqrt{5} \end{vmatrix}$$

$$= \sqrt{3}\times 5\begin{vmatrix} 1 & 2 & 1 \\ \sqrt{5} & \sqrt{5} & \sqrt{2} \\ \sqrt{3} & \sqrt{3} & \sqrt{5} \end{vmatrix} + 0 \qquad [\because C_1 \text{ and } C_3 \text{ are identical}]$$

$$= 5\sqrt{3}\begin{vmatrix} 1 & 2 & 1 \\ \sqrt{5} & \sqrt{5} & \sqrt{2} \\ \sqrt{3} & \sqrt{3} & \sqrt{5} \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$,

$$\text{then } \Delta = 5\sqrt{3}\begin{vmatrix} 1 & 1 & 1 \\ \sqrt{5} & 0 & \sqrt{2} \\ \sqrt{3} & 0 & \sqrt{5} \end{vmatrix}$$

Expanding along C_2, then

$$\Delta = 5\sqrt{3}\cdot(-1)\begin{vmatrix} \sqrt{5} & \sqrt{2} \\ \sqrt{3} & \sqrt{5} \end{vmatrix} = -5\sqrt{3}(5-\sqrt{6})$$

$$= -25\sqrt{3} + 15\sqrt{2}$$

$$= 15\sqrt{2} - 25\sqrt{3}$$

92. Given that, a, b and c are p th, q th and r th terms of HP $\Rightarrow \frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ are p th, q th and r th terms of an AP. Let A and D are the first term and common difference of AP, then

$$\frac{1}{a} = A + (p-1)D \qquad \ldots(i)$$

$$\frac{1}{b} = A + (q-1)D \qquad \ldots(ii)$$

$$\frac{1}{c} = A + (r-1)D \qquad \ldots(iii)$$

Now, given determinant is

$$\Delta = \begin{vmatrix} bc & ca & ab \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = abc\begin{vmatrix} \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$$

On substituting the values of $\frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ from Eqs. (i), (ii) and (iii) in Δ, then

$$\Delta = abc\begin{vmatrix} A+(p-1)D & A+(q-1)D & A+(r-1)D \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$$

Applying $R_1 \to R_1 - (A-D)R_3 - DR_2$, then

$$\Delta = abc\begin{vmatrix} 0 & 0 & 0 \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = 0$$

93. Let $z = \begin{vmatrix} -5 & 3+5i & \frac{3}{2}-4i \\ 3-5i & 8 & 4+5i \\ \frac{3}{2}+4i & 4-5i & 9 \end{vmatrix}$

Then, $\bar{z} = \begin{vmatrix} -5 & 3-5i & \frac{3}{2}+4i \\ 3+5i & 8 & 4-5i \\ \frac{3}{2}-4i & 4+5i & 9 \end{vmatrix}$ [i.e., conjugate of z]

$= \begin{vmatrix} -5 & 3+5i & \frac{3}{2}-4i \\ 3-5i & 8 & 4+5i \\ \frac{3}{2}+4i & 4-5i & 9 \end{vmatrix}$

[interchanging rows into columns]

$\Rightarrow \quad \bar{z} = z$

Hence, z is purely real.

94. LHS $= \begin{vmatrix} ah+bg & g & ab+ch \\ bf+ba & f & hb+bc \\ af+bc & c & bg+fc \end{vmatrix}$

$$= b\begin{vmatrix} ah+bg & g & a \\ bf+ba & f & h \\ af+bc & c & g \end{vmatrix} + c\begin{vmatrix} ah+bg & g & h \\ bf+ba & f & b \\ af+bc & c & f \end{vmatrix}$$

In second determinant, applying $C_1 \to C_1 - bC_2 - aC_3$, then

$$= \begin{vmatrix} ah+bg & bg & a \\ bf+ba & bf & h \\ af+bc & bc & g \end{vmatrix} + c\begin{vmatrix} 0 & g & h \\ 0 & f & b \\ 0 & c & f \end{vmatrix}$$

In first determinant, applying $C_2 \to C_2 - C_1$, then

$$= \begin{vmatrix} ah+bg & -ah & a \\ bf+ba & -ba & h \\ af+bc & -af & q \end{vmatrix} + 0 = a\begin{vmatrix} ah+bg & a & h \\ bf+ba & h & b \\ af+bc & g & f \end{vmatrix} = \text{RHS}$$

95. Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix}$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, then

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ 1+\sin A & \sin B-\sin A & \sin C-\sin A \\ \sin A+\sin^2 A & (\sin B-\sin A)(\sin B+\sin A+1) & (\sin C-\sin A)(\sin C+\sin A+1) \end{vmatrix}$$

Expanding along R_1, then

$$\Delta = \begin{vmatrix} \sin B-\sin A & \sin C-\sin A \\ (\sin B-\sin A)(\sin B+\sin A+1) & (\sin C-\sin A)(\sin C+\sin A+1) \end{vmatrix}$$

$$= (\sin B-\sin A)(\sin C-\sin A)\begin{vmatrix} 1 & 1 \\ \sin B+\sin A+1 & \sin C+\sin A+1 \end{vmatrix}$$

$$= (\sin B-\sin A)(\sin C-\sin A)(\sin C-\sin B)$$

But, given $\Delta = 0$

$\therefore \quad (\sin B-\sin A)(\sin C-\sin A)(\sin C-\sin B) = 0$

$\therefore \quad \sin B-\sin A = 0$ or $\sin C-\sin A = 0$

or $\quad \sin C-\sin B = 0$

$\Rightarrow \quad \sin B = \sin A$ or $\sin C = \sin A$ or $\sin C = \sin B$

$B = A$ or $C = A$ or $C = B$

In all the three cases, we will have an isosceles triangle.

96. Let $\Delta = \begin{vmatrix} \beta\gamma & \beta\gamma'+\beta'\gamma & \beta'\gamma' \\ \gamma'\alpha & \gamma'\alpha'+\gamma'\alpha & \gamma'\alpha' \\ \alpha\beta & \alpha\beta'+\alpha'\beta & \alpha'\beta' \end{vmatrix}$

Taking $\beta'\gamma'$, $\gamma'\alpha'$ and $\alpha'\beta'$ common from R_1, R_2 and R_3 respectively, then

$$\Delta = (\beta'\gamma')(\gamma'\alpha')(\alpha'\beta')\begin{vmatrix} \frac{\beta}{\beta'}\frac{\gamma}{\gamma'} & \frac{\beta}{\beta'}+\frac{\gamma}{\gamma'} & 1 \\ \frac{\gamma}{\gamma'}\frac{\alpha}{\alpha'} & \frac{\gamma}{\gamma'}+\frac{\alpha}{\alpha'} & 1 \\ \frac{\alpha}{\alpha'}\frac{\beta}{\beta'} & \frac{\alpha}{\alpha'}+\frac{\beta}{\beta'} & 1 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

Then, $\Delta = (\alpha'\beta'\gamma')^2\begin{vmatrix} \frac{\beta}{\beta'}\frac{\gamma}{\gamma'} & \frac{\beta}{\beta'}+\frac{\gamma}{\gamma'} & 1 \\ \frac{\gamma}{\gamma'}\left(\frac{\alpha}{\alpha'}-\frac{\beta}{\beta'}\right) & \left(\frac{\alpha}{\alpha'}-\frac{\beta}{\beta'}\right) & 0 \\ \frac{\beta}{\beta'}\left(\frac{\alpha}{\alpha'}-\frac{\gamma}{\gamma'}\right) & \left(\frac{\alpha}{\alpha'}-\frac{\gamma}{\gamma'}\right) & 0 \end{vmatrix}$

$$= (\alpha'\beta'\gamma')^2\left(\frac{\alpha}{\alpha'}-\frac{\beta}{\beta'}\right)\left(\frac{\alpha}{\alpha'}-\frac{\gamma}{\gamma'}\right)\begin{vmatrix} \frac{\beta}{\beta'}\frac{\gamma}{\gamma'} & \frac{\beta}{\beta'}+\frac{\gamma}{\gamma'} & 1 \\ \frac{\gamma}{\gamma'} & 1 & 0 \\ \frac{\beta}{\beta'} & 1 & 0 \end{vmatrix}$$

Expanding along C_3, then

$$\Delta = (\alpha'\beta'\gamma')^2\left(\frac{\alpha}{\alpha'}-\frac{\beta}{\beta'}\right)\left(\frac{\alpha}{\alpha'}-\frac{\gamma}{\gamma'}\right)\left(\frac{\gamma}{\gamma'}-\frac{\beta}{\beta'}\right)$$

$$= (\alpha'\beta'\gamma')^2\left(\frac{\alpha}{\alpha'}-\frac{\beta}{\beta'}\right)\left(\frac{\beta}{\beta'}-\frac{\gamma}{\gamma'}\right)\left(\frac{\gamma}{\gamma'}-\frac{\alpha}{\alpha'}\right)$$

$$= (\alpha'\beta'\gamma')^2\frac{(\alpha\beta'-\alpha'\beta)(\beta\gamma'-\beta'\gamma)(\gamma\alpha'-\gamma'\alpha)}{(\alpha'\beta'\gamma')^2}$$

Hence, $\Delta = (\alpha\beta'-\alpha'\beta)(\beta\gamma'-\beta'\gamma)(\gamma\alpha'-\gamma'\alpha)$

97. Since, $y = \frac{u}{v}$

$$\therefore \quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{vu' - uv'}{v^2}$$

$$\Rightarrow \quad v^2\frac{dy}{dx} = vu' - uv' \qquad \ldots(i)$$

On differentiating both sides w.r.t. x, we get

$$v^2\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot 2vv' = (vu'' + u'v') - (uv'' + v'u')$$

$$\Rightarrow \quad v^2\frac{d^2y}{dx^2} + 2vv'\frac{dy}{dx} = vu'' - uv''$$

On multiplying both sides by v, then

$$v^3\frac{d^2y}{dx^2} + 2v'\left(v^2\frac{dy}{dx}\right) = v^2u'' - uvv''$$

$$\Rightarrow \quad v^3\frac{d^2y}{dx^2} + 2v'(vu' - uv') = v^2u'' - uvv'' \qquad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad v^3\frac{d^2y}{dx^2} = 2uv^2 - uvv'' - 2vu'v' + v^2u'' \qquad \ldots(ii)$$

and $\begin{vmatrix} v & v & 0 \\ v' & v' & v \\ v'' & v'' & 2v'' \end{vmatrix} = u(2v'^2 - vu'') - v(2u'v' - u''v)$

$$= 2uv'^2 - 4vv'' - 2vu'v' + v^2u'' \qquad \ldots(iii)$$

From Eqs. (ii) and (iii), we get

$$v^3\frac{d^2y}{dx^2} = \begin{vmatrix} u & v & 0 \\ u' & v' & v \\ u' & v'' & 2v' \end{vmatrix}$$

98. Here, we have to prove that $\Delta(x)$ is independent of x. So, it is sufficient to prove that $\Delta'(x) = 0$

Now, $\Delta(x) = \begin{vmatrix} \sin(x+\alpha) & \cos(x+\alpha) & a + x\sin\alpha \\ \sin(x+\beta) & \cos(x+\beta) & b + x\sin\beta \\ \sin(x+\gamma) & \cos(x+\gamma) & c + x\sin\gamma \end{vmatrix}$

On differentiating w.r.t. x, we get

$$\Delta'(x) = \begin{vmatrix} \cos(x+\alpha) & \cos(x+\alpha) & a + x\sin\alpha \\ \cos(x+\beta) & \cos(x+\beta) & b + x\sin\beta \\ \cos(x+\gamma) & \cos(x+\gamma) & c + x\sin\gamma \end{vmatrix}$$

$$+ \begin{vmatrix} \sin(x+\alpha) & -\sin(x+\alpha) & a + x\sin\alpha \\ \sin(x+\beta) & -\sin(x+\beta) & b + x\sin\beta \\ \sin(x+\gamma) & -\sin(x+\gamma) & c + x\sin\gamma \end{vmatrix}$$

$$+ \begin{vmatrix} \sin(x+\alpha) & \cos(x+\alpha) & \sin\alpha \\ \sin(x+\beta) & \cos(x+\beta) & \sin\beta \\ \sin(x+\gamma) & \cos(x+\gamma) & \sin\gamma \end{vmatrix}$$

$$= 0 - 0 + \begin{vmatrix} \sin(x+\alpha) & \cos(x+\alpha) & \sin\alpha \\ \sin(x+\beta) & \cos(x+\beta) & \sin\beta \\ \sin(x+\gamma) & \cos(x+\gamma) & \sin\gamma \end{vmatrix}$$

$$= \begin{vmatrix} \sin(x+\alpha) & \cos(x+\alpha) & \sin\alpha \\ \sin(x+\beta) & \cos(x+\beta) & \sin\beta \\ \sin(x+\gamma) & \cos(x+\gamma) & \sin\gamma \end{vmatrix}$$

Applying $C_1 \to C_1 - (\cos x)\,C_3$ and $C_2 \to C_2 + (\sin x)C_3$, we get

$$\Delta'(x) = \begin{vmatrix} \sin x\cos\alpha & \cos x\cos\alpha & \sin\alpha \\ \sin x\cos\beta & \cos x\cos\beta & \sin\beta \\ \sin x\cos\gamma & \cos x\cos\gamma & \sin\gamma \end{vmatrix}$$

$$= \sin x\cdot\cos x \begin{vmatrix} \cos\alpha & \cos\alpha & \sin\alpha \\ \cos\beta & \cos\beta & \sin\beta \\ \cos\gamma & \cos\gamma & \sin\gamma \end{vmatrix}$$

$$= \sin x\cdot\cos x \times 0 \qquad [\because C_1 \text{ and } C_2 \text{ are identical}]$$

$$= 0$$

Thus, $\Delta(x)$ is independent of x.

99. Let $\Delta = \begin{vmatrix} {}^xC_1 & {}^xC_2 & {}^xC_3 \\ {}^yC_1 & {}^yC_2 & {}^yC_3 \\ {}^zC_1 & {}^zC_2 & {}^zC_3 \end{vmatrix} = \begin{vmatrix} x & \frac{x(x-1)}{1\cdot 2} & \frac{x(x-1)(x-2)}{1\cdot 2\cdot 3} \\ y & \frac{y(y-1)}{1\cdot 2} & \frac{y(y-1)(y-2)}{1\cdot 2\cdot 3} \\ z & \frac{z(z-1)}{1\cdot 2} & \frac{z(z-1)(z-2)}{1\cdot 2\cdot 3} \end{vmatrix}$

$$= \frac{xyz}{12}\begin{vmatrix} 1 & x-1 & x^2-3x+2 \\ 1 & y-1 & y^2-3y+2 \\ 1 & z-1 & z^2-3z+2 \end{vmatrix}$$

Applying $C_2 \to C_2 + C_1$, then

$$\Delta = \frac{xyz}{12}\begin{vmatrix} 1 & x & x^2-3x+2 \\ 1 & y & y^2-3y+2 \\ 1 & z & z^2-3z+2 \end{vmatrix}$$

Applying $C_3 \to C_3 + 3C_2 - 2C_1$, then

$$\Delta = \frac{xyz}{12}\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = \frac{1}{12}xyz(x-y)(y-z)(z-x)$$

100. (i) $\because \quad f(x) = \begin{vmatrix} 1+\sin^2 x & \cos^2 x & 4\sin 2x \\ \sin^2 x & 1+\cos^2 x & 4\sin 2x \\ \sin^2 x & \cos^2 x & 1+4\sin 2x \end{vmatrix}$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$f(x) = \begin{vmatrix} 1+\sin^2 x & \cos^2 x & 4\sin 2x \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

Applying $C_2 \to C_2 + C_1$, then

$$f(x) = \begin{vmatrix} 1+\sin^2 x & 2 & 4\sin 2x \\ \vdots & & \\ -1 & \ldots\ 0\ \ldots & 0 \\ \vdots & & \\ -1 & -1 & 1 \end{vmatrix}$$

Expanding along R_2, then

$$f(x) = \begin{vmatrix} 2 & 4\sin 2x \\ -1 & 1 \end{vmatrix} = 2 + 4\sin 2x$$

$\therefore$ Maximum value of

$$f(x) = 2 + 4(1) = 6$$

(ii) $\because \quad \Delta = \begin{vmatrix} \sin^2 A & \sin A \cos A & \cos^2 A \\ \sin^2 B & \sin B \cos B & \cos^2 B \\ \sin^2 C & \sin C \cos C & \cos^2 C \end{vmatrix}$

$$= \cos^2 A \cos^2 B \cos^2 C \begin{vmatrix} \tan^2 A & \tan A & 1 \\ \tan^2 B & \tan B & 1 \\ \tan^2 C & \tan C & 1 \end{vmatrix}$$

$$= -\cos^2 A \cos^2 B \cos^2 C \begin{vmatrix} 1 & \tan A & \tan^2 A \\ 1 & \tan B & \tan^2 B \\ 1 & \tan C & \tan^2 C \end{vmatrix}$$

$$= -\cos^2 A \cos^2 B \cos^2 C (\tan A - \tan B)$$
$$(\tan B - \tan C)(\tan C - \tan A)$$

$$= -\sin(A - B)\sin(B - C)\sin(C - A)$$

$$= \sin(A - B)\sin(B - C)\sin(A - C) \geq 0 \quad [\because A \geq B \geq C]$$

$\therefore \quad \Delta \geq 0$

Hence, minimum value of Δ is 0.

101. Let $f(x) = \begin{vmatrix} x^2 - 4x + 6 & 2x^2 + 4x + 10 & 3x^2 - 2x + 16 \\ x - 2 & 2x + 2 & 3x - 1 \\ 1 & 2 & 3 \end{vmatrix}$

On differentiating w.r.t. x, we get

$$f'(x) = \begin{vmatrix} 2x - 4 & 4x + 4 & 6x - 2 \\ x - 2 & 2x + 2 & 3x - 1 \\ 1 & 2 & 3 \end{vmatrix}$$

$$+ \begin{vmatrix} x^2 - 4x + 6 & 2x^2 + 4x + 10 & 3x^2 - 2x + 16 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{vmatrix}$$

$$+ \begin{vmatrix} x^2 - 4x + 6 & 2x^2 + 4x + 10 & 3x^2 - 2x + 16 \\ x - 2 & 2x + 2 & 3x - 1 \\ 0 & 0 & 0 \end{vmatrix}$$

$f'(x) = 0, \forall\ x \in R$ and $f(x) =$ Constant

As, $\quad f(0) = \begin{vmatrix} 6 & 10 & 16 \\ -2 & 2 & -1 \\ 1 & 2 & 3 \end{vmatrix} = 2 \therefore \ f(x) = 2$

Now, $\quad I = \int_{-3}^{3} \frac{x^2 \sin x}{1 + x^6} f(x)\, dx = 2 \int_{-3}^{3} \frac{x^2 \sin x}{1 + x^6}\, dx$

Let $\quad g(x) = \frac{x^2 \sin x}{1 + x^6}$

$\therefore \quad g(-x) = \frac{-x^2 \sin x}{1 + x^6} = -g(x)$

Hence, g is an odd function.

$\therefore \quad I = 0$

102. Since, $\quad Y = X$ and $Z = tX$

$$Y_1 = sX_1 + Xs_1 \qquad \ldots(i)$$

$$Y_2 = sX_2 + Xs_2 + 2X_1 s_1 \qquad \ldots(ii)$$

$$Z_1 = tX_1 + Xt_1 \qquad \ldots(iii)$$

and $\quad Z_2 = tX_2 + Xt_2 + 2X_1 t_1 \qquad \ldots(iv)$

$$\text{LHS} = \begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix}$$

$$= \begin{vmatrix} X & sX & tX \\ X_1 & sX_1 + Xs_1 & tX_1 + Xt_1 \\ X_2 & sX_2 + Xs_2 + 2X_1 s_1 & tX_2 + Xt_2 + 2X_1 t_1 \end{vmatrix}$$

[using Eqs. (i), (ii), (iii) and (iv)]

Applying $C_2 \to C_2 - sC_1$ and $C_3 \to C_3 - tC_1$, then

$$= \begin{vmatrix} X & 0 & 0 \\ X_1 & Xs_1 & Xt_1 \\ X_2 & Xs_2 + 2X_1 s_1 & Xt_2 + 2X_1 t_1 \end{vmatrix}$$

Expanding w.r.t. R_1, then

$$= X^2 \begin{vmatrix} s_1 & t_1 \\ Xs_2 + 2X_1 s_1 & Xt_2 + 2X_1 t_1 \end{vmatrix}$$

Applying $R_2 \to R_2 - 2X_1 R_1$, then

$$= X^2 \begin{vmatrix} s_1 & t_1 \\ Xs_2 & Xt_2 \end{vmatrix} = X^3 \begin{vmatrix} s_1 & t_1 \\ s_2 & t_2 \end{vmatrix} = \text{RHS}$$

103. Given determinant may be expressed as

$$\Delta = \begin{vmatrix} f & g & h \\ xf' + f & xg' + g & xh' + h \\ (x^2 f'' + 4xf' + 2f) & (x^2 g'' + 4xg' + 2g) & (x^2 h'' + 4xh' + 2h) \end{vmatrix}$$

Now, applying $R_3 \to R_3 - 4R_2 + 2R_1$, then

$$\Delta = \begin{vmatrix} f & g & h \\ xf' + f & xg' + g & xh' + h \\ x^2 f'' & x^2 g'' & x^2 h'' \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$, then $\Delta = \begin{vmatrix} f & g & h \\ xf' & xg' & xh' \\ x^2 f'' & x^2 g'' & x^2 h'' \end{vmatrix}$

$$\Rightarrow \quad \Delta = x \begin{vmatrix} f & g & h \\ f' & g' & h' \\ x^2 f'' & x^2 g'' & x^2 h'' \end{vmatrix}$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ x^3 f'' & x^3 g'' & x^3 h'' \end{vmatrix}$$

$$\therefore \quad \Delta' = \begin{vmatrix} f' & g & h' \\ f' & g' & h' \\ x^3 f'' & x^3 g'' & x^3 h'' \end{vmatrix} + \begin{vmatrix} f & g & h \\ f'' & g'' & h'' \\ x^3 f'' & x^3 g'' & x^3 h'' \end{vmatrix}$$

$$+ \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3 f'')' & (x^3 g'')' & (x^3 h'')' \end{vmatrix}$$

$$= 0 + 0 + \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3 f'')' & (x^3 g'')' & (x^3 h'')' \end{vmatrix}$$

Hence, $\Delta' = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3 f'')' & (x^3 g'')' & (x^3 h'')' \end{vmatrix}$

104. Let the given determinant be equal to zero. Then, there exist x, y and z not all zero, such that

$$a_1 x + a_2 y + a_3 z = 0,\ b_1 x + b_2 y + b_3 z = 0$$

and $c_1 x + c_2 y + c_3 z = 0$

Assume that, $|x| \geq |y| \geq |z|$ and $x \neq 0$. Then, from $a_1 x = (-a_2 y) + (-a_3 z)$

$$\therefore \quad |a_1 x| = |-a_2 y - a_3 z| \leq |a_2 y| + |a_3 z|$$

$$\Rightarrow \quad |a_1|\ |x| \leq |a_2|\,|y| + |a_3|\,|z|$$

But $x \neq 0$ i.e. $|a_1| \leq |a_2| + |a_3|$

Similarly, $|b_2| \leq |b_1| + |b_3|$

$|c_3| \leq |c_1| + |c_2|$

which is contradiction. Hence, the assumption that the determinant is zero must be wrong.

105. $\text{LHS} = \begin{vmatrix} (a-a_1)^{-2} & (a-a_1)^{-1} & a_1^{-1} \\ (a-a_2)^{-2} & (a-a_2)^{-1} & a_2^{-1} \\ (a-a_3)^{-2} & (a-a_3)^{-1} & a_3^{-1} \end{vmatrix}$

$$= (a-a_1)^{-2}(a-a_2)^{-2}(a-a_3)^{-2} \begin{vmatrix} 1 & (a-a_1) & a_1^{-1}(a-a_1)^2 \\ 1 & (a-a_2) & a_2^{-1}(a-a_2)^2 \\ 1 & (a-a_3) & a_3^{-1}(a-a_3)^2 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to -R_3 - R_1$, then

$$\text{LHS} = \frac{1}{\prod (a-a_i)^2} \begin{vmatrix} 1 & (a-a_1) & a_1^{-1}(a-a_1)^2 \\ 0 & (a_1-a_2) & \dfrac{(a^2-a_1a_2)(a_1-a_2)}{a_1a_2} \\ 0 & (a_1-a_3) & \dfrac{(a^2-a_1a_3)(a_1-a_3)}{a_1a_3} \end{vmatrix}$$

Expanding w.r.t. 1st column, then

$$\text{LHS} = \frac{1}{\prod (a-a_i)^2} \begin{vmatrix} (a_1-a_2) & \dfrac{(a^2-a_1a_2)(a_1-a_2)}{a_1a_2} \\ (a_1-a_3) & \dfrac{(a^2-a_1a_3)(a_1-a_3)}{a_1a_3} \end{vmatrix}$$

$$= \frac{(a_1-a_2)(a_1-a_3)}{\prod (a-a_i)^2} \begin{vmatrix} 1 & \dfrac{a^2-a_1a_2}{a_1a_2} \\ 1 & \dfrac{a^2-a_1a_3}{a_1a_3} \end{vmatrix}$$

$$= \frac{(a_1-a_2)(a_1-a_3)\,a^2(a_2-a_3)}{a_1a_2a_3 \prod (a-a_i)^2} = \frac{-a^2 \prod (a_i - a_j)}{\prod a_i \prod (a-a_i)^2}$$

Numerator $= -a^2 (a_1-a_2)(a_2-a_3)(a_3-a_1)$

The resulting expression has negative sign.

106. The given system of equation will have a non-trivial solution in the determinant of coefficients.

$$\therefore \quad \Delta = \begin{vmatrix} a-t & b & c \\ b & c-t & a \\ c & a & b-t \end{vmatrix}$$

$\Delta = 0$ is a cubic equation in t.

So, it has in general three solutions t_1, t_2 and t_3.

Let $\Delta = a_0 t^3 + a_1 t^2 + a_2 t + a_3$

Clearly, $a_0 =$ Coefficient of $t^3 = -1$,

so $t_1 t_2 t_3 = -\dfrac{a_3}{a_0} = -\dfrac{a_3}{-1} = a_3 =$ Constant term in the expansion of Δ. i.e. Δ (at $t = 0$)

$$\therefore \quad t_1 t_2 t_3 = a_3 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

107. (i) Eliminating a, b and c from given equations, we obtain

$$\begin{vmatrix} -1 & \dfrac{y}{z} & \dfrac{z}{y} \\ -1 & \dfrac{z}{x} & \dfrac{x}{z} \\ -1 & \dfrac{x}{y} & \dfrac{y}{z} \end{vmatrix} = 0$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$\begin{vmatrix} -1 & \dfrac{y}{z} & \dfrac{z}{y} \\ 0 & \dfrac{z}{x} - \dfrac{y}{z} & \dfrac{x}{z} - \dfrac{z}{y} \\ 0 & \dfrac{x}{y} - \dfrac{y}{z} & \dfrac{y}{x} - \dfrac{z}{y} \end{vmatrix} = 0$$

Expanding along C_1, then

$$-\left(\frac{z}{x} - \frac{y}{x}\right)\left(\frac{y}{x} - \frac{z}{y}\right) + \left(\frac{x}{y} - \frac{y}{z}\right)\left(\frac{x}{z} - \frac{z}{y}\right) = 0$$

$$\Rightarrow \quad \frac{yz}{x^2} + \frac{zx}{y^2} + \frac{xy}{z^2} + 1 = 0$$

(ii) To eliminate x, y and z.

Let $\alpha = \dfrac{y}{z}$, $\beta = \dfrac{z}{x}$ and $\gamma = \dfrac{x}{y}$ in the given equations,

$$b\alpha + \frac{c}{\alpha} = a, \qquad \ldots(\text{i})$$

$$c\beta + \frac{a}{\beta} = b \qquad \ldots(\text{ii})$$

and $$a\gamma + \frac{b}{\gamma} = c \qquad \ldots(\text{iii})$$

Also, $\alpha\beta\gamma = 1$

From Eqs. (i), (ii) and (iii), we get

$$\left(b\alpha + \frac{c}{\alpha}\right)\left(c\beta + \frac{a}{\beta}\right)\left(a\gamma + \frac{b}{\gamma}\right) = abc$$

$$\Rightarrow \quad 2abc + ac^2 \frac{\beta\gamma}{\alpha} + a^2 b \frac{\alpha\gamma}{\beta} + b^2 c \frac{\alpha\beta}{\gamma} + a^2 c \frac{\gamma}{\alpha\beta} + bc^2 \frac{\beta}{\gamma\alpha} + ab^2 \frac{\alpha}{\beta\gamma} = abc$$

$$\Rightarrow \quad ac^2 \frac{1}{\alpha^2} + a^2 b \frac{1}{\beta^2} + b^2 c \frac{1}{\gamma^2} + a^2 c\gamma^2 + bc^2\beta^2 + ab^2\alpha^2 = -abc \qquad [\because \alpha\beta\gamma = 1]$$

$$\Rightarrow a\left(\frac{c^2}{\alpha^2} + b^2\alpha^2\right) + b\left(\frac{a^2}{\beta^2} + \beta^2 c^2\right) + c\left(\frac{b^2}{\gamma^2} + a^2\gamma^2\right) = -abc \qquad \ldots(\text{iv})$$

On squaring Eqs. (i), (ii) and (iii), we get

$$b^2\alpha^2 + \frac{c^2}{\alpha^2} = a^2 - 2bc,\ c^2\beta^2 + \frac{a^2}{\beta^2} = b^2 - 2ca \text{ and}$$

$$a^2\gamma^2 + \frac{b^2}{\gamma^2} = c^2 - 2ab$$

On putting these values in Eq. (iv), we get

$$a(a^2 - 2bc) + b(b^2 - 2ca) + c(c^2 - 2ab) = -abc$$

$$a^3 + b^3 + c^3 = 5abc$$

108. Here, $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$. According to the question, x, y and z not all zero. Hence, the given system of equations has non-trivial solution.

$$\Delta = 0$$

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

$$\Rightarrow \quad \frac{1}{2}(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2] = 0$$

$$\therefore \quad a + b + c = 0$$

or $$(a-b)^2 + (b-c)^2 + (c-a)^2 = 0$$

***Case* I** If $a + b + c = 0$

From first two equations,

$$ax + by - (a+b)z = 0$$

$$bx - (a+b)y + ax = 0$$

[by cross-multiplication law]

$$\therefore \quad \frac{x}{ab - (a+b)^2} = \frac{y}{-b(a+b) - a^2} = \frac{z}{-a(a+b) - b^2}$$

$$\Rightarrow \quad \frac{x}{-(a^2 + ab + b^2)} = \frac{y}{-(a^2 + ab + b^2)} = \frac{z}{-(a^2 + ab + b^2)}$$

$$\therefore \quad x : y : z = 1 : 1 : 1$$

***Case* II** If $(a-b)^2 + (b-c)^2 + (c-a)^2 = 0$

It is possible only, when

$$a - b = 0,\ b - c = 0 \text{ and } c - a = 0$$

Then, $$a = b = c$$

In this case all the three equations reduce in the forms

$$x + y + z = 0 \qquad \ldots(i)$$

Then, Eq. (i) will be satisfied, if

$$x = k,\ y = k\omega,\ z = k\omega^2$$

or $$x = k,\ y = k\omega^2,\ z = k\omega$$

where ω is the cube root of unity.

Then, $$x : y : z = 1 : \omega : \omega^2 \text{ or } 1 : \omega^2 : \omega$$

Hence, combined both cases, we get

$$x : y : z = 1 : 1 : 1$$

or $$1 : \omega : \omega^2$$

or $$1 : \omega^2 : \omega$$

109. Applying $C_1 \to C_1 + C_2 + C_3$, then

$$f(x) = \begin{vmatrix} 1 & (1+b^2)x & (1+c^2)x \\ 1 & 1+b^2x & (1+c^2)x \\ 1 & (1+b^2)x & 1+c^2x \end{vmatrix} \qquad [\because a^2 + b^2 + c^2 + 2 = 0]$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$= \begin{vmatrix} 1 & (1+b^2)x & (1+c^2)x \\ 0 & 1-x & 0 \\ 0 & 0 & 1-x \end{vmatrix} = (1-x^2)$$

Hence, degree of $f(x) = 2$

110. For no solution or infinitely many solutions

$$\begin{vmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix} = 0$$

Applying $C_1 \to C_1 + C_2 + C_3$, then

$$\begin{vmatrix} \alpha+2 & 1 & 1 \\ \alpha+2 & \alpha & 1 \\ \alpha+2 & 1 & \alpha \end{vmatrix} = 0$$

Applying $R_1 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$\begin{vmatrix} \alpha+2 & 1 & 1 \\ 0 & \alpha-1 & 0 \\ 0 & 0 & \alpha-1 \end{vmatrix} = 0 \Rightarrow (\alpha-1)^2(\alpha+2) = 0$$

$$\therefore \quad \alpha = 1, -2$$

For $\alpha = 1$, clearly there an infinitely many solutions and when we put $\alpha = -2$ in given system of equations and adding them together LHS $\neq$ RHS. i.e., no solution.

111. $\because a_1, a_2, a_3, \ldots$ are in GP.

$\therefore$ Using $a_n = a_1 r^{n-1}$, we get the given determinant, as

$$\begin{vmatrix} \log(a_1r^{n-1}) & \log(a_1r^n) & \log(a_1r^{n+1}) \\ \log(a_1r^{n+2}) & \log(a_1r^{n+3}) & \log(a_1r^{n+4}) \\ \log(a_1r^{n+5}) & \log(a_1r^{n+6}) & \log(a_1r^{n+7}) \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$ and using $\log m - \log n = \log\left(\frac{m}{n}\right)$, we get

$$\begin{vmatrix} \log(a_1r^{n-1}) & \log r & 2\log r \\ \log(a_1r^{n+2}) & \log r & 2\log r \\ \log(a_1r^{n+5}) & \log r & 2\log r \end{vmatrix} = 0$$

[$\because C_2$ and C_3 are proportional]

112. Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 0 & x & 0 \\ 0 & 0 & y \end{vmatrix} = xy$$

113. $\because$ $$D = \begin{vmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = 0$$

and $$D_1 = \begin{vmatrix} -1 & -2 & 3 \\ k & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = (3-k) = 0, \text{ if } k = 3$$

$$D_2 = \begin{vmatrix} 1 & -1 & 3 \\ -1 & k & -2 \\ 1 & -3 & 4 \end{vmatrix} = (k-3) = 0, \text{ if } k = 3$$

$$D_3=\begin{vmatrix}-1 & -2 & 3\\ k & 1 & -2\\ 1 & -3 & 4\end{vmatrix}=(k=3)=0, \text{ if } k=3$$

$\therefore$ System of equations has no solution for $k\neq 3$.

114. The system of equations

$x-cy-bz=0,\ -cx+y-az=0$ and $-bx-ay+z=0$

have non-trivial solution, if

$$\begin{vmatrix}1 & -c & -b\\ -c & 1 & -a\\ -b & -a & 1\end{vmatrix}=0$$

$$\Rightarrow \quad 1+2(-a)(-b)(-c)-a^2-b^2-c^2=0$$

or $$a^2+b^2+c^2+2abc=1$$

115. $$\begin{vmatrix}a & a+1 & a-1\\ -b & b+1 & b-1\\ c & c-1 & c+1\end{vmatrix}+(-1)^n\begin{vmatrix}a+1 & b+1 & c-1\\ a-1 & b-1 & c+1\\ a & -b & c\end{vmatrix}=0$$

$$\Rightarrow\begin{vmatrix}a & a+1 & a-1\\ -b & b+1 & b-1\\ c & c-1 & c+1\end{vmatrix}+(-1)^n\begin{vmatrix}a+1 & a-1 & a\\ b+1 & b-1 & -b\\ c-1 & c+1 & c\end{vmatrix}=0$$

[by property]

$$\Rightarrow\begin{vmatrix}a & a+1 & a-1\\ -b & b+1 & b-1\\ c & c-1 & c+1\end{vmatrix}+(-1)^{n+2}\begin{vmatrix}a & a+1 & a-1\\ -b & b+1 & b-1\\ c & c-1 & c+1\end{vmatrix}=0$$

116. Applying $R_1\to R_1+R_3$, then

$$f(\theta)=\begin{vmatrix}\theta & \cdots & 0 & \cdots & 2\\ & & & & \vdots\\ -\tan\theta & & 1 & & \tan\theta\\ & & & & \vdots\\ -1 & & -\tan\theta & & 1\end{vmatrix}$$

$$=2(1+\tan^2\theta)=2\sec^2\theta\geq 2$$

$\therefore f(\theta)\in[2,\infty)$

117. Non-zero solution means non-trivial solution.

For non-trivial solution of the given system of linear equations

$$\begin{vmatrix}4 & k & 2\\ k & 4 & 1\\ 2 & 2 & 1\end{vmatrix}=0$$

$$\Rightarrow\ 4(4-2)-k(k-2)+(2k-8)=0$$

$$\Rightarrow\quad -k^2+6k-8=0$$

$$\Rightarrow\quad k^2-6k+8=0$$

$$\Rightarrow\quad (k-2)(k-4)=0$$

$$\therefore\quad k=2,4$$

Clearly, there exist values of k.

118. For trivial solution $\begin{vmatrix}1 & -k & 1\\ k & 3 & -k\\ 3 & 1 & -1\end{vmatrix}\neq 0$

$$\Rightarrow 1(-3+k)+k(-k+3k)+1(k-9)\neq 0$$

$$\Rightarrow\quad 2k^2+2k-12\neq 0$$

$$\Rightarrow\quad k^2+k-6\neq 0$$

$$\Rightarrow\quad (k+3)(k-2)\neq 0$$

$$\Rightarrow\quad k\neq 2,-3$$

or $$k\in R-\{2,-3\}$$

119. $\Delta=\begin{vmatrix}k+1 & 8\\ k & k=3\end{vmatrix}=(k+1)(k+3)-8k=k^2-4k+3$

$\therefore\quad \Delta=(k-1)(k-3)$

$\Delta_1=\begin{vmatrix}4k & 8\\ 3k-1 & k+3\end{vmatrix}=4k^2+12k-24k+8=4k^2-12k+8$

$$\Delta_1=4(k-1)(k-2)$$

and $\Delta_2=\begin{vmatrix}k+1 & 4k\\ k & 3k-1\end{vmatrix}=(k+1)(3k-1)-4k^2=-k^2+2k+1$

$\therefore\quad \Delta_2=-(k-1)^2$

As given no solutions

$\Rightarrow\quad \Delta_1$ and $\Delta_2\neq 0$

but $\Delta=0$

$k=3$

120. $\because\begin{vmatrix}3 & 1+f(1) & 1+f(2)\\ 1+f(1) & 1+f(2) & 1+f(3)\\ 1+f(2) & 1+f(3) & 1+f(4)\end{vmatrix}$

$$=\begin{vmatrix}1+1+1 & 1+\alpha+\beta & 1+\alpha^2+\beta^2\\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3\\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4\end{vmatrix}$$

$$=\begin{vmatrix}1 & 1 & 1\\ 1 & \alpha & \beta\\ 1 & \alpha^2 & \beta^2\end{vmatrix}\times\begin{vmatrix}1 & 1 & 1\\ 1 & \alpha & \beta\\ 1 & \alpha^2 & \beta^2\end{vmatrix}=\begin{vmatrix}1 & 1 & 1\\ 1 & \alpha & \beta\\ 1 & \alpha^2 & \beta^2\end{vmatrix}^2$$

$$=\{(1-\alpha)(1-\beta)(\alpha-\beta)\}^2$$

$$=(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$$

So, $k=1$.

121. The given system can be written as

$$(2-\lambda)x_1-2x_2+x_3=0$$

$$2x_1-(3+\lambda)x_2+2x_3=0$$

$$-x_1+2x_2-\lambda x_3=0$$

For non-trivial solutions, $\Delta=0$

$$\begin{vmatrix}2-\lambda & -2 & 1\\ 2 & -(3+\lambda) & 2\\ -1 & 2 & -\lambda\end{vmatrix}=0$$

$$\Rightarrow (2-\lambda)(\lambda^2+3\lambda-4)+2(-2\lambda+2)+1(4-3-\lambda)=0$$

$$\Rightarrow\quad \lambda^3+\lambda^2-5\lambda+3=0$$

$$\Rightarrow\quad \lambda=1,1,-3$$

Hence, λ has two values.

122. Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ 2\alpha+3 & 4\alpha+3 & 6\alpha+3 \\ 4\alpha+8 & 8\alpha+8 & 12\alpha+8 \end{vmatrix} = -648\alpha$$

Applying $R_3 \to R_3 - 2R_2$, then

$$\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ 2\alpha+3 & 4\alpha+3 & 6\alpha+3 \\ 2 & 2 & 2 \end{vmatrix} = -648\alpha$$

Applying $C_2 \to C_2 - \frac{1}{2}(C_1 + C_3)$, then

$$\begin{vmatrix} (1+\alpha)^2 & \cdots & -\alpha^2 & \cdots & (1+3\alpha)^2 \\ 2\alpha+3 & & 4\alpha+3 & & 6\alpha+3 \\ 2 & & 0 & & 2 \end{vmatrix} = -648\alpha$$

$\Rightarrow \quad \alpha^2(4\alpha + 6 - 12\alpha - 6) = -648\alpha$

$\Rightarrow \quad -8\alpha^3 = -648\alpha$

$\Rightarrow \quad \alpha^3 - 81\alpha = 0$

$\therefore \quad \alpha = 0, 9, -9$

123. For non-trivial solution

$$\begin{bmatrix} 1 & \lambda & -1 \\ \lambda & -1 & -1 \\ 1 & 1 & -\lambda \end{bmatrix} = 0$$

$\Rightarrow \quad 1(\lambda+1) - \lambda(-\lambda^2+1) - 1(\lambda+1) = 0$

$\Rightarrow \quad \lambda(\lambda^2 - 1) = 0$

$\Rightarrow \quad \lambda = 0, \ \pm 1$

124.

$$x^3 \begin{bmatrix} 1 & 1 & 1+x^3 \\ 2 & 4 & 1+8x^3 \\ 3 & 9 & 1+27x^3 \end{bmatrix} = 10$$

$\Rightarrow$

$$x^3 \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 3 & 9 & 1 \end{bmatrix} + x^6 \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 27 \end{bmatrix} = 10$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, then

$$x^3 \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & -1 \\ 3 & 6 & -2 \end{bmatrix} + x^6 \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 6 \\ 3 & 6 & 24 \end{bmatrix} = 10$$

$\Rightarrow \quad 2x^3 + 12x^6 = 10 \ \text{ or } 6x^6 + x^3 - 5 = 0$

or $\quad (6x^3 - 5)(x^3 + 1) = 0$

$\Rightarrow \quad x^3 = \frac{5}{6} \ \text{ or } \ x^3 = -1$

$\therefore \quad x = \left(\frac{5}{6}\right)^{1/3}, -1$

i.e. Two distinct values of x.

125.

$$\Delta = \begin{vmatrix} a & 2 \\ 3 & -2 \end{vmatrix} = -2a - 6,$$

$$\Delta_1 = \begin{vmatrix} \lambda & 2 \\ \mu & -2 \end{vmatrix} = -2(\lambda + \mu)$$

or $\quad \Delta_2 = \begin{vmatrix} a & \lambda \\ 3 & \mu \end{vmatrix} = \alpha\mu - 3\lambda$

System has unique solution for $\Delta \neq 0$

$\therefore \ a \neq -3$ for all values λ and μ

System has infinitely many solution for

$$\Delta = \Delta_1 = \Delta_2 = 0$$

$\therefore \ a = -3, \ \lambda + \mu = 0, \ a\mu - 3\lambda = 0$

and system has no solution

$$\Delta = 0 \Rightarrow a = -3$$

and $\quad \lambda + \mu \neq 0$

126. $\because \ \Delta = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ a & b & 1 \end{bmatrix} = 1(a-b) - 1(1-a) + 1(b-a^2) = -(a-1)^2$

$$\Delta_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ 0 & b & 1 \end{bmatrix} = 1(a-b) - 1(1) + 1(b) = (a-1)$$

$$\Delta_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ a & 0 & 1 \end{bmatrix} = 1(1) - 1(1-a) + 1(0-a) = 0$$

and $\Delta_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ a & b & 0 \end{bmatrix} = 1(-b) - 1(-a) + 1(b-a^2) = -a(a-1)$

For $a = 1, \Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$

and for $b = 1$ only

$$x + y + z = 1$$
$$x + y + z = 1$$
$$x + y + z = 0$$

i.e. no solution ($\because$ RHS are not equal)

Hence, for no solution $b = 1$ only.

CHAPTER

08

Matrices

Learning Part

Session 1

- Definition
- Types of Matrices
- Difference Between a Matrix and a Determinant
- Equal Matrices
- Operations of Matrices
- Various Kinds of Matrices

Session 2

- Transpose of a Matrix
- Symmetric Matrix
- Orthogonal Matrix
- Complex Conjugate (or Conjugate) of a Matrix
- Hermitian Matrix
- Unitary Matrix
- Determinant of a Matrix
- Singular and Non-Singular Matrices

Session 3

- Adjoint of a Matrix
- Elementary Row Operations
- Matrix Polynomial
- Inverse of a Matrix
- Equivalent Matrices
- Use of Mathematical Induction

Session 4

- Solutions of Linear Simultaneous Equations Using Matrix Method

Practice Part

- JEE Type Examples
- Chapter Exercises

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J. J. Sylvester was the first to use the word ''Matrix" in 1850 and later on in 1858 Arthur Cayley developed the theory of matrices in a systematic way. 'Matrices' is a powerful tool in mathematics and its study is becoming important day by day due to its wide applications in almost every branch of science. This mathematical tool is not only used in certain branches of sciences but also in genetics, economics, sociology, modern psychology and industrial management.

Session 1

Definition, Types of Matrices, Difference Between a Matrix and a Determinant, Equal Matrices, Operations of Matrices, Various Kinds of Matrices

Definition

A set of mn numbers (real or complex) arranged in the form of a rectangular array having m rows and n columns is called a matrix of order $m \times n$ or an $m \times n$ matrix (which is read as m by n matrix).

An $m \times n$ matrix is usually written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{m_1} & a_{m_2} & a_{m_3} & \dots & a_{mn} \end{bmatrix}$$

In a compact form the above matrix is represented by $[a_{ij}]$, $i = 1, 2, 3, \dots, m$, $j = 1, 2, 3, \dots, n$ or simply by $[a_{ij}]_{m \times n}$, where the symbols a_{ij} represent any numbers (a_{ij} lies in the ith row (from top) and jth column (from left)).

Notations A matrix is denoted by capital letter such as A, B, C, ..., X, Y, Z.

Note

1. A matrix may be represented by the symbols $[a_{ij}], (a_{ij}), \| a_{ij} \|$ or by a single capital letter A (say)
   $$A = [a_{ij}]_{m \times n} \text{ or } (a_{ij})_{m \times n} \text{ or } \| a_{ij} \|$$
   Generally, the first system is adopted.
2. The numbers a_{11}, a_{12}, ..., etc., of rectangular array are called the elements or entries of the matrix.
3. A matrix is essentially an arrangement of elements and has no value.
4. The plural of 'matrix' is 'matrices'.

Example 1. If a matrix has 12 elements, what are the possible orders it can have? What will be the possible orders if it has 7 elements?

Sol. We know that, if a matrix is of order $m \times n$, it has mn elements. Thus, to find all possible orders of a matrix with 12 elements, we will find all ordered pairs of natural numbers, whose product is 12.

Thus, all possible ordered pairs are (1, 12), (12, 1), (2, 6), (6, 2), (3, 4), (4, 3).

Hence, possible orders are 1×12, 12×1, 2×6, 6×2, 3×4 and 4×3.

If the matrix has 7 elements, then the possible orders will be 1×7 and 7×1.

Example 2. Construct a 2×3 matrix $A = [a_{ij}]$, whose elements are given by

(i) $a_{ij} = \dfrac{(i+2j)^2}{2}$. (ii) $a_{ij} = \dfrac{1}{2} \left| 2i - 3j \right|$.

(iii) $a_{ij} = \begin{cases} i - j, & i \geq j \\ i + j, & i < j \end{cases}$.

(iv) $a_{ij} = \left[\dfrac{i}{j} \right]$,
where [.] denotes the greatest integer function.

(v) $a_{ij} = \left\{ \dfrac{2i}{3j} \right\}$,
where {.} denotes the fractional part function.

(vi) $a_{ij} = \left(\dfrac{3i + 4j}{2} \right)$,
where (.) denotes the least integer function.

Sol. We have, $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2\times 3}$

(i) Since, $a_{ij}=\dfrac{(i+2j)^2}{2}$, therefore

$$a_{11}=\frac{(1+2)^2}{2}=\frac{9}{2}, a_{12}=\frac{(1+4)^2}{2}=\frac{25}{2},$$

$$a_{13}=\frac{(1+6)^2}{2}=\frac{49}{2}, a_{21}=\frac{(2+2)^2}{2}=8,$$

$$a_{22}=\frac{(2+4)^2}{2}=18 \text{ and } a_{23}=\frac{(2+6)^2}{2}=32$$

Hence, the required matrix is $A=\begin{bmatrix} \frac{9}{2} & \frac{25}{2} & \frac{49}{2} \\ 8 & 18 & 32 \end{bmatrix}$.

(ii) Since, $a_{ij}=\dfrac{1}{2}\left|\,2i-3j\,\right|$, therefore

$$a_{11}=\frac{1}{2}\left|\,2-3\,\right|=\frac{1}{2}\left|\,-1\,\right|=\frac{1}{2},$$

$$a_{12}=\frac{1}{2}\left|\,2-6\,\right|=\frac{1}{2}\left|\,-4\,\right|=\frac{4}{2}=2,$$

$$a_{13}=\frac{1}{2}\left|\,2-9\,\right|=\frac{1}{2}\left|\,-7\,\right|=\frac{7}{2},$$

$$a_{21}=\frac{1}{2}\left|\,2-3\,\right|=\frac{1}{2}\left|\,-1\,\right|=\frac{1}{2},$$

$$a_{22}=\frac{1}{2}\left|\,2-6\,\right|=\frac{1}{2}\left|\,-4\,\right|=\frac{4}{2}=2$$

and $a_{23}=\dfrac{1}{2}\left|\,4-9\,\right|=\dfrac{1}{2}\left|\,-5\,\right|=\dfrac{5}{2}$

Hence, the required matrix is $A=\begin{bmatrix} \frac{1}{2} & 2 & \frac{7}{2} \\ \frac{1}{2} & 2 & \frac{5}{2} \end{bmatrix}$.

(iii) Since, $a_{ij}=\begin{cases} i-j, & i\geq j \\ i+j, & i<j \end{cases}$, therefore

$$a_{11}=1-1=0, a_{12}=1+2=3, a_{13}=1+3=4,$$
$$a_{21}=2-1=1, a_{22}=2-2=0 \text{ and } a_{23}=2+3=5$$

Hence, the required matrix is

$$A=\begin{bmatrix} 0 & 3 & 4 \\ 1 & 0 & 5 \end{bmatrix}$$

(iv) Since, $a_{ij}=\left[\dfrac{i}{j}\right]$, therefore $\quad[\because [x]\leq x\,]$

$$a_{11}=\left[\frac{1}{1}\right]=[1]=1, a_{12}=\left[\frac{1}{2}\right]=[0.5]=0,$$

$$a_{13}=\left[\frac{1}{3}\right]=[0.33]=0, a_{21}=\left[\frac{2}{1}\right]=[2]=2$$

and $a_{22}=\left[\dfrac{2}{2}\right]=[1]=1$ and $a_{23}=\left[\dfrac{2}{3}\right]=[0.67]=0$

Hence, the required matrix is $A=\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix}$

(v) Since, $a_{ij}=\left\{\dfrac{2i}{3j}\right\}$, therefore $\quad[\because 0\leq\{x\}<1]$

$$a_{11}=\left\{\frac{2}{3}\right\}=\frac{2}{3}, a_{12}=\left\{\frac{2}{6}\right\}=\left\{\frac{1}{3}\right\}=\frac{1}{3},$$

$$a_{13}=\left\{\frac{2}{9}\right\}=\frac{2}{9}, a_{21}=\left\{\frac{4}{3}\right\}=\left\{1+\frac{1}{3}\right\}=\frac{1}{3},$$

$$a_{22}=\left\{\frac{4}{6}\right\}=\left\{\frac{2}{3}\right\}=\frac{2}{3} \text{ and } a_{23}=\left\{\frac{4}{9}\right\}=\frac{4}{9}$$

Hence, the required matrix is $A=\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{9} \\ \frac{1}{3} & \frac{2}{3} & \frac{4}{9} \end{bmatrix}$.

(vi) Since, $a_{ij}=\left(\dfrac{3i+4j}{2}\right)$, therefore $\quad[\because (x)\geq x\,]$

$$a_{11}=\left(\frac{3+4}{2}\right)=\left(\frac{7}{2}\right)=(3.5)=4,$$

$$a_{12}=\left(\frac{3+8}{2}\right)=\left(\frac{11}{2}\right)=(5.5)=6,$$

$$a_{13}=\left(\frac{3+12}{2}\right)=\left(\frac{15}{2}\right)=(7.5)=8,$$

$$a_{21}=\left(\frac{6+4}{2}\right)=\left(\frac{10}{2}\right)=(5)=5,$$

$$a_{22}=\left(\frac{6+8}{2}\right)=\left(\frac{14}{2}\right)=(7)=7$$

and $a_{23}=\left(\dfrac{6+12}{2}\right)=\left(\dfrac{18}{2}\right)=(9)=9$

Hence, the required matrix is

$$A=\begin{bmatrix} 4 & 6 & 8 \\ 5 & 7 & 9 \end{bmatrix}$$

Types of Matrices

1. Row Matrix or Row Vector

A matrix is said to be row matrix or row vector, if it contains only one row, i.e. a matrix $A=[a_{ij}]_{m\times n}$ is said to be row matrix, if $m=1$.

For example,

(i) $A=[a_{11}\quad a_{12}\quad a_{13}\quad \ldots\quad a_{1n}]_{1\times n}$

(ii) $B=[3\quad 5\quad -7\quad 9]_{1\times 4}$

are called row matrices.

2. Column Matrix or Column Vector

A matrix is said to be column matrix or column vector, if it contains only one column, i.e., a matrix $A=[a_{ij}]_{m\times n}$ is said to be column matrix, if $n=1$. *For example,*

(i) $A=\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{mn} \end{bmatrix}_{m\times 1}$ (ii) $B=\begin{bmatrix} 7 \\ 0 \\ -8 \\ 2 \\ 1 \end{bmatrix}_{5\times 1}$

are called column matrices.

3. Rectangular Matrix

A matrix is said to be rectangular matrix, if the number of rows and the number of columns are not equal i.e., a matrix $A=[a_{ij}]_{m\times n}$ is called a rectangular matrix, iff $m \neq n$. *For example,*

(i) $A=\begin{bmatrix} 1 & 3 & 4 & 5 \\ 2 & 0 & -3 & 8 \\ 7 & 4 & 2 & 5 \end{bmatrix}_{3\times 4}$ (ii) $B=\begin{bmatrix} 2 & -3 \\ 3 & 0 \\ 4 & 8 \end{bmatrix}_{3\times 2}$

are called rectangular matrices.

4. Square Matrix

A matrix is said to be a square matrix, if the number of rows and the number of columns are equal i.e., a matrix $A=[a_{ij}]_{m\times n}$ is called a square matrix, iff $m=n$.

For example,

(i) $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$ (ii) $B=\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2}$

are called square matrices.

Remark

If $A=[a_{ij}]$ is a square matrix of order n, then elements (entries) $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ are said to constitute the *diagonal* of the matrix A. The line along which the diagonal elements lie is called principal or leading diagonal. Thus, if $A=\begin{bmatrix} 1 & 4 & 0 \\ 8 & 3 & -2 \\ 9 & 2 & 5 \end{bmatrix}$, then the elements of the diagonal of A are 1, 3, 5.

5. Diagonal Matrix

A square matrix is said to be a diagonal matrix, if all its non-diagonal elements are zero. Thus, $A=[a_{ij}]_{n\times n}$ is called a diagonal matrix, if $a_{ij}=0$, when $i \neq j$.

For example,

(i) $A=[2]$ (ii) $B=\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ (iii) $C=\begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix}$

are diagonal matrices of order 1, 2 and 3, respectively. A diagonal matrix of order n having $d_1, d_2, d_3, \ldots, d_n$ as diagonal elements may be denoted by diag $(d_1, d_2, d_3, .., d_n)$.

Thus, $A=$ diag (2), $B=$ diag $(-1, 2)$ and $C=$ diag (3, 5, 7).

Remark

(i) No element of principal diagonal in a diagonal matrix is zero.

(ii) Minimum number of zero in a diagonal matrix is given by $n(n-1)$, where n is order of matrix.

6. Scalar Matrix

A diagonal matrix is said to be a scalar matrix, if its diagonal elements are equal. Thus, $A=[a_{ij}]_{n\times n}$ is called scalar matrix, if

$$a_{ij}=\begin{cases} 0, \text{ if } i \neq j \\ k, \text{ if } i=j \end{cases}, \text{ where } k \text{ is scalar.}$$

For example,

(i) $[7]$ (ii) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

are scalar matrices of order 1, 2 and 3, respectively. They can be written as diag (7), diag (2, 2) and diag (5, 5, 5), respectively.

7. Unit or Identity Matrix

A diagonal matrix is said to be an identity matrix, if its diagonal elements are equal to 1.

Thus, $A=[a_{ij}]_{n\times n}$ is called unit or identity matrix, if

$$a_{ij}=\begin{cases} 0, \text{ if } i \neq j \\ 1, \text{ if } i=j \end{cases}$$

A unit matrix of order n is denoted by I_n or I. *For example,*

(i) $I_1=[1]$ (ii) $I_2=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (iii) $I_3=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

are identity matrices of order 1, 2 and 3, respectively.

8. Singleton Matrix

A matrix is said to be singleton matrix, if it has only one element i.e. a matrix $A=[a_{ij}]_{m\times n}$ is said to be singleton matrix, if $m=n=1$.

For example, $[3], [k], [-2]$ are singleton matrices.

9. Triangular Matrix

A square matrix is called a triangular matrix, if its each element above or below the principal diagonal is zero. It is of two types:

(a) **Upper Triangular Matrix** A square matrix in which all elements below the principal diagonal are zero is called an upper triangular matrix i.e., a matrix $A=[a_{ij}]_{n\times n}$ is said to be an upper triangular matrix, if $a_{ij}=0$, when $i>j$.

For example,

(i) $\begin{bmatrix} 3 & -2 & 4 & 1 \\ 0 & 2 & -3 & 2 \\ 0 & 0 & 7 & 5 \\ 0 & 0 & 0 & 8 \end{bmatrix}_{4\times 4}$

(ii) $\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & 0 & a_{33} & a_{34} & a_{35} \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & a_{55} \end{bmatrix}_{5\times 5}$

are upper triangular matrices.

(b) **Lower Triangular Matrix** A square matrix in which all elements above the principal diagonal are zero is called a lower triangular matrix i.e., a matrix $A=[a_{ij}]_{n\times n}$ is said to be a lower triangular matrix, if $a_{ij}=0$, when $i<j$. *For example*,

(i) $\begin{bmatrix} 7 & 0 & 0 \\ 5 & 4 & 0 \\ 2 & 3 & 4 \end{bmatrix}_{3\times 3}$ (ii) $\begin{bmatrix} 10 & 0 & 0 & 0 \\ 8 & 9 & 0 & 0 \\ 5 & 6 & 7 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix}_{4\times 4}$

are lower triangular matrices.

Note

Minimum number of zeroes in a triangular matrix is given by $\frac{n(n-1)}{2}$, where n is order of matrix.

10. Horizontal Matrix

A matrix is said to be horizontal matrix, if the number of rows is less than the number of columns i.e., a matrix $A=[a_{ij}]_{m\times n}$ is said to horizontal matrix, iff $m<n$.

For example, $A=\begin{bmatrix} 2 & 3 & 4 & 5 \\ 8 & 9 & 7 & -2 \\ 2 & -2 & -3 & 4 \end{bmatrix}_{3\times 4}$ is a horizontal matrix. [$\because$ number of rows (3) < number of columns (4)]

11. Vertical Matrix

A matrix is said to be vertical matrix, if the number of rows is greater than the number of columns i.e., a matrix $A=[a_{ij}]_{m\times n}$ is said to vertical matrix, iff $m>n$.

For example, $A=\begin{bmatrix} 2 & 3 & 4 \\ 0 & -1 & 7 \\ 3 & 5 & 4 \\ 2 & 7 & 9 \\ -1 & 2 & -5 \end{bmatrix}_{5\times 3}$ is a vertical matrix.

[$\because$ number of rows (5) > number of columns (3)]

12. Null Matrix or Zero Matrix

A matrix is said to be null matrix or zero matrix, if all elements are zero i.e., a matrix $A=[a_{ij}]_{m\times n}$ is said to be a zero or null matrix, iff $a_{ij}=0,\ \forall\ i,\ j$. It is denoted by O.

For example,

(i) $O_{2\times 3}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (ii) $O_{3\times 3}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

are called the null matrices.

13. Sub-Matrix

A matrix which is obtained from a given matrix by deleting any number of rows and number of columns is called a sub-matrix of the given matrix.

For example, $\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$ is a sub-matrix of $\begin{bmatrix} 8 & 9 & 5 \\ 2 & 3 & 4 \\ 3 & -2 & 5 \end{bmatrix}$.

14. Trace of a Matrix

The sum of all diagonal elements of a square matrix $A=[a_{ij}]_{n\times n}$ (say) is called the **trace** of a matrix A and is denoted by $\text{Tr}\,(A)$.

Thus, $$\text{Tr}\,(A)=\sum_{i=1}^{n} a_{ii}$$

For example, If $A=\begin{bmatrix} 2 & -7 & 9 \\ 0 & 3 & 2 \\ 8 & 9 & 4 \end{bmatrix}$, then

$$\text{Tr}\,(A)=2+3+4=9$$

Properties of Trace of a Matrix

Let $A=[a_{ij}]_{n\times n}$, $B=[b_{ij}]_{n\times n}$ and k is a scalar, then

(i) $\text{Tr}\,(kA)=k\cdot\text{Tr}\,(A)$

(ii) $\text{Tr}\,(A\pm B)=\text{Tr}\,(A)\pm\text{Tr}\,(B)$

(iii) $\text{Tr}\,(AB)=\text{Tr}\,(BA)$

(iv) $\text{Tr}\,(A)=\text{Tr}\,(A')$

(v) $\text{Tr}\,(I_n)=n$

(vi) $\text{Tr}\,(AB)\neq\text{Tr}\,(A)\,\text{Tr}\,(B)$

(vii) $\text{Tr}\,(A)=\text{Tr}\,(C\,A\,C^{-1})$,
where C is a non-singular square matrix of order n.

15. Determinant of Square Matrix

Let $A=[a_{ij}]_{n\times n}$ be a matrix. The determinant formed by the elements of A is said to be the determinant of matrix A. This is denoted by $|A|$.

For example,

If $A=\begin{bmatrix}3 & 4 & 5\\ 6 & 7 & 8\\ 2 & -3 & 5\end{bmatrix}$, then $|A|=\begin{vmatrix}3 & 4 & 5\\ 6 & 7 & 8\\ 2 & -3 & 5\end{vmatrix}=-39.$

Remark

1. If $A_1, A_2, A_3, \ldots, A_n$ are square matrices of the same order, then $|A_1A_2A_3\ldots A_n|=|A_1||A_2||A_3|\ldots|A_n|$.
2. If k is a scalar and A is a square matrix of order n, then $|kA|=k^n|A|$

16. Comparable Matrices

Two matrices $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{p\times q}$ are said to be comparable, if $m=p$ and $n=q$.

For example,

The matrices $\begin{bmatrix}a & b & c\\ d & e & f\end{bmatrix}$ and $\begin{bmatrix}p & q & r\\ s & t & u\end{bmatrix}$ are comparable

but the matrices $\begin{bmatrix}1 & 2\\ 4 & 8\end{bmatrix}$ and $\begin{bmatrix}2 & 4 & 6\\ 5 & 3 & 1\end{bmatrix}$ are not comparable.

Difference Between a Matrix and a Determinant

(i) A matrix cannot be reduced to a number but determinant can be reduced to a number.

(ii) The number of rows may or may not be equal to the number of columns in matrices but in determinant the number of rows is equal to the number of columns.

(iii) On interchanging the rows and columns, a different matrix is formed but in determinant it does not change the value.

(iv) A square matrix A such that $|A|\neq 0$, is called a non-singular matrix. If $|A|=0$, then the matrix A is called a singular matrix.

(v) Matrices represented by [ ], ( ), $\|\ \|$ but determinant is represented by $|\ |$.

Equal Matrices

Two matrices are said to be equal, if

(i) they are of the same order i.e., if they have same number of rows and columns.

(ii) the elements in the corresponding positions of the two matrices are equal.

Thus, if $A=[a_{ij}]_{m\times n}$, $B=[b_{ij}]_{p\times q}$, then $A=B$, iff

(i) $m=p, n=q$ (ii) $a_{ij}=b_{ij}, \forall\ i, j$

For example, If $A=\begin{bmatrix}-1 & 2 & 4\\ 3 & 0 & 5\end{bmatrix}_{2\times 3}$ and

$B=\begin{bmatrix}a & b & c\\ d & e & f\end{bmatrix}_{2\times 3}$ are equal matrices, then

$a=-1, b=2, c=4\ d=3, e=0, f=5$

Example 3. If $\begin{bmatrix}x+3 & 2y+x\\ z-1 & 4w-8\end{bmatrix}=\begin{bmatrix}-x-1 & 0\\ 3 & 2w\end{bmatrix}$, then find the value of $|x+y|+|z+w|$.

Sol. As the given matrices are equal so their corresponding elements are equal.

$$x+3=-x-1 \Rightarrow 2x=-4 \quad \ldots(i)$$

$$\therefore\quad x=-2$$

$$2y+x=0$$

$$\Rightarrow\quad 2y-2=0 \quad [\text{from Eq. (i)}]$$

$$\Rightarrow\quad y=1 \quad \ldots(ii)$$

$$z-1=3$$

$$\Rightarrow\quad z=4 \quad \ldots(iii)$$

$$4w-8=2w$$

$$\Rightarrow\quad 2w=8$$

$$\therefore\quad w=4 \quad \ldots(iv)$$

Hence, $|x+y|+|z+w|=|-2+1|+|4+4|=1+8=9$

Example 4. If $\begin{bmatrix}2\alpha+1 & 3\beta\\ 0 & \beta^2-5\beta\end{bmatrix}=\begin{bmatrix}\alpha+3 & \beta^2+2\\ 0 & -6\end{bmatrix}$, find the equation whose roots are α and β.

Sol. The given matrices will be equal, iff

$$2\alpha+1=\alpha+3\Rightarrow\alpha=2$$

$$3\beta=\beta^2+2\Rightarrow\beta^2-3\beta+2=0$$

$$\therefore\quad \beta=1,2 \text{ and } \beta^2-5\beta=-6 \quad \ldots(i)$$

$$\Rightarrow\quad \beta^2-5\beta+6=0$$

$$\therefore\quad \beta=2,3 \quad \ldots(ii)$$

From Eqs. (i) and (ii), we get $\beta=2$

$$\Rightarrow\quad \alpha=2, \beta=2$$

$\therefore$ Required equation is $x^2-(2+2)x+2\cdot 2=0$

$$\Rightarrow\quad x^2-4x+4=0$$

Operations of Matrices

Addition of Matrices

Let A, B be two matrices, each of order $m\times n$. Then, their sum $A+B$ is a matrix of order $m\times n$ and is obtained by adding the corresponding elements of A and B.

Thus, if $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{m\times n}$, then $A+B=[a_{ij}+b_{ij}]_{m\times n}, \forall\, i, j$

Example 5. Given, $A=\begin{bmatrix}1 & 3 & 5\\ -2 & 0 & 2\\ 0 & 4 & -3\end{bmatrix}, B=\begin{bmatrix}0 & 3\\ -2 & 0\\ 0 & -4\end{bmatrix}$ and $C=\begin{bmatrix}4 & 1 & -2\\ 3 & 2 & 1\\ 2 & -1 & 7\end{bmatrix}$. Find (whichever defined)

(i) $A+B$. (ii) $A+C$.

Sol. (i) Given, A is a matrix of the type 3×3 and B is a matrix of the type 3×2.
Since, A and B are not of the same type.
$\therefore$ Sum $A+B$ is not defined.

(ii) As A and C are two matrices of the same type, therefore the sum $A+C$ is defined.

$$\therefore\ A+C=\begin{bmatrix}1 & 3 & 5\\ -2 & 0 & 2\\ 0 & 4 & -3\end{bmatrix}+\begin{bmatrix}4 & 1 & -2\\ 3 & 2 & 1\\ 2 & -1 & 7\end{bmatrix}$$

$$=\begin{bmatrix}1+4 & 3+1 & 5-2\\ -2+3 & 0+2 & 2+1\\ 0+2 & 4-1 & -3+7\end{bmatrix}=\begin{bmatrix}5 & 4 & 3\\ 1 & 2 & 3\\ 2 & 3 & 4\end{bmatrix}$$

Example 6. If $a, b; b, c$ and c, a are the roots of $x^2-4x+3=0$, $x^2-8x+15=0$ and $x^2-6x+5=0$, respectively. Compute $\begin{bmatrix}a^2+c^2 & a^2+b^2\\ b^2+c^2 & a^2+c^2\end{bmatrix}+\begin{bmatrix}2ac & -2ab\\ -2bc & -2ac\end{bmatrix}$.

Sol. $\because$ $x^2-4x+3=0$

$\Rightarrow$ $(x-1)(x-3)=0$ $\quad\therefore\ x=1, 3$

$x^2-8x+15=0$

$\Rightarrow$ $(x-3)(x-5)=0$ $\quad\therefore\ x=3, 5$

and $x^2-6x+5=0$

$\Rightarrow$ $(x-5)(x-1)=0$ $\quad\therefore\ x=5, 1$

It is clear that $a=1$, $b=3$ and $c=5$

Now, $\begin{bmatrix}a^2+c^2 & a^2+b^2\\ b^2+c^2 & a^2+c^2\end{bmatrix}+\begin{bmatrix}2ac & -2ab\\ -2bc & -2ac\end{bmatrix}$

$$=\begin{bmatrix}a^2+c^2+2ac & a^2+b^2-2ab\\ b^2+c^2-2bc & a^2+c^2-2ac\end{bmatrix}=\begin{bmatrix}(a+c)^2 & (a-b)^2\\ (b-c)^2 & (a-c)^2\end{bmatrix}$$

$$=\begin{bmatrix}(1+5)^2 & (1-3)^2\\ (3-5)^2 & (1-5)^2\end{bmatrix}=\begin{bmatrix}36 & 4\\ 4 & 16\end{bmatrix}$$

Properties of Matrix Addition

Property 1 Addition of matrices is commutative,
i.e. $A+B=B+A$
where A and B are any two $m\times n$ matrices, i.e. matrices of the same order.

Property 2 Addition of matrices is associative
i.e. $(A+B)+C=A+(B+C)$
where A, B and C are any three matrices of the same order $m\times n$ (say).

Property 3 Existence of additive identity
i.e. $A+O=A=O+A$
where A be any $m\times n$ matrix and O be the $m\times n$ null matrix. The null matrix O is the identity element for matrix addition.

Property 4 Existence of additive inverse
If A be any $m\times n$ matrix, then there exists another $m\times n$ matrix B, such that $A+B=O=B+A$
where O is the $m\times n$ null matrix.
Here, the matrix B is called the additive inverse of the matrix A or the negative of A.

Property 5 Cancellation laws
If A, B and C are matrices of the same order $m\times n$ (say),
then $A+B=A+C\Rightarrow B=C$ [left cancellation law]
and $B+A=C+A\Rightarrow B=C$ [right cancellation law]

Scalar Multiplication

Let $A=[a_{ij}]_{m\times n}$ be a matrix and k be any number called a scalar. Then, the matrix obtained by multiplying every element of A by k is called the scalar multiple of A by k and is denoted by kA.
Thus, $kA=[ka_{ij}]_{m\times n}$

Properties of Scalar Multiplication

If $A=[a_{ij}]_{m\times n}$, $B=[b_{ij}]_{m\times n}$ are two matrices and k, l are scalars , then

(i) $k(A+B)=kA+kB$ (ii) $(k+l)A=kA+lA$

(iii) $(kl)A=k(lA)=l(kA)$

(iv) $(-k)A=-(kA)=k(-A)$

(v) $1A=A, (-1)A=-A$

Example 7. Determine the matrix A, when $A=4\begin{bmatrix}1 & 2 & 3\\ -1 & -2 & -3\\ 4 & 2 & 6\end{bmatrix}+2\begin{bmatrix}5 & 4 & 1\\ 3 & 2 & 4\\ 3 & 8 & 2\end{bmatrix}$.

Sol. $$A=\begin{bmatrix}4&8&12\\-4&-8&-12\\16&8&24\end{bmatrix}+\begin{bmatrix}10&8&2\\6&4&8\\6&16&4\end{bmatrix}$$

$$=\begin{bmatrix}4+10&8+8&12+2\\-4+6&-8+4&-12+8\\16+6&8+16&24+4\end{bmatrix}=\begin{bmatrix}14&16&14\\2&-4&-4\\22&24&28\end{bmatrix}$$

Example 8. If $A=\begin{bmatrix}0&2\\3&-4\end{bmatrix}$ and $kA=\begin{bmatrix}0&3a\\2b&24\end{bmatrix}$, then find the value of $b-a-k$.

Sol. We have, $$A=\begin{bmatrix}0&2\\3&-4\end{bmatrix}\Rightarrow kA=\begin{bmatrix}0&2k\\3k&-4k\end{bmatrix}$$

But $$kA=\begin{bmatrix}0&3a\\2b&24\end{bmatrix}$$

$$\therefore\quad\begin{bmatrix}0&2k\\3k&-4k\end{bmatrix}=\begin{bmatrix}0&3a\\2b&24\end{bmatrix}$$

$$\Rightarrow\quad 2k=3a,\ 3k=2b,\ -4k=24$$

$$\Rightarrow\quad k=-6,\ a=-4,\ b=-9$$

Hence, $$b-a-k=-9-(-4)-(-6)=-9+4+6=1$$

Subtraction of Matrices

Let A, B be two matrices, each of order $m\times n$. Then, their subtraction $A-B$ is a matrix of order $m\times n$ and is obtained by subtracting the corresponding elements of A and B. Thus, if $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{m\times n}$,

then $$A-B=[a_{ij}-b_{ij}]_{m\times n},\ \forall\ i,j$$

For example, If $A=\begin{bmatrix}2&3\\4&5\\6&7\end{bmatrix}$ and $B=\begin{bmatrix}a&b\\c&d\\e&f\end{bmatrix}$,

then $$A-B=\begin{bmatrix}2&3\\4&5\\6&7\end{bmatrix}-\begin{bmatrix}a&b\\c&d\\e&f\end{bmatrix}=\begin{bmatrix}2-a&3-b\\4-c&5-d\\6-e&7-f\end{bmatrix}$$

Example 9. Given, $A=\begin{bmatrix}1&2&-3\\5&0&2\\1&-1&1\end{bmatrix}$ and $B=\begin{bmatrix}3&-1&2\\4&2&5\\2&0&3\end{bmatrix}$. Find the matrix C such that $A+2C=B$.

Sol. Given, $A+2C=B$

$$2C=B-A=\begin{bmatrix}3&-1&2\\4&2&5\\2&0&3\end{bmatrix}-\begin{bmatrix}1&2&-3\\5&0&2\\1&-1&1\end{bmatrix}$$

$$=\begin{bmatrix}3-1&-1-2&2+3\\4-5&2-0&5-2\\2-1&0+1&3-1\end{bmatrix}$$

$$\therefore\quad 2C=\begin{bmatrix}2&-3&5\\-1&2&3\\1&1&2\end{bmatrix}\Rightarrow C=\frac{1}{2}\begin{bmatrix}2&-3&5\\-1&2&3\\1&1&2\end{bmatrix}$$

$$=\begin{bmatrix}1&-3/2&5/2\\-1/2&1&3/2\\1/2&1/2&1\end{bmatrix}$$

Example 10. Solve the following equations for X and Y. $2X-Y=\begin{bmatrix}3&-3&0\\3&3&2\end{bmatrix}, 2Y+X=\begin{bmatrix}4&1&5\\-1&4&-4\end{bmatrix}$

Sol. Given, $$2X-Y=\begin{bmatrix}3&-3&0\\3&3&2\end{bmatrix}$$

On multiplying both sides by 2, we get

$$4X-2Y=2\begin{bmatrix}3&-3&0\\3&3&2\end{bmatrix};\ 4X-2Y=\begin{bmatrix}6&-6&0\\6&6&4\end{bmatrix}\quad\text{...(i)}$$

also given $$X+2Y=\begin{bmatrix}4&1&5\\-1&4&-4\end{bmatrix}\quad\text{...(ii)}$$

Adding Eqs. (i) and (ii), we get

$$5X=\begin{bmatrix}6&-6&0\\6&6&4\end{bmatrix}+\begin{bmatrix}4&1&5\\-1&4&-4\end{bmatrix}$$

$$=\begin{bmatrix}6+4&-6+1&0+5\\6-1&6+4&4-4\end{bmatrix}=\begin{bmatrix}10&-5&5\\5&10&0\end{bmatrix}$$

$$\therefore\quad X=\frac{1}{5}\begin{bmatrix}10&-5&5\\5&10&0\end{bmatrix}\Rightarrow\quad X=\begin{bmatrix}2&-1&1\\1&2&0\end{bmatrix}$$

Putting the value of X in Eq. (ii), we get

$$\begin{bmatrix}2&-1&1\\1&2&0\end{bmatrix}+2Y=\begin{bmatrix}4&1&5\\-1&4&-4\end{bmatrix}$$

$$\Rightarrow\quad 2Y=\begin{bmatrix}4&1&5\\-1&4&-4\end{bmatrix}-\begin{bmatrix}2&-1&1\\1&2&0\end{bmatrix}$$

$$=\begin{bmatrix}4-2&1+1&5-1\\-1-1&4-2&-4-0\end{bmatrix}=\begin{bmatrix}2&2&4\\-2&2&-4\end{bmatrix}$$

$$\therefore\quad Y=\begin{bmatrix}1&1&2\\-1&1&-2\end{bmatrix}$$

Hence, $$X=\begin{bmatrix}2&-1&1\\1&2&0\end{bmatrix}\text{ and }Y=\begin{bmatrix}1&1&2\\-1&1&-2\end{bmatrix}$$

Remark

If two matrices A and B are of the same order, then only their addition and subtraction is possible and these matrices are said to be conformable for addition or subtraction. On the other hand, if the matrices A and B are of different orders, then their addition and subtraction is not possible and these matrices are called non-conformable for addition and subtraction.

Multiplication

Conformable for Multiplication

If A and B be two matrices which are said to be conformable for the product AB. If the number of columns in A (called the pre-factor) is equal to the number of rows in B (called the post-factor) otherwise non-conformable for multiplication. Thus,

(i) AB is defined, if number of columns in A = number of rows in B.

(ii) BA is defined, if number of columns in B = number of rows in A.

Multiplication of Matrices

Let $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{n\times p}$ be two matrices, then the product AB is defined as the matrix $C=[C_{ij}]_{m\times p}$,

where $C_{ij}=\sum_{j=1}^{n} a_{ij}\, b_{jk}, 1\le i\le m, 1\le k\le p$

$$=a_{i1}\, b_{1k}+a_{i2}\, b_{2k}+a_{i3}\, b_{3k}+\ldots+a_{in}\, b_{nk}$$

i.e., (i,k) th entry of the product AB is the sum of the product of the corresponding elements of the ith row of A (pre-factor) and kth column of B (post-factor).

Note

In the product AB, $\begin{cases} A = \text{Pre-factor} \\ B = \text{Post-factor} \end{cases}$

Example 11. If $A=\begin{bmatrix}0&1&2\\1&2&3\\2&3&4\end{bmatrix}$ and $B=\begin{bmatrix}1&-2\\-1&0\\2&-1\end{bmatrix}$, obtain the product AB and explain why BA is not defined?

Sol. Here, the number of columns in $A=3=$ the number of rows in B. Therefore, the product AB is defined.

$$AB=\begin{bmatrix}0&1&2\\1&2&3\\2&3&4\end{bmatrix}\begin{matrix}R_1\\R_2\\R_3\end{matrix}\times\begin{matrix}C_1&C_2\end{matrix}\begin{bmatrix}1&-2\\-1&0\\2&-1\end{bmatrix}$$

R_1, R_2, R_3 are rows of A and C_1, C_2 are columns of B.

$$\therefore\quad AB=\begin{bmatrix}R_1C_1&R_1C_2\\R_2C_1&R_2C_2\\R_3C_1&R_3C_2\end{bmatrix}_{3\times 2}$$

$$\begin{bmatrix}\boxed{0\ 1\ 2}\begin{bmatrix}1\\-1\\2\end{bmatrix} & \boxed{0\ 1\ 2}\begin{bmatrix}-2\\0\\-1\end{bmatrix}\\ \boxed{1\ 2\ 3}\begin{bmatrix}1\\-1\\2\end{bmatrix} & \boxed{1\ 2\ 3}\begin{bmatrix}-2\\0\\-1\end{bmatrix}\\ \boxed{2\ 3\ 4}\begin{bmatrix}1\\-1\\2\end{bmatrix} & \boxed{2\ 3\ 4}\begin{bmatrix}-2\\0\\-1\end{bmatrix}\end{bmatrix}_{3\times 2}$$

For convenience of multiplication we write columns in horizontal rectangles.

$$\begin{bmatrix}\{\boxed{0\ 1\ 2}\}\{\boxed{1\ -1\ 2}\} & \{\boxed{0\ 1\ 2}\}\{\boxed{-2\ 0\ -1}\}\\ \{\boxed{1\ 2\ 3}\}\{\boxed{1\ -1\ 2}\} & \{\boxed{1\ 2\ 3}\}\{\boxed{-2\ 0\ -1}\}\\ \{\boxed{2\ 3\ 4}\}\{\boxed{1\ -1\ 2}\} & \{\boxed{2\ 3\ 4}\}\{\boxed{-2\ 0\ -1}\}\end{bmatrix}$$

$$=\begin{bmatrix}0\times1+1\times(-1)+2\times2 & 0\times(-2)+1\times0+2\times(-1)\\ 1\times1+2\times(-1)+3\times2 & 1\times(-2)+2\times0+3\times(-1)\\ 2\times1+3\times(-1)+4\times2 & 2\times(-2)+3\times0+4\times(-1)\end{bmatrix}_{3\times2}$$

$$=\begin{bmatrix}0-1+4 & 0+0-2\\ 1-2+6 & -2+0-3\\ 2-3+8 & -4+0-4\end{bmatrix}_{3\times2}=\begin{bmatrix}3&-2\\5&-5\\7&-8\end{bmatrix}_{3\times2}$$

Since, the number of columns of B is 2 and the number of rows of A is 3, BA is not defined ($\because 2\neq 3$).

Remark

Verification for the product to be correct .

From above example

$$\begin{bmatrix}0&1&2\\1&2&3\\2&3&4\end{bmatrix}\times\begin{bmatrix}1&-2\\-1&0\\2&-1\end{bmatrix}=\begin{bmatrix}3&-2\\5&-5\\7&-8\end{bmatrix}$$

Sum $\underline{3\ \ 6\ \ 9}$ $\qquad$ $\underline{15\ -15}$

Now, $\boxed{3\,6\,9}\begin{bmatrix}1\\-1\\2\end{bmatrix}$ $\begin{matrix}=3\times1+6\times(-1)+9\times2\\=3-6+18\\=15\end{matrix}$

and $\boxed{3\,6\,9}\begin{bmatrix}-2\\0\\-1\end{bmatrix}$ $\begin{matrix}=3(-2)+6\times0+9\times(-1)\\=-6+0-9\\=-15\end{matrix}$

Example 12. If $A=\begin{bmatrix}0&-\tan(\alpha/2)\\\tan(\alpha/2)&0\end{bmatrix}$ and I is a 2×2 unit matrix, prove that

$$I+A=(I-A)\begin{bmatrix}\cos\alpha&-\sin\alpha\\\sin\alpha&\cos\alpha\end{bmatrix}.$$

Sol. Since, $I=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ and given $A=\begin{bmatrix}0&-\tan(\alpha/2)\\\tan(\alpha/2)&0\end{bmatrix}$

$$\therefore\quad I+A=\begin{bmatrix}1&-\tan(\alpha/2)\\\tan(\alpha/2)&1\end{bmatrix}\qquad\ldots(i)$$

$$\text{RHS}=(I-A)\begin{bmatrix}\cos\alpha&-\sin\alpha\\\sin\alpha&\cos\alpha\end{bmatrix}$$

$$=\begin{bmatrix}1&\tan(\alpha/2)\\-\tan(\alpha/2)&1\end{bmatrix}\begin{bmatrix}\cos\alpha&-\sin\alpha\\\sin\alpha&\cos\alpha\end{bmatrix}$$

$$= \begin{bmatrix} 1 & \tan(\alpha/2) \\ -\tan(\alpha/2) & 1 \end{bmatrix} \begin{bmatrix} \frac{1-\tan^2(\alpha/2)}{1+\tan^2(\alpha/2)} & \frac{-2\tan(\alpha/2)}{1+\tan^2(\alpha/2)} \\ \frac{2\tan(\alpha/2)}{1+\tan^2(\alpha/2)} & \frac{1-\tan^2(\alpha/2)}{1+\tan^2(\alpha/2)} \end{bmatrix}$$

Let $\tan(\alpha/2) = \lambda$, then

$$\text{RHS} = \begin{bmatrix} 1 & \lambda \\ -\lambda & 1 \end{bmatrix} \begin{bmatrix} \frac{1-\lambda^2}{1+\lambda^2} & \frac{-2\lambda}{1+\lambda^2} \\ \frac{2\lambda}{1+\lambda^2} & \frac{1-\lambda^2}{1+\lambda^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1-\lambda^2+2\lambda^2}{1+\lambda^2} & \frac{-2\lambda+\lambda(1-\lambda^2)}{1+\lambda^2} \\ \frac{-\lambda(1-\lambda^2)+2\lambda}{1+\lambda^2} & \frac{2\lambda^2+1-\lambda^2}{1+\lambda^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1+\lambda^2}{1+\lambda^2} & \frac{-\lambda(1+\lambda^2)}{1+\lambda^2} \\ \frac{\lambda(1+\lambda^2)}{1+\lambda^2} & \frac{1+\lambda^2}{1+\lambda^2} \end{bmatrix} = \begin{bmatrix} 1 & -\lambda \\ \lambda & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\tan(\alpha/2) \\ \tan(\alpha/2) & 1 \end{bmatrix} [\because \lambda = \tan(\alpha/2)]$$

$= I + A$ [from Eq. (i)]

$=$ LHS

Pre-multiplication and Post-multiplication of Matrices

The matrix AB is the matrix B pre-multiplied by A and the matrix BA is the matrix B post-multiplied by A.

Properties of Multiplication of Matrices

Property 1 Multiplication of matrices is not commutative

i.e. $AB \neq BA$

Note

1. If $AB = -BA$, then A and B are said to anti-commute.
2. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$, then $AB = BA = \begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix}$.

Observe that multiplication of diagonal matrices of same order will be commutative.

Property 2 Matrix multiplication associative if conformability assumed.

i.e. $A(BC) = (AB)\,C$

Property 3 Matrix multiplication is distributive with respect to addition. i.e. $A(B + C) = AB + AC$, whenever both sides of equality are defined.

Property 4 If A is an $m \times n$ matrix, then $I_m\ A = A = A\,I_n$.

Property 5 If product of two matrices is a zero matrix, it is not necessary that one of the matrices is a zero matrix.

For example,

$$\text{(i)} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \times \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1\cdot(-1)+1.1 & 1\cdot1+1\cdot(-1) \\ 2\cdot(-1)+2.1 & 2\cdot1+2\cdot(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

$$\text{(ii)} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0\cdot0+0\cdot0 & 0\cdot0+0\cdot1 \\ 1\cdot0+0\cdot0 & 1\cdot0+0\cdot1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

None of the matrices on the LHS is a null matrix whereas their product is a null matrix.

Note If A and B are two non-zero matrices such that $AB = 0$, then A and B are called the divisors of zero. Also, if

$AB = 0 \Rightarrow |AB| = 0 \Rightarrow |A||B| = 0$

$\Rightarrow$ $|A| = 0$ or $|B| = 0$ but not the converse.

Property 6 Multiplication of a matrix A by a null matrix conformable with A for multiplication.

For example, If $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix}_{3\times2}$ and $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times3}$,

then $AO = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times3}$, which is a 3×3 null matrix.

Property 7 Multiplication of a matrix by itself

The product of $A\,A\,A$ ... m times $= A^m$ and $(A^m)^n = A^{mn}$

Note

1. If I be unit matrix, then $I^2 = I^3 = \ldots = I^m = I\ (m \in I_+)$
2. If A and B are two matrices of the same order, then
   (i) $(A+B)^2 = A^2 + AB + BA + B^2$
   (ii) $(A-B)^2 = A^2 - AB - BA + B^2$
   (iii) $(A-B)(A+B) = A^2 + AB - BA + B^2$
   (iv) $(A+B)(A-B) = A^2 - AB + BA - B^2$
   (v) $A(-B) = (-A)(B) = -AB$

Example 13. If $A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}$, verify that $(AB)\,C = A(BC)$ and $A(B + C) = AB + AC$.

Sol. We have, $AB = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 1\cdot2+2\cdot2 & 1\cdot1+2\cdot3 \\ (-2)\cdot2+3\cdot2 & (-2)\cdot1+3\cdot3 \end{bmatrix} = \begin{bmatrix} 6 & 7 \\ 2 & 7 \end{bmatrix}$$

$$BC=\begin{bmatrix}2&1\\2&3\end{bmatrix}\times\begin{bmatrix}-3&1\\2&0\end{bmatrix}=\begin{bmatrix}2\cdot(-3)+1\cdot2&2\cdot1+1\cdot0\\2\cdot(-3)+3\cdot2&2\cdot1+3\cdot0\end{bmatrix}$$

$$=\begin{bmatrix}-6+2&2+0\\-6+6&2+0\end{bmatrix}=\begin{bmatrix}-4&2\\0&2\end{bmatrix}$$

$$AC=\begin{bmatrix}1&2\\-2&3\end{bmatrix}\times\begin{bmatrix}-3&1\\2&0\end{bmatrix}=\begin{bmatrix}1\cdot(-3)+2\cdot2&1\cdot1+2\cdot0\\(-2)\cdot(-3)+3\cdot2&(-2)\cdot1+3\cdot0\end{bmatrix}$$

$$=\begin{bmatrix}-3+4&1+0\\6+6&-2+0\end{bmatrix}=\begin{bmatrix}1&1\\12&-2\end{bmatrix}$$

$$B+C=\begin{bmatrix}2&1\\2&3\end{bmatrix}+\begin{bmatrix}-3&1\\2&0\end{bmatrix}=\begin{bmatrix}2-3&1+1\\2+2&3+0\end{bmatrix}=\begin{bmatrix}-1&2\\4&3\end{bmatrix}$$

$$\text{Now, } (AB)C=\begin{bmatrix}6&7\\2&7\end{bmatrix}\times\begin{bmatrix}-3&1\\2&0\end{bmatrix}=\begin{bmatrix}-18+14&6+0\\-6+14&2+0\end{bmatrix}$$

$$=\begin{bmatrix}-4&6\\8&2\end{bmatrix} \quad \text{...(i)}$$

$$A(BC)=\begin{bmatrix}1&2\\-2&3\end{bmatrix}\times\begin{bmatrix}-4&2\\0&2\end{bmatrix}$$

$$=\begin{bmatrix}-4+0&2+4\\8+0&-4+6\end{bmatrix}=\begin{bmatrix}-4&6\\8&2\end{bmatrix} \quad \text{...(ii)}$$

Thus, from Eqs. (i) and (ii), we get, $(AB)C=A(BC)$

$$\text{Now, } A(B+C)=\begin{bmatrix}1&2\\-2&3\end{bmatrix}\times\begin{bmatrix}-1&2\\4&3\end{bmatrix}=\begin{bmatrix}-1+8&2+6\\2+12&-4+9\end{bmatrix}$$

$$=\begin{bmatrix}7&8\\14&5\end{bmatrix} \quad \text{...(iii)}$$

$$\text{and} \quad AB+AC=\begin{bmatrix}6&7\\2&7\end{bmatrix}+\begin{bmatrix}1&1\\12&-2\end{bmatrix}=\begin{bmatrix}6+1&7+1\\2+12&7-2\end{bmatrix}$$

$$=\begin{bmatrix}7&8\\14&5\end{bmatrix} \quad \text{...(iv)}$$

Thus, from Eqs. (iii) and (iv), we get

$$A(B+C)=AB+AC$$

Example 14. If $A=\begin{bmatrix}0&1&0\\0&0&1\\p&q&r\end{bmatrix}$, show that

$A^3=pI+qA+rA^2$.

Sol. We have, $A^2=A\cdot A$

$$=\begin{bmatrix}0&1&0\\0&0&1\\p&q&r\end{bmatrix}\times\begin{bmatrix}0&1&0\\0&0&1\\p&q&r\end{bmatrix}$$

$$=\begin{bmatrix}0&0&1\\p&q&r\\pr&p+qr&q+r^2\end{bmatrix}$$

$$\therefore \quad A^3=A^2\cdot A=\begin{bmatrix}0&0&1\\p&q&r\\pr&p+qr&q+r^2\end{bmatrix}\times\begin{bmatrix}0&1&0\\0&0&1\\p&q&r\end{bmatrix}$$

$$=\begin{bmatrix}p&q&r\\pr&p+qr&q+r^2\\pq+r^2p&pr+q^2+qr^2&p+2qr+r^3\end{bmatrix} \quad \text{...(i)}$$

$$\text{and } pI+qA+rA^2=p\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}+q\begin{bmatrix}0&1&0\\0&0&1\\p&q&r\end{bmatrix}$$

$$+r\begin{bmatrix}0&0&1\\p&q&r\\pr&p+qr&q+r^2\end{bmatrix}$$

$$=\begin{bmatrix}p&0&0\\0&p&0\\0&0&p\end{bmatrix}+\begin{bmatrix}0&q&0\\0&0&q\\pq&q^2&qr\end{bmatrix}+\begin{bmatrix}0&0&r\\pr&qr&r^2\\pr^2&pr+qr^2&qr+r^3\end{bmatrix}$$

$$=\begin{bmatrix}p+0+0&0+q+0&0+0+r\\0+0+pr&p+0+qr&0+q+r^2\\0+pq+pr^2&0+q^2+pr+qr^2&p+qr+qr+r^3\end{bmatrix}$$

$$=\begin{bmatrix}p&q&r\\pr&p+qr&q+r^2\\pq+pr^2&q^2+pr+qr^2&p+2qr+r^3\end{bmatrix} \quad \text{...(ii)}$$

Thus, from Eqs. (i) and (ii), we get $A^3=pI+qA+rA^2$

Example 15. ***Find*** x***, so that***

$$[1\ x\ 1]\begin{bmatrix}1&3&2\\0&5&1\\0&3&2\end{bmatrix}\begin{bmatrix}1\\1\\x\end{bmatrix}=\boldsymbol{O}.$$

Sol. We have, $[1\quad x\quad 1]\begin{bmatrix}1&3&2\\0&5&1\\0&3&2\end{bmatrix}\begin{bmatrix}1\\1\\x\end{bmatrix}=O$

$$\Rightarrow \quad [1\quad 5x+6\quad x+4]\begin{bmatrix}1\\1\\x\end{bmatrix}=O$$

$$\Rightarrow \quad [1+5x+6+x^2+4x]=O$$

or $$x^2+9x+7=0$$

$$\therefore \quad x=\frac{-9\pm\sqrt{(81-28)}}{2} \Rightarrow x=\frac{-9\pm\sqrt{53}}{2}$$

Various Kinds of Matrices

Idempotent Matrix

A square matrix A is called idempotent provided it satisfies the relation $A^2 = A$.

Note

$A^n = A, \forall\, n \geq 2, n \in N$.

Example 16. Show that the matrix $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is idempotent.

Sol. $A^2 = A \cdot A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \times \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 2\cdot 2 + (-2)\cdot(-1) + (-4)\cdot 1 & 2\cdot(-2) + (-2)\cdot 3 + (-4)\cdot(-2) & 2\cdot(-4) + (-2)\cdot 4 + (-4)\cdot(-3) \\ (-1)\cdot 2 + 3\cdot(-1) + 4\cdot 1 & (-1)\cdot(-2) + 3\cdot 3 + 4\cdot(-2) & (-1)\cdot(-4) + 3\cdot 4 + 4\cdot(-3) \\ 1.2 + (-2)\cdot(-1) + (-3)\cdot 1 & 1\cdot(-2) + (-2)\cdot 3 + (-3)\cdot(-2) & 1\cdot(-4) + (-2)\cdot 4 + (-3)\cdot(-3) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A$$

Hence, the matrix A is idempotent.

Periodic Matrix

A square matrix A is called periodic, if $A^{k+1} = A$, where k is a positive integer. If k is the least positive integer for which $A^{k+1} = A$, then k is said to be **period** of A.

For $k = 1$, we get $A^2 = A$ and we called it to be **idempotent matrix.**

Note

Period of an idempotent matrix is 1.

Nilpotent Matrix

A square matrix A is called nilpotent matrix of order m provided it satisfies the relation $A^k = O$ and $A^{k-1} \neq O$, where k is positive integer and O is null matrix and k is the order of the nilpotent matrix A.

Example 17. ***Show that*** $\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ ***is nilpotent matrix of order 3.***

Sol. Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

$$\therefore\ A^2 = A \cdot A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+5-6 & 1+2-3 & 3+6-9 \\ 5+10-12 & 5+4-6 & 15+12-18 \\ -2-5+6 & -2-2+3 & -6-6+9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$$

$$\therefore\ A^3 = A^2 \cdot A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+0 & 0+0+0 & 0+0+0 \\ 3+15-18 & 3+6-9 & 9+18-27 \\ -1-5+6 & -1-2+3 & -3-6+9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

$\therefore \quad A^3 = O$ i.e., $A^k = O$

Here, $k = 3$

Hence, the matrix A is nilpotent of order 3.

Involutory Matrix

A square matrix A is called involutory provided it satisfies the relation $A^2 = I$, where I is identity matrix.

Note $A = A^{-1}$ for an involutory matrix.

Example 18. ***Show that the matrix*** $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ is involutory.

Sol. $A^2 = A \cdot A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix} \times \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 25-24+0 & 40-40+0 & 0+0+0 \\ -15+15+0 & -24+25+0 & 0+0+0 \\ -5+6-1 & -8+10-2 & 0+0+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence, the given matrix A is involutory.

Exercise for Session 1

1. If $A=\begin{bmatrix}\alpha & 2\\ 2 & \alpha\end{bmatrix}$ and $|A^3| = 125$, α is equal to

(a) ± 2 (b) ± 3
(c) ± 5 (d) 0

2. If $A=\begin{bmatrix}1 & -1\\ 2 & -1\end{bmatrix}$, $B=\begin{bmatrix}a & 1\\ b & -1\end{bmatrix}$ and $(A+B)^2 = A^2 + B^2$, the value of $a+b$ is

(a) 4 (b) 5
(c) 6 (d) 7

3. If $A=\begin{bmatrix}1 & 2\\ 2 & 3\end{bmatrix}$ and $A^2 - \lambda A - I_2 = O$, then λ is equal to

(a) -4 (b) -2
(c) 2 (d) 4

4. Let $A=\begin{bmatrix}0 & \alpha\\ 0 & 0\end{bmatrix}$ and $(A+I)^{50} - 50A = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$, the value of $a+b+c+d$, is

(a) 1 (b) 2
(c) 4 (d) None of these

5. If $A=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$, then $A^2 = I$ is true for

(a) $\theta = 0$ (b) $\theta = \frac{\pi}{4}$
(c) $\theta = \frac{\pi}{2}$ (d) None of these

6. If $\begin{bmatrix}\alpha & \beta\\ \gamma & -\alpha\end{bmatrix}$ is to be the square root of two rowed unit matrix, then α, β and γ should satisfy the relation

(a) $1-\alpha^2+\beta\gamma = 0$ (b) $\alpha^2+\beta\gamma-1=0$
(c) $1+\alpha^2+\beta\gamma = 0$ (d) $1-\alpha^2-\beta\gamma = 0$

7. If $A=\begin{bmatrix}1 & 0\\ 1/2 & 1\end{bmatrix}$, then A^{100} is equal to

(a) $\begin{bmatrix}1 & 0\\ 25 & 0\end{bmatrix}$ (b) $\begin{bmatrix}1 & 0\\ 50 & 1\end{bmatrix}$
(c) $\begin{bmatrix}1 & 0\\ (1/2)^{100} & 1\end{bmatrix}$ (d) None of these

8. If the product of n matrices $\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}\cdots\begin{bmatrix}1 & n\\ 0 & 1\end{bmatrix}$ is equal to the matrix $\begin{bmatrix}1 & 378\\ 0 & 1\end{bmatrix}$, the value of n is equal to

(a) 26 (b) 27
(c) 377 (d) 378

9. If A and B are two matrices such that $AB = B$ and $BA = A$, then $A^2 + B^2$ is equal to

(a) $2AB$ (b) $2BA$
(c) $A+B$ (d) AB

Session 2

Transpose of a Matrix, Symmetric Matrix, Orthogonal Matrix, Complex Conjugate (or Conjugate) of a Matrix, Hermitian Matrix, Unitary Matrix, Determinant of a Matrix, Singular and Non-Singular Matrices,

Transpose of a Matrix

Let $A=[a_{ij}]_{m\times n}$ be any given matrix, then the matrix obtained by interchanging the rows and columns of A is called the transpose of A. Transpose of the matrix A is denoted by A' or A^T or A^t. In other words, if $A=[a_{ij}]_{m\times n}$, then $A'=[a_{ji}]_{n\times m}$.

For example,

If $$A=\begin{bmatrix} 2 & 3 & 4 & 5 \\ -2 & -1 & 4 & 8 \\ 7 & 5 & 3 & 1 \end{bmatrix}_{3\times 4},$$

then $$A'=\begin{bmatrix} 2 & -2 & 7 \\ 3 & -1 & 5 \\ 4 & 4 & 3 \\ 5 & 8 & 1 \end{bmatrix}_{4\times 3}$$

Properties of Transpose Matrices

If A' and B' denote the transpose of A and B respectively, then

(i) $(A')'=A$

(ii) $(A\pm B)'=A'\pm B'$; A and B are conformable for matrix addition.

(iii) $(kA)'=kA'$; k is a scalar.

(iv) $(AB)'=B'A'$; A and B are conformable for matrix product AB.

In general, $(A_1A_2A_3\ldots A_{n-1}A_n)'=A'_n\ A'_{n-1}\cdots A'_3\ A'_2\ A'_1$ (reversal law for transpose).

Remark

$I'=I$, where I is an identity matrix.

Example 19. ***If*** $A=\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, ***find the values of*** θ ***satisfying the equation*** $A^T+A=I_2$.

Sol. We have, $A^T+A=I_2$

$$\Rightarrow \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}+\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2\cos\theta & 0 \\ 0 & 2\cos\theta \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \cos\theta=\frac{1}{2}=\cos\left(\frac{\pi}{3}\right) \Rightarrow \theta=2n\pi\pm\frac{\pi}{3},\ n\in \mathrm{I}$$

Symmetric Matrix

A square matrix $A=[a_{ij}]_{n\times n}$ is said to be symmetric, if $A'=A$ i.e., $a_{ij}=a_{ji}$, $\forall\ i, j$.

For example,

If $A=\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$, then $A'=\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

Here, A is symmetric matrix as $A'=A$.

Note

1. Maximum number of distinct entries in any symmetric matrix of order n is $\frac{n(n+1)}{2}$.
2. For any square matrix A with real number entries, then $A+A'$ is a symmetric matrix.

Proof $(A+A')'=A'+(A')'=A'+A=A+A'$

Skew-Symmetric Matrix

A square matrix $A=[a_{ij}]_{n\times n}$ is said to be skew-symmetric matrix, if $A'=-A$, i.e. $a_{ij}=-a_{ji}$, $\forall\ i, j$. (the pair of conjugate elements are additive inverse of each other)

Now, if we put $i=j$, we have $a_{ii}=-a_{ii}$.

Therefore, $2a_{ii}=0$ or $a_{ii}=0$, $\forall\ i$'s.

This means that all the diagonal elements of a skew-symmetric matrix are zero, but not the converse.

For example,

If $A=\begin{bmatrix}0 & h & g\\ -h & 0 & f\\ -g & -f & 0\end{bmatrix}$, then

$$A'=\begin{bmatrix}0 & -h & -g\\ h & 0 & -f\\ g & f & 0\end{bmatrix}=-\begin{bmatrix}0 & h & g\\ -h & 0 & f\\ -g & -f & 0\end{bmatrix}=-A$$

Here, A is skew-symmetric matrix as $A'=-A$.

Note

1. Trace of a skew-symmetric matrix is always 0.
2. For any square matrix A with real number entries, then $A-A'$ is a skew-symmetric matrix.

   Proof $(A-A')'=A'-(A')'=A'-A=-(A-A')$
3. Every square matrix can be uniquely expressed as the sum of a symmetric and a skew-symmetric matrix.

   i.e. If A is a square matrix, then we can write

   $$A=\frac{1}{2}(A+A')+\frac{1}{2}(A-A')$$

Example 20. ***The square matrix*** $A=[a_{ij}]_{m\times m}$ ***given by*** $a_{ij}=(i-j)^n$***, show that*** A ***is symmetric and skew-symmetric matrices according as*** n ***is even or odd, respectively.***

Sol. $\because a_{ij}=(i-j)^n=(-1)^n(j-i)^n$

$$=(-1)^n\ a_{ji}=\begin{cases}a_{ji}\ , n \text{ is even integer}\\ -a_{ji}, n \text{ is odd integer}\end{cases}$$

Hence, A is symmetric if n is even and skew-symmetric if n is odd integer.

Example 21. Express A as the sum of a symmetric and a skew-symmetric matrix, where $A=\begin{bmatrix}3 & 5\\ -1 & 2\end{bmatrix}$.

Sol. We have,

$$A=\begin{bmatrix}3 & 5\\ -1 & 2\end{bmatrix}, \text{ then } A'=\begin{bmatrix}3 & -1\\ 5 & 2\end{bmatrix}$$

Let $P=\frac{1}{2}(A+A')=\frac{1}{2}\begin{bmatrix}6 & 4\\ 4 & 4\end{bmatrix}=\begin{bmatrix}3 & 2\\ 2 & 2\end{bmatrix}=P'$

Thus, $P=\frac{1}{2}(A+A')$ is a symmetric matrix.

Also, let $Q=\frac{1}{2}(A-A')=\frac{1}{2}\begin{bmatrix}0 & 6\\ -6 & 0\end{bmatrix}=\begin{bmatrix}0 & 3\\ -3 & 0\end{bmatrix}$

Then, $Q'=\begin{bmatrix}0 & -3\\ 3 & 0\end{bmatrix}=-\begin{bmatrix}0 & 3\\ -3 & 0\end{bmatrix}=-Q$

Thus, $Q=\frac{1}{2}(A-A')$ is a skew-symmetric matrix.

Now, $P+Q=\begin{bmatrix}3 & 2\\ 2 & 2\end{bmatrix}+\begin{bmatrix}0 & 3\\ -3 & 0\end{bmatrix}=\begin{bmatrix}3 & 5\\ -1 & 2\end{bmatrix}=A$

Hence, A is represented as the sum of a symmetric and a skew-symmetric matrix.

Properties of Symmetric and Skew-Symmetric Matrices

(i) If A be a square matrix, then AA' and $A'A$ are symmetric matrices.

(ii) All positive integral powers of a symmetric matrix are symmetric, because

$$(A^n)'=(A')^n$$

(iii) All positive odd integral powers of a skew-symmetric matrix are skew-symmetric and positive even integral powers of a skew-symmetric matrix are symmetric, because $(A^n)'=(A')^n$

(iv) If A be a symmetric matrix and B be a square matrix of order that of A, then $-A, kA, A', A^{-1}, A^n$ and $B'AB$ are also symmetric matrices, where $n\in N$ and k is a scalar.

(v) If A be a skew-symmetric matrix, then
   (a) A^{2n} is a symmetric matrix for $n\in N$.
   (b) A^{2n+1} is a skew-symmetric matrix for $n\in N$.
   (c) kA is a skew-symmetric matrix, where k is scalar.
   (d) $B'AB$ is also skew-symmetric matrix, where B is a square matrix of order that of A.

(vi) If A and B are two symmetric matrices, then
   (a) $A\pm B, AB+BA$ are symmetric matrices.
   (b) $AB-BA$ is a skew-symmetric matrix
   (c) AB is a symmetric matrix, iff $AB=BA$
   (where A and B are square matrices of same order)

(vii) If A and B are two skew-symmetric matrices, then
   (a) $A\pm B, AB-BA$ are skew-symmetric matrices.
   (b) $AB+BA$ is a symmetric matrix.
   (where A and B are square matrices of same order)

(viii) If A be a skew-symmetric matrix and C is a column matrix, then $C'AC$ is a zero matrix, where $C'AC$ is conformable.

Orthogonal Matrix

A square matrix A is said to be orthogonal matrix, iff $AA'=I$, where I is an identity matrix.

Note

1. If $AA'=I$, then $A^{-1}=A'$.
2. If A and B are orthogonal, then AB is also orthogonal.
3. If A is orthogonal, then A^{-1} and A' are also orthogonal.

Example 22. If $\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$ is orthogonal, then find the value of $2\alpha^2 + 6\beta^2 + 3\gamma^2$.

Sol. Let $A = \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$, then $A' = \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix}$

Since, A is orthogonal.

$\therefore \quad AA' = I$

$$\Rightarrow \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}\begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4\beta^2 + \gamma^2 & 2\beta^2 - \gamma^2 & -2\beta^2 + \gamma^2 \\ 2\beta^2 - \gamma^2 & \alpha^2 + \beta^2 + \gamma^2 & \alpha^2 - \beta^2 - \gamma^2 \\ -2\beta^2 + \gamma^2 & \alpha^2 - \beta^2 - \gamma^2 & \alpha^2 + \beta^2 + \gamma^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equating the corresponding elements, we get

$$4\beta^2 + \gamma^2 = 1 \quad \text{...(i)}$$

$$2\beta^2 - \gamma^2 = 0 \quad \text{...(ii)}$$

and $$\alpha^2 + \beta^2 + \gamma^2 = 1 \quad \text{...(iii)}$$

From Eqs. (i) and (ii), we get

$$\beta^2 = \frac{1}{6} \text{ and } \gamma^2 = \frac{1}{3}$$

From Eq. (iii),

$$\alpha^2 = 1 - \beta^2 - \gamma^2 = 1 - \frac{1}{6} - \frac{1}{3} = \frac{1}{2}$$

Hence, $2\alpha^2 + 6\beta^2 + 3\gamma^2 = 2 \times \frac{1}{2} + 6 \times \frac{1}{6} + 3 \times \frac{1}{3} = 3$

Aliter

The rows of matrix A are unit orthogonal vectors

$$\vec{R}_1 \cdot \vec{R}_2 = 0 \Rightarrow 2\beta^2 - \gamma^2 = 0 \Rightarrow 2\beta^2 = \gamma^2 \quad \text{...(i)}$$

$$\vec{R}_2 \cdot \vec{R}_3 = 0 \Rightarrow \alpha^2 - \beta^2 - \gamma^2 = 0 \Rightarrow \beta^2 + \gamma^2 = \alpha^2 \quad \text{...(ii)}$$

and $$\vec{R}_3 \cdot \vec{R}_3 = 1 \Rightarrow \alpha^2 + \beta^2 + \gamma^2 = 1 \quad \text{...(iii)}$$

From Eqs. (i), (ii) and (iii), we get

$$\alpha^2 = \frac{1}{2}, \beta^2 = \frac{1}{6} \text{ and } \gamma^2 = \frac{1}{3}$$

$\therefore \quad 2\alpha^2 + 6\beta^2 + 3\gamma^2 = 3$

Example 23. ***If*** $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ ***is a matrix satisfying*** $AA' = 9I_3$, ***find the value of*** $|a| + |b|$.

Sol. Since, $AA' = 9I_3$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix} = 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 9 & 0 & a + 2b + 4 \\ 0 & 9 & 2a - 2b + 2 \\ a + 2b + 4 & 2a - 2b + 2 & a^2 + b^2 + 4 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

Equating the corresponding elements, we get

$$a + 2b + 4 = 0 \quad \text{...(i)}$$

$$2a - 2b + 2 = 0 \quad \text{...(ii)}$$

and $$a^2 + b^2 + 4 = 9 \quad \text{...(iii)}$$

From Eqs. (i) and (ii), we get

$$a = -2 \text{ and } b = -1$$

Hence, $|a| + |b| = |-2| + |-1| = 2 + 1 = 3$

Complex Conjugate (or Conjugate) of a Matrix

If a matrix A is having complex numbers as its elements, the matrix obtained from A by replacing each element of A by its conjugate ($\overline{a \pm ib} = a \mp ib$, where $i = \sqrt{-1}$) is called the conjugate of matrix A and is denoted by $\overline{A}$.

For example, If $A = \begin{bmatrix} 2 + 5i & 3 - i & 7 \\ -2i & 6 + i & 7 - 5i \\ 1 - i & 3 & 6i \end{bmatrix}$, where $i = \sqrt{-1}$,

then $$\overline{A} = \begin{bmatrix} 2 - 5i & 3 + i & 7 \\ 2i & 6 - i & 7 + 5i \\ 1 + i & 3 & -6i \end{bmatrix}$$

Note

If all elements of A are real, then $\overline{A} = A$.

Properties of Complex Conjugate of a Matrix

If A and B are two matrices of same order, then

(i) $(\overline{\overline{A}}) = A$

(ii) $(\overline{A + B}) = \overline{A} + \overline{B}$, where A and B being conformable for addition.

(iii) $(\overline{kA}) = k\,\overline{A}$, where k is real.

(iv) $(\overline{AB}) = \overline{A}\,\overline{B}$, where A and B being conformable for multiplication.

Conjugate Transpose of a Matrix

The conjugate of the transpose of a matrix A is called the conjugate transpose of A and is denoted by A^{θ} i.e.
$A^{\theta} = \text{Conjugate of } A' = (\overline{A'})$

For example,

If $A = \begin{bmatrix} 2+4i & 3 & 5-9i \\ 4 & 5+2i & 3i \\ 2 & -5 & 4-i \end{bmatrix}$,

where $i = \sqrt{-1}$,

then $A^{\theta} = (\overline{A'}) = \begin{bmatrix} 2-4i & 4 & 2 \\ 3 & 5-2i & -5 \\ 5+9i & -3i & 4+i \end{bmatrix}$

Properties of Transpose Conjugate Matrix

If A and B are two matrices of same order, then

(i) $(\overline{A})' = (\overline{A'})$ (ii) $(A^{\theta})^{\theta} = A$

(iii) $(A+B)^{\theta} = A^{\theta} + B^{\theta}$, where A and B being conformable for addition.

(iv) $(kA)^{\theta} = k\,A^{\theta}$, where k is real.

(v) $(AB)^{\theta} = B^{\theta}A^{\theta}$, where A and B being conformable for multiplication

Hermitian Matrix

A square matrix $A = [a_{ij}]_{n\times n}$ is said to be hermitian, if $A^{\theta} = A$ i.e., $a_{ij} = \overline{a}_{ji}, \forall\, i, j$. If we put $j = i$, we have $a_{ii} = \overline{a}_{ii}$ $\Rightarrow a_{ii}$ is purely real for all i's.

This means that all the diagonal elements of a hermitian matrix must be purely real.

For example,

If $A = \begin{bmatrix} \alpha & \lambda + i\mu & \theta + i\phi \\ \lambda - i\mu & \beta & x + iy \\ \theta - i\phi & x - iy & \gamma \end{bmatrix}$

where $\alpha, \beta, \gamma, \lambda, \mu, \theta, \phi, x, y \in R$ and $i = \sqrt{-1}$, then

$$A' = \begin{bmatrix} \alpha & \lambda - i\mu & \theta - i\phi \\ \lambda + i\mu & \beta & x - iy \\ \theta + i\phi & x + iy & \gamma \end{bmatrix}$$

$$\therefore \quad A^{\theta} = (\overline{A'}) = \begin{bmatrix} \alpha & \lambda + i\mu & \theta + i\phi \\ \lambda - i\mu & \beta & x + iy \\ \theta - i\phi & x - iy & \gamma \end{bmatrix} = A$$

Here, A is hermitian matrix as $A^{\theta} = A$.

Note

For any square matrix A with complex number entries, then $A + A^{\theta}$ is a Hermitian matrix.

Proof $(A + A^{\theta})^{\theta} = A^{\theta} + (A^{\theta})^{\theta} = A^{\theta} + A = A + A^{\theta}$

Skew-Hermitian Matrix

A square matrix $A = [a_{ij}]_{n\times n}$ is said to be skew-hermitian matrix. If $A^{\theta} = -A$, i.e. $a_{ij} = -\overline{a}_{ij}, \forall\, i, j$. If we put $j = i$, we have $a_{ii} = -\overline{a}_{ii} \Rightarrow a_{ii} + \overline{a}_{ii} = 0 \Rightarrow a_{ii}$ is purely imaginary for all i's. This means that all the diagonal elements of a skew-hermitian matrix must be purely imaginary or zero.

For example,

If $A = \begin{bmatrix} 2i & -2-3i & -2+i \\ 2-3i & -i & 3i \\ 2+i & 3i & 0 \end{bmatrix}$, where $i = \sqrt{-1}$,

then $A' = \begin{bmatrix} 2i & 2-3i & 2+i \\ -2-3i & -i & 3i \\ -2+i & 3i & 0 \end{bmatrix}$

$$\therefore \quad A^{\theta} = (\overline{A'}) = \begin{bmatrix} -2i & 2+3i & 2-i \\ -2+3i & i & -3i \\ -2-i & -3i & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 2i & -2-3i & -2+i \\ 2-3i & -i & 3i \\ 2+i & 3i & 0 \end{bmatrix} = -A$$

Hence, A is skew-hermitian matrix.

Note

1. For any square matrix A with complex number entries, then $A - A^{\theta}$ is a skew-hermitian matrix.

   Proof $(A - A^{\theta})^{\theta} = (A^{\theta}) - (A^{\theta})^{\theta} = A^{\theta} - A = -(A - A^{\theta})$

2. Every square matrix (with complex elements) can be uniquely expressed as the sum of a hermitian and a skew-hermitian matrix i.e.
   If A is a square matrix, then we can write
   $$A = \frac{1}{2}(A + A^{\theta}) + \frac{1}{2}(A - A^{\theta})$$

Example 24. *Express A as the sum of a hermitian and a skew-hermitian matrix, where* $A = \begin{bmatrix} 2+3i & 7 \\ 1-i & 2i \end{bmatrix}, i = \sqrt{-1}$.

Sol. We have, $A = \begin{bmatrix} 2+3i & 7 \\ 1-i & 2i \end{bmatrix}$, then $A^{\theta} = (\overline{A'}) = \begin{bmatrix} 2-3i & 1+i \\ 7 & -2i \end{bmatrix}$

$$\text{Let } P = \frac{1}{2}(A + A^{\theta}) = \frac{1}{2}\begin{bmatrix} 4 & 8+i \\ 8-i & 0 \end{bmatrix} = \begin{bmatrix} 2 & 4+\frac{i}{2} \\ 4-\frac{i}{2} & 0 \end{bmatrix} = P^{\theta}$$

Thus, $P = \frac{1}{2}(A + A^{\theta})$ is a hermitian matrix.

Also, let $Q = \frac{1}{2}(A - A^{\theta}) = \frac{1}{2}\begin{bmatrix} 6i & 6-i \\ -6-i & 4i \end{bmatrix}$

$$= \begin{bmatrix} 3i & 3-\frac{i}{2} \\ -3-\frac{i}{2} & 2i \end{bmatrix} = -\begin{bmatrix} -3i & -3+\frac{i}{2} \\ 3+\frac{i}{2} & -2i \end{bmatrix} = -Q^{\theta}$$

Thus, $Q = \frac{1}{2}(A - A^{\theta})$ is a skew-hermitian matrix.

Now, $P + Q = \begin{bmatrix} 2 & 4+\frac{i}{2} \\ 4-\frac{i}{2} & 0 \end{bmatrix} + \begin{bmatrix} 3i & 3-\frac{i}{2} \\ -3-\frac{i}{2} & 2i \end{bmatrix}$

$$= \begin{bmatrix} 2+3i & 7 \\ 1-i & 2i \end{bmatrix} = A$$

Hence, A is represented as the sum of a hermitian and a skew-hermitian matrix.

Properties of Hermitian and Skew-Hermitian Matrices

(i) If A be a square matrix, then AA^{θ} and $A^{\theta}A$ are hermitian matrices.

(ii) If A is a hermitian matrix, then
- (a) iA is skew-hermitian matrix, where $i = \sqrt{-1}$.
- (b) iff $\overline{A}$ is hermitian matrix.
- (c) kA is hermitian matrix, where $k \in R$.

(iii) If A is a skew-hermitian matrix, then
- (a) iA is hermitian matrix, where $i = \sqrt{-1}$.
- (b) iff $\overline{A}$ is skew-hermitian matrix.
- (c) kA is skew-hermitian matrix, where $k \in R$.

(iv) If A and B are hermitian matrices of same order, then
- (a) $k_1A + k_2B$ is also hermitian, where $k_1, k_2 \in R$.
- (b) AB is also hermitian, if $AB = BA$.
- (c) $AB + BA$ is a hermitian matrix.
- (d) $AB - BA$ is a skew-hermitian matrix.

(v) If A and B are skew-hermitian matrices of same order, then $k_1A + k_2B$ is also skew-hermitian matrix.

Unitary Matrix

A square matrix A is said to be unitary matrix iff $AA^{\theta} = I$, where I is an identity matrix.

Note

1. If $AA^{\theta} = I$, then $A^{-1} = A^{\theta}$
2. If A and B are unitary, then AB is also unitary.
3. If A is unitary, then A^{-1} and A' are also unitary.

Example 25. Verify that the matrix $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is unitary, where $i = \sqrt{-1}$.

Sol. Let $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$, then

$$A^{\theta} = (\overline{A'}) = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$\therefore \quad AA^{\theta} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \times \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence, A is unitary matrix.

Determinant of a Matrix

Let A be a square matrix, then the determinant formed by the elements of A without changing their respective positions is called the determinant of A and is denoted by $\det A$ or $|A|$.

i.e., If $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$, then $|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

Properties of the Determinant of a Matrix

If A and B are square matrices of same order, then

(i) $|A|$ exists $\Leftrightarrow$ A is a square matrix.

(ii) $|A'| = |A|$

(iii) $|AB| = |A||B|$ and $|AB| = |BA|$

(iv) If A is orthogonal matrix, then $|A| = \pm 1$

(v) If A is skew-symmetric matrix of odd order, then $|A| = 0$

(vi) If A is skew-symmetric matrix of even order, then $|A|$ is a perfect square.

(vii) $|kA| = k^n|A|$, where n is order of A and k is scalar.

(viii) $|A^n| = |A|^n$, where $n \in N$

(ix) If $A = \text{diag}(a_1, a_2, a_3, \ldots, a_n)$, then $|A| = a_1 \cdot a_2 \cdot a_3 \ldots a_n$

Example 26. If A, B and C are square matrices of order n and $\det(A) = 2$, $\det(B) = 3$ and $\det(C) = 5$, then find the value of $10\det(A^3B^2C^{-1})$.

Sol. Given, $|A| = 2$, $|B| = 3$ and $|C| = 5$.

Now, $10\det(A^3B^2C^{-1}) = 10 \times |A^3B^2C^{-1}|$

$$= 10 \times |A^3| \times |B^2| \times |C^{-1}| = 10 \times |A|^3 \times |B|^2 \times |C|^{-1}$$
$$= \frac{10 \times |A|^3 \times |B^2|}{|C|} = \frac{10 \times 2^3 \times 3^2}{5} = 144$$

Example 27. ***If*** $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$, $abc = 1$, $A^T A = I$, ***then find the value of*** $a^3 + b^3 + c^3$.

Sol. $\because$ $A^T A = I$

$\Rightarrow$ $|A^T A| = |I| \Rightarrow |A^T||A| = 1$

$\Rightarrow$ $|A||A| = 1$ $[\because |A^T| = |A|]$

$\Rightarrow$ $|A| = \pm 1$

$\Rightarrow$ $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \pm 1$

$\Rightarrow$ $3abc - (a^3 + b^3 + c^3) = \pm 1$

or $3 - (a^3 + b^3 + c^3) = \pm 1$ $[\because abc = 1]$

or $a^3 + b^3 + c^3 = 3 \pm 1 = 2 \text{or } 4$

Singular and Non-Singular Matrices

A square matrix A is said to be a singular, if $|A| = 0$ and a square matrix A is said to be non-singular, if $|A| \neq 0$.

For example,

(i) $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 2 & 4 & 6 \end{bmatrix}$ is a singular matrix.

Since, $|A| = 0$.

(ii) $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ is a non-singular matrix.

Since, $|A| = 10 - 12 = -2 \neq 0$

Example 28. If $\omega \neq 1$ is a complex cube root of unity, then prove that

$$\begin{bmatrix} 1 + 2\omega^{2017} + \omega^{2018} & \omega^{2018} & 1 \\ 1 & 1 + \omega^{2018} + 2\omega^{2017} & \omega^{2017} \\ \omega^{2017} & \omega^{2018} & 2 + \omega^{2017} + 2\omega^{2018} \end{bmatrix}$$ ***is singular.***

Sol. Let $A = \begin{bmatrix} 1 + 2\omega^{2017} + \omega^{2018} & \omega^{2018} & 1 \\ 1 & 1 + \omega^{2018} + 2\omega^{2017} & \omega^{2017} \\ \omega^{2017} & \omega^{2018} & 2 + \omega^{2017} + 2\omega^{2018} \end{bmatrix}$

$\because$ $\omega^3 = 1 \Rightarrow \omega^{2017} = \omega$

and $\omega^{2018} = \omega^2$, then

$$A = \begin{bmatrix} 1 + 2\omega + \omega^2 & \omega^2 & 1 \\ 1 & 1 + \omega^2 + 2\omega & \omega \\ \omega & \omega^2 & 2 + \omega + 2\omega^2 \end{bmatrix}$$
$$= \begin{bmatrix} \omega & \omega^2 & 1 \\ 1 & \omega & \omega \\ \omega & \omega^2 & -\omega \end{bmatrix} \quad [\because 1 + \omega + \omega^2 = 0]$$

Now, $|A| = \begin{vmatrix} \omega & \omega^2 & 1 \\ 1 & \omega & \omega \\ \omega & \omega^2 & -\omega \end{vmatrix} = \omega \begin{vmatrix} \omega & \omega & 1 \\ 1 & 1 & \omega \\ \omega & \omega & -\omega \end{vmatrix} = 0$ $[\because C_1 = C_2]$

Thus, $|A| = 0$.

Hence, A is singular matrix.

Example 29. Find the real values of x for which the matrix $\begin{bmatrix} x+1 & 3 & 5 \\ 1 & x+3 & 5 \\ 1 & 3 & x+5 \end{bmatrix}$ is non-singular.

Sol. Let $A = \begin{bmatrix} x+1 & 3 & 5 \\ 1 & x+3 & 5 \\ 1 & 3 & x+5 \end{bmatrix}$

$\therefore$ $|A| = \begin{vmatrix} x+1 & 3 & 5 \\ 1 & x+3 & 5 \\ 1 & 3 & x+5 \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, then

$$|A| = \begin{vmatrix} x+9 & 3 & 5 \\ x+9 & x+3 & 5 \\ x+9 & 3 & x+5 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$|A| = \begin{vmatrix} x+9 & 3 & 5 \\ 0 & x & 0 \\ 0 & 0 & x \end{vmatrix} = x^2(x+9)$$

$\because$ A is non-singular.

$\therefore$ $|A| \neq 0 \Rightarrow x^2(x+9) \neq 0$

$\therefore$ $x \neq 0, -9$

Hence, $x \in R - \{0, -9\}$.

Exercise for Session 2

1 If $A=\begin{bmatrix} 4 & x+2 \\ 2x-3 & x+1 \end{bmatrix}$ is symmetric, then x is equal to

(a) 2 (b) 3
(c) 4 (d) 5

2 If A and B are symmetric matrices, then ABA is

(a) symmetric matrix (b) skew-symmetric matrix
(c) diagonal matrix (d) scalar matrix

3 If A and B are symmetric matrices of the same order and $P = AB + BA$ and $Q = AB - BA$, then $(PQ)'$ is equal to

(a) PQ (b) QP
(c) $-QP$ (d) None of these

4 If A is a skew-symmetric matrix and n is odd positive integer, then A^n is

(a) a skew-symmetric matrix (b) a symmetric matrix
(c) a diagonal matrix (d) None of these

5 If A is symmetric as well as skew-symmetric matrix, then A is

(a) diagonal matrix (b) null matrix
(c) triangular matrix (d) None of these

6 If A is square matrix order 3, then $|(A - A')^{2015}|$ is

(a) $|A|$ (b) $|A'|$
(c) 0 (d) None of these

7 The maximum number of different elements required to form a symmetric matrix of order 6 is

(a) 15 (b) 17
(c) 19 (d) 21

8 If A and B are square matrices of order 3×3 such that A is an orthogonal matrix and B is a skew-symmetric matrix, then which of the following statement is true?

(a) $|AB| = 1$ (b) $|AB| = 0$
(c) $|AB| = -1$ (d) None of these

9 The matrix $A=\begin{bmatrix} i & 1-2i \\ -1-2i & 0 \end{bmatrix}$, where $i=\sqrt{-1}$, is

(a) symmetric (b) skew-symmetric
(c) hermitian (d) skew-hermitian

10 If A and B are square matrices of same order such that $A^* = A$ and $B^* = B$, where A^* denotes the conjugate transpose of A, then $(AB - BA)^*$ is equal to

(a) null matrix (b) $AB - BA$
(c) $BA - AB$ (d) None of these

11 If matrix $A=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ -i & a \end{bmatrix}, i=\sqrt{-1}$ is unitary matrix, a is equal to

(a) 2 (b) -1
(c) 0 (d) 1

12 If A is a 3×3 matrix and $\det(3A) = k\{\det(A)\}$, k is equal to

(a) 9 (b) 6
(c) 1 (d) 27

13 If A and B are square matrices of order 3 such that $|A| = -1, |B| = 3$, then $|3AB|$ is equal to

(a) -9 (b) -81
(c) -27 (d) 81

14 If A is a square matrix such that $A^2 = A$, then $\det(A)$ is equal to

(a) 0 or 1 (b) -2 or 2

(c) -3 or 3 (d) None of these

15 If I is a unit matrix of order 10, the determinant of I is equal to

(a) 10 (b) 1

(c) $\frac{1}{10}$ (d) 9

16 If $A_i = \begin{bmatrix} 2^{-i} & 3^{-i} \\ 3^{-i} & 2^{-i} \end{bmatrix}$, then $\sum_{i=1}^{\infty} \det(A_i)$ is equal to

(a) $\frac{3}{4}$ (b) $\frac{5}{24}$

(c) $\frac{5}{4}$ (d) $\frac{7}{144}$

17 The number of values of x for which the matrix $A = \begin{bmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1-x \end{bmatrix}$ is singular, is

(a) 0 (b) 1

(c) 2 (d) 3

18 The number of values of x in the closed interval $[-4, -1]$, the matrix $\begin{bmatrix} 3 & -1+x & 2 \\ 3 & -1 & x+2 \\ x+3 & -1 & 2 \end{bmatrix}$ is singular, is

(a) 0 (b) 1

(c) 2 (d) 3

19 The values of x for which the given matrix $\begin{bmatrix} -x & x & 2 \\ 2 & x & -x \\ x & -2 & -x \end{bmatrix}$ will be non-singular are

(a) $-2 \le x \le 2$ (b) for all x other than 2 and -2

(c) $x \ge 2$ (d) $x \le -2$

Session 3

Adjoint of a Matrix, Inverse of a Matrix (Reciprocal Matrix), Elementary Row Operations (Transformations), Equivalent Matrices, Matrix Polynomial, Use of Mathematical Induction,

Adjoint of a Matrix

Let $A=[a_{ij}]$ be a square matrix of order n and let C_{ij} be cofactor of a_{ij} in A. Then, the transpose of the matrix of cofactors of elements of A is called the adjoint of A and is denoted by adj (A).

Thus, $\text{adj}\,(A)=[C_{ij}]'$

$\Rightarrow \quad (\text{adj}\,A)_{ij}=C_{ji}=\text{Cofactor of } a_{ji} \text{ in } A$

i.e. if $\quad A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

$$\text{adj } A=\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}'=\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

where C_{ij} denotes the cofactor of a_{ij} in A.

Here, $\quad C_{11}=\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}=a_{22}a_{33}-a_{23}\,a_{32},$

$$C_{12}=-\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}=a_{31}a_{23}-a_{33}a_{21},$$

$$C_{13}=\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}=a_{21}a_{32}-a_{31}a_{22},$$

$$C_{21}=-\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}=a_{13}a_{32}-a_{12}a_{33},$$

$$C_{22}=\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}=a_{11}a_{33}-a_{31}a_{13},$$

$$C_{23}=-\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}=a_{12}a_{31}-a_{11}a_{32},$$

$$C_{31}=\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}=a_{12}a_{23}-a_{22}a_{13},$$

$$C_{32}=-\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}=a_{13}a_{21}-a_{11}a_{23}$$

and $\quad C_{33}=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=a_{11}a_{22}-a_{21}a_{12}$

Rule to Write Cofactors of an Element a_{ij}

Cross the row and column intersection at the element a_{ij} and the determinant which is left be denoted by D, then

$$\text{Cofactors of } a_{ij}=\begin{cases} D, & \text{if } i+j=\text{even integer} \\ -D, & \text{if } i+j=\text{odd integer} \end{cases}$$

Example 30. Find the cofactor of a_{23} in $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & -1 \\ 1 & -3 & 5 \end{bmatrix}$.

Sol. Let $\quad A=\begin{bmatrix} 3 & 1 & 4 \\ 0 & \cdots 2 & \cdots -1 \\ 1 & -3 & 5 \end{bmatrix}$

$\therefore \quad$ Cofactor of $a_{23}=-D \qquad [\because 2+3=\text{odd}]$

where $D=\begin{vmatrix} 3 & 1 \\ 1 & -3 \end{vmatrix}$

[after crossing the 2nd row and 3rd column]

$=-9-1=-10$

Hence, cofactor of $a_{23}=-(-10)=10$

Note

The adjoint of a square matrix of order 2 is obtained by interchanging the diagonal elements and changing signs of off-diagonal elements.

If $\quad A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$(\text{adj } A)=\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example 31. ***Find the adjoint of the matrix***

$$A=\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}.$$

Sol. If C be the matrix of cofactors of the element in $|A|$, then

$$C=\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{vmatrix} 5 & 0 \\ 4 & 3 \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 0 & 5 \\ 2 & 4 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 5 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 5 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 12 & 0 & -10 \\ 6 & -3 & 0 \\ -15 & 0 & 5 \end{bmatrix}$$

$$\Rightarrow \text{adj}A = C' = \begin{bmatrix} 12 & 6 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5 \end{bmatrix}$$

Maha Shortcut for Adjoint

(Goyal's Method)

This method applied only for third order square matrix.

Method : Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$

Step I Write down the three rows of A and rewrite first two rows.
i.e.

1 2 3
0 5 0
2 4 3
1 2 3
0 5 0

Step II After Step I, rewrite first two columns.
i.e.

1 2 3 1 2
0 5 0 0 5
2 4 3 2 4
1 2 3 1 2
0 5 0 0 5

Step III After Step II, deleting first row and first column, then we get all elements of adj A i.e.,

1 ⋯ 2 ⋯ 3 ⋯ 1 ⋯ 2
⋮
0 5 0 0 5
⋮ ⤭ ⤭ ⤭ ⇒ 15 0 −10 first column of adj A
2 4 3 2 4
⋮ ⤭ ⤭ ⤭ ⇒ 6 −3 0 second column of adj A
1 2 3 1 2
⋮ ⤭ ⤭ ⤭ ⇒ −15 0 5 third column of adj A
0 5 0 0 5

$$\text{or} \quad \text{adj } A = \begin{bmatrix} 15 & 6 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5 \end{bmatrix}$$

Properties of Adjoint Matrix

Property 1 If A be a square matrix order n, then

$$A(\text{adj } A) = (\text{adj } A)A = |A| I_n$$

i.e., the product of a matrix and its adjoint is commutative.

Deductions of Property 1

Deduction 1 If A be a square singular matrix of order n, then $A(\text{adj } A) = (\text{adj } A)\, A = O$ [null matrix]

Since, for singular matrix, $|A| = 0$.

Deduction 2 If A be a square non-singular matrix of order n, then $|\text{adj } A| = |A|^{n-1}$

Since, for non-singular matrix, $|A| \neq 0$.

Proof $\because$ $A(\text{adj } A) = |A| I_n$

Taking determinant on both sides, then

$$|A(\text{adj } A)| = ||A| I_n|$$

$$\Rightarrow \quad |A||\text{adj } A| = |A|^n |I_n| = |A|^n \qquad [\because |I_n| = 1]$$

$$\Rightarrow \quad |\text{adj } A| = |A|^{n-1} \qquad [\because |A| \neq 0]$$

Note

In general $\underbrace{|\text{adj (adj (adj} \ldots \text{(adj } A)))|}_{\text{adj repeat } m \text{ times}} = |A|^{(n-1)^m}$

Property 2 If A and B are square matrices of order n, then

$$\text{adj}(AB) = (\text{adj } B)(\text{adj } A)$$

Property 3 If A is a square matrix of order n, then

$$(\text{adj } A)' = \text{adj } A'$$

Property 4 If A be a square non-singular matrix of order n, then $\text{adj}(\text{adj} A) = |A|^{n-2} A$

Proof $\because$ $A(\text{adj } A) = |A| I_n$ ...(i)

Replace A by adj A, then

$$(\text{adj } A)(\text{adj (adj } A)) = |\text{adj } A| I_n$$
$$= |A|^{n-1} I_n \quad [\because |\text{adj } A| = |A|^{n-1}]$$
$$= I_n |A|^{n-1}$$

Pre-multiplying both sides by matrix A, then

$$A(\text{adj } A)(\text{adj (adj } A)) = AI_n |A|^{n-1} = A|A|^{n-1}$$

$$\Rightarrow \quad |A| I_n (\text{adj}(\text{adj} A)) = A|A|^{n-1}$$

$$\Rightarrow \quad (\text{adj (adj } A)) = A|A|^{n-2}$$

or $\text{adj}(\text{adj} A) = |A|^{n-2} A$

Property 5 If A be a square non-singular matrix of order n, then

$$|\text{adj (adj } A)| = |A|^{(n-1)^2}$$

Proof $\because$ $\text{adj (adj } A) = |A|^{n-2} A$

Taking determinant on both sides, then

$$|\text{adj (adj } A)| = ||A|^{n-2} A|$$

$$= |A|^{n(n-2)} |A| \qquad [\because |kA| = k^n |A|]$$
$$= |A|^{n^2-2n+1} = |A|^{(n-1)^2}$$

Note

In general, $\underbrace{|\text{adj (adj (adj ... (adj } A))) |}_{\text{adj } m \text{ times}} = |A|^{(n-1)^m}$

Property 6 If A be a square matrix of order n and k is a scalar, then

$$\text{adj}\,(kA) = k^{n-1} \cdot (\text{adj}\, A)$$

Proof $\because$ $A(\text{adj}\, A) = |A| I_n$ ...(i)

Replace A by kA, then

$$kA(\text{adj}\,(kA)) = |kA| I_n = k^n |A| I_n$$
$$\Rightarrow \quad A(\text{adj}\,(kA)) = k^{n-1} |A| I_n$$
$$= k^{n-1} A(\text{adj}\, A) \qquad [\text{from Eq. (i)}]$$

Hence, $\text{adj}\,(kA) = k^{n-1}(\text{adj}\, A)$

Property 7 If A be a square matrix of order n and $m \in N$, then $(\text{adj}\, A^m) = (\text{adj}\, A)^m$

Property 8 If A and B be two square matrices of order n such that B is the adjoint of A and k is a scalar, then

$$|AB + kI_n| = (|A| + k)^n$$

Proof $\because$ $B = \text{adj}\ A$

$\therefore$ $AB = A(\text{adj}\ A) = |A| I_n$

$$\text{LHS} = |AB + kI_n| = ||A| I_n + kI_n| = |(|A| + k) I_n|$$
$$= (|A| + k)^n |I_n| = (|A| + k)^n = \text{RHS}$$

Property 9 Adjoint of a diagonal matrix is a diagonal matrix.

i.e. If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, then $\text{adj}\, A = \begin{bmatrix} bc & 0 & 0 \\ 0 & ca & 0 \\ 0 & 0 & ab \end{bmatrix}$

Note

$\text{adj}\,(I_n) = I_n$.

Example 32. If $A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$, find the values of

(i) $|A||\text{adj}\ A|$ (ii) $|\text{adj}(\text{adj}\ (\text{adj}\ A))|$
(iii) $|\text{adj}\ (3A)|$ (iv) $\text{adj}\ (\text{adj}\ A)$

Sol. $\because$ $A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

$\therefore |A| = (-1)(1-1) - (1)(-1-1) + (1)(1+1) = 4 \neq 0$

$\Rightarrow$ A is non-singular.

(i) $|A||\text{adj}\ A| = |A||A|^2 \qquad [\because n = 3]$
$= |A|^3 = 4^3 = 64$

(ii) $|\text{adj}\,(\text{adj}\,(\text{adj}\ A))| = |A|^{(3-1)^3} = |A|^8 = 4^8 = 2^{16}$

(iii) $|\text{adj}\,(3A)| = |3^2\ \text{adj}\ A| = (3^2)^3 |\text{adj}\ A|$
$= 3^6 |A|^2 = 729 \times 4^2 = 11664$

(iv) $\text{adj}(\text{adj}\, A) = |A|^2 A = 16A$

Example 33. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ and B is the adjoint of A, find the value of $|AB + 2I|$, where I is the identity matrix of order 3.

Sol. $\because$ $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

$\therefore$ $|A| = \begin{vmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{vmatrix}$

$= 3(-3+4) + 3(2-0) + 4(-2-0) = 1 \neq 0$

$\therefore$ $|AB + 2I| = (|A| + 2)^3$ [by property 8]

$= (1+2)^3 = 3^3 = 27$

Inverse of a Matrix
(Reciprocal Matrix)

A square matrix A (non-singular) of order n is said to be invertible, if there exists a square matrix B of the same order such that $AB = I_n = BA$,

then B is called the inverse (reciprocal) of A and is denoted by A^{-1}. Thus, $A^{-1} = B \Leftrightarrow AB = I_n = BA$

We have, $A(\text{adj}\ A) = |A| I_n$

$\Rightarrow$ $A^{-1} A(\text{adj}\ A) = A^{-1} I_n |A|$

$\Rightarrow$ $I_n(\text{adj}\ A) = A^{-1} |A| I_n$

$\therefore$ $A^{-1} = \dfrac{\text{adj}\, A}{|A|}$; provided $|A| \neq 0$

Note The necessary and sufficient condition for a square matrix A to be invertible is that $|A| \neq 0$.

Properties of Inverse of a Matrix

Property 1 (Uniqueness of inverse) Every invertible matrix possesses a unique inverse.

Proof Let A be an invertible matrix of order $n \times n$. Let B and C be two inverses of A. Then,

$$AB = BA = I_n \qquad \text{...(i)}$$

and $AC = CA = I_n$ ...(ii)

Now, $AB = I_n$

$\Rightarrow$ $C(AB) = CI_n$ [pre-multiplying by C]

$\Rightarrow$ $(CA)B = CI_n$ [by associativity]

$\Rightarrow$ $I_n B = CI_n$ [$\because CA = I_n$ by Eq. (ii)]

$\Rightarrow$ $B = C$

Hence, an invertible matrix possesses a unique inverse.

Property 2 (Reversal law) If A and B are invertible matrices of order $n \times n$, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof It is given that A and B are invertible matrices.

$$|A| \neq 0 \text{ and } |B| \neq 0 \Rightarrow |A||B| \neq 0$$

$\Rightarrow$ $|AB| \neq 0$

Hence, AB is an invertible matrix.

Now, $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$ [by associativity]

$= (A I_n)A^{-1}$ [$\because BB^{-1} = I_n$]

$= A A^{-1}$ [$\because A I_n = A$]

$= I_n$ [$\because A A^{-1} = I_n$]

Also, $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B$ [by associativity]

$= B^{-1}(I_n B)$ [$\because A^{-1}A = I_n$]

$= B^{-1}B$ [$\because I_n B = B$]

$= I_n$ [$\because B^{-1}B = I_n$]

Thus, $(AB)(B^{-1}A^{-1}) = I_n = (B^{-1}A^{-1})(AB)$

Hence, $(AB)^{-1} = B^{-1}A^{-1}$

Note

If $A, B, C, \ldots, Y, Z$ are invertible matrices, then

$(ABC \ldots YZ)^{-1} = Z^{-1}Y^{-1} \ldots C^{-1}B^{-1}A^{-1}$ [reversal law]

Property 3 Let A be an invertible matrix of order n, then A' is also invertible and $(A')^{-1} = (A^{-1})'$.

Proof $\because A$ is invertible matrix

$\therefore$ $|A| \neq 0 \Rightarrow |A'| \neq 0$ [$\because |A| = |A'|$]

Hence, A^{-1} is also invertible.

Now, $AA^{-1} = I_n = A^{-1}A$

$\Rightarrow$ $(AA^{-1})' = (I_n)' = (A^{-1}A)'$

$\Rightarrow$ $(A^{-1})'A' = I_n = A'(A^{-1})'$

[by reversal law for transpose]

$\Rightarrow$ $(A')^{-1} = (A^{-1})'$ [by definition of inverse]

Property 4 Let A be an invertible matrix of order n and $k \in N$, then

$$(A^k)^{-1} = (A^{-1})^k = A^{-k}$$

Proof We have,

$$(A^k)^{-1} = \underbrace{(A \times A \times A \times \ldots \times A)^{-1}}_{\text{repeat } k \text{ times}}$$

$$= \underbrace{A^{-1} \times A^{-1} \times A^{-1} \times \ldots \times A^{-1}}_{\text{repeat } k \text{ times}}$$

[by reversal law for inverse]

$$= (A^{-1})^k = A^{-k}$$

Property 5 Let A be an invertible matrix of order n, then $(A^{-1})^{-1} = A$.

Proof We have, $A^{-1}A = I_n$

$\therefore$ Inverse of $A^{-1} = A \Rightarrow (A^{-1})^{-1} = A$

Note

$I_n^{-1} = I_n$ as $I_n^{-1} I_n = I_n$

Property 6 Let A be an invertible matrix of order n, then $|A^{-1}| = \dfrac{1}{|A|}$.

Proof $\because A$ is invertible, then $|A| \neq 0$.

Now, $A A^{-1} = I_n = A^{-1}A$

$\Rightarrow$ $|A A^{-1}| = |I_n|$

$\Rightarrow$ $|A||A^{-1}| = 1$

[$\because |AB| = |A||B|$ and $|I_n| = 1$]

$\Rightarrow$ $|A^{-1}| = \dfrac{1}{|A|}$ [$\because |A| \neq 0$]

Property 7 Inverse of a non-singular diagonal matrix is a diagonal matrix.

i.e. If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ and $|A| \neq 0$, then

$$A^{-1} = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}$$

Note

The inverse of a non-singular square matrix A of order 2 is obtained by interchanging the diagonal elements and changing signs of off-diagonal elements and dividing by $|A|$.

For example,

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $|A| = (ad - bc) \neq 0$, then

$$A^{-1} = \frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example 34. Compute the inverse of the matrix $A=\begin{bmatrix}0&1&2\\1&2&3\\3&1&1\end{bmatrix}$.

Sol. We have, $A=\begin{bmatrix}0&1&2\\1&2&3\\3&1&1\end{bmatrix}$

Then, $|A|=\begin{vmatrix}0&1&2\\1&2&3\\3&1&1\end{vmatrix}=0\cdot(2-3)-1(1-9)+2(1-6)$

$=-2\neq 0$

$\therefore$ A^{-1} exists.

Now, cofactors along $R_1=-1, 8, -5$

cofactors along $R_2=1, -6, 3$

cofactors along $R_3=-1, 2, -1$

Let C is a matrix of cofactors of the elements in $|A|$

$\therefore \quad C=\begin{bmatrix}-1&8&-5\\1&-6&3\\-1&2&-1\end{bmatrix}$

$\therefore \quad \text{adj } A=C'=\begin{bmatrix}-1&1&-1\\8&-6&2\\-5&3&-1\end{bmatrix}$

Hence, $A^{-1}=\dfrac{\text{adj } A}{|A|}=-\dfrac{1}{2}\begin{bmatrix}-1&1&-1\\8&-6&2\\-5&3&-1\end{bmatrix}$

$=\begin{bmatrix}\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}\\-4&3&-1\\\frac{5}{2}&-\frac{3}{2}&\frac{1}{2}\end{bmatrix}$

Example 35. If A and B are symmetric non-singular matrices of same order, $AB=BA$ and $A^{-1}B^{-1}$ exist, prove that $A^{-1}B^{-1}$ is symmetric.

Sol. $\because \quad A'=A,\ B'=B$ and $|A|\neq 0, |B|'\neq 0$

$\therefore \quad (A^{-1}B^{-1})'=(B^{-1})'(A^{-1})'$

[by reversal law of transpose]

$=(B')^{-1}(A')^{-1}$ [by property 3]

$=B^{-1}A^{-1}$ [$\because A'=A$ and $B'=B$]

$=(AB)^{-1}$ [by reversal law of inverse]

$=(BA)^{-1}$ [$\because AB=BA$]

$=A^{-1}B^{-1}$ [by reversal law of inverse]

Hence, $A^{-1}B^{-1}$ is symmetric.

Example 36. Matrices A and B satisfy $AB=B^{-1}$, where $B=\begin{bmatrix}2&-2\\-1&0\end{bmatrix}$, find the value of λ for which $\lambda A-2B^{-1}+I=O$, without finding B^{-1}.

Sol. $\because \quad AB=B^{-1}$ or $AB^2=I$

Now, $\lambda A-2B^{-1}+I=O$

$\Rightarrow \quad \lambda AB-2B^{-1}B+IB=O$ [post-multiplying by B]

$\Rightarrow \quad \lambda AB-2I+B=O$

$\Rightarrow \quad \lambda AB^2-2IB+B^2=O$

[again post-multiplying by B]

$\Rightarrow \quad \lambda AB^2-2B+B^2=O$

$\Rightarrow \quad \lambda I-2B+B^2=O$ [$\because AB^2=I$]

$\Rightarrow \lambda\begin{bmatrix}1&0\\0&1\end{bmatrix}-2\begin{bmatrix}2&-2\\-1&0\end{bmatrix}+\begin{bmatrix}2&-2\\-1&0\end{bmatrix}\begin{bmatrix}2&-2\\-1&0\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix}\lambda&0\\0&\lambda\end{bmatrix}-\begin{bmatrix}4&-4\\-2&0\end{bmatrix}+\begin{bmatrix}6&-4\\-2&2\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix}\lambda+2&0\\0&\lambda+2\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$

$\Rightarrow \quad \lambda+2=0$

$\therefore \quad \lambda=-2$

Example 37. If A, B and C are three non-singular square matrices of order 3 satisfying the equation $A^2=A^{-1}$ and let $B=A^8$ and $C=A^2$, find the value of $\det(B-C)$.

Sol. $\because \quad B=A^8=(A^2)^4=(A^{-1})^4$ [$\because A^{-1}=A^2$]

$=(A^4)^{-1}=(A^{2\cdot 2})^{-1}$

$=((A^2)^2)^{-1}=((A^2)^{-1})^2$

$=((A^{-1})^{-1})^2=A^2=C$

So, $B=C\Rightarrow B-C=0$

$\therefore \quad \det(B-C)=0$

Elementary Row Operations
(Transformations)

The following three types of operations (transformations) on the rows of a given matrix are known as elementary row operation (transformations).

(i) The interchange of ith and jth rows is denoted by $R_i\leftrightarrow R_j$ or R_{ij}.

(ii) The multiplication of the ith row by a constant $k(k\neq 0)$ is denoted by $R_i\to kR_i$ or $R_i(k)$

(iii) The addition of the ith row to the elements of the jth row multiplied by constant $k(k \neq 0)$ is denoted by $R_i \to R_i + kR_j$ or $R_{ij}(k)$.

Note

Similarly, we can define the three column operations, $C_{ij}(C_i \leftrightarrow C_j)$, $C_i(k)(C_i \to kC_i)$ and $C_{ij}(k)(C_i \to C_i + kC_j)$.

Equivalent Matrices

Two matrices are said to be equivalent if one is obtained from the other by elementary operations (transformations). The symbol ~ is used for equivalence.

Properties of Equivalent Matrices

(i) If A and B are equivalent matrices, there exist non-singular matrices P and Q such that $B = PAQ$

(ii) If A and B are equivalent matrices such that $B = PAQ$, then $P^{-1}BQ^{-1} = A$

(iii) Every non-singular square matrix can be expressed as the product of elementary matrices.

Example 38. Transform $\begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & 10 \\ 3 & 8 & 4 \end{bmatrix}$ into a unit matrix.

Sol. Let $A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & 10 \\ 3 & 8 & 4 \end{bmatrix}$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$, we get

$$A \sim \begin{bmatrix} 1 & 3 & 3 \\ 0 & -2 & 4 \\ 0 & -1 & -5 \end{bmatrix}$$

Applying $R_2 \to \left(-\frac{1}{2}\right)R_2$ and $R_2 \to (-1)R_2$, we get

$$A \sim \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & -2 \\ 0 & 1 & 5 \end{bmatrix}$$

Applying $R_1 \to R_1 - 3R_2$ and $R_3 \to R_3 - R_2$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 9 \\ 0 & 1 & -2 \\ 0 & 0 & 7 \end{bmatrix}$$

Applying $R_3 \to \left(\frac{1}{7}\right)R_3$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 9 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $R_1 \to R_1 - 9R_3$ and $R_2 \to R_2 + 2R_3$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, $A \sim I$

Example 39. Given $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$. Find P such that $BPA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Sol. Given, $BPA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$$\therefore \quad P = B^{-1}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}A^{-1} \qquad \text{...(i) [by property]}$$

$$\because \quad B = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \Rightarrow B^{-1} = \frac{1}{(-1)}\begin{bmatrix} 4 & -3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix}$$

$$\therefore \quad B^{-1} = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix} \qquad \text{...(ii)}$$

and $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix}$

$$\therefore \quad |A| = 1(4-3) - 1(2-2) + 1(6-8) = -1 \neq 0$$

$\Rightarrow A^{-1}$ exists.

Now, adj $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 1 \\ -2 & -1 & 2 \end{bmatrix}$ [by shortcut method]

$$\therefore \quad A^{-1} = \frac{\text{adj } A}{|A|} = \begin{bmatrix} -1 & -2 & 3 \\ 0 & 1 & -1 \\ 2 & 1 & -2 \end{bmatrix} \qquad \text{...(iii)}$$

Substituting the values of A^{-1} and B^{-1} from Eqs. (ii) and (iii) in Eq. (i), then

$$P = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & -2 & 3 \\ 0 & 1 & -1 \\ 2 & 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -4 & 7 & -7 \\ 3 & -5 & 5 \end{bmatrix}$$

To Compute the Inverse of a Non-Singular Matrix by Elementary Operations
(Gauss-Jordan Method)

If A be a non-singular matrix of order n, then write $A = I_n A$.

If A is reduced to I_n by elementary operations (LHS), then suppose I_n is reduced to P(RHS) and not change A in RHS, then after elementary operations, we get

$$I_n = PA,$$

then P is the inverse of A

$$\therefore \quad P = A^{-1}$$

Example 40. ***Find the inverse of the matrix*** $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$***, using elementary row operations.***

Sol. Let $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

$$\therefore \quad |A| = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{vmatrix} = 1(3-1) - 2(2+1) + 5(2+3) = 21 \neq 0$$

$\therefore \quad A^{-1}$ exists.

We write $A = IA$

$$\Rightarrow \quad \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 + R_1$, we get

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & -9 \\ 0 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to (-1)R_2$ and $R_3 \to \left(\frac{1}{3}\right)R_3$, we get

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 9 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} A$$

Applying $R_1 \to R_1 - 2R_2$ and $R_3 \to R_3 - R_2$, we get

$$\begin{bmatrix} 1 & 0 & -13 \\ 0 & 1 & 9 \\ 0 & 0 & -7 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 0 \\ 2 & -1 & 0 \\ -\frac{5}{3} & 1 & \frac{1}{3} \end{bmatrix} A$$

Applying $R_3 \to \left(-\frac{1}{7}\right)R_3$, we get

$$\begin{bmatrix} 1 & 0 & -13 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 0 \\ 2 & -1 & 0 \\ \frac{5}{21} & -\frac{1}{7} & -\frac{1}{21} \end{bmatrix} A$$

Applying $R_2 \to R_2 - 9R_3$ and $R_1 \to R_1 + 13R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{21} & \frac{1}{7} & -\frac{13}{21} \\ -\frac{1}{7} & \frac{2}{7} & \frac{3}{7} \\ \frac{5}{21} & -\frac{1}{7} & -\frac{1}{21} \end{bmatrix} A$$

Hence, $$A^{-1} = \begin{bmatrix} \frac{2}{21} & \frac{1}{7} & -\frac{13}{21} \\ -\frac{1}{7} & \frac{2}{7} & \frac{3}{7} \\ \frac{5}{21} & -\frac{1}{7} & -\frac{1}{21} \end{bmatrix}$$

Matrix Polynomial

Let $f(x) = a_0 x^m + a_1 x^{m-1} + a_2 x^{m-2} + \ldots + a_{m-1}x + a_m$ be a polynomial in x and let $A = [a_{ij}]_{n \times n}$, then expression of the form

$f(A) = a_0 A^m + a_1 A^{m-1} + a_2 A^{m-2} + \ldots + a_{m-1} A + a_m I_n$ is called a matrix polynomial.

Thus, to obtain $f(A)$ replace x by A in $f(x)$ and the constant term is multiplied by the identity matrix of order equal to that of A.

For example, If $f(x) = x^2 - 7x + 32$ is a polynomial in x and A is a square matrix of order 3, then $f(A) = A^2 - 7A + 32 I_3$ is a matrix polynomial.

Note

1. The polynomial equation $f(x) = 0$ is satisfied by the matrix $A = [a_{ij}]_{n \times n}$, then $f(A) = 0$.
2. Let $A = [a_{ij}]_{n \times n}$ satisfies the equation $a_0 + a_1 x + a_2 x^2 + \ldots + a_r x^r = 0$, then A is invertible of $a_0 \neq 0, |A| = 0$ and its inverse is given by $$A^{-1} = \frac{1}{a_0}(a_1 I_n + a_2 A + \ldots + a_r A^{r-1}).$$

Example 41. If $A = \begin{bmatrix} k & l \\ m & n \end{bmatrix}$ and $kn \neq lm$, show that $A^2 - (k+n)A + (kn - lm)I = O$. Hence, find A^{-1}.

Sol. We have, $A = \begin{bmatrix} k & l \\ m & n \end{bmatrix}$, then $|A| = \begin{vmatrix} k & l \\ m & n \end{vmatrix}$

$$= kn - ml \neq 0 \qquad \text{[given]}$$

$\therefore\ A^{-1}$ exists.

Now, $A^2 = A \cdot A = \begin{bmatrix} k & l \\ m & n \end{bmatrix}\begin{bmatrix} k & l \\ m & n \end{bmatrix} = \begin{bmatrix} k^2 + lm & kl + ln \\ mk + nm & ml + n^2 \end{bmatrix}$

$\therefore\ A^2 - (k+n)A + (kn - lm)I$

$$= \begin{bmatrix} k^2 + lm & kl + ln \\ mk + nm & ml + n^2 \end{bmatrix} - (k+n)\begin{bmatrix} k & l \\ m & n \end{bmatrix} + (kn - lm)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} k^2 + lm & kl + ln \\ mk + nm & ml + n^2 \end{bmatrix} - \begin{bmatrix} k^2 + nk & kl + nl \\ km + nm & kn + n^2 \end{bmatrix} + \begin{bmatrix} kn - lm & 0 \\ 0 & kn - lm \end{bmatrix}$$

$$= \begin{bmatrix} k^2 + lm - k^2 - nk + kn - lm & kl + ln - kl - ln \\ mk + nm - km - nm & ml + n^2 - kn - n^2 + kn - lm \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

As $A^2 - (k+n)A + (kn - lm)I = O$

$\Rightarrow\quad (kn - lm)I = (k+n)A - A^2$

$\Rightarrow\quad (kn - lm)IA^{-1} = ((k+n)A - A^2)A^{-1}$

$\Rightarrow\quad (kn - lm)A^{-1} = (k+n)AA^{-1} - A(AA^{-1})$

$= (k+n)I - AI \qquad [\because AA^{-1} = I]$

$= (k+n)I - A$

$= (k+n)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} k & l \\ m & n \end{bmatrix}$

$= \begin{bmatrix} k+n & 0 \\ 0 & k+n \end{bmatrix} - \begin{bmatrix} k & l \\ m & n \end{bmatrix}$

$\Rightarrow\quad (kn - lm)A^{-1} = \begin{bmatrix} n & -l \\ -m & k \end{bmatrix}$

Hence, $A^{-1} = \dfrac{1}{(kn - lm)}\begin{bmatrix} n & -l \\ -m & k \end{bmatrix}$

Example 42. If $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$, find the value of $|a| + |b|$ such that $A^2 + aA + bI = O$. Hence, find A^{-1}.

Sol. We have, $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$, then $|A| = \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} = 3 - 2 = 1 \neq 0$

$\therefore\quad A^{-1}$ exists.

Now, $A^2 = A \cdot A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 4 \\ 8 & 3 \end{bmatrix}$

Since, $A^2 + aA + bI = O$

$$\Rightarrow \begin{bmatrix} 11 & 4 \\ 8 & 3 \end{bmatrix} + a\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 11 + 3a + b & 4 + a \\ 8 + 2a & 3 + a + b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Equating the corresponding elements, we get

$$11 + 3a + b = 0 \qquad \text{...(i)}$$

$$4 + a = 0 \qquad \text{...(ii)}$$

$$8 + 2a = 0 \qquad \text{...(iii)}$$

$$3 + a + b = 0 \qquad \text{...(iv)}$$

From Eqs. (ii) and (iv), we get $a = -4$ and $b = 1$

$\therefore\quad |a| + |b| = |-4| + |1| = 4 + 1 = 5$

As $A^2 + aA + bI = O$

$\Rightarrow\quad A^2 - 4A + I = O \Rightarrow I = 4A - A^2$

$\Rightarrow\quad IA^{-1} = (4A - A^2)A^{-1}$

$\Rightarrow\quad A^{-1} = 4(AA^{-1}) - A(AA^{-1})$

$= 4I - AI = 4I - A$

$$= 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$\therefore\ A^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$

Use of Mathematical Induction

Example 43. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, show that

$$(aI + bA)^n = a^n I + na^{n-1} bA,\ \forall\ n \in N.$$

Sol. Let $P(n) : (aI + bA)^n = a^n I + na^{n-1}bA$

Step I For $n = 1$,

LHS $= (aI + bA)^1 = aI + bA$

and RHS $= a^1\ I + 1 \cdot a^0\ bA = aI + bA$

LHS = RHS

Therefore, $P(1)$ is true.

Step II Assume that $P(k)$ is true, then

$$P(k) : (aI + bA)^k = a^k\ I + ka^{k-1}\ bA$$

Step III For $n = k + 1$, we have to prove that

$$P(k+1) : (aI + bA)^{k+1} = a^{k+1}\ I + (k+1)\ a^k bA$$

LHS $= (aI + bA)^{k+1} = (aI + bA)^k\ (aI + bA)$

$= (a^k I + ka^{k-1}bA)(aI + bA) \qquad$ [from step II]

$= a^{k+1}\ I^2 + a^k b\,(IA) + ka^k b\,(AI) + k\ a^{k-1}b^2\ A^2$

$= a^{k+1}\ I + (k+1)a^k b\ A + 0$

$[\because AI = A,\ A^2 = 0 \text{ and } I^2 = I]$

$= a^{k+1}I + (k+1)a^k bA = \text{RHS}$

Therefore, $P(k+1)$ is true.

Hence, by the principal of mathematical2 induction $P(n)$ is true for all $n \in N$.

Example 44. *If* $A=\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}$, *use mathematical induction to show that* $A^n=\begin{bmatrix}1+2n & -4n\\ n & 1-2n\end{bmatrix}$, $\forall$ $n\in N$.

Sol. Let $P(n):A^n=\begin{bmatrix}1+2n & -4n\\ n & 1-2n\end{bmatrix}$

Step I For $n=1$,

$\text{LHS}=A^1=A$

and $\text{RHS}=\begin{bmatrix}1+2 & -4\\ 1 & 1-2\end{bmatrix}=\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}=A$

$\Rightarrow$ LHS = RHS

Therefore, $P(1)$ is true.

Step II Assume that $P(k)$ is true, then

$P(k):A^k=\begin{bmatrix}1+2k & -4k\\ k & 1-2k\end{bmatrix}$

Step III For $n=k+1$, we have to prove that

$P(k+1):A^{k+1}=\begin{bmatrix}3+2k & -4(k+1)\\ k+1 & -1-2k\end{bmatrix}$

$\text{LHS}=A^{k+1}=A^k\cdot A$

$=\begin{bmatrix}1+2k & -4k\\ k & 1-2k\end{bmatrix}\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}$ [from step II]

$=\begin{bmatrix}3(1+2k)-4k & -4(1+2k)+4k\\ 3k+1(1-2k) & -4k-1(1-2k)\end{bmatrix}$

$=\begin{bmatrix}3+2k & -4(k+1)\\ k+1 & -1-2k\end{bmatrix}=\text{RHS}$

Therefore, $P(k+1)$ is true.

Hence, by the principal of mathematical induction $P(n)$ is true for all $n\in N$.

Exercise for Session 3

1 If $A=\begin{bmatrix}-1 & -2 & -2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{bmatrix}$, then adj A equals to

(a) A (b) A^T (c) $3A$ (d) $3A^T$

2 If A is a 3×3 matrix and B is its adjoint such that $|B|=64$, then $|A|$ is equal to

(a) 64 (b) $\pm$ 64 (c) $\pm$ 8 (d) 18

3 For any 2×2 matrix A, if $A(\text{adj } A)=\begin{bmatrix}10 & 0\\ 0 & 10\end{bmatrix}$, then $|A|$ is equal to

(a) 0 (b) 10 (c) 20 (d) 100

4 If A is a singular matrix, then adj A is

(a) singular (b) non-singular (c) symmetric (d) not defined

5 If $A=\begin{bmatrix}1 & 2 & -1\\ -1 & 1 & 2\\ 2 & -1 & 1\end{bmatrix}$, then det (adj (adj A)) is

(a) 14^4 (b) 14^3 (c) 14^2 (d) 14

6 If $k\in R_0$, then det $(\text{adj}\,(k\,I_n))$ is equal to

(a) k^{n-1} (b) $k^{n(n-1)}$ (c) k^n (d) k

7 With $1,\omega,\omega^2$ as cube roots of unity, inverse of which of the following matrices exists?

(a) $\begin{bmatrix}1 & \omega\\ \omega & \omega^2\end{bmatrix}$ (b) $\begin{bmatrix}\omega^2 & 1\\ 1 & \omega\end{bmatrix}$

(c) $\begin{bmatrix} \omega & \omega^2 \\ \omega^2 & 1 \end{bmatrix}$ (d) None of these

8 If the matrix A is such that $A\begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 7 & 7 \end{bmatrix}$, then A is equal to

(a) $\begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix}$

9 If $A = \begin{bmatrix} \cos x & \sin x & 0 \\ -\sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} = f(x)$, then A^{-1} is equal to

(a) $f(-x)$ (b) $f(x)$
(c) $-f(x)$ (d) $-f(-x)$

10 The element in the first row and third column of the inverse of the matrix $\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, is

(a) -2 (b) 0 (c) 1 (d) None of these

11 If $A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$, then $(A(\text{adj } A)\, A^{-1})\, A$ is equal to

(a) $\begin{bmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & \frac{1}{6} & -\frac{1}{6} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \end{bmatrix}$ (c) $2\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ (d) $2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

12 A is an involutory matrix given by $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$, then the inverse of $\frac{A}{2}$ will be

(a) $2A$ (b) $\frac{A^{-1}}{2}$
(c) $\frac{A}{2}$ (d) A^2

13 If A satisfies the equation $x^3 - 5x^2 + 4x + \lambda = 0$, then A^{-1} exists, if

(a) $\lambda \neq 1$ (b) $\lambda \neq 2$
(c) $\lambda \neq -1$ (d) $\lambda \neq 0$

14 A square non-singular matrix A satisfies the equation $x^2 - x + 2 = 0$, then A^{-1} is equal to

(a) $I - A$ (b) $(I - A)/2$
(c) $I + A$ (d) $(I + A)/2$

15 Matrix A is such that $A^2 = 2A - I$, where I is the identity matrix, then for $n \geq 2$, A^n is equal to

(a) $nA - (n - 1)I$ (b) $nA - I$
(c) $2^{n-1}A - (n - 1)I$ (d) $2^{n-1}A - I$

16 If $X = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, the value of X^n is

(a) $\begin{bmatrix} 3n & -4n \\ n & -n \end{bmatrix}$ (b) $\begin{bmatrix} 2n + n & 5 - n \\ n & -n \end{bmatrix}$
(c) $\begin{bmatrix} 3^n & (-4)^n \\ 1^n & (-1)^n \end{bmatrix}$ (d) None of these

Session 4

Solutions of Linear Simultaneous Equations Using Matrix Method

Solutions of Linear Simultaneous Equations Using Matrix Method

Let us consider a system of n linear equations in n unknowns say $x_1, x_2, x_3, ..., x_n$ given as below

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n &= b_3 \\ \ldots \quad \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots &\\ \ldots \quad \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots &\\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n &= b_n \end{aligned}\right\} \quad \ldots(i)$$

If $b_1 = b_2 = b_3 = \ldots = b_n = 0$, then the system of Eq.(i) is called a system of homogeneous linear equations and if atleast one of $b_1, b_2, b_3, ..., b_n$ is non-zero, then it is called a system of non-homogeneous linear equation. We write the above system of Eq. (i) in the matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \ldots \\ \ldots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \ldots \\ \ldots \\ b_n \end{bmatrix}$$

$$\Rightarrow \qquad AX = B \qquad \ldots(ii)$$

where $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{bmatrix},$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \ldots \\ \ldots \\ x_n \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \ldots \\ \ldots \\ b_n \end{bmatrix}$$

Pre-multiplying Eq. (ii) by A^{-1}, we get

$$A^{-1}(AX) = A^{-1}B \Rightarrow (A^{-1}A)X = A^{-1}B$$

$$\Rightarrow \qquad IX = A^{-1}B$$

$$\Rightarrow \qquad X = A^{-1}B = \frac{(\text{adj } A)B}{|A|}$$

Types of Equations

(1) **When system of equations is non-homogeneous**

(i) If $|A| \neq 0$, then the system of equations is consistent and has a unique solution given by $X = A^{-1}B$.

(ii) If $|A| = 0$ and $(\text{adj}A) \cdot B \neq 0$, then the system of equations is inconsistent and has no solution.

(iii) If $|A| = 0$ and $(\text{adj}A) \cdot B = O$, then the system of equations is consistent and has an infinite number of solutions.

(2) **When system of equations is homogeneous**

(i) If $|A| \neq 0$, then the system of equations has only trivial solution and it has one solution.

(ii) If $|A| = 0$, then the system of equations has non-trivial solution and it has infinite solutions.

(iii) If number of equations < number of unknowns, then it has non-trivial solution.

Note

Non-homogeneous linear equations can also be solved by Cramer's rule, this method has been discussed in the chapter on determinants.

Example 45. Solve the system of equations $x + 2y + 3z = 1$, $2x + 3y + 2z = 2$ and $3x + 3y + 4z = 1$ with the help of matrix inversion.

Sol. We have,

$$x + 2y + 3z = 1,\ 2x + 3y + 2z = 2 \text{ and } 3x + 3y + 4z = 1$$

The given system of equations in the matrix form are written as below.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$AX = B$$

$$\Rightarrow \quad X = A^{-1}B \qquad \ldots(i)$$

where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$$|A| = 1(12-6) - 2(8-6) + 3(6-9)$$
$$= 6 - 4 - 9 = -7 \neq 0$$

$\therefore$ A^{-1} exists and has unique solution.

Let C be the matrix of cofactors of elements in $|A|$.

Now, cofactors along $R_1 = 6, -2, -3$

cofactors along $R_2 = 1, -5, 3$

and cofactors along $R_3 = -5, 4, -1$

$$\therefore \quad C = \begin{bmatrix} 6 & -2 & -3 \\ 1 & -5 & 3 \\ -5 & 4 & -1 \end{bmatrix}$$

$$\therefore \quad \text{adj } A = C^T$$

$$\Rightarrow \quad \text{adj } A = \begin{bmatrix} 6 & -2 & -3 \\ 1 & -5 & 3 \\ -5 & 4 & -1 \end{bmatrix}^T = \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix}$$

$$\Rightarrow \quad A^{-1} = \frac{\text{adj } A}{|A|} = -\frac{1}{7}\begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{6}{7} & -\frac{1}{7} & \frac{5}{7} \\ \frac{2}{7} & \frac{5}{7} & -\frac{4}{7} \\ \frac{3}{7} & -\frac{3}{7} & \frac{1}{7} \end{bmatrix}$$

From Eq. (i), $X = A^{-1}B$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{6}{7} & -\frac{1}{7} & \frac{5}{7} \\ \frac{2}{7} & \frac{5}{7} & -\frac{4}{7} \\ \frac{3}{7} & -\frac{3}{7} & \frac{1}{7} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{7} \\ \frac{8}{7} \\ -\frac{2}{7} \end{bmatrix}$$

Hence, $x = -\frac{3}{7}, y = \frac{8}{7}$ and $z = -\frac{2}{7}$ is the required solution.

Example 46. Solve the system of equations $x + y + z = 6$, $x + 2y + 3z = 14$ and $x + 4y + 7z = 30$ with the help of matrix method.

Sol. We have, $x + y + z = 6$,

$$x + 2y + 3z = 14$$

and $$x + 4y + 7z = 30$$

The given system of equations in the matrix form are written as below :

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$$

$$AX = B \qquad \ldots(i)$$

where, $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$

$$|A| = 1(14-12) - 1(7-3) + 1(4-2) = 2 - 4 + 2 = 0$$

$\therefore$ The equation either has no solution or an infinite number of solutions. To decide about this, we proceed to find

(adj A) B.

Let C be the matrix of cofactors of elements in $|A|$.

Now, cofactors along $R_1 = 2, -4, 2$

cofactors along $R_2 = -3, 6, -3$

and cofactors along $R_3 = 1, -2, 1$

$$\therefore \quad C = \begin{bmatrix} 2 & -4 & 2 \\ -3 & 6 & -3 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \text{adj} = C^T = \begin{bmatrix} 2 & -3 & 1 \\ -4 & 6 & -2 \\ 2 & -3 & 1 \end{bmatrix},$$

then $$(\text{adj } A)\, B = \begin{bmatrix} 2 & -3 & 1 \\ -4 & 6 & -2 \\ 2 & -3 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = O$$

Hence, both conditions $|A| = 0$ and $(\text{adj } A)\, B = O$ are satisfied, then the system of equations is consistent and has an infinite number of solutions.

Proceed as follows :

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 1 & 2 & 3 & \vdots & 14 \\ 1 & 4 & 7 & \vdots & 30 \end{bmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 0 & 1 & 2 & \vdots & 8 \\ 0 & 2 & 4 & \vdots & 16 \end{bmatrix}$$

Applying $R_3 \to R_3 - 2R_2$, then

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 0 & 1 & 2 & \vdots & 8 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

Then, Eq. (i) reduces to

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x + y + z \\ y + 2z \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix}$$

On comparing $x + y + z = 6$ and $y + 2z = 8$

Taking $z = k \in R$, then $y = 8 - 2k$ and $x = k - 2$.

Since, k is arbitrary, hence the number of solutions is infinite.

Example 47. Solve the system of equations $x + 3y - 2z = 0$, $2x - y + 4z = 0$ and $x - 11y + 14z = 0$.

Sol. We have, $x + 3y - 2z = 0$

$$2x - y + 4z = 0$$

$$x - 11y + 14z = 0$$

The given system of equations in the matrix form are written as below.

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$AX = O \quad \ldots(i)$$

where $A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore \quad |A| = 1(-14 + 44) - 3(28 - 4) - 2(-22 + 1)$

$= 30 - 72 + 42 = 0$

and therefore the system has a non-trivial solution. Now, we may write first two of the given equations

$$x + 3y = 2z \text{ and } 2x - y = -4z$$

Solving these equations in terms of z, we get

$$x = -\frac{10}{7}z \text{ and } y = \frac{8}{7}z$$

Putting $x = -\frac{10}{7}z$ and $y = \frac{8}{7}z$ in third equation of the given system,

we get, LHS $= -\frac{10}{7}z - \frac{88}{7}z + 14z = 0 =$ RHS

Now, if $z = 7k$, then $x = -10k$ and $y = 8k$.

Hence, $x = -10k$, $y = 8k$ and $z = 7k$ (where k is arbitrary) are the required solutions.

Example 48. Solve the system of equations

$$2x + 3y - 3z = 0,$$

$$3x - 3y + z = 0$$

and $$3x - 2y - 3z = 0$$

Sol. We have, $2x + 3y - 3z = 0$

$$3x - 3y + z = 0$$

$$3x - 2y - 3z = 0$$

The given system of equations in the matrix form are written as below.

$$\begin{bmatrix} 2 & 3 & -3 \\ 3 & -3 & 1 \\ 3 & -2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$AX = O \quad \ldots(i)$$

where $A = \begin{bmatrix} 2 & 3 & -3 \\ 3 & -3 & 1 \\ 3 & -2 & -3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore \quad |A| = 2(9 + 2) - 3(-9 - 3) - 3(-6 + 9)$

$= 22 + 36 - 9 = 49 \neq 0$

Hence, the equations have a unique trivial solution $x = 0$, $y = 0$ and $z = 0$ only.

Echelon Form of a Matrix

A matrix A is said to be in echelon form, if

(i) The first non-zero element in each row is 1.

(ii) Every non-zero row in A preceds every zero-row.

(iii) The number of zeroes before the first non-zero element in 1st, 2nd, 3rd, ... rows should be in increasing order.

For example,

(i) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 4 & 3 \\ 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Rank of Matrix

The rank of a matrix is said to be r, if

(i) It has atleast minors of order r is different from zero.

(ii) All minors of A of order higher than r are zero.

The rank of A is denoted by $\rho(A)$.

Note

1. The rank of a zero matrix is zero and the rank of an identity matrix of order n is n.
2. The rank of a matrix in echelon form is equal to the number of non-zero rows of the matrix.
3. The rank of a non-singular matrix ($|A| \neq 0$) of order n is n.

Properties of Rank of Matrices

(i) If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$, then $\rho(A + B) \leq \rho(A) + \rho(B)$

(ii) If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$, then $\rho(AB) \leq \rho(A)$ and $\rho(AB) \leq \rho(B)$

(iii) If $A = [a_{ij}]_{n \times n}$, then $\rho(A) = \rho(A')$

Example 49. Find the rank of $\begin{bmatrix} 3 & -1 & 2 \\ -3 & 1 & 2 \\ -6 & 2 & 4 \end{bmatrix}$.

Sol. We have,

Let $A = \begin{bmatrix} 3 & -1 & 2 \\ -3 & 1 & 2 \\ -6 & 2 & 4 \end{bmatrix}$

Applying $R_2 \to R_2 + R_1$ and $R_3 \to R_3 + 2R_1$, we get

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 8 \end{bmatrix}$$

Applying $R_3 \to R_3 - 2R_2$, we get

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

Applying $R_1 \to \left(\frac{1}{3}\right) R_1$ and $R_2 \to \left(\frac{1}{4}\right) R_2$, then

$$A = \begin{bmatrix} 1 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

This is Echelon form of matrix A.

$\therefore$ Rank = Number of non-zero rows $\Rightarrow \rho(A) = 2$

Aliter $|A| = \begin{vmatrix} 3 & -1 & 2 \\ -3 & 1 & 2 \\ -6 & 2 & 4 \end{vmatrix}$

$$= 3(4-4) + 1(-12+12) + 2(-6+6) = 0$$

$\therefore$ Rank of $A \neq 3$ but less than 3.

There will be ${}^3C_2 \times {}^3C_2 = 9$ square minors of order 2. Now, we consider of there minors.

(i) $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$ (ii) $\begin{vmatrix} 3 & 2 \\ -6 & 4 \end{vmatrix} = 24 \neq 0$

Hence, all minors are not zero.

Hence, rank of A is 2. $\Rightarrow \rho(A) = 2$

Solutions of Linear Simultaneous Equations Using Rank Method

Let us consider a system of n linear equations in n unknowns say $x_1, x_2, x_3, ..., x_n$ given as below.

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n &= b_3 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n &= b_m \end{aligned}\right\} \quad \ldots(i)$$

We write the above system of Eq. (i) in the matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots & \ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \ldots \\ \ldots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \ldots \\ \ldots \\ b_m \end{bmatrix}$$

$$\Rightarrow \quad AX = B \quad \ldots(ii)$$

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots & \ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & \ldots & a_{mn} \end{bmatrix}$,

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \ldots \\ \ldots \\ x_n \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \ldots \\ \ldots \\ b_m \end{bmatrix}$

The matrix A is called the coefficient matrix and the matrix

$$C = [A : B] = \left[\begin{array}{cccccc:c} a_{11} & a_{12} & a_{13} & \ldots & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \ldots & \ldots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \ldots & \ldots & a_{3n} & b_3 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & \ldots & a_{mn} & b_m \end{array}\right]$$

is called the augmented matrix of the given system of equations.

Types of Equations

1. **Consistent Equation** If $\rho(A) = \rho(C)$
   (i) **Unique Solution** If $\rho(A) = \rho(C) = n$, where $n =$ number of knowns.
   (ii) **Infinite Solution** If $\rho(A) = \rho(C) = r$, where $r < n$.
2. **Inconsistent Equation** If $\rho(A) \neq \rho(C)$, then no solution.

Example 50. Determine for what values of λ and μ the following system of equations

$$x + y + z = 6,$$
$$x + 2y + 3z = 10$$

and $x+2y+\lambda z=\mu$

have (i) no solution? (ii) a unique solution?
(iii) an infinite number of solutions?

Sol. We can write the above system of equations in the matrix form

$$\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 2 & \lambda\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}6\\ 10\\ \mu\end{bmatrix}$$

$\Rightarrow \quad AX=B$

where $A=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 2 & \lambda\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ and $B=\begin{bmatrix}6\\ 10\\ \mu\end{bmatrix}$

$\therefore$ The augmented matrix

$$C=[A:B]=\begin{bmatrix}1 & 1 & 1 & \vdots & 6\\ 1 & 2 & 3 & \vdots & 10\\ 1 & 2 & \lambda & \vdots & \mu\end{bmatrix}$$

Applying $R_2\to R_2-R_1$ and $R_3\to R_3-R_1$, we get

$$C=\begin{bmatrix}1 & 1 & 1 & \vdots & 6\\ 0 & 1 & 2 & \vdots & 4\\ 0 & 1 & \lambda-1 & \vdots & \mu-6\end{bmatrix}$$

Applying $R_3\to R_3-R_2$, we get

$$C=\begin{bmatrix}1 & 1 & 1 & \vdots & 6\\ 0 & 1 & 2 & \vdots & 4\\ 0 & 0 & \lambda-3 & \vdots & \mu-10\end{bmatrix}$$

(i) No solution $\rho(A)\neq\rho(C)$
i.e. $\lambda-3=0$ and $\mu-10\neq 0$
$\therefore \quad \lambda=3$ and $\mu\neq 10$

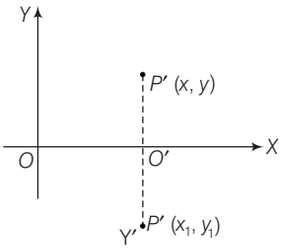
(ii) A unique solution $\rho(A)=\rho(C)=3$
i.e., $\lambda-3\neq 0$ and $\mu\in R$
$\therefore \quad \lambda\neq 3$ and $\mu\in R$

(iii) Infinite number of solutions
$\rho(A)=\rho(C)\ (\angle 3)$
i.e. $\lambda-3=0$ and $\mu-10=0$
$\therefore \quad \lambda=3$ and $\mu=10$

Reflection Matrix

(i) Reflection in the X-axis

Let $P(x,y)$ be any point and $P'(x_1,y_1)$ be its image after reflection in the X-axis, then



$$\begin{cases}x_1=x\\ y_1=-y\end{cases}\quad [O' \text{ is the mid-point of } P \text{ and } P']$$

These may be rewritten as

$$\begin{cases}x_1=1\cdot x+0\cdot y\\ y_1=0\cdot x+(-1)\cdot y\end{cases}$$

These system of equations in the matrix form are written as below.

$$\begin{bmatrix}x_1\\ y_1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$

Thus, **the matrix $\begin{bmatrix}\mathbf{1} & \mathbf{0}\\ \mathbf{0} & \mathbf{-1}\end{bmatrix}$ describes the reflection of a point $P(x,y)$ in the X-axis.**

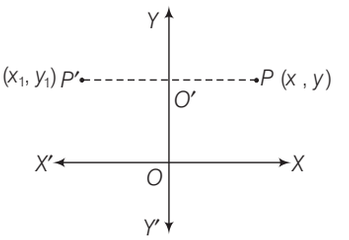
(ii) Reflection in the Y-axis

Let $P(x,y)$ be any point and $P'(x_1,y_1)$ be its image after reflection in the Y-axis, then

$$\begin{cases}x_1=-x\\ y_1=y\end{cases}\quad [O' \text{ is the mid-point of } P \text{ and } P']$$

These may be written as

$$\begin{cases}x_1=(-1)\cdot x+0\cdot y\\ y_1=\ 0\cdot x+1\cdot y\end{cases}$$



These system of equations in the matrix form are written as below.

$$\begin{bmatrix}x_1\\ y_1\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$

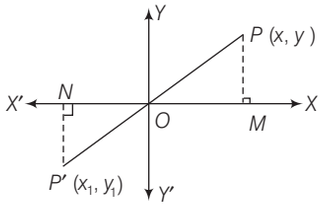
Thus, **the matrix $\begin{bmatrix}\mathbf{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{1}\end{bmatrix}$ describes the reflection of a point $P(x,y)$ in the Y-axis.**

(iii) Reflection through the origin

Let $P(x,y)$ be any point and $P'(x_1,y_1)$ be its image after reflection through the origin, then

$$\begin{cases}x_1=-x\\ y_1=-y\end{cases}\quad [O' \text{ is the mid-point of } P \text{ and } P']$$

These may be written as



$$\begin{cases} x_1 = (-1)\, x + 0 \cdot y \\ y_1 = 0 \cdot x + (-1) \cdot y \end{cases}$$

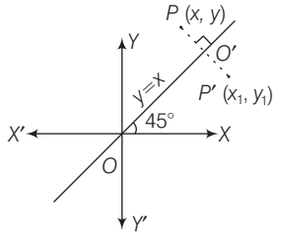
These system of equations in the matrix form are written as below.

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Thus, **the matrix** $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ **describes the reflection of a point** $P(x, y)$ **through the origin.**

(iv) Reflection in the line $y = x$

Let $P(x, y)$ be any point and $P'(x_1, y_1)$ be its image after reflection in the line $y = x$, then



$$\begin{cases} x_1 = y \\ y_1 = x \end{cases} \quad [O' \text{ is the mid-point of } P \text{ and } P']$$

These may be written as

$$\begin{cases} x_1 = 0 \cdot x + 1 \cdot y \\ y_1 = 1 \cdot x + 0 \cdot y \end{cases}$$

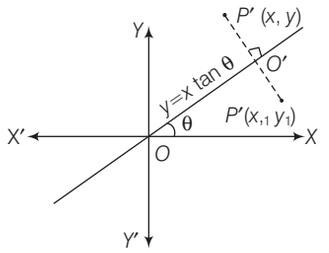
These system of equations in the matrix form are written as below.

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Thus, **the matrix** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **describes the reflection of a point** $P(x, y)$ **in the line** $y = x$**.**

(v) Reflection in the line $y = x \tan\theta$

Let $P(x, y)$ be any point and $P'(x_1, y_1)$ be its image after reflection in the line $y = x \tan\theta$, then



$$\begin{cases} x_1 = x\cos 2\theta + y \sin 2\theta \\ y_1 = x \sin 2\theta - y \cos 2\theta \end{cases}$$

$[O' \text{ is the mid-point of } P \text{ and } P']$

These may be written as

$$\begin{cases} x_1 = x \cdot \cos 2\theta + y \cdot \sin 2\theta \\ y_1 = x \cdot \sin 2\theta + y \cdot (-\cos 2\theta) \end{cases}$$

These system of equations in the matrix form are written as below.

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Thus, the matrix $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$ describes the reflection of a point $P(x, y)$ in the line $y = x \tan\theta$.

Note

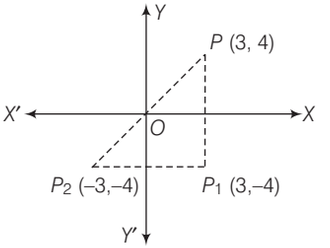
By putting $\theta = 0, \frac{\pi}{2}, \frac{\pi}{4}$, we can get the reflection matrices in the X-axis, Y-axis and the line $y = x$, respectively.

Example 51. The point $P(3, 4)$ undergoes a reflection in the X-axis followed by a reflection in the Y-axis. Show that their combined effect is the same as the single reflection of $P(3, 4)$ in the origin.

Sol. Let $P_1(x_1, y_1)$ be the image of $P(3, 4)$ after reflection in the X-axis. Then,

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

Therefore, the image of $P(3, 4)$ after reflection in the X-axis is $P_1(3, -4)$.



Now, let $P_2(x_2, y_2)$ be the image of $P_1(3, -4)$ after reflection in the Y-axis, then

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$$

Therefore, the image of $P_1(3, -4)$ after reflection in the Y-axis is $P_2(-3, -4)$.

Further, let $P_3(x_3, y_3)$ be the image of $P(3, 4)$ in the origin O. Then,

$$\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$$

Therefore, the image of $P(3, 4)$ after reflection in the origin is $P_3(-3, -4)$. It is clear that $P_2 = P_3$

Hence, the image of P_2 of P often successive reflections in their X-axis and Y-axis is the same as P_3, which is single reflection of P in the origin.

Example 52. Find the image of the point $(-2, -7)$ under the transformations $(x, y) \to (x - 2y, -3x + y)$.

Sol. Let (x_1, y_1) be the image of the point (x, y) under the given transormations, then

$$\begin{cases} x_1 = x - 2y = 1 \cdot x + (-2) \cdot y \\ y_1 = -3x + y = (-3) \cdot x + 1 \cdot y \end{cases}$$

$$\Rightarrow \quad \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -7 \end{bmatrix} = \begin{bmatrix} -2+14 \\ 6-7 \end{bmatrix} = \begin{bmatrix} 12 \\ -1 \end{bmatrix}$$

Therefore, the required image is $(12, -1)$.

Example 53. The image of the point $A(2, 3)$ by the line mirror $y = x$ is the point B and the image of B by the line mirror $y = 0$ is the point (α, β). Find α and β.

Sol. Let $B(x_1, y_1)$ be the image of the point $A(2,3)$ about the line $y = x$, then

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Therefore, the image of $A(2, 3)$ by the line mirror $y = x$ is $B(3, 2)$.

Given, image of B by the line mirror $y = 0$ (X-axis) is (α, β), then

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

On comparing, we get $\alpha = 3$ and $\beta = -2$.

Example 54. Find the image of the point $(-\sqrt{2}, \sqrt{2})$ by the line mirror $y = x \tan\left(\frac{\pi}{8}\right)$.

Sol. Let (x_1, y_1) be the image of $(-\sqrt{2}, \sqrt{2})$ about the line $y = x \tan\left(\frac{\pi}{8}\right)$.

On comparing $y = x \tan\left(\frac{\pi}{8}\right)$ by $y = x \tan\theta$

$$\therefore \qquad \theta = \frac{\pi}{8}$$

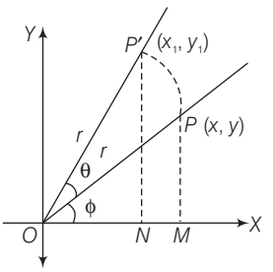
Now,
$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

On comparing $x_1 = 0$ and $y_1 = -2$.

Therefore, the required image is $(0, -2)$.

Rotation Through an Angle θ



Let $P(x, y)$ be any point such that $OP = r$ and $\angle POX = \phi$. Let OP rotate through an angle θ in the anti-clockwise direction such that $P'(x_1, y_1)$ is the new position.

$$\therefore \qquad OP' = r, \qquad [\because OP = OP']$$

then
$$\begin{cases} x_1 = x\cos\theta - y\sin\theta \\ x_1 = x\sin\theta + y\cos\theta \end{cases}$$

These system of equations in the matrix form are written as below.

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Thus, the matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ describes a rotation of a line segment through an angle θ.

Remember Use of complex number

$$OP' = OP\, e^{i\theta},\ i = \sqrt{-1}$$

$$(x_1 + iy_1) = (x + iy)(\cos\theta + i\sin\theta)$$

$$= (x\cos\theta - y\sin\theta) + i(x\sin\theta + y\cos\theta)$$

$$\therefore \qquad x_1 = x\cos\theta - y\sin\theta$$

and
$$y_1 = x\sin\theta + y\cos\theta$$

Example 55. Find the matrices of transformation T_1T_2 and T_2T_1 when T_1 is rotation through an angle 60° and T_2 is the reflection in the Y-axis. Also, verify that $T_1T_2 \neq T_2T_1$.

Sol. $T_1 = \begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$

and $T_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore \quad T_1T_2 = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} \times \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1+0 & 0-\sqrt{3} \\ -\sqrt{3}+0 & 0+1 \end{bmatrix}$

$$= \frac{1}{2}\begin{bmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \quad \text{...(i)}$$

and $T_2T_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \times \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1+0 & \sqrt{3}+0 \\ 0+\sqrt{3} & 0+1 \end{bmatrix}$

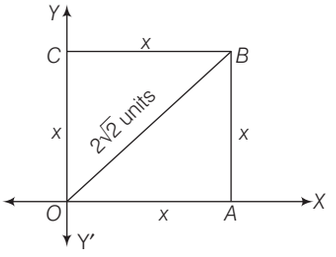
$$= \frac{1}{2}\begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \quad \text{...(ii)}$$

It is clear from Eqs.(i) and (ii), then

$$T_1T_2 \neq T_2T_1$$

Example 56. Write down 2×2 matrix A which corresponds to a counterclockwise rotation of 60° about the origin. In the diagram the square $OABC$ has its diagonal OB of $2\sqrt{2}$ units in length. The square is rotated counterclockwise about O through 60°. Find the coordiates of the vertices of the square after rotating.

Sol. The matrix describes a rotation through an angle 60° in counterclockwise direction is



$$\begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$$

Since, each side of the square be x,

then $\quad x^2 + x^2 = (2\sqrt{2})^2$

$\Rightarrow \quad 2x^2 = 8 \Rightarrow x^2 = 4$

$\therefore \quad x = 2$ units

Therefore, the coordinates of the vertices O, A, B and C are (0, 0), (2, 0), (2, 2) and (0, 2), respectively. Let after rotation A map into A', B map into B', C map into C' but the O map into itself.

If coordinates of A', B' and C' are (x', y'), (x'', y'') and (x''', y'''), respectively.

$\therefore \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 \\ 2\sqrt{3} \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$

$\therefore \quad x' = 1, y' = \sqrt{3} \Rightarrow A(2, 0) \to A'(1, \sqrt{3})$

and $\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2-2\sqrt{3} \\ 2\sqrt{3}+2 \end{bmatrix} = \begin{bmatrix} 1-\sqrt{3} \\ \sqrt{3}+1 \end{bmatrix}$

$\therefore \quad x'' = 1 - \sqrt{3}, y'' = \sqrt{3} + 1$

$\Rightarrow \quad B(2, 2) \to B'(1 - \sqrt{3}, \sqrt{3} + 1)$

$\begin{bmatrix} x''' \\ y''' \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -2\sqrt{3} \\ 2 \end{bmatrix} = \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix}$

$\therefore \quad x''' = -\sqrt{3}, y''' = 1$

$\Rightarrow \quad C(0, 2) \to C'(-\sqrt{3}, 1)$

Eigen Values or Characteristic roots and Characteristic Vectors of a square matrix

Let X be any non-zero vector satisfying

$$AX = \lambda X \quad \text{...(i)}$$

where λ is any scalar, then λ is said to be eigen value or characteristic root of square matrix A and the vector X is called eigen vector or characteristic vector of matrix A.

Now, from Eq. (i), we have

$$(A - \lambda I)\, X = O$$

Since, $X \neq O$, we deduce that the matrix $(A - \lambda I)$ is singular, so that its determinant is 0

i.e.

$$|A - \lambda I| = 0 \quad \text{...(ii)}$$

is called characteristic equation of matrix A.

If A be $n \times n$ matrix, then equation $|A - \lambda I| = 0$ reduces to polynomial equation of nth from degree in λ, which given n values of λ i.e., matrix A will have n characteristic roots or eigen values.

Important Properties of Eigen Values

(i) Any square matrix A and its transpose A^T have the same eigen values.

(ii) The sum of the eigen values of a matrix is equal to the trace of the matrix.

(iii) The product the eigen values of a matrix A is equal to the determinant of A.

(iv) If $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \ldots, \lambda_n$ are the eigen values of A, then the eigen values of

(a) kA are $k\lambda_1, k\lambda_2, k\lambda_3, k\lambda_4, \ldots, k\lambda_n$.

(b) A^m are $\lambda_1^m, \lambda_2^m, \lambda_3^m, \lambda_4^m, \ldots, \lambda_n^m$.

(c) A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}, \frac{1}{\lambda_4}, \ldots, \frac{1}{\lambda_n}$.

Remark

1. All the eigen values of a real symmetric matrix are real and the eigen vectors corresponding to two distinct eigen values are orthogonal.
2. All the eigen values of a real skew-symmetric matrix are purely imaginary or zero. An odd order skew-symmetric matrix is singular and hence has zero as an eigen value.

Example 57. Let matrix $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$, find the non-zero column vector X such that $AX = \lambda X$ for some scalar λ.

Sol. The characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \quad \begin{bmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -4 & -3-\lambda \end{bmatrix} = 0$$

$$\Rightarrow \quad \lambda^3 - 4\lambda^2 - \lambda + 4 = 0$$

or $\quad (\lambda + 1)(\lambda - 1)(\lambda - 4) = 0$

The eigen values are $\lambda = -1, 1, 4$

If $\lambda = -1$, we get $5x + 6y + 6z = 0$, $x + 4y + 2z = 0$

and $\quad -x - 4y - 2z = 0$

Giving $\quad \frac{x}{6} = \frac{y}{2} = \frac{z}{-7}$, $X = \begin{bmatrix} 6 \\ 2 \\ -7 \end{bmatrix}$

If $\lambda = 1$, we get $3x + 6y + 6z = 0$, $x + 2y + 2z = 0$

and $\quad -x - 4y - 4z = 0$

Giving, $\quad \frac{x}{0} = \frac{y}{1} = \frac{z}{-1}$, $X = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

If $\lambda = 4$, we get $0 \cdot x + 6y + 6z = 0$, $x - y + 2z = 0$

and $-x - 4y - 7z = 0$

Giving, $\quad \frac{x}{3} = \frac{y}{1} = \frac{3}{-1}$, $x = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$

Hence, vector are $X = \begin{bmatrix} 6 \\ 2 \\ -7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$.

Example 58. If A and P are the square matrices of the same order and if P be invertible, show that the matrices A and $P^{-1}AP$ have the same characteristic roots.

Sol. Let $\quad P^{-1}AP = B$

$$\therefore \quad |B - \lambda I| = |P^{-1}AP - \lambda I|$$

$$= |P^{-1}AP - P^{-1}\lambda P| \qquad [\because P^{-1}P = I]$$

$$= |P^{-1}(A - \lambda I)P|$$

$$= |P^{-1}||A - \lambda I||P|$$

$$= \frac{1}{|P|}|A - \lambda I||P| = |A - \lambda I|$$

Example 59. Show that the characteristic roots of an idempotent matrix are either zero or unity.

Sol. Let A be an idempotent matrix, then

$$A^2 = A \qquad \ldots(i)$$

If λ be an eigen value of the matrix A corresponding to eigen vector X, so that

$$AX = \lambda X \qquad \ldots(ii)$$

where $\quad X \neq 0$

From Eq. (ii), $\quad A(AX) = A(\lambda X)$

$\Rightarrow \quad (AA)X = \lambda(AX)$

$\Rightarrow \quad A^2X = \lambda(\lambda X)$ [from Eq. (ii)]

$\Rightarrow \quad AX = \lambda^2 X$ [from Eq. (i)]

$\Rightarrow \quad \lambda X = \lambda^2 X$ [from Eq. (ii)]

$\Rightarrow \quad (\lambda - \lambda^2)X = 0$

$\Rightarrow \quad \lambda - \lambda^2 = 0$ $[\because X \neq 0]$

$\therefore \quad \lambda = 0$

or $\quad \lambda = 1$

Example 60. If $3, -2$ are the eigen values of a non-singular matrix A and $|A| = 4$, find the eigen values of adj (A).

Sol. $\because A^{-1} = \frac{\text{adj } A}{|A|}$, if λ is eigen value of A, then λ^{-1} is eigen value of A^{-1}.

Thus, for adj $(A)X = (A^{-1}X)|A| = |A|\lambda^{-1}I$

Thus, eigen value corresponding to $\lambda = 3$ is $\frac{4}{3}$ and

corresponding to $\lambda = -2$ is $\frac{4}{-2} = -2$

Cayley-Hamilton Theorem

Every square matrix A satisfies its characteristic equation $|A - \lambda I| = 0$

i.e., $\quad a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n = 0$

$\therefore$ By Cayley-Hamilton theorem

$$a_0A^n + a_1A^{n-1} + a_2A^{n-2} + \ldots + a_nI = O$$

$$\Rightarrow A^{-1} = -\left\{\frac{a_0}{a_n}A^{n-1} + \frac{a_1}{a_n}A^{n-2} + \frac{a_2}{a_n}A^{n-3} + \ldots + \frac{a_{n-1}}{a_n}I\right\}$$

Example 61. Find the characteristic equation of the matrix $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and hence find its inverse using Cayley-hamilton theorem.

Sol. Characteristic equation is

$$|A - \lambda I| = 0 \Rightarrow \begin{bmatrix} 2-\lambda & 1 \\ 3 & 2-\lambda \end{bmatrix} = 0$$

$$\Rightarrow \quad (2-\lambda)^2 - 3 = 0$$

$$\Rightarrow \quad \lambda^2 - 4\lambda + 1 = 0$$

$\therefore$ By Cayley-hamilton theorem,

$$A^2 - 4A + I = O \text{ or } I = 4A - A^2$$

Multiplying by A^{-1}, we get

$$A^{-1} = 4A^{-1}A - A^{-1}AA$$

$$= 4I - IA = 4I - A$$

$$= 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

Exercise for Session 4

1 If the system of equations $ax + y = 1, x + 2y = 3, 2x + 3y = 5$ are consistent, then a is given by

(a) 0 (b) 1 (c) 2 (d) None of these

2 The system of equations $x + y + z = 2, 2x + y - z = 3, 3x + 2y + \lambda z = 4$ has unique solution if

(a) $\lambda \neq 0$ (b) $-1 < \lambda < 1$ (c) $\lambda = 0$ (d) $-2 < \lambda < 2$

3 The value of a for which the following system of equations $a^3x + (a+1)^3y + (a+2)^3z = 0$, $ax + (a+1)y + (a+2)z = 0, x + y + z = 0$ has a non-trivial solution is equal to

(a) 2 (b) 1 (c) 0 (d) –1

4 The number of solutions of the set of equations

$\frac{2x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0, -\frac{x^2}{a^2} + \frac{2y^2}{b^2} - \frac{z^2}{c^2} = 0, -\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{2z^2}{c^2} = 0$ is

(a) 6 (b) 7 (c) 8 (d) 9

5 The matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is the matrix reflection in the line

(a) $x = 1$ (b) $x + y = 1$ (c) $y = 1$ (d) $x = y$

6 The matrix S is rotation through an angle 45° and G is the reflection about the line $y = 2x$, then $(SG)^2$ is equal to

(a) $7I$ (b) $5I$ (c) $3I$ (d) I

7 If $A = \begin{bmatrix} -2 & 3 \\ -1 & 1 \end{bmatrix}$, then A^3 is equal to

(a) $2A$ (b) A (c) $2I$. (d) I

8 If $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ and the sum of eigen values of A is m and product of eigen values of A is n, then $m + n$ is equal to

(a) 10 (b) 12 (c) 14 (d) 16

9 If $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$ and θ be the angle between the two non-zero column vectors X such that $AX = \lambda X$ for some scalar λ, then $9\sec^2\theta$ is equal to

(a) 13 (b) 12 (c) 11 (d) 10

Shortcuts And Important Results To Remember

1 $|A|$ exists $\Leftrightarrow$ A is square matrix.

2 No element of principal diagonal in a diagonal matrix is zero.

3 If A is a diagonal matrix of order n, then

(a) Number of zeroes in A is $n(n-1)$

(b) If $d_1, d_2, d_3, \ldots, d_n$ are diagonal elements, then

$$A = \text{diag}\{d_1, d_2, d_3, \ldots, d_n\}$$

and $$|A| = d_1 d_2 d_3 \ldots d_n$$

$$A^{-1} = \text{diag}(d_1^{-1}, d_2^{-1}, d_3^{-1}, \ldots, d_n^{-1})$$

(c) Diagonal matrix is both upper and lower triangular.

(d) $\text{diag}\{a_1, a_2, a_3, \ldots, a_n\} \times \text{diag}\{b_1, b_2, b_3, \ldots, b_n\}$

$$= \text{diag}\{a_1b_1, a_2b_2, a_3b_3, \ldots, a_nb_n\}$$

4 If $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, then $A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ and $B^k = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$, $\forall\, k \in N$.

5 If A and B are square matrices of order n, then

(a) $|kA| = k^n |A|$, k is scalar

(b) $|AB| = |A||B|$

(c) $|kAB| = k^n |A||B|$, k is scalar

(d) $|AB| = |BA|$

(e) $|A^T| = |A| = |A^\theta|$, where A^θ is conjugate transpose matrix of A

(f) $|A|^m = |A^m|$, $m \in N$

6 Minimum number of zeroes in a triangular matrix is given by $\frac{n(n-1)}{2}$, where n is order of matrix.

7 If A is a skew-symmetric matrix of odd order, then $|A| = 0$ and of even order is a non-zero perfect square.

8 If A is involutory matrix, then

(a) $|A| = \pm 1$

(b) $\frac{1}{2}(I + A)$ and $\frac{1}{2}(I - A)$ are idempotent and $\frac{1}{2}(I + A).\frac{1}{2}(I - A) = 0$

9 If A is orthogonal matrix, then $|A| = \pm 1$

10 To obtain an orthogonal matrix B from a skew-symmetric matrix A, then

$B = (I - A)^{-1}(I + A)$ or $B = (I - A)(I + A)^{-1}$

11 The sum of two orthogonal matrices is not orthogonal while the sum of two symmetric (skew-symmetric) matrices is symmetric (skew-symmetric)

12 The product of two orthogonal matrices is orthogonal while the product of two symmetric (skew-symmetric) matrices need not be symmetric (skew-symmetric)

13 The adjoint of a square matrix of order 2 can be easily obtained by interchanging the principal diagonal elements and changing the sign of the other diagonal.

i.e., If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\text{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

14 If $|A| \neq 0$, then $|A^{-1}| = \frac{1}{|A|}$.

15 If A and B are invertible matrices such that $AB = C$, then $|B| = \frac{|C|}{|A|}$.

16 Commutative law does not necessarily hold for matrices.

17 If $AB = -BA$, then matrices A and B are called anti-commutative matrices.

18 If $AB = O$, it is not necessary that atleast one of the matrix should be zero matrix.

For example, If $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$, then $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ while neither A nor B is the null matrix.

19 If A, B and C are invertible matrices, then

(a) $(AB)^{-1} = B^{-1}A^{-1}$

(b) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

20 If B is a non-singular matrix and A is any square matrix, then $\det(B^{-1}AB) = \det(A)$

21 If A is a non-singular square matrix of order n, then $\text{adj}(\text{adj } A) = |A|^{n-2} A$

22 If A is a non-singular square matrix of order n, then

$$\underbrace{|\text{adj}(\text{adj}(\text{adj} \ldots (\text{adj}(\text{adj } A))))|}_{m \text{ times}} = |A|^{(n-1)^m}$$

23 If $A = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}$; $A^m = O$, $\forall\, m \geq 2$

$\therefore \quad (A + I)^n = I + nA$

24 If A and B are two symmetric matrices, then $A \pm B$, $AB + BA$ are symmetric matrices and $AB - BA$ is a skew-symmetric matrix.

25 If A and B are two square matrices of order n and λ be a scalar, then

(i) $\text{Tr}(\lambda A) = \lambda \text{Tr}(A)$

(ii) $\text{Tr}(A \pm B) = \text{Tr}(A) \pm \text{Tr}(B)$

(iii) $\text{Tr}(AB) = \text{Tr}(BA)$

(iv) $\text{Tr}(A) = \text{Tr}(A')$

(v) $\text{Tr}(I_n) = n$

(vi) $\text{Tr}(O) = 0$

(vii) $\text{Tr}(AB) \neq \text{Tr}(A) \cdot \text{tr}(B)$

contd...

26 If rank of a matrix A is denoted by $\rho(A)$, then

(i) $\rho(A) = 0$, if A is zero matrix.

(ii) $\rho(A) = 1$, if every element of A is same.

(iii) If A and B are square matrices of order n each and $\rho(A) = \rho(B) = n$, then $\rho(AB) = n$

(iv) If A is a square matrix of order n and $\rho(A) = n - 1$, then $\rho(\text{adj } A) = 1$ and if $\rho(A) < n - 1$, then $\rho(\text{adj } A) = 0$

27 System of planes

$$a_{11}x + a_{12}y + a_{13}z = b_1,$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

and $$a_{31}x + a_{32}y + a_{33}z = b_3$$

Augmented matrix $C = [A : B]$ and if Rank of $A = r$ and Rank of $C = s$, then

(i) If $r = s = 1$, then planes are coincident

(ii) If $r = 1$, $s = 2$, then planes are parallel

(iii) If $r = s = 2$, then planes intersect along a single straight line

(iv) If $r = 2$, $s = 3$, then planes form a triangular prism

(v) If $r = s = 3$, then planes meet at a single point

28 If P is an orthogonal matrix, then $\det(P) = \pm 1$

(i) P represents a reflection about a line, then $\det(P) = -1$.

(ii) P represents a rotation about a point, then $\det(P) = 1$.

29 **Cayley-Hamilton Theorem** : Every matrix satisfies its characteristic equation.

For Example, Let A be a square matrix, then $|A - \lambda I| = 0$ is the characteristic equation for A.

If $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$ is the characteristic equation for A, then $A^3 - 6A^2 + 11A - 6I = O$. Roots of characteristic equation for A are called eigen values of A or characteristic roots of A or latent roots of A. If λ is a characteristic root of A, then λ^{-1} is characteristic root of A^{-1}.

JEE Type Solved Examples :
Single Option Correct Type Questions

▪ This section contains **10 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

● ***Ex. 1*** *If A is a square matrix of order 2 such that $A\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}-1\\2\end{bmatrix}$ and $A^2\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$. The sum of elements and product of elements of A are S and P, then $S+P$ is*

(a) -1 (b) 2 (c) 4 (d) 5

Sol. (*d*) Let $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$

From first part, $A\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}-1\\2\end{bmatrix}$ ...(i)

$\Rightarrow$ $\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}-1\\2\end{bmatrix}$

or $a-b=-1$ ...(ii)

and $c-d=2$ ...(iii)

From second part,

$$A^2\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}\Rightarrow A\left(A\begin{bmatrix}1\\-1\end{bmatrix}\right)=\begin{bmatrix}1\\0\end{bmatrix}$$

From Eq. (i), we get

$$A\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}\Rightarrow\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$$

or $-a+2b=1$ ...(iv)

and $-c+2d=0$ ...(v)

From Eqs. (ii) and (iv), we get

$$a=-1,\ b=0$$

and from Eqs. (iii) and (v), we get

$$c=4,\ d=2$$

$\therefore$ $S=a+b+c+d=5$

and $P=abcd=0$

Hence, $S+P=5$

● ***Ex. 2*** *If P is an orthogonal matrix and $Q=PAP^T$ and $B=P^TQ^{1000}P$, then B^{-1} is, where A is involutory matrix*

(a) A (b) A^{1000} (c) I (d) None of these

Sol. (*c*) Given, P is orthogonal

$\therefore$ $P^TP=I$ ...(i)

and $Q=PAP^T$ ...(ii)

Now, $B=P^TQ^{1000}P=P^T(PAP^T)^{1000}P$ [from Eq. (ii)]

$$=P^TPAP^T(PAP^T)^{999}P$$
$$=IAP^T\cdot PAP^T(PAP^T)^{998}P$$
$$=AIAP^T(PAP^T)^{998}P$$
$$=A^2P^TPAP^T(PAP^T)^{997}P$$
$$=A^2IAP^T(PAP^T)^{997}P$$
$$=A^3P^T(PAP^T)^{997}P$$
$$\ldots\quad\ldots\quad\ldots$$
$$=A^{1000}P^TP=A^{1000}=I\quad[\because A\text{ is involutory}]$$

Hence, $B^{-1}=I^{-1}=I$

● ***Ex. 3*** *If A is a diagonal matrix of order 3×3 is commutative with every square matrix of order 3×3 under multiplication and trace $(A)=12$, then*

(a) $|A|=64$ (b) $|A|=16$

(c) $|A|=12$ (d) $|A|=4$

Sol. (*a*) A diagonal matrix is commutative with every square matrix, if it is scalar matrix so every diagonal element is 4.

$\therefore$ $A=\begin{bmatrix}4 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\end{bmatrix}$

$\Rightarrow$ $|A|=4\cdot4\cdot4=64$

● ***Ex. 4*** *If $A=[a_{ij}]_{4\times4}$, such that $a_{ij}=\begin{cases}2, & \text{when } i=j\\ 0, & \text{when } i\neq j\end{cases}$, then $\left\{\dfrac{\det(\text{adj}(\text{adj}\,A))}{7}\right\}$ is [when $\{\cdot\}$ represents fractional part function]*

(a) $\frac{1}{7}$ (b) $\frac{2}{7}$ (c) $\frac{3}{7}$ (d) $\frac{4}{7}$

Sol. (*a*) $\because$ $A=\begin{bmatrix}2 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 2\end{bmatrix}$

$\therefore$ $|A|=\begin{vmatrix}2 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 2\end{vmatrix}=2^4=16$

$\because$ $\det(\text{adj}(\text{adj}\,A))=|\text{adj}(\text{adj}\,A)|=|A|^{3^2}=|A|^9$

$$=(2^4)^9=2^{36}=(2^3)^{12}=(1+7)^{12}$$
$$=1+{}^{12}C_1(7)+{}^{12}C_2(7)^2+\ldots$$

$$\frac{\det(\text{adj}(\text{adj}\,A))}{7}=\frac{1}{7}+\text{Positive integer}$$

$\therefore$ $\left\{\dfrac{\det(\text{adj}(\text{adj}\,A))}{7}\right\}=\dfrac{1}{7}$

● ***Ex. 5*** *If* $A=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$ *and* $\det(A^n - I) = 1-\lambda^n$, $n \in N$, *then the value of* λ, *is*

(a) 1 (b) 2 (c) 3 (d) 4

Sol. (*b*) $\because$ $A=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$

$\therefore$ $A^2=\begin{bmatrix}2 & 2\\ 2 & 2\end{bmatrix}=2\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}=2A$

$\Rightarrow$ $A^3 = A^2 \cdot A = 2A^2 = 2^2 A$

Similarly, $A^n = 2^{n-1}A$

$\therefore$ $A^n - I = \begin{bmatrix}2^{n-1} & 2^{n-1}\\ 2^{n-1} & 2^{n-1}\end{bmatrix} - \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$$= \begin{bmatrix}2^{n-1}-1 & 2^{n-1}\\ 2^{n-1} & 2^{n-1}-1\end{bmatrix}$$

$\Rightarrow$ $\det(A^n - I) = (2^{n-1}-1)^2 - (2^{n-1})^2$

$= 1 - 2^n = 1 - \lambda^n$ [given]

$\therefore$ $\lambda = 2$

● ***Ex. 6*** *If* $A=\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}$ *and* $f(x)=\dfrac{1+x}{1-x}$, *then* $f(A)$ *is*

(a) $\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$ (b) $\begin{bmatrix}2 & 2\\ 2 & 2\end{bmatrix}$

(c) $\begin{bmatrix}-1 & -1\\ -1 & -1\end{bmatrix}$ (d) None of these

Sol. (*c*) $\because$ $f(x)=\dfrac{1+x}{1-x}$

$\Rightarrow$ $(1-x)f(x) = 1+x$

$\Rightarrow$ $(I-A)f(A) = (I+A)$

$\Rightarrow$ $f(A) = (I-A)^{-1}(I+A)$

$$= \left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}\right)^{-1}\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}\right)$$

$$= \begin{bmatrix}0 & -2\\ -2 & 0\end{bmatrix}^{-1}\begin{bmatrix}2 & 2\\ 2 & 2\end{bmatrix} = -\frac{1}{4}\begin{bmatrix}0 & 2\\ 2 & 0\end{bmatrix}\begin{bmatrix}2 & 2\\ 2 & 2\end{bmatrix}$$

$$= -\frac{1}{4}\begin{bmatrix}4 & 4\\ 4 & 4\end{bmatrix} = \begin{bmatrix}-1 & -1\\ -1 & -1\end{bmatrix}$$

● ***Ex. 7*** *The number of solutions of the matrix equation* $X^2=\begin{bmatrix}1 & 1\\ 2 & 3\end{bmatrix}$ *is*

(a) more than 2 (b) 2

(c) 0 (d) 1

Sol. (*a*) Let $X=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$

$\therefore$ $X^2=\begin{bmatrix}a^2+bc & b(a+d)\\ c(a+d) & bc+d^2\end{bmatrix}=\begin{bmatrix}1 & 1\\ 2 & 3\end{bmatrix}$ [given]

$\Rightarrow$ $a^2 + bc = 1, b(a+d) = 1,$

$c(a+d) = 2$ and $bc + d^2 = 3$

$\Rightarrow$ $d^2 - a^2 = 2$

$\Rightarrow$ $d - a = \dfrac{2}{d+a} = 2b$ and $d + a = \dfrac{1}{b}$

$\therefore$ $2d = 2b + \dfrac{1}{b}$ and $2a = \dfrac{1}{b} - 2b$

Also, $c = 2b$

Now, from $bc + d^2 = 3$

$\Rightarrow$ $2b^2 + \left(b + \dfrac{1}{2b}\right)^2 = 3 \Rightarrow 3b^2 + \dfrac{1}{4b^2} - 2 = 0$

$\Rightarrow$ $12b^4 - 8b^2 + 1 = 0$

or $(6b^2 - 1)(2b^2 - 1) = 0$

$\Rightarrow$ $b = \pm\dfrac{1}{\sqrt{6}}$ or $b = \pm\dfrac{1}{\sqrt{2}}$

Therefore, matrices are

$$\begin{bmatrix}0 & \frac{1}{\sqrt{2}}\\ \sqrt{2} & \sqrt{2}\end{bmatrix}, \begin{bmatrix}0 & -\frac{1}{\sqrt{2}}\\ -\sqrt{2} & -\sqrt{2}\end{bmatrix}, \begin{bmatrix}\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}}\\ \frac{2}{\sqrt{6}} & \frac{4}{\sqrt{6}}\end{bmatrix} \text{ and } \begin{bmatrix}-\frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}}\\ -\frac{2}{\sqrt{6}} & -\frac{4}{\sqrt{6}}\end{bmatrix}.$$

● ***Ex. 8*** *For a matrix* $A=\begin{bmatrix}1 & 2r-1\\ 0 & 1\end{bmatrix}$, *the value of* $\prod_{r=1}^{50}\begin{bmatrix}1 & 2r-1\\ 0 & 1\end{bmatrix}$ *is equal to*

(a) $\begin{bmatrix}1 & 100\\ 0 & 1\end{bmatrix}$ (b) $\begin{bmatrix}1 & 4950\\ 0 & 1\end{bmatrix}$ (c) $\begin{bmatrix}1 & 5050\\ 0 & 1\end{bmatrix}$ (d) $\begin{bmatrix}1 & 2500\\ 0 & 1\end{bmatrix}$

Sol. (*d*) $\prod_{r=1}^{50}\begin{bmatrix}1 & 2r-1\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 1+3+5+\ldots+99\\ 0 & 1\end{bmatrix}$

$$=\begin{bmatrix}1 & (50)^2\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 2500\\ 0 & 1\end{bmatrix}$$

● ***Ex. 9*** *If* $A_1, A_2, A_3, \ldots, A_{2n-1}$ *are n skew-symmetric matrices of same order, then* $B=\sum_{r=1}^{n}(2r-1)(A_{2r-1})^{2r-1}$ *will be*

(a) symmetric
(b) skew-symmetric
(c) neither symmetric nor skew-symmetric
(d) data not adequate

Sol. $(b)\ \because\ B = A_1 + 3A_3^3 + 5A_5^5 + \ldots + (2n-1)(A_{2n-1})^{2n-1}$

$$\therefore\ B^T = (A_1 + 3A_3^3 + 5A_5^5 + \ldots + (2n-1)(A_{2n-1})^{2n-1})^T$$

$$= A_1^T + 3(A_3^T)^3 + 5(A_5^T)^5 + \ldots + (2n-1)(A_{2n-1}^T)^{2n-1}$$

$$= -A_1 + 3(-A_3)^3 + 5(-A_5)^5 + \ldots + (2n-1)(-A_{2n-1})^{2n-1}$$

$$= -(A_1 + 3A_3^3 + 5A_5^5 + \ldots + (2n-1)A_{2n-1}^{2n-1})$$

$$= -B$$

Hence, B is skew-symmetric.

● ***Ex. 10*** *Elements of a matrix A of order* 10×10 *are defined as* $a_{ij} = \omega^{i+j}$ *(where* ω *is cube root of unity), then trace* (A) *of the matrix is*

(a) 0 (b) 1 (c) 3 (d) None of these

Sol. $(d)\ \text{tr}(A) = \sum_{i=j=1}^{10} a_{ij} = \sum_{i=j=1}^{10} \omega^{i+j} = \sum_{i=1}^{10} \omega^{2i}$

$$= \omega^2 + \omega^4 + \omega^6 + \omega^8 + \ldots + \omega^{20}$$

$$= (\omega^2 + \omega + 1) + (\omega^2 + \omega + 1) + (\omega^2 + \omega + 1) + \omega^{20}$$

$$= 0 + 0 + 0 + \omega^2 = \omega^2$$

JEE Type Solved Examples :
More than One Correct Option Type Questions

■ This section contains **5 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of **which more than one** may be correct.

● ***Ex. 11*** *If* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *(where* $bc \neq 0$*) satisfies the equations* $x^2 + k = 0$*, then*

(a) $a + d = 0$ (b) $k = -|A|$

(c) $k = |A|$ (d) None of these

Sol. (a, c) We have, $A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix}$

As A satisfies $x^2 + k = 0$, therefore

$$A^2 + kI = 0$$

$$\Rightarrow \quad \begin{bmatrix} a^2 + bc + k & (a+d)b \\ (a+d)c & bc + d^2 + k \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad a^2 + bc + k = 0, (a+d)b = 0,$$

$$(a+d)c = 0 \text{ and } bc + d^2 + k = 0$$

As $bc \neq 0 \Rightarrow b \neq 0, c \neq 0$

So, $a + d = 0 \ \Rightarrow\ a = -d$

Also, $k = -(a^2 + bc) = -(-ad + bc) = (ad - bc) = |A|$

● ***Ex. 12*** *If* $A = [a_{ij}]_{n \times n}$ *and f is a function, we define* $f(A) = [f(a_{ij})]_{n \times n}$*. Let* $A = \begin{bmatrix} \frac{\pi}{2} - \theta & \theta \\ -\theta & \frac{\pi}{2} - \theta \end{bmatrix}$*, then*

(a) $\sin A$ is invertible (b) $\sin A = \cos A$

(c) $\sin A$ is orthogonal (d) $\sin 2A = 2 \sin A \cos A$

Sol. $(a, c)\ \ \sin A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ and $\cos\theta = \begin{bmatrix} \sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$

$$\therefore \quad |\sin A| = \cos^2\theta + \sin^2\theta = 1 \neq 0$$

Hence, $\sin A$ is invertible.

$$\text{Also, } (\sin A)(\sin A)^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence, $\sin A$ is orthogonal.

$$\text{Also, } 2\sin A \cos A = 2\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$$

$$= 2\begin{bmatrix} \sin 2\theta & 1 \\ \cos 2\theta & 0 \end{bmatrix} \neq \sin 2A$$

● ***Ex. 13*** *Let A and B are two square idempotent matrices such that* $AB \pm BA$ *is a null matrix, the value of* $\det(A - B)$ *can be equal*

(a) -1 (b) 0

(c) 1 (d) 2

Sol. (a, b, c)

$$\because (A - B)^2 = A^2 - AB - BA + B^2$$

$$= A + B \qquad [\because AB + BA = 0 \text{ and } A^2 = A, B^2 = B]$$

$$\therefore \quad |A - B|^2 = |A + B| \qquad \ldots(i)$$

$$\text{and } (A + B)^2 = A^2 + AB + BA + B^2$$

$$= A + B \quad [\because AB + BA = 0 \text{ and } A^2 = A, B^2 = B]$$

$$\Rightarrow \quad |A + B|^2 = |A + B|$$

$$\Rightarrow \quad |A + B|(|A + B| - 1) = 0$$

$$\therefore \quad |A + B| = 0, 1$$

From Eq. (i),

$$|A-B|^2 = 0, 1 \Rightarrow |A-B| = 0, \pm 1$$

or $$\det(A-B) = 0, -1, 1$$

● ***Ex. 14*** *If $AB = A$ and $BA = B$, then*

(a) $A^2B = A^2$ (b) $B^2A = B^2$

(c) $ABA = A$ (d) $BAB = B$

Sol. (*a, b, c, d*)

We have, $A^2B = A(AB) = A \cdot A = A^2$,

$B^2A = B(BA) = BB = B^2$,

$ABA = A(BA) = AB = A$, $BAB = B(AB) = BA = B$

● ***Ex. 15*** *If A is a square matrix of order 3 and I is an Identity matrix of order 3 such that $A^3 - 2A^2 - A + 2I = 0$, then A is equal to*

(a) I (b) $2I$ (c) $\begin{bmatrix} 2 & -1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Sol. (*a, b, d*) It is clear that $A = I$ and $A = 2I$ satisfy the given equaion $A^3 - 2A^2 - A + 2I = 0$ and the characteristic equation of the matrix in (c) is

$$\begin{vmatrix} 2-\lambda & -1 & 2 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda^3 - 2\lambda^2 + \lambda - 2 = 0,$$

giving $$A^3 - 2A^2 + A - 2I = 0$$

$$\neq A^3 - 2A^2 - A + 2I = 0$$

and the characteristic equation of the matrix in (d) is

$$\begin{vmatrix} 2-\lambda & 1 & -2 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda^3 - 2\lambda^2 - \lambda + 2 = 0,$$

giving $$A^3 - 2A^2 - A + 2I = 0$$

JEE Type Solved Examples : Passage Based Questions

■ This section contains **2 solved passages**. Base upon each of the passage **3 multiple choice question** have to be answered. Each of these question has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I

(Ex. Nos. 16 to 18)

If $A_0 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ and $B_0 = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$ and

$B_n = \text{adj}(B_{n-1}), n \in N$ and I is an identity matrix of order 3.

16. $\det(A_0 + A_0^2 B_0^2 + A_0^3 + A_0^4 B_0^4 + \ldots$ upto 12 terms) is equal to

(a) 1200 (b) −960

(c) 0 (d) −9600

17. $B_2 + B_3 + B_4 + \ldots + B_{50}$ is equal to

(a) B_0 (b) $7B_0$

(c) $49B_0$ (d) $49I$

18. For a variable matrix X, the equation $A_0 X = B_0$ will have

(a) unique solution

(b) infinite solution

(c) finitely many solution

(d) no solution

Sol. (Ex. Nos 16 to 18)

$$\because \quad A_0 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \Rightarrow |A_0| = 0$$

and $$\text{adj}\, B_0 = \begin{bmatrix} -4 & 1 & 4 \\ -3 & 0 & 4 \\ -3 & 1 & 3 \end{bmatrix}^T = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix} = B_0$$

$$\because \quad B_n = \text{adj}(B_{n-1}), n \in N$$

$$\therefore \quad B_1 = \text{adj}(B_0) = B_0$$

$$\Rightarrow \quad B_2 = \text{adj}(B_1) = \text{adj}(B_0) = B_0,$$

Similarly $B_3 = B_0, B_4 = B_0, \ldots$

$$\therefore \quad B_n = B_0 \ \forall\ n \in N$$

16. (*c*) $\det(A_0 + A_0^2 B_0^2 + A_0^3 + A_0^4 B_0^4 + \ldots$ upto 12 terms) $= \det$

$\{A_0 (I + A_0 B_0^2 + A_0^2 + A_0^3 B_0^4 + \ldots \text{upto} 12 \text{ terms})\}$

$= |A_0|(I + A_0 B_0^2 + A_0^2 + A_0^3 B_0^4 + \ldots$ upto 12 terms)

$= 0 \qquad [\because |A_0| = 0]$

17. (*c*) $B_2 + B_3 + B_4 + \ldots + B_{50} = B_0 + B_0 + B_0 + \ldots + B_0 = 49B_0$

18. (*d*) $\because |A_0| = 0$

$\Rightarrow$ A_0^{-1} is not possible.

Hence, system of equation $A_0 X = B_0$ has no Sol.

Passage II
(Ex. Nos. 19 to 21)

Let $A=\begin{bmatrix}1&0&0\\1&0&1\\0&1&0\end{bmatrix}$ satisfies $A^n=A^{n-2}+A^2-I$ for $n\geq 3$ and consider a matrix $\underset{3\times 3}{U}$ with its columns as U_1, U_2, U_3, such that

$$A^{50}U_1=\begin{bmatrix}1\\25\\25\end{bmatrix},\ A^{50}U_2=\begin{bmatrix}0\\1\\0\end{bmatrix} \text{ and } A^{50}U_3=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

19. The value of $|A^{50}|$ equals

(a) -1 (b) 0 (c) 1 (d) 25

20. Trace of A^{50} equals

(a) 0 (b) 1 (c) 2 (d) 3

21. The value of $|U|$ equals

(a) -1 (b) 0 (c) 1 (d) 2

Sol. (Ex. Nos. 19 to 21)

$\because \quad A^n=A^{n-2}+A^2-I \Rightarrow A^{50}=A^{48}+A^2-I$

Further, $A^{48}=A^{46}+A^2-I$

$A^{46}=A^{44}+A^2-I$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$A^4=A^2+A^2-I^4$

On adding all, we get

$$A^{50}=25A^2-24I \qquad ...(i)$$

19. (*c*) $|A^{50}|=|A|^{50}=\begin{vmatrix}1&0&0\\1&0&1\\0&1&0\end{vmatrix}^{50}=(-1)^{50}=1$

20. (*d*) $\because A^2=\begin{bmatrix}1&0&0\\1&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}1&0&0\\1&0&1\\0&1&0\end{bmatrix}=\begin{bmatrix}1&0&0\\1&1&0\\1&0&1\end{bmatrix}$

$$\therefore\ A^{50}=25A^2-24I=\begin{bmatrix}25&0&0\\25&25&0\\25&0&25\end{bmatrix}-\begin{bmatrix}24&0&0\\0&24&0\\0&0&24\end{bmatrix}$$ [from Eq. (i)]

$$=\begin{bmatrix}1&0&0\\25&1&0\\25&0&1\end{bmatrix} \qquad ...(ii)$$

Hence, trace of $A^{50}=1+1+1=3$

21. (*c*) Let $U_1=\begin{bmatrix}x\\y\\z\end{bmatrix}$

Given, $A^{50}U_1=\begin{bmatrix}1\\25\\25\end{bmatrix}\Rightarrow\begin{bmatrix}1&0&0\\25&1&0\\25&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\25\\25\end{bmatrix}$ [from Eq. (ii)]

$\Rightarrow \begin{bmatrix}x\\25x+y\\25x+z\end{bmatrix}=\begin{bmatrix}1\\25\\25\end{bmatrix}$, we get $x=1, y=20$ and $z=0$

$\therefore \quad U_1=\begin{bmatrix}1\\0\\0\end{bmatrix}$, similarly $U_2=\begin{bmatrix}0\\1\\0\end{bmatrix}$

and $U_3=\begin{bmatrix}0\\0\\1\end{bmatrix} \Rightarrow U=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=I$

$\therefore \quad |U|=1$

JEE Type Solved Examples :
Single Integer Answer Type Questions

- This section contains **2 examples.** The answer to each example is a **single digit integer** ranging from **0** to **9** (both inclusive).

● ***Ex. 22*** *Let A be a* 3×3 *diagonal matrix which commutes with every* 3×3 *matrix. If* $\det(A)=8$*, then tr A is*

Sol. (6) Let $A=\begin{bmatrix}\alpha&0&0\\0&\beta&0\\0&0&\gamma\end{bmatrix}$

$$\therefore\ \begin{bmatrix}\alpha&0&0\\0&\beta&0\\0&0&\gamma\end{bmatrix}\begin{bmatrix}a&h&g\\h&b&f\\g&f&c\end{bmatrix}=\begin{bmatrix}a&h&g\\h&b&f\\g&f&c\end{bmatrix}\begin{bmatrix}\alpha&0&0\\0&\beta&0\\0&0&\gamma\end{bmatrix}$$

$\Rightarrow \quad \alpha=\beta=\gamma$

$\therefore \quad A=\begin{bmatrix}\alpha&0&0\\0&\alpha&0\\0&0&\alpha\end{bmatrix}$

$\Rightarrow \quad \det(A)=\alpha^3=8$ [given]

$\therefore \quad \alpha=2$

$\therefore \quad A=\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}$

$\Rightarrow \quad \text{tr } A=2+2+2=6$

● ***Ex. 23*** *Let A and B be two non-singular matrices such that $A \neq I$, $B^3 = I$ and $AB = BA^2$, where I is the identity matrix, the least value of k such that $A^k = I$ is*

Sol. (7) Given, $AB = BA^2 \Rightarrow B = A^{-1}BA^2 \Rightarrow B^3 = I$

$\Rightarrow (A^{-1}BAA)(A^{-1}BAA)(A^{-1}BAA) = I$

$\Rightarrow (A^{-1}BA)(BA)(BAA) = I \quad [\because A^{-1}A = I]$

$\Rightarrow A^{-1}B(AB)(AB)AA = I$

$\Rightarrow A^{-1}B(BA^2)(BA^2)AA = I \quad [\because AB = BA^2]$

$\Rightarrow A^{-1}BBA(AB)A^4 = I$

$\Rightarrow A^{-1}BBA(BA^2)A^4 = I \quad [\because AB = BA^2]$

$\Rightarrow A^{-1}BB(AB)A^6 = I$

$\Rightarrow A^{-1}BB(BA^2)A^6 = I \quad [\because AB = BA^2]$

$\Rightarrow A^{-1}B^3A^8 = I$

$\Rightarrow (A^{-1}I)A^8 = I \quad . \quad [\because B^3 = I]$

$\Rightarrow A^{-1}A^8 = I$

$\Rightarrow A^7 = I = A^k \quad [\because A^k = I]$

$\Rightarrow A^k = A^7$

$\therefore$ Least value of k is 7.

JEE Type Solved Examples : Matching Type Questions

■ This section contains **2 examples.** Example 24 have three statements (A, B and C) given in **Column I** and four statements (p, q, r and s) in **Column II** and example 25 have three statements (A, B and C) given in **Column I** and five statements (p, q, r, s and t). In **Column II** any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II.**

● ***Ex. 24***

	Column I		Column II
(A)	If A is a square matrix of order 3 and $\det(A) = 3$, then $\det(6A^{-1})$ is divisible by	(p)	3
(B)	If A is a square matrix of order 3 and $\det(A) = \frac{1}{4}$, then det [adj (adj (2A))] is divisible by	(q)	4
(C)	If A and B are square matrices of odd order and $(A+B)^2 = A^2 + B^2$, if det $(A) = 2$, then det(B) is divisible by	(r)	5
		(s)	6

Sol. (A) → (p, q, s); (B) →(q); (C) →(p, q, r, s)

(A) $\det(6A^{-1}) = 6^3 \det(A^{-1}) = \dfrac{216}{\det(A)} = \dfrac{216}{3} = 72$

(B) $\det[\text{adj (adj } (2A))] = [\det(2A)]^4 = [2^3 \det(A)]^4$

$= 2^{12}[\det(A)]^4$

$= 2^{12}\left(\dfrac{1}{4}\right)^4 = 2^4 = 16$

(C) $\because \quad (A+B)^2 = A^2 + AB + BA + B^2$

$\Rightarrow A^2 + B^2 = A^2 + AB + BA + B^2 \quad [\because (A+B)^2 = A^2 + B^2]$

$\Rightarrow AB + BA = O \Rightarrow AB = -BA$

$\therefore \det(AB) = \det(-BA) = -\det(BA)$

$\Rightarrow \det(A)\cdot\det(B) = -\det(B)\cdot\det(A)$

$\Rightarrow 2\det(A)\cdot\det(B) = 0 \Rightarrow 4\det(B) = 0 \quad [\because \det(A) = 2]$

$\therefore \det(B) = 0$

● ***Ex. 25***

	Column I		Column II
(A)	If $\begin{bmatrix}1 & 2 & a\\ 0 & 1 & 4\\ 0 & 0 & 1\end{bmatrix}^n = \begin{bmatrix}1 & 18 & 2007\\ 0 & 1 & 36\\ 0 & 0 & 1\end{bmatrix}$, then $(n+a)$ is divisible by	(p)	4
(B)	If A is a square matrix of order 3 such that $\lvert A\rvert = a$, $B = \text{adj}(A)$ and $\lvert B\rvert = b$, then $(ab^2 + a^2b + 1)\lambda$ is divisible by, where $\frac{1}{2}\lambda = \frac{a}{b} + \frac{a^2}{b^3} + \frac{a^3}{b^5} + \ldots$ upto ∞ and $a = 3$	(q)	6
(C)	Let $A = \begin{bmatrix}a & b & c\\ p & q & r\\ 1 & 1 & 1\end{bmatrix}$ and $B = A^2$. If $(a-b)^2 + (p-q)^2 = 25$, $(b-c)^2 + (q-r)^2 = 36$ and $(c-a)^2 + (r-p)^2 = 49$, then det $\left(\frac{B}{2}\right)$ is divisible by	(r)	10
		(s)	12
		(t)	15

Sol. (A) → (p, r); (B) →(t); (C) → (q, s)

(A) $\because \quad A = \begin{bmatrix}1 & 2 & a\\ 0 & 1 & 4\\ 0 & 0 & 1\end{bmatrix}$

$\therefore \quad A^2 = \begin{bmatrix}1 & 2 & a\\ 0 & 1 & 4\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & a\\ 0 & 1 & 4\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 4 & 2a+8\\ 0 & 1 & 8\\ 0 & 0 & 1\end{bmatrix}$

$$\Rightarrow \quad A^3 = \begin{bmatrix} 1 & 4 & 2a+8 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & a \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 3a+24 \\ 0 & 1 & 12 \\ 0 & 0 & 1 \end{bmatrix}$$

Similarly, we get

$$A^n = \begin{bmatrix} 1 & 2n & na + 8\sum_{r=0}^{n-1} r \\ 0 & 1 & 4n \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 18 & 2007 \\ 0 & 1 & 36 \\ 0 & 0 & 1 \end{bmatrix} \text{ [given]}$$

$\Rightarrow \quad 2n = 18 \Rightarrow n = 9$

$\therefore \quad na + 8\sum_{r=0}^{n-1} r = 2007 \Rightarrow 9a + 8\sum_{r=0}^{8} r = 2007$

$\Rightarrow \quad 9a + 8\cdot\left(\frac{8\times 9}{2}\right) = 2007 \Rightarrow 9a = 2007 - 288 = 1719$

$\therefore \quad a = 191$

Hence, $n + a = 9 + 191 = 200$

(B) $B = \text{adj } A$

$\Rightarrow \quad b = |B| = |\text{adj}\, A| = |A|^2 = a^2 = 9 \Rightarrow a = 3, b = 9$

and $\quad \frac{1}{2}\lambda = \frac{3}{9} + \frac{3^2}{9^3} + \frac{3^3}{9^5} + \ldots + \infty$

$$= \frac{1}{3} + \frac{1}{81} + \frac{1}{27\times 81} + \ldots + \infty = \frac{\frac{1}{3}}{1 - \frac{1}{27}} = \frac{9}{26}$$

$$\Rightarrow \qquad \lambda = \frac{9}{13}$$

Now, $(ab^2 + a^2b + 1)\lambda = (3\times 81 + 9\times 9 + 1)\times \frac{9}{13} = 225$

(C) $\det(A) = \begin{vmatrix} a & b & c \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a & p & 1 \\ b & q & 1 \\ c & r & 1 \end{vmatrix} = 2\times\frac{1}{2}\begin{vmatrix} a & p & 1 \\ b & q & 1 \\ c & r & 1 \end{vmatrix}$

$= 2 \times$ Area of the triangle with vertices $(a, p), (b, q)$ and (c, r) with sides 5, 6, 7

$= 2\times\sqrt{s(s-a)(s-b)(s-c)} = 2\times 6\sqrt{6} = 12\sqrt{6}$

Hence, $\det\left(\frac{B}{2}\right) = \left(\frac{1}{2}\right)^3 \det(B) = \frac{1}{8}\det(A^2)$

$$= \frac{1}{8}(\det A)^2 = \frac{1}{8}(12\sqrt{6})^2 = 108$$

JEE Type Solved Examples :
Statement I and II Type Questions

■ **Direction** example numbers 26 and 27 are Assertion-Reason type examples. Each of these examples contains two statements:

Statement-1 (Assertion) and **Statement-2** (Reason)

Each of these examples also has four alternative choices, **ONLY ONE** of which is the correct answer. You have to select the correct choice as given below.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement-1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

● ***Ex. 26*** **Statement-1** *A is singular matrix of order $n \times n$, then adj A is singular.*

Statement-2 $|\text{adj}\, A| = |A|^{n-1}$

Sol. (*d*) If A is non-singular matrix of order $n \times n$, then

$$|\text{adj}\, A| = |A|^{n-1}$$

Hence, Statement-1 is false and Statement-2 is true.

● ***Ex. 27*** **Statement-1** *If A and B are two matrices such that $AB = B$, $BA = A$, then $A^2 + B^2 = A + B$.*

Statement-2 *A and B are idempotent matrices, then $A^2 = A$, $B^2 = B$.*

Sol. (*b*) $\because AB = B$

$\Rightarrow \quad B(AB) = B\cdot B$

$\Rightarrow \quad (BA)\,B = B^2$ [by associative law]

$\Rightarrow \quad AB = B^2$ [$\because BA = A$]

$\Rightarrow \quad B = B^2$ [$\because AB = B$]

and $\quad BA = A$

$\Rightarrow \quad A(BA) = A\cdot A$

$\Rightarrow \quad (AB)\,A = A^2$ [by associative law]

$\Rightarrow \quad BA = A^2$ [$\because AB = B$]

$\Rightarrow \quad A = A^2$ [$\because BA = A$]

Hence, $\therefore \quad A^2 + B^2 = A + B$

Here, both statements are true and Statement-2 is not a correct explanation for Statement-1.

Subjective Type Examples

■ In this section, there are **12 subjective** solved examples.

● ***Ex. 28*** *If $A^n = 0$, then evaluate*

(i) $I + A + A^2 + A^3 + \ldots + A^{n-1}$

(ii) $I - A + A^2 - A^3 + \ldots + (-1)^{n-1} A^{n-1}$ for odd 'n', where I is the identity matrix having the same order of A.

Sol. (i) $A^n = 0 \Rightarrow A^n - I = -I$

$$\Rightarrow \quad A^n - I^n = -I \Rightarrow I^n - A^n = I$$

$$\Rightarrow (I - A)(I + A + A^2 + A^3 + \ldots + A^{n-1}) = I$$

$$\Rightarrow (I + A + A^2 + A^3 + \ldots + A^{n-1}) = (I - A)^{-1} I = (I - A)^{-1}$$

(ii) $A^n = 0 \Rightarrow A^n + I = I$

$$\Rightarrow A^n + I^n = I$$

$$\Rightarrow I^n + A^n = I$$

$$\Rightarrow (I + A)(I - A + A^2 - A^3 + \ldots + A^{n-1}) = I \quad [\because n \text{ is odd}]$$

$$\Rightarrow I - A + A^2 - A^3 + \ldots + A^{n-1} = (I + A)^{-1} I = (I + A)^{-1}$$

● ***Ex. 29*** *If A is idempotent matrix, then show that $(A + I)^n = I + (2^n - 1)A, \forall\, n \in N$, where I is the identity matrix having the same order of A.*

Sol. $\because$ A is idempotent matrix

$$\therefore \quad A^2 = A,$$

similarly $A = A^2 = A^3 = A^4 = \ldots = A^n$ ...(i)

Now, $(A + I)^n = (I + A)^n$

$$= I + {}^nC_1 A + {}^nC_2 A^2 + {}^nC_3 A^3 + \ldots + {}^nC_n A^n$$

$$= I + ({}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_n)A \quad \text{[from Eq. (i)]}$$

$$= I + (2^n - 1)A$$

Hence, $(A + I)^n = I + (2^n - 1)A, \forall\, n \in N$.

● ***Ex. 30*** *If the matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (a, b, c, d not all simultaneously zero) commute, find the value of $\dfrac{d-b}{a+c-b}$. Also, show that the matrix which commutes with A is of the form $\begin{bmatrix} \alpha - \beta & \frac{2\beta}{3} \\ \beta & \alpha \end{bmatrix}$.*

Sol. Given, $AB = BA$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a + 2c & b + 2d \\ 3a + 4c & 3b + 4d \end{bmatrix} = \begin{bmatrix} a + 3b & 2a + 4b \\ c + 3d & 2c + 4d \end{bmatrix}$$

On comparing, we get

$$a + 2c = a + 3b$$

$$\Rightarrow \quad b = \frac{2c}{3} \qquad \ldots(i)$$

$$b + 2d = 2a + 4b$$

$$\Rightarrow \quad d = a + \frac{3b}{2} \qquad \ldots(ii)$$

$$\Rightarrow \quad 3a + 4c = c + 3d$$

$$d = a + c \qquad \ldots(iii)$$

and $\quad 3b + 4d = 2c + 4d$

$$\Rightarrow \quad b = \frac{2c}{3} \qquad \ldots(iv)$$

$$\Rightarrow \quad \frac{d-b}{a+c-b} = \frac{d-b}{d-b} = 1 \qquad \text{[from Eq. (iii)]}$$

Now, $\quad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d - c & \frac{2c}{3} \\ c & d \end{bmatrix}$

If $c = \beta$ and $d = \alpha$, then $B = \begin{bmatrix} \alpha - \beta & \frac{2\beta}{3} \\ \beta & \alpha \end{bmatrix}$

● ***Ex. 31*** *Given the matrix $A = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$ and X be the solution set of the equation $A^x = A$, where $x \in N - \{1\}$. Evaluate $\prod\left(\dfrac{x^3 + 1}{x^3 - 1}\right)$, where the continued extends for all $x \in X$.*

Sol. $\because$ $A^2 = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}\begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$

$= A$

$$\therefore \quad A^2 = A^3 = A^4 = A^5 = \ldots = A$$

but given $\quad A^x = A$

$$\Rightarrow \quad x = 2, 3, 4, 5, \ldots \qquad [\because x \neq 1, \text{ given}]$$

$$\therefore \quad \prod\left(\frac{x^3 + 1}{x^3 - 1}\right) = \prod\left(\frac{x + 1}{x - 1}\right)\prod\left(\frac{x^2 - x + 1}{x^2 + x + 1}\right)$$

On putting $\quad x = 2, 3, 4, 5, \ldots$

$$\prod\left(\frac{x^3+1}{x^3-1}\right)=\lim_{n\to\infty}\prod_{x=2}^{n}\left(\frac{x+1}{x-1}\right)\prod_{x=2}^{n}\left(\frac{x^2-x+1}{x^2+x+1}\right)$$

$$=\lim_{n\to\infty}\left(\frac{3\cdot4\cdot5\ldots(n-1)n(n+1)}{1\cdot2\cdot3\ldots(n-3)(n-2)(n-1)}\right)\times\lim_{n\to\infty}\left(\frac{3\cdot7\cdot\ldots(n^2-n+1)}{7\cdot13\ldots(n^2-n+1)(n^2+n+1)}\right)$$

$$=\lim_{n\to\infty}\frac{n(n+1)}{2}\times\frac{3}{(n^2+n+1)}$$

$$=\frac{3}{2}\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{1}{n}+\frac{1}{n^2}\right)}=\frac{3}{2}\cdot\frac{(1+0)}{(1+0+0)}=\frac{3}{2}$$

● ***Ex. 32*** *If P is a non-singular matrix, with $\text{adj}(P^{-1})$ in terms of 'P ', then show that $\text{adj}(Q^{-1}BP^{-1})=PAQ$. Given that, $\text{adj}(B)=A$ and $|P|=|Q|=1$.*

Sol. $\because\ \text{adj}(P^{-1})=|P|(P^{-1})^{-1}=|P|P=P \qquad [\because |P|=1]$

and $\text{adj}(Q^{-1}BP^{-1})=\text{adj}(P^{-1})\cdot\text{adj}B\cdot\text{adj}(Q^{-1})$

$$=\frac{P}{|P|}A\cdot\frac{Q}{|Q|}=PAQ \qquad [\because |P|=|Q|=1]$$

● ***Ex. 33*** *Let A and B be matrices of order n. Prove that if $(I-AB)$ is invertible, $(I-BA)$ is also invertible and $(I-BA)^{-1}=I+B(I-AB)^{-1}A$, where I be the identity matrix of order n.*

Sol. Here, $I-BA=BIB^{-1}-BABB^{-1}=B(I-AB)B^{-1}$ ...(i)

Hence, $|I-BA|=|B||I-AB||B^{-1}|=|B||I-AB|\frac{1}{|B|}$

$=|I-AB|$

If $|I-AB|\neq0$, then $|I-BA|\neq0$

i.e. if $(I-AB)$ is invertible, then $(I-BA)$ is also invertible.

Now, $(I-BA)[I+B(I-AB)^{-1}A]$

$=(I-BA)+(I-BA)B(I-AB)^{-1}A$ [using Eq. (i)]

$=(I-BA)+B(I-AB)B^{-1}B(I-AB)^{-1}A$

$=(I-BA)+B(I-AB)(I-AB^{-1})A$

$=(I-BA)+BA=I$

Hence, $(I-BA)^{-1}=I+B(I-AB)^{-1}A$.

● ***Ex. 34*** *Prove that the inverse of* $\begin{bmatrix}A & O\\ B & C\end{bmatrix}$ *is* $\begin{bmatrix}A^{-1} & O\\ -C^{-1}BA^{-1} & C^{-1}\end{bmatrix}$, *where A, C are non-singular matrices and O is null matrix and find the inverse.* $\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\1&1&1&1\end{bmatrix}$

Sol. We have, First part $\begin{bmatrix}A & O\\ B & C\end{bmatrix}\begin{bmatrix}A^{-1} & O\\ -C^{-1}BA^{-1} & C^{-1}\end{bmatrix}$

$$=\begin{bmatrix}AA^{-1} & O\\ BA^{-1}-CC^{-1}BA^{-1} & CC^{-1}\end{bmatrix}$$

$$=\begin{bmatrix}I & O\\ BA^{-1}-BA^{-1} & I\end{bmatrix}=\begin{bmatrix}I & O\\ 0 & I\end{bmatrix}$$

Hence, $\begin{bmatrix}A^{-1} & O\\ -C^{-1}BA^{-1} & C^{-1}\end{bmatrix}$ is the inverse of $\begin{bmatrix}A & O\\ B & C\end{bmatrix}$.

Second part $\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\1&1&1&1\end{bmatrix}=\begin{bmatrix}A & O\\ B & C\end{bmatrix}$

where $A=\begin{bmatrix}1&0\\1&1\end{bmatrix}$, $B=\begin{bmatrix}1&1\\1&1\end{bmatrix}$, $C=\begin{bmatrix}1&0\\1&1\end{bmatrix}$ and $O=\begin{bmatrix}0&0\\0&0\end{bmatrix}$

and $A^{-1}=\begin{bmatrix}1&0\\-1&1\end{bmatrix}$, $C^{-1}=\begin{bmatrix}1&0\\-1&1\end{bmatrix}$

Now, $C^{-1}BA^{-1}=\begin{bmatrix}1&0\\-1&1\end{bmatrix}\begin{bmatrix}1&1\\1&1\end{bmatrix}\begin{bmatrix}1&0\\-1&1\end{bmatrix}=\begin{bmatrix}0&1\\0&0\end{bmatrix}$

$\therefore$ Inverse of $\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\1&1&1&1\end{bmatrix}$ is $\begin{bmatrix}1&0&0&0\\-1&1&0&0\\0&-1&1&0\\0&0&-1&1\end{bmatrix}$

● ***Ex. 35*** *Let* $A=\begin{bmatrix}3&a&-1\\2&5&c\\b&8&2\end{bmatrix}$ *is symmetric and* $B=\begin{bmatrix}d&3&a\\b-a&e&-2b-c\\-2&6&-f\end{bmatrix}$ *is skew-symmetric, find AB. If AB is symmetric or skew-symmetric or neither of them. Justify your answer.*

Sol. $\because\ A$ is symmetric

$\therefore\quad c=8, b=-1$ and $a=2$ ...(i)

and B is skew-symmetric

$\therefore\quad d=e=f=0$ and $2b+c=6, a=2, b-a=-3$ ...(ii)

From Eqs. (i) and (ii), we get

$a=2, b=-1, c=8, d=0, e=0, f=0$

$\therefore\quad A=\begin{bmatrix}3&2&-1\\2&5&8\\-1&8&2\end{bmatrix}$ and $B=\begin{bmatrix}0&3&2\\-3&0&-6\\-2&6&0\end{bmatrix}$

$\Rightarrow\quad AB=\begin{bmatrix}-4&3&-6\\-31&54&-26\\-28&9&-50\end{bmatrix}$

which neither symmetric nor skew-symmetric.

● ***Ex. 36*** *If B, C are square matrices of order n and if $A = B + C$, $BC = CB$, $C^2 = 0$, show that for any positive integer p, $A^{p+1} = B^p[B + (p+1)C]$.*

Sol. $\because\ A = B + C \Rightarrow A^{p+1} = (B + C)^{p+1}$

$$= {}^{p+1}C_0 B^{p+1} + {}^{p+1}C_1 B^p C + {}^{p+1}C_2 B^{p-1} C^2 + \ldots + {}^{p+1}C_{p+1} C^{p+1}$$

$$= B^{p+1} + {}^{p+1}C_1 B^p C + 0 + 0 + \ldots \quad [\because C^2 = 0 \Rightarrow C^2 = C^3 = \ldots = 0]$$

$$= B^p[B + (p+1)C]$$

Hence, $A^{p+1} = B^p[B + (p+1)C]$

● ***Ex. 37*** *If there an three square matrices A, B, C of same order satisfying the equation $A^2 = A^{-1}$ and let $B = A^{2^n}$ and $C = A^{2^{(n-2)}}$, prove that $\det(B - C) = 0$.*

Sol. $\because\ B = A^{2^n} = A^{2\cdot 2^{n-1}} = (A^2)^{2^{n-1}} = (A^{-1})^{2^{n-1}}\ [\because A^2 = A^{-1}]$

$$= (A^{2^{n-1}})^{-1} = (A^{2\cdot 2^{n-2}})^{-1} = [(A^2)^{2^{n-2}}]^{-1}$$

$$= [(A^{-1})^{2^{n-2}}]^{-1} = ((A^{-1})^{-1})^{2^{n-2}} = A^{2^{n-2}} = C$$

$\Rightarrow\quad B - C = 0 \Rightarrow \det(B - C) = 0$

● ***Ex. 38*** *Construct an orthogonal matrix using the skew-symmetric matrix $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$.*

Sol. $\because\ A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \Rightarrow I - A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

$$\Rightarrow\quad (I - A)^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \text{ and } (I + A) = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

Let B be the orthogonal matrix from a skew-symmetric matrix A, then $B = (I - A)^{-1}(I + A)$

$$= \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix} = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & -\frac{3}{5} \end{bmatrix}$$

● ***Ex. 39*** *If $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 4 & 1 \\ -2 & -4 & -1 \end{bmatrix}$ and X, Y are two non-zero column vectors such that $AX = \lambda X$, $AY = \mu Y$, $\lambda \neq \mu$, find angle between X and Y.*

Sol. $\because AX = \lambda X \Rightarrow (A - \lambda I)X = 0$

$\because\quad X \neq 0$

$\therefore\quad \det(A - \lambda I) = 0$

$$\Rightarrow\quad \begin{vmatrix} 3-\lambda & 2 & 2 \\ 2 & 4-\lambda & 1 \\ -2 & -4 & -1-\lambda \end{vmatrix} = 0$$

Applying $R_3 \to R_3 + R_2$, then

$$\begin{vmatrix} 3-\lambda & 2 & 2 \\ 2 & 4-\lambda & 1 \\ 0 & -\lambda & -\lambda \end{vmatrix} = 0$$

Applying $C_2 \to C_2 - C_3$, then

$$\Rightarrow\quad \begin{vmatrix} 3-\lambda & 0 & 2 \\ 2 & 3-\lambda & 1 \\ 0 & 0 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow\quad -\lambda(3-\lambda)^2 = 0$$

$$\Rightarrow\quad \lambda = 0, 3$$

It is clear that $\lambda = 0, \mu = 3$

For $\lambda = 0$, $AX = 0 \Rightarrow \begin{bmatrix} 3 & 2 & 2 \\ 2 & 4 & 1 \\ -2 & -4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow\quad 3x + 2y + 2z = 0$ and $2x + 4y + z = 0$

$\therefore\quad \frac{x}{-6} = \frac{y}{1} = \frac{z}{8}$

So, $X = \begin{bmatrix} -6 \\ 1 \\ 8 \end{bmatrix}$

For $\mu = 3$, $(A - 3I)Y = 0$

$$\Rightarrow\quad \begin{bmatrix} 0 & 2 & 2 \\ 2 & 1 & 1 \\ -2 & -4 & -4 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow\quad 0\cdot\alpha + 2\beta + 2\gamma = 0$ and $2\alpha + \beta + \gamma = 0$

$\therefore\quad \frac{\alpha}{0} = \frac{\beta}{4} = \frac{\gamma}{-4}$

$\Rightarrow\quad \frac{\alpha}{0} = \frac{\beta}{-1} = \frac{\gamma}{1}$

So, $Y = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

If θ be the angle between X and Y, then

$$\cos\theta = \frac{0\cdot(-6) + (-1)\cdot 1 + 1\cdot 8}{\sqrt{(0+1+1)}\sqrt{36+1+64}} = \frac{7}{\sqrt{202}}$$

$\therefore\quad \theta = \cos^{-1}\left(\frac{7}{\sqrt{202}}\right)$

Matrices Exercise 1 :
Single Option Correct Type Questions

■ This section contains **30 multiple choice questions.** Each question has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct

1. If $A^5 = O$ such that $A^n \neq I$ for $1 \leq n \leq 4$, then $(I - A)^{-1}$ is equal to
(a) A^4 (b) A^3
(c) $I + A$ (d) None of these

2. Let $A = \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$ and suppose that $\det(A) = 2$, then $\det(B)$ equals, where $B = \begin{bmatrix} 4x & 2a & -p \\ 4y & 2b & -q \\ 4z & 2c & -r \end{bmatrix}$
(a) -2 (b) -8 (c) -16 (d) 8

3. If both $A - \frac{1}{2}I$ and $A + \frac{1}{2}I$ are orthogonal matrices, then
(a) A is orthogonal
(b) A is skew-symmetric matrix of even order
(c) $A^2 = \frac{3}{4}I$
(d) None of the above

4. Let $a = \lim_{x \to 1}\left(\frac{x}{\ln x} - \frac{1}{x \ln x}\right)$, $b = \lim_{x \to 0}\left(\frac{x^3 - 16x}{4x + x^2}\right)$, $c = \lim_{x \to 0}\left(\frac{\ln(1 + \sin x)}{x}\right)$ and $d = \lim_{x \to -1} \frac{(x+1)^3}{3[\sin(x+1) - (x+1)]}$, then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is
(a) idempotent (b) involutory
(c) non-singular (d) nilpotent

5. Let $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$. If θ is the angle between the two non-zero column vectors X such that $AX = \lambda X$ for some scalar λ, then $\tan\theta$ is equal to
(a) 3 (b) 5
(c) 7 (d) 9

6. If a square matrix A is involutory, then A^{2n+1} is equal to
(a) I (b) A
(c) A^2 (d) $(2n + 1)A$

7. If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then $\lim_{n \to \infty} \frac{A^n}{n}$ is (where $\theta \in R$)
(a) a zero matrix (b) an identity matrix
(c) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$

8. The rank of the matrix $\begin{bmatrix} -1 & 2 & 5 \\ 2 & -4 & a-4 \\ 1 & -2 & a+1 \end{bmatrix}$ is (where $a = -6$)
(a) 1 (b) 2 (c) 3 (d) 4

9. A is an involutory matrix given by $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$, the inverse of $\frac{A}{2}$ will be
(a) $2A$ (b) $\frac{A^{-1}}{2}$ (c) $\frac{A}{2}$ (d) A^2

10. Let A be a nth order square matrix and B be its adjoint, then $|AB + kI_n|$, is (where k is a scalar quantity)
(a) $(|A| + k)^{n-2}$ (b) $(|A| + k)^n$
(c) $(|A| + k)^{n-1}$ (d) $(|A| + k)^{n+1}$

11. If A and B are two square matrices such that $B = -A^{-1}BA$, then $(A + B)^2$ is equal to
(a) O (b) $A^2 + B^2$
(c) $A^2 + 2AB + B^2$ (d) $A + B$

12. If matrix $A = [a_{ij}]_{3\times 3}$, matrix $B = [b_{ij}]_{3\times 3}$, where $a_{ij} + a_{ji} = 0$ and $b_{ij} - b_{ji} = 0$, then $A^4 \cdot B^3$ is
(a) skew-symmetric matrix (b) singular
(c) symmetric (d) zero matrix

13. Let A be a $n \times n$ matrix such that $A^n = \alpha A$, where α is a real number different from 1 and -1. The matrix $A + I_n$ is
(a) singular (b) invertible
(c) scalar matrix (d) None of these

14. If $A = \begin{bmatrix} \frac{-1 + i\sqrt{3}}{2i} & \frac{-1 - i\sqrt{3}}{2i} \\ \frac{1 + i\sqrt{3}}{2i} & \frac{1 - i\sqrt{3}}{2i} \end{bmatrix}$, $i = \sqrt{-1}$ and $f(x) = x^2 + 2$, then $f(A)$ equals to
(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\left(\frac{3 - i\sqrt{3}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
(c) $\left(\frac{5 - i\sqrt{3}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $(2 + i\sqrt{3})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

15. The number of solutions of the matrix equation $X^2 = I$ other than I is
(a) 0 (b) 1
(c) 2 (d) more than 2

16. If A and B are square matrices such that $A^{2006} = 0$ and $AB = A + B$, then $\det(B)$ equals to
(a) -1 (b) 0
(c) 1 (d) None of these

17. If $P = \begin{bmatrix} \cos\frac{\pi}{6} & \sin\frac{\pi}{6} \\ -\sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{bmatrix}, A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $Q = PAP^T$, then $P^T Q^{2007} P$ is equal to
(a) $\begin{bmatrix} 1 & \sqrt{3}/2 \\ 0 & 2007 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2007 \\ 0 & 1 \end{bmatrix}$
(c) $\begin{bmatrix} \sqrt{3}/2 & 2007 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1 & 2007 \end{bmatrix}$

18. There are two possible values of A in the solution of the matrix equation $\begin{bmatrix} 2A+1 & -5 \\ -4 & A \end{bmatrix}^{-1} \begin{bmatrix} A-5 & B \\ 2A-2 & C \end{bmatrix} = \begin{bmatrix} 14 & D \\ E & F \end{bmatrix}$, where A, B, C, D, E, F are real numbers. The absolute value of the difference of these two solutions, is
(a) $\frac{8}{3}$ (b) $\frac{11}{3}$ (c) $\frac{1}{3}$ (d) $\frac{19}{3}$

19. If $f(\theta) = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta & \cos\theta \\ \sin\theta & -\cos\theta & 0 \end{bmatrix}$, then $f\left(\frac{\pi}{7}\right)$ is
(a) symmetric (b) skew-symmetric
(c) singular (d) non-singular

20. In a square matrix A of order 3 the elements a_{ii}'s are the sum of the roots of the equation $x^2 - (a+b)x + ab = 0$; $a_{i,i+1}$'s are the product of the roots, $a_{i,i-1}$'s are all unity and the rest of the elements are all zero.

The value of the $\det(A)$ is equal to
(a) 0 (b) $(a+b)^3$
(c) $a^3 - b^3$ (d) $(a^2+b^2)(a+b)$

21. If A and B are two non-singular matrices of the same order such that $B^r = I$ for some positive integer $r > 1$. Then $A^{-1}B^{r-1}A - A^{-1}B^{-1}A$ is equal to
(a) I (b) $2I$ (c) 0 (d) $-I$

22. If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, C = ABA^T$, then $A^T C^n A, n \in I^+$ equals to
(a) $\begin{bmatrix} -n & 1 \\ 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -n \\ 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 \\ 1 & -n \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 \\ -n & 1 \end{bmatrix}$

23. If A is a square matrix of order 3 such that $|A| = 2$, then $|(\text{adj } A^{-1})^{-1}|$ is
(a) 1 (b) 2 (c) 4 (d) 8

24. If A and B are different matrices satisfying $A^3 = B^3$ and $A^2B = B^2A$, then
(a) $\det(A^2 + B^2)$ must be zero
(b) $\det(A - B)$ must be zero
(c) $\det(A^2 + B^2)$ as well as $\det(A - B)$ must be zero
(d) atleast one of $\det(A^2 + B^2)$ or $\det(A - B)$ must be zero

25. If A is a skew-symmetric matrix of order 2 and B, C are matrices $\begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}, \begin{bmatrix} 9 & -4 \\ -2 & 1 \end{bmatrix}$ respectively, then
$A^3(BC) + A^5(B^2C^2) + A^7(B^3C^3) + \ldots + A^{2n+1}(B^nC^n)$, is
(a) a symmetric matrix (b) a skew-symmetric matrix
(c) an identity matrix (d) None of these

26. If $A = \begin{bmatrix} a & b & c \\ x & y & z \\ p & q & r \end{bmatrix}, B = \begin{bmatrix} q & -b & y \\ -p & a & -x \\ r & -c & z \end{bmatrix}$ and if A is invertible, then which of the following is not true?
(a) $|A| = |B|$
(b) $|A| = -|B|$
(c) $|\text{adj } A| = |\text{adj } B|$
(d) A is invertible $\Leftrightarrow B$ is invertible

27. Let three matrices $A = \begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$, then $\text{tr}(A) + \text{tr}\left(\frac{ABC}{2}\right) + \text{tr}\left(\frac{A(BC)^2}{4}\right) + \text{tr}\left(\frac{A(BC)^3}{8}\right) + \ldots + \infty$ equals to
(a) 4 (b) 9 (c) 12 (d) 6

28. If A is non-singular and $(A - 2I)(A - 4I) = O$, then $\frac{1}{6}A + \frac{4}{3}A^{-1}$ is equal to
(a) O (b) I
(c) $2I$ (d) $6I$

29. If $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & a & 1 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & b \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$, then
(a) $a = 1, b = -1$ (b) $a = 2, b = -\frac{1}{2}$
(c) $a = -1, b = 1$ (d) $a = \frac{1}{2}, b = \frac{1}{2}$

30. Given the matrix $A = \begin{bmatrix} x & 3 & 2 \\ 1 & y & 4 \\ 2 & 2 & z \end{bmatrix}$. If $xyz = 60$ and $8x + 4y + 3z = 20$, then $A(\text{adj } A)$ is equal to
(a) $64I$ (b) $88I$ (c) $68I$ (d) $34I$

Matrices Exercise 2 :
More than One Correct Option Type Questions

- This section contains **15 multiple choice questions**. Each question has four choices (a), (b), (c) and (d) out of which **MORE THAN ONE** may be correct.

31. If $A=\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$, then

(a) $A^3=9A$ (b) $A^3=27A$

(c) $A+A=A^2$ (d) A^{-1} does not exist

32. A square matrix A with elements from the set of real numbers is said to be orthogonal if $A'=A^{-1}$. If A is an orthogonal matrix, then

(a) A' is orthogonal (b) A^{-1} is orthogonal

(c) adj $A=A'$ (d) $|A^{-1}|=1$

33. Let $A=\begin{bmatrix}1&2&2\\2&1&2\\2&2&1\end{bmatrix}$, then

(a) $A^2-4A-5I_3=O$ (b) $A^{-1}=\frac{1}{5}(A-4I_3)$

(c) A^3 is not invertible (d) A^2 is invertible

34. D is a 3×3 diagonal matrix. Which of the following statements are not true?

(a) $D^T=D$

(b) $AD=DA$ for every matrix A of order 3×3

(c) D^{-1} if exists is a scalar matrix

(d) None of the above

35. The rank of the matrix $\begin{bmatrix}-1&2&5\\2&-4&a-4\\1&-2&a+1\end{bmatrix}$, is

(a) 2, if $a=-6$ (b) 2, if $a=1$

(c) 1, if $a=2$ (d) 1, if $a=-6$

36. If $A=\begin{bmatrix}3&-3&4\\2&-3&4\\0&-1&1\end{bmatrix}$, then

(a) adj (adj A) $=A$ (b) $|$ adj (adj $(A)|=1$

(c) $|$ adj $(A)|=1$ (d) None of these

37. If B is an idempotent matrix and $A=I-B$, then

(a) $A^2=A$ (b) $A^2=I$

(c) $AB=O$ (d) $BA=O$

38. If A is a non-singular matrix, then

(a) A^{-1} is symmetric if A is symmetric

(b) A^{-1} is skew-symmetric if A is symmetric

(c) $|A^{-1}|=|A|$

(d) $|A^{-1}|=|A|^{-1}$

39. Let A and B are two matrices such that $AB=BA$, then for every $n\in N$

(a) $A^nB=BA^n$

(b) $(AB)^n=A^nB^n$

(c) $(A+B)^n={}^nC_0A^n+{}^nC_1A^{n-1}B+\ldots+{}^nC_nB^n$

(d) $A^{2n}-B^{2n}=(A^n-B^n)(A^n+B^n)$

40. If A and B are 3×3 matrices and $|A|\neq0$, which of the following are true?

(a) $|AB|=0\Rightarrow|B|=0$

(b) $|AB|=0\Rightarrow B=0$

(c) $|A^{-1}|=|A|^{-1}$

(d) $|A+A|=2|A|$

41. If A is a matrix of order $m\times m$ such that $A^2+A+2I=O$, then

(a) A is non-singular (b) A is symmetric

(c) $|A|\neq0$ (d) $A^{-1}=-\frac{1}{2}(A+I)$

42. If $A^2-3A+2I=0$, then A is equal to

(a) I (b) $2I$

(c) $\begin{bmatrix}3&-2\\1&0\end{bmatrix}$ (d) $\begin{bmatrix}3&1\\-2&0\end{bmatrix}$

43. If A and B are two matrices such that their product AB is a null matrix, then

(a) det $A\neq0\Rightarrow B$ must be a null matrix

(b) det $B\neq0\Rightarrow A$ must be a null matrix

(c) atleast one of the two matrices must be singular

(d) if neither det A nor det B is zero, then the given statement is not possible

44. If D_1 and D_2 are two 3×3 diagonal matrices where none of the diagonal elements is zero, then

(a) D_1D_2 is a diagonal matrix

(b) $D_1D_2=D_2D_1$

(c) $D_1^2+D_2^2$ is a diagonal matrix

(d) None of the above

45. Let, $C_k={}^nC_k$ for $0\le k\le n$ and $A_k=\begin{bmatrix}C_{k-1}^2&0\\0&C_k^2\end{bmatrix}$ for $k\ge1$ and

$A_1+A_2+A_3+\ldots+A_n=\begin{bmatrix}k_1&0\\0&k_2\end{bmatrix}$, then

(a) $k_1=k_2$ (b) $k_1+k_2=2$

(c) $k_1={}^{2n}C_n-1$ (d) $k_2={}^{2n}C_{n+1}$

Matrices Exercise 3 : Passage Based Questions

- This section contains **6 passages**. Based upon each of the passage **3 multiple choice questions** have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I (Q. Nos. 46 to 48)

Suppose A and B be two non-singular matrices such that $AB = BA^m$, $B^n = I$ and $A^p = I$, where I is an identity matrix.

46. If $m = 2$ and $n = 5$, then p equals to
(a) 30 (b) 31
(c) 33 (d) 81

47. The relation between m, n and p, is
(a) $p = mn^2$ (b) $p = m^n - 1$
(c) $p = n^m - 1$ (d) $p = m^{n-1}$

48. Which of the following ordered triplet (m, n, p) is false?
(a) (3, 4, 80) (b) (6, 3, 215)
(c) (8, 3, 510) (d) (2, 8, 255)

Passage II (Q. Nos. 49 to 51)

Let $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$ is an orthogonal matrix and $abc = \lambda\,(< 0)$.

49. The value of $a^2b^2 + b^2c^2 + c^2a^2$, is
(a) 2λ (b) -2λ
(c) λ^2 (d) $-\lambda$

50. The value of $a^3 + b^3 + c^3$, is
(a) λ (b) 2λ
(c) 3λ (d) None of these

51. The equation whose roots are a, b, c, is
(a) $x^3 - 2x^2 + \lambda = 0$ (b) $x^3 - \lambda x^2 + \lambda x + \lambda = 0$
(c) $x^3 - 2x^2 + 2\lambda x + \lambda = 0$ (d) $x^3 \pm x^2 - \lambda = 0$

Passage III (Q. Nos. 52 to 53)

Let $A = [a_{ij}]_{3\times 3}$. If tr is arithmetic mean of elements of rth row and $a_{ij} + a_{jk} + a_{ki} = 0$ holds for all $1 \le i, j, k \le 3$.

52. $\sum\limits_{1\le i}\sum\limits_{j\le 3} a_{ij}$ is not equal to
(a) $t_1 + t_2 + t_3$ (b) zero
(c) $(\det(A))^2$ (d) $t_1t_2t_3$

53. Matrix A is
(a) non-singular
(b) symmetric
(c) skew-symmetric
(d) neither symmetric nor skew-symmetric

Passage IV (Q. Nos. 54 to 56)

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ be a square matrix and C_1, C_2, C_3 be three column matrices satisfying $AC_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $AC_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$ and $AC_3 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ of matrix B. If the matrix $C = \frac{1}{3}(A \cdot B)$.

54. The value of $\det(B^{-1})$, is
(a) 2 (b) $\frac{1}{2}$ (c) 3 (d) $\frac{1}{3}$

55. The ratio of the trace of the matrix B to the matrix C, is
(a) $-\frac{9}{5}$ (b) $-\frac{5}{9}$ (c) $-\frac{2}{3}$ (d) $-\frac{3}{2}$

56. The value of $\sin^{-1}(\det A) + \tan^{-1}(9 \det C)$, is
(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{4}$ (d) π

Passage V (Q. Nos. 57 to 59)

If A is symmetric and B skew-symmetric matrix and $A + B$ is non-singular and $C = (A + B)^{-1}(A - B)$.

57. $C^T(A + B)C$ equals to
(a) $A + B$ (b) $A - B$
(c) A (d) B

58. $C^T(A - B)C$ equals to
(a) $A + B$ (b) $A - B$ (c) A (d) B

59. $C^T AC$ equals to
(a) $A + B$ (b) $A - B$
(c) A (d) B

Passage VI (Q. Nos. 60 to 61)

Let A be a square matrix of order 3 satisfies the matrix equation $A^3 - 6A^2 + 7A - 8I = O$ and $B = A - 2I$. Also, $\det A = 8$.

60. The value of $\det(\text{adj}(I - 2A^{-1}))$ is equal to
(a) $\frac{25}{16}$ (b) $\frac{125}{64}$
(c) $\frac{64}{125}$ (d) $\frac{16}{25}$

61. If $\text{adj}\left(\left(\frac{B}{2}\right)^{-1}\right) = \left(\frac{p}{q}\right)B$, where $p, q \in N$, the least value of $(p + q)$ is equal to
(a) 7 (b) 9 (c) 29 (d) 41

Matrices Exercise 4 :
Single Integer Answer Type Questions

- This section contains **10 questions**. The answer to each question is a **single digit integer**, ranging from 0 to 9 (both inclusive).

62. Let A, B, C, D be (not necessarily) real matrices such that $A^T = BCD; B^T = CDA; C^T = DAB$ and $D^T = ABC$ for the matrix $S = ABCD$, the least value of k such that $S^k = S$ is

63. If $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$ and a function $f(x)$ is defined as $f(x) = \det(A^T A^{-1})$ and if $f(f(f(f\ldots f(x))))$ is $(n \geq 2)\,\lambda$, the value of 2^{λ} is

64. If the matrix $A = \begin{bmatrix} \lambda_1^2 & \lambda_1\lambda_2 & \lambda_1\lambda_3 \\ \lambda_2\lambda_1 & \lambda_2^2 & \lambda_2\lambda_3 \\ \lambda_3\lambda_1 & \lambda_3\lambda_2 & \lambda_3^2 \end{bmatrix}$ is idempotent, the value of $\lambda_1^2 + \lambda_2^2 + \lambda_3^2$ is

65. Let A be a 3×3 matrix given by $A = [a_{ij}]$. If for every column vector X, $X^T AX = O$ and $a_{23} = -1008$, the sum of the digits of a_{32} is

66. Let X be the solution set of the equation $A^x = I$, where $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$ and I is the corresponding unit matrix and $x \subseteq N$, the minimum value of $\Sigma(\cos^x\theta + \sin^x\theta)$, $\theta \in R - \left\{\frac{n\pi}{2}, n \in I\right\}$ is

67. If A is an idempotent matrix and I is an identity matrix of the same order, then the value of n, such that $(A + I)^n = I + 127\,A$ is

68. Suppose $a, b, c \in R$ and $abc = 1$, if $A = \begin{bmatrix} 3a & b & c \\ b & 3c & a \\ c & a & 3b \end{bmatrix}$ is such that $A^T A = 4^{1/3}\,I$ and $|A| > 0$, the value of $a^3 + b^3 + c^3$ is

69. If $A = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$ and $(A^8 + A^6 + A^4 + A^2 + I)\,V = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$, where V is a vertical vector and I is the 2×2 identity matrix and if λ is sum of all elements of vertical vector V, the value of 11λ is

70. Let the matrix A and B be defined as $A = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 \\ 7 & 3 \end{bmatrix}$, then the absolute value of $\det(2A^9 B^{-1})$ is

71. Let $A = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}$ and $(A + I)^{70} - 70A = \begin{bmatrix} a-1 & b-1 \\ c-1 & d-1 \end{bmatrix}$, the value of $a + b + c + d$ is

Matrices Exercise 5 :
Matching Type Questions

- This section contains **4 questions**. Question 72 has four statements (A, B, C and D) given in **Column I** and four statements (p, q, r and s) in **Column II** and questions 73 to 75 have four statements (A, B, C and D) given in **Column I** and five statements (p, q, r, s and t) in **Column II**. Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II**.

72. Suppose a, b, c are three distinct real numbers and $f(x)$ is a real quadratic polynomial such that

$$\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix}\begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2 + 3a \\ 3b^2 + 3b \\ 3c^2 + 3c \end{bmatrix}.$$

Column I		Column II	
(A)	x-coordinate(s) of the point of intersection of $y = f(x)$ with the X-axis is	(p)	–2
(B)	Area (in sq units) bounded by $y = \frac{3}{2} f(x)$ and the X-axis is	(q)	1
(C)	Maximum value of $f(x)$ is	(r)	2
(D)	Length (in unit) of the intercept made by $y = f(x)$ on the X-axis is	(s)	4

73. If A is non-singular matrix of order $n \times n$,

	Column I		Column II
(A)	adj (A^{-1}) is	(p)	$A\,(\det A)^{n-2}$
(B)	det (adj (A^{-1})) is	(q)	$(\det A)^{n-1}\,(\text{adj } A)$
(C)	adj (adj A) is	(r)	$\dfrac{\text{adj (adj } A)}{(\det A)^{n-1}}$
(D)	adj (A det (A)) is	(s)	$(\det A)^{1-n}$
		(t)	$\dfrac{A}{(\det A)}$

74.

	Column I		Column II
(A)	If A is a diagonal matrix of order 3×3 is commutative with every square matrix of order 3×3 under multiplication and tr $(A) = 12$, then $\lvert A \rvert$ is divisible by	(p)	3
		(q)	4
(B)	Let $a, b, c \in R^+$ and the system of equations $(1-a)x + y + z = 0$, $x + (1-b)y + z = 0$, $x + y + (1-c)z = 0$ has infinitely many solutions. If λ be the minimum value of $a\,b\,c$, then λ is divisible by	(r)	6
(C)	Let $A = [a_{ij}]_{3\times 3}$ be a matrix whose elements are distinct integers from 1, 2, 3, ..., 9. The matrix is formed so that the sum of the numbers is every row, column and each diagonal is a multiple of 9. If number of all such possible matrices is λ, then λ is divisible by	(s)	8
(D)	If the equations $x + y = 1$, $(c+2)x + (c+4)y = 6$, $(c+2)^2x + (c+4)^2 y = 36$ are consistent and c_1, c_2 $(c_1 > c_2)$ are two values of c, then $c_1\,c_2$ is divisible by	(t)	9

75.

	Column I		Column II
(A)	If C is skew-symmetric matrix of order n and X is $n \times 1$ column matrix, then X^TCX is	(p)	invertible
(B)	If A is skew - symmetric, then $I - A$ is, where I is an identity matrix of order A.	(q)	singular
(C)	If $S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $A = \begin{bmatrix} b+c & c-a & b-a \\ c-b & c+a & a-b \\ b-c & a-c & a+b \end{bmatrix}$ $(a, b, c \neq 0)$, then SAS^{-1} is	(r)	symmetric
(D)	If A, B, C are the angles of a triangle, then the matrix $A = \begin{bmatrix} \sin 2A & \sin C & \sin B \\ \sin C & \sin 2B & \sin A \\ \sin B & \sin A & \sin 2C \end{bmatrix}$ is	(s)	non-singular
		(t)	non-invertible

Matrices Exercise 6 : Statement I and II Type Questions

■ **Directions** (Q. Nos. 76 to 85) are Assertion-Reason type questions. Each of these questions contains two statements:
Statement-1 (Assertion) and **Statement-2** (Reason)
Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

76. **Statement-1** If matrix $A = [a_{ij}]_{3\times 3}$, $B = [b_{ij}]_{3\times 3}$, where $a_{ij} + a_{ji} = 0$ and $b_{ij} - b_{ji} = 0$, then A^4B^5 is non-singular matrix.
Statement-2 If A is non-singular matrix, then $\lvert A \rvert \neq 0$.

77. **Statement-1** If A and B are two square matrices of order $n \times n$ which satisfy $AB = A$ and $BA = B$, then $(A + B)^7 = 2^6\,(A + B)$.
Statement-2 A and B are unit matrices.

78. **Statement-1** For a singular matrix A, if $AB = AC \Rightarrow B = C$
Statement-2 If $\lvert A \rvert = 0$, then A^{-1} does not exist.

79. **Statement-1** If A is skew-symmetric matrix of order 3, then its determinant should be zero.
Statement-2 If A is square matrix, $\det(A) = \det(A') = \det(-A')$.

80. Let A be a skew-symmetric matrix, $B = (I - A)(I + A)^{-1}$ and X and Y be column vectors conformable for multiplication with B.
Statement-1 $(BX)^T\,(BY) = X^TY$
Statement-2 If A is skew-symmetric, then $(I + A)$ is non-singular.

81. **Statement-1** Let a 2×2 matrix A has determinant 2. If $B = 9A^2$, the determinant of B^T is equal to 36.

Statement-2 If A, B and C are three square matrices such that $C = AB$, then $|C| = |A||B|$.

82. **Statement-1** If $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$, then $A^3 + A^2 + A = I$.

Statement-2 If $\det(A - \lambda I) = C_0\lambda^3 + C_1\lambda^2 + C_2\lambda + C_3 = 0$, then $C_0A^3 + C_1A^2 + C_3A + C_3I = O$.

83. **Statement-1** $A = [a_{ij}]$ be a matrix of order 3×3, where $a_{ij} = \dfrac{i-j}{i+2j}$ cannot be expressed as a sum of symmetric and skew-symmetric matrix.

Statement-2 Matrix $A = [a_{ij}]_{n\times n}$, $a_{ij} = \dfrac{i-j}{i+2j}$ is neither symmetric nor skew-symmetric.

84. **Statement-1** If A, B, C are matrices such that $|A_{3\times 3}| = 3$, $|B_{3\times 3}| = -1$ and $|C_{2\times 2}| = 2$, $|2ABC| = -12$.

Statement-2 For matrices A, B, C of the same order $|ABC| = |A||B||C|$.

85. **Statement-1** The determinant of a matrix $A = [a_{ij}]_{n\times n}$, where $a_{ij} + a_{ji} = 0$ for all i and j is zero.

Statement-2 The determinant of a skew-symmetric matrix of odd order is zero.

Matrices Exercise 7 : Subjective Type Questions

- In this section, there are **12 subjective** questions.

86. If S is a real skew-symmetric matrix, the show that $I - S$ is non-singular and matrix $A = (I+S)(I-S)^{-1} = (I-S)^{-1}(I+S)$ is orthogonal.

87. If M is a 3×3 matrix, where $\det M = I$ and $MM^T = I$, where I is an identity matrix, prove that $\det(M - I) = 0$.

88. If $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, $B = \begin{bmatrix} \cos 2\beta & \sin 2\beta \\ \sin 2\beta & -\cos 2\beta \end{bmatrix}$, where $0 < \beta < \dfrac{\pi}{2}$, then prove that $BAB = A^{-1}$. Also, find the least value of α for which $BA^4B = A^{-1}$.

89. Find the product of two matrices

$$A = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} B = \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$$

Show that, AB is the zero matrix if θ and ϕ differ by an odd multiple of $\dfrac{\pi}{2}$.

90. Show that the matrix $\begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$ is orthogonal,

if $l_1^2 + m_1^2 + n_1^2 = \Sigma l_1^2 = 1 = \Sigma l_2^2 = \Sigma l_3^2$ and $l_1l_2 + m_1m_2 + n_1n_2 = \Sigma l_1l_2 = 0 = \Sigma\, l_2l_3 = \Sigma l_3l_1$.

91. A finance company has offices located in every division, every district and every taluka in a certain state in India. Assume that there are five divisions, thirty districts and 200 talukas in the state. Each office has one head clerk, one cashier, one clerk and one peon. A divisional office has, in addition, one office superintendent, two clerks, one typist and one poen. A district office, has in addition, one clerk and one peon. The basic monthly salaries are as follows:

Office superintendent ₹ 500, Head clerk ₹ 200, cashier ₹ 175, clerks and typist ₹ 150 and peon ₹ 100. Using matrix notation find

(i) the total number of posts of each kind in all the offices taken together,

(ii) the total basic monthly salary bill of each kind of office

(iii) the total basic monthly salary bill of all the offices taken together.

92. In a development plan of a city, a contractor has taken a contract to construct certain houses for which he needs building materials like stones, sand etc. There are three firms A, B, C that can supply him these materials. At one time these firms A, B, C supplied him 40, 35 and 25 truck loads of stones and 10, 5 and 8 truck loads of stone and sand, respectively. If the cost of one truck load of stone and sand are ₹ 1200 and 500 respectively, find the total amount paid by the contractor to each of these firms A, B, C separately.

93. Show that the matrix $A = \begin{bmatrix} 1 & a & \alpha & a\alpha \\ 1 & b & \beta & b\beta \\ 1 & c & \gamma & c\gamma \end{bmatrix}$ is of rank 3 provided no two of a, b, c are equal and no two of α, β, γ are equal.

94. By the method of matrix inversion, solve the system.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x & u \\ y & v \\ z & w \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 52 & 15 \\ 0 & -1 \end{bmatrix}$$

95. If $x_1 = 3y_1 + 2y_2 - y_3$, $\quad y_1 = z_1 - z_2 + z_3$

$x_2 = -y_1 + 4y_2 + 5y_3$, $y_2 = z_2 + 3z_3$

$x_3 = y_1 - y_2 + 3y_3$, $\quad y_3 = 2z_1 + z_2$

express x_1, x_2, x_3 in terms of z_1, z_2, z_3.

96. For what values of k the set of equations
$2x-3y+6z-5t=3, y-4z+t=1,$
$4x-5y+8z-9t=k$ has
(i) no solution? (ii) infinite number of solutions?

97. Let A, B, U, V and X be the matrices defined as follows.

$$A=\begin{bmatrix}a & 1 & 0\\ 1 & b & d\\ 1 & b & c\end{bmatrix}, B=\begin{bmatrix}a & 1 & 1\\ 0 & d & c\\ f & g & h\end{bmatrix}, U=\begin{bmatrix}f\\ g\\ h\end{bmatrix}, V=\begin{bmatrix}a^2\\ 0\\ 0\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$$

If $AX=U$ has infinitely many solutions, show that $BX=V$ cannot have a unique solution. If $afd\neq 0$, show that $BX=V$ has no solution.

Matrices Exercise 8 :
Questions Asked in Previous 13 Year's Exam

■ This section contains questions asked in **IIT-JEE, AIEEE, JEE Main & JEE Advanced** from year **2005** to year **2017**.

98. $A=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & -2 & 4\end{bmatrix}; I=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ and

$A^{-1}=\frac{1}{6}[A^2+cA+dI]$ where $c, d\in R$, the pair of values (c, d) **[IIT- JEE 2005, 3M]**

(a) (6, 11) (b) (6, − 11) (c) (− 6, 11) (d) (− 6, − 11)

99. If $P=\begin{bmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}, A=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$ and $Q=PAP^T$, the $P(Q^{2005})P^T$ equal to **[IIT- JEE 2005, 3M]**

(a) $\begin{bmatrix}1 & 2005\\ 0 & 1\end{bmatrix}$ (b) $\begin{bmatrix}\sqrt{3}/1 & 2005\\ 1 & 0\end{bmatrix}$

(c) $\begin{bmatrix}1 & 2005\\ \sqrt{3}/2 & 1\end{bmatrix}$ (d) $\begin{bmatrix}1 & \sqrt{3}/2\\ 0 & 2005\end{bmatrix}$

100. If $A=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$ and $I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, which one of the following holds for all $n\geq 1$, (by the principal of mathematical induction) **[AIEEE 2005, 3M]**

(a) $A^n=nA+(n-1)I$ (b) $A^n=2^{n-1}A+(n-1)I$

(c) $A^n=nA-(n-1)I$ (d) $A^n=2^{n-1}A-(n-1)I$

101. If $A^2-A+I=0$, then A^{-1} is equal to **[AIEEE 2005, 3M]**

(a) A^{-2} (b) $A+I$ (c) $I-A$ (d) $A-I$

102. If $A=\begin{pmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 3 & 2 & 1\end{pmatrix}, U_1, U_2$ and U_3 are column matrices satisfying $AU_1=\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}, AU_2=\begin{pmatrix}2\\ 3\\ 0\end{pmatrix}$ and $AU_3=\begin{pmatrix}2\\ 3\\ 1\end{pmatrix}$ and U is 3×3 matrix when columns are U_1, U_2, U_3, then answer the following questions

(i) The value of $|U|$ is
(a) 3 (b) − 3 (c) 3/2 (d) 2

(ii) The sum of the elements of U^{-1} is
(a) − 1 (b) 0 (c) 1 (d) 3

(iii) The value of $(3\ 2\ 0)\,U\begin{pmatrix}3\\ 2\\ 0\end{pmatrix}$ is **[IIT- JEE 2006, 5+5+5M]**

(a) 5 (b) 5/2 (c) 4 (d) 3/2

103. Let $A=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$ and $B=\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}, a, b\in N$. Then, **[AIEEE 2006, 4½M]**

(a) there cannot exist any B such that $AB=BA$
(b) there exist more than one but finite number of B's such that $AB=BA$
(c) there exists exactly one B such that $AB=BA$
(d) there exist infinitely among B's such that $AB=BA$

104. If A and B are square matrices of size $n\times n$ such that $A^2-B^2=(A-B)(A+B)$, which of the following will be always true? **[AIEEE 2006, 3M]**

(a) $A=B$ (b) $AB=BA$
(c) Either of A or B is a zero matrix
(d) Either of A or B is identity matrix

105. Let $A=\begin{bmatrix}5 & 5\alpha & \alpha\\ 0 & \alpha & 5\alpha\\ 0 & 0 & 5\end{bmatrix}$. If $|A^2|=25$, then $|\alpha|$ equals to **[AIEEE 2007, 3M]**

(a) 5^2 (b) 1 (c) 1/5 (d) 5

106. Let A and B be 3×3 matrices of real numbers, where A is symmetric, B is skew-symmetric and $(A+B)(A-B)=(A-B)(A+B)$. If $(AB)^t=(-1)^k AB$, where $(AB)^t$ is the transpose of matrix AB, the value of k is **[IIT- JEE 2008, 1½M]**

(a) 0 (b) 1 (c) 2 (d) 3

107. Let A be a square matrix all of whose entries are integers. Which one of the following is true? **[AIEEE 2008, 3M]**

(a) If det $A\neq\pm 1$, then A^{-1} exists and all its entries are non-integers
(b) If det $A=\pm 1$, then A^{-1} exists and all its entries are integers
(c) If det $A=\pm 1$, then A^{-1} need not exist
(d) If det $A=\pm 1$, then A^{-1} exists but all its entries are not necessarily integers

108. Let A be a 2×2 matrix with real entries. Let I be the 2×2 identity matrix. Denote by $\text{tr}(A)$, the sum of diagonal entries of A. Assume that $A^2 = I$. **[AIEEE 2008, 3M]**

Statement-1 If $A \neq I$ and $A \neq -I$, then $\det A = -1$.

Statement-2 If $A \neq I$ and $A \neq -1$, then $\text{tr}(A) \neq 0$.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1

(c) Statement-1 is true, Statement-2 is false

(d) Statement-1 is false, Statement-2 is true

109. Let A be the set of all 3×3 symmetric matrices all of whose entries are either 0 or 1. Five of these entries are 1 and four of them are 0. **[IIT- JEE 2009, 4+4+4M]**

(i) The number of matrices in A is

(a) 12 (b) 6

(c) 9 (d) 3

(ii) The number of matrices A for which the system of linear equations $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has a unique solution, is

(a) less than 4 (b) atleast 4 but less than 7

(c) atleast 7 but less than 10

(d) atleast 10

(iii) The number of matrices A in which the system of linear equations $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is inconsistent is

(a) 0 (b) more than 2

(c) 2 (d) 1

110. Let A be a 2×2 matrix

Statement-1 $\text{adj}\,(\text{adj}\,A) = A$

Statement-2 $|\text{adj}\,A| = |A|$ **[AIEEE 2009, 4M]**

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1

(c) Statement-1 is true, Statement-2 is false

(d) Statement-1 is false, Statement-2 is true

111. The number of 3×3 matrices A whose are either 0 or 1 and for which the system $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has exactly two distinct solutions, is **[IIT- JEE 2010, 3M]**

(a) 0 (b) $2^9 - 1$

(c) 168 (d) 2

112. Let p be an odd prime number and T_p be the following set of 2×2 matrices.

$$T_p = \left\{ A = \begin{bmatrix} a & b \\ c & a \end{bmatrix};\ a, b, c, \in \{0, 1, 2, \ldots, p-1\} \right\}$$

[IIT- JEE 2010, 3+3+3M]

(i) The number of A in T_p such that A is either symmetric or skew-symmetric or both and $\det(A)$ divisible by p, is

(a) $(p-1)^2$ (b) $2(p-1)$

(c) $(p-1)^2 + 1$ (d) $2p - 1$

(ii) The number of A in T_p such that the trace of A is not divisible by p but $\det(A)$ is divisible by p, is

[**Note** The trace of a matrix is the sum of its diagonal entries]

(a) $(p-1)(p^2 - p + 1)$ (b) $p^3 - (p-1)^2$

(c) $(p-1)^2$ (d) $(p-1)(p^2 - 2)$

(iii) The number of A in T_p such that $\det(A)$ is not divisible by p, is

(a) $2p^2$ (b) $p^3 - 5p$ (c) $p^3 - 3p$ (d) $p^3 - p^2$

113. Let k be a positive real number and let

$$A = \begin{bmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 2k-1 & \sqrt{k} \\ 1-2k & 0 & 2\sqrt{k} \\ -\sqrt{k} & -2\sqrt{k} & 0 \end{bmatrix}.$$

If $\det(\text{adj}\,A) + \det(\text{adj}\,B) = 10^6$, then $[k]$ is equal to **[IIT- JEE 2010, 3M]**

Note adj M denotes the adjoint of a square matrix M and $[k]$ denotes the largest integer less than or equal to k}.

114. The number of 3×3 non-singular matrices, with four entries as 1 and all other entries as 0, is **[AIEEE 2010, 8M]**

(a) 5 (b) 6

(c) atleast 7 (d) less than 4

115. Let A be a 2×2 matrix with non-zero entries and let $A^2 = I$, where I is 2×2 identity matrix. Define $\text{Tr}(A)$ = sum of diagonal elements of A and $|A|$ = determinant of matrix A.

Statement-1 $\text{Tr}(A) = 0$

Statement-2 $|A| = 1$ **[AIEEE 2010, 4M]**

(a) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.

(b) Statement-1 is true, Statement-2 is false.

(c) Statement-1 is false, Statement-2 is true.

(d) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

116. Let M and N be two 3×3 non-singular skew-symmetric matrices such that $MN = NM$. If P^T denotes the transpose of P, then $M^2N^2(M^TN)^{-1}(MN^{-1})^T$ is equal to **[IIT- JEE 2011, 4M]**

(a) M^2 (b) $-N^2$ (c) $-M^2$ (d) MN

117. Let a, b and c be three real numbers satisfying

$$[a\ b\ c]\begin{bmatrix}1 & 9 & 7\\ 8 & 2 & 7\\ 7 & 3 & 7\end{bmatrix} = [0\ 0\ 0]. \qquad \ldots(E)$$

(i) If the point $P(a, b, c)$, with reference to (E), lies on the plane $2x + y + z = 1$, then the value of $7a + b + c$ is

(a) 0 (b) 12 (c) 7 (d) 6

(ii) Let ω be a solution of $x^3 - 1 = 0$ with $\text{Im}(\omega) > 0$. If $a = 2$ with b and c satisfying (E), the value of $\frac{3}{\omega^a} + \frac{1}{\omega^b} + \frac{3}{\omega^c}$ is equal to

(a) -2 (b) 2 (c) 3 (d) -3

(iii) Let $b = 6$ with a and c satisfying (E). If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then $\sum_{n=0}^{\infty}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)^n$ is **[IIT- JEE 2011, 3+3+3M]**

(a) 6 (b) 7

(c) $\frac{6}{7}$ (d) ∞

118. Let $\omega \neq 1$ be a cube root of unity and S be the set of all non-singular matrices of the form $\begin{bmatrix}1 & a & b\\ \omega & 1 & c\\ \omega^2 & \omega & 1\end{bmatrix}$, where each of a, b and c is either ω or ω^2. The number of distinct matrices in the set S is **[IIT- JEE 2011, 3M]**

(a) 2 (b) 6

(c) 4 (d) 8

119. Let M be a 3×3 matrix satisfying $M\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}-1\\2\\3\end{bmatrix}$,

$$M\begin{bmatrix}1\\-1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\-1\end{bmatrix} \text{ and } M\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\12\end{bmatrix}.$$

The sum of the diagonal entries of M is **[IIT- JEE 2011, 4M]**

120. Let A and B are symmetric matrices of order 3.

Statement-1 $A(BA)$ and $(AB)A$ are symmetric matrices.

Statement-2 AB is symmetric matrix, if matrix multiplication of A with B is commutative.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1

(b) Statement-1 is true, Statement-2 is false

(c) Statement-1 is false, Statement-2 is true

(d) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1 **[AIEEE 2011, 4M]**

121. Let $P = [a_{ij}]$ be a 3×3 matrix and $Q = [b_{ij}]$, where $b_{ij} = 2^{i+j}\ a_{ij}$ for $1 \le i, j \le 3$. If the determinant of P is 2, the determinant of the matrix Q is **[IIT- JEE 2012, 3M]**

(a) 2^{11} (b) 2^{12}

(c) 2^{13} (d) 2^{10}

122. If P is a 3×3 matrix such that $P^T = 2P + I$, where P^T is the transpose of P and I is the 3×3 identity matrix, then there exists a column matrix $X = \begin{bmatrix}x\\y\\z\end{bmatrix} \neq \begin{bmatrix}0\\0\\0\end{bmatrix}$ such that **[IIT- JEE 2012, 3M]**

(a) $PX = \begin{bmatrix}0\\0\\0\end{bmatrix}$ (b) $PX = X$ (c) $PX = 2X$ (d) $PX = -X$

123. If the adjoint of a 3×3 matrix P is $\begin{bmatrix}1 & 4 & 4\\ 2 & 1 & 7\\ 1 & 1 & 3\end{bmatrix}$, then the possible value(s) of the determinant of P is (are) **[IIT- JEE 2012, 4M]**

(a) -2 (b) -1 (c) 1 (d) 2

124. If $A = \begin{pmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 3 & 2 & 1\end{pmatrix}$, u_1 and u_2 are the column matrices such that $Au_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$ and $Au_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$, then $u_1 + u_2$ is equal to **[AIEEE 2012, 4M]**

(a) $\begin{pmatrix}-1\\-1\\0\end{pmatrix}$ (b) $\begin{pmatrix}1\\-1\\-1\end{pmatrix}$ (c) $\begin{pmatrix}-1\\1\\0\end{pmatrix}$ (d) $\begin{pmatrix}-1\\1\\-1\end{pmatrix}$

125. Let P and Q be 3×3 matrices with $P \neq Q$. If $P^3 = Q^3$ and $P^2Q = Q^2P$, the determinant of $(P^2 + Q^2)$ is equal to **[AIEEE 2012, 4M]**

(a) 0 (b) -1 (c) -2 (d) 1

126. If $P = \begin{bmatrix}1 & \alpha & 3\\ 1 & 3 & 3\\ 2 & 4 & 4\end{bmatrix}$ is the adjoint of a 3×3 matrix A and $|A| = 4$, then α is equal to **[JEE Main 2013, 4M]**

(a) 11 (b) 5 (c) 0 (d) 4

127. For 3×3 matrices M and N, which of the following statement(s) is (are) **not** correct?

(a) N^TMN is symmetric or skew-symmetric, according as M is symmetric or skew-symmetric

(b) $MN - NM$ is skew-symmetric for all symmetric matrices M and N

(c) MN is symmetric for all symmetric matrices M and N

(d) $(\text{adj}\,M)(\text{adj}\ N) = \text{adj}\,(MN)$ for all invertible matrices M and N **[JEE Advanced 2013, 4M]**

128. Let ω be a complex cube root of unity with $\omega \neq 1$ and $P = [p_{ij}]$ be a $n \times n$ matrix with $p_{ij} = \omega^{i+j}$. Then, $p^2 \neq 0$, when n is equal to **[JEE Advanced 2013, 3M]**

(a) 55 (b) 56 (c) 57 (d) 58

129. If A is a 3×3 non-singular matrix such that $AA' = A'A$ and $B = A^{-1}A'$, then BB' equals to **[JEE Main 2014, 4M]**

(a) B^{-1} (b) $(B^{-1})'$ (c) $I + B$ (d) I

130. Let M be a 2×2 symmetric matrix with integer entries. Then, M is invertible, if

(a) the first column of M is the transpose of the second row of M

(b) the second row of M is the transpose of the first column of M

(c) m is a diagonal matrix with non-zero entries in the main diagonal

(d) the product of entries in the main diagonal of M is not the square of an integer **[JEE Advanced 2014, 3M]**

131. Let M and N be two 3×3 matrices such that $MN = NM$. Further, if $M \neq N^2$ and $M^2 = N^4$, then

(a) determinant of $(M^2 + MN^2)$ is 0

(b) there is a 3×3 non-zero matrix U such that $(M^2 + MN^2)U$ is the zero matrix

(c) determinant of $(M^2 + MN^2) \geq 1$

(d) for a 3×3 matrix U, if $(M^2 + MN^2)U$ equals the zero matrix, then U is the zero matrix **[JEE Advanced 2014, 3M]**

132. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ is a matrix satisfying the equation $AA^T = 9I$, where I is 3×3 identity matrix, then the ordered pair (a, b) is equal to **[JEE Main 2015, 4M]**

(a) (2, 1) (b) (−2, −1) (c) (2, −1) (d) (−2, 1)

133. Let X and Y be two arbitrary 3×3 non-zero, skew-symmetric matrices and Z be an arbitrary 3×3 non-zero, symmetric matrix. Then, which of the following matrices is (are) skew-symmetric? **[JEE Advanced 2015, 4M]**

(a) $Y^3Z^4 - Z^4Y^3$ (b) $X^{44} + Y^{44}$

(c) $X^4Z^3 - Z^3X^4$ (d) $X^{23} + Y^{23}$

134. If $A = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}$ and $A\,\text{adj}A = AA^T$, then $5a + b$ is equal to **[JEE Main 2016, 4M]**

(a) 5 (b) 13

(c) 4 (d) −1

135. Let $P = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & -5 & 0 \end{bmatrix}$, where $\alpha \in R$. Suppose $Q = [q_{ij}]$ is a matrix such that $PQ = kI$, where $k \in R, k \neq 0$ and I is the identity matrix of order 3. If $q_{23} = -\frac{k}{8}$ and $\det.(Q) = \frac{k^2}{2}$, then **[JEE Advanced 2016, 4M]**

(a) $\alpha = 0, k = 8$ (b) $4\alpha - k + 8 = 0$

(c) det (Padj (Q)) $= 2^9$ (d) det (Q adj (P)) $= 2^{13}$

136. Let $z = \frac{-1 + \sqrt{3}i}{2}$, where $i = \sqrt{-1}$, and $r, s = \{1, 2, 3\}$. Let $P = \begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix}$ and I be the identity matrix of oreder 2. Then the total number of ordered pairs (r, s) for which $p^2 = -I$ is **[JEE Advanced 2016, 3M]**

(a) $\frac{1}{2}|\mathbf{a} - \mathbf{b}|$ (b) $\frac{1}{2}|\mathbf{a} + \mathbf{b}|$

(c) $|\mathbf{a} - \mathbf{b}|$ (d) $|\mathbf{a} + \mathbf{b}|$

137. Let $P = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}$ and I be the identity matrix of order 3. If $Q = [q_{ij}]$ is a matrix such that $P^{50} - Q = I$, then $\frac{q_{31} + q_{32}}{q_{21}}$ equals **[JEE Advanced 2016, 3M]**

(a) 52 (b) 103

(c) 201 (d) 205

138. If $A = \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$, then adj $(3A^2 + 12A)$ is equal to **[JEE Main 2017, 4M]**

(a) $\begin{bmatrix} 72 & -63 \\ -84 & 51 \end{bmatrix}$ (b) $\begin{bmatrix} 72 & -84 \\ -63 & 51 \end{bmatrix}$

(c) $\begin{bmatrix} 51 & 63 \\ 84 & 72 \end{bmatrix}$ (d) $\begin{bmatrix} 51 & 84 \\ 63 & 72 \end{bmatrix}$

Answers

Exercise for Session 1

1. (b) **2.** (b) **3.** (d) **4.** (b) **5.** (a) **6.** (b)
7. (b) **8.** (b) **9.** (c)

Exercise for Session 2

1. (d) **2.** (a) **3.** (c) **4.** (a) **5.** (b) **6.** (c)
7. (d) **8.** (b) **9.** (d) **10.** (c) **11.** (b) **12.** (d)
13. (b) **14.** (a) **15.** (b) **16.** (b) **17.** (c) **18.** (b)
19. (b)

Exercise for Session 3

1. (d) **2.** (c) **3.** (b) **4.** (d) **5.** (a) **6.** (b)
7. (d) **8.** (c) **9.** (a) **10.** (d) **11.** (c) **12.** (a)
13. (d) **14.** (b) **15.** (a) **16.** (d)

Exercise for Session 4

1. (a) **2.** (a) **3.** (d) **4.** (d) **5.** (d) **6.** (d)
7. (d) **8.** (b) **9.** (d)

Chapter Exercises

1. (d) **2.** (c) **3.** (b) **4.** (d) **5.** (c) **6.** (b)
7. (a) **8.** (a) **9.** (a) **10.** (b) **11.** (b) **12.** (b)
13. (b) **14.** (d) **15.** (d) **16.** (b) **17.** (b) **18.** (d)
19. (d) **20.** (d) **21.** (c) **22.** (d) **23.** (c) **24.** (d)
25. (b) **26.** (a) **27.** (d) **28.** (b) **29.** (a) **30.** (c)
31. (a, d) **32.** (a, b, d) **33.** (a, b, d) **34.** (b, c)
35. (b, d) **36.** (a, b, c) **37.** (a, c, d) **38.** (a, d)
39. (a,c,d) **40.** (a, c) **41.** (a, c, d) **42.** (a,b,c,d)
43. (c, d) **44.** (a, b, c) **45.** (a, c)
46. (b) **47.** (b) **48.** (c) **49.** (b) **50.** (d) **51.** (d)
52. (d) **53.** (c) **54.** (d) **55.** (a) **56.** (c) **57.** (a)
58. (b) **59.** (c) **60.** (a) **61.** (a)
62. (3) **63.** (2) **64.** (1) **65.** (9) **66.** (2) **67.** (7)
68. (9) **69.** (1) **70.** (2) **71.** (6)
72. (A) → (p, r); (B) → (s); (C) → (q); (D) → (s)
73. (A) → (r, t); (B) → (s); (C) → (p); (D) → (q)
74. (A) → (q, s); (B) → (p, t); (C) → (p, q, r, s); (D) → (q, s)
75. (A) → (q, t); (B) → (p, s); (C) → (p, r, s); (D) → (q, r, t)
76. (d) **77.** (c) **78.** (d) **79.** (c) **80.** (a) **81.** (d)
82. (d) **83.** (d) **84.** (d) **85.** (a)

88. $\alpha = \dfrac{2\pi}{3}$ **89.** $\begin{bmatrix} \cos\theta\cos\phi\cos(\theta \sim \phi) & \cos\theta\sin\phi\cos(\theta \sim \phi) \\ \sin\theta\cos\phi\cos(\theta \sim \phi) & \sin\theta\sin\phi\cos(\theta \sim \phi) \end{bmatrix}$

91. (i) Number of posts in all the offices taken together are 5 office superintendents; 235 head clerks; 235 cashiers; 275 clerks; 5 typists and 270 peons.
(ii) Total basic monthly salary bill of each division or district and taluka offices an ₹1675, ₹875 and ₹625, respectively.
(iii) Total basic monthly salary bill of all the offices taken together is ₹ 159625.

92. ₹53000; ₹44500; ₹34000, respectively

94. $x = 1, u = -1, y = 3, v = 2, z = 5, w = 1$

95. $x_1 = z_1 - 2z_2 + 9z_3, x_2 = 9z_1 + 10z_2 + 11z_3, x_3 = 7z_1 + z_2 - 2z_3$

96. (i) $k \neq 7$ (ii) $k = 7$

98. (c) **99.** (a) **100.** (c)
101. (c) **102.** (i) (a), (ii) (b), (iii) (a) **103.** (b) **104.** (b)
105. (c) **106.** (b,d) **107.** (d) **108.** (c)
109. (i) (a), (ii) (b), (iii) (b)
110. (b) **111.** (a)
112. (i) (d), (ii) (c), (iii) (d) **113.** (4) **114.** (c)
115. (b) **116.** (c)
117. (i) (d), (ii) (a), (iii) (b)
118. (a) **119.** (9) **120.** (a) **121.** (c)
122. (d) **123.** (a, d) **124.** (b) **125.** (a) **126.** (a) **127.** (c,d)
128. (a,b,d) **129.** (d) **130.** (c, d) **131.** (a, b) **132.** (b) **133.** (c, d)
134. (a) **135.** (b,c) **136.** (1) **137.** (b) **138.** (c)

Solutions

1. $\because \quad A^4(I-A) = A^4I - A^5 = A^4 - 0 = A^4 \neq I$

$A^3(I-A) = A^3I - A^4 = A^3 - A^4 \neq I$

$(I+A)(I-A) = I^2 - A^2 = I - A^2 \neq I$

2. $\because \det(B) = \begin{vmatrix} 4x & 2a & -p \\ 4y & 2b & -q \\ 4z & 2c & -r \end{vmatrix} = -8 \begin{vmatrix} x & a & p \\ y & b & q \\ z & c & r \end{vmatrix}$

$= -8 \begin{vmatrix} x & y & z \\ a & b & c \\ p & q & r \end{vmatrix} = 8 \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}$ [by property]

$= -8 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = -8 \det(A) = -16$

3. $\because \quad \left(A - \frac{1}{2}I\right)\left(A - \frac{1}{2}I\right)^T = I$ ...(i)

and $\left(A + \frac{1}{2}I\right)\left(A + \frac{1}{2}I\right)^T = I$ ...(ii)

$\Rightarrow \quad \left(A - \frac{1}{2}I\right)\left(A^T - \frac{I}{2}\right) = I$

and $\left(A + \frac{1}{2}I\right)\left(A^T + \frac{1}{2}I\right) = I$

$\Rightarrow \quad A + A^T = 0$ [subtracting the two results]

$\Rightarrow \quad A^T = -A$

$\therefore$ A is skew-symmetric matrix.

From first result, we get

$AA^T = \frac{3}{4}I$

$\Rightarrow \quad A^2 = -\frac{3}{4}I$

$\therefore \quad |A^2| = \left|-\frac{3}{4}I\right|$

$\therefore \quad |A|^2 = \left(-\frac{3}{4}\right)^n$

$\Rightarrow$ n is even.

4. $\because \quad a = \lim_{x\to 1}\left(\frac{x}{\ln x} - \frac{1}{x \ln x}\right) = \lim_{x\to 1}\left(\frac{x^2-1}{x \ln x}\right)$

$= \lim_{x\to 1}\left(\frac{2x}{1 + \ln x}\right)$ [by L'Hospital's Rule]

$b = \lim_{x\to 0}\left(\frac{x^3 - 16x}{4x + x^2}\right)$

$= \lim_{x\to 0}\left(\frac{x(x+4)(x-4)}{x(x+4)}\right) = \lim_{x\to 0}(x-4) = -4$

$c = \lim_{x\to 0}\frac{\ln(1+\sin x)}{x}$

$= \lim_{x\to 0}\frac{\ln(1+\sin x)}{\sin x} \cdot \lim_{x\to 0}\frac{\sin x}{x} = 1 \cdot 1 = 1$

$d = \lim_{x\to -1}\frac{(x+1)^3}{3[\sin(x+1) - (x+1)]}$

$= \lim_{x\to -1}\frac{3(x+1)^2}{3[\cos(x+1) - 1]}$ [using L'Hospital's Rule]

$= -\lim_{x\to -1}\frac{1}{\frac{[1-\cos(x+1)]}{(x+1)^2}} = -2$

Let, $A = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \Rightarrow A^2 = 0$

5. $\because \quad (A - \lambda I)X = 0$

$\therefore \quad |A - \lambda I| = 0$

$\Rightarrow \quad \begin{vmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow \quad \lambda^2 - 3\lambda - 10 = 0$

$\therefore \quad \lambda = -2, 5$

For $\lambda = -2 \quad \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$

For $\lambda = 5 \quad \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore \cos\theta = \frac{4\cdot 1 + (-3)\cdot 1}{\sqrt{(16+9)}\sqrt{(1+1)}} = \frac{1}{5\sqrt{2}}$

$\therefore \tan\theta = \sqrt{(\sec^2\theta - 1)} = \sqrt{49} = 7$

6. $\because A^{2n+1} = (A^2)^n \cdot A = (I)^n \cdot A = IA = A$

7. $\because \quad A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$\therefore \quad A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$

$\Rightarrow \lim_{n\to\infty}\frac{A^n}{n} = \begin{bmatrix} \lim_{n\to\infty}\frac{\cos n\theta}{n} & \lim_{n\to\infty}\frac{\sin n\theta}{n} \\ -\lim_{n\to\infty}\frac{\sin n\theta}{n} & \lim_{n\to\infty}\frac{\cos n\theta}{n} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$=$ a zero matrix [$\because -1 < \sin\infty < 1$ and $-1 < \cos\infty < 1$]

8. Let $A = \begin{bmatrix} -1 & 2 & 5 \\ 2 & -4 & -10 \\ 1 & -2 & -5 \end{bmatrix}$ [$\because a = -6$]

Applying $R_2 \to R_2 + 2R_1$ and $R_3 \to R_3 + R_1$, then

$A = \begin{bmatrix} -1 & 2 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \rho(A) = 1$

9. $\because$ A is involutory

$\therefore \quad A^2 = I \Rightarrow A = A^{-1}$

$\therefore \quad \left(\frac{A}{2}\right)^{-1} = 2A^{-1} = 2A$

10. $\because \quad B = \text{adj } A$

$\Rightarrow \quad AB = A(\text{adj } A) = |A| I_n$

$\therefore \quad AB + kI_n = |A| I_n + kI_n = (|A| + k) I_n$

$\Rightarrow \quad |AB + kI_n| = |(|A| + k) I_n| = (|A| + k)^n$

11. $\because \quad B = -A^{-1}BA$

$\Rightarrow \quad AB = -BA$

$\Rightarrow \quad AB + BA = 0$

Now, $(A + B)^2 = (A + B)(A + B)$

$= A^2 + AB + BA + B^2$

$= A^2 + 0 + B^2$

$= A^2 + B^2$

12. Since, A is skew-symmetric.

$\therefore \quad |A| = 0$

$\Rightarrow \quad |A^4 B^3| = |A^4||B^3| = |A|^4 |B|^3 = 0$

13. Let $B = A + I_n$

$\therefore \quad A = B - I_n$

Given, $A^n = \alpha A$

$\Rightarrow \quad (B - I_n)^n = \alpha (B - I_n)$

$\Rightarrow \quad B^n - {}^nC_1 B^{n-1} + {}^nC_2 B^{n-2} + \ldots + (-1)^n I_n$

$= \alpha B - \alpha I_n$

$\Rightarrow B(B^{n-1} - {}^nC_1 B^{n-2} + {}^nC_2 B^{n-3} + \ldots + (-1)^{n-1} I_n - \alpha I_n)$

$= [(-1)^{n+1} - \alpha] I_n \neq 0 \qquad [\because \alpha \neq \pm 1]$

Hence, B is invertible.

14. $\because \ \omega = \dfrac{-1 + i\sqrt{3}}{2}$ and $\omega^2 = \dfrac{-1 - i\sqrt{3}}{2}$

Also, $\omega^3 = 1$ and $\omega + \omega^2 = -1$

Thus, $A = \begin{bmatrix} -i\omega & -i\omega^2 \\ i\omega^2 & i\omega \end{bmatrix}$

$\therefore \quad A^2 = \begin{bmatrix} -i\omega & -i\omega^2 \\ i\omega^2 & i\omega \end{bmatrix}\begin{bmatrix} -i\omega & -i\omega^2 \\ i\omega^2 & i\omega \end{bmatrix} = \begin{bmatrix} -\omega^2 + \omega & 0 \\ 0 & -\omega^2 + \omega \end{bmatrix}$

Now, $f(A) = A^2 + 2I = \begin{bmatrix} -\omega^2 + \omega & 0 \\ 0 & -\omega^2 + \omega \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$= \begin{bmatrix} -\omega^2 + \omega + 2 & 0 \\ 0 & -\omega^2 + \omega + 2 \end{bmatrix}$

$= (-\omega^2 + \omega + 2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (2 + i\sqrt{3})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

15. $\because \quad X^2 = I \Rightarrow (X^{-1}X) X = X^{-1}I$

$\Rightarrow \quad IX = X^{-1}$

$\Rightarrow \quad X = X^{-1}$

which is self invertible involutory matrix.

There are many such matrices which are inverse of their own.

16. $\because \quad AB = A + B$

$\Rightarrow \quad B = AB - A = A(B - I)$

$\Rightarrow \quad \det(B) = \det(A) \cdot \det(B - I) = 0$

$[\because A^{2006} = 0 \Rightarrow \det A^{2006} = 0] \ [\therefore \det A = 0]$

17. We have, $P^T = P^{-1} \qquad [\because PP^T = I]$

Now, $Q = PAP^T = PAP^{-1}$

$\therefore \quad Q^{2007} = PA^{2007}P^{-1}$

$\therefore \quad P^T Q^{2007} P = P^{-1}(PA^{2007}P^{-1}) P$

$= A^{2007} = \begin{bmatrix} 1 & 2007 \\ 0 & 1 \end{bmatrix}$

$\left[\because A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \ldots\right]$

18. $\because \quad \begin{bmatrix} A - 5 & B \\ 2A - 2 & C \end{bmatrix} = \begin{bmatrix} 2A + 1 & -5 \\ -4 & A \end{bmatrix}\begin{bmatrix} 14 & D \\ E & F \end{bmatrix}$

$\Rightarrow \quad A - 5 = 28A + 14 - 5E$

$\Rightarrow \quad 5E = 27A + 19 \qquad \ldots(i)$

$2A - 2 = -56 + AE$

$\Rightarrow \quad AE = 2A + 54 \qquad \ldots(ii)$

From Eq. (i), we get

$5AE = 27A^2 + 19A$

$\Rightarrow \quad 5(2A + 54) = 27A^2 + 19A \qquad$ [from Eq. (ii)]

$\Rightarrow \quad 27A^2 + 9A - 270 = 0$

$\Rightarrow \quad 9(A - 3)(3A + 10) = 0$

$\therefore \quad A = 3, A = -\dfrac{10}{3}$

$\therefore$ Absolute value of difference

$= \left|3 + \dfrac{10}{3}\right| = \dfrac{19}{3}$

19. $\because |f(\theta)| = \begin{vmatrix} \cos^2\theta & \cos\theta \sin\theta & -\sin\theta \\ \cos\theta \sin\theta & \sin^2\theta & \cos\theta \\ \sin\theta & -\cos\theta & 0 \end{vmatrix}$

On multiplying in R_3 by $\cos\theta$ and then take common $\cos\theta$ from C_1, then

$$|f(\theta)| = \begin{vmatrix} \cos\theta & \cos\theta \sin\theta & -\sin\theta \\ \sin\theta & \sin^2\theta & \cos\theta \\ \sin\theta & -\cos^2\theta & 0 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_3$, we get

$$|f(\theta)| = \begin{vmatrix} \cos\theta & \cos\theta\sin\theta & -\sin\theta \\ 0 & 1 & \cos\theta \\ \sin\theta & -\cos^2\theta & 0 \end{vmatrix} = 1$$

Applying $C_2 \to C_2 - \sin\theta\, C_1$, then

$$|f(\theta)| = \begin{vmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & \cos\theta \\ \sin\theta & -1 & 0 \end{vmatrix} = 1$$

$\therefore \quad f\left(\dfrac{\pi}{7}\right)$ is non-singular matrix.

20. $\because\ a_{11} = a_{22} = a_{33} = a + b,$

$a_{12} = a_{23} = ab,\ a_{21} = a_{32} = 1,\ a_{13} = a_{31} = 0$

$$\therefore \quad A = \begin{bmatrix} a+b & ab & 0 \\ 1 & a+b & ab \\ 0 & 1 & a+b \end{bmatrix}$$

$$\Rightarrow |A| = \begin{vmatrix} a+b & ab & 0 \\ 1 & a+b & ab \\ 0 & 1 & a+b \end{vmatrix}$$

$$= (a+b)[(a+b)^2 - ab] - ab(a+b) = (a+b)(a^2+b^2)$$

21. Given, $B^r = I \Rightarrow B^r B^{-1} = IB^{-1}$

$$\Rightarrow \quad B^{r-1} = B^{-1}$$

$$\therefore \quad A^{-1}B^{r-1}A = A^{-1}B^{-1}A$$

$$\Rightarrow \quad A^{-1}B^{r-1}A - A^{-1}B^{-1}A = 0$$

22. Here, $A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$

$$\Rightarrow \quad AA^T = I$$

$\because \quad C = ABA^T \Rightarrow A^TC = BA^T$

Now, $A^TC^nA = A^TC \cdot C^{n-1}A$

$$= BA^TC^{n-1}A = BA^TC\,C^{n-2}A$$

$$= B^2A^TC^{n-2}A$$

$$\ldots \quad \ldots \quad \ldots$$

$$= B^{n-1}A^TCA = B^{n-1}(BA^T)\,A$$

$$= B^nA^TA = B^nI = B^n = \begin{bmatrix} 1 & 0 \\ -n & 1 \end{bmatrix}$$

23. $\because\ |\text{adj}\,A^{-1}| = |A^{-1}|^2 = \dfrac{1}{|A|^2}$

$$\therefore\ |(\text{adj}\,A^{-1})^{-1}| = \frac{1}{|\text{adj}\,A^{-1}|} = |A|^2 = 2^2 = 4$$

24. $\because\ A^3 - A^2B = B^3 - B^2A$

$$\Rightarrow \quad A^2(A-B) = B^2(B-A)$$

or $\quad (A^2 + B^2)(A-B) = 0$

or $\det(A^2+B^2)\cdot\det(A-B) = 0$

Either $\det(A^2+B^2) = 0$ or $\det(A-B) = 0$

25. Let, $A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$,

$$BC = \begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}\begin{bmatrix} 9 & -4 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\therefore\ B^2C^2 = (BC)^2 = I^2 = I$$

Similarly, $B^2C^2 = B^3C^3 = \ldots = B^nC^n = I$

Let, $\quad D = A^3(BC) + A^5(B^2C^2) + A^7(B^3C^3)$

$$+ \ldots + A^{2n+1}(B^nC^n)$$

$$= A^3 + A^5 + A^7 + \ldots + A^{2n+1}$$

$$= A(A^2 + A^4 + A^6 + \ldots + A^{2n})$$

Let, $\quad A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$

$$\Rightarrow \quad A^2 = \begin{bmatrix} -a^2 & 0 \\ 0 & -a^2 \end{bmatrix} = -a^2I$$

$$\therefore \quad D = IA(-a^2 + a^4 - a^6 + \ldots + (-1)^n a^{2n}) \qquad [a > 0]$$

$$= A(-a^2 + a^4 - a^6 + \ldots + (-1)^n a^{2n})$$

Hence, D is skew-symmetric.

26. $\because \quad |B| = \begin{vmatrix} q & -b & y \\ -p & a & -x \\ r & -c & z \end{vmatrix}$

Applying $R_2 \to (-1)\,R_2$, then

$$|B| = \begin{vmatrix} q & -b & y \\ p & -a & x \\ r & -c & z \end{vmatrix}$$

Applying $C_2 \to (-1)\,C_2$, then

$$|B| = \begin{vmatrix} q & b & y \\ p & a & x \\ r & c & z \end{vmatrix} = |B^T| = \begin{vmatrix} q & p & r \\ b & a & c \\ y & x & z \end{vmatrix}$$

$$= -\begin{vmatrix} b & a & c \\ q & p & r \\ y & x & z \end{vmatrix} \qquad [R_1 \leftrightarrow R_2]$$

$$= \begin{vmatrix} b & a & c \\ y & x & z \\ q & p & r \end{vmatrix} \qquad [R_2 \leftrightarrow R_30]$$

$$= -\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = -|A|$$

$$\Rightarrow \quad |B| = -|A|$$

Also, $|\text{adj}\,B| = |B|^2$

$$= |A|^2 = |\text{adj}\,A| \qquad [\because |A| \neq 0, \text{ then } |B| \neq 0]$$

27. $\because \quad BC = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$$\therefore \quad \text{tr}(A) + \text{tr}\left(\frac{ABC}{2}\right) + \text{tr}\left(\frac{A(BC)^2}{4}\right) + \text{tr}\left(\frac{A(BC)^3}{8}\right) + \ldots$$

$$= \text{tr}(A) + \text{tr}\left(\frac{A}{2}\right) + \text{tr}\left(\frac{A}{2^2}\right) + \text{tr}\left(\frac{A}{2^3}\right) + \ldots \text{ upto } \infty$$

$$= \text{tr}(A) + \frac{1}{2}\text{tr}(A) + \frac{1}{2^2}\text{tr}(A) + \ldots \text{ upto } \infty$$

$$= \frac{\text{tr}(A)}{1 - \left(\frac{1}{2}\right)} = 2\,\text{tr}(A) = 2(2+1) = 6$$

28. We have, $\quad (A - 2I)(A - 4I) = 0$

$$\Rightarrow \quad A^2 - 4A - 2A + 8I^2 = 0$$

$$\Rightarrow \quad A^2 - 6A + 8I = 0$$

$$\Rightarrow \quad A^{-1}(A^2 - 6A + 8I) = A^{-1}0$$

$$\Rightarrow \quad A - 6I + 8A^{-1} = 0$$

$$\Rightarrow \quad \frac{1}{6}A + \frac{4}{3}A^{-1} = I$$

29. We have, $AA^{-1} = I$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & a & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & 6 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & b+1 \\ 0 & 1 & 2(b+1) \\ 4(1-a) & 3(a-1) & ab+2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

On comparing, we get

$$b + 1 = 0, ab + 2 = 1, a - 1 = 0$$

$$\therefore \quad a = 1, b = -1$$

30. $\because \quad A(\text{adj } A) = |A| I$ ...(i)

Now, $$|A| = \begin{vmatrix} x & 3 & 2 \\ 1 & y & 4 \\ 2 & 2 & z \end{vmatrix}$$

$$= x(yz - 8) - 3(z - 8) + 2(2 - 2y)$$

$$= xyz - (8x + 4y + 3z) + 28$$

$$= 60 - 20 + 28 = 68$$

From Eq. (i), $A(\text{adj } A) = 68I$

31. Here, $|A| = 0$

$\therefore A^{-1}$ does not exist.

Now, $$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} = 3A$$

$$\therefore \quad A^3 = A^2 \cdot 2 = 3A \cdot A = 3A^2 = 3(3A) = 9A$$

32. $\because \quad A' = A^{-1} \Rightarrow AA' = I$ ...(i)

Now, $(A')' A' = I$

$\therefore A'$ is orthogonal

From Eq. (i), $(AA')^{-1} = I^{-1}$

$$\Rightarrow \quad (A')^{-1} A^{-1} = I$$

$$\Rightarrow \quad (A^{-1})'(A^{-1}) = I$$

$\therefore A^{-1}$ is orthogonal

Since, $\text{adj } A = A^{-1}|A| \neq A'$

and $$|A^{-1}| = \frac{1}{|A|} = \pm 1 \qquad [\text{for orthogonal } |A| = \pm 1]$$

33. $\because$ $$A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

We have, $A^2 - 4A - 5I_3$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$$\Rightarrow \quad 5I_3 = A^2 - 4A = A(A - 4I_3)$$

$$\Rightarrow \quad I_3 = \frac{1}{5}A(A - 4I_3)$$

$$\therefore \quad A^{-1} = \frac{1}{5}(A - 4I_3)$$

Since, $|A| = 5$

$$\therefore \quad |A^3| = |A|^3 = 125 \neq 0$$

$\Rightarrow A^3$ is invertible

Similarly, A^2 is invertible.

34. Let, $D = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = D^T$ and let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

$$\therefore \quad DA = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{bmatrix} aa_1 & aa_2 & aa_3 \\ bb_1 & bb_2 & bb_3 \\ cc_1 & cc_2 & cc_3 \end{bmatrix}$$

$$AD = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} a_1a & a_2b & a_3c \\ b_1a & b_2b & b_3c \\ c_1a & c_2b & c_3c \end{bmatrix} \neq DA$$

and $$D^{-1} = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}$$

$$|D^{-1}| = \frac{1}{abc} \neq 0 \qquad [\because a \neq 0, b \neq 0, c \neq 0]$$

35. Let $$A = \begin{bmatrix} -1 & 2 & 5 \\ 2 & -4 & a-4 \\ 1 & -2 & a+1 \end{bmatrix}$$

Applying $R_2 \to R_2 + 2R_1$ and $R_3 \to R_3 + R_1$, then

$$A = \begin{bmatrix} -1 & 2 & 5 \\ 0 & 0 & a+6 \\ 0 & 0 & a+6 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_2$, then

$$A = \begin{bmatrix} -1 & 2 & 5 \\ 0 & 0 & a+6 \\ 0 & 0 & 0 \end{bmatrix}$$

For $a = -6$, $\rho(A) = 1$

For $a = 1, 2$, $\rho(A) = 2$

36. Here, $|A| = \begin{vmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{vmatrix}$

$= 3(-3+4) + 3(2-0) + 4(-2+0) = 1 \neq 0$

$\because \quad \text{adj}(\text{adj}\, A) = |A|^{3-2} A = A$ ...(i)

and $\quad |\text{adj}(A)| = |A|^{3-1} = |A|^2 = 1^2 = 1$

Also, $|\text{adj}(\text{adj}(A))| = |A| = 1$ [from Eq. (i)]

37. $\because \quad A = I - B$

$\Rightarrow \quad A^2 = I^2 + B^2 - 2B = I - B = A$ [$\because B$ is idempotent]

and $\quad AB = B - B^2 = B - B = 0$ [null matrix]

and $\quad BA = B - B^2 = B - B = 0$ [null matrix]

38. $\because \quad |A| \neq 0 \Rightarrow A^{-1}$ is also symmetric, if A is symmetric

and $|A^{-1}| = \dfrac{1}{|A|} = |A|^{-1}$

39. $\because \quad A^2B = A(AB) = A(BA) = (AB)A = (BA)A = BA^2$

Similarly, $A^3B = BA^3$

In general, $A^nB = BA^n, \forall\, n \geq 1$

and $\quad (A+B)^n = {}^nC_0A^n + {}^nC_1A^{n-1}B + {}^nC_2A^{n-2}B^2 + \ldots + {}^nC_nB^n$

Also, $(A^n - B^n)(A^n + B^n) = A^nA^n + A^nB^n - B^nA^n - B^nB^n$

$= A^{2n} - B^{2n}$ [$\because AB = BA$]

40. $|AB| = 0 \Rightarrow |A||B| = 0$

$\therefore \quad |B| = 0$ as $|A| \neq 0$

Also, $\quad |A^{-1}| = |A|^{-1}$

41. Here, $A(A+I) = -2I$ ...(i)

$\Rightarrow \quad |A(A+I)| = |-2I| = (-2)^m \neq 0$

Thus, $\quad |A| \neq 0,$

also, $\quad I = -\dfrac{1}{2}A(A+I)$ [from Eq. (i)]

$\therefore \quad A^{-1} = -\dfrac{1}{2}(A+I)$

42. $\because \quad A^2 - 3A + 2I = 0$ ...(i)

$\Rightarrow \quad A^2 - 3AI + 2I^2 = 0$

$\Rightarrow \quad (A-I)(A-2I) = 0$

$\therefore \quad A = I$ or $A = 2I$

Characteristic Eq. (i) is

$\lambda^2 - 3\lambda + 2 = 0 \Rightarrow \lambda = 1, 2$

It is clear that alternate (c) and (d) have the characteristic equation $\lambda^2 - 3\lambda + 2 = 0$.

43. $\because \quad AB = 0$

$\Rightarrow \quad |AB| = 0 \Rightarrow |A||B| = 0$

or $\quad (\det A)(\det B) = 0$

$\Rightarrow$ Either $\det A = 0$ or $\det B = 0$

Hence, atleast one of the two matrices must be singular otherwise this statement is not possible.

44. Let $D_1 = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$ and $D_2 = \begin{bmatrix} d_4 & 0 & 0 \\ 0 & d_5 & 0 \\ 0 & 0 & d_6 \end{bmatrix}$

$\therefore D_1D_2 = \begin{bmatrix} d_1d_4 & 0 & 0 \\ 0 & d_2d_5 & 0 \\ 0 & 0 & d_3d_6 \end{bmatrix} = D_2D_1$

and $\quad D_1^2 + D_2^2 = \begin{bmatrix} d_1^2 & 0 & 0 \\ 0 & d_2^2 & 0 \\ 0 & 0 & d_3^2 \end{bmatrix} + \begin{bmatrix} d_4^2 & 0 & 0 \\ 0 & d_5^2 & 0 \\ 0 & 0 & d_6^2 \end{bmatrix}$

$= \begin{bmatrix} d_1^2 + d_4^2 & 0 & 0 \\ 0 & d_2^2 + d_5^2 & 0 \\ 0 & 0 & d_3^2 + d_6^2 \end{bmatrix}$

45. $A_1 + A_2 + A_3 + \ldots + A_n = \begin{bmatrix} C_0^2 & 0 \\ 0 & C_1^2 \end{bmatrix} + \begin{bmatrix} C_1^2 & 0 \\ 0 & C_2^2 \end{bmatrix} + \begin{bmatrix} C_2^2 & 0 \\ 0 & C_3^2 \end{bmatrix} + \ldots + \begin{bmatrix} C_{n-1}^2 & 0 \\ 0 & C_n^2 \end{bmatrix}$

$= \begin{bmatrix} C_0^2 + C_1^2 + C_2^2 + \ldots + C_{n-1}^2 & 0 \\ 0 & C_1^2 + C_2^2 + C_3^2 + \ldots + C_n^2 \end{bmatrix}$

$= \begin{bmatrix} {}^{2n}C_n - 1 & 0 \\ 0 & {}^{2n}C_n - 1 \end{bmatrix} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$ [given]

$\therefore \quad k_1 = k_2 = {}^{2n}C_n - 1$

Passage (Q. Nos. 46 to 48)

$\because \quad AB = BA^m$

$\Rightarrow \quad B = A^{-1}BA^m$

$\therefore \quad B^n = \underbrace{(A^{-1}BA^m)(A^{-1}BA^m)\ldots(A^{-1}BA^m)}_{n \text{ times}}$

$= A^{-1}\underbrace{BA^{m-1}BA^{m-1}\ldots BA^{m-1}BA^{m-1}}_{n \text{ times}}A$ ...(i)

Given, $\quad AB = BA^m$

$\Rightarrow \quad AAB = ABA^m = BA^{2m} \Rightarrow AAAB = BA^{3m}$

Similarly, $\quad A^xB = BA^{mx}\ \forall m \in N$

From Eq. (i), we get

$B^n = A^{-1}BA^{m-1}\underbrace{BA^{m-1}BA^{m-1}\ldots BA^{m-1}BA^{m-1}}_{(n-1) \text{ times}}A$

$= A^{-1}B(A^{m-1}B)A^{m-1}\underbrace{BA^{m-1}\ldots BA^{m-1}BA^{m-1}}_{(n-2) \text{ times}}A$

$= A^{-1}BBA^{(m-1)m}A^{m-1}\underbrace{BA^{m-1}\ldots BA^{m-1}BA^{m-1}}_{(n-2) \text{ times}}A$

$= A^{-1}B^2A^{(m^2-1)}\underbrace{BA^{m-1}\ldots BA^{m-1}BA^{m-1}}_{(n-2) \text{ times}}A$

$\ldots \qquad \ldots \qquad \ldots \quad \ldots$

$= A^{-1}B^m(A)^{(m^n-1)}A$

$I = A^{-1}IA^{(m^n-1)}A$ [$\because B^n = I$]

$I = A^{-1}A^{(m^n-1)}A = A^{-1}A^{m^n}$

$\Rightarrow \quad I = A^{(m^n-1)}$

$\therefore \quad p = m^n - 1$ ...(ii) [$\because A^p = I$]

46. Put $m = 2, n = 5$ in Eq. (ii), we get

$$p = 2^5 - 1 = 31$$

47. From Eq. (ii), we get

$$p = m^n - 1$$

48. From Eq. (ii), we get

$$510 \neq 8^3 - 1$$

Passage (Q. Nos. 49 to 51)

$\because$ A is an orthogonal matrix

$\therefore$ $AA^T = I$

$$\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} = 1\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a^2+b^2+c^2 & ab+bc+ca & ab+bc+ca \\ ab+bc+ca & a^2+b^2+c^2 & ab+bc+ca \\ ab+bc+ca & ab+bc+ca & a^2+b^2+c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By equality of matrices, we get

$$a^2 + b^2 + c^2 = 1 \quad \text{...(i)}$$

$$ab + bc + ca = 0 \quad \text{...(ii)}$$

$$(a + b + c)^2 + a^2 = b^2 + c^2 + 2(ab + bc + ca)$$

$$= 1 + 0 = 1$$

$\therefore$ $a + b + c = \pm 1$ ...(iii)

49. $\because a^2b^2 + b^2c^2 + c^2a^2 = (ab + bc + ca)^2 - 2abc(a + b + c)$

$$= 0 - 2abc(\pm 1) = \mp 2\lambda \qquad [\because abc = \lambda]$$

$$= -2\lambda \qquad [\because \lambda < 0]$$

50. $\because$ $a^3 + b^3 + c^3 - 3abc = (a + b + c)$

$$(a^2 + b^2 + c^2 - ab - bc - ca)$$

$\Rightarrow$ $a^3 + b^3 + c^3 - 3\lambda = (\pm 1)(1 - 0)$

[from Eqs. (i), (ii) and (iii) and $abc = \lambda$]

$\Rightarrow$ $a^3 + b^3 + c^3 = 3\lambda \pm 1$

51. Equation whose roots are a, b, c is

$$x^3 - (a + b + c)x^2 + (ab + bc + ca)x - abc = 0$$

$\Rightarrow$ $x^3 - (\pm 1)x^2 + 0 - \lambda = 0$

$\therefore$ $x^3 \pm x^2 - \lambda = 0$

Passage (Q. Nos. 52 to 53)

$\therefore$ $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$\Rightarrow$ $t_1 = \dfrac{a_{11} + a_{12} + a_{13}}{3} = 0, \qquad [\because a_{ij} + a_{jk} + a_{ki} = 0]$

$$t_2 = \frac{a_{21} + a_{22} + a_{23}}{3} = 0$$

and $t_3 = \dfrac{a_{31} + a_{32} + a_{33}}{3} = 0$

52. $\sum\limits_{1 \le i,\ j \le 3} a_{ij} = 3(t_1 + t_2 + t_3) = 0 = t_1 + t_2 + t_3$

$$\neq t_1t_2t_3 \qquad [\because t_1 = 0, t_2 = 0, t_3 = 0]$$

and $\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$= \begin{vmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} = 0$$

$\therefore$ $(\det A)^2 = 0$

53. $\because$ $a_{11} + a_{11} + a_{11} = 0, a_{11} + a_{12} + a_{21} = 0,$

$a_{11} + a_{13} + a_{31} = 0, a_{22} + a_{22} + a_{22} = 0,$

$a_{22} + a_{12} + a_{21} = 0, a_{22} + a_{23} + a_{32} = 0,$

$a_{33} + a_{13} + a_{31} = 0, a_{33} + a_{23} + a_{32} = 0$

and $a_{33} + a_{12} + a_{21} = 0$, we get

$a_{11} = a_{22} = a_{33} = 0$

and $a_{12} = -a_{21}, a_{23} = -a_{32}, a_{13} = -a_{31}$

Hence, A is skew-symmetric matrix.

Passage (Q. Nos. 54 to 56)

Let $B = \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix}$

$\therefore$ $C_1 = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}, C_2 = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$ and $C_3 = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix}$

$\Rightarrow$ $AC_1 = \begin{bmatrix} \alpha_1 \\ 2\alpha_1 + \alpha_2 \\ 3\alpha_1 + 2\alpha_2 + \alpha_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow$ $\alpha_1 = 1, \alpha_2 = -2, \alpha_3 = 1$

$\Rightarrow$ $AC_2 = \begin{bmatrix} \beta_1 \\ 2\beta_1 + \beta_2 \\ 3\beta_1 + 2\beta_2 + \beta_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$

$\Rightarrow$ $\beta_1 = 2, \beta_2 = -1, \beta_3 = -4$

and $AC_3 = \begin{bmatrix} \gamma_1 \\ 2\gamma_1 + \gamma_2 \\ 3\gamma_1 + 2\gamma_2 + \gamma_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

$\Rightarrow$ $\gamma_1 = 2, \gamma_2 = -1, \gamma_3 = -3$

$\therefore$ $B = \begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & -1 \\ 1 & -4 & -3 \end{bmatrix}$

$\Rightarrow$ $\det B = \begin{vmatrix} 1 & 2 & 2 \\ -2 & -1 & -1 \\ 1 & -4 & -3 \end{vmatrix}$

$$= 1(3 - 4) - 2(6 + 1) + 2(8 + 1) = 3$$

and $C = \dfrac{1}{3}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & -1 \\ 1 & -4 & -3 \end{bmatrix}$

$$= \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad \det C = \begin{vmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & 1 & 1 \\ 0 & 0 & \frac{1}{3} \end{vmatrix} = \frac{1}{9}$$

54. $\det(B^{-1}) = \dfrac{1}{\det B} = \dfrac{1}{3}$

55. $\dfrac{\text{Trace of } B}{\text{Trace of } C} = \dfrac{(-3)}{\left(\frac{5}{3}\right)} = -\dfrac{9}{5}$

56. $\sin^{-1}(\det A) + \tan^{-1}(9 \det C) = \sin^{-1}(1) + \tan^{-1}(1)$

$$= \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$$

Passage (Q. Nos. 57 to 59)

Given, $A^T = A$, $B^T = -B$, $\det(A + B) \neq 0$

and $C = (A + B)^{-1}(A - B)$

$\Rightarrow \quad (A + B)C = A - B$ ...(i)

Also, $(A + B)^T = A - B$ ...(ii)

and $(A - B)^T = A + B$ ...(iii)

57. $C^T(A + B)C = C^T[(A + B)C]$

$= C^T(A - B)$ [from Eq. (i)]

$= C^T(A + B)^T$ [from Eq. (ii)]

$= [(A + B)C]^T$

$= (A - B)^T$ [from Eq. (i)]

$= A + B$ [from Eq. (iii)]

58. $C^T(A - B)C = [C^T(A + B)^T]C$ [from Eq. (ii)]

$= [(A + B)C]^T C$

$= (A - B)^T C$ [from Eq. (i)]

$= (A + B)C$ [from Eq. (iii)]

$= A - B$ [from Eq. (i)]

59. $C^T AC = C^T\left(\dfrac{A + B + A - B}{2}\right)C$

$= \dfrac{1}{2}C^T(A + B)C + \dfrac{1}{2}C^T(A - B)C$

$= \dfrac{1}{2}(A + B) + \dfrac{1}{2}(A - B)$ [from Q13 and Q15]

$= A$

Passage (Q. Nos. 60 to 61)

$\because \quad B = A - 2I$

$\therefore \quad A^{-1}B = I - 2A^{-1}$ ...(i)

60. $\det[\text{adj}\,(I - 2A^{-1})] = \det[\text{adj}(A^{-1}B)]$ [from Eq. (i)]

$= |\text{adj}(A^{-1}B)|$

$= |A^{-1}B|^2 = (|A^{-1}||B|)^2 = \left(\dfrac{|B|}{|A|}\right)^2$ ...(ii)

From Eq. (i), we get $B = A - 2I$

$\therefore \quad B^3 = (A - 2I)^3 = A^3 - 6A^2 + 12A - 8I$

$= 5A \qquad [\because A^3 - 6A^2 + 7A - 8I = 0]$

$\Rightarrow \quad |B^3| = |5A|$

$\Rightarrow \quad |B|^3 = 5^3|A|$

$\Rightarrow \quad |B|^3 = 5^3 \times 8$

$\Rightarrow \quad |B|^3 = (10)^3$

$\therefore \quad |B| = 10$

From Eq. (ii), we get

$$\det[\text{adj}(I - 2A^{-1})] = \left(\frac{|B|}{|A|}\right)^2 = \left(\frac{10}{8}\right)^2 = \frac{25}{16}$$

61. $\text{adj}\left[\left(\dfrac{B}{2}\right)^{-1}\right] = \dfrac{\frac{B}{2}}{\left|\frac{B}{2}\right|} = \dfrac{\frac{B}{2}}{\frac{1}{8}|B|} = \dfrac{4B}{|B|} = \dfrac{4}{10}B \qquad [\because |B| = 10]$

$= \dfrac{2}{5}B = \dfrac{p}{q}B$ [given]

$\therefore \quad p = 2$ and $q = 5$

Hence, $p + q = 7$

62. $S = ABCD = A(BCD) = AA^T$ ...(i)

$\therefore \quad S^3 = (ABCD)(ABCD)(ABCD)$

$= (ABC)(DAB)(CDA)(BCD)$

$= D^T C^T B^T A^T = (BCD)^T A^T$

$= (A^T)^T A^T = AA^T = S$

$\Rightarrow \quad S^3 = S$

Hence, least value of k is 3.

63. $\because \quad A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$

$\therefore \quad \det A = \begin{vmatrix} 1 & \tan x \\ -\tan x & 1 \end{vmatrix} = (1 + \tan^2 x) = \sec^2 x$

$\Rightarrow \quad \det A^T = \det A = \sec^2 x$

Now, $f(x) = \det(A^T A^{-1}) = (\det A^T)(\det A^{-1})$

$= (\det A^T)(\det A)^{-1} = \dfrac{\det A^T}{\det A} = 1$

$\therefore \quad \underbrace{\lambda = f(f(f(f \ldots f(x))))}_{n \text{ times}} = 1 \qquad [\because f(x) = 1]$

Hence, $2^\lambda = 2^1 = 2$

64. $\because A^2 = A \cdot A = \begin{bmatrix} \lambda_1^2 & \lambda_1\lambda_2 & \lambda_1\lambda_3 \\ \lambda_2\lambda_1 & \lambda_2^2 & \lambda_2\lambda_3 \\ \lambda_3\lambda_1 & \lambda_3\lambda_2 & \lambda_3^2 \end{bmatrix}\begin{bmatrix} \lambda_1^2 & \lambda_1\lambda_2 & \lambda_1\lambda_3 \\ \lambda_2\lambda_1 & \lambda_2^2 & \lambda_2\lambda_3 \\ \lambda_3\lambda_1 & \lambda_3\lambda_2 & \lambda_3^2 \end{bmatrix}$

$$= \begin{bmatrix} \lambda_1^2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) & \lambda_1\lambda_2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) & \lambda_1\lambda_3(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) \\ \lambda_1\lambda_2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) & \lambda_2^2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) & \lambda_2\lambda_3(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) \\ \lambda_1\lambda_3(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) & \lambda_3\lambda_2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) & \lambda_3^2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) \end{bmatrix}$$

$$= (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)\, A$$

Given, A is idempotent

$\Rightarrow \quad A^2 = A$

$\therefore \quad \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$

65. Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and given $X^T AX = O$

$$\Rightarrow \quad [x_1\ x_2\ x_3] \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = O$$

$$\Rightarrow \quad [x_1\ x_2\ x_3] \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{bmatrix} = O$$

$$\Rightarrow \quad a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{21}x_1x_2 + a_{22}x_2^2 + a_{23}x_2x_3 + a_{31}x_1x_3 + a_{32}x_2x_3 + a_{33}x_3^2 = 0$$

$$\Rightarrow \quad a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + (a_{12} + a_{21})\, x_1x_2 + (a_{23} + a_{32})\, x_2x_3 + (a_{31} + a_{13})\, x_3x_1 = 0$$

it is true for every x_1, x_2, x_3, then

$$a_{11} = a_{22} = a_{33} = 0 \text{ and } a_{12} = -a_{21},\ a_{23} = -a_{32},\ a_{13} = -a_{31}$$

Now, as $a_{23} = -1008 \Rightarrow a_{32} = 1008$

$\therefore$ Sum of digits $= 1 + 0 + 0 + 8 = 9$

66. $\because A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$

$$\therefore \quad A^2 = A \cdot A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$\Rightarrow A^2 = I \Rightarrow A^4 = A^6 = A^8 = \ldots = I$

Now, $\quad A^x = I$

$\Rightarrow \quad x = 2, 4, 6, 8, \ldots$

$$\therefore \sum (\cos^x \theta + \sin^x \theta) = (\cos^2 \theta + \sin^2 \theta) + (\cos^4 \theta + \sin^4 \theta) + (\cos^6 \theta + \sin^6 \theta) + \ldots$$

$$= (\cos^2 \theta + \cos^4 \theta + \cos^6 \theta + \ldots) + (\sin^2 \theta + \sin^4 \theta + \sin^6 \theta + \ldots)$$

$$= \frac{\cos^2 \theta}{1 - \cos^2 \theta} + \frac{\sin^2 \theta}{1 - \sin^2 \theta}$$

$$= \cot^2 \theta + \tan^2 \theta \geq 2$$

Hence, minimum value of $\sum (\cos^x \theta + \sin^x \theta)$ is 2.

67. $\because A$ is idempotent matrix

$\therefore \quad A^2 = A$

$\Rightarrow \quad A = A^2 = A^3 = A^4 = A^5 = \ldots$ ...(i)

Now, $(A + I)^n = (I + A)^n$

$$= I + {}^nC_1\, A + {}^nC_2\, A^2 + {}^nC_3\, A^3 + \ldots + {}^nC_n A^n$$

$$= I + ({}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_n)\, A$$

[from Eq.(i)]

$\Rightarrow \quad (A + I)^n = I + (2^n - 1)A$ ...(ii)

Given, we get

$(A + I)^n = I + 127\, A$ ...(iii)

From Eqs (ii) and (iii), we get

$$2^n - 1 = 127$$

$\Rightarrow \quad 2^n = 128 = 2^7$

$\therefore \quad n = 7$

68. $\because \quad A = \begin{bmatrix} 3a & b & c \\ b & 3c & a \\ c & a & 3b \end{bmatrix}$

$$\therefore \det(A) = \begin{vmatrix} 3a & b & c \\ b & 3c & a \\ c & a & 3b \end{vmatrix} = 29abc - 3(a^3 + b^3 + c^3)$$

Or

$$|A| = 29abc - 3(a^3 + b^3 + c^3) \quad \ldots(i)$$

Given, $\quad A^TA = 4^{1/3}\, I$

$\Rightarrow \quad |A^T\, A| = |4^{1/3}\, I|$

$\Rightarrow \quad |A^T|\,|A| = (4^{1/3})^3\, |I|$

$\Rightarrow \quad |A|\,|A| = 4 \cdot 1$

$\Rightarrow \quad |A|^2 = 4$

$\therefore \quad |A| = 2 \quad [\because |A| > 0]$

From Eq.(i), we get

$$2 = 29abc - 3(a^3 + b^3 + c^3)$$

$\Rightarrow \quad 2 = 29 - 3(a^3 + b^3 + c^3) \quad [\because abc = 1]$

$\therefore \ a^3 + b^3 + c^3 = 9$

69. $\because \quad A = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$

$$\therefore A^2 = A \cdot A = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = 3I$$

$\Rightarrow \quad A^4 = (A^2)^2 = 9I,\ A^6 = 27I,\ A^8 = 81I$

Now, $\quad (A^8 + A^6 + A^4 + A^2 + I)\, V = (121)\, IV = (121)\, V$ ...(i)

Given, $(A^8 + A^6 + A^4 + A^2 + I)\, V = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$ ...(ii)

From Eqs.(i) and (ii), $(121)\, V = \begin{bmatrix} 0 \\ 11 \end{bmatrix} \Rightarrow V = \begin{bmatrix} 0 \\ \frac{1}{11} \end{bmatrix}$

$\therefore$ Sum of elements of $V = 0 + \frac{1}{11} = \frac{1}{11} = \lambda$ [given]

$\therefore \quad 11\, \lambda = 1$

70. $\because A = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 \\ 7 & 3 \end{bmatrix}$

$\therefore \quad \det A = -1$ and $\det B = 2$

Now, $\det(2A^9\, B^{-1}) = 2^2 \cdot \det(A^9) \cdot \det(B^{-1})$

$$= 2^2 \cdot (\det A)^9 \cdot (\det B)^{-1}$$

$$= 2^2 \cdot (-1)^9 \cdot (2)^{-1} = -2$$

Hence, absolute value of $\det(2A^9B^{-1}) = 2$

71. $\because \quad A=\begin{bmatrix}0 & \alpha\\ 0 & 0\end{bmatrix}$

$\therefore \quad A^2=A\cdot A=\begin{bmatrix}0 & \alpha\\ 0 & 0\end{bmatrix}\begin{bmatrix}0 & \alpha\\ 0 & 0\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=0$

$\Rightarrow \quad A^2=A^3=A^4=A^5=\ldots=0$

Now, $(A+I)^{70}=(I+A)^{70}$

$=I+{}^{70}C_1\,A+{}^{70}C_2\,A^2+{}^{70}C_3\,A^3+..+{}^{70}C_{70}\,A^{70}$

$=I+70\,A+0+0+\ldots=I+70A$

$\Rightarrow (A+I)^{70}-70A=I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}a-1 & b-1\\ c-1 & d-1\end{bmatrix}$ [given]

$\therefore \quad a-1=1, b-1=0, c-1=0, d-1=1$

$\Rightarrow \quad a=2, b=1, c=1, d=2$

Hence, $a+b+c+d=6$

72. (A) → (p, r); (B) → (s); (C) → (q); (D) → (s)

On comparing, we get

$\{4f(-1)-3\}a^2+\{4f(1)-3\}a+f(2)=0$

$\{4f(-1)-3\}b^2+\{4f(1)-3\}b+f(2)=0,$

and $\{4f(-1)-3\}c^2+\{4f(1)-3\}C+f(2)=0$

It is clear that a, b, c are the roots of

$\{4f(-1)-3\}x^2+\{4f(1)-3\}x+f(2)=0$, then

$4f(-1)-3=0, 4f(1)-3=0, f(2)=0$

$\Rightarrow \quad f(-1)=\frac{3}{4}, f(1)=\frac{3}{4}, f(2)=0$

Let $f(x)=(x-2)(ax+b)$

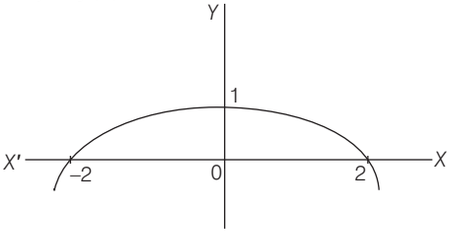
Now, $f(-1)=\frac{3}{4} \Rightarrow (-3)(-a+b)=\frac{3}{4} \Rightarrow a-b=\frac{1}{4}$

$f(1)=\frac{3}{4} \Rightarrow (-1)(a+b)=\frac{3}{4} \Rightarrow a+b=-\frac{3}{4}$

$\therefore \quad a=-\frac{1}{4}, b=-\frac{1}{2}$

$\Rightarrow \quad f(x)=\frac{1}{4}(4-x^2)$

Graph of $y=f(x)$



(A) x-coordinates of the point intersection of $y=f(x)$ with the X-axis are -2 and 2.

(B) Area $=\frac{3}{2}\int_{-2}^{2}\frac{1}{4}(4-x^2)dx=\frac{3}{4}\int_0^2(4-x^2)dx$

$=\frac{3}{4}\left[4x-\frac{x^3}{3}\right]_0^2=\frac{3}{4}\times\frac{16}{3}=4$

(C) Maximum value of $f(x)$ is 1.

(D) Length of intercept on the X-axis is 4.

73. (A) → (r, t); (B) → (s); (C) → (p); (D) → (q)

(A) $\text{adj}\,(A^{-1})=(A^{-1})^{-1}\det(A^{-1})=\frac{A}{\det(A)}$

Also, $\frac{\text{adj}\,(\text{adj}\,A)}{(\text{adj}\,A)^{n-1}}=\frac{A[\det(A)]^{n-2}}{(\det A)^{n-1}}=\frac{A}{\det(A)}$

(B) $\det(\text{adj}\,(A^{-1}))=(\det A^{-1})^{n-1}$

$=\frac{1}{(\det A)^{n-1}}=(\det A)^{1-n}$

(C) $\text{adj}\,[\text{adj}\,A]=A(\det A)^{n-2}$

(D) $\text{adj}\,(A\det A)=(\det A)^{n-1}(\text{adj}\,A)$

74. (A)→(q, s); (B)→(p, t); (C)→(p, q, r, s); (D)→(q, s)

(A) A diagonal matrix is commutative with every square matrix, if it is scalar matrix, so every diagonal element is 4.

Therefore, $|A|=\begin{vmatrix}4 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\end{vmatrix}=64$

(B) $\begin{vmatrix}1-a & 1 & 1\\ 1 & 1-b & 1\\ 1 & 1 & 1-c\end{vmatrix}=0$

Applying $R_1\to R_1-R_3$ and $R_2\to R_2-R_3$, then

$\begin{vmatrix}-a & 0 & c\\ 0 & -b & c\\ 1 & 1 & 1-c\end{vmatrix}=0$

$\Rightarrow \quad -a(-b+bc-c)-0+c(b)=0$

$ab+bc+ca=abc$ ...(i)

Now, $AM\geq GM$

$\Rightarrow \quad \frac{ab+bc+ca}{3}\geq(ab\cdot bc\cdot ca)^{\frac{1}{3}}$

$\Rightarrow \quad \frac{abc}{3}\geq(abc)^{\frac{2}{3}}$ [from Eq. (i)]

$\Rightarrow \quad (abc)^{\frac{1}{3}}\geq 3$

$\therefore \quad abc\geq 27$

Hence, $\lambda=27$

(C) $\because \quad A=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$

Given, $\sum_{k=1}^{3}a_{ik}=9\lambda_i, \forall\, i\in\{1,2,3\};$

$\sum_{k=1}^{3}a_{kj}=9\mu_j, \forall\, j\in\{1,2,3\}$ and

$a_{11}+a_{22}+a_{33}=9\upsilon$; where $\lambda_i, \mu_j, \nu\in\{1,2\}$

Following types of matrices are possible:

$A=\begin{bmatrix}1 & & \\ & 3 & \\ & & 5\end{bmatrix}; B=\begin{bmatrix}2 & & \\ & 3 & \\ & & 4\end{bmatrix}; C=\begin{bmatrix}7 & & \\ & 3 & \\ & & 8\end{bmatrix};$

$D=\begin{bmatrix}6 & & \\ & 3 & \\ & & 9\end{bmatrix}; E=\begin{bmatrix}1 & & \\ & 6 & \\ & & 2\end{bmatrix}; F=\begin{bmatrix}3 & & \\ & 6 & \\ & & 9\end{bmatrix};$

$$G=\begin{bmatrix}4&&\\&6&\\&&8\end{bmatrix};\ H=\begin{bmatrix}5&&\\&6&\\&&7\end{bmatrix};\ I=\begin{bmatrix}1&&\\&9&\\&&8\end{bmatrix};$$

$$\mathcal{J}=\begin{bmatrix}2&&\\&9&\\&&7\end{bmatrix};\ K=\begin{bmatrix}3&&\\&9&\\&&6\end{bmatrix};\ L=\begin{bmatrix}4&&\\&9&\\&&5\end{bmatrix}$$

Now, if we interchange 1 and 5 to obtain

$$A_1=\begin{bmatrix}5&4&9\\7&3&8\\6&2&1\end{bmatrix}$$

Also, $$A^T=\begin{bmatrix}1&8&9\\2&3&4\\6&7&5\end{bmatrix}$$

and $$A_1^T=\begin{bmatrix}5&7&6\\4&3&2\\9&8&1\end{bmatrix}$$

Then, from A we get four matrices A, A_1, A^T, A_1^T.

Similarly, from $B, C, D, \ldots, K, L$ we get 4 matrices.

Thus, total $12\times 4=48$ matrices. Hence, $\lambda=48$.

(D) For consistent, $\begin{vmatrix}1&1&1\\c+2&c+4&6\\(c+2)^2&(c+4)^2&36\end{vmatrix}=0$

Applying $C_2\to C_2-C_1$, we get

$$\begin{vmatrix}1&0&1\\c+2&2&6\\(c+2)^2&4c+12&36\end{vmatrix}=0$$

$$\Rightarrow\quad 2\begin{vmatrix}1&0&1\\c+2&1&6\\(c+2)^2&2c+6&36\end{vmatrix}=0$$

$$\Rightarrow\quad -12c-0+1[(c+2)(2c+6)-(c+2)^2]=0$$

$$\Rightarrow\quad c^2-6c+8=0$$

$$\Rightarrow\quad c=2,4$$

$$\therefore\quad c_1=4,\ c_2=2$$

$$\Rightarrow\quad c_1^{c_2}=4^2=16$$

75. (A) $\to$(q, t); (B) $\to$(p, s); (C) $\to$(p, r, s); (D) $\to$(q, r, t)

(A) Here, X is a $n\times 1$ matrix, C is $n\times n$ matrix and X^T is a $1\times n$ matrix.

Hence, X^TCX is a 1×1 matrix.

Let $X^TCX=[\lambda]$, then

$$(X^TCX)^T=X^TC^T(X^T)^T=X^T(-C)X=-X^TCX$$

$$\therefore\quad [\lambda]=-[\lambda]$$

$$\Rightarrow\quad \lambda=0$$

$$\Rightarrow\quad X^TCX=O$$

i.e., X^TCX is null matrix.

(B) Consider the homogeneous system

$$(I-A)X=O$$

$$\Rightarrow\quad AX=IX=X \qquad \ldots(i)$$

Now, $$(AX)^T=X^TA^T\Rightarrow X^T=-X^TA$$

$$\Rightarrow\quad X^TX=-X^TAX=-X^TX \qquad \text{[from Eq. (i)]}$$

$$\Rightarrow\quad 2X^TX=O\quad\Rightarrow\quad |X|=O$$

$(I-A)X=O$ has only trivial solution

$\therefore$ $I-A$ is non-singular

$\Rightarrow$ $(I-A)$ is invertible

(C) $\because$ $$S=\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}$$

$$\Rightarrow\quad S^{-1}=\frac{1}{2}\begin{bmatrix}-1&1&1\\1&-1&1\\1&1&-1\end{bmatrix}$$

We have, $$SA=\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}\begin{bmatrix}b+c&c-a&b-a\\c-b&c+a&a-b\\b-c&a-c&a+b\end{bmatrix}$$

$$=\begin{bmatrix}0&2a&2a\\2b&0&2b\\2c&2c&0\end{bmatrix}$$

$$\therefore\quad SAS^{-1}=\begin{bmatrix}0&2a&2a\\2b&0&2b\\2c&2c&0\end{bmatrix}\frac{1}{2}\begin{bmatrix}-1&1&1\\1&-1&1\\1&1&-1\end{bmatrix}$$

$$=\begin{bmatrix}0&a&a\\b&0&b\\c&c&0\end{bmatrix}\begin{bmatrix}-1&1&1\\1&-1&1\\1&1&-1\end{bmatrix}=\begin{bmatrix}2a&0&0\\0&2b&0\\0&0&2c\end{bmatrix}$$

$$\therefore\quad |SAS^{-1}|=8abc\neq 0$$

(D) $\because$ $$A=\begin{bmatrix}\sin 2A&\sin C&\sin B\\\sin C&\sin 2B&\sin A\\\sin B&\sin A&\sin 2C\end{bmatrix}$$

$$|A|=\begin{bmatrix}2ak\cos A&ck&bk\\ck&2bk\cos B&ak\\bk&ak&2ck\cos C\end{bmatrix}$$

$$=k^3\begin{bmatrix}a\cos A+a\cos A&a\cos B+b\cos A&a\cos C+c\cos A\\a\cos B+b\cos A&b\cos B+b\cos B&b\cos C+c\cos B\\a\cos C+c\cos A&b\cos C+c\cos B&c\cos C+c\cos C\end{bmatrix}$$

$$=k^3\begin{vmatrix}a&\cos A&0\\b&\cos B&0\\c&\cos C&0\end{vmatrix}\times\begin{vmatrix}\cos A&a&0\\\cos B&b&0\\\cos C&c&0\end{vmatrix}=k^3\cdot 0\cdot 0=0$$

76. Since, matrix A is skew-symmetric

$$\therefore\quad |A|=0$$

$$\therefore\quad |A^4B^5|=0$$

$\Rightarrow A^4B^5$ is singular matrix.

Statement-1 is false and Statement-2 is true.

77. $\because \quad AB = A,\ BA = B \Rightarrow A^2 = A$ and $B^2 = B$

$\therefore \quad (A+B)^2 = A^2 + B^2 + AB + BA = A + B + A + B$

$= 2(A+B)$

$(A+B)^3 = (A+B)^2 . (A+B)$

$= 2(A+B)^2 = 2^2 (A+B)$

$\therefore \quad (A+B)^7 = 2^6 (A+B)$

Statement-1 is true and Statement-2 is false.

78. A^{-1}exists only for non-singular matrix

$AB = AC \Rightarrow A^{-1}(AB) = A^{-1}(AC)$

$\Rightarrow \quad (A^{-1}\ A)\ B = (A^{-1}\ A)\ C$

$\Rightarrow \quad IB = IC$

$\Rightarrow \quad B = C$, if A^{-1} exist

$\therefore \quad |A| \neq 0$

Statement-1 is false and Statement-2 is true.

79. Statement-2 is false

$\because \quad \det(A^{-1}) \neq \det(-A')$

$[\because \det(-A') = (-1)^3 \det(A') = -\det(A')]$

but in Statement-1

$A' = -A \Rightarrow A = -A'$

$\therefore \quad \det(A) = \det(-A')$

$= -\det A' = -\det(A)$

$\Rightarrow \quad 2\det(A) = 0$

$\therefore \quad \det(A) = 0$

Then, Statement-1 is true.

80. $\because (BX)^T (BY) = \{(I-A)(I+A)^{-1} X\}^T (I-A)(I+A)^{-1} Y$

$= X^T \{(I+A)^{-1}\}^T (I-A)^T (I-A)(I+A)^{-1} Y$

$= X^T (I+A^T)^{-1} (I-A^T)(I-A)(I+A)^{-1} Y$

$= X^T (I-A)^{-1} (I+A)(I-A)(I+A)^{-1} Y$

$= X^T (I-A)^{-1} (I-A)(I+A)(I+A)^{-1} Y$

$[\because A^T = -A$ and $(I-A)(I+A) = (I+A)(I-A)]$

$= X^T \cdot I \cdot I \cdot Y = X^T Y$

Both Statements are true; Statement-2 is correct explanation for Statement-1.

81. $\because \quad |A| = 2$

and $\quad B = 9A^2$ (given)

$\therefore \quad |B| = |9A^2| = 9^2 |A|^2$

$= 81 \times 4 = 324 \Rightarrow |B^T| = |B| = 324$

Hence, Statement-1 is false but Statement-2 is true.

82. $\therefore \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 & -1 \\ 1 & -1-\lambda & 0 \\ 1 & 0 & -1-\lambda \end{vmatrix} = 0$

$\Rightarrow \quad (1-\lambda)(1+\lambda)^2 - 1 - \lambda - 1 - \lambda = 0$

$\Rightarrow \quad \lambda^3 + \lambda^2 + \lambda + 1 = 0$

$\Rightarrow \quad A^3 + A^2 + A + I = 0$

$\Rightarrow \quad A^3 + A^2 + A = -I$

Statement-1 is false but Statement-2 is true.

83. $A = \begin{bmatrix} 0 & -\frac{1}{5} & -\frac{2}{7} \\ \frac{1}{4} & 0 & -\frac{1}{8} \\ \frac{2}{5} & \frac{1}{7} & 0 \end{bmatrix}$

which is neither symmetric nor skew-symmetric. Infact every square matrix can be expressed as a sum of symmetric and skew-symmetric matrix. Hence, Statement-1 is false and Statement-2 is true.

84. ABC is not defined, as order of A, B and C are such that they are not conformable for multiplication.

Hence, Statement-1 is false and Statement-2 is true.

85. $\because \quad A^T = -A$

$\Rightarrow \quad |A^T| = |-A|$

$= (-1)^5 |A| = -|A|$

$\Rightarrow \quad |A| = -|A|$

$\Rightarrow \quad 2|A| = 0$

$\therefore \quad |A| = 0$

Both Statements are true but Statement-2 is a correct explanation of Statement-1.

86. $\because S$ is skew-symmetric matrix

$\therefore \quad S^T = -S$ ...(i)

First we will show that $I - S$ is non-singular. The equality $|I - S| = 0 \Rightarrow I$ is a characteristic root of the matrix S but this is not possible, for a real skew-symmetric matrix can have zero or purely imaginary numbers as its characteristic roots. Thus, $|I - S| \neq 0$ i.e., $I - S$ is non-singular.

We have,

$A^T = \{(I+S)(I-S)^{-1}\}^T = \{(I-S)^{-1}(I+S)\}^T$

$= ((I-S)^{-1})^T (I+S)^T = (I+S)^T \{(I-S)^{-1}\}^T$

$= ((I-S)^T)^{-1}(I+S)^T = (I+S)^T ((I-S)^T)^{-1}$

$= (I^T - S^T)^{-1}(I^T + S^T) = (I^T + S^T)(I^T - S^T)^{-1}$

$= (I+S)^{-1}(I-S) = (I-S)(I+S)^{-1}$ [from Eq. (i)]

$\therefore \quad A^T A = (I+S)^{-1}(I-S)(I+S)(I-S)^{-1}$

$= (I-S)(I+S)^{-1}(I-S)^{-1}(I+S)$

$= (I+S)^{-1}(I+S)(I-S)(I-S)^{-1}$

$= (I-S)(I-S)^{-1}(I+S)^{-1}(I+S)$

$= I \cdot I = I \cdot I = I = I$

Hence, A is orthogonal.

87. $\because \quad MM^T = I$ ...(i)

Let $\quad B = M - I$ ...(ii)

$\therefore \quad B^T = M^T - I^T = M^T - M^T M$ [from Eq. (i)]

$= M^T (I - M) = -M^T B$ [from Eq. (ii)]

Now, $\det(B^T) = \det(-M^T B)$

$= (-1)^3 \det(M^T)\det(B) = -\det(M^T)\det(B)$

$\Rightarrow \quad \det(B) = -\det(M)\det(B) = -\det(B)$

$\therefore \quad \det(B) = 0$

$\Rightarrow \quad \det(M - I) = 0$

88. $\because \quad BAB = A^{-1}$

$\Rightarrow \quad ABAB = I$

$\Rightarrow \quad (AB)^2 = I$

Now, $AB = \begin{bmatrix} \cos(\alpha + 2\beta) & \sin(\alpha + 2\beta) \\ \sin(\alpha + 2\beta) & -\cos(\alpha + 2\beta) \end{bmatrix}$

and $(AB)^2 = (AB)(AB) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \quad [\because (AB)(AB) = I\,]$

Also, $BA^4B = A^{-1}$

or $A^4B = B^{-1}A^{-1} = (AB)^{-1} = AB$

or $A^4 = A \qquad \ldots(i)$

Now, $A^2 = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$= \begin{bmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{bmatrix}$

Similarly, $A^4 = \begin{bmatrix} \cos 4\alpha & -\sin 4\alpha \\ \sin 4\alpha & \cos 4\alpha \end{bmatrix}$

Hence, from Eq. (i)

$$\begin{bmatrix} \cos 4\alpha & -\sin 4\alpha \\ \sin 4\alpha & \cos 4\alpha \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

or $4\alpha = 2\pi + \alpha$

$\therefore \quad \alpha = \dfrac{2\pi}{3}$

89. $AB = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}\begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$

$= \begin{bmatrix} \cos^2\theta\cos^2\phi + \cos\theta\cos\phi\sin\theta\sin\phi & \cos^2\theta\cos\phi\sin\phi + \sin^2\phi\sin\theta\cos\theta \\ \cos^2\phi\cos\theta\sin\theta + \sin^2\theta\sin\phi\cos\phi & \cos\theta\cos\phi\sin\theta\sin\phi + \sin^2\theta\sin^2\phi \end{bmatrix}$

$= \begin{bmatrix} \cos\theta\cos\phi(\cos\theta\cos\phi + \sin\theta\sin\phi) & \cos\phi\sin\phi(\cos\theta\cos\phi + \sin\theta\sin\phi) \\ \sin\theta\cos\phi(\cos\theta\cos\phi + \sin\theta\sin\phi) & \sin\theta\sin\phi(\cos\theta\cos\phi + \sin\theta\sin\phi) \end{bmatrix}$

$= \begin{bmatrix} \cos\theta\cos\phi\cos(\theta - \phi) & \cos\theta\sin\phi\cos(\theta - \phi) \\ \sin\theta\cos\phi\cos(\theta - \phi) & \sin\theta\sin\phi\cos(\theta - \phi) \end{bmatrix}$

Clearly, AB is the zero matrix, if $\cos(\theta \sim \phi) = 0$ i.e., $\theta - \phi$ is an odd multiple of $\dfrac{\pi}{2}$.

90. Let $A = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$

$\therefore \quad A^T = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}$

Now, $AA^T = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \times \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}$

$= \begin{bmatrix} \Sigma l_1^2 & \Sigma l_1 l_2 & \Sigma l_3 l_1 \\ \Sigma l_1 l_2 & \Sigma l_2^2 & \Sigma l_2 l_3 \\ \Sigma l_3 l_1 & \Sigma l_2 l_3 & \Sigma l_3^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

Hence, matrix A is orthogonal.

91. Let us use the symbols Div, Dis, Tal for division, district, taluka respectively and O, H, C, Cl, T and P for office superintendent, Head clerk, Cashier, Clerk, Typist and Peon respectively.

Then, the number of offices can be arranged as elements of a row matrix A and the composition of staff in various offices can be arranged in a 3×6 matrix B (say).

$\therefore \quad A = \begin{matrix} \text{Div} & \text{Dis} & \text{Tal} \end{matrix}$

$\begin{bmatrix} 5 & 30 & 200 \end{bmatrix}$

and $B = \begin{matrix} \text{O} & \text{H} & \text{C} & \text{Cl} & \text{T} & \text{P} \end{matrix}$

$\begin{bmatrix} 1 & 1 & 1 & 2+1 & 1 & 1+1 \\ 0 & 1 & 1 & 1+1 & 0 & 1+1 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$

or $B = \begin{matrix} \text{O} & \text{H} & \text{C} & \text{Cl} & \text{T} & \text{P} \end{matrix}$

$\begin{bmatrix} 1 & 1 & 1 & 3 & 1 & 2 \\ 0 & 1 & 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$

The basic monthly salaries of various types of employees of these offices correspond to the elements of the column matrix C.

$\therefore \quad C = \begin{matrix} \text{O} \\ \text{H} \\ \text{C} \\ \text{Cl} \\ \text{T} \\ \text{P} \end{matrix} \begin{bmatrix} 500 \\ 200 \\ 175 \\ 150 \\ 150 \\ 100 \end{bmatrix}$

(i) Total number of Posts = AB

$$= \begin{matrix} & \text{O} & \text{H} & \text{C} & \text{Cl} & \text{T} & \text{P} \\ \text{Div}\ \ \text{Dis}\ \ \text{Tal} & & & & & & \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 3 & 1 & 2 \\ 0 & 1 & 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{matrix} \text{O} & \text{H} & \text{C} & \text{Cl} & \text{T} & \text{P} \end{matrix}$$

$$[5 \quad 235 \quad 235 \quad 275 \quad 5 \quad 270]$$

i.e., Required number of posts in all the offices taken together are 5 office Suprintendents, 235 Head Clerks, 235 Cashiers, 275 Clerks, 5 Typists and 270 Peons.

(ii) The total basic monthly salary bill of each kind of office = BC

$$= \begin{bmatrix} \text{O} & \text{H} & \text{C} & \text{Cl} & \text{T} & \text{P} \\ 1 & 1 & 1 & 3 & 1 & 2 \\ 0 & 1 & 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix} \times \begin{matrix} \begin{bmatrix} 500 \\ 200 \\ 175 \\ 150 \\ 150 \\ 100 \end{bmatrix} \end{matrix} \begin{matrix} \text{O} \\ \text{H} \\ \text{C} \\ \text{Cl} \\ \text{T} \\ \text{P} \end{matrix}$$

$$= \begin{bmatrix} 500 + 200 + 175 + 3 \times 150 + 1 \times 150 + 2 \times 100 \\ 0 + 1 \times 200 + 1 \times 175 + 2 \times 150 + 0 + 2 \times 100 \\ 0 + 1 \times 200 + 1 \times 175 + 1 \times 150 + 0 + 1 \times 100 \end{bmatrix}$$

$$= \begin{bmatrix} 1675 \\ 875 \\ 625 \end{bmatrix}$$

i.e., The total basic monthly salary bill of each divisional, district and taluka offices are ₹ 1675, ₹ 875 and ₹ 625, respectively.

(iii) The total basic monthly salary bill of all the offices taken together

$$= ABC = A(BC)$$

$$= [5 \quad 30 \quad 200] \times \begin{bmatrix} 1675 \\ 875 \\ 625 \end{bmatrix}$$

$$= [5 \times 1675 + 30 \times 875 + 200 \times 625]$$

$$= [159625]$$

Hence, total basic monthly salary bill of all the offices taken together is ₹ 159625.

92. The total load of stone and sand supplied by A can be represented by row matrix X_1 and cost of one truck load of stone and sand can be represented by column matrix Y_1.

$$\therefore \quad X_1 = [40 \quad 10], \; Y_1 = \begin{bmatrix} 1200 \\ 500 \end{bmatrix}$$

Total amount paid by contractor to $A = X_1 Y_1$

$$= [40 \quad 10] \begin{bmatrix} 1200 \\ 500 \end{bmatrix}$$

$$= [48000 + 5000]$$

$$= [53000]$$

$\therefore$ Amount paid by contractor to A is ₹ 53000.

Similarly for B, $X_2 = [35 \quad 5]$, $Y_2 = \begin{bmatrix} 1200 \\ 500 \end{bmatrix}$

Total amount paid by contractor to $B = X_2 Y_2$

$$= [35 \quad 5] \begin{bmatrix} 1200 \\ 500 \end{bmatrix} = [42000 + 2500]$$

$$= [44500]$$

$\therefore$ Amount paid by contractor to B is ₹ 44500.

Similarly for C,

$$X_3 = [25 \quad 8], \; Y_3 = \begin{bmatrix} 1200 \\ 500 \end{bmatrix}$$

Total amount paid by contractor to $C = X_3 Y_3$

$$= [25 \quad 8] \begin{bmatrix} 1200 \\ 500 \end{bmatrix}$$

$$= [30000 + 4000] = [34000]$$

$\therefore$ Amount paid by contractor to C is ₹ 34000.

93. We have, $A = \begin{bmatrix} 1 & a & \alpha & a\alpha \\ 1 & b & \beta & b\beta \\ 1 & c & \gamma & c\gamma \end{bmatrix}$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$A = \begin{bmatrix} 1 & a & \alpha & a\alpha \\ 0 & b-a & \beta-\alpha & b\beta - a\alpha \\ 0 & c-a & \gamma-\alpha & c\gamma - a\alpha \end{bmatrix}$$

Applying $C_2 \to C_2 - aC_1$, $C_3 \to C_3 - \alpha C_1$ and $C_4 \to C_4 - a\alpha C_1$, we get

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & b-a & \beta-\alpha & b\beta - a\alpha \\ 0 & c-a & \gamma-\alpha & c\gamma - a\alpha \end{bmatrix}$$

Applying $C_4 \to C_4 - \alpha C_2 - bC_3$, we get

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & b-a & \beta-\alpha & 0 \\ 0 & c-a & \gamma-\alpha & (c-b)(\gamma-\alpha) \end{bmatrix}$$

For $\rho(A) = 3$

$c - a \neq 0, \gamma - \alpha \neq 0, c - b \neq 0, b - a \neq 0, \beta - \alpha \neq 0$

i.e., $a \neq b, b \neq c, c \neq a$ and $\alpha \neq \beta, \beta \neq \gamma, \gamma \neq \alpha$

94. We have, $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x & u \\ y & v \\ z & w \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 52 & 15 \\ 0 & -1 \end{bmatrix}$

or $AX = B$

or $X = A^{-1}B$ ...(i)

Where, $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x & u \\ y & v \\ z & \omega \end{bmatrix}$ and $B = \begin{bmatrix} 9 & 2 \\ 52 & 15 \\ 0 & -1 \end{bmatrix}$

$$\therefore \quad |A| = 1(-5-7) - 1(-2-14) + 1(2-10)$$

$$= -12 + 16 - 8 = -4 \neq 0$$

Let C be the matrix of cofactors of elements of $|A|$.

$$\therefore \quad C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{vmatrix} 5 & 7 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 7 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 5 \\ 2 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 5 & 7 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 7 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 16 & -8 \\ 2 & -3 & 1 \\ 2 & -5 & 3 \end{bmatrix}$$

$$\therefore \quad \text{adj } A = C' = \begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{\text{adj } A}{|A|} = -\frac{1}{4} \begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix}$$

Now, $A^{-1}B = -\frac{1}{4} \begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix} \times \begin{bmatrix} 9 & 2 \\ 52 & 15 \\ 0 & -1 \end{bmatrix}$

$$= -\frac{1}{4}\begin{bmatrix} -4 & 4 \\ -12 & -8 \\ -20 & -4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ 5 & 1 \end{bmatrix}$$

From Eq. (i) $X = A^{-1}B$

$$\Rightarrow \quad \begin{bmatrix} x & u \\ y & v \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ 5 & 1 \end{bmatrix}$$

On equating the corresponding elements, we have

$x = 1, u = -1$

$y = 3, v = 2$

$z = 5, w = 1$

95. Since, $x_1 = 3y_1 + 2y_2 - y_3$

$$\Rightarrow \quad [x_1] = [3 \quad 2 \quad -1]\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Putting the values of y_1, y_2, y_3, we get

$$[x_1] = [3 \quad 2 \quad -1]\begin{bmatrix} z_1 - z_2 + z_3 \\ 0 + z_2 + 3z_3 \\ 2z_1 + z_2 + 0 \end{bmatrix}$$

$$= [3 \quad 2 \quad -1]\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 3 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$= [3 + 0 - 2 \quad -3 + 2 - 1 \quad 3 + 6 + 0]\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$= [1 \quad -2 \quad 9]\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$[x_1] = [z_1 - 2z_2 + 9z_3]$$

$\therefore \quad x_1 = z_1 - 2z_2 + 9z_3$ ...(i)

Further, $x_2 = -y_1 + 4y_2 + 5y_3$

$$\Rightarrow \quad [x_2] = [-1 \quad 4 \quad 5]\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Putting the values of y_1, y_2, y_3, we get

$$[x_2] = [-1 \quad 4 \quad 5]\begin{bmatrix} z_1 - z_2 + z_3 \\ 0 + z_2 + 3z_3 \\ 2z_1 + z_2 + 0 \end{bmatrix}$$

$$= [-1 \quad 4 \quad 5]\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 3 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$= [-1 + 0 + 10 \quad 1 + 4 + 5 \quad -1 + 12 + 0]\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$= [9 \quad 10 \quad 11]\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = [9z_1 + 10z_2 + 11z_3]$$

Hence, $x_2 = 9z_1 + 10z_2 + 11z_3$ ...(ii)

Further, $x_3 = y_1 - y_2 + 3y_3$

$$\therefore \quad [x_3] = [1 \quad -1 \quad 3]\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Putting the values of y_1, y_2, y_3, we get

$$\Rightarrow \quad [x_3] = [1 \quad -1 \quad 3]\begin{bmatrix} z_1 - z_2 + z_3 \\ 0 + z_2 + 3z_3 \\ 2z_1 + z_2 + 0 \end{bmatrix}$$

$$= [1 \quad -1 \quad 3]\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 3 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$= [1 - 0 + 6 \quad -1 - 1 + 3 \quad 1 - 3 + 0]\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$= [7 \quad 1 \quad -2]\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = [7z_1 + z_2 - 2z_3]$$

$\therefore \quad x_3 = 7z_1 + z_2 - 2z_3$ ...(iii)

Hence, from Eqs. (i), (ii) and (iii), we get

$x_1 = z_1 - 2z_2 + 9z_3,\ x_2 = 9z_1 + 10z_2 + 11z_3,\ x_3 = 7z_1 + z_2 - 2z_3$

96. Given equations can be written as,

$2x - 3y + 6z = 5t + 3$

$y - 4z = 1 - t$

$4x - 5y + 8z = 9t + k$

which is of the form $AX = B$.

Let C be the augmented matrix, then

$$C = [A : B] = \begin{bmatrix} 2 & -3 & 6 & \vdots & 5t + 3 \\ 0 & 1 & -4 & \vdots & 1 - t \\ 4 & -5 & 8 & \vdots & 9t + k \end{bmatrix}$$

Applying $R_3 \to R_3 - 2R_1$, then

$$C = \begin{bmatrix} 2 & -3 & 6 & \vdots & 5t + 3 \\ 0 & 1 & -4 & \vdots & 1 - t \\ 0 & 1 & -4 & \vdots & -t + k - 6 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_2$, then

$$C = \begin{bmatrix} 2 & -3 & 6 & \vdots & 5t + 3 \\ 0 & 1 & -4 & \vdots & 1 - t \\ 0 & 0 & 0 & \vdots & k - 7 \end{bmatrix}$$

(i) For no solution

$R_A \neq R_C$

$\therefore \quad k \neq 7$

(ii) For infinite number of solutions

$R_A = R_C$

$\therefore \quad k = 7$

97. $AX = U$ has infinite many solutions

$\Rightarrow \quad |A| = 0 = |A_1| = |A_2| = |A_3|$

Now, $|A| = 0$

$$\Rightarrow \quad \begin{vmatrix} a & 1 & 0 \\ 1 & b & d \\ 1 & b & c \end{vmatrix} = 0 \Rightarrow (ab - 1)(c - d) = 0$$

$\Rightarrow \quad ab = 1$ or $c = d$ ...(i)

and $|A_1| = 0$

$\Rightarrow \begin{vmatrix} f & 1 & 0 \\ g & b & d \\ h & b & c \end{vmatrix} = 0$

$\Rightarrow fb(c-d) - gc + hd = 0$

$\Rightarrow fb(c-d) = gc - hd$ ...(ii)

$\Rightarrow |A_2| = 0$

$\Rightarrow \begin{vmatrix} a & f & 0 \\ 1 & g & d \\ 1 & h & c \end{vmatrix} = 0$

$\Rightarrow a(gc-dh) - f(c-d) = 0$

$\Rightarrow a(gc-dh) = f(c-d)$ ...(iii)

$|A_3| = 0$

$\Rightarrow \begin{vmatrix} a & 1 & f \\ 1 & b & g \\ 1 & b & h \end{vmatrix} = 0$

$\Rightarrow (h-g)(ab-1) = 0$

$\Rightarrow h = g$ or $ab = 1$ ...(iv)

Taking $c = d \Rightarrow h = g$ and $ab \neq 1$ (from Eqs. (i), (ii) and (iv))

Now, taking $BX = V$,

Then, $|B| = \begin{vmatrix} a & 1 & 1 \\ 0 & d & c \\ f & g & h \end{vmatrix} = 0$

[$\because$ In view of $c = d$ and $g = h$, c_2 and c_3 are identical]

$\Rightarrow$ $BX = V$ has no unique solution.

and $|B_1| = \begin{vmatrix} a^2 & 1 & 1 \\ 0 & d & c \\ 0 & g & h \end{vmatrix} = 0$ [$\because c = d, g = h$]

$|B_2| = \begin{vmatrix} a & a^2 & 1 \\ 0 & 0 & c \\ f & 0 & h \end{vmatrix} = a^2 fc = a^2 df$ [$\because c = d$]

and $|B_3| = \begin{vmatrix} a & 1 & a^2 \\ 0 & d & 0 \\ f & g & 0 \end{vmatrix} = -a^2 df$

If $a^2 df \neq 0$, then $|B_2| = |B_3| \neq 0$

Hence, no solution exist.

98. Given, $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}$, $A^{-1} = \frac{1}{6}\begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 2 & 1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 5 \\ 0 & -10 & 14 \end{bmatrix}$$

$$cA = \begin{bmatrix} c & 0 & 0 \\ 0 & c & c \\ 0 & -2c & 4c \end{bmatrix}; dI = \begin{bmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{bmatrix}$$

$\therefore$ By $A^{-1} = \frac{1}{6}[A^2 + cA + dI]$

$\Rightarrow 6 = 1 + c + d$ [By equality of matrices]

$\therefore (-6, 11)$ satisfy the relation.

99. If $Q = PAP^T$

then $P^T Q = AP^T$ [$\because PP^T = I$]

$\Rightarrow P^T Q^{2005} P = AP^T Q^{2004} P$

$= A^2 P^T Q^{2003} P = A^3 P^T Q^{2002} P$

$= A^{2004} P^T (QP)$

$= A^{2004} P^T (PA)$ [$Q = PAP^T \Rightarrow QP = PA$]

$= A^{2005}$

$\therefore A^{2005} = \begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$

100. $A^2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

$A^3 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

$\therefore A^n = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}$

$nA = \begin{bmatrix} n & 0 \\ n & n \end{bmatrix}, (n-1)\, I = \begin{bmatrix} n-1 & 0 \\ 0 & n-1 \end{bmatrix}$

$nA - (n-1)\, I = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix} = A^n$

101. $A^2 - A + I = 0$

$\Rightarrow I = A - A^2 \Rightarrow I = A(I - A)$

$\Rightarrow A^{-1} I = A^{-1}(A(I - A)) \Rightarrow A^{-1} = I - A$

102. (i) Let U_1 be $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ so that $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

Similarly, $U_2 = \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}, U_3 = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}$

Hence, $U = \begin{pmatrix} 1 & 2 & 2 \\ -2 & -1 & -1 \\ 1 & -4 & -3 \end{pmatrix}$

$\therefore |U| = 3$

(ii) $\because \quad \text{Adj}U = \begin{pmatrix} -1 & -2 & 0 \\ -7 & -5 & -3 \\ 9 & -6 & 3 \end{pmatrix}$

$\therefore \quad U^{-1} = \dfrac{\text{Adj}U}{|U|} = \dfrac{\text{Adj}U}{3}$

$\Rightarrow$ sum of the elements of

$$U^{-1} = \frac{1}{3}(-1-2+0-7-5-3+9+6+3) = 0$$

(iii) The value of

$$(3\ 2\ 0)\, U \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = (3\ 2\ 0) \begin{pmatrix} 1 & 2 & 2 \\ -2 & -1 & -1 \\ 1 & -6 & -3 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$$

$$= (-1\ \ 4\ \ 4) \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$$

$$= (-3+8+0) = 5$$

103. $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, B = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$

$\Rightarrow \quad AB = \begin{pmatrix} a & 2b \\ 3a & 4b \end{pmatrix}$

and $\quad BA = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} a & 2a \\ 3b & 4b \end{pmatrix}$

Hence, $AB = BA$ only when $a = b$.

104. $A^2 - B^2 = (A-B)(A+B)$

$\Rightarrow \quad A^2 - B^2 = A^2 + AB - BA - B^2$

$\Rightarrow \quad AB = BA$

105. $A = \begin{bmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5\alpha \\ 0 & 0 & 5 \end{bmatrix} \Rightarrow |A \cdot A| = |A||A| = (25\alpha)^2 = 25$

$\Rightarrow \quad \alpha^2 = \dfrac{1}{25}$

$\Rightarrow \quad \alpha = \pm \dfrac{1}{5}$

106. $\because \quad A^t = A, B^t = -B$

Given, $\quad (A+B)(A-B) = (A-B)(A+B)$

$\Rightarrow \quad A^2 - AB + BA - B^2 = A^2 + AB - BA - B^2$

$\Rightarrow \quad AB = BA$

Also, given $\quad (AB)^t = (-1)^k\, AB$

$\Rightarrow \quad B^t A^t = (-1)^k\, AB$

$\Rightarrow \quad -BA = (-1)^k\, AB$

$\Rightarrow \quad (-1) = (-1)^k \qquad [\because AB = BA]$

$\therefore \quad k = 1, 3, 5, \ldots$

107. Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 1/2 \end{bmatrix}$

$$\det A = \begin{vmatrix} 2 & 1 \\ 0 & 1/2 \end{vmatrix} = 1$$

and $\quad A^{-1} = \begin{bmatrix} 1/2 & -1 \\ 0 & 2 \end{bmatrix}$

and let $\quad A = \begin{bmatrix} 3 & 0 \\ -3 & -1/3 \end{bmatrix},$

$$\det A = \begin{vmatrix} 3 & 0 \\ -3 & -1/3 \end{vmatrix} = -1$$

and $\quad A^{-1} = \begin{bmatrix} 1/3 & 0 \\ -3 & -3 \end{bmatrix}$

108. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ or $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Then, $A^2 = I$

$\therefore \quad \det A = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = -1$ and $\text{tr}(A) = 0$

109. (i) If two zero's are the entries in the diagonal, then

$${}^3C_2 \times {}^3C_1 = 9$$

If the entries in the principal diagonal is1, then

$${}^3C_1 = 3$$

$\Rightarrow$ Total matrix $= 9 + 3 = 12$

(ii) $\begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 1 \end{bmatrix}$ either $b = 0$ or $c = 0 \Rightarrow |A| \neq 0$

$\Rightarrow$ 2 matrices

$\begin{bmatrix} 0 & a & b \\ a & 1 & c \\ b & c & 0 \end{bmatrix}$ either $a = 0$ or $c = 0 \Rightarrow |A| \neq 0$

$\Rightarrow$ 2 matrices

$\begin{bmatrix} 1 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix}$ either $a = 0$ or $b = 0 \Rightarrow |A| \neq 0$

$\Rightarrow$ 2 matrices

$$\begin{bmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{bmatrix}$$

If $a = b = 0 \Rightarrow |A| = 0$

If $a = c = 0 \Rightarrow |A| = 0$

If $b = c = 0 \Rightarrow |A| = 0$

$\Rightarrow$ There will be only 6 matrices.

(iii) The six matrix A for which $|A| = 0$ are

$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \Rightarrow$ inconsistent

$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow$ inconsistent

$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow$ infinite solutions

$$\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \Rightarrow \text{inconsistent}$$

$$\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 1\end{bmatrix} \Rightarrow \text{inconsistent}$$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 1 & 1\end{bmatrix} \Rightarrow \text{infinite solutions}$$

110. $|\text{adj } A| = |A|^{n-1} = |A|^{2-1} = |A|$

$$\text{adj}(\text{adj } A) = |A|^{n-2} A$$

$$= |A|^{2-2} A = |A|^0 A = A$$

111. Three planes cannot meet only at two distinct points.

Hence, number of matrices = 0

112. If A is symmetric matrix, then $b = c$

$$\therefore \quad \det(A) = \begin{vmatrix} a & b \\ b & a \end{vmatrix} = a^2 - b^2 = (a+b)(a-b)$$

$a, b, c \in \{0, 1, 2, 3, \ldots, p-1\}$

Number of numbers of type

$$np = 1$$
$$np + 1 = 1$$
$$np + 2 = 1$$
$$\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots$$
$$np + (p-1) = 1 \ \forall\ n \in I$$

(i) as $\det(A)$ is divisible by $p \Rightarrow$ either $a + b$ divisible by p corresponding number of ways $= (p-1)$ [excluding zero] or $(a - b)$ is divisible by p corresponding number of ways $= p$
Total Number of ways $= 2p - 1$

(ii) as $\text{Tr}(A)$ not divisible by $p \Rightarrow a \neq 0$
$\det(A)$ is divisible by $p \Rightarrow a^2 - bc$ divisible by p
Number of ways of selection of a, b, c

$$= (p-1)[(p-1) \times 1] = (p-1)^2$$

(iii) Total number of $A = p \times p \times p = p^3$
Number of A such that $\det(A)$ divisible by p

$$= (p-1)^2 + \text{ number of } A \text{ in which } a = 0$$
$$= (p-1)^2 + p + p - 1 = p^2$$

Required number $= p^3 - p^2$

113. $|A| = (2k-1)(-1 + 4k^2) + 2\sqrt{k}(2\sqrt{k} + 4k\sqrt{k})$

$$+ 2\sqrt{k}(4k\sqrt{k} + 2\sqrt{k})(2k-1)(4k^2-1)$$
$$+ 4k + 8k^2 + 8k^2 + 4k$$

$$= (2k-1)(4k^2-1) + 8k + 16k^2$$
$$= 8k^3 - 4k^2 - 2k + 1 + 8k + 16k^2$$
$$= 8k^3 + 12k^2 + 6k + 1$$

$|B| = 0$ as B is skew-symmetric matrix of odd order.

$$\Rightarrow \quad (8k^3 + 12k^2 + 6k + 1)^2 = (10^3)^2$$
$$\Rightarrow \quad (2k+1)^3 = 10^3$$
$$\Rightarrow \quad 2k + 1 = 10$$
$$\Rightarrow \quad k = 4.5$$
$$\Rightarrow \quad [k] = 4$$

114. First row with exactly one zero

$\therefore$ Total number of cases = 6

First row 2 zeroes, we get more cases.

$\therefore$ Total we get more than 7.

115. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $abcd \neq 0$

$$A^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\Rightarrow \quad A^2 = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix}$$

$$\Rightarrow \quad a^2 + bc = 1, \ bc + d^2 = 1$$
$$ab + bd = ac + cd = 0$$
$$c \neq 0 \text{ and } b \neq 0$$

and $\quad a + d = 0$

Trace $\quad A = a + d = 0$

$$|A| = ad - bc = -a^2 - bc = 1$$

116. $MN = NM$

$$M^2N^2(M^TN)^{-1}(MN^{-1})^T M^2N^2N^{-1}(M^T)^{-1}(N^{-1})^T \cdot M^T$$
$$= M^2N \cdot (M^T)^{-1}(N^{-1})^T M^T = -M^2 \cdot N(M)^{-1}(N^T)^{-1} M^T$$
$$= +M^2NM^{-1}N^{-1}M^T = -M \cdot NMM^{-1}N^{-1} M$$
$$= -MNN^{-1}M = -M^2$$

Note

A skew-symmetric matrix of order 3 cannot be non-singular hence the question is wrong.

117. (i) $a + 8b + 7c = 0; \quad 9a + 2b + 3c = 0$

$$7a + 7b + 7c = 0$$

Solving these equations, we get

$$b = 6a$$
$$\Rightarrow \quad c = -7a$$

Now, $\quad 2x + y + z = 0$

$$\Rightarrow \quad 2a + 6a + (-7a) = 1$$
$$\Rightarrow \quad a = 1, b = 6, c = -7$$
$$\therefore \quad 7a + b + c = 7 + 6 - 7 = 6$$

(ii) $\because a = 2$ with b and c satisfying (E)

$\therefore \quad 2 + 8b + 7c = 0, 18 + 2b + 3c = 0$

and $\quad 2 + b + c = 0$

we get $\quad b = 12$ and $c = -14$

Hence, $\dfrac{3}{\omega^a} + \dfrac{1}{\omega^b} + \dfrac{3}{\omega^c} = \dfrac{3}{\omega^2} + \dfrac{1}{\omega^{12}} + \dfrac{3}{\omega^{-14}}$

$$= \frac{3\omega}{\omega^3} + \frac{1}{1} + 3\omega^{14}$$
$$= 3\omega + 1 + 3\omega^2$$
$$= 1 + 3(\omega + \omega^2)$$
$$= 1 + 3(-1) = -2$$

(iii) $\because b = 6$, with a and c satisfying (E)

$\therefore \quad a + 48 + 7c = 0, 9a + 12 + 3c = 0, a + 6 + c = 0$

we get $\quad a = 1, c = -7$

Given, α, β are the roots of $ax^2 + bx + c = 0$

$$\therefore \quad \alpha + \beta = -\frac{b}{a} = -6,$$

$$\alpha\beta = \frac{c}{a} = -7$$

Now, $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = \dfrac{\alpha + \beta}{\alpha\beta} = \dfrac{-6}{-7} = \dfrac{6}{7}$

$$\therefore \quad \sum_{n=0}^{\infty} \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)^n = \sum_{n=0}^{\infty} \left(\frac{6}{7}\right)^n$$

$$= 1 + \left(\frac{6}{7}\right) + \left(\frac{6}{7}\right)^2 + \ldots \infty$$

$$= \frac{1}{1 - 6/7} = 7$$

118. For the given matrix to be non singular

$$\therefore \quad \begin{vmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{vmatrix} \neq 0$$

$$\Rightarrow \quad 1 - (a + c)\,\omega + ac\omega^2 \neq 0$$

$$\Rightarrow \quad (1 - a\omega)(1 - c\omega) \neq 0$$

$$\Rightarrow \quad a \neq \omega^2 \text{ and } c \neq \omega^2$$

$\because a, b$ and c are complex cube roots of unity.

$\therefore a$ and c can take only one value i.e., ω while b can take two values i.e., ω and ω^2.

$\therefore$ Total number of distinct = 2

119. Let $M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$$M\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow b = -1, e = 2, h = 3$$

$$M\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \Rightarrow a = 0, d = 3, g = 2$$

$$M\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix} \Rightarrow g + h + i = 12 \Rightarrow i = 7$$

$\therefore$ Sum of diagonal elements $= a + e + i = 0 + 2 + 7 = 9$

120. Since, A and B are symmetric matrices

$\therefore \quad A' = A$ and $B' = B$

Statement-1 Let $P = A(BA)$

$\therefore \quad P' = (A(BA))' = (BA)'\,A'$

$= (A'\,B')\,A'$

$= (AB)\,A \qquad [\because A' = A, B' = B]$

$= A(BA) \qquad$ [By associative law]

$= P$

$\Rightarrow A(BA)$ is symmetric

Now, let $\quad Q = (AB)\,A$

$Q' = ((AB)\,A)'$

$= A'(AB)' = A'(B'\,A')$

$= A(BA) \qquad [\because A' = A, B' = B]$

$= (AB)\,A \qquad$ [By associative law]

$= Q$

$\Rightarrow (AB)\,A$ is symmetric.

$\therefore$ Statement-1 is true.

Statement - 2 $(AB)' = B'\,A' = BA \qquad [\because A' = A, B' = B]$

$= AB \qquad [\because AB = BA]$

$\Rightarrow AB$ is symmetric matrix

$\therefore$Statement-2 is true.

Hence, both Statements are true, Statement-2 is not a correct explanation for Statement-1.

121. We have, $|Q| = \begin{vmatrix} 2^2a_{11} & 2^3a_{12} & 2^4a_{13} \\ 2^3a_{21} & 2^4a_{22} & 2^5a_{23} \\ 2^4a_{31} & 2^5a_{32} & 2^6a_{33} \end{vmatrix}$

$$= 2^2 \cdot 2^3 \cdot 2^4 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 2a_{21} & 2a_{22} & 2a_{23} \\ 2^2a_{31} & 2^2a_{32} & 2^2a_{33} \end{vmatrix}$$

$$= 2^9 \cdot 2 \cdot 2^2 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 2^{12}\,|P|$$

$\therefore \quad |Q| = 2^{12} \times 2 = 2^{13}$

122. $\because \quad P^T = 2P + I \qquad \ldots$ (i)

$\therefore \quad (P^T)^T = (2P + I)^T$

$\Rightarrow \quad P = 2\,P^T + I \qquad \ldots$ (ii)

From Eqs. (i) and (ii), we get

$P = 2(2P + I) + I$

$\Rightarrow \quad P = -I$

$\therefore \quad PX = -IX = -X$

123. Given, adj $P = \begin{bmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{bmatrix}$

$$\Rightarrow \quad |\text{adj}P| = \begin{vmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{vmatrix}$$

$= 1(-4) - 4(-1) + 4(1) = 4$

$\Rightarrow \quad |P|^{3-1} = 4$

$\Rightarrow \quad |P| = \pm 2$

124. Let $u_1 + u_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Now, $\quad Au_1 + Au_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$$\Rightarrow \quad A(u_1+u_2)=\begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}1&0&0\\2&1&0\\3&2&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}x\\2x+y\\3x+2y+z\end{pmatrix}=\begin{pmatrix}1\\1\\0\end{pmatrix}$$

$\therefore \quad x=1, 2x+y=1$

and $\quad 3x+2y+z=0$

$\Rightarrow \quad x=1, y=-1, z=-1$

Hence, $u_1+u_2=\begin{pmatrix}1\\-1\\-1\end{pmatrix}$.

125. Given, $P^3=Q^3$ ...(i)

and $P^2Q=Q^2P$ ... (ii)

Subtracting Eq. (i) and (ii), we get

$$P^3-P^2Q=Q^3-Q^2P$$

$$P^2(P-Q)=-Q^2(P-Q)$$

$$\Rightarrow \quad (P^2+Q^2)(P-Q)=O$$

$$\Rightarrow \quad |(P^2+Q^2)(P-Q)|=|O|$$

$$\Rightarrow \quad |P^2+Q^2||P-Q|=0$$

$\therefore \quad |P^2+Q^2|=0 \quad [\because P\neq Q]$

126. Given, adj $A=P$

$\therefore \quad |\text{adj } A|=|P|$

$\Rightarrow \quad |A|^{3-1}=|P| \quad [\because |A|=4]$

$\Rightarrow \quad 16=|P|$

$$\Rightarrow \quad 16=\begin{vmatrix}1&\alpha&3\\1&3&3\\2&4&4\end{vmatrix}$$

$\Rightarrow \quad 16=1(0)-\alpha(4-6)+3(4-6)$

$\Rightarrow \quad 16=2\alpha-6$

$\Rightarrow \quad 2\alpha=22$

$\therefore \quad \alpha=11$

127. (*a*) $(N^TMN)^T=N^TM^T(N^T)^T=N^TM^TN=N^TMN$

or $-N^TMN$ According as M is symmetric or skew-symmetric.

$\therefore$ Correct.

(b) $(MN-NM)^T=(MN)^T-(NM)^T=N^TM^T-M^TN^T$

$=NM-MN \quad [\because M, N \text{ are symmetric}]$

$=-(MN-NM)$

$\therefore$ correct

(c) $(MN)^T=N^TM^T=NM\neq MN \quad [\because M, N \text{ are symmetric}]$

$\therefore$ Incorrect.

(d) $(\text{adj}M)(\text{adj}N)=\text{adj}(NM)\neq\text{adj}(MN)$

$\therefore$ Incorrect.

128. $P=[p_{ij}]_{n\times n}=[\omega^{i+j}]_{n\times n}=\begin{bmatrix}\omega^2&\omega^3&\omega^4&\dots&\omega^{1+n}\\\omega^3&\omega^4&\omega^5&\dots&\omega^{2+n}\\\omega^4&\omega^5&\omega^6&\dots&\omega^{3+n}\\\vdots&\vdots&\vdots&&\vdots\\\omega^{n+1}&\omega^{n+2}&\omega^{n+3}&\dots&\omega^{2n}\end{bmatrix}$

$$\therefore \quad P^2=\begin{bmatrix}0&0&0&\dots&0\\0&0&0&\dots&0\\0&0&0&\dots&0\\\vdots&\vdots&\vdots&&\vdots\\0&0&0&\dots&0\end{bmatrix}=0$$

If n is multiple of 3, so for $P^2\neq 0$, n should not be a multiple of 3, i.e. n can take values 55, 56 and 58.

129. $B=A^{-1}A'$

$B'=(A^{-1}A')'=A(A^{-1})'$

Now, $BB'=(A^{-1}A')A(A^{-1})'=A^{-1}(A'A)(A^{-1})'$

$=A^{-1}(AA')(A^{-1})' \quad [\because A'A=AA']$

$=(A^{-1}A)A'(A^{-1})'$

$=(IA')(A^{-1})'=A'(A^{-1})'=(A^{-1}A)'=I'=I$

130. Let $M=\begin{bmatrix}a&b\\b&c\end{bmatrix}$, where $a, b, c\in I$

M is invertible if $\begin{vmatrix}a&b\\b&c\end{vmatrix}\neq 0$

$\Rightarrow \quad ac-b^2\neq 0$

(a) $\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}b\\c\end{bmatrix}\Rightarrow a=b=c \Rightarrow ac-b^2=0$

$\therefore$ Option (a) is incorrect

(b) $[b\ c]=[a\ b]\Rightarrow a=b=c\Rightarrow ac-b^2=0$

$\therefore$ Option (b) is incorrect

(c) $M=\begin{bmatrix}a&0\\0&c\end{bmatrix}$, then $|M|=ac\neq 0$

$\therefore$ M is invertible

$\therefore$ Option (c) is correct.

(d) As $ac\neq(\text{Integer})^2 \Rightarrow ac\neq b^2$

$\therefore$ Option (d) is correct.

131. Given, $MN=NM$, $M\neq N^2$ and $M^2=N^4$

Then, $\quad M^2=N^4$

$\Rightarrow \quad (M+N^2)(M-N^2)=0$

$\therefore \quad M+N^2=0 \quad [\because M\neq N^2]$

$\Rightarrow \quad |M+N^2|=0$

(a) $|M^2+MN^2|=|M||M+N^2|=0$

$\therefore$ Option (a) is correct.

(b) $(M^2+MN^2)U=M(M+N^2)U=0$

$\therefore$ Option (b) is correct.

(c) $\because\ |M^2 + MN^2| = 0$ from option (a)

$\therefore \quad |M^2 + MN^2| \neq 1$

$\therefore$ Option (c) is incorrect.

(d) If $AX = 0$ and $|A| = 0$, then X can be non-zero.

132. $\because\ AA^T = 9I$

$$\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ a & 2 & b\end{bmatrix}\begin{bmatrix}1 & 2 & a\\ 2 & 1 & 2\\ 2 & -2 & b\end{bmatrix} = 9\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}9 & 0 & a+4+2b\\ 0 & 9 & 2a+2-2b\\ a+4+2b & 2a+2-2b & a^2+4+b^2\end{bmatrix} = \begin{bmatrix}9 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 9\end{bmatrix}$$

On comparing, we get

$$a + 2b + 4 = 0 \quad \text{... (i)}$$
$$2a - 2b + 2 = 0 \quad \text{... (ii)}$$

From Eqs. (i) and (ii), we get

$$a = -2,$$
$$b = -1$$

$\therefore$ Ordered pair is $(-2, -1)$.

133. $\because\ X^T = -X, Y^T = -Y, Z^T = Z$

(a) $(Y^3 Z^4 - Z^4 Y^3)^T = (Y^3 Z^4)^T - (Z^4 Y^3)^T$

$= (Z^4)^T (Y^3)^T - (Y^3)^T (Z^4)^T$

$= (Z^T)^4 (Y^T)^3 - (Y^T)^3 (Z^T)^4$

$= -Z^4 Y^3 + Y^3 Z^4$

$= Y^3 Z^4 - Z^4 Y^3$

Option (a) is incorrect.

(b) $X^{44} + Y^{44}$ is symmetric matrix. Option b is incorrect.

(c) $(X^4 Z^3 - Z^3 X^4)^T = (X^4 Z^3)^T - (Z^3 X^4)^T$

$= (Z^3)^T (X^4)^T - (X^4)^T (Z^3)^T$

$= (Z^T)^3 (X^T)^4 - (X^T)^4 (Z^T)^3$

$= Z^3 X^4 - X^4 Z^3$

$= -(X^4 Z^3 - Z^3 X^4)$

$\therefore$ Option (c) is correct.

(d) $X^{23} + Y^{23}$ is skew-symmetric matrix. Option (d) is correct.

134. $\because \quad A \text{ adj } A = AA^T$

$\Rightarrow \quad A^{-1}(A \text{ adj } A) = A^{-1}(AA^T)$

$\Rightarrow \quad (A^{-1}A) \text{ adj } A = (A^{-1}A)A^T$

$\Rightarrow \quad I(\text{adj } A) = IA^T$

or $\quad \text{adj } A = A^T$

or $\quad \begin{bmatrix}2 & b\\ -3 & 5a\end{bmatrix} = \begin{bmatrix}5a & 3\\ -b & 2\end{bmatrix}$

$\Rightarrow \quad 5a = 2 \text{ and } b = 3$

$\therefore \quad 5a + b = 5$

135. $\because PQ = kI \Rightarrow \dfrac{P.Q}{k} = I \Rightarrow P^{-1} = \dfrac{Q}{k}$ ...(i)

Also $\quad |P| = 12\alpha + 20$ ...(ii)

and $\quad \text{given } q_{23} = \dfrac{-k}{8}$

Comparing the third element of 2^{nd} row on both sides,

we get $\quad \dfrac{1}{(12\alpha + 20)}(-(3\alpha + 4)) = \dfrac{1}{k} \times \dfrac{-k}{8}$

$\Rightarrow \quad 24\alpha + 32 = 12\alpha + 20$

$\alpha = -1$ ...(iii)

From (ii), $\quad |P| = 8$ ...(iv)

Also $\quad PQ = kI$

$\Rightarrow \quad |PQ| = |kI|$

$\Rightarrow \quad |P||Q| = k^3$

$\Rightarrow \quad 8 \times \dfrac{k^2}{2} = k^3 \qquad \left(\because |P| = 8, |Q| = \dfrac{k^2}{2}\right)$

$\therefore \quad k = 4$ ...(v)

(b) $4\alpha - k + 8 = -4 - 4 + 8 = 0$

(c) $\det(P \text{ adj } (Q)) = |P|\,|\text{adj } Q| = |P||Q|^2 = 8 \times 8^2 = 2^9$

(d) $\det(Q \text{adj } (P)) = |Q|\,|\text{adj } P| = |Q||P|^2 = 8 \times 8^2 = 2^9$

136. $\because Z = \dfrac{-1 + \sqrt{3}i}{2} = \omega$ ...(i)

$\Rightarrow \quad \omega^3 = 1 \text{ and } 1 + \omega + \omega^2 = 0$

Now, $\quad P = \begin{bmatrix}(-\omega)^r & \omega^{2s}\\ \omega^{2s} & \omega^r\end{bmatrix}$

$\therefore \quad P^2 = \begin{bmatrix}(-\omega)^r & \omega^{2s}\\ \omega^{2s} & \omega^r\end{bmatrix}\begin{bmatrix}(-\omega)^r & \omega^{2s}\\ \omega^{2s} & \omega^r\end{bmatrix}$

$= \begin{bmatrix}\omega^{2r} + \omega^{4s} & \omega^{2r}((-\omega)^r + \omega^r)\\ \omega^{2s}((-\omega)^r + \omega^r) & \omega^{4s} + \omega^{2r}\end{bmatrix}$

$= \begin{bmatrix}\omega^{2r} + \omega^s & \omega^{2s}((-\omega)^r + \omega^r)\\ \omega^{2s}((-\omega)^r + \omega^s) & \omega^s + \omega^{2r}\end{bmatrix} (\because \omega^3 = 1)$

$\because \quad P^2 = -I = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$ ...(ii)

Form Eqs. (i) and (ii), we get

$$\omega^{2r} + \omega^s = -1$$

and $\quad \omega^{2s}((-\omega)^r + \omega^r) = 0$

$\Rightarrow r$ is odd and $s = r$ but not a multiple of 3. Which is possible when $r = s = 1$

$\therefore$ Only one pair is there.

137. $P=\begin{bmatrix}1&0&0\\4&1&0\\16&4&1\end{bmatrix}=I+\begin{bmatrix}0&0&0\\4&0&0\\16&4&0\end{bmatrix}=I+A$

Let $A=\begin{bmatrix}0&0&0\\4&0&0\\16&4&0\end{bmatrix}$

$\Rightarrow A^2=\begin{bmatrix}0&0&0\\0&0&0\\16&0&0\end{bmatrix}$ and $A^3=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$

$\Rightarrow A^n$ is a null matrix $\forall n\geq 3$

$\therefore \quad P^{50}=(I+A)^{50}=I+50A+\frac{50\times 49}{2}A^2$

$\Rightarrow \quad Q+\mathcal{J}=I+50A+25\times 49A^2$

or $\quad Q=50A+25\times 49A^2$

$$=\begin{bmatrix}0&0&0\\200&0&0\\800&200&0\end{bmatrix}+\begin{bmatrix}0&0&0\\0&0&0\\19600&0&0\end{bmatrix}$$

$\therefore \quad \begin{bmatrix}q_{11}&q_{12}&q_{13}\\q_{21}&q_{22}&q_{23}\\q_{31}&q_{32}&q_{33}\end{bmatrix}=\begin{bmatrix}0&0&0\\200&0&0\\20400&200&0\end{bmatrix}$

On comparing, we get

$q_{21}=q_{32}=200,\ q_{31}=20400$

$\therefore \quad \frac{q_{31}+q_{32}}{q_{21}}=\frac{20400+200}{200}$

$=102+1=103$

138. $\because A^2=\begin{bmatrix}2&-3\\-4&1\end{bmatrix}\begin{bmatrix}2&-3\\-4&1\end{bmatrix}=\begin{bmatrix}16&-9\\-12&13\end{bmatrix}$

$\therefore \quad 3A^2+12A=3\begin{bmatrix}16&-9\\-12&13\end{bmatrix}+12\begin{bmatrix}2&-3\\-4&1\end{bmatrix}$

$=\begin{bmatrix}72&-63\\-84&51\end{bmatrix}$

$\Rightarrow \quad \text{adj}\,(3A^2+12A)=\begin{bmatrix}51&63\\84&72\end{bmatrix}$

CHAPTER

09

Probability

Learning Part

Session 1

- Some Basic Definitions
- Mathematical or Priori or Classical Definition of Probability
- Odds in Favour and Odds Against the Event

Sesstion 2

- Some Important Symbols
- Conditional Probability

Sesstion 3

- Total Probability Theorem
- Baye's Theorem or Inverse Probability

Sesstion 4

- Binomial Theorem on Probability
- Poisson Distribution
- Expectation
- Multinomial Theorem
- Uncountable Uniform Spaces

Practice Part

- JEE Type Examples
- Chapter Exercises

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Session 1

Some Basic Definitions, Mathematical or Priori or Classical Definition of Probability, Odds in Favour and Odds Against the Event

Some Basic Definitions

1. Random Experiment

An experiment whose outcome cannot be predicted with certainty, is called a random experiment.

Or

If in each trial of an experiment, which when repeated under identical conditions, the outcome is not unique but the outcome in a trial is one of the several possible outcomes, then such an experiment is known as a random experiment.

For example,

(i) "Throwing an unbiased die" is a random experiment because when a die is thrown, we cannot say with certainty which one of the numbers 1, 2, 3, 4, 5 and 6 will come up.

(ii) "Tossing of a fair coin" is a random experiment because when a coin is tossed, we cannot say with certainty whether either a head or a tail will come up.

(iii) "Drawing a card from a well-shuffled pack of cards'' is a random experiment.

Remark

1. A die is a solid cube which has six faces and numbers 1, 2, 3, 4, 5 and 6 marked on the faces, respectively. In throwing or rolling a die, then any one number can be on the uppermost face.
2. (i) A pack of cards consists of 52 cards in 4 suits, i.e (a) Spades (♠) (b) Clubs (♣) (c) Hearts (♥) (d) Diamonds (♦). Each suit consists of 13 cards. Out of these, spades and clubs are black faced cards, while hearts and diamonds are red faced cards. The King, Queen, Jack (or Knave) are called face cards or honour cards.
   (ii) **Game of bridge** It is played by 4 players, each player is given 13 cards.
   (iii) **Game of whist** It is played by two pairs of persons.

2. Sample Space

The set of all possible results of a random experiment is called the sample space of that experiment and it is generally denoted by S.

Each element of a sample space is called a **sample point.**

For example,

(i) If we toss a coin, there are two possible results, namely a head (H) or a tail (T).
So, the sample space in this experiment is given by
$$S = \{H, T\}.$$

(ii) When two coins are tossed, the sample space
$$S = \{HH, HT, TH, TT\}$$o
where, HH denotes the head on the first coin and head on the second coin. Similarly, HT denotes the head on the first coin and tail on the second coin.

(iii) When we throw a die, then any one of the numbers 1, 2, 3, 4, 5 and 6 will come up. So, the sample space
$$S = \{1, 2, 3, 4, 5, 6\}.$$

3. Elementary Event

An event having only a single sample point is called an elementary or simple event.

For example, When two coins are tossed, the sample space, $S = \{HH, HT, TH, TT\}$, then the event, $E_1 = \{HH\}$ of getting both the heads is a simple event.

4. Mixed Event or Compound Event or Composite Event

An event other than elementary or simple event is called mixed event.

For example,

(i) When two coins are tossed, the sample space
$$S = \{HH, HT, TH, TT\}$$
Then, the event $E = \{HH, HT, TH\}$ of getting atleast one head, is a mixed event.

(ii) When a die is thrown, the sample space
$$S = \{1, 2, 3, 4, 5, 6\}$$
Let $A = \{2, 4, 6\}$ = the event of occurrence of an even number
and $B = \{3, 6\}$ = the event of occurrence of a number divisible by 3.
Here, A and B are mixed events.

5. Equally likely Events

The given events are said to be equally likely, if none of them is expected to occur in preference to the other.

For example,

(i) When an unbiased coin is tossed, then occurrence of head or tail are equally likely cases and there is no reason to expect a 'head' or a 'tail' in preference to the other.

(ii) When an unbiased die is thrown, all the six faces 1, 2, 3, 4, 5 and 6 are equally likely to come up. There is no reason to expect 1 or 2 or 3 or 4 or 5 or 6 in preference to the other.

6. Independent Events

Two events are said to be independent, if the occurrence of one does not depend on the occurrence of the other.

For example, When an unbiased die is thrown, then the sample space $S = \{1, 2, 3, 4, 5, 6\}$

Let $E_1 = \{1, 3, 5\}$ = the event of occurrence of an odd number and $E_2 = \{2, 4, 6\}$ = the event of occurrence of an even number. Clearly, the occurrence of odd number does not depend on the occurrence of even number. So, E_1 and E_2 are independent events.

7. Complementary Event

Let E be an event and S be the sample space for a random experiment, then complement of E is denoted by E' or E^c or $\bar{E}$. Clearly, E' means E does not occur.

Thus, E' occurs $\Leftrightarrow$ E does not occur.

For example, When an unbiased die is thrown, then the sample space $S = \{1, 2, 3, 4, 5, 6\}$.

If $E = \{1, 4, 6\}$, then $E' = \{2, 3, 5\}$

8. Mutually Exclusive Events

A set of events is said to be mutually exclusive, if occurrence of one of them precludes the occurrence of any of the remaining events. If a set of events $E_1, E_2, ..., E_n$ for mutually exclusive events.

Then, $E_1 \cap E_2 \cap \ldots \cap E_n = \phi$

For example, If we thrown an unbiased die, then the sample space $S = \{1, 2, 3, 4, 5, 6\}$ in which

$E_1 = \{1, 2, 3\}$ = the event of occurrence of a number less than 4 and $E_2 = \{5, 6\}$ = the event of occurrence of a number greater than 4. Clearly, $E_1 \cap E_2 = \phi$

So, E_1 and E_2 are mutually exclusive.

9. Exhaustive Events

A set of events is said to be exhaustive, if the performance of the experiment results in the occurrence of atleast one of them. If a set of events $E_1, E_2, ..., E_n$ for exhaustive events.

Then, $E_1 \cup E_2 \cup \ldots \cup E_n = S$

For example, If we thrown an unbiased die, then sample space $S = \{1, 2, 3, 4, 5, 6\}$ in which

$E_1 = \{1, 2, 3, 4\}$ = the event of occurrence of a number less than 5 and $E_2 = \{3, 4, 5, 6\}$ = the event of occurrence of a number greater than 2.

Then, $E_1 \cup E_2 = \{1, 2, 3, 4, 5, 6\}$ and $E_1 \cap E_2 = \{3, 4\}$

So, $E_1 \cup E_2 = S$ and $E_1 \cap E_2 \neq \phi$

Hence, E_1 and E_2 are exhaustive events.

10. Mutually Exclusive and Exhaustive Events

A set of events is said to be mutually exclusive and exhaustive, if above two conditions are satisfied. If a set of events $E_1, E_2, \ldots, E_n$ for mutually exclusive and exhaustive events.

Then, $E_1 \cup E_2 \cup \ldots \cup E_n = S$ and $E_1 \cap E_2 \cap \ldots \cap E_n = \phi$

For example, If we thrown an unbiased die, then sample space

$S = \{1, 2, 3, 4, 5, 6\}$ in which

$E_1 = \{1, 3, 5\}$ = the event of occurrence of an odd number and $E_2 = \{2, 4, 6\}$ = the event of occurrence of an even number.

Then, $E_1 \cup E_2 = \{1, 2, 3, 4, 5, 6\}$ and $E_1 \cap E_2 = \phi$

So, $E_1 \cup E_2 = S$ and $E_1 \cap E_2 = \phi$.

Hence, E_1 and E_2 are mutually exclusive and exhaustive events.

Mathematical or Priori
or Classical Definition of Probability

The probability of an event E to occur is the ratio of the number of cases in its favour to the total number of cases (equally likely).

$$\therefore \quad P(E) = \frac{n(E)}{n(S)} = \frac{\text{Number of cases favourable to event E}}{\text{Total number of cases}}$$

Range of Value of $P(E)$

Probability of occurrence of an event is a number lying between 0 and 1.

Proof Let S be the sample space and E be an event. Then,

$$E \subseteq S \qquad \ldots(i)$$

Also, $$\phi \subseteq S \qquad \ldots(ii)$$

where ϕ is a null set. From Eqs. (i) and (ii), we get

$$\phi \subseteq S \supseteq E \Rightarrow \; n(\phi) \leq n(E) \leq n(S)$$

$$\Rightarrow \qquad 0 \leq \frac{n(E)}{n(S)} \leq 1 \qquad [\because \; n(\phi) = 0]$$

$$\Rightarrow \qquad 0 \leq P(E) \leq 1$$

Remark

1. For impossible event ϕ; $P(\phi) = 0$
2. For sure event S, $P(S) = 1$

Relationship between $P(E)$ and $P(E')$

If E is any event and E' be the complement of event E, then

$$P(E) + P(E') = 1$$

Proof Let S be the sample space, then

$$E' = S - E$$

$$\Rightarrow \quad n(E') = n(S) - n(E)$$

$$\Rightarrow \quad \frac{n(E')}{n(S)} = 1 - \frac{n(E)}{n(S)}$$

$$\Rightarrow \quad P(E') = 1 - P(E)$$

$$\text{i.e.} \quad P(E) + P(E') = 1$$

Odds in Favour and Odds Against the Event

Let S be the sample space. If a is the number of cases favourable to the event E, b is the number of cases favourable to the event E', the **odds in favour** of E are defined by $a : b$ and **odds against** of E are $b : a$.

i.e. odds in favour of event E is

$$\frac{a}{b} = \frac{n(E)}{n(E')} = \frac{\frac{n(E)}{n(S)}}{\frac{n(E')}{n(S)}} = \frac{P(E)}{P(E')} \Rightarrow \frac{P(E')}{P(E)} = \frac{b}{a}$$

$$\Rightarrow \quad \frac{P(E') + P(E)}{P(E)} = \frac{b + a}{a}$$

$$\Rightarrow \quad \frac{1}{P(E)} = \frac{b + a}{a}$$

$$\Rightarrow \quad P(E) = \frac{a}{a + b} \text{ and } P(E') = \frac{b}{a + b}$$

Remark

We use the sign '+' for the operation 'or' and '×' for the operation 'and' in order to solve the problems on definition of probability.

Example 1. If three coins are tossed, represent the sample space and the event of getting atleast two heads, then find the number of elements in them.

Sol. Let S be the sample space and E be the event of occurrence of atleast two heads and let H denote the occurrence of head and T denote the occurrence of tail, when one coin is tossed.

Then, $S = \{H, T\} \times \{H, T\} \times \{H, T\}$

$$S = \{(H, H, H), (H, H, T), (H, T, H), (T, H, H), (T, T, H), (T, H, T), (H, T, T), (T, T, T)\}$$

and $E = \{(H, H, H), (H, H, T), (H, T, H), (T, H, H)\}$

Also, $n(S) = 8$ and $n(E) = 4$

Example 2. One ticket is drawn at random from a bag containing 24 tickets numbered 1 to 24. Represent the sample space and the event of drawing a ticket containing number which is a prime. Also, find the number of elements in them.

Sol. Let S be the sample space and E be the event of occurrence a prime number.

Then, $S = \{1, 2, 3, 4, 5, ..., 24\}$

and $E = \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$

Also, $n(S) = 24$ and $n(E) = 9$

Example 3. Two dice are thrown simultaneously. What is the probability obtaining a total score less than 11?

Sol. Let S be the sample space and E be the event of obtaining a total less than 11.

Then, $S = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\} \Rightarrow n(S) = 6 \times 6 = 36$

Let E' be the event of obtaining a total score greater than or equal to 11.

Also, $E' = \{(5, 6), (6, 5), (6, 6)\}$; $\therefore\ n(E') = 3$

Then, probability of obtaining a total score greater than or equal to 11,

$$P(E') = \frac{n(E')}{n(S)} = \frac{3}{36} = \frac{1}{12}$$

$$\therefore \quad P(E) = 1 - P(E') = 1 - \frac{1}{12} = \frac{11}{12}$$

Hence, required probability is $\frac{11}{12}$.

Example 4. ***If a leap-leap year is selected at random, then what is the chance it will contain 53 Sunday?***

Sol. A leap-leap year has 367 days i.e., 52 complete week and three days more. These three days will be three consecutive days of a week. A leap-leap year will have 53 Sundays, if out of the three consecutive days of a week selected at random one is a Sunday.

Let be the sample space and E be the event that out of the three consecutive days of a week one is Sunday, then

S = {(Sunday, Monday, Tuesday), (Monday, Tuesday, Wednesday), (Tuesday, Wednesday, Thursday), (Wednesday, Thursday, Friday), (Thursday, Friday, Saturday), (Friday, Saturday, Sunday), (Saturday, Sunday, Monday)}; $n(S) = 7$

and E = {(Sunday, Monday, Tuesday), (Friday, Saturday, Sunday), (Saturday, Sunday, Monday)}

$$\therefore \quad n(E) = 3$$

Now, required probability, $P(E) = \frac{n(E)}{n(S)} = \frac{3}{7}$

Example 5. ***From a pack of 52 playing cards, three cards are drawn at random. Find the probability of drawing a King, a Queen and a Knave.***

Sol. Let S be the sample space and E be the event that out of the three cards drawn one is a King, one is a Queen and one is a Knave.

$\therefore$ $n(S)$ = Total number of selecting 3 cards out of 52 cards
$= {}^{52}C_3$

and $n(E)$ = Number of selecting 3 cards out of one is King, one is Queen and one is Knave $= {}^4C_1 \cdot {}^4C_1 \cdot {}^4C_1 = 64$

$\therefore$ Required probability, $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{64}{{}^{52}C_3} = \dfrac{64}{\dfrac{52 \cdot 51 \cdot 50}{1 \cdot 2 \cdot 3}} = \dfrac{16}{5525}$

Example 6. ***A bag contains 8 red and 5 white balls. Three balls are drawn at random. Find the probability that***

(i) all the three balls are white.
(ii) all the three balls are red.
(iii) one ball is red and two balls are white.

Sol. Let S be the sample space, E_1 be the event of getting 3 white balls, E_2 be the event of getting 3 red balls and E_3 be the event of getting one red ball and two white balls.

$\therefore$ $n(S)$ = Number of ways of selecting 3 balls out of $13(8+5) = {}^{13}C_3 = \dfrac{13 \cdot 12 \cdot 11}{1 \cdot 2 \cdot 3} = 286$

(i) $n(E_1)$ = Number of ways of selecting 3 white balls out of 5
$= {}^5C_3 = {}^5C_2 = \dfrac{5 \cdot 4}{1 \cdot 2} = 10$

$\therefore$ P (getting 3 white balls) $= \dfrac{n(E_1)}{n(S)} = \dfrac{10}{286} = \dfrac{5}{143}$

(ii) $n(E_2)$ = Number of ways of selecting 3 red balls out of 8
$= {}^8C_3 = \dfrac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} = 56$

$\therefore$ P (getting 3 red balls) $= \dfrac{n(E_2)}{n(S)}$
$= \dfrac{56}{286} = \dfrac{28}{143}$

(iii) $n(E_3)$ = Number of ways of selecting 1 red ball out of 8 and 2 black balls out of 5 $= {}^8C_1 \cdot {}^5C_2 = 8 \cdot 10 = 80$

$\therefore$ P (getting 1 red and 2 black balls)
$= \dfrac{n(E_3)}{n(S)} = \dfrac{80}{286} = \dfrac{40}{143}$

Exercise for Session 1

1. A problem in mathematics is given to three students and their respective probabilities of solving the problem are $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$. The probability that the problem is solved, is

(a) $\frac{3}{4}$ (b) $\frac{1}{2}$ (c) $\frac{2}{3}$ (d) $\frac{1}{3}$

2. A dice is thrown 3 times and the sum of the 3 numbers thrown is 15. The probability that the first throw was a four, is

(a) $\frac{1}{5}$ (b) $\frac{1}{4}$ (c) $\frac{1}{6}$ (d) $\frac{2}{5}$

3. Three faces of a fair dice are yellow, two faces red and one blue. The dice is tossed three times. The probability that the colours yellow, red and blue appear in the first, second and third toss respectively, is

(a) $\frac{1}{6}$ (b) $\frac{1}{12}$ (c) $\frac{1}{24}$ (d) $\frac{1}{36}$

4. *A* speaks truth in 75% of cases and *B* in 80% of cases. The percentage of cases they are likely to contradict each other in stating the same fact, is

(a) 30% (b) 35% (c) 45% (d) 25%

5. An unbiased dice with faces marked 1, 2, 3, 4, 5, 6 is rolled four times. Out of four face values obtained the probability that the minimum face value is not less than 2 and the maximum face value is not greater than 5, is

(a) $\frac{16}{81}$ (b) $\frac{1}{81}$ (c) $\frac{80}{81}$ (d) $\frac{65}{81}$

6. Three numbers are chosen at random without replacement from {1, 2, 3, ..., 10}. The probability that the minimum of the chosen number is 3 or their maximum is 7, is

(a) $\frac{11}{20}$ (b) $\frac{7}{20}$ (c) $\frac{11}{40}$ (d) $\frac{7}{40}$

7. Seven white balls and three black balls are randomly placed in a row. The probability that no two black balls are placed adjacently, is

(a) $\frac{1}{2}$ (b) $\frac{7}{15}$ (c) $\frac{2}{15}$ (d) $\frac{1}{3}$

8. Two numbers are selected randomly from the set $S = \{1, 2, 3, 4, 5, 6\}$ without replacement. The probability that minimum of the two numbers is less than 4, is

(a) $\frac{1}{15}$ (b) $\frac{14}{15}$ (c) $\frac{1}{5}$ (d) $\frac{4}{5}$

9. If $\frac{1+3p}{3}, \frac{1-p}{4}$ and $\frac{1-2p}{2}$ are the probabilities of the three mutually exclusive events, then $p \in$

(a) $[0, 1]$ (b) $\left[0, \frac{1}{2}\right]$ (c) $\left[\frac{1}{3}, 1\right]$ (d) $\left[\frac{1}{3}, \frac{1}{2}\right]$

10. Three identical dice are rolled once. The probability that the same number will appear on each of them, is

(a) $\frac{1}{6}$ (b) $\frac{1}{36}$ (c) $\frac{1}{18}$ (d) $\frac{3}{28}$

11. If the letters of the word ASSASSIN are written down in a row, the probability that no two S's occur together, is

(a) $\frac{1}{35}$ (b) $\frac{1}{21}$ (c) $\frac{1}{14}$ (d) $\frac{1}{28}$

12. A box contains 2 black, 4 white and 3 red balls. One ball is drawn at random from the box and kept aside. From the remaining balls in the box another ball is drawn and kept beside the first. This process is repeated till all the balls are drawn from the box. The probability that the balls drawn from the box are in the sequence 2 black, 4 white and 3 red, is

(a) $\frac{1}{126}$ (b) $\frac{1}{630}$ (c) $\frac{1}{1260}$ (d) $\frac{1}{2520}$

13. If three distinct numbers are chosen randomly from the first 100 natural numbers, then the probability that all three of them are divisible by both 2 and 3, is

(a) $\frac{4}{55}$ (b) $\frac{4}{35}$ (c) $\frac{4}{33}$ (d) $\frac{4}{1155}$

14. There are 2 vans each having numbered seats, 3 in the front and 4 at the back. There are 3 girls and 9 boys to be seated in the vans. The probability of 3 girls sitting together in a back row on adjacent seats, is

(a) $\frac{1}{13}$ (b) $\frac{1}{39}$ (c) $\frac{1}{65}$ (d) $\frac{1}{91}$

15. *A* and *B* stand in a ring along with 10 other persons. If the arrangement is at random, then the probability that there are exactly 3 persons between *A* and *B*, is

(a) $\frac{1}{11}$ (b) $\frac{2}{11}$ (c) $\frac{3}{11}$ (d) $\frac{4}{11}$

16. The first 12 letters of English alphabet are written down at random in a row. The probability that there are exactly 4 letters between *A* and *B*, is

(a) $\frac{7}{33}$ (b) $\frac{7}{66}$ (c) $\frac{7}{99}$ (d) $\frac{5}{33}$

17. Six boys and six girls sit in a row randomly. The probability that the six girls sit together or the boys and girls sit alternately, is

(a) $\frac{3}{308}$ (b) $\frac{1}{100}$ (c) $\frac{2}{205}$ (d) $\frac{4}{407}$

18. If from each of three boxes containing 3 white and 1 black, 2 white and 2 black, 1 white and 3 black balls, one ball is drawn, the probability of drawing 2 white and 1 black ball, is

(a) $\frac{13}{32}$ (b) $\frac{1}{4}$ (c) $\frac{1}{32}$ (d) $\frac{3}{16}$

19. The probability that a year chosen at random has 53 Sundays, is

(a) $\frac{5}{7}$ (b) $\frac{3}{7}$ (c) $\frac{5}{28}$ (d) $\frac{3}{28}$

20. If the letters of the word MATHEMATICS are arranged arbitrarily, the probability that C comes before E, E before H, H before I and I before S, is

(a) $\frac{3}{10}$ (b) $\frac{1}{20}$ (c) $\frac{1}{120}$ (d) $\frac{1}{720}$

Session 2

Some Important Symbols, Conditional Probability

Some Important Symbols

If A, B and C are any three events, then

(i) $A \cap B$ or AB denotes the event of simultaneous occurrence of both the events A and B.

(ii) $A \cup B$ or $A + B$ denotes the event of occurrence of atleast one of the events A or B.

(iii) $A - B$ denotes the occurrence of event A but not B.

(iv) $\overline{A}$ denotes the not occurrence of event A.

(v) $A \cap \overline{B}$ denotes the occurrence of event A but not B.

(vi) $\overline{A} \cap \overline{B} = (\overline{A \cup B})$ denotes the occurrence of neither A nor B.

(vii) $A \cup B \cup C$ denotes the occurrence of atleast one event A, B or C.

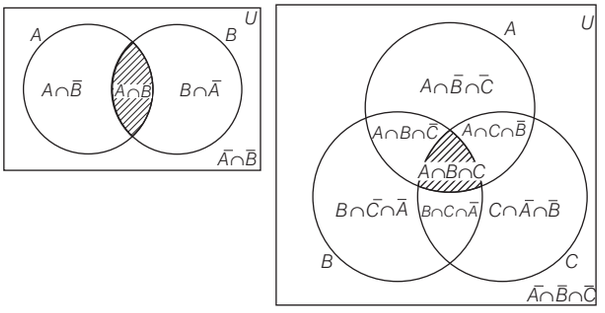
(viii) $(A \cap \overline{B}) \cup (\overline{A} \cap B)$ denotes the occurrence of exactly one of A and B.

(ix) $A \cap B \cap C$ denotes the occurrence of all three A, B and C.

(x) $(A \cap B \cap \overline{C}) \cup (A \cap \overline{B} \cap C) \cup (\overline{A} \cap B \cap C)$ denotes the occurrence of exactly two of A, B and C.

Remark

Remember with the help of figures



Important Results

1. If A and B are arbitrary events, then

(a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof Let S be the sample space. Since, we know that

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$\Rightarrow \quad \frac{n(A \cup B)}{n(S)} = \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)}$$

$$\Rightarrow \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Remark

If A and B are mutually exclusive events, then $A \cap B = \phi$.
Hence, $P(A \cap B) = 0$.
$\therefore P(A \cup B) = P(A) + P(B)$

(b) P (exactly one of A, B occurs)

$$= P(A \cap \overline{B}) + P(\overline{A} \cap B)$$
$$= P(A) - P(A \cap B) + P(B) - P(A \cap B)$$
$$= P(A) + P(B) - 2P(A \cap B)$$
$$= P(A \cup B) - P(A \cap B)$$

(c) P (neither A nor B)

$$= P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)$$

Remark

$P(\overline{A} \cup \overline{B}) = 1 - P(A \cap B)$

2. If A, B and C are three events, then

(a) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B)$
$- P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

Remark

If A, B and C are mutually exclusive events, then

$$A \cap B = \phi,\ B \cap C = \phi,\ C \cap A = \phi,\ A \cap B \cap C = \phi$$
$$\Rightarrow P(A \cap B) = 0, P(B \cap C) = 0, P(C \cap A) = 0, P(A \cap B \cap C) = 0$$
$$\therefore \quad P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

General form of Addition Theorem of Probability

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \ldots + (-1)^{n-1} P(A_1 \cap A_2 \cap \ldots \cap A_n)$$

Remark

If $A_1, A_2, \ldots, A_n$ are mutually exclusive events, then

$$\sum_{i<j} P(A_i \cap A_j) = 0, \quad \sum_{i<j<k} P(A_i \cap A_j \cap A_k) = 0$$

and $\quad P(A_1 \cap A_2 \cap \ldots \cap A_n) = 0$

$$\therefore \quad P(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{i=1}^{n} P(A_i)$$

(b) P (atleast two of A, B, C occur)
$= P(A \cap B) + P(B \cap C) + P(C \cap A) - 2P(A \cap B \cap C)$

(c) P (exactly two of A, B, C occur)
$= P(A \cap B) + P(B \cap C) + P(C \cap A) - 3P(A \cap B \cap C)$

(d) P (exactly one of A, B, C occur)
$= P(A) + P(B) + P(C) - 2P(A \cap B) - 2P(B \cap C) - 2P(C \cap A) + 3P(A \cap B \cap C)$

3. (a) If $A_1, A_2, ..., A_n$ are independent events, then
$P(A_1 \cap A_2 \cap ... \cap A_n) = P(A_1)\, P(A_2) ... P(A_n)$

(b) If $A_1, A_2, ..., A_n$ are mutually exclusive events, then
$P(A_1 \cup A_2 \cup ... \cup A_n) = P(A_1) + P(A_2) + ... + P(A_n)$

(c) If $A_1, A_2, ..., A_n$ are exhaustive events, then
$P(A_1 \cup A_2 \cup ... \cup A_n) = 1$

(d) If $A_1, A_2, ..., A_n$ are mutually exclusive and exhaustive events, then
$P(A_1 \cup A_2 \cap ... \cap A_n) = P(A_1) + P(A_2) + ... + P(A_n) = 1$

4. If $A_1, A_2, ..., A_n$ are n events, then

(a) $P(A_1 \cup A_2 \cup ... \cup A_n) \le P(A_1) + P(A_2) + ... + P(A_n)$

(b) $P(A_1 \cap A_2 \cap ... \cap A_n) \ge 1 - P(\overline{A}_1) - P(\overline{A}_2) - ... - P(\overline{A}_n)$

Important Result

If E_1 and E_2 are independent events, then

(a) E_1 and $\overline{E}_2$ are independent events.

(b) $\overline{E}_1$ and E_2 are independent events.

(c) $\overline{E}_1$ and $\overline{E}_2$ are independent events.

Proof Given, E_1 and E_2 are independent events, then
$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$

(a) $P(E_1 \cap \overline{E}_2) = P(E_1) - P(E_1 \cap E_2)$
$= P(E_1) - P(E_1) \cdot P(E_2)$
$= P(E_1)[1 - P(E_2)] = P(E_1) \cdot P(\overline{E}_2)$

So, E_1 and $\overline{E}_2$ are independent events.

(b) Same as in part (i).

(c) $P(\overline{E}_1 \cap \overline{E}_2) = P(\overline{E_1 \cup E_2})$
$= 1 - P(E_1 \cup E_2) = 1 - [P(E_1) + P(E_2) - P(E_1 \cap E_2)]$
$= 1 - P(E_1) - P(E_2) + P(E_1) \cdot P(E_2)$
$= P(\overline{E}_1) - P(E_2)[1 - P(E_1)]$
$= P(\overline{E}_1) - P(E_2) \cdot P(\overline{E}_1) = P(\overline{E}_1)[1 - P(E_2)]$
$= P(\overline{E}_1)\, P(\overline{E}_2)$

Remark

If $E_1, E_2, ..., E_n$ are independent events,
then $P(E_1 \cup E_2 \cup ... \cup E_n)$
$= 1 - P(E_1 \cup E_2 \cup ... \cup E_n)' = 1 - P(E_1' \cap E_2' \cap ... \cap E_n')$
$= 1 - P(E_1') \cdot P(E_2') ... P(E_n')$

Example 7. For a post, three persons A, B and C appear in the interiew. The probability of A being selected is twice that of B and the probability of B being selected is thrice that of C. What are the individual probabilities of A, B and C being selected?

Sol. Let E_1, E_2 and E_3 be the events of selection of A, B and C respectively.

Let $P(E_3) = x$.

Then, $P(E_2) = 3P(E_3) = 3x$ and $P(E_1) = 2P(E_2) = 6x$

Since, E_1, E_2 and E_3 are mutually exclusive and exhaustive events.

$\therefore \quad P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) = 1$

$\therefore \quad P(E_1) + P(E_2) + P(E_3) = 1$

$\Rightarrow \quad 6x + 3x + x = 1$

$\therefore \quad x = \frac{1}{10}$

Hence, $P(E_1) = 6x = \frac{6}{10} = \frac{3}{5}$

$P(E_2) = 3x = \frac{3}{10}$ and $P(E_3) = x = \frac{1}{10}$

Example 8. If A and B are independent events, the probability that both A and B occur is $\frac{1}{8}$ and the probability that none of them occurs is $\frac{3}{8}$. Find the probability of the occurrence of A.

Sol. We have,

$$P(A \cap B) = \frac{1}{8} \Rightarrow P(A)\, P(B) = \frac{1}{8} \quad ...(i)$$

[$\because A$ and B are independent]

and $P(\overline{A} \cap \overline{B}) = \frac{3}{8} \Rightarrow P(\overline{A})\, P(\overline{B}) = \frac{3}{8}$

$\Rightarrow \quad (1 - P(A))(1 - P(B)) = \frac{3}{8}$

$\Rightarrow \quad 1 - P(A) - P(B) + \frac{1}{8} = \frac{3}{8}$ [from Eq. (i)]

$\Rightarrow \quad P(A) + P(B) = \frac{3}{4} \quad ...(ii)$

The quadratic equation whose roots are $P(A)$ and $P(B)$ is
$x^2 - [P(A) + P(B)]x + P(A) \cdot P(B) = 0$

$\Rightarrow \quad x^2 - \frac{3}{4}x + \frac{1}{8} = 0$ [from Eqs. (i) and (ii)]

or $8x^2 - 6x + 1 = 0$ or $x = \frac{1}{2}, \frac{1}{4}$

Hence, $P(A) = \frac{1}{2}$ or $\frac{1}{4}$

Example 9. A and B are two candidates seeking admission in IIT. The probability that A is selected is 0.5 and the probability that both A and B are selected is atmost 0.3. Is it possible that the probability of B getting selected is 0.9?

Sol. Let E_1 and E_2 are the events of A and B selected, respectively.

Given, $P(E_1 \cap E_2) \le 0.3$ and $P(E_1) = 0.5$

Since, $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$\because \quad P(E_1 \cup E_2) \le 1$

$\therefore \quad P(E_1) + P(E_2) - P(E_1 \cap E_2) \le 1$

$\Rightarrow \quad P(E_1) + P(E_2) \le 1 + P(E_1 \cap E_2)$

$\Rightarrow \quad 0.5 + P(E_2) \le 1 + 0.3 \Rightarrow P(E_2) \le 0.8$

Hence, $P(E_2) \ne 0.9$

Example 10. Let A, B and C be three events. If the probability of occurring exactly one event out of A and B is $1 - a$, out of B and C is $1 - 2a$, out of C and A is $1 - a$ and that of occurring three events simultaneously is a^2, then prove that the probability that atleast one out of A, B and C will occur is greater than $\frac{1}{2}$.

Sol. Given,

$$P(A) + P(B) - 2P(A \cap B) = 1 - a \quad \text{...(i)}$$

$$P(B) + P(C) - 2P(B \cap C) = 1 - 2a \quad \text{...(ii)}$$

and $P(C) + P(A) - 2P(C \cap A) = 1 - a$ ...(iii)

$\therefore \quad P(A \cap B \cap C) = a^2$ ...(iv)

$$\therefore P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

$$= \frac{1}{2}\{P(A) + P(B) - 2P(A \cap B) + P(B) + P(C) - 2P(B \cap C) + P(C) + P(A) - 2P(C \cap A)\} + P(A \cap B \cap C)$$

$$= \frac{1}{2}\{1 - a + 1 - 2a + 1 - a\} + a^2 \text{ [from Eqs. (i), (ii), (iii) and (iv)]}$$

$$= \frac{3}{2} - 2a + a^2 = (a-1)^2 + \frac{1}{2} > \frac{1}{2} \quad [\because a \ne 1]$$

Example 11. If A, B and C are three events, such that $P(A) = 0.3$, $P(B) = 0.4$, $P(C) = 0.8$, $P(AB) = 0.08$, $P(AC) = 0.28$, $P(ABC) = 0.09$. If $P(A \cup B \cup C) \ge 0.75$, then show that $P(BC)$ lies in the interval $0.23 \le x \le 0.48$.

Sol. Let $P(BC) = x$

Since, $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(CA) + P(ABC)$

$\therefore \quad = 0.3 + 0.4 + 0.8 - 0.08 - x - 0.28 + 0.09 = 1.23 - x$

But given that, $P(A \cup B \cup C) \ge 0.75$ and $P(A \cup B \cup C) \le 1$

$\therefore \quad 0.75 \le 1.23 - x \le 1 \Rightarrow -0.75 \ge -1.23 + x \ge -1$

or $1.23 - 0.75 \ge x \ge 1.23 - 1$ or $0.23 \le x \le 0.48$

Conditional Probability

The probability of occurrence of an event E_1, given that E_2 has already occurred is called the conditional probability of occurrence of E_1 on the condition that E_2 has already occurred, it is denoted by $P\left(\frac{E_1}{E_2}\right)$.

Thus, $P\left(\frac{E_1}{E_2}\right) = \frac{P(E_1 \cap E_2)}{P(E_2)}, E_2 \ne \phi = \frac{\frac{n(E_1 \cap E_2)}{n(S)}}{\frac{n(E_2)}{n(S)}}$

$\Rightarrow \quad = \frac{n(E_1 \cap E_2)}{n(E_2)}$

Remark

1. If E_1 and E_2 are independent events, then $P\left(\frac{E_2}{E_1}\right) = P(E_2)$
2. If E_1 and E_2 are two events such that $E_2 \ne \phi$, then $P\left(\frac{E_1}{E_2}\right) + P\left(\frac{\bar{E}_1}{E_2}\right) = 1$
3. If $E_1, E_2, E_3, \ldots, E_4$ are independent events, then $P(E_1 \cup E_2 \cup E_3 \cup \ldots \cup E_n) = 1 - P(\bar{E}_1) \cdot P(\bar{E}_2) \cdot P(\bar{E}_3) \ldots P(\bar{E}_n)$
4. If E_1, E_2 and E_3 are three events such that $E_1 \ne \phi$, $E_1E_2 \ne \phi$, then $P(E_1 \cap E_2 \cap E_3) = P(E_1) \cdot P\left(\frac{E_2}{E_1}\right) \cdot P\left(\frac{E_3}{E_1E_2}\right)$

Generalised form

If $E_1, E_2, E_3, \ldots, E_n$ are n events such that $E_1 \ne \phi$, $E_1E_2 \ne \phi$, $E_1E_2E_3 \ne \phi$, ..., $E_1E_2E_3 \ldots E_{n-1} \ne \phi$, then $P(E_1 \cap E_2 \cap E_3 \cap \ldots \cap E_n)$

$$= P(E_1) \cdot P\left(\frac{E_2}{E_1}\right) \cdot P\left(\frac{E_3}{E_1E_2}\right) \cdot P\left(\frac{E_4}{E_1E_2E_3}\right) \ldots P\left(\frac{E_n}{E_1E_2E_3 \ldots E_{n-1}}\right)$$

Example 12. Two dice are thrown. Find the probability that the sum of the numbers coming up on them is 9, if it is known that the number 5 always occurs on the first dice.

Sol. Let S be the sample space

$\therefore \quad S = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$

$\therefore \quad n(S) = 36$

and let $E_1 \equiv$ The event that the sum of the numbers coming up is 9.

and $E_2 \equiv$ The event of occurrence of 5 on the first dice.

$\therefore \quad E_1 \equiv \{(3, 6), (6, 3), (4, 5), (5, 4)\}$

$\therefore \quad n(E_1) = 4$

and $E_2 = \{(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)\}$

$\therefore \quad n(E_2) = 6$

$E_1 \cap E_2 = \{(5,4)\}$

$\therefore \quad n(E_1 \cap E_2) = 1$

Now, $P(E_1 \cap E_2) = \dfrac{n(E_1 \cap E_2)}{n(S)} = \dfrac{1}{36}$

and $P(E_2) = \dfrac{n(E_2)}{n(S)} = \dfrac{6}{36} = \dfrac{1}{6}$

$\therefore$ Required probability,

$$P\left(\frac{E_1}{E_2}\right) = \frac{P(E_1 \cap E_2)}{P(E_2)} = \frac{\frac{1}{36}}{\frac{1}{6}} = \frac{1}{6}$$

Aliter $P\left(\dfrac{E_1}{E_2}\right) = \dfrac{n(E_1 \cap E_2)}{n(E_2)} = \dfrac{1}{6}$

Example 13. In a class, 30% students fail in English; 20% students fail in Hindi and 10% students fail in English and Hindi both. A student is chosen at random, then what is the probability that he will fail in English, if he has failed in Hindi?

Sol. Let S be the sample space.

If $n(S) = 100$, then

$E_1 \equiv$ The event that the student chosen fail in English

$\therefore n(E_1) = 30$

and $E_2 \equiv$ The event that the student chosen fail in Hindi

$\therefore \quad n(E_2) = 20$ and $n(E_1 \cap E_2) = 10$

$\therefore \quad P(E_2) = \dfrac{n(E_2)}{n(S)}$

$= \dfrac{20}{100} = \dfrac{1}{5}$

and $P(E_1 \cap E_2) = \dfrac{n(E_1 \cap E_2)}{n(S)} = \dfrac{10}{100} = \dfrac{1}{10}$

$\therefore$ Required probability, $P\left(\dfrac{E_1}{E_2}\right) = \dfrac{P(E_1 \cap E_2)}{P(E_2)} = \dfrac{\frac{1}{10}}{\frac{1}{5}} = \dfrac{1}{2}$

Aliter $P\left(\dfrac{E_1}{E_2}\right) = \dfrac{n(E_1 \cap E_2)}{n(E_2)}$

$= \dfrac{10}{20} = \dfrac{1}{2}$

Exercise for Session 2

1 If $P(A) = 0{\cdot}8$, $P(B) = 0{\cdot}5$, then $P(A \cap B)$ lies in the interval

(a) $[0{\cdot}2, 0{\cdot}5]$ (b) $[0{\cdot}2, 0{\cdot}3]$ (c) $[0{\cdot}3, 0{\cdot}5]$ (d) $[0{\cdot}1, 0{\cdot}5]$

2 If $P(A) = \dfrac{1}{4}$, $P(B) = \dfrac{1}{13}$ and $P(A \cap B) = \dfrac{1}{52}$, then the value of $P(\bar{A} \cap \bar{B})$, is

(a) $\dfrac{3}{13}$ (b) $\dfrac{5}{13}$ (c) $\dfrac{7}{13}$ (d) $\dfrac{9}{13}$

3 If A and B are independent events such that $P(\bar{A} \cap B) = \dfrac{2}{15}$ and $P(A \cap \bar{B}) = \dfrac{1}{6}$, then $P(B)$ is

(a) $\dfrac{1}{5}$ (b) $\dfrac{1}{6}$ (c) $\dfrac{4}{5}$ (d) $\dfrac{5}{6}$

4 If A and B are two events such that $P(A \cup B) = \dfrac{5}{6}$, $P(A) = \dfrac{1}{3}$, $P(B) = \dfrac{3}{4}$, then A and B are

(a) mutually exclusive (b) dependent (c) independent (d) None of these

5 If A, B and C are mutually exclusive and exhaustive events associated with a random experiment. If $P(B) = \dfrac{3}{2}P(A)$ and $P(C) = \dfrac{1}{2}P(B)$, then $P(A)$ is equal to

(a) $\dfrac{2}{13}$ (b) $\dfrac{4}{13}$ (c) $\dfrac{6}{13}$ (d) $\dfrac{8}{13}$

6 If A and B are two events, then $P(A) + P(B) = 2P(A \cap B)$ if and only if

(a) $P(A) + P(B) = 1$ (b) $P(A) = P(B)$ (c) $P(A) + P(B) > 1$ (d) None of these

7 If A and B are two events such that $P(A \cap B) = \dfrac{1}{4}$, $P(\bar{A} \cap \bar{B}) = \dfrac{1}{5}$ and $P(A) = P(B) = p$, then p is equal to

(a) $\dfrac{17}{40}$ (b) $\dfrac{19}{40}$ (c) $\dfrac{21}{40}$ (d) $\dfrac{23}{40}$

8 If A and B are two events such that $P(A \cup B) = \frac{3}{4}$, $P(A \cap B) = \frac{1}{4}$, $P(\bar{A}) = \frac{2}{3}$. Then $(\bar{A} \cap B)$ is equal to

(a) $\frac{5}{12}$ (b) $\frac{3}{8}$ (c) $\frac{5}{8}$ (d) $\frac{1}{4}$

9 If $P(B) = \frac{3}{4}$, $P(A \cap B \cap \bar{C}) = \frac{1}{3}$ and $P(\bar{A} \cap B \cap \bar{C}) = \frac{1}{3}$, then $P(B \cap C)$ is equal to

(a) $\frac{1}{12}$ (b) $\frac{1}{6}$ (c) $\frac{1}{15}$ (d) $\frac{1}{9}$

10 If A and B are two events such that $P(A) > 0$ and $P(B) \neq 1$, then $P\left(\frac{\bar{A}}{\bar{B}}\right)$ is equal to

(a) $1 - P\left(\frac{A}{B}\right)$ (b) $1 - P\left(\frac{A}{\bar{B}}\right)$ (c) $\frac{1 - P(A \cup B)}{P(B)}$ (d) $\frac{P(\bar{A})}{P(B)}$

11 If $P(A) = \frac{3}{8}$, $P(B) = \frac{5}{8}$ and $P(A \cup B) = \frac{3}{4}$, then $P\left(\frac{\bar{A}}{\bar{B}}\right)$ is equal to

(a) $\frac{1}{4}$ (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $\frac{3}{4}$

12 If two events A and B are such that $P(\bar{A}) = 0 \cdot 3$, $P(B) = 0 \cdot 4$ and $P(A \cap \bar{B}) = 0 \cdot 5$, then $P\left(\frac{B}{A \cup \bar{B}}\right)$ is equal to

(a) $\frac{1}{4}$ (b) $\frac{1}{5}$ (c) $\frac{2}{5}$ (d) $\frac{3}{5}$

13 Two dice are thrown. The probability that the number appeared have a sum of 8. If it is known that the second die always exhibits 4, is

(a) $\frac{5}{6}$ (b) $\frac{1}{6}$ (c) $\frac{2}{3}$ (d) $\frac{1}{3}$

14 A is targetting to B, B and C are targetting to A. The probability of hitting the target by A, B and C are $\frac{2}{3}, \frac{1}{2}, \frac{1}{3}$ respectively. If A is hit, the probability that B hits the target and C does not, is

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) $\frac{2}{3}$ (d) $\frac{3}{4}$

15 If A and B are two events such that $A \cap B \neq \phi$, $P\left(\frac{A}{B}\right) = P\left(\frac{B}{A}\right)$. Then,

(a) $A = B$ (b) $P(A) = P(B)$
(c) A and B are independent (d) All of these

Session 3

Total Probability Theorem, Baye's Theorem or Inverse Probability

Total Probability Theorem

Let $E_1, E_2, \ldots, E_n$ be n mutually exclusive and exhaustive events i.e., $E_i \cap E_j = \phi$ for $i \neq j$ and $\bigcup_{i=1}^{n} E_i = S$.

Suppose that, $P(E_i) > 0, \forall\ 1 \le i \le n$

Then for any event E

$$P(E) = \sum_{i=1}^{n} P(E_i) \cdot P\left(\frac{E}{E_i}\right)$$

Proof Since, $E_1, E_2, \ldots, E_n$ are disjoint

$\therefore E \cap E_1, E \cap E_2, \ldots, E \cap E_n$ are also disjoint.

Now, $$E = E \cap S = E \cap \left(\bigcup_{i=1}^{n} E_i\right) = \bigcup_{i=1}^{n} (E \cap E_i)$$

$\therefore$ $$P(E) = \sum_{i=1}^{n} P(E \cap E_i) = \sum_{i=1}^{n} P(E_i) \cdot P\left(\frac{E}{E_i}\right)$$

Example 14. The probability that certain electronic component fails, when first used is 0.10. If it does not fail immediately, then the probability that it lasts for one year is 0.99. What is the probability that a new component will last for one year?

Sol. Given probability of electronic component fails, when first used $= 0.10$

i.e., $P(\underline{F}) = 0.10$

$\therefore$ $P(F) = 1 - P(F) = 0.90$

and let $P(Y)$ = Probability of new component to last for one year

$\because$ $P(F) + P(\overline{F}) = 1$

Obviously, the two events are mutually exclusive and exhaustive

$\therefore$ $$P\left(\frac{Y}{F}\right) = 0 \text{ and } P\left(\frac{Y}{\overline{F}}\right) = 0.99$$

$\therefore$ $$P(Y) = P(F) \cdot P\left(\frac{Y}{F}\right) + P(\overline{F}).P\left(\frac{Y}{\overline{F}}\right)$$

$$= 0.10 \times 0 + 0.90 \times 0.99$$

$$= 0 + (0.9)(0.99) = 0.891$$

Example 15. Three groups A, B and C are contesting for positions on the Board of Directors of a company. The probabilities of their winning are 0.5, 0.3 and 0.2, respectively. If the group A wins, then the probability of introducing a new product is 0.7 and the corresponding probabilities for groups B and C are 0.6 and 0.5, respectively. Find the probability that the new product will be introduced.

Sol. Given, $P(A) = 0.5$, $P(B) = 0.3$ and $P(C) = 0.2$

$\therefore$ $P(A) + P(B) + P(C) = 1$

Then, events A, B, C are exhaustive.

If $P(E)$ = Probability of introducing a new product, then as given

$$P\left(\frac{E}{A}\right) = 0.7,\ P\left(\frac{E}{B}\right) = 0.6 \text{ and } P\left(\frac{E}{C}\right) = 0.5$$

$$P(E) = P(A) \cdot P\left(\frac{E}{A}\right) + P(B) \cdot P\left(\frac{E}{B}\right) + P(C) \cdot P\left(\frac{E}{C}\right)$$

$$= 0.5 \times 0.7 + 0.3 \times 0.6 + 0.2 \times 0.5$$

$$= 0.35 + 0.18 + 0.10 = 0.63$$

Example 16. An urn contains 2 white and 2 black balls. A ball is drawn at random. If it is white, it is not replace into urn, otherwise it is replaced along with another ball of the same colour. The process is repeated, find the probability that the third ball drawn is black.

Sol. For the first two draw, the balls taken out may be

Let E_1 = White and White; E_2 = White and Black

E_3 = Black and White; E_4 = Black and Black

$\therefore$ $$P(E_1) = P(W) \cdot P\left(\frac{W}{W}\right) = \frac{2}{4} \cdot \frac{1}{3} = \frac{1}{6}$$

$$P(E_2) = P(W) \cdot P\left(\frac{B}{W}\right) = \frac{2}{4} \cdot \frac{2}{3} = \frac{1}{3}$$

$$P(E_3) = P(B) \cdot P\left(\frac{W}{B}\right) = \frac{2}{4} \cdot \frac{2}{5} = \frac{1}{5}$$

and $$P(E_4) = P(B) \cdot P\left(\frac{B}{B}\right) = \frac{2}{4} \cdot \frac{3}{5} = \frac{3}{10}$$

$$\therefore\ P(E_1)+P(E_2)+P(E_3)+P(E_4)=\frac{1}{6}+\frac{1}{3}+\frac{1}{5}+\frac{3}{10}$$
$$=\frac{10+20+12+18}{60}=1$$

Then, events E_1, E_2, E_3 and E_4 are exhaustive. Obviously, these events are mutually exclusive, then

$$P\left(\frac{B}{E_1}\right)=\frac{2}{2}=1;\ P\left(\frac{B}{E_2}\right)=\frac{3}{4}$$

$$P\left(\frac{B}{E_3}\right)=\frac{3}{4}\text{ and }P\left(\frac{B}{E_4}\right)=\frac{4}{6}=\frac{2}{3}$$

$\therefore$ Required probability,

$$P(B)=P(E_1)\cdot P\left(\frac{B}{E_1}\right)+P(E_2)\cdot P\left(\frac{B}{E_2}\right)$$
$$+P(E_3)\cdot P\left(\frac{B}{E_3}\right)+P(E_4)\cdot P\left(\frac{B}{E_4}\right)$$
$$=\frac{1}{6}\times 1+\frac{1}{3}\times\frac{3}{4}+\frac{1}{5}\times\frac{3}{4}+\frac{3}{10}\times\frac{2}{3}$$
$$=\frac{1}{6}+\frac{1}{4}+\frac{3}{20}+\frac{1}{5}$$
$$=\frac{10+15+9+12}{60}=\frac{46}{60}=\frac{23}{30}$$

Baye's Theorem or Inverse Probability

If an event E can occur only with one of the n mutually exclusive and exhaustive events $E_1, E_2, E_3,..., E_n$ and the probabilities $P(E/E_1), P(E/E_2), ..., P(E/E_n)$ are known, then

$$P\left(\frac{E_k}{E}\right)=\frac{P(E_k)\cdot P\left(\frac{E}{E_k}\right)}{\sum_{k=1}^{n}P(E_k)\cdot P\left(\frac{E}{E_k}\right)}$$

Proof The event E occurs with one of the n mutually exclusive and exhaustive events $E_1, E_2, E_3,..., E_n$, then

$$E=EE_1+EE_2+EE_3+...+EE_n$$
$$\Rightarrow\quad P(E)=P(EE_1)+P(EE_2)+P(EE_3)+...+P(EE_n)$$
$$=\sum_{k=1}^{n}P(EE_k)=\sum_{k=1}^{n}P(E_k)\cdot P\left(\frac{E}{E_k}\right)$$

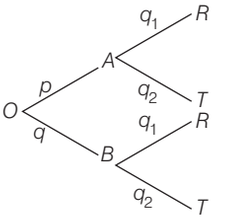
$$\therefore\ P\left(\frac{E_k}{E}\right)=\frac{P(E_k\ E)}{P(E)}=\frac{P(E_k)\cdot P\left(\frac{E}{E_k}\right)}{\sum_{k=1}^{n}P(E_k)\cdot P\left(\frac{E}{E_k}\right)}$$

Remark

The probabilities $P(E_k)$ and $P\left(\frac{E_k}{E}\right)$ are known as **priori** and **posteriori** probabilities, respectively.

Remarks We can visualise a tree structure here

$$P(A)=p,\ P(B)=q$$
$$P\left(\frac{R}{A}\right)=p_1,\ P\left(\frac{T}{A}\right)=q_1$$
$$P\left(\frac{R}{B}\right)=p_2,\ P\left(\frac{T}{B}\right)=q_2$$



If we are to find $P\left(\frac{A}{R}\right)$, we go

$$\therefore\quad P\left(\frac{A}{R}\right)=\frac{P(A)\cdot P\left(\frac{R}{A}\right)}{P(A)\cdot P\left(\frac{R}{A}\right)+P(B)\cdot P\left(\frac{R}{B}\right)}$$

Example 17. A bag A contains 2 white and 3 red balls and a bag B contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and it is found to be red. Then, find the probability that it was drawn from the bag B.

Sol. Let $E_1 \equiv$ The event of ball being drawn from bag A.

$E_2 \equiv$ The event of ball being drawn from bag B.

and $E \equiv$ The event of ball being red.

Since, both the bag are equally likely to be selected, therefore

$$P(E_1)=P(E_2)=\frac{1}{2}\ \text{ and }\quad P\left(\frac{E}{E_1}\right)=\frac{3}{5}\text{ and }P\left(\frac{E}{E_2}\right)=\frac{5}{9}$$

$\therefore$ Required probability,

$$P\left(\frac{E_2}{E}\right)=\frac{P(E_2)\cdot P\left(\frac{E}{E_2}\right)}{P(E_1)\cdot P\left(\frac{E}{E_1}\right)+P(E_2)\cdot P\left(\frac{E}{E_2}\right)}$$
$$=\frac{\frac{1}{2}\times\frac{5}{9}}{\frac{1}{2}\times\frac{3}{5}+\frac{1}{2}\times\frac{5}{9}}=\frac{\frac{5}{9}}{\frac{3}{5}+\frac{5}{9}}=\frac{25}{52}$$

Example 18. A man is known to speak the truth 3 out of 4 times. He throws a die and reports that it is a six. Find the probability that it is actually a six.

Sol. Let E_1 be the event that the man reports that it is a six and E be the event that a six occurs.

Then, $P(E)=\frac{1}{6}$

$\therefore \quad P(\bar{E})=1-P(E)=1-\frac{1}{6}=\frac{5}{6}$

$\therefore \quad P\left(\frac{E_1}{E}\right)=P$ (man speaking the truth) $=\frac{3}{4}$

and $P\left(\frac{E_1}{\bar{E}}\right)=P$ (man not speaking the truth) $=1-\frac{3}{4}=\frac{1}{4}$

Clearly, $\left(\frac{E}{E_1}\right)$ is the event that it is actually a six, when it is known that the man reports a six.

$$P\left(\frac{E}{E_1}\right)=\frac{P(E)\cdot P\left(\frac{E_1}{E}\right)}{P(E)\cdot P\left(\frac{E_1}{E}\right)+P(\bar{E})\cdot P\left(\frac{E_1}{E}\right)}$$

$$=\frac{\frac{1}{6}\times\frac{3}{4}}{\frac{1}{6}\times\frac{3}{4}+\frac{5}{6}\times\frac{1}{4}}=\frac{3}{8}$$

Example 19. In a test, an examinee either guesses or copies or knows the answer to a multiple choice question with four choices. The probability that he makes a guess is $\frac{1}{3}$ and the probability that he copies the answer is $\frac{1}{6}$. The probability that his answer is correct given that he copied it is $\frac{1}{8}$. Find the probability that he knew the answer to the question given that he correctly answered it.

Sol. Lest E_1 be the event that the answer is guessed, E_2 be the event that the answer is copied, E_3 be the event that the examinee knows the answer and E be the event that the examinee answers correctly.

Given, $P(E_1)=\frac{1}{3}$, $P(E_2)=\frac{1}{6}$

Assume that events E_1, E_2 and E_3 are exhaustive

$\therefore \quad P(E_1)+P(E_2)+P(E_3)=1$

$\therefore \quad P(E_3)=1-P(E_1)-P(E_2)=1-\frac{1}{3}-\frac{1}{6}=\frac{1}{2}$

Now, $P\left(\frac{E}{E_1}\right)$

$\equiv$ Probability of getting correct answer by guessing

$=\frac{1}{4}$ [since 4 alternatives]

$P\left(\frac{E}{E_2}\right)\equiv$ Probability of answering correctly by copying $=\frac{1}{8}$

and $P\left(\frac{E}{E_3}\right)\equiv$ Probability of answering correctly by knowing = 1

Clearly, $\left(\frac{E_3}{E}\right)$ is the event he knew the answer to the question, given that he correctly answered it.

$$\therefore \quad P\left(\frac{E_3}{E}\right)=\frac{P(E_3)\cdot P\left(\frac{E}{E_3}\right)}{P(E_1)\cdot P\left(\frac{E}{E_1}\right)+P(E_2)\cdot P\left(\frac{E}{E_2}\right)+P(E_2)\cdot P\left(\frac{E}{E_3}\right)}$$

$$=\frac{\frac{1}{2}\times 1}{\frac{1}{3}\times\frac{1}{4}+\frac{1}{6}\times\frac{1}{8}+\frac{1}{2}\times 1}=\frac{24}{29}$$

Example 20. A and B are two independent witnesses (i.e., there is no collusion between them) in a case. The probability that A will speak the truth is x and the probability that B will speak the truth is y. A and B agree in a certain statements. Show that the probability that the statements is true, is $\frac{xy}{1-x-y+2xy}$.

Sol. Let E_1 be the event that both A and B speak the truth, E_2 be the event that both A and B tell a lie and E be the event that A and B agree in a certain statements.

And also, let C be the event that A speak the truth and D be the event that B speaks the truth.

$\therefore \quad E_1=C\cap D$

[$\because C$ and D are independent events]

and $E_2=\bar{C}\cap\bar{D}$

then, $P(E_1)=(C\cap D)=P(C)\cdot P(D)=xy$

and $P(E_2)=P(\bar{C}\cap\bar{D})=P(\bar{C})\,P(\bar{D})$

$=\{1-P(C)\}\{1-P(D)\}=(1-x)(1-y)$

$=1-x-y-xy$

Now, $P\left(\frac{E}{E_1}\right)\equiv$ Probability that A and B will agree, when both of them speak the truth $=1$

and $P\left(\frac{E}{E_2}\right)=$ Probability that A and B will agree, when both of them tell a lie = 1

Clearly, $\left(\frac{E_1}{E}\right)$ be the event that the statement is true

$$\therefore \quad P\left(\frac{E_1}{E}\right)=\frac{P(E_1)\cdot P\left(\frac{E}{E_1}\right)}{P(E_1)\cdot P\left(\frac{E}{E_1}\right)+P(E_2)\cdot P\left(\frac{E}{E_2}\right)}$$

$$=\frac{xy\cdot 1}{xy\cdot 1+(1-x-y+xy)\cdot 1}=\frac{xy}{1-x-y+2xy}$$

Exercise for Session 3

1. A bag A contains 3 white and 2 black balls and another bag B contains 2 white and 4 black balls. A bag and a ball out of it are picked at random. The probability that the ball is white, is

(a) $\frac{2}{7}$ (b) $\frac{7}{9}$ (c) $\frac{4}{15}$ (d) $\frac{7}{15}$

2. There are two bags, one of which contains 3 black and 4 white balls, while the other contains 4 black and 3 white balls. A die is cast. If the face 1 or 3 turns up a ball is taken out from the first bag and if any other face turns up, a ball is taken from the second bag. The probability of choosing a black ball, is

(a) $\frac{7}{15}$ (b) $\frac{8}{15}$ (c) $\frac{10}{21}$ (d) $\frac{11}{21}$

3. There are two groups of subjects, one of which consists of 5 Science subjects and 3 Engineering subjects and the other consists of 3 Science and 5 Engineering subjects. An unbiased die is cast. If number 3 or 5 turns up, a subject from group I is selected, otherwise a subject is selected from group II. The probability that an Engineering subject is selected ultimately, is

(a) $\frac{7}{13}$ (b) $\frac{9}{17}$ (c) $\frac{13}{24}$ (d) $\frac{11}{20}$

4. Urn A contains 6 red and 4 white balls and urn B contains 4 red and 6 white balls. One ball is drawn at random from urn A and placed in urn B. Then a ball is drawn from urn B and placed in urn A. Now, if one ball is drawn from urn A, the probability that it is red, is

(a) $\frac{6}{11}$ (b) $\frac{17}{50}$ (c) $\frac{16}{55}$ (d) $\frac{32}{55}$

5. A box contains N coins, of which m are fair and the rest are biased. The probability of getting head when a fair coin is tossed is $\frac{1}{2}$, while it $\frac{2}{3}$ when a biased coin is tossed. A coin is drawn from the box at random and is tossed twice. The first time it shows head and the second time it shows tail. The probability that the coin drawn is fair, is

(a) $\frac{5m}{m+8N}$ (b) $\frac{3m}{m+8N}$ (c) $\frac{7m}{m+8N}$ (d) $\frac{9m}{m+8N}$

6. A pack of playing cards was found to contain only 51 cards. If the first 13 cards which are examined are all red, then the probability that the missing card is black, is

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{15}{26}$ (d) $\frac{16}{39}$

7. A purse contains n coins of unknown values. A coin is drawn from it at random and is found to be a rupee. Then the chance that it is the only rupee coin in the purse, is

(a) $\frac{1}{n}$ (b) $\frac{2}{n+1}$ (c) $\frac{1}{n(n+1)}$ (d) $\frac{2}{n(n+1)}$

8. A card is lost from a pack of 52 playing cards. From the remainder of the pack, one card is drawn and is found to be a spade. The probability that the missing card is a spade, is

(a) $\frac{2}{17}$ (b) $\frac{3}{17}$ (c) $\frac{4}{17}$ (d) $\frac{5}{17}$

9. A person is known to speak the truth 4 times out of 5. He throws a die and reports that it is an ace. The probability that it is actually an ace, is

(a) $\frac{1}{3}$ (b) $\frac{2}{9}$ (c) $\frac{4}{9}$ (d) $\frac{5}{9}$

10. Each of the n urns contains 4 white and 6 black balls, the $(n+1)$ th urn contains 5 white and 5 black balls. Out of $(n+1)$ urns an urn is chosen at random and two balls are drawn from it without replacement. Both the balls are found to be black. If the probability that the $(n+1)$th urn was chosen to drawn the balls is $\frac{1}{16}$, the value of n, is

(a) 10 (b) 11 (c) 12 (d) 13

Session 4

Binomial Theorem on Probability, Poisson Distribution, Expectation, Multinomial Theorem, Uncountable Uniform Spaces (Geometrical Problems)

Binomial Theorem on Probability

Suppose, a binomial experiment has probability of success p and that of failure q (i.e., $p+q=1$). If E be an event and let $X=$ number of successes i.e., number of times event E occurs in n trials. Then, the probability of occurrence of event E exactly r times in n trials is denoted by $P(X=r)$ or $P(r)$ and is given by $P(X=r)$

or $P(r) = {}^nC_r\, p^r q^{n-r}$

$= (r+1)$ th terms in the expansion of $(q+p)^n$

where, $r=0, 1, 2, 3, \ldots, n$.

Remark

1. The probability of getting atleast k success is
$$P(r \geq k) = \sum_{r=k}^{n} {}^nC_r p^r q^{n-r}.$$
2. The probability of getting atmost k success is
$$P(0 \leq r \leq k) = \sum_{r=0}^{k} {}^nC_r p^r q^{n-r}.$$
3. The probability distribution of the random variable X is as given below

X	0	1	2	…	r	…	n
$P(X)$	q^n	${}^nC_1\, pq^{n-1}$	${}^nC_2\, p^2 q^{n-2}$	…	${}^nC_r\, p^r q^{n-r}$		p^n

4. The mean, the variance and the standard deviation of binomial distribution are np, npq, $\sqrt{npq}$.
5. **Mode of binomial distribution** Mode of Binomial distribution is the value of r when $P(X=r)$ is maximum.
$$(n+1)\,p - 1 \leq r \leq (n+1)p$$

Example 21. If on an average, out of 10 ships, one is drowned, then what is the probability that out of 5 ships, atleast 4 reach safely?

Sol. Let p be the probability that a ship reaches safely.

$\therefore \quad p = \dfrac{9}{10}$

$\therefore \quad q =$ Probability that a ship is drowned $= 1 - p = 1 - \dfrac{9}{10}$

$\therefore \qquad q = \dfrac{1}{10}$

Let X be the random variable, showing the number of ships reaching safely.

Then, P (atleast 4 reaching safely) $= P(X=4 \text{ or } X=5)$

$= P(X=4) + P(X=5)$

$$= {}^5C_4\left(\frac{9}{10}\right)^4\left(\frac{1}{10}\right)^{5-4} + {}^5C_5\left(\frac{9}{10}\right)^5\left(\frac{1}{10}\right)^{5-5}$$

$$= \frac{5\times 9^4}{10^5} + \frac{9^5}{10^5} = \frac{9^4\times 14}{10^5}$$

Example 22. Numbers are selected at random one at a time, from the numbers 00, 01, 02, ..., 99 with replacement. An event E occurs, if and only if the product of the two digits of a selected number is 18. If four numbers are selected, then find the probability that E occurs atleast 3 times.

Sol. Out of the numbers 00, 01, 02, ..., 99, those numbers the product of whose digits is 18 are 29, 36, 63, 92 i.e., only 4.

$$p = P(E) = \frac{4}{100} = \frac{1}{25},\ q = P(\bar{E}) = 1 - \frac{1}{25} = \frac{24}{25}$$

Let X be the random variable, showing the number of times E occurs in 4 selections.

Then, $P(E$ occurs atleast 3 times$) = P(X=3 \text{ or } X=4)$

$= P(X=3) + P(X=4) = {}^4C_3\, p^3 q^1 + {}^4C_4\, p^4 q^0$

$$= 4p^3 q + p^4 = 4\times\left(\frac{1}{25}\right)^3 \times \frac{24}{25} + \left(\frac{1}{25}\right)^4$$

$$= \frac{97}{390625}$$

Example 23. A man takes a step forward with probability 0.4 and backward with probability 0.6. Then, find the probability that at the end of eleven steps he is one step away from the starting point.

Sol. Since, the man is one step away from starting point mean that either

(i) man has taken 6 steps forward and 5 steps backward.

(ii) man has taken 5 steps forward and 6 steps backward.

Taking, movement 1 step forward as success and 1 step backward as failure.

$\therefore \quad p$ = Probability of success = 0.4

and q = Probability of failure = 0.6

$\therefore$ Required probability = $P(X = 6 \text{ or } X = 5)$

$$= P(X = 6) + P(X = 5) = {}^{11}C_6\ p^6q^5 + {}^{11}C_5\ p^5q^6$$

$$= {}^{11}C_5(p^6q^5 + p^5q^6)$$

$$= \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}\{(0 \cdot 4)^6(0 \cdot 6)^5 + (0 \cdot 4)^5(0 \cdot 6)^6\}$$

$$= \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}(0 \cdot 24)^5 = 0 \cdot 37$$

Hence, the required probability is 0·37.

Example 24. Find the minimum number of tosses of a pair of dice, so that the probability of getting the sum of the digits on the dice equal to 7 on atleast one toss, is greater than 0.95. (Given, $\log_{10} 2 = 0 \cdot 3010$, $\log_{10} 3 = 0 \cdot 4771$)

Sol. The sample space,

$$S = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$$

$\therefore \quad n(S) = 36$ and let E be the event getting the sum of digits on the dice equal to 7, then

$$E = \{(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)\}$$

$\therefore \quad n(E) = 6$

p = Probability of getting the sum 7

$$p = \frac{6}{36} = \frac{1}{6} \quad \therefore \quad q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$$

$\because$ Probability of not throwing the sum 7 in first m trials = q^m

$\therefore$ P (atleast one 7 in m throws) = $1 - q^m = 1 - \left(\frac{5}{6}\right)^m$

According to the question, $1 - \left(\frac{5}{6}\right)^m > 0 \cdot 95$

$$\Rightarrow \quad \left(\frac{5}{6}\right)^m < 1 - 0 \cdot 95 \ \Rightarrow \ \left(\frac{5}{6}\right)^m < 0.05$$

$$\Rightarrow \quad \left(\frac{5}{6}\right)^m < \frac{1}{20}$$

Taking logarithm,

$$\Rightarrow \quad m\{\log_{10} 5 - \log_{10} 6\} < \log_{10} 1 - \log_{10} 20$$

$$\Rightarrow m\{1 - \log_{10} 2 - \log_{10} 2 - \log_{10} 3\} < 0 - \log_{10} 2 - \log_{10} 10$$

$$\Rightarrow \quad m\{1 - 2\log_{10} 2 - \log_{10} 3\} < -\log_{10} 2 - 1$$

$$\Rightarrow \quad m\{1 - 0.6020 - 0.4771\} < -0.3010 - 1$$

$$\Rightarrow \quad -0.079\,m < -1.3010$$

$$\Rightarrow \quad m > \frac{1.3010}{0.079} = 16.44$$

$$\therefore \quad m > 16.44$$

Hence, the least number of trials is 17.

Example 25. Write probability distribution, when three coins are tossed.

Sol. Let X be a random variable denoting the number of heads occurred, then $P(X = 0)$ = Probability of occurrence of zero head

$$= P(TTT) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

$P(X = 1)$ = Probability of occurrence of one head

$$= P(HTT) + P(THT) + P(TTH)$$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{8}$$

$P(X = 2)$ = Probability of occurrence of two heads

$$= P(HHT) + P(HTH) + P(THH)$$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{8}$$

$P(X = 3)$ = Probability of occurrence of three heads

$$= P(HHH) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

Thus, the probability distribution when three coins are tossed is as given below

X	0	1	2	3
$P(X)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

another form,

$$X : \begin{pmatrix} 0 & 1 & 2 & 3 \\ \frac{1}{3} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{pmatrix}$$

$$P(X) :$$

Example 26. The mean and variance of a binomial variable X are 2 and 1, respectively. Find the probability that X takes values greater than 1.

Sol. Given, mean, $np = 2$ ...(i)

and variance, $npq = 1$ ...(ii)

On dividing Eq. (ii) by Eq. (i), we get $q = \frac{1}{2}$

$$\therefore \quad p = 1 - q = \frac{1}{2}$$

From Eq. (i), $n \times \frac{1}{2} = 2 \ \therefore \ n = 4$

The binomial distribution is $\left(\frac{1}{2} + \frac{1}{2}\right)^4$

Now, $P(X > 1) = P(X = 2) + P(X = 3) + P(X = 4)$

$$= {}^4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 + {}^4C_4\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^3 + {}^4C_4\left(\frac{1}{2}\right)^4$$

$$= \frac{6 + 4 + 1}{16} = \frac{11}{16}$$

Aliter $P(X > 1) = 1 - \{P(X = 0) + P(X = 1)\}$

$$= 1 - \left\{{}^4C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^4 + {}^4C_3\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3\right\} = 1 - \left(\frac{1 + 4}{16}\right) = \frac{11}{16}$$

Poisson Distribution

It is the limiting case of binomial distribution under the following conditions :

(i) Number of trails are very large i.e. $n \to \infty$

(ii) $p \to 0$

(iii) $nq \to \lambda$, a finite quantity (λ is called parameter)

(a) Probability of r success for poisson distribution is given by $P(X = r) = \dfrac{e^{-\lambda}\lambda^r}{r!}, r = 0, 1, 2, \ldots$

(b) Recurrence formula for poisson distribution is given by $P(r+1) = \dfrac{\lambda}{(r+1)} P(r)$

Remark

1. For poisson distribution, mean = variance = $\lambda = np$
2. If X and Y are independent poisson variates with parameters λ_1 and λ_2, then $X + Y$ has poisson distribution with parameter $\lambda_1 + \lambda_2$.

Expectation

If p be the probability of success of a person in any venture and m be the sum of money which he will receive in case of success, the sum of money denoted by pm is called his expectation.

Example 27. A random variable X has Poisson's distribution with mean 3. Then find the value of $P(X > 2.5)$

Sol. $P(X > 2.5) = 1 - P(X = 0) - P(X = 1) - P(X = 2)$

$\because \quad P(X = k) = e^{-\lambda} \cdot \dfrac{\lambda^k}{k!}$

$\therefore \quad P(X > 2.5) = 1 - \dfrac{e^{-\lambda}}{0!} - \dfrac{e^{-\lambda} \cdot \lambda^1}{1!} - \dfrac{e^{-\lambda} \cdot \lambda^2}{2!}$

$= 1 - e^{-\lambda}\left(1 + \lambda + \dfrac{\lambda^2}{2}\right)$

$= 1 - e^{-3}\left(1 + 3 + \dfrac{9}{2}\right) \qquad (\because \ \lambda = np = 3)$

$= 1 - \dfrac{17}{2e^3}$

Example 28. *A and B throw with one die for a stake of ₹ 11 which is to be won by the player who first throw 6. If A has the first throw, then what are their respective expectations?*

Sol. Since, A can win the game at the 1st, 3rd, 5th,..., trials.

If p be the probability of success and q be the probability of fail, then

$$p = \frac{1}{6} \quad \text{and} \quad q = \frac{5}{6}$$

$P(A \text{ wins at the first trial}) = \dfrac{1}{6}$

$P(A \text{ wins at the 3rd trials}) = \dfrac{5}{6} \cdot \dfrac{5}{6} \cdot \dfrac{1}{6}$

$P(A \text{ wins at the 5th trials}) = \dfrac{5}{6} \cdot \dfrac{5}{6} \cdot \dfrac{5}{6} \cdot \dfrac{5}{6} \cdot \dfrac{1}{6}$ and so on.

Therefore, $P(A \text{ wins}) = \dfrac{1}{6} + \left(\dfrac{5}{6}\right)^2 \dfrac{1}{6} + \left(\dfrac{5}{6}\right)^4 \cdot \dfrac{1}{6} + \ldots \infty$

$$= \frac{\frac{1}{6}}{1 - \left(\frac{5}{6}\right)^2} = \frac{6}{11}$$

Similarly, $P(B \text{ wins}) = \dfrac{5}{6} \cdot \dfrac{1}{6} + \left(\dfrac{5}{6}\right)^3 \dfrac{1}{6} + \left(\dfrac{5}{6}\right)^5 \dfrac{1}{6} + \ldots \infty$

$$= \frac{\frac{5}{6} \cdot \frac{1}{6}}{1 - \left(\frac{5}{6}\right)^2} = \frac{5}{11}$$

Hence, expectations of A and B are ₹ $\dfrac{6}{11} \times 11$ and ₹ $\dfrac{5}{11} \times 11$, respectively. i.e. Expectations of A and B are ₹ 6 and ₹ 5, respectively.

Multinomial Theorem

If a dice has m faces marked 1, 2, 3,..., m and if such n dice are thrown, then the probability that the sum of the numbers of the upper faces is equal to r is given by the coefficient of x^r in $\dfrac{(x + x^2 + \ldots + x^m)^n}{m^n}$.

Example 29. A person throws two dice, one the common cube and the other a regular tetrahedron, the number on the lowest face being taken in the case of the tetrahedron, then find the probability that the sum of the numbers appearing on the dice is 6.

Sol. Let S be the sample space, then

$$S = \{1, 2, 3, 4\} \times \{1, 2, 3, 4, 5, 6\}$$

$\therefore \quad n(S) = 24$

If E be the event that the sum of the numbers on dice is 6.

Then, $n(E)$ = Coefficient of x^6 in

$(x^1 + x^2 + x^3 + x^4) \times (x^1 + x^2 + x^3 + x^4 + x^5 + x^6)$

$= 1 + 1 + 1 + 1 = 4$

$\therefore$ Required probability, $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{4}{24} = \dfrac{1}{6}$

Example 30. Five ordinary dice are rolled at random and the sum of the numbers shown on them is 16. What is the probability that the numbers shown on each is any one from 2, 3, 4 or 5?

Sol. If the integers x_1, x_2, x_3, x_4 and x_5 are shown on the dice, then $x_1 + x_2 + x_3 + x_4 + x_5 = 16$

where, $1 \le x_i \le 6 \qquad (i = 1, 2, 3, 4, 5)$

The number of total solutions of this equation.

$= \text{Coefficient of } x^{16} \text{ in } (x^1 + x^2 + x^3 + x^4 + x^5 + x^6)^5$

$= \text{Coefficient of } x^{16} \text{ in } x^5 (1 + x + x^2 + x^3 + x^4 + x^5)^5$

$= \text{Coefficient of } x^{11} \text{ in } (1 + x + x^2 + x^3 + x^4 + x^5)^5$

$= \text{Coefficient of } x^{11} \text{ in } \left\{\left(\frac{1 - x^6}{1 - x}\right)^5\right\}$

$= \text{Coefficient of } x^{11} \text{ in } (1 - x^6)^5 (1 - x)^{-5}$

$= \text{Coefficient of } x^{11} \text{ in}$

$(1 - 5x^6 + \ldots)(1 + {}^5C_1 x + {}^6C_2 x^2 + \ldots$

$+ {}^9C_{C_5} x^5 + \ldots + {}^{15}C_{11} x^{11} + \ldots)$

$= {}^{15}C_{11} - 5 \cdot {}^9C_5$

$= {}^{15}C_4 - 5 \cdot {}^9C_4 = \frac{15 \cdot 14 \cdot 13 \cdot 12}{1 \cdot 2 \cdot 3 \cdot 4} - 5 \cdot \frac{9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4} = 735$

If S be the sample space

$\therefore \qquad n(S) = 735$

Let E be the occurrence event, then

$n(E) =$ The number of integral solutions of

$x_1 + x_2 + x_3 + x_4 + x_5 = 16,$

where $2 \le x_i \le 5 \qquad (i = 1, 2, 3, 4, 5)$

$= \text{Coefficient of } x^{16} \text{ in } (x^2 + x^3 + x^4 + x^5)^5$

$= \text{Coefficient of } x^{16} \text{ in } x^{10}(1 + x + x^2 + x^3)^5$

$= \text{Coefficient of } x^6 \text{ in } (1 + x + x^2 + x^3)^5$

$= \text{Coefficient of } x^6 \text{ in } \left\{\left(\frac{1 - x^4}{1 - x}\right)^5\right\}$

$= \text{Coefficient of } x^6 \text{ in } (1 - x^4)^5 (1 - x)^{-5}$

$= \text{Coefficient of } x^6 \text{ in}$

$(1 - 5x^4 + \ldots)(1 + {}^5C_1 x + {}^6C_2 x^2 + \ldots + {}^{10}C_6 x^6 + \ldots)$

$= {}^{10}C_6 - 5 \cdot {}^6C_2 = {}^{10}C_4 - 5 \cdot {}^6C_2$

$= \frac{10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4} - 5 \cdot \frac{6 \cdot 5}{1 \cdot 2} = 210 - 75 = 135$

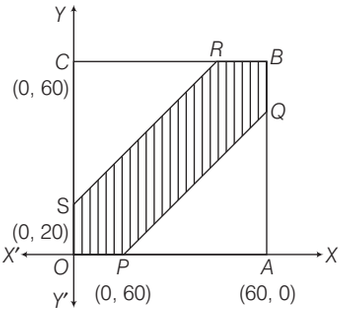
$\therefore$ The required probability, $P(E) = \frac{n(E)}{n(S)} = \frac{135}{735} = \frac{9}{49}$

Uncountable Uniform Spaces

(Geometrical Problems)

Example 31. Two persons A and B agree to meet at a place between 11 to 12 noon. The first one to arrive waits for 20 min and then leave. If the time of their arrival be independent and at random, then what is the probability that A and B meet?

Sol. Let A and B arrive at the place of their meeting x minutes and y minutes after 11 noon.



The given condition $\Rightarrow$ their meeting is possible only if

$|x - y| \le 20 \qquad \ldots(i)$

$OABC$ is a square, where $A \equiv (60, 0)$ and $C \equiv (0, 60)$

Considering the equality part of Eq. (i)

i.e., $\qquad |x - y| = 20$

$\therefore$ The area representing the favourable cases

$=$ Area $OPQBRSO$

$=$ Area of square $OABC -$ Area of $\Delta PAQ -$ Area of ΔSRC

$= (60)(60) - \frac{1}{2}(40)(40) - \frac{1}{2}(40)(40)$

$= 3600 - 1600 = 2000$ sq units

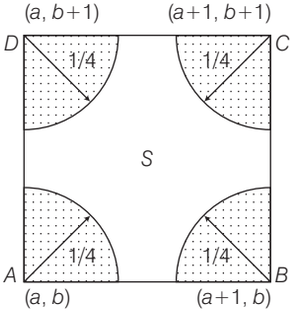
Total way $=$ Area of square $OABC = (60)(60) = 3600$ sq units

Required probability $= \frac{2000}{3600} = \frac{5}{9}$

Example 32. Consider the cartesian plane R^2 and let X denote the subset of points for which both coordinates are integers. A coin of diameter $\frac{1}{2}$ is tossed randomly onto the plane. Find the probability p that the coin covers a point of X.

Sol. Let S denote the set of points inside a square with corners

$(a, b), (a, b + 1), (a + 1, b), (a + 1, b + 1) \in X$



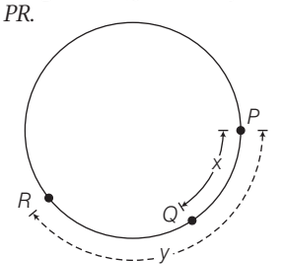
Let P denotes the set of points in S with distance less than $\frac{1}{4}$ from any corner point. (observe that the area of P is equal to the area inside a circle of

radius $\frac{1}{4}$). Thus a coin, whose centre falls in S, will cover a point of X if and only if its centre falls in a point of P.

Hence, $p = \frac{\text{area of } P}{\text{area of } S} = \frac{\pi\left(\frac{1}{4}\right)^2}{1} = \frac{\pi}{15} \approx 0.2$

Example 33. Three points P, Q and R are selected at random from the circumference of a circle. Find the probability p that the points lie on a semi-circle.

Sol. Let the length of the circumference is $2s$. Let x denote the clockwise arc length of PQ and let y denote the clockwise arc length of PR.



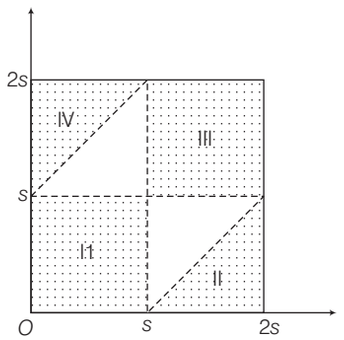
Thus, $0 < x < 2s$ and $0 < y < 2s$

Let A denotes the subset of S for which any of the following conditions holds:

(i) $x, y < s$ (ii) $x < s$ and $y - x > s$

(iii) $x, y > s$ (iv) $y < s$ and $x - y > s$

Then, A consists of those points for which P, Q and R lie on a semi-circle. Thus,



$$p = \frac{\text{area of } A}{\text{area of } S} = \frac{3s^2}{4s^2} = \frac{3}{4}$$

Example 34. A wire of length l is cut into three pieces. Find the probability that the three pieces form a triangle.

Sol. Let the lengths of three parts of the wire be x, y and $l - (x + y)$. Then, $x > 0$, $y > 0$

and $l - (x + y) > 0$

i.e., $x + y < l$ or $y < l - x$

Since, in a triangle, the sum of any two sides is greater than third side, so

$$x + y > l - (x + y) \Rightarrow y > \frac{l}{2} - x$$

and $$x + l - (x + y) > y \Rightarrow y < \frac{l}{2}$$

and $$y + l - (x + y) > x \Rightarrow x < \frac{l}{2}$$

$$\Rightarrow \quad \frac{l}{2} - x < y < \frac{l}{2} \text{ and } 0 < x < \frac{l}{2}$$

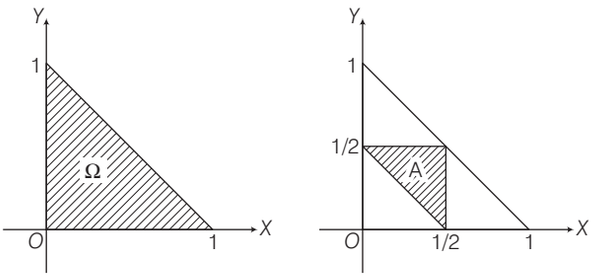
So, required probability $= \dfrac{\int_0^{l/2}\int_{l/2-x}^{l/2} dy\,dx}{\int_0^{l}\int_0^{l-x} dy\,dx}$

$$= \frac{\int_0^{l/2}\left\{\frac{l}{2} - \left(\frac{l}{2} - x\right)\right\}dx}{\int_0^{l}(l-x)dx} = \frac{\int_0^{l/2} x\,dx}{\int_0^{l}(l-x)\,dx} = \frac{l^2/8}{l^2/2} = \frac{1}{4}$$

Aliter

The elementary event w is characterised by two parameters x and y [since $z = l - (x + y)$]. We depict the event by a point on x, y plane. The conditions $x > 0$, $y > 0$, $x + y < l$ are imposed on the quantities x and y, the sample space is the interior of a right angled triangle with unit legs

i.e. $S_\Omega = \frac{1}{2}$.



The condition A requiring that a triangle could be formed from the segments $x, y, l - (x + y)$ reduces to the following two conditions: (1) The sum of any two sides is larger than the third side, (2) The difference between any two sides is smaller than the third side. This condition is associated with the triangular domain A with area.

$$S_A = \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) = \frac{1}{8} \;\therefore\; P(A) = \frac{S_A}{S_\Omega} = \frac{\left(\frac{1}{8}\right)}{\left(\frac{1}{2}\right)} = \frac{1}{4}$$

Exercise for Session 4

1 A coin is tossed three times. The probability of getting exactly 2 heads is

(a) $\frac{1}{4}$ (b) $\frac{1}{8}$ (c) $\frac{3}{8}$ (d) $\frac{5}{8}$

2 A coin is tossed 4 times. The probability that atleast one head turns up is

(a) $\frac{1}{16}$ (b) $\frac{1}{8}$ (c) $\frac{7}{8}$ (d) $\frac{15}{16}$

3 The following is the probability distribution of a random variable X.

X	1	2	3	4	5
$P(X)$	0.1	0.2	k	0.3	$2k$

The value of k is

(a) $\frac{4}{15}$ (b) $\frac{1}{15}$ (c) $\frac{1}{5}$ (d) $\frac{2}{15}$

4 A random variable X has the distribution

X	2	3	4
$P(X = x)$	0.3	0.4	0.3

Then, variance of the distribution, is

(a) 0·6 (b) 0·7 (c) 0·77 (d) 1·55

5. In a box containing 100 bulbs, 10 bulbs are defective. Probability that out of a sample of 5 bulbs, none is defective, is

(a) 10^{-5} (b) 2^{-5} (c) $(0 \cdot 9)^5$ (d) $0 \cdot 9$

6. A pair of dice is rolled together till a sum of either 5 or 7 is obtained. The probability that 5 comes before 7, is

(a) $\frac{2}{5}$ (b) $\frac{2}{7}$ (c) $\frac{3}{7}$ (d) None of these

7. If X follows the binomial distribution with parameters $n = 6$ and p and $9P(X = 4) = P(X = 2)$, then p is

(a) $\frac{1}{4}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{2}{3}$

8. If probability of a defective bolt is 0.1, then mean and standard deviation of distribution of bolts in a total of 400, are

(a) 30, 3 (b) 40, 5 (c) 30, 4 (d) 40, 6

9. The mean and variance of a binomial distribution are $\frac{5}{4}$ and $\frac{15}{16}$ respectively, then value of p, is

(a) $\frac{1}{2}$ (b) $\frac{15}{16}$ (c) $\frac{1}{4}$ (d) $\frac{3}{4}$

10. The mean and variance of a binomial distribution are 6 and 4, then n is

(a) 9 (b) 12 (c) 18 (d) 10

11. A die is thrown 100 times. Getting an even number is considered a success. Variance of number of successes, is

(a) 10 (b) 20 (c) 25 (d) 50

12. 10% of tools produced by a certain manufacturing process turn out to be defective. Assuming binomial distribution, the probability of 2 defective in sample of 10 tools chosen at random, is

(a) 0.368 (b) 0.194 (c) 0.271 (d) None of these

13. If X follows a binomial distribution with parameters $n = 100$ and $p = \frac{1}{3}$, then $P(X = r)$ is maximum, when r equals

(a) 16 (b) 32 (c) 33 (d) None of these

14. The expected value of the number of points, obtained in a single throw of die, is

(a) $\frac{3}{2}$ (b) $\frac{5}{2}$ (c) $\frac{7}{2}$ (d) $\frac{9}{2}$

15. Two points P and Q are taken at random on a line segment OA of length a. The probability that $PQ > b$, where $0 < b < a$, is

(a) $\frac{b}{a}$ (b) $\frac{b^2}{a^2}$ (c) $\left(\frac{a-b}{a}\right)^2$ (d) $\left(\frac{a-2b}{a-b}\right)^2$

Shortcuts and Important Results to Remember

1 If n letters corresponding to n envelopes are placed in the envelopes at random, then

(i) probability that all letters are in right envelopes $= \frac{1}{n!}$.

(ii) probability that all letters are not in right envelopes $= 1 - \frac{1}{n!}$.

(iii) probability that no letter is in right envelopes $= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots + (-1)^n \frac{1}{n!}$.

(iv) probability that exactly r letters are in right envelopes $= \frac{1}{r!}\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots + (-1)^{n-r} \frac{1}{(n-r)!}\right]$.

2 When two dice are thrown, the probability of getting a total r (sum of numbers on upper faces), is

(i) $\frac{(r-1)}{36}$, if $2 \le r \le 7$ (ii) $\frac{(13-r)}{36}$, if $8 \le r \le 12$

3 When three dice are thrown, the probability of getting a total r (sum of numbers on upper faces), is

(i) $\frac{{}^{r-1}C_2}{216}$, if $3 \le r \le 8$ (ii) $\frac{25}{216}$, if $r = 9$

(iii) $\frac{27}{216}$, if $r = 10, 11$ (iv) $\frac{25}{216}$, if $r = 12$

(v) $\frac{{}^{(20-r)}C_2}{216}$, if $13 \le r \le 18$

4 If A and B are two finite sets (Let $n(A) = n$ and $n(B) = m$) and if a mapping is selected at random from the set of all mappings from A to B, the probability that the mapping is

(i) a one-one function is $\frac{{}^mP_n}{m^n}$.

(ii) a one-one onto function is $\frac{n!}{m^n}$.

(iii) a many one function is $1 - \frac{{}^mP_n}{m^n}$.

5 If r squares are selected from a chess board of size 8×8, then the probability that they lie on a diagonal line, is $\frac{4({}^7C_r + {}^6C_r + {}^5C_r + \ldots + {}^rC_r) + 2({}^8C_r)}{{}^{64}C_r}$ for $1 \le r \le 7$.

6 If n objects are distributed among n persons, then the probability that atleast one of them will not get anything, is $\frac{n^n - n!}{n^n}$.

7 Points about coin, dice and playing cards:

(a) **Coin** If 'one' coin is tossed n times 'n' coins are tossed once, then number of simple events (or simple points) in the space of the experiment is 2^n. All these events are equally likely.

(b) **Dice** If 'one' die is thrown 'n' times or 'n' dice are thrown once, then number of simple events (or simple points) in the space of the experiment is 6^n (here dice is cubical). All events are equally likely.

(c) **Playing Cards** A pack of playing cards has 52 cards. There are four suits Spade (♠ black face), Heart (♥ red face), Diamond (♦ red face) and Club (♣ black face) each having 13 cards. In 13 cards of each suit, there are 3 face (or court) cards namely King, Queen and Jack (or knave), so there are in all 12 face cards 4 King, 4 Queen and 4 Jacks (or knaves). 4 of each suit namely Ace (or Ekka), King, Queen and Jack (or knave).

(i) **Game of bridge** It is played by 4 players, each player is given 13 cards.

(ii) **Game of whist** It is played by two pairs of persons.

(iii) If two cards (one after the other) can be drawn out of a well-shuffled pack of 52 cards, then number of ways; (×) With replacement is $52 \times 52 = (52)^2 = 2704$ (β) Without replacement is $52 \times 51 = 2652$.

(iv) Two cards (simultaneously) can be drawn out of a well-shuffled pack of 52 cards, then number of ways is ${}^{52}C_2 = \frac{52 \times 51}{2} = 1326$.

8 Out of $(2n + 1)$ tickets consecutively numbered, three are drawn at random, then the probability that the numbers on them are in AP, is $\frac{3n}{4n^2 - 1}$.

9 Out of $3n$ consecutive integers, three are selected at random, then the probability that their sum is divided by 3, is $\frac{(3n^2 - 3n + 2)}{(3n-1)(3n-2)}$.

10 Two numbers a and b are chosen at random from the set $\{1, 2, 3, \ldots, 5n\}$, the probability that $a^4 - b^4$ is divisible by 5, is $\frac{17n - 5}{5(5n-1)}$.

11 Two numbers a and b are chosen at random from the set $\{1, 2, 3, \ldots, 3n\}$ the probability that $a^2 - b^2$ is divisible by 3, is $\frac{(5n-3)}{3(3n-1)}$.

12 Two numbers a and b are chosen at random from the set $\{1, 2, 3, \ldots, 3n\}$, the probability that $a^3 + b^3$ is divisible by 3, is $\frac{1}{3}$.

13 There are n stations between two cities A and B. A train is to stop at three of these n stations. The probability that no two of these three stations are consecutive, is $\frac{(n-3)(n-4)}{n(n-1)}$.

JEE Type Solved Examples :
Single Option Correct Type Questions

■ This section contains **10 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

● ***Ex. 1*** *The probability that in a year of* 22nd *century chosen at random, there will be* 53 *Sundays, is*

(a) $\frac{3}{28}$ (b) $\frac{2}{28}$ (c) $\frac{7}{28}$ (d) $\frac{5}{28}$

Sol. (d) In the 22nd century, there are 25 leap years viz. 2100, 2104, 2108,..., 2196 and 75 non-leap years.

Consider the following events:

E_1 = Selecting a leap year from 22nd century

E_2 = Selecting a non-leap year from 22nd century

E = There are 53 Sundays in a year of 22nd century

We have,

$$P(E_1) = \frac{25}{100},\ P(E_2) = \frac{75}{100},\ P\left(\frac{E}{E_1}\right) = \frac{2}{7} \text{ and } P\left(\frac{E}{E_2}\right) = \frac{1}{7}$$

Required probability $= P(E) = P((E \cap E_1) \cup (E \cap E_2))$

$$= P(E \cap E_1) + P(E \cap E_2)$$

$$= P(E_1).P\left(\frac{E}{E_1}\right) + P(E_2).P\left(\frac{E}{E_2}\right)$$

$$= \frac{25}{100} \times \frac{2}{7} + \frac{75}{100} \times \frac{1}{7} = \frac{5}{28}$$

● ***Ex. 2*** *In a convex hexagon two diagonals are drawn at random. The probability that the diagonals intersect at an interior point of the hexagon, is*

(a) $\frac{5}{12}$ (b) $\frac{7}{12}$ (c) $\frac{2}{5}$ (d) $\frac{3}{5}$

Sol. (a) We have,

Number of diagonals of a hexagon $= {}^6C_2 - 6 = 9$

$\therefore$ $n(s)$ = Total number of selections of two diagonals

$= {}^9C_2 = 36$

and $n(E)$ = The number of selections of two diagonals which intersect at an interior point

= The number of selections of four vertices $= {}^6C_4 = 15$

Hence, required probability $= \frac{n(E)}{n(S)} = \frac{15}{36} = \frac{5}{12}$

● ***Ex. 3*** *If three integers are chosen at random from the set of first* 20 *natural numbers, the chance that their product is a multiple of* 3*, is*

(a) $\frac{1}{57}$ (b) $\frac{13}{19}$

(c) $\frac{2}{19}$ (d) $\frac{194}{285}$

Sol. (d) $n(S)$ = Total number of ways of selecting 3 integers from 20 natural numbers $= {}^{20}C_3 = 1140$.

Their product is multiple of 3 means atleast one number is divisible by 3. The number which are divisible by 3 are 3, 6, 9, 12, 15 and 16.

$\therefore$ $n(E)$ = The number of ways of selecting atleast one of them multiple of 3

$$= {}^6C_1 \times {}^{14}C_2 + {}^6C_2 \times {}^{14}C_1 \times {}^6C_3 = 776$$

$\therefore$ Required probability $= \frac{n(E)}{n(S)}$

$$= \frac{776}{1140} = \frac{194}{285}$$

● ***Ex. 4*** *If three numbers are selected from the set of the first 20 natural numbers, the probability that they are in GP, is*

(a) $\frac{1}{285}$ (b) $\frac{4}{285}$

(c) $\frac{11}{1140}$ (d) $\frac{1}{71}$

Sol. (c) $n(S)$ = Total number of ways of selecting 3 numbers from first 20 natural numbers $= {}^{20}C_3 = 1140$

Three numbers are in GP, the favourable cases are 1, 2, 4; 1, 3, 9; 1, 4, 16; 2, 4, 8; 2, 6, 18; 3, 6, 12; 4, 8, 16; 5, 10, 20; 4, 6, 9; 8, 12, 18; 9, 12, 16

$\therefore$ $n(E)$ = The number of favourable cases = 11

$\therefore$ Required probability $= \frac{n(E)}{n(S)} = \frac{11}{1140}$

● ***Ex. 5*** *Two numbers b and c are chosen at random with replacement from the numbers* 1, 2, 3, 4, 5, 6, 7, 8 *and* 9*. The probability that* $x^2 + bx + c > 0$ *for all* $x \in R$*, is*

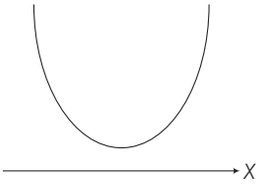
(a) $\frac{17}{123}$ (b) $\frac{32}{81}$

(c) $\frac{82}{125}$ (d) $\frac{45}{143}$

Sol. (b) Here, $x^2 + bx + c > 0, \forall\ x \in R$

$\therefore$ $D < 0$

$\Rightarrow$ $b^2 < 4c$



Value of b	Possible values of c		
1	$1 < 4c$	$\Rightarrow c > \frac{1}{4} \Rightarrow$	{1, 2, 3, 4, 5, 6, 7, 8, 9}
2	$4 < 4c$	$\Rightarrow c > 1 \Rightarrow$	{2, 3, 4, 5, 6, 7, 8, 9}
3	$9 < 4c$	$\Rightarrow c > \frac{9}{4} \Rightarrow$	{3, 4, 5, 6, 7, 8, 9}
4	$16 < 4c$	$\Rightarrow c > 4 \Rightarrow$	{5, 6, 7, 8, 9}
5	$25 < 4c$	$\Rightarrow c > 6.25 \Rightarrow$	{7, 8, 9}
6	$36 < 4c$	$\Rightarrow c > 9 \Rightarrow$	Impossible
7	Impossible		
8	Impossible		
9	Impossible		

$n(E)$ = Number of favourable cases = 9 + 8 + 7 + 5 + 3 = 32

$n(S)$ = Total ways = $9 \times 9 = 81$

$\therefore$ Required probability = $\frac{n(E)}{n(S)} = \frac{32}{81}$

● **Ex. 6** *Three dice are thrown. The probability of getting a sum which is a perfect square, is*

(a) $\frac{2}{5}$ (b) $\frac{9}{20}$

(c) $\frac{1}{4}$ (d) None of these

Sol. (d) $n(S)$ = Total number of ways = $6 \times 6 \times 6 = 216$

The sum of the numbers on three dice varies from 3 to 18 and among these 4, 9 and 16 are perfect squares.

$\therefore n(E)$ = Number of favourable ways

= Coefficient of x^4 in $(x + x^2 + \ldots + x^6)^3$ + Coefficient of x^9 in $(x + x^2 + \ldots + x^6)^3$ + Coefficient of x^{16} in $(x + x^2 + \ldots + x^6)^3$

= Coefficient of x in $(1 + x + \ldots + x^5)^3$ + Coefficient of x^6 in $(1 + x + x^2 + \ldots + x^5)^3$ + Coefficient of x^{13} in $(1 + x + x^2 + \ldots + x^5)^3$

= Coefficient of x in $(1 - x^6)^3(1 - x)^{-3}$ + Coefficient of x^6 in $(1 - x^6)^3(1 - x)^{-3}$ + Coefficient of x^{13} in $(1 - x^6)^3(1 - x)^{-3}$

= Coefficient of x in $(1)(1 + {}^3C_1 x + \ldots)$ + Coefficient of x^6 in $(1 - 3x^6)(1 + {}^3C_1 x + \ldots)$ + Coefficient of x^{13} in $(1 - 3x^6 + 3x^{12} + \ldots); (1 + {}^3C_1 x + \ldots)$

$= {}^3C_1 + ({}^8C_6 - 3) + ({}^{15}C_{13} - 3 \times {}^9C_7 + 9)$

$= {}^3C_1 + ({}^8C_2 - 3) + ({}^{15}C_2 - 3 \times {}^9C_2 + 9)$

$= 3 + 25 + 6$

$= 34$

$\therefore$ Required probability = $\frac{n(E)}{n(S)} = \frac{34}{216} = \frac{17}{108}$

● **Ex. 7** *A quadratic equation is chosen from the set of all quadratic equations which are unchanged by squaring their roots. The chance that the chosen equation has equal roots, is*

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$

(c) $\frac{1}{4}$ (d) None of these

Sol. (a) Let α and β be the roots of the quadratic equation.

According to question,

$\alpha + \beta = \alpha^2 + \beta^2$ and $\alpha\beta = \alpha^2\beta^2 \Rightarrow \alpha\beta(\alpha\beta - 1) = 0$

$\Rightarrow \alpha\beta = 1$ or $\alpha\beta = 0$

$\Rightarrow \alpha = 1, \beta = 1; \alpha = \omega, \beta = \omega^2$ [cube roots and unity]

$\alpha = 1, \beta = 0; \alpha = 0, \beta = 0$

$\therefore n(S)$ = Number of quadratic equations which are unchanged by squaring their roots = 4

and $n(E)$ = Number of quadratic equations have equal roots = 2

$\therefore$ Required probability = $\frac{n(E)}{n(S)} = \frac{2}{4} = \frac{1}{2}$

● **Ex. 8** *Three-digit numbers are formed using the digits* 0, 1, 2, 3, 4, 5 *without repetition of digits. If a number is chosen at random, then the probability that the digits either increase or decrease, is*

(a) $\frac{1}{10}$ (b) $\frac{2}{11}$ (c) $\frac{3}{10}$ (d) $\frac{4}{11}$

Sol. (c) $n(S)$ = Total number of three digit numbers

$= {}^6P_3 - {}^5P_2 = 120 - 20 = 100$

$n(E)$ = Number of numbers with digits either increase or decrease

= Number of numbers with increasing digits + Number of numbers with decreasing digits

$= {}^5C_3 + {}^6C_3 = 10 + 20 = 30$

$\therefore$ Required probability = $\frac{n(E)}{n(S)} = \frac{30}{100} = \frac{3}{10}$

● **Ex. 9** *If X follows a binomial distribution with parameters* $n = 8$ *and* $p = \frac{1}{2}$, *then* $p(|x - 4| \le 2)$ *is equal to*

(a) $\frac{121}{128}$ (b) $\frac{119}{128}$ (c) $\frac{117}{128}$ (d) $\frac{115}{128}$

Sol. (b) Here, $p = \frac{1}{2}, n = 8$

$\therefore \quad q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

$\therefore$ The binomial distribution is $\left(\frac{1}{2} + \frac{1}{2}\right)^8$

Also, $|x - 4| \le 2$

$\Rightarrow \quad -2 \le x - 4 \le 2 \Rightarrow 2 \le x \le 6$

$$\therefore\ p(|x-4|\le 2) = p(x=2)+p(x=3)+p(x=4)+p(x=5)+p(x=6)$$

$$= {}^8C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^6 + {}^8C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^5 + {}^8C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^4 + {}^8C_5\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^3 + {}^8C_6\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^2$$

$$= \frac{{}^8C_2 + {}^8C_3 + {}^8C_4 + {}^8C_5 + {}^8C_6}{2^8}$$

$$= \frac{238}{256} = \frac{119}{128}$$

● ***Ex. 10*** *A doctor is called to see a sick child. The doctor knows (prior to the visit) that* 90% *of the sick children in that neighbourhood are sick with the flue, denoted by F, while* 10% *are sick with the measles, denoted by M. A well-known symptom of measles is a rash, denoted by R. The probability of having a rash for a child sick with the measles is* 0.95. *However, occasionally children with the flue also develop a rash with conditional probability* 0.08. *Upon examination the child, the doctor finds a rash, then the probability that the child has the measles, is*

(a) $\frac{89}{167}$ (b) $\frac{91}{167}$ (c) $\frac{93}{167}$ (d) $\frac{95}{167}$

Sol. (d) $\because P(F) = 0\cdot90,\ P(M) = 0\cdot10,$

$$P\left(\frac{R}{F}\right) = 0\cdot08,\ P\left(\frac{R}{M}\right) = 0\cdot95$$

$$\therefore\ P\left(\frac{M}{R}\right) = \frac{P(M)\cdot P\left(\frac{R}{M}\right)}{P(M)\cdot P\left(\frac{R}{M}\right) + P(F)\cdot P\left(\frac{R}{F}\right)}$$

$$= \frac{0\cdot10\times 0\cdot95}{0\cdot10\times 0\cdot95 + 0\cdot90\times 0\cdot08} = \frac{0\cdot095}{0\cdot167} = \frac{95}{167}$$

JEE Type Solved Examples :
More than One Correct Option Type Questions

■ This section contains **5 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of **which more than one** may be correct.

● ***Ex. 11*** *Let p_n denote the probability of getting n heads, when a fair coin is tossed m times. If p_4, p_5, p_6 are in AP, then values of m can be*

(a) 5 (b) 7 (c) 10 (d) 14

Sol. (b, d) $\because\ p_4 = {}^mC_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^{m-4} = {}^mC_4\left(\frac{1}{2}\right)^m$

$$p_5 = {}^mC_5\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^{m-5} = {}^mC_5\left(\frac{1}{2}\right)^m$$

and $$p_6 = {}^mC_6\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^{m-6} = {}^mC_6\left(\frac{1}{2}\right)^m$$

According to the question, p_4, p_5, p_6 are in AP

$$\therefore\quad 2p_5 = p_4 + p_6$$

$$\Rightarrow\quad 2\times {}^mC_5\left(\frac{1}{2}\right)^m = {}^mC_4\left(\frac{1}{2}\right)^m + {}^mC_6\left(\frac{1}{2}\right)^m$$

or $$2\times {}^mC_5 = {}^mC_4 + {}^mC_6$$

or $$2 = \frac{{}^mC_4}{{}^mC_5} + \frac{{}^mC_6}{{}^mC_5} \Rightarrow 2 = \frac{5}{m-5+1} + \frac{m-6+1}{6}$$

$$\Rightarrow\quad 2 = \frac{5}{m-4} + \frac{m-5}{6} \Rightarrow (m^2-2)\,m + 98 = 0$$

$$\Rightarrow\quad (m-14)(m-7) = 0$$

$$\Rightarrow\quad m = 7 \text{ or } 14$$

● ***Ex. 12*** *A random variable X follows binomial distribution with mean a and variance b. Then,*

(a) $a > b > 0$ (b) $\frac{a}{b} > 1$

(c) $\frac{a^2}{a-b}$ is an integer (d) $\frac{a^2}{a+b}$ is an integer

Sol. (a, b, c) Suppose, $X \sim B(n, p)$ i.e. $(q+p)^n$

Here, $np = a$ and $npq = b$

$$\therefore\quad q = \frac{b}{a}, \text{ then } p = 1-q = 1-\frac{b}{a}$$

Now, $0 < q < 1 \Rightarrow 0 < \frac{b}{a} < 1 \Rightarrow a > b > 0$ [alternate (a)]

and $\frac{a}{b} > 1$ [alternate (b)]

Also, $\frac{a^2}{a-b} = \frac{(np)^2}{np-npq} = \frac{np}{1-q} = \frac{np}{p} = n =$ Integer [alternate (c)]

● ***Ex. 13*** *If $A_1, A_2, ..., A_n$ are n independent events, such that $P(A_i) = \frac{1}{i+1}$, $i = 1, 2, .., n$, then the probability that none of $A_1, A_2, A_3, ..., A_n$ occur, is*

(a) $\frac{n}{n+1}$ (b) $\frac{1}{n+1}$

(c) less than $\frac{1}{n}$ (d) greater than $\frac{1}{n+2}$

Sol. (b, c, d) $\because A_1, A_2, A_3, \ldots, A_n$ are n independent, then

Required probability $= P(A_1' \cap A_2' \cap A_3' \cap \ldots \cap A_n')$

$= P(A_1').P(A_2').P(A_3') \ldots P(A_n')$

$= (1 - P(A_1))(1 - P(A_2))(1 - P(A_3)) \ldots (1 - P(A_n))$

$= \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right) \ldots \left(1 - \frac{1}{n+1}\right)$

$= \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \ldots \times \frac{n}{n+1} = \frac{1}{n+1}$

$\because n + 2 > n + 1 > n; \ \therefore \ \frac{1}{n+2} < \frac{1}{n+1} < \frac{1}{n}$

● ***Ex. 14*** *A and B are two events, such that* $P(A \cup B) \geq \frac{3}{4}$ *and* $\frac{1}{8} \leq P(A \cap B) \leq \frac{3}{8}$, *then*

(a) $P(A) + P(B) \leq \frac{11}{8}$ (b) $P(A) \cdot P(B) \leq \frac{3}{8}$

(c) $P(A) + P(B) \geq \frac{7}{8}$ (d) None of these

Sol. (a, c) $\because \ \frac{3}{4} \leq P(A \cup B) \leq 1$

$\Rightarrow \quad \frac{3}{4} \leq P(A) + P(B) - P(A \cap B) \leq 1$

As the minimum value of $P(A \cap B) = \frac{1}{8}$, we get

$P(A) + P(B) - \frac{1}{8} \geq \frac{3}{4} \Rightarrow P(A) + P(B) \geq \frac{7}{8}$ [alternate (c)]

As the maximum value of $P(A \cap B) = \frac{3}{8}$, we get

$P(A) + P(B) - \frac{3}{8} \leq 1 \Rightarrow P(A) + P(B) \leq \frac{11}{8}$ [alternate (a)]

● ***Ex. 15*** *A, B, C and D cut a pack of* 52 *cards successively in the order given. If the person who cuts a spade first receives* ₹ 350, *then the expectations of*

(a) B is ₹ 96 (b) D is ₹ 54

(c) $(A + C)$ is ₹ 200 (d) $(B - D)$ is ₹ 56

Sol. (a, b, c) Let E be the event of any one cutting a spade in one cut and let S be the sample space, then

$n(E) = {}^{13}C_1 = 13$ and $n(S) = {}^{52}C_1 = 52$

$\therefore \quad p = P(E) = \frac{n(E)}{n(S)} = \frac{1}{4}$ and $q = p(\bar{E}) = 1 - p = \frac{3}{4}$

The probability of A winning (when A starts the game)

$$= p + q^4 p + q^8 p + \ldots \infty = \frac{p}{1 - q^4} = \frac{\frac{1}{4}}{1 - \left(\frac{3}{4}\right)^4} = \frac{64}{175}$$

$\therefore \quad E(A) = ₹\,350 \times \frac{64}{175} = ₹\,128$

$E(B) = ₹128 \times q = ₹128 \times \frac{3}{4} = ₹\,96$

$E(C) = ₹96 \times q = ₹96 \times \frac{3}{4} = ₹\,72$

and $E(D) = ₹72 \times q = ₹72 \times \frac{3}{4} = ₹\,54$

$\therefore \ E(A + C) = ₹\,200$ and $E(B - D) = ₹\,42$.

JEE Type Solved Examples : Passage Based Questions

■ This section contains **3 solved passages** based upon each of the passage **3 multiple choice** examples have to be answered. Each of these examples has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I

(Ex. Nos. 16 to 18)

Each coefficient in the equation $ax^2 + bx + c = 0$ *is determined by throwing an ordinary die.*

16. *The probability that roots of quadratic are real and distinct, is*

(a) $\frac{5}{216}$ (b) $\frac{19}{108}$ (c) $\frac{173}{216}$ (d) $\frac{17}{108}$

Sol. (b) For roots of $ax^2 + bx + c = 0$ to be real and distinct,

$$b^2 - 4ac > 0$$

Value of b	Possible values of a and c
1, 2	No values of a and c
3	(1, 1), (1, 2), (2, 1)
4	(1, 1), (1, 2), (2, 1), (1, 3), (3, 1)
5	(1, 1), (1, 2), (2, 1), (1, 3), (3, 1), (2, 2), (1, 4), (4, 1), (1, 5), (5, 1), (2, 3), (3, 2), (1, 6), (6, 1)
6	(1, 1), (1, 2), (2, 1), (1, 3), (3, 1), (2, 2), (1, 4), (4, 1), (1, 5), (5, 1), (2, 3), (3, 2), (1, 6), (6, 1), (2, 4), (4, 2)

If E be the event of favourable cases, then $n(E) = 38$

Total ways, $n(S) = 6 \times 6 \times 6 = 216$

Hence, the required probability, $p_1 = \frac{n(E)}{n(S)} = \frac{38}{216} = \frac{19}{108}$

17. *The probability that roots of quadratic are equal, is*

(a) $\frac{5}{216}$ (b) $\frac{7}{216}$ (c) $\frac{11}{216}$ (d) $\frac{17}{216}$

Sol. (a) For roots of $ax^2 + bx + c = 0$ to be equal $b^2 = 4ac$

i.e. b^2 must be even.

Value of b	Possible values of a and c
2	(1, 1)
4	(2, 2), (1, 4), (4, 1)
6	(3, 3)

If E be the event of favourable cases, then $n(E) = 5$

Total ways, $n(S) = 6 \times 6 \times 6 = 216$

Hence, the required probability, $p_2 = \dfrac{n(E)}{n(S)} = \dfrac{5}{216}$

18. *The probability that roots of quadratic are imaginary, is*

(a) $\dfrac{103}{216}$ (b) $\dfrac{133}{216}$ (c) $\dfrac{157}{216}$ (d) $\dfrac{173}{216}$

Sol. (d) Let p_3 = Probability that roots of $ax^2 + bx + c = 0$ are imaginary

$= 1 -$ (Probability that roots of $ax^2 + bx + c = 0$ are real)

$= 1 - (p_1 + p_2)$ [from above]

$= 1 - \dfrac{43}{216} = \dfrac{173}{216}$

Passage II

(Ex. Nos. 19 to 21)

A box contains n coins. Let $P(E_i)$ be the probability that exactly i out of n coins are biased. If $P(E_i)$ is directly proportional to $i(i+1); 1 \le i \le n$.

19. *Proportionality constant k is equal to*

(a) $\dfrac{3}{n(n^2+1)}$ (b) $\dfrac{1}{(n^2+1)(n+2)}$

(c) $\dfrac{3}{n(n+1)(n+2)}$ (d) $\dfrac{1}{(n+1)(n+2)(n+3)}$

Sol. (c) $\because P(E_i) \propto i(i+1)$

$\Rightarrow P(E_i) = k\,i(i+1)$, where k is proportionality constant.

We have, $P(E_1) + P(E_2) + P(E_3) + \ldots + P(E_n) = 1$

($\because E_1, E_2, \ldots, E_n$ are mutually exclusive and exhaustive events)

$$\Rightarrow \sum_{i=1}^{n} P(E_i) = 1$$

$$\Rightarrow k \sum_{i=1}^{n} (i^2 + i) = 1$$

$$\Rightarrow k\,[\Sigma n^2 + \Sigma n] = 1$$

$$\Rightarrow k\left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right] = 1$$

$$\therefore \quad k = \frac{3}{n(n+1)(n+2)} \quad \ldots(i)$$

20. *If P be the probability that a coin selected at random is biased, then $\lim_{x \to \infty} P$ is*

(a) $\dfrac{1}{4}$ (b) $\dfrac{3}{4}$ (c) $\dfrac{3}{5}$ (d) $\dfrac{7}{8}$

Sol. (b) $\because P = P(E) = \sum_{i=1}^{n} \cdot P(E_i)\, P\left(\dfrac{E}{E_i}\right)$ ...(ii)

$$= \sum_{i=1}^{n} k\, i(i+1) . \frac{i}{n}$$

$$= \frac{k}{n} \sum_{i=1}^{n} (i^3 + i^2) = \frac{k}{n} [\Sigma n^3 + \Sigma n^2]$$

$$= \frac{k}{n}\left[\left(\frac{n(n+1)^2}{2}\right) + \frac{n(n+1)(2n+1)}{6}\right]$$

$$= \frac{k\,(n+1)\,(n+2)\,(3n+1)}{12}$$

$$= \frac{3}{n(n+1)(n+2)} \cdot \frac{(n+1)\,(n+2)\,(3n+1)}{12}$$ [from Eq. (i)]

$$= \frac{3n+1}{4n} = \frac{3}{4} + \frac{1}{4n}$$

$$\therefore \lim_{n \to \infty} P = \lim_{n \to \infty}\left[\frac{3}{4} + \frac{1}{3n}\right] = \frac{3}{4} + 0 = \frac{3}{4}$$

21. *If a coin is selected at random is found to be biased, the probability that it is the only biased coin the box, is*

(a) $\dfrac{1}{(n+1)(n+2)(n+3)(n+4)}$ (b) $\dfrac{12}{n(n+1)(n+2)(3n+1)}$

(c) $\dfrac{24}{n(n+1)(n+2)(2n+1)}$ (d) $\dfrac{24}{n(n+1)(n+2)(3n+1)}$

Sol. (d) $P\left(\dfrac{E_1}{E}\right) = \dfrac{P(E_1) \cdot P\left(\dfrac{E}{E_1}\right)}{\sum_{i=1}^{n} P(E_i) \cdot P\left(\dfrac{E}{E_i}\right)} = \dfrac{2k \times \dfrac{1}{n}}{P(E)}$ [from Eq. (ii)]

$$= \frac{\dfrac{2k}{n}}{\left(\dfrac{3n+1}{4n}\right)} = \frac{8k}{(3n+1)}$$

$$= \frac{24}{n\,(n+1)\,(n+2)\,(3n+1)}$$ [from Eq. (i)]

Passage III

(Ex. Nos. 22 to 24)

Let S be the set of the first 21 natural numbers, then the probability of

22. *Choosing $\{x, y\} \subseteq S$, such that $x^3 + y^3$ is divisible by 3, is*

(a) $\dfrac{1}{6}$ (b) $\dfrac{1}{5}$ (c) $\dfrac{1}{4}$ (d) $\dfrac{1}{3}$

Sol. (d) $\because S = \{1, 2, 3, 4, 5, \ldots, 21\}$

Total number of ways choosing x and y is

$${}^{21}C_2 = \frac{21 \cdot 20}{1 \cdot 2} = 210$$

Now, arrange the given numbers as below:

1	4	7	10	13	16	19
2	5	8	11	14	17	20
3	6	9	12	15	18	21

We see that, $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$ will be divisible by 3 in the following cases:

One of two numbers belongs to the first row and one of the two numbers belongs to the second row or both numbers occurs in third row.

$\therefore$ Number of favourable cases $= ({}^7C_1)({}^7C_1) + {}^7C_2 = 70$

$\therefore$ Required probability $= \frac{70}{210} = \frac{1}{3}$

23. *Choosing $\{x, y, z\} \subseteq S$, such that x, y, z are in AP, is*

(a) $\frac{5}{133}$ (b) $\frac{10}{133}$ (c) $\frac{3}{133}$ (d) $\frac{2}{133}$

Sol. (b) Given, x, y, z are in AP

$\therefore \quad 2y = x + z$

It is clear that sum of x and z is even.

$\therefore$ x and z both are even or odd out of set S.

i.e., 11 numbers (1, 3, 5,..., 21) are odd and 10 numbers (2, 4, 6,..., 20) are even.

$\therefore$ Number of favourable cases

$$= {}^{21}C_2 + {}^{10}C_2 = \frac{11 \cdot 10}{1 \cdot 2} + \frac{10 \cdot 9}{1 \cdot 2} = 100$$

and total number of ways choosing x, y and z is

$${}^{21}C_3 = \frac{21 \cdot 20 \cdot 19}{1 \cdot 2 \cdot 3} = 1330$$

$\therefore$ Required probability $= \frac{100}{1330} = \frac{10}{133}$

24. *Choosing $\{x, y, z\} \subseteq S$, such that x, y, z are not consecutive, is*

(a) $\frac{17}{70}$ (b) $\frac{34}{70}$ (c) $\frac{51}{70}$ (d) $\frac{34}{35}$

Sol. (c) Given, x, $y$and z are not consecutive.

$\therefore$ Number of favourable ways $= {}^{21-3+1}C_3$

$$= {}^{19}C_3 = \frac{19 \cdot 18 \cdot 17}{1 \cdot 2 \cdot 3} = 969$$

and total number of ways $= {}^{21}C_3 = \frac{21 \cdot 20 \cdot 19}{1 \cdot 2 \cdot 3} = 1330$

$\therefore$ Required probability $= \frac{969}{1330} = \frac{51}{70}$

JEE Type Solved Examples :
Single Integer Answer Type Questions

- This section contains **2 examples.** The answer to each example is a **single digit integer** ranging from **0** to **9** (both inclusive).

● **Ex. 25** *The altitude through A of ΔABC meets BC at D and the circumscribed circle at E. If $D \equiv (2, 3)$, $E \equiv (5, 5)$, the ordinate of the orthocentre being a natural number. If the probability that the orthocentre lies on the lines $y = 1;\ y = 2;\ y = 3 \ldots\ldots y = 10$ is $\frac{m}{n}$, where m and n are relative primes, the value of $m + n$ is*

Sol. (8) Let the orthocentre be $O(x, y)$.

It is clear from the OE is perpendicular bisector of line BC.

$\therefore \quad OD = DE$

$\Rightarrow \sqrt{(x-2)^2 + (y-3)^2} = \sqrt{(5-2)^2 + (5-3)^2}$

$\Rightarrow \quad (x-2)^2 + (y-3)^2 = (5-2)^2 + (5-3)^2$

$\Rightarrow \quad x^2 + y^2 - 4x - 6y = 0 \Rightarrow x = 2 \pm \sqrt{13 - (y-3)^2}$

$\Rightarrow y$ can take the values as 1, 2, 3, 4, 5, 6

$\therefore \quad$ Required probability $= \frac{6}{10} = \frac{3}{5} = \frac{m}{n}$ [given]

$\Rightarrow \quad m = 3$ and $n = 5$

$\therefore \quad m + n = 8$

● **Ex. 26** *The digits 1, 2, 3, 4, 5, 6, 7, 8 and 9 are written in random order to form a nine digit number. Let p be the probability that this number is divisible by 36, the value of $9p$ is*

Sol. (2) $\because$ $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$, a number consisting all these digits will be divisible by 9. Thus, the number will be divisible by 36, if and only if it is divisible by 4. The number formed by its last two digits must be divisible by 4. The possible values of the last pair to the following:

12, 16, 24, 28, 32, 36, 48, 52, 56, 64, 68, 72, 76, 84, 92, 96.

i.e., There are 16 ways of choosing last two digits.

The remaining digits can be arranged in ${}^7P_7 = 7!$ ways.

Therefore, number of favourable ways $= 16 \times 7!$

and number of total ways $= 9!$

$\therefore$ Required probability, $p = \frac{16 \times 7!}{9!} = \frac{16}{9 \times 8} = \frac{2}{9}$

$\Rightarrow \quad 9p = 2$

JEE Type Solved Examples : Matching Type Questions

■ This section contains **2 examples.** Examples 27 and 28 have four statements (A, B, C and D) given in **Column I** and four statements (p, q, r and s) in **Column II.** Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II.**

● ***Ex. 27*** *If n positive integers taken at random are multiplied together.*

	Column I		Column II
(A)	The probability that the last digit is 1, 3, 7 or 9 is $P(n)$, then 100 $P(2)$ is divisible by	(p)	3
(B)	The probability that the last digit is 2, 4, 6 or 8 is $Q(n)$, then 100 $Q(2)$ is divisible by	(q)	4
(C)	The probability that the last digit is 5 is $R(n)$, then 100 $R(2)$ is divisible by	(r)	6
(D)	The probability that the last digit is zero is $S(n)$, then 100 $S(2)$ is divisible by	(s)	9

Sol. A → (q); B → (p, q, r); C → (p, s); D → (p, s)

Let n positive integers be $x_1, x_2, x_3, \ldots, x_n$

Let $a = x_1 . x_2 . x_3 \ldots x_n$

Since, the last digit in each of the numbers $x_1, x_2, \ldots, x_n$ can be any one of the digits

0, 1, 2,..., 9 (total 10)

$\therefore \quad n(S) = 10^n$

Let E_1, E_2, E_3 and E_4 are the events given in A, B, C and D, respectively.

(A) $n(E_1) = 4^n \Rightarrow P(E_1) = \left(\frac{4}{10}\right)^n = P(n)$ [given]

$\therefore \quad 100\, P(2) = 16$

(B) $n(E_2) = n$ (last digit is 1 or 2 or 3 or 4 or 6 or 7 or 8 or 9) $- n(E_1) = 8^n - 4^n$

$\Rightarrow \quad P(E_2) = \frac{8^n - 4^n}{10^n} = Q(n)$ [given]

$\therefore \quad 100\, Q(2) = 64 - 16 = 48$

(C) $n(E_3) = n$ (last digit is 1 or 3 or 5 or 7 or 9) $- n(E_1)$

$= 5^n - 4^n$

$\Rightarrow \quad P(E_3) = \frac{5^n - 4^n}{10^n} = R(n)$ [given]

$\therefore \; 100\, R(2) = 25 - 16 = 9$

(D) $n(E_4) = n(S) - n$ (last digit is 1 or 2 or 3 or 4 or 6 or 7 or 8 or 9) $- n(E_3) = 10^n - 8^n - (5^n - 4^n)$

$\therefore \quad P(E_4) = \frac{10^n - 8^n - 5^n + 4^n}{10^n} = S(n)$ [given]

$\therefore \quad 100\, S(2) = 27$

● ***Ex. 28*** *If A and B are two independent events, such that* $P(A) = \frac{1}{3}$ *and* $P(B) = \frac{1}{4}$.

	Column I		Column II
(A)	If $P\left(\frac{A}{B}\right) = \lambda_1$, then $12\lambda_1$ is	(p)	a prime number
(B)	If $P\left(\frac{A}{A \cup B}\right) = \lambda_2$, then $9\lambda_2$ is	(q)	a composite number
(C)	If $P[(A \cap \bar{B}) \cup (\bar{A} \cap B)] = \lambda_3$, then $12\lambda_3$ is	(r)	a natural number
(D)	If $P(\bar{A} \cup B) = \lambda_4$, then $12\lambda_4$ is	(s)	a perfect number

Sol. A → (q, r); B (q, r, s); C → (p, r); D → (q, r)

$\because$ A and B are independent events.

$\therefore \quad P(A \cap B) = P(A) \cdot P(B) = \frac{1}{12},$

$P(A \cap \bar{B}) = P(A) \cdot P(\bar{B}) = \frac{1}{3} \times \left(1 - \frac{1}{4}\right) = \frac{1}{4},$

$P(\bar{A} \cap B) = P(\bar{A}) \cdot P(B) = \left(1 - \frac{1}{3}\right) \cdot \frac{1}{4} = \frac{1}{6}$

(A) $P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{12}}{\frac{1}{4}} = \frac{1}{3} = \lambda_1$ [given]

$\therefore \quad 12\lambda_1 = 4$ [natural number and composite number]

(B) $P\left(\frac{A}{A \cup B}\right) = \frac{P(A \cap (A \cup B))}{P(A \cup B)}$

$= \frac{P(A)}{P(A \cup B)} = \frac{P(A)}{P(A) + P(B)} - P(A \cap B)$

$= \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{4} - \frac{1}{12}} = \frac{2}{3} = \lambda_2$ [given]

$\therefore \quad 9\lambda_2 = 6$

[natural number, composite number and perfect number]

(C) $P(A \cap \bar{B}) \cup (\bar{A} \cap B)) = P(A \cap \bar{B}) + P(\bar{A} \cap B)$

$= \frac{1}{4} + \frac{1}{6} = \frac{5}{12} = \lambda_3$ [given]

$\therefore \quad 12\lambda_3 = 5$ [prime number and natural number]

(D) $P(\bar{A} \cup B) = P(\bar{A}) + P(B) - P(\bar{A} \cap B)$

$= \left(1 - \frac{1}{3}\right) + \frac{1}{4} - \frac{1}{6} = \frac{3}{4} = \lambda_4$ [given]

$\therefore \quad 12\lambda_4 = 9$ [natural number and composite number]

JEE Type Solved Examples :
Statement I and II Type Questions

■ **Directions** Example numbers 29 and 30 are Assertion-Reason type examples. Each of these examples contains two statements:
Statement-1 (Assertion) and **Statement-2** (Reason)
Each of these examples also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement-1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

● ***Ex. 29.*** *A man P speaks truth with probability p and another man Q speaks truth with probability* $2p$.

Statement-1 *If P and Q contradict each other with probability* $\frac{1}{2}$*, then there are two values of p.*

Statement-2 *A quadratic equation with real coefficients has two real roots.*

Sol. (c) Let E_1 be the event that P speaks the truth, then $P(E_1) = p$ and let E_2 be the event that Q speaks the truth, then $P(E_2) = 2p$.

Statement-1 If P and Q contradict each other with probability $\frac{1}{2}$, then $P(E_1)\cdot P(E_2') + P(E_1')\cdot P(E_2) = \frac{1}{2}$

$\Rightarrow p\cdot(1-2p) + (1-p)\cdot 2p = \frac{1}{2} \Rightarrow 8p^2 - 6p + 1 = 0$

$\Rightarrow (2p-1)(4p-1) = 0 \Rightarrow p = \frac{1}{2}$ and $p = \frac{1}{4}$

$\therefore$ Statement-1 is true.

Statement-2 Let quadratic equation $ax^2 + bx + c = 0$, where $a, b, c \in R$

If $b^2 - 4ac < 0$

then, roots are imaginary.

$\therefore$ Statement-2 is false.

● ***Ex. 30*** *A fair die thrown twice. Let* (a, b) *denote the outcome in which the first throw shows a and the second shows b. Let A and B be the following two events:*

$A = \{(a, b) | \ a \text{ is even}\}$, $B = \{(a, b) | \ b \text{ is even}\}$

Statement-1 If $C = \{(a, b) | \ a + b \text{ is odd}\}$, then $P(A \cap B \cap C) = \frac{1}{8}$.

Statement-2 If $D = \{(a, b) | \ a + b \text{ is even}\}$, then $P[(A \cap B \cap D) | (A \cup B)] = \frac{1}{3}$

Sol. (c) If a and b are both even, then $a + b$ is even, therefore $P(A \cap B \cap C) = 0$

$\therefore$ Statement-1 is false.

Also, $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{2}$, $P(A \cap B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$= \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4}$

$\therefore P[(A \cap B \cap D) | (A \cup B)] = \frac{P((A \cap B \cap D) \cap (A \cup B))}{P(A \cup B)}$

$= \frac{P(A \cap B)}{P(A \cup B)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$ $[\because A \cap B \subseteq D]$

$\therefore$ Statement-2 is true.

Subjective Type Examples

■ In this section, there are **24 subjective** solved examples.

● ***Ex. 31*** *Three critics review a book. Odds in favour of the book are* $5:2$, $4:3$ *and* $3:4$ *respectively for the three critics. Find the probability that majority are in favour of the book.*

Sol. Let the critics be E_1, E_2 and E_3. Let $P(E_1)$, $P(E_2)$ and $P(E_3)$ denotes the probabilities of the critics E_1, E_2 and E_3 to be in favour of the book. Since, the odds in favour of the book for the critics E_1, E_2 and E_3 are $5:2$, $4:3$ and $3:4$, respectively.

$\therefore \quad P(E_1) = \frac{5}{5+2} = \frac{5}{7}$, $P(E_2) = \frac{4}{4+3} = \frac{4}{7}$

and $\quad P(E_3) = \frac{3}{3+4} = \frac{3}{7}$

Clearly, the event of majority being in favour = the event of atleast two critics being in favour

$\therefore$ The required probability

$= P(E_1E_2\bar{E}_3) + P(\bar{E}_1E_2E_3) + P(E_1\bar{E}_2E_3) + P(E_1E_2E_3)$

$= P(E_1)\cdot P(E_2)\cdot P(\bar{E}_3) + P(\bar{E}_1)\cdot P(E_2)\cdot P(E_3)$

$+ P(E_1)\cdot P(\bar{E}_2)\cdot P(E_3) + P(E_1)\cdot P(E_2)\cdot P(E_3)$

$[\because E_1, E_2$ and E_3 are independent$]$

$$= \frac{5}{7}\cdot\frac{4}{7}\cdot\left(1-\frac{3}{7}\right)+\left(1-\frac{5}{7}\right)\cdot\frac{4}{7}\cdot\frac{3}{7}+\frac{5}{7}\cdot\left(1-\frac{4}{7}\right)\cdot\frac{3}{7}+\frac{5}{7}\cdot\frac{4}{7}\cdot\frac{3}{7}$$

$$= \frac{1}{7^3}[80+24+45+60] = \frac{209}{343}$$

● ***Ex. 32*** *A has 3 shares is a lottery containing 3 prizes and 9 blanks; B has 2 shares in a lottery containing 2 prizes and 6 blanks. Compare their chances of success.*

Sol. Let E_1 and E_2 be the events of success of A and B, respectively. Therefore, E'_1 and E'_2 are the events of unsuccess of A and B, respectively.

Since, A has 3 shares in a lottery containing 3 prizes and 9 blanks, therefore A will draw 3 tickets out of 12 tickets (3 prizes + 9 blanks). Then, A will get success if he draws atleast one prize out of 3 draws. Similarly, B will get success if he draws atleast one prize out of 2 draws.

$$\therefore \quad P(E'_1) = \frac{{}^9C_3}{{}^{12}C_3} = \frac{\frac{9\cdot8\cdot7}{1\cdot2\cdot3}}{\frac{12\cdot11\cdot10}{1\cdot2\cdot3}} = \frac{21}{55}$$

$$\therefore \quad P(E_1) = 1 - P(E'_1) = 1 - \frac{21}{55} = \frac{34}{55}$$

$$\text{Again,} \quad P(E'_2) = \frac{{}^6C_2}{{}^8C_2} = \frac{\frac{6\cdot5}{1\cdot2}}{\frac{8\cdot7}{1\cdot2}} = \frac{15}{28}$$

$$\therefore \quad P(E_2) = 1 - P(E'_2) = 1 - \frac{15}{28} = \frac{13}{28}$$

$$\text{Hence,} \quad \frac{P(E_1)}{P(E_2)} = \frac{\frac{34}{55}}{\frac{13}{28}} = \frac{952}{715}$$

$$\therefore \quad P(E_1) : P(E_2) = 952 : 715$$

● ***Ex. 33*** *A bag contains a white and b black balls. Two players A and B alternately draw a ball from the bag, replacing the ball each time after the draw till one of them draws a white ball and wins the game. If A begins the game and the probability of A winning the game is three times that of B, show that $a : b = 2 : 1$.*

Sol. Let E_1 denote the event of drawing a white ball at any draw and E_2 that for a black ball and let E be the event for A winning the game

$$\therefore P(E_1) = \frac{a}{a+b} \text{ and } P(E_2) = \frac{b}{a+b}$$

$$\therefore P(E) = P(E_1 \text{ or } E_2E_2E_1 \text{ or } E_2E_2E_2E_2E_1 \text{ or } \ldots)$$

$$= P(E_1) + P(E_2E_2E_1) + P(E_2E_2E_2E_2E_1) + \ldots$$

$$= P(E_1) + P(E_2)\,P(E_2)\,P(E_1) + P(E_2)\,P(E_2)\,P(E_2)\,P(E_2)\,P(E_1) + \ldots$$

[$\because E_1$ and E_2 are independent]

$$= \frac{P(E_1)}{1-\{P(E_2)\}^2} \qquad \text{[sum of infinite GP]}$$

$$= \frac{\frac{a}{a+b}}{1-\left(\frac{b}{a+b}\right)^2} = \frac{a(a+b)}{a^2+2ab} \quad \therefore \; P(E) = \frac{a+b}{a+2b}$$

Then, $P(E')$ is the probability for B winning the game

$$\therefore \quad P(E') = 1 - P(E) = 1 - \frac{a+b}{a+2b} = \frac{b}{a+2b}$$

According to the problem, $P(E) = 3P(E')$

$$\Rightarrow \qquad \frac{a+b}{a+2b} = \frac{3b}{a+2b} \quad \Rightarrow \alpha + \beta = 3\beta \Rightarrow \alpha = 2\beta$$

$$\therefore \qquad \frac{a}{b} = \frac{2}{1} \qquad \Rightarrow \alpha : \beta = 2 : 1$$

● ***Ex. 34*** *Five persons entered the lift cabin on the ground floor of an 8 floors house. Suppose that each of them, independently and with equal probability can leave the cabin at any floor beginning with the first. Find out the probability of all five persons leaving at different floors.*

Sol. Let S be the sample space and E be the event that the five persons get down at different floors.

Total number of floors excluding the ground floor = 7

Since, each of the 5 persons can get down at any one of the 7 floors in 7 ways.

$\therefore$ $n(S)$ = Total number of ways in which the 5 persons can get down = 7^5

and $n(E)$ = number of ways in which the 5 persons can get down at 5 different floors out of 7 floors = ${}^7P_5$

$\therefore$ Required probability, $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{{}^7P_5}{7^5}$

● ***Ex. 35*** *Let X be a set containing n elements. Two subsets A and B of X are chosen at random. Find the probability that $A \cup B = X$.*

Sol. Let $X = \{x_1, x_2, \ldots, x_n\}$

For each $x_i \in X$ $(1 \le i \le n)$, we have the following four choices

(i) $x_i \in A$ and $x_i \in B$ (ii) $x_i \in A$ and $x_i \notin B$

(iii) $x_i \notin A$ and $x_i \in B$ (iv) $x_i \notin A$ and $x_i \notin B$

Let S be the sample space and E be the event favourable for the occurrence of $A \cup B = X$.

$$\therefore \qquad n(S) = 4^n$$

$$\text{and} \qquad n(E) = 3^n \qquad [\because \text{case (iv)} \notin X]$$

Hence, the required probability,

$$P(E) = \frac{n(E)}{n(S)} \Rightarrow = \frac{3^n}{4^n} = \left(\frac{3}{4}\right)^n$$

● **Ex. 36** *Two persons each makes a single throw with a pair of dice. Find the probability that the throws are unequal.*

Sol. Let E be the event that the throws of the two persons are unequal. Then, E' be the event that the throws of the two persons are equal.

$\therefore$ The total number of cases for E' is $(36)^2$

i.e., $n(S) = (36)^2$ [$\because S$ be the sample space]

We now proceed to find out the number of favourable cases for E'. Suppose

$$(x + x^2 + x^3 + \ldots + x^6)^2 = a_2 x^2 + a_3 x^3 + \ldots + a_{12} x^{12}$$

The number of favourable ways of $E' = a_2^2 + a_3^2 + \ldots + a_{12}^2$

$\therefore$ $n(E')$ = coefficient of constant term in

$$(a_2 x^2 + a_3 x^3 + \ldots + a_{12} x^{12}) \times \left(\frac{a_2}{x^2} + \frac{a_3}{x^3} + \ldots + \frac{a_{12}}{x^{12}}\right)$$

$$= \text{coefficient of constant term in } \frac{(1 - x^6)^2}{(1 - x)^2} \times \frac{\left(1 - \frac{1}{x^6}\right)^2}{\left(1 - \frac{1}{x}\right)^2}$$

$$= \text{coefficient of } x^{10} \text{ in } (1 - x^6)^4 (1 - x)^{-4}$$

$$= \text{coefficient of } x^{10} \text{ in } (1 - 4x^6 + \ldots)(1 + {}^4C_1 x + {}^5C_2 x^2 + \ldots + {}^{13}C_{10} x^{10} + \ldots)$$

$$= {}^{13}C_{10} - 4 \cdot {}^7C_4$$

$$= {}^{13}C_3 - 4 \cdot {}^7C_3 = \frac{13 \cdot 12 \cdot 11}{1 \cdot 2 \cdot 3} - 4 \cdot \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} = 146$$

$\therefore$ $P(E') = \dfrac{n(E')}{n(S)} = \dfrac{146}{(36)^2} = \dfrac{73}{648}$

Hence, required probability,

$$P(E) = 1 - P(E') = 1 - \frac{73}{648} = \frac{575}{648}$$

● **Ex. 37** *If X and Y are independent binomial variates $B(5, 1/2)$ and $B(7, 1/2)$, find the value of $P(X + Y = 3)$.*

Sol. We have,

$$P(X + Y = 3) = P(X = 0, Y = 3) + P(X = 1, Y = 2) + P(X = 2, Y = 1) + P(X = 3, Y = 0)$$

$$= P(X = 0)\, P(Y = 3) + P(X = 1)\, P(Y = 2) + P(X = 2)\, P(Y = 1) + P(X = 3)\, P(Y = 0)$$

[$\because X$ and Y are independent]

$$= {}^5C_0 \left(\frac{1}{2}\right)^5 \cdot {}^7C_3 \left(\frac{1}{2}\right)^7 + {}^5C_1 \left(\frac{1}{2}\right)^5 {}^7C_2 \left(\frac{1}{2}\right)^7 + {}^5C_2 \left(\frac{1}{2}\right)^5 {}^7C_1 \left(\frac{1}{2}\right)^7 + {}^5C_3 \left(\frac{1}{2}\right)^5 {}^7C_0 \left(\frac{1}{2}\right)^7$$

$$= \left(\frac{1}{2}\right)^{12} [(1)(35) + (5)(21) + (10)(7) + (10)(1)]$$

$$= \frac{220}{2^{12}} = \frac{55}{1024}$$

● **Ex. 38** *If $a \in [-20, 0]$, find the probability that the graph of the function $y = 16x^2 + 8(a + 5)\, x - 7a - 5$ is strictly above the X-axis.*

Sol. Since, the graph $y = 16x^2 + 8(a + 5)\, x - 7a - 5$ is strictly above the X-axis, therefore $y > 0$ for all x

$\Rightarrow$ $16x^2 + 8(a + 5)\, x - 7a - 5 > 0, \forall x$

$\therefore$ Discriminant < 0

$\Rightarrow$ $64(a + 5)^2 - 4 \cdot 16(-7a - 5) < 0$

$\Rightarrow a^2 + 17a + 30 < 0 \Rightarrow (a + 15)(a + 2) < 0$

$\Rightarrow$ $-15 < a < -2$

$\therefore$ Required probability $= \dfrac{\int_{-15}^{-2} dx}{\int_{-20}^{0} dx} = \dfrac{13}{20}$

● **Ex. 39** *3 distinct integers are selected at random from* 1, 2, 3,..., 20. *Find out the probability that the sum is divisible by* 5.

Sol. The number of wayds choosing 3 distinct integers from 1, 2, 3, ..., 20 is

$${}^{20}C_3 = \frac{20 \cdot 19 \cdot 18}{1 \cdot 2 \cdot 3} = 20 \times 57 = 1140$$

Now, arrange the given numbers as below:

1	6	11	16
2	7	12	17
3	8	13	18
4	9	14	19
5	10	15	20

We see that the sum of three digits divisible by 5 in the following cases :

Two number from 1st row and one number from 3rd row or one number from 2nd row and two numbers from 4th row or three numbers from 5th row or one number from each (1st row, 4th row, 5th row) or one number from each (2nd row, 3rd row, 5th row).

Then, the number of favourable ways

$$= {}^4C_2 \times {}^4C_1 + {}^4C_1 \times {}^4C_2 + {}^4C_3 + {}^4C_1 \times {}^4C_1 \times {}^4C_1 + {}^4C_1 \times {}^4C_1 \times {}^4C_1$$

$$= 24 + 24 + 4 + 64 + 64 = 180$$

Hence, the required probability $= \dfrac{180}{1140} = \dfrac{3}{19}$

Note

If divisible by 4, then take four rows and if divisible by 3, then take three rows, etc.

● ***Ex. 40*** *5 girls and 10 boys sit at random in a row having 15 chairs numbered as 1 to 15. Find the probability that end seats are occupied by the girls and between any two girls odd number of boys sit.*

Sol. There are four gaps in between the girls where the boys can sit. Let the number of boys in these gaps be $2a+1$, $2b+1$, $2c+1$, $2d+1$, then

$$2a+1+2b+1+2c+1+2d+1=10$$

or $$a+b+c+d=3$$

The number of solutions of above equation

$$= \text{coefficient of } x^3 \text{ in } (1-x)^{-4} = {}^6C_3 = 20$$

Thus, boys and girls can sit in $20\times 10!\times 5!$ ways. Total ways $= 15!$

Hence, the required probability $= \dfrac{20\times 10!\times 5!}{15!}$

● ***Ex. 41*** *A four digit number (numbered from 0000 to 9999) is said to be lucky if sum of its first two digits is equal to the sum of its last two digits. If a four-digit number is picked up at random, find the probability that it is lucky number.*

Sol. The total number of ways of choosing a four digit number is $10^4 = 10000$. Let a_k denote the number of distinct non-negative integral solutions of the equation $x+y=k$ $(0\le k\le 18)$

$\therefore$ The number of favourable cases $= a_0^2 + a_1^2 + \ldots + a_{18}^2$

Suppose, $(1+x+x^2+\ldots+x^9)^2$

$$= a_0 + a_1x + a_2x^2 + \ldots + a_{18}x^{18}$$

Thus, $a_0^2 + a_1^2 + \ldots + a_{18}^2$ = coefficient of constant term in

$$(a_0 + a_1x + \ldots + a_{18}x^{18}) \times \left(a_0 + \frac{a_1}{x} + \ldots + \frac{a_{18}}{x^{18}}\right)$$

= coefficient of constant term in

$$(1+x+x^2+\ldots+x^9)^2 \times \left(1+\frac{1}{x}+\frac{1}{x^2}+\ldots+\frac{1}{x^9}\right)^2$$

= coefficient of x^{18} in $(1+x+x^2+\ldots+x^9)^4$

= coefficient of x^{18} in $(1-x^{10})^4\,(1-x)^{-4}$

= coefficient of x^{18} in $(1-4x^{10})\,(1+{}^4C_1x+{}^5C_2x^2+\ldots)$

$= {}^{21}C_{18} - 4\cdot{}^{11}C_8 = 1330 - 660 = 670$

Hence, the required probability $= \dfrac{670}{10000} = 0.067$

● ***Ex. 42***

(i) If four squares are chosen at random on a chess board, find the probability that they lie on a diagonal line.

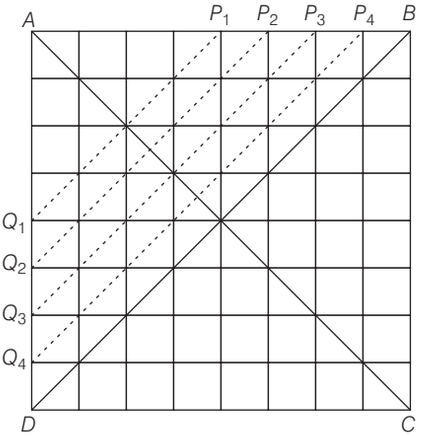
(ii) If two squares are chosen at random on a chess board, what is the probability that they have exactly one corner in common?

(iii) If nine squares are chosen at random on a chess board, what is the probability that they form a square of size 3×3?

Sol. (i) Total number of ways $= {}^{64}C_4$

The chess board can be divided into two parts by a diagonal line BD. Now, if we begin to select four squares from the diagonal P_1Q_1, P_2Q_2, …, BD, then we can find number of squares selected

$$= 2({}^4C_4 + {}^5C_4 + {}^6C_4 + {}^7C_4) = 112$$



Similarly, number of squares for the diagonals chosen parallel to $AC = 112$

$\therefore$ Total favourable ways $= 364$

$\therefore$ Required probability $= \dfrac{364}{{}^{64}C_4}$.

(ii) Total ways $= 64\times 63$

Now, if first square is in one of the four corners, then the second square can be chosen in just one way $= (4)(1) = 4$

If the first square is one of the 24 non-corner squares along the sides of the chess board, the second square can be chosen in two ways $= (24)(2) = 48$.

Now, if the first square is any of the 36 remaining squares, the second square can be chosen in four ways

$$= (36)(4) = 144$$

$\therefore$ Favourable ways $= 4 + 48 + 144 = 196$

$\therefore$ Required probability $= \dfrac{196}{64\times 63} = \dfrac{7}{144}$.

(iii) Total ways $= {}^{64}C_9$

A chess board has 9 horizontal and 9 vertical lines. We see that a square of size 3×3 can be formed by choosing four consecutive horizontal and vertical lines.

Hence, favourable ways $= ({}^6C_1)({}^6C_1) = 36$

$\therefore$ Required probability $= \dfrac{36}{{}^{64}C_9}$.

● ***Ex. 43*** *Out of $(2n+1)$ tickets consecutively numbered, three are drawn at random. Find the chance that the numbers on them are in AP.*

Sol. Let us consider first $(2n+1)$ natural numbers as $(2n+1)$ consecutive numbers.

Let S be the sample space and E be the event of favourable cases.

$$\therefore \quad n(S) = {}^{2n+1}C_3$$

$$= \frac{(2n+1)\,2n(2n-1)}{1\cdot 2\cdot 3} = \frac{n(4n^2-1)}{3}$$

Let the three numbers drawn be a, b, c where $a < b < c$.

Common difference d	Triplets (a, b, c)	Number of Triplets
1	$(1, 2, 3), (2, 3, 4), \ldots, (2n-1, 2n, 2n+1)$	$2n-1$
2	$(1, 3, 5), (2, 4, 6), \ldots, (2n-3, 2n-1, 2n+1)$	$2n-3$
3	$(1, 4, 7), (2, 5, 8), \ldots, (2n-5, 2n-2, 2n+1)$	$2n-5$
...	.....	...
...	.....	...
...	.....	...
$n-1$	$(1, n, 2n-1), (2, n+1, 2n), (3, n+2, 2n+1)$	3
n	$(1, n+1, 2n+1)$	1

$$\therefore \quad n(E) = 1 + 3 + \ldots + (2n-5) + (2n-3) + (2n-1)$$

$$= \frac{n}{2}\{1 + 2n - 1\} = n^2$$

$$\therefore \text{ Required probability, } P(E) = \frac{n(E)}{n(S)}$$

$$= \frac{n^2}{\frac{n(4n^2-1)}{3}} = \frac{3n}{4n^2-1}$$

Aliter Let S be the sample space and E be the event of favourable cases.

$$\therefore \quad n(S) = {}^{2n+1}C_3 = \frac{(2n+1)\,2n(2n-1)}{1\cdot 2\cdot 3} = \frac{n(4n^2-1)}{3}$$

Let the three numbers a, b, c are drawn where $a < b < c$ and given a, b, c are in AP.

$$\therefore \quad b = \frac{a+c}{2} \text{ or } 2b = a + c \qquad \ldots(i)$$

It is clear from Eq. (i), a and c are both odd or both even.

Out of $(2n+1)$ tickets consecutively numbers either $(n+1)$ of them will be odd and n of them will be even (if the numbers begin with an odd number) or $(n+1)$ of them will be even and n of them will be odd (if the number begin with an even number).

$$\therefore \quad n(E) = {}^{n+1}C_2 + {}^{n}C_2$$

$$= \frac{(n+1)\,n}{2} + \frac{n(n-1)}{2} = n^2$$

$\therefore$ Required probability,

$$P(E) = \frac{n(E)}{n(S)} = \frac{n^2}{\frac{n(4n^2-1)}{3}} = \frac{3n}{4n^2-1}$$

● ***Ex. 44*** *Out of $3n$ consecutive integers, three are selected at random. Find the chance that their sum is divisible by 3.*

Sol. Let $3n$ consecutive integers (start with the integer m) are

$$m, m+1, m+2, \ldots, m+3n-1$$

Now, we write these $3n$ numbers in 3 rows as follows :

$$m, m+3, m+6, \ldots, m+3n-3$$

$$m+1, m+4, m+7, \ldots, m+3n-2$$

$$m+2, m+5, m+8, \ldots, m+3n-1$$

The total number of ways of choosing 3 integers out of $3n$ is

$${}^{3n}C_3 = \frac{3n(3n-1)(3n-2)}{1\cdot 2\cdot 3}$$

$$= \frac{n(3n-1)(3n-2)}{2}$$

The sum of the three numbers shall be divisible by 3 if and only if either all the three numbers are from the same row or all the three numbers are from different rows.

Therefore, the number of favourable ways are

$$3({}^{n}C_3) + ({}^{n}C_1)({}^{n}C_1)({}^{n}C_1)$$

$$= \frac{3n(n-1)(n-2)}{1\cdot 2\cdot 3} + n^3 = \frac{3n^3 - 3n^2 + 2n}{2}$$

$\therefore$ The required probability

$$= \frac{\text{Favourable ways}}{\text{Total ways}}$$

$$= \frac{\frac{3n^3 - 3n^2 + 2n}{2}}{\frac{n(3n-1)(3n-2)}{2}}$$

$$= \frac{3n^2 - 3n + 2}{(3n-1)(3n-2)}$$

● ***Ex. 45*** *If $6n$ tickets numbered 0, 1, 2, ..., $6n-1$ are placed in a bag and three are drawn out, show that the chance that the sum of the numbers on them is equal to $6n$ is $\dfrac{3n}{(6n-1)(6n-2)}$.*

Sol. Total number of ways to selecting 3 tickets from $6n$ tickets

$$= {}^{6n}C_3 = n(6n-1)(6n-2) \quad \text{...(i)}$$

For the sum of these tickets of be $6n$, we have the following pattern :

Lowest number	Numbers	Ways
0	$(0, 1, 6n-1), (0, 2, 6n-2) \ldots (0, 3n-1, 3n+1)$	$(3n-1)$
1	$(1, 2, 6n-3), (1, 3, 6n-4) \ldots (1, 3n-1, 3n)$	$(3n-2)$
2	$(2, 3, 6n-5), (2, 4, 6n-6) \ldots (2, 3n-2, 3n)$	$(3n-4)$
3	$(3, 4, 6n-7), (3, 5, 6n-8) \ldots (3, 3n-2, 3n-1)$	$(3n-5)$
4		
5		
:		
$(2n-2)$	$(2n-2, 2n-1, 2n+3), (2n-2, 2n, 2n+2)$	2
$(2n-1)$	$(2n-1, 2n, 2n+1)$	1

Lowest number cannot be greater than $(2n-1)$ as their sum will become $> 6n$.

$\therefore$ Favourable ways $= 1 + 2 + \ldots + (3n-5) + (3n-4) + (3n-2) + (3n-1)$

Adding Ist with last, 2nd with last one, respectively

$$= [1 + 3n - 1] + [2 + 3n - 2] + \ldots \text{upto } n \text{ terms}$$

$$= 3n + 3n + \ldots n \text{ terms}$$

$$= 3n(n) = 3n^2$$

Hence, probability $= \dfrac{3n^2}{n(6n-1)(6n-2)}$

$$= \frac{3n}{(6n-1)(6n-2)}$$

Probability Exercise 1 :
Single Option Correct Type Questions

- This section contains **30 multiple choice questions.** Each question has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct

1. There are two vans each having numbered seats, 3 in the front and 4 at the back. There are 3 girls and 9 boys to be seated in the vans. The probability of 3 girls sitting together in a back row on adjacent seats, is

(a) $\frac{1}{13}$ (b) $\frac{1}{39}$ (c) $\frac{1}{65}$ (d) $\frac{1}{91}$

2. The probability that a year chosen at random has 53 Sundays, is

(a) $\frac{1}{7}$ (b) $\frac{2}{7}$ (c) $\frac{3}{28}$ (d) $\frac{5}{28}$

3. The probability that a leap year selected at random contains either 53 Sundays or 53 Mondays, is

(a) $\frac{1}{7}$ (b) $\frac{2}{7}$ (c) $\frac{3}{7}$ (d) $\frac{4}{7}$

4. A positive integer N is selected so as to be $100 < N < 200$. Then, the probability that it is divisible by 4 or 7, is

(a) $\frac{7}{33}$ (b) $\frac{17}{33}$ (c) $\frac{32}{99}$ (d) $\frac{34}{99}$

5. Two numbers a and b are selected at random from $1, 2, 3, \ldots, 100$ and are multiplied. Then, the probability that the product ab is divisible by 3, is

(a) $\frac{67}{150}$ (b) $\frac{83}{150}$ (c) $\frac{67}{75}$ (d) $\frac{8}{75}$

6. Three different numbers are selected at random from the set $A = \{1, 2, 3, \ldots, 10\}$. The probability that the product of two of the numbers is equal to third, is

(a) $\frac{3}{4}$ (b) $\frac{1}{40}$ (c) $\frac{1}{8}$ (d) $\frac{39}{40}$

7. The numbers $1, 2, 3, \ldots, n$ are arranged in a random order. Then, the probability that the digits $1, 2, 3, \ldots, k (k < n)$ appears as neighbours in that order, is

(a) $\frac{1}{n!}$ (b) $\frac{k!}{n!}$ (c) $\frac{(n-k)!}{n!}$ (d) $\frac{(n-k+1)!}{n!}$

8. The numbers $1, 2, 3, \ldots, n$ are arranged in a random order. Then, the probability that the digits $1, 2, 3, \ldots, k (k < n)$ appears as neighbours, is

(a) $\frac{(n-k)!}{n!}$ (b) $\frac{(n-k+1)}{{}^{n}C_{k}}$

(c) $\frac{(n-k)}{{}^{n}C_{k}}$ (d) $\frac{k!}{n!}$

9. Four identical dice are rolled once. The probability that atleast three different numbers appear on them, is

(a) $\frac{13}{42}$ (b) $\frac{17}{42}$ (c) $\frac{23}{42}$ (d) $\frac{25}{42}$

10. Three of the six vertices of a regular hexagon are chosen at random. The probability that the triangle formed by these vertices is equilateral, is

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{10}$ (d) $\frac{1}{20}$

11. Two small squares on a chess board are chosen at random. Then, the probability that they have a common side, is

(a) $\frac{1}{3}$ (b) $\frac{1}{9}$ (c) $\frac{1}{18}$ (d) $\frac{5}{18}$

12. A letter is known to have come from CHENNAI, JAIPUR, NAINITAL, DUBAI and MUMBAI. On the post mark only two consecutive letters AI are legible. Then the probability that it come from MUMBAI, is

(a) $\frac{42}{319}$ (b) $\frac{84}{403}$ (c) $\frac{39}{331}$ (d) $\frac{42}{331}$

13. Let a die is loaded in such a way that prime number faces are twice as likely to occur as a non-prime number faces. Then, the probability that an odd number will be show up when the die is tossed, is

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{4}{9}$ (d) $\frac{5}{9}$

14. One ticket is selected at random from 100 tickets numbered $00, 01, 02, \ldots, 99$.
Suppose, X and Y are the sum and product of the digit found on the ticket $P(X = 7 / Y = 0)$ is given by

(a) $\frac{2}{3}$ (b) $\frac{2}{19}$ (c) $\frac{1}{50}$ (d) None of these

15. All the spades are taken out from a pack of cards. From these cards, cards are drawn one by one without replacement till the ace of spades comes. The probability that the ace comes in the 4th draw, is

(a) $\frac{1}{13}$ (b) $\frac{12}{13}$

(c) $\frac{4}{13}$ (d) None of these

16. A number is selected at random from the first twenty-five natural numbers. If it is a composite number, then it is divided by 5. But if it is not a composite number, then it is divided by 2. The probability that there will be no remainder in the division, is

(a) $\frac{11}{30}$ (b) 0.4

(c) 0.2 (d) None of these

17. If a bag contains 50 tickets, numbered 1, 2, 3, …, 50 of which five are drawn at random and arranged in ascending order of magnitude $(x_1 < x_2 < x_3 < x_4 < x_5)$. The probability that $x_3 = 30$, is

(a) $\frac{{}^{20}C_2 \times {}^{29}C_2}{{}^{50}C_5}$ (b) $\frac{{}^{20}C_2}{{}^{50}C_5}$

(c) $\frac{{}^{29}C_2}{{}^{50}C_5}$ (d) None of these

18. India play two matches each with West Indies and Australia. In any match the probabilities of India getting points 0, 1 and 2 are 0.45, 0.05 and 0.50, respectively. Assuming that the outcomes are independent, then the probability of India getting atleast 7 points, is

(a) 0.8750 (b) 0.0875 (c) 0.0625 (d) 0.0250

19. Three six faced dice are tossed together, then the probability that exactly two of the three numbers are equal, is

(a) $\frac{165}{216}$ (b) $\frac{177}{216}$ (c) $\frac{51}{216}$ (d) $\frac{90}{216}$

20. Three six-faced fair dice are thrown together. The probability that the sum of the numbers appearing on the dice is $k (3 \le k \le 8)$, is

(a) $\frac{(k-1)(k-2)}{432}$ (b) $\frac{k(k-1)}{432}$

(c) $\frac{k^2}{432}$ (d) None of these

21. A book contains 1000 pages. A page is chosen at random. The probability that the sum of the digits of the marked number on the page is equal to 9, is

(a) $\frac{23}{500}$ (b) $\frac{11}{200}$

(c) $\frac{7}{100}$ (d) None of these

22. A bag contains four tickets numbered 00, 01, 10 and 11. Four tickets are chosen at random with replacement, then the probability that sum of the numbers on the tickets is 23, is

(a) $\frac{3}{32}$ (b) $\frac{1}{64}$ (c) $\frac{5}{256}$ (d) $\frac{7}{256}$

23. Fifteen coupons are numbered 1 to 15. Seven coupons are selected at random, one at a time with replacement. Then, the probability that the largest number appearing on a selected coupon be 9, is

(a) $\left(\frac{1}{15}\right)^7$ (b) $\left(\frac{8}{15}\right)^7$

(c) $\left(\frac{3}{5}\right)^7$ (d) None of these

24. A box contains tickets numbered 1 to 20. 3 tickets are drawn from the box with replacement. The probability that the largest number on the tickets is 7, is

(a) $\frac{7}{20}$ (b) $1 - \left(\frac{7}{20}\right)^3$

(c) $\frac{2}{19}$ (d) None of these

25. An unbiased die with faces marked 1, 2, 3, 4, 5 and 6 is rolled four times. Out of four face values obtained, then the probability that the minimum face value is not less than 2 and the maximum face value is not greater than 5, is

(a) $\frac{16}{81}$ (b) $\frac{1}{81}$ (c) $\frac{80}{81}$ (d) $\frac{65}{81}$

26. A bag contains four tickets marked with numbers 112, 121, 211 and 222. One ticket is drawn at random from the bag. Let $E_i (i = 1, 2, 3)$ denote the event that ith digit on the ticket is 2. Then, which of the following is not true?

(a) E_1 and E_2 are independent
(b) E_2 and E_3 are independent
(c) E_3 and E_1 are independent
(d) E_1, E_2 and E_3 are not independent

27. Two non-negative integers are chosen at random. The probability that the sum of the square is divisible by 10, is

(a) $\frac{17}{100}$ (b) $\frac{9}{50}$ (c) $\frac{7}{50}$ (d) $\frac{9}{16}$

28. Two positive real numbers x and y satisfying $x \le 1$ and $y \le 1$ are chosen at random. The probability that $x + y \le 1$, given that $x^2 + y^2 \le 1/4$, is

(a) $\frac{8-\pi}{16-\pi}$ (b) $\frac{4-\pi}{16-\pi}$

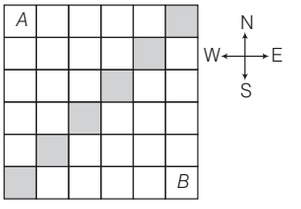
(c) $\frac{4-\pi}{8-\pi}$ (d) None of these

29. If the sides of a triangle are decided by the throw of a single dice thrice, the probability that triangle is of maximum area given that it is an isosceles triangle, is

(a) $\frac{1}{7}$ (b) $\frac{1}{27}$

(c) $\frac{1}{14}$ (d) None of these

30. A and B are persons standing in corner square as shown in the figure. They start to move on same time with equal speed, if A can move only in East or South direction and B can move only in North or West direction. If in each step they reach in next square and their choice of direction are equality. If it is given that A and B meet in shaded region, then the probability that they have met in the top most shaded square, is



(a) $\frac{1}{6}$ (b) $\frac{{}^5C_2}{{}^{10}C_3}$ (c) $\frac{1}{{}^{10}C_5}$ (d) $\frac{1}{2^5 \times 6}$

Probability Exercise 2 :
More than One Correct Option Type Questions

- This section contains **15 multiple choice questions**. Each question has four choices (a), (b), (c) and (d) out of which **MORE THAN ONE** may be correct.

31. For two given events A and B, $P(A \cap B)$ is
(a) not less than $P(A) + P(B) - 1$
(b) not greater than $P(A) + P(B)$
(c) equal to $P(A) + P(B) - P(A \cup B)$
(d) equal to $P(A) + P(B) + P(A \cup B)$

32. If E and F are independent events such that $0 < P(E) < 1$ and $0 < P(F) < 1$, then
(a) E and F are mutually exclusive
(b) E and $\overline{F}$ (complement of the event F) are independent
(c) $\overline{E}$ and $\overline{F}$ are independent
(d) $P(E / F) + P(\overline{E} / F) = 1$

33. For any two events A and B in a sample space:
(a) $P\left(\frac{A}{B}\right) \geq \frac{P(A) + P(B) - 1}{P(B)}$, $P(B) \neq 0$, is always true
(b) $P(A \cap \overline{B}) = P(A) - P(A \cap B)$, does not hold
(c) $P(A \cup B) = 1 - P(\overline{A})\, P(\overline{B})$, if A and B are independent
(d) $P(A \cup B) = 1 - P(\overline{A})\, P(\overline{B})$, if A and B are disjoint

34. Let E and F be two independent events. Then, the probability that both E and F happens is $\frac{1}{12}$ and the probability that neither E nor F happens is $\frac{1}{2}$. Then,
(a) $P(E) = 1/3$, $P(F) = 1/4$ (b) $P(E) = 1/2$, $P(F) = 1/6$
(c) $P(E) = 1/6$, $P(F) = 1/2$ (d) $P(E) = 1/4$, $P(F) = 1/3$

35. If $\overline{E}$ and $\overline{F}$ are the complementary events of events E and F, respectively and if $0 < P(F) < 1$, then
(a) $P(\overline{E} / F) + P(\overline{E} / F) = 1$ (b) $P(E / F) + P(E / \overline{F}) = 1$
(c) $P(\overline{E} / F) + P(E / \overline{F}) = 1$ (d) $P(E / \overline{F}) + P(\overline{E} / \overline{F}) = 1$

36. Let $0 < P(A) < 1$, $0 < P(B) < 1$ and $P(A \cup B) = P(A) + P(B) - P(A)\, P(B)$. Then,
(a) $P(B - A) = P(B) - P(A)$ (b) $P(A' \cup B') = P(A') + P(B')$
(c) $P((A \cup B)') = P(A')\, P(B')$ (d) $P(A / B) = P(A)$

37. If A and B are two events, then the probability that exactly one of them occurs is given by
(a) $P(A) + P(B) - 2P(A \cap B)$
(b) $P(A \cap B') + P(A' \cap B)$
(c) $P(A \cup B) - P(A \cap B)$
(d) $P(A') + P(B') - 2P(A' \cap B')$

38. If A and B are two independent events such that $P(A) = 1/2$ and $P(B) = 1/5$. Then,
(a) $P(A \cup B) = 3/5$ (b) $P(A / B) = 1/2$
(c) $P(A / A \cup B) = 5/6$ (d) $P(A \cap B) / (A' \cup B') = 0$

39. A student appears for tests I, II and III. The student is successful if he passes either in tests I and II or tests I and III. The probabilities of the student passing in tests I, II and III are p, q and $1/2$, respectively. If the probability that the student is successful is $1/2$, then
(a) $p = 1$, $q = 0$ (b) $p = 2/3$, $q = 1/2$
(c) $p = 3/5$, $q = 2/3$ (d) infinitely values of p and q

40. Let X be a set containing n elements. If two subsets A and B of X are picked at random, then the probability that A and B have same number of elements, is
(a) $\frac{{}^{2n}C_n}{2^{2n}}$ (b) $\frac{1}{{}^{2n}C_n}$ (c) $\frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2^n \cdot (n!)}$ (d) $\frac{3^n}{4^n}$

41. Suppose m boys and m girls take their seats randomly around a circle. The probability of their sitting is $({}^{2m-1}C_m)^{-1}$, when
(a) no two boys sit together (b) no two girls sit together
(c) boys and girls sit alternatively
(d) all the boys sit together

42. The probabilities that a student passes in Mathematics, Physics and Chemistry are m, p and c, respectively. In these subjects, the student has a 75% chance of passing in atleast one, a 50% chance of passing in atleast two and a 40% chance of passing in exactly two. Which of the following relations are true?
(a) $p + m + c = 19/20$ (b) $p + m + c = 27/20$
(c) $pmc = 1/10$ (d) $pmc = 1/4$

43. $(n \geq 5)$ persons are sitting in a row. Three of these are selected at random, the probability that no two of the selected persons are sit together, is
(a) $\frac{{}^{n-3}P_2}{{}^{n}P_2}$ (b) $\frac{{}^{n-3}C_2}{{}^{n}C_2}$ (c) $\frac{(n-3)(n-4)}{n(n-1)}$ (d) $\frac{{}^{n-3}C_2}{{}^{n}P_2}$

44. Given that $x \in [0, 1]$ and $y \in [0, 1]$. Let A be the event of (x, y) satisfying $y^2 \leq x$ and B be the event of (x, y) satisfying $x^2 \leq y$, then not true, is
(a) $P(A \cap B) = \frac{1}{3}$
(b) A and B are exhaustive
(c) A and B are mutually exclusive
(d) A and B are independent

45. If the probability of chosing an integer 'k' out of $2n$ integers $1, 2, 3, \ldots, 2n$ is inversely proportional to $k^4 (1 \leq k \leq n)$. If α is the probability that chosen number is odd and β is the probability that chosen number is even, then
(a) $\alpha > \frac{1}{2}$ (b) $\alpha > \frac{2}{3}$ (c) $\beta < \frac{1}{2}$ (d) $\beta < \frac{2}{3}$

Probability Exercise 3 : Passage Based Questions

- This section contains **9 passages**. Based upon each of the passage **3 multiple choice questions** have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I

(Q. Nos. 46 to 48)

If p and q are chosen randomly from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ with replacement.

46. The probability that roots of $x^2 + px + q = 0$ are real and distinct, is

(a) 0.38 (b) 0.03 (c) 0.59 (d) 0.89

47. The probability that roots of $x^2 + px + q = 0$ are equal, is

(a) 0.58 (b) 0.55 (c) 0.38 (d) 0.03

48. The probability that roots of $x^2 + px + q = 0$ are imaginary, is

(a) 0.62 (b) 0.38 (c) 0.59 (d) 0.89

Passage II

(Q. Nos. 49 to 51)

A chess game between two grandmasters X and Y is won by whoever first wins a total of two games. X 's chances of winning, drawing or loosing any particular game are a, b and c, respectively. The games are independent and $a + b + c = 1$.

49. The probability that X wins the match after $(n+1)$ th game $(n \geq 1)$, is

(a) $na^2 b^{n-1}$ (b) $na^2 b^{n-2}(b + (n-1)c)$

(c) $na^2 bc^{n-1}$ (d) $na\, b^{n-1}(b + nc)$

50. The probability that Y wins the match after the 4th game, is

(a) $abc(2a + 3b)$ (b) $bc^2(a + 3b)$

(c) $2ac^2(b + c)$ (d) $3bc^2(2a + b)$

51. The probability that X wins the match, is

(a) $\frac{a^2(a+2c)}{(a+c)^3}$ (b) $\frac{a^3}{(a+c)^3}$ (c) $\frac{a^2(a+3c)}{(a+c)^3}$ (d) $\frac{c^3}{(a+c)^3}$

Passage III

(Q. Nos. 52 to 54)

There are n students in a class. Let $P(E_\lambda)$ be the probability that exactly λ out of n pass the examination. If $P(E_\lambda)$ is directly proportional to $\lambda^2 (0 \leq \lambda \leq n)$.

52. Proportionality constant k is equal to

(a) $\frac{1}{\Sigma n}$ (b) $\frac{1}{\Sigma n^2}$ (c) $\frac{1}{\Sigma n^3}$ (d) $\frac{1}{\Sigma n^4}$

53. If $P(A)$ be the probability that a student selected at random has passed the examination, then $\lim_{x \to \infty} P(A)$, is

(a) 0.25 (b) 0.50

(c) 0.75 (d) 0.35

54. If a selected student has been found to pass the examination, then the probability that he is the only student to have passed the examination, is

(a) $\frac{1}{\Sigma n}$ (b) $\frac{1}{\Sigma n^2}$

(c) $\frac{1}{\Sigma n^3}$ (d) $\frac{1}{\Sigma n^4}$

Passage IV

(Q. Nos. 55 to 57)

A cube having all of its sides painted is cut to be two horizontal, two vertical and other two planes, so as to form 27 cubes all having the same dimensions of these cubes, a cube is selected at random.

55. If P_1 be the probability that the cube selected having atleast one of its sides painted, then the value of $27P_1$, is

(a) 14 (b) 18 (c) 22 (d) 26

56. If P_2 be the probability that the cube selected has two sides painted, then the value of $27P_2$, is

(a) 3 (b) 8 (c) 12 (d) 17

57. If P_3 be the probability that the cube selected has none of its sides painted, then the value of $27P_3$, is

(a) 1 (b) 2

(c) 3 (d) 5

Passage V

(Q. Nos. 58 to 60)

A JEE aspirant estimates that she will be successful with an 80% chance, if she studies 10 h per day with a 60% chance, if she studies 7 h per day and with a 40% chance if she studies 4 h per day. She further believes that she will study 10 h, 7 h and 4 h per day with probabilities 0.1, 0.2, and 0.7, respectively.

58. The probability that she will be successful, is

(a) 0.28 (b) 0.38 (c) 0.48 (d) 0.58

59. Given that she is successful, the chances that she studied for 4 h, is

(a) $\frac{1}{12}$ (b) $\frac{5}{12}$

(c) $\frac{7}{12}$ (d) $\frac{11}{12}$

60. Given that she does not achieve success, the chance that she studied for 4 h, is

(a) $\frac{15}{26}$ (b) $\frac{17}{26}$ (c) $\frac{19}{26}$ (d) $\frac{21}{26}$

Passage VI

(Q. Nos. 61 to 63)

Suppose E_1, E_2 and E_3 be three mutually exclusive events such that $P(E_i) = p_i$ for $i = 1, 2, 3$.

61. If p_1, p_2 and p_3 are the roots of $27x^3 - 27x^2 + ax - 1 = 0$, the value of a is

(a) 3 (b) 6 (c) 9 (d) 12

62. P (none of E_1, E_2 , E_3) equals

(a) 0
(b) $p_1 + p_2 + p_3$
(c) $(1 - p_1)(1 - p_2)(1 - p_3)$
(d) None of the above

63. $P(E_1 \cap \bar{E}_2) + P(E_2 \cap \bar{E}_3) + P(E_3 \cap \bar{E}_1)$ equals

(a) $p_1(1 - p_2) + p_2(1 - p_3) + p_3(1 - p_1)$
(b) $p_1p_2 + p_2p_3 + p_3p_1$
(c) $p_1 + p_2 + p_3$
(d) None of the above

Passage VII

(Q. Nos. 64 to 66)

Let $A = \{1, 2, 3\}$ and $B = \{-2, -1, 0, 1, 2, 3\}$.

64. The probability of increasing functions from A to B, is

(a) $\frac{1}{27}$ (b) $\frac{1}{18}$ (c) $\frac{5}{54}$ (d) $\frac{7}{54}$

65. The probability of non-decreasing functions from A to B, is

(a) $\frac{5}{27}$ (b) $\frac{7}{27}$
(c) $\frac{1}{3}$ (d) $\frac{11}{27}$

66. The probability of onto functions from B to B, such that $f(i) \neq i$, $i = -2, -1, 0, 1, 2, 3$, is

(a) $\frac{53}{144}$ (b) $\frac{35}{144}$ (c) $\frac{29}{72}$ (d) $\frac{25}{72}$

Passage VIII

(Q. Nos. 67 to 69)

A random variable X takes values $0, 1, 2, 3, \ldots$ with probability proportional to $(x+1)\left(\frac{1}{5}\right)^x$.

67. $P(X = 0)$ equals

(a) $\frac{2}{25}$ (b) $\frac{4}{25}$ (c) $\frac{9}{25}$ (d) $\frac{16}{25}$

68. $P(X \geq 2)$ equals

(a) $\frac{11}{25}$ (b) $\frac{13}{25}$ (c) $\frac{11}{125}$ (d) $\frac{13}{125}$

69. The expectation of X i.e., $E(X)$ is equal to

(a) $\frac{1}{4}$ (b) 2 (c) $\frac{1}{2}$ (d) 4

Passage IX

(Q. Nos. 70 to 72)

Let $n = 10\lambda + r$, where $\lambda, r \in N$, $0 \leq r \leq 9$. A number a is chosen at random from the set $\{1, 2, 3, \ldots, n\}$ and let p_n denote the probability that $(a^2 - 1)$ is divisible by 10.

70. If $r = 0$, then np_n equals

(a) 2λ (b) $(\lambda + 1)$
(c) $(2\lambda + 1)$ (d) λ

71. If $r = 9$, then np_n equals

(a) 2λ (b) $2(\lambda + 1)$
(c) $(2\lambda + 1)$ (d) λ

72. If $1 \leq r \leq 8$, then np_n equals

(a) $(2\lambda - 1)$ (b) 2λ
(c) $(2\lambda + 1)$ (d) λ

Probability Exercise 4 : Single Integer Answer Type Questions

- This section contains **10 questions**. The answer to each question is a **single digit integer**, ranging from **0 to 9** (both inclusive).

73. A bag contains $(n + 1)$ coins. It is known that one of these coins shows heads on both sides, whereas the other coins are fair. One coin is selected at random and tossed. If the probability that the toss result in heads is $\frac{7}{12}$, then the value of n is

74. A determinant of the second order is made with the elements 0 and 1. If $\frac{m}{n}$ be the probability that the determinant made is non-negative, where m and n are relative primes, then the value of $n - m$ is

75. Three students appear in an examination of Mathematics. The probabilities of their success are $\frac{1}{3}, \frac{1}{4}$ and $\frac{1}{5}$, respectively. If the probability of success of atleast two is $\frac{\lambda}{12}$, then the value of λ is

76. A die is rolled three times, if p be the probability of getting a large number than the previous number, then the value of $54p$ is

77. In a multiple choice question, there are five alternative answers of which one or more than one are correct. A candidate will get marks on the question, if he ticks all the correct answers. If he decides to tick answers at random, then the least number of choices should he be allowed, so that the probability of his getting marks on the question exceeds $\frac{1}{8}$ is

78. There are n different objects $1, 2, 3, \ldots, n$ distributed at random in n places marked $1, 2, 3, \ldots, n$. If p be the probability that atleast three of the objects occupy places corresponding to their number, then the value of $6p$ is

79. A sum of money is rounded off to the nearest rupee, if $\left(\frac{m}{n}\right)^2$ be the probability that the round off error is atleast ten paise, where m and n are positive relative primes, then the value of $(n - m)$ is

80. A special die is so constructed that the probabilities of throwing 1, 2, 3, 4, 5 and 6 are $(1-k)/6, (1+2k)/6, (1-k)/6, (1+k)/6, (1-2k)/6$ and $(1+k)/6$, respectively. If two such dice are thrown and the probability of getting a sum equal to lies between $\frac{1}{9}$ and $\frac{2}{9}$, then the integral value of k is

81. Seven digits from the numbers 1, 2, 3, 4, 5, 6, 7, 8 and 9 are written in random order. If the probability that this seven-digit number divisible by 9 is p, then the value of $18p$ is

82. 8 players $P_1, P_2, P_3, \ldots, P_8$ play a knock out tournament. It is known that all the players are of equal strength. The tournament is held in three rounds where the players are paired at random in each round. If it is given that P_1 wins in the third round. If p be the probability that P_2 loses in the second round, then the value of $7p$ is

Probability Exercise 5 : Matching Type Questions

- This section contains **6 questions**. Questions 83 to 88 have four statements (A, B C and D) given in **Column I** and four statements (p, q, r and s) in **Column II**. Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II**.

83.

	Column I		**Column II**
(A)	If $P(\overline{A}) = 0.3$, $P(B) = 0.4$ and $P(A\overline{B}) = 0.5$ and $P[B/(A \cup \overline{B})] = \lambda_1$, then $\frac{1}{\lambda_1}$ is	(p)	A prime number
(B)	The coefficient of a quadratic equation $ax^2 + bx + c = 10 (a \neq b \neq c)$ are chosen from first three prime numbers, then the probability that roots of the equation are real is λ_2, then $\frac{1}{\lambda_2}$ is	(q)	A composite number
(C)	A fair coin is tossed repeatedly. If tail appears on first four tosses, then the probability of head appearing on the fifth toss is λ_3, then $\frac{1}{\lambda_3}$ is	(r)	A natural number
(D)	Three persons A, B and C are to speak at a function along with 6 other persons. If the persons speak in random order, then the probability that A speaks before B and B speaks before C is λ_4, then $\frac{1}{\lambda_4}$ is	(s)	A perfect number

84. A and B are two events, such that $P(A) = \frac{3}{5}$ and $P(B) = \frac{2}{3}$

Column I		Column II	
(A)	$P(A \cap B) \in$	(p)	$\left[\frac{2}{3}, 1\right]$
(B)	$P(A \cup B) \in$	(q)	$\left[\frac{4}{9}, 1\right]$
(C)	$P(A / B) \in$	(r)	$\left[\frac{2}{5}, \frac{9}{10}\right]$
(D)	$P(B / A) \in$	(s)	$\left[\frac{4}{15}, \frac{3}{5}\right]$

85. Three players A, B and C alternatively throw a die in that order, the first player to throw a 6 being deemed the winner. A's die is fair whereas B and C throw dice with probabilities p_1 and p_2 respectively, of throwing a 6.

Column I		Column II	
(A)	If $p_1 = \frac{1}{5}$, $p_2 = \frac{1}{4}$ and probability that A wins the game is $\frac{1}{\lambda_1}$, then λ_1 is divisor of	(p)	6
(B)	If $p_1 = \frac{1}{5}$, $p_2 = \frac{1}{4}$ and probability that C wins the game is $\frac{1}{\lambda_2}$, then λ_2 is divisor of	(q)	8
(C)	If $P(A \text{ wins}) = P(B \text{ wins})$ and $\frac{1}{p_1} = \lambda_3$, then λ_3 is divisor of	(r)	12
(D)	If game is equiprobable to all the three players and $\frac{1}{p_1} = \lambda_4$, then λ_4 is divisor of	(s)	15

86. Two numbers a and b are chosen at random from the set $\{1, 2, 3, 4, \ldots, 9\}$ with replacement. The probability that the equation $x^2 + \sqrt{2}(a - b)x + b = 0$ has

Column I		Column II	
(A)	Real and distinct roots is p_1, then the value of $[9p_1]$, where $[\cdot]$ denotes the greatest integer function, is	(p)	2
(B)	Imaginary roots is p_2, then the value of $[9p_2]$, where $[\cdot]$ denotes the greatest integer function, is	(q)	3
(C)	Equal roots is p_3, then the value of $[81p_3]$, where $[\cdot]$ denotes the greatest integer function, is	(r)	4
(D)	Real roots is p_4, then the value of $[9p_4]$, where $[\cdot]$ denotes the greatest integer function, is	(s)	5

87. Three numbers are chosen at random without replacement from the set $\{|x| 1 \le x \le 10, x \in N\}$

Column I		Column II	
(A)	Let p_1 be the probability that the minimum of the chosen numbers is 3 and maximum is 7, then the value of $\frac{2}{5p_1}$, is	(p)	10
(B)	Let p_2 be the probability that the minimum of the chosen numbers is 4 or their maximum is 8, then the value of $80p_2$, is	(q)	14
(C)	Let p_3 be the probability that their minimum is 3, given that their maximum is 7, then the value of $\frac{2}{p_3}$, is	(r)	16
(D)	Let p_4 be the probability that their minimum is 4, given that their maximum is 8, then the value of $\frac{2}{p_4}$, is	(s)	22

88.

Column I		Column II	
(A)	If the integers m and n are chosen at random between 1 and 100, then the probability that a number of the form $7^m + 7^n$ is divisible by 5, is	(p)	$\frac{1}{7}$
(B)	A second order determinant is written down at random using the numbers 1, −1 as elements. The probability that the value of the determinant is non-zero, is	(q)	$\frac{1}{5}$
(C)	The probability of a number n showing in a throw of a die marked 1 to 6 is proportional to n. Then, the probability of the number 3 showing in a throw, is	(r)	$\frac{2}{5}$
(D)	A pair of dice is rolled together till a sum of either 5 or 7 is obtained. Then, the probability that 5 comes before 7, is	(s)	$\frac{1}{2}$

Probability Exercise 6 :
Statement I and II Type Questions

■ **Directions** (Q. Nos. 89 to 100) are Assertion-Reason type questions. Each of these questions contains two statements:
Statement-1 (Assertion) and
Statement-2 (Reason) Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1

(c) Statement1 is true, Statement-2 is false

(d) Statement-1 is false, Statement-2 is true

89. **Statement-1** If 10 coins are thrown simultaneously, then the probability of appearing exactly four heads is equal to probability of appearing exactly six heads.

Statement-2 ${}^{n}C_{r} = {}^{n}C_{s} \Rightarrow$ either $r = s$ or $r + s = n$ and $P(H) = P(T)$ in a single trial.

90. **Statement-1** If A is any event and $P(B) = 1$, then A and B are independent.

Statement-2 $P(A \cap B) = P(A) \cdot P(B)$, if A and B are independent.

91. **Statement-1** If A and B be the events in a sample space, such that $P(A) = 0.3$ and $P(B) = 0.2$, then $P(A \cap \overline{B})$ cannot be found.
Statement-2 $P(A \cap \overline{B}) = P(A) - P(A \cap B)$

92. **Statement-1** Let A and B be two events, such that $P(A \cup B) = P(A \cap B)$, then

$$P(A \cap B') = P(A' \cap B) = 0$$

Statement-2 Let A and B be two events, such that $P(A \cup B) = P(A \cap B)$, then $P(A) + P(B) = 1$

93. A fair die is rolled once.
Statement-1 The probability of getting a composite number is $\frac{1}{3}$, is

Statement-2 There are three possibilities for the obtained number.

(i) The number is prime number.

(ii) The number is a composite number and

(iii) The number is 1.

Hence, probabilities of getting a prime number is $\frac{1}{3}$.

94. From a well shuffled pack of 52 playing cards, a card is drawn at random. Two events A and B are defined as
A : Red card is drawn
B : Card drawn is either a Diamond or Heart
Statement-1 $P(A + B) = P(AB)$
Statement-2 $A \subseteq B$ and $B \subseteq A$

95. **Statement-1** The probability that A and B can solve a problem is $\frac{1}{2}$ and $\frac{1}{3}$ respectively, then the probability that problem will be solved is $\frac{5}{6}$.

Statement-2 Above mentioned events are independent events.

96. **Statement-1** Out of 21 tickets with numbers 1 to 21, 3 tickets are drawn at random, the chance that the numbers on them are in AP is $\frac{10}{133}$.

Statement-2 Out of $(2n + 1)$ tickets consecutively numbered three are drawn at random, the chance that the numbers on them are in AP is $(4n - 10)/(4n^2 - 1)$.

97. **Statement-1** If A and B are two events, such that $0 < P(A)$, $P(B) < 1$, then $P\left(\frac{A}{\overline{B}}\right) + P\left(\frac{\overline{A}}{\overline{B}}\right) = \frac{3}{2}$

Statement-2 If A and B are two events, such that $0 < P(A)$, $P(B) < 1$, then
$P(A / B) = \frac{P(A \cap B)}{P(B)}$ and $P(\overline{B}) = P(A \cap \overline{B}) + P(\overline{A} \cap \overline{B})$

98. In a T-20 tournament, there are five teams. Each team plays one match against every other team.

Each team has 50% chance of winning any game it plays. No match ends in a tie.

Statement-1 The probability that there is an undefeated team in the tournament is $\frac{5}{16}$.

Statement-2 The probability that there is a winless team in the tournament is $\frac{3}{16}$.

99. **Statement-1** If p is chosen at random in the closed interval $[0, 5]$, then the probability that the equation $x^2 + px + \frac{1}{4}(p + 2) = 0$ has real is $\frac{3}{5}$.

Statement-2 If discriminant ≥ 0, then roots of the quadratic equation are always real.

100. Let a sample space S contains n elements. Two events A and B are defined on S and $B \neq \phi$.

Statement-1 The conditional probability of the event A given B, is the ratio of the number of elements in AB divided by the number of elements in B.

Statement-2 The conditional probability model given B, is equally likely model on B.

Probability Exercise 7 : Subjective Type Questions

- In this section, there are **24 subjective** questions.

101. A five digit number is formed by the digits 1, 2, 3, 4 and 5 without repetition. Find the probability that the number formed is divisible by 4.

102. A dice is rolled three times, then find the probability of getting a large number than the previous number.

103. A car is parked among N cars standing in a row but not at either end. On his return, the owner finds that exactly r of the N places are still occupied. What is the probability that both the places neighbouring his car are empty?

104. Two teams A and B play a tournament. The first one to win $(n+1)$ games win the series. The probability that A wins a game is p and that B wins a game is q (no ties). Find the probability that A wins the series.

Hence or otherwise prove that $\sum_{r=0}^{n} {}^{n+1}C_r \cdot \frac{1}{2^{n+r}} = 1$.

105. An artillery target may be either at point I with probability $\frac{8}{9}$ or at point II with probability $\frac{1}{9}$. We have 21 shells each of which can be fired either at point I or II. Each shell may hit the target independently of the other shell with probability $\frac{1}{2}$.

How many shells must be fired at point I to hit the target with maximum probability?

106. There are 6 red and 8 green balls in a bag. 5 balls are drawn at random and placed in a red box. The remaining balls are placed in a green box. What is the probability that the number of red balls in the green box plus the number of green balls in the red box is not a prime number?

107. An urn contains 'a' green and 'b' pink balls $k\ (< a, b)$ balls are drawn and laid a side, their colour being ignored. Then, one more ball is drawn. Find the probability that it is green.

108. A fair coin is tossed 12 times. Find the probability that two heads do not occur consecutively.

109. Given that $x + y = 2a$, where a is constant and that all values of x between 0 and $2a$ are equally likely, then show that the chance that $xy > \frac{3}{4}a^2$, is $\frac{1}{2}$.

110. A chess game between Kamsky and Anand is won by whoever first wins a out of 2 games. Kamsky's chance of winning, drawing or loosing a particular game are 2. The games are independent and $p + q + r = 1$. Prove that the probability that Kamsky wins the match is $\frac{p^2(p+3r)}{(p+r)^3}$.

111. Of three independent events, the chance that only the first occurs is a, the other that only the second occurs is b and the chance of only third occurs is c. Show that the cases of three events are respectively $a/(a+x)$, $b/(b+x)$, $c/(c+x)$, where x is a root of the equation $(a+x)(b+x)(c+x) = x^2$.

112. A is a set containing n elements. A subset P of A is chosen at random and the set A is reconstructed by replacing the elements of P. Another subset Q of A is now chosen at random. Find the probability that $P \cup Q$ contains exactly r elements, with $1 \leq r \leq n$.

113. An electric component manufactured by 'RASU electronics' is tested for its defectiveness by a sophisticated testing device. Let A denote the even ''the device is defective'' and B the event ''the testing device reveals the component to be defective.'' Suppose, $P(A) = \alpha$ and $P(B/A) = P(B'/A') = 1 - \alpha$, where $0 < \alpha < 1$. Show that the probability that the component is not defective, given that the testing device reveals it to be defective is independent of α.

114. A bag contains n white and n red balls. Pairs of balls are drawn without replacement until the bag is empty. Show that the probability that each pair consists of one white and one red ball is $2^n / ({}^{2n}C_n)$.

115. If m things are distributed among 'a' men and 'b' women, then show that the probability that the number of things received by men is odd, is $\frac{1}{2}\frac{\{(b+a)^m - (b-a)^m\}}{(b+a)^m}$.

Probability Exercise 8 :
Questions Asked in Previous 13 Year's Exam

■ This section contains questions asked in **IIT-JEE, AIEEE, JEE Main & JEE Advanced** from year **2005** to year **2017**.

116. A person goes to office either by car, scooter, bus or train. The probability of which being $\frac{1}{7}, \frac{3}{7}, \frac{2}{7}$ and $\frac{1}{7}$, respectively. Probability that he reaches office late, if he takes car, scooter, bus or train is $\frac{2}{9}, \frac{1}{9}, \frac{4}{9}$ and $\frac{1}{9}$, respectively. Given that he reached office in time, then what is the probability that he travelled by a car.
[IIT-JEE 2005, 2M]

117. A six faced fair die is thrown until 1 comes. Then, the probability that 1 comes in even number of trials, is
[IIT-JEE 2005, 3M]
(a) $\frac{5}{11}$ (b) $\frac{5}{6}$ (c) $\frac{6}{11}$ (d) $\frac{1}{6}$

118. Let A and B be two events such that $P(\overline{A \cup B}) = \frac{1}{6}$, $P(A \cap B) = \frac{1}{4}$ and $P(\overline{A}) = \frac{1}{4}$, where $\overline{A}$ stands for complement of event A. Then, events A and B are
[IIT-JEE 2005, 3M]
(a) independent but not equally likely
(b) mutually exclusive and independent
(c) equally likely and mutually exclusive
(d) equally likely but not independent

119. Three houses are available in a locality. Three persons apply for the houses. Each applies for one house without consulting others. The probability that all the three apply for the same house, is **[AIEEE 2005, 3M]**
(a) $\frac{8}{9}$ (b) $\frac{7}{9}$ (c) $\frac{2}{9}$ (d) $\frac{1}{9}$

120. A random variable X has Poisson's distribution with mean 2. Then, $P(X > 1.5)$ is equal to **[AIEEE 2005, 3M]**
(a) $1 - \frac{3}{e^2}$ (b) $\frac{3}{e^2}$ (c) $\frac{2}{e^2}$ (d) 0

121. There are n urns each containing $(n + 1)$ balls such that the ith urn contains i white balls and $(n + 1 - i)$ red balls. Let u_i be the event of selecting ith urn, $i = 1, 2, 3, \ldots, n$ and w denotes the event of getting a white ball.
[IIT-JEE 2006, 5+5+5M]

(i) If $P(u_1) \propto i$, where $i = 1, 2, 3, \ldots, n$, then $\lim_{n \to \infty} P(w)$, is
(a) 1 (b) $\frac{2}{3}$ (c) $\frac{3}{4}$ (d) $\frac{1}{4}$

(ii) If $P(u_i) = c$, where c is a constant, then $P\left(\frac{u_n}{w}\right)$, is
(a) $\frac{2}{n+1}$ (b) $\frac{1}{n+1}$ (c) $\frac{n}{n+1}$ (d) $\frac{1}{2}$

(iii) If n is even and E denotes the event of choosing even numbered urn $\left(P(u_i) = \frac{1}{n}\right)$, then the value of $P\left(\frac{w}{E}\right)$, is
(a) $\frac{n+2}{2n+1}$ (b) $\frac{n+2}{2(n+1)}$ (c) $\frac{n}{n+1}$ (d) $\frac{1}{n+1}$

122. At a telephone enquiry system, the number of phone calls regarding relevant enquiry follow Poisson's distribution with an average of 5 phone calls during 10 min time interval. The probability that there is atmost one phone call during a 10 min time period, is **[AIEEE 2006, 4, 5M]**
(a) $\frac{6}{5^e}$ (b) $\frac{5}{6}$ (c) $\frac{6}{55}$ (d) $\frac{6}{e^5}$

123. One Indian and four American men and their wives are to be seated randomly around a circular table. Then, the conditional probability that the Indian man is seated adjacent to his wife given that each American man is seated adjacent to his wife, is **[IIT-JEE 2007, 3M]**
(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{2}{5}$ (d) $\frac{1}{5}$

124. Let $H_1, H_2, \ldots, H_n$ be mutually exclusive events with $P(H_i) > 0, i = 1, 2, \ldots, n$. Let E be any other event with $0 < P(E) < 1$.

Statement-1 $P(H_i / E) > P(E / H_i)\, P(H_i)$, for $i = 1, 2, \ldots, n$.

Statement-2 $\sum_{i=1}^{n} P(H_i) = 1$
[IIT-JEE 2007, 3M]
(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement-1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

125. Let E^c denote the complement of an event E. Let E, F and G be pairwise independent events with $P(G) > 0$ and $P(E \cap F \cap G) = 0$, then $P(E^c \cap F^c / G)$, is **[IIT-JEE 2007, 3M]**
(a) $P(E^c) + P(F^c)$ (b) $P(E^c) - P(F^c)$
(c) $P(E^c) - P(F)$ (d) $P(E) - P(F^c)$

126. A pair of fair dice is thrown independently three times. Then, the probability of getting a score of exactly 9 twice, is **[AIEEE 2007, 3M]**
(a) $\frac{1}{729}$ (b) $\frac{8}{9}$ (c) $\frac{8}{729}$ (d) $\frac{8}{243}$

127. Two aeroplanes I and II bomb a target in successions. The probabilities of I and II scoring a hit correctly are 0.3 and 0.2, respectively. The second plane will bomb only if the first misses the target. The probability that the target is hit by the second plane, is **[AIEEE 2007, 3M]**
(a) 0.06 (b) 0.14 (c) 0.2 (d) 0.7

128. An experiment has 10 equally likely outcomes. Let A and B be two non-empty events of the experiment. If A consists of 4 outcomes, then the number of outcomes that B must have, so that A and B are independent, is **[IIT-JEE 2008, 3M]**

(a) 2, 4 or 8 (b) 3, 6 or 9 (c) 4 or 8 (d) 5 or 10

129. Consider the system of equations $ax + by = 0$ and $cx + dy = 0$, where $a, b, c, d \in \{0, 1\}$.

Statement-1 The probability that the system of equations has a unique solution is 3/8 and **[IIT-JEE 2008, 3M]**

Statement-2 The probability that the system of equations has a solution is 1.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.

(c) Statement-1 is true, Statement-2 is false.

(d) Statement-1 is false, Statement-2 is true.

130. A die is thrown. Let A be the event that the number obtained is greater than 3. Let B be the event that the number obtained is less than 5. Then $P(A \cup B)$ is **[AIEEE 2008, 3M]**

(a) 0 (b) 1 (c) $\frac{2}{5}$ (d) $\frac{3}{5}$

131. It is given that the events A and B are such that $P(A) = \frac{1}{4}$, $P\left(\frac{A}{B}\right) = \frac{1}{2}$ and $P\left(\frac{B}{A}\right) = \frac{2}{3}$. Then $P(B)$ is **[AIEEE 2008, 3M]**

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{1}{2}$ (d) $\frac{1}{6}$

■ **Passage for Question Nos. 132 to 134**

A fair die is tossed repeatedly until a six is obtained. Let X denote the number of tosses required.

132. The probability that $X = 3$ is

(a) $\frac{25}{216}$ (b) $\frac{25}{36}$ (c) $\frac{5}{36}$ (d) $\frac{125}{216}$

133. The probability that $X \geq 3$ is

(a) $\frac{125}{216}$ (b) $\frac{25}{36}$ (c) $\frac{5}{36}$ (d) $\frac{25}{216}$

134. The conditional probability that $X \geq 6$ given $X > 3$, is

(a) $\frac{125}{216}$ (b) $\frac{25}{216}$ (c) $\frac{5}{36}$ (d) $\frac{25}{36}$ **[IIT-JEE 2009, 4+4+4M]**

135. In a binomial distribution $B\left(n, p = \frac{1}{4}\right)$, if the probability of atleast one success is greater than or equal to $\frac{9}{10}$, then n is greater than **[AIEEE 2009, 4M]**

(a) $\frac{4}{\log_{10} 4 - \log_{10} 3}$ (b) $\frac{1}{\log_{10} 4 - \log_{10} 3}$

(c) $\frac{1}{\log_{10} 4 + \log_{10} 3}$ (d) $\frac{9}{\log_{10} 4 - \log_{10} 3}$

136. One ticket is selected at random from 50 tickets numbered 00, 01, 02, ..., 49. Then, the probability that the sum of the digits on the selected ticket is 8, given that the product of these digits is zero, is **[AIEEE 2009, 4M]**

(a) $\frac{1}{50}$ (b) $\frac{1}{14}$

(c) $\frac{1}{7}$ (d) $\frac{5}{14}$

137. Let ω be a complex cube root of unity with $\omega \neq 1$. A fair die is thrown three times. If r_1, r_2 and r_3 are the numbers obtained on the die, then the probability that $\omega^{r_1} + \omega^{r_2} + \omega^{r_3} = 0$, is **[IIT-JEE 2010, 3M]**

(a) $\frac{1}{18}$ (b) $\frac{1}{9}$ (c) $\frac{2}{9}$ (d) $\frac{1}{36}$

138. A signal which can be green or red with probability $\frac{4}{5}$ and $\frac{1}{5}$ respectively, is received by station A and then transmitted to station B. The probability of each station receiving the signal correctly is $\frac{3}{4}$. If the signal received at station B is green, then the probability that the original signal was green, is **[IIT-JEE 2010, 5M]**

(a) $\frac{3}{5}$ (b) $\frac{6}{7}$ (c) $\frac{20}{23}$ (d) $\frac{9}{20}$

139. Four numbers are chosen at random (without replacement) from the set $\{1, 2, 3, \ldots, 20\}$.

Statement-1 The probability that the chosen numbers, when arranged in some order will form an AP is $\frac{1}{85}$.

Statement-2 If the four chosen number form an AP, then the set of all possible values of common difference is $\{\pm 1, \pm 2, \pm 3, \pm 4, \pm 5\}$. **[AIEEE 2010, 8M]**

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

(b) Statement-1 is true, Statement-2 is false

(c) Statement-1 is false, Statement-2 is true

(d) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1

140. An urn contains nine balls of which three are red, four are blue and two are green. Three balls are drawn at random without replacement from the urn. The probability that the three balls have different colour, is **[AIEEE 2010, 4M]**

(a) $\frac{2}{7}$ (b) $\frac{1}{21}$ (c) $\frac{2}{23}$ (d) $\frac{1}{3}$

■ **Passage for Question Nos. 141 and 142**

Let U_1 and U_2 be two urns such that U_1 contains 3 white balls and 2 red balls and U_2 contains only 1 white ball. A fair coin is tossed. If head appears, then 1 ball is drawn at random from U_1 and put into U_2. However, if tail appears, then 2 balls are drawn at random from U_1 and put into U_2. Now, 1 ball is drawn at random from U_2.

141. The probability of the drawn ball from U_2 being white, is

(a) $\frac{13}{30}$ (b) $\frac{23}{30}$ (c) $\frac{19}{30}$ (d) $\frac{11}{30}$

142. Given that the drawn ball from U_2 is white, then the probability that head appeared on the coin, is **[IIT-JEE 2011, 3+3M]**

(a) $\frac{17}{23}$ (b) $\frac{11}{23}$ (c) $\frac{15}{23}$ (d) $\frac{12}{23}$

143. Let E and F be two independent events. The probability that exactly one of them occurs is $\frac{11}{25}$ and the probability of none of them occurring is $\frac{2}{25}$. If $P(T)$ denotes the probability of occurrence of the event T, then **[IIT-JEE 2011, 4M]**

(a) $P(E)=\frac{4}{5}, P(F)=\frac{3}{5}$ (b) $P(E)=\frac{1}{5}, P(F)=\frac{2}{5}$

(c) $P(E)=\frac{2}{5}, P(F)=\frac{1}{5}$ (d) $P(E)=\frac{3}{5}, P(F)=\frac{4}{5}$

144. Consider 5 independent Bernoulli's trials each with probability of success P. If the probability of atleast one failure is greater than or equal to $\frac{31}{32}$, then P lies in the interval **[AIEEE 2011, 4M]**

(a) $\left(\frac{3}{4}, \frac{11}{12}\right]$ (b) $\left[0, \frac{1}{2}\right]$ (c) $\left(\frac{11}{12}, 1\right]$ (d) $\left(\frac{1}{2}, \frac{3}{4}\right]$

145. If C and D are two events, such that $C \subset D$ and $P(D) \neq 0$, then the correct statement among the following, is **[AIEEE 2011, 4M]**

(a) $P\left(\frac{C}{D}\right) \geq P(C)$ (b) $P\left(\frac{C}{D}\right) < P(C)$

(c) $P\left(\frac{C}{D}\right) = \frac{P(D)}{P(C)}$ (d) $P\left(\frac{C}{D}\right) = P(C)$

146. Let A, B and C are pairwise independent events with $P(C) > 0$ and $P(A \cap B \cap C) = 0$. Then, $P\left(\frac{(A^c \cap B^c)}{C}\right)$ is

(a) $P(A^c) - P(B)$ (b) $P(A) - P(B^c)$

(c) $P(A^c) + P(B^c)$ (d) $P(A^c) - P(B^c)$

147. A ship is fitted with three engines E_1, E_2 and E_3. The engines function independently of each other with respective probabilities $\frac{1}{2}, \frac{1}{4}$ and $\frac{1}{4}$, respectively. For the ship to be operational atleast two of its engines must function. Let X denote the event that the ship is operational and let X_1, X_2 and X_3 denote respectively the events that the engines E_1, E_2 and E_3 are functioning. Which of the following is (are) true? **[IIT-JEE 2012, 4M]**

(a) $P[X_1^c / X] = \frac{3}{16}$

(b) P [exactly two engines of the ship are functioning $/X] = \frac{7}{8}$

(c) $P[X / X_2] = \frac{5}{16}$

(d) $P[X / X_1] = \frac{7}{16}$

148. Four fair dice D_1, D_2, D_3 and D_4 each having six faces numbered 1, 2, 3, 4, 5 and 6 are rolled simultaneously. The probability that D_4 shows a number appearing on one of D_1, D_2 and D_3, is **[IIT-JEE 2012, 3M]**

(a) $\frac{91}{216}$ (b) $\frac{108}{216}$ (c) $\frac{25}{216}$ (d) $\frac{127}{216}$

149. Let X and Y be two events, such that $P(X / Y) = \frac{1}{2}$, $P(Y / X) = \frac{1}{3}$ and $P(X \cap Y) = \frac{1}{6}$. Which of the following is (are) correct? **[IIT-JEE 2012, 4M]**

(a) $P(X \cup Y) = \frac{2}{3}$ (b) X and Y are independent

(c) X and Y are not independent

(d) $P(X^c \cap Y) = \frac{1}{3}$

150. Three numbers are chosen at random without replacement from {1, 2, 3, …, 8}. The probability that their minimum is 3, given that their maximum is 6, is **[AIEEE 2012, 4M]**

(a) $\frac{1}{4}$ (b) $\frac{2}{5}$ (c) $\frac{3}{8}$ (d) $\frac{1}{5}$

151. A multiple choice examination has 5 questions. Each question has three alternative answers of which exactly one is correct. The probability that a student will get 4 or more correct answers just by guessing, is **[JEE Main 2013, 4M]**

(a) $\frac{13}{3^5}$ (b) $\frac{11}{3^5}$ (c) $\frac{10}{3^5}$ (d) $\frac{17}{3^5}$

152. Four persons independently solve a certain problem correctly with probabilities $\frac{1}{2}, \frac{3}{4}, \frac{1}{4}$ and $\frac{1}{8}$. Then, the probability that the problem is solved correctly by atleast one of them, is **[JEE Advanced 2013, 2M]**

(a) $\frac{235}{256}$ (b) $\frac{21}{256}$ (c) $\frac{3}{256}$ (d) $\frac{253}{256}$

153. Of the three independent events E_1, E_2 and E_3, the probability that only E_1 occurs α, only E_2 occurs is β and only E_3 occurs is γ. Let the probability p that none of events E_1, E_2 or E_3 occurs satisfy the equation $(\alpha - 2\beta)p = \alpha\beta$ and $(\beta - 3\gamma)p = 2\beta\gamma$. All the given probabilities are assumed to lie in the interval $(0, 1)$.

Then, $\dfrac{\text{Probability of occurrence of } E_1}{\text{Probability of occurrence of } E_3}$ is

[JEE Advanced 2013, 4M]

■ **Passage for Question Nos. 154 and 155**

A box B_1 contains 1 white ball, 3 red balls and 2 black balls. Another box B_2 contains 2 white balls, 3 red balls and 4 black balls. A third box B_3 contains 3 white balls, 4 red balls and 5 black balls.

154. If 2 balls are drawn (without replacement) from a randomly selected box and one of the balls is white and the other ball is red, then the probability that these 2 balls are drawn from box B_2 , is

(a) $\dfrac{116}{181}$ (b) $\dfrac{126}{181}$ (c) $\dfrac{65}{181}$ (d) $\dfrac{55}{181}$

155. If 1 ball is drawn from each of the boxes B_1, B_2 and B_3, then the probability that all 3 drawn balls are of the same colour, is **[JEE Advanced 2013, 3+3M]**

(a) $\dfrac{82}{648}$ (b) $\dfrac{90}{648}$ (c) $\dfrac{558}{648}$ (d) $\dfrac{566}{648}$

156. Let A and B be two events, such that $P(\overline{A \cup B}) = \dfrac{1}{6}$, $P(A \cap B) = \dfrac{1}{4}$ and $P(\overline{A}) = \dfrac{1}{4}$, where $\overline{A}$ stands for the complement of the event A. Then, the events A and B are

(a) independent but not equally likely **[JEE Main 2014, 4M]**
(b) independent and equally likely
(c) mutually exclusive and independent
(d) equally likely but not independent

157. Three boys and two girls stand in a queue. The probability that the number of boys ahead of every girl is atleast one more than the number of girls ahead of her, is **[JEE Advanced 2014, 3M]**

(a) $\dfrac{1}{2}$ (b) $\dfrac{1}{3}$ (c) $\dfrac{2}{3}$ (d) $\dfrac{3}{4}$

■ **Passage for Question Nos. 158 and 159**

Box 1 contains three cards bearing numbers 1, 2, 3, box 2 contains five cards bearing numbers 1, 2, 3, 4, 5 and box 3 contains seven cards bearing numbers 1, 2, 3, 4, 5, 6, 7. A card is drawn from each of the boxes. Let x_i be the number on the card drawn from the ith box, $i = 1, 2, 3$.

158. The probability that $x_1 + x_2 + x_3$ is odd, is

(a) $\dfrac{29}{105}$ (b) $\dfrac{53}{105}$ (c) $\dfrac{57}{105}$ (d) $\dfrac{1}{2}$

159. The probability that x_1, x_2 and x_3 are in arithmetic progression, is **[JEE Advanced 2014, 3+3M]**

(a) $\dfrac{9}{105}$ (b) $\dfrac{10}{105}$ (c) $\dfrac{11}{105}$ (d) $\dfrac{7}{105}$

160. If 12 identical balls are to be placed in 3 identical boxes, then the probability that one of the boxes contains exactly 3 balls, is **[JEE Main 2015, 4M]**

(a) $220\left(\dfrac{1}{3}\right)^{12}$ (b) $22\left(\dfrac{1}{3}\right)^{11}$ (c) $\dfrac{55}{3}\left(\dfrac{2}{3}\right)^{11}$ (d) $55\left(\dfrac{2}{3}\right)^{10}$

161. The minimum number of times a fair coin needs to be tossed, so that the probability of getting atleast two heads is atleast 0.96, is **[JEE Advanced 2015, 4M]**

■ **Passage for Question Nos. 162 and 163**

Let n_1 and n_2 be the number of red and black balls respectively, in box I. Let n_3 and n_4 be the number of red and black balls respectively, in box II.

162. One of the two boxes, box I and box II, was selected at random and a ball was drawn randomly out of this box. The ball was found to be red. If the probability that this red ball was drawn from box II is $1/3$, then the correct option(s) with the possible values of n_1, n_2, n_3 and n_4 is (are)

(a) $n_1 = 3, n_2 = 3, n_3 = 5, n_4 = 15$ (b) $n_1 = 3, n_2 = 6, n_3 = 10, n_4 = 50$
(c) $n_1 = 8, n_2 = 6, n_3 = 5, n_4 = 20$ (d) $n_1 = 6, n_2 = 12, n_3 = 5, n_4 = 20$

163. A ball is drawn at random from box I and transferred to box II. If the probability of drawing a red ball from box I, after this transfer is $\dfrac{1}{3}$, then correct option(s) with possible values of n_1 and n_2 is (are) **[JEE Advanced 2015, 4+4M]**

(a) $n_1 = 4$ and $n_2 = 6$ (b) $n_1 = 2$ and $n_2 = 3$
(c) $n_1 = 10$ and $n_2 = 20$ (d) $n_1 = 3$ and $n_2 = 6$

164. Let two fair six-faced dice A and B be thrown simultaneously. If E_1 is the event that die A shows up four, E_2 is the event that die B shows up two and E_3 is the event that the sum of numbers on both dice is odd, then which of the following statements is NOT true? **[JEE Main 2016, 4M]**

(a) E_2 and E_3 are independent (b) E_1 and E_3 are independent
(c) E_1, E_2 and E_3 are independent (d) E_1 and E_2 are independent

165. A computer producing factory has only two plants T_1 and T_2. Plant T_1 produces 20% and plant T_2 produces 80% of the total computers produced 7% of computers produced in the factory turn out to be defective. It is known that P (computer terms out to be defective given that it is produced in plant T_1) = $10P$ (computer terms out to be defective given that it is produced in plant T_2), when $P(E)$ denotes the probability of an event E. A computer produced in the factory is randomly selected and it does not turn out to be defective. Then, the probability that it is produced in plant T_2 is

[JEE Advanced 2016, 3M]

(a) $\dfrac{36}{73}$ (b) $\dfrac{47}{79}$ (c) $\dfrac{78}{93}$ (d) $\dfrac{75}{83}$

■ **Passage for Question Nos. 166 and 167**

Football teams T_1 and T_2 have to play two games against each other. It is assumed that the outcomes of the two games are independent. The probabilities of T_1 winning, drawing and losing a game against T_2 are $\frac{1}{2}, \frac{1}{6}$ and $\frac{1}{3}$ respectively. Each team gets 3 points for a win, 1 point for a draw and 0 point for a loss in a game. Let X and Y denote the total points scored by teams T_1 and T_2 respectively, after two games. **[JEE Advanced 2016, 3+3M]**

(a) $\frac{1}{4}$ (b) $\frac{5}{12}$ (c) $\frac{1}{2}$ (d) $\frac{7}{12}$

167. $P(X = Y)$ is

(a) $\frac{11}{36}$ (b) $\frac{1}{3}$ (c) $\frac{13}{36}$ (d) $\frac{1}{2}$

168. A box contains 15 green and 10 yellow balls. If 10 balls are randomly drawn, one-by-one with replacement, then the variance of the number of green balls drawn is **[JEE Main 2017, 4M]**

(a) $\frac{6}{25}$ (b) $\frac{12}{5}$ (c) 6 (d) 4

169. If two different numbers are taken from the set $\{0, 1, 2, 3, \ldots, 10\}$, then the probability that their sum as well as absolute difference are both multiple of 4, is **[JEE Main 2017, 4M]**

(a) $\frac{7}{55}$ (b) $\frac{6}{55}$ (c) $\frac{12}{55}$ (d) $\frac{14}{45}$

170. For three events A, B and C.

P(Exactly one of A or B or C occurs)

= P(Exactly one of B or C occurs)

= P(Exactly one of C or A occurs) = $\frac{1}{4}$

and P (All the three events occur simultaneously) = $\frac{1}{16}$,

Then the probability that atleast one of the events occurs, is **[JEE Main 2017, 4M]**

(a) $\frac{3}{16}$ (b) $\frac{7}{32}$

(c) $\frac{7}{16}$ (d) $\frac{7}{64}$

Answers

Exercise for Session 1

1. (a) 2. (a) 3. (d) 4. (b) 5. (a) 6. (c)
7. (b) 8. (d) 9. (d) 10. (d) 11. (c) 12. (c)
13. (d) 14. (d) 15. (b) 16. (b) 17. (a) 18. (a)
19. (c) 20. (c)

Exercise for Session 2

1. (c) 2. (d) 3. (b) 4. (c) 5. (b) 6. (b)
7. (c) 8. (a) 9. (a) 10. (b) 11. (c) 12. (a)
13. (b) 14. (b) 15. (b)

Exercise for Session 3

1. (d) 2. (d) 3. (c) 4. (d) 5. (d) 6. (b)
7. (d) 8. (c) 9. (c) 10. (a)

Exercise for Session 4

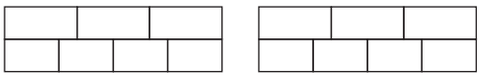
1. (c) 2. (d) 3. (d) 4. (a) 5. (c) 6. (a)
7. (a) 8. (d) 9. (c) 10. (c) 11. (c) 12. (b)
13. (c) 14. (c) 15. (a)

Chapter Exercise

1. (d) 2. (d) 3. (c) 4. (d) 5. (b) 6. (b)
7. (d) 8. (b) 9. (d) 10. (c) 11. (c) 12. (b)
13. (d) 14. (b) 15. (a) 16. (c) 17. (a) 18. (b)
19. (d) 20. (a) 21. (b) 22. (a) 23. (c) 24. (d)
25. (a) 26. (d) 27. (b) 28. (a) 29. (b) 30. (c)
31. (a,b,c) 32. (b,c,d) 33. (a,c) 34. (a,d) 35. (a,b) 36. (c, d)
37. (a,b,c,d) 38. (a,b,c,d) 39. (a,b,c) 40. (a,c)
41. (a,b,c) 42. (b,c) 43. (a,b,c,d) 44. (b,c,d)
45. (a,c) 46. (c) 47. (d) 48. (b) 49. (b) 50. (d)
51. (c) 52. (b) 53. (c) 54. (c) 55. (d) 56. (c)
57. (a) 58. (c) 59. (c) 60. (d) 61. (c) 62. (a)
63. (c) 64. (c) 65. (b) 66. (a) 67. (d) 68. (d)
69. (c) 70. (a) 71. (b) 72. (c) 73. (5) 74. (3)
75. (2) 76. (5) 77. (4)
78. (1) 79. (1) 80. (0) 81. (2) 82. (2)
83. (A) → (q,r);(B) →(p,r);(C) → (p,r);(D) → (q,r,s)
84. (A) → (s);(B) → (p);(C) → (r);(D) → (q)
85. (A) → (p,r,s);(B) → (p,r,s);(C) → (s);(D) → (q,r)
86. (A) → (q);(B) → (r);(C) → (p);(D) → (s)
87. (A) → (r);(B) → (s);(C) → (p);(D) → (q)
88. (A) → (q);(B) → (s);(C) → (p);(D) → (r)
89. (a) 90. (a) 91. (a) 92. (c) 93. (c)
94. (a) 95. (d) 96. (c) 97. (d) 98. (c)
99. (a) 100. (a)
101. $\left(\frac{1}{5}\right)$ 102. $\left(\frac{5}{54}\right)$ 103. $\left(\frac{(N-r)(N-r-1)}{(N-1)(N-2)}\right)$
105. (12) 106. $\left(\frac{213}{1001}\right)$ 107. $\left(\frac{a}{a+b}\right)$
108. $\left(\frac{377}{4096}\right)$ 109. $\left(\frac{1}{2}\right)$ 112. $\left(\frac{{}^nC_r\, 3^r}{4^n}\right)$
116. $\frac{1}{7}$ 117. (a) 118. (a) 119. (d) 120. (a)
121. (i) (b), (ii) (a), (iii) (b) 122. (d) 123. (c) 124. (d)
125. (c) 126. (d) 127. (b) 128. (d) 129. (b) 130. (b)
131. (a) 132. (a) 133. (b) 134. (d) 135. (b) 136. (b)
137. (c) 138. (c) 139. (b) 140. (a) 141. (b) 142. (d)
143. (a, d) 144. (b) 145. (a) 146. (a) 147. (b,d) 148. (a)
149. (a,b) 150. (d) 151. (b) 152. (a) 153. (b) 154. (d)
155. (a) 156. (a) 157. (a) 158. (b) 159. (b) 160. (c)
161. (8) 162. (a,b) 163. (c,d) 164. (c) 165. (c) 166. (b)
167. (c) 168. (b) 169. (c) 170. (c)

Solutions

1.



$n(S)$ = Number of total ways = ${}^{14}P_{12} = \frac{14!}{2!} = 7 \times 13!$

The girls can be seated together in the back seats leaving a corner seat in $4 \times 3! = 24$ ways and the boys can be seated in the remaining 11 seats in

${}^{11}P_9 = \frac{11!}{2!} = \frac{1}{2} \times 11!$ ways

$\therefore$ $n(E)$ = Number of favourable ways $= 24 \times \frac{1}{2} \times 11! = 12!$

The required probability $= \frac{n(E)}{n(S)} = \frac{12!}{7 \times 13!} = \frac{1}{91}$

2. **For non-leap year**

The probability of 53 Sundays $= \frac{1}{7}$

For leap year

The probability of 53 Sundays $= \frac{2}{7}$

$\therefore$ Required probability $= \frac{3}{4} \times \frac{1}{7} + \frac{1}{4} \times \frac{2}{7} = \frac{5}{28}$

3. Let us consider two events:

A : The leap year contains 53 Sundays.

B : The leap year contains 53 Mondays. We have,

$P(A) = \frac{2}{7}$, $P(B) = \frac{2}{7}$ and $P(A \cap B) = \frac{1}{7}$

$\therefore$ Required probability $= P(A \cup B)$

$= P(A) + P(B) - P(A \cap B)$

$= \frac{2}{7} + \frac{2}{7} - \frac{1}{7} = \frac{3}{7}$

4. Let us consider two events:

A : Numbers divisible by 4.

B : Numbers divisible by 7.

We have, $A = \{104, 108, \ldots, 196\}$

$\Rightarrow$ $n(A) = 24$

$B = \{105, 112, \ldots, 196\}$

$n(B) = 14$ and $A \cap B = \{112, 140, 168, 196\}$

$\Rightarrow$ $n(A \cap B) = 4$

$n(E)$ = Number of favourable ways $n(A \cup B)$

$= n(A) + n(B) - n(A \cap B) = 34$

$n(S)$ = Total number of ways $= 99$

$\therefore$ Required probability $= \frac{n(E)}{n(S)} = \frac{34}{99}$

5. $n(S)$ = Total number of ways = ${}^{100}C_2 = 50 \times 99$

The product is divisible by 3, if atleast one of the two numbers is divisible by 3.

Let $n(E)$ = Number of ways, if atleast one of the two numbers is divisible by 3.

and $n(\bar{E})$ = Number of ways, if none of the two numbers chosen is divisible by 3.

$= {}^{67}C_2 = \frac{67 \times 66}{1 \times 2} = 67 \times 33$

$\therefore$ Required probability $= \frac{n(E)}{n(S)} = \frac{n(S) - n(\bar{E})}{n(S)}$

$= 1 - \frac{67 \times 33}{50 \times 99} = \frac{83}{150}$

6. $n(S)$ = Total number of ways = ${}^{10}C_3 = \frac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3} = 120$

The product of two numbers is equal to third number, the favourable cases are 2, 3, 6; 2, 4, 8; 2, 5, 10

$\therefore$ $n(E)$ = The number of favourable cases = 3

$\therefore$ Required probability $= \frac{n(E)}{n(S)} = \frac{3}{120} = \frac{1}{40}$

7. $n(S)$ = Total number of ways = ${}^{n}P_n = n!$

Considering digits 1, 2, 3, 4, …, k as one digit, we have $(n - k + 1)$ digits which can be arranged $= (n - k + 1)!$

$\therefore$ $n(E)$ = Number of favourable ways $= (n - k + 1)!$

Hence, required probability $= \frac{n(E)}{n(S)} = \frac{(n - k + 1)!}{n!}$

8. $n(S)$ = Total number of ways = ${}^{n}P_n = n!$

The number of ways in which the digits 1, 2, 3, 4, …, k $(k < n)$ occur together $= k!(n - k + 1)!$

Hence, required probability $= \frac{n(E)}{n(S)} = \frac{k!(n - k + 1)!}{n!} = \frac{(n - k + 1)}{{}^{n}C_k}$

9. Let a, b, c and d are four different numbers out of {1, 2, 3, 4, 5, 6}.

$\Rightarrow$ (a, a, a, a) can appear in ${}^{6}C_1 = 6$ ways

$\Rightarrow$ (a, a, a, b) can appear in $2 \times {}^{6}C_2 = 30$ ways

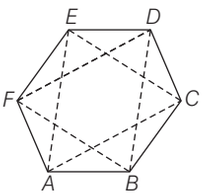
$\Rightarrow$ (a, a, b, b) can appear in ${}^{6}C_2 = 15$ ways

$\Rightarrow$ (a, a, b, c) can appear in $3 \times {}^{6}C_3 = 60$ ways

$\Rightarrow$ (a, b, c, d) can appear in ${}^{6}C_4 = 15$ ways

$\therefore$ Required probability $= \frac{60 + 15}{6 + 30 + 15 + 60 + 15} = \frac{75}{126} = \frac{25}{42}$

10. Let $ABCDEF$ be the regular hexagon.



Number of total triangles $= {}^{6}C_3 = 20$ ways

For the favourable event, the vertices should be either A, C, E or B, D, F

$\therefore$ The required probability $= \dfrac{\text{Favourable ways}}{\text{Total ways}} = \dfrac{2}{20} = \dfrac{1}{10}$

11. Total number of ways to choose two squares

$$= {}^{64}C_2 = \frac{64 \cdot 63}{2} = 32 \cdot 63$$

For favourable ways we must chosen two consecutive small squares for any row or any columns.

$\therefore$ Number of favourable ways $= 7 \cdot 8 + 8 \cdot 7 = 2 \cdot 8 \cdot 7$

$\therefore$ Required probability $= \dfrac{2 \cdot 8 \cdot 7}{32 \cdot 63} = \dfrac{1}{18}$

12. In the word MUMBAI, there are 5 adjacent pairs of letters of which only one gives AI.

$$\therefore \text{ Required probability} = \frac{\frac{1}{5}}{\frac{1}{6}+\frac{1}{5}+\frac{1}{7}+\frac{1}{4}+\frac{1}{5}} = \frac{84}{403}$$

13. Numbers on die are 1, 2, 3, 4, 5, 6.

Prime numbers are 2, 3, 5 and non-prime numbers are 1, 4, 6.

Now, let weight assigned to non-prime numbers is λ, then weight assigned to prime number is 2λ.

$\therefore \lambda + 2\lambda + 2\lambda + \lambda + 2\lambda + \lambda = 1$

$$\Rightarrow \qquad \lambda = \frac{1}{9}$$

$\therefore$ Probability that an odd number will be show up when the die is tossed 1 or 3 or 5.

$$\lambda + 2\lambda + 2\lambda = 5\lambda = \frac{5}{9}$$

14. We have,

$$(X = 7) = \{07, 16, 25, 34, 43, 52, 61, 70\}$$

and $\quad (Y = 0) = \{00, 01, 02, \ldots, 10, 20, 30, \ldots, 90\}$

Thus, $\quad (X = 7) \cap (Y = 0) = \{07, 70\}$

$$\therefore \qquad P\left(\frac{X=7}{Y=0}\right) = \frac{P\{(X=7) \cap (Y=0)\}}{P(Y=0)} = \frac{2}{19}$$

15. The probability of not drawing the ace in the first draw, in the second draw and in the third draw are (here all spades i.e., 13 cards) $\dfrac{12}{13}, \dfrac{11}{12}, \dfrac{10}{11}$, respectively.

Probability of drawing ace of spades in the 4th draw

$$= \frac{1}{10} \text{ (only one ace and remaining cards = 10)}$$

$\therefore$ Required probability $= \dfrac{12}{13} \times \dfrac{11}{12} \times \dfrac{10}{11} \times \dfrac{1}{10} = \dfrac{1}{13}$

16. $n(S) =$ Total number of ways $= {}^{25}C_1 = 25$

Set of composite numbers $= \{4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25\}$ and set of non-composite numbers $= \{1, 2, 3, 5, 7, 11, 13, 17, 19, 23\}$

Now, set of composite numbers of the form $5k\,(k \in N)$ $= \{10, 15, 20, 25\}$

and set of non-composite numbers of the form $2k\,(k \in N) = \{2\}$

$$\therefore \text{ Required prabability} = \frac{{}^4C_1 + {}^1C_1}{{}^{25}C_1} = \frac{5}{25} = 0.2$$

17. $n(S) =$ Total number of ways $= {}^{50}C_5$

Now, x_3 is fixed to be 30 and x_1, x_2 (two numbers) are to be chosen from first 29 numbers and x_4, x_5 (two numbers) from last 20 numbers are to be chosen.

$\therefore \quad n(E) =$ Number of favourable ways $= {}^{29}C_2 \times {}^{20}C_2$

Hence, required probability $= \dfrac{n(E)}{n(S)} = \dfrac{{}^{29}C_2 \times {}^{20}C_2}{{}^{50}C_5}$

18. The points are 2, 2, 2, 2 or 2, 2, 2, 1

$\therefore$ Required probability

$$= (0.5)^4 + {}^4C_1 \times (0.5)^3 \times (0.05)^1 = 0.0875$$

19. $n(S) =$ Total number of ways $= 6^3 = 216$

$n(E) =$ Number of favourable ways $= 2 \times {}^6C_2 \times \dfrac{3!}{2!} = 90$

$\therefore$ Required probability $= \dfrac{n(E)}{n(S)} = \dfrac{90}{216}$

20. $n(S) =$ Total number of ways $= 6 \times 6 \times 6 = 216$

$n(E) =$ Number of favourable cases

$= $ Coefficient of x^k in $(x + x^2 + x^3 + x^4 + x^5 + x^6)^3$

$= $ Coefficient of x^{k-3} in $(1 + x + x^2 + x^3 + x^4 + x^5)^3$

$= $ Coefficient of x^{k-3} in $(1 - x^6)^3 (1 - x)^{-3}$

$= $ Coefficient of x^{k-3} in $(1 - x)^{-3}$ $\qquad [\because\ 0 \le k - 3 \le 5]$

$= $ Coefficient of x^{k-3} in $(1 + {}^3C_1 x + \ldots)$

$= {}^{k-1}C_{k-3} = {}^{k-1}C_2 = \dfrac{(k-1)(k-2)}{2}$

$\therefore$ Required probability $= \dfrac{n(E)}{n(S)} = \dfrac{(k-1)(k-2)}{432}$

21. $n(S) =$ Total number of ways $= 1000$

The favourable cases that the sum of the digits of the marked number on the page is equal to 9 are one digit number or two digits numbers or three digits numbers, if three digit number is abc. Then, $a + b + c = 9;\ 0 \le a, b, c \le 9$

$\therefore \qquad n(E) =$ Number of favourable ways

$=$ Number of solutions of the equation

$= {}^{9+3-1}C_{3-1} = {}^{11}C_2 = 55$

$\therefore$ Required probability $= \dfrac{n(E)}{n(S)} = \dfrac{55}{1000} = \dfrac{11}{200}$

22. $n(S) =$ The total number of ways of choosing the tickets

$= 4 \times 4 \times 4 \times 4 = 256$

$n(E) =$ The number of ways in which the sum can be 23

$= $ Coefficient of x^{23} in $(1 + x + x^{10} + x^{11})^4$

$= $ Coefficient of x^{23} in $(1 + x^4) + (1 + x^{10})^4$

$= $ Coefficient of x^{23} in $(1 + 4x + 6x^2 + 4x^3 + x^4) \times (1 + 4x^{10} + 6x^{20})$

$= 4 \times 6 = 24$

The probability of required event $= \dfrac{n(E)}{n(S)} = \dfrac{24}{256} = \dfrac{3}{32}$

23. Total coupons $= 15$

$1 \le$ selected coupon number ≤ 9 i.e., 1, 2, 3, 4, 5, 6, 7, 8, 9

$\therefore$ Probability of one selected coupon $= \frac{9}{15} = \frac{3}{5}$

Hence, the required probability

$$= \left(\frac{3}{5}\right) \times \left(\frac{3}{5}\right) \times \ldots \times 7 \text{ times} = \left(\frac{3}{5}\right)^7$$

24. Let X denote the largest number on the 3 tickets drawn.

We have, $P(X \leq 7) = \left(\frac{7}{20}\right)^3$ and $P(X \leq 6) = \left(\frac{6}{20}\right)^3$

Thus, $P(X = 7) = P(X \leq 7) - P(X \leq 6) = \left(\frac{7}{20}\right)^3 - \left(\frac{6}{20}\right)^3$

25. Total number = 6 (i.e., 1, 2, 3, 4, 5, 6)

Favourable number = 2, 3, 4, 5 = 4

Probability of favourable number in one draw $= \frac{4}{6} = \frac{2}{3}$

$\therefore$ Required probability $= \left(\frac{2}{3}\right)^4 = \frac{16}{81}$

26. We have, $P(E_i) = \frac{2}{4} = \frac{1}{2}$ for $i = 1, 2, 3$

Also, for $i \neq j$, $P(E_i \cap E_j) = \frac{1}{4} = P(E_i)\, P(E_j)$

Therefore, E_i and E_j are independent for $i \neq j$.

Also, $P(E_1 \cap E_2 \cap E_3) = \frac{1}{4} \neq P(E_1) P(E_2) P(E_3)$

$\therefore$ E_1, E_2 and E_3 are not independent.

27. Let x and y are two non-negative integers are chosen such that $x^2 + y^2$ is divisible by 10.

By the division algorithm, there exist integers x_1, y_1, a_1 and b_1 such that $x = 10x_1 + a_1$ and $y = 10y_1 + b_1$ with $0 \leq a, b_1 \leq 9$.

Thus, we can write

$x^2 + y^2 = 100(x_1^2 + y_1^2) + 20(a_1x_1 + b_1y_1) + (a_1^2 + b_1^2)$

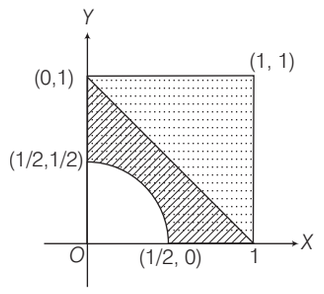
We see that $x^2 + y^2$ will be divisible by 10 if and only if $a_1^2 + b_1^2$ is divisible by 10. Now, there are 10 choices each for a_1 and b_1, so that there are $10 \times 10 = 100$ ways of choosing them. The pairs (a_1, b_1) for which $a_1^2 + b_1^2$ is divisible by 10 are follows:

(0, 0), (1, 3), (1, 7), (2, 4), (2, 6), (3, 1), (3, 9), (4, 2), (4, 8), (5, 5), (6, 2), (6, 8), (7, 1), (7, 9), (8, 4), (8, 6), (9, 3), (9, 7)

Therefore, 18 distinct ways.

$\therefore$ Required probability $= \frac{18}{100} = \frac{9}{50}$

28. Required probability $= \frac{\text{Area of strips region}}{\text{Area of dotted region}}$



$$= \frac{\frac{1}{2} \times 1 \times 1 - \frac{1}{4} \times \pi \left(\frac{1}{2}\right)^2}{1 \times 1 - \frac{1}{4} \times \pi \left(\frac{1}{2}\right)^2} = \frac{8 - \pi}{16 - \pi}$$

29. When the two equal sides are 1 each, then third side could be only 1.

When the two equal sides are 2 each, then third side can take values 1, 2, 3.

When two equal sides are 3 each, then third side can take values 1, 2, 3, 4, 5. When the two equal sides are 4 each, then third side can take values 1, 2, 3, 4, 5, 6 same in the case when two equal sides are 5 and 6.

$\therefore$ Total number of triangles = 1 + 3 + 5 + 6 + 6 + 6 = 27

Required probability $= \frac{1}{27}$

30. Required probability

$$= \frac{\left(\frac{1}{2}\right)^5 \times \left(\frac{1}{2}\right)^5}{\left(\sum_{r=0}^{5} {}^5C_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{5-r}\right)^2} = \frac{1}{\left(\sum_{r=0}^{5} {}^5C_r\right)^2} = \frac{1}{{}^{10}C_5}$$

31. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$\therefore 0 \leq P(A \cup B) \leq 1 - 1 \leq -P(A \cup B) \leq 0$

$P(A) + P(B) - 1 \leq P(A) + P(B) - P(A \cup B) \leq P(A) + P(B)$

32. E and F are independent events. Then,

$$P(E \cap F) = P(E) \cdot P(F) \qquad \ldots(i)$$

Option (a) is obviously not true. So, check for options (b), (c) and (d)

$$\begin{aligned} P(E \cap \bar{F}) &= P(E) - P(E \cap F) \\ &= P(E) - P(E) \cdot P(F) \qquad \text{[from Eq. (i)]} \\ &= P(E)[1 - P(F)] \\ &= P(E) \cdot P(\bar{F}) \end{aligned}$$

$\therefore$ E and $\bar{F}$ are independent events.

Now, $P(\bar{E} \cap \bar{F}) = P(\overline{E \cup F})$

$$\begin{aligned} &= 1 - P(E \cup F) \\ &= 1 - [P(E) + P(F) - P(E \cap F)] \\ &= [1 - P(E)] - P(F) + P(E) \cdot P(F) \\ &= P(\bar{E}) - P(F)[1 - P(E)] \\ &= P(\bar{E})\,[1 - P(F)] \\ &= P(\bar{E}) \cdot P(\bar{F}) \end{aligned}$$

$\therefore$ $\bar{E}$ and $\bar{F}$ are independent events.

Again, $P\left(\frac{E}{F}\right) + P\left(\frac{\bar{E}}{F}\right) = \frac{P(E \cap F)}{P(F)} + \frac{P(\bar{E} \cap F)}{P(F)}$

$$\begin{aligned} &= \frac{P(E) \cdot P(F)}{P(F)} + \frac{P(\bar{E}) \cdot P(F)}{P(F)} \\ &= P(E) + P(\bar{E}) = 1 \end{aligned}$$

33. We know that, $P(A \cap B) \geq P(A) + P(B) - 1$ ...(i)

$$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} \qquad [P(B) \neq 0]$$

$$\Rightarrow \quad P\left(\frac{A}{B}\right) \geq \frac{P(A)+P(B)-1}{P(B)} \qquad \text{[from Eq. (i)]}$$

Option (a) is true.

$$P(A \cap \overline{B}) = P(A) - P(A \cap B)$$

Option (b) is not true.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

If A and B are independent events, then $P(A \cap B) = P(A) \cdot P(B)$.

Then, $P(A \cup B) = P(A) + P(B) - P(A) \cdot P(B)$

$$= P(A) + P(B)\,[1 - P(A)] + 1 - 1$$

$$= 1 + P(B)\,P(\overline{A}) - P(\overline{A}) \qquad [\because P(\overline{A}) = 1 - P(A)]$$

$$= 1 + P(\overline{A})[P(B) - 1] = 1 - P(\overline{A}) \cdot P(\overline{B})$$

Option (c) is true.

If A and B are disjoint, then $P(A \cap B) = 0$.

Then, $P(A \cup B) = 1 - P(\overline{A})\,P(\overline{B})$ does not hold.

34. E and F are two independent events

$$P(E \cap F) = P(E) \cdot P(F) \qquad \ldots(i)$$

$$P(E \cap F) = \frac{1}{12} \qquad \ldots(ii)$$

$$P(\overline{E} \cap \overline{F}) = \frac{1}{2}$$

$$\Rightarrow \quad 1 - P(E \cup F) = \frac{1}{2}$$

$$\Rightarrow \quad P(E) + P(F) - P(E \cap F) = \frac{1}{2}$$

$$\Rightarrow \quad P(E) + P(F) - \frac{1}{12} = \frac{1}{2}$$

$$\Rightarrow \quad P(E) + P(F) = \frac{1}{2} + \frac{1}{12}$$

$$\Rightarrow \quad P(E) + P(F) = \frac{7}{12} \qquad \ldots(iii)$$

From Eqs. (i) and (ii), we get,

$$P(E) \cdot P(F) = \frac{1}{12}$$

$$P(E) = \frac{1}{12P(F)}$$

Put this value in Eq. (iii) we get

$$P(F) + \frac{1}{12P(F)} = \frac{7}{12}$$

Let $P(F) = x$

Then, $x + \frac{1}{12x} = \frac{7}{12}$

$$\Rightarrow \quad \frac{12x^2 + 1}{12x} = \frac{7}{12} \Rightarrow 12x^2 - 7x + 1 = 0$$

$$12x^2 - 4x - 3x + 1 = 0$$

$$\Rightarrow \quad 4x(3x-1) - 1(3x-1) = 0$$

$$(3x-1)(4x-1) = 0 \Rightarrow x = \frac{1}{3} \text{ or } \frac{1}{4}$$

$$\therefore \quad P(E) = \frac{1}{4}, P(F) = \frac{1}{3}$$

or $P(E) = \frac{1}{3}, P(F) = \frac{1}{4}$

35. $P\left(\frac{E}{F}\right) + P\left(\frac{\overline{E}}{F}\right) = \frac{P(E \cap F)}{P(F)} + \frac{P(\overline{E} \cap F)}{P(F)}$

$$= \frac{P(E \cap F) + P(\overline{E} \cap F)}{P(F)} = \frac{P(E \cap F) + P(F) - P(E \cap F)}{P(F)}$$

$$= \frac{P(F)}{P(F)} = 1$$

and $P\left(\frac{E}{F}\right) + P\left(\frac{\overline{E}}{F}\right) = \frac{P(E \cap \overline{F})}{P(\overline{F})} + \frac{P(\overline{E} \cap \overline{F})}{P(\overline{F})}$

$$= \frac{[P(E) - P(E \cap F)] + [1 - P(E \cup F)]}{P(\overline{F})}$$

$$= \frac{[P(E \cup F) - P(F)] + [1 - P(E \cup F)]}{P(\overline{F})}$$

$$= \frac{1 - P(F)}{P(\overline{F})} = \frac{P(\overline{F})}{P(\overline{F})} = 1$$

36. $P(B - A) = P(B \cap \overline{A}) = P(B) - P(A \cap B)$

Option (a) is not correct.

$P(A' \cup B') = 1 - P(A \cap B) = 1 - P(A) \cdot P(B)$ [by given condition]

$P(A') + P(B') = 1 - P(A) + 1 - P(B) = 2 - [P(A) + P(B)]$

Option (b) is not correct $P[(A \cup B)'] = 1 - P(A \cup B)$

$$= 1 - [P(A) + P(B) - P(A) \cdot P(B)]$$ [by given condition]

$$= 1 - P(A) - P(B)\,[1 - P(A)]$$

$$= P(\overline{A}) - P(B) \cdot P(\overline{A}) = P(\overline{A}) \cdot P(\overline{B})$$

Option (c) is correct.

$$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{P(A) \cdot P(B)}{P(B)} = P(A)$$

37. Required probability $= P(A - B) + P(B - A)$

$$= P(A \cap \overline{B}) + P(B \cap \overline{A})$$

$$= P(A) - P(A \cap B) + P(B) - P(A \cap B)$$

$$= P(A) + P(B) - 2P(A \cap B)$$

So, options (a) and (b) are true.

$P(A - B) + P(B - A) = P(A \cup B) - P(A \cap B)$ [by venn diagram]

So, option (c) is also true.

$P(A') + P(B') - 2P(A' \cap B') = 1 - P(A) + 1 - P(B) - 2 + 2P(A \cup B)$

$$= 2P(A \cup B) - P(A) - P(B)$$

$$= 2P(A) + 2P(B) - 2P(A \cap B) - P(A) - P(B)$$

$$= P(A) + P(B) - 2P(A \cap B)$$

$\therefore$ Option (d) is also true.

38. A and B are independent events, then $P(A \cap B) = P(A) \cdot P(B)$

$$P(A) = \frac{1}{2} \text{ and } P(B) = \frac{1}{5}$$

$$P(A \cap B) = \frac{1}{10}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{1}{2} + \frac{1}{5} - \frac{1}{10} = \frac{5 + 2 - 1}{10} = \frac{6}{10} = \frac{3}{5}$$

$$P\left(\frac{A}{B}\right) = P(A) = \frac{1}{2}$$

$$P\left(\frac{A}{A \cup B}\right) = \frac{P[A \cap (A \cup B)]}{P(A \cup B)} = \frac{P(A)}{P(A \cup B)} = \frac{\frac{1}{2}}{\frac{3}{5}} = \frac{5}{6}$$

$$P\left(\frac{A \cap B}{A' \cup B'}\right) = P\left(\frac{A \cap B}{(A \cap B)'}\right)$$

$$= \frac{P[(A \cap B) \cap (A \cap B)']}{P(A \cap B)'} = 0$$

39. Let A, B and C be the event that the student is successful in tests I, II and III, respectively.

P (the student is successful)

$= P(A \cap B \cap C') + P(A \cap B' \cap C) + P(A \cap B \cap C)$

$= P(A) \cdot P(B) \cdot P(C') + P(A) \cdot P(B') \cdot P(C) + P(A) \cdot P(B) \cdot P(C)$

$\therefore$ A, B and C are independent events.

$$= pq\left(1 - \frac{1}{2}\right) + p(1-q)\left(\frac{1}{2}\right) + pq\,\frac{1}{2}$$

$$= \frac{1}{2}[pq + p(1-q) + pq] = \frac{1}{2}p(1+q)$$

$$\therefore \qquad \frac{1}{2} = \frac{1}{2}p(1+q) \implies p(1+q) = 1$$

which is satisfied for all pairs of values in (a), (b) and (c). Also, it is satisfied for infinitely many values as p and q. For instance, when $p = \frac{n}{n+1}$ and $q = \frac{1}{n}$, where n is any positive integer.

40. Total number of subset of set contain n elements $= 2^n$

Number of ways choosing A and $B = 2^n \cdot 2^n = 2^{2n}$.

The number of subset of x which contains exactly r elements

$= {}^nC_r$

$\therefore$ The number of ways of choosing A and B, so that they have the same number of elements

$$= ({}^nC_0)^2 + ({}^nC_1)^2 + \ldots + ({}^nC_n)^2 = {}^{2n}C_n$$

$$= \frac{1 \cdot 2 \cdot 3 \ldots (2n-1)(2n)}{n!\,n!} = \frac{2^n(1 \cdot 3 \cdot 5 \ldots (2n-1))}{n!}$$

41. The number of ways in which m boys and m girls can take their seats around a circle is $(2m-1)!$

(a) We make the girls sit first around the circle. This can be done in $(m-1)!$ ways, after this boys can take their seats in $(m!)$ ways.

$\therefore$ Favourable number of ways $= m!(m-1)!$

Required probability $= \dfrac{m!(m-1)!}{(2m-1)!} = \dfrac{1}{{}^{2m-1}C_m}$

(b) Similarly as (a)

(c) Similarly as (a)

(d) Required probability $= \dfrac{m!m!}{(2m-1)!} \neq \dfrac{1}{{}^{2m-1}C_m}$

42. According to the question,

$$(m + p + c) - mp - mc - pc + mpc = \frac{3}{4} \qquad \ldots(i)$$

$$mp(1-c) + mc(1-p) + pc(1-m) = \frac{2}{5}$$

or $$mp + mc + pc - 3mpc = \frac{2}{5} \qquad \ldots(ii)$$

Also, $$mp + pc + mc - 2mpc = \frac{1}{2} \qquad \ldots(iii)$$

From Eqs. (ii) and (iii), we get

$$mpc = \frac{1}{2} - \frac{2}{5} = \frac{1}{10}$$

$$\therefore \qquad mp + mc + pc = \frac{2}{5} + \frac{3}{10} = \frac{7}{10}$$

$$m + p + c = \frac{3}{4} + \frac{7}{10} - \frac{1}{10} = \frac{27}{20}$$

43. Favourable number of cases $= {}^{n-3}P_2$

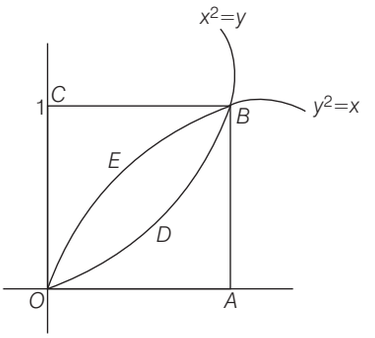
Total number of cases $= {}^nP_2$

$$\therefore \text{ Required probability} = \frac{{}^{n-3}P_2}{{}^nP_2} = \frac{{}^{n-3}C_2 \times 2!}{{}^nC_2 \times 2!} = \frac{{}^{n-3}C_2}{{}^nC_2}$$

$$= \frac{\frac{(n-3)(n-4)}{2 \times 1}}{\frac{n(n-1)}{2 \times 1}} = \frac{(n-3)(n-4)}{n(n-1)}$$

44. A = The event of (x, y) belonging to the area $OEBAO$

B = The event of (x, y) belonging to the area $ODBCO$



$$\therefore \qquad P(A) = \frac{\text{area of } OEBAO}{\text{area of } OABCO} = \frac{\int_0^1 \sqrt{x}\,dx}{1 \times 1} = \frac{2}{3}$$

and $$P(B) = \frac{\text{area of } ODBCO}{\text{area of } OABCO} = \frac{\int_0^1 \sqrt{y}\,dy}{1 \times 1} = \frac{2}{3}$$

and $$P(A \cap B) = \frac{\text{area of } ODBEO}{\text{area of } OABCO} = \frac{\int_0^1 \sqrt{x}\,dx - \int_0^1 x^2 dx}{1 \times 1} = \frac{1}{3}$$

$\therefore$ $P(A) + P(B) = \frac{2}{3} + \frac{2}{3} \neq 1$.

So, A and B are not exhaustive,

$$P(A) \cdot P(B) = \frac{2}{3} \cdot \frac{2}{3} \neq \frac{1}{3}.$$

So, A and B are not independent

and $P(A \cup B) = 1$, $P(A) + P(B) \neq P(A \cup B)$.

So, A and B are not mutually exclusive.

45. Let $P(k) \propto \frac{1}{k^4} \Rightarrow P(k) = \frac{\lambda}{k^4}$

where λ is proportionality constant

Then $\sum_{k=1}^{2n} \frac{\lambda}{k^4} = 1 \Rightarrow \lambda \sum_{k=1}^{2n} \frac{1}{k^4} = 1$

$\therefore \quad \alpha = \sum_{k=1}^{n} P(2k-1) = \lambda \sum_{k=1}^{n} \frac{1}{(2k-1)^4}$

and $\quad \beta = \sum_{k=1}^{n} P(2k) = \lambda \sum_{k=1}^{n} \frac{1}{(2k)^4} < \lambda \sum_{k=1}^{n} \frac{1}{(2k-1)^4}$

$\Rightarrow \quad \beta < \alpha$ and $\alpha + \beta = 1$

Then, $1 - \alpha < \alpha$ and $\beta < 1 - \beta$

$\therefore \quad \alpha > \frac{1}{2}$ and $\beta < \frac{1}{2}$

46. Roots of $x^2 + px + q = 0$ are real and distinct, if $p^2 > 4q$.

Value of p	Possible values of q
1	No value
2	No value
3	1, 2
4	1, 2, 3
5	1, 2, 3, 4, 5, 6
6	1, 2, 3,..., 8
7	1, 2, 3,..., 10
8	1, 2, 3,..., 10
9	1, 2, 3,..., 10
10	1, 2, 3,..., 10

$\therefore$ Number of favourable ways
$= 2 + 3 + 6 + 8 + 10 + 10 + 10 + 10 = 59$
and total ways $= 10 \times 10 = 100$

Hence, the required probability $= \frac{59}{100} = 0.59$

47. Roots of $x^2 + px + q = 0$ are equal, if $p^2 = 4q$
i.e., p^2 must be even.

Value of p	Possible values of q
2	1
4	4
6	9
8	No value
10	No value

$\therefore$ Number of favourable ways $= 1 + 1 + 1 = 3$
and total ways $= 10 \times 10 = 100$

Hence, the required probability $= \frac{3}{100} = 0.03$

48. Roots of $x^2 + px + q = 0$ are imaginary, if $p^2 < 4q$

Hence, the required probability $= 1 -$(Probability that roots of $x^2 + px + q = 0$ are real) $= 1 - (0.59 - 0.03) = 1 - 0.62$

$= 0.38$

49. X can win after the $(n+1)$ th game in the following two mutually exclusive ways.

(i) X wins exactly one of the first n games draws $(n-1)$ games and wins the $(n+1)$ th game.

$\therefore$ Probability, $P_1 = ({}^nP_1\, a\, b^{n-1})a = na^2b^{n-1}$

(ii) X losses exactly one of the first n games, wins exactly one of the first n games and draws $(n-2)$ games and wins the $(n+1)$ th game.

$\therefore$ Probability, $P_2 = ({}^nP_2(ac)b^{n-2})a = n(n-1)a^2b^{n-2}c$

Hence, the probability that X wins two match after $(n+1)$ th game.

$$P_n = P_1 + P_2 = na^2b^{n-2}[b + (n-1)c]$$

50. Put $n = 3$ in solution of question 4 and interchange a and c, then required probability $= 3c^2 \cdot b^1(b + 2a) = 3bc^2(2a + b)$

51. The probability that X wins the match

$$= \sum_{n=1}^{\infty} Pn = \sum_{n=1}^{\infty} na^2b^{n-1} + \sum_{n=1}^{\infty} n(n-1)a^2b^{n-2}c$$

$$= \frac{a^2}{b}\sum_{n=1}^{\infty} nb^n + \frac{a^2c}{b^2}\sum_{n=1}^{\infty} n(n-1)b^n$$

$$= \frac{a^2}{b}\cdot\frac{b}{(1-b)^2} + \frac{a^2c}{b^2}\cdot\frac{2b^2}{(1-b)^3}$$

[sum of infinite AGS]

$$= \frac{a^2(1-b+2c)}{(1-b)^3} = \frac{a^2(a+3c)}{(a+c)^3} \quad \left[\because a+b+c=1 \;\because 1-b=a+c\right]$$

52. $\because P(E_\lambda) \propto \lambda^2$

$\Rightarrow P(E_\lambda) = k\lambda^2$, where k is proportionality constant.

$\because E_0, E_1, E_2, \ldots, E_n$ are mutually exclusive and exhaustive events.
We have, $P(E_0) + P(E_1) + P(E_2) + \ldots + P(E_n) = 1$

$$0 + k(1)^1 + k(2)^2 + \ldots + k(n)^2 = 1$$

$$\Rightarrow \quad k(\sum n^2) = 1$$

$$\therefore \quad k = \frac{1}{\sum n^2} \qquad \ldots(i)$$

53. $P(A) = \sum_{\lambda=0}^{n} P(E_\lambda)\cdot P\left(\frac{A}{E_\lambda}\right)$

$$= \sum_{\lambda=0}^{n}\left(k\lambda^2 \times \frac{\lambda}{n}\right) = \frac{k}{n}\sum_{\lambda=1}^{n}\lambda^3 = \frac{k}{n}\sum_{\lambda=0}^{n}\lambda^3$$

$$= \frac{k}{n}\sum n^3 = \frac{1}{n}\times\frac{1}{\sum n^2}\times\sum n^3 \qquad \text{[from Eq. (i)]}$$

$$= \frac{\left(\frac{n(n+1)}{2}\right)^2}{\frac{n\cdot n(n+1)(2n+1)}{6}} = \frac{3}{2}\cdot\left(\frac{n+1}{2n+1}\right) = \frac{3}{4}\cdot\frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{1}{2n}\right)}$$

$$\therefore \lim_{n\to\infty} P(A) = \frac{3}{4}\lim_{n\to\infty}\left(\frac{1+\frac{1}{n}}{1+\frac{1}{2n}}\right) = \frac{3}{4}\cdot\left(\frac{1+0}{1+0}\right) = \frac{3}{4} = 0.75$$

54. $$P\left(\frac{E_1}{A}\right) = \frac{P(E_1)\cdot P\left(\frac{A}{E_1}\right)}{\sum_{\lambda=0}^{n} P(E_\lambda)\cdot P\left(\frac{A}{E_\lambda}\right)} = \frac{P(E_1)\cdot P\left(\frac{A}{E_1}\right)}{P(A)}$$

$$= \frac{\frac{6}{n(n+1)(2n+1)}\times\frac{1}{n}}{\frac{3n(n+1)}{2(2n+1)}} = \frac{4}{[n(n+1)]^2} = \frac{1}{\sum n^3}$$

55. The number of cubes having atleast one side painted is

$9 + 9 + 3 + 3 + 1 + 1 = 26$ and total cubes $= 27$

$\therefore$ Required probability, $P_1 = \frac{26}{27} \Rightarrow 27P_1 = 26$

56. The number of cubes having two sides painted is

$4 + 4 + 1 + 1 + 1 + 1 = 12$ and total cubes $= 27$

$\therefore$ Required probability, $P_2 = \frac{12}{27} \Rightarrow 27P_2 = 12$

57. Required probability, $P_3 = 1 - p_1 = 1 - \frac{26}{27} = \frac{1}{27} \Rightarrow 27P_3 = 1$

58. A : She gets a success

E_1 : She studies 10 h

$\therefore \quad P(E_1) = 0.1$

E_2 : She studies 7 h

$\therefore \quad P(E_2) = 0.2$

and E_3: She studies 4h

$\therefore \quad P(E_3) = 0.7$

and $P\left(\frac{A}{E_1}\right) = 0.80$, $P\left(\frac{A}{E_2}\right) = 0.60$ and $P\left(\frac{A}{E_3}\right) = 0.40$

$$\therefore \quad P(A) = P(E_1)\cdot P\left(\frac{A}{E_1}\right) + P(E_2)\cdot P\left(\frac{A}{E_2}\right) + P(E_3)\cdot P\left(\frac{A}{E_3}\right)$$

$$= 0.1\times 0.80 + 0.2\times 0.60 + 0.7\times 0.40 = 0.48$$

59. $$P\left(\frac{E_3}{A}\right) = \frac{P(E_3\cap A)}{P(A)} = \frac{P(E_3)\cdot P\left(\frac{A}{E_3}\right)}{P(A)} = \frac{0.7\times 0.40}{0.48} = \frac{7}{12}$$

60. $$P\left(\frac{E_3}{A}\right) = \frac{P(E_3\cap \bar{A})}{P(\bar{A})} = \frac{P(E_3) - P(E_3\cap A)}{1 - P(A)}$$

$$= \frac{0.7 - 0.28}{1 - 0.48} = \frac{0.42}{0.52} = \frac{21}{26}$$

61. $\because p_1$, p_2 and p_3 are mutually exclusive events.

$$\therefore \quad p_1 + p_2 + p_3 = 1$$

Also, p_1, p_2 and p_3 are the roots of

$$27x^3 - 27x^2 + ax - 1 = 0$$

$$\therefore \quad p_1 + p_2 + p_3 = 1 \qquad \ldots(i)$$

$$p_1p_2 + p_2p_3 + p_3p_1 = \frac{a}{27} \qquad \ldots(ii)$$

and $$p_1p_2p_3 = \frac{1}{27}$$

Now, AM of $p_1, p_2, p_3 = \frac{p_1 + p_2 + p_3}{3} = \frac{1}{3}$

and GM of $p_1, p_2, p_3 = (p_1p_2p_3)^{1/3} = \left(\frac{1}{27}\right)^{1/3} = \frac{1}{3}$

Here, AM = GM

$$\therefore \quad p_1 = p_2 = p_3 = \frac{1}{3}$$

From Eq. (ii), we get

$$\frac{1}{3}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3} = \frac{a}{27} \Rightarrow a = 9$$

62. P (none of E_1, E_2, E_3) $= 1 - P(E_1 \cup E_2 \cup E_3)$

$$= 1 - [P(E_1) + P(E_2) + P(E_3)]$$

$$= 1 - (p_1 + p_2 + p_3) = 0$$

[$\because E_1$, E_2 and E_3 are mutually exclusive]

63. $P(E_1 \cup \bar{E}_2) + P(E_2 \cap \bar{E}_3) + P(E_3 \cap \bar{E}_1)$

$= P(E_1) - P(E_1 \cap E_2) + P(E_2) - P(E_2 \cap E_3)$

$\quad + P(E_3) - P(E_3 \cap E_1)$

$= P(E_1) - 0 + P(E_2) - 0 + P(E_3) - 0$

$= p_1 + p_2 + p_3$ [$\because E_1$, E_2 and E_3 are mutually exclusive]

64. The number of increasing functions $= {}^6C_3 = 20$

and the number of total functions $= 6^3 = 216$

$\therefore$ Required probability $= \frac{20}{216} = \frac{5}{54}$

65. The number of non-decreasing functions $= {}^{6+3-1}C_3 = {}^8C_3 = 56$

the number of total functions $= 6^3 = 216$

$\therefore$ Required probability $= \frac{56}{216} = \frac{7}{27}$

66. The number of onto functions such that $f(i) \neq i$ is

$$6!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right) = 265$$

and number of total functions $= 6! = 720$

$\therefore$ Required probability $= \frac{265}{720} = \frac{53}{144}$

67. $\because P(X = x) \propto (x+1)\left(\frac{1}{5}\right)^x$

$$P(X = x) = k(x+1)\left(\frac{1}{5}\right)^x$$

We have, $k\left[1 + 2\left(\frac{1}{5}\right) + 3\left(\frac{1}{5}\right)^2 + \ldots\right] = 1$

$$\Rightarrow \quad k\left[\frac{1}{\left(1-\frac{1}{5}\right)^2}\right] = 1 \Rightarrow k = \frac{16}{25}$$

Now, $P(X = 0) = k(1)\left(\frac{1}{5}\right)^0 = k = \frac{16}{25}$

68. $P(X \geq 2) = 1 - [P(X = 0) + P(X = 1)]$

$$= 1 - k\left[1 + \frac{2}{5}\right] = 1 - \frac{7k}{5} = 1 - \frac{7}{5} \times \frac{16}{25} = \frac{13}{125}$$

69. $E(X) = \sum_{x=0}^{\infty} x\, P(X = x) = k \sum_{x=0}^{\infty} x(x+1)\left(\frac{1}{5}\right)^x$

$$= k\left[(1)(2)\left(\frac{1}{5}\right)^1 + (2)(3)\left(\frac{1}{5}\right)^2 + (3)(4)\left(\frac{1}{5}\right)^3 + \ldots + \infty\right]$$

$$E(X) = k\left[\frac{2}{5} + \frac{6}{25} + \frac{12}{125} + \ldots + \infty\right] \quad \ldots(i)$$

and $\frac{1}{5}E(X) = k\left[\frac{2}{25} + \frac{6}{125} + \frac{12}{625} + \ldots + \infty\right]$ ...(ii)

On subtracting Eq. (ii) from Eq. (i), we get

$$\frac{4}{5}E(X) = k\left[\frac{2}{5} + \frac{4}{25} + \frac{6}{125} + \ldots + \infty\right] \quad \ldots(iii)$$

On multiplying both sides by $\frac{1}{5}$ in Eq. (i), we get

$$\frac{4}{25}E(X) = k\left[\frac{2}{25} + \frac{4}{125} + \frac{6}{625} + \ldots + \infty\right] \quad \ldots(iv)$$

On subtracting Eq. (iv) from Eq. (iii), we get

$$\frac{16}{25}E(X) = k\left[\frac{2}{5} + \frac{2}{25} + \frac{2}{125} + \ldots + \infty\right]$$

$$= k\left[\frac{\frac{2}{5}}{1 - \frac{1}{5}}\right] = \frac{k}{2} = \frac{16}{25} \times \frac{1}{2} \qquad \left[\because k = \frac{16}{25}\right]$$

$\therefore \quad E(x) = \frac{1}{2}$

70. $\because (a^2 - 1)$ is divisible by 10, if and only if last digit of a is 1 or 9.

If $r = 0$, then there are 2λ ways to choose a.

$\therefore \quad p_n = \frac{2\lambda}{n} \Rightarrow n p_n = 2\lambda$

71. If $r = 9$, then there are $2(\lambda + 1)$ ways to choose a.

$\therefore \quad p_n = \frac{2(\lambda + 1)}{n}$

$\Rightarrow \quad n p_n = 2(\lambda + 1)$

72. If $1 \leq r \leq 8$, then there are $2(\lambda + 1)$ ways to choose a.

$\therefore \quad p_n = \frac{2(\lambda + 1)}{n}$

$\Rightarrow \quad x p_n = 2(\lambda + 1)$

73. $\frac{7}{12} = \frac{1}{(n+1)} \cdot 1 + \frac{n}{(n+1)} \cdot \frac{1}{2}$

$\Rightarrow \frac{7}{6} = \frac{(n+2)}{(n+1)} \Rightarrow n = 5$

74. Let S be the sample space, then

$n(S)$ = Total number of determinants that can be made with 0 and 1 $= 2 \times 2 \times 2 \times 2 = 16$

$\because \begin{vmatrix} a & b \\ c & d \end{vmatrix}$, each element can be replaced by two types

i.e., 0 and 1

and let E be the event that the determinant made is non-negative.

Also, E' be the event that the determinant is negative.

$\therefore \quad E' = \left\{\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}\right\}$

$\therefore \quad P(E') = 3$

then $P(E') = \frac{n(E')}{n(S)} = \frac{3}{16}$

Hence, the required probability,

$$P(E) = 1 - P(E')$$

$$= 1 - \frac{3}{16} = \frac{13}{16} = \frac{m}{n} \qquad \text{[given]}$$

$\Rightarrow \quad m = 13$ and $n = 16$, then $n - m = 3$

75. Let E_1, E_2 and E_3 be the events that first, second and third student get success. Then,

$$P(E_1) = \frac{1}{3},\ P(E_2) = \frac{1}{4} \text{ and } P(E_3) = \frac{1}{5}$$

Given, probability of success of atleast two $= \frac{\lambda}{12}$

$\Rightarrow \quad P(E_1 \cap E_2 \cap \bar{E}_3) + P(\bar{E}_1 \cap E_2 \cap E_3)$

$$+ P(E_1 \cap \bar{E}_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3) = \frac{\lambda}{12}$$

$\Rightarrow \quad P(E_1) \cdot P(E_2) \cdot P(\bar{E}_3) + P(\bar{E}_1) \cdot P(E_2) \cdot P(E_3)$

$$+ P(E_1) \cdot P(\bar{E}_2) \cdot P(E_3) + P(E_1) \cdot P(E_2) \cdot P(E_3) = \frac{\lambda}{12}$$

$\Rightarrow \quad \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{4}{5} + \frac{2}{3} \cdot \frac{1}{4} \cdot \frac{1}{5} + \frac{1}{3} \cdot \frac{3}{4} \cdot \frac{1}{5} + \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{5} = \frac{\lambda}{12}$

$\Rightarrow \quad \frac{10}{60} = \frac{\lambda}{12}$

$\therefore \quad \lambda = 2$

76. Let S be the sample space and E be the event of getting a large number than the previous number.

$\therefore \quad n(S) = 6 \times 6 \times 6 = 216$

Now, we count the number of favourable ways. Obviously, the second number has to be greater than 1. If the second number is $i (i > 1)$, then the number of favourable ways $= (i - 1) \times (6 - i)$

$\therefore \quad n(E)$ = Total number of favourable ways

$$= \sum_{i=1}^{6} (i - 1) \times (6 - i)$$

$$= 0 + 1 \times 4 + 2 \times 3 + 3 \times 2 + 4 \times 1 + 5 \times 0$$

$$= 4 + 6 + 6 + 4 = 20$$

Therefore, the required probability, $P(E) = \frac{n(E)}{n(S)} = \frac{20}{216}$

$$= \frac{5}{54} = p \qquad \text{[given]}$$

$\therefore \quad 54p = 5$

77. The number of ways to answer a question $= 2^5 - 1 = 31$.

i.e., In 31 ways only one correct.

Let number of choices $= n$

Now, according to the question $\frac{n}{31} > \frac{1}{8}$

$\Rightarrow \quad n > \frac{31}{8} \Rightarrow n > 3.8$

$\therefore$ Least value of $n = 4$

78. Let E_i denote the event that the ith object goes to the ith place, we have

$$P(E_i \cap E_j \cap E_k) = \frac{(n-3)!}{n!} \text{ for } i < j < k$$

Since, we can choose 3 places out of n is ${}^nC_3$ ways, the probability of the required event is

$$p = {}^nC_3 \cdot \frac{(n-3)!}{n!} = \frac{n!}{3!(n-3)!} \cdot \frac{(n-3)!}{n!} = \frac{1}{6}$$

$\therefore \quad 6p = 1$

79. The sample space is

$S = \{-0.50, -0.49, -0.48, \ldots, -0.01, 0.00, 0.01, \ldots, 0.49\}$

Let E be the event that the round off error is atleast 10 paise, then E' is the event that the round off error is atmost a paise.

$\therefore \quad E' = \{-0.09, -0.08, \ldots, -0.01, 0.00, 0.01, \ldots, 0.09\}$

$\therefore \quad n(E') = 19$ and $n(S) = 100$

$\therefore \; P(E') = \frac{n(E')}{n(S)} = \frac{19}{100}$

$\therefore$ Required probability, $P(E) = 1 - P(E') = 1 - \frac{19}{100}$

$$= \frac{81}{100} = \left(\frac{m}{n}\right)^2$$

$\therefore \quad m = 9$ and $n = 10$

$\Rightarrow \quad n - m = 1$

80. Let E_1, E_2, E_3, E_4, E_5 and E_6 be the events of occurrence of 1, 2, 3, 4, 5 and 6 on the dice respectively and let E be the event of getting a sum of numbers equal to 9.

$\therefore \quad P(E_1) = \frac{1-k}{6}; P(E_2) = \frac{1+2k}{6}; P(E_3) = \frac{1-k}{6};$

$P(E_4) = \frac{1+k}{6}; P(E_5) = \frac{1-2k}{6}; P(E_6) = \frac{1+k}{6}$

and $\frac{1}{9} \le P(E) \le \frac{2}{9}$

Then, $E \equiv \{(3, 6), (6, 3), (4, 5), (5, 4)\}$

Hence, $P(E) = P(E_3E_6) + P(E_6E_3) + P(E_4E_5) + P(E_5E_4)$

$= P(E_3)P(E_6) + P(E_6)P(E_3) + P(E_4)P(E_5) + P(E_5)P(E_4)$

$= 2P(E_3)P(E_6) + 2P(E_4)P(E_5)$

[since E_1, E_2, E_3, E_4, E_5 and E_6 are independent]

$$= 2\left(\frac{1-k}{6}\right)\left(\frac{1+k}{6}\right) + 2\left(\frac{1+k}{6}\right)\left(\frac{1-2k}{6}\right)$$

$$= \frac{1}{18}[2 - k - 3k^2]$$

Since, $\frac{1}{9} \le P(E) \le \frac{2}{9}$

$\Rightarrow \quad \frac{1}{9} \le \frac{1}{18}[2 - k - 3k^2] \le \frac{2}{9}$

$\Rightarrow \quad 2 \le 2 - k - 3k^2 \le 4$

$\Rightarrow \quad 2 \le 2 - k - 3k^2$ and $2 - k - 3k^2 \le 4$

$\Rightarrow \quad 3k\left(k + \frac{1}{3}\right) \le 0$ and $3k^2 + k + 2 \ge 0$

$\Rightarrow \quad -\frac{1}{3} \le k \le 0$ and $k \in R$

$\therefore \quad -\frac{1}{3} \le k \le 0$

Hence, integral value of k is 0.

81. Let $a_1, a_2, a_3, a_4, a_5, a_6$ and a_7 be the seven digits and the remaining two be a_8 and a_9.

Let $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 = 9k, k \in I$ ...(i)

Also, $a_1 + a_2 + a_3 + a_4 + \ldots + a_9 = 1 + 2 + 3 + 4 + \ldots + 9$

$$= \frac{9 \times 10}{2} = 45 \quad \ldots(ii)$$

On subtracting Eq. (i) from Eq. (ii), we get

$a_8 + a_9 = 45 - 9k$ ...(iii)

Since, $a_1 + a_2 + a_3 + a_4 + \ldots + a_9$ and $a_1 + a_2 + \ldots + a_7$ are divisible by 9, if and only if $a_8 + a_9$ is divisible by 9. Let S be the sample space and E be the event that the sum of the digits a_8 and a_9 is divisible by 9.

$\because \quad a_8 + a_9 = 45 - 9k$

Maximum value of $a_8 + a_9 = 17$ and minimum value of

$a_8 + a_9 = 3$

$\therefore \quad 3 \le 45 - 9k \le 17$

$\Rightarrow \quad -42 \le -9k \le -28 \;\Rightarrow\; \frac{42}{9} \le k \ge \frac{28}{9}$

or $\frac{28}{9} \le k \ge \frac{42}{9}$

Hence, $k = 4$ [$\because k$ is positive integer]

$\therefore$ From Eq. (iii), we get

$a_8 + a_9 = 45 - 9 \times 4$

$\therefore \quad a_8 + a_9 = 9$

Now, possible pair of (a_8, a_9) can be$\{(1, 8), (2, 7), (3, 6), (4, 5)\}$

$\therefore \quad E = \{(1, 8), (2, 7), (3, 6), (4, 5)\}$

$n(E) = 4$ and $n(S) = {}^9C_2 = 36$

$\therefore$ Required probability, $P(E) = \frac{n(E)}{n(S)} = \frac{4}{36} = \frac{1}{9} = p$ [given]

$\therefore \quad 18p = 2$

82. Let A be the event of P_1 winning in third round and B be the event of P_2 winning in first round but loosing in second round.

We have, $P(A) = \frac{1}{{}^8C_1} = \frac{1}{8}$

$P(B \cap A)$ = Probability of both P_1 and P_2 winning in first round $\times$ Probability of P_1 winning and P_2 loosing in second round $\times$ Probability of P_1 winning in third round

$$= \frac{{}^{8-2}C_{4-2}}{{}^8C_4} \times \frac{{}^{4-2}C_{2-1}}{{}^4C_2} \times \frac{{}^{2-1}C_{1-1}}{{}^2C_1}$$

$$= \frac{{}^6C_2}{{}^8C_4} \times \frac{{}^2C_1}{{}^4C_2} \times \frac{{}^1C_0}{{}^2C_1} = \frac{1}{28}$$

Hence, $P\left(\frac{B}{A}\right)=\frac{P(B\cap A)}{P(A)}=\frac{\frac{1}{28}}{\frac{1}{8}}=\frac{2}{7}=p$ [given]

$\therefore \quad 7p=2$

83. (A) $\rightarrow$(q,r); (B) $\rightarrow$ (p,r); (C) $\rightarrow$ (p,r); (D) $\rightarrow$ (q,r,s)

(A) $P(\overline{A})=0.3\Rightarrow P(A)=1-P(\overline{A})=0.7,\ P(B)=0.4,$

$\Rightarrow \quad 0.5=0.7-P(AB)$

$\therefore \quad P(AB)=0.2$

$$\Rightarrow P\left[\frac{B}{(A\cup\overline{B})}\right]=\frac{P[B\cap(A\cup\overline{B})]}{P(A\cup\overline{B})}=\frac{P(A\cap B)}{P(A)+P(\overline{B})-P(A\overline{B})}$$

$$=\frac{0.2}{0.7+0.6-0.5}=\frac{1}{4}=\lambda_1 \quad \text{[given]}$$

$\therefore \quad \frac{1}{\lambda_1}=4$ [composite number and natural number]

(B) First three prime numbers are 2, 3 and 5.

For roots to be real $D\geq 0$

Thus, real roots are obtained by $b=5, a=2, c=3$

and $b=5, a=3, c=2$

i.e., two ways.

Total ways of choosing $a, b, c=3\times 2\times 1=6$

$\therefore$ Required probability $=\frac{2}{6}=\frac{1}{3}=\lambda_2$ [given]

$\therefore \quad \frac{1}{\lambda_2}=3$

[prime number and natural number]

(C) Here, tossing of the coin is an independent event. Thus, the result of 5th trial is independent of outcome of previous trials.

$\therefore \quad \lambda_3=\frac{1}{2} \Rightarrow \frac{1}{\lambda_3}=2$

[prime number and natural number]

(D) Clearly, $n(S)={}^9P_9=9!$

Now, 3 positions out of 9 positions can be chosen in ${}^9C_3$ ways and at these positions A, B and C can speak in required order, further remaining 6 persons can speak in 6! ways.

$\therefore$ Required probability $=\frac{{}^9C_3\times 6!}{9!}$

$$=\frac{9!\times 6!}{3!\times 6!\times 9!}=\frac{1}{6}=\lambda_4 \quad \text{[given]}$$

$\therefore \quad \frac{1}{\lambda_4}=6$

[a composite number, a natural number and a perfect number]

84. (A) $\rightarrow$ (s); (B) $\rightarrow$ (p); (C) $\rightarrow$ (r); (D) $\rightarrow$ (q)

(A) $\frac{3}{5}\geq P(A\cap B)\geq P(A)+P(B)-1=\frac{3}{5}+\frac{2}{3}-1=\frac{4}{15}$

$\therefore \quad \frac{3}{5}\geq P(A\cap B)\geq\frac{4}{15}$

$\Rightarrow \quad P(A\cap B)\in\left[\frac{4}{15},\frac{3}{5}\right]$

(B) $\because P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$$=\frac{3}{5}+\frac{2}{3}-P(A\cap B)=\frac{19}{15}-P(A\cap B)$$

$\Rightarrow \quad \frac{2}{3}\leq P(A\cup B)\leq 1 \quad \left[\because P(A\cap B)\in\left[\frac{4}{15},\frac{3}{5}\right]\right]$

$\therefore \quad P(A\cup B)\in\left[\frac{2}{3},1\right]$

(C) $P\left(\frac{A}{B}\right)=\frac{P(A\cap B)}{P(B)}=\frac{3}{2}P(A\cap B)$

$\Rightarrow \quad \frac{2}{5}\leq P\left(\frac{A}{B}\right)\leq\frac{9}{10}\left[\because P(A\cap B)\in\left[\frac{4}{15},\frac{3}{5}\right]\right]$

$\therefore \quad P\left(\frac{A}{B}\right)\in\left[\frac{2}{5},\frac{9}{10}\right]$

(D) $P\left(\frac{B}{A}\right)=\frac{P(B\cap A)}{P(A)}=\frac{5}{3}P(A\cap B)$

$\Rightarrow \quad \frac{4}{9}\leq P\left(\frac{B}{A}\right)\leq 1 \quad \left[\because P(A\cap B)\in\left[\frac{4}{15},\frac{3}{5}\right]\right]$

$\therefore \quad P\left(\frac{B}{A}\right)\in\left[\frac{4}{9},1\right]$

85. (A) $\rightarrow$ (p,r,s) ; (B) $\rightarrow$ (p,r,s); (C) $\rightarrow$ (s); (D) $\rightarrow$ (q,r)

Let $q_1=1-p_1$ and $q_2=1-p_2$

(A) $\because A$ can win the game at the 1st, 4th, 7th,... trials.

$$\therefore P(A \text{ wins})=\frac{1}{6}+\frac{5}{6}(q_1)(q_2)\left(\frac{1}{6}\right)+\left(\frac{5}{6}\right)^2(q_1)^2(q_2)^2\left(\frac{1}{6}\right)+\ldots$$

$$=\frac{\frac{1}{6}}{1-\frac{5}{6}q_1q_2}=\frac{1}{6-5q_1q_2}=\frac{1}{6-5\left(\frac{4}{5}\right)\left(\frac{3}{4}\right)}=\frac{1}{\lambda_1} \text{ [given]}$$

$\therefore \quad \lambda_1=3$

(B) $P(C \text{ wins})=\frac{5}{6}.q_1.p_2+\left(\frac{5}{6}\right)^2.q_1^2.q_2p_2+\ldots$

$$=\frac{\frac{5}{6}q_1.p_2}{1-\frac{5}{6}q_1q_2}=\frac{5q_1p_2}{6-5q_1q_2}=\frac{5\times\frac{4}{5}\times\frac{1}{4}}{6-5\times\frac{4}{5}\times\frac{3}{4}}=\frac{1}{3}$$

$$=\frac{1}{\lambda_2} \quad \text{[given]}$$

$\therefore \quad \lambda_2=3$

(C) $\because$ $P(A \text{ wins})=P(B \text{ wins})$

$$\Rightarrow \quad \frac{1}{6-5q_1q_2}=\frac{5p_1}{6-5q_1q_2}$$

$\therefore \quad p_1=\frac{1}{5}=\frac{1}{\lambda_3}$ [given]

$\therefore \quad \lambda_3=5$

(D) $P(A \text{ wins}) = P(B \text{ wins}) = P(C \text{ wins})$

$$\Rightarrow \quad \frac{1}{6-5q_1q_2} = \frac{5p_1}{6-5q_1q_2} = \frac{5q_1p_2}{6-5\,q_1q_2}$$

$$\Rightarrow \quad 1 = 5p_1 = 5q_1p_2$$

$$\Rightarrow \quad p_1 = \frac{1}{5}, \frac{1}{p_2} = 5q_1 = 5\left(1-\frac{1}{5}\right) = 4 = \lambda_4 \qquad \text{[given]}$$

$$\therefore \quad \lambda_4 = 4$$

86. (A) → (q); (B) → (r); (C) → (p); (D) → (s)

(A) $\because a, b \in \{1, 2, 3, \ldots, 9\}$

For real and distinct roots $D > 0$

i.e., $2(a-b)^2 > 4b \Rightarrow (a-b)^2 > 2b$

The possible pairs are

b	a	Total pairs of a and b
1	3, 4, 5, ..., 9	7
2	5, 6, ..., 9	5
3	6, 7, 8, 9	4
4	1, 7, 8, 9	4
5	1, 9	2
6	1, 2	2
7	1, 2, 3	3
8	1, 2, 3	3
9	1, 2, 3, 4	4
		34

$n(S) = 9 \times 9 = 81$ and $n(E) = 34$

$$\therefore \quad p_1 = \frac{34}{81} \Rightarrow 9p_1 = \frac{34}{9}$$

$$\therefore \quad [9p_1] = \left[\frac{34}{9}\right] = 3$$

(B) For imaginary roots,

$$p_2 = 1 - p_4 = 1 - \frac{5}{9} = \frac{4}{9}$$

$$\therefore \quad [9p_2] = 4$$

(C) For equal roots, there are only 2 possible pairs are

$$b = 2, a = 4 \text{ and } b = 8, a = 4$$

$$\therefore \quad n(S) = 81, n(E) = 2$$

$$\therefore \quad p_3 = \frac{2}{81}$$

$$\Rightarrow \quad [81p_3] = 2$$

(D) For real roots,

$$p_4 = 1 - (p_1 + p_3) = 1 - \left(\frac{34}{81} + \frac{2}{81}\right)$$

$$= 1 - \frac{36}{81} = 1 - \frac{4}{9} = \frac{5}{9}$$

$$\therefore \quad [9p_4] = 5$$

87. (A) → (r); B → (s); (C) → (p); D → (q)

(A) $n(S) = {}^{10}C_3 = 120$ and $n(E) = {}^3C_1 = 3$, because on selection 3 and 7, we have to select one from 4, 5 and 6.

$$\therefore \quad P(E) = \frac{n(E)}{n(S)} = \frac{3}{120} = \frac{1}{40} = p_1 \qquad \text{[given]}$$

$$\Rightarrow \quad \frac{2}{5p_1} = 16$$

(B) The probability of 4 being the minimum number

$= \frac{{}^6C_2}{{}^{10}C_3} = \frac{1}{8}$ [because after selecting 4 any two can be selected from 5, 6, 7, 8, 9, 10]

and the probability of being the maximum number

$= \frac{{}^7C_2}{{}^{10}C_3} = \frac{7}{40}$ [because after selecting 8 any two can be selected from 1, 2, 3, 4, 5, 6, 7]

and the probability of 4 being the minimum number and 8 being the maximum number

$= \frac{{}^3C_1}{{}^{10}C_3} = \frac{1}{40}$ [because on selecting 4, 8 and we have to select one from 5, 6, 7]

$\therefore$ Required probability

$$= P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{1}{8} + \frac{7}{40} - \frac{1}{40} = \frac{11}{40} = p_2 \qquad \text{[given]}$$

$$\therefore \quad 80p_2 = 22$$

(C) Let A = {maximum of three numbers is 7}

$\therefore \quad A = \{1, 2, 3, 4, 5, 6, 7\}$

and B = {minimum of three numbers is 3}

$\therefore \quad B = \{3, 4, 5, 6, 7, 8, 9, 10\}$

and $A \cap B = \{3, 4, 5, 6, 7\}$

$$\therefore \quad P\left(\frac{B}{A}\right) = \frac{P(B \cap A)}{P(A)} = \frac{{}^3C_1}{{}^6C_2} = \frac{1}{5} = p_3 \qquad \text{[given]}$$

$$\therefore \quad \frac{2}{p_3} = 10$$

(D) Let A = { maximum of three numbers is 8 }

$\therefore \quad A = \{1, 2, 3, 4, 5, 6, 7, 8\}$

and B = { minimum of three numbers is 4 }

$\therefore \quad B = \{4, 5, 6, 7, 8, 9, 10\}$ and $A \cap B = \{4, 5, 6, 7, 8\}$

$$\therefore \quad P\left(\frac{B}{A}\right) = \frac{P(B \cap A)}{P(A)} = \frac{{}^3C_1}{{}^7C_2} = \frac{1}{7} = p_4 \qquad \text{[given]}$$

$$\therefore \quad \frac{2}{p_4} = 14$$

88. (A) → (q); (B) → (s); C → (p); (D) → (r)

(A) We know that 7^λ, $\lambda \in N$ has 1, 3, 7, 9 at the unit's place for $\lambda = 4k, 4k-1, 4k-2, 4k-3$ respectively, when $k = 1, 2, 3, \ldots$

Clearly, $7^m + 7^n$ will be divisible by 5, if 7^m has 3 or 7 in the unit's place and 7^n has 7 or 3 in the unit's place or 7^m has 1 or 9 in the unit's place and 7^n has 9 or 1 in the unit's place.

$\therefore$ For any choice of m, n the digit in the unit's place of $7^m + 7^n$ is 2, 4, 6, 8, 0

It is divisible by 5 only when this digit is 0.

$\therefore$ Required probability $= \frac{1}{5}$

(B) $n(S) = 2 \times 2 \times 2 \times 2 = 16$

[because each of the four places in determinant can be filled in 2 ways]

The zero determinants are

$$\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} -1 & -1 \\ -1 & -1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix}, \begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix}, \begin{vmatrix} -1 & 1 \\ -1 & 1 \end{vmatrix}, \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix}$$

Number of zero determinants = 8, number of non-zero determinants

$= 16 - 8 = 8 = n(E)$ [say]

$\therefore$ Required probability $= \frac{n(E)}{n(S)} = \frac{8}{16} = \frac{1}{2}$

(C) $\because P(E_n) \propto n$

$\Rightarrow P(E_n) = kn$, where k is proportionality constant.

Clearly,

$P(E_1) + P(E_2) + P(E_3) + P(E_4) + P(E_5) + P(E_6) = 1$

$\Rightarrow \quad k(1 + 2 + 3 + 4 + 5 + 6) = 1 \Rightarrow k = \frac{1}{21}$

$\therefore$ Required probability $= P(E_3) = 3k = 3 \times \frac{1}{21} = \frac{1}{7}$

(D) 5 can be thrown in 4 ways and 7 can be thrown in 6 ways in a single throw of two dice.

Number of ways of throwing neither 5 nor 7

$= 36 - (4 + 6) = 26$

Probability of throwing a sum of 5 in a throw $= \frac{4}{36} = \frac{1}{9}$

and probability of throwing neither 5 nor 7 $= \frac{26}{36} = \frac{13}{18}$

$\therefore$ Required probability

$$= \frac{1}{9} + \frac{13}{18}\left(\frac{1}{9}\right) + \left(\frac{13}{18}\right)^2\left(\frac{1}{9}\right) + \ldots = \frac{\frac{1}{9}}{1 - \frac{13}{18}} = \frac{2}{5}$$

89. $\text{In}\left(\frac{1}{2} + \frac{1}{2}\right)^{10}$

Probability of appearing exactly four heads

$$= {}^{10}C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^6 = {}^{10}C_{10-4}\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^4$$

$$= {}^{10}C_6\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^4$$

= Probability of appearing exactly six heads. Both statements are true,

Statement-2 is a correct explanation for Statement-1.

90. If A and B are independent, then

$P(A \cap B) = P(A) \cdot P(B) = P(A)$ [$\because P(B) = 1$] ...(i)

and $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= P(A) + P(B) - P(A)$ [from Eq. (i)]

$= P(B) = 1$ which is true.

Hence, both statements are true and Statement-2 is a correct explanation for Statement-1.

91. $\because \quad P(A \cap \overline{B}) = P(A) - P(A \cap B)$

$\Rightarrow \quad P(A \cap \overline{B}) = 0.3 - P(A \cap B)$

$\therefore P(A \cap \overline{B})$ cannot be found. Hence, both statements are true and Statement-2 is a correct explanation for Statement-1.

92. $\because P(A \cup B) = P(A \cap B)$

$\Rightarrow P(A) + P(B) - P(A \cap B) = P(A \cap B)$

$\Rightarrow (P(A) - P(A \cap B)) + (P(B) - P(A \cap B)) = 0$

$\Rightarrow \quad P(A \cap B') + P(A' \cap B) = 0$ ...(i)

$\because \quad A \cap B \subseteq A$ and $A \cap B \subseteq B$

$\Rightarrow \quad P(A \cap B) \le P(A)$ and $P(A \cap B) \le P(B)$

$\Rightarrow P(A) - P(A \cap B) \ge 0$ and $P(B) - P(A \cap B) \ge 0$

$\Rightarrow \quad P(A \cap B') \ge 0$ ...(ii)

and $P(A' \cap B) \ge 0$ ...(iii)

From Eqs. (i), (ii) and (iii), we get

$P(A \cap B') = 0$ and $P(A' \cap B) = 0$

or $P(A \cap B') = P(A' \cap B) = 0$

$\Rightarrow$ Statement-1 is true and Statement-2 is false.

93. **Statement-1** There are six equally likely possibilities of which only 2 are favourable (4 and 6)

$\therefore$ Probability that the obtained number is composite $= \frac{2}{6} = \frac{1}{3}$

$\therefore$ Statement-1 is true.

Statement-2 As the given 3 possibilities are not equally likely.

$\therefore$ Statement-2 is false.

94. Total cards = 52 = 26 Red + 26 Black

13 Diamond 13 Heart

Given A : Red card is drawn

B : Card drawn is either a diamond or heart

It is clear that $A \subseteq B$ and $B \subseteq A$

$\therefore$ Statement-2 is true.

and $P(A + B) = P(A \cup B) = P(A \cup A) = P(A)$ [$\because A \subseteq B$ and $B \subseteq A$]

and $P(AB) = P(A \cap B) = P(A \cap A) = P(A)$ [$\because A \subseteq B$ and $B \subseteq A$]

$\therefore \quad P(A + B) = P(AB)$

Statement-1 is true.

Hence, both statements are true and Statement-2 is a correct explanation for Statement-1.

95. Required probability = 1 − Problem will not be solved

$= 1 - P(\overline{A} \cap \overline{B}) = 1 - P(\overline{A})P(\overline{B}) = 1 - (1 - P(A))(1 - P(B))$

$= 1 - \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) = 1 - \frac{1}{2} \times \frac{2}{3} = 1 - \frac{1}{3} = \frac{2}{3}$

$\therefore$ Statement-1 is false and Statement-2 is true.

96. Total ways $= {}^{2n+1}C_3 = \dfrac{(2n+1).2n.(2n-1)}{1\cdot 2\cdot 3} = \dfrac{n(4n^2-1)}{3}$

Let the three numbers a, b, c are drawn, where $a < b < c$ and given a, b and c are in AP.

$\therefore \quad 2b = a + c$...(i)

It is clear from Eq. (i) that a and c both are odd or both are even.

$\therefore$ Favourable ways $= {}^{n+1}C_2 + {}^{n}C_2$

$$= \frac{(n+1)n}{1\cdot 2} + \frac{n(n-1)}{1\cdot 2} = n^2$$

$\therefore$ Required probability $= \dfrac{n^2}{\dfrac{n(4n^2-1)}{3}} = \dfrac{3n}{(4n^2-1)}$

$\Rightarrow$ Statement-2 is false.

In Statement-1, $2n + 1 = 21$

$\Rightarrow \quad n = 10$

$\because$ Required probability $= \dfrac{3\times 10}{4(10)^2 - 1} = \dfrac{30}{399} = \dfrac{10}{133}$

$\therefore$ Statement-1 is true.

97. In Statement-2 $P\left(\dfrac{A}{B}\right) = \dfrac{P(A\cap B)}{P(B)}$ [by definition]

$P(\overline{B}) = P((A\cup\overline{A})\cap\overline{B}) = P((A\cap\overline{B})\cup(\overline{A}\cap\overline{B}))$

$= P(A\cap\overline{B}) + P(\overline{A}\cap\overline{B})$ ...(i)

$\therefore$ Statement-2 is true.

In Statement-1

$$P\left(\frac{A}{\overline{B}}\right) + P\left(\frac{\overline{A}}{\overline{B}}\right) = \frac{P(A\cap\overline{B})}{P(\overline{B})} + \frac{P(\overline{A}\cap\overline{B})}{P(\overline{B})}$$

$$= \frac{P(A\cap\overline{B}) + P(\overline{A}\cap\overline{B})}{P(\overline{B})} = \frac{P(\overline{B})}{P(\overline{B})} = 1 \quad \text{[From Eq. (i)]}$$

$\therefore$ Statement-1 is false.

98. The total number of matches played in the tournament

$= {}^5C_2 = 10$

The probability that a particular team (say A) wins all its 4 matches $= \left(\dfrac{1}{2}\right)^4 = \dfrac{1}{16}$

$\therefore$ Probability that team is undefeated in the tournament

$$= {}^5C_1\left(\frac{1}{2}\right)^4 = \frac{5}{16}$$

$\Rightarrow$ Statement-1 is true.

Similarly, the probability that there is an winless team $= \dfrac{5}{16}$

$\Rightarrow$ Statement-2 is false.

99. For real roots, $D \geq 0$

$\Rightarrow \quad p^2 - 4\cdot 1\cdot\dfrac{1}{4}(p+2) \geq 0$

$\Rightarrow \quad p^2 - p - 2 \geq 0$

$\Rightarrow \quad (p-2)(p+1) \geq 0$

$\Rightarrow \quad p \leq -1$ or $p \geq 2$

But $p \in [0, 5]$.

So, $E = [2, 5]$

$n(E)$ = length of the interval $[2, 5] = 3$

and $n(S)$ = length of the interval $[0, 5] = 5$

$\therefore$ Required probability $= \dfrac{n(E)}{n(S)} = \dfrac{3}{5}$

Hence, both statements are true and Statement-2 is a correct explanation for Statement-1

100. $P\left(\dfrac{A}{B}\right) = \dfrac{P(A\cap B)}{P(B)} = \dfrac{\dfrac{n(A\cap B)}{n(S)}}{\dfrac{n(B)}{n(S)}} = \dfrac{n(A\cap B)}{n(B)} = \dfrac{n(AB)}{n(B)}$

Hence, both statements are true and Statement-2 is a correct explanation for Statement-1.

101. Let S be the sample space, then

$n(S)$ = Total number of numbers of five digits formed with the digits 1, 2, 3, 4 and 5 without repetition $= {}^5P_5 = 5! = 120$

We know that, a number is divisible by 4 if the last two digits of the number is divisible by 4.

Then, for divisible by 4, last two digits 12 or 24 or 32 or 52.

Let E be the event that the number formed is divisible by 4.

$\therefore \quad n(E) = 3! \times 4 = 24$

$\therefore$ Required probability, $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{24}{120} = \dfrac{1}{5}$

102. Let S be the sample space and E be the event of getting a large number than the previous number.

$\therefore \quad n(S) = 6\times 6\times 6 = 216$

Now, we count the number of favourable ways. Obviously, the second number has to be greater than 1. If the second number is $i (i > 1)$, then the number of favourable ways $= (i-1)\times(6-i)$

$\therefore$ $n(E)$ = Total number of favourable ways

$$= \sum_{i=1}^{6}(i-1)\times(6-i)$$

$$= 0 + 1\times 4 + 2\times 3 + 3\times 2 + 4\times 1 + 5\times 0$$

$$= 4 + 6 + 6 + 4 = 20$$

Therefore, the required probability,

$$P(E) = \frac{n(E)}{n(S)} = \frac{20}{216} = \frac{5}{54}$$

103. Finding r cars in N places, there are $(r-1)$ cars other than his own in $(N-1)$ places.

$\therefore$ Total number of ways $= {}^{N-1}C_{r-1} = \dfrac{(n-1)!}{(r-1)!\,(N-r)!}$

Now, the $(r-1)$ cars must be parked in $N-3$ places (because neighbouring slots are empty).

$\therefore$ Number of favourable ways $= {}^{N-3}C_{r-1}$

$$= \frac{(N-3)!}{(r-1)!\,(N-r-2)!}$$

$\therefore$ Required probability $= \dfrac{\text{Favourable ways}}{\text{Total ways}}$

$$= \frac{(N-3)!}{(r-1)!\,(N-r-2)!}\times\frac{(r-1)!\,(N-r)!}{(N-1)!}$$

$$= \frac{(N-r)\,(N-r-1)}{(N-1)\,(N-2)}$$

104. A wins the series in $(n + r + 1)$ games (say). He wins the $(n + r + 1)$ th game and
n out of the first $(n + r)$ games.

$$\therefore \qquad P(A) = \sum_{r=0}^{n} ({}^{n+r}C_n)\, q^r p^{n+1} \qquad [\text{where } p + q = 1]$$

Similarly, $\quad P(B) = \sum_{r=0}^{n} ({}^{n+r}C_n)\, q^{n+1} p^r$

Now, $P(A) + P(B) = 1$

$$\therefore \qquad \sum_{r=0}^{n} [q^r p^{n+1} + q^{n+1} p^r]\, {}^{n+r}C_n = 1$$

Now, put $p = q = \frac{1}{2}$

$$\therefore \qquad \sum_{r=0}^{n} {}^{n+r}C_n \left[\frac{1}{2^{n+r+1}} + \frac{1}{2^{n+r+1}}\right] = 1$$

$$\Rightarrow \qquad \sum_{r=0}^{n} ({}^{n+r}C_n)\, \frac{1}{2^{n+r}} = 1$$

105. Let A denotes the event that the target is hit when x shells are fired at point I.
Let $E_1 (E_2)$ denote the event.
We have, $P(E_1) = \frac{8}{9}$, $P(E_2) = \frac{1}{9}$

$$\Rightarrow \qquad P\left(\frac{A}{E_1}\right) = 1 - \left(\frac{1}{2}\right)^x \text{ and } P\left(\frac{A}{E_2}\right) = 1 - \left(\frac{1}{2}\right)^{21-x}$$

$$\text{Now,} \quad P(A) = \frac{8}{9}\left[1 - \left(\frac{1}{2}\right)^x\right] + \frac{1}{9}\left[1 - \left(\frac{1}{2}\right)^{21-x}\right]$$

$$\Rightarrow \qquad \frac{dP(A)}{dx} = \frac{8}{9}\left[\left(\frac{1}{2}\right)^x \log 2\right] + \frac{1}{9}\left[-\left(\frac{1}{2}\right)^{21-x} \log 2\right]$$

Now, we must have $\frac{dP(A)}{dx} = 0 \Rightarrow \quad x = 12$, also $\frac{d^2P(A)}{dx^2} < 0$

Hence, $P(A)$ is maximum where $x = 12$.

106. The composition of the balls in the red box and in the green box; and the sum suggested in the problem may be one of the following:

Red box		**Green box**		**Sum of Green in Red and Red in Green**
Red	**Green**	**Green**	**Red**	
0	5	3	6	11
1	4	4	5	9
2	3	5	4	7
3	2	6	3	5
4	1	7	2	3
5	0	8	1	1

Of these the 2nd and the last correspond to the sum being NOT a prime number. Hence, the required probability

$$= \frac{{}^6C_1 \times {}^8C_4 \times {}^6C_5 \times {}^8C_0}{{}^{14}C_5} = \frac{420 + 6}{2002} = \frac{213}{1001}$$

107. Let E_i denote the event that out of the first k balls drawn, i balls are green. Let A denote the event that $(k + 1)$th ball drawn is also green. We have, now $P(E_i) = \frac{{}^aC_1 \times {}^bC_{k-i}}{{}^{a+b}C_k}$

Here, $0 \le i \le k$ and $P\left(\frac{A}{E_i}\right) = \frac{a - i}{a + b - k}$

$$\text{Now,} \qquad P(A) = \sum_{j=0}^{k} \frac{{}^aC_j \times {}^bC_{k-j}}{{}^{a+b}C_k} \times \frac{a - j}{a + b - k}$$

Also, $(1 + x)^{a-1} (1 + x)^b$

$$= [{}^{a-1}C_0 + {}^{a-1}C_1 x + \ldots + {}^{a-1}C_{a-1} x^{a-1}] \times [{}^bC_0 + {}^bC_1 x + \ldots + {}^bC_b x^b]$$

$$\Rightarrow \sum_{j=0}^{n} ({}^{a-1}C_j)\, ({}^bC_{k-j}) = \text{coefficient of } x^k$$

$$\text{Hence,} \qquad P(A) = \frac{a}{a + b}$$

108. Total number of outcomes $= 2 \times 2 \times 2 \ldots 12$ times $= 4096$

Let a_n denote the number of outcomes in which two consecutive heads do not occur when the fair coin is tossed n times.

$$\Rightarrow \qquad a_1 = 2,\ a_2 = 3$$

For $n \ge 3$, if the last outcome is T, then we cannot have two consecutive heads in the first $(n - 1)$ tosses. This can happen in a_{n-1} ways. If the last outcome is H, we must have T the $(n - 1)$th toss and we cannot have two consecutive heads in the first $(n - 2)$ tosses. This can happen in a_{n-2} ways.

$$\Rightarrow \qquad a_n = a_{n-1} + a_{n-2} \text{ for } n \ge 3$$

$$\Rightarrow \qquad a_{10} = 144,\ a_{11} = 233$$

$$\Rightarrow \qquad a_{12} = 377$$

Hence, the required probability is $\frac{377}{4096}$.

109. Let PQ be a diameter of a circle with centre O and radius a. Take a point A at random in PQ.
Let, $AP = x$, $AQ = y$, then $x + y = 2a$ and all values of x between 0 and $2a$ are equally likely.
Draw the ordinate AB, then $AB^2 = AP, AQ = xy$
If P', Q' are the mid-points of OP, OQ, the ordinates at these points are equal to $a\sqrt{\frac{3}{4}}$.
Hence, $AB > a\sqrt{\frac{3}{4}}$ if and only if, A lies in $P'Q'$.
Hence, the chance that $xy > \frac{3}{4}a^2$ is $\frac{A'B'}{AB}$ i.e., $\frac{1}{2}$.

110. (i) Kamsky wins one of the first n games and draws the remaining $[(n - 1)$ or
(ii) Kamsky wins exactly one of the first n games and draws the remaining$]$ $n - 2$. We have,

$$P(i) = {}^nP_1 pq^{n-1}$$

and $\qquad P(ii) = {}^nP_2\, pq^{n-2} r$

$\Rightarrow$ The probability that Kamsky wins this match is

$$\sum_{n=1}^{\infty} p^2 [nq^{n-1} + n(n-1)\, rq^{n-2}]$$

$$= p^2 \sum_{n=1}^{\infty} nq^{n-1} + p^2 r \sum_{n=1}^{\infty} n(n-1)\, q^{n-2}$$

Differentiating both sides w.r.t. q, we get

$$\sum_{n=1}^{\infty} nq^{n-1} = \frac{1}{(1-q)^2} \text{ and } \sum_{n=1}^{\infty} n(n-1)\, q^{n-2} = \frac{2}{(1-q)^3}$$

Thus, the probability that Kamsky wins the match is

$$\frac{p^2}{(1-q)^2} + \frac{2p^2 r}{(1-q)^3} = \frac{p^2(p+3r)}{(p+r)^3}$$

because $p + q + r = 1$.

111. Let A, B and C be three independent events having probabilities p, q and r, respectively.

Then, according to the question, we have

P (only the first occurs) $= P(A \cap \overline{B} \cap \overline{C})$

[$\because A, B, C$ are independent]

$$= P(A) \cdot P(\overline{B}) \cdot P(\overline{C})$$

$$= p(1-q)(1-r) = a \quad \text{...(i)}$$

P (only the second occurs) $= P(\overline{A} \cap B \cap \overline{C})$

$$= P(\overline{A}) \cdot P(B) \cdot P(\overline{C})$$

$$= (1-p)\, q(1-r) = b \quad \text{...(ii)}$$

and P (only the third occurs) $= P(\overline{A} \cap \overline{B} \cap C)$

$$= P(\overline{A}) \cdot P(\overline{B}) \cdot P(C)$$

$$= (1-p)(1-q)\, r = c \quad \text{...(iii)}$$

Multiplying Eqs. (i), (ii) and (iii), then

$$pqr\, \{(1-p)(1-q)(1-r)\}^2 = abc$$

or $$\frac{abc}{pqr} = [(1-p)(1-q)(1-r)]^2 = x^2 \quad \text{[say] ...(iv)}$$

$$(1-p)(1-q)(1-r) = x \quad \text{...(v)}$$

Dividing Eq. (i) by Eq. (v), then

$$\frac{p}{1-p} = \frac{a}{x} \text{ or } px = a - ap$$

$\therefore$ $$p = \frac{a}{(a+x)}$$

Similarly, $q = \dfrac{b}{b+x}$ and $r = \dfrac{c}{c+x}$

Replacing the values of p, q and r in Eq. (iv), then

$$\left\{\left(1 - \frac{a}{a+x}\right)\left(1 - \frac{b}{b+x}\right)\left(1 - \frac{c}{c+x}\right)\right\}^2 = x^2$$

$\Rightarrow$ $$\frac{(x^3)^2}{(a+x)^2(b+x)^2(c+x)^2} = x^2$$

$\Rightarrow$ $$\frac{x^3}{(a+x)(b+x)(c+x)} = x$$

or $$(a+x)(b+x)(c+x) = x^2$$

Hence, x is a root of the equation $(a+x)(b+x)(c+x) = x^2$

112. Let $A = \{a_1, a_2, \ldots, a_n\}$

For each $a_i \in A$ $(1 \le i \le n)$, we have the following choices.

(i) $a_i \in P$ and $a_i \in Q$ (ii) $a_i \in P$ and $a_i \notin Q$

(iii) $a_i \notin P$ and $a_i \in Q$ (iv) $a_i \notin P$ and $a_i \notin Q$

Therefore, for one element a_i of A, total number of cases is 4.

Let S be the sample space

$\therefore$ $$n(S) = 4^n$$

and number of cases in which $a_i \in P \cup Q$ is 3, since case $4 \notin P \cup Q$ and let E be the event of favourable cases, then

$n(E)$ = number of ways in which exactly r elements of A will belong to $P \cup Q$

$$= {}^nC_r (3)^r\, 1^{n-r} = {}^nC_r 3^r$$

$\therefore$ Required probability, $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{{}^nC_r 3^r}{4^n}$

113. Given that $P(A) = \alpha$, $P(B/A) = P(B'/A') = 1 - \alpha$

Thus, $P(A') = 1 - P(A) = 1 - \alpha$

and $P(B/A') = 1 - P(B'/A') = 1 - (1-\alpha) = \alpha$ ...(i)

$\therefore$ $$P(A'/B) = \frac{P(A' \cap B)}{P(B)}$$

$$= \frac{P(B) - P(A \cap B)}{P(B)} = \frac{P(B) - P(A) \cdot P(B/A)}{P(B)}$$

$$\left[\because P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)}\right]$$

$$= \frac{P(B) - \alpha(1-\alpha)}{P(B)} \quad \text{...(ii)}$$

But $$P(B) = P(A) \cdot P(B/A) + P(A') \cdot P(B/A')$$

$$= \alpha(1-\alpha) + (1-\alpha) \cdot \alpha \quad \text{[from Eq. (i)]}$$

$$= 2\alpha(1-\alpha) \quad \text{...(3)}$$

Putting the value of $P(B)$ from Eq. (iii) in Eq. (ii), then

$$P\left(\frac{A'}{B}\right) = \frac{2\alpha(1-\alpha) - \alpha(1-\alpha)}{2\alpha(1-\alpha)} = \frac{\alpha(1-\alpha)}{2\alpha(1-\alpha)} = \frac{1}{2}$$

which is independent of α.

114. Let S be the sample space and E be the event that each of the n pairs of balls drawn consists of one white and one red ball.

$\therefore$ $$n(S) = ({}^{2n}C_2)({}^{2n-2}C_2)({}^{2n-4}C_2) \ldots ({}^4C_2)({}^2C_2)$$

$$= \left\{\frac{(2n)(2n-1)}{1 \cdot 2}\right\}\left\{\frac{(2n-2)(2n-3)}{1 \cdot 2}\right\}\left\{\frac{(2n-4)(2n-5)}{1 \cdot 2}\right\} \ldots \left\{\frac{4 \cdot 3}{1 \cdot 2}\right\}\left\{\frac{2 \cdot 1}{1 \cdot 2}\right\}$$

$$= \frac{1 \cdot 2 \cdot 3 \cdot 4 \ldots (2n-1)\, 2n}{2^n} = \frac{2n!}{2^n}$$

and $$n(E) = ({}^nC_1 \cdot {}^nC_1)({}^{n-1}C_1 \cdot {}^{n-1}C_1)({}^{n-2}C_1 \cdot {}^{n-2}C_1) \ldots ({}^2C_1 \cdot {}^2C_1)({}^1C_1 \cdot {}^1C_1)$$

$$= n^2 \cdot (n-1)^2 \cdot (n-2)^2 \ldots 2^2 \cdot 1^2 = [1 \cdot 2 \cdot 3 \ldots (n-1)\, n]^2 = (n!)^2$$

$\therefore$ Required probability,

$$P(E) = \frac{n(E)}{n(S)} = \frac{(n!)^2}{(2n)!/2^n} = \frac{2^n}{\dfrac{2n!}{(n!)^2}} = \frac{2^n}{{}^{2n}C_n}$$

115. Let p be the probability that any one thing is received by a men and q be the probability that any one thing is received by a women.

$\therefore$ $$p = \frac{a}{a+b} \text{ and } q = \frac{b}{a+b}$$

Clearly, $p + q = 1$ i.e., $q = 1 - p$

Out of m things, if r are received by a men, then the rest $(m-r)$ will be received by women.

The probability for this to happen is given by

$$P(r) = {}^{m}C_r p^r q^{m-r} \quad [r = 0, 1, ..., m]$$

The probability P that odd number of things are received by men is given by

$$P = P(1) + P(3) + P(5) + \ldots$$
$$= {}^{m}C_1 pq^{m-1} + {}^{m}C_3 p^3 q^{m-3} + {}^{m}C_5 p^5 q^{m-5} + \ldots \quad ...(i)$$

We know that,

$$(q+p)^m = q^m + {}^{m}C_1 q^{m-1} p + {}^{m}C_2 q^{m-2} p^2 + \ldots + p^m \quad ...(ii)$$

and $(q-p)^m = q^m - {}^{m}C_1 q^{m-1} p + {}^{m}C_2 q^{m-2} p^2 - \ldots + (-1)^m p^m$ ...(iii)

Subtracting Eq.(iii) from Eq. (ii), then

$(q+p)^m - (q-p)^m = 2\{{}^{m}C_1 q^{m-1} p + {}^{m}C_3 q^{m-3} p^3 + \ldots\}$

$= 2P$ [from Eq. (i)]

$$\therefore \quad P = \frac{1}{2}\{(q+p)^m - (q-p)^m\}$$

$$= \frac{1}{2}\left\{1 - \left(\frac{b-a}{b+a}\right)^m\right\} = \frac{1}{2}\left\{\frac{(b+a)^m - (b-a)^m}{(b+a)^m}\right\}$$

116. $\because P(C) = \frac{1}{7}, P(S) = \frac{3}{7}, P(B) = \frac{2}{7}, P(T) = \frac{1}{7}$

Let E be the event that person reaches late

$$\therefore \; P\left(\frac{E}{C}\right) = \frac{2}{9}, P\left(\frac{E}{S}\right) = \frac{1}{9}, P\left(\frac{E}{B}\right) = \frac{4}{9}, P\left(\frac{E}{T}\right) = \frac{1}{9}$$

To find $P\left(\frac{C}{E}\right)$ [$\because$ reaches in time = not late]

Using Baye's Theorem

$$P\left(\frac{C}{E}\right) = \frac{P(C).P\left(\frac{\overline{E}}{C}\right)}{P(C).P\left(\frac{\overline{E}}{C}\right) + P(S).P\left(\frac{\overline{E}}{S}\right) + P(B).P\left(\frac{\overline{E}}{B}\right) + P(T).P\left(\frac{\overline{E}}{T}\right)}$$

$$= \frac{\frac{1}{7}\times\left(1-\frac{2}{9}\right)}{\frac{1}{7}\times\left(1-\frac{2}{9}\right) + \frac{3}{7}\times\left(1-\frac{1}{9}\right) + \frac{2}{7}\times\left(1-\frac{4}{9}\right) + \frac{1}{7}\times\left(1-\frac{1}{9}\right)}$$

$$= \frac{7}{7+3\times8+2\times5+8} = \frac{7}{49} = \frac{1}{7}$$

117. Probability of getting 1 is $\frac{1}{6}$ and probability of not getting 1 is $\frac{5}{6}$.

Then, getting 1 in even number of chances = getting 1 in 2nd chance or in 4th chance or in 6th chance and so on.

$\therefore$ Required probability

$$= \frac{5}{6}\times\frac{1}{6} + \left(\frac{5}{6}\right)^3 \times\frac{1}{6} + \left(\frac{5}{6}\right)^6 \times\frac{1}{6} + \ldots\infty$$

$$= \frac{\frac{5}{36}}{1-\frac{25}{36}} = \frac{5}{11}$$

118. $P(\overline{A \cup B}) = \frac{1}{6}$; $P(A \cap B) = \frac{1}{4}$,

$$P(\overline{A}) = \frac{1}{4} \Rightarrow P(A) = \frac{3}{4},$$

$$P(\overline{A \cup B}) = 1 - P(A \cup B)$$

$$= 1 - P(A) - P(B) + P(A \cap B)$$

$$\Rightarrow \quad \frac{1}{6} = \frac{1}{4} - P(B) + \frac{1}{4}$$

$$\Rightarrow \quad P(B) = \frac{1}{3}$$

Since, $P(A \cap B) = P(A)\cdot P(B)$ and $P(A) \neq P(B)$

$\therefore$ A and B are independent but not equally likely.

119. For a particular house being selected, probability $= \frac{1}{3}$

Probability (all the persons apply for the same house)

$$= \left(\frac{1}{3}\times\frac{1}{3}\times\frac{1}{3}\right)3 = \frac{1}{9}$$

120. $P(X > 1.5) = 1 - P(X = 0) - P(X = 1)$

$$P(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}$$

$$\therefore \quad P(x > 1.5) = 1 - \frac{1}{e^2} - \frac{2}{e^2} = 1 - \frac{3}{e^2} \qquad [\because \lambda = np = 2]$$

121. (i) $P(u_i) = ki, \Sigma P(u_i) = 1 \Rightarrow k = \frac{2}{n(n+1)}$

$$\lim_{n\to\infty} P(w) = \lim_{n\to\infty}\sum_{i=1}^{\infty}\frac{2i^2}{n(n+1)^2}$$

$$= \lim_{n\to\infty}\frac{2n(n+1)(2n+1)}{n(n+1)^2 6} = \frac{2}{3}$$

(ii) $$P\left(\frac{u_n}{w}\right) = \frac{P(u_n \cap w)}{P(w)} = \frac{P(u_n)\cdot P\left(\frac{w}{u_n}\right)}{\sum_{i=1}^{n} P(u_n)\cdot P\left(\frac{w}{u_i}\right)}$$

$$\frac{c\frac{n}{n+1}}{c\sum_{i=1}^{n}\frac{i}{(n+1)}} = \frac{n.2}{n(n+1)} = \frac{2}{n+1}$$

(iii) $E = u_2 \cup u_4 \cup u_6 \cup \ldots \cup u_n$

$$P(E) = P(u_2) + P(u_4) + \ldots + P(u_n)$$

$$= \frac{1}{n} + \frac{1}{n} + \ldots + \frac{1}{n} = \frac{1}{n}\frac{n}{2} = \frac{1}{2}$$

$$P\left(\frac{w}{E}\right) = \frac{P(w \cap E)}{P(E)}$$

$$= \frac{P(w \cap u_2) + P(w \cap u_4) + \ldots + P(w \cap u_n)}{\frac{1}{2}}$$

$$= 2\left[\frac{1}{n}.\frac{2}{n+1} + \frac{1}{n}.\frac{4}{n+1} + \ldots + \frac{1}{n}.\frac{n}{n+1}\right]$$

$$= \frac{2}{n}.\frac{\frac{n}{4}(2+n)}{n+1} = \frac{n+2}{2(n+1)}$$

122. $P(X=r)=\dfrac{e^{-m}m^r}{r!}$

$=P(X\le 1)=P(X=0)+P(X=1)$

$=e^{-5}+5\times e^{-5}=\dfrac{6}{e^5}$ $[\because m=\text{mean}=5]$

123. Let E be the event when each American man is seated adjacent to his wife and A be the event when Indian man is seated adjacent to his wife.

Now, $n(A\cap E)=(4!)\times(2!)^5$ and $n(E)=(5!)\times(2!)^4$

$\Rightarrow \quad P\left(\dfrac{A}{E}\right)=\dfrac{P(A\cap E)}{P(E)}=\dfrac{n(A\cap E)}{n(E)}=\dfrac{(4!)\times(2!)^5}{(5!)\times(2!)^4}=\dfrac{2}{5}$

124. **Statement-1** If $P(H_i\cap E)=0$ for some i, then

$$P\left(\frac{H_i}{E}\right)=P\left(\frac{E}{H_i}\right)=0$$

If $P(H_i\cap E)\ne 0,\ \forall\ i=1,2,3,\ldots,n$, then

$P\left(\dfrac{H_i}{E}\right)=\dfrac{P(H_i\cap E)}{P(E)}=\dfrac{P(H_i\cap E)}{P(H_i)}\times\dfrac{P(H_i)}{P(E)}$

$=P\left(\dfrac{E}{H_i}\right)\times\dfrac{P(H_i)}{P(E)}>P\left(\dfrac{E}{H_i}\right).P(H_i)$ [as $0<P(E)<1$]

Hence, Statement-1 may not always be true.

Statement -2 Clearly, $H_1\cup H_2\cup\ldots\cup H_n=S$ (Sample space)

$\Rightarrow\ P(H_1)+P(H_2)+\ldots+P(H_n)=1$

125. $P\left(\dfrac{E^c\cap F^c}{G}\right)=\dfrac{P(E^c\cap F^c\cap G)}{P(G)}$

$=\dfrac{P(G)-P(E\cap G)-P(F\cap G)}{P(G)}$

$=\dfrac{P(G)-P(E)\cdot P(G)-P(F)\cdot P(G)}{P(G)}$

$=1-P(E)-P(F)=P(E^c)-P(F)$

126. Probability of getting sum of nine in a single thrown $=\dfrac{1}{9}$

$\therefore$ Probability of getting sum nine exactly two times out of three draws $={}^3C_2\left(\dfrac{1}{9}\right)^2\left(\dfrac{8}{9}\right)=\dfrac{8}{243}$

127. $P(I)=0.3,\ P(II)=0.2$

$\therefore$ Required probability

$=P(\bar{I})P(II)=(1-P(I))P(II)=(1-0.3)\times 0.2=0.7\times 0.2=0.14$

128. Since, $P(A)=\dfrac{2}{5}$

For independent events,

$$P(A\cap B)=P(A)P(B)\ \Rightarrow\ P(A\cap B)\le\frac{2}{5}$$

$\Rightarrow \quad P(A\cap B)=\dfrac{1}{10},\dfrac{2}{10},\dfrac{3}{10},\dfrac{4}{10}$

[maximum 4 outcomes may be in $A\cap B$]

(i) Now, $P(A\cap B)=\dfrac{1}{10}$

$\Rightarrow \quad P(A)\cdot P(B)=\dfrac{1}{10}$

$\Rightarrow \quad P(B)=\dfrac{1}{10}\times\dfrac{5}{2}=\dfrac{1}{4}$, not possible

(ii) Now, $P(A\cap B)=\dfrac{2}{10}\ \Rightarrow\ \dfrac{2}{5}\times P(B)=\dfrac{2}{10}$

$\Rightarrow \quad P(B)=\dfrac{5}{10}$, outcomes of $B=5$

(iii) Now, $P(A\cap B)=\dfrac{3}{10}$

$\Rightarrow \quad P(A)P(B)=\dfrac{3}{10}\ \Rightarrow\ \dfrac{2}{5}\times P(B)=\dfrac{3}{10}$

$P(B)=\dfrac{3}{4}$, not possible

(iv) Now, $P(A\cap B)=\dfrac{4}{10}\ \Rightarrow\ P(A)\cdot P(B)=\dfrac{4}{10}$

$\Rightarrow \quad P(B)=1$, outcomes of $B=10$.

129. For unique solution $\begin{vmatrix}a & b\\ c & d\end{vmatrix}\ne 0$,

where $a, b, c, d\in\{0,1\}$. Total cases = 16

Favourable cases = 6 (Either $ad=1, bc=0$ or $ad=0, bc=1$)

Probability that system of equations has unique solution is $\dfrac{6}{16}=\dfrac{3}{8}$ and system of equations has either unique solution or infinite solutions, so that probability for system to have a solution is 1.

130. $A=\{4,5,6\},\ B=\{1,2,3,4\}$

$\therefore \quad A\cup B=\{1,2,3,4,5,6\}$

$\Rightarrow \quad n(A\cup B)=6$ and total ways $=6$

$\therefore \quad P(A\cup B)=\dfrac{6}{6}=1$

131. $\because \quad P\left(\dfrac{A}{B}\right)=\dfrac{1}{2}\ \Rightarrow\ \dfrac{P(A\cap B)}{P(B)}=\dfrac{1}{2}$ ...(i)

and $P\left(\dfrac{B}{A}\right)=\dfrac{2}{3}\ \Rightarrow\ \dfrac{P(A\cap B)}{P(A)}=\dfrac{2}{3}$ ...(ii)

On dividing Eq. (i) by Eq. (ii), we get

$$\frac{P(A)}{P(B)}=\frac{3}{4}\ \Rightarrow\ P(B)=\frac{4}{3}P(A)$$

$=\dfrac{4}{3}\times\dfrac{1}{4}=\dfrac{1}{3}\quad\left[\because P(A)=\dfrac{1}{4}\right]$

132. $P(X=3)=\left(\dfrac{5}{6}\right)\left(\dfrac{5}{6}\right)\dfrac{1}{6}=\dfrac{25}{216}$

133. $P(X\ge 3)=1-P(X\le 2)=1-\left\{\dfrac{1}{6}+\dfrac{5}{6}\times\dfrac{1}{6}\right\}=1-\dfrac{11}{36}=\dfrac{25}{36}$

134. $P(X\ge 6)=\dfrac{5^5}{6^6}+\dfrac{5^6}{6^7}+\ldots\infty=\dfrac{\frac{5^5}{6^6}}{1-\frac{5}{6}}=\left(\dfrac{5}{6}\right)^5$

and $P(X>3)=\dfrac{5^3}{6^4}+\dfrac{5^4}{6^5}+\dfrac{5^5}{6^6}+\ldots\infty=\dfrac{\frac{5^3}{6^4}}{1-\frac{5}{6}}=\left(\dfrac{5}{6}\right)^3$

Hence, the conditional probability $=\dfrac{\left(\frac{5}{6}\right)^5}{\left(\frac{5}{6}\right)^3}=\dfrac{25}{36}$

135. $1-q^n \geq \frac{9}{10} \Rightarrow q^n \leq \frac{1}{10} \Rightarrow \left(\frac{3}{4}\right)^n \leq \frac{1}{10}$

$\Rightarrow n(\log_{10}3 - \log_{10}4) \leq 0-1 \Rightarrow n \geq \frac{1}{(\log_{10}4 - \log_{10}3)}$

136. $S = \{00, 01, 02,..., 49\}$

Let A be the event that sum of the digits on the selected ticket is 8, then$A = \{08, 17, 26, 26, 35, 44\}$

and let B be the event that the product of the digits is zero, then

$B = \{00, 01, 02, ..., 09, 10, 20, 30, 40\}$

$\therefore \quad A \cap B = \{08\}$

$\therefore$ Required probability $= P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{50}}{\frac{14}{50}} = \frac{1}{14}$

137. Required probability $= \frac{2 \times 2 \times 2(3!)}{6^3} = \frac{2}{9}$

138. Let E_1 denote original signal is green, E_2 denote original signal is red and E denote signal received at station B is green.

$$\therefore \quad P\left(\frac{E_1}{E}\right) = \frac{P(E_1)\cdot P\left(\frac{E}{E_1}\right)}{P(E_1)\cdot P\left(\frac{E}{E_1}\right) + P(E_2)\cdot P\left(\frac{E}{E_2}\right)}$$

$$= \frac{\frac{4}{5}\left[\left(\frac{3}{4}\right)^2 + \left(\frac{1}{4}\right)^2\right]}{\frac{4}{5}\left[\left(\frac{3}{4}\right)^2 + \left(\frac{1}{4}\right)^2\right] + \frac{1}{5}\left[\frac{3}{4}\times\frac{1}{4} + \frac{1}{4}\times\frac{3}{4}\right]} = \frac{40}{46} = \frac{20}{23}$$

139. Total ways $= {}^{20}C_4 = \frac{20 \,.\, 19 \,.\, 18 \,.\, 17}{1 \,.\, 2 \,.\, 3 \,.\, 4} = 4845$

Statement-1

Common difference (d)	Number of cases
1	17
2	14
3	11
4	8
5	5
6	2

$\therefore$ Number of favourable cases $= 17 + 14 + 11 + 8 + 5 + 2$

$= 57$

$\therefore$ Required probability $= \frac{57}{4845} = \frac{1}{85}$

Statement-1 is true and Statement-2 is false.

140. Total ways, ${}^9C_3 = \frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} = 84$

Favourable ways $= {}^3C_1 \times {}^4C_1 \times {}^2C_1 = 24$

$\therefore$ Required probability $= \frac{24}{84} = \frac{2}{7}$

141. $H \rightarrow 1$ ball from U_1 to U_2

$T \rightarrow 2$ balls from U_1 to U_2

E : 1 ball drawn from U_2

$$\therefore \; P(W \text{ from } U_2) = \frac{1}{2}\times\left(\frac{3}{5}\times 1\right) + \frac{1}{2}\times\left(\frac{2}{5}\times\frac{1}{2}\right)$$

$$+\frac{1}{2}\times\left(\frac{{}^3C_2}{{}^5C_2}\times\frac{1}{3}\right) + \frac{1}{2}\times\left(\frac{{}^3C_1 \,.\, {}^2C_1}{{}^5C_2}\times\frac{2}{3}\right) = \frac{23}{30}$$

142. $$P\left(\frac{H}{W}\right) = \frac{P(H)\times P\left(\frac{W}{H}\right)}{P(H)\times P\left(\frac{W}{H}\right) + P(T)\times P\left(\frac{W}{T}\right)} = \frac{\frac{1}{2}\left(\frac{3}{2}\times 1 + \frac{2}{5}\times\frac{1}{2}\right)}{\frac{23}{30}}$$

$$= \frac{12}{23}$$

143. Let $P(E) = e$ and $P(F) = f$

$$\Rightarrow \quad P(E \cup F) - P(E \cap F) = \frac{11}{25}$$

$$\Rightarrow \quad P(E) + P(F) - 2P(E \cap F) = \frac{11}{25}$$

$$\Rightarrow \quad e + f - 2ef = \frac{11}{25} \qquad \text{...(i)}$$

$$P(\overline{E} \cap \overline{F}) = \frac{2}{25} \Rightarrow P(\overline{E}).\, P(\overline{F}) = \frac{2}{25} \Rightarrow (1-e)(1-f) = \frac{2}{25}$$

$$\Rightarrow \quad e + f - ef = \frac{23}{25} \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get $ef = \frac{12}{25}$ and $e + f = \frac{7}{5}$

On solving, we get $e = \frac{4}{5}$, $f = \frac{3}{5}$ or $e = \frac{3}{5}$, $f = \frac{4}{5}$

144. Given probability of atleast one failure $\geq \frac{31}{32}$

$$\Rightarrow \quad 1 - P(X = 0) \geq \frac{31}{32}$$

$$\Rightarrow \; 1 - {}^5C_0 \cdot Q^0 \cdot P^5 \geq \frac{31}{32} \qquad [\because (P+Q)^5]$$

$$\Rightarrow \quad P^5 \leq \frac{1}{32}$$

$$\therefore \quad P \leq \frac{1}{2} \text{ and } P \geq 0 \;\Rightarrow\; P \in \left[0, \frac{1}{2}\right]$$

145. We have, $C \subset D \Rightarrow C \cap D = C$

$$\Rightarrow P\left(\frac{C}{D}\right) = \frac{P(C \cap D)}{P(D)} = \frac{P(C)}{P(D)} \geq P(C) \qquad [\because\, 0 < P(D) \leq 1]$$

146. $$P\left(\frac{A^c \cap B^c}{C}\right) = \frac{P(A^c \cap B^c \cap C)}{P(C)}$$

$$= \frac{P(C) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)}{P(C)}$$

$$= \frac{P(C) - P(A)\cdot P(C) - P(B)\cdot P(C) + 0}{P(C)}$$

[$\because$ A, B, C are pairwise independent]

$$= 1 - P(A) - P(B) = P(A^c) - P(B)$$

147. $P(X) = P(X_1 \cap X_2 \cap X_3) + P(\overline{X}_1 \cap X_2 \cap X_3)$
$+ P(X_1 \cap \overline{X}_2 \cap X_3) + P(X_1 \cap X_2 \cap \overline{X}_3)$
$= \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{4} = \frac{1}{4}$

(a) $P\left(\frac{\overline{X}_1}{X}\right) = \frac{\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4}}{\frac{1}{4}} = \frac{1}{8}$

(b) P (exactly two engines of the ship are functioning / X)
$= \frac{\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{4}}{1/4} = \frac{7}{8}$

(c) $P\left(\frac{X}{X_2}\right)$ = Probability that X occurs given that engine E_2 has started $\frac{P(X_1 \cap X_2 \cap X_3) + P(\overline{X}_1 \cap X_2 \cap X_3)}{P(X_1 \cap X_2 \cap X_3) + P(\overline{X}_1 \cap X_2 \cap X_3)}$
$+ P(X_1 \cap X_2 \cap \overline{X}_3)$
$+P(X_1 \cap X_2 \cap \overline{X}_3) + P(\overline{X}_1 \cap X_2 \cap \overline{X}_3) = \frac{5}{8}$

(d) $P\left(\frac{X}{X_1}\right)$ = Probability that X occurs given that engine E_1 has started $\frac{P(X_1 \cap X_2 \cap X_3)}{P(X_1 \cap X_2 \cap X_3) + P(X_1 \cap \overline{X}_2 \cap X_3)}$
$+ P(X_1 \cap \overline{X}_2 \cap X_3) + P(X_1 \cap X_2 \cap \overline{X}_3)$
$+P(X_1 \cap X_2 \cap \overline{X}_3) + P(X_1 \cap \overline{X}_2 \cap \overline{X}_3) = \frac{7}{16}$

148. ***Case* I** When D_1, D_2, D_3 all show different number and one of the number is shown by D_4, $P(E_1) = \frac{{}^6C_3 \times 3!}{216} \times \frac{3}{6} = \frac{60}{216}$

***Case* II** When D_1, D_2, D_3 all show same number and that number is shown byD_4, $P(E_2) = 6 \times \left(\frac{1}{6}\right)^4 = \frac{1}{216}$

***Case* III** When two numbers shown by D_1, D_2, D_3 are same and one is different and one of the number is shown by D_4,

$$P(E_3) = \frac{{}^6C_1 \times {}^5C_1}{216} \times \frac{3!}{2!} \times \frac{2}{6} = \frac{30}{216}$$

$\therefore$ Required probability $= P(E_1) + P(E_2) + P(E_3) = \frac{91}{216}$

149. $P\left(\frac{X}{Y}\right) = \frac{P(X \cap Y)}{P(Y)} = \frac{1}{2} \Rightarrow P(Y) = \frac{1}{3}$

$P\left(\frac{Y}{X}\right) = \frac{P(X \cap Y)}{P(X)} = \frac{1}{3} \Rightarrow P(X) = \frac{1}{2}$

(a) $P(X \cup Y) = P(X) + P(Y) - P(X \cap Y) = \frac{2}{3}$

(b) $\because P(X \cap Y) = P(X) \cdot P(Y)$, they are independent. Also, X^c and Y will be independent

Now, $P(X^c \cap Y) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6} \neq \frac{1}{3}$.

150. $\because S = \{1, 2, 3, \ldots, 8\}$

Let A : Maximum of three numbers is 6

$\therefore A = \{1, 2, 3, 4, 5, 6\}$

and B : Maximum of three numbers is 3

$\therefore B = \{3, 4, 5, 6, 7, 8\}$ and $A \cap B = \{3, 4, 5, 6\}$

$$\Rightarrow P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{n(A \cap B)}{n(S)}}{\frac{n(B)}{n(S)}} = \frac{n(A \cap B)}{n(B)} = \frac{{}^2C_1}{{}^5C_2} = \frac{1}{5}$$

151. Here, $p = \frac{1}{3}$ and $q = 1 - \frac{1}{3} = \frac{2}{3}$

$\therefore$ Required probability $= P(X = 4) + P(X = 5)$ $\left[\because \left(\frac{2}{3} + 1\right)^5\right]$

$$= {}^5C_4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right) + {}^5C_5 \left(\frac{1}{3}\right)^5 = \frac{11}{3^5}$$

152. Probability of solving the problem correctly by atleast one of them $= 1 - \left(\left(1 - \frac{1}{2}\right)\left(1 - \frac{3}{4}\right)\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{8}\right)\right) = \frac{235}{256}$

153. Let x, y and z be the probability of occurrence of E_1, E_2 and E_3, respectively.

Then, $\alpha = x(1-y)(1-z) = \frac{px}{(1-x)}$ $[\because p = (1-x)(1-y)(1-z)]$

Similarly, $\beta = \frac{py}{(1-y)}$ and $\gamma = \frac{pz}{(1-z)}$

Now, $(\alpha - 2\beta)\, p = \alpha\beta$

$$\Rightarrow \frac{p}{\beta} - \frac{2p}{\alpha} = 1$$

$$\Rightarrow \frac{(1-y)}{y} - \frac{2(1-x)}{x} = 1$$

$$\Rightarrow x = 2y \quad \ldots(i)$$

and $(\beta - 3\gamma)\, p = 2\beta$

$$\Rightarrow \frac{p}{\gamma} - \frac{3p}{\beta} = 2$$

$$\Rightarrow \frac{(1-z)}{z} - \frac{3(1-y)}{y} = 2$$

$$\Rightarrow y = 3z \quad \ldots(ii)$$

From Eqs. (i) and (ii), we get $x = 6z$ $\therefore$ $\frac{x}{z} = 6$

154. Let E = Event that one ball is white and the other ball is red.

Then, $P\left(\frac{B_2}{E}\right) = \frac{\frac{{}^2C_1 \times {}^3C_1}{{}^9C_2}}{\frac{{}^1C_1 \times {}^3C_1}{{}^6C_2} + \frac{{}^2C_1 \times {}^3C_1}{{}^9C_2} + \frac{{}^3C_1 \times {}^4C_1}{{}^{12}C_2}} = \frac{55}{181}$

155. Required probability $= \prod_{i=1}^{3} P(W_i) + \prod_{i=1}^{3} P(R_i) + \prod_{i=1}^{3} P(B_i)$

$$= \frac{1}{6} \times \frac{2}{9} \times \frac{3}{12} + \frac{3}{6} \times \frac{3}{9} \times \frac{4}{12} + \frac{2}{6} \times \frac{4}{9} \times \frac{5}{12}$$

$$= \frac{82}{648}$$

156. $\because P(A \cup B) = 1 - P(\overline{A \cup B}) = 1 - \frac{1}{6} = \frac{5}{6}$

$$P(A) = 1 - P(\overline{A}) = 1 - \frac{1}{4} = \frac{3}{4}$$

and $P(B) = P(A \cup B) - P(A) + P(A \cap B)$

$$= \frac{5}{6} - \frac{3}{4} + \frac{1}{4} = \frac{1}{3}$$

$\Rightarrow A$ and B are not equally likely.

Also, $P(A \cap B) = \frac{1}{4} = \frac{3}{4} \times \frac{1}{3} = P(A)\,.\,P(B)$

$\therefore$ A and B are independent.

Hence, A and B are independent but not equally likely.

157. $n(S) = 5! = 120$ and possible favourable cases are

$(B, G, G, B, B), (G, G, B, B, B), (G, B, G, B, B),$

$(G, B, B, G, B), (B, G, B, G, B)$

$\therefore$ Number of favourable cases $= n(E) = 5 \times 12 = 60$

$\therefore$ Required probability,

$$P(E) = \frac{n(E)}{n(S)} = \frac{60}{120} = \frac{1}{2}$$

158. $n(S) = 3 \times 5 \times 7 = 105$; $x_1 + x_2 + x_3 =$ odd

***Case* I** All three odd $= 2 \times 3 \times 4 = 24$

***Case* II** Two even and one odd

$= 1 \times 2 \times 4 + 1 \times 3 \times 3 + 2 \times 2 \times 3 = 29 \therefore n(E) = 24 + 29 = 53$

Required probability, $P(E) = \frac{n(E)}{n(S)} = \frac{53}{105}$

159. x_1, x_2, x_3 are in AP.

AP with common difference $= 1$, $(1, 2, 3)$ $(2, 3, 4)$ $(3, 4, 5)$

AP with common difference $= 2$, $(1, 3, 5)$, $(2, 4, 6)$, $(3, 5, 7)$

AP with common difference $= 3$, $(1, 4, 7)$

AP with common difference $= 0$, $(1, 1, 1)$, $(2, 2, 2)$ $(3, 3, 3)$

$\therefore$ $n(E) = 10$

$\therefore$ Required probability, $P(E) = \frac{n(E)}{n(S)} = \frac{10}{105}$

160. We have mentioned that boxes are different and one particular box has 3 balls.

Then number of ways $= \frac{{}^{12}C_3 \times 2^9}{3^{12}} = \frac{55}{3}\left(\frac{2}{3}\right)^{11}$

161. Using Binomial distribution

$$P(X \geq 2) = 1 - P(X = 0) - P(X = 1)$$

$$= 1 - \left(\frac{1}{2}\right)^n - \left[{}^nC_1 \cdot \left(\frac{1}{2}\right) \cdot \left(\frac{1}{2}\right)^{n-1}\right]$$

$$= 1 - \frac{1}{2^n} - {}^nC_1 \cdot \frac{1}{2^n} = 1 - \left(\frac{1+n}{2^n}\right)$$

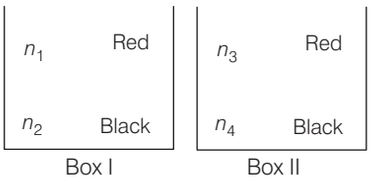
Given, $P(X \geq 2) \geq 0.96$

$$1 - \frac{(n+1)}{2^n} \geq \frac{24}{25}$$

$$\Rightarrow \quad \frac{(n+1)}{2^n} \leq \frac{1}{25}$$

$$\Rightarrow \quad n = 8$$

162.



Let A = Drawing red ball

$$\therefore \quad P(A) = P(B_1) \cdot P(A/B_1) + P(B_2) \cdot P(A/B_2)$$

$$= \frac{1}{2}\left(\frac{n_1}{n_1 + n_2}\right) + \frac{1}{2}\left(\frac{n_3}{n_3 + n_4}\right)$$

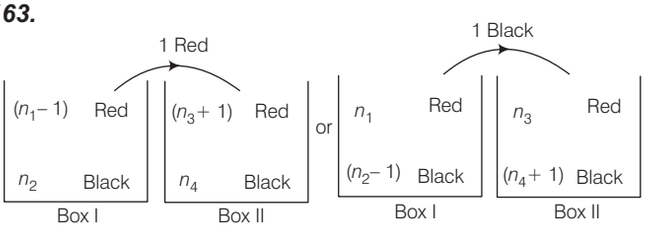
Given, $P(B_2/A) = \frac{1}{3} \Rightarrow \frac{P(B_2) \cdot P(B_2 \cap A)}{P(A)} = \frac{1}{3}$

$$\Rightarrow \quad \frac{\frac{1}{2}\left(\frac{n_3}{n_3 + n_4}\right)}{\frac{1}{2}\left(\frac{n_1}{n_1 + n_2}\right) + \frac{1}{2}\left(\frac{n_3}{n_3 + n_4}\right)} = \frac{1}{3}$$

$$\Rightarrow \quad \frac{n_3(n_1 + n_2)}{n_1(n_3 + n_4) + n_3(n_1 + n_2)} = \frac{1}{3}$$

Now, check options, then clearly options (a) and (b) satisfy.

163.



$\therefore$ P (drawing red ball from B_1) $= \frac{1}{3}$

$$\Rightarrow \quad \left(\frac{n_1 - 1}{n_1 + n_2 - 1}\right)\left(\frac{n_1}{n_1 + n_2}\right) + \left(\frac{n_2}{n_1 + n_2}\right)\left(\frac{n_1}{n_1 + n_2 - 1}\right) = \frac{1}{3}$$

$$\Rightarrow \quad \frac{n_1^2 + n_1 n_2 - n_1}{(n_1 + n_2)(n_1 + n_2 - 1)} = \frac{1}{3}$$

Clearly, options (c) and (d) satisfy.

164. $\because$ $P(E_1) = \frac{1}{6}, P(E_2) = \frac{1}{6},$

$$P(E_3) = \frac{2 + 4 + 6 + 4 + 2}{36} = \frac{1}{2}$$

Also, $P(E_1 \cap E_2) = \frac{1}{36},$

$$P(E_1 \cap E_3) = \frac{1}{12}, P(E_3 \cap E_1) = \frac{1}{12}$$

and $P(E_1 \cap E_2 \cap E_3) = 0 \neq P(E_1).P(E_2).P(E_3)$

Hence, E_1, E_2, E_3 are not independent

165. $P(T_1) = \frac{200}{100} = \frac{1}{5}, P(T_2) = \frac{80}{100} = \frac{4}{5},$

$P(D) = \frac{7}{100}$. Let $P\left(\frac{D}{T_2}\right) = x$, then

$$P\left(\frac{D}{T_1}\right) = 10 \cdot P\left(\frac{D}{T_2}\right) = 10x$$

$\because$ $P(D) = P(T_1) \times P\left(\frac{D}{T_1}\right) + P(T_2) \times P\left(\frac{D}{T_2}\right)$

$\Rightarrow$ $\frac{7}{100} = \frac{1}{5} \times 10x + \frac{4}{5} \times x$

$\therefore$ $x = \frac{1}{40}$

$\therefore$ $P\left(\frac{T_2}{\overline{D}}\right) = \frac{P(T_2) \times P\left(\frac{\overline{D}}{T_2}\right)}{P(T_1) \times P\left(\frac{\overline{D}}{T_1}\right) + P(T_2) \times P\left(\frac{\overline{D}}{T_2}\right)}$

$$= \frac{P(T_2) \times P\left(\frac{\overline{D}}{T_2}\right)}{P(\overline{D})}$$

$$= \frac{\frac{4}{5} \times \frac{39}{40}}{\frac{93}{100}}$$

$$= \frac{78}{93}$$

166. $P(x > y) = P(T_1$ wins 2 games or T_1 wins either of the matches and other is draw)

$$= \left(\frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{6} + \frac{1}{6} \times \frac{1}{2}\right)$$

$$= \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$$

167. $P(x = y) = P(T_1$ and T_2 win alternately)

$+ P$ (Both matches are draws)

$$= \left(\frac{1}{2} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{2}\right) + \left(\frac{1}{6} \times \frac{1}{6}\right)$$

$$= \frac{1}{3} + \frac{1}{36} = \frac{13}{36}$$

168. $P = \frac{15}{25} = \frac{3}{5}, Q = 1 - P = 1 - \frac{3}{5} = \frac{2}{5}$

and $n = 10$

$\therefore$ Variance $= nPQ = 10 \times \frac{3}{5} \times \frac{2}{5} = \frac{12}{5}$

169. **Cases**: (0, 4), (0, 8), (2, 6), (2, 10), (4, 8), (6, 10)

$\therefore$ Required probability $= \frac{6}{{}^1C_2} = \frac{6}{55}$

170. $\because$ $P(A) + P(B) - 2P(A \cap B) = \frac{1}{4}$ ...(i)

$P(B) + P(C) - 2P(B \cap C) = \frac{1}{4}$ ...(ii)

$P(C) + P(A) - 2P(C \cap A) = \frac{1}{4}$ ...(iii)

and $P(A \cap B \cap C) = \frac{1}{16}$ ...(iv)

Now, adding Eqs. (i), (ii) and (iii), then

$P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) = \frac{3}{8}$ ...(v)

On adding Eqs. (iv) and (v), we get

$$P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C) = \frac{3}{8} + \frac{1}{16}$$

or $P(A \cup B \cup C) = \frac{7}{16}$

CHAPTER

10

Mathematical Induction

Learning Part

- Introduction
- Statement
- Mathematical Statement

Practice Part

- JEE Type Examples
- Chapter Exercises

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Introduction

There are two basic processes of reasoning which are commonly used to draw mathematical or scientific conclusions. Reasoning or drawing conclusions can be classified in two categories:

(i) Inductive reasoning
(ii) Deductive reasoning

(i) **Inductive reasoning** This is the process of reasoning from particular to general.
The numbers 324, 576, 603, 732 are all divisible by 3. From these particular results, we can hope to have a general result that all numbers of 3-digits are divisible by 3. But this is not true, because 692 is not divisible by 3.
If at all this conjuctive were true, we would have to establish its validity either by verifying the conjuctive for all possible 3-digit numbers or by using some mathematical process. The process of reasoning a valid general result from particular results is called inductive reasoning.

(ii) **Deductive reasoning** This is the process of reasoning from general to particular.
The sum of first n natural numbers is $\frac{n(n+1)}{2}$. This is a general result. From this, we can deduce that the sum of first 100 natural numbers is $5050\left(=\frac{100(100+1)}{2}\right)$. This process of reasoning a valid particular result from general result is called deductive reasoning.
The principle of mathematical induction is a mathematical process which is used to establish the validity of a general result involving **natural numbers**.

Statement

A sentence or description which is either definitely true or definitely false is called a **statement**. *For example,*

1. Mumbai is the capital of Maharashtra is a true statement.
2. There are 30 days in February is a false statement.
3. Umang is a good boy is not a statement (as it is not a definite sentence. One day whose name is Umang may be a good boy while the other boy whose name is also Umang may not be a good boy. Also, the word 'good' is not well-defined).

Mathematical Statement

A statement involving mathematical relation or relations is called **mathematical statement**.

A statement concerning the natural number 'n' is generally denoted by $P(n)$.

For example, If $P(n)$ denotes the statement "$n(n+1)$ is divisible by 2",

then $P(3)$: "$3(3+1)=12$ is divisible by 2"

and $P(8)$: "$8(8+1)=72$ is divisible by 2", etc.

Here, $P(3)$ and $P(8)$ are both true.

First Principle of Mathematical Induction

To prove that $P(n)$ is true for all natural numbers $n \geq i$, we proceed as follows:

***Step* I** (*Verification Step*) : Verify $P(n)$ for $n = i$.

***Step* II** (*Assumption Step*) : Assume $P(n)$ is true for $n = k > i$.

***Step* III** (*Induction Step*) : Using results in Step I and Step II, prove that $P(k+1)$ is true.

Remark

If $P(n)$ is true for $n = 1$ (i.e., for $i =$).

Second Principle of Mathematical Induction

Sometimes the above procedure will not work. Then, we consider the alternative principle called the second principle of mathematical induction, which consists of the following steps:

***Step* I** (*Verification Step*) : Verify $P(n)$ for $n = i$.

***Step* II** (*Assumption Step*) : Assume $P(n)$ is true for $i < n \leq k$.

***Step* III** (*Induction Step*) : Prove $P(n)$ for $n = k+1$.

Remark

The second principle of mathematical induction is useful to prove recurrence relations which involve three successive terms,

For example, statements of the type

$$pT_{n+1} = qT_n + rT_{n-1}$$

Different Types of Problems of Mathematical Induction

Type I These problems are of the **Identity Type.** Examples of this type are as follows:

Example 1. Prove by mathematical induction that $1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$ for every natural number n.

Sol. Let $P(n) : 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$ ...(i)

***Step* I** For $n = 1$, LHS of Eq. (i) $= 1^3 = 1$

and RHS of Eq. (i) $= \left[\frac{1(1+1)}{2}\right]^2 = 1^2 = 1$

$\therefore$ LHS = RHS

Therefore, $P(1)$ is true.

***Step* II** Assume $P(k)$ is true, then

$$P(k) : 1^3 + 2^3 + 3^3 + \ldots + k^3 = \left[\frac{k(k+1)}{2}\right]^2$$

***Step* III** For $n = k + 1$,

$$P(k+1) : 1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 = \left[\frac{(k+1)(k+2)}{2}\right]^2$$

$$\text{LHS} = 1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 = \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3$$

[by assumption step]

$$= \frac{(k+1)^2}{4}[k^2 + 4(k+1)] = \frac{(k+1)^2(k^2+4k+4)}{4} = \frac{(k+1)^2(k+2)^2}{4} = \left[\frac{(k+1)(k+2)}{2}\right]^2 = \text{RHS}$$

Therefore, $P(k+1)$ is true. Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in N$.

Example 2. Prove by mathematical induction that $1\cdot2\cdot3 + 2\cdot3\cdot4 + \ldots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$ for every natural number.

Sol. Let $P(n) : 1\cdot2\cdot3 + 2\cdot3\cdot4 + \ldots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$ ...(i)

***Step* I** For $n = 1$, LHS of Eq. (i) $= 1\cdot2\cdot3 = 6$

and RHS of Eq. (i) $= \frac{1\cdot(1+1)(1+2)(1+3)}{4} = 6$

$\therefore$ LHS = RHS

Therefore, $P(1)$ is true.

***Step* II** Assume that $P(k)$ is true, then

$$P(k) : 1\cdot2\cdot3 + 2\cdot3\cdot4 + \ldots + k(k+1)(k+2) = \frac{k(k+1)(k+2)(k+3)}{4}$$

***Step* III** For $n = k + 1$

$$P(k+1) : 1\cdot2\cdot3 + 2\cdot3\cdot4 + \ldots + k(k+1)(k+2) + (k+1)(k+2)(k+3) = \frac{(k+1)(k+2)(k+3)(k+4)}{4}$$

$$\therefore \text{LHS} = 1\cdot2\cdot3 + 2\cdot3\cdot4 + \ldots + k(k+1)(k+2) + (k+1)(k+2)(k+3) = \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3)$$

[by assumption step]

$$= \frac{(k+1)(k+2)(k+3)}{4}(k+4) = \frac{(k+1)(k+2)(k+3)(k+4)}{4} = \text{RHS}$$

Therefore, $P(k+1)$ is true. Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$.

Example 3. Prove by mathematical induction that $\frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + \ldots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$, $\forall\, n \in N$.

Sol. Let $P(n) : \frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + \ldots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$ ...(i)

***Step* I** For $n = 1$,

LHS of Eq. (i) $= \frac{1}{1\cdot2\cdot3} = \frac{1}{6}$

and RHS of Eq. (i) $= \frac{1\cdot(1+3)}{4(1+1)(1+2)} = \frac{1}{6}$

Therefore, $P(1)$ is true.

***Step* II** Assume that $P(k)$ is true, then

$$P(k): \frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\ldots+\frac{1}{k(k+1)(k+2)}$$
$$=\frac{k(k+3)}{4(k+1)(k+2)}$$

***Step* III** For $n=k+1$,

$$P(k+1): \frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\ldots+\frac{1}{k(k+1)(k+2)}$$
$$+\frac{1}{(k+1)(k+2)(k+3)}=\frac{(k+1)(k+4)}{4(k+2)(k+3)}$$

$$\therefore \text{ LHS}=\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\ldots+\frac{1}{k(k+1)(k+2)}$$
$$+\frac{1}{(k+1)(k+2)(k+3)}$$
$$=\frac{k(k+3)}{4(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)}$$
[by assumption step]
$$=\frac{k(k+3)^2+4}{4(k+1)(k+2)(k+3)}$$
$$=\frac{k^3+6k^2+9k+4}{4(k+1)(k+2)(k+3)}$$
$$=\frac{(k+1)^2(k+4)}{4(k+1)(k+2)(k+3)}$$
$$=\frac{(k+1)(k+4)}{4(k+2)(k+3)}=\text{RHS}$$

Therefore, $P(k+1)$ is true. Hence, by the principle of mathematical induction $P(n)$ is true for all $n\in N$.

Example 4. *Prove by mathematical induction that* $\sum_{r=0}^{n} r\,{}^nC_r = n\cdot 2^{n-1}$, $\forall\, n\in N$.

Sol. Let $P(n): \sum_{r=0}^{n} r\,{}^nC_r = n\cdot 2^{n-1}$

***Step* I** For $n=1$,

$$\text{LHS of Eq. (i)}=\sum_{r=0}^{1} r\cdot {}^1C_r = 0+1\cdot {}^1C_1 = 1$$

and RHS of Eq. (i) $=1\cdot 2^{1-1}=2^0=1$

Therefore, $P(1)$ is true.

***Step* II** Assume that $P(k)$ is true, then $P(k)$: $\sum_{r=0}^{k} r\cdot {}^kC_r = k\cdot 2^{k-1}$

***Step* III** For $n=k+1$

$$P(k+1): \sum_{r=0}^{k+1} r\cdot {}^{k+1}C_r = (k+1)\cdot 2^k$$

$$\therefore \text{ LHS}=\sum_{r=0}^{k+1} r\cdot {}^{k+1}C_r = 0+\sum_{r=1}^{k+1} r\cdot {}^{k+1}C_r$$
$$=\sum_{r=1}^{k+1} r\cdot {}^{k+1}C_r = \sum_{r=1}^{k} r\cdot {}^{k+1}C_r + (k+1)^{k+1}C_{k+1}$$
$$=\sum_{r=1}^{k} r\,({}^kC_r + {}^kC_{r-1})+(k+1)$$
$$=\sum_{r=1}^{k} r\cdot {}^kC_r + \sum_{r=1}^{k} r\cdot {}^kC_{r-1}+(k+1)$$
$$=\sum_{r=0}^{k} r\cdot {}^kC_r + \sum_{r=0}^{k} r\cdot {}^kC_{r-1}+(k+1)$$
$$=\sum_{r=0}^{k} r\cdot {}^kC_r + \sum_{r=0}^{k+1} r\cdot {}^kC_{r-1}$$
$$=\sum_{r=0}^{k} r\cdot {}^kC_r + \sum_{r=0}^{k} (r+1)\cdot {}^kC_r$$
$$=\sum_{r=0}^{k} r\cdot {}^kC_r + \sum_{r=0}^{k} r\cdot {}^kC_r + \sum_{r=0}^{k} {}^kC_r$$
$$=P(k)+P(k)+2^k \qquad \text{[by assumption step]}$$
$$=k\cdot 2^{k-1}+k\cdot 2^{k-1}+2^k = 2\cdot k\cdot 2^{k-1}+2^k$$
$$=k\cdot 2^k+2^k=(k+1)\cdot 2^k=\text{RHS}$$

Therefore, $P(k+1)$ is true. Hence, by the principle of mathematical induction $P(n)$ is true for all $n\in N$.

Type II These problems are of the **Divisibility Type**. Examples of this type are as follows:

Example 5. Use the principle of mathematical induction to show that $5^{2n+1}+3^{n+2}\cdot 2^{n-1}$ divisible by 19 for all natural numbers n.

Sol. Let $P(n)=5^{2n+1}+3^{n+2}\cdot 2^{n-1}$

***Step* I** For $n=1$, $P(1)=5^{2+1}+3^{1+2}\cdot 2^{1-1}$

$=125+27=152$, which is divisible by 19.

Therefore, the result is true for $n=1$.

***Step* II** Assume that the result is true for $n=k$, i.e.,

$P(k)=5^{2k+1}+3^{k+2}\cdot 2^{k-1}$ is divisible by 19.

$\Rightarrow$ $P(k)=19r$, where r is an integer.

***Step* III** For $n=k+1$,

$$P(k+1)=5^{2(k+1)+1}+3^{k+1+2}\cdot 2^{k+1-1}$$
$$=5^{2k+3}+3^{k+3}\cdot 2^k$$
$$=25\cdot 5^{2k+1}+3\cdot 3^{k+2}\cdot 2\cdot 2^{k-1}$$
$$=25\cdot 5^{2k+1}+6\cdot 3^{k+2}\cdot 2^{k-1}$$

Now, $5^{2k+1}+3^{k+2}\cdot 2^{k-1}\;\overline{)\;25\cdot 5^{2k+1}+6\cdot 3^{k+2}\cdot 2^{k-1}\;(25}$

$25\cdot 5^{2k+1}+25\cdot 3^{k+2}\cdot 2^{k-1}$

$- \qquad -$

$-19\cdot 3^{k+2}\cdot 2^{k-1}$

$\therefore \quad 25\cdot 5^{2k+1} + 6\cdot 3^{k+2}\cdot 2^{k-1}$

$= 25\cdot(5^{2k+1} + 3^{k-2}\cdot 2^{k-2}) - 19\cdot 3^{k+2}\cdot 2^{k-1}$

i.e., $P(k+1) = 25\,P(k) - 19\cdot 3^{k+2}\cdot 2^{k-1}$

But we know that $P(k)$ is divisible by 19. Also, $19\cdot 3^{k+2}\cdot 2^{k-1}$ is clearly divisible by 19.

Therefore, $P(k+1)$ is divisible by 19. This shows that the result is true for $n = k+1$.

Hence, by the principle of mathematical induction, the result is true for all $n \in N$.

Example 6. Use the principle of mathematical induction to show that $a^n - b^n$ is divisible by $a-b$ for all natural numbers n.

Sol. Let $P(n) = a^n - b^n$

***Step* I** For $n = 1$,

$P(1) = a - b$, which is divisible by $a - b$.

Therefore, the result is true for $n = 1$.

***Step* II** Assume that the result is true for $n = k$,

i.e., $P(k) = a^k - b^k$ is divisible by $a - b$.

$\Rightarrow \quad P(k) = (a-b)\,r$, where r is an integer.

***Step* III** For $n = k+1$,

$\therefore \quad P(k+1) = a^{k+1} - b^{k+1}$

Now, $a^k - b^k \overline{)a^{k+1} - b^{k+1}}(a$

$\underline{-a^{k+1} \mp ab^k}$

$ab^k - b^{k+1} = b^k(a-b)$

$\therefore \quad a^{k+1} - b^{k+1} = a(a^k - b^k) + b^k(a-b)$

i.e., $P(k+1) = a\,P(k) + b^k\,(a-b)$

But we know that $P(k)$ is divisible by $a-b$. Also, $b^k\,(a-b)$ is clearly divisible by $a-b$.

Therefore, $P(k+1)$ is divisible by $a-b$.

This shows that the result is true for $n = k+1$.

Hence, by the principle of mathematical induction, the result is true for all $n \in N$.

Type III These problems are of the **Inequality Type**. Examples of this type are as follows:

Example 7. ***Using mathematical induction, show that*** $\tan n\alpha > n\tan\alpha$***, where*** $0 < \alpha < \dfrac{\pi}{4(n-1)}, \forall\, n \in N$ ***and*** $n > 1$.

Sol. Let $P(n): \tan n\alpha > n\tan\alpha$

***Step* I** For $n = 2, \quad \tan 2\alpha > 2\tan\alpha$

$\Rightarrow \quad \dfrac{2\tan\alpha}{1-\tan^2\alpha} - 2\tan\alpha > 0$

$\Rightarrow \quad 2\tan\alpha\left(\dfrac{1-(1-\tan^2\alpha)}{1-\tan^2\alpha}\right) > 0$

$\Rightarrow \quad \tan^2\alpha\cdot\tan 2\alpha > 0 \quad \left[\because 0 < \alpha < \dfrac{\pi}{4} \text{ for } n = 2\right]$

$\Rightarrow \quad \tan 2\alpha > 0 \quad \left[\because 0 < 2\alpha < \dfrac{\pi}{2}\right]$

which is true ($\because$ in first quadrant, $\tan 2\alpha$ is positive)

Therefore, $P(2)$ is true.

***Step* II** Assume that $P(k)$ is true, then $P(k)$: $\tan k\alpha > k\tan\alpha$

***Step* III** For $n = k+1$, we shall prove that

$\tan(k+1)\alpha > (k+1)\tan\alpha$

$\because \quad \tan(k+1)\alpha = \dfrac{\tan k\alpha + \tan\alpha}{1-\tan k\alpha\tan\alpha}$ ...(i)

when $0 < \alpha < \dfrac{\pi}{4k}$ or $0 < k\alpha < \dfrac{\pi}{4}$

i.e. $0 < \tan k\alpha < 1$, also $0 < \tan\alpha < 1$

$\therefore \quad \tan k\alpha\tan\alpha < 1$

$1 - \tan k\alpha\tan\alpha > 0$ and $1 - \tan k\alpha\tan\alpha < 1$ ...(ii)

From Eqs. (i) and (ii), we get

$\tan(k+1)\alpha > \dfrac{\tan k\alpha + \tan\alpha}{1}$

$> \tan k\alpha + \tan\alpha > k\tan\alpha + \tan\alpha$ [by assumption step]

$\therefore \quad \tan(k+1)\alpha > (k+1)\tan\alpha$

Therefore, $P(k+1)$ is true. Hence by the principle of mathematical induction $P(n)$ is true for all $n \in N$.

Example 8. Show using mathematical induction that $n! < \left(\dfrac{n+1}{2}\right)^n$, where $n \in N$ and $n > 1$.

Sol. Let $P(n): n! < \left(\dfrac{n+1}{2}\right)^n$

***Step* I** For $n = 2, 2! < \left(\dfrac{2+1}{2}\right)^2 \Rightarrow 2 < \dfrac{9}{4}$

$\Rightarrow \quad 2 < 2\cdot 25$, which is true.

Therefore, $P(2)$ is true.

***Step* II** Assume that $P(k)$ is true, then

$P(k): k! < \left(\dfrac{k+1}{2}\right)^k$

***Step* III** For $n = k+1$, we shall prove that

$P(k+1): (k+1)! < \left(\dfrac{k+2}{2}\right)^{k+1}$

From assumption step $k! < \dfrac{(k+1)^k}{2^k}$

$$\Rightarrow \quad (k+1)k! < \frac{(k+1)^{k+1}}{2^k}$$

$$\Rightarrow \quad (k+1)! < \frac{(k+1)^{k+1}}{2^k} \quad \ldots(i)$$

Let us assume, $\frac{(k+1)^{k+1}}{2^k} < \left(\frac{k+2}{2}\right)^{k+1}$ ...(ii)

$$\Rightarrow \quad \left(\frac{k+2}{k+1}\right)^{k+1} > 2 \Rightarrow \left(1+\frac{1}{k+1}\right)^{k+1} > 2$$

$$\Rightarrow \quad 1+(k+1)\cdot\frac{1}{(k+1)} + {}^{k+1}C_2\left(\frac{1}{k+1}\right)^2 + \ldots > 2$$

$$\Rightarrow \quad 1+1+{}^{k+1}C_2\left(\frac{1}{k+1}\right)^2 + \ldots > 2$$

which is true, hence Eq. (ii) is true.

From Eqs. (i) and (ii), we get

$$(k+1)! < \frac{(k+1)^{k+1}}{2^k} < \left(\frac{k+2}{2}\right)^{k+1}$$

$$\Rightarrow \quad (k+1)! < \left(\frac{k+2}{2}\right)^{k+1}$$

Therefore, $P(k+1)$ is true. Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$.

Type IV These problems are of the **Second principle of induction**. Examples of this type are as follows:

Example 9. If $a+b=c+d$ and $a^2+b^2=c^2+d^2$, then show by mathematical induction

$$a^n+b^n=c^n+d^n$$

Sol. $P(n): a^n+b^n=c^n+d^n$

***Step* I** For $n=1$ and $n=2$,

$P(1): a+b=c+d$ and $P(2): a^2+b^2=c^2+d^2$

which are true (from given conditions).

Therefore, $P(1)$ and $P(2)$ are true.

***Step* II** Assume $P(k-1)$ and $P(k)$ to be true

$$\therefore \quad a^{k-1}+b^{k-1}=c^{k-1}+d^{k-1} \quad \ldots(i)$$

and $$a^k+b^k=c^k+d^k \quad \ldots(ii)$$

***Step* III** For $n=k+1$,

$P(k+1): a^{k+1}+b^{k+1}=c^{k+1}+d^{k+1}$

$$\therefore \quad \text{LHS} = a^{k+1}+b^{k+1}$$

$$= (a+b)(a^k+b^k) - ab^k - ba^k$$

$$= (a+b)(a^k+b^k) - ab(a^{k-1}+b^{k-1})$$

[given $a+b=c+d$ and $a^2+b^2=c^2+d^2$, then $ab=cd$]

$$= (c+d)(c^k+d^k) - cd(c^{k-1}+d^{k-1})$$

[from Eqs. (i) and (ii)]

$$= c^{k+1}+d^{k+1} = \text{RHS}$$

Therefore, $P(k+1)$ is true. Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$.

Example 10. ***Let*** $I_m = \int_0^{\pi}\left(\frac{1-\cos mx}{1-\cos x}\right)dx$ ***use mathematical induction to prove that*** $I_m = m\pi$, $m = 0, 1, 2, \ldots$

Sol. $\because \quad I_m = \int_0^{\pi}\left(\frac{1-\cos mx}{1-\cos x}\right)dx$

***Step* I** For $m=1$, $I_1 = \int_0^{\pi}\left(\frac{1-\cos x}{1-\cos x}\right)dx$

$\therefore \quad I_1 = \pi$ and for $m=2$,

$$I_2 = \int_0^{\pi}\left(\frac{1-\cos 2x}{1-\cos x}\right)dx$$

$$= \int_0^{\pi}\frac{2\sin^2 x(1+\cos x)}{(1-\cos x)(1+\cos x)}dx$$

$$= \int_0^{\pi}\frac{2\sin^2 x(1+\cos x)}{\sin^2 x}dx = 2\int_0^{\pi}(1+\cos x)dx$$

$$= 2[x+\sin x]_0^{\pi} = 2[(\pi+0)-(0+0)] = 2\pi$$

which are true, therefore I_1 and I_2 are true.

***Step* II** Assume I_{k-1} and I_k to be true

$$\therefore \quad I_{k-1} = (k-1)\pi \quad \ldots(i)$$

and $$I_k = k\pi \quad \ldots(ii)$$

***Step* III** For $m=k+1$,

$$I_{k+1} = \int_0^{\pi}\frac{1-\cos(k+1)x}{1-\cos x}dx$$

$$\therefore \quad I_{k+1} - I_k = \int_0^{\pi}\frac{\cos kx - \cos(k+1)x}{1-\cos x}dx$$

$$= \int_0^{\pi}\frac{2\sin\left(\frac{2k+1}{2}\right)x\cdot\sin\left(\frac{x}{2}\right)}{2\sin^2\left(\frac{x}{2}\right)}dx$$

$$= \int_0^{\pi}\frac{\sin\left(\frac{2k+1}{2}\right)x}{\sin\left(\frac{x}{2}\right)}dx \quad \ldots(iii)$$

Similarly, $I_k - I_{k-1} = \int_0^{\pi}\frac{\sin\left(\frac{2k-1}{2}\right)x}{\sin\left(\frac{x}{2}\right)}dx \quad \ldots(iv)$

On subtracting Eq. (iv) from Eq. (iii), we get

$$I_{k+1} - 2I_k + I_{k-1} = \int_0^{\pi} \frac{\sin\left(\frac{2k+1}{2}\right)x - \sin\left(\frac{2k-1}{2}\right)x}{\sin\left(\frac{x}{2}\right)} dx$$

$$= \int_0^{\pi} \frac{2\cos(kx)\sin\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)} dx = 2\int_0^{\pi} \cos kx\, dx = 2\left[\frac{\sin kx}{k}\right]_0^{\pi} = 0$$

$$\Rightarrow \quad I_{k+1} = 2I_k - I_{k-1} = 2k\pi - (k-1)\pi \quad \text{[by assumption step]}$$

$$= k\pi + \pi = (k+1)\pi$$

This shows that the result is true for $m = k + 1$. Hence, by the principle of mathematical induction the result is true for all $m \in N$.

Type V These problems are of the **Recursion Type**. Examples of this type are as follows:

Example 11. ***Given*** $u_{n+1} = 3u_n - 2u_{n-1}$ ***and*** $u_0 = 2$, $u_1 = 3$. ***Prove that*** $u_n = 2^n + 1$, $\forall\, n \in N$.

Sol. $\because\ u_{n+1} = 3u_n - 2u_{n-1}$ ...(i)

***Step* I** Given, $u_1 = 3 = 2 + 1 = 2^1 + 1$ which is true for $n = 1$.

Putting $n = 1$ in Eq. (i), we get

$$u_{1+1} = 3u_1 - 2u_{1-1}$$

$$\Rightarrow \quad u_2 = 3u_1 - 2u_0 = 3\cdot 3 - 2\cdot 2 = 5 = 2^2 + 1$$

which is true for $n = 2$.

Therefore, the result is true for $n = 1$ and $n = 2$.

***Step* II** Assume it is true for $n = k$, then it is also true for $n = k - 1$.

Then, $u_k = 2^k + 1$ ...(ii)

and $u_{k-1} = 2^{k-1} + 1$ ...(iii)

***Step* III** Putting $n = k$ in Eq. (i), we get

$$u_{k+1} = 3u_k - 2u_{k-1}$$

$$= 3(2^k + 1) - 2(2^{k-1} + 1) \quad \text{[from Eqs. (ii) and (iii)]}$$

$$= 3\cdot 2^k + 3 - 2\cdot 2^{k-1} - 2 = 3\cdot 2^k + 3 - 2^k - 2$$

$$= (3-1)2^k + 1 = 2\cdot 2^k + 1 = 2^{k+1} + 1$$

This shows that the result is true for $n = k + 1$. Hence, by the principle of mathematical induction the result is true for all $n \in N$.

Example 12. Let $u_1 = 1$, $u_2 = 2$, $u_3 = \frac{7}{2}$ and $u_{n+3} = 3u_{n+2} - \left(\frac{3}{2}\right)u_{n+1} - u_n$. Use the principle of mathematical induction to show that $u_n = \frac{1}{3}\left[2^n + \left(\frac{1+\sqrt{3}}{2}\right)^n + \left(\frac{1-\sqrt{3}}{2}\right)^n\right] \forall\, n \geq 1$.

Sol. $\because\ u_n = \frac{1}{3}\left[2^n + \left(\frac{1+\sqrt{3}}{2}\right)^n + \left(\frac{1-\sqrt{3}}{2}\right)^n\right]$ ...(i)

***Step* I** For $n = 1$, $u_1 = \frac{1}{3}\left[2^1 + \left(\frac{1+\sqrt{3}}{2}\right)^1 + \left(\frac{1-\sqrt{3}}{2}\right)^1\right]$

$$= \frac{1}{3}[2 + 1] = 1$$

which is true for $n = 1$ and for $n = 2$,

$$u_2 = \frac{1}{3}\left[2^2 + \left(\frac{1+\sqrt{3}}{2}\right)^2 + \left(\frac{1-\sqrt{3}}{2}\right)^2\right]$$

$$= \frac{1}{3}\left[4 + \left(\frac{4+2\sqrt{3}}{4}\right) + \left(\frac{4-2\sqrt{3}}{4}\right)\right] = \frac{1}{3}[6] = 2$$

which is true for $n = 2$.

Therefore, the result is true for $n = 1$ and $n = 2$.

***Step* II** Assume it is true for $n = k$, then it is also true for $n = k - 1$, $k - 2$

$$\therefore \quad u_k = \frac{1}{3}\left[2^k + \left(\frac{1+\sqrt{3}}{2}\right)^k + \left(\frac{1-\sqrt{3}}{2}\right)^k\right] \quad ...(ii)$$

$$u_{k-1} = \frac{1}{3}\left[2^{k-1} + \left(\frac{1+\sqrt{3}}{2}\right)^{k-1} + \left(\frac{1-\sqrt{3}}{2}\right)^{k-1}\right] \quad ...(iii)$$

$$u_{k-2} = \frac{1}{3}\left[2^{k-2} + \left(\frac{1+\sqrt{3}}{2}\right)^{k-2} + \left(\frac{1-\sqrt{3}}{2}\right)^{k-2}\right] \quad ...(iv)$$

***Step* III** Given that, $u_{n+3} = 3u_{n+2} - \left(\frac{3}{2}\right)u_{n+1} - u_n$

Replace n by $k - 2$

Then, $u_{k+1} = 3u_k - \frac{3}{2}u_{k-1} - u_{k-2}$

$$= \frac{1}{3}\left[3\cdot 2^k + 3\left(\frac{1+\sqrt{3}}{2}\right)^k + 3\left(\frac{1-\sqrt{3}}{2}\right)^k\right]$$

$$+ \frac{1}{3}\left[-\frac{3}{2}\cdot 2^{k-1} - \frac{3}{2}\left(\frac{1+\sqrt{3}}{2}\right)^{k-1} - \frac{3}{2}\left(\frac{1-\sqrt{3}}{2}\right)^{k-1}\right]$$

$$+ \frac{1}{3}\left[-2^{k-2} - \left(\frac{1+\sqrt{3}}{2}\right)^{k-2} - \left(\frac{1-\sqrt{3}}{2}\right)^{k-2}\right]$$

$$= \frac{1}{3}\left[3\cdot 2^k - 3\cdot 2^{k-2} - 2^{k-2} + \left(\frac{1+\sqrt{3}}{2}\right)^k\right.$$

$$- \frac{3}{2}\left(\frac{1+\sqrt{3}}{2}\right)^{k-1} - \left(\frac{1+\sqrt{3}}{2}\right)^{k-2} + 3\left(\frac{1-\sqrt{3}}{2}\right)^k$$

$$\left. - \frac{3}{2}\left(\frac{1-\sqrt{3}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{3}}{2}\right)^{k-2}\right]$$

$$= \frac{1}{3}\left[2^{k-2}(3\cdot 4-3-1)+\left(\frac{1+\sqrt{3}}{2}\right)^{k-2}\left[3\left(\frac{1+\sqrt{3}}{2}\right)^{2}-\frac{3}{2}\left(\frac{1+\sqrt{3}}{2}\right)-1\right]+\left(\frac{1-\sqrt{3}}{2}\right)^{k-2}\left[3\left(\frac{1-\sqrt{3}}{2}\right)^{2}-\frac{3}{2}\left(\frac{1-\sqrt{3}}{2}\right)-1\right]\right]$$

$$= \frac{1}{3}\left[2^{k-2}\cdot 8+\left(\frac{1+\sqrt{3}}{2}\right)^{k-2}\left[\frac{3(1+\sqrt{3})^{2}-3(1+\sqrt{3})-4}{4}\right]+\left(\frac{1-\sqrt{3}}{2}\right)^{k-2}\left[\frac{3(1-\sqrt{3})^{2}-3(1-\sqrt{3})-4}{4}\right]\right]$$

$$= \frac{1}{3}\left[2^{k+1}+\left(\frac{1+\sqrt{3}}{2}\right)^{k-2}\left[\frac{10+6\sqrt{3}}{8}\right]+\left(\frac{1-\sqrt{3}}{2}\right)^{k-2}\left[\frac{10-6\sqrt{3}}{8}\right]\right]$$

$$= \frac{1}{3}\left[2^{k+1}+\left(\frac{1+\sqrt{3}}{2}\right)^{k-2}\left(\frac{1+\sqrt{3}}{2}\right)^{3}+\left(\frac{1-\sqrt{3}}{2}\right)^{k-2}\left(\frac{1-\sqrt{3}}{2}\right)^{3}\right]$$

$$= \frac{1}{3}\left[2^{k+1}+\left(\frac{1+\sqrt{3}}{2}\right)^{k+1}+\left(\frac{1-\sqrt{3}}{2}\right)^{k+1}\right]$$

This shows that the result is true for $n = k + 1$. Hence, by the principle of mathematical induction the result is true for all $n \in N$.

Shortcuts and Important Results to Remember

1. Principle of mathematical induction is widely used in proving identities, theorems, divisibility of an expression by a number or by another expression, inequalities, etc.
2. Principle of mathematical induction can only help in verifying an established result. It cannot discover a new formula.
3. If $f(k)$ is divisible by a number p and it is to be proved that $f(k + 1)$ is divisible by p, sometimes it is easier to show that $f(k + 1) - f(k)$ is divisible by p.
4. $\underbrace{aaaaa\ldots a}_{m \text{ times}} = a\,(1 + 10 + 10^2 + 10^3 + \ldots + 10^{m-1})$
$= \dfrac{a\,(10^m - 1)}{9}$, $\forall\ 1 \le a \le 9$ and $a \in N$.

JEE Type Solved Examples :
Single Option Correct Type Questions

▪ This section contains **5 multiple choice examples**. Each example has four choices (a), (b), (c) and (d), out of which **ONLY ONE** is correct.

● ***Ex. 1*** *When P is a natural number, then* $P^{n+1} + (P+1)^{2n-1}$ *is divisible by*

(a) P (b) $P^2 + P$ (c) $P^2 + P + 1$ (d) $P^2 - 1$

Sol. (c) For $n = 1$, we get

$$P^{n+1} + (P+1)^{2n-1} = P^2 + (P+1)^1 = P^2 + P + 1,$$

which is divisible by $P^2 + P + 1$, so result is true for $n = 1$.

Let us assume that the given result is true for $n = m \in N$.

i.e., $P^{m+1} + (P+1)^{2m-1}$ is divisible by $P^2 + P + 1$.

i.e., $P^{m+1} + (P+1)^{2m-1} = k(P^2 + P + 1), \forall\ k \in N$ ...(i)

Now, $P^{(m+1)+1} + (P+1)^{2(m+1)-1}$

$= P^{m+2} + (P+1)^{2m+1}$

$= P^{m+2} + (P+1)^2 (P+1)^{2m-1}$

$= P^{m+2} + (P+1)^2 [k(P^2 + P + 1) - P^{m+1}]$

[by using Eq. (i)]

$= P^{m+2} + (P+1)^2 \cdot k(P^2 + P + 1) - (P+1)^2 (P)^{m+1}$

$= P^{m+1} [P - (P+1)^2] + (P+1)^2 \cdot k(P^2 + P + 1)$

$= P^{m+1} [P - P^2 - 2P - 1] + (P+1)^2 \cdot k(P^2 + P + 1)$

$= -P^{m+1} [P^2 + P + 1] + (P+1)^2 \cdot k(P^2 + P + 1)$

$= (P^2 + P + 1) [k \cdot (P+1)^2 - P^{m+1}]$

which is divisible by $P^2 + P + 1$, so the result is true for $n = m + 1$. Therefore, the given result is true for all $n \in N$ by induction.

● ***Ex. 2*** *Let* $P(n)$ *denote the statement that* $n^2 + n$ *is odd. It is seen that* $P(n) \Rightarrow P(n+1)$, $P(n)$ *is true for all*

(a) $n > 1$ (b) n

(c) $n > 2$ (d) None of these

Sol. (d) $P(n) = n^2 + n$. It is always odd (statement) but square of any odd number is always odd and also, sum of two odd numbers is always even. So, for no any 'n' for which this statement is true.

● ***Ex. 3*** *For a positive integer n,*

let $a(n) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{2^n - 1}$. *Then*

(a) $a(100) > 100$

(b) $a(100) < 200$

(c) $a(200) \le 100$

(d) $a(200) > 100$

Sol. (d) It can be proved with the help of mathematical induction that

$$\frac{n}{2} < a(n) \le n$$

$$\Rightarrow \quad \frac{200}{2} < a(200)$$

$$\Rightarrow \quad a(200) > 100$$

● ***Ex. 4*** *Let* $S(k) = 1 + 3 + 5 + \ldots + (2k-1) = 3 + k^2$. *Which of the following is true?*

(a) Principle of mathematical induction can be used to prove the formula

(b) $S(k) \Rightarrow S(k+1)$

(c) $S(k) \not\Rightarrow S(k+1)$

(d) $S(1)$ is correct

Sol. (c) We have, $S(k) = 1 + 3 + 5 + \ldots + (2k-1) = 3 + k^2$,

$S(1) \Rightarrow 1 = 4$, which is not true

and $S(2) \Rightarrow 4 = 7$, which is not true.

Hence, induction cannot be applied and $S(k) \ne S(k+1)$.

● ***Ex. 5*** $10^n + 3(4^{n+2}) + 5$ *is divisible by* ($n \in N$)

(a) 7 (b) 5

(c) 9 (d) 17

Sol. (c) $10^n + 3(4^{n+2}) + 5$

Taking $n = 2$,

$$10^2 + 3 \times 4^4 + 5 = 100 + 768 + 5 = 873$$

Therefore, this is divisible by 9.

JEE Type Solved Examples :
Statement I and II Type Questions

■ **Directions** Example numbers 6 and 7 are Assertion-Reason type Examples. Each of these Examples contains two statements :
Statement-1 (Assertion) and
Statement-2 (Reason)
Each of these examples also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below:

(a) Statement-1 is true, Statement-2 is true Statement-2 is correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true, Statement-2 is not the correct explanation for Statement-1
(c) Statement-1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

● ***Ex. 6*** **Statement-1** *For all natural number n,*

$$1+2+\ldots+n<(2n+1)^2$$

Statement-2 *For all natural numbers,* $(2n+3)^2<7(n+1)$

Sol. (*b*) Let $P(n):1+2+3+\ldots+n<(2n+1)^2$

***Step* I** For $n=1$,

$P(1):1<(2+1)^2 \Rightarrow 1<9$

which is true.

***Step* II** Assume $P(n)$ is true for $n=k$, then

$P(k):1+2+\ldots+k<(2k+1)^2$

***Step* III** For $n=k+1$, we shall prove that

$P(k+1):1+2+3+\ldots+k+(k+1)<(2k+3)^2$

From assumption step

$1+2+3+\ldots+k+(k+1)<(2k+1)^2+k+1$

$=4k^2+5k+2$

$=(2k+3)^2-7(k+1)<(2k+3)^2 \qquad [\because 7(k+1)>0]$

$\therefore$ $P(k+1)$ is true.

Here, both Statements are true but Statement-2 is not correct explanation of Statement-1.

● ***Ex. 7*** **Statement-1** *For all natural numbers n,*

$$7+77+777+\ldots\underbrace{777\ldots7}_{n \text{ digits}}=\frac{7}{81}(10^{n+1}-9n-10)$$

Statement-2 *For all natural numbers,*

$$\underbrace{777\ldots7}_{n \text{ digits}}=7+7\times10+7\times10^2+\ldots+7\times10^n$$

Sol. (*c*) $\because$ $\underbrace{777\ldots7}_{n \text{ digits}}=7\underbrace{(111\ldots1)}_{n \text{ digits}}$

$=7\,(1+10+10^2+\ldots+10^{n-1})$

$=7+7\times10+7\times10^2+\ldots+7\times10^{n-1}$

$\neq 7+7\times10+7\times10^2+\ldots+7\times10^n$

$\therefore$ Statement-2 is false.

Now, let $P(n):7+77+777+\ldots+\underbrace{777\ldots7}_{n \text{ digits}}=\frac{7}{81}$

$(10^{n+1}-9n-10)$

***Step* I** For $n=1$,

LHS $=7$ and RHS $=\frac{7}{81}(10^2-9-10)=7$

$\therefore$ LHS = RHS

which is true for $n=1$.

***Step* II** Assume $P(n)$ is true for $n=k$, then

$P(k):7+77+777+\ldots+\underbrace{777\ldots7}_{k \text{ digits}}$

$=\frac{7}{81}(10^{k+1}-9k-10)$

***Step* III** For $n=k+1$,

$P(k+1):7+77+777+\ldots+\underbrace{777\ldots7}_{k \text{ digits}}+\underbrace{777\ldots7}_{(k+1) \text{ digits}}$

$=\frac{7}{81}[10^{k+2}-9(k+1)-10]$

LHS $=7+77+777+\ldots+\underbrace{777\ldots7}_{k \text{ digits}}+\underbrace{777\ldots7}_{(k+1) \text{ digits}}$

$=\frac{7}{81}(10^{k+1}-9k-10)+7(1+10+10^2+\ldots+10^k)$

[from assumption step]

$=\frac{7}{81}(10^{k+1}-9k-10)+\frac{7(10^{k+1}-1)}{10-1}$

$=\frac{7}{81}(10^{k+1}-9k-10+9\cdot10^{k+1}-9)$

$=\frac{7}{81}[10^{k+1}(1+9)-9(k+1)-10]$

$=\frac{7}{81}[10^{k+2}-9(k+1)-10]$

= RHS

Therefore, $P(k+1)$ is true. Hence, by mathematical induction $P(n)$ is true for all natural numbers.

Hence, Statement-1 is true and Statement-2 is false.

Subjective Type Examples

■ In this section, there are **8 subjective** solved examples.

● ***Ex. 8*** *Prove by induction that the integer next greater than* $(3+\sqrt{5})^n$ *is divisible by* 2^n *for all* $n \in N$.

Sol. Let $\alpha = 3+\sqrt{5}$ and $\beta = 3-\sqrt{5}$

$\therefore \quad 0 < \beta^n < 1, \forall\, n \in N$

$\Rightarrow \quad \alpha+\beta = 6, \alpha\beta = 4$ ...(i)

Then, α and β are the roots of

$$x^2 - 6x + 4 = 0$$

$$\Rightarrow \quad \left.\begin{aligned}\alpha^2 &= 6\alpha - 4\\ \beta^2 &= 6\beta - 4\end{aligned}\right] \quad \text{...(ii)}$$

$\therefore \quad \alpha^2+\beta^2 = 6(\alpha+\beta) - 8 = 28$ ...(iii)

$\therefore \quad \alpha^n+\beta^n = (3+\sqrt{5})^n + (3-\sqrt{5})^n$

$= 2[3^n + {}^nC_2 3^{n-2}\cdot 5 + {}^nC_4 3^{n-4}\cdot 5^2 + \ldots]$

= Even integer.

As, $0 < \beta^n < 1$, $\alpha^n+\beta^n$ is the even integer next greater than α^n.

***Step* I** For $n = 1$,

$\alpha+\beta = 6$ [from Eq. (i)]

divisible by 2^1

and for $n = 2, \alpha^2+\beta^2 = 28$ [from Eq. (iii)]

divisible by 2^2

which is true for $n = 1, 2$.

***Step* II** Assume it is true for $n = k$.

i.e., $\alpha^k+\beta^k$ is divisible by 2^k.

***Step* III** For $n = k+1$,

the integer next greater than α^{k+1} is $\alpha^{k+1}+\beta^{k+1}$

$= \alpha^2\cdot\alpha^{k-1} + \beta^2\cdot\beta^{k-1}$

$= (6\alpha-4)\cdot\alpha^{k-1} + (6\beta-4)\cdot\beta^{k-1}$ [from Eq. (ii)]

$= 6(\alpha^k+\beta^k) - 4(\alpha^{k-1}+\beta^{k-1})$

= 3 (divisible by 2^{k+1}) − (divisible by 2^{k+1})

= Divisible by 2^{k+1}

This shows that the result is true for $n = k+1$. Hence, the integer next greater than α^{k+1} is divisible by 2^{k+1}.

● ***Ex. 9*** *Using mathematical induction, show that*

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{(n+1)^2}\right) = \frac{n+2}{2(n+1)}, \forall\, n \in N.$$

Sol. Let $P(n): \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{(n+1)^2}\right)$

$$= \frac{n+2}{2(n+1)} \quad \text{...(i)}$$

***Step* I** For $n = 1$,

LHS of Eq. (i) $= 1-\frac{1}{2^2} = \frac{3}{4}$ and RHS of Eq. (i)

$$= \frac{3}{2\cdot 2} = \frac{3}{4}.$$

Therefore, $P(1)$ is true.

***Step* II** Assume it is true for $n = k$, then

$$P(k): \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{(k+1)^2}\right) = \frac{k+2}{2(k+1)}$$

***Step* III** For $n = k+1$,

$$P(k+1): \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{(k+1)^2}\right)\left(1-\frac{1}{(k+2)^2}\right) = \frac{k+3}{2(k+2)}$$

$$\therefore \quad \text{LHS} = \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{(k+1)^2}\right)\left(1-\frac{1}{(k+2)^2}\right)$$

$$= \frac{k+2}{2(k+1)}\left(1-\frac{1}{(k+2)^2}\right) \quad \text{[by assumption step]}$$

$$= \frac{(k+2)}{2(k+1)}\cdot\frac{[(k+2)^2-1)]}{(k+2)^2} = \frac{k^2+4k+3}{2(k+1)(k+2)}$$

$$= \frac{(k+1)(k+3)}{2(k+1)(k+2)} = \frac{(k+3)}{2(k+2)} = \text{RHS}$$

This shows that the result is true for $n = k+1$. Hence, by the principle of mathematical induction, the result is true for all $n \in N$.

● ***Ex. 10*** *Using the principle of mathematical induction to show that*

$$\tan^{-1}\left(\frac{x}{1+1\cdot 2\cdot x^2}\right) + \tan^{-1}\left(\frac{x}{1+2\cdot 3\cdot x^2}\right) + \ldots + \tan^{-1}\left(\frac{x}{1+n(n+1)\,x^2}\right) = \tan^{-1}(n+1)\,x - \tan^{-1}x, \forall\, x \in N.$$

Sol. Let $P(n): \tan^{-1}\left(\frac{x}{1+1\cdot 2\cdot x^2}\right)+\tan^{-1}\left(\frac{x}{1+2\cdot 3\cdot x^2}\right)+\ldots+\tan^{-1}\left(\frac{x}{1+n(n+1)\,x^2}\right)=\tan^{-1}(n+1)x-\tan^{-1}x$ ...(i)

***Step* I** For $n=1$,

LHS of Eq. (i) $=\tan^{-1}\left(\frac{x}{1+1\cdot 2\cdot x^2}\right)$

$=\tan^{-1}\left(\frac{2x-x}{1+2x\cdot x}\right)=\tan^{-1}2x-\tan^{-1}x$

= RHS of Eq. (i)

Therefore, $P(1)$ is true.

***Step* II** Assume it is true for $n=k$.

$P(k): \tan^{-1}\left(\frac{x}{1+1\cdot 2\,x^2}\right)+\tan^{-1}\left(\frac{x}{1+2\cdot 3x^2}\right)+\ldots+\tan^{-1}\left(\frac{x}{[1+k(k+1)\,x^2]}\right)=\tan^{-1}(k+1)\,x-\tan^{-1}x$

***Step* III** For $n=k+1$,

$P(k+1): \tan^{-1}\left(\frac{x}{1+1\cdot 2\cdot x^2}\right)+\tan^{-1}\left(\frac{x}{1+2\cdot 3\cdot x^2}\right)+\ldots+\tan^{-1}\left(\frac{x}{1+k(k+1)\,x^2}\right)+\tan^{-1}\left(\frac{x}{1+(k+1)(k+2)\,x^2}\right)$

$=\tan^{-1}(k+2)\,x-\tan^{-1}x$

$\therefore$ LHS $=\tan^{-1}\left(\frac{x}{1+1\cdot 2\cdot x^2}\right)+\tan^{-1}\left(\frac{x}{1+2\cdot 3\cdot x^2}\right)+\ldots+\tan^{-1}\left(\frac{x}{1+k(k+1)\,x^2}\right)+\tan^{-1}\left(\frac{x}{1+(k+1)(k+2)\,x^2}\right)$

$=\tan^{-1}(k+1)\,x-\tan^{-1}x+\tan^{-1}\left(\frac{x}{1+(k+1)(k+2)\,x^2}\right)$ [by assumption step]

$=\tan^{-1}(k+1)\,x-\tan^{-1}x+\tan^{-1}\left(\frac{(k+2)x-(k+1)x}{1+(k+2)\,x\,(k+1)\,x}\right)$

$=\tan^{-1}(k+1)\,x-\tan^{-1}x+\tan^{-1}(k+2)\,x-\tan^{-1}(k+1)\,x$

$=\tan^{-1}(k+2)x-\tan^{-1}x=$ RHS

This shows that the result is true for $n=k+1$. Hence, by the principle of mathematical induction, the result is true for all $n\in N$.

● ***Ex. 11*** *Use the principle of mathematical induction to prove that for all $n\in N$.*

$$\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\ldots+\sqrt{2}}}}=2\cos\left(\frac{\pi}{2^{n+1}}\right)$$

when the LHS contains n radical sign.

Sol. Let $P(n)=\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\ldots+\sqrt{2}}}}=2\cos\left(\frac{\pi}{2^{n+1}}\right)$ ...(i)

***Step* I** For $n=1$,

LHS of Eq. (i) $=\sqrt{2}$ and RHS Eq. (i) $=2\cos\left(\frac{\pi}{2^2}\right)$

$=2\cos\left(\frac{\pi}{4}\right)=2\cdot\frac{1}{\sqrt{2}}=\sqrt{2}$

Therefore, $P(1)$ is true.

***Step* II** Assume it is true for $n=k$,

$P(k)=\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\ldots+\sqrt{2}}}}}_{k \text{ radical sign}}=2\cos\left(\frac{\pi}{2^{k+1}}\right)$

***Step* III** For $n=k+1$,

$\therefore\ P(k+1)=\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\ldots+\sqrt{2}}}}}_{(k+1) \text{ radical sign}}$

$=\sqrt{\{2+P(k)\}}$

$=\sqrt{2+2\cos\left(\frac{\pi}{2^{k+1}}\right)}$ [by assumption step]

$=\sqrt{2\left(1+\cos\left(\frac{\pi}{2^{k+1}}\right)\right)}$

$=\sqrt{2\left(1+2\cos^2\left(\frac{\pi}{2^{k+2}}\right)-1\right)}$

$=\sqrt{4\cos^2\left(\frac{\pi}{2^{k+2}}\right)}=2\cos\left(\frac{\pi}{2^{k+2}}\right)$

This shows that the result is true for $n=k+1$. Hence, by the principle of mathematical induction, the result is true for all $n\in N$.

● ***Ex. 12*** *Prove by mathematical induction that*

$$\frac{1}{1+x}+\frac{2}{1+x^2}+\frac{4}{1+x^4}+\ldots+\frac{2^n}{1+x^{2^n}}=\frac{1}{x-1}+\frac{2^{n+1}}{1-x^{2^{n+1}}}$$

where, $|x|\neq 1$ *and n is non-negative integer.*

Sol. Let $P(n): \frac{1}{1+x}+\frac{2}{1+x^2}+\frac{4}{1+x^4}+\ldots+\frac{2^n}{1+x^{2^n}}=\frac{1}{x-1}+\frac{2^{n+1}}{1-x^{2^{n+1}}}$ ...(i)

***Step* I** For $n = 1$,

$$\text{LHS of Eq. (i)} = \frac{1}{1+x} + \frac{2}{1+x^2}$$

$$= \frac{1}{x-1} - \frac{1}{x-1} + \frac{1}{1+x} + \frac{2}{1+x^2}$$

$$= \frac{1}{x-1} + \left(\frac{1}{1-x} + \frac{1}{1+x}\right) + \frac{2}{1+x^2}$$

$$= \frac{1}{x-1} + \frac{2}{1-x^2} + \frac{2}{1+x^2}$$

$$= \frac{1}{x-1} + 2\left(\frac{2}{(1-x^2)(1+x^2)}\right) = \frac{1}{x-1} + \frac{2^2}{1-x^2}$$

= RHS of Eq. (i)

***Step* II** Assume it is true for $n = k$, then

$$P(k): \frac{1}{1+x} + \frac{2}{1+x^2} + \frac{4}{1+x^4} + \ldots + \frac{2^k}{1+x^{2^k}}$$

$$= \frac{1}{x-1} + \frac{2^{k+1}}{1-x^{2^{k+1}}}$$

***Step* III** For $n = k + 1$,

$$P(k+1): \frac{1}{1+x} + \frac{2}{1+x^2} + \frac{4}{1+x^4} + \ldots + \frac{2^k}{1+x^{2^k}} + \frac{2^{k+1}}{1+x^{2^{k+1}}}$$

$$= \frac{1}{x-1} + \frac{2^{k+2}}{1-x^{2^{k+1}}}$$

$$\text{LHS} = \frac{1}{1+x} + \frac{2}{1+x^2} + \frac{4}{1+x^4} + \ldots + \frac{2k}{1+x^{2^k}} + \frac{2^{k+1}}{1+x^{2^{k+1}}}$$

$$= \frac{1}{x-1} + \frac{2^{k+1}}{1-x^{2^{k+1}}} + \frac{2^{k+1}}{1+x^{2^{k+1}}}$$

[by assumption step]

$$= \frac{1}{x-1} + 2^{k+1}\left(\frac{2}{(1-x^{2^{k+1}})(1+x^{2^{k+1}})}\right)$$

$$= \frac{1}{x-1} + \frac{2^{k+2}}{1-x^{2^{k+2}}} = \text{RHS}$$

This shows that the result is true for $n = k + 1$. Hence, by the principle of mathematical induction, the result is true for all $n \in N$.

● ***Ex. 13*** *Using the principle of mathematical induction to prove that* $\int_0^{\pi/2} \frac{\sin^2 nx}{\sin x}\, dx = 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1}$

Sol. Let $P(n): \int_0^{\pi/2} \frac{\sin^2 nx}{\sin x}\, dx = 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1}$ ...(i)

***Step* I** For $n = 1$,

$$\text{LHS of Eq. (i)} = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x}\, dx = \int_0^{\pi/2} \sin x\, dx$$

$$= -[\cos x]_0^{\pi/2} = -(0-1) = 1$$

and RHS of Eq. (i) = 1

Therefore, $P(1)$ is true.

***Step* II** Assume it is true for $n = k$, then

$$P(k): \int_0^{\pi/2} \frac{\sin^2 kx}{\sin x}\, dx = 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2k-1}$$

***Step* III** For $n = k + 1$,

$$P(k+1): \int_0^{\pi/2} \frac{\sin^2 (k+1)x}{\sin x}\, dx = 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2k-1} + \frac{1}{2k+1}$$

$$\text{LHS} = \int_0^{\pi/2} \frac{\sin^2 (k+1)x}{\sin x}\, dx$$

$$= \int_0^{\pi/2} \frac{\sin^2 (k+1)x - \sin^2 kx + \sin^2 kx}{\sin x}\, dx$$

$$= \int_0^{\pi/2} \frac{\sin^2 (k+1)x - \sin^2 kx}{\sin x}\, dx + \int_0^{\pi/2} \frac{\sin^2 kx}{\sin x}\, dx$$

$$= \int_0^{\pi/2} \frac{\sin (2k+1)x \sin x}{\sin x}\, dx + P(k)$$

[by assumption step]

$$= \int_0^{\pi/2} \sin (2k+1)x\, dx + P(k)$$

$$= -\left[\frac{\cos (2k+1)x}{2k+1}\right]_0^{\pi/2} + P(k)$$

$$= -\frac{1}{(2k+1)}\left[\cos\left(\pi k + \frac{\pi}{2}\right) - 1\right] + P(k)$$

$$= -\frac{1}{(2k+1)}[-\sin \pi k - 1] + P(k)$$

$$= -\frac{1}{2k+1}[-0-1] + P(k)$$

$$= \frac{1}{(2k+1)} + 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{(2k-1)}$$

[by assumption step]

$$= 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{(2k-1)} + \frac{1}{(2k+1)}$$

= RHS

This shows that the result is true for $n = k + 1$. Hence, by the principle of mathematical induction, the result is true for all $n \in N$.

● **Ex. 14** *Use induction to show that for all $n \in N$.*

$$\sqrt{a+\sqrt{a+\sqrt{a+\ldots+\sqrt{a}}}} < \frac{1+\sqrt{(4a+1)}}{2}$$

where 'a' is fixed positive number and n radical signs are taken on LHS.

Sol. Let $P(n): \sqrt{a+\sqrt{a+\sqrt{a+\ldots+\sqrt{a}}}} < 1+\sqrt{\frac{(4a+1)}{2}}$

***Step* I** For $n = 1$, then $\sqrt{a} < \frac{1+\sqrt{(4a+1)}}{2}$

$\Rightarrow \quad 2\sqrt{a} < 1+\sqrt{(4a+1)}$

$\Rightarrow \quad 4a < 1+4a+1+2\sqrt{(4a+1)}$

$\Rightarrow \quad 2\sqrt{(4a+1)}+2>0$ which is true.

Therefore, $P(1)$ is true.

***Step* II** Assume it is true for $n = k$,

$$P(k): \underbrace{\sqrt{a+\sqrt{a+\sqrt{a+\ldots+\sqrt{a}}}}}_{k \text{ radical signs}} < \frac{1+\sqrt{(4a+1)}}{2}$$

***Step* III** For $n = k+1$,

$$P(k+1): \underbrace{\sqrt{a+\sqrt{a+\sqrt{a+\ldots+\sqrt{a}}}}}_{(k+1) \text{ radical signs}} < \frac{1+\sqrt{(4a+1)}}{2}$$

From assumption step

$$\underbrace{\sqrt{a+\sqrt{a+\sqrt{a+\ldots+\sqrt{a}}}}}_{k \text{ radical signs}} < \frac{1+\sqrt{4a+1}}{2}$$

$$a+\underbrace{\sqrt{a+\sqrt{a+\sqrt{a+\ldots+\sqrt{a}}}}}_{k \text{ radical signs}} < a+\frac{1+\sqrt{(4a+1)}}{2}$$

$$\Rightarrow \underbrace{\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a+\ldots+\sqrt{a}}}}}}_{(k+1) \text{ radical signs}} < \sqrt{a+\frac{1+\sqrt{(4a+1)}}{2}}$$

$$= \sqrt{\frac{2a+1+\sqrt{(4a+1)}}{2}} = \sqrt{\frac{4a+2+2\sqrt{(4a+1)}}{4}}$$

$$= \sqrt{\frac{(\sqrt{(4a+1)})^2+1+2\sqrt{(4a+1)}}{4}}$$

$$= \sqrt{\left(\frac{1+\sqrt{(4a+1)}}{2}\right)^2} = \frac{1+\sqrt{(4a+1)}}{2}$$

$$\therefore \underbrace{\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a+\ldots+\sqrt{a}}}}}}_{(k+1) \text{ radical signs}} < \frac{1+\sqrt{(4a+1)}}{2}$$

which is true for $n = k+1$.

Hence, by the principle of mathematical induction, the result is true for all $n \in N$.

● **Ex. 15** *Prove by induction that*

$$\left\{\prod_{r=0}^{n} f_r(x)\right\}' = \sum_{i=1}^{n}\{f_1(x) f_2(x)\ldots f_i'(x)\ldots f_n(x)\},$$

where dash denotes derivative with respect to x.

Sol. Let $P(n): \left\{\prod_{r=0}^{n} f_r(x)\right\}' = \sum_{i=1}^{n}\{f_1(x)f_2(x)\ldots f_i'(x)\ldots f_n(x)\}$ ...(i)

***Step* I** For $n = 1$,

LHS of Eq. (i) $= \left\{\prod_{r=1}^{1} f_r(x)\right\}' = \{f_1(x)\}' = f_1'(x)$

RHS of Eq. (i) $= \sum_{i=1}^{1}\{f_1(x)f_2(x)\ldots f_i'(x)\ldots f_1(x)\}$

$= f_1'(x)$

which is true for $n = 1$.

***Step* II** Assume it is true for $n = k$, then

$$P(k): \left\{\prod_{r=1}^{k} f_r(x)\right\}' = \sum_{i=1}^{k}\{f_1(x) f_2(x)\ldots f_i'(x)\ldots f_k(x)\}$$

***Step* III** For $n = k+1$,

$$P(k+1): \left\{\prod_{r=1}^{k+1} f_r(x)\right\}' = \sum_{i=1}^{k+1}\{f_1(x) f_2(x)\ldots f_i'(x)\ldots f_k(x)\}$$

$$\text{LHS} = \left\{\prod_{r=1}^{(k+1)} f_r(x)\right\}' = \left\{\prod_{r=1}^{k} f_r(x)\cdot f_{k+1}(x)\right\}'$$

$$= \prod_{r=1}^{k} f_r(x)\cdot f'_{k+1}(x) + f_{k+1}(x)\left\{\prod_{r=1}^{k} f_r(x)\right\}'$$

$$= \prod_{r=1}^{k} f_r(x)\cdot f'_{k+1}(x) + f_{k+1}(x)\cdot\sum_{i=1}^{k}\{f_1(x)\cdot f_2(x)\ldots f_i'(x)\ldots f_k(x)\}$$

[by assumption step]

$$= \{f_1(x) f_2(x)\ldots f_k(x)\} f'_{k+1}(x) + f_{k+1}(x)\sum_{i=1}^{k}\{f_1(x) f_2(x)\ldots f_i'(x)\ldots f_k(x)\}$$

$$= \sum_{i=1}^{k+1}\{f_1(x) f_2(x)\ldots f_i'(x)\ldots f_{k+1}(x)\}$$

= RHS

This shows that the result is true for $n = k+1$. Hence, by the principle of mathematical induction, the result is true for all $n \in N$.

Mathematical Induction Exercise 1 :
Single Option Correct Type Questions

- This section contains 3 **multiple choice questions**. Each question has four choices (a), (b), (c) and (d), out of which **ONLY ONE** is correct.

1. If $a_n = \sqrt{7+\sqrt{7+\sqrt{7+\ldots}}}$ having n radical signs. Then, by mathematical induction which one is true?
(a) $a_n > 7, \forall\ n \geq 1$
(b) $a_n > 3, \forall\ n \geq 1$
(c) $a_n < 4, \forall\ n \geq 1$
(d) $a_n < 3, \forall\ n \geq 1$

2. If $P(n) = 2 + 4 + 6 + \ldots + 2n, n \in N$, then $P(k) = k(k+1) + 2$
$\Rightarrow P(k+1) = (k+1)(k+2) + 2, \forall\ k \in N$. So, we can conclude that $P(n) = n(n+1) + 2$ for
(a) all $n \in N$ (b) $n > 1$
(c) $n > 2$ (d) Nothing can be said

3. The value of the natural number n such that the inequality $2^n > 2n + 1$ is valid, is
(a) for $n \geq 3$ (b) for $n < 3$ (c) for all n (d) for mn

Mathematical Induction Exercise 2 :
Statement I and II Type Questions

- **Directions** Question Number 4 to 6 Assertion-Reason type questions. Each of these questions contains two statements.
Statement-1 (Assertion) and
Statement-2 (Reason)
Each of these questions also four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below:
  (a) Statement-1 is true, Statement-2 is true; Statement-2 is correct explanation for Statement-1
  (b) Statement-1 is true, Statement-2 is true; Statement-2 is not correct explanation for Statement-1
  (c) Statement-1 is true, Statement-2 is false
  (d) Statement-1 is false, Statement-2 is true

4. **Statement-1** If $a_1 = 1, a_2 = 5$, then $a_n = 3^n - 2^n, \forall\ n \in N$ and $n \geq 1$.
Statement-2 $a_{n+2} = 5a_{n+1} - 6a_n, n \geq 1$.

5. **Statement-1** For all natural numbers n, $2 \cdot 7^n + 3 \cdot 5^n - 5$ is divisible by 24.
Statement-2 If $f(x)$ is divisible by x, then $f(x+1) - f(x)$ is divisible by $x + 1, \forall\ x \in N$.

6. **Statement-1** For all natural numbers n,
$0 \cdot 5 + 0 \cdot 55 + 0 \cdot 555 + \ldots$ upto n terms $= \frac{5}{9}\left\{n - \frac{1}{9}\left(1 - \frac{1}{10^n}\right)\right\}$
Statement-2 $a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(1-r^n)}{(1-r)}$, for $0 < r < 1$.

Mathematical Induction Exercise 3 :
Subjective Type Questions

- In this section, there are **10 subjective** questions.

7. Prove the following by using induction for all $n \in N$.
(i) $11^{n+2} + 12^{2n+1}$ is divisible by 133.
(ii) $n^7 - n$ is divisible by 42.
(iii) $3^{2n} + 24n - 1$ is divisible by 32.
(iv) $n(n+1)(n+5)$ is divisible by 6.
(v) $(25)^{n+1} - 24n + 5735$ is divisible by $(24)^2$.
(vi) $x^{2n} - y^{2n}$ is divisible by $x + y$.

8. Prove by induction that if n is a positive integer not divisible by 3, then $3^{2n} + 3^n + 1$ is divisible by 13.

9. Prove by induction that the product of three consecutive positive integers is divisible by 6.

10. Prove by induction that the sum of three successive natural numbers is divisible by 9.

11. Prove by induction that the even power of every odd integer when divided by 8 leaves the remainder 1.

12. Prove the following by using induction for all $n \in N$:

(i) $1+2+3+\ldots+n=\dfrac{n(n+1)}{2}$

(ii) $1^2+2^2+3^2+\ldots+n^2=\dfrac{n(n+1)(2n+1)}{6}$

(iii) $1\cdot3+3\cdot5+5\cdot7+\ldots+(2n-1)(2n+1)=\dfrac{n(4n^2+6n-1)}{3}$

(iv) $\dfrac{1}{2\cdot5}+\dfrac{1}{5\cdot8}+\dfrac{1}{8\cdot11}+\ldots+\dfrac{1}{(3n-1)(3n+2)}=\dfrac{n}{6n+4}$

(v) $1\cdot4\cdot7+2\cdot5\cdot8+3\cdot6\cdot9+\ldots$ upto n terms $=\dfrac{n}{4}(n+1)(n+6)(n+7)$

(vi) $\dfrac{1^2}{1\cdot3}+\dfrac{2^2}{3\cdot5}+\ldots+\dfrac{n^2}{(2n-1)(2n+1)}=\dfrac{n(n+1)}{2(2n+1)}$

13. Let $a_0=2, a_1=5$ and for $n\geq2, a_n=5a_{n-1}-6a_{n-2}$, then prove by induction that $a_n=2^n+3^n, \forall\ n\geq0, n\in N$.

14. If $a_1=1, a_{n+1}=\dfrac{1}{n+1}a_n, n\geq1$, then prove by induction that $a_{n+1}=\dfrac{1}{(n+1)!}, n\in N$.

15. If a, b, c, d, e and f are six real numbers such that

$$a+b+c=d+e+f$$
$$a^2+b^2+c^2=d^2+e^2+f^2$$

and $a^3+b^3+c^3=d^3+e^3+f^3$, prove by mathematical induction that

$a^n+b^n+c^n=d^n+e^n+f^n, \forall\ n\in N$.

16. Using mathematical induction, prove that

$$\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)+\ldots+\tan^{-1}\left(\frac{1}{n^2+n+1}\right)=\tan^{-1}\left(\frac{n}{n+2}\right).$$

Mathematical Induction Exercise 4 : Questions Asked in Previous 13 Year's Exam

- This section contains questions asked in **IIT-JEE, AIEEE, JEE Main & JEE Advanced** from year **2005** to year **2017**.

17. **Statement-1** For every natural number $n\geq2\ \dfrac{1}{\sqrt{1}}+\dfrac{1}{\sqrt{2}}+\ldots+\dfrac{1}{n}>\sqrt{n}$,

Statement-2 For every natural number $n\geq2\ \sqrt{n(n+1)}<n+1$

(a) Statement-1 is true, Statement-2 is true; Statement-2 is correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement-1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true **[AIEEE 2008, 3M]**

18. **Statement-1** For each natural number $n, (n+1)^7-n^7-1$ is divisible by 7.

Statement-2 For each natural number n, n^7-n is divisible by 7.

(a) Statement-1 is false, Statement-2 is true
(b) Statement-1 is true, Statement-2 is true; Statement-2 is correct explanation for Statement-1
(c) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(d) Statement-1 is true, Statement-2 is false **[AIEEE 2011, 4M]**

Answers

Chapter Exercise

1. (c) **2.** (d) **3.** (a) **4.** (a) **5.** (c) **6.** (b) **17.** (b) **18.** (b)

Solutions

1. Let $P(n): a_n = \sqrt{7+\sqrt{7+\sqrt{7+\ldots}}}$ (n radical sign)

***Step* I** For $n = 1$,

$P(1): a_1 = \sqrt{7} < 4$

***Step* II** Assume that $a_k < 4$ for all natural number, $n = k$

***Step* III** For $n = k+1$,

$P(k+1): a_{k+1} = \sqrt{7+\sqrt{7+\sqrt{7+\ldots}}}$

[$(k+1)$ radical sign]

$= \sqrt{7+a_k} < \sqrt{7+4}$ [$\because a_k < 4$]

< 4 [by assumption]

This shows that, $a_{k+1} < 4$, i.e. the result is true for $n = k+1$.
Hence, by the principle of mathematical induction

$a_n < 4, \forall n \geq 1$

2. It is obvious.

3. Check through options, the condition $2^n > 2n+1$ is valid for $n \geq 3$.

4. Let $P(n): a_n = 3^n - 2^n$

***Step* I** For $n = 1$,

LHS $= a_1 = 1$ [given]

and RHS $= 3^1 - 2^1 = 1$

$\therefore$ LHS = RHS

Hence, $P(1)$ is true.

For $n = 2$,

LHS $= a_2 = 5$ [given]

and RHS $= 3^2 - 2^2 = 5$

$\therefore$ LHS = RHS

Hence, $P(2)$ is also true.

Thus, $P(1)$ and $P(2)$ are true.

***Step* II** Let $P(k)$ and $P(k-1)$ are true

$\therefore$ $a_k = 3^k - 2^k$ and $a_{k-1} = 3^{k-1} - 2^{k-1}$

***Step* III** For $n = k+1$,

$a_{k+1} = 5a_k - 6a_{k-1}$ [from Statement-2]

$= 5(3^k - 2^k) - 6(3^{k-1} - 2^{k-1})$

$= 5\cdot 3^k - 5\cdot 2^k - 2\cdot 3^k + 3\cdot 2^k$

$= 3\cdot 3^k - 2\cdot 2^k = 3^{k+1} - 2^{k+1}$

which is true for $n = k+1$.

Hence, both statements are true and Statement-2 is a correct explanation of Statement-1.

5. Let $P(n): 2\cdot 7^n + 3\cdot 5^n - 5$

***Step* I** For $n = 1$,

$P(1): 2\cdot 7^1 + 3\cdot 5^1 - 5$

: 24 is divisible by 24.

***Step* II** Assume $P(k)$ is divisible by 24, then

$P(k): 2\cdot 7^k + 3\cdot 5^k - 5 = 24\lambda$, λ is positive integer.

***Step* III** For $n = k+1$,

$P(k+1) - P(k) = (2\cdot 7^{k+1} + 3\cdot 5^{k+1} - 5) - (2\cdot 7^k + 3\cdot 5^k - 5)$

$= 2\cdot 7^k(7-1) + 3\cdot 5^k(5-1)$

$= 12(7^k + 5^k)$

$=$ divisible by 24

$= 24\mu, \forall \mu \in I$

[$\because 7^k + 5^k$ is always divisible by 24]

$\therefore$ $P(k+1) = P(k) + 24\mu = 24\lambda + 24\mu$

$= 24(\lambda + \mu)$

Hence, $P(k+1)$ is divisible by 24.

Hence, Statement-1 is true and Statement-2 is false.

6. ***Step* I** For $n = 1$,

LHS $= 0.5$ and RHS $= \frac{5}{9}\left\{1 - \frac{1}{9}\left(1 - \frac{1}{10}\right)\right\} = \frac{5}{9}\left(1 - \frac{1}{10}\right) = \frac{5}{10} = 0.5$

$\therefore$ LHS = RHS

which is true for $n = 1$.

***Step* II** Assume it is true for $n = k$, then $0.5 + 0.55 + 0.555 + \ldots$ + upto k terms

$= \frac{5}{9}\left\{k - \frac{1}{9}\left(1 - \frac{1}{10^k}\right)\right\}$

***Step* III** For $n = k+1$,

LHS $= 0.5 + 0.55 + 0.555 + \ldots +$ upto $(k+1)$ terms

$= \frac{5}{9}\left\{k - \frac{1}{9}\left(1 - \frac{1}{10^k}\right)\right\} + (k+1)$ th terms

$= \frac{5}{9}\left\{k - \frac{1}{9}\left(1 - \frac{1}{10^k}\right)\right\} + \underbrace{0.555\ldots5}_{(k+1) \text{ times}}$

$= \frac{5}{9}\left\{k - \frac{1}{9}\left(1 - \frac{1}{10^k}\right)\right\} + \frac{1}{10^{k+1}}\left(\underbrace{555\ldots5}_{(k+1) \text{ times}}\right)$

$= \frac{5}{9}\left\{k - \frac{1}{9}\left(1 - \frac{1}{10^k}\right)\right\} + \frac{5}{10^{k+1}}$

$(1 + 10 + 10^2 + \ldots + 10^k)$

$= \frac{5}{9}\left\{k - \frac{1}{9}\left(1 - \frac{1}{10^k}\right)\right\} + \frac{5\cdot(10^{k+1} - 1)}{10^{k+1}\cdot(10-1)}$

$= \frac{5}{9}\left\{k - \frac{1}{9}\left(1 - \frac{1}{10^k}\right) + \frac{10^{k+1} - 1}{10^{k+1}}\right\}$

$= \frac{5}{9}\left\{(k+1) - \frac{1}{9} + \frac{1}{9\cdot 10^k} - \frac{1}{10^{k+1}}\right\}$

$= \frac{5}{9}\left\{(k+1) - \frac{1}{9} + \frac{(10-9)}{9\cdot 10^{k+1}}\right\}$

$= \frac{5}{9}\left\{(k+1) - \frac{1}{9}\left(1 - \frac{1}{10^{k+1}}\right)\right\} =$ RHS

which is true for $n = k+1$.

Hence, both statements are true but Statement-2 is not a correct explanation for Statement-1.

7. (i) Let $P(n) = 11^{n+2} + 12^{2n+1}$

***Step* I** For $n = 1$,

$$P(1) = 11^{1+2} + 12^{2\times 1+1} = 11^3 + 12^3$$
$$= (11 + 12)(11^2 - 11 \times 12 + 12^2)$$
$$= 23 \times 133, \text{ which is divisible by 133.}$$

Therefore, the result is true for $n = 1$.

***Step* II** Assume that the result is true for $n = k$, then

$P(k) = 11^{k+2} + 12^{2k+1}$ is divisible by 133.

$\Rightarrow$ $P(k) = 133r$, where r is an integer.

***Step* III** For $n = k + 1$,

$$\therefore\ P(k+1) = 11^{(k+1)+2} + 12^{2(k+1)+1} = 11^{k+3} + 12^{2k+3}$$
$$= 11^{(k+1)+1} \cdot 11 + 12^{2k+1} \cdot 12^2$$
$$= 11 \cdot 11^{k+2} + 144 \cdot 12^{2k+1}$$

Now, $11^{k+2} + 12^{2k+1}\overline{)11 \cdot 11^{k+2} + 144 \cdot 12^{2k+1}}(11$

$$11 \cdot 11^{k+2} + 11 \cdot 12^{2k+1}$$
$$- \quad -$$
$$133 \cdot 12^{2k+1}$$

$$\therefore\ 11 \cdot 11^{(k+2)} + 144 \cdot 12^{2k+1}$$
$$= 11(11^{k+2} + 12^{2k+1}) + 133 \cdot 12^{2k+1}$$

i.e. $P(k+1) = 11P(k) + 133 \cdot 12^{2k+1}$

But we know that, $P(k)$ is divisible by 133. Also, $133 \cdot 12^{2k+1}$ is divisible by 133.

Hence, $P(k + 1)$ is divisible by 133. This shows that, the result is true for $n = k + 1$.

Hence, by the principle of mathematical induction, the result is true for all $n \in N$.

(ii) Let $P(n) = n^7 - n$

***Step* I** For $n = 1$,

$P(1) = 1^7 - 1 = 0$, which is divisible by 42.

Therefore, the result is true for $n = 1$.

***Step* II** Assume that the result is true for $n = k$. Then,

$P(k) = k^7 - k$ is divisible by 42.

$\Rightarrow$ $P(k) = 42r$, where r is an integer.

***Step* III** For $n = k + 1$,

$$P(k+1) = (k+1)^7 - (k+1)$$
$$= (1+k)^7 - (k+1)$$
$$= 1 + {}^7C_1 k + {}^7C_2 k^2 + {}^7C_3 k^3 + {}^7C_4 k^4 + {}^7C_5 k^5 + {}^7C_6 k^6 + {}^7C_7 k^7 - (k+1)$$
$$= (k^7 - k) + ({}^7C_1 k + {}^7C_2 k^2 + {}^7C_3 k^3 + {}^7C_4 k^4 + {}^7C_5 k^5 + {}^7C_6 k^6)$$

But by assumption $k^7 - k$ is divisible by 42.

Also, ${}^7C_1 k + {}^7C_2 k^2 + {}^7C_3 k^3 + {}^7C_4 k^4 + {}^7C_5 k^5 + {}^7C_6 k^6$

is divisible by 42. [$\because$ ${}^7C_r$, $1 \le r \le 6$ is divisible by 7]

Hence, $P(k + 1)$ is divisible by 42. This shows that, the result is true for $n = k + 1$.

$\therefore$ By the principle of mathematical induction, the result is true for all $n \in N$.

(iii) Let $P(n) = 3^{2n} + 24n - 1$

***Step* I** For $n = 1$,

$$P(1) = 3^{2\times 1} + 24 \times 1 - 1 = 3^2 + 24 - 1 = 9 + 24 - 1$$
$$= 32, \text{ which is divisible by 32.}$$

Therefore, the result is true for $n = 1$.

***Step* II** Assume that the result is true for $n = k$. Then,

$P(k) = 3^{2k} + 24k - 1$ is divisible by 32.

$\Rightarrow$ $P(k) = 32r$, where r is an integer.

***Step* III** For $n = k + 1$,

$$P(k+1) = 3^{2(k+1)} + 24(k+1) - 1$$
$$= 3^{2k+2} + 24k + 24 - 1$$
$$= 3^2 \cdot 3^{2k} + 24k + 23$$
$$= 9 \cdot 3^{2k} + 24k + 23$$

Now, $3^{2k} + 24k - 1\overline{)9 \times 3^{2k} + 24k + 23}(9$

$$9 \cdot 3^{2k} + 216k - 9$$
$$- \quad - \quad +$$
$$-192k + 32$$

$$\therefore\ P(k+1) = 9(3^{2k} + 24k - 1) - 32(6k - 1)$$
$$= 9P(k) - 32(6k - 1)$$

$\therefore$ $P(k+1) = 9(32r) - 32(6k - 1)$ [by assumption step]

$$= 32(9r - 6k + 1),$$

which is divisible by 32, as $9r - 6k + 1$ is an integer.

Therefore, $P(k + 1)$ is divisible by 32. Hence, by the principle of mathematical induction $P(n)$ is divisible by 32, $\forall\ n \in N$.

(iv) Let $P(n) = n(n + 1)(n + 5)$

***Step* I** For $n = 1$,

$$P(1) = 1 \cdot (1+1)(1+5) = 1 \cdot 2 \cdot 6$$
$$= 12, \text{ which is divisible by 6.}$$

Therefore, the result is true for $n = 1$.

Step II Assume that the result is true for $n = k$. Then,

$\Rightarrow$ $P(k) = k(k + 1)(k + 5)$ is divisible by 6.
$P(k) = 6r$, r is an integer.

Step III For $n = k + 1$,

$$P(k+1) = (k+1)(k+1+1)(k+1+5)$$
$$= (k+1)(k+2)(k+6)$$

Now, $P(k+1) - P(k) = (k+1)(k+2)(k+6) - k(k+1)(k+5)$

$$= (k+1)\{k^2 + 8k + 12 - k^2 - 5k\}$$
$$= (k+1)(3k+12)$$
$$= 3(k+1)(k+4)$$

$\Rightarrow$ $P(k+1) = P(k) + 3(k+1)(k+4)$

which is divisible by 6 as $P(k)$ is divisible by 6 [by assumption step]

and clearly $3(k + 1)(k + 4)$ is divisible by 6, $\forall\ k \in N$.

Hence, the result is true for $n = k + 1$.

Therefore, by the principle of mathematical induction, the result is true for all $n \in N$.

(v) Let $P(n) = (25)^{n+1} - 24n + 5735$

***Step* I** For $n = 1$,

$$P(1) = (25)^2 - 24 + 5735 = 625 - 24 + 5735 = 6336$$
$$= 11 \times (24)^2, \text{ which is divisible by } (24)^2.$$

Therefore, the result is true for $n = 1$.

***Step* II** Assume that the result is true for $n = k$. Then,

$P(k) = (25)^{k+1} - 24k + 5735$ is divisible by $(24)^2$.

$\Rightarrow$ $P(k) = (24)^2 r$, where r is an integer.

***Step* III** For $n = k + 1$,

$$P(k+1) = (25)^{(k+1)+1} - 24(k+1) + 5735$$
$$= (25)^{k+2} - 24k + 5711$$
$$= (25)(25)^{k+1} - 24k + 5711$$

Now, $P(k+1) - P(k)$

$= \{(25)(25)^{k+1} - 24k + 5711\} - \{(25)^{k+1} - 24k + 5735\}$

$= (24)(25)^{k+1} - 24$

$= 24\{(25)^{k+1} - 1\}$

$\Rightarrow$ $P(k+1) = P(k) + 24\{(25)^{k+1} - 1\}$

But by assumption $P(k)$ is divisible by $(24)^2$. Also, $24\{(25)^{k+1} - 1\}$ is clearly divisible by $(24)^2$, $\forall\ k \in N$. This shows that, the result is true for $n = k + 1$.

Hence, by the principle of mathematical induction, result is true for all $n \in N$.

(vi) Let $P(n) = x^{2n} - y^{2n}$

***Step* I** For $n = 1$,

$$P(1) = x^2 - y^2$$
$= (x - y)(x + y)$ which is divisible by $(x + y)$.

Therefore, the result is true for $n = 1$.

***Step* II** Assume that the result is true for $n = k$. Then,

$P(k) = x^{2k} - y^{2k}$ is divisible by $x + y$.

$\Rightarrow$ $P(k) = (x + y)r$, where r is an integer.

***Step* III** For $n = k + 1$,

$$= x^2 \cdot x^{2k} - y^2 \cdot y^{2k}$$
$$= x^2 x^{2k} - x^2 y^{2k} + x^2 y^{2k} - y^2 y^{2k}$$
$$= x^2(x^{2k} - y^{2k}) + y^{2k}(x^2 - y^2)$$
$$= x^2(x + y)r + y^{2k}(x - y)(x + y)$$

[by assumption step]

$$= (x + y)\{x^2 r + y^{2k}(x - y)\}$$

which is divisible by $(x + y)$ as $x^2 r + y^{2k}(x - y)$ is an integer.

This shows that the result is true for $n = k + 1$. Hence, by the principle of mathematical induction, the result is true for all $n \in N$.

8. Let $P(n) = 3^{2n} + 3^n + 1$, $\forall\ n$ is a positive integer not divisible by 3.

***Step* I** For $n = 1$,

$$P(1) = 3^2 + 3 + 1 = 9 + 3 + 1$$
$$= 13, \text{ which is divisible by } 13.$$

Therefore, $P(1)$ is true.

***Step* II** Assume $P(n)$ is true for $n = k$, k is a positive integer not divisible by 3, then

$P(k) = 3^{2k} + 3^k + 1$, is divisible by 13.

$\Rightarrow$ $P(k) = 13r$, where r is an integer.

***Step* III** For $n = k + 1$,

$$P(k+1) = 3^{2(k+1)} + 3^{k+1} + 1$$
$$= 3^2 \cdot 3^{2k} + 3 \cdot 3^k + 1$$

Now, $3^{2k} + 3^k + 1 \overline{)\ 3^2 \cdot 3^{2k} + 3 \cdot 3^k + 1\ (3^2}$

$3^2 \cdot 3^{2k} + 3^2 \cdot 3^k + 3^2$

$- \quad - \quad -$

$-6 \cdot 3^k - 8$

$\Rightarrow$ $P(k+1) = 3^2(3^{2k} + 3^k + 1) - 6 \cdot 3^k - 8$

$$= 9P(k) - 2(3^{k+1} + 4)$$
$$= 9(13r) - 2(3^{k+1} + 4) \qquad \text{[by assumption step]}$$

which is divisible by 13 as $3^{k+1} + 4$ is also divisible by 13, $\forall\ k \in N$ and not divisible by 3. This shows that the result is true for $n = k + 1$. Hence, by the principle of mathematical induction, the result is true for all natural numbers not divisible by 3.

9. Let $P(n) = n(n+1)(n+2)$, where n is a positive integer.

***Step* I** For $n = 1$,

$$P(1) = 1(1+1)(1+2) = 1 \cdot 2 \cdot 3$$
$$= 6, \text{ which is divisible by } 6.$$

Therefore, the result is true for $n = 1$.

***Step* II** Let us assume that the result is true for $n = k$, where k is a positive integer.

Then, $P(k) = k(k+1)(k+2)$ is divisible by 6.

$\Rightarrow$ $P(k) = 6r$, where r is an integer.

$a^2\ a^k + b^2 b^k \overline{)\ a^2\ }$ [infact positive integer]

***Step* III** For $n = k + 1$, where k is a positive integer.

$$P(k+1) = (k+1)(k+1+1)(k+2+1)$$
$$= (k+1)(k+2)(k+3)$$

Now, $P(k+1) - P(k) = (k+1)(k+2)(k+3) - k(k+1)(k+2)$

$$= (k+1)(k+2)(k+3-k)$$
$$= 3(k+1)(k+2)$$

$\Rightarrow$ $P(k+1) = P(k) + 3(k+1)(k+2)$

But we know that, $P(k)$ is divisible by 6. Also, $3(k+1)(k+2)$ is divisible by 6 for all positive integer. This shows that the result is true for $n = k + 1$. Hence, by the principle of mathematical induction, the result is true for all positive integer.

10. Let $P(n) = n^3 + (n+1)^3 + (n+2)^3$, where $n \in N$.

***Step* I** For $n = 1$,

$$P(1) = 1^3 + 2^3 + 3^3 = 1 + 8 + 27$$
$$= 36, \text{ which is divisible by } 9.$$

***Step* II** Assume that $P(n)$ is true for $n = k$, then

$P(k) = k^3 + (k+1)^3 + (k+2)^3$, where $k \in N$.

$\Rightarrow$ $P(k) = 9r$, where r is a positive integer.

***Step* III** For $n = k + 1$,

$$P(k+1) = (k+1)^3 + (k+2)^3 + (k+3)^3$$

Now, $P(k+1) - P(k) = (k+1)^3 + (k+2)^3 + (k+3)^3 - \{k^3 + (k+1)^3 + (k+2)^3\}$

$$= (k+3)^3 - k^3$$

$$= k^3 + 9k^2 + 27k + 27 - k^3$$

$$= 9(k^2 + 3k + 3)$$

$$\Rightarrow \quad P(k+1) = P(k) + 9(k^2 + 3k + 3)$$

$$= 9r + 9(k^2 + 3k + 3)$$

$$= 9(r + k^2 + 3k + 3)$$

which is divisible by 9 as $(r + k^2 + 3k + 3)$ is a positive integer.

Hence, by the principle mathematical induction, $P(n)$ is divisible by 9 for all $n \in N$.

11. Let $P(n) : (2r+1)^{2n}$, $\forall\ n \in N$ and $r \in I$.

***Step* I** For $n = 1$,

$$P(1) : (2r+1)^2 = 4r^2 + 4r + 1$$

$$= 4r(r+1) + 1 = 8p + 1,\ p \in I$$

[$\because r(r+1)$ is an even integer]

Therefore, $P(1)$ is true.

***Step* II** Assume $P(n)$ is true for $n = k$, then

$P(k) : (2r+1)^{2k}$ is divisible by 8 leaves remainder 1.

$\Rightarrow \quad P(k) = 8m + 1$, $n \in I$, where m is a positive integer.

***Step* III** For $n = k + 1$,

$\therefore \quad P(k+1) = (2r+1)\,2(k+1)$

$$= (2r+1)^{2k}(2r+1)^2$$

$$= (8m+1)(8p+1) \qquad \text{[from Steps I and II]}$$

$$= 64\ mp + 8\ (m + p) + 1$$

$$= 8\ (8mp + m + p) + 1$$

which is true for $n = k + 1$ as $8mp + m + p$ is an integer. Hence, by the principle of mathematical induction, when $P(n)$ is divided by 8 leaves the remainder 1 for all $n \in N$.

12. (i) Let $P(n) : 1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$ ...(i)

***Step* I** For $n = 1$,

LHS of Eq. (i) $= 1$

RHS of Eq. (i) $= \dfrac{1(1+1)}{2} = 1$

LHS = RHS

Therefore, $P(1)$ is true.

***Step* II** Let us assume that the result is true for $n = k$. Then,

$$P(k) : 1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}$$

***Step* III** For $n = k + 1$, we have to prove that

$$P(k+1) = 1 + 2 + 3 + \ldots + k + (k+1) = \frac{(k+1)(k+2)}{2}$$

$$\text{LHS} = 1 + 2 + 3 + \ldots + k + (k+1)$$

$$= \frac{k(k+1)}{2} + k + 1 \qquad \text{[by assumption step]}$$

$$= (k+1)\left(\frac{k}{2} + 1\right) = (k+1)\left(\frac{k+2}{2}\right)$$

$$= \frac{(k+1)(k+2)}{2}$$

$$= \text{RHS}$$

This shows that the result is true for $n = k + 1$. Therefore, by the principle of mathematical induction, the result is true for all $n \in N$.

(ii) Let $P(n) : 1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{(n+1)(2n+1)}{6}$ ...(i)

***Step* I** For $n = 1$,

LHS of Eq. (i) $= 1^2 = 1$

RHS of Eq. (i) $= \dfrac{1(1+1)(2\times 1+1)}{6}$

$$= \frac{1\cdot 2\cdot 3}{6} = 1$$

LHS = RHS

Therefore, $P(1)$ is true.

***Step* II** Let us assume that the result is true for $n = k$. Then,

$$P(k) : 1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

***Step* III** For $n = k + 1$, we have to prove that

$$P(k+1) : 1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$\text{LHS} = 1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \qquad \text{[by assumption step]}$$

$$= (k+1)\left\{\frac{k(2k+1)}{6} + (k+1)\right\}$$

$$= (k+1)\left\{\frac{2k^2 + 7k + 6}{6}\right\}$$

$$= (k+1)\left\{\frac{(k+2)(2k+3)}{6}\right\} = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \text{RHS}$$

This shows that the result is true for $n = k + 1$. Therefore, by the principle of mathematical induction, the result is true for all $n \in N$.

(iii) Let $P(n) : 1\cdot 3 + 3\cdot 5 + 5\cdot 7 + \ldots + (2n-1)(2n+1)$

$$= \frac{n(4n^2 + 6n - 1)}{3} \qquad \text{...(i)}$$

***Step* I** For $n = 1$,

LHS of Eq. (i) $= 1\cdot 3 = 3$

RHS of Eq. (i) $= \dfrac{1(4\times 1^2 + 6\times 1 - 1)}{3} = \dfrac{4+6-1}{3} = 3$

$\therefore \quad$ LHS = RHS

Therefore, $P(1)$ is true.

***Step* II** Assume that the result is true for $n = k$. Then,

$$P(k) : 1\cdot 3 + 3\cdot 5 + 5\cdot 7 + \ldots + (2k-1)(2k+1) = \frac{k(4k^2 + 6k - 1)}{3}$$

***Step* III** For $n = k + 1$, we have to prove that

$$P(k+1): 1\cdot 3 + 3\cdot 5 + 5\cdot 7 + \ldots + (2k-1)(2k+1) + (2k+1)(2k+3)$$
$$= \frac{(k+1)\left[4(k+1)^2 + 6(k+1) - 1\right]}{3}$$
$$= \frac{(k+1)(4k^2 + 14k + 9)}{3}$$

$$\text{LHS} = 1\cdot 3 + 3\cdot 5 + 5\cdot 7 + \ldots + (2k-1)(2k+1) + (2k+1)(2k+3)$$
$$= \frac{k(4k^2 + 6k - 1)}{3} + (2k+1)(2k+3) \quad \text{[by assumption step]}$$
$$= \frac{4k^3 + 6k^2 - k}{3} + (4k^2 + 8k + 3)$$
$$= \frac{4k^3 + 18k^2 + 23k + 9}{3}$$
$$= \frac{(k+1)(4k^2 + 14k + 9)}{3} = \text{RHS}$$

This shows that the result is true for $n = k + 1$. Therefore, by the principle of mathematical induction, the result is true for all $n \in N$.

(iv) Let $P(n): \dfrac{1}{2\cdot 5} + \dfrac{1}{5\cdot 8} + \dfrac{1}{8\cdot 11} + \ldots + \dfrac{1}{(3n-1)(3n+2)}$

$$= \frac{n}{6n+4} \quad \ldots(i)$$

***Step* I** For $n = 1$,

$$\text{LHS of Eq. (i)} = \frac{1}{2\cdot 5} = \frac{1}{10}$$
$$\text{RHS of Eq. (i)} = \frac{1}{6\times 1 + 4} = \frac{1}{10}$$
$$\text{LHS} = \text{RHS}$$

Therefore, $P(1)$ is true.

***Step* II** Let us assume that the result is true for $n = k$. Then,

$$P(k): \frac{1}{2\cdot 5} + \frac{1}{5\cdot 8} + \frac{1}{8\cdot 11} + \ldots + \frac{1}{(3k-1)(3k+2)} = \frac{k}{6k+4}$$

***Step* III** For $n = k + 1$, we have to prove that

$$P(k+1): \frac{1}{2\cdot 5} + \frac{1}{5\cdot 8} + \frac{1}{8\cdot 11} + \ldots + \frac{1}{(3k-1)(3k+2)} + \frac{1}{(3k+2)(3k+5)}$$
$$= \frac{(k+1)}{6(k+1)+4} = \frac{(k+1)}{6k+10}$$

$$\text{LHS} = \frac{1}{2\cdot 5} + \frac{1}{5\cdot 8} + \frac{1}{8\cdot 11} + \ldots + \frac{1}{(3k-1)(3k+2)} + \frac{1}{(3k+2)(3k+5)}$$
$$= \frac{k}{6k+4} + \frac{1}{(3k+2)(3k+5)} \quad \text{[by assumption step]}$$
$$= \frac{k(3k+5)+2}{2(3k+2)(3k+5)} = \frac{3k^2 + 5k + 2}{2(3k+2)(3k+5)}$$
$$= \frac{(k+1)(3k+2)}{2(3k+2)(3k+5)} = \frac{k+1}{6k+10}$$
$$= \text{RHS}$$

This shows that the result is true for $n = k + 1$. Therefore, by the principle of mathematical induction, the result is true for all $n \in N$.

(v) Let $P(n): 1\cdot 4\cdot 7 + 2\cdot 5\cdot 8 + 3\cdot 6\cdot 9 + \ldots$ + upto n terms

$$= \frac{n}{4}(n+1)(n+6)(n+7)$$

i.e., $P(n): 1\cdot 4\cdot 7 + 2\cdot 5\cdot 8 + 3\cdot 6\cdot 9 + \ldots + n(n+3)(n+6)$

$$= \frac{n}{4}(n+1)(n+6)(n+7) \quad \ldots(i)$$

***Step* I** For $n = 1$,

$$\text{LHS of Eq. (i)} = 1\cdot 4\cdot 7 = 28$$
$$\text{RHS of Eq. (i)} = \frac{1}{4}(1+1)(1+6)(1+7) = \frac{2\cdot 7\cdot 8}{4} = 28$$
$$\text{LHS} = \text{RHS}$$

Therefore, $P(1)$ is true.

***Step* II** Let us assume that the result is true for $n = k$. Then,

$$P(k): 1\cdot 4\cdot 7 + 2\cdot 5\cdot 8 + 3\cdot 6\cdot 9 + \ldots + k(k+3)(k+6) = \frac{k}{4}(k+1)(k+6)(k+7)$$

***Step* III** For $n = k + 1$, we have to prove that

$$P(k+1): 1\cdot 4\cdot 7 + 2\cdot 5\cdot 8 + 3\cdot 6\cdot 9 + \ldots + k(k+3)(k+6) + (k+1)(k+4)(k+7)$$
$$= \frac{(k+1)}{4}(k+2)(k+7)(k+8)$$

$$\text{LHS} = 1\cdot 4\cdot 7 + 2\cdot 5\cdot 8 + 3\cdot 6\cdot 9 + \ldots + k(k+3)(k+6) + (k+1)(k+4)(k+7)$$
$$= \frac{k}{4}(k+1)(k+6)(k+7) + (k+1)(k+4)(k+7) \quad \text{[by assumption step]}$$
$$= (k+1)(k+7)\left\{\frac{k}{4}(k+6) + (k+4)\right\}$$
$$= (k+1)(k+7)\left\{\frac{k^2 + 6k + 4k + 16}{4}\right\}$$
$$= (k+1)(k+7)\left\{\frac{k^2 + 10k + 16}{4}\right\}$$
$$= (k+1)(k+7)\left\{\frac{(k+2)(k+8)}{4}\right\}$$
$$= \frac{(k+1)}{4}(k+2)(k+7)(k+8) = \text{RHS}$$

This shows that the result is true for $n = k + 1$. Hence, by the principle of mathematical induction, the result is true for all $n \in N$.

(vi) Let $P(n): \dfrac{1^2}{1\cdot 3} + \dfrac{2^2}{3\cdot 5} + \ldots + \dfrac{n^2}{(2n-1)(2n+1)}$

$$= \frac{n(n+1)}{2(2n+1)} \quad \ldots(i)$$

***Step* I** For $n = 1$,

$$\text{LHS of Eq. (i)} = \frac{1^2}{1\cdot 3} = \frac{1}{3}$$

$$\text{RHS of Eq. (i)} = \frac{1(1+1)}{2(2\times 1+1)} = \frac{2}{2(3)} = \frac{1}{3}$$

$$\text{LHS} = \text{RHS}$$

Therefore, $P(1)$ is true.

***Step* II** Let us assume that the result is true for $n = k$, then

$$P(k) = \frac{1^2}{1\cdot 3} + \frac{2^2}{3\cdot 5} + \ldots + \frac{k^2}{(2k-1)(2k+1)} = \frac{k(k+1)}{2(2k+1)}$$

***Step* III** For $n = k + 1$, we have to prove that

$$P(k+1): \frac{1^2}{1\cdot 3} + \frac{2^2}{3\cdot 5} + \ldots + \frac{k^2}{(2k-1)(2k+1)} + \frac{(k+1)^2}{(2k+1)(2k+3)} = \frac{(k+1)(k+2)}{2(2k+3)}$$

$$\text{LHS} = \frac{1^2}{1\cdot 3} + \frac{2^2}{3\cdot 5} + \ldots + \frac{k^2}{(2k-1)(2k+1)} + \frac{(k+1)^2}{(2k+1)(2k+3)}$$

$$= \frac{k(k+1)}{2(2k+1)} + \frac{(k+1)^2}{(2k+1)(2k+3)} \quad \text{[by assumption step]}$$

$$= \frac{(k+1)}{(2k+1)}\left\{\frac{k}{2} + \frac{k+1}{(2k+3)}\right\} = \frac{(k+1)}{(2k+1)}\left\{\frac{2k^2+5k+2}{2(2k+3)}\right\}$$

$$= \frac{(k+1)}{(2k+1)}\cdot\frac{(k+2)(2k+1)}{2(2k+3)} = \frac{(k+1)(k+2)}{2(2k+3)}$$

$$= \text{RHS}$$

This shows that, the result is true for $n = k + 1$. Therefore, by the principle of mathematical induction the result is true for all $n \in N$.

13. Let $P(n): a_n = 2^n + 3^n, \forall\ n \geq 0, n \in N$

and $a_0 = 2, a_1 = 5$ and for $n \geq 2$; $a_n = 5a_{n-1} - 6a_n - 2$

***Step* I** For $n = 0$,

$$a_0 = 2^0 + 3^0 = 1 + 10 = 2$$

which is true as $a_0 = 2$. [given]

Also, for $n = 1$, $\quad a_1 = 2^1 + 3^1 = 2 + 3 = 5$

which is also true as $a_1 = 5$. [given]

Hence, $P(0)$ and $P(1)$ are true.

***Step* II** Assume that $P(k-1)$ and $P(k)$ are true. Then,

$$a_{k-1} = 2^{k-1} + 3^{k-1} \quad \ldots(i)$$

where $\quad a_{k-1} = 5a_{k-2} - 6a_{k-3}$ and $a_k = 2^k + 3^k \quad \ldots(ii)$

where $\quad a_k = 5a_{k-1} - 6a_{k-2}$

***Step* III** For $n = k + 1$,

$$P(k+1): a_{k+1} = 2^{k+1} + 3^{k+1},\ \forall\ k \geq 0, k \in N.$$

where $\quad a_{k+1} = 5a_k - 6a_{k-1}$

Now, $\quad a_{k+1} = 5a_k - 6a_{k-1}$

$$= 5(2^k + 3^k) - 6(2^{k-1} + 3^{k-1})$$

[by using Eqs. (i) and (ii)]

$$= 5\cdot 2^k + 5\cdot 3^k - 6\cdot 2^{k-1} - 6\cdot 3^{k-1}$$

$$= 2^{k-1}(5\cdot 2 - 6) + 3^{k-1}(5\cdot 3 - 6)$$

$$= 2^{k-1}\cdot 4 + 3^{k-1}\cdot 9 = 2^{k+1} + 3^{k+1}$$

$$\Rightarrow \quad a_{k+1} = 2^{k+1} + 3^{k+1}$$

where $\quad a_{k+1} = 5a_k - 6a_{k-1}$

This shows that the result is true for $n = k + 1$. Hence, by the second principle of mathematical induction, the result is true for $n \in N, n \geq 0$.

14. Let $P(n): a_{n+1} = \frac{1}{(n+1)!}, n \in N \quad \ldots(i)$

where $a_1 = 1$ and $a_{n+1} = \frac{1}{(n+1)}a_n, n \geq 1 \quad \ldots(ii)$

***Step* I** For $n = 1$, from Eq. (i), we get

$$a_2 = \frac{1}{(1+1)!} = \frac{1}{2!}$$

But from Eq. (ii), we get $a_2 = \frac{1}{(1+1)}, a_1 = \frac{1}{2}(1) = \frac{1}{2}$

which is true.

Also, for $n = 2$ from Eq. (i), we get

$$a_3 = \frac{1}{3!} = \frac{1}{6}$$

But from Eq. (ii), we get $\quad a_3 = \frac{1}{3}, a_2 = \frac{1}{3}\cdot\frac{1}{2} = \frac{1}{6}$

which is also true.

Hence, $P(1)$ and $P(2)$ are true.

***Step* II** Assume that $P(k-1)$ and $P(k)$ are true. Then,

$$P(k-1): a_k = \frac{1}{k!} \quad \ldots(ii)$$

where, $a_k = \frac{1}{k}a_{k-1}, \quad k \geq 1 \quad \ldots(iv)$

and $\quad P(k): a_{k+1} = \frac{1}{(k+1)!} \quad \ldots(v)$

where $\quad a_{k+1} = \frac{1}{k+1}a_k, k \geq 1 \quad \ldots(vi)$

***Step* III** For $n = k + 1$,

$$P(k+1): a_{k+2} = \frac{1}{(k+2)!} \quad \ldots(vii)$$

where $\quad a_{k+2} = \frac{1}{(k+2)}a_{k+1} \quad \ldots(viii)$

Now, LHS of Eq. (vii) $= a_{k+2}$

$$= \frac{1}{(k+2)}a_{k+1} \quad \text{[using Eq. (viii)]}$$

$$= \frac{1}{(k+2)}\cdot\frac{1}{(k+1)}a_k \quad \text{[using Eq. (vi)]}$$

$$= \frac{1}{k+2}\cdot\frac{1}{k+1}\cdot\frac{1}{k}a_{k-1} \quad \text{[using Eq. (iv)]}$$

$$= \frac{1}{k+2}\cdot\frac{1}{k+1}\cdot\frac{1}{k}\cdot\frac{1}{k!} \quad \text{[using Eq. (iii)]}$$

$$= \frac{1}{(k+2)!} = \text{RHS of Eq. (vii)}$$

This shows that the result is true for $n = k + 1$. Hence, by the second principle of mathematical induction, the result is true for all $n \geq 1, n \in N$.

15. Let $P(n): a^n + b^n + c^n = d^n + e^n + f^n, \forall\, n \in N$ ...(i)

where $a + b + c = d + e + f$ ...(ii)

$a^2 + b^2 + c^2 = d^2 + e^2 + f^2$ ...(iii)

and $a^3 + b^3 + c^3 = d^3 + e^3 + f^3$ ...(iv)

***Step* I** For $n = 1$ from Eq. (i), we get

$P(1): a + b + c = d + e + f$ [given]

Hence, the result is true for $n = 1$.

Also, for $n = 2$ from Eq. (i), we get

$P(2): a^2 + b^2 + c^2 = d^2 + e^2 + f^2$ [given]

Hence, the result is true for $n = 2$.

Also, for $x = 3$, from Eq. (i), we get

$P(3): a^3 + b^3 + c^3 = d^3 + e^3 + f^3$ [given]

Hence, the result is true for $n = 3$.

Therefore, $P(1)$, $P(2)$ and $P(3)$ are true.

***Step* II** Assume that $P(k-2)$, $P(k-1)$ and $P(k)$ are true, then

$P(k-2): a^{k-2} + b^{k-2} + c^{k-2} = d^{k-2} + e^{k-2} + f^{k-2}$ ...(v)

$P(k-1): a^{k-1} + b^{k-1} + c^{k-1} = d^{k-1} + e^{k-1} + f^{k-1}$ ...(vi)

and $P(k): a^k + b^k + c^k = d^k + e^k + f^k$ ...(vii)

***Step* III** For $xn = k + 1$, we shall to prove that

$P(k+1): a^{k+1} + b^{k+1} + c^{k+1} = d^{k+1} + e^{k+1} + f^{k+1}$

LHS $= a^{k+1} + b^{k+1} + c^{k+1}$

$= (a^k + b^k + c^k)(a + b + c) - (a^{k-1} + b^{k-1} + c^{k-1})$

$(ab + bc + ca) + abc\,(a^{k-2} + b^{k-2} + c^{k-2})$

$= (d^k + e^k + f^k)(d + e + f) - (d^{k-1} + e^{k-1} + f^{k-1})$

$(de + ef + fd) + def(d^{k-2} + e^{k-2} + f^{k-2})$

[using Eqs. (ii), (iii), (iv), (v), (vi), (vii)]

$\therefore \quad (a + b + c)^2 = (d + e + f)^2$

$\Rightarrow \quad a^2 + b^2 + c^2 + 2(ab + bc + ca)$

$= d^2 + e^2 + f^2 + 2(de + ef + fd)$

$\Rightarrow \quad ab + bc + ca = de + ef + fd$

$[\because a^2 + b^2 + c^2 = d^2 + e^2 + f^2]$

and $a^3 + b^3 + c^3 - 3abc$

$= (d + e + f)(d^2 + e^2 + f^2 - de - ef - fd)$

$= d^3 + e^3 + f^3 - 3def$

$\Rightarrow \quad abc = def \qquad [\because a^3 + b^3 + c^3 = d^3 + e^3 + f^3]$

$= d^{k+1} + e^{k+1} + f^{k+1} = \text{RHS}$

This shows that the result is true for $n = k + 1$. Hence, by second principle of mathematical induction, the result is true for all $n \in N$.

16. Let $P(n): \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \ldots + \tan^{-1}\left(\frac{1}{n^2 + n + 1}\right)$

$= \tan^{-1}\left(\frac{n}{n+2}\right)$ ...(i)

***Step* I** For $n = 1$,

LHS of Eq. (i) $= \tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\left(\frac{1}{1+2}\right)$

$= \text{RHS of Eq. (i)}$

Therefore, $P(1)$ is true.

***Step* II** Assume that $P(k)$ is true. Then,

$P(k): \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \ldots + \tan^{-1}\left(\frac{1}{k^2 + k + 1}\right)$

$= \tan^{-1}\left(\frac{k}{k+2}\right)$

***Step* III** For $n = k + 1$,

$P(k+1): \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \ldots + \tan^{-1}\left(\frac{1}{k^2 + k + 1}\right)$

$+ \tan^{-1}\left(\frac{1}{(k+1)^2 + (k+1) + 1}\right)$

$= \tan^{-1}\left(\frac{k+1}{k+3}\right)$ ...(ii)

LHS of Eq. (ii)

$= \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \ldots + \tan^{-1}\left(\frac{1}{k^2 + k + 1}\right)$

$+ \tan^{-1}\left(\frac{1}{(k+1)^2 + (k+1) + 1}\right)$

$= \tan^{-1}\left(\frac{k}{k+2}\right) + \tan^{-1}\left(\frac{1}{(k+1)^2 + (k+1) + 1}\right)$

[by assumption step]

$= \tan^{-1}\left(\frac{k}{1 + (k+1)}\right) + \tan^{-1}\left(\frac{1}{k^2 + 3k + 3}\right)$

$= \tan^{-1}\left(\frac{k}{1 + (k+1)}\right) + \tan^{-1}\left(\frac{1}{1 + (k+1)(k+2)}\right)$

$= \tan^{-1}\left(\frac{(k+1) - 1}{1 + (k+1)\cdot 1}\right) + \tan^{-1}\left(\frac{(k+2) - (k+1)}{1 + (k+2)(k+1)}\right)$

$= \tan^{-1}(k+1) - \tan^{-1}1 + \tan^{-1}(k+2) - \tan^{-1}(k+1)$

$= \tan^{-1}(k+2) - \tan^{-1}1$

$= \tan^{-1}\left(\frac{k+2-1}{1 + (k+2)\cdot 1}\right) = \tan^{-1}\left(\frac{k+1}{k+3}\right) = \text{RHS of Eq. (ii)}$

This shows that the result is true for $n = k + 1$. Hence, by the principle of mathematical induction, the result is true for all $n \in N$.

17. Let $P(n) = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}}$

$\therefore \quad P(2) = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = 1.707 > \sqrt{2}$

Let us assume that

$$P(k) = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{k}} > \sqrt{k} \text{ is true for } n = k + 1.$$

$$\text{LHS} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}$$

$$> \sqrt{k} + \frac{1}{\sqrt{k+1}} = \frac{\sqrt{k(k+1)+1}}{\sqrt{(k+1)}} > \frac{k+1}{\sqrt{(k+1)}}$$

$$[\because \sqrt{k(k+1)+1} > k, \forall\ k \geq 0]$$

$$\therefore \quad P(k+1) > \sqrt{(k+1)}$$

By mathematical induction Statement-1 is true, $\forall\ n \geq 2$.

Now, let $\alpha(n) = \sqrt{n(n+1)}$

$$\therefore \qquad \alpha(2) = \sqrt{2(2+1)} = \sqrt{6} < 3$$

Let us assume that

$$\alpha(k) = \sqrt{k(k+1)} < (k+1) \text{ is true}$$

for $n = k + 1$

$$\text{LHS} = \sqrt{(k+1)(k+2)} < (k+2)$$

$$[\because (k+1) < (k+2)]$$

By mathematical induction Statement-2 is true but Statement-2 is not a correct explanation for Statement-1.

18. Let $P(n) = n^7 - n$

By mathematical induction for $n = 1$, $P(1) = 0$, which is divisible by 7

for $n = k$, $P(k) = k^7 - k$

Assume $P(k)$ is divisible by 7

$$\therefore \qquad k^7 - k = 7\lambda,\ \lambda \in I \qquad \ldots(i)$$

For $n = k + 1$,

$$P(k+1) = (k+1)^7 - (k+1)$$

$$= ({}^7C_0 k^7 + {}^7C_1 k^6 + {}^7C_2 k^5 + {}^7C_3 k^4 + \ldots + {}^7C_6 k + {}^7C_7) - (k+1)$$

$$= (k^7 - k) + 7(k^6 + 3k^5 + \ldots + k)$$

$$= 7\lambda + 7(k^6 + 3k^5 + \ldots + k) = \text{Divisible by 7}$$

$\therefore$ Statement-2 is true.

Also, let $F(n) = (n+1)^7 - n^7 - 1$

$$= \{(n+1)^7 - (n+1)\} - (n^7 - n)$$

$$= \text{Divisible by 7 from Statement-2}$$

Hence, both statements are true and Statement-2 is correct explanation of Statement-1.

CHAPTER

11

Sets, Relations and Functions

Learning Part

Session 1

- Definition of Set
- Representation of Set
- Different Types of Sets
- Laws and Theorems
- Venn Diagrams (Euler-Venn Diagrams)

Session 2

- Ordered Pair
- Definition of Relation
- Ordered Relation
- Composition of Two Relations

Session 3

- Definition of Functions
- Domain, Codomain and Range
- Composition of Mapping
- Equivalence Classes
- Partition of Set
- Congruences

Practice Part

- JEE Type Examples
- Chapter Exercises

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Session 1

Definition of Set, Representation of Set, Different Types of Sets, Laws and Theorems, Venn Diagram (Euler-Venn Diagrams)

Introduction

The concept of set is fundamental in modern Mathematics. Today this concept is being used in different branches of Mathematics and widely used in the foundation of relations and functions. The theory of sets was developed by German Mathematician **Georg Cantor** (1845-1918).

Definition of Set

A set is well-defined collection of distinct objects. Sets are usually denoted by capital letters
$A, B, C, X, Y, Z, \ldots$.

Examples of sets

(i) The set of all complex numbers.

(ii) The set of vowels in the alphabets of English language.

(iii) The set of all natural numbers.

(iv) The set of all triangles in a plane.

(v) The set of all states in India.

(vi) The set of all months in year which has 30 days.

(vii) The set of all stars in space.

Elements of the Set

The elements of the set are denoted by small letters in the alphabets of English language, i.e. $a, b, c, x, y, z, \ldots$. If x is an element of a set A, we write $x \in A$ (read as 'x belongs to A').

If x is not an element of A, then we write $x \notin A$ (read as 'x does not belong to A').

For example,

If $A = \{1, 2, 3, 4, 5\}$, then $3 \in A, 6 \notin A$.

Representation of a Set

There are two methods for representing a set.

1. Tabulation or Roster or Enumeration Method

Under this method, the elements are enclosed in curly brackets or braces { } after separating them by commas.

Remark

1. The order of writing the elements of a set is immaterial, so $\{a, b, c\}, \{b, a, c\}, \{c, a, b\}$ all denote the same set.
2. An element of a set is not written more than once, i.e. the set {1, 2, 3, 4, 3, 3, 2, 1, 2, 1, 4} is identical with the set {1, 2, 3, 4}.

For example,

1. If A is the set of prime numbers less than 10, then
$$A = \{2, 3, 5, 7\}$$
2. If A is the set of all even numbers lying between 2 and 20, then
$$A = \{4, 6, 8, 10, 12, 14, 16, 18\}$$

2. Set Builder Method

Under this method, the stating properties which its elements are to satisfy, then we write
$$A = \{x\ P(x)\} \quad \text{or} \quad A = \{x : P(x)\}$$
and read as 'A is the set of elements x, such that x has the property P'.

Remark

1. " : " or " | " means 'such that'.
2. The other names of this method are property method, rule method and symbolic method.

For example,

1. If $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$, then we can write
$$A = \{x \in N : x \leq 8\}.$$
2. A is the set of all odd integers lying between 2 and 51, then
$$A = \{x : 2 < x < 51, x \text{ is odd}\}.$$

Some Standard Sets

- N denotes set of all natural numbers $= \{1, 2, 3, \ldots\}$.
- Z or I denotes set of all integers
$= \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.
- Z_0 or I_0 denotes set of all integers excluding zero
$= \{\ldots, -3, -2, -1, 1, 2, 3, \ldots\}$.
- Z^+ or I^+ denotes set of all positive integers
$= \{1, 2, 3, \ldots\} = N$.
- E denotes set of all even integers
$= \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$.
- O denotes set of all odd integers
$= \{\ldots, -5, -3, -1, 1, 3, 5, \ldots\}$.
- W denotes set of all whole numbers $= \{0, 1, 2, 3, \ldots\}$.
- Q denotes set of all rational numbers $= \{x : x = p/q$, where p and q are integers and $q \neq 0\}$.
- Q_0 denotes set of all non-zero rational numbers $\{x : x = p/q$, where p and q are integers and $p \neq 0$ and $q \neq 0\}$.
- Q^+ denotes set of all positive rational numbers $= \{x : x = p/q$, where p and q are both positive or negative integers$\}$
- R denotes set of all real numbers.
- R_0 denotes set of all non-zero real numbers.
- R^+ denotes set of all positive real numbers.
- $R - Q$ denotes set of all irrational numbers.
- C denotes set of all complex numbers
$= \{a + ib : a, b \in R$ and $i = \sqrt{-1}\}$.
- C_0 denotes set of all non-zero complex numbers
$= \{a + ib : a, b \in R_0$ and $i = \sqrt{-1}\}$.
- N_a denotes set of all natural numbers which are less than or equal to a, where a is positive integer
$= \{1, 2, 3, \ldots, a\}$.

Different Types of Sets

1. Null Set or Empty Set or Void Set

A set having no element is called a null set or empty set or void set. It is denoted by ϕ or $\{\ \}$.

Remark

1. ϕ is called the null set.
2. ϕ is unique.
3. ϕ is a subset of every set.
4. ϕ is never written within braces i.e., $\{\phi\}$ is not the null set.
5. $\{0\}$ is not an empty set as it contains the element 0 (zero).

For example,

1. $\{x : x \in N, 4 < x < 5\} = \phi$
2. $\{x : x \in R, x^2 + 1 = 0\} = \phi$
3. $\{x : x^2 = 25, x$ is even number$\} = \phi$

2. Singleton or Unit Set

A set having one and only one element is called singleton or unit set.

For example, $\{x : x - 3 = 4\}$ is a singleton set.

Since, $x - 3 = 4 \Rightarrow x = 7$

$\therefore$ $\{x : x - 3 = 4\} = \{7\}$

3. Subset

If every element of a set A is also an element of a set B, then A is called the subset of B, we write $A \subseteq B$ (read as A is subset of B or A is contained in B).

Thus, $A \subseteq B \Leftrightarrow [x \in A \Rightarrow x \in B]$

Remark

1. Every set is a subset of itself
i.e., $A \subseteq A$.
2. If $A \subseteq B$, $B \subseteq C$, then $A \subseteq C$.

For example,

1. If $A = \{2, 3, 4\}$ and $B = \{5, 4, 2, 3, 1\}$, then $A \subseteq B$.
2. The sets $\{a\}, \{b\}, \{a, b\}, \{b, c\}$ are the subsets of the set $\{a, b, c\}$.

4. Total Number of Subsets

If a set A has n elements, then the number of subsets of $A = 2^n$.

Example 1. Write the letters of the word ALLAHABAD in set form and find the number of subsets in it and write all subsets.

Sol. There are 5 different letters in the word ALLAHABAD i.e., A,L,H,B,D, then set is {A, B, D, H, L}, then number of subsets $= 2^5 = 32$ and all subsets are

ϕ, {A}, {B}, {D}, {H}, {L}, {A, B}, {A, D}, {A, H}, {A, L}, {B, D}, {B, H}, {B, L}, {D, H}, {D, L}, {H, L}, {A, B, D}, {A, B, H}, {A, B, L}, {A, D, H}, {A, D, L}, {A, H, L}, {B, D, H}, {B, D, L}, {B, H, L}, {D, H, L}, {A, B, D, H}, {A, B, D, L}, {B, D, H, L}, {A, D, H, L}, {A, B, H, L}, {A, B, D, H, L}.

5. Equal Sets

Two sets A and B are said to be equal, if every element of A is an element of B, and every element of B is an element of A. If A and B are equal, we write $A = B$.

It is clear that $A \subseteq B$ and $B \subseteq A \Leftrightarrow A = B$.

For example,

1. The sets {1, 2, 5} and {5, 2, 1} are equal.
2. $\{1, 2, 3\} = \{x : x^3 - 6x^2 + 11x - 6 = 0\}$

6. Power Set

The set of all the subsets of a given set A is said to be the power set A and is denoted by $P(A)$ or 2^A.

Symbolically, $P(A) = \{x : x \subseteq A\}$

Thus, $x \in P(A) \Leftrightarrow x \subseteq A$.

Remark

1. ϕ and A are both elements of $P(A)$.
2. If $A = \phi$, then $P(\phi) = \{\phi\}$, a singleton but ϕ is a null set.
3. If $A = \{a\}$, then $P(A) = \{\phi, \{a\}\}$

   For example, If $A = \{a, b, c\}$, then

   $P(A)$ or $2^A = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}\}$

   Also, $n(P(A))$ or $n(2^A) = 2^3 = 8$
4. Since, $P(\phi) = \{\phi\}$

   $\therefore$ $P(P(\phi)) = \{\phi, \{\phi\}\}$

   and $P(P(P(\phi))) = \{\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$
5. If A has n elements, then $P(A)$ has 2^n elements.

7. Super Set

The statement $A \subseteq B$ can be rewritten as $B \supseteq A$, then B is called the super set of A and is written as $B \supset A$.

8. Proper Subset

A set A is said to be proper subset of a set B, if every element of A is an element of B and B has atleast one element which is not an element of A and is denoted by $A \subset B$ (read as "A is a proper subset of B").

For example,

1. If $A = \{1, 2, 4\}$ and $B = \{5, 1, 2, 4, 3\}$, then $A \subset B$
   Since, $3, 5 \notin A$.
2. If $A = \{a, b, c\}$ and $B = \{c, b, a\}$, then $A \not\subset B$ (since, B does not contain any element which is not in A).
3. $N \subset I \subset Q \subset R \subset C$

9. Finite and Infinite Sets

A set in which the process of counting of elements comes to an end is called a finite set, otherwise it is called an infinite set.

For example,

1. Each one of the following sets is a finite set.
   (i) Set of universities in India.
   (ii) Set of Gold Medalist students in Civil Branch, sec A in A.M.I.E. (India).
   (iii) Set of natural numbers less than 500.
2. Each one of the following is an infinite set.
   (i) Set of all integers.
   (ii) Set of all points in a plane.
   (iii) $\{x : x \in R, 1 < x < 2\}$
   (iv) Set of all concentric circles with centre as origin.

10. Cardinal Number of a Finite Set

The number of distinct elements in a finite set A is called cardinal number and the cardinal number of a set A is denoted by $n(A)$.

For example,

If $A = \{-3, -1, 8, 9, 13, 17\}$, then $n(A) = 6$.

11. Comparability of Sets

Two sets A and B are said to be comparable, if either $A \subset B$ or $B \subset A$ or $A = B$, otherwise A and B are said to be incomparable.

For example,

1. The sets $A = \{1,2,3\}$ and $B = \{1,2,4,6\}$ are incomparable (since $A \not\subset B$ or $B \not\subset A$ or $A \neq B$)
2. The sets $A = \{1, 2, 4\}$ and $B = \{1, 4\}$ are comparable (since $B \subset A$).

12. Universal Set

All the sets under consideration are likely to be subsets of a set is called the universal set and is denoted by Ω or S or U.

For example,

1. The set of all letters in alphabet of English language $U = \{a, b, c, \ldots, x, y, z\}$ is the universal set of vowels in alphabet of English language.
   i.e., $A = \{a, e, i, o, u\}$.
2. The set of all integers $I = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$ is the universal set of all even integers
   i.e., $\{0, \pm 2, \pm 4, \pm 6, \ldots\}$

13. Union of Sets

The union of two sets A and B is the set of all those elements which are either in A or in B or in both. This set is denoted by $A \cup B$ or $A + B$ (read as 'A union B' or 'A cup B' or 'A join B').

Symbolically, $A \cup B = \{x : x \in A \text{ or } x \in B\}$

or $A \cup B = \{x : x \in A \vee x \in B\}$

Clearly, $x \in A \cup B \Leftrightarrow x \in A$ or $x \in B$

For example,

1. If $A = \{1, 2, 3, 4\}$ and $B = \{4, 5, 6\}$,
   then $A \cup B = \{1, 2, 3, 4, 4, 5, 6\} = \{1, 2, 3, 4, 5, 6\}$
2. If $A = \{1,2,3\}$, $B = \{2, 3, 4, 5\}$, $C = \{7, 8\}$,
   then $A \cup B \cup C = \{1, 2, 3, 4, 5, 7, 8\}$

Remark

The union of a finite number of sets $A_1, A_2, A_3, \ldots, A_n$ is represented by $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n$ or $\bigcup_{i=1}^{n} A_i$.

Symbolically, $\bigcup_{i=1}^{n} A_i = \{x : x \in A_i \text{ for atleast one } i\}$

14. Intersection of Sets

The intersection of two sets A and B is the set of all elements which are common in A and B. This set is denoted by $A \cap B$ or AB (read as 'A intersection B' or 'A cap B' or 'A meet B').

Symbolically, $A \cap B = \{x : x \in A \text{ and } x \in B\}$

or $A \cap B = \{x : x \in A \wedge x \in B\}$

Clearly, $x \in A \cap B \Leftrightarrow x \in A$ and $x \in B$

For example,

1. If $A = \{1, 2, 3\}$ and $B = \{3, 4, 5, 6\}$, then $A \cap B = \{3\}$.
2. If $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$ and $C = \{3, 4, 5\}$, then $A \cap B \cap C = \{3\}$

Remark

The intersection of a finite number of sets $A_1, A_2, A_3, \ldots, A_n$ represented by

$$A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n \quad \text{or} \quad \bigcap_{i=1}^{n} A_i.$$

Symbolically, $\bigcap_{i=1}^{n} A_i = \{x : x \in A_i \text{ for all } i\}$

15. Disjoint Sets

If the two sets A and B have no common element.

i.e., $A \cap B = \phi$, then the two sets A and B are called disjoint or mutually exclusive events.

For example, If $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$, then $A \cap B = \phi$

Hence, A and B are disjoint sets.

Remark

If $S = \{a_1, a_2, a_3, \ldots, a_n\}$, so

number of ordered pairs of disjoint sets of S is $\frac{3^n + 1}{2}$.

($\because$ each element in either (A) or (B) or neither

$\therefore$ Total ways $= 3^n$ i.e., $A = B$, iff $A = B = \phi$ (1 case) otherwise A and B are interchangeable.

$\therefore$ Number of ordered pairs of disjoint sets of $S = 1 + \frac{3^n - 1}{2} = \frac{3^n + 1}{2}$.

16. Difference of Sets

If A and B be two given sets, then the set of all those elements of A which do not belong to B is called difference of sets A and B. It is written as $A - B$. It is also denoted by $A \sim B$ or $A \backslash B$ or ${}_{C_A}B$ (complement of B in A).

Symbolically, $A - B = \{x : x \in A \text{ and } x \notin B\}$

Clearly, $x \in A - B \Leftrightarrow x \in A$ and $x \notin B$.

Remark

1. $A - B \neq B - A$
2. The sets $A - B$, $B - A$ and $A \cap B$ are disjoint sets.
3. $A - B \subseteq A$ and $B - A \subseteq B$
4. $A - \phi = A$ and $A - A = \phi$

For example,

If $A = \{1, 2, 3, 4\}$ and $B = \{4, 5, 6, 7\}$, then $A - B = \{1, 2, 3\}$.

17. Symmetric Difference of Two Sets

Let A and B be two sets. The symmetric difference of sets A and B is the set $(A - B) \cup (B - A)$ or $(A \cup B) - (A \cap B)$ and is denoted by $A \Delta B$ or $A \oplus B$ (A direct sum B).

i.e., $A \oplus B$ or $A \Delta B = (A - B) \cup (B - A)$

and $A \oplus B$ or $A \Delta B = (A \cup B) - (A \cap B)$

Remark

1. $A \Delta B = \{x : x \in A \text{ and } x \notin B\}$
   or $A \Delta B = \{x : x \in B \text{ and } x \notin A\}$
2. $A \Delta B = B \Delta A$ (commutative)

For example,

Let $A = \{1,2,3,4,5\}$ and $B = \{1,3,5,7\}$,

then $A - B = \{2, 4\}$, $B - A = \{7\}$

$\therefore$ $A \Delta B = (A - B) \cup (B - A) = \{2, 4, 7\}$

18. Complement Set

Let U be the universal set and A be a set, such that $A \subset U$. Then, the complement of A with respect to U is denoted by A' or A^c or $C(A)$ or $U - A$.

Symbolically, A' or A^c or $C(A) = \{x : x \in U \text{ and } x \notin A\}$.

Clearly, $x \in A' \Leftrightarrow x \notin A$.

Remark

1. $U' = \phi$ and $\phi' = U$
2. $A \cup A' = U$ and $A \cap A' = \phi$

For example,

Let $U = \{1, 2, 3, 4, 5, 6, 7\}$ and $A = \{1, 3, 5, 7\}$.

Then, $A' = U - A = \{2, 4, 6\}$

Laws and Theorems

1. **Idempotent Laws**
For **any set A,**
(i) $A \cup A = A$ (ii) $A \cap A = A$
Proof
(i) Let $x \in A \cup A \Leftrightarrow x \in A$ or $x \in A$
$\Leftrightarrow x \in A$
Hence, $A \cup A = A$
(ii) Let $x \in A \cap A \Leftrightarrow x \in A$ and $x \in A$
$\Leftrightarrow x \in A$
Hence, $A \cap A = A$

2. **Identity Laws**
For **any set A,**
(i) $A \cup \phi = A$ (ii) $A \cap \phi = \phi$
(iii) $A \cup U = U$ (iv) $A \cap U = A$
Proof
(i) Let $x \in A \cup \phi \Leftrightarrow x \in A$ and $x \in \phi$
$\Leftrightarrow x \in A$
Hence, $A \cup \phi = A$
(ii) Let $x \in A \cap \phi \Leftrightarrow x \in A$ and $x \in \phi$
$\Leftrightarrow x \in \phi$
Hence, $A \cap \phi = \phi$
(iii) Let $x \in A \cup U \Leftrightarrow x \in A$ or $x \in U$
$\Leftrightarrow x \in U$
Hence, $A \cup U = U$
(iv) Let $x \in A \cap U \Leftrightarrow x \in A$ and $x \in U$
$\Leftrightarrow x \in A$
Hence, $A \cap U = A$

3. **Commutative Laws**
For any two sets A and B, we have
(i) $A \cup B = B \cup A$ (ii) $A \cap B = B \cap A$
Proof
(i) Let $x \in A \cup B \Leftrightarrow x \in A$ or $x \in B$
$\Leftrightarrow x \in B$ or $x \in A$
$\Leftrightarrow x \in B \cup A$
$\therefore \quad x \in A \cup B \Leftrightarrow x \in B \cup A$
Hence, $A \cup B = B \cup A$
(ii) Let $x \in A \cap B \Leftrightarrow x \in A$ and $x \in B$
$\Leftrightarrow x \in B$ and $x \in A$
$\Leftrightarrow x \in B \cap A$
$\therefore \quad x \in A \cap B \Leftrightarrow x \in B \cap A$
Hence, $A \cap B = B \cap A$.

4. **Associative Laws**
For any three sets A, B and C, we have
(i) $A \cup (B \cup C) = (A \cup B) \cup C$
(ii) $A \cap (B \cap C) = (A \cap B) \cap C$
Proof
(i) Let $x \in A \cup (B \cup C) \Leftrightarrow x \in A$ or $x \in B \cup C$
$\Leftrightarrow x \in A$ or $(x \in B$ or $x \in C)$
$\Leftrightarrow (x \in A$ or $x \in B)$ or $x \in C$
$\Leftrightarrow x \in A \cup B$ or $x \in C$
$\Leftrightarrow x \in (A \cup B) \cup C$
$\therefore \quad x \in A \cup (B \cup C) \Leftrightarrow x \in (A \cup B) \cup C$
Hence, $A \cup (B \cup C) = (A \cup B) \cup C$.
(ii) Let $x \in A \cap (B \cap C) \Leftrightarrow x \in A$ and $x \in B \cap C$
$\Leftrightarrow x \in A$ and $(x \in B$ and $x \in C)$
$\Leftrightarrow (x \in A$ and $x \in B)$ and $x \in C$
$\Leftrightarrow x \in A \cap B$ and $x \in C$
$\Leftrightarrow x \in (A \cap B) \cap C$
Hence, $A \cap (B \cap C) = (A \cap B) \cap C$.

5. **Distributive Laws**
For any three sets A, B and C, we have
(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Proof
(i) Let $x \in A \cup (B \cap C) \Leftrightarrow x \in A$ or $x \in B \cap C$
$\Leftrightarrow x \in A$ or $(x \in B$ and $x \in C)$
$\Leftrightarrow (x \in A$ or $x \in B)$ and $(x \in A$ or $x \in C)$
$\Leftrightarrow x \in A \cup B$ and $x \in A \cup C$
$\Leftrightarrow x \in [(A \cup B) \cap (A \cup C)]$
$\therefore \quad x \in A \cup (B \cap C) \Leftrightarrow x \in (A \cup B) \cap (A \cup C)$
Hence, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
(ii) Let $x \in A \cap (B \cup C) \Leftrightarrow x \in A$ and $x \in B \cup C$
$\Leftrightarrow x \in A$ and $(x \in B$ or $x \in C)$
$\Leftrightarrow (x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$
$\Leftrightarrow x \in A \cap B$ or $x \in A \cap C$
$\Leftrightarrow x \in (A \cap B) \cup (A \cap C)$
$\therefore \quad x \in A \cap (B \cup C) \Leftrightarrow x \in (A \cap B) \cup (A \cap C)$
Hence, $A \cap (B \cup C) = (A \cup B) \cap (A \cup C)$.

6. **For any two sets A and B, we have**
(i) $P(A) \cap P(B) = P(A \cap B)$
(ii) $P(A) \cup P(B) \subseteq P(A \cup B)$
where, $P(A)$ is the power set of A.
Proof
(i) Let $x \in P(A) \cap P(B) \Leftrightarrow x \in P(A)$ or $x \in P(B)$
$\Leftrightarrow x \subseteq A$ or $x \subseteq B$
$\Leftrightarrow x \subseteq A \cap B$
$\Leftrightarrow x \in P(A \cap B)$
Hence, $P(A) \cap P(B) = P(A \cap B)$

(ii) Let $x \in P(A) \cup P(B) \Leftrightarrow x \in P(A)$ or $x \in P(B)$
$\Leftrightarrow x \subseteq A$ or $x \subseteq B$
$\Leftrightarrow x \subseteq A \cup B$
$\Leftrightarrow x \in P(A \cup B)$
Hence, $P(A) \cup P(B) \subseteq P(A \cup B)$

7. If A is any set, then $\boldsymbol{(A')' = A}$
Proof Let $x \in (A')' \Leftrightarrow x \notin A' \Leftrightarrow x \in A$
Hence, $(A')' = A$

8. **De-Morgan's Laws**
For any three sets A, B and C, we have
(i) $\boldsymbol{(A \cup B)' = A' \cap B'}$
(ii) $\boldsymbol{(A \cap B)' = A' \cup B'}$
(iii) $\boldsymbol{A - (B \cup C) = (A - B) \cap (A - C)}$
(iv) $\boldsymbol{A - (B \cap C) = (A - B) \cup (A - C)}$

Proof

(i) Let $x \in (A \cup B)' \Leftrightarrow x \notin A \cup B$
$\Leftrightarrow x \notin A$ and $x \notin B$
$\Leftrightarrow x \in A'$ and $x \in B'$
$\Leftrightarrow x \in A' \cap B'$
$\therefore \quad x \in (A \cup B)' \Leftrightarrow x \in A' \cap B'$
Hence, $(A \cup B)' = A' \cap B'$.

(ii) Let $x \in (A \cap B)' \Leftrightarrow x \notin A \cap B$
$\Leftrightarrow x \notin A$ or $x \notin B$
$\Leftrightarrow x \in A'$ or $x \in B'$
$\Leftrightarrow x \in A' \cup B'$
$\therefore \quad x \in (A \cap B)' \Leftrightarrow x \in A' \cup B'$
Hence, $(A \cap B)' = A' \cup B'$.

(iii) Let $x \in A - (B \cup C) \Leftrightarrow x \in A$ and $x \notin B \cup C$
$\Leftrightarrow x \in A$ and $(x \notin B$ and $x \notin C)$
$\Leftrightarrow (x \in A$ and $x \notin B)$ and $(x \in A$ and $x \notin C)$
$\Leftrightarrow x \in (A - B)$ and $x \in (A - C)$
$\Leftrightarrow x \in (A - B) \cap (A - C)$
Hence, $A - (B \cup C) = (A - B) \cap (A - C)$.

(iv) Let $x \in A - (B \cap C) \Leftrightarrow x \in A$ and $x \notin (B \cap C)$
$\Leftrightarrow x \in A$ and $(x \notin B$ or $x \notin C)$
$\Leftrightarrow (x \in A$ and $x \notin B)$ or $(x \in A$ and $x \notin C)$
$\Leftrightarrow x \in (A - B)$ or $x \in (A - C)$
$\Leftrightarrow x \in (A - B) \cup (A - C)$
Hence, $A - (B \cap C) = (A - B) \cup (A - C)$.

Aliter

$A - (B \cap C) = A \cap (B \cap C)' \quad [\because A - B = A \cap B']$
$= A \cap (B' \cap C)'$
$= (A \cap B') \cup (A \cap C')$
$= (A - B) \cup (A - C)$

More Results on Operations on Sets

For any two sets A and B, we have

1. $\boldsymbol{A \subseteq A \cup B, B \subseteq A \cup B, A \cap B \subseteq A, A \cap B \subseteq B}$

2. $\boldsymbol{A - B = A \cap B'}$
Proof
Let $x \in A - B \Leftrightarrow x \in A$ and $x \notin B$
$\Leftrightarrow x \in A$ and $x \in B'$
$\Leftrightarrow x \in A \cap B'$
Hence, $A - B = A \cap B'$

3. $\boldsymbol{(A - B) \cup B = A \cup B}$
Proof $(A - B) \cup B = (A \cap B') \cup B$
$= (A \cup B) \cap (B' \cup B)$ [from distributive law]
$= (A \cup B) \cap U$
$= A \cup B$
Hence, $(A - B) \cup B = A \cup B$

4. $\boldsymbol{(A - B) \cap B = \phi}$
Proof $(A - B) \cap B = (A \cap B') \cap B$
$= A \cap (B' \cap B)$ [from associative law]
$= A \cap \phi = \phi$
Hence, $(A - B) \cap B = \phi$

5. $\boldsymbol{A \subseteq B \Leftrightarrow B' \subseteq A'}$
Proof Only if part Let $A \subseteq B$ ...(i)
To prove $B' \subseteq A'$
Let $x \in B' \Rightarrow x \notin B$
$\Rightarrow x \notin A$ $[\because A \subseteq B]$
$\Rightarrow x \in A'$
Thus, $x \in B' \Rightarrow x \in A'$ $[\because B \subseteq A]$
Hence, $B' \subseteq A'$ ...(ii)
If part Let $B' \subseteq A'$ ...(iii)
To prove $A \subseteq B$
Let $x \in A \Rightarrow x \notin A'$
$\Rightarrow x \notin B'$ [from Eq.(iii)]
$\Rightarrow x \in B$
Hence, $A \subseteq B$ ...(iv)
From Eqs. (ii) and (iv), we get $A \subseteq B \Leftrightarrow B' \subseteq A'$

6. $\boldsymbol{A - B = B' - A'}$
Proof $A - B = (A \cap B')$
$= B' \cap A = B' \cap (A')' = B' - A'$
Hence, $A - B = B' - A'$

7. $\boldsymbol{(A \cup B) \cap (A \cup B') = A}$
Proof $(A \cup B) \cap (A \cup B') = A \cup (B \cap B')$
[by distributive law]
$= A \cup \phi = A$
Hence, $(A \cup B) \cap (A \cup B') = A$

8. $\boldsymbol{A \cup B = (A - B) \cup (B - A) \cup (A \cap B)}$
Proof $(A - B) \cup (B - A) \cup (A \cap B)$
$= [(A \cup B) - (A \cap B)] \cup (A \cap B)$

$=[(A \cup B) \cap (A \cap B)'] \cup (A \cap B)$
$=[(A \cup B) \cup (A \cap B)] \cap [(A \cap B)' \cup (A \cap B)]$
$=(A \cup B) \cap U = A \cup B$

Hence, $A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$

9. $\boldsymbol{A - (A - B) = A \cap B}$

Proof $A - (A - B) = A - (A \cap B')$
$= A \cap (A \cap B')'$
$= A \cap (A' \cup B)$
$= (A \cap A') \cup (A \cap B)$
$= \phi \cup (A \cap B) = A \cap B$

Hence, $A - (A - B) = A \cap B$

10. $\boldsymbol{A - B = B - A \Leftrightarrow A = B}$

Proof **Only if part** Let $A - B = B - A$ ...(i)

To prove $A = B$

Let $x \in A \Leftrightarrow (x \in A$ and $x \notin B)$ or $(x \in A$ and $x \in B)$
$\Leftrightarrow x \in (A - B)$ or $x \in (A \cap B)$
$\Leftrightarrow x \in (B - A)$

or $x \in A \cap B$ [from Eq. (i)]
$\Leftrightarrow (x \in B$ and $x \notin A)$ or $(x \in B$ and $x \in A)$
$\Leftrightarrow x \in B$

Hence, $A = B$

If part Let $A = B$

To prove $A - B = B - A$

Now, $A - B = A - A = \phi$ $[\because B = A]$

and $B - A = A - A = \phi$ $[\because B = A]$

$\therefore$ $A - B = B - A$

Hence, $A = B \Rightarrow A - B = B - A$

11. $\boldsymbol{A \cup B = A \cap B \Leftrightarrow A = B}$

Proof **Only if part** Let $A \cup B = A \cap B$

Now, $x \in A \Rightarrow x \in A \cup B$

$\Rightarrow$ $x \in A \cap B$ $[\because A \cup B = A \cap B]$

$\Rightarrow$ $x \in B$

Thus, $A \subseteq B$ ...(i)

Again, $y \in B \Rightarrow y \in A \cup B$

$\Rightarrow$ $y \in A \cap B$ $[\because A \cup B = A \cap B]$

$\Rightarrow$ $y \in A$

Thus, $B \subseteq A$ ...(ii)

From Eqs. (i) and (ii), we have $A = B$

Thus, $A \cup B = A \cap B \Rightarrow A = B$.

If part Let $A = B$ ...(iii)

To prove $A \cup B = A \cap B$

Now, $A \cup B = A \cup A = A$ $[\because B = A]$...(iv)

and $A \cap B = A \cap A = A$ $[\because B = A]$...(v)

From Eqs. (iv) and (v), we have $A \cup B = A \cap B$

Hence, $A \cup B = A \cap B \Leftrightarrow A = B$

Example 2. ***Let*** A, B ***and*** C ***be three sets such that*** $A \cup B = A \cup C$ ***and*** $A \cap B = A \cap C$***. Show that*** $B = C$***.***

Sol. Given, $A \cup B = A \cup C$ ...(i)

and $A \cap B = A \cap C$ ...(ii)

To prove $B = C$.

From Eq. (i), $(A \cup B) \cap C = (A \cup C) \cap C$

$\Rightarrow$ $(A \cap C) \cup (B \cap C) = (A \cap C) \cup (C \cup C)$

$\Rightarrow$ $(A \cap B) \cup (B \cap C) = (A \cap C) \cup C$
$[\because A \cap C = A \cap B]$

$\Rightarrow$ $(A \cap B) \cup (B \cap C) = C$ $[\because A \cap C \subseteq C]$

Thus, $C = (A \cap B) \cup (B \cap C)$ ...(iii)

Again, from Eq. (i), $(A \cup B) \cap B = (A \cup C) \cap B$

$\Rightarrow$ $(A \cap B) \cup (B \cap B) = (A \cap B) \cup (C \cap B)$

$\Rightarrow$ $(A \cap B) \cup B = (A \cap B) \cup (B \cap C)$

$\Rightarrow$ $B = (A \cap B) \cup (B \cap C)$
$[\because A \cap B \subseteq B]$

Thus, $B = (A \cap B) \cup (B \cap C)$ ...(iv)

From Eqs. (iii) and (iv), we have $B = C$.

Example 3. Let A and B be any two sets. If for some set X, $A \cap X = B \cap X = \phi$ and $A \cup X = B \cup X$, prove that $A = B$.

Sol. Given, $A \cap X = B \cap X = \phi$ ...(i)

and $A \cup X = B \cup X$ ...(ii)

From Eq. (ii), $A \cap (A \cup X) = A \cap (B \cup X)$

$\Rightarrow$ $A = (A \cap B) \cup (A \cap X)$
$[\because A \subseteq A \cup X \ \therefore A \cap (A \cup X) = A]$

$\Rightarrow$ $A = (A \cap B) \cup \phi$ $[\because A \cap X = \phi]$

$\Rightarrow$ $A = (A \cap B)$

$\Rightarrow$ $A \subseteq B$ ...(iii)

Again, $A \cup X = B \cup X$

$\Rightarrow$ $B \cap (A \cup X) = B \cap (B \cup X)$

$\Rightarrow$ $(B \cap A) \cup (B \cap X) = B$
$[\because B \subseteq B \cup X \ \therefore B \cap (B \cup X) = B]$

$\Rightarrow$ $(B \cap A) \cup \phi = B$ $[\because B \cap X = \phi]$

$\Rightarrow$ $B \cap A = B$

$\Rightarrow$ $B \subseteq A$ ...(iv)

From Eqs. (iii) and (iv), we have $A = B$.

Example 4. If A and B are any two sets, prove that $P(A) = P(B) \Rightarrow A = B$.

Sol. Given, $P(A) = P(B)$ ...(i)

To prove $A = B$

Let $x \in A \Rightarrow$ there exists a subset X of A such that $x \in X$.

Now, $X \subseteq A \Rightarrow X \in P(A)$

$\Rightarrow$ $X \in P(B)$ [from Eq. (i)]

$\Rightarrow \quad X \subseteq B$

$\Rightarrow \quad x \in B \qquad [\because x \in X]$

Thus, $x \in A \Rightarrow x \in B$

$\therefore \quad A \subseteq B \qquad \ldots(ii)$

Let $y \in B \Rightarrow$ there exists a subset Y of B such that $y \in Y$.

Now, $Y \subseteq B \Rightarrow Y \in P(B)$

$\Rightarrow \quad Y \in P(A) \qquad [\because P(B) = P(A)]$

$\Rightarrow \quad Y \subseteq A$

$\Rightarrow \quad y \in A \qquad [\because y \in Y]$

Thus, $y \in B \Rightarrow y \in A$

$\therefore \quad B \subseteq A \qquad \ldots(iii)$

From Eqs. (ii) and (iii), we have $A = B$

Use of Sets in Logical Problems

M = Set of students which have Mathematics.

P = Set of students which have Physics.

C = Set of students which have Chemistry.

Applying the different operations on the above sets, then we get following important results.

M' = Set of students which have no Mathematics.

P' = Set of students which have no Physics.

C' = Set of students which have no Chemistry.

$M \cup P$ = Set of students which have atleast one subject Mathematics or Physics.

$P \cup C$ = Set of students which have atleast one subject Physics or Chemistry.

$C \cup M$ = Set of students which have atleast one subject Chemistry or Mathematics.

$M \cap P$ = Set of students which have both subjects Mathematics and Physics.

$P \cap C$ = Set of students which have both subjects Physics and Chemistry.

$C \cap M$ = Set of students which have both subjects Chemistry and Mathematics.

$M \cap P'$ = Set of students which have Mathematics but not Physics.

$P \cap C'$ = Set of students which have Physics but not Chemistry.

$C \cap M'$ = Set of students which have Chemistry but not Mathematics.

$(M \cup P)'$ = Set of students which have not both subjects Mathematics and Physics.

$(P \cup C)'$ = Set of students which have not both subjects Physics and Chemistry.

$(C \cup M)'$ = Set of students which have not both subjects Chemistry and Mathematics.

$(M \cap P \cap C)$ = Set of students which have all three subjects Mathematics, Physics and Chemistry.

$(M \cup P \cup C)$ = Set of all students which have three subjects.

Cardinal Number of Some Sets

If A, B and C are finite sets and U be the finite universal set, then

(i) $n(A') = n(U) - n(A)$

(ii) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(iii) $n(A \cup B) = n(A) + n(B)$, if A and B are disjoint non-void sets.

(iv) $n(A \cap B') = n(A) - n(A \cap B)$

(v) $n(A' \cap B') = n(A \cup B)' = n(U) - n(A \cup B)$

(vi) $n(A' \cup B') = n(A \cap B)' = n(U) - n(A \cap B)$

(vii) $n(A - B) = n(A) - n(A \cap B)$

(viii) $n(A \cap B) = n(A \cup B) - n(A \cap B') - n(A' \cap B)$

(ix) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$

(x) If $A_1, A_2, A_3, \ldots, A_n$ are disjoint sets, then
$n(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n) = n(A_1) + n(A_2) + n(A_3) + \ldots + n(A_n)$

Example 5. If A and B be two sets containing 6 and 3 elements respectively, what can be the minimum number of elements in $A \cup B$? Also, find the maximum number of elements in $A \cup B$.

Sol. We have, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$,

$n(A \cup B)$ is minimum or maximum according as $n(A \cap B)$ is maximum or minimum, respectively.

***Case* I** If $n(A \cap B)$ is minimum i.e., $n(A \cap B) = 0$ such that

$A = \{a, b, c, d, e, f\}$ and $B = \{g, h, i\}$

$\therefore \quad n(A \cup B) = n(A) + n(B) = 6 + 3 = 9$

***Case* II** If $n(A \cap B)$ is maximum i.e., $n(A \cap B) = 3$, such that

$A = \{a, b, c, d, e, f\}$ and $B = \{d, a, c\}$

$\therefore\ n(A \cup B) = n(A) + n(B) - n(A \cap B) = 6 + 3 - 3 = 6$

Example 6. Suppose $A_1, A_2, \ldots, A_{30}$ are thirty sets each with five elements and $B_1, B_2, \ldots, B_n$ are n sets each with three elements.

Let $\bigcup_{i=1}^{30} A_i = \bigcup_{j=1}^{n} B_j = S$

Assume that each element of S belongs to exactly ten of the A_i's and exactly to nine of the B_j's. Find n.

Sol. Given, A's are thirty sets with five elements each, so

$$\sum_{i=1}^{30} n(A_i) = 5 \times 30 = 150 \quad \ldots(i)$$

If the m distinct elements in S and each element of S belongs to exactly 10 of the A_i's, so we have

$$\sum_{i=1}^{30} n(A_i) = 10m \quad \ldots(ii)$$

$\therefore$ From Eqs. (i) and (ii), we get $10m = 150$

$$\therefore \quad m = 15 \quad \ldots(iii)$$

Similarly, $\sum_{j=1}^{n} n(B_j) = 3n$ and $\sum_{j=1}^{n} n(B_j) = 9m$

$$\therefore \quad 3n = 9m \implies n = \frac{9m}{3} = 3m$$

$$= 3 \times 15 = 45 \quad \text{[from Eq. (iii)]}$$

Hence, $n = 45$

Example 7. In a group of 1000 people, there are 750 who can speak Hindi and 400 who can speak Bengali. How many can speak Hindi only? How many can speak Bengali only? How many can speak both Hindi and Bengali?

Sol. Let H and B be the set of those people who can speak Hindi and Bengali respectively, then according to the problem, we have

$$n(H \cup B) = 1000,$$
$$n(H) = 750, n(B) = 400$$

We know that,

$$n(H \cup B) = n(H) + n(B) - n(H \cap B)$$
$$1000 = 750 + 400 - n(H \cap B)$$
$$\therefore \quad n(H \cap B) = 150$$

$\therefore$ Number of people speaking Hindi and Bengali both is 150.

Also, $n(H \cap B') = n(H) - n(H \cap B)$

$$= 750 - 150$$
$$= 600$$

Thus, number of people speaking Hindi only is 600.

Again, $n(B \cap H') = n(B) - n(B \cap H) = 400 - 150 = 250$

Thus, number of people speaking Bengali only is 250.

Example 8. A survey of 500 television watchers produced the following information, 285 watch football, 195 watch hockey, 115 watch basketball, 45 watch football and basketball, 70 watch football and hockey, 50 watch hockey and basketball, 50 do not watch any of the three games. How many watch all the three games? How many watch exactly one of the three games?

Sol. Let F, H and B be the sets of television watchers who watch Football, Hockey and Basketball, respectively.

Then, according to the problem, we have

$$n(U) = 500, \ n(F) = 285, n(H) = 195,$$
$$n(B) = 115, \ n(F \cap B) = 45,$$
$$n(F \cap H) = 70, n(H \cap B) = 50$$

and $n(F' \cup H' \cup B') = 50,$

where U is the set of all the television watchers.

Since, $n(F' \cup H' \cup B') = n(U) - n(F \cup H \cup B)$

$$\implies 50 = 500 - n(F \cup H \cup B)$$
$$\implies n(F \cup H \cup B) = 450$$

We know that,

$$n(F \cup H \cup B) = n(F) + n(H) + n(B) - n(F \cap H) - n(H \cap B) - n(B \cap F) + n(F \cap H \cap B)$$
$$\implies 450 = 285 + 195 + 115 - 70 - 50 - 45 + n(F \cap H \cap B)$$
$$\therefore \quad n(F \cap H \cap B) = 20$$

which is the number of those who watch all the three games. Also, number of persons who watch football only

$$= n(F \cap H' \cap B')$$
$$= n(F) - n(F \cap H) - n(F \cap B) + n(F \cap H \cap B)$$
$$= 285 - 70 - 45 + 20 = 190$$

The number of persons who watch hockey only

$$= n(H \cap F' \cap B')$$
$$= n(H) - n(H \cap F) - n(H \cap B) + n(H \cap F \cap B)$$
$$= 195 - 70 - 50 + 20 = 95$$

and the number of persons who watch basketball only

$$= n(B \cap H' \cap F')$$
$$= n(B) - n(B \cap H) - n(B \cap F) + n(H \cap F \cap B)$$
$$= 115 - 50 - 45 + 20 = 40$$

Hence, required number of those who watch exactly one of the three games

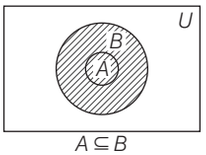
$$= 190 + 95 + 40 = 325$$

Venn Diagrams
(Euler-Venn Diagrams)

The diagram drawn to represent sets are called Venn diagrams or Euler Venn diagrams. Here, we represent the universal set U by points within rectangle and the subset A of the set U is represented by the interior of a circle. If a set A is a subset of a set B, then the circle representing A is drawn inside the circle representing B. If A and B are not equal but they have some common elements, then to represent A and B by two intersecting circles.

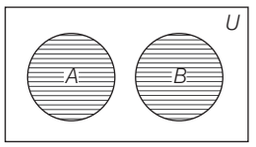
Venn Diagrams in Different Situations

1. Subset

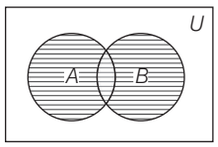


$A \subseteq B$

2. Union of sets

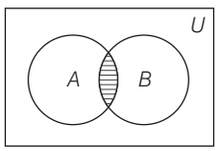


$A \cup B$, when $A \cap B = \phi$

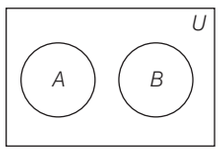


$A \cup B$, when neither $A \subseteq B$ nor $B \subseteq A$

3. Intersection of sets

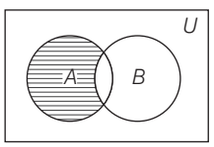


$A \cap B$, when neither $A \subseteq B$ nor $B \subseteq A$

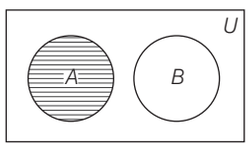


$A \cap B$, when $A \cap B = \phi$ (no shaded one)

4. Difference of sets

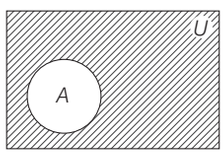


$A - B$, when neither $A \subseteq B$ nor $B \subseteq A$



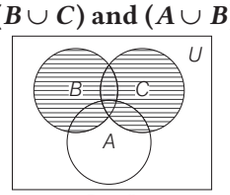
$A - B$, when $A - B = \phi$

5. Complement set

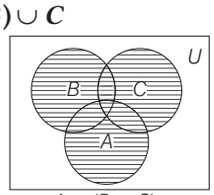


A' = shaded one

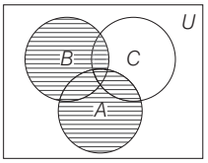
6. $A \cup (B \cup C)$ and $(A \cup B) \cup C$



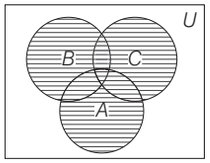
$B \cup C$



$A \cup (B \cup C)$



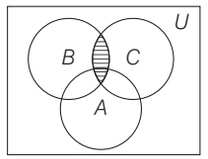
$A \cup B$



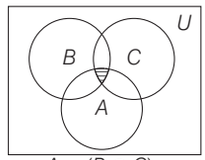
$(A \cup B) \cup C$

Hence, $A \cup (B \cup C) = (A \cup B) \cup C$ which is associative law for union.

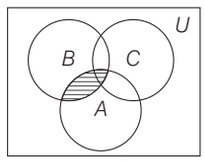
7. $A \cap (B \cap C)$ and $(A \cap B) \cap C$



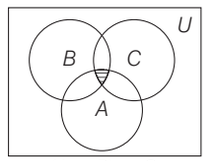
$B \cap C$



$A \cap (B \cap C)$



$A \cap B$



$(A \cap B) \cap C$

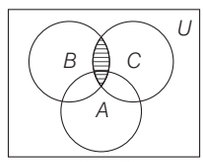
Hence, $A \cap (B \cap C) = (A \cap B) \cap C$ which is associative law for intersection.

8. Distributive law

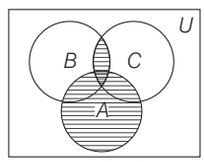
(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

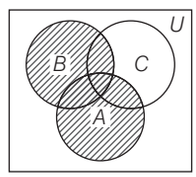
(i)



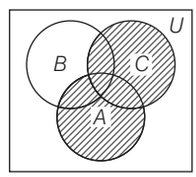
$(B \cap C)$



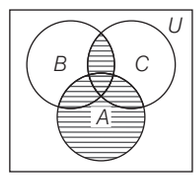
$A \cup (B \cap C)$



$(A \cup B)$



$(A \cup C)$

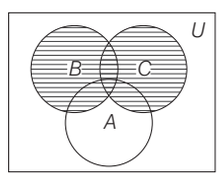


$(A \cup B) \cap (A \cup C)$

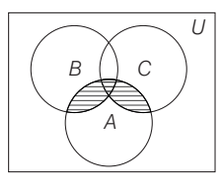
It is clear from diagrams that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

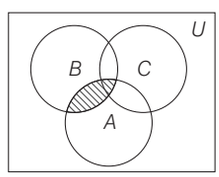
(ii)



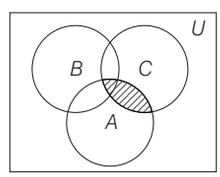
$(B \cup C)$



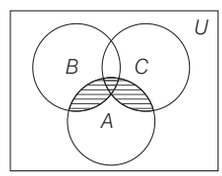
$A \cap (B \cup C)$



$(A \cap B)$



$(A \cap C)$

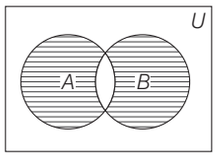


$(A \cap B) \cup (A \cap C)$

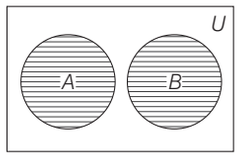
It is clear from diagrams that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

9. Symmetric difference



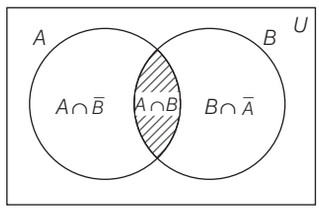
$A \Delta B$, when neither $A \subseteq B$ nor $B \subseteq A$ and $A \cap B \neq \phi$

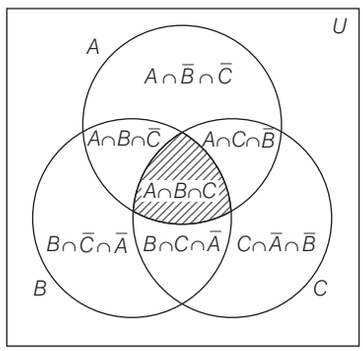


$A \Delta B$, when $A \cap B = \phi$

Remark

Remember with the help of figures.





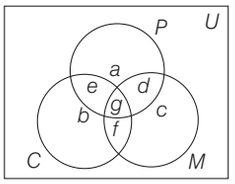
Example 9. A class has 175 students. The following table shows the number of students studying one or more of the following subjects in this case.

Subjects	Number of students
Mathematics	100
Physics	70
Chemistry	46
Mathematics and Physics	30
Mathematics and Chemistry	28
Physics and Chemistry	23
Mathematics, Physics and Chemistry	18

How many students are enrolled in Mathematics alone, Physics alone and Chemistry alone? Are there students who have not offered any one of these subjects?

Sol. Let P, C and M denotes the sets of students studying Physics, Chemistry and Mathematics, respectively.

Let a, b, c, d, e, f, g denote the elements (students) contained in the bounded region as shown in the diagram.



Then, $a + d + e + g = 170$

$c + d + f + g = 100$

$b + e + f + g = 46$

$d + g = 30$

$e + g = 23$

$f + g = 28$

$g = 18$

After solving, we get $g = 18$, $f = 10$, $e = 5$, $d = 12$, $a = 35$, $b = 13$ and $c = 60$

$\therefore \quad a + b + c + d + e + f + g = 153$

So, the number of students who have not offered any of these three subjects $= 175 - 153 = 22$

Number of students studying Mathematics only, $c = 60$

Number of students studying Physics only, $a = 35$

Number of students studying Chemistry only, $b = 13$

Aliter

Let P, C and M be the sets of students studying Physics, Chemistry and Mathematics, respectively. Then, we are given that

$n(P) = 70, n(C) = 46, n(M) = 100$

$n(M \cap P) = 30, n(M \cap C) = 28$

$n(P \cap C) = 23$

and $n(M \cap P \cap C) = 18$

$\therefore$ The number of students enrolled in Mathematics only

$= n(M \cap P' \cap C') = n(M \cap (P \cup C)')$

[by De-Morgan's law]

$= n(M) - n(M \cap (P \cup C))$

$= n(M) - \{n[(M \cap P) \cup (M \cap C)]\}$

[by distributive law]

$= n(M) - n(M \cap P) - (M \cap C) + n(M \cap P \cap C)$

$= 100 - 30 - 28 + 18 = 60$

Similarly, the number of students enrolled in Physics only,

$n(P \cap M' \cap C')$

$= n(P) - n(P \cap M) - n(P \cap C) + n(P \cap M \cap C)$

$= 70 - 30 - 23 + 18 = 35$

and the number of students enrolled in Chemistry only,

$n(C \cap M' \cap P') = n(C) - n(C \cap M) - n(C \cap P) + n$

$(C \cap M \cap P)$

$= 46 - 28 - 23 + 18 = 13$

and the number of students who have not offered any of the three subjects,

$n(M' \cap P' \cap C') = n(M \cap P \cap C)'$ [by De-Morgan's law]

$= n(U) - n(M \cup P \cup C)$

$= n(U) - \{n(M) + n(P) + n(C) - n(M \cap P)$

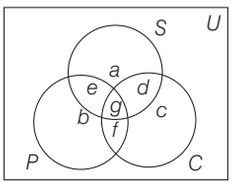
$- n(M \cap C) - n(P \cap C) + n(P \cap C \cap M)\}$

$= 175 - \{100 + 70 + 46 - 30 - 28 - 23 + 8\}$

$= 175 - 153 = 22$

Example 10. In a pollution study of 1500 Indian rivers the following data were reported. 520 were polluted by sulphur compounds, 335 were polluted by phosphates, 425 were polluted by crude oil, 100 were polluted by both crude oil and sulphur compounds, 180 were polluted by both sulphur compounds and phosphates, 150 were polluted by both phosphates and crude oil and 28 were polluted by sulphur compounds, phosphates and crude oil. How many of the rivers were polluted by atleast one of the three impurities?
How many of the rivers were polluted by exactly one of the three impurities?

Sol. Let S, P and C denote the sets of rivers polluted by sulphur compounds, by phosphates and by crude oil respectively, and let a, b, c, d, e, f, g denote the elements (impurities) contained in the bounded region as shown in the diagram.



Then, $a + d + e + g = 520$

$c + d + f + g = 425$

$b + e + f + g = 335 \implies d + g = 100$

$e + g = 180 \implies f + g = 150$

$g = 28$

After solving, we get

$g = 28, f = 122, e = 152, b = 33, d = 72, c = 203$ and $a = 268$

The number of rivers were polluted by atleast one of the three impurities

$= (a + b + c + d + e + f + g) = 878$

and the number of rivers were polluted by exactly one of the three impurities,

$a + b + c = 268 + 33 + 203 = 504$

Aliter

Let S, P and C denote the sets of rivers polluted by sulphur compounds, by phosphates and by crude oil, respectively.

Then, we are given that

$n(S) = 520, n(P) = 335, n(C) = 425, n(C \cap S) = 100,$

$n(S \cap P) = 180, n(P \cap C) = 150$ and $n(S \cap P \cap C) = 28.$

The number of rivers polluted by atleast one of the three impurities,

$n(S \cup P \cup C)$

$= n(S) + n(P) + n(C) - n(S \cap P)$

$- n(P \cap C) - n(C \cap S) + n(S \cap P \cap C)$

$= 520 + 335 + 425 - 180 - 150 - 100 + 28 = 878$

and the number of rivers polluted by exactly one of the three impurities,

$n\{(S \cap P' \cap C') \cup (P \cap C' \cap S') \cup (C \cap P' \cap S')$

$= n\{(S \cap (P \cup C)')\} \cup \{P \cap (C \cup S)')\} \cup \{C \cap P \cup S)'\}$

$= n(S \cap (P \cup C)') + n(P \cap (C \cup S)') + n(C \cap (P \cup S)')$

$= n(S) - n(S \cap P) - n(S \cap C)$

$+ n(S \cap P \cap C) + n(P) - n(P \cap C) - n(P \cap S)$

$+ n(S \cap P \cap C)$

$+ n(C) - n(C \cap P) - n(C \cap S) + n(S \cap P \cap C)$

$= n(S) + n(P) + n(C) - 2n(S \cap P) - 2n(S \cap C)$

$- 2n(P \cap C) + 3n(S \cap P \cap C)$

$= 520 + 335 + 425 - 360 - 200 - 300 + 84 = 504$

Exercise for Session 1

1. If $X = \{4^n - 3n - 1 : n \in N\}$ and $y = \{9(n-1) : n \in N\}$, then $X \cup Y$ equals

(a) X (b) Y (c) N (d) None of these

2. If $N_a = \{an : n \in N\}$, then $N_5 \cap N_7$ equals

(a) N (b) N_5 (c) N_7 (d) N_{35}

3. If A and B are two sets, then $A \cap (A \cup B)'$ equals

(a) A (b) B (c) ϕ (d) None of these

4. If U be the universal set and $A \cup B \cup C = U$, then $[(A - B) \cup (B - C) \cup (C - A)']$ equals

(a) $A \cup B \cup C$ (b) $A \cap B \cap C$ (d) $A \cup (B \cap C)$ (d) $A \cap (B \cup C)$

5. If A and B are two sets, then $(A - B) \cup (B - A) \cup (A \cap B)$ equals

(a) $A \cup B$ (b) $A \cap B$ (c) A (d) B'

6. If $A = \{x : x$ is a multiple of 4$\}$ and $B = \{x : x$ is a multiple of 6$\}$, then $A \subset B$ consists of all multiple of

(a) 4 (b) 8 (c) 12 (d) 16

7. A set contains $2n + 1$ elements. The number of subsets of this set containing more than n elements equals

(a) 2^{n-1} (b) 2^n (c) 2^{n+1} (d) 2^{2n}

8. If $A = \{\phi, \{\phi\}\}$, then the power set of A is

(a) A (b) $\{\phi, \{\phi\}, A\}$

(c) $\{\phi, \{\phi\}, \{\{\phi\}\}, A\}$ (d) None of these

9. Given $n(U) = 20, n(A) = 12, n(B) = 9, n(A \cap B) = 4$, where U is the universal set, A and B are subsets of U, then $n((A \cup B)')$ equals

(a) 3 (b) 9 (c) 11 (d) 17

10. A survey shows that 63% of the Indians like cheese, whereas 76% like apples. If x % of the Indians like both cheese and apples, then x can be

(a) 40 (b) 65 (c) 39 (d) None of these

11. In a class of 55 students, the number of students studying different subjects are 23 in Mathematics, 24 in Physics, 19 in Chemistry, 12 in Mathematics and Physics, 9 in Mathematics and Chemistry, 7 in Physics and Chemistry and 4 in all the three subjects. The number of students who have taken exactly one subject is

(a) 6 (b) 7 (c) 9 (d) 22

Session 2

Ordered Pair, Definition of Relation, Ordered Relation, Composition of Two Relations

Ordered Pair

If A be a set and $a, b \in A$, then the ordered pair of elements a and b in A denoted by (a, b), where a is called the first coordinate and b is called the second coordinate.

Remark

1. Ordered pairs (a, b) and (b, a) are different.
2. Ordered pairs (a, b) and (c, d) are equal iff $a = c$ and $b = d$.

Cartesian Product of Two Sets

The cartesian product to two sets A and B is the set of all those ordered pairs whose first coordinate belongs to A and second coordinate belongs to B. This set is denoted by $A \times B$ (read as 'A cross B' or 'product set of A and B').

Symbolically, $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$

or $A \times B = \{(a, b) : a \in A \wedge b \in B\}$

Thus, $(a, b) \in A \times B \Leftrightarrow a \in A \wedge b \in B$

Similarly, $B \times A = \{(b, a) : b \in B \wedge a \in A\}$

Remark

1. $A \times B \neq B \times A$
2. If A has p elements and B has q elements, then $A \times B$ has pq elements.
3. If $A = \phi$ and $B = \phi$, then $A \times B = \phi$.
4. Cartesian product of n sets $A_1, A_2, A_3, \ldots, A_n$ is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ $a_i \in A_i$, $i = 1, 2, 3, \ldots, n$ and is denoted by $A_1 \times A_2 \times \ldots \times A_n$ or $\prod_{i=1}^{n} A_i$.

Example 11. ***If*** $A = \{1, 2, 3\}$ ***and*** $B = \{4, 5\}$***, find*** $A \times B$***,*** $B \times A$ ***and show that*** $A \times B \neq B \times A$***.***

Sol. $A \times B = \{1, 2, 3\} \times \{4, 5\} = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}$

and $B \times A = \{4, 5\} \times \{1, 2, 3\} = \{(4, 1), (4, 2), (4, 3), (5, 1), (5, 2), (5, 3)\}$

It is clear that $A \times B \neq B \times A$.

Example 12. If A and B be two sets and $A \times B = \{(3, 3), (3, 4), (5, 2), (5, 4)\}$, find A and B.

Sol. A = First coordinates of all ordered pairs = $\{3, 5\}$

and B = Second coordinates of all ordered pairs = $\{2, 3, 4\}$

Hence, $A = \{3, 5\}$ and $B = \{2, 3, 4\}$

Important Theorems on Cartesian Product

If A, B and C are three sets, then

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(iii) $A \times (B - C) = (A \times B) - (A \times C)$

(iv) $(A \times B) \cap (S \times T) = (A \cap S) \times (B \cap T)$, where S and T are two sets.

(v) If $A \subseteq B$, then $(A \times C) \subseteq (B \times C)$

(vi) If $A \subseteq B$, then $(A \times B) \cap (B \times A) = A^2$

(vii) If $A \subseteq B$ and $C \subseteq D$, then $A \times C \subseteq B \times D$

Example 13. ***If*** A ***and*** B ***are two sets given in such a way that*** $A \times B$ ***consists of 6 elements and if three elements of*** $A \times B$ ***are (1, 5), (2, 3) and (3, 5), what are the remaining elements?***

Sol. Since, $(1, 5), (2, 3), (3, 5) \in A \times B$, then clearly $1, 2, 3 \in A$ and $3, 5 \in B$.

$A \times B = \{1, 2, 3\} \times (3, 5)$

$= (1, 3), (1, 5), (2, 3), (2, 5), (3, 3), (3, 5)$

Hence, the remaining elements are $(1, 3), (2, 5), (3, 3)$.

Relations

Introduction of Relation

We use sentences depending upon the relationship of an object to the other object in our daily life such as

(i) 'Ram, Laxman, Bharat, Shatrughan' were the sons of Dashrath.
(ii) 'Sita' was the wife of Ram.
(iii) 'Laxman' was the brother of Ram.
(iv) 'Dashrath' was the father of Ram.
(v) 'Kaushaliya' was the mother of Ram.

If Ram, Laxman, Bharat, Shatrughan, Sita, Kaushaliya and Dashrath are represented by a, b, c, d, e, f and y respectively and A represents the set, then

$$A = \{a, b, c, d, e, f, y\}$$

Here, we see that any two elements of set A are related many ways, i.e. a, b, c, d are sons of y. 'a' is the son of y is represented by aRy. Similarly, b is the son of y, c is the son of y and d is also son of y are represented as $b\,R\,y$, $c\,R\,y$ and $d\,R\,y$, respectively.

If we write here $y\,R\,a$ it means that y is the son of a which is impossible, since a is the son of y. Hence, y and a cannot be related like this. Its generally represented as $y \not R a$. Hence, we can say that a and y are in definite order. a comes before R and y after R. Therefore, aRy may be represented as a order pair (a, y). Similarly, bRy, cRy and dRy are represented by $(b, y), (c, y)$ and (d, y), respectively. If all pairs will represented by a set, then we see that first element of each pair is the son of second element. Hence, the set of these pairs may be represented by set R, then

$$R = \{(a, y), (b, y), (c, y), (d, y)\}$$

Symbolically, $R = \{(x, y) : x, y \in A$, where x is son of $y\}$

It is clear that R is subset of $A \times A$

i.e., $R \subseteq A \times A$

Corollary In above example, if

$A = \{a, b, c, d\}$ and $B = \{e, f, y\}$, then

$R = \{(x, z) : x \in A, z \in B$, where x is son of $z\}$

It is clear that $R \subseteq A \times B$.

Definition of Relation

A relation (or binary relation) R, from a non-empty set A to another non-empty set B, is a subset of $A \times B$.

i.e., $R \subseteq A \times B$ or $R \subseteq \{(a, b) : a \in A, b \in B\}$

Now, if (a, b) be an element of the relation R, then we write aRb (read as 'a is related to b')

i.e., $(a, b) \in R \Leftrightarrow aRb$

and if (a, b) is not an element of the relation R, then we write $a \not R b$ (read as 'a is not related to b'),

i.e. $(a, b) \notin R \Leftrightarrow a \not R b$.

Remark

1. Any subset of $A \times A$ is said to be a relation on A.
2. If A has m elements and B has n elements, then $A \times B$ has $m \times n$ elements and total number of different relations from A to B is 2^{mn}.
3. If $R = A \times B$, then Domain $R = A$ and Range $R = B$.
4. The domain as well as range of the empty set ϕ is ϕ.
5. If $A = \text{Dom}\, R$ and $B = \text{Ran}\, R$, then we write $B = R\,[A]$.

For example,

Let $A = \{1, 2, 3\}$ and $B = \{3, 5, 7\}$, then
$A \times B = \{(1, 3), (1, 5), (1, 7), (2, 3), (2, 5), (2, 7), (3, 3), (3, 5), (3, 7)\}$.
But $R \subseteq A \times B$
i.e., every subset of $A \times B$ is a relation from A to B. If we consider the relation, $R = \{(1, 5), (1, 7), (3, 5), (3, 7)\}$
Then, $1\,R\,5; 1\,R\,7; 3\,R\,5; 3\,R\,7$
Also, $1\,R\,3; 2\,R\,3; 2\,R\,5; 2\,R\,7; 3\,R\,3;$
Clearly, Domain $R = \{1, 3\}$ and Range $R = \{5, 7\}$

For example,

Let $A = \{1, 2, 3\}$ and $B = \{4, 5\}$, then number of different relations from A to B is $2^{3\times 2} = 2^6 = 64$ because A has 3 elements and B has 2 elements.

Types of Relations from One Set to Another Set

1. Empty Relation

A relation R from A to B is called an empty relation or a void relation from A to B if $R = \phi$.

For example,

Let $A = \{2, 4, 6\}$ and $B = \{7, 11\}$
Let $R = \{(a, b) : a \in A, b \in B$ and $a - b$ is even$\}$
As, none of the numbers $2 - 7, 2 - 11, 4 - 7, 4 - 11, 6 - 7, 6 - 11$ is an even number, $R = \phi$.
Hence, R is an empty relation.

2. Universal Relation

A relation R from A to B is said to be the universal relation, if $R = A \times B$.

For example, Let $A = \{1, 2\}$, $B = \{1, 3\}$ and
$R = \{(1, 1), (1, 3), (2, 1), (2, 3)\}$
Here, $R = A \times B$
Hence, R is the universal relation from A to B.

Types of Relations on a Set

1. Empty Relation

A relation R on a set A is said to be an empty relation or a void relation, if $R = \phi$.

For example,

Let $A = \{1,3\}$ and $R = \{(a, b) : a, b \in A$ and $a + b$ is odd$\}$

Hence, R contains no element, therefore R is an empty relation on A.

2. Universal Relation

A relation R on a set A is said to be universal relation on A, if $R = A \times A$.

For example,

Let $A = \{1, 2\}$

and $R = [(1, 1), (1, 2), (2, 1), (2, 2)]$

Here, $R = A \times A$

Hence, R is the universal relation on A.

3. Identity Relation

A relation R on a set A is said to be the identity relation on A, if

$$R = [(a, b) : a \in A, b \in A \text{ and } a = b]$$

Thus, identity relation, $R = [(a, a) : \forall\, a \in A]$

Identity relation on set A is also denoted by I_A.

Symbolically, $I_A = [(a, a) : a \in A]$

For example,

Let $A = \{1, 2, 3\}$

Then, $I_A = \{(1, 1), (2, 2), (3, 3)\}$

Remark

In an identity relation on A every element of A should be related to itself only.

4. Inverse Relation

If R is a relation from a set A to a set B, then inverse relation of R to be denoted by R^{-1}, is a relation from B to A.

Symbolically, $R^{-1} = \{(b, a) : (a, b) \in R\}$

Thus, $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}, \forall\, a \in A, b \in B.$

Remark

1. Dom (R^{-1}) = Range (R) and Range (R^{-1}) = Dom (R)
2. $(R^{-1})^{-1} = R$

For example,

If $R = \{(1, 2), (3, 4), (5, 6)\}$, then

$R^{-1} = \{(2, 1), (4, 3), (6, 5)\}$

$\therefore (R^{-1})^{-1} = \{(1, 2), (3, 4), (5, 6)\} = R$

Here, dom $(R) = \{1, 3, 5\}$, range $(R) = \{2, 4, 6\}$

and dom $(R^{-1}) = \{2, 4, 6\}$, range $(R^{-1}) = \{1, 3, 5\}$

Clearly, dom (R^{-1}) = range (R)

and range (R^{-1}) = dom (R).

Various Types of Relations

1. Reflexive Relation

A relation R on a set A is said to be reflexive, if $a\, R\, a, \forall\, a \in A$

i.e., if $(a, a) \in R, \forall\, a \in A$

For example,

Let $A = \{1, 2, 3\}$

$R_1 = \{(1, 1), (2, 2), (3, 3)\}$

$R_2 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3)\}$

and $R_3 = \{(1, 1), (2, 2), (2, 3), (3, 2)\}$

Here, R_1 and R_2 are reflexive relations on A, R_3 is not a reflexive relation on A as $(3, 3) \notin R_3$, i.e. $3 \not R_3\ 3$.

Remark

The identity relation is always a reflexive relation but a reflexive relation may or may not be the identity relation. It is clear in the above example given, R_1 is both reflexive and identity relation on A but R_2 is a reflexive relation on A but not an identity relation on A.

2. Symmetric Relation

A relation R on a set A is said to be symmetric relation, if

$$a\, R\, b \Rightarrow b\, R\, a, \forall\, a, b \in A$$

i.e., if $(a, b) \in R \Rightarrow (b, a) \in R, \forall\, a, b \in A$

For example,

Let $A = \{1, 2, 3\}$

$R_1 = \{(1, 2), (2, 1)\}$

$R_2 = \{(1, 2), (2, 1), (1, 3), (3, 1)\}$

and $R_3 = \{(2, 3), (1, 3), (3, 1)\}$

Here, R_1 and R_2 are symmetric relations on A but R_3 is not a symmetric relation on A because $(2, 3) \in R_3$ and $(3, 2) \notin R_3$.

3. Anti-symmetric Relation

A relation R on a set A is said to be anti-symmetric,

if $a\, R\, b, b\, R\, a \Rightarrow a = b, \forall\, a, b \in A$

i.e., $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b, \forall\, a, b \in A$

For example,

Let R be the relation in N (natural number) defined by, ''x is divisor of y'', then R is anti-symmetric because x divides y and y divides $x \Rightarrow x = y$

4. Transitive Relation

A relation R on a set A is said to be a transitive relation, if $a\,R\,b$ and $b\,R\,c \Rightarrow aRc, \forall\ a, b, c \in A$

i.e., $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R, \forall\ a, b, c \in A$

For example,

Let $A = \{1, 2, 3\}$

$R_1 = \{(1, 2), (2, 3), (1, 3), (3, 2)\}$

$R_2 = \{(2, 3), (3, 1)\}$

$R_3 = \{(1, 3), (3, 2), (1, 2)\}$

Then, R_1 is not transitive relation on A because $(2, 3) \in R_1$ and $(3, 2) \in R_1$ but $(2, 2) \notin R_1$. Again, R_2 is not transitive relation on A because $(2, 3) \in R_2$ and $(3, 1) \in R_2$ but $(2, 1) \notin R_2$. Finally R_3 is a transitive relation.

Example 14. ***Let*** $A = \{1, 2, 3\}$ ***and*** $R = \{(a, b) : a, b \in A,$ a ***divides*** b ***and*** b ***divides*** $a\}$***. Show that*** R ***is an identity relation on*** A***.***

Sol. Given, $A = \{1, 2, 3\}$

$a \in A, b \in B, a$ divides b and b divides a.

$\Rightarrow \quad a = b$

$\therefore \quad R = \{(a, a), a \in A\} = \{(1, 1), (2, 2), (3, 3)\}$

Hence, R is the identity relation on A.

Example 15. Let $A = \{3, 5\}, B = \{7, 11\}$.
Let $R = \{(a, b) : a \in A, b \in B, a - b$ is even$\}$.
Show that R is an universal relation from A to B.

Sol. Given, $A = \{3, 5\}, B = \{7, 11\}$.

Now, $R = \{(a, b) : a \in A, b \in B$ and $a - b$ is even$\}$
$= \{(3, 7), (3, 11), (5, 7), (5, 11)\}$

Also, $A \times B = \{(3, 7), (3, 11), (5, 7), (5, 11)\}$

Clearly, $R = A \times B$

Hence, R is an universal relation from A to B.

Example 16. Prove that the relation R defined on the set N of natural numbers by $xRy \Leftrightarrow 2x^2 - 3xy + y^2 = 0$ is not symmetric but it is reflexive.

Sol. (i) $2x^2 - 3x \cdot x + x^2 = 0, \forall\ x \in N$.

$\therefore\ x\ R\ x, \forall\ x \in N$, i.e. R is reflexive.

(ii) For $x = 1, y = 2, 2x^2 - 3xy + y^2 = 0$

$\therefore\ 1R2$

But $2 \cdot 2^2 - 3 \cdot 2 \cdot 1 + 1^2 = 3 \neq 0$

So, 2 is not related to 1 i.e., $2 \not R 1$

$\therefore$ R is not symmetric.

Example 17. Let N be the set of natural numbers and relation R on N be defined by $x\,R\,y \Leftrightarrow x$ divides y, $\forall\ x, y \in N$.
Examine whether R is reflexive, symmetric, anti-symmetric or transitive.

Sol. (i) x divides x i.e., $x\,R\,x, \forall\ x \in N$

$\therefore$ R is reflexive.

(ii) 1 divides 2 i.e., $1\ R\ 2$ but $2 \not R\ 1$ as 2 does not divide 1.

(iii) x divides y and y divides $x \Rightarrow x = y$

i.e., $x\,R\,y$ and $y\,R\,x \Rightarrow x = y$

$\therefore$ R is anti-symmetric relation.

(iv) $x\,R\,y$ and $y\,R\,z \Rightarrow x$ divides y and y divides z.

$\Rightarrow kx = y$ and $k'y = 2$, where k, k' are positive integers.

$\Rightarrow\ kk'x = z\ \Rightarrow\ x$ divides $z\ \Rightarrow\ x\,R\,z$

$\therefore$ R is transitive.

Equivalence Relation

A relation R on a set A is said to be an equivalence relation on A, when R is (i) reflexive (ii) symmetric and (iii) transitive. The equivalence relation denoted by ~.

Example 18. N ***is the set of natural numbers. The relation*** R ***is defined on*** $N \times N$ ***as follows***

$(a, b)\,R\,(c, d) \Leftrightarrow a + d = b + c$

Prove that R ***is an equivalence relation.***

Sol. (i) $(a, b)\,R\,(a, b) \Rightarrow a + b = b + a$

$\therefore$ R is reflexive.

(ii) $(a, b)\,R\,(c, d) \Rightarrow a + d = b + c$

$\Rightarrow \quad c + b = d + a\ \Rightarrow\ (c, d)\,R\,(a, b)$

$\therefore$ R is symmetric.

(iii) $(a, b)\,R\,(c, d)$ and $(c, d)\,R\,(e, f) \Rightarrow a + d = b + c$ and $c + f = d + e$

$\Rightarrow\ a + d + c + f = b + c + d + e$

$\Rightarrow \quad a + f = b + e\ \Rightarrow (a, b)\,R\,(e, f)$

$\therefore$ R is transitive.

Thus, R is an equivalence relation on $N \times N$.

Example 19. A relation R on the set of complex numbers is defined by $z_1\,R\,z_2 \Leftrightarrow \dfrac{z_1 - z_2}{z_1 + z_2}$ is real, show that R is an equivalence relation.

Sol. (i) $z_1 R z_1 \Rightarrow \dfrac{z_1 - z_1}{z_1 + z_1}, \forall z_1 \in C\ \Rightarrow\ 0$ is real

$\therefore$ R is reflexive.

(ii) $z_1 R z_2 \Rightarrow \dfrac{z_1 - z_2}{z_1 + z_2}$ is real

$\Rightarrow -\left(\dfrac{z_2 - z_1}{z_1 + z_2}\right)$ is real $\Rightarrow \left(\dfrac{z_2 - z_1}{z_1 + z_2}\right)$ is real

$\Rightarrow z_2 R z_1, \forall z_1, z_2 \in C$

$\therefore$ R is symmetric.

(iii) $\because z_1 R z_2 \Rightarrow \dfrac{z_1 - z_2}{z_1 + z_2}$ is real

$\Rightarrow \left(\overline{\dfrac{z_1 - z_2}{z_1 + z_2}}\right) = -\left(\dfrac{z_1 - z_2}{z_1 + z_2}\right)$

$\Rightarrow \left(\dfrac{\bar{z}_1 - \bar{z}_2}{\bar{z}_1 + \bar{z}_2}\right) + \left(\dfrac{z_1 - z_2}{z_1 + z_2}\right) = 0$

$\Rightarrow 2(z_1\bar{z}_1 - z_2\bar{z}_2) = 0 \Rightarrow |z_1|^2 = |z_2|^2$ ...(i)

Similarly, $z_2 R z_2 \Rightarrow |z_2|^2 = |z_3|^2$ ...(ii)

From Eqs. (i) and (ii), we get

$z_1 R z_2, z_2 R z_3$

$\Rightarrow |z_1|^2 = |z_3|^2$

$\Rightarrow z_1 R z_3$

$\therefore R$ is transitive.

Hence, R is an equivalence relation.

Ordered Relation

A relation R is called ordered, if R is transitive but not an equivalence relation.

Symbolically, $a R b, b R c \Rightarrow a R c, \forall a, b, c \in A$

For example,

Let $R = \{(1,2),(2,1),(2,3),(3,2),(1,3)\}$.

Here, R is symmetric.

Since, $(1,2) \in R \Rightarrow (2,1) \in R, (2,3) \in R \Rightarrow (3,2) \in R$

and R is transitive.

Since, $(1,2) \in R, (2,3) \in R \Rightarrow (1,3) \in R$

but R is not reflexive.

Since, $(1,1) \notin R, (2,2) \notin R, (3,3) \notin R$

Hence, R is not an equivalence relation.

$\therefore$ R is an ordered relation.

Partial Order Relation

A relation R is called partial order relation, if R is reflexive, transitive and anti-symmetric at the same time.

For example,

Let $R = \{(1,1),(2,2),(3,3),(1,2),(2,3),(1,3)\}$

$\therefore$ $R^{-1} = \{(1,1),(2,2),(3,3),(2,1),(3,2),(3,1)\}$

$R \cap R^{-1} = \{(1,1),(2,2),(3,3)\}$ = Identity

$\therefore$ R is anti-symmetric.

It is clear that R is reflexive.

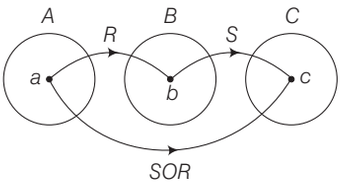
Since, $(1,1) \in R, (2,2) \in R, (3,3) \in R$ and R is transitive.

Since, $(1,2) \in R$ and $(2,3) \in R \Rightarrow (1,3) \in R$

Hence, R is partial order relation.

Composition of Two Relations

If A, B and C are three sets such that $R \subseteq A \times B$ and $S \subseteq B \times C$, then $(SOR)^{-1} = R^{-1}OS^{-1}$. It is clear that aRb, $bSc \Rightarrow aSORc$.



More generally,

$(R_1 O R_2 O R_3 O \ldots O R_n)^{-1} = R_n^{-1} O \ldots O R_3^{-1} O R_2^{-1} O R_1^{-1}$

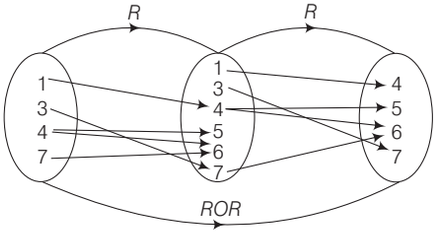
Example 20. ***Let R be a relation such that $R = \{(1,4),(3,7),(4,5),(4,6),(7,6)\}$, find***

(i) $R^{-1}OR^{-1}$ and (ii) $(R^{-1}OR)^{-1}$.

Sol. (i) We know that, $(ROR)^{-1} = R^{-1}OR^{-1}$

Dom $(R) = \{1, 3, 4, 7\}$

Range $(R) = \{4, 5, 6, 7\}$



We see that,

$1 \longrightarrow 4 \longrightarrow 5 \Rightarrow (1,5) \in ROR$

$1 \longrightarrow 4 \longrightarrow 6 \Rightarrow (1,6) \in ROR$

$3 \longrightarrow 7 \longrightarrow 6 \Rightarrow (3,6) \in ROR$

$\therefore$ $ROR = \{(1,5),(1,6),(3,6)\}$

Then, $R^{-1}OR^{-1} = (ROR)^{-1}$

$= \{(5,1),(6,1),(6,3)\}$

(ii) We know that, $(R^{-1}OR)^{-1} = R^{-1}O(R^{-1})^{-1} = R^{-1}OR$

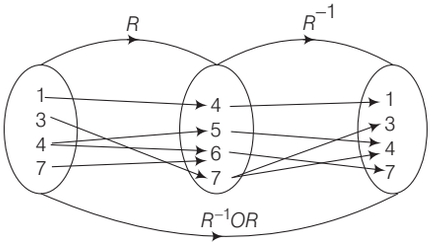
Since,

$R = \{(1,4),(3,7),(4,5),(4,6),(7,6)\}$

$\therefore$ $R^{-1} = \{(4,1),(7,3),(5,4),(6,4),(6,7)\}$

$\therefore$ Dom $(R) = \{1, 3, 4, 7\}$, Range $(R) = \{4, 5, 6, 7\}$

$\text{Dom}(R^{-1}) = \{4, 5, 6, 7\}$, $\text{Range}(R^{-1}) = \{1, 3, 4, 7\}$



We see that,

$1 \xrightarrow{R} 4 \xrightarrow{R^{-1}} 1 \Rightarrow (1, 1) \in R^{-1}OR$

$3 \xrightarrow{R} 7 \xrightarrow{R^{-1}} 3 \Rightarrow (3, 3) \in R^{-1}OR$

$4 \xrightarrow{R} 5 \xrightarrow{R^{-1}} 4 \Rightarrow (4, 4) \in R^{-1}OR$

$4 \xrightarrow{R} 6 \xrightarrow{R^{-1}} 4 \Rightarrow (4, 4) \in R^{-1}OR$

$4 \xrightarrow{R} 6 \xrightarrow{R^{-1}} 7 \Rightarrow (4, 7) \in R^{-1}OR$

$7 \xrightarrow{R} 6 \xrightarrow{R^{-1}} 4 \Rightarrow (7, 4) \in R^{-1}OR$

$7 \xrightarrow{R} 6 \xrightarrow{R^{-1}} 7 \Rightarrow (7, 7) \in R^{-1}OR$

$\therefore\ R^{-1}OR = \{(1, 1), (3, 3), (4, 4), (7, 7), (4, 7), (7, 4)\}$

Hence, $(R^{-1}OR)^{-1} = R^{-1}OR = \{(1, 1), (3, 3)$
$(4, 4), (7, 7), (4, 7), (7, 4)\}$

Theorems on Binary Relations

If R is a relation on a set A, then

(i) R is reflexive $\Rightarrow R^{-1}$ is reflexive.

(ii) R is symmetric $\Rightarrow R^{-1}$ is symmetric.

(iii) R is transitive $\Rightarrow R^{-1}$ is transitive.

Exercise for Session 2

1. If $A = \{2,3,5\}$, $B = \{2,5,6\}$, then $(A - B) \times (A \cap B)$ is

(a) {(3, 2), (3, 3), (3, 5)} (b) {(3, 2), (3, 5), (3, 6)}
(c) {(3, 2), (3, 5)} (d) None of these

2. If $n(A) = 4$, $n(B) = 3$, $n(A \times B \times C) = 24$, then $n(C)$ equals

(a) 1 (b) 2 (c) 17 (d) 288

3. The relation R defined on the set of natural numbers as $\{(a, b) : a \text{ differs from } b \text{ by } 3\}$ is given by

(a) {(1, 4), (2, 5), (3, 6), ...} (b) {(4, 1), (5, 2), (6, 3), ...} (c) {(1, 3), (2, 6), (3, 9), ...} (d) None of these

4. Let A be the non-void set of the children in a family. The relation 'x is a brother of y' on A, is

(a) reflexive (b) anti-symmetric (c) transitive (d) equivalence

5. Let $n(A) = n$, then the number of all relations on A, is

(a) 2^n (b) $2^{n!}$ (c) 2^{n^2} (d) None of these

6. If $S = \{1, 2, 3, \ldots, 20\}$, $K = \{a, b, c, d\}$, $G = \{b, d, e, f\}$. The number of elements of $(S \times K) \cup (S \times G)$ is

(a) 40 (b) 100 (c) 120 (d) 140

7. The relation R is defined on the set of natural numbers as $\{(a, b) : a = 2b\}$, then R^{-1} is given by

(a) {(2, 1) (4, 2) (6, 3), …} (b) {(1, 2) (2, 4) (3, 6), …} (c) R^{-1} is not defined (d) None of these

8. The relation $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)\}$ on set $A = \{1, 2, 3\}$ is

(a) reflexive but not symmetric (b) reflexive but not transitive
(c) symmetric and transitive (d) Neither symmetric nor transitive

9. The number of equivalence relations defined in the set $S = \{a, b, c\}$ is

(a) 5 (b) 3! (c) 2^3 (d) 3^3

10. If R be a relation $<$ from $A = \{1, 2, 3, 4\}$ to $B = \{1, 3, 5\}$, i.e. $(a, b) \in R \Leftrightarrow a < b$, then ROR^{-1}, is

(a) {(1, 3), (1, 5), (2, 3), (2, 5), (3,5), (4, 5)} (b) {(3, 1), (5, 1), (3, 2), (5, 2), (5, 3), (5, 4)}
(c) {(3, 3), (3, 5), (5, 3), (5, 5)} (d) {(3, 3), (3, 4), (4, 5)}

Session 3

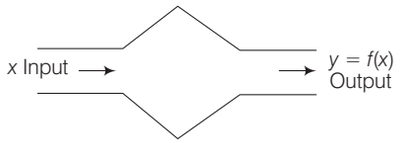
Definition of Functions, Domain, Codomain and Range, Composition of Mapping, Equivalence Classes, Partition of Set, Congruences

Functions

Introduction

If two variable quantities x and y according to some law are so related that corresponding to each value of x (considered only real), which belongs to set E, there corresponds one and only one finite value of the quantity y (i.e., unique value of y). Then, y is said to be a function (single valued) of x, defined by $y = f(x)$, where x is the **argument** or **independent variable** and y is the **dependent variable** defined on the set E.

For example, If r is the radius of the circle and A its area, then r and A are related by $A = \pi r^2$ or $A = f(r)$. Then, we say that the area A of the circle is the function of the radius r. **Graphically**,



Where, y is the image of x and x is the pre-image of y under f.

Remark

1. If to each value of x, which belongs to set E there corresponds one or more than one values of the quantity y. Then, y is called the multiple valued function of x defined on the set E.
2. The word 'FUNCTION' is used only for single valued function. *For example,* $y = \sqrt{x}$ is single valued functions but $y^2 = x$ is a multiple valued function.
   $\therefore\ y^2 = x \Rightarrow y = \pm\sqrt{x}$ for one value of x, y gives two values.

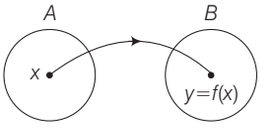
Definition of Functions

If A and B be two non-empty sets, then a function from A to B associates to each element x in A, a unique element $f(x)$ in B and is written as

$$f : A \to B \text{ or } A \xrightarrow{f} B$$

which is read as f is a mapping from A to B.

The other terms used for functions are **operators** or **transformations.**



Remark

1. If $x \in A$, $y = [f(x)] \in B$, then $(x, y) \in f$.
2. If $(x_1, y_1) \in f$ and $(x_2, y_2) \in f$, then $y_1 = y_2$.

Domain, Codomain and Range

Domain The set of A is called the domain of f (denoted by D_f).

Codomain The set of B is called the codomain of f (denoted by C_f).

Range The range of f denoted by R_f is the set consisting of all the images of the elements of the domain A.

Range of $f = [f(x) : x \in A]$

The range of f is always a subset of codomain B.

Onto and Into Mappings

In the mapping $f : A \to B$ such

$$f(A) = B$$

i.e., Range = Codomain

Then, the function is **Onto** and if $f(A) \subset B$, i.e. Range $\subset$ Codomain, then the function is **Into**.

Remark

Onto functions is also known as **surjective.**

Method to Test Onto or Into Mapping

Let $f : A \to B$ be a mapping. Let y be an arbitrary element in B and then $y = f(x)$, where $x \in A$. Then, express x in terms of y.

Now, if $x \in A, \forall\, y \in B$, then f is onto

and if $x \notin A, \forall\, y \in B$, then f is into.

For into mapping Find an element of B which is not f-image of any element of A.

One-one and Many-one Mapping

(i) The mapping $f : A \to B$ is called one-one mapping, if no two different elements of A have the same image in B. Such a mapping is also known as **injective mapping** or an **injection** or **monomorphism.**

Method to Test One-one If $x_1, x_2 \in A$,

then $f(x_1) = f(x_2)$

$\Rightarrow$ $x_1 = x_2$ and $x_1 \neq x_2$

$\Rightarrow$ $f(x_1) \neq f(x_2)$

(ii) The mapping $f : A \to B$ is called many-one mapping, if two or more than two different elements in A have the same image in B.

Method to Test Many-one

If $x_1, x_2 \in A$, then $f(x_1) = f(x_2)$

$\Rightarrow$ $x_1 \neq x_2$

From above classification, we conclude that function is of four types

(i) One-one onto (bijective)

(ii) One-one into

(iii) Many-one onto

(iv) Many-one into

Number of Functions (Mappings) at One Place in a Table

Let $f : A \to B$ be a mapping such that A and B are finite sets having m and n elements respectively, then

	Description of mappings
(i)	Total number of mappings from A to B
(ii)	Total number of one-one mappings from A to B
(iii)	Total number of many-one mappings from A to B
(iv)	Total number of onto (surjective) mappings from A to B
(v)	Total number of one-one onto (bijective) mappings from A to B
(vi)	Total number of into mappings from A to B

Example 21. Let N be the set of all natural numbers. Consider $f : N \to N : \ f(x) = 2x, \forall\, x \in N$. Show that f is one-one into.

Sol. Let $x_1, x_2 \in N$, then

$f(x_1) = f(x_2)$

$\Rightarrow$ $2x_1 = 2x_2 \ \Rightarrow \ x_1 = x_2$

$\therefore$ f is one-one.

Let $y = 2x$, then $x = \dfrac{y}{2}$

Now, if we put $y = 5$, then $x = \dfrac{5}{2} \notin N$.

This show that $5 \in N$ has no pre-image in N. So, f is into.

Hence, f is one-one and into.

Example 22. Show that the mapping $f : R \to R : f(x) = \cos x, \forall\, x \in R$ is neither one-one nor onto.

Sol. Let $x_1, x_2 \in R$.

Then, $f(x_1) = f(x_2) \ \Rightarrow \cos x_1 = \cos x_2$

$\Rightarrow$ $x_1 = 2n\pi \pm x_2 \ \Rightarrow \ x_1 \neq x_2$

$\therefore$ f is not one-one.

Let $y = \cos x$, but $-1 \leq \cos x \leq 1$

$\therefore$ $y \in [-1, 1]$

$[-1, 1] \subset R$

So, f is into (not onto).

Hence, f is neither one-one nor onto.

Constant Mapping

The mapping $f : A \to B$ is known as a constant mapping, if the range of B has only one element.

For all $x \in A$, $f(x) = a$, where as $a \in B$.

Identity Mapping

The mapping $f : A \to B$ is known as an identity mapping, if $f(a) = a, \forall\, a \in A$ and it is denoted by I_A.

Remark

I_A is bijective or bijection.

Equal Mapping

Let A and B be two mappings are $f : A \to B$ and $g : A \to B$ such that

$$f(x) = g(x), \forall\, x \in A$$

Then, the mappings f and g are equal and written as $f = g$.

Inclusion Mapping

The mapping $f : A \to B$ is known as inclusion mapping.

If $A \subseteq B$, then $f(a) = a, \forall\, a \in A$.

Equivalent or Equipotent or Equinumerous Set

The mapping $f : A \to B$ is known as equivalent sets, if A and B are both one-one and onto and written as $A \sim B$ which is read as 'A wiggle B'.

Inverse Mapping

If $f : A \to B$ be one-one and onto mapping, let $b \in B$, then there exist exactly one element $a \in A$ such that $f(a) = b$, so we may define

$$f^{-1} : B \to A : f^{-1}(b) = a$$
$$\Leftrightarrow \qquad f(a) = b$$

The function f^{-1} is called the inverse of f. A functions is invertible iff f is one-one onto.

Remark

1. $f^{-1}(b) \subseteq A$
2. If $f : A \to B$ and $g : B \to A$, then f and g are said to be invertible.

Example 23. Let $f : R \to R$ be defined by $f(x) = \cos(5x + 2)$. Is f invertible? Justify your answer.

Sol. For invertible of f, f must be bijective (i.e., one-one onto).

If $x_1, x_2 \in R$,

then $f(x_1) = f(x_2)$

$\Rightarrow \cos(5x_1 + 2) = \cos(5x_2 + 2)$

$\Rightarrow 5x_1 + 2 = 2n\pi \pm (5x_2 + 2)$

$\Rightarrow x_1 \neq x_2$

$\therefore$ f is not one-one.

But $-1 \le \cos(5x + 2) \le 1$

$\therefore -1 \le f(x) \le 1$

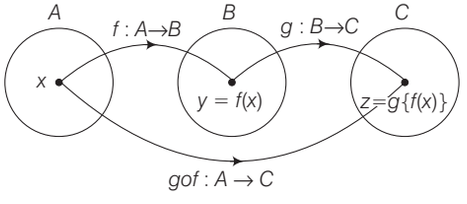
$\text{Range} = [-1, 1] \subset R$

$\therefore$ f is into mapping.

Hence, the function $f(x)$ is no bijective and so it is not invertible.

Composition of Mapping

Let A, B and C be three non-empty sets. Let $f : A \to B$ and $g : B \to C$ be two mappings, then $gof : A \to C$. This function is called the product or composite of f and g, given by $(gof)x = g\{f(x)\}, \forall\ x \in A$.



Important Remarks

1. (i) $(fog)x = f\{g(x)\}$ (ii) $(fof)x = f\{f(x)\}$
   (iii) $(gog)x = g\{g(x)\}$ (iv) $(fg)x = f(x) \cdot g(x)$
   (v) $(f \pm g)x = f(x) \pm g(x)$ (vi) $\left(\frac{f}{g}\right)x = \frac{f(x)}{g(x)}; g(x) \neq 0$
2. Let $h : A \to B$, $g : B \to C$ and $f : C \to D$ be any three functions. Then, $(fog)oh = fo(goh)$.
3. Let $f : A \to B$, $g : B \to C$ be two functions, then
   (i) f and g are injective $\Rightarrow gof$ is injective.
   (ii) f and g are surjective $\Rightarrow gof$ is surjective.
   (iii) f and g are bijective $\Rightarrow gof$ is bijective.
4. An injective mapping from a finite set to itself in bijective.

Example 24. If $f : R \to R$ and $g : R \to R$ be two mapping such that $f(x) = \sin x$ and $g(x) = x^2$, then
(i) prove that $fog \neq gof$.
(ii) find the values of $(fog)\frac{\sqrt{\pi}}{2}$ and $(gof)\left(\frac{\pi}{3}\right)$.

Sol. (i) Let $x \in R$

$\therefore \quad (fog)x = f\{g(x)\} \qquad [\because g(x) = x^2]$

$= f\{x^2\} = \sin x^2 \qquad \ldots(i)$

$[\because f(x) = \sin x]$

and $(gof)x = g\{f(x)\}$

$= g(\sin x) \qquad [\because f(x) = \sin x]$

$= \sin^2 x \qquad \ldots(ii)$

$[\because g(x) = x^2]$

From Eqs. (i) and (ii), we get $(fog)x \neq (gof)x, \forall\ x \in R$

Hence, $fog \neq gof$

(ii) From Eq. (i), $(fog)x = \sin x^2$

$\therefore \quad (fog)\frac{\sqrt{\pi}}{2} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

and from Eq. (ii), $(gof)x = \sin^2 x$

$\therefore \quad (gof)\frac{\pi}{3} = \sin^2\frac{\pi}{3} = \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}$

Example 25. If the mapping f and g are given by

$f = \{(1, 2), (3, 5), (4, 1)\}$

and $g = \{(2, 3), (5, 1), (1, 3)\}$,

write down pairs in the mapping fog and gof.

Sol. Domain $f = \{1, 3, 4\}$, Range $f = \{2, 5, 1\}$

Domain $g = \{2, 5, 1\}$, Range $g = \{1, 3\}$

$\because$ Range f = Dom $g = \{(2, 5, 1)\}$

$\therefore$ gof mapping is defined.

Then, gof mapping defined following way

$$\{1, 3, 4\} \xrightarrow{f} \{2, 5, 1\} \xrightarrow{g} \{1, 3\}$$
$$gof$$

We see that, $f(1) = 2,\ f(3) = 5,\ f(4) = 1$

and $g(2) = 3,\ g(5) = 1,\ g(1) = 3$

$\therefore \quad (gof)(1) = g\{f(1)\} = g(2) = 3$

$(gof)(3) = g\{f(3)\} = g(5) = 1$

$$(gof)(4) = g\{f(4)\} = g(1) = 3$$

Hence, $gof = \{(1, 3), (3, 1), (4, 3)\}$

Now, since Range of $f \subset$ Dom f

$\therefore$ fog is defined.

Then, fog mapping defined following way

$$\{2, 5, 1\} \xrightarrow{g} \{1, 3, 4\} \xrightarrow{f} \{2, 5, 1\}$$

$$fog$$

We see that, $g(2) = 3,\ g(5) = 1,\ g(1) = 3$

$$f(1) = 2,\ f(3) = 5,\ f(4) = 1$$

$\therefore$ $(fog)(2) = f\{g(2)\} = f(3) = 5$

$(fog)(5) = f\{g(5)\} = f(1) = 2$

$(fog)(1) = f\{g(1)\} = f(3) = 5$

Hence, $fog = \{(2,5),(5,2),(1,5)\}$

Equivalence Classes

If R be an equivalence relation on a set A, then $[a]$ is equivalence class of a with respect to R.

Symbolically, X_a or $[a] = \{x : x \in X, x\ R\ a\}$.

Remark

1. Square brackets[ ] are used to denote the equivalence classes.
2. $a \in [a]$ and $a \in [b] \Rightarrow [a] = [b]$
3. Either $[a] = [b]$ or $[a] \cap [b] = \phi$
4. Equivalence class of a also denoted by $E(a)$ or $\bar{a}$.
5. If $a \sim b$, $\frac{(a-b)}{m} = k$, the total number of equivalence class is m.

Example 26. Let $I = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \ldots\}$ and $R = \{(a,b) : (a-b)/4 = k, k \in I\}$ is an equivalence relation, find equivalence class.

Sol. Given, $\frac{a-b}{4} = k$

$\Rightarrow$ $a = 4k + b$, where $0 \le b < 4$

It is clear b has only value in 0, 1, 2, 3.

(i) Equivalence class of $[0] = \{x : x \in I \text{ and } x \sim 0\}$
$= \{x : x - 0 = 4k\} = \{0, \pm 4, \pm 8, \pm 12, \ldots)$
where, $k = 0, \pm 1, \pm 2, \pm 3, \ldots$

(ii) Equivalence class of $[1] = \{x : x \in I \text{ and } x \sim 1\}$
$= \{x : x - 1 = 4k\} = \{x : x = 4k + 1\}$
$= \{\ldots, -11, -7, -3, 1, 5, 9, \ldots\}$

(iii) Equivalence class of $[2] = \{x : x \in I \text{ and } x \sim 2\}$
$= \{x : x - 2 = 4k\} = \{x : x = 4k + 2\}$
$= \{\ldots, -10, -6, -2, 2, 6, 10, \ldots\}$

(iv) Equivalence class of $[3] = \{x : x \in I \text{ and } x \sim 3\}$
$= \{x : x - 3 = 4k\} = \{x : x = 4k + 3\}$
$= \{\ldots, -9, -5, -1, 5, 9, 13, \ldots\}$

Continue this process, we see that the equivalence class

$$[4] = [0], [5] = [1], [6] = [2], [7] = [3], [8] = [0]$$

Hence, total equivalence relations are [0], [1], [2], [3] and also clear

(i) $I = [0] \cup [1] \cup [2] \cup [3]$
(ii) every equivalence is a non-empty.
(iii) for any two equivalence classes $[a] \cap [b] = \phi$.

Partition of a Set

If A be a non-empty set, then a partition of A, if

(i) A is a collection of non-empty disjoint subsets of A.
(ii) union of collection of non-empty sets is A.

i.e., If A be a non-empty set and A_1, A_2, A_3, A_4 are subsets of A, then the set $\{A_1, A_2, A_3, A_4\}$ is called partition, if

(i) $A_1 \cup A_2 \cup A_3 \cup A_4 = A$
(ii) $A_1 \cap A_2 \cap A_3 \cap A_4 = \phi$

For example,

If $A = \{0, 1, 2, 3, 4\}$ and $A_1 = \{0\}, A_2 = \{1\}, A_3 = \{4\}$ and $A_4 = \{2, 3\}$, then we see that for $P = \{A_1, A_2, A_3, A_4\}$

(i) all A_1, A_2, A_3, A_4 are non-empty subset of A
(ii) $A_1 \cup A_2 \cup A_3 \cup A_4 = \{0, 1, 2, 3, 4\} = A$ and
(iii) $A_i \cap A_j \ne \phi, \forall\ i \ne j\ (i, j = 1, 2, 3, 4)$
Hence, from definition $P = \{A_1, A_2, A_3, A_4\}$ is partition of A.

Congruences

Let m be a positive integer, then two integers a and b are said to be congruent modulo m, if $a - b$ is divisible by m.

i.e., $m \overline{)\ a - b\ (\lambda}$

$$\begin{array}{r} a - b \\ \underline{-\ \ +} \\ 0 \end{array}$$

$\therefore$ $a - b = m\lambda$, where λ is a positive integer.

The congruent modulo 'm' is defined on all $a\ b \in I$ by $a \equiv b \pmod m$, if $a - b = m\lambda$, $\lambda \in I_+$.

Example 27. Find congruent solutions of $155 \equiv 7 \pmod 4$.

Sol. Since, $\left(\frac{155-7}{4} = \frac{148}{4} = 37\right)$

and $a = 155, b = 7, m = 4$

$\therefore$ $\lambda = \frac{a-b}{4} = \frac{155-7}{4} = \frac{148}{4}$

[here, $a = 155, b = 7$]

$= 37$ (integer)

Example 28. Find all congruent solutions of $8x \equiv 6 \pmod{14}$.

Sol. Given, $8x \equiv 6 \pmod{14}$

$$\therefore \quad \lambda = \frac{8x-6}{14}, \text{ where } \lambda \in I_+$$

$$\therefore \quad 8x = 14\lambda + 6$$

$$\Rightarrow \quad x = \frac{14\lambda + 6}{8}$$

$$\Rightarrow \quad x = \frac{7\lambda + 3}{4}$$

$$= \frac{4\lambda + 3(\lambda + 1)}{4}$$

$$x = \lambda + \frac{3}{4}(\lambda + 1), \text{ where } \lambda \in I_+$$

and here greatest common divisor of 8 and 14 is 2, so there are two required solutions.

For $\lambda = 3$ and 7, $x = 6$ and 13.

Exercise for Session 3

1. The values of b and c for which the identity $f(x+1) - f(x) = 8x + 3$ is satisfied, where $f(x) = bx^2 + cx + d$, are

(a) $b = 2, c = 1$ (b) $b = 4, c = -1$ (c) $b = -1, c = 4$ (d) $b = -1, c = 1$

2. If $f(x) = \frac{x-1}{x+1}$, then $f(ax)$ in terms of $f(x)$ is equal to

(a) $\frac{f(x)+a}{1+af(x)}$ (b) $\frac{(a-1)f(x)+a+1}{(a+1)f(x)+a-1}$ (c) $\frac{(a+1)f(x)+a-1}{(a-1)f(x)+a+1}$ (d) None of these

3. If f be a function satisfying $f(x+y) = f(x) + f(y)$, $\forall\, x, y \in R$. If $f(1) = k$, then $f(n)$, $n \in N$ is equal to

(a) k^n (b) nk (c) k (d) None of these

4. If $g = \{(1,1), (2,3), (3,5), (4,7)\}$ is a function described by the formula $g(x) = \alpha x + \beta$, what values should be assigned to α and β?

(a) $\alpha = 1, \beta = 1$ (b) $\alpha = 2, \beta = -1$ (c) $\alpha = 1, \beta = -2$ (d) $\alpha = -2, \beta = -1$

5. The values of the parameter α for which the function $f(x) = 1 + \alpha x$, $\alpha \neq 0$ is the inverse of itself, is

(a) -2 (b) -1 (c) 1 (d) 2

6. If $f(x) = (a - x^n)^{1/n}$, where $a > 0$ and $n \in N$, then $fof(x)$ is equal to

(a) a (b) x (c) x^n (d) a^n

7. If $f(x) = (ax^2 + b)^3$, the function g such that $f(g(x)) = g(f(x))$, is given by

(a) $g(x) = \left(\frac{b - x^{1/3}}{a}\right)^{1/2}$ (b) $g(x) = \frac{1}{(ax^2+b)^3}$ (c) $g(x) = (ax^2 + b)^{1/3}$ (d) $g(x) = \left(\frac{x^{1/3} - b}{a}\right)^{1/2}$

8. Which of the following functions from I to itself are bijections?

(a) $f(x) = x^3$ (b) $f(x) = x + 2$ (c) $f(x) = 2x + 1$ (d) $f(x) = x^2 + x$

9. Let $f : R - \{n\} \to R$ be a function defined by $f(x) = \frac{x-m}{x-n}$, where $m \neq n$. Then,

(a) f is one-one onto (b) f is one-one into (c) f is many-one onto (d) f is many-one into

10. If $f(x + 2y, x - 2y) = xy$, then $f(x, y)$ equals

(a) $\frac{x^2 - y^2}{8}$ (b) $\frac{x^2 - y^2}{4}$ (c) $\frac{x^2 + y^2}{4}$ (d) $\frac{x^2 - y^2}{2}$

Shortcuts and Important Results to Remember

1. Every set is a subset of itself.
2. Null set is a subset of every set.
3. The set $\{0\}$ is not an empty set as it contains one element 0. The set $\{\phi\}$ is not an empty set as it contains one element ϕ.
4. The order of finite set A of n elements is denoted by $O(A)$ or $n(A)$.
5. Number of subsets of a set containing n elements is 2^n.
6. Number of proper subsets of a set containing n elements is $2^n - 1$.
7. If $A = \phi$, then $P(A) = \phi$; $\therefore\ n(P(A)) = 1$.
8. The order of an infinite set is undefined.
9. A natural number p is a prime number, if p is greater than one and its factors are 1 and p only.
10. Finite sets are equivalent sets only, when they have equal number of elements.
11. Equal sets are equivalent sets but equivalent sets may not be equal sets.
12. If A is any set, then $A \subseteq A$ is true but $A \subset A$ is false.
13. If $A \subseteq B$, then $A \cup B = B$
14. $A \subset B \Leftrightarrow A \subseteq B$ and $A \neq B$
15. $x \notin A \cup B \Leftrightarrow x \notin A$ and $x \notin B$
16. $x \notin A \cap B \Leftrightarrow x \notin A$ or $x \notin B$
17. If $A_1, A_2, \ldots, A_n$ is a finite family of sets, then their union is denoted by $\bigcup_{i=1}^{n} A_i$ or $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n$.
18. If $A_1, A_2, A_3, \ldots, A_n$ is a finite family of sets, then their intersection is denoted by $\bigcap_{i=1}^{n} A_i$ or $A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n$.
19. $R - Q$ is the set of all irrational numbers.
20. Total number of relations from set A to set B is equal to $2^{n(A)n(B)}$.
21. The universal relation on a non-empty set is always reflexive, symmetric and transitive.
22. The identity relation on a non-empty set is always anti-symmetric.
23. The identity relation on a set is also called the diagonal relation on A.
24. For two relations R and S, the composite relations RoS, SoR may be void relations.
25. Every polynomial function $f : R \to R$ of degree odd is ONTO.
26. Every polynomial function $f : R \to R$ of degree even is INTO.
27. (i) The number of onto functions that can be defined from a finite set A containing n elements onto a finite set B containing 2 elements $= 2^n - 2$
    (ii) The number of onto functions that can be defined from a finite set A containing n elements onto a finite set B containing 3 elements $= 3^n - 3 \cdot 2^n + 3$
28. If a set A has n elements, then the number of binary relations on $A = n^{n^2}$.
29. If $fog = gof$, then either $f^{-1} = g$ or $g^{-1} = f$.
30. If f and g are bijective functions such that $f : A \to B$ and $g : B \to C$, then $gof : A \to C$ is bijective. Also, $(gof)^{-1} = f^{-1}og^{-1}$.
31. Let $f : A \to B$, $g : B \to C$ be two functions, then
    (i) f and g are injective $\Rightarrow gof$ is injective
    (ii) f and g are surjective $\Rightarrow gof$ is surjective
    (iii) f and g are bijective $\Rightarrow gof$ is bijective
32. Let $f : A \to B$, $g : B \to C$ be two functions, then
    (i) $gof : A \to C$ is injective $\Rightarrow f : A \to B$ is injective
    (ii) $gof : A \to C$ is surjective $\Rightarrow g : B \to C$ is surjective
    (iii) $gof : A \to C$ is injective and $g : B \to C$ is surjective $\Rightarrow$ $g : B \to C$ is injective
    (iv) $gof : A \to C$ is surjective and $g : B \to C$ is injective $\Rightarrow$ $f : A \to B$ is surjective
33. An injective mapping from a finite set to itself is bijective.

JEE Type Solved Examples :
Single Option Correct Type Questions

▪ This section contains **6 multiple choice examples**. Each example has four choices (a), (b), (c) and (d), out of which **ONLY ONE** is correct.

● ***Ex. 1*** *Two finite sets have m and n elements. The total number of subsets of the first set is 56 more than the total number of subsets of the second set. The values of m and n are*

(a) 7, 6 (b) 6, 3 (c) 5, 1 (d) 8, 7

Sol. (*b*) Since, $2^m - 2^n = 56 = 8 \times 7 = 2^3 \times 7$

$\Rightarrow \quad 2^n(2^{m-n} - 1) = 2^3 \times 7$

$\Rightarrow \quad n = 3$ and $2^{m-n} = 8 = 2^3 \Rightarrow n = 3$ and $m - n = 3$

$\Rightarrow \quad n = 3$ and $m - 3 = 3 \quad \Rightarrow n = 3$ and $m = 6$

● ***Ex. 2*** *If $aN = \{ax : x \in N\}$ and $bN \cap cN = dN$, where b, $c \in N$ are relatively prime, then*

(a) $d = bc$ (b) $c = bd$

(c) $b = cd$ (d) None of these

Sol. (a) bN = The set of positive integral multiples of b

cN = The set of positive integral multiples of c

$\therefore\ bN \cap cN$ = The set of positive integral multiples of bc

$= bcN \quad [\because\ b$ and c are prime$]$

$\therefore \quad d = bc$

● ***Ex. 3*** *In a town of 10000 families, it was found that 40% families buy newspaper A, 20% families buy newspaper B and 10% families buy newspaper C, 5% families buy newspapers A and B, 3% buy newspapers B and C and 4% buy newspapers A and C. If 2% families buy all the three newspapers, then number of families which buy A only is*

(a) 3100 (b) 3300 (c) 2900 (d) 1400

Sol. (*b*) $n(A) = 40\%$ of $10000 = 4000$

$n(B) = 20\%$ of $10000 = 2000$

$n(C) = 10\%$ of $10000 = 1000$

$n(A \cap B) = 5\%$ of $10000 = 500$

$n(B \cap C) = 3\%$ of $10000 = 300$

$n(C \cap A) = 4\%$ of $10000 = 400$

$n(A \cap B \cap C) = 2\%$ of $10000 = 200$

We want to find $n(A \cap B^c \cap C^c) = n[A \cap (B \cup C)^c]$

$= n(A) - n[A \cap (B \cup C)] = n(A) - n[(A \cap B) \cup (A \cap C)]$

$= n(A) - [n(A \cap B) + n(A \cap C) - n(A \cap B \cap C)]$

$= 4000 - [500 + 400 - 200] = 4000 - 700 = 3300$

● ***Ex. 4*** *Let R be the relation on the set R of all real numbers defined by aRb iff $|a - b| \le 1$. Then, R is*

(a) reflexive and symmetric (b) symmetric only

(c) transitive only (d) anti-symmetric only

Sol. (a) $\because\ |a - a| = 0 < 1 \ \Rightarrow\ a\ R\ a, \forall\ a \in R$

$\therefore\ R$ is reflexive.

Again, $aRb \ \Rightarrow\ |a - b| \le 1$

$\Rightarrow \quad |b - a| \le 1 \Rightarrow bRa$

$\therefore\ R$ is symmetric.

Again, $1\ R\ 2$ and $2R1$ but $2 \ne 1$

$\therefore\ R$ is not anti-symmetric.

Further, $1R2$ and $2R3$ but $1 \not R 3 \quad [\because\ |1 - 3| = 2 > 1]$

$\therefore\ R$ is not transitive.

● ***Ex. 5*** *The relation R defined on A = {1, 2, 3} by aRb, if $|a^2 - b^2| \le 5$. Which of the following is false?*

(a) $R = \{(1, 1), (2, 2), (3, 3), (2, 1), (1, 2), (2, 3), (3, 2)\}$

(b) $R^{-1} = R$

(c) Domain of $R = \{1, 2, 3\}$

(d) Range of $R = \{5\}$

Sol. (*d*) Let $a = 1$

Then, $|a^2 - b^2| \le 5 \ \Rightarrow\ |1 - b^2| \le 5$

$\Rightarrow \quad |b^2 - 1| \le 5 \ \Rightarrow\ b = 1, 2$

Let $a = 2$

Then, $|a^2 - b^2| \le 5$

$\Rightarrow \quad |4 - b^2| \le 5 \ \Rightarrow\ |b^2 - 4| \le 5$

$\therefore \quad b = 1, 2, 3$

Let $a = 3$

Then, $|a^2 - b^2| \le 5$

$\Rightarrow \quad |9 - b^2| \le 5 \ \Rightarrow\ |b^2 - 9| \le 5 \ \Rightarrow\ b = 2, 3$

$\therefore \quad R = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3)\}$

$R^{-1} = \{(y, x) : (x, y) \in R\}$

$= \{(1, 1), (2, 1), (1, 2), (2, 2), (3, 2), (2, 3), (3, 3)\} = R$

Domain of $R = \{x : (x, y) \in R\} = \{1, 2, 3\}$

Range of $R = \{y : (x, y) \in R\} = \{1, 2, 3\}$

● ***Ex. 6*** *If $f(x) = \dfrac{1}{(1-x)}$, $g(x) = f\{f(x)\}$ and $h(x) = f[f\{f(x)\}]$. Then the value of $f(x) \cdot g(x) \cdot h(x)$ is*

(a) 6 (b) −1 (c) 1 (d) 2

Sol. (*b*) $\because\ g(x) = f\{f(x)\} = f\left(\dfrac{1}{1-x}\right) = \dfrac{1}{1 - \dfrac{1}{1-x}} = \dfrac{x-1}{x}$

and $h(x) = f[f\{f(x)\}] = f(g(x))$

$= \dfrac{1}{1 - g(x)} = \dfrac{1}{1 - \dfrac{x-1}{x}} = x$

$\therefore\ f(x) \cdot g(x) \cdot h(x) = \dfrac{1}{(1-x)} \cdot \dfrac{(x-1)}{x} \cdot x = -1$

JEE Type Solved Examples :
More than One Correct Option Type Questions

■ This section contains **3 multiple choice examples**. Each example has four choices (a), (b), (c) and (d) out of which **MORE THAN ONE** may be correct.

● ***Ex. 7*** *If I is the set of integers and if the relation R is defined over I by aRb, iff $a-b$ is an even integer, $a, b \in I$, the relation R is*

(a) reflexive (b) anti-symmetric
(c) symmetric (d) equivalence

Sol. (a, c, d)

$aRb \Leftrightarrow a-b$ is an even integer, $a, b \in I$

$a - a = 0$ (even integer)

$\therefore \quad (a, a) \in R, \ \forall \ a \in I$

$\therefore$ R is reflexive relation.

Let $(a, b) \in R \Rightarrow (a-b)$ is an even integer.

$\Rightarrow \quad -(b-a)$ is an even integer.

$\Rightarrow \quad (b-a)$ is an even integer.

$\Rightarrow \quad (b, a) \in R$

$\therefore$ R is symmetric relation.

Now, let $(a, b) \in R$ and $(b, c) \in R$

Then, $(a-b)$ is an even integer and $(b-c)$ is an even integer.

So, let $\quad a - b = 2x_1, \ x_1 \in I$

and $\quad b - c = 2x_2, \ x_2 \in I$

$\therefore \quad (a-b)+(b-c) = 2(x_1 + x_2)$

$\Rightarrow \quad (a-c) = 2(x_1 + x_2) \Rightarrow a - c = 2x_3$

$\therefore$ $(a-c)$ is an even integer.

$\therefore$ aRb and $bRc \Rightarrow aRc$ So, R is transitive relation.

Hence, R is an equivalence relation.

● ***Ex. 8*** *If $f(x) = \dfrac{a-x}{a+x}$, the domain of $f^{-1}(x)$ contains*

(a) $(-\infty, \infty)$ (b) $(-\infty, -1)$
(c) $(-1, \infty)$ (d) $(0, \infty)$

Sol. (b, c, d)

Let $\quad y = f(x) = \dfrac{a-x}{a+x} \Rightarrow ay + xy = a - x$

$\therefore \quad x = \dfrac{a(1-y)}{(1+y)} = f^{-1}(y) \Rightarrow f^{-1}(x) = \dfrac{a(1-x)}{(1+x)}$

$\therefore$ $f^{-1}(x)$ is not defined for $x = -1$.

Domain of $f^{-1}(x)$ belongs to $(-\infty, -1) \cup (-1, \infty)$.

Now, for $a = -1$, given function $f(x) = -1$, which is constant.

Then, $f^{-1}(x)$ is not defined.

$\therefore \quad a \neq -1$

● ***Ex. 9*** *If $f(x) = \dfrac{\sin([x]\pi)}{x^2 + x + 1}$, where $[\cdot]$ denotes the greatest integer function, then*

(a) f is one-one
(b) f is not one-one and non-constant
(c) f is constant function (d) f is zero function

Sol. (c, d)

$\because \quad \sin([x]\pi) = 0$

$\therefore \quad f(x) = 0 \quad [\because [x] \text{ is an integer}]$

$\Rightarrow$ $f(x)$ is a constant function and also $f(x)$ is a zero function.

JEE Type Solved Examples :
Passage Based Questions

■ This section contains **2 solved passages** based upon each of the passage **3 multiple choice** examples have to be answered. Each of these examples has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I
(Ex. Nos. 10 to 12)

If $A = \{x : |x| < 2\}$, $B = \{x : |x-5| \leq 2\}$,
$C = \{x : |x| > x\}$ and $D = \{x : |x| < x\}$

● ***10.*** *The number of integral values in $A \cup B$ is*

(a) 4 (b) 6
(c) 8 (d) 10

● ***11.*** *The number of integral values in $A \cap C$ is*

(a) 1 (b) 2
(c) 3 (d) 0

● ***12.*** *The number of integral values in $A \cap D$ is*

(a) 2 (b) 4
(c) 6 (d) 0

Sol. (Ex. Nos. 10 to 12)

$$A = \{x : |x| < 2\} = \{x : -2 < x < 2\} = (-2, 2)$$

$$B = \{x : |x - 5| \le 2\} = \{x : -2 \le x - 5 \le 2\}$$

$$= \{x : 3 \le x \le 7\} = [3, 7]$$

$$C = \{x : |x| > x\} = \{x : x < 0\} = (-\infty, 0)$$

and $D = \{x : |x| < x\} = \phi$

10. (*c*) $A \cup B = (-2, 2) \cup [3, 7]$

Integral values in $A \cup B$ are $-1, 0, 1, 3, 4, 5, 6, 7$.

$\therefore$ Number of integral values in $A \cup B$ is 8.

11. (*a*) $A \cap C = (-2, 2) \cap (-\infty, 0) = (-2, 0)$

Integral value in $A \cap C$ is -1.

$\therefore$ Number of integral values in $A \cap C$ is 1.

12. (*d*) $A \cap D = (-2, 2) \cap \phi = \phi$

$\therefore$ Number of integral values in $A \cap D$ is 0.

Passage II

(Ex. Nos. 13 to 15)

If $A = \{x : x^2 - 2x + 2 > 0\}$ *and* $B = \{x : x^2 - 4x + 3 \le 0\}$

● ***13.*** *$A \cap B$ equals*

(a) $[1, \infty)$ (b) $[1, 3]$

(c) $(-\infty, 3]$ (d) $(-\infty, 1) \cup (3, \infty)$

● ***14.*** *$A - B$ equals*

(a) $(-\infty, \infty)$ (b) $(1, 3)$

(c) $(3, \infty)$ (d) $(-\infty, 1) \cup (3, \infty)$

● ***15.*** *$A \cup B$ equals*

(a) $(-\infty, 1)$ (b) $(3, \infty)$

(c) $(-\infty, \infty)$ (d) $(1, 3)$

Sol. (Ex. Nos. 13 to 15)

$$A = \{x : x^2 - 2x + 2 > 0\} = \{x : (x - 1)^2 + 1 > 0\} = (-\infty, \infty)$$

$$B = \{x : x^2 - 4x + 3 \le 0\} = \{x : (x - 1)(x - 3) \le 0\}$$

$$= \{x : 1 \le x \le 3\} = [1, 3]$$

13. (*b*) $A \cap B = (-\infty, \infty) \cap [1, 3] = [1, 3]$

14. (*d*) $A - B = (-\infty, \infty) - [1, 3] = (-\infty, 1) \cup (3, \infty)$

15. (*c*) $A \cup B = (-\infty, \infty) \cup [1, 3] = (-\infty, \infty)$

JEE Type Solved Examples :
Single Integer Answer Type Questions

■ This section contains **2 examples.** The answer to each example is a **single digit integer** ranging from **0** to **9** (both inclusive).

● ***Ex. 16*** *If $f : R^+ \to A$, where $A = \{x : -5 < x < \infty\}$ is defined by $f(x) = x^2 - 5$ and if $f^{-1}(13) = \{-\lambda\sqrt{(\lambda - 1)}, \lambda\sqrt{(\lambda - 1)}\}$, the value of λ is*

Sol. (*3*) $f^{-1}(13) = \{x : f(x) = 13\} = \{x : x^2 - 5 = 13\}$

$$= \{x : x^2 = 18\} = \{x : x = \pm 3\sqrt{2}\}$$

$$= \{-3\sqrt{2}, 3\sqrt{2}\}$$

$$= \{-\lambda\sqrt{(\lambda - 1)}, \lambda\sqrt{(\lambda - 1)}\} \quad \text{[given]}$$

$\therefore \quad \lambda = 3$

● ***Ex. 17*** *If $A = \{2, 3\}$, $B = \{4, 5\}$ and $C = \{5, 6\}$, then $n\{(A \times B) \cup (B \times C)\}$ is*

Sol. (*8*) $\because A \times B = \{2, 3\} \times \{4, 5\}$

$$= \{(2, 4), (2, 5), (3, 4), (3, 5)\}$$

and $B \times C = \{4, 5\} \times \{5, 6\}$

$$= \{(4, 5), (4, 6), (5, 5), (5, 6)\}$$

$\therefore \quad (A \times B) \cup (B \times C) = \{(2, 4), (2, 5), (3, 4), (3, 5),$

$(4, 5), (4, 6), (5, 5), (5\ 6)\}$

Now, $n\{(A \times B) \cup (B \times C)\} = 8$

JEE Type Solved Examples : Matching Type Questions

■ This section contains **1 examples.** Example 18 have three statements (A, B and C) given in Column I and four statements (p, q, r and s) in **Column II.** Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II.**

● ***Ex. 18***

	Column I		Column II
(A)	$R = \{(x, y) : x < y\ ;\ x, y \in N\}$	(p)	Reflexive
(B)	$S = \{(x, y) : x + y = 10\ ;\ x, y \in N\}$	(q)	Symmetric
(C)	$T = \{(x, y) : x = y$ or $x - y = 1\ ;\ x, y \in N\}$	(r)	Transitive
(D)	$U = \{(x, y) : x^y = y^x;\ x, y \in N\}$	(s)	Equivalence

Sol. (A) → (r); (B) → (q); (C) → (p); (D) → (p, q, r, s)

(A) $\because \quad R = \{(x, y) : x < y;\ x, y \in N\}$

$x \not< x \ \therefore (x, x) \notin R$

So, R is not reflexive.

Now, $(x, y) \in R \Rightarrow x < y \not\Rightarrow y < x \Rightarrow (y, x) \notin R$

$\therefore$ R is not symmetric.

Let $(x, y) \in R$ and $(y, z) \in R$

$\Rightarrow \quad x < y$ and $y < z \Rightarrow x < z \Rightarrow (x, z) \in R$

$\therefore R$ is transitive.

(B) $\because\ S = \{(x, y) : x + y = 10\ ;\ x, y \in N\}$

$\therefore \quad x + x = 10 \Rightarrow 2x = 10 \Rightarrow x = 5$

So, each element of N is not related to itself by the relation $x + y = 10$.

$\therefore$ S is not reflexive.

Now, $(x, y) \in S \Rightarrow x + y = 10 \Rightarrow y + x = 10$

$\Rightarrow \quad (y, x) \in S$

$\therefore$ S is symmetric relation.

Now, let $(3, 7) \in S$ and $(7, 3) \in S \Rightarrow (3, 3) \notin S$

$\therefore$ S is not transitive.

(C) $\because\ T = \{(x, y) : x = y$ or $x - y = 1;\ x, y \in N\}$

$\therefore \quad x = x$

So, $\quad (x, x) \in T, \forall\ x \in N$

$\therefore \quad T$ is reflexive.

Let $\quad (3, 2) \in T$ and $3 - 2 = 1$

$\not\Rightarrow \quad 2 - 3 = -1 \Rightarrow (2, 3) \notin T$

$\therefore \quad T$ is not symmetric.

Now, let $\quad (3, 2) \in T$ and $(2, 1) \in T$

$\therefore \quad 3 - 2 = 1$ and $2 - 1 = 1$

Then, $\quad (3, 1) \notin T \qquad [\because 3 - 1 = 2 \neq 1]$

$\therefore \quad T$ is not transitive.

(D) $U = \{(x, y) : x^y = y^x;\ x, y \in N\}$

$\because \quad x^x = x^x$

$\therefore \quad (x, x) \in U$

$\therefore U$ is reflexive.

Now, $(x, y) \in U \Rightarrow x^y = y^x$

$\Rightarrow \quad y^x = x^y \Rightarrow (y, x) \in U$

$\therefore U$ is symmetric.

Now, let $(x, y) \in U$ and $(y, z) \in U$

$\Rightarrow \quad x^y = y^x$ and $y^z = z^y$

Now, $\quad (x^y)^z = (y^x)^z$

$\Rightarrow \quad (x^z)^y = (y^z)^x \Rightarrow (x^z)^y = (z^y)^x$

$\Rightarrow \quad (x^z)^y = (z^x)^y \Rightarrow x^z = z^x \Rightarrow (x, z) \in U$

$\therefore \quad U$ is transitive.

Hence, U is an equivalence relation.

JEE Type Solved Examples : Statement I and II Type Questions

■ **Directions** Example numbers 19 and 20 are Assertion-Reason type examples. Each of these examples contains two statements:

Statement-1 (Assertion) and

Statement-2 (Reason)

Each of these examples also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below:

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1

(c) Statement-1 is true; Statement-2 is false

(d) Statement-1 is false; Statement-2 is true

● ***Ex. 19*** **Statement-1** *If $A \cup B = A \cup C$ and $A \cap B = A \cap C$, then $B = C$.*

Statement-2 $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Sol. (*a*) We have, $B = B \cup (A \cap B)$

$= B \cup (A \cap C) \qquad [\because A \cap B = A \cap C]$

$= (A \cup C) \cap (B \cup C)$ $[\because A \cup B = A \cup C]$
$= (A \cap B) \cup C$
$= (A \cap C) \cup C$ $[\because A \cap B = A \cap C]$
$= C$

Hence, Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation of Statement-1.

● ***Ex. 20*** **Statement-1** *If U is universal set and $B = U - A$, then $n(B) = n(U) - n(A)$.*
Statement-2 *For any three arbitrary sets A, B and C, if $C = A - B$, then $n(C) = n(A) - n(B)$.*

Sol. (c) $\because$ $B = U - A = A'$
$\therefore$ $n(B) = n(A') = n(U) - n(A)$
So, Statement-1 is true.
But for any three arbitrary sets A, B and C, we cannot always have
$$n(C) = n(A) - n(B)$$
if $C = A - B$
As it is not specified A is universal set or not. In case not conclude
$$n(C) = n(A) - n(B)$$
Hence, Statement-2 is false.

Subjective Type Examples

■ In this section, there are **12 subjective** solved examples.

● ***Ex. 21.*** *If $A = A \cup B$, prove that $B = A \cap B$.*

Sol. $\because A = A \cap B$
$\therefore$ $A \subseteq A \cup B$ and $A \cup B \subseteq A$
Now, let $x \in B \Leftrightarrow x \in A \cup B$ [by definition of union]
$\Leftrightarrow$ $x \in A$ $[\because A \subseteq A \cup B]$
$\Leftrightarrow$ $x \in A \cap B$ $[\because A \subseteq A \cup B$, then also $A \subseteq A \cap B]$
$\therefore$ $B \subseteq A \cap B$ and $A \cap B \subseteq B$
Hence, $A \cap B = B$ **Hence proved.**

● ***Ex. 22*** *Find the smallest and largest sets of Y such that $Y \cup \{1, 2\} = \{1, 2, 3, 5, 9\}$.*

Sol. Smallest set of Y has three elements and largest set of Y has five elements, since RHS set has five elements.
$\therefore$ Smallest set of Y is $\{3, 5, 9\}$
and largest set of Y is $\{1, 2, 3, 5, 9\}$.

● ***Ex. 23*** *If P, Q and R are the subsets of a set A, then prove that $R \times (P^c \cup Q^c)^c = (R \times P) \cap (R \times Q)$.*

Sol. We know that from De-Morgan's law,
$$A^c \cap B^c = (A \cup B)^c \quad \ldots(i)$$
Replacing A by P^c and B by Q^c, then Eq. (i) becomes
$(P^c)^c \cap (Q^c)^c = (P^c \cup Q^c)^c$
$\Rightarrow$ $P \cap Q = (P^c \cup Q^c)^c$ $[\because (A^c)^c = A]$ ...(ii)
$\therefore R \times (P^c \cup Q^c)^c = R \times (P \cap Q)$ [from Eq. (ii)]
$= (R \times P) \cap (R \times Q)$ [by cartesian product]
Hence, $R \times (P^c \cup Q^c)^c = (R \times P) \cap (R \times Q)$

● ***Ex. 24*** *Check the following relations R and ρ for reflexive, symmetry and transitivity.*

(i) aRb iff b is divisible by a, where a and b are natural numbers.

(ii) $\alpha\rho\beta$ iff α is perpendicular to β, where α and β are straight lines in a plane.

Sol. (i) The relation R is reflexive, since a is divisible by a, R is not symmetric because b is divisible by a but a is not divisible by b. i.e., $aRb \nRightarrow bRa$
Again, R is transitive, since b is divisible by a and c is divisible by b, then always c is divisible by a.

(ii) The relation ρ is not reflexive as no line can be perpendicular to itself. The relation ρ is symmetric, since a line α is perpendicular to β, then β is perpendicular to α and the relation ρ is not transitive, since a line α is perpendicular to β and if β is perpendicular to γ (new line), then α is not perpendicular to γ (since, α is parallel to γ).

● ***Ex. 25*** *Let $f:[0,1] \to [0,1]$ be defined by $f(x) = \dfrac{1-x}{1+x}$; $0 \le x \le 1$ and $g:[0,1] \to [0,1]$ be defined by $g(x) = 4x(1-x), 0 \le x \le 1$.*
Determine the functions *fog* and *gof*.
Note that $[0,1]$ stands for the set of all real members x that satisfy the condition $0 \le x \le 1$.

Sol. $(fog)x = f\{g(x)\} = f\{4x(1-x)\}$ $[\because g(x) = 4x(1-x)]$
$$= \frac{1 - 4x(1-x)}{1 + 4x(1-x)} \qquad \left[\because f(x) = \frac{1-x}{1+x}\right]$$
$$= \frac{1 - 4x + 4x^2}{1 + 4x - 4x^2} = \frac{(2x-1)^2}{1 + 4x - 4x^2}$$

and $(gof)x = g\{f(x)\} = g\left\{\frac{1-x}{1+x}\right\}$ $\left[\because f(x) = \frac{1-x}{1+x}\right]$

$$= 4\left(\frac{1-x}{1+x}\right)\left(1-\frac{1-x}{1+x}\right) = 4\left(\frac{1-x}{1+x}\right)\left(\frac{2x}{1+x}\right)$$

$$= \frac{8x(1-x)}{(1+x)^2}$$

● ***Ex. 26*** *If A, B are two sets, prove that* $A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$.

Hence or otherwise prove that

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

where, $n(A)$ denotes the number of elements in A.

Sol. Let $x \in A \cup B \Rightarrow x \in A$ or $x \in B$

$\Leftrightarrow (x \in A$ and $x \notin B)$ or $(x \in B$ and $x \notin A)$

or $(x \in A$ and $x \in B)$ [from definition of union]

$\Leftrightarrow$ $x \in (A - B)$ or $x \in (B - A)$ or $x \in A \cap B$

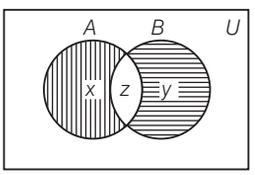
$\Leftrightarrow$ $x \in (A - B) \cup (B - A) \cup (A \cap B)$

$\therefore$ $A \cup B \subseteq (A - B) \cup (B - A) \cup (A \cap B)$

and $(A - B) \cup (B - A) \cup (A \cap B) \subseteq A \cup B$

Hence, $A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$

Let the common elements in A and B are z and only element of A are x (represented by vertical lines in the Venn diagram) and only element of B are y (represented by horizontal lines in the Venn diagram)



$\therefore$ $n(A)$ = Total elements of $A = x + z$

$n(B)$ = Total elements of $B = y + z$

$n(A \cap B)$ = Common elements in A and $B = z$

Now, $n(A \cup B)$ = Total elements in complete region of A and B

$$= x + y + z$$
$$= (x + z) + (y + z) - z$$
$$= n(A) + n(B) - n(A \cap B)$$

Hence, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

● ***Ex. 27*** *Let* $A = \{\theta: 2\cos^2\theta + \sin\theta \le 2\}$ *and* $B = \{\theta: \pi / 2 \le \theta \le 3\pi / 2\}$. Then find $A \cap B$.

Sol. $\because$ $2\cos^2\theta + \sin\theta \le 2$

$\therefore$ $2(1 - \sin^2\theta) + \sin\theta \le 2$

$\Rightarrow$ $2\sin^2\theta - \sin\theta \ge 0$

$\Rightarrow$ $\sin\theta(2\sin\theta - 1) \ge 0$

$\Rightarrow$ $\sin\theta\left(\sin\theta - \frac{1}{2}\right) \ge 0$

$\therefore$ $\sin\theta \le 0$ and $\sin\theta \ge \frac{1}{2}$

Now, the values of θ which lie in the interval $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$.

$\left[\because B = \left\{\theta: \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\right\}\right]$

So, θ satisfy $\sin\theta \le 0$ in the interval $\pi \le \theta \le \frac{3\pi}{2}$

and θ satisfy $\sin\theta \ge \frac{1}{2}$ in the interval $\frac{\pi}{2} \le \theta \le \frac{5\pi}{6}$.

$\therefore$ $A \cap B = \left\{\theta: \pi \le \theta \le \frac{3\pi}{2}\right\}$

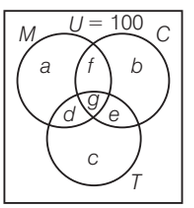
and $A \cap B = \left\{\theta: \frac{\pi}{2} \le \theta \le \frac{5\pi}{6}\right\}$

Hence, $A \cap B = \left\{\theta: \frac{\pi}{2} \le \theta \le \frac{5\pi}{6} \text{ or } \pi \le \theta \le \frac{3\pi}{2}\right\}$

$$= \left\{\theta: \theta \in \left[\frac{\pi}{2}, \frac{5\pi}{6}\right] \cup \left[\pi, \frac{3\pi}{2}\right]\right\}$$

● ***Ex. 28*** *An investigator interviewed* 100 *students to determine their preferences for the three drinks; milk* (M)*, coffee* (C) *and tea* (T)*. He reported the following:* 10 *students has all the three drinks M, C, T*; 20 *had M and C*; 30 *had C and T,* 25 *had M and T*; 12 *had M only;* 5 *had C only and* 8 *had T only. Using a Venn diagram, find how many did not take any of the three drinks?*

Sol. Given, M, C and T are the sets of drinks; milk, coffee and tea, respectively. Let us denote the number of drinks (students) contained in the bounded region as shown in the diagram by a, b, c, d, e, f and g, respectively.



Then, $g = 10$

$g + f = 20 \Rightarrow f = 10$ $[\because g = 10]$

$g + e = 30 \Rightarrow e = 20$

$d + g = 25 \Rightarrow d = 15$

and $a = 12, b = 5, c = 8$

Thus, total number of students taking drinks M or C or T

$$= a + b + c + d + e + f + g$$
$$= 12 + 5 + 8 + 15 + 20 + 10 + 10 = 80$$

Hence, the number of students taking none of them drinks

$$= 100 - 80 = 20$$

● ***Ex. 29*** *In a certain city, only two newspapers A and B are published. It is known that 25% of the city population reads A and 20% reads B, while 8% reads A and B. It is also known that 30% of those who read A but not B, look into advertisements and 40% of those who read B but not A, look into advertisements while 50% of those who read both A and B, look into advertisements. What per cent of the population read on advertisement?*

Sol. Let C = Set of people who read paper A
and D = Set of people who read paper B

Given, $n(C) = 25, n(D) = 20, n(C \cap D) = 8$

$\therefore \quad n(C \cap D') = n(C) - n(C \cap D)$

$= 25 - 8 = 17$

But total number of people who read A but not $B = 30\%$

$\therefore$ Percentage of people reading A but not $B = 30\%$ of 17

$$= \frac{30 \times 17}{100} = \frac{51}{10}$$

and $n(C' \cap D) = n(D) - n(C \cap D) = 20 - 8 = 12$

Also, total number of people who read B but not $A = 40\%$

$\therefore$ Percentage of people reading B but not $A = 40\%$ of 12

$$= \frac{40 \times 12}{100} = \frac{24}{5}$$

and given total people who read A and $B = 50\%$

$\therefore$ Total number of people who read A and $B = 50\%$ of 8

$$= \frac{50 \times 8}{100} = 4$$

$\therefore$ Percentage of people reading an advertisement

$$= \frac{51}{10} + \frac{24}{5} + 4 = 13.9\%$$

● ***Ex. 30*** *An analysis of 100 personal injury claims made upon a motor insurance company revealed that loss or injury in respect of an eye, an arm, a leg occurred in 30, 50 and 70 cases, respectively. Claims involving this loss or injury to two of these members numbered 44. How many claims involved loss or injury to all the three, we must assume that one or another of three members was mentioned in each of the 100 claims?*

Sol. Let the set of people having injuries in eyes, arms or legs be denoted by E, A and L, respectively. Then, according to the problem, we have

$$n(E \cup A \cup L) = 30, n(E) = 30$$

$$n(A) = 50, n(L) = 70$$

and $n[(E \cap A \cap L') \cup (E \cap A' \cap L) \cup (E' \cap A \cap L) = 44$

or $n(E \cap A \cap L') + n(E \cap A' \cap L) + n(E' \cap A \cap L) = 44$

[$\because$ each case is mutually exclusive]

or $n(E \cap A) - n(E \cap A \cap L) + n(E \cap L) - n(E \cap A \cap L) + n(E \cap L) - n(E \cap A \cap L) = 44$

$\Rightarrow \quad n(E \cap A) + n(E \cap L) + n(A \cap L) - 3n(E \cap A \cap L) = 44$ ...(i)

$\because \quad n(E \cup A \cup L) = 100$

$\therefore n(E) + n(A) + n(L) - n(E \cap A) - n(A \cap L) - n(E \cap L) + n(E \cap A \cap L) = 100$

$\Rightarrow \quad 30 + 50 + 70 - \{44 + 3n(E \cap A \cap L)\} + n(E \cap A \cap L) = 100$ [from Eq. (i)]

$\Rightarrow \quad 6 - 2n(E \cap A \cap L) = 0$

$\therefore \quad n(E \cap L \cap A) = 3$

Hence, there are three claims involved in loss or injury to all the three.

Aliter

Let E = Set of people having injuries in eyes

$\therefore \quad n(E) = 30$

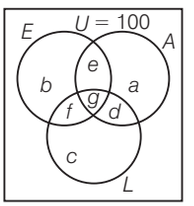
A = Set of people having injuries in arms

$\therefore \quad n(A) = 50$

and L = Set of people having injuries in legs

$\therefore \quad n(L) = 70$

Let us denote the number of injuries contained in the bounded region as shown in the diagram by a, b, c, d, e, f and g, respectively.



Then, $b + e + f + g = 30$ ...(i)

$a + d + e + g = 50$ ...(ii)

$c + d + f + g = 70$ ...(iii)

$d + e + f = 44$ ...(iv)

and $a + b + c + d + e + f + g = 100$ ...(v)

On adding Eqs. (i), (ii) and (iii), we get

$$a + b + c + 2(d + e + f) + 3g = 150$$

$\Rightarrow \quad 100 - d - e - f - g + 2(d + e + f) + 3g = 150$ [from Eq. (v)]

$\Rightarrow \quad d + e + f + 2g = 50$

$\Rightarrow \quad 44 + 2g = 50$ [from Eq. (iv)]

$\therefore \quad g = 3$

Hence, there are three claims involved loss or injury to all the three.

● ***Ex. 31*** *N is the set of natural number. The relation R is defined on $N \times N$ as follows:*

$$(a, b)\, R\, (c, d) \Leftrightarrow ad(b + c) = bc(a + d)$$

Prove that R is an equivalence relation.

Sol. **Reflexive**

Since, $(a, b)\ R\ (a, b) \Leftrightarrow ab(b + a) = ba(a + b), \forall\ a, b \in N$ is true.

Hence, R is reflexive.

Symmetric $(a, b)\ R\ (c, d)$

$\Leftrightarrow \quad ad(b + c) = bc(a + d)$

$\Leftrightarrow \quad bc(a + d) = ad(b + c)$

$\Leftrightarrow \quad cb(d + a) = da(c + b)$

$\Leftrightarrow \quad (c, d)\ R\ (a, b)$

Hence, R is symmetric.

Transitive

Since, $(a, b)\ R\ (c, d) \Leftrightarrow ad(b + c) = bc(a + d)$

$$\Leftrightarrow \quad \frac{b + c}{bc} = \frac{a + d}{ad}$$

$$\Leftrightarrow \quad \frac{1}{c} + \frac{1}{b} = \frac{1}{d} + \frac{1}{a}$$

$$\Leftrightarrow \quad \frac{1}{a} - \frac{1}{b} = \frac{1}{c} - \frac{1}{d}$$

$$\therefore \quad (a, b)\ R\ (c, d) \ \Leftrightarrow \ \frac{1}{a} - \frac{1}{b} = \frac{1}{c} - \frac{1}{d} \qquad \text{...(i)}$$

and similarly $(c, d)\ R\ (e, f) \Leftrightarrow \frac{1}{c} - \frac{1}{d} = \frac{1}{e} - \frac{1}{f}$ ...(ii)

From Eqs. (i) and (ii),

$$(a,b)R(c,d) \text{ and } (c,d)R(e,f) \Leftrightarrow \frac{1}{a} - \frac{1}{b} = \frac{1}{e} - \frac{1}{f}$$

$$\Leftrightarrow (a, b)\ R\ (e, f)$$

So, R is transitive. Hence, R is an equivalence relation.

● ***Ex. 32*** *The sets S and E are defined as given below:*

$S = \{(x, y): |x - 3| < 1 \text{ and } |y - 3| < 1\}$ *and*

$E = \{(x, y): 4x^2 + 9y^2 - 32x - 54y + 109 \le 0\}$.

Show that $S \subset E$.

Sol. **Graph of *S***

$\because\ |x - 3| < 1 \ \Rightarrow -1 < (x - 3) < 1 \ \Rightarrow\ 2 < x < 4$

Similarly, $|y - 3| < 1 \ \Rightarrow\ 2 < y < 4$

So, S consists of all points inside the square (not on $x \neq 2, 4$ and $y \neq 2, 4$) bounded by the lines $x = 2$, $y = 2$, $x = 4$ and $y = 4$.

Graph of *E*

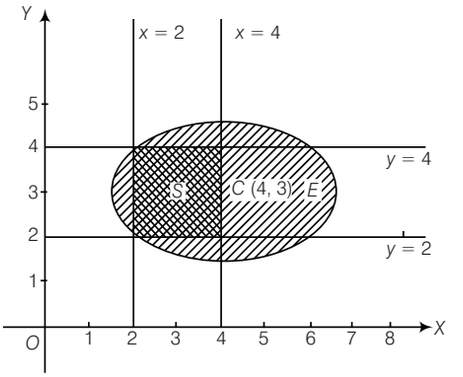
$\because \quad 4x^2 + 9y^2 - 32x - 54y + 109 \le 0$

$\Rightarrow \quad 4(x^2 - 8x) + 9(y^2 - 6y) + 109 \le 0$

$\Rightarrow \quad 4(x - 4)^2 + 9(y - 3)^2 \le 36$

$$\Rightarrow \quad \frac{(x - 4)^2}{3^2} + \frac{(y - 3)^2}{2^2} \le 1$$

So, E consists of all points inside and on the ellipse with centre $(4, 3)$ and semi-major and semi-minor axes are 3 and 2, respectively.



From the above graph, it is evident that the double hatched (which is S) is within the region represented by E. i.e., $S \subset E$

Sets, Relations and Functions Exercise 1 :
Single Option Correct Type Questions

■ This section contains **39 multiple choice questions.** Each question has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct

1. If A and B are two sets, then $A \cap (A \cup B)$ equals
(a) A (b) B
(c) ϕ (d) None of these

2. If R is a relation from a set A to a set B and S is a relation from a set B to set C, then the relation SoR
(a) is from A to C (b) is from C to A
(c) does not exist (d) None of these

3. Let $R = \{(1, 3), (2, 2), (3, 2)\}$ and $S = \{(2, 1), (3, 2), (2, 3)\}$ be two relations on set $A = \{(1, 2, 3)\}$. Then, RoS is equal
(a) $\{(2, 3), (3, 2), (2, 2)\}$ (b) $\{(1, 3), (2, 2), (3, 2), (2, 1), (2, 3)\}$
(c) $\{(3, 2), (1, 3)\}$ (d) $\{(2, 3) (3, 2)\}$

4. If X and Y are two sets, then $X \cap (Y \cap X)'$ equals
(a) X (b) Y
(c) ϕ (d) None of these

5. For real numbers x and y, we write $x\ R\ y \Leftrightarrow x - y + \sqrt{2}$ is an irrational number. Then, the relation R is
(a) reflexive (b) symmetric
(c) transitive (d) None of these

6. Let $f(x) = (x + 1)^2 - 1, (x \geq -1)$. Then, the set $S = \{x : f(x) = f^{-1}(x)\}$ is
(a) $\left\{0, -1, \frac{-3 + i\sqrt{3}}{2}, \frac{-3 - i\sqrt{3}}{2}\right\}, i = \sqrt{-1}$
(b) $\{0, 1, -1\}$ (c) $\{0, -1\}$
(d) empty

7. The number of elements of the power set of a set containing n elements is
(a) 2^{n-1} (b) 2^n (c) $2^n - 1$ (d) 2^{n+1}

8. Which one of the following is not true?
(a) $A - B \subseteq A$ (b) $B' - A' \subseteq A$
(c) $A \subseteq A - B$ (d) $A \cap B' \subseteq A$

9. If $A = \{1, 2, 3\}$ and $B = \{3, 8\}$, then $(A \cup B) \times (A \cap B)$ is
(a) $\{(3, 1), (3, 2), (3, 3), (3, 8)\}$ (b) $\{(1, 3), (2, 3), (3, 3), (8, 3)\}$
(c) $\{(1, 2), (2, 2), (3, 3), (8, 8)\}$ (d) $\{(8, 3), (8, 2), (8, 1), (8, 8)\}$

10. Let $A = \{p, q, r\}$. Which of the following is not an equivalence relation on A?
(a) $R_1 = \{(p, q), (q, r), (p, r), (p, p)\}$
(b) $R_2 = \{(r, q), (r, p), (r, r), (q, q)\}$
(c) $R_3 = \{(p, p), (q, q), (r, r), (p, q)\}$
(d) None of the above

11. Let $A = \{x : x$ is a multiple of 3$\}$ and $B = \{x : x$ is a multiple of 5$\}$, then $A \cap B$ is given by
(a) $\{3, 6, 9\}$ (b) $\{5, 10, 15, 20, \ldots\}$
(c) $\{15, 30, 45, \ldots\}$ (d) None of these

12. Let $A = \{1, 2, 3\}$, $B = \{3, 4\}$ and $C = \{4, 5, 6\}$, then $A \cup (B \cap C)$ is
(a) $\{3\}$ (b) $\{1, 2, 3, 4\}$
(c) $\{1, 2, 5, 6\}$ (d) $\{1, 2, 3, 4, 5, 6\}$

13. Let $A = \{x, y, z\}$, $B = \{u, v, w\}$ and $f : A \to B$ be defined by $f(x) = u$, $f(y) = v, f(z) = w$. Then, f is
(a) surjective but not injective
(b) injective but not surjective
(c) bijective
(d) None of the above

14. If $A = \{2, 4\}$ and $B = \{3, 4, 5\}$, then $(A \cap B) \times (A \cup B)$ is
(a) $\{(2, 2), (3, 4), (4, 2), (5, 4)\}$ (b) $\{(2, 3), (4, 3), (4, 5)\}$
(c) $\{(2, 4), (3, 4), (4, 4), (4, 5)\}$ (d) $\{(4, 2), (4, 3), (4, 4), (5, 4)\}$

15. In the set $X = \{a, b, c, d\}$, which of the following functions in X?
(a) $R_1 = \{(b, a) (a, b), (c, d), (a, c)\}$
(b) $R_2 = \{(a, d) (d, c), (b, b), (c, c)\}$
(c) $R_3 = \{(a, b) (b, c), (c, d), (b, d)\}$
(d) $R_4 = \{(a, a) (b, b), (c, c), (a, d)\}$

16. The composite mapping fog of the map $f : R \to R$, $f(x) = \sin x$ and $g : R \to R$, $g(x) = x^2$ is
(a) $x^2 \sin x$ (b) $(\sin x)^2$ (c) $\sin x^2$ (d) $\sin x / x^2$

17. Which of the following is the empty set?
(a) $\{x : x$ is a real number and $x^2 - 1 = 0\}$
(b) $\{x : x$ is a real number and $x^2 + 1 = 0\}$
(c) $\{x : x$ is a real number and $x^2 - 9 = 0\}$
(d) $\{x : x$ is a real number and $x^2 = x + 2\}$

18. In order that a relation R defined on a non-empty set A is an equivalence relation. It is sufficient, if R
(a) is reflexive
(b) is symmetric
(c) is transitive
(d) possesses all the above three properties

19. Let $A = \{p, q, r, s\}$ and $B = \{1, 2, 3\}$. Which of the following relations from A to B is not a function?
(a) $R_1 = \{(p, 1), (q, 2), (r, 1), (s, 2)\}$
(b) $R_2 = \{(p, 1), (q, 2), (r, 1), (s, 1)\}$
(c) $R_3 = \{(p, 1), (q, 2), (r, 2), (r, 2)\}$
(d) $R_4 = \{(p, 2), (q, 3), (r, 2), (s, 2)\}$

20. n/m means that n is factor of m, then the relation f is
(a) reflexive and symmetric
(b) transitive and symmetric
(c) reflexive, transitive and symmetric
(d) reflexive, transitive and not symmetric

21. The solution of $8x \equiv 6 \pmod{14}$ are
(a) [8],[6] (b) [8],[14]
(c) [6],[13] (d) [8],[14],[16]

22. Let A be a set containing 10 distinct elements, the total number of distinct functions from A to A is
(a) 10! (b) 10^{10} (c) 2^{10} (d) $2^{10} - 1$

23. Let A and B be two non-empty subsets of set X such that A is not a subset of B, then
(a) A is a subset of the complement of B
(b) B is a subset of A
(c) A and B are disjoint
(d) A and the complement of B are non-disjoint

24. f and h are function from $A \to B$, where $A = \{a, b, c, d\}$ and $B = \{s, t, u\}$ defined as follows
$f(a) = t, f(b) = s, f(c) = s$
$f(d) = u, h(a) = s, h(b) = t$
$h(c) = s, h(a) = u, h(d) = u$
Which one of the following statement is true?
(a) f and h are functions
(b) f is a function and h is not a function
(c) f and h are not functions
(d) None of the above

25. Let I be the set of integer and $f : I \to I$ be defined as $f(x) = x^2$, $x \in I$, the function is
(a) bijection (b) injection
(c) surjection (d) None of these

26. Which of the four statements given below is different from others?
(a) $f : A \to B$
(b) $f : x \to f(x)$
(c) f is a mapping of A into B
(d) f is a function of A into B

27. The number of surjections from $A = \{1, 2, \ldots, n\}$, $n \geq 2$ onto $B = \{a, b\}$ is
(a) ${}^nP_2$ (b) $2^n - 2$
(c) $2^n - 1$ (d) None of these

28. Let $f : R \to R$ be defined by $f(x) = 3x - 4$, then $f^{-1}(x)$ is
(a) $\frac{1}{3}(x + 4)$ (b) $\frac{1}{3}x - 4$
(c) $3x + 4$ (d) not defined

29. $f : R \to R$ is a function defined by $f(x) = 10x - 7$. If $g = f^{-1}$, then $g(x)$ equals
(a) $\frac{1}{10x - 7}$ (b) $\frac{1}{10x + 7}$ (c) $\frac{x + 7}{10}$ (d) $\frac{x - 7}{10}$

30. Let R be a relation defined by $R = \{(a, b) : a \geq b\}$, where a and b are real numbers, then R is
(a) reflexive, symmetric and transitive
(b) reflexive, transitive but not symmetric
(c) symmetric, transitive but not reflexive
(d) neither transitive, nor reflexive, not symmetric

31. If sets A and B are defined as
$A = \{(x, y) : y = e^x, x \in R\}$ and $B = \{(x, y) : y = x, x \in R\}$.
(a) $B \subset A$ (b) $A \subset B$
(c) $A \cap B = \phi$ (d) $A \cup B$

32. If $f : A \to B$ is a bijective function, then $f^{-1}of$ is equal to
(a) fof^{-1}
(b) f
(c) f^{-1}
(d) I_A(the identity map of the set A)

33. If $f(y) = \frac{y}{\sqrt{(1 - y^2)}}$, $g(y) = \frac{y}{\sqrt{(1 + y^2)}}$, then $(fog)y$ is equal to
(a) $\frac{y}{\sqrt{(1 - y^2)}}$ (b) $\frac{y}{\sqrt{(1 + y^2)}}$ (c) y (d) $\frac{(1 - y^2)}{\sqrt{(1 - y^2)}}$

34. If $f : R \to R$ is defined by $f(x) = 2x + |x|$, then $f(3x) - f(-x) - 4x$ equals
(a) $f(x)$ (b) $-f(x)$ (c) $f(-x)$ (d) $2f(x)$

35. Let R and S be two non-void relations on a set A. Which of the following statement is false?
(a) R and S are transitive $\Rightarrow R \cup S$ is transitive.
(b) R and S are transitive $\Rightarrow R \cap S$ is symmetric.
(c) R and S are symmetric $\Rightarrow R \cup S$ is symmetric.
(d) R and S are reflexive $\Rightarrow R \cap S$ is reflexive.

36. Let $f : R \to R$, $g : R \to R$ be two functions given by $f(x) = 2x - 3$, $g(x) = x^3 + 5$. Then, $(fog)^{-1}(x)$ is equal to
(a) $\left(\frac{x + 7}{2}\right)^{1/3}$ (b) $\left(x - \frac{7}{2}\right)^{1/3}$ (c) $\left(\frac{x - 2}{7}\right)^{1/3}$ (d) $\left(\frac{x - 7}{2}\right)^{1/3}$

37. If $f(x) = ax + b$ and $g(x) = cx + d$, then $f(g(x)) = g(f(x)) \Leftrightarrow$
(a) $f(a) = g(c)$ (b) $f(b) = g(b)$
(c) $f(d) = g(b)$ (d) $f(c) = g(a)$

38. If $f : R \to R$, $g : R \to R$ be two given functions, then $f(x) = 2\min(f(x) - g(x), 0)$ equals
(a) $f(x) + g(x) - |g(x) - f(x)|$
(b) $f(x) + g(x) + |g(x) - f(x)|$
(c) $f(x) - g(x) + |g(x) - f(x)|$
(d) $f(x) - g(x) - |g(x) - f(x)|$

39. Let $f : R \to R$, $g : R \to R$ be two given functions, such that f is injective and g is surjective, then which of the following is injective?
(a) gof (b) fog (c) gog (d) fof

Sets, Relations and Functions Exercise 2 : More than One Correct Option Type Questions

- This section contains **3 multiple choice questions**. Each question has four choices (a), (b), (c) and (d) out of which **MORE THAN ONE** may be correct.

40. Let L be the set of all straight lines in the Euclidean plane. Two lines l_1 and l_2 are said to be related by the relation R iff l_1 is parallel to l_2. Then, the relation R is

(a) reflexive (b) symmetric
(c) transitive (d) equivalence

41. Let $X = \{1, 2, 3, 4, 5, 6\}$ and $Y = \{1, 3, 5, 7, 9\}$. Which of the following is/are relations from X to Y?

(a) $R_1 = \{(x, y) : y = 2 + x, x \in X, y \in Y\}$
(b) $R_2 = \{(1, 1), (2, 1), (3, 3), (4, 3), (5, 5)\}$
(c) $R_3 = \{(1, 1), (1, 3), (3, 5), (3, 7), (5, 7)\}$
(d) $R_4 = \{(1, 3), (2, 5), (2, 4), (7, 9)\}$

42. Let the function $f : R - \{-b\} \to R - \{1\}$ be defined by $f(x) = \dfrac{x + a}{x + b} (a \neq b)$, then

(a) f is one-one but not onto
(b) f is onto but not one-one
(c) f is both one-one and onto
(d) $f^{-1}(2) = a - 2b$

Sets, Relations and Functions Exercise 3 : Passage Based Questions

- This section contains **2 passages**. Based upon each of the passage **3 multiple choice questions** have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I
(Q. Nos. 43 to 45)

Let f and g be real valued functions defined as

$$f(x) = \begin{cases} 7x^2 + x - 8, & x \le 1 \\ 4x + 5, & 1 < x \le 7 \\ 8x + 3, & x > 7 \end{cases} \quad g(x) = \begin{cases} |x|, & x < -3 \\ 0, & -3 \le x < 2 \\ x^2 + 4, & x \ge 2 \end{cases}$$

43. The value of $(gof)(0) + (fog)(-3)$ is

(a) -8 (b) 0
(c) 8 (d) 16

44. The value of $2(fog)(7) - (gof)(6)$ is

(a) 9 (b) 11
(c) 13 (d) 15

45. The value of $4(gof)(2) - (fog)(9)$ is

(a) 0 (b) 2 (c) 5 (d) 9

Passage II
(Q. Nos. 46 to 48)

R_1 on Z defined by $(a, b) \in R_1$ iff $|a - b| \le 7$, R_2 on Q defined by $(a, b) \in R_2$ iff $ab = 4$ and R_3 on R defined by $(a, b) \in R_3$ iff $a^2 - 4ab + 3ab^2 = 0$.

46. Relation R_1 is

(a) reflexive and symmetric (b) symmetric and transitive
(c) reflexive and transitive (d) equivalence

47. Relation R_2 is

(a) reflexive (b) symmetric
(c) transitive (d) equivalence

48. Relation R_3 is

(a) reflexive (b) symmetric
(c) transitive (d) equivalence

Sets, Relations and Functions Exercise 4 : Single Integer Answer Type Questions

- This section contains **6 questions**. The answer to each question is a **single digit integer**, ranging from 0 to 9 (both inclusive).

49. In a group of 45 students, 22 can speak Hindi only and 12 can speak English only . If $(2\lambda + 1)$ student can speak both Hindi and English, the value of λ is

50. If $A = \left\{x \mid \cos x > -\dfrac{1}{2} \text{ and } 0 \le x \le \pi\right\}$ and $B = \left\{x \mid \sin x > \dfrac{1}{2} \text{ and } \dfrac{\pi}{3} \le x \le \pi\right\}$ and if $\pi\lambda \le A \cap B < \pi\mu$, the value of $(\lambda + \mu)$ is

51. If $S = R$, $A = \{x : -3 \le x < 7\}$ and $B = \{x : 0 < x < 10\}$, the number of positive integers in $A \Delta B$ is

52. Two finite sets have m and n elements. The total number of subsets of the first set is 48 more than the total number of subsets of the second set. The value of $m - n$ is

53. If two sets A and B are having 99 elements in common, the number of elements common to each of the sets $A \times B$ and $B \times A$ are $121\lambda^2$, the value of λ is

Sets, Relations and Functions Exercise 5 : Matching Type Questions

- This section contains **2 questions**. Questions 54 and 55 have three statements (A, B and C) given in **Column I** and four statements (p, q, r and s) in **Column II** and questions 70 and 71 have four statements (A, B, C and D) given in **Column I** and five statements (p, q, r, s and t) in **Column II**. Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II**.

54. The functions defined have domain R.

Column I		Column II	
(A)	$7x + 1$	(p)	onto $[-1, 1]$ but not one-one $[0, \pi]$
(B)	$\cos x$	(q)	one-one on $[0, \pi]$ but not onto R
(C)	$\sin x$	(r)	one-one and onto R
(D)	$1 + \ln x$	(s)	one-one on $(0, \infty)$

55. The domain of the function $f(x)$ is denoted by D_f.

Column I		Column II	
(A)	$f(x) = \sqrt{(3 - x)} + \sin^{-1}\left(\frac{3 - 2x}{5}\right)$, then D_f is	(p)	$\bigcup_{k \in I} [2k\pi, (2k + 1)\pi]$
(B)	$f(x) = \log_{10}(1 - \log_{10}(x^2 - 5x + 16))$, then D_f is	(q)	$[-4, -\pi] \cup [0, \pi]$
(C)	$f(x) = \cos^{-1}\left(\frac{2}{2 + \sin x}\right)$, then D_f is	(r)	$(2, 3)$
(D)	$f(x) = \sqrt{(\sin x)} + \sqrt{(16 - x^2)}$, then D_f is	(s)	$[-1, 3]$

Sets, Relations and Functions Exercise 6 : Statement I and II Type Questions

- **Directions** Question numbers 56 to 59 are Assertion-Reason type questions. Each of these questions contains two statements :
  (a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.
  (b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.
  (c) Statement-1 is true, Statement-2 is false.
  (d) Statement-1 is false, Statement-2 is true.

56. **Statement-1** If a set A has n elements, then the number of binary relations on $A = n^{n^2}$.
Statement-2 Number of possible relations from A to $A = 2^{n^2}$.

57. **Statement-1** If $A = \{x \mid g(x) = 0\}$ and $B = \{x \mid f(x) = 0\}$, then $A \cap B$ be a root of $\{f(x)\}^2 + \{g(x)\}^2 = 0$.
Statement-2 $x \in A \cap B \Rightarrow x \in A$ or $x \in B$.

58. **Statement-1** $P(A) \cap P(B) = P(A \cap B)$, where $P(A)$ is power set of set A.
Statement-2 $P(A) \cup P(B) = P(A \cup B)$

59. **Statement-1** If Sets A and B have three and six elements respectively, then the minimum number of elements in $A \cup B$ is 6.
Statement-2 $A \cap B = 3$.

Sets, Relations and Functions Exercise 7 : Subjective Type Questions

- In this section, there are **15 subjective** questions.

60. Let $A = \{x : x \text{ is a natural number}\}$,
$B = \{x : x \text{ is an even natural number}\}$,
$C = \{x : x \text{ is an odd natural number}\}$
and $D = \{x : x \text{ is a prime number}\}$.
Find
(i) $A \cap B$ (ii) $A \cap C$
(iii) $B \cap D$ (iv) $C \cap D$

61. Let U be the set of all people and $M = \{\text{Males}\}$,
$S = \{\text{College students}\}$,
$T = \{\text{Teenagers}\}$, $W = \{\text{People having height more than five feet}\}$.
Express each of the following in the notation of set theory.
(i) College student having heights more than five feet.
(ii) People who are not teenagers and have their height less five feet.
(iii) All people who are neither males nor teenagers nor college students.

62. The set X consists of all points within and on the unit circle $x^2 + y^2 = 1$, whereas the set Y consists of all points on and inside the rectangular boundary $x = 0$, $x = 1$, $y = -1$ and $y = 1$. Determine $X \cup Y$ and $X \cap Y$. Illustrate your answer by diagrams.

63. In a group of children, 35 play football out of which 20 play football only, 22 play hockey; 25 play cricket out of which 11 play cricket only. Out of these 7 play cricket and football but not hockey, 3 play football and hockey but not cricket and 12 play football and cricket both.
How many play all the three games? How many play cricket and hockey but not football, how many play hockey only? What is the total number of children in the group?

64. Of the members of three athletic team in a certain school, 21 are on the basketball team, 26 on the hockey team and 29 on the football team. 14 play hockey and basketball, 15 play hockey and football, 12 play football and basketball and 8 play all the three games. How many members are there in all?

65. In a survey of 200 students of higher secondary school, it was found that 120 studied Mathematics; 90 studies Physics and 70 studied Chemistry; 40 studied Mathematics and Physics; 3 studied Physics and Chemistry; 50 studied Chemistry and Mathematics and 20 studied none of these subjects. Find the number of students who studied all the three subjects.

66. In a survey of population of 450 people, it is found that 205 can speak English, 210 can speak Hindi and 120 people can speak Tamil. If 100 people can speak both Hindi and English; 80 people can speak both English and Tamil, 35 people can speak Hindi and Tamil and 20 people can speak all the three languages, find the number of people who can speak English but not a Hindi or Tamil. Find also the number of people who can speak neither English nor Hindi nor Tamil.

67. A group of 123 workers went to a canteen for cold drinks, ice-cream and tea, 42 workers took ice-cream, 36 tea and 30 cold drinks. 15 workers purchased ice-cream and tea, 10 ice-cream and cold drinks, and 4 cold drinks and tea but not ice-cream, 11 took ice-cream and tea but not cold drinks. Determine how many workers did not purchase anything?

68. Let n be a fixed positive integer. Define a relation R on I (the set of all integers) as follows:
$a\ R\ b$ iff $n|(a - b)$ i.e., iff $(a - b)$ is divisible by n. Show that R is an equivalence relations on I.

69. N is the set of positive integers. The relation R is defined on $N \times N$ as follows:
$$(a, b)\ R\ (c, d) \Leftrightarrow ad = bc$$
Prove that R is an equivalence relation.

70. The following relations are defined on the set of real numbers.
(i) $a\ R\ b \Leftrightarrow |a - b| > 0$
(ii) $a\ R\ b \Leftrightarrow |a| = |b|$
(iii) $a\ R\ b \Leftrightarrow |a| \geq |b|$
(iv) $a\ R\ b \Leftrightarrow 1 + ab > 0$
(v) $a\ R\ b \Leftrightarrow |a| \leq b$
Find whether these relations are reflexive, symmetric or transitive.

71. Let $A = \{x : -1 \leq x \leq 1\} = B$ for each of the following functions from A to B. Find whether it is surjective, injective or bijective
(i) $f(x) = \dfrac{x}{2}$
(ii) $g(x) = |x|$
(iii) $h(x) = x|x|$
(iv) $k(x) = x^2$
(v) $l(x) = \sin \pi x$

72. If the functions f and g defined from the set of real numbers R to R such that $f(x) = e^x$ and $g(x) = 3x - 2$, then find functions fog and gof. Also, find the domain of the functions $(fog)^{-1}$ and $(gof)^{-1}$.

73. If $f(x) = \dfrac{x^2 - x}{x^2 + 2x}$, then find the domain and range of f. Show that f is one-one. Also, find the function $\dfrac{d(f^{-1}(x))}{dx}$ and its domain.

74. If the functions f, g and h are defined from the set of real numbers R to R such that

$$f(x) = x^2 - 1,\ g(x) = \sqrt{(x^2 + 1)},$$

$$h(x) = \begin{cases} 0, & \text{if } x \le 0 \\ x, & \text{if } x \ge 0 \end{cases}.$$

Then, find the composite function $hofog$ and determine whether the function fog is invertible and h is the identity function.

Sets, Relations and Functions Exercise 8 : Questions Asked in Previous 13 Year's Exam

■ This section contains questions asked in **IIT-JEE, AIEEE, JEE Main & JEE Advanced** from year **2005** to year **2017**.

75. Let $R = \{(3,3), (6,6), (9,9), (12,12), (6,12), (3,9)$ be a relation on the set $A = \{3, 6, 9, 12\}$.

The relation is **[AIEEE 2005, 3M]**

(a) an equivalence relation
(b) reflexive and symmetric only
(c) reflexive and transitive only
(d) reflexive only

76. Let W denotes the words in the English dictionary. Define the relation R by $R = \{(x, y) \in W \times W\}$ the words x and y have atleast one letter in common, then R is **[AIEEE 2006, 3M]**

(a) not reflexive, symmetric and transitive
(b) reflexive, symmetric and not transitive
(c) reflexive, symmetric and transitive
(d) reflexive, not symmetric and transitive

77. Let R be the real line, consider the following subsets of the plane $R \times R$ such that **[AIEEE 2008, 3M]**

$S = \{(x, y) : y = x + 1 \text{ and } 0 < x < 2\}$

$T = \{(x, y) : x - y \text{ is an integer}\}$.

Which one of the following is true?

(a) Both S and T are equivalence relations on R
(b) S is an equivalence relation on R but T is not
(c) T is an equivalence relation on R but S is not
(d) Neither S nor T is an equivalence relations on R

78. If A, B and C are three sets such that $A \cap B = A \cap C$ and $A \cup B = A \cup C$, then **[AIEEE 2009, 4M]**

(a) $A \cap B = \phi$ (b) $A = B$
(c) $A = C$ (d) $B = C$

79. Let $S = \{1, 2, 3, 4\}$. The total number of unordered pair of disjoint subsets of S is equal to **[IIT-JEE 2010, 5M]**

(a) 25 (b) 34 (c) 42 (d) 41

80. Consider the following relations.

$R = \{(x, y) \mid x, y$ are real numbers and $x = wy$ for some rational number $w\}$

$S = \left\{\left(\dfrac{m}{n}, \dfrac{p}{q}\right) \middle| m, n, p \text{ and } q \text{ are integers such that } n, q \ne 0 \text{ and } qm = pn\right\}$, then **[AIEEE 2010, 4M]**

(a) neither R nor S is an equivalence relation
(b) S is an equivalence relation but R is not an equivalence relation
(c) R and S both are equivalence relations
(d) R is an equivalence relation but S is not an equivalence relation

81. Let $P = \{\theta : \sin\theta - \cos\theta = \sqrt{2}\cos\theta\}$ and $Q = \{\theta : \sin\theta + \cos\theta = \sqrt{2}\sin\theta\}$ be two sets. Then, **[IIT-JEE 2011, 3M]**

(a) $P \subset Q$ and $A - P \ne \phi$ (b) $Q \not\subset P$
(c) $P \not\subset Q$ (d) $P = Q$

82. Let $f(x) = x^2$ and $g(x) = \sin x$ for all $x \in R$. Then, the set of all x satisfying $(fogogof)(x) = (gogof)(x)$, where $(fog)(x) = f(g(x))$ is **[IIT-JEE 2011, 3M]**

(a) $\pm\sqrt{n\pi}$, $n \in \{0, 1, 2, \ldots\}$

(b) $\pm\sqrt{n\pi}$, $n \in \{1, 2, 3, \ldots\}$

(c) $\dfrac{\pi}{2} + 2n\pi$, $n \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

(d) $2n\pi$, $n \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

83. Let R be the set of real numbers.

Statement-1 $A = \{(x, y) \in R \times R : y - x$ is an integer$\}$ is an equivalence relation on R.

Statement-2 $B = \{(x, y) \in R \times R : x = \alpha y$ for some rational number $\alpha\}$ is an equivalence relation on R. **[AIEEE 2011, 4M]**

(a) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is false
(c) Statement-1 is false, Statement-2 is true
(d) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

84. Let A and B be two sets containing 2 elements and 4 elements, respectively. The number of subsets of $A \times B$ having 3 or more elements, is **[JEE Main 2013, 4M]**
(a) 220 (b) 219 (c) 211 (d) 256

85. If $X = \{4^n - 3n - 1 : n \in N\}$ and $Y = \{9(n-1) : n = N\}$, where N is the set of natural numbers, then $X \cup Y$ is equal to **[JEE Main 2014, 4M]**
(a) X (b) Y (c) N (d) $Y - X$

86. Let A and B be two sets containing four and two elements, respectively. Then, the number of subsets of the set $A \times B$, each having atleast three elements is **[JEE Main 2015, 4M]**
(a) 275 (b) 510 (c) 219 (d) 256

Answers

Exercise for Session 1

1. (b) **2.** (d) **3.** (c) **4.** (b) **5.** (a) **6.** (c)
7. (d) **8.** (c) **9.** (a) **10.** (c) **11.** (d)

Exercise for Session 2

1. (c) **2.** (b) **3.** (b) **4.** (c) **5.** (c) **6.** (c)
7. (b) **8.** (a) **9.** (a) **10.** (c)

Exercise for Session 3

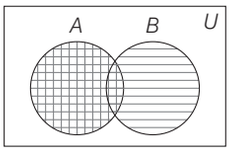
1. (b) **2.** (c) **3.** (b) **4.** (b) **5.** (b) **6.** (b)
7. (d) **8.** (b) **9.** (b) **10.** (a)

Chapter Exerises

1. (a) **2.** (a) **3.** (a) **4.** (d) **5.** (a) **6.** (c)
7. (b) **8.** (c) **9.** (b) **10.** (d) **11.** (c) **12.** (b)
13. (c) **14.** (d) **15.** (b) **16.** (c) **17.** (b) **18.** (d)
19. (c) **20.** (d) **21.** (c) **22.** (b) **23.** (d) **24.** (b)
25. (d) **26.** (b) **27.** (b) **28.** (a) **29.** (c) **30.** (b)
31. (b) **32.** (d) **33.** (c) **34.** (d) **35.** (a) **36.** (d)
37. (c) **38.** (d) **39.** (d) **40.** (a,b,c,d) **41.** (a,b,c)
42. (c,d) **43.** (b) **44.** (a) **45.** (d) **46.** (a) **47.** (b)
48. (a) **49.** (5) **50.** (1) **51.** (3) **52.** (2) **53.** (9)
54. (A) $\to$ (r); (B) $\to$ (q); (C) $\to$ (p); (D) $\to$ (s)
55. (A) $\to$ (s); (B) $\to$ (r); (C) $\to$ (p); (D) $\to$ (q)
56. (b) **57.** (c) **58.** (c) **59.** (a)
60. (i) B (ii) C (iii) $\{2\}$ (iv) $\{x : x$ is an odd prime, natural number$\}$
61. (i) $S \cap W$ (ii) $T' \cap W'$ (iii) $(M \cup T \cup S)'$
62. $X \cup Y = \{(x, y) : x^2 + y^2 \le 1$ or $0 \le x \le 1$ and $-1 \le y \le 1\}$
$X \cap Y = \{(x, y) : x^2 + y^2 \le 1$ and $x \ge 0\}$
63. 5, 2, 12, 60 **64.** 43 **65.** 20 **66.** 45, 110
67. 44
70. (i) Not reflexive, symmetric, not transitive
(ii) Reflexive, symmetric, transitive
(iii) Reflexive, not symmetric, transitive
(iv) Reflexive, symmetric, not transitive
(v) Not reflexive, not symmetric, transitive
71. (i) Injective (ii) Injective
(iii) Bijective (iv) Not injective
(v) Surjective
72. $(fog)x = e^{3x-2}$; $x \in R$ $(gof)x = 3e^x - 2$; $x \in R$
Domain of $(fog)^{-1}(x) = (0, \infty)$.
Domain of $(gof)^{-1}(x) = (-2, \infty)$.
73. $\frac{3}{(1-x)^2}$, $R - \{1\}$
$\frac{df^{-1}(x)}{dx} = \frac{3}{(1-x)^2}$, Domain of $\frac{df^{-1}(x)}{dx} = R - \{1\}$
74. $(hofog)\,x = \begin{cases} 0, & x^2 \le 0 \\ x^2, & x^2 \ge 0 \end{cases}$, h is not an identity function and fog is not invertible.
75. (c) **76.** (b) **77.** (c) **78.** (d) **79.** (d) **80.** (b)
81. (d) **82.** (a) **83.** (a) **84.** (b) **85.** (b) **86.** (c)

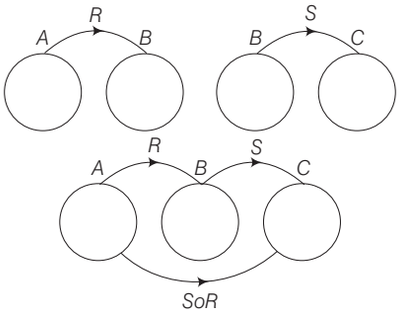
Solutions

1. By Venn diagram,



It is clear that $A \cap (A \cup B) = A$

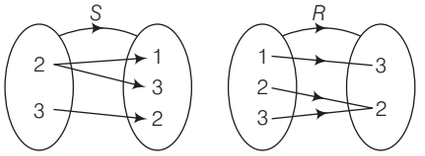
2.



SoR is the relation from A to C.

3. $R = \{(1, 3), (2, 2), (3, 2)\}$

$S = \{(2, 1), (3, 2), (2, 3)\}$



$RoS = \{(2, 3), (2, 2), (3, 2)\}$

4. $X \cap (Y \cap X)' = X \cap (Y' \cup X')$

$= (X \cap Y') \cup (X \cap X')$

$= (X \cap Y') \cup \phi = X \cap Y'$

$= X - (X \cap Y)$

5. $xRy \Leftrightarrow (x - y + \sqrt{2})$ is an irrational number.

Let $(x, x) \in R$.

Then, $x - x + \sqrt{2} = \sqrt{2}$ which is an irrational number.

$\therefore \quad x\ R\ x, \forall\ x \in R$

$\therefore$ R is an reflexive relation.

$x\ R\ y \Rightarrow (x - y + \sqrt{2})$ is an irrational number.

$\Rightarrow\ -(y - x - \sqrt{2})$ is an irrational number.

$\Rightarrow (y - x - \sqrt{2})$ is an irrational number.

$y\ R\ x \Rightarrow (y - x + \sqrt{2})$ is an irrational number.

So, $xRy \not\Rightarrow yRx \therefore R$ is not a symmetric relation.

Let $(1, 2) \in R$, then $(1 - 2 + \sqrt{2})$ is an irrational number.

$\Rightarrow \quad (\sqrt{2} - 1)$ is an irrational number.

and $(2, 3) \in R$, then $(2 \ -3 + \sqrt{2})$ is an irrational number.

$\Rightarrow \quad (\sqrt{2} - 1)$ is an irrational number.

$(1, 3) \in R \Rightarrow (1 - 3 + \sqrt{2})$ is an irrational number.

$\Rightarrow \quad (\sqrt{2} - 2)$ is an irrational number.

So, $(1, 2) \in R$ and $(2, 3) \in R \not\Rightarrow (1, 3) \in R$ (by any way)

$\therefore$ R is not transitive relation.

6. $f(x) = (x + 1)^2 - 1 \qquad [\because x \geq 1]$

$= x^2 + 1 + 2x - 1 = x^2 + 2x$

$S = \{x : f(x) \equiv f^{-1}(x)\}$

S is the set of point of intersection of $(y = x)$ and tf.

Now, solve $y = x$ and $f(x) = x^2 + 2x$

$$x^2 + 2x = x$$

$$x^2 + x = 0$$

$$x(x + 1) = 0$$

$$x = 0 \text{ or } x = -1$$

7. Let set A contains n elements.

Power set of A is the set of all subsets.

$\therefore$ Number of subsets of $A = {}^nC_o + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n$

$\therefore$ Power set of A contains 2^n elements.

8. By Venn diagram, it is clear that

$A - B \subseteq A$ and $B' - A' \subseteq A$ and $A \cap B' \subseteq A$

but $A \not\subseteq A - B$

9. $A = \{1, 2, 3\}$

$$B = \{3, 8\}$$

$$A \cup B = \{1, 2, 3, 8\}$$

$$A \cap B = \{3\}$$

$$(A \cup B) \times (A \cap B) = \{1, 2, 3, 8\} \times \{3\}$$

$$= \{(1, 3), (2, 3), (3, 3) (8, 3)\}$$

10. $A = \{p, q, r\}$

$R_1 = \{(p, q), (q,r), (p, r), (p, p)\}$

$(q, q) \notin R_1$, so R_1 is not reflexive relation.

So, R_1 is not an equivalence relation.

$$R_2 = \{(r, q), (r, p), (r, r), (q, q)\}$$

Here, $(p, p) \notin R_2$, so R_2 is not reflexive relation.

So, R_2 is not an equivalence relation.

$$R_3 = \{(p, p), (q, q), (r, r), (p, q)\}$$

R_3 is an reflexive relation.

$(p, a) \in R_3$ but $(q, p) \notin R_3$

R_3 is not symmetric relation.

So, R_3 is not equivalence relation.

11. $A = \{x : x$ is a multiple of 3$\}$

$A = \{x : x = 3m, m \in N\}$

$B = \{x : x$ is a multiple of 5$\}$

$B = \{x : x = 5n, n \in N\}$

$A \cap B = \{x : x$ is a multiple of both 3 and 5$\}$

$$= \{15, 30, 45, \ldots\}$$

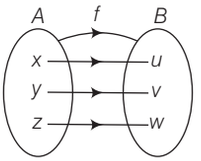
12. $A = \{1, 2, 3\}, B = \{3, 4\}, C = \{4, 5, 6\}$

$\Rightarrow \quad B \cap C = \{4\}$

and $A \cup (B \cap C) = \{1, 2, 3, 4\}$

13. $A = \{x, y, z\}, B = \{u, v, w\}$

Now, $f : A \to B$



f is one-one and f is onto.

14. $A = \{2, 4\}$

$B = \{3, 4, 5\}$

$A \cap B = \{4\}$

$A \cup B = \{2, 3, 4, 5\}$

$(A \cap B) \times (A \cup B) = \{(4, 2), (4, 3), (4, 4), (4, 5)\}$

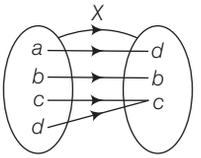
15. $X = \{a, b, c, d\}$

$R_1 = \{(b, a), (a, b), (c, d), (a, c)\}$

$(a, b) \in R_1$ and $(a, c) \in R$,

$\therefore R_1$ is not a function.

$R_2 = \{(a, d), (d, c), (b, b), (c, c)\}$



Hence, R_2 is a function.

16. $f : R \to R$

$\Rightarrow \quad f(x) = \sin x$ and $g : R \to R$

$\Rightarrow \quad g(x) = x^2$

Range of g is $R^+ \cup \{0\}$, which is the subset of domain of f.

$\therefore$ Composition of fog is possible.

$$fog = f(g(x)) = f(x^2)$$
$$= \sin x^2$$

17. $x^2 - 1 = 0$

$\Rightarrow \quad x = -1, 1$

$\therefore$ x is real, q $\quad x^2 + 1 = 0$

$\Rightarrow \quad x = \pm i$

$\therefore$ x is not real, $\quad x^2 - 9 = 0$

$\Rightarrow \quad x = \pm 3$

$\therefore$ x is real $\quad x^2 - x - 2 = 0$

$\Rightarrow \quad x = 2, -1$

$\therefore$ x is real.

18. By definition for equivalent relation.

R should be reflexive, symmetric, transitive.

19. $\because$ x - coordinates of two brackets are same.

20. $\frac{n}{m}$ means that n is a factor of m.

So, f is reflexive.

$\because$ A number is a factor of itself.

Now, if n is a factor of m, then m is not a factor of n

$\therefore$ f is not symmetric. Let n is a factor of m and m is a factor of s, then it is true that n is a factor of s.

$\therefore$ f is transitive.

21. $\lambda = \frac{8x - 6}{14}$, where $\lambda \in I_+$

$\therefore \quad 8x = 14\lambda + 6 \Rightarrow x = \frac{14\lambda + 6}{8}$

$\Rightarrow \quad x = \frac{7\lambda + 3}{4} = \lambda + \frac{3}{4}(\lambda + 1)$, when $\lambda \in I$

and here greatest common divisor of 8 and 14 is 2, so there are two required solutions.

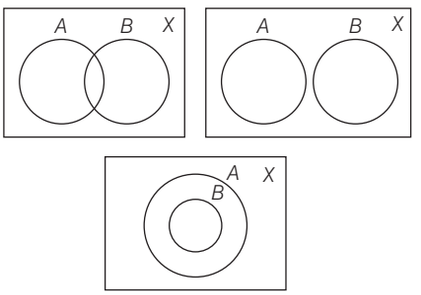
for $\lambda = 3$ and $\lambda = 7$, $x = 6, 13$ or $x = [6][13]$

22. $n(A) = 10$

Total number of distinc Functions from A to $A = 10^{10}$.

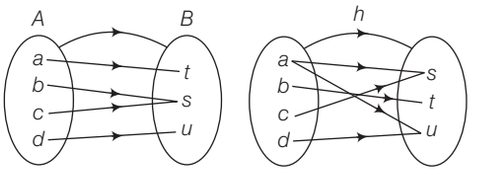
23. $A \subseteq X$ and $B \subseteq X$ and $A \subseteq B$

In all 3 possible cases,



24. $A = \{a, b, c, d\}$

$B = \{s, t, u\}$



It is clear that f is a function.

But in relation h, a have h image s and u.

So, h is not a function.

25. $f(x) = x^2, x \in Z$

$f(1) = 1$

$f(-1) = 1$

$\therefore$ f is not one-one

Range of f is set of whole number.

Which is a subset of Z.

$\therefore f$ is not onto.

26. It is obvious.

27. $A = \{1, 2, ..., n\}\ n \geq 2$

$B = \{a, b\}$

Number of into functions from A to $B = 2$

Total Numer of functions from A to $B = [n(B)]^{n(A)} = 2^n$

$\therefore$ Total Number of onto functions from A to $B = 2^n - 2$

28. $f : R \to R$

$\Rightarrow \qquad f(x) = 3x - 4$

f is one-one onto function.

$\therefore$ Let $\qquad y = 3x - 4$

$$x = \frac{y+4}{3}$$

Replace x by $y \Rightarrow y = \frac{x+4}{3} = f^{-1}(x)$

29. $f : R \to R$

$\Rightarrow \qquad f(x) = 10x - 7$

It is clear that f is one-one and onto.

$\therefore$ Let $\qquad y = 10x - 7$

$\therefore \qquad x = \frac{y+7}{10} = f^{-1}(y)$

$$g(x) = f^{-1}(x) = \frac{x+7}{10}$$

30. $R = \{(a, b) : a \geq b\}$

We know that, $a \geq a$

$\therefore \qquad (a, a) \in R, \forall\ a \in R$

R is a reflexive relation.

Let $(a, b) \in R$

$\Rightarrow \qquad a \geq b$

$\not\Rightarrow \qquad b \leq a$

$\not\Rightarrow \qquad (b, a) \in R$

So, R is not symmetric relation.

Now, let $(a, b) \in R$ and $(b, c) \in R$.

$\Rightarrow \qquad a \geq b$ and $b \geq c$

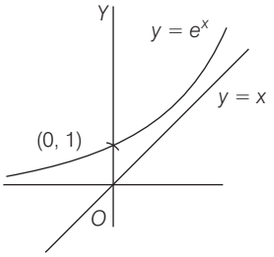
$\Rightarrow \qquad a \geq c$

$\Rightarrow \qquad (a, c) \in R$

$\therefore$ R is a transitive relation.

31. $A = \{(x, y) : y = e^x,\ x \in R\}$

$B = \{(x, y) : y = x,\ x \in R\}$

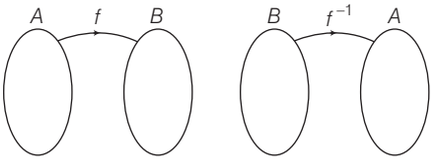


$\therefore \qquad A \subset B$

32. $f : A \to B$

f is a function, then f^{-1} is also a bijective function.

Composite function $(f^{-1}\ of) = I_A$



33. $f(y) = \frac{y}{\sqrt{(1-y^2)}},\ g(y) = \frac{y}{\sqrt{(1+y^2)}}$

and $\qquad (fog)y = f(g(y)) = f\left(\frac{y}{\sqrt{(1+y^2)}}\right)$

$$= \frac{\frac{y}{\sqrt{(1+y^2)}}}{\sqrt{1 - \frac{y^2}{(1+y^2)}}} = \frac{\frac{y}{\sqrt{(1+y^2)}}}{\frac{1}{\sqrt{(1+y^2)}}} = y$$

34. $f : R \to R$

$$f(x) = 2x + |x|$$

When $x \geq 0$, then $f(x) = 2x + x = 3x$

When $x < 0$, then $f(x) = 2x - x = x$

Now, when $x \geq 0$

$f(3x) - f(-x) - 4x = 3(3x) - (-x) - 4x = 9x + x - 4x$

$= 6x \qquad [\because x \geq 0]$

$= 2(3x) = 2f(x) \qquad [\therefore -x \leq 0]$

When $x < 0$,

$f(3x) - f(-x) - 4x = 3x - (-3x) - 4x = 2x = 2f(x)$

35. Let $A = \{1, 2, 3\}$, $R = \{(1, 1)\ (1, 2)\}$

and $S = \{(2, 2), (2, 3)\}$

be the transitive relation on A.

Then, $R \cup S = \{(1, 1,)\ (1, 2)(2, 2)\ (2, 3)\}$

$R \cup S$ is not transitive, because $(1, 2) \in R \cup S$

and $(2, 3) \in R \cup S$ but $(1, 3) \notin R \cup S$.

36. $f : R \to R$

$g : R \to R$

$f(x) = 2x - 3$

$g(x) = x^3 + 5$

$\Rightarrow \qquad (fog)(x) = f(g(x)) = f(x^3 + 5) = 2(x^3 + 5) - 3$

$= 2x^3 + 7$

Now, let $\ y = 2x^3 + 7$

$2x^3 = y - 7$

$$x = \left(\frac{y-7}{2}\right)^{1/3}$$

Replacing x by y, we get

$$y = \left(\frac{x-7}{2}\right)^{1/3}$$

$\therefore \qquad (fog)^{-1}(x) = \left(\frac{x-7}{2}\right)^{1/3}$

37. $f(x) = ax + b$

$g(x) = cx + d$

$f(g(x)) = g(f(x))$

$f(cx + d) = g(ax + b)$

$a(cx + d) + b = c(ax + b) + d$

$acx + ad + b = acx + bc + d$

$ad + b = cb + d$

$f(d) = g(b)$

38. $f : R \to R,\ g : R \to R$

$f(x) = 2 \min f(x) - g(x), 0)$

Let $f(x) - g(x) > 0$, then

$F(x) = f(x) - g(x) - |f(x) - g(x)|$ and $f(x) - g(x) < 0$, then

$F(x) = 2[f(x) - g(x)] = [f(x) - g(x)] - |f(x) - g(x)|$

39. $f : R \to R$ and $g : R \to R$ such that f is injective and $+g$ is surjective.

Then, g may be one-one or many-one.

If g is one-one, then gof is one-one.

fog is one-one

gog is one-one

But if g is many-one, then gof is not one-one.

fog is not one-one.

gog is many-one

Now, fof is one-one

40. Relation R on the set of all straight lines in the plane is of parallel line.

A line is parallel to itself. So, R is reflexive.

If l_1 is parallel to l_2, then l_2 is parallel to l_1.

$\therefore$ R is symmetric relation. $[l_1, l_2 \in L]$

Let $l_1, l_2, l_3 \in L$

l_1 is parallel to l_2 and l_2 is parallel to l_3.

Then, l_1 is parallel to l_3.

$\therefore$ R is transitive relation.

So, R is equivalence relation.

41. $X = \{1, 2, 3, 4, 5\}$

$Y = \{1, 3, 5, 7, 9\}$

(a) $R_1 = \{(x, y) : y = 2 + x,\ x \in X,\ y \in Y\}$

$x = 1$	$y = 2$
$x = 2$	$y = 4$
$x = 3$	$y = 5$
$x = 4$	$y = 6$
$x = 5$	$y = 7$

So, R_1 is a relation from X to Y.

(b) $R_2 = \{(1, 1), (2, 1), (3, 3)\ (4, 3), (5, 5)\}$

$R_2 \subseteq X \times Y$

(c) $R_3 = \{(1, 1), (1, 3), (3, 5)\ (5, 7)\}$

$R_3 \subseteq X \times Y$

(d) $R_4 \not\subseteq X \times Y$

42. $f : R - \{-b\} \to R - \{1\}$

$$f(x) = \frac{x + a}{x + b} \qquad [a \neq b]$$

Let $x_1, x_2 \in D_f$

$$f(x_1) = f(x_2)$$

$$\Rightarrow \quad \frac{x_1 + a}{x_1 + b} = \frac{x_2 + a}{x_2 + b}$$

$\Rightarrow x_1x_2 + bx_1 + ax_2 + ab = x_1x_2 + ax_1 + bx_2 + ab$

$\Rightarrow \quad b(x_1 - x_2) = a(x_1 - x_2)$

$\Rightarrow \quad (x_1 - x_2)(b - a) = 0$

$\Rightarrow \quad x_1 = x_2 \qquad [\because a \neq b]$

$\therefore f$ is one-one function.

Now, let $\quad y = \dfrac{x + a}{x + b}$

$xy + by = x + a$

$x(y - 1) = a - by$

$x = \dfrac{a - by}{y - 1}$ and $f^{-1}(y) = \dfrac{a - by}{y - 1}$

$\because \quad y \in R - \{1\}$

$\therefore$ x is defined, $\forall y \in R - \{1\}$

$$f^{-1}(2) = \frac{a - 2b}{2 - 1} = a - 2b$$

Sol. (Q. Nos. 43 to 45)

43. $(gof)(0) = g(f(0)) = g(7(0)^2 + 0 - 8)$

$= g(-8) = |-8| = 8$

and $\quad (fog)(-3) = f(g(-3)) = f(0) = 7(0)^2 + 0 - 8 = -8$

$\therefore \quad (gof)(0) + (fog)(-3) = -8 + 8 = 0$

44. $(fog)(7) = f(g(7)) = f(7^2 + 4) = f(53)$

$= 8(53) + 3 = 427$

and $\quad (gof)(6) = g(f(6)) = g(4 \times 6 + 5) + g(29)$

$= (29)^2 + 4 = 845$

$\therefore \quad 2(fog)(7) - (gof)(6) = 2 \times 427 - 845 = 9$

45. $(gof)(2) = g(f(2)) = g(4 \times 2 + 5) = g(13)$

$= (13)^2 + 4 = 173$

and $\quad (fog)(g) = f(g(9)) = f(9^2 + 4) = f(85)$

$= 8 \times 85 + 3 = 683$

$\therefore \quad 4(gof)(2) - (fog)(9) = 4 \times 173 - 683 = 9$

Sol. (Q. Nos. 46-48)

46. We have, $(a, b) \in R_1$ iff $|a - b| \leq 7$, where $a, b \in z$

Reflexivity Let $a \in z$

$\Rightarrow \quad a - a = 0$

$\Rightarrow \quad |a - a| \leq 7$

$\Rightarrow \quad 0 \leq 7$

$\Rightarrow \quad (a, a) \in R_1$

$\therefore$The relation R_1 is reflexive.

Symmetry

$(a, b) \in R_1$

$\Rightarrow \quad |a - b| \leq 7 \Rightarrow |-(b - a)| \leq 7$

$\Rightarrow \quad |b - a| \leq 7 \Rightarrow (b, a) \in R_1$

$\therefore$The relation R_1 is symmetric.

Transitivity We have $(2, 6), (6, 10) \in R_1$ because

$|2 - 6| = 4 \leq 7$ and $|6 - 10| = 4 \leq 7$

Also, $\quad |2 - 10| = 8 \not\leq 7$

$\therefore \quad (2, 10) \notin R_1$

Hence, the relation R_1 is not transitive.

47. We have $(a, b) \in R_2$ iff $ab = 4$, where $a, b \in Q$

Reflexivity $5 \in Q$ and $(5)(5) = 25 \neq 4$

$\therefore \quad (5, 5) \notin R_2$

The relation R_2 is not reflexive.

Symmetry

$(a, b) \in R_2$

$\Rightarrow \quad ab = 4 \Rightarrow ba = 4$

$\Rightarrow \quad (b, a) \in R_2$

$\therefore$ The relation R_2 is symmetric.

Transitivity We have $\left(8, \frac{1}{2}\right), \left(\frac{1}{2}, 8\right) \in R_2$ because

$$8\left(\frac{1}{2}\right) = 4 \quad \text{and} \quad \left(\frac{1}{2}\right)(8) = 4$$

Also, $8(8) = 64 \neq 4$

$\therefore \quad (8, 8) \notin R_2$

$\therefore$ The relation R_2 is not transitive.

48. We have, $(a, b) \in R_3$ iff $a^2 - 4ab + 3b^2 = 0$ where $a, b \in R$

Reflexivity

$\therefore \quad a^2 - 4a \cdot a + 3d^2 = 4a^2 - 4a^2 = 0$

$\therefore \quad (a, a) \in R_3$

$\therefore$ The relation R_3 is reflexive.

Symmetry

$(a, b) \in R_3$

$\Rightarrow \quad a^2 - 4ab + 3b^2 = 0$, we get $a = b$ and $a = 3b$

and $(b, a) \in R_3$

$\Rightarrow \quad b^2 - 4ab + 3a^2 = 0$

we get $b = a$ and $b = 3a$

$\therefore \quad (a, b) \in R_3 \Rightarrow (b, a) \in R_3$

$\therefore$ The relation R_3 is not symmetric.

Transitivity We have $(3, 1), \left(1, \frac{1}{3}\right) \in R_3$

because $(3)^2 - 4(3)(1) + 3(1)^2 = 9 - 12 + 3 = 0$

and $(1)^2 - 4(1)\left(\frac{1}{3}\right) + 3\left(\frac{1}{3}\right)^2 = 1 - \frac{4}{3} + \frac{1}{3} = 0$

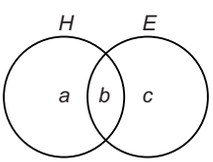
Also, $\left(3, \frac{1}{3}\right) \notin R_3$, because

$$(3)^2 - 4 \cdot (3)\left(\frac{1}{3}\right) + 3\left(\frac{1}{3}\right)^2 = 9 - 4 + \frac{1}{3} = \frac{16}{3} \neq 0$$

$\therefore$ The relation R_3 is not transitive.

49. Given, $a = 22$,

$c = 12$



and $a + b + c = 45$

$\Rightarrow \quad 22 + b + 12 = 45$

$\therefore \quad b = 11 = 2\lambda + 1$

$\Rightarrow \quad \lambda = 5$

50. $\therefore \quad \cos x > -\frac{1}{2}$ and $0 \leq x \leq \pi$

$\Rightarrow \quad -\frac{2\pi}{3} < x < \frac{2\pi}{3}$ and $0 \leq x \leq \pi$

$\Rightarrow \quad 0 \leq x < \frac{2\pi}{3}$

$\therefore \quad A = \left[0, \frac{2\pi}{3}\right]$

Again, $\sin x > \frac{1}{2}$ and $\frac{\pi}{3} \leq x \leq \pi$

$\Rightarrow \quad \frac{\pi}{6} < x < \frac{5\pi}{6}$ and $\frac{\pi}{3} \leq x \leq \pi$

$\Rightarrow \quad \frac{\pi}{3} \leq x < \frac{5\pi}{6}$

$B = \left[\frac{\pi}{3}, \frac{5\pi}{6}\right)$

Now, $A \cap B = \left[\frac{\pi}{3}, \frac{2\pi}{3}\right)$

$\frac{\pi}{3} \leq A \cap B < \frac{2\pi}{3}$

Here $\lambda = \frac{1}{3}$ and $\mu = \frac{2}{3}$

$\lambda + \mu = 1$

51. Here, $A = [-3, 7), B = (0, 10)$

and $S = (-\infty, \infty)$

$\therefore \quad A - B = [-3, 0]$ and $B - A = [7, 10)$

$\therefore \quad A \Delta B = (A - B) \cup (B - A) = [-3, 0] \cup [7, 10)$

$\therefore$ Positive integers are 7, 8, 9.

Number of positive integers = 3

52. As $2^m - 2^n = 48 = 16 \times 3 = 2^4 \times 3$

$\Rightarrow \quad 2^n(2^{m-n} - 1) = 2^4 (2^2 - 1)$

$\therefore \quad n = 4$ and $m - n = 2$

$n = 4$ and $m = 6$

Now, $m - n = 2$

53. $n((A \times B) \cap (B \times A)) = n((A \cap B) \times (B \cap A))$

$= n(A \cap B) \cdot n(B \cap A)$

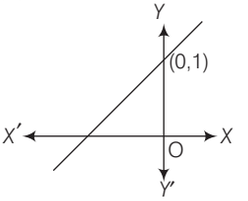
$= n(A \cap B) \cdot n(A \cap B)$

$= 99 \times 99 = 121 \times 9^2$

$\therefore \quad \lambda = 9$

54. (A) $y = 7x + 1$

$f(x) = 7x + 1$



Let $x_1, x_2 \in D_f$,

then $f(x_1) = f(x_2)$

$\Rightarrow \quad 7x_1 + 1 = 7x_2 + 1 \Rightarrow x_1 = x_2$

f is one-one, $\forall\, x \in R$

Now, $y = 7x + 1 \Rightarrow x = \dfrac{y-1}{7}$

for each $y \in R$, we get $x \in R$

f is onto function

(B) $y = \cos x$

for $x \in [0, \pi], y \in [-1, 1]$

$\therefore f$ is one-one on $[0\ \pi]$,

$\forall\ x \in R, y \in [-1, 1]$

y is not onto R.

(C) $y = \sin x$ or $f(x) = \sin x$

for $x \in [0, \pi], y \in [0, 1]$

$$f\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \text{ and } f\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$\therefore f$ is not one-one on $(0, \pi)$,

$\forall\ x \in R$ and $y \in [-1, 1]$

$\therefore f$ is onto $[-1, 1]$.

(D) $y = 1 + \text{In}\, x$ and $f(x) = 1 + \text{In}\, x$

y is defined for $x \in (0, \infty)$

Let $x_1, x_2 \in D_f$

then $f(x_1) = f(x_2)$

$\Rightarrow 1 + \text{In}\, x_1 = 1 + \text{In}\, x_2$

$\Rightarrow x_1 = x_2$

$\therefore f$ is one-one, $\forall\ x \in (0, \infty)$

55. **(A)** Let $y = \sqrt{3 - x} + \sin^{-1}\left(\dfrac{3-2x}{5}\right)$

For y to be defined $3 - x \geq 0$ on $-1 \leq \dfrac{3-2x}{5} \leq 1$

$x \leq 3$ ...(i)

$-5 \leq 3 - 2x \leq 5$

and $-1 \leq x \leq 4$ ...(ii)

From Eqs. (i) and (ii), we get

$x \in [-1, 3]$

(B) Let $y = \log_{10}\{1 - \log_{10}(x^2 - 5x + 16)\}$ for y to be defined

$x^2 - 5x + 16 > 0$ and $1 - \log_{10}(x^2 - 5x + 16) > 0$

$\left(x - \dfrac{5}{2}\right)^2 + \dfrac{39}{4} > 0$ and $\log_{10}(x^2 - 5x + 16) < 1$

which is true, $\forall\ x \in R$ ...(i)

$\Rightarrow x^2 - 5x + 16 < 10$

$\Rightarrow x^2 - 5x + 6 < 0 \Rightarrow (x-3)(x-2) < 0$

$\Rightarrow 2 < x < 3$ ...(ii)

From Eqs. (i) and (ii), $x \in (2, 3)$

(C) Let $y = \cos^{-1}\dfrac{2}{2 + \sin x}$, for y to be defined

$$-1 \leq \frac{2}{2 + \sin x} \leq 1 \qquad \left[\because -1 < \sin x \leq 1 \atop 1 < 2 + \sin x \leq 3\right]$$

Multiplying by $(2 + \sin x)$

$-(2 + \sin x) \leq 2 \leq 2 + \sin x$

$\Rightarrow -2 - \sin x \leq 2 \quad | \quad 2 \leq 2 + \sin x$

$\Rightarrow -\sin x \leq 4 \quad | \quad \sin x \geq 0$

$\Rightarrow \sin x \geq -4 \quad | \quad 2n\pi \leq x \leq (2n+1)\pi, n \in z$ ...(i)

We know that $\sin x \in [-1, 1]$

$\therefore x \in R$ ... (ii)

From Eqs. (i) and (ii); $x \in [2k\pi, (2k+1)\pi]$

Domain $= \underset{k \in I}{\cup} [2k\pi, (2k + 1\pi)]$

(D) $y = \sqrt{\sin x} + \sqrt{16 - x^2}$ for y to be defined

$\sin x \geq 0 \quad | \quad 16 - x^2 \geq 0$

$x \in [2k\pi, (2k+1)\pi], k \in I$ ...(i) $\quad | \quad -4 \leq x \leq 4$ ...(ii)

From Eqs. (i) and (ii), we get

$x \in [-4, -\pi] \cup [0, \pi]$

56. Let

$A = \{a_1, a_2, a_3, \ldots, a_n\}$

Then, the number of binary relations on $A = n^{(n \times n)} = n^{n^2}$

and number of relations form A to $A = 2^{n \times n} = 2^{n^2}$

Both statements are true but Statement-2 is not a correct explanation for Statement-1.

57. Let $\alpha \in (A \cap B) \Rightarrow \alpha \in A$ and $\alpha \in B$

$\Rightarrow g(\alpha) = 0$

and $f(\alpha) = 0$

$\Rightarrow \{f(\alpha)\}^2 + \{g(\alpha)\}^2 = 0$

$\Rightarrow \alpha$ is a root of $\{f(x)\}^2 + \{g(x)\}^2 = 0$

Hence, Statement-1 is true and Statement-2 is false.

58. Let $x \in P(A \cap B)$

$\Leftrightarrow x \subseteq (A \cap B)$

$\Leftrightarrow x \subseteq A$ and $x \subseteq B$

$\Leftrightarrow x \in P(A)$ and $x \in P(B)$

$\Leftrightarrow x \in P(A) \cap P(B)$

$\therefore P(A \cap B) \subseteq P(A) \cap P(B)$

and $P(A) \cap P(B) \subseteq P(A \cap B)$

Hence, $P(A) \cap P(B) = P(A \cap B)$

Now, consider sets $A = \{1\}, B = \{2\} \Rightarrow A \cup B = \{1, 2\}$

$\therefore P(A) = \{\phi, \{1\}\}, P(B) = \{\phi, \{2\}\}$.

and $P(A \cup B) = \{\phi\ \{1\}, \{2\}, \{1, 2\} \neq P(A) \cup P(B)\}$

Hence, Statement-1 is true and Statement-2 is false.

59. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$= 3 + 6 - n(A \cap B) = 9 - n(A \cap B)$

As maximum number of element in $(A \cap B) = 3$

$\therefore$ Minimum number of elements in $(A \cap B) = 9 - 3 = 6$

Both statements are true; Statement-2 is a correct explanation for Statement-1.

60. $A = \{x : x \text{ is a natural number}\}$

$B = \{x : x \text{ is an even natural number}\}$

$C = \{x : x \text{ is an odd natural number}\}$

$D = \{x : x \text{ is a prime number}\}$

(i) $A \cap B = \{x : x = 2n, n \in N\} = B$

(ii) $A \cap C = \{x : x$ is an odd natural number$\} = C$

(iii) $B \cap D = \{x : x$ is prime natural number$\} = \{2\}$

(iv) $C \cap D = \{x : x$ is odd prime natural number$\}$

61. $U =$ Set of all people

$M =$ {Males}

$S =$ {College students}

$T =$ {Teenagers}

$W =$ {People having height more than 5 feet}

(i) College students having heights more than 5 feet $= S \cap W$

(ii) People who are not teenagers and having their heights less than 5 feet $= T' \cap W'$

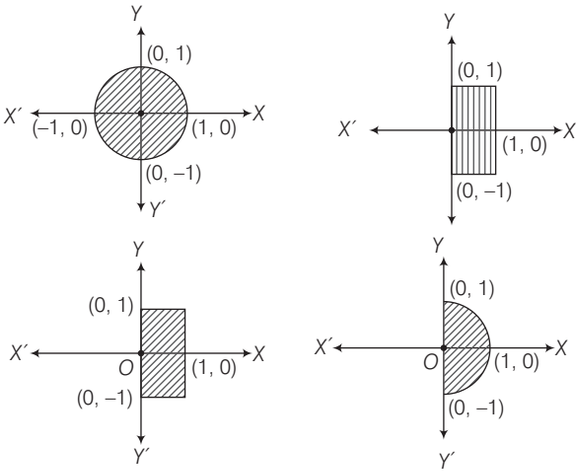
(iii) All people who are neither males nor teenagers nor college students $= (M \cup T \cup S)'$

62. $X = \{(x, y) : x^2 + y^2 \le 1\}$

$Y = \{(x, y) : 0 \le x \le 1, -1 \le y \le 1\}$

$X \cup Y = \{(x, y) : x^2 + y^2 \le 1$ or $0 \le x \le 1$ and $-1 \le y \le 1\}$

$X \cap Y = \{(x, y) : x^2 + y^2 \le 1$ and $x \ge 0\}$

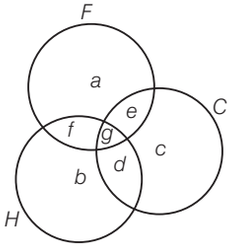


63. Given, $a = 20$ ...(i)

$e + f + g = 15$ ...(ii)

$b + d + f + g = 22$ ...(iii)

$c = 11$ ...(iv)



$c + d + e + g = 25$ ...(v)

$\Rightarrow \quad d + e + g = 14$ ...(vi)

$e = 7$ ...(vii)

$f = 3$ ...(viii)

From Eqs. (vii), (viii) and (ix),

$e + g = 12$ ...(ix)

$e = 7, f = 3, g = 5$

From Eq. (vi), $d = 2$

From Eq. (iii) $b + 2 + 3 + 5 = 22$

$\therefore \quad b = 12$

Hence, $a = 20, b = 12, c = 11, d = 2, e = 7, f = 3, g = 5$

Number of children play all the three games $= g = 5$

Number of children play cricket and hockey but not football $= d = 2$

Number of children play hockey only $= b = 12$

Total number of children in the group

$= a + b + c + d + e + f + g = 60$

64. $a + f + e + g = 21$ ...(i)

$b + d + f + g = 26$ ...(ii)

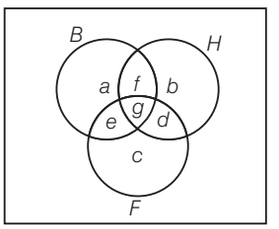
$c + d + e + g = 29$ ...(iii)

$f + g = 14$ ...(iv)

$g + d = 15$ ...(v)

$e + g = 12$ ...(vi)

$g = 8$ ...(vii)



From Eqs. (vii) and (vi), $e = 4$

Form Eqs. (vii) and (v), $d = 7$

From Eqs. (vii) and (iv), $f = 6$

From Eq. (iii), $c + 7 + 4 + 8 = 29 \Rightarrow c = 29 - 19 = 10 = c$

From Eq. (ii), $b + 7 + 6 + 8 = 26 \Rightarrow b = 26 - 21 \Rightarrow \ b = 5$

From Eq. (i), $a + 6 + 4 + 8 = 21 \Rightarrow a = 21 - 18 \Rightarrow \ a = 3$

$n(B) + n(H) + n(F) = a + b + c + d + e + f + g$

$= 3 + 5 + 10 + 7 + 4 + 6 + 8 = 43$

65. $a + e + f + g = 120$ ...(i)

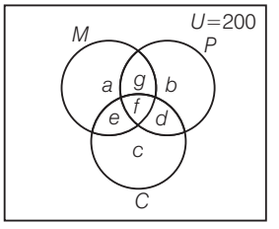
$b + d + f + g = 90$ ...(ii)

$e + f + c + d = 70$ ...(iii)

$g + f = 40$ ...(iv)

$f + d = 30$ ...(v)

$e + f = 50$ ...(vi)



$U - (a + b + c + d + e + f + g) = 20$

$\Rightarrow \quad a + b + c + d + e + f + g = 180$ ...(vii)

From Eqs. (i) and (iv), $a + e = 80$ ...(viii)

From Eqs. (ii) and (iv), $b + d = 50$ ...(ix)

From Eqs. (iii) and (v), $e + c = 40$ ...(x)

from Eqs (viii), (ix) & (x), $a + b + c + d + e + e = 197$ ...(xi)

from (xi), (vii) and (iv), $197 - e + 40 = 180$

$170 - e + 40 = 180$

$e = 210 - 180 = 30$

From Eq. (vi), $e + f = 50$

$\Rightarrow \quad 30 + f = 50$

$\Rightarrow \quad f = 20$

66. $b + e + f + g = 205$ ...(i)

$a + d + f + g = 210$ ...(ii)

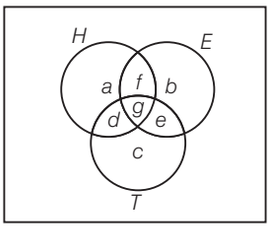
$c + d + e + g = 120$ ...(iii)

$f + g = 100$ ...(iv)

$e + g = 800$ ...(v)

$d + g = 35$ ... (vi)

$g = 20$ ...(vii)



From Eqs. (vi) and (vii), $d = 15$

From Eqs. (vii) and (v), $e = 60$

From Eqs. (vii) and (iv), $f = 80$

From Eq. (i), $b + 60 + 80 + 20 = 205 \Rightarrow b = 205 - 160$

$\Rightarrow b = 45$ = Can speak English but not Hindi or Tamil.

From Eq. (ii) $a + 15 + 80 + 20 = 210$

$\Rightarrow \quad a + 115 = 210 \Rightarrow a = 95$

From Eq. (iii), $c + 15 + 60 + 20 = 120$

$\Rightarrow \quad c = 120 - 95 \Rightarrow c = 25$

People who can speak neither E nor H nor T

$= 450 - (95 + 45 + 25 + 15 + 60 + 80 + 20)$

$= 450 - 340 = 110$

67. $c + f + g + e = 42$ ...(i)

$b + d + g + e = 36$ ...(ii)

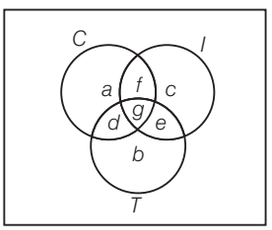
$a + f + d + g = 30$ ...(iii)

$g + e = 15$ ...(iv)

$f + g = 10$ ...(v)

$d = 4$ ...(vi)

$e = 11$ ...(vii)



From (iv) and (vii), $g + 11 = 15 \Rightarrow g = 4$ ...(viii)

From (v) and (viii), $f + 4 = 10 \Rightarrow f = 6$ ...(ix)

From (i), $c + 6 + 4 + 11 = 42 \Rightarrow c = 21$ ...(x)

From (ii), $b + 4 + 4 + 11 = 36 \Rightarrow b = 17$ ...(xi)

From (iii), $a + 6 + 4 + 4 = 30 \Rightarrow a = 16$ ...(xii)

Number of required persons

$= 123 - (16 + 17 + 21 + 4 + 11 + 6 + 4)$

$= 123 - 79$

$= 44$

68. aRb iff $n\,|(a - b)|$ i.e. $(a - b)$ is divisible by n.

Reflexivity $a - a = 0$ which is divisible by n.

So, $(a, a) \in R, \forall\ a \in I$

$\therefore$ R is reflexive relation.

Symmetry Let $(a, b) \in R$

Then, $(a, b) \in R \Rightarrow (a - b)$ is divisible by n.

$\Rightarrow \quad -(b - a)$ is divisible by n.

$\Rightarrow \quad (b - a)$ is divisible by n.

$\Rightarrow \quad (b, a) \in R$

$\therefore$ R is symmetric relation.

Transitivity Let $(a, b) \in R, (b, c) \in R$, then $(a - b)$ and $(b - c)$ are divisible by n.

$\Rightarrow \quad a - b = nk_1$ and $b - c = nk_2 \qquad [k_1, k_2 \in I]$

$\Rightarrow \quad (a - b) + (b - c) = n(k_1 + k_2)$

$\Rightarrow \quad a - c = n\,(k_1 + k_2)$

$\Rightarrow$ $(a - c)$ is divisible by n.

$\Rightarrow \quad (a, c) \in R$

$\therefore$ R is transitive relation.

$\therefore$ R is an equivalence relation.

69. R defined on $N \times N$ such that

$(a, b)\ R\ (c, d) \Leftrightarrow ad = bc$

Reflexivity Let $(a, b) \in N \times N$

$\Rightarrow \quad a, b \in N \ \Rightarrow \ ab = ba$

$\Rightarrow \quad (a, b)\ R\ (a, b)$

$\therefore$ R is reflexive on, $N \times N$.

Symmetry Let $(a, b), (c, d) \in N \times N$,

then $(a, b)\ R\ (c, d) \Rightarrow ad = bc$

$\Rightarrow \quad cb = da$

$\Rightarrow \quad (c, d)\ R\ (a, b)$

$\therefore$ R is symmetric on $N \times N$.

Transitivity Let $(a, b), (c, d), (e, f) \in N \times N$

Then, $(a, b)\ R\ (c, d) \Rightarrow ad = bc$ ...(i)

$(c, d)\ R\ (e, f) \Rightarrow cf = de$ ...(ii)

From Eqs. (i) and (ii), $(ad)\,(cf) = (bc)\,(de)$

$\Rightarrow \quad af = be$

$\Rightarrow \quad (a, b)\ R\ (e, f)$

$\therefore$ R is transitive relation on $N \times N$.

$\therefore$ R is equivalence relation on $N \times N$.

70. (i) $aRb \Leftrightarrow |a-b| > 0$

Reflexivity $a - a = 0$

$\therefore \quad (a, a) \notin R$

$\therefore$ R is not reflexive

Symmetry $(a, b) \in R \Rightarrow |a-b| > 0$

$\Rightarrow \quad |-(b-a) > 0|$

$\Rightarrow \quad |b-a| > 0$

$\Rightarrow \quad (b, a) \in R$

$\therefore$ R is symmetric relation

Transitivity $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow \quad |a-b| > 0$ and $|b-c| > 0$

$\Rightarrow \quad |a-b| + |b-c| > 0$ [by addition]

Now, let $a > b$ and $b > c$, then $a > c$

$|a-b| + |b-c| = a - b + b - c = a - c > 0$

$\Rightarrow \quad |a-c| > 0$

If $a < b$ and $b > c$, then

$|a-b| + |b-c| = -(a-b) + (b-c) = -a + 2b - c$

$\not\Rightarrow \quad |a-c| > 0$

$\therefore$ R is not transitive relation.

(ii) $aRb \Leftrightarrow |a| = |b|$

Reflexivity We have, $|a| = |a|$

$\Rightarrow \quad aRa \forall a$

$\therefore$ R is reflecxive relation.

Symmetry $aRb \Rightarrow |a| = |b|$

$\Rightarrow \quad |b| = |a|$

$\Rightarrow \quad bRa$

$\therefore$ R is symmetric relation.

Transitivity $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow \quad |a| = |b|$ and $|b| = |c|$

$\Rightarrow \quad |a| = |c|$

$\Rightarrow \quad (a, c) \in R$

$\therefore$ R is transitive relation.

(iii) $aRb \Leftrightarrow |a| \geq |b|$

Reflexivity For any $a \in R$, we have $|a| \geq |a|$

So, $aRa \ \forall \ a$

$\therefore$ R is reflexive relation.

Symmetry $aRb \Rightarrow |a| \geq |b|$

$\Rightarrow \quad |b| \leq |a|$

$\therefore$ R is not symmetric relation.

Transitivity aRb and $bRc \Rightarrow |a| \geq |b|$ and $|b| \geq |c|$

$\Rightarrow \quad |a| \geq |c|$

$\Rightarrow \quad a \ R \ c$

$\therefore$ R is transitive relation.

(iv) $aRb \Leftrightarrow 1 + ab > 0, \forall \ a, b \in R$

Reflexivity Let $a \notin R \Rightarrow 1 + a \cdot a = 1 + a^2 > 0$

$\Rightarrow \quad (a, a) \in R$

$\therefore$ R is reflexive on R.

Symmetry Let $(a, b) \in R$, then $(a, b) \in R$

$\Rightarrow \quad 1 + ab > 0$

$\Rightarrow \quad 1 + ba > 0$

$\Rightarrow \quad (b, a) \in R$

$\therefore$ R is symmetric on R.

Transitivity We observe that $\left(1, \frac{1}{2}\right) \in R$ and $\left(\frac{1}{2}, -1\right) \in R$ but $(1, -1) \notin R$ because

$1 + (1)(-1) = 0 \not> 0$

$\therefore$ R is not transitive on R.

(v) $aRb \Leftrightarrow |a| \leq b$

Reflexivity Let $-1 \in R$, then $|-1| \not\leq (-1)$

$\therefore$ R is not reflexive relation

Symmetry Now, let $-3 \, R \, 4$, then $|4| \not\leq -3$

$\Rightarrow \quad 4 \not R -3$

$\therefore R$ is not symmetric relation

Transitivity aRb and $bRc \Rightarrow |a| \leq b$ and $|b| \leq c$

Then, $|a| \leq c \ \Rightarrow aRc$

$\therefore$ R is transitive relation.

71. $A = \{x : -1 \leq x \leq\}$

$B = \{x : -1 \leq x \leq\}$

(i) $f(x) = \frac{x}{2}$

Let $x_1, x_2 \in A$

$\therefore \quad f(x_1) = f(x_2) \Rightarrow \frac{x_1}{2} = \frac{x_2}{2}$

$\Rightarrow \quad x_1 = x_2$

$\therefore$ f is one-one function.

Now, let $y = \frac{x}{2} \Rightarrow x = 2y$

$\Rightarrow \quad -1 \leq y \leq 1$

$\Rightarrow \quad -2 \leq 2y \leq 2 \Rightarrow -2 \leq x \leq 2$

Let $x \in [-1, 1]$

$\therefore$ There are some value of y for which x does not exist. So, f not onto.

(ii) $g(x) = |x|$

For $x = -1, g(-1) = 1$

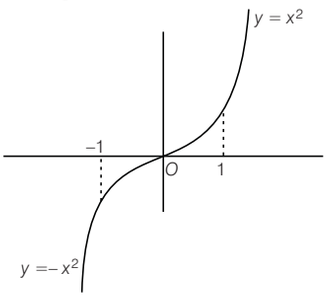
and for $x = 1, g(1) = 1$

$\therefore$ f is not one-one function

Let $y = |x|$, then $y \geq 0$

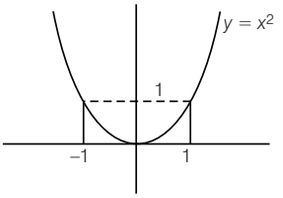
$\therefore$ f is not onto.

(iii) $h(x) = x|x| = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$



From figure, it is clear tat h is one-one and onto i.e., bijective.

(iv) $k(x) = x^2$



$k(1) = 1$

and $k(-1) = 1$

So, k is many-one function.

From figure, $y \in (0, 1)$

$\therefore$ y is not onto function.

(v) $y = l(x) = \sin \pi x$

for $x = 1$, $l(1) = \sin \pi = 0$

for $x = -1$, $l(-1) = \sin(-\pi) = 0$

$\therefore l$ is not one-one.

Now, $-1 \le x \le 1$

$\Rightarrow$ $-\pi \le \pi\ x \le \pi$

$\Rightarrow$ $-1 \le \sin \pi\ x \le 1$

$\therefore$ y is onto function.

Hence, l is surjective function.

72. $(fog)x = f(3x - 2) = e^{3x-2}$

and $(gof)x = g(e^x) = 3e^x - 2$

Let $(fog)x = y \Rightarrow e^{3x-2} = y$

$\Rightarrow$ $3x - 2 = \log_e y \Rightarrow x = \dfrac{2 + \log_e y}{3}$

$\Rightarrow (fog)^{-1}(y) = \dfrac{2 + \log_e y}{3}$

$\Rightarrow y > 0$ So, domain of $(fog)^{-1}$ is $(0, \infty)$.

Now, again let $(gof)x = 3e^x - 2$

$\Rightarrow$ $y = 3e^x - 2 \Rightarrow e^x = \dfrac{y + 2}{3}$

$\therefore$ $x = \log_e\left(\dfrac{y + 2}{3}\right)$

$\Rightarrow$ $(gof)^{-1}(y) = \log_e\left(\dfrac{y + 2}{3}\right)$

Clearly, $y + 2 > 0 \Rightarrow y > -2$

$\therefore$ Domain of $(gof)^{-1}$ is $(-2, \infty)$.

73. $f(x) = \dfrac{x^2 - x}{x^2 + 2x}$ ...(i)

$f(x) = \dfrac{x(x-1)}{x(x+2)}$

$f(x) = \dfrac{(x-1)}{(x+2)},\ x \ne 0$ ...(ii)

$D_f = \{x : x^2 + 2x \ne 0\}$ [from Eq. (i)]

$= \{x : x \in R - \{0, -2\}\}$

Now, let $y = \dfrac{x-1}{x+2}$

$\Rightarrow$ $yx + 2y = x - 1 \Rightarrow x(y - 1) = -(1 + 2y)$

$\Rightarrow$ $x = \dfrac{1 + 2y}{1 - y}$

Now, for $y = 1$, x is not defined.

Now, $x = 0,\ f(x) = -\dfrac{1}{2}$

$\therefore$ $R_f = R - \left\{1, -\dfrac{1}{2}\right\}$

Now, let $x_1, x_2 \in D_f$

Then, $f(x_1) = f(x_2) \Rightarrow \dfrac{x_1 - 1}{x_1 + 2} = \dfrac{x_2 - 1}{x_2 + 2}$

$\Rightarrow$ $x_1x_2 + 2x_1 - x_2 - 2 = x_1x_2 - x_1 + 2x_2 - 2$

$\Rightarrow$ $x_1 = x_2$

$\therefore f$ is one-one function.

Now, let $y = \dfrac{x-1}{x+2}$

Then, $x = \dfrac{1 + 2y}{1 - y}$

$\Rightarrow$ $f^{-1}(y) = \dfrac{1 + 2y}{1 - y}$ $[\because f(x) = y \Rightarrow x = f^{-1}(y)]$

Replace y by x, we get $f^{-1}(x) = \dfrac{1 + 2x}{1 - x}$

$\Rightarrow$ $\dfrac{d}{dx}\{f^{-1}(x)\} = \dfrac{(1-x)\,2 - (1+2x)(-1)}{(1-x)^2}$

$= \dfrac{2 - 2x + 1 + 2x}{(1-x)^2}$

$\Rightarrow$ $\dfrac{d}{dx}\{f^{-1}(x)\} = \dfrac{3}{(1-x)^2}$

$\therefore$ Domain of $\dfrac{d}{dx}\{f^{-1}(x)\} = R - \{1\}$

74. $f(x) = x^2 - 1$

$g(x) = \sqrt{x^2 + 1};\ h(x) = \begin{cases} 0, & \text{if } x \le 0 \\ x, & \text{if } x \ge 0 \end{cases}$

$\therefore$ $(hofog)(x) = (hof)\{g(x)\}$

$= (hof)\sqrt{x^2 + 1}$

$= h\{f\sqrt{(x^2 + 1)}\}$

$= h\{\sqrt{(x^2 + 1)^2} - 1\} = h(x^2 + 1 - 1)$

$= h(x^2) = x^2$ $[\because x^2 \ge 0]$

and $(fog)(x) = f\{g(x)\}$

$= f(\sqrt{x^2 + 1}) = \left(\sqrt{x^2 + 1}\right)^2 - 1 = x^2 + 1 - 1 = x^2$

Let $y = (fog)x = x^2,\ \forall\ x \in R$

If $x = 1$, then $y = 1$

If $x = -1$, then $y = 1$

So, fog is not one-one, so it is not invertible $h(x) = \begin{cases} 0, & x \le 0 \\ x, & x \ge 0 \end{cases}$

For $x = -1$, $h(-1) = 0$ and for $x = -2$, $h(-2) = 0$

$\therefore$ h is not identity function.

75. Here, (3, 3), (6, 6), (9, 9), (12, 12) So, it is Reflexive and
(3, 6), (6, 12), (3, 12) So, it is Transitive
Here, reflexive and transitive only.

76. Clearly, $(x, x) \in R, \forall\ x \in W$
So, R is reflexive.
Let $(x, y) \in R$, then $(y, x) \in R$ as x and y have atleast one letter in common. So, R is symmetric. But R is not transitive.
e.g. Let x = INDIA, y = BOMBAY and z = JUHU
Then, $(x, y) \in R$ and $(y, z) \in R$ but $(x, z) \notin R$

77. $T = \{(x, y) : x - y \in I\}$
As $0 \in I$, so T is a reflexive relation.
If $x - y \in I \Rightarrow y - x \in I$
$\therefore T$ is symmetric also.
If $x - y = I$ and $y - z = I2$
Then, $x - z = (x - y) + (y - z) = I_1 + I_2 \in I$
$\therefore$ T is also transitive.
Hence, T is an equivalence relation. Clearly, $x \neq x + 1 \Rightarrow (x, x) \notin S$
$\therefore S$ is not reflexive.

78. $\because\ A \cap B = A \cap C \Rightarrow B = C$ and $A \cup B = A \cup C \Rightarrow B = C$
Hence, $B = C$

79. For disoint sets, $A \cap B = \phi$
Each element in either A or B or neither.
$\therefore$ Total ways $= 3^4 = 81$; $A = B$ iff $A = B = \phi$
Otherwise, A and B are interchangable
$\therefore$ Number of unordered pair for disoint subsets of

$$S = \frac{3^4 + 1}{2} = 41$$

80. xRy need not implies yRx.

$S : \frac{m}{n} S \frac{P}{q} \Leftrightarrow qm = pm \Rightarrow \frac{m}{s} S \frac{m}{n}$ is reflexive.

$\frac{m}{n} S \frac{p}{q} \Rightarrow \frac{p}{q} S \frac{m}{n}$ is symmetric.

and $\frac{m}{n} S \frac{p}{q}, \frac{p}{q} S \frac{r}{t} \Rightarrow qm = pn,\ pt = qr$

$\Rightarrow$ $mt = nr \Rightarrow \frac{m}{n} S \frac{r}{t}$ is transitive.

$\therefore$ S is an equivalence relation.

81. $P : \sin\theta - \cos\theta = \sqrt{2}\cos\theta \Rightarrow \tan\theta = \sqrt{2} + 1$

$Q : \sin\theta + \cos\theta = \sqrt{2}\sin\theta \Rightarrow \tan\theta = \frac{1}{\sqrt{2} - 1} = \sqrt{2} + 1$

$\therefore$ $P = Q$

82. $\because (fogogof)(x) = (gogof)(x)$

$\therefore$ $(\sin\sin x^2)^2 = \sin\sin x^2 \Rightarrow \sin\sin x^2 = 0$ or 1

$\Rightarrow$ $x = +\sqrt{n\pi}, n \in \{0, 1, 2, 3, \ldots\}$

83. **Statement-1** $A = \{(x, y) \in R \times R : y - x$ is an integer$\}$

(a) **Reflexive** $xRy : (x - x)$ is an integer
which is true.
Hence it is reflexive.

(b) **Symmetric** $xRy : (x - y)$ is an integer.
$\Rightarrow -(y - x)$ is also an integer.
$\therefore$ $(y - x)$ is also an integer.
$\Rightarrow$ $y\ R\ x$
Hence, it is symmetric.

(c) **Transitive** $x\ R\ y$ and $y\ R\ z$
$\Rightarrow (x - y)$ and $(y - z)$ are integere and.
$\Rightarrow (x - y) + (y - z)$ is an integer.
$\Rightarrow (x - z)$ is an integer.
$\Rightarrow x\,R\,z$
$\therefore$ It is transitive
Hence, it is equvalence relation.

Statement-2
$B = \{(x, y) \in R \times R : x = \alpha y$ for some reational number $\alpha\}$
If $\alpha = 1$, then
$xRy : x = y$ (To check equivalence)

(a) **Reflexive** $xRx : x = x$ (True)
$\therefore$Reflexive

(b) **Symmetric** $xRy : x = y \Rightarrow y = x \Rightarrow yRx$
$\therefore$ Symmetric

(c) **Transitive** xRy and $yRz \Rightarrow x = y$
and $y = z \Rightarrow x = z \Rightarrow xRz$
$\therefore$ Transitive
Hence, it is equivalence relation.
$\therefore$Both are true but Statement-2 is not correct explanation of Statement-2

84. $\because A \times B$ has 8 elements.
$\therefore$Number of subsets $= 2^8 = 256$
Number of subsets with zero element $= {}^8C_0 = 1$
Number of subsets with one element $= {}^8C_1 = 8$
Number of subsets with one elements $= {}^8C_2 = 28$
Hence, Number of subsets of $A \times B$ having 3 or more elements
$= 256 - (1 + 8 + 28) = 256 - 37 = 219$

85. Since, $4^n - 3n - 1 = (1 + 3)^n - 3n - 1$
$= (1 + {}^nC_1 \cdot 3 + {}^nC_2 \cdot 3^2 + {}^nC_3 \cdot 3^3 + \ldots + {}^nC_n \cdot 3^n) - 3n - 1$
$= 3^2({}^nC_2 + {}^nC_3 \cdot 3 + \ldots + {}^nC_n \cdot 3^{n-2})$
$\Rightarrow 4^n - 3n - 1$ is a multiple of 9 for $n \geq 2$
For $n = 1$, $4^n - 3n - 1 = 4 - 3 - 1 = 0$
For $n = 2$, $4^n - 3n - 1 = 16 - 6 - 1 = 9$
$\therefore 4^n - 3n - 1$ is multiple of 9 for all $n \in N$.
It is clear that X contains elements, which are multiples of 9 and Y contains all multiples of 9.
$\therefore$ $X \subseteq Y$ i.e., $X \cup Y = Y$

86. $n(A) = 4, n(B) = 2 \Rightarrow n(A \times B) = 8$
The number of subsets of $A \times B$ having at least three elements
$= {}^8C_3 + {}^8C_4 + {}^8C_5 + \ldots + {}^8C_8$
$= 2^8 - ({}^8C_0 + {}^8C_1 + {}^8C_2)$
$= 256 - (1 + 8 + 28) = 219$

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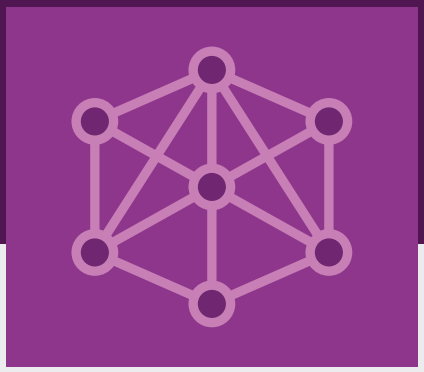

Skills in Mathematics for

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With Sessionwise Theory & Exercises

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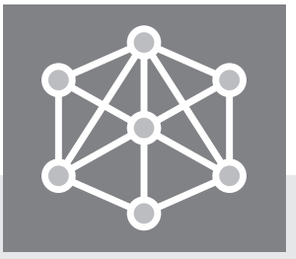

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PREFACE

IF YOU CONTINUOUSLY PUT YOUR EFFORTS ON AN ASPECT, YOU HAVE VERY GOOD CHANCE TO GET POSITIVE OUTCOME i.e. ***SUCCESS***

It is a matter of great pride and honour for me to have received such an overwhelming response to the previous editions of this book from the readers. In a way, this has inspired me to revise this book thoroughly as per the changed pattern of JEE Main & Advanced. I have tried to make the contents more relevant as per the needs of students, many topics have been re-written, a lot of new problems of new types have been added in etcetc. All possible efforts are made to remove all the printing errors that had crept in previous editions. The book is now in such a shape that the students would feel at ease while going through the problems, which will in turn clear their concepts too.

A Summary of changes that have been made in Revised & Enlarged Edition

- Theory has been completely updated so as to accommodate all the changes made in JEE Syllabus & Pattern in recent years.
- The most important point about this new edition is, now the whole text matter of each chapter has been divided into small sessions with exercise in each session. In this way the reader will be able to go through the whole chapter in a systematic way.
- Just after completion of theory, Solved Examples of all JEE types have been given, providing the students a complete understanding of all the formats of JEE questions & the level of difficulty of questions generally asked in JEE.
- Along with exercises given with each session, a complete cumulative exercises have been given at the end of each chapter so as to give the students complete practice for JEE along with the assessment of knowledge that they have gained with the study of the chapter.
- Last 13 Years questions asked in JEE Main & Adv, IIT-JEE & AIEEE have been covered in all the chapters.

However I have made the best efforts and put my all Coordinate Geometry teaching experience in revising this book. Still I am looking forward to get the valuable suggestions and criticism from my own fraternity i.e. the fraternity of JEE teachers.

I would also like to motivate the students to send their suggestions or the changes that they want to be incorporated in this book. All the suggestions given by you all will be kept in prime focus at the time of next revision of the book.

Dr. SK Goyal

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- when $b>a$ $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,
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PRACTICE PART

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- Chapter Exercises

SYLLABUS

JEE MAIN

Cartesian system of rectangular coordinates in a plane, distance formula, section formula, locus and its equation, translation of axes, slope of a line, parallel and perpendicular lines, intercepts of a line on the coordinate axes.

Straight Lines

Various forms of equations of a line, intersection of lines, angles between two lines, conditions for concurrence of three lines, distance of a point from a line, equations of internal and external bisectors of angles between two lines, coordinates of centroid, orthocentre and circumcentre of a triangle, equation of family of lines passing through the point of intersection of two lines.

Circles, Conic Sections

Standard form of equation of a circle, general form of the equation of a circle, its radius and centre, equation of a circle when the end points of a diameter are given, points of intersection of a line and a circle with the centre at the origin and condition for a line to be tangent to a circle, equation of the tangent. Sections of cones, equations of conic sections (parabola, ellipse and hyperbola) in standard forms, condition for $y=mx+c$ to be a tangent and point (s) of tangency.

JEE ADVANCED

Analytical Geometry

Two Dimensions Cartesian Coordinates, distance between two points, section formulae, shift of origin.

Equation of a straight line in various forms, angle between two lines, distance of a point from a line. Lines through the point of intersection of two given lines, equation of the bisector of the angle between two lines, concurrency of lines, centroid, orthocentre, incentre and circumcentre of a triangle.

Equation of a circle in various forms, equations of tangent, normal and chord.

Parametric equations of a circle, intersection of a circle with a straight line or a circle, equation of a circle through the points of intersection of two circles and those of a circle and a straight line.

Equations of a parabola, ellipse and hyperbola in standard form, their foci, directrices and eccentricity, parametric equations, equations of tangent and normal.

CHAPTER

01

Coordinate System and Coordinates

Learning Part

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- Introduction
- Rectangular Cartesian Coordinates of a Point
- Relation between the Polar and Cartesian Coordinates
- Coordinate Axes
- Polar Coordinates of a Point

Session 2

- Distance between Two Points
- Distance between Two Points in Polar Coordinates
- Choice of Axes

Session 3

- Section formulae
- Incentre
- Area of Triangle
- Centroid of a Triangle
- Some Standard Results

Session 4

- Locus and Its Equation
- Change of Axes or the Transformations of Axes
- Removal of the Term xy from $F(x, y) = ax^2 + 2hxy + by^2$ without Changing the Origin
- Position of a Point Which Lies Inside a Triangle

Practice Part

- JEE Type Examples
- Chapter Exercises

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Session 1

Introduction, Coordinate Axes, Rectangular Cartesian Coordinates of a Point, Polar Coordinates of a Point, Relation between the Polar and Cartesian Coordinates

Introduction

The great philospher and mathematician of France **Rane Descartes** (1596-1665) published a book 'La Geometric' in 1637.

Descartes gave a new idea i.e. each point in a plane is expressed by an ordered pair of algebraic real numbers like $(x, y), (r, \theta)$ etc., called coordinates of the point.

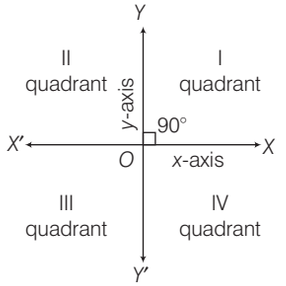
The point (x, y) is called cartesian coordinates and (r, θ) is called polar coordinates of the point. Then represents different forms of equations which are developed for all types of straight lines and curves.

Thus the Coordinate Geometry (or Analytical Geometry) is that branch of mathematics in which geometrical problems are solved with the help of Algebra.

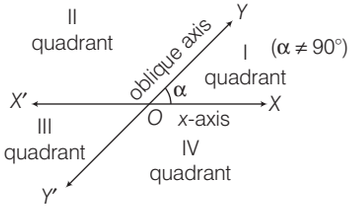
Coordinate Axes

The position of a point in a plane is determined with reference to two intersecting straight lines called the **coordinate axes** and their point of intersection is called the **origin** of coordinates.

If these two axes of reference (generally we call them x and y axes) cut each other at right angle, they are called **rectangular axes** otherwise they are called **oblique axes**. The axes divide the coordinate plane in four quadrants.



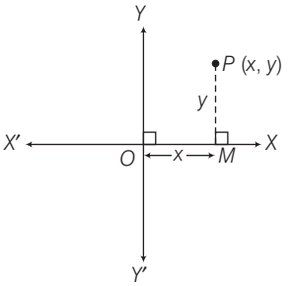
(a) **Rectangular axes**



(b) **Oblique axes**

Rectangular Cartesian Coordinates of a Point

Let $X'OX$ and $Y'OY$ be two perpendicular axes in the plane of paper intersecting at O. Let P be any point in the plane of the paper. Draw PM perpendicular to OX. Then the lengths OM and PM are called the rectangular cartesian coordinates or briefly the coordinates of P.



Let $OM = x$ and $MP = y$

Then, the position of the point P in the plane with respect to the coordinate axes is represented by the ordered pair (x, y). The ordered pair (x, y) is called the coordinates of point P.

i.e. $OM = x$-coordinate or abscissa of the point P

and $MP = y$-coordinate or ordinate of the point P.

Remarks

1. The ordinate of every point on X-axis is 0.
2. The abscissa of every point on Y-axis is 0.
3. The abscissa and ordinate of the origin $O(0, 0)$ are both zero.
4. The abscissa and ordinate of a point are at perpendicular distance from Y-axis and X-axis respectively.
5. Table for conversion sign of coordinates :

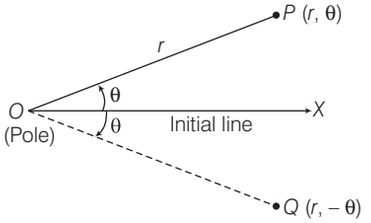
Quadrants	XOY (I)	$X'OY$ (II)	$X'OY'$ (III)	XOY' (IV)
Sign of x coordinates	+	−	−	+
Sign of y coordinates	+	+	−	−
Sign of (x, y)	(+, +)	(−, +)	(−, −)	(+, −)

6. Equation of X-axis, $y = 0$ and equation of Y-axis, $x = 0$.

Polar Coordinates of a Point

If $OP = r$ (radius vector)

and $\angle XOP = \theta$ (vectorial angle)



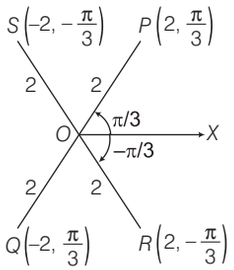
Then, the ordered pair of real numbers (r, θ) called the polar coordinates of the point P.

Remarks

1. r may be positive or negative according as θ is measured in anticlockwise or clockwise direction. θ lies between $-\pi$ to π i.e. $-\pi < \theta \leq \pi$. If it is greater than π, then we subtract 2π from it and if it is less than $-\pi$, then we add 2π, to it. It is also known as principal value of P.
2. Always taken θ in radian.

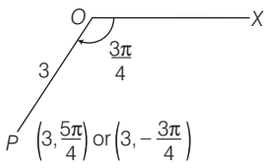
Example 1. Draw the polar coordinates $\left(2, \frac{\pi}{3}\right), \left(-2, \frac{\pi}{3}\right), \left(-2, -\frac{\pi}{3}\right)$ and $\left(2, -\frac{\pi}{3}\right)$ on the plane.

Sol.



Example 2. Draw the polar coordinate $\left(3, \frac{5\pi}{4}\right)$ on the plane.

Sol. Here, $\theta = \frac{5\pi}{4} > \pi$



then, $\theta - 2\pi = \frac{5\pi}{4} - 2\pi = -\frac{3\pi}{4}$

$\therefore \quad \left(3, \frac{5\pi}{4}\right)$ is same $\left(3, -\frac{3\pi}{4}\right)$

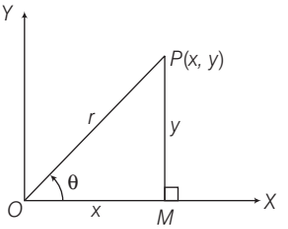
Relation between the Polar and Cartesian Coordinates

Let $P(x, y)$ be the cartesian coordinates with respect to axes OX and OY and (r, θ) be its polar coordinates with respect to pole O and initial line OX.

It is clear from figure

$$OM = x = r\cos\theta \quad \ldots (i)$$

and $$MP = y = r\sin\theta \quad \ldots (ii)$$



Squaring and adding Eqs. (i) and (ii), we get

$$x^2 + y^2 = r^2 \quad \text{or} \quad r = \sqrt{(x^2 + y^2)}$$

Dividing Eqs. (ii) by (i), then

$$\tan\theta = \left(\frac{y}{x}\right) \text{ or } \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

i.e. $$(r\cos\theta, r\sin\theta) \Rightarrow (x, y) \quad \ldots (iii)$$

and $$\left(\sqrt{(x^2 + y^2)}, \tan^{-1}\left(\frac{y}{x}\right)\right) \Rightarrow (r, \theta) \quad \ldots (iv)$$

If r and θ are known then we can find (x, y) from Eq. (iii) and if x and y are known then we can find (r, θ) from Eq. (iv).

$\because \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$

If $\alpha = \tan^{-1}\left|\frac{y}{x}\right|$

Then, values of θ in four quadrants

Quadrant	I	II	III	IV
θ	α	$\pi - \alpha$	$-\pi + \alpha$	$-\alpha$

Example 3. Find the cartesian coordinates of the points whose polar coordinates are

(i) $\left(5, \pi - \tan^{-1}\left(\frac{4}{3}\right)\right)$ (ii) $\left(5\sqrt{2}, \frac{\pi}{4}\right)$

Sol. (i) Given, $r = 5, \theta = \pi - \tan^{-1}\left(\frac{4}{3}\right)$

Now, $x = r\cos\theta = 5\cos\left(\pi - \tan^{-1}\left(\frac{4}{3}\right)\right)$

$= -5\cos\left(\tan^{-1}\left(\frac{4}{3}\right)\right)$

$= -5\cos\left(\cos^{-1}\left(\frac{3}{5}\right)\right) = -5\times\frac{3}{5} = -3$

and $y = r\sin\theta = 5\sin\left(\pi - \tan^{-1}\left(\frac{4}{3}\right)\right)$

$= 5\sin\left(\tan^{-1}\left(\frac{4}{3}\right)\right)$

$= 5\sin\left(\sin^{-1}\left(\frac{4}{5}\right)\right) = 5\times\frac{4}{5} = 4$

Hence, cartesian coordinates of the given point will be $(-3, 4)$.

(ii) Given, $r = 5\sqrt{2}, \theta = \frac{\pi}{4}$

Now, $x = r\cos\theta = 5\sqrt{2}\cos\left(\frac{\pi}{4}\right) = 5\sqrt{2}\times\frac{1}{\sqrt{2}} = 5$

and $y = r\sin\theta = 5\sqrt{2}\sin\left(\frac{\pi}{4}\right) = 5\sqrt{2}\times\frac{1}{\sqrt{2}} = 5$

Hence, cartesian coordinates of the given point, will be $(5, 5)$.

Example 4. Find the polar coordinates of the points whose cartesian coordinates are

(i) $(-2, -2)$ (ii) $(-3, 4)$

Sol. (i) Given, $x = -2, y = -2$

$\therefore$ $r = \sqrt{(x^2 + y^2)} = \sqrt{(4+4)} = 2\sqrt{2}$

and $\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{-2}{-2}\right| = \tan^{-1}1 = \frac{\pi}{4}$

Since, point $(-2, -2)$ lies in III quadrant.

$\therefore$ $\theta = -\pi + \alpha = -\pi + \frac{\pi}{4} = -\frac{3\pi}{4}$

Hence, polar coordinates of the given points will be $\left(2\sqrt{2}, -\frac{3\pi}{4}\right)$.

Remark : If we find θ, from the equation,

$\tan\theta = \frac{y}{x} = \frac{-2}{-2} = 1$

then, $\theta = \frac{\pi}{4}$ and then

$(x, y) = (r\cos\theta, r\sin\theta) = \left(2\sqrt{2}\times\frac{1}{\sqrt{2}}, 2\sqrt{2}\times\frac{1}{\sqrt{2}}\right)$

$= (2, 2) \neq (-2, -2)$

(ii) Given, $x = -3, y = 4$

$\therefore$ $r = \sqrt{(x^2 + y^2)} = \sqrt{(9+16)} = 5$

and $\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{4}{-3}\right| = \tan^{-1}\left(\frac{4}{3}\right)$

Since, point $(-3, 4)$ lies in II quadrant

$\therefore$ $\theta = \pi - \alpha = \pi - \tan^{-1}\left(\frac{4}{3}\right)$

Hence, polar coordinates of the given points, will be $\left(5, \pi - \tan^{-1}\left(\frac{4}{3}\right)\right)$.

Example 5. Transform the equation $r^2 = a^2\cos 2\theta$ into cartesian form.

Sol. $\because$ $r = \sqrt{(x^2 + y^2)}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

or $r^2 = (x^2 + y^2)$ and $\tan\theta = \frac{y}{x}$

Given, $r^2 = a^2\cos 2\theta = a^2\left(\frac{1 - \tan^2\theta}{1 + \tan^2\theta}\right)$

or $(x^2 + y^2) = a^2\left(\frac{1 - \frac{y^2}{x^2}}{1 + \frac{y^2}{x^2}}\right)$

or $(x^2 + y^2)^2 = a^2(x^2 - y^2)$

This is the required equation in cartesian form.

Aliter :

$r^2 = a^2\cos 2\theta$

or $r^2 = a^2(\cos^2\theta - \sin^2\theta)$

$\because$ $x = r\cos\theta$ and $y = r\sin\theta$

and $r^2 = x^2 + y^2$

then $r^2 = a^2\left(\frac{x^2}{r^2} - \frac{y^2}{r^2}\right)$

or $r^4 = a^2(x^2 - y^2)$

or $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

Example 6. Transform the equation $x^2 + y^2 = ax$ into polar form.

Sol. $\because$ $x = r\cos\theta,\ y = r\sin\theta$

Given, $x^2 + y^2 = ax$

$\Rightarrow$ $r^2 = a(r\cos\theta)$

or $r = a\cos\theta$

This is the required equation in polar form.

Exercise for Session 1

1. The polar coordinates of the point whose cartesian coordinates are $(-1, -1)$ is

(a) $\left(\sqrt{2}, \frac{\pi}{4}\right)$ (b) $\left(\sqrt{2}, \frac{3\pi}{4}\right)$ (c) $\left(\sqrt{2}, -\frac{\pi}{4}\right)$ (d) $\left(\sqrt{2}, -\frac{3\pi}{4}\right)$

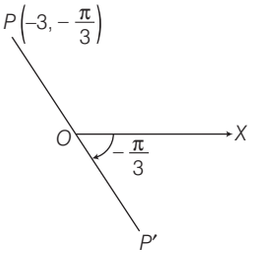
2. The cartesian coordinates of the point whose polar coordinates are $\left(13, \pi - \tan^{-1}\left(\frac{5}{12}\right)\right)$ is

(a) (12, 5) (b) (–12, 5) (c) (–12, – 5) (d) (12, – 5)

3. The transform equation of $r^2 \cos^2\theta = a^2 \cos 2\theta$ to cartesian form is $(x^2 + y^2)\, x^2 = a^2\lambda$, then value of λ is

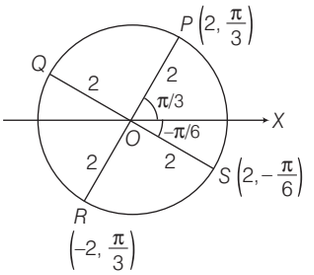
(a) $y^2 - x^2$ (b) $x^2 - y^2$ (c) xy (d) x^2y^2

4. The coordinates of P' in the figure is



(a) $\left(3, \frac{\pi}{3}\right)$ (b) $\left(3, -\frac{\pi}{3}\right)$ (c) $\left(-3, -\frac{\pi}{3}\right)$ (d) $\left(-3, \frac{\pi}{3}\right)$

5. The cartesian coordinates of the point Q in the figure is



(a) $(\sqrt{3}, 1)$ (b) $(-\sqrt{3}, 1)$ (c) $(-\sqrt{3}, -1)$ (d) $(\sqrt{3}, -1)$

6. A point lies on X-axis at a distance 5 units from Y-axis. What are its coordinates ?

7. A point lies on *Y-axis at a distance 4 units from X*-axis. What are its coordinates ?

8. A point lies on negative direction of X-axis at a distance 6 units from Y-axis. What are its coordinates ?

9. Transform the equation $y = x \tan\alpha$ to polar form.

10. Transform the equation $r = 2a \cos\theta$ to cartesian form.

Session 2

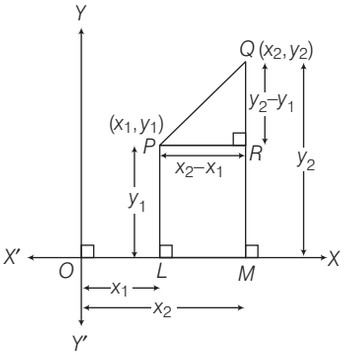
Distance between Two Points, Choice of Axes, Distance between Two Points in Polar Coordinates

Distance Between Two Points

Theorem : The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Proof : Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be any two points in the plane. Let us assume that the points P and Q are both in 1st quadrant (for the sake of exactness).



From P and Q draw PL and QM perpendiculars to X-axis. From P draw PR perpendicular to QM and join PQ. Then

$$OL = x_1,\ OM = x_2,\ PL = y_1,\ QM = y_2$$

$\therefore$ $PR = LM = OM - OL = x_2 - x_1$

and $QR = QM - RM = QM - PL = y_2 - y_1$

Since, PRQ is a right angled triangle, therefore by pythagoras theorem.

$$(PQ)^2 = (PR)^2 + (QR)^2$$

$\therefore$ $|PQ| = \sqrt{(PR)^2 + (QR)^2}$ ($\because PQ$ is always positive)

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$\therefore$ The distance PQ between the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

or $\sqrt{(\text{difference in x coordinates})^2 + (\text{difference in y coordinates})^2}$

or $\sqrt{(\text{difference of abscissaes})^2 + (\text{difference of ordinates})^2}$

Notations : We shall denote the distance between two points P and Q of the coordinate plane, either by $|PQ|$ or by $\overline{PQ}$.

***Corollary* 1 :** The above formula is true for all positions of the points (i.e. either point or both points are not in the 1st quadrant) keeping in mind, the proper signs of their coordinates.

***Corollary* 2 :** The distance of the point $P(x, y)$ from the origin $O(0, 0)$ is given by

$$|OP| = \sqrt{(x - 0)^2 + (y - 0)^2} = \sqrt{(x^2 + y^2)}$$

***Corollary* 3 :** The above formula can also be used as

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

***Corollary* 4 :** (i) If PQ is parallel to X-axis, then $y_1 = y_2$ and so

$$|PQ| = \sqrt{(x_2 - x_1)^2} = |x_2 - x_1|$$

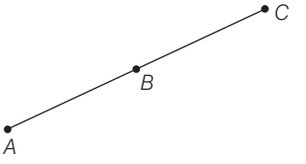
(ii) If PQ is parallel to Y-axis, then $x_1 = x_2$ and so

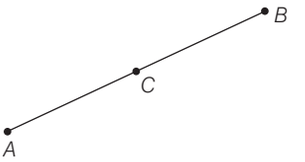
$$|PQ| = \sqrt{(y_2 - y_1)^2} = |y_2 - y_1|$$

***Corollary* 5 :** If distance between two points is given, then use $\pm$ sign.

Remarks

1. If three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear, then $|AB| \pm |BC| = |AC|$





2. When three points are given and it is required to :

(i) an **Isosceles triangle**, show that two of its sides (or two angles) are equal.

(ii) an **Equilateral triangle**, show that its all sides are equal or each angle is of 60°.

(iii) a **Right angle triangle**, show that the sum of the squares of two sides is equal to the square of the third side.

(iv) an **Isosceles right angled triangle**, show that two of its sides are equal and the sum of the squares of two equal sides is equal to the square of the third side.

(v) a **Scalene triangle**, show that its all sides are unequal.

3. When four points are given and it is required to

(i) a **Square**, show that the four sides are equal and the diagonals are also equal.

(ii) a **Rhombus, (or equilateral trapezium)** show that the four sides are equal and the diagonals are not equal.

(iii) a **Rectangle**, show that the opposite sides are equal and the diagonals are also equal.

(iv) a **Parallelogram**, show that the opposite sides are equal and the diagonals are not equal.

(v) a **Trapezium**, show that the two sides are parallel and the other two sides are not parallel.

(vi) An Isosceles Trapezium, show that the two sides are parallel and the other two sides are not parallel but equal.

4. If A, B, C be the vertices of a triangle and we have to find the coordinates of the circumcentre then, let the circumcentre be $P(x, y)$ and use $PA^2 = PB^2$ and $PA^2 = PC^2$ this will give two equations in x and y then solve these two equations and (x, y).

Important Remarks for Objective Questions

(i) If (x_1, y_1) and (x_2, y_2) are the ends of the hypotenuse of a right angled isosceles triangle, then the third vertex is given by

$$\left(\frac{(x_1 + x_2) \pm (y_1 - y_2)}{2}, \frac{(y_1 + y_2) \mp (x_1 - x_2)}{2}\right)$$

(ii) If two vertices of an equilateral triangle are (x_1, y_1) and (x_2, y_2), then coordinates of the third vertex are

$$\left(\frac{x_1 + x_2 \mp \sqrt{3}(y_2 - y_1)}{2}, \frac{y_1 + y_2 \pm \sqrt{3}(x_2 - x_1)}{2}\right)$$

Example 7. Prove that the distance of the point $(a\cos\alpha, a\sin\alpha)$ from the origin is independent of α.

Sol. Let $P \equiv (a\cos\alpha, a\sin\alpha)$ and $O \equiv (0, 0)$

$$\text{then} \quad |OP| = \sqrt{(a\cos\alpha - 0)^2 + (a\sin\alpha - 0)^2}$$
$$= \sqrt{(a^2\cos^2\alpha + a^2\sin^2\alpha)}$$
$$= \sqrt{a^2(\cos^2\alpha + \sin^2\alpha)} = \sqrt{a^2}$$
$$= |a|, \text{ which is independent of } \alpha.$$

Example 8. Find the distance between the points $(a\cos\alpha, a\sin\alpha)$ and $(a\cos\beta, a\sin\beta)$, where $a > 0$.

Sol. Let $P \equiv (a\cos\alpha, a\sin\alpha)$ and $Q \equiv (a\cos\beta, a\sin\beta)$

$$\text{then} \quad |PQ| = \sqrt{(a\cos\alpha - a\cos\beta)^2 + (a\sin\alpha - a\sin\beta)^2}$$
$$= \sqrt{a^2\{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2\}}$$
$$= \sqrt{a^2\{\cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta + \sin^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta\}}$$
$$= \sqrt{a^2\{1 + 1 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)\}}$$
$$= \sqrt{a^2(2 - 2\cos(\alpha - \beta))}$$
$$= \sqrt{2a^2(1 - \cos(\alpha - \beta))}$$
$$= \sqrt{\left(2a^2 \cdot 2\sin^2\left(\frac{\alpha - \beta}{2}\right)\right)}$$
$$= \sqrt{4a^2\sin^2\left(\frac{\alpha - \beta}{2}\right)}$$
$$= \left|2a\sin\left(\frac{\alpha - \beta}{2}\right)\right|$$
$$= 2a\left|\sin\left(\frac{\alpha - \beta}{2}\right)\right| \qquad (\because a > 0)$$

Example 9. If the point (x, y) be equidistant from the points $(6, -1)$ and $(2, 3)$, prove that $x - y = 3$.

Sol. Let $P \equiv (x, y)$, $A \equiv (6, -1)$ and $B \equiv (2, 3)$

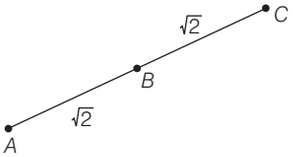
By the given condition, $PA = PB$

$$\Rightarrow \quad \sqrt{(x-6)^2 + (y+1)^2} = \sqrt{(x-2)^2 + (y-3)^2}$$
$$\text{or} \quad (x-6)^2 + (y+1)^2 = (x-2)^2 + (y-3)^2$$
$$\text{or} \quad x^2 - 12x + 36 + y^2 + 2y + 1 = x^2 - 4x + 4 + y^2 - 6y + 9$$
$$\text{or} \quad 8x - 8y = 24$$
$$\text{or} \quad x - y = 3$$

Example 10. Using distance formula, show that the points $(1, 5), (2, 4)$ and $(3, 3)$ are collinear.

Sol. Let $A \equiv (1, 5)$, $B \equiv (2, 4)$ and $C \equiv (3, 3)$ be the given points, then

$$|AB| = \sqrt{(1-2)^2 + (5-4)^2} = \sqrt{2}$$



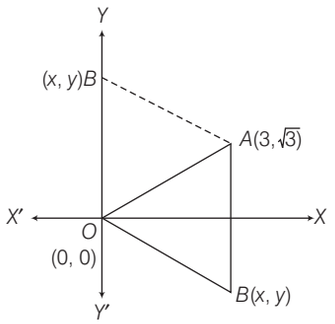
$$|BC| = \sqrt{(2-3)^2 + (4-3)^2} = \sqrt{2}$$
$$\text{and} \quad |AC| = \sqrt{(1-3)^2 + (5-3)^2} = 2\sqrt{2}$$
$$\text{Clearly,} \quad |AB| + |BC| = \sqrt{2} + \sqrt{2} = 2\sqrt{2} = |AC|.$$

Hence, A, B, C are collinear.

Example 11. An equilateral triangle has one vertex at the point $(0, 0)$ and another at $(3, \sqrt{3})$. Find the coordinates of the third vertex.

Sol. Let $O \equiv (0, 0)$ and $A \equiv (3, \sqrt{3})$ be the given points and let $B \equiv (x, y)$ be the required point. Then

$$OA = OB = AB$$



$$\Rightarrow \quad (OA)^2 = (OB)^2 = (AB)^2$$

$$\Rightarrow \quad (3-0)^2 + (\sqrt{3}-0)^2 = (x-0)^2 + (y-0)^2$$

$$= (x-3)^2 + (y-\sqrt{3})^2$$

$$\Rightarrow \quad 12 = x^2 + y^2 = x^2 + y^2 - 6x - 2\sqrt{3}y + 12$$

Taking first two members then

$$x^2 + y^2 = 12 \qquad \ldots(i)$$

and taking last two members, then

$$6x + 2\sqrt{3}y = 12 \text{ or } y = \sqrt{3}(2 - x) \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$x^2 + 3(2-x)^2 = 12$$

or $$4x^2 - 12x = 0$$

$$\Rightarrow \quad x = 0, 3$$

Putting $x = 0, 3$ in Eq. (ii), we get $y = 2\sqrt{3}, -\sqrt{3}$

Hence, the coordinates of the third vertex B are $(0, 2\sqrt{3})$ or $(3, -\sqrt{3})$.

Short Cut Method : According to important note :

$$\left(\frac{x_1 + x_2 \mp \sqrt{3}(y_2 - y_1)}{2}, \frac{y_1 + y_2 \pm \sqrt{3}(x_2 - x_1)}{2}\right)$$

i.e. $$\left(\frac{0 + 3 \mp \sqrt{3}(\sqrt{3} - 0)}{2}, \frac{0 + \sqrt{3} \pm \sqrt{3}(3 - 0)}{2}\right)$$

or $$\left(\frac{3 \mp 3}{2}, \frac{\sqrt{3} \pm 3\sqrt{3}}{2}\right)$$

$$\Rightarrow \quad (0, 2\sqrt{3}) \text{ or } (3, -\sqrt{3})$$

Example 12. Show that four points $(1, -2)$, $(3, 6)$, $(5, 10)$ and $(3, 2)$ are the vertices of a parallelogram.

Sol. Let $A \equiv (1, -2)$, $B \equiv (3, 6)$, $C \equiv (5, 10)$ and $D \equiv (3, 2)$ be the given points. Then

$$|AB| = \sqrt{(1-3)^2 + (-2-6)^2} = \sqrt{4 + 64} = 2\sqrt{17}$$

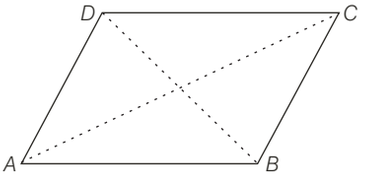
$$|BC| = \sqrt{(3-5)^2 + (6-10)^2} = \sqrt{4 + 16} = 2\sqrt{5}$$

$$|CD| = \sqrt{(5-3)^2 + (10-2)^2} = \sqrt{4 + 64} = 2\sqrt{17}$$

$$|AD| = \sqrt{(1-3)^2 + (-2-2)^2} = \sqrt{4 + 16} = 2\sqrt{5}$$

$$|AC| = \sqrt{(1-5)^2 + (-2-10)^2} = \sqrt{16 + 144} = 4\sqrt{10}$$

and $$|BD| = \sqrt{(3-3)^2 + (6-2)^2} = 4$$



Clearly, $|AB| = |CD|$, $|BC| = |AD|$ and $|AC| \neq |BD|$

Hence, $ABCD$ is a parallelogram.

Example 13. Let the opposite angular points of a square be $(3, 4)$ and $(1, -1)$. Find the coordinates of the remaining angular points.

Sol. Let $A(3, 4)$ and $C(1, -1)$ be the given angular points of a square $ABCD$ and let $B(x, y)$ be the unknown vertex. Then

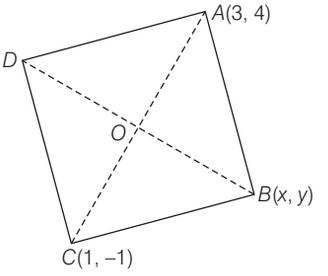
$$AB = BC$$

$$\Rightarrow \quad (AB)^2 = (BC)^2$$

$$\Rightarrow \quad (x-3)^2 + (y-4)^2 = (x-1)^2 + (y+1)^2$$

$$\Rightarrow \quad 4x + 10y - 23 = 0$$

$$\Rightarrow \quad x = \left(\frac{23 - 10y}{4}\right) \qquad \ldots(i)$$



Also, in ΔABC,

$$(AB)^2 + (BC)^2 = (AC)^2$$

$$\Rightarrow \quad (x-3)^2 + (y-4)^2 + (x-1)^2 + (y+1)^2$$

$$= (3-1)^2 + (4+1)^2$$

$$\Rightarrow \quad x^2 + y^2 - 4x - 3y - 1 = 0 \qquad \ldots(ii)$$

Substituting the value of x from Eqs. (i) into (ii), we get

$$\left(\frac{23 - 10y}{4}\right)^2 + y^2 - 4\left(\frac{23 - 10y}{4}\right) - 3y - 1 = 0$$

$$\Rightarrow \quad 4y^2 - 12y + 5 = 0$$

or $$(2y - 1)(2y - 5) = 0$$

$$\therefore \quad y = \frac{1}{2} \text{ or } \frac{5}{2}$$

Putting $y = \frac{1}{2}$ in Eq. (i), we get $x = \frac{9}{2}$,

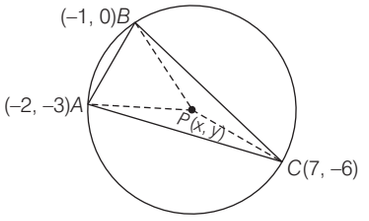
and putting $y = \frac{5}{2}$ in Eq. (i), we get $x = -\frac{1}{2}$

Hence, the required vertices of the square are $\left(\frac{9}{2}, \frac{1}{2}\right)$ and $\left(-\frac{1}{2}, \frac{5}{2}\right)$.

Example 14. Find the circumcentre of the triangle whose vertices are $(-2, -3)$, $(-1, 0)$ and $(7, -6)$. Also find the radius of the circumcircle.

Sol. Let $A \equiv (-2, -3)$, $B \equiv (-1, 0)$ and $C \equiv (7, -6)$.

Let $P \equiv (x, y)$ be the circumcentre of ΔABC.



Since, P is the circumcentre

$$\therefore \quad |PA| = |PB| = |PC| \Rightarrow (PA)^2 = (PB)^2 = (PC)^2$$

$$(x+2)^2 + (y+3)^2 = (x+1)^2 + (y-0)^2$$

$$= (x-7)^2 + (y+6)^2$$

$$\Rightarrow \quad x^2 + y^2 + 4x + 6y + 13 = x^2 + y^2 + 2x + 1$$

$$= x^2 + y^2 - 14x + 12y + 85$$

Taking first two members, we get

$$x + 3y + 6 = 0 \qquad \ldots(i)$$

and taking 1st and last member then, we get

$$3x - y - 12 = 0 \qquad \ldots(ii)$$

Solving Eqs. (i) and (ii), we get

$$x = 3, y = -3$$

Hence, circumcentre is $(3, -3)$.

Radius of the circumcircle

$$= PB = \sqrt{(3+1)^2 + (-3-0)^2}$$

$$= \sqrt{16+9} = 5 \text{ units}$$

Example 15. If the line segment joining the points $A(a,b)$ and $B(c,d)$ subtends an angle θ at the origin O, prove that

$$\cos\theta = \frac{ac+bd}{\sqrt{(a^2+b^2)(c^2+d^2)}}$$

or $$OA.OB\cos\theta = ac + bd$$

Sol. Let $OA = r_1$ and $OB = r_2$

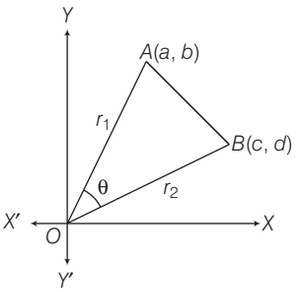
Now, $$r_1 = |OA| = \sqrt{(a^2 + b^2)} \qquad \ldots(i)$$

and $$r_2 = |OB| = \sqrt{(c^2 + d^2)} \qquad \ldots(ii)$$

Also, $$|AB| = \sqrt{(a-c)^2 + (b-d)^2}$$

$$= \sqrt{a^2 + b^2 + c^2 + d^2 - 2ac - 2bd}$$

$$= \sqrt{r_1^2 + r_2^2 - 2(ac + bd)}$$ [from Eqs. (i) and (ii)]



By using **Cosine formula** in ΔAOB

$$\cos\theta = \frac{(OA)^2 + (OB)^2 - (AB)^2}{2 \cdot OA \,.\, OB}$$

$$= \frac{r_1^2 + r_2^2 - (r_1^2 + r_2^2 - 2(ac + bd))}{2r_1 r_2}$$

$$= \frac{2(ac+bd)}{2r_1 r_2} = \frac{(ac+bd)}{r_1 r_2} \qquad \ldots(iii)$$

$$= \frac{(ac+bd)}{\sqrt{(a^2+b^2)}\sqrt{(c^2+d^2)}}$$ [from Eqs. (i) and (ii)]

$$= \frac{(ac+bd)}{\sqrt{(a^2+b^2)(c^2+d^2)}}$$

Also from Eq. (iii),

$$r_1 r_2 \cos\theta = ac + bd \quad \text{or} \quad OA.OB\cos\theta = ac + bd$$

Choice of Axes

For simplification we carefully choose the axes or the origin. Some situations are given below :

(i) If two lines are perpendicular then point of intersection is taken as origin and these lines must be taken as the coordinate axes.

(ii) If two fixed points A and B are given then we take $|AB| = 2a$ and the mid-point of AB as origin 'O', line AOB as X-axis and the line perpendicular to AB through O is taken as Y-axis then the coordinates of the fixed points are $(\pm a, 0)$. Similarly if AOB as Y-axis and the line perpendicular to AB through O is taken as X-axis then the coordinates of the fixed points are $(0, \pm a)$.

(iii) If there is a symmetry of any kind then take the coordinates of the points in a general way i.e. (x_i, y_i), $i = 1, 2, 3, \ldots$ etc.

Example 16. Show that the triangle, the coordinates of whose vertices are given by integers, can never be an equilateral triangle.

Sol. Let $A \equiv (0, 0)$, $B \equiv (a, 0)$ and $C \equiv (b, c)$ be the vertices of equilateral triangle ABC where a, b, c are integers then,

$$|AB| = |BC| = |CA|$$

$$\Rightarrow \quad (AB)^2 = (BC)^2 = (CA)^2$$

$$\Rightarrow \quad a^2 = (a-b)^2 + c^2 = b^2 + c^2$$

From first two members, we get

$$b^2 + c^2 = 2ab \qquad \ldots(i)$$

and taking first and third members, then

$$b^2 + c^2 = a^2 \qquad \ldots(ii)$$

From Eqs. (i) and (ii) we get

$$a = 2b \qquad (\because a \neq 0)$$

From Eq. (ii), $b^2 + c^2 = (2b)^2$

or $c^2 = 3b^2$

or $c = \pm b\sqrt{3}$

which is impossible, since b and c are integers.

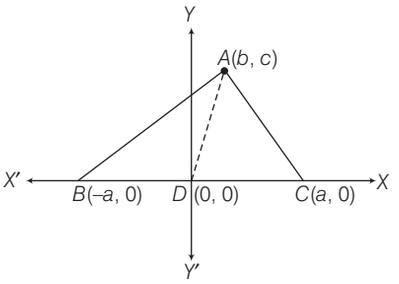
Example 17. In any triangle ABC, show that

$$AB^2 + AC^2 = 2(AD^2 + BD^2)$$

where, D is the middle point of BC.

Sol. Let D as the origin and DC and DY as the X and Y-axes respectively. Let $BC = 2a$, then

$B \equiv (-a, 0)$, $C \equiv (a, 0)$ and let $A \equiv (b, c)$



Now, LHS $= AB^2 + AC^2$

$$= (b+a)^2 + (c-0)^2 + (b-a)^2 + (c-0)^2$$

$$= 2(a^2 + b^2 + c^2) \qquad \ldots(i)$$

and RHS $= 2(AD^2 + BD^2)$

$$= 2\{(b-0)^2 + (c-0)^2 + a^2\}$$

$$= 2(a^2 + b^2 + c^2) \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we get

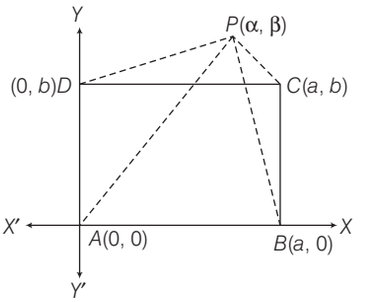
$$AB^2 + AC^2 = 2(AD^2 + BD^2)$$

Example 18. Let $ABCD$ be a rectangle and P be any point in its plane. Show that $PA^2 + PC^2 = PB^2 + PD^2$

Sol. Let A as the origin and AB and AD as the X and Y-axes respectively. Let $AB = a$ and $AD = b$, then

$B \equiv (a, 0)$, $D \equiv (0, b)$ and $C \equiv (a, b)$

Let $P \equiv (\alpha, \beta)$



Now, LHS $= PA^2 + PC^2$

$$= (\alpha - 0)^2 + (\beta - 0)^2 + (\alpha - a)^2 + (\beta - b)^2$$

$$= 2\alpha^2 + 2\beta^2 - 2a\alpha - 2b\beta + a^2 + b^2 \qquad \ldots(i)$$

and RHS $= PB^2 + PD^2$

$$= (\alpha - a)^2 + (\beta - 0)^2 + (\alpha - 0)^2 + (\beta - b)^2$$

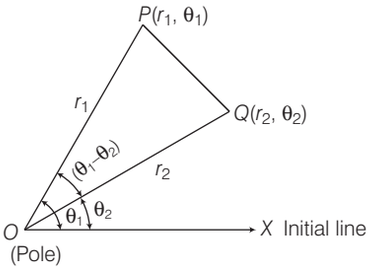
$$= 2\alpha^2 + 2\beta^2 - 2a\alpha - 2b\beta + a^2 + b^2 \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$PA^2 + PC^2 = PB^2 + PD^2$$

Distance between Two Points in Polar Coordinates

Let O be the pole and OX be the initial line. Let P and Q be two given points whose polar coordinates are (r_1, θ_1) and (r_2, θ_2) respectively.



Then, $OP = r_1$, $OQ = r_2$

and $\angle POX = \theta_1$, $\angle QOX = \theta_2$

$\therefore$ $\angle POQ = \theta_1 - \theta_2$

By using **Cosine formula** in ΔPOQ,

$$\cos(\angle POQ) = \frac{(OP)^2 + (OQ)^2 - (PQ)^2}{2(OP)(OQ)}$$

or $$\cos(\theta_1 - \theta_2) = \frac{r_1^2 + r_2^2 - (PQ)^2}{2\, r_1 r_2}$$

$\therefore$ $$|PQ| = \sqrt{(r_1^2 + r_2^2 - 2\, r_1 r_2 \cos(\theta_1 - \theta_2))}$$

Remark

Always taking θ_1 and θ_2 in radians.

Example 19. Prove that the points $(0, 0)$, $\left(3, \frac{\pi}{2}\right)$ and $\left(3, \frac{\pi}{6}\right)$ are the vertices of an equilateral triangle.

Sol. Let $A \equiv (0, 0)$, $B \equiv \left(3, \frac{\pi}{2}\right)$ and $C \equiv \left(3, \frac{\pi}{6}\right)$

Here, given coordinates are in polar form

$\therefore$ $$|AB| = \sqrt{(0^2 + 3^2 - 2 \cdot 0 \cdot 3 \cos\left(\frac{\pi}{2} - 0\right)} = 3 \text{ units}$$

$$|BC| = \sqrt{\left(3^2 + 3^2 - 2 \cdot 3 \cdot 3 \cos\left(\frac{\pi}{2} - \frac{\pi}{6}\right)\right)}$$

$$= \sqrt{\left(18 - 18 \sin\frac{\pi}{6}\right)} = \sqrt{(18 - 9)} = 3 \text{ units}$$

and $$|CA| = \sqrt{\left(3^2 + 0^2 - 2 \cdot 3 \cdot 0 \cos\left(\frac{\pi}{6} - 0\right)\right)} = 3 \text{ units}$$

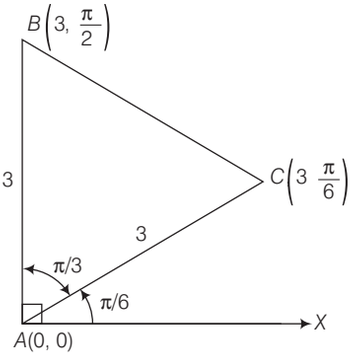
$\therefore$ $$|AB| = |BC| = |CA|$$

Hence, points A, B, C are the vertices of an equilateral triangle.

Aliter :

$\because$ $$\angle BAX = \frac{\pi}{2}$$

and $$\angle CAX = \frac{\pi}{6}$$



$\therefore$ $$\angle BAC = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$$

$\because$ In $\Delta\, ABC$

$$AB = AC$$

$\therefore$ $$\angle ACB = \angle ABC = \alpha \quad \text{(say)}$$

$\therefore$ $$\alpha + \alpha + \frac{\pi}{3} = \pi$$

or $$\alpha = \frac{\pi}{3}$$

Hence, $$|AB| = |BC| = |CA|.$$

Exercise for Session 2

1. If the distance between the points $(a, 2)$ and $(3, 4)$ be 8, then a equals to

(a) $2 + 3\sqrt{15}$ (b) $2 - 3\sqrt{15}$ (c) $2 \pm 3\sqrt{15}$ (d) $3 \pm 2\sqrt{15}$

2. The three points $(-2, 2)$, $(8, -2)$ and $(-4, -3)$ are the vertices of

(a) an isosceles triangle (b) an equilateral triangle (c) a right angled triangle (d) None of these

3. The distance between the points $\left(3, \frac{\pi}{4}\right)$ and $\left(7, \frac{5\pi}{4}\right)$ is

(a) 8 (b) 10 (c) 12 (d) 14

4. Let $A(6, -1)$, $B(1, 3)$ and $C(x, 8)$ be three points such that $AB = BC$, then the value of x are

(a) 3, 5 (b) –3, 5 (c) 3, – 5 (d) –3, – 5

5. The points $(a + 1, 1)$, $(2a + 1, 3)$ and $(2a + 2, 2a)$ are collinear, if

(a) $a = -1, 2$ (b) $a = \frac{1}{2}, 2$ (c) $a = 2, 1$ (d) $a = -\frac{1}{2}, 2$

6. If $A \equiv (3, 4)$ and B is a variable point on the lines $|x| = 6$. If $AB \leq 4$ then the number of positions of B with integral coordinates is

(a) 5 (b) 6 (c) 10 (d) 12

7. The number of points on X-axis which are at a distance c units $(c < 3)$ from $(2, 3)$ is

(a) 1 (b) 2 (c) 0 (d) 3

8. The point on the axis of y which its equidistant from $(-1, 2)$ and $(3, 4)$, is

(a) (0, 3) (b) (0, 4) (c) (0, 5) (d) (0, – 6)

9. Find the distance between the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$, where t_1 and t_2 are the roots of the equation $x^2 - 2\sqrt{3}x + 2 = 0$ and $a > 0$.

10. If $P(at^2, 2at)$, $Q\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$ and $S(a, 0)$ be any three points, show that $\frac{1}{SP} + \frac{1}{SQ}$ is independent of t.

11. Prove that the points $(3, 4)$, $(8, -6)$ and $(13, 9)$ are the vertices of a right angled triangle.

12. Show that the points $(0, -1)$, $(6, 7)$, $(-2, 3)$ and $(8, 3)$ are the vertices of a rectangle.

13. Find the circumcentre and circumradius of the triangle whose vertices are $(-2, 3)$, $(2, -1)$ and $(4, 0)$.

14. The vertices of a triangle are $A(1, 1)$, $B(4, 5)$ and $C(6, 13)$. Find $\cos A$.

15. Two opposite vertices of a square are $(2, 6)$ and $(0, -2)$. Find the coordinates of the other vertices.

16. If the point (x, y) is equidistant from the points $(a + b, b - a)$ and $(a - b, a + b)$, prove that $bx = ay$.

17. If a and b are real numbers between 0 and 1 such that the points $(a, 1)$, $(1, b)$ and $(0, 0)$ form an equilateral triangle, find a and b.

18. An equilateral triangle has one vertex at $(3, 4)$ and another at $(-2, 3)$. Find the coordinates of the third vertex.

19. If P be any point in the plane of square $ABCD$, prove that

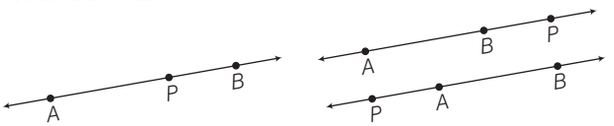
$$PA^2 + PC^2 = PB^2 + PD^2$$

Session 3

Section Formula, Centroid of a Triangle, Incentre, Some Standard Results, Area of Triangle

Section Formula

Definition : If P be any point on the line AB between A and B then we say that P divides segment AB *internally* in the ratio $AP : PB$.



Also, if P be any point on the line AB but not between A and B (P may be to the right or the left of the points A, B) then P divides AB *externally* in the ratio $AP : PB$

Note

$$\frac{AP}{PB} = \begin{cases} \text{Positive, in internally division} \\ \text{Negative, in externally division} \end{cases}$$

(i) Formula for Internal Division

Theorem : If the point $P(x, y)$ divides the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ internally in the ratio $m : n$, then prove that

$$x = \frac{mx_2 + nx_1}{m + n}$$

$$y = \frac{my_2 + ny_1}{m + n}$$

Proof : The given points are $A(x_1, y_1)$ and $B(x_2, y_2)$. Let us assume that the points A and B are both in 1st quadrant (for the sake of exactness). Since $P(x, y)$ divides AB internally in the ratio $m : n$ i.e. $AP : PB = m : n$. From A, B and P draw AL, BM and PN perpendiculars to X-axis. From A and P draw AH and PJ perpendiculars to PN and BM respectively, then

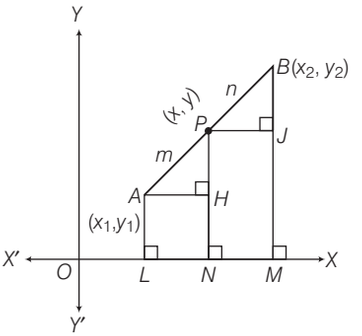
$OL = x_1, ON = x, OM = x_2, AL = y_1, PN = y$ and $BM = y_2$

$$\therefore \quad AH = LN = ON - OL = x - x_1$$

$$PJ = NM = OM - ON = x_2 - x$$

$$PH = PN - HN = PN - AL = y - y_1$$

and $$BJ = BM - JM = BM - PN = y_2 - y$$



Clearly, the Δs AHP and PJB are similar and therefore, their sides are proportional

$$\therefore \quad \frac{AH}{PJ} = \frac{PH}{BJ} = \frac{AP}{PB}$$

or $$\frac{x - x_1}{x_2 - x} = \frac{y - y_1}{y_2 - y} = \frac{m}{n}$$

(i) (ii) (iii)

From Eqs. (i) and (iii), we have

$$\frac{x - x_1}{x_2 - x} = \frac{m}{n}$$

$$\Rightarrow \quad nx - nx_1 = mx_2 - mx$$

$$\Rightarrow \quad (m + n)\, x = mx_2 + nx_1$$

$$\therefore \quad x = \frac{mx_2 + nx_1}{m + n}$$

and from Eqs. (ii) and (iii), we have

$$\frac{y - y_1}{y_2 - y} = \frac{m}{n}$$

$$\Rightarrow \quad ny - ny_1 = my_2 - my$$

$$\Rightarrow \quad (m + n)\, y = my_2 + ny_1$$

$$\therefore \quad y = \frac{my_2 + ny_1}{m + n}$$

Thus, the coordinates of P are

$$\left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n} \right).$$

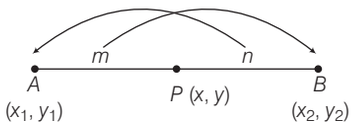
***Corollary* 1 :** The above section formula is true for all positions of the points (i.e. either point or both points are not in the 1st quadrant), keeping in mind, the proper signs of their coordinates.

***Corollary* 2 :** If P is the mid-point of AB then $m = n$, the coordinates of the middle-point of AB are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

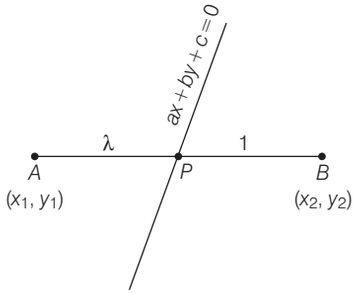
Remarks

1. If $P(\alpha, \beta)$ be the mid-point of AB and if coordinates of A are (λ, μ) then the coordinates of B are $(2\alpha - \lambda, 2\beta - \mu)$, i.e. (Double the x-co-ordinate of mid point – x-coordinate of given point, Double the y-co-ordinate of mid point – y-coordinate of given point).
2. The following diagram will help to remember the section formula.



3. For finding ratio, use ratio $\lambda : 1$, then coordinates of P are $\left(\frac{x_1 + \lambda x_2}{1 + \lambda}, \frac{y_1 + \lambda y_2}{1 + \lambda}\right)$. If λ is positive then divides internally and if λ is negative, then divides externally.
4. The straight line $ax + by + c = 0$ divides the joint of points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio

$$\frac{AP}{PB} = \frac{\lambda}{1} = -\frac{(ax_1 + by_1 + c)}{(ax_2 + by_2 + c)}$$



If ratio is positive, then divides internally and if ratio is negative then divides externally.

Proof : Coordinates of P are $\left(\frac{x_1 + \lambda x_2}{1 + \lambda}, \frac{y_1 + \lambda y_2}{1 + \lambda}\right)$

$\because P$ lies on the line $ax + by + c = 0$, then

$$a\left(\frac{x_1 + \lambda x_2}{1 + \lambda}\right) + b\left(\frac{y_1 + \lambda y_2}{1 + \lambda}\right) + c = 0$$

or $$(ax_1 + by_1 + c) + \lambda(ax_2 + by_2 + c) = 0$$

or $$\frac{\lambda}{1} = -\frac{(ax_1 + by_1 + c)}{(ax_2 + by_2 + c)}$$

5. The line joining the points (x_1, y_1) and (x_2, y_2) is divided by the X-axis in the ratio $-\frac{y_1}{y_2}$ and by Y-axis in the ratio $-\frac{x_1}{x_2}$.
6. In square, rhombus, rectangle and parallelogram diagonals bisect to each other.

Example 20. Find the coordinates of the point which divides the line segment joining the points $(5, -2)$ and $(9, 6)$ in the ratio $3 : 1$.

Sol. Let the required point be (x, y), then

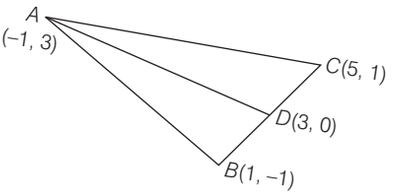
$$x = \left(\frac{3 \times 9 + 1 \times 5}{3 + 1}\right) = 8$$

and $$y = \left(\frac{3 \times 6 + 1 \times (-2)}{3 + 1}\right) = 4$$

Thus, the coordinates of the required point are $(8, 4)$.

Example 21. Find the length of median through A of a triangle whose vertices are $A(-1, 3), B(1, -1)$ and $C(5, 1)$.

Sol. Let D be the mid-point of BC, then coordinates of D are $\left(\frac{1+5}{2}, \frac{-1+1}{2}\right)$ i.e. $(3, 0)$



$\therefore$ Median $AD = \sqrt{(3+1)^2 + (0-3)^2}$

$= \sqrt{16 + 9} = \sqrt{25}$

$= 5$ units

Example 22. Determine the ratio in which $y - x + 2 = 0$ divides the line joining $(3, -1)$ and $(8, 9)$.

Sol. Suppose the line $y - x + 2 = 0$ divides the line segment joining $A(3, -1)$ and $B(8, 9)$ in the ratio $\lambda : 1$ at point P, then the coordinates of the point P are $\left(\frac{8\lambda + 3}{\lambda + 1}, \frac{9\lambda - 1}{\lambda + 1}\right)$.

But P lies on $y - x + 2 = 0$ therefore

$$\left(\frac{9\lambda - 1}{\lambda + 1}\right) - \left(\frac{8\lambda + 3}{\lambda + 1}\right) + 2 = 0$$

$\Rightarrow$ $9\lambda - 1 - 8\lambda - 3 + 2\lambda + 2 = 0$

$\Rightarrow$ $3\lambda - 2 = 0$ or $\lambda = \frac{2}{3}$

So, the required ratio is $\frac{2}{3} : 1$, i.e. $2 : 3$ (internally) since here λ is positive.

Shortcut method

According to Remark 4 :

$$\lambda = -\left(\frac{-1 - 3 + 2}{9 - 8 + 2}\right) = \frac{2}{3}$$

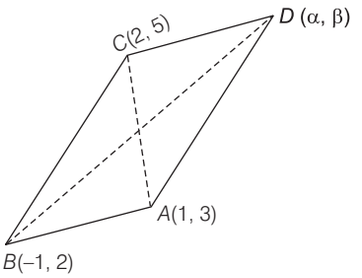
or $\lambda : 1 = 2 : 3$

Example 23. The coordinates of three consecutive vertices of a parallelogram are (1, 3), (–1, 2) and (2, 5). Then find the coordinates of the fourth vertex.

Sol. Let the fourth vertex be $D(\alpha, \beta)$. Since $ABCD$ is a parallelogram, the diagonals bisect to each other.
i.e. mid-point of BD = mid-point of AC

$$\therefore \quad \left(\frac{\alpha-1}{2}, \frac{\beta+2}{2}\right) = \left(\frac{2+1}{2}, \frac{5+3}{2}\right)$$

or $$\left(\frac{\alpha-1}{2}, \frac{\beta+2}{2}\right) = \left(\frac{3}{2}, 4\right)$$



On equating abscissaes and ordinates, we get

$$\frac{\alpha-1}{2} = \frac{3}{2} \quad \text{or} \quad \alpha - 1 = 3 \quad \text{or} \quad \alpha = 4$$

and $$\frac{\beta+2}{2} = 4 \quad \text{or} \quad \beta + 2 = 8 \quad \text{or} \quad \beta = 6$$

Hence, the coordinates of the fourth vertex $D(\alpha, \beta)$ is $(4, 6)$.

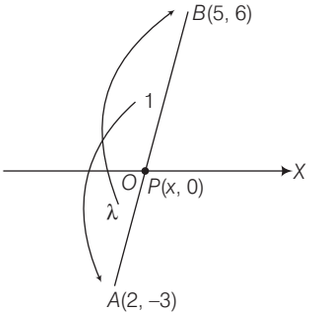
Example 24. In what ratio does X-axis divide the line segment joining $(2, -3)$ and $(5, 6)$?

Sol. Let the given points be $A(2, -3)$ and $B(5, 6)$. Let AB be divided by the X-axis at $P(x, 0)$ in the ratio $\lambda : 1$ internally. Considering the ordinate of P, then

$$0 = \frac{\lambda \times 6 + 1 \times (-3)}{\lambda + 1}$$

or $$\lambda = \frac{1}{2}$$

$\therefore$ The ratio is $\frac{1}{2} : 1$ i.e. $1 : 2$ (Internally)



Shortcut Method

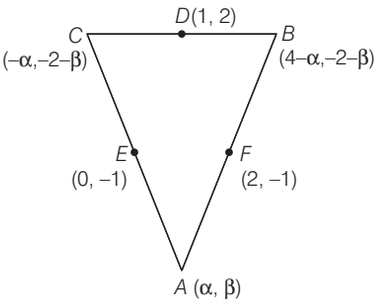
According to Remark 5 :

$$\frac{\lambda}{1} = -\frac{y_1}{y_2} = \frac{-(-3)}{6} = \frac{1}{2}$$

$\therefore$ The ratio is $\frac{1}{2} : 1$ i.e. $1 : 2$ (internally)

Example 25. The mid-points of the sides of a triangle are $(1, 2), (0, -1)$ and $(2, -1)$. Find the coordinates of the vertices of a triangle with the help of two unknowns.

Sol. Let $D(1, 2)$, $E(0, -1)$ and $F(2, -1)$ be the mid-points of BC, CA and AB respectively.
Let the coordinates of A be (α, β) then coordinates of B and C are $(4 - \alpha, -2 - \beta)$ and $(-\alpha, -2 - \beta)$ respectively (see note 1)
$\because$ D is the mid-point of B and C



then $$1 = \frac{4 - \alpha - \alpha}{2}$$

$\Rightarrow$ $$1 = 2 - \alpha \quad \text{or} \quad \alpha = 1$$

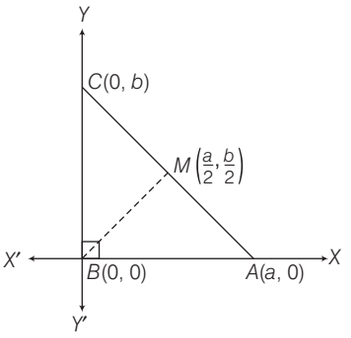
and $$2 = \frac{-2 - \beta - 2 - \beta}{2}$$

$\Rightarrow$ $$2 = -2 - \beta \quad \text{or} \quad \beta = -4$$

Hence, coordinates of A, B and C are $(1, -4), (3, 2)$ and $(-1, 2)$ respectively.

Example 26. Prove that in a right angled triangle the mid-point of the hypotenuse is equidistant from its vertices.

Sol. Let the given right angled triangle be ABC, with right angled at B. We take B as the origin and BA and BC as the X and Y-axes respectively.
Let $BA = a$ and $BC = b$
then $A \equiv (a, 0)$ and $C \equiv (0, b)$
Let M to be the mid-point of the hypotenuse AC, then coordinates of M are $\left(\frac{a}{2}, \frac{b}{2}\right)$



$$\therefore \quad |AM| = \sqrt{\left(a-\frac{a}{2}\right)^2+\left(0-\frac{b}{2}\right)^2} = \frac{\sqrt{(a^2+b^2)}}{2} \quad \text{... (i)}$$

$$|BM| = \sqrt{\left(0-\frac{a}{2}\right)^2+\left(0-\frac{b}{2}\right)^2} = \frac{\sqrt{(a^2+b^2)}}{2} \quad \text{...(ii)}$$

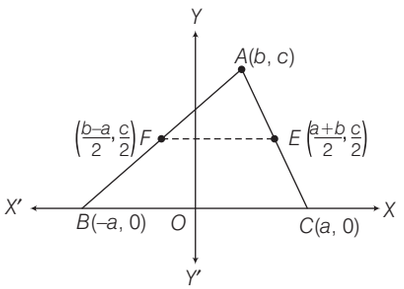
and $$|CM| = \sqrt{\left(0-\frac{a}{2}\right)^2+\left(b-\frac{b}{2}\right)^2} = \frac{\sqrt{(a^2+b^2)}}{2} \quad \text{...(iii)}$$

From Eqs. (i), (ii) and (iii), we get

$$|AM| = |BM| = |CM|$$

Example 27 Show that the line joining the mid-points of any two sides of a triangle is half the third side.

Sol. We take O as the origin and OC and OY as the X and Y-axes respectively.



Let $BC = 2a$, then $B \equiv (-a, 0)$, $C \equiv (a, 0)$

Let $A \equiv (b, c)$, if E and F are the mid-points of sides AC and AB respectively.

Then, $$E \equiv \left(\frac{a+b}{2}, \frac{c}{2}\right) \text{ and } F \equiv \left(\frac{b-a}{2}, \frac{c}{2}\right)$$

Now, $$FE = \sqrt{\left(\frac{a+b}{2}-\frac{b-a}{2}\right)^2+\left(\frac{c}{2}-\frac{c}{2}\right)^2} = a$$

$$= \frac{1}{2}(2a) = \frac{1}{2}(BG)$$

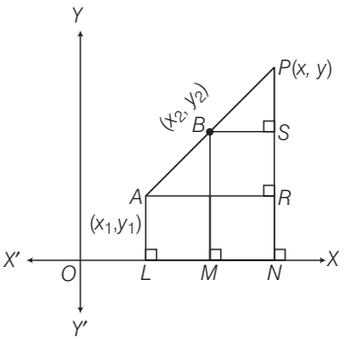
Hence, the line joining the mid-points of any two sides of a triangle is half the third side.

(ii) Formula for External Division

Theorem : If the point $P(x, y)$ divides the line joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ externally in the ratio $m : n$ then prove that

$$x = \frac{mx_2 - nx_1}{m-n}, \; y = \frac{my_2 - ny_1}{m-n}$$

Proof : The given points are $A(x_1, y_1)$ and $B(x_2, y_2)$. Let us assume that the points A and B are both in the 1st quadrant (for the sake of exactness). Let $P(x, y)$ be the point which divides AB externally in the ratio $m : n$, so that $\frac{AP}{BP} = \frac{m}{n}$.



From A, B and P draw AL, BM and PN perpendiculars on X-axis. Also, from A and B draw AR and BS perpendiculars on PN,

then $$AR = LN = ON - OL = x - x_1$$

$$BS = MN = ON - OM = x - x_2$$

$$PR = PN - RN = PN - AL = y - y_1$$

and $$PS = PN - SN = PN - BM = y - y_2$$

Clearly, the Δs APR and BPS are similar and therefore their sides are proportional.

$$\therefore \quad \frac{AP}{PB} = \frac{AR}{BS} = \frac{PR}{PS}$$

or $$\frac{m}{n} = \frac{x-x_1}{x-x_2} = \frac{y-y_1}{y-y_2}$$

(i) (ii) (iii)

From Eqs. (i) and (ii), we have

$$\frac{m}{n} = \frac{x-x_1}{x-x_2}$$

$$\Rightarrow \quad mx - mx_2 = nx - nx_1$$

$$\Rightarrow \quad (m-n)x = mx_2 - nx_1$$

or $$x = \frac{mx_2 - nx_1}{m-n}$$

Also, from Eqs. (i) and (iii), we have

$$\frac{m}{n}=\frac{y-y_1}{y-y_2}$$

$$\Rightarrow \quad my-my_2=ny-ny_1$$

$$\Rightarrow \quad (m-n)\,y=my_2-ny_1$$

or $$y=\frac{my_2-ny_1}{m-n}$$

Thus, the coordinates of P are $\left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}\right)$.

(Here, $m \neq n$)

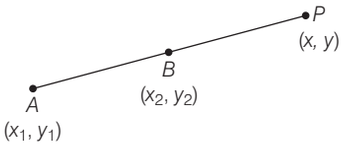
***Corollary* 1 :** The above formula is true for all positions of the points, keeping in mind, the proper signs of their coordinates.

***Corollary* 2 :** The above coordinates can also be expressed as

$$\left(\frac{mx_2+(-n)\,x_1}{m+(-n)}, \frac{my_2+(-n)\,y_1}{m+(-n)}\right)$$

and this can be thought of as the coordinates of the point dividing AB internally in the ratio $m:-n$

***Corollary* 3 :** $\because \quad \frac{AP}{PB}=\frac{m}{n}$



or $$\frac{AP}{PB}-1=\frac{m}{n}-1$$

or $$\frac{AP-PB}{PB}=\frac{m-n}{n}$$

or $$\frac{AB}{PB}=\frac{m-n}{n}$$

Now, we can say that B divides AP in the ratio $m-n:n$ internally.

i.e. $$x_2=\frac{(m-n)\,x+nx_1}{(m-n)+n} \quad \Rightarrow \quad x=\frac{mx_2-nx_1}{m-n}$$

and $$y_2=\frac{(m-n)\,y+ny_1}{(m-n)+n} \quad \Rightarrow \quad y=\frac{my_2-ny_1}{m-n}$$

***Corollary* 4 :** (for proving A, B and C are collinear)

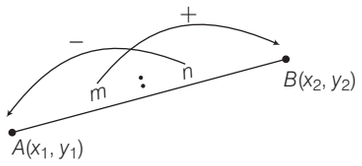
If A, B, C three points are collinear then let C divides AB in the ratio $\lambda:1$ internally.

If $\lambda =+$ ve rational, then divide internally

and if $\lambda =-$ ve rational, then divide externally.

Remarks

1. The following diagram will help to remember the section formula



2. Let $\frac{m}{n}=\lambda$, then $\left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}\right)$

or $$\left(\frac{\frac{m}{n}x_2-x_1}{\frac{m}{n}-1}, \frac{\frac{m}{n}y_2-y_1}{\frac{m}{n}-1}\right) \quad \text{or} \quad \left(\frac{\lambda x_2-x_1}{\lambda-1}, \frac{\lambda y_2-y_1}{\lambda-1}\right)$$

Example 28. Find the coordinates of a point which divides externally the line joining $(1,-3)$ and $(-3,9)$ in the ratio $1:3$.

Sol. Let the coordinates of the required point be $P(x, y)$,

Then, $$x=\left(\frac{1\times(-3)-3\times 1}{1-3}\right)$$

and $$y=\left(\frac{1\times 9-3\times(-3)}{1-3}\right)$$

i.e. $$x=3 \quad \text{and} \quad y=-9$$

Hence, the required point is $(3,-9)$.

Example 29. The line segment joining $A(6,3)$ to $B(-1,-4)$ is doubled in length by having its length added to each end. Find the coordinates of the new ends.

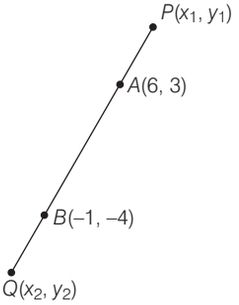
Sol. Let P and Q be the required new ends

Let the coordinates of P be (x_1, y_1)

Given, $$AB=2AP$$

$$\Rightarrow \quad \frac{AB}{AP}=\frac{2}{1}$$

i.e. A divides BP internally in the ratio $2:1$.



Then, $$6=\frac{2\times x_1+1\times(-1)}{2+1}$$

$\Rightarrow \quad 19 = 2x_1$ or $x_1 = \frac{19}{2}$

and $\quad 3 = \frac{2 \times y_1 + 1 \times (-4)}{2+1}$

$\Rightarrow \quad 13 = 2y_1$ or $y_1 = \frac{13}{2}$

$\therefore$ Coordinates of P are $\left(\frac{19}{2}, \frac{13}{2}\right)$.

Also, let coordinates of Q be (x_2, y_2)

Given, $\quad AB = 2BQ \Rightarrow \frac{AB}{BQ} = \frac{2}{1}$

i.e. B divides AQ internally in the ratio 2 : 1

Then $\quad -1 = \frac{2 \times x_2 + 1 \times 6}{2+1}$

$\Rightarrow \quad -9 = 2x_2$ or $x_2 = -\frac{9}{2}$

and $\quad -4 = \frac{2 \times y_2 + 1 \times 3}{2+1}$

$\Rightarrow \quad -15 = 2y_2$ or $y_2 = -\frac{15}{2}$

$\therefore$ Coordinates of Q are $\left(-\frac{9}{2}, -\frac{15}{2}\right)$

Aliter : $\because \quad AB = 2AP$

$\Rightarrow \quad \frac{AB}{AP} = \frac{2}{1} \Rightarrow \frac{AB}{AP} + 1 = \frac{2}{1} + 1$

$\Rightarrow \quad \frac{AB + AP}{AP} = \frac{3}{1} \Rightarrow \frac{BP}{AP} = \frac{3}{1}$

$\therefore$ P divides AB externally in the ratio 1 : 3

Then, $\quad x_1 = \frac{1 \times (-1) - 3 \times 6}{1-3} = \frac{19}{2}$

and $\quad y_1 = \frac{1 \times (-4) - 3 \times 3}{1-3} = \frac{13}{2}$

$\therefore$ Coordinates of P are $\left(\frac{19}{2}, \frac{13}{2}\right)$

Also, $\quad AB = 2BQ$

$\Rightarrow \quad \frac{AB}{BQ} = \frac{2}{1} \Rightarrow \frac{AB}{BQ} + 1 = \frac{2}{1} + 1$

$\Rightarrow \quad \frac{AB + BQ}{BQ} = \frac{3}{1} \Rightarrow \frac{AQ}{BQ} = \frac{3}{1}$

$\therefore$ Q divides AB externally in the ratio 3 : 1

then, $\quad x_2 = \frac{3 \times (-1) - 1 \times 6}{3-1} = -\frac{9}{2}$

and $\quad y_2 = \frac{3 \times (-4) - 1 \times 3}{3-1} = -\frac{15}{2}$

$\therefore$ Coordinates of Q are $\left(-\frac{9}{2}, -\frac{15}{2}\right)$.

Example 30. Using section formula show that the points $(1, -1), (2, 1)$ and $(4, 5)$ are collinear.

Sol. Let $A \equiv (1, -1)$, $B \equiv (2, 1)$ and $C \equiv (4, 5)$

Suppose C divides AB in the ratio $\lambda : 1$ internally, then

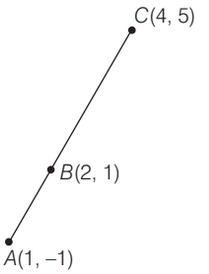
$$4 = \frac{\lambda \times 2 + 1 \times 1}{\lambda + 1}$$

$\Rightarrow \quad 4\lambda + 4 = 2\lambda + 1$

or $\quad \lambda = -\frac{3}{2}$

i.e. C divides AB in the ratio 3 : 2 (externally).

Hence, A, B, C are collinear.



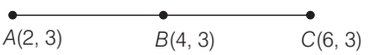
Example 31. Find the ratio in which the point $(2, y)$ divides the line segment joining $(4, 3)$ and $(6, 3)$ and hence find the value of y.

Sol. Let $A \equiv (4, 3)$, $B \equiv (6, 3)$ and $P \equiv (2, y)$

Let P divides AB internally in the ratio $\lambda : 1$

then, $\quad 2 = \frac{6\lambda + 4}{\lambda + 1} \Rightarrow 2\lambda + 2 = 6\lambda + 4$

$\Rightarrow \quad -4\lambda = 2$ or $\lambda = -\frac{1}{2}$



$\therefore$ P divides AB externally in the ratio 1 : 2 ($\because \lambda$ is negative)

Now, $y = \frac{1 \times 3 - 2 \times 3}{1 - 2} = 3$

(iii) Harmonic Conjugates

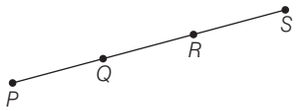
If four points in a line, then the system is said to form a range. Let four points say P, Q, R, S.

If the range (PQ, RS) has a cross ratio equal to -1, then it is called harmonic.

i.e. $\quad \frac{PR}{RQ} \cdot \frac{SQ}{SP} = -1 \Rightarrow \frac{PR}{RQ} = -\frac{SP}{SQ} = \lambda$ (say)

$\therefore \quad \frac{PR}{RQ} = \frac{\lambda}{1} \Rightarrow PR : RQ = \lambda : 1$ (internally)

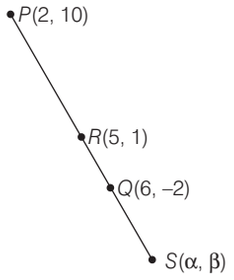
and $\quad \frac{SP}{SQ} = -\frac{\lambda}{1} \Rightarrow PS : SQ = \lambda : 1$ (externally)



Hence, R and S are called the **harmonic conjugates** to each other with respect to the points P and Q.

Example 32 Find the harmonic conjugates of the point $R(5,1)$ with respect to the points $P(2,10)$ and $Q(6,-2)$.

Sol. Let $S(\alpha, \beta)$ (be the harmonic conjugates of the point $R(5,1)$). Suppose R divides PQ in the ratio $\lambda : 1$ internally, then S divides PQ in the ratio $\lambda : 1$ externally, then



$$5 = \frac{6\lambda + 2}{\lambda + 1} \Rightarrow 5\lambda + 5 = 6\lambda + 2$$

$\therefore \quad \lambda = 3$

Also, $\quad 1 = \dfrac{-2\lambda + 10}{\lambda + 1}$

$$\lambda + 1 = -2\lambda + 10 \Rightarrow 3\lambda = 9$$

$\therefore \quad \lambda = 3$

Now, $\quad \alpha = \dfrac{3 \times 6 - 1 \times 2}{3 - 1} = 8$

and $\quad \beta = \dfrac{3 \times (-2) - 1 \times 10}{3 - 1}$

$$= -8$$

Hence, harmonic conjugates of $R(5,1)$ is $S(8,-8)$.

Centroid of a Triangle

Definition : The point of intersection of the medians of a triangle is called the centroid of the triangle and it divides the median internally in the ratio 2 : 1.

Theorem : Prove that the coordinates of the centroid of the triangle whose vertices are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

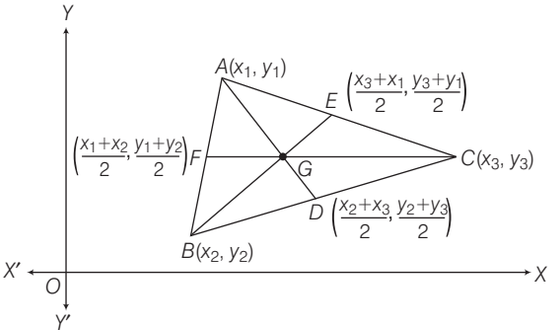
Also, deduce that the medians of a triangle are concurrent.

Proof : Let $A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$ and $C \equiv (x_3, y_3)$ be the vertices of the triangle ABC. Let us assume that the points A, B and C are in the 1st quadrant (for the sake of exactness) whose medians are AD, BE and CF respectively so D, E and F are respectively the mid-points of BC, CA and AB then the coordinates of D, E, F are

$$D \equiv \left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$$

$$E \equiv \left(\frac{x_3 + x_1}{2}, \frac{y_3 + y_1}{2}\right)$$

and $\quad F \equiv \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$



The coordinates of a point dividing AD in the ratio 2 : 1 are

$$\left(\frac{2 \cdot \left(\dfrac{x_2 + x_3}{2}\right) + 1 \cdot x_1}{2+1}, \frac{2 \cdot \left(\dfrac{y_2 + y_3}{2}\right) + 1 \cdot y_1}{2+1}\right)$$

or $\quad \left(\dfrac{x_1 + x_2 + x_3}{3}, \dfrac{y_1 + y_2 + y_3}{3}\right)$

and the coordinates of a point dividing BE in the ratio 2 : 1 are

$$\left(\frac{2 \cdot \left(\dfrac{x_3 + x_1}{2}\right) + 1 \cdot x_2}{2+1}, \frac{2 \cdot \left(\dfrac{y_3 + y_1}{2}\right) + 1 \cdot y_2}{2+1}\right)$$

or $\quad \left(\dfrac{x_1 + x_2 + x_3}{3}, \dfrac{y_1 + y_2 + y_3}{3}\right)$

Similarly the coordinates of a point dividing CF in the ratio 2 : 1 are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

$\therefore$ The common point which divides AD, BE and CF in the ratio 2 : 1 is

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

Hence, medians of a triangle are concurrent and the coordinates of the centroid are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

Important Theorem

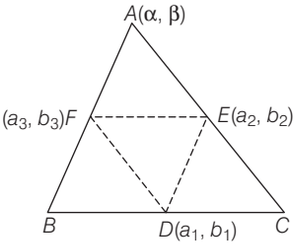
Centroid of the triangle obtained by joining the middle points of the sides of a triangle is the same as the centroid of the original triangle.

Or

If $(a_1, b_1), (a_2, b_2)$ and (a_3, b_3) are the mid-points of the sides of a triangle, then its centroid is given by

$$\left(\frac{a_1 + a_2 + a_3}{3}, \frac{b_1 + b_2 + b_3}{3}\right)$$

Proof : Let D, E, F are the mid-points of BC, CA and AB respectively now let coordinates of A are (α, β) then coordinates of B and C are $(2a_3 - \alpha, 2b_3 - \beta)$ and $(2a_2 - \alpha, 2b_2 - \beta)$ are respectively.



$\because$ $D(a_1, b_1)$ is the mid-point of B and C, then

$$2a_1 = 2a_3 - \alpha + 2a_2 - \alpha \quad \Rightarrow \quad \alpha = a_2 + a_3 - a_1$$

and $$2b_1 = 2b_3 - \beta + 2b_2 - \beta \quad \Rightarrow \quad \beta = b_2 + b_3 - b_1$$

Now, coordinates of B are $(2a_3 - \alpha, 2b_3 - \beta)$

or $$(a_3 + a_1 - a_2, b_3 + b_1 - b_2)$$

and coordinates of C are $(2a_2 - \alpha, 2b_2 - \beta)$

or $$(a_2 + a_1 - a_3, b_2 + b_1 - b_3)$$

Hence, coordinates of A, B and C are

$$A \equiv (a_2 + a_3 - a_1, b_2 + b_3 - b_1),$$
$$B \equiv (a_3 + a_1 - a_2, b_3 + b_1 - b_2)$$

and $$C \equiv (a_2 + a_1 - a_3, b_2 + b_1 - b_3)$$

$\therefore$ Coordinates of centroid of triangle ABC are

$$\left(\frac{a_1 + a_2 + a_3}{3}, \frac{b_1 + b_2 + b_3}{3}\right)$$

which is same as the centroid of triangle DEF.

***Corollary* 1 (Finger Rule) :** If mid-points of the sides of a triangle are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3), then coordinates of the original triangle are

$$(x_2 + x_3 - x_1, y_2 + y_3 - y_1),$$
$$(x_3 + x_1 - x_2, y_3 + y_1 - y_2)$$

and $$(x_1 + x_2 - x_3, y_1 + y_2 - y_3).$$

***Corollary* 2 :** If two vertices of a triangle are (x_1, y_1) and (x_2, y_2) and the coordinates of centroid are (α, β), then coordinates of the third vertex are

$$(3\alpha - x_1 - x_2, 3\beta - y_1 - y_2)$$

***Corollary* 3 :** According to important theorem Δs ABC and DEF are similar

$$\therefore \quad \frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta DEF} = \frac{(BC)^2}{(EF)^2}$$

$$= \frac{4\{(a_2 - a_3)^2 + (b_2 - b_3)^2\}}{\{(a_2 - a_3)^2 + (b_2 - b_3)^2\}} = 4$$

$\therefore$ Area of $\Delta ABC = 4 \times$ Area of ΔDEF

i.e. Area of a triangle is four times the area of the triangle formed by joining the mid-points of its sides.

Example 33. Two vertices of a triangle are $(-1, 4)$ and $(5, 2)$. If its centroid is $(0, -3)$, find the third vertex.

Sol. Let the third vertex be (x, y) then the coordinates of the centroid of triangle are

$$\left(\frac{-1 + 5 + x}{3}, \frac{4 + 2 + y}{3}\right) \text{ i.e. } \left(\frac{4 + x}{3}, \frac{6 + y}{3}\right)$$

Now, $$\left(\frac{4 + x}{3}, \frac{6 + y}{3}\right) = (0, -3)$$

$$\Rightarrow \quad \frac{4 + x}{3} = 0 \quad \text{and} \quad \frac{6 + y}{3} = -3$$

$$\Rightarrow \quad 4 + x = 0 \quad \text{and} \quad y + 6 = -9$$

or $$x = -4 \quad \text{and} \quad y = -15$$

Hence, the third vertex is $(-4, -15)$.

Shortcut Method

According to corollary 2

$$(x, y) = (3 \times 0 - (-1) - 5, 3 \times (-3) - 4 - 2)$$
$$= (-4, -15)$$

Example 34. ***The vertices of a triangle are*** $(1, 2), (h, -3)$ ***and*** $(-4, k)$. ***Find the value of*** $\sqrt{\{(h+k)^2 + (h+3k)^2\}}$. ***If the centroid of the triangle be at the point*** $(5, -1)$.

Sol. Here, $$\frac{1 + h - 4}{3} = 5 \quad \text{and} \quad \frac{2 - 3 + k}{3} = -1$$

then, we get $h = 18, k = -2$

$$\therefore \quad = \sqrt{(h + k)^2 + (h + 3k)^2}$$
$$= \sqrt{(18 - 2)^2 + (18 - 6)^2}$$
$$= \sqrt{(16^2 + 12^2)} = 20$$

Example 35. If $D(-2, 3)$, $E(4, -3)$ and $F(4, 5)$ are the mid-points of the sides BC, CA and AB of triangle ABC, then find $\sqrt{(|AG|^2 + |BG|^2 - |CG|^2)}$ where, G is the centroid of ΔABC.

Sol. Let the coordinates of A be (α, β)

then coordinates of B are $(8 - \alpha, 10 - \beta)$

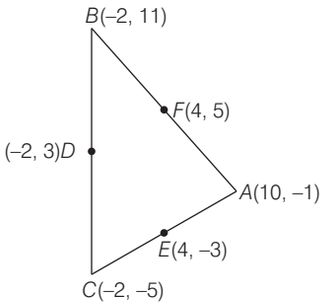
and coordinates of C are $(8 - \alpha, -6 - \beta)$

$\because$ D is the mid-point of BC, then

$$\frac{8 - \alpha + 8 - \alpha}{2} = -2$$

and $$\frac{10 - \beta - 6 - \beta}{2} = 3$$

i.e. $\alpha = 10$ and $\beta = -1$



$\therefore$ Coordinates of A, B, C are $(10, -1)$, $(-2, 11)$ and $(-2, -5)$ respectively.

Now, coordinates of centroid

$$G \equiv \left(\frac{10 - 2 - 2}{3}, \frac{-1 + 11 - 5}{3}\right)$$

i.e. $$G \equiv \left(2, \frac{5}{3}\right)$$

$\therefore$ $$AG = \sqrt{(10 - 2)^2 + \left(-1 - \frac{5}{3}\right)^2}$$

$$= \sqrt{\left(64 + \frac{64}{9}\right)} = \frac{8}{3}\sqrt{(10)}$$

$$BG = \sqrt{(-2 - 2)^2 + \left(11 - \frac{5}{3}\right)^2}$$

$$= \sqrt{16 + \frac{(28)^2}{9}} = \frac{4}{3}\sqrt{(58)}$$

and $$CG = \sqrt{(-2 - 2)^2 + \left(-5 - \frac{5}{3}\right)^2}$$

$$= \sqrt{\left(16 + \frac{400}{9}\right)} = \frac{4}{3}\sqrt{(34)}$$

Hence, $\sqrt{(|AG|^2 + |BG|^2 - |CG|^2)}$

$$= \sqrt{\left(\frac{64}{9} \times 10 + \frac{16}{9} \times 58 - \frac{16}{9} \times 34\right)}$$

$$= \sqrt{\frac{32}{9}(20 + 29 - 17)}$$

$$= \sqrt{\left(\frac{32}{9} \times 32\right)} = \frac{32}{3}$$

Example 36. If G be the centroid of the ΔABC and O be any other point in the plane of the triangle ABC, then show that

$OA^2 + OB^2 + OC^2 = GA^2 + GB^2 + GC^2 + 3GO^2$.

Sol. Let G be the origin and GO be X-axis.

$O \equiv (a, 0)$, $A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$

and $C \equiv (x_3, y_3)$

Now, LHS $= OA^2 + OB^2 + OC^2$

$$= (x_1 - a)^2 + y_1^2 + (x_2 - a)^2 + y_2^2 + (x_3 - a)^2 + y_3^2$$

$$= (x_1^2 + x_2^2 + x_3^2) + (y_1^2 + y_2^2 + y_3^2)$$

$$- 2a(x_1 + x_2 + x_3) + 3a^2$$

$$= \sum x_1^2 + \sum y_1^2 - 0 + 3a^2 \qquad \left(\because \frac{x_1 + x_2 + x_3}{3} = 0 \text{ i.e., } x_1 + x_2 + x_3 = 0\right)$$

$$= \sum x_1^2 + \sum y_1^2 + 3a^2 \qquad \text{...(i)}$$

and RHS $= GA^2 + GB^2 + GC^2 + 3GO^2$

$$= x_1^2 + y_1^2 + x_2^2 + y_2^2 + x_3^2 + y_3^2 + 3\{(a - 0)^2\}$$

$$= \sum x_1^2 + \sum y_1^2 + 3a^2 \qquad \text{... (ii)}$$

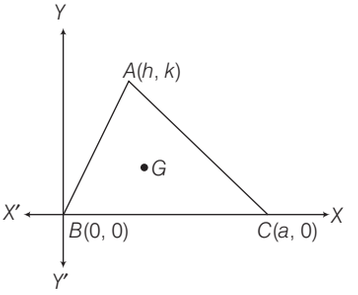
Hence, from Eqs. (i) and (ii), we get

$OA^2 + OB^2 + OC^2 = GA^2 + GB^2 + GC^2 + 3GO^2$.

Example 37. If G be the centroid of ΔABC, show that

$$AB^2 + BC^2 + CA^2 = 3(GA^2 + GB^2 + GC^2).$$

Sol. We take B as the origin and BC and BY as the X and Y-axes respectively.



Let $BC = a$, then $B \equiv (0, 0)$ and $C \equiv (a, 0)$
and let $A \equiv (h, k)$
then, coordinates of G will be

$$\left(\frac{h+0+a}{3}, \frac{k+0+0}{3}\right), \text{ i.e. } \left(\frac{h+a}{3}, \frac{k}{3}\right)$$

Take ΔABC as in 1st quadrant (for the sake of exactness).

Now, LHS $= (AB)^2 + (BC)^2 + (CA)^2$

$$= (h-0)^2 + (k-0)^2 + a^2 + (h-a)^2 + (k-0)^2$$

$$= 2h^2 + 2k^2 - 2ah + 2a^2 \quad \text{... (i)}$$

$$\text{RHS} = 3((GA)^2 + (GB)^2 + (GC)^2)$$

$$= 3\left\{\left(\frac{a+h}{3} - h\right)^2 + \left(\frac{k}{3} - k\right)^2 + \left(\frac{a+h}{3} - 0\right)^2 + \left(\frac{k}{3} - 0\right)^2 + \left(\frac{a+h}{3} - a\right)^2 + \left(\frac{k}{3} - 0\right)^2\right\}$$

$$= \frac{3}{9}\{(a-2h)^2 + (-2k)^2 + (a+h)^2 + k^2 + (h-2a)^2 + k^2\}$$

$$= \frac{1}{3}\{6a^2 + 6h^2 + 6k^2 - 6ah\}$$

$$= 2h^2 + 2k^2 - 2ah + 2a^2 \quad \text{... (ii)}$$

Hence, from Eqs. (i) and (ii), we get

$$AB^2 + BC^2 + CA^2 = 3(GA^2 + GB^2 + GC^2)$$

Example 38. The vertices of a triangle are $(1, a), (2, b)$ and $(c^2, -3)$

(i) Prove that its centroid can not lie on the Y-axis.

(ii) Find the condition that the centroid may lie on the X-axis.

Sol. Centroid of the triangle is

$$G \equiv \left(\frac{1+2+c^2}{3}, \frac{a+b-3}{3}\right) \text{ i.e. } \left(\frac{3+c^2}{3}, \frac{a+b-3}{3}\right)$$

(i) $\because$ G will lie on Y-axis, then

$$\frac{3+c^2}{3} = 0$$

$$\Rightarrow \quad c^2 = -3$$

or $\quad c = \pm i\sqrt{3}$

$\because$ Both values of c are imaginary.

Hence, G can not lie on Y-axis.

(ii) $\because$ G will lie on X-axis, then $\frac{a+b-3}{3} = 0$

$$\Rightarrow \quad a + b - 3 = 0$$

or $\quad a + b = 3$

Incentre

Definition : The point of intersection of internal angle bisectors of triangle is called the incentre of the triangle.

Theorem : Prove that the coordinates of the incentre of a triangle whose vertices are $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ are

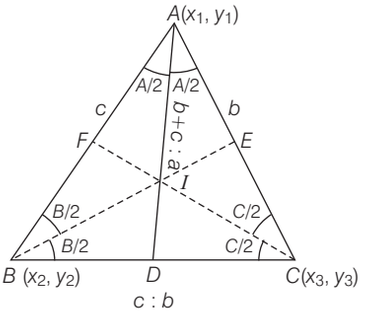
$$\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$$

where, a, b, c are the lengths of sides BC, CA and AB respectively.

Also, prove that the internal bisectors of the angles of a triangle are concurrent.

Proof : Given $A \equiv (x_1, y_1), B(x_2, y_2), C \equiv (x_3, y_3)$ be the vertices of ΔABC and $BC = a, CA = b$ and $AB = c$. Let AD be the bisector of A. We know that the bisector of an angle of a triangle divides the opposite side in the ratio of the sides containing the triangle.

$$\therefore \quad \frac{BD}{DC} = \frac{AB}{AC} = \frac{c}{b} \quad \text{... (i)}$$



Hence, D divides BC in the ratio $c : b$

$\therefore$ Coordinates of D are $\left(\frac{cx_3 + bx_2}{c+b}, \frac{cy_3 + by_2}{c+b}\right)$

From Eq. (i), $\quad \frac{DC}{BD} = \frac{b}{c}$ or $\frac{DC}{BD} + 1 = \frac{b}{c} + 1$

or $\quad \frac{DC + BD}{BD} = \left(\frac{b+c}{c}\right)$ or $\frac{a}{BD} = \left(\frac{b+c}{c}\right)$

$$\therefore \quad BD = \frac{ac}{(b+c)}$$

Also, in ΔABD, BI is the bisector of B.

Then, $\quad \frac{AI}{ID} = \frac{AB}{BD} = \frac{c}{\left(\frac{ac}{b+c}\right)} = \frac{b+c}{a}$

$\therefore$ I divides AD in the ratio $b+c:a$

$\therefore$ Coordinates of I are

$$\left(\frac{(b+c)\cdot\frac{cx_3+bx_2}{c+b}+a\cdot x_1}{b+c+a},\ \frac{(b+c)\cdot\frac{cy_3+by_2}{c+b}+b\cdot y_1}{b+c+a}\right)$$

i.e. $$\left(\frac{ax_1+bx_2+cx_3}{a+b+c},\frac{ay_1+by_2+cy_3}{a+b+c}\right)$$

Similarly we can show that the coordinates of the point which divides BE internally in the ratio $c+a:b$ and the coordinates of the point which divides CF internally in the ratio $a+b:c$ will be each

$$\left(\frac{ax_1+bx_2+cx_3}{a+b+c},\frac{ay_1+by_2+cy_3}{a+b+c}\right)$$

and $$CE=\frac{ab}{(c+a)},\ AE=\frac{bc}{(c+a)},$$

$$AF=\frac{bc}{a+b},\ BF=\frac{ac}{a+b}$$

Thus, the three internal bisectors of the angles of a triangle meet in a point I.

$$I\equiv\left(\frac{ax_1+bx_2+cx_3}{a+b+c},\frac{ay_1+by_2+cy_3}{a+b+c}\right)$$

***Corollary* 1 :** If ΔABC is equilateral, then $a=b=c$

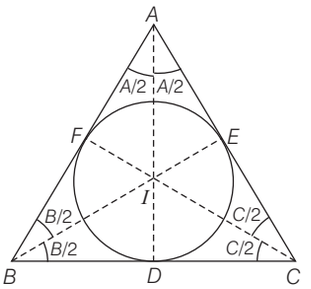
$$\text{incentre}=\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right)=\text{centroid}$$

i.e. incentre and centroid coincide in equilateral, triangle.

***Corollary* 2 :** $AE=AF=s-a$

$BD=BF=s-b$

$CD=CE=s-c$



where, $$s=\frac{a+b+c}{2}$$

and $$|BC|=a,|CA|=b,|AB|=c$$

Proof : Let $AE=AF=\alpha$

($\because$ Lengths of tangents are equal from a point to a circle)

$BD=BF=\beta$

$CD=CE=\gamma$

Also, $a=BC=BD+DC=\beta+\gamma$ ....(i)

$b=CA=CE+AE=\gamma+\alpha$ ...(ii)

and $c=AB=AF+BF=\alpha+\beta$ ...(iii)

Adding all, we get

$a+b+c=2(\alpha+\beta+\gamma)$

or $2s=2(\alpha+\beta+\gamma)$

$\therefore$ $s=\alpha+\beta+\gamma$

From Eqs. (i), (ii) and (iii), we get

$\alpha=s-a,\beta=s-b,\gamma=s-c$

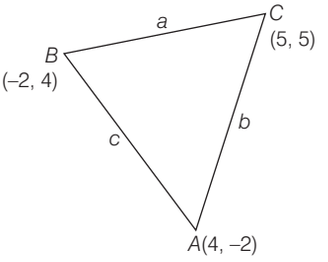
Example 39. Find the coordinates of incentre of the triangle whose vertices are $(4,-2)$, $(-2,4)$ and $(5,5)$.

Sol. Let $A(4,-2)$, $B(-2,4)$ and $C(5,5)$ be the vertices of the given triangle. Then

$$a=BC=\sqrt{(-2-5)^2+(4-5)^2}=\sqrt{50}=5\sqrt{2}$$

$$b=CA=\sqrt{(5-4)^2+(5+2)^2}=\sqrt{50}=5\sqrt{2}$$

and $$c=AB=\sqrt{(4+2)^2+(-2-4)^2}=\sqrt{72}=6\sqrt{2}$$



Let (x,y) be the coordinates of incentre of ΔABC. Then

$$x=\frac{ax_1+bx_2+cx_3}{a+b+c}$$

$$=\frac{5\sqrt{2}\times4+5\sqrt{2}\times(-2)+6\sqrt{2}\times5}{5\sqrt{2}+5\sqrt{2}+6\sqrt{2}}$$

$$=\frac{20\sqrt{2}-10\sqrt{2}+30\sqrt{2}}{16\sqrt{2}}$$

$$=\frac{40}{16}=\frac{5}{2}$$

and $$y=\frac{ay_1+by_2+cy_3}{a+b+c}$$

$$=\frac{5\sqrt{2}\times(-2)+5\sqrt{2}\times4+6\sqrt{2}\times5}{5\sqrt{2}+5\sqrt{2}+6\sqrt{2}}$$

$$=\frac{40}{16}=\frac{5}{2}$$

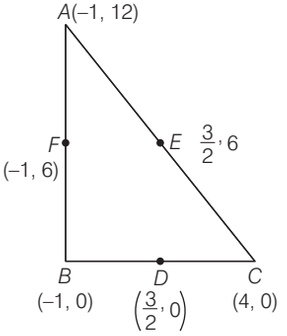
$\therefore$ The coordinates of the incentre are $\left(\frac{5}{2},\frac{5}{2}\right)$

Example 40. *If* $\left(\frac{3}{2}, 0\right), \left(\frac{3}{2}, 6\right)$ *and* $(-1, 6)$ *are mid-points of the sides of a triangle, then find*

(i) Centroid of the triangle

(ii) Incentre of the triangle

Sol. Let $A \equiv (\alpha, \beta)$, then coordinates of $B \equiv (-2-\alpha, 12-\beta)$ and coordinates of $C \equiv (3-\alpha, 12-\beta)$. But mid-point of BC is $\left(\frac{3}{2}, 0\right)$



then $3 = -2-\alpha+3-\alpha$

$\Rightarrow \quad \alpha = -1$

and $0 = 12-\beta+12-\beta$

$\Rightarrow \quad \beta = 12$

$\therefore$ Coordinates of vertices are

$A \equiv (-1, 12)$, $B \equiv (-1, 0)$ and $C \equiv (4, 0)$

(i) Centroid : The centroid of ΔABC is

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

or $\left(\frac{-1-1+4}{3}, \frac{12+0+0}{3}\right)$ i.e. $\left(\frac{2}{3}, 4\right)$

(ii) Incentre : We have

$$a = BC = \sqrt{(-1-4)^2+(0-0)^2} = 5$$

$$b = CA = \sqrt{(4+1)^2+(0-12)^2} = 13$$

and $c = AB = \sqrt{(-1+1)^2+(12-0)^2} = 12$

$\therefore$ The incentre of ΔABC is

$$\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$$

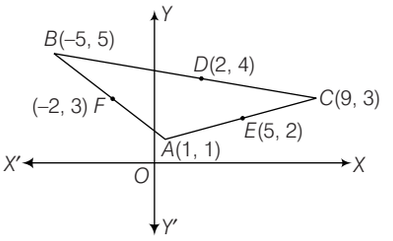
or $\left(\frac{5\times(-1)+13\times(-1)+12\times4}{5+13+12}, \frac{5\times12+13\times0+12\times0}{5+13+12}\right)$

i.e. $(1, 2)$

Example 41. If a vertex of a triangle be (1, 1) and the middle points of two sides through it be (– 2, 3) and (5, 2), then find the centroid and the incentre of the triangle.

Sol. Let coordinate of A be (1, 1) and mid-points of AB and AC are F and E.

$\therefore \quad F \equiv (-2, 3)$ and $E \equiv (5, 2)$



Hence, coordinates of B and C are $(2\times(-2)-1, (2\times3-1)$ and $(2\times2-(-5), 2\times4-5)$ respectively.

i.e. $B \equiv (-5, 5)$ and $C \equiv (9, 3)$

Then, centroid is $\left(\frac{1-5+9}{3}, \frac{1+5+3}{3}\right)$ i.e., $\left(\frac{5}{3}, 3\right)$

Also, $a = |BC| = \sqrt{(-5-9)^2+(5-3)^2} = \sqrt{200} = 10\sqrt{2}$

$b = |CA| = \sqrt{(9-1)^2+(3-1)^2} = \sqrt{68} = 2\sqrt{17}$

and $c = |AB| = \sqrt{(1+5)^2+(1-5)^2} = \sqrt{52} = 2\sqrt{13}$

Then, incentre is

$$\left(\frac{10\sqrt{2}\times1+2\sqrt{17}\times(-5)+2\sqrt{13}\times9}{10\sqrt{2}+2\sqrt{17}+2\sqrt{13}}, \frac{10\sqrt{2}\times1+2\sqrt{17}\times5+2\sqrt{13}\times3}{10\sqrt{2}+2\sqrt{17}+2\sqrt{13}}\right)$$

i.e. $\left(\frac{5\sqrt{2}-5\sqrt{17}+9\sqrt{13}}{5\sqrt{2}+\sqrt{17}+\sqrt{13}}, \frac{5\sqrt{2}+5\sqrt{17}+3\sqrt{13}}{5\sqrt{2}+\sqrt{17}+\sqrt{13}}\right)$

Example 42. If G be the centroid and I be the incentre of the triangle with vertices $A(-36, 7), B(20, 7)$ and $C(0, -8)$ and $GI = \frac{25}{3}\sqrt{(205)}\,\lambda$, then find the value of λ.

Sol. Coordinates of centroid are

$$G \equiv \left(-\frac{16}{3}, 2\right)$$

and $a = |BC| = \sqrt{(20-0)^2+(7+8)^2}$

$= \sqrt{625} = 25$

$b = |CA| = \sqrt{(0+36)^2+(-8-7)^2}$

$= \sqrt{1521} = 39$

$c = |AB| = \sqrt{(-36-20)^2+(7-7)^2}$

$= \sqrt{(56)^2} = 56$

Therefore, the coordinates of incentre are

$$I \equiv \left(\frac{25\times(-36)+39\times20+56\times0}{25+39+56}, \frac{25\times7+39\times7+56\times(-8)}{25+39+56}\right)$$

i.e., $I \equiv (-1, 0)$

$\therefore \quad GI = \sqrt{\left(-\frac{16}{3}+1\right)^2 + (2-0)^2} = \frac{\sqrt{(205)}}{3}$

but given $\quad GI = \frac{25}{3}\sqrt{(205)}\,\lambda$

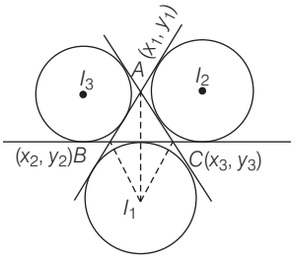
$\therefore \quad \frac{1}{3}\sqrt{(205)} = \frac{25}{3}\sqrt{(205)}\,\lambda$

$\Rightarrow \quad \lambda = \frac{1}{25}$

Some Standard Results

1. Excentres of a Triangle

This is the point of intersection of the external bisectors of the angles of a triangle.



The circle opposite to the vertex A is called the escribed circle opposite A or the circle escribed to the side BC. If I_1 is the point of intersection of internal bisector of $\angle BAC$ and external bisector of $\angle ABC$ and $\angle ACB$, then

$$I_1 \equiv \left(\frac{ax_1 - bx_2 - cx_3}{a-b-c}, \frac{ay_1 - by_2 - cy_3}{a-b-c}\right)$$

or $$I_1 \equiv \left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{-ay_1 + by_2 + cy_3}{-a+b+c}\right)$$

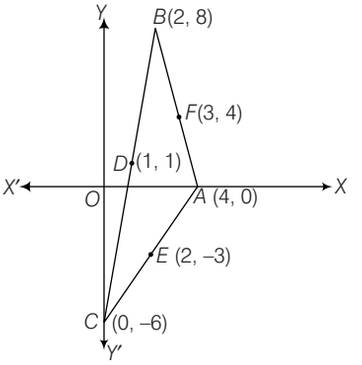
Similarly, $$I_2 \equiv \left(\frac{ax_1 - bx_2 + cx_3}{a-b+c}, \frac{ay_1 - by_2 + cy_3}{a-b+c}\right)$$

and $$I_3 \equiv \left(\frac{ax_1 + bx_2 - cx_3}{a+b-c}, \frac{ay_1 + by_2 - cy_3}{a+b-c}\right)$$

where, $|BC| = a, |CA| = b$ and $|AB| = c$

Example 43. If the coordinates of the mid-points of sides BC, CA and AB of triangle ABC are $(1, 1), (2, -3)$ and $(3, 4)$, then find the excentre opposite to the vertex A.

Sol. Let $D(1, 1)$, $E(2, -3)$ and $F(3, 4)$ are the mid-points of the sides of the triangle BC, CA and AB respectively. Let $A \equiv (\alpha, \beta)$



then $\quad B \equiv (6-\alpha, 8-\beta)$

and $\quad C \equiv (4-\alpha, -6-\beta)$

Also, D is the mid-point of B and C, then

$$1 = \frac{6-\alpha+4-\alpha}{2} \Rightarrow \alpha = 4$$

and $$1 = \frac{8-\beta-6-\beta}{2} \Rightarrow \beta = 0$$

$\therefore$ $A \equiv (4, 0)$, $B \equiv (2, 8)$ and $C \equiv (0, -6)$, then

$$a = |BC| = \sqrt{(2-0)^2 + (8+6)^2} = \sqrt{200} = 10\sqrt{2}$$

$$b = |CA| = \sqrt{(0-4)^2 + (-6-0)^2} = \sqrt{52} = 2\sqrt{13}$$

and $$c = |AB| = \sqrt{(4-2)^2 + (0-8)^2} = \sqrt{68} = 2\sqrt{17}$$

Hence, the coordinates of the excentre opposite to A are

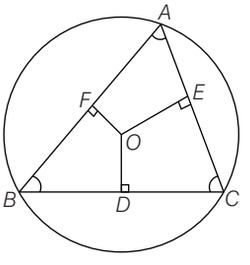
$$\left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{ay_1 + by_2 + cy_3}{-a+b+c}\right)$$

i.e., $$\left(\frac{-10\sqrt{2}\times 4 + 2\sqrt{13}\times 2 + 2\sqrt{17}\times 0}{-10\sqrt{2} + 2\sqrt{13} + 2\sqrt{17}}, \frac{-10\sqrt{2}\times 0 + 2\sqrt{13}\times 8 + 2\sqrt{17}\times(-6)}{-10\sqrt{2} + 2\sqrt{13} + 2\sqrt{17}}\right)$$

or $$\left(\frac{-20\sqrt{2} + 2\sqrt{13}}{-5\sqrt{2} + \sqrt{13} + \sqrt{17}}, \frac{8\sqrt{13} - 6\sqrt{17}}{-5\sqrt{2} + \sqrt{13} + \sqrt{17}}\right)$$

2. Circumcentre of a Triangle

The circumcentre of a triangle is the point of intersection of the perpendicular bisectors of the sides of a triangle (i.e., the lines through the mid-point of a side and perpendicular to it). Let $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of ΔABC and if angles of ΔABC are given, then coordinates of circumcentre



are $\left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}\right)$

Or

$\left(\frac{ax_1 \cos A + bx_2 \cos B + cx_3 \cos C}{a\cos A + b\cos B + c\cos C}, \frac{ay_1 \cos A + by_2 \cos B + cy_3 \cos C}{a\cos A + b\cos B + c\cos C}\right)$

where, $|BC| = a, |CA| = b$ and $|AB| = c$

Example 44. In a ΔABC with vertices $A(1, 2)$, $B(2, 3)$ and $C(3, 1)$ and $\angle A = \angle B = \cos^{-1}\left(\frac{1}{\sqrt{10}}\right)$, $\angle C = \cos^{-1}\left(\frac{4}{5}\right)$, then find the circumcentre of ΔABC.

Sol. Since, $\angle A = \angle B = \cos^{-1}\left(\frac{1}{\sqrt{10}}\right)$

$\Rightarrow \quad \cos A = \cos B = \frac{1}{\sqrt{10}}$

then, $\sin A = \sin B = \frac{3}{\sqrt{10}}$

$\therefore \quad \sin 2A = \sin 2B = 2 \times \frac{3}{\sqrt{10}} \times \frac{1}{\sqrt{10}} = \frac{3}{5}$

and $\angle C = \cos^{-1}\left(\frac{4}{5}\right)$

$\Rightarrow \quad \cos C = \frac{4}{5}$ then, $\sin C = \frac{3}{5}$

$\therefore \quad \sin 2C = 2 \times \frac{3}{5} \times \frac{4}{5} = \frac{24}{25}$

Let the circumcenter be (x, y), then

$$x = \frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C} = \frac{1 \times \frac{3}{5} + 2 \times \frac{3}{5} + 3 \times \frac{24}{25}}{\frac{3}{5} + \frac{3}{5} + \frac{24}{25}} = \frac{13}{6}$$

and $$y = \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C} = \frac{2 \times \frac{3}{5} + 3 \times \frac{3}{5} + 1 \times \frac{24}{25}}{\frac{3}{5} + \frac{3}{5} + \frac{24}{25}} = \frac{11}{6}$$

Hence, coordinates of circumcenter are $\left(\frac{13}{6}, \frac{11}{6}\right)$.

Two Important Tricks for Circumcentre

(a) If angles of triangle ABC are not given and the vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are given, then the circumcentre of the ΔABC is given by

$$\left(\frac{(x_1 + x_2) + \lambda(y_1 - y_2)}{2}, \frac{(y_1 + y_2) - \lambda(x_1 - x_2)}{2}\right)$$

Here, we observe that

$$P = \begin{bmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{bmatrix}$$

$$\therefore \quad \lambda = \frac{\vec{\mathbf{R}}_1 \cdot \vec{\mathbf{R}}_2}{|P|}$$

(b) If the angle C is given instead of coordinates of the vertex C and the vertices $A(x_1, y_1)$, $B(x_2, y_2)$ of ΔABC are given, then the circumcentre of ΔABC is given by

$$\left(\frac{(x_1 + x_2) \pm \cot C(y_1 - y_2)}{2}, \frac{(y_1 + y_2) \pm \cot C(x_1 - x_2)}{2}\right)$$

Remark

Circumcentre of the right angled triangle ABC, right angled at A is $\frac{B + C}{2}$.

Example 45. Find the circumcentre of the triangle whose vertices are (2, 2), (4, 2) and (0, 4).

Sol. Let the given points are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) respectively.

for the matrix

$$P = \begin{bmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_2 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 4 & -2 \end{bmatrix}$$

$$\therefore \quad \lambda = \frac{\vec{\mathbf{R}}_1 \cdot \vec{\mathbf{R}}_2}{|P|} = \frac{2 \times 4 + (-2) \times (-2)}{2 \times (-2) - 4 \times (-2)} = \frac{12}{4} = 3$$

$\therefore$ Circumcentre of the triangle

$$\equiv \left(\frac{(x_1 + x_2) + \lambda(y_1 - y_2)}{2}, \frac{(y_1 + y_2) - \lambda(x_1 - x_2)}{2}\right)$$

$$\equiv \left(\frac{2 + 4 + 3(2 - 2)}{2}, \frac{2 + 2 - 3(2 - 4)}{2}\right)$$

$$\equiv (3, 5)$$

Example 46. Find the circumcentre of triangle ABC if $A \equiv (7, 4)$, $B \equiv (3, -2)$ and $\angle C = \frac{\pi}{3}$.

Sol. Here, $x_1 = 7, y_1 = 4, x_2 = 3, y_2 = -2$ and $\angle C = \frac{\pi}{3}$

$\therefore$ The circumcentre of ΔABC

$$\equiv \left(\frac{(x_1 + x_2) \pm \cot C(y_1 - y_2)}{2}, \frac{(y_1 + y_2) \mp \cot C(x_1 - x_2)}{2} \right)$$

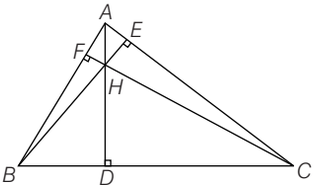
$$\equiv \left(\frac{(7+3) \pm \frac{1}{\sqrt{3}}(4+2)}{2}, \frac{(4-2) \mp \frac{1}{\sqrt{3}}(7-3)}{2} \right)$$

$$\equiv \left(5 + \sqrt{3}, 1 - \frac{2}{\sqrt{3}} \right) \text{ or } \left(5 - \sqrt{3}, 1 + \frac{2}{\sqrt{3}} \right)$$

3. Orthocentre of a Triangle

The orthocentre of a triangle is the point of intersection of altitudes
(i.e., the lines through the vertices and perpendicular to opposite sides).

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of ΔABC and if angles of ΔABC are given, then coordinates of orthocentre are



$$\left(\frac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \frac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C} \right)$$

or $$\left(\frac{ax_1 \sec A + bx_2 \sec B + cx_3 \sec C}{a \sec A + b \sec B + c \sec C}, \frac{ay_1 \sec A + by_2 \sec B + cy_3 \sec C}{a \sec A + b \sec B + c \sec C} \right)$$

where, $|BC| = a, |CA| = b$ and $|AB| = c$

Important trick for orthocentre :

orthocentre of the triangle whose vertices are (0, 0), (x_1, y_1) is given by

$$(\lambda(y_2 - y_1), -\lambda(x_2 - x_1))$$

where, $$\lambda = \frac{x_1 x_2 + y_1 y_2}{x_1 y_2 - x_2 y_1}$$

Remarks

1. The orthocentre of a triangle having vertices (α, β), (β, α) and (α, α) is (α, α).
2. The orthocentre of a triangle having vertices is $\left(-\frac{1}{\alpha\beta\gamma}, -\alpha\beta\gamma \right)$
3. The orthocentre of right angled triangle ABC, right angled at A is A.

Example 47. Find the orthocentre of ΔABC if $A \equiv (0, 0), B \equiv (3, 5)$ and $C \equiv (4, 7)$.

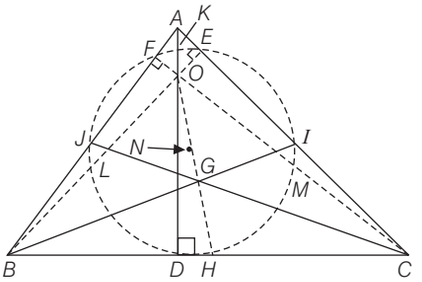
Sol. Here, $x_1 = 3, y_1 = 5, x_2 = 4, y_2 = 7$

$$\therefore \quad \lambda = \frac{3 \times 4 + 5 \times 7}{3 \times 7 - 4 \times 5} = 47$$

$\Rightarrow$ Orthocentre of $\Delta ABC \equiv (47(7-5), -47(4-3))$
$\equiv (94, -47)$

4. Nine Point Centre of a Triangle

If a circle passing through the feet of perpendiculars (i.e., D, E, F) mid-points of sides BC, CA, AB respectively (i.e., H, I, J) and the $n\,x$
mid-points of the line joining the orthocentre O to the angular points A, B, C (i.e., K, L, M) thus the nine points $D, E, F, H, I, J, K, L, M$ all lie on a circle.



This circle is known as nine point circle and its centre is called the nine point centre. The nine-point centre of a triangle is collinear with the circumcentre and the orthocentre and bisects the segments joining them and radius of nine point circle of a triangle is half the radius of the circumcircle.

***Corollary* 1 :** The orthocentre, the nine point centre, the centroid and the circumcentre therefore all lie on a straight line.

Corollary* 2 :** If O is orthocentre, N is nine point centre, G is centroid and C is circumcentre, then to remember it see ***ONGC (i.e. Oil Natural Gas Corporation) in left of G are 2 and in right is 1, therefore G divides O and C in the ratio 2 : 1 (internally).

***Corollary* 3 :** N is the mid-point of O and C

***Corollary* 4 :** Radius of nine point circle $= \frac{1}{2} \times$ Radius of circumcircle

Remarks

1. The distance between the orthocentre and circumcentre in an equilateral triangle is zero.
2. If the circumcentre and centroid of a triangle are respectively (α, β) (γ, δ) then orthocentre will be $(3\gamma - 2\alpha, 3\delta - 2\beta)$.

Example 48. If a triangle has its orthocentre at (1,1) and circumcentre at $\left(\frac{3}{2}, \frac{3}{4}\right)$, then find the centroid and nine point centre.

Sol. Since, centroid divides the orthocentre and circumcentre in the ratio 2 : 1 (internally) and if centroid $G(x, y)$, then

$$O(1,1) \overset{2}{\text{———}} G(x,y) \overset{1}{\text{———}} C\left(\frac{3}{2}, \frac{3}{4}\right)$$

$$x = \frac{2 \times \frac{3}{2} + 1 \times 1}{2+1} = \frac{4}{3}$$

and $$y = \frac{2 \times \frac{3}{4} + 1 \times 1}{2+1} = \frac{5}{6}$$

$\therefore$ Centroid is $\left(\frac{4}{3}, \frac{5}{6}\right)$ and nine point centre is the mid-point of orthocentre and circumcentre.

$\therefore$ Nine point centre is $\left(\frac{1+\frac{3}{2}}{2}, \frac{1+\frac{3}{4}}{2}\right)$, i.e. $\left(\frac{5}{4}, \frac{7}{8}\right)$.

Example 49. The vertices of a triangle are $A(a, a\tan\alpha), B(b, b\tan\beta)$ and $C(c, c\tan\gamma)$. If the circumcentre of ΔABC coincides with the origin and $H(\bar{x}, \bar{y})$ is the orthocentre, then show that $\frac{\bar{y}}{\bar{x}} = \left(\frac{\sin\alpha + \sin\beta + \sin\gamma}{\cos\alpha + \cos\beta + \cos\gamma}\right)$.

Sol. If R be the circumradius and O be the circumcentre

$\therefore$ $OA = OB = OC = R$

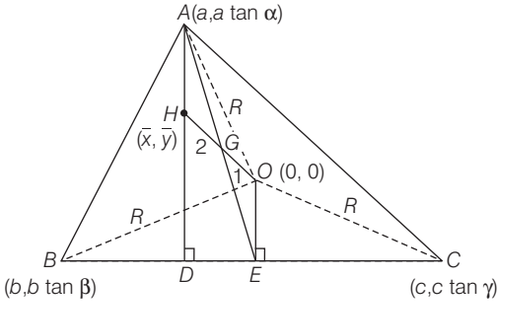
or $\sqrt{(a^2 + a^2\tan^2\alpha)} = \sqrt{(b^2 + b^2\tan^2\beta)}$

$= \sqrt{(c^2 + c^2\tan^2\gamma)} = R$

or $a\sec\alpha = b\sec\beta = c\sec\gamma = R$

or $a = R\cos\alpha, b = R\cos\beta, c = R\cos\gamma$

then, $a\tan\alpha = R\cos\alpha \cdot \frac{\sin\alpha}{\cos\alpha} = R\sin\alpha$



Similarly,

$b\tan\beta = R\sin\beta$ and $c\tan\gamma = R\sin\gamma$

$\therefore$ Coordinates of the vertices of a triangle are :

$A(R\cos\alpha, R\sin\alpha), B(R\cos\beta, R\sin\beta)$

and $C(R\cos\gamma, R\sin\gamma)$

$\therefore$ Centroid

$$(G) \equiv \left(\frac{R(\cos\alpha + \cos\beta + \cos\gamma)}{3}, \frac{R(\sin\alpha + \sin\beta + \sin\gamma)}{3}\right)$$

Since, G divides H and O in the ratio 2 : 1 (internally), then

$$\frac{R}{3}(\cos\alpha + \cos\beta + \cos\gamma) = \frac{2\cdot 0 + 1\cdot\bar{x}}{2+1}$$

or $$\frac{R}{3}(\cos\alpha + \cos\beta + \cos\gamma) = \frac{\bar{x}}{3} \quad \text{...(i)}$$

and $$\frac{R}{3}(\sin\alpha + \sin\beta + \sin\gamma) = \frac{2\cdot 0 + 1\cdot\bar{y}}{2+1}$$

or $$\frac{R}{3}(\sin\alpha + \sin\beta + \sin\gamma) = \frac{\bar{y}}{3} \quad \text{...(ii)}$$

Dividing Eqs. (ii) by (i), then we get

$$\frac{\bar{y}}{\bar{x}} = \left(\frac{\sin\alpha + \sin\beta + \sin\gamma}{\cos\alpha + \cos\beta + \cos\gamma}\right)$$

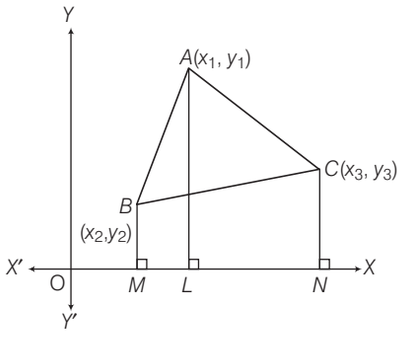
Area of a Triangle

Theorem : The area of a triangle, the coordinates of whose vertices are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) is

$$\frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$$

or $$\frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$$

Proof : Let ABC be a triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$. Let us assume that the points A, B and C are in 1st quadrant (for the sake of exactness). Draw AL, BM and CN perpendicular on X-axis. Let Δ be the required area of the triangle ABC, then



Δ = Area of triangle ABC

= [Area of trapezium $ABML$ + Area of trapezium $ALNC$ − Area of trapezium $BMNC$]

[$\because$ Area of trapezium $= \frac{1}{2}$ (Sum of parallel sides) × (Distance between them)]

$$\therefore \Delta = \left[\frac{1}{2}(BM + AL)(ML) + \frac{1}{2}(AL + CN)(LN) - \frac{1}{2}(BM + CN)(MN)\right]$$

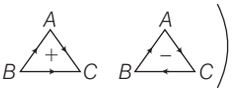
$$= \left[\frac{1}{2}(BM + AL)(OL - OM) + \frac{1}{2}(AL + CN)(ON - (OL) - \frac{1}{2}(BM + CN)(ON - OM)\right]$$

$$= \left[\frac{1}{2}(y_2 + y_1)(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)(x_3 - x_1) - \frac{1}{2}(y_2 + y_3)(x_3 - x_2)\right]$$

$$= \frac{1}{2}[x_1(y_2 + y_1 - y_1 - y_3) + x_2(-y_2 - y_1 + y_2 + y_3) + x_3(y_1 + y_3 - y_2 - y_3)]$$

$$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

The area of triangle ABC will come out to be a positive quantity only when the vertices A, B, C are taken in anticlockwise direction and if points A, B, C are taken in clockwise direction then the area will be negative and if the points A, B, C are taken arbitrary then the area will be positive or negative, the numerical value being the same in all cases.

Thus in general (i.e., 

)

$$\text{Area of } \Delta ABC = \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$$

This expression can be written in determinant form as follows

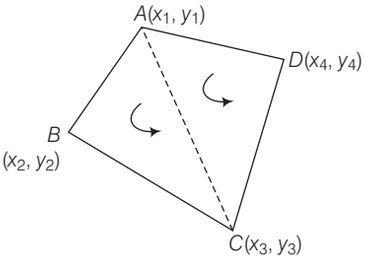
$$\frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$$

***Corollary* 1 :** Area of triangle can also be found by easy method

$$\Delta = \left|\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}\right|$$

and **Area of quadrilateral *ABCD* :** The area of a quadrilateral can be found out by dividing the quadrilateral into two triangles.

$\therefore$ Area of quadrilateral $ABCD$



= Area of ΔABC + Area of ΔDAC

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_3 & y_3 \\ x_1 & y_1 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_4 & y_4 \\ x_1 & y_1 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_3 & y_3 \\ x_4 & y_4 \end{vmatrix}$$

$$= \left|\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_3 & y_3 \\ x_4 & y_4 \end{vmatrix}\right| + \frac{1}{2}\begin{vmatrix} x_4 & y_4 \\ x_1 & y_1 \end{vmatrix}$$

$$\left(\because \begin{vmatrix} x_3 & y_3 \\ x_1 & y_1 \end{vmatrix} = -\begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix}\right)$$

$\therefore$ **Area of polygon whose vertices are $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n)$ is**

$$\frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \begin{vmatrix} x_3 & y_3 \\ x_4 & y_4 \end{vmatrix} + \ldots\ldots + \begin{vmatrix} x_n & y_n \\ x_1 & y_1 \end{vmatrix}\right|$$

Or

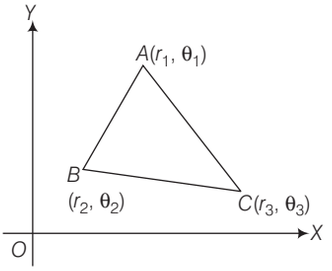
Stair Method Repeat first coordinates one time in last for down arrow use positive sign and for up arrow use negative sign.

$\therefore$ Area of polygon $= \frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ \cdots & \cdots \\ \cdots & \cdots \\ x_n & y_n \\ x_1 & y_1 \end{vmatrix}\right|$

$$= \frac{1}{2}|\{(x_1y_2 + x_2y_3 + \ldots + x_ny_1) - (y_1x_2 + y_2x_3 + \ldots + y_nx_1)\}|$$

***Corollary* 2 :** If the coordinates of the vertices of the triangle are given in polar form i.e.,

$A(r_1, \theta_1), B(r_2, \theta_2), C(r_3, \theta_3)$.



Then, area of triangle

$$= \frac{1}{2} | r_1 r_2 \sin(\theta_1 - \theta_2) + r_2 r_3 \sin(\theta_2 - \theta_3) + r_3 r_1 \sin(\theta_3 - \theta_1) |$$

$$= \frac{1}{2} | \sum r_1 r_2 \sin(\theta_1 - \theta_2) |$$

***Corollary* 3 :** If $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are the sides of a triangle, then the area of the triangle is given by (without solving the vertices)

$$\Delta = \frac{1}{2 | C_1 C_2 C_3 |} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$$

where, C_1, C_2, C_3 are the cofactors of c_1, c_2, c_3 in the determinant

Here, $$C_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2)$$

$$C_2 = \begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix} = (a_3 b_1 - a_1 b_3)$$

and $$C_3 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = (a_1 b_2 - a_2 b_1)$$

and $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = c_1 C_1 + c_2 C_2 + c_3 C_3$$

Or

$$\text{Area of triangle} = \frac{\Delta^2}{2 | \Delta_1 \Delta_2 \Delta_3 |}$$

where, $$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \Delta_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \Delta_2 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

and $$\Delta_3 = \begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix}$$

***Corollary* 4 :** Area of the triangle formed by the lines of the form $y = m_1x + c_1, y = m_2x + c_2$ and $y = m_3x + c_3$ is

$$\Delta = \frac{1}{2} \left| \frac{(c_2 - c_3)^2}{(m_2 - m_3)} + \frac{(c_3 - c_1)^2}{(m_3 - m_1)} + \frac{(c_1 - c_2)^2}{(m_1 - m_2)} \right|$$

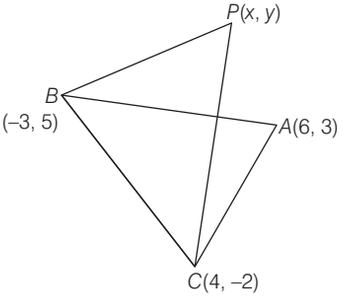
Remarks

1. If area of a triangle is given then, use ± sign.
2. The points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear, then area of $(\Delta ABC) = 0$.
3. Four given points will be collinear, then area of the quadrilateral is zero.
4. Area of the triangle formed by the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) is $\Delta = \frac{1}{2} \left| \begin{vmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{vmatrix} \right|$
5. If one vertex (x_3, y_3) is at (0, 0) then, $\Delta = \frac{1}{2} | x_1y_2 - x_2y_1 |$

Example 50. The coordinates of A, B, C are $(6, 3), (-3, 5)$ and $(4, -2)$ respectively and P is any points (x, y). Show that the ratio of the areas of the triangles PBC and ABC is $\frac{|x + y - 2|}{7}$.

Sol. We have

$$\frac{\text{Area of } \Delta PBC}{\text{Area of } \Delta ABC} = \frac{\frac{1}{2} | \{x(5 + 2) - 3(-2 - y) + 4(y - 5)\} |}{\frac{1}{2} | \{6(5 + 2) - 3(-2 - 3) + 4(3 - 5)\} |}$$



$$= \frac{| 7x + 7y - 14 |}{| 49 |} = \frac{7 | x + y - 2 |}{49} = \frac{| x + y - 2 |}{7}$$

Example 51. Find the area of the pentagon whose vertices are $A(1, 1), B(7, 21), C(7 - 3), D(12, 2)$ and $E(0, -3)$.

Sol. The required area

$$= \frac{1}{2} \left| \begin{vmatrix} 1 & 1 \\ 7 & 21 \end{vmatrix} + \begin{vmatrix} 7 & 21 \\ 7 & -3 \end{vmatrix} + \begin{vmatrix} 7 & -3 \\ 12 & 2 \end{vmatrix} + \begin{vmatrix} 12 & 2 \\ 0 & -3 \end{vmatrix} + \begin{vmatrix} 0 & -3 \\ 1 & 1 \end{vmatrix} \right|$$

$$= \frac{1}{2} | (21 - 7) + (-21 - 147) + (14 + 36) + (-36 - 0) + (0 + 3) |$$

$$= \frac{1}{2} | -137 | = \frac{137}{2} \text{ sq units}$$

Example 52. Show that the points $(a,0),(0,b)$ and (1, 1) are collinear, if $\frac{1}{a}+\frac{1}{b}=1$

Sol. Let $A\equiv(a,0)$, $B\equiv(0,b)$ and $C\equiv(1,1)$

Now, points A,B,C will be collinear, if area of $\Delta ABC=0$

or $$\frac{1}{2}\left|\begin{vmatrix}a&0\\0&b\end{vmatrix}+\begin{vmatrix}0&b\\1&1\end{vmatrix}+\begin{vmatrix}1&1\\a&0\end{vmatrix}\right|=0$$

$\Rightarrow$ $$|(ab-0)+(0-b)+(0-a)|=0$$

or $$ab-a-b=0$$

or $$a+b=ab \text{ or } \frac{1}{a}+\frac{1}{b}=1$$

Example 53. Prove that the coordinates of the vertices of an equilateral triangle can not all be rational.

Sol. Let $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ be the vertices of a triangle ABC. If possible let x_1,y_1,x_2,y_2,x_3,y_3 be all rational.

Now, area of $\Delta ABC=\frac{1}{2}|x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)|$

$=$ Rational ...(i)

Since, ΔABC is equilateral

$\therefore$ Area of $\Delta ABC=\frac{\sqrt{3}}{4}(\text{side})^2$

$=\frac{\sqrt{3}}{4}(AB)^2$

$=\frac{\sqrt{3}}{4}\{(x_1-x_2)^2+(y_1-y_2)^2\}$

$=$ Irrational ...(ii)

From Eqs. (i) and (ii),

Rational = Irrational

which is contradiction.

Hence, x_1,y_1,x_2,y_2,x_3,y_3 cannot all be rational.

Example 54. The coordinates of two points A and B are (3, 4) and $(5,-2)$ respectively. Find the coordinates of any point P if $PA=PB$ and area of ΔAPB is 10.

Sol. Let coordinates of P be (h,k).

$\because$ $PA=PB \Rightarrow (PA)^2=(PB)^2$

$\Rightarrow$ $(h-3)^2+(k-4)^2=(h-5)^2+(k+2)^2$

$\Rightarrow$ $(h-3)^2-(h-5)^2+(k-4)^2-(k+2)^2=0$

$\Rightarrow$ $(2h-8)(2)+(2k-2)(-6)=0$

$\Rightarrow$ $(h-4)-3(k-1)=0$

$\Rightarrow$ $h-3k-1=0$ ...(i)

Now, Area of $\Delta PAB=\frac{1}{2}\left|\begin{vmatrix}h&k&1\\3&4&1\\5&-2&1\end{vmatrix}\right|=10$

or $6h+2k-26=\pm20$

$\Rightarrow$ $6h+2k-46=0$ or $6h+2k-6=0$

$\Rightarrow$ $3h+k-23=0$ or $3h+k-3=0$

Solving $h-3k-1=0$ and $3h+k-23=0$,

we get $h=7,k=2$

Solving $h-3k-1=0$ and $3h+k-3=0$,

we get $h=1,k=0$

Hence, the coordinates of P are (7, 2) or (1, 0).

Example 55. Find the area of the triangle formed by the straight lines $7x-2y+10=0$, $7x+2y-10=0$ and $9x+y+2=0$ (without solving the vertices of the triangle).

Sol. The given lines are :

$$7x-2y+10=0$$
$$7x+2y-10=0$$
$$9x+y+2=0$$

$\therefore$ Area of triangle $\Delta=\frac{1}{2|C_1C_2C_3|}\begin{vmatrix}7&-2&10\\7&2&-10\\9&1&2\end{vmatrix}^2$ ...(i)

where, $C_1=\begin{vmatrix}7&2\\9&1\end{vmatrix}=7-18=-11,$

$C_2=\begin{vmatrix}9&1\\7&-2\end{vmatrix}=-18-7=-25$

and $C_3=\begin{vmatrix}7&-2\\7&2\end{vmatrix}=14+14=28,$

and $\begin{vmatrix}7&-2&10\\7&2&-10\\9&1&2\end{vmatrix}=10C_1-10C_2+2C_3$

$=10\times(-11)-10\times(-25)+2\times28=196$

$\therefore$ From Eq. (i), $\Delta=\frac{1}{2|-11\times(-25)\times28|}\times(196)^2$

$=\frac{196\times196}{2\times11\times25\times28}=\frac{686}{275}$ sq units

Example 56. If Δ_1 is the area of the triangle with vertices $(0,0),(a\tan\alpha,b\cot\alpha),(a\sin\alpha,b\cos\alpha)$; Δ_2 is the area of the triangle with vertices $(a,b),(a\sec^2\alpha,b\,\text{cosec}^2\alpha),(a+a\sin^2\alpha,b+b\cos^2\alpha)$ and Δ_3 is the area of the triangle with vertices $(0,0)$, $(a\tan\alpha,-b\cot\alpha),(a\sin\alpha,b\cos\alpha)$. Show that there is no value of α for which Δ_1,Δ_2 and Δ_3 are in GP.

Sol. We have, $\Delta_1 = \frac{1}{2}|(a\tan\alpha)(b\cos\alpha) - (a\sin\alpha)(b\cot\alpha)|$
($\because$ one vertex is (0,0))

$= \frac{1}{2}|ab||\sin\alpha - \cos\alpha|$ ... (i)

and $\Delta_2 = \frac{1}{2}\left|\begin{vmatrix} a-(a+a\sin^2\alpha) & a\sec^2\alpha-(a+a\sin^2\alpha) \\ b-(b+b\cos^2\alpha) & b\,\text{cosec}^2\,\alpha-(b+b\cos^2\alpha)\end{vmatrix}\right|$

(See remark 4)

$= \frac{1}{2}\left|\begin{vmatrix} -a\sin^2\alpha & a(\tan^2\alpha-\sin^2\alpha) \\ -b\cos^2\alpha & b(\cot^2\alpha-\cos^2\alpha)\end{vmatrix}\right|$

$= \frac{1}{2}|ab|\times\left|\begin{vmatrix} -\sin^2\alpha & \sin^2\alpha(\sec^2\alpha-1) \\ -\cos^2\alpha & \cos^2\alpha(\text{cosec}^2\alpha-1)\end{vmatrix}\right|$

$= \frac{1}{2}|ab|\times\left|\begin{vmatrix} -\sin^2\alpha & \sin^2\alpha\tan^2\alpha \\ -\cos^2\alpha & \cos^2\alpha\cot^2\alpha\end{vmatrix}\right|$

$= \frac{1}{2}|ab|\times|-\sin^2\alpha\cos^2\alpha\cot^2\alpha+\sin^2\alpha\cos^2\alpha\tan^2\alpha|$

$= \frac{1}{2}|ab|\times|-\cos^4\alpha+\sin^4\alpha|$

$= \frac{1}{2}|ab|\times|\sin^2\alpha+\cos^2\alpha|\times|\sin^2\alpha-\cos^2\alpha|$

$= \frac{1}{2}|ab|\times|1|\times|-\cos 2\alpha|$

$= \frac{1}{2}|ab|\times|\cos 2\alpha|$ ... (ii)

and $\Delta_3 = \frac{1}{2}|(a\tan\alpha)(b\cos\alpha)-(-b\cot\alpha)(a\sin\alpha)|$

$= \frac{1}{2}|ab||\sin\alpha+\cos\alpha|$ ... (iii)

Since, $\Delta_1, \Delta_2, \Delta_3$ are in GP, then $\Delta_1\Delta_3 = \Delta_2^2$

$\Rightarrow \frac{1}{2}|ab||\sin\alpha-\cos\alpha|\times\frac{1}{2}|ab||\sin\alpha+\cos\alpha|$

$= \frac{1}{4}|ab|^2|\cos 2\alpha|^2$ [from Eqs. (i), (ii) and (iii)]

$\Rightarrow |\sin^2\alpha-\cos^2\alpha| = |\cos 2\alpha|^2$

$\Rightarrow |-\cos 2\alpha| = |\cos 2\alpha|^2$

$\Rightarrow |\cos 2\alpha| = |\cos 2\alpha|^2$

$\Rightarrow |\cos 2\alpha|(1-|\cos 2\alpha|) = 0$

$\therefore 1-|\cos 2\alpha| = 0$ ($\because |\cos 2\alpha| \neq 0$)

$\Rightarrow |\cos 2\alpha| = 1$

or $\cos 2\alpha = \pm 1$ or $\cos 2\alpha = 1$

and $\cos 2\alpha = -1$

or $2\alpha = 2n\pi, 2\alpha = (2p+1)\pi$

or $\alpha = n\pi, \alpha = p\pi + \frac{\pi}{2}; n, p \in I$

For these values of α the vertices of the given triangles are not defined. Hence Δ_1, Δ_2 and Δ_3 cannot be in GP for any value of α.

Exercise for Session 3

1. The coordinates of the middle points of the sides of a triangle are (4, 2), (3, 3) and (2, 2), then coordinates of centroid are

(a) (3, 7 / 3) (b) (3, 3) (c) (4, 3) (d) (3, 4)

2. The incentre of the triangle whose vertices are $(-36, 7), (20, 7)$ and $(0, -8)$ is

(a) (0, – 1) (b) (–1, 0) (c) (1, 1) (d) $\left(\frac{1}{2}, 1\right)$

3. If the orthocentre and centroid of a triangle are $(-3, 5)$ and (3, 3) then its circumcentre is

(a) (6, 2) (b) (3, – 1) (c) (–3, 5) (d) (–3, 1)

4. An equilateral triangle has each side equal to a. If the coordinates of its vertices are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) then the square of the determinant $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1\end{vmatrix}$ equals

(a) $3a^4$ (b) $\frac{3a^4}{2}$ (c) $\frac{3}{4}a^4$ (d) $\frac{3}{8}a^4$

5. The vertices of a triangle are $A(0,0), B(0,2)$ and $C(2,0)$. The distance between circumcentre and orthocentre is

(a) $\sqrt{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) 2 (d) $\frac{1}{2}$

6. $A(a, b)$, $B(x_1, y_1)$ and $C(x_2, y_2)$ are the vertices of a triangle. If a, x_1, x_2 are in GP with common ratio r and b, y_1, y_2 are in GP with common ratio s, then area of ΔABC is

(a) $ab(r-1)(s-1)(s-r)$ (b) $\frac{1}{2}ab(r+1)(s+1)(s-r)$

(c) $\frac{1}{2}ab(r-1)(s-1)(s-r)$ (d) $ab(r+1)(s+1)(r-s)$

7. The points $(x+1, 2), (1, x+2), \left(\frac{1}{x+1}, \frac{2}{x+1}\right)$ are collinear, then x is equal to

(a) -4 (b) -8 (c) 4 (d) 8

8. The vertices of a triangle are $(6, 0), (0, 6)$ and $(6, 6)$. Then distance between its circumcentre and centroid, is

(a) $2\sqrt{2}$ (b) 2 (c) $\sqrt{2}$ (d) 1

9. The nine point centre of the triangle with vertices $(1, \sqrt{3}), (0, 0)$ and $(2, 0)$ is

(a) $\left(1, \frac{\sqrt{3}}{2}\right)$ (b) $\left(\frac{2}{3}, \frac{1}{\sqrt{3}}\right)$ (c) $\left(\frac{2}{3}, \frac{\sqrt{3}}{2}\right)$ (d) $\left(1, \frac{1}{\sqrt{3}}\right)$

10. The vertices of a triangle are $(0, 0), (1, 0)$ and $(0, 1)$. Then excentre opposite to $(0, 0)$ is

(a) $\left(1-\frac{1}{\sqrt{2}}, 1+\frac{1}{\sqrt{2}}\right)$ (b) $\left(1+\frac{1}{\sqrt{2}}, 1+\frac{1}{\sqrt{2}}\right)$ (c) $\left(1+\frac{1}{\sqrt{2}}, 1-\frac{1}{\sqrt{2}}\right)$ (d) $\left(1-\frac{1}{\sqrt{2}}, 1-\frac{1}{\sqrt{2}}\right)$

11. If α, β, γ are the real roots of the equation $x^3 - 3px^2 + 3qx - 1 = 0$, then find the centroid of the triangle whose vertices are $\left(\alpha, \frac{1}{\alpha}\right), \left(\beta, \frac{1}{\beta}\right)$ and $\left(\gamma, \frac{1}{\gamma}\right)$.

12. If centroid of a triangle be $(1,4)$ and the coordinates of its any two vertices are $(4, -8)$ and $(-9, 7)$, find the area of the triangle.

13. Find the centroid and incentre of the triangle whose vertices are $(1, 2), (2, 3)$ and $(3, 4)$.

14. Show that the area of the triangle with vertices $(\lambda, \lambda - 2), (\lambda + 3, \lambda)$ and $(\lambda + 2, \lambda + 2)$ is independent of λ.

15. Prove that the points $(a, b + c), (b, c + a)$ and $(c, a + b)$ are collinear.

16. Prove that the points $(a, b), (c, d)$ and $(a - c, b - d)$ are collinear, if $ad = bc$.

17. If the points $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) are collinear, show that $\sum\left(\frac{y_1 - y_2}{x_1 x_2}\right) = 0$, i.e.

$$\frac{y_1 - y_2}{x_1 x_2} + \frac{y_2 - y_3}{x_2 x_3} + \frac{y_3 - y_1}{x_3 x_1} = 0$$

18. The coordinates of points A, B, C and D are $(-3, 5), (4, -2), (x, 3x)$ and $(6, 3)$ respectively and $\frac{\Delta ABC}{\Delta BCD} = \frac{2}{3}$, find x.

19. Find the area of the hexagon whose vertices taken in order are $(5, 0), (4, 2), (1, 3), (-2, 2), (-3, -1)$ and $(0, -4)$.

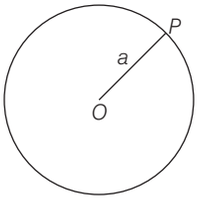
Session 4

Locus and Its Equation, Change of Axes the Transformation of Axes, Removal of the Term *xy* from $F(x, y) = ax^2 + 2hxy + by^2$ without Changing the Origin, Position of a Point which lies Inside a Triangle

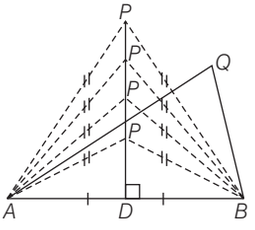
Locus and Its Equation

Locus : The locus of a moving point is the path traced out by that point under one or more given conditions.

For example 1. If a point P moves in a plane such that whose distance from a fixed point O (say) in the plane is always constant distance a. Thus the locus of the moving point P is clearly a circle with centre O and radius a.



For example 2. If a point P moves in a plane such that whose distance from two fixed points A and B (say) are always equal i.e. $PA = PB$



(The point P cannot be at Q because $AQ \neq BQ$)

Obviously all the positions of the moving point P lies on the right bisector of AB. Thus the locus of the moving point P is the right bisector of AB.

Equation of a Locus

A relation $f(x, y) = 0$ between x and y which is satisfied by each point on the locus and such that each point satisfying the equation is on the locus is called the ***equation of the locus***.

How to Find the Locus of a Point

Let (x_1, y_1) be the coordinates of the moving point say P. Now, apply the geometrical conditions on x_1, y_1 . This gives a relation between x_1 and y_1. Now replace x_1 by x and y_1 by y in the eliminant and resulting equation would be the equation of the locus.

***Corollary* 1 :** If x and y are not there in the question, the coordinates of P may also be taken as (x, y).

***Corollary* 2 :** If coordinates and equation are not given in the question, suitable choice of origin and axes may be made.

***Corollary* 3 :** To find the locus of the point of intersection of two straight lines, eliminate the parameter or parameters from the given lines. If more than one parameter, then additional condition or conditions will also be given.

Note

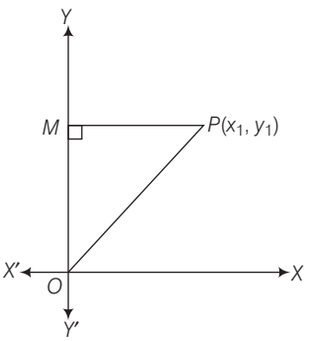
Simplify the equation by squaring both sides if square roots are there and taking LCM to remove the denominators.

Example 57. Find the locus of a point which moves such that its distance from the point (0, 0) is twice its distance from the *Y*-axis.

Sol. Let $P(x_1, y_1)$ be the moving point whose locus is required.

By hypothesis $|OP| = 2|PM|$ ($\because$ P lies in any quadrant)

$\Rightarrow \sqrt{(x_1^2 + y_1^2)} = 2|x_1|$



Squaring both sides, then

$$x_1^2 + y_1^2 = 4x_1^2$$

$$\Rightarrow \quad 3x_1^2 - y_1^2 = 0$$

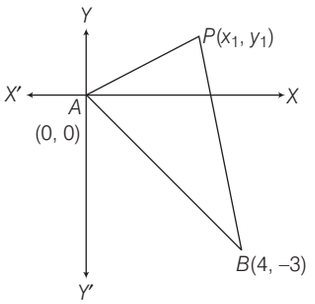
Changing (x_1, y_1) to (x, y), then

$$3x^2 - y^2 = 0$$

which is the required locus of P.

Example 58. Find the locus of the moving point P such that $2PA = 3PB$, where A is $(0, 0)$ and B is $(4, -3)$.

Sol. Let $P(x_1, y_1)$ be the moving point whose locus is required.



By hypothesis,

$$2PA = 3PB \text{ or } 4(PA)^2 = 9(PB)^2$$

$$\Rightarrow \quad 4\{x_1^2 + y_1^2\} = 9\{(x_1 - 4)^2 + (y_1 + 3)^2\}$$

$$\Rightarrow \quad 4(x_1^2 + y_1^2) = 9(x_1^2 + y_1^2 - 8x_1 + 6y_1 + 25)$$

or $\quad 5x_1^2 + 5y_1^2 - 72x_1 + 54y_1 + 225 = 0$

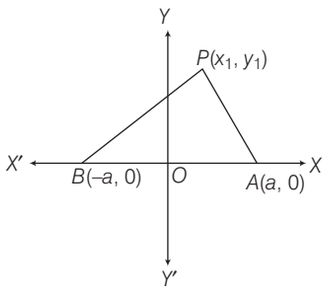
Changing (x_1, y_1) to (x, y), then

$$5x^2 + 5y^2 - 72x + 54y + 225 = 0$$

which is the required locus of P.

Example 59. A point moves so that the sum of the squares of its distances from two fixed points $A(a, 0)$ and $B(-a, 0)$ is constant and equal to $2c^2$, find the locus of the point.

Sol. Let $P(x_1, y_1)$ be the moving point whose locus is required.



By hypothesis, $\quad (PA)^2 + (PB)^2 = 2c^2$

$$\Rightarrow \quad (x_1 - a)^2 + (y_1 - 0)^2 + (x_1 + a)^2 + (y_1 - 0)^2 = 2c^2$$

$$\Rightarrow \quad 2x_1^2 + 2y_1^2 + 2a^2 = 2c^2$$

or $\quad x_1^2 + y_1^2 = c^2 - a^2$

Changing (x_1, y_1) to (x, y), then

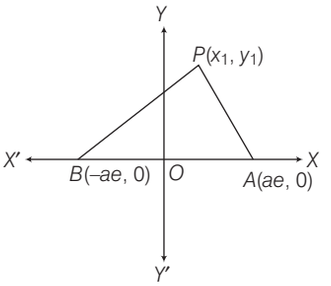
$$x^2 + y^2 = c^2 - a^2$$

which is the required locus of P.

Example 60. A point moves such that the sum of its distances from two fixed points $(ae, 0)$ and $(-ae, 0)$ is always $2a$. Prove that the equation of the locus is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \textit{where} \quad b^2 = a^2(1 - e^2)$$

Sol. Let $P(x_1, y_1)$ be the moving point whose locus is required and $A(ae, 0)$ and $B(-ae, 0)$ be the given fixed points.



By hypothesis, $\quad |PA| + |PB| = 2a$

or $\quad \sqrt{(x_1 - ae)^2 + (y_1 - 0)^2} + \sqrt{(x_1 + ae)^2 + (y_1 - 0)^2} = 2a$

or $\quad \sqrt{(x_1^2 + y_1^2 - 2aex_1 + a^2e^2)} + \sqrt{(x_1^2 + y_1^2 + 2aex_1 + a^2e^2)}$

$$= 2a \qquad \text{...(i)}$$

Let $\quad l = x_1^2 + y_1^2 - 2aex_1 + a^2e^2$

and $\quad m = x_1^2 + y_1^2 + 2aex_1 + a^2e^2 \qquad (l - m \text{ method})$

then, Eq. (i) can be written as

$$\sqrt{l} + \sqrt{m} = 2a \qquad \text{...(ii)}$$

and $\quad l - m = -4aex_1$

or $\quad (\sqrt{l} + \sqrt{m})(\sqrt{l} - \sqrt{m}) = -4aex_1$

or $\quad 2a(\sqrt{l} - \sqrt{m}) = -4aex_1 \qquad [\text{from Eq. (ii)}]$

or $\quad \sqrt{l} - \sqrt{m} = -2ex_1 \qquad \text{...(iii)}$

Adding Eqs. (ii) and (iii), then

$$2\sqrt{l} = 2a - 2ex_1 \text{ or } \sqrt{l} = a - ex_1$$

Squaring both sides,

$$l = a^2 - 2aex_1 + e^2x_1^2$$

$$\Rightarrow \quad x_1^2 + y_1^2 - 2aex_1 + a^2e^2 = a^2 - 2aex_1 + e^2x_1^2$$

$$\Rightarrow \quad (1 - e^2)x_1^2 + y_1^2 = a^2(1 - e^2)$$

or $\quad \dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{a^2(1 - e^2)} = 1$

or $\quad \dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} = 1 \qquad [\because b^2 = a^2(1 - e^2)]$

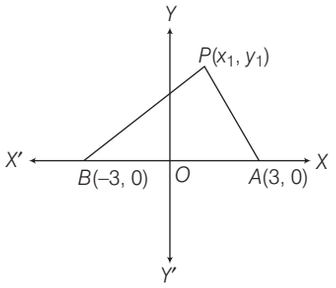
Changing (x_1, y_1) to (x, y), then

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

which is the required locus of P.

Example 61. Find the equation of the locus of a point which moves so that the difference of its distances from the points $(3, 0)$ and $(-3, 0)$ is 4 units.

Sol. Let $P(x_1, y_1)$ be the moving point whose locus is required and $A(3, 0)$ and $B(-3, 0)$ be the given fixed points.



By hypothesis

$$|PB| - |PA| = 4 \text{ (assume } |PB| > |PA|)$$

$$\Rightarrow \quad \sqrt{(x_1+3)^2 + (y_1-0)^2} - \sqrt{(x_1-3)^2 + (y_1-0)^2} = 4$$

$$\Rightarrow \quad \sqrt{(x_1^2 + y_1^2 + 6x_1 + 9)} = 4 + \sqrt{(x_1^2 + y_1^2 - 6x_1 + 9)}$$

Squaring both sides then,

$$x_1^2 + y_1^2 + 6x_1 + 9 = 16 + x_1^2 + y_1^2 - 6x_1 + 9 + 8\sqrt{(x_1^2 + y_1^2 - 6x_1 + 9)}$$

or $$(12x_1 - 16) = 8\sqrt{(x_1^2 + y_1^2 - 6x_1 + 9)}$$

or $$(3x_1 - 4) = 2\sqrt{(x_1^2 + y_1^2 - 6x_1 + 9)}$$

Again, squaring both sides, then

$$9x_1^2 - 24x_1 + 16 = 4x_1^2 + 4y_1^2 - 24x_1 + 36$$

or $$5x_1^2 - 4y_1^2 = 20$$

$$\Rightarrow \quad \frac{x_1^2}{4} - \frac{y_1^2}{5} = 1$$

Changing (x_1, y_1) by (x, y), then

$$\frac{x^2}{4} - \frac{y^2}{5} = 1$$

which is the required locus of P.

Aliter ($l - m$ method) :

Since, $$|PB| - |PA| = 4$$

$$\Rightarrow \quad \sqrt{(x_1+3)^2 + (y_1-0)^2} - \sqrt{(x_1-3)^2 + (y_1-0)^2} = 4$$

$$\Rightarrow \quad \sqrt{(x_1^2 + y_1^2 + 6x_1 + 9)} - \sqrt{(x_1^2 + y_1^2 - 6x_1 + 9)} = 4 \quad \text{...(i)}$$

Let $$l = x_1^2 + y_1^2 + 6x_1 + 9$$

and $$m = x_1^2 + y_1^2 - 6x_1 + 9$$

then, Eq. (i) can be written as

$$\sqrt{l} - \sqrt{m} = 4 \quad \text{... (ii)}$$

and $$l - m = 12x_1$$

$$\Rightarrow \quad (\sqrt{l} + \sqrt{m})(\sqrt{l} - \sqrt{m}) = 12x_1$$

$$\Rightarrow \quad (\sqrt{l} + \sqrt{m})(4) = 12x_1 \quad \text{[from Eq. (ii)]}$$

$$\Rightarrow \quad \sqrt{l} + \sqrt{m} = 3x_1 \quad \text{...(iii)}$$

Adding Eqs. (ii) and (iii),

$$2\sqrt{l} = (3x_1 + 4)$$

Squaring both sides,

$$4l = 9x_1^2 + 24x_1 + 16$$

$$\Rightarrow \quad 4(x_1^2 + y_1^2 + 6x_1 + 9) = 9x_1^2 + 24x_1 + 16$$

$$\Rightarrow \quad 5x_1^2 - 4y_1^2 = 20 \quad \text{or} \quad \frac{x_1^2}{4} - \frac{y_1^2}{5} = 1$$

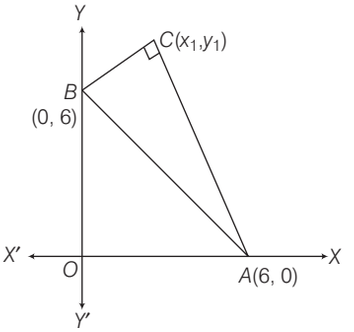
Changing (x_1, y_1) by (x, y), then

$$\frac{x^2}{4} - \frac{y^2}{5} = 1$$

which is the required locus of P.

Example 62. The ends of the hypotenuse of a right angled triangle are $(6, 0)$ and $(0, 6)$. Find the locus of the third vertex.

Sol. Let $C(x_1, y_1)$ be the moving point (third vertex) whose locus is required and $A(6, 0)$ and $B(0, 6)$ be the given vertices.



By hypothesis

$$(AC)^2 + (BC)^2 = (AB)^2 \qquad (\because \angle ACB = 90°)$$

$$\Rightarrow \quad (x_1-6)^2 + (y_1-0)^2 + (x_1-0)^2 + (y_1-6)^2 = 6^2 + 6^2$$

$$\Rightarrow \quad 2x_1^2 + 2y_1^2 - 12x_1 - 12y_1 = 0$$

or $$x_1^2 + y_1^2 - 6x_1 - 6y_1 = 0$$

Changing (x_1, y_1) by (x, y), then

$$x^2 + y^2 - 6x - 6y = 0$$

which is the required locus of third vertex C.

Aliter 1. Slope of $AC \times$ slope of $BC = -1$

$$\Rightarrow \quad \left(\frac{y_1 - 0}{x_1 - c}\right) \times \left(\frac{y_1 - r}{x_1 - 0}\right) = -1$$

or $$x_1^2 + y_1^2 - 6x_1 - 6y_1 = 0$$

Aliter 2. Mid-point of AB is $M \equiv (3, 3)$

$$\because \quad MA = MB = MC \Rightarrow (MA)^2 \equiv (MC)^2$$

$$(3-0)^2 + (3-6)^2 = (x_1-3)^2 (y_1-3)^2$$

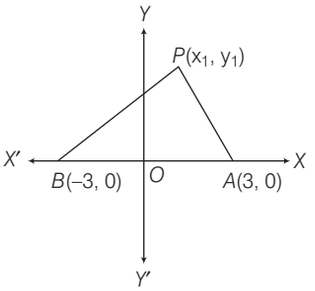
or $$x_1^2 + y_1^2 - 6x_1 - 6y_1 = 0$$

$\therefore$ Required locus is

$$x^2 + y^2 - 6x - 6y = 0$$

Example 63. Find the equation of the locus of a point which moves so that the sum of their distances from $(3, 0)$ and $(-3, 0)$ is less than 9.

Sol. Let $P(x_1, y_1)$ be the moving point whose locus is required and $A(3, 0)$ and $B(-3, 0)$ are the given points.



By hypothesis,

$$|PA| + |PB| < 9$$

$$\Rightarrow \quad \sqrt{\{(x_1-3)^2 + (y_1-0)^2\}} + \sqrt{\{(x_1+3)^2 + (y_1-0)^2\}} < 9$$

$$\Rightarrow \quad \sqrt{(x_1^2 + y_1^2 - 6x_1 + 9)} < 9 - \sqrt{(x_1^2 + y_1^2 + 6x_1 + 9)}$$

On squaring, we get

$$(x_1^2 + y_1^2 - 6x_1 + 9) < (81 + x_1^2 + y_1^2 + 6x_1 + 9) - 18\sqrt{(x_1^2 + y_1^2 + 6x_1 + 9)}$$

$$(\because a < b \Rightarrow a^2 < b^2 \text{ provided } a > 0)$$

$$\Rightarrow \quad -12x_1 - 81 < -18\sqrt{(x_1^2 + y_1^2 + 6x_1 + 9)}$$

$$\Rightarrow \quad (4x_1 + 27) > 6\sqrt{(x_1^2 + y_1^2 + 6x_1 + 9)}$$

$$(\because \text{If } a > b, \text{ then } -a < -b)$$

On squaring, we get

$$16x_1^2 + 729 + 216x > 36x_1^2 + 36y_1^2 + 216x_1 + 324$$

$$\Rightarrow \quad 20x_1^2 + 36y_1^2 < 405$$

Changing (x_1, y_1) by (x, y), then

$$20x^2 + 36y^2 < 405$$

which is the required locus of P.

Example 64. Find the locus of a point whose coordinate are given by $x = t + t^2$, $y = 2t + 1$, *where t is variable.*

Sol. Given, $x = t + t^2$ ...(i)

and $y = 2t + 1$ ...(ii)

From Eq. (ii), $t = \left(\frac{y-1}{2}\right)$ ...(iii)

On eliminating t from Eqs. (i) and (iii), we get required locus as

$$x = \left(\frac{y-1}{2}\right) + \left(\frac{y-1}{2}\right)^2$$

$$\Rightarrow \quad x = \left(\frac{y-1}{2}\right)\left(1 + \frac{y-1}{2}\right)$$

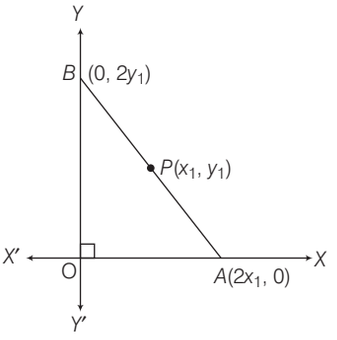
$$\Rightarrow \quad x = \left(\frac{y-1}{2}\right)\left(\frac{y+1}{2}\right)$$

or $4x = y^2 - 1$

or $y^2 = 4x + 1$

Example 65. A stick of length l rests against the floor and a wall of a room. If the stick begins to slide on the floor, find the locus of its middle point.

Sol. Let the cross section of the floor and wall be taken as the coordinate axes and AB be one of the position of the stick. Let the mid-point of AB be $P(x_1, y_1)$, then coordinates of A and B are $(2x_1, 0)$ and $(0, 2y_1)$ respectively.



But given, $|AB| = l$

$$\Rightarrow \quad (AB)^2 = l^2$$

$$\Rightarrow \quad (2x_1 - 0)^2 + (0 - 2y_1)^2 = l^2$$

$$\Rightarrow \quad 4x_1^2 + 4y_1^2 = l^2$$

Changing (x_1, y_1) by (x, y), then

$$4(x^2 + y^2) = l^2$$

which is the required locus of P.

Aliter :

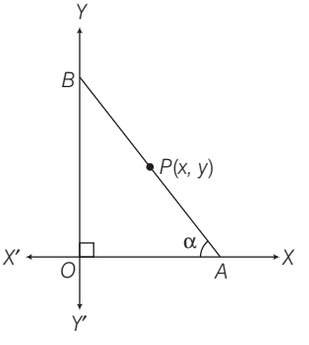
Since, $|AB| = l$

Let $\angle OAB = \alpha$

$\therefore$ $OA = l\cos\alpha$ and $OB = l\sin\alpha$

then, $A \equiv (l\cos\alpha, 0)$

and $B \equiv (0, l\sin\alpha)$



Let $P(x, y)$ be the mid-point of AB, then

$$2x = l\cos\alpha \quad ...(i)$$

$$2y = l\sin\alpha \quad ...(ii)$$

Squaring and adding Eqs. (i) and (ii), then

$$4(x^2 + y^2) = l^2$$

which is the required locus of P.

Example 66. Find the locus of the point of intersection of the lines $x\cos\alpha + y\sin\alpha = a$ and $x\sin\alpha - y\cos\alpha = b$, where α is variable.

Sol. Given equations are

$$x\cos\alpha + y\sin\alpha = a \quad ...(i)$$

and $$x\sin\alpha - y\cos\alpha = b \quad ...(ii)$$

Here, α is a variable, on eliminating α. Squaring and adding Eqs. (i) and (ii), we get required locus as

$$(x\cos\alpha + y\sin\alpha)^2 + (x\sin\alpha - y\cos\alpha)^2 = a^2 + b^2$$

or $$(x^2\cos^2\alpha + y^2\sin^2\alpha + 2xy\cos\alpha\sin\alpha) + (x^2\sin^2\alpha + y^2\cos^2\alpha - 2xy\sin\alpha\cos\alpha) = a^2 + b^2$$

$$\Rightarrow \quad x^2(\cos^2\alpha + \sin^2\alpha) + y^2(\sin^2\alpha + \cos^2\alpha) = a^2 + b^2$$

$$\Rightarrow \quad x^2 + y^2 = a^2 + b^2$$

Example 67. A variable line cuts X-axis at A, Y-axis at B, where $OA = a, OB = b$ (O as origin) such that $a^2 + b^2 = 1$.

Find the locus of

(i) centroid of ΔOAB

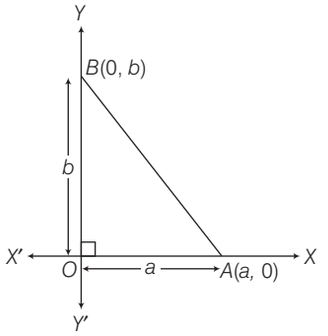
(ii) circumcentre of ΔOAB

Sol. (i) Coordinates of A and B are $(a, 0)$ and $(0, b)$ respectively.

If centroid of ΔOAB be $G(x, y)$

then $$x = \frac{0 + a + 0}{3} \quad \Rightarrow \quad a = 3x$$

and $$y = \frac{0 + 0 + b}{3} \quad \Rightarrow \quad b = 3y$$



On substituting in $a^2 + b^2 = 1$, we get

$$(3x)^2 + (3y)^2 = 1$$

or $$x^2 + y^2 = \frac{1}{9}$$

which is the required locus of G.

(ii) If the circumcentre be $C(x, y)$. Since in semicircle angle $90°$, then AB is the diameter of the circumcircle OAB.

$\therefore$ Circumcentre is the mid-point of AB.

Then $$x = \frac{a + 0}{2} \quad \Rightarrow \quad a = 2x$$

and $$y = \frac{0 + b}{2} \quad \Rightarrow \quad b = 2y$$

On substituting in $a^2 + b^2 = 1$, we get

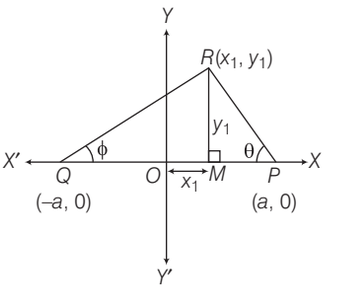
$$(2x)^2 + (2y)^2 = 1$$

or $$x^2 + y^2 = \frac{1}{4}$$

which is the required locus of C.

Example 68. Two points P and Q are given, R is a variable point on one side of the line PQ such that $\angle RPQ - \angle RQP$ is a positive constant 2α. Find the locus of the point R.

Sol. Let the X-axis along QP and the middle point of PQ is origin and let coordinates of moving point R be (x_1, y_1).



Let $$OP = OQ = a$$

then coordinates of P and Q are $(a, 0)$ and $(-a, 0)$ respectively.

Draw RM perpendicular on QP

$\therefore$ $$OM = x_1 \quad \text{and} \quad MR = y_1$$

and let $$\angle RPM = \theta \quad \text{and} \quad \angle RQM = \phi$$

Now, in ΔRMP,

$$\tan\theta = \frac{RM}{MP} = \frac{RM}{OP - OM} = \frac{y_1}{a - x_1} \quad ...(i)$$

and in ΔRMQ,

$$\tan\phi = \frac{RM}{QM} = \frac{RM}{OQ + OM} = \frac{y_1}{a + x_1} \quad ...(ii)$$

But given

$$\angle RPQ - \angle RQP = 2\alpha \quad \text{(constant)}$$

$$\Rightarrow \quad \theta - \phi = 2\alpha$$

$$\therefore \quad \tan(\theta - \phi) = \tan 2\alpha$$

$$\Rightarrow \quad \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi} = \tan 2\alpha$$

$$\Rightarrow \quad \frac{\frac{y_1}{a-x_1}-\frac{y_1}{a+x_1}}{1+\left(\frac{y_1}{a-x_1}\right).\left(\frac{y_1}{a+x_1}\right)}=\tan 2\alpha$$

[from Eqs. (i) and (ii)]

$$\Rightarrow \quad \frac{2x_1y_1}{a^2-x_1^2+y_1^2}=\tan 2\alpha$$

or $\quad a^2-x_1^2+y_1^2=2x_1y_1\cot 2\alpha$

or $\quad x_1^2-y_1^2+2x_1y_1\cot 2\alpha=a^2$

Hence, locus of the point $R(x_1, y_1)$ is

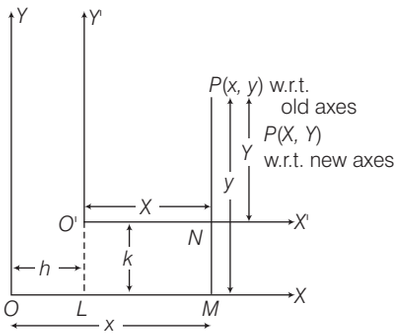
$$x^2-y^2+2xy\cot 2\alpha=a^2$$

Change of Axes OR the Transformations of Axes

In coordinate geometry we have discussed the coordinates of a point or the equation of a curve are always considered on taking a fixed point O as the origin and two perpendicular straight lines through O as the coordinates axes. For convenient the coordinates of the point or the equation of the curve changes when either the origin is changed or the direction of axes or both are suitably. These processes in coordinate geometry are known as the *transformations or change of axes.* This process of transformation of coordinates will be of great advantage to solve most of the problems very easily.

(i) Change of origin OR Shifting of origin (Translation of Axes)

To change the origin of coordinates to another point (h, k) *whereas the directions of axes remain unaltered.*



Let O be the origin of coordinates and OX, OY be the original coordinate axes. Let O' be the new origin and (h, k) its coordinates referred to the original axes. Draw two lines $O'X'$ and $O'Y'$ through O' and parallel to OX and OY respectively. Let $P(x, y)$ be any point referred to the original axes OX, OY. Again suppose that the coordinates of the same point P referred to the new axes $O'X', O'Y'$ are (X, Y).

From O' draw $O'L$ perpendicular to OX. from P draw PM perpendicular to OX to meet $O'X'$ in N. Then

$$OL=h, O'L=k, OM=x, PM=y, O'N=X$$

and $\quad PN=Y$

we have $\quad x=OM=OL+LM=OL+O'N=h+X$

i.e. $\quad \boldsymbol{x=X+h}$ ...(i)

and $\quad y=PM=PN+NM=PN+O'L=Y+k$

i.e. $\quad \boldsymbol{y=Y+k}$ ...(ii)

from Eqs. (i) and (ii),

$$\boldsymbol{X=x-h} \quad \text{and} \quad \boldsymbol{Y=y-k}$$

Thus, if origin is shifted to point (h, k) without rotation of axes, then new equation of curve can be obtained by putting $x+h$ in place of x and $y+k$ in place of y.

Remarks

1. In this case axes are shifted parallel to themselves, then it is also called **Transformation by parallel axes.**
2. **Inverse translation or shifting the origin back :** Some times it is required to shift the new origin back. Then putting $x-h$ in place of x and $y-k$ in place of y in any equation of curve referred to the new origin to get the corresponding equation referred to the old origin.
3. The above transformation is true whether the axes be rectangular or oblique.

Example 69. Find the equation of the curve $2x^2+y^2-3x+5y-8=0$ when the origin is transferred to the point $(-1, 2)$ without changing the direction of axes.

Sol. Here, we want to shift the origin to the point $(-1, 2)$ without changing the direction of axes. Then we replace x by $x-1$ and y by $y+2$ in the equation of given curve, then the transformed equation is

$$2(x-1)^2+(y+2)^2-3(x-1)+5(y+2)-8=0$$

$$\Rightarrow \quad 2x^2+y^2-7x+9y+11=0$$

Example 70. The equation of a curve referred to the new axes, axes retaining their direction and origin is $(4, 5)$ is $x^2+y^2=36$. Find the equation referred to the original axes.

Sol. Here we want to shift the $(4, 5)$ to the origin without changing the direction of axes. Then we replace x by $x-4$ and y by $y-5$ in the equation of given curve then the required equation is

$$(x-4)^2+(y-5)^2=36$$

$$\Rightarrow \quad x^2+y^2-8x-10y+5=0$$

Example 71. Shift the origin to a suitable point so that the equation $y^2 + 4y + 8x - 2 = 0$ will not contain term in y and the constant.

Sol. Let the origin be shifted to the point (h, k) without changing the direction of axes. Then we replace x by $x + h$ and y by $y + k$ in the equation of the given curve then the transformed equation is

$$(y + k)^2 + 4(y + k) + 8(x + h) - 2 = 0$$

$$\Rightarrow \quad y^2 + (2k + 4)y + 8x + (k^2 + 4k + 8h - 2) = 0$$

Since, this equation is required to be free from the term containing y and the constant, we have

$$2k + 4 = 0 \quad \text{and} \quad k^2 + 4k + 8h - 2 = 0$$

$$\therefore \quad k = -2 \quad \text{and} \quad h = \frac{3}{4}$$

Hence, the point to which the origin be shifted is $\left(\frac{3}{4}, -2\right)$.

Example 72. At what point the origin be shifted, if the coordinates of a point $(-1, 8)$ become $(-7, 3)$?

Sol. Let the origin be shifted to the point (h, k) without changing the direction of axes. Then we replace x by $x + h$ and y by $y + k$ and we get new co-ordinates. Here, given old coordinates and new coordinates are $(-1, 8)$ and $(-7, 3)$ respectively.

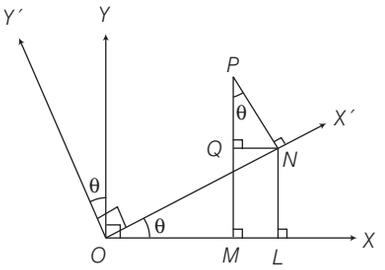
We have $\quad -1 + h = -7 \quad$ and $\quad 8 + k = 3$

$$\Rightarrow \quad h = -6 \quad \text{and} \quad k = -5$$

Hence, the origin must be shifted to $(-6, -5)$.

(ii) Rotation of Axes (Change of Directions of Axes)

To find the change in the coordinates of a point when the directions of axes are rotated through an angle θ the origin being fixed.



Let OX and OY be the original system of coordinate axes. Let OX' and OY' be the new axes obtained by rotating the original axes through an angle θ. Let P be a point in the plane whose coordinates are (x, y) and (X, Y) referred to old and new axes respectively. Draw PM and PN perpendiculars to OX and OX' and also NL and NQ perpendiculars to OX and PM. We have

$$OM = x, PM = y, ON = X, PN = Y$$

Now, $\quad x = OM = OL - ML$

$$\left(\because \text{Angle between any two lines = Angle between their perpendiculars i.e. } \angle XOX' = \angle NPM = \theta\right)$$

$$= OL - QN = ON\cos\theta - PN\sin\theta$$

$$= X\cos\theta - Y\sin\theta$$

i.e. $\quad \boldsymbol{x = X\cos\theta - Y\sin\theta}$ ...(i)

and $\quad y = PM = PQ + QM = PQ + NL$

$$= PN\cos\theta + ON\sin\theta$$

$$= Y\cos\theta + X\sin\theta$$

i.e. $\quad \boldsymbol{y = X\sin\theta + Y\cos\theta}$ ...(ii)

Now, multiplying Eqs. (i) by $\cos\theta$ and Eq. (ii) by $\sin\theta$ and adding we get

$$\boldsymbol{X = x\cos\theta + y\sin\theta} \quad \text{...(iii)}$$

Also, subtracting the product of Eq. (i) by $\sin\theta$ from the product of Eq. (ii) by $\cos\theta$, we get

$$\boldsymbol{Y = -x\sin\theta + y\cos\theta} \quad \text{...(iv)}$$

also $x^2 + y^2 = X^2 + Y^2 = OP^2$ are unchanged i.e. the distance of the point P from the origin O remains unaffected by the rotation of axes.

Rule : When the axes are rotated through θ, replace (x, y) by $(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$.

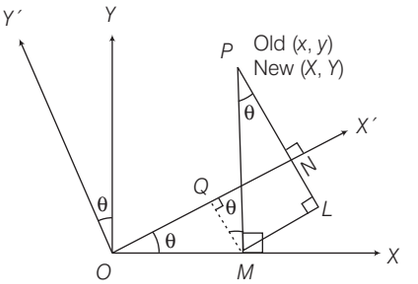
Shifting the coordinate axes back : Some times it is required to shift the new coordinates axes back. Then replace (x, y) by

$$(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta).$$

Independent Proof (Aliter)

The relations $X = x\cos\theta + y\sin\theta$ and $Y = -x\sin\theta + y\cos\theta$ can be obtained independently.

Proof : Draw PM and PN perpendiculars to OX and OX' and also ML and MQ perpendiculars to PN and OX' respectively.



We have, $\quad OM = x, PM = y, ON = X$

and $\quad PN = Y$

Also, angle between any two lines

= Angle between their perpendiculars lines

i.e. $\angle XOX' = \angle MPN = \angle PMQ = \theta$

$\therefore$ $X = ON = OQ + QN$

$= OQ + ML$

$= OM\cos\theta + PM\sin\theta$

$(\because OQ = OM\cos\theta$ and $ML = PM\sin\theta)$

$= x\cos\theta + y\sin\theta$

i.e. $\boldsymbol{X = x\cos\theta + y\sin\theta}$

and $Y = PN = PL - NL = PL - QM$

$= PM\cos\theta - OM\sin\theta$

$(\because PL = PM\cos\theta$ and $QM = OM\sin\theta)$

$= y\cos\theta - x\sin\theta$

i.e. $\boldsymbol{Y = -x\sin\theta + y\cos\theta}$

Remark :

The results Eqs. (i), (ii), (iii) and (iv) can be conveniently remembered by the following methods.

(i) Light heavy method : Let x, y be light and X, Y be heavy then heavy X, Y down and x, y up then

	x	y
X	$\cos\theta$	$\sin\theta$
Y	$-\sin\theta$	$\cos\theta$

third row is obtained by differentiating second row with respect to θ.

For remembrance C S (Civil Services)

$\downarrow$ $\downarrow$

$\cos\theta$ $\sin\theta$ $\left(\text{i.e. } \begin{matrix} C \to \cos\theta \\ S \to \sin\theta \end{matrix}\right)$

$\downarrow$ $\downarrow$

$-\sin\theta$ $\cos\theta$ $\left(\begin{matrix} (\cos\theta)' = -\sin\theta \\ (\sin\theta)' = \cos\theta \end{matrix}\right)$

(a) Finding x and y in terms of X and Y

x = Sum of the products of the elements in the left most column with the corresponding elements of the first column

i.e. $x = X\cos\theta - Y\sin\theta$

and y = Sum of the products of the elements in the left most column with the corresponding elements of the second column.

i.e. $y = X\sin\theta + Y\cos\theta$

Hence, $\left.\begin{matrix} x = X\cos\theta - Y\sin\theta \\ y = X\sin\theta + Y\cos\theta \end{matrix}\right\}$

(b) Finding X and Y in terms of x and y

X = Sum of the products of the elements of top-row with the corresponding elements of first row.

i.e. $X = x\cos\theta + y\sin\theta$

and Y = Sum of the products of the elements of top-row with the corresponding elements of second row.

i.e. $Y = -x\sin\theta + y\cos\theta$

Hence, $\left.\begin{matrix} X = x\cos\theta + y\sin\theta \\ Y = -x\sin\theta + y\cos\theta \end{matrix}\right\}$

(ii) Matrix method :

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} \text{ (say)}$$

and $$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = A'\begin{bmatrix} X \\ Y \end{bmatrix}$$

where, A' is the transpose matrix of A.

(iii) Complex number method :

Let $z = x + iy$

and $Z = X + iY$, where $i = \sqrt{-1}$

then $z = Ze^{i\theta}$ ...(i)

i.e. $(x + iy) = (X + iY)(\cos\theta + i\sin\theta)$

On comparing real and imaginary parts, we get

$\left.\begin{matrix} x = X\cos\theta - Y\sin\theta \\ y = X\sin\theta + Y\cos\theta \end{matrix}\right\}$

Again from Eq. (i), $Z = ze^{-i\theta}$

i.e. $(X + iY) = (x + iy)(\cos\theta - i\sin\theta)$

On comparing real and imaginary parts, we get

$\left.\begin{matrix} X = x\cos\theta + y\sin\theta \\ Y = -x\sin\theta + y\cos\theta \end{matrix}\right\}$

Example 73. If the axes are turned through 45°, find the transformed form of the equation $3x^2 + 3y^2 + 2xy = 2$.

Sol. Here, $\theta = 45°$ so $\sin\theta = \cos\theta = \frac{1}{\sqrt{2}}$

Replacing (x, y) by $(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$

i.e. $\left(\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}}\right)$

Then, $3x^2 + 3y^2 + 2xy = 2$ becomes

$$3\left(\frac{x-y}{\sqrt{2}}\right)^2 + 3\left(\frac{x+y}{\sqrt{2}}\right)^2 + 2\left(\frac{x-y}{\sqrt{2}}\right)\left(\frac{x+y}{\sqrt{2}}\right) = 2$$

$\Rightarrow \quad 3(2x^2 + 2y^2) + 2(x^2 - y^2) = 4$

$\Rightarrow \quad 8x^2 + 4y^2 = 4$

or $\quad 2x^2 + y^2 = 1$

which is free from the term containing xy.

Example 74. Prove that if the axes be turned through $\frac{\pi}{4}$ the equation $x^2 - y^2 = a^2$ is transformed to the form $xy = \lambda$. Find the value of λ.

Sol. Here, $\theta = \frac{\pi}{4}$ so $\sin\theta = \cos\theta = \frac{1}{\sqrt{2}}$

Replacing (x, y) by $(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$

i.e. $\left(\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}}\right)$

then, $x^2 - y^2 = a^2$ becomes

$$\left(\frac{x-y}{\sqrt{2}}\right)^2 - \left(\frac{x+y}{\sqrt{2}}\right)^2 = a^2$$

$$\Rightarrow \quad \left(\frac{x-y}{\sqrt{2}} + \frac{x+y}{\sqrt{2}}\right)\left(\frac{x-y}{\sqrt{2}} - \frac{x+y}{\sqrt{2}}\right) = a^2$$

$$\Rightarrow \quad \left(\frac{2x}{\sqrt{2}}\right)\left(\frac{-2y}{\sqrt{2}}\right) = a^2$$

or $\quad xy = -\frac{a^2}{2}$

Comparing it with $xy = \lambda$, then we get $\lambda = -\frac{a^2}{2}$.

Example 75. Through what angle should the axes be rotated so that the equation $9x^2 - 2\sqrt{3}\,xy + 7y^2 = 10$ may be changed to $3x^2 + 5y^2 = 5$?

Sol. Let angle be θ then replacing (x, y) by $(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$

then, $\quad 9x^2 - 2\sqrt{3}xy + 7y^2 = 10$ becomes

$$9(x\cos\theta - y\sin\theta)^2 - 2\sqrt{3}$$
$$(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta)$$
$$+ 7(x\sin\theta + y\cos\theta)^2 = 10$$

$$\Rightarrow \quad x^2(9\cos^2\theta - 2\sqrt{3}\sin\theta\cos\theta + 7\sin^2\theta)$$
$$+ 2xy(-9\sin\theta\cos\theta - \sqrt{3}\cos 2\theta + 7\sin\theta\cos\theta)$$
$$+ y^2(9\cos^2\theta + 2\sqrt{3}\sin\theta\cos\theta + 7\cos^2\theta) = 10$$

On comparing with $3x^2 + 5y^2 = 5$ (coefficient of xy)

we get $-9\sin\theta\cos\theta - \sqrt{3}\cos 2\theta + 7\sin\theta\cos\theta = 0$

or $\quad \sin 2\theta = -\sqrt{3}\cos 2\theta$

or $\quad \tan 2\theta = -\sqrt{3} = \tan(180^\circ - 60^\circ)$

or $\quad 2\theta = 120^\circ$

$\therefore \quad \theta = 60^\circ$

Example 76. If (x, y) and (X, Y) be the coordinates of the same point referred to two sets of rectangular axes with the same origin and if $ux + vy$, when u and v are independent of X and Y become $VX + UY$, show that $u^2 + v^2 = U^2 + V^2$

Sol. Let the axes rotate an angle θ and if (x, y) be the point with respect to old axes and (X, Y) be the co-ordinates with respect to new axes, then

$$x + iy = (X + iY)e^{i\theta} = (X + iY)(\cos\theta + i\sin\theta)$$

On comparing real and imaginary parts, we get

$$\begin{cases} x = X\cos\theta - Y\sin\theta \\ y = X\sin\theta + Y\cos\theta \end{cases}$$

Then, $ux + vy = u(X\cos\theta - Y\sin\theta) + v(X\sin\theta + Y\cos\theta)$

$$= (u\cos\theta + v\sin\theta)X + (-u\sin\theta + v\cos\theta)Y$$

but given new curve $VX + UY$

then, $VX + UY = (u\cos\theta + v\sin\theta)$
$X + (-u\sin\theta + v\cos\theta)Y$

On comparing the coefficients of X and Y, we get

$$V = u\cos\theta + v\sin\theta \quad \text{...(i)}$$

and $\quad U = -u\sin\theta + v\cos\theta \quad \text{...(ii)}$

Squaring and adding Eqs. (i) and (ii), we get

$$V^2 + U^2 = (u\cos\theta + v\sin\theta)^2 + (-u\sin\theta + v\cos\theta)^2$$
$$= u^2 + v^2$$

Hence, $\quad u^2 + v^2 = U^2 + V^2$

Aliter 1 (By matrix method) :

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X\cos\theta - Y\sin\theta \\ X\sin\theta + Y\cos\theta \end{bmatrix} \quad \text{...(i)}$$

$\because \quad ux + vy = [u \quad v]\begin{bmatrix} x \\ y \end{bmatrix} = [u\ v]\begin{bmatrix} X\cos\theta + Y\sin\theta \\ X\sin\theta + Y\cos\theta \end{bmatrix}$

[from (i)]

$$= uX\cos\theta - uY\sin\theta + vX\sin\theta + vY\cos\theta$$
$$= X(u\cos\theta + v\sin\theta) + Y(-u\sin\theta + v\cos\theta)$$

but given new curve $VX + UY$

Then, $VX + UY = X(u\cos\theta + v\sin\theta) + Y$
$(-u\sin\theta + v\cos\theta)$

On comparing the coefficients of X and Y, we get

$$V = u\cos\theta + v\sin\theta \quad \text{...(ii)}$$

$$U = -u\sin\theta + v\cos\theta \quad \text{...(iii)}$$

Squaring and adding Eqs. (ii) and (iii), we get

$$u^2 + v^2 = U^2 + V^2$$

Aliter 2 (Best approach) :

$$ux + vy = R_e((u - iv)(x + iy))$$
$$= R_e((u - iv)(X + iY)e^{i\theta}) \quad \text{...(i)}$$

and $\quad VX + UY = R_e((V - iU)(X + iY)) \quad \text{...(ii)}$

From Eqs. (i) and (ii), we get

$$V - iU = (u - iv)e^{i\theta}$$

Taking modulus both sides, then

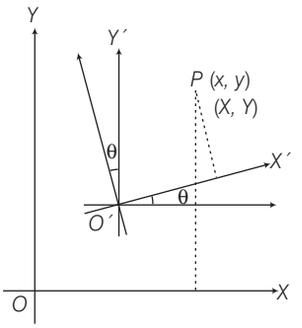
$$|V - iU| = |u - iv||e^{i\theta}|$$

$$\Rightarrow \quad \sqrt{(V^2 + U^2)} = \sqrt{u^2 + v^2} \cdot 1$$

or $$u^2 + v^2 = U^2 + V^2$$

(iii) Double Transformation (Origin Shifted and Axes Rotated)

If origin is shifted to the point. $O'(h, k)$ and at the same time the directions of axes are rotated through an angle θ in the anticlockwise sense such that new coordinates of $P(x, y)$ become (X, Y).



Then, we get $x = h + X\cos\theta - Y\sin\theta$ ...(i)

and $y = k + X\sin\theta + Y\cos\theta$ ...(ii)

In practice we have to replace x by $h + x\cos\theta - y\sin\theta$ and y by $k + x\sin\theta + y\cos\theta$.

Again, if we want to shift the coordinate axes back to their original positions, then we obtained X and Y by solving Eqs. (i) and (ii), then

$$X = (x - h)\cos\theta + (y - k)\sin\theta$$

and $$Y = -(x - h)\sin\theta + (y - k)\cos\theta$$

Example 77. What does the equation $2x^2 + 4xy - 5y^2 + 20x - 22y - 14 = 0$ becomes when referred to rectangular axes through the point $(-2, -3)$, the new axes being inclined at an angle of 45° *with the old ?*

Sol. Let $O'(-2, -3)$ be the new origin and axes are rotated about O' through an angle $45°$ in anticlockwise direction then replacing x and y by

$$-2 + x\cos 45° - y\sin 45°$$

and $$-3 + x\sin 45° + y\cos 45°$$

i.e. $-2 + \left(\frac{x - y}{\sqrt{2}}\right)$ and $-3 + \left(\frac{x + y}{\sqrt{2}}\right)$ respectively in the given curve, then the new equation of curve will be

$$2\left\{-2 + \left(\frac{x - y}{\sqrt{2}}\right)\right\}^2 + 4\left\{-2 + \left(\frac{x - y}{\sqrt{2}}\right)\right\}\left\{-3 + \left(\frac{x + y}{\sqrt{2}}\right)\right\}$$

$$-5\left\{-3 + \left(\frac{x + y}{\sqrt{2}}\right)\right\}^2 + 20\left\{-2 + \left(\frac{x - y}{\sqrt{2}}\right)\right\}$$

$$-22\left\{-3 + \left(\frac{x + y}{\sqrt{2}}\right)\right\} - 14 = 0$$

$$\Rightarrow \quad x^2 - 14xy - 7y^2 - 2 = 0$$

Removal of the Term xy from $f(x, y) = ax^2 + 2hxy + by^2$ without Changing the Origin

Clearly, $h \neq 0$

Rotating the axes through an angle θ, we have

$$x = X\cos\theta - Y\sin\theta$$

and $$y = X\sin\theta + Y\cos\theta$$

$\because$ $$f(x, y) = ax^2 + 2hxy + by^2$$

After rotation, new equation is

$$F(X, Y) = (a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta)X^2$$

$$+ 2\{(b - a)\cos\theta\sin\theta + h(\cos^2\theta - \sin^2\theta)\}XY$$

$$+ (a\sin^2\theta - 2h\cos\theta\sin\theta + b\cos^2\theta)Y^2$$

Now, coefficient of $XY = 0$

Then, we get $\cot 2\theta = \frac{a - b}{2h}$

Remark

Usually, we use the formula, $\tan 2\theta = \frac{2h}{a - b}$ for finding the angle of rotation, θ.

However, if $a = b$, we use $\cot 2\theta = \frac{a - b}{2h}$ as in this case $\tan 2\theta$ is not defined.

Example 78. Given the equation $4x^2 + 2\sqrt{3}xy + 2y^2 = 1$, through what angle should the axes be rotated so that the term in xy be wanting from the transformed equation.

Sol. Comparing the given equation with $ax^2 + 2hxy + by^2$, we get $a = 4$, $h = \sqrt{3}$ $b = 2$. If axes are to be rotated at θ, then

$$\tan 2\theta = \frac{2h}{a - b} = \frac{2\sqrt{3}}{2} = \sqrt{3} = \tan\frac{\pi}{3}$$

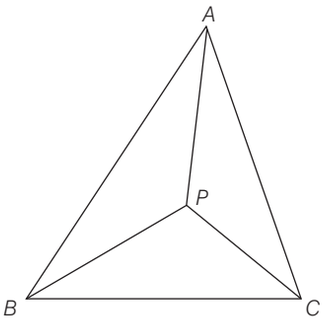
$$2\theta = \frac{\pi}{3}, \pi + \frac{\pi}{3} \Rightarrow 2\theta = \frac{\pi}{3}, \frac{4\pi}{3}$$

$\therefore$ $$\theta = \frac{\pi}{6}, \frac{2\pi}{3}$$

Position of a Point which Lies Inside a Triangle

If any point say (P) lies within the triangle ABC,

then $\Delta_1 + \Delta_2 + \Delta_3 = \Delta$



where, $\Delta =$ Area of triangle ABC,

$\Delta_1 =$ Area of ΔPBC,

$\Delta_2 =$ Area of ΔPCA,

$\Delta_3 =$ Area of ΔPAB

Also, $\Delta_1 \neq 0, \Delta_2 \neq 0, \Delta_3 \neq 0, \Delta \neq 0$

(Each individual area must be non-zero)

Example 79. Find λ if $(\lambda, \lambda+1)$ is an interior point of ΔABC where, $A \equiv (0,3); B \equiv (-2,0)$ and $C \equiv (6,1)$.

Sol. The point $P(\lambda, \lambda+1)$ will be inside the triangle ABC, then

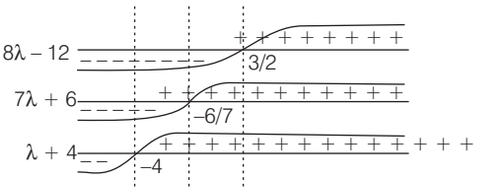
Area of ΔPBC + Area of ΔPCA + Area of ΔPAB = Area of ΔABC

$$\Rightarrow \frac{1}{2}\left|\begin{vmatrix} \lambda & \lambda+1 & 1 \\ -2 & 0 & 1 \\ 6 & 1 & 1 \end{vmatrix}\right| + \frac{1}{2}\left|\begin{vmatrix} \lambda & \lambda+1 & 1 \\ 6 & 1 & 1 \\ 0 & 3 & 1 \end{vmatrix}\right| + \frac{1}{2}\left|\begin{vmatrix} \lambda & \lambda+1 & 1 \\ 0 & 3 & 1 \\ -2 & 0 & 1 \end{vmatrix}\right|$$

$$= \frac{1}{2}\left|\begin{vmatrix} 0 & 3 & 1 \\ -2 & 0 & 1 \\ 6 & 1 & 1 \end{vmatrix}\right|$$

$$\Rightarrow \quad |7\lambda + 6| + |8\lambda - 12| + |\lambda + 4| = 22$$

For $\lambda < -4$:



Then, $-(7\lambda + 6) - (8\lambda - 12) - (\lambda + 4) = 22$

$\Rightarrow \quad -16\lambda = 20 \quad \therefore \quad \lambda = -\frac{5}{4}$

which is impossible.

For $-4 \leq \lambda < -\frac{6}{7}$:

Then, $-(7\lambda + 6) - (8\lambda - 12) + (\lambda + 4) = 22$

$\Rightarrow \quad -14\lambda = 12 \quad \therefore \quad = -\frac{6}{7}$

which is impossible.

For $-\frac{6}{7} \leq \lambda < \frac{3}{2}$:

Then, $(7\lambda + 6) - (8\lambda - 12) + \lambda + 4 = 22 \Rightarrow 22 = 22$

$\because$ at $\lambda = -\frac{6}{7}$, area of $\Delta PBC = 0$

$\therefore \quad \lambda \neq \frac{-6}{7}$

$\therefore \quad \lambda \in \left(-\frac{6}{7}, \frac{3}{2}\right)$

For $\lambda \geq \frac{3}{2}$:

Then, $7\lambda + 6 + 8\lambda - 12 + \lambda + 4 = 22 \Rightarrow 16\lambda = 24$

$\therefore \quad \lambda = \frac{3}{2}$

$\because$ at $\lambda = \frac{3}{2}$, area of $\Delta PCA = 0$

$\therefore \quad \lambda \neq \frac{3}{2}$

Hence, value of $\lambda \in \left(-\frac{6}{7}, \frac{3}{2}\right)$.

Exercise for Session 4

1. The equation of the locus of points equidistant from $(-1,-1)$ and $(4,2)$ is

(a) $3x-5y-7=0$ (b) $5x+3y-9=0$ (c) $4x+3y+2=0$ (d) $x-3y+5=0$

2. The equation of the locus of a point which moves so that its distance from the point $(ak,0)$ is k times its distance from the point $\left(\frac{a}{k},0\right)$, $(k\neq 1)$ is

(a) $x^2-y^2=a^2$ (b) $2x^2-y^2=2a^2$ (c) $xy=a^2$ (d) $x^2+y^2=a^2$

3. If the coordinates of a variable point P be $\left(t+\frac{1}{t},t-\frac{1}{t}\right)$, where t is the variable quantity, then the locus of P is

(a) $xy=8$ (b) $2x^2-y^2=8$ (c) $x^2-y^2=4$ (d) $2x^2+3y^2=5$

4. If the coordinates of a variable point P be $(\cos\theta+\sin\theta,\sin\theta-\cos\theta)$, where θ is the parameter, then the locus of P is

(a) $x^2-y^2=4$ (b) $x^2+y^2=2$ (c) $xy=3$ (d) $x^2+2y^2=3$

5. If a point moves such that twice its distance from the axis of x exceeds its distance from the axis of y by 2, then its locus is

(a) $x-2y=2$ (b) $x+2y=2$ (c) $2y-x=2$ (d) $2y-3x=5$

6. The equation $4xy-3x^2=a^2$ become when the axes are turned through an angle $\tan^{-1}2$ is

(a) $x^2+4y^2=a^2$ (b) $x^2-4y^2=a^2$ (c) $4x^2+y^2=a^2$ (d) $4x^2-y^2=a^2$

7. Transform the equation $x^2-3xy+11x-12y+36=0$ to parallel axes through the point $(-4,1)$ becomes $ax^2+bxy+1=0$ then $b^2-a=$

(a) $\frac{1}{4}$ (b) $\frac{1}{16}$ (c) $\frac{1}{64}$ (d) $\frac{1}{256}$

8. Find the equation of the locus of all points equidistant from the point $(2,4)$ and the Y-axis.

9. Find the equation of the locus of the points twice as far from $(-a,0)$ as from $(a,0)$.

10. OA and OB are two perpendicular straight lines. A straight line AB is drawn in such a manner that $OA+OB=8$. Find the locus of the mid point of AB.

11. The ends of a rod of length l move on two mutually perpendicular lines. Find the locus of the point on the rod which divides it in the ratio 1 : 2.

12. The coordinates of three points O, A, B are $(0,0)$, $(0,4)$ and $(6,0)$ respectively. A point P moves so that the area of ΔPOA is always twice the area of ΔPOB. Find the equation to both parts of the locus of P.

13. What does the equation $(a-b)(x^2+y^2)-2abx=0$ become, if the origin be moved to the point $\left(\frac{ab}{a-b},0\right)$?

14. The equation $x^2+2xy+4=0$ is transformed to the parallel axes through the point $(6,\lambda)$. For what value of λ its new form passes through the new origin ?

15. Show that if the axes be turned through $7\frac{1}{2}^\circ$; the equation $\sqrt{3}x^2+(\sqrt{3}-1)xy-y^2=0$ become free of xy in its new form.

16. Find the angle through which the axes may be turned so that the equation $Ax+By+C=0$ may reduce to the form $x=$ constant, and determine the value of this constant.

17. Transform $12x^2+7xy-12y^2-17x-31y-7=0$ to rectangular axes through the point $(1,-1)$ inclined at an angle $\tan^{-1}\left(\frac{4}{3}\right)$ to the original axes.

Shortcuts and Important Results to Remember

1 If D, E, F are the mid-points of the sides BC, CA, AB of ΔABC, the
$A = E + F - D$
$B = F + D - E$
and $C = D + E - F$

2 Orthocentre, nine point centre, centroid, circumcentre of a triangle are collinear. Centroid divides the line joining orthocentre and circumcentre in the ration 2 :1 (Internally) and nine point centre is the mid-point of orthocentre and circumcentre.

3 The circumcentre of a right angled triangle is the mid-point of the hypotenuse.

4 In an equilateral triangle orthocentre, nine point centre, centroid, circumcentre, incentre coincide.

5 The distance between the orthocentre and circumcentre in an equilateral triangle is zero.

6 The orthocentre of a triangle having vertices (α, β), (β, α) and (α, α) is (α, α).

7 Orthocentre of the triangle formed by the points $\left(\alpha, \frac{1}{\alpha}\right); \left(\beta, \frac{1}{\beta}\right); \left(\gamma, \frac{1}{\gamma}\right)$ is $\left(-\frac{1}{\alpha\beta\gamma}, -\alpha\beta\gamma\right)$

i.e. all points and orthocentre lie on $xy = 1$.

8 **Points in a triangle :** Centroid (G), Incentre (I), Excentres (I_1, I_2, I_3), Orthocentre (O), Circumcentre (C) are given by $\left(\frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}, \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3}\right)$, where

	m_1	m_2	m_3
G	1	1	1
I	$\sin A$	$\sin B$	$\sin C$
I_1	$-\sin A$	$\sin B$	$\sin C$
I_2	$\sin A$	$-\sin B$	$\sin C$
I_3	$\sin A$	$\sin B$	$-\sin C$
O	$\tan A$	$\tan B$	$\tan C$
C	$\sin 2A$	$\sin 2B$	$\sin 2B$

and vertices

$A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$, $C \equiv (x_3, y_3)$ and A, B, C are the angles of ΔABC.

9 If the circumcentre and centroid of a triangle are respectively (α, β), (γ, δ), then orthocentre will be $(3\gamma - 2\alpha, 3\delta - 2\beta)$.

10 If $ABCD$ is a parallelogram, then $D = A - B + C$.

11 If D, E, F are the mid-points of the sides BC, CA, AB of ΔABC, then the centroid of ΔABC = centroid of ΔDEF. If area of $\Delta ABC = \Delta$, then area of ΔAFE = area of ΔBDF = area of ΔCED = area of $\Delta DEF = \frac{\Delta}{4}$ and area of parallelogram $CEFD$ = area of parallelogram $BDEF$ = area of parallelogram $AEDF = \frac{\Delta}{2}$

12 Orthocentre of the right angle triangled ABC, right angled at A is A.

13 Circumcentre of the right angled triangle ABC, right angled at A is $\frac{B + C}{2}$.

14 X-axis divides the line segment joining (x_1, y_1), (x_2, y_2) in the ratio $-y_1 : y_2$ and Y-axis divides the same line segment in the ratio $-x_1 : x_2$.

15 Area of the triangle formed by $(x_1, y_{1)}, (x_2\ y_2), (x_3, y_3)$ is $\frac{1}{2}\left|\begin{vmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{vmatrix}\right|$.

16 The area of the triangle formed by $y = m_1x + c_1$, $y = m_2x + c_2$, $y = m_3\ x + c_3$ is $\frac{1}{2} \Sigma \frac{(c_1 - c_2)^2}{|m_1 - m_2|}$.

17 Area of the quadrilateral formed by $(x_1, y_1)(x_2, y_{2)}\ (x_3, y_3), (x_4, y_4)$ is
$$\frac{1}{2}\left|\begin{vmatrix} x_1 - x_3 & x_2 - x_4 \\ y_1 - y_3 & y_2 - y_4 \end{vmatrix}\right|$$

18 If $(x_1, y_1), (x_2, y_2)$ are the ends of the hypotenuse of a right angled isosceles triangle, then the third vertex is given by $\left(\frac{x_1 + x_2 \pm (y_1 - y_2)}{2}, \frac{y_1 + y_2 \mp (x_1 - x_2)}{2}\right)$

19 Given the two vertices (x_1, y_1) and (x_2, y_2) of an equilateral triangle, then its third vertex is given by $\left(\frac{x_1 + x_2 \pm \sqrt{3}(y_1 - y_2)}{2}, \frac{y_1 + y_2 \mp \sqrt{3}(x_1 - x_2)}{2}\right)$

20 Circumcentre of the triangle formed by the points $(x_1, y_1), (x_{2,}\ y_2)$ and (x_3, y_3) is same as that of triangle formed by the points $(0, 0)$, $(x_2 - x_1, y_2 - y_1), (x_3 - x_1, y_3 - y_1)$.

JEE Type Solved Examples :
Single Option Correct Type Questions

This section contains **5 multiple choice examples**. Each example has four choice (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

● ***Ex. 1.*** *Locus of centroid of the triangle whose vertices are* $(a\cos t, a\sin t), (b\sin t, -b\cos t)$ *and* $(1, 0)$*, where* t *is a parameter, is*

(a) $(3x-1)^2 + (3y)^2 = a^2 - b^2$

(b) $(3x-1)^2 + (3y)^2 = a^2 + b^2$

(c) $(3x+1)^2 + (3y)^2 = a^2 + b^2$

(d) $(3x+1)^2 + 3y^2 = a^2 - b^2$

Sol. (b) Let $A \equiv (a\cos t, a\sin t)$, $B \equiv (b\sin t, -b\cos t)$ and $C \equiv (1, 0)$

$$\therefore \text{Centroid} \equiv \left(\frac{a\cos t + b\sin t + 1}{3}, \frac{a\sin t - b\cos t}{3}\right)$$

Let $$x = \frac{a\cos t + b\sin t + 1}{3} \Rightarrow 3x - 1 = a\cos t + b\sin t \quad \ldots(i)$$

and $$y = \frac{a\sin t - b\cos t}{3} \Rightarrow 3y = a\sin t - b\cos t \quad \ldots(ii)$$

On squaring and adding Eqs. (i) and (ii), we get

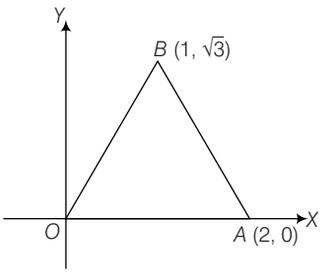
$$(3x-1)^2 + (3y)^2 = a^2 + b^2$$

which is locus of centroid.

● ***Ex. 2.*** *The incentre of triangle with vertices* $(1, \sqrt{3}), (0, 0)$ *and* $(2, 0)$ *is*

(a) $\left(1, \frac{\sqrt{3}}{2}\right)$ (b) $\left(\frac{2}{3}, \frac{1}{\sqrt{3}}\right)$ (c) $\left(\frac{2}{3}, \frac{\sqrt{3}}{2}\right)$ (d) $\left(1, \frac{1}{\sqrt{3}}\right)$

Sol. (d)



Let $O \equiv (0, 0)$, $A \equiv (2, 0)$

and $B \equiv (1, \sqrt{3})$

$\because$ $OA = 2 = OB = AB$

$\Rightarrow \Delta OAB$ is an equilateral.

$\therefore$ Incentre = Centroid

$$\equiv \left(\frac{0+2+1}{3}, \frac{0+0+\sqrt{3}}{3}\right)$$

$$\therefore \text{Incentre} \equiv \left(1, \frac{1}{\sqrt{3}}\right)$$

● ***Ex. 3.*** *Orthocentre of triangle with vertices* $(0, 0)$*, (3, 4) and (4, 0) is*

(a) $\left(3, \frac{5}{4}\right)$ (b) (3, 12)

(c) $\left(3, \frac{3}{4}\right)$ (d) (3, 9)

Sol. (c) Denote the points are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) from the matrix

$$P = \begin{bmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 0 \\ -1 & 4 \end{bmatrix}$$

$$\therefore \quad \lambda = \frac{\vec{\mathbf{R}}_1 \cdot \vec{\mathbf{R}}_2}{|P|}$$

$$= \frac{(-4)(-1) + 0}{-16} = -\frac{1}{4}$$

$\therefore$ Circumcentre of the triangle

$$\equiv \left(\frac{x_1 + x_2 + \lambda(y_1 - y_2)}{2}, \frac{y_1 + y_2 - \lambda(x_1 - x_2)}{2}\right)$$

$$\equiv \left(\frac{3 - 4\lambda}{2}, \frac{4 + 3\lambda}{2}\right) \equiv \left(2, \frac{13}{8}\right)$$

i.e. $$C \equiv \left(2, \frac{13}{8}\right)$$

and centroid $G \equiv \left(\frac{7}{3}, \frac{4}{3}\right)$

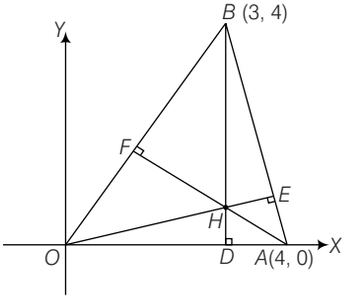
$\therefore$ Orthocentre

$$H \equiv \left(3 \times \frac{7}{3} - 2 \times 2, 3 \times \frac{4}{3} - 2 \times \frac{13}{8}\right)$$

$$\equiv \left(3, \frac{3}{4}\right)$$

Aliter : Let orthocentre

$$H \equiv (\alpha, \beta)$$



$\therefore$ Slope of $BH \times$ slope of $OA = -1$

$$\Rightarrow \quad \left(\frac{\beta - 4}{\alpha - 3}\right) \times \left(\frac{0}{4}\right) = -1$$

$\therefore \quad \alpha - 3 = 0$

$\Rightarrow \quad \alpha = 3$ ...(i)

and slope of $AH \times$ slope of $OB = -1$

$\Rightarrow \quad \left(\frac{\beta - 0}{\alpha - 4}\right) \times \left(\frac{4}{3}\right) = -1$

From Eq. (i), $\beta = \frac{3}{4}$

Hence, orthocentre is $\left(3, \frac{3}{4}\right)$.

● ***Ex. 4.*** *If x_1, x_2, x_3 as well as y_1, y_2, y_3 are in GP, with the same common ratio, then the points $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3)*

(a) lie on a straight line

(b) lie on an ellipse

(c) lie on a circle

(d) are vertices of a triangle

Sol. (a) Let common ratio of GP is r, then $x_2 = x_1 r$, $x_3 = x_1 r^2$, $y_2 = y_1 r$ and $y_3 = y_1 r^2$.

Let $A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$ and $C \equiv (x_3, y_3)$

$\therefore$ Area of $\Delta ABC = \frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$

$= \frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_1 r & y_1 r & 1 \\ x_1 r^2 & y_1 r^2 & 1 \end{vmatrix}\right|$

$= \frac{1}{2}|x_1 y_1|\left|\begin{vmatrix} 1 & 1 & 1 \\ r & r & 1 \\ r^2 & r^2 & 1 \end{vmatrix}\right| = 0$ ($\because C_1$ and C_2 are identical)

$\Rightarrow$ Points A, B, C are collinear.

● ***Ex. 5.*** *Let A be the image of $(2, -1)$ with respect to Y-axis. Without transforming the origin, coordinate axis are turned at an angle 45° in the clockwise direction. Then, the coordiates of A in the new system are*

(a) $\left(-\frac{1}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)$ (b) $\left(-\frac{3}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

(c) $\left(\frac{1}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$ (d) $\left(\frac{3}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

Sol. (a) Since, the image of (h, k) w.r.t. Y-axis is $(-h, k)$.

$\therefore$ Coordinate of A are $(-2, -1)$.

If (X, Y) are the coordinates of A w.r.t. the new coordinate axes obtained by turning the axes through an angle 45° in the clockwise direction, then

$$X = -2\cos(-45°) - \sin(-45°) = -\frac{2}{\sqrt{2}} + \frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}$$

and $$Y = 2\sin(-45°) - \cos(-45°) = -\frac{2}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{3}{\sqrt{2}}$$

$\therefore$ Required coordinates are $\left(-\frac{1}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)$.

JEE Type Solved Examples :
More than One Correct Option Type Questions

■ This section contains **3 multiple choice examples**. Each example has four choices (a), (b), (c) and (d). Out of which **MORE THAN ONE** may be correct.

● ***Ex. 6.*** *Let $S_1, S_2, \ldots$, be squares such that for each $n \geq 1$, the length of a side of S_n equals the length of a diagonal of S_{n+1}. If the length of a side S_1 is 10 cm, then for which of the following value of n is the area of S_n less than 1 sq cm?*

(a) 7 (b) 8 (c) 9 (d) 10

Sol. (b,c,d)

If a be the side of the square, then diagonal $d = a\sqrt{2}$ by hypothesis

$$a_n = \sqrt{2}\, a_{n+1}$$

$\Rightarrow \quad a_{n+1} = \frac{a_n}{\sqrt{2}} = \frac{a_{n-1}}{(\sqrt{2})^2} = \frac{a_{n-2}}{(\sqrt{2})^3} = \ldots = \frac{a_1}{(\sqrt{2})^n}$

$\Rightarrow \quad a_n = \frac{a_1}{(\sqrt{2})^{n-1}} = \frac{10}{(2)^{(n-1)/2}}$

$\because$ Area of $S_n < 1 \Rightarrow a_n^2 < 1$

$\Rightarrow \quad \frac{100}{2^{n-1}} < 1$

$\Rightarrow \quad 2^{n-1} > 100 > 2^6$

$\Rightarrow \quad n - 1 > 6$

$\Rightarrow \quad n > 7$

$\therefore \quad n = 8, 9, 10, \ldots$

● ***Ex 7.*** *If each of the vertices of a triangle has integral coordinates, then the triangles may be*

(a) right angled (b) equilateral

(c) isosceles (d) scalene

Sol. (a,c,d) Let $A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$ and $C \equiv (x_3, y_3)$ be the vertices of triangle ABC. Given $x_1, y_1, x_2, y_2, x_3, y_3$ be all integers.

Now, area of $\Delta ABC = \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$

$=$ Rational ...(i)

If ΔABC is equilateral then,

$$\text{Area of } \Delta ABC = \frac{\sqrt{3}}{4}(\text{side})^2$$

$$= \frac{\sqrt{3}}{4}\{(x_1 - x_2)^2 + (y_1 - y_2)^2\}$$

$$= \text{Irrational} \quad \ldots(\text{ii})$$

It is clear from Eqs. (i) and (ii), ΔABC can not be equilateral.

● ***Ex. 8.*** *ABC is an isosceles triangle. If the coordinates of the base are $B(1, 3)$ and $C(-2, 7)$. The coordinates of vertex A can be*

(a) (1, 6) (b) $\left(-\frac{1}{2}, 5\right)$ (c) $\left(\frac{5}{6}, 6\right)$ (d) $\left(-7, \frac{1}{8}\right)$

Sol. (b,c,d) Let vertex of the ΔABC be $A(x, y)$

$$\therefore \quad AB = AC$$

$$\Rightarrow \quad (AB)^2 = (AC)^2$$

$$\Rightarrow \quad (x-1)^2 + (y-3)^2 = (x+2)^2 + (y-7)^2$$

$$\Rightarrow \quad 6x - 8y + 43 = 0 \quad \ldots(\text{i})$$

Here, use observe that the coordinates $\left(-\frac{1}{2}, 5\right)$, $\left(\frac{5}{6}, 6\right)$ and $\left(-7, \frac{1}{8}\right)$ satisfy the Eq. (i).

JEE Type Solved Examples : Paragraph Based Questions

■ This section contains **one solved paragraph** based **3 multiple choice questions**. Each of these questions has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Paragraph

(Q. Nos. 9 to 11)

If $A\left(\alpha, \frac{1}{\alpha}\right)$, $B\left(\beta, \frac{1}{\beta}\right)$, $C\left(\gamma, \frac{1}{\gamma}\right)$ be the vertices of a ΔABC, where α, β are the roots of $x^2 - 6ax + 2 = 0$; β, γ are the roots of $x^2 - 6bx + 3 = 0$ and γ, α are the roots of $x^2 - 6cx + 6 = 0$; a, b, c being positive.

9. The value of $a + b + c$ is

(a) 1 (b) 2

(c) 3 (d) 5

10. The coordinates of centroid of ΔABC is

(a) $\left(1, \frac{11}{9}\right)$ (b) $\left(\frac{1}{3}, \frac{11}{18}\right)$

(c) $\left(2, \frac{11}{18}\right)$ (d) $\left(\frac{2}{3}, \frac{11}{19}\right)$

11. The coordinates of orthocentre of ΔABC is

(a) $\left(-\frac{1}{2}, -2\right)$ (b) $\left(-\frac{1}{3}, -3\right)$

(c) $\left(-\frac{1}{5}, -5\right)$ (d) $\left(-\frac{1}{6}, -6\right)$

Sol. $\because \alpha, \beta$ are the roots of $x^2 - 6ax + 2 = 0$

$$\therefore \quad \alpha + \beta = 6a \quad \ldots(\text{i})$$

$$\text{and} \quad \alpha\beta = 2 \quad \ldots(\text{ii})$$

Again, β, γ are the roots of $x^2 - 6bx + 3 = 0$

$$\therefore \quad \beta + \gamma = 6b \quad \ldots(\text{iii})$$

$$\text{and} \quad \beta\gamma = 3 \quad \ldots(\text{iv})$$

Again, γ, α are the roots of $x^2 - 6cx + 6 = 0$

$$\therefore \quad \gamma + \alpha = 6c \quad \ldots(\text{v})$$

$$\text{and} \quad \gamma\alpha = 6 \quad \ldots(\text{vi})$$

from Eqs. (ii), (iv) and (vi), we get

$$\alpha\beta \cdot \beta\gamma \cdot \gamma\alpha = 2 \cdot 3 \cdot 6$$

$$\Rightarrow \quad \alpha\beta\gamma = 6$$

$$\therefore \quad \alpha = 2, \beta = 1, \gamma = 3$$

9. (b) Adding Eqs. (i) , (iii) and (v), we get

$$2(\alpha + \beta + \gamma) = 6(a + b + c)$$

$$\text{or} \quad a + b + c = \frac{1}{3}(2 + 1 + 3) = 2$$

10. (c) Centroid of $\Delta ABC \equiv \left(\frac{\Sigma\alpha}{3}, \frac{\Sigma\alpha\beta}{3\alpha\beta\gamma}\right)$

$$\equiv \left(\frac{2+1+3}{3}, \frac{2\cdot 1 + 1\cdot 3 + 3\cdot 2}{3\cdot 2\cdot 1\cdot 3}\right)$$

$$\equiv \left(2, \frac{11}{18}\right)$$

11. (d) Orthocentre of $\Delta ABC \equiv \left(-\frac{1}{\alpha\beta\gamma}, -\alpha\beta\gamma\right)$

$$\equiv \left(-\frac{1}{6}, -6\right)$$

JEE Type Solved Examples :
Single Integer Answer Type Questions

- This section contains **one solved example**. The answer to this example is **a single digit integer** ranging from 0 to 9 (both inclusive).

● ***Ex.12.*** *If the points* $(-2, 0), \left(-1, \frac{1}{\sqrt{3}}\right)$ *and* $(\cos\theta, \sin\theta)$ *are collinear, then the number of values of* $\theta \in [0, 2\pi]$ *is*

Sol. (1) Since, the given points are collinear, then

$$\begin{vmatrix} -2 & 0 & 1 \\ -1 & \frac{1}{\sqrt{3}} & 1 \\ \cos\theta & \sin\theta & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad -2\left(\frac{1}{\sqrt{3}} - \sin\theta\right) - 0 + 1\left(-\sin\theta - \frac{\cos\theta}{\sqrt{3}}\right) = 0$$

$$\Rightarrow \quad \sqrt{3}\sin\theta - \cos\theta = 2$$

$$\Rightarrow \quad \frac{\sqrt{3}}{2}\sin\theta - \frac{1}{2}\cos\theta = 1$$

or $$\sin\left(\theta - \frac{\pi}{6}\right) = 1$$

or $$\theta - \frac{\pi}{6} = 2n\pi + \frac{\pi}{2}$$

for $n = 0$, $$\theta = \frac{\pi}{6} + \frac{\pi}{2} = \frac{2\pi}{3} \in [0, 2\pi]$$

Number of values θ is 1.

JEE Type Solved Examples :
Matching Type Questions

- This section contains **one solved example**. Which has four statements (A, B, C and D) given in **Column I** and four statements (p, q, r and s) in **Column II**. Any given statements in **Column I** can have correct matching with one or more statement(s) given in **Column II**.

● ***Ex. 13.*** *Match the following*

	Column I		Column II
A.	The points $(\lambda + 1, 1), (2\lambda + 1, 3)$ and $(2\lambda + 2, 2\lambda)$ are collinear then number of values of λ is	(p)	a prime number
B.	Area of ΔABC is 20 sq units, where A, B and C are (4, 6), (10, 14) and (x, y) respectively. AC is perpendicular to BC, then number of positions of C is	(q)	an odd number
C.	In a ΔABC coordinates of orthocentre, centroid and vertex A are respectively $(2, 2), (2, 1)$ and $(0, 2)$. The x-coordinate of vertex B is	(r)	a composite number
D.	A man starts from $P(-3, 4)$ and reaches the point $Q(0, 1)$ touching the X-axis at $R(\lambda, 0)$ such that $PR + RQ$ is minimum, then $10\lvert\lambda\rvert$ is	(s)	a perfect number

Sol. (A) $\rightarrow$ (p); (B) $\rightarrow$ (r); (C) $\rightarrow$ (p, q); (D) $\rightarrow$ (r, s)

(A) $\because$ Points are collinear

$$\therefore \quad \begin{vmatrix} \lambda + 1 & 1 & 1 \\ 2\lambda + 1 & 3 & 1 \\ 2\lambda + 2 & 2\lambda & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad (\lambda + 1)(3 - 2\lambda) - 1(2\lambda + 1 - 2\lambda - 2) + 1(4\lambda^2 + 2\lambda - 6\lambda - 6) = 0$$

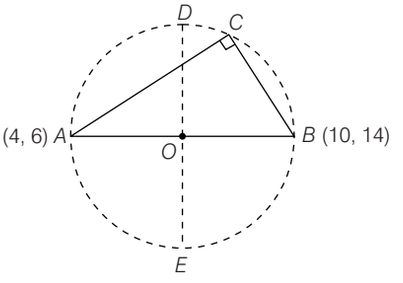
or $$2\lambda^2 - 3\lambda - 2 = 0$$

or $$(2\lambda + 1)(\lambda - 2) = 0$$

$$\therefore \quad \lambda = 2, -\frac{1}{2}$$

Number of values of λ is 2.

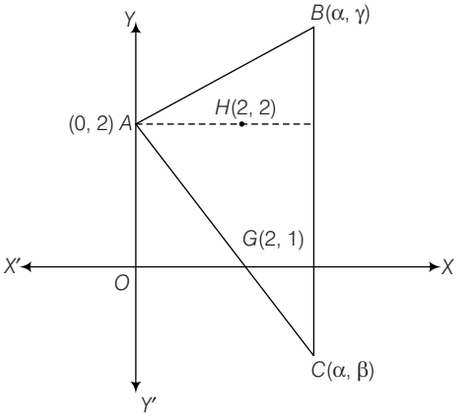
(B) $\because$ Area of triangles ABC is 20 sq units.



$\because$ C can not be at D and E

$\therefore$ Four positions are possible two above AB and two below AB.

(C)

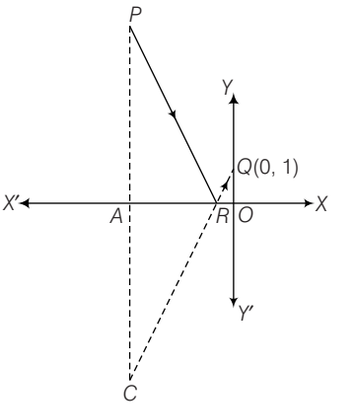


$\because \quad \frac{\alpha+\alpha+0}{3}=2$

$\therefore \quad \alpha=3$

$\Rightarrow$ x-coordinates of vertex $B=3$

(D) For $PR + RQ$ to be minimum, it should be the path of light



$\because \quad \angle PRA = \angle QRO$

From similar triangles PAR and QOR

$$\frac{AR}{RO}=\frac{PA}{QO} \quad \text{or} \quad \frac{\lambda+3}{0-\lambda}=\frac{4}{1}$$

or $\quad \lambda=-\frac{3}{5} \quad \therefore \quad 10|\lambda|=6.$

JEE Type Solved Examples :
Statement I and II Type Questions

■ **Directions** (Ex. Nos. 14 and 15) are Assertion-Reason type examples. Each of these examples contains two statements.

Statement I (Assertion) and **Statement II** (Reason)

Each of these examples also has four alternative choices, (a), (b), (c) and (d) only one out of which is the correct answer. You have to select the correct choice as given below :

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I

(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I

(c) Statement I is true, Statement II is false

(d) Statement I is false, Statement II is true

● ***Ex. 14.*** **Statement I** *The area of the triangle formed by the points $A(100,102)$, $B(102,105)$, $C(104,107)$ is same as the area formed by $A'(0,0)$, $B'(2,3)$, $C'(4,5)$.*

Statement II *The area of the triangle is constant with respect to translation :*

Sol. (a) Area of triangle is unaltered by shifting origin to any point. If origin is shifted to (100, 102), then A, B, C becomes $A'(0,0)$, $B'(2,3)$, $C'(4,5)$ respectively. Hence, both statements are true and Statement II is correct explanation for Statement I.

● ***Ex. 15.*** **Statement I** *If centroid and circumcentre of a triangle are known its orthocentre can be found*

Statement II *Centroid, orthocentre and circumcentre of a triangle are collinear.*

Sol. (b) $\because$ Centroid divides orthocentre and circumcentre in the ratio 2 : 1 (internally).

$\therefore$ We can find easily orthocentre.

$\Rightarrow$ Statement I is true, and centroid, orthocentre and circumcentre are collinear Statement II is true but Statement II is not correct explanation for Statement I.

Subjective Type Examples

■ In this section, there are **7 subjective solved examples**.

● ***Ex. 16.*** *The four points $A(\alpha,0), B(\beta,0), C(\gamma,0)$ and $D(\delta,0)$ are such that α,β are the roots of equation $ax^2+2hx+b=0$ and γ,δ are those of equation $a'x^2+2h'x+b'=0$. Show that the sum of the ratios in which C and D divide AB is zero, if $ab'+a'b=2hh'$.*

Sol. Since, α,β are the roots of $ax^2+2hx+b=0$

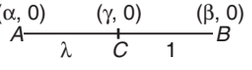
$$\therefore \quad \alpha+\beta=-\frac{2h}{a} \quad \text{and} \quad \alpha\beta=\frac{b}{a} \qquad \ldots(i)$$

and γ,δ are the roots of $a'x^2+2h'x+b'=0$

then, $$\gamma+\delta=-\frac{2h'}{a'} \quad \text{and} \quad \gamma\delta=\frac{b'}{a'} \qquad \ldots(ii)$$

Let C divides AB in the ratio $\lambda:1$

then

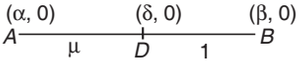


$$\gamma=\frac{\lambda\cdot\beta+1\cdot\alpha}{\lambda+1}$$

$$\therefore \quad \lambda=\frac{\gamma-\alpha}{\beta-\gamma}$$

and let D divides AB in the ratio $\mu:1$

then



$$\delta=\frac{\mu\cdot\beta+1\cdot\alpha}{\mu+1}$$

$$\therefore \quad \mu=\frac{\delta-\alpha}{\beta-\delta}$$

but given $\lambda+\mu=0$

$$\Rightarrow \quad \frac{\gamma-\alpha}{\beta-\gamma}+\frac{\delta-\alpha}{\beta-\delta}=0$$

$$\Rightarrow \quad (\alpha+\beta)(\gamma+\delta)-2\alpha\beta-2\gamma\delta=0$$

$$\Rightarrow \quad \left(-\frac{2h}{a}\right)\left(-\frac{2h'}{a'}\right)-\frac{2b}{a}-\frac{2b'}{a'}=0$$

[from Eqs. (i) and (ii)]

or $$ab'+a'b=2hh'$$

● ***Ex. 17.*** *If m_1 and m_2 are the roots of the equation $x^2+(\sqrt{3}+2)x+(\sqrt{3}-1)=0$ Show that the area of the triangle formed by the lines $y=m_1x, y=m_2x$ and $y=c$ is $\left(\frac{\sqrt{33}+\sqrt{11}}{4}\right)c^2$.*

Sol. Since, m_1 and m_2 are the roots of the equation

$$x^2+(\sqrt{3}+2)x+(\sqrt{3}-1)=0$$

then $$m_1+m_2=-(\sqrt{3}+2),\ m_1m_2=(\sqrt{3}-1)$$

$$\therefore \quad m_1-m_2=\sqrt{(m_1+m_2)^2-4m_1m_2}$$

$$=\sqrt{(3+4+4\sqrt{3}-4\sqrt{3}+4)}=\sqrt{11}$$

and coordinates of the vertices of the given triangle are $(0,0)$, $(c/m_1,c)$ and $(c/m_2,c)$.

Hence, the required area of triangle $=\frac{1}{2}\left|\frac{c}{m_1}\times c-\frac{c}{m_2}\times c\right|$

$$=\frac{1}{2}c^2\left|\left(\frac{1}{m_1}-\frac{1}{m_2}\right)\right|=\frac{1}{2}c^2\frac{|m_2-m_1|}{|m_1m_2|}$$

$$=\frac{1}{2}c^2\frac{\sqrt{11}}{(\sqrt{3}-1)}$$

$$=\frac{1}{2}c^2\cdot\frac{\sqrt{11}(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}$$

$$=\left(\frac{\sqrt{33}+\sqrt{11}}{4}\right)c^2$$

● ***Ex. 18.*** *If x coordinates of two points B and C are the roots of equation $x^2+4x+3=0$ and their y coordinates are the roots of equation $x^2-x-6=0$. If x coordinate of B is less than x coordinate of C and y coordinate of B is greater than the y coordinate of C and coordinates of a third point A be $(3,-5)$, find the length of the bisector of the interior angle at A.*

Sol. $\because \quad x^2+4x+3=0 \Rightarrow x=-1,-3$

and $\quad x^2-x-6=0 \Rightarrow x=-2,3$

Also, given that x and y coordinates of B are respectively less than and greater than the corresponding coordinates of C.

$$\therefore \quad B\equiv(-3,3) \quad \text{and} \quad C\equiv(-1,-2)$$

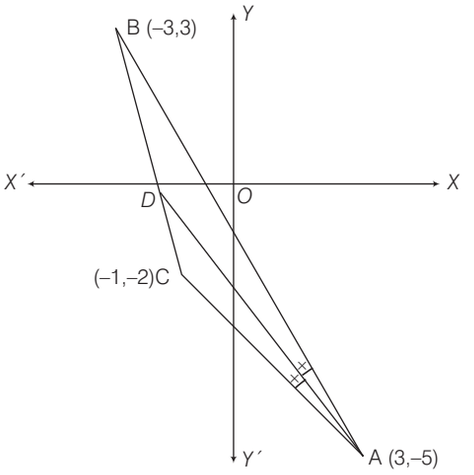
Now, $$AB=\sqrt{(3+3)^2+(-5-3)^2}=10$$

and $$AC=\sqrt{(3+1)^2+(-5+2)^2}=5$$

$$\therefore \quad \frac{AB}{AC}=\frac{2}{1}$$

Let AD be the bisector of $\angle BAC$, then

$$\frac{BD}{DC}=\frac{AB}{AC}=\frac{2}{1}$$



Thus, D divides BC internally in the ratio 2 : 1

$$\therefore \quad D \equiv \left(\frac{2(-1)+1(-3)}{2+1}, \frac{2(-2)+1(3)}{2+1} \right)$$

Thus, $$D \equiv \left(-\frac{5}{3}, -\frac{1}{3} \right)$$

Now, $$AD = \sqrt{\left(3+\frac{5}{3}\right)^2 + \left(-5+\frac{1}{3}\right)^2}$$

$$= \sqrt{\frac{196}{9}+\frac{196}{9}} = \frac{14\sqrt{2}}{3} \text{ units}$$

● ***Ex. 19.*** *The distance between two parallel lines is unity. A point P lies between the lines at a distance a from one of them. Find the length of a side of an equilateral triangle PQR, vertex Q of which lies on one of the parallel lines and vertex R lies on the other line.*

Sol. Let $PQ = QR = RP = r$

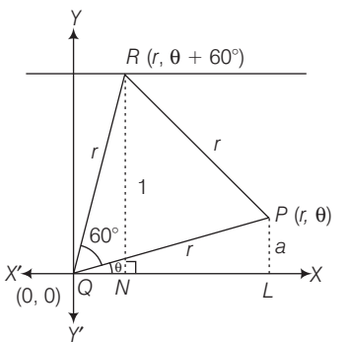
and $\angle PQL = \theta$

then, $\angle XQR = \theta + 60°$

Given, $PL = a$ and $RN = 1$ unit

In ΔPQL, $\sin\theta = \frac{PL}{QP} = \frac{a}{r}$

$$\therefore \quad a = r\sin\theta \qquad \ldots(i)$$



and in ΔQRN, $\sin(\theta+60°) = \frac{RN}{QR} = \frac{1}{r}$

$$\therefore \quad r\sin(\theta+60°) = 1$$

$$r\{\sin\theta\cos60° + \cos\theta\sin60°\} = 1$$

$$\Rightarrow \quad r\left\{\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta\right\} = 1$$

$$\Rightarrow \quad r\left\{\frac{1}{2}\times\frac{a}{r} + \frac{\sqrt{3}}{2}\times\sqrt{1-\frac{a^2}{r^2}}\right\} = 1 \qquad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad \frac{a}{2} + \frac{\sqrt{3}}{2}\sqrt{(r^2-a^2)} = 1$$

$$\Rightarrow \quad \frac{\sqrt{3}}{2}\sqrt{(r^2-a^2)} = 1 - \frac{a}{2}$$

or $$\frac{3}{4}(r^2-a^2) = 1 + \frac{a^2}{4} - a$$

$$\Rightarrow \quad 3r^2 - 3a^2 = 4 + a^2 - 4a$$

$$\Rightarrow \quad 3r^2 = 4a^2 - 4a + 4$$

$$= 4(a^2 - a + 1)$$

$$\therefore \quad r = \frac{2}{\sqrt{3}}\sqrt{(a^2-a+1)}$$

Hence, length of side of an equilateral triangle

$$= \frac{2}{\sqrt{3}}\sqrt{(a^2-a+1)} \text{ units.}$$

● ***Ex. 20.*** *In a ΔABC, $A \equiv (\alpha, \beta)$, $B \equiv (1, 2)$, $C \equiv (2, 3)$ and point A lies on the line $y = 2x + 3$ where $\alpha, \beta \in I$. If the area of ΔABC be such that $[\Delta] = 2$, where $[.]$ denotes the greatest integer function, find all possible coordinates of A.*

Sol. $\because (\alpha, \beta)$ lies on $y = 2x + 3$

then $\beta = 2\alpha + 3$

Thus, the coordinates of A are $(\alpha, 2\alpha+3)$ ...(i)

$$\Delta = \frac{1}{2}\left|\left\{\begin{vmatrix}\alpha & 2\alpha+3\\ 1 & 2\end{vmatrix} + \begin{vmatrix}1 & 2\\ 2 & 3\end{vmatrix} + \begin{vmatrix}2 & 3\\ \alpha & 2\alpha+3\end{vmatrix}\right\}\right|$$

$$= \frac{1}{2}|2\alpha - (2\alpha+3) + 3 - 4 + 4\alpha + 6 - 3\alpha|$$

$$= \frac{1}{2}|\alpha + 2|$$

but $[\Delta] = 2$

$$\Rightarrow \quad \left[\frac{1}{2}|\alpha+2|\right] = 2$$

$$\Rightarrow \quad 2 \le \frac{|\alpha+2|}{2} < 3$$

$$\Rightarrow \quad 4 \le |\alpha+2| < 6$$

$$\Rightarrow \quad 4 \le \alpha + 2 < 6$$

and $$-6 < \alpha + 2 \le -4$$

$$\Rightarrow \quad 2 \le \alpha < 4 \quad \text{and} \quad -8 < \alpha \le -6$$

$$\because \quad \alpha \in I$$

$$\therefore \quad \alpha = 2, 3, -7, -6$$

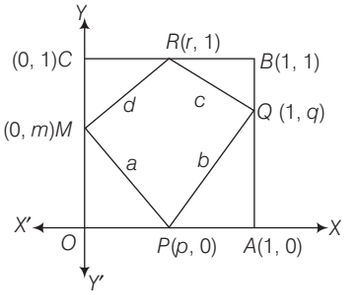
Hence, possible coordinates of A are $(2,7), (3,9), (-7,-11)$ and $(-6,-9)$.

● ***Ex. 21.*** *Let S be the square of unit area. Consider any quadrilateral which has one vertex on each side of S. If a,b,c and d denote the lengths of the sides of the quadrilateral, prove that* $2 \le a^2 + b^2 + c^2 + d^2 \le 4$.

Sol. Given S be the area of square with vertices
$O(0, 0), A(1, 0), B(1, 1), C(0, 1)$.

Let $PQRM$ be the quadrilateral with vertices
$P(p, 0), Q(1, q), R(r, 1)$ and $M(0, m)$

and sides $MP = a, PQ = b, QR = c, RM = d$



Then, $a^2 = p^2 + m^2$

$b^2 = (1-p)^2 + q^2$

$c^2 = (1-q)^2 + (1-r)^2$

$d^2 = r^2 + (1-m)^2$

$\therefore\ a^2 + b^2 + c^2 + d^2 = p^2 + (1-p)^2 + q^2 + (1-q)^2 + r^2 + (1-r)^2 + m^2 + (1-m)^2$

$= 2[p^2 + q^2 + r^2 + m^2 - p - q - r - m + 2]$

$= 2\left[\left(p - \frac{1}{2}\right)^2 + \left(q - \frac{1}{2}\right)^2 + \left(r - \frac{1}{2}\right)^2 + \left(m - \frac{1}{2}\right)^2 + 1\right] \ge 2$

$\Rightarrow\quad a^2 + b^2 + c^2 + d^2 \ge 2$ ...(i)

also, since $0 \le x \le 1$

$\therefore\quad x^2 \le 1$

$\therefore\quad a^2 \le 1, b^2 \le 1, c^2 \le 1, d^2 \le 1$

$\Rightarrow\quad a^2 + b^2 + c^2 + d^2 \le 4$ ...(ii)

From Eqs. (i) and (ii), we get

$2 \le a^2 + b^2 + c^2 + d^2 \le 4$

● ***Ex. 22.*** *The circumcentre of a triangle with vertices* $(a, a\tan\alpha), B(b, b\tan\beta)$ *and* $C(c, c\tan\gamma)$ *lies at the origin, where* $0 < \alpha, \beta, \gamma < \pi/2$ *and* $a + \beta + \gamma = \pi$. *Show that its orthocentre lies on the line*

$$4\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right)\cos\left(\frac{\gamma}{2}\right)x - 4\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right)y = y$$

Sol. Since, the circumcentre of the triangle is at the origin O, we have $OA = OB = OC = R$, where R is the circumradius of the circumcircle.

$\therefore\quad (OA)^2 = R^2 \Rightarrow a^2 + a^2\tan^2\alpha = R^2 \Rightarrow a = R\cos\alpha$

Therefore, the coordinates of A are $(R\cos\alpha, R\sin\alpha)$.

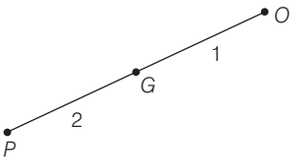
Similarly, the coordinates of B are $(R\cos\beta, R\sin\beta)$ and those of C are $(R\cos\gamma, R\sin\gamma)$.

Thus the coordinates of centroid G of ΔABC are

$$\left(\frac{R}{3}(\cos\alpha + \cos\beta + \cos\gamma), \frac{R}{3}(\sin\alpha + \sin\beta + \sin\gamma)\right)$$

Now, if $P(h, k)$ is the orthocentre of ΔABC, then

$$\frac{R}{3}(\cos\alpha + \cos\beta + \cos\gamma) = \frac{1\cdot h + 2\cdot 0}{1+2} \quad (\because PG : GO = 2:1)$$



$\Rightarrow\quad R(\cos\alpha + \cos\beta + \cos\gamma) = h$ ...(i)

and $\frac{R}{3}(\sin\alpha + \sin\beta + \sin\gamma) = \frac{1\cdot k + 2\cdot 0}{1+2}$

$\Rightarrow\quad R(\sin\alpha + \sin\beta + \sin\gamma) = k$ ...(ii)

Dividing (ii) by (i), then

$\Rightarrow\quad \frac{\sin\alpha + \sin\beta + \sin\gamma}{\cos\alpha + \cos\beta + \cos\gamma} = \frac{k}{h}$

$\Rightarrow\quad \frac{4\cos(\alpha/2)\cos(\beta/2)\cos(\gamma/2)}{1 + 4\sin(\alpha/2)\sin(\beta/2)\sin(\gamma/2)} = \frac{k}{h}$

(Because, $\alpha + \beta + y = \pi$ by identity)

Hence, the orthocentre $P(h, k)$ lies on the line

$$4\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right)\cos\left(\frac{\gamma}{2}\right)x - 4\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right)y = y$$

Coordinate System and Coordinates Exercise 1 : Single Option Correct Type Questions

■ This section contains **15 multiple choice questions**. Each question has four choices (a), (b), (c), (d) out of which **ONLY ONE** is correct.

1. Vertices of a variable triangle are $(3, 4)$, $(5\cos\theta, 5\sin\theta)$ and $(5\sin\theta, -5\cos\theta)$, where $\theta \in R$. Locus of its orthocentre is
(a) $x^2 + y^2 + 6x + 8y - 25 = 0$
(b) $x^2 + y^2 - 6x + 8y - 25 = 0$
(c) $x^2 + y^2 + 6x - 8y - 25 = 0$
(d) $x^2 + y^2 - 6x - 8y - 25 = 0$

2. If a rod AB of length 2 units slides on coordinate axes in the first quadrant. An equilateral triangle ABC is completed with C on the side away from O. Then, locus of C is
(a) $x^2 + y^2 - xy + 1 = 0$
(b) $x^2 + y^2 - xy\sqrt{3} + 1 = 0$
(c) $x^2 + y^2 + xy\sqrt{3} - 1 = 0$
(d) $x^2 + y^2 - xy\sqrt{3} - 1 = 0$

3. The sides of a triangle are $3x + 4y$, $4x + 3y$ and $5x + 5y$ units, where $x > 0$, $y > 0$. The triangle is
(a) right angled (b) acute angled
(c) obtuse angled (d) isosceles

4. Let P and Q be the points on the line joining $A(-2, 5)$ and $B(3, 1)$ such that $AP = PQ = QB$. Then, the mid-point of PQ is
(a) $\left(\frac{1}{2}, 3\right)$ (b) $\left(-\frac{1}{4}, 4\right)$
(c) $(2, 3)$ (d) $(-1, 4)$

5. A triangle ABC right angled at A has points A and B as $(2, 3)$ and $(0, -1)$ respectively. If $BC = 5$ units, then the point C is
(a) $(4, 2)$ (b) $(-4, 2)$
(c) $(-4, 4)$ (d) $(4, -4)$

6. The locus of a point P which divides the line joining $(1, 0)$ and $(2\cos\theta, 2\sin\theta)$ internally in the ratio $2:3$ for all θ is
(a) a straight line (b) a circle
(c) a pair of straight lines (d) a parabola

7. The points with the coordinates $(2a, 3a)$, $(3b, 2b)$ and (c, c) are collinear
(a) for no value of a, b, c (b) for all values of a, b, c
(c) if $a, \frac{c}{5}, b$ are in HP (d) if $a, \frac{2c}{5}, b$ are in HP

8. The vertices of a triangle are $(0, 3)$, $(-3, 0)$ and $(3, 0)$. The coordinates of its orthocentre are
(a) $(0, -2)$ (b) $(0, 2)$
(c) $(0, 3)$ (d) $(0, -3)$

9. ABC is an equilateral triangle such that the vertices B and C lie on two parallel lines at a distance 6. If A lies between the parallel lines at a distance 4 from one of them, then the length of a side of the equilateral triangle is
(a) 8 (b) $\sqrt{\frac{88}{3}}$
(c) $\frac{4\sqrt{7}}{\sqrt{3}}$ (d) None of these

10. A, B, C are respectively the points $(1, 2)$, $(4, 2)$, $(4, 5)$. If T_1, T_2 are the points of trisection of the line segment AC and S_1, S_2 are the points of trisection of the line segment BC, the area of the quadrilateral $T_1S_1S_2T_2$ is
(a) 1 (b) $\frac{3}{2}$ (c) 2 (d) $\frac{5}{2}$

11. (i) The points $(-1, 0)$, $(4, -2)$ and $(\cos 2\theta, \sin 2\theta)$ are collinear
(ii) The points $(-1, 0)$, $(4, -2)$ and $\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}, \frac{2\tan\theta}{1+\tan^2\theta}\right)$ are collinear
(a) both statements are equivalent
(b) statemetn (i) has more solution than statement (ii) for θ
(c) statement (ii) has more solution than statement (i) for θ
(d) None of the above

12. If $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ are the values of n for which $\sum_{r=0}^{n-1} x^{2r}$ is divisible by $\sum_{r=0}^{n-1} x^r$, then the triangle having vertices (α_1, β_1), (α_2, β_2) and (α_3, β_3) cannot be
(a) an isosceles triangle
(b) a right angled isosceles triangle
(c) a right angled triangle
(d) an equilateral triangle

13. A triangle ABC with vertices $A(-1, 0)$, $B\left(-2, \frac{3}{4}\right)$ and $C\left(-3, -\frac{7}{6}\right)$ has its orthocentre at H. Then, the orthocentre of triangle BCH will be
(a) $(-3, -2)$ (b) $(1, 3)$
(c) $(-1, 2)$ (d) None of these

14. If $\sum_{i=1}^{4}(x_i^2 + y_i^2) \le 2x_1x_3 + 2x_2x_4 + 2y_2y_3 + 2y_1y_4$, the points $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$ are
(a) the vertices of a rectangle
(b) collinear
(c) the vertices of a trapezium
(d) None of the above

15. Without change of axes the origin is shifted to (h, k), then from the equation $x^2 + y^2 - 4x + 6y - 7 = 0$, then term containing linear powers are missing, then point (h, k) is
(a) (3, 2) (b) (−3, 2)
(c) (2, −3) (d) (−2, −3)

Coordinate System and Coordinates Exercise 2 : More than One Correct Option Type Questions

- This section contains **7 multiple choice questions**. Each questions has four choices (a), (b), (c), (d) out of which **MORE THAN ONE** may be correct.

16. If $(-6, -4), (3, 5), (-2, 1)$ are the vertices of a parallelogram, then remaining vertex can be
(a) $(0, -1)$ (b) $(-1, 0)$
(c) $(-11, -8)$ (d) $(7, 10)$

17. If the point $P(x, y)$ be equidistant from the points $A(a + b, a - b)$ and $B(a - b, a + b)$ then
(a) $ax = by$
(b) $bx = ay$
(c) $x^2 - y^2 = 2(ax + by)$
(d) P can be (a, b)

18. If the coordinates of the vertices of a triangle are rational numbers, then which of the following points of the triangle will always have rational coordinates
(a) centroid (b) incentre
(c) circumcentre (d) orthocentre

19. The points $A(-4, -1)$, $B(-2, -4)$, $C(4, 0)$ and $D(2, 3)$ are the vertices of a
(a) parallelogram (b) rectangle
(c) rhombus (d) square

20. The medians AD and BE of the triangle with vertices $A(0, b)$, $B(0, 0)$ and $C(a, 0)$ are mutually perpendicular if
(a) $b = a\sqrt{2}$ (b) $a = b\sqrt{2}$
(c) $b = -a\sqrt{2}$ (d) $a = -b\sqrt{2}$

21. The points $A(x, y)$, $B(y, z)$ and $C(z, x)$ represents the vertices of a right angled triangle, if
(a) $x = y$ (b) $y = z$
(c) $z = x$ (d) $x = y = z$

22. Let the base of a triangle lie along the line $x = a$ and be of length $2a$. The area of this triangles is a^2, if the vertex lies on the line
(a) $x = -a$ (b) $x = 0$
(c) $x = \frac{a}{2}$ (d) $x = 2a$

Coordinate System and Coordinates Exercise 3 : Paragraph Based Questions

- This section contains **2 solved paragraphs** based upon each of the paragraph, **3 multiple choice questions** have to be answered. Each of these question has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Paragraph I

(Q. Nos. 23 to 25)

ABC is a triangle right angled at A, $AB = 2AC$. $A \equiv (1, 2)$, $B \equiv (-3, 1)$. ACD is an equilateral triangle. The vertices of two triangles are in anticlockwise sense. $BCEF$ is a square with vertices in clockwise sense.

23. If area of ΔACF is S, then the value of $8S$ is
(a) 42 (b) 51
(c) 62 (d) 102

24. Length of DE is
(a) $\sqrt{17}\sqrt{(4 - \sqrt{3})}$
(b) $\frac{\sqrt{17}}{2}\sqrt{(8 + \sqrt{3})}$
(c) $\frac{\sqrt{17}}{2}\sqrt{(4 + \sqrt{3})}$
(d) $\sqrt{15}\sqrt{(4 + \sqrt{3})}$

25. The y-coordinate of the centroid of the square $BCEF$ is
(a) $-\frac{1}{4}$ (b) $-\frac{3}{4}$
(c) $-\frac{5}{4}$ (d) $-\frac{7}{4}$

Paragraph II

(Q. Nos. 26 to 28)

Let $O(0,0)$, $A(2,0)$ *and* $B\left(1,\frac{1}{\sqrt{3}}\right)$ *be the vertices of a triangle.*

Let R be the region consisting of all those points P inside ΔOAB *satisfying.*

$d(P,OA) \le min\ \{d(P,OB), d(P,AB)\}$, *where, d denotes the distance from the point P to the corresponding line. Let M be the peak of region R.*

26. Length of perpendicular from M to OA is equal to

(a) $\sqrt{3}$ (b) $\frac{1}{\sqrt{3}}$ (c) 3 (d) $2-\sqrt{3}$

27. The perimeter of region R is equal to

(a) $4-\sqrt{3}$ (b) $4+\sqrt{3}$

(c) $4+3\sqrt{3}$ (d) $2+4\sqrt{(2-\sqrt{3})}$

28. The area of region R is equal to

(a) $2-\sqrt{3}$ (b) $2+\sqrt{3}$ (c) $2\sqrt{3}$ (d) $4+\sqrt{3}$

Coordinate System and Coordinates Exercise 4 : Single Integer Answer Type Questions

- This section contains **5 questions**. The answer to each example is **a single digit integer** ranging from 0 to 9 (both inclusive).

29. If the area of the triangle formed by the points $(2a, b), (a+b, 2b+a)$ and $(2b, 2a)$ be Δ_1 and the area of the triangle whose vertices are $(a+b, a-b), (3b-a, b+3a)$ and $(3a-b, 3b-a)$ be Δ_2, then the value of Δ_2/Δ_1 is

30. The diameter of the nine point circle of the triangle with vertices $(3, 4), (5\cos\theta, 5\sin\theta)$ and $(5\sin\theta, -5\cos\theta)$, where $\theta \in R$, is

31. The ends of the base of an isosceles triangle are $(2\sqrt{2}, 0)$ and $(0, \sqrt{2})$. One side is of length $2\sqrt{2}$. If Δ be the area of triangle, then the value of $[\Delta]$ is (where $[\cdot]$ denotes the greatest integer function)

32. If (x, y) is the incentre of the triangle formed by the points $(3, 4), (4, 3)$ and $(1, 2)$, then the value of x^2 is

33. Let P and Q be points on the line joining $A(-2, 5)$ and $B(3, 1)$ such that $AP = PQ = QB$. If mid-point of PQ is (a, b), then the value of $\frac{b}{a}$ is

Coordinate System and Coordinates Exercise 5 : Matching Type Questions

- The section contains **2 questions**. Each question has four statements (A, B, C and D) given in **Column I** and four statements (p, q, r and s) in **Column II**. Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II**.

34. Consider the triangle with vertices $A(0, 0)$, $B(5, 12)$ and $C(16, 12)$.

	Column I	Column II	
A.	If (λ, μ) are the coordinates of centroid of triangle ABC, then $(\lambda+\mu)$ is divisible by	(p)	3
B.	If (λ, μ) are the coordinates of circumcentre of triangle ABC, then 2λ is divisible by	(q)	5
C.	If (λ, μ) are the coordinates of incentre of triangle ABC, then μ is divisible by	(r)	7
D.	If (λ, μ) are the coordinates of excentre opposite to vertex B, then $\lambda+\mu$ is divisible by	(s)	9

35. The vertices of a triangle are $A(-10, 8)$, $B(14, 8)$ and $C(-10, 26)$. Let G, I, H, O be the centroid, incentre, orthocentre, circumcentre respectively of ΔABC.

	Column I	Column II	
A.	The inradius r is	(p)	a prime number
B.	The circumradius R is	(q)	an even number
C.	The area of ΔIGH is	(r)	a composite number
D.	The area of ΔOGI is	(s)	a perfect number

Coordinate System and Coordinates Exercise 6 : Statement I and II Type Questions

■ **Directions** (Q. Nos. 36 to 39) are Assertion-Reason type questions. Each of these questions contains two statements :

Statement I (Assertion) and

Statement II (Reason)

Each of these question also has four alternative choices (a), (b), (c) and (d), only one out of which is the correct answer.

You have to select the correct choice as given below

(a) Statement I is true, statement II is true; statement II is a correct explanation for statement I

(b) Statement I is true, statement II is true; statement II is not a correct explanation for statement I

(c) Statement I is true, statement II is false

(d) Statement I is false, statement II is true

36. The vertices of a triangle an $A(1, 2)$, $B(-1, 3)$ and $C(3, 4)$. Let D, E, F divide BC, CA, AB respectively in the same ratio.

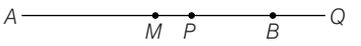
Statement I The centroid of triangle DEF is $(1, 3)$.

Statement II The triangle ABC and DEF have the same centroid.

37. **Statement I** Let the vertices of a ΔABC be $A(-5, -2)$, $B(7, 6)$ and $C(5, -4)$. Then, the coordinates of the circumcentre are $(1, 2)$.

Statement II In a right angled triangle, the mid point of the hypotenuse is the circumcentre of the triangle.

38. A line segment AB is divided internally and externally in the same ratio at P and Q respectively and M is the mid-point of AB.



Statement I MP, MB, MQ are in G.P

Statement II AP, AB and AQ are in H.P.

39. **Statement I** Transformation of the equation $x^2 - 3xy + 11x - 12y + 36 = 0$ to parallel axes through the point $(-4, 1)$ becomes $ax^2 + bxy + 1 = 0$, then $b^2 - a = \frac{1}{64}$.

Statement II If the axes turned through an angle θ, then the equation $f(x, y) = 0$ is transformed by replacing (x, y) by $((x\cos\theta - y\sin\theta), (x\sin\theta + y\cos\theta))$.

Coordinate System and Coordinates Exercise 7 : Subjective Type Questions

■ In this section, there are **7 subjective questions**.

40. If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertices of a ΔABC and (x, y) be a point on the internal bisector of angle A, then prove that

$$b\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + c\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

where, $AC = b$ and $AB = c$.

41. If a, b, c be the pth, qth and rth terms respectively of a HP, show that the points $(bc, p), (ca, q)$ and (ab, r) are collinear.

42. A line L intersects three sides BC, CA and AB of a triangle in P, Q, R respectively, show that

$$\frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AR}{RB} = -1$$

43. If the points, $\left(\frac{a^3}{a-1}, \frac{a^2-3}{a-1}\right), \left(\frac{b^3}{b-1}, \frac{b^2-3}{b-1}\right)$ and $\left(\frac{c^3}{c-1}\ \frac{c^2-3}{c-1}\right)$ are collinear for three distinct values a, b, c and $a \neq 1, b \neq 1$ and $c \neq 1$, then show that

$$abc - (bc + ca + ab) + 3(a + b + c) = 0$$

44. Show that the area of the triangle whose sides are $a_r x + b_r y + c_r = 0, r = 1, 2, 3$ is $\frac{\Delta^2}{2|C_1 C_2 C_3|}$, where C_1, C_2 and C_3 are the cofactors of c_1, c_2 and c_3 respectively in the determinant

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

45. If $A_1, A_2, A_3, ..., A_n$ are n points in a plane whose coordinates are $(x_1, y_1), (x_2, y_2), (x_3, y_3), ..., (x_n, y_n)$ respectively. A_1A_2 is bisected in the point G_1; G_1A_3 is divided at G_2 in the ratio $1:2$; G_2A_4 is divided at G_3 in the ratio $1:3$; G_3A_5 at G_4 in the $1:4$ and so on until all the points are exhausted. Show that the coordinates of the final point so obtained are

$$\frac{x_1 + x_2 + ... + x_n}{n} \quad \text{and} \quad \frac{y_1 + y_2 + ... + y_n}{n}$$

46. If by change of axes without change of origin, the expression $ax^2 + 2hxy + by^2$ becomes $a_1x_1^2 + 2h_1x_1y_1 + b_1y_1^2$, prove that

(i) $a + b = a_1 + b_1$

(ii) $ab - h^2 = a_1b_1 - h_1^2$

(iii) $(a - b)^2 + 4h^2 = (a_1 - b_1)^2 + 4h_1^2$

Coordinate System and Coordinates Exercise 8 : Questions Asked in Previous 13 Year's Exams

- This section contains questions asked in **IIT-JEE, AIEEE, JEE Main & JEE Advanced** from year **2005** to **2017**.

47. If a vertex of a triangle is (1, 1) and the mid-points of two side through this vertex are $(-1, 2)$ and $(3, 2)$, then the centroid of the triangle is **[AIEEE 2005, 3M]**

(a) $\left(\frac{1}{3}, \frac{7}{3}\right)$ (b) $\left(1, \frac{7}{3}\right)$

(c) $\left(-\frac{1}{3}, \frac{7}{3}\right)$ (d) $\left(-1, \frac{7}{3}\right)$

48. Let $O(0, 0)$, $P(3, 4)$, $Q(6, 0)$ be the vertices of the triangle OPQ. The point R inside the triangle OPQ is such that the triangles OPR, PQR, OQR are of equal area. The coordinates of R are **[IIT-JEE 2007, 3M]**

(a) $\left(\frac{4}{3}, 3\right)$ (b) $\left(3, \frac{2}{3}\right)$

(c) $\left(3, \frac{4}{3}\right)$ (d) $\left(\frac{4}{3}, \frac{2}{3}\right)$

49. Let $A(h, k)$, $B(1, 1)$ and $C(2, 1)$ be the vertices of a right angled triangle with AC as its hypotenuse. If the area of the triangle is 1, then the set of values which 'k' can take is given by **[AIEEE 2007, 3M]**

(a) $\{1, 3\}$ (b) $\{0, 2\}$

(c) $(-1, 3\}$ (d) $\{-3, -2\}$

50. Three distinct points A, B and C are given in the 2-dimensional coordinates plane such that the ratio of the distance of any one of them from the point $(1, 0)$ to the distance from the point $(-1, 0)$ is equal to $\frac{1}{3}$. Then, the circumcentre of the triangle ABC is at the point **[AIEEE 2009, 4M]**

(a) $\left(\frac{5}{4}, 0\right)$ (b) $\left(\frac{5}{2}, 0\right)$

(c) $\left(\frac{5}{3}, 0\right)$ (d) $(0, 0)$

51. The x-coordinate of the incentre of the triangle that has the coordinates of mid-points of its sides are (0, 1), (1, 1) and (1, 0) is **[JEE Main 2013, 4M]**

(a) $2 + \sqrt{2}$ (b) $2 - \sqrt{2}$

(c) $1 + \sqrt{2}$ (d) $1 - \sqrt{2}$

52. The number of points, having both coordinates are integers, that lie in the interior of the triangle with vertices (0, 0), (0, 41) and (41, 0) is **[JEE Main 2015, 4M]**

(a) 820 (b) 780

(c) 901 (d) 861

53. Let k be an integer such that the triangle with vertices $(k, -3k)$, $(5, k)$ and $(-k, 2)$ has area 28 sq units. Then, the orthocentre of this triangle is at the point **[JEE Main 2017, 4M]**

(a) $\left(2, \frac{1}{2}\right)$ (b) $\left(2, -\frac{1}{2}\right)$

(c) $\left(1, \frac{3}{4}\right)$ (d) $\left(1, -\frac{3}{4}\right)$

Answers

Exercise for Session 1

1. (d) 2. (b) 3. (b)
4. (b) 5. (b) 6. $(\pm 5, 0)$
7. $(0, \pm 4)$ 8. $(-6, 0)$
9. $\theta = \alpha$ 10. $x^2 + y^2 = 2ax$

Exercise for Session 2

1. (d) 2. (c) 3. (b) 4. (b) 5. (d)
6. (a) 7. (c) 8. (c) 9. $8a$ 13. $\left(\frac{3}{2}, \frac{5}{2}\right)$ and $\frac{5\sqrt{2}}{2}$
14. $\frac{63}{65}$ 15. (5, 1) and (−3, 3) 17. $a = b = 2 - \sqrt{3}$
18. $\left(\frac{1+\sqrt{3}}{2}, \frac{7-5\sqrt{3}}{2}\right)$ or $\left(\frac{1-\sqrt{3}}{2}, \frac{7+5\sqrt{3}}{2}\right)$

Exercise for Session 3

1. (a) 2. (b) 3. (a) 4. (c) 5. (a) 6. (c)
7. (a) 8. (c) 9. (d) 10. (b) 11. (p, q)
12. $\frac{333}{2}$ sq units 13. Centroid = incentre $\equiv (2, 3)$
18. $-\frac{3}{67}, \frac{45}{17}$ 19. 34 sq units

Exercise for Session 4

1. (b) 2. (d) 3. (c) 4. (b) 5. (c) 6. (b)
7. (c) 8. $y^2 - 8y - 4x + 20 = 0$
9. $3x^2 + 3y^2 - 10ax + 3a^2 = 0$ 10. $x + y = 4$
11. $9x^2 + 36y^2 = 4l^2$ 12. $x^2 - 9y^2 = 0$
13. $(a - b)^2 (x^2 + y^2) = a^2 b^4$ 14. $-3\frac{1}{3}$
16. $\tan^{-1}\left(\frac{B}{A}\right), -\frac{C}{\sqrt{(A^2 + B^2)}}$ 17. $xy = 0$

Chapter Exercises

1. (d) 2. (d) 3. (c) 4. (a) 5. (a) 6. (b)
7. (d) 8. (c) 9. (c) 10. (b) 11. (b) 12. (d)
13. (d) 14. (a) 15. (c) 16. (b,c,d) 17. (b,d) 18. (a,c,d)
19. (a,b) 20. (b,d) 21. (a,b,c) 22. (b,d) 23. (b) 24. (b)
25. (d) 26. (d) 27. (d) 28. (a) 29. (4) 30. (5)
31. (3) 32. (9) 33. (6)
34. $(A) \to (p,q)$; $(B) \to (p,r)$; $(C) \to (p,s)$; $(D) \to (p)$
35. $(A) \to (q,r,s)$; $(B) \to (r)$; $(C) \to (q,r,s)$; $(D) \to (p)$
36. (a) 37. (a) 38. (a) 39. (b) **47.** (b) 48. (c)
49. (c) 50. (a) 51. (b) 52. (b) 53. (a)

Solutions

1. Since, the distance of all points from $O(0, 0) = 5$

$\therefore$ $O(0, 0)$ be the circumcentre.

Centroid of triangle (G)

$$\equiv \left(\frac{3 + 5\cos\theta + 5\sin\theta)}{3}, \frac{4 + 5\sin\theta - 5\cos\theta}{3}\right)$$

Let $H(\alpha, \beta)$ be the orthocentre of the triangle, then

$$\alpha = 3 + 5\cos\theta + 5\sin\theta$$

$\Rightarrow$ $(\alpha - 3) = 5\cos\theta + 5\sin\theta$ ...(i)

and $\beta = 4 + 5\sin\theta - 5\cos\theta$

$\Rightarrow$ $(\beta - 4) = 5\sin\theta - 5\cos\theta$ ...(ii)

On squaring and adding Eqs. (i) and (ii), we get

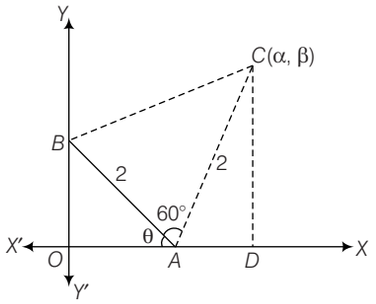
$$(\alpha - 3)^2 + (\beta - 4)^2 = 25 + 25$$

$\Rightarrow$ $\alpha^2 + \beta^2 - 6\alpha - 8\beta - 25 = 0$

Hence, the locus of $H(\alpha, \beta)$ is

$$x^2 + y^2 - 6x - 8y - 25 = 0$$

2.



Let $AB = AC = 2$

and $\angle BAO = \theta$

Let $C \equiv (\alpha, \beta)$

$\therefore$ $\alpha = 2\cos\theta + 2\cos(180° - (60° + \theta))$

$= 2\cos\theta - 2\cos(60° + \theta)$

$$= 2\cos\theta - 2\left\{\cos\theta \times \frac{1}{2} - \sin\theta \times \frac{\sqrt{3}}{2}\right\}$$

$= \cos\theta + \sqrt{3}\sin\theta$ ...(i)

and $\beta = 2\sin(180° - (60° + \theta))$

$= 2\sin(60° + \theta)$

$$= 2\left(\sin\theta \times \frac{1}{2} + \cos\theta \times \frac{\sqrt{3}}{2}\right)$$

$= \sin\theta + \sqrt{3}\cos\theta$ ...(ii)

Eliminating θ from Eqs. (i) and (ii), we get

$$\alpha^2 + \beta^2 - \alpha\beta\sqrt{3} - 1 = 0$$

Hence, the locus of $H(\alpha, \beta)$ is

$$x^2 + y^2 - xy\sqrt{3} - 1 = 0$$

3. Let $a = 3x + 4y$, $b = 4x + 3y$, $c = 5x + 5y$

$\therefore$ $$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$= \frac{(3x + 4y)^2 + (4x + 3y)^2 - (5x + 5y)^2}{2(3x + 4y)(4x + 3y)}$$

$$= -\frac{xy}{(3x + 4y)(4x + 3y)} < 0$$

$\Rightarrow$ $C > 90°$

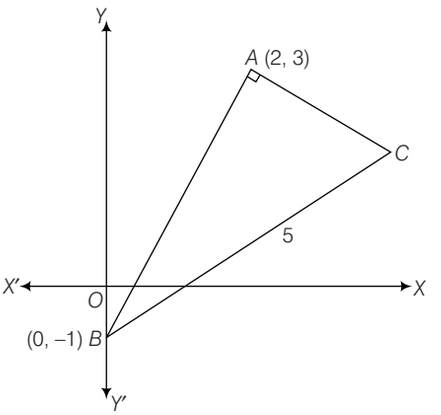
Hence, triangle is obtuse angled.

4.

$\because$ $AP = PQ = QB$

$\therefore$ mid-point of PQ = mid-point of $AB = \left(\frac{1}{2}, 3\right)$

5. $z_A = 2 + 3i$; $z_B = -i$ and $AB = \sqrt{(4 + 16)} = \sqrt{20} = 2\sqrt{5}$



$\therefore$ $(AC) = \sqrt{(BC)^2 - (AC)^2} = \sqrt{5}$

$\therefore$ $$\frac{z_C - z_A}{z_B - z_A} = \frac{AC}{AB}e^{i\pi/2} = \frac{i}{2}$$

$\Rightarrow$ $z_C - (2 + 3i) = \frac{i}{2}(-i - 2 - 3i)$

or $z_C = 2 + 3i - i + 2 = 4 + 2i$

$\therefore$ $C \equiv (4, 2)$

6. Let $P \equiv (h, k)$ and $A \equiv (1, 0)$, $B \equiv (2\cos\theta, 2\sin\theta)$

$\because$ $PA : PB = 2 : 3$

$\therefore$ $h = \frac{4\cos\theta + 3}{5} \Rightarrow 5h - 3 = 4\cos\theta$ ...(i)

and $k = \frac{4\sin\theta + 0}{5} \Rightarrow 5k = 4\sin\theta$ ...(ii)

OA squaring and adding Eqs. (i) and (ii), we get

$$(5h - 3)^2 + (5k)^2 = 16$$

$\therefore$ Locus of C is $(5x - 3)^2 + (5y)^2 = 16$

or $25(x^2 + y^2) - 30x - 7 = 0$

which is a circle.

7. $\frac{1}{2}\begin{vmatrix} 2a & 3a & 1 \\ 3b & 2b & 1 \\ c & c & 1 \end{vmatrix} = 0$

$\Rightarrow \quad 2a(2b-c) - 3a(3b-c) + 1(3bc - 2bc) = 0$

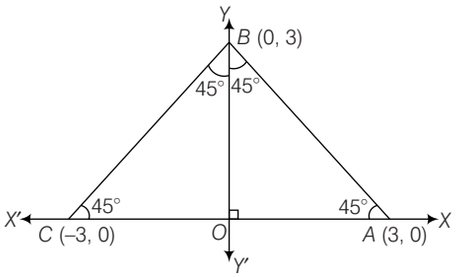
or $\quad -5ab + ac + bc = 0$

or $\quad c = \frac{5ab}{a+b}$

or $\quad \frac{2c}{5} = \frac{2ab}{a+b}$

Hence, $a, \frac{2c}{5}, b$ are in HP.

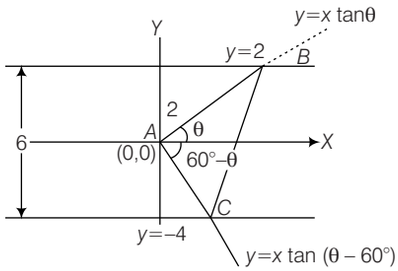
8.



It is clear from the figure orthocentre $\equiv (0, 3)$

9. Let, $A(0, 0)$

Coordinate of B are $(2\cot\theta, 2)$ and coordinate of C are $\{4\cot(60° - \theta), -4\}$



$\therefore \quad AB = AC$

$\Rightarrow \quad (AB)^2 = (AC)^2$

$4\cot^2\theta + 4 = 16\cot^2(60° - \theta) + 16$

$4\operatorname{cosec}^2\theta = 16\operatorname{cosec}^2(60° - \theta)$

$\Rightarrow \quad \sin(60° - \theta) = 2\sin\theta$

$\Rightarrow \quad \left(\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta\right) = 2\sin\theta$

$\therefore \quad \tan\theta = \frac{\sqrt{3}}{5}$

$\therefore$ The required length

$= AB = \sqrt{(4\cot^2\theta + 4)}$

$= 2\sqrt{\frac{28}{3}} = \frac{4\sqrt{7}}{\sqrt{3}}$

10.

$\because \quad A \equiv (1, 2), B \equiv (4, 2), C \equiv (4, 5) \qquad (\because AT_1 = T_1T_2 = T_2C)$

T_1 divides AC in the ratio 1 : 2 internally, then

$$T_1 \equiv \left(\frac{1\times 4 + 2\times 1}{1+2}, \frac{1\times 5 + 2\times 2}{1+2}\right)$$

or $\quad T_1 \equiv (2, 3)$

and T_2 is the mid-point of T_1 and C, then

$$T_2 \equiv \left(\frac{2+4}{2}, \frac{3+5}{2}\right)$$

or $\quad T_2 \equiv (3, 4)$

S_1 divides BC in the ratio 1 : 2 (internally), then

$(\because BS_1 = S_1S_2 = S_2C)$

$$S_1 \equiv \left(\frac{1\times 4 + 2\times 4}{1+2}, \frac{1\times 5 + 2\times 2}{1+2}\right)$$

or $\quad S_1 \equiv (4, 3)$

and S_2 is the mid-point of S_1 and C, then

$$S_2 \equiv \left(\frac{4+4}{2}, \frac{3+5}{2}\right)$$

or $\quad S_2 = (4, 4)$

Area of quadrilateral $T_1S_1S_2T_2$

$$= \frac{1}{2}\begin{vmatrix} 2 & 3 \\ 4 & 3 \\ 4 & 4 \\ 3 & 4 \\ 2 & 3 \end{vmatrix}$$

$= \frac{1}{2}\left|(6 + 16 + 16 + 9) - (12 + 12 + 12 + 8)\right|$

$= \frac{3}{2}$ sq units.

11. **Statement (i)** :

$\because$ Points $(-1, 0), (4, -2)$ and $(\cos2\theta, \sin2\theta)$ are collinear.

$\therefore \quad \begin{vmatrix} -1-\cos2\theta & 0-\sin2\theta \\ 4-\cos2\theta & -2-\sin2\theta \end{vmatrix} = 0$

$\Rightarrow \quad 2 + \sin2\theta + 2\cos2\theta + \sin2\theta\cos2\theta + 4\sin2\theta - \sin2\theta\cos2\theta = 0$

$\Rightarrow \quad 5\sin2\theta + 2(1 + \cos2\theta) = 0$

or $\quad 10\sin\theta\cos\theta + 4\cos^2\theta = 0$

or $\quad 2\cos\theta(5\sin\theta + 2\cos\theta) = 0$

or $\quad \cos\theta = 0$ and $\tan\theta = -\frac{2}{5}$

or $\quad \theta = m\pi + \frac{\pi}{2}$

and $\quad \theta = n\pi + \tan^{-1}\left(\frac{-2}{5}\right); m, n \in I$

Statement (ii) :

$\because$ Points $(-1, 0), (4, -2)$ and $\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}, \frac{2\tan\theta}{1+\tan^2\theta}\right)$ are collinear.

$$\therefore \quad \begin{vmatrix} -1-\frac{1-\tan^2\theta}{1+\tan^2\theta} & 0-\frac{2\tan\theta}{1+\tan^2\theta} \\ 4-\frac{1-\tan^2\theta}{1+\tan^2\theta} & -2-\frac{2\tan^2\theta}{1+\tan^2\theta} \end{vmatrix} = 0$$

$\Rightarrow \; 10\tan^3\theta + 4\tan^2\theta + 10\tan\theta + 4 = 0$

or $\quad (2\tan^2\theta + 2)(5\tan\theta + 2) = 0$

or $\quad \tan^2\theta \neq -1$

$\therefore \quad \tan\theta = -\frac{2}{5}$

$\Rightarrow \; \theta = p\pi + \tan^{-1}\left(\frac{-2}{5}\right), p \in I$

12. $\because \quad \frac{\sum_{r=0}^{n-1} x^{2r}}{\sum_{r=0}^{n-1} x^r} = \text{integer}$

$$\Rightarrow \quad \frac{\frac{1\cdot(1-x^{2n})}{(1-x^2)}}{\frac{1\cdot(1-x^n)}{(1-x)}} = \text{integer}$$

$$\Rightarrow \quad \frac{1+x^n}{1+x} = \text{integer}$$

n must be 1, 3, 5, 7, 9, 11, ...

$\therefore$ vertices are (1, 7), (3, 9), (5, 11)

Here, $(AB)^2 = 8, (BC)^2 = 8, (CA)^2 = 32$

$\therefore$ triangle cannot be an equilateral.

13. The orthocentre of ΔBCH is the vertex $A(-1, 0)$.

14. Given, $x_1^2 + y_1^2 + x_2^2 + y_2^2 + x_3^2 + y_3^2 + x_4^2 + y_4^2$

$\leq 2x_1x_3 + 2x_2x_4 + 2y_2y_3 + 2y_1y_4$

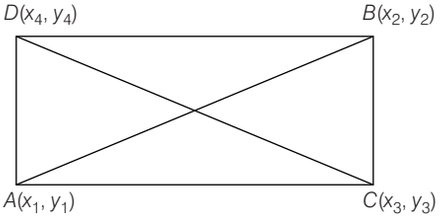
$\Rightarrow \; (x_1 - x_3)^2 + (x_2 - x_4)^2 + (y_2 - y_3)^2 + (y_1 - y_4)^2 \leq 0$

or $\quad (x_1 - x_3)^2 + (x_2 - x_4)^2 + (y_2 - y_3)^2 + (y_1 - y_4)^2 = 0$

or $\quad x_1 = x_3, x_2 = x_4, y_2 = y_3, y_1 = y_4$

or $\quad \frac{x_1 + x_2}{2} = \frac{x_3 + x_4}{2}$

and $\quad \frac{y_1 + y_2}{2} = \frac{y_3 + y_4}{2}$



Hence, AB and CD bisect each other

Therefore, $ACBD$ is a parallelogram. Also,

$(AB)^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 = (x_3 - x_4)^2 + (y_3 - y_4)^2 = (CD)^2$

Thus, $ACBD$ is a parallelogram and $AB = CD$.

Hence, it is a rectangle.

15. $(x + h)^2 + (y + k)^2 - 4(x + h) + 6(y + k) - 7 = 0$

$\Rightarrow \; x^2 + y^2 + 2x(h-2) + 2y(k+3) + (h^2 + k^2 - 4h + 6k - 7) = 0$

According to question

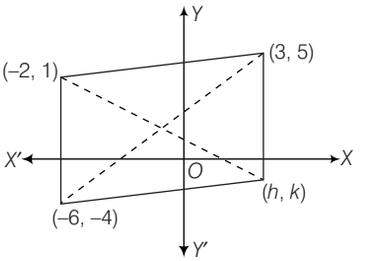
$$h - 2 = 0, k + 3 = 0$$

$\therefore \quad h = 2, k = -3$

hence, $\quad (h, k) \equiv (2, -3)$.

16. If the remaining vertex is (h, k), then

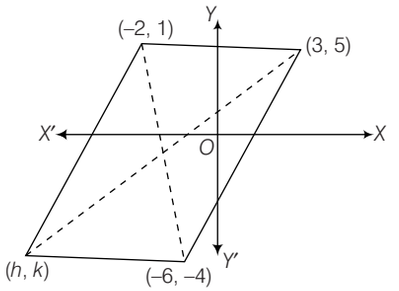
Case I



$h - 2 = 3 - 6, k + 1 = 5 - 4 \quad$ or $\quad h = -1, k = 0$

$\therefore$ Fourth vertex is $(-1, 0)$.

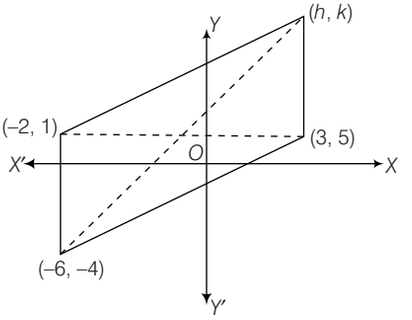
Case II



$h + 3 = -2 - 6, k + 5 = 1 - 4 \quad$ or $\quad h = -11, k = -8$

$\therefore$ Fourth vertex is $(-11, -8)$.

Case III



$$h - 6 = -2 + 3, k - 4 = 1 + 5$$

or $\quad h = 7, k = 10$

$\therefore$ Fourth vertex is (7, 10).

17. $(PA)^2 = (PB)^2$

$\Rightarrow \; (x - a - b)^2 + (y - a + b)^2 = (x - a + b)^2 + (y - a - b)^2$

$\Rightarrow \; (x - a + b)^2 - (x - a - b)^2 = (y - a + b)^2 - (y - a - b)^2$

$\Rightarrow \quad (2x - 2a)(2b) = (2y - 2a)(2b)$

or $\quad x = y$

and P is the mid-point of AB i.e., (a, a)

If P be (a, b), then $bx = ay \qquad (\because a = b)$

18. Let vertices of triangle ABC are

$$A \equiv (x_1, y_1),\ B \equiv (x_2, y_2) \text{ and } C \equiv (x_3, y_3)$$

where, $x_1, x_2, x_3, y_1, y_2, y_3 \in Q$

$\therefore$ $a = BC = \sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2}$

$=$ rational or irrational

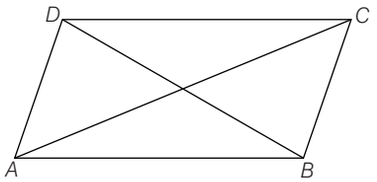
similarly, $b = CA,\ c = AB$

Then, incentre

$$\equiv \left(\frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + cy_3}{a + b + c} \right)$$

$\therefore$ Incentre has rational or irrational coordinates but centroid, circumcentre and orthocentre have always rational coordinates.

19. Since, $A \equiv (-4, -1),\ B \equiv (-2, -4),\ C \equiv (4, 0),\ D \equiv (2, 3)$



$\therefore$ $(AB)^2 = (-4 + 2)^2 + (-1 + 4)^2 = 13$

$(BC)^2 = (-2 - 4)^2 + (-4 - 0)^2 = 52$

$(CD)^2 = (4 - 2)^2 + (0 - 3)^2 = 13$

and $(DA)^2 = (2 + 4)^2 + (3 + 1)^2 = 52$

Clearly, $AB = CD$ and $BC = AD$

$\therefore$ $ABCD$ is a parallelogram.

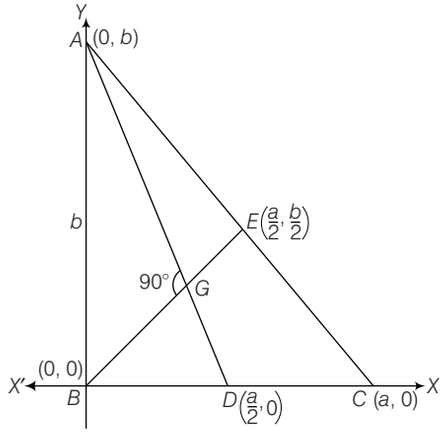
Again, $(AC)^2 = (-4 - 4)^2 + (-1 - 0)^2 = 65$

and $(BD)^2 = (-2 - 2)^2 + (-4 - 3)^2 = 65$

$\therefore$ $AC = BD$

Hence, $ABCD$ is a rectangle.

20.



$\because$ $BG = \frac{2}{3} BE = \frac{2}{3} \times \frac{\sqrt{(a^2 + b^2)}}{2}$

$\Rightarrow$ $(BG)^2 = \frac{1}{9}(a^2 + b^2)$ ...(i)

and $AG = \frac{2}{3} AD = \frac{2}{3} \times \sqrt{\left(\frac{a^2}{4} + b^2\right)}$

$\Rightarrow$ $(AG)^2 = \frac{1}{9}(a^2 + 4b^2)$ ...(ii)

Now, in right angled triangle ABG,

$(BA)^2 = (AG)^2 + (BG)^2$

$\Rightarrow$ $b^2 = \frac{1}{9}(a^2 + 4b^2) + \frac{1}{9}(a^2 + b^2)$ [from Eqs. (i) and (ii)]

$\Rightarrow$ $a^2 = 2b^2$

$\therefore$ $a = \pm b\sqrt{2}$

21. $\because$ $(AB)^2 = (x - y)^2 + (y - z)^2$

$(BC)^2 = (y - z)^2 + (z - x)^2$

and $(CA)^2 = (z - x)^2 + (x - y)^2$

Case I If $x = y$, then

$(AB)^2 + (CA)^2 = (BC)^2$

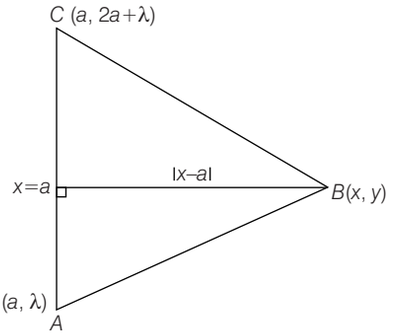
Case II If $y = z$, then

$(AB)^2 + (BC)^2 = (CA)^2$

Case III If $z = x$, then

$(BC)^2 + (CA)^2 = (AB)^2$

22.



Let $A \equiv (a, \lambda)$

then, $B \equiv (a, 2a + \lambda)$

and $C \equiv (x, y)$

Area of $\Delta ABC = a^2$ (given)

$\Rightarrow$ $\frac{1}{2} \times 2a \times |x - a| = a^2$

$\Rightarrow$ $|x - a| = a$

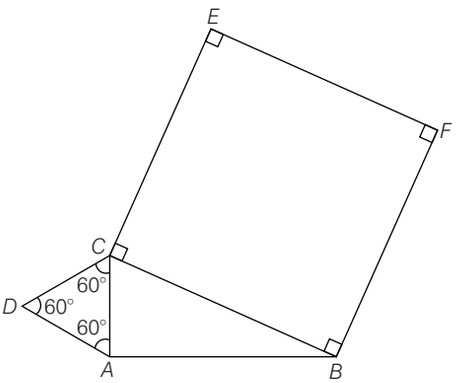
$\Rightarrow$ $x - a = \pm a$

$\Rightarrow$ $x = 0,\ x = 2a$

Sol. (Q. Nos. 23 to 25)

$$AB = \sqrt{(1 + 3)^2 + (2 - 1)^2} = \sqrt{17}$$

$$AC = \frac{1}{2} AB = \frac{1}{2}\sqrt{17}$$



For coordinates of C:

$$\frac{z_c - z_A}{z_B - z_A} = \frac{AC}{AB} e^{i\pi/2} = \frac{i}{2}$$

$$\Rightarrow\ z_c - (1 + 2i) = \frac{i}{2}(-3 + i - 1 - 2i)$$

$$= \frac{i}{2}(-4 - i) = -2i + \frac{1}{2}$$

or $$z_c = \frac{3}{2}$$

$$\therefore\quad C \equiv \left(\frac{3}{2}, 0\right)$$

For coordinates of D:

$$\frac{z_D - z_A}{z_C - z_A} = e^{i\pi/3}$$

$$\Rightarrow\ z_D - (1 + 2i) = \left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\left(\frac{1}{2} - 2i\right)$$

$$= \frac{1}{4} + \sqrt{3} - i + \frac{i\sqrt{3}}{4}$$

or $$z_D = \left(\frac{5}{4} + \sqrt{3}\right) + i\left(1 + \frac{\sqrt{3}}{4}\right)$$

$$\therefore\quad D \equiv \left(\frac{5}{4} + \sqrt{3}, 1 + \frac{\sqrt{3}}{4}\right)$$

For coordinates of F:

$$\frac{z_F - z_B}{z_C - z_B} = e^{-i\pi/2} = -i$$

$$\Rightarrow\ z_F - (-3 + i) = -i\left(\frac{9}{2} - i\right)$$

$$\Rightarrow\quad z_F = -4 - \frac{7}{2}i$$

$$\therefore\quad F \equiv \left(-4, -\frac{7}{2}\right), \text{ then}$$

$$E \equiv \left(\frac{1}{2}, -\frac{9}{2}\right)$$

23. $$S = \frac{1}{2}\left|\begin{vmatrix} 1 & 2 & 1 \\ \frac{3}{2} & 0 & 1 \\ -4 & -\frac{7}{2} & 1 \end{vmatrix}\right| = \frac{51}{8}$$

$$\therefore\quad 8S = 51$$

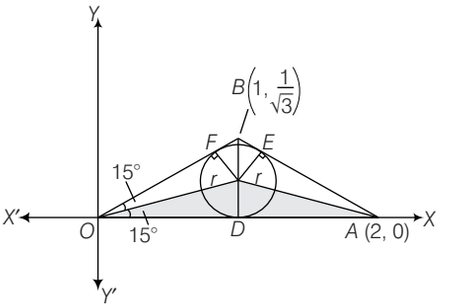
24. $$DE = \sqrt{\left(\frac{5}{4} + \sqrt{3} - \frac{1}{2}\right)^2 + \left(1 + \frac{\sqrt{3}}{4} + \frac{9}{2}\right)^2}$$

$$= \sqrt{\left(\frac{3}{4} + \sqrt{3}\right)^2 + \left(\frac{11}{2} + \frac{\sqrt{3}}{4}\right)^2} = \frac{\sqrt{17}}{2}\sqrt{(8 + \sqrt{3})}$$

25. y-coordinate of the centroid of the square

$$BCEF = \frac{0 - \frac{7}{2}}{2} = -\frac{7}{4}$$

■ ***Sol.*** (Q. Nos. 26 to 28)



Given, $O \equiv (0, 0)\ A \equiv (2, 0)$

and $$B \equiv \left(1, \frac{1}{\sqrt{3}}\right)$$

Let $P \equiv (x, y)$

If I be the incentre of ΔOAB.

If in radius = r, then

$ID = IE = IF = r$, *if* P at I,

then $d(P, OA) = d(P, OB) = d(P, AB) = r$,

But $d(P, OA) \le \min.\{d(P, OB), d(P, AB)\}$

which is possible only P lies in the ΔOIA

$\therefore$ M be the peak of region R.

$\therefore\quad M = I$

26. Length of perpendicular from M to $OA = ID = r = OD \tan 15°$

$$= 1 \times (2 - \sqrt{3})$$

$$= 2 - \sqrt{3}$$

27. Perimeter of region

$$R = IO + OA + AI$$

$$= r\,\text{cosec}\,15° + 2 + r\,\text{cosec}\,15°$$

$$= 2r\ \text{cosec}\,15° + 2$$

$$= 2(2 - \sqrt{3}) \cdot \frac{2\sqrt{2}}{(\sqrt{3} - 1)} + 2$$

$$= (\sqrt{3} - 1)^2 \cdot \frac{2\sqrt{2}}{(\sqrt{3} - 1)} + 2$$

$$= 2\sqrt{2}(\sqrt{3} - 1) + 2$$

$$\Rightarrow\quad = 2 + 4\sqrt{(2 - \sqrt{3})}$$

28. Area of region $R = \frac{1}{2} \times OA \times r$

$$= \frac{1}{2} \times 2 \times (2 - \sqrt{3}) = (2 - \sqrt{3})$$

29. We know that the area of the triangle formed by joining the mid-points of any triangle is one fourth of that triangle, then

$$\Delta_2 = 4\Delta_1$$

$$\therefore\quad \frac{\Delta_2}{\Delta_1} = 4$$

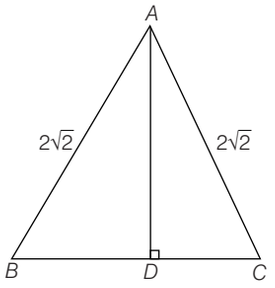
30. Since, the distance of all points from $O(0, 0) = 5$

$\therefore$Circumradius $(R) = 5$

Hence, diameter of nine point circle = $R = 5$

31. Let $B \equiv (2\sqrt{2}, 0), C \equiv (0, \sqrt{2})$

$\therefore \quad BC = \sqrt{(8+2)} = \sqrt{10}$



and $\quad AD = \sqrt{(AB)^2 - (BD)^2}$

$$= \sqrt{\left(8 - \frac{10}{4}\right)} = \sqrt{\frac{11}{2}}$$

$\therefore \quad \Delta = \frac{1}{2} \cdot BC \cdot AD$

$$= \frac{1}{2} \cdot \sqrt{10} \cdot \frac{\sqrt{11}}{\sqrt{2}} = \frac{\sqrt{55}}{2}$$

$\Rightarrow \quad [\Delta] = 3$

32. Let $A \equiv (3, 4), B \equiv (4, 3)$ and $C \equiv (1, 2)$

$\therefore \quad a = |BC| = \sqrt{(9+1)} = \sqrt{10}$

$b = |CA| = \sqrt{(4+4)} = 2\sqrt{2}$

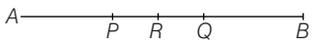
$c = |AB| = \sqrt{(1+1)} = \sqrt{2}$

$$\Rightarrow \quad x = \frac{\sqrt{10} \times 3 + 2\sqrt{2} \times 4 + \sqrt{2} \times 1}{\sqrt{10} + 2\sqrt{2} + \sqrt{2}}$$

$$= \frac{3(\sqrt{10} + 3\sqrt{2})}{(\sqrt{10} + 3\sqrt{2})} = 3$$

$\therefore \quad x^2 = 9$

33.



$\because \quad AP = PQ = QB$

Let R is the mid-point of PQ such that $PR = RQ$.

Now, $\quad AR = AP + PR = QB + RQ = RB$

$\Rightarrow \quad AR = RB$

$\therefore$ R bisects the line segment AB, then

$$R \equiv \left(\frac{-2+3}{2}, \frac{5+1}{2}\right)$$

$$\equiv \left(\frac{1}{2}, 3\right) = (a, b) \qquad \text{(given)}$$

$\therefore \quad a = \frac{1}{2}, \; b = 3 \Rightarrow \frac{b}{a} = 6$

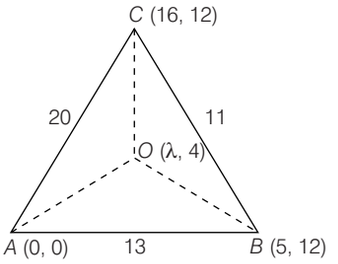
34. **(A)** Centroid of ΔABC is

$$\left(\frac{0+5+16}{3}, \frac{0+12+12}{3}\right) \equiv (7, 8)$$

Here, $\quad \lambda = 7, \mu = 8$

$\therefore \quad \lambda + \mu = 15$

(B)



$(OA)^2 = (OB)^2 = (OC)^2$

$\Rightarrow \quad \lambda^2 + \mu^2 = (\lambda - 5)^2 + (\mu - 12)^2 = (\lambda - 16)^2 + (\mu - 12)^2$

$\Rightarrow \quad (\lambda - 5)^2 + (\mu - 12)^2 = (\lambda - 16)^2 + (\mu - 12)^2$

or $\quad (\lambda - 5) = -(\lambda - 16)$

$\therefore \quad \lambda = \frac{21}{2} \Rightarrow 2\lambda = 21$

(C) Incentre of ΔABC is

$$\left(\frac{11 \times 0 + 13 \times 16 + 20 \times 5}{11 + 13 + 20}, \frac{11 \times 0 + 13 \times 12 + 20 \times 12}{11 + 13 + 20}\right)$$

$$\equiv (7, 9)$$

Here, $\quad \mu = 9$

(D) Excentre opposite to vertex B is

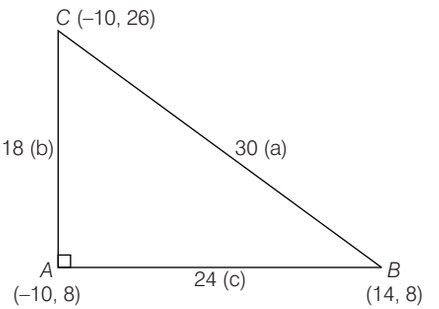
$$\left(\frac{11 \times 0 + 13 \times 16 - 20 \times 5}{11 + 13 - 20}, \frac{11 \times 0 + 13 \times 12 - 20 \times 12}{11 + 13 - 20}\right)$$

$$\equiv (27, -21)$$

Here, $\quad \lambda = 27, \mu = -21$

$\therefore \quad \lambda + \mu = 6$

35.



$$\Delta = \frac{1}{2}\left|\begin{vmatrix} -10 & 8 & 1 \\ 14 & 8 & 1 \\ -10 & 26 & 1 \end{vmatrix}\right| = 216$$

$$s = \frac{a+b+c}{2} = \frac{30+18+24}{2} = 36$$

$$G \equiv \left(\frac{-10+14-10}{3}, \frac{8+8+26}{3}\right)$$

$$\equiv (-2, 14)$$

$$I \equiv \left(\frac{30 \times -10 + 24 \times -10 + 18 \times 14}{30+24+18}, \frac{30 \times 8 + 24 \times 26 + 18 \times 8}{30+24+18}\right)$$

$$\equiv (-4, 14)$$

$H = A \equiv (-10, 8)$

$O =$ mid-point of $BC \equiv (2, 17)$

(A) $r = \dfrac{\Delta}{s} = \dfrac{216}{36} = 6$

(B) $R = \dfrac{BC}{2} = \dfrac{30}{2} = 15$

(C) Area of $\Delta IGH = \dfrac{1}{2}\left|\begin{vmatrix} -4 & 14 & 1 \\ -2 & 14 & 1 \\ -10 & 8 & 1 \end{vmatrix}\right| = 6$

(D) Area of $\Delta OGI = \dfrac{1}{2}\left|\begin{vmatrix} 2 & 17 & 1 \\ -2 & 14 & 1 \\ -4 & 14 & 1 \end{vmatrix}\right| = 3$

36. The centroid of triangle ABC is $G(1, 3)$. If D divided BC in the ratio $\lambda : 1$, then

$$D \equiv \left(\frac{3\lambda - 1}{\lambda + 1}, \frac{4\lambda + 3}{\lambda + 1}\right), E \equiv \left(\frac{\lambda + 3}{\lambda + 1}, \frac{3\lambda + 4}{\lambda + 1}\right)$$

and $$F \equiv \left(\frac{-\lambda + 1}{\lambda + 1}, \frac{3\lambda + 2}{\lambda + 1}\right)$$

$\therefore$ The centroid of triangle DEF is $(1, 3)$.

Both statements are true and statement II is correct explanation of statement I.

37. $\because$ $|AB| = \sqrt{(-5-7)^2 + (-2-6)^2} = 4\sqrt{13}$

$$|BC| = \sqrt{(7-5)^2 + (6+4)^2} = 2\sqrt{26}$$

and $$|CA| = \sqrt{(5+5)^2 + (-4+2)^2} = 2\sqrt{26}$$

$\because$ $(BC)^2 + (CA)^2 = (AB)^2$ and $|BC| = |CA|$

$\Rightarrow \Delta ABC$ is right angled isosceles triangle and right angle at C.

$\therefore$ Circumcentre is mid-point of A, $B \equiv (1, 2)$. Both statements are true and statement II is correct explanation of statement I.

38. Let $A \equiv (0, 0)$, $M \equiv (h, k)$, $B \equiv (2h, 2k)$ and let same ratio $\lambda : 1$

$$P \equiv \left(\frac{2h\lambda}{\lambda + 1}, \frac{2k\lambda}{\lambda + 1}\right), Q = \left(\frac{2h\lambda}{\lambda - 1}, \frac{2k\lambda}{\lambda - 1}\right)$$

$$AP = \frac{2\lambda}{(\lambda + 1)}\sqrt{(h^2 + k^2)},\ AB = 2\sqrt{(h^2 + k^2)},$$

$$AQ = \frac{2\lambda}{(\lambda - 1)}\sqrt{(h^2 + k^2)}$$

$\because$ $$\frac{AP - AB}{AB - AQ} = \frac{\lambda - 1}{\lambda + 1} = \frac{AP}{AQ}$$

$\therefore$ AP, AB and AQ are in HP.

$\Rightarrow$ $AM + MP$, $AM + MB$ and $AM + MQ$ are in HP

$\therefore$ MP, MB and MQ are in GP.

Both statements are true and statement II is correct explanation of statement I.

39. Replacing x by $(x - 4)$ and y by $y + 1$ is

$x^2 - 3xy + 11x - 12y + 36 = 0$, then

$(x-4)^2 - 3(x-4)(y+1) + 11(x-4) - 12(y+1) + 36 = 0$

$\Rightarrow$ $$x^2 - 3xy + 8 = 0$$

or $$\frac{x^2}{8} - \frac{3xy}{8} + 1 = 0$$

comparing with $ax^2 + bxy + 1 = 0$.

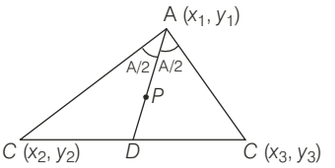
$\therefore$ $$a = \frac{1}{8}, b = -\frac{3}{8}$$

$\therefore$ $$b^2 - a = \frac{9}{64} - \frac{1}{8} = \frac{1}{64}$$

$\Rightarrow$ Statement I is true.

Hence, both statements are true and statement II is not correct explanation of statement I.

40. Let AD be the internal bisector of angle A.



Then, $$\frac{BD}{DC} = \frac{AB}{AC} = \frac{c}{b}$$

$\therefore$ $$D \equiv \left(\frac{cx_3 + bx_2}{c + b}, \frac{cy_3 + by_2}{c + b}\right)$$

Let $P \equiv (x, y)$. According to question, $P(x, y)$ lies on AD, therefore P, A, D are collinear

$\therefore$ $$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ \dfrac{cx_3 + bx_2}{c + b} & \dfrac{cy_3 + by_2}{c + b} & 1 \end{vmatrix} = 0$$

or $$\frac{1}{(c+b)}\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ cx_3 + bx_2 & cy_3 + by_2 & c + b \end{vmatrix} = 0$$

or $$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ cx_3 & cy_3 & c \end{vmatrix} + \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ bx_2 & by_2 & b \end{vmatrix} = 0$$

(using the addition property of determinants)

or $$c\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} + b\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Hence, $$b\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + c\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

41. Let first term and common difference of corresponding AP are A and D.

Since, pth term of HP $= a$

$\therefore$ pth term of AP $= \dfrac{1}{a}$

$\Rightarrow$ $$A + (p-1)D = \frac{1}{a} \quad \text{...(i)}$$

Similarly, $$A + (q-1)D = \frac{1}{b} \quad \text{...(ii)}$$

and $A + (r-1)\,D = \frac{1}{c}$ ...(iii)

Subtracting Eq. (ii) from Eq. (i),

$$(p-q)\,D = \frac{1}{a} - \frac{1}{b} \quad \text{...(iv)}$$

and subtracting Eq. (iii) from Eq. (ii),

$$(q-r)\,D = \frac{1}{b} - \frac{1}{c} \quad \text{...(v)}$$

Dividing Eq. (iv) by Eq. (v),

$$\frac{p-q}{q-r} = \frac{\frac{1}{a} - \frac{1}{b}}{\frac{1}{b} - \frac{1}{c}} = \frac{(b-a)}{ba} \times \frac{bc}{(c-b)}$$

$$\Rightarrow \quad \frac{p-q}{q-r} = \frac{bc-ac}{ac-ab} \quad \text{...(vi)}$$

Let $A\,(bc, p)$, $B\,(ca, q)$, $C\,(ab, r)$

$$\text{Slope of } AB = \frac{q-p}{ca-bc}$$

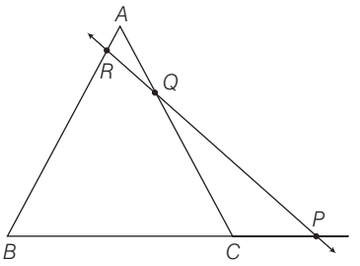
and $$\text{Slope of } BC = \frac{r-q}{ab-ca}$$

$$= \frac{q-p}{ca-bc} \quad \text{[from Eq. (vi)]}$$

Since, Slope of AB = slope of BC

Hence, A, B, C are collinear.

42. Suppose the equation of the line L is $ax + by + c = 0$. Let the coordinates of A, B, C be respectively $(x_1, y_1)\,; (x_2, y_2)\,; (x_3, y_3)$. Suppose P divides BC in the ratio $\lambda : 1$, then coordinates of P are $\left(\frac{\lambda x_3 + x_2}{\lambda + 1}, \frac{\lambda y_3 + y_2}{\lambda + 1}\right)$ since P lies on L.



$$\Rightarrow \quad a\left(\frac{\lambda x_3 + x_2}{\lambda + 1}\right) + b\left(\frac{\lambda y_3 + y_2}{\lambda + 1}\right) + c = 0$$

$$\Rightarrow \quad a(\lambda x_3 + x_2) + b(\lambda y_3 + y_2) + c(\lambda + 1) = 0$$

$$\Rightarrow \quad \lambda(ax_3 + by_3 + c) + (ax_2 + by_2 + c) = 0$$

$$\therefore \quad \frac{\lambda}{1} = \frac{BP}{CP} = -\frac{(ax_2 + by_2 + c)}{(ax_3 + by_3 + c)}$$

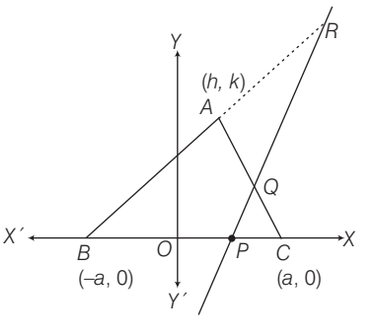
Similarly, $$\frac{CQ}{QA} = -\frac{(ax_3 + by_3 + c)}{(ax_1 + by_1 + c)}$$

and $$\frac{AR}{RB} = -\frac{(ax_1 + by_1 + c)}{(ax_2 + by_2 + c)}$$

Hence, $$\frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AR}{RB} = -1$$

Aliter :

Let middle point O of BC as the origin, BC as the X-axis and yy' as Y-axis.



Let $BC = 2a$

$\therefore \quad OB = OC = a$

Then, $B \equiv (-a, 0)$

and $C \equiv (a, 0)$

Let $A \equiv (h, k)$

Let P, Q, R divides BC, CA and AB in the ratio $\lambda : 1, \mu : 1$ and $\nu : 1$

i.e. $$\frac{BP}{PC} = \lambda, \frac{CQ}{QA} = \mu, \frac{AR}{RB} = \nu$$

Case I : Two sides divided internally and one side divided externally.

Hence, λ, μ are positive and ν is negative

Now, $$P \equiv \left(\frac{\lambda a - a}{\lambda + 1}, 0\right); Q \equiv \left(\frac{\mu h + a}{\mu + 1}, \frac{\mu k}{\mu + 1}\right)$$

and $$R \equiv \left(\frac{-a\nu + h}{\nu + 1}, \frac{k}{\nu + 1}\right)$$

Since, points P, Q, R are collinear.

$$\therefore \quad \frac{1}{2}\begin{vmatrix} \frac{\lambda a - a}{\lambda + 1} & 0 & 1 \\ \frac{\mu h + a}{\mu + 1} & \frac{\mu k}{\mu + 1} & 1 \\ \frac{-a\nu + h}{\nu + 1} & \frac{k}{\nu + 1} & 1 \end{vmatrix} = 0$$

or $$\begin{vmatrix} \lambda a - a & 0 & \lambda + 1 \\ \mu h + a & \mu k & \mu + 1 \\ -a\nu + h & k & \nu + 1 \end{vmatrix} = 0$$

or $$k\begin{vmatrix} \lambda a - a & 0 & \lambda + 1 \\ \mu h + a & \mu & \mu + 1 \\ -a\nu + h & 1 & \nu + 1 \end{vmatrix} = 0$$

Applying $R_2 \to R_2 - \mu R_3$, then

$$k\begin{vmatrix} \lambda a - a & 0 & \lambda + 1 \\ a + a\mu\nu & 0 & 1 - \mu\nu \\ -a\nu + h & 1 & \nu + 1 \end{vmatrix} = 0$$

or $$-k\begin{vmatrix} \lambda a - a & \lambda + 1 \\ a + a\mu\nu & 1 - \mu\nu \end{vmatrix} = 0$$

or $$-ka\begin{vmatrix}\lambda-1 & \lambda+1\\ 1+\mu\nu & 1-\mu\nu\end{vmatrix}=0$$

or $$-ka\{(\lambda-1)(1-\mu\nu)-(\lambda+1)(1+\mu\nu)\}=0$$

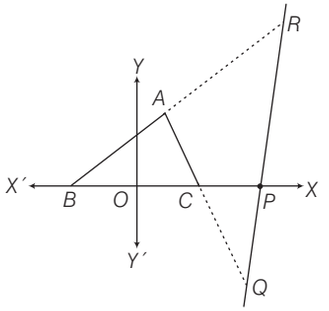
or $$-ka\{-2\lambda\mu\nu-2\}$$

$\Rightarrow$ $$2ka(\lambda\mu\nu+1)=0$$

or $$\lambda\mu\nu=-1 \qquad (\because\ ka\neq 0)$$

or $$\frac{BP}{PC}\cdot\frac{CQ}{QA}\cdot\frac{AR}{RB}=-1$$

Case II : All the three sides will be divided externally



Here, λ, μ and ν are negative.

Now, in case I formula for internal division will be used then we can show that

$$\lambda\mu\nu=-1$$

i.e. $$\frac{BP}{PC}\cdot\frac{CQ}{QA}\cdot\frac{AR}{RB}=-1$$

43. Since, given points are collinear, then

$$\frac{1}{2}\begin{vmatrix}\frac{a^3}{a-1} & \frac{a^2-3}{a-1} & 1\\ \frac{b^3}{b-1} & \frac{b^2-3}{b-1} & 1\\ \frac{c^3}{c-1} & \frac{c^2-3}{c-1} & 1\end{vmatrix}=0$$

$\Rightarrow$ $$\frac{1}{(a-1)(b-1)(c-1)}\begin{vmatrix}a^3 & a^2-3 & a-1\\ b^3 & b^2-3 & b-1\\ c^3 & c^2-3 & c-1\end{vmatrix}=0$$

or $$\begin{vmatrix}a^3 & a^2-3 & a-1\\ b^3 & b^2-3 & b-1\\ c^3 & c^2-3 & c-1\end{vmatrix}=0$$

$\Rightarrow$ $$\begin{vmatrix}a^3 & a^2 & a\\ b^3 & b^2 & b\\ c^3 & c^2 & c\end{vmatrix}+\begin{vmatrix}a^3 & a^2 & -1\\ b^3 & b^2 & -1\\ c^3 & c^2 & -1\end{vmatrix}+\begin{vmatrix}a^3 & -3 & a\\ b^3 & -3 & b\\ c^3 & -3 & c\end{vmatrix}+\begin{vmatrix}a^3 & -3 & -1\\ b^3 & -3 & -1\\ c^3 & -3 & -1\end{vmatrix}=0$$

or $$-abc\begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}+\begin{vmatrix}1 & a^2 & a^3\\ 1 & b^2 & b^3\\ 1 & c^2 & c^3\end{vmatrix}-3\begin{vmatrix}1 & a & a^3\\ 1 & b & b^3\\ 1 & c & c^3\end{vmatrix}+0=0$$

or $$-abc(a-b)(b-c)(c-a)+(ab+bc+ca)(a-b)(b-c)(c-a)-3(a+b+c)(a-b)(b-c)(c-a)=0$$

$\Rightarrow$ $$-(a-b)(b-c)(c-a)\{abc-(ab+bc+ca)+3(a+b+c)\}=0$$

$\therefore$ $$abc-(ab+bc+ca)+3(a+b+c)=0$$

$$\left.\begin{array}{l}\because\ a,b,c \text{ are distinct}\\ \therefore a\neq b,\ b\neq c,\ c\neq a\\ \therefore(a-b)(b-c)(c-a)\neq 0\end{array}\right\}$$

a, b, c are distinct.

Aliter : Suppose the given points lie on the line $lx+my+n=0$

then, $$l\left(\frac{t^3}{t-1}\right)+m\left(\frac{t^2-3}{t-1}\right)+n=0$$

where, $t=a, b, c$

$\Rightarrow$ $$lt^3+mt^2+nt-(3m+n)=0 \qquad ...(i)$$

i.e. a, b, c are the roots of Eq. (i)

then, $$a+b+c=-\frac{m}{l}$$

$$ab+bc+ca=\frac{n}{l}$$

$$abc=\frac{(3m+n)}{l}$$

Now, $$abc-(bc+ca+ab)+3(a+b+c)$$

$$=\frac{(3m+n)}{l}-\frac{n}{l}-\frac{3m}{l}=0$$

Hence, $$abc-(bc+ca+ab)+3(a+b+c)=0$$

44. $\because\ \Delta^c=\Delta^2$

where, Δ^c be the determinant of cofactors of Δ.

Let, $$\Delta^c=\begin{vmatrix}A_1 & B_1 & C_1\\ A_2 & B_2 & C_2\\ A_3 & B_3 & C_3\end{vmatrix}$$

then, $$A_1=b_2c_3-b_3c_2,\ B_1=a_3c_2-a_2c_3,\ C_1=a_2b_3-a_3b_2$$

$$A_2=b_3c_1-b_1c_3,\ B_2=c_3a_1-c_1a_3,\ C_2=a_3b_1-a_1b_3$$

$$A_3=b_1c_2-b_2c_1,\ B_3=a_2c_1-a_1c_2,\ C_3=a_1b_2-a_2b_1$$

Given lines are

$$a_1x+b_1y+c_1=0 \qquad ...(i)$$

$$a_2x+b_2y+c_2=0 \qquad ...(ii)$$

$$a_3x+b_3y+c_3=0 \qquad ...(iii)$$

Let P be the point of intersection of Eqs. (ii) and (iii), then

$$\frac{x}{b_2c_3-b_3c_2}=\frac{y}{c_2a_3-c_3a_2}=\frac{1}{a_2b_3-a_3b_2}$$

i.e. $$\frac{x}{A_1}=\frac{y}{B_1}=\frac{1}{C_1}$$

$\therefore$ $$P\equiv\left(\frac{A_1}{C_1},\frac{B_1}{C_1}\right)$$

Similarly, if Q and R be the points of intersections of Eqs. (iii) and (i) and Eqs. (i) and (ii) respectively, then

$$Q\equiv\left(\frac{A_2}{C_2},\frac{B_2}{C_2}\right),\ R\equiv\left(\frac{A_3}{C_3},\frac{B_3}{C_3}\right)$$

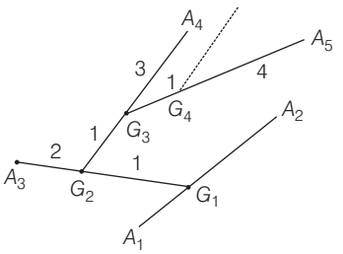
$$\therefore \text{ Area of } \Delta PQR = \frac{1}{2}\left|\begin{vmatrix} \frac{A_1}{C_1} & \frac{B_1}{C_1} & 1 \\ \frac{A_2}{C_2} & \frac{B_2}{C_2} & 1 \\ \frac{A_3}{C_3} & \frac{B_3}{C_3} & 1 \end{vmatrix}\right| = \frac{1}{2\,|\,C_1C_2C_3|}\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

$$= \frac{1}{2\,|\,C_1C_2C_3|}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2 = \frac{\Delta^2}{2\,|\,C_1C_2C_3|}$$

45. Since, G_1 is the middle point of A_1A_2 where

$$A_1 \equiv (x_1, y_1) \quad \text{and} \quad A_2 \equiv (x_2, y_2)$$

$$\therefore \quad G_1 \equiv \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$



Since, G_2 divided G_1A_3 internally in the ratio 1:2

$$\therefore \quad G_2 \equiv \left(\frac{2\cdot\left(\frac{x_1 + x_2}{2}\right) + 1.\,x_3}{2+1}, \frac{2\cdot\left(\frac{y_1 + y_2}{2}\right) + 1.y_3}{2+1}\right)$$

$$\text{i.e.} \quad G_2 \equiv \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

Again G_3 divides G_2A_4 internally in the ratio 1:3

$$\therefore \quad G_3 \equiv \left(\frac{3\cdot\left(\frac{x_1 + x_2 + x_3}{3}\right) + 1.x_4}{3+1}, \frac{3\cdot\left(\frac{y_1 + y_2 + y_3}{3}\right) + 1.y_4}{3+1}\right)$$

$$\text{i.e.} \quad G_3 \equiv \left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}\right)$$

Proceeding in this way, we can show that

$$G_{n-1} \equiv \left(\frac{x_1 + x_2 + \ldots + x_n}{n}, \frac{y_1 + y_2 + \ldots + y_n}{n}\right)$$

or we can say that the coordinates of the final point are

$$\frac{x_1 + x_2 + \ldots + x_n}{n} \text{ and } \frac{y_1 + y_2 + \ldots + y_n}{n}$$

46. Let the axes be rotated through an angle α, since (x, y) be the coordinates with respect to old axes and (x_1, y_1) be the coordinates with respect to new axes, then

$$x = x_1\cos\alpha - y_1\sin\alpha$$

and $$y = x_1\sin\alpha + y_1\cos\alpha$$

Now, the expression $ax^2 + 2hxy + by^2$ becomes

$$a(x_1\cos\alpha - y_1\sin\alpha)^2 + 2h(x_1\cos\alpha - y_1\sin\alpha)$$
$$(x_1\sin\alpha + y_1\cos\alpha) + b(x_1\sin\alpha + y_1\cos\alpha)^2$$

$$= x_1^2(a\cos^2\alpha + h\sin2\alpha + b\sin^2\alpha)$$
$$+ x_1y_1(2h\cos2\alpha - a\sin2\alpha + b\sin2\alpha)$$
$$+ y_1^2(a\sin^2\alpha - h\sin2\alpha + b\cos^2\alpha) \quad \ldots(i)$$

But, from question the expression $ax^2 + 2hxy + by^2$ transforms into

$$a_1x_1^2 + 2h_1x_1y_1 + b_1y_1^2 \quad \ldots(ii)$$

Therefore, the expressions (i) and (ii) are the same. Hence, equating the coefficients, we get

$$a_1 = a\cos^2\alpha + h\sin2\alpha + b\sin^2\alpha \quad \ldots(iii)$$

$$2h_1 = 2h\cos2\alpha - a\sin2\alpha + b\sin2\alpha \quad \ldots(iv)$$

$$b_1 = a\sin^2\alpha - h\sin2\alpha + b\cos^2\alpha \quad \ldots(v)$$

On adding Eqs. (iii) and (v), we get

$$a_1 + b_1 = a(\cos^2\alpha + \sin^2\alpha) + b(\sin^2\alpha + \cos^2\alpha)$$

or $$a_1 + b_1 = a + b \quad \ldots(vi)$$

Again, $$a_1 - b_1 = a(\cos^2\alpha - \sin^2\alpha) + 2h\sin2\alpha + b(\sin^2\alpha - \cos^2\alpha)$$

$$\Rightarrow \quad (a_1 - b_1) = (a - b)\cos2\alpha + 2h\sin2\alpha$$

$$(a_1 - b_1)^2 = (a - b)^2\cos^2 2\alpha + 4h^2\sin^2 2\alpha + 4h(a - b)\sin2\alpha\cos2\alpha \quad \ldots(vii)$$

Again, from Eq. (iv)

$$2h_1 = 2h\cos2\alpha - (a - b)\sin2\alpha$$

$$\therefore \quad 4h_1^2 = 4h^2\cos^2 2\alpha + (a - b)^2\sin^2 2\alpha - 4h(a - b)\sin2\alpha\cos2\alpha \quad \ldots(viii)$$

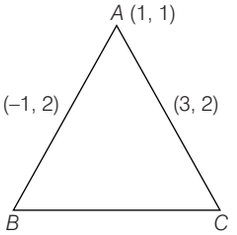
On adding Eqs. (vii) and (viii), we get

$$(a_1 - b_1)^2 + 4h_1^2 = (a - b)^2 + 4h^2$$

Again, $$4a_1b_1 - 4h_1^2 = (a_1 + b_1)^2 - (a_1 - b_1)^2 - 4h^2$$
$$= (a + b)^2 - (a - b)^2 - 4h^2 \quad [\text{from Eq. (vi)}]$$

Hence, $$a_1b_1 - h_1^2 = ab - h^2$$

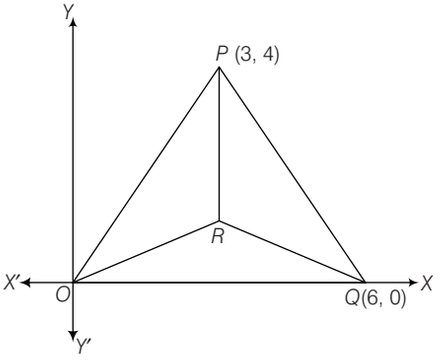
47. Vertex of triangle is (1, 1) and mid-point of sides through this vertex are (−1,2) and (3, 2)



$\Rightarrow$ Vertex B and C come out to be (−3, 3) and (5, 3)

$\therefore$ Centroid is $\frac{1-3+5}{3}, \frac{1+3+5}{3} \Rightarrow \left(1, \frac{7}{3}\right)$

48.

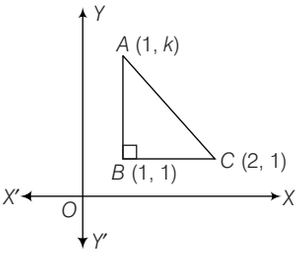


$\because$ Area of (ΔOPR) = Area of (ΔPQR) = Area of (ΔOQR)

$\therefore$ By geometry R should be the centroid of ΔOPQ

$$\therefore \quad R \equiv \left(\frac{0+3+6}{3}, \frac{0+4+0}{3}\right) \equiv \left(3, \frac{4}{3}\right)$$

49. Given, the vertices of a right angled triangle are $A(1, k)$, $B(1, 1)$ and $C(2, 1)$ and area of $\Delta ABC = 1$ sq unit



We know that, area of right angled triangle

$$= \frac{1}{2} \times BC \times AB = \frac{1}{2}(1)\left|(k-1)\right|$$

$$= \frac{1}{2}\left|k-1\right| \Rightarrow \pm(k-1) = 2$$

$$\Rightarrow \quad k = -1, 3$$

$$\therefore \quad K = \{-1, 3\}$$

50. Let $P \equiv (1, 0)$ and $Q \equiv (-1, 0)$

Given that, $\dfrac{AP}{AQ} = \dfrac{BP}{BQ} = \dfrac{CP}{CQ} = \dfrac{1}{3}$

$$\Rightarrow \quad 3AP = AQ \text{ or } 9(AP)^2 = (AQ)^2$$

Let $A \equiv (x, y)$, then

$$a((x-1)^2 + (y-0)^2) = (x+1)^2 + (y-0)^2$$

$$\Rightarrow \quad 8x^2 + 8y^2 - 20x + 8 = 0$$

$$\Rightarrow \quad x^2 + y^2 - \frac{5}{2}x + 1 = 0 \qquad \ldots(i)$$

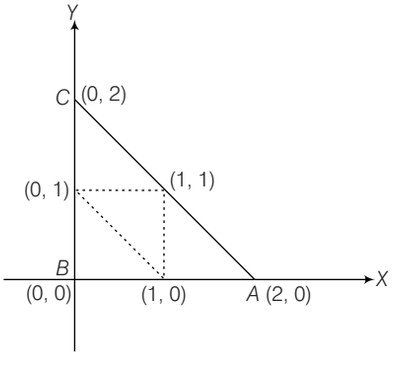
Circumcentre of ΔABC = Centre of circle Eq. (i)

$$= \left(\frac{5}{4}, 0\right)$$

51. From the figure, we have

$$a = 2, b = 2\sqrt{2}, c = 2$$

$$x_1 = 0, x_2 = 0, x_3 = 2 \quad (\because BC = a, CA = b, AB = c)$$



Now, x-coordinate of incentre is given as

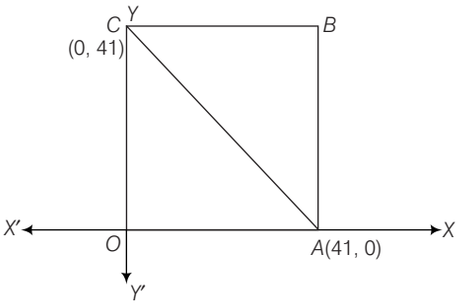
$$\frac{ax_1 + bx_2 + cx_3}{a+b+c}$$

$\Rightarrow$ x-coordinate of incentre

$$= \frac{2 \times 0 + 20\sqrt{2} \times 0 + 2 \times 2}{2 + 2 + 2\sqrt{2}}$$

$$= \frac{2}{2+\sqrt{2}} = 2 - \sqrt{2}$$

52.



Total number of integral points inside the square

$$OABC = 40 \times 40 = 1600.$$

Number of integral points on AC = Number of integral points on $OB = 40$ (namely (1, 1), (2, 2), (3, 3), ... , (40, 40)

Hence, number of integral points inside the

$$\Delta OAC = \frac{1600 - 40}{2} = 780$$

Aliter : $x > 0, y > 0$ $x + y < 41$ or $0 < x + y < 41$

$\therefore$ Number of integral points inside the $\Delta OAC = {}^{40}C_2 = 780$.

53. $\dfrac{1}{2}\left|\begin{vmatrix} k & -3k & 1 \\ 5 & k & 1 \\ -k & 2 & 1 \end{vmatrix}\right| = 28$

$$\Rightarrow \quad 5k^2 + 13k + 10 = \pm 56$$

$$\Rightarrow \quad 5k^2 + 13k - 46 = 0$$

or $\quad 5k^2 + 13k + 66 = 0$

$$\therefore \quad 5k^2 + 13k - 46 = 0$$

and $\quad 5k^2 + 13k + 66 \neq 0 \qquad (\because D < 0)$

$$\Rightarrow \quad k = 2, -\frac{23}{5}$$

$$\therefore \quad k = 2, k \neq -\frac{23}{5} \qquad (\because k \in I)$$

$\therefore$ Vertices an $A \equiv (2, -6)$, $B \equiv (5, 2)$, $C \equiv (-2, 2)$.

Denote the points are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) from the matrix $P = \begin{bmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{bmatrix} = \begin{bmatrix} 4 & -8 \\ 7 & 0 \end{bmatrix}$

$$\therefore \quad \lambda = \frac{\vec{\mathbf{R}}_1 \cdot \vec{\mathbf{R}}_2}{|P|} = \frac{28}{56} = \frac{1}{2}$$

$\therefore$ Circumcentre of triangle is $\left(\dfrac{7 + \frac{1}{2} \times -8}{2}, \dfrac{-4 - \frac{1}{2} \times -3}{2}\right)$

or $\left(\dfrac{3}{2}, -\dfrac{5}{4}\right)$ and centroid is $\left(\dfrac{5}{3}, -\dfrac{2}{3}\right)$

then, orthocentre

$$= \left(\frac{5}{3} \times 3 - 2 \times \frac{3}{2}, -\frac{2}{3} \times 3 + 2 \times \frac{5}{4}\right) \text{ or } \left(2, \frac{1}{2}\right),$$

CHAPTER

02

The Straight Lines

Learning Part

Session 1

- Definition
- Slope or Gradient of a Line
- Lines Parallel to Coordinate Axes
- Different Forms of the Equation of a Straight Line
- The Distance Form or Symmetric Form or Parametric Form of a Line
- Angle of Inclination of a Line
- Angle Between Two Lines
- Intercepts of a Line on Axes
- Reduction of General Equation to Standard Form

Session 2

- Position of Two Points Relative to a Given Line
- Equations of Lines Parallel and Perpendicular to a Given Line
- Distance Between Two Parallel Lines
- Position of a Point which Lies Inside a Triangle
- Distance of a Point From a Line
- Area of Parallelogram

Session 3

- Points of Intersection of Two Lines
- Family of Lines
- Concurrent Lines
- How to Find Circumcentre and Orthocentre by Slopes

Session 4

- Equations of Straight Lines Passing Through a Given Point and Making a Given Angle with a Given Line
- Equation of the Bisectors
- Equation of that Bisector of the Angle Between Two Lines Which Contains a Given Point
- A Line Equally Inclined with Two Lines
- Bisector of the Angle Containing The Origin
- How to Distinguish the Acute (Internal) and Obtuse (External) Angle Bisectors

Session 5

- The Foot of Perpendicular Drawn from the Point (x_1, y_1) to the Line $ax + by + c = 0$
- Image or Reflection of a Point (x_1, y_1) in Different Cases
- Image or Reflection of a Point (x_1, y_1) about a Line Mirror
- Use of Image or Reflection

Session 6

- Reflection of Light
- Condition of Collineirty If Three Given Points are in Cyclic Order
- Refraction of Light

Practice Part

- JEE Type Examples
- Chapter Exercises

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Session 1

Definition, Angle of Inclination of a Line, Slope or Gradient of a Line, Angle Between Two Lines, Lines Parallel to Coordinate Axes, Intercepts of a Line on Axes, Different Forms of the Equation of A Straight Line, Reduction of General Equation to Standard Form, The Distance Form or Symmetric Form or Parametric Form of a Line

Definition

A straight line defined as the curve which is such that the line segment joining any two points on it lies wholly on it.

Theorem : Show that the general equation of the first degree in x, y represents a straight line.

Proof : The general equation of the first degree is

$$ax + by + c = 0 \quad \text{...(i)}$$

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the coordinates of any two points on the curve given by Eq. (i), then

$$ax_1 + by_1 + c = 0 \quad \text{...(ii)}$$

$$ax_2 + by_2 + c = 0 \quad \text{...(iii)}$$

Multiplying Eq. (iii) by λ and adding to Eq. (ii), we have

$$a(x_1 + \lambda x_2) + b(y_1 + \lambda y_2) + c(1 + \lambda) = 0$$

or $$a\left(\frac{x_1 + \lambda x_2}{1 + \lambda}\right) + b\left(\frac{y_1 + \lambda y_2}{1 + \lambda}\right) + c = 0 \quad (\lambda \neq -1)$$

This relation shows that the point

$$\left(\frac{x_1 + \lambda x_2}{1 + \lambda}, \frac{y_1 + \lambda y_2}{1 + \lambda}\right) \text{lies on Eq. (i).}$$

But from previous chapter we know that this point divides the join of $P(x_1, y_1)$ and $Q(x_2, y_2)$ is the ratio $\lambda : 1$.

Since, λ can have any value, so each point on the line PQ lies on Eq. (i) i.e. the line wholly lies on Eq. (i). Hence, by the definition of the straight line as given above we conclude that Eq. (i) represents a straight line.

Hence, the general equation of first degree in x, y *viz* $ax + by + c = 0$ represents a straight line.

Remarks

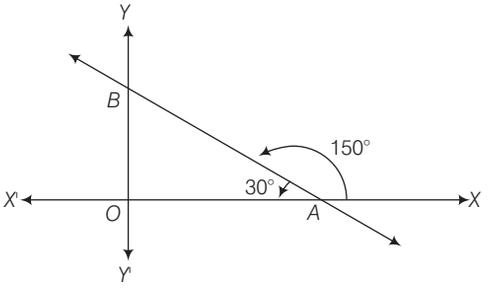
1. The number of **arbitrary constants** in the equation of a straight line is two (we observe three constants a, b and c in the equation $ax + by + c = 0$ of a straight line. The given equation of line can be rewritten as $\left(\frac{a}{c}\right)x + \left(\frac{b}{c}\right)y + 1 = 0$ or $px + qy + 1 = 0$ where $p = \frac{a}{c}$ and $q = \frac{b}{c}$.

   Thus, we have only two arbitrary constants p and q in the equation of a straight line.

   Hence, to completely determine the equation of a straight line, we require two conditions to determine the two unknowns in general.
2. A straight line is briefly written as a 'line.'
3. The equation of a straight line is the relation between x and y, which is satisfied by the coordinates of each and every point on the line.

Angle of Inclination of a Line

The angle of inclination of a line is the measure of the angle between the X-axis and the line measured in the anticlockwise direction.



Here, angle of inclination of line $AB = 150°$.

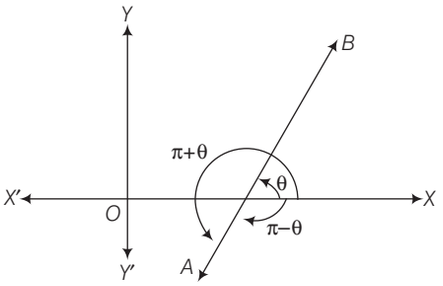
Remarks

1. When two lines are parallel, they have the same inclination.
2. The inclination of a line which is parallel to *X*-axis or coinciding with X-axis is 0°.
3. The angle of inclination of the line lies between 0° and 180° i.e. $0 < \theta \leq \pi$ and $\theta \neq \frac{\pi}{2}$.

Slope or Gradient of a Line

If inclination of a line is $(\theta \neq 90°)$, then $\tan\theta$ is called the slope or gradient of the line. It is usually denoted by *m*.

θ is positive or negative according as it is measured in anticlockwise or clockwise direction.



i.e. Slope of $AB = m$ of $AB = m(AB)$

$= \tan\theta$ or $\tan[-(\pi - \theta)]$

$= \tan(\pi + \theta)$

$=$ slope of $BA = m$ of BA

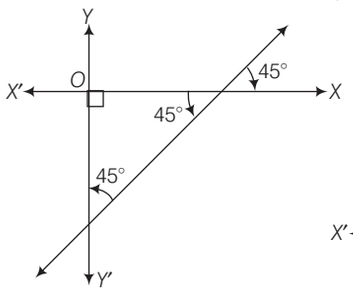
$= m(BA)$

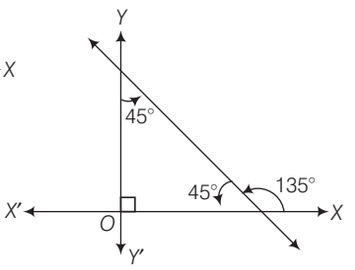
$\therefore \quad m(AB) = m(BA)$

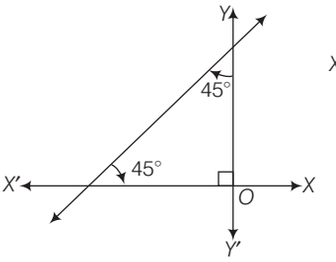
Hence, we do not take into consideration the direction of a line segment while talking of its slope.

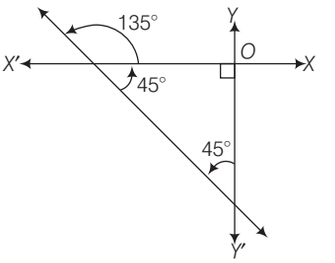
Remarks

1. Slope of a line is not the angle but is the tangent of the inclination of the line.
2. If a line is parallel to *X*-axis, then its slope = tan0° = 0.
3. Slope of a line parallel to *Y*-axis or perpendicular to *X*-axis is not defined. Whenever we say that the slope of a line is not defined.
4. If a line is equally inclined to the axes, then it will make an angle of 45° or 135° with the positive direction of *X*-axis. Slope in this case will be tan 45° or tan135°. i.e. ± 1





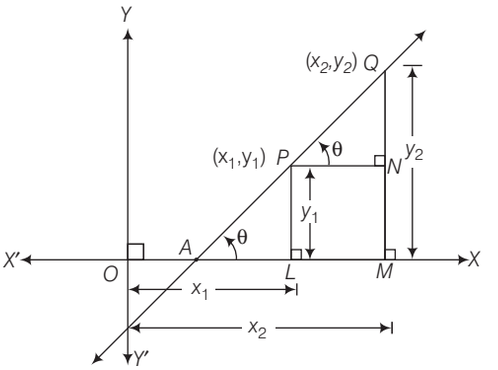




Theorem : If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points on a line *l*, then the slope *m* of the line *l* is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}, x_1 \neq x_2$$

If $x_1 = x_2$, then *m* is not defined. In that case the line is perpendicular to *X*-axis.



Proof : Given $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points on a line *l*, let line *l* makes an angle θ with positive direction of *X*-axis. Draw *PL*, *QM* perpendiculars on *X*-axis and

$PN \perp QM$

Then, $PN = LM = OM - OL = x_2 - x_1$

and $QN = QM - NM = QM - PL$

$= y_2 - y_1$

Also, $\angle QAM = \angle QPN = \theta$

Now, in ΔQPN

$$\tan\theta = \frac{QN}{PN} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Difference of ordinates}}{\text{Difference of abscissaes}}$$

or $$m = \frac{y_2 - y_1}{x_2 - x_1}$$

If $x_1 = x_2$, then $\tan\theta = \infty$ or $\theta = \frac{\pi}{2}$ i.e. *m* is not defined or the line is perpendicular to *X*-axis.

Remarks

1. When the two lines are parallel, then their slopes are equal i.e. $m_1 = m_2$.
2. If three points *A*, *B*, *C* are collinear, then slope of *AB* = slope of *BC* = slope of *AC*.

Example 1. Find the inclination of the line whose slope is $-\frac{1}{\sqrt{3}}$.

Sol. Let α be the inclination of a line then its slope $= \tan\alpha$

$$\therefore \quad \tan\alpha = -\frac{1}{\sqrt{3}} = -\tan 30^\circ$$

$$= \tan(180^\circ - 30^\circ) = \tan 150^\circ$$

$$\Rightarrow \quad \alpha = 150^\circ$$

Example 2. Find the slope of the line through the points $(4,-6),(-2,-5)$.

Sol. Slope of the line $m = \frac{-5-(-6)}{-2-(4)} = -\frac{1}{6}$

Example 3. Determine λ, so that 2 is the slope of the line through $(2,5)$ and $(\lambda,3)$.

Sol. Slope of the line joining $(2,5)$ and $(\lambda,3)$

$$= \frac{3-5}{\lambda-2} = \frac{-2}{\lambda-2} = 2 \qquad \text{(given)}$$

$$\Rightarrow \quad -2 = 2\lambda - 4$$

$$\Rightarrow \quad 2\lambda = 2$$

$$\therefore \quad \lambda = 1$$

Example 4. Show that the line joining the points $(2,-3)$ and $(-5,1)$ is parallel to the line joining $(7,-1)$ and $(0,3)$.

Sol. Slope of the line joining the points $(2,-3)$ and $(-5,1)$ is

$$m_1 = \frac{1-(-3)}{-5-2} = -\frac{4}{7}$$

and slope of the line joining the points $(7,-1)$ and $(0,3)$ is

$$m_2 = \frac{3-(-1)}{0-7} = \frac{-4}{7}$$

Here, $\quad m_1 = m_2$

Hence, lines are parallel.

Example 5. ***Find whether the points*** $(-a,-b),[-(s+1)a,-(s+1)b]$ ***and*** $[(t-1)a,(t-1)b]$ ***are collinear ?***

Sol. Let $A \equiv (-a,-b)$, $B \equiv [-(s+1)a,-(s+1)b]$ and $C \equiv ((t-1)a,(t-1)b)$

Then, slope of $AB = \frac{-(s+1)b+b}{-(s+1)a+a} = \frac{b}{a}$

and slope of $BC = \frac{(t-1)b+(s+1)b}{(t-1)a+(s+1)a} = \frac{b}{a}$

Hence, given points are collinear.

Example 6. For what value of k the points $(k,2-2k),(-k+1,2k)$ and $(-4-k,6-2k)$ are collinear ?

Sol. Let $A \equiv (k,2-2k)$, $B \equiv (-k+1,2k)$ and $C \equiv (-4-k,6-2k)$ are collinear, then

Slope of AB = Slope of AC

$$\Rightarrow \quad \frac{2k-(2-2k)}{-k+1-k} = \frac{6-2k-(2-2k)}{-4-k-k}$$

$$\Rightarrow \quad \frac{4k-2}{-2k+1} = \frac{4}{-4-2k} \qquad \left(k \neq \frac{1}{2} (\because \text{Denominator} \neq 0)\right)$$

$$\Rightarrow \quad (4k-2)(-4-2k) = 4(-2k+1)$$

$$\Rightarrow \quad (2k-1)(-2-k) - (-2k+1) = 0$$

$$\Rightarrow \quad (2k-1)(-2-k+1) = 0$$

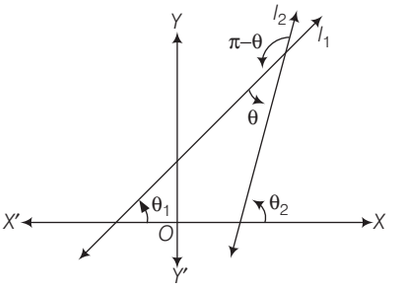
$$\therefore \quad k \neq \frac{1}{2}, \quad \therefore \quad k = -1$$

Angle Between Two Lines

Theorem : The acute angle θ between the lines having slopes m_1 and m_2 is given by

$$\theta = \tan^{-1}\left|\frac{m_1 - m_2}{1+m_1 m_2}\right|$$

Proof : Let l_1 and l_2 be two non-perpendicular lines, neither of which is parallel to the Y-axis.



Let m_1 and m_2 be the slopes of two given lines l_1 and l_2 respectively. Let θ_1 and θ_2 be the inclinations of these lines.

$$\therefore \quad m_1 = \tan\theta_1 \text{ and } m_2 = \tan\theta_2$$

Let θ and $\pi - \theta$ be the angles between the lines $\left(\theta \neq \frac{\pi}{2}\right)$.

Then, $\quad \theta_2 = \theta + \theta_1$ or $\theta = \theta_2 - \theta_1$

$$\Rightarrow \quad \tan\theta = \tan(\theta_2 - \theta_1)$$

$$\Rightarrow \quad \tan\theta = \left(\frac{\tan\theta_2 - \tan\theta_1}{1+\tan\theta_2\tan\theta_1}\right) = \left(\frac{m_2 - m_1}{1+m_1 m_2}\right) \quad \text{...(i)}$$

Also, $\quad \tan(\pi-\theta) = -\tan\theta = -\left(\frac{m_2 - m_1}{1+m_2 m_1}\right)$ ...(ii)

From Eqs. (i) and (ii) the angle between two lines of slopes m_1 and m_2 is given by

$$\tan\theta = \pm\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$$

$$\Rightarrow \qquad \theta = \tan^{-1}\left[\pm\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)\right]$$

Hence, the acute angle between the lines l_1 and l_2 is given by

$$\theta = \tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|.$$

***Corollary* 1 :** If two lines, whose slopes are m_1 and m_2 are parallel,

iff $\qquad \theta = 0^\circ \text{ (or } \pi) \Leftrightarrow \tan\theta = 0$

$$\Leftrightarrow \quad m_1 = m_2$$

Thus, when two lines are parallel, their slopes are equal.

***Corollary* 2 :** If two lines, whose slopes are m_1 and m_2 are perpendicular,

iff $\qquad \theta = \frac{\pi}{2}\left(\text{or} -\frac{\pi}{2}\right) \Leftrightarrow \cot\theta = 0$

$$\Leftrightarrow \quad m_1 m_2 = -1$$

Thus, when two lines are perpendicular, the product of their slopes is -1. The slope of each is the negative reciprocal of the slope of other i.e. if m is the slope of a line, then the slope of a line perpendicular to it is $-\frac{1}{m}$.

Example 7. Find the angle between the lines joining the points $(0, 0), (2, 3)$ and $(2, -2), (3, 5)$.

Sol. Let the given points be $A \equiv (0, 0)$, $B \equiv (2, 3)$, $C \equiv (2, -2)$ and $D \equiv (3, 5)$. Let m_1 and m_2 be the slopes of the lines AB and CD respectively.

$$\therefore \qquad m_1 = \frac{3-0}{2-0} = \frac{3}{2} \text{ and } m_2 = \frac{5-(-2)}{3-2} = 7$$

Let θ be the acute angle between the lines

$$\therefore \qquad \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{\frac{3}{2} - 7}{1 + \left(\frac{3}{2}\right)\cdot 7}\right|$$

$$= \left|\frac{-11}{23}\right| = \frac{11}{23}$$

$$\therefore \qquad \theta = \tan^{-1}\left(\frac{11}{23}\right)$$

Example 8. The angle between two lines is $\frac{\pi}{4}$ and the slope of one of them is $\frac{1}{2}$. Find the slope of the other line.

Sol. If θ be the acute angle between the lines with slopes m_1 and m_2, then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

Let $\qquad \theta = \frac{\pi}{4}$ and $m_1 = \frac{1}{2}$

then $\qquad \tan\frac{\pi}{4} = \left|\frac{\frac{1}{2} - m_2}{1 + \frac{1}{2}\cdot m_2}\right|$

$$\Rightarrow \qquad 1 = \left|\frac{1 - 2m_2}{2 + m_2}\right|$$

$$\Rightarrow \qquad \frac{1 - 2m_2}{2 + m_2} = \pm 1$$

Taking positive sign then,

$$1 - 2m_2 = 2 + m_2$$

$$\therefore \qquad m_2 = -\frac{1}{3}$$

and taking negative sign then,

$$1 - 2m_2 = -2 - m_2$$

$$\therefore \qquad m_2 = 3$$

Hence, the slope of the other line is either $-\frac{1}{3}$ or 3.

Example 9. Without using pythagoras theorem, show that the points $A(-1, 3), B(0, 5)$ and $C(3, 1)$ are the vertices of a right angled triangle.

Sol. In ΔABC, we have

$$\text{Slope of side } AB = \frac{5-3}{0-(-1)} = 2 = m_1 \qquad \text{(say)}$$

$$\text{Slope of side } BC = \frac{1-5}{3-0} = -\frac{4}{3} = m_2 \qquad \text{(say)}$$

and $\qquad \text{Slope of side } CA = \frac{3-1}{-1-3} = -\frac{1}{2} = m_3 \qquad \text{(say)}$

Clearly, $\qquad m_1 m_3 = 2 \times \left(-\frac{1}{2}\right) = -1$

$\therefore$ AB and CA are perpendicular to each other i.e.

$$\angle BAC = 90^\circ$$

Hence, the given points are the vertices of a right angled triangle.

Example 10. ***A line passes through the points*** $A(2,-3)$ ***and*** $B(6,3)$ ***. Find the slopes of the lines which are***

(i) parallel to AB (ii) perpendicular to AB

Sol. Let m be the slope of AB. Then $m = \dfrac{3-(-3)}{6-2} = \dfrac{6}{4} = \dfrac{3}{2}$

(i) Let m_1 be the slope of a line parallel to AB, then

$$m_1 = m = \frac{3}{2}$$

(ii) The slope of a line perpendicular to AB is

$$-\frac{1}{m} = -\frac{1}{\frac{3}{2}} = -\frac{2}{3}$$

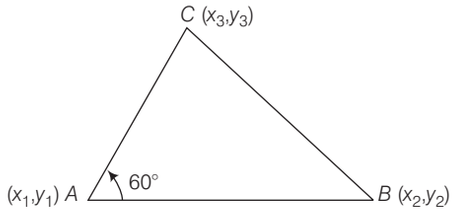
Example 11. Show that the triangle which has one of the angles as 60°, can not have all vertices with integral coordinates.

Sol. Let ABC be a triangle whose vertices are $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$. Assume $x_1, x_2, x_3, y_1, y_2, y_3$ all are integers.

Let $\angle BAC = 60^\circ$

$$\text{Slope of } AC = \frac{y_3 - y_1}{x_3 - x_1} = m_1 \quad \text{(say)}$$

and $$\text{Slope of } AB = \frac{y_2 - y_1}{x_2 - x_1} = m_2 \quad \text{(say)}$$



Here, m_1 and m_2 are rational numbers
($\because x_1, x_2, x_3, y_1, y_2, y_3$ are integers)

$$\therefore \quad \tan(\angle BAC) = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$= \text{Rational} \quad (\because m_1 \text{ and } m_2 \text{ are rational})$$

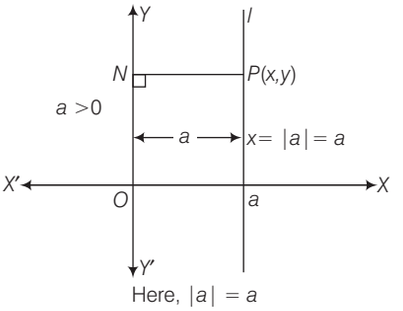
But $\tan(\angle BAC) = \tan 60^\circ = \sqrt{3} =$ Irrational

$\because$ Rational number $\neq$ Irrational number

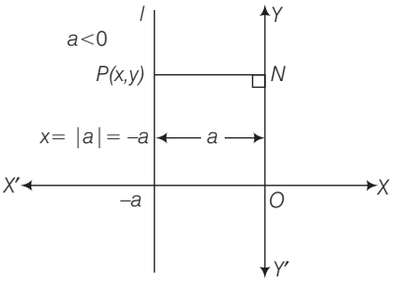
Which is contradiction so our assumption that the vertices are integers is wrong. Hence, the triangle having one angle of 60° can not have all vertices with integral coordinates.

Lines Parallel to Coordinate Axes

(i) **Equation of a line parallel to *Y*-axis :** Let l be a straight line parallel to Y-axis and at a distance a from it, a being the directed distance of the line from the Y-axis . Therefore, the line lies on the right of Y-axis if $a > 0$ and if $a < 0$, then the line would lies on the left of Y-axis.



Here, $|a| = a$



Here, $|a| = -a$

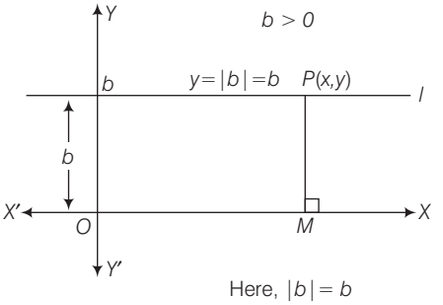
Let $P(x, y)$ be any point on the line l, then

$x = a$ is the required equation. (Here, $|a| = a$)

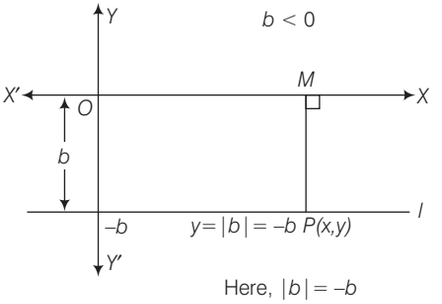
Remarks

1. In particular equation of Y-axis is $x = 0$ ($\because a = 0$)
2. A line is parallel to Y-axis, at a distance from it and is on the negative side of Y-axis, then its equation is $x = -a$.

(ii) **Equation of a line parallel to *X*-axis :** Let l be a straight line parallel to X-axis and at a distance b from it, b being the directed distance of the line from the X-axis. Therefore, the line lies above the X-axis, if $b > 0$ and if $b < 0$, then the line would lie below the X-axis.



Here, $|b| = b$



Here, $|b| = -b$

Let $P(x, y)$ be any point on the line l, then $y = b$ is the required equation (Here, $|b| = b$).

Remarks

1. In particular equation of X-axis is $y = 0$ $(\because b = 0)$
2. A line parallel to X-axis at a distance b from it and is on the negative side of X-axis, then its equation is $y = -b$.

Example 12. Find the equation of the straight line parallel to Y-axis and at a distance (i) 3 units to the right (ii) 2 units to the left.

Sol. (i) Equation of straight line parallel to Y-axis at a distance a units to the right is $x = a$.

$\therefore$ Required equation is $x = 3$

(ii) Equation of straight line parallel to Y-axis at a distance a units to the left is $x = -a$.

$\therefore$ Required equation is $x = -2$.

Example 13. ***Find the equation of the straight line parallel to X-axis and at a distance***

(i) 5 units above the X-axis

(ii) 9 *units below the X-axis.*

Sol. (i) Equation of a straight line parallel to X-axis at a distance b units above the X-axis is $y = b$.

$\therefore$ Required equation is $y = 5$

(ii) Equation of a straight line parallel to X-axis at a distance b units below the X-axis is $y = -b$.

$\therefore$ Required equation is $y = -9$

Example 14. Find the equation of the straight line which passes through the point $(2, -3)$ and is

(i) parallel to the X-axis

(ii) perpendicular to the X-axis

Sol. (i) Let equation of any line parallel to X-axis is

$$y = b \qquad \text{....(i)}$$

Since, it passes through the point $(2, -3)$.

Putting $y = -3$ in Eq. (i), then

$$b = -3$$

Hence, required equation of the line is $y = -3$.

Eq. (i) Let equation of any line perpendicular to X-axis = Equation of any line parallel to Y-axis is

$$x = a \qquad \text{...(ii)}$$

Since, it passes through the point $(2, -3)$ putting $x = 2$ in Eq. (ii)

Then, $2 = a \Rightarrow a = 2$

Hence, required equation of the line $x = 2$.

Example 15. Find the equation of a line which is equidistant from the lines $x = -\frac{7}{2}$ and $x = \frac{15}{2}$.

Sol. Since, the given (both) lines are parallel to Y-axis and the required line is equidistant from these lines, so it is also parallel to Y-axis. Let equation of any line parallel to Y-axis is

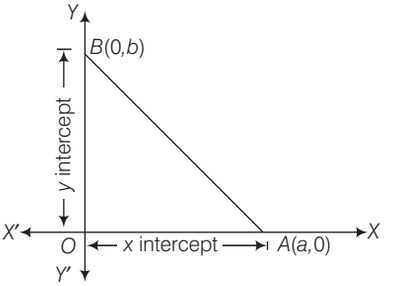
$$x = a$$

Here, $a = \dfrac{\left(-\frac{7}{2} + \frac{15}{2}\right)}{2} = \dfrac{8}{4} = 2$ units

Hence, its equation is $x = 2$.

Intercepts of a Line on Axes

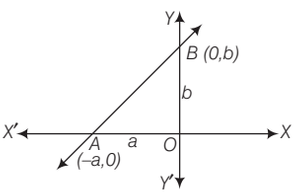
If a line cuts X-axis at $A(a, 0)$ and the Y-axis at $B(0, b)$ then OA and OB are known as the intercepts of the line on X-axis and Y-axis respectively. $|a|$ is called the length of intercept of the line on X-axis. Intercept of a line on X-axis may be positive or negative and $|b|$ is called the length of intercept of the line on Y-axis. Intercept of a line on Y-axis may be positive or negative.



Remark

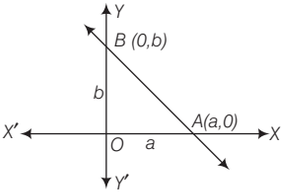
If a line parallel to Y-axis, then its intercept on Y-axis is not defined and if a line parallel to X-axis, then its intercept on X-axis is not defined.

Intercepts in II quadrant



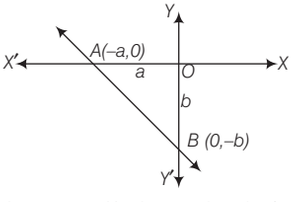
Intercept on X-axis $= -a$, length of intercept on X-axis $= |a|$
Intercept on Y-axis $= b$, length of intercept on Y-axis $= |b|$

Intercepts in I quadrant



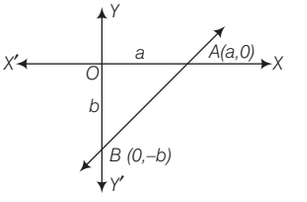
Intercept on X-axis $= a$, length of intercept on X-axis $= |a|$
Intercept on Y-axis $= b$, length of intercept on Y-axis $= |b|$

Intercepts in III quadrant



Intercept on X-axis $= -a$, length of intercept on X-axis $= |a|$
Intercept on Y-axis $= -b$, length of intercept on Y-axis $= |b|$

Intercepts in IV quadrant



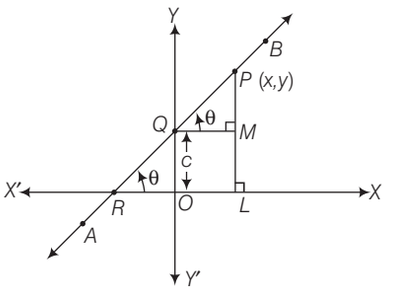
Intercept on X-axis $= a$, length of intercept on X-axis $= |a|$
Intercept on Y-axis $= -b$, length of intercept on Y-axis $= |b|$

Different Forms of the Equation of a Straight Line

(i) Slope-Intercept Form

Theorem : The equation of the straight line whose slope is m and which cuts an intercept c on the Y-axis is

$$y = mx + c$$



Proof : Let AB be a line whose slope is m and which cuts an intercept c on Y-axis. Let $P(x, y)$ be any point on the line. Draw $PL \perp$ to X-axis and $QM \perp$ to PL.

Then, from figure,

$$\angle PRL = \angle PQM = \theta, OQ = c$$

$$\therefore \quad PM = PL - ML = PL - OQ = y - c$$

and $$QM = OL = x$$

Now in ΔPQM, $$\tan\theta = \frac{PM}{QM}$$

$$\Rightarrow \quad m = \frac{y-c}{x} \Rightarrow y = mx + c$$

which is the required equation of the line.

Remarks

1. If the line passes through the origin, then $c = 0$ ($\because 0 = m.0 + c \Rightarrow c = 0$) and hence equation of the line will become $y = mx$.
2. Equation of any line may be taken as $y = mx + c$.
3. If the line is parallel to X-axis, then $\theta = 0°$
   i.e. $m = \tan 0° = 0$. Hence, equation of the line parallel to X-axis is $y = c$.

Example 16. If the straight line $y = mx + c$ passes through the points $(2, 4)$ and $(-3, 6)$, find the values of m and c.

Sol. Since, $(2, 4)$ lies on $y = mx + c$

$$\therefore \quad 4 = 2m + c \qquad ...(i)$$

Again, $(-3, 6)$ lies on $y = mx + c$

$$\therefore \quad 6 = -3m + c \qquad ...(ii)$$

On solving Eqs. (i) and (ii), we get $\quad m = -\frac{2}{5}, c = \frac{24}{5}$

Example 17. What are the inclination to the X-axis and intercept on Y-axis of the line

$$3y = \sqrt{3}x + 6 \ ?$$

Sol. The given equation can be written as

$$y = \frac{x}{\sqrt{3}} + 2 \qquad ...(i)$$

Now, comparing Eq. (i) with $y = mx + c$, then we get

$$m = \frac{1}{\sqrt{3}}$$

Let θ be the inclination to the X-axis, then

$$\tan\theta = \tan 30°$$

$$\therefore \quad \theta = 30° \quad \text{and} \quad c = 2.$$

Example 18. Find the equation of the straight line cutting off an intercept of 3 units on negative direction of Y-axis and inclined at an angle $\tan^{-1}\left(\frac{3}{5}\right)$ to the axis of x.

Sol. Here, $\quad c = -3$ and $\theta = \tan^{-1}\left(\frac{3}{5}\right)$

or $\quad \tan\theta = \frac{3}{5} = m$

Hence, the equation of the line

$$y = mx + c$$

i.e. $\quad y = \frac{3}{5}x - 3$

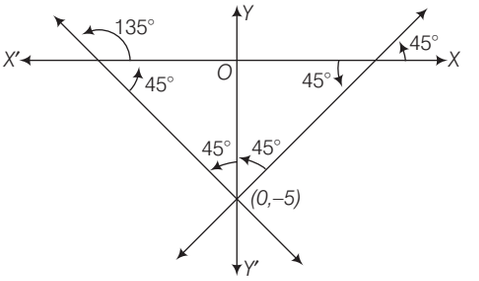
or $\quad 3x - 5y - 15 = 0$

Example 19. ***Find the equation to the straight line cutting off an intercept of 5 units on negative direction of Y-axis and being equally inclined to the axes.***

Sol. Here, $\quad c = -5$

$$m = \tan 45° \quad \text{or} \quad \tan 135°$$

i.e. $\quad m = \pm 1$



Hence, required equation is

$$y = (\pm 1)\, x - 5$$

or $\quad y = \pm x - 5$

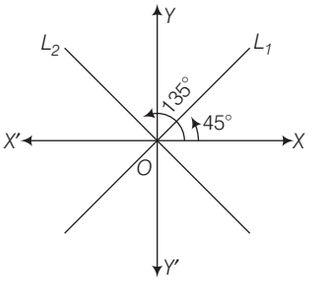
Example 20. Find the equations of the bisectors of the angle between the coordinate axes.

Sol. Let L_1 and L_2 be the straight lines bisecting the co-ordinate axes.

Both L_1 and L_2 pass through origin

$\therefore$ Equation of line through origin is $y = mx$

for L_1, $m = \tan 45° = 1$



$\therefore$ Equation of line L_1 is $y = x$

i.e. $x - y = 0$

For L_2, $m = \tan 135° = -1$

$\therefore$ Equation of line L_2 is $y = -x$

i.e. $x + y = 0$

Hence, equations of the bisectors of the angle between the coordinate axes are $x \pm y = 0$.

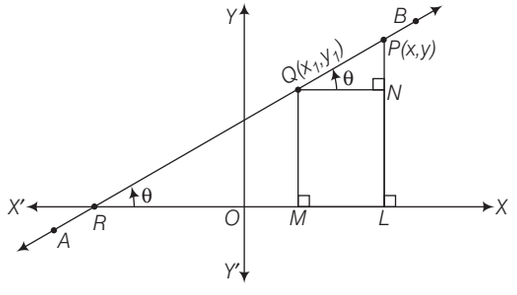
(ii) The Point-Slope Form of a Line

Theorem : The equation of the straight line which passes through the point (x_1, y_1) and has the slope 'm' is

$$y - y_1 = m(x - x_1)$$

Proof : Let AB be a straight line whose slope is m and which pass through the point $Q(x_1, y_1)$. Let the line AB cuts X-axis at R and $\angle BRX = \theta$, then

$$\tan\theta = m$$



Let $P(x, y)$ be any point on the line AB. Draws PL and QM perpendiculars from P and Q on X-axis respectively. Also draw QN perpendicular from Q on PL, then from figure

$$\angle PRL = \angle PQN = \theta,\ OL = x,\ OM = x_1,\ PL = y,\ QM = y_1$$

Then, $QN = ML = OL - OM = x - x_1$

and $PN = PL - NL = PL - QM = y - y_1$

Now, in triangle PQN,

$$\tan\theta = \frac{PN}{QN} = \frac{y - y_1}{x - x_1}$$

$$\therefore \quad m = \frac{y - y_1}{x - x_1}$$

or $$y - y_1 = m(x - x_1)$$

which is the required equation of the line.

Aliter : Let the equation of the required line be

$$y = mx + c \qquad \text{...(i)}$$

where, m is the slope of the line.

Since line Eq. (i) passes through the point (x_1, y_1), therefore

$$y_1 = mx_1 + c \qquad \text{...(ii)}$$

Subtracting (ii) form (i), we get

$$\boldsymbol{y - y_1 = m(x - x_1)}$$

which is the required equation of the line.

Corollary **:** If the line passes through the origin, then putting $x_1 = 0$ and $y_1 = 0$ in $y - y_1 = m(x - x_1)$.

It becomes $y = mx$, which is the equation of the line passing through the origin and having slope m.

Remark

The equation $y - y_1 = m(x - x_1)$ is called **point-slope** form or **one point** form of the equation.

Example 21. ***Find the equation of a line which makes an angle of 135° with the positive direction of X-axis and passes through the point*** $(3, 5)$ **.**

Sol. The slope of the line $= m = \tan 135° = -1$

Here $x_1 = 3, y_1 = 5$.

$\therefore$ The required equation of the line is

$$y - 5 = -1(x - 3)$$

or $$x + y - 8 = 0$$

Example 22. Find the equation of the straight line bisecting the segment joining the points $(5, 3)$ and $(4, 4)$ and making an angle of 45° with the positive direction of X-axis.

Sol. Here, m = slope of the line $= \tan 45° = 1$.

Let A be the mid-point of $(5, 3)$ and $(4, 4)$. Then, the coordinates of A are

$$\left(\frac{5+4}{2}, \frac{3+4}{2}\right) \quad \text{i.e.} \quad \left(\frac{9}{2}, \frac{7}{2}\right).$$

Hence, the required equation of the line is

$$y - \frac{7}{2} = 1\left(x - \frac{9}{2}\right)$$

or $$x - y - 1 = 0$$

Example 23. Find the equation of the right bisector of the line joining (1, 1) and (3, 5) .

Sol. Let m be the slope of the line joining (1, 1) and (3, 5).

Then, $m = \frac{5-1}{3-1} = \frac{4}{2} = 2$

$\therefore$ Slope (M) of right bisector of the join of (1, 1)

and $(3, 5) = -\frac{1}{m}$

$\therefore$ $M = -\frac{1}{2}$

Mid-point of the join of (1, 1) and (3, 5) is $\left(\frac{1+3}{2}, \frac{1+5}{2}\right)$

i.e. (2, 3) .

Hence, equation of the right bisector passing through (2, 3) and having slope $M = -\frac{1}{2}$ is

$$y - 3 = -\frac{1}{2}(x - 2)$$

or $x + 2y - 8 = 0$

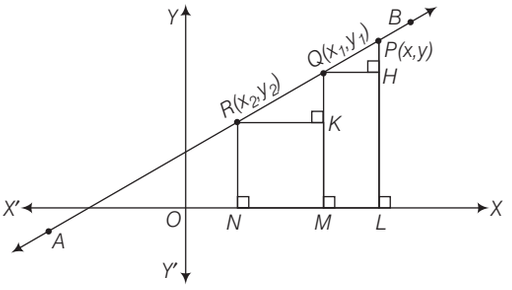
(iii) The Two-Point Form of a Line

Theorem : The equation of a line passing through two given points (x_1, y_1) and (x_2, y_2) is given by

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

Proof :

Let AB be a line which passes through two points $Q(x_1, y_1)$ and $R(x_2, y_2)$. Let $P(x, y)$ be any point on the line AB.



Draws PL, QM and RN are perpendiculars from P, Q and R on X-axis respectively. Also draws QH and RK are perpendiculars on PL and QM respectively. Then from figure

$$ON = x_2, OM = x_1, OL = x, RN = y_2,$$
$$QM = y_1 \text{ and } PL = y$$

then, $RK = NM = OM - ON = x_1 - x_2$

$$QH = ML = OL - OM = x - x_1$$
$$QK = QM - KM = QM - RN = y_1 - y_2$$

and $PH = PL - HL = PL - QM = y - y_1$

Now, triangles PHQ and QKR are similar, then

$$\frac{PH}{QK} = \frac{QH}{RK}$$

$$\Rightarrow \quad \frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}$$

or $$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

which is the required equation of the line.

Aliter I : Let the equation of the required line be

$$y = mx + c \qquad \text{...(i)}$$

where m is the slope of the line.

Since, line Eq. (i) passes through the points (x_1, y_1) and (x_2, y_2) therefore

$$y_1 = mx_1 + c \qquad \text{...(ii)}$$

and $$y_2 = mx_2 + c \qquad \text{...(iii)}$$

Now, subtracting Eqs. (ii) from (i), we get

$$y - y_1 = m(x - x_1) \qquad \text{...(iv)}$$

and subtracting Eqs. (ii) from (iii), we get

$$y_2 - y_1 = m(x_2 - x_1) \qquad \text{...(v)}$$

Dividing Eqs. (iv) by (v) then, we get

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

or $$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

which is the required equation of the line.

Aliter II : Since points, $P(x, y), Q(x_1, y_1)$ and $R(x_2, y_2)$ are collinear then area of $\Delta PQR = 0$

i.e. $$\frac{1}{2}\left|\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}\right| = 0 \text{ or } \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

which is the required equation of the line.

Example 24. ***Find the equation to the straight line joining the points*** $\left(at_1, \frac{a}{t_1}\right)$ ***and*** $\left(at_2, \frac{a}{t_2}\right)$.

Sol. The equation of the line joining the points $\left(at_1, \frac{a}{t_1}\right)$ and $\left(at_2, \frac{a}{t_2}\right)$ is

$$y - \frac{a}{t_1} = \left(\frac{\frac{a}{t_2} - \frac{a}{t_1}}{at_2 - at_1}\right)(x - at_1)$$

$$\Rightarrow \quad y - \frac{a}{t_1} = -\frac{a(t_1 - t_2)}{at_1t_2(t_1 - t_2)}(x - at_1)$$

or $$y - \frac{a}{t_1} = -\frac{1}{t_1t_2}(x - at_1)$$

or $$t_1t_2y - at_2 = -x + at_1$$

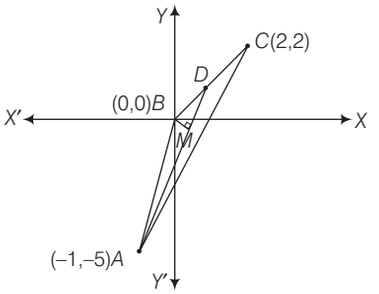
or $$x + t_1t_2y = a(t_1 + t_2)$$

which is the required equation of the line.

Example 25. Let ABC be a triangle with $A(-1,-5)$, $B(0,0)$ and $C(2,2)$ and let D be the middle point of BC. Find the equation of the perpendicular drawn from B to AD.

Sol. $\because$ D is the middle point of BC.

$\therefore$ Coordinates of D are $\left(\frac{0+2}{2}, \frac{0+2}{2}\right)$



i.e. $D(1, 1)$

Slope of median $AD = \frac{1+5}{1+1} = 3$

$\therefore$ Slope of BM which is perpendicular to $AD = -\frac{1}{3}$.

Hence, equation of the line BM is

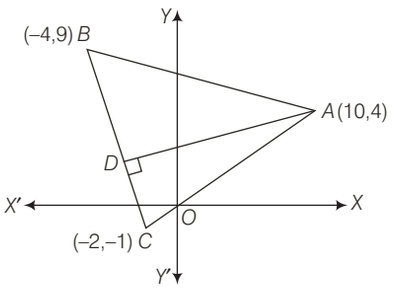
$$y - 0 = -\frac{1}{3}(x - 0) \Rightarrow x + 3y = 0$$

which is the required equation of the line.

Example 26. The vertices of a triangle are $A(10,4), B(-4,9)$ and $C(-2,-1)$. Find the equation of the altitude through A.

Sol. $\because$ Slope of $BC = \frac{-1-9}{-2+4} = \frac{-10}{2} = -5$

$\therefore$ Slope of altitude $AD = -\frac{1}{-5} = \frac{1}{5}$



Hence, equation of altitude AD which passes through $(0, 4)$ and having slope $\frac{1}{5}$ is

$$y - 4 = \frac{1}{5}(x - 10)$$

or $$x - 5y + 10 = 0$$

Example 27. ***Find the equations of the medians of a triangle, the coordinates of whose vertices are*** $(-1,6), (-3,-9)$ ***and*** $(5,-8)$**.**

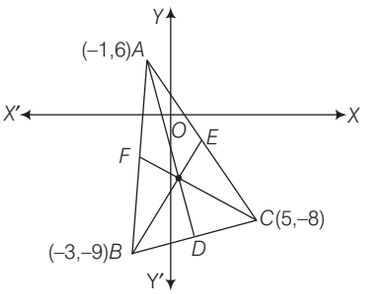
Sol. Let $A(-1, 6)$, $B(-3, -9)$ and $C(5, -8)$ be the vertices of ΔABC. Let D, E and F be the mid-points of the sides BC, CA and AB respectively.

Coordinates of $D \equiv \left(\frac{-3+5}{2}, \frac{-9-8}{2}\right)$

i.e. $\left(1, -\frac{17}{2}\right)$

Coordinates of $E \equiv \left(\frac{5-1}{2}, \frac{-8+6}{2}\right)$

i.e. $(2, -1)$



and coordinates of $F \equiv \left(\frac{-1-3}{2}, \frac{6-9}{2}\right)$

i.e. $(-2, -3/2)$

$\therefore$ Equation of the median AD = Equation of line through $(-1, 6)$ and $\left(1, -\frac{17}{2}\right)$ is

$$y - 6 = \frac{-\frac{17}{2} - 6}{1+1}(x + 1)$$

$$\Rightarrow \quad y - 6 = -\frac{29}{4}(x + 1) \quad \text{or} \quad 29x + 4y + 5 = 0$$

Equation of median BE is

$$y + 9 = \frac{-1+9}{2+3}(x + 3) \quad \text{or} \quad 8x - 5y - 21 = 0$$

and equation of median CF is

$$y + 8 = \frac{-\frac{3}{2} + 8}{-2-5}(x - 5)$$

or $$13x + 14y + 47 = 0$$

Example 28. Find the ratio in which the line segment joining the points (2, 3) and (4, 5) is divided by the line joining (6, 8) and (–3, –2) .

Sol. The equation of line passing through (6, 8) and (–3, – 2) is

$$y-8=\frac{-2-8}{-3-6}(x-6)$$

$\Rightarrow \quad 9y-72=10x-60$

or $\quad 10x-9y+12=0 \quad ...(i)$

Let the required ratio be $\lambda : 1$.

Now, the coordinates of the point P which divide the line segment joining the points (2, 3) and (4, 5) in the ratio $\lambda : 1$ is

$$P\left(\frac{2+4\lambda}{1+\lambda}, \frac{3+5\lambda}{1+\lambda}\right)$$

Clearly P lies on Eq. (i), then

$$10\left(\frac{2+4\lambda}{1+\lambda}\right)-9\left(\frac{3+5\lambda}{1+\lambda}\right)+12=0$$

or $\quad 20+40\lambda-27-45\lambda+12+12\lambda=0$

or $\quad 7\lambda+5=0 \quad \text{or} \quad \lambda=-\frac{5}{7}$

$\therefore$ The required ratio $=\lambda : 1=-\frac{5}{7}:1=-5:7$

Hence, the required ratio is 5 : 7 (externally).

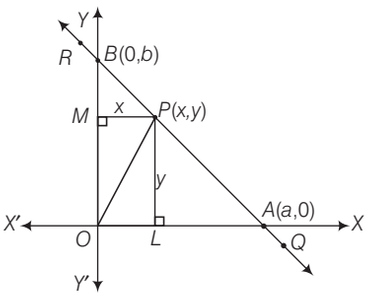
(iv) The Intercept Form of a Line :

Theorem : The equation of the straight line which cuts off intercepts of lengths of a and b on X-axis and Y-axis respectively, is

$$\frac{x}{a}+\frac{y}{b}=1$$

Proof : Let QR be a line which cuts off intercepts $OA = a$ and $OB = b$ on the X-axis and Y-axis respectively, where $a \neq 0$. The line is non vertical, because b is finite. Let $P(x, y)$ be a general point on the line.

Draws PL and PM perpendiculars on X-axis and Y-axis respectively. Then $PL = y$ and $OL = x$ also join OP. Clearly,



Area of ΔOAB = Area of ΔOAP + Area of ΔOPB

$\Rightarrow \quad \frac{1}{2}\cdot OA\cdot OB=\frac{1}{2}\cdot OA\cdot PL+\frac{1}{2}\cdot OB\cdot PM$

$\Rightarrow \quad \frac{1}{2}\cdot OA\cdot OB=\frac{1}{2}\cdot OA\cdot PL+\frac{1}{2}\cdot OB\cdot OL$

or $\quad \frac{1}{2}ab=\frac{1}{2}ay+\frac{1}{2}bx$

$\Rightarrow \quad ab=ay+bx$

or $\quad \frac{x}{a}+\frac{y}{b}=1$

which is the required equation of the line.

Aliter I : Equation of the line through $A(a, 0)$ and $B(0, b)$ is

$$y-0=\frac{b-0}{0-a}(x-a)$$

or $\quad -ay=bx-ab$

or $\quad bx+ay=ab$

or $\quad \frac{x}{a}+\frac{y}{b}=1$

which is the required equation of the line.

Aliter II : Points $A(a, 0)$, $P(x, y)$ and $B(0, b)$ are collinear, we have

slope of AB = slope of AP

$\Rightarrow \quad \frac{b-0}{0-a}=\frac{y-0}{x-a} \Rightarrow bx-ab=-ay$

$\Rightarrow \quad bx+ay=ab \Rightarrow \frac{x}{a}+\frac{y}{b}=1$

or

$$\begin{vmatrix} a & 0 & 1 \\ x & y & 1 \\ 0 & b & 1 \end{vmatrix}=0$$

$\Rightarrow \quad a(y-b)-0+1\cdot(bx)=0$

or $\quad bx+ay=ab$

or $\quad \frac{x}{a}+\frac{y}{b}=1$

which is the required equation of the line.

Example 29. Find the equation of the line through (2, 3) so that the segment of the line intercepted between the axes is bisected at this point.

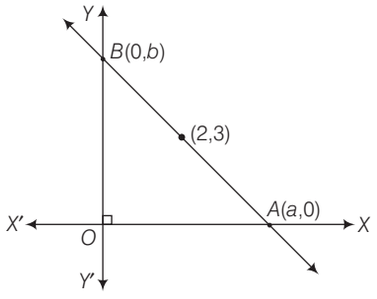
Sol. Let the required line segment be AB.

Let O be the origin and $OA = a$ and $OB = b$.

Then the coordinates of A and B are $(a, 0)$ and $(0, b)$ respectively.

$\therefore \quad \frac{a+0}{2} = 2 \Rightarrow a = 4$

and $\quad \frac{0+b}{2} = 3 \Rightarrow b = 6$



Hence, the equation of the required line is

$$\frac{x}{4} + \frac{y}{6} = 1$$

i.e. $\quad 3x + 2y = 12$

Example 30. Find the equation to the straight line which passes through the points $(3, 4)$ and have intercepts on the axes :

(a) equal in magnitude but opposite in sign

(b) such that their sum is 14

Sol. (a) Let intercepts on the axes be a and $-a$ respectively.

$\therefore$ The equation of the line in intercept form is

$$\frac{x}{a} + \frac{y}{-a} = 1 \quad \text{or} \quad x - y = a \qquad \text{...(i)}$$

Since, Eq. (i) passes through $(3, 4)$, then

$$3 - 4 = a$$

$\therefore \quad a = -1$

From Eq. (i), $\quad x - y + 1 = 0$

which is the required equation of the line.

(b) Let the equation of the line be $\frac{x}{a} + \frac{y}{b} = 1$

This passes through $(3, 4)$.

Therefore $\quad \frac{3}{a} + \frac{4}{b} = 1 \qquad \text{...(ii)}$

It is given that $a + b = 14$

$\therefore \quad b = 14 - a$

Putting $b = 14 - a$ in Eq. (ii), we get

$$\frac{3}{a} + \frac{4}{14-a} = 1$$

$\Rightarrow \quad 42 - 3a + 4a = 14a - a^2$

$\Rightarrow \quad a^2 - 13a + 42 = 0$

$\Rightarrow \quad (a-7)(a-6) = 0$

$\therefore \quad a = 6, 7$

Then, $\quad b = 8, 7 \qquad (\because b = 14 - a)$

Hence, the required equations are

$$\frac{x}{6} + \frac{y}{8} = 1 \quad \text{and} \quad \frac{x}{7} + \frac{y}{7} = 1$$

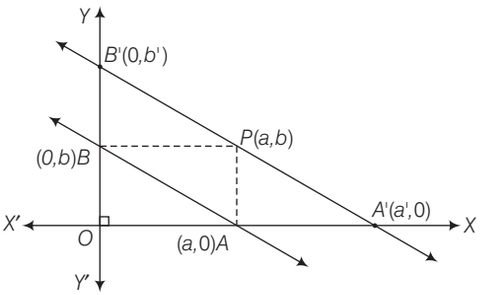
i.e. $\quad 4x + 3y = 24$ and $x + y = 7$

Example 31. Find the equation of the straight line through the point $P(a, b)$ parallel to the line $\frac{x}{a} + \frac{y}{b} = 1$. Also find the intercepts made by it on the axes.

Sol. Let the line $\frac{x}{a} + \frac{y}{b} = 1$

meets the axes in A and B respectively. So that

$$OA = a, OB = b$$



Let the required parallel line meet in A' and B' respectively, so that

$OA' = a'$ (say)

and $\quad OB' = b'$ (say)

$\therefore$ Equation of required line is

$$\frac{x}{a'} + \frac{y}{b'} = 1 \qquad \text{...(i)}$$

Since, $\Delta's$ OAB and $OA'B'$ are similar, then

$$\frac{OA'}{OA} = \frac{OB'}{OB}$$

i.e. $\quad \frac{a'}{a} = \frac{b'}{b} = \lambda$ (say)

$\Rightarrow \quad a' = a\lambda, b' = b\lambda$

Substituting these values in Eq. (i), then

$$\frac{x}{a\lambda} + \frac{y}{b\lambda} = 1 \qquad \text{...(ii)}$$

It passes through (a, b), then

$$\frac{a}{a\lambda} + \frac{b}{b\lambda} = 1$$

$\Rightarrow \quad \frac{2}{\lambda} = 1$ or $\lambda = 2$

From Eq. (ii) required equation is

$$\frac{x}{2a} + \frac{y}{2b} = 1$$

Evidently intercepts on the axes are $2a$ and $2b$.

(v) The Normal Form or Perpendicular Form of a Line

Theorem : The equation of the straight line upon which the length of perpendicular from the origin is p and this normal makes an angle α with the positive direction of X-axis is

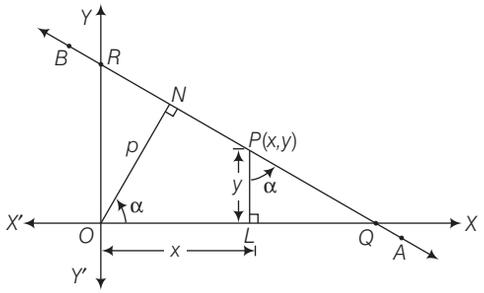
$$x\cos\alpha + y\sin\alpha = p.$$

Proof : Let AB be a line such that the length of perpendicular from O to the line be p

i.e. $ON = p$

and $\angle NOX = \alpha$

Let $P(x, y)$ be any point on the line. Draw PL perpendicular from P on X-axis.



Let line AB cuts X and Y-axes at Q and R respectively.

Now, $\angle NQO = 90° - \alpha$

$\therefore$ $\angle LPQ = 90° - (90° - \alpha) = \alpha$

In ΔPLQ, $\tan\alpha = \dfrac{LQ}{PL} = \dfrac{LQ}{y}$

$\therefore$ $LQ = y\tan\alpha$ ...(i)

Also, in ΔONQ, $\cos\alpha = \dfrac{ON}{OQ}$

$\Rightarrow$ $\cos\alpha = \dfrac{p}{OL + LQ}$

$\Rightarrow$ $OL\cos\alpha + LQ\cos\alpha = p$

$\Rightarrow$ $x\cos\alpha + y\tan\alpha\cos\alpha = p$

$(\because OL = x \text{ and } LQ = y\tan\alpha)$

$\therefore$ $\boldsymbol{x\cos\alpha + y\sin\alpha = p}$

which is the required equation of the line AB.

Aliter I : $\because \angle NOQ = \alpha$

then $\angle NOR = 90° - \alpha$

Now, in ΔONQ, $\sec\alpha = \dfrac{OQ}{ON} = \dfrac{OQ}{p}$

or $OQ = p\sec\alpha$

Also in ΔONR, $\sec(90° - \alpha) = \dfrac{OR}{ON}$

$\Rightarrow$ $\operatorname{cosec}\alpha = \dfrac{OR}{p}$

or $OR = p\operatorname{cosec}\alpha$

Thus, AB makes intercepts $p\sec\alpha$ and $p\operatorname{cosec}\alpha$ on X-axis and Y-axis respectively.

$\therefore$ Equation of AB is $\dfrac{x}{p\sec\alpha} + \dfrac{y}{p\operatorname{cosec}\alpha} = 1$

or $\boldsymbol{x\cos\alpha + y\sin\alpha = p}$

which is the required equation of the line AB.

Aliter II : The points $Q(p\sec\alpha, 0)$, $P(x, y)$ and $R(0, p\operatorname{cosec}\alpha)$ are collinear, then

$$\begin{vmatrix} p\sec\alpha & 0 & 1 \\ x & y & 1 \\ 0 & p\operatorname{cosec}\alpha & 1 \end{vmatrix} = 0$$

or $p\sec\alpha(y - p\operatorname{cosec}\alpha) - 0 + 1(px\operatorname{cosec}\alpha) = 0$

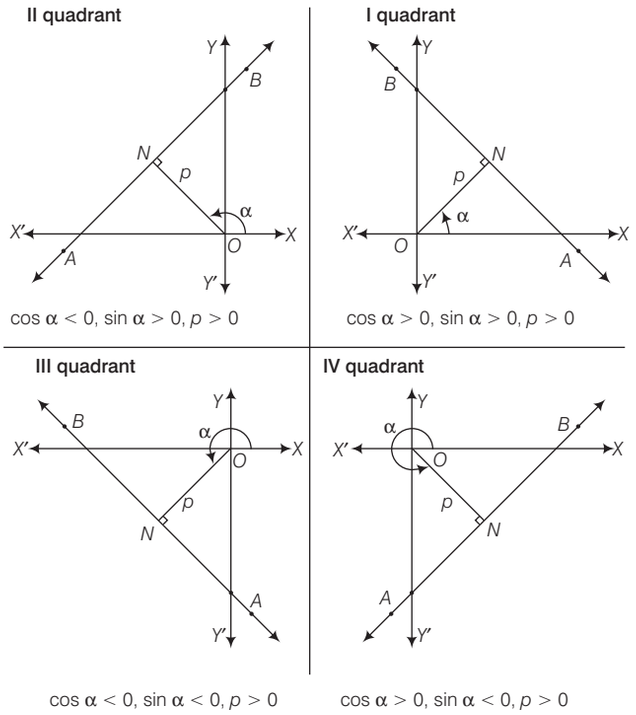
or $p(y\sin\alpha - p) + px\cos\alpha = 0$

or $\boldsymbol{x\cos\alpha + y\sin\alpha = p}$

which is the required equation of the line AB.

Remarks

1. Here, p is always taken as positive and α is measured from positive direction of X-axis in anticlockwise direction between 0 and 2π (i.e. $0 \le \alpha < 2\pi$).
2. (Coefficient of $x)^2$+ (Coefficient of $y)^2 = \cos^2\alpha + \sin^2\alpha = 1$
3. $\cos\alpha$ and $\cos(90° - \alpha)$ are the direction cosines of ON.
4.



***Corollary* 1 :** If $\alpha = 0°$, then equation $x\cos\alpha + y\sin\alpha = p$ becomes $x\cos 0° + y\sin 0° = p$

i.e. $x = p$ (Equation of line parallel to Y-axis)

***Corollary* 2 :** If $\alpha = \frac{\pi}{2}$, then equation $x\cos\alpha + y\sin\alpha = p$ becomes $x\cos\left(\frac{\pi}{2}\right) + y\sin\left(\frac{\pi}{2}\right) = p$

i.e. $y = p$ (Equation of line parallel to X-axis).

***Corollary* 3 :** If $\alpha = 0°$, $p = 0$ then equation $x\cos\alpha + y\sin\alpha = p$ becomes $x\cos 0° + y\sin 0° = 0$

i.e. $x = 0$ (Equation of Y-axis)

***Corollary* 4 :** If $\alpha = \frac{\pi}{2}$, $p = 0$ then equation $x\cos\alpha + y\sin\alpha = p$ becomes $x\cos\left(\frac{\pi}{2}\right) + y\sin\left(\frac{\pi}{2}\right) = 0$

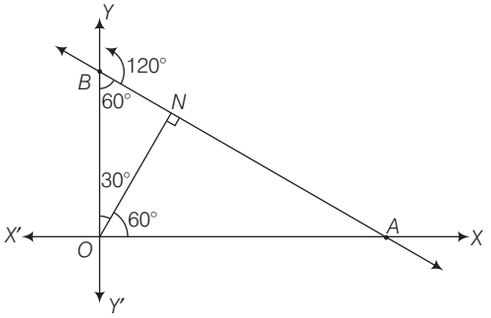
i.e. $y = 0$ (Eq. (i) of X-axis).

Example 32. The length of perpendicular from the origin to a line is 9 and the line makes an angle of 120° with the positive direction of Y-axis. Find the equation of the line.

Sol. Here, $\alpha = 60°$ and $p = 9$.

$\therefore$ Equation of the required line is

$x\cos 60° + y\sin 60° = 9$



$\Rightarrow \quad x\left(\frac{1}{2}\right) + y\left(\frac{\sqrt{3}}{2}\right) = 9$

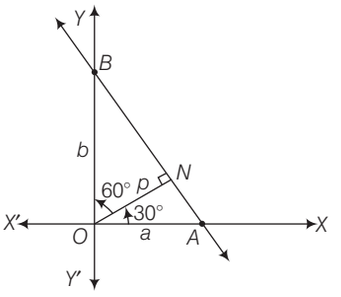
or $\quad x + y\sqrt{3} = 18$

Example 33. Find the equation of the straight line on which the perpendicular from origin makes an angle of 30° with X-axis and which forms a triangle of area $\left(\frac{50}{\sqrt{3}}\right)$ sq units with the coordinates axes.

Sol. Let $\angle NOA = 30°$

Let $ON = p > 0, OA = a, OB = b$

In ΔONA, $\quad \cos 30° = \frac{ON}{OA} = \frac{p}{a}$



$\Rightarrow \quad \frac{\sqrt{3}}{2} = \frac{p}{a}$ or $a = \frac{2p}{\sqrt{3}}$

and in ΔONB, $\cos 60° = \frac{ON}{OB} = \frac{p}{b}$

$\Rightarrow \quad \frac{1}{2} = \frac{p}{b}$ or $b = 2p$

$\because \quad$ Area of $\Delta OAB = \frac{1}{2}ab = \frac{1}{2}\left(\frac{2p}{\sqrt{3}}\right)$

$(2p) = \frac{2p^2}{\sqrt{3}}$

$\therefore \quad \frac{2p^2}{\sqrt{3}} = \frac{50}{\sqrt{3}}$ (given)

$\Rightarrow \quad p^2 = 25$

or $\quad p = 5 \quad (\because p > 0)$

$\therefore$ Using $x\cos\alpha + y\sin\alpha = p$, the equation of the line AB is

$x\cos 30° + y\sin 30° = 5$

or $\quad x\sqrt{3} + y = 10$

Reduction of General Equation to Standard Form

Let $Ax + By + C = 0$ be the general equation of a straight line where A and B are not both zero.

(i) Reduction of 'Slope-Intercept' Form

Given equation is $Ax + By + C = 0$

$\Rightarrow \quad By = -Ax - C$

$\Rightarrow \quad y = \left(-\frac{A}{B}\right)x + \left(-\frac{C}{B}\right)$ (Assuming $B \neq 0$)

Comparing it with $y = mx + c$

we get $\quad$ slope $(m) = -\frac{A}{B} = -\frac{\text{coefficient of } x}{\text{coefficient of } y}$

and $\quad y$ intercept $(c) = -\frac{C}{B} = -\frac{\text{constant term}}{\text{coefficient of } y}$

***Corollary* 1 :** Find angle between the lines $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$.

Slope of the line

$$A_1x + B_1y + C_1 = 0 \text{ is } -\frac{A_1}{B_1} = m_1 \qquad \text{(say)}$$

and slope of the line

$$A_2x + B_2y + C_2 = 0 \text{ is } -\frac{A_2}{B_2} = m_2 \qquad \text{(say)}$$

If θ is the angle between the two lines, then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| = \left|\frac{\left(-\frac{A_1}{B_1}\right) - \left(-\frac{A_2}{B_2}\right)}{1 + \left(-\frac{A_1}{B_1}\right)\left(-\frac{A_2}{B_2}\right)}\right|$$

$$= \left|\frac{A_1B_2 - A_2B_1}{A_1A_2 + B_1B_2}\right|$$

$$\therefore \qquad \theta = \tan^{-1}\left|\frac{A_1B_2 - A_2B_1}{A_1A_2 + B_1B_2}\right|$$

***Corollary* 2 :** Find the condition of (i) parallelism (ii) perpendicularity of the lines

$$A_1x + B_1y + C_1 = 0$$

and $\qquad A_2x + B_2y + C_2 = 0$

(i) If the two lines are parallel, $\theta = 0°$

$$\therefore \qquad \tan\theta = \tan 0° = 0$$

$$\Rightarrow \qquad \left|\frac{A_1B_2 - A_2B_1}{A_1A_2 + B_1B_2}\right| = 0$$

$$\Rightarrow \qquad A_1B_2 - A_2B_1 = 0$$

or $\qquad \frac{A_1}{A_2} = \frac{B_1}{B_2}$ **(Remember)**

which is required condition of parallelism.

(ii) If the two lines are perpendicular, $\theta = 90°$

$$\therefore \qquad \tan\theta = \tan 90° = \infty$$

$$\Rightarrow \qquad \left|\frac{A_1B_2 - A_2B_1}{A_1A_2 + B_1B_2}\right| = \infty$$

$$\Rightarrow \qquad A_1A_2 + B_1B_2 = 0 \qquad \textbf{(Remember)}$$

which is required condition of perpendicularity.

Remark

If two lines are coincident, then

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} \qquad \textbf{(Remember)}$$

(ii) Reduction to 'Intercept' Form

Given equation is $Ax + By + C = 0$

$$\Rightarrow \qquad Ax + By = -C$$

$$\Rightarrow \qquad \frac{A}{-C}x + \frac{B}{-C}y = 1 \qquad \text{(Assuming } C \neq 0\text{)}$$

$$\Rightarrow \qquad \frac{x}{(-C/A)} + \frac{y}{(-C/B)} = 1 \quad \text{(Assuming } A \neq 0, B \neq 0\text{)}$$

Comparing with $\frac{x}{a} + \frac{y}{b} = 1$

we get, $\quad x\text{-intercept } (a) = -\frac{C}{A} = -\frac{\text{constant term}}{\text{coefficient of } x}$

and $\quad y\text{-intercept } (b) = -\frac{C}{B} = -\frac{\text{constant term}}{\text{coefficient of } y}$

(iii) Reduction to 'Normal' Form

Given equation is $Ax + By + C = 0$. Let its **normal form** be $x\cos\alpha + y\sin\alpha = p$.

Clearly, equations $Ax + By + C = 0$ and $x\cos\alpha + y\sin\alpha = p$ represent the same line.

Therefore, $\qquad \frac{A}{\cos\alpha} = \frac{B}{\sin\alpha} = \frac{C}{-p}$

$$\Rightarrow \qquad \cos\alpha = -\frac{Ap}{C}$$

and $\qquad \sin\alpha = -\frac{Bp}{C}$ ...(i)

$$\cos^2\alpha + \sin^2\alpha = 1$$

$$\left(-\frac{Ap}{C}\right)^2 + \left(-\frac{Bp}{C}\right)^2 = 1$$

or $\qquad p^2 = \frac{C^2}{A^2 + B^2}$

$$\Rightarrow \qquad p = \frac{|C|}{\sqrt{(A^2 + B^2)}}$$

From Eq. (i), $\cos\alpha = -\frac{|C|}{C} \cdot \frac{A}{\sqrt{(A^2 + B^2)}}$,

$$\sin\alpha = -\frac{|C|}{C} \cdot \frac{B}{\sqrt{(A^2 + B^2)}}.$$

Putting the values of $\cos\alpha, \sin\alpha$ and p in $x\cos\alpha + y\sin\alpha = p$, we get

$$x\left(\frac{-|C|}{C}\cdot\frac{A}{\sqrt{(A^2+B^2)}}\right)x+y\left(\frac{-|C|}{C}\cdot\frac{B}{\sqrt{(A^2+B^2)}}\right)$$
$$=\frac{|C|}{\sqrt{(A^2+B^2)}}$$

$$\Rightarrow\quad\left(-\frac{A}{\sqrt{(A^2+B^2)}}\right)x+\left(-\frac{B}{\sqrt{(A^2+B^2)}}\right)y=\frac{C}{\sqrt{(A^2+B^2)}}$$

This is the normal form of the line $Ax+By+C=0$.

Rule : First shift the constant term on the RHS and make it positive, if it is not so by multiplying the whole equation by '-1' and then divide both sides by

$$\sqrt{(\text{coefficient of } x)^2+(\text{coefficient of } y)^2}$$

Example 34. Reduce $x+\sqrt{3}y+4=0$ to the :

(i) Slope-intercept form and find its slope and y-intercept

(ii) Intercept form and find its intercepts on the axes

(iii) Normal form and find the values of p and α

Sol. (i) Given equation is $x+\sqrt{3}\,y+4=0$

$\Rightarrow\quad \sqrt{3}y=-x-4$

$\Rightarrow\quad y=\left(-\frac{1}{\sqrt{3}}\right)x+\left(-\frac{4}{\sqrt{3}}\right)$

which is in the slope-intercept form $y=mx+c$

Where slope $(m)=-\frac{1}{\sqrt{3}}$ and y-intercept $(c)=-\frac{4}{\sqrt{3}}$

(ii) Given equation is

$x+\sqrt{3}y+4=0$

$\Rightarrow\quad x+\sqrt{3}y=-4$

$\Rightarrow\quad \frac{x}{-4}+\frac{\sqrt{3}y}{-4}=1$

$\Rightarrow\quad \frac{x}{-4}+\frac{y}{-4/\sqrt{3}}=1$

which is in the intercept form $\frac{x}{a}+\frac{y}{b}=1$

where x-intercept $(a)=-4$ and y-intercept $(b)=-\frac{4}{\sqrt{3}}$

(iii) Given equation is $x+\sqrt{3}y+4=0$

$\Rightarrow\quad x+\sqrt{3}y=-4$

$\Rightarrow\quad -x-\sqrt{3}y=4$ (RHS made positive)

Dividing both sides by $\sqrt{(-1)^2+(-\sqrt{3})^2}=2$, we get

$$\left(-\frac{1}{2}\right)x+\left(-\frac{\sqrt{3}}{2}\right)y=2$$

Which is the normal form $x\cos\alpha+y\sin\alpha=p$.

where, $\cos\alpha=-\frac{1}{2}=-\cos 60^\circ=\cos(180^\circ-60^\circ)$

or $\cos(180^\circ+60^\circ)$

$\therefore\quad \alpha=120^\circ$ or 240°

and $\sin\alpha=-\frac{\sqrt{3}}{2}=-\sin 60^\circ=\sin(180^\circ+60^\circ)$

or $\sin(360^\circ-60^\circ)$

$\therefore\quad \alpha=240^\circ$ or 300°

Hence, $\alpha=240^\circ,\ p=2$

$\therefore$ Required normal form is

$$x\cos 240^\circ+y\sin 240^\circ=2$$

Example 35. Find the measure of the angle of intersection of the lines whose equations are $3x+4y+7=0$ and $4x-3y+5=0$.

Sol. Given lines are $3x+4y+7=0,\ 4x-3y+5=0$. Comparing the given lines with $A_1x+B_1y+C_1=0,\ A_2x+B_2y+C_2=0$ respectively, we get

$A_1=3,\ B_1=4$ and $A_2=4,\ B_2=-3$

$\because\quad A_1A_2+B_1B_2=3\times 4+4(-3)=0$

Hence, the given lines are perpendicular.

Example 36. Find the angle between the lines

$(a^2-ab)\,y=(ab+b^2)\,x+b^3$

and $(ab+a^2)\,y=(ab-b^2)\,x+a^3$

where $a>b>0$.

Sol. The given equations of lines can be written as

$$(ab+b^2)\,x-(a^2-ab)\,y+b^3=0 \quad ...(i)$$

and $$(ab-b^2)\,x-(ab+a^2)\,y+a^3=0 \quad ...(ii)$$

Comparing the given lines (i) and (ii) with the lines

$A_1x+B_1y+C_1=0$ and $A_2x+B_2y+C_2=0$

respectively, we get

$A_1=ab+b^2,\ B_1=-(a^2-ab)$

and $A_2=ab-b^2,\ B_2=-(ab+a^2)$

Let θ be the acute angle between the lines, then

$$\tan\theta=\left|\frac{A_1B_2-A_2B_1}{A_1A_2+B_1B_2}\right|$$

$$\tan\theta=\left|\frac{(ab+b^2)\times(-(ab+a^2))-(ab-b^2)\times(-(a^2-ab))}{(ab+b^2)(ab-b^2)+(a^2-ab)(ab+a^2)}\right|$$

$$=\left|\frac{-\{a^2b^2+a^3b+ab^3+a^2b^2-a^3b+a^2b^2+a^2b^2-b^3a\}}{(a^2b^2-b^4+a^4-a^2b^2)}\right|$$

$$=\left|\frac{-4a^2b^2}{a^4-b^4}\right|=\frac{4a^2b^2}{a^4-b^4}$$

$$\therefore\ \theta=\tan^{-1}\left(\frac{4a^2b^2}{a^4-b^4}\right)$$

Example 37. Two equal sides of an isosceles triangle are given by the equations $7x - y + 3 = 0$ and $x + y - 3 = 0$ and its third side passes through the point $(1, -10)$. Determine the equation of the third side.

Sol. Given equations

$$7x - y + 3 = 0 \quad \text{...(i)}$$

and $$x + y - 3 = 0 \quad \text{...(ii)}$$

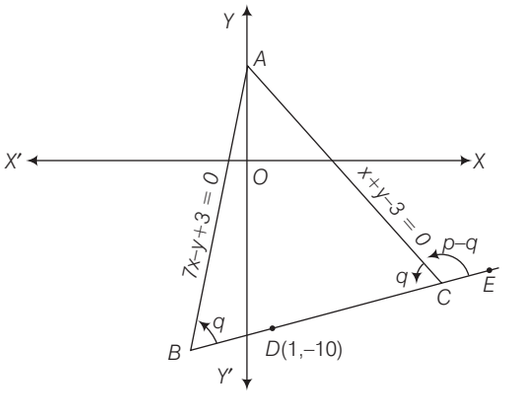
represents two equal sides AB and AC of an isosceles triangle ABC. Since its third side passes through $D(1, -10)$ then its equation is

$$y + 10 = m(x - 1) \quad \text{...(iii)}$$

$\because$ $AB = AC$

Let $\angle ABC = \angle ACB = \theta$

then $\angle ACE = \pi - \theta$



From Eqs. (i) and (ii), slopes of AB and AC are

$$m_1 = 7 \quad \text{and} \quad m_2 = -1$$

respectively.

$$\therefore \quad \tan\theta = \frac{7 - m}{1 + 7m}$$

and $$\tan(\pi - \theta) = \frac{-1 - m}{1 + (-1)\, m} = -\left(\frac{1 + m}{1 - m}\right)$$

$$\Rightarrow \quad -\tan\theta = -\left(\frac{1 + m}{1 - m}\right) \Rightarrow \tan\theta = \left(\frac{1 + m}{1 - m}\right)$$

$$\therefore \quad \frac{7 - m}{1 + 7m} = \frac{1 + m}{1 - m}$$

$$\Rightarrow \quad (7 - m)(1 - m) = (1 + 7m)(1 + m)$$

$$\Rightarrow \quad 6m^2 + 16m - 6 = 0$$

or $$3m^2 + 8m - 3 = 0 \text{ or } (3m - 1)(m + 3) = 0$$

$$\Rightarrow \quad m = \frac{1}{3}, -3$$

Hence from Eq. (iii), the third side BC has two equations

$$y + 10 = \frac{1}{3}(x - 1) \text{ and } y + 10 = -3(x - 1)$$

or $$x - 3y - 31 = 0 \quad \text{and} \quad 3x + y + 7 = 0$$

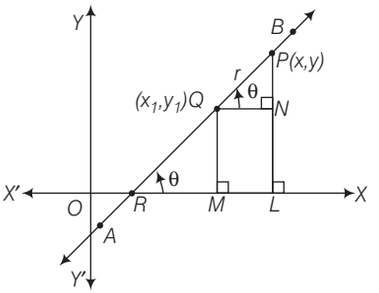
The Distance form or Symmetric form or Parametric form of a line

Theorem : The equation of the straight line passing through (x_1, y_1) and making an angle θ with the positive direction of X-axis is

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$$

where, r is the directed distance between the points (x, y) and (x_1, y_1).

Proof : Let AB be a line which passes through the point $Q(x_1, y_1)$ and meet X-axis at R and makes an angle θ with the positive direction of X-axis.



Let $P(x, y)$ be any point on the line at a distance r from Q. Draws PL and QM are perpendiculars from P and Q on X-axis respectively and draw QN perpendicular on PL. Then,

$$QN = ML = OL - OM = x - x_1$$

and $$PN = PL - NL = PL - QM = y - y_1$$

from ΔPQN,

$$\cos\theta = \frac{QN}{PQ} = \frac{x - x_1}{r} \text{ or } \frac{x - x_1}{\cos\theta} = r \quad \text{...(i)}$$

and $$\sin\theta = \frac{PN}{PQ} = \frac{y - y_1}{r} \text{ or } \frac{y - y_1}{\sin\theta} = r \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$$

***Corollary* 1 :** $\because$ $\dfrac{x - x_1}{\cos\theta} = \dfrac{y - y_1}{\sin\theta} = r$, then

$$\left.\begin{aligned} x &= x_1 + r\cos\theta \\ y &= y_1 + r\sin\theta \end{aligned}\right\}$$

parametric equations of straight line AB.

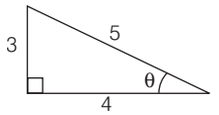
***Corollary* 2 :** If P point above Q then r is positive then coordinates of P are $(x_1 + r\cos\theta, y_1 + r\sin\theta)$ and if P below Q then r is negative then coordinates of P are $(x_1 - r\cos\theta, y_1 - r\sin\theta)$.

Example 38. ***The slope of a straight line through*** $A(3,2)$ ***is*** $\frac{3}{4}$***. Find the coordinates of the points on the line that are 5 units away from*** A***.***

Sol. Let straight line makes an angle θ with positive direction of X-axis,

then $\tan\theta = \frac{3}{4}$

$\therefore$ $\sin\theta = \frac{3}{5}$ and $\cos\theta = \frac{4}{5}$



$\therefore$ Equation of the straight line through $A(3,2)$ in parametric form is

$$\frac{x-3}{\cos\theta} = \frac{y-2}{\sin\theta} = \pm 5$$

$\therefore$ $x = 3 \pm 5\cos\theta = 3 \pm 5 \times \frac{4}{5} = 3 \pm 4 = 7 \text{ or } -1$

and $y = 2 \pm 5\sin\theta = 2 \pm 5 \times \frac{3}{5} = 2 \pm 3 = 5 \text{ or } -1$

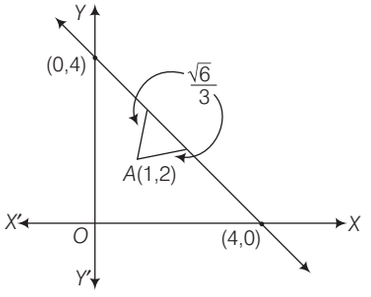
Hence, the coordinates of the points are $(7,5)$ and $(-1,-1)$.

Example 39. ***Find the direction in which a straight line must be drawn through the point*** $(1,2)$ ***so that its point of intersection with the line*** $x+y=4$ ***may be at a distance*** $\frac{1}{3}\sqrt{6}$ ***from this point.***

Sol. Let the straight line makes an angle θ with the positive direction of X-axis.

$\therefore$ Equation of the line through $(1,2)$ in parametric form is

$$\frac{x-1}{\cos\theta} = \frac{y-2}{\sin\theta} = \frac{1}{3}\sqrt{6}$$



or $x = 1 + \frac{\sqrt{6}}{3}\cos\theta$ and $y = 2 + \frac{\sqrt{6}}{3}\sin\theta$

Since, the point $\left(1 + \frac{\sqrt{6}}{3}\cos\theta, 2 + \frac{\sqrt{6}}{3}\sin\theta\right)$ lies on the line

$x + y = 4$

$\therefore$ $1 + \frac{\sqrt{6}}{3}\cos\theta + 2 + \frac{\sqrt{6}}{3}\sin\theta = 4$

$\Rightarrow$ $\frac{\sqrt{6}}{3}(\cos\theta + \sin\theta) = 1$

$\Rightarrow$ $\cos\theta + \sin\theta = \frac{3}{\sqrt{6}} = \frac{\sqrt{3}}{\sqrt{2}}$

$\Rightarrow$ $\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta = \frac{\sqrt{3}}{2}$

$\Rightarrow$ $\cos\left(\theta - \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{6}\right)$

$\theta - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{6}; n \in I$

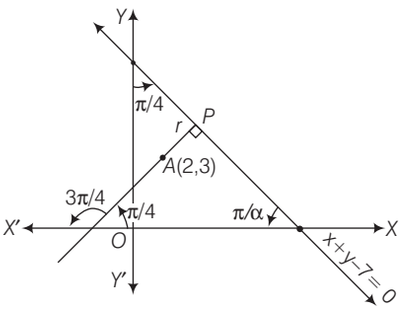
for $n = 0$, $\theta = \pm\frac{\pi}{6} + \frac{\pi}{4}$ $(\because 0 \le \theta < \pi)$

$= 15°, 75°$

Example 40. A line through $(2,3)$ makes an angle $\frac{3\pi}{4}$ with the negative direction of X-axis. Find the length of the line segment cut off between $(2,3)$ and the line $x+y-7=0$.

Sol. $\because$ Line makes an angle $\frac{3\pi}{4}$ with the negative direction of X-axis.

$\therefore$ Line makes an angle $\frac{\pi}{4}$ with the positive direction of X-axis.



$\therefore$ The equation of the line through $(1,2)$ in parametric form is

$$\frac{x-2}{\cos\left(\frac{\pi}{4}\right)} = \frac{y-3}{\sin\left(\frac{\pi}{4}\right)} = r$$

i.e. $\frac{x-2}{\frac{1}{\sqrt{2}}} = \frac{y-3}{\frac{1}{\sqrt{2}}} = r$ ...(i)

$\therefore$ $x = 2 + \frac{r}{\sqrt{2}}$ and $y = 3 + \frac{r}{\sqrt{2}}$

Let the line (i) meet the line $x + y - 7 = 0$ in P

$\therefore$ Coordinates of $P\left(2 + \frac{r}{\sqrt{2}}, 3 + \frac{r}{\sqrt{2}}\right)$ lies on $x + y - 7 = 0$

then $2 + \frac{r}{\sqrt{2}} + 3 + \frac{r}{\sqrt{2}} - 7 = 0$

or $\frac{2r}{\sqrt{2}} = 2$ or $r = \sqrt{2}$

$\therefore$ $AP = \sqrt{2}$

Example 41. Find the distance of the point $(2, 3)$ from the line $2x - 3y + 9 = 0$ measured along the line $2x - 2y + 5 = 0$.

Sol. Since, slope of the line $2x - 2y + 5 = 0$ is 1, its makes an angle $\frac{\pi}{4}$ with positive direction of X-axis.

The equation of the line through $(2, 3)$ and making an angle $\frac{\pi}{4}$ in parametric form

$$\frac{x-2}{\cos\left(\frac{\pi}{4}\right)} = \frac{y-3}{\sin\left(\frac{\pi}{4}\right)} = r \text{ or } \frac{x-2}{\frac{1}{\sqrt{2}}} = \frac{y-3}{\frac{1}{\sqrt{2}}} = r$$

Coordinates of any point on this line are $\left(2 + \frac{r}{\sqrt{2}}, 3 + \frac{r}{\sqrt{2}}\right)$.

This point lies on the line $2x - 3y + 9 = 0$

$$\Rightarrow \quad 2\left(2 + \frac{r}{\sqrt{2}}\right) - 3\left(3 + \frac{r}{\sqrt{2}}\right) + 9 = 0$$

$$\Rightarrow \quad -\frac{r}{\sqrt{2}} + 4 = 0$$

$$\therefore \quad r = 4\sqrt{2}$$

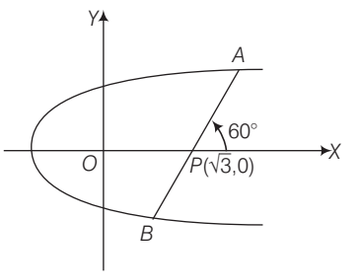
Example 42. If the line $y - \sqrt{3}x + 3 = 0$ cuts the parabola $y^2 = x + 2$ at A and B, then find the value of $PA.\ PB$ {where $P \equiv (\sqrt{3}, 0)$}.

Sol. Slope of line $y - \sqrt{3}x + 3 = 0$ is $\sqrt{3}$

If line makes an angle θ with X-axis, then $\tan\theta = \sqrt{3}$

$$\therefore \quad \theta = 60°$$

$$\frac{x - \sqrt{3}}{\cos 60°} = \frac{y - 0}{\sin 60°} = r$$



$$\Rightarrow \left(\sqrt{3} + \frac{r}{2}, \frac{r\sqrt{3}}{2}\right) \text{ be a point on the parabola } y^2 = x + 2$$

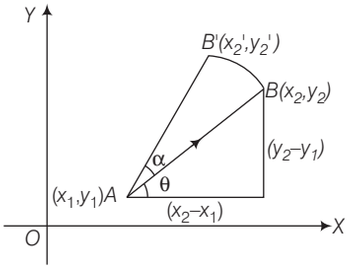
then, $\frac{3}{4}r^2 = \sqrt{3} + \frac{r}{2} + 2 \Rightarrow 3r^2 - 2r - 4(2 + \sqrt{3}) = 0$

$$\therefore \quad PA \cdot PB = r_1 r_2 = \left|\frac{-4(2+\sqrt{3})}{3}\right| = \frac{4(2+\sqrt{3})}{3}$$

Special Corollaries

(i) **Angle made by AB with positive X-axis (where A and B are given points)** :be two points and let AB makes an angle θ with the positive direction of X-axis and let d be the distance between A and B. Then Let $A(x_1, y_1)$ and $B(x_2, y_2)$

$$\cos\theta = \frac{x_2 - x_1}{d}, \sin\theta = \frac{y_2 - y_1}{d}$$



where, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = AB$

and $x_2 = x_1 + d\cos\theta, y_2 = y_1 + d\sin\theta$

If AB rotates an angle α about A, then new coordinates of B are

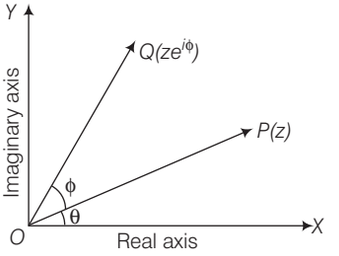
$$x_2' = x_1 + d\cos(\theta + \alpha),$$
$$y'_2 = y_1 + d\sin(\theta + \alpha)$$

and here, $AB = AB' = d$.

(ii) **Complex number as a rotating arrow in Argand plane** :

Let $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$, where $i = \sqrt{-1}$ ... (i)

be a complex number representing a point P in the Argand plane.



Then, $OP = |z| = r$ and $\angle POX = \theta$

Now, consider complex number $z_1 = ze^{i\phi}$

or $z_1 = re^{i\theta} \cdot e^{i\phi} = r \cdot e^{i(\theta+\phi)}$ [from Eq. (i)]

Clearly the complex number z_1 represents a point Q in the Argand plane, when

$$OQ = r \quad \text{and} \quad \angle QOX = \theta + \phi$$

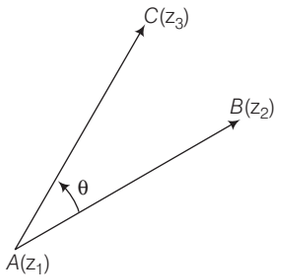
Clearly multiplication of z with $e^{i\phi}$ rotates the vector $\vec{OP}$ through angle ϕ in anti-clockwise sense. Similarly multiplication of z with $e^{-i\phi}$ will rotate the vector $\vec{OP}$ in clockwise sense.

Remark

If z_1, z_2 and z_3 are the affixes of the three points A, B and C such that $AC = AB$ and $\angle CAB = \theta$. Therefore

$$\vec{AB} = z_2 - z_1, \vec{AC} = z_3 - z_1$$

Then $\vec{AC}$ will be obtained by rotating $\vec{AB}$ through an angle θ in anticlockwise sense and therefore



$$\vec{AC} = \vec{AB}e^{i\theta}$$

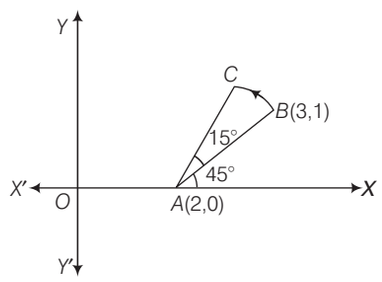
or $$(z_3 - z_1) = (z_2 - z_1)\, e^{i\theta} \text{ or } \left(\frac{z_3 - z_1}{z_2 - z_1}\right) = e^{i\theta}$$

Example 43. The line joining the points $A(2, 0)$ and $B(3, 1)$ is rotated about A in the anticlockwise direction through an angle of 15°. Find the equation of the line in the new position. If B goes to C in the new position, what will be the coordinates of C ?

Sol. By special corollary (i)

Here $$AB = \sqrt{(2-3)^2 + (0-1)^2} = \sqrt{2}$$

and slope of $$AB = \frac{1-0}{3-2} = 1 = \tan 45°$$



$$\therefore \quad \angle BAX = 45°$$

Now line AB is rotated through an angle of 15°

$$\Rightarrow \quad \angle CAX = 45° + 15° = 60°$$

and $$AB = AC = \sqrt{2}$$

Equation of line AC in parametric form is

$$\left.\begin{aligned} x &= 2 + r\cos 60° \\ y &= 0 + r\sin 60° \end{aligned}\right\} \quad \text{...(i)}$$

Since, $AC = r = \sqrt{2}$

Put $r = \sqrt{2}$ in Eq. (i), then

$$x = 2 + \sqrt{2}\cdot\frac{1}{2} = \frac{4+\sqrt{2}}{2}$$

and $$y = \sqrt{2}\,.\,\frac{\sqrt{3}}{2} = \frac{\sqrt{6}}{2}$$

Equation of the line AC is

$$\frac{x-2}{y} = \cot 60° = \frac{1}{\sqrt{3}}$$

or $$x\sqrt{3} - y - 2\sqrt{3} = 0$$

and coordinates of C are $\left(\frac{4+\sqrt{2}}{2}, \frac{\sqrt{6}}{2}\right)$.

Aliter (By special corollary (ii))

$\because$ $A \equiv (2, 0)$, $B \equiv (3, 1)$, let $C \equiv (x, y)$

$\therefore$ $z_A = 2, z_B = 3 + i,\ z_C = x + iy$, where $i = \sqrt{-1}$

$$\frac{z_C - z_A}{z_B - z_A} = e^{i\frac{5\pi}{12}} \qquad \left(\because 15° = \frac{5\pi}{12}\right)$$

$$\Rightarrow \quad z_C - 2 = (1+i)(\cos 15° + i\sin 15°)$$

or $$z_C = 2 + (1+i)\left(\frac{\sqrt{3}+1}{2\sqrt{2}} + i\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)\right)$$

$$= \left(2 + \frac{\sqrt{3}+1}{2\sqrt{2}} - \frac{\sqrt{3}-1}{2\sqrt{2}}\right) + i\left(\frac{\sqrt{3}+1}{2\sqrt{2}} + \frac{\sqrt{3}-1}{2\sqrt{2}}\right)$$

$$= \left(2 + \frac{1}{\sqrt{2}}\right) + i\left(\frac{\sqrt{3}}{\sqrt{2}}\right) = \frac{4+\sqrt{2}}{2} + i\left(\frac{\sqrt{6}}{2}\right)$$

$$\therefore \quad C \equiv \left(\frac{4+\sqrt{2}}{2}, \frac{\sqrt{6}}{2}\right)$$

and equation of AC

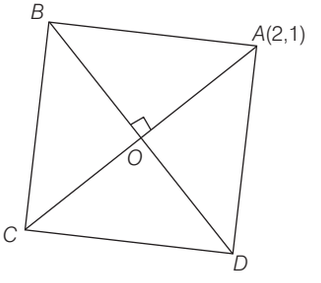
$$y - 0 = \tan 60°\,(x - 2) \Rightarrow x\sqrt{3} - y - 2\sqrt{3} = 0$$

Example 44. The centre of a square is at the origin and one vertex is $A(2, 1)$. Find the coordinates of other vertices of the square.

Sol. [By special corollary (ii)]

$\because$ $A \equiv (2, 1)$

$\therefore$ $z_A = 2 + i$, where $i = \sqrt{-1}$



Now, in triangle AOB,

$$OA = OB, \angle AOB = 90° = \frac{\pi}{2}$$

$\therefore$ $z_B = z_A e^{i\frac{\pi}{2}} = iz_A = 2i - 1$

$\therefore$ $B \equiv (-1, 2)$

$\because$ O is the mid-point of AC and BD

$\therefore$ $C \equiv (-2, -1)$ and $D \equiv (1, -2)$.

Example 45. The extremities of the diagonal of a square are $(1,1)$, $(-2,-1)$. Obtain the other two vertices and the equation of the other diagonal.

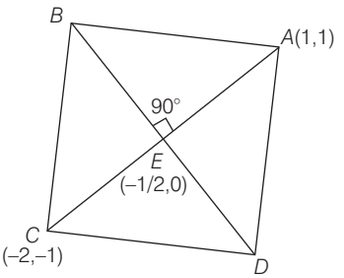
Sol. **(By special corollary (ii))**

$\because$ $A \equiv (1, 1)$

$\therefore$ $z_A = 1 + i$, where $i = \sqrt{-1}$

and $C \equiv (-2, -1)$

$\therefore$ $z_C = -2 - i$



then centre of square $E \equiv \left(-\frac{1}{2}, 0\right)$

$\therefore$ $z_E = -\frac{1}{2}$

Now, in ΔAEB, $(EA = EB)$

$$\frac{z_B - z_E}{z_A - z_E} = e^{i\frac{\pi}{2}} = i$$

$\Rightarrow$ $z_B + \frac{1}{2} = i\left(1 + i + \frac{1}{2}\right)$

$\therefore$ $z_B = -\frac{3}{2} + \frac{3}{2}i$

$B \equiv \left(-\frac{3}{2}, \frac{3}{2}\right)$

then $D \equiv \left(-1 + \frac{3}{2}, -\frac{3}{2}\right)$

or $D \equiv \left(\frac{1}{2}, -\frac{3}{2}\right)$

Hence, equation of other diagonal BD is

$$y - 0 = \frac{\frac{3}{2} - 0}{-\frac{3}{2} + \frac{1}{2}}\left(x + \frac{1}{2}\right)$$

$\Rightarrow$ $6x + 4y + 3 = 0$

Exercise for Session 1

1. The distance of the point (3, 5) from the line $2x + 3y - 14 = 0$ measured parallel to the line $x - 2y = 1$ is

(a) $\frac{7}{\sqrt{5}}$ (b) $\frac{7}{\sqrt{13}}$ (c) $\sqrt{5}$ (d) $\sqrt{13}$

2. The lines $x\cos\alpha + y\sin\alpha = p_1$ and $x\cos\beta + y\sin\beta = p_2$ will be perpendicular, if

(a) $\alpha = \beta$ (b) $|\alpha - \beta| = \pi/2$

(c) $\alpha = \pi/2$ (d) $\alpha \pm \beta = \pi/2$

3. If each of the points $(x_1, 4)$, $(-2, y_1)$ lies on the line joining the points $(2, -1)$, $(5, -3)$, then the point $P(x_1, y_1)$ lies on the line

(a) $6(x + y) - 25 = 0$ (d) $2x + 6y + 1 = 0$

(c) $2x + 3y - 6 = 0$ (d) $6(x + y) + 25 = 0$

4. The equation of the straight line passing through the point (4, 3) and making intercepts on the coordinate axes whose sum is –1 is

(a) $\frac{x}{2} + \frac{y}{3} = -1$ and $\frac{x}{-2} + \frac{y}{1} = -1$ (b) $\frac{x}{2} - \frac{y}{3} = -1$ and $\frac{x}{-2} + \frac{y}{1} = -1$

(c) $\frac{x}{2} + \frac{y}{3} = 1$ and $\frac{x}{-2} + \frac{y}{1} = 1$ (d) $\frac{x}{2} - \frac{y}{3} = 1$ and $\frac{x}{-2} + \frac{y}{1} = 1$

5. If the straight lines $ax + by + c = 0$ and $x\cos\alpha + y\sin\alpha = c$ enclose an angle $\pi/4$ between them and meet the straight line $x\sin\alpha - y\cos\alpha = 0$ in the same point, then

(a) $a^2 + b^2 = c^2$ (b) $a^2 + b^2 = 2$ (c) $a^2 + b^2 = 2c^2$ (d) $a^2 + b^2 = 4$

6. The angle between the lines $2x - y + 3 = 0$ and $x + 2y + 3 = 0$ is

(a) 30° (b) 45° (c) 60° (d) 90°

7. The inclination of the straight line passing through the point (–3, 6) and the mid-point of the line joining the points (4, –5) and (–2, 9) is

(a) $\pi/4$ (b) $\pi/2$ (c) $3\pi/4$ (d) π

8. A square of side a lies above the *X*-axis and has one vertex at the origin. The side passing through the origin makes an angle $\pi/6$ with the positive direction of *X*-axis. The equation of its diagonal not passing through the origin is

(a) $y(\sqrt{3} - 1) - x(1 - \sqrt{3}) = 2a$ (b) $y(\sqrt{3} + 1) + x(1 - \sqrt{3}) = 2a$

(c) $y(\sqrt{3} + 1) + x(1 + \sqrt{3}) = 2a$ (d) $y(\sqrt{3} + 1) + x(\sqrt{3} - 1) = 2a$

9. $A(1,3)$ and $C(7,5)$ are two opposite vertices of a square. The equation of side through *A* is

(a) $x + 2y - 7 = 0$ (b) $x - 2y + 5 = 0$

(c) $2x + y - 5 = 0$ (d) $2x - y + 1 = 0$

10. The equation of a straight line passing through the point $(-5, 4)$ and which cuts off an intercept of $\sqrt{2}$ units between the lines $x + y + 1 = 0$ and $x + y - 1 = 0$ is

(a) $x - 2y + 13 = 0$ (b) $2x - y + 14 = 0$

(c) $x - y + 9 = 0$ (d) $x - y + 10 = 0$

11. Equation to the straight line cutting off an intercept 2 from negative direction of the axis of *y* and inclined at 30° to the positive direction of axis of *x* is

(a) $y + x - \sqrt{3} = 0$ (b) $y - x + 2 = 0$

(c) $y - x\sqrt{3} - 2 = 0$ (d) $y\sqrt{3} - x + 2\sqrt{3} = 0$

12. What is the value of *y* so that the line through $(3, y)$ and (2, 7) is parallel to the line through (–1, 4) and (0, 6) ?

13. A straight line is drawn through the point $P(2,2)$ and is inclined at an angle of 30° with the *X*-axis. Find the coordinates of two points on it at a distance 4 from *P* on either side of *P*.

14. If the straight line through the point $P(3,4)$ makes an angle $\frac{\pi}{6}$ with *X*-axis and meets the line $12x + 5y + 10 = 0$ at *Q*, find the length of *PQ*.

15. Find the distance of the point (2, 3) from the line $2x - 3y + 9 = 0$ measured along the line $x - y + 1 = 0$.

16. A line is such that its segment between the straight line $5x - y - 4 = 0$ and $3x + 4y - 4 = 0$ is bisected at the point $(1,5)$. Obtain the equation.

17. The side *AB* and *AC* of a ΔABC are respectively $2x + 3y = 29$ and $x + 2y = 16$. If the mid-point of *BC* is (5, 6), then find the equation of *BC*.

18. A straight line through $A(-15, -10)$ meets the lines $x - y - 1 = 0$, $x + 2y = 5$ and $x + 3y = 7$ respectively at *A*, *B* and *C*. If $\frac{12}{AB} + \frac{40}{AC} = \frac{52}{AD}$, prove that the line passes through the origin.

Session 2

Position of Two Points Relative to a Given Line, Position of a Point Which Lies Inside a Triangle, Equations of Lines Parallel and Perpendicular to a Given Line, Distance of a Point From a Line, Distance Between Two Parallel Lines, Area of Parallelogram,

Position of Two Points Relative to a Given Line

Theorem : The points $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on the same or opposite sides of the line $ax + by + c = 0$ according as

$$\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c} > 0 \text{ or } < 0 .$$

Proof : Let the line PQ be divided by the line $ax + by + c = 0$ in the ratio $\lambda : 1$ (internally) at the point R.

$\therefore$ The coordinates of R are $\left(\frac{x_1 + \lambda x_2}{1+\lambda}, \frac{y_1 + \lambda y_2}{1+\lambda}\right)$.

The point of R lies on the line $ax + by + c = 0$

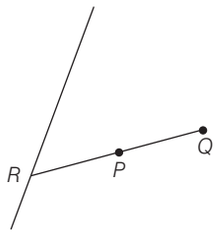
then $\quad a\left(\frac{x_1 + \lambda x_2}{1+\lambda}\right) + b\left(\frac{y_1 + \lambda y_2}{1+\lambda}\right) + c = 0$

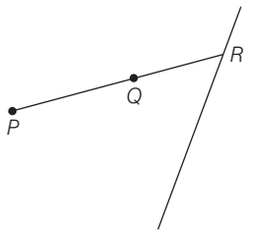
$\Rightarrow \quad \lambda(ax_2 + by_2 + c) + (ax_1 + by_1 + c) = 0$

$\Rightarrow \quad \lambda = -\left(\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}\right) \qquad (\because ax_2 + by_2 + c \neq 0)$

Case I : Let P and Q are on same side of the line $ax + by + c = 0$.

$\therefore$ R divides PQ externally.





$\therefore$ λ is negative

$\Rightarrow \quad -\left(\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}\right) < 0$

$\Rightarrow \quad \left(\frac{ax_1 + by_1 + c}{a x_2 + by_2 + c}\right) > 0$

or $\quad \frac{f(x_1, y_1)}{f(x_2, y_2)} > 0$

where, $f(x, y) \equiv ax + by + c$.

Case II : Let P and Q are on opposite sides of the line $ax + by + c = 0$

$\therefore$ R divides PQ internally.

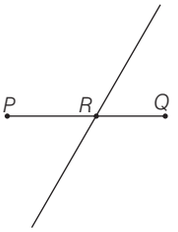
$\therefore$ λ is positive

$\Rightarrow \quad -\left(\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}\right) > 0$

$\Rightarrow \quad \left(\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}\right) < 0$

or $\quad \frac{f(x_1, y_1)}{f(x_2, y_2)} < 0$

where, $f(x, y) = ax + by + c$



Remarks

1. The side of the line where origin lies is known as origin side.
2. A point (α, β) will lie on origin side of the line $ax + by + c = 0$, if $a\alpha + b\beta + c$ and c have same sign.
3. A point (α, β) will lie on non-origin side of the line $ax + by + c = 0$, if $a\alpha + b\beta + c$ and c have opposite sign.

Example 46. Are the points $(2,1)$ and $(-3,5)$ on the same or opposite side of the line $3x-2y+1=0$?

Sol. Let $f(x,y)\equiv 3x-2y+1$

$$\therefore \quad \frac{f(2,1)}{f(-3,5)}=\frac{3(2)-2(1)+1}{3(-3)-2(5)+1}=-\frac{5}{18}<0$$

Therefore, the two points are on the opposite sides of the given line.

Example 47. Is the point $(2,-7)$ lies on origin side of the line $2x+y+2=0$?

Sol. Let $f(x,y)\equiv 2x+y+2$

$$\therefore \quad f(2,-7)=2(2)-7+2=-1$$

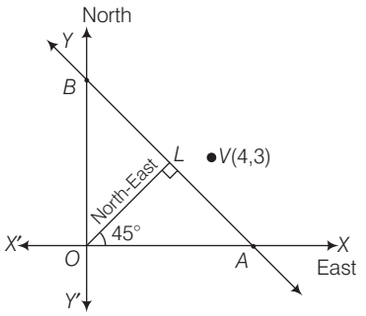
$f(2,-7)<0$ and constant $2>0$

Hence, the point $(2,-7)$ lies on non-origin side.

Example 48. A straight canal is at a distance of $4\frac{1}{2}$ km from a city and the nearest path from the city to the canal is in the north-east direction. Find whether a village which is at 3 km north and 4 km east from the city lies on the canal or not. If not, then on which side of the canal is the village situated ?

Sol. Let $O(0,0)$ be the given city and AB be the straight canal.

Given, $OL=\frac{9}{2}$ km



$\therefore$ Equation of AB

i.e. Equation of canal is

$$x\cos 45^\circ + y\sin 45^\circ = \frac{9}{2}$$

or
$$x+y=\frac{9}{\sqrt{2}} \qquad \text{...(i)}$$

Let V be the given village, then $V\equiv(4,3)$

Putting $x=4$ and $y=3$ in Eq. (i),

then $4+3=\frac{9}{\sqrt{2}}$, i.e. $7=\frac{9}{\sqrt{2}}$ which is impossible.

Hence, the given village V does not lie on the canal.

Also if $f(x,y)\equiv x+y-\frac{9}{\sqrt{2}}$

$$\therefore \quad \frac{f(4,3)}{f(0,0)}=\left(\frac{4+3-\frac{9}{\sqrt{2}}}{0+0-\frac{9}{\sqrt{2}}}\right)=-\left(\frac{7\sqrt{2}-9}{9}\right)<0$$

Hence, the village is on that side of the canal on which origin or the city lies.

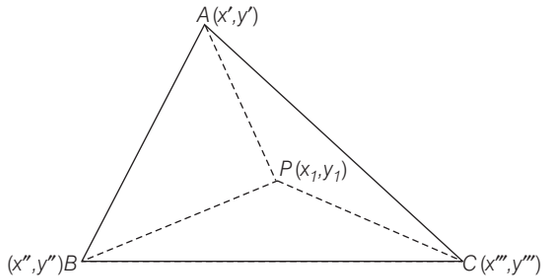
Position of a Point Which Lies Inside a Triangle

Let $P(x_1,y_1)$ be the point and equations of the sides of a triangle are

$$BC: a_1x+b_1y+c_1=0$$

$$CA: a_2x+b_2y+c_2=0$$

and
$$AB: a_3x+b_3y+c_3=0$$



First find the coordinates of A, B and C say,

$$A\equiv(x',y');\ B\equiv(x'',y'') \text{ and } C\equiv(x''',y''')$$

and if coordinates of A, B, C are given, then find equations of BC, CA and AB.

If $P(x_1,y_1)$ lies inside the triangle, then P and A must be on the same side of BC, P and B must be on the same side of AC, P and C must be on the same side of AB , then

$$\frac{a_1x_1+b_1y_1+c_1}{a_1x'+b_1y'+c_1}>0 \qquad \text{...(i)}$$

$$\frac{a_2x_1+b_2y_1+c_2}{a_2x''+b_2y''+c_2}>0 \qquad \text{...(ii)}$$

and
$$\frac{a_3x_1+b_3y_1+c_3}{a_3x'''+b_3y'''+c_3}>0 \qquad \text{...(iii)}$$

The required values of $P(x_1,y_1)$ must be intersection of these inequalities Eqs. (i), (ii) and (iii).

Aliter (Best Method) : First draw the exact diagram of the problem. If the point $P(x_1, y_1)$ move on the line $y = ax + b$ for all x_1, then

$$P \equiv (x_1, ax_1 + b)$$

and the portion DE of the line $y = ax + b$ (Excluding D and E) lies within the triangle. Now line $y = ax + b$ cuts any two sides out of three sides, then find coordinates of D and E.

$$D \equiv (\alpha, \beta)$$

and $E \equiv (\gamma, \delta)$ (say)

then $\alpha < x_1 < \gamma$

and $\beta < ax_1 + b < \delta$

Example 49. For what values of the parameter t does the point $P(t, t+1)$ lies inside the triangle ABC where $A \equiv (0, 3), B \equiv (-2, 0)$ and $C \equiv (6, 1)$.

Sol. Equations of sides

$$BC : x - 8y + 2 = 0$$

$$CA : x + 3y - 9 = 0$$

and $AB : 3x - 2y + 6 = 0$

Since, $P(t, t+1)$ lies inside the triangle ABC, then P and A must be on the same side of BC

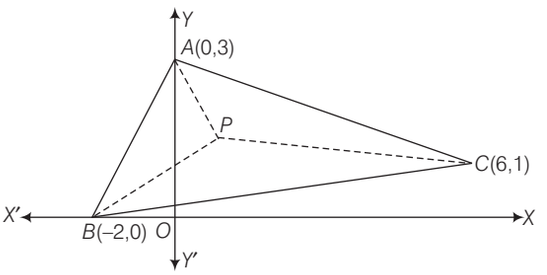
$$\therefore \quad \frac{\text{value of } (x - 8y + 2) \text{ at } P(t, t+1)}{\text{value of } (x - 8y + 2) \text{ at } A(0, 3)} > 0$$

i.e. $$\frac{t - 8(t+1) + 2}{0 - 24 + 2} > 0$$

or $$\frac{-7t - 6}{-22} > 0$$

or $$7t + 6 > 0$$

$$\therefore \quad t > -\frac{6}{7} \qquad \text{...(i)}$$



and P, B must be on the same side of CA

$$\therefore \quad \frac{\text{value of } (x + 3y - 9) \text{ at } P(t, t+1)}{\text{value of } (x + 3y - 9) \text{ at } B(-2, 0)} > 0$$

i.e. $$\frac{t + 3(t+1) - 9}{-2 + 0 - 9} > 0$$

or $$\frac{4t - 6}{-11} > 0$$

or $$4t - 6 < 0$$

$$\therefore \quad t < \frac{3}{2} \qquad \text{...(ii)}$$

and P, C must be on the same side of AB

$$\therefore \quad \frac{\text{value of } (3x - 2y + 6) \text{ at } P(t, t+1)}{\text{value of } (3x - 2y + 6) \text{ at } C(6, 1)} > 0$$

i.e. $$\frac{3t - 2(t+1) + 6}{18 - 2 + 6} > 0$$

or $$\frac{t + 4}{22} > 0$$

or $$t + 4 > 0$$

$$\therefore \quad t > -4 \qquad \text{...(iii)}$$

From Eqs. (i), (ii) and (iii), we get

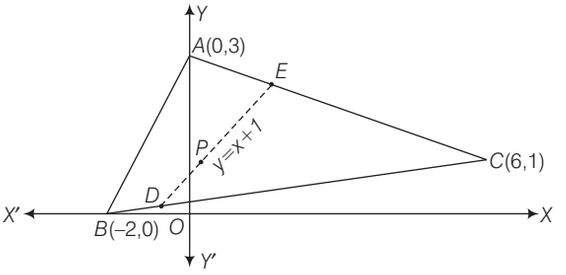
$$-\frac{6}{7} < t < \frac{3}{2}$$

i.e. $$t \in \left(-\frac{6}{7}, \frac{3}{2}\right)$$

Aliter : First draw the exact diagram of ΔABC, the point $P(t, t+1)$ move on the line

$$y = x + 1$$

for all t.



Now, D and E are the intersection of

$$y = x + 1, x - 8y + 2 = 0$$

and $y = x + 1, \quad x + 3y - 9 = 0$

respectively.

$$\therefore \quad D \equiv \left(-\frac{6}{7}, \frac{1}{7}\right)$$

and $$E \equiv \left(\frac{3}{2}, \frac{5}{2}\right)$$

Thus, the points on the line $y = x + 1$ whose x-coordinates lies between $-\frac{6}{7}$ and $\frac{3}{2}$ lie within the triangle ABC.

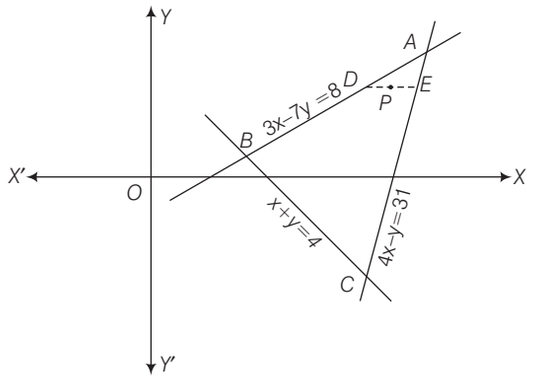
Hence, $$-\frac{6}{7} < t < \frac{3}{2}$$

i.e. $$t \in \left(-\frac{6}{7}, \frac{3}{2}\right)$$

Example 50. Find λ if $(\lambda, 2)$ is an interior point of ΔABC formed by $x + y = 4$, $3x - 7y = 8$ and $4x - y = 31$.

Sol. Let $P \equiv (\lambda, 2)$

First draw the exact diagram of ΔABC, the point $P(\lambda, 2)$ move on the line $y = 2$ for all λ.



Now, D and E are the intersection of

$$3x - 7y = 8, y = 2$$

and $4x - y = 31, y = 2$ respectively.

$$\therefore \quad D \equiv \left(\frac{22}{3}, 2\right) \text{ and } E \equiv \left(\frac{33}{4}, 2\right)$$

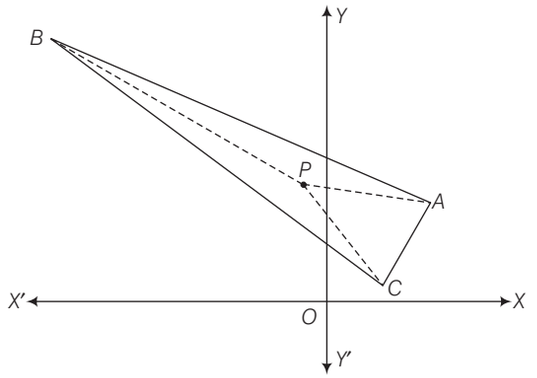
Thus, the points on the line $y = 2$ whose x-coordinates lies between $\frac{22}{3}$ and $\frac{33}{4}$ lie within the ΔABC.

Hence, $$\frac{22}{3} < \lambda < \frac{33}{4}$$

i.e. $$\lambda \in \left(\frac{22}{3}, \frac{33}{4}\right)$$

Example 51. Determine all values of α for which the point (α, α^2) lies inside the triangle formed by the lines $2x + 3y - 1 = 0$, $x + 2y - 3 = 0$ and $5x - 6y - 1 = 0$.

Sol. The coordinates of the vertices are



$$A\left(\frac{5}{4}, \frac{7}{8}\right), B(-7, 5) \text{ and } C\left(\frac{1}{3}, \frac{1}{9}\right)$$

$\because P(\alpha, \alpha^2)$ lies inside the ΔABC, then

(i) A and P must lie on the same side of BC

(ii) B and P must lie on the same side of CA

(iii) C and P must lie on the same side of AB, then

$$\frac{\frac{5}{2} + \frac{21}{8} - 1}{2\alpha + 3\alpha^2 - 1} > 0$$

$$\Rightarrow \quad \frac{33}{2\alpha + 3\alpha^2 - 1} > 0$$

or $$3\alpha^2 + 2\alpha - 1 > 0$$

$$\Rightarrow \quad (\alpha + 1)\left(\alpha - \frac{1}{3}\right) > 0$$

$$\Rightarrow \quad \alpha \in (-\infty, -1) \cup \left(\frac{1}{3}, \infty\right) \quad \text{...(i)}$$

and $$\frac{-35 - 30 - 1}{5\alpha - 6\alpha^2 - 1} > 0$$

$$\Rightarrow \quad 5\alpha - 6\alpha^2 - 1 < 0$$

$$\Rightarrow \quad \left(\alpha - \frac{1}{3}\right)\left(\alpha - \frac{1}{2}\right) > 0$$

$$\therefore \quad \alpha \in (-\infty, 1/3) \cup (1/2, \infty) \quad \text{..(ii)}$$

and $$\frac{\frac{1}{3} + \frac{2}{9} - 3}{\alpha + 2\alpha^2 - 3} > 0$$

$$\Rightarrow \quad \alpha + 2\alpha^2 - 3 < 0$$

$$\Rightarrow \quad (2\alpha + 3)(\alpha - 1) < 0$$

$$\therefore \quad \alpha \in (-3/2, 1) \quad \text{...(iii)}$$

From Eq. (i), Eq. (ii) and Eq. (iii), we get

$$\alpha \in (-3/2, -1) \cup (1/2, 1).$$

Aliter : Let $P(\alpha, \alpha^2)$ first draw the exact diagram of ΔABC.

The point $P(\alpha, \alpha^2)$ move on the curve $y = x^2$ for all α.

Now, intersection of $y = x^2$

and $$2x + 3y - 1 = 0$$

or $$2x + 3x^2 - 1 = 0$$

$$\therefore \quad x = -1, x = \frac{1}{3}$$

Let intersection points

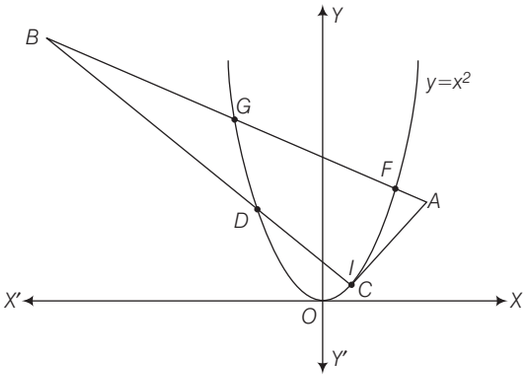
$$D \equiv (-1, 1) \text{ and } E \equiv \left(\frac{1}{3}, \frac{1}{9}\right)$$

intersection of $y = x^2$

and $$x + 2y - 3 = 0$$

or $$x + 2x^2 - 3 = 0$$

$$\therefore \quad x = 1, x = -3/2$$



Let intersection points

$$F \equiv (1, 1) \text{ and } G \equiv \left(-\frac{3}{2}, \frac{9}{4}\right)$$

and intersection of $y = x^2$ and $5x - 6y - 1 = 0$

or $\quad 5x - 6x^2 - 1 = 0$

$\therefore \quad x = \frac{1}{3}, x = \frac{1}{2}$

Let intersection points

$$H \equiv \left(\frac{1}{3}, \frac{1}{9}\right) \text{ and } I \equiv \left(\frac{1}{2}, \frac{1}{4}\right).$$

Thus the points on the curve $y = x^2$ whose x-coordinate lies between $-3/2$ & -1 and $\frac{1}{2}$ & 1 lies within the triangle ABC.

Hence, $\quad -\frac{3}{2} < \alpha < -1 \text{ or } \frac{1}{2} < \alpha < 1$

i.e. $\quad \alpha \in \left(-\frac{3}{2}, -1\right) \cup \left(\frac{1}{2}, 1\right)$

Equations of Lines Parallel and Perpendicular to a Given Line

Theorem 1: The equation of line parallel to $ax + by + c = 0$ is $ax + by + \lambda = 0$, where λ is some constant.

Proof : Let the equation of any line parallel to

$$ax + by + c = 0 \quad \text{...(i)}$$

be $\quad a_1 x + b_1 y + c_1 = 0 \quad \text{...(ii)}$

then $\quad \frac{a_1}{a} = \frac{b_1}{b} = k \quad \text{(say)}$

$\therefore \quad a_1 = ak, b_1 = bk$

Then from Eq. (ii),

$$akx + bky + c = 0$$

Dividing it by k, then

$$ax + by + \frac{c}{k} = 0$$

or $\quad ax + by + \lambda = 0 \quad \left(\text{writing } \lambda \text{ for } \frac{c}{k}\right)$

Hence, any line parallel to $ax + by + c = 0$ is

$$ax + by + \lambda = 0$$

where λ is some constant.

Aliter : The given line is

$$ax + by + c = 0 \quad \text{...(i)}$$

Its slope $= -\frac{a}{b}$

Thus, any line parallel to Eq. (i) is given by

$$y = \left(-\frac{a}{b}\right)x + \lambda_1$$

$\Rightarrow \quad ax + by - b\lambda_1 = 0$

$\Rightarrow \quad ax + by + \lambda = 0 \quad (\text{writing } \lambda \text{ for } -b\lambda_1)$

where, λ is some constant.

Corollary **:** The equation of the line parallel to $ax + by + c = 0$ and passing through (x_1, y_1) is

$$a(x - x_1) + b(y - y_1) = 0$$

Working Rule :

(i) Keep the terms containing x and y unaltered.

(ii) Change the constant.

(iii) The constant λ is determined from an additional condition given in the problem.

Theorem 2 : The equation of the line perpendicular to the line $ax + by + c = 0$ is

$bx - ay + \lambda = 0$, where λ is some constant.

Proof : Let the equation of any line perpendicular to

$$ax + by + c = 0 \quad \text{... (i)}$$

be $\quad a_1 x + b_1 y + c_1 = 0 \quad \text{...(ii)}$

then $\quad aa_1 + bb_1 = 0$

or $\quad aa_1 = -bb_1$

$\Rightarrow \quad \frac{a_1}{b} = \frac{b_1}{-a} = k \quad \text{(say)}$

$\therefore \quad a_1 = bk, b_1 = -ak$

Then, from Eq. (ii), $bkx - aky + c_1 = 0$ dividing it by k, then

$$bx - ay + \frac{c_1}{k} = 0$$

or $bx - ay + \lambda = 0$ $\left(\text{writing } \lambda \text{ for } \frac{c_1}{k}\right)$

Hence, any line perpendicular to $ax + by + c = 0$ is

$$bx - ay + \lambda = 0$$

where, λ is some constant.

Aliter : The given line is

$$ax + by + c = 0 \quad ...(i)$$

$$\text{Its slope} = -\frac{a}{b}$$

Slope of perpendicular line of Eq. (i) is $\frac{b}{a}$.

Thus any line perpendicular to Eq. (i) is given by

$$y = \left(\frac{b}{a}\right)x + \lambda_1$$

$\Rightarrow$ $bx - ay + a\lambda_1 = 0$

or $bx - ay + \lambda = 0$ (writing λ for $a\lambda_1$)

where, λ is some constant.

***Corollary* 1 :** The equation of the line through (x_1, y_1) and perpendicular to $ax + by + c = 0$ is

$$b(x - x_1) - a(y - y_1) = 0$$

***Corollary* 2 :** Also equation of the line perpendicular to $ax + by + c = 0$ is written as

$$\frac{x}{a} - \frac{y}{b} + k = 0, \text{ where } k \text{ is some constant.}$$

Working Rule :

(i) Interchange the coefficients of x and y and changing sign of one of these coefficients.

(ii) Changing the constant term.

(iii) The value of λ can be determined from an additional condition given in the problem.

Example 52. Find the general equation of the line which is parallel to $3x - 4y + 5 = 0$. Also find such line through the point $(-1, 2)$.

Sol. Equation of any parallel to $3x - 4y + 5 = 0$ is

$$3x - 4y + \lambda = 0 \quad ...(i)$$

which is general equation of the line.

Also Eq. (i) passes through $(-1, 2)$, then

$$3(-1) - 4(2) + \lambda = 0$$

$\therefore$ $\lambda = 11$

Then from Eq. (i) required line is

$$3x - 4y + 11 = 0$$

Example 53. Find the general equation of the line which is perpendicular to $x + y + 4 = 0$. Also find such line through the point $(1, 2)$.

Sol. Equation of any line perpendicular to $x + y + 4 = 0$ is

$$x - y + \lambda = 0 \quad ...(i)$$

which is general equation of the line.

Also Eq. (i) passes through $(1, 2)$, then

$$1 - 2 + \lambda = 0$$

$\therefore$ $\lambda = 1$

Then from Eq. (i), required line is

$$x - y + 1 = 0$$

Example 54. Show that the equation of the line passing through the point $(a\cos^3\theta, a\sin^3\theta)$ and perpendicular to the line

$x \sec\theta + y \operatorname{cosec}\theta = a$ is

$x\cos\theta - y\sin\theta = a\cos 2\theta$

Sol. The given equation $x \sec\theta + y \operatorname{cosec}\theta = a$ can be written as

$$x\sin\theta + y\cos\theta = a\sin\theta\cos\theta \quad ...(i)$$

$\therefore$ equation of perpendicular line of Eq. (i) is

$$x\cos\theta - y\sin\theta = \lambda \quad ...(ii)$$

Also it is pass through $(a\cos^3\theta, a\sin^3\theta)$

$\therefore$ $a\cos^3\theta\cdot\cos\theta - a\sin^3\theta\cdot\sin\theta = \lambda$

$\Rightarrow$ $\lambda = a(\cos^4\theta - \sin^4\theta)$

$= a(\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta)$

$= a\cdot 1\cdot\cos 2\theta = a\cos 2\theta$

From Eq. (ii), the required equation of the line is

$$x\cos\theta - y\sin\theta = a\cos 2\theta$$

Aliter : (From corollary (2) of Theorem (2)

Equation of any line perpendicular to the line

$x\sec\theta + y\operatorname{cosec}\theta = a$, is

$$\frac{x}{\sec\theta} - \frac{y}{\operatorname{cosec}\theta} = k$$

or $x\cos\theta - y\sin\theta = k \quad ...(iii)$

Also, it pass through $(a\cos^3\theta, a\sin^3\theta)$

$\therefore$ $a\cos^3\theta\cdot\cos\theta - a\sin^3\theta\cdot\sin\theta = k$

or $k = a(\cos^4\theta - \sin^4\theta)$

$= a(\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta)$

$= a\cdot 1\cdot\cos 2\theta$

$= a\cos 2\theta$

From Eq. (iii), the required equation of the line is

$$x\cos\theta - y\sin\theta = a\cos 2\theta$$

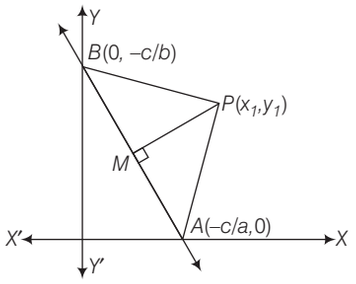
Distance of a Point From a Line

Theorem : The length of perpendicular from a point (x_1, y_1) to the line $ax + by + c = 0$ is

$$\frac{|ax_1 + by_1 + c|}{\sqrt{(a^2 + b^2)}}$$

Proof : Given line is $ax + by + c = 0$

$$\Rightarrow \quad \frac{x}{\left(-\frac{c}{a}\right)} + \frac{y}{\left(-\frac{c}{b}\right)} = 1$$



Let the given line intersects the X-axis and Y-axis at A and B respectively, then coordinates of A and B are $\left(-\frac{c}{a}, 0\right)$ and $\left(0, -\frac{c}{b}\right)$ respectively.

Draw PM perpendicular to AB.

Now, Area of ΔPAB

$$= \frac{1}{2}\left| x_1\left(0 + \frac{c}{b}\right) - \frac{c}{a}\left(-\frac{c}{b} - y_1\right) + 0\,(y_1 - 0)\right|$$

$$= \frac{1}{2}\left|\frac{c}{ab}\right| |ax_1 + by_1 + c| \qquad \ldots(i)$$

Let $PM = p$

Also, area of ΔPAB

$$= \frac{1}{2} \cdot AB \cdot PM = \frac{1}{2}\sqrt{\left\{\left(-\frac{c}{a} - 0\right)^2 + \left(0 + \frac{c}{b}\right)^2\right\}} \cdot p$$

$$= \frac{1}{2}\left|\frac{c}{ab}\right| \sqrt{(a^2 + b^2)} \cdot p \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we have

$$\frac{1}{2}\left|\frac{c}{ab}\right| \sqrt{(a^2 + b^2)} \cdot p = \frac{1}{2}\left|\frac{c}{ab}\right| |ax_1 + by_1 + c|$$

or $$p = \frac{|ax_1 + by_1 + c|}{\sqrt{(a^2 + b^2)}}$$

Aliter I : Let PM makes an angle θ with positive direction of X-axis.

Then, equation of PM in distance form will be

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = p \qquad (\because PM = p)$$

Therefore coordinates of M will be

$$(x_1 + p\cos\theta, y_1 + p\sin\theta)$$

Since, M lies on $ax + by + c = 0$, then

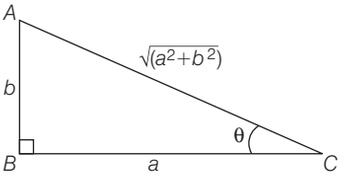
$$a\,(x_1 + p\cos\theta) + b\,(y_1 + p\sin\theta) + c = 0$$

or $$p\,(a\cos\theta + b\sin\theta) = -(ax_1 + by_1 + c) \qquad \ldots(iii)$$

Since, slope of $AB = -\frac{a}{b}$

$\therefore$ Slope of $PM = \frac{b}{a}$

$\therefore$ $\tan\theta = \frac{b}{a}$ ($\because$ PM makes an angle θ with positive direction of X-axis)



then $$\sin\theta = \frac{b}{\sqrt{(a^2 + b^2)}}$$

and $$\cos\theta = \frac{a}{\sqrt{(a^2 + b^2)}}$$

Now, from Eq. (iii),

$$p\left(\frac{a^2}{\sqrt{(a^2 + b^2)}} + \frac{b^2}{\sqrt{(a^2 + b^2)}}\right) = -(ax_1 + by_1 + c)$$

or $$p = -\frac{(ax_1 + by_1 + c)}{\sqrt{(a^2 + b^2)}}$$

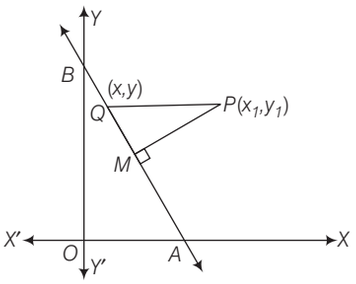
Since, p is positive

$\therefore$ $$p = \frac{|ax_1 + by_1 + c|}{\sqrt{(a^2 + b^2)}}$$

Aliter II : Let $Q\,(x, y)$ be any point on the line

$$ax + by + c = 0$$

Hence, the length of perpendicular from P on AB will be least value of PQ.



Let $z = (PQ)^2$

$= (x - x_1)^2 + (y - y_1)^2$ ...(iv)

$= (x - x_1)^2 + \left(-\frac{c}{b} - \frac{ax}{b} - y_1\right)^2$

$\left(\because ax + by + c = 0 \;\; \therefore y = -\frac{c}{b} - \frac{ax}{b}\right)$

$\therefore \quad \frac{dz}{dx} = 2(x - x_1) + 2\left(-\frac{c}{b} - \frac{ax}{b} - y_1\right)\left(-\frac{a}{b}\right)$

and $\frac{d^2z}{dx^2} = 2 + 2\left(-\frac{a}{b}\right)\left(-\frac{a}{b}\right) = 2\left(1 + \frac{a^2}{b^2}\right) = \text{positive}$

$\because$ z is minimum

$\therefore$ PQ is also minimum.

For maximum or minimum, $\frac{dz}{dx} = 0$

$2(x - x_1) + 2\left(-\frac{c}{b} - \frac{ax}{b} - y_1\right)\left(-\frac{a}{b}\right) = 0$

or $2(x - x_1) + 2(y - y_1)\left(-\frac{a}{b}\right) = 0 \quad \left(\because y = -\frac{c}{b} - \frac{ax}{b}\right)$

or $\frac{(x - x_1)}{a} = \frac{(y - y_1)}{b} = \frac{a(x - x_1) + b(y - y_1)}{a \cdot a + b \cdot b}$

$= \frac{(ax + by + c) - (ax_1 + by_1 + c)}{(a^2 + b^2)}$

(by law of proportion)

$= \frac{0 - (ax_1 + by_1 + c)}{(a^2 + b^2)} \quad (\because ax + by + c = 0)$

$\Rightarrow \quad (x - x_1) = -\frac{a(ax_1 + by_1 + c)}{(a^2 + b^2)}$

and $(y - y_1) = -\frac{b(ax_1 + by_1 + c)}{(a^2 + b^2)}$

$\therefore$ From Eq. (iv),

$PQ = \sqrt{(ax_1 + by_1 + c)^2 \frac{(a^2 + b^2)}{(a^2 + b^2)^2}} = \frac{|ax_1 + by_1 + c|}{\sqrt{(a^2 + b^2)}}$

$\because$ Least value of PQ is PM

$\therefore \quad p = PM = \frac{|ax_1 + by_1 + c|}{\sqrt{(a^2 + b^2)}}$

Aliter III : Let $M \equiv (h, k)$

Since, $M(h, k)$ lies on AB,

$ah + bk + c = 0$ ..(v)

Now, AB and PM are perpendicular to each other, then

(slope of PM) $\times$ (slope of AB) $= -1$

$\Rightarrow \quad \frac{y_1 - k}{x_1 - h} \times \left(-\frac{a}{b}\right) = -1$

$\Rightarrow \quad \frac{x_1 - h}{a} = \frac{y_1 - k}{b} = \frac{a(x_1 - h) + b(y_1 - k)}{a \cdot a + b \cdot b}$

(by law of proportion)

$= \frac{(ax_1 + by_1 + c) - (ah + bk + c)}{a^2 + b^2}$

$= \frac{ax_1 + by_1 + c}{a^2 + b^2}$ [from Eq. (v)]

$\therefore \quad (PM)^2 = (x - x_1)^2 + (y - y_1)^2$

$= \left(\frac{ax_1 + by_1 + c}{a^2 + b^2}\right)^2 (a^2 + b^2)$

$\therefore$ Length of perpendicular

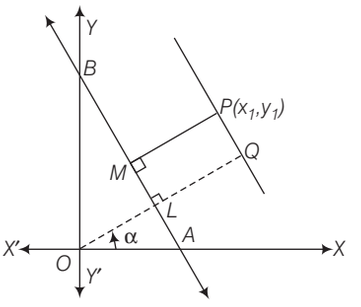
$PM = \pm \frac{(ax_1 + by_1 + c)}{\sqrt{(a^2 + b^2)}}$

Hence, $PM = p = \frac{|ax_1 + by_1 + c)}{\sqrt{(a^2 + b^2)}}$

Aliter IV : Equation of AB in normal form is

$\frac{a}{\sqrt{(a^2 + b^2)}} x + \frac{b}{\sqrt{(a^2 + b^2)}} y = \frac{-c}{\sqrt{(a^2 + b^2)}}$

$\Rightarrow \quad OL = -\frac{c}{\sqrt{(a^2 + b^2)}}$



Equation of line parallel to AB and passes through (x_1, y_1) is

$$a(x - x_1) + b(y - y_1) = 0$$

or $$ax + by = ax_1 + by_1$$

Normal form is

$$\frac{a}{\sqrt{(a^2 + b^2)}} x + \frac{b}{\sqrt{(a^2 + b^2)}} y = \frac{ax_1 + by_1}{\sqrt{(a^2 + b^2)}}$$

$$\Rightarrow \quad OQ = \frac{ax_1 + by_1}{\sqrt{(a^2 + b^2)}}$$

$$\therefore \quad PM = QL = OQ - OL = \frac{ax_1 + by_1 + c}{\sqrt{(a^2 + b^2)}}$$

Hence, required perpendicular distance

$$p = \frac{|ax_1 + by_1 + c|}{\sqrt{(a^2 + b^2)}}$$

Aliter V : The equation of line through $P(x_1, y_1)$ and perpendicular to $ax + by + c = 0$ is

$$b(x - x_1) - a(y - y_1) = 0 \qquad ...(vi)$$

If this perpendicular meet the line $ax + by + c = 0$ in $M(x_2, y_2)$ then (x_2, y_2) lie on both the lines $ax + by + c = 0$ and Eq. (vi), then

$b(x_2 - x_1) - a(y_2 - y_1) = 0, ax_2 + by_2 + c = 0$

$ax_2 + by_2 + c = a(x_2 - x_1) + b(y_2 - y_1) + ax_1 + by_1 + c = 0$

or $$b(x_2 - x_1) - a(y_2 - y_1) = 0 \qquad ...(vii)$$

and $$a(x_2 - x_1) + b(y_2 - y_1) = -(ax_1 + by_1 + c) \qquad ...(viii)$$

On squaring and adding Eqs. (vii) and (viii), we get

$$(a^2 + b^2)((x_2 - x_1)^2 + (y_2 - y_1)^2) = (ax_1 + by_1 + c)^2$$

or $$PM = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \frac{|ax_1 + by_1 + c|}{\sqrt{(a^2 + b^2)}}$$

Hence, length of perpendicular

$$PM = p = \frac{|ax_1 + by_1 + c|}{\sqrt{(a^2 + b^2)}}$$

***Corollary* 1 :** The length of perpendicular from the origin to the line $ax + by + c = 0$ is

$$\frac{|a \cdot 0 + b \cdot 0 + c|}{\sqrt{(a^2 + b^2)}} \quad \text{i.e.} \quad \frac{|c|}{\sqrt{(a^2 + b^2)}}$$

***Corollary* 2 :** The length of perpendicular from (x_1, y_1) to the line $x \cos\alpha + y \sin\alpha = p$ is

$$\frac{|x_1 \cos\alpha + y_1 \sin\alpha - p|}{\sqrt{(\cos^2\alpha + \sin^2\alpha)}} = |x_1 \cos\alpha + y_1 \sin\alpha - p|$$

Working Rule :

(i) Put the point (x_1, y_1) for (x, y) on the LHS while the RHS is zero.

(ii) Divide LHS after Eq. (i) by $\sqrt{(a^2 + b^2)}$, where a and b are the coefficients of x and y respectively.

Example 55. Find the sum of the abscissas of all the points on the line $x + y = 4$ that lie at a unit distance from the line $4x + 3y - 10 = 0$.

Sol. Any point on the line $x + y = 4$ can be taken as $(x_1, 4 - x_1)$. As it is at a unit distance from the line $4x + 3y - 10 = 0$, we get

$$\frac{|4x_1 + 3(4 - x_1) - 10|}{\sqrt{(4^2 + 3^2)}} = 1$$

$$\Rightarrow \quad |x_1 + 2| = 5 \Rightarrow x_1 + 2 = \pm 5$$

$$\Rightarrow \quad x_1 = 3 \quad \text{or} \quad -7$$

$\therefore$ Required sum $= 3 - 7 = -4$.

Example 56. If p and p' are the length of the perpendiculars from the origin to the straight lines whose equations are $x \sec\theta + y \operatorname{cosec}\theta = a$ and $x \cos\theta - y \sin\theta = a \cos 2\theta$, then find the value of $4p^2 + p'^2$.

Sol. We have, $p = \frac{|-a|}{\sqrt{(\sec^2\theta + \operatorname{cosec}^2\theta)}}$

$$\therefore \quad p^2 = \frac{a^2}{\sec^2\theta + \operatorname{cosec}^2\theta} = \frac{a^2 \sin^2\theta \cos^2\theta}{1}$$

$$\Rightarrow \quad 4p^2 = a^2 \sin^2 2\theta \qquad ...(i)$$

and $$p' = \frac{|-a \cos 2\theta|}{\sqrt{(\cos^2\theta + \sin^2\theta)}} = |-a \cos 2\theta|$$

$$\therefore \quad (p')^2 = a^2 \cos^2 2\theta \qquad ...(ii)$$

$\therefore$ Adding Eqs. (i) and (ii), we get

$$4p^2 + p'^2 = a^2$$

Example 57. ***If p is the length of the perpendicular from the origin to the line $\frac{x}{a} + \frac{y}{b} = 1$, then prove that $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2}$.***

Sol. p = length of perpendicular from origin to

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$= \frac{|0 + 0 - 1|}{\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2}} = \frac{1}{\sqrt{\left(\frac{1}{a^2} + \frac{1}{b^2}\right)}}$$

or $$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} \quad \text{or} \quad \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2}$$

Example 58. Prove that no line can be drawn through the point $(4, -5)$ so that its distance from $(-2, 3)$ will be equal to 12.

Sol. Suppose, if possible.

Equation of line through $(4, -5)$ with slope of m is

$$y + 5 = m(x - 4)$$

$$\Rightarrow \quad mx - y - 4m - 5 = 0$$

Then, $$\frac{|m(-2) - 3 - 4m - 5|}{\sqrt{m^2 + 1}} = 12$$

$$\Rightarrow \quad |-6m - 8| = 12\sqrt{(m^2 + 1)}$$

On squaring, $$(6m + 8)^2 = 144(m^2 + 1)$$

$$\Rightarrow \quad 4(3m + 4)^2 = 144(m^2 + 1)$$

$$\Rightarrow \quad (3m + 4)^2 = 36(m^2 + 1)$$

$$\Rightarrow \quad 27m^2 - 24m + 20 = 0 \qquad \text{...(i)}$$

Since, the discriminant of Eq. (i) is $(-24)^2 - 4 \cdot 27 \cdot 20 = -1584$ which is negative, there is no real value of m. Hence no such line is possible.

Distance between Two Parallel Lines

Let the two parallel lines be

$$ax + by + c = 0 \quad \text{and} \quad ax + by + c_1 = 0$$

The distance between the parallel lines is the perpendicular distance of any point on one line from the other line.

Let (x_1, y_1) be any point on $ax + by + c = 0$

$$\therefore \quad ax_1 + by_1 + c = 0 \qquad \text{...(i)}$$

Now, perpendicular distance of the point (x_1, y_1) from the line $ax + by + c_1 = 0$ is

$$\frac{|ax_1 + by_1 + c_1|}{\sqrt{(a^2 + b^2)}} = \frac{|c_1 - c|}{\sqrt{(a^2 + b^2)}} \quad \text{[from Eq. (i)]}$$

This is required distance between the given parallel lines.

Aliter I : The distance between the lines is

$$d = \frac{\lambda}{\sqrt{(a^2 + b^2)}}$$

(i) $\lambda = |c_1 - c|$, if both the lines are on the same side of the origin.

(ii) $\lambda = |c_1| + |c|$, if the lines are on the opposite side of the origin.

Aliter II : Find the coordinates of any point on one of the given lines, preferably putting $x = 0$ or $y = 0$. Then the perpendicular distance of this point from the other line is the required distance between the lines.

Example 59. Find the distance between the lines $5x - 12y + 2 = 0$ and $5x - 12y - 3 = 0$.

Sol. The distance between the lines

$5x - 12y + 2 = 0$ and $5x - 12y - 3 = 0$ is

$$\frac{|2 - (-3)|}{\sqrt{(5)^2 + (-12)^2}} = \frac{5}{13}$$

Aliter I : The constant term in both equations are 2 and -3 which are of opposite sign. Hence origin lies between them.

$\therefore$ Distance between lines is $\dfrac{|2| + |-3|}{\sqrt{(5)^2 + (-12)^2}} = \dfrac{5}{13}$

Aliter II : Putting $y = 0$ in $5x - 12y - 3 = 0$ then $x = \dfrac{3}{5}$

$\therefore \quad \left(\dfrac{3}{5}, 0\right)$ lie on $5x - 12y - 3 = 0$

Hence, distance between the lines

$5x - 12y + 2 = 0$ and $(5x - 12y - 3 = 0)$

= Distance from $\left(\dfrac{3}{5}, 0\right)$ to the line $5x - 12y + 2 = 0$

$$= \frac{\left|5 \times \frac{3}{5} - 0 + 2\right|}{\sqrt{5^2 + (-12)^2}} = \frac{5}{13}$$

Example 60. Find the equations of the line parallel to $5x - 12y + 26 = 0$ and at a distance of 4 units from it.

Sol. Equation of any line parallel to $5x - 12y + 26 = 0$ is

$$5x - 12y + \lambda = 0 \qquad \text{...(i)}$$

Since, the distance between the parallel lines is 4 units, then

$$\frac{|\lambda - 26|}{\sqrt{(5)^2 + (-12)^2}} = 4$$

or $\quad |\lambda - 26| = 52 \quad$ or $\quad \lambda - 26 = \pm 52$

or $\quad \lambda = 26 \pm 52 \quad \therefore \quad \lambda = -26$ or 78

Substituting the values of λ in Eq. (i), we get

$$5x - 12y - 26 = 0$$

and $\quad 5x - 12y + 78 = 0$

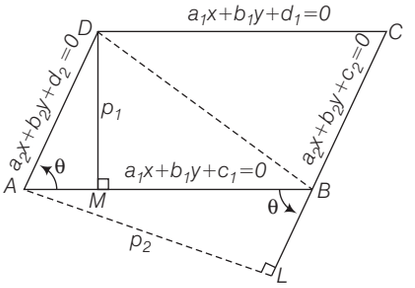
Area of Parallelogram

Theorem : Area of parallelogram $ABCD$ whose sides AB, BC, CD and DA are represented by $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_1x + b_1y + d_1 = 0$ and $a_2x + b_2y + d_2 = 0$ is

$$\frac{p_1 p_2}{\sin\theta} \quad \text{or} \quad \frac{|c_1 - d_1||c_2 - d_2|}{\left|\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\right|}$$

where, p_1 and p_2 are the distances between parallel sides and θ is the angle between two adjacent sides.

Proof : Since, p_1 and p_1 are the distances between the pairs of parallel sides of the parallelogram and θ is the angle between two adjacent sides, then



Area of parallelogram $ABCD$

$$= 2 \times \text{Area of } \Delta ABD$$

$$= 2 \times \frac{1}{2} \times AB \times p_1$$

$$= AB \times p_1$$

$$= \frac{p_2}{\sin\theta} \times p_1 \qquad \left(\because \text{in } \Delta ABL, \sin\theta = \frac{p_2}{AB} \right)$$

$$= \frac{p_1 p_2}{\sin\theta} \qquad \ldots(i)$$

Now, p_1 = Distance between parallel sides AB and DC

$$= \frac{|c_1 - d_1|}{\sqrt{(a_1^2 + b_1^2)}}$$

and p_2 = Distance between parallel sides AD and BC

$$= \frac{|c_2 - d_2|}{\sqrt{(a_2^2 + b_2^2)}}$$

Also, $\tan\theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right| = \left| \frac{-\frac{a_1}{b_1} - \left(-\frac{a_2}{b_2}\right)}{1 + \left(-\frac{a_1}{b_1}\right)\left(-\frac{a_2}{b_2}\right)} \right|$

$$\left(\because \; m_1 = \text{slope of } AB = -\frac{a_1}{b_1} \right.$$

$$\left. \text{and } m_2 = \text{slope of } AD = -\frac{a_2}{b_2} \right)$$

$$= \left| \frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_1 b_2} \right|$$

$$\therefore \qquad \sin\theta = \frac{|a_1 b_2 - a_2 b_1|}{\sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}}$$

Now, substitute the values of p_1, p_2 and $\sin\theta$ in Eq. (i)

$$\therefore \text{ Area of parallelogram } ABCD = \frac{|c_1 - d_1||c_2 - d_2|}{|a_1 b_2 - a_2 b_1|} = \frac{|c_1 - d_1||c_2 - d_2|}{\left| \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \right|}$$

Corollaries :

1. If $p_1 = p_2$, then $ABCD$ becomes a rhombus

$$\therefore \text{ Area of rhombus } ABCD = \frac{p_1^2}{\sin\theta} = \frac{(c_1 - d_1)^2}{|a_1 b_2 - a_2 b_1| \sqrt{\left(\frac{a_1^2 + b_1^2}{a_2^2 + b_2^2} \right)}}$$

2. If d_1 and d_2 are the lengths of two perpendicular diagonals of a rhombus, then

$$\text{Area of rhombus} = \frac{1}{2} d_1 d_2$$

3. Area of the parallelogram whose sides are $y = mx + a$, $y = mx + b$, $y = nx + c$ and $y = nx + d$ is $\frac{|a - b||c - d|}{|m - n|}$.

Example 61. Show that the area of the parallelogram formed by the lines $x + 3y - a = 0$, $3x - 2y + 3a = 0$, $x + 3y + 4a = 0$ and $3x - 2y + 7a = 0$ is $\frac{20}{11} a^2$ sq units.

Sol. Required area of the parallelogram

$$= \frac{|-a - 4a||3a - 7a|}{\left| \begin{vmatrix} 1 & 3 \\ 3 & -2 \end{vmatrix} \right|} = \frac{20}{11} a^2 \text{ sq units}$$

Example 62. Show that the area of the parallelogram formed by the lines

$$x\cos\alpha + y\sin\alpha = p,\ x\cos\alpha + y\sin\alpha = q,$$

$$x\cos\beta + y\sin\beta = r,\ x\cos\beta + y\sin\beta = s \text{ is}$$

$$|(p - q)(r - s)\,\text{cosec}\,(\alpha - \beta)|.$$

Sol. The equation of sides of the parallelogram are

$$x\cos\alpha + y\sin\alpha - p = 0,$$

$$x\cos\alpha + y\sin\alpha - q = 0,$$

$$x\cos\beta + y\sin\beta - r = 0,$$

and $$x\cos\beta + y\sin\beta - s = 0$$

$\therefore$ Required area of the parallelogram

$$= \frac{|-p-(-q)|\,|-r-(-s)|}{\left|\begin{vmatrix}\cos\alpha & \sin\alpha\\ \cos\beta & \sin\beta\end{vmatrix}\right|} = \frac{|p-q|\,|r-s|}{|\sin(\beta-\alpha)|}$$

$$= |(p-q)(r-s)\operatorname{cosec}(\alpha-\beta)|$$

Example 63. Prove that the diagonals of the parallelogram formed by the lines

$$\frac{x}{a}+\frac{y}{b}=1,\ \frac{x}{b}+\frac{y}{a}=1,\ \frac{x}{a}+\frac{y}{b}=2 \text{ and } \frac{x}{b}+\frac{y}{a}=2$$

are at right angles. Also find its area ($a \neq b$).

Sol. The distance between the parallel sides

$$\frac{x}{a}+\frac{y}{b}=1 \quad \text{and} \quad \frac{x}{a}+\frac{y}{b}=2$$

is $$\frac{|2-1|}{\sqrt{\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}} = \frac{1}{\sqrt{\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}} = p_1 \quad \text{(say)}$$

and the distance between the parallel sides

$$\frac{x}{b}+\frac{y}{a}=1 \quad \text{and} \quad \frac{x}{b}+\frac{y}{a}=2$$

is $$\frac{|2-1|}{\sqrt{\left(\frac{1}{b^2}+\frac{1}{a^2}\right)}} = \frac{1}{\sqrt{\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}} = p_2 \quad \text{(say)}$$

Here, $p_1 = p_2$.

$\therefore$ Parallelogram is a rhombus.

But we know that diagonals of rhombus are perpendicular to each other.

$\therefore$ Area of the rhombus $= \dfrac{|(-1+2)(-1+2)|}{\left|\begin{vmatrix}\frac{1}{a} & \frac{1}{b}\\ \frac{1}{b} & \frac{1}{a}\end{vmatrix}\right|}$

$$= \frac{a^2b^2}{|b^2-a^2|}(a \neq b)$$

Example 64. Show that the four lines $ax \pm by \pm c = 0$ enclose a rhombus whose area is $\dfrac{2c^2}{|ab|}$.

Sol. The given lines are

$$ax+by+c=0 \quad \text{...(i)}$$

$$ax+by-c=0 \quad \text{...(ii)}$$

$$ax-by+c=0 \quad \text{...(iii)}$$

and $$ax-by-c=0 \quad \text{...(iv)}$$

Distance between the parallel lines Eqs. (i) and (ii) is $\dfrac{2c}{\sqrt{(a^2+b^2)}} = p_1$ (say) and distance between the parallel lines Eqs. (iii) and (iv) is

$$\frac{2c}{\sqrt{(a^2+b^2)}} = p_2 \quad \text{(say)}$$

Here, $p_1 = p_2$

$\therefore$ it is a rhombus.

$\therefore$ Area of rhombus $= \dfrac{|(c+c)(c+c)|}{\left|\begin{vmatrix}a & b\\ a & -b\end{vmatrix}\right|} = \dfrac{4c^2}{|-2ab|} = \dfrac{2c^2}{|ab|}$

Exercise for Session 2

1. The number of lines that are parallel to $2x+6y-7=0$ and have an intercept 10 between the coordinate axes is

(a) 1 (b) 2 (c) 4 (d) infinitely many

2. The distance between the lines $4x+3y=11$ and $8x+6y=15$ is

(a) $\frac{7}{2}$ (b) $\frac{7}{5}$ (c) $\frac{7}{10}$ (d) $\frac{9}{10}$

3. If the algebraic sum of the perpendicular distances from the points (2, 0), (0, 2) and (1, 1) to a variable straight line is zero, then the line passes through the point

(a) (1, 1) (b) (–1, 1) (c) (–1, –1) (d) (1, –1)

4. If the quadrilateral formed by the lines $ax+by+c=0$, $a'x+b'y+c'=0$, $ax+by+c'=0$ and $a'x+b'y+c'=0$ have perpendicular diagonals, then

(a) $b^2+c^2=b'^2+c'^2$ (b) $c^2+a^2=c'^2+a'^2$ (c) $a^2+b^2=a'^2+b'^2$ (d) None of these

5. The area of the parallelogram formed by the lines $3x-4y+1=0$, $3x-4y+3=0$, $4x-3y-1=0$ and $4x-3y-2=0$, is

(a) $\frac{1}{7}$ sq units (b) $\frac{2}{7}$ sq units (c) $\frac{3}{7}$ sq units (d) $\frac{4}{7}$ sq units

6. Area of the parallelogram formed by the lines $y = mx$, $y = mx + 1$, $y = nx$ and $y = nx + 1$ equals

(a) $\frac{|m+n|}{(m-n)^2}$ (b) $\frac{2}{|m+n|}$ (c) $\frac{1}{|m+n|}$ (d) $\frac{1}{|m-n|}$

7. The coordinates of a point on the line $y = x$ where perpendicular distance from the line $3x + 4y = 12$ is 4 units, are

(a) $\left(\frac{3}{7}, \frac{5}{7}\right)$ (b) $\left(\frac{3}{2}, \frac{3}{2}\right)$ (c) $\left(-\frac{8}{7}, -\frac{8}{7}\right)$ (d) $\left(\frac{32}{7}, -\frac{32}{7}\right)$

8. A line passes through the point $(2, 2)$ and is perpendicular to the line $3x + y = 3$, then its y-intercept is

(a) $-\frac{2}{3}$ (b) $\frac{2}{3}$ (c) $-\frac{4}{3}$ (d) $\frac{4}{3}$

9. If the points (1, 2) and (3, 4) were to be on the opposite side of the line $3x - 5y + a = 0$, then

(a) $7 < a < 11$ (b) $a = 7$ (c) $a = 11$ (d) $a < 7$ or $a > 11$

10. The lines $y = mx$, $y + 2x = 0$, $y = 2x + k$ and $y + mx = k$ form a rhombus if m equals

(a) –1 (b) $\frac{1}{2}$ (c) 1 (d) 2

11. The points on the axis of x, whose perpendicular distance from the straight line $\frac{x}{a} + \frac{y}{b} = 1$ is a

(a) $\frac{b}{a}(a \pm \sqrt{(a^2 + b^2)}, 0)$ (b) $\frac{a}{b}(b \pm \sqrt{(a^2 + b^2)}, 0)$

(c) $\frac{b}{a}(a + b, 0)$ (d) $\frac{a}{b}(a \pm \sqrt{(a^2 + b^2)}, 0)$

12. The three sides of a triangle are given by $(x^2 - y^2)(2x + 3y - 6) = 0$. If the point $(-2, a)$ lies inside and $(b, 1)$ lies outside the triangle, then

(a) $a \in \left(2, \frac{10}{3}\right); b \in (-1, 1)$ (b) $a \in \left(-2, \frac{10}{3}\right); b \in \left(-1, \frac{9}{2}\right)$

(c) $a \in \left(1, \frac{10}{3}\right); b \in (-3, 5)$ (d) None of these

13. Are the points (3, 4) and (2, –6) on the same or opposite sides of the line $3x - 4y = 8$?

14. If the points (4, 7) and $(\cos\theta, \sin\theta)$, where $0 < \theta < \pi$, lie on the same side of the line $x + y - 1 = 0$, then prove that θ lies in the first quadrant.

15. Find the equations of lines parallel to $3x - 4y - 5 = 0$ at a unit distance from it.

16. Show that the area of the parallelogram formed by the lines $2x - 3y + a = 0, 3x - 2y - a = 0, 2x - 3y + 3a = 0$ and $3x - 2y - 2a = 0$ is $\frac{2a^2}{5}$ sq units.

17. A line 'L' is drawn from $P(4, 3)$ to meet the lines $L_1 : 3x + 4y + 5 = 0$ and $L_2 : 3x + 4y + 15 = 0$ at point A and B respectively. From 'A' a line, perpendicular to L is drawn meeting the line L_2 at A_1. Similarly from point 'B' a line, perpendicular to L is drawn meeting the line L_1 at B_1. Thus a parallelogram AA_1BB_1 is formed. Find the equation (s) of 'L' so that the area of the parallelogram AA_1BB_1 is least.

18. The vertices of a ΔOBC are $O(0, 0)$, $B(-3, -1)$, $C(-1, -3)$. Find the equation of the line parallel to BC and intersecting the sides OB and OC and whose perpendicular distance from the origin is $\frac{1}{2}$.

Session 3

Point of Intersection of Two Lines, Concurrent Lines Family of Lines, How to Find Circumcentre and Orthocentre by Slopes

Points of Intersection of Two Lines

Let $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ be two non-parallel lines. If (x_1, y_1) be the coordinates of their point of intersection,

then $a_1x_1 + b_1y_1 + c_1 = 0$ and $a_2x_1 + b_2y_1 + c_2 = 0$

Solving these two by cross multiplication, then

$$\frac{x_1}{b_1c_2 - b_2c_1} = \frac{y_1}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

we get $$(x_1, y_1) \equiv \left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$$

$$\equiv \left(\frac{\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}}, \frac{\begin{vmatrix} c_1 & c_2 \\ a_1 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}}\right)$$

Remarks

1. Here lines are not parallel, they have unequal slopes, then, $a_1b_2 - a_2b_1 \neq 0$
2. In solving numerical questions, we should not be remember the coordinates (x_1, y_1) given above, but we solve the equations directly.

Concurrent Lines

The three given lines are concurrent, if they meet in a point. Hence to prove that three given lines are concurrent, we proceed as follows :

I Method : Find the point of intersection of any two lines by solving them simultaneously. If this point satisfies the third equation also, then the given lines are concurrent.

II Method : The three lines $a_ix + b_iy + c_i = 0, i = 1, 2, 3$ are concurrent if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

III Method : The condition for the lines $P = 0, Q = 0$ and $R = 0$ to be concurrent is that three constants l, m, n (not all zeros at the same time) can be obtained such that

$$lP + mQ + nR = 0$$

Remarks

1. The reader is advised to follow method I in numerical problems.
2. For finding unknown quantity applying method II.

Example 65. Show that the lines

$2x + 3y - 8 = 0$, $x - 5y + 9 = 0$ and $3x + 4y - 11 = 0$

are concurrent.

Sol. **I Method :** Solving the first two equations, we see that their point of intersection is $(1, 2)$ which also satisfies the third equation

$$3 \times 1 + 4 \times 2 - 11 = 0$$

Hence the given lines are concurrent.

II Method : We have $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 8 \\ 1 & -5 & 9 \\ 3 & 4 & -11 \end{vmatrix}$

Applying $C_3 \to C_3 + C_1 + 2C_2$

$$= \begin{vmatrix} 2 & 3 & 0 \\ 1 & -5 & 0 \\ 3 & 4 & 0 \end{vmatrix} = 0$$

Hence the given lines are concurrent.

III Method : Suppose

$$l(2x + 3y - 8) + m(x - 5y + 9) + n(3x + 4y - 11) = 0$$

$$\Rightarrow \quad x(2l + m + 3n) + y(3l - 5m + 4n) + (-8l + 9m - 11n) = 0$$

$$= 0 \cdot x + 0 \cdot y + 0$$

On comparing,

$$2l + m + 3n = 0, 3l - 5m + 4n = 0, -8l + 9m - 11n = 0$$

After solving, we get $l = 19, m = 1, n = -13$

Hence, $19(2x + 3y - 8) + (x - 5y + 9) - 13(3x + 4y - 11) = 0$

Hence the given lines are concurrent.

Example 66. If the lines $ax+y+1=0$, $x+by+1=0$ and $x+y+c=0$ (a,b and c being distinct and different from 1) are concurrent, then find the value of $\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}$.

Sol. The given lines are concurrent, then

$$\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix}=0 \Rightarrow \begin{vmatrix} a & 1-a & 1-a \\ 1 & b-1 & 0 \\ 1 & 0 & c-1 \end{vmatrix}=0$$

(applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$)

Expanding along first row

$\Rightarrow \quad a(b-1)(c-1)-(1-a)(c-1)-(1-a)(b-1)=0$

$\Rightarrow \quad a(1-b)(1-c)+(1-a)(1-c)+(1-a)(1-b)=0$

Dividing by $(1-a)(1-b)(1-c)$, then

$$\frac{a}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}=0$$

$$\Rightarrow \quad -1+\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}=0$$

Hence, $$\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}=1$$

Example 67. Show that the three straight lines $2x-3y+5=0$, $3x+4y-7=0$ and $9x-5y+8=0$ meet in a point.

Sol. If we multiply these three equations by 3, 1 and -1, we have

$$3(2x-3y+5)+(3x+4y-7)-(9x-5y+8)=0$$

which is an identity.

Hence, three lines meet in a point.

Family of Lines

Theorem : Any line through the point of intersection of the lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ can be represented by the equation

$$(a_1x+b_1y+c_1)+\lambda(a_2x+b_2y+c_2)=0$$

where λ is a parameter which depends on the other property of line.

Proof : The equations of the lines are

$$a_1x+b_1y+c_1=0 \quad \text{...(i)}$$

and $$a_2x+b_2y+c_2=0 \quad \text{...(ii)}$$

Multiplying μ and ν in Eqs. (i) and (ii) and adding, we get

$$\mu(a_1x+b_1y+c_1)+\nu(a_2x+b_2y+c_2)=0$$

where μ, ν are any constants not both zero.

Dividing both sides by μ, then

$$(a_1x+b_1y+c_1)+\frac{\nu}{\mu}(a_2x+b_2y+c_2)=0$$

$$\Rightarrow (a_1x+b_1y+c_1)+\lambda(a_2x+b_2y+c_2)=0 \left(\text{where, } \lambda=\frac{\nu}{\mu}\right)$$

It is a first degree equation in x and y. So it represents family of lines through the point of intersection of Eqs. (i) and (ii).

Thus, the family of straight lines through the intersection of lines

$$L_1 \equiv a_1x+b_1y+c_1=0$$

and $$L_2 \equiv a_2x+b_2y+c_2=0 \text{ is}$$

$$(a_1x+b_1y+c_1)+\lambda(a_2x+b_2y+c_2)=0$$

i.e. $$L_1+\lambda L_2=0$$

Corollaries **:**

1. The equation $L_1+\lambda L_2=0$ or $\mu L_1+\nu L_2=0$ represent a line passing through the intersection of the lines $L_1=0$ and $L_2=0$ which is a fixed point, where λ, μ, ν are constants.
2. For finding fixed point, the number of constants in family of lines are one or two. If number of constants more than two, then convert in two or one constant form.

Example 68. Find the equation of the straight line passing through the point $(2,1)$ and through the point of intersection of the lines $x+2y=3$ and $2x-3y=4$.

Sol. Equation of any straight line passing through the intersection of the lines $x+2y=3$ and $2x-3y=4$ is

$$\lambda(x+2y-3)+(2x-3y-4)=0 \quad \text{...(i)}$$

Since, it passes through the point $(2, 1)$

$\therefore \quad \lambda(2+2-3)+(4-3-4)=0$

$\Rightarrow \quad \lambda-3=0$

$\therefore \quad \lambda=3$

Now, substituting this value of λ in (i), we get

$$3(x+2y-3)+(2x-3y-4)=0$$

i.e. $$5x+3y-13=0$$

which is the equation of required line.

Example 69. The family of lines $x(a+2b)+y(a+3b)=a+b$ passes through the point for all values of a and b. Find the point.

Sol. The given equation can be written as

$$a(x+y-1)+b(2x+3y-1)=0$$

which is equation of a line passing through the point of intersection of the lines $x+y-1=0$ and $2x+3y-1=0$. The point of intersection of these lines is $(2,-1)$. Hence the given family of lines passes through the point $(2,-1)$ for all values of a and b.

Example 70. If $3a+2b+6c=0$ the family of straight lines $ax+by+c=0$ passes through a fixed point. Find the coordinates of fixed point.

Sol. Given, $3a+2b+6c=0$

or $$\frac{a}{2}+\frac{b}{3}+c=0 \quad \text{...(i)}$$

and family of straight lines is

$$ax+by+c=0 \quad \text{...(ii)}$$

Subtracting Eqs. (i) from (ii), then

$$a\left(x-\frac{1}{2}\right)+b\left(y-\frac{1}{3}\right)=0$$

which is equation of a line passing through the point of intersection of the lines

$$x-\frac{1}{2}=0 \quad \text{and} \quad y-\frac{1}{3}=0$$

$\therefore$ The coordinates of fixed point are $\left(\frac{1}{2},\frac{1}{3}\right)$.

Example 71. If $4a^2+9b^2-c^2+12ab=0$, then the family of straight lines $ax+by+c=0$ is either concurrent at ... or at ....

Sol. Given, $4a^2+9b^2-c^2+12ab=0$

or $$(2a+3b)^2-c^2=0$$

or $$c=\pm(2a+3b) \quad \text{...(i)}$$

and family of straight lines is

$$ax+by+c=0 \quad \text{...(ii)}$$

Substituting the value of c from Eqs. (i) in (ii), then

$$ax+by\pm(2a+3b)=0$$

$\Rightarrow$ $$a(x\pm 2)+b(y\pm 3)=0$$

Taking '+' sign : $a(x+2)+b(y+3)=0$

which is equation of a line passing through the point of intersection of the lines $x+2=0$ and $y+3=0$

$\therefore$ coordinates of fixed point are $(-2,-3)$.

Taking '–' sign : $a(x-2)+b(y-3)=0$

which is equation of a line passing through the point of intersection of the lines

$$x-2=0 \quad \text{and} \quad y-3=0$$

$\therefore$ coordinates of fixed point are $(2,3)$

Hence, the family of straight lines $ax+by+c=0$ is either concurrent at $(-2,-3)$ or *at* $(2,3)$.

Example 72. Find the equation of the line passing through the point of intersection of the lines $x+5y+7=0, 3x+2y-5=0$

and (a) parallel to the line $7x+2y-5=0$

(b) perpendicular to the line $7x+2y-5=0$

Sol. Any line passing through the point of intersection of the given lines is

$$(x+5y+7)+\lambda(3x+2y-5)=0$$

$\Rightarrow$ $$x(1+3\lambda)+y(5+2\lambda)+(7-5\lambda)=0 \quad \text{...(i)}$$

Its slope $=-\dfrac{(1+3\lambda)}{(5+2\lambda)}$

(a) Line Eq. (i) is to be parallel to $7x+2y-5=0$

then $$-\frac{(1+3\lambda)}{(5+2\lambda)}=-\frac{7}{2}$$

$\Rightarrow$ $$2+6\lambda=35+14\lambda$$

$\Rightarrow$ $$8\lambda=-33$$

$\Rightarrow$ $$\lambda=-\frac{33}{8}$$

Substituting this value of λ in Eq. (i), we get the required equation as $7x+2y-17=0$

(b) Line, (i) is to be perpendicular to $7x+2y-5=0$

$\therefore$ $$-\frac{(1+3\lambda)}{(5+2\lambda)}\times-\left(\frac{7}{2}\right)=-1$$

or $$7+21\lambda=-10-4\lambda$$

$\therefore$ $$\lambda=-\frac{17}{25}$$

Substituting this value of λ is Eq. (i), we get the required equation as

$$2x-7y-20=0.$$

Aliter :

The point of intersection of the given lines

$x+5y-7=0$ and $3x+2y-5=0$ is $(3,-2)$.

$\therefore$ Equation of line through $(3,-2)$ is

$$y+2=m(x-3) \quad \text{...(ii)}$$

(a) Line (ii) is parallel to $7x+2y-5=0$

$\therefore$ $$m=-\frac{7}{2}$$

Hence, the equation of the required line is

$$y+2=-\frac{7}{2}(x-3)$$

or $$7x+2y-17=0$$

(b) Line (ii) is perpendicular to $7x+2y-5=0$

then $$m\times\left(-\frac{7}{2}\right)=-1$$

or $$m=\frac{2}{7}$$

Hence, the equation of the required line is

$$y+2=\frac{2}{7}(x-3)$$

or $$2x-7y-20=0$$

Example 73. Find the equation of straight line which passes through the intersection of the straight lines

$$3x - 4y + 1 = 0 \quad \textit{and} \quad 5x + y - 1 = 0$$

and cuts off equal intercepts from the axes.

Solution : Equation of any line passing through the intersection of the given lines is

$$(3x - 4y + 1) + \lambda (5x + y - 1) = 0$$

$$\Rightarrow \quad x(3 + 5\lambda) + y(-4 + \lambda) + (1 - \lambda) = 0 \qquad ...(i)$$

$$\Rightarrow \quad \frac{x}{\left(\frac{\lambda - 1}{3 + 5\lambda}\right)} + \frac{y}{\left(\frac{\lambda - 1}{\lambda - 4}\right)} = 1$$

but given x-intercept = y-intercept

i.e. $$\left(\frac{\lambda - 1}{3 + 5\lambda}\right) = \left(\frac{\lambda - 1}{\lambda - 4}\right)$$

$$\Rightarrow \quad \frac{1}{3 + 5\lambda} = \frac{1}{\lambda - 4}$$

$(\lambda \neq 1 \because$ if $\lambda = 1$ then line (i) pass through origin)

$$\therefore \quad \lambda - 4 = 3 + 5\lambda$$

or $$4\lambda = -7$$

$$\therefore \quad \lambda = -\frac{7}{4}$$

Substituting the value of λ in Eq. (i), we get required equation is $23x + 23y = 11$.

Example 74. If t_1 and t_2 are roots of the equation $t^2 + \lambda t + 1 = 0$, where λ is an arbitrary constant. Then prove that the line joining the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ always passes through a fixed point. Also find that point.

Sol. $\because t_1$ and t_2 are the roots of the equation $t^2 + \lambda t + 1 = 0$

$$\therefore \quad t_1 + t_2 = -\lambda \quad \text{and} \quad t_1 t_2 = 1 \qquad ...(i)$$

Equation of the line joining the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is

$$y - 2at_1 = \frac{2at_2 - 2at_1}{at_2^2 - at_1^2}(x - at_1^2)$$

$$\Rightarrow \quad y - 2at_1 = \frac{2}{(t_2 + t_1)}(x - at_1^2)$$

$$\Rightarrow \quad y(t_1 + t_2) - 2at_1 t_2 - 2at_1^2 = 2x - 2at_1^2$$

$$\Rightarrow \quad y(t_1 + t_2) - 2at_1 t_2 = 2x$$

$$\Rightarrow \quad y(-\lambda) - 2a = 2x \qquad \text{[from Eq. (i)]}$$

or $$(x + a) + \lambda\left(\frac{y}{2}\right) = 0$$

which is equation of a line passing through the point of intersection of the lines $x + a = 0$ and $\frac{y}{2} = 0$.

$\therefore$ coordinates of fixed point are $(-a, 0)$.

Example 75. A variable straight line through the point of intersection of the lines $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{b} + \frac{y}{a} = 1$ meets the coordinate axes in A and B. Show that the locus of the mid-point of AB is the curve $2xy(a + b) = ab(x + y)$.

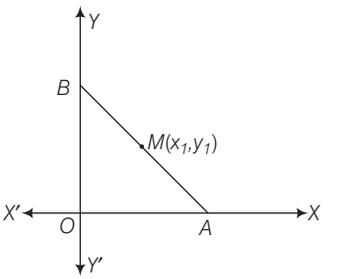
Sol. Any line through the point of intersection of given lines is

$$\left(\frac{x}{a} + \frac{y}{b} - 1\right) + \lambda\left(\frac{x}{b} + \frac{y}{a} - 1\right) = 0$$

$$x\left(\frac{1}{a} + \frac{\lambda}{b}\right) + y\left(\frac{1}{b} + \frac{\lambda}{a}\right) = (1 + \lambda)$$

$$\Rightarrow \quad x\left(\frac{b + a\lambda}{ab}\right) + y\left(\frac{a + b\lambda}{ab}\right) = (1 + \lambda)$$

$$\Rightarrow \quad \frac{x}{\left\{\frac{ab(1 + \lambda)}{b + a\lambda}\right\}} + \frac{y}{\left\{\frac{ab(1 + \lambda)}{a + b\lambda}\right\}} = 1$$



This meets the X-axis at

$$A \equiv \left(\frac{ab(1 + \lambda)}{b + a\lambda}, 0\right)$$

and meets the Y-axis at

$$B \equiv \left(0, \frac{ab(1 + \lambda)}{a + b\lambda}\right)$$

Let the mid-point of AB is $M(x_1, y_1)$, then

$$x_1 = \frac{ab(1 + \lambda)}{2(b + a\lambda)} \quad \text{and} \quad y_1 = \frac{ab(1 + \lambda)}{2(a + b\lambda)}$$

$$\therefore \quad \frac{1}{x_1} + \frac{1}{y_1} = \frac{2(b + a\lambda)}{ab(1 + \lambda)} + \frac{2(a + b\lambda)}{ab(1 + \lambda)}$$

$$= \frac{2}{ab(1 + \lambda)}(b + a\lambda + a + b\lambda)$$

$$= \frac{2}{ab(1 + \lambda)}(a + b)(1 + \lambda)$$

$$\Rightarrow \quad \frac{(x_1 + y_1)}{x_1 y_1} = \frac{2(a + b)}{ab}$$

$$\Rightarrow \quad 2x_1 y_1 (a + b) = ab(x_1 + y_1)$$

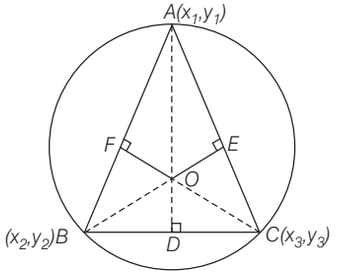
Hence, the locus of mid point of AB is

$$2xy(a + b) = ab(x + y).$$

How to Find Circumcentre and Orthocentre by Slopes

(i) Circumcentre

The circumcentre of a triangle is the point of intersection of the perpendicular bisectors of the sides of a triangle. It is the centre of the circle which passes through the vertices of the triangle and so its distance from the vertices of the triangle is the same and this distance is known as the **circumradius** of the triangle.



Let $O(x, y)$ be the circumcentre.

If D, E and F are the mid points of BC, CA and AB respectively and $OD \perp BC, OE \perp CA$ and $OF \perp AB$

$\therefore$ slope of $OD \times$ slope of $BC = -1$

and slope of $OE \times$ slope of $CA = -1$

and slope of $OF \times$ slope of $AB = -1$

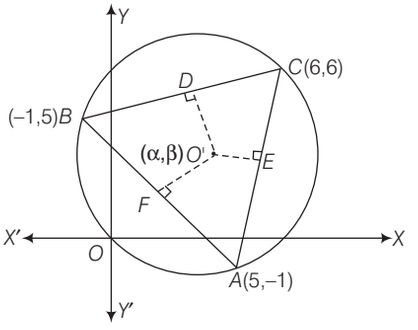
Solving any two, we get (x, y).

Example 76. Find the coordinates of the circumcentre of the triangle whose vertices are $A(5, -1), B(-1, 5)$ and $C(6, 6)$. Find its radius also.

Sol. Let circumcentre be $O'(\alpha, \beta)$ and mid points of sides BC, CA and AB are $D\left(\frac{5}{2}, \frac{11}{2}\right)$, $E\left(\frac{11}{2}, \frac{5}{2}\right)$ and $F(2, 2)$ respectively. Since $O'D \perp BC$.

$\therefore$ Slope of $O'D \times$ slope of $BC = -1$

$$\Rightarrow \quad \frac{\beta - \frac{11}{2}}{\alpha - \frac{5}{2}} \times \frac{6-5}{6+1} = -1$$



$$\Rightarrow \quad \frac{2\beta - 11}{7(2\alpha - 5)} = -1$$

$$\Rightarrow \quad 2\beta - 11 = -14\alpha + 35$$

$$\Rightarrow \quad 14\alpha + 2\beta = 46$$

$$\therefore \quad 7\alpha + \beta = 23 \qquad \ldots(i)$$

and $O'E \perp CA$

$\therefore$ Slope of $O'E \times$ Slope of $CA = -1$

$$\Rightarrow \quad \frac{\beta - \frac{5}{2}}{\alpha - \frac{11}{2}} \times \frac{-1-6}{5-6} = -1$$

$$\Rightarrow \quad \frac{2\beta - 5}{2\alpha - 11} \times \frac{7}{1} = -1$$

$$\Rightarrow \quad 14\beta - 35 = -2\alpha + 11$$

$$\therefore \quad 2\alpha + 14\beta = 46$$

$$\therefore \quad \alpha + 7\beta = 23 \qquad \ldots(ii)$$

Solving Eqs. (i) and (ii), we get

$$\alpha = \beta = \frac{23}{8}$$

$\therefore$ Circumcentre $= \left(\frac{23}{8}, \frac{23}{8}\right)$

$\therefore$ Circumradius $= O'A = O'B = O'C$

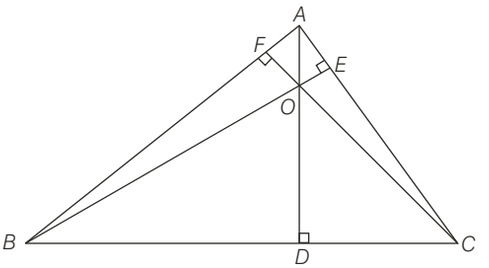
$$= O'C = \sqrt{(\alpha - 6)^2 + (\beta - 6)^2}$$

$$= \sqrt{\left(\frac{23}{8} - 6\right)^2 + \left(\frac{23}{8} - 6\right)^2}$$

$$= \sqrt{\left(\frac{25}{8}\right)^2 + \left(\frac{25}{8}\right)^2} = \frac{25\sqrt{2}}{8} \text{ units.}$$

(ii) Orthocentre

The orthocentre of a triangle is the point of intersection of altitudes.



Here O is the orthocentre since $AD \perp BC, BE \perp CA$ and $CF \perp AB$, then $OA \perp BC, OB \perp CA$, and $OC \perp AB$

Solving any two we can get coordinates of O.

Remarks

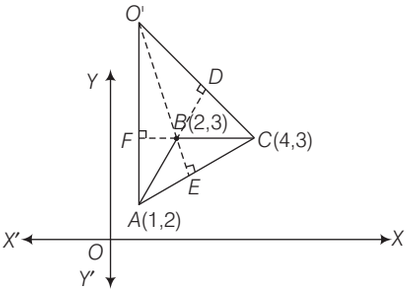
1. If any two lines out of three lines i.e. AB, BC and CA are perpendicular, then orthocentre is the point of intersection of two perpendicular lines.
2. Firstly find the slope of lines BC, CA and AB.

Example 77. Find the orthocentre of the triangle formed by the lines $xy = 0$ and $x + y = 1$.

Sol. Three sides of the triangle are $x = 0$, $y = 0$ and $x + y = 1$. The coordinates of the vertices are $O(0, 0)$, $A(1, 0)$ and $B(0, 1)$. The triangle OAB is a right angled triangle having right angle at O. Therefore $O(0, 0)$ is the orthocentre. Since we know that the point of intersection of two perpendicular lines is the orthocentre of the triangle OAB.

Example 78. Find the orthocentre of the triangle ABC whose angular points are $A(1, 2)$, $B(2, 3)$ and $C(4, 3)$..

Sol. Now, Slope of $BC = \dfrac{3-3}{4-2} = 0$



Slope of $CA = \dfrac{2-3}{1-4} = \dfrac{1}{3}$

and Slope of $AB = \dfrac{3-2}{2-1} = 1$

Let orthocentre be $O'(\alpha, \beta)$ then

Slope of $O'A \times$ slope of $BC = -1$

$$\frac{2-\beta}{1-\alpha} \times 0 = -1$$

$$\Rightarrow \quad \frac{0}{1-\alpha} = -1$$

$$\Rightarrow \quad 1 - \alpha = 0$$

$$\therefore \quad \alpha = 1$$

and Slope of $OB \times$ slope of $CA = -1$

$$\Rightarrow \quad \frac{3-\beta}{2-\alpha} \times \frac{1}{3} = -1$$

$$\Rightarrow \quad 3 - \beta = 3\alpha - 6$$

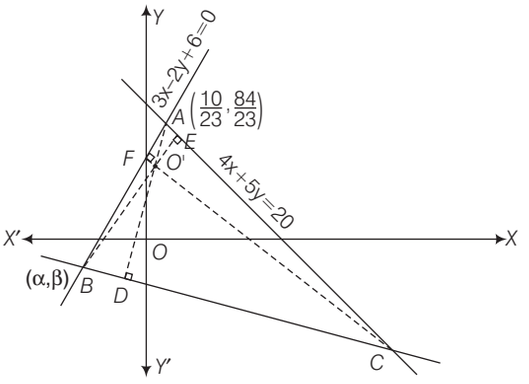
$$\Rightarrow \quad 3\alpha + \beta = 9$$

$$\therefore \quad \beta = 6 \qquad (\because \alpha = 1)$$

Hence, orthocentre of the given triangle is $(1, 6)$.

Example 79. The equations of two sides of a triangle are $3x - 2y + 6 = 0$ and $4x + 5y = 20$ and the orthocentre is $(1, 1)$. Find the equation of the third side.

Sol. Let $3x - 2y + 6 = 0$ and $4x + 5y = 20$ are the equations of the sides AB and AC. The point of intersection of AB and AC is $\left(\dfrac{10}{23}, \dfrac{84}{23}\right)$. Let slope of BC is m. Since $O'A \perp BC$



$\therefore$ Slope of $O'A \times$ Slope of $BC = -1$

$$\Rightarrow \quad \frac{\frac{84}{23} - 1}{\frac{10}{23} - 1} \times m = -1$$

$$\Rightarrow \quad \frac{61}{-13} m = -1$$

$$\therefore \quad m = \frac{13}{61}$$

Let the vertex B is (α, β).

(α, β) lies on $3x - 2y + 6 = 0$

$$\therefore \quad 3\alpha - 2\beta + 6 = 0 \qquad \text{...(i)}$$

and $O'B \perp AC$

$\therefore$ Slope of $O'B \times$ slope of $AC = -1$

$$\frac{\beta - 1}{\alpha - 1} \times \left(-\frac{4}{5}\right) = -1$$

$$\Rightarrow \quad 4\beta - 4 = 5\alpha - 5$$

$$\Rightarrow \quad 5\alpha - 4\beta - 1 = 0 \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get

$$\alpha = -13 \quad \text{and} \quad \beta = -\frac{33}{2}$$

Since, third side passes through $\left(-13, -\dfrac{33}{2}\right)$ with slope $\dfrac{13}{61}$, therefore its equation is

$$y + \frac{33}{2} = \frac{13}{61}(x + 13)$$

$$\Rightarrow \quad 122y + 33 \times 61 = 26x + 2 \times 169$$

$$\Rightarrow \quad 26x - 122y - 1675 = 0$$

Aliter : The equation of line through A. i.e. point of intersection of AB and AC is

$$(3x - 2y + 6) + \lambda\,(4x + 5y - 20) = 0 \qquad \text{...(i)}$$

it passes through $(1, 1)$, then

$$(3 - 2 + 6) + \lambda\,(4 + 5 - 20) = 0$$

$$\Rightarrow \quad 7 - 11\lambda = 0$$

$$\therefore \quad \lambda = \frac{7}{11}$$

From Eq. (i), $(3x-2y+6)+\frac{7}{11}(4x+5y-20)=0$

$\Rightarrow \quad 61x+13y-74=0$

$\therefore \quad$ Slope of $AD=-\frac{61}{13}$

$\Rightarrow \quad$ Slope of $BC=\frac{13}{61}$

If coordinates of $B(\alpha,\beta)$, B lies on AB

$\therefore \quad 3\alpha-2\beta+6=0 \quad ...(ii)$

and $O'B\perp CA$

then $\frac{\beta-1}{\alpha-1}\times\left(-\frac{4}{5}\right)=-1$

$\Rightarrow \quad 5\alpha-4\beta-1=0 \quad ...(iii)$

Solving Eqs. (ii) and (iii), we get

$$\alpha=-13 \quad \text{and} \quad \beta=-\frac{33}{2}$$

$\therefore$ Equation of third side i.e. BC is

$$y+\frac{33}{2}=\frac{13}{61}(x+13)$$

$\therefore \quad 26x-122y-1675=0$

Example 80. If the orthocentre of the triangle formed by the lines $2x+3y-1=0$, $x+2y-1=0$, $ax+by-1=0$ is at origin, then find (a,b).

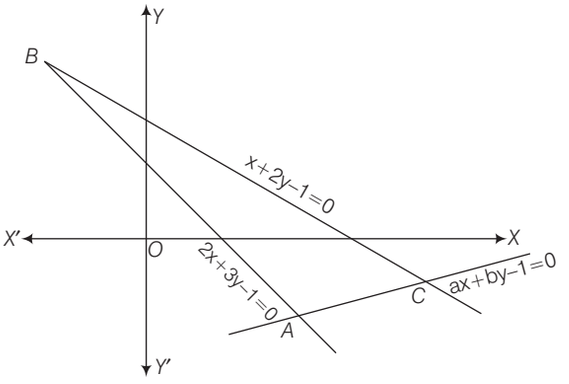
Sol. The equation of a line through A i.e. the point of intersection of AB and AC, is

$$(2x+3y-1)+\lambda(ax+by-1)=0 \quad ...(i)$$

It passes through $O(0,0)$, then

$$-1-\lambda=0$$

$\therefore \quad \lambda=-1$



From Eq. (i),

$$2x+3y-1-ax-by+1=0$$

$\Rightarrow \quad (2-a)x+(3-b)y=0$

Since, $AD\perp BC$

$\therefore \quad -\frac{(2-a)}{(3-b)}\times\left(-\frac{1}{2}\right)=-1$

$\Rightarrow \quad 2-a=-6+2b$

$\Rightarrow \quad a+2b=8 \quad ...(ii)$

Similarly, $BE\perp AC$, we get

$$a+b=0 \quad ...(iii)$$

Solving Eqs. (ii) and (iii), we get

$$b=8 \quad \text{and} \quad a=-8$$

$\therefore \quad (a,b)$ is $(-8,8)$.

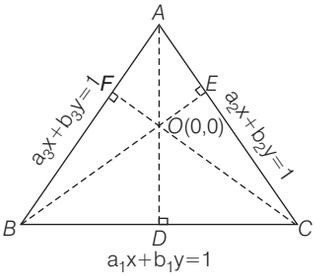
Example 81. If the equations of the sides of a triangle are $a_r x+b_r y=1; r=1,2,3$ and the orthocentre is the origin, then prove that

$$a_1a_2+b_1b_2=a_2a_3+b_2b_3=a_3a_1+b_3b_1.$$

Sol. The equation of the line through A, i.e. the point of intersection of AB and AC is

$$(a_2x+b_2y-1)+\lambda(a_3x+b_3y-1)=0 \quad ...(i)$$

It passes through $(0,0)$, then



$$-1-\lambda=0$$

$\therefore \quad \lambda=-1$

From Eq. (i), $a_2x+b_2y-1-a_3x-b_3y+1=0$

$\therefore \quad (a_2-a_3)x+(b_2-b_3)y=0$

Since, $AD\perp BC$

$\therefore$ Slope of $AD\times$ slope of $BC=-1$

$$-\frac{(a_2-a_3)}{(b_2-b_3)}\times\left(-\frac{a_1}{b_1}\right)=-1$$

$\Rightarrow \quad a_1a_2-a_3a_1=-b_1b_2+b_1b_3$

$\Rightarrow \quad a_1a_2+b_1b_2=a_3a_1+b_3b_1 \quad ...(ii)$

Similarly, $BE\perp CA$, then we get

$$a_1a_2+b_1b_2=a_2a_3+b_2b_3 \quad ...(iii)$$

From Eqs. (ii) and (iii), we get

$$a_1a_2+b_1b_2=a_2a_3+b_2b_3=a_3a_1+b_3b_1$$

Exercise for Session 3

1. The locus of the point of intersection of lines $x\cos\alpha + y\sin\alpha = a$ and $x\sin\alpha - y\cos\alpha = b$ (α is a parameter) is

(a) $2(x^2 + y^2) = a^2 + b^2$ (b) $x^2 - y^2 = a^2 - b^2$

(c) $x^2 + y^2 = a^2 + b^2$ (d) $x^2 - y^2 = a^2 + b^2$

2. If a, b, c are in AP then $ax + by + c = 0$ represents

(a) a straight line (b) a family of concurrent lines

(c) a family of parallel lines (d) None of these

3. If the lines $x + 2ay + a = 0$, $x + 3by + b = 0$ and $x + 4cy + c = 0$ are concurrent, then a, b, c are in

(a) AP (b) GP

(c) HP (d) AGP

4. The set of lines $ax + by + c = 0$, where $3a + 2b + 4c = 0$ is concurrent at the point

(a) $\left(\frac{3}{4}, \frac{1}{2}\right)$ (b) $\left(\frac{1}{2}, \frac{3}{4}\right)$

(c) $\left(-\frac{3}{4}, -\frac{1}{2}\right)$ (d) $\left(-\frac{1}{2}, -\frac{3}{4}\right)$

5. If the lines $ax + y + 1 = 0$, $x + by + 1 = 0$ and $x + y + c = 0$, (a, b and c being distinct and different from 1) are concurrent, then the value of $\frac{a}{a-1} + \frac{b}{b-1} + \frac{c}{c-1}$ is

(a) –2 (b) –1

(c) 1 (d) 2

6. If $u \equiv a_1x + b_1y + c_1 = 0$ and $v \equiv a_2x + b_2y + c_2 = 0$ and $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, then $u + kv = 0$ represents

(a) $u = 0$ (b) a family of concurrent lines

(c) a family of parallel lines (d) None of these

7. The straight lines $x + 2y - 9 = 0$, $3x + 5y - 5 = 0$ and $ax + by - 1 = 0$ are concurrent, if the straight line $35x - 22y + 1 = 0$ passes through the point

(a) (a, b) (b) (b, a)

(c) $(a, -b)$ (d) $(-a, b)$

8. If the straight lines $x + y - 2 = 0$, $2x - y + 1 = 0$ and $ax + by - c = 0$ are concurrent, then the family of lines $2ax + 3by + c = 0$ (a, b, c are non-zero) is concurrent at

(a) (2, 3) (b) $\left(\frac{1}{2}, \frac{1}{3}\right)$

(c) $\left(-\frac{1}{6}, -\frac{5}{9}\right)$ (d) $\left(\frac{2}{3}, -\frac{7}{5}\right)$

9. The straight line through the point of intersection of $ax + by + c = 0$ and $a'x + b'y + c' = 0$ are parallel to Y-axis has the equation

(a) $x(ab' - a'b) + (cb' - c'b) = 0$ (b) $x(ab' + a'b) + (cb' + c'b) = 0$

(c) $y(ab' - a'b) + (c'a - ca') = 0$ (d) $y(ab' + a'b) + (c'a + ca') = 0$

10. If the equations of three sides of a triangle are $x + y = 1, 3x + 5y = 2$ and $x - y = 0$, then the orthocentre of the triangle lies on the line/lines

(a) $5x - 3y = 1$ (b) $5y - 3x = 1$

(c) $2x - 3y = 1$ (d) $5x - 3y = 2$

11. Find the equation of the line through the intersection of $2x - 3y + 4 = 0$ and $3x + 4y - 5 = 0$ and perpendicular to $6x - 7y + c = 0$

(a) $199y + 120x = 125$ (b) $199y - 120x = 125$

(c) $119x + 102y = 125$ (d) $119x - 102y = 125$

12. The locus of the point of intersection of the lines $\frac{x}{a} - \frac{y}{b} = m, \frac{x}{a} + \frac{y}{b} = \frac{1}{m}$ is

(a) a circle (b) an ellipse

(c) a hyperbola (d) a parabola

13. Find the condition on a and b, such that the portion of the line $ax + by - 1 = 0$, intercepted between the lines $ax + y + 1 = 0$ and $x + by = 0$ subtends a right angled at the origin.

14. If the lines $(a - b - c)x + 2ay + 2a = 0, 2bx + (b - c - a)y + 2b = 0$ and $(2c + 1)x + 2cy + c - a - b = 0$ are concurrent, then prove that either $a + b + c = 0$ or $(a + b + c)^2 + 2a = 0$.

15. Prove that the lines $ax + by + c = 0, bx + cy + a = 0$ and $cx + ay + b = 0$ are concurrent if $a + b + c = 0$ or $a + b\omega + c\omega^2 = 0$ or $a + b\omega^2 + c\omega = 0$, where ω is a complex cube root of unity.

16. Find the equation of the straight line which passes through the intersection of the lines $x - y - 1 = 0$ and $2x - 3y + 1 = 0$ and is parallel to (i) X-axis (ii) Y-axis (iii) $3x + 4y = 14$.

17. Let a, b, c be parameters. Then, the equation $ax + by + c = 0$ will represent a family of straight lines passing through a fixed-point, if there exists a linear relation between a, b and c.

18. Prove that the family of lines represented by $x(1 + \lambda) + y(2 - \lambda) + 5 = 0, \lambda$ being arbitrary, pass through a fixed point. Also find the fixed point.

19. Prove that $\left(-a, -\frac{a}{2}\right)$ is the orthocentre of the triangle formed by the lines $y = m_i x + \frac{a}{m_i}, i = 1, 2, 3; m_1, m_2, m_3$ being the roots of the equation $x^3 - 3x^2 + 2 = 0$.

Session 4

Equations of Straight Lines Passing Through a Given Point and Making a Given Angle with a Given Line, A Line Equally Inclined With Two Lines, Equation of the Bisectors, Bisector of the Angle Containing the Origin, Equation of that Bisector of the Angle Between Two Lines which Contains a Given Point, How to Distinguish the Acute (Internal) and Obtuse (External) Angle Bisectors

Equations of Straight Lines Passing Through a Given Point and Making a Given Angle with a Given Line

Theorem : Prove that the equations of the straight lines which pass through a given point (x_1, y_1) and make a given angle α with the given straight line $y = mx + c$ are

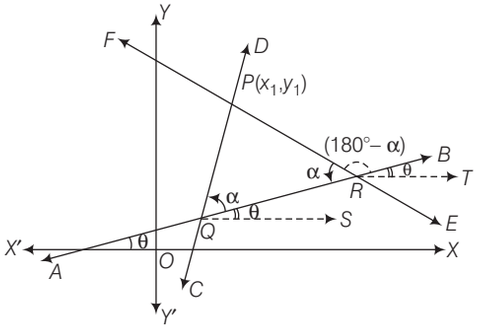
$$y - y_1 = \tan(\theta \pm \alpha)(x - x_1)$$

where, $m = \tan\theta$.

Proof : Let AB be the given line which makes an angle θ with X-axis.

$\therefore$ $m = \tan\theta$

Let CD and EF are two required lines which make angle α with the given line. Let these lines meet the given line AB at Q and R respectively



$\therefore$ $\angle DQS = \angle PQR + \angle RQS = (\alpha + \theta)$

$\therefore$ Equation of line CD is

$$y - y_1 = \tan(\theta + \alpha)(x - x_1) \quad \ldots(i)$$

and $\angle FRT = \angle FRB + \angle BRT$

$$= 180° - \alpha + \theta$$

$$= 180° + (\theta - \alpha)$$

$\therefore$ Equation of line EF is

$$y - y_1 = \tan(180° + \theta - \alpha)(x - x_1)$$

or $$y - y_1 = \tan(\theta - \alpha)(x - x_1) \quad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$y - y_1 = \tan(\theta \pm \alpha)(x - x_1)$$

These are the equations of the two required lines.

Example 82. Find the equations of the straight lines passing through the point $(2, 3)$ and inclined at $\pi/4$ radians to the line $2x + 3y = 5$.

Sol. Let the line $2x + 3y = 5$ make an angle θ with positive X-axis.

Then $$\tan\theta = -\frac{2}{3}$$

Now $$\tan\theta \cdot \tan\frac{\pi}{4} = -\frac{2}{3} \times 1$$

$$= -\frac{2}{3} \neq \pm 1$$

Slopes of required lines are

$$\tan\left(\theta + \frac{\pi}{4}\right) \text{ and } \tan\left(\theta - \frac{\pi}{4}\right)$$

$$\therefore \quad \tan\left(\theta+\frac{\pi}{4}\right)=\frac{\tan\theta+\tan\left(\frac{\pi}{4}\right)}{1-\tan\theta\tan\left(\frac{\pi}{4}\right)}=\frac{\left(-\frac{2}{3}\right)+1}{1-\left(-\frac{2}{3}\right)(1)}=\frac{1}{5}$$

and $$\tan\left(\theta-\frac{\pi}{4}\right)=\frac{\tan\theta-\tan\left(\frac{\pi}{4}\right)}{1+\tan\theta\tan\left(\frac{\pi}{4}\right)}$$

$$=\frac{\left(-\frac{2}{3}\right)-1}{1+\left(-\frac{2}{3}\right)(1)}=-5$$

$\therefore$ Equations of required lines are

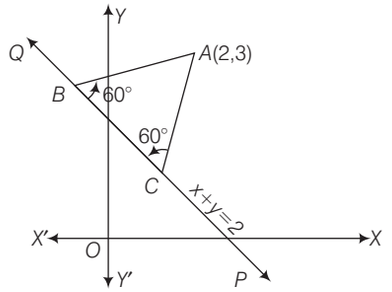
$$y-3=\frac{1}{5}(x-2) \text{ and } y-3=-5(x-2)$$

i.e. $x-5y+13=0$ and $5x+y-13=0$

Example 83. A vertex of an equilateral triangle is $(2, 3)$ and the opposite side is $x+y=2$. Find the equations of the other sides.

Sol. Let $A(2,3)$ be one vertex and $x+y=2$ be the opposite side of an equilateral triangle. Clearly remaining two sides pass through the point $A(2,3)$ and make an angle $60°$ with $x+y=2$

$\because$ Slope of $x+y=2$ is -1



Let $\tan\theta=-1$

$\therefore \quad \theta=135°$

$\therefore$ Equations of the other two sides are

$$y-3=\tan(135°\pm60°)(x-2)$$

i.e. sides are

$y-3=\tan(195°)(x-2)$ (taking '+' sign)

$\Rightarrow \quad y-3=\tan(180°+15°)(x-2)$

$\Rightarrow \quad y-3=\tan 15°(x-2)$

$\Rightarrow \quad y-3=(2-\sqrt{3})(x-2)$

$\Rightarrow \quad (2-\sqrt{3})x-y=1-2\sqrt{3}$

and $y-3=\tan(75°)(x-2)$ (taking '–' sign)

$\Rightarrow \quad y-3=\cot 15°(x-2)$

$\Rightarrow \quad y-3=(2+\sqrt{3})(x-2)$

$\Rightarrow \quad (2+\sqrt{3})x-y=1+2\sqrt{3}$

Hence, equations of other sides are

$$(2-\sqrt{3})x-y=1-2\sqrt{3}$$

and $$(2+\sqrt{3})x-y=1+2\sqrt{3}$$

Example 84. The straight lines $3x+4y=5$ and $4x-3y=15$ intersect at a point A. On these lines, the points B and C are chosen so that $AB=AC$. Find the possible equations of the line BC passing through the point $(1,2)$.

Sol. Clearly $\angle BAC=90°$

$\because \quad AB=AC$

$\therefore \quad \angle ABC=\angle BCA=45°$

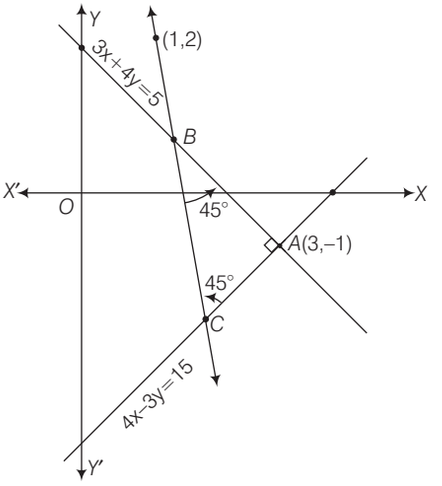
$\alpha=45°$

$\because$ Slope of $3x+4y=5$ is $-\frac{3}{4}$

Let $\tan\theta=-\frac{3}{4}$

So, possible equations of BC are given by

$$y-2=\tan(\theta\pm\alpha)(x-1)$$



$$\Rightarrow \quad y-2=\left(\frac{\tan\theta\pm\tan\alpha}{1\mp\tan\theta\tan\alpha}\right)(x-1)$$

$$\Rightarrow \quad y-2=\left(\frac{-\frac{3}{4}\pm1}{1\mp\left(-\frac{3}{4}\right)(1)}\right)(x-1)$$

$$\Rightarrow \quad y-2=\left(\frac{-3\pm4}{4\mp(-3)}\right)(x-1)$$

$$\Rightarrow \quad y-2=\frac{1}{4-(-3)}(x-1) \quad \text{(taking upper sign)}$$

or $\quad x-7y+13=0$

and $y-2=\dfrac{(-3-4)}{(4+(-3))}(x-1)$ (taking below sign)

or $7x+y-9=0$

Hence, possible equation of the line BC are $x-7y+13=0$ and $7x+y-9=0$

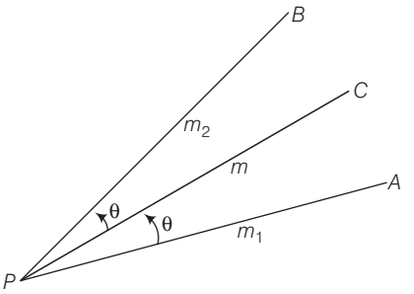
A Line Equally Inclined with Two Lines

Theorem : If two lines with slopes m_1 and m_2 be equally inclined to a line with slope m, then

$$\left(\frac{m_1-m}{1+mm_1}\right)=-\left(\frac{m_2-m}{1+mm_2}\right)$$

Proof : Let be two lines of slopes m_1 and m_2 intersecting at a point P.

Let $\angle CPA=\angle BPC=\theta$



$\therefore$ $\tan(\angle CPA)=\left(\dfrac{m-m_1}{1+mm_1}\right)$

or $\tan\theta=\left(\dfrac{m-m_1}{1+mm_1}\right)$ $(\because m>m_1)$ ...(i)

and $\tan(\angle BPC)=\left(\dfrac{m_2-m}{1+m_2m}\right)$

or $\tan\theta=\left(\dfrac{m_2-m}{1+m_2m}\right)$ $(\because m_2>m)$ ... (ii)

From Eqs. (i) and (ii), we get

$$\left(\frac{m-m_1}{1+mm_1}\right)=\left(\frac{m_2-m}{1+m_2m}\right) \text{ or } \left(\frac{m_1-m}{1+mm_1}\right)=-\left(\frac{m_2-m}{1+mm_2}\right)$$

Remarks

1. The above equation gives two values of m which are the slopes of the lines parallel to the bisectors of the angles between the two given lines.
2. Sign of m in both brackets is same.

Example 85. Find the equations to the straight lines passing through the point $(2, 3)$ and equally inclined to the lines $3x-4y-7=0$ and $12x-5y+6=0$.

Sol. Let m be the slope of the required line. Then its equation is

$$y-3=m(x-2) \quad \text{...(i)}$$

It is given that line (i) is equally inclined to the lines $3x-4y-7=0$ and $12x-5y+6=0$ then

$$\Rightarrow \left(\frac{\frac{3}{4}-m}{1+\frac{3}{4}m}\right)=-\left(\frac{\frac{12}{5}-m}{1+\frac{12}{5}m}\right)$$

$$\left(\begin{array}{c}\text{slope of } 3x-4y-7=0 \text{ is } \dfrac{3}{4}\\ \text{and slope of } 12x-5y+6=0 \text{ is } \dfrac{12}{5}\end{array}\right)$$

$$\Rightarrow \left(\frac{3-4m}{4+3m}\right)=-\left(\frac{12-5m}{5+12m}\right)$$

$$\Rightarrow (3-4m)(5+12m)+(4+3m)(12-5m)=0$$

$$\Rightarrow 63m^2-32m-63=0$$

$$\Rightarrow (7m-9)(9m+7)=0$$

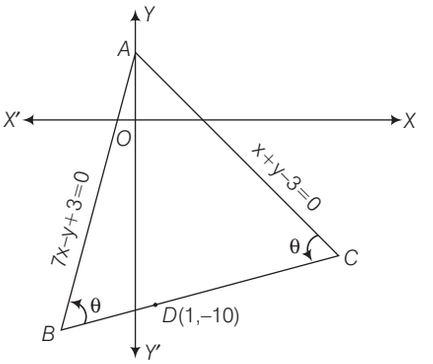
$$\therefore \quad m=\frac{9}{7},-\frac{7}{9}$$

Putting these values of m in Eq. (i) we obtain the equations of required lines as $9x-7y+3=0$ and $x+9y-41=0$.

Example 86. Two equal sides of an isosceles triangle are given by the equations $7x-y+3=0$ and $x+y-3=0$ and its third side passes through the point $(1,-10)$. Determine the equation of the third side.

Sol. Let m be the slope of BC. Since $AB=AC$.

Therefore BC makes equal angles with AB and AC.



Then $\left(\dfrac{7-m}{1+7m}\right)=-\left(\dfrac{-1-m}{1+(-1)m}\right)$

$$\Rightarrow (7-m)(1-m)-(1+7m)(1+m)=0$$

$$\Rightarrow 6m^2+16m-6=0$$

$$\Rightarrow 3m^2+8m-3=0$$

$\Rightarrow \quad (3m-1)(m+3)=0$

$\Rightarrow \quad m=\frac{1}{3}, -3$

Equation of third side BC is $y+10=m(x-1)$

i.e. $y+10=\frac{1}{3}(x-1)$ and $y+10=-3(x-1)$

or $x-3y-31=0$ and $3x+y+7=0$

Equation of the Bisectors

Theorem : Prove that the equation of the bisectors of the angles between the lines

$$a_1x+b_1y+c_1=0 \quad \text{and} \quad a_2x+b_2y+c_2=0$$

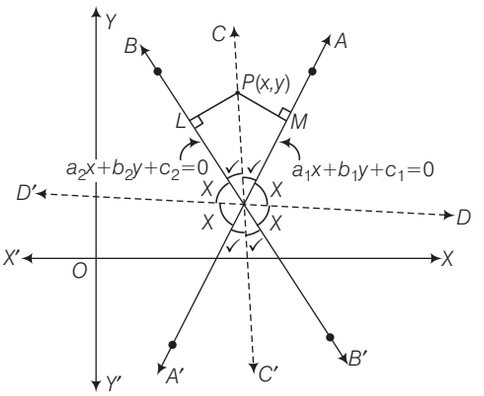
are given by $\dfrac{(a_1x+b_1y+c_1)}{\sqrt{(a_1^2+b_1^2)}}=\pm\dfrac{(a_2x+b_2y+c_2)}{\sqrt{(a_2^2+b_2^2)}}$

Proof : Let the given lines be AA' and BB' whose equations are

$$a_1x+b_1y+c_1=0 \qquad ...(i)$$

and $$a_2x+b_2y+c_2=0 \qquad ...(ii)$$

Since bisectors of the angles between the two lines are the locus of a point which moves in a plane such that whose distance from two lines are equal.



Let CC' and DD' be the two bisectors of the angle between the lines AA' and BB'. Let $P(x, y)$ be any point on CC', then

Length of the perpendicular from P on AA'

= length of the perpendicular from P on BB'

$$\therefore \quad \frac{|a_1x+b_1y+c_1|}{\sqrt{(a_1^2+b_1^2)}}=\frac{|a_2x+b_2y+c_2|}{\sqrt{(a_2^2+b_2^2)}}$$

or $$\frac{(a_1x+b_1y+c_1)}{\sqrt{(a_1^2+b_1^2)}}=\pm\frac{(a_2x+b_2y+c_2)}{\sqrt{(a_2^2+b_2^2)}}$$

These are the required equations of the bisectors.

Note

The two bisectors are perpendicular to each other.

Example 87. Find the equations of the bisectors of the angles between the straight lines $3x-4y+7=0$ and $12x+5y-2=0$.

Sol. The equations of the bisectors of the angles between $3x-4y+7=0$ and $12x+5y-2=0$ are

$$\frac{(3x-4y+7)}{\sqrt{(3)^2+(-4)^2}}=\pm\frac{(12x+5y-2)}{\sqrt{(12)^2+(5)^2}}$$

or $$\frac{(3x-4y+7)}{5}=\pm\frac{(12x+5y-2)}{13}$$

or $$(39x-52y+91)=\pm(60x+25y-10)$$

Taking the positive sign, we get

$$21x+77y-101=0$$

as one bisector.

Taking the negative sign, we get $99x-77y+81=0$ as the second bisector.

Bisector of the Angle Containing the Origin

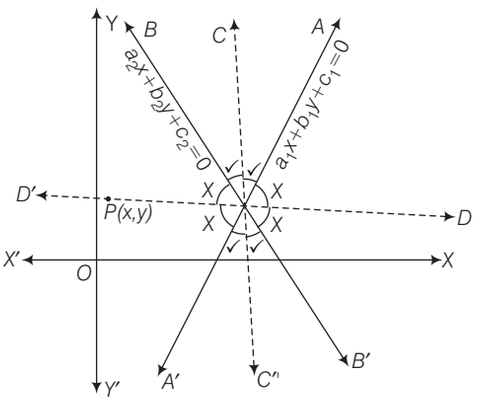
Let equations of lines be

$$a_1x+b_1y+c_1=0 \qquad ...(i)$$

$$a_2x+b_2y+c_2=0 \qquad ...(ii)$$

where c_1 and c_2 are positive.

Let $P(x, y)$ be taken on the bisector of the angle which contains the origin.



(i) Let $P(x, y)$ lies on DD', then either $O(0,0)$ and $P(x, y)$ will lie on the same side of the two lines Eqs. (i) and (ii), then

$$\frac{a_1x+b_1y+c_1}{0+0+c_1}>0$$

and $$\frac{a_2x+b_2y+c_2}{0+0+c_2}>0$$

or $$a_1x+b_1y+c_1>0$$

and $$a_2x+b_2y+c_2>0 \qquad (\because c_1, c_2>0)$$

If the origin $O(0,0)$ and $P(x,y)$ lie on the opposite side of the two lines Eqs. (i) and (ii), then

$$\frac{a_1x+b_1y+c_1}{0+0+c_1}<0 \quad \text{and} \quad \frac{a_2x+b_2y+c_2}{0+0+c_2}<0$$

or $\quad a_1x+b_1y+c_1<0 \quad \text{and} \quad a_2x+b_2y+c_2<0$

$(\because c_1, c_2 > 0)$

Then equation of bisectors will be

$$\frac{|a_1x+b_1y+c_1|}{\sqrt{(a_1^2+b_1^2)}}=\frac{|a_2x+b_2y+c_2|}{\sqrt{(a_2^2+b_2^2)}}$$

Case I : If $a_1x+b_1y+c_1>0$ and $a_2x+b_2y+c_2>0$

then $\quad \frac{(a_1x+b_1y+c_1)}{\sqrt{(a_1^2+b_1^2)}}=\frac{(a_2x+b_2y+c_2)}{\sqrt{(a_2^2+b_2^2)}}$

Case II : If $a_1x+b_1y+c_1<0$ and $a_2x+b_2y+c_2<0$

then $\quad -\frac{(a_1x+b_1y+c_1)}{\sqrt{(a_1^2+b_1^2)}}=-\frac{(a_2x+b_2y+c_2)}{\sqrt{(a_2^2+b_2^2)}}$

i.e. $\quad \frac{(a_1x+b_1y+c_1)}{\sqrt{(a_1^2+b_1^2)}}=\frac{(a_2x+b_2y+c_2)}{\sqrt{(a_2^2+b_2^2)}}$

Thus is both cases equation of the bisector containing the origin, when c_1 and c_2 are positive is

$$\frac{(a_1x+b_1y+c_1)}{\sqrt{(a_1^2+b_1^2)}}=\frac{(a_2x+b_2y+c_2)}{\sqrt{(a_2^2+b_2^2)}}$$

and equation of the bisector of the angle between the lines

$$a_1x+b_1y+c_1=0 \quad \text{and} \quad a_2x+b_2y+c_2=0$$

which does not contain the origin when c_1 and c_2 are positive is

$$\frac{(a_1x+b_1y+c_1)}{\sqrt{(a_1^2+b_1^2)}}=-\frac{(a_2x+b_2y+c_2)}{\sqrt{(a_2^2+b_2^2)}}$$

Working Rule :

(i) First re-write the equations of the two lines so that their constant terms are positive.

(ii) The bisector of the angle containing the origin and does not containing the origin, then taking +ve and − ve sign in

$$\frac{(a_1x+b_1y+c_1)}{\sqrt{(a_1^2+b_1^2)}}=\pm\frac{(a_2x+b_2y+c_2)}{\sqrt{(a_2^2+b_2^2)}}$$ respectively.

Example 88. Find the equations of angular bisector bisecting the angle containing the origin and not containing the origin of the lines $4x+3y-6=0$ and $5x+12y+9=0$.

Sol. Firstly make the constant terms (c_1, c_2) positive, then

$$-4x-3y+6=0 \text{ and } 5x+12y+9=0$$

$\therefore$ The equation of the bisector bisecting the angle containing origin is

$$\frac{(-4x-3y+6)}{\sqrt{(-4^2)+(-3)^2}}=\frac{(5x+12y+9)}{\sqrt{(5)^2+(12)^2}}$$

$$\Rightarrow \quad \left(\frac{-4x-3y+6}{5}\right)=\left(\frac{5x+12y+9}{13}\right)$$

$$\Rightarrow \quad -52x-39y+78=25x+60y+45$$

$$\Rightarrow \quad 77x+99y-33=0 \text{ or } 7x+9y-3=0$$

and the equation of the bisector bisecting the angle not containing origin is

$$\frac{(-4x-3y+6)}{\sqrt{(-4)^2+(-3)^2)}}=-\frac{(5x+12y+9)}{\sqrt{(5^2)+(12)^2}}$$

$$\Rightarrow \quad \left(\frac{-4x-3y+6}{5}\right)=-\left(\frac{5x+12y+9}{13}\right)$$

$$\Rightarrow \quad -52x-39y+78=-25x-60y-45$$

$$\Rightarrow \quad 27x-21y-123=0 \text{ or } 9x-7y-41=0$$

Equation of that Bisector of the Angle between Two Lines which Contains a Given Point

Let the equations of the two lines be

$$a_1x+b_1y+c_1=0 \quad \text{...(i)}$$

and $\quad a_2x+b_2y+c_2=0 \quad \text{...(ii)}$

The equation of the bisector of the angle between the two lines containing the points (h,k) will be

$$\frac{(a_1x+b_1y+c_1)}{\sqrt{(a_1^2+b_1^2)}}=\frac{(a_2x+b_2y+c_2)}{\sqrt{(a_2^2+b_2^2)}}$$

or $\quad \frac{(a_1x+b_1y+c_1)}{\sqrt{(a_1^2+b_1^2)}}=-\frac{(a_2x+b_2y+c_2)}{\sqrt{(a_2^2+b_2^2)}}$

according as $a_1h+b_1k+c_1$ and $a_2h+b_2k+c_2$ are of the same sign or opposite sign.

Example 89. Find the bisector of the angle between the lines $2x+y-6=0$ and $2x-4y+7=0$ which contains the point (1, 2).

Sol. Value of $2x+y-6$ at (1, 2) is − 2 (negative)

and value of $2x-4y+7$ at (1, 2) is 1 (positive)

i.e. opposite sign.

$\therefore$ Equation of bisector containing the point (1, 2) is

$$\frac{(2x+y-6)}{\sqrt{(2^2+1^2)}}=-\frac{(2x-4y+7)}{\sqrt{(2)^2+(-4)^2}}$$

$$\Rightarrow \quad 2(2x+y-6)+(2x-4y+7)=0$$

or $\quad 6x-2y-5=0$

How to Distinguish the Acute (Internal) and Obtuse (External) Angle Bisectors?

Let the equations of the two lines be

$$a_1x + b_1y + c_1 = 0 \quad ...(i)$$

and $$a_2x + b_2y + c_2 = 0 \quad ...(ii)$$

where, $c_1 > 0, c_2 > 0$.

$\because$ Equations of bisectors are

$$\frac{(a_1x + b_1y + c_1)}{\sqrt{(a_1^2 + b_1^2)}} = \pm \frac{(a_2x + b_2y + c_2)}{\sqrt{(a_2^2 + b_2^2)}} \quad ...(iii)$$

when Eq. (iii) be simplified, let the bisectors be

$$p_1x + q_1y + r_1 = 0 \quad ...(iv)$$

and $$p_2x + q_2y + r_2 = 0 \quad ...(v)$$

Since the two bisectors are at right angles, the angle α between the acute (internal) bisector and any one of the given lines must lie between 0 and 45° i.e. $0 < \alpha < 45°$.

$\therefore$ $$0 < \tan\alpha < 1$$

If m_1 and m_2 are the slopes of Eqs. (i) and (iii) respectively.

Then, $$m_1 = -\frac{a_1}{b_1} \quad \text{and} \quad m_2 = -\frac{p_1}{q_1}$$

$\therefore$ $$\tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1\, m_2}\right|$$

$$= \left|\frac{\left(-\frac{a_1}{b_1}\right) - \left(-\frac{p_1}{q_1}\right)}{1 + \left(-\frac{a_1}{b_1}\right)\left(-\frac{p_1}{q_1}\right)}\right| = \left|\frac{a_1q_1 - b_1p_1}{b_1q_1 + a_1p_1}\right|$$

Hence, if $0 < \tan\alpha < 1$, $p_1x + q_1y + r_1 = 0$ is the acute (internal) bisector and if $\tan\alpha > 1$, $p_2x + q_2y + r_2 = 0$ is the obtuse (external) bisector.

Shortcut Method for finding Acute (Internal) and Obtuse (External) Angle Bisectors

Let the equations of the two lines be

$$a_1x + b_1y + c_1 = 0$$

$$a_2x + b_2y + c_2 = 0$$

Taking $$c_1 > 0, c_2 > 0 \text{ and } a_1b_2 \neq a_2b_1$$

Then equations of the bisectors are

$$\frac{(a_1x + b_1y + c_1)}{\sqrt{(a_1^2 + b_1^2)}} = \pm \frac{(a_2\ x + b_2\ y + c_2)}{\sqrt{(a_2^2 + b_2^2)}}$$

Conditions	Acute angle bisector	Obtuse angle bisector
$a_1a_2 + b_1b_2 > 0$	−	+
$a_1a_2 + b_1b_2 < 0$	+	−

Remarks

1. Bisectors are perpendiculars to each other.
2. `+' sign gives the bisector of the angle containing origin.
3. If $a_1a_2 + b_1b_2 > 0$ then the origin lies in obtuse angle and if $a_1a_2 + b_1b_2 < 0$, then the origin lies in acute angle.

Explanation : Equations of given lines in normal form will be respectively

$$-\frac{a_1x}{\sqrt{(a_1^2 + b_1^2)}} - \frac{b_1y}{\sqrt{(a_1^2 + b_1^2\ )}} = \frac{c_1}{\sqrt{(a_1^2 + b_1^2)}} \ (\because c_1 > 0)$$

and $$-\frac{a_2x}{\sqrt{(a_2^2 + b_2^2)}} - \frac{b_2y}{\sqrt{(a_2^2 + b_2^2)}} = \frac{c_2}{\sqrt{(a_2^2 + b_2^2)}} \ (\because c_2 > 0)$$

If $$\cos\alpha = -\frac{a_1}{\sqrt{(a_1^2 + b_1^2)}} \quad \text{then} \quad \sin\alpha = -\frac{b_1}{\sqrt{(a_1^2 + b_1^2)}}$$

and $$\cos\beta = -\frac{a_2}{\sqrt{(a_2^2 + b_2^2)}} \quad \text{then} \quad \sin\beta = -\frac{b_2}{\sqrt{(a_2^2 + b_2^2)}}$$

Now, $$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$= \frac{(a_1a_2 + b_1b_2)}{\sqrt{(a_1^2 + b_1^2)}\sqrt{(a_2^2 + b_2^2)}}$$

$$\cos(\alpha - \beta) > 0 \quad \text{or} \quad < 0$$

according as $(\alpha - \beta)$ is acute or obtuse.

i.e. $$a_1a_2 + b_1b_2 > 0 \quad \text{or} \quad < 0$$

Hence, bisector of the angle between the lines will be the bisector of the acute or obtuse angle according as origin lies in the acute or obtuse angle according as $a_1a_2 + b_1b_2 < 0$ or >0.

Example 90. Find the equation of the bisector of the obtuse angle between the lines $3x - 4y + 7 = 0$ and $12x + 5y - 2 = 0$.

Sol. Firstly make the constant terms (c_1, c_2) positive

$$3x - 4y + 7 = 0 \quad \text{and} \quad -12x - 5y + 2 = 0$$

$\because$ $a_1a_2 + b_1b_2 = (3)(-12) + (-4)(-5) = -36 + 20 = -16$

$\therefore$ $a_1a_2 + b_1b_2 < 0$

Hence "–" sign gives the obtuse bisector.

$\therefore$ Obtuse bisector is

$$\frac{(3x - 4y + 7)}{\sqrt{(3)^2 + (-4)^2}} = -\frac{(-12x - 5y + 2)}{\sqrt{(-12)^2 + (-5)^2}}$$

$\Rightarrow$ $13(3x - 4y + 7) = -5(-12x - 5y + 2)$

$\Rightarrow$ $21x + 77y - 101 = 0$ is the obtuse angle bisector.

Example 91. Find the bisector of acute angle between the lines $x + y - 3 = 0$ and $7x - y + 5 = 0$.

Sol. Firstly, make the constant terms (c_1, c_2) positive then

$-x - y + 3 = 0$ and $7x - y + 5 = 0$

$\because \quad a_1a_2 + b_1b_2 = (-1)(7) + (-1)(-1) = -7 + 1 = -6$

i.e. $a_1a_2 + b_1b_2 < 0$

Hence "+" sign gives the acute bisector.

$\therefore$ Acute bisector is $\dfrac{-x - y + 3}{\sqrt{(-1)^2 + (-1)^2}} = + \dfrac{7x - y + 5}{\sqrt{(7)^2 + (-1)^2}}$

$\Rightarrow \quad \dfrac{-x - y + 3}{\sqrt{2}} = \dfrac{7x - y + 5}{5\sqrt{2}}$

$\Rightarrow \quad -5x - 5y + 15 = 7x - y + 5$

$\therefore \quad 12x + 4y - 10 = 0 \quad$ or $\quad 6x + 2y - 5 = 0$

is the acute angle bisector.

Example 92. Find the coordinates of incentre of the triangle. The equation of whose sides are

$AB : x + y - 1 = 0, BC : 7x - y - 15 = 0$

and $CA : x - y - 1 = 0$.

Sol. Firstly, make the constant terms $(c_1, c_2,$ and $c_3)$ positive

i.e. $AB : -x - y + 1 = 0$ ...(i)

$BC : -7x + y + 15 = 0$ ...(ii)

$CA : -x + y + 1 = 0$ ...(iii)

$\because$ The incentre of triangle is the point of intersection of internal or acute angle bisectors.

Internal bisector of *AB* and *BC* :

$-x - y + 1 = 0$

$-7x + y + 15 = 0$

$\because \quad a_1a_2 + b_1b_2 = (-1)(-7) + (-1)(1) = 6 > 0$

$\therefore$ Acute or internal bisector is

$$\frac{(-x - y + 1)}{\sqrt{(-1)^2 + (-1)^2}} = -\frac{(-7x + y + 15)}{\sqrt{(-7)^2 + (1)^2}}$$

$$\Rightarrow \quad \frac{(-x - y + 1)}{\sqrt{2}} = -\frac{(-7x + y + 15)}{5\sqrt{2}}$$

$\Rightarrow \quad -5x - 5y + 5 = 7x - y - 15$

or $\quad 12x + 4y - 20 = 0$

or $\quad 3x + y - 5 = 0$ ...(iv)

Internal bisector of BC and CA :

$-7x + y + 15 = 0$

$-x + y + 1 = 0$

$\because \quad a_1a_2 + b_1b_2 = (-7)(-1) + (1)(1) = 8 > 0$

$\because$ Acute or internal bisector is

$$\frac{(-7x + y + 15)}{\sqrt{(-7)^2 + (1)^2}} = -\frac{(-x + y + 1)}{\sqrt{(-1)^2 + (1)^2}}$$

$$\Rightarrow \quad \frac{-7x + y + 15}{5\sqrt{2}} = \frac{(x - y - 1)}{\sqrt{2}}$$

$\Rightarrow \quad -7x + y + 15 = 5x - 5y - 5$

or $\quad 12x - 6y - 20 = 0$

or $\quad 6x - 3y - 10 = 0$ ...(v)

Finally, solve Eqs. (iv) and (v), we get

$$x = \frac{5}{3} \text{ and } y = 0$$

Hence coordinates of incentre are $\left(\frac{5}{3}, 0\right)$.

Exercise for Session 4

1. The straight lines $2x + 11y - 5 = 0, 24x + 7y - 20 = 0$ and $4x - 3y - 2 = 0$

(a) form a triangle

(b) are only concurrent

(c) are concurrent with one line bisecting the angle between the other two

(d) None of the above

2. The line $x + 3y - 2 = 0$ bisects the angle between a pair of straight lines of which one has the equation $x - 7y + 5 = 0$. The equation of other line is

(a) $3x + 3y - 1 = 0$ (b) $x - 3y + 2 = 0$ (c) $5x + 5y + 3 = 0$ (d) $5x + 5y - 3 = 0$

3. P is a point on either of the two lines $y - \sqrt{3}\,|x| = 2$ at a distance of 5 units from their point of intersection. The coordinates of the foot of the perpendicular from P on the bisector of the angle between them are

(a) $\left(0, \dfrac{4 + 5\sqrt{3}}{2}\right)$ or $\left(0, \dfrac{4 - 5\sqrt{3}}{2}\right)$ depending on which the point P is taken

(b) $\left(0, \dfrac{4 + 5\sqrt{3}}{2}\right)$ (c) $\left(0, \dfrac{4 - 5\sqrt{3}}{2}\right)$ (d) $\left(\dfrac{5}{2}, \dfrac{5\sqrt{3}}{2}\right)$

4. In a triangle ABC, the bisectors of angles B and C lie along the lines $x = y$ and $y = 0$. If A is (1, 2), then the equation of line BC is

(a) $2x + y = 1$ (b) $3x - y = 5$ (c) $x - 2y = 3$ (d) $x + 3y = 1$

5. In ΔABC, the coordinates of the vertex A are (4, –1) and lines $x - y - 1 = 0$ and $2x - y = 3$ are the internal bisecters of angles B and C. Then, the radius of the incircle of triangle ABC is

(a) $\frac{5}{\sqrt{5}}$ (b) $\frac{3}{\sqrt{5}}$ (c) $\frac{6}{\sqrt{5}}$ (d) $\frac{7}{\sqrt{5}}$

6. The equation of the straight line which bisects the intercepts made by the axes on the lines $x + y = 2$ and $2x + 3y = 6$ is

(a) $2x = 3$ (b) $y = 1$ (c) $2y = 3$ (d) $x = 1$

7. The equation of the bisector of the acute angle between the lines $2x - y + 4 = 0$ and $x - 2y = 1$ is

(a) $x + y + 5 = 0$ (b) $x - y + 1 = 0$ (c) $x - y = 5$ (d) $x - y + 5 = 0$

8. The equation of the bisector of that angle between the lines $x + y = 3$ and $2x - y = 2$ which contains the point (1, 1) is

(a) $(\sqrt{5} - 2\sqrt{2})x + (\sqrt{5} + \sqrt{2})y = 3\sqrt{5} - 2\sqrt{2}$ (b) $(\sqrt{5} + 2\sqrt{2})x + (\sqrt{5} - \sqrt{2})y = 3\sqrt{5} + 2\sqrt{2}$

(c) $3x = 10$ (d) $3x - 5y + 2 = 0$

9. Find the equations of the two straight lines through (7, 9) and making an angle of 60° with the line $x - \sqrt{3}y - 2\sqrt{3} = 0$.

10. Equation of the base of an equilateral triangle is $3x + 4y = 9$ and its vertex is at the point (1, 2). Find the equations of the other sides and the length of each side of the triangle.

11. Find the coordinates of those points on the line $3x + 2y = 5$ which are equidistant from the lines $4x + 3y - 7 = 0$ and $2y - 5 = 0$.

12. Two sides of rhombus $ABCD$ are parallel to the lines $y = x + 2$ and $y = 7x + 3$. If the diagonal of the rhombus intersect at the point (1, 2) and the vertex A lies on Y-axis, find the possible coordinates of A.

13. The bisector of two lines L_1 and L_2 are given by $3x^2 - 8xy - 3y^2 + 10x + 20y - 25 = 0$. If the line L_1 passes through origin, find the equation of line L_2.

14. Find the equation of the bisector of the angle between the lines $x + 2y - 11 = 0$ and $3x - 6y - 5 = 0$ which contains the point (1, –3).

15. Find the equation of the bisector of the angle between the lines $2x - 3y - 5 = 0$ and $6x - 4y + 7 = 0$ which is the supplement of the angle containing the point (2, –1).

Session 5

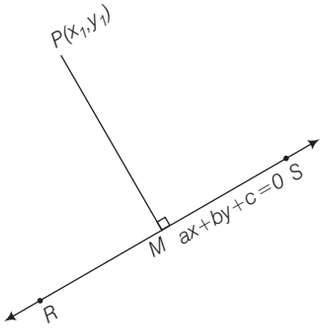
The Foot of Perpendicular Drawn from the Point (x_1, y_1) to the Line $ax + by + c = 0$, Image or Reflection of a Point (x_1, y_1) about a Line Mirror, Image or Reflection of a Point In Different Cases, Use of Image or Reflection

The Foot of Perpendicular Drawn from the Point (x_1, y_1) to the Line $ax + by + c = 0$

Let $P \equiv (x_1, y_1)$ and let M be the foot of perpendicular drawn from P on $ax + by + c = 0$.

In order to find the coordinates of M, find the equation of the line PM which is perpendicular to RS and passes through $P(x_1, y_1)$, i.e. $bx - ay = bx_1 - ay_1$

or $b(x - x_1) - a(y - y_1) = 0$ and solving it with $ax + by + c = 0$, then we get coordinates of M.



Aliter I : Let the coordinates of M are (x_2, y_2) then $M(x_2, y_2)$ lies on $ax + by + c = 0$

$$\Rightarrow \quad ax_2 + by_2 + c = 0 \qquad ...(i)$$

and $\because PM \perp RS$

then (Slope of PM) (Slope of RS) $= -1$

$$\Rightarrow \quad \left(\frac{y_2 - y_1}{x_2 - x_1}\right) \times \left(-\frac{a}{b}\right) = -1$$

or $$bx_2 - ay_2 = bx_1 - ay_1 \qquad ...(ii)$$

Solving Eqs. (i) and (ii), we get (x_2, y_2).

Aliter II : Let the coordinates of M are (x_2, y_2)

$\because \quad PM \perp RS$

and M lies on $\quad ax + by + c = 0$

i.e. $$ax_2 + by_2 + c = 0 \qquad ...(iii)$$

and $$\left(\frac{y_2 - y_1}{x_2 - x_1}\right) \times \left(-\frac{a}{b}\right) = -1 \qquad (\because PM \perp RS)$$

or $$\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b}$$

or $$\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = \frac{a(x_2 - x_1) + b(y_2 - y_1)}{(a^2 + b^2)}$$

(By ratio proportion method)

$$= \frac{(ax_2 + by_2) - (ax_1 + by_1)}{a^2 + b^2}$$

$$= \frac{-c - (ax_1 + by_1)}{a^2 + b^2} \qquad \text{[from Eq. (iii)]}$$

or $$\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = -\frac{(ax_1 + by_1 + c)}{(a^2 + b^2)}$$

Example 93. Find the coordinates of the foot of the perpendicular drawn from the point (2, 3) to the line $y = 3x + 4$.

Sol. Given line is

$$3x - y + 4 = 0 \qquad ...(i)$$

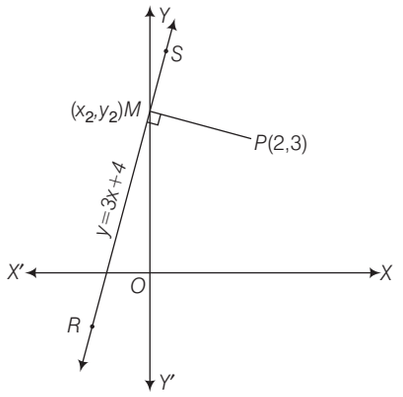
Let Eq. $P \equiv (2, 3)$.

Let, M be the foot of perpendicular drawn from P on RS.

Then equation of PM passes through $P(2, 3)$ and perpendicular to RS is

$$x + 3y - (2 + 3 \times 3) = 0$$

i.e. $$x + 3y - 11 = 0 \qquad ...(ii)$$



Solving Eqs. (i) and (ii), we get

$$x = -\frac{1}{10}, y = \frac{37}{10}$$

$$\therefore \quad M \equiv \left(-\frac{1}{10}, \frac{37}{10}\right)$$

Aliter I : Let the coordinates of M be (x_2, y_2) then $M(x_2, y_2)$ lies on $3x - y + 4 = 0$

$$\Rightarrow \quad 3x_2 - y_2 + 4 = 0 \qquad \ldots(iii)$$

and $\because PM \perp RS$

$\therefore$ (Slope of PM) $\times$ (Slope of RS) $= -1$

$$\Rightarrow \quad \left(\frac{y_2 - 3}{x_2 - 2}\right) \times (3) = -1$$

$$\text{or} \quad x_2 + 3y_2 - 11 = 0 \qquad \ldots(iv)$$

Solving Eqs. (iii) and (iv), we get

$$x_2 = -\frac{1}{10}, y_2 = \frac{37}{10}$$

$$\therefore \quad M \equiv \left(-\frac{1}{10}, \frac{37}{10}\right)$$

Aliter II : By Ratio Proportion Method :

$$\frac{x_2 - 2}{3} = \frac{y_2 - 3}{-1} = \frac{-(3 \times 2 - 3 + 4)}{3^2 + (-1)^2}$$

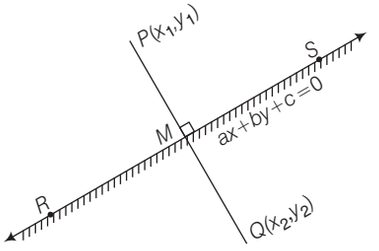
$$\Rightarrow \quad \frac{x_2 - 2}{3} = \frac{y_2 - 3}{-1} = -\frac{7}{10}$$

$$x_2 = -\frac{1}{10} \text{ and } y_2 = \frac{37}{10}$$

$$\therefore \quad M \equiv \left(-\frac{1}{10}, \frac{37}{10}\right)$$

Image or Reflection of a Point (x_1, y_1) About a Line Mirror

Let $Q \equiv (x_2, y_2)$ be the image of $P \equiv (x_1, y_1)$ then find coordinates of the foot of perpendicular M drawn from the point $P(x_1, y_1)$ on RS and use fact that M is the mid- point P and Q.



$$\text{i.e.} \quad M \equiv \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Aliter I : Ratio Proportion Method :

$$\because \quad PQ \perp RS$$

$$\therefore \quad (\text{ Slope of } PQ) \times (\text{ Slope of } RS) = -1$$

$$\text{or} \quad \left(\frac{y_2 - y_1}{x_2 - x_1}\right) \times \left(-\frac{a}{b}\right) = -1$$

$$\text{or} \quad \frac{(x_2 - x_1)}{a} = \frac{(y_2 - y_1)}{b}$$

$$\text{or} \quad \frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = \frac{a(x_2 - x_1) + b(y_2 - y_1)}{a^2 + b^2}$$

$$= \frac{a(2x - x_1 - x_1) + b(2y - y_1 - y_1)}{a^2 + b^2}$$

$$\left(\because \begin{array}{l} M(x, y) \text{ is mid-point of } P \text{ and } Q \\ \therefore\ x_2 = 2x - x_1 \text{ and } y_2 = 2y - y_1 \end{array}\right)$$

$$= \frac{-2ax_1 - 2by_1 + 2(ax + by)}{a^2 + b^2}$$

$$= \frac{-2ax_1 - 2by_1 + 2(-c)}{a^2 + b^2} \quad (\because ax + by = -c)$$

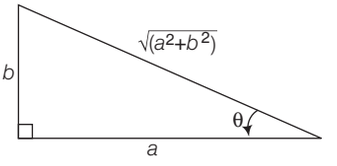
$$= \frac{-2(ax_1 + by_1 + c)}{(a^2 + b^2)}$$

$$\text{i.e.} \quad \frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = -\frac{2(ax_1 + by_1 + c)}{(a^2 + b^2)}$$

Aliter II : By Distance form or Symmetric form or parametric form :

$$\because \quad \text{Slope of } RS = -\frac{a}{b}$$

$$\therefore \quad \text{Slope of } PQ = \frac{b}{a}$$



Let $\tan\theta = \frac{b}{a}$

$\therefore \quad \sin\theta = \frac{b}{\sqrt{(a^2+b^2)}}$

and $\cos\theta = \frac{a}{\sqrt{(a^2+b^2)}}$

Put the equation of the mirror line such that the coefficient of y becomes negative.

Suppose if $b > 0$

then $ax + by + c = 0$

becomes $-ax - by - c = 0$

and $p = PM =$ Directed distance from $P(x_1, y_1)$ on $-ax - by - c = 0$ (i.e. p +ve or −ve)

$$= \left(\frac{-ax_1 - by_1 - c}{\sqrt{(a^2+b^2)}}\right)$$

$\therefore \quad PQ = 2PM = 2p = 2\left(\frac{-ax_1 - by_1 - c}{\sqrt{(a^2+b^2)}}\right) = r$

$\Rightarrow$ Required image has the coordinates $(x_1 + r\cos\theta, y_1 + r\sin\theta)$.

Example 94. Find the image of the point $(4, -13)$ with respect to the line mirror $5x + y + 6 = 0$.

Sol. Let, $P \equiv (4, -13)$ and Let, $Q \equiv (x_2, y_2)$ be mirror image P with respect to line mirror $5x + y + 6 = 0$.

Let, $M(\alpha, \beta)$ be the foot of perpendicular from $P(4, -13)$ on the line mirror $5x + y + 6 = 0$, then

$$\frac{\alpha - 4}{5} = \frac{\beta + 13}{1} = \frac{-(5\times 4 - 13 + 6)}{5^2 + 1^2}$$

or $\frac{\alpha - 4}{5} = \frac{\beta + 13}{1} = -\frac{1}{2}$

$\therefore \quad M \equiv \left(\frac{3}{2}, -\frac{27}{2}\right)$

$\because$ M is the mid-point of P and Q, then

$$Q \equiv (x_2, y_2) \equiv \left(2\times\frac{3}{2} - 4, 2\times\left(-\frac{27}{2}\right) + 13\right)$$

i.e. $Q \equiv (-1, -14)$

Aliter I :

By Ratio Proportion Method : Let, $Q(x_2, y_2)$ the image of $P(4, -13)$ with respect to line mirror $5x + y + 6 = 0$, then

$$\frac{x_2 - 4}{5} = \frac{y_2 + 13}{1} = -\frac{2(5\times 4 - 13 + 6)}{5^2 + 1^2} = -1$$

or $x_2 - 4 = -5$ and $y_2 + 13 = -1$

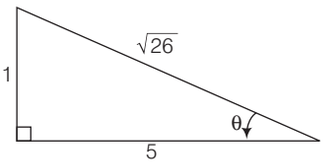
$\therefore \quad x_2 = -1$ and $y_2 = -14$

Hence $Q \equiv (-1, -14)$.

Aliter II :

By distance form or Symmetric form or Parametric form : Let, $P \equiv (4, -13)$ and Q be the image of P with respect to line mirror $(RS)\ 5x + y + 6 = 0$

$\because$ Slope of $RS = -5$



$\therefore$ Slope of $PQ = \frac{1}{5} = \tan\theta$

$\therefore \quad \sin\theta = \frac{1}{\sqrt{26}}$ and $\cos\theta = \frac{5}{\sqrt{26}}$

Now, put the equation of the mirror line such that the coefficient of y becomes negative.

Then, $5x + y + 6 = 0$ becomes $-5x - y - 6 = 0$ and

$p = \perp$ Directed distance from $P(4, -13)$ on $(-5x - y - 6 = 0)$

$$= \frac{-5\times 4 + 13 - 6}{\sqrt{(-5)^2 + (-1)^2}} = -\frac{13}{\sqrt{26}}$$

$\therefore \quad PQ = r = 2p = -\frac{26}{\sqrt{26}} = -\sqrt{26}$

Hence, required image has the coordinates

$$Q \equiv (4 - \sqrt{26}\cos\theta, -13 - \sqrt{26}\sin\theta)$$

i.e. $\left(4 - \sqrt{26}\times\frac{5}{\sqrt{26}}, -13 - \sqrt{26}\times\frac{1}{\sqrt{26}}\right)$

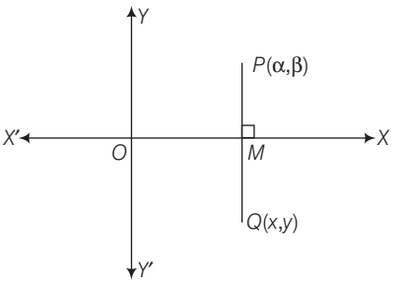
i.e. $(4 - 5, -13 - 1)$

Hence, $Q \equiv (-1, -14)$

Image or Reflection of a Point in Different Cases

(i) The image or reflection of a point with respect to *X*-axis

Let $P(\alpha, \beta)$ be any point and $Q(x, y)$ be its image about X-axis, then (M is the mid-point of P and Q)



$x = \alpha$ and $y = -\beta$

$\therefore$ $Q \equiv (\alpha, -\beta)$

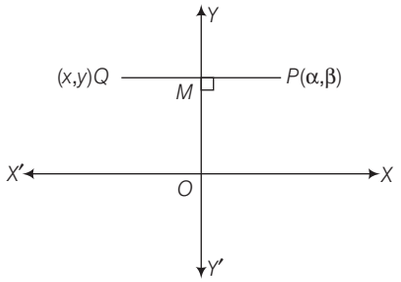
i.e. sign change of ordinate.

Remark

The image of the line $ax + by + c = 0$ about X-axis is $ax - by + c = 0$

(ii) The image or reflection of a point with respect to Y-axis

Let $P(\alpha, \beta)$ be any point and $Q(x, y)$ be its image about Y-axis, then (M is the mid-point of PQ)



$x = -\alpha$ and $y = \beta$

$\therefore$ $Q \equiv (-\alpha, \beta)$

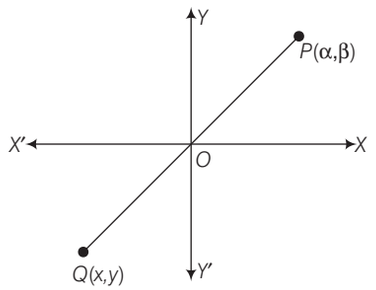
i.e. sign change of abscissae.

Remark

The image of the line $ax + by + c = 0$ about Y-axis is $-ax + by + c = 0$

(iii) The image or reflection of a point with respect to origin

Let $P(\alpha, \beta)$ be any point and $Q(x, y)$ be its image about the origin (O is the mid point of PQ), then



$x = -\alpha$ and $y = -\beta$

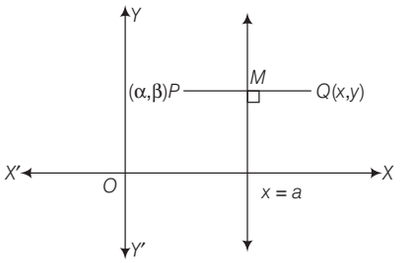
$\therefore$ $Q \equiv (-\alpha, -\beta)$

i.e. sign change of abscissae and ordinate.

Remark

The image of the line $ax + by + c = 0$ about origin is $-ax - by + c = 0$.

(iv) The image or reflection of a point with respect to the line $x = a$



Let $P(\alpha, \beta)$ be any point and $Q(x, y)$ be its image about the line $x = a$, then $y = \beta$

$\therefore$ Coordinates of M are (a, β)

$\because$ M is the mid-point of PQ

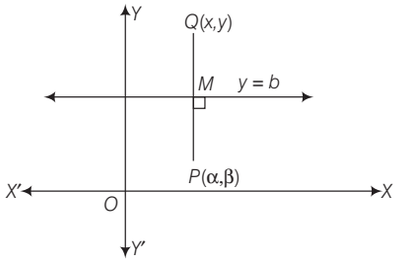
$\therefore$ $Q \equiv (2a - \alpha, \beta)$

Remark

The image of the line $ax + by + c = 0$ about the line $x = \lambda$ is $a(2\lambda - x) + by + c = 0$

(v) The image or reflection of a point with respect to the line $y = b$

Let $P(\alpha, \beta)$ be any point and $Q(x, y)$ be its image about the line y =b, then $x = \alpha$



$\therefore$ Co-ordinates of M are (α, b)

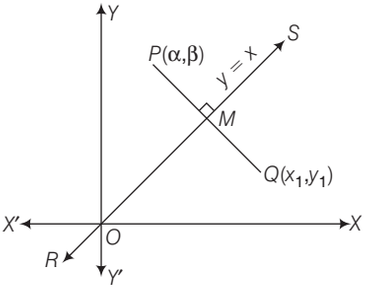
$\because$ M is the mid-point of PQ $\therefore$ $Q \equiv (\alpha, 2b - \beta)$

Remark

The image of the line $ax + by + c = 0$ about the line $y = \mu$ is $ax + b(2\mu - y) + c = 0$.

(vi) The image or reflection of a point with respect to the line y = x

Let $P(\alpha, \beta)$ be any point and $Q(x_1, y_1)$ be its image about the line $y = x$ (RS), then $PQ \perp RS$



$\therefore$ (Slope of PQ) $\times$ (Slope of RS) $=-1$

or $$\frac{y_1-\beta}{x_1-\alpha}\times 1=-1$$

or $$x_1-\alpha=\beta-y_1 \quad ...(i)$$

and mid--point of PQ lie on $y=x$

i.e. $$\left(\frac{y_1+\beta}{2}\right)=\left(\frac{x_1+\alpha}{2}\right)$$

or $$x_1+\alpha=\beta+y_1 \quad ...(ii)$$

Solving Eqs. (i) and (ii), we get $x_1=\beta$ and $y_1=\alpha$

$\therefore$ $Q\equiv(\beta,\alpha)$ i.e. interchange of x and y.

Remark

The image of the line $ax + by + c = 0$ about the line $y = x$ is $ay + bx + c = 0$.

(vii) The image or reflection of a point with respect to the line $y = x\tan\theta$

Let $P(\alpha,\beta)$ be any point and $Q(x_1, y_1)$ be its image about the line $y = x\tan\theta$ (RS), then $PQ \perp RS$

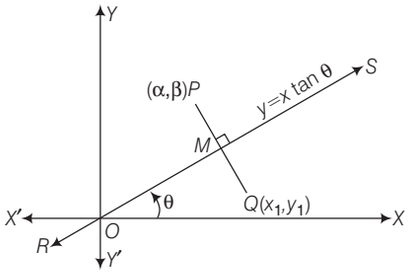
$\therefore$ (Slope of PQ) $\times$ (Slope of RS) $=-1$

or $$\frac{y_1-\beta}{x_1-\alpha}\times\tan\theta=-1$$

$\Rightarrow$ $$y_1-\beta=(\alpha-x_1)\cot\theta \quad ...(i)$$

and mid-point of PQ lie on $y = x\tan\theta$

i.e. $$\left(\frac{y_1+\beta}{2}\right)=\left(\frac{x_1+\alpha}{2}\right)\tan\theta$$



or $$y_1+\beta=(x_1+\alpha)\tan\theta \quad ...(ii)$$

Solving Eqs. (i) and (ii), we get

$$x_1=\alpha\cos 2\theta+\beta\sin 2\theta$$

and $$y_1=\alpha\sin 2\theta-\beta\cos 2\theta$$

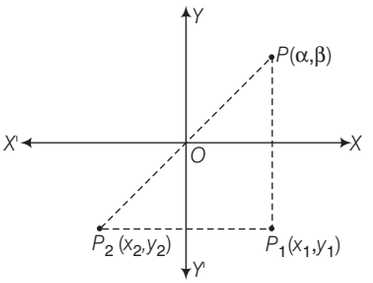
$\therefore$ $$Q\equiv(\alpha\cos 2\theta+\beta\sin 2\theta,\ \alpha\sin 2\theta-\beta\cos 2\theta)$$

Example 95. The point $P(\alpha,\beta)$ undergoes a reflection in the X-axis followed by a reflection in the Y-axis. Show that their combined effect is the same as the single reflection of $P(\alpha,\beta)$ in the origin when $\alpha,\beta>0$.

Sol. Let $P_1(x_1, y_1)$ be the image of (α,β) after reflection in the X-axis. Then

$$x_1=\alpha \quad \text{and} \quad y_1=-\beta \quad ...(i)$$

Now, let $P_2(x_2,y_2)$ be the image of $P_1(x_1, y_1)$ in the Y-axis. Then



$$x_2=-x_1,\ y_2=y_1$$

$\Rightarrow$ $$x_2=-\alpha,\ y_2=-\beta \quad \text{[from Eq. (i)]} \quad ...(ii)$$

further let $P_3(x_3, y_3)$ be the image of $P(\alpha,\beta)$ in the origin O. Then

$$x_3=-\alpha,\ y_3=-\beta \quad ...(iii)$$

From Eqs. (ii) and (iii) ,we get

$$x_3=x_2 \text{ and } y_3=y_2.$$

Hence the image of P_2 of P after successive reflection in their X-axis and Y-axis is the same as the single reflection of P in the origin.

Example 96. Find the image of the point $(-2,-7)$ under the transformations $(x, y)\to(x-2y,-3x+y)$.

Sol. Let (x_1, y_1) be the image of the point (x, y) under the given transformation. Then

$$x_1=x-2y=(-2)-2(-7)=12$$

$\therefore$ $x_1=12$ and $y_1=-3x+y=-3(-2)-7=-1$

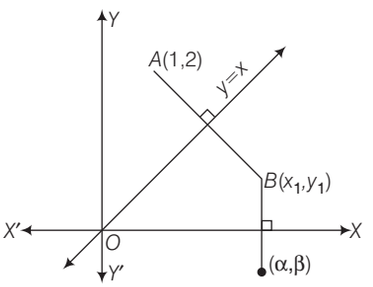
$\therefore$ $y_1=-1$

Hence, the image is $(12, -1)$.

Example 97. The image of the point $A(1,2)$ by the line mirror $y = x$ is the point B and the image of B by the line mirror $y = 0$ is the point (α,β). Find α and β.

Sol. Let (x_1, y_1) be the image of the point $(1, 2)$ about the line $y = x$.

Then $$x_1=2,\ y_1=1 \quad ...(i)$$



Also given image of $B(x_1, y_1)$ by the line mirror $y = 0$ is (α,β). Then $\alpha = x_1 = 2$

and $$\beta=-y_1=-1 \quad \text{[from Eq. (i)]}$$

Hence, $\alpha = 2$ and $\beta = -1$

Example 98. The point (4, 1) undergoes the following three transformations successively :

(i) Reflection about the line $y = x$.

(ii) Translation through a distance 2 units along the positive direction of X-axis.

(iii) Rotation through an angle $\pi/4$ about the origin in the anticlockwise direction.

Then, find the coordinates of the final position.

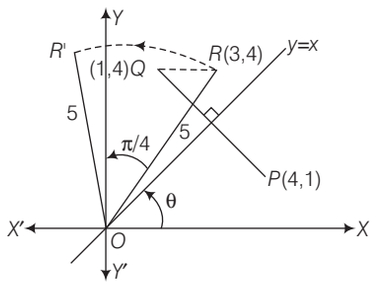
Sol. Let $Q(x_1, y_1)$ be the reflection of P about the line $y = x$.

Then

$$\left.\begin{aligned} x_1 &= 1 \\ y_1 &= 4 \end{aligned}\right\}$$

$\therefore$ Coordinates of Q is (1, 4).

Given that Q move 2 units along the positive direction of X-axis.



$\therefore$ Coordinates of R is $(x_1 + 2, y_1)$

or $R(3, 4)$

If OR makes an angle θ , then

$$\tan\theta = \frac{4}{3}$$

$$\therefore \quad \sin\theta = \frac{4}{5} \quad \text{and} \quad \cos\theta = \frac{3}{5}$$

After rotation of $\frac{\pi}{4}$ let new position of R is R' and

$$OR = OR' = \sqrt{3^2 + 4^2} = 5$$

$\therefore$ OR' makes an angle $(\pi/4 + \theta)$ with X-axis.

Coordinates of $R'\left(OR'\cos\left(\frac{\pi}{4}+\theta\right), OR'\sin\left(\frac{\pi}{4}+\theta\right)\right)$

i.e. $R'\left(OR'\left(\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta\right),\right.$

$\left. OR'\sin\left(\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\right)\right)$

$$\Rightarrow \quad R'\left(5\left(\frac{3}{5\sqrt{2}} - \frac{4}{5\sqrt{2}}\right), 5\left(\frac{3}{5\sqrt{2}} + \frac{4}{5\sqrt{2}}\right)\right)$$

$$\Rightarrow \quad R'\left(-\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$$

Aliter (Use of complex number) :

Let Q be the reflection of $P(4, 1)$ about the line $y = x$, then $Q \equiv (1, 4)$

$\because$ Q move 2 units along the +ve direction of X-axis, if new point is ,R then $R \equiv (3, 4)$.

If $R(3, 4) = R(z_1)$

when $z_1 = (3 + 4i)$

then $R'(x, y) = R'(z_2)$

$\therefore$ $z_2 = z_1 e^{i\pi/4}$ $\left(\because \angle ROR' = \frac{\pi}{4}\right)$

$$= (3 + 4i)\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

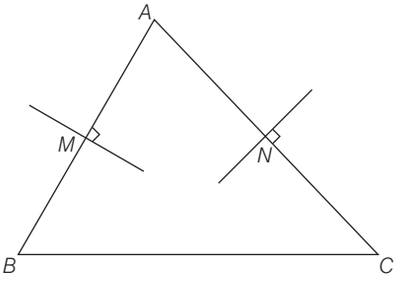
$$= (3 + 4i)\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \left(-\frac{1}{\sqrt{2}} + \frac{7i}{\sqrt{2}}\right)$$

Hence, new coordinates are $\left(-\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$.

Use of Image or Reflection

To make problems simpler and easier use Image or reflection.

Types of problems : (i) If vertex of a ΔABC and equations of perpendicular bisectors of AB and AC are given, then B and C are the images or reflections of A about the perpendicular bisectors of AB and AC (where M and N are the mid-points of AB and AC).



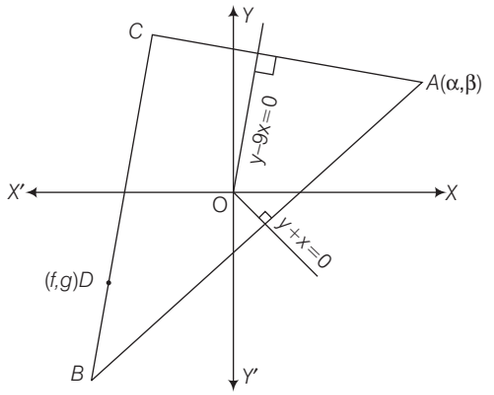
Example 99. The base of a triangle passes through a fixed point (f, g) and its sides are respectively bisected at right angles by the lines $y + x = 0$ and $y - 9x = 0$. Determine the locus of its vertex.

Sol. Let $A \equiv (\alpha, \beta)$ the image of $A(\alpha, \beta)$ about $y + x = 0$ is B, then $B \equiv (-\beta, -\alpha)$ and if image of $A(\alpha, \beta)$ about $y - 9x = 0$ is $C(x_2, y_2)$, then

$$\frac{x_2 - \alpha}{-9} = \frac{y_2 - \beta}{1} = \frac{-2(\beta - 9\alpha)}{1 + 81}$$

$$\therefore \quad x_2 = \frac{9\beta - 40\alpha}{41} \text{ and } y_2 = \frac{40\beta + 9\alpha}{41}$$

$$\therefore \quad C \equiv \left(\frac{9\beta - 40\alpha}{41}, \frac{40\beta + 9\alpha}{41}\right)$$



Hence, B, D, C are collinear, then

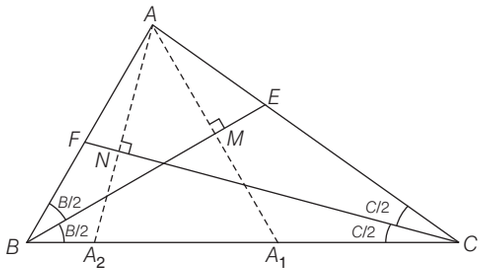
$$\frac{1}{2}\begin{vmatrix} -\beta & -\alpha & 1 \\ f & g & 1 \\ \frac{9\beta - 40\alpha}{41} & \frac{40\beta + 9\alpha}{41} & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad 4(\alpha^2 + \beta^2) + (4g + 5f)\alpha + (4f - 5g)\beta = 0$$

Hence, locus of vertex is

$$4(x^2 + y^2) + (4g + 5f)x + (4f - 5g)y = 0$$

(ii) The images or reflections of vertex A of a ΔABC about the angular bisectors of angles B and C lie on the side BC. (By congruence) A_1 and A_2 are the images of A about the angle bisectors BE and CF respectively, where M and N are the mid-points of AA_1 and AA_2.



Example 100. Find the equations of the sides of the triangle having $(3, -1)$ as a vertex, $x - 4y + 10 = 0$ and $6x + 10y - 59 = 0$ being the equations of an angle bisector and a median respectively drawn from different vertices.

Sol. Let BE be the angle bisector and CF be the median. Given equations of BE and CF are $x - 4y + 10 = 0$ and $6x + 10y - 59 = 0$ respectively.

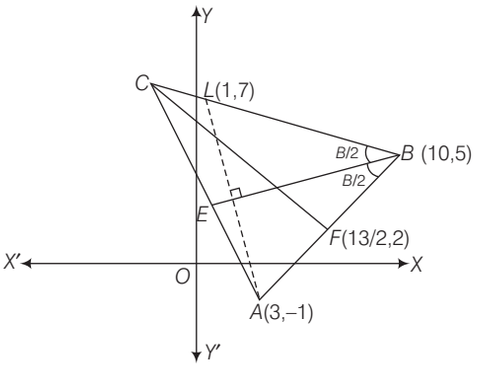
Since, image of A with respect to BE lie on BC. If image of A is $L(h, k)$.

then $$\frac{h-3}{1} = \frac{k+1}{-4} = \frac{-2(3+4+10)}{1^2 + (-4)^2} = -2$$

i.e. $h = 1, k = 7$

$\therefore \quad L \equiv (1, 7)$

$\because$ F be the mid-point of AB.



Let $F \equiv (\alpha, \beta)$

then $B \equiv (2\alpha - 3, 2\beta + 1)$

$\because$ B lie on BE, then

$$(2\alpha - 3) - 4(2\beta + 1) + 10 = 0$$

i.e. $2\alpha - 8\beta + 3 = 0$ ...(i)

and F lie on CF, then

$$6\alpha + 10\beta - 59 = 0 \quad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get

$$\alpha = \frac{13}{2}, \beta = 2$$

then $F \equiv \left(\frac{13}{2}, 2\right)$

and $B = (10, 5)$

Equation of AB is

$$y + 1 = \frac{2+1}{\frac{13}{2} - 3}(x - 3)$$

$$\Rightarrow \quad y + 1 = \frac{6}{7}(x - 3)$$

or $6x - 7y - 25 = 0$

Equation of BC is

$$y - 5 = \frac{7-5}{1-10}(x - 10)$$

$$\Rightarrow \quad y - 5 = -\frac{2}{9}(x - 10)$$

or $2x + 9y - 65 = 0$

$\because$ CA is the family of lines of CB and CF

then $(2x + 9y - 65) + \lambda(6x + 10y - 59) = 0$ ...(iii)

it pass through $A(3, -1)$

then $(6 - 9 - 65) + \lambda(18 - 10 - 59) = 0$

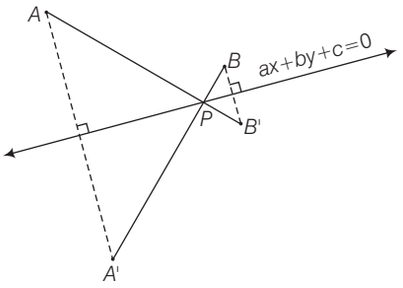
$\therefore \quad \lambda = -\frac{4}{3}$

From Eq. (iii), we get equation of AC is

$$18x + 13y - 41 = 0$$

(iii) Optimization (Minimization or Maximization)

(a) **Minimization :** Let A and B are two given points on the same side of $ax + by + c = 0$. Suppose we want to determine a point P on $ax + by + c = 0$ such that $PA + PB$ is minimum. Then find the image of A or B about the line $ax + by + c = 0$ (say A' or B') then join B' with A or A' with B wherever it intersects $ax + by + c = 0$ is the required point.



$\therefore \quad PA + PB = PA + PB'$

or $\quad PA + PB = PA' + PB$

Remark

By triangle in equality

Sum of two sides of a triangle > Third side

i.e. $|PA + PB| = |PA + PB'| = |AB'|$ (minimum value).

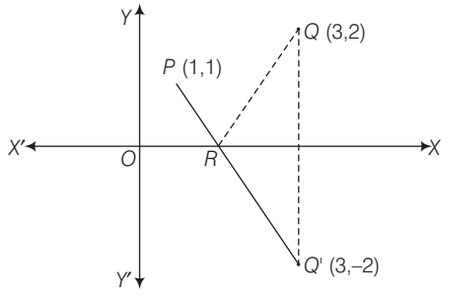
Example 101. Find a point R on the X-axis such that $PR + RQ$ is the minimum, when $P \equiv (1, 1)$ and $Q \equiv (3, 2)$.

Sol. Since P and Q lie on the same side of X-axis.

The image of $Q(3, 2)$ about X-axis is $Q'(3, -2)$ then the equation of line PQ' is

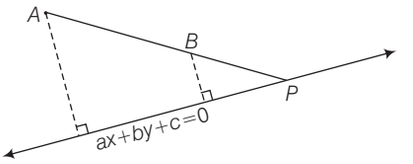
$$y - 1 = \frac{-2-1}{3-1}(x - 1)$$

$\Rightarrow \quad 3x + 2y - 5 = 0$



This line meets X-axis at $R\left(\frac{5}{3}, 0\right)$ which is the required point.

(b) **Maximization :** Let A and B are two given points on the same side of $ax + by + c = 0$. Suppose we want to determine a point P on $ax + by + c = 0$ such that $|PA - PB|$ is maximum, then find the equation of line AB wherever it intersects $ax + by + c = 0$ is the required point.



Remark

By triangle inequality

Difference of two sides of a triangle < Third side

i.e. $\quad |PA - PB| = |AB|$ (maximum value)

Example 102. Find a point P on the line $3x + 2y + 10 = 0$ such that $|PA - PB|$ is maximum where A is (4, 2) and B is (2, 4).

Sol. Let, $\quad L(x, y) = 3x + 2y + 10$

$\therefore \quad L(4, 2) = 12 + 4 + 10 = 26$

and $\quad L(2, 4) = 6 + 8 + 10 = 24$

$\therefore$ A and B lie on the same side of the line

$$3x + 2y + 10 = 0 \qquad \text{...(i)}$$

Equation of line AB is

$$y - 2 = \frac{4-2}{2-4}(x - 4)$$

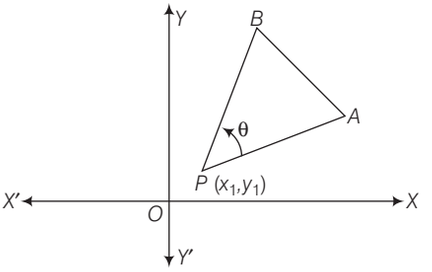
or $\quad x + y - 6 = 0 \qquad$ ...(ii)

This line Eq. (ii) meets Eq. (i) at $P \equiv (-22, 28)$ which is the required point.

Aliter

Let $\quad$ P be (x_1, y_1) and $\angle APB = \theta$

then $\quad \cos\theta = \dfrac{(PA)^2 + (PB)^2 - (AB)^2}{2PA \cdot PB}$



since $\cos\theta \le 1$

$$\Rightarrow \quad \frac{(PA)^2 + (PB)^2 - (AB)^2}{2PA \cdot PB} \le 1$$

$$\Rightarrow \quad (PA - PB)^2 \le (AB)^2$$

$$\Rightarrow \quad |PA - PB| \le |AB|$$

$$\Rightarrow \quad |PA - PB| \le 2\sqrt{2}$$

Maximum value of $|PA - PB|$ is $2\sqrt{2}$

when, $\theta = 0$.

i.e. P lies on the line AB as well as on the given line.

$\therefore$ Equation of AB is

$$y - 2 = \frac{4-2}{2-4}(x - 4)$$

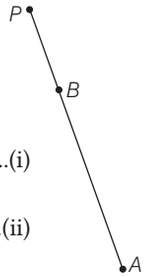
$$\Rightarrow \quad y - 2 = -x + 4$$

$$\Rightarrow \quad x + y = 6 \qquad \text{...(i)}$$

and given line

$$3x + 2y + 10 = 0 \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get $P(-22, 28)$.



Exercise for Session 5

1. The coordinates of the foot of the perpendicular from $(2, 3)$ to the line $3x + 4y - 6 = 0$ are

(a) $\left(-\frac{14}{25}, -\frac{27}{25}\right)$ (b) $\left(\frac{14}{25}, -\frac{17}{25}\right)$

(c) $\left(-\frac{14}{25}, \frac{17}{25}\right)$ (d) $\left(\frac{14}{25}, \frac{27}{25}\right)$

2. If the foot of the perpendicular from the origin to a straight line is at the point $(3, -4)$. Then the equation of the line is

(a) $3x - 4y = 25$ (b) $3x - 4y + 25 = 0$

(c) $4x + 3y - 25 = 0$ (d) $4x - 3y + 25 = 0$

3. The coordinates of the foot of the perpendicular from (a, 0) on the line $y = mx + \frac{a}{m}$ are

(a) $\left(0, -\frac{1}{a}\right)$ (b) $\left(0, \frac{a}{m}\right)$ (c) $\left(0, -\frac{a}{m}\right)$ (d) $\left(0, \frac{1}{a}\right)$

4. If the equation of the locus of a point equidistant from the points (a_1, b_1) and (a_2, b_2) is $(a_1 - a_2)x + (b_1 - b_2)y + c = 0$, then the value of c is

(a) $a_1^2 - a_2^2 + b_1^2 - b_2^2$ (b) $\sqrt{(a_1^2 + b_1^2 - a_2^2 - b_2^2}$

(c) $\frac{1}{2}(a_1^2 + a_2^2 + b_1^2 + b_2^2)$ (d) $\frac{1}{2}(a_2^2 + b_2^2 - a_1^2 - b_1^2)$

5. The image of the point (3, 8) in the line $x + 3y = 7$ is

(a) (1, 4) (b) (4, 1) (c) (–1, – 4) (d) (–4, – 1)

6. The image of the point (4, –3) with respect to the line $y = x$ is

(a) (–4, – 3) (b) (3, 4) (c) (–4, 3) (d) (–3, 4)

7. The coordinates of the image of the origin O with respect to the straight line $x + y + 1 = 0$ are

(a) $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ (b) (–2, – 2)

(c) (1, 1) (d) (–1, – 1)

8. If $(-2, 6)$ is the image of the point (4, 2) with respect to the line $L = 0$, then $L \equiv$

(a) $6x - 4y - 7 = 0$ (b) $2x - 3y - 5 = 0$

(c) $3x - 2y + 5 = 0$ (d) $3x - 2y + 10 = 0$

9. The image of $P(a, b)$ on the line $y = -x$ is Q and the image of Q on the line $y = x$ is R. Then the mid-point of PR is

(a) $(a + b, a + b)$ (b) $\left(\frac{a+b}{2}, \frac{b+2}{2}\right)$

(c) $(a - b, b - a)$ (d) $(0, 0)$

10. The nearest point on the line $3x - 4y = 25$ from the origin is

(a) $(3, 4)$ (b) $(3, -4)$ (c) $(3, 5)$ (d) $(-3, 5)$

11. Consider the points $A(0, 1)$ and $B(2, 0)$, P be a point on the line $4x + 3y + 9 = 0$. The coordinates of P such that $|PA - PB|$ is maximum are

(a) $\left(-\frac{12}{5}, \frac{17}{5}\right)$ (b) $\left(-\frac{84}{5}, \frac{13}{5}\right)$ (c) $\left(-\frac{6}{5}, \frac{17}{5}\right)$ (d) $(0, -3)$

12. Consider the points $A(3, 4)$ and $B(7, 13)$. If P is a point on the line $y = x$ such that $PA + PB$ is minimum, then the coordinates of P are

(a) $\left(\frac{12}{7}, \frac{12}{7}\right)$ (b) $\left(\frac{13}{7}, \frac{13}{7}\right)$ (c) $\left(\frac{31}{7}, \frac{31}{7}\right)$ (d) $(0, 0)$

13. The image of a point $P(2, 3)$ in the line mirror $y = x$ is the point Q and the image of Q in the line mirror $y = 0$ is the point $R(x, y)$. Find the coordinates of R.

14. The equations of perpendicular bisector of the sides AB and AC of a ΔABC are $x - y + 5 = 0$ and $x + 2y = 0$ respectively. If the point A is $(1, -2)$, find the equation of line BC.

15. In a ΔABC, the equation of the perpendicular bisector of AC is $3x - 2y + 8 = 0$. If the coordinates of the point A and B are $(1, -1)$ and $(3, 1)$ respectively, find the equation of the line BC and the centre of the circumcircle of ΔABC.

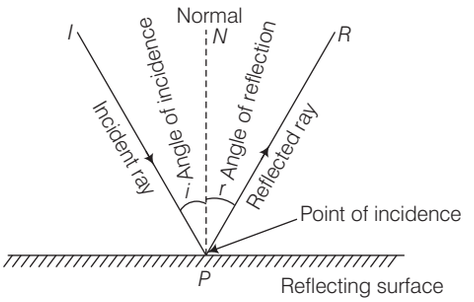
16. Is there a real value of λ for which the image of the point $(\lambda, \lambda - 1)$ by the line mirror $3x + y = 6\lambda$ is the point $(\lambda^2 + 1, \lambda)$? If so find λ.

Session 6

Reflection of Light, Refraction of Light, Conditions of Collinearity if Three Given Points be in Cyclic Order

Reflection of Light

When a ray of light falls on a smooth polished surface (Mirror) separating two media, a part of it is reflected back into the first medium.



IP is the incident ray and PR is the reflected ray. A perpendicular drawn to the surface, at the point of incidence P is called the normal. Hence PN is the normal. The angle between the incident ray and the normal ($\angle IPN$) is called the **angle of incidence** which is represented by $\angle i$

i.e. $\angle IPN = \angle i$ = Angle of incidence and the angle between the reflected ray and the normal ($\angle IPR$) is called the **angle of reflection** which is represented by $\angle r$.

i.e. $\angle IPR = \angle r$ = Angle of reflection.

Laws of Reflection :

(i) The incident ray, normal and the reflected ray to a surface at the point of incidence all lie in the same plane.

(ii) The angle of incidence = angle of reflection

i.e. $\angle i = \angle r$

Example 103. A ray of light is sent along the line $x - 2y - 3 = 0$. Upon reaching the line $3x - 2y - 5 = 0$, the ray is reflected from it. Find the equation of the line containing the reflected ray.

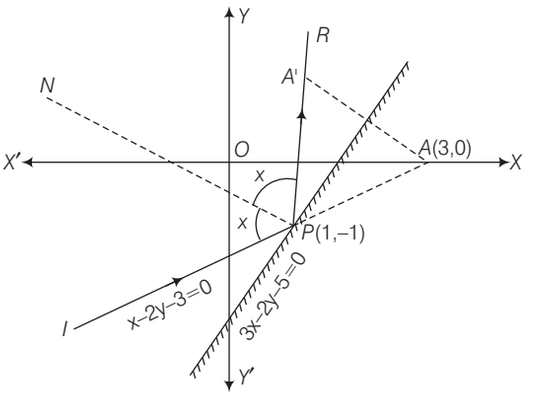
Sol. To get coordinates of point P, we solve the given equation of lines together as

$$x - 2y - 3 = 0$$

$$3x - 2y - 5 = 0$$

$\therefore \quad x = 1, y = -1$

$\therefore$ Coordinates of P are $(1, -1)$.



Let slope of reflected ray be m.

Since, slope of line mirror is 3/2.

$\therefore$ Slope of $PN = -2/3$ and

slope of $IP = 1/2$, line PN is equally inclined to IP and PR, then

$$\left(\frac{m-\left(-\frac{2}{3}\right)}{1+m\left(-\frac{2}{3}\right)}\right) = -\left(\frac{\frac{1}{2}-\left(-\frac{2}{3}\right)}{1+\frac{1}{2}\left(-\frac{2}{3}\right)}\right)$$

$$\Rightarrow \quad \frac{3m+2}{3-2m} = -\frac{7}{4}$$

$$\Rightarrow \quad 12m + 8 = -21 + 14m$$

$$\therefore \quad 2m = 29$$

$$\Rightarrow \quad m = \frac{29}{2}$$

$\therefore$ Equation of reflected ray $y + 1 = \frac{29}{2}(x - 1)$

$$\Rightarrow \quad 2y + 2 = 29x - 29$$

$$\Rightarrow \quad 29x - 2y - 31 = 0.$$

Aliter (Image method) : Take A (3, 0) be any point on IP and if $A'(\alpha, \beta)$ be the image of A about the mirror line $3x - 2y - 5 = 0$, then

$$\frac{\alpha - 3}{3} = \frac{\beta - 0}{-2} = \frac{-2(9 - 0 - 5)}{9 + 4}$$

$$\therefore \qquad \alpha = \frac{15}{13} \quad \text{and} \quad \beta = \frac{16}{13}$$

$$\therefore \qquad A' \equiv \left(\frac{15}{13}, \frac{16}{13}\right)$$

$\therefore$ Equation of $A'P$ is the equation of the reflected ray then its equation is,

$$y + 1 = \frac{\left(\frac{16}{13} + 1\right)}{\left(\frac{15}{13} - 1\right)}(x - 1)$$

$$\Rightarrow \qquad y + 1 = \frac{29}{2}(x - 1) \text{ or } 29x - 2y - 31 = 0$$

Example 104. A light beam, emanating from the point (3, 10) reflects from the straight line $2x + y - 6 = 0$ and, then passes through the point (7, 2). Find the equations of the incident and reflected beams.

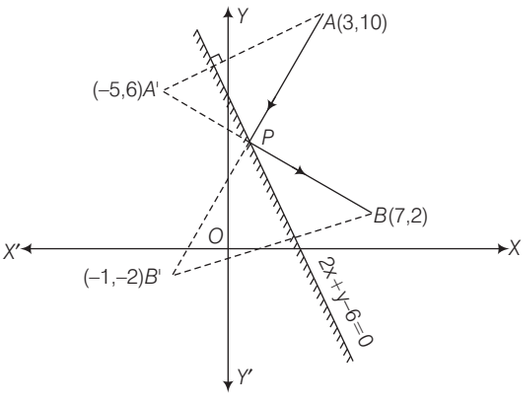
Sol. Let images of A and B about the line $2x + y - 6 = 0$ are $A'(\alpha, \beta)$ and $B'(\gamma, \delta)$ respectively.

Then, $$\frac{\alpha - 3}{2} = \frac{\beta - 10}{1} = \frac{-2(6 + 10 - 6)}{2^2 + 1^2}$$

$$= -4$$

$$\therefore \qquad \alpha = -5, \beta = 6$$

i.e. $$A' \equiv (-5, 6)$$



and $$\frac{\gamma - 7}{2} = \frac{\delta - 2}{1} = \frac{-2(14 + 2 - 6)}{2^2 + 1^2} = -4$$

$$\therefore \qquad \gamma = -1, \delta = -2$$

i.e. $$B' \equiv (-1, -2).$$

$\therefore$ Equation of incident ray AB' is

$$y + 2 = \frac{10 + 2}{3 + 1}(x + 1) \text{ or } 3x - y + 1 = 0$$

and equation of reflected ray $A'B$ is

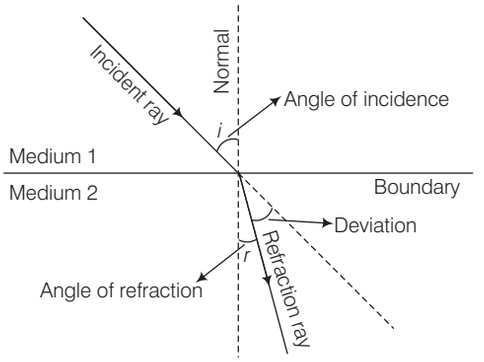
$$y - 6 = \frac{2 - 6}{7 + 5}(x + 5)$$

$$\Rightarrow \qquad y - 6 = -\frac{1}{3}(x + 5)$$

or $$x + 3y - 13 = 0$$

Refraction of Light

When a ray of light falls on the boundary separating the two transparent media, there is a change in direction of ray. This phenomenon is called refraction.



Laws of Refraction

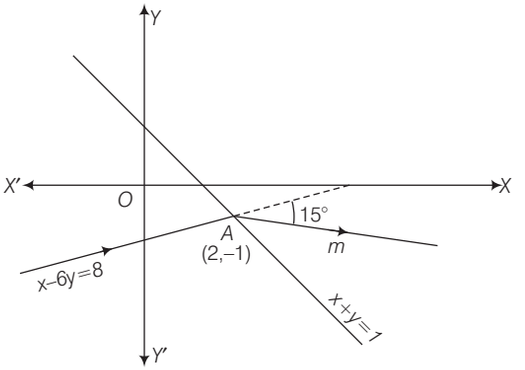
(i) The incident ray, normal and the refracted ray to the surface separating the two transparent media all lie in the same plane.

(ii) The ratio of sine of angle of incidence to the sine of the angle of refraction is constant for the two given media. The constant is called the refractive index of medium 2 with respect to medium 1.

i.e. $$_1\mu_2 = \frac{\sin i}{\sin r}$$

Example 105. A ray of light is sent along the line $x - 6y = 8$. After refracting across the line $x + y = 1$ it enters the opposite side after turning by 15° away from the line $x + y = 1$. Find the equation of the line along which the refracted ray travels.

Sol. The point of intersection of $x - 6y = 8$ and $x + y = 1$ is $A \equiv (2, -1)$. Let the required ray have the slope $= m$, then



$$\tan 15° = \left|\frac{m - \frac{1}{6}}{1 + \frac{m}{6}}\right|$$

$$\Rightarrow \qquad 2-\sqrt{3}=\pm\left(\frac{6m-1}{m+6}\right)$$

then, $$\frac{6m-1}{m+6}=2-\sqrt{3} \text{ or } \sqrt{3}-2$$

$$\Rightarrow \qquad m=\frac{70-37\sqrt{3}}{13}$$

or $$m=\frac{37\sqrt{3}-70}{61}$$

Let the angle between $x+y=1$ and the line through $A(2,-1)$ with the slope $\frac{70-37\sqrt{3}}{13}$ be α, then

$$\tan\alpha=\left|\frac{\frac{70-37\sqrt{3}}{13}-(-1)}{1-\left(\frac{70-37\sqrt{3}}{13}\right)}\right|=\left|\frac{83-37\sqrt{3}}{37\sqrt{3}-57}\right|$$

$$=\frac{83-37\sqrt{3}}{37\sqrt{3}-57}$$

and if angle between $x+y=1$ and the line through $A(2,-1)$ with the slope $\frac{37\sqrt{3}-70}{61}$ be β, then

$$\tan\beta=\left|\frac{\frac{37\sqrt{3}-70}{61}-(-1)}{1-\left(\frac{37\sqrt{3}-70}{61}\right)}\right|=\left|\frac{37\sqrt{3}-9}{131-37\sqrt{3}}\right|$$

$$=\frac{37\sqrt{3}-9}{131-37\sqrt{3}}$$

Here $\tan\alpha>\tan\beta, \therefore \alpha>\beta$

therefore the slope of the refracted ray $=\frac{70-37\sqrt{3}}{13}$

$\therefore$ The equation of the refracted ray is

$$y+1=\frac{(70-37\sqrt{3})}{13}(x-2)$$

$$\Rightarrow \qquad 13y+13=(70-37\sqrt{3})\,x-140+74\sqrt{3}$$

or $$(70-37\sqrt{3})\,x-13y-153+74\sqrt{3}=0$$

Conditions of Collinearity if Three given points are in Cyclic Order

Let the three given points

$$A\equiv(f(a),g(a)),\ B\equiv(f(b),g(b))$$

and $C\equiv(f(c),g(c))$ lie on the line

$lx+my+n=0$, where l,m and n are constants.

Then $$lf(t)+m\,g(t)+n=0 \qquad \text{...(i)}$$

where, $t=a,b,c$

i.e. a, b, c are the roots of the Eq. (i).

In this case Eq. (i) must be cubic in t.

$$At^3+Bt^2+Ct+D=0 \qquad \text{(say)}$$

then $$a+b+c=-\frac{B}{A}, ab+bc+ca=\frac{C}{A}$$

and $$abc=-\frac{D}{A}$$

which are the required conditions.

Example 106. If the points $\left(\frac{a^3}{a-1},\frac{a^2-3}{a-1}\right),\left(\frac{b^3}{b-1},\frac{b^2-3}{b-1}\right)$ and $\left(\frac{c^3}{c-1},\frac{c^2-3}{c-1}\right)$ are collinear for three distinct values a,b,c and different from 1, then show that $abc-(bc+ca+ab)+3(a+b+c)=0$.

Sol. Let the three given points lie on the line

$$lx+my+n=0, \qquad \text{where } l, m \text{ and } n \text{ are constants.}$$

Then, $$l\left(\frac{t^3}{t-1}\right)+m\left(\frac{t^2-3}{t-1}\right)+n=0$$

$$\Rightarrow \qquad lt^3+mt^2+nt-(3m+n)=0$$

for $t=a, b, c$

i.e. a, b, c are the roots of

$$lt^3+mt^2+nt-3m-n=0$$

then $$a+b+c=-\frac{m}{l}, ab+bc+ca=\frac{n}{l}$$

and $$abc=\left(\frac{3m+n}{l}\right)$$

Now, $abc-(bc+ca+ab)+3(a+b+c)$

$$=\left(\frac{3m+n}{l}\right)-\frac{n}{l}-\frac{3m}{l}=0$$

Hence, $abc-(bc+ca+ab)+3(a+b+c)=0$

Example 107. If t_1,t_2 and t_3 are distinct, the points $(t_1,2at_1+at_1^3),(t_2,2at_2+at_2^3)$ and $(t_3,2at_3+at_3^3)$ are collinear, then prove that $t_1+t_2+t_3=0$.

Sol. Let the three given points lie on the line $lx+my+n=0$, where l, m and n are constants. Then,

$$l(t)+m(2at+at^3)+n=0$$

$$\Rightarrow \qquad (am)\,t^3+(2am+l)\,t+n=0 \qquad \text{...(i)}$$

for $t=t_1,t_2,t_3$

i.e., t_1, t_2, t_3 are the roots of Eq. (i), then

$$t_1+t_2+t_3=0$$

Exercise for Session 6

1. A ray of light passing through the point (1, 2) is reflected on the X-axis at a point P and passes through the point (5, 3). The abscissae of the point P is

(a) 3 (b) $\frac{13}{3}$ (c) $\frac{13}{5}$ (d) $\frac{13}{4}$

2. The equation of the line segment AB is $y = x$. If A and B lie on the same side of the line mirror $2x - y = 1$, then the image of AB has the equation

(a) $x + y = 2$ (b) $8x + y = 9$ (c) $7x - y = 6$ (d) None of these

3. A ray of light travelling along the line $x + y = 1$ is incident on the X-axis and after refraction it enters the other side of the X-axis by turning $\pi / 6$ away from the X-axis. The equation of the line along which the refracted ray travels is

(a) $x + (2 - \sqrt{3})\, y = 1$ (b) $x\,(2 + \sqrt{3}) + y = 2 + \sqrt{3}$
(c) $(2 - \sqrt{3})x + y = 1$ (d) $x + (2 + \sqrt{3})y = (2 + \sqrt{3})$

4. All the points lying inside the triangle formed by the points $(0, 4)$, $(2, 5)$ and $(6, 2)$ satisfy

(a) $3x + 2y + 8 \geq 0$ (b) $2x + y - 10 \geq 0$
(c) $2x - 3y - 11 \geq 0$ (d) $-2x + y - 3 \geq 0$

5. Let O be the origin and let $A(1, 0)$, $B(0, 1)$ be two points. If $P(x, y)$ is a point such that $xy > 0$ and $x + y < 1$ then

(a) P lies either inside in ΔOAB or in third quadrant (b) P cannot be inside in ΔOAB
(c) P lies inside the ΔOAB (d) None of these

6. A ray of light coming along the line $3x + 4y - 5 = 0$ gets reflected from the line $ax + by - 1 = 0$ and goes along the line $5x - 12y - 10 = 0$ then

(a) $a = \frac{64}{115}, b = \frac{112}{15}$ (b) $a = -\frac{64}{115}, b = \frac{8}{115}$
(c) $a = \frac{64}{115}, b = \frac{8}{115}$ (d) $a = -\frac{64}{115}, b = \frac{-8}{115}$

7. Two sides of a triangle have the joint equation $x^2 - 2xy - 3y^2 + 8y - 4 = 0$. The third side, which is variable, always passes through the point $(-5, -1)$. Find the range of values of the slope of the third side, so that the origin is an interior point of the triangle.

8. Determine the range of values of $\theta \in [0, 2\pi]$ for which $(\cos\theta, \sin\theta)$ lies inside the triangle formed by the lines $x + y - 2 = 0$, $x - y - 1 = 0$ and $6x + 2y - \sqrt{10} = 0$.

9. Let $P(\sin\theta, \cos\theta)$, where $0 \leq \theta \leq 2\pi$ be a point and let OAB be a triangle with vertices (0, 0), $\left(\frac{\sqrt{3}}{2}, 0\right)$ and $\left(0, \frac{\sqrt{3}}{2}\right)$. Find θ if P lies inside the ΔOAB.

10. Find all values of θ for which the point $(\sin^2\theta, \sin\theta)$ lies inside the square formed by the line $xy = 0$ and $4xy - 2x - 2y + 1 = 0$.

11. Determine whether the point $(-3, 2)$ lies inside or outside the triangle whose sides are given by the equations $x + y - 4 = 0$, $3x - 7y + 8 = 0$, $4x - y - 31 = 0$.

12. A ray of light is sent along the line $x - 2y + 5 = 0$, upon reaching the line $3x - 2y + 7 = 0$, the ray is reflected from it. Find the equation of the line containing the reflected ray.

Shortcuts and Important Results to Remember

1. Area of parallelogram formed by the lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + d_1 = 0$, $a_1x + b_1y + c_2 = 0$ and $a_2x + b_2y + d_2 = 0$ is

$$\frac{|c_1 - c_2||d_1 - d_2|}{\left|\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\right|}$$

2. Area of parallelogram formed by the lines $y = m_1x + c_1$, $y = m_2x + d_1$, $y = m_1x + c_2$ and $y = m_2x + d_2$ is

$$\frac{|c_1 - c_2||d_1 - d_2|}{|m_1 - m_2|}$$

3. If $A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$ and $C \equiv (x_3, y_3)$ are the vertices of a ΔABC, then angle A is acute or obtuse according as

$$(x_1 - x_2)(x_1 - x_3) + (y_1 - y_2)(y_1 - y_3) > 0 \quad \text{or} \quad < 0$$

Similarly, for $\angle B$

$$(x_2 - x_3)(x_2 - x_1) + (y_2 - y_3)(y_2 - y_1) > 0 \quad \text{or} \quad < 0$$

and for $\angle C$

$$(x_3 - x_1)(x_3 - x_2) + (y_3 - y_1)(y_3 - y_2) > 0 \quad \text{or} \quad < 0$$

4. If the origin lies in the acute angle or obtuse angle between the lines

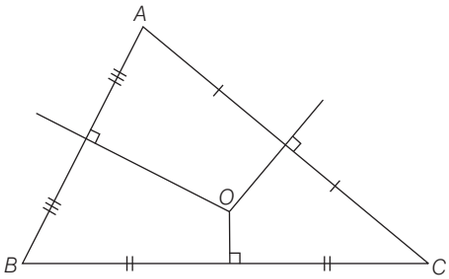
$$a_1x + b_1y + c_1 = 0$$

and $\quad a_2x + b_2y + c_2 = 0$

according as $\quad (a_1a_2 + b_1b_2)c_1c_2 < 0$ or > 0.

5. If $A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$ and $C \equiv (x_3, y_3)$ are the vertices of a ΔABC then the equations of the right bisectors (perpendicular bisectors) of the sides BC, CA and AB are

$$y(y_2 - y_3) + x(x_2 - x_3) = \left(\frac{x_2^2 - x_3^2}{2}\right) + \left(\frac{y_2^2 - y_3^2}{2}\right);$$



$$y(y_3 - y_1) + x(x_3 - x_1) = \left(\frac{x_3^2 - x_1^2}{2}\right) + \left(\frac{y_3^2 - y_1^2}{2}\right)$$

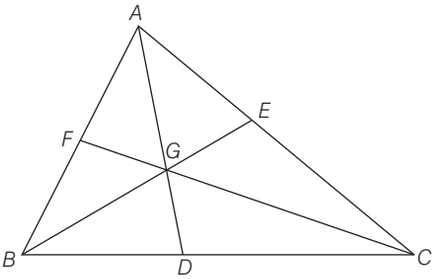
and $\quad y(y_1 - y_2) + x(x_1 - x_2) = \left(\frac{x_1^2 - x_2^2}{2}\right) + \left(\frac{y_1^2 - y_2^2}{2}\right)$

respectively. Where O is the circumcentre of ΔABC.

6. If $A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$ and $C \equiv (x_3, y_3)$ are the vertices of a ΔABC then the equations of medians AD, BE and CF are

$$y(x_2 + x_3 - 2x_1) - x(y_2 + y_3 - 2y_1)$$
$$= y_1(x_2 + x_3) - x_1(y_2 + y_3);$$
$$y(x_3 + x_1 - 2x_2) - x(y_3 + y_1 - 2y_2)$$
$$= y_2(x_3 + x_1) - x_2(y_3 + y_1)$$

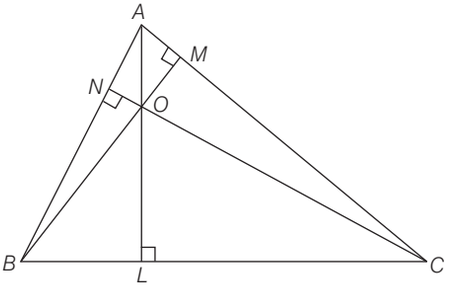
and $y(x_1 + x_2 - 2x_3) - x(y_1 + y_2 - 2y_3) = y_3(x_1 + x_2) - x_3(y_1 + y_2)$ respectively.



Where, G is the centroid of ΔABC.

7. If $A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$ and $C \equiv (x_3, y_3)$ are the vertices of a ΔABC then the equations of the altitudes AL, BM and CN are

$$y(y_2 - y_3) + x(x_2 - x_3) = y_1(y_2 - y_3) + x_1(x_2 - x_3);$$
$$y(y_3 - y_1) + x(x_3 - x_1) = y_2(y_3 - y_1) + x_2(x_3 - x_1)$$



and $\quad y(y_1 - y_2) + x(x_1 - x_2) = y_3(x_1 - x_2) + x_3(x_1 - x_2)$

respectively. Where O is the orthocentre of ΔABC.

8. If sides of a triangle ABC are represented by

$$BC : a_1x + b_1y + c_1 = 0,$$
$$CA : a_2x + b_2y + c_2 = 0$$

and $\quad AB : a_3x + b_3y + c_3 = 0$

then $\quad |BC| : |CA| : |AB|$

$$= \sqrt{(a_1^2 + b_1^2)}\left|\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}\right|$$

$$: \sqrt{(a_2^2 + b_2^2)}\left|\begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix}\right| : \sqrt{(a_3^2 + b_3^2)}\left|\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\right|$$

JEE Type Solved Examples :
Single Option Correct Type Questions

■ This section contains **10 multiple choice examples**. Each example has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

● ***Ex. 1*** *A rectangle ABCD has its side AB parallel to $y = x$ and vertices A, B and D lie on $y = 1$, $x = 2$ and $x = -2$ respectively, then locus of vertex C is*

(a) $x = 5$ (b) $x - y = 5$

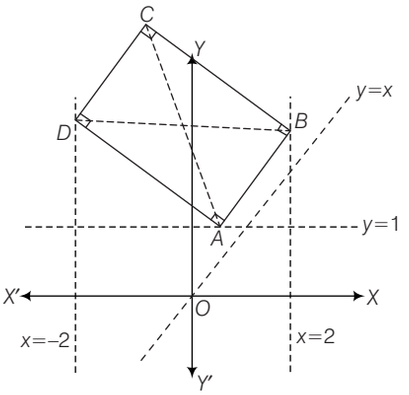
(c) $y = 5$ (d) $x + y = 5$

Sol. (c) Since AB is parallel to $y = x$.

$\therefore$ Equation of AB is $y = x + a$ (say)

$\because$ A lies on $y = 1$

$\therefore$ $A \equiv (1 - a, 1)$



Again, B lies on $x = 2$

$\therefore$ $B = (2, 2 + a)$

$\Rightarrow$ Equation of AD is

$y - 1 = -[x - (1 - a)]$ or $y = 2 - x - a$

$\because$ D lies on $x = -2$

$\therefore$ $D \equiv (-2, 4 - a)$

Let $C \equiv (h, k)$

$\because$ Diagonals of rectangle bisects to each other

$\therefore$ $h + 1 - a = 2 - 2 \Rightarrow a = 1 + h$

and $k + 1 = 2 + a + 4 - a$

$\Rightarrow$ $k = 5$

$\therefore$ Locus of C is $y = 5$

● ***Ex. 2*** *The line $(\lambda + 1)^2 x + \lambda y - 2\lambda^2 - 2 = 0$ passes through a point regardless of the value λ. Which of the following is the line with slope 2 passing through the point?*

(a) $y = 2x - 8$ (b) $y = 2x - 5$

(c) $y = 2x - 4$ (d) $y = 2x + 8$

Sol. (a)

$\because$ $(\lambda + 1)^2 x + \lambda y - 2\lambda^2 - 2 = 0$

or $(\lambda^2 + 2\lambda + 1)x + \lambda y - 2\lambda^2 - 2 = 0$

or $(\lambda^2 + 1)(x - 2) + \lambda(2x + y) = 0$

$\therefore$ For fixed point

$x - 2 = 0$ and $2x + y = 0$

$\therefore$ Fixed point is $(2, -4)$

$\therefore$ Equation of required line is $y + 4 = 2(x - 2)$

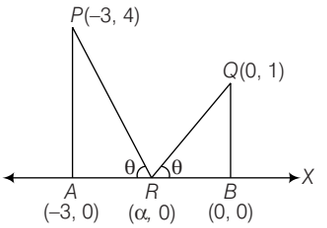
or $y = 2x - 8$

● ***Ex. 3*** *A man starts from the point $P(-3, 4)$ and reaches point $Q(0, 1)$ touching X-axis at R such that $PR + RQ$ is minimum, then the point R is*

(a) $\left(\frac{3}{5}, 0\right)$ (b) $\left(-\frac{3}{5}, 0\right)$ (c) $\left(-\frac{2}{5}, 0\right)$ (d) $(-2, 0)$

Sol. (b) Let $R = (\alpha, 0)$

For $PR + PQ$ to be minimum it should be the path of light and thus we have



$\Delta APR \sim \Delta BQR$

$\Rightarrow$ $\frac{AR}{RB} = \frac{PA}{QB} \Rightarrow \frac{\alpha + 3}{0 - \alpha} = \frac{4}{1} \Rightarrow \alpha = -\frac{3}{5}$

Hence, $R \equiv \left(-\frac{3}{5}, 0\right)$

● ***Ex. 4*** *If the point $P(a, a^2)$ lies inside the triangle formed by the lines $x = 0$, $y = 0$ and $x + y = 2$, then exhaustive range of 'a' is*

(a) $(0, 1)$ (b) $(1, \sqrt{2})$

(c) $(\sqrt{2} - 1, 1)$ (d) $(\sqrt{2} - 1, 2)$

Sol. (a) Since the point $P(a, a^2)$ lies on $y = x^2$

Solving, $y = x^2$

and $x + y = 2$, we get

$x^2 + x - 2 = 0$

$\Rightarrow$ $(x + 2)(x - 1) = 0$

$\Rightarrow$ $x = -2, 1$

It is clear from figure,

$A \equiv (1, 1)$

also $a > 0$ for I quadrant.

$\therefore$ $a \in (0, 1)$

● **Ex. 5** *If* $5a + 4b + 20c = t$, *then the value of* t *for which the line* $ax + by + c - 1 = 0$ *always passes through a fixed point is*

(a) 0 (b) 20

(c) 30 (d) None of these

Sol. (b) Equation of line $\frac{ax}{c-1} + \frac{by}{c-1} + 1 = 0$ has two independent parameters. It can pass through a fixed point if it contains only one independent parameter. Now there must be one relation between $\frac{a}{c-1}$ and $\frac{b}{c-1}$ independent of a, b and c so that $\frac{a}{c-1}$ can be expressed in terms of $\frac{b}{c-1}$ and straight line contains only one independent parameter. Now that given relation can be expressed as $\frac{5a}{c-1} + \frac{4b}{c-1} = \frac{t-20c}{c-1}$.

RHS is independent of c if $t = 20$.

● **Ex. 6** *If the straight lines,* $ax + amy + 1 = 0$, $bx + (m+1)by + 1 = 0$ *and* $cx + (m+2)cy + 1 = 0$, $m \neq 0$ *are concurrent, then* a, b, c *are in*

(a) AP only for $m = 1$ (b) AP for all m

(c) GP for all m (d) HP for all m

Sol. (d) The three lines are concurrent if

$$\begin{vmatrix} a & am & 1 \\ b & m+1 & 1 \\ c & (m+2)c & 1 \end{vmatrix} = 0$$

Applying $C_2 \to C_2 - mC_1$, then

$$\begin{vmatrix} a & 0 & 1 \\ b & b & 1 \\ c & 2c & 1 \end{vmatrix} = 0$$

or $a(b - 2c) - 0 + 1(2bc - bc) = 0$

or $b = \frac{2ac}{a+c}$, which is independent of m.

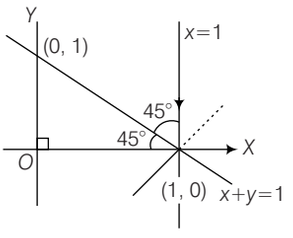
$\therefore$ a, b, c are in HP for all m.

● **Ex. 7** *If a ray travelling the line* $x = 1$ *gets reflected the line* $x + y = 1$, *then the equation of the line along which the reflected ray travels is*

(a) $y = 0$ (b) $x - y = 1$

(c) $x = 0$ (d) None of these

Sol. (a) Reflected ray is X-axis.



$\therefore$ Equation $y = 0$

● **Ex. 8** *Through the point* $P(\alpha, \beta)$, *when* $\alpha\beta > 0$, *the straight line* $\frac{x}{a} + \frac{y}{b} = 1$ *is drawn so as to form with coordinate axes a triangle of area* Δ. *If* $ab > 0$, *then the least value of* Δ *is*

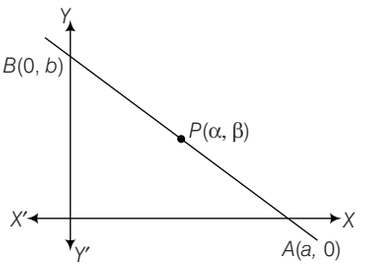
(a) $\alpha\beta$ (b) $2\alpha\beta$

(c) $4\alpha\beta$ (d) $8\alpha\beta$

Sol. (b) Given line is

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad \ldots(i)$$

$\therefore$ $A \equiv (a, 0), B \equiv (0, b)$



$\because$ Area of $(\Delta AOB) = \Delta$

$\therefore$ $\frac{1}{2}|ab| = \Delta \Rightarrow ab = 2\Delta$ $(\because ab > 0)$

Since, the line (i) passes through the point $P(\alpha, \beta)$.

$\therefore$ $\frac{\alpha}{a} + \frac{\beta}{b} = 1 \Rightarrow \frac{\alpha}{a} + \frac{a\beta}{2\Delta} = 1$ $\left(\because b = \frac{2\Delta}{a}\right)$

$\Rightarrow$ $\alpha^2\beta - 2a\Delta + 2\Delta\alpha = 0$

$\because a$ is real

$\therefore$ $D \geq 0$

$\Rightarrow$ $4\Delta^2 - 4\beta(2\Delta\alpha) \geq 0$ or $\Delta \geq 2\alpha\beta$

$\therefore$ Least value of Δ is $2\alpha\beta$.

● **Ex. 9** *The coordinates of the point* P *on the line* $2x + 3y + 1 = 0$, *such that* $|PA - PB|$ *is maximum, where* A *is* $(2, 0)$ *and* B *is* $(0, 2)$ *is*

(a) $(5, -3)$ (b) $(7, -5)$

(c) $(9, -7)$ (d) $(11, -9)$

Sol. (b) $|PA - PB| \leq |AB|$

Maximum value of $|PA - PB|$ is $|AB|$, which is possible only when P, A, B are collinear

if $P(x, y)$, then equation AB is

$$\frac{x}{2} + \frac{y}{2} = 1$$

$\Rightarrow$ $x + y = 2$ ...(i)

Now solving Eq. (i)

and $2x + 3y + 1 = 0$ ...(ii)

Then, we get,

$$x = 7, y = -5$$

$\therefore$ $P \equiv (7, -5)$

● ***Ex. 10*** *Equation of the straight line which belongs to the system of straight lines* $a(2x+y-3)+b(3x+2y-5)=0$ *and is farthest from the point (4, −3) is*

(a) $4x+11y-15=0$ (b) $3x-4y+1=0$

(c) $7x+y-8=0$ (d) None of these

Sol. (b) The system of straight lines $a(2x+y-3)+b(3x+2y-5)=0$ passes through the point of intersection of the lines $2x+y-3=0$ and $3x+2y-5=0$ i.e. (1, 1)

The line of this family which is farthest from $(4,-3)$ is the line through (1, 1) and perpendicular to the line joining (1, 1) and $(4,-3)$

∴ The required line is

$$y-1=\frac{3}{4}(x-1)$$

or $$3x-4y+1=0$$

JEE Type Solved Examples :
More than One Correct Option Type Questions

■ This section contains **5 multiple choice examples**. Each example has four choices (a), (b), (c) and (d) out of which **MORE THAN ONE** may be correct.

● ***Ex. 11*** *The vertices of a square inscribed in the triangle with vertices* $A(0,0)$, $B(2,1)$ *and* $C(3,0)$, *given that two of its vertices are on the side AC, are*

(a) $\left(\frac{3}{2},0\right)$ (b) $\left(\frac{3}{2},\frac{3}{4}\right)$ (c) $\left(\frac{9}{4},\frac{3}{4}\right)$ (d) $\left(\frac{9}{4},0\right)$

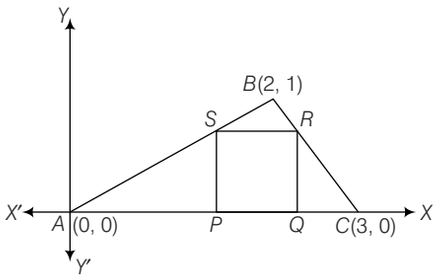
Sol. (a, b, c, d)

Let $PQRS$ be a square inscribed in ΔABC and

$$PQ=QR=RS=SP=\lambda \quad \text{(say)}$$

Let $P\equiv(a,0)$,

∴ $Q\equiv(a+\lambda,\lambda)$, $R\equiv(a+\lambda,\lambda)$ and $S\equiv(a,\lambda)$



Now equation of AB is

$$x-2y=0 \quad \text{...(i)}$$

and equation of BC is

$$x+y-3=0 \quad \text{...(ii)}$$

∵ S lies on AB, then

$$a-2\lambda=0 \quad \text{...(iii)}$$

and R lies on BC, then

$$a+\lambda+\lambda-3=0 \text{ or } a+2\lambda-3=0 \quad \text{...(iv)}$$

From Eqs. (iii) and (iv), we get $a=\frac{3}{2}, \lambda=\frac{3}{4}$

Hence, $P\equiv\left(\frac{3}{2},0\right)$, $Q\equiv\left(\frac{9}{4},0\right)$,

$R\equiv\left(\frac{9}{4},\frac{3}{2}\right)$, $S=\left(\frac{3}{2},\frac{3}{4}\right)$

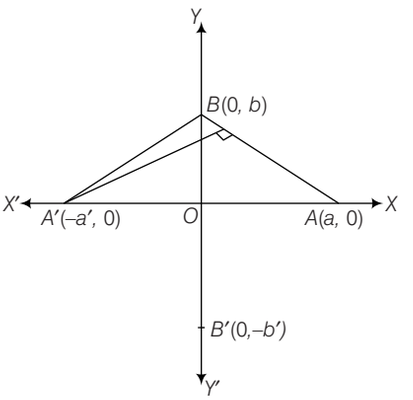
● ***Ex. 12*** *Line* $\frac{x}{a}+\frac{y}{b}=1$ *cuts the coordinate axes at* $A(a,0)$ *and* $B(0,b)$ *and the line* $\frac{x}{a'}+\frac{y}{b'}=-1$ *at* $A'(-a',0)$ *and* $B'(0,-b')$. *If the points A, B, A′, B′ are concyclic, then the orthocentre of the triangle ABA′ is*

(a) (0, 0) (b) $(0, b')$

(c) $\left(0,\frac{-aa'}{b}\right)$ (d) $\left(0,\frac{bb'}{a}\right)$

Sol. (b, c)

∵ A, B, A', B' are concyclic then,



$$OA\cdot OA'=OB\cdot OB'$$

or $$(a)\cdot(-a')=(b)\cdot(-b')$$

or $$aa'=bb' \quad \text{...(i)}$$

The equation of altitude through A' is

$$y-0=\frac{a}{b}(x+a')$$

It intersects the altitude

$$x=0 \text{ at } y=\frac{aa'}{b}$$

∴ Orthocentre is $\left(0,\frac{aa'}{b}\right)$ or $(0,b')$ [from Eq. (i)]

● ***Ex. 13*** *Two straight lines $u=0$ and $v=0$ passes through the origin and angle between them is $\tan^{-1}\left(\frac{7}{9}\right)$. If the ratio of the slope of $v=0$ and $u=0$ is $\frac{9}{2}$, then their equations are*

(a) $y=3x$ and $3y=2x$
(b) $2y=3x$ and $3y=x$
(c) $y+3x=0$ and $3y+2x=0$
(d) $2y+3x=0$ and $3y+x=0$

Sol. (a, b, c, d)

Let the slope of $u=0$ be m, then the slope of $v=0$ is $\frac{9m}{2}$.

Therefore, $$\left|\frac{m-\frac{9m}{2}}{1+m\times\frac{9m}{2}}\right|=\frac{7}{9}$$

or $$\left|\frac{-7m}{2+9m^2}\right|=\frac{7}{9}$$

$\Rightarrow$ $$9m^2+9m+2=0 \quad \text{or} \quad 9m^2-9m+2=0$$

$\Rightarrow$ $$m=-\frac{2}{3},-\frac{1}{3} \text{ or } m=\frac{2}{3},\frac{1}{3}$$

Therefore, the equation of lines are
(i) $2x+3y=0$ and $3x+y=0$
(ii) $x+3y=0$ and $3x+2y=0$
(iii) $2x=3y$ and $3x=y$
(iv) $x=3y$ and $3x=2y$

● ***Ex. 14*** *Two sides of a rhombus OABC (lying entirely in the first or third quadrant) of are equal to 2 sq units are $y=\frac{x}{\sqrt{3}}$, $y=\sqrt{3}x$. Then the possible coordinates of B is/are (O being the origin)*

(a) $(1+\sqrt{3},1+\sqrt{3})$ (b) $(-1-\sqrt{3},-1-\sqrt{3})$
(c) $(3+\sqrt{3},3+\sqrt{3})$ (d) $(\sqrt{3}-1,\sqrt{3}-1)$

Sol. (a, b)

Here, $\angle COA=30°$

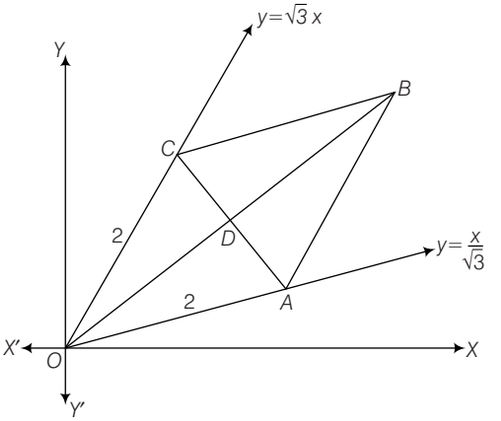
Let $OA=AB=BC=CO=x$

$\because$ Area of rhombus $OABC$

$$=2\times\frac{1}{2}\times x\times x\sin 30°$$

$$=\frac{x^2}{2}=2 \qquad \text{[given]}$$

$\therefore$ $x=2$



Coordinates of A and C are $(\sqrt{3},1)$ and $(1,\sqrt{3})$ in I quadrant and in III quadrant are $(-\sqrt{3},-1)$ and $(-1,-\sqrt{3})$

Hence, coordinates of B are $(\sqrt{3}+1,\sqrt{3}+1)$ and $(-\sqrt{3}-1,-\sqrt{3}-1)$

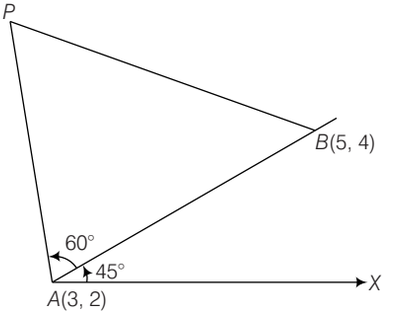
● ***Ex. 15*** *A and B are two fixed points whose coordinates are (3, 2) and (5, 4) respectively. The coordinates of a point P, if ABP is an equilateral triangle are*

(a) $(4-\sqrt{3},3+\sqrt{3})$ (b) $(4+\sqrt{3},3-\sqrt{3})$
(c) $(3-\sqrt{3},4+\sqrt{3})$ (d) $(3+\sqrt{3},4-\sqrt{3})$

Sol. (a, b)

$\because$ $AB=AP=BP=2\sqrt{2}$

$\therefore$ Coordinates of P are $(3+2\sqrt{2}\cos 105°,\ 2+2\sqrt{2}\sin 105°)$



or $(3-(\sqrt{3}-1),2+\sqrt{3}+1)$

or $(4-\sqrt{3},3+\sqrt{3})$

If P below AB, then coordinates of P are

$(3+2\sqrt{2}\cos 15°,2-2\sqrt{2}\sin 15°)$

or $[(3+\sqrt{3}+1,2-(\sqrt{3}-1)]$

or $(4+\sqrt{3},3-\sqrt{3})$

JEE Type Solved Examples : Paragraph Based Questions

- This section contains **2 solved paragraphs** based upon each of the paragraph **3 multiple choice questions** have to be answered. Each of these question has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Paragraph I

(Q. Nos. 16 to18)

Let $d(P, OA) \le \min\cdot\{d(P, AB), d(P, BC), d(P, OC)\}$, where d denotes the distance from the point to the corresponding the line and R be the region consisting of all those points P inside the rectangle OABC such that $O \equiv (0, 0)$, $A \equiv (3, 0)$, $B \equiv (3, 2)$ and $C \equiv (0, 2)$. Let M be the peak of region R.

16. Length of the perpendicular from M to OA is

(a) 4 (b) 3 (c) 2 (d) 1

17. If λ be the perimeter of region R, then λ is

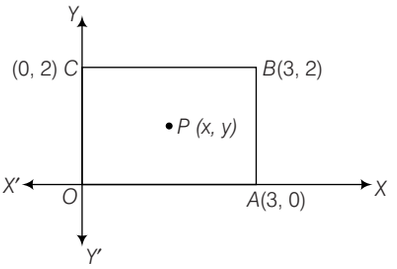
(a) $4 - \sqrt{2}$ (b) $4 + \sqrt{2}$ (c) $4 + 2\sqrt{2}$ (d) 10

18. If Δ be the area of region R, then Δ is

(a) 2 (b) 4 (c) 6 (d) 8

Sol. Let $P \equiv (x, y)$

$\because$ $d(P, OA) \le \min\cdot\{d(P, AB), d(P, BC), d(P, OC)\}$



$\Rightarrow$ $|y| \le \min\{|3 - x|, |2 - y|, |x|\}$

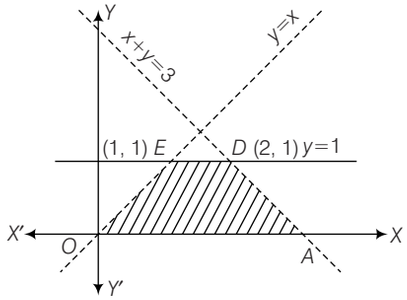
As the rectangle $OABC$ lies in I quadrant,

$\therefore$ $y \le \min\cdot[3 - x, 2 - y. x]$

We draw the graph of

$y = 3 - x, y = 2 - y, y = x$

or $x + y = 3, y = 1, y = x$



$\therefore$ $E \equiv (1, 1)$ and $D \equiv (2, 1)$

16. (d) $\because$ M lie on $y = 1$

$\therefore$ Length of $\perp$ from M to OA is 1.

17. (c) λ = Perimeter of region R

$= OA + AD + DE + EO$

$= 3 + \sqrt{2} + 1 + \sqrt{2}$

$= 4 + 2\sqrt{2}$

18. (a) Δ = Area of region R

$= \frac{1}{2}(OA + ED) \times 1$

$= \frac{1}{2}(3 + 1) = 2$

Paragraph II

(Q. Nos. 19 to 21)

A variable straight line 'L' is drawn through $O(0, 0)$ to meet this lines $L_1 : y - x - 10 = 0$ and $L_2 : y - x - 20 = 0$ at the points A and B respectively.

19. A point P is taken on 'L' such that $\frac{2}{OP} = \frac{1}{OA} + \frac{1}{OB}$, then the locus of P is

(a) $3x + 3y - 40 = 0$ (b) $3x + 3y + 40 = 0$

(c) $3x - 3y - 40 = 0$ (d) $3x - 3y + 40 = 0$

20. A point P is taken on 'L' such that $(OP)^2 = OA \cdot OB$, then the locus of P is

(a) $(y - x)^2 = 25$ (b) $(y - x)^2 = 50$

(c) $(y - x)^2 = 100$ (d) $(y - x)^2 = 200$

21. A point P is taken on 'L' such that $\frac{1}{(OP)^2} = \frac{1}{(OA)^2} + \frac{1}{(OB)^2}$, then locus of P is

(a) $(y - x)^2 = 32$ (b) $(y - x)^2 = 64$

(c) $(y - x)^2 = 80$ (d) $(y - x)^2 = 100$

Sol. Let the equation of line 'L' through origin is

$$\frac{x - 0}{\cos\theta} = \frac{y - 0}{\sin\theta} = r$$

$\therefore$ $P \equiv (r\cos\theta, r\sin\theta)$

Let $OA = r_1$ and $OB = r_2$

$\therefore$ $A \equiv (r_1\cos\theta, r_1\sin\theta)$

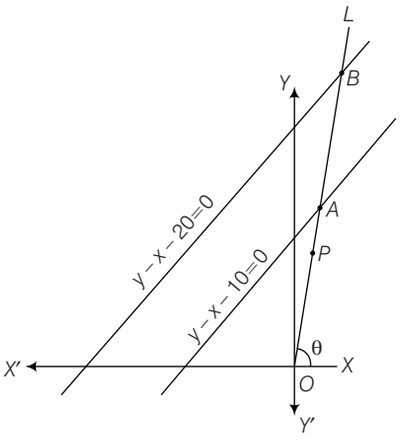
and $B \equiv (r_2\cos\theta, r_2\sin\theta)$

A lies on $L_1 : y - x - 10 = 0$

$\therefore$ $r_1\sin\theta - r_1\cos\theta - 10 = 0$

$\Rightarrow$ $r_1 = \frac{10}{\sin\theta - \cos\theta}$ ...(i)

$\Rightarrow$ B lies on $L_1 : y - x - 20 = 0$



$\therefore \quad r_2 \sin\theta - r_2 \cos\theta - 20 = 0$

$\Rightarrow \quad r_2 = \dfrac{20}{\sin\theta - \cos\theta}$ ...(ii)

19. (d) $\because \quad \dfrac{2}{OP} = \dfrac{1}{OA} + \dfrac{1}{OB} \Rightarrow \dfrac{2}{r} = \dfrac{1}{r_1} + \dfrac{1}{r_2}$

$\Rightarrow \quad \dfrac{2}{r} = \dfrac{\sin\theta - \cos\theta}{10} + \dfrac{\sin\theta - \cos\theta}{20}$ [from Eqs. (i) and (ii)]

or $\quad 2 = \dfrac{r\sin\theta - r\cos\theta}{10} + \dfrac{r\sin\theta - r\cos\theta}{20}$

or $\quad 2 = \dfrac{y - x}{10} + \dfrac{y - x}{20}$ $[\because P \equiv (r\cos, r\sin\theta)]$

$\therefore$ Locus of P is $3x - 3y + 40 = 0$

20. (d) $\because (OP)^2 = OA \cdot OB$

$\Rightarrow \quad r^2 = r_1 \cdot r_2$

$\Rightarrow \quad r^2 = \dfrac{10}{(\sin\theta - \cos\theta)} \cdot \dfrac{20}{(\sin\theta - \cos\theta)}$ [from Eqs. (i) and (ii)]

or $\quad (r\sin\theta - r\cos\theta)^2 = 200$

$\therefore$ Locus of P is $(y - x)^2 = 200$

21. (c) $\because \quad \dfrac{1}{(OP)^2} = \dfrac{1}{(OA)^2} + \dfrac{1}{(OB)^2}$

$\Rightarrow \quad \dfrac{1}{r^2} = \dfrac{1}{r_1^2} + \dfrac{1}{r_2^2}$

$\Rightarrow \quad \dfrac{1}{r^2} = \dfrac{(\sin\theta - \cos\theta)^2}{100} + \dfrac{(\sin\theta - \cos\theta)^2}{400}$

or $\quad 400 = 4(r\sin\theta - r\cos\theta)^2 + (r\sin\theta - r\cos\theta)^2$

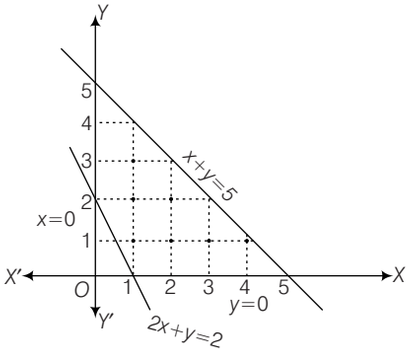
$\therefore$ Locus of P is $(y - x)^2 = 80$

JEE Type Solved Examples :
Single Integer Answer Type Questions

- This section contains **2 examples**. The answer to each example is **a single digit integer,** ranging from 0 to 9 (both inclusive).

● ***Ex. 22*** *$P(x, y)$ is called a natural point if $x, y \in N$. The total number of points lying inside the quadrilateral formed by the lines $2x + y = 2$, $x = 0$, $y = 0$ and $x + y = 5$ is*

Sol. (6) First, we construct the graph of the given quadrilateral.



It is clear from the graph that there are six points lying inside the quadrilateral.

● ***Ex. 23*** *The distance of the point (x, y) from the origin is defined as $d = \max\cdot\{|x|, |y|\}$. Then the distance of the common point for the family of lines $x(1 + \lambda) + \lambda y + 2 + \lambda = 0$ (λ being parameter) from the origin is*

Sol. (2) Given family of lines is

$$x(1 + \lambda) + \lambda y + 2 + \lambda = 0$$

$\Rightarrow \quad (x + 2) + \lambda(x + y + 1) = 0$

for common point or fixed point

$$x + 2 = 0$$

and $\quad x + y + 1 = 0$

or $\quad x = -2, y = 1$

$\therefore$ Common point is $(-2, 1)$

or $\quad d = \max\{|-2|, |1|\}$

$= \max\{2, 1\} = 2$

JEE Type Solved Examples :
Matching Type Questions

- This section contains **2 examples**. Examples 24 and 25 has four statements (A, B, C and D) given in **Column I** and four statements (p, q, r and s) in **Column II**. Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II**.

● ***Ex. 24*** *Consider the following linear equations x and y*

$$ax + by + c = 0$$
$$bx + cy + a = 0$$
$$cx + ay + b = 0$$

	Column I		Column II
(A)	$a+b+c \neq 0$ and $a^2+b^2+c^2 = ab+bc+ca$	(p)	Lines are sides of a triangle
(B)	$a+b+c = 0$ and $a^2+b^2+c^2 \neq ab+bc+ca$	(q)	Lines are different and concurrent
(C)	$a+b+c \neq 0$ and $a^2+b^2+c^2 \neq ab+bc+ca$	(r)	Number of pair (x, y) satisfying the equations are infinite
(D)	$a+b+c = 0$ and $a^2+b^2+c^2 = ab+bc+ca$	(s)	Lines are identical

Sol. (A) $\rightarrow (r, s)$; (B) $\rightarrow (q)$; (C) $\rightarrow (p)$; (D) $\rightarrow (r)$

(A) if $a+b+c \neq 0$ and $a^2+b^2+c^2 = ab+bc+ca$

or $\frac{1}{2}\{(a-b)^2+(b-c)^2+(c-a)^2\} = 0$

or $a-b=0, b-c=0, c-a=0$

or $a=b=c \Rightarrow$ All the lines an identical

and number of pair (x, y) are infinite.

(B) If $a+b+c=0$ and $a^2+b^2+c^2 \neq ab+bc+ca$

$\Rightarrow a+b+c=0$, but a, b, c are not simultaneously equal.

Hence, lines are different and concurrent.

(C) If $a+b+c \neq 0$ and $a^2+b^2+c^2 \neq ab+bc+ca$

$$\Rightarrow \Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \neq 0$$ and a, b, c are not all simultaneously equal.

$\therefore$ Lines are sides of a triangle.

(D) If $a+b+c=0$ and $a^2+b^2+c^2 = ab+bc+ca$

$\Rightarrow \Delta = 0$ and $a=b=c$

$\therefore$ Equations are satisfied for any (x, y).

● ***Ex. 25*** *The equation of the sides of a triangle are* $x+2y+1=0, 2x+y+2=0$ *and* $px+qy+1=0$ *and area of triangle is* Δ.

	Column I		Column II
(A)	$p=2, q=3$, then 8Δ is divisible by	(p)	3
(B)	$p=3, q=2$, then 8Δ is divisible by	(q)	4
(C)	$p=3, q=4$, then 10Δ is divisible by	(r)	6
(D)	$p=4, q=3$, then 20Δ is divisible by	(s)	9

Sol. (A) $\rightarrow (p)$; (B) $\rightarrow (p, q, r)$; (C) $\rightarrow (p, r)$; (D) $\rightarrow (p, s)$

$\because$ $\Delta = \frac{D^2}{2|C_1\ C_2\ C_3|}$, where

$$D = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ p & q & 1 \end{vmatrix} = 3(p-1)$$

and C_1, C_2, C_3 are co-factors of third column, then

$C_1 = 2q-p, C_2 = 2p-q, C_3 = -3$

$\therefore$ $\Delta = \frac{3(p-1)^2}{2|2q-p||2p-q|}$

(A) for $p=2, q=3$

$\Rightarrow \Delta = \frac{3}{8}$

$\therefore 8\Delta = 3$

(B) for $p=3, q=2$

$\Rightarrow \Delta = \frac{3}{2}$

$\therefore 8\Delta = 12$

(C) for $p=3, q=4$

$\Rightarrow \Delta = \frac{6}{10}$

$\therefore 10\Delta = 6$

(D) for $p=4, q=3$

$\Rightarrow \Delta = \frac{27}{20}$

$\therefore 20\Delta = 27$

JEE Type Solved Examples :
Statement I and II Type Questions

■ **Directions** (Ex. Nos. 26 and 27) are Assertion-Reason type examples. Each of these examples contains two statements.

Statement I (Assertion) and **Statement II** (Reason)

Each of these examples also has four alternative choices. Only one of which is the correct answer. You have to select the correct choice as given below.

(a) Statement I is true, statement II is true; statement II is a correct explanation for statement I.

(b) Statement I is true, statement II is true; statement II is not a correct explanation for statement I.

(c) Statement I is true, statement II is false.

(d) Statement I is false, statement II is true.

● ***Ex. 26*** *Consider the lines,* $L_1 : \frac{x}{3} + \frac{y}{4} = 1; L_2 : \frac{x}{4} + \frac{y}{3} = 1;$ $L_3 : \frac{x}{3} + \frac{y}{4} = 2$ *and* $L_4 : \frac{x}{4} + \frac{y}{3} = 2$

Statement I : The quadrilateral formed by these four lines is a rhombus.

Statement II : If diagonals of a quadrilateral formed by any four lines are unequal and intersect at right angle, then it is a rhombus.

Sol. (c) $\because$ L_1, L_3 are parallel.

$\therefore$ Distance between L_1 and $L_3 = \dfrac{1}{\sqrt{\left(\frac{1}{9} + \frac{1}{16}\right)}} = \dfrac{12}{5}$ and

L_2, L_4 are parallel.

$\therefore$ Distance between L_2 and $L_4 = \dfrac{1}{\sqrt{\left(\frac{1}{16} + \frac{1}{9}\right)}} = \dfrac{12}{5}$

$\therefore$ Distance between L_1 and L_3 = Distance between L_2 and L_4.

$\therefore$ Quadrilateral formed by L_1, L_2, L_3, L_4 is a rhombus.

Hence, statement I is true and statement II is false.

● ***Ex. 27***

Statement I : Incentre of the triangle formed by the lines whose sides are $3x + 4y = 0; 5x - 12y = 0$ and $y - 15 = 0$ is the point P whose coordinates are (1, 8).

Statement II : Point P equidistant from the three lines forming the triangle.

Sol. (b) Let $L_1 \equiv 3x + 4y = 0$,

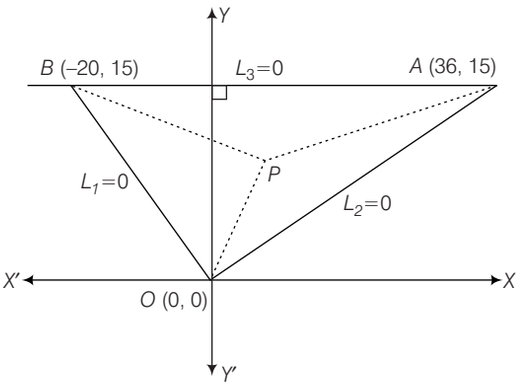
$L_2 \equiv 5x - 12y = 0$ and $L_3 \equiv y - 15 = 0$

Length of $\perp$ from P to $L_1 = \dfrac{3 + 32}{5} = 7$

Length of $\perp$ from P to $L_2 = \dfrac{|5 - 96|}{13} = 7$

and Length of $\perp$ from P to $L_3 = \dfrac{|8 - 15|}{1} = 7$

$\therefore$ Statement II is true



Also, Area of $\Delta OPA = \dfrac{1}{2} \times OA \times 7$

$= \dfrac{1}{2} \times 39 \times 7 = \Delta_1$

Area of $\Delta OPB = \dfrac{1}{2} \times OB \times 7$

$= \dfrac{1}{2} \times 25 \times 7 = \Delta_2$

and Area of $\Delta APB = \dfrac{1}{2} \times AB \times 7$

$= \dfrac{1}{2} \times 56 \times 7 = \Delta_3$

$\therefore$ $\Delta_1 + \Delta_2 + \Delta_3 = \dfrac{7}{2}(39 + 25 + 56)$

$= \dfrac{7 \times 120}{2} = \dfrac{1}{2} \times 56 \times 15 =$ Area of ΔAOB

$\Rightarrow P$ inside the triangle.

Hence, both statements are true and statement II is not correct explanation of statement I.

Subjective Type Examples

■ In this section, there are **15 subjective solved examples**.

● ***Ex. 28.*** *If x-coordinates of two points B and C are the roots of equation $x^2 + 4x + 3 = 0$ and their y-coordinates are the roots of equation $x^2 - x - 6 = 0$. If x-coordinate of B is less than x-coordinate of C and y-coordinate of B is greater than the y-coordinate of C and coordinates of a third point A be $(3, -5)$. Find the length of the bisector of the interior angle at A .*

Sol. $\because \quad x^2 + 4x + 3 = 0 \Rightarrow \quad x = -1, -3$

and $\quad x^2 - x - 6 = 0 \Rightarrow \quad x = -2, 3$

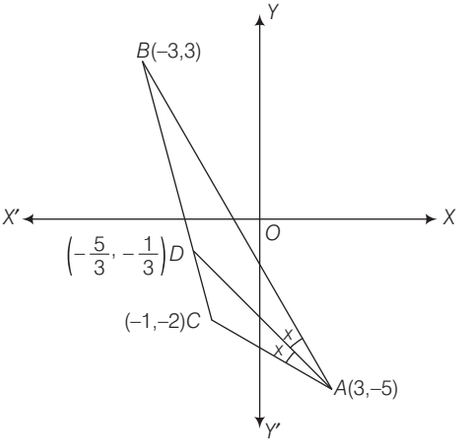
Also given that x and y-coordinates of B are respectively less than and greater than the corresponding coordinates of C.

$\therefore \quad B \equiv (-3, 3) \quad \text{and} \quad C \equiv (-1, -2)$

Now $\quad AB = \sqrt{(3+3)^2 + (-5-3)^2} = 10$

and $\quad AC = \sqrt{(3+1)^2 + (-5+2)^2} = 5$

$\therefore \quad \dfrac{AB}{AC} = \dfrac{2}{1}$



Let AD be the bisector of $\angle BAC$, then

$$\frac{BD}{DC} = \frac{AB}{AC} = \frac{2}{1}$$

Thus D divides BC internally in the ratio 2 : 1

$\therefore \quad D \equiv \left(\dfrac{2(-1) + 1(-3)}{2+1}, \dfrac{2(-2) + 1(3)}{2+1}\right)$

Thus, $\quad D \equiv \left(-\dfrac{5}{3}, -\dfrac{1}{3}\right)$

Now, $\quad AD = \sqrt{\left(3 + \dfrac{5}{3}\right)^2 + \left(-5 + \dfrac{1}{3}\right)^2}$

$$= \sqrt{\frac{196}{9} + \frac{196}{9}} = \frac{14\sqrt{2}}{3} \text{ units.}$$

● ***Ex. 29.*** *Let the sides of a parallelogram be $u = p, u = q, v = r$ and $v = s$ where, $u = lx + my + n$ and $v = l'x + m'y + n'$. Show that the equation of the diagonal through the point of intersection of $u = p$ and $v = r$ and $u = q$ and $v = s$, is given by* $\begin{vmatrix} u & v & 1 \\ p & r & 1 \\ q & s & 1 \end{vmatrix} = 0$

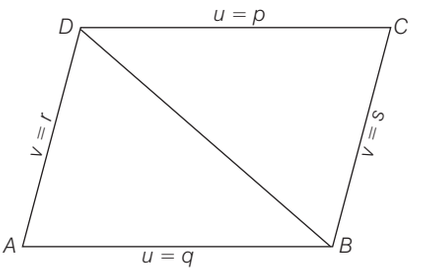
Sol. Equation of the line through point of intersection D of lines $u - p = 0$ and $v - r = 0$ is

$$(u - p) + \lambda (v - r) = 0 \qquad \ldots(i)$$

it is also passing through $u = q$ and $v = s$, then Eq. (i) becomes

$$(q - p) + \lambda (s - r) = 0$$

$\therefore \quad \lambda = -\dfrac{(q-p)}{(s-r)} \qquad \ldots(ii)$



From Eqs. (i) and (ii), we get

$$(u - p) - \frac{(q-p)}{(s-r)}(v - r) = 0$$

$$\Rightarrow \; u(s - r) - p(s - r) - v(q - p) + r(q - p) = 0$$

$$\Rightarrow \quad u(r - s) - v(p - q) + ps - qr = 0$$

$$\Rightarrow \quad \begin{vmatrix} u & v & 1 \\ p & r & 1 \\ q & s & 1 \end{vmatrix} = 0$$

● ***Ex. 30.*** *The vertices B and C of a triangle ABC lie on the lines $3y = 4x$ and $y = 0$ respectively and the side BC passes through the point $\left(\dfrac{2}{3}, \dfrac{2}{3}\right)$. If ABOC is a rhombus, O being the origin, find the equation of the line BC and the coordinates of A.*

Sol. Let the side of the rhombus be a

$\therefore \quad OB = BA = AC = CO = a$

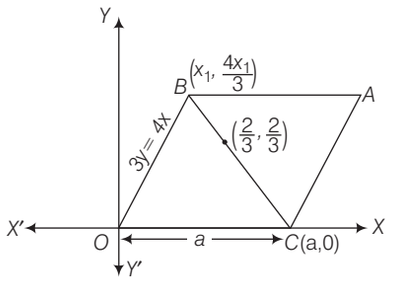
Co-ordinates of A is $\left(a + x_1, \dfrac{4x_1}{3}\right)$

$\therefore \quad (OB)^2 = (BA)^2 = (AC)^2 = (CO)^2 = a^2 \Rightarrow (OB)^2 = a^2$

$$\Rightarrow \quad x_1^2 + \frac{16x_1^2}{9} = a^2 \Rightarrow \frac{25x_1^2}{9} = a^2 \Rightarrow a = \frac{5x_1}{3}$$

$$\therefore \quad C \equiv \left(\frac{5x_1}{3}, 0\right), \quad A \equiv \left(\frac{8x_1}{3}, \frac{4x_1}{3}\right),$$

$$B \equiv \left(x_1, \frac{4x_1}{3}\right), \quad O \equiv (0, 0)$$



$\therefore$ Equation of BC

$$y - 0 = \frac{\frac{4x_1}{3} - 0}{x_1 - \frac{5x_1}{3}}\left(x - \frac{5x_1}{3}\right)$$

$$\Rightarrow \quad y = -2x + \frac{10x_1}{3} \qquad \ldots(1)$$

it is passing through $\left(\frac{2}{3}, \frac{2}{3}\right)$, then

$$\frac{2}{3} = -\frac{4}{3} + \frac{10x_1}{3}$$

$$\Rightarrow \quad 2 = \frac{10x_1}{3}$$

$$\therefore \quad x_1 = \frac{3}{5}$$

Hence, coordinates are

$$C \equiv (1, 0), \quad A \equiv \left(\frac{8}{5}, \frac{4}{5}\right)$$

$$B \equiv \left(\frac{3}{5}, \frac{4}{5}\right), \quad O \equiv (0, 0)$$

From Eq. (i), equation of BC is

$$y = -2x + 2$$

$$\Rightarrow \quad 2x + y = 2$$

● ***Ex. 31*** *The ends AB of a straight line segment of constant length c slide upon the fixed rectangular axes OX and OY respectively. If the rectangle OAPB be completed, then show that the locus of the foot of perpendicular drawn from P to AB is* $x^{2/3} + y^{2/3} = c^{2/3}$.

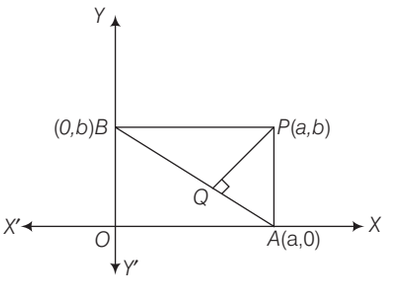
Sol. Let $A \equiv (a, 0)$, $B \equiv (0, b)$ then $P \equiv (a, b)$

Since $AB = c$

$$\sqrt{a^2 + b^2} = c$$

or $$a^2 + b^2 = c^2 \qquad \ldots(i)$$

and let $Q \equiv (x_1, y_1)$



$\because$ $PQ \perp AB$

$\therefore$ Slope of $PQ \times$ Slope of $AB = -1$

$$\Rightarrow \quad \left(\frac{b - y_1}{a - x_1}\right) \times \left(\frac{b - 0}{0 - a}\right) = -1$$

$$\Rightarrow \quad ax_1 - by_1 = a^2 - b^2 \qquad \ldots(ii)$$

$\because$ Equation of AB is $\frac{x}{a} + \frac{y}{b} = 1$

But Q lies on AB then $\frac{x_1}{a} + \frac{y_1}{b} = 1$

$$\Rightarrow \quad bx_1 + ay_1 = ab \qquad \ldots(iii)$$

From Eqs. (ii) and (iii), we get

$$x_1 = \frac{a^3}{a^2 + b^2}, \; y_1 = \frac{b^3}{a^2 + b^2}$$

Now, $$x_1^{2/3} + y_1^{2/3} = \frac{(a^2 + b^2)}{(a^2 + b^2)^{2/3}}$$

$$= (a^2 + b^2)^{1/3} = c^{2/3} \qquad \text{[from Eq. (i)]}$$

Hence, required locus is $x^{2/3} + y^{2/3} = c^{2/3}$.

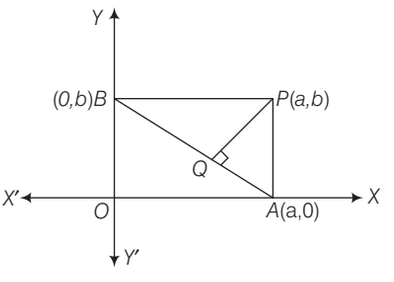
Aliter :

Since, $AB = c$

Let $\angle BAO = \theta$

$\therefore$ $OA = c\cos\theta$ and $OB = c\sin\theta$

$\because$ $OB = PA = c\sin\theta$



In ΔPQA, $$\sin\theta = \frac{QA}{PA} = \frac{QA}{c\sin\theta}$$

$\therefore$ $$QA = c\sin^2\theta$$

Now, in ΔQAM, $$\sin\theta = \frac{QM}{QA} = \frac{QM}{c\sin^2\theta}$$

$\therefore$ $$QM = c\sin^3\theta$$

and $$\cos\theta = \frac{MA}{QA} = \frac{MA}{c\sin^2\theta}$$

$\Rightarrow$ $$MA = c\sin^2\theta\cos\theta$$

If coordinates of Q be (x_1, y_1)

then $x_1 = OM = OA - MA = c\cos\theta - c\sin^2\theta\cos\theta$

$x_1 = c\cos^3\theta$ ...(i)

and $y_1 = MQ = c\sin^3\theta$ ...(ii)

From Eqs. (i) and (ii),

$$x_1^{2/3} + y_1^{2/3} = c^{2/3}\cos^2\theta + c^{2/3}\sin^2\theta = c^{2/3}$$

$\Rightarrow$ $x_1^{2/3} + y_1^{2/3} = c^{2/3}$

Hence, locus of Q is $x^{2/3} + y^{2/3} = c^{2/3}$

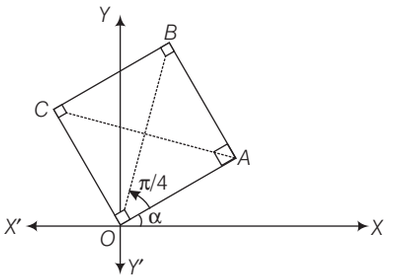
● ***Ex. 32*** *A square lies above the X-axis and has one vertex at the origin. The side passing through the origin makes an angle* $\alpha\ (0 < \alpha < \pi/4)$ *with the positive direction of the X-axis. Prove that the equation of its diagonals are*

$$y(\cos\alpha - \sin\alpha) = x(\sin\alpha + \cos\alpha)$$

and $y(\sin\alpha + \cos\alpha) + x(\cos\alpha - \sin\alpha) = a$

where, is the length of each side of the square.

Sol. Here, $OA = AB = BC = CO = a$



Equation of OB is $y - 0 = \tan\left(\frac{\pi}{4} + \alpha\right)(x - 0)$

$\Rightarrow$ $y\cos\left(\frac{\pi}{4} + \alpha\right) = x\sin\left(\frac{\pi}{4} + \alpha\right)$

$\Rightarrow$ $y(\cos\alpha - \sin\alpha) = x(\cos\alpha + \sin\alpha)$

Hence, equation of diagonal OB is

$$x(\cos\alpha + \sin\alpha) + y(\sin\alpha - \cos\alpha) = 0$$

Coordinates of A are $(a\cos\alpha, a\sin\alpha)$

Diagonal AC is perpendicular to the diagonal OB,

$$\text{Slope of } AC = -\frac{1}{\text{slope of } OB}$$

$$= -\frac{1}{\tan\left(\frac{\pi}{4} + \alpha\right)} = -\cot\left(\frac{\pi}{4} + \alpha\right)$$

Hence, equation of diagonal AC is

$$y - a\sin\alpha = -\cot\left(\frac{\pi}{4} + \alpha\right)(x - a\cos\alpha)$$

$\Rightarrow$ $y - a\sin\alpha = -\left(\frac{\cos\alpha - \sin\alpha}{\cos\alpha + \sin\alpha}\right)(x - a\cos\alpha)$

$\Rightarrow$ $y(\cos\alpha + \sin\alpha) - a\sin\alpha\cos\alpha - a\sin^2\alpha$

$$= -x(\cos\alpha - \sin\alpha) + a\cos^2\alpha - a\sin\alpha\cos\alpha$$

$\Rightarrow$ $x(\cos\alpha - \sin\alpha) + y(\cos\alpha + \sin\alpha) = a$

● ***Ex. 33*** *In a* ΔABC, $A \equiv (\alpha, \beta)$, $B \equiv (1, 2)$, $C \equiv (2, 3)$ *and point A lies on the line* $y = 2x + 3$, *where* $\alpha, \beta \in I$. *If the area of* ΔABC *be such that* $[\Delta] = 2$, *where [.] denotes the greatest integer function, find all possible coordinates of A.*

Sol. $\because$ (α, β) lies on $y = 2x + 3$

then $\beta = 2\alpha + 3$

Thus ,the coordinates of A are $(\alpha, 2\alpha + 3)$

$$\Delta = \frac{1}{2}\left|\begin{vmatrix}\alpha & 2\alpha+3\\ 1 & 2\end{vmatrix} + \begin{vmatrix}1 & 2\\ 2 & 3\end{vmatrix} + \begin{vmatrix}2 & 3\\ \alpha & 2\alpha+3\end{vmatrix}\right|$$

$$= |2\alpha - 2\alpha - 3 + 3 - 4 + 4\alpha + 6 - 3\alpha|$$

$$\Delta = \frac{1}{2}|\alpha + 2|$$

But $[\Delta] = \left[\frac{1}{2}|\alpha + 2|\right] = 2$ (given)

$\therefore$ $2 \le \frac{|\alpha + 2|}{2} < 3$

$\Rightarrow$ $4 \le |\alpha + 2| < 6$

$\Rightarrow$ $4 \le \alpha + 2 < 6$ or $-6 < \alpha + 2 \le -4$

$\Rightarrow$ $2 \le \alpha < 4$ or $-8 < \alpha \le -6$

$\therefore$ $\alpha = 2, 3, -7, -6$

then $\beta = 7, 9, -11, -9$

Hence, coordinates of A are $(2, 7), (3, 9), (-7, -11)$ and $(-6, -9)$.

● ***Ex. 34*** *Find the values of non-negative real numbers* $\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3$ *such that the algebraic sum of the perpendiculars drawn from points* $(\lambda_1, 4), (\lambda_2, 5), (\lambda_3, -3), (2, \mu_1), (3, \mu_2)$ *and* $(7, \mu_3)$ *on a variable line passing through* $(2, 1)$ *is zero.*

Sol. Let the equation of the variable line be $ax + by + c = 0$. It is given that

$$\Rightarrow \frac{(a\lambda_1 + 4b + c)}{\sqrt{(a^2 + b^2)}} + \frac{(a\lambda_2 + 5b + c)}{\sqrt{(a^2 + b^2)}} + \frac{(a\lambda_3 - 3b + c)}{\sqrt{(a^2 + b^2)}} + \frac{(2a + b\mu_1 + c)}{\sqrt{(a^2 + b^2)}} + \frac{(3a + b\mu_2 + c)}{\sqrt{(a^2 + b^2)}} + \frac{(7a + b\mu_3 + c)}{\sqrt{(a^2 + b^2)}} = 0$$

$\Rightarrow$ $a(\lambda_1 + \lambda_2 + \lambda_3 + 12) + b(\mu_1 + \mu_2 + \mu_3 + 6) + 6c = 0$

$\Rightarrow$ $a\left(\frac{\lambda_1 + \lambda_2 + \lambda_3 + 12}{6}\right) + b\left(\frac{\mu_1 + \mu_2 + \mu_3 + 6}{6}\right) + c = 0$ ...(i)

But the line passes through (2,1), therefore

$$2a + b + c = 0 \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{\lambda_1 + \lambda_2 + \lambda_3 + 12}{6} = 2 \text{ and } \frac{\mu_1 + \mu_2 + \mu_3 + 6}{6} = 1$$

$\Rightarrow$ $\lambda_1 + \lambda_2 + \lambda_3 = 0$ and $\mu_1 + \mu_2 + \mu_3 = 0$

$\Rightarrow$ $\lambda_1 = \lambda_2 = \lambda_3$ and $\mu_1 = \mu_2 = \mu_3$ $[\because \lambda_i, \mu_i \ge 0$ for all i $]$

$\lambda_1 = \lambda_2 = \lambda_3 = \alpha$ (say)

and $\mu_1 = \mu_2 = \mu_3 = \beta$ (say)

where $\alpha \ge 0$ and $\beta \ge 0$.

● **Ex. 35** *The three sides of a triangle are* $L_r \equiv x\cos\theta_r + y\sin\theta_r - p_r = 0$, *where r= 1,2,3. Show that the orthocentre is given by* $L_1\cos(\theta_2 - \theta_3) = L_2\cos(\theta_3 - \theta_1) = L_3\cos(\theta_1 - \theta_2)$.

Sol. The given lines are

$$L_1 \equiv x\cos\theta_1 + y\sin\theta_1 - p_1 = 0$$
$$L_2 \equiv x\cos\theta_2 + y\sin\theta_2 - p_2 = 0$$
$$L_3 \equiv x\cos\theta_3 + y\sin\theta_3 - p_3 = 0$$

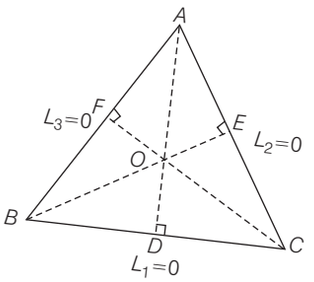
Now, equation of AD is $L_2 + \lambda L_3 = 0$ ...(i)

$\Rightarrow (x\cos\theta_2 + y\sin\theta_2 - p_2) + \lambda(x\cos\theta_3 + y\sin\theta_3 - p_3) = 0$

$\Rightarrow x(\cos\theta_2 + \lambda\cos\theta_3) + y(\sin\theta_2 + \lambda\sin\theta_3) - (p_2 + \lambda p_3) = 0$

$\therefore$ Slope of $AD = -\dfrac{(\cos\theta_2 + \lambda\cos\theta_3)}{(\sin\theta_2 + \lambda\sin\theta_3)}$

and Slope of $BC = -\dfrac{\cos\theta_1}{\sin\theta_1}$



Since, $AD \perp BC$

$\therefore$ Slope of $BC \times$ slope of $AD = -1$

$\Rightarrow -\dfrac{\cos\theta_1}{\sin\theta_1} \times \left[-\dfrac{(\cos\theta_2 + \lambda\cos\theta_3)}{(\sin\theta_2 + \lambda\sin\theta_3)}\right] = -1$

$\Rightarrow \cos\theta_1\cos\theta_2 + \lambda\cos\theta_3\cos\theta_1 = -\sin\theta_1\sin\theta_2 - \lambda\sin\theta_3\sin\theta_1$

$\Rightarrow \cos(\theta_1 - \theta_2) + \lambda\cos(\theta_3 - \theta_1) = 0$

$\therefore \lambda = -\dfrac{\cos(\theta_1 - \theta_2)}{\cos(\theta_3 - \theta_1)}$

Now from Eq. (i),

$$L_2 - \frac{\cos(\theta_1 - \theta_2)}{\cos(\theta_3 - \theta_1)} L_3 = 0$$

$\therefore$ $L_2\cos(\theta_3 - \theta_1) = L_3\cos(\theta_1 - \theta_2)$ ...(ii)

Similarly, we can obtain equation of altitude BE as

$$L_3\cos(\theta_1 - \theta_2) = L_1\cos(\theta_2 - \theta_3) \quad ...(iii)$$

From Eqs. (ii) and (iii), we get

$$L_1\cos(\theta_2 - \theta_3) = L_2\cos(\theta_3 - \theta_1) = L_3\cos(\theta_1 - \theta_2)$$

● **Ex. 36** *Let (h, k) be a fixed point, where* $h > 0, k > 0$. *A straight line passing through this point cuts the positive direction of the co-ordinate axes at the points P and Q. Find the minimum area of the triangle OPQ ,O being the origin.*

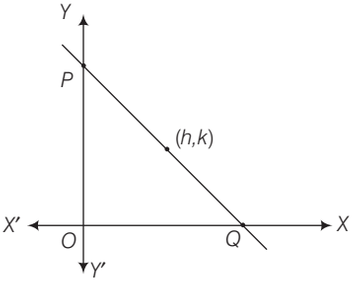
Sol. Equation of any line passing through the fixed point (h, k) and having slope m can be taken as

$$y - k = m(x - h) \quad ...(i)$$

$\Rightarrow \dfrac{y}{m} - \dfrac{k}{m} = x - h$ or $x - \dfrac{y}{m} = h - \dfrac{k}{m}$

$\Rightarrow \dfrac{x}{\left(h - \dfrac{k}{m}\right)} - \dfrac{y}{m\left(h - \dfrac{k}{m}\right)} = 1$

$\Rightarrow \dfrac{x}{\left(h - \dfrac{k}{m}\right)} + \dfrac{y}{(k - mh)} = 1$



$\therefore$ $OQ = h - \dfrac{k}{m}$ and $OP = k - mh$

Area of triangle $OPQ = \dfrac{1}{2}.OQ.OP = \dfrac{1}{2}.\left(h - \dfrac{k}{m}\right)(k - mh)$

$\Rightarrow = \dfrac{1}{2}\left(2hk - mh^2 - \dfrac{k^2}{m}\right)$

$\Rightarrow A(m) = hk - \dfrac{mh^2}{2} - \dfrac{k^2}{2m}$ ...(ii)

To minimise, $\dfrac{dA}{dm} = 0$

$\Rightarrow 0 - \dfrac{h^2}{2} + \dfrac{k^2}{2m^2} = 0 \Rightarrow m = \pm\dfrac{k}{h}$

$\therefore \left.\dfrac{d^2A}{dm^2}\right| = -\dfrac{2k^2}{2m^3}$

$$\left.\frac{d^2A}{dm^2}\right|_{m=-k/h} = \frac{h^3}{k} > 0 \qquad (\because h > 0, k > 0)$$

Hence for $m = -\dfrac{k}{h}$, $A(m)$ is minimum . Put $m = -\dfrac{k}{h}$ in Eq. (ii), we get minimum area.

$\Rightarrow$ Minimum area of $\Delta OPQ = hk + \dfrac{hk}{2} + \dfrac{hk}{2} = 2hk$

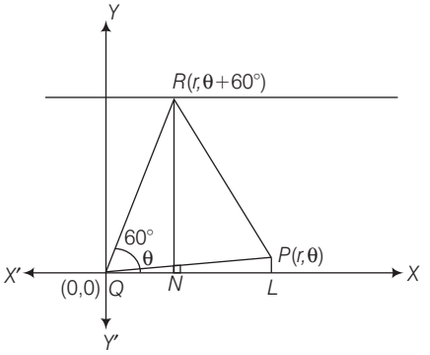
● **Ex. 37** *The distance between two parallel lines is unity. A point P lies between the lines at a distance a from one of them. Find the length of a side of an eqilateral triangle PQR, vertex Q of which lies on one of the parallel lines and vertex R lies on the other line.*

Sol. Let $PQ = QR = RP = r$

and $\angle PQL = \theta$

then $\angle XQR = \theta + 60°$

Given $PL = a$ and $RN = 1$ unit



In ΔPQL

$$\sin\theta = \frac{PL}{QP} = \frac{a}{r}$$

$\therefore$ $a = r\sin\theta$ ...(i)

and in ΔQRN,

$$r\sin(\theta + 60°) = \frac{RN}{QR} = \frac{1}{r}$$

$$\sin(\theta + 60°) = 1$$

$$(\sin\theta \cos 60° + \cos\theta \sin 60°) = 1$$

$\Rightarrow$ $$r\left(\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta\right) = 1$$

$\Rightarrow$ $$r\left(\frac{1}{2}\times\frac{a}{r} + \frac{\sqrt{3}}{2}\times\sqrt{1-\frac{a^2}{r^2}}\right) = 1 \quad \text{[from Eq. (i)]}$$

$\Rightarrow$ $$\frac{a}{2} + \frac{\sqrt{3}}{2}\sqrt{(r^2 - a^2)} = 1$$

$\Rightarrow$ $$\frac{\sqrt{3}}{2}\sqrt{(r^2 - a^2)} = 1 - \frac{a}{2}$$

or $$\frac{3}{4}(r^2 - a^2) = 1 + \frac{a^2}{4} - a$$

$\Rightarrow$ $$3r^2 - 3a^2 = 4 + a^2 - 4a$$

$\Rightarrow$ $$3r^2 = 4(a^2 - a + 1)$$

$\therefore$ $$r = \frac{2}{\sqrt{3}}\sqrt{(a^2 - a + 1)}$$

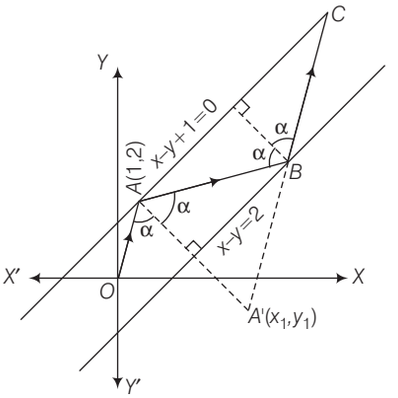
● ***Ex. 38*** *A ray of light travelling along the line OA (O being origin) is reflected by the line mirror $x - y + 1 = 0$, is the point of incidence being A (1, 2) the reflected ray, travelling along AB is again reflected by the line mirror $x - y = 2$, the point of incidence being B. If this reflected ray moves along BC, find the equation of the line BC.*

Sol. Since, slope of $OA = \frac{2-0}{1-0} = 2 = m_1$ (say)

and slope of normal to $x - y + 1 = 0$ is

$$-1 = m_2 \quad \text{(say)}$$

Now, let slope of AB is m



$\because$ OA and AB are equally inclined to normal of $x - y + 1 = 0$ then,

$$\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right) = -\left(\frac{m - m_2}{1 + m m_2}\right)$$

$\Rightarrow$ $$\left(\frac{2-(-1)}{1+2(-1)}\right) = -\left(\frac{m-(-1)}{1+m(-1)}\right)$$

$\Rightarrow$ $3(1 - m) = m + 1$

or $$m = \frac{1}{2}$$

$\therefore$ Equation of AB is

$$y - 2 = \frac{1}{2}(x - 1)$$

or $$x - 2y + 3 = 0$$

Now, solving $x - 2y + 3 = 0$

and $$x - y = 2$$

then, we get $x = 7, y = 5$

i.e. $$B \equiv (7, 5)$$

$\because$ BC is parallel to OA

$\therefore$ Equation of BC is

$$y - 5 = m_1(x - 7)$$

$\Rightarrow$ $$y - 5 = 2(x - 7)$$

i.e. $$2x - y - 9 = 0$$

Aliter :

If image of $A(1, 2)$ with respect to line mirror $x - y = 2$ be $A'(x_1, y_1)$, then

$$\frac{x_1 - 1}{1} = \frac{y_1 - 2}{-1} = \frac{-2(1 - 2 - 2)}{1 + 1}$$

or $$x_1 = 4, y_1 = -1$$

i.e. $$A' \equiv (4, -1)$$

$\because$ BC is parallel to OA

then equation of BC = equation of $A'C$ is

$$y - y_1 = 2(x - x_1)$$

$\Rightarrow$ $$y + 1 = 2(x - 4)$$

or $$2x - y - 9 = 0$$

● ***Ex. 39*** *Consider two lines $L_1 \equiv x - y = 0$ and $L_2 \equiv x + y = 0$ and a moving point $P(x, y)$. Let $d(P, L_i)$, $i = 1, 2$ represents the distance of the point 'P' from L_i. If point 'P' moves in certain region 'R' in such a way $\sum_{i=1}^{2} d(P, L_i) \in [2, 4]$.*

Find the area of region 'R'.

Sol. $\because$ Given $\sum_{i=1}^{2} d(P, L_i) \in [2, 4]$

$$\Rightarrow \quad 2 \le \sum_{i=1}^{2} d(P, L_i) \le 4$$

$$\Rightarrow \quad 2 \le d(P, L_1) + d(P, L_2) \le 4$$

$$\Rightarrow \quad 2 \le \frac{|x-y|}{\sqrt{2}} + \frac{|x+y|}{\sqrt{2}} \le 4$$

$$\Rightarrow \quad 2\sqrt{2} \le |x-y| + |x+y| \le 4\sqrt{2}$$

Case I : If $x - y > 0$, $x + y > 0$

then $\quad 2\sqrt{2} \le (x-y) + (x+y) \le 4\sqrt{2}$

$\Rightarrow \quad 2\sqrt{2} \le 2x \le 4\sqrt{2}$ or $\sqrt{2} \le x \le 2\sqrt{2}$

Case II : If $x - y > 0$, $x + y < 0$

then $\quad 2\sqrt{2} \le (x-y) - (x+y) \le 4\sqrt{2}$

$\Rightarrow \quad 2\sqrt{2} \le -2y \le 4\sqrt{2}$

or $\quad -\sqrt{2} \ge y \ge -2\sqrt{2}$ or $-2\sqrt{2} \le y \le -\sqrt{2}$

Case III : If $x - y < 0$, $x + y > 0$

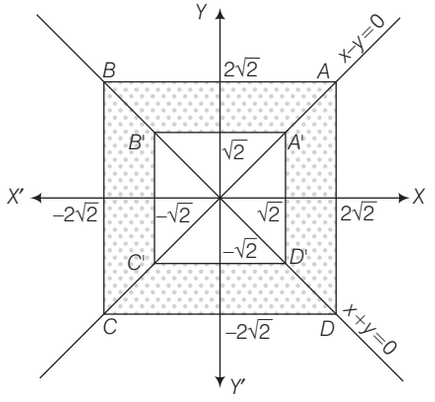
then $\quad 2\sqrt{2} \le -(x-y) + (x+y) \le 4\sqrt{2}$

$\Rightarrow \quad 2\sqrt{2} \le 2y \le 4\sqrt{2}$ or $\sqrt{2} \le y \le 2\sqrt{2}$

Case IV : If $x - y < 0$, $x + y < 0$

then $\quad 2\sqrt{2} \le -(x-y) - (x+y) \le 4\sqrt{2}$

$\Rightarrow \quad 2\sqrt{2} \le -2x \le 4\sqrt{2}$



or $\quad -\sqrt{2} \ge x \ge -2\sqrt{2}$

or $\quad -2\sqrt{2} \le x \le -\sqrt{2}$

Combining all cases, we get

$$x \in [-2\sqrt{2}, -\sqrt{2}] \cup [\sqrt{2}, 2\sqrt{2}]$$

and $\quad y \in [-2\sqrt{2}, -\sqrt{2}] \cup [\sqrt{2}, 2\sqrt{2}]$

Hence, area of the required region

$$= (4\sqrt{2})^2 - (2\sqrt{2})^2$$

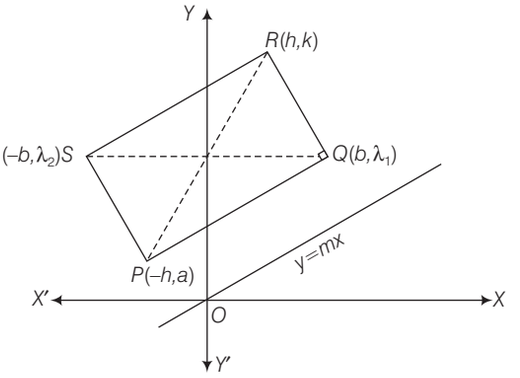
$$= 32 - 8 = 24 \text{ sq units.}$$

● ***Ex. 40*** *A rectangle PQRS has its side PQ parallel to the line $y = mx$ and vertices P, Q and S on the lines $y = a$, $x = b$ and $x = -b$ respectively. Find the locus of the vertex R.*

Sol. $\because$ PQ is parallel to $y = mx$

$\therefore$ Equation of PQ is $y = mx + \lambda$ ...(i)

$\because$ Diagonals bisect to each other



$\therefore$ x-coordinate of P is $-h$.

Suppose y-coordinate of Q and S are λ_1 and λ_2 respectively.

$\because$ Eq. (i) pass through $P(-h, a)$

then $\quad a = -mh + \lambda$

$\therefore \quad \lambda = a + mh$

$\Rightarrow \quad y = mx + a + mh$

Q also lie on it, then

$$\lambda_1 = mb + a + mh$$

$\therefore \quad Q \equiv (b, mb + a + mh)$

Also, slope of $PQ \times$ slope of $QR = -1$

$$m \times \frac{(k - (mb + a + mh))}{(h - b)} = -1$$

$$\Rightarrow \quad mk - m^2b - am - m^2h = -h + b$$

$$\Rightarrow \quad (m^2 - 1)h - mk + b(m^2 + 1) + am = 0$$

$\therefore$ Locus of R is

$$(m^2 - 1)x - my + b(m^2 + 1) + am = 0$$

● ***Ex. 41.*** *For points $P \equiv (x_1, y_1)$ and $Q \equiv (x_2, y_2)$ of the coordinate plane, a new distance $d(P, Q)$ is defined by $d(P, Q) = |x_1 - x_2| + |y_1 - y_2|$. Let $O \equiv (0, 0)$ and $A \equiv (3, 2)$. Prove that the set of the points in the first quadrant which are equidistant (with respect to the new distance) from O and A consists of the union of a line segment of finite length and an infinite ray. Sketch this set in a labelled diagram.*

Sol. Let $P(x, y)$ be any point in the first quadrant, we have

$x > 0$, $y > 0$

$$d(P, Q) = |x - 0| + |y - 0| = |x| + |y| = x + y$$

and $\quad d(P, A) = |x - 3| + |y - 2|$

Given, $\quad d(P, Q) = d(P, A)$

$$x + y = |x - 3| + |y - 2| \quad \text{...(i)}$$

Case I : $0 \le x < 3,\ 0 \le y < 2$

then Eq. (i) becomes

$$x + y = 3 - x + 2 - y$$

or $$x + y = 5/2$$

Case II : $0 \le x < 3,\ y \ge 2$

then Eq. (i) becomes

$$x + y = 3 - x + y - 2$$

or $$x = \frac{1}{2}$$

Case III : $x \ge 3,\ 0 \le y < 2$

then Eq. (i) becomes

$$x + y = x - 3 + 2 - y$$

$\Rightarrow$ $$y = -\frac{1}{2}$$ (Impossible)

Case IV : $x \ge 3,\ y \ge 2$

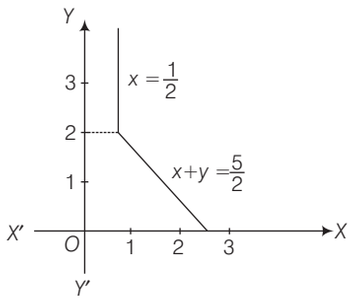
$$x + y = x - 3 + y - 2$$

$\Rightarrow$ $0 = -5$ (Impossible)

Combining all cases, then

$$x + y = 5/2;\ 0 \le x < 3,\ 0 \le y < 2$$

and $$x = \frac{1}{2};\ 0 \le x < 3,\ y \ge 2$$



The labelled diagram is given in adjoining figure.

● ***Ex. 42.*** *A line through the variable point $A(k+1, 2k)$ meets the lines $7x + y - 16 = 0, 5x - y - 8 = 0, x - 5y + 8 = 0$ at B, C, D respectively, prove that AC, AB, AD are in HP.*

Sol. Given lines are

$$7x + y - 16 = 0 \quad . \quad ..(i)$$

$$5x - y - 8 = 0 \quad ...(ii)$$

$$x - 5y + 8 = 0 \quad ...(iii)$$

Let the equation of line passing through $A(k+1, 2k)$ making an angle θ with the + ve direction of X-axis, be

$$\frac{x-(k+1)}{\cos\theta} = \frac{y-2k}{\sin\theta} = r_1, r_2, r_3 \quad (\text{if } AB = r_1,\ AC = r_2,\ AD = r_3)$$

$\therefore$ $$B \equiv [(k+1) + r_1 \cos\theta, 2k + r_1 \sin\theta]$$

$$C \equiv [(k+1) + r_2 \cos\theta, 2k + r_2 \sin\theta]$$

$$D \equiv [(k+1) + r_3 \cos\theta, 2k + r_3 \sin\theta]$$

Points B, C, D satisfying Eqs. (i), (ii) and (iii) respectively

then $$r_1 = \frac{9(1-k)}{7\cos\theta + \sin\theta}$$

$$r_2 = \frac{3(1-k)}{5\cos\theta - \sin\theta}$$

and $$r_3 = \frac{9(1-k)}{5\sin\theta - \cos\theta}$$

$\therefore$ $$\frac{1}{r_2} + \frac{1}{r_3} = \frac{(5\cos\theta - \sin\theta)}{3(1-k)} + \frac{(5\sin\theta - \cos\theta)}{9(1-k)}$$

$$= \frac{15\cos\theta - 3\sin\theta + 5\sin\theta - \cos\theta}{9(1-k)}$$

$$= \frac{14\cos\theta + 2\sin\theta}{9(1-k)} = \frac{2}{r_1}$$

Hence r_2, r_1, r_3 are in HP

i.e. AC, AB, AD are in HP.

The Straight Lines Exercise 1 :
Single Option Correct Type Questions

■ This section contains **30 multiple choice questions**. Each question has four choices (a), (b), (c), (d) out of which **ONLY ONE** is correct.

1. The straight line $y = x - 2$ rotates about a point where it cuts X-axis and becomes perpendicular on the straight line $ax + by + c = 0$, then its equation is
(a) $ax + by + 2a = 0$ (b) $ay - bx + 2b = 0$
(c) $ax + by + 2b = 0$ (d) None of these

2. If $\frac{2}{1!9!} + \frac{2}{3!7!} + \frac{1}{5!5!} = \frac{2^m}{n!}$, then orthocentre of the triangle having sides $x - y + 1 = 0$, $x + y + 3 = 0$ and $2x + 5y - 2 = 0$ is
(a) $(2m - 2n, m - n)$ (b) $(2m - 2n, n - m)$
(c) $(2m - n, m + n)$ (d) $(2m - n, m - n)$

3. If $f(x + y) = f(x)f(y) \forall\ x, y \in R$ and $f(1) = 2$, then area enclosed by $3|x| + 2|y| \le 8$ is
(a) $f(4)$ sq units (b) $\frac{1}{2}f(6)$ sq units
(c) $\frac{1}{3}f(6)$ sq units (d) $\frac{1}{3}f(5)$ sq units

4. The graph of the function $y = \cos x \cos(x + 2) - \cos^2(x + 1)$ is
(a) a straight line passing through $(0, -\sin^2 1)$ with slope 2
(b) a straight line passing through $(0, 0)$
(c) a parabola with vertex $(1, -\sin^2 1)$
(d) a straight line passing through the point $\left(\frac{\pi}{2}, -\sin^2 1\right)$ are parallel to the X-axis.

5. A line passing through the point (2, 2) and the axes enclose an area λ. The intercepts on the axes made by the line are given by the two roots of
(a) $x^2 - 2|\lambda|x + |\lambda| = 0$ (b) $x^2 + |\lambda|x + 2|\lambda| = 0$
(c) $x^2 - |\lambda|x + 2|\lambda| = 0$ (d) None of these

6. The set of value of 'b' for which the origin and the point (1, 1) lie on the same side of the straight line $a^2x + aby + 1 = 0 \forall a \in R, b > 0$ are
(a) $b \in (2, 4)$ (b) $b \in (0, 2)$
(c) $b \in [0, 2]$ (d) None of these

7. Line L has intercepts a and b on the co-ordinates axes, when the axes are rotated through a given angle; keeping the origin fixed, the same line has intercepts p and q, then
(a) $a^2 + b^2 = p^2 + q^2$ (b) $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2} + \frac{1}{q^2}$
(c) $a^2 + p^2 = b^2 + q^2$ (d) $\frac{1}{a^2} + \frac{1}{p^2} = \frac{1}{b^2} + \frac{1}{q^2}$

8. If the distance of any point (x, y) from the origin is defined as $d(x, y) = \max\{|x|, |y|\}$, $d(x, y) = a$ non-zero constant, then the locus is
(a) a circle (b) a straight line
(c) a square (d) a triangle

9. If p_1, p_2, p_3 be the perpendiculars from the points $(m^2, 2m), (mm', m + m')$ and $(m'^2, 2m')$ respectively on the line $x\cos\alpha + y\sin\alpha + \frac{\sin^2\alpha}{\cos\alpha} = 0$, then p_1, p_2, p_3 are in
(a) AP (b) GP
(c) HP (d) None of these

10. $ABCD$ is a square whose vertices A, B, C and D are (0, 0), (2, 0), (2, 2) and (0, 2) respectively. This square is rotated in the xy plane with an angle of 30° in anti-clockwise direction about an axis passing through the vertex A the equation of the diagonal BD of this rotated square is ..... . If E is the centre of the square, the equation of the circumcircle of the triangle ABE is
(a) $\sqrt{3}x + (1 - \sqrt{3})y = \sqrt{3}$, $x^2 + y^2 = 4$
(b) $(1 + \sqrt{3})x - (1 - \sqrt{2})y = 2$, $x^2 + y^2 = 9$
(c) $(2 - \sqrt{3})x + y = 2(\sqrt{3} - 1)$, $x^2 + y^2 - x\sqrt{3} - y = 0$
(d) None of the above

11. The point (4, 1) undergoes the following three successive transformations
(i) reflection about the line $y = x - 1$.
(ii) translation through a distance 1 unit along the positive direction of X-axis.
(iii) rotation through an angle $\frac{\pi}{4}$ about the origin in the anti-clockwise direction
Then, the coordinates of the final point are
(a) (4, 3) (b) $\left(\frac{7}{2}, \frac{7}{2}\right)$
(c) $(0, 3\sqrt{2})$ (d) (3, 4)

12. If the square $ABCD$, where $A(0, 0)$, $B(2, 0)$, $C(2, 2)$ and $D(0, 2)$ undergoes the following three transformations successively
(i) $f_1(x, y) \to (y, x)$
(ii) $f_2(x, y) \to (x + 3y, y)$
(iii) $f_3(x, y) \to \left(\frac{x - y}{2}, \frac{x + y}{2}\right)$
then the final figure is a
(a) square (b) parallelogram
(c) rhombus (d) None of these

13. The line $x + y = a$ meets the axes of x and y at A and B respectively. A triangle AMN is inscribed in the triangle OAB, O being the origin, with right angle at N, M and N lie respectively on OB and AB. If the area of the triangle AMN is $\frac{3}{8}$ of the area of the triangle OAB, then $\frac{AN}{BN}$ is equal to

(a) 1 (b) 2 (c) 3 (d) 4

14. If $P(1, 0)$, $Q(-1, 0)$ and $R(2, 0)$ are three given points, then the locus of point S satisfying the relation $(SQ)^2 + (SR)^2 = 2(SP)^2$ is

(a) a straight line parallel to X-axis
(b) a circle through the origin
(c) a circle with centre at the origin
(d) a straight line parallel to Y-axis

15. If $A\left(\frac{\sin\alpha}{3} - 1, \frac{\cos\alpha}{2} - 1\right)$ and $B(1, 1)$, $\alpha \in [-\pi, \pi]$ are two points on the same side of the line $3x - 2y + 1 = 0$, then α belongs to the interval

(a) $\left(-\pi, -\frac{3\pi}{4}\right] \cup \left(\frac{\pi}{4}, \pi\right)$ (b) $[-\pi, \pi]$

(c) ϕ (d) None of these

16. The line $x + y = 1$ meets X-axis at A and Y-axis at B, P is the mid-point of AB. P_1 is the foot of the perpendicular from P to OA; M_1 is that of P_1 from OP ; P_2 is that of M_1 from OA ; M_2 is that of P_2 from OP ; P_3 is that of M_2 from OA and so on. If P_n denotes the nth foot of the perpendicular on OA form M_{n-1}, then OP_n is equal to

(a) $\frac{1}{2n}$ (b) $\frac{1}{2^n}$

(c) $2^n - 1$ (d) $2^n + 3$

17. The line $x = c$ cuts the triangle with corners (0, 0); (1, 1) and (9, 1) into two regions. For the area of the two regions to be the same, then c must be equal to

(a) $\frac{5}{2}$ (b) 3

(c) $\frac{7}{2}$ (d) 3 or 15

18. If the straight lines $x + 2y = 9$, $3x - 5y = 5$ and $ax + by = 1$ are concurrent, then the straight line $5x + 2y = 1$, passes through the point

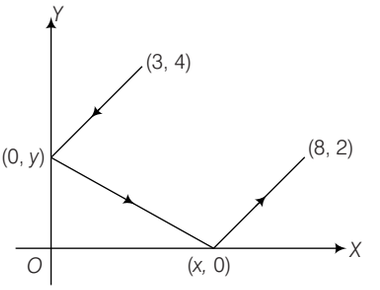
(a) $(a, -b)$ (b) $(-a, b)$
(c) (a, b) (d) $(-a, -b)$

19. The ends of the base of the isosceles triangle are at (2, 0) and (0, 1) and the equation of one side is $x = 2$, then the orthocentre of the triangle is

(a) $\left(\frac{3}{4}, \frac{3}{2}\right)$ (b) $\left(\frac{5}{4}, 1\right)$

(c) $\left(\frac{3}{4}, 1\right)$ (d) $\left(\frac{4}{3}, \frac{7}{12}\right)$

20. Suppose that a ray of light leaves the point (3, 4), reflects off the Y-axis towards the X-axis, reflects off the X-axis, and finally arrives at the point (8, 2). The value of x is



(a) $4\frac{1}{2}$ (b) $4\frac{1}{3}$ (c) $4\frac{2}{3}$ (d) $5\frac{1}{3}$

21. m, n are two integers with $0 < n < m$. A is the point (m, n) on the cartesian plane. B is the reflection of A in the line $y = x$. C is the reflection of B in the Y-axis, D is the reflection of C in the X-axis and E is the reflection of D in the Y-axis. The area of the pentagon $ABCDE$ is

(a) $2m(m + n)$ (b) $m(m + 3n)$
(c) $m(2m + 3n)$ (d) $2m(m + 3n)$

22. A straight line L with negative slope passes through the point (8, 2) and cuts the positive coordinates axes at points P and Q. As L varies, the absolute minimum value of $OP + OQ$ is (O is origin)

(a) 10 (b) 18 (c) 16 (d) 12

23. Drawn from origin are two mutually perpendicular lines forming an isosceles triangle together with the straight line $2x + y = a$, then the area of this triangle is

(a) $\frac{a^2}{2}$ sq units (b) $\frac{a^2}{3}$ sq units

(c) $\frac{a^2}{5}$ sq units (d) None of these

24. The number of integral values of m for which the x-coordinate of the point of intersection of the lines $3x + 4y = 9$ and $y = mx + 1$ is also an integer is

(a) 2 (b) 0 (c) 4 (d) 1

25. A ray of light coming from the point (1, 2) is reflected at a point A on the X-axis and then passes through the point (5, 3). The coordinates of the point A are

(a) $\left(\frac{13}{5}, 0\right)$ (b) $\left(\frac{5}{13}, 0\right)$

(c) (–7, 0) (d) None of these

26. Consider the family of lines $5x + 3y - 2 + \lambda(3x - y - 4) = 0$ and $x - y + 1 + \mu(2x - y - 2) = 0$. Equation of straight line that belong to both families is $ax + by - 7 = 0$, then $a + b$ is

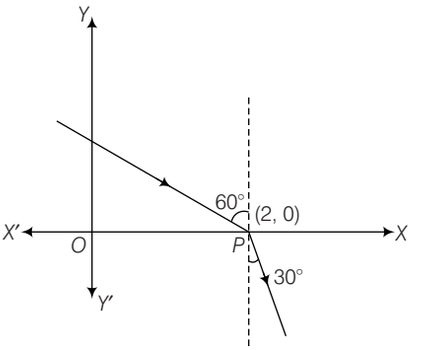
(a) 1 (b) 3 (c) 5 (d) 7

27. In ΔABC equation of the right bisectors of the sides AB and AC are $x + y = 0$ and $x - y = 0$ respectively. If $A \equiv (5, 7)$, then equation of side BC is
(a) $7y = 5x$ (b) $5x = y$
(c) $5y = 7x$ (d) $5y = x$

28. Two particles start from the point $(2, -1)$, one moving 2 units along the line $x + y = 1$ and the other 5 units along the line $x - 2y = 4$. If the particles move towards increasing y, then their new positions are
(a) $(2 - \sqrt{2}, \sqrt{2} - 1); (2\sqrt{2} + 2, \sqrt{5} - 1)$
(b) $(2\sqrt{2} + 2, \sqrt{5} - 1); (2\sqrt{2}, \sqrt{2} + 1)$
(c) $(2 + \sqrt{2}, \sqrt{2} + 1); (2\sqrt{2} + 2, \sqrt{5} + 1)$
(d) $(2 - \sqrt{2}, \sqrt{5} - 1); (\sqrt{2} - 1, 2\sqrt{2} + 2)$

29. Let P be $(5, 3)$ and a point R on $y = x$ and Q on the X-axis be such that $PQ + QR + RP$ is minimum, then the coordinates of Q are
(a) $\left(\frac{17}{8}, 0\right)$ (b) $\left(\frac{17}{4}, 0\right)$
(c) $\left(\frac{17}{2}, 0\right)$ (d) $(17, 0)$

30. In the adjacent figure combined equation of the incident and refracted ray is



(a) $(x-2)^2 + y^2 + \frac{4}{\sqrt{3}}(x-2)y = 0$
(b) $(x-2)^2 + y^2 - \frac{4}{\sqrt{3}}(x-2)y = 0$
(c) $(x-2)^2 + y^2 + \frac{y}{\sqrt{3}}(x-2) = 0$
(d) $(x-2)^2 + y^2 - \frac{y}{\sqrt{3}}(x-2) = 0$

The Straight Lines Exercise 2 :
More than One Correct Option Type Questions

- The section contains **15 multiple choice questions**. Each question has four choices (a), (b), (c), and (d) out of which **MORE THAN ONE** may be correct.

31. The point of intersection of the lines $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{b} + \frac{y}{a} = 1$ lies on
(a) $x - y = 0$
(b) $(x + y)(a + b) = 2ab$
(c) $(lx + my)(a + b) = (l + m)ab$
(d) $(lx - my)(a + b) = (l - m)ab$

32. The equations $(b - c)x + (c - a)y + a - b = 0$ and $(b^3 - c^3)x + (c^3 - a^3)y + a^3 - b^3 = 0$ will represent the same line, if
(a) $b = c$ (b) $c = a$
(c) $a = b$ (d) $a + b + c = 0$

33. The area of a triangle is 5. Two of its vertices are (2, 1) and (3, –2). The third vertex lies on $y = x + 3$. The coordinates of the third vertex cannot be
(a) $\left(\frac{-3}{2}, \frac{3}{2}\right)$ (b) $\left(\frac{3}{4}, \frac{-3}{2}\right)$
(c) $\left(\frac{7}{2}, \frac{13}{2}\right)$ (d) $\left(\frac{-1}{4}, \frac{11}{4}\right)$

34. If the lines $x - 2y - 6 = 0, 3x + y - 4 = 0$ and $\lambda x + 4y + \lambda^2 = 0$ are concurrent, then
(a) $\lambda = 2$ (b) $\lambda = -3$ (c) $\lambda = 4$ (d) $\lambda = -4$

35. Equation of a straight line passing through the point of intersection of $x - y + 1 = 0$ and $3x + y - 5 = 0$ are perpendicular to one of them is
(a) $x + y + 3 = 0$ (b) $x + y - 3 = 0$
(c) $x - 3y - 5 = 0$ (d) $x - 3y + 5 = 0$

36. If one vertex of an equilateral triangle of side a lies at the origin and the other lies on the line $x - \sqrt{3}y = 0$, the coordinates of the third vertex are
(a) $(0, a)$ (b) $\left(\frac{\sqrt{3}a}{2}, \frac{-a}{2}\right)$ (c) $(0, -a)$ (d) $\left(\frac{-\sqrt{3}a}{2}, \frac{a}{2}\right)$

37. If the line $ax + by + c = 0, bx + cy + a = 0$ and $cx + ay + b = 0$ are concurrent $(a + b + c \neq 0)$ then
(a) $a^3 + b^3 + c^3 - 3abc = 0$ (b) $a = b$
(c) $a = b = c$ (d) $a^2 + b^2 + c^2 - bc - ca - ab = 0$

38. $A(1, 3)$ and $C(7, 5)$ are two opposite vertices of a square. The equation of a side through A is
(a) $x + 2y - 7 = 0$ (b) $x - 2y + 5 = 0$
(c) $2x + y - 5 = 0$ (d) $2x - y + 1 = 0$

39. If $6a^2 - 3b^2 - c^2 + 7ab - ac + 4bc = 0$, then the family of lines $ax + by + c = 0$ is concurrent at
(a) $(-2, -3)$ (b) $(3, -1)$
(c) $(2, 3)$ (d) $(-3, 1)$

40. Consider the straight lines $x + 2y + 4 = 0$ and $4x + 2y - 1 = 0$. The line $6x + 6y + 7 = 0$ is
(a) bisector of the angle including origin
(b) bisector of acute angle
(c) bisector of obtuse angle
(d) None of the above

41. Two roads are represented by the equations $y - x = 6$ and $x + y = 8$. An inspection bungalow has to be so constructed that it is at a distance of 100 from each of the roads. Possible location of the bungalow is given by
(a) $(100\sqrt{2} + 1, 7)$ (b) $(1 - 100\sqrt{2}, 7)$
(c) $(1, 7 + 100\sqrt{2})$ (d) $(1, 7 - 100\sqrt{2})$

42. If (a, b) be an end of a diagonal of a square and the other diagonal has the equation $x - y = a$, then another vertex of the square can be
(a) $(a - b, a)$ (b) $(a, 0)$
(c) $(0, -a)$ (d) $(a + b, b)$

43. Consider the equation $y - y_1 = m(x - x_1)$. If m and x_1 are fixed and different lines are drawn for different values of y_1, then
(a) the lines will pass through a fixed point
(b) there will be a set of parallel lines
(c) all the lines intersect the line $x = x_1$
(d) all the lines will be parallel to the line $y = x_1$

44. Let $L_1 \equiv ax + by + a\sqrt[3]{b} = 0$ and $L_2 \equiv bx - ay + b\sqrt[3]{a} = 0$ be two straight lines. The equations of the bisectors of the angle formed by the foci whose equations are $\lambda_1 L_1 - \lambda_2 L_2 = 0$ and $\lambda_1 L_1 + \lambda_2 L_2 = 0$, λ_1 and λ_2 being non-zero real numbers, are given by
(a) $L_1 = 0$ (b) $L_2 = 0$
(c) $\lambda_1 L_1 + \lambda_2 L_2 = 0$ (d) $\lambda_2 L_1 - \lambda_1 L_2 = 0$

45. The equation of the bisectors of the angles between the two intersecting lines $\frac{x-3}{\cos\theta} = \frac{y+5}{\sin\theta}$ and $\frac{x-3}{\cos\phi} = \frac{y+5}{\sin\phi}$ are $\frac{x-3}{\cos\alpha} = \frac{y+5}{\sin\alpha}$ and $\frac{x-3}{\beta} = \frac{y+5}{\gamma}$, then
(a) $\alpha = \frac{\theta + \phi}{2}$ (b) $\beta = -\sin\alpha$
(c) $\gamma = \cos\alpha$ (d) $\beta = \sin\alpha$

The Straight Lines Exercise 3 : Paragraph Based Questions

- The section contains **5 Paragraphs** based upon each of the paragraphs **3 multiple choice questions** have to be answered. Each of these questions has four choices (a), (b), (c), and (d) out of which **ONLY ONE** is correct.

Paragraph I

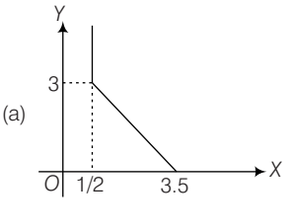
(Q. Nos. 46 to 48)

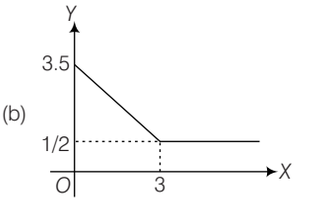
For points $P \equiv (x_1, y_1)$ and $Q = (x_2, y_2)$ of the coordinate plane, a new distance $d(P, Q)$ is defined by $d(P, Q) = |x_1 - x_2| + |y_1 - y_2|$
Let $O \equiv (0, 0)$, $A \equiv (1, 2)$, $B \equiv (2, 3)$ and $C \equiv (4, 3)$ are four fixed points on x-y plane.

46. Let $R(x, y)$, such that R is equidistant from the point O and A with respect to new distance and if $0 \le x < 1$ and $0 \le y < 2$, then R lie on a line segment whose equation is
(a) $x + y = 3$ (b) $x + 2y = 3$
(c) $2x + y = 3$ (d) $2x + 2y = 3$

47. Let $S(x, y)$, such that S is equidistant from points O and B with respect to new distance and if $x \ge 2$ and $0 \le y < 3$, then locus of S is
(a) a line segment of finite length
(b) a line of infinite length
(c) a ray of finite length
(d) a ray of infinite length

48. Let $T(x, y)$, such that T is equidistant from point O and C with respect to new distance and if T lie in first quadrant, then T consists of the union of a line segment of finite length and an infinite ray whose labelled diagram is

(a)


(b)


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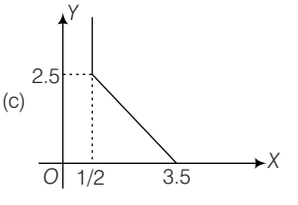

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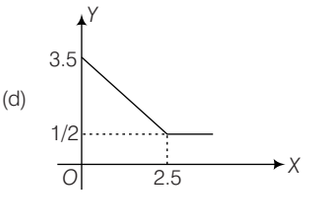


Paragraph II

(Q. Nos. 49 to 51)

In a triangle ABC, if the equation of sides AB, BC and CA are $2x - y + 4 = 0$, $x - 2y - 1 = 0$ *and* $x + 3y - 3 = 0$ *respectively.*

49. Tangent of internal angle A is equal to

(a) -7 (b) -3

(c) $\frac{1}{2}$ (d) 7

50. The equation of external bisector of angle B is

(a) $x - y - 1 = 0$ (b) $x - y + 1 = 0$

(c) $x + y - 5 = 0$ (d) $x + y + 5 = 0$

51. The image of point B w.r.t the side CA is

(a) $\left(-\frac{3}{5}, \frac{26}{5}\right)$ (b) $\left(-\frac{3}{5}, -\frac{26}{5}\right)$

(c) $\left(\frac{3}{5}, -\frac{26}{5}\right)$ (d) $\left(\frac{3}{5}, \frac{26}{5}\right)$

Paragraph III

(Q. Nos. 52 to 54)

$A(1, 3)$ *and* $C\left(-\frac{2}{5}, \frac{-2}{5}\right)$ *are the vertices of a triangle ABC and the equation of the angle bisector of* $\angle ABC$ *is* $x + y = 2$.

52. Equation of BC is

(a) $7x + 3y - 4 = 0$ (b) $7x + 3y + 4 = 0$

(c) $7x - 3y + 4 = 0$ (d) $7x - 3y - 4 = 0$

53. Coordinates of vertex B are

(a) $\left(\frac{3}{10}, \frac{17}{10}\right)$ (b) $\left(\frac{17}{10}, \frac{3}{10}\right)$

(c) $\left(-\frac{5}{2}, \frac{9}{2}\right)$ (d) $\left(\frac{9}{2}, -\frac{5}{2}\right)$

54. Equation of AB is

(a) $3x + 7y = 24$

(b) $3x + 7y + 24 = 0$

(c) $13x + 7y + 8 = 0$

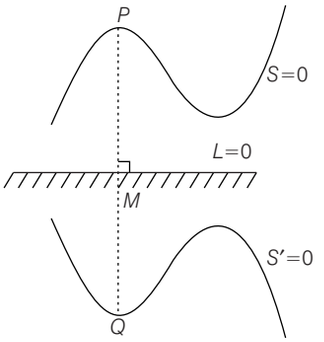
(d) $13x - 7y + 8 = 0$

Paragraph IV

(Q. Nos. 55 to 57)

Let $S' = 0$ *be the image or reflection of the curve* $S = 0$ *about line mirror* $L = 0$. *Suppose P be any point on the curve* $S = 0$ *and Q be the image or reflection about the line mirror* $L = 0$, *then Q will lie on* $S' = 0$.

How to find the image or reflection of a curve?



Let the given curve be $S : f(x, y) = 0$ *and line mirror* $L : ax + by + c = 0$. *We take a point P on the given curve in parametric form. Suppose Q be the image or reflection of point P about line mirror* $L = 0$, *which again contains the same parameter. Let* $Q \equiv (\phi(t), \psi(t))$, *where t is parameter. Now let* $x = \phi(t)$ *and* $y = \psi(t)$

Eliminating t, we get the equation of the reflected curve S′.

55. The image of the line $3x - y = 2$ in the line $y = x - 1$ is

(a) $x + 3y = 2$ (b) $3x + y = 2$

(c) $x - 3y = 2$ (d) $x + y = 2$

56. The image of the circle $x^2 + y^2 = 4$ in the line $x + y = 2$ is

(a) $x^2 + y^2 - 2x - 2y = 0$ (b) $x^2 + y^2 - 4x - 4y + 6 = 0$

(c) $x^2 + y^2 - 2x - 2y + 2 = 0$ (d) $x^2 + y^2 - 4x - 4y + 4 = 0$

57. The image of the parabola $x^2 = 4y$ in the line $x + y = a$ is

(a) $(x - a)^2 = 4(a - y)$ (b) $(y - a)^2 = 4(a - x)$

(c) $(x - a)^2 = 4(a + y)$ (d) $(y - a)^2 = 4(a + x)$

Paragraph V

(Q. Nos. 58 to 60)

In a ΔABC, the equation of the side BC is $2x - y = 3$ and its circumcentre and orhtocentre are (2, 4) and (1, 2) respectively.

58. Circumradius of ΔABC is

(a) $\sqrt{\frac{61}{5}}$ (b) $\sqrt{\frac{51}{5}}$ (c) $\sqrt{\frac{41}{5}}$ (d) $\sqrt{\frac{43}{5}}$

59. $\sin B \cdot \sin C =$

(a) $\frac{9}{2\sqrt{61}}$ (b) $\frac{9}{4\sqrt{61}}$ (c) $\frac{9}{\sqrt{61}}$ (d) $\frac{9}{5\sqrt{61}}$

60. The distance of orthocentre from vertex A is

(a) $\frac{1}{\sqrt{5}}$ (b) $\frac{6}{\sqrt{5}}$ (c) $\frac{3}{\sqrt{5}}$ (d) $\frac{2}{\sqrt{5}}$

The Straight Lines Exercise 4 : Single Integer Answer Type Questions

- The section contains **10 questions**. The answer to eaeh question is a **single digit integer**, ranging from 0 to 9 (both inclusive).

61. The number of possible straight lines passing through (2, 3) and forming a triangle with the coordinate axes, whose area is 12 sq units, is

62. The portion of the line $ax + 3y - 1 = 0$, intercepted between the lines $ax + y + 1 = 0$ and $x + 3y = 0$ subtend a right angle at origin, then the value of $|a|$ is

63. Let ABC be a triangle and $A \equiv (1, 2)$, $y = x$ be the perpendicular bisector of AB and $x - 2y + 1 = 0$ be the angle bisector of $\angle C$. If the equation of BC is given by $ax + by - 5 = 0$, then the value of $a - 2b$ is

64. A lattice point in a plane is a point for which both coordinates are integers. If n be the number of lattice points inside the triangle whose sides are $x = 0$, $y = 0$ and $9x + 223y = 2007$, then tens place digit in n is

65. The number of triangles that the four lines $y = x + 3$, $y = 2x + 3$, $y = 3x + 2$ and $y + x = 3$ form is

66. In a plane there are two families of lines : $y = x + n$, $y = -x + n$, where $n \in \{0, 1, 2, 3, 4\}$. The number of squares of the diagonal of length 2 formed by these lines is

67. Given $A(0, 0)$ and $B(x, y)$ with $x \in (0, 1)$ and $y > 0$. Let the slope of line AB be m_1. Point C lies on line $x = 1$ such that the slope of BC is equal to m_2, where $0 < m_2 < m_1$. If the area of triangle ABC can be expressed as $(m_1 - m_2)f(x)$ and the largest possible value of $f(x)$ is λ, then the value of $\frac{1}{\lambda}$ is

68. If $(\lambda, \lambda + 1)$ is an interior point of ΔABC, where $A \equiv (0, 3)$, $B \equiv (-2, 0)$ and $C \equiv (6, 1)$, then the number of integral values of λ is

69. For all real values of a and b, lines $(2a + b)x + (a + 3b)y + (b - 3a) = 0$ and $\lambda x + 2y + 6 = 0$ and $\lambda x + 2y + 6 = 0$ are concurrent, then the value of $|\lambda|$ is

70. If from point (4, 4) perpendiculars to the straight lines $3x + 4y + 5 = 0$ and $y = mx + 7$ meet at Q and R and area of triangle PQR is maximum, then the value of $3m$ is

The Straight Lines Exercise 5 : Matching Type Questions

- The section contains **5 questions**. Questions 1, 2 and 3 have four statement (A, B, C and D) given in **Column I** and four statements (p, q, r and s) in **Column II** and questions 74 and 75 have three statements (A, B and C) given in **Column I** and five statements (p, q, r, s and t) in **Column II**. Any given statement in **Column I** can have correct matching with one or more statement (s) given in **Column II**.

71. Let L_1, L_2, L_3 be three straight lines a plane and n be the number of circles touching all the lines.

	Column I		Column II
(A)	The lines are concurrent, then $n + 1$ is a	(p)	natural number
(B)	The lines are parallel, then $2n + 3$ is a	(q)	prime number
(C)	Two lines are parallel, then $n + 2$ is a	(r)	composite number
(D)	The lines are neither concurrent nor parallel, then $n + 2$ is a	(s)	perfect number

72. Match the Columns

	Column I		Column II
(A)	Lines $x - 2y - 6 = 0$, $3x + y - 4 = 0$ and $\lambda x + 4y + \lambda^2 = 0$ are concurrent, then the value of $\lvert\lambda\rvert$ is	(p)	2
(B)	The variable straight lines $3x(a + 1) + 4y(a - 1) - 3(a - 1) = 0$ for different value of 'a' passes through a fixed point (p, q) if $\lambda = p - q$, then the value of $4\lvert\lambda\rvert$	(q)	3
(C)	If the line $x + y - 1 - \left\lvert\frac{\lambda}{2}\right\rvert = 0$ passing through the intersection of $x - y + 1 = 0$ and $3x + y - 5 = 0$, is perpendicular to one of them, then the value of $\lvert\lambda + 1\rvert$ is	(r)	4
(D)	If the line $y - x - 1 + \lambda = 0$ is equidistant from the points $(1, -2)$ and $(3, 4)$, then the value of $\lvert\lambda\rvert$ is	(s)	5

73. Consider the triangle formed by the lines $y + 3x + 2 = 0$, $3y - 2x - 5 = 0$ and $4y + x - 14 = 0$

	Column I		Column II
(A)	If $(0, \lambda)$ lies inside the triangle, then integral values are less than $\lvert 3\lambda \rvert$	(p)	4
(B)	If $(1, \lambda)$ lies inside the triangle, then integral values are less than $\lvert 3\lambda \rvert$	(q)	5
(C)	If $(\lambda, 2)$ lies inside the triangle, then integral values of $\lvert 6\lambda \rvert$ are	(r)	6
(D)	If $(\lambda, 7/2)$ lies inside the triangle, then integral value of $\lvert 6\lambda \rvert$ are	(s)	7

74. Match the following

	Column I		Column II
(A)	The area bounded by the curve max. $\{\lvert x \rvert, \lvert y \rvert\} = 1$ is	(p)	0
(B)	If the point (a, a) lies between the lines $\lvert x + y \rvert = 6$, then $[\lvert a \rvert]$ is (where $[.]$ denotes the greatest integer function)	(q)	1
(C)	Number of integral values of b for which the origin and the point (1, 1) lie on the same side of the st. line $a^2x + aby + 1 = 0$ for all $a \in R \sim \{0\}$ is	(r)	2
		(s)	3
		(t)	4

75. Match the following

	Column I		Column II
(A)	If the distance of any point (x, y) from origin is defined as $d(x, y) = 2\lvert x \rvert + 3\lvert y \rvert$. If perimeter and area of figure bounded by $d(x, y) = 6$ are λ unit and μ sq units respectively, then	(p)	(λ, μ) lies on $x = 3y$
(B)	If the vertices of a triangle are (6, 0), (0, 6) and (6, 6). If distance between circumcentre and orthocentre and distance between circumcentre and centroid are λ unit and μ unit respectively, then	(q)	(λ, μ) lies on $x^2 - y^2 = 64$
(C)	The ends of the hypotenuse of a right angled triangle are (6, 0) and (0, 6). If the third vertex is (λ, μ), then	(r)	(λ, μ) lies on $x^2 + y^2 - 6x - 6y = 0$
		(s)	(λ, μ) lies on $x^2 - 16y = 16$
		(t)	(λ, μ) lies on $x^2 - y^2 = 16$

The Straight Lines Exercise 6 : Statement I and II Type Questions

■ **Directions** (Q. Nos 76 to 83) are Assertion-Reason type questions. Each of these question contains two statements.

Statement I (Assertion) and

Statement II (Reason)

Each of these questions has four alternative choices, only one of which is the correct answer.

You have to select the correct choice.

(a) Statement I is true, statement II is true; statement II is a correct explanation for statement I

(b) Statement I is true, statement II is true; statement II is not a correct explanation for statement I

(c) Statement I is true, statement II is false

(d) Statement I is false, statement II is true

76. **Statement I** The lines $x(a + 2b) + y(a + 3b) = a + b$ are concurrent at the point $(2, -1)$

Statement II The lines $x + y - 1 = 0$ and $2x + 3y - 1 = 0$ intersect at the point $(2, -1)$

77. **Statement I** The points (3, 2) and (1, 4) lie on opposite side of the line $3x - 2y - 1 = 0$

Statement II The algebraic perpendicular distance from the given point to the line have opposite sign.

78. **Statement I** If sum of algebraic distances from points $A(1, 2)$, $B(2, 3)$, $C(6, 1)$ is zero on the line $ax + by + c = 0$, then $2a + 3b + c = 0$

Statement II The centroid of the triangle is (3, 2)

79. **Statement I** Let $A \equiv (0, 1)$ and $B \equiv (2, 0)$ and P be a point on the line $4x + 3y + 9 = 0$, then the co-ordinates of P such that $\lvert PA - PB \rvert$ is maximum is $\left(-\frac{12}{5}, \frac{17}{5}\right)$.

Statement II $\lvert PA - PB \rvert \le \lvert AB \rvert$

80. **Statement I** The incentre of a triangle formed by the line $x\cos\left(\frac{\pi}{9}\right) + y\sin\left(\frac{\pi}{9}\right) = \pi$,

$x\cos\left(\frac{8\pi}{9}\right) + y\sin\left(\frac{8\pi}{9}\right)$

$= \pi$ and $x\cos\left(\frac{13\pi}{9}\right) + y\sin\left(\frac{13\pi}{9}\right) = \pi$ is (0, 0).

Statement II Any point equidistant from the given three non-concurrent straight lines in the plane is the incentre of the triangle.

81. **Statement I** Reflection of the point (5, 1) in the line $x + y = 0$ is $(-1, -5)$.

Statement II Reflection of a point $P(\alpha, \beta)$ in the line $ax + by + c = 0$ is $Q(\alpha', \beta')$, if $\left(\frac{\alpha + \alpha'}{2}, \frac{\beta + \beta'}{2}\right)$ lies on the line.

82. **Statement I** The internal angle bisector of angle C of a triangle ABC with sides AB, AC and BC as $y = 0$, $3x + 2y = 0$, and $2x + 3y + 6 = 0$, respectively, is $5x + 5y + 6 = 0$.

Statement II The image of point A with respect to $5x + 5y + 6 = 0$ lies on the side BC of the triangle.

83. **Statement I** If the point $(2a - 5, a^2)$ is on the same side of the line $x + y - 3 = 0$ as that of the origin, then $a \in (2, 4)$.

Statement II The point (x_1, y_1) and (x_2, y_2) lie on the same or opposite sides of the line $ax + by + c = 0$, as $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same or opposite signs.

The Straight Lines Exercise 7 :
Subjective Type Questions

- In this section, there are **15 subjective questions**.

84. If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertices of a triangle, then show that the equation of the line joining A and the circumcentre is given by

$$(\sin 2B)\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + (\sin 2C)\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

85. Find the coordinates of the point at unit distance from the lines

$3x - 4y + 1 = 0, 8x + 6y + 1 = 0$.

86. A variable line makes intercepts on the coordinate axes, the sum of whose squares is constant and equal to k^2. Show that the locus of the foot of the perpendicular from the origin to this line is $(x^2 + y^2)^2 (x^{-2} + y^{-2}) = k^2$.

87. A variable line intersects n lines $y = mx, (m = 1, 2, 3, \ldots, n)$ in the points $A_1, A_2, A_3, \ldots, A_n$ respectively.

If $\sum_{p=1}^{n} \frac{1}{OA_p} = c$ (constant). Show that line passes through a fixed point. Find the coordinates of this fixed point (O being origin).

88. Given n straight lines and a fixed point O. A straight line is drawn through O meeting these lines in the points $R_1, R_2, R_3, \ldots, R_n$ and a point R is taken on it such that

$$\frac{n}{OR} = \sum_{r=1}^{n} \frac{1}{OR_r}$$

Prove that the locus of R is a straight line.

89. Prove that all lines represented by the equation

$$(2\cos\theta + 3\sin\theta)x + (3\cos\theta - 5\sin\theta)y = 5\cos\theta - 2\sin\theta$$

pass through a fixed point for all θ. What are the coordinates of this fixed point and its reflection in the line $x + y = \sqrt{2}$? Prove that all lines through reflection point can be represented by equation

$$(2\cos\theta + 3\sin\theta)x + (3\cos\theta - 5\sin\theta)y = (\sqrt{2} - 1)(5\cos\theta - 2\sin\theta)$$

90. P is any point on the line $x - a = 0$. If A is the point $(a, 0)$ and PQ, the bisector of the angle OPA, meets the X-axis in Q. Prove that the locus of the foot of the perpendicular from Q on OP is $(x - a)^2 (x^2 + y^2) = a^2 y^2$.

91. Having given the bases and the sum of the areas of a number of triangles is constant, which have a common vertex. Show that the locus of this vertex is a straight line.

92. $A(3, 0)$ and $B(6, 0)$ are two fixed points and $U(\alpha, \beta)$ is a variable point on the plane. AU and BU meet the y-axis at C and D respectively and AD meets OU at V. Prove that CV passes through (2, 0) for any position of U in the plane.

93. A variable line is drawn through O to cut two fixed straight lines L_1 and L_2 in R and S. A point P is chosen on the variable line such that $\frac{m+n}{OP} = \frac{m}{OR} + \frac{n}{OS}$. Show that the locus of P is a straight line passing through the point of intersection of L_1 and L_2.

94. A line through $A(-5, -4)$ meets the lines $x + 3y + 2 = 0, 2x + y + 4 = 0$ and $x - y - 5 = 0$ at the points B, C and D respectively, if

$$\left(\frac{15}{AB}\right)^2 + \left(\frac{10}{AC}\right)^2 = \left(\frac{6}{AD}\right)^2$$

find the equation of the line.

95. Two fixed straight lines X-axis and $y = mx$ are cut by a variable line in the points $A(a, 0)$ and $B(b, mb)$ respectively. P and Q are the feet of the perpendiculars drawn from A and B upon the lines $y = mx$ and X-axis. Show that, if AB passes through a fixed point (h, k), then PQ will also pass through a fixed point. Find the fixed point.

96. Find the equation of straight lines passing through point $(2, 3)$ and having an intercept of length 2 units between the straight lines $2x + y = 3, 2x + y = 5$.

97. Let $O(0, 0)$, $A(2, 0)$ and $B\left(1, \frac{1}{\sqrt{3}}\right)$ be the vertices of a triangle. Let R be the region consisting of all those points P inside ΔOAB which satisfy

$d(P, OA) \leq \min\{d(P, OB), d(P, AB)\}$

where d denotes the distance from the point to the corresponding line. Sketch the region R and find its area.

98. Two triangles ABC and PQR are such that the perpendiculars from A to QR, B to RP and C to PQ are concurrent. Show that the perpendicular from P to BC, Q to CA and R to AB are also concurrent.

The Straight Lines Exercise 8 : Questions Asked in Previous 13 Year's Exams

- This section contains questions asked in **IIT-JEE, AIEEE, JEE Main & JEE Advanced** from year **2005** to **2017**.

99. The line parallel to the X-axis and passing through the intersection of the lines $ax + 2by + 3b = 0$ and $bx - 2ay - 3a = 0$, where $(a, b) \neq (0, 0)$ is **[AIEEE 2005, 3M]**

(a) below the X-axis at a distance of $\frac{3}{2}$ from it

(b) below the X-axis at a distance of $\frac{2}{3}$ from it

(c) above the X-axis at a distance of $\frac{3}{2}$ from it

(d) above the X-axis at a distance of $\frac{2}{3}$ from it

100. A straight line through the point $A(3, 4)$ is such that its intercept between the axes is bisected at A. Its equation is **[AIEEE 2006, 4.5M]**

(a) $x + y = 7$ (b) $3x - 4y + 7 = 0$

(c) $4x + 3y = 24$ (d) $3x + 4y = 25$

101. If (a, a^2) falls inside the angle made by the lines $y = \frac{x}{2}$, $x > 0$ and $y = 3x$, $x > 0$, then a belong to **[AIEEE 2006, 6M]**

(a) $\left(0, \frac{1}{2}\right)$ (b) $(3, \infty)$

(c) $\left(\frac{1}{2}, 3\right)$ (d) $\left(-3, -\frac{1}{2}\right)$

102. Lines $L_1 : y - x = 0$ and $L_2 : 2x + y = 0$ intersect the line $L_3 : y + 2 = 0$ at P and Q respectively. The bisector of the acute angle between L_1 and L_2 intersects L_3 at R.

[IIT-JEE 2007, 3M]

Statement I The ratio $PR : RQ$ equals $2\sqrt{2} : \sqrt{5}$ because

Statement II In any triangle, bisector of an angle divides the triangle into two similar triangles.

(a) Statement I is true, statement II is true; statement II is not a correct explanation for statement I

(b) Statement I is true, statement II is true; statement II is not a correct explanation for statement I

(c) Statement I is true, statement II is false

(d) Statement I is false, statement II is true

103. Let $P = (-1, 0)$, $Q = (0, 0)$ and $R = (3, 3\sqrt{3})$ be three point. The equation of the bisector of the angle PQR is **[AIEEE 2007, 3M]**

(a) $\frac{\sqrt{3}}{2}x + y = 0$ (b) $x + \sqrt{3}y = 0$

(c) $\sqrt{3}x + y = 0$ (d) $x + \frac{\sqrt{3}}{2}y = 0$

104. Consider the lines given by

$$L_1 : x + 3y - 5 = 0$$
$$L_2 : 3x - ky - 1 = 0$$
$$L_3 : 5x + 2y - 12 = 0$$

Match the statements/Expressions in **Column I** with the statements/Expressions in **Column II**

	Column I		Column II
(A)	L_1, L_2, L_3 are concurrent, if	(p)	$k = -9$
(B)	one of L_1, L_2, L_3 is parallel to at least one of the other two, if	(q)	$k = -\frac{6}{5}$
(C)	L_1, L_2, L_3 form a triangle, if	(r)	$k = \frac{5}{6}$
(D)	L_1, L_2, L_3 do not form a triangle, if	(s)	$k = 5$

[IIT-JEE 2008, 6M]

105. The perpendicular bisector of the line segment joining $P(1, 4)$ and $Q(k, 3)$ has y-intercept -4. Then a possible value of k is **[AIEEE 2008, 3M]**

(a) 1 (b) 2 (c) -2 (d) -4

106. The lines $p(p^2+1)x - y + q = 0$ and $(p^2+1)^2 x + (p^2+1)y + 2q = 0$ are perpendicular to a common line for **[AIEEE 2009, 4M]**

(a) exactly one values of p (b) exactly two values of p
(c) more than two values of p (d) no value of p

107. The line L given by $\frac{x}{5} + \frac{y}{b} = 1$ passes through the point (13, 32). The line K is parallel to L and has the equation $\frac{x}{c} + \frac{y}{3} = 1$. Then the distance between L and K is **[AIEEE 2010, 4M]**

(a) $\sqrt{17}$ (b) $\frac{17}{\sqrt{15}}$
(c) $\frac{23}{\sqrt{17}}$ (d) $\frac{23}{\sqrt{15}}$

108. A straight line L through the point $(3, -2)$ is inclined at an angle 60° to the line $\sqrt{3}x + y = 1$. If L also intersects the X-axis, then the equation of L is **[IIT-JEE 2011, 3M]**

(a) $y + \sqrt{3}x + 2 - 3\sqrt{3} = 0$ (b) $y - \sqrt{3}x + 2 + 3\sqrt{3} = 0$
(c) $\sqrt{3}y - x + 3 + 2\sqrt{3} = 0$ (d) $\sqrt{3}y + x - 3 + 2\sqrt{3} = 0$

109. The lines $L_1 : y - x = 0$ and $L_2 : 2x + y = 0$ intersect the line $L_3 : y + 2 = 0$ at P and Q respectively. The bisector of the acute angle between L_1 and L_2 intersects L_3 at R. **[AIEEE 2011, 4M]**

Statement I : The ratio $PR : RQ$ equals $2\sqrt{2} : \sqrt{5}$

Statement II : In any triangle, bisector of an angle divides the triangle into two similar triangles.

(a) Statement I is true, statement II is true; statement II is not a correct explanation for statement I.
(b) Statement I is true, statement II is false.
(c) Statement I is false, statement II is true.
(d) Statement I is true, statement II is true; statement II is a correct explanation for statement I

110. If the line $2x + y = k$ passes through the point which divides the line segment joining the points (1, 1) and (2, 4) in the ratio 3 : 2, then k equals **[AIEEE 2012, 4M]**

(a) $\frac{29}{5}$ (b) 5
(c) 6 (d) $\frac{11}{5}$

111. A ray of light along $x + \sqrt{3}y = \sqrt{3}$ gets reflected upon reaching X-axis, the equation of the reflected ray is **[JEE Main 2013, 4M]**

(a) $y = x + \sqrt{3}$ (b) $\sqrt{3}y = x - \sqrt{3}$
(c) $y = \sqrt{3}x - \sqrt{3}$ (d) $\sqrt{3}y = x - 1$

112. For $a > b > c > 0$, the distance between (1, 1) and the point of intersection of the lines $ax + by + c = 0$ and $bx + ay + c = 0$ is less than $2\sqrt{2}$. Then **[JEE Advanced 2013, 3M]**

(a) $a + b - c > 0$ (b) $a - b + c < 0$
(c) $a - b + c > 0$ (d) $a + b - c < 0$

113. Let PS be the median of the triangle with vertices $P(2, 2)$, $Q(6, -1)$ and $R(7, 3)$. The equation of the line passing through $(1, -1)$ and parallel to PS is **[JEE Main 2014, 4M]**

(a) $4x + 7y + 3 = 0$ (b) $2x - 9y - 11 = 0$
(c) $4x - 7y - 11 = 0$ (d) $2x + 9y + 7 = 0$

114. Let a, b, c and d be non-zero numbers. If the point of intersection of the lines $4ax + 2ay + c = 0$ and $5bx + 2by + d = 0$ lies in the fourth quadrant and is equidistant from the two axes, then **[JEE Main 2014, 4M]**

(a) $3bc - 2ad = 0$ (b) $3bc + 2ad = 0$
(c) $2bc - 3ad = 0$ (d) $2bc + 3ad = 0$

115. For a point P in the plane, let $d_1(P)$ and $d_2(P)$ be the distance of the point P from the lines $x - y = 0$ and $x + y = 0$ respectively. The area of the region R consisting of all points P lying in the first quadrant of the plane and satisfying $2 \le d_1(P) + d_2(P) \le 4$, is **[JEE Advanced 2014, 3M]**

116. The number of points, having both co-ordinates as integers, that lie in the interior of the triangle with vertices (0, 0), (0, 41) and (41, 0) is **[JEE Advanced 2015, 4M]**

(a) 820 (b) 780
(c) 901 (d) 861

117. Two sides of a rhombus are along the lines, $x - y + 1 = 0$ and $7x - y - 5 = 0$. If its diagonals intersect at $(-1, -2)$, then which one of the following is a vertex of this rhombus? **[JEE Main 2016, 4M]**

(a) $\left(\frac{1}{3}, -\frac{8}{3}\right)$ (b) $\left(-\frac{10}{3}, -\frac{7}{3}\right)$
(c) $(-3, -9)$ (d) $(-3, -8)$

Answers

Exercise for Session 1

1. (c) **2.** (b) **3.** (b) **4.** (d) **5.** (b)
6. (d) **7.** (c) **8.** (d) **9.** (a,d) **10.** (c)
11. (d) **12.** $y = 9$ **13.** $(2 + 2\sqrt{3}, 4)$ and $(2 - 2\sqrt{3}, 0)$
14. $PQ = \dfrac{132}{12\sqrt{3} + 5}$ **15.** $4\sqrt{2}$ units
16. $83x - 35y + 92 = 0$ **17.** $x + y - 11 = 0$

Exercise for Session 2

1. (b) **2.** (c) **3.** (a) **4.** (c) **5.** (b) **6.** (d)
7. (c,d) **8.** (d) **9.** (d) **10.** (d) **11.** (b) **12.** (d)
13. The two points are on the opposite side of the given line.
15. $3x - 4y = 0$ and $3x - 4y - 10 = 0$
17. $7x + y - 31 = 0$ **18.** $2x + 2y + \sqrt{2} = 0$

Exercise for Session 3

1. (c) **2.** (b) **3.** (c) **4.** (a) **5.** (d) **6.** (a)
7. (a) **8.** (c) **9.** (a) **10.** (a,b) **11.** (c) **12.** (c)
13. $2a + b^2 + b = 0$ **16.** (i) $y = 3$ (ii) $x = 4$, (iii) $3x + 4y = 24$
18. $\left(\dfrac{-5}{3}, \dfrac{-5}{3}\right)$

Exercise for Session 4

1. (c) **2.** (d) **3.** (b) **4.** (b) **5.** (c) **6.** (b)
7. (c) **8.** (a) **9.** $x = 7$ and $x + \sqrt{3}y = 7 + 9\sqrt{3}$
10. $x(4\sqrt{3} + 3) + y(4 - 3\sqrt{3}) = 11 - 2\sqrt{3}$ and
$y(4 + 3\sqrt{3}) - x(4\sqrt{3} - 3) = 11 + 2\sqrt{3}$, $\dfrac{4\sqrt{3}}{15}$
11. $\left(-\dfrac{1}{14}, \dfrac{73}{28}\right)$ and $\left(\dfrac{1}{16}, \dfrac{77}{32}\right)$
12. $\left(0, \dfrac{5}{2}\right)$ and $(0, 0)$ **13.** $x + 2y - 6 = 0$
14. $3x = 19$ **15.** $10x - 10y - 3 = 0$

Exercise for Session 5

1. (d) **2.** (a) **3.** (b) **4.** (d) **5.** (c)
6. (d) **7.** (d) **8.** (c) **9.** (d) **10.** (b)
11. (b) **12.** (c) **13.** $(3, -2)$ **14.** $14x + 23y - 40 = 0$
15. $4x - y + 6 = 0, \left(-\dfrac{4}{5}, \dfrac{14}{5}\right)$ **16.** (2)

Exercise for Session 6

1. (c) **2.** (c) **3.** (a,b) **4.** (a) **5.** (a)
6. (c) **7.** $m \in \left(-1, \dfrac{1}{5}\right)$
8. $\theta \in \left(0, \dfrac{5\pi}{6} - \tan^{-1} 3\right)$ **9.** $\theta \in \left(0, \dfrac{\pi}{12}\right) \cup \left(\dfrac{5\pi}{12}, \dfrac{\pi}{2}\right)$
10. $\theta \in \left\{\underset{n=z}{\cup}\left(2n\pi, 2n\pi + \dfrac{\pi}{6}\right)\right\} \cup \left\{\underset{m=z}{\cup}\left(2m\pi + \dfrac{5\pi}{6}, 2m\pi\right)\right\}$
11. Outside **12.** $29x - 2y + 33 = 0$

Chapter Exercises

1. (b) **2.** (a) **3.** (c) **4.** (d) **5.** (c) **6.** (b)
7. (b) **8.** (b) **9.** (b) **10.** (c) **11.** (c) **12.** (b)
13. (c) **14.** (d) **15.** (a) **16.** (b) **17.** (b) **18.** (c)
19. (b) **20.** (b) **21.** (b) **22.** (b) **23.** (c) **24.** (a)
25. (a) **26.** (b) **27.** (a) **28.** (a) **29.** (b) **30.** (a)
31. (a,b,c,d) **32.** (a,b,c,d) **33.** (a,c) **34.** (a,d)
35. (b,d) **36.** (a,b,c,d) **37.** (a,c,d) **38.** (a,d) **39.** (a,b) **40.** (a,b)
41. (a,b,c,d) **42.** (b,d) **43.** (a,b,c) **44.** (a,b) **45.** (a,b,c)
46. (d) **47.** (d) **48.** (a) **49.** (a) **50.** (d) **51.** (a)
52. (b) **53.** (c) **54.** (a) **55.** (c) **56.** (d) **57.** (b)
58. (a) **59.** (a) **60.** (b) **61.** (3) **62.** (6) **63.** (5)
64. (8) **65.** (3) **66.** (9) **67.** (8) **68.** (2) **69.** (2)
70. (4) **71.** (A) → (p); (B) → (p,q); (C) → (p,r) (D) → (p,r,s)
72. (A) → (p,r); (B) → (q); (C) → (q,s) (D) → (p)
73. (A) → (p,q); (B) → (p, q,r,s); (C) → (p,q,r,s); (D) → (p,q,r,s)
74. (A) → (t); (B) → (p,q,r); (C) → (s)
75. (A) → (q,s); (B) → (p,t); (C) → (r) **76.** (a) **77.** (a)
78. (d) **79.** (d) **80.** (c) **81.** (b) **82.** (b) **83.** (d)
85. $\left(\dfrac{6}{5}, \dfrac{-1}{10}\right), \left(-\dfrac{2}{5}, \dfrac{-13}{10}\right), \left(0, \dfrac{3}{2}\right), \left(\dfrac{-8}{5}, \dfrac{3}{10}\right)$
87. $\left(\dfrac{\pm\sum_{p=1}^{n}\dfrac{1}{\sqrt{(1+p^2)}}}{c}, \dfrac{\pm\sum_{p=1}^{n}\dfrac{p}{\sqrt{(1+p^2)}}}{c}\right)$ **94.** $2x + 3y + 22 = 0$
95. $\left(\dfrac{h + mk}{1 + m^2}, \dfrac{mh - k}{1 + m^2}\right)$ **96.** $3x + 4y - 18 = 0$ and $x - 2 = 0$
97. $(2 - \sqrt{3})$ sq units. **99.** (a) **100.** (c) **101.** (c) **102.** (c) **103.** (c)
103. (c) **104.** (A) → (s); (B) → (p,q); (C) → (r); (D) → (p,q,s)
105.(a) **106.** (a) **107.** (c) **108.** (b) **109.** (b) **110.** (c) **111.** (b)
112.(a) **113.** (d) **114.** (a) **115.** (6) **116.** (b) **117.** (a)

Solutions

1. Equation of line passing through (2, 0) and perpendicular to $ax + by + c = 0$

Then, required equation is

$$y - 0 = \frac{b}{a}(x - 2)$$

$$ay = bx - 2b$$

$$\Rightarrow \quad ay - bx + 2b = 0$$

2. $\because \frac{2}{1!9!} + \frac{2}{3!7!} + \frac{1}{5!5!} = \frac{2^m}{n!}$

$$\frac{1}{10!}\left\{\frac{2 \times 10!}{1!9!} + \frac{2 \times 10!}{3!7!} + \frac{10!}{5!5!}\right\} = \frac{2^m}{n!}$$

$$\Rightarrow \quad \frac{1}{10!}\{2^{10}C_1 + 2^{10}C_3 + {}^{10}C_5\} = \frac{2^m}{n!}$$

$$\frac{1}{10!}\{{}^{10}C_1 + {}^{10}C_3 + {}^{10}C_5 + {}^{10}C_7 + {}^{10}C_9\} = \frac{2^m}{n!}$$

$$\Rightarrow \quad \frac{1}{10!}(2)^{10-1} = \frac{2^m}{n!}$$

$$\therefore \quad m = 9 \text{ and } n = 10$$

Hence, $x - y + 1 = 0$ and $x + y + 3 = 0$ are perpendicular to each other, then orthocentre is the point of intersection which is $(-2, -1)$

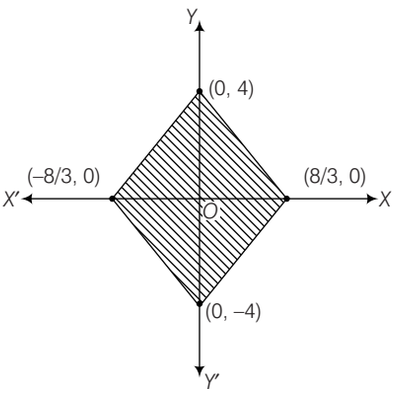
$\therefore \quad -2 = 2m - 2n$ and $-1 = m - n$

$\therefore$ Point is $(2m - 2n, m - n)$.

3. $\therefore$ Required area

$$= 4 \times \frac{1}{2}\left(\frac{8}{3} \times 4\right) = \frac{64}{3} = \frac{2^6}{3} \qquad \ldots(i)$$

$\because \quad f(x + y) = f(x)f(y)$



$$\therefore \quad f(2) = f(1)f(1) = 2^2$$

$$f(3) = f(1 + 2) = f(1)f(2) = 2^3$$

.....

.....

$$\therefore \quad f(n) = 2^n$$

$$\therefore \quad \text{Area} = \frac{2^6}{3} = \frac{f(6)}{3} \text{ sq units}$$

4. We have, $y = \cos x \cos(x + 2) - \cos^2(x + 1)$

$$y = \frac{1}{2}\{2\cos x \cos(x + 2) - 2\cos^2(x + 1)\}$$

$$= \frac{1}{2}\{\cos(2x + 2) + \cos 2 - 1 - \cos(2x + 2)\}$$

$$= \frac{1}{2}(\cos 2 - 1)$$

$$= \frac{1}{2}(1 - 2\sin^2 1 - 1)$$

$$= -\sin^2 1$$

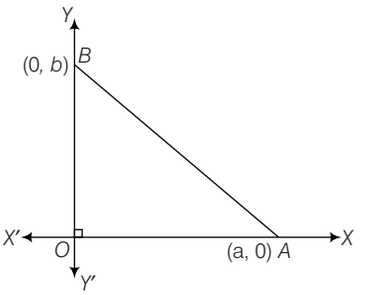
which is a straight line passing through $(\lambda, -\sin^2 1)$; $\forall\ \lambda \in R$ and parallel to the X-axis.

5. Let line $\frac{x}{a} + \frac{y}{b} = 1$ ...(i)

Its passes through (2, 2), then

$$\frac{2}{a} + \frac{2}{b} = 1$$

$$\Rightarrow \quad 2(a + b) = ab \qquad \ldots(ii)$$



$\because$ Area of $\Delta AOB = \frac{1}{2}ab = |\lambda|$ (given)

$$\therefore \quad ab = 2|\lambda|$$

from Eq. (ii), $\quad a + b = |\lambda|$

Hence, required equation is

$$x^2 - (a + b)x + ab = 0$$

or $\quad x^2 - |\lambda| x + 2|\lambda| = 0$

6. $\frac{\text{Value of } (a^2x + aby + 1) \text{ at } (1, 1)}{\text{Value of } (a^2x + aby + 1) \text{ at } (0, 0)} > 0$

or $\quad \frac{a^2 + ab + 1}{1} > 0;\ \forall\ a \in R$

or $\quad a^2 + ab + 1 > 0;\ \forall\ a \in R$

$$\therefore \quad D < 0$$

$$\Rightarrow \quad b^2 - 4 < 0$$

$$\Rightarrow \quad -2 < b < 2 \text{ but } b > 0$$

$$\therefore \quad 0 < b < 2$$

i.e. $\quad b \in (0, 2)$

7. Equation of L is $\frac{x}{a}+\frac{y}{b}=1$ and let the axis be rotated through an angle θ and let (X, Y) be the new coordinates of any point $P(x, y)$ in the plane, then
$x = X\cos\theta - Y\sin\theta$, $y = X\sin\theta + Y\cos\theta$, the equation of the line with reference to original coordinates is

$$\frac{x}{a}+\frac{y}{b}=1$$

i.e. $$\frac{X\cos\theta - Y\sin\theta}{a}+\frac{X\sin\theta + Y\cos\theta}{b}=1 \quad \ldots(i)$$

and with reference to new coordinates is

$$\frac{X}{p}+\frac{Y}{q}=1 \quad \ldots(ii)$$

Comparing Eqs. (i) and (ii), we get

$$\frac{\cos\theta}{a}+\frac{\sin\theta}{b}=\frac{1}{p} \quad \ldots(iii)$$

and $$-\frac{\sin\theta}{a}+\frac{\cos\theta}{b}=\frac{1}{q} \quad \ldots(iv)$$

Squaring and adding Eqs. (iii) and (iv), we get

$$\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{p^2}+\frac{1}{q^2}$$

8. $d(x, y) = \max\{|x|, |y|\}$ ...(i)

but $d(x, y) = a$ ...(ii)

From Eqs. (i) and (ii), we get

$a = \max\{|x|, |y|\}$

if $|x| > |y|$, then $a = |x|$

$\therefore$ $x = \pm a$

and if $|y| > |x|$, then $a = |y|$

$\therefore$ $y = \pm a$

Therefore locus represents a straight line.

9. $P_1 = |m^2\cos\alpha + 2m\sin\alpha + \frac{\sin^2\alpha}{\cos\alpha}|$

$$= \frac{(m\cos\alpha + \sin\alpha)^2}{|\cos\alpha|}$$

$$p_2 = \left|mm'\cos\alpha + (m+m')\sin\alpha + \frac{\sin^2\alpha}{\cos\alpha}\right|$$

$$= \frac{|(m\cos\alpha + \sin\alpha)|\,|m'\cos\alpha + \sin\alpha|}{|\cos\alpha|}$$

and $p_3 = |m'^2\cos\alpha + 2m'\sin\alpha + \frac{\sin^2\alpha}{\cos\alpha}|$

$$= \frac{(m'\cos\alpha + \sin\alpha)^2}{|\cos\alpha|}$$

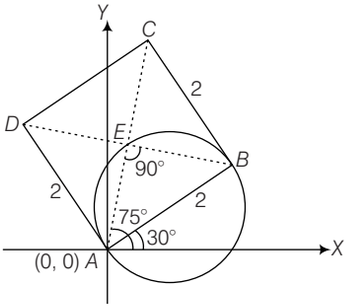
$\because$ $p_2^2 = p_1 p_3$

Hence, p_1, p_2, p_3 are in GP.

10. Side of the square = 2 unit

Coordinates of B, C and D are $(\sqrt{3}, 1)$, $(\sqrt{3}-1, \sqrt{3}+1)$ and $(-1, \sqrt{3})$ respectively.

Slope of $BD = \frac{\sqrt{3}-1}{-1-\sqrt{3}} = \frac{(\sqrt{3}-1)(\sqrt{3}-1)}{-2} = \sqrt{3}-2$



$\therefore$ Equation of BD is

$$y-1 = (\sqrt{3}-2)(x-\sqrt{3})$$

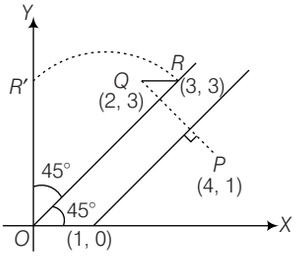
$\Rightarrow$ $$(2-\sqrt{3})x + y = 2(\sqrt{3}-1)$$

and equation of the circumcircle of the triangle ABE (Apply diametric form as AB is diameter)

$$(x-0)(x-\sqrt{3}) + (y-0)(y-1) = 0$$

$\Rightarrow$ $$x^2 + y^2 - x\sqrt{3} - y = 0$$

11. If (α, β) be the image of $(4, 1)$ w.r.t $y = x-1$, then $(\alpha, \beta) = (2, 3)$, say point Q



After translation through a distance 1 unit along the positive direction of X-axis at the point whose coordinate are $R \equiv (3, 3)$.

After rotation through are angle $\frac{\pi}{4}$ about the origin in the anticlockwise direction, then R goes to R' such that

$$OR = OR' = 3\sqrt{2}$$

$\therefore$ The coordinates of the final point are $(0, 3\sqrt{2})$.

12. $\because$ $A \equiv (0, 0)$; $B \equiv (2, 0)$; $C \equiv (2, 2)$; $D \equiv (0, 2)$

(i) $f_1(x, y) \to (y, x)$, then

$A \equiv (0, 0)$; $B \equiv (0, 2)$; $C \equiv (2, 2)$, $D \equiv (2, 0)$

(ii) $f_2(x, y) \to (x + 3y, y)$, then

$A \equiv (0, 0)$; $B \equiv (6, 2)$; $C \equiv (8, 2)$, $D \equiv (2, 0)$

(iii) $f_3(x, y) \to \left(\frac{x-y}{2}, \frac{x+y}{2}\right)$, then

$A \equiv (0, 0)$; $B \equiv (2, 4)$; $C \equiv (3, 5)$, $D \equiv (1, 1)$

Now, $AB = DC = 2\sqrt{5}$, $AD = BC = \sqrt{2}$

and $AC = \sqrt{34}$, $BD = \sqrt{10}$

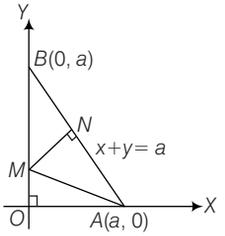
i.e. $AC \neq BD$

$\therefore$ Final figure is a parallelogram.

13. Let $\frac{AN}{BN} = \lambda$

Then, coordinate of N are $\left(\frac{a}{1+\lambda}, \frac{a\lambda}{1+\lambda}\right)$

$\because$ Slope of $AB = -1$



$\therefore$ Slope of $MN = 1$

$\therefore$ Equation on MN is

$$y - \frac{a\lambda}{1+\lambda} = x - \frac{a}{1+\lambda} \Rightarrow x - y = a\left(\frac{1-\lambda}{\lambda+1}\right)$$

So, the coordinates of M are $\left(0, a\left(\frac{\lambda-1}{\lambda+1}\right)\right)$

Therefore, area of $\Delta AMN = \frac{3}{8}$ area of ΔOAB

$$\Rightarrow \quad \frac{1}{2}\cdot AN \cdot MN = \frac{3}{8}\cdot\frac{1}{2}a\cdot a$$

$$\Rightarrow \quad \frac{1}{2}\cdot\left|\frac{a\lambda\sqrt{2}}{1+\lambda}\cdot\frac{a\sqrt{2}}{1+\lambda}\right| = \frac{3}{8}\cdot\frac{1}{2}a\cdot a$$

$$\Rightarrow \quad \frac{a^2\lambda}{(1+\lambda)^2} = \frac{3}{8}\cdot\frac{1}{2}a^2$$

$$\therefore \quad \lambda = 3 \quad \text{or} \quad \lambda = \frac{1}{3}$$

For $\lambda = \frac{1}{3}$, then M lies outside the segment OB and hence the required value of $\lambda = 3$.

14. Let $S = (x, y)$, given $(SQ)^2 + (SR)^2 = 2(SP)^2$

$$\Rightarrow \quad (x+1)^2 + y^2 + (x-2)^2 + y^2 = 2[(x-1^2) + y^2]$$

$$\Rightarrow \quad 2x^2 + 2y^2 - 2x + 5 = 2(x^2 + y^2 - 2x + 1)$$

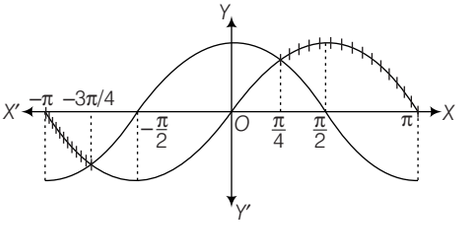
$$\Rightarrow \quad 2x + 3 = 0 \Rightarrow x = -\frac{3}{2}$$

A straight line parallel to Y-axis.

15. $$\frac{\text{Value of}(3x-2y+1) \text{ at } A}{\text{Value of } (3x-2y+1) \text{ at } B} > 0$$

$$\Rightarrow \quad \frac{(\sin\alpha - 3) - (\cos\alpha - 2) + 1}{(3-2+1)} > 0$$

$$\Rightarrow \quad \sin\alpha - \cos\alpha > 0 \Rightarrow \sin\alpha > \cos\alpha$$



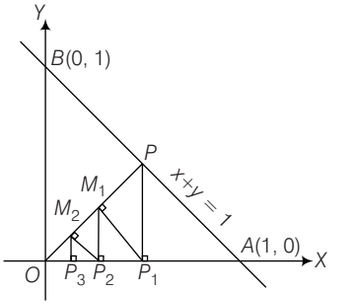
It is clear from the figure

$$\alpha \in \left(-\pi, \frac{-3\pi}{4}\right) \cup \left(\frac{\pi}{4}, \pi\right).$$

16. $\because$ Equation of AB is $x + y = 1$, then coordinates of A and B are (1, 0) and (0, 1) respectively.

$\therefore$ Coordinates of P are $\left(\frac{1}{2}, \frac{1}{2}\right)$

$\because$ PP_1 is perpendicular to OA



Equation of OP is $y = x$

Then, $$OP_1 = PP_1 = \frac{1}{2}$$

We have, $$(OM_{n-1})^2 = (OP_n)^2 + (P_nM_{n-1})^2$$

$$= 2(OP_n)^2 \qquad \{\because y = x\}$$

$$= 2\alpha_n^2 \text{ (say)}$$

Also, $$(OP_{n-1})^2 = (OM_{n-1})^2 + (P_{n-1}M_{n-1})^2$$

$$\alpha_{n-1}^2 = 2\alpha_n^2 + \frac{1}{2}\alpha_{n-1}^2$$

$$\Rightarrow \quad \frac{1}{2}\alpha_{n-1}^2 = 2\alpha_n^2$$

$$\Rightarrow \quad \alpha_n = \frac{1}{2}\alpha_{n-1}$$

$$\therefore \quad OP_n = \alpha_n = \frac{1}{2}\alpha_{n-1}$$

$$= \frac{1}{2^2}\alpha_{n-2} = \frac{1}{2^3}\alpha_{n-3}$$

.....

.....

.....

$$= \frac{1}{2^{n-1}}\alpha_1$$

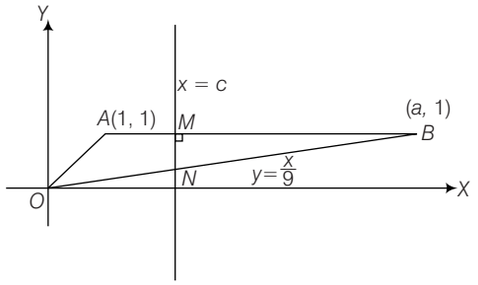
$$= \frac{1}{2^{n-1}}\left(\frac{1}{2}\right) = \frac{1}{2^n}.$$

17. Let $O \equiv (0, 0)$, $A \equiv (1, 1)$ and $B \equiv (9, 1)$

Area of $\Delta OAB = \frac{1}{2} \times 8 \times 1 = 4$

It is clear that $1 < c < 9$

and $$M \equiv (c, 1) \text{ and } N \equiv \left(c, \frac{c}{9}\right)$$



$\therefore$ Area of $\Delta BMN = 2$ (given)

$\Rightarrow \quad \frac{1}{2} \times (9 - c) \times \left(1 - \frac{c}{9}\right) = 2$

or $\quad (9 - c)^2 = 36$

or $\quad 9 - c = \pm 6 \Rightarrow c = 3$ or 15

but $\quad 1 < c < 9$

$\therefore \quad c = 3$

18. The three lines are concurrent if

$$\begin{vmatrix} 1 & 2 & -9 \\ 3 & -5 & -5 \\ a & b & -1 \end{vmatrix} = 0$$

or $\quad 5a + 2b = 1$

which is three of the line $5x + 2y = 1$ passes through (a, b).

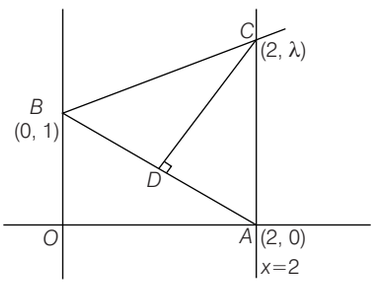
19. $\because BC = AC$

$\Rightarrow \quad 2^2 + (\lambda - 1)^2 = \lambda^2$

$\Rightarrow \quad 4 = \lambda^2 - (\lambda - 1)^2$

$= (2\lambda - 1)(1)$

$\therefore \quad \lambda = \frac{5}{2}$



$\because$ Equation. of AB is $\frac{x}{2} + \frac{y}{1} = 1$, $D \equiv \left(1, \frac{1}{2}\right)$ (mid-point of AB)

$\therefore$ Equation of CD is $2x - y = \mu$

$\because$ CD pass through D, thus

$$2 - \frac{1}{2} = \mu \text{ or } \mu = \frac{3}{2}$$

$\therefore$ Equation of CD is $\quad 2x - y = \frac{3}{2}$ ...(i)

and Eq. (i) of line $\perp$ to AC and pass through B is $y = 1$ ...(ii)

from Eqs. (i) and (ii), we get

Orthocentre $\equiv \left(\frac{5}{4}, 1\right)$

20. Let $A \equiv (3, 4)$, $B \equiv (0, y)$, $C \equiv (x, 0)$, $D \equiv (8, 2)$

$\therefore$ Slope of $AB = -$ Slope of BC

$\Rightarrow \quad \frac{y - 4}{0 - 3} = -\left(\frac{0 - y}{x - 0}\right)$

or $\quad 4x - xy = 3y$ ...(i)

and slope of $BC = -$ slope of CD

$\Rightarrow \quad \left(\frac{0 - y}{x - 0}\right) = -\left(\frac{2 - 0}{8 - x}\right)$

or $\quad 2x + xy = 8y$ ...(ii)

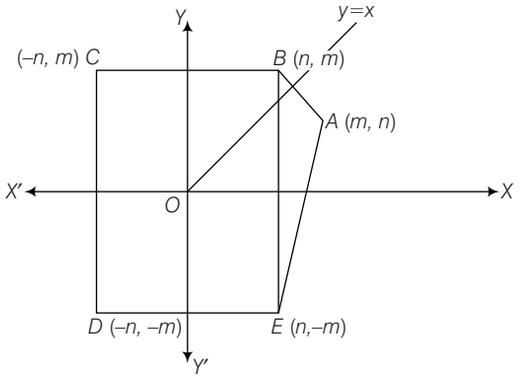
adding Eqs. (i) and (ii), we get

$$6x = 11y \quad \text{...(iii)}$$

from Eqs. (ii) and (iii), we get

$$x = \frac{13}{3} = 4\frac{1}{3}$$

21.



Area of rectangle $BCDE = (2n)(2m)$

$= 4mn$

and area of $\Delta ABE = \frac{1}{2} \times 2m \times (m - n)$

$= m(m - n)$

$\therefore$ Area of pentagon $= 4mn + m(m - n)$

$= m(m + 3n)$

22. The equation of the line L, be $y - 2 = m(x - 8)$, $m < 0$

coordinates of P and Q are $P\left(8 - \frac{2}{m}, 0\right)$ and $Q(0, 2 - 8m)$.

So, $\quad OP + OQ = 8 - \frac{2}{m} + 2 - 8m$

$= 10 + \frac{2}{(-m)} + 8(-m) \geq$

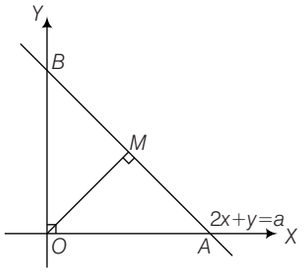
$10 + 2\sqrt{\frac{2}{(-m)} \times 8(-m)} \geq 18$

So, absolute minimum value of $OP + OQ = 18$

23. Let the two perpendiculars through the origin intersect $2x + y = a$ at A and B so that the triangle OAB is isosceles.

OM = length of perpendicular from O to

AB, $OM = \frac{a}{\sqrt{5}}$.



Also, $AM = MB = OM$

$\Rightarrow \quad AB = \dfrac{2a}{\sqrt{5}}$

Area of $\Delta OAB = \dfrac{1}{2} \cdot AB \cdot OM$

$= \dfrac{1}{2} \cdot \dfrac{2a}{\sqrt{5}} \cdot \dfrac{a}{\sqrt{5}} = \dfrac{a^2}{5}$ sq units

24. Solving given equations, we get

$$x = \frac{5}{3 + 4m}$$

x is an integer, if $3 + 4m = 1, -1, 5, -5$

or $\quad m = \dfrac{-2}{4}, \dfrac{-4}{4}, \dfrac{2}{4}, \dfrac{-8}{4}$

or $\quad m = -\dfrac{1}{2}, -1, \dfrac{1}{2}, -2$

Hence, m has two integral values.

25. Let the coordinates of A be $(a, 0)$. Then the slope of the reflected ray is

$$\frac{3-0}{5-a} = \tan\theta \qquad \text{(say) ...(i)}$$

Then the slope of the incident ray

$$= \frac{2-0}{1-a} = \tan(\pi - \theta)$$

From Eqs. (i) and (ii), we get

$$\tan\theta + \tan(\pi - \theta) = 0$$

$\Rightarrow \quad \dfrac{3}{5-a} + \dfrac{2}{1-a} = 0$

$\Rightarrow \quad 3 - 3a + 10 - 2a = 0$

$$a = \frac{13}{5}$$

Thus, the coordinate of A is $\left(\dfrac{13}{5}, 0\right)$

26. Lines $5x + 3y - 2 + \lambda(3x - y - 4) = 0$ are concurrent at $(1, -1)$ and lines

$x - y + 1 + \mu(2x - y - 2) = 0$ are concurrent at $(3, 4)$.

Thus equation of line common to both family is

$$y + 1 = \frac{4+1}{3-1}(x - 1)$$

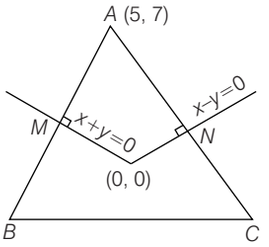
or $\quad 5x - 2y - 7 = 0$

$\therefore \quad a = 5, b = -2 \Rightarrow a + b = 3$

27. $\because$ B is the reflection of $A(5, 7)$ w.r.t the line $x + y = 0$

$\therefore \quad B \equiv (-7, -5)$

and C is the reflection of $A(5, 7)$ w.r.t the line $x - y = 0$



$\therefore \quad C \equiv (7, 5)$

$\therefore$ Equation of BC is $y + 5 = \dfrac{5+5}{7+7}(x + 7)$ or $7y = 5x$

28. Let $P \equiv (2, -1)$

$P(2, -1)$ goes 2 units along $x + y = 1$ upto A and 5 units along $x - 2y = 4$ upto B.

Now, slope of $x + y = -1$ is $-1 = \tan\theta$ (say)

$\therefore \quad \theta = 135°$

and slope $x - 2y = 4$ is $\dfrac{1}{2} = \tan\phi$ (say)

$\therefore \quad \sin\phi = \dfrac{1}{\sqrt{5}}, \cos\phi = \dfrac{2}{\sqrt{5}}$

The coordinates of A

i.e. $\quad (2 + 2\cos 135°, -1 + 2\sin 135°)$

or $\quad (2 - \sqrt{2}, \sqrt{2} - 1)$

The coordinates of B

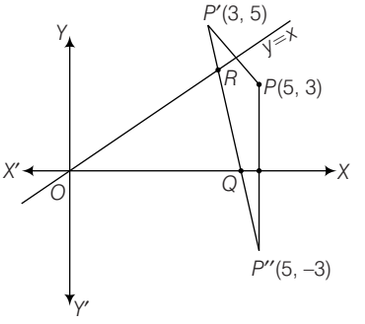
i.e. $(2 + 5\cos\phi, -1 + 5\sin\phi)$ or $(2 + 2\sqrt{5}, \sqrt{5} - 1)$

29. $\because \quad P \equiv (5, 3)$

Let P' and P'' be the images of P w.r.t $y = x$ and $y = 0$ (X-axis) respectively, then $P' \equiv (3, 5)$ and $P'' \equiv (5, -3)$

$\because \quad PQ + QR + RP$ is minimum

$\therefore \quad P', R, Q, P''$ are collinear.



$\therefore$ Equation of $P'P''$ is

$$y + 3 = \left(\frac{5+3}{3-5}\right)(x - 5)$$

or $\quad 4x + y = 17$

$\therefore \quad Q \equiv \left(\dfrac{17}{4}, 0\right)$ $(\because Q$ on Y-axis)

30. Equation of incident ray is

$$y - 0 = \tan(90° + 60°)(x - 2)$$

or $$y = -\frac{1}{\sqrt{3}}(x - 2)$$

or $$(x - 2) + y\sqrt{3} = 0$$

and equation of refracted ray is

$$y - 0 = -\tan 60°(x - 2)$$

or $$y = -\sqrt{3}(x - 2)$$

or $$(x - 2) + \frac{y}{\sqrt{3}} = 0$$

$\therefore$ Combined equation is

$$[(x - 2) + y\sqrt{3}]\left((x - 2) + \frac{y}{\sqrt{3}}\right) = 0$$

i.e. $$(x - 2)^2 + y^2 + \frac{4}{\sqrt{3}}(x - 2)y = 0$$

31. Point of intersection of $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{b} + \frac{y}{a} = 1$ is $P\left(\frac{ab}{a+b}, \frac{ab}{a+b}\right)$, this point P satisfies alternates (a), (b), (c) and (d).

32. The two lines will be identical if their exists some real number k such that

$b^3 - c^3 = k(b - c)$, $c^3 - a^3 = k(c - a)$ and $a^3 - b^3 = k(a - b)$

$\Rightarrow$ $b - c = 0$ or $b^2 + c^2 + bc = k$

$c - a = 0$ or $c^2 + a^2 + ca = k$

and $a - b = 0$ or $a^2 + b^2 + ab = k$

$\Rightarrow$ $a = b$ or $b = c$ or $c = a$

or $b^2 + c^2 + bc = c^2 + a^2 + ca$

$\Rightarrow$ $b = c$ or $c = a$

or $a = b$ or $a + b + c = 0$

33. As the third vertex lies on the line $y = x + 3$, its coordinates are of the form $(x, x + 3)$. The area of the triangle with vertices $(2, 1)$, $(3, -2)$ and $(x, x + 3)$ is given by

$$\frac{1}{2}\left|\begin{vmatrix} x & x+3 & 1 \\ 2 & 1 & 1 \\ 3 & -2 & 1 \end{vmatrix}\right| = |2x - 2| = 5 \qquad \text{(given)}$$

$\therefore$ $2x - 2 = \pm 5 \Rightarrow x = \frac{-3}{2}, \frac{7}{2}$

Thus, the coordinates of the third vertex are $\left(\frac{7}{2}, \frac{13}{2}\right)$ or $\left(\frac{-3}{2}, \frac{3}{2}\right)$.

34. $\begin{vmatrix} 1 & -2 & -6 \\ 3 & 1 & -4 \\ \lambda & 4 & \lambda^2 \end{vmatrix} = 0$

$\Rightarrow$ $\lambda^2 + 2\lambda - 8 = 0$

$\therefore$ $(\lambda + 4)(\lambda - 2) = 0$

$\Rightarrow$ $\lambda = -4, 2$

35. Equation of any line through the point of intersection of the given lines is $(3x + y - 5) + \lambda(x - y + 1) = 0$.

Since this line is perpendicular to one of the given lines

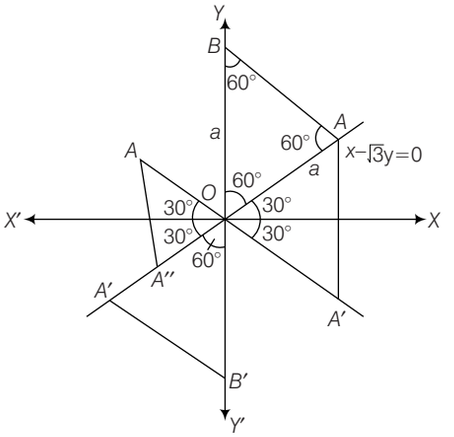
$$\frac{3 + \lambda}{\lambda - 1} = -1 \text{ or } \frac{1}{3}$$

$\Rightarrow \lambda = -1$ or -5, therefore the required straight line is

$$x + y - 3 = 0$$

or $$x - 3y + 5 = 0$$

36. If B lies on Y-axis, then coordinates of B are $(0, a)$ or $(0, -a)$



If third vertex in IV quadrant or in II quadrant, then its coordinates are $(a\cos 30°, -a\sin 30°)$ and $(-a\cos 30°, a\sin 30°)$

i.e. $$\left(\frac{a\sqrt{3}}{2}, -\frac{a}{2}\right) \text{ and } \left(-\frac{a\sqrt{3}}{2}, \frac{a}{2}\right),$$

37. Since, $ax + by + c = 0$, $bx + cy + a = 0$ and $cx + ay + b = 0$ are concurrent

$\therefore$ $$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

$\Rightarrow$ $3abc - a^3 - b^3 - c^3 = 0$

$\Rightarrow$ $-(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = 0$

$a + b + c \neq 0$

$\therefore$ $a^2 + b^2 + c^2 - ab - bc - ca = 0$

$$\frac{1}{2}\{(a - b)^2 + (b - c)^2 + (c - a)^2\} = 0$$

As a, b, c are real numbers

$\therefore$ $b - c = 0, c - a = 0, a - b = 0$

$\Rightarrow$ $a = b = c$

38. $\because$ $E \equiv (4, 4)$

$\therefore z_C = 7 + 5i, z_E = 4 + 4i$

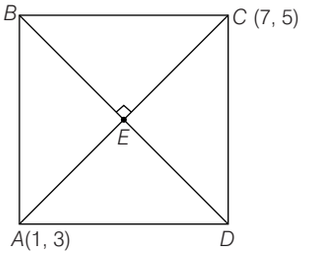
Now, (in ΔBEC)

$$\frac{z_B - z_E}{z_C - z_E} = e^{i\frac{\pi}{2}} = i$$

$\Rightarrow$ $z_B - 4 - 4i = i(7 + 5i - 4 - 4i)$

or $z_B = 3 + 7i$

$\therefore$ $B \equiv (3, 7)$, then $D \equiv (5, 1)$



Equation of AB is

$$y - 3 = \frac{7-3}{3-1}(x-1) \text{ or } 2x - y + 1 = 0$$

and equation of AD is

$$y - 3 = \frac{1-3}{5-1}(x-1) \text{ or } x + 2y - 7 = 0$$

39. Given,

$$6a^2 - 3b^2 - c^2 + 7ab - ac + 4bc = 0$$

$$\Rightarrow \quad 6a^2 + (7b - c)a - (3b^2 - 4bc + c^2) = 0$$

$$\Rightarrow \quad a = \frac{-(7b-c) \pm \sqrt{(7b-c)^2 + 24\,(3b^2 - 4bc + c^2)}}{12}$$

$$\Rightarrow \qquad 12a + 7b - c = \pm (11b - 5c)$$

$$\Rightarrow \qquad 12a - 4b + 4c = 0$$

or $\qquad 12a + 18b - 6c = 0$

$$\Rightarrow \qquad 3a - b + c = 0$$

or $\qquad -2a - 3b + c = 0$

Hence $(3, -1)$ or $(-2, -3)$ lies on the line $ax + by + c = 0$,

40. $x + 2y + 4 = 0$ and $4x + 2y - 1 = 0$

$$\Rightarrow \qquad x + 2y + 4 = 0$$

and $\qquad -4x - 2y + 1 = 0$

Here, $\qquad (1)(-4) + (2)(-2) = -8 < 0$

$\therefore$ Bisector of the angle including the acute angle bisectors and origin is

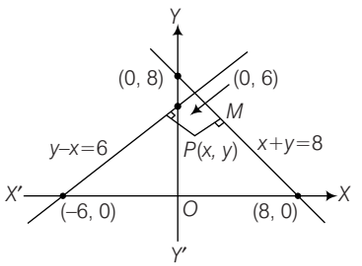
$$\frac{x + 2y + 4}{\sqrt{5}} = \frac{(-4x - 2y + 1)}{2\sqrt{5}}$$

$$\Rightarrow \qquad 6x + 6y + 7 = 0$$

41. Let position of bunglow is $P(x_1, y_1)$, then $PM = 100$ and $PN = 100$

$\therefore$ $\qquad \dfrac{x_1 + y_1 - 8}{\sqrt{2}} = \pm\, 100$

and $\qquad \dfrac{x_1 - y_1 + 6}{\sqrt{2}} = \pm\, 100$



After solving, we get

$$x_1 = 1 \pm 100\sqrt{2}, 1$$

and $\qquad y_1 = 7, 7 \pm 100\sqrt{2}$

Hence, $\qquad (1 + 100\sqrt{2}, 7), (1 - 100\sqrt{2}, 7),$

$$(1, 7 + 100\sqrt{2}), (1, 7 - 100\sqrt{2})$$

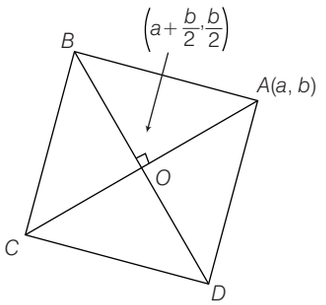
42. Equation of the other diagonal is $x + y = \lambda$ which pass through (a, b), then

$$a + b = \lambda$$

$\therefore$ Equation of other diagonal is

$$x + y = a + b$$

i.e. then centre of the square is the point of intersection of $x - y = a$ and $x + y = a + b$ is $\left(a + \dfrac{b}{2}, \dfrac{b}{2}\right)$, then vertex



$$C \equiv (2a + b - a, b - b)$$

$\therefore$ $\qquad C \equiv (a + b, 0)$

If $\qquad B \equiv z$

Then, $\qquad \dfrac{z - \left(a + \dfrac{b}{2} + \dfrac{ib}{2}\right)}{(a + ib) - \left(a + \dfrac{b}{2} + \dfrac{ib}{2}\right)} = \dfrac{BO}{AO} e^{i\frac{\pi}{2}} = i \quad (\because BO = AO)$

$$\Rightarrow \qquad z - \left(a + \frac{b}{2} + \frac{ib}{2}\right) = i\left(-\frac{b}{2} + \frac{ib}{2}\right) = -\frac{ib}{2} - \frac{b}{2}$$

$\therefore$ $\qquad z = a$

$\therefore$ $\qquad B \equiv (a, 0)$

then, $\qquad D \equiv (a + b, b)$

Hence, other vertices are $(a + b, 0), (a, 0)$ and $(a + b, b)$.

43. $(y - y_1) - m(x - x_1) = 0$ is family of lines

$\therefore$ $\qquad y - y_1 = 0, x - x_1 = 0$

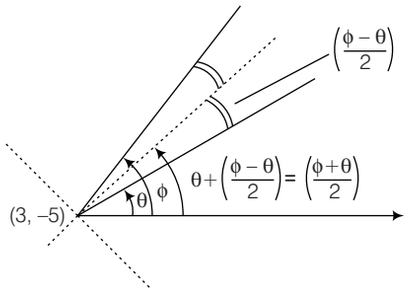
Then, $\qquad y = y_1$ and $x = x_1$

44. Given lines $L_1 = 0$ and $L_2 = 0$ are perpendicular and given bisectors are $\lambda_1 L_1 - \lambda_2 L_2 = 0$ and $\lambda_1 L_1 + \lambda_2 L_2 = 0$

$\therefore$ bisectors are perpendicular to each other.

Hence, bisectors of $\lambda_1 L_1 - \lambda_2 L_2 = 0$ and $\lambda_1 L_1 + \lambda_2 L_2 = 0$ are $L_1 = 0$ and $L_2 = 0$.

45. $\therefore$ One bisector makes an angle $\left(\dfrac{\theta + \phi}{2}\right)$ with X-axis, then other bisector makes an angle $90° + \left(\dfrac{\theta + \phi}{2}\right)$ with X-axis.



$\therefore$ Equations of bisectors are

$$\frac{x-3}{\cos\left(\frac{\theta+\phi}{2}\right)}=\frac{y+5}{\sin\left(\frac{\theta+\phi}{2}\right)} \quad \ldots(i)$$

and $$\frac{x-3}{\cos\left(\frac{\pi}{2}+\frac{\theta+\phi}{2}\right)}=\frac{y+5}{\sin\left(\frac{\pi}{2}+\frac{\theta+\phi}{2}\right)}$$

$\Rightarrow$ $$\frac{x-3}{-\sin\left(\frac{\theta+\phi}{2}\right)}=\frac{y+5}{\cos\left(\frac{\theta+\phi}{2}\right)} \quad \ldots(ii)$$

But given bisector are $\frac{x-3}{\cos\alpha}=\frac{y+5}{\sin\alpha}$

$\therefore$ $\alpha=\frac{\theta+\phi}{2}$ and $\frac{x-3}{\beta}=\frac{y+5}{\gamma}$ [from Eq. (i)]...(iii)

$\therefore$ $\beta=-\sin\left(\frac{\theta+\phi}{2}\right)=-\sin\alpha$ [from Eq. (ii)]

and $\gamma=\cos\left(\frac{\theta+\phi}{2}\right)=\cos\alpha$

46. $\because OR = AR$

$\Rightarrow$ $|x-0|+|y-0|=|x-1|+|y-2|$

$\Rightarrow$ $|x|+|y|=|x-1|+|y-2|$

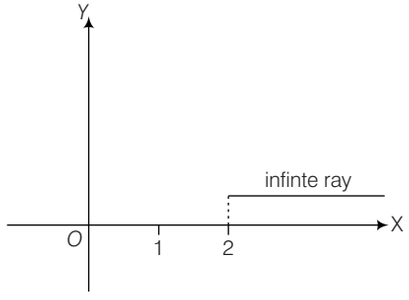
$\because$ $0 \le x < 1$ and $0 \le y < 2$

$\therefore$ $x+y=-(x-1)-(y-2)$

$\Rightarrow$ $2x+2y=3$

47. $OS = BS$

$\Rightarrow$ $|x-0|+|y-0|=|x-2|+|y-3|$



$\Rightarrow$ $|x|+|y|=|x-2|+|y-3|$

$\because$ $x \ge 2$ and $0 \le y < 3$

$\therefore$ $x+y=x-2+3-y$

$\Rightarrow$ $2y=1$

$\therefore$ $y=\frac{1}{2}$

48. $\because OT = CT$

$\Rightarrow$ $|x-0|+|y-0|=|x-4|+|y-3|$

$\because$ $x \ge 0, y \ge 0$

$\Rightarrow$ $x+y=|x-4|+|y-3|$

Case I : If $0 \le x \le 4$ and $0 \le y \le 3$

$x+y=4-x+3-y$

$\Rightarrow$ $x+y=\frac{7}{2}$

Case II : If $0 \le x \le 4$ and $y \ge 3$

$x+y=4-x+y-3$

$\Rightarrow$ $x=\frac{1}{2}$

Case III : If $x \ge 4$ and $0 \le y \le 3$

$x+y=x-4+3-y$

$y=-1/2$ (impossible)

Case IV : If $x \ge 4$ and $y \ge 3$

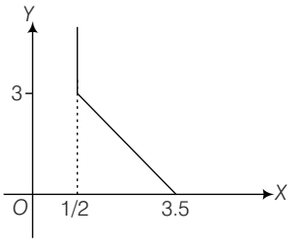
$x+y=x-4+y-3$

$\Rightarrow$ $0=-7$ (impossible)

Combining all cases, we get

$x+y=\frac{7}{2}, \forall\ 0 \le x \le 4$ and $0 \le y \le 3$

and $x=\frac{1}{2}, \forall\ 0 \le x \le 4$ and $y \ge 3$

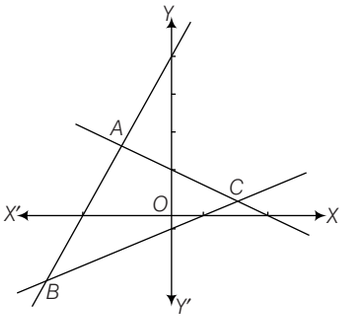


Sol. (Q. Nos. 49 to 51)

$AB : 2x-y+4=0,$

$BC : x-2y-1=0$

and $CA : x+3y-3=0$



$\therefore$ $m_{AB}=m_1=2$

$m_{BC}=m_2=\frac{1}{2}$

and $m_{CA}=m_3=-\frac{1}{3}$

$\because$ $m_1 > m_2 > m_3$

49. $\because \angle A$ is obtuse

$$\therefore \quad \tan A = \frac{m_3 - m_1}{1 + m_3 m_1}$$

$$= \frac{-\frac{1}{3} - 2}{1 - \frac{2}{3}} = -7$$

50. For external bisector of B

$$AB : 2x - y + 4 = 0$$

$$BC : -x + 2y + 1 = 0$$

$$\because \quad (2)(-1) + (-1)(2) = -4 < 0$$

$\therefore$ External bisector of B is

$$\left(\frac{2x - y + 4}{\sqrt{5}}\right) = -\frac{(-x + 2y + 1)}{\sqrt{5}}$$

or $$x + y + 5 = 0$$

51. Let (α, β) be the image of $B(-3, -2)$ w.r.t. the line $x + 3y - 3 = 0$, then

$$\frac{\alpha + 3}{1} = \frac{\beta + 2}{3} = \frac{-2(-3 - 6 - 3)}{1 + 9}$$

or $$\frac{\alpha + 3}{1} = \frac{\beta + 2}{3} = \frac{12}{5}$$

or $$\alpha = -\frac{3}{5} \text{ and } \beta = \frac{26}{5}$$

$\therefore$ Required image is $\left(-\frac{3}{5}, \frac{26}{5}\right)$,

Sol. (Q. Nos. 52 to 54)

Let $$B \equiv (\lambda, 2 - \lambda) \qquad (\because B \text{ lies on } x + y = 2)$$

Slope of line $AB = m_1 = \frac{1 + \lambda}{1 - \lambda}$

and Slope of line $BC = m_2 = \frac{5\lambda - 12}{-5\lambda - 2}$

$$= \frac{12 - 5\lambda}{2 + 5\lambda}$$

Let slope of bisector $(x + y = 2) = m_3 = -1$

Now, $$\frac{m_3 - m_1}{1 + m_3 m_1} = \frac{m_2 - m_3}{1 + m_2 m_3}$$

$$\Rightarrow \quad \frac{-1 - \frac{1 + \lambda}{1 - \lambda}}{1 - \frac{1 + \lambda}{1 - \lambda}} = \frac{\frac{12 - 5\lambda}{2 + 5\lambda} + 1}{1 - \frac{12 - 5\lambda}{2 + 5\lambda}}$$

or $$\frac{-2}{-2\lambda} = \frac{14}{-10 + 10\lambda}$$

or $$14\lambda = -10 + 10\lambda$$

$\therefore$ $$\lambda = \frac{-5}{2} \qquad \ldots(i)$$

52. Equation of BC is

$$y - (2 - \lambda) = \frac{-\frac{2}{5} - (2 - \lambda)}{-\frac{2}{5} - \lambda}(x - \lambda)$$

or $$y - 2 - \frac{5}{2} = \frac{-\frac{2}{5} - \frac{9}{2}}{-\frac{2}{5} + \frac{5}{2}}\left(x + \frac{5}{2}\right)$$

or $$7x + 3y + 4 = 0$$

53. Coordinates of vertex B are $(\lambda, 2 - \lambda)$

i.e. $$\left(-\frac{5}{2}, \frac{9}{2}\right) \qquad \text{[from Eq. (i)]}$$

54. $A \equiv (1, 3)$ and $B \equiv \left(-\frac{5}{2}, \frac{9}{2}\right)$

$\therefore$ Equation of AB is

$$y - 3 = \frac{\frac{9}{2} - 3}{-\frac{5}{2} - 1}(x - 1)$$

or $$3x + 7y = 24$$

55. Any point on the line $3x - y = 2$ is $(t, 3t - 2)$, t being parameter.
If (x, y) be image of the point $(t, 3t - 2)$ in the line $y = x - 1$ or $x - y - 1 = 0$, then

$$\frac{x - t}{1} = \frac{y - (3t - 2)}{-1}$$

$$= -\frac{2(t - 3t + 2 - 1)}{1 + 1}$$

$$\Rightarrow \quad \frac{x - t}{1} = \frac{y - 3t + 2}{-1} = 2t - 1$$

or $$x - t = 2t - 1$$

$$\Rightarrow \quad x + 1 = 3t \qquad \ldots(i)$$

and $$y - 3t + 2 = -2t + 1$$

$$\Rightarrow \quad y + 1 = t \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$x + 1 = 3(y + 1)$$

$$\Rightarrow \quad x - 3y = 2$$

56. Any point on the circle $x^2 + y^2 = 4$ is $(2\cos\theta, 2\sin\theta)$, θ being parameter.
If (x, y) be image of the point $(2\cos\theta, 2\sin\theta)$, in the line $x + y = 2$, then

$$\frac{x - 2\cos\theta}{1} = \frac{y - 2\sin\theta}{1}$$

$$= \frac{-2(2\cos\theta + 2\sin\theta - 2)}{1 + 1}$$

or $$x - 2\cos\theta = y - 2\sin\theta$$

$$= -2\cos\theta - 2\sin\theta + 2 \qquad \ldots(i)$$

or $$x - 2\cos\theta = -2\cos\theta - 2\sin\theta + 2$$

$$\Rightarrow \quad x - 2 = -2\sin\theta$$

and $$y - 2\sin\theta = -2\cos\theta - 2\sin\theta + 2$$

$$\Rightarrow \quad y - 2 = -2\cos\theta \qquad \ldots(ii)$$

From Eqs. (i) and (ii),

$$(x - 2)^2 + (y - 2)^2 = 4$$

$$\Rightarrow \quad x^2 + y^2 - 4x - 4y + 4 = 0$$

57. Any point on the parabola $x^2 = 4y$ is $(2t, t^2)$, t being parameter.

If (x, y) be image of the point $(2t, t^2)$ in the $x + y = a$, then

$$\frac{x-2t}{1} + \frac{y-t^2}{1} = \frac{-2(2t + t^2 - a)}{1+1}$$

$$= -2t - t^2 + a$$

or $\quad x - 2t = -2t - t^2 + a$

$\Rightarrow \quad x - a = -t^2 \quad \ldots(i)$

and $\quad y - t^2 = -2t - t^2 + a$

$\Rightarrow \quad y - a = -2t \quad \ldots(ii)$

From Eqs. (i) and (ii) we get

$$(y-a)^2 = 4t^2 = -4(x-a)$$

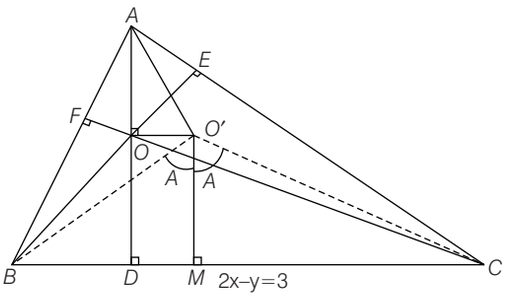
or $\quad (y-a)^2 = 4(a-x)$

Sol. (Q. Nos. 58 to 60)

Given orthocentre $O \equiv (1, 2)$

and circumcentre

$$O' = (2, 4)$$



$\because$ Slope of $OO' =$ Slope of $(2x - y = 3)$

and $\quad OD = O'M = \frac{3}{\sqrt{5}}$

Let R be the circumradius

$\therefore \quad O'M = R\cos A$

$\Rightarrow \quad R\cos A = \frac{3}{\sqrt{5}} \quad \ldots(i)$

58. $R = AO' = \sqrt{(AO)^2 + (OO')^2}$

$$= \sqrt{(2R\cos A)^2 + 5}$$

$$= \sqrt{\left(\frac{6}{\sqrt{5}}\right)^2 + 5} \qquad \text{[from Eq. (i)]}$$

$$= \sqrt{\frac{61}{5}}$$

59. $\because OD = 2R\cos B\cos C$

$\therefore \quad 2R\cos B\cos C = \frac{3}{\sqrt{5}}$

$= R\cos A \qquad$ [from Eq. (i)] ...(ii)

$\Rightarrow \quad \cos A = 2\cos B\cos C$

$\Rightarrow \quad -\cos(B + C) = 2\cos B\cos C \qquad (\because A + B + C = \pi)$

$\Rightarrow \quad -(\cos B\cos C - \sin B\sin C) = 2\cos B\cos C$

or $\quad \sin B\sin C = 3\cos B\cos C$

$$= 3 \times \frac{3}{2R\sqrt{5}}$$

$$= \frac{9}{2\sqrt{61}} \qquad \left(\because R = \sqrt{\frac{61}{5}}\right)$$

60. $\because AO = 2R\cos A$

$$= 2 \times \frac{3}{\sqrt{5}} \qquad \text{[from Eq. (i)]}$$

$$= \frac{6}{\sqrt{5}}$$

61. The equation of straight line through (2, 3) with slope m is

$$y - 3 = m(x - 2)$$

or $\quad mx - y = 2m - 3$

or $\quad \frac{x}{\left(\frac{2m-3}{m}\right)} + \frac{y}{(3-2m)} = 1$

Here, $\quad OA = \frac{2m-3}{m}$ or $OB = 3 - 2m$

$\because$ The area of $\Delta OAB = 12$

$\Rightarrow \quad \left|\frac{1}{2} \times OA \times OB\right| = 12$

or $\quad \frac{1}{2}\left(\frac{2m-3}{m}\right)(3-2m) = \pm 12$

or $\quad (2m-3)^2 = \pm 24m$

Taking positive sign, we get $4m^2 - 36m + 9 = 0$

Here $D > 0$, This is a quadratic in m which given two value of m, and taking negative sign, we get $(2m + 3)^2 = 0$.

This gives one line of m as $\frac{-3}{2}$.

Hence, three straight lines are possible.

62. $\because$ Point of intersection of $ax + 3y - 1 = 0$ and $ax + y + 1 = 0$ is $A\left(-\frac{2}{a}, 1\right)$ and point of intersection of $ax + 3y - 1 = 0$ and $x + 3y = 0$ is $B\left(\frac{1}{a-1}, -\frac{1}{3(a-1)}\right)$

$\Rightarrow \quad$ Slope of OA is $m_{OA} = -\frac{a}{2}$

and $\quad$ Slope of OB is $m_{OB} = -\frac{1}{3}$

$\because \quad m_{OA} \times m_{OB} = -1$

$\therefore \quad -\frac{a}{2} \times -\frac{1}{3} = -1$

or $\quad a = -6$

$\therefore \quad |a| = 6$

63. Here, B is the image of A w.r.t line $y = x$

$\therefore$ $B \equiv (2, 1)$ and C is the image of A w.r.t line $x - 2y + 1 = 0$ if $C \equiv (\alpha, \beta)$, then

$$\frac{\alpha - 1}{1} = \frac{\beta - 2}{-2} = \frac{-2(1 - 4 + 1)}{1 + 4}$$

or $\alpha = \frac{9}{5}$ and $\beta = \frac{2}{5}$

$\therefore$ $C \equiv \left(\frac{9}{5}, \frac{2}{5}\right)$

$\Rightarrow$ Equation of BC is

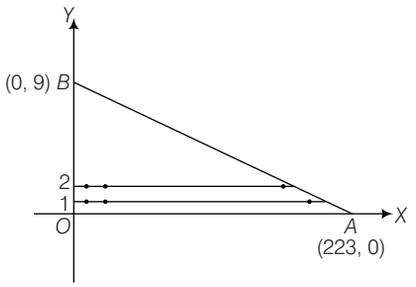
$$y - 1 = \frac{\left(\frac{2}{5} - 1\right)}{\left(\frac{9}{5} - 2\right)}(x - 2)$$

or $3x - y - 5 = 0$ ($\because$ Eq. of BC is $ax + by - 5 = 0$)

Here, $a = 3, b = -1$

$\therefore$ $a - 2b = 5$

64. On the line $y = 1$, the number of lattice points is $\left\lceil \frac{2007 - 223}{9} \right\rceil = 198$



Hence, the total number of points

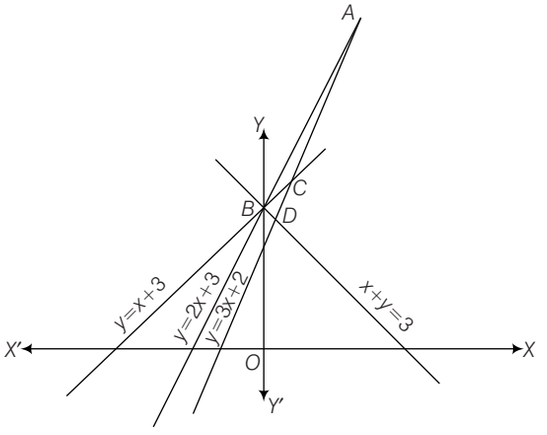
$$= \sum_{y=1}^{8} \left\lceil \frac{2007 - 223y}{9} \right\rceil$$

$$= 198 + 173 + 148 + 123 + 99 + 74 + 49 + 24 = 888$$

Hence, tens place digit is 8.

65. A rough sketch of the lines is given.

There are three triangle namely ABC, BCD and ABD



66. Let a be the length of side of square

$\therefore$ $a^2 + a^2 = 2^2 \Rightarrow a = \sqrt{2}$

i.e. distance between parallel lines is $\sqrt{2}$

Now, let two lines of family $y = x + n$ are $y = x + n$, and $y = x + n_2$, where

$n_1, n_2 \in \{0, 1, 2, 3, 4\}$

$\therefore$ $\frac{|n_1 - n_2|}{\sqrt{2}} = \sqrt{2}$

or $|n_1 - n_2| = 2$

$\Rightarrow$ $\{n_1, n_2\}$ are $\{0, 2\}$, $\{1, 3\}$ and $\{2, 4\}$

Hence, both the family have three such pairs. So, the number of squares possible is $3 \times 3 = 9$.

67. Let the coordinate of C be $(1, c)$, then

$$m_2 = \frac{c - y}{1 - x}$$

or $m_2 = \frac{c - m_1 x}{1 - x}$ ($\because$ slope of $AB = m_1$)

$\Rightarrow$ $m_2(1 - x) = c - m_1 x$

or $c = (m_1 - m_2)x + m_2$

Now, the area of ΔABC is $\frac{1}{2}|cx - y|$

$$= \frac{1}{2}|((m_1 - m_2)x + m_2)x - m_1 x| \quad (\because y = m_1 x)$$

$$= \frac{1}{2}(m_1 - m_2)(x - x^2) \quad [\because m_1 > m_2 \text{ and } x \in (0, 1)]$$

Hence, $f(x) = \frac{1}{2}(x - x^2)$

$\therefore$ $\frac{df(x)}{dx} = \frac{1}{2}(1 - 2x)$

and $\frac{d^2 f(x)}{dx^2} = -1 < 0$

For maximum of

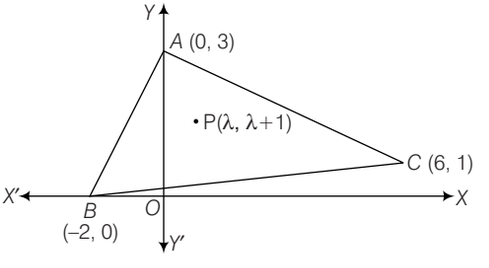
$$f(x), \frac{df(x)}{dx} = 0 \Rightarrow x = \frac{1}{2}$$

$\therefore$ $f(x)|_{\max} = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right)$

$$= \frac{1}{8} = \lambda \quad \text{(given)}$$

$\Rightarrow$ $\frac{1}{\lambda} = 8$

68. Equation of AB is $3x - 2y + 6 = 0$



Equation of BC is $x - 8y + 2 = 0$,
Equation of CA is $x + 3y - 9 = 0$
Let $P \equiv (\lambda, \lambda + 1)$
$\because$ B and P lie on one side of AC, then

$$\frac{\lambda + 3(\lambda + 1) - 9}{-2 + 0 - 9} > 0$$

or $4\lambda - 6 < 0$

or $\lambda < \frac{3}{2}$ (i)

and C and P lie on one side of AB, then

$$\frac{3\lambda - 2(\lambda + 1) + 6}{18 - 2 + 6} > 0$$

or $\lambda + 4 > 0$

or $\lambda > -4$ ...(ii)

Finally, A and P lie on one side of BC, then

$$\frac{\lambda - 8(\lambda + 1) + 2}{0 - 24 + 2} > 0$$

or $-7\lambda - 6 < 0$

or $\lambda > -\frac{6}{7}$ ...(iii)

From Eqs. (i), (ii) and (iii), we get

$$-\frac{6}{7} < \lambda < \frac{3}{2}$$

Integral values of λ are 0 and 1.
Hence, number of integral values of λ is 2.

69. Lines

$$(2a + b)x + (a + 3b)y + b - 3a = 0$$

or $a(2x + y - 3) + b(x + 3y + 1) = 0$

are concurrent at the point of intersection of lines $2x + y - 3 = 0$ and $x + 3y + 1 = 0$ which is $(2, -1)$.
Now, line $\lambda x + 2y + 6 = 0$ must pass through $(2, -1)$, therefore,

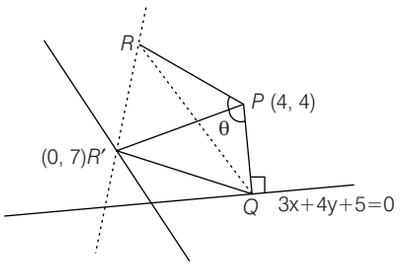
$$2\lambda - 2 + 6 \text{ or } \lambda = -2$$

$\therefore$ $|\lambda| = 2$

70. Since, PQ is of fixed length.

Area of $\Delta PQR = \frac{1}{2}|PQ|\,|RP|\sin\theta$

This will be maximum, if $\sin\theta = 1$ and RP is maximum.

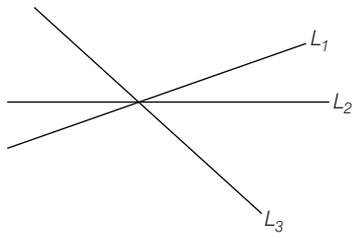


Since, line $y = mx + 7$ rotates about $(0, 7)$, if PR' is perpendicular to the line than PR' is maximum value of PR.

$\therefore$ $m = -\left(\frac{4 - 0}{4 - 7}\right) = \frac{4}{3}$

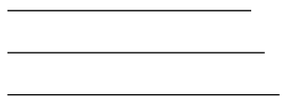
Hence, $3m = 4$

71. **(A)**



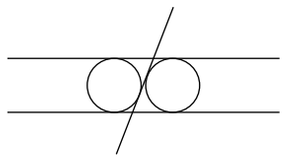
In this case no circle
$\therefore$ $n = 0 \Rightarrow n + 1 = 1$

(B)



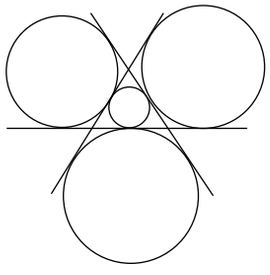
In this case no circle
$\therefore$ $n = 0 \Rightarrow 2n + 3 = 3$

(C)



In this case two circle which are touching all three lines
$\therefore$ $n = 2 \Rightarrow n + 2 = 4$

(D)



In this case four circle which are touching all three lines
$\therefore$ $n = 4 \Rightarrow n + 2 = 6$

72. **(A)** The given lines an concurrent. So,

$$\begin{vmatrix} 1 & -2 & -6 \\ 3 & 1 & -4 \\ \lambda & 4 & \lambda^2 \end{vmatrix} = 0$$

or $\lambda^2 + 2\lambda - 8 = 0$

or $\lambda = 2, -4$

$\therefore$ $|\lambda| = 2, 4$

(B) Given family is

$$3x(a + 1) + 4y(a - 1) - 3(a - 1) = 0$$

or $a(3x + 4y - 3) + (3x - 4y + 3) = 0$

for fixed point=

$$3x + 4y - 3 = 0$$

and $3x - 4y + 3 = 0$

$\therefore$ $x = 0, y = \frac{3}{4}$

Fixed point is $\left(0, \frac{3}{4}\right)$,

Here $p = 0, q = \frac{3}{4}$

$\therefore \quad 4|\lambda| = 4|p - q| = 3$

(C) The point of intersection of $x - y + 1 = 0$ and $3x + y - 5 = 0$ is $(1, 2)$. It lies on the line

$$x + y - 1 - \left|\frac{\lambda}{2}\right| = 0$$

$\Rightarrow \quad 1 + 2 - 1 - \left|\frac{\lambda}{2}\right| = 0$

or $\quad |\lambda| = 4 \text{ or } \lambda = -4, 4$

$\therefore \quad \lambda + 1 = -3, 5 \text{ or } |\lambda + 1| = 3, 5$

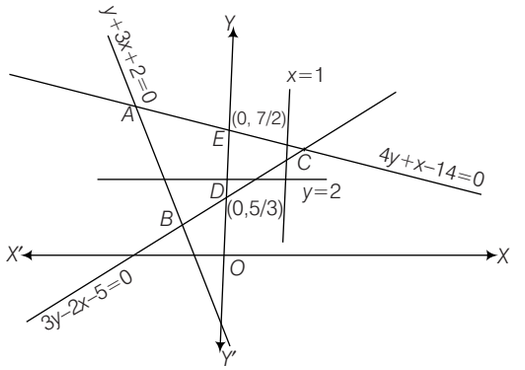
(D) The mid-point of $(1, -2)$ and $(3, 4)$ will satisfy

$$y - x - 1 + \lambda = 0$$

or $\quad 1 - 2 - 1 + \lambda = 0$

$\therefore \quad \lambda = 2 \text{ or } |\lambda| = 2$

73.



(A) The points on the line $x = 0$, whose y-coordinate lies between $\frac{5}{3}$ and $\frac{7}{2}$ inside the triangle ABC.

$\therefore \quad \frac{5}{3} < \lambda < \frac{7}{2} \text{ or } 5 < 3\lambda < 10.5$

$\therefore \quad |3\lambda| = 6, 7, 8, 9, 10$

(B) $\because C \equiv (2, 3)$

The points on the line $x = 1$, whose y-coordinate lies between

$\frac{8}{3}$ (put $x = 1$ in $3y - 2x - 5 = 0$)

and $\quad \frac{13}{4}$ (put $x = 1$ in $4y + x - 14 = 0$)

$\therefore \quad \frac{8}{3} < \lambda < \frac{13}{4} \text{ or } 8 < 3\lambda < 9.75$

$\therefore \quad |3\lambda| = 9$

(C) $\because B \equiv (-1, 1)$

The point on the line $y = 2$, whose x-coordinate lies between

$\frac{-4}{3}$ (put $y = 2$ in $y + 3x + 2 = 0$)

and $\quad \frac{1}{2}$ (put $y = 2$ in $3y - 2x - 5 = 0$)

$\therefore \quad \frac{-4}{3} < \lambda < \frac{1}{2} \text{ or } -8 < 6\lambda < 3$

Integral values of 6λ are

$-7, -6, -5, -4, -3, -2, -1, 0, 1, 2$

$\therefore \quad |6\lambda| = 7, 6, 5, 4, 3, 2, 1, 0$

(D) $\because A \equiv (-2, 4)$

The points on the line $y = \frac{7}{2}$, whose x-coordinates lies between

0 (put $y = \frac{7}{2}$ in $4y + x - 14 = 0$)

and $\quad \frac{-11}{6}$ (put $y = \frac{7}{2}$ in $y + 3x + 2 = 0$)

$\therefore \quad \frac{-11}{6} < \lambda < 0$

or $\quad -11 < 6\lambda < 0$

Integral value of 6λ are

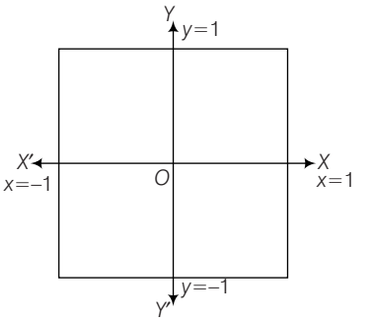
$-10, -9, -8, -7, -6, -3, -2, -1$

$\therefore \quad |6\lambda| = 10, 9, 8, 7, 6, 5, 4, 3, 2, 1$

74. **(A)** $\because \max.\{|x|, |y|\} = 1$

If $\quad |x| = 1$ and if $|y| = 1$

then $x = \pm 1$ and $y = \pm 1$



$\therefore \quad$ Required area $= 2 \times 2 = 4$ sq units

(B) The line $y = x$ cuts the lines $|x + y| = 6$

i.e. $\quad x + y = \pm 6$

at $\quad x = \pm 3, y = \pm 3$

or $\quad (-3, -3)$ and $(3, 3)$

then $\quad -3 < a < 3$

$\therefore \quad 0 \le |a| < 3$

$\therefore \quad [|a|] = 0, 1, 2$

(C) Since $(0, 0)$ and $(1, 1)$ lie on the same side.

So, $\quad a^2 + ab + 1 > 0$

$\because$ Coefficient of a^2 is > 0

$\therefore \quad D < 0$

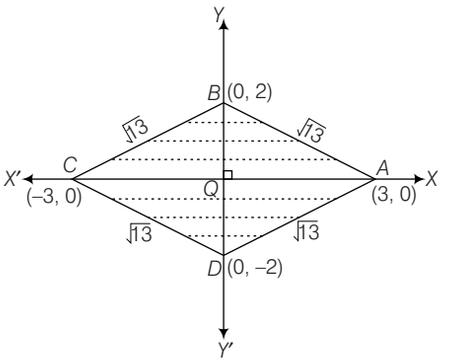
$b^2 - 4 < 0 \text{ or } -2 < b < 2$

$\Rightarrow \quad b = -1, 0, 1$

$\therefore$ Number of values of b is 3.

75. **(A)** $\because\ d(x, y) = 2|x| + 3|y| = 6$ (given)

$\therefore \qquad \dfrac{|x|}{3} + \dfrac{|y|}{2} = 1$



$\therefore$ Perimeter, $\lambda = 4\sqrt{13}$

and area, $\mu = 4 \times \dfrac{1}{2} \times 3 \times 2 = 12$

then $\dfrac{\lambda^2}{16} - \mu = 1$

and $\lambda^2 - \mu^2 = 64$

Hence, locus of (λ, μ) are

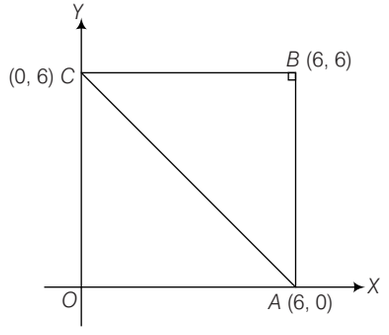
$x^2 - 16y = 16$

and $x^2 - y^2 = 64$

(B) It is clear that orthocentre is (6, 6)

$O' \equiv (6, 6)$,

Circumcentre is $C' \equiv (3, 3)$ and centroid is $G' \equiv (4, 4)$



$\therefore \qquad \lambda = O'C' = \sqrt{(0-3)^2 + (6-3)^2}$

$= \sqrt{9+9} = 3\sqrt{2}$

and $\mu = C'G' = \sqrt{(4-3)^2 + (4-3)^2}$

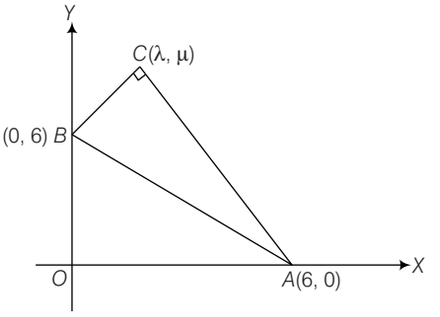
$= \sqrt{1+1} = \sqrt{2}$

$\therefore \quad \lambda^2 - \mu^2 = 16$ and $\lambda = 3\mu$

Hence, locus of (λ, μ) are

$x^2 - y^2 = 16$ and $x = 3y$

(C) $\therefore$ Slope of $AC \times$ slope of $BC = -1$



$\Rightarrow \qquad \left(\dfrac{\mu - 0}{\lambda - 6}\right) \times \left(\dfrac{\mu - 6}{\lambda - 0}\right) = -1$

$\Rightarrow \qquad \mu^2 - 6\mu = -\lambda^2 + 6\lambda$

or $\qquad \lambda^2 + \mu^2 - 6\lambda - 6\mu = 0$

Hence, locus of (λ, μ) is

$x^2 + y^2 - 6x - 6y = 0$

76. $\because \qquad x(a + 2b) + y(a + 3b) = a + b$

$\Rightarrow \qquad a(x + y - 1) + b(2x + 3y - 1) = 0$

then $\qquad x + y - 1 = 0$ and $2x + 3y - 1 = 0$

$\therefore$ point of intersection is $(2, -1)$

Hence, both statement are true and statement II is correct explanation for statement I.

77. $\because$ Algebraic perpendicular from (3, 2) to the line

$3x - 2y + 1 = 0$ is $\dfrac{9 - 4 + 1}{\sqrt{(9+4)}}$ i.e. $\dfrac{6}{\sqrt{13}} = p_1$ (say)

and algebraic perpendicular distance from (1, 4) to the line $3x - 2y + 1 = 0$ is

$\dfrac{3 - 8 + 1}{\sqrt{9+4}}$ i.e. $\dfrac{-4}{\sqrt{13}} = p_2$ (say)

$\therefore \qquad p_1 p_2 = \dfrac{6}{\sqrt{13}} \times \dfrac{-4}{\sqrt{13}} = \dfrac{-24}{13} < 0$

Hence, both statements are true and statement II is a correct explanation for statement I.

78. Sum of algebraic distances from points $A(1, 2)$, $B(2, 3)$, $C(6, 1)$ to the line $ax + by + c = 0$ is zero (given), then

$$\frac{a + 2b + c}{\sqrt{(a^2 + b^2)}} + \frac{(2a + 3b + c)}{\sqrt{(a^2 + b^2)}} + \frac{(6a + b + c)}{\sqrt{(a^2 + b^2)}} = 0$$

$\Rightarrow \qquad 9a + 6b + 3c = 0$

or $\qquad 3a + 2b + c = 0$

$\therefore$ Statement I is false.

Also, centroid of ΔABC is $\left(\dfrac{1 + 2 + 6}{3}, \dfrac{2 + 3 + 1}{3}\right)$

i.e. (3, 2)

$\therefore$ Statement II is true.

79. Equation of AB is

$$y-1=\frac{0-1}{2-0}(x-0) \Rightarrow x+2y-2=0$$

$\because \quad |PA-PB| \le |AB|$

$\Rightarrow |PA-PB|$ to be maximum, then A, B and P must be collinear.

Solving $\quad x+2y-2=0$

and $\quad 4x+3y+9=0,$

we get, $\quad P \equiv \left(\frac{24}{5}, \frac{17}{5}\right)$

Hence, Statement I is false and Statement II is obviously true.

80. Statement II is false as the point satisfying such a property can be the excentre of the triangle.

Let $\quad L_1 \equiv x\cos\left(\frac{\pi}{9}\right)+y\sin\left(\frac{\pi}{9}\right)-\pi=0,$

$$L_2 \equiv x\cos\left(\frac{8\pi}{9}\right)+y\sin\left(\frac{8\pi}{9}\right)-\pi=0 \text{ and}$$

$$L_3 \equiv x\cos\left(\frac{13\pi}{9}\right)+y\sin\left(\frac{8\pi}{9}\right)-\pi=0$$

and $\quad P \equiv (0, 0)$

Length $\perp$ from P to L_1 = Length of $\perp$ from P to L_2 = Length of $\perp$ from P to $L_3 = \pi$ and P lies inside the triangle.

$\therefore P(0, 0)$ is incentre of triangle.

Hence, statement I is true and statement II is false.

81. $\because$ Mid-point of $(5, 1)$ and $(-1, -5)$ i.e. $(2, -2)$ lies on $x+y=0$ and (slope of $x+y=0$) $\times$ (slope of line joining $(5, 1)$ and $(-1, -5)$) $= (-1)\times\frac{-6}{-6}$

$\therefore$ Statement I is true.

Statement II is also true.

Hence, both statements are true but statement II is not correct explanation of statement I.

82. Equation of AC and BC are $3x+2y=0$ and $2x+3y+6=0$

$\because (3)(2)+(2)(3)=12>0$

$\therefore$ Internal angle bisector of C is

$$\left(\frac{3x+2y}{\sqrt{13}}\right)=-\left(\frac{2x+3y+6}{\sqrt{13}}\right)$$

or $\quad 5x+5y+6=0$

$\Rightarrow$ Statement I is true.

Also, the image of A about the angle bisectors of angle B and C lie on the side BC. (by congruence).

$\therefore$ Statement II is true.

Both statements are true and statement II is not correct explanation of statement I.

83. $\because$ Points (x_1, y_1) and (x_2, y_2) lie on the same or opposite sides of the line

$$ax+by+c=0, \text{ as}$$

$$\frac{ax_1+by_1+c}{ax_2+by_2+c}>0 \text{ or } <0$$

$\therefore$ Statement II is true.

Also, $(2a-5, a^2)$ and $(0, 0)$ on the same side of $x+y-3=0$, then

$$\frac{2a-5+a^2-3}{0+0-3}>0$$

$\Rightarrow \quad a^2+2a-8<0$

or $\quad (a+4)(a-2)<0$

$\therefore \quad a \in (-4, 2)$

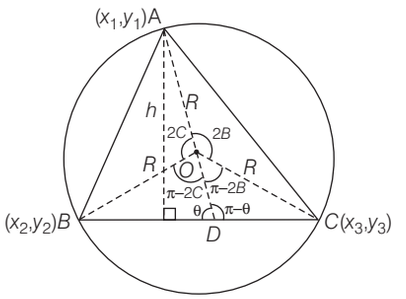
$\Rightarrow$ Statement I is false

Hence, statement I is false and statement II is true.

84. In ΔOBD, $\quad \frac{BD}{\sin(\pi-2C)}=\frac{R}{\sin\theta}$ ...(i)

In ΔODC, $\quad \frac{DC}{\sin(\pi-2B)}=\frac{R}{\sin(\pi-\theta)}$ ...(ii)

From Eqs. (i) and (ii), $\quad \frac{BD}{DC}=\frac{\sin 2C}{\sin 2B}$



$\therefore$ Coordinates of D are

$$\left(\frac{x_2\sin 2B+x_3\sin 2C}{\sin 2B+\sin 2C}, \frac{y_2\sin 2B+y_3\sin 2C}{\sin 2B+\sin 2C}\right)$$

Let (x, y) be any point on AD, then equation of AD is

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ \frac{x_2\sin 2B+x_3\sin 2C}{\sin 2B+\sin 2C} & \frac{y_2\sin 2B+y_3\sin 2C}{\sin 2B+\sin 2C} & 1 \end{vmatrix}=0$$

or $$\begin{vmatrix} x & y & \\ x_1 & y_1 & \\ x_2\sin 2B+x_3\sin 2C & y_2\sin 2B+y_3\sin 2C & \end{vmatrix}$$

$$\begin{vmatrix} & & 1 \\ & & 1 \\ & & \sin 2B+\sin 2C \end{vmatrix}=0$$

or $$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2\sin 2B & y_2\sin 2B & \sin 2B \end{vmatrix}$$

$$+\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3\sin 2C & y_3\sin 2C & \sin 2C \end{vmatrix}=0$$

or $$(\sin 2B)\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}+(\sin 2C)\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=0$$

85. Let (x_1, y_1) be the coordinates of a point at unit distance from each of the given lines.

$\Rightarrow \quad \dfrac{|3x_1 - 4y_1 + 1|}{\sqrt{3^2 + 4^2}} = 1$ and $\dfrac{|8x_1 + 6y_1 + 1|}{\sqrt{8^2 + 6^2}} = 1$

$\Rightarrow \quad 3x_1 - 4y_1 + 1 = \pm 5$ and $8x_1 + 6y_1 + 1 = \pm 10$

$\Rightarrow \quad 3x_1 - 4y_1 - 4 = 0$ ...(i)

or $\quad 3x_1 - 4y_1 + 6 = 0$ ...(ii)

$\quad 8x_1 + 6y_1 - 9 = 0$ ...(iii)

or $\quad 8x_1 + 6y_1 + 11 = 0$ ...(iv)

$(1) \cap (3)$

$\Rightarrow \quad x_1 / 60 = y_1 / -5 = 1 / 50,$

$\therefore \quad (x_1, y_1) = \left(\dfrac{6}{5}, -\dfrac{1}{10}\right)$

$(1) \cap (4)$

$\Rightarrow \quad x_1 / -20 = y_1 / -65 = 1 / 50,$

$\therefore \quad (x_1, y_1) = \left(-\dfrac{2}{5}, -\dfrac{13}{10}\right)$

$(2) \cap (3)$

$\Rightarrow \quad x_1 / 0 = y_1 / 75 = 1 / 50, \therefore (x_1, y_1) = (0, 3/2)$

$(2) \cap (4)$

$\Rightarrow \quad x_1 / -80 = y_1 / 15 = 1 / 50, \therefore (x_1, y_1) = \left(-\dfrac{8}{5}, \dfrac{3}{10}\right)$

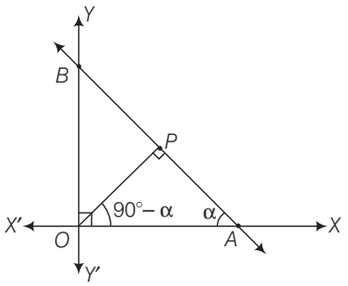
Hence, the required four points have the coordinates

$$\left(\frac{6}{5}, -\frac{1}{10}\right), \left(-\frac{2}{5}, -\frac{13}{10}\right), \left(0, \frac{3}{2}\right), \left(-\frac{8}{5}, \frac{3}{10}\right).$$

86. Let $\angle OAB = \alpha$

$\therefore \quad OA = AB \cos\alpha$ and $OB = AB \sin\alpha$

$\because \quad (OA)^2 + (OB)^2 = k^2$



i.e. $\quad (AB)^2 (\cos^2\alpha + \sin^2\alpha) = k^2$

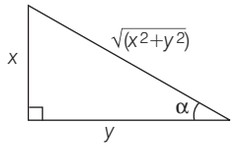
or $\quad AB = k$

then $\quad OA = k \cos\alpha$ and $OB = k \sin\alpha$

$\therefore$ Equation of AB is $\dfrac{x}{k\cos\alpha} + \dfrac{y}{k\sin\alpha} = 1$

or $\quad \dfrac{x}{\cos\alpha} + \dfrac{y}{\sin\alpha} = k$ ...(i)

Let P be the foot of perpendicular from O on AB.



$\therefore$ Equation of OP is $y = x \tan(90° - \alpha)$

or $\quad \cot\alpha = \dfrac{y}{x}$

$\therefore \quad \sin\alpha = \dfrac{x}{\sqrt{(x^2 + y^2)}}$

and $\quad \cos\alpha = \dfrac{y}{\sqrt{(x^2 + y^2)}}$ ...(ii)

Substituting the values of $\sin\alpha$ and $\cos\alpha$ from Eq. (i) in (i) then we get the required locus of P

$\therefore \quad \dfrac{x}{y / \sqrt{(x^2 + y^2)}} + \dfrac{y}{x / \sqrt{(x^2 + y^2)}} = k$

$\Rightarrow \quad (x^2 + y^2)\sqrt{(x^2 + y^2)} = kxy$

Squaring both sides, we get

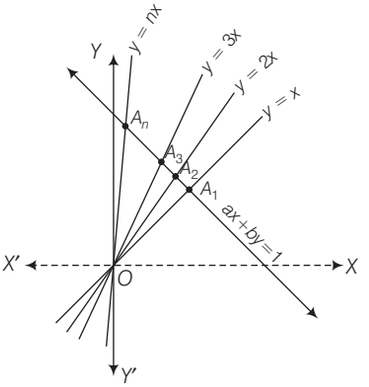
$$(x^2 + y^2)^2 (x^2 + y^2) = k^2 x^2 y^2$$

or $\quad (x^2 + y^2)^2 \left(\dfrac{x^2}{x^2y^2} + \dfrac{y^2}{x^2y^2}\right) = k^2$

or $\quad (x^2 + y^2)^2 (x^{-2} + y^{-2}) = k^2$

87. Let the equation of variable line be $ax + by = 1$. Then the coordinates of A_p will be

$$A_p \equiv \left(\frac{1}{a + bp}, \frac{p}{a + bp}\right)$$



$\therefore \quad OA_p = \dfrac{\sqrt{(1 + p^2)}}{|a + bp|}$ ...(i)

Given, $\quad \sum_{p=1}^{n} \dfrac{1}{OA_p} = c$

$\Rightarrow \quad \sum_{p=1}^{n} \dfrac{|a + bp|}{\sqrt{(1 + p^2)}} = c$ [from Eq. (1)]

$\Rightarrow \quad a\left(\pm \sum_{p=1}^{n} \dfrac{1}{\sqrt{(1 + p^2)}}\right) + b\left(\pm \sum_{p=1}^{n} \dfrac{p}{\sqrt{(1 + p^2)}}\right) = c$

or $\quad a\left(\pm \dfrac{\sum_{p=1}^{n} \dfrac{1}{\sqrt{(1 + p^2)}}}{c}\right) + b\left(\pm \dfrac{\sum_{p=1}^{n} \dfrac{p}{\sqrt{(1 + p^2)}}}{c}\right) = 1$

So, line always passes through a fixed point whose coordinates are

$$\left(\frac{\pm \sum_{p=1}^{n} \frac{1}{\sqrt{(1+p^2)}}}{c}, \frac{\pm \sum_{p=1}^{n} \frac{p}{\sqrt{(1+p^2)}}}{c} \right)$$

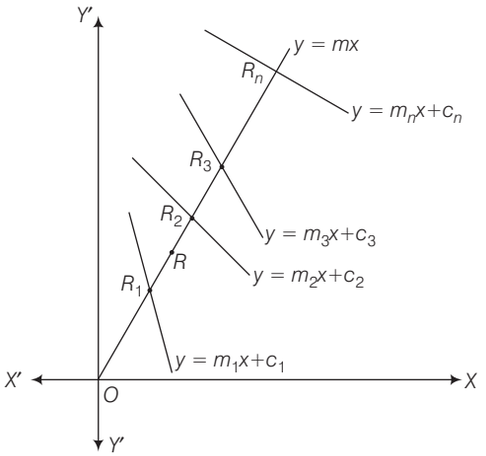
88. Let the equation of given n lines be

$$y = m_r x + c_r,$$

where $r = 1, 2, 3, \ldots, n$ ...(i)

Let equation of line through origin O is

$$y = mx \quad \text{...(ii)}$$



Solving Eqs. (i) and (ii) , we get

$$R_r \equiv \left(\frac{c_r}{m - m_r}, \frac{mc_r}{m - m_r} \right)$$

$$\therefore \quad OR_r = \sqrt{\left(\frac{c_r}{m - m_r} \right)^2 + \left(\frac{m\, c_r}{m - m_r} \right)^2}$$

$$= \left| \frac{c_r}{m - m_r} \right| \sqrt{(1 + m^2)} \quad \text{...(iii)}$$

Let $R \equiv (x_1, y_1)$

$$\therefore \quad y_1 = mx_1 \Rightarrow m = \frac{y_1}{x_1} \quad \text{...(iv)}$$

Given, $\frac{n}{OR} = \sum_{r=1}^{n} \frac{1}{OR_r}$

$$\Rightarrow \quad \frac{n}{\sqrt{(x_1^2 + y_1^2)}} = \sum_{r=1}^{n} \left| \frac{m - m_r}{c_r} \right| \frac{1}{\sqrt{(1 + m^2)}} \quad \text{[from Eq . (iii)]}$$

$$= \frac{1}{\sqrt{(1 + m^2)}} \left\{ m \left(\sum_{r=1}^{n} \left(\pm \frac{1}{c_r} \right) \right) + \sum_{r=1}^{n} \left(\mp \frac{m_r}{c_r} \right) \right\}$$

$$= \frac{1}{\sqrt{(1 + m^2)}} (ma + b)$$

$$\left\{ \text{where } a = \sum_{r=1}^{n} \left(\pm \frac{1}{c_r} \right) \text{ and } b = \sum_{r=1}^{n} \left(\mp \frac{m_r}{c_r} \right) \right.$$

then $\frac{n}{\sqrt{(x_1^2 + y_1^2)}} = \frac{\frac{y_1}{x_1} a + b}{\sqrt{1 + \left(\frac{y_1}{x_1} \right)^2}}$ [from Eq. (iv)]

$$\Rightarrow \quad n = ay_1 + bx_1$$

Hence, locus of point R is $bx + ay = n$.

89. First equation can be expressed as

$$(2x + 3y - 5) \cos\theta + (3x - 5y + 2) \sin\theta = 0$$

$$\Rightarrow \quad (2x + 3y - 5) + (3x - 5y + 2) \tan\theta = 0$$

It is clear that these lines will pass through the point of intersection of the lines

$$\left.\begin{aligned} 2x + 3y - 5 = 0 \\ 3x - 5y + 2 = 0 \end{aligned}\right\} \quad \text{...(i)}$$

for all values of θ.

Solving the system of Eq. (i), we get (1, 1).

Hence, the fixed point is P (1, 1). Let Q (α, β) be the reflection of P (1, 1) in the line $x + y = \sqrt{2}$.

Then $\frac{\alpha - 1}{1} = \frac{\beta - 1}{1} = \frac{-2(1 + 1 - \sqrt{2})}{1^2 + 1^2} = \sqrt{2} - 2$

$$\therefore \quad \alpha = \sqrt{2} - 1, \beta = \sqrt{2} - 1$$

i.e. $Q \equiv (\sqrt{2} - 1, \sqrt{2} - 1)$

If the required family of lines is

$$(2 \cos\theta + 3 \sin\theta)\, x + (3 \cos\theta - 5 \sin\theta)\, y = \lambda$$

in order that each member of the family pass through Q, we have

$$\lambda = (\sqrt{2} - 1)(2 \cos\theta + 3 \sin\theta + 3 \cos\theta - 5 \sin\theta)$$

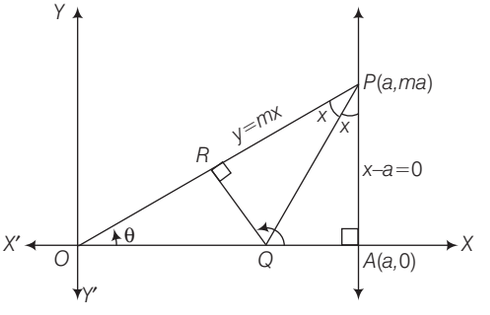
$$\lambda = (\sqrt{2} - 1)(5 \cos\theta - 2 \sin\theta)$$

Hence, equation of required family is

$$(2 \cos\theta + 3 \sin\theta)\, x + (3 \cos\theta - 5 \sin\theta)\, y = (\sqrt{2} - 1)(5 \cos\theta - 2 \sin\theta).$$

90. Let R (h, k) be the foot of perpendicular from Q on OP.

Let equation of OP be



$$y = mx$$

then $k = mh$

or $m = \frac{k}{h}$ ...(i)

and coordinates of $P \equiv (a, ma)$

$\because$ PQ is the bisector of OPA

$\therefore$ $\angle APQ = \angle RPQ$

and $\angle PAQ = \angle QRP = 90°$

$\therefore$ $PA = PR$

then $|ma| = \sqrt{(h-a)^2 + (k-ma)^2}$

From Eq. (i), $\left|\frac{ak}{h}\right| = \sqrt{(h-a)^2 + \left(k - \frac{ak}{h}\right)^2}$

$\Rightarrow$ $a|k| = |(h-a)|\sqrt{(h^2+k^2)}$

Hence, required locus is

$$(x-a)^2(x^2+y^2) = a^2y^2$$

91. Let the coordinates of the vertex be (h, k) and equations of the bases be

$x\cos\alpha_r + y\sin\alpha_r - p_r = 0$ where $r = 1, 2, 3, ..., n$

and their lengths be respectively $l_1, l_2, l_3, ..., l_n$.

$\because$ Length of perpendicular from (h, k) on $x\cos\alpha_r + y\sin\alpha_r - p_r = 0$ is

$$\frac{|h\cos\alpha_r + k\sin\alpha_r - p_r|}{\sqrt{(\cos^2\alpha + \sin^2\alpha)}},$$

i.e. $|h\cos\alpha_r + k\sin\alpha_r - p_r|$

Given, sum of areas of all triangles = constant

then

$$\sum_{r=1}^{n} \frac{1}{2} l_r \cdot |h\cos\alpha_r + k\sin\alpha_r - p_r| = C'$$

$\Rightarrow$ $\sum_{r=1}^{n} \frac{1}{2} . l_r . (\pm (h\cos\alpha_r + k\sin\alpha_r - p_r)) = C'$

$\Rightarrow$ $h\left(\sum_{r=1}^{n} \pm \frac{1}{2} l_r \cos\alpha_r\right) + k\left(\sum_{r=1}^{n} \pm \frac{1}{2} l_r . \sin\alpha_r\right)$

$$= \sum_{r=1}^{n} \pm \frac{1}{2} l_r . p_r + C'$$

$\Rightarrow$ $Ah + Bk = -C$

$\therefore$ Required locus is

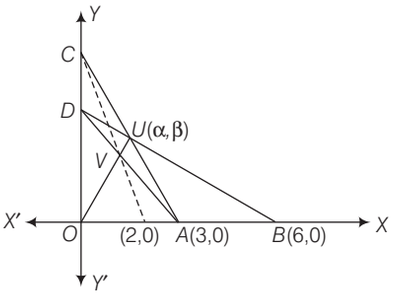
$$Ax + By + C = 0$$

where A, B, C are constants.

92. The equation of BU is

$$y - \beta = \frac{0-\beta}{6-\alpha}(x-\alpha)$$

So that the coordinates of D are $\left(0, \frac{6\beta}{6-\alpha}\right)$



Similarly, the coordinates of C are $\left(0, \frac{3\beta}{3-\alpha}\right)$

Now, the equation of AD is

$$\frac{x}{3} + \frac{(6-\alpha)}{6\beta} y = 1 \quad ...(i)$$

and the equation of OU is

$$\beta x = \alpha y \quad ...(ii)$$

Solving Eqs. (i) and (ii), we get

$$x = \frac{6\alpha}{6+\alpha}, \; y = \frac{6\beta}{6+\alpha}$$

Hence, coordinates of V are $\left(\frac{6\alpha}{6+\alpha}, \frac{6\beta}{6+\alpha}\right)$

Then, the equation of CV is

$$y - \frac{3\beta}{3-\alpha} = \frac{\frac{6\beta}{6+\alpha} - \frac{3\beta}{3-\alpha}}{\frac{6\alpha}{6+\alpha} - 0}(x-0)$$

$\Rightarrow$ $y - \frac{3\beta}{3-\alpha} = \frac{-9\alpha\beta}{6\alpha(3-\alpha)} x$

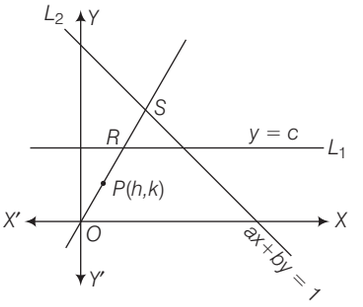
$\Rightarrow$ $y = \frac{3\beta}{(3-\alpha)}\left(1 - \frac{x}{2}\right)$

which pass through the point $(2, 0)$ for all values of (α, β).

93. Let the equation of the variable line through `O' be

$$\frac{x}{\cos\theta} = \frac{y}{\sin\theta}$$

and let $OR = r_1$, $OS = r_2$ and $OP = r_3$



Then coordinates of R, S and P are :

$R(r_1\cos\theta, r_1\sin\theta)$, $S(r_2\cos\theta, r_2\sin\theta)$, $P(r_3\cos\theta, r_3\sin\theta)$

R lies on L_1 and S lies on L_2.

Let $L_1 \equiv y - c = 0$

and $L_2 \equiv ax + by - 1 = 0$

$\therefore$ $r_1\sin\theta = c$ and $ar_2\cos\theta + br_2\sin\theta = 1$

$\therefore$ $r_1 = \frac{c}{\sin\theta}$ and $r_2 = \frac{1}{a\cos\theta + b\sin\theta}$

From the given condition

$$\frac{m+n}{r_3} = \frac{m}{r_1} + \frac{n}{r_2}$$

$\Rightarrow$ $\frac{m+n}{r_3} = \frac{m\sin\theta}{c} + n(a\cos\theta + b\sin\theta)$ ...(i)

Let the coordinates of P be (h, k), then

$$h = r_3 \cos\theta,\ k = r_3 \sin\theta$$

From Eq. (i), $\quad m + n = \dfrac{mr_3 \sin\theta}{c} + n(ar_3 \cos\theta + br_3 \sin\theta)$

$$\Rightarrow \quad m + n = \frac{mk}{c} + n(ah + bk)$$

Locus of P is $\ n(ax + by) + \dfrac{my}{c} = (m + n)$

$$\Rightarrow \quad n(ax + by - 1) + \frac{m}{c}(y - c) = 0$$

$$\Rightarrow \quad (ax + by - 1) + \frac{m}{nc}(y - c) = 0$$

$$\Rightarrow \quad L_2 + \lambda L_1 = 0 \quad \left(\text{where, } \lambda = \frac{m}{nc}\right)$$

Hence, locus of P is a point of intersection of L_1 and L_2.

94. The given lines are

$$x + 3y + 2 = 0 \quad \text{...(i)}$$

$$2x + y + 4 = 0 \quad \text{...(ii)}$$

$$x - y - 5 = 0 \quad \text{...(iii)}$$

Equation of the line passing through $A(-5, -4)$ and making an angle θ with the positive direction of X-axis is

$$\frac{x + 5}{\cos\theta} = \frac{y + 4}{\sin\theta} = r \ (AB,\ AC,\ AD) \quad \text{...(iv)}$$

$\therefore$ Points $(-5 + AB\cos\theta, -4 + AB\sin\theta)$, $(-5 + AC\cos\theta, -4 + AC\sin\theta)$ and $(-5 + AD\cos\theta, -4 + AD\sin\theta)$ lie on Eqs. (i), (ii) and (iii) respectively.

$$(-5 + AB\cos\theta) + 3(-4 + AB\sin\theta) + 2 = 0$$

$$\Rightarrow \quad AB(\cos\theta + 3\sin\theta) = 15$$

$$\Rightarrow \quad \frac{15}{AB} = \cos\theta + 3\sin\theta$$

Similarly, $\quad \dfrac{10}{AC} = 2\cos\theta + \sin\theta$

and $\quad \dfrac{6}{AD} = \cos\theta - \sin\theta$

From given condition

$$\left(\frac{15}{AB}\right)^2 + \left(\frac{10}{AC}\right)^2 = \left(\frac{6}{AD}\right)^2$$

we get $(\cos\theta + 3\sin\theta)^2 + (2\cos\theta + \sin\theta)^2 = (\cos\theta - \sin\theta)^2$

$$\Rightarrow \quad 4\cos^2\theta + 9\sin^2\theta + 12\sin\theta\cos\theta = 0$$

$$\Rightarrow \quad (2\cos\theta + 3\sin\theta)^2 = 0$$

$$\therefore \quad \tan\theta = -\frac{2}{3}$$

Hence the equation of the line from Eq. (iv) is

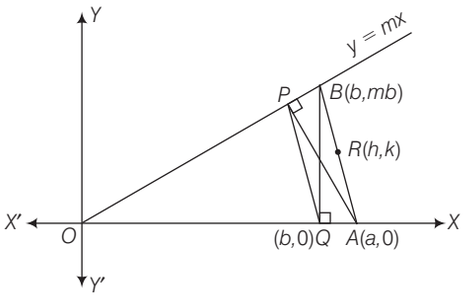
$$y + 4 = -\frac{2}{3}(x + 5) \Rightarrow 2x + 3y + 22 = 0$$

95. $\because$ A, R and B are collinear

then, $\quad \dfrac{k - 0}{h - a} = \dfrac{mb - 0}{b - a}$

$$\therefore \quad \frac{a}{b}k - am + mh - k = 0 \quad \text{...(i)}$$

Let $P \equiv (\alpha, \beta)$



$\because$ P be the foot of perpendicular from A on $y = mx$, then

$$\frac{\alpha - a}{-m} = \frac{\beta - 0}{1} = \frac{-(0 - ma)}{(1 + m^2)}$$

$$\therefore \quad \alpha = \frac{a}{1 + m^2},\ \beta = \frac{am}{1 + m^2}$$

i.e. $\quad P \equiv \left(\dfrac{a}{1 + m^2}, \dfrac{am}{1 + m^2}\right)$

$\therefore$ Equation of PQ is

$$y - 0 = \frac{\frac{am}{1 + m^2} - 0}{\frac{a}{1 + m^2} - b}(x - b)$$

$$\Rightarrow \quad \frac{a}{b}(y - mx) + am - (1 + m^2)y = 0 \quad \text{...(ii)}$$

Adding Eqs. (i) and (ii), then

$$\frac{a}{b}(y - mx + k) + (mh - k - (1 + m^2)y) = 0$$

$$\Rightarrow \quad (mh - k - (1 + m^2)y) + \lambda(y - mx + k) = 0$$

$$\left(\text{where, } \lambda = \frac{a}{b}\right)$$

Hence PQ pass through a fixed point.

For fixed point

$$mh - k - (1 + m^2)y = 0,\ y - mx + k = 0$$

$$y = \frac{mh - k}{(1 + m^2)},\ x = \frac{h + mk}{(1 + m^2)}$$

Hence, fixed point is $\left(\dfrac{h + mk}{1 + m^2}, \dfrac{mh - k}{1 + m^2}\right)$.

96. Given lines are parallel and distance between them < 2

Given lines are

$$2x + y = 3 \quad \text{...(i)}$$

and $\quad 2x + y = 5 \quad \text{...(ii)}$

Equation of any line through Eqs. (ii) and (iii) is

$$y - 3 = m(x - 2)$$

or $\quad y = mx - 2m + 3 \quad \text{...(iii)}$

Let line (iii) cut lines (i) and (ii) at A and B respectively.

Solving Eqs. (i) and (iii), we get

$$A \equiv \left(\frac{2m}{m + 2}, \frac{6 - m}{m + 2}\right)$$

and solving Eqs. (ii) and (iii), we get

$$B \equiv \left(\frac{2m+2}{m+2}, \frac{m+6}{m+2}\right)$$

According to question $AB = 2$

$$\Rightarrow \quad (AB)^2 = 4$$

$$\Rightarrow \quad \left(\frac{2}{m+2}\right)^2 + \left(\frac{2m}{m+2}\right)^2 = 4$$

$$\Rightarrow \quad 1 + m^2 = m^2 + 4m + 4 \qquad \text{...(iv)}$$

Case I : When m is finite (line is not perpendicular to X-axis) then from Eq. (iv).

$$1 = 4m + 4$$

$$\therefore \quad m = -\frac{3}{4}$$

Case II : When m is infinite (line is perpendicular to X-axis) then from Eq. (iv),

$$\frac{1}{m^2} + 1 = 1 + \frac{4}{m} + \frac{4}{m^2}$$

$$0 + 1 = 1 + 0 + 0$$

$$1 = 1 \text{ which is true}$$

Hence $m \to \infty$ acceptable.

Hence, equation of the required lines are

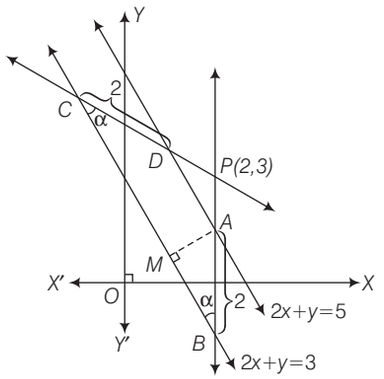
$$y - 3 = -\frac{3}{4}(x - 2)$$

and $$\frac{y-3}{\infty} = x - 2 \quad \Rightarrow x - 2 = 0$$

i.e. $3x + 4y = 18$ and $x - 2 = 0$

Aliter I :

$\because$ $2x + y = 3$ cuts Y-axis at $(0, 3)$ and line $2x + y = 5$ cuts Y-axis at $(0, 5)$



Therefore intercept on Y-axis is 2.

Also, $AM =$ distance between parallel lines

$$= \frac{|-5+3|}{\sqrt{2^2+1^2}} = \frac{2}{\sqrt{5}}$$

$$\therefore \quad MB = \sqrt{(AB)^2 - (AM)^2} = \sqrt{4 - \frac{4}{5}} = \frac{4}{\sqrt{5}}$$

then $\tan\alpha = \frac{AM}{MB} = \frac{1}{2}$

Also $\tan\theta = -2$ (slope of $2x + y = 5$)

Now, equation of required lines are

$$y - 3 = \tan(\theta \pm \alpha)(x - 2)$$

$$\Rightarrow \quad y - 3 = \left(\frac{\tan\theta \pm \tan\alpha}{1 \mp \tan\theta \tan\alpha}\right)(x - 2)$$

$$\Rightarrow \quad y - 3 = \frac{(-2) \pm \frac{1}{2}}{1 \mp (-2)\left(\frac{1}{2}\right)}(x-2)$$

$$\Rightarrow \quad y - 3 = \frac{\left(-2 \pm \frac{1}{2}\right)}{1 \mp (-1)}(x - 2)$$

$$\Rightarrow (1 \mp (-1))(y - 3) = \left(-2 \pm \frac{1}{2}\right)(x - 2)$$

$$\Rightarrow \quad x - 2 = 0 \quad \text{and} \quad 2y - 6 = -\frac{3}{2}(x - 2)$$

i.e. $x - 2 = 0$ and $3x + 4y - 18 = 0$

Aliter II : Any line through $(2, 3)$ is

$$\frac{x-2}{\cos\theta} = \frac{y-3}{\sin\theta} = r$$

Suppose this line cuts $2x + y = 5$ and $2x + y = 3$ at D and C respectively but given $DC = 2$

then $D \equiv (2 + r\cos\theta, 3 + r\sin\theta)$

and $C \equiv (2 + (r+2)\cos\theta, 3 + (r+2)\sin\theta)$

$\because$ D and C lies on

$$2x + y = 5 \quad \text{and} \quad 2x + y = 3$$

then $2(2 + r\cos\theta) + (3 + r\sin\theta) = 5$ ... (v)

and $2(2 + (r+2)\cos\theta) + (3 + (r+2)\sin\theta) = 3$ ... (vi)

Subtracting Eq. (v) from Eq. (vi), then

$$4\cos\theta + 2\sin\theta = -2$$

or $$2\cos\theta + \sin\theta = -1$$

$$\Rightarrow \quad 2\left(\frac{1 - \tan^2\left(\frac{\theta}{2}\right)}{1 + \tan^2\left(\frac{\theta}{2}\right)}\right) + \left(\frac{2\tan\left(\frac{\theta}{2}\right)}{1 + \tan^2\left(\frac{\theta}{2}\right)}\right) = -1$$

$$\Rightarrow \quad 2 - 2\tan^2\left(\frac{\theta}{2}\right) + 2\tan\left(\frac{\theta}{2}\right) = -1 - \tan^2\left(\frac{\theta}{2}\right)$$

$$\Rightarrow \quad \tan^2\left(\frac{\theta}{2}\right) - 2\tan\left(\frac{\theta}{2}\right) - 3 = 0$$

$$\therefore \quad \tan\left(\frac{\theta}{2}\right) = -1 \text{ or } 3$$

$$\therefore \quad \tan\theta = \infty \quad \text{or} \quad -\frac{3}{4}$$

$\therefore$ Required lines are

$$y - 3 = \infty(x - 2)$$

and $$y - 3 = -\frac{3}{4}(x - 2)$$

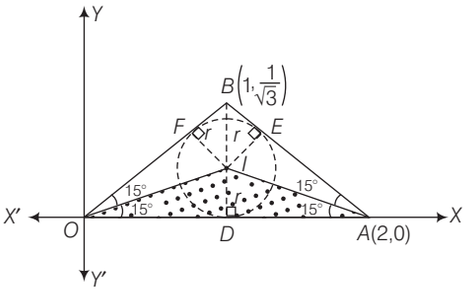
i.e. $x - 2 = 0$

and $3x + 4y - 18 = 0$

97. If I be the incentre of ΔOAB.

If inradius $= r$

then $ID = IE = IF = r$



If P at I,then

$$d(P, OA) = d(P, OB) = d(P, AB) = r$$

But $d(P,OA) \le \min\{d(P, OB), d(P, AB)\}$

which is possible only when P lies in the ΔOIA.

$$\therefore \quad \tan 15^\circ = \frac{ID}{OD} = \frac{r}{1}$$

$$\Rightarrow \quad r = (2 - \sqrt{3})$$

$\therefore$ Required area $= \frac{1}{2} \cdot 2 \cdot r = r = (2 - \sqrt{3})$ sq units.

98. Let $A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$ and $C \equiv (x_3, y_3)$ are the vertices of a triangle ABC and $P \equiv (a_1, b_1)$, $Q \equiv (a_2, b_2)$ and $R \equiv (a_3, b_3)$ are the vertices of triangle PQR.

Equation of perpendicular from A to QR is

$$y - y_1 = -\frac{(a_2 - a_3)}{(b_2 - b_3)}(x - x_1)$$

or $(a_2 - a_3)\, x + (b_2 - b_3)\, y - x_1 (a_2 - a_3) - y_1 (b_2 - b_3) = 0$ ...(i)

Similarly, equations of perpendiculars from B to RP and C to PQ are respectively,

$(a_3 - a_1)\, x + (b_3 - b_1)\, y - x_2 (a_3 - a_1) - y_2 (b_3 - b_1) = 0$ ...(ii)

and $(a_1 - a_2)\, x + (b_1 - b_2)\, y - x_3 (a_1 - a_2) - y_3 (b_1 - b_2) = 0$ ...(iii)

Given that lines (i), (ii) and (iii) are concurrent, then adding, we get

$$(x_2 - x_3)\, a_1 + (x_3 - x_1)\, a_2 + (x_1 - x_2)\, a_3 + (y_2 - y_3)\, b_1 + (y_3 - y_1) b_2 + (y_1 - y_2)\, b_3 = 0 \quad \text{...(iv)}$$

Now, equation of perpendicular from P to BC is

$$y - b_1 = -\frac{(x_2 - x_3)}{(y_2 - y_3)}(x - a_1)$$

or $(x_2 - x_3)\, x + (y_2 - y_3)\, y - a_1 (x_2 - x_3) - b_1 (y_2 - y_3) = 0$ ...(v)

Similarly, equations of perpendiculars from Q to CA and R to AB are respectively,

$$(x_3 - x_1)\, x + (y_3 - y_1)\, y - a_2 (x_3 - x_1) - b_2 (y_3 - y_1) = 0 \quad \text{...(vi)}$$

and $(x_1 - x_2)\, x + (y_1 - y_2)\, y - a_3 (x_1 - x_2) - b_3 (y_1 - y_2) = 0$ ...(vii)

Adding Eqs. (v), (vi) and (vii), we get

LHS = 0 (identically) [from Eq. (iv)]

Hence perpendiculars from P to BC, Q to CA and R to AB are concurrent.

99. The line passing through the intersection of lines $ax + 2by + 3b = 0$ and $bx - 2ay - 3a = 0$ is

$$ax + 2by + 3b + \lambda(bx - 2ay - 3a) = 0$$

$$\Rightarrow (a + b\lambda)x + (2b - 2a\lambda)y + 3b - 3\lambda a = 0$$

As this line is parallel to X-axis.

$$\therefore \quad a + b\lambda = 0 \Rightarrow \lambda = -\frac{a}{b}$$

$$\Rightarrow \quad ax + 2by + 3a - \frac{a}{b}(bx - 2ay - 3a) = 0$$

$$\Rightarrow \quad ax + 2by + 3b - ax + \frac{2a^2}{b}y + \frac{3a^2}{b} = 0$$

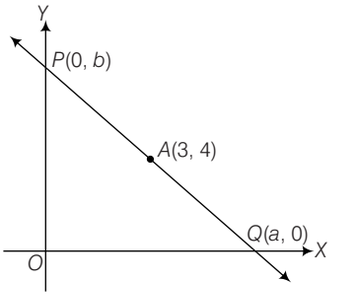
$$y\left(2b + \frac{2a^2}{b}\right) + 3b + \frac{3a^2}{b} = 0$$

$$y\left(\frac{2b^2 + 2a^2}{b}\right) = -\left(\frac{3b^2 + 3a^2}{b}\right)$$

$$y = \frac{-3(a^2 + b^2)}{2(b^2 + a^2)} = \frac{-3}{2}$$

So, it is $\frac{3}{2}$ units below X-axis.

100.



$\because$ A is the mid-point of PQ, therefore

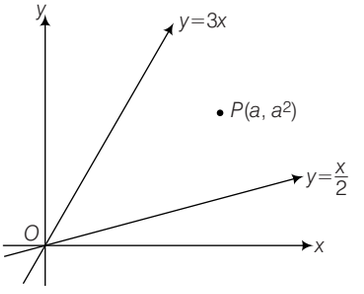
$$\frac{a + 0}{2} = 3, \frac{0 + b}{2} = 4$$

$$\Rightarrow \quad a = 6, b = 8$$

$\therefore$Equation of line is $\frac{x}{6} + \frac{y}{8} = 1$

or $4x + 3y = 24$

101. Clearly for point P,

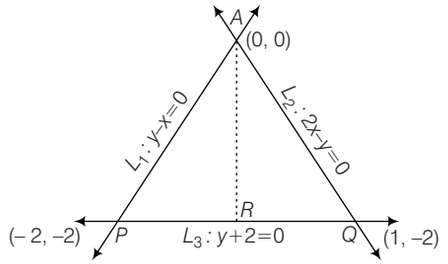


$$a^2 - 3a < 0 \text{ and } a^2 - \frac{a}{2} > 0$$

$$\Rightarrow \quad \frac{1}{2} < a < 3$$

102. Point of intersection of L_1 and L_2 is $A(0, 0)$.

Also $P(-2, -2)$, $Q(1, -2)$



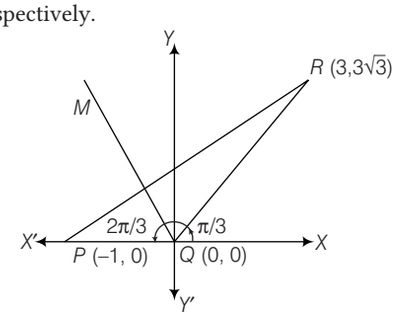
$\because$ AR is the bisector of $\angle PAQ$, therefore R divides PQ in the same ratio as $AP : AQ$.

Thus $PR : RQ = AP : AQ = 2\sqrt{2} : \sqrt{5}$

$\therefore$ Statement I is true.

Statement II is clearly false.

103. Given : The coordinates of points P, Q, R are $(-1, 0)$, $(0, 0)$, $(3, 3\sqrt{3})$ respectively.



Slope of equation $QR = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{3\sqrt{3}}{3}$

$\Rightarrow \quad \tan\theta = \sqrt{3} \Rightarrow \theta = \dfrac{\pi}{3}$

$\Rightarrow \quad \angle RQX = \dfrac{\pi}{3}$

$\therefore \quad \angle RQP = \pi - \dfrac{\pi}{3} = \dfrac{2\pi}{3}$

Let QM bisects the $\angle PQR$,

$\therefore$ Slope of the line $QM = \tan\dfrac{2\pi}{3} = -\sqrt{3}$

$\therefore$ Equation of line QM is $(y - 0) = -\sqrt{3}(x - 0)$

$\Rightarrow \quad y = -\sqrt{3}x \Rightarrow \sqrt{3}x + y = 0$

104. **(A)** $\because$ L_1, L_2, L_3 are concurrent, then

$$\begin{vmatrix} 1 & 3 & -5 \\ 3 & -k & -1 \\ 5 & 2 & -12 \end{vmatrix} = 0 \Rightarrow k = 5$$

(B) slope of (L_1) = slope of (L_2)

$\Rightarrow \quad -\dfrac{1}{3} = \dfrac{3}{k} \quad \therefore \; k = -9$

and slope of (L_3) = slope of (L_2)

$\Rightarrow \quad -\dfrac{5}{2} = \dfrac{3}{k} \quad \therefore \; k = -\dfrac{6}{5}$

(C) Lines are not concurrent or not parallel, then

$$k \neq 5, k \neq -9, k \neq -\frac{6}{5}$$

$\therefore \quad k = \dfrac{5}{6}$

(D) The given lines do not form a triangle if they are concurrent or any two of them are parallel.

$\therefore \quad k = 5, k = -9, k = -\dfrac{6}{5}$

105. Slope of $PQ = \dfrac{3-4}{k-1} = \dfrac{-1}{k-1}$

$\therefore$ Slope of perpendicular bisector of $PQ = (k-1)$

Also mid-point of $PQ \left(\dfrac{k+1}{2}, \dfrac{7}{2}\right)$

$\therefore$ Equation of perpendicular bisector is

$$y - \frac{7}{2} = (k-1)\left(x - \frac{k+1}{2}\right)$$

$\Rightarrow \quad 2y - 7 = 2(k-1)x - (k^2 - 1)$

$\Rightarrow \quad 2(k-1)x - 2y + (8 - k^2) = 0$

$\therefore \quad Y$-intercept $= -\dfrac{8-k^2}{-2} = -4$

$\Rightarrow \quad 8 - k^2 = -8$ or $k^2 = 16 \Rightarrow k = \pm 4$

106. If the line $p(p^2 + 1)x - y + q = 0$

and $\quad (p^2 + 1)^2 x + (p^2 + 1)y + 2q = 0$

are perpendicular to a common line, then these lines must be parallel to each other,

$\therefore \quad m_1 = m_2 \Rightarrow -\dfrac{p(p^2+1)}{-1} = -\dfrac{(p^2+1)^2}{p^2+1}$

$\Rightarrow \quad (p^2 + 1)(p + 1) = 0$

$\Rightarrow \quad p = -1$

$\therefore p$ can have exactly one value.

107. Slope of line $L = -\dfrac{b}{5}$

Slope of line $K = -\dfrac{3}{c}$

Line L is parallel to line K.

$\Rightarrow \quad \dfrac{b}{5} = \dfrac{3}{c} \Rightarrow bc = 15$

$(13, 32)$ is a point on L.

$\therefore \quad \dfrac{13}{5} + \dfrac{32}{b} = 1 \Rightarrow \dfrac{32}{b} = -\dfrac{8}{5}$

$\Rightarrow \quad b = -20 \Rightarrow c = -\dfrac{3}{4}$

Equation of $K : y - 4x = 3$

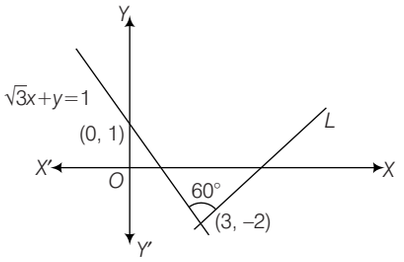
$\Rightarrow \quad 4x - y + 3 = 0$

Distance between L and $K = \dfrac{|52 - 32 + 3|}{\sqrt{17}}$

$= \dfrac{23}{\sqrt{17}}$

108. Let the slope of line L be m.

Then $$\left|\frac{m+\sqrt{3}}{1-\sqrt{3}m}\right| = \sqrt{3}$$



$$\Rightarrow \quad m+\sqrt{3} = \pm(\sqrt{3}-3m)$$

$$\Rightarrow \quad 4m = 0 \text{ or } 2m = 2\sqrt{3}$$

$$\Rightarrow \quad m = 0 \text{ or } m = \sqrt{3}$$

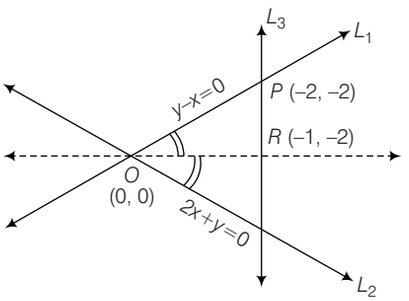
$\because$ L intersects X-axis,

$$\therefore \quad m = \sqrt{3}$$

$\therefore$ Equation of L is $y+2 = \sqrt{3}(x-3)$

or $$\sqrt{3}x - y - (2+3\sqrt{3}) = 0$$

109.



$$L_1 : y - x = 0, \quad L_2 : 2x + y = 0, \quad L_3 : y + 2 = 0$$

On solving the equation of lines L_1 and L_2, we get their point of intersection (0, 0) i.e. origin O.

On solving the equation of lines L_1 and L_3,

we get $P = (-2, -2)$

Similarly, we get $Q = (-1, -2)$

We know that bisector of an angle of a triangle, divide the opposite side the triangle in the ratio of the sides including the angle [Angle Bisector Theorem of a Triangle]

$$\therefore \quad \frac{PR}{RQ} = \frac{OP}{OQ} = \frac{\sqrt{(-2)^2+(-2)^2}}{\sqrt{(-1)^2+(-2)^2}} = \frac{2\sqrt{2}}{\sqrt{5}}$$

110. Let the joining points be $A(1, 1)$ and $B(2, 4)$.

Let point C divides line AB in the ratio 3 : 2. So, by section formula we have

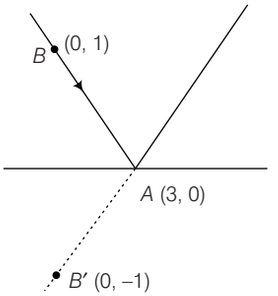
$$C = \left(\frac{3\times 2 + 2\times 1}{3+2}, \frac{3\times 4 + 2\times 1}{3+2}\right) = \left(\frac{8}{5}, \frac{14}{5}\right)$$

Since Line $2x + y = k$ passes through $C\left(\frac{8}{5}, \frac{14}{5}\right)$

$\therefore C$ satisfies the equation $2x + y = k$.

$$\Rightarrow \quad \frac{2+8}{5} + \frac{14}{5} = k \Rightarrow k = 6$$

111. Suppose $B(0, 1)$ be any point on given line and coordinate of A is $(\sqrt{3}, 0)$. So, equation of



Reflected ray is $\frac{-1-0}{0-\sqrt{3}} = \frac{y-0}{x-\sqrt{3}}$

$$\Rightarrow \quad \sqrt{3}y = x - \sqrt{3}$$

112. The intersection point of two lines is $\left(\frac{-c}{a+b}, \frac{-c}{a+b}\right)$

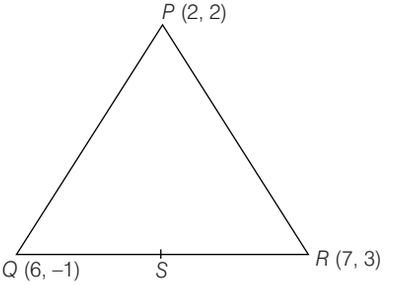
Distance between (1, 1) and $\left(\frac{-c}{a+b}, \frac{-c}{a+b}\right) < 2\sqrt{2}$

$$\Rightarrow \quad 2\left(1+\frac{c}{a+b}\right)^2 < 8$$

$$\Rightarrow \quad 1 + \frac{c}{a+b} < 2$$

$$\Rightarrow \quad a + b - c > 0$$

113. Let P, Q, R, be the vertices of ΔPQR



Since, PS is the median, S is mid-point of QR

So, $$S = \left(\frac{7+6}{2}, \frac{3-1}{2}\right) = \left(\frac{13}{2}, 1\right)$$

Now, slope of $PS = \frac{2-1}{2-\frac{13}{2}} = -\frac{2}{9}$

Since, required line is parallel to PS therefore slope of required line = slope of PS Now, eqn of line passing through $(1, -1)$ and having slope $-\frac{2}{9}$ is

$$y - (-1) = -\frac{2}{9}(x-1)$$

$$9y + 9 = -2x + 2$$

$$\Rightarrow \quad 2x + 9y + 7 = 0$$

114. Given lines are

$$4ax + 2ay + c = 0$$
$$5bx + 2by + d = 0$$

The point of intersection will be

$$\frac{x}{2ad - 2bc} = \frac{-y}{4ad - 5bc} = \frac{1}{8ab - 10ab}$$

$$\Rightarrow \quad x = \frac{2(ad - bc)}{-2ab} = \frac{bc - ad}{ab}$$

$$\Rightarrow \quad y = \frac{5bc - 4ad}{-2ab} = \frac{4ad - 5bc}{2ab}$$

$\because$ Point of intersection is in fourth quadrant so x is positive and y is negative.

Also distance from axes is same

So $x = -y$ ($\because$ distance from X-axis is $-y$ as y is negative)

$$\frac{bc - ad}{ab} = \frac{5bc - 4ad}{2ab} \Rightarrow 3bc - 2ad = 0$$

115. Let the point P be (x, y)

Then $d_1(P) = \left|\frac{x - y}{\sqrt{2}}\right|$ and $d_2(P) = \left|\frac{x + y}{\sqrt{2}}\right|$

For P lying in first quadrant $x > 0, y > 0$.

Also $2 \le d_1(P) + d_2(P) \le 4$

$$\Rightarrow \quad 2 \le \left|\frac{x - y}{\sqrt{2}}\right| + \left|\frac{x + y}{\sqrt{2}}\right| \le 4$$

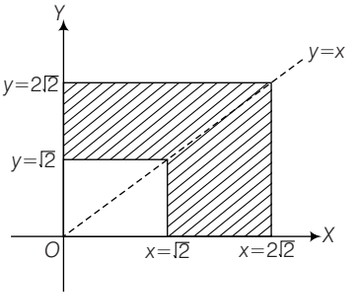
If $x > y$, then

$$2 \le \frac{x - y + x + y}{\sqrt{2}} \le 4 \text{ or } \sqrt{2} \le x \le 2\sqrt{2}$$

If $x < y$, then

$$2 \le \frac{y - x + x + y}{2} \le 4 \text{ or } \sqrt{2} \le y \le 2\sqrt{2}$$

The required region is the shaded region in the figure given below.



$\therefore$ Required area $= (2\sqrt{2})^2 - (\sqrt{2})^2 = 8 - 2 = 6$ sq units

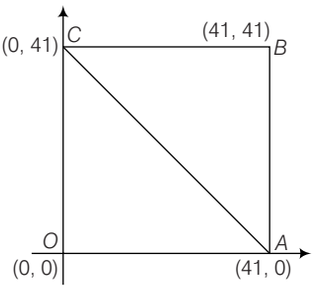
116. Total number of integral points inside the square $OABC$

$= 40 \times 40 = 1600$

Number of integral points on AC

= Number of integral points on OB

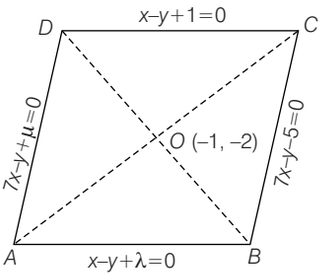
$= 40$ [namely (1, 1), (2, 2) ... (40, 40)]



$\therefore$ Number of integral points inside the ΔOAC

$$= \frac{1600 - 40}{2} = 780$$

117.



Let other two sides of rhombus are

$$x - y + \lambda = 0$$

and $7x - y + \mu = 0$

then O is equidistant from AB and DC and from AD and BC

$\therefore \quad |-1 + 2 + 1| = |-1 + 2 + \lambda| \Rightarrow \lambda = -3$

and $\quad |-7 + 2 - 5| = |-7 + 2 + \mu| \Rightarrow \mu = 15$

$\therefore$Other two sides are

$$x - y - 3 = 0$$

and $7x - y + 15 = 0$

On solving the equation of sides pairwise, we get the vertices as $\left(\frac{1}{3}, \frac{-8}{3}\right)$, $(1, 2)$, $\left(\frac{-7}{3}, \frac{-4}{3}\right)$, $(-3, -6)$

CHAPTER

03

Pair of Straight Lines

Learning Part

Session 1

- Introduction
- Homogeneous Equation in Two Variables

Session 2

- Angle between the Pair of Lines $ax^2 + 2hxy + by^2$

Session 3

- Bisectors of the Angle between the Lines Given by a Homogeneous Equation

Session 4

- General Equation of Second Degree
- Important Theorems

Session 5

- To Find the Point of Intersection of Lines Represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ with the Help of Partial Differentiation
- Removal of First Degree Term
- Equation of the Lines Joining the Origin to the Points of Intersection of a Given Line and a Given Curve

Practice Part

- JEE Type Examples
- Chapter Exercises

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Session 1

Introduction, Homogeneous Equation in Two Variables

Introduction

Let the equation of two lines be

$$ax + by + c = 0 \quad \text{...(i)}$$

and $$a_1x + b_1y + c_1 = 0 \quad \text{...(ii)}$$

Hence, $(ax + by + c)(a_1x + b_1y + c_1) = 0$ is called the joint equation of lines Eqs. (i) and (ii) and conversely, if joint equation of two lines be

$$(ax + by + c)(a_1x + b_1y + c_1) = 0,$$

then their separate equations will be

$$ax + by + c = 0 \quad \text{and} \quad a_1x + b_1y + c_1 = 0$$

Remark

In order to find the joint equation of two lines, make RHS of two lines equal to zero and then multiply the two equations.

Example 1 Find the joint equation of lines $y = x$ and $y = -x$.

Sol. The given lines can be rewritten as

$$x - y = 0 \quad \text{and} \quad x + y = 0$$

$\therefore$ Joint equation of lines is $(x - y)(x + y) = 0$

or $$x^2 - y^2 = 0$$

Wrong process : Since, the lines are

$$y = x \quad \text{and} \quad y = -x$$

Then joint equation is $$y^2 = -x^2$$

$\Rightarrow$ $$x^2 + y^2 = 0$$

This process is wrong, since RHS of two equations are not equal to zero.

Remark

In order to find the separate equations of two lines when their joint equation is given, first of all make RHS equal to zero and then resolve LHS into two linear factors or use Shri Dharacharya method. The two factors equated to zero will give the separate equations of lines.

Example 2 Find the separate equation of lines represented by the equation $x^2 - 6xy + 8y^2 = 0$.

Sol. Separate equation of lines represented by the equation

$$x^2 - 6xy + 8y^2 = 0$$

i.e. $$(x - 4y)(x - 2y) = 0$$

$\Rightarrow$ $x - 4y = 0$ and $x - 2y = 0$

Aliter :

We have, $$x^2 - 6xy + 8y^2 = 0$$

By Shri Dharacharya method, $x = \dfrac{6y \pm \sqrt{(-6y)^2 - 4 \cdot 8y^2}}{2}$

$\Rightarrow$ $$x = 3y \pm y\sqrt{(9 - 8)}$$

$$x = 3y \pm y$$

$\therefore$ $$x = 4y \quad \text{and} \quad x = 2y$$

Hence, the lines are

$$x - 4y = 0 \quad \text{and} \quad x - 2y = 0.$$

Homogeneous Equation in Two Variables

An equation of the form

$$a_0y^n + a_1y^{n-1}x + a_2y^{n-2}x^2 + ... + a_nx^n = 0 \quad \text{...(i)}$$

in which the sum of the powers of x and y in every term is the same (here n), is called a **homogeneous** equation (of degree n).

We will prove that Eq. (i) represents n straight lines passing through the origin.

$$a_0y^n + a_1y^{n-1}x + a_2y^{n-2}x^2 + ... + a_nx^n = 0$$

Dividing each term by x^n, we get

$$a_0\left(\frac{y}{x}\right)^n + a_1\left(\frac{y}{x}\right)^{n-1} + a_2\left(\frac{y}{x}\right)^{n-2} + ... + a_n = 0$$

Above is an equation of nth degree in $\dfrac{y}{x}$. Let the roots of this equation be $m_1, m_2, m_3, ..., m_n$.

Then, the above equation will be identical with

$$a_0\left(\frac{y}{x} - m_1\right)\left(\frac{y}{x} - m_2\right)\left(\frac{y}{x} - m_3\right)...\left(\frac{y}{x} - m_n\right) = 0$$

$\Rightarrow a_0(y - m_1x)(y - m_2x)(y - m_3x)....(y - m_nx) = 0$

Hence, Eq. (i) represents n straight lines

$$y - m_1 x = 0, y - m_2 x = 0,$$
$$y - m_3 x = 0, \ldots, y - m_n x = 0$$

all of which clearly pass through the origin.

Corollary : Since, $ax^2 + 2hxy + by^2 = 0$ is a homogeneous equation of second degree, it represents two straight lines through origin. The given equation is

$$ax^2 + 2hxy + by^2 = 0 \quad \ldots(i)$$

Dividing by x^2, we get

$$a + 2h\left(\frac{y}{x}\right) + b\left(\frac{y}{x}\right)^2 = 0$$

$$\Rightarrow \quad b\left(\frac{y}{x}\right)^2 + 2h\left(\frac{y}{x}\right) + a = 0 \quad \ldots(ii)$$

Putting $\frac{y}{x} = m$

then, $bm^2 + 2hm + a = 0 \quad \ldots(iii)$

If m_1 and m_2 be two roots, then

$$m_1 + m_2 = -\frac{2h}{b} = -\frac{\text{coefficient of } xy}{\text{coefficient of } y^2}$$

and $$m_1 m_2 = \frac{a}{b} = \frac{\text{coefficient of } x^2}{\text{coefficient of } y^2}$$

$$\therefore \quad |m_1 - m_2| = \sqrt{\{(m_1 + m_2)^2 - 4m_1 m_2\}}$$
$$= \frac{2}{|b|}\sqrt{(h^2 - ab)}$$

Thus, $y = m_1 x$ and $y = m_2 x$ are two straight lines which are given by Eq. (i). Also, from Eq. (iii),

$$m = \frac{-2h \pm 2\sqrt{(h^2 - ab)}}{2b}$$
$$= \frac{-h \pm \sqrt{(h^2 - ab)}}{b} = \frac{y}{x} \qquad \left[\because m = \frac{y}{x}\right]$$

$$\therefore \quad by = \{-h + \sqrt{(h^2 - ab)}\}x$$

and $$by = \{-h - \sqrt{(h^2 - ab)}\}x$$

are two lines represented by Eq. (i).

(i) The lines are real and distinct, if $h^2 - ab > 0$.
(ii) The lines are coincident, if $h^2 - ab = 0$.
(iii) The lines are imaginary, if $h^2 - ab < 0$.

Remarks

1. In further discussions, we will consider only real cases.
2. **Two very useful identities** When lines represented by $ax^2 + 2hxy + by^2 = 0$ pass through the origin, let their equations be
$y = m_1 x$ and $y = m_2 x$
then, $(y - m_1 x)$ and $(y - m_2 x)$
must be factors of $ax^2 + 2hxy + by^2 = 0$
then $ax^2 + 2hxy + by^2 = b(y - m_1 x)(y - m_2 x)$
[Making coefficient of y^2 equal on both sides]
Now, comparing both sides, we get
$2h = -b(m_1 + m_2)$ and $a = bm_1 m_2$
$\therefore \quad m_1 + m_2 = -\frac{2h}{b}$ and $m_1 m_2 = \frac{a}{b}$

Example 3 Find the condition that the slope of one of the lines represented by $ax^2 + 2hxy + by^2 = 0$ should be n times the slope of the other.

Sol. Let the lines represented by
$ax^2 + 2hxy + by^2 = 0$ are $y = m_1 x$ and $y = m_2 x$.

Therefore, $m_1 + m_2 = -\frac{2h}{b} \quad \ldots(i)$

and $m_1 m_2 = \frac{a}{b} \quad \ldots(ii)$

Given, $m_2 = nm_1$

From Eq. (i), $m_1 + nm_1 = -\frac{2h}{b}$

$\therefore \quad m_1 = -\frac{2h}{b(1+n)} \quad \ldots(iii)$

and from Eq. (ii), $m_1(nm_1) = \frac{a}{b}$

$\therefore \quad nm_1^2 = \frac{a}{b}$

$\Rightarrow \quad n\left\{\frac{-2h}{b(1+n)}\right\}^2 = \frac{a}{b}$ [from Eq. (iii)]

$\Rightarrow \quad \frac{4nh^2}{b^2(1+n)^2} = \frac{a}{b}$

$\Rightarrow \quad 4nh^2 = ab(1+n)^2 \quad \ldots(iv)$

This is the required condition.

Corollary : If slope of one line is double of the other, then put $n = 2$ in Eq. (iv), we have

$$8h^2 = 9ab.$$

Example 4 If the slope of one of the lines represented by $ax^2 + 2hxy + by^2 = 0$ be the nth power of the other, prove that, $(ab^n)^{\frac{1}{n+1}} + (a^n b)^{\frac{1}{n+1}} + 2h = 0$.

Sol. Let m and m^n be the slopes of the lines represented by
$ax^2 + 2hxy + by^2 = 0$

then $m + m^n = -\frac{2h}{b} \quad \ldots(i)$

and $m \cdot m^n = \frac{a}{b} \Rightarrow m^{n+1} = \frac{a}{b}$

$\Rightarrow m = \left(\frac{a}{b}\right)^{\frac{1}{(n+1)}}$ ...(ii)

Substituting the value of m from Eq. (ii) in Eq. (i), then

$$\left(\frac{a}{b}\right)^{\frac{1}{(n+1)}} + \left(\frac{a}{b}\right)^{\left(\frac{n}{n+1}\right)} = -\frac{2h}{b}$$

$\Rightarrow a^{\frac{1}{n+1}} \cdot b^{\frac{n}{n+1}} + a^{\frac{n}{n+1}} \cdot b^{\frac{1}{n+1}} + 2h = 0$

Corollary **:** If slope of one line is square of the other, then put $n = 2$, then

$$(ab^2)^{\frac{1}{3}} + (a^2b)^{\frac{1}{3}} = -2h$$

On cubing both sides, we get

$$ab^2 + a^2b + 3(ab^2)^{\frac{1}{3}}(a^2b)^{\frac{1}{3}}\left((ab^2)^{\frac{1}{3}} + (a^2b)^{\frac{1}{3}}\right) = -8h^3$$

$\Rightarrow ab(a+b) + 3ab(-2h) = -8h^3$

$\therefore \frac{(a+b)}{h} + \frac{8h^2}{ab} = 6.$

Example 5 Find the product of the perpendiculars drawn from the point (x_1, y_1) on the lines $ax^2 + 2hxy + by^2 = 0$.

Sol. Let the lines represented by $ax^2 + 2hxy + by^2 = 0$ be $y = m_1x$ and $y = m_2x$.

Therefore, $m_1 + m_2 = -\frac{2h}{b}$ and $m_1m_2 = \frac{a}{b}$

The lengths of the perpendiculars from (x_1, y_1) on these lines are

$$\frac{|y_1 - m_1x_1|}{\sqrt{1+m_1^2}} \quad \text{and} \quad \frac{|y_1 - m_2x_1|}{\sqrt{1+m_2^2}}$$

Their product $= \frac{|y_1 - m_1x_1|}{\sqrt{1+m_1^2}} \times \frac{|y_1 - m_2x_1|}{\sqrt{1+m_2^2}}$

$$= \frac{|(y_1 - m_1x_1)(y_1 - m_2x_1)|}{\sqrt{(1+m_1^2)(1+m_2^2)}}$$

$$= \frac{|y_1^2 - (m_1+m_2)x_1y_1 + m_1m_2x_1^2|}{\sqrt{1+m_1^2+m_2^2+m_1^2m_2^2}}$$

$$= \frac{|y_1^2 - (m_1+m_2)x_1y_1 + m_1m_2x_1^2|}{\sqrt{1+(m_1+m_2)^2 - 2m_1m_2 + (m_1m_2)^2}}$$

$$= \frac{|y_1^2 + \frac{2h}{b}x_1y_1 + \frac{a}{b}x_1^2|}{\sqrt{\left(1 + \frac{4h^2}{b^2} - \frac{2a}{b} + \frac{a^2}{b^2}\right)}}$$

$$= \frac{|ax_1^2 + 2hx_1y_1 + by_1^2|}{\sqrt{\{(a-b)^2 + 4h^2\}}}$$

Example 6 Find the condition that one of the lines given by $ax^2 + 2hxy + by^2 = 0$ may be perpendicular to one of the lines given by $a'x^2 + 2h'xy + b'y^2 = 0$.

Sol. Since, both pair are passing through origin, let $y = mx$ be one of the lines represented by

$$ax^2 + 2hxy + by^2 = 0$$

then, $ax^2 + 2hx(mx) + b(mx)^2 = 0$

$\Rightarrow bm^2 + 2hm + a = 0$ ...(i)

then, $y = -\frac{1}{m}x$ be one of the line represented by

$$a'x^2 + 2h'xy + b'y^2 = 0$$

$$\left[\because y = -\frac{1}{m}x \text{ is perpendicular to } y = mx \text{ and passing through origin}\right]$$

then $a'x^2 + 2h'x\left(-\frac{1}{m}x\right) + b'\left(-\frac{1}{m}x\right)^2 = 0$

$\Rightarrow a'm^2 - 2h'm + b' = 0$ ...(ii)

On solving Eqs. (i) and (ii), by cross-multiplication rule i.e.

$$\begin{array}{cccccc} m^2 & m & 1 & m^2 & m \\ b & 2h & a & b & 2h \\ a' & -2h' & b' & a' & -2h' \end{array}$$

$$\Rightarrow \frac{m^2}{2hb' + 2h'a} = \frac{m}{aa' - bb'} = \frac{1}{-2h'b - 2a'h}$$

$\therefore m^2 = -\frac{(hb' + h'a)}{(h'b + a'h)}, m = \frac{(bb' - aa')}{2(a'h + h'b)}$

On eliminating m, we obtain

$$4(ha' + h'b)(h'a + hb') + (bb' - aa')^2 = 0.$$

Example 7 Show that the centroid (x', y') of the triangle with sides $ax^2 + 2hxy + by^2 = 0$ and $lx + my = 1$, is given by

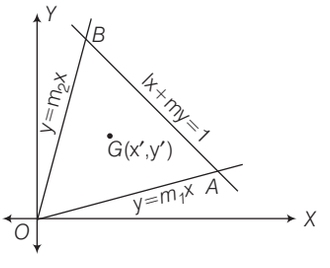
$$\frac{x'}{bl - hm} = \frac{y'}{am - hl} = \frac{2}{3(am^2 - 2hlm + bl^2)}.$$

Sol. Let the lines represented by

$$ax^2 + 2hxy + by^2 = 0$$

be $y = m_1x$

and $y = m_2x$.



Therefore, $m_1 + m_2 = -\frac{2h}{b}$

and $m_1 m_2 = \frac{a}{b}$

Coordinates of A and B are

$$\left(\frac{1}{l+mm_1}, \frac{m_1}{l+mm_1}\right) \text{ and } \left(\frac{1}{l+mm_2}, \frac{m_2}{l+mm_2}\right)$$

Since, centroid $= (x', y')$,

then, $x' = \left(\frac{\frac{1}{l+mm_1}+\frac{1}{l+mm_2}+0}{3}\right)$

$$= \left(\frac{2l + m(m_1+m_2)}{3\{l^2 + ml(m_1+m_2) + m^2 m_1 m_2)\}}\right)$$

$$= \left(\frac{2l - \frac{2hm}{b}}{3\left(l^2 - \frac{2hml}{b} + \frac{m^2 a}{b}\right)}\right)$$

$$= \frac{2}{3}\cdot\frac{(bl - hm)}{(am^2 - 2hlm + bl^2)} \quad \ldots(i)$$

and $y' = \left(\frac{\frac{m_1}{l+mm_1}+\frac{m_2}{l+mm_2}+0}{3}\right)$

$$= \left(\frac{l(m_1+m_2) + 2mm_1m_2}{3\{l^2 + lm(m_1+m_2) + m^2 m_1 m_2)\}}\right)$$

$$= \left(\frac{-\frac{2hl}{b}+\frac{2ma}{b}}{3\left(l^2 - \frac{2hlm}{b} + \frac{m^2 a}{b}\right)}\right)$$

$$= \frac{2}{3}\cdot\frac{(am - hl)}{(am^2 - 2hlm + bl^2)} \quad \ldots(ii)$$

From Eqs. (i) and (ii), we get

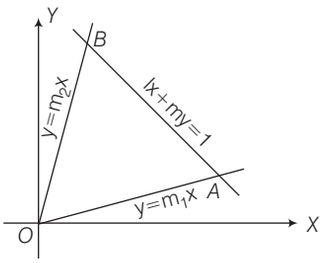
$$\frac{x'}{bl - hm} = \frac{y'}{am - hl} = \frac{2}{3(am^2 - 2hlm + bl^2)}$$

Example 8 Show that the area of the triangle formed by the lines $ax^2 + 2hxy + by^2 = 0$ and $lx + my + n = 0$ is $\frac{n^2\sqrt{(h^2 - ab)}}{|(am^2 - 2hlm + bl^2)|}$.

Sol. Let equation of lines represented by

$ax^2 + 2hxy + by^2 = 0$ be $y = m_1 x$ and $y = m_2 x$

therefore, $m_1 + m_2 = -\frac{2h}{b}$ and $m_1 m_2 = \frac{a}{b}$.



Coordinates of A and B are

$$\left(\frac{-n}{l+mm_1}, \frac{-nm_1}{l+mm_1}\right) \text{ and } \left(\frac{-n}{l+mm_2}, \frac{-nm_2}{l+mm_2}\right),$$

respectively.

Then, required area

$$= \frac{1}{2}\left|\left(\frac{-n}{l+mm_1}\right)\left(\frac{-nm_2}{l+mm_2}\right) - \left(\frac{-n}{l+mm_2}\right)\left(\frac{-nm_1}{l+mm_1}\right)\right|$$

[$\because$ if coordinates are $(0,0), (x_1, y_1)$ and, (x_2, y_2), then area $= \frac{1}{2}|x_1 y_2 - x_2 y_1|$]

$$= \frac{1}{2}\left|\frac{n^2(m_2 - m_1)}{|l^2 + lm(m_1+m_2) + m^2 m_1 m_2|}\right|$$

$$= \frac{1}{2}\left|\frac{n^2\sqrt{(m_1+m_2)^2 - 4m_1m_2}}{|l^2 + lm(m_1+m_2) + m^2 m_1 m_2|}\right|$$

$$= \frac{1}{2}\left|\frac{n^2\sqrt{\left(\frac{4h^2}{b^2} - \frac{4a}{b}\right)}}{l^2 - \frac{2hlm}{b} + \frac{m^2 a}{b}}\right|$$

$$= \frac{n^2\sqrt{(h^2 - ab)}}{|(am^2 - 2hlm + bl^2)|}$$

Example 9 Show that the two straight lines $x^2(\tan^2\theta + \cos^2\theta) - 2xy\tan\theta + y^2\sin^2\theta = 0$ move with the axis of x angles such that the difference of their tangents is 2.

Sol. Given equation is

$$x^2(\tan^2\theta + \cos^2\theta) - 2xy\tan\theta + y^2\sin^2\theta = 0 \quad \ldots(i)$$

and homogeneous equation of second degree

$$ax^2 + 2hxy + by^2 = 0 \quad \ldots(ii)$$

On comparing Eqs. (i) and (ii), we get

$$a = \tan^2\theta + \cos^2\theta$$

$$h = -\tan\theta$$

and $b = \sin^2\theta$

Let separate lines of Eq. (ii) are

$$y = m_1 x$$

and $y = m_2 x$

where, $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$

therefore, $m_1 + m_2 = -\frac{2h}{b} = \frac{2\tan\theta}{\sin^2\theta}$

and $m_1 \cdot m_2 = \frac{a}{b} = \frac{\tan^2\theta + \cos^2\theta}{\sin^2\theta}$

$\therefore \quad m_1 \sim m_2 = \sqrt{(m_1 + m_2)^2 - 4m_1m_2}$

$\Rightarrow \quad \tan\theta_1 \sim \tan\theta_2 = \sqrt{\frac{4\tan^2\theta}{\sin^4\theta} - \frac{4(\tan^2\theta + \cos^2\theta)}{\sin^2\theta}}$

$= \frac{2}{\sin^2\theta}\sqrt{\tan^2\theta - \sin^2\theta(\tan^2\theta + \cos^2\theta)}$

$= \frac{2\sin\theta}{\sin^2\theta}\sqrt{(\sec^2\theta - \tan^2\theta - \cos^2\theta)}$

$= \frac{2\sin\theta}{\sin^2\theta}\sqrt{(1 - \cos^2\theta)}$

$= \frac{2}{\sin\theta}\sin\theta = 2$

Exercise for Session 1

1. The lines given by the equation $(2y^2 + 3xy - 2x^2)(x + y - 1) = 0$ form a triangle which is

(a) equilateral (b) isosceles

(c) right angled (d) obtuse angled

2. Area of the triangle formed by the lines $y^2 - 9xy + 18x^2 = 0$ and $y = 9$ is

(a) 27/4 (b) 0

(c) 9 / 4 (d) 27

3. The equation $3x^2 + 2hxy + 3y^2 = 0$ represents a pair of straight lines passing through the origin. The two lines are

(a) real and distinct, if $h^2 > 3$ (b) real and distinct, if $h^2 > 9$

(c) real and coincident, if $h^2 = 3$ (d) real and coincident, if $h^2 > 3$

4. If one of the lines of the pair $ax^2 + 2hxy + by^2 = 0$ bisects the angle between positive directions of the axes, then a, b, h satisfy the relation

(a) $a + b = 2|h|$ (b) $a + b = -2h$

(c) $a - b = 2|h|$ (d) $(a - b)^2 = 4h^2$

5. If the slope of one of the lines given by $a^2x^2 + 2hxy + b^2y^2 = 0$ be three times of the other, then h is equal to

(a) $2\sqrt{3}ab$ (b) $-2\sqrt{3}ab$

(c) $\frac{2}{\sqrt{3}}ab$ (d) $-\frac{2}{\sqrt{3}}ab$

6. Find the separate equations of two straight lines whose joint equation is $ab(x^2 - y^2) + (a^2 - b^2)xy = 0$.

7. Find the coordinates of the centroid of the triangle whose sides are $12x^2 - 20xy + 7y^2 = 0$ and $2x - 3y + 4 = 0$.

8. If the lines $ax^2 + 2hxy + by^2 = 0$ be two sides of a parallelogram and the line $lx + my = 1$ be one of its diagonal, show that the equation of the other diagonal is $y(bl - hm) = x(am - hl)$.

9. Find the condition that one of the lines given by $ax^2 + 2hxy + by^2 = 0$ may coincide with one of the lines given by $a'x^2 + 2h'xy + b'y^2 = 0$.

Session 2

Angle between the Pair of Lines $ax^2 + 2hxy + by^2$

Angle between the Pair of Lines $ax^2 + 2hxy + by^2$

Theorem The angle θ between the pair of lines represented by $ax^2 + 2hxy + by^2 = 0$

is given by $\theta = \tan^{-1}\left\{\dfrac{2\sqrt{(h^2 - ab)}}{|a + b|}\right\}$.

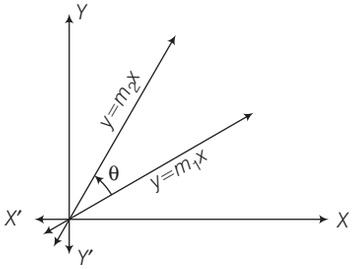
Proof Let $y = m_1 x$ and $y = m_2 x$ be the lines represented by

$$ax^2 + 2hxy + by^2 = 0.$$

Then, $m_1 + m_2 = -\dfrac{2h}{b}, m_1 m_2 = \dfrac{a}{b}$

Since, θ be the angle between the lines

$$y = m_1 x \quad \text{and} \quad y = m_2 x.$$



Then, $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right| = \dfrac{\sqrt{(m_1 + m_2)^2 - 4m_1 m_2}}{|1 + m_1 m_2|}$

$$= \frac{\sqrt{\left(\dfrac{-2h}{b}\right)^2 - 4\left(\dfrac{a}{b}\right)}}{\left|1 + \dfrac{a}{b}\right|} = \frac{2\sqrt{(h^2 - ab)}}{|a + b|}$$

$\therefore \quad \theta = \tan^{-1}\left\{\dfrac{2\sqrt{(h^2 - ab)}}{|a + b|}\right\}$

Remark

$$\theta = \sin^{-1}\left\{\frac{2\sqrt{h^2 - ab}}{\sqrt{(a - b)^2 + 4h^2}}\right\}$$

Corollary 1 *Condition for the lines to be perpendicular.*

The lines are perpendicular if the angle between them is $\dfrac{\pi}{2}$.

i.e. $\theta = \dfrac{\pi}{2}$

$\Rightarrow \quad \cot\theta = \cot\dfrac{\pi}{2}$

$\Rightarrow \quad \cot\theta = 0$

$\Rightarrow \quad \dfrac{|a + b|}{2\sqrt{(h^2 - ab)}} = 0 \quad \Rightarrow \quad a + b = 0$

i.e. Coefficient of x^2 + Coefficient of $y^2 = 0$

Hence, the lines represented by $ax^2 + 2hxy + by^2 = 0$ are perpendicular, iff $a + b = 0$ i.e. coefficient of x^2 + coefficient of $y^2 = 0$.

Remark

Pair of any two perpendicular lines through the origin.

$\because$ Lines represented by $ax^2 + 2hxy + by^2 = 0$

be perpendicular, then $a + b = 0$ or $b = -a$

Hence, the equation becomes $ax^2 + 2hxy - ay^2 = 0$

$\Rightarrow \quad x^2 + \left(\dfrac{2h}{a}\right)xy - y^2 = 0$

$\Rightarrow \quad x^2 + pxy - y^2 = 0,$ (Remember)

where, p is any constant.

Corollary 2 *Pair of lines perpendicular to the lines represented by*

$$ax^2 + 2hxy + by^2 = 0$$

and through origin.

Let lines represented by

$ax^2 + 2hxy + by^2 = 0$ be $y = m_1 x$ and $y = m_2 x$

then $\quad m_1 + m_2 = -\dfrac{2h}{b}$

and $\quad m_1 m_2 = \dfrac{a}{b}$

lines perpendiculars to $y = m_1 x$

and $y = m_2 x$

and passing through origin are

$$y = -\frac{1}{m_1} x$$

and $$y = -\frac{1}{m_2} x$$

then, pair is $$\left(y + \frac{x}{m_1}\right)\left(y + \frac{x}{m_2}\right) = 0$$

$$\Rightarrow \quad x^2 + xy\,(m_1 + m_2) + m_1 m_2\, y^2 = 0$$

$$\Rightarrow \quad x^2 - \frac{2hxy}{b} + \frac{a}{b} y^2 = 0$$

$$\therefore \quad bx^2 - 2hxy + ay^2 = 0$$

Aid to memory For perpendicular pairs interchange the coefficients of x^2 and y^2 and change the sign of xy.

***Corollary* 3** *Condition for the lines to be coincident.*

The lines are coincident, if the angle between them is $0°$ (or π)

i.e. $$\theta = 0 \text{ (or } \pi)$$

$$\therefore \quad \tan\theta = 0°$$

$$\Rightarrow \quad \frac{2\sqrt{(h^2 - ab)}}{|a + b|} = 0$$

$$\Rightarrow \quad h^2 - ab = 0$$

$$\Rightarrow \quad h^2 = ab$$

Hence, the lines represented by $ax^2 + 2hxy + by^2 = 0$ are coincident, iff $h^2 = ab$, then $ax^2 + 2hxy + by^2$ is a perfect square.

Remark

The parallel lines will be coincident only as both pass through a point.

Example 10 Find the angle between the lines $(x^2 + y^2)\sin^2\alpha = (x\cos\beta - y\sin\beta)^2$.

Sol. Given equation is

$$(x^2 + y^2)\sin^2\alpha = (x\cos\beta - y\sin\beta)^2$$

$$\Rightarrow \quad x^2(\sin^2\alpha - \cos^2\beta) + 2xy\sin\beta\cos\beta + y^2(\sin^2\alpha - \sin^2\beta) = 0 \quad \text{...(i)}$$

The homogeneous equation of second degree is

$$ax^2 + 2hxy + by^2 = 0 \quad \text{...(ii)}$$

On comparing Eqs. (i) and (ii), we get

$a = \sin^2\alpha - \cos^2\beta$, $h = \sin\beta\cos\beta$,

$b = \sin^2\alpha - \sin^2\beta$

Let the angle between the lines representing by Eq. (i) is θ.

$$\therefore \tan\theta = 2\frac{\sqrt{h^2 - ab}}{|a + b|}$$

$$= 2\frac{\sqrt{\sin^2\beta\cos^2\beta - (\sin^2\alpha - \cos^2\beta)(\sin^2\alpha - \sin^2\beta)}}{|\sin^2\alpha - \cos^2\beta + \sin^2\alpha - \sin^2\beta|}$$

$$= 2\frac{\sqrt{\{\sin^2\beta\cos^2\beta - \sin^4\alpha + \sin^2\alpha\sin^2\beta + \sin^2\alpha\cos^2\beta - \sin^2\beta\cos^2\beta\}}}{|(2\sin^2\alpha - 1)|}$$

$$= 2\frac{\sqrt{\sin^2\alpha(1 - \sin^2\alpha)}}{|-\cos 2\alpha|}$$

$$= \frac{2\sin\alpha\cos\alpha}{|-\cos 2\alpha|} = \tan 2\alpha$$

$$\therefore \quad \theta = 2\alpha$$

Example 11 Show that the angle between the lines given by $(a + 2hm + bm^2)x^2 + 2\{(b - a)m - (m^2 - 1)h\}xy + (am^2 - 2hm + b)y^2 = 0$ is the same whatever be the value of m.

Sol. Given equation is

$$(a + 2hm + bm^2)x^2 + 2\{(b - a)m - (m^2 - 1)h\}xy + (am^2 - 2hm + b)y^2 = 0 \quad \text{...(i)}$$

The homogeneous equation of second degree

$$Ax^2 + 2Hxy + By^2 = 0 \quad \text{...(ii)}$$

On comparing Eqs. (i) and (ii), we get

$A = a + 2hm + bm^2$, $H = (b - a)m - (m^2 - 1)h$,

$B = am^2 - 2hm + b$

Let the angle between the lines representing by Eq. (i) is θ.

$$\therefore \quad \tan\theta = \frac{2\sqrt{(H^2 - AB)}}{|A + B|}$$

$$= \frac{2\sqrt{\{(b - a)m - (m^2 - 1)h\}^2 - (a + 2hm + bm^2)(am^2 - 2hm + b)}}{|a + 2hm + bm^2 + am^2 - 2hm + b|}$$

$$= \frac{2\sqrt{(m^2 + 1)^2(h^2 - ab)}}{|a + b|(m^2 + 1)} = \frac{2\sqrt{(h^2 - ab)}}{|a + b|}$$

$$\Rightarrow \quad \theta = \tan^{-1}\left\{\frac{2\sqrt{(h^2 - ab)}}{|a + b|}\right\}$$

which is independent of m. Hence, the angle between the lines representing by Eq. (i) is same for all values of m.

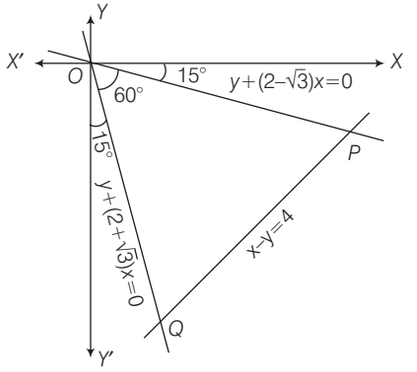
Example 12 Show that the straight lines $x^2 + 4xy + y^2 = 0$ and the line $x - y = 4$ form an equilateral triangle.

Sol. Equation $x^2 + 4xy + y^2 = 0$ ...(i)

is a homogeneous equation of second degree in x and y. Therefore, it represents two lines OP and OQ through the origin.

Equation, $x - y = 4$ ...(ii)

represent the line PQ.



Let $\angle POQ = \theta$.

then, $\tan\theta = \dfrac{2\sqrt{[(2)^2 - 1\cdot 1]}}{|1+1|} = \sqrt{3}$

$\therefore \quad \theta = 60°$

From Eq. (i), $x^2 + 4xy + y^2 = 0$

$\Rightarrow \quad \left(\dfrac{y}{x}\right)^2 + 4\left(\dfrac{y}{x}\right) + 1 = 0$

$\Rightarrow \quad \dfrac{y}{x} = \dfrac{-4 \pm \sqrt{(4^2 - 4)}}{2} = -2 \pm \sqrt{3}$

$\Rightarrow \quad y = (-2 \pm \sqrt{3})\, x$

i.e. $OP : y + (2 - \sqrt{3})\, x = 0$

and $OQ : y + (2 + \sqrt{3})\, x = 0$

$\because$ Slope of $PQ = 1$ and Slope of $OP = -(2 - \sqrt{3})$.

If $\angle OPQ = \alpha$

then $\tan\alpha = \left|\dfrac{1 - (-2 + \sqrt{3})}{1 - 2 + \sqrt{3}}\right| = \left|\dfrac{3 - \sqrt{3}}{-1 + \sqrt{3}}\right| = \sqrt{3}$

$\therefore \quad \alpha = 60°$

Hence, $\angle OQP = 180° - (60° + 60°) = 60°$

Hence, ΔOPQ is an equilateral triangle.

Example 13 Show that the condition that two of the three lines represented by $ax^3 + bx^2y + cxy^2 + dy^3 = 0$ may be at right angles is $a^2 + ac + bd + d^2 = 0$.

Sol. The given equation being homogeneous of third degree represents three straight lines through the origin. Since, two of these lines are to be at right angles.

Let pair of these lines be $(x^2 + pxy - y^2)$, p is constant and the other factor is $(ax - dy)$.

Hence, $ax^3 + bx^2y + cxy^2 + dy^3 = (x^2 + pxy - y^2)(ax - dy)$

Comparing the coefficients of similar terms, we get

$b = ap - d$ ...(i)

$c = -pd - a$ ...(ii)

Multiplying Eq. (i) by d and Eq. (ii) by a and adding, we get

$bd + ac = -d^2 - a^2$

$\Rightarrow \quad a^2 + ac + bd + d^2 = 0$

Aliter :

Let $y = m_1x$, $y = m_2x$ and $y = m_3x$ be the lines represented by the equation

$ax^3 + bx^2y + cxy^2 + dy^3 = 0.$

Then

$ax^3 + bx^2y + cxy^2 + dy^3 = d\,(y - m_1x)\,(y - m_2x)\,(y - m_3x)$ ...(i)

On equating the coefficients of x^3, x^2y and xy^2 on both sides, we get

$m_1 + m_2 + m_3 = -\dfrac{c}{d}$

$m_1m_2 + m_2m_3 + m_3m_1 = \dfrac{b}{d}$

and $m_1m_2m_3 = -\dfrac{a}{d}$ ...(ii)

Let the perpendicular lines be $y = m_1x$ and $y = m_2x$, then

$m_1m_2 = -1$

From Eq. (ii), $m_3 = \dfrac{a}{d}$ ...(iii)

On putting $y = m_3x$ in Eq. (i), we get

$ax^3 + bm_3x^3 + cm_3^2x^3 + dm_3^3x^3 = 0$

$\Rightarrow \quad dm_3^3 + cm_3^2 + bm_3 + a = 0$

$\Rightarrow \quad d\left(\dfrac{a}{d}\right)^3 + c\left(\dfrac{a}{d}\right)^2 + b\left(\dfrac{a}{d}\right) + a = 0$ [from Eq. (iii)]

Hence, $a^2 + ac + bd + d^2 = 0$

Exercise for Session 2

1. The angle between the pair of straight lines $y^2 \sin^2\theta - xy \sin^2\theta + x^2(\cos^2\theta - 1) = 0$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{2\pi}{3}$

2. The angle between the lines given by the equation $ay^2 - (1+\lambda^2)\,xy - ax^2 = 0$ is same as the angle between the lines

(a) $5x^2 + 2xy - 3y^2 = 0$ (b) $x^2 - 2xy - 3y^2 = 0$

(c) $x^2 - y^2 = 100$ (d) $xy = 0$

3. Which of the following pair of straight lines intersect at right angles ?

(a) $2x^2 = y\,(x + 2y)$ (b) $(x + y)^2 = x\,(y + 3x)$

(c) $2y\,(x + y) = xy$ (d) $y = \mp 2x$

4. If $h^2 = ab$, then the lines represented by $ax^2 + 2hxy + by^2 = 0$ are

(a) parallel (b) perpendicular

(c) coincident (d) None of these

5. Equation $ax^3 - 9x^2y - xy^2 + 4y^3 = 0$ represents three straight lines. If the two of the lines are perpendicular, then a is equal to

(a) –5 (b) 5

(c) –4 (d) 4

6. Find the angle between the lines whose joint equation is $2x^2 - 3xy + y^2 = 0$.

7. Show that the lines $(1 - \cos\theta \tan\alpha)\,y^2 - (2\cos\theta + \sin^2\theta \tan\alpha)\,xy + \cos\theta\,(\cos\theta + \tan\alpha)\,x^2 = 0$ include an angle α between them.

8. Find the angle between the lines represented by the equation $x^2 - 2pxy + y^2 = 0$.

9. Show that the lines $x^2 - 4xy + y^2 = 0$ and $x + y = 1$ form an equilateral triangle and find its area.

10. Prove that the triangle formed by the lines $ax^2 + 2hxy + by^2 = 0$ and $lx + my = 1$ is isosceles, if $h\,(l^2 - m^2) = (a - b)\,m$.

Session 3

Bisectors of the Angle between the Lines Given by a Homogeneous Equation

Bisectors of the Angle between the Lines Given by a Homogeneous Equation

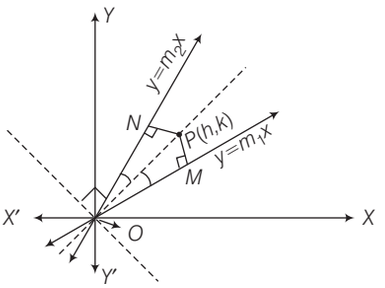
Theorem The joint equation of the bisectors of the angles between the lines represented by the equation

$$ax^2 + 2hxy + by^2 = 0 \quad \text{is} \quad \frac{x^2 - y^2}{a - b} = \frac{xy}{h}.$$

Proof Let the lines represented by $ax^2 + 2hxy +$ be $y - m_1 x = 0$ and $y - m_2 x = 0$, then

$$m_1 + m_2 = -\frac{2h}{b} \quad \text{and} \quad m_1 m_2 = \frac{a}{b}$$

Since, the bisectors of the angles between the lin locus of a point which is equidistant from the tw lines.



Let $P(h, k)$ be a point on a bisector of the angle between the given lines. Then, $PM = PN$

$$\Rightarrow \quad \frac{|k - m_1 h|}{\sqrt{(1 + m_1^2)}} = \frac{|k - m_2 h|}{\sqrt{(1 + m_2^2)}}$$

$$\Rightarrow \quad \frac{(k - m_1 h)}{\sqrt{(1 + m_1^2)}} = \pm \frac{(k - m_2 h)}{\sqrt{(1 + m_2^2)}}$$

Hence, the locus of a $P(h, k)$ is

$$\frac{(y - m_1 x)}{\sqrt{(1 + m_1^2)}} = \pm \frac{(y - m_2 x)}{\sqrt{(1 + m_2^2)}}$$

$\therefore$ The pair of bisectors is

$$\left(\frac{(y - m_1 x)}{\sqrt{(1 + m_1^2)}} + \frac{(y - m_2 x)}{\sqrt{(1 + m_2^2)}}\right)\left(\frac{(y - m_1 x)}{\sqrt{(1 + m_1^2)}} - \frac{(y - m_2 x)}{\sqrt{(1 + m_2^2)}}\right) = 0$$

$$\Rightarrow \quad \frac{(y - m_1 x)^2}{(1 + m_1^2)} - \frac{(y - m_2 x)^2}{(1 + m_2^2)} = 0$$

$$\Rightarrow \quad (1 + m_2^2)(y^2 + m_1^2 x^2 - 2m_1 xy) - (1 + m_1^2)(y^2 + m_2^2 x^2 - 2m_2 xy) = 0$$

$$m_1^2)y^2 - (m_2^2 - m_1^2)x^2 + 2xy(m_2 - m_1) - 2m_1 m_2 (m_2 - m_1)xy = 0$$

$$+ m_1)(y^2 - x^2) + 2xy - 2m_1 m_2 xy = 0 \qquad [\because m_1 - m_2 \neq 0]$$

$$\Rightarrow \quad (x^2 - y^2)\left(-\frac{2h}{b}\right) = 2xy\left(1 - \frac{a}{b}\right) \qquad \left[\because m_1 + m_2 = -\frac{2h}{b},\ m_1 m_2 = \frac{a}{b}\right]$$

$$\therefore \quad \frac{x^2 - y^2}{a - b} = \frac{xy}{h} \qquad [b \neq 0]$$

Aliter :

Let the equation $ax^2 + 2hxy + by^2 = 0$ represent two lines L_1OM_1 and L_2OM_2 making angles θ_1 and θ_2 with the positive direction of X-axis.

If slopes of L_1OM_1 and L_2OM_2 are m_1 and m_2, then

$$m_1 = \tan\theta_1 \quad \text{and} \quad m_2 = \tan\theta_2$$

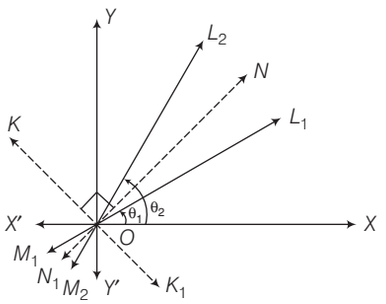
and $$m_1 + m_2 = -\frac{2h}{b}, m_1 m_2 = \frac{a}{b} \qquad \text{...(i)}$$

Let NON_1 and KOK_1 are the required bisectors,

Since $$\angle NOL_1 = \angle NOL_2 = \frac{\theta_2 - \theta_1}{2}$$

$$\angle NOX = \theta_1 + \frac{\theta_2 - \theta_1}{2} = \frac{\theta_1 + \theta_2}{2}$$

Since, $$\angle NOK = \frac{\pi}{2}$$



$$\therefore \qquad \angle KOX = \frac{\pi}{2} + \angle NOX = \frac{\pi}{2} + \left(\frac{\theta_1 + \theta_2}{2}\right)$$

Equation of bisectors are $y = x \tan\left(\frac{\theta_1 + \theta_2}{2}\right)$

$$\Rightarrow \qquad y - x \tan\left(\frac{\theta_1 + \theta_2}{2}\right) = 0 \qquad \text{...(ii)}$$

and
$$y = x \tan\left(\frac{\pi}{2} + \frac{\theta_1 + \theta_2}{2}\right)$$

$$\Rightarrow \qquad y = -x \cot\left(\frac{\theta_1 + \theta_2}{2}\right)$$

$$\Rightarrow \qquad y + x \cot\left(\frac{\theta_1 + \theta_2}{2}\right) = 0 \qquad \text{...(iii)}$$

$\therefore$ Pair of bisectors

$$\left[y - x \tan\left(\frac{\theta_1 + \theta_2}{2}\right)\right]\left[y + x \cot\left(\frac{\theta_1 + \theta_2}{2}\right)\right] = 0$$

$$\Rightarrow \quad y^2 - x^2 + xy\left(\cot\left(\frac{\theta_1 + \theta_2}{2}\right) - \tan\left(\frac{\theta_1 + \theta_2}{2}\right)\right) = 0$$

$$\Rightarrow \qquad x^2 - y^2 = xy\left(\frac{1 - \tan^2\left(\frac{\theta_1 + \theta_2}{2}\right)}{\tan\left(\frac{\theta_1 + \theta_2}{2}\right)}\right)$$

$$\Rightarrow \qquad x^2 - y^2 = xy\left(\frac{2}{\tan(\theta_1 + \theta_2)}\right)$$

$$\Rightarrow \qquad x^2 - y^2 = 2xy\left(\frac{1 - \tan\theta_1 \tan\theta_2}{\tan\theta_1 + \tan\theta_2}\right)$$

$$\Rightarrow \qquad x^2 - y^2 = 2xy\left(\frac{1 - m_1 m_2}{m_1 + m_2}\right)$$

$$\Rightarrow \qquad x^2 - y^2 = 2xy\left(\frac{1 - a/b}{-2h/b}\right)$$

$$\therefore \qquad \frac{(x^2 - y^2)}{(a - b)} = \frac{xy}{h}$$

Remark

The joint equation of the bisectors is $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$

or $\qquad hx^2 - (a - b)\,xy - hy^2 = 0$

i.e. coefficient of x^2 + coefficient of $y^2 = 0$.

Hence, the bisectors of the angle between the lines are always perpendicular to each other.

Corollaries

1. If $a = b$, the bisectors are $x^2 - y^2 = 0$

   i.e. $\qquad x - y = 0,\ x + y = 0$

2. If $h = 0$, the bisectors are $xy = 0$

   i.e. $\qquad x = 0,\ y = 0$

Example 14 Find the equation of the bisectors of the angle between the lines represented by $3x^2 - 5xy + 4y^2 = 0$.

Sol. Given equation is

$$3x^2 - 5xy + 4y^2 = 0 \qquad \text{...(i)}$$

Comparing it with the equation

$$ax^2 + 2hxy + by^2 = 0 \qquad \text{...(ii)}$$

then $\qquad a = 3,\ h = -\frac{5}{2},\ b = 4$

Hence, the equation of bisectors of the angle between the pair of the lines (i) is

$$\frac{x^2 - y^2}{3 - 4} = \frac{xy}{-5/2}$$

$$\Rightarrow \qquad \frac{x^2 - y^2}{-1} = \frac{2xy}{-5}$$

$$\therefore \qquad 5x^2 - 2xy - 5y^2 = 0$$

Example 15 Show that the line $y = mx$ bisects the angle between the lines

$$ax^2 - 2hxy + by^2 = 0,$$

if $\qquad h(1 - m^2) + m(a - b) = 0.$

Sol. Equation of pair of bisectors of angles between lines $ax^2 - 2hxy + by^2 = 0$ is

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{-h}$$

$$\Rightarrow \qquad -h(x^2 - y^2) = (a - b)\,xy \qquad \text{...(i)}$$

But $y = mx$ is one of these lines, then it will satisfy it. Substituting $y = mx$ in Eq. (i),

$$-h(x^2 - m^2x^2) = (a - b)x \cdot mx$$

Dividing by x^2, $h(1 - m^2) + m(a - b) = 0$

Example 16 If pairs of straight lines $x^2 - 2pxy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ be such that each pair bisects the angle between the other pair, then prove that $pq = -1$.

Sol. According to the question, the equation of the bisectors of the angle between the lines

$$x^2 - 2pxy - y^2 = 0 \quad \text{...(i)}$$

is $$x^2 - 2qxy - y^2 = 0 \quad \text{...(ii)}$$

$\therefore$ The equation of bisectors of the angle between the lines (i) is

$$\frac{x^2 - y^2}{1 - (-1)} = \frac{xy}{-p}$$

$$\Rightarrow \quad -px^2 - 2xy + py^2 = 0 \quad \text{...(iii)}$$

Since, Eqs. (ii) and (iii) are identical, comparing Eqs. (ii) and (iii), we get

$$\frac{1}{-p} = \frac{-2q}{-2} = \frac{-1}{p} \Rightarrow pq = -1$$

Remark

By taking the bisectors of the angles between the pair of lines (ii), we will get the same result.

Example 17 Prove that the angle between one of the lines given by $ax^2 + 2hxy + by^2 = 0$ and one of the lines $ax^2 + 2hxy + by^2 + \lambda(x^2 + y^2) = 0$ is equal to angle between other two lines of the system.

Sol. The equation of the bisectors of the angle between the lines

$$ax^2 + 2hxy + by^2 = 0 \quad \text{...(i)}$$

is $$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$

and the equation of bisectors of the angle between the lines

$$ax^2 + 2hxy + by^2 + \lambda(x^2 + y^2) = 0 \quad \text{...(ii)}$$

$$\Rightarrow \quad (a + \lambda)x^2 + 2hxy + (b + \lambda)y^2 = 0$$

is $$\frac{x^2 - y^2}{(a + \lambda) - (b + \lambda)} = \frac{xy}{h} \Rightarrow \frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$

$\therefore$ Bisectors of angles between lines given by Eqs. (i) and (ii) are the same. Hence the result.

Example 18 Show that the pair of lines given by $a^2x^2 + 2h(a+b)xy + b^2y^2 = 0$ is equally inclined to the pair given by $ax^2 + 2hxy + by^2 = 0$.

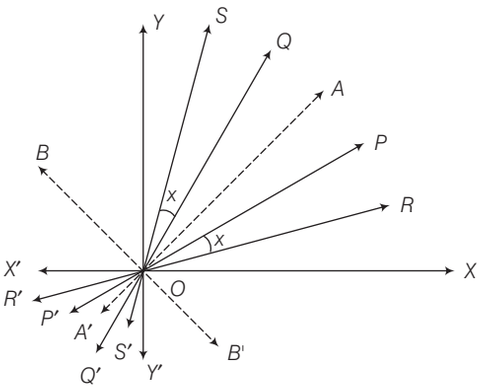
Sol. Given pair of lines are

$$a^2x^2 + 2h(a + b)xy + b^2y^2 = 0 \quad \text{...(i)}$$

and $$ax^2 + 2hxy + by^2 = 0 \quad \text{...(ii)}$$

Equation of bisectors of first pair is

$$\frac{x^2 - y^2}{a^2 - b^2} = \frac{xy}{h(a + b)} \Rightarrow \frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$



which are also the bisectors of the second pair.

Let $P'OP$, $Q'OQ$, $R'OR$, $S'OS$ be the lines of the Eqs. (i) and (ii) pairs respectively and $A'OA$ and $B'OB$ be their bisectors. We have,

$$\angle ROA = \angle SOA$$

and $$\angle POA = \angle QOA$$

$$\Rightarrow \quad \angle ROA - \angle POA = \angle SOA - \angle QOA$$

$$\Rightarrow \quad \angle ROP = \angle SOQ$$

Hence the result.

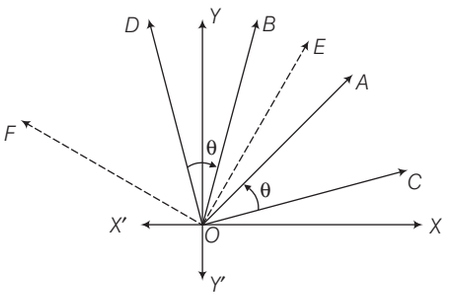
Example 19 If the lines represented by $x^2 - 2pxy - y^2 = 0$ are rotated about the origin through an angle θ, one in clockwise direction and the other in anti-clockwise direction, then find the equation of the bisectors of the angle between the lines in the new position.

Sol. Since, lines represented by $x^2 - 2pxy - y^2 = 0$ are perpendicular to each other. The bisectors of the angles between the lines in new position are same as the bisectors of the angles between their old positions. i.e.

OC, OD ; OA, OB be the old and new pairs respectively and OE and OF be their bisectors, we have

$$\angle COE = \angle DOE$$

and $$\angle COA = \angle DOB = \theta \quad \text{[given]}$$



$$\Rightarrow \quad \angle COE - \angle COA = \angle DOE - \angle DOB$$

$$\Rightarrow \quad \angle AOE = \angle BOE$$

Therefore, the required equation is $\frac{x^2 - y^2}{1 - (-1)} = \frac{xy}{-p}$

i.e. $px^2 + 2xy - py^2 = 0$.

Exercise for Session 3

1. If coordinate axes are the angle bisectors of the pair of lines $ax^2 + 2hxy + by^2 = 0$, then

(a) $a = b$ (b) $h = 0$

(c) $a^2 + b = 0$ (d) $a + b^2 = 0$

2. If the line $y = mx$ is one of the bisector of the lines $x^2 + 4xy - y^2 = 0$, then the value of m is

(a) $\frac{\sqrt{5} - 1}{2}$ (b) $\frac{\sqrt{5} + 1}{2}$

(c) $-\left(\frac{\sqrt{5} + 1}{2}\right)$ (d) $-\left(\frac{\sqrt{5} - 1}{2}\right)$

3. If one of the lines of $my^2 + (1 - m^2)\,xy - mx^2 = 0$ is a bisector of the angle between the lines $xy = 0$, then $\cos^{-1}(m)$ is

(a) 0 (b) $\pi/2$

(c) π (d) $3\pi / 2$

4. The bisectors of the angles between the lines $(ax + by)^2 = c\,(bx - ay)^2, c > 0$ are respectively parallel and perpendicular to the line

(a) $bx - ay + \mu = 0$ (b) $ax + by + \lambda = 0$

(c) $ax - by + \nu = 0$ (d) $bx + ay + \tau = 0$

5. If the pairs of straight lines $ax^2 + 2hxy - ay^2 = 0$ and $bx^2 + 2gxy - by^2 = 0$ be such that each bisects the angles between the other, then prove that $hg + ab = 0$.

6. Prove that the lines $2x^2 + 6xy + y^2 = 0$ are equally inclined to the lines $4x^2 + 18xy + y^2 = 0$.

7. Show that the lines bisecting the angle between the bisectors of the angles between the lines $ax^2 + 2hxy + by^2 = 0$ are given by $(a - b)(x^2 - y^2) + 4hxy = 0$.

8. Prove that the bisectors of the angle between the lines $ax^2 + acxy + cy^2 = 0$ and $\left(3 + \frac{1}{c}\right)x^2 + xy + \left(3 + \frac{1}{a}\right)y^2 = 0$ are always the same.

9. The lines represented by $x^2 + 2\lambda xy + 2y^2 = 0$ and the lines represented by $(1 + \lambda)x^2 - 8xy + y^2 = 0$ are equally inclined, find the values of λ.

Session 4

General Equation of Second Degree, Important Theorems

General Equation of Second Degree

The equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

is the general equation of second degree and represents a conics (pair of straight lines, circle, parabola, ellipse, hyperbola). It contains six constants a, b, c, f, g, h.

i.e. $a =$ coefficient of x^2, $b =$ coefficient of y^2,

$c =$ constant term, $g =$ half the coefficient of x,

$f =$ half the coefficient of y,

$h =$ half the coefficient of xy.

Theorem The necessary and sufficient condition for

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

to represent a pair of straight lines is that

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \text{ or } \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

Proof **Necessary condition :** Let the equation be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \ldots(i)$$

represent a pair of lines. Assuming that these lines are not parallel, we suppose further that their point of intersection is (x_1, y_1). Shifting the origin at (x_1, y_1) without rotating the coordinate axes, we have the Eq. (i) transforms to

$$a(X + x_1)^2 + 2h(X + x_1)(Y + y_1) + b(Y + y_1)^2 + 2g(X + x_1) + 2f(Y + y_1) + c = 0 \quad \ldots(ii)$$

Now this Eq. (ii) represents a pair of lines through the new origin and consequently, it is homogeneous in X and Y.

Hence, the coefficients of X and Y and the constant term in Eq. (ii) must vanish separately.

i.e. coefficient of $X =$ coefficient of $Y =$ constant term $= 0$

$$\Rightarrow \quad ax_1 + hy_1 + g = 0 \quad \ldots(iii)$$

$$hx_1 + by_1 + f = 0 \quad \ldots(iv)$$

and $$ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0 \quad \ldots(v)$$

Now, $ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0$

$$\Rightarrow \quad x_1(ax_1 + hy_1 + g) + y_1(hx_1 + by_1 + f) + (gx_1 + fy_1 + c) = 0$$

$$\Rightarrow \quad x_1 \cdot 0 + y_1 \cdot 0 + gx_1 + fy_1 + c = 0 \quad \text{[from Eqs. (iii) and (iv)]}$$

$$\Rightarrow \quad gx_1 + fy_1 + c = 0 \quad \ldots(vi)$$

On eliminating x_1, y_1 from Eqs. (iii), (iv) and (vi), we get the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$\therefore \; abc + 2fgh - af^2 - bg^2 - ch^2 = 0,$

as the required condition.

Remarks

1. **Without using determinant** On solving Eqs. (iii) and (iv), we get

$$(x_1, y_1) = \left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2} \right)$$

and then substituting the values of x_1 and y_1 in Eqs. (vi), we obtain

$$g\left(\frac{hf - bg}{ab - h^2} \right) + f\left(\frac{gh - af}{ab - h^2} \right) + c = 0$$

$$\Rightarrow \quad abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

2. By making $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

homogeneous with the help of a new variable z, i.e.

$ax^2 + 2hxy + by^2 + 2gxz + 2fyz + cz^2 = 0$

Let $f(x, y, z) \equiv ax^2 + 2hxy + by^2 + 2gxz + 2fyz + cz^2 = 0$

$$\therefore \quad \frac{\partial f}{\partial x} = 2ax + 2hy + 2gz = 0$$

$$\frac{\partial f}{\partial y} = 2hx + 2by + 2fz = 0$$

$$\frac{\partial f}{\partial z} = 2gx + 2fy + 2cz = 0$$

and finally putting $z = 1$, we obtain equations

$$ax + hy + g = 0, \; hx + by + f = 0, \; gx + fy + c = 0$$

which are same as Eqs. (iii), (iv) and (vi), respectively.

3. If $ab - h^2 = 0$, the lines given by Eq. (i) are parallel. In this case, the method followed in the above proof fails and we follow the following method.

Aliter I : (Proof)

Let the lines represented by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad ...(i)$$

be $lx + my + n = 0$ and $l'x + m'y + n' = 0$

then $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

$$\equiv (lx + my + n)(l'x + m'y + n') \quad ...(ii)$$

Comparing the coefficients of similar terms in both sides of Eq. (ii), we get

$$\left.\begin{aligned} &ll' = a, mm' = b, nn' = c \\ &lm' + l'm = 2h, \ln' + l'n = 2g, \\ &mn' + m'n = 2f \end{aligned}\right\} \quad ...(iii)$$

We now eliminate l, m, n, l', m' and n' from these equations,

we have $\begin{vmatrix} l & l' & 0 \\ m & m' & 0 \\ n & n' & 0 \end{vmatrix} \times \begin{vmatrix} l' & l & 0 \\ m' & m & 0 \\ n' & n & 0 \end{vmatrix} = 0$

[$\because$ each determinant $= 0$]

$$\Rightarrow \begin{vmatrix} 2ll' & lm' + l'm & ln' + l'n \\ ml' + m'l & 2mm' & mn' + m'n \\ nl' + n'l & nm' + n'm & 2nn' \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{vmatrix} = 0$$

$$\Rightarrow \Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$\therefore \quad \Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0,$

which is the required necessary condition.

Remark

Without using determinant

Now, $(lm' + l'm)(\ln' + l'n)(mn' + m'n) = 2h \cdot 2g \cdot 2f$

$\Rightarrow \quad 2ll'mm'nn' + ll'(m^2n'^2 + m'^2n^2)$

$+ mm'(n^2l'^2 + n'^2l^2) + nn'(l^2m'^2 + l'^2m^2) = 8fgh$

$\Rightarrow \quad 2ll'mm'nn' + ll'\{(mn' + m'n)^2 - 2mm'nn'\}$

$+ mm'\{(nl' + n'l)^2 - 2nn'll'\} + nn'\{(lm' + l'm)^2 - 2ll'mm'\}$

$= 8fgh$ [from Eq. (iii)]

$\Rightarrow 2abc + a(4f^2 - 4bc) + b(4g^2 - 4ca) + c(4h^2 - 4ab) = 8fgh$

$\therefore \quad abc + 2fgh - af^2 - bg^2 - ch^2 = 0,$

which is the required necessary condition.

Aliter II : (Proof)

Given equation is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad ...(i)$$

Case I If $a \neq 0$, then writing Eq. (i) as a quadratic equation in x, we get

$$ax^2 + 2x(hy + g) + by^2 + 2fy + c = 0$$

Solving, we have

$$x = \frac{-2(hy + g) \pm \sqrt{4(hy + g)^2 - 4a(by^2 + 2fy + c)}}{2a}$$

$$\therefore x = \frac{-(hy + g) \pm \sqrt{\{(h^2 - ab)y^2 + 2(gh - af)y + (g^2 - ac)\}}}{a}$$

Eq. (i), will represent two straight lines, if LHS of Eq. (i), can be resolved into two linear factors, therefore the expression under the square root should be a perfect square.

[$\because Ax^2 + Bx + C = 0$ is a perfect square $\Leftrightarrow B^2 - 4AC = 0$]

Hence, $4(gh - af)^2 - 4(h^2 - ab)(g^2 - ac) = 0$

or $\quad abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \quad ...(ii)$

Ths is called discriminant of the Eq. (i).

Case II If $a = 0, b \neq 0$, then writing Eq. (i) as a quadratic equation in y

i.e. $\quad by^2 + 2y(hx + f) + 2gx + c = 0$

and proceeding above we get the condition

$$2fgh - bg^2 - ch^2 = 0$$

which is condition obtained by putting $a = 0$ in Eq. (ii).

Case III If $a = 0, b = 0$ but $h \neq 0$, then Eq. (i) becomes

$$2hxy + 2gx + 2fy + c = 0$$

Multiplying by $\frac{h}{2}$ [$\because h \neq 0$]

$$\Rightarrow \quad h^2xy + hgx + hfy + \frac{ch}{2} = 0$$

$$\Rightarrow \quad (hx + f)(hy + g) = fg - \frac{ch}{2}$$

Above equation represents two straight lines, if

$$fg - \frac{ch}{2} = 0 \quad \Rightarrow \quad 2fgh - ch^2 = 0$$

which is condition obtained by putting $a = 0, b = 0$ in Eq. (ii).

Hence in each case, the condition that

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents two straight lines is

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

which is the required necessary condition.

Sufficient condition (Conversely)

Here, we have to show that the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines.

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

i.e. lines $ax + hy + g = 0, hx + by + f = 0, \ gx + fy + c = 0$ are concurrent.

Let the point of concurrency be (x_1, y_1).

Then, $ax_1 + hy_1 + g = 0$ ...(iii)

$hx_1 + by_1 + f = 0$ ...(iv)

and $gx_1 + fy_1 + c = 0$ ...(v)

Now, shifting the origin at (x_1, y_1) without rotating the coordinate axes the equation

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ reduces to

$$a(X + x_1)^2 + 2h(X + x_1)(Y + y_1) + b(Y + y_1)^2 + 2g(X + x_1) + 2f(Y + y_1) + c = 0$$

$$\Rightarrow \ aX^2 + 2hXY + bY^2 + 2X(ax_1 + hy_1 + g) + 2Y(hx_1 + by_1 + f) + x_1(ax_1 + hy_1 + g) + y_1(hx_1 + by_1 + f) + (gx_1 + fy_1 + c) = 0$$

$$\Rightarrow \ aX^2 + 2hXY + bY^2 + 0 + 0 + 0 + 0 + 0 = 0$$

[from Eqs. (iii), (iv) and (v)]

i.e. $aX^2 + 2hXY + bY^2 = 0$

It is homogeneous equation of second degree. So, it represents a pair of straight lines through the new origin.

Hence, the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines, if

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$$

Some useful identities If $y = m_1x + c_1, y = m_2x + c_2$ be lines represented by Eq. (i). Then,

$$\begin{aligned} ax^2 + 2hxy + by^2 + 2gx + 2fy + c &= b(y - m_1x - c_1)(y - m_2x - c_2) \\ &= b(y^2 - (m_1 + m_2)xy + m_1m_2x^2 + (m_1c_2 + m_2c_1)x - (c_1 + c_2)y + c_1c_2) \end{aligned}$$

On equating coefficients, we get

$$m_1 + m_2 = -\frac{2h}{b}, m_1m_2 = \frac{a}{b}, m_1c_2 + m_2c_1 = \frac{2g}{b},$$

$$c_1 + c_2 = -\frac{2f}{b}, c_1c_1 = \frac{c}{b}$$

These five relations are very useful to solve many problems.

Important Theorems

Theorem 1 The angle between the lines represented by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

is given by

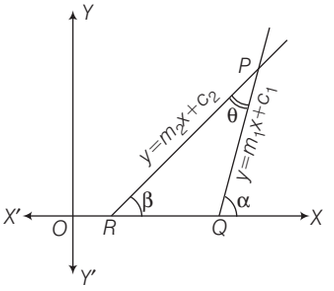
$$\theta = \tan^{-1}\left\{\frac{2\sqrt{(h^2 - ab)}}{|a + b|}\right\}.$$

Proof Let $y = m_1x + c_1$

and $y = m_2x + c_2$

be the lines represented by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$



where, $m_1 = \tan\alpha, m_2 = \tan\beta$

Then, $ax^2 + 2hxy + by^2 + 2gx + 2fy + c \equiv (y - m_1x - c_1)(y - m_2x - c_2)$

Comparing coefficients of like powers, we obtain

$$m_1 + m_2 = -\frac{2h}{b}, m_1m_2 = \frac{a}{b}$$

Now, if θ be the acute angle between the lines $y = m_1x + c_1$ and $y = m_2x + c_2$, then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| = \frac{\sqrt{(m_1 + m_2)^2 - 4m_1m_2}}{|1 + m_1m_2|}$$

$$= \frac{\sqrt{\left(-\frac{2h}{b}\right)^2 - 4\left(\frac{a}{b}\right)}}{\left|1 + \frac{a}{b}\right|} = \frac{2\sqrt{(h^2 - ab)}}{|a + b|}$$

$$\therefore \quad \theta = \tan^{-1}\left\{\frac{2\sqrt{(h^2 - ab)}}{|a + b|}\right\}$$

***Corollary* 1.** The angle between the lines represented by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

is the same as the angle between the lines represented by

$$ax^2 + 2hxy + by^2 = 0$$

***Corollary* 2.** The lines represented by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

are perpendicular iff $a + b = 0$ and parallel iff $h^2 = ab$.

Theorem 2 The lines represented by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

will be coincident, if $h^2 - ab = 0, g^2 - ac = 0$

and $f^2 - bc = 0.$

Proof Let the lines represented by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

be $lx + my + n = 0$ and $l'x + m'y + n' = 0$

then $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

$$\equiv (lx + my + n)(l'x + m'y + n')$$

Comparing the coefficients of similar terms in both sides, we get

$$\left.\begin{aligned} &ll' = a, mm' = b, nn' = c \\ &lm' + l'm = 2h, \ln' + l'n = 2g \\ &mn' + m'n = 2f \end{aligned}\right\}$$

$\because$ Lines $lx + my + n = 0$ and $l'x + m'y + n' = 0$ are coincident, then $\frac{l}{l'} = \frac{m}{m'} = \frac{n}{n'}$.

Taking the ratios in pairs, then

$$lm' - l'm = 0, mn' - m'n = 0, \ln' - l'n = 0$$

$$\Rightarrow \quad \sqrt{(lm' + l'm)^2 - 4ll'mm'} = 0,$$

$$\sqrt{(mn' + m'n)^2 - 4mm'nn'} = 0$$

and $\sqrt{(ln' + l'n)^2 - 4ll'nn'} = 0$

i.e. $\sqrt{(4h^2 - 4ab)} = 0, \sqrt{(4f^2 - 4bc)} = 0$

and $\sqrt{(4g^2 - 4ac)} = 0$

i.e. $h^2 - ab = 0, f^2 - bc = 0, g^2 - ac = 0$

Theorem 3 The point of intersection of the lines represented by

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is

$$\left(\sqrt{\left(\frac{f^2 - bc}{h^2 - ab}\right)}, \sqrt{\left(\frac{g^2 - ca}{h^2 - ab}\right)}\right) \text{or} \left(\frac{bg - hf}{h^2 - ab}, \frac{af - gh}{h^2 - ab}\right).$$

Proof Let the lines represented by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

be $lx + my + n = 0$ and $l'x + m'y + n' = 0$

then $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

$$\equiv (lx + my + n)(l'x + m'y + n')$$

Comparing the coefficients of similar terms in both sides

then $ll' = a, mm' = b, nn' = c$

$$lm' + l'm = 2h, \ln' + l'n = 2g, mn' + m'n = 2f$$

$$\Rightarrow \quad (lm' - l'm) = \sqrt{(lm' + l'm)^2 - 4ll'mm'} = 2\sqrt{(h^2 - ab)}$$

$$\Rightarrow \quad (nl' - n'l) = \sqrt{(ln' + l'n)^2 - 4ll'nn'} = 2\sqrt{(g^2 - ac)}$$

and $(mn' - m'n) = \sqrt{(mn' + m'n)^2 - 4mm'nn'}$

$$= 2\sqrt{(f^2 - bc)}$$

Now, solving $lx + my + n = 0$ and $l'x + m'y + n' = 0$

then $\frac{x}{(mn' - m'n)} = \frac{y}{(nl' - n'l)} = \frac{1}{(lm' - l'm)}$

$$\Rightarrow \quad \frac{x}{2\sqrt{(f^2 - bc)}} = \frac{y}{2\sqrt{(g^2 - ac)}} = \frac{1}{2\sqrt{(h^2 - ab)}}$$

$$\therefore \quad (x, y) = \left(\sqrt{\left(\frac{f^2 - bc}{h^2 - ab}\right)}, \sqrt{\left(\frac{g^2 - ac}{h^2 - ab}\right)}\right)$$

Also, $x = \sqrt{\left(\frac{f^2 - bc}{h^2 - ab}\right)} = \frac{\sqrt{(f^2 - bc)(h^2 - ab)}}{(h^2 - ab)}$

$$= \frac{\sqrt{f^2h^2 - abf^2 - bch^2 + b(abc)}}{(h^2 - ab)}$$

$$= \frac{\sqrt{f^2h^2 - abf^2 - bch^2 + b(af^2 + bg^2 + ch^2 - 2fgh)}}{(h^2 - ab)}$$

$$[\because abc + 2fgh - af^2 - bg^2 - ch^2 = 0]$$

$$= \frac{\sqrt{(f^2h^2 + b^2g^2 - 2bfgh)}}{(h^2 - ab)} = \frac{\sqrt{(bg - hf)^2}}{(h^2 - ab)}$$

$$= \left(\frac{bg - hf}{h^2 - ab}\right)$$

Similarly, $y = \left(\frac{af - gh}{h^2 - ab}\right)$

Hence, $(x, y) = \left(\frac{bg - hf}{h^2 - ab}, \frac{af - gh}{h^2 - ab}\right).$

Remembering Method (For second point)

Since, $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

taking first two rows (repeat first column)

$$\begin{matrix} a & h & g & a \\ h & b & f & h \end{matrix}$$

$\Rightarrow \quad ab - h^2, hf - bg, gh - af$

$\Rightarrow \quad h^2 - ab, bg - hf, af - gh$

$\Rightarrow \quad 1, \dfrac{bg - hf}{h^2 - ab}, \dfrac{af - gh}{h^2 - ab}$

Hence, point of intersection is $\left(\dfrac{bg - hf}{h^2 - ab}, \dfrac{af - gh}{h^2 - ab}\right)$.

OR

Cofactors of third column are C_{13}, C_{23}, C_{33}

$\because \quad C_{13} = \begin{vmatrix} h & b \\ g & f \end{vmatrix} = hf - bg$

$C_{23} = -\begin{vmatrix} a & h \\ g & f \end{vmatrix} = hg - af$

and $\quad C_{33} = \begin{vmatrix} a & h \\ h & b \end{vmatrix} = ab - h^2$

Point of intersection is $\left(\dfrac{C_{13}}{C_{33}}, \dfrac{C_{23}}{C_{33}}\right)$ i.e.

$$\left(\frac{hf - bg}{ab - h^2}, \frac{hg - af}{ab - h^2}\right) \quad \text{or} \quad \left(\frac{bg - hf}{h^2 - ab}, \frac{af - hg}{h^2 - ab}\right).$$

Remembering Mehod (For first point)

Cofactors of leading diagonal are

C_{11}, C_{22}, C_{33}

$\because \quad C_{11} = \begin{vmatrix} b & f \\ f & c \end{vmatrix} = bc - f^2,$

$C_{22} = \begin{vmatrix} a & g \\ g & c \end{vmatrix} = ac - g^2$

and $\quad C_{33} = \begin{vmatrix} a & h \\ h & b \end{vmatrix} = ab - h^2$

$\therefore$ Point of intersection are $\left(\sqrt{\dfrac{C_{11}}{C_{33}}}, \sqrt{\dfrac{C_{22}}{C_{33}}}\right)$

i.e. $\quad \left(\sqrt{\left(\dfrac{bc - f^2}{ab - h^2}\right)}, \sqrt{\left(\dfrac{ac - g^2}{ab - h^2}\right)}\right)$

or $\quad \left(\sqrt{\left(\dfrac{f^2 - bc}{h^2 - ab}\right)}, \sqrt{\left(\dfrac{g^2 - ac}{h^2 - ab}\right)}\right)$

Theorem 4 The pair of bisectors of the lines represented by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

is $\quad \dfrac{(x - \alpha)^2 - (x - \beta)^2}{(a - b)} = \dfrac{(x - \alpha)(y - \beta)}{h}$

where (α, β) be the point of intersection of the pair of straight lines represented by Eq. (i).

Proof Since (α, β) be the point of intersection of the lines represented by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \ldots(i)$$

Shifting the origin at (α, β) without rotating the coordinate axes, the Eq. (i) reduces to

$$a(X + \alpha)^2 + 2h(X + \alpha)(Y + \beta) + b(Y + \beta)^2 + 2g(X + \alpha) + 2f(Y + \beta) + c = 0$$

$[\because x = X + \alpha \text{ and } y = Y + \beta]$

$$\Rightarrow \quad (aX^2 + 2hXY + bY^2) + 2X(a\alpha + h\beta + g) + 2Y(h\alpha + b\beta + f) + a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c = 0 \ldots(ii)$$

This equation represents a pair of straight lines passing through the new origin. So, it must be homogeneous equation of second degree in X and Y.

$\therefore \quad a\alpha + h\beta + g = 0 \qquad \ldots(iii)$

$h\alpha + b\beta + f = 0 \qquad \ldots(iv)$

and $a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c = 0 \qquad \ldots(v)$

Now, from Eq. (ii), $\quad aX^2 + 2hXY + bY^2 = 0 \qquad \ldots(vi)$

The equation of the bisectors of the angles between the lines given by Eq. (vi) is

$$\frac{X^2 - Y^2}{a - b} = \frac{XY}{h} \qquad \ldots(vii)$$

[with reference to new origin]

Replacing X by $x - \alpha$ and Y by $y - \beta$ in Eq. (vii), then

$$\frac{(x - \alpha)^2 - (y - \beta)^2}{(a - b)} = \frac{(x - \alpha)(y - \beta)}{h}$$

[with reference to old origin]

which is the required equation of the bisectors of the angles between the lines given by Eq. (i).

Remark

If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

represents two straight lines, then the equation of lines through the origin and parallel to them is $ax^2 + 2hxy + by^2 = 0$.

Example 20 For what value of λ does the equation $12x^2 - 10xy + 2y^2 + 11x - 5y + \lambda = 0$ represent a pair of straight lines? Find their equations and the angle between them.

Sol. Comparing the given equation with the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

we get $a = 12, h = -5, b = 2, g = \frac{11}{2}, f = -\frac{5}{2}$ and $c = \lambda$

If the given equation represents a pair of straight lines, then

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$\Rightarrow \quad 12 \times 2 \times \lambda + 2 \times \left(-\frac{5}{2}\right) \times \frac{11}{2} \times (-5) - 12 \times \frac{25}{4} - 2 \times \frac{121}{4} - \lambda \times 25 = 0$$

$\therefore \quad \lambda = 2$, also $h^2 - ab = 25 - 24 = 1 > 0$

$\therefore$ The given equation will represent a pair of straight lines, if $\lambda = 2$.

To find the two lines

First method

Substituting $\lambda = 2$ in the given equation, we get

$$12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0 \qquad ...(i)$$

Since, $12x^2 - 10xy + 2y^2 = 2(3x - y)(2x - y)$

factors of Eq. (i) can be taken as

$2(3x - y + l)(2x - y + m)$

$$= 12x^2 - 10xy + 2y^2 + 2(2l + 3m)x + 2(-l - m)y + 2lm$$

On comparing, $2l + 3m = \frac{11}{2}, l + m = \frac{5}{2}, lm = 1$

Solving, we get $l = 2, m = \frac{1}{2}$.

Thus, the factors of Eq. (i) are

$$2(3x - y + 2)\left(2x - y + \frac{1}{2}\right) = 0$$

or $(3x - y + 2)(4x - 2y + 1) = 0.$

$\therefore$ The two straight lines represented by the given equation are

$$3x - y + 2 = 0 \quad \text{and} \quad 4x - 2y + 1 = 0.$$

Second Method

Writing Eq. (i) as quadratic equation in x, we get

$$12x^2 + (-10y + 11)x + 2y^2 - 5y + 2 = 0$$

$\therefore$

$$x = \frac{-(-10y + 11) \pm \sqrt{(-10y + 11)^2 - 48(2y^2 - 5y + 2)}}{24}$$

i.e. $$24x = (10y - 11) \pm \sqrt{(4y^2 + 20y + 25)}$$

$$= (10y - 11) \pm (2y + 5)$$

$\therefore \quad 24x = 12y - 6$, i.e. $4x - 2y + 1 = 0$

and $24x = 8y - 16$, i.e. $3x - y + 2 = 0$

are the required lines.

To find the angle between the lines

If θ be the angle between the lines, then

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{|a + b|} = \frac{2\sqrt{25 - 24}}{|12 + 2|} = \frac{1}{7}$$

$\therefore \quad \theta = \tan^{-1}\left(\frac{1}{7}\right).$

Example 21 ***Prove that the equation $8x^2 + 8xy + 2y^2 + 26x + 13y + 15 = 0$ represents a pair of parallel straight lines. Also, find the perpendicular distance between them.***

Sol. Given equation is

$$8x^2 + 8xy + 2y^2 + 26x + 13y + 15 = 0 \qquad ...(i)$$

Writing Eq. (i) as quadratic equation in x, we get

$$8x^2 + 2x(4y + 13) + 2y^2 + 13y + 15 = 0$$

$$\therefore \quad x = \frac{-2(4y + 13) \pm \sqrt{4(4y + 13)^2 - 32(2y^2 + 13y + 15)}}{16}$$

$$\Rightarrow \quad x = \frac{-(4y + 13) \pm \sqrt{(4y + 13)^2 - 8(2y^2 + 13y + 15)}}{8}$$

$$\Rightarrow \quad x = \frac{-(4y + 13) \pm 7}{8}$$

$\Rightarrow \quad 8x = -4y - 13 + 7$, i.e. $4x + 2y + 3 = 0$

and $8x = -4y - 13 - 7$, i.e. $2x + y + 5 = 0$

i.e. the given Eq. (i) represents two straight lines

$$2x + y + 5 = 0$$

and $$4x + 2y + 3 = 0$$

i.e. $$2x + y + \frac{3}{2} = 0$$

both lines are parallel.

$\therefore$ Distance between them $= \frac{\left|5 - \frac{3}{2}\right|}{\sqrt{2^2 + 1^2}} = \frac{7}{2\sqrt{5}}$

Aliter : Here, $\Delta = 8 \times 2 \times 15 + 2 \times \frac{13}{2} \times 13 \times 4 - 8 \times \left(\frac{13}{2}\right)^2 - 2 \times (13)^2 - 15 \times (4)^2 = 0$

and $h^2 = (4)^2 = 16 = 8 \times 2 = ab$

$\therefore$ Given equation

$$8x^2 + 8xy + 2y^2 + 26x + 13y + 15 = 0 \qquad ...(i)$$

represents two parallel straight lines.

Since, $8x^2 + 8xy + 2y^2 = 2(2x + y)^2$

factors of Eq. (i) can be taken as

$$2(2x + y + l)(2x + y + m)$$

$$= 8x^2 + 8xy + 2y^2 + 2(2m + 2l)x + 2(m + l)y + 2lm$$

On comparing, we get $l+m=\frac{13}{2}$ and $lm=\frac{15}{2}$

$\therefore$ Distance between them

$$=\frac{|l-m|}{\sqrt{(2^2+1^2)}}=\frac{\sqrt{(l+m)^2-4lm}}{\sqrt{5}}$$

$$=\frac{\sqrt{\frac{169}{4}-\frac{60}{2}}}{\sqrt{5}}=\frac{7}{2\sqrt{5}}$$

Remark

For comparing coefficients write equation in form

$$\begin{matrix}2x & + & y & + & l\\ 2x & + & y & + & m\end{matrix}$$

coefficient *of* x is $2m+2l$, cofficient of y is $l+m$ and coefficients of constant term is lm.

i.e. $l+m=\frac{13}{2}, lm=\frac{15}{2}$

Example 22 ***Find the combined equation of the straight lines passing through the point (1, 1) and parallel to the lines represented by the equation*** $x^2-5xy+4y^2+x+2y-2=0$.

Sol. Given equation of lines is

$$x^2-5xy+4y^2+x+2y-2=0. \quad ...(i)$$

Since, $x^2-5xy+4y^2=(x-4y)(x-y)$

Factors of Eq. (i) taken as $(x-4y+l)(x-y+m)$.

Now, equation of line through (1, 1) and parallel to

$$x-4y+l=0 \text{ is } x-4y+\lambda=0$$

i.e. $1-4+\lambda=0$

$\therefore$ $\lambda=3$

then line is $x-4y+3=0$ ...(ii)

and equation of line through (1, 1) and parallel to

$$x-y+m=0 \text{ is } x-y+\mu=0$$

i.e. $1-1+\mu=0$

$\therefore$ $\mu=0$

then line is $x-y=0$ ...(iii)

Hence, equation of lines Eqs. (ii) and (iii) is

$$(x-4y+3)(x-y)=0$$

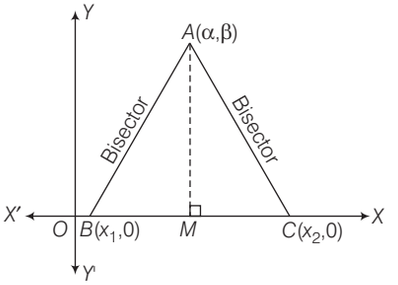
i.e. $x^2-5xy+4y^2+3x-3y=0$

Example 23 ***If*** $ax^2+2hxy+by^2+2gx+2fy+c=0$ ***represents a pair of lines, prove that the area of the triangle formed by their bisectors and axis of x is*** $\frac{\sqrt{(a-b)^2+4h^2}}{|2h|}\cdot\left|\frac{ca-g^2}{ab-h^2}\right|$.

Sol. Given $ax^2+2hxy+by^2+2gx+2fy+c=0$ ...(i)

The point of intersection of the lines given by Eq. (i) are

$$\alpha=\sqrt{\left(\frac{f^2-bc}{h^2-ab}\right)}, \beta=\sqrt{\left(\frac{g^2-ca}{h^2-ab}\right)}$$



Hence, equation of the bisectors of the lines given by Eq. (i) is

$$\frac{(x-\alpha)^2-(y-\beta)^2}{a-b}=\frac{(x-\alpha)(y-\beta)}{h}$$

For X-axis, $y=0$.

$\therefore$ $$\frac{(x-\alpha)^2-\beta^2}{a-b}=\frac{-\beta(x-\alpha)}{h}$$

or $h(x-\alpha)^2+\beta(x-\alpha)(a-b)-h\beta^2=0$ ...(ii)

Eq. (ii) is a quadratic in $(x-\alpha)$ and let two values of x be x_1 and x_2, so that its roots are

$$x_1-\alpha \text{ and } x_2-\alpha$$

$\therefore$ $(x_1-\alpha)+(x_2-\alpha)=$ Sum of roots $=\frac{-\beta(a-b)}{h}$

$(x_1-\alpha)(x_2-\alpha)=$ Products of roots $=-\beta^2$

$\therefore$ $|x_2-x_1|=|(x_2-\alpha)-(x_1-\alpha)|$

$$=\sqrt{[(x_2-\alpha)+(x_1-\alpha)]^2-4(x_2-\alpha)(x_1-\alpha)}$$

$\therefore$ $$|x_2-x_1|=\sqrt{\left\{\frac{\beta^2(a-b)^2}{h^2}+4\beta^2\right\}}$$

$$=\left|\frac{\beta}{h}\right|\sqrt{(a-b)^2+4h^2}$$

$\therefore$ Area of $\Delta ABC=\frac{1}{2}|BC||AM|$

$$=\frac{1}{2}|x_2-x_1||\beta|$$

$$=\frac{1}{2}\left|\frac{\beta}{h}\right|\sqrt{(a-b)^2+4h^2}\times|\beta|$$

$$=\frac{\sqrt{(a-b)^2+4h^2}}{|2h|}\cdot\beta^2$$

$$=\frac{\sqrt{(a-b)^2+4h^2}}{|2h|}\cdot\frac{ca-g^2}{ab-h^2}$$

Exercise for Session 4

1. If $\lambda x^2 + 10xy + 3y^2 - 15x - 21y + 18 = 0$ represents a pair of straight lines. Then, the value of λ is

(a) – 3 (b) 3 (c) 4 (d) – 4

2. The point of intersection of the straight lines given by the equation $3y^2 - 8xy - 3x^2 - 29x + 3y - 18 = 0$ is

(a) $\left(1, \frac{1}{2}\right)$ (b) $\left(1, -\frac{1}{2}\right)$ (c) $\left(-\frac{3}{2}, \frac{5}{2}\right)$ (d) $\left(-\frac{3}{2}, -\frac{5}{2}\right)$

3. If the equation $12x^2 + 7xy - py^2 - 18x + qy + 6 = 0$ represents two perpendicular lines, then the value of p and q are

(a) 12, 1 (b) 12, – 1 (c) 12, $\frac{23}{2}$ (d) 12, $-\frac{23}{2}$

4. If the angle between the two lines represented by $2x^2 + 5xy + 3y^2 + 7y + 4 = 0$ is $\tan^{-1}(m)$, then m is equal to

(a) $-\frac{1}{5}$ (b) $\frac{1}{5}$ (c) $-\frac{3}{5}$ (d) $\frac{3}{5}$

5. The equation of second degree $x^2 + 2\sqrt{2}xy + 2y^2 + 4x + 4\sqrt{2}\,y + 1 = 0$ represents a pair of straight lines, the distance between them is

(a) 2 (b) $2\sqrt{3}$ (c) 4 (d) $4\sqrt{3}$

6. Find the area of the parallelogram formed by the lines
$2x^2 + 5xy + 3y^2 = 0$ and $2x^2 + 5xy + 3y^2 + 3x + 4y + 1 = 0$.

7. Find the locus of the incentre of the triangle formed by
$xy - 4x - 4y + 16 = 0$ and $x + y = a$ ($a > 4, a \neq 4\sqrt{2}$ and a is the parameter).

8. If the equation $2hxy + 2gx + 2fy + c = 0$ represents two straight lines, then show that they form a rectangle of area $\frac{|fg|}{h^2}$ with the coordinate axes.

9. Find the area of the triangle formed by the lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and axis of x.

10. Find the equations of the straight lines passing through the point (1, 1) and parallel to the lines represented by the equation $x^2 - 5xy + 4y^2 + x + 2y - 2 = 0$.

Session 5

To Find the Point of Intersection of Lines Represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ with the Help of Partial Differentiation, Removal of First Degree Terms, Equation of the Lines Joining the Origin to the Points of Intersection of a Given Line and a Given Curve

To Find the Point of Intersection of Lines Represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Let $\phi(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

$\therefore \quad \dfrac{\partial \phi}{\partial x} = 2ax + 2hy + 2g$ [treating y as constant]

and $\quad \dfrac{\partial \phi}{\partial y} = 2hx + 2by + 2f$ [treating x as constant]

For point of intersection $\dfrac{\partial \phi}{\partial x} = 0$ and $\dfrac{\partial \phi}{\partial y} = 0$,

we obtain $\quad ax + hy + g = 0$ and $hx + by + f = 0$

Solving them $\quad \dfrac{x}{fh - bg} = \dfrac{y}{gh - af} = \dfrac{1}{ab - h^2}$

$$(x, y) = \left(\frac{bg - fh}{h^2 - ab}, \frac{af - gh}{h^2 - ab} \right)$$

Working rule In practice, therefore, the general equation of second degree $\phi = 0$, represents a pair of straight lines, we solve its partial derivatives $\dfrac{\partial \phi}{\partial x} = 0, \dfrac{\partial \phi}{\partial y} = 0$ for their intersecting point $(x, y) = (\alpha, \beta)$.

Remembering Method (without use of partial derivatives)

Since, $\quad \Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

from first two rows

$a \quad h \quad g \quad \Rightarrow \quad ax + hy + g = 0$

and $\quad h \quad b \quad f \quad \Rightarrow \quad hx + by + f = 0$ and then solve.

Example 24 Find the point of intersection of lines represented by $2x^2 - 7xy - 4y^2 - x + 22y - 10 = 0$.

Sol. Let $\quad \phi \equiv 2x^2 - 7xy - 4y^2 - x + 22y - 10 = 0$

$\therefore \quad \dfrac{\partial \phi}{\partial x} \equiv 4x - 7y - 1 = 0$

and $\quad \dfrac{\partial \phi}{\partial y} \equiv -7x - 8y + 22 = 0$

then, the point of intersection is $(x, y) = (2, 1)$.

Removal of First Degree Terms

Let point of intersection of lines represented by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \ldots(i)$$

is (α, β).

Here, $(\alpha, \beta) = \left(\dfrac{bg - fh}{h^2 - ab}, \dfrac{af - gh}{h^2 - ab} \right)$

For removal of first degree terms, shift the origin to (α, β).

i.e. Replacing x by $(X + \alpha)$ and y be $(Y + \beta)$ in Eq. (i).

Aliter : Direct equation after removal of first degree terms is

$aX^2 + 2hXY + bY^2 + (g\alpha + f\beta + c) = 0$

where, $\quad \alpha = \dfrac{bg - fh}{h^2 - ab}$

and $\quad \beta = \dfrac{af - gh}{h^2 - ab}$.

Example 25 Find the new equation of curve $12x^2 + 7xy - 12y^2 - 17x - 31y - 7 = 0$ after removing the first degree terms.

Sol. Let $\phi \equiv 12x^2 + 7xy - 12y^2 - 17x - 31y - 7 = 0$ ...(i)

$$\therefore \quad \frac{\partial \phi}{\partial x} = 24x + 7y - 17 = 0$$

and $$\frac{\partial \phi}{\partial y} \equiv 7x - 24y - 31 = 0$$

Their point of intersection is $(x, y) \equiv (1, -1)$

Here, $\alpha = 1, \beta = -1$

Shift the origin to $(1, -1)$ then replacing $x = X + 1$ and $y = Y - 1$ in Eq. (i) the required equation is

$$12(X+1)^2 + 7(X+1)(Y-1) - 12(Y-1)^2 - 17(X+1) - 31(Y-1) - 7 = 0$$

i.e. $12X^2 + 7XY - 12Y^2 = 0$

Aliter :

Here, $\alpha = 1$ and $\beta = -1$

and $$g = -\frac{17}{2}, f = -\frac{31}{2}, c = -7$$

$$\therefore \quad g\alpha + f\beta + c = -\frac{17}{2} \times 1 - \frac{31}{2} \times -1 - 7 = 0$$

$\therefore$ Removed equation is

$$aX^2 + 2hXY + by^2 + (g\alpha + f\beta + c) = 0$$

$$12X^2 + 7XY - 12Y^2 + 0 = 0$$

$$12X^2 + 7XY - 12Y^2 = 0.$$

Example 26 Transform the equation $x^2 + 4xy + y^2 - 2x + 2y + 4 = 0$ into the form $\frac{Y'^2}{b^2} - \frac{X'^2}{a^2} = 1$.

Sol. To remove the first degree terms, we shift the origin to the point (α, β).

Then, $$\alpha = \frac{bg - fh}{h^2 - ab} = \frac{1 \times (-1) - 1 \times 2}{4 - 1 \times 1} = -1$$

and $$\beta = \frac{af - gh}{h^2 - ab} = \frac{1 \times 1 - (-1) \times (2)}{4 - 1 \times 1} = 1$$

then, the transformed equation is

$$X^2 + 4XY + Y^2 + (g\alpha + f\beta + c) = 0$$

$$\Rightarrow X^2 + 4XY + Y^2 + (-1 \times (-1) + 1 \times 1 + 4) = 0$$

$$\therefore \quad X^2 + 4XY + Y^2 + 6 = 0 \qquad ...(i)$$

Now, to remove the XY term from Eq. (i), we rotate the axes through an angle θ given by

$$\cot 2\theta = \frac{a - b}{2h} = 0 \qquad [\because \text{here } a = b]$$

$$\Rightarrow \quad \cot 2\theta = 0$$

$$\Rightarrow \quad 2\theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$

$$\Rightarrow \quad \theta = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}$$

Taking $\theta = \frac{\pi}{4}$ $\left[\text{we can also take } \theta = \frac{3\pi}{4}\right]$

$$\Rightarrow \quad \cos\theta = \frac{1}{\sqrt{2}}, \sin\theta = \frac{1}{\sqrt{2}}$$

Now, if (X', Y') be the coordinates of the point when the axes are rotated through $\theta = \frac{\pi}{4}$, we have

$$X = \frac{X' - Y'}{\sqrt{2}}, \; Y = \frac{X' + Y'}{\sqrt{2}}$$

then Eq. (i) becomes

$$\frac{Y'^2}{6} - \frac{X'^2}{2} = 1 \Rightarrow \frac{Y'^2}{b^2} - \frac{X'^2}{a^2} = 1$$

where, $a^2 = 2$ and $b^2 = 6$.

Equation of the Lines Joining the Origin to the Points of Intersection of a Given Line and a Given Curve

Theorem The combined equation of the straight lines joining the origin to the points of intersection of a second degree curve

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

and a straight line $lx + my + n = 0$ is :

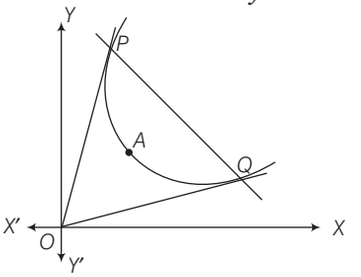
$$ax^2 + 2hxy + by^2 + 2gx\left(\frac{lx + my}{-n}\right) + 2fy\left(\frac{lx + my}{-n}\right) + c\left(\frac{lx + my}{-n}\right)^2 = 0$$

Proof The equation of the curve (PAQ) is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad ...(i)$$

and the equation of the line PQ be

$$lx + my + n = 0 \qquad ...(ii)$$



From the equation of the line Eq. (ii), find the value of '1' in terms of x and y,

i.e. $$\frac{lx + my}{-n} = 1 \qquad ...(iii)$$

Now, the Eq. (i) can be written as

or $ax^2 + 2hxy + by^2 + (2gx + 2fy)(1) + c(1)^2 = 0$

or $ax^2 + 2hxy + by^2 + (2gx + 2fy)\left(\frac{lx + my}{-n}\right) + c\left(\frac{lx + my}{-n}\right)^2 = 0$ ...(iv)

[replacing 1 by $\frac{lx + my}{-n}$ from Eq. (iii)]

Hence, the Eq. (iv) is homogeneous equation of second degree. Above Eq. (iv) on simplification will be of the form $Ax^2 + 2Hxy + By^2 = 0$ and will represent the required straight lines. If θ be the angle between them, then

$$\theta = \tan^{-1}\left(\frac{2\sqrt{(H^2 - AB)}}{|A + B|}\right)$$

Hence, the equation of pairs of straight lines passing through the origin and the points of intersection of a curve and a line is obtained by making the curve homogeneous with the help of the line.

Example 27 Prove that the angle between the lines joining the origin to the points of intersection of the straight line $y = 3x + 2$ with the curve $x^2 + 2xy + 3y^2 + 4x + 8y - 11 = 0$ is $\tan^{-1}\left(\frac{2\sqrt{2}}{3}\right)$.

Sol. Equation of curve is $x^2 + 2xy + 3y^2 + 4x + 8y - 11 = 0$ ...(i)

and line $y = 3x + 2 \Rightarrow \frac{y - 3x}{2} = 1$ ...(ii)

Making Eq. (i) homogeneous with the help of Eq. (ii), then

$$x^2 + 2xy + 3y^2 + 4x\left(\frac{y - 3x}{2}\right) + 8y\left(\frac{y - 3x}{2}\right) - 11\left(\frac{y - 3x}{2}\right)^2 = 0$$

$$\Rightarrow \quad x^2 + 2xy + 3y^2 + 2xy - 6x^2 + 4y^2 - 12xy - \frac{11}{4}(y - 3x)^2 = 0$$

$$\Rightarrow \quad -5x^2 - 8xy + 7y^2 - \frac{11}{4}(y^2 - 6xy + 9x^2) = 0$$

$$\Rightarrow \quad -20x^2 - 32xy + 28y^2 - 11y^2 + 66xy - 99x^2 = 0$$

$$\Rightarrow \quad 119x^2 - 34xy - 17y^2 = 0$$

$$\Rightarrow \quad 7x^2 - 2xy - y^2 = 0 \qquad ...(iii)$$

This is the equation of lines joining the origin to the points of intersection of Eqs. (i) and (ii).

Comparing Eq. (iii) with $ax^2 + 2hxy + by^2 = 0$

$$a = 7, h = -1, b = -1$$

If θ is the acute angle between pair of lines of Eq. (iii), then

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{|a + b|} = \frac{2\sqrt{(1 + 7)}}{|7 - 1|} = \frac{2\sqrt{8}}{6} = \frac{4\sqrt{2}}{6} = \frac{2\sqrt{2}}{3}$$

$$\therefore \quad \theta = \tan^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

Example 28 Find the condition that the pair of straight lines joining the origin to the intersections of the line $y = mx + c$ and the circle $x^2 + y^2 = a^2$ may be at right angles.

Sol. The equations of the line and the circle are

$$y = mx + c \qquad ...(i)$$

and $x^2 + y^2 = a^2$ ...(ii)

The pair of straight lines joining the origin to the intersections of Eqs. (i) and (ii), is obtained by making homogeneous Eq. (ii) with the help of Eq. (i).

$$\because \quad y = mx + c \Rightarrow \frac{y - mx}{c} = 1$$

$$\therefore \quad x^2 + y^2 = a^2(1)^2 \Rightarrow x^2 + y^2 = a^2\left(\frac{y - mx}{c}\right)^2$$

$$\Rightarrow x^2(c^2 - a^2m^2) + 2ma^2xy + y^2(c^2 - a^2) = 0 \qquad ...(iii)$$

The lines given by Eq. (iii), are at right angles, then coefficient of x^2 + coefficient of $y^2 = 0$

$$\Rightarrow \quad c^2 - a^2m^2 + c^2 - a^2 = 0$$

$$\therefore \quad 2c^2 = a^2(1 + m^2)$$

which is the required condition.

Example 29 Prove that the pair of lines joining the origin to the intersection of the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ by the line $lx + my + n = 0$ are coincident, if a $a^2l^2 + b^2m^2 = n^2$.

Sol. The given curve is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ...(i)

and line $lx + my + n = 0$

$$\Rightarrow \quad lx + my = -n$$

$$\Rightarrow \quad \frac{lx + my}{-n} = 1 \qquad ...(ii)$$

Making Eq. (i) homogeneous with the help of Eq. (ii), then

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \left(\frac{lx + my}{-n}\right)^2$$

$$\Rightarrow \quad \frac{n^2x^2}{a^2} + \frac{n^2y^2}{b^2} = l^2x^2 + m^2y^2 + 2lmxy$$

$$\Rightarrow \quad \left(\frac{n^2}{a^2} - l^2\right)x^2 - 2lmxy + \left(\frac{n^2}{b^2} - m^2\right)y^2 = 0 \qquad ...(iii)$$

This is of the form $Ax^2 + 2Hxy + By^2 = 0$,

then $$A = \frac{n^2}{a^2} - l^2,\ H = -lm$$

and $$B = \frac{n^2}{b^2} - m^2$$

The lines given by Eq. (iii) will be coincident, if

$$H^2 - AB = 0 \Rightarrow H^2 = AB$$

$$\Rightarrow \quad l^2m^2 = \left(\frac{n^2}{a^2} - l^2\right)\left(\frac{n^2}{b^2} - m^2\right)$$

$$\Rightarrow \quad l^2m^2 = \frac{n^4}{a^2b^2} - \frac{n^2m^2}{a^2} - \frac{l^2n^2}{b^2} + l^2m^2$$

$$\Rightarrow \quad \frac{n^4}{a^2b^2} = \frac{n^2m^2}{a^2} + \frac{n^2l^2}{b^2}$$

$$\Rightarrow \quad n^2 = b^2m^2 + a^2l^2$$

$$\Rightarrow \quad a^2l^2 + b^2m^2 = n^2$$

Example 30 Show that the straight lines joining the origin to the points of intersection of curves

$$ax^2 + 2hxy + by^2 + 2gx = 0$$

and $$a'x^2 + 2h'xy + b'y^2 + 2g'x = 0$$

are at right angles, if $g'(a+b) = g(a'+b')$.

Sol. The two curves meet in two points and the required lines joining the origin to these points will be obtained by making one equation homogeneous with the help of the other.

$$ax^2 + 2hxy + by^2 + 2gx = 0 \quad \text{...(i)}$$

$$a'x^2 + 2h'xy + b'y^2 + 2g'x = 0 \quad \text{...(ii)}$$

Multiplying Eq. (i) by g' and Eq. (ii) by g and subtracting, we get $(ag' - a'g)x^2 + (2hg' - 2h'g)xy + (bg' - b'g)y^2 = 0$

If the lines are at right angles, then coefficient of x^2 + coefficient of $y^2 = 0$

$\Rightarrow \quad ag' - a'g + bg' - b'g = 0$

$\therefore \quad (a + b)g' = (a' + b')g.$

Exercise for Session 5

1. If the straight lines joining origin to the points of intersection of the line $x + y = 1$ with the curve $x^2 + y^2 + x - 2y - m = 0$ are perpendicular to each other, then the value of m should be

(a) $-\frac{1}{2}$ (b) 0 (c) $\frac{1}{2}$ (d) 1

2. The pair of straight lines joining the origin to the common points of $x^2 + y^2 = 4$ and $y = 3x + c$ are perpendicular, if c^2 is equal to

(a) –1 (b) 6 (c) 13 (d) 20

3. Mixed term xy is to be removed from the general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, one should rotate the axes through an angle θ, then $\tan 2\theta$ is equal to

(a) $\frac{(a - b)}{2h}$ (b) $\frac{2h}{(a + b)}$ (c) $\frac{(a + b)}{2h}$ (d) $\frac{2h}{(a - b)}$

4. The lines joining the origin to the points of intersection of $2x^2 + 3xy - 4x + 1 = 0$ and $3x + y = 1$ are given by

(a) $x^2 - y^2 - 5xy = 0$ (b) $x^2 - y^2 + 5xy = 0$ (c) $x^2 + y^2 - 5xy = 0$ (d) $x^2 + y^2 + 5xy = 0$

5. The equation of the line joining the origin to the point of intersection of the lines $2x^2 + xy - y^2 + 5x - y + 2 = 0$ is

(a) $x + y = 0$ (b) $x - y = 0$ (c) $x - 2y = 0$ (d) $2x + y = 0$

6. Find the equation of the lines joining the origin to the points of intersection of $3x - 2y = 1$ with $3x^2 + 5xy - 3y^2 + 2x + 3y = 0$ and show that they are at right angles.

7. If the straight line joining the origin and the points of intersection of $y = mx + 1$ and $x^2 + y^2 = 1$ be perpendicular to each other, then find the value of m.

8. Prove that the straight lines joining the origin to the points of intersection of the straight line $kx + hy = 2hk$ with the curve $(x - h)^2 + (y - k)^2 = c^2$ are at right angles, if $h^2 + k^2 = c^2$.

9. Show that for all values of λ, the lines joining the origin to the points common to $x^2 + 2hxy - y^2 + gx + fy = 0$ and $fx - gy = \lambda$ are at right angles.

10. Find the equations of the straight lines joining the origin to the points of intersection of $x^2 + y^2 - 4x - 2y = 0$ and $x^2 + y^2 - 2x - 4y = 4$.

Shortcuts and Important Results to Remember

1 If slope of one of the lines represented by $ax^2 + 2hxy + by^2 = 0$ should be n times the slope of the other, then $4nh^2 = ab(1+n)^2$.

2 If the slope of one of the lines represented by $ax^2 + 2hxy + by^2 = 0$ be the nth power of the other, then $(ab^n)^{1/n+1} + (a^n b)^{1/n+1} + 2h = 0$

3 If two of the three lines represented by $ax^3 + bx^2y + cxy^2 + dy^3 = 0$ may be at right angles, then $a^2 + ac + bd + d^2 = 0$

4 If pairs of straight lines $x^2 + 2m_1xy - y^2 = 0$ and $x^2 + 2m_2xy - y^2 = 0$ be such that each pair bisects the angle between the other pair, then $m_1m_2 = -1$

5 If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of parallel lines, then

(i) $h^2 = ab$, $bg^2 = af^2$.

(ii) the distance between them $= 2\sqrt{\dfrac{(g^2 - ac)}{a(a+b)}}$.

6 If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and $ax^2 + 2hxy + by^2 - 2gx - 2fy + c = 0$ each represent a pair of lines, then the area of the parallelogram enclosed by them is $\dfrac{2|c|}{\sqrt{(h^2 - ab)}}$.

JEE Type Solved Examples :
Single Option Correct Type Questions

■ This section contains **6 multiple choice examples**. Each example has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

● **Ex. 1** *If the pairs of lines* $x^2 + 2xy + \lambda y^2 = 0$ *and* $\lambda x^2 + 2xy + y^2 = 0$ *have exactly one line in common, then the joint equation of the other two lines is given by*

(a) $3x^2 + 8xy - 3y^2 = 0$ (b) $3x^2 + 10xy + 3y^2 = 0$

(c) $x^2 + 2xy - 3y^2 = 0$ (d) $3x^2 + 2xy - y^2 = 0$

Sol. (b) Let $y = mx$, be a line common to the given pairs of lines, then

$$\lambda m^2 + 2m + 1 = 0 \text{ and } m^2 + 2m + \lambda = 0$$

$$\Rightarrow \quad \frac{m^2}{2(\lambda - 1)} = \frac{m}{(1 - \lambda^2)} = \frac{1}{2(\lambda - 1)}$$

$$\Rightarrow \quad m^2 = 1 \text{ and } m = -\frac{1}{2}(\lambda + 1)$$

$$\Rightarrow \quad (\lambda + 1)^2 = 4 \Rightarrow \lambda + 1 = 2, -2$$

$$\therefore \quad \lambda = 1 \text{ or } -3$$

But for $\lambda = 1$, the two pairs have both the lines common.

So, $\lambda = -3$ and the slope m of the line common to both the pairs is 1. Now,

$$x^2 + 2xy + \lambda y^2 = x^2 + 2xy - 3y^2 = (x - y)(x + 3y)$$

and $\lambda x^2 + 2xy + y^2 = -3x^2 + 2xy + y^2 = -(x - y)(3x + y)$

Hence, the joint equation of other two lines is

$$(x + 3y)(3x + y) = 0 \text{ or } 3x^2 + 10xy + 3y^2 = 0.$$

● **Ex. 2** *The combined equation of the lines* l_1 *and* l_2 *is* $2x^2 + 6xy + y^2 = 0$ *and that of the lines* m_1 *and* m_2 *is* $4x^2 + 18xy + y^2 = 0$. *If the angle between* l_1 *and* m_2 *is* α, *then the angle between* l_2 *and* m_1 *will be*

(a) $\frac{\pi}{2} - \alpha$ (b) $\frac{\pi}{4} + \alpha$ (c) α (d) 2α

Sol. (c) The combined equation of the bisectors of the angles between the lines of the first pair is

$$\frac{x^2 - y^2}{2 - 1} = \frac{xy}{3} \Rightarrow x^2 - y^2 = \frac{1}{3}xy$$

and the combined equation of the bisectors of the angles between the lines of the second pair is

$$\frac{x^2 - y^2}{4 - 1} = \frac{xy}{9} \Rightarrow x^2 - y^2 = \frac{1}{3}xy$$

It is clear that the two pairs are equally inclined to each other.

Hence, the angle between l_2 and m_1 is α.

● **Ex. 3** *The pair of lines* $\sqrt{3}x^2 - 4xy + \sqrt{3}y^2 = 0$ *are rotated about the origin by* $\frac{\pi}{6}$ *in the anti-clockwise sense. The equation of the pair in the new position is*

(a) $x^2 - \sqrt{3}xy = 0$ (b) $y^2 - \sqrt{3}xy = 0$

(c) $\sqrt{3}x^2 - xy = 0$ (d) $\sqrt{3}y^2 - xy = 0$

Sol. (c) The given equation of pair of straight lines can be written as

$$(\sqrt{3}x - y)(x - \sqrt{3}y) = 0$$

$$\Rightarrow \quad y = \sqrt{3}x \text{ and } y = \frac{1}{\sqrt{3}}x$$

$$\Rightarrow \quad y = x \tan 60^\circ \text{ and } y = x \tan 30^\circ$$

After rotation, the separate equations are

$$y = x \tan 90^\circ \text{ and } y = x \tan 60^\circ$$

$$\Rightarrow \quad x = 0 \text{ and } y = x\sqrt{3}$$

Hence, the combined equation in the new position is

$$x(\sqrt{3}x - y) = 0$$

or $$\sqrt{3}x^2 - xy = 0.$$

● **Ex. 4** *If the pair of lines* $ax^2 - 2xy + by^2 = 0$ *and* $bx^2 - 2xy + ay^2 = 0$ *be such that each pair bisects the angle between the other pair, then* $|a - b|$ *equals to*

(a) 1 (b) 2

(c) 3 (d) 4

Sol. (b) According to the example, the equation of the bisectors of the angle between the lines

$$ax^2 - 2xy + by^2 = 0 \quad \text{...(i)}$$

is $$bx^2 - 2xy + ay^2 = 0 \quad \text{...(ii)}$$

∴ The equation of bisectors of the angle between the lines Eq. (i) is

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{-1}$$

$$\Rightarrow \quad x^2 + (a - b)xy - y^2 = 0 \quad \text{...(iii)}$$

Since, Eqs. (ii) and (iii) are identical, comparing Eqs. (ii) and (iii), we get

$$\frac{b}{1} = \frac{-2}{a - b} = \frac{a}{-1}$$

$$\Rightarrow \quad (a - b)b = -2$$

and $$(a - b)a = 2$$

$$\therefore \quad (a - b)^2 = 4$$

or $$|a - b| = 2$$

● **Ex. 5** *The equation of line which is parallel to the line common to the pair of lines given by* $3x^2 + xy - 4y^2 = 0$ *and* $6x^2 + 11xy + 4y^2 = 0$ *and at a distance of* 2 *units from it is*

(a) $3x - 4y = -10$ (b) $x - y = 2$

(c) $3x + 4y = 10$ (d) $2x + y = -2$

Sol. (c) We have, $3x^2 + xy - 4y^2 = 0$

or $(x - y)(3x + 4y) = 0$ ...(i)

and $6x^2 + 11xy + 4y^2 = 0$

or $(2x + y)(3x + 4y) = 0$ ...(ii)

Equation of line common to Eqs. (i) and (ii) is

$$3x + 4y = 0 \quad \text{...(iii)}$$

Equation of any line parallel to Eq. (iii) is

$$3x + 4y = \lambda$$

Since, its distance from Eq. (iii) is 2, we have

$$\frac{|\lambda - 0|}{\sqrt{(3^2 + 4^2)}} = 2 \text{ or } \lambda = \pm 10$$

Hence, required lines are $3x + 4y = \pm 10$.

● **Ex. 6** *The lines joining the origin to the points of intersection of* $x^2 + y^2 + 2gx + c = 0$ *and* $x^2 + y^2 + 2fy - c = 0$ *are at right angles, if*

(a) $g^2 + f^2 = c$ (b) $g^2 - f^2 = c$

(c) $g^2 - f^2 = 2c$ (d) $g^2 + f^2 = c^2$

Sol. (c) Given,

$$x^2 + y^2 + 2gx + c = 0 \quad \text{...(i)}$$

and $x^2 + y^2 + 2fy - c = 0$ ...(ii)

On subtracting Eq. (ii) from Eq. (i), we get

$$2gx - 2fy + 2c = 0$$

or $$\frac{fy - gx}{c} = 1 \quad \text{...(iii)}$$

On adding Eqs. (i) and (ii), we get

$$2(x^2 + y^2 + gx + fy) = 0$$

or $x^2 + y^2 + gx + fy = 0$ ...(iv)

Homogenising Eq. (iv) with the help of Eq. (iii), then

$$x^2 + y^2 + (gx + fy)\left(\frac{fy - gx}{c}\right) = 0$$

∴ The lines will be at right angles, when

$$\left(1 - \frac{g^2}{c}\right) + \left(1 + \frac{f^2}{c}\right) = 0$$

$$\Rightarrow \quad g^2 - f^2 = 2c$$

JEE Type Solved Examples :
More than One Correct Option Type Questions

■ This section contains **3 multiple choice examples**. Each example has four choices (a), (b), (c) and (d) out of which **MORE THAN ONE** may be correct.

● **Ex. 7** *The lines joining the origin to the point of intersection of* $3x^2 + mxy - 4x + 1 = 0$ *and* $2x + y - 1 = 0$ *are at right angles. Which of the following is/are possible value of m?*

(a) -4 (b) 3 (c) 4 (d) 7

Sol. (a,b,c,d) Given line is $2x + y = 1$ ...(i)

and curve is $3x^2 + mxy - 4x + 1 = 0$ ...(ii)

Homogenising Eq. (ii) with the help of Eq. (i), then

$$3x^2 + mxy - 4x(2x + y) + (2x + y)^2 = 0$$

or $$-x^2 + mxy + y^2 = 0$$

the lines are at right angles as $a + b = 0$, when $h^2 > ab$

i.e. $$\frac{m^2}{4} + 1 > 0$$

which is true for all $m \in R$.

● **Ex. 8** *The lines* $(lx + my)^2 - 3(mx - ly)^2 = 0$ *and* $lx + my + n = 0$ *form*

(a) an isosceles triangle (b) a right angled triangle

(c) an equilateral triangle (d) None of these

Sol. (a,c) $(lx + my)^2 - 3(mx - ly)^2 = 0$

$$\Rightarrow (l^2 - 3m^2)x^2 + 8mlxy + (m^2 - 3l^2)y^2 = 0$$

$$\Rightarrow \{(l + m\sqrt{3})x + (m - l\sqrt{3})y\}\{(l - \sqrt{3}m)x + (m + \sqrt{3}l)y\} = 0$$

Let slope of $(l + m\sqrt{3})x + (m - l\sqrt{3})y = 0$ be m_1 and slope of $(l - m\sqrt{3})x + (m + l\sqrt{3})y = 0$ be m_2 and slope of $lx + my + n = 0$ is m_3.

Now, $$\tan\theta_1 = \left|\frac{m_3 - m_1}{1 + m_3 m_1}\right| = \left|\frac{-\frac{l}{m} + \frac{l + m\sqrt{3}}{m - l\sqrt{3}}}{1 + \frac{l}{m}\cdot\left(\frac{l + m\sqrt{3}}{m - l\sqrt{3}}\right)}\right|$$

$$= \left|\frac{\sqrt{3}(l^2 + m^2)}{(l^2 + m^2)}\right| = \sqrt{3}$$

$$\Rightarrow \quad \theta_1 = 60^\circ$$

Similarly, $\tan\theta_2 = \left|\frac{m_2 - m_1}{1 + m_2 m_1}\right| = \sqrt{3}$

$\Rightarrow \quad \theta_2 = 60°$

Hence, an equilateral triangle is formed which is also an isosceles one.

● ***Ex. 9*** *If the equation $ax^2 - 6xy + y^2 + bx + cy + d = 0$ represents a pair of lines whose slopes are m and m^2, then the value(s) of a is/are*

(a) -27 (b) -8
(c) 8 (d) 27

Sol. (a,c) $\because$ m and m^2 are the roots of the equation

$$\left(\frac{y}{x}\right)^2 - 6\left(\frac{y}{x}\right) + a = 0$$

$\therefore \quad m + m^2 = 6$ and $m \cdot m^2 = a$ ...(i)

Now, $(m + m^2)^3 = (6)^3$

$\Rightarrow m^3 + m^6 + 3m \cdot m^2 (m + m^2) = 216$

$\Rightarrow \quad a + a^2 + 3a(6) = 216$ [from Eq. (i)]

$\Rightarrow \quad a^2 + 19a - 216 = 0$

$\Rightarrow \quad (a + 27)(a - 8) = 0$

$\therefore \quad a = -27, 8$

JEE Type Solved Examples : Paragraph Based Questions

■ This section contains **one solved paragraph** based upon each paragraph **2 multiple choice questions** have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Paragraph

(Q. Nos. 10 to11)

Consider the equation of a pair of straight lines as $\lambda xy - 8x + 9y - 12 = 0$

10. The value of λ is

(a) 0 (b) 2
(c) 4 (d) 6

11. The point of intersection of lines is (α, β), then the equation whose roots are α, β, is

(a) $4x^2 + x - 8 = 0$ (b) $6x^2 + x - 12 = 0$
(c) $4x^2 - x - 8 = 0$ (d) $6x^2 - x - 12 = 0$

Sol. Given equation is,

$$\lambda xy - 8x + 9y - 12 = 0$$

Here, $a = 0, b = 0, c = -12,$

$$f = \frac{9}{2}, g = -4 \text{ and } h = \frac{\lambda}{2}$$

10. (d) For $\Delta = 0$,

$$0 + 2 \times \frac{9}{2} \times -4 \times \frac{\lambda}{2} - 0 - 0 + 12 \times \frac{\lambda^2}{4} = 0$$

$\Rightarrow \quad 3\lambda^2 - 18\lambda = 0$

$\therefore \quad \lambda = 0, 6$

Hence, $\lambda = 6$ [$\because$ for $\lambda = 0$, it will give an equation of first degree]

11. (b) Let $f(x,y) = 6xy - 8x + 9y - 12$ [$\because \lambda = 6$]

$\therefore \quad \frac{\partial f}{\partial x} = 6y - 8$ and $\frac{\partial f}{\partial y} = 6x + 9$

For point of intersection $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$, we get

$$x = -\frac{3}{2}, y = \frac{4}{3}$$

$\therefore \quad \alpha = -\frac{3}{2}$ and $\beta = \frac{4}{3}$

Hence, required equation is

$$x^2 - \left(-\frac{3}{2} + \frac{4}{3}\right)x - 2 = 0$$

$\Rightarrow \quad 6x^2 + x - 12 = 0$

JEE Type Solved Examples :
Single Integer Answer Type Questions

- This section contains **2 examples**. The answer to each example is **a single digit integer**, ranging from 0 to 9 (both inclusive).

● ***Ex. 12*** *If the sum of the slopes of the lines given by* $x^2 - 2cxy - 7y^2 = 0$ *is four times their product, then the value of* c *is*

Sol. (2) Given, $m_1 + m_2 = 4m_1m_2$

$$\Rightarrow \quad -\frac{2h}{b} = 4 \cdot \frac{a}{b} \Rightarrow h = -2a$$

$$\therefore \quad -c = -2 \times 1 \Rightarrow c = 2$$

● ***Ex. 13*** *If one of the lines given by* $6x^2 - xy + 4cy^2 = 0$ *is* $3x + 4y = 0$, *then the value of* $|c|$ *is*

Sol. (3) $\because 3x + 4y = 0$ is one of the lines given by $6x^2 - xy + 4cy^2 = 0$, then

$$6x^2 - x\left(-\frac{3x}{4}\right) + 4c\left(-\frac{3x}{4}\right)^2 = 0 \qquad \left[\because y = -\frac{3x}{4}\right]$$

$$\Rightarrow \quad 6 + \frac{3}{4} + \frac{9c}{4} = 0 \; ; \quad \therefore \; c = -3$$

Hence, $|c| = 3$

JEE Type Solved Examples :
Matching Type Questions

- This section contains **one solved example**. This example has four statements (A, B, C and D) is **Column I** and four statements (p, q, r and s) in **Column II**. Any given statements in **Column I** can have correct matching with one or more statement(s) given in **Column II**.

● ***Ex. 14*** *Match the following*

	Column I		Column II
(A)	If the slope of one of the lines represented by $ax^2 + 2hxy + by^2 = 0$ is the square of the other, then $\frac{a+b}{h} + \frac{8h^2}{ab}$ is a	(p)	Odd prime number
(B)	The product of perpendiculars drawn from the point (1, 2) to the pair of lines $x^2 + 4xy + y^2 = 0$ is λ units, then $[\lambda]$ is a (where $[\cdot]$ denotes the greatest integer function).	(q)	Composite number
(C)	Distance between two lines represented by the line pair $x^2 - 4xy + 4y^2 + x - 2y - 6 = 0$ is λ unit, then $[\lambda]$ is a (where $[\cdot]$ denotes the greatest integer function).	(r)	Even prime number
(D)	If the pairs $a_1x^2 + 2h_1xy + b_1y^2 = 0$ and $a_2x^2 + 2h_2xy + b_2y^2 = 0$ have one line common, then $\frac{(a_1b_2 - a_2b_1)^2}{(b_1h_2 - b_2h_1)(a_2h_1 - a_1h_2)}$ is a	(s)	Perfect number

Sol. (A) $\rightarrow$ (q, s); (B) $\rightarrow$ (p); (C) $\rightarrow$ (r); (D) $\rightarrow$ (q)

(A) Let m and m^2 be the slopes of the lines represented by $ax^2 + 2hxy + by^2 = 0$, then

$$m + m^2 = -\frac{2h}{b} \qquad \text{...(i)}$$

and $$m \cdot m^2 = \frac{a}{b} \qquad \text{...(ii)}$$

From Eq. (i), $(m + m^2)^3 = \left(-\frac{2h}{b}\right)^3$

$$\Rightarrow m^3 + m^6 + 3m \cdot m^2(m + m^2) = -\frac{8h^3}{b^3}$$

$$\Rightarrow \frac{a}{b} + \frac{a^2}{b^2} + \frac{3a}{b}\left(-\frac{2h}{b}\right) = -\frac{8h^3}{b^3} \quad \text{[from Eqs. (i) and (ii)]}$$

$$\Rightarrow \quad ab + a^2 - 6ah = -\frac{8h^3}{b}$$

$$\therefore \quad \frac{a+b}{h} + \frac{8h^2}{ab} = 6$$

(B) Let $y = m_1x$ and $y = m_2x$ lines represented by $x^2 + 4xy + y^2 = 0$, then $m_1 + m_2 = -4$, $m_1m_2 = 1$. Then, product of perpendiculars drawn from the point (1, 2)

$$= \frac{|(2 - m_1)(2 - m_2)|}{\sqrt{(1 + m_1^2)}\sqrt{(1 + m_2^2)}}$$

$$= \frac{|4 - 2(m_1 + m_2) + m_1m_2|}{\sqrt{1 + (m_1 + m_2)^2 - 2m_1m_2 + (m_1m_2)^2}}$$

$$= \frac{|4 + 8 + 1|}{\sqrt{(1 + 16 - 2 + 1)}} = \frac{13}{4} \text{ units}$$

$\because \quad \lambda = \frac{13}{4}$

$\therefore \quad [\lambda] = 3$

(C) Given, line pair is $x^2 - 4xy + 4y^2 + x - 2y - 6 = 0$

$\Rightarrow \quad (x - 2y)^2 + (x - 2y) - 6 = 0$

$\Rightarrow \quad (x - 2y + 3)(x - 2y - 2) = 0$

$\therefore$ Lines are $x - 2y + 3 = 0$ and $x - 2y - 2 = 0$

Hence, distance between lines $= \dfrac{|3 - (-2)|}{\sqrt{1 + 4}} = \sqrt{5}$ unit

$\therefore \quad \lambda = \sqrt{5}$

Hence, $[\lambda] = [\sqrt{5}] = 2$

(D) Let $y = mx$ be the common line, then

$b_1m^2 + 2h_1m + a_1 = 0$ and $b_2m^2 + 2h_2m + a_2 = 0$.

$$\Rightarrow \quad \begin{vmatrix} b_1 & 2h_1 \\ b_2 & 2h_2 \end{vmatrix} \times \begin{vmatrix} 2h_1 & a_1 \\ 2h_2 & a_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2$$

$$\Rightarrow \quad 4(b_1h_2 - b_2h_1)(a_2h_1 - a_1h_2) = (a_1b_2 - a_2b_1)^2$$

$$\therefore \quad \frac{(a_1b_2 - a_2b_1)^2}{(b_1h_2 - b_2h_1)(a_2h_1 - a_1h_2)} = 4$$

JEE Type Solved Examples :
Statement I and II Type Questions

■ **Directions** (Ex. Nos. 15-16) Each of these examples contains two statements.

Statement I (Assertion) and

Statement II (Reason)

Each of these examples also has four alternative choices, only one of which is the correct answer. You have to select the correct choice, as given below :

(a) Statement I is true, Statement II is true and Statement II is a correct explanation for Statement I

(b) Statement I is true, Statement II is true and Statement II is not a correct explanation for Statement I

(c) Statement I is true, Statement II is false

(d) Statement I is false, Statement II is true

Ex. 15 **Statement I** *The combined equation of l_1, l_2 is $3x^2 + 6xy + 2y^2 = 0$ and that of m_1, m_2 is $5x^2 + 18xy + 2y^2 = 0$. If angle between l_1, m_2 is θ, then angle between l_2, m_1 is θ.*

Statement II *If the pairs of lines $l_1l_2 = 0, m_1m_2 = 0$ are equally inclinded that angle between l_1 and m_2 = angle between l_2 and m_1.*

Sol. (a) The pair of bisectors of $3x^2 + 6xy + 2y^2 = 0$ is

$\dfrac{x^2 - y^2}{3 - 2} = \dfrac{xy}{3}$, i.e. $3x^2 - xy - 3y^2 = 0$ and pair of bisectors of

$5x^2 + 18xy + 2y^2 = 0$ is $\dfrac{x^2 - y^2}{5 - 2} = \dfrac{xy}{9}$

i.e. $3x^2 - xy - 3y^2 = 0$ are coincides.

$\therefore$ Angle between l_1, m_2 is same as angle between l_2, m_1.

$\therefore$ Both statements are true and Statement II is a correct explanation for Statement I.

Ex. 16 **Statement I** *The equation $2x^2 - 3xy - 2y^2 + 5x - 5y + 3 = 0$ represents a pair of perpendicular straight lines.*

Statement II *A pair of lines given by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are perpendicular if $a + b = 0$.*

Sol. (d) Here, $\Delta \neq 0$, $h^2 > ab$ and $a + b = 0$

$\therefore \quad 2x^2 - 3xy - 2y^2 + 5x - 5y + 3 = 0$

represents a rectangular hyperbola.

$\therefore$ Statement I is false and Statement II is true.

Subjective Type Examples

■ In this section, there are **10 subjective solved examples**.

● ***Ex. 17*** *If the lines represented by $2x^2 - 5xy + 2y^2 = 0$ be the two sides of a parallelogram and the line $5x + 2y = 1$ be one of its diagonal. Find the equation of the other diagonal, and area of the parallelogram.*

Sol. Given pair of lines, $2x^2 - 5xy + 2y^2 = 0$

i.e. $\quad (x - 2y)(2x - y) = 0$

$\therefore \quad x - 2y = 0$ and $2x - y = 0$

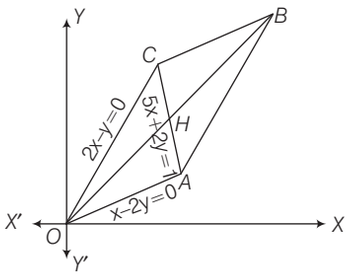
and the diagonal $5x + 2y = 1$ does not pass through origin, hence it is AC.

On solving, $x - 2y = 0$ and $5x + 2y = 1$, we get

$$A = \left(\frac{1}{6}, \frac{1}{12}\right)$$

and solving $2x - y = 0$ and $5x + 2y = 1$, we get

$$C \equiv \left(\frac{1}{9}, \frac{2}{9}\right).$$



Since, diagonals of parallelogram bisect each other, if bisect at H.

Then, $$H \equiv \left\{\frac{1}{2}\left(\frac{1}{6}+\frac{1}{9}\right), \frac{1}{2}\left(\frac{1}{12}+\frac{2}{9}\right)\right\}$$

i.e. $$H = \left(\frac{5}{36}, \frac{11}{72}\right)$$

Hence, the equation of other diagonal which passes through O and H is

$$y - 0 = \frac{\frac{11}{72} - 0}{\frac{5}{36} - 0}(x - 0)$$

$$\Rightarrow \quad y = \frac{11}{10}x$$

$$\Rightarrow \quad 11x - 10y = 0$$

Now, area of parallelogram = $2 \times$ Area of ΔOAC

$$= 2 \times \frac{1}{2}\left|\frac{1}{6} \times \frac{2}{9} - \frac{1}{9} \times \frac{1}{12}\right| = \frac{1}{36} \text{ sq units}$$

● ***Ex. 18*** *Prove that the equation* $(a + 2h + b)x^2 - 2(a - b)xy + (a - 2h + b)y^2 = 0$ *represents a pair of lines each inclined at an angle of* $45°$ *to one or other of the lines given by,* $ax^2 + 2hxy + by^2 = 0$.

Sol. Given equation is

$$ax^2 + 2hxy + by^2 = 0 \quad \ldots(i)$$

Let the lines represented by Eq. (i) are

$$y - m_1 x = 0 \quad \ldots(ii)$$

and $$y - m_2 x = 0 \quad \ldots(iii)$$

therefore, $m_1 + m_2 = -\frac{2h}{b}, m_1 m_2 = \frac{a}{b}$

If $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$, the equation of lines through the origin making $45°$ with the lines Eqs. (ii) and (iii) will be

$$y - x\tan(\theta_1 - 45°) = 0$$

and $$y - x\tan(\theta_2 - 45°) = 0$$

Their combined equation is

$$\{y - x\tan(\theta_1 - 45°)\}\{y - x\tan(\theta_2 - 45°)\} = 0$$

$$\Rightarrow \quad \left(y - \frac{m_1 - 1}{1 + m_1}x\right)\left(y - \frac{m_2 - 1}{1 + m_2}x\right) = 0$$

$$\Rightarrow \quad \{(1 + m_1)y - (m_1 - 1)x\}\{(1 + m_2)y - (m_2 - 1)x\} = 0$$

$$\Rightarrow \quad (1 + m_1)(1 + m_2)y^2 - xy\{(1 + m_1)(m_2 - 1) + (m_1 - 1)(1 + m_2)\} + (m_1 - 1)(m_2 - 1)x^2 = 0$$

$$\Rightarrow \quad (1 + m_1 + m_2 + m_1 m_2)y^2 - 2xy(m_1 m_2 - 1) + \{m_1 m_2 - (m_1 + m_2) + 1\}x^2 = 0$$

$$\Rightarrow \quad \left(1 - \frac{2h}{b} + \frac{a}{b}\right)y^2 - 2xy\left(\frac{a}{b} - 1\right) + \left(\frac{a}{b} + \frac{2h}{b} + 1\right)x^2 = 0$$

$$\therefore \quad (a + 2h + b)x^2 - 2(a - b)xy + (a - 2h + b)y^2 = 0$$

● ***Ex. 19*** *If* $u \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ *represents a pair of straight lines, prove that the equation of the third pair of straight lines passing through the points where these meet the axes is* $cu + 4(fg - ch)xy = 0$.

Sol. $u \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

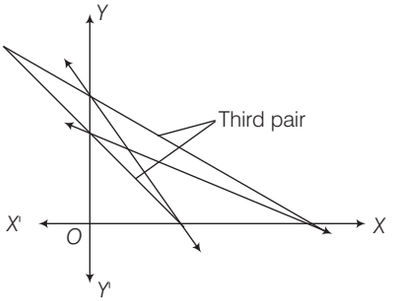
$\because$ u represents a pair of straight line, then $\Delta = 0$

$$\therefore \quad abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \quad \ldots(i)$$

Combined equation of axes is $xy = 0$

Now, the curve through the intersection of $u = 0$ and $xy = 0$ is

$$u + \lambda xy = 0 \quad \ldots(ii)$$



i.e. $$ax^2 + 2hxy + by^2 + 2gx + 2fy + c + \lambda xy = 0$$

$$\Rightarrow \quad ax^2 + xy(2h + \lambda) + by^2 + 2gx + 2fy + c = 0$$

If it represents a pair of lines, then

$$abc + 2fg\left(h + \frac{\lambda}{2}\right) - af^2 - bg^2 - c\left(h + \frac{\lambda}{2}\right)^2 = 0$$

$$\Rightarrow (abc + 2fgh - af^2 - bg^2 - ch^2) + \lambda(fg - ch) - \frac{c\lambda^2}{4} = 0$$

$$\Rightarrow \quad 0 + \lambda(fg - ch) - \frac{c\lambda^2}{4} = 0 \quad \text{[from Eq. (i)]}$$

$$\therefore \quad \lambda = \frac{4(fg - ch)}{c} \quad \ldots(iii)$$

Hence, the equation of third pair from Eqs. (ii) and (iii) is

$$u + \frac{4(fg - ch)}{c}xy = 0$$

$$\therefore \quad cu + 4(fg - ch)xy = 0$$

● **Ex. 20** *If the equation* $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ *represents a pair of parallel lines, prove that*
(i) $h = \sqrt{ab}$ *and* $g\sqrt{b} = f\sqrt{a}$ *or* $(h = -\sqrt{ab}$ *and* $g\sqrt{b} = -f\sqrt{a})$.
(ii) the distance between them is $2\sqrt{\left(\frac{g^2 - ac}{a(a+b)}\right)}$.

Sol. Given equation is,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \ldots(i)$$

Let the equation of the parallel lines represented by Eq. (i) be

$$lx + my + n = 0 \quad \text{and} \quad lx + my + n_1 = 0.$$

Then, $(lx + my + n)(lx + my + n_1)$

$$\equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c$$

Equating the coefficients, we get

$$l^2 = a \quad \ldots(ii)$$

$$m^2 = b \quad \ldots(iii)$$

$$nn_1 = c \quad \ldots(iv)$$

$$2lm = 2h \quad \ldots(v)$$

$$(n + n_1)l = 2g \quad \ldots(vi)$$

$$m(n + n_1) = 2f \quad \ldots(vii)$$

Eq. (i) From Eq. (v)

$\Rightarrow \quad h = lm = \pm\sqrt{ab}$

Now, $\quad h = \sqrt{ab}$ or $h = -\sqrt{ab}$

$\because \quad abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$\Rightarrow abc + 2fg\sqrt{ab} - af^2 - bg^2 - c \cdot ab = 0$

[substituting the value of $h = \sqrt{ab}$]

$\Rightarrow \quad -(f\sqrt{a} - g\sqrt{b})^2 = 0 \Rightarrow f\sqrt{a} = g\sqrt{b}$

Thus, the given equation represents a pair of lines.

Also, if $h = -\sqrt{ab}$, then $g\sqrt{b} = -f\sqrt{a}$

(ii) The distance between parallel lines

$$= \frac{|n - n_1|}{\sqrt{(l^2 + m^2)}} = \frac{\sqrt{(n + n_1)^2 - 4nn_1}}{\sqrt{l^2 + m^2}}$$

$$= \frac{\sqrt{\left(\frac{4g^2}{l^2} - 4c\right)}}{\sqrt{(a+b)}} \quad \text{[from Eqs. (vi), (ii) and (iii)]}$$

$$= 2\sqrt{\left(\frac{g^2 - ac}{a(a+b)}\right)} \quad [\because l^2 = a]$$

Remark

In some books, the conditions for parallel lines are stated as $h^2 = ab$ and $bg^2 = af^2$ we show that by an example that these conditions are not sufficient because

$$h^2 = ab \Rightarrow h = \pm\sqrt{ab}$$

and $\quad bg^2 = af^2 \Rightarrow g\sqrt{b} = \pm f\sqrt{a}$

$\Rightarrow \quad h = \sqrt{ab}, g\sqrt{b} = f\sqrt{a}$ or $(h = -\sqrt{ab}, g\sqrt{b} = -f\sqrt{a})$

Consider, for example, $4x^2 + 4xy + y^2 + 4x - 2y + 5 = 0$

Here, $a = 4, h = 2, b = 1, g = 2, f = -1, c = 5$ and

$h = \sqrt{ab}, g\sqrt{b} = -f\sqrt{a}$

But $abc + 2fgh - af^2 - bg^2 - ch^2$

$= 4 \times 1 \times 5 + 2 \times (-1) \times 2 \times 2 - 4 \times (-1)^2 - 1 \times (2)^2 - 5 \times (2)^2$

$= 20 - 8 - 4 - 4 - 20 = -16 \neq 0$

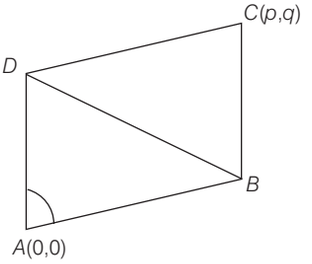
Hence, the Eq. (i) does not represent a pair of lines at all.

● **Ex. 21** *A parallelogram is formed by the lines* $ax^2 + 2hxy + by^2 = 0$ *and the lines through* (p, q) *parallel to them. Show that the equation of the diagonal of the parallelogram which does not pass through origin is* $(2x - p)(ap + hq) + (2y - q)(hp + bq) = 0$.
Show also that the area of the parallelogram is $|(ap^2 + 2hpq + bq^2)| / 2\sqrt{(h^2 - ab)}$.

Sol. The combined equation of AB and AD is

$$S_1 \equiv ax^2 + 2hxy + by^2 = 0$$

Now, equation of lines through (p, q) and parallel to $S_1 \equiv 0$ is $S_2 \equiv a(x - p)^2 + 2h(x - p)(y - q) + b(y - q)^2 = 0$



Hence, the equation of diagonal BD is $S_1 - S_2 = 0$

$$a(-2xp + p^2) + 2h(-py - qx + pq) + b(-2qy + q^2) = 0$$

$$-ap(2x - p) - hq(2x - p) - hp(2y - q) - bq(2y - q) = 0$$

[$\because 2hpq$ is written as $hpq + hpq$]

Hence, diagonal of BD is

$$(2x - p)(ap + hq) + (2y - q)(bq + hp) = 0$$

Area of parallelogram $= \frac{p_1 p_2}{\sin\theta}$

$\Rightarrow p_1 p_2$ = product of perpendiculars from (p, q) on AB and AD whose combined equation is

$$ax^2 + 2hxy + by^2 = 0$$

$$\therefore \quad p_1 p_2 = \frac{|ap^2 + 2hpq + bq^2|}{\sqrt{\{(a-b)^2 + 4h^2\}}}$$

$$\therefore \quad \tan\theta = \frac{2\sqrt{(h^2 - ab)}}{|(a+b)|}$$

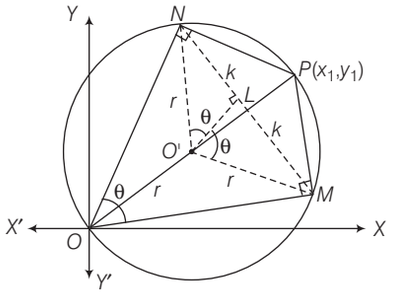
$$\therefore \quad \sin\theta = \frac{2\sqrt{(h^2 - ab)}}{\sqrt{(a-b)^2 + 4h^2}}$$

$$\therefore \text{ Required area} = \frac{|ap^2 + 2hpq + bq^2|}{2\sqrt{(h^2 - ab)}}$$

● ***Ex. 22*** *A point moves so that the distance between the feet of perpendiculars drawn from it to the lines* $ax^2 + 2hxy + by^2 = 0$ *is a constant* $2k$. *Show that the equation of its locus is* $(x^2 + y^2)(h^2 - ab) = k^2\{(a-b)^2 + 4h^2\}$.

Sol. $\because \angle ONP = \angle OMP = \frac{\pi}{2}$

$\therefore$ O, M, P, N are concyclic with diameter OP.



Let $P \equiv (x_1, y_1)$

$\therefore$ Coordinates of centre are $O' = \left(\frac{x_1}{2}, \frac{y_1}{2}\right)$

$\therefore$ Radius $r = O'N = OM = OO'$

$$= \sqrt{\left(\frac{x_1}{2}\right)^2 + \left(\frac{y_1}{2}\right)^2} = \frac{\sqrt{(x_1^2 + y_1^2)}}{2}$$

Let $\angle MON = \theta$

$\therefore \quad \angle MO'N = 2\theta \Rightarrow \angle NO'L = \angle MO'L = \theta$

and given $MN = 2k$, then $NL = LM = k$

Now, in $\Delta NO'L$,

$$\sin\theta = \frac{NL}{O'N} = \frac{k}{r} = \frac{k}{\frac{\sqrt{(x_1^2 + y_1^2)}}{2}} = \frac{2k}{\sqrt{(x_1^2 + y_1^2)}} \quad \text{...(i)}$$

But angle between the lines represented by

$ax^2 + 2hxy + by^2 = 0$ is $\tan\theta = \frac{2\sqrt{(h^2 - ab)}}{|a+b|}$.

$$\therefore \quad \sin\theta = \frac{2\sqrt{(h^2 - ab)}}{\sqrt{(a-b)^2 + 4h^2}} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{2\sqrt{(h^2 - ab)}}{\sqrt{(a-b)^2 + 4h^2}} = \frac{2k}{\sqrt{(x_1^2 + y_1^2)}}$$

On squaring both sides, we get

$$(x_1^2 + y_1^2)(h^2 - ab) = k^2\{(a-b)^2 + 4h^2\}$$

Hence, locus of $P(x_1, y_1)$ is

$$(x^2 + y^2)(h^2 - ab) = k^2\{(a-b)^2 + 4h^2\}.$$

● ***Ex. 23*** *Show that if two of the lines* $ax^3 + bx^2y + cxy^2 + dy^3 = 0 (a \neq 0)$ *make complementary angles with X-axis in anti-clockwise sense, then* $a(a-c) + d(b-d) = 0$.

Sol. The given equation can be written as

$$dm^3 + cm^2 + bm + a = 0, \text{ where } m = \frac{y}{x}$$

Let its roots be m_1, m_2, m_3.

$$\therefore \quad m_1 m_2 m_3 = -\frac{a}{d} \quad \text{...(i)}$$

If $m_1 = \tan\alpha$, then $m_2 = \tan(90° - \alpha)$

[$\because$ two lines makes complementary angles with X-axis]

$$\therefore \quad m_2 = \cot\alpha \text{ then } m_1 m_2 = 1$$

From Eq. (i), $\quad m_3 = -\frac{a}{d}$

Since, m_3 is root of the above cubic, we have

$$d\left(-\frac{a^3}{d^3}\right) + c\left(\frac{a^2}{d^2}\right) + b\left(-\frac{a}{d}\right) + a = 0$$

$$\Rightarrow \quad \left(-\frac{a^3}{d^2}\right) + \left(\frac{ca^2}{d^2}\right) - \left(\frac{ab}{d}\right) + a = 0$$

$$\Rightarrow \quad -a^3 + ca^2 - abd + ad^2 = 0$$

On dividing each by a, we get

$$-a^2 + ca - bd + d^2 = 0$$

$$\therefore \quad a(a-c) + d(b-d) = 0.$$

● ***Ex. 24*** *Show that the equation* $a(x^4 + y^4) - 4bxy(x^2 - y^2) + 6cx^2y^2 = 0$ *represents two pairs of lines at right angles and that if* $2b^2 = a^2 + 3ac$, *the two pairs will coincide.*

Sol. Given equation is

$$a(x^4 + y^4) - 4bxy(x^2 - y^2) + 6cx^2y^2 = 0 \quad \text{...(i)}$$

Eq. (i) is a homogeneous equation of fourth degree and since it represents two pairs at right angles. i.e. sum of the coefficients of x^2 and y^2 should be zero.

Let $a(x^4 + y^4) - 4bxy(x^2 - y^2) + 6cx^2y^2$

$$= (ax^2 + pxy - ay^2)(x^2 + qxy - y^2),$$

where, p and q are constants.

On comparing similar powers, we get

$$p + aq = -4b \quad \text{...(ii)}$$

$$-2a + pq = 6c \quad \text{...(iii)}$$

Again, given two pairs coincide, then

$$\frac{p}{a} = q$$

or $$p = aq \quad \ldots(iv)$$

From Eqs. (ii) and (iv),

$$q = -\frac{2b}{a}$$

and $$p = -2b$$

Substituting the values of p and q in Eq. (iii), we get

$$-2a + \frac{4b^2}{a} = 6c$$

$$\Rightarrow \quad -a^2 + 2b^2 = 3ac$$

$$\therefore \quad 2b^2 = a^2 + 3ac$$

● ***Ex. 25*** *Show that the locus of a point such that the product of the perpendiculars let fall from it on three lines represented by* $ay^3 + by^2x + cyx^2 + dx^3 = 0$ *is constant* $= k^3$, *is*

$ay^3 + by^2x + cyx^2 + dx^3 = k^3\sqrt{(a-c)^2 + (b-d)^2}$.

Sol. Let the three represented lines are

$$y = m_1x,\ y = m_2x \text{ and } y = m_3x$$

$$ay^3 + by^2x + cyx^2 + dx^3 = a(y - m_1x)(y - m_2x)(y - m_3x)$$

On comparing similar powers on both sides

$$\Rightarrow \quad m_1 + m_2 + m_3 = -\frac{b}{a} \quad \ldots(i)$$

$$m_1m_2 + m_2m_3 + m_3m_1 = \frac{c}{a} \quad \ldots(ii)$$

$$m_1m_2m_3 = -\frac{d}{a} \quad \ldots(iii)$$

Let the point $P(x_1, y_1)$, from given condition

$$\frac{(y_1 - m_1x_1)}{\sqrt{(1+m_1^2)}} \cdot \frac{(y_1 - m_2x_1)}{\sqrt{(1+m_2^2)}} \cdot \frac{(y_1 - m_3x_1)}{\sqrt{(1+m_3^2)}} = k^3$$

$$\Rightarrow \quad \frac{1}{a}(ay_1^3 + by_1^2x_1 + cy_1x_1^2 + dx_1^3)$$

$$= k^3\sqrt{(1+m_1^2)(1+m_2^2)(1+m_3^2)}$$

$$[\because ay^3 + by^2x + cyx^2 + dx^3$$

$$= a(y - m_1x)(y - m_2x)(y - m_3x)]$$

$$\Rightarrow (ay_1^3 + by_1^2x_1 + cy_1x_1^2 + dx_1^3)$$

$$= ak^3\sqrt{(1 + \Sigma m_1^2 + \Sigma m_1^2m_2^2 + m_1^2m_2^2m_3^2)}$$

$$= ak^3\sqrt{\{1 + (\Sigma m_1)^2 - 2\Sigma m_1m_2 + (\Sigma m_1m_2)^2 - 2m_1m_2m_3\Sigma m_1 + (m_1m_2m_3)^2\}}$$

$$= ak^3\sqrt{1 + \frac{b^2}{a^2} - \frac{2c}{a} + \frac{c^2}{a^2} + \frac{2d}{a} \cdot \left(-\frac{b}{a}\right) + \left(-\frac{d}{a}\right)^2}$$

$$= k^3\sqrt{(a^2 + b^2 - 2ac + c^2 - 2bd + d^2)}$$

$$= k^3\sqrt{\{(a-c)^2 + (b-d)^2\}}$$

Hence, locus of P is

$$(ay^3 + by^2x + cyx^2 + dx^3) = k^3\sqrt{\{(a-c)^2 + (b-d)^2\}}$$

● ***Ex. 26*** *If one of the lines given by the equation* $ax^2 + 2hxy + by^2 = 0$ *coincides with one of the lines given by* $a'x^2 + 2h'xy + b'y^2 = 0$ *and the other lines represented by them be perpendicular, then .*

$$\frac{ha'b'}{b'-a'} = \frac{h'ab}{b-a} = \frac{1}{2}\sqrt{(-aa'bb')}$$

Sol. Let the two lines represented by

$$ax^2 + 2hxy + by^2 = 0 \quad \ldots(i)$$

be $$y = m_1x$$

and $$y = m_2x$$

$$\therefore \quad m_1 + m_2 = -\frac{2h}{b} \quad \ldots(ii)$$

and $$m_1m_2 = \frac{a}{b} \quad \ldots(iii)$$

and the lines represented by

$$a'x^2 + 2h'xy + b'y^2 = 0 \quad \ldots(iv)$$

be $$y = -\frac{1}{m_1}x \quad \text{and} \quad y = m_2x$$

$$\therefore \quad -\frac{1}{m_1} + m_2 = -\frac{2h'}{b'} \quad \ldots(v)$$

and $$-\left(\frac{1}{m_1}\right)m_2 = \frac{a'}{b'} \quad \ldots(vi)$$

From Eqs. (iii) and (vi), we get

$$-m_2^2 = \frac{aa'}{bb'} \ \Rightarrow \ m_2 = \frac{\sqrt{(-aba'b')}}{bb'}$$

From Eqs. (iii), we get $\quad m_1 = -\frac{\sqrt{(-aba'b')}}{a'b}$

Substituting these values of m_1 and m_2 in Eqs. (ii) and (v), we get

$$-\frac{\sqrt{-aba'b'}}{a'b} + \frac{\sqrt{-aba'b'}}{bb'} = -\frac{2h}{b}$$

$$\Rightarrow \quad \frac{ha'b'}{b'-a'} = \frac{1}{2}\sqrt{-aa'bb'} \quad \ldots(vii)$$

and $$\frac{a'b}{\sqrt{-aba'b'}} + \frac{\sqrt{-aba'b'}}{bb'} = -\frac{2h'}{b'}$$

$$\frac{h'ab}{b-a} = \frac{1}{2}\sqrt{-aba'b'} \quad \ldots(viii)$$

From Eqs. (vii) and (viii), we get

$$\frac{ha'b'}{b'-a'} = \frac{h'ab}{b-a} = \frac{1}{2}\sqrt{-aba'b'}$$

Pair of Straight Lines Exercise 1 :
Single Option Correct Type Questions

- This section contains **12 multiple choice questions**. Each question has four choices (a), (b), (c), (d) out of which **ONLY ONE** is correct.

1. If the sum of the slopes of the lines given by $4x^2 + 2\lambda xy - 7y^2 = 0$ is equal to the product of the slopes, then λ is equal to
(a) -4 (b) -2 (c) 2 (d) 4

2. The equation $3ax^2 + 9xy + (a^2 - 2)y^2 = 0$ represents two perpendicular straight lines for
(a) only one value of a (b) for all values of a
(c) for only two values of a (d) for no value of a

3. The image of the pair of lines represented by $ax^2 + 2hxy + by^2 = 0$ by the line mirror $y = 0$ is
(a) $ax^2 - 2hxy - by^2 = 0$ (b) $bx^2 - 2hxy + ay^2 = 0$
(c) $bx^2 + 2hxy + ay^2 = 0$ (d) $ax^2 - 2hxy + by^2 = 0$

4. Number of points lying on the line $7x + 4y + 2 = 0$ which is equidistant from the lines $15x^2 + 56xy + 48y^2 = 0$ is
(a) 0 (b) 1
(c) 2 (d) 4

5. Orthocentre of the triangle formed by the lines $xy - 3x - 5y + 15 = 0$ and $3x + 5y = 15$ is
(a) $(-5, -3)$ (b) $(5, 3)$
(c) $(-3, -5)$ (d) $(3, 5)$

6. Two of the straight lines given by $3x^3 + 3x^2y - 3xy^2 + dy^3 = 0$ are at right angles, if d equal to
(a) -4 (b) -3
(c) -2 (d) -1

7. Two lines are given by $(x - 2y)^2 + \lambda(x - 2y) = 0$. The value of $|\lambda|$ so that the distance between them is 3, is
(a) $\sqrt{5}$ (b) $2\sqrt{5}$
(c) $3\sqrt{5}$ (d) $4\sqrt{5}$

8. The four straight lines given by the equations $12x^2 + 7xy - 12y^2 = 0$ and $12x^2 + 7xy - 12y^2 - x + 7y - 1 = 0$ lie along the sides of a
(a) square (b) rhombus
(c) rectangle (d) parallelogram

9. Distance between the parallel lines $4x^2 + 20xy + 25y^2 + 2x + 5y - 12 = 0$ is
(a) $\frac{3}{\sqrt{29}}$ (b) $\frac{5}{\sqrt{29}}$
(c) $\frac{7}{\sqrt{29}}$ (d) $\frac{9}{\sqrt{29}}$

10. The point of intersection of the two lines given by $2x^2 - 5xy + 2y^2 + 3x + 3y + 1 = 0$ is
(a) $(-2, 2)$ (b) $(-3, 3)$
(c) $(3, 3)$ (d) $(2, 2)$

11. If $\alpha, \beta > 0$ and $\alpha < \beta$ and $\alpha x^2 + 4\gamma xy + \beta y^2 + 4p(x + y + 1) = 0$ represents a pair of straight lines, then
(a) $\alpha \le p \le \beta$ (b) $p \le \alpha$
(c) $p \ge \alpha$ (d) $p \le \alpha$ or $p \ge \beta$

12. If the equation of the pair of straight lines passing through the point $(1, 1)$, one making an angle θ with the positive direction of the X-axis and the other making the same angle with the positive direction of the Y-axis, is $x^2 - (a+2)xy + y^2 + a(x + y - 1) = 0$, $a \neq -2$, then the value of $\sin 2\theta$ is
(a) $a - 2$ (b) $a + 2$
(c) $\frac{2}{(a + 2)}$ (d) $\frac{2}{a}$

Pair of Straight Lines Exercise 2 :
More than One Option Correct Type Questions

- This section contains **6 multiple choice questions**. Each questions has four choices (a), (b), (c), (d) out of which **MORE THAN ONE** may be correct.

13. The equation of image of pair of lines $y = |x - 1|$ in Y-axis is
(a) $y = |x + 1|$ (b) $y = |x - 1| + 3$
(c) $x^2 - y^2 + 2x + 1 = 0$ (d) $x^2 - y^2 + 2x - 1 = 0$

14. The equation $ax^2 + by^2 + cx + cy = 0$ represent a pair of straight lines, if
(a) $a + b = 0$
(b) $c = 0$
(c) $a + c = 0$
(d) $c(a + b) = 0$

15. If $x^2 + \alpha y^2 + 2\beta y = a^2$ represents a pair of perpendicular straight lines, then
(a) $\alpha = 1, \beta = a$
(b) $\alpha = 1, \beta = -a$
(c) $\alpha = -1, \beta = -a$
(d) $\alpha = -1, \beta = a$

16. If the pair of lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ intersect on Y-axis, then
(a) $f^2 = bc$
(b) $abc = 2fgh$
(c) $bg^2 \neq ch^2$
(d) $2fgh = bg^2 + ch^2$

17. Two pair of straight lines have the equations $y^2 + xy - 12x^2 = 0$ and $ax^2 + 2hxy + by^2 = 0$. One line will be common among them, if
(a) $a = -3(2h + 3b)$ (b) $a = 8(h - 2b)$
(c) $a = 2(b + h)$ (d) $a = -3(b + h)$

18. The combined equation of three sides of a triangle is $(x^2 - y^2)(2x + 3y - 6) = 0$. If $(-2, a)$ is an interior and $(b, 1)$ is an exterior point of the triangle, then
(a) $2 < a < \frac{10}{3}$ (b) $-2 < a < \frac{10}{3}$
(c) $-1 < b < \frac{9}{2}$ (d) $-1 < b < 1$

Pair of Straight Lines Exercise 3 : Paragraph Based Questions

- This section contains **3 paragraphs** based upon each of the paragraph, **3 multiple choice questions** have to be answered. Each of these question has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Paragraph I

(Q. Nos. 19 to 21)

Consider the equation of a pair of straight lines as $x^2 - 3xy + \lambda y^2 + 3x - 5y + 2 = 0$

19. The value of λ is
(a) 1 (b) 2
(c) 3 (d) 4

20. The point of intersection of lines is (α, β), then the value of $\alpha^2 + \beta^2$ is
(a) 2 (b) 5
(c) 10 (d) 17

21. The angle between the lines is θ, then the value of $\cos 2\theta$ is
(a) $\frac{1}{3}$ (b) $\frac{2}{3}$
(c) $\frac{3}{5}$ (d) $\frac{4}{5}$

Paragraph II

(Q. Nos. 22 to 24)

Let $f_1(x, y) \equiv ax^2 + 2hxy + by^2 = 0$ *and let* $f_{i+1}(x, y) = 0$ *denote the equation of the bisectors of* $f_i(x, y) = 0$ *for all* $i = 1, 2, 3, \ldots$

22. $f_3(x, y) = 0$ is
(a) $hx^2 - (a - b)xy - hy^2 = 0$
(b) $(a - b)x^2 + 4hxy - (a - b)y^2 = 0$
(c) $ax^2 + 2hxy + by^2 = 0$
(d) None of the above

23. If $f_{i+1}(x, y) = 0$ represents the equation of a pair of perpendicular lines, then $f_3(x, y) = 0$ is same as
(a) $f_1(x, y) = 0$
(b) $f_2(x, y) = 0$
(c) $hx^2 - (a - b)xy - hy^2 = 0$
(d) None of the above

24. The value of $\sum_{n=2}^{5} \frac{f_{n+2}(x, y)}{f_n(x, y)}$ is
(a) 14 (b) 4
(c) 54 (d) 6

Paragraph III

(Q. Nos. 25 to 27)

Consider a pair of perpendicular straight lines $2x^2 + 3xy + by^2 - 11x + 13y + c = 0$

25. The value of c is
(a) -2 (b) 2
(c) -3 (d) 3

26. The value of $|b + 2c|$ is
(a) 4 (b) 6
(c) 8 (d) 10

27. If point of intersection of lines is C and points of intersection of the lines with the X-axis are A and B, if distance between the orthocentre and the circumcentre of ΔABC is λ, then $[\lambda]$ is (where $[.]$ denotes the greatest integer function
(a) 2 (b) 3
(c) 4 (d) 5

Pair of Straight Lines Exercise 4 :
Single Integer Answer Type Questions

- This section contains **5 questions**. The answer to each question is **a single digit integer**, ranging from 0 to 9 (both inclusive).

28. Equation $\lambda x^3 - 10x^2y - xy^2 + 4y^3 = 0$ represented three straight lines, out of these three, two lines makes equal angle with $y = x$ and $\lambda > 0$, then the value of λ is

29. Area enclosed by curves $y^2 - 5xy + 6x^2 + 3x - y = 0$ and $y^2 - 5xy + 6x^2 + 2x - y = 0$ is λ sq units, then the value of λ is

30. The lines represented by $x^2 + 2\lambda xy + 2y^2 = 0$ and $(\lambda + 1)$ $x^2 - 8xy + y^2 = 0$ are equally inclined, then the value of $|\lambda|$ is

31. If the lines joining the origin to the intersection of the line $y = nx + 2$ and the curve $x^2 + y^2 = 1$ are at right angles, then the value of n^2 is

32. If area of the triangle formed by the line $x + y = 3$ and the angle bisectors of the pair of straight lines $x^2 - y^2 + 2y = 1$ is λ sq units, then the value of λ is

Pair of Straight Lines Exercise 5 :
Matching Type Questions

- This section contains **one question.** This question has four statements (A,B,C and D) given in **Column I** and four statements (p, q, r and s) in **Column II**. Any given statements in **Column I** can have correct matching with one or more statement(s) given in **Column II**.

33. Match the following

	Column I		Column II
(A)	The pair of lines joining the origin to the points of intersection of the curve $9x^2 + 16y^2 = 144$ by the line $2x + 2y + \lambda = 0$ are coincident, then $\|\lambda\|$ is divisible by	(p)	2
(B)	If the straight lines joining the origin to the points of intersection of the straight line $4x + 3y = 24$ and the curve $(x-3)^2 + (y-4)^2 = \lambda^2$, are at right angles, then $\|\lambda\|$ is divisible by	(q)	3
(C)	The two line pairs $y^2 - 4y + 3 = 0$ and $x^2 + 4xy + 4y^2 - 5x - 10y + 4 = 0$ enclose a 4 sided convex polygon, if area of polygon is λ sq units, then λ is divisible by	(r)	5
(D)	If the pairs of lines $3x^2 - 2pxy - 3y^2 = 0$ and $5x^2 - 2qxy - 5y^2 = 0$ are such that each pair bisects the angle between the other pair. If $\lambda = \|pq\|$, then λ is divisible by	(s)	6

Pair of Straight Lines Exercise 6 :
Statement I and II Types Questions

- **Directions** (Q. Nos. 34-37) are Assertion-Reason type questions. Each of these questions contains two statements:
  Statement I (Assertion) and **Statement II** (Reason)
  Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below :
  (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
  (b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
  (c) Statement I is true, Statement II is false
  (d) Statement I is false, Statement II is true

34. **Statement I** The four straight lines given by $6x^2 + 5xy - 6y^2 = 0$ and $6x^2 + 5xy - 6y^2 - x + 5y - 1 = 0$ are the sides of a square.

Statement II The lines represented by general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are perpendicular if $a + b = 0$.

35. **Statement I** Two of the straight lines represented by $dx^3 + cx^2y + bxy^2 + ay^3 = 0$ will be at right angles if $d^2 + bd + bc + a^2 = 0$.

Statement II Product of the slopes of two perpendicular lines is -1.

36. **Statement I** If $\alpha\beta = -1$, then the pair of straight lines $x^2 - 2\alpha xy - y^2 = 0$ and $y^2 + 2\beta xy - x^2 = 0$ are the angle bisector of each other.

Statement II Pair of angle bisector lines of the pair of lines $ax^2 + 2hxy + by^2 = 0$ is $h(x^2 - y^2) = (a - b)xy$.

37. **Statement I** If $a + b = -2h$, then one line of the pair of lines $ax^2 + 2hxy + by^2 = 0$ bisects the angle between coordinate axes in positive quadrant.

Statement II If $ax + y(2h + a) = 0$ is a factor of $ax^2 + 2hxy + by^2 = 0$, then $b + 2h + a = 0$.

Pair of Straight Lines Exercise 7 :
Subjective Type Questions

- In this section, there are **7 subjective questions**.

38. Prove that the straight lines represented by $(y - mx)^2 = a^2(1 + m^2)$ and $(y - nx)^2 = a^2(1 + n^2)$ form a rhombus.

39. Prove that the equation $m(x^3 - 3xy^2) + y^3 - 3x^2y = 0$ represents three straight lines equally inclined to each other.

40. Show that the straight lines $(A^2 - 3B^2)x^2 + 8ABxy + (B^2 - 3A^2)y^2 = 0$ form with the line $Ax + By + C = 0$ an equilateral triangle whose area is $\frac{C^2}{\sqrt{3}(A^2 + B^2)}$

41. Find the equations of the diagonals of the parallelogram formed by the lines $L^2 - aL = 0$ and $L'^2 - aL' = 0$, where $L = x\cos\theta + y\sin\theta - p$ *and* $L' = x\cos\theta' + y\sin\theta' - p'$.

42. If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and $ax^2 + 2hxy + by^2 - 2gx - 2fy + c = 0$ each represents a pair of lines, then prove that the area of the parallelogram enclosed by them is $\frac{2|c|}{\sqrt{(h^2 - ab)}}$.

43. Prove that lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are equidistant from the origin, if $f^4 - g^4 = c(bf^2 - ag^2)$. Also, find the product of their distances from the origin.

44. Prove that if two of the lines represented by $ax^4 + bx^3y + cx^2y^2 + dxy^3 + ay^4 = 0$ bisects the angle between the other two, then $c + 6a = 0$ and $b + d = 0$.

Pair of Straight Lines Exercise 8 : Question Asked in Previous 13 Years Exams

- This section contains questions asked in **IIT-JEE, AIEEE, JEE Main & JEE Advanced** from year **2005** to **2017**.

45. If the pair of lines $ax^2+2(a+b)xy+by^2=0$ lie long diameters of a circle and divide the circle into four sectors such that the area of one of the sector is thrice the area of the another sector, then **[AIEEE 2005, 3M]**

(a) $3a^2+2ab+3b^2=0$ (b) $3a^2+10ab+3b^2=0$

(c) $3a^2-2ab+3b^2=0$ (d) $3a^2-10ab+3b^2=0$

46. If one of the lines of $my^2+(1-m^2)xy-mx^2=0$ is a bisector of the angle between the lines $xy=0$, then m is **[AIEEE 2007, 3M]**

(a) $-\frac{1}{2}$ (b) -2

(c) 1 (d) 2

Answers

Exercise for Session 1

1. (c) **2.** (a) **3.** (b) **4.** (b) **5.** (c, d)

6. $bx+ay=0,\ ax-by=0$ **7.** $\left(\frac{8}{3},\frac{8}{3}\right)$

9. $(ab'-a'b)^2=4(ah'-a'h)(h'b-hb')$

Exercise for Session 2

1. (b) **2.** (c, d) **3.** (a) **4.** (c) **5.** (b, c)

6. $\tan^{-1}\left(\frac{1}{3}\right)$ **8.** $\sec^{-1}(\pm p)$ **9.** $\frac{1}{6}\sqrt{3}$ sq units

Exercise for Session 3

1. (b) **2.** (a, c) **3.** (a, c) **4.** (b) **9.** ± 2

Exercise for Session 4

1. (b) **2.** (d) **3.** (a, d) **4.** (b) **5.** (a)

6. 1 sq unit **7.** $x-y=0$ **9.** $\frac{|g^2-ac|}{|a|\sqrt{h^2-ab}}$

10. $x^2-5xy+4y^2+3x-3y=0$

Exercise for Session 5

1. (c) **2.** (d) **3.** (d) **4.** (a) **5.** (b)

6. $9x^2+10xy-9y^2=0$ **7.** ± 1 **10.** $x^2-xy-2y^2=0$

Chapter Exercises

1. (b) **2.** (c) **3.** (d) **4.** (c) **5.** (b) **6.** (b)

7. (c) **8.** (a) **9.** (c) **10.** (c) **11.** (d) **12.** (c)

13. (a, c) **14.** (a, b, d) **15.** (c, d) **16.** (a, d) **17.** (a, b) **18.** (a, d)

19. (b) **20.** (c) **21.** (d) **22.** (b) **23.** (a) **24.** (b)

25. (a) **26.** (b) **27.** (c) **28.** (7) **29.** (1) **30.** (2)

31. (7) **32.** (2)

33. (A) $\rightarrow$ (p, r); (B) $\rightarrow$ (r); (C) $\rightarrow$ (p, q, s); (D) $\rightarrow$ (q, r)

34. (b) **35.** (b) **36.** (a) **37.** (b)

41. $\begin{cases} x(\cos\theta-\cos\theta')+y(\sin\theta-\sin\theta')-p+p'=0 \\ \text{and } x(\cos\theta+\cos\theta')+y(\sin\theta+\sin\theta')-p-p'-a=0 \end{cases}$

43. $\frac{|c|}{\sqrt{(a-b)^2+4h^2}}$

45. (a) **46.** (c)

Solutions

1. $\because m_1 + m_2 = m_1 m_2 \Rightarrow \dfrac{-2\lambda}{-7} = \dfrac{4}{-7}$

$\therefore \quad \lambda = -2$

2. For perpendicular lines

$3a + (a^2 - 2) = 0$

$\Rightarrow \quad a = \dfrac{-3 \pm \sqrt{(9+8)}}{6} = \dfrac{-3 \pm \sqrt{17}}{6}$

3. For mirror image with respect to $y = 0$ replace y by $-y$, then image of the pair of lines $ax^2 + 2hxy + by^2 = 0$ is

$$ax^2 + 2hx(-y) + b(-y)^2 = 0$$

or $$ax^2 - 2hxy + by^2 = 0$$

4. $\because \quad 15x^2 + 56xy + 48y^2 = 0$ ...(i)

$\Rightarrow \quad (3x + 4y)(5x + 12y) = 0$

Equation of lines represented by Eq. (i) are

$3x + 4y = 0$ and $5x + 12y = 0$

Let any point on $7x + 4y + 2 = 0$ is $\left(\lambda, \dfrac{-7\lambda - 2}{4}\right)$.

According to questions,

$$\left|\frac{3\lambda + 4\left(\frac{-7\lambda-2}{4}\right)}{5}\right| = \left|\frac{5\lambda + 12\left(\frac{-7\lambda-2}{4}\right)}{13}\right|$$

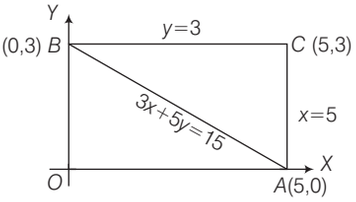
$$\Rightarrow \quad \frac{1}{5}|-4\lambda - 2| = \frac{1}{13}|-16\lambda - 6|$$

$$\Rightarrow \quad \frac{|2\lambda + 1|}{5} = \frac{|8\lambda + 3|}{13}$$

$$\Rightarrow \quad 13(2\lambda + 1) = \pm 5(8\lambda + 3)$$

$$\therefore \quad \lambda = -\frac{1}{7}, -\frac{14}{33}$$

5. Given lines are $xy - 3x - 5y + 15 = 0$



$\Rightarrow \quad (x - 5)(y - 3) = 0$

$\therefore \quad x = 5$ and $y = 3$

Hence, orthocentre is (5, 3).

6. $3x^3 + 3x^2y - 3xy^2 + dy^3 = (x^2 + pxy - y^2)(3x - dy)$

On comparing coefficients of x^2y and xy^2, we get

$$3p - d = 3$$

and $$-3 - pd = -3$$

$\therefore \quad p = 0, \ d = -3$

7. $\because \quad (x - 2y)^2 + \lambda(x - 2y) = 0$

$\Rightarrow \quad (x - 2y)(x - 2y + \lambda) = 0$

$\therefore$ Lines are $x - 2y = 0$ and $x - 2y + \lambda = 0$

Distance between lines = 3 (given)

$\Rightarrow \quad \dfrac{|\lambda - 0|}{\sqrt{(1+4)}} = 3$

$\therefore \quad |\lambda| = 3\sqrt{5}$

8. $\therefore \quad 12x^2 + 7xy - 12y^2 = 0$ ...(i)

$\Rightarrow \quad (3x + 4y)(4x - 3y) = 0$

$\therefore$ Lines represented by Eq. (i) are

$3x + 4y = 0$ and $4x - 3y = 0$

and $12x^2 + 7xy - 12y^2 - x + 7y - 1 = 0$ ...(ii)

$\Rightarrow \quad (3x + 4y - 1)(4x - 3y + 1) = 0$

$\therefore$ Lines represented by Eq. (ii) are

$3x + 4y - 1 = 0$ and $4x - 3y + 1 = 0$

Distance between parallel lines $3x + 4y = 0$ and

$3x + 4y - 1 = 0$ is $\dfrac{1}{5}$.

And distance between parallel lines $4x - 3y = 0$ and

$4x - 3y + 1 = 0$ is $\dfrac{1}{5}$.

Hence, all sides along a square.

9. Given,

$$4x^2 + 20xy + 25y^2 + 2x + 5y - 12 = 0 \quad \text{...(i)}$$

$$\Rightarrow \quad (2x + 5y)^2 + (2x + 5y) - 12 = 0$$

$$\Rightarrow \quad (2x + 5y + 4)(2x + 5y - 3) = 0$$

$\therefore$ Lines represented by Eq. (i) are

$2x + 5y + 4 = 0$ and $2x + 5y - 3 = 0$

Hence, distance between parallel lines $= \dfrac{|4 - (-3)|}{\sqrt{4 + 25}} = \dfrac{7}{\sqrt{29}}$.

10. Let $f(x, y) \equiv 2x^2 - 5xy + 2y^2 + 3x + 3y + 1 = 0$

$\therefore \quad \dfrac{\partial f}{\partial x} = 4x - 5y + 3$ and $\dfrac{\partial f}{\partial y} = -5x + 4y + 3$

For point of intersection $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$

$\Rightarrow \ 4x - 5y + 3 = 0$ and $-5x + 4y + 3 = 0$, we get

$x = 3, y = 3$

$\therefore$ Point of intersection is (3, 3).

11. $\alpha x^2 + 4\gamma\, xy + \beta y^2 + 4p(x + y + 1) = 0$

represents a pair of straight lines.

$\therefore \ \Delta = 0$

$\Rightarrow 4\alpha\beta p + 16p^2\gamma - 4p^2\alpha - 4p^2\beta - 16\gamma^2 p = 0$

$\Rightarrow \quad (16p)\gamma^2 - 16p^2\gamma + 4p(p\alpha + p\beta - \alpha\beta) = 0$

$\therefore \quad B^2 - 4AC \geq 0$

$\Rightarrow \quad (16p^2)^2 - 4 \cdot (16p) \cdot 4p\,(p\alpha + p\beta - \alpha\beta) \geq 0$

$\Rightarrow \quad p^2(p^2 - p\alpha - p\beta + \alpha\beta) \geq 0$

$\Rightarrow \quad (p - \alpha)(p - \beta) \geq 0$

$\therefore \quad p \leq \alpha$ or $p \geq \beta \quad [\because \alpha < \beta]$

12. Equation of first line is $y-1=\tan\theta(x-1)$

Equation of second line is $y-1=\cot\theta(x-1)$

So, their joint equation is

$$[(y-1)-\tan\theta(x-1)][(y-1)-\cot\theta(x-1)]=0$$

$$\Rightarrow (y-1)^2-(x-1)(y-1)(\tan\theta+\cot\theta)+(x-1)^2=0$$

$$x^2-(\tan\theta+\cot\theta)xy+y^2+(\tan\theta+\cot\theta-2)(x+y-1)=0$$

On comparing with the given equation, we get

$$\tan\theta+\cot\theta=a+2$$

or $$\frac{1}{\sin\theta\cos\theta}=a+2$$

or $$\sin 2\theta=\frac{2}{(a+2)}$$

13. For image w.r.t., Y-axis replace x by $-x$, then required image of lines is

$$y=|-x-1| \text{ or } y=|x+1|$$

and on squaring both sides, then

$$y^2=x^2+2x+1$$

$$\Rightarrow x^2-y^2+2x+1=0$$

14. Given equation is

$$ax^2+by^2+cx+cy=0 \quad ...(i)$$

Eq. (i) represents a pair of straight lines.

$$\therefore \Delta=0$$

$$\Rightarrow 0+2\cdot 0-a\times\frac{c^2}{4}-b\times\frac{c^2}{4}=0$$

or $$c^2(a+b)=0$$

$$\therefore c=0 \text{ or } a+b=0$$

15. Given, equation is

$$x^2+\alpha y^2+2\beta y-a^2=0 \quad ...(i)$$

Equation (i) represents a pair of perpendicular straight lines

$\therefore \Delta=0$ and coefficient of x^2 + coefficient of $y^2=0$

$$\Rightarrow -a^2\alpha-\beta^2=0 \text{ and } 1+\alpha=0$$

$$\therefore \alpha=-1 \text{ and } \beta=\pm a$$

16. Given pair of lines is

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad ...(i)$$

$\because$ Point of intersection of lines represented by (i) is

$$\left(\frac{hf-bg}{ab-h^2},\frac{gh-af}{ab-h^2}\right) \text{ or } \left(\sqrt{\left(\frac{f^2-bc}{h^2-ab}\right)},\sqrt{\left(\frac{g^2-ac}{h^2-ab}\right)}\right)$$

For Y-axis put $x=0$

$\therefore$ $hf=bg$, $f^2=bc$ and $h^2\neq ab$

Also, $\Delta=0$

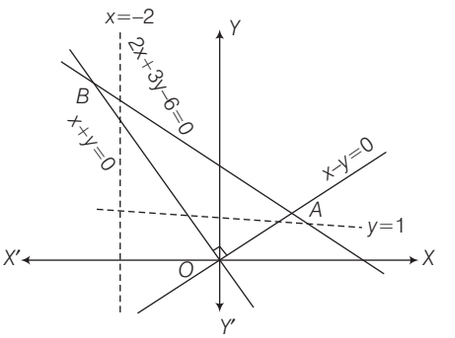
$\therefore$ $abc+2fgh-af^2-bg^2-ch^2=0$

For $$f^2=bc, 2fgh=bg^2+ch^2$$

17. Let $y=mx$ be the common line, then $m^2+m-12=0$ and $bm^2+2hm+a=0$, then from first equation $m=-4, 3$

Substitute $m=-4$ in second equation, then $a=8(h-2b)$ and substitute $m=3$ in second equation, then $a=-3(2h+3b)$.

18. The separate equations of the sides are $x+y=0$, $x-y=0$ and $\frac{x}{3}+\frac{y}{2}=1$.



Intersection of $x=-2$ with $y=-x$ and $2x+3y-6=0$ gives the range of values of a.

$$\therefore 2<a<\frac{10}{3}$$

and intersection of $y=1$ with $y=x$ and $2x+3y-6=0$ given the range of values of b

$$-1<b<1 \quad [\because (b,1) \text{ is exterior point}]$$

Sol. (Q. Nos. 19 to 21)

Given equation is $x^2-3xy+\lambda y^2+3x-5y+2=0$

Here, $a=1, b=\lambda, c=2,$

$$f=-\frac{5}{2}, g=\frac{3}{2} \text{ and } h=-\frac{3}{2}.$$

19. $\Delta=0$

$$1\times\lambda\times 2+2\times-\frac{5}{2}\times\frac{3}{2}\times-\frac{3}{2}-1\times\frac{25}{4}-\lambda\times\frac{9}{4}-2\times\frac{9}{4}=0$$

$$\Rightarrow 2\lambda+\frac{45}{4}-\frac{25}{4}-\frac{9\lambda}{4}-\frac{9}{2}=0$$

$$\Rightarrow -\frac{\lambda}{4}+\frac{1}{2}=0$$

$$\therefore \lambda=2$$

20. Let $f(x,y)=x^2-3xy+2y^2+3x-5y+2$ $[\because \lambda=2]$

$$\therefore \frac{\partial f}{\partial x}=2x-3y+3, \frac{\partial f}{\partial y}=-3x+4y-5$$

For point of intersection $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$, we get

$$x=-3, y=-1$$

$$\therefore \alpha=-3 \text{ and } \beta=-1$$

$\therefore$ The value of $\alpha^2+\beta^2$ is 10.

21. $$\tan\theta=\frac{2\sqrt{(h^2-ab)}}{(a+b)}=\frac{2\sqrt{\left(\frac{9}{4}-2\right)}}{1+2}=\frac{1}{3}$$

$$\therefore \cos 2\theta=\frac{1-\tan^2\theta}{1+\tan^2\theta}=\frac{1-\frac{1}{9}}{1+\frac{1}{9}}=\frac{4}{5}$$

Solutions (Q. Nos. 22 to 24)

22. $\because \quad f_1(x, y) \equiv ax^2 + 2hxy + by^2 = 0$ ...(i)

bisectors of $f_1(x, y)$ is $f_2(x, y)$

$\therefore$ Equation of bisectors of (i) is $\dfrac{x^2 - y^2}{a - b} = \dfrac{xy}{h}$

$\Rightarrow \quad hx^2 - (a - b)xy - hy^2 = 0$

$\therefore \quad f_2(x, y) \equiv hx^2 - (a - b)xy - hy^2 = 0$ ...(ii)

Now, equation of bisectors of (ii) is

$$\frac{x^2 - y^2}{h - (-h)} = \frac{xy}{-\dfrac{(a - b)}{2}}$$

$\Rightarrow \quad \dfrac{x^2 - y^2}{2h} = -\dfrac{2xy}{(a - b)}$

$\Rightarrow \quad (a - b)x^2 + 4hxy - (a - b)y^2 = 0$

$\therefore \quad f_3(x, y) \equiv (a - b)x^2 + 4hxy - (a - b)y^2 = 0$

23. $f_2(x, y) \equiv bx^2 - 2hxy + ay^2 = 0$

and $f_3(x, y) \equiv ax^2 + 2hxy + by^2 = 0$

is same as $f_1(x, y) = 0$

24. For all $n \geq 2$

$$f_{n+2}(x, y) = f_n(x, y)$$

$\therefore \quad \dfrac{f_{n+2}(x, y)}{f_n(x, y)} = 1$

Now, $\displaystyle\sum_{n=2}^{5} \frac{f_{n+2}(x, y)}{f_n(x, y)} = \sum_{n=2}^{5} 1 = 1 + 1 + 1 + 1 = 4$

Sol. (Q. Nos. 25 to 27)

For perpendicular straight lines, then
coefficient of x^2 + coefficient of $y^2 = 0$

$\Rightarrow \quad 2 + b = 0$

$\therefore \quad b = -2$

25. $2x^2 + 3xy - 2y^2 - 11x + 13y + c = 0$

represents a pair of straight lines, then we have

$$2 \times -2 \times c + 2 \times \frac{13}{2} \times -\frac{11}{2} \times \frac{3}{2} - 2 \times \frac{169}{4} + 2 \times \frac{121}{4} - c \times \frac{9}{4} = 0$$

$\Rightarrow \quad -4c - \dfrac{429}{4} - \dfrac{169}{2} + \dfrac{121}{2} - \dfrac{9c}{4} = 0$

$\Rightarrow \quad c = -2$

26. $|b + 2c| = |-2 - 4| = 6$

27. Now, pair of lines is $2x^2 + 3xy - 2y^2 - 11x + 13y - 2 = 0$

Let $f(x, y) \equiv 2x^2 + 3xy - 2y^2 - 11x + 13y - 2 = 0$

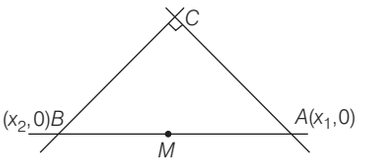
$\therefore \quad \dfrac{\partial f}{\partial x} = 4x + 3y - 11$ and $\dfrac{\partial f}{\partial y} = 3x - 4y + 13$

For point of intersection, $\dfrac{\partial f}{dx} = 0$ and $\dfrac{\partial f}{\partial y} = 0$

$\Rightarrow \quad 4x + 3y - 11 = 0$ and $3x - 4y + 13 = 0$

$\therefore \quad C \equiv \left(\dfrac{1}{5}, \dfrac{19}{5}\right)$

Also, intersection on X-axis put $y = 0$.



$\therefore \quad 2x^2 - 11x + 2 = 0 \Rightarrow x_1 + x_2 = \dfrac{11}{2}$

$\therefore$ Circumcentre of $\Delta ABC \Rightarrow M\left(\dfrac{x_1 + x_2}{2}, 0\right)$

$\therefore \quad M \equiv \left(\dfrac{11}{4}, 0\right)$

and orthocentre is $C \equiv \left(\dfrac{1}{5}, \dfrac{19}{5}\right)$

$\therefore$ Required distance $= \sqrt{\left(\dfrac{11}{4} - \dfrac{1}{5}\right)^2 + \left(0 - \dfrac{19}{5}\right)^2}$

$\therefore \quad \lambda = \sqrt{\dfrac{8377}{400}} = \sqrt{20.94} = 4.57$

$$[\lambda] = [4.57] = 4$$

28. Given, $4y^3 - xy^2 - 10x^2y + \lambda x^3 = 0$

$\Rightarrow \quad 4\left(\dfrac{y}{x}\right)^3 - \left(\dfrac{y}{x}\right)^2 - 10\left(\dfrac{y}{x}\right) + \lambda = 0$

Let $y = mx$, then

$$4m^3 - m^2 - 10m + \lambda = 0$$

and let m_1, m_2, m_3 be the slopes of three lines, then

$$m_1 + m_2 + m_3 = \frac{1}{4} \quad \text{...(i)}$$

$$m_1m_2 + m_2m_3 + m_3m_1 = -\frac{10}{4} \quad \text{...(ii)}$$

$$m_1m_2m_3 = -\frac{\lambda}{4} \quad \text{...(iii)}$$

Since, two lines are equally inclined with $y = x$, so let $m_2m_3 = 1$

From Eq. (iii), $m_1 = -\dfrac{\lambda}{4}$

From Eq. (i), $m_2 + m_3 = \dfrac{\lambda + 1}{4}$ ...(iv)

and from Eq. (ii), $1 + \left(-\dfrac{\lambda}{4}\right)\left(\dfrac{\lambda + 1}{4}\right) = -\dfrac{10}{4}$

$\Rightarrow \quad 16 - \lambda^2 - \lambda = -40$

$\Rightarrow \quad \lambda^2 + \lambda - 56 = 0$

$\Rightarrow \quad (\lambda + 8)(\lambda - 7) = 0$

$\lambda = -8, 7$

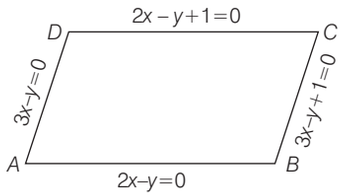
Hence, $\lambda = 7 \quad [\because \lambda > 0]$

29. $\because y^2 - 5xy + 6x^2 + 3x - y = 0$

$\Rightarrow \quad (3x - y)(2x - y + 1) = 0$

and $y^2 - 5xy + 6x^2 + 2x - y = 0$

$\Rightarrow \quad (3x - y + 1)(2x - y) = 0$



$\therefore$ Required Area $= \dfrac{(1-0)(1-0)}{\begin{vmatrix} 3 & -1 \\ 2 & -1 \end{vmatrix}} = 1$ sq unit

$\therefore \quad \lambda = 1$

30. Given pairs are

$$x^2 + 2\lambda xy + 2y^2 = 0 \quad \text{...(i)}$$

and $$(\lambda + 1)x^2 - 8xy + y^2 = 0 \quad \text{...(ii)}$$

$\because$ Equations of angle bisectors of (i) and (ii) are equal

$\Rightarrow \dfrac{x^2 - y^2}{1 - 2} = \dfrac{xy}{\lambda}$ and $\dfrac{x^2 - y^2}{(\lambda + 1) - 1} = \dfrac{xy}{-4}$ must be same

$\Rightarrow \quad \dfrac{\lambda}{-1} = -\dfrac{4}{\lambda} \Rightarrow \lambda^2 = 4 \Rightarrow \lambda = \pm 2$

$\therefore \quad |\lambda| = 2$

31. Given, $$\frac{y - nx}{2} = 1 \quad \text{...(i)}$$

and $$x^2 + y^2 = 1 \quad \text{...(ii)}$$

Homogenising Eq. (ii) with Eq. (i), we get

$$x^2 + y^2 = \left(\frac{y - nx}{2}\right)^2$$

$$\Rightarrow \quad \left(\frac{n^2}{4} - 1\right)x^2 - nxy + \left(\frac{1}{4} - 1\right)y^2 = 0$$

or $$(n^2 - 4)x^2 - 4nxy - 3y^2 = 0$$

The lines will be at right angles, when

$$(n^2 - 4) + (-3) = 0 \Rightarrow n^2 = 7$$

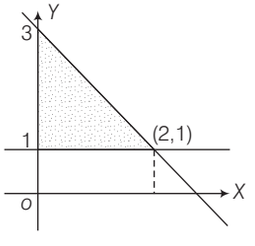
32. $\because \quad x^2 - y^2 + 2y = 1$

$$\Rightarrow \quad x^2 = (y^2 - 2y + 1) = (y - 1)^2$$

$$\Rightarrow \quad x = \pm (y - 1)$$

i.e. $\quad x - y + 1 = 0$ and $x + y - 1 = 0$

$\therefore$ Bisectors are $y = 1$ and $x = 0$



So, area between $x = 0$, $y = 1$ and $x + y = 3$ is given by

$$\frac{1}{2} \times 2 \times 2 = 2 \text{ sq units}$$

$\therefore \quad \lambda = 2$

33. (A) $\rightarrow$ (p, r); (B) $\rightarrow$ (r); (C) $\rightarrow$ (p, q, s); (D) $\rightarrow$ (q, r)

(A) Given $$\frac{2x + 2y}{-\lambda} = 1 \quad \text{...(i)}$$

and $$9x^2 + 16y^2 = 144 \quad \text{...(ii)}$$

Homogenising Eq. (ii) with Eq. (i), we get

$$9x^2 + 16y^2 = 144\left(\frac{2x + 2y}{-\lambda}\right)^2$$

$$\Rightarrow \quad (9\lambda^2 - 576)x^2 - 1152xy + (16\lambda^2 - 576)y^2 = 0$$

The lines are coincident, then

$$(-576)^2 = (9\lambda^2 - 576)(16\lambda^2 - 576)$$

$$\Rightarrow \quad 144\lambda^4 - 576\lambda^2(25) = 0$$

or $$\lambda^2 = 100$$

$\therefore \quad |\lambda| = 10$

(B) Given, $$\frac{4x + 3y}{24} = 1 \quad \text{...(i)}$$

and $$(x - 3)^2 + (y - 4)^2 = \lambda^2$$

or $$x^2 + y^2 - (6x + 8y) + 25 - \lambda^2 = 0 \quad \text{...(ii)}$$

Homogenising Eq. (ii) with Eq. (i), we get

$$x^2 + y^2 - (6x + 8y)\frac{(4x + 3y)}{24} + (25 - \lambda^2)\left(\frac{4x + 3y}{24}\right)^2 = 0$$

$$\Rightarrow \quad 16(25 - \lambda^2)x^2 + 9(25 - \lambda^2)y^2 + (-600 - 24\lambda^2)xy = 0$$

The lines will be at right angles, when

$$16(25 - \lambda^2) + 9(25 - \lambda^2) = 0$$

$$\Rightarrow \quad 625 - 25\lambda^2 = 0$$

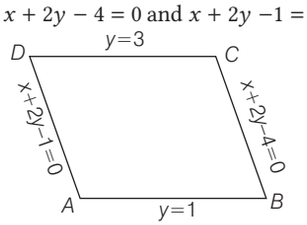
$$\Rightarrow \quad \lambda^2 = 25$$

$\therefore \quad |\lambda| = 5$

(C) The lines pairs are $(y - 1)(y - 3) = 0$ i.e. $y = 1$ and $y = 3$
the other line pair is $x^2 + 4xy + 4y^2 - 5x - 10y + 4 = 0$

i.e. $\quad (x + 2y - 4)(x + 2y - 1) = 0$

or $\quad x + 2y - 4 = 0$ and $x + 2y - 1 = 0$



Required area $= \dfrac{(3-1)(4-1)}{\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}} = 6$ sq units

$\therefore \quad \lambda = 6$

(D) Equation of the bisectors of $3x^2 - 2pxy - 3y^2 = 0$ is

$$\frac{x^2 - y^2}{3 - (-3)} = \frac{xy}{-p} \Rightarrow px^2 + 6xy - py^2 = 0$$

Which is same as $5x^2 - 2qxy - 5y^2 = 0$

i.e. $$\frac{p}{5} = \frac{6}{-2q} = \frac{-p}{-5} \Rightarrow pq = -15$$

$\therefore \quad \lambda = |pq| = |-15| = 15$

34. $\because$ Lines represented by $6x^2 + 5xy - 6y^2 = 0$ are $l_1 : 2x + 3y = 0$, $l_2 : 3x - 2y = 0$

and lines represented by

$$6x^2 + 5xy - 6y^2 - x + 5y - 1 = 0$$

are $l_3 : 2x + 3y - 1 = 0$

$l_4 : 3x - 2y + 1 = 0$

$\because$ $l_1 \perp l_2, l_2 \perp l_3, l_3 \perp l_4$ and $l_4 \perp l_1$.

Also, l_1, l_2 intersect at $(0, 0)$ and $(0, 0)$ is equidistant from l_3 and l_4. These lines form sides of square.

$\therefore$ Statement I is true.

Statement II is true but Statement II is not a correct explanation for Statement I.

35. $dx^3 + cx^2y + bxy^2 + ay^3 = (x^2 + pxy - y^2)(dx - ay)$

On comparing, we get

$$c = -a + pd \implies p = \frac{c + a}{d} \quad \text{...(i)}$$

and $$b = -ap - d \implies p = \frac{-b - d}{a} \quad \text{...(ii)}$$

From Eq. (i) and Eq. (ii),

$$\frac{c + a}{d} = \frac{-b - d}{a}$$

$$\Rightarrow \quad d^2 + bd + ac + a^2 = 0$$

$\therefore$ Statement I is true.

Hence, both statements are true but Statement II is not correct explanation for Statement I.

36. Pair of bisectors between the lines $x^2 - 2\alpha xy - y^2 = 0$ is

$$\frac{x^2 - y^2}{1 - (-1)} = \frac{xy}{-\alpha}$$

$$\Rightarrow \quad \alpha x^2 + 2xy - \alpha y^2 = 0 \quad \text{...(i)}$$

Comparing it with $y^2 + 2\beta xy - x^2 = 0$, then

$$-\frac{\alpha}{1} = \frac{1}{\beta} = \frac{\lambda}{-1}$$

$$\Rightarrow \quad \alpha\beta = -1$$

Hence, both statements are true and Statement II is correct explanation for Statement I.

37. Put $2h = -(a + b)$ in $ax^2 + 2hxy + by^2 = 0$

$$\Rightarrow \quad ax^2 - (a + b)xy + by^2 = 0$$

$$\Rightarrow \quad (x - y)(ax - by) = 0$$

$\Rightarrow$ One of the line bisects the angle between coordinate axes in positive quadrant.

Also, put $b = -2h - a$ in $ax - by$, we have

$$ax - by = ax - (-2h - a)y$$
$$= ax + (2h + a)y$$

Hence, $ax + (2h + a)y$ is a factor of

$$ax^2 + 2hyx + by^2 = 0$$

$\therefore$ Both statements are true but Statement II is not correct explanation for Statement I.

38. From $(y - mx)^2 = a^2(1 + m^2)$

$$\Rightarrow \quad y - mx = \pm a\sqrt{(1 + m^2)}$$

$\therefore$ Lines $\quad y - mx = a\sqrt{1 + m^2}$ ...(i)

and $\quad y - mx = -a\sqrt{1 + m^2}$ ...(ii)

and from $\quad (y - nx)^2 = a^2(1 + n^2)$

$$\Rightarrow \quad y - nx = \pm a\sqrt{(1 + n^2)}$$

$\therefore$ Lines $\quad y - nx = a\sqrt{1 + n^2}$ ...(iii)

and $\quad y - nx = -a\sqrt{1 + n^2}$ ...(iv)

Since, lines (i) and (ii) are parallel, the distance between lines (i) and (ii) is

$$\frac{|a\sqrt{1 + m^2} + a\sqrt{1 + m^2}|}{\sqrt{1 + m^2}} = |2a|$$

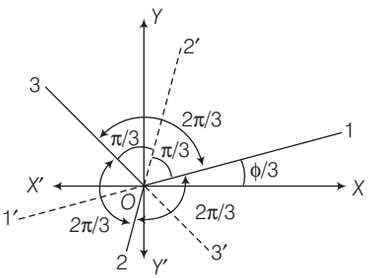
Similarly, (iii) and (iv) are parallel lines and the distance between them $= |2a|$. Therefore, distance between parallel lines are same.

Hence, lines (i), (ii), (iii) and (iv) form a rhombus.

39. Dividing by x^3, the given equation can be written as

$$m\left(1 - 3\left(\frac{y}{x}\right)^2\right) + \left(\frac{y}{x}\right)^3 - 3\left(\frac{y}{x}\right) = 0$$

writing $\frac{y}{x} = \tan\theta$, we have



$$m(1 - 3\tan^2\theta) + \tan^3\theta - 3\tan\theta = 0$$

$$\therefore \quad m = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = \tan 3\theta$$

Let $\quad m = \tan\phi$,

then $\quad \tan 3\theta = \tan\phi$

$$\therefore \quad 3\theta = \phi, \quad \phi + \pi, \quad \phi + 2\pi$$

$$\Rightarrow \quad \theta = \frac{\phi}{3}, \quad \frac{\phi}{3} + \frac{\pi}{3}, \quad \frac{\phi}{3} + \frac{2\pi}{3}$$

Thus, the given equation represents three lines through the origin and they are inclined at angles $\frac{\phi}{3}, \frac{\phi}{3} + \frac{\pi}{3}, \frac{\phi}{3} + \frac{2\pi}{3}$ with the X-axis. Clearly, they are equally inclined to each other.

40. Given pair of lines is

$$(A^2 - 3B^2)x^2 + 8ABxy + (B^2 - 3A^2)y^2 = 0 \quad \ldots(i)$$

Given line is $Ax + By + C = 0$ ...(ii)

Comparing Eq. (i) with $ax^2 + 2hxy + by^2 = 0$

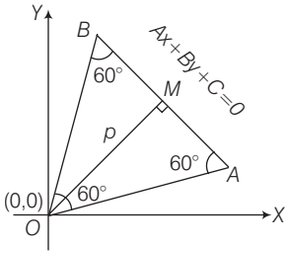
$$a = A^2 - 3B^2,\ h = 4AB,\ b = B^2 - 3A^2$$

If θ be the acute angle between the lines represented by Eq. (i), then

$$\tan\theta = \frac{2\sqrt{(h^2 - ab)}}{|a + b|} = \frac{2\sqrt{16A^2B^2 - (A^2 - 3B^2)(B^2 - 3A^2)}}{|(A^2 - 3B^2 + B^2 - 3A^2)|}$$

$$= \frac{2\sqrt{(3A^4 + 3B^4 + 6A^2B^2)}}{|-2(A^2 + B^2)|} = \frac{\sqrt{3}(A^2 + B^2)}{(A^2 + B^2)} = \sqrt{3}$$

$\therefore \quad \theta = 60°$



Eq. (i) can be written as

$$(B^2 - 3A^2)\left(\frac{y}{x}\right)^2 + 8AB\left(\frac{y}{x}\right) + (A^2 - 3B^2) = 0$$

Solving for y/x, we have

$$\frac{y}{x} = \frac{-8AB \pm \sqrt{64A^2B^2 - 4(B^2 - 3A^2)(A^2 - 3B^2)}}{2(B^2 - 3A^2)}$$

$$= \frac{-4AB \pm \sqrt{16A^2B^2 - (A^2B^2 - 3B^4 - 3A^4 + 9A^2B^2)}}{(B^2 - 3A^2)}$$

$$\Rightarrow \quad \frac{y}{x} = \frac{-4AB \pm \sqrt{3}(A^2 + B^2)}{(B^2 - 3A^2)}$$

$$\Rightarrow \quad y = \left\{\frac{-4AB \pm \sqrt{3}\,(A^2 + B^2)}{B^2 - 3A^2}\right\}x$$

Taking the positive sign, slope of one of the lines

$$= \frac{\sqrt{3}\,(A^2 + B^2) - 4AB}{B^2 - 3A^2}$$

and slope of the line $Ax + By + C = 0$ is $-A/B$.

Acute angle between these lines

$$= \tan^{-1}\left|\frac{\dfrac{\sqrt{3}(A^2 + B^2) - 4AB}{B^2 - 3A^2} + \dfrac{A}{B}}{1 - \left(\dfrac{A}{B}\right)\left[\dfrac{\sqrt{3}(A^2 + B^2) - 4AB}{B^2 - 3A^2}\right]}\right|$$

$$= \tan^{-1}\left|\frac{-3A^2 + \sqrt{3}\,A^2B - 3AB^2 + \sqrt{3}B^3}{-\sqrt{3}A^3 + A^2B - \sqrt{3}AB^2 + B^3}\right|$$

$$= \tan^{-1}(\sqrt{3}) = 60°$$

then remaining third angle is $180° - (60° + 60°) = 60°$.

$\therefore$ The ΔAOB is equilateral.

In ΔOAB,

Length of $\perp$ from O on AB $= p = OM = \dfrac{|0 + 0 + C|}{\sqrt{(A^2 + B^2)}}$

$$p = \frac{|C|}{\sqrt{(A^2 + B^2)}}$$

and in ΔOAM,

$$\sin 60° = \frac{p}{OA} \quad \Rightarrow \quad OA = \frac{2p}{\sqrt{3}}$$

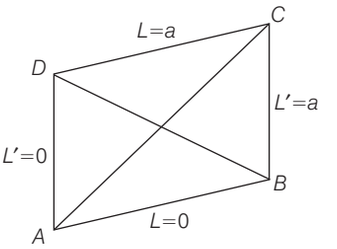
$$\therefore \qquad OA = \frac{2|C|}{\sqrt{3}\sqrt{(A^2 + B^2)}}$$

$\therefore$ Area of the equilateral triangle $= \dfrac{\sqrt{3}}{4}(\text{side})^2 = \dfrac{\sqrt{3}}{4}(OA)^2$

$$= \frac{\sqrt{3}}{4} \times \frac{4|C|^2}{3 \times (A^2 + B^2)} = \frac{C^2}{\sqrt{3}(A^2 + B^2)}$$

41. Lines of parallelogram are

$$L = 0,\ L = a$$
$$L' = 0,\ L' = a$$



where, $L = x\cos\theta + y\sin\theta - p$

$$L' = x\cos\theta' + y\sin\theta' - p'$$

Equation of the line AC, through the point of intersection of $L = 0$ and $L' = 0$ is

$$L + \lambda L' = 0$$

It also passes through

$$L = a \text{ and } L' = a, \quad \ldots(i)$$

then, $\quad a + \lambda a = 0$

$\therefore \qquad \lambda = -1$

Hence, diagonal of AC is $L - L' = 0$

i.e. $\quad x(\cos\theta - \cos\theta') + y(\sin\theta - \sin\theta') - p + p' = 0$

Equation of the line BD through point of intersection of $L = 0$ and $L' = a$ is

$$L + \lambda(L' - a) = 0$$

it also passes through $L = a$ and $L' = 0$.

$\therefore \quad a + \lambda(0 - a) = 0$

$\therefore \quad \lambda = 1$

Equation of diagonal BD is $L + L' - a = 0$

$\Rightarrow x(\cos\theta + \cos\theta') + y(\sin\theta + \sin\theta') - p - p' - a = 0$

42. Let the first equation represents lines

$$(lx + my + n)(l_1x + m_1y + n_1) = 0$$

then the second equation represents lines

$$(lx + my - n)(l_1x + m_1y - n_1) = 0$$

On comparing coefficients, $ll_1 = a, mm_1 = b, nn_1 = c$

$$mn_1 + m_1n = 2f, nl_1 + n_1l = 2g, \; lm_1 + l_1 m = 2h.$$

Let the angle between two non-parallel lines be θ, then

$$\tan\theta = \frac{2\sqrt{(h^2 - ab)}}{|a + b|}$$

$$\therefore \quad \sin\theta = \frac{2\sqrt{(h^2 - ab)}}{\sqrt{(a - b)^2 + 4h^2}}$$

$\therefore$ Area of parallelogram $= \dfrac{p_1p_2}{\sin\theta}$,

where, p_1 and p_2 are distance between two parallel sides.

$$\therefore \quad p_1 = \frac{|n - (-n)|}{\sqrt{l^2 + m^2}} \text{ and } p_2 = \frac{|n_1 - (-n_1)|}{\sqrt{l_1^2 + m_1^2}}$$

$$\therefore \quad p_1p_2 = \frac{|4nn_1|}{\sqrt{(l^2l_1^2 + l^2m_1^2 + m^2l_1^2 + m^2m_1^2)}}$$

$$= \frac{4|nn_1|}{\sqrt{\{l^2l_1^2 + m^2m_1^2 + (lm_1 + l_1m)^2 - 4ll_1mm_1\}}}$$

$$= \frac{4|c|}{\sqrt{a^2 + b^2 + 4h^2 - 2ab}} = \frac{4|c|}{\sqrt{(a - b)^2 + 4h^2}}$$

$$\therefore \quad \text{Area} = \frac{p_1p_2}{\sin\theta}$$

$$= \frac{4|c|}{(\sqrt{(a - b)^2 + 4h^2})\left(\dfrac{2\sqrt{h^2 - ab}}{\sqrt{(a - b)^2 + 4h^2}}\right)}$$

$$= \frac{2|c|}{\sqrt{(h^2 - ab)}}$$

43. Given, equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents two lines

$$lx + my + n = 0 \quad \text{and} \quad l_1x + m_1y + n_1 = 0$$

then, $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

$$= (lx + my + n)(l_1x + m_1y + n_1)$$

Comparing the coefficients of similar terms, we get

$$\left.\begin{aligned} &ll_1 = a, mm_1 = b, nn_1 = c \\ &lm_1 + ml_1 = 2h, n_1l + nl_1 = 2g, nm_1 + n_1m = 2f \end{aligned}\right\} \quad \ldots(\text{i})$$

Since, the lines $lx + my + n = 0$ and $l_1x + m_1y + n_1 = 0$ are equidistant from the origin, then

$$\frac{|n|}{\sqrt{l^2 + m^2}} = \frac{|n_1|}{\sqrt{l_1^2 + m_1^2}}$$

$$\Rightarrow \quad n^2l_1^2 + n^2m_1^2 = n_1^2l^2 + n_1^2m^2$$

$$\Rightarrow \quad n^2l_1^2 - n_1^2l^2 = n_1^2m^2 - n^2m_1^2$$

$$\Rightarrow (nl_1 - n_1l)(nl_1 + n_1l) = (n_1m - nm_1)(n_1m + nm_1)$$

On squaring both sides, we get

$$(nl_1 - n_1l)^2 (nl_1 + n_1l)^2 = (n_1m - nm_1)^2(n_1m + nm_1)^2$$

$$\Rightarrow \quad [(nl_1 + n_1l)^2 - 4ll_1nn_1][nl_1 + n_1l]^2$$

$$= [(n_1m + nm_1)^2 - 4mm_1nn_1][n_1m + nm_1]^2$$

$$\Rightarrow \quad (4g^2 - 4ac)(4g^2) = (4f^2 - 4bc)(4f^2)$$

[using the six relations]

$$\Rightarrow \quad (g^2 - ac)g^2 = (f^2 - bc)f^2 \qquad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad f^4 - g^4 = c(bf^2 - ag^2)$$

Also, product of distances from origin to the lines represented by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

is $\dfrac{|n|}{\sqrt{(l^2 + m^2)}} \cdot \dfrac{|n_1|}{\sqrt{(l_1^2 + m_1^2)}}$

$$= \frac{|nn_1|}{\sqrt{(l^2l_1^2 + l^2m_1^2 + m^2l_1^2 + m^2m_1^2)}}$$

$$= \frac{|nn_1|}{\sqrt{(l^2l_1^2 + l^2m_1^2 + m^2l_1^2 + m^2m_1^2)}}$$

$$= \frac{|nn_1|}{\sqrt{(l^2l_1^2 + m^2m_1^2 + (lm_1 + ml_1)^2 - 2ll_1mm_1)}}$$

$$= \frac{|c|}{\sqrt{(a^2 + b^2 + 4h^2 - 2ab)}} \qquad \text{[from Eq. (i)]}$$

$$= \frac{|c|}{\sqrt{(a - b)^2 + 4h^2}}$$

44. We know that bisectors are mutually perpendicular to each other, then

$$ax^4 + bx^3y + cx^2y^2 + dxy^3 + ay^4 = 0$$

represents two pairs of mutually perpendicular lines.

Let $ax^4 + bx^3y + cx^2y^2 + dxy^3 + ay^4$

$$= (ax^2 + pxy - ay^2)(x^2 + qxy - y^2)$$

where, p and q are constants.

Comparing the coefficients of similar terms, we get

$$b = aq + p \qquad \ldots(\text{i})$$

$$c = pq - 2a \qquad \ldots(\text{ii})$$

$$d = -p - aq \qquad \ldots(\text{iii})$$

On adding Eqs. (i) and (iii), we have

$$b + d = 0$$

From Eq. (ii), $pq = c + 2a$ ...(iv)

But given that bisectors of one pair are given by the other, i.e.

$$\frac{x^2 - y^2}{a - (-a)} = \frac{xy}{(p/2)}$$

$$\Rightarrow \quad \frac{x^2 - y^2}{4xy} = \frac{a}{p} \qquad \text{...(v)}$$

is the same as the other pair.

$$x^2 + qxy - y^2 = 0$$

or $$\frac{x^2 - y^2}{4xy} = -\frac{q}{4} \qquad \text{...(vi)}$$

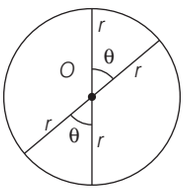
From Eqs. (v) and (vi), $\frac{a}{p} = -\frac{q}{4}$

$$\therefore \quad pq = -4a$$

Again from Eq. (iv), $-4a = c + 2a$

$$\therefore \quad c + 6a = 0$$

45. Let A be the area of small sector, then area of major sector is $3A$.



$$\therefore \quad A + 3A + A + 3A = \pi r^2$$

$$\Rightarrow \quad A = \frac{\pi}{8} r^2$$

$$\Rightarrow \quad \frac{1}{2} r^2\theta = \frac{\pi}{8} r^2 \quad [\because \text{area of sector} = \frac{1}{2} r^2\theta]$$

$$\therefore \quad \theta = \frac{\pi}{4}$$

$\therefore$ Angle between lines represented by

$ax^2 + 2(a + b)xy + by^2 = 0$ is $\frac{\pi}{4}$.

$$\Rightarrow \quad \tan\frac{\pi}{4} = \frac{2\sqrt{(a+b)^2 - ab}}{|a + b|}$$

$$\Rightarrow \quad 1 = \frac{2\sqrt{(a+b)^2 - ab}}{|a + b|}$$

$$\Rightarrow \quad (a + b)^2 = 4(a + b)^2 - 4ab$$

$$\therefore \quad 3a^2 + 2ab + 3b^2 = 0$$

46. The equation of the bisectors of the lines $xy = 0$ are $y = \pm x$.

Putting $y = \pm x$ in

$my^2 + (1 - m^2)xy - mx^2 = 0$, we get

$$\pm (1 - m^2)x^2 = 0$$

$$\Rightarrow \quad m^2 = 1$$

$\therefore$

CHAPTER

04

Circle

Learning Part

Session 1

- Definition
- Locus of the Mid-point of the Chords of the Circle that Subtends an Angle of 2θ at its Centre
- Equation of Circle in Different Forms

Session 2

- Diametric Form of a Circle
- Equation of Circle Passing Through Three Non-Collinear Points

Session 3

- Intercepts Made on the Axes by a Circle
- Position of a Point with Respect to Circle
- Different Forms of the Equations of a Circle
- Maximum and Minimum Distance of a Point from the Circle

Session 4

- Intersection of a Line and a Circle
- The Length of Intercept Cut-off from a Line by a Circle
- Tangent to a Circle at a Given Point
- Normal to a Circle at a Given Point
- Product of the Algebraical Distances *PA* and *PB* is Constant when from *P*, *A* Secant be Drawn to Cut the Circle in the Points *A* and *B*

Session 5

- Tangents from a Point to the Circle
- Power of a Point with Respect to a Circle
- Chord Bisected at a Given Point
- Director Circle
- Length of the Tangent from a Point to a Circle
- Chord of Contact
- Pair of Tangents

Session 6

- Diameter of a Circle
- Common Tangents to Two Circles
- Family of Circles
- Two Circles Touching Each Other
- Common Chord of Two Circles

Session 7

- Angle of Intersection of Two Circles
- Radical Centre
- Limiting Point
- Radical Axis
- Co-axial System of Circles
- Image of the Circle by the Line Mirror

Practice Part

- JEE Type Examples
- Chapter Exercises

Arihant on Your Mobile !

Exercises with the symbol can be practised on your mobile. See inside cover page to activate for free.

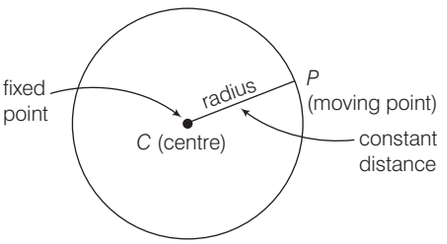
Session 1

Definition, Equation of Circle in Different Forms, Locus of the Mid-point of the Chords of the Circle that Subtends an Angle of 2θ at its Centre

Definition

A circle is the locus of a point which moves in a plane, so that its distance from a *fixed point* in the plane is always *constant*.

The fixed point is called the *centre* of the circle and the constant distance is called its *radius*.



i.e. CP = constant distance = Radius

Equation of a Circle : The curve traced by the moving point is called its *circumference*. i.e. the equation of any circle is satisfied by co-ordinates of all points on its circumference.

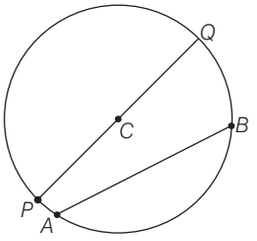
OR

The equation of the circle is meant the equation of the circumference.

OR

It is the set of all points lying on the circumference of the circle.

Chord and Diameter : The line joining any two points on the circumference is called a chord. If any chord passing through its centre is called its diameter.



AB = Chord, PQ = Diameter

where, C is centre of the circle.

Equation of Circle in Different Forms

1. Centre-radius form

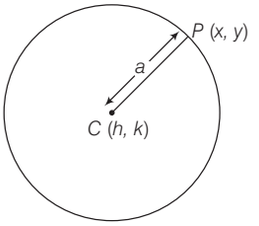
Let a be the *radius* and $C(h, k)$ be the *centre* of any circle.

If $P(x, y)$ be any point on the circumference.

Then, $CP = a \Rightarrow (CP)^2 = a^2$

$\Rightarrow$ $(x-h)^2 + (y-k)^2 = a^2$

This equation is known as the central form of the equation of a circle.



Remark

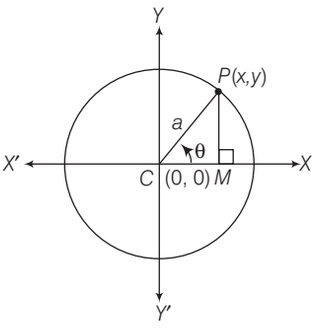
When, $C(h, k) = C(0, 0)$, then equation of circle becomes $x^2 + y^2 = a^2$ which is known as **standard form** of the circle.

2. Parametric form

If the radius of a circle whose centre is at $C(0, 0)$ makes an angle θ with the positive direction of X-axis, then θ is called the parameter.

Let $CP = a$

$\therefore$ $CM = x, PM = y \Rightarrow x = a\cos\theta, y = a\sin\theta$



Hence, $(a\cos\theta, a\sin\theta)$ or 'θ' are the parametric coordinates of the circle $x^2 + y^2 = a^2$ and $x = a\cos\theta$ and $y = a\sin\theta$ are called parametric equations of the circle $x^2 + y^2 = a^2$ with parameters a and θ. $(0 \le \theta < 2\pi)$.

Remarks

1. The parametric coordinates of any point on the circle $(x-h)^2 + (y-k)^2 = a^2$ are given by $(h + a\cos\theta, k + a\sin\theta)$ $(0 \le \theta < 2\pi)$ and parametric equations of the circle $(x-h)^2 + (y-k)^2 = a^2$ are $x = h + a\cos\theta$, $y = k + a\sin\theta$.
2. Equation of the chord of the circle $x^2 + y^2 = a^2$ joining $(a\cos\alpha, a\sin\alpha)$ and $(a\cos\beta, a\sin\beta)$ is
$$x\cos\left(\frac{\alpha+\beta}{2}\right) + y\sin\left(\frac{\alpha+\beta}{2}\right) = a\cos\left(\frac{\alpha-\beta}{2}\right).$$

General form The equation of the circle with centre (h, k) and radius a is $(x-h)^2 + (y-k)^2 = a^2$

or $$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - a^2 = 0 \quad \text{...(i)}$$

which is of the form

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...(ii)}$$

This is known as the **general equation** of a circle comparing Eqs. (i) and (ii), we get

$$h = -g, k = -f \quad \text{and} \quad a = \sqrt{(g^2 + f^2 - c)}$$

$\therefore$ Coordinates of the centre are $(-g, -f)$ and

$$\text{Radius} = \sqrt{(g^2 + f^2 - c)}\ (g^2 + f^2 \ge c)$$

Remarks

1. **Rule for finding the centre and radius of a circle**
   (i) Make the coefficients of x^2 and y^2 equal to 1 and right hand side equal to zero.
   (ii) Then, coordinates of centre will be (α, β), where, $\alpha = -\frac{1}{2}$ (coefficient of x) and $\beta = -\frac{1}{2}$ (coefficient of y)
   (iii) Radius $= \sqrt{\alpha^2 + \beta^2 - (\text{constant term})}$
2. **Conditions for a circle** A general equation of second degree
   $$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$
   in x, y represent a circle, if
   (i) coefficient of x^2 = coefficient of y^2
   i.e. $a = b$
   (ii) coefficient of xy is zero
   i.e. $h = 0$
3. **Nature of the circle** Radius of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $\sqrt{(g^2 + f^2 - c)}$
   Now, the following cases are possible :
   (i) If $g^2 + f^2 - c > 0$, then the radius of circle will be real. Hence in this case, real circle is possible.
   (ii) If $g^2 + f^2 - c = 0$, then the radius of circle will be real. Hence in this case, circle is called a point circle.
   (iii) If $g^2 + f^2 - c < 0$, then the radius of circle will be imaginary number. Hence in this case, circle is called a virtual circle or imaginary circle.
4. **Concentric circle** Two circles having the same centre $C(h, k)$ but different radii r_1 and r_2 respectively are called concentric circles. Thus, the circles $(x-h)^2 + (y-k)^2 = r_1^2$ and $(x-h)^2 + (y-k)^2 = r_2^2, r_1 \ne r_2$ are concentric circles. Therefore, the equations of concentric circles differ only in constant terms.

Example 1. Find the centre and radius of the circle
$$2x^2 + 2y^2 = 3x - 5y + 7$$

Sol. The given equation of circle is
$$2x^2 + 2y^2 = 3x - 5y + 7$$
or $$x^2 + y^2 - \frac{3}{2}x + \frac{5}{2}y - \frac{7}{2} = 0$$
If centre is (α, β), then
$$\alpha = -\frac{1}{2}\left(-\frac{3}{2}\right) = \frac{3}{4}$$
and $$\beta = -\frac{1}{2}\left(\frac{5}{2}\right) = -\frac{5}{4}$$
$\therefore$ Centre of circle is (α, β) i.e. $\left(\frac{3}{4}, -\frac{5}{4}\right)$
and radius of the circle
$$= \sqrt{\alpha^2 + \beta^2 - (\text{constant term})}$$
$$= \sqrt{\frac{9}{16} + \frac{25}{16} + \frac{7}{2}} = \sqrt{\frac{9 + 25 + 56}{16}} = \frac{3\sqrt{10}}{4}$$

Example 2. ***Prove that the radii of the circles*** $x^2 + y^2 = 1, x^2 + y^2 - 2x - 6y = 6$ and $x^2 + y^2 - 4x - 12y = 9$ ***are in AP.***

Sol. Given circles are $$x^2 + y^2 = 1 \quad \text{...(i)}$$
$$x^2 + y^2 - 2x - 6y - 6 = 0 \quad \text{...(ii)}$$
and $$x^2 + y^2 - 4x - 12y - 9 = 0 \quad \text{...(iii)}$$
Let r_1, r_2 and r_3 be the radii of the circles Eqs. (i), (ii) and (iii), respectively.
Then, $r_1 = 1$
$$r_2 = \sqrt{(-1)^2 + (-3)^2 + 6} = 4$$
and $$r_3 = \sqrt{(-2)^2 + (-6)^2 + 9} = 7$$
Clearly, $r_2 - r_1 = 4 - 1 = 3 = r_3 - r_2$
Hence, r_1, r_2, r_3 are in AP.

Example 3. ***Find the equation of the circle whose centre is the point of intersection of the lines*** $2x - 3y + 4 = 0$ ***and*** $3x + 4y - 5 = 0$ ***and passes through the origin.***

Sol. The point of intersection of the lines $2x - 3y + 4 = 0$ and $3x + 4y - 5 = 0$ is $\left(-\frac{1}{17}, \frac{22}{17}\right)$.

Therefore, the centre of the circle is at $\left(-\frac{1}{17}, \frac{22}{17}\right)$.

Since, the origin lies on the circle, its distance from the centre of the circle is radius of the circle, therefore,

$$r = \sqrt{\left(-\frac{1}{17} - 0\right)^2 + \left(\frac{22}{17} - 0\right)^2} = \sqrt{\frac{485}{289}}$$

$\therefore$ The equation of the circle becomes

$$\left(x + \frac{1}{17}\right)^2 + \left(y - \frac{22}{17}\right)^2 = \frac{485}{289}$$

or $\quad 17\,(x^2 + y^2) + 2x - 44y = 0$

Aliter : $\because$ Point of intersection of the lines $2x - 3y + 4 = 0$ and $3x + 4y - 5 = 0$ is $\left(-\frac{1}{17}, \frac{22}{17}\right)$.

Therefore, the centre of the circle is at $\left(-\frac{1}{17}, \frac{22}{17}\right)$

Let required circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \ldots(i)$$

Here, $-g = -\frac{1}{17}, -f = \frac{22}{17}, c = 0$

[$\because$ Circle passes through origin]

From Eq. (i), $\quad x^2 + y^2 + \frac{2x}{17} - \frac{44}{17}y = 0$

or $\quad 17(x^2 + y^2) + 2x - 44y = 0$

Example 4. *Find the equation of the circle concentric with the circle* $x^2 + y^2 - 8x + 6y - 5 = 0$ *and passing through the point* $(-2, -7)$ *.*

Sol. The given equation of circle is

$$x^2 + y^2 - 8x + 6y - 5 = 0$$

Therefore, the centre of the circle is at $(4, -3)$. Since, the required circle is concentric with this circle, therefore, the centre of the required circle is also at $(4, -3)$. Since, the point $(-2, -7)$ lies on the circle, the distance of the centre from this point is the radius of the circle. Therefore, we get

$$r = \sqrt{(4 + 2)^2 + (-3 + 7)^2} = \sqrt{52}$$

Hence, the equation of the circle becomes

$$(x - 4)^2 + (y + 3)^2 = 52$$

or $\quad x^2 + y^2 - 8x + 6y - 27 = 0$

Aliter : Equation of concentric circle is

$$x^2 + y^2 - 8x + 6y + \lambda = 0 \qquad \ldots(i)$$

which pass through $(-2, -7)$, then

$$4 + 49 + 16 - 42 + \lambda = 0$$

$\therefore \quad \lambda = -27$

From Eq. (i), required circle is

$$x^2 + y^2 - 8x + 6y - 27 = 0$$

Example 5. A circle has radius 3 units and its centre lies on the line $y = x - 1$. Find the equation of the circle if it passes through (7, 3).

Sol. Let the centre of the circle be (h, k). Since, the centre lies on $y = x - 1$, we get

$$k = h - 1 \qquad \ldots(i)$$

Since, the circle passes through the point $(7, 3)$, therefore the distance of the centre from this point is the radius r of the circle. We have,

$$r = \sqrt{(h - 7)^2 + (k - 3)^2}$$

or $\quad 3 = \sqrt{(h - 7)^2 + (h - 1 - 3)^2}$ [from Eq. (i)]

$\Rightarrow \quad 9 = (h - 7)^2 + (h - 4)^2$

$\Rightarrow \quad h^2 - 11h + 28 = 0$

or $\quad (h - 7)(h - 4) = 0$

or $\quad h = 7 \quad$ and $\quad h = 4$

For $h = 7$, we get $k = 6$ from Eq. (i)

and for $h = 4$, we get $k = 3$, from Eq. (i).

Hence, there are two circles which satisfy the given conditions. They are

$$(x - 7)^2 + (y - 6)^2 = 9$$

or $\quad x^2 + y^2 - 14x - 12y + 76 = 0$

and $\quad (x - 4)^2 + (y - 3)^2 = 9$

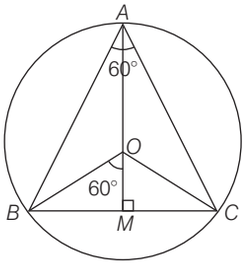
or $\quad x^2 + y^2 - 8x - 6y + 16 = 0$

Example 6. Find the area of an equilateral triangle inscribed in the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.

Sol. Given circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \ldots(i)$$

Let O be the centre and ABC be an equilateral triangle inscribed in the circle Eq. (i).



$$O \equiv (-g, -f)$$

and $\quad OA = OB = OC = \sqrt{g^2 + f^2 - c} \qquad \ldots(ii)$

In ΔBOM, $\sin 60° = \frac{BM}{OB}$

$\Rightarrow$ $BM = OB \sin 60° = (OB)\frac{\sqrt{3}}{2}$

$\therefore$ $BC = 2BM = \sqrt{3}\,(OB)$ ...(iii)

$\therefore$ Area of $\Delta ABC = \frac{\sqrt{3}}{4}(BC)^2$

$= \frac{\sqrt{3}}{4} 3\,(OB)^2$ [from Eq. (iii)]

$= \frac{3\sqrt{3}}{4}(g^2 + f^2 - c)$ sq units.

Example 7. Find the parametric form of the equation of the circle

$$x^2 + y^2 + px + py = 0.$$

Sol. Equation of the circle can be re-written in the form

$$\left(x + \frac{p}{2}\right)^2 + \left(y + \frac{p}{2}\right)^2 = \frac{p^2}{2}$$

Therefore, the parametric form of the equation of the given circle is

$$x = -\frac{p}{2} + \frac{p}{\sqrt{2}}\cos\theta = \frac{p}{2}(-1 + \sqrt{2}\cos\theta)$$

and $$y = -\frac{p}{2} + \frac{p}{\sqrt{2}}\sin\theta = \frac{p}{2}(-1 + \sqrt{2}\sin\theta)$$

where, $0 \le \theta < 2\pi$.

Example 8. If the parametric of form of a circle is given by

(a) $x = -4 + 5\cos\theta$ and $y = -3 + 5\sin\theta$

(b) $x = a\cos\alpha + b\sin\alpha$ and $y = a\sin\alpha - b\cos\alpha$

find its cartesian form.

Sol. (a) The given equations are

$$x = -4 + 5\cos\theta$$

and $$y = -3 + 5\sin\theta$$

or $$(x + 4) = 5\cos\theta$$ ...(i)

and $$(y + 3) = 5\sin\theta$$ ...(ii)

Squaring and adding Eqs. (i) and (ii), then

$$(x + 4)^2 + (y + 3)^2 = 5^2$$

or $$(x + 4)^2 + (y + 3)^2 = 25$$

(b) The given equations are

$$x = a\cos\alpha + b\sin\alpha$$ ...(iii)

$$y = a\sin\alpha - b\cos\alpha$$ ...(iv)

Squaring and adding Eqs. (iii) and (iv), then

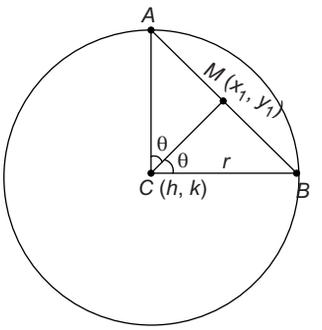
$$x^2 + y^2 = (a\cos\alpha + b\sin\alpha)^2 + (a\sin\alpha - b\cos\alpha)^2$$

$\Rightarrow$ $x^2 + y^2 = a^2 + b^2$

Locus of the Mid-point of the Chords of the Circle that Subtends an Angle of 2θ at its Centre

Let mid-point $M(x_1, y_1)$ and centre, radius of circle are (h,k), r respectively, then

$$\cos\theta = \frac{CM}{r} = \frac{\sqrt{(x_1 - h)^2 + (y_1 - k)^2}}{r}$$



$\therefore$ Required locus is $\frac{(x-h)^2 + (y-k)^2 - r^2}{r^2} = -\sin^2\theta$

Remembering Method :

First make coefficient of x^2 = coefficient of $y^2 = 1$

and RHS of circle is zero, then $\frac{\text{LHS of circle}}{(\text{radius})^2} = -\sin^2\theta$

Example 9. Find the locus of mid-points of the chords of the circle $4x^2 + 4y^2 - 12x + 4y + 1 = 0$ that subtend an angle of $\frac{2\pi}{3}$ at its centre.

Sol. Here, $2\theta = \frac{2\pi}{3} \Rightarrow \theta = \frac{\pi}{3}$

Equation of circle can be written as

$$x^2 + y^2 - 3x + y + \frac{1}{4} = 0$$

$\therefore$ Required locus is

$$\frac{x^2 + y^2 - 3x + y + \frac{1}{4}}{\left(\sqrt{\frac{9}{4} + \frac{1}{4} - \frac{1}{4}}\right)^2} = -\sin^2\left(\frac{\pi}{3}\right) = -\frac{3}{4}$$

$\Rightarrow$ $$x^2 + y^2 - 3x + y + \frac{1}{4} = -\frac{27}{16}$$

or $$16(x^2 + y^2) - 48x + 16y + 31 = 0$$

Exercise for Session 1

1. If $x^2 + y^2 - 2x + 2ay + a + 3 = 0$ represents a real circle with non-zero radius, then most appropriate is

(a) $a \in (-\infty, -1)$ (b) $a \in (-1, 2)$

(c) $a \in (2, \infty)$ (d) $a \in (-\infty, -1) \cup (2, \infty)$

2. If the equation $ax^2 + (2-b)xy + 3y^2 - 6bx + 30y + 6b = 0$ represents a circle, then $a^2 + b^2$ is

(a) 5 (b) 13

(c) 25 (d) 41

3. The equation of the circle passing through $(4, 5)$ having the centre at $(2, 2)$ is

(a) $x^2 + y^2 + 4x + 4y - 5 = 0$ (b) $x^2 + y^2 - 4x - 4y - 5 = 0$

(c) $x^2 + y^2 - 4x - 13 = 0$ (d) $x^2 + y^2 - 4x - 4y + 5 = 0$

4. Equation of the diameter of the circle is given by $x^2 + y^2 - 12x + 4y + 6 = 0$ is given by

(a) $x + y = 0$ (b) $x + 3y = 0$

(c) $x = y$ (d) $3x + 2y = 0$

5. If the lines $3x - 4y + 4 = 0$ and $6x - 8y - 7 = 0$ are tangents to a circle, then the diameter of the circle is

(a) $\frac{3}{2}$ (b) 3

(c) $\frac{5}{2}$ (d) 5

6. Area of a circle in which a chord of length $\sqrt{2}$ makes an angle $\frac{\pi}{2}$ at the centre is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$

(c) π (d) 2π

7. The lines $2x - 3y - 5 = 0$ and $3x - 4y = 7$ are diameters of a circle of area 154 sq units, then the equation of the circle is :

(a) $x^2 + y^2 + 2x - 2y - 62 = 0$ (b) $x^2 + y^2 + 2x - 2y - 47 = 0$

(c) $x^2 + y^2 - 2x + 2y - 62 = 0$ (d) $x^2 + y^2 - 2x + 2y - 47 = 0$

8. If the lines $2x + 3y + 1 = 0$ and $3x - y - 4 = 0$ lie along diameters of a circle of circumference 10π, then the equation of the circle is

(a) $x^2 + y^2 - 2x + 2y - 23 = 0$ (b) $x^2 + y^2 - 2x - 2y - 23 = 0$

(c) $x^2 + y^2 + 2x + 2y - 23 = 0$ (d) $x^2 + y^2 + 2x - 2y - 23 = 0$

9. The triangle PQR is inscribed in the circle $x^2 + y^2 = 25$. If Q and R have coordinates $(3, 4)$ and $(-4, 3)$ respectively, then $\angle QPR$ is equal to

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

10. If a circle is concentric with the circle $x^2 + y^2 - 4x - 6y + 9 = 0$ and passes through the point $(-4, -5)$, then its equation is

(a) $x^2 + y^2 + 4x + 6y - 87 = 0$ (b) $x^2 + y^2 - 4x + 6y + 87 = 0$

(c) $x^2 + y^2 - 4x - 6y - 87 = 0$ (d) $x^2 + y^2 + 4x + 6y + 87 = 0$

11. Let AB be a chord of the circle $x^2 + y^2 = r^2$ subtending a right angle at the centre. Then, the locus of the centroid of the ΔPAB as P moves on the circle is

(a) a parabola (b) a circle

(c) an ellipse (d) a pair of straight lines

12. Let PQ and RS be tangents extremities of the diameter PR of a circle of radius r. If PS and RQ intersect at a point X on the circumference of the circle, then $2r$ equals

(a) $\sqrt{PQ \cdot RS}$ (b) $\dfrac{PQ + RS}{2}$

(c) $\dfrac{2PQ.RS}{PQ + RS}$ (d) $\sqrt{\dfrac{(PQ)^2 + (RS)^2}{2}}$

13. Find the centre and radius of the circle $5x^2 + 5y^2 + 4x - 8y = 16$.

14. Prove that the centres of the circles $x^2 + y^2 = 1, x^2 + y^2 + 6x - 2y - 1 = 0$ and $x^2 + y^2 - 12x + 4y = 1$ are collinear.

15. Find the equation of the circle whose centre is $(1, 2)$ and which passes through the point of intersection of $3x + y = 14$ and $2x + 5y = 18$.

16. Find the equation of the circle passing through the centre of the circle $x^2 + y^2 - 4x - 6y = 8$ and being concentric with the circle $x^2 + y^2 - 2x - 8y = 5$.

17. Prove that the locus of the centre of the circle $\dfrac{1}{2}(x^2 + y^2) + x\cos\theta + y\sin\theta - 4 = 0$ is $x^2 + y^2 = 1$.

18. Find the equation of the following curves in cartesian form. If the curve is a circle, then find its centre and radius

$x = -1 + 2\cos\alpha,\ y = 3 + 2\sin\alpha \qquad (0 \le \alpha < 2\pi)$

Session 2

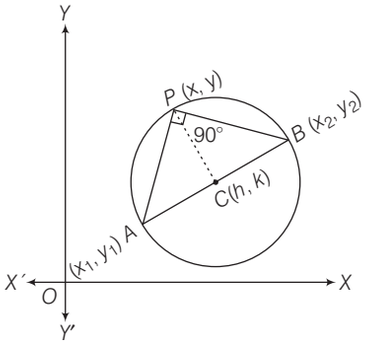
Diametric Form of a Circle, Equation of Circle Passing Through Three Non-Collinear Points

Diametric Form of a Circle

Theorem : The equation of the circle on the line segment joining (x_1, y_1) and (x_2, y_2) as diameter is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.$$

Proof : Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the end points of a diameter and let $P(x, y)$ be any point on the circle



Now, $\quad$ Slope of $AP = \dfrac{y - y_1}{x - x_1}$

and $\quad$ Slope of $BP = \dfrac{y - y_2}{x - x_2}$

Since, $\angle APB = 90°$

$\therefore$ Slope of $AP \times$ Slope of $BP = -1$

$$\Rightarrow \quad \frac{(y - y_1)}{(x - x_1)} \times \frac{(y - y_2)}{(x - x_2)} = -1$$

$$\Rightarrow \quad (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

Remark

The diameteric form of a circle can also be written as

$$x^2 + y^2 - x(x_1 + x_2) - y(y_1 + y_2) + x_1x_2 + y_1y_2 = 0$$

or $\quad x^2 + y^2 - x$ (sum of abscissae) $- y$ (sum of ordinates) + product of abscissae + product of ordinates $= 0$

Example 10. Find the equation of the circle the end points of whose diameter are the centres of the circles $x^2 + y^2 + 6x - 14y = 1$ and $x^2 + y^2 - 4x + 10y = 2$.

Sol. The centres of the given circles

$$x^2 + y^2 + 6x - 14y - 1 = 0$$

and $x^2 + y^2 - 4x + 10y - 2 = 0$ are $(-3, 7)$ and $(2, -5)$, respectively.

According to the question, the points $(-3, 7)$ and $(2, -5)$ are the extremities of the diameter of required circle.

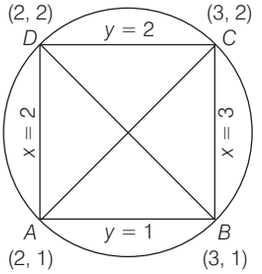
Hence, equation of circle is

$$(x + 3)(x - 2) + (y - 7)(y + 5) = 0$$

$$\Rightarrow \quad x^2 + y^2 + x - 2y - 41 = 0$$

Example 11. The sides of a square are $x = 2, x = 3, y = 1$ and $y = 2$. Find the equation of the circle drawn on the diagonals of the square as its diameter.

Sol. Let $ABCD$ be a square and equation of its sides AB, BC, CD and DA are $y = 1$, $x = 3$, $y = 2$, and $x = 2$, respectively.



Then, $A \equiv (2, 1)$, $B \equiv (3, 1)$, $C \equiv (3, 2)$ and $D \equiv (2, 2)$

Since, diagonals of squares are the diameters of the circle, then equation of circle is

$$(x - 2)(x - 3) + (y - 1)(y - 2) = 0$$

$$\Rightarrow \quad x^2 + y^2 - 5x - 3y + 8 = 0 \text{ (If } AC \text{ as diameter).}$$

Example 12. ***The abscissae of two points*** A ***and*** B ***are the roots of the equation*** $x^2 + 2ax - b^2 = 0$ ***and their ordinates are the roots of the equation*** $x^2 + 2px - q^2 = 0$***. Find the equation and the radius of the circle with*** AB ***as diameter.***

Sol. Given equations are

$$x^2 + 2ax - b^2 = 0 \qquad \text{...(i)}$$

and $x^2 + 2px - q^2 = 0$ ...(ii)

Let the roots of the Eq. (i) be α and β and those of Eq. (ii) be γ and δ. Then,

$$\left.\begin{aligned}\alpha + \beta &= -2a\\ \alpha\beta &= -b^2\end{aligned}\right\} \text{ and } \left.\begin{aligned}\gamma + \delta &= -2p\\ \gamma\delta &= -q^2\end{aligned}\right\}$$

Let $A \equiv (\alpha, \gamma)$ and $B \equiv (\beta, \delta)$.

Now, equation of circle whose diameter is AB will be

$$(x-\alpha)(x-\beta) + (y-\gamma)(y-\delta) = 0$$

$\Rightarrow$ $x^2 + y^2 - (\alpha+\beta)x - (\gamma+\delta)y + \alpha\beta + \gamma\delta = 0$

$\Rightarrow$ $x^2 + y^2 + 2ax + 2py - b^2 - q^2 = 0$

and radius $= \sqrt{(a^2 + p^2 + b^2 + q^2)}$

Equation of Circle Passing Through Three Non-Collinear Points

Let the equation of circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad ...(i)$$

If three points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ lie on the circle Eq. (i), their coordinates must satisfy its equation. Hence, solving equations

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0, \quad ...(ii)$$

$$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0, \quad ...(iii)$$

and $$x_3^2 + y_3^2 + 2gx_3 + 2fy_3 + c = 0, \quad ...(iv)$$

g, f, c are obtained from Eqs. (ii), (iii) and (iv). Then, to find the circle Eq. (i).

Aliter : Eliminate g, f, c from Eqs. (i), (ii), (iii) and (iv) with the help of determinant

$$\begin{vmatrix} x^2+y^2 & x & y & 1\\ x_1^2+y_1^2 & x_1 & y_1 & 1\\ x_2^2+y_2^2 & x_2 & y_2 & 1\\ x_3^2+y_3^2 & x_3 & y_3 & 1\end{vmatrix} = 0$$

which is the required equation of circle

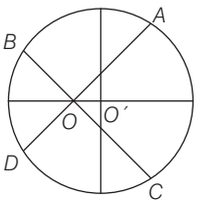
Applying $R_1 \to R_1 - R_4$, $R_2 \to R_2 - R_4$ and $R_3 \to R_3 - R_4$ then, we get

$$\begin{vmatrix} x^2+y^2-x_3^2-y_3^2 & x-x_3 & y-y_3 & 0\\ x_1^2+y_1^2-x_3^2-y_3^2 & x_1-x_3 & y_1-y_3 & 0\\ x_2^2+y_2^2-x_3^2-y_3^2 & x_2-x_3 & y_2-y_3 & 0\\ x_3^2+y_3^2 & x_3 & y_3 & 1\end{vmatrix} = 0$$

i.e. $$\begin{vmatrix} x^2+y^2-x_3^2-y_3^2 & x-x_3 & y-y_3\\ x_1^2+y_1^2-x_3^2-y_3^2 & x_1-x_3 & y_1-y_3\\ x_2^2+y_2^2-x_3^2-y_3^2 & x_2-x_3 & y_2-y_3\end{vmatrix} = 0$$

Remarks

1. **Cyclic quadrilateral** If all four vertices of a quadrilateral lie on a circle, then the quadrilateral is called a cyclic quadrilateral. The four vertices are said to be concyclic.
2. **Concyclic points** If A, B, C, D are concyclic, then
$$OA \cdot OD = OB \cdot OC$$
where, O' be the centre of the circle.



Example 13. Find the equation of the circle which passes through the points $(4, 1), (6, 5)$ and has its centre on the line $4x + y = 16$.

Sol. Let the equation of circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad ...(i)$$

Since, the centre of Eq. (i) i.e. $(-g, -f)$ lies on $4x + y = 16$

then, $-4g - f = 16$

or $4g + f + 16 = 0$ ...(ii)

Since, the points $(4, 1)$ and $(6, 5)$ lie on circle $x^2 + y^2 + 2gx + 2fy + c = 0$, we get the equations

$$16 + 1 + 8g + 2f + c = 0$$

or $17 + 8g + 2f + c = 0$ ...(iii)

and $36 + 25 + 12g + 10f + c = 0$

or $61 + 12g + 10f + c = 0$ ...(iv)

Subtracting Eq. (iii) from Eq. (iv), then

$$44 + 4g + 8f = 0 \quad ...(v)$$

Solving Eqs. (ii) and (v), we get

$$f = -4 \quad \text{and} \quad g = -3$$

Now, from Eq. (iii), $17 - 24 - 8 + c = 0$

$\Rightarrow$ $c = 15$

Hence, the equation of circle becomes

$$x^2 + y^2 - 6x - 8y + 15 = 0$$

Example 14. Find the equation of the circle passing through the three non-collinear points $(1, 1), (2, -1)$ and $(3, 2)$.

Sol. Let the equation of circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad ...(i)$$

Since, the three given points lie on circle Eq. (i), we get

$$1 + 1 + 2g + 2f + c = 0$$

or $2g + 2f + c + 2 = 0$ ...(ii)

$\Rightarrow$ $4 + 1 + 4g - 2f + c = 0$

or $4g - 2f + c + 5 = 0$ ...(iii)

$\Rightarrow$ $9 + 4 + 6g + 4f + c = 0$

or $6g + 4f + c + 13 = 0$ ...(iv)

Subtracting Eq. (ii) from Eq. (iii) and subtracting Eq. (iii) from Eq. (iv), then

$2g - 4f + 3 = 0$ ...(v)

and $2g + 6f + 8 = 0$ ...(vi)

Solving Eq. (v) and Eq. (vi), we get

$$f = -\frac{1}{2} \quad \text{and} \quad g = -\frac{5}{2}$$

Now, from Eq. (ii), $-5 - 1 + c + 2 = 0$

$\therefore$ $c = 4$

Hence, from Eq. (i), equation of circle is

$$x^2 + y^2 - 5x - y + 4 = 0$$

Aliter I Equation of circle passing through three points $(1, 1), (2, -1)$ and $(3, 2)$ is

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ 1^2 + 1^2 & 1 & 1 & 1 \\ 2^2 + (-1)^2 & 2 & -1 & 1 \\ 3^2 + 2^2 & 3 & 2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x^2 + y^2 & x & y & 1 \\ 2 & 1 & 1 & 1 \\ 5 & 2 & -1 & 1 \\ 13 & 3 & 2 & 1 \end{vmatrix} = 0$$

Applying $R_1 \to R_1 - R_2, R_3 \to R_3 - R_2$ and $R_4 \to R_4 - R_2$, then

$$\Rightarrow \quad \begin{vmatrix} x^2 + y^2 - 2 & x - 1 & y - 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 2 & \ldots\ 1 & \ldots\ 1 & \ldots\ 1 \\ \vdots & \vdots & \vdots & \vdots \\ 3 & 1 & -2 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 11 & 2 & 1 & 0 \end{vmatrix} = 0$$

Expand with respect to fourth column, then

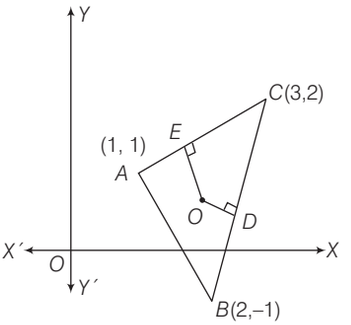
$$\begin{vmatrix} x^2 + y^2 - 2 & x - 1 & y - 1 \\ 3 & 1 & -2 \\ 11 & 2 & 1 \end{vmatrix} = 0$$

Expand with respect to first now, then

$$(x^2 + y^2 - 2)(5) - (x - 1)(25) + (y - 1)(-5) = 0$$

or $x^2 + y^2 - 5x - y + 4 = 0$

Aliter II The centre of the circumcircle is the point of intersection of the right bisectors of the sides of the triangle and the radius is the distance of the circumcentre from any of the vertices of the triangle.



Let D and E are the mid-points of BC and CA, then

$$D \equiv \left(\frac{5}{2}, \frac{1}{2}\right) \quad \text{and} \quad E \equiv \left(2, \frac{3}{2}\right)$$

Slope of $BC = \dfrac{2 - (-1)}{3 - 2} = 3$

$\therefore$ Slope of $OD = -\dfrac{1}{3}$

$\therefore$ Equation of OD, $y - \dfrac{1}{2} = -\dfrac{1}{3}\left(x - \dfrac{5}{2}\right)$

$\Rightarrow$ $6y - 3 = -2x + 5$

$\therefore$ $2x + 6y - 8 = 0$ ...(i)

or $x + 3y - 4 = 0$

and Slope of $CA = \dfrac{1 - 2}{1 - 3} = \dfrac{1}{2}$

$\therefore$ Slope of $OE = -2$

$\therefore$ Equation of OE,

$$y - \frac{3}{2} = -2(x - 2)$$

$\Rightarrow$ $2y - 3 = -4x + 8$

$\Rightarrow$ $4x + 2y - 11 = 0$ ...(ii)

Solving Eq. (i) and Eq. (ii), we get $x = \dfrac{5}{2}$ and $y = \dfrac{1}{2}$

$\therefore$ Circumcentre is $\left(\dfrac{5}{2}, \dfrac{1}{2}\right)$ and radius

$$OC = \sqrt{\left(3 - \frac{5}{2}\right)^2 + \left(2 - \frac{1}{2}\right)^2} = \sqrt{\frac{5}{2}}$$

$\therefore$ Equation of circle is

$$(x - 5/2)^2 + (y - 1/2)^2 = 5/2$$

$\Rightarrow$ $x^2 + y^2 - 5x - y + 4 = 0$

Example 15. Show that the four points $(1, 0), (2, -7), (8, 1)$ and $(9, -6)$ are concyclic.

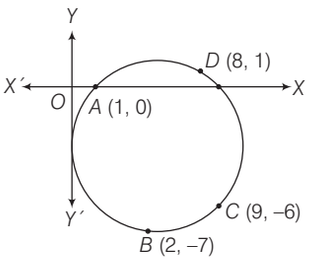
Sol. Since, the given four points are concyclic, we are to show that they lie on a circle. Let the general equation of circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0$$ ...(i)

has three parameters, it is sufficient to obtain the equation of the circle passing through any three of these points. For concyclic, the fourth point should lie on this circle.

Let three points $A(1, 0)$, $B(2, -7)$ and $D(8, 1)$ lie on Eq. (i), then

$$1 + 0 + 2g + 0 + c = 0 \quad \text{or} \quad 1 + 2g + c = 0 \qquad \text{...(ii)}$$



$$(2)^2 + (-7)^2 + 2g(2) + 2f(-7) + c = 0$$

or $$53 + 4g - 14f + c = 0 \qquad \text{...(iii)}$$

and $$(8)^2 + (1)^2 + 2g(8) + 2f(1) + c = 0$$

$$\Rightarrow \quad 65 + 16g + 2f + c = 0 \qquad \text{...(iv)}$$

Now, subtracting Eq. (ii) from Eq. (iii), we get

$$52 + 2g - 14f = 0$$

or $$26 + g - 7f = 0 \qquad \text{...(v)}$$

and subtracting Eq. (iii) from Eq. (iv), we get

$$12 + 12g + 16f = 0$$

$$\Rightarrow \quad 3 + 3g + 4f = 0 \qquad \text{...(vi)}$$

Solving Eq. (v) and Eq. (vi), we get

$$g = -5 \quad \text{and} \quad f = 3$$

From Eq. (ii), $1 - 10 + c = 0$

$$\therefore \quad c = 9$$

Therefore, equation of circle passing through these points is

$$x^2 + y^2 - 10x + 6y + 9 = 0$$

Substituting the fourth point in the equation of this circle, we get

$$(9)^2 + (-6)^2 - 10(9) + 6(-6) + 9 = 0$$

Hence, the point $C(9, -6)$ lies on the circle, that is, the four points are concyclic.

Exercise for Session 2

1. If the line $x + 2\lambda y + 7 = 0$ is a diameter of the circle $x^2 + y^2 - 6x + 2y = 0$, then the value of λ is

(a) 1 (b) 3
(c) 5 (d) 7

2. If one end of a diameter of the circle $2x^2 + 2y^2 - 4x - 8y + 2 = 0$ is $(-1, 2)$, then the other end of the diameter is

(a) (2, 1) (b) (3, 2)
(c) (4, 3) (d) (5, 4)

3. If a circle passes through the points (0,0), $(a, 0)$ and $(0, b)$, then centre of the circle is

(a) (a, b) (b) $\left(\frac{a}{2}, \frac{b}{2}\right)$
(c) $\left(\frac{a}{2}, \frac{b}{4}\right)$ (d) $\left(\frac{a}{4}, \frac{b}{2}\right)$

4. A circle passes through the points $(-1, 3)$ and $(5, 11)$ and its radius is 5. Then, its centre is

(a) $(-5, 0)$ (b) $(-5, 7)$
(c) $(2, 7)$ (d) $(5, 0)$

5. The radius of the circle, having centre at $(2, 1)$ whose one of the chord is a diameter of the circle $x^2 + y^2 - 2x - 6y + 6 = 0$ is

(a) 3 (b) 2
(c) 1 (d) $\sqrt{3}$

6. The centre of the circle inscribed in the square formed by the lines $x^2 - 8x + 12 = 0$ and $y^2 - 14y + 45 = 0$ is

(a) (4, 7) (b) (7, 4)
(c) (9, 4) (d) (4, 9)

7. $ABCD$ is a square whose side is a. The equation of the circle circumscribing the square, taking AB and AD as the axes of reference is

(a) $x^2 + y^2 + ax + ay = 0$ (b) $x^2 + y^2 - ax + ay = 0$

(c) $x^2 + y^2 - ax - ay = 0$ (d) $x^2 + y^2 + ax - ay = 0$

8. The locus of the centre of the circle for which one end of the diameter is (3, 3) while the other end lies on the line $x + y = 4$ is

(a) $x + y = 3$ (b) $x + y = 5$

(c) $x + y = 7$ (d) $x + y = 9$

9. The equation of the circle which passes through (1, 0) and (0, 1) and has its radius as small as possible is

(a) $x^2 + y^2 + x + y = 0$ (b) $x^2 + y^2 - x + y = 0$

(c) $x^2 + y^2 + x - y = 0$ (d) $x^2 + y^2 - x - y = 0$

10. If the points $(2, 0), (0, 1), (4,5)$ and $(0, c)$ are concyclic, then the value of c is

(a) 1 (b) -1

(c) $\frac{14}{3}$ (d) $-\frac{14}{3}$

11. The point on a circle nearest to the point $P(2,1)$ is at a distance of 4 units and farthest point is (6, 5), then the centre of the circle is

(a) $(3+\sqrt{2}, 2+\sqrt{2})$ (b) $(2+\sqrt{2}, 3+\sqrt{2})$

(c) $(4+\sqrt{2}, 3+\sqrt{2})$ (d) $(3+\sqrt{2}, 4+\sqrt{2})$

12. The intercept on the line $y = x$ by the circle $x^2 + y^2 - 2x = 0$ is AB. Equation of the circle on AB as a diameter is

(a) $x^2 + y^2 - x - y = 0$ (b) $x^2 + y^2 - x + y = 0$

(c) $x^2 + y^2 + x + y = 0$ (d) $x^2 + y^2 + x - y = 0$

13. Find the equation of the circle, the end points of whose diameter are $(2, -3)$ and $(-2, 4)$. Find the centre and radius.

14. If (4, 1) be an extremity of a diameter of the circle $x^2 + y^2 - 2x + 6y - 15 = 0$, find the coordinates of the other extremity of the diameter.

15. Find the equation of the circle drawn on the diagonal of the rectangle as its diameter whose sides are $x = 4, x = -2, y = 5$ and $y = -2$.

16. Find the equation of the circle which passes through the points $(1, 1), (2, 2)$ and whose radius is 1.

17. Find the equation of the circle which passes through the points $(3, 4), (3, -6)$ and $(1, 2)$.

Session 3

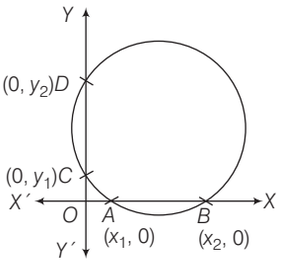
Intercepts Made on the Axes by a Circle, Different Forms of the Equations of a Circle, Position of a Point with Respect to Circle, Maximum and Minimum Distance of a Point from the Circle

Intercepts Made on the Axes by a Circle

Let the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ ...(i)

Length of intercepts on X-axis and Y-axis are

$|AB| = |x_2 - x_1|$ and $|CD| = |y_2 - y_1|$ respectively.



The circle intersects the X-axis, when $y = 0$

then $x^2 + 2gx + c = 0$

Since, the circle intersects the X-axis at $A(x_1, 0)$ and $B(x_2, 0)$

then, $x_1 + x_2 = -2g, x_1 x_2 = c$

$\therefore \quad |AB| = |x_2 - x_1| = \sqrt{(x_2 + x_1)^2 - 4x_1 x_2}$

$= 2\sqrt{(g^2 - c)}$

and the circle intersects the Y-axis, when $x = 0$, then

$y^2 + 2fy + c = 0$

Since, the circle intersects the Y-axis at $C(0, y_1)$ and $D(0, y_2)$

then, $y_1 + y_2 = -2f, y_1 y_2 = c$

$\therefore \quad |CD| = |y_2 - y_1| = \sqrt{(y_2 + y_1)^2 - 4y_2 y_1}$

$= 2\sqrt{(f^2 - c)}$

Remarks

1. Intercepts are always positive.
2. If circle touches X-axis, then $|AB| = 0$

   $\therefore \quad c = g^2$

   and if circle touches Y-axis, then $|CD| = 0$

   $\therefore \quad c = f^2$
3. If circle touches both axes, then $|AB| = 0 = |CD|$

   $\therefore \quad c = g^2 = f^2$

Example 16. Find the equation of the circle whose diameter is the line joining the points $(-4, 3)$ and $(12, -1)$. Find also the intercept made by it on Y-axis.

Sol. Equation of circle having $(-4, 3)$ and $(12, -1)$ as the ends of a diameter is

$(x + 4)(x - 12) + (y - 3)(y + 1) = 0$

$\Rightarrow \quad x^2 + y^2 - 8x - 2y - 51 = 0$ ...(i)

Comparing Eq. (i) with standard equation of circle

$x^2 + y^2 + 2gx + 2fy + c = 0$

then, $g = -4, f = -1, c = -51$

$\therefore$ Intercept on Y-axis $= 2\sqrt{(f^2 - c)} = 2\sqrt{(1 + 51)} = 4\sqrt{13}$.

Different Forms of the Equations of a Circle

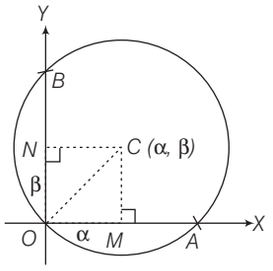
(i) When the circle passes through the origin (0, 0) and has intercepts 2α and 2β on the X-axis and Y-axis, respectively

Here, $OA = 2\alpha, OB = 2\beta$

then, $OM = \alpha$ and $ON = \beta$

Centre of the circle is $C(\alpha, \beta)$ and radius

$OC = \sqrt{(\alpha^2 + \beta^2)}$



then, equation of circle is

$$(x-\alpha)^2+(y-\beta)^2=\alpha^2+\beta^2$$

or $$x^2+y^2-2\alpha x-2\beta y=0$$

Remark

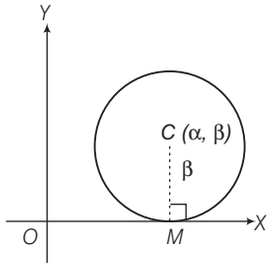
If a circle is passing through origin, then constant term is absent i.e. $x^2+y^2+2gx+2fy=0$

(ii) When the circle touches *X*-axis

Let (α,β) be the centre of the circle, then radius $=|\beta|$

$\therefore$ Equation of circle is

$$(x-\alpha)^2+(y-\beta)^2=\beta^2$$



$$\Rightarrow \quad x^2+y^2-2\alpha x-2\beta y+\alpha^2=0$$

Remark

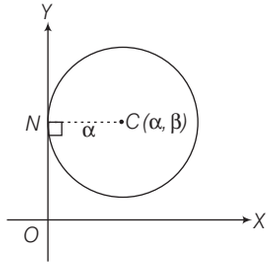
If the circle $x^2+y^2+2gx+2fy+c=0$ touches the *X*-axis, then

$$|-f|=\sqrt{g^2+f^2-c} \quad \text{or} \quad c=g^2$$

(iii) When the circle touches *Y*-axis

Let (α,β) be the centre of the circle, then

$$\text{radius}=|\alpha|$$



$\therefore$ Equation of circle is

$$(x-\alpha)^2+(y-\beta)^2=\alpha^2$$

$$\Rightarrow \quad x^2+y^2-2\alpha x-2\beta y+\beta^2=0$$

Remark

If the circle $x^2+y^2+2gx+2fy+c=0$ touches the *Y*-axis, then

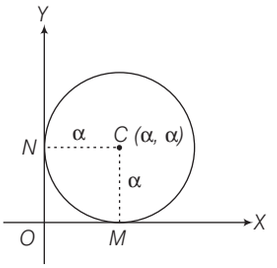
$$|-g|=\sqrt{g^2+f^2-c}$$

or $$c=f^2$$

(iv) When the circle touches both axes

Here, $$|OM|=|ON|$$

Since, length of tangents are equal from any point on circle.



$\therefore$ Let centre is (α,α) also radius $=\alpha$

$\therefore$ Equation of circle is $(x-\alpha)^2+(y-\alpha)^2=\alpha^2$

$$\Rightarrow \quad x^2+y^2-2\alpha x-2\alpha y+\alpha^2=0$$

Remarks

1. If the circle $x^2+y^2+2gx+2fy+c=0$ touches both the axes, then $|-g|=|-f|=\sqrt{g^2+f^2-c}$

$\therefore \quad c=g^2=f^2$

$\therefore \quad g=f=\pm\sqrt{c}$

$\therefore$ Equation of circle is

$$x^2+y^2\pm2\sqrt{c}\,x\pm2\sqrt{c}\,y+c=0$$

$$\Rightarrow \quad (x\pm\sqrt{c})^2+(y\pm\sqrt{c})^2=c^2$$

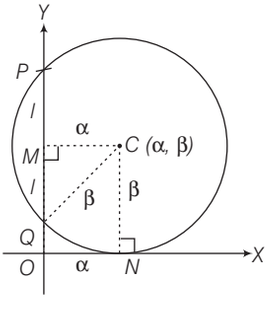
2. If $\alpha>0$, then centres for I, II, III and IV quadrants are $(\alpha,\alpha),(-\alpha,\alpha),(-\alpha,-\alpha)$ and $(\alpha,-\alpha)$, respectively. Then, equation of circles in these quadrants are

$$(x-\alpha)^2+(y-\alpha)^2=\alpha^2,(x+\alpha)^2+(y-\alpha)^2=\alpha^2,$$

$(x+\alpha)^2+(y+\alpha)^2=\alpha^2$ and $(x-\alpha)^2+(y+\alpha)^2=\alpha^2$, respectively.

(v) When the circle touches *X*-axis and cut-off intercepts on *Y*-axis of length 2*l*

Let centre be (α,β)



$\therefore$ radius $= \beta$

$CQ = CN = \beta$

In $\Delta CMQ, \beta^2 = \alpha^2 + l^2, \alpha = \sqrt{(\beta^2 - l^2)}$ (for I quadrant)

$\therefore$ Equation of circle is

$$[x - \sqrt{(\beta^2 - l^2)}]^2 + (y - \beta)^2 = \beta^2$$

Remark

$\because$ Length of intercepts on Y-axis of the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ is $2l = 2\sqrt{(f^2 - c)}$

i.e. $l^2 = f^2 - c$

and also circle touches X-axis

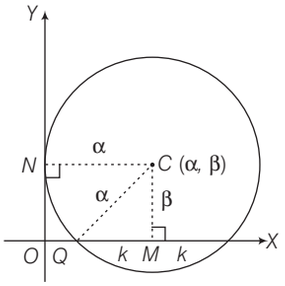
then, $c = g^2$

$\therefore$ $l^2 = f^2 - g^2$ or $l^2 = (-f)^2 - (-g)^2$

$\therefore$ Locus of centre is $y^2 - x^2 = l^2$ (rectangular hyperbola)

(vi) When the circle touches *Y*-axis and cut-off intercept on *X*-axis of length $2k$

Let centre be (α, β)



$\therefore$ radius $= \alpha$

$CN = CQ = \alpha$

In ΔCMQ, $\alpha^2 = \beta^2 + k^2$

$\beta = \sqrt{(\alpha^2 - k^2)}$ (for I quadrant)

$\therefore$ Equation of circle is

$$(x - \alpha)^2 + (y - \sqrt{\alpha^2 - k^2})^2 = \alpha^2$$

Remarks

$\because$ Length of intercept on X-axis of the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ is $2k = 2\sqrt{(g^2 - c)}$

i.e. $k^2 = g^2 - c$

and also circle touches Y-axis

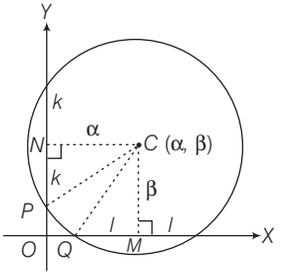
then, $c = f^2$

$\therefore$ $k^2 = g^2 - f^2 = (-g)^2 - (-f)^2$

$\therefore$ Locus of centre is $x^2 - y^2 = k^2$ (rectangular hyperbola)

(vii) When the circle cut-off intercepts on *X*-axis and *Y*-axis of lengths $2l$ and $2k$ and not passing through origin

Let centre be (α, β)



$\therefore$ radius $= CP = CQ = \lambda$ (say)

$(CP)^2 = (CQ)^2 = \lambda^2$

$\alpha^2 + k^2 = \beta^2 + l^2 = \lambda^2$

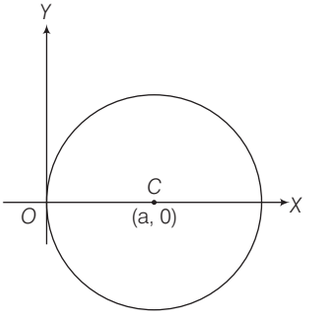
$\therefore$ $\alpha = \sqrt{\lambda^2 - k^2}$ and $\beta = \sqrt{\lambda^2 - l^2}$

$\therefore$ Equation of circle is (for I quadrant)

$$(x - \sqrt{\lambda^2 - k^2})^2 + (y - \sqrt{\lambda^2 - l^2})^2 = \lambda^2$$

(viii) When the circle passes through the origin and centre lies on *X*-axis

Let centre of circle be $C(a, 0)$



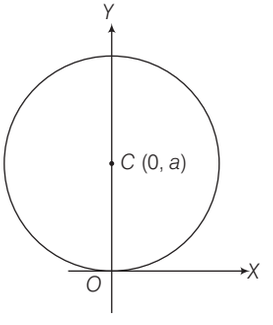
$\therefore$ radius $= a$

$\therefore$ Equation of circle is

$$(x - a)^2 + (y - 0)^2 = a^2 \text{ or } x^2 + y^2 - 2ax = 0$$

(ix) When the circle passes through origin and centre lies on *Y*-axis

Let centre of circle be $C(0, a)$



$\therefore$ radius $= a$

$\therefore$ Equation of circle is

$$(x-0)^2 + (y-a)^2 = a^2$$

or $$x^2 + y^2 - 2ay = 0$$

Example 17. Find the equation of the circle which touches the axis of y at a distance of 4 units from the origin and cuts the intercept of 6 units from the axis of x.

Sol. $\because$ $CM = NO = 4$

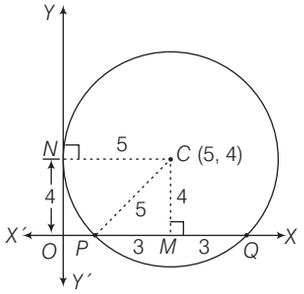
In ΔPCM, $(PC)^2 = (3)^2 + (4)^2$

$\therefore$ $PC = 5$

radius of circle $= 5$

$\therefore$ $NC = 5$

Centre of circle is (5, 4).



$\therefore$ Equation of circle, if centre in I quadrant

$$(x-5)^2 + (y-4)^2 = 25$$

If centre in II, III and IV quadrant, then equations are

$$(x+5)^2 + (y-4)^2 = 25,$$

$$(x+5)^2 + (y+4)^2 = 25$$

and $$(x-5)^2 + (y+4)^2 = 25$$

Hence, there are 4 circles which satisfy the given conditions. They are

$$(x \pm 5)^2 + (y \pm 4)^2 = 25$$

or $$x^2 + y^2 \pm 10x \pm 8y + 16 = 0$$

Aliter : Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...(i)}$$

Since, the circle touches the Y-axis

$\therefore$ $c = f^2$

or $f = \pm\sqrt{c}$

Also given the circle makes an intercept of 6 units along X-axis. Therefore,

$$2\sqrt{g^2 - c} = 6$$

or $$g^2 - c = 9$$

or $$g = \pm\sqrt{(c+9)}$$

From Eq. (i), the equation of circle can be written as

$$x^2 + y^2 \pm 2\sqrt{(c+9)}\, x \pm 2\sqrt{c}y + c = 0$$

The circle touches the Y-axis

$\therefore$ $x = 0$

$\therefore$ $y^2 \pm 2\sqrt{c}y + c = 0$

or $(y \pm \sqrt{c})^2 = 0$

$\therefore$ $y \pm \sqrt{c} = 0$

$$y = \mp\sqrt{c}$$

Since, the circle touches the Y-axis at a distance of 4 units from the origin, we have

$$y = \mp\sqrt{c} = 4$$

or $c = 16$

therefore, $f = \pm\sqrt{c} = \pm 4$

and $g = \pm\sqrt{c+9} = \pm\sqrt{16+9} = \pm 5$

Hence, there are 4 circles which satisfy the given conditions. They are

$$x^2 + y^2 \pm 10x \pm 8y + 16 = 0$$

Example 18. Find the equation of the circle which passes through the origin and makes intercepts of length a and b on the X and Y axes, respectively.

Sol. Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...(i)}$$

Since, the circle passes through the origin, we get $c = 0$ and given the intercepts on X and Y axes are a and b

then, $2\sqrt{(g^2 - c)} = a$

or $2\sqrt{(g^2 - 0)} = a$

$\therefore$ $g = \pm a/2$

and $2\sqrt{(f^2 - c)} = b$

or $2\sqrt{(f^2 - 0)} = b$

$\therefore$ $f = \pm b/2$

Hence, the equation of circle from Eq. (i) becomes

$$x^2 + y^2 \pm ax \pm by = 0$$

Example 19. Find the equation of the circle which touches the axes and whose centre lies on the line $x - 2y = 3$.

Sol. Since, the circle touches both the axes, let the radius of the circle by a, then

Case I If centre (a, a) but given centre lies on

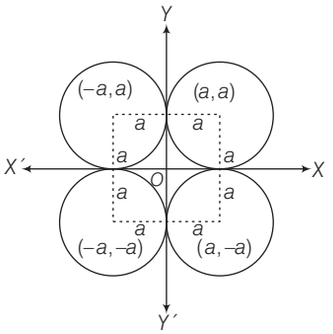
$$x - 2y = 3$$

$\therefore$ $a - 2a = 3$

$\therefore$ $a = -3$

$\therefore$ Centre $= (-3, -3)$

and radius = $|-3| = 3$



∴ Equation of circle is

$$(x+3)^2 + (y+3)^2 = 3^2$$

and $x^2 + y^2 + 6x + 6y + 9 = 0$

Case II If centre $(-a, a)$ but centre lies on $x - 2y = 3$

∴ $-a - 2a = 3$

∴ $a = -1$

then, centre $= (1, -1)$ and radius $= |-1| = 1$

∴ Equation of circle is $(x-1)^2 + (y+1)^2 = 1$

or $x^2 + y^2 - 2x + 2y + 1 = 0$

Case III If the centre $= (-a, -a)$

but centre lies on $x - 2y = 3$

∴ $-a + 2a = 3$

∴ $a = 3$

then centre $(-3, -3)$ and radius $= |3| = 3$

∴ Equation of circle is

$$(x+3)^2 + (y+3)^2 = 3^2$$

or $x^2 + y^2 + 6x + 6y + 9 = 0$

Case IV If centre $= (a, -a)$ but centre lies on $x - 2y = 3$

or $a + 2a = 3$

∴ $a = 1$

then centre $= (1, -1)$ and radius $= 1$

∴ Equation of circle is

$$(x-1)^2 + (y+1)^2 = 1$$

or $x^2 + y^2 - 2x + 2y + 1 = 0$

Aliter I : Since, the circle touches both the axes, therefore its centre will be $(a, \pm a)$ and radius will be $|a|$, where a is positive or negative number.

Case I If centre $= (a, a)$

Since, centre lies on $x - 2y = 3$

∴ $a - 2a = 3$

$a = -3$

∴ Centre of circle is $(-3, -3)$ and radius $= |-3| = 3$.

Hence, equation of circle will be

$$(x+3)^2 + (y+3)^2 = 3^2$$

or $x^2 + y^2 + 6x + 6y + 9 = 0$

Case II If centre $= (a, -a)$

Since, centre lies on $x - 2y = 3$

∴ $a + 2a = 3$

∴ $a = 1$

∴ Centre of circle is $(1, -1)$ and radius $= |1| = 1$

Hence, equation of circle will be

$$(x-1)^2 + (y+1)^2 = 1$$

or $x^2 + y^2 - 2x + 2y + 1 = 0$

Aliter II : Let the equation of circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...(i)}$$

centre $= (-g, -f)$

Since, centre $(-g, -f)$ lies on $x - 2y = 3$

or $-g + 2f = 3$ ...(ii)

Since, circle touches both axes

∴ $g^2 = f^2 = c$ or $g = \pm f$

if $g = f$, then from Eq. (ii), $-f + 2f = 3$

∴ $f = 3$ and $g = 3$

but $c = f^2 = g^2 = 9$

∴ Equation of circle from Eq. (i) is

$$x^2 + y^2 + 6x + 6y + 9 = 0$$

and if $g = -f$, then from Eq. (ii)

$$f + 2f = 3$$

∴ $f = 1$ and $g = -1$

but $c = g^2 = f^2 = 1$

∴ Equation of circle from Eq. (i) is

$$x^2 + y^2 - 2x + 2y + 1 = 0$$

Aliter III : Since, centre of circle lies on $x - 2y = 3$, also since circle touches the axes, therefore, its centre will lie on the line $y = x$ or $y = -x$

Case I When the centre lies on the line $y = x$

but $x - 2y = 3$

or $x - 2x = 3$

∴ $x = -3 = y$

Hence, the centre $= (-3, -3)$ and radius $= |-3| = 3$

Therefore, the equation of circle in this case will be

$$(x+3)^2 + (y+3)^2 = 3^2$$

or $x^2 + y^2 + 6x + 6y + 9 = 0$

Case II When the centre lies on the line $y = -x$

but $x - 2y = 3$

or $x + 2x = 3$

∴ $x = 1$ then $y = -1$

∴ Centre of circle $(1, -1)$ and radius is $|1| = 1$

Hence, equation of circle will be

$$(x-1)^2 + (y+1)^2 = 1$$

or $x^2 + y^2 - 2x + 2y + 1 = 0$

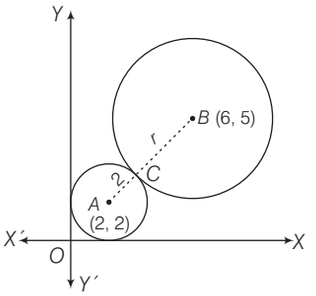
Example 20. A circle of radius 2 lies in the first quadrant and touches both the axes of coordinates. Find the equation of the circle with centre at (6, 5) and touching the above circle externally.

Sol. Given, $AC = 2$ units

and $A \equiv (2, 2), B \equiv (6, 5)$

then $AB = \sqrt{(2-6)^2 + (2-5)^2}$

$= \sqrt{16+9} = 5$



Since $AC + CB = AB$

$\therefore \quad 2 + CB = 5$

$\therefore \quad CB = 3$

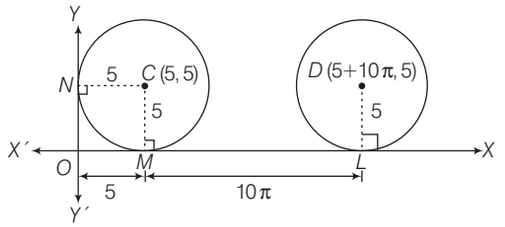
Hence, equation of required circle with centre at (6, 5) and radius 3 is

$$(x-6)^2 + (y-5)^2 = 3^2$$

or $x^2 + y^2 - 12x - 10y + 52 = 0$

Example 21. A circle of radius 5 units touches the coordinate axes in first quadrant. If the circle makes one complete roll on X-axis along the positive direction of X-axis, find its equation in the new position.

Sol. Let C be the centre of the circle in its initial position and D be its centre in the new position.



Since, the circle touches the coordinates axes in first quadrant and the radius of circle be 5 units.

$\therefore$ Centre of circle is (5, 5)

Moving length of circle = circumference of the circle

$$= 2\pi r = 2\pi(5) = 10\pi$$

Now, centre of circle in new position is $(5 + 10\pi, 5)$ and radius is 5 units, therefore, its equation will be

$$(x - 5 - 10\pi)^2 + (y-5)^2 = 5^2$$

or $x^2 + y^2 - 10(1 + 2\pi)x - 10y + 100\pi^2 + 100\pi + 25 = 0$

Position of a Point with Respect to Circle

Theorem : A point (x_1, y_1) lies outside, on or inside a circle

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$$

according as $S_1 >, =, \text{or} < 0$

where, $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c.$

Proof : Let $P(x_1, y_1)$ be the given point and let C be the centre of the circle

Then, $C \equiv (-g, -f)$

$\therefore \quad CP = \sqrt{(x_1+g)^2 + (y_1+f)^2}$

If r be the radius of the circle, then

$$r = \sqrt{(g^2 + f^2 - c)}$$

The point P lies outside, on or inside the circle according as

$CP >, =, \quad \text{or} \quad < r$

$\Rightarrow \quad (CP)^2 >, =, \quad \text{or} \quad < r^2$

$\Rightarrow \quad (x_1+g)^2 + (y_1+f)^2 >, =, \quad \text{or} \quad < g^2 + f^2 - c$

$\Rightarrow \quad x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c >, =, \quad \text{or} \quad < 0$

$\Rightarrow \quad S_1 >, =, \quad \text{or} \quad < 0$

where, $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c.$

Example 22. Discuss the position of the points (1, 2) and (6, 0) with respect to the circle

$$x^2 + y^2 - 4x + 2y - 11 = 0.$$

Sol. Let $S \equiv x^2 + y^2 - 4x + 2y - 11 = 0$ for the point (1, 2)

$$S_1 = 1^2 + 2^2 - 4 \cdot 1 + 2 \cdot 2 - 11 = -6$$

$\therefore \quad S_1 < 0$

and for the point (6, 0)

$$S_2 = 6^2 + 0 - 4 \cdot 6 + 2 \cdot 0 - 11$$

$$= 36 - 24 - 11$$

$$= 36 - 35 = 1$$

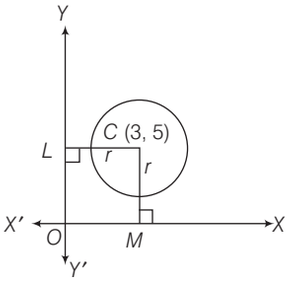
$\therefore \quad S_2 > 0$

Hence, the point (1, 2) lies inside the circle and the point (6, 0) lies outside the circle.

Example 23. The circle $x^2 + y^2 - 6x - 10y + \lambda = 0$ does not touch or intersect the coordinate axes and the point (1, 4) is inside the circle. Find the range of values of λ.

Sol. Let $S \equiv x^2 + y^2 - 6x - 10y + \lambda = 0$

$\because$ Point (1, 4) is inside the circle, then $S_1 < 0$



$$1 + 16 - 6 - 40 + \lambda < 0$$

$$\Rightarrow \quad \lambda < 29 \qquad \text{...(i)}$$

Centre and radius of the circle are (3, 5) and $\sqrt{(34 - \lambda)}$, respectively.

$\because$ Circle does not touch or intersect the coordinate axes.

$\therefore \quad 5 > r$ and $3 > r$

or $\quad 5 > \sqrt{(34 - \lambda)}$ and $3 > \sqrt{(34 - \lambda)}$

$\Rightarrow \quad 25 > 34 - \lambda$ and $9 > 34 - \lambda$

$\Rightarrow \quad \lambda > 9$ and $\lambda > 25$

$\therefore \quad \lambda > 25 \qquad \text{...(ii)}$

Also, $\quad 34 - \lambda > 0$

$\therefore \quad \lambda < 34 \qquad \text{...(iii)}$

From Eqs. (i), (ii) and (iii), we get $25 < \lambda < 29$

Maximum and Minimum Distance of a Point from the Circle

Let any point $P(x_1, y_1)$ and circle

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{...(i)}$$

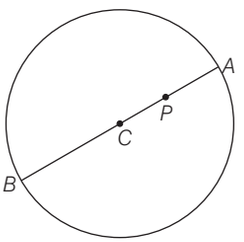
The centre and radius of the circle are

$$C(-g, -f) \quad \text{and} \quad \sqrt{(g^2 + f^2 - c)} \text{ respectively}$$

Case I If *P* inside the circle

In this case $S_1 < 0$

$\because \quad r = \sqrt{(g^2 + f^2 - c)} = CA = CB$



The **minimum distance** of P from circle $= PA = CA - CP$

$$= r - CP$$

and the **maximum distance** of P from circle $= PB$

$$= CB + CP = r + CP$$

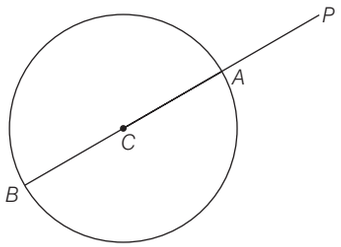
Case II If *P* outside the circle

In this case $S_1 > 0$ the **minimum distance** of P from circle

$$= PA = CP - CA = CP - r$$

and the **maximum distance** of P from the circle

$$= PB = CP + CB = r + CP$$



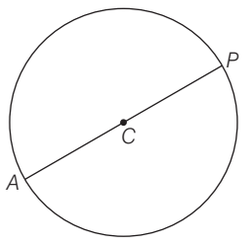
Case III If *P* on the circle

In this case $S_1 = 0$

the **minimum distance** of P from the circle $= 0$

and the **maximum distance** of P from the circle

$$= PA = 2r$$



Remark

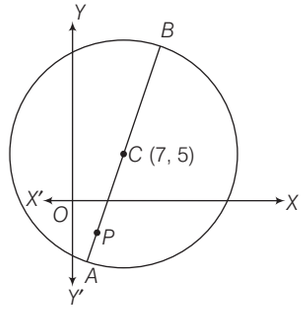
If point P inside or outside or on the circle and centre of circle at C and radius r, then minimum distance of P from the circle $= |CP - r|$ and maximum distance of P from the circle $= CP + r$

Example 24. *Find the shortest and largest distance from the point* $(2, -7)$ *to the circle*

$$x^2 + y^2 - 14x - 10y - 151 = 0$$

Sol. Let $\quad S \equiv x^2 + y^2 - 14x - 10y - 151 = 0$

$\therefore \quad S_1 = (2)^2 + (-7)^2 - 14(2) - 10(-7) - 151 = -56 < 0$



$\therefore$ $P(2,-7)$ inside the circle

radius of the circle, $r = \sqrt{(-7)^2 + (-5)^2 + 151} = 15$

$\because$ Centre of circle $C \equiv (7,5)$

$\therefore$ $CP = \sqrt{(7-2)^2 + (5+7)^2} = 13$

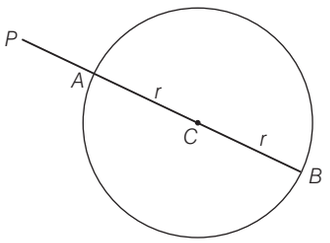
$\therefore$ **Shortest distance** $= PA = r - CP = 15 - 13 = 2$

and **Largest distance** $= PB = r + CP = 15 + 13 = 28$

Example 25. Find the points on the circle $x^2 + y^2 - 2x + 4y - 20 = 0$ which are farthest and nearest to the point $(-5,6)$.

Sol. The given circle is $S \equiv x^2 + y^2 - 2x + 4y - 20 = 0$

Let $P \equiv (-5,6)$



For the point P

$S_1 = 25 + 36 + 10 + 24 - 20$

$= 75 > 0$

$\therefore$ Point $P(-5,6)$ lies outside the circle.

The centre and radius of the circle are $(1,-2)$ and 5, respectively.

$\because$ $CP = \sqrt{(1+5)^2 + (-2-6)^2} = 10$

Now, point A divides CP in the ratio

$$\frac{AP}{AC} = \frac{CP - r}{r} = \frac{10-5}{5} = 1$$

$\therefore$ A is mid-point of CP.

$\therefore$ $A \equiv \left(\frac{1-5}{2}, \frac{-2+6}{2}\right)$

or $A \equiv (-2,2)$

and C is the mid-point of AB.

$\therefore$ $B \equiv (2\times 1 - (-2), 2\times -2 - 2)$

or $B \equiv (4,-6)$

Hence, point $A(-2,2)$ is nearest to P and $B(4,-6)$ is farthest from P.

Exercise for Session 3

1. The length of intercept, the circle $x^2 + y^2 + 10x - 6y + 9 = 0$ makes on the X-axis is

(a) 2 (b) 4 (c) 6 (d) 8

2. The circle $x^2 + y^2 + 4x - 7y + 12 = 0$ cuts an intercept on Y-axis is of length

(a) 1 (b) 3 (c) 5 (d) 7

3. The locus of the centre of a circle which passes through the origin and cuts-off a length $2b$ from the line $x = c$ is

(a) $y^2 + 2cx = b^2 + c^2$ (b) $x^2 + cx = b^2 + c^2$ (c) $y^2 + 2cy = b^2 + c^2$ (d) $x^2 + cy = b^2 + c^2$

4. If a straight line through $C(-\sqrt{8}, \sqrt{8})$ making an angle of 135° with the X-axis cuts the circle $x = 5\cos\theta$, $y = 5\sin\theta$ at points A and B, then the length of AB is

(a) 3 (b) 5 (c) 8 (d) 10

5. If a circle of constant radius $3k$ passes through the origin and meets the axes at A and B, the locus of the centroid of ΔOAB is

(a) $x^2 + y^2 = k^2$ (b) $x^2 + y^2 = 2k^2$ (c) $x^2 + y^2 = 3k^2$ (d) $x^2 + y^2 = 4k^2$

6. The centre of the circle touching Y-axis at $(0,3)$ and making an intercept of 2 units on positive X-axis is

(a) $(10, \sqrt{3})$ (b) $(\sqrt{3}, 10)$ (c) $(\sqrt{10}, 3)$ (d) $(3, \sqrt{10})$

7. A circle passes through the points $A(1,0)$ and $B(5,0)$ and touches the Y-axis at $C(0,\lambda)$. If $\angle ACB$ is maximum, then

(a) $|\lambda| = \sqrt{5}$ (b) $|\lambda| = 2\sqrt{5}$ (c) $|\lambda| = 3\sqrt{5}$ (d) $|\lambda| = 4\sqrt{5}$

8. The equation of a circle whose centre is $(3, -1)$ and which intercept chord of 6 units length on straight line $2x - 5y + 18 = 0$ is

(a) $x^2 + y^2 - 6x + 2y - 28 = 0$ (b) $x^2 + y^2 + 6x - 2y - 28 = 0$

(c) $x^2 + y^2 + 4x - 2y + 24 = 0$ (d) $x^2 + y^2 + 2x - 2y - 12 = 0$

9. The locus of the centre of a circle which touches externally the circle $x^2 + y^2 - 6x - 6y + 14 = 0$ and also touches the *Y*-axis, is given by the equation

(a) $x^2 - 6x - 10y + 14 = 0$ (b) $x^2 - 10x - 6y + 14 = 0$

(c) $y^2 - 6x - 10y + 14 = 0$ (d) $y^2 - 10x - 6y + 14 = 0$

10. The locus of the centre of a circle of radius 2 which rolls on the outside of circle $x^2 + y^2 + 3x - 6y - 9 = 0$ is

(a) $x^2 + y^2 + 3x - 6y + 5 = 0$ (b) $x^2 + y^2 + 3x - 6y - 31 = 0$

(c) $x^2 + y^2 + 3x - 6y + 11 = 0$ (d) $x^2 + y^2 + 3x - 6y - 36 = 0$

11. The point $([\lambda + 1], [\lambda])$ is lying inside the circle $x^2 + y^2 - 2x - 15 = 0$. Then, the set of all values of λ is (where [.] represents the greatest integer function)

(a) $[-2, 3]$ (b) $(-2, 3)$ (c) $[-2, 0) \cup (0, 3)$ (d) $[0, 3)$

12. The greatest distance of the point $(10, 7)$ from the circle $x^2 + y^2 - 4x - 2y - 20 = 0$ is

(a) 5 (b) 10 (c) 15 (d) 20

13. Find equations to the circles touching *Y*-axis at $(0, 3)$ and making intercept of 8 units on the *X*-axis.

14. Show that the circle $x^2 + y^2 - 2ax - 2ay + a^2 = 0$ touches both the coordinate axes.

15. If the point $(\lambda, -\lambda)$ lies inside the circle $x^2 + y^2 - 4x + 2y - 8 = 0$, then find range of λ.

16. Find the equation of the circle which passes through the origin and cuts-off chords of lengths 4 and 6 on the positive side of the *X*-axis and *Y*-axis, respectively.

Session 4

Intersection of a Line and a Circle, Product of the Algebraical Distances *PA* and *PB* is Constant when from *P*, *A* Secant be Drawn to Cut the Circle in the Points *A* and *B*, The Length of Intercept Cut-off from a Line by a Circle, Tangent to a Circle at a Given Point, Normal to a Circle at a Given Point

Intersection of a Line and a Circle

Let the equation of the circle be

$$x^2 + y^2 = a^2 \quad \text{...(i)}$$

and the equation of the line be

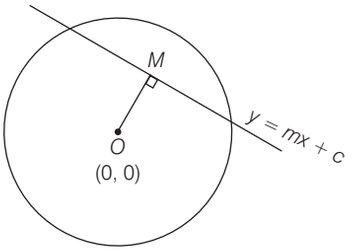
$$y = mx + c \quad \text{...(ii)}$$

From Eq. (i) and Eq. (ii)

$$x^2 + (mx + c)^2 = a^2$$

or $$(1 + m^2)\, x^2 + 2mcx + c^2 - a^2 = 0 \quad \text{...(iii)}$$

Case I When points of intersection are real and distinct, then Eq. (iii) has two distinct roots.



$$\therefore \quad B^2 - 4AC > 0$$

or $$4m^2c^2 - 4(1 + m^2)(c^2 - a^2) > 0$$

or $$a^2 > \frac{c^2}{1 + m^2}$$

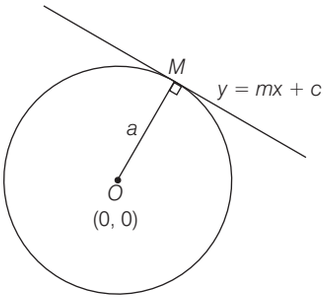
or $$a > \frac{|c|}{\sqrt{(1 + m^2)}} = \text{length of perpendicular}$$

from $(0, 0)$ to $y = mx + c$

$\Rightarrow$ $a >$ length of perpendicular from $(0, 0)$ to $y = mx + c$

Thus, a line intersects a given circle at two distinct points if radius of circle is greater than the length of perpendicular from centre of the circle to the line.

Case II When the points of intersection are coincident, then Eq. (iii) has two equal roots



$$\therefore \quad B^2 - 4AC = 0$$

$$\Rightarrow \quad 4m^2c^2 - 4(1 + m^2)(c^2 - a^2) = 0$$

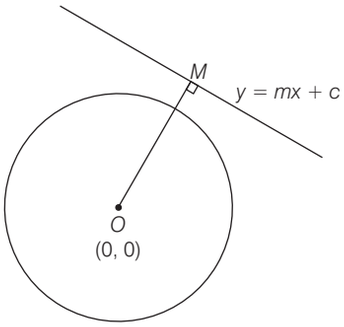
$$\therefore \quad a^2 = \frac{c^2}{(1 + m^2)}$$

or $$a = \frac{|c|}{\sqrt{(1 + m^2)}}$$

$a =$ length of the perpendicular from the point $(0, 0)$ to $y = mx + c$

Thus, a line touches the circle if radius of circle is equal to the length of perpendicular from centre of the circle to the line.

Case III When the points of intersection are imaginary. In this case (iii) has imaginary roots



$$\therefore \quad B^2 - 4AC < 0$$

$$4m^2c^2 - 4(1+m^2)(c^2 - a^2) = 0$$

$$\therefore \quad a^2 < \frac{c^2}{1+m^2}$$

or $a < \dfrac{|c|}{\sqrt{1+m^2}}$ = length of perpendicular from $(0,0)$ to $y = mx + c$

or $a <$ length of perpendicular from $(0,0)$ to $y = mx + c$

Thus, a line does not intersect a circle if the radius of circle is less than the length of perpendicular from centre of the circle to the line.

Example 26. *Find the points of intersection of the line* $2x + 3y = 18$ *and the circle* $x^2 + y^2 = 25$.

Sol. We have, $2x + 3y = 18$ ....(i)

and $x^2 + y^2 = 25$ ...(ii)

From Eq. (i), $y = \dfrac{18 - 2x}{3}$

Substituting in Eq. (ii), then $x^2 + \left(\dfrac{18-2x}{3}\right)^2 = 25$

$\Rightarrow \quad 9x^2 + 4(9 - x)^2 = 225$

$\Rightarrow \quad 9x^2 + 4(81 - 18x + x^2) = 225$

$\Rightarrow \quad 13x^2 - 72x + 324 - 225 = 0$

$\Rightarrow \quad 13x^2 - 72x + 99 = 0$

$\Rightarrow \quad (x - 3)(13x - 33) = 0$

$\Rightarrow \quad x = 3 \quad \text{or} \quad x = \dfrac{33}{13}$

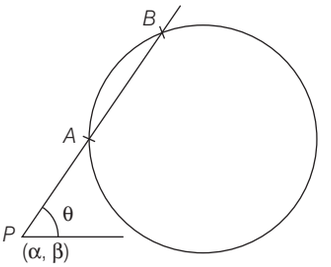
From Eq. (i), $y = 4 \quad \text{or} \quad y = \dfrac{56}{13}$

Hence, the points of intersection of the given line and the given circle are $(3,4)$ and $\left(\dfrac{33}{13}, \dfrac{56}{13}\right)$.

Product of the Algebraical Distances *PA* and *PB* is Constant when from *P*, A Secant be Drawn to Cut the Circle in the Points *A* and *B*

If a straight line through $P(\alpha, \beta)$ makes an angle θ with the positive direction of X-axis, then its equation is

$$\frac{x - \alpha}{\cos\theta} = \frac{y - \beta}{\sin\theta} = r$$



where, r is the algebraical distance of the point (x, y) from the point $P(\alpha, \beta)$.

$\therefore \quad (x, y) = (\alpha + r\cos\theta, \beta + r\sin\theta)$

If this point lies on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

or $(\alpha + r\cos\theta)^2 + (\beta + r\sin\theta)^2 + 2g(\alpha + r\cos\theta) + 2f(\beta + r\sin\theta) + c = 0$

$\Rightarrow \quad r^2 + 2r(\alpha\cos\theta + \beta\sin\theta + g\cos\theta + f\sin\theta) + (\alpha^2 + \beta^2 + 2g\alpha + 2f\beta + c) = 0$

This is quadratic equation in r, then PA and PB are the roots of this equation.

$\therefore \quad PA \cdot PB = \alpha^2 + \beta^2 + 2g\alpha + 2f\beta + c = \text{constant}$

Since, RHS is independent of θ.

Remark

Secants are drawn from a given point A to cut a given circle at the pairs of points P_1, Q_1 ; P_2, Q_2;...; P_n, Q_n, then

$$AP_1 \cdot AQ_1 = AP_2 \cdot AQ_2 = \ldots = AP_n \cdot AQ_n$$

The Length of Intercept Cut-off from a Line by a Circle

Theorem : The length of the intercept cut-off from the line $y = mx + c$ by the circle $x^2 + y^2 = a^2$ is

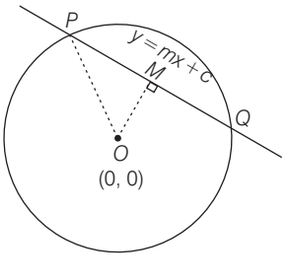
$$2\sqrt{\left\{\frac{a^2(1+m^2) - c^2}{(1+m^2)}\right\}}$$

Proof : Draw OM perpendicular to PQ

Now, OM = length of perpendicular from $O(0,0)$ to

$$(y = mx + c) = \frac{|c|}{\sqrt{(1+m^2)}}$$

and OP = radius of the circle = a



In ΔOPM, $PM = \sqrt{(OP)^2 - (OM)^2}$

$$= \sqrt{a^2 - \frac{c^2}{(1+m^2)}} = \sqrt{\left\{\frac{a^2(1+m^2) - c^2}{1+m^2}\right\}}$$

$$\therefore \quad PQ = 2PM = 2\sqrt{\left\{\frac{a^2(1+m^2) - c^2}{1+m^2}\right\}}$$

Remarks

1. If the line $y = mx + c$ touches the circle $x^2 + y^2 = a^2$, then intercepted length is zero

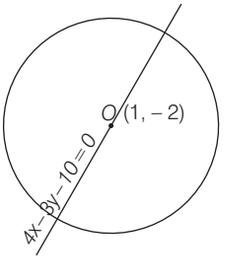
   i.e. $PQ = 0 \Rightarrow 2\sqrt{\left\{\frac{a^2(1+m^2) - c^2}{1+m^2}\right\}} = 0$

   $\therefore \quad c^2 = a^2(1+m^2)$

   which is the required condition for tangency.
2. If a line touches the circle, then length of perpendicular from the centre upon the line is equal to the radius of the circle.

Example 27. *Find the length of the intercept on the straight line* $4x - 3y - 10 = 0$ *by the circle* $x^2 + y^2 - 2x + 4y - 20 = 0$.

Sol. Centre and radius of the circle $x^2 + y^2 - 2x + 4y - 20 = 0$ are $(1, -2)$ and $\sqrt{1 + 4 + 20} = 5$ respectively.



Let OM be the perpendicular from O on the line

$$4x - 3y - 10 = 0$$

then $$OM = \frac{|4 \times 1 - 3 \times (-2) - 10|}{\sqrt{4^2 + (-3)^2}} = 0$$

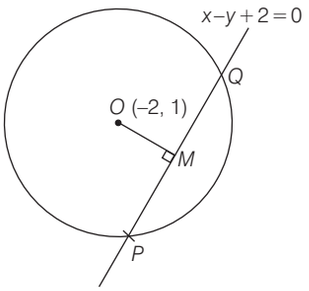
Hence, line $4x - 3y - 10 = 0$ passes through the centre of the circle.

Hence, intercepted length = diameter of the circle

$$= 2 \times 5 = 10$$

Example 28. Find the coordinates of the middle point of the chord which the circle $x^2 + y^2 + 4x - 2y - 3 = 0$ cuts-off the line $x - y + 2 = 0$.

Sol. Centre and radius of the circle $x^2 + y^2 + 4x - 2y - 3 = 0$ are $(-2, 1)$ and $\sqrt{4 + 1 + 3} = 2\sqrt{2}$ respectively.



Draw perpendicular from O upon $x - y + 2 = 0$ is OM.

Equation of OM which is perpendicular to $x - y + 2 = 0$ is $x + y = \lambda$, it passes through $(-2, 1)$

Then, $-2 + 1 = \lambda$

$\therefore \quad \lambda = -1$

then equation of OM is $x + y + 1 = 0$

Since, M is the mid-point of PQ which is point of intersection of $x - y + 2 = 0$ and $x + y + 1 = 0$, coordinates of M is $\left(-\frac{3}{2}, \frac{1}{2}\right)$.

Aliter : Let $M \equiv (\alpha, \beta)$, then

$$\frac{\alpha + 2}{1} = \frac{\beta - 1}{-1} = -\frac{(-2 - 1 + 2)}{1 + 1}$$

(Here, M is foot of perpendicular)

$$\Rightarrow \quad \frac{\alpha + 2}{1} = \frac{\beta - 1}{-1} = \frac{1}{2}$$

or $$\alpha = -\frac{3}{2} \text{ and } \beta = \frac{1}{2}$$

$$\therefore \quad M \equiv \left(-\frac{3}{2}, \frac{1}{2}\right)$$

Example 29. For what value of λ will the line $y = 2x + \lambda$ be a tangent to the circle $x^2 + y^2 = 5$?

Sol. Comparing the given line with $y = mx + c$, we get

$m = 2$, $c = \lambda$ and given circle with $x^2 + y^2 = a^2$

then $$a^2 = 5$$

$\because$ Condition for tangency is

$$c^2 = a^2(1 + m^2)$$

$\Rightarrow$ $\lambda^2 = 5(1 + 4)$

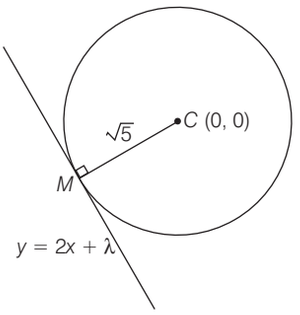
$\lambda^2 = 25$

$\therefore$ $\lambda = \pm 5$

Aliter : Since, line $y = 2x + \lambda$

or $2x - y + \lambda = 0$

is the tangent to the circle $x^2 + y^2 = 5$ then length of perpendicular from centre upon the line is equal to the radius of the circle



$\therefore$ $|CM| = \sqrt{5}$

or $\dfrac{|0 - 0 + \lambda|}{\sqrt{4 + 1}} = \sqrt{5}$

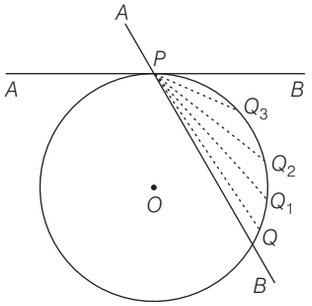
$\Rightarrow$ $\dfrac{|\lambda|}{\sqrt{5}} = \sqrt{5}$

$\Rightarrow$ $|\lambda| = 5$

or $\lambda = \pm 5$

Tangent to a Circle at a Given Point

Let PQ be a chord and AB be a secant passing through P. Let P be the fixed point and move along the circle towards P, then the secant PQ turns about P. In the limit, when Q coincides with P, then the secant AB becomes a tangent to the circle at the point P.



Different forms of the equations of tangents

1. Point form :

Theorem : The equation of tangent at the point $P(x_1, y_1)$ to a circle

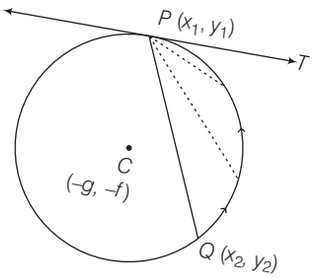
$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ is}$$

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

Proof : Since, $P(x_1, y_1)$ be a point on the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...(i)}$$

Let $Q(x_2, y_2)$ be any other point on the circle Eq. (i). Since, points $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on the circle, therefore



$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \quad \text{...(ii)}$$

and $$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0 \quad \text{...(iii)}$$

On subtracting Eq. (ii) from Eq. (iii), we have

$$(x_2^2 - x_1^2) + (y_2^2 - y_1^2) + 2g$$

$$(x_2 - x_1) + 2f(y_2 - y_1) = 0$$

$\Rightarrow$ $(x_2 - x_1)(x_2 + x_1 + 2g) + (y_2 - y_1)$

$$(y_2 + y_1 + 2f) = 0$$

$\Rightarrow$ $$\left(\frac{y_2 - y_1}{x_2 - x_1}\right) = -\left(\frac{x_1 + x_2 + 2g}{y_1 + y_2 + 2f}\right) \quad \text{...(iv)}$$

Now, the equation of the chord PQ is

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1) \quad \text{...(v)}$$

Putting the value of $\left(\dfrac{y_2 - y_1}{x_2 - x_1}\right)$ from Eq. (iv) in Eq. (v), then equation PQ becomes

$$y - y_1 = -\left(\frac{x_1 + x_2 + 2g}{y_1 + y_2 + 2f}\right)(x - x_1) \quad \text{...(vi)}$$

Now, when $Q \to P$ (along the circle), line PQ becomes tangent at P, we have $x_2 \to x_1, y_2 \to y_1$. So, the equation of tangent at $P(x_1, y_1)$ is :

$$y - y_1 = -\left(\frac{x_1 + x_1 + 2g}{y_1 + y_1 + 2f}\right)(x - x_1)$$

$$\Rightarrow \quad y - y_1 = -\left(\frac{x_1 + g}{y_1 + f}\right)(x - x_1)$$

$$\Rightarrow \quad (y - y_1)(y_1 + f) + (x - x_1)(x_1 + g) = 0$$

$$\Rightarrow \quad xx_1 + yy_1 + gx + fy = x_1^2 + y_1^2 + gx_1 + fy_1$$

On adding $gx_1 + fy_1 + c$ to both sides, we get

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$$
$$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \qquad \text{[from Eq. (ii)]}$$

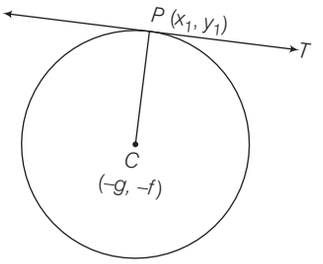
$$\Rightarrow \quad xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

This is the required equation of the tangent PT to the circle at the point (x_1, y_1).

Aliter : Since, circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

$P(x_1, y_1)$ lie on the circle

$$\therefore \quad x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \qquad \text{...(i)}$$



Its centre is $C(-g, -f)$

$$\therefore \quad \text{Slope of } CP = \frac{y_1 - (-f)}{x_1 - (-g)} = \frac{y_1 + f}{x_1 + g}$$

Since, tangent PT is perpendicular to CP.

$$\therefore \quad \text{Slope of tangent} = -\left(\frac{x_1 + g}{y_1 + f}\right)$$

$\therefore$ Equation of tangent at $P(x_1, y_1)$ is

$$y - y_1 = -\left(\frac{x_1 + g}{y_1 + f}\right)(x - x_1)$$

$$\Rightarrow \quad (y - y_1)(y_1 + f) + (x_1 + g)(x - x_1) = 0$$

$$\Rightarrow \quad xx_1 + yy_1 + gx + fy = x_1^2 + y_1^2 + gx_1 + fy_1$$

On adding $gx_1 + fy_1 + c$ to both sides, we get

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$$
$$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \qquad \text{[from Eq. (i)]}$$

or $\quad xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

This is the required equation of the tangent PT to the circle at the point $P(x_1, y_1)$.

Remarks

1. For equation of tangent of circle at (x_1, y_1), substitute xx_1 for x^2, yy_1 for y^2, $\frac{x + x_1}{2}$ for x, $\frac{y + y_1}{2}$ for y and $\frac{xy_1 + x_1y}{2}$ for xy and keep the constant as such.
2. This method of tangent at (x_1, y_1) is applied only for any conics of second degree. i.e. equation of tangent of $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ at (x_1, y_1) is $axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0$

Wrong process : Mostly students use wrong process

Suppose any curve

$$ax^3 + by^3 = c$$

or $\quad a(x)(x^2) + b(y)(y^2) = c$

Equation of tangent at (x_1, y_1)

$$\Rightarrow \quad a\left(\frac{x + x_1}{2}\right)xx_1 + b\left(\frac{y + y_1}{2}\right)yy_1 = c^2$$

which is a second degree conic not the equation of tangent.

Reason : This method is applicable only for second degree conic, its a third degree conic. (find its tangent only by calculus)

Example 30. *Prove that the tangents to the circle $x^2 + y^2 = 25$ at $(3, 4)$ and $(4, -3)$ are perpendicular to each other.*

Sol. The equations of tangents to $x^2 + y^2 = 25$ at $(3, 4)$ and $(4, -3)$ are

$$3x + 4y = 25 \qquad \text{...(i)}$$

and $\quad 4x - 3y = 25 \qquad \text{...(ii)}$

respectively.

Now, $\quad$ slope of Eq. (i) $= -\frac{3}{4} = m_1$ (say)

and $\quad$ slope of Eq. (ii) $= \frac{4}{3} = m_2$ (say)

Clearly, $\quad m_1 m_2 = -1$

Hence, Eq. (i) and Eq. (ii) are perpendicular to each other.

Example 31. Find the equation of tangent to the circle $x^2 + y^2 - 2ax = 0$ at the point $[a(1 + \cos\alpha), a\sin\alpha]$.

Sol. The equation of tangent of $x^2 + y^2 - 2ax = 0$ at $[a(1 + \cos\alpha), a\sin\alpha]$ is

$$x \cdot a(1 + \cos\alpha) + y \cdot a\sin\alpha - a[x + a(1 + \cos\alpha)] = 0$$

$$\Rightarrow \quad ax\cos\alpha + ay\sin\alpha - a^2(1 + \cos\alpha) = 0$$

or $\quad x\cos\alpha + y\sin\alpha = a(1 + \cos\alpha)$

Example 32. Show that the circles $x^2 + y^2 - 4x + 6y + 8 = 0$ and $x^2 + y^2 - 10x - 6y + 14 = 0$ touch at $(3, -1)$.

Sol. Equation of tangent at $(3, -1)$ of the circle $x^2 + y^2 - 4x + 6y + 8 = 0$ is

$$3x + (-1)\,y - 2(x + 3) + 3(y - 1) + 8 = 0$$

or $$x + 2y - 1 = 0 \quad \text{...(i)}$$

and equation of tangent at $(3, -1)$ of the circle $x^2 + y^2 - 10x - 6y + 14 = 0$ is

$$3 \cdot x + (-1) \cdot y - 5(x + 3) - 3(y - 1) + 14 = 0$$

or $$-2x - 4y + 2 = 0$$

or $$x + 2y - 1 = 0 \quad \text{...(ii)}$$

which is the same as Eq (i).

Hence, the given circles touch at $(3, -1)$.

2. Parametric form :

Theorem : The equation of tangent to the circle $x^2 + y^2 = a^2$ at the point $(a\cos\theta, a\sin\theta)$ is $x\cos\theta + y\sin\theta = a$

Proof : The equation of tangent of $x^2 + y^2 = a^2$ at (x_1, y_1) is $xx_1 + yy_1 = a^2$ (using point form of the tangent)

Putting $x_1 = a\cos\theta, y_1 = a\sin\theta$

then, we get $x\cos\theta + y\sin\theta = a$

***Corollary* 1 :** Equation of chord joining $(a\cos\theta, a\sin\theta)$ and $(a\cos\phi, a\sin\phi)$ is

$$x\cos\left(\frac{\theta+\phi}{2}\right) + y\sin\left(\frac{\theta+\phi}{2}\right) = a\cos\left(\frac{\theta-\phi}{2}\right)$$

***Corollary* 2 :** Point of intersection of tangents at $(a\cos\theta, a\sin\theta)$ and $(a\cos\phi, a\sin\phi)$ is

$$\left(\frac{a\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}, \frac{a\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}\right)$$

Remembering method :

$$\because \quad x\cos\left(\frac{\theta+\phi}{2}\right) + y\sin\left(\frac{\theta+\phi}{2}\right) = a\cos\left(\frac{\theta-\phi}{2}\right)$$

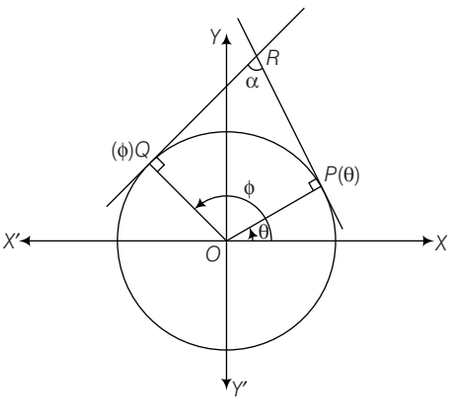
or $$x\left\{\frac{a\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}\right\} + y\left\{\frac{a\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}\right\} = a^2$$

We get $$\left(\frac{a\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}, \frac{a\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}\right)$$

***Corollary* 3 :** The angle between a pair of tangents from a point P to the circle $x^2 + y^2 = a^2$ is α. Then, the locus of the point P is

$$x^2 + y^2 = \frac{a^2}{\sin^2\left(\frac{\alpha}{2}\right)}$$

Proof



$$\because \quad \phi - \theta + \alpha = 180^\circ$$

$$\therefore \quad \frac{\theta - \phi}{2} = -\left(90^\circ - \frac{\alpha}{2}\right)$$

or $$\cos\left(\frac{\theta-\phi}{2}\right) = \sin\left(\frac{\alpha}{2}\right)$$

Now, point of intersection is

$$\left(\frac{a\cos\left(\frac{\theta+\phi}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}, \frac{a\sin\left(\frac{\theta+\phi}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}\right)$$

Let $$x = \frac{a\cos\left(\frac{\theta+\phi}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)} \text{ and } y = \frac{a\sin\left(\frac{\theta+\phi}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}$$

$$\therefore \quad x^2 + y^2 = \frac{a^2}{\sin^2\left(\frac{\alpha}{2}\right)}$$

Remarks

1. The angle between a pair of tangents from a point P to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is 2θ, then the locus of P is $x^2 + y^2 + 2gx + 2fy + c = (g^2 + f^2 - c)\cot^2\theta$
2. If angle between a pair of tangents from a point P to the circle $x^2 + y^2 = a^2$ is $\frac{\pi}{2}$, then the locus of P is
$$x^2 + y^2 = 2a^2 \qquad \left(\text{Here, } \alpha = \frac{\pi}{2}\right)$$
which is **director circle** of $x^2 + y^2 = a^2$.
($\because$ locus of point of intersection of perpendicular tangents is **director circle**)
3. The equation of the tangent to the circle $(x-a)^2 + (y-b)^2 = r^2$ at the point $(a + r\cos\theta, b + r\sin\theta)$ is
$$(x-a)\cos\theta + (y-b)\sin\theta = r.$$

Example 33. ***The angle between a pair of tangents from a point P to the circle $x^2 + y^2 = 25$ is $\frac{\pi}{3}$. Find the equation of the locus of the point P.***

Sol. Here, $\alpha = \frac{\pi}{3}$

$\therefore$ Required locus is $x^2 + y^2 = \frac{25}{\sin^2\left(\frac{\pi}{6}\right)} = 100$

Example 34. ***The angle between a pair of tangents from a point P to the circle $x^2 + y^2 - 6x - 8y + 9 = 0$ is $\frac{\pi}{3}$. Find the equation of the locus of the point P.***

Sol. Here, $2\theta = \frac{\pi}{3}$ or $\theta = \frac{\pi}{6}$

$\therefore$ Required locus is

$$x^2 + y^2 - 6x - 8y + 9 = (9 + 16 - 9)\cot^2\frac{\pi}{6}$$

or $\quad x^2 + y^2 - 6x - 8y + 9 = 16 \times 3$

or $\quad x^2 + y^2 - 6x - 8y - 39 = 0$

3. Slope form :

Theorem : The equation of a tangent of slope m to the circle $x^2 + y^2 = a^2$ is $y = mx \pm a\sqrt{(1+m^2)}$ and the coordinates of the point of contact are

$$\left(\pm\frac{am}{\sqrt{(1+m^2)}}, \mp\frac{a}{\sqrt{(1+m^2)}}\right)$$

Proof : Let $y = mx + c$ is the tangent of the circle $x^2 + y^2 = a^2$.

$\therefore$ Length of perpendicular from centre of circle $(0,0)$ on $(y = mx + c)$ = radius of circle

$$\therefore \quad \frac{|c|}{\sqrt{(1+m^2)}} = a \quad \Rightarrow \quad c = \pm a\sqrt{(1+m^2)}$$

On substituting this value of c in $y = mx + c$, we get

$$y = mx \pm a\sqrt{(1+m^2)} \qquad \ldots(i)$$

which are the required equations of tangents.

Also, let (x_1, y_1) be the point of contact, then equation of tangent at (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is

$$xx_1 + yy_1 = a^2 \qquad \ldots(ii)$$

On comparing Eq. (i) and Eq. (ii), we get

$$\frac{x_1}{m} = \frac{y_1}{-1} = \frac{a^2}{\pm a\sqrt{(1+m^2)}}$$

$$\Rightarrow \quad \frac{x_1}{m} = -\frac{y_1}{1} = \pm\frac{a}{\sqrt{(1+m^2)}}$$

$$\Rightarrow \quad x_1 = \pm\frac{am}{\sqrt{1+m^2}} \quad \text{and} \quad y_1 = \mp\frac{a}{\sqrt{(1+m^2)}}$$

Hence, $(x_1, y_1) = \left(\pm\frac{am}{\sqrt{(1+m^2)}}, \mp\frac{a}{\sqrt{(1+m^2)}}\right)$

Corollary : It also follows that $y = mx + c$ is a tangent to $x^2 + y^2 = a^2$, if $c^2 = a^2(1+m^2)$ which is condition of tangency.

Remarks

1. The reason why there are two equations $y = mx \pm a\sqrt{1+m^2}$, there are two tangents, both are parallel and at the ends of diameter.
2. The line $ax + by + c = 0$ is tangent to the circle $x^2 + y^2 = r^2$ if and only if $c^2 = r^2(a^2 + b^2)$.
3. If the line $y = mx + c$ is the tangent to the circle $x^2 + y^2 = r^2$, then point of contact is given by $\left(-\frac{mr^2}{c}, \frac{r^2}{c}\right)$
4. If the line $ax + by + c = 0$ is the tangent to the circle $x^2 + y^2 = r^2$, then point of contact is given by $\left(-\frac{ar^2}{c}, -\frac{br^2}{c}\right)$.
5. The condition that the line $lx + my + n = 0$ touches the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $(lg + mf - n)^2 = (l^2 + m^2)(g^2 + f^2 - c)$.
6. Equation of tangent of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ in terms of slope is $y + f = m(x + g) \pm \sqrt{(g^2 + f^2 - c)(1 + m^2)}$
7. The equation of tangents of slope m to the circle $(x-a)^2 + (y-b)^2 = r^2$ are given by $(y-b) = m(x-a) \pm r\sqrt{(1+m^2)}$
and the coordinates of the points of contact are $\left(a \pm \frac{mr}{\sqrt{(1+m^2)}}, b \mp \frac{r}{\sqrt{(1+m^2)}}\right)$

Example 35. Find the equations of the tangents to the circle $x^2 + y^2 = 9$, which

(i) are parallel to the line $3x + 4y - 5 = 0$

(ii) are perpendicular to the line $2x + 3y + 7 = 0$

(iii) make an angle of 60^o with the X-axis

Sol. (i) Slope of $3x + 4y - 5 = 0$ is $-\frac{3}{4}$

Let $m = -\frac{3}{4}$

and equation of circle is $x^2 + y^2 = 9$

$\therefore$ Equations of tangents

$$y = -\frac{3}{4}x \pm 3\sqrt{\left(1 + \left(-\frac{3}{4}\right)^2\right)}$$

$\Rightarrow \quad 4y = -3x \pm 15$ or $3x + 4y \pm 15 = 0$

(ii) Slope of $2x + 3y + 7 = 0$ is $-\frac{2}{3}$

$\therefore$ Slope of perpendicular to $2x + 3y + 7 = 0$ is $\frac{3}{2} = m$ (say)

and given circle is $x^2 + y^2 = 9$

$\therefore$ Equations of tangents perpendicular to $2x + 3y + 7 = 0$ is

$$y = \frac{3}{2}x \pm 3\sqrt{1 + \left(\frac{3}{2}\right)^2}$$

$\Rightarrow \quad 2y = 3x \pm 3\sqrt{13}$

or $\quad 3x - 2y \pm 3\sqrt{13} = 0$

(iii) Since, tangent make an angle $60°$ with the X-axis

$\therefore \quad m = \tan 60° = \sqrt{3}$

and given circle $\quad x^2 + y^2 = 9$

$\therefore$ Equation of tangents $y = \sqrt{3}x \pm 3\sqrt{1 + (\sqrt{3})^2}$

or $\quad \sqrt{3}\,x - y \pm 6 = 0$

Aliter :

(i) Let tangent parallel to $3x + 4y - 5 = 0$ is

$$3x + 4y + \lambda = 0 \qquad \ldots(i)$$

and circle $\quad x^2 + y^2 = 9$

then perpendicular distance from $(0, 0)$ on Eq. (i) = radius

$$\frac{|\lambda|}{\sqrt{(3^2 + 4^2)}} = 3$$

or $\quad |\lambda| = 15$

$\therefore \quad \lambda = \pm 15$

From Eq. (i), equations of tangents are

$$3x + 4y \pm 15 = 0$$

(ii) Let tangent perpendicular to $2x + 3y + 7 = 0$ is

$$3x - 2y + \lambda = 0 \qquad \ldots(ii)$$

and circle $\quad x^2 + y^2 = 9$

then, perpendicular distance from $(0, 0)$ on Eq. (ii) = radius

$$\frac{|\lambda|}{\sqrt{3^2 + (-2)^2}} = 3$$

or $\quad |\lambda| = 3\sqrt{13}$

or $\quad \lambda = \pm 3\sqrt{13}$

From Eq. (ii), equations of tangents are

$$3x - 2y \pm 3\sqrt{13} = 0$$

(iii) Let equation of tangent which makes an angle of $60°$ with the X-axis is

$$y = \sqrt{3}x + c \qquad \ldots(iii)$$

or $\quad \sqrt{3}x - y + c = 0$

and circle $\quad x^2 + y^2 = 9$

then, perpendicular distance from $(0, 0)$ to Eq. (iii) = radius

$$\frac{|c|}{\sqrt{(\sqrt{3})^2 + (-1)^2}} = 3$$

or $\quad |c| = 6$

or $\quad c = \pm 6$

From Eq. (iii), equations of tangents are

$$\sqrt{3}x - y \pm 6 = 0$$

Example 36. Prove that the line $lx + my + n = 0$ touches the circle $(x - a)^2 + (y - b)^2 = r^2$ if $(al + bm + n)^2 = r^2(l^2 + m^2)$.

Sol. If the line $lx + my + n = 0$ touches the circle $(x - a)^2 + (y - b)^2 = r^2$, then length of the perpendicular from the centre = radius

$$\frac{|la + mb + n|}{\sqrt{(l^2 + m^2)}} = r$$

$\Rightarrow \quad (la + mb + n)^2 = r^2(l^2 + m^2)$

Aliter :

Here, line is $lx + my + n = 0$ and circle is $(x - a)^2 + (y - b)^2 = r^2$. Here, centre of circle (a, b) shift at $(0, 0)$, then replacing x by $x + a$ and y by $y + b$ in the equation of straight line $lx + my + n = 0$ and circle $(x - a)^2 + (y - b)^2 = r^2$, the new form of straight line and circle are

$$l(x + a) + m(y + b) + n = 0$$

or $\quad lx + my + (al + mb + n) = 0 \qquad \ldots(i)$

and $\quad x^2 + y^2 = r^2 \qquad \ldots(ii)$

respectively.

On comparing Eq. (i) with $y = Mx + C$

then $\quad M = -\frac{l}{m}$

and $\quad C = -\frac{(al + bm + n)}{m}$

Since, Eq. (i) is the tangent of Eq. (ii), then

$$C^2 = r^2 (1 + M^2)$$

or $$\frac{(al + bm + n)^2}{m^2} = r^2 \left(1 + \frac{l^2}{m^2}\right)$$

or $$(al + bm + n)^2 = r^2 (l^2 + m^2)$$

Example 37. Show that the line $3x - 4y = 1$ touches the circle $x^2 + y^2 - 2x + 4y + 1 = 0$. Find the coordinates of the point of contact.

Sol. The centre and radius of the circle

$$x^2 + y^2 - 2x + 4y + 1 = 0 \text{ are } (1, -2)$$

and $\sqrt{(-1) + (2)^2 - 1} = 2$ respectively.

Since, length of perpendicular from centre $(1, -2)$ on $3x - 4y = 1$ is

$$\frac{|3 \times 1 - 4 \times (-2) - 1|}{\sqrt{(3)^2 + (-4)^2}} = \frac{10}{5}$$

$$= 2 = \text{radius of the circle}$$

Hence, $3x - 4y = 1$ touches the circle

$$x^2 + y^2 - 2x + 4y + 1 = 0$$

Second part : Let point of contact is (x_1, y_1), then tangent at (x_1, y_1) on $x^2 + y^2 - 2x + 4y + 1 = 0$ is

$$xx_1 + yy_1 - (x + x_1) + 2(y + y_1) + 1 = 0$$

$$\Rightarrow \quad x(x_1 - 1) + y(y_1 + 2) - x_1 + 2y_1 + 1 = 0 \quad \text{...(i)}$$

and given line $\quad 3x - 4y - 1 = 0 \quad$ ...(ii)

Since, Eq. (i) and Eq. (ii) are identical, then comparing Eq. (i) and Eq. (ii), we get

$$\frac{x_1 - 1}{3} = \frac{y_1 + 2}{-4} = \frac{-x_1 + 2y_1 + 1}{-1}$$

or $$x_1 = -\frac{1}{5} \quad \text{and} \quad y_1 = -\frac{2}{5}$$

$\therefore$ Point of contact is $\left(-\frac{1}{5}, -\frac{2}{5}\right)$.

Aliter for second part : Since, perpendicular line to tangent always passes through the centre of the circle, perpendicular line to

$$3x - 4y = 1 \quad \text{...(i)}$$

is $$4x + 3y = \lambda \quad \text{...(ii)}$$

which passes through $(1, -2)$, then

$$4 - 6 = \lambda$$

$\therefore$ $$\lambda = -2$$

From Eq. (ii), $\quad 4x + 3y = -2 \quad$ ...(iii)

Solving Eq. (i) and Eq. (iii), we get the point of contact i.e. $x = -\frac{1}{5}$ and $y = -\frac{2}{5}$

Hence, point of contact is $\left(-\frac{1}{5}, -\frac{2}{5}\right)$.

Example 38. If $lx + my = 1$ touches the circle $x^2 + y^2 = a^2$, prove that the point (l, m) lies on the circle $x^2 + y^2 = a^{-2}$.

Sol. Since, $lx + my = 1$ touches the circle $x^2 + y^2 = a^2$.

Then, length of perpendicular from $(0, 0)$ on $lx + my = 1$ is equal to radius

then, $\frac{|-1|}{\sqrt{l^2 + m^2}} = a$ or $l^2 + m^2 = a^{-2}$

Hence, locus of (l, m) is $x^2 + y^2 = a^{-2}$

Aliter : Let the point of contact of line $lx + my = 1$ and circle $x^2 + y^2 = a^2$ is (x_1, y_1), then tangent of circle at (x_1, y_1) is $xx_1 + yy_1 = a^2$

Since, $\quad xx_1 + yy_1 = a^2 \quad$ and $\quad lx + my = 1$

are identical, then $\frac{x_1}{l} = \frac{y_1}{m} = \frac{a^2}{1}$

$\therefore$ $$x_1 = la^2, y_1 = ma^2$$

but (x_1, y_1) lie on $x^2 + y^2 = a^2$

then, $$l^2a^4 + m^2a^4 = a^2$$

$\therefore$ $$l^2 + m^2 = a^{-2}$$

$\therefore$ Locus of (l, m) is $x^2 + y^2 = a^{-2}$

Example 39. Show that the line $(x - 2)\cos\theta + (y - 2)\sin\theta = 1$ touches a circle for all values of θ. Find the circle.

Sol. Given line is $(x - 2)\cos\theta + (y - 2)\sin\theta$

$$1 = \cos^2\theta + \sin^2\theta$$

On comparing

$$x - 2 = \cos\theta \quad \text{...(i)}$$

and $$y - 2 = \sin\theta \quad \text{...(ii)}$$

Squaring and adding Eq. (i) and Eq. (ii), then

$$(x - 2)^2 + (y - 2)^2 = \cos^2\theta + \sin^2\theta$$

$$\Rightarrow \quad (x - 2)^2 + (y - 2)^2 = 1$$

or $$x^2 + y^2 - 4x - 4y + 7 = 0$$

Aliter : Since, tangent at $(\cos\theta, \sin\theta)$ of

$$x^2 + y^2 = 1 \quad \text{...(i)}$$

is $$x\cos\theta + y\sin\theta = 1 \quad \text{...(ii)}$$

replacing x by $x - 2$ and y by $y - 2$ in Eqs. (i) and (ii), then

$$(x - 2)^2 + (y - 2)^2 = 1 \quad \text{...(iii)}$$

$$(x - 2)\cos\theta + (y - 2)\sin\theta = 1 \quad \text{...(iv)}$$

Hence, Eq. (iv) touches the circle Eq. (iii).

$\therefore$ Equation of circle is

$$(x - 2)^2 + (y - 2)^2 = 1$$

or $$x^2 + y^2 - 4x - 4y + 7 = 0$$

Normal to a Circle at a Given Point

The normal of a circle at any point is a straight line which is perpendicular to the tangent at the point and always passes through the centre of the circle.

Different form of the Equation of Normals

1. Point form :

Theorem : The equation of normal at the point $P(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

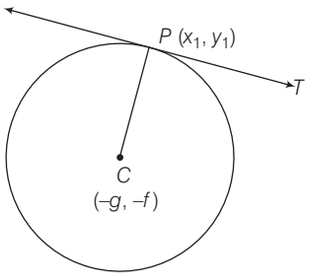
$$\frac{x - x_1}{x_1 + g} = \frac{y - y_1}{y_1 + f}$$

Proof :

Equation of the given circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad ...(i)$$

Its centre C is $(-g, -f)$



Let $P(x_1, y_1)$ be the given point.

$\because$ Normal of the circle at $P(x_1, y_1)$ passes through centre $C(-g, -f)$ of the circle.

Then, equation of normal CP passes through the points $C(-g, -f)$ and $P(x_1, y_1)$ is

$$y - y_1 = \frac{(y_1 + f)}{(x_1 + g)}(x - x_1)$$

or $$\frac{x - x_1}{x_1 + g} = \frac{y - y_1}{y_1 + f}$$

This is the required equation of normal at $P(x_1, y_1)$ of the given circle.

Remark

Easy method to find normal at (x_1, y_1) of second degree conics

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad ...(i)$$

then, according to determinant $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

write first two rows as $ax_1 + hy_1 + g$ and $hx_1 + by_1 + f$

Then, normal at (x_1, y_1) of conic (i)

$$\frac{x - x_1}{ax_1 + hy_1 + g} = \frac{y - y_1}{hx_1 + by_1 + f}$$

***Corollary* 1 :** Equation of normal of $x^2 + y^2 = a^2$ at (x_1, y_1) is

$$\frac{x - x_1}{1 \cdot x_1 + 0 + 0} = \frac{y - y_1}{0 + 1 \cdot y_1 + 0}$$

(Here, $g, f = 0$ and $a = b = 1$)

$\Rightarrow$ $$\frac{x - x_1}{x_1} = \frac{y - y_1}{y_1}$$

or $$\frac{x}{x_1} = \frac{y}{y_1}$$

***Corollary* 2 :** Equation of normal of $x^2 + y^2 + 2gx + 2fy + c = 0$ at (x_1, y_1) is

$$\frac{x - x_1}{x_1 + g} = \frac{y - y_1}{y_1 + f}$$ (Here, $a = b = 1$ and $h = 0$)

Remarks

1. Normal always passes through the centre of the circle. Just write the equation of the line joining (x_1, y_1) and the centre of the circle.
2. The equations of the normals show that they pass through the centre i.e. the normals are the radii which we know from *Euclidean geometry*.

Example 40. ***Find the equation of the normal to the circle*** $x^2 + y^2 = 2x$***, which is parallel to the line*** $x + 2y = 3$.

Sol. Given circle is $x^2 + y^2 - 2x = 0$

Centre of given circle is $(1, 0)$

Since, normal is parallel to $x + 2y = 3$

let the equation of normal is $x + 2y = \lambda$

Since, normal passes through the centre of the circle i.e. $(1, 0)$

then $1 + 0 = \lambda$

$\therefore$ $\lambda = 1$

then, equation of normal is $x + 2y = 1$

or $x + 2y - 1 = 0$

Aliter Equation of normal at (x_1, y_1) of $x^2 + y^2 - 2x = 0$ is

$$\frac{x - x_1}{x_1 - 1} = \frac{y - y_1}{y_1 - 0}$$

or $$\text{Slope} = \frac{y_1}{x_1 - 1} = m_1 \qquad \text{(say)}$$

Since normal is parallel to $x + 2y = 3$

$\therefore$ Slope $= -\frac{1}{2} = m_2$ (say)

but given $m_1 = m_2$

$$\frac{y_1}{x_1 - 1} = -\frac{1}{2} \text{ or } x_1 + 2y_1 - 1 = 0$$

$\therefore$ Locus of (x_1, y_1) is $x + 2y - 1 = 0$

Example 41. Find the equation of the normal to the circle $x^2 + y^2 - 5x + 2y - 48 = 0$ at the point $(5, 6)$.

Sol. Equation of the normal at $(5, 6)$ is

$$\frac{x-5}{5-\frac{5}{2}} = \frac{y-6}{6+1} \Rightarrow \frac{x-5}{\frac{5}{2}} = \frac{y-6}{7} \Rightarrow \frac{2x-10}{5} = \frac{y-6}{7}$$

$\Rightarrow$ $14x - 70 = 5y - 30$

$\therefore$ $14x - 5y - 40 = 0$

Aliter I : Since, centre of the circle $x^2 + y^2 - 5x + 2y - 48 = 0$ is $\left(\frac{5}{2}, -1\right)$, normal at (5,6) is the equation of a line, which passes through $\left(\frac{5}{2}, -1\right)$ and (5, 6) is

$$y + 1 = \frac{6+1}{5-\frac{5}{2}}\left(x - \frac{5}{2}\right) \Rightarrow y + 1 = \frac{14}{5}\left(x - \frac{5}{2}\right)$$

$\Rightarrow$ $y + 1 = \frac{7}{5}(2x - 5)$

$\Rightarrow$ $5y + 5 = 14x - 35$ or $14x - 5y - 40 = 0$

Aliter II : Equation of tangent at (5, 6) is

$$5 \cdot x + 6 \cdot y - \frac{5}{2}(x + 5) + (y + 6) - 48 = 0$$

$\Rightarrow$ $10x + 12y - 5x - 25 + 2y + 12 - 96 = 0$

$\Rightarrow$ $5x + 14y - 109 = 0$

Slope of tangent $= -\frac{5}{14}$

$\therefore$ Slope of normal $= \frac{14}{5}$

$\therefore$ Equation of normal at (5, 6) with slope $\frac{14}{5}$ is

$$y - 6 = \frac{14}{5}(x - 5)$$

$\Rightarrow$ $5y - 30 = 14x - 70$

or $14x - 5y - 40 = 0$

2. Parametric form

Since, parametric coordinates of circle $x^2 + y^2 = a^2$ is $(a\cos\theta, a\sin\theta)$.

$\therefore$ Equation of normal at $(a\cos\theta, a\sin\theta)$ is

$$\frac{x}{a\cos\theta} = \frac{y}{a\sin\theta}$$

or $$\frac{x}{\cos\theta} = \frac{y}{\sin\theta} \text{ or } y = x\tan\theta$$

or $y = mx$, where $m = \tan\theta$

which is slope form of normal.

Exercise for Session 4

1. The length of the chord cut-off by $y = 2x + 1$ from the circle $x^2 + y^2 = 2$ is

(a) $\frac{5}{6}$ (b) $\frac{6}{5}$ (c) $\frac{6}{\sqrt{5}}$ (d) $\frac{\sqrt{5}}{6}$

2. Circle $x^2 + y^2 - 4x - 8y - 5 = 0$ will intersect the line $3x - 4y = \lambda$ in two distinct points, if

(a) $-10 < \lambda < 5$ (b) $9 < \lambda < 20$ (c) $-35 < \lambda < 15$ (d) $-16 < \lambda < 30$

3. If the line $3x - 4y + \lambda = 0$, $(\lambda > 0)$ touches the circle $x^2 + y^2 - 4x - 8y - 5 = 0$ at (a, b), then $\lambda + a + b$ is equal to

(a) -22 (b) -20 (c) 20 (d) 22

4. Tangent which is parallel to the line $x - 3y - 2 = 0$ of the circle $x^2 + y^2 - 4x + 2y - 5 = 0$, has point/points of contact

(a) (1, – 2) (b) (–1, 2) (c) (3, 4) (d) (3, – 4)

5. If a circle, whose centre is $(-1,1)$ touches the straight line $x + 2y = 12$, then the co-ordinates of the point of contact are

(a) $\left(-\frac{7}{2}, -4\right)$ (b) $\left(-\frac{18}{5}, -\frac{21}{5}\right)$ (c) $(2, -7)$ (d) $(-2, -5)$

6. The area of the triangle formed by the tangent at the point (a, b) to the circle $x^2 + y^2 = r^2$ and the coordinate axes is

(a) $\frac{r^4}{2ab}$ (b) $\frac{r^4}{2|ab|}$ (c) $\frac{r^4}{ab}$ (d) $\frac{r^4}{|ab|}$

7. The equation of the tangent to the circle $x^2 + y^2 + 4x - 4y + 4 = 0$ which make equal intercepts on the positive coordinate axes is

(a) $x + y = 2$ (b) $x + y = 2\sqrt{2}$ (c) $x + y = 4$ (d) $x + y = 8$

8. If $a > 2b > 0$, then the positive value of m for which $y = mx - b\sqrt{(1+m^2)}$ is a common tangent to $x^2 + y^2 = b^2$ and $(x-a)^2 + y^2 = b^2$ is

(a) $\frac{2b}{\sqrt{(a^2-4b^2)}}$ (b) $\frac{\sqrt{(a^2-4b^2)}}{2b}$ (c) $\frac{2b}{a-2b}$ (d) $\frac{b}{a-2b}$

9. The angle between a pair of tangents from a point P to the circle $x^2 + y^2 = 16$ is $\frac{\pi}{3}$ and locus of P is $x^2 + y^2 = r^2$, then value of r is

(a) 5 (b) 6 (c) 7 (d) 8

10. The normal at the point $(3, 4)$ on a circle cuts the circle at the point $(-1, -2)$. Then, the equation of the circle is

(a) $x^2 + y^2 + 2x - 2y - 13 = 0$ (b) $x^2 + y^2 - 2x - 2y - 11 = 0$

(c) $x^2 + y^2 - 2x + 2y + 12 = 0$ (d) $x^2 + y^2 - 2x - 2y + 14 = 0$

11. The line $ax + by + c = 0$ is a normal to the circle $x^2 + y^2 = r^2$. The portion of the line $ax + by + c = 0$ intercepted by this circle is of length

(a) $\sqrt{r}$ (b) r (c) r^2 (d) $2r$

12. If the line $ax + by + c = 0$ touches the circle $x^2 + y^2 - 2x = \frac{3}{5}$ and is normal to the circle $x^2 + y^2 + 2x - 4y + 1 = 0$, then (a, b) are

(a) (1, 3) (b) (3, 1) (c) (1, 2) (d) (2, 1)

13. Show that for all values of θ, $x \sin\theta - y \cos\theta = a$ touches the circle $x^2 + y^2 = a^2$.

14. Find the equation of the tangents to the circle $x^2 + y^2 - 2x - 4y - 4 = 0$ which are (i) parallel (ii) perpendicular to the line $3x - 4y - 1 = 0$.

15. Find the equation of the family of circle which touch the pair of straight lines $x^2 - y^2 + 2y - 1 = 0$.

16. Find the value of λ so that the line $3x - 4y = \lambda$ may touch the circle $x^2 + y^2 - 4x - 8y - 5 = 0$.

17. Show that the area of the triangle formed by the positive X-axis, the normal and tangent to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$ is $2\sqrt{3}$.

Session 5

Tangents from a Point to the Circle, Length of the Tangent from a Point to a Circle, Power of a Point with Respect to a Circle, Chord of Contact, Chord Bisected at a Given Point, Pair of Tangents, Director Circle

Tangent from a Point to the Circle

Theorem : From a given point two tangents can be drawn to a circle which are real, coincident or imaginary according as the given point lies outside,on or inside the circle .

Proof : If circle is $x^2 + y^2 = a^2$ ...(i)

any tangent to the circle Eq. (i) is

$$y = mx + a\sqrt{(1+m^2)} \quad \text{...(ii)}$$

If outside point is (x_1, y_1)

then, $y_1 = mx_1 + a\sqrt{1+m^2}$

or $(y_1 - mx_1)^2 = a^2(1+m^2)$

or $y_1^2 + m^2x_1^2 - 2mx_1y_1 = a^2 + a^2m^2$

$\Rightarrow$ $m^2(x_1^2 - a^2) - 2mx_1y_1 + y_1^2 - a^2 = 0$ ...(iii)

which is quadratic in m which gives two values of m.

(real coincident or imaginary) corresponding to any value of x_1 and y_1.

The tangents are real, coincident or imaginary according as the values of m obtained from Eq. (iii) are real, coincident or imaginary.

or Discriminant $>, =,$ or <0

$\Rightarrow$ $4x_1^2y_1^2 - 4(x_1^2 - a^2)(y_1^2 - a^2) >, =, \text{or} < 0$

$\Rightarrow$ $(x_1^2 + y_1^2 - a^2) >, = \text{or} < 0$

i.e. $P(x_1, y_1)$ lies outside, on or inside the circle $x^2 + y^2 = a^2$.

If P outside the circle, then substituting these values of m in Eq. (ii), we get the equation of tangents.

Aliter :

First write equation of line through (x_1, y_1) say

$$y - y_1 = m(x - x_1) \quad \text{...(i)}$$

which is tangent of the circle $x^2 + y^2 = a^2$, then

length of perpendicular from centre $(0, 0)$ to Eq. (i) = radius of the circle

then, $\dfrac{|mx_1 - y_1|}{\sqrt{(1+m^2)}} = a$

or $(mx_1 - y_1)^2 = a^2(1+m^2)$

or $m^2x_1^2 - 2mx_1y_1 + y_1^2 = a^2 + a^2m^2$

$\Rightarrow$ $m^2(x_1^2 - a^2) - 2mx_1y_1 + y_1^2 - a^2 = 0$

which is quadratic in m which gives two values of m.

Example 42. Find the equations of the tangents to the circle $x^2 + y^2 = 16$ drawn from the point $(1, 4)$.

Sol. Given circle is

$$x^2 + y^2 = 16 \quad \text{...(i)}$$

Any tangent of Eq. (i) in terms of slope is

$$y = mx + 4\sqrt{(1+m^2)} \quad \text{...(ii)}$$

which passes through $(1, 4)$

then, $4 = m + 4\sqrt{(1+m^2)}$

$\Rightarrow$ $(4 - m)^2 = 16(1 + m^2)$

$\Rightarrow$ $15m^2 + 8m = 0$

$\therefore$ $m = 0, -\dfrac{8}{15}$

From Eq. (ii), equations of tangents drawn from (1, 4) are

$$y = 4$$

and $$y = -\frac{8}{15}x + 4\sqrt{\left(1 + \frac{64}{225}\right)}$$

or $8x + 15y = 68$ respectively.

Aliter : Equation of line through (1, 4) is $y - 4 = m(x - 1)$

$\Rightarrow$ $$mx - y + 4 - m = 0 \quad ...(i)$$

Then, perpendicular length from centre (0, 0) to $mx - y + 4 - m = 0$ is equal to radius

then, $$\frac{|4 - m|}{\sqrt{m^2 + 1}} = 4$$

or $$(4 - m)^2 = 16(m^2 + 1)$$

$\Rightarrow$ $$15m^2 + 8m = 0$$

$\therefore$ $$m = 0, -\frac{8}{15}$$

From Eq. (i), equation of tangents from (1, 4) are $y = 4$ and $8x + 15y = 68$, respectively.

Example 43. The angle between a pair of tangents from a point P to the circle $x^2 + y^2 + 4x - 6y + 9\sin^2\alpha + 13\cos^2\alpha = 0$ is 2α. Find the equation of the locus of the point P.

Sol. Let coordinates of P be (x_1, y_1) and given circle is

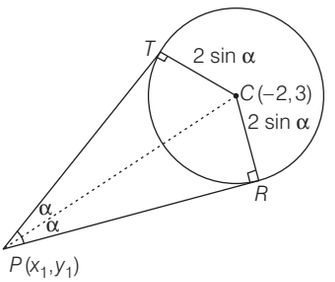
$$x^2 + y^2 + 4x - 6y + 9\sin^2\alpha + 13\cos^2\alpha = 0$$

or $$(x + 2)^2 + (y - 3)^2 - 4 - 9 + 9\sin^2\alpha + 13\cos^2\alpha = 0$$

$\Rightarrow$ $$(x + 2)^2 + (y - 3)^2 + 9\sin^2\alpha - 13(1 - \cos^2\alpha) = 0$$

$\Rightarrow$ $$(x + 2)^2 + (y - 3)^2 = 4\sin^2\alpha$$

$\therefore$ Centre and radius are $(-2, 3)$ and $2\sin\alpha$, respectively.



Distance between $P(x_1, y_1)$ and centre of circle $C(-2, 3)$ is

$$CP = \sqrt{(x_1 + 2)^2 + (y_1 - 3)^2}$$

In ΔPCT, $$\sin\alpha = \frac{CT}{CP} = \frac{2\sin\alpha}{\sqrt{(x_1 + 2)^2 + (y_1 - 3)^2}}$$

or $$\sqrt{(x_1 + 2)^2 + (y_1 - 3)^2} = 2$$

or $$(x_1 + 2)^2 + (y_1 - 3)^2 = 4$$

The required locus of $P(x_1, y_1)$ is

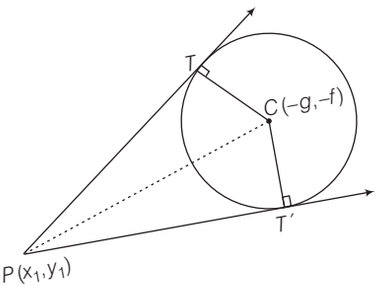
$$(x + 2)^2 + (y - 3)^2 = 4$$

Length of the Tangent from a Point to a Circle

Theorem : The length of tangent from the point $P(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$\sqrt{(x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c)} = \sqrt{S_1}$$

Proof : Let PT and PT' be two tangents from the given point $P(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$. Then, the centre and radius are $C(-g, -f)$ and $\sqrt{(g^2 + f^2 - c)}$ $(= CT = CT')$ respectively.



In ΔPCT,

$$PT = \sqrt{(PC)^2 - (CT)^2}$$

$$= \sqrt{(x_1 + g)^2 + (y_1 + f)^2 - g^2 - f^2 + c}$$

$$= \sqrt{(x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c)} = \sqrt{S_1} = PT'$$

where, $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

Remarks

1. To find length of tangent
   let $S = x^2 + y^2 + 2gx + 2fy + c$
   then, $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$
   where, $P(x_1, y_1)$
   $\therefore$ length of tangent $= \sqrt{S_1}$
2. For S_1 first write the equation of circle in general form i.e. coefficient of x^2 = coefficient of $y^2 = 1$ and making RHS of circle is zero, then let LHS by S.

Example 44. ***Find the length of tangents drawn from the point*** $(3, -4)$ ***to the circle*** $2x^2 + 2y^2 - 7x - 9y - 13 = 0$.

Sol. The equation of the given circle is

$$2x^2 + 2y^2 - 7x - 9y - 13 = 0$$

Re-writing the given equation of the circle

i.e. $$x^2 + y^2 - \frac{7}{2}x - \frac{9}{2}y - \frac{13}{2} = 0$$

Let $S = x^2 + y^2 - \frac{7}{2}x - \frac{9}{2}y - \frac{13}{2}$

$\therefore \quad S_1 = (3)^2 + (-4)^2 - \frac{7}{2}\times 3 - \frac{9}{2}\times(-4) - \frac{13}{2}$

$= 25 - \frac{21}{2} + 18 - \frac{13}{2} = 43 - 17 = 26$

$\therefore$ Length of tangent $= \sqrt{S_1} = \sqrt{26}$

Example 45. If the length of tangent from (f, g) to the circle $x^2 + y^2 = 6$ be twice the length of the tangent from (f, g) to circle $x^2 + y^2 + 3x + 3y = 0$, then find the value of $f^2 + g^2 + 4f + 4g$.

Sol. According to the question

$$\sqrt{(g^2 + f^2 - 6)} = 2\sqrt{(f^2 + g^2 + 3f + 3g)}$$

On squaring $g^2 + f^2 - 6 = 4f^2 + 4g^2 + 12f + 12g$

or $3f^2 + 3g^2 + 12f + 12g + 6 = 0$

or $f^2 + g^2 + 4f + 4g + 2 = 0$

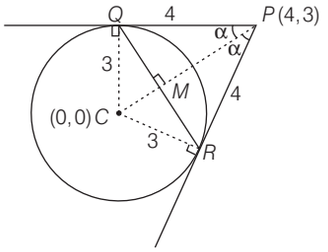
or $f^2 + g^2 + 4f + 4g = -2$

Example 46. Show that the area of the triangle formed by tangents from the point (4, 3) to the circle $x^2 + y^2 = 9$ and the line segment joining their points of contact is $7\frac{17}{25}$ square units in length.

Sol. Since, $PQ = PR = \sqrt{4^2 + 3^2 - 9} = 4$ units

$\therefore \quad \angle CPQ = \angle CPR = \alpha$ (let)

$\therefore \quad PC = \sqrt{(4-0)^2 + (3-0)^2} = 5$ units



$\therefore$ In $\Delta PQC, \quad \tan\alpha = \frac{3}{4},$

$\therefore \quad \sin\alpha = \frac{3}{5}$

and $\cos\alpha = \frac{4}{5}$

In $\Delta PMQ, \quad \cos\alpha = \frac{PM}{4} = \frac{4}{5}$

$\therefore \quad PM = \frac{16}{5}$

and $\sin\alpha = \frac{QM}{4} = \frac{3}{5}$

$\therefore \quad QM = \frac{12}{5}$

$\therefore$ Area of $\Delta PQR = \frac{1}{2}\cdot QR \cdot PM$

$= \frac{1}{2}(2QM)\cdot PM = (QM)(PM)$

$= \left(\frac{12}{5}\right)\left(\frac{16}{5}\right) = \frac{192}{25}$

$= 7\frac{17}{25}$ sq units

Example 47. Show that the length of the tangent from any point on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ to the circle $x^2 + y^2 + 2gx + 2fy + c_1 = 0$ is $\sqrt{(c_1 - c)}$.

Sol. Let (x_1, y_1) be any point on

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

then $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$ ...(i)

$\therefore$ Length of tangent from (x_1, y_1) to the circle

$x^2 + y^2 + 2gx + 2fy + c_1 = 0$ is

$\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c_1}$

$= \sqrt{(-c + c_1)} = \sqrt{(c_1 - c)}$ [From Eq. (i)]

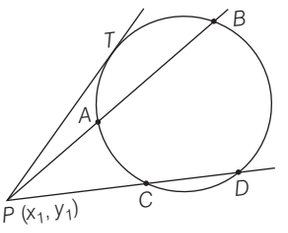
Power of a Point With Respect to a Circle

Theorem : The power of a point $P(x_1, y_1)$ with respect to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ is } S_1$$

where, $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

Proof : Let $P(x_1, y_1)$ be a point outside the circle and PAB and PCD drawn two secants. The power of $P(x_1, y_1)$ with respect to



$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$$

is equal to $PA \cdot PB$ which is

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = S_1$$

$\therefore$ Power remains constant for the circle

i.e. independent of A and B.

$\therefore\ PA \cdot PB = PC \cdot PD = (PT)^2 = S_1 = (\sqrt{S_1})^2$

$\therefore\ PA \cdot PB = (\sqrt{S_1})^2$ = square of the length of tangent.

Remark

If P outside, inside or on the circle, then $PA \cdot PB$ is + ve, − ve or zero, respectively.

Example 48. ***Find the power of point*** $(2, 4)$ ***with respect to the circle*** $x^2 + y^2 - 6x + 4y - 8 = 0$

Sol. The power of the point $(2, 4)$ with respect to the circle

$$x^2 + y^2 - 6x + 4y - 8 = 0 \text{ is } (\sqrt{S_1})^2 \text{ or } S_1$$

where, $S = x^2 + y^2 - 6x + 4y - 8$

$$\therefore \quad S_1 = (2)^2 + (4)^2 - 6 \times 2 + 4 \times 4 - 8$$
$$= 4 + 16 - 12 + 16 - 8 = 16$$

[$\because (2, 4)$ is outside from the circle $x^2 + y^2 - 6x + 4y - 8 = 0$]

Example 49. ***Show that the locus of the point, the powers of which with respect to two given circles are equal, is a straight line.***

Sol. Let the given circles be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...(i)}$$

and $$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \quad \text{...(ii)}$$

Let $P(x_1, y_1)$ be a point, the powers of which with respect to the circles Eqs. (i) and (ii) are equal. Then,

$$\therefore \quad [\sqrt{(x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c)}]^2$$
$$= [\sqrt{(x_1^2 + y_1^2 + 2g_1x_1 + 2f_1y_1 + c_1)}]^2$$

or $$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$
$$= x_1^2 + y_1^2 + 2g_1x_1 + 2f_1y_1 + c_1$$

$$\Rightarrow \quad 2(g - g_1)x_1 + 2(f - f_1)y_1 + c - c_1 = 0$$

then, locus of $P(x_1, y_1)$ is

$$2(g - g_1)x + 2(f - f_1)y + c - c_1 = 0$$

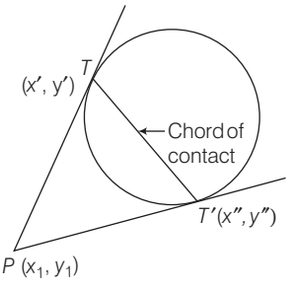
which is a straight line.

Chord of Contact

From any external point, two tangents can be drawn to a given circle. The chord joining the points of contact of the two tangents is called the chord of contact of tangents.

Theorem : The equation of the chord of contact of tangents drawn from a point (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$.

Sol. Let $T(x', y')$ and $T'(x'', y'')$ be the points of contact of tangents drawn from $P(x_1, y_1)$ to $x^2 + y^2 = a^2$.



Then, equations of tangents PT and PT' are

$$xx' + yy' = a^2 \text{ and } xx'' + yy'' = a^2 \text{ respectively.}$$

Since, both tangents pass through $P(x_1, y_1)$, then

$$x_1x' + y_1y' = a^2$$

and $$x_1x'' + y_1y'' = a^2$$

$\because$ Points $T(x', y')$ and $T'(x'', y'')$ lie on

$$xx_1 + yy_1 = a^2$$

$\therefore$ Equation of **chord of contact** TT' is $xx_1 + yy_1 = a^2$

Remark

Equation of chord of contact like as equation of tangent at that point but point different.

Now, for chord of contact at (x_1, y_1), replacing x^2 by xx_1, y^2 by yy_1, x by $\frac{x + x_1}{2}$, y by $\frac{y + y_1}{2}$

and xy by $\frac{xy_1 + x_1y}{2}$.

Corollary 1 : If R is the radius of the circle and L is the length of the tangent from $P(x_1, y_1)$ on $S = 0$.

Here, $L = \sqrt{S_1}$, then

(a) Length of chord of contact $TT' = \dfrac{2LR}{\sqrt{(R^2 + L^2)}}$

(b) Area of triangle formed by the pair of tangents and its chord of contact $= \dfrac{RL^3}{R^2 + L^2}$

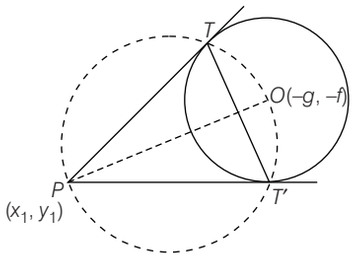
(c) Angle between the pair of tangents from $P(x_1, y_1)$ $= \tan^{-1}\left(\dfrac{2RL}{L^2 - R^2}\right)$

Corollary 2 : Equation of the circle circumscribing the triangle PTT' is

$$(x - x_1)(x + g) + (y - y_1)(y + f) = 0,$$

where, $O(-g, -f)$ is the centre of the circle

$$x^2 + y^2 + 2gx + 2fx + c = 0$$



i.e. Required circle always passes through the centre of the given circle (Here, OP is the diameter of the required circle).

Example 50. If the pair of tangents are drawn from the point $(4,5)$ to the circle $x^2+y^2-4x-2y-11=0$, then,

(i) Find the length of chord of contact.

(ii) Find the area of the triangle formed by a pair of tangents and their chord of contact.

(iii) Find the angle between the pair of tangents.

Sol. Here, $P \equiv (4,5)$,

$$R = \sqrt{((2)^2+(1)^2+11)} = 4$$

and $L = \sqrt{S_1} = \sqrt{((4)^2+(5)^2-4\times4-2\times5-11)} = 2$

(i) Length of chord of contact

$$= \frac{2LR}{\sqrt{(R^2+L^2)}} = \frac{2\times2\times4}{\sqrt{(4)^2+(2)^2}} = \frac{8}{\sqrt{5}} \text{ unit}$$

(ii) Area of triangle $= \frac{RL^3}{R^2+L^2} = \frac{4\times8}{16+4} = \frac{8}{5}$ sq units

(iii) Angle between the pair of tangents

$$= \pi + \tan^{-1}\left(\frac{2\times4\times2}{2^2-4^2}\right)$$

$$= \pi - \tan^{-1}\left(\frac{4}{3}\right) \qquad (\because L < R)$$

Example 51. Tangents PQ, PR are drawn to the circle $x^2+y^2=36$ from the point $P(-8,2)$ touching the circle at Q, R respectively. Find the equation of the circumcircle of ΔPQR.

Sol. Here, $P \equiv (-8,2)$ and $O \equiv (0,0)$

$\therefore$ Equation of the required circle is

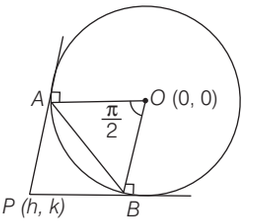
$$(x-(-8))(x-0)+(y-2)(y-0)=0$$

or $x^2+y^2+8x-2y=0$ ($\because OP$ is the diameter)

Example 52. Find the condition that chord of contact of any external point (h,k) to the circle $x^2+y^2=a^2$ should subtend right angle at the centre of the circle .

Sol. Equation of chord of contact AB is $hx+ky=a^2$

For equation of pair of tangents of OA and OB, make homogeneous $x^2+y^2=a^2$ with the help of $hx+ky=a^2$ or $\frac{hx+ky}{a^2}=1$



then, $$x^2+y^2 = a^2\left(\frac{hx+ky}{a^2}\right)^2$$

or $$a^2(x^2+y^2) = (hx+ky)^2$$

or $$x^2(a^2-h^2) - 2hkxy + y^2(a^2-k^2) = 0$$

but $$\angle AOB = \frac{\pi}{2}$$

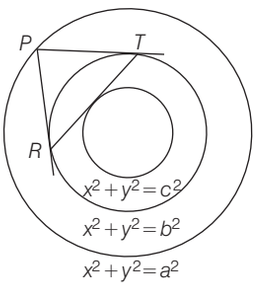
$\therefore$ Coefficient of x^2 + Coefficient of $y^2 = 0$

$\Rightarrow$ $a^2-h^2+a^2-k^2=0$ or $h^2+k^2=2a^2$

Example 53. The chord of contact of tangents drawn from a point on the circle $x^2+y^2=a^2$ to the circle $x^2+y^2=b^2$ touches the circle $x^2+y^2=c^2$. Show that a,b,c are in GP.

Sol. Let $P(a\cos\theta, a\sin\theta)$ be a point on the circle $x^2+y^2=a^2$.

Then, equation of chord of contact of tangents drawn from $P(a\cos\theta, a\sin\theta)$ to the circle $x^2+y^2=b^2$ is



$$ax\cos\theta + ay\sin\theta = b^2 \qquad \text{...(i)}$$

This touches the circle $x^2+y^2=c^2$ ...(ii)

$\therefore$ Length of perpendicular from $(0,0)$ to Eq. (i) = radius of Eq. (ii)

$\therefore$ $$\frac{|0+0-b^2|}{\sqrt{(a^2\cos^2\theta + a^2\sin^2\theta)}} = c$$

or $$b^2 = ac$$

$\Rightarrow$ a, b, c are in GP.

Chord Bisected at a Given Point

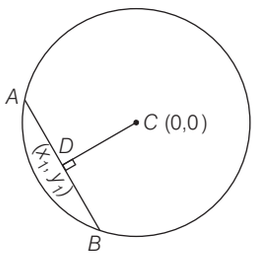
Theorem : The equation of the chord of the circle $x^2 + y^2 = a^2$ bisected at the point (x_1, y_1) is given by

$$xx_1 + yy_1 - a^2 = x_1^2 + y_1^2 - a^2$$

or $$T = S_1$$

Proof : Let any chord AB of the circle $x^2 + y^2 = a^2$ be bisected at $D(x_1, y_1)$.

If centre of circle is represented by C



then, slope of $DC = \dfrac{0 - y_1}{0 - x_1} = \dfrac{y_1}{x_1}$

$\therefore$ Slope of the chord AB is $-\dfrac{x_1}{y_1}$

then, equation of AB is $y - y_1 = -\dfrac{x_1}{y_1}(x - x_1)$

or $$yy_1 - y_1^2 = -xx_1 + x_1^2$$

or $$xx_1 + yy_1 = x_1^2 + y_1^2$$

or $$\boldsymbol{xx_1 + yy_1 - a^2 = x_1^2 + y_1^2 - a^2} \text{ or } \boldsymbol{T = S_1}$$

Remarks

1. The equation of chord of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, which is bisected at (x_1, y_1) ; is $T = S_1$
   where, $T = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$
   and $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$
2. The chord bisected at point (x_1, y_1) is the farthest from the centre among all the chords passing through the point (x_1, y_1). Also, for such chord, the length of the chord is minimum.

Example 54. ***Find the equation of the chord of*** $x^2 + y^2 - 6x + 10y - 9 = 0$ ***which is bisected at*** $(-2, 4)$.

Sol. The equation of the required chord is

$$-2x + 4y - 3(x - 2) + 5(y + 4) - 9 = 4 + 16 + 12 + 40 - 9$$

$\Rightarrow$ $$-5x + 9y - 46 = 0$$

or $$5x - 9y + 46 = 0$$

Example 55. Find the middle point of the chord intercepted on line $lx + my + n = 0$ by the circle $x^2 + y^2 = a^2$.

Sol. Let (x_1, y_1) be the middle point of the chord intercepted by the circle $x^2 + y^2 = a^2$ on the line $lx + my + n = 0$.

Then, equation of the chord of the circle $x^2 + y^2 = a^2$, whose middle points is (x_1, y_1), is

$$xx_1 + yy_1 - a^2 = x_1^2 + y_1^2 - a^2$$

or $$xx_1 + yy_1 = x_1^2 + y_1^2 \quad \text{...(i)}$$

Clearly, $lx + my + n = 0$ and Eq. (i) represented the same line,

$$\frac{x_1}{l} = \frac{y_1}{m} = \frac{x_1^2 + y_1^2}{-n} = \lambda \quad \text{(say)}$$

$\therefore$ $$\left.\begin{aligned} x_1 &= l\lambda \\ y_1 &= m\lambda \end{aligned}\right\} \quad \text{...(ii)}$$

and $$x_1^2 + y_1^2 = -n\lambda$$

or $$l^2\lambda^2 + m^2\lambda^2 = -n\lambda \quad \text{[from (ii)]}$$

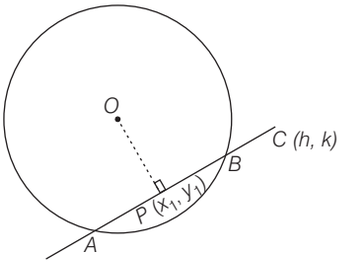
$\therefore$ $$\lambda = -\frac{n}{l^2 + m^2}$$

So, from Eq. (ii), $$x_1 = -\frac{nl}{l^2 + m^2}, y_1 = -\frac{mn}{l^2 + m^2}$$

Hence, the required point is $\left(-\dfrac{nl}{l^2 + m^2}, -\dfrac{nm}{l^2 + m^2}\right)$.

Example 56. Through a fixed point (h, k), secants are drawn to the circle $x^2 + y^2 = r^2$. Show that the locus of mid-point of the portions of secants intercepted by the circle is $x^2 + y^2 = hx + ky$.

Sol. Let $P(x_1, y_1)$ be the middle point of any chord AB, which passes through the point $C(h, k)$.



Equation of chord AB is $T = S_1$

$\therefore$ $$xx_1 + yy_1 - r^2 = x_1^2 + y_1^2 - r^2$$

or $$x_1^2 + y_1^2 = xx_1 + yy_1$$

But since AB passes through $C(h, k)$, then

$$x_1^2 + y_1^2 = hx_1 + ky_1$$

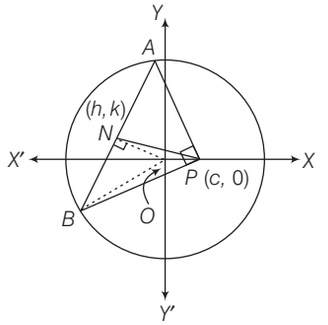
$\therefore$ Locus of $P(x_1, y_1)$ is $x^2 + y^2 = hx + ky$

Example 57. Find the locus of middle points of chords of the circle $x^2 + y^2 = a^2$, which subtend right angle at the point $(c, 0)$.

Sol. Let $N(h, k)$ be the middle point of any chord AB, which subtend a right angle at $P(c, 0)$.

Since, $\angle APB = 90°$

$\therefore$ $NA = NB = NP$



(since distance of the vertices from middle point of the hypotenuse are equal)

or $(NA)^2 = (NB)^2 = (h - c)^2 + (k - 0)^2$ ...(i)

But also $\angle BNO = 90°$

$\therefore$ $(OB)^2 = (ON)^2 + (NB)^2$

$\Rightarrow$ $-(NB)^2 = (ON)^2 - (OB)^2$

$\Rightarrow$ $-[(h - c)^2 + (k - 0)^2] = (h^2 + k^2) - a^2$

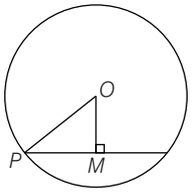
or $2(h^2 + k^2) - 2ch + c^2 - a^2 = 0$

$\therefore$ Locus of $N(h, k)$ is

$$2(x^2 + y^2) - 2cx + c^2 - a^2 = 0$$

Example 58. Find the equation of the chord of the circle $x^2 + y^2 = r^2$ passing through the point $(2,3)$ farthest from the centre.

Sol. Let $P \equiv (2,3)$ be the given point and M be the middle point of chord of circle $x^2 + y^2 = r^2$ through P.



Then, $(OM)^2 = (OP)^2 - (PM)^2$

If OM maximum, then PM is minimum. i.e. P coincides with M, which is middle point of the chord.

Hence, the equation of the chord is

$$T = S_1$$

i.e. $2x + 3y - r^2 = 2^2 + 3^2 - r^2$

or $2x + 3y = 13$

Pair of Tangents

Theorem : The combined equation of the pair of tangents drawn from a point $P(x_1, y_1)$ to the circle $x^2 + y^2 = a^2$ is

$$(x^2 + y^2 - a^2)(x_1^2 + y_1^2 - a^2) = (xx_1 + yy_1 - a^2)^2$$

or $SS_1 = T^2$

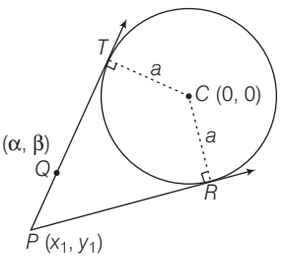
where, $S = x^2 + y^2 - a^2, S_1 = x_1^2 + y_1^2 - a^2$

and $T = xx_1 + yy_1 - a^2$

Proof : The given circle is $x^2 + y^2 = a^2$

Its centre and radius are $C(0, 0)$ and a respectively . Given external point be $P(x_1, y_1)$.

From point $P(x_1, y_1)$ two tangents PT and PR be drawn to the circle, touching circle at T and R respectively.



Let $Q(\alpha, \beta)$ on PT, then equation of PQ is

$$y - y_1 = \frac{\beta - y_1}{\alpha - x_1}(x - x_1)$$

or $y(\alpha - x_1) - x(\beta - y_1) - \alpha y_1 + \beta x_1 = 0$

Length of perpendicular from $C(0, 0)$ on $PT = a$ (radius)

$$\Rightarrow \quad \frac{|\beta x_1 - \alpha y_1|}{\sqrt{(\alpha - x_1)^2 + (\beta - y_1)^2}} = a$$

or $(\beta x_1 - \alpha y_1)^2 = a^2\{(\alpha - x_1)^2 + (\beta - y_1)^2\}$

$\therefore$ Locus of $Q(\alpha, \beta)$ is

$$(yx_1 - xy_1)^2 = a^2\{(x - x_1)^2 + (y - y_1)^2\}$$

$\Rightarrow$ $y^2x_1^2 + x^2y_1^2 - 2xy\,x_1y_1 = a^2$

$$\{x^2 + x_1^2 - 2xx_1 + y^2 + y_1^2 - 2yy_1\}$$

$\Rightarrow y^2(x_1^2 - a^2) + x^2(y_1^2 - a^2) - a^2(x_1^2 + y_1^2)$

$$= 2xyx_1y_1 - 2a^2xx_1 - 2a^2yy_1$$

On adding both sides, $(x^2x_1^2 + y^2y_1^2 + a^4)$, then

$y^2(x_1^2 + y_1^2 - a^2) + x^2(x_1^2 + y_1^2 - a^2) - a^2(x_1^2 + y_1^2 - a^2)$

$$= (xx_1 + yy_1 - a^2)^2$$

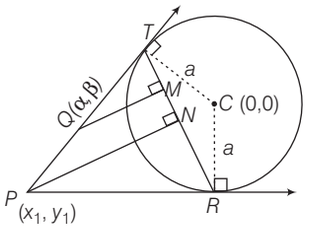
$$\Rightarrow \quad (x^2 + y^2 - a^2)(x_1^2 + y_1^2 - a^2) = (xx_1 + yy_1 - a^2)^2$$

This is the required equation of pair of tangents drawn from (x_1, y_1) to circle $x^2 + y^2 = a^2$.

Aliter : Let circle be $x^2 + y^2 = a^2$ with centre $C(0, 0)$ and radius a. Length of tangents from $P(x_1, y_1)$ and $Q(\alpha, \beta)$ are

$$PT = \sqrt{x_1^2 + y_1^2 - a^2}$$

and $$QT = \sqrt{\alpha^2 + \beta^2 - a^2}$$



Now, equation of TR (chord of contact is)

$$xx_1 + yy_1 - a^2 = 0$$

$$\therefore \quad PN = \frac{|x_1^2 + y_1^2 - a^2|}{\sqrt{(x_1^2 + y_1^2)}} \text{ and } QM = \frac{|\alpha x_1 + \beta y_1 - a^2|}{\sqrt{(x_1^2 + y_1^2)}}$$

But from similar Δ^s PNT and QMT,

$$\frac{PN}{QM} = \frac{PT}{QT} \Rightarrow \frac{(PN)^2}{(QM)^2} = \frac{(PT)^2}{(QT)^2}$$

$$\Rightarrow \quad \frac{\dfrac{(x_1^2 + y_1^2 - a^2)^2}{(x_1^2 + y_1^2)}}{\dfrac{(\alpha x_1 + \beta y_1 - a^2)^2}{(x_1^2 + y_1^2)}} = \frac{x_1^2 + y_1^2 - a^2}{\alpha^2 + \beta^2 - a^2}$$

$$\Rightarrow \quad (x_1^2 + y_1^2 - a^2)(\alpha^2 + \beta^2 - a^2) = (\alpha x_1 + \beta y_1 - a^2)^2$$

$\therefore$ Locus of $Q(\alpha, \beta)$ is

$$(x^2 + y^2 - a^2)(x_1^2 + y_1^2 - a^2)$$
$$= (xx_1 + yy_1 - a^2)^2$$

This is the required equation of pair of tangents drawn from (x_1, y_1) to circle $x^2 + y^2 = a^2$.

Corollary **:** The angle between the two tangents from (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is $2\tan^{-1}\left(\dfrac{a}{\sqrt{S_1}}\right)$, where $S_1 = x_1^2 + y_1^2 - a^2$.

Remarks

1. Equation of pair of tangents in notation form is $SS_1 = T^2$
   where, $S \equiv x^2 + y^2 - a^2$
   $S_1 \equiv x_1^2 + y_1^2 - a^2, T \equiv xx_1 + yy_1 - a^2$
2. When circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ and tangents are drawn from (x_1, y_1), then pair of tangents is
   $(x^2 + y^2 + 2gx + 2fy + c)(x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c)$
   $= [xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c]^2$
   where, $S \equiv x^2 + y^2 + 2gx + 2fy + c$,
   $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$,
   and $T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$

Advised for Students

Students are advised that, if they do not want to use the formula $(SS_1 = T^2)$, then use the following method :

Let $y - y_1 = m(x - x_1)$ be any line through $P(x_1, y_1)$.

Then, use condition of tangency $p = r$ i.e.

Length of perpendicular from the centre of circle or this line = radius of the circle.

Gives the values of m. In such away, we can find the equations of tangents from P.

Example 59. Find the equations of the tangents from the point $A(3, 2)$ to the circle $x^2 + y^2 + 4x + 6y + 8 = 0$.

Sol. Combined equation of the pair of tangents drawn from $A(3, 2)$ to the given circle $x^2 + y^2 + 4x + 6y + 8 = 0$ can be written in the usual notation.

$$T^2 = SS_1 \text{ namely}$$

$$\Rightarrow \quad [3x + 2y + 2(x + 3) + 3(y + 2) + 8]^2$$
$$= [x^2 + y^2 + 4x + 6y + 8][9 + 4 + 12 + 12 + 8]$$

$$\Rightarrow \quad (5x + 5y + 20)^2 = 45(x^2 + y^2 + 4x + 6y + 8)$$

$$\Rightarrow \quad 5(x + y + 4)^2 = 9(x^2 + y^2 + 4x + 6y + 8)$$

$$\Rightarrow \quad 5(x^2 + y^2 + 2xy + 8x + 8y + 16)$$
$$= 9(x^2 + y^2 + 4x + 6y + 8)$$

$$\Rightarrow \quad 4x^2 + 4y^2 - 10xy - 4x + 14y - 8 = 0$$

or $$2x^2 + 2y^2 - 5xy - 2x + 7y - 4 = 0$$

or $$(2x - y - 4)(x - 2y + 1) = 0$$

Hence, the required tangents to the circle from $A(3, 2)$ are

$$2x - y - 4 = 0 \quad \text{and} \quad x - 2y + 1 = 0$$

Aliter : Let $S \equiv x^2 + y^2 + 4x + 6y + 8 = 0$

Centre $C(-2, -3)$ and radius $= \sqrt{5}$

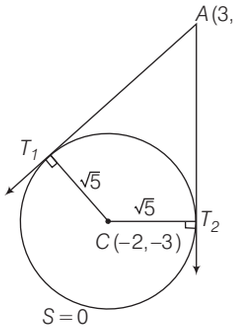
Let the slope of a tangent from A to '$S=0$' be m, then equation of tangent is

$$y-2=m(x-3)$$

or $$mx-y+2-3m=0 \quad \ldots(i)$$

Length of perpendicular from $C(-2,-3)$ on Eq. (i)

= radius of circle.



$$\Rightarrow \quad \frac{|-2m+3+2-3m|}{\sqrt{(m^2+1)}}=\sqrt{5}$$

or $$(5m-5)^2=5(m^2+1)$$

$$\Rightarrow \quad 25m^2-50m+25=5m^2+5$$

$$\Rightarrow \quad 20m^2-50m+20=0$$

or $$2m^2-5m+2=0$$

or $$(2m-1)(m-2)=0$$

$$\therefore \quad m=\frac{1}{2} \text{ or } m=2$$

Substituting these values of m in Eq. (i), we get the equations of two tangents are $x-2y+1=0$ and $2x-y-4=0$.

Director Circle

Director circle : The locus of the point of intersection of two perpendicular tangents to a given circle is known as its director circle.

Theorem : The equation of the director circle of the circle $x^2+y^2=a^2$ is

$$x^2+y^2=2a^2$$

Proof : The equation of any tangent to the circle $x^2+y^2=a^2$

is $$y=mx+a\sqrt{(1+m^2)} \quad \ldots(i)$$

Let $P(h,k)$ be the point of intersection of tangents, then $P(h,k)$ lies on Eq. (i)

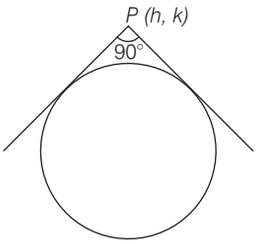
$$\therefore \quad k=mh+a\sqrt{(1+m^2)}$$

or $$(k-mh)^2=a^2(1+m^2)q$$

or $$m^2(h^2-a^2)-2mkh+k^2-a^2=0$$

This is quadratic equation in m, let two roots are m_1 and m_2.

But tangents are perpendiculars, then



$$m_1m_2=-1$$

$$\Rightarrow \quad \frac{k^2-a^2}{h^2-a^2}=-1 \text{ or } k^2-a^2=-h^2+a^2$$

or $$h^2+k^2=2a^2$$

Hence, locus of $P(h,k)$ is $\boldsymbol{x^2+y^2=2a^2}$

Aliter : The combined equation of the pair of tangents drawn from (h,k) to $x^2+y^2=a^2$ is

$$SS_1=T^2$$

where, $$S=x^2+y^2-a^2$$

$$S_1=h^2+k^2-a^2$$

and $$T=hx+ky-a^2$$

$$\therefore \quad (x^2+y^2-a^2)(h^2+k^2-a^2)=(hx+ky-a^2)^2$$

This equation will represent a pair of perpendicular lines if, coefficient of x^2 + coefficient of $y^2=0$

$$\Rightarrow \quad h^2+k^2-a^2-h^2+h^2+k^2-a^2-k^2=0$$

$$\Rightarrow \quad h^2+k^2-2a^2=0 \text{ or } h^2+k^2=2a^2$$

Hence, the locus of (h,k) is $\boldsymbol{x^2+y^2=2a^2}$

Remarks

1. The equation of the director circle of the circle $(x-h)^2+(y-k)^2=a^2$ is $(x-h)^2+(y-k)^2=2a^2$
2. The equation of the director circle of the circle $x^2+y^2+2gx+2fy+c=0$ is $x^2+y^2+2gx+2fy+c=g^2+f^2-c$
3. If two tangents are drawn from a point on the director circle to the circle, then angle between tangents is 90°.

Example 60. If two tangents are drawn from a point on the circle $x^2+y^2=50$ to the circle $x^2+y^2=25$, then find the angle between the tangents.

Sol. $\because x^2+y^2=50$ is the director circle of $x^2+y^2=25$

Hence, angle between tangents = 90°

Exercise for Session 5

1. If the tangent at the point P on the circle $x^2 + y^2 + 6x + 6y = 2$ meets the straight line $5x - 2y + 6 = 0$ at a point Q on the Y-axis, then the length PQ is

(a) 4 (b) $2\sqrt{5}$ (c) 5 (d) $3\sqrt{5}$

2. If the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is touched by $y = x$ at P such that $OP = 6\sqrt{2}$, where O is origin, then the value of c is

(a) 36 (b) 72 (c) 144 (d) 288

3. The chord of contact of tangents from a point P to a circle passes through Q. If l_1 and l_2 are the lengths of tangents from P and Q to the circle, then PQ is equal to

(a) $\frac{l_1 + l_2}{2}$ (b) $\frac{l_1 - l_2}{2}$ (c) $\sqrt{(l_1^2 + l_2^2)}$ (d) $\sqrt{(l_1^2 - l_2^2)}$

4. If the chord of contact of tangents from a point (x_1, y_1) to the circle $x^2 + y^2 = a^2$ touches the circle $(x - a)^2 + y^2 = a^2$, then the locus of (x_1, y_1) is

(a) a circle (b) a parabola (c) an ellipse (d) a hyperbola

5. The locus of the mid-points of a chord of the circle $x^2 + y^2 = 4$, which subtends a right angle at the origin is

(a) $x + y = 1$ (b) $x^2 + y^2 = 1$ (c) $x + y = 2$ (d) $x^2 + y^2 = 2$

6. The length of tangents from $P(1,-1)$ and $Q(3,3)$ to a circle are $\sqrt{2}$ and $\sqrt{6}$ respectively, then the length of tangent from $R(-2,-7)$ to the same circle is

(a) $\sqrt{41}$ (b) $\sqrt{51}$ (c) $\sqrt{61}$ (d) $\sqrt{71}$

7. If the angle between the tangents drawn to $x^2 + y^2 + 2gx + 2fy + c = 0$ from $(0,0)$ is $\frac{\pi}{2}$, then

(a) $g^2 + f^2 = 3c$ (b) $g^2 + f^2 = 2c$ (c) $g^2 + f^2 = 5c$ (d) $g^2 + f^2 = 4c$

8. The chords of contact of the pair of tangents drawn from each point on the line $2x + y = 4$ to the circle $x^2 + y^2 = 1$ pass through a fixed point

(a) (2, 4) (b) $\left(-\frac{1}{2}, -\frac{1}{4}\right)$ (c) $\left(\frac{1}{2}, \frac{1}{4}\right)$ (d) (–2, – 4)

9. The length of tangent from (0, 0) to the circle $2(x^2 + y^2) + x - y + 5 = 0$ is

(a) $\sqrt{5}$ (b) $\sqrt{\left(\frac{5}{2}\right)}$ (c) $\frac{\sqrt{5}}{2}$ (d) $\sqrt{2}$

10. The perpendicular tangents to the circle $x^2 + y^2 = a^2$ meet at P. Then, the locus of P has the equation

(a) $x^2 + y^2 = 2a^2$ (b) $x^2 + y^2 = 3a^2$ (c) $x^2 + y^2 = 4a^2$ (d) $x^2 + y^2 = 5a^2$

11. The tangents to $x^2 + y^2 = a^2$ having inclinations α and β intersect at P. If $\cot\alpha + \cot\beta = 0$, then the locus of P is

(a) $x + y = 0$ (b) $x - y = 0$ (c) $xy = 0$ (d) $xy = 1$

12. The exhaustive range of values of a such that the angle between the pair of tangents drawn from (a,a) to the circle $x^2 + y^2 - 2x - 2y - 6 = 0$ lies in the range $\left(\frac{\pi}{3}, \pi\right)$ is

(a) (–1, 3) (b) $(-5,-3) \cup (3, 5)$ (c) (–3, 5) (d) $(-3,-1) \cup (3, 5)$

13. Distances from the origin to the centres of the three circles $x^2 + y^2 - 2\lambda x = c^2$, where c is a constant and λ is available, are in GP. Prove that the lengths of tangents drawn from any point on the circle $x^2 + y^2 = c^2$ to the three circles are also in GP.

14. Find the area of the quadrilateral formed by a pair of tangents from the point $(4, 5)$ to the circle $x^2 + y^2 - 4x - 2y - 11 = 0$ and a pair of its radii.

15. If the length of the tangent from a point (f, g) to the circle $x^2 + y^2 = 4$ be four times the length of the tangent from it to the circle $x^2 + y^2 = 4x$, show that $15f^2 + 15g^2 - 64f + 4 = 0$.

16. Find the equation of that chord of the circle $x^2 + y^2 = 15$ which is bisected at $(3, 2)$.

17. The chords of contact of the pair of tangents to the circle $x^2 + y^2 = 1$ drawn from any point on the line $2x + y = 4$ pass through the point (α, β), then find $\alpha^2 + \beta^2$.

Session 6

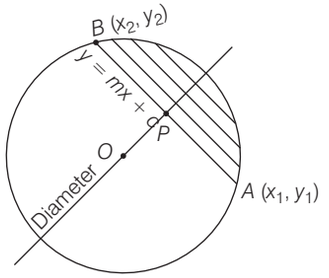
Diameter of a Circle, Two Circles Touching Each Other, Common Tangents to Two Circles, Common Chord of Two Circles, Family of Circles

Diameter of a Circle

The locus of the middle points of a system of parallel chords of a circle is called a diameter of the circle.

Let the circle be $x^2 + y^2 = a^2$ and equation of parallel chord is

$$y = mx + c$$



Let $P(h, k)$ be the middle point of the chord $y = mx + c$.
Since, P is the mid-point of $A(x_1\ y_1)$ and $B(x_2, y_2)$, then

$$\frac{x_1 + x_2}{2} = h \quad \text{and} \quad \frac{y_1 + y_2}{2} = k$$

or $x_1 + x_2 = 2h$ and $y_1 + y_2 = 2k$ ...(i)

$\because$ $P(h, k)$ lie on $y = mx + c$

then, $k = mh + c$

or $k - mh = c$ ...(ii)

Substituting $y = mx + c$ in $x^2 + y^2 = a^2$

then, $x^2 + (mx + c)^2 = a^2$

or $(1 + m^2)\, x^2 + 2mcx + c^2 - a^2 = 0$ ...(iii)

Let x_1, x_2 are roots of Eq. (iii), then

$$x_1 + x_2 = -\frac{2mc}{1 + m^2}$$

$\Rightarrow$ $2h = -\frac{2m}{(1 + m^2)}(k - mh)$ [from Eq. (i) and Eq. (ii)]

$\Rightarrow$ $h + m^2 h = -mk + m^2 h \quad \Rightarrow \quad h + mk = 0$

Hence, locus of (h, k) is $x + my = 0$

Aliter : Let (h, k) be the middle point of the chord $y = mx + c$ of the circle $x^2 + y^2 = a^2$

then, $T = S_1 \quad \Rightarrow \quad xh + ky = h^2 + k^2$

$$\text{slope} = -\frac{h}{k} = m \quad \Rightarrow \quad h + mk = 0$$

Hence, locus of mid-point is $x + my = 0$.

Remark

The diameter of circle always passes through the centre of the circle and perpendicular to the parallel chords.

Let circle is $x^2 + y^2 = a^2$ and parallel chord be $y = mx + c$, then equation of line $\perp$ to $y = mx + c$ is

$$my + x + \lambda = 0 \qquad \ldots(i)$$

which passes through origin (centre)

then, $0 + 0 + \lambda = 0 \quad \therefore \quad \lambda = 0$

Then, equation of diameter from Eq. (i) is $x + my = 0$.

Example 61. Find the equation of the diameter of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ which corresponds to the chord $ax + by + d = 0$.

Sol. The diameter of circle passes through the centre of the circle and perpendicular to the chord $ax + by + d = 0$ is

$$bx - ay + \lambda = 0 \qquad \ldots(i)$$

which passes through centre of circle i.e. $(-g, -f)$

Then, $-bg + af + \lambda = 0$

$\therefore$ $\lambda = bg - af$

From Eq. (i), the equation of the diameter is

$$bx - ay + bg - af = 0$$

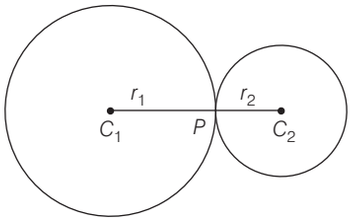
Two Circles Touching Each Other

1. When two circles touch each other externally

Then, distance between their centres = sum of their radii

i.e. $|C_1C_2| = r_1 + r_2$

In such cases, the point of contact P divides the line joining C_1 and C_2 internally in the ratio $r_1 : r_2$



$$\Rightarrow \qquad \frac{C_1P}{C_2P} = \frac{r_1}{r_2}$$

If $C_1 \equiv (x_1, y_1)$ and $C_2 \equiv (x_2, y_2)$

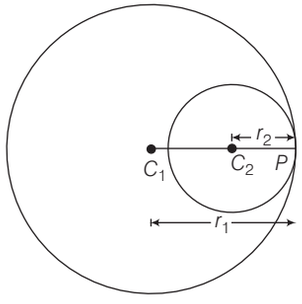
then, coordinate of P is $\left(\frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \frac{r_1y_2 + r_2y_1}{r_1 + r_2} \right)$

2. When two Circles Touch each other Internally

Then, distance between their centres =Difference of their radii

i.e. $|C_1C_2| = |r_1 - r_2|$

In such cases, the point of contact P divides the line joining C_1 and C_2 externally in the ratio $r_1 : r_2$



$$\Rightarrow \qquad \frac{C_1P}{C_2P} = \frac{r_1}{r_2}$$

If $C_1 \equiv (x_1, y_1)$ and $C_2 \equiv (x_2, y_2)$

then, coordinates of P is $\left(\frac{r_1x_2 - r_2x_1}{r_1 - r_2}, \frac{r_1y_2 - r_2y_1}{r_1 - r_2} \right)$

Example 62. ***Examine if the two circles*** $x^2 + y^2 - 2x - 4y = 0$ ***and*** $x^2 + y^2 - 8y - 4 = 0$ ***touch each other externally or internally.***

Sol. Given circles are

$$x^2 + y^2 - 2x - 4y = 0 \qquad \text{...(i)}$$

and $$x^2 + y^2 - 8y - 4 = 0 \qquad \text{...(ii)}$$

Let centres and radii of circles Eqs. (i) and (ii) are represented by C_1, r_1 and C_2, r_2, respectively.

$\therefore$ $C_1 \equiv (1, 2), r_1 = \sqrt{(1 + 4)}$ or $r_1 = \sqrt{5}$

and $C_2 \equiv (0, 4), r_2 = \sqrt{0 + 16 + 4}$ or $r_2 = 2\sqrt{5}$

Now, $C_1C_2 = \sqrt{(1 - 0)^2 + (2 - 4)^2}$

$C_1C_2 = \sqrt{5} = r_2 - r_1$

Hence, the two circles touch each other internally.

Example 63. ***Prove that the circles*** $x^2 + y^2 + 2ax + c^2 = 0$ ***and*** $x^2 + y^2 + 2by + c^2 = 0$ ***touch each other, if*** $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$.

Sol. Given circles are

$$x^2 + y^2 + 2ax + c^2 = 0 \qquad \text{...(i)}$$

and $$x^2 + y^2 + 2by + c^2 = 0 \qquad \text{...(ii)}$$

Let C_1 and C_2 be the centres of circles Eqs. (i) and (ii), respectively and r_1 and r_2 be their radii, then

$C_1 = (-a, 0), C_2 = (0, -b)$,

$r_1 = \sqrt{(a^2 - c^2)}$, $r_2 = \sqrt{(b^2 - c^2)}$

Here, we do not find the two circles touch each other internally or externally.

For touch, $|C_1C_2| = |r_1 \pm r_2|$

or $\sqrt{(a^2 + b^2)} = |\sqrt{(a^2 - c^2)} \pm \sqrt{(b^2 - c^2)}|$

On squaring

$$a^2 + b^2 = a^2 - c^2 + b^2 - c^2 \pm 2\sqrt{(a^2 - c^2)}\sqrt{(b^2 - c^2)}$$

or $$c^2 = \pm \sqrt{a^2b^2 - c^2(a^2 + b^2) + c^4}$$

Again, squaring,

$$c^4 = a^2b^2 - c^2(a^2 + b^2) + c^4$$

or $c^2(a^2 + b^2) = a^2b^2$ or $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$

Common Tangents to Two Circles

Different Cases of Intersection of Two Circles :

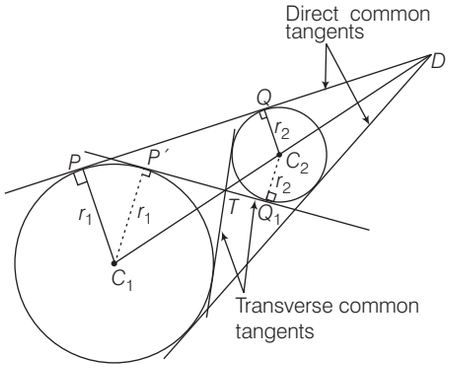
Let the two circles be

$$(x - x_1)^2 + (y - y_1)^2 = r_1^2 \qquad \text{...(i)}$$

and $$(x - x_2)^2 + (y - y_2)^2 = r_2^2 \qquad \text{...(ii)}$$

with centres $C_1(x_1, y_1)$ and $C_2(x_2, y_2)$ and radii r_1 and r_2 respectively. Then following cases may arise :

Case I : When $|C_1C_2| > r_1 + r_2$ i.e. the distance between the centres is greater than the sum of their radii.



$\because$ $C_1 \equiv (x_1, y_1)$, $C_2 \equiv (x_2, y_2)$

$\therefore$ $D \equiv \left(\dfrac{r_1x_2 - r_2x_1}{r_1 - r_2}, \dfrac{r_1y_2 - r_2y_1}{r_1 - r_2} \right) \equiv (\alpha, \beta)$ (say)

and $T \equiv \left(\dfrac{r_1x_2 + r_2x_1}{r_1 + r_2}, \dfrac{r_1y_2 + r_2y_1}{r_1 + r_2} \right)$

$\equiv (\gamma, \delta)$ (say)

How to find direct common tangents Let equation of common tangent through $D(\alpha, \beta)$ is

$$y - \beta = m(x - \alpha) \quad \text{...(i)}$$

Now, length of $\perp$ from C_1 or C_2 on Eq. (i) $= r_1$ or r_2

Then, we get two values of m.

Substituting the values of m in Eq. (i), we get two direct common tangents.

How to find transverse common tangents Let equation of common tangent through $T(\gamma, \delta)$ is

$$y - \delta = M(x - \gamma) \quad \text{...(ii)}$$

Now, length of $\perp$ from C_1 or C_2 on Eq. (i) $= r_1$ or r_2

then, we get two values of M.

Substituting the values of M in Eq. (i), we get two transverse common tangents.

Remark

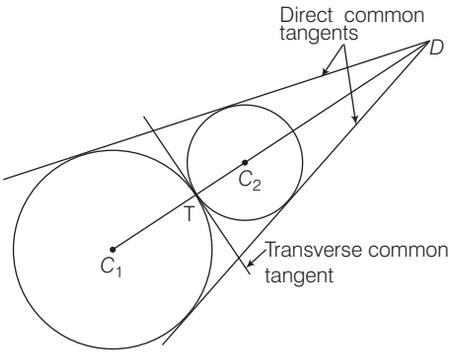
In this case circles neither cut nor touch i.e. Number of solutions of two circles is zero.

Case II : When $|C_1C_2| = r_1 + r_2$

i.e. the distance between the centres is equal to the sum of their radii.

In this case two direct common tangents are real and distinct while the transverse tangents are coincident.

How to find transverse common tangent



$\because$ Equation of circles are

$$S_1 \equiv (x - x_1)^2 + (y - y_1)^2 - r_1^2 = 0$$

and $S_2 \equiv (x - x_2)^2 + (y - y_2)^2 - r_2^2 = 0$

then, equation of common tangent is

$$S_1 - S_2 = 0$$

which is same as equation of common chord.

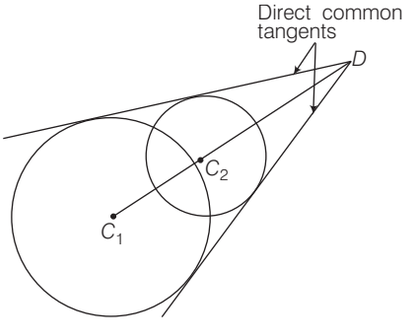
Remark

In this case circles touch at one point i.e. Number of solutions of two circles is one.

Case III : When $|r_1 - r_2| < |C_1C_2| < r_1 + r_2$

i.e. the distance between the centres is less than sum of their radii and greater than difference of their radii.

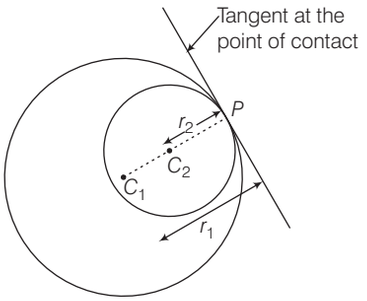
In this case two direct common tangents are real and distinct while the transverse tangents are imaginary.



Remark

In this case circles cuts at two points i.e. Number of solutions of two circles is two.

Case IV : When $|C_1C_2| = |r_1 - r_2|$, i.e. the distance between the centres is equal to the difference of their radii.



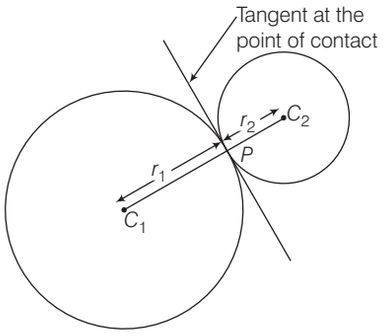
In this case two tangents are real and coincident while the other two tangents are imaginary.

If circles are represented by $S_1 = 0$ and $S_2 = 0$, then equation of common tangent is $S_1 - S_2 = 0$.

Remark

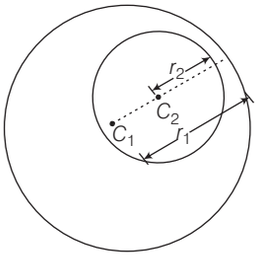
If circles touch each other externally, i.e. $|C_1C_2| = r_1 + r_2$, then equation of tangent at the point of contact is

$$S_1 - S_2 = 0$$



In this case circles touch at one point.
i.e. Number of solutions of two circles is one.

Case V : When $|C_1C_2| < |r_1 - r_2|$, i.e. the distance between the centres is less than the difference of their radii.



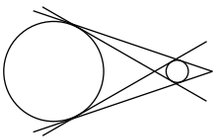
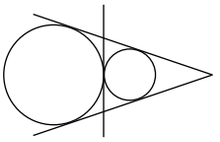
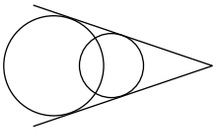
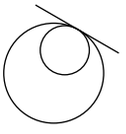
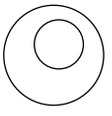
In this case, all the four common tangents are imaginary.

Remark

In this case circles neither cut nor touch each other i.e. Number of solution of two circles is zero.

Common Tangents to two Circles

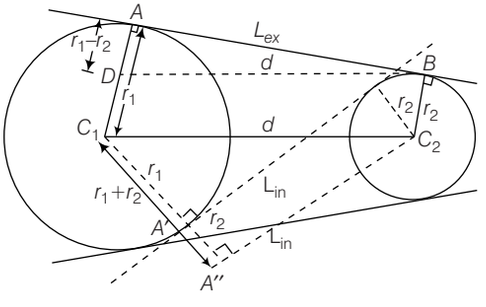
If two circles with centres C_1 and C_2 and their radii are r_1 and r_2, then

	Condition	Figure	Number of common tangents
(i)	$\|C_1C_2\| > r_1 + r_2$		4
(ii)	$\|C_1C_2\| = r_1 + r_2$		3
(iii)	$\|r_1 - r_2\| < \|C_1C_2\| < r_1 + r_2$		2
(iv)	$\|C_1C_2\| = \|r_1 - r_2\|$		1
(v)	$\|C_1C_2\| < \|r_1 - r_2\|$		0

Length of External Common Tangent and Internal Common Tangent to Two Circles

Length of external common tangent $L_{ex} = \sqrt{d^2 - (r_1 - r_2)^2}$

and length of internal common tangent



$L_{in} = \sqrt{d^2 - (r_1 + r_2)^2}$ (Applicable only when $d > r_1 + r_2$)

where, d is the distance between the centres of two circles and r_1, r_2 are the radii of two circles, when $|C_1C_2| = d$.

Angle between Direct Common Tangents (DCT) and Transverse Common Tangents (TCT)

Case I : If $d > r_1 + r_2$, then

Angle between DCT $= 2\sin^{-1}\left(\frac{|r_1 - r_2|}{d}\right)$

And angle between TCT $= 2\sin^{-1}\left(\frac{r_1 + r_2}{d}\right)$

Case II : If $d = r_1 + r_2$, then

angle between DCT $= 2\sin^{-1}\left(\frac{|r_1 - r_2|}{r_1 + r_2}\right)$

and angle between TCT $= \pi$

Case III : If $|r_1 - r_2| < d < r_1 + r_2$, then

angle between DCT $= 2\sin^{-1}\left(\frac{|r_1 - r_2|}{d}\right)$

Here, transverse common tangents are not possible .

Case IV : If $d = |r_1 - r_2|$

Angle between DCT $= \pi$

Here, transverse common tangents are not possible.

Case V : If $d < |r_1 - r_2|$

Here, tangents are not possible.

Example 64. Find all the common tangents to the circles

$x^2 + y^2 - 2x - 6y + 9 = 0$ and $x^2 + y^2 + 6x - 2y + 1 = 0$.

Sol. The given circles are

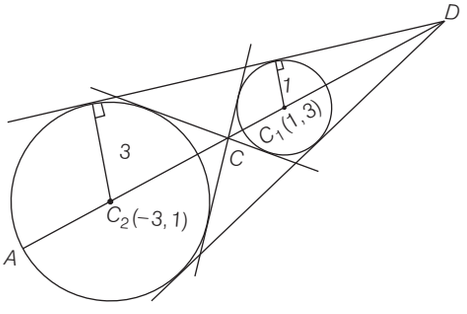
$$x^2 + y^2 - 2x - 6x + 9 = 0$$

$$\Rightarrow \quad (x-1)^2 + (y-3)^2 = 1 \qquad \ldots(i)$$

and $\quad x^2 + y^2 + 6x - 2y + 1 = 0$

$$\Rightarrow \quad (x+3)^2 + (y-1)^2 = 9 \qquad \ldots(ii)$$

Centres and radii of circles Eq. (i) and Eq. (ii) are



$C_1(1, 3), r_1 = 1$

and $C_2(-3, 1), r_2 = 3$ respectively.

$\therefore \quad C_1C_2 = \sqrt{(16 + 4)} = 2\sqrt{5}$

$\therefore \quad C_1C_2 > r_1 + r_2$

Hence, the circles do not intersect to each other.

The direct common tangents meet AB produced at D, then point D will divide C_2C_1 in the ratio 3 : 1 (externally).

Coordinates of D are $\left(\frac{3(1) - 1(-3)}{3 - 1}, \frac{3(3) - 1(1)}{3 - 1}\right)$ or (3, 4)

and the point C divide C_2C_1 in the ratio 3 : 1 (internally)

then coordinates of C are $\left(\frac{3(1) + 1(-3)}{3 + 1}, \frac{3(3) + 1(1)}{3 + 1}\right)$ or

(0, 5/2)

Direct tangents : Any line through (3, 4) is

$$y - 4 = m(x - 3)$$

$$\Rightarrow \quad mx - y + 4 - 3m = 0 \qquad \ldots(i)$$

Apply the usual condition of tangency to any of the circle

$$\frac{m - 3 + 4 - 3m}{\sqrt{m^2 + 1}} = \pm 1$$

$$\Rightarrow \quad (-2m + 1)^2 = m^2 + 1$$

$$\Rightarrow \quad 3m^2 - 4m = 0$$

$$\Rightarrow \quad m = 0, m = 4/3$$

$\therefore$ Equations of direct common tangents are

$$y = 4 \quad \text{and} \quad 4x - 3y = 0$$

Transverse tangents : Any line through $C\ (0,\ 5/2)$ is

$$y - 5/2 = mx$$

or $\quad mx - y + 5/2 = 0 \qquad \ldots(ii)$

Apply the usual condition of tangency to any of the circle

$$\therefore \quad \frac{m \cdot 1 - 3 + 5/2}{\sqrt{m^2 + 1}} = \pm 1$$

$$\Rightarrow \quad m^2 + \frac{1}{4} - m = m^2 + 1$$

$$\Rightarrow \quad 0 \cdot m^2 - m - \frac{3}{4} = 0$$

$$\therefore \quad m = \infty \quad \text{and} \quad m = -3/4$$

Hence, equations of transverse tangents are

$$x = 0 \quad \text{and} \quad 3x + 4y - 10 = 0$$

Example 65. Show that the common tangents to the circles $x^2 + y^2 - 6x = 0$ and $x^2 + y^2 + 2x = 0$ form an equilateral triangle.

Sol. The given circles are

$$x^2 + y^2 - 6x = 0$$

or $\quad (x-3)^2 + (y-0)^2 = 9 \qquad \ldots(i)$

and $\quad x^2 + y^2 + 2x = 0$

or $\quad (x+1)^2 + (y-0)^2 = 1 \qquad \ldots(ii)$

Centres and radii of circles Eqs. (i) and (ii) are $C_1(3, 0), r_1 = 3$ and $C_2(-1, 0), r_2 = 1$, respectively.

$\because \quad C_1C_2 = \sqrt{[3 - (-1)]^2 + 0} = 4$

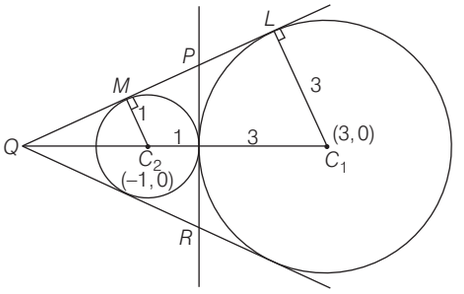
$\therefore \quad C_1C_2 = r_1 + r_2$

Hence, the two circles touch each other externally, therefore, there will be three common tangents. Equation of the common tangent at the point of contact is $S_1 - S_2 = 0$

$\Rightarrow \quad (x^2 + y^2 - 6x) - (x^2 + y^2 + 2x) = 0$

$\Rightarrow \quad -8x = 0$

$\therefore \quad x = 0$



Let the coordinates of Q be (h, k), then

$$\frac{QC_2}{QC_1} = \frac{C_2M}{C_1L} = \frac{1}{3}$$

$\therefore \quad QC_2 : QC_1 = 1 : 3$

$\therefore \quad h = \dfrac{1\cdot(3) - 3\cdot(-1)}{1-3} = -3 \quad \text{and} \quad k = 0$

$\therefore \quad Q \equiv (-3, 0)$

Equation of line passing through $Q(-3, 0)$ is

$$y - 0 = m(x + 3)$$

or $\quad mx - y + 3m = 0 \quad \ldots(iii)$

where, m is the slope of direct tangents since Eq. (iii) is the common tangent (direct) of the circles Eqs. (i) and (ii), then Length of perpendicular from centre of Eq. (ii) i.e. $(-1, 0)$ to the Eq. (iii) = radius of circle Eq. (ii)

$$\Rightarrow \quad \frac{|-m - 0 + 3m|}{\sqrt{m^2 + 1}} = 1 \quad \text{or} \quad 4m^2 = m^2 + 1$$

$$\Rightarrow \quad 3m^2 = 1$$

$$\therefore \quad m = \pm \frac{1}{\sqrt{3}}$$

From Eq. (iii), common tangents are (direct)

$$y = \frac{x}{\sqrt{3}} + \sqrt{3} \quad \text{and} \quad y = -\frac{x}{\sqrt{3}} - \sqrt{3}$$

Hence, all common tangents are $\quad x = 0 \quad \ldots(iv)$

$$y = \frac{x}{\sqrt{3}} + \sqrt{3} \quad \ldots(v)$$

and $\quad y = -\dfrac{x}{\sqrt{3}} - \sqrt{3} \quad \ldots(vi)$

Let P,Q,R be the point of intersection of lines Eqs.(iv), (v); (v), (vi) and (iv), (vi) respectively, then

$$P \equiv (0, \sqrt{3}); Q \equiv (-3, 0) \quad \text{and} \quad R \equiv (0, -\sqrt{3})$$

Now, $\quad PQ = QR = RP = 2\sqrt{3}$

Hence, ΔPQR is an equilateral triangle thus common tangents form an equilateral triangle.

Example 66. Find the number of common tangents to the circles $x^2 + y^2 - 8x + 2y + 8 = 0$ and $x^2 + y^2 - 2x - 6y - 15 = 0$.

Sol. For $x^2 + y^2 - 8x + 2y + 8 = 0$

$$C_1 \equiv (4, -1), r_1 = \sqrt{(16 + 1 - 8)} = 3$$

and for $x^2 + y^2 - 2x - 6y - 15 = 0$

$$C_2 \equiv (1, 3), r_2 = \sqrt{(1 + 9 + 15)} = 5$$

Now, $\quad |C_1C_2|$ = Distance between centres

$$= \sqrt{(4-1)^2 + (-1-3)^2} = 5$$

and $\quad r_1 + r_2 = 3 + 5 = 8$

$$|r_1 - r_2| = |3 - 5| = 2$$

or $\quad |r_1 - r_2| < |C_1C_2| < r_1 + r_2$

Hence, the two circles intersect at two distinct points. Therefore, two tangents can be drawn.

Example 67. Find the lengths of external and internal common tangents and also find the angle between external common tangents and internal common tangents of the circles $x^2 + y^2 + 2x - 8y + 13 = 0$ and $x^2 + y^2 - 8x - 2y + 8 = 0$.

Sol. The given circles are $x^2 + y^2 + 2x - 8y + 13 = 0$

$$\Rightarrow \quad (x+1)^2 + (y-4)^2 = 2^2 \quad \ldots(i)$$

and $\quad x^2 + y^2 - 8x - 2y + 8 = 0$

$$\Rightarrow \quad (x-4)^2 + (y-1)^2 = 3^2 \quad \ldots(ii)$$

Centres and radii of circles Eqs. (i) and (ii) are $C_1(-1, 4)$, $r_1 = 2$ and $C_2(4, 1)$, $r_2 = 3$ respectively.

$\therefore \quad |C_1C_2| = d = \sqrt{(25 + 9)} = \sqrt{34}$

$\Rightarrow \quad d > r_1 + r_2$

Hence, the circles do not intersect to each other.

$\therefore \quad L_{ex} = \sqrt{d^2 - (r_1 - r_2)^2} = \sqrt{34 - 1} = \sqrt{33}$

and $\quad L_{in} = \sqrt{d^2 - (r_1 + r_2)^2} = \sqrt{(34 - 25)} = 3$

Angle between external common tangents

$$= 2\sin^{-1}\left(\frac{|r_1 - r_2|}{d}\right) = 2\sin^{-1}\left(\frac{1}{\sqrt{33}}\right)$$

and angle between internal common tangent

$$= 2\sin^{-1}\left(\frac{r_1 + r_2}{d}\right) = 2\sin^{-1}\left(\frac{5}{\sqrt{33}}\right)$$

Common Chord of Two Circles

The chord joining the points of intersection of two given circles is called their common chord.

Theorem : The equation of common chord of two circles

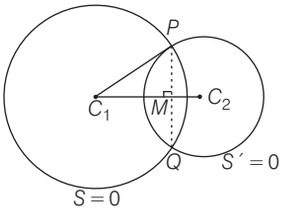
$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$$

and $S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0$

is $2x(g - g') + 2y(f - f') + c - c' = 0$

i.e. $S - S' = 0$

Proof : $\because$ $S = 0$ and $S' = 0$ be two intersecting circles.



Then, $S - S' = 0$ or

$2x(g - g') + 2y(f - f') + c - c' = 0$

is a first degree equation in x and y.

So, it represent a straight line. Also, this equation satisfied by the intersecting points of two given circles $S = 0$ and $S' = 0$.

Hence, $S - S' = 0$ represents the common chord of circles $S = 0$ and $S' = 0$

Length of common chord :

We have, $PQ = 2(PM)$ ($\because M$ is mid-point of PQ)

$$= 2\sqrt{\{(C_1P)^2 - (C_1M)^2\}}$$

where, C_1P = radius of the circle ($S = 0$)

and $C_1M =$ length of perpendicular from C_1 on common chord PQ.

Corollary 1 : The common chord PQ of two circles becomes of the maximum length when it is a diameter of the smaller one between them.

Corollary 2 : Circle on the common chord a diameter, then centre of the circle passing through P and Q lie on the common chord of two circles i.e.

$$S - S' = 0$$

Corollary 3 : If the length of common chord is zero, then the two circles touch each other and the common chord becomes the common tangent to the two circles at the common point of contact.

Example 68. *Prove that the length of the common chord of the two circles :*

$(x-a)^2 + (y-b)^2 = c^2$

and $(x-b)^2 + (y-a)^2 = c^2$ is $\sqrt{4c^2 - 2(a-b)^2}$.

Find also the condition when the given circles touch.

Sol. The equation of circles are

$$S_1 \equiv (x-a)^2 + (y-b)^2 - c^2 = 0 \quad \text{...(i)}$$

and $$S_2 \equiv (x-b)^2 + (y-a)^2 - c^2 = 0 \quad \text{...(ii)}$$

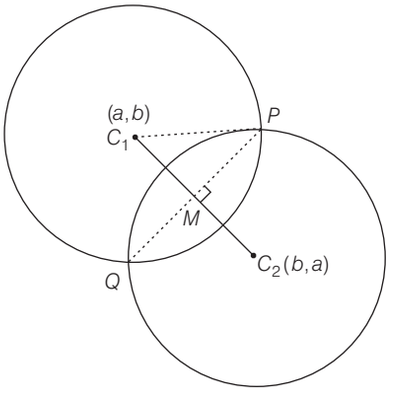
then equation of common chord is $S_1 - S_2 = 0$

$\Rightarrow$ $(x-a)^2 - (x-b)^2 + (y-b)^2 - (y-a)^2 = 0$

or $(2x - a - b)(-a + b) + (2y - b - a)(-b + a) = 0$

$\Rightarrow$ $2x - a - b - 2y + b + a = 0$

$\Rightarrow$ $x - y = 0$



Now, C_1M = Length of perpendicular from C_1 (a, b) on $PQ(x - y = 0) = \dfrac{|a-b|}{\sqrt{2}}$

and C_1P = radius of the circle Eq. (i) = c

$\therefore$ In ΔPC_1M, $PM = \sqrt{(PC_1)^2 - (C_1M)^2}$

$$= \sqrt{c^2 - \frac{(a-b)^2}{2}}$$

$\therefore$ $$PQ = 2PM = 2\sqrt{c^2 - \frac{(a-b)^2}{2}}$$

$$= \sqrt{4c^2 - 2(a-b)^2}$$

Also, when the circles touch, then chord PQ becomes the tangent and $PQ = 0$.

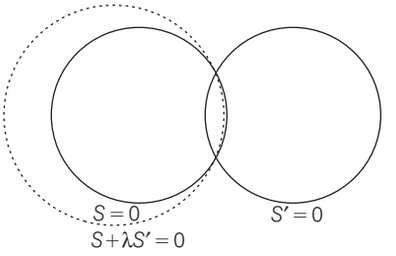
$\therefore$ The condition of tangency is $4c^2 - 2(a-b)^2 = 0$.

i.e. $2c^2 = (a-b)^2$

Family of Circles

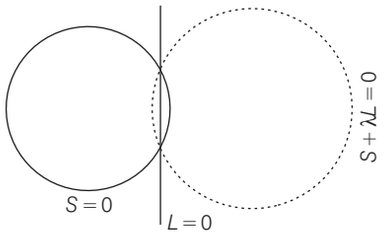
1. The equation of the family of circles passing through the point of intersection of two given circles $S = 0$ and $S' = 0$ is given as

$$S + \lambda S' = 0 \quad (\text{where, } \lambda \text{ is a parameter, } \lambda \neq -1)$$



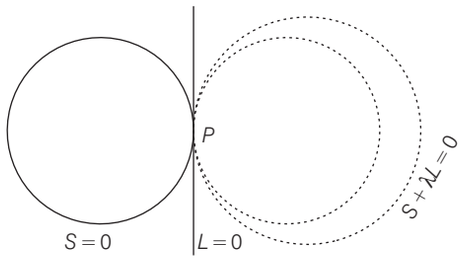
2. The equation of the family of circles passing through the point of intersection of circle $S=0$ and a line $L=0$ is given as

$$S+\lambda L=0 \qquad \text{(where, } \lambda \text{ is a parameter)}$$

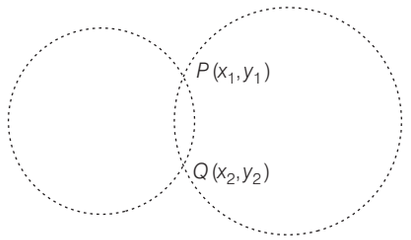


3. The equation of the family of circles touching the circle $S=0$ and the line $L=0$ at their point of contact P is

$$S+\lambda L=0 \qquad \text{(where, } \lambda \text{ is a parameter)}$$



4. The equation of a family of circles passing through two given points $P(x_1, y_1)$ and $Q(x_2, y_2)$ can be written in the form



$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+\lambda\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0$$

(where, λ is a parameter)

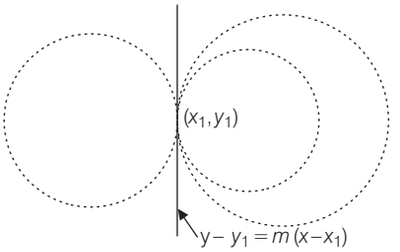
5. The equation of family of circles which touch $y-y_1=m(x-x_1)$ at (x_1, y_1) for any finite m is

$$(x-x_1)^2+(y-y_1)^2+\lambda\{(y-y_1)-m(x-x_1)\}=0$$

and if m is infinite, the family of circles is

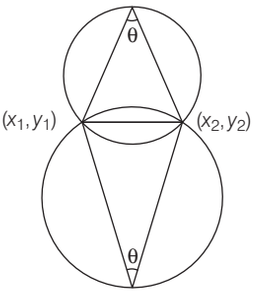
$$(x-x_1)^2+(y-y_1)^2+\lambda(x-x_1)=0$$

(where, λ is a parameter)



6. Equation of the circles given in diagram are

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)\pm\cot\theta\{(x-x_1)(y-y_2)-(x-x_2)(y-y_1)\}=0$$



Example 69. ***Find the equation of the circle passing through*** (1, 1) ***and the points of intersection of the circles***

$x^2+y^2+13x-3y=0$ and $2x^2+2y^2+4x-7y-25=0$.

Sol. The given circles are

$$x^2+y^2+13x-3y=0 \qquad \text{...(i)}$$

and $2x^2+2y^2+4x-7y-25=0$

or $$x^2+y^2+2x-\frac{7}{2}y-\frac{25}{2}=0 \qquad \text{...(ii)}$$

Equation of any circle passing through the point of intersection of the circles Eqs. (i) and (ii) is

$$(x^2+y^2+13x-3y)+\lambda\left(x^2+y^2+2x-\frac{7}{2}y-\frac{25}{2}\right)=0 \quad \text{...(iii)}$$

Its passes through (1, 1), then

$$(1+1+13-3)+\lambda\left(1+1+2-\frac{7}{2}-\frac{25}{2}\right)=0$$

$\Rightarrow \quad 12+\lambda(-12)=0 \ \therefore \ \lambda=1$

Substituting the value of λ in Eq. (iii), the required equation is

$$x^2+y^2+13x-3y+x^2+y^2+2x-\frac{7}{2}y-\frac{25}{2}=0$$

$\Rightarrow \quad 2x^2+2y^2+15x-\frac{13}{2}y-\frac{25}{2}=0$

$\Rightarrow \quad 4x^2+4y^2+30x-13y-25=0$

Example 70. Find the equation of the circle passing through the point of intersection of the circles $x^2 + y^2 - 6x + 2y + 4 = 0$, $x^2 + y^2 + 2x - 4y - 6 = 0$ and with its centre on the line $y = x$.

Sol. Equation of any circle through the points of intersection of given circles is

$$(x^2 + y^2 - 6x + 2y + 4) + \lambda (x^2 + y^2 + 2x - 4y - 6) = 0$$

$$\Rightarrow \quad x^2(1 + \lambda) + y^2 (1 + \lambda) - 2x (3 - \lambda) + 2y (1 - 2\lambda) + (4 - 6\lambda) = 0$$

$$\text{or} \quad x^2 + y^2 - \frac{2x(3 - \lambda)}{(1 + \lambda)} + \frac{2y(1 - 2\lambda)}{(1 + \lambda)} + \frac{(4 - 6\lambda)}{(1 + \lambda)} = 0 \qquad ...(i)$$

Its centre $\left\{\frac{3 - \lambda}{1 + \lambda}, \frac{2\lambda - 1}{1 + \lambda}\right\}$ lies on the line $y = x$

$$\text{then} \quad \frac{2\lambda - 1}{1 + \lambda} = \frac{3 - \lambda}{1 + \lambda} \Rightarrow \lambda \neq -1$$

$$\therefore \quad 2\lambda - 1 = 3 - \lambda \text{ or } 3\lambda = 4$$

$$\therefore \quad \lambda = 4/3$$

$\therefore$ Substituting the value of $\lambda = 4/3$ in Eq. (i), we get the required equation is

$$7x^2 + 7y^2 - 10x - 10y - 12 = 0$$

Example 71. ***Find the equation of the circle passing through the points of intersection of the circles*** $x^2 + y^2 - 2x - 4y - 4 = 0$ ***and*** $x^2 + y^2 - 10x - 12y + 40 = 0$ ***and whose radius is 4.***

Sol. Equation of the any circle through the points of intersection of given circles is

$$(x^2 + y^2 - 2x - 4y - 4) + \lambda(x^2 + y^2 - 10x - 12y + 40) = 0$$

$$\Rightarrow \quad x^2(1 + \lambda) + y^2(1 + \lambda) - 2x(1 + 5\lambda) - 2y(2 + 6\lambda) - 4 + 40\lambda = 0$$

$$\text{or} \quad x^2 + y^2 - 2x\frac{(1 + 5\lambda)}{(1 + \lambda)} - 2y\frac{(2 + 6\lambda)}{(1 + \lambda)} + \frac{(40\lambda - 4)}{(1 + \lambda)} = 0 \qquad ...(i)$$

Its radius

$$\sqrt{\left(\frac{1 + 5\lambda}{1 + \lambda}\right)^2 + \left(\frac{2 + 6\lambda}{1 + \lambda}\right)^2 - \left(\frac{40\lambda - 4}{1 + \lambda}\right)} = 4 \qquad \text{(given)}$$

$$\Rightarrow \quad \frac{(1 + 5\lambda)^2 + (2 + 6\lambda)^2 - (40\lambda - 4)(1 + \lambda)}{(1 + \lambda)^2} = 16$$

$$\Rightarrow \quad 5\lambda^2 - 34\lambda - 7 = 0$$

$$\text{or} \quad (\lambda - 7)(5\lambda + 1) = 0$$

$$\therefore \quad \lambda = 7 \quad \text{or} \quad \lambda = -\frac{1}{5}$$

Substituting the values of λ in Eq. (i), the required circles are

$$2x^2 + 2y^2 - 18x - 22y + 69 = 0$$

and

$$x^2 + y^2 - 2y - 15 = 0$$

Example 72. Find the equation of the circle passing through points of intersection of the circle $x^2 + y^2 - 2x - 4y + 4 = 0$ and the line $x + 2y = 4$ which touches the line $x + 2y = 0$.

Sol. Equation of any circle through points of intersection of the given circle and the line is

$$(x^2 + y^2 - 2x - 4y + 4) + \lambda(x + 2y - 4) = 0$$

$$\text{or} \quad x^2 + y^2 + (\lambda - 2)x + (2\lambda - 4)y + 4(1 - \lambda) = 0 \qquad ...(i)$$

It will touch the line $x + 2y = 0$, then solution of Eq. (i) and $x = -2y$ be unique.

Hence, the roots of the equation

$$(-2y)^2 + y^2 + (\lambda - 2)(-2y) + (2\lambda - 4)y + 4(1 - \lambda) = 0$$

$$\text{or} \quad 5y^2 + 4(1 - \lambda) = 0$$

must be equal.

Then, $0 - 4 \cdot 5 \cdot 4(1 - \lambda) = 0$ or $1 - \lambda = 0$ or $\lambda = 1$

From Eq. (i), the required circle is $x^2 + y^2 - x - 2y = 0$

Example 73. Find the circle whose diameter is the common chord of the circles $x^2 + y^2 + 2x + 3y + 1 = 0$ and $x^2 + y^2 + 4x + 3y + 2 = 0$.

Sol. Given circles are

$$S \equiv x^2 + y^2 + 2x + 3y + 1 = 0$$

and

$$S' \equiv x^2 + y^2 + 4x + 3y + 2 = 0$$

Hence, their common chord is $S - S' = 0$

$$\Rightarrow \quad -2x - 1 = 0 \quad \text{or} \quad 2x + 1 = 0 \qquad ...(i)$$

Now, the required circle must pass through the point of intersection of S and S'.

Hence, its equation is $S + \lambda S' = 0$

$$\Rightarrow \quad (x^2 + y^2 + 2x + 3y + 1) + \lambda (x^2 + y^2 + 4x + 3y + 2) = 0$$

$$\Rightarrow \quad x^2 (1 + \lambda) + y^2 (1 + \lambda) + 2x (1 + 2\lambda) + 3y (1 + \lambda) + (1 + 2\lambda) = 0$$

$$\text{or} \quad x^2 + y^2 + 2x \frac{(1 + 2\lambda)}{(1 + \lambda)} + 3y + \frac{(1 + 2\lambda)}{(1 + \lambda)} = 0 \qquad ...(ii)$$

Its centre is $\left(-\frac{1 + 2\lambda}{1 + \lambda}, -\frac{3}{2}\right)$

But from Eq. (i), $2x + 1 = 0$ is a diameter of this circle.

Hence, its centre must lie on this line

$$\therefore \quad -2\left(\frac{1 + 2\lambda}{1 + \lambda}\right) + 1 = 0$$

$$\Rightarrow \quad -2 - 4\lambda + 1 + \lambda = 0$$

$$\Rightarrow \quad -1 - 3\lambda = 0$$

$$\therefore \quad \lambda = -\frac{1}{3}$$

Hence, from Eq. (ii), the required circle is

$$2x^2 + 2y^2 + 2x + 6y + 1 = 0$$

Example 74. If two curves, whose equations are $ax^2+2hxy+by^2+2gx+2fy+c=0$ and $a'x^2+2h'xy+b'y^2+2g'x+2f'y+c'=0$ intersect in four concyclic points, prove that $\frac{a-b}{h}=\frac{a'-b'}{h'}$

Sol. The equation of family of curves passing through the points of intersection of two curves is

$$(ax^2+2hxy+by^2+2gx+2fy+c)+\lambda(a'x^2+2h'xy+b'y^2+2g'x+2f'y+c')=0$$

or $$(a+\lambda a')x^2+2xy(h+h'\lambda)+(b+\lambda b')y^2+2x(g+\lambda g')+2y(f+\lambda f')+(c+\lambda c')=0 \quad ...(i)$$

Four concyclic points lie on a circle, then Eq. (i) represents a circle. Then,

coefficient of x^2 = coefficient of y^2 and coefficient of $xy=0$

$\Rightarrow$ $a+\lambda a'=b+\lambda b'$

or $(a-b)=-\lambda(a'-b')$ ...(ii)

and $2(h+h'\lambda)=0$

or $\lambda=-\frac{h}{h'}$ ...(iii)

Substituting the value of λ from Eq. (iii) in Eq. (ii), then

$$\frac{a-b}{h}=\frac{a'-b'}{h'}$$

Exercise for Session 6

1. Circles $x^2+y^2-2x-4y=0$ and $x^2+y^2-8y-4=0$

(a) touch each other internally
(b) touch each other externally
(c) cuts each other at two points
(d) None of these

2. The number of common tangents that can be drawn to the circles $x^2+y^2-4x-6y-3=0$ and $x^2+y^2+2x+2y+1=0$ is

(a) 1 (b) 2 (c) 3 (d) 4

3. If one of the circles $x^2+y^2+2ax+c=0$ and $x^2+y^2+2bx+c=0$ lies within the other, then

(a) $ab>0, c>0$
(b) $ab>0,\ c<0$
(c) $ab<0,\ c>0$
(d) $ab<0,\ c<0$

4. The condition that the circle $(x-3)^2+(y-4)^2=r^2$ lies entirely within the circle $x^2+y^2=R^2$ is

(a) $R+r\le 7$
(b) $R^2+r^2<49$
(c) $R^2-r^2<25$
(d) $R-r>5$

5. The circles whose equations are $x^2+y^2+c^2=2ax$ and $x^2+y^2+c^2-2by=0$ will touch one another externally, if

(a) $\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{a^2}$
(b) $\frac{1}{c^2}+\frac{1}{a^2}=\frac{1}{b^2}$
(c) $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2}$
(d) $\frac{1}{b^2}+\frac{1}{c^2}=\frac{2}{a^2}$

6. Two circles with radii r_1 and r_2, $r_1>r_2\ge 2$, touch each other externally. If θ be the angle between the direct common tangents, then

(a) $\theta=\sin^{-1}\left(\frac{r_1+r_2}{r_1-r_2}\right)$
(b) $\theta=2\sin^{-1}\left(\frac{r_1-r_2}{r_1+r_2}\right)$
(c) $\theta=\sin^{-1}\left(\frac{r_1-r_2}{r_1+r_2}\right)$
(d) None of these

7. The circles $x^2+y^2-10x+16=0$ and $x^2+y^2=r^2$ intersect each other in two distinct points if

(a) $r<2$ (b) $r>8$ (c) $2<r<8$ (d) $2\le r\le 8$

8. If the circle $x^2+y^2+4x+22y+c=0$ bisects the circumference of the circle $x^2+y^2-2x+8y-d=0$, then $c+d$ is equal to

(a) 40 (b) 50 (c) 60 (d) 70

9. Two circles $x^2 + y^2 = 6$ and $x^2 + y^2 - 6x + 8 = 0$ are given. Then, the equation of the circle through their points of intersection and the point (1, 1) is

(a) $x^2 + y^2 - 6x + 4 = 0$ (b) $x^2 + y^2 - 3x + 1 = 0$

(c) $x^2 + y^2 - 4x + 2 = 0$ (d) $x^2 + y^2 - 2x + 1 = 0$

10. The equation of the circle described on the common chord of the circles $x^2 + y^2 + 2x = 0$ and $x^2 + y^2 + 2y = 0$ as diameter is

(a) $x^2 + y^2 + x - y = 0$ (b) $x^2 + y^2 - x + y = 0$

(c) $x^2 + y^2 - x - y = 0$ (d) $x^2 + y^2 + x + y = 0$

11. The equation of the diameter of the circle $3(x^2 + y^2) - 2x + 6y - 9 = 0$ which is perpendicular to the line $2x + 3y = 12$ is

(a) $3x - 2y + 3 = 0$ (b) $3x - 2y - 3 = 0$

(c) $3x - 2y + 1 = 0$ (d) $3x - 2y - 1 = 0$

12. If the curves $ax^2 + 4xy + 2y^2 + x + y + 5 = 0$ and $ax^2 + 6xy + 5y^2 + 2x + 3y + 8 = 0$ intersect at four concyclic points, then the value of a is

(a) -6 (b) -4 (c) 4 (d) 6

13. Find the equation of the circle passing through the points of intersection of $x^2 + y^2 + 13x - 3y = 0$ and $2x^2 + 2y^2 + 4x - 7y - 25 = 0$ and the point (1, 1).

14. Show that the common chord of the circles $x^2 + y^2 - 6x - 4y + 9 = 0$ and $x^2 + y^2 - 8x - 6y + 23 = 0$ pass through the centre of the second circle and find its length.

15. Prove that the circles $x^2 + y^2 + 2ax + 2by = 0$ and $x^2 + y^2 + 2a_1x + 2b_1y = 0$ touch each other, if $ab_1 = a_1b$.

16. Find the equations of common tangents to the circles $x^2 + y^2 - 24x + 2y + 120 = 0$ and $x^2 + y^2 + 20x - 6y - 116 = 0$.

Session 7

Angle of Intersection of Two Circles, Radical Axis, Radical Centre, Co-axial System of Circles, Limiting Point, Image of the Circle by the Line Mirror

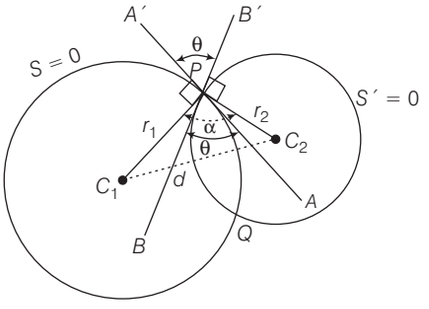
Angle of Intersection of Two Circles

Let the two circles

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$$

and $\quad S' \equiv x^2 + y^2 + 2g_1 x + 2f_1 y + c_1 = 0$

intersect each other at the points P and Q.The angle θ between two circles $S = 0$ and $S' = 0$ is defined as the angle between the tangents to the two circles at the point of intersection.



C_1 and C_2 are the centres of circles

$S = 0$ and $S' = 0$, then

$$C_1 \equiv (-g, -f) \quad \text{and} \quad C_2 \equiv (-g_1, -f_1)$$

and radii of circles $S = 0$ and $S' = 0$ are

$$r_1 = \sqrt{(g^2 + f^2 - c)} \text{ and } r_2 = \sqrt{(g_1^2 + f_1^2 - c_1)}$$

Let $\quad d = |C_1C_2| =$ Distance between their centres

$$= \sqrt{(-g + g_1)^2 + (-f + f_1)^2}$$

$$= \sqrt{(g^2 + f^2 + g_1^2 + f_1^2 - 2gg_1 - 2ff_1)}$$

Now, in ΔC_1PC_2, $\cos\alpha = \left(\dfrac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right)$ $\left(\because \alpha + \theta + 90° + 90° = 360°\right)$

or $\quad \cos(180° - \theta) = \left(\dfrac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right) \quad (\therefore \alpha = 180° - \theta)$

$$\therefore \quad \cos\theta = \left|\frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right| \quad \ldots(i)$$

Orthogonal Intersection of Circles

If the angle between the circles is 90°, i.e. $\theta = 90°$, then the circles are said to be **orthogonal circles** or we say that the circles cut each other **orthogonally**.

Then, from Eq. (i), $0 = \dfrac{r_1^2 + r_2^2 - d^2}{2r_1r_2}$

or $\quad r_1^2 + r_2^2 - d^2 = 0$ or $r_1^2 + r_2^2 = d^2$

$$\Rightarrow \quad g^2 + f^2 - c + g_1^2 + f_1^2 - c_1 = g^2 + f^2 + g_1^2 + f_1^2 - 2gg_1 - 2ff_1$$

or $\quad 2gg_1 + 2ff_1 = c + c_1$

Remark

Equation of a circle cutting the three circles $x^2 + y^2 + 2g_ix + 2f_iy + c_i = 0$ $(i = 1, 2, 3)$ orthogonally is

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ -c_1 & g_1 & f_1 & -1 \\ -c_2 & g_2 & f_2 & -1 \\ -c_3 & g_3 & f_3 & -1 \end{vmatrix}$$

Example 75. Find the angle between the circles

$$S : x^2 + y^2 - 4x + 6y + 11 = 0$$

and $\quad S' : x^2 + y^2 - 2x + 8y + 13 = 0$

Sol. Centres and radii of circles S and S' are

$$C_1(2, -3),\ r_1 = \sqrt{2},\ C_2(1, -4),\ r_2 = 2.$$

Distance between centres, $d = |C_1C_2|$

$$= \sqrt{(2 - 1)^2 + (-3 + 4)^2} = \sqrt{2}$$

If angle between the circles is θ, then

$$\cos\theta = \left|\frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right|$$

$$\cos\theta = \left|\frac{2+4-2}{2\cdot\sqrt{2}\cdot 2}\right| = \frac{1}{\sqrt{2}}$$

$\therefore \quad \theta = 45°$

Example 76. Show that the circles $x^2 + y^2 - 6x + 4y + 4 = 0$ and $x^2 + y^2 + x + 4y + 1 = 0$ cut orthogonally.

Sol. Comparing the given circles by general equation of circles

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

and $\quad x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$

then, $\quad g = -3,\ f = 2,\ c = 4$

and $\quad g_1 = \frac{1}{2},\ f_1 = 2,\ c_1 = 1$

Then, given circles cut orthogonally, if

$$2gg_1 + 2ff_1 = c + c_1$$

We have, $\quad 2\times(-3)\times\frac{1}{2} + 2\times 2\times 2 = 4 + 1$

$\Rightarrow \quad -3 + 8 = 5$ or $5 = 5$.

Hence, the given circles cut each other orthogonally.

Example 77. Find the equation of the circle which cuts the circle $x^2 + y^2 + 5x + 7y - 4 = 0$ orthogonally, has its centre on the line $x = 2$ and passes through the point $(4, -1)$.

Sol. Let the required circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \ldots(i)$$

Since, $(4, -1)$ lie on Eq. (i), then

$$17 + 8g - 2f + c = 0 \quad \ldots(ii)$$

Centre of Eq. (i) is $(-g, -f)$

Since, centre lie on $x = 2$ then $-g = 2$

$\therefore \quad g = -2 \quad \ldots(iii)$

From Eq. (ii), $1 - 2f + c = 0 \quad \ldots(iv)$

and given circle is

$$x^2 + y^2 + 5x + 7y - 4 = 0 \quad \ldots(v)$$

Given the circles Eqs. (i) and (v) cut each other orthogonally,

$\therefore \quad 2g\times\frac{5}{2} + 2f\times\frac{7}{2} = c - 4$

or $\quad 5g + 7f = c - 4$

$\quad -10 + 7f = c - 4$ [from Eq. (iii)]

or $\quad -6 + 7f - c = 0 \quad \ldots(vi)$

Solving Eqs. (iv) and (vi), we get

$$f = 1 \quad \text{and} \quad c = 1$$

Substituting the values of g, f, c in Eq. (i), we get

$$x^2 + y^2 - 4x + 2y + 1 = 0$$

Example 78. Find the equations of the two circles which intersect the circles $x^2 + y^2 - 6y + 1 = 0$ and $x^2 + y^2 - 4y + 1 = 0$ orthogonally and touch the line $3x + 4y + 5 = 0$.

Sol. Let the required circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \ldots(i)$$

and given circles are $x^2 + y^2 - 6y + 1 = 0 \quad \ldots(ii)$

and $\quad x^2 + y^2 - 4y + 1 = 0 \quad \ldots(iii)$

Since, Eq. (i) cuts Eq. (ii) and Eq. (iii) orthogonally

$\therefore \quad 2g\times 0 + 2f\times(-3) = c + 1$

or $\quad -6f = c + 1 \quad \ldots(iv)$

and $\quad 2g\times 0 + 2f\times(-2) = c + 1$

or $\quad -4f = c + 1 \quad \ldots(v)$

Solving, Eqs. (iv) and (v), we get

$$f = 0 \quad \text{and} \quad c = -1$$

From Eq. (i), $\quad x^2 + y^2 + 2gx - 1 = 0 \quad \ldots(vi)$

centre and radius of Eq. (vi) are $(-g, 0)$ and $\sqrt{(g^2+1)}$, respectively.

Since, $3x + 4y + 5 = 0$ is tangent of Eq. (vi), then length of perpendicular from $(-g, 0)$ to this line = radius of circle

or $\quad \frac{|-3g + 0 + 5|}{\sqrt{(9+16)}} = \sqrt{(g^2+1)}$

$$|-3g + 5| = 5\sqrt{(g^2+1)}$$

or $\quad (-3g+5)^2 = 25(g^2+1)$

or $\quad 9g^2 + 25 - 30g = 25g^2 + 25$

or $\quad 16g^2 + 30g = 0$

$\therefore \quad g = 0 \quad \text{and} \quad g = -\frac{15}{8}$

Equations of circles are from Eq. (vi),

$$x^2 + y^2 - 1 = 0 \quad \text{and} \quad x^2 + y^2 - \frac{15}{4}x - 1 = 0$$

or $\quad x^2 + y^2 - 1 = 0 \quad \text{and} \quad 4x^2 + 4y^2 - 15x - 4 = 0.$

Example 79. ***Prove that the two circles, which pass through (0, a) and*** $(0, -a)$ ***and touch the line*** $y = mx + c$ ***, will cut orthogonally, if*** $c^2 = a^2(2 + m^2)$

Sol. Let the equation of the circles be

$$x^2 + y^2 + 2gx + 2fy + d = 0 \quad \ldots(i)$$

Since, these circles pass through $(0, a)$ and $(0, -a)$, then

$$a^2 + 2fa + d = 0 \quad \ldots(ii)$$

and $\quad a^2 - 2fa + d = 0 \quad \ldots(iii)$

Solving, Eq. (ii) and Eq. (iii), we get $f = 0$ and $d = -a^2$.

Substituting these values of f and d in Eq. (i), we obtain

$$x^2 + y^2 + 2gx - a^2 = 0 \quad \ldots(iv)$$

Now, $y = mx + c$ touch this circle, therefore, length of the perpendicular from the centre = radius

$$\frac{|-mg - 0 + c|}{\sqrt{1 + m^2}} = \sqrt{(g^2 + a^2)}$$

$$(c - mg)^2 = (1 + m^2)(g^2 + a^2)$$

or $$g^2 + 2mcg + a^2(1 + m^2) - c^2 = 0$$

Let g_1, g_2 are the roots of this equation

$$\therefore \quad g_1 g_2 = a^2(1 + m^2) - c^2 \qquad ...(v)$$

Now, the equations of the two circles represented by Eq. (iv) are

$$x^2 + y^2 + 2g_1 x - a^2 = 0$$

and $$x^2 + y^2 + 2g_2 x - a^2 = 0.$$

These two circles will be cuts orthogonal, if

$$2g_1 g_2 + 0 = -a^2 - a^2$$

or $$g_1 g_2 = -a^2 \qquad ...(vi)$$

From Eqs. (v) and (vi),

$$-a^2 = a^2(1 + m^2) - c^2$$

or $$c^2 = a^2(2 + m^2)$$

which is the required condition.

Example 80. Find the equation of the circle which cuts orthogonally each of the three circles given below :

$x^2 + y^2 - 2x + 3y - 7 = 0$, $x^2 + y^2 + 5x - 5y + 9 = 0$

and $x^2 + y^2 + 7x - 9x + 29 = 0$.

Sol. Let the required circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad ...(i)$$

Since, it is orthogonal to three given circles respectively, therefore

$$2g \times (-1) + 2f \times \frac{3}{2} = c - 7$$

or $$-2g + 3f = c - 7 \qquad ...(ii)$$

$$2g \times \frac{5}{2} + 2f \times \left(-\frac{5}{2}\right) = c + 9$$

or $$5g - 5f = c + 9 \qquad ...(iii)$$

and $$2g \times \frac{7}{2} + 2f \times \left(-\frac{9}{2}\right) = c + 29$$

or $$7g - 9f = c + 29 \qquad ...(iv)$$

Subtracting, Eq. (ii) from Eq. (iii),

$$7g - 8f = 16 \qquad ...(v)$$

and subtracting Eq. (iii) from Eq. (iv),

$$2g - 4f = 20 \qquad ...(vi)$$

Solving Eq. (v) and Eq. (vi), we get

$$g = -8 \quad \text{and} \quad f = -9$$

Putting the values of g and f in Eq. (iii)

$$-40 + 45 = c + 9$$

$$\Rightarrow \quad 5 = c + 9$$

or $$c = -4$$

Substituting the values of g, f, c in Eq. (i), then required circle is

$$x^2 + y^2 - 16x - 18y - 4 = 0$$

Radical Axis

The radical axis of two circles is the locus of a point which moves such that the lengths of the tangents drawn from it to the two circles are equal.

Consider, $$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0 \qquad ...(i)$$

and $$S' \equiv x^2 + y^2 + 2g_1 x + 2f_1 y + c_1 = 0 \qquad ...(ii)$$

Let $P(x_1, y_1)$ be a point such that

$$|PA| = |PB|$$

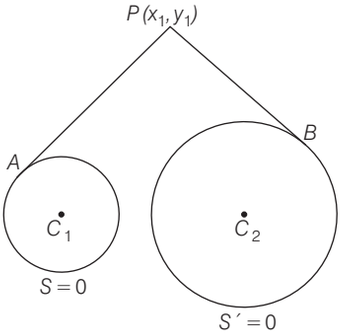
$$\Rightarrow \quad \sqrt{(x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c)} = \sqrt{(x_1^2 + y_1^2 + 2g_1 x_1 + 2f_1 y_1 + c_1)}$$

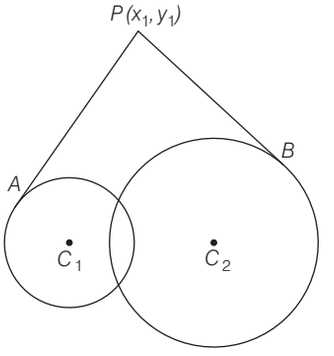
On squaring, $$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = x_1^2 + y_1^2 + 2g_1 x_1 + 2f_1 y_1 + c_1$$

$$\Rightarrow \quad 2(g - g_1)x_1 + 2(f - f_1)y_1 + c - c_1 = 0$$

$\therefore$ Locus of $P(x_1, y_1)$ is

$$2(g - g_1)x + 2(f - f_1)y + c - c_1 = 0$$

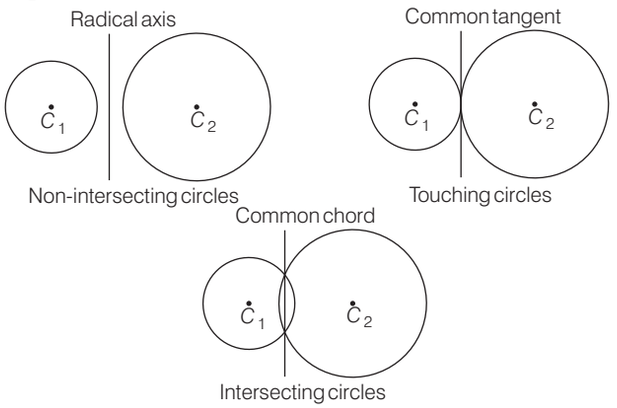




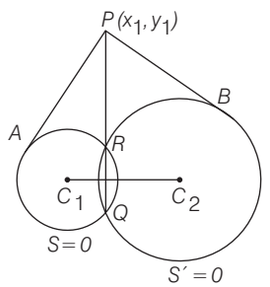
which is the required equation of radical axis of the given circles. Clearly this is a straight line.

Some Properties of the Radical Axis

(i) **The radical axis and common chord are identical :** Since, the radical axis and common chord of two circles $S=0$ and $S'=0$ are the same straight line $S-S'=0$, they are identical. The only difference is that the common chord exists only if the circles intersect in two real points, while the radical axis exists for all pair of circles irrespective of their position.



The position of the radical axis of the two circles geometrically is shown below:



From Euclidian geometry

$$(PA)^2 = PR \cdot PQ = (PB)^2$$

(ii) **The radical axis is perpendicular to the straight line which joins the centres of the circles :**

Consider, $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ ...(i)

and $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ ...(ii)

Since, $C_1 \equiv (-g, -f)$ and $C_2 \equiv (-g_1, -f_1)$ are the centres of the circles Eqs. (i) and (ii), then slope of

$$C_1C_2 = \frac{-f_1 + f}{-g_1 + g} = \frac{f - f_1}{g - g_1} = m_1 \quad \text{(say)}$$

Equation of the radical axis is

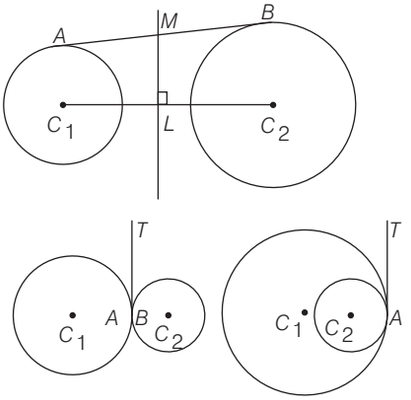
$$2(g - g_1)x + 2(f - f_1)y + c - c_1 = 0$$

Slope of radical axis is $-\dfrac{(g - g_1)}{f - f_1} = m_2$ (say)

$\therefore$ $m_1m_2 = -1$

Hence, C_1C_2 and radical axis are perpendicular to each other.

(iii) **The radical axis bisects common tangents of two circles :** Let AB be the common tangent. If it meets the radical axis LM in M, then MA and MB are two tangents to the circles. Hence, $MA = MB$ since lengths of tangents are equal from any point on radical axis. Hence, radical axis bisects the common tangent AB.



If the two circles touch each other externally or internally, then A and B coincide. In this case the common tangent itself becomes the radical axis.

(iv) **The radical axis of three circles taken in pairs are concurrent :** Let the equations of three circles be

$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \quad \text{...(i)}$$

$$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0 \quad \text{...(ii)}$$

$$S_3 \equiv x^2 + y^2 + 2g_3x + 2f_3y + c_3 = 0 \quad \text{...(iii)}$$

The radical axis of the above three circles taken in pairs are given by

$$S_1 - S_2 \equiv 2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0 \quad \text{...(iv)}$$

$$S_2 - S_3 \equiv 2x(g_2 - g_3) + 2y(f_2 - f_3) + c_2 - c_3 = 0 \quad \text{...(v)}$$

$$S_3 - S_1 \equiv 2x(g_3 - g_1) + 2y(f_3 - f_1) + c_3 - c_1 = 0 \quad \text{...(vi)}$$

Adding Eqs. (iv), (v) and (vi), we find LHS vanished identically. Thus, the three lines are concurrent.

(v) **If two circles cut a third circle orthogonally, the radical axis of the two circles will pass through the centre of the third circle.**

OR

The locus of the centre of a circle cutting two given circles orthogonally is the radical axis of the two circles.

Let $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ ...(i)

$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ ...(ii)

$S_3 \equiv x^2 + y^2 + 2g_3x + 2f_3y + c_3 = 0$ ...(iii)

Since, Eqs. (i) and (ii) both cut Eq. (iii) orthogonally

$\therefore \quad 2g_1g_3 + 2f_1f_3 = c_1 + c_3$

and $\quad 2g_2g_3 + 2f_2f_3 = c_2 + c_3$

Subtracting, we get

$$2g_3(g_1 - g_2) + 2f_3(f_1 - f_2) = c_1 - c_2 \quad \text{...(iv)}$$

Now, radical axis of Eqs. (i) and (ii) is

$$S_1 - S_2 = 0$$

or $\quad 2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$

Since, it will pass through the centre of Eq. (iii) circle

$\therefore -2g_3(g_1 - g_2) - 2f_3(f_1 - f_2) + c_1 - c_2 = 0$

or $2g_3(g_1 - g_2) + 2f_3(f_1 - f_2) = c_1 - c_2$ ...(v)

which is true by Eq. (iv),

Remark

Radical axis need not always pass through the mid-point of the line joining the centres of the two circles.

Example 81. If two circles $x^2 + y^2 + 2gx + 2fy = 0$ and $x^2 + y^2 + 2g'x + 2f'y = 0$ touch each other, then $f'g = fg'$.

Sol. If two circles touch each other, then their radical axis is their common tangent.

$\therefore$ Radical axis of two circles is

$$(x^2 + y^2 + 2gx + 2fy) - (x^2 + y^2 + 2g'x + 2f'y) = 0$$

or $\quad 2x(g - g') + 2y(f - f') = 0$

or $\quad x(g - g') + y(f - f') = 0$ ...(i)

If this touches the circle $x^2 + y^2 + 2gx + 2fy = 0$, then the length of perpendicular from its centre $(-g, -f)$ to (i) =radius $\sqrt{(g^2 + f^2)}$ of the circle

i.e. $\quad \dfrac{|-g(g - g') - f(f - f')|}{\sqrt{(g - g')^2 + (f - f')^2}} = \sqrt{g^2 + f^2}$

or $\quad \{(-(g^2 + f^2) + gg' + ff')\}^2$

$$= (g^2 + f^2)\{(g - g')^2 + (f + f')^2\}$$

or $\quad (g^2 + f^2)^2 + (gg' + ff')^2 - 2(g^2 + f^2)(gg' + ff')$

$$= (g^2 + f^2)\{(g^2 + f^2) + (g'^2 + f'^2) - 2(gg' + ff')\}$$

or $(gg' + ff')^2 = (g^2 + f^2)(g'^2 + f'^2)$

On simplifying, $\quad 2gg'ff' = g^2f'^2 + f^2g'^2$

or $\quad (gf' - g'f)^2 = 0$

or $\quad gf' = g'f$

Aliter : If two circles touch each other, then

distance between their centres = sum or difference of their radii

$$\sqrt{(g - g')^2 + (f - f')^2} = \sqrt{(g^2 + f^2)} \pm \sqrt{(g'^2 + f'^2)}$$

or $\quad \sqrt{(g^2 + f^2 + g'^2 + f'^2 - 2gg' - 2ff')}$

$$= \sqrt{(g^2 + f^2)} \pm \sqrt{(g'^2 + f'^2)}$$

On squaring, we have

$$g^2 + f^2 + g'^2 + f'^2 - 2gg' - 2ff'$$

$$= g^2 + f^2 + g'^2 + f'^2 \pm 2\sqrt{(g^2 + f^2)}\sqrt{(g'^2 + f'^2)}$$

or $(gg' + ff') = \pm\sqrt{(g^2 + f^2)(g'^2 + f'^2)}$

Again, on squaring both sides, we get

$$g^2g'^2 + f^2f'^2 + 2gg'ff' = g^2g'^2 + g^2f'^2 + f^2g'^2 + f^2f'^2$$

or $\quad g^2f'^2 + f^2g'^2 - 2gg'ff' = 0$

or $\quad (gf' - g'f)^2 = 0$

or $\quad gf' - g'f = 0$

or $\quad gf' = g'f$

Example 82. A and B are two fixed points and P moves so that $PA = nPB$. Show that locus of P is a circle and for different values of n all the circles have a common radical axis.

Sol. Let $A \equiv (a, 0)$, $B \equiv (-a, 0)$ and $P \equiv (h, k)$

$\therefore \quad PA = \sqrt{(h - a)^2 + k^2}$

$$PB = \sqrt{(h + a)^2 + k^2}$$

Since, $\quad PA = nPB$

or $\quad (PA)^2 = n^2(PB)^2$

$\Rightarrow \quad \{(h - a)^2 + k^2\} = n^2\{(h + a)^2 + k^2\}$

$\Rightarrow \quad (h^2 + k^2 - 2ah + a^2) = n^2(h^2 + k^2 + 2ah + a^2)$

$\Rightarrow \quad (1 - n^2)h^2 + (1 - n^2)k^2 - 2ah(1 + n^2) + (1 - n^2)a^2 = 0$

or $\quad h^2 + k^2 - 2ah\dfrac{(1 + n^2)}{(1 - n^2)} + a^2 = 0$

$\therefore$ Locus of P is $\quad x^2 + y^2 - \left(\dfrac{1 + n^2}{1 - n^2}\right)2ax + a^2 = 0$

which is a circle. For different values of n.

If two different values of n are n_1 and n_2, then circles are

$$x^2 + y^2 - \left(\frac{1 + n_1^2}{1 - n_1^2}\right)2ax + a^2 = 0 \quad \text{...(i)}$$

and $\quad x^2 + y^2 - \left(\dfrac{1 + n_2^2}{1 - n_2^2}\right)2ax + a^2 = 0$ ...(ii)

$\therefore$ Radical axis of Eqs. (i) and (ii) is

$$2ax\left\{\frac{1+n_2^2}{1-n_2^2}-\frac{1+n_1^2}{1-n_1^2}\right\}=0$$

or $x=0$ or Y-axis.

Hence, for different values of n the circles have a common radical axis.

Example 83. Show that the difference of the squares of the tangents to two coplanar circles from any point P in the plane of the circles varies as the perpendicular from P on their radical axis. Also, prove that the locus of a point such that the difference of the squares of the tangents from it to two given circles is constant is a line parallel to their radical axis.

Sol. Let the two circles be

$$S_1 \equiv x^2+y^2+2g_1x+2f_1y+c_1=0 \quad ...(i)$$

and $$S_2 \equiv x^2+y^2+2g_2x+2f_2y+c_2=0 \quad ...(ii)$$

and let $P=(h,k)$

$\therefore$ Radical axis of Eqs. (i) and (ii) is

$$2(g_1-g_2)x+2(f_1-f_2)y+c_1-c_2=0 \quad ...(iii)$$

Let length of tangents from $P(h,k)$ on Eqs. (i) and (ii) are l_1 and l_2, then

$$l_1=\sqrt{S_1}=\sqrt{(h^2+k^2+2g_1h+2f_1k+c_1)}$$

and $$l_2=\sqrt{S_2}=\sqrt{(h^2+k^2+2g_2h+2f_2k+c_2)}$$

According to the question,

$$l_1^2-l_2^2=2(g_1-g_2)h+2(f_1-f_2)k+c_1-c_2 \quad ...(iv)$$

Let p be the perpendicular distance from $P(h,k)$ on Eq. (iii),

$$\therefore \quad p=\frac{|2(g_1-g_2)h+2(f_1-f_2)k+c_1-c_2|}{\sqrt{4(g_1-g_2)^2+4(f_1-f_2)^2}} \quad ...(v)$$

From Eqs. (iv) and (v), we get

$$p=\frac{|l_1^2-l_2^2|}{2\sqrt{(g_1-g_2)^2+(f_1-f_2)^2}}$$

or $$\frac{|l_1^2-l_2^2|}{p}=2\sqrt{(g_1-g_2)^2+(f_1-f_2)^2}=\text{constant}$$

$\therefore \quad |l_1^2-l_2^2| \propto p$

Locus of $P(h,k)$ in Eq. (iv) is

$$2(g_1-g_2)x+2(f_1-f_2)y+c_1-c_2=(l_1^2-l_2^2)$$

a line which is parallel to Eq. (iii).

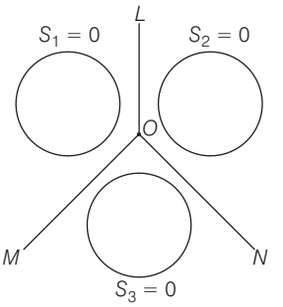
Radical Centre

The radical axes of three circles, taken in pairs, meet in a point, which is called their radical centre. Let the three circles be

$$S_1=0 \quad ...(i)$$

$$S_2=0 \quad ...(ii)$$

$$S_3=0 \quad ...(iii)$$



Let OL, OM and ON be radical axes of the pair sets of circles

$$\{S_1=0, S_2=0\}, \{S_3=0, S_1=0\}$$

and $$\{S_2=0, S_3=0\} \text{ respectively.}$$

Equations of OL, OM and ON are respectively

$$S_1-S_2=0 \quad ...(iv)$$

$$S_3-S_1=0 \quad ...(v)$$

$$S_2-S_3=0 \quad ...(vi)$$

Let the straight lines Eqs. (iv) and (v) i.e. OL and OM meet in O. The equation of any straight line passing through O is

$$(S_1-S_2)+\lambda(S_3-S_1)=0$$

where λ is any constant.

For $\lambda=1$ this equation becomes

$$S_2-S_3=0$$

which is, by Eq. (vi), equation of ON.

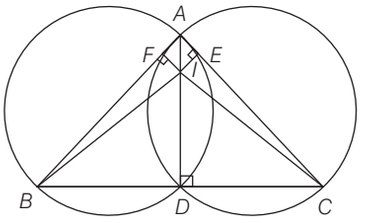
Thus, the third radical axis also passes through the point where the Eqs. (iv) and (v) meet. In the above figure O is the **radical centre.**

Properties of Radical Centre

1. Coordinates of radical centre can be found by solving the equations

$$S_1=S_2=S_3=0$$

2. **The radical centre of three circles described on the sides of a triangle as diameters is the orthocentre of the triangle :**



Draw perpendicular from A on BC.

$\therefore \quad \angle ADB = \angle ADC = \pi/2$

Therefore, the circles whose diameters are AB and AC passes through D and A. Hence, AD is their radical axis. Similarly, the radical axis of the circles on AB and BC as diameters is the perpendicular line from B on CA and radical axis of the circles on BC and CA as diameters is the perpendicular line from C on AB. Hence, the radial axis of three circles meet in a point. This point I is radical centre but here radical centre is the point of intersection of altitudes i.e. AD, BE and CF. Hence, radical centre = orthocentre.

3. The radical centre of three given circles will be the centre of a fourth circle which cuts all the three circles orthogonally and the radius of the fourth circle is the length of tangent drawn from radical centre of the three given circles to any of these circles.

   Let the fourth circle be $(x-h)^2+(y-k)^2=r^2$, where (h, k) is centre of this circle and r be the radius. The centre of circle is the radical centre of the given circles and r is the length of tangent from (h, k) to any of the given three circles.

Example 84. Find the radical centre of circles $x^2+y^2+3x+2y+1=0$, $x^2+y^2-x+6y+5=0$ and $x^2+y^2+5x-8y+15=0$. Also, find the equation of the circle cutting them orthogonally.

Sol. **:** Given circles are

$$S_1 \equiv x^2+y^2+3x+2y+1=0$$
$$S_2 \equiv x^2+y^2-x+6y+5=0$$
$$S_3 \equiv x^2+y^2+5x-8y+15=0$$

Equations of two radical axes are

$$S_1-S_2 \equiv 4x-4y-4=0 \quad \text{or} \quad x-y-1=0$$

and $\quad S_2-S_3 \equiv -6x+14y-10=0 \quad \text{or} \quad 3x-7x+5=0$

Solving them the radical centre is (3, 2) also, if r is the length of the tangent drawn from the radical centre (3, 2) to any one of the given circles, say S_1, we have

$$r=\sqrt{S_1}=\sqrt{3^2+2^2+3\cdot 3+2\cdot 2+1}=\sqrt{27}$$

Hence, (3, 2) is the centre and $\sqrt{27}$ is the radius of the circle intersecting them orthogonally.

$\therefore$ Its equation is

$(x-3)^2+(y-2)^2=r^2=27$ or $x^2+y^2-6x-4y-14=0$

Aliter : Let $x^2+y^2+2gx+2fy+c=0$ be the equation of the circle cutting the given circles orthogonally.

$$\therefore \quad 2g\left(\frac{3}{2}\right)+2f(1)=c+1$$

or $\quad 3g+2f=c+1 \quad \ldots(i)$

$$2g\left(-\frac{1}{2}\right)+2f(3)=c+5$$

or $\quad -g+6f=c+5 \quad \ldots(ii)$

and $\quad 2g\left(\frac{5}{2}\right)+2f(-4)=c+15$

or $\quad 5g-8f=c+15 \quad \ldots(iii)$

Solving, Eqs. (i), (ii) and (iii), we get

$$g=-3,\ f=-2 \text{ and } c=-14$$

$\therefore$ Equation of required circle is

$$x^2+y^2-6x-4y-14=0$$

Example 85. ***Find the radical centre of three circles described on the three sides*** $4x-7y+10=0$, $x+y-5=0$ ***and*** $7x+4y-15=0$ ***of a triangle as diameters.***

Sol. Since, the radical centre of three circles described on the sides of a triangle as diameters is the orthocentre of the triangle.

$\therefore$ Radical centre = orthocentre

Given sides are $\quad 4x-7y+10=0 \quad \ldots(i)$

$$x+y-5=0 \quad \ldots(ii)$$
$$7x+4y-15=0 \quad \ldots(iii)$$

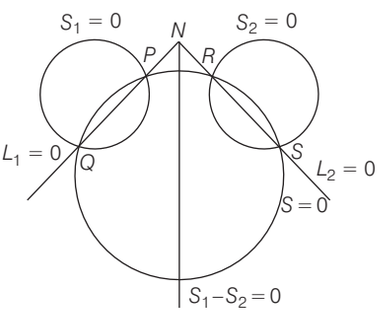
Since, lines Eqs. (i) and (iii) are perpendiculars the point of intersection of Eqs. (i) and (iii) is (1, 2), the orthocentre of the triangle. Hence, radical centre is (1, 2).

Example 86. Prove that if the four points of intersection of the circles $x^2+y^2+ax+by+c=0$ and $x^2+y^2+a'x+b'y+c'=0$ by the lines $Ax+By+C=0$ and $A'x+B'y+C'=0$ respectively are concyclic, then

$$\begin{vmatrix} a-a' & b-b' & c-c' \\ A & B & C \\ A' & B' & C' \end{vmatrix}=0$$

Sol. The given circles and given lines are

$$S_1 \equiv x^2+y^2+ax+by+c=0$$
$$S_2 \equiv x^2+y^2+a'x+b'y+c'=0$$
$$L_1 \equiv Ax+By+C=0$$
$$L_2 \equiv A'x+B'y+C'=0$$



Let $S_1 = 0$ meet $L_1 = 0$ at two points P and Q and $S_2 = 0$ meet $L_2 = 0$ at two points R and S.

Further P,Q, R and S are given to be concyclic. Let the circle through them is

$$x^2 + y^2 + 2gx + 2fy + \lambda = 0 \quad \text{...(i)}$$

Radical axis of $S_1 = 0$ and $S_2 = 0$ is

$$S_1 - S_2 = 0$$

$$\Rightarrow \quad (a - a')x + (b - b')y + c - c' = 0 \quad \text{...(ii)}$$

The radical axis of $S_1 = 0$ and $S = 0$ is $L_1 = 0$

or $\quad Ax + By + C = 0 \quad \text{...(iii)}$

and radical axis of $S_2 = 0$ and $S = 0$ is $L_2 = 0$

or $\quad A'x + B'y + C' = 0 \quad \text{...(iv)}$

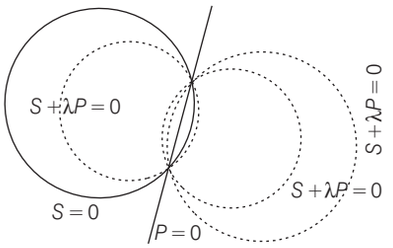
Since, the radical axes of any three circles taken in pairs are concurrent. (i.e. lines Eqs. (ii), (iii) and (iv) are concurrent).

we have $\begin{vmatrix} a - a' & b - b' & c - c' \\ A & B & C \\ A' & B' & C' \end{vmatrix} = 0$

Co-axial System of Circles

A system (or a family) of circles, every pair of which have the same radical axis, are called co-axial circles.

(1) The equation of a system of co-axial circles, when the equation of the radical axis and of one circle of the system are



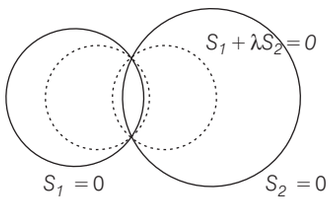
$$P \equiv lx + my + n = 0$$

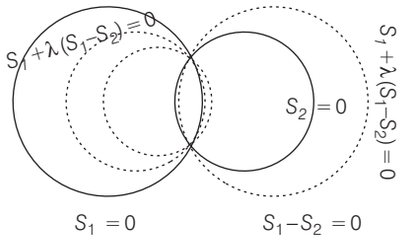
and $\quad S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$

respectively, is

$$S + \lambda P = 0 \ (\lambda \text{ is an arbitrary constant})$$

(2) The equation of a co-axial system of circles, where the equation of any two circles of the system are





$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$

and $\quad S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

respectively is

$$S_1 + \lambda(S_1 - S_2) = 0 \qquad (\lambda \neq -1)$$

or $\quad S_2 + \lambda_1(S_1 - S_2) = 0 \qquad (\lambda_1 \neq -1)$

Other form $S_1 + \lambda S_2 = 0 \qquad (\lambda \neq -1)$

(3) The equation of a system of co-axial circles in the simplest form is

$$x^2 + y^2 + 2gx + c = 0$$

where, g is variable and c, a constant.

The common radical axis is the Y-axis (since centre on X-axis) and the equation of a system of other co-axial circles in the simplest form is

$$x^2 + y^2 + 2fy + c = 0$$

where, f is variable and c, a constant (since centre on Y-axis). The common radical axis is the X-axis.

Example 87. *Find the equation of the system of circles co-axial with the circles*

$$x^2 + y^2 + 4x + 2y + 1 = 0$$

and $x^2 + y^2 - 2x + 6y - 6 = 0$

Also, find the equation of that particular circle whose centre lies on the radical axis.

Sol. Given circles are

$$S_1 \equiv x^2 + y^2 + 4x + 2y + 1 = 0$$

and $\quad S_2 \equiv x^2 + y^2 - 2x + 6y - 6 = 0$

$\therefore$ Radical axis is $\quad S_1 - S_2 = 0$

i.e. $\quad 6x - 4y + 7 = 0 \quad \text{...(i)}$

Now, system of co-axial circle is

$$S_1 + \lambda (S_1 - S_2) = 0$$

$$\Rightarrow \quad (x^2 + y^2 + 4x + 2y + 1) + \lambda(6x - 4y + 7) = 0$$

$$\Rightarrow \quad x^2 + y^2 + 2x(2 + 3\lambda) + 2y(1 - 2\lambda) + 1 + 7\lambda = 0 \quad \text{...(ii)}$$

Its centre $[-(2 + 3\lambda), -(1 - 2\lambda)]$ lies on Eq. (i)

$\therefore \quad 6 \times -(2 + 3\lambda) - 4 \times -(1 - 2\lambda) + 7 = 0$

or $\quad -12 - 18\lambda + 4 - 8\lambda + 7 = 0$

or $\quad -26\lambda - 1 = 0$

$$\therefore \qquad \lambda = -\frac{1}{26}$$

Substituting the value of λ in Eq. (ii), the equation of circle is

$$x^2 + y^2 + 2x\left(2 - \frac{3}{26}\right) + 2y\left(1 + \frac{2}{26}\right) + 1 - \frac{7}{26} = 0$$

$$\Rightarrow \qquad 26(x^2 + y^2) + 98x + 56y + 9 = 0$$

Example 88. Prove that the tangents from any point of a fixed circle of co-axial system to two other fixed circles of the system are in a constant ratio.

Sol. Let the equations of the circles be $x^2 + y^2 + 2g_i x + c = 0$, $i = 1,2,3$. Since, all the three circles are fixed g_1, g_2 and g_3 are constants.

Let $P(h, k)$ be any point on the first circle, so that

$$h^2 + k^2 + 2g_1 h + c = 0 \qquad \ldots(i)$$

Let PQ and PR be the tangents from P on the other two circles

$$\therefore \qquad PQ = \sqrt{(h^2 + k^2 + 2g_2 h + c)}$$

and $$PR = \sqrt{(h^2 + k^2 + 2g_3 h + c)}$$

$$\therefore \qquad \frac{(PQ)^2}{(PR)^2} = \frac{h^2 + k^2 + 2g_2 h + c}{h^2 + k^2 + 2g_3 h + c}$$

$$= \frac{-2g_1 h + 2g_2 h}{-2g_1 h + 2g_3 h} \qquad \text{[from Eq. (i)]}$$

$$= \frac{g_2 - g_1}{g_3 - g_1} = \text{constant}$$

because g_1, g_2, g_3 are constants.

Example 89. If A, B, C be the centres of three co-axial circles and t_1, t_2, t_3 be the lengths of the tangents to them from any point, prove that

$$\overline{BC} \cdot t_1^2 + \overline{CA} \cdot t_2^2 + \overline{AB}\ t_3^2 = 0$$

Sol. Let the equations of three circles are $x^2 + y^2 + 2g_i x + c = 0$, $i = 1, 2, 3,$.

According to the question

$$A \equiv (-g_1, 0),\ B \equiv (-g_2, 0),\ C \equiv (-g_3, 0)$$

Let any point be $P(h, k)$

$$\therefore \qquad t_1 = \sqrt{h^2 + k^2 + 2g_1 h + c}$$

$$t_2 = \sqrt{h^2 + k^2 + 2g_2 h + c}$$

$$t_3 = \sqrt{h^2 + k^2 + 2g_3 h + c}$$

and $$\overline{AB} = (g_1 - g_2)$$

$$\overline{BC} = (g_2 - g_3)$$

and $$\overline{CA} = (g_3 - g_1)$$

Now, $\overline{BC} \cdot t_1^2 + \overline{CA} \cdot t_2^2 + \overline{AB}\ t_3^2$

$$= \Sigma (g_2 - g_3)(h^2 + k^2 + 2g_1 h + c)$$

$$= (h^2 + k^2 + c)\ \Sigma\ (g_2 - g_3) + 2h \Sigma g_1 (g_2 - g_3)$$

$$= (h^2 + k^2 + c)(g_2 - g_3 + g_3 - g_1 + g_1 - g_2)$$

$$+ 2h\ \{g_1(g_2 - g_3) + g_2(g_3 - g_1) + g_3(g_1 - g_2)\}$$

$$= (h^2 + k^2 + c)(0) + 2h(0) = 0$$

which proves the result.

Limiting Point

Limiting points of system of co-axial circles are the centres of the point circles belonging to the family (Circles whose radii are zero are called **point circles).**

1. Limiting points of the co-axial system

Let the circle is

$$x^2 + y^2 + 2gx + c = 0 \qquad \ldots(i)$$

where, g is variable and c is constant.

$\therefore$ Centre and the radius of Eq. (i) are $(-g, 0)$ and $\sqrt{(g^2 - c)}$, respectively. Let

$$\sqrt{g^2 - c} = 0$$

$$\therefore \qquad g = \pm\sqrt{c}$$

Thus, we get the two limiting points of the given co-axial system as

$$(\sqrt{c}, 0) \ \text{ and } \ (-\sqrt{c}, 0)$$

Clearly the above limiting points are real and distinct, real and coincident or imaginary according as $c >, =, < 0$

2. System of co-axial circles whose two limiting point are given :

Let (α, β) and (γ, δ) be the two given limiting points. Then, the corresponding point circles with zero radii are

$$(x - \alpha)^2 + (y - \beta)^2 = 0$$

and $$(x - \gamma)^2 + (y - \delta)^2 = 0$$

or $$x^2 + y^2 - 2\alpha x - 2\beta y + \alpha^2 + \beta^2 = 0$$

and $$x^2 + y^2 - 2\gamma x - 2\delta y + \gamma^2 + \delta^2 = 0$$

The equation of co-axial system is

$$(x^2 + y^2 - 2\alpha x - 2\beta y + \alpha^2 + \beta^2)$$

$$+\lambda(x^2 + y^2 - 2\gamma x - 2\delta y + \gamma^2 + \delta^2) = 0$$

where, $\lambda \neq -1$ is a variable parameter.

$$\Rightarrow x^2(1+\lambda)+y^2(1+\lambda)-2x(\alpha+\gamma\lambda)$$
$$-2y(\beta+\delta\lambda)+(\alpha^2+\beta^2)+\lambda(\gamma^2+\delta^2)=0$$

or $$x^2+y^2-\frac{2(\alpha+\gamma\lambda)}{(1+\lambda)}x-2\frac{(\beta+\delta\lambda)}{(1+\lambda)}y$$
$$+\frac{(\alpha^2+\beta^2)+\lambda(\gamma^2+\delta^2)}{(1+\lambda)}=0$$

Centre of this circle is $\left(\frac{(\alpha+\gamma\lambda)}{(1+\lambda)},\frac{(\beta+\delta\lambda)}{(1+\lambda)}\right)$ ...(i)

For limiting point,

Radius
$$=\sqrt{\frac{(\alpha+\gamma\lambda)^2}{(1+\lambda)^2}+\frac{(\beta+\delta\lambda)^2}{(1+\lambda)^2}-\frac{(\alpha^2+\beta^2)+\lambda(\gamma^2+\delta^2)}{(1+\lambda)}}=0$$

After solving, find λ. Substituting value of λ in Eq. (i), we get the limiting point of co-axial system.

Example 90. ***Find the coordinates of the limiting points of the system of circles determined by the two circles***

$x^2+y^2+5x+y+4=0$ and $x^2+y^2+10x-4y-1=0$

Sol. The given circles are

$$S_1\equiv x^2+y^2+5x+y+4=0$$

and $$S_2\equiv x^2+y^2+10x-4y-1=0$$

$\therefore$ Equation of the co-axial system of circles is $S_1+\lambda S_2=0$

or $$(x^2+y^2+5x+y+4)$$
$$+\lambda(x^2+y^2+10x-4y-1)=0$$

or $$x^2(1+\lambda)+y^2(1+\lambda)+5x(1+2\lambda)$$
$$+y(1-4\lambda)+(4-\lambda)=0$$

or $$x^2+y^2+\frac{5(1+2\lambda)}{(1+\lambda)}x+\frac{(1-4\lambda)}{(1+\lambda)}y+\frac{(4-\lambda)}{(1+\lambda)}=0$$

The centre of this circles is

$$\left(\frac{-5(1+2\lambda)}{2(1+\lambda)},-\frac{(1-4\lambda)}{2(1+\lambda)}\right)$$ ...(i)

Radius $=\sqrt{\frac{25(1+2\lambda)^2}{4(1+\lambda)^2}+\frac{(1-4\lambda)^2}{4(1+\lambda)^2}-\frac{(4-\lambda)}{(1+\lambda)}}=0$

or $25(1+2\lambda)^2+(1-4\lambda)^2-4(4-\lambda)(1+\lambda)=0$

or $25(4\lambda^2+4\lambda+1)+(16\lambda^2-8\lambda+1)-4(-\lambda^2+3\lambda+4)=0$

or $120\lambda^2+80\lambda+10=0$ or $12\lambda^2+8\lambda+1=0$

or $(6\lambda+1)(2\lambda+1)=0$

i.e. $$\lambda=-\frac{1}{6} \text{ and } -\frac{1}{2}$$

Substituting these values of λ in Eq. (i), we get the points $(-2,-1)$ and $(0,-3)$ which are the required limiting points.

Example 91. If the origin be one limiting point of a system of co-axial circles of which $x^2+y^2+3x+4y+25=0$ is a member, find the other limiting point.

Sol. Equation of circle with origin as limiting point is

$$(x-0)^2+(y-0)^2=0 \text{ or } x^2+y^2=0$$

belongs to the system of co-axial circles of which one member is

$$x^2+y^2+3x+4y+25=0$$

Hence, the equation of the whole system is

$$(x^2+y^2+3x+4y+25)+\lambda(x^2+y^2)=0$$

or $$x^2(1+\lambda)+y^2(1+\lambda)+3x+4y+25=0$$

or $$x^2+y^2+\frac{3}{(1+\lambda)}x+\frac{4}{(1+\lambda)}y+\frac{25}{(1+\lambda)}=0$$ ...(i)

$\therefore$ Its centre $=\left(-\frac{3}{2(1+\lambda)},-\frac{2}{(1+\lambda)}\right)$ ...(ii)

Radius of Eq. (i) can be zero for limiting point, then

$$\frac{9}{4(1+\lambda)^2}+\frac{4}{(1+\lambda)^2}-\frac{25}{(1+\lambda)}=0$$
$$9+16-100(1+\lambda)=0$$

$\Rightarrow$ $$1+\lambda=\frac{1}{4} \text{ or } \lambda=-\frac{3}{4}$$

From Eq. (ii), $$\left(\frac{-3}{2(1-3/4)},\frac{-2}{(1-3/4)}\right)$$

or $(-6,-8)$ is the other limiting point of the system.

Example 92. Prove that the limiting points of the system

$$x^2+y^2+2gx+c+\lambda(x^2+y^2+2fx+k)=0$$

subtend a right angle at the origin, if $\frac{c}{g^2}+\frac{k}{f^2}=2$.

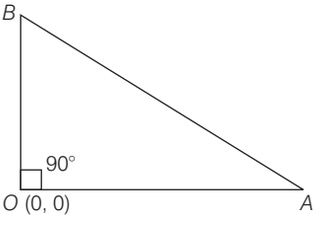
Sol. The given circle is

$$x^2+y^2+2gx+c+\lambda(x^2+y^2+2fy+k)=0$$

or $(1+\lambda)x^2+(1+\lambda)y^2+2gx+2fy\lambda+c+k\lambda=0$

or $$x^2+y^2+\frac{2g}{(1+\lambda)}x+\frac{2f\lambda}{(1+\lambda)}y+\frac{c+k\lambda}{(1+\lambda)}=0$$ ...(i)

Its centre is $$\left(\frac{-g}{1+\lambda},\frac{-f\lambda}{1+\lambda}\right)$$ ...(ii)



Radius of circle Eq. (i) is =0

$$\Rightarrow \quad \sqrt{\frac{g^2}{(1+\lambda)^2}+\frac{f^2\lambda^2}{(1+\lambda)^2}-\frac{(c+k\lambda)}{(1+\lambda)}}=0$$

or $\quad \lambda^2(f^2-k)-\lambda(k+c)+g^2-c=0$

which is a quadratic in λ. Let roots be λ_1 and λ_2.

$$\therefore \quad \lambda_1+\lambda_2=\frac{k+c}{f^2-k} \quad \text{and} \quad \lambda_1\lambda_2=\frac{g^2-c}{f^2-k}$$

then limiting points are [from Eq. (ii)]

$$A\left(\frac{-g}{1+\lambda_1},\frac{-f\lambda_1}{1+\lambda_1}\right) \quad \text{and} \quad B\left(\frac{-g}{1+\lambda_2},\frac{-f\lambda_2}{1+\lambda_2}\right)$$

But given that *AB* subtend a right angle at the origin.

$\therefore$ Slope of *OA* × Slope of *OB* = − 1

$$\Rightarrow \quad \left(\frac{\frac{-f\lambda_1}{1+\lambda_1}}{\frac{-g}{1+\lambda_1}}\right)\times\left(\frac{\frac{-f\lambda_2}{1+\lambda_2}}{\frac{-g}{1+\lambda_2}}\right)=-1$$

or $$\frac{f\lambda_1}{g}\times\frac{f\lambda_2}{g}=-1$$

or $$f^2\lambda_1\lambda_2+g^2=0$$

or $$f^2\frac{(g^2-c)}{(f^2-k)}+g^2=0$$

or $$2g^2f^2-cf^2-kg^2=0$$

or $$2=\frac{c}{g^2}+\frac{k}{f^2}$$

Example 93. Find the radical axis of co-axial system of circles whose limiting points are (–1,2) and (2, 3).

Sol. Equations of circles with limiting points are (–1, 2) and (2,3) are

$$(x+1)^2+(y-2)^2=0$$

or $$x^2+y^2+2x-4y+5=0 \quad \ldots(i)$$

and $$(x-2)^2+(y-3)^2=0$$

or $$x^2+y^2-4x-6y+13=0 \quad \ldots(ii)$$

respectively.

$\therefore$ Radical axis of circles Eqs. (i) and (ii) is

$$(x^2+y^2+2x-4y+5)$$
$$-(x^2+y^2-4x-6y+13)=0$$

or $$6x+2y-8=0$$

or $$3x+y-4=0$$

Example 94. Find the equation of the circle which passes through the origin and belongs to the co-axial of circles whose limiting points are (1,2) and (4,3).

Sol. Equations of circles whose limiting points are (1,2) and (4,3) are

$$(x-1)^2+(y-2)^2=0$$

or $$x^2+y^2-2x-4y+5=0 \quad \ldots(i)$$

and $$(x-4)^2+(y-3)^2=0$$

or $$x^2+y^2-8x-6y+25=0 \quad \ldots(ii)$$

Therefore, the corresponding system of co-axial circles is

$$(x^2+y^2-2x-4y+5)$$
$$+\lambda(x^2+y^2-8x-6y+25)=0 \quad \ldots(iii)$$

It passes through origin, then

$$5+25\lambda=0$$

$$\therefore \quad \lambda=-\frac{1}{5}$$

Substituting the value of λ in Eq. (iii), the required circle is

$$5(x^2+y^2-2x-4y+5)$$
$$-(x^2+y^2-8x-6y+25)=0$$

or $$4x^2+4y^2-2x-14y=0$$

or $$2x^2+2y^2-x-7y=0$$

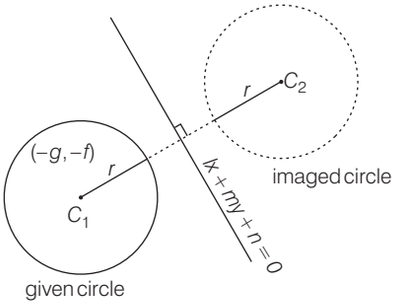
Image of the Circle by the Line Mirror

Let the circle be $x^2+y^2+2gx+2fy+c=0$ and line mirror is $lx+my+n=0$ in this condition, radius of circle remains unchanged but centres changes. Let the centre of imaged circle be (x_1, y_1).

Then, $$\frac{x_1-(-g)}{l}=\frac{y_1-(-f)}{m}=\frac{-2(-lg-mf+c)}{(l^2+m^2)}$$

we get, $$x_1=\frac{(l^2g-m^2g+2mlf-2nl)}{(l^2+m^2)}$$

and $$y_1=\frac{(m^2f-l^2f+2mlg-2mn)}{(l^2+m^2)}$$



$\therefore$ Required imaged circle is $(x-x_1)^2+(y-y_1)^2=r^2$

where, $r=\sqrt{(g^2+f^2-c)}$

Example 95. ***Find the equation of the image of the circle*** $x^2 + y^2 + 16x - 24y + 183 = 0$ ***by the line mirror*** $4x + 7y + 13 = 0$.

Sol. The given circle and line are

$$x^2 + y^2 + 16x - 24y + 183 = 0 \quad \ldots(i)$$

and $$4x + 7y + 13 = 0 \quad \ldots(ii)$$

Centre and radius of circle Eq. (i) are $(-8,12)$ and 5, respectively. Let the centre of the imaged circle be (x_1, y_1). Hence, (x_1, y_1) be the image of the point $(-8, 12)$ with respect to the line $4x + 7y + 13 = 0$, then

$$\frac{x_1 - (-8)}{4} = \frac{y_1 - 12}{7} = \frac{-2(4(-8) + 7(12) + 13)}{(4^2 + 7^2)}$$

$$\Rightarrow \quad \frac{x_1 + 8}{4} = \frac{y_1 - 12}{7} = -2$$

$\therefore \; x_1 = -16, y_1 = -2$

$\therefore$ Equation of the imaged circle is $(x + 16)^2 + (y + 2)^2 = 5^2$

or $$x^2 + y^2 + 32x + 4y + 235 = 0$$

Exercise for Session 7

1. The circles $x^2 + y^2 + x + y = 0$ and $x^2 + y^2 + x - y = 0$ intersect at an angle of

(a) $\pi / 6$ (b) $\pi / 4$ (c) $\pi / 3$ (d) $\pi / 2$

2. If the circles of same radius a and centres at (2, 3) and (5, 6) cut orthogonally, then a equals to

(a) 1 (b) 2 (c) 3 (d) 4

3. If the circles $x^2 + y^2 + 2x + 2ky + 6 = 0$ and $x^2 + y^2 + 2ky + k = 0$ intersect orthogonally, k is

(a) 2 or $-\frac{3}{2}$ (b) -2 or $-\frac{3}{2}$ (c) 2 or $\frac{3}{2}$ (d) -2 or $\frac{3}{2}$

4. If a circle passes through the point (a,b) and cuts the circle $x^2 + y^2 = 4$ orthogonally, then the locus of its centre is

(a) $2ax + 2by + (a^2 + b^2 + 4) = 0$ (b) $2ax + 2by - (a^2 + b^2 + 4) = 0$

(c) $2ax - 2by + (a^2 + b^2 + 4) = 0$ (d) $2ax - 2by - (a^2 + b^2 + 4) = 0$

5. The locus of the centre of the circle which cuts orthogonally the circle $x^2 + y^2 - 20x + 4 = 0$ and which touches $x = 2$ is

(a) $x^2 = 16y$ (b) $x^2 = 16y + 4$

(c) $y^2 = 16x$ (d) $y^2 = 16x + 4$

6. The equation of a circle which cuts the three circles $x^2 + y^2 - 3x - 6y + 14 = 0$, $x^2 + y^2 - x - 4y + 8 = 0$ and $x^2 + y^2 + 2x - 6y + 9 = 0$ orthogonally is

(a) $x^2 + y^2 - 2x - 4y + 1 = 0$ (b) $x^2 + y^2 + 2x + 4y + 1 = 0$

(c) $x^2 + y^2 - 2x + 4y + 1 = 0$ (d) $x^2 + y^2 - 2x - 4y - 1 = 0$

7. The equation of radical axis of the circles $x^2 + y^2 + x - y + 2 = 0$ and $3x^2 + 3y^2 - 4x - 12 = 0$ is

(a) $2x^2 + 2y^2 - 5x + y - 14 = 0$ (b) $7x - 3y + 18 = 0$

(c) $5x - y + 14 = 0$ (d) None of these

8. The radical centre of the circles $x^2 + y^2 = 1, x^2 + y^2 + 10y + 24 = 0$ and $x^2 + y^2 - 8x + 15 = 0$ is

(a) $(2, 5/2)$ (b) $(-2, 5/2)$

(c) $(-2, -5/2)$ (d) $(2, -5/2)$

9. If $(1,2)$ is a limiting point of the co-axial system of circles containing the circle $x^2 + y^2 + x - 5y + 9 = 0$, then the equation of the radical axis is

(a) $x - 9y + 4 = 0$ (b) $3x - y + 4 = 0$

(c) $x + 3y - 4 = 0$ (d) $9x + y - 4 = 0$

10. The limiting points of the system of circles represented by the equation $2(x^2+y^2)+\lambda x+\frac{9}{2}=0$ are

(a) $\left(\pm\frac{3}{2},0\right)$ (b) $(0,0)$ and $\left(\frac{9}{2},0\right)$

(c) $\left(\pm\frac{9}{2},0\right)$ (d) $(\pm3,\ 0)$

11. One of the limiting points of the co-axial system of circles containing the circles $x^2+y^2-4=0$ and $x^2+y^2-x-y=0$ is

(a) $(\sqrt{2},\sqrt{2})$ (b) $(-\sqrt{2},\sqrt{2})$

(c) $(-\sqrt{2},-\sqrt{2})$ (d) None of these

12. The point (2, 3) is a limiting point of a co-axial system of circles of which $x^2+y^2=9$ is a member. The coordinates of the other limiting point is given by

(a) $\left(\frac{18}{13},\frac{27}{13}\right)$ (b) $\left(\frac{9}{13},\frac{6}{13}\right)$

(c) $\left(\frac{18}{13},-\frac{27}{13}\right)$ (d) $\left(-\frac{18}{13},-\frac{9}{13}\right)$

13. Two circles are drawn through the points $(a,5a)$ and $(4a,a)$ to touch the *Y*-axis. Prove that they intersect at angle $\tan^{-1}\left(\frac{40}{9}\right)$.

14. Find the equation of the circle which cuts orthogonally the circle $x^2+y^2-6x+4y-3=0$, passes through (3,0) and touches the axis of *y*.

15. Tangents are drawn to the circles $x^2+y^2+4x+6y-19=0$, $x^2+y^2=9$ from any point on the line $2x+3y=5$. Prove that their lengths are equal.

16. Find the coordinates of the point from which the lengths of the tangents to the following three circles be equal $3x^2+3y^2+4x-6y-1=0$, $2x^2+2y^2-3x-2y-4=0$ and $2x^2+2y^2-x+y-1=0$

17. Find the equation of a circle which is co-axial with the circles $x^2+y^2+4x+2y+1=0$ and $x^2+y^2-x+3y-\frac{3}{2}=0$ and having its centre on the radical axis of these circles.

18. Find the radical axis of a co-axial system of circles whose limiting points are (1, 2) and (3 , 4).

Shortcuts and Important Results to Remember

1 If the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ cut the X-axis and Y-axis in four concyclic points, then $a_1a_2 = b_1b_2$.

2 If two conic sections $a_1x^2 + 2h_1xy + b_1y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $a_2x^2 + 2h_2xy + b_2y^2 + 2g_2x + 2f_2y + c_2 = 0$ will intersect each other in four concyclic points, if $\frac{a_1 - b_1}{a_2 - b_2} = \frac{h_1}{h_2}$.

3 If the circle $S_1 = 0$, bisects the circumference of the circle $S_2 = 0$, then their common chord will be the diameter of the circle $S_2 = 0$.

4 The radius of the director circle of a given circle is $\sqrt{2}$ times the radius of the given circle.

5 The point of intersection of the tangents at the points $P(a\cos\alpha, a\sin\alpha)$ and $Q(a\cos\beta, a\sin\beta)$ on the circle $x^2 + y^2 = a^2$ is

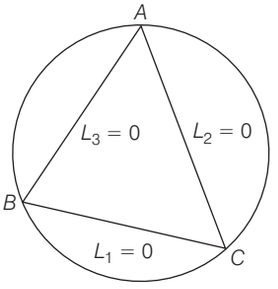
$$\left(\frac{a\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}, \frac{a\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}\right)$$

6 If the tangent to the circle $x^2 + y^2 = r^2$ at the point (a,b) meets the coordinates axes at the points A and B and O is the origin, then the area of the ΔOAB is $\frac{r^4}{2ab}$.

7 The length of the common chord of the circles $x^2 + y^2 + ax + by + c = 0$ and $x^2 + y^2 + bx + ay + c = 0$ is $\sqrt{\frac{1}{2}(a+b)^2 - 4c}$.

8 The length of the common chord of the circles $(x-a)^2 + y^2 = a^2$ and $x^2 + (y-b)^2 = b^2$ is $\frac{2ab}{\sqrt{a^2 + b^2}}$.

9 Family of circles circumscribing a triangle whose sides are given by $L_1 = 0, L_2 = 0$ and $L_3 = 0$ is given by $L_1L_2 + \lambda L_2L_3 + \mu L_3L_1 = 0$ provided coefficient of $xy = 0$ and coefficient of x^2 = coefficient of y^2.

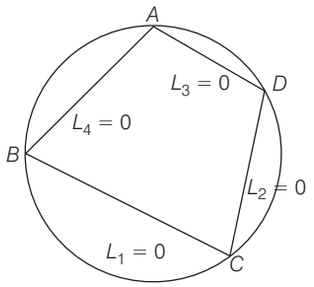


Remark

Equation of the circle circumscribing the triangle formed by the lines $a_rx + b_ry + c_r = 0$, where $r = 1, 2, 3$, is :

$$\begin{vmatrix} \frac{a_1^2 + b_1^2}{a_1x + b_1y + c_1} & a_1 & b_1 \\ \frac{a_2^2 + b_2^2}{a_2x + b_2y + c_2} & a_2 & b_2 \\ \frac{a_3^2 + b_3^2}{a_3x + b_3y + c_3} & a_3 & b_3 \end{vmatrix} = 0$$

10 Equation of circle circumscribing a quadrilateral whose sides in order are represented by the lines $L_1 = 0, L_2 = 0, L_3 = 0$ and $L_4 = 0$ is given by $L_1L_3 + \lambda L_2L_4 = 0$



provided coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$.

11 The locus of the middle point of a chord of a circle subtending a right angle at a given point will be a circle.

12 The length of an equilateral triangle inscribed in the circle $x^2 + y^2 = a^2$ is $a\sqrt{3}$.

13 The distance between the chord of contact of tangents to $x^2 + y^2 + 2gx + 2fy + c = 0$ from the origin and the point (g,f) is $\frac{|g^2 + f^2 - c|}{2\sqrt{(g^2 + f^2)}}$.

14 The shortest chord of a circle passing through a point P inside the circle is the chord whose middle point is P.

15 The length of transverse common tangent $<$ the length of direct common tangent.

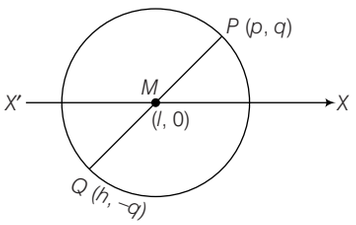
JEE Type Solved Examples :
Single Option Correct Type Questions

■ This section contains **10 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

● ***Ex. 1*** *Two distinct chords drawn from the point (p,q) on the circle $x^2+y^2=px+qy$, where $pq\neq 0$, are bisected by the X-axis. Then,*

(a) $|p|=|q|$ (b) $p^2=8q^2$ (c) $p^2<8q^2$ (d) $p^2>8q^2$

Sol. (d)



Suppose chord bisect at $M(\lambda,0)$, then other end point of chord is $(h,-q)$

where, $\lambda=\dfrac{p+h}{2}$

which lie on $x^2+y^2=px+qy$

or $h^2+q^2=ph-q^2$

$\Rightarrow$ $h^2-ph+2q^2=0$

for two distinct chords, $B^2-4AC>0$

or $p^2-4\cdot1\cdot2q^2>0$

or $p^2>8q^2$

● ***Ex. 2*** *The values of λ for which the circle $x^2+y^2+6x+5+\lambda(x^2+y^2-8x+7)=0$ dwindles into a point are*

(a) $1\pm\dfrac{\sqrt{2}}{3}$ (b) $2\pm\dfrac{2\sqrt{2}}{3}$ (c) $2\pm\dfrac{4\sqrt{2}}{3}$ (d) $1\pm\dfrac{4\sqrt{2}}{3}$

Sol. (c) The given circle is

$$x^2+y^2+6x+5+\lambda(x^2+y^2-8x+7)=0$$

or $x^2(1+\lambda)+y^2(1-\lambda)+(6-8\lambda)x+(5+7\lambda)=0$

$\Rightarrow$ $x^2+y^2+\left(\dfrac{6-8\lambda}{1+\lambda}\right)x+\left(\dfrac{5+7\lambda}{1+\lambda}\right)=0$

This will dwindle into a point circle, then radius of the circle = 0

$$\sqrt{\left(\frac{3-4\lambda}{1+\lambda}\right)^2+0-\left(\frac{5+7\lambda}{1+\lambda}\right)}=0$$

$\Rightarrow$ $(3-4\lambda)^2-(5+7\lambda)(1+\lambda)=0$

$\Rightarrow$ $9-16\lambda^2-24\lambda-5-5\lambda-7\lambda-7\lambda^2=0$

$\Rightarrow$ $9\lambda^2-36\lambda+4=0$

$$\lambda=\frac{36\pm\sqrt{(36)^2-4.9.4}}{2.9}$$

$\therefore$ $\lambda=2\pm\dfrac{4\sqrt{2}}{3}$

● ***Ex. 3*** *If $f(x+y)=f(x)\cdot f(y)$ for all x and y, $f(1)=2$ and $\alpha_n=f(n), n\in N$, then the equation of the circle having (α_1,α_2) and (α_3,α_4) as the ends of its one diameter is*

(a) $(x-2)(x-8)+(y-4)(y-16)=0$

(b) $(x-4)(x-8)+(y-2)(y-16)=0$

(c) $(x-2)(x-16)+(y-4)(y-8)=0$

(d) $(x-6)(x-8)+(y-5)(y-6)=0$

Sol. (a) $\because$ $f(x+y)=f(x).f(y)$ ...(i)

$\because$ $f(1)=2$

In Eq. (i), Put $x=y=1$,

then $f(2)=f(1).f(1)=2^2$

Now, in Eq. (i), $x=1, y=2$, then

$$f(3)=f(1)f(2)=2.2^2=2^3$$

Hence, $f(n)=2^n$

$\therefore$ $\alpha_n=f(n)=2^n\forall n\in N$

$(\alpha_1,\alpha_2)\equiv(2,4)$

and $(\alpha_3,\alpha_4)\equiv(8,16)$

Equation of circle in diametric form is

$(x-2)(x-8)+(y-4)(y-16)=0$

● ***Ex. 4*** *Two circles of radii a and b touching each other externally, are inscribed in the area bounded by $y=\sqrt{(1-x^2)}$ and the X-axis. If $b=\dfrac{1}{2}$, then a is equal to*

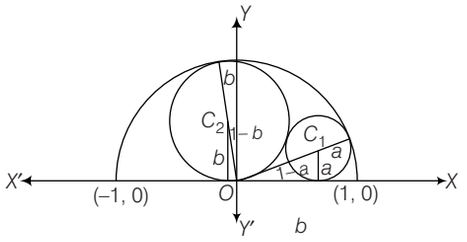
(a) $\dfrac{1}{4}$ (b) $\dfrac{1}{8}$

(c) $\dfrac{1}{2}$ (d) $\dfrac{1}{\sqrt{2}}$

Sol. (a) Let the centres of circles be C_1 and C_2, then

$C_1\equiv(\sqrt{(1-2a)},a)$

and $C_2\equiv(\sqrt{(1-2b)},b)$



Now, $C_1C_2 = a + b$

$$\Rightarrow \quad \left((\sqrt{(1-2a)})^2 + \left(a - \frac{1}{2}\right)^2\right) = \left(a + \frac{1}{2}\right)^2 \qquad \left[\because b = \frac{1}{2}\right]$$

or $$1 - 2a + \left(a - \frac{1}{2}\right)^2 = \left(a + \frac{1}{2}\right)^2$$

or $$1 - 2a + a^2 + \frac{1}{4} - a = a^2 + \frac{1}{4} + a$$

or $$a = \frac{1}{4}$$

● ***Ex. 5*** *There are two circles whose equations are* $x^2 + y^2 = 9$ *and* $x^2 + y^2 - 8x - 6y + n^2 = 0$, $n \in I$. *If the two circles having exactly two common tangents, then the number of possible values of n is*

(a) 2 (b) 7 (c) 8 (d) 9

Sol. (d) Given circles are $S_1 : x^2 + y^2 - 9 = 0$

Its centre C_1:(0,0) and radius $r_1 = 3$

and $S_2 : x^2 + y^2 - 8x - 6y + n^2 = 0$

Its centre C_2:(4,3) and radius $r_2 = \sqrt{(25 - n^2)}$

Here, $25 - n^2 > 0 \Rightarrow -5 < n < 5$ ...(i)

For exactly two common tangents,

$$r_1 + r_2 > C_1C_2$$

$$\Rightarrow \quad 3 + \sqrt{(25 - n^2)} > \sqrt{(4^2 + 3^2)}$$

$$\Rightarrow \quad \sqrt{(25 - n^2)} > 2$$

$$\Rightarrow \quad 25 - n^2 > 4$$

or $$n^2 < 21$$

or $$-\sqrt{21} < n < \sqrt{21} \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$-\sqrt{21} < n < \sqrt{21}$$

But $n \in I$. So, $n = -4, -3, -2, -1, 0, 1, 2, 3, 4$

Hence, number of possible values of n is 9.

● ***Ex. 6*** *Suppose* $f(x, y) = 0$ *is the equation of a circle such that* $f(x, 1) = 0$ *has equal roots (each equal to 2) and* $f(1, x) = 0$ *also has equal roots (each equal to zero). The equation of circle is*

(a) $x^2 + y^2 + 4x + 3 = 0$ (b) $x^2 + y^2 + 4y + 3 = 0$

(c) $x^2 + y^2 + 4x - 3 = 0$ (d) $x^2 + y^2 - 4x + 3 = 0$

Sol. (d) Let $f(x, y) = x^2 + y^2 + 2gx + 2fy + c$

$$\Rightarrow \quad f(x, 1) = x^2 + 1 + 2gx + 2f + c \equiv (x - 2)^2 \qquad \text{(given)}$$

then, $$g = -2,\ 2f + c = 3 \qquad \text{...(i)}$$

Also, $$f(1, x) = 1 + x^2 + 2g + 2fx + c \equiv (x - 0)^2 \qquad \text{(given)}$$

then, $$f = 0, 2g + c = -1 \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$g = -2, f = 0, c = 3$$

Thus, equation of circle is

$$x^2 + y^2 - 4x + 3 = 0$$

● ***Ex. 7*** *A variable circle C has the equation* $x^2 + y^2 - 2(t^2 - 3t + 1)x - 2(t^2 + 2t)y + t = 0$, *where t is a parameter. If the power of point* (a, b) *w.r.t. the circle C is constant, then the ordered pair* (a, b) *is*

(a) $\left(\frac{1}{10}, -\frac{1}{10}\right)$ (b) $\left(\frac{1}{10}, \frac{1}{10}\right)$

(c) $\left(-\frac{1}{10}, \frac{1}{10}\right)$ (d) $\left(-\frac{1}{10}, -\frac{1}{10}\right)$

Sol. (c) $\because C : x^2 + y^2 - 2(t^2 - 3t + 1)x - 2(t^2 + 2t)y + t = 0$

given power of circle = constant

$$\therefore \quad a^2 + b^2 - 2(t^2 - 3t + 1)a - 2(t^2 + 2t)b + t = \text{constant}$$

$$\Rightarrow \quad -2(a + b)t^2 + (6a - 4b + 1)t + (a^2 + b^2 - 2a) = \text{constant}$$

$\because$ Power of circle is constant, then

$$a + b = 0 \text{ and } 6a - 4b + 1 = 0$$

or $b = -a$, then $6a + 4a + 1 = 0$

$$\therefore \quad a = -\frac{1}{10},\ b = \frac{1}{10}$$

Hence, required ordered pair is $\left(-\frac{1}{10}, \frac{1}{10}\right)$

● ***Ex. 8*** *If the radii of the circles* $(x - 1)^2 + (y - 2)^2 = 1$ *and* $(x - 7)^2 + (y - 10)^2 = 4$ *are increasing uniformly w.r.t. time as* 0.3 unit/s *and* 0.4 unit/s *respectively, then they will touch each other at t equals to*

(a) 45 s (b) 90 s

(c) 11 s (d) 135 s

Sol. (b) Given circles are $S_1 : (x - 1)^2 + (y - 2)^2 = 1$

Its centre C_1:(1,2) and radius $r_1 = 1$

and $S_2 : (x - 7)^2 + (y - 10)^2 = 4$

Its centre C_2:(7,10) and radius $r_2 = 2$

$\because \quad C_1C_2 = 10 > r_1 + r_2$

Hence, the two circles are separated.

The radii of the two circles at time t are $(1 + 0.3t)$ and $(2 + 0.4t)$

For the two circles touch each other, then

$$C_1C_2 = |(1+0.3t) \pm (2+0.4t)|$$

$\Rightarrow$ $10 = |3+0.7t|$ or $10 = |-1-0.1t|$

$\Rightarrow$ $0.7t+3 = \pm 10$ or $-1-0.1t = \pm 10$

$\Rightarrow$ $t = 10$ or $t = 90$ $[\because t > 0]$

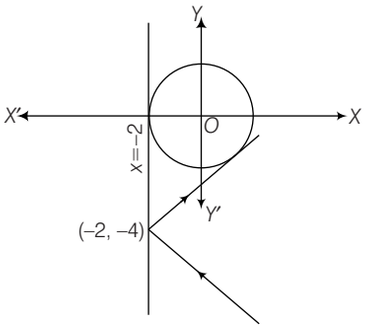
● ***Ex. 9*** *A light ray gets reflected from $x = -2$. If the reflected ray touches the circle $x^2 + y^2 = 4$ and the point of incident is $(-2,-4)$, then the equation of the incident ray is*

(a) $4y + 3x + 22 = 0$ (b) $3y + 4x + 20 = 0$

(c) $4y + 2x + 20 = 0$ (d) $y + x + 6 = 0$

Sol. (a) Any tangent of $x^2 + y^2 = 4$ is $y = mx \pm 2\sqrt{(1+m^2)}$.

If it passes through $(-2,-4)$, then $-4 = -2m \pm 2\sqrt{(1+m^2)}$



or $(m-2)^2 = 1 + m^2$

or $m = \infty,\ m = 3/4$

Hence, the slope of the reflected ray is 3/4.

Thus, the equation of the incident ray is

$$y + 4 = -\frac{3}{4}(x+2)$$

i.e. $4y + 3x + 22 = 0$

● ***Ex. 10*** *If a circle having centre at (α,β) radius r completely lies with in two lines $x + y = 2$ and $x + y = -2$, then, $\min.(|\alpha+\beta+2|, |\alpha+\beta-2|)$ is*

(a) greater than $\sqrt{2}r$

(b) less than $\sqrt{2}r$

(c) greater than $2r$

(d) less than $2r$

Sol. (a) Minimum distance of the centre from line > radius of circle i.e. $\min.\left\{\frac{|\alpha+\beta+2|}{\sqrt{2}}, \frac{|\alpha+\beta-2|}{\sqrt{2}}\right\} > r$

or $\min.\{|\alpha+\beta+2|, |\alpha+\beta-2|\} > \sqrt{2}r$

JEE Type Solved Examples :
More than One Correct Option Type Questions

■ This section contains **5 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which MORE THAN ONE may be correct.

● ***Ex. 11*** *If point $P(x,y)$ is called a lattice point, if $x, y \in I$. Then, the total number of lattice points in the interior of the circle $x^2 + y^2 = a^2, a \neq 0$ cannot be*

(a) 202 (b) 203 (c) 204 (d) 205

Sol. (a, b, c) Given circle is $x^2 + y^2 = a^2$ ...(i)

Clearly (0, 0) will belong the interior of circle Eq. (i). Also, other points interior to circle Eq. (i) will have the coordinates of the form

$$(\pm\lambda, 0), (0, \pm\lambda), \text{ where } \lambda^2 < a^2$$

and $(\pm\lambda, \pm\mu)$ and $(\pm\mu, \pm\lambda)$, where $\lambda^2 + \mu^2 < a^2$ and $\lambda, \mu \in I$

$\therefore$ Number of lattice points in the interior of the circle will be of the form $1 + 4r + 8t$, where $r, t = 0,1,2,\ldots$

$\therefore$ Number of such points must be of the form $4n + 1$, where $n = 0,1,2,\ldots$

● ***Ex. 12*** *Let x, y be real variable satisfying $x^2 + y^2 + 8x - 10y - 40 = 0$. Let $a = \max.\{\sqrt{(x+2)^2 + (y-3)^2}\}$ and $b = \min.\{\sqrt{(x+2)^2 + (y-3)^2}\}$, then*

(a) $a + b = 18$ (b) $a - b = 4\sqrt{2}$

(c) $a + b = 4\sqrt{2}$ (d) $a \cdot b = 73$

Sol. (a, b, d) Given circle is

$$x^2 + y^2 + 8x - 10y - 40 = 0$$

The centre and radius of the circle are $(-4,5)$ and 9, respectively.

Distance of the centre $(-4,5)$ from $(-2,3)$ is

$$\sqrt{(4+4)} = 2\sqrt{2}.$$

Therefore, $a = 2\sqrt{2} + 9$

and $b = -2\sqrt{2} + 9$

$\therefore$ $a + b = 18,\ a - b = 4\sqrt{2},\ ab = 73$

● ***Ex. 13*** *The equation of the tangents drawn from the origin to the circle* $x^2 + y^2 - 2rx - 2hy + h^2 = 0$, *are*

(a) $x = 0$

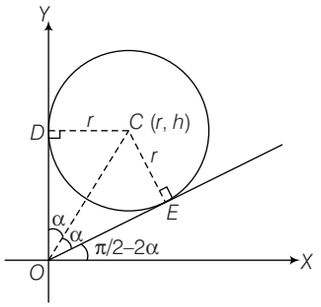
(b) $y = 0$

(c) $(h^2 - r^2)x - 2rhy = 0$

(d) $(h^2 - r^2)x + 2rhy = 0$

Sol. (a, c) The given equation is $(x - r)^2 + (y - h)^2 = r^2$

tangents are $x = 0$



and $y = x\tan\left(\frac{\pi}{2} - 2\alpha\right) = x\cot 2\alpha$

$$= \frac{x(1 - \tan^2\alpha)}{2\tan\alpha}$$

$$y = \frac{x\left(1 - \frac{r^2}{h^2}\right)}{2\left(\frac{r}{h}\right)} \qquad \left(\because \text{in } \Delta ODC, \tan\alpha = \frac{r}{h}\right)$$

or $(h^2 - r^2)x - 2rhy = 0$

● ***Ex. 14*** *Point M moved on the circle* $(x - 4)^2 + (y - 8)^2 = 20$. *Then it broke away from it and moving along a tangent to the circle cut the X-axis at point* $(-2, 0)$. *The coordinates of the point on the circle at which the moving point broke away is*

(a) $\left(\frac{42}{5}, \frac{36}{5}\right)$ (b) $\left(-\frac{2}{5}, \frac{44}{5}\right)$

(c) $(6, 4)$ (d) $(2, 4)$

Sol. (b, c) Given circle is

$$(x - 4)^2 + (y - 8)^2 = 20$$

or $x^2 + y^2 - 8x - 16y + 60 = 0$ ...(i)

Equation of chord of contact from $(-2, 0)$ is

$$-2 \cdot x + 0 \cdot y - 4(x - 2) - 8(y + 0) + 60 = 0$$

or $3x + 4y - 34 = 0$ ...(ii)

Solving Eqs. (i) and (ii), we get

$$x^2 + \left(\frac{34 - 3x}{4}\right)^2 - 8x - 16\left(\frac{34 - 3x}{4}\right) + 60 = 0$$

or $5x^2 - 28x - 12 = 0$

or $(x - 6)(5x + 2) = 0$

or $x = 6, -\frac{2}{5}$

Therefore, the points are $(6, 4)$ and $\left(-\frac{2}{5}, \frac{44}{5}\right)$.

● ***Ex. 15*** *The equations of four circles are* $(x \pm a)^2 + (y \pm a)^2 = a^2$. *The radius of a circle touching all the four circles is*

(a) $(\sqrt{2} - 1)a$ (b) $2\sqrt{2}\,a$

(c) $(\sqrt{2} + 1)a$ (d) $(2 + \sqrt{2})a$

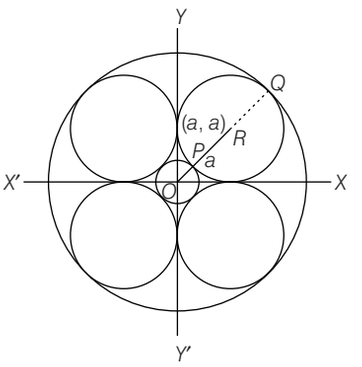
Sol. (a, c) Radius of inner circle $= OR - a$

$$= \sqrt{(a^2 + a^2)} - a$$

$$= a(\sqrt{2} - 1)$$

Radius of outer circle $= OR + RQ$

$$= a\sqrt{2} + a = a(\sqrt{2} + 1)$$



JEE Type Solved Examples :
Paragraph Based Questions

- This section contains **2 solved paragraphs** based upon each of the paragraph **3 multiple choice** questions have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Paragraph I

(Q. Nos. 16 to 18)

Consider the relation $4l^2 - 5m^2 + 6l + 1 = 0$, where $l, m \in R$.

16. The line $lx + my + 1 = 0$ touches a fixed circle whose equation is

(a) $x^2 + y^2 - 4x - 5 = 0$ (b) $x^2 + y^2 + 6x + 6 = 0$

(c) $x^2 + y^2 - 6x + 4 = 0$ (d) $x^2 + y^2 + 4x - 4 = 0$

17. Tangents PA and PB are drawn to the above fixed circle from the point P on the line $x + y - 1 = 0$. Then, the chord of contact AB passes through the fixed point

(a) $\left(\frac{1}{2}, -\frac{5}{2}\right)$ (b) $\left(\frac{1}{3}, \frac{4}{3}\right)$ (c) $\left(-\frac{1}{2}, \frac{3}{2}\right)$ (d) $\left(\frac{1}{2}, \frac{5}{2}\right)$

18. The number of tangents which can be drawn from the point $(2, -3)$ are

(a) 0 (b) 1 (c) 2 (d) 1 or 2

Sol.

16. (c) Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \ldots(i)$$

The line $lx + my + 1 = 0$ touch circle Eq. (i), then

$$\frac{|-lg - mf + 1|}{\sqrt{(l^2 + m^2)}} = \sqrt{(g^2 + f^2 - c)}$$

$$\Rightarrow \quad (lg + mf - 1)^2 = (l^2 + m^2)(g^2 + f^2 - c)$$

or $(f^2 - c)l^2 + (g^2 - c)m^2 - 2gflm + 2gl + 2fm - 1 = 0 \quad \ldots(ii)$

But the given condition is

$$4l^2 - 5m^2 + 6l + 1 = 0 \quad \ldots(iii)$$

Comparing Eqs. (ii) and (iii), we get

$$\frac{f^2 - c}{4} = \frac{g^2 - c}{-5} = \frac{-2gf}{0} = \frac{g}{3} = \frac{2f}{0} = \frac{-1}{1}$$

Then, we get $g = -3$, $f = 0$, $c = 4$

Substituting these values in Eq. (i), the equation of the circle is

$$x^2 + y^2 - 6x + 4 = 0$$

17. (a) Let any point on the line $x + y - 1 = 0$ is

$$P(\lambda, 1 - \lambda), \lambda \in R.$$

Then, equation of AB is

$$\lambda x + (1 - \lambda)y - 3(x + \lambda) + 4 = 0$$

$$\Rightarrow \quad (-3x + y + 4) + \lambda(x - y - 3) = 0$$

for fixed point $-3x + y + 4 = 0$, $x - y - 3 = 0$

$$\therefore \quad x = \frac{1}{2}, y = -\frac{5}{2}$$

$\therefore$ Fixed point is $\left(\frac{1}{2}, \frac{-5}{2}\right)$

18. (c) Let $S \equiv x^2 + y^2 - 6x + 4 = 0$.

$$\therefore \quad S_1 = (2)^2 + (-3)^2 - 6(2) + 4$$

$$= 4 + 9 - 12 - 4$$

$$= 5 > 0$$

Therefore, point $(2, -3)$ lies outside the circle from which two tangents can be drawn.

Paragraph II

(Q. Nos. 19 to 21)

If α- chord of a circle be that chord which subtends an angle α at the centre of the circle.

19. If $x + y = 1$ is α-chord of $x^2 + y^2 = 1$, then α is equal to

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{3\pi}{4}$

20. If slope of a $\frac{\pi}{3}$-chord of $x^2 + y^2 = 4$ is 1, then its equation is

(a) $x - y + \sqrt{6} = 0$ (b) $x - y + \sqrt{3} = 0$

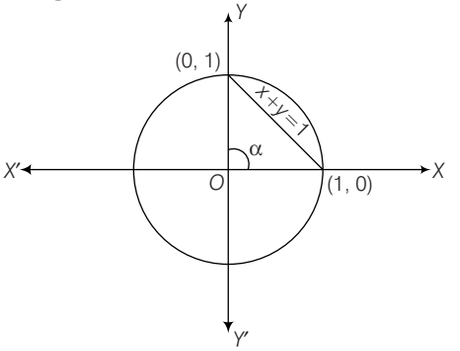
(c) $x - y - \sqrt{3} = 0$ (d) $x - y - 2\sqrt{3} = 0$

21. Distance of $\frac{2\pi}{3}$ — chord of $x^2 + y^2 + 2x + 4y + 1 = 0$ from the centre is

(a) $\frac{1}{\sqrt{2}}$ (b) 1 (c) $\sqrt{2}$ (d) 2

Sol.

19. (c) From figure

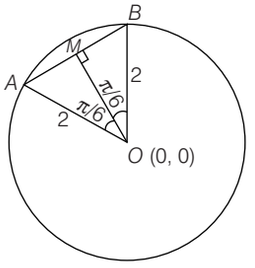


$$\alpha = \frac{\pi}{2}$$

20. (a) ∵ Slope of chord is 1.

Let the equation of chord be $x - y + \lambda = 0$.

∵ $$OM = 2\cos\left(\frac{\pi}{6}\right) = \sqrt{3}$$



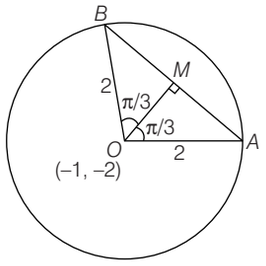
∴ $$\frac{|0 - 0 + \lambda|}{\sqrt{2}} = \sqrt{3}$$

⇒ $$\lambda = \pm\sqrt{6}$$

Hence, equation of chords are

$$x - y \pm \sqrt{6} = 0.$$

21. (b) From figure,



$$OM = 2\cos\left(\frac{\pi}{3}\right) = 1$$

JEE Type Solved Examples :
Single Integer Answer Type Questions

■ This section contains **2 examples.** The answer to each example is a **single digit integer,** ranging from 0 to 9 (both inclusive).

● ***Ex. 22*** *A circle with centre in the first quadrant is tangent to $y = x + 10$, $y = x - 6$ and the Y-axis. Let (p, q) be the centre of the circle. If the value of $(p + q) = a + b\sqrt{a}$, when $a, b \in Q$, then the value of $|a - b|$ is*

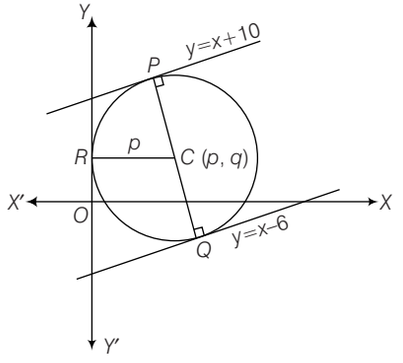
Sol. (6) ∵ $$CP = CR$$

⇒ $$\frac{|p - q + 10|}{\sqrt{2}} = p$$

or $$p - q + 10 = p\sqrt{2} \quad \ldots(i)$$

and $$CP = CQ$$

$$\frac{p - q + 10}{\sqrt{2}} = -\left(\frac{p - q - 6}{\sqrt{2}}\right) \text{ or } p - q = -2 \quad \ldots(ii)$$



From Eqs. (i) and (ii), we get

$$p = 4\sqrt{2} \text{ and } q = 4\sqrt{2} + 2$$

Now, $$p + q = 2 + 8\sqrt{2} = a + b\sqrt{2}$$ (given)

∴ $$a = 2, b = 8$$

Hence, $|a - b| = |2 - 8| = 6$

● ***Ex. 23*** *If the circles $x^2 + y^2 + (3 + \sin\theta)x + 2\cos\phi y = 0$ and $x^2 + y^2 + (2\cos\phi)x + 2\lambda y = 0$ touch each other, then the maximum value of λ is*

Sol. (1) Since, both the circles are passing through the origin (0, 0), the equation of tangent at (0,0) of first circle will be same as that of the tangent at (0, 0) of second circle.

Equation of tangent at (0, 0) of first circle is

$$(3 + \sin\theta)x + (2\cos\phi)y = 0 \quad \ldots(i)$$

Equaton of tangent at (0, 0) of second circle is

$$(2\cos\phi)x + 2\lambda y = 0 \quad \ldots(ii)$$

Therefore, Eqs. (i) and (ii) must be identical, then

$$\frac{3 + \sin\theta}{2\cos\phi} = \frac{2\cos\phi}{2\lambda}$$

or $$\lambda = \frac{2\cos^2\phi}{(3 + \sin\theta)}$$

or $$\lambda_{max} = 1 \qquad (\text{when } \sin\theta = -1 \text{ and } \cos\phi = 1)$$

JEE Type Solved Examples : Matching Type Questions

■ This section contains **2 examples.** Examples 24 and 25 have four statements (A, B, C and D) given in **Column I** and four statements (p, q, r and s) in **Column II**. Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II.**

● ***Ex. 24.*** *Consider the circles C_1 of radius a and C_2 of radius b, $b > a$ both lying the first quadrant and touching the coordinate axes.*

	Column I		Column II
(A)	C_1 and C_2 touch each other and $\frac{b}{a} = \lambda + \sqrt{\mu}$, $\lambda \in$ prime number and $\mu \in$ whole number, then	(p)	$\lambda + \mu$ is a prime number
(B)	C_1 and C_2 cut orthogonally and $\frac{b}{a} = \lambda + \sqrt{\mu}$, $\lambda \in$ prime number and $\mu \in$ whole number, then	(q)	$\lambda + \mu$ is a composite number
(C)	C_1 and C_2 intersect so that the common chord is longest and $\frac{b}{a} = \lambda + \sqrt{\mu}$, $\lambda \in$ prime number and $\mu \in$ whole number, then	(r)	$2\lambda + \mu$ is a perfect number
(D)	C_2 passes through the centre of C_1 and $\frac{b}{a} = \lambda + \sqrt{\mu}$, $\lambda \in$ prime number and $\mu \in$ whole number, then	(s)	$\lvert\lambda - \mu\rvert$ is a prime number

Sol. (A)→ (p, s); (B) → (p); (C) → (p, r, s); (D) → (q, r)

$\because \quad C_1 : x^2 + y^2 - 2ax - 2ay + a^2 = 0$

Centre : (a,a) and radius : a

and $\quad C_2 : x^2 + y^2 - 2bx - 2by + b^2 = 0$

Centre : (b,b) and radius : b

(A) $\because$ C_1 and C_2 touch each other, then

$$\sqrt{2}(b-a) = b + a \Rightarrow \frac{b}{a} = (\sqrt{2}+1)^2 = 3 + \sqrt{8}$$

$$\Rightarrow \quad \lambda = 3, \mu = 8$$

(B) $\because$ C_1 and C_2 intersect orthogonally, then

$$2(b-a)^2 = b^2 + a^2$$

$$\Rightarrow \quad a^2 + b^2 - 4ab = 0$$

or $$\left(\frac{b}{a}\right)^2 - 4\left(\frac{b}{a}\right) + 1 = 0$$

$$\therefore \quad \frac{b}{a} = \frac{4 \pm \sqrt{(16-4)}}{2} = 2 + \sqrt{3}$$

$$\Rightarrow \quad \lambda = 2, \mu = 3]$$

(C) $\because$ C_1 and C_2 intersect, the common chord is

$$2(b-a)(x+y) = b^2 - a^2$$

given common chord is longest, then passes through (a,a)

$$\Rightarrow \quad 2(b-a)(2a) = b^2 - a^2$$

or $$(b-3a)(b-a) = 0$$

$$\because \quad b - a \neq 0 \qquad [b > a]$$

$$\therefore \quad b - 3a = 0$$

or $$\frac{b}{a} = 3 \Rightarrow \lambda = 3, \mu = 0$$

(D) $\because$ C_2 passes through (a,a), then $a^2 + a^2 - 2ab - 2ab + b^2 = 0$

or $b^2 - 4ab + 2a^2 = 0$

or $$\left(\frac{b}{a}\right)^2 - 4\left(\frac{b}{a}\right) + 2 = 0$$

or $$\frac{b}{a} = \frac{4 \pm \sqrt{(16-8)}}{2} = 2 + \sqrt{2}$$

$$\Rightarrow \quad \lambda = 2, \mu = 2$$

● ***Ex. 25.*** *Match the following*

	Column I		Column II
(A)	The circles $x^2 + y^2 + 2x + c = 0$ $(c > 0)$ and $x^2 + y^2 + 2y + c = 0$ touch each other, then the value of $2c$ is	(p)	1
(B)	The circles $x^2 + y^2 + 2x + 3y + c = 0$ $(c > 0)$ and $x^2 + y^2 - x + 2y + c = 0$ intersect orthogonally, then the value of $2c$ is	(q)	2
(C)	The circle $x^2 + y^2 = 9$ is contains the circle $x^2 + y^2 - 2x + 1 - c^2 = 0$ $(c > 0)$, then $2c$ can be	(r)	3
(D)	The circle $x^2 + y^2 = 9$ is contains in the circle $x^2 + y^2 - 2x + 1 - \frac{c^2}{4} = 0$ $(c > 0)$, then $(c-6)$ can be	(s)	4

Sol. (A) → (p); (B) → (q); (C) → (p, q, r); (D) → (r, s)

(A) The circles

$$S_1 : (x+1)^2 + y^2 = (\sqrt{(1-c)})^2$$

Centre $C_1 : (-1, 0)$, radius $r_1 : \sqrt{(1-c)}$

and $$S_2 : x^2 + (y+1)^2 = (\sqrt{(1-c)})^2$$

Centre $C_2 : (0, -1)$, radius : $r_2 = \sqrt{(1-c)}$

Now, $\quad C_1C_2 = \sqrt{2}$ and $r_1 = r_2$

$\therefore$ The circles will touch externally only and $C_1C_2 = r_1 + r_2$

$$\Rightarrow \quad \sqrt{2} = 2\sqrt{(1-c)} \text{ or } 2c = 1$$

(B) The circles $S_1:(x+1)^2+\left(y+\frac{3}{2}\right)^2=\left(\sqrt{\left(\frac{13}{4}-c\right)}\right)^2$

Centre $C_1:\left(-1,-\frac{3}{2}\right)$, radius $r_1:\sqrt{\left(\frac{13}{4}-c\right)}$

and $S_2:\left(x-\frac{1}{2}\right)^2+(y+1)^2=\left(\sqrt{\left(\frac{5}{4}-c\right)}\right)^2$

Centre $C_2:\left(\frac{1}{2},-1\right)$, radius $r_2:\sqrt{\left(\frac{5}{4}-c\right)}$

For intersect orthogonally

$$(C_1C_2)^2=r_1^2+r_2^2$$

$$\Rightarrow \quad \left(\frac{3}{2}\right)^2+\left(\frac{1}{2}\right)^2=\frac{13}{4}-c+\frac{5}{4}-c$$

or $$2c=2$$

(C) The circles

$$S_1:x^2+y^2=3^2$$

Centre $C_1:(0,0)$, radius $r_1:3$

and $$S_2:(x-1)^2+y^2=c^2$$

Centre $C_2:(1,0)$, radius $r_2:c$

Now, S_2 will be contained in S_1, then

$$C_1C_2<r_1-r_2$$

or $$1<3-c \text{ or } c<2\Rightarrow 2c<4$$

(D) The circles

$$S_1:x^2+y^2=9$$

Centre $C_1:(0,0)$, radius $r_1:3$ and

$$S_2:(x-1)^2+y^2=\left(\frac{c}{2}\right)^2$$

Centre $C_2:(1,0)$, radius $r_2:\frac{c}{2}$

Now, S_1 will be contained in S_2,

then, $$r_2-r_1>C_1C_2$$

$$\Rightarrow \quad \frac{c}{2}-3>1 \text{ or } c>8$$

$$\therefore \quad (c-6)>2$$

JEE Type Solved Examples :
Statement I and II Type Questions

- **Directions** (Ex. Nos. 26 and 27) are Assertion-Reason Type examples. Each of these examples contains two statements :
  Statement I (Assertion) and **Statement II** (Reason)

  Each of these examples also has four alternative choices only one of which is the correct answer. You have to select the correct choice as given below :
  (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
  (b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
  (c) Statement I is true, Statement II is false
  (d) Statement I is false, Statement II is true

- ***Ex. 26*** *C_1 is a circle of radius 2 touching X-axis and Y-axis. C_2 is another circle of radius greater than 2 and touching the axes as well as the circle C_1.*

Statement I *Radius of Circle $C_2=\sqrt{2}(\sqrt{2}+1)(\sqrt{2}+2)$*

Statement II *Centres of both circles always lie on the line $y=x$.*

Sol. (c) $C_1:(x-2)^2+(y-2)^2=2^2$

$$C_2:(x-r)^2+(y-r)^2=r^2 \qquad (r>2)$$

According to question,

$$\sqrt{(r-2)^2+(r-2)^2}=r+2$$

$$(r-2)^2+(r-2)^2=(r+2)^2$$

$$r^2-12r+4=0$$

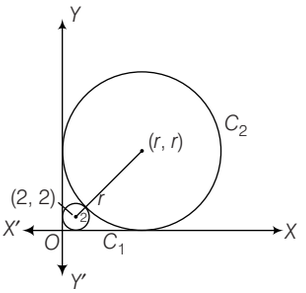
$$r=\frac{12\pm\sqrt{(144-16)}}{2}$$

$$=6\pm 4\sqrt{2}$$

$$\therefore \quad r=6+4\sqrt{2} \qquad [\because r>2]$$

$$=2(\sqrt{2}+1)^2$$

$$=\sqrt{2}(\sqrt{2}+1)(2+\sqrt{2})$$



∴Statement I is true and Statement II is always not true (where circles in II or IV quadrants)

- ***Ex. 27*** *From the point $P(\sqrt{2},\sqrt{6})$ tangents PA and PB are drawn to the circle $x^2+y^2=4$*

Statement I *Area of the quadrilateral OAPB (O being origin) is 4.*

Statement II *Tangents PA and PB are perpendicular to each other and therefore quadrilateral OAPB is a square.*

Sol. (a) Clearly, $P(\sqrt{2},\sqrt{6})$ lies on $x^2+y^2=8$, which is the director circle of $x^2+y^2=4$.

Therefore, tangents *PA* and *PB* are perpendicular to each other. So, *OAPB* is a square.

Hence, area of $OAPB=(\sqrt{S_1})^2=S_1$

$$=(\sqrt{2})^2+(\sqrt{6})^2-4=4$$

∴Both statements are true and statement II is correct explanation of statement I.

Subjective Type Examples

■ In this section, there are **16 subjective solved examples.**

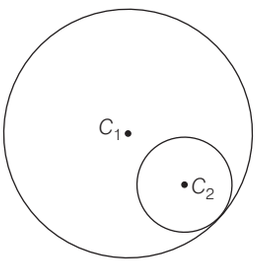
● ***Ex. 28*** *Find the equation of a circle having the lines* $x^2 + 2xy + 3x + 6y = 0$ *as its normals and having size just sufficient to contain the circle*

$$x(x-4) + y(y-3) = 0.$$

Sol. Given pair of normals is $x^2 + 2xy + 3x + 6y = 0$

or $(x + 2y)(x + 3) = 0$

$\therefore$ Normals are $x + 2y = 0$ and $x + 3 = 0$ the point of intersection of normals $x + 2y = 0$ and $x + 3 = 0$ is the centre of required circle, we get centre $C_1 \equiv (-3, 3/2)$ and other circle is



$$x(x-4) + y(y-3) = 0$$

or $x^2 + y^2 - 4x - 3y = 0$ ...(i)

its centre $C_2 \equiv (2, 3/2)$ and radius $r = \sqrt{4 + \frac{9}{4}} = \frac{5}{2}$

Since, the required circle just contains the given circle(i), the given circle should touch the required circle internally from inside.

$\Rightarrow$ radius of the required circle $= |C_1 - C_2| + r$

$$= \sqrt{(-3-2)^2 + \left(\frac{3}{2} - \frac{3}{2}\right)^2} + \frac{5}{2}$$

$$= 5 + \frac{5}{2} = \frac{15}{2}$$

Hence, equation of required circle is

$$(x+3)^2 + (y - 3/2)^2 = \left(\frac{15}{2}\right)^2$$

or $x^2 + y^2 + 6x - 3y - 54 = 0$

● ***Ex. 29*** *Let a circle be given by*

$$2x(x-a) + y(2y-b) = 0 \qquad (a \neq 0,\ b \neq 0)$$

Find the condition on a and b if two chords, each bisected by the X-axis, can be drawn to the circle from $(a, b/2)$.

Sol. The given circle is $2x(x-a) + y(2y-b) = 0$

or $x^2 + y^2 - ax - by/2 = 0$

Let AB be the chord which is bisected by X-axis at a point M. Let its coordinates be $M(h, 0)$

and let $S \equiv x^2 + y^2 - ax - by/2 = 0$

$\therefore$ Equation of chord AB is $T = S_1$

$$hx + 0 - \frac{a}{2}(x+h) - \frac{b}{4}(y+0) = h^2 + 0 - ah - 0$$

Since, its passes through $(a, b/2)$ we have

$$ah - \frac{a}{2}(a+h) - \frac{b^2}{8} = h^2 - ah$$

$$\Rightarrow \quad h^2 - \frac{3ah}{2} + \frac{a^2}{2} + \frac{b^2}{8} = 0$$

Now, there are two chords bisected by the X-axis, so there must be two distinct real roots of h.

$\therefore$ $B^2 - 4AC > 0$

$$\Rightarrow \quad \left(\frac{-3a}{2}\right)^2 - 4 \cdot 1 \cdot \left(\frac{a^2}{2} + \frac{b^2}{8}\right) > 0$$

$$\Rightarrow \quad a^2 > 2b^2.$$

Aliter : Given circle is

$$2x(x-a) + y(2y-b) = 0$$

or $x^2 + y^2 - ax - \frac{by}{2} = 0$ ...(i)

Let chords bisected at $M(h, 0)$ but given chords can be drawn $A\left(a, \frac{b}{2}\right)$ then chord cut the circle at $B(\lambda, -b/2)$

$\because$ Mid-point of ordinates of A and B is origin.

$\therefore$ $B(\lambda, b/2)$ lies on Eq. (i)

$\therefore$ $\lambda^2 + \frac{b^2}{4} - a\lambda + \frac{b^2}{4} = 0$

or $\lambda^2 - a\lambda + \frac{b^2}{2} = 0$

$\because$ λ is real

$\therefore$ $B^2 - 4AC > 0$ or $a^2 - 4 \cdot \frac{b^2}{2} > 0$ or $a^2 > 2b^2$

● ***Ex. 30*** *Let* C_1 *and* C_2 *be two circles with* C_2 *lying inside* C_1. *A circle* C *lying inside* C_1 *touches* C_1 *internally and* C_2 *externally. Identify the locus of the centre of* C.

Sol. Let the given circles $C_1$and C_2 have centres O_1 and O_2 with radii r_1 and r_2, respectively. Let centre of circle C is at O radius is r.

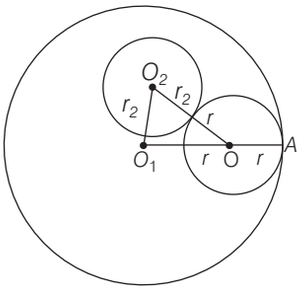
$\because$ $OO_2 = r + r_2$

$OO_1 = r_1 - r$

$\Rightarrow$ $OO_1 + OO_2 = r_1 + r_2$

which is greater than O_1O_2 as $O_1O_2 < r_1 + r_2$.

$\therefore$ Locus of O is an ellipse with foci O_1 and O_2.



Aliter :

Let $O_1 \equiv (0, 0)$, $O_2 \equiv (a, b)$ and $O \equiv (h, k)$

$\therefore$ $C_1 : x^2 + y^2 = r_1^2$

$C_2 : (x - a)^2 + (y - b)^2 = r_2^2$

$C : (x - h)^2 + (y - k)^2 = r^2$

$\Rightarrow$ $OO_2 = r + r_2$

$\Rightarrow$ $\sqrt{(h - a)^2 + (k - b)^2} = r + r_2$ ...(i)

and $OO_1 = r_1 - r$

$\Rightarrow$ $\sqrt{(h^2 + k^2)} = r_1 - r$ ...(ii)

On adding Eqs. (i) and (ii) we get

$$\sqrt{(h - a)^2 + (k - b)^2} + \sqrt{(h^2 + k^2)} = r_1 + r_2$$

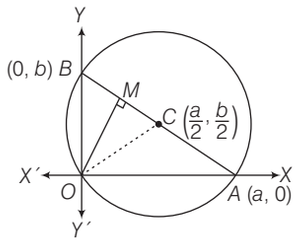
$\therefore$ Locus of O is $\sqrt{(x - a)^2 + (y - b)^2} + \sqrt{(x^2 + y^2)} = r_1 + r_2$

which represents an ellipse with foci are at (a,b) and $(0, 0)$.

● ***Ex. 31*** *A circle of constant radius r passes through the origin O, and cuts the axes at A and B. Show that the locus of the foot of the perpendicular from O to AB is*

$$(x^2 + y^2)^2 (x^{-2} + y^{-2}) = 4r^2$$

Sol. Let the coordinates of A and B are $(a, 0)$ and $(0, b)$.



$\therefore$ Equation of AB is $\frac{x}{a} + \frac{y}{b} = 1$ ...(i)

Centre of circle lie on line AB, since AB is diameter of the circle ($\because \angle AOB = \pi / 2$)

$\therefore$ Coordinate of centre C is $C \equiv \left(\frac{a}{2}, \frac{b}{2}\right)$

Since, the radius of circle = r

$\therefore$ $r = AC = CB = OC$

$$= \sqrt{\left(0 - \frac{a}{2}\right)^2 + \left(0 - \frac{b}{2}\right)^2} = \sqrt{\frac{a^2 + b^2}{4}}$$

$\therefore$ $a^2 + b^2 = 4r^2$ ...(ii)

Equation of OM which is $\perp$ to AB is

$$ax - by = \lambda$$

It passes through (0, 0)

$\therefore$ $0 = \lambda$

$\therefore$ Equation of OM is

$$ax - by = 0$$ ...(iii)

On solving Eq. (i) and Eq. (iii), we get

$$a = \frac{x^2 + y^2}{x} \text{ and } b = \frac{x^2 + y^2}{y}$$

Substituting the values of a and b in Eq. (ii), we get

$$(x^2 + y^2)^2 \left(\frac{1}{x^2} + \frac{1}{y^2}\right) = 4r^2$$

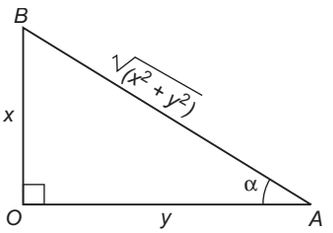
or $(x^2 + y^2)^2 (x^{-2} + y^{-2}) = 4r^2$

which is the required locus.

Aliter :

$\because$ AB is the diameter of circle. If $\angle OAB = \alpha$, then

$$OA = 2r \cos \alpha, OB = 2r \sin \alpha$$



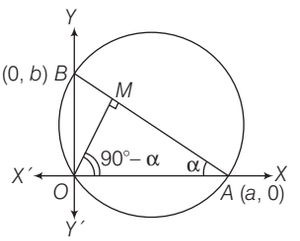
Equation of AB is

$$\frac{x}{2r \cos \alpha} + \frac{y}{2r \sin \alpha} = 1$$

$\Rightarrow$ $\frac{x}{\cos \alpha} + \frac{y}{\sin \alpha} = 2r$ ...(i)

and equation of OM is $y = x \tan (90° - \alpha)$

$\Rightarrow$ $\cot \alpha = \frac{y}{x}$



$\therefore$ $\sin \alpha = \frac{x}{\sqrt{(x^2 + y^2)}}$

and $\cos \alpha = \frac{y}{\sqrt{(x^2 + y^2)}}$

Then, from Eq. (i),

$$\frac{x}{y}\sqrt{(x^2+y^2)}+\frac{y}{x}\sqrt{(x^2+y^2)}=2r$$

$$\Rightarrow \quad \frac{(x^2+y^2)\sqrt{(x^2+y^2)}}{xy}=2r$$

On squaring, we have $(x^2+y^2)^2\,\frac{(x^2+y^2)}{x^2\,y^2}=4r^2$

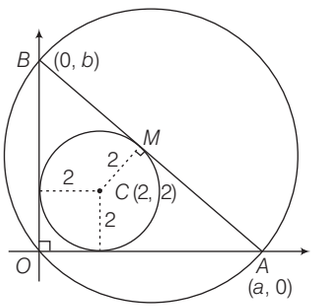
$$\Rightarrow \quad (x^2+y^2)^2\,(x^{-2}+y^{-2})=4r^2$$

● ***Ex. 32*** *The circle $x^2+y^2-4x-4y+4=0$ is inscribed in a triangle which has two of its sides along the coordinate axes. The locus of the circumcentre of the triangle is $x+y-xy+k\,(x^2+y^2)^{1/2}=0$. Find k.*

Sol. The given circle is $x^2+y^2-4x-4y+4=0$. This can be re-written as $(x-2)^2+(y-2)^2=4$ which has centre $C\,(2,2)$ and radius 2.

Let the equation of third side is

$$\frac{x}{a}+\frac{y}{b}=1 \qquad \text{(equation of } AB\text{)}$$



Length of perpendicular from (2, 2) on AB = radius = CM

$$\therefore \quad \frac{\left|\frac{2}{a}+\frac{2}{b}-1\right|}{\sqrt{\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}}=2$$

Since, origin and (2, 2) lie on the same side of AB

$$\therefore \quad -\frac{\left(\frac{2}{a}+\frac{2}{b}-1\right)}{\sqrt{\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}}=2$$

$$\text{or} \quad \frac{2}{a}+\frac{2}{b}-1=-2\sqrt{\left(\frac{1}{a^2}+\frac{1}{b^2}\right)} \qquad \ldots(i)$$

Since, $\quad \angle AOB=\frac{\pi}{2}$

Hence, AB is the diameter of the circle passing through ΔOAB, mid-point of AB is the centre of the circle i.e. $\left(\frac{a}{2},\frac{b}{2}\right)$.

Let centre be $(h,k)\equiv\left(\frac{a}{2},\frac{b}{2}\right)$ then $a=2h$ and $b=2k$.

Substituting the values of a and b in Eq. (i), then

$$\frac{2}{2h}+\frac{2}{2k}-1=-2\sqrt{\left(\frac{1}{4h^2}+\frac{1}{4k^2}\right)}$$

$$\Rightarrow \quad \frac{1}{h}+\frac{1}{k}-1=-\sqrt{\left(\frac{1}{h^2}+\frac{1}{k^2}\right)}$$

$$\text{or} \quad h+k-hk+\sqrt{(h^2+k^2)}=0$$

$\therefore$ Locus of $M\,(h,k)$ is

$$x+y-xy+\sqrt{(x^2+y^2)}=0$$

Hence, the required value of k is 1.

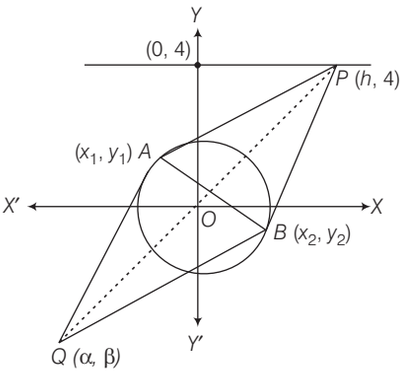
● ***Ex. 33*** *P is a variable on the line $y=4$. Tangents are drawn to the circle $x^2+y^2=4$ from P to touch it at A and B. The parallelogram PAQB is completed . Find the equation of the locus of Q.*

Sol. Let $P\,(h,4)$ be a variable point. Given circle is

$$x^2+y^2=4 \qquad \ldots(i)$$

Draw tangents from $P\,(h,4)$ and complete parallelogram $PAQB$.

Equation of the diagonal AB which is chord of contact of $x^2+y^2=4$ is $hx+4y=4$ $\ldots(ii)$



Let coordinates of A and B are (x_1,y_1) and (x_2,y_2), respectively.

Since, $A\,(x_1,y_1)$ and $B\,(x_2,y_2)$ lies on Eq. (ii)

$\therefore \quad hx_1+4y_1=4 \quad$ and $\quad hx_2+4y_2=4$

$\therefore \quad h\,(x_1+x_2)+4\,(y_1+y_2)=8 \qquad \ldots(iii)$

Since, $PAQB$ is parallelogram

$\therefore$ Mid-point of AB = Mid-point of PQ

$$\Rightarrow \quad \frac{x_1+x_2}{2}=\frac{\alpha+h}{2}$$

$$\text{and} \quad \frac{y_1+y_2}{2}=\frac{\beta+4}{2} \qquad \ldots(iv)$$

Eliminating x from Eqs. (i) and (ii), then

$$\left(\frac{4-4y}{h}\right)^2 + y^2 = 4$$

$$\Rightarrow \quad 16 + 16y^2 - 32y + h^2y^2 = 4h^2$$

$$\Rightarrow \quad (16 + h^2)\, y^2 - 32y + 16 - 4h^2 = 0$$

$$\therefore \quad y_1 + y_2 = \frac{32}{16 + h^2} \qquad \ldots(v)$$

From Eqs. (iii) and (v), we get

$$x_1 + x_2 = \frac{8h}{16 + h^2} \qquad \ldots(vi)$$

From Eqs. (iv) and (vi)

$$\beta + 4 = \frac{32}{16 + h^2}$$

or $$(16 + h^2)(\beta + 4) = 32 \qquad \ldots(vii)$$

From Eqs. (iv) and (vi)

$$\alpha + h = \frac{8h}{16 + h^2}$$

or $$(16 + h^2)(\alpha + h) = 8h \qquad \ldots(viii)$$

Dividing Eq. (viii) by Eq. (vii), then

$$\frac{\alpha + h}{\beta + 4} = \frac{h}{4} \quad \text{or} \quad h = \frac{4\alpha}{\beta}$$

Substituting the value of h in Eq. (vii) then

$$\left(16 + \frac{16\alpha^2}{\beta^2}\right)(\beta + 4) = 32$$

$$\Rightarrow \quad (\alpha^2 + \beta^2)(\beta + 4) = 2\beta^2$$

Hence, locus of $Q(\alpha, \beta)$ is $(x^2 + y^2)(y + 4) = 2y^2$

● ***Ex. 34*** *Show that the circumcircle of the triangle formed by the lines $ax + by + c = 0$; $bx + cy + a = 0$ and $cx + ay + b = 0$ passes through the origin if $(b^2 + c^2)(c^2 + a^2)(a^2 + b^2) = abc(b + c)(c + a)(a + b)$.*

Sol. Equation of conic is

$$(bx + cy + a)(cx + ay + b) + \lambda\,(cx + ay + b)(ax + by + c) + \mu\,(ax + by + c)(bx + cy + a) = 0 \qquad \ldots(i)$$

where, λ and μ are constants.

Eq. (i) represents a circle if the coefficient of x^2 and y^2 are equal and the coefficient of xy is zero such that

$$bc + \lambda ca + \mu ab = ca + \lambda ab + \mu bc$$

or $$(a - b)\, c + \lambda\,(b - c)\, a + \mu\,(c - a)\, b = 0 \qquad \ldots(ii)$$

and $$(c^2 + ab) + \lambda\,(a^2 + bc) + \mu\,(b^2 + ac) = 0 \qquad \ldots(iii)$$

on solving Eq. (ii) and Eq. (iii) by cross multiplication rule, we get

$$\frac{1}{(c^2 - ab)(a^2 + b^2)} = \frac{\lambda}{(a^2 - bc)(b^2 + c^2)} = \frac{\mu}{(b^2 - ac)(c^2 + a^2)}$$

$$\therefore \quad \lambda = \frac{(a^2 - bc)(b^2 + c^2)}{(c^2 - ab)(a^2 + b^2)}$$

and $$\mu = \frac{(b^2 - ac)(c^2 + a^2)}{(c^2 - ab)(a^2 + b^2)} \qquad \ldots(iv)$$

and given, Eq. (i) passes through the origin then

$$ab + bc\lambda + ca\,\mu = 0 \qquad \ldots(v)$$

From Eqs. (iv) and (v), we get

$$ab + \frac{bc\,(a^2 - bc)(b^2 + c^2)}{(c^2 - ab)(a^2 + b^2)} + \frac{ca\,(b^2 - ac)(c^2 + a^2)}{(c^2 - ab)(a^2 + b^2)} = 0$$

$$\Rightarrow \quad (c^2 - ab)(a^2 + b^2)\, ab + (a^2 - bc)(b^2 + c^2)bc + (b^2 - ca)(c^2 + a^2)\, ca = 0$$

$$\Rightarrow \quad abc^2\,(a^2 + b^2) + a^2bc\,(b^2 + c^2) + b^2ca\,(c^2 + a^2) = a^2b^2\,(a^2 + b^2) + b^2c^2\,(b^2 + c^2) + c^2a^2\,(c^2 + a^2)$$

$$\Rightarrow \quad abc\,\{c\,(a^2 + b^2) + a\,(b^2 + c^2) + b\,(c^2 + a^2)\} = a^2b^2\,(a^2 + b^2) + b^2c^2\,(b^2 + c^2) + c^2a^2\,(c^2 + a^2)$$

$$\Rightarrow \quad abc\,\{(a + b)(b + c)(c + a) - 2abc\} = a^2b^2\,(a^2 + b^2) + b^2c^2\,(b^2 + c^2) + c^2a^2\,(c^2 + a^2)$$

$$\Rightarrow \quad abc\,(a + b)(b + c)(c + a) = 2a^2b^2c^2 + a^2b^2\,(a^2 + b^2) + b^2c^2\,(b^2 + c^2) + c^2a^2\,(c^2 + a^2)$$

$$\Rightarrow \quad abc\,(a + b)(b + c)(c + a) = (a^2 + b^2)(b^2 + c^2)(c^2 + a^2)$$

Hence, $$(a^2 + b^2)(b^2 + c^2)(c^2 + a^2) = abc\,(a + b)(b + c)(c + a)$$

● ***Ex. 35*** *If four points P, Q, R, S in the plane be taken and the square of the length of the tangents from P to the circle on QR as diameter be denoted by $\{P, QR\}$, show that*

$$\{P, RS\} - \{P, QS\} + \{Q, PR\} - \{Q, RS\} = 0$$

Sol. Let $P \equiv (x_1, y_1)$, $Q \equiv (x_2, y_2)$, $R \equiv (x_3, y_3)$ and $S \equiv (x_4, y_4)$.

Equation of circle with RS as diameter is

$$(x - x_3)(x - x_4) + (y - y_3)(y - y_4) = 0$$

$$\therefore \quad \{P, RS\} = (x_1 - x_3)(x_1 - x_4) + (y_1 - y_3)(y_1 - y_4)$$

Now, equation of circle with QS as diameter is

$$(x - x_2)(x - x_4) + (y - y_2)(y - y_4) = 0$$

$$\therefore \quad \{P, QS\} = (x_1 - x_2)(x_1 - x_4) + (y_1 - y_2)(y_1 - y_4)$$

Equation of circle with PR as diameter is

$$(x - x_1)(x - x_3) + (y - y_1)(y - y_3) = 0$$

$\therefore$ $\{Q, PR\} = (x_2 - x_1)(x_2 - x_3) + (y_2 - y_1)(y_2 - y_3)$

Equation of circle with RS as diameter is

$$(x - x_3)(x - x_4) + (y - y_3)(y - y_4) = 0$$

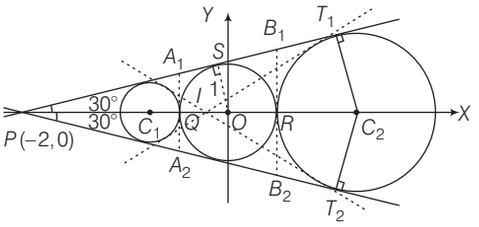
$\therefore$ $\{Q, RS\} = (x_2 - x_3)(x_2 - x_4) + (y_2 - y_3)(y_2 - y_4)$

Hence, $\{P, RS\} - \{P, QS\} + \{Q, PR\} - \{Q, RS\} = 0$

● ***Ex. 36*** *Let T_1, T_2 be two tangents drawn from $(-2, 0)$ on the circle $C : x^2 + y^2 = 1$. Determine the circles touching C and having T_1, T_2 as their pair of tangents. Further, find the equations of all possible common tangents to these circles when taken two at a time.*

Sol. In figure $OS = 1$, $OP = 2$

$\therefore$ $\sin \angle SPO = \dfrac{1}{2} = \sin 30°$



$\therefore$ $\angle SPO = 30°$

$\because$ $PA_1 = PA_2 \Rightarrow \angle PA_1A_2 = \angle PA_2A_1$

$\Rightarrow$ ΔPA_1A_2 is an equilateral triangle.

Therefore, centre C_1 is centroid of ΔPA_1A_2, C_1 divides PQ in the ratio 2 : 1.

$\therefore$ $C_1 \equiv \left(-\dfrac{4}{3}, 0\right)$ and its radius $= C_1Q = \dfrac{1}{3}$

$\Rightarrow$ $C_1 : (x + 4/3)^2 + y^2 = \left(\dfrac{1}{3}\right)^2$ ...(i)

The other circle C_2 touches the equilateral triangle PB_1B_2 externally.

its radius is given by $= \dfrac{\Delta}{s - a}$, where $B_1B_2 = a$

$$= \frac{\frac{\sqrt{3}}{4}a^2}{\frac{3a}{2} - a} = \frac{\sqrt{3}}{2}a$$

but $\tan 30° = \dfrac{a/2}{3} \Rightarrow a = \dfrac{6}{\sqrt{3}}$

$\therefore$ Radius $= \dfrac{\sqrt{3}}{2} \cdot \dfrac{6}{\sqrt{3}} = 3$

$\Rightarrow$ coordinates of C_2 are (4, 0)

$\therefore$ Equation of $C_2 : (x - 4)^2 + y^2 = 3^2$ ...(ii)

Equations of common tangents to circle (i) and circle C are

$$x = -1, y = \pm \frac{1}{\sqrt{3}}(x + 2), \{T_1 \text{ and } T_2\}$$

and equations of common tangents to circle (ii) and circle C are

$$x = 1, y = \pm \frac{1}{\sqrt{3}}(x + 2) (\{T_1 \text{ and } T_2\}$$

To find the remaining two transverse common tangents to Eqs. (i) and (ii). If I divides C_1 and C_2 in the ratio $r_1 : r_2 = 1/3 : 3 = 1 : 9$.

Therefore coordinates of I are $(-4/5, 0)$.

Equation of any line through I is $y - 0 = m(x + 4/5)$. If it will touch Eq. (ii)

then $\dfrac{|m(4 + 4/5) - 0|}{\sqrt{(1 + m^2)}} = 3$

$\Rightarrow$ $\left(\dfrac{24}{5}\right)^2 m^2 = 9(1 + m^2)$

$\Rightarrow$ $64m^2 = 25 + 25m^2$

$\Rightarrow$ $39m^2 = 25 \Rightarrow m = \pm \dfrac{5}{\sqrt{39}}$

Therefore, equations of transverse common tangents are

$$y = \pm \frac{5}{\sqrt{39}}(x + 4/5)$$

● ***Ex. 37*** *Find the equation of the circle of minimum radius which contains the three circles*

$$x^2 - y^2 - 4y - 5 = 0$$

$$x^2 + y^2 + 12x + 4y + 31 = 0$$

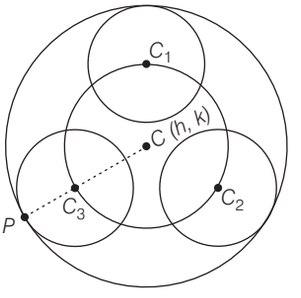
and $$x^2 + y^2 + 6x + 12y + 36 = 0$$

Sol. The coordinates of the centres and radii of three given circles are as given below :

$C_1 \equiv (0, 2)$; $r_1 = 3$

$C_2 \equiv (-6, -2)$; $r_2 = 3$

and $C_3 \equiv (-3, -6)$; $r_3 = 3$



Let $C \equiv (h, k)$ be the centre of the circle passing through the centres C_1 (0, 2), C_2 (−6, −2) and C_3 (−3, −6).

Then, $CC_1 = CC_2 = CC_3$

$\Rightarrow$ $(CC_1)^2 = (CC_2)^2 = (CC_3)^2$

$$\Rightarrow \quad (h-0)^2+(k-2)^2=(h+6)^2+(k+2)^2$$
$$=(h+3)^2+(k+6)^2$$
$$\Rightarrow \quad -4k+4=12h+4k+40=6h+12k+45$$
$$\Rightarrow \quad 12h+8k+36=0$$
or $$3h+2k+9=0 \qquad ...(i)$$
and $$6h-8k-5=0 \qquad ...(ii)$$

On solving Eqs. (i) and (ii), we get $h=-\frac{31}{18}, k=-\frac{23}{12}$

Now, $CP=CC_3+C_3P=CC_3+3$

$$=\sqrt{\left(-3+\frac{31}{18}\right)^2+\left(-6+\frac{23}{12}\right)^2}+3=\left(\frac{5}{36}\sqrt{(949)}+3\right)$$

Hence, equation of required circle is

$$\left(x+\frac{31}{18}\right)^2+\left(y+\frac{23}{12}\right)^2=\left(3+\frac{5}{\sqrt{36}}\sqrt{949}\right)^2$$

Remark

If radii of three given circles are distinct say $r_1 < r_2 < r_3$ then the radius of the required circle will be equal to (CC_1 or CC_2 or CC_3) + r_3 ($\because CC_1 = CC_2 = CC_3$)

● ***Ex. 38*** *Find the point P on the circle* $x^2+y^2-4x-6y+9=0$ *such that*

(i) $\angle POX$ is minimum,

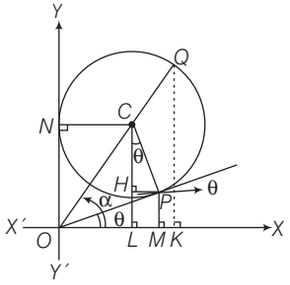
(ii) *OP is maximum, when O is the origin and OX is the X-axis.*

Sol. Given circle is

$$x^2+y^2-4x-6y+9=0$$
or $$(x-2)^2+(y-3)^2=2^2 \qquad .... (i)$$

Its centre is $C\equiv(2,3)$ and radius $r=2$

Eq. (i) Let OP and ON be the two tangents from O to the circle Eq. (i), then $OP=ON=3$



then $\angle POX$ is minimum when OP is tangent to the circle Eq. (i) at P

Let $\angle POX=\theta$

$$\therefore \quad P\equiv(OP\cos\theta, OP\sin\theta)$$
i.e. $$P\equiv(3\cos\theta, 3\sin\theta) \qquad ...(ii)$$

From figure, $OM=OL+LM=NC+HP=NC+CP\sin\theta$

$$\Rightarrow \quad OP\cos\theta=NC+CP\sin\theta$$
$$\Rightarrow \quad 3\cos\theta=2+2\sin\theta$$
$$\Rightarrow \quad 9(1-\sin^2\theta)=4(1+\sin\theta)^2$$
$$\Rightarrow \quad 9(1-\sin\theta)=4(1+\sin\theta) \qquad (\because \sin\theta\neq -1)$$
$$\therefore \quad \sin\theta=\frac{5}{13} \quad \text{and} \quad \cos\theta=\frac{12}{13}$$

From Eq. (ii), $P\equiv\left(3\times\frac{12}{13}, 3\times\frac{5}{13}\right)$ i.e. $P\equiv\left(\frac{36}{13}, \frac{15}{13}\right)$

Eq. (ii) OP will be maximum, if P becomes the point extended part of OC cuts the circle. Let this point be Q

then maximum value of $OP=OQ=OC+CQ=(\sqrt{13}+2)$

Let $\angle COX=\alpha$

then, $Q\equiv(OQ\cos\alpha, OQ\sin\alpha)$

$$\equiv((2+\sqrt{13})\cos\alpha, (2+\sqrt{13})\sin\alpha) \qquad ... (iii)$$

Now, in ΔCOL, $\cos\alpha=\frac{OL}{OC}=\frac{NC}{OC}=\frac{2}{\sqrt{13}}$

$$\therefore \quad \sin\alpha=\frac{3}{\sqrt{13}}$$

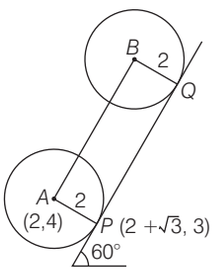
Now, from Eq. (iii), $Q\equiv\left(2+\frac{4}{\sqrt{13}}, 3+\frac{6}{\sqrt{13}}\right)$

● ***Ex. 39*** *The circle* $x^2+y^2-4x-8y+16=0$ *rolls up the tangent to it at* $(2+\sqrt{3}, 3)$ *by* 2 *units, assuming the X-axis as horizontal, find the equation of the circle in the new position.*

Sol. Given circle is

$$x^2+y^2-4x-8y+16=0 \qquad ...(i)$$

Let $P\equiv(2+\sqrt{3}, 3)$



Equation of tangent to the circle Eq. (i) at $P(2+\sqrt{3}, 3)$ is

$$(2+\sqrt{3})x+3y-2(x+2+\sqrt{3})-4(y+3)+16=0$$
or $$\sqrt{3}x-y-2\sqrt{3}=0 \qquad ...(ii)$$

Let A and B be the centres of the circles in old and new positions, then

$$B\equiv(2+2\cos 60^\circ, 4+2\sin 60^\circ)$$
($\because AB$ makes an angle 60° with X-axis)
or $$B\equiv(3, 4+\sqrt{3})$$

and radius $= \sqrt{2^2 + 4^2 - 16} = 2$

$\therefore$ Equation of the required circle is

$$(x-3)^2 + (y-4-\sqrt{3})^2 = 2^2$$

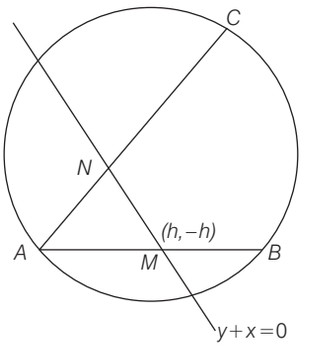
or $$x^2 + y^2 - 6x - 2(4+\sqrt{3})y + 24 + 8\sqrt{3} = 0$$

● ***Ex. 40*** *Find the intervals of the values of 'a' for which the line $y + x = 0$ bisects two chords drawn from a point $\left(\frac{1+\sqrt{2}a}{2}, \frac{1-\sqrt{2}a}{2}\right)$ to the circle*

$2x^2 + 2y^2 - (1+\sqrt{2}\,a)x - (1-\sqrt{2}a)y = 0.$

Sol. The point $A\left(\frac{1+\sqrt{2}a}{2}, \frac{1-\sqrt{2}a}{2}\right)$ lies on the given circle as its coordinate satisfy the equation of the circle. Let AB and AC are two chords drawn from A. Let M and N are the mid-points of AB and AC.

Let coordinate of M be $(h, -h)$ and coordinate of B is (α, β), then



$$h = \frac{\alpha + \frac{1+\sqrt{2}a}{2}}{2}$$

and $$-h = \frac{\beta + \frac{1-\sqrt{2}a}{2}}{2}$$

$\therefore$ $$\alpha = 2h - \frac{1-\sqrt{2}a}{2}$$

and $$\beta = -2h - \frac{1-\sqrt{2}a}{2}$$

Since, $B(\alpha, \beta)$ lies on the given circle, we have

$\Rightarrow$ $$2\left[2h - \frac{1+\sqrt{2}a}{2}\right]^2 + 2\left[-2h - \frac{1-\sqrt{2}a}{2}\right]^2 - (1+\sqrt{2}a)\left[2h - \frac{1+\sqrt{2}a}{2}\right] - (1-\sqrt{2}a)\left[-2h - \frac{1-\sqrt{2}a}{2}\right] = 0$$

$\Rightarrow$ $$16h^2 - 4h(1+\sqrt{2}a) + 4h(1-\sqrt{2}a) + \frac{(1+\sqrt{2}a)^2}{2} + \frac{(1-\sqrt{2}a)^2}{2} - 2h(1+\sqrt{2}a) + \frac{(1+\sqrt{2}a)^2}{2} + 2h(1-\sqrt{2}a) + \frac{(1-\sqrt{2}a)^2}{2} = 0$$

$\Rightarrow$ $$16h^2 - 12\sqrt{2}\,ah + (1+\sqrt{2}a)^2 + (1-\sqrt{2}a)^2 = 0$$

$\Rightarrow$ $$16h^2 - 12\sqrt{2}\,ah + 2 + 4a^2 = 0$$

or $$8h^2 - 6\sqrt{2}\,ah + 1 + 2a^2 = 0$$

Hence, for two real and different values of h, we must have

$$(-6\sqrt{2}a)^2 - 4 \cdot 8(1+2a^2) > 0$$

or $$72a^2 - 32(1+2a^2) > 0$$

$\Rightarrow$ $$8a^2 - 32 > 0$$

$\Rightarrow$ $$a^2 - 4 > 0$$

$$(a+2)(a-2) > 0$$

Hence, the required value of a (from wavy curve)

$$a \in (-\infty, -2) \cup (2, \infty)$$

Aliter : Equation of chord AB whose mid-point is $(h, -h)$ is

$$T = S_1$$

$$2xh - 2yh - (1+\sqrt{2}a)\left(\frac{x+h}{2}\right) - (1-\sqrt{2}a)\left(\frac{y-h}{2}\right) = 2h^2 + 2h^2 - (1+\sqrt{2}a)h + (1-\sqrt{2}a)h$$

$\Rightarrow$ $$4xh - 4yh - (1+\sqrt{2}a)(x+h) - (1-\sqrt{2}a)(y-h) = 8h^2 - 2(1+\sqrt{2}a)h + 2(1-\sqrt{2}a)h$$

$\Rightarrow$ $$x[4h - (1+\sqrt{2}a)] - y[4h + (1-\sqrt{2}a)] - h(1+\sqrt{2}a) + h(1-\sqrt{2}a) = 8h^2 - 2(1+\sqrt{2}a)h + 2(1-\sqrt{2}a)h$$

or $$8h^2 - (1+\sqrt{2}a)h + (1-\sqrt{2}a)h - x[4h - (1+\sqrt{2}a)] + y[4h + (1-\sqrt{2}a)] = 0$$

It passes through $A\left(\frac{1+\sqrt{2}a}{2}, \frac{1-\sqrt{2}a}{2}\right)$, then

$$8h^2 - 2\sqrt{2}ah - \left(\frac{1+\sqrt{2}a}{2}\right)[4h - (1+\sqrt{2}a)] + \left(\frac{1-\sqrt{2}a}{2}\right)[4h + (1-\sqrt{2}a)] = 0$$

or $$8h^2 - 2\sqrt{2}ah - 2h(1+\sqrt{2}a) + \frac{(1+\sqrt{2}a)^2}{2} + 2h(1-\sqrt{2}a) + \frac{(1-\sqrt{2}a)^2}{2} = 0$$

or $$8h^2 - 6\sqrt{2}ah + 1 + 2a^2 = 0$$

Hence, for two real and different values of h, we must have

$$(-6\sqrt{2}a)^2 - 4\cdot 8\cdot(1+2a^2) > 0$$

or $$a^2 - 4 > 0$$

$$\therefore \quad (a+2)(a-2) > 0$$

$$\therefore \quad a \in (-\infty, -2) \cup (2, \infty)$$

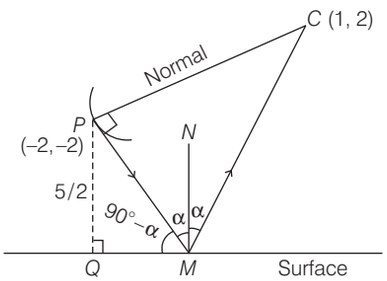
● ***Ex. 41*** *A ball moving around the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ in anti-clockwise direction leaves it tangentially at the point $P(-2, -2)$. After getting reflected from a straight line, it passes through the centre of the circle. Find the equation of the straight line if its perpendicular distance from P is 5/2. You can assume that the angle of incidence is equal to the angle of reflection.*

Sol. Radius of the circle $= CP = \sqrt{9+16} = 5$

Let the equation is of surface is $y = mx + c$

given $$PQ = \frac{5}{2}$$

$$\therefore \quad \frac{-2m + 2 + c}{\sqrt{(1+m^2)}} = \pm\frac{5}{2} \qquad \ldots(i)$$



Tangent at P strikes it at the point M and after reflection passes through the centre $C(1, 2)$.

Let MN be the normal at M.

$$\angle PMN = \angle NMC = \alpha$$

In ΔPCM, $$\tan 2\alpha = \frac{PC}{PM}$$

$$\Rightarrow \quad \tan 2\alpha = \frac{5}{PM}$$

$$\Rightarrow \quad PM = 5\cot 2\alpha \qquad \ldots(ii)$$

and in ΔPQM

$$\sin(90°-\alpha) = \frac{5/2}{PM}$$

$$\therefore \quad PM = \frac{5}{2\cos\alpha} \qquad \ldots(iii)$$

From Eqs. (ii) and (iii), $5\cot 2\alpha = \dfrac{5}{2\cos\alpha}$

$$\Rightarrow \quad 2\cot 2\alpha\cos\alpha = 1$$

$$\Rightarrow \quad \frac{2\cos 2\alpha}{\sin 2\alpha}.\cos\alpha = 1$$

$$\Rightarrow \quad \frac{2(1-2\sin^2\alpha)\cos\alpha}{2\sin\alpha\cos\alpha} = 1$$

$$\Rightarrow \quad 1 - 2\sin^2\alpha = \sin\alpha$$

$$\Rightarrow \quad 2\sin^2\alpha + \sin\alpha - 1 = 0$$

$$\Rightarrow \quad (2\sin\alpha - 1)(\sin\alpha + 1) = 0$$

$$\Rightarrow \quad \sin\alpha \neq -1$$

$$\therefore \quad \sin\alpha = \frac{1}{2}$$

$$\therefore \quad \alpha = 30°$$

Tangent at $P(-2,-2)$ is

$$-2x - 2y - (x-2) - 2(y-2) - 20 = 0$$

$$\Rightarrow \quad 3x + 4y + 14 = 0$$

Slope of $PM = -3/4$

$\because$ $\angle PMQ = 90° - \alpha = 90° - 30° = 60°$

$$\therefore \quad \tan 60° = \left|\frac{m + 3/4}{1 - 3m/4}\right|, \quad \sqrt{3} = \frac{4m+3}{4-3m}$$

$$\therefore \quad m = \frac{4\sqrt{3}-3}{4+3\sqrt{3}}$$

From Eq. (i) $\pm\dfrac{5}{2} = \dfrac{2(1-m)+c}{\sqrt{1+m^2}}$

we get $$c = \frac{11+2\sqrt{3}}{4+3\sqrt{3}} \quad \text{or} \quad \frac{-39+2\sqrt{3}}{4+3\sqrt{3}}$$

c being intercept on Y-axis made by surface is clearly –ve. Hence, the required line is

$$y = \left(\frac{4\sqrt{3}-3}{4+3\sqrt{3}}\right)x + \left(\frac{-39+2\sqrt{3}}{4+3\sqrt{3}}\right)$$

$$\Rightarrow \quad (4\sqrt{3}-3)x - (4+3\sqrt{3})y - (39-2\sqrt{3}) = 0.$$

● ***Ex. 42*** *Find the limiting points of the circles $(x^2 + y^2 + 2gx + c) + \lambda(x^2 + y^2 + 2fy + d) = 0$ and show that the square of the distance between them is*

$$\frac{(c-d)^2 - 4f^2g^2 + 4cf^2 + 4dg^2}{f^2 + g^2}$$

Sol. The given circles are

$$(x^2 + y^2 + 2gx + c) + \lambda(x^2 + y^2 + 2fy + d) = 0$$

$$\Rightarrow \quad x^2 + y^2 + \frac{2g}{1+\lambda}x + \frac{2f\lambda}{1+\lambda}y + \frac{(c+\lambda d)}{1+\lambda} = 0$$

Centre of the circle $\left(\dfrac{-g}{1+\lambda}, \dfrac{-f\lambda}{1+\lambda}\right)$

Equating the radius of this circle to zero, we get

$$\frac{g^2}{(1+\lambda)^2} + \frac{f^2\lambda^2}{(1+\lambda)^2} - \frac{(c+\lambda d)}{(1+\lambda)} = 0$$

$$\Rightarrow \quad g^2 + f^2\lambda^2 - (c + \lambda d)(1 + \lambda) = 0$$

$$\Rightarrow \quad (f^2 - d)\lambda^2 - (c + d)\lambda + g^2 - c = 0$$

Let the roots be λ_1 and λ_2

then $\lambda_1 + \lambda_2 = \dfrac{(c+d)}{(f^2-d)}, \lambda_1\lambda_2 = \dfrac{g^2-c}{f^2-d}$

$$\therefore \quad (\lambda_1 - \lambda_2) = \sqrt{(\lambda_1 + \lambda_2)^2 - 4\lambda_1\lambda_2}$$

$$= \sqrt{\frac{(c+d)^2}{(f^2-d)^2} - \frac{4(g^2-c)}{(f^2-d)}}$$

$$= \frac{\sqrt{(c+d)^2 - 4f^2g^2 + 4cf^2 + 4dg^2}}{(f^2-d)} \quad ...(i)$$

$$\therefore \quad \lambda_1 = \frac{(c+d) + \sqrt{(c-d)^2 - 4f^2g^2 + 4cf^2 + 4dg^2}}{2(f^2-d)} \quad ...(ii)$$

and $\lambda_2 = \dfrac{(c+d) - \sqrt{\{(c-d)^2 - 4f^2g^2 + 4cf^2 + 4dg^2\}}}{2(f^2-d)}$ ...(iii)

Hence, limiting points are

$$\left(\frac{-g}{1+\lambda_1}, \frac{-f\lambda_1}{1+\lambda_1}\right) \quad \text{and} \quad \left(\frac{-g}{1+\lambda_2}, \frac{-f\lambda_2}{1+\lambda_2}\right)$$

Substituting the values of λ_1 and λ_2 from Eqs. (ii) and (iii) square of the distance between limiting points

$$= \left(\frac{-g}{1+\lambda_1} + \frac{g}{1+\lambda_2}\right)^2 + \left(\frac{-f\lambda_1}{1+\lambda_1} + \frac{f\lambda_2}{1+\lambda_2}\right)^2$$

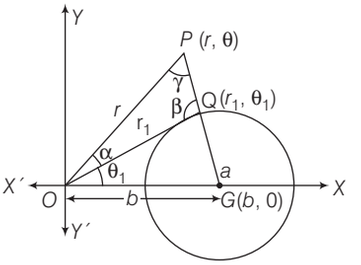
$$= \frac{(g^2+f^2)(\lambda_1-\lambda_2)^2}{[1+(\lambda_1+\lambda_2)+\lambda_1\lambda_2]^2}$$

$$= \frac{(g^2+f^2)\dfrac{\{(c-d)^2 - 4f^2g^2 + 4cf^2 + 4dg^2\}}{(f^2-d)^2}}{\left(\dfrac{g^2+f^2}{f^2-d}\right)^2}$$

$$\Rightarrow \quad \frac{[(c-d)^2 - 4f^2g^2 + 4cf^2 + 4dg^2]}{(g^2+f^2)}$$

● ***Ex. 43*** *One vertex of a triangle of given species is fixed and another moves along circumference of a fixed circle. Prove that the locus of the remaining vertex is a circle and find its radius.*

Sol. Let OPQ be a triangle of given species. Then the angles α, β, γ will be fixed.



Let the polar coordinates of Q be (r_1, θ_1), we have to find the locus of $P(r, \theta)$. In $\angle OCQ$

$$\cos\theta_1 = \frac{r_1^2 + b^2 - a^2}{2r_1 b} \quad ...(i)$$

$$\because \quad \theta = \alpha + \theta_1, \quad \therefore \ \theta_1 = \theta - \alpha \quad ...(ii)$$

using sine rule in ΔOPQ

$$\frac{r}{\sin\beta} = \frac{r_1}{\sin\gamma}$$

$$\therefore \quad r_1 = \frac{r\sin\gamma}{\sin\beta} \quad ...(iii)$$

Substituting the values of θ_1 and r_1 from Eqs. (ii)and (iii) in Eq. (i)

$$2b\frac{r\sin\gamma}{\sin\beta}\cos(\theta-\alpha) = \frac{r^2\sin^2\gamma}{\sin^2\beta} + b^2 - a^2$$

$$\Rightarrow \quad \frac{a^2\sin^2\beta}{\sin^2\gamma} = r^2 + \frac{b^2\sin^2\beta}{\sin^2\gamma} - 2b\frac{r\sin\beta}{\sin\gamma}\cos(\theta-\alpha)$$

This is an equation of circle in polar form with radius $\dfrac{\sin\beta}{\sin\gamma}$.

Circle Exercise 1 :
Single Option Correct Type Questions

■ This section contains **30 multiple choice questions.** Each question has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct

1. The sum of the square of the length of the chord intercepted by the line $x+y=n, n\in N$ on the circle $x^2+y^2=4$ is

(a) 11 (b) 22
(c) 33 (d) None of these

2. Tangents are drawn to the circle $x^2+y^2=50$ from a point 'P' lying on the X-axis. These tangents meet the Y-axis at points 'P_1' and 'P_2'. Possible coordinates of 'P' so that area of triangle PP_1P_2 is minimum, is

(a) $(10,0)$ (b) $(10\sqrt{2},0)$ (c) $(-10\sqrt{2},0)$ (d) $(10\sqrt{3},0)$

3. Equation of chord AB of circle $x^2+y^2=2$ passing through $P(2,2)$ such that $\frac{PB}{PA}=3$, is given by

(a) $x=3y$ (b) $x=y$
(c) $y-2=\sqrt{3}(x-2)$ (d) $y-3=\sqrt{3}(x-1)$

4. If r_1 and r_2 are the radii of smallest and largest circles which passes through (5, 6) and touches the circle $(x-2)^2+y^2=4$, then r_1r_2 is

(a) $\frac{4}{41}$ (b) $\frac{41}{4}$ (c) $\frac{5}{41}$ (d) $\frac{41}{5}$

5. Equation of circle $S(x,y)=0, (S(2,3)=16)$ which touches the line $3x+4y-7=0$ at $(1,1)$ is given by

(a) $x^2+y^2+x+2y-5=0$
(b) $x^2+y^2+2x+2y-7=0$
(c) $x^2+y^2+4x-6y+13=0$
(d) $x^2+y^2-4x+6y-7=0$

6. If $P(2,8)$ is an interior point of a circle $x^2+y^2-2x+4y-\lambda=0$ which neither touches nor intersects the axes, then set for λ is

(a) $(-\infty,-1)$ (b) $(-\infty,-4)$
(c) $(96,\infty)$ (d) ϕ

7. The difference between the radii of the largest and smallest circles which have their centre on the circumference of the circle $x^2+y^2+2x+4y-4=0$ and pass through the point (a,b) lying outside the given circle is

(a) 6 (b) $\sqrt{(a+1)^2+(b+2)^2}$
(c) 3 (d) $\sqrt{(a+1)^2+(b+2)^2}-3$

8. The number of rational point(s) (a point (a,b) is rational, if a and b both are rational numbers) on the circumference of a circle having centre (π, e) is

(a) atmost one (b) atleast two
(c) exactly two (d) infinite

9. Three sides of a triangle have the equations $L_r\equiv y-m_rx-c_r=0; r=1,2,3$. Then $\lambda L_2L_3+\mu L_3L_1+\nu L_1L_2=0$, where $\lambda\neq 0,\mu\neq 0,\nu\neq 0$ is the equation of circumcircle of triangle, if

(a) $\lambda(m_2+m_3)+\mu(m_3+m_1)+\nu(m_1+m_2)=0$
(b) $\lambda(m_2m_3-1)+\mu(m_3m_1-1)+\nu(m_1m_2-1)=0$
(c) Both (a) and (b)
(d) None of the above

10. $f(x,y)\equiv x^2+y^2+2ax+2by+c=0$ represent a circle. If $f(x,0)=0$ has equal roots, each being 2 and $f(0,y)=0$ has 2 and 3 as its roots, then the centre of the circle is

(a) $\left(2,\frac{5}{2}\right)$ (b) Data are not consistent
(c) $\left(-2,-\frac{5}{2}\right)$ (d) Data are inconsistent

11. If $(1+\alpha x)^n=1+8x+24x^2+\ldots$ and a line through $P(\alpha,n)$ cuts the circle $x^2+y^2=4$ in A and B, then $PA.PB$ is equal to

(a) 4 (b) 8 (c) 16 (d) 32

12. A region in the xy-plane is bounded by the curve $y=\sqrt{(25-x^2)}$ and the line $y=0$. If the point $(a,a+1)$ lies in the interior of the region, then

(a) $a\in(-4,3)$ (b) $a\in(-\infty,-1)\cup(3,\infty)$
(c) $a\in(-1,3)$ (d) None of these

13. $S(x,y)=0$ represents a circle. The equation $S(x,2)=0$ gives two identical solutions $x=1$ and the equation $S(1,y)=0$ gives two distinct solutions $y=0,2$, then the equation of the circle is

(a) $x^2+y^2+2x-2y+1=0$ (b) $x^2+y^2-2x+2y+1=0$
(c) $x^2+y^2-2x-2y-1=0$ (d) $x^2+y^2-2x-2y+1=0$

14. Let $0<\alpha<\frac{\pi}{2}$ be a fixed angle. If $P=(\cos\theta,\sin\theta)$ and $Q=(\cos(\alpha-\theta),\sin(\alpha-\theta))$, then Q is obtained from P by

(a) clockwise rotation around origin through an angle α
(b) anti-clockwise rotation around origin through an angle α
(c) reflection in the line through origin with slope $\tan\alpha$
(d) reflection in the line through origin with slope $\tan\left(\frac{\alpha}{2}\right)$

15. The number of points (x, y) having integral coordinates satisfying the condition $x^2 + y^2 < 25$ is
(a) 69 (b) 80 (c) 81 (d) 77

16. The point $([P+1], [P])$, (where $[.]$ denotes the greatest integer function) lying inside the region bounded by the circle $x^2 + y^2 - 2x - 15 = 0$ and $x^2 + y^2 - 2x - 7 = 0$, then
(a) $P \in [-1, 0) \cup [0, 1) \cup [1, 2)$ (b) $P \in [-1, 2) - \{0, 1\}$
(c) $P \in (-1, 2)$ (d) None of these

17. A point P lies inside the circles $x^2 + y^2 - 4 = 0$ and $x^2 + y^2 - 8x + 7 = 0$. The point P starts moving under the conditions that its path encloses greatest possible area and it is at a fixed distance from any arbitrarily chosen fixed point in its region. The locus of P is
(a) $4x^2 + 4y^2 - 12x + 1 = 0$ (b) $4x^2 + 4y^2 + 12x - 1 = 0$
(c) $x^2 + y^2 - 3x - 2 = 0$ (d) $x^2 + y^2 - 3x + 2 = 0$

18. The set of values of 'c' so that the equations $y = |x| + c$ and $x^2 + y^2 - 8|x| - 9 = 0$ have no solution is
(a) $(-\infty, -3) \cup (3, \infty)$ (b) $(-3, 3)$
(c) $(-\infty, -5\sqrt{2}) \cup (5\sqrt{2}, \infty)$ (d) $(5\sqrt{2} - 4, \infty)$

19. If a line segment $AM = a$ moves in the plane XOY remaining parallel to OX so that the left end point A slides along the circle $x^2 + y^2 = a^2$, the locus of M is
(a) $x^2 + y^2 = 4a^2$ (b) $x^2 + y^2 = 2ax$
(c) $x^2 + y^2 = 2ay$ (d) $x^2 + y^2 - 2ax - 2ay = 0$

20. The four points of intersection of the lines $(2x - y + 1)(x - 2y + 3) = 0$ with the axes lie on a circle whose centre is at the point
(a) $\left(-\frac{7}{4}, \frac{5}{4}\right)$ (b) $\left(\frac{3}{4}, \frac{5}{4}\right)$ (c) $\left(\frac{9}{4}, \frac{5}{4}\right)$ (d) $\left(0, \frac{5}{4}\right)$

21. The number of integral values of λ for which $x^2 + y^2 + \lambda x + (1 - \lambda)y + 5 = 0$ is the equation of a circle whose radius cannot exceed 5, is
(a) 14 (b) 18 (c) 16 (d) None of these

22. Let $\phi(x, y) = 0$ be the equation of a circle. If $\phi(0, \lambda) = 0$ has equal roots $\lambda = 2, 2$ and $\phi(\lambda, 0) = 0$ has roots $\lambda = \frac{4}{5}, 5$, then the centre of the circle is
(a) $\left(2, \frac{29}{10}\right)$ (b) $\left(\frac{29}{10}, 2\right)$ (c) $\left(-2, \frac{29}{10}\right)$ (d) None of these

23. The locus of the point of intersection of the tangents to the circle $x = r\cos\theta, y = r\sin\theta$ at points whose parametric angles differ by $\frac{\pi}{3}$ is
(a) $x^2 + y^2 = 4(2 - \sqrt{3})r^2$ (b) $3(x^2 + y^2) = 1$
(c) $x^2 + y^2 = (2 - \sqrt{3})r^2$ (d) $3(x^2 + y^2) = 4r^2$

24. One of the diameter of the circle circumscribing the rectangle $ABCD$ is $4y = x + 7$. If A and B are the points $(-3, 4)$ and $(5, 4)$ respectively, then the area of the rectangle is
(a) 16 sq units (b) 24 sq units
(c) 32 sq units (d) None of these

25. A, B, C and D are the points of intersection with the coordinate axes of the lines $ax + by = ab$ and $bx + ay = ab$, then
(a) A, B, C, D are concyclic
(b) A, B, C, D form a parallelogram
(c) A, B, C, D form a rhombus
(d) None of the above

26. α, β and γ are parametric angles of three points P, Q and R respectively, on the circle $x^2 + y^2 = 1$ and A is the point $(-1, 0)$. If the lengths of the chords AP, AQ and AR are in GP, then $\cos\left(\frac{\alpha}{2}\right), \cos\left(\frac{\beta}{2}\right)$ and $\cos\left(\frac{\gamma}{2}\right)$ are in
(a) AP (b) GP
(c) HP (d) None of these

27. The equation of the circle passing through (2, 0) and (0, 4) and having the minimum radius is
(a) $x^2 + y^2 = 20$
(b) $x^2 + y^2 - 2x - 4y = 0$
(c) $(x^2 + y^2 - 4) + \lambda(x^2 + y^2 - 16) = 0$
(d) None of the above

28. A circle of radius unity is centred at the origin. Two particles start moving at the same time from the point (1, 0) and move around the circle in opposite direction. One of the particle moves anticlockwise with constant speed v and the other moves clockwise with constant speed $3v$. After leaving (1, 0), the two particles meet first at a point P and continue until they meet next at point Q. The coordinates of the point Q are
(a) (1, 0) (b) (0, 1) (c) (−1, 0) (d) (0, −1)

29. The circle $x^2 + y^2 = 4$ cuts the line joining the points $A(1, 0)$ and $B(3, 4)$ in two points P and Q. Let $\frac{BP}{PA} = \alpha$ and $\frac{BQ}{QA} = \beta$, then α and β are roots of the quadratic equation
(a) $x^2 + 2x + 7 = 0$ (b) $3x^2 + 2x - 21 = 0$
(c) $2x^2 + 3x - 27 = 0$ (d) None of these

30. The locus of the mid-points of the chords of the circle $x^2 + y^2 + 4x - 6y - 12 = 0$ which subtend an angle of $\frac{\pi}{3}$ radians at its circumference is
(a) $(x + 2)^2 + (y - 3)^2 = 6.25$ (b) $(x - 2)^2 + (y + 3)^2 = 6.25$
(c) $(x + 2)^2 + (y - 3)^2 = 18.75$ (d) $(x + 2)^2 + (y + 3)^2 = 18.75$

Circle Exercise 2 :
More than One Correct Option Type Questions

This section contains **15 multiple choice questions**. Each question has four choices (a), (b), (c) and (d) out of which **MORE THAN ONE** may be correct.

31. If OA and OB are two perpendicular chords of the circle $r = a\cos\theta + b\sin\theta$ passing through origin, then the locus of the mid-point of AB is

(a) $x^2 + y^2 = a + b$ (b) $x = \frac{a}{2}$

(c) $x^2 - y^2 = a^2 - b^2$ (d) $y = \frac{b}{2}$

32. If A and B are two points on the circle $x^2 + y^2 - 4x + 6y - 3 = 0$ which are farthest and nearest respectively, from the point (7, 2), then

(a) $A \equiv (2 - 2\sqrt{2}, -3 - 2\sqrt{2})$

(b) $A \equiv (2 + 2\sqrt{2}, -3 + 2\sqrt{2})$

(c) $B \equiv (2 + 2\sqrt{2}, -3 + 2\sqrt{2})$

(d) $B \equiv (2 - 2\sqrt{2}, -3 - 2\sqrt{2})$

33. If the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ cuts each of the circles $x^2 + y^2 - 4 = 0$, $x^2 + y^2 - 6x - 8y + 10 = 0$ and $x^2 + y^2 + 2x - 4y - 2 = 0$ at the extremities of a diameter, then

(a) $c = -4$ (b) $g + f = c - 1$

(c) $g^2 + f^2 - c = 17$ (d) $gf = 6$

34. The possible value of $\lambda(\lambda > 0)$ such that the angle between the pair of tangents from point $(\lambda, 0)$ to the circle $x^2 + y^2 = 4$ lies in interval $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$ is

(a) $\left(\frac{4}{\sqrt{3}}, 2\sqrt{2}\right)$ (b) $(0, \sqrt{2})$

(c) (1,2) (d) $\left(\frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right)$

35. If a chord of the circle $x^2 + y^2 - 4x - 2y - c = 0$ is trisected at the points $\left(\frac{1}{3}, \frac{1}{3}\right)$ and $\left(\frac{8}{3}, \frac{8}{3}\right)$, then

(a) $c = 10$ (b) $c = 20$

(c) $c = 15$ (d) $c^2 - 40c + 400 = 0$

36. From the point $A(0,3)$ on $x^2 + 4x + (y-3)^2 = 0$, a chord AB is drawn and extended to a point M, such that $AM = 2AB$. An equation of the locus of M is

(a) $x^2 + 6x + (y-2)^2 = 0$

(b) $x^2 + 8x + (y-3)^2 = 0$

(c) $x^2 + y^2 + 8x - 6y + 9 = 0$

(d) $x^2 + y^2 + 6x - 4y + 4 = 0$

37. An equation of a circle touching the axes of coordinates and the line $x\cos\alpha + y\sin\alpha = 2$ can be

(a) $x^2 + y^2 - 2gx - 2gy + g^2 = 0$, where $g = \frac{2}{(\cos\alpha + \sin\alpha + 1)}$

(b) $x^2 + y^2 - 2gx - 2gy + g^2 = 0$, where $g = \frac{2}{(\cos\alpha + \sin\alpha - 1)}$

(c) $x^2 + y^2 - 2gx + 2gy + g^2 = 0$, where $g = \frac{2}{(\cos\alpha - \sin\alpha + 1)}$

(d) $x^2 + y^2 - 2gx + 2gy + g^2 = 0$, where $g = \frac{2}{(\cos\alpha - \sin\alpha - 1)}$

38. If α is the angle subtended at $P(x_1, y_1)$ by the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, then

(a) $\cot\alpha = \frac{\sqrt{S_1}}{\sqrt{(g^2 + f^2 - c)}}$ (b) $\cot\frac{\alpha}{2} = \frac{\sqrt{S_1}}{\sqrt{(g^2 + f^2 - c)}}$

(c) $\tan\alpha = \frac{2\sqrt{(g^2 + f^2 - c)}}{\sqrt{S_1}}$ (d) $\alpha = 2\tan^{-1}\left(\frac{\sqrt{(g^2 + f^2 - c)}}{\sqrt{S_1}}\right)$

39. The equation of the circle which touches the axis of coordinates and the line $\frac{x}{3} + \frac{y}{4} = 1$ and whose centre lies in the first quadrant is $x^2 + y^2 - 2\lambda x - 2\lambda y + \lambda^2 = 0$, then λ is equal to

(a) 1 (b) 2

(c) 3 (d) 6

40. If P is a point on the circle $x^2 + y^2 = 9$, Q is a point on the line $7x + y + 3 = 0$, and the line $x - y + 1 = 0$, is the perpendicular bisector of PQ, then the coordinates of P are

(a) (3, 0) (b) $\left(\frac{72}{25}, -\frac{21}{25}\right)$

(c) (0, 3) (d) $\left(-\frac{72}{25}, \frac{21}{25}\right)$

41. If a circle passes through the point $\left(3, \sqrt{\frac{7}{2}}\right)$ and touches $x + y = 1$ and $x - y = 1$, then the centre of the circle is

(a) (4,0) (b) (4, 2) (c) (6, 0) (d) (7, 9)

42. The equation of a circle C_1 is $x^2 + y^2 = 4$. The locus of the intersection of orthogonal tangents to the circle is the curve C_2 and the locus of the intersection of perpendicular tangents to the curve C_2 is the curve C_3. Then,

(a) C_3 is a circle

(b) the area enclosed by the curve C_3 is 8π

(c) C_2 and C_3 are circles with the same centre

(d) None of the above

43. The equation of a tangent to the circle $x^2+y^2=25$ passing through $(-2,11)$ is
(a) $4x+3y=25$ (b) $3x+4y=38$
(c) $24x-7y+125=0$ (d) $7x+24y=230$

44. Consider the circles
$C_1 \equiv x^2+y^2-2x-4y-4=0$ and
$C_2 \equiv x^2+y^2+2x+4y+4=0$
and the line $L \equiv x+2y+2=0$, then
(a) L is the radical axis of C_1 and C_2
(b) L is the common tangent of C_1 and C_2
(c) L is the common chord of C_1 and C_2
(d) L is perpendicular to the line joining centres of C_1 and C_2

45. A square is inscribed in the circle $x^2+y^2-10x-6y+30=0$. One side of the square is parallel to $y=x+3$, then one vertex of the square is
(a) (3, 3) (b) (7,3)
(c) $(6,3-\sqrt{3})$ (d) $(6,3+\sqrt{3})$

Circle Exercise 3 : Paragraph Based Questions

This section contains **7 paragraphs** based upon each of the paragraph **3 multiple choice questions** have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Paragraph I

(Q. Nos. 46 to 48)

Consider the circle $S: x^2+y^2-4x-1=0$ and the line $L: y=3x-1$. If the line L cuts the circle at A and B.

46. Length of the chord AB is
(a) $\sqrt{5}$ (b) $\sqrt{10}$ (c) $2\sqrt{5}$ (d) $5\sqrt{2}$

47. The angle subtended by the chord AB is the minor arc of S is
(a) $\frac{\pi}{4}$ (b) $\frac{2\pi}{3}$ (c) $\frac{3\pi}{4}$ (d) $\frac{5\pi}{6}$

48. Acute angle between the line L and the circle S is
(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

Paragraph II

(Q. Nos. 49 to 51)

P is a variable point on the line $L=0$. Tangents are drawn to the circle $x^2+y^2=4$ from P to touch it at Q and R. The parallelogram $PQSR$ is completed.

49. If $L \equiv 2x+y-6=0$, then the locus of the circumcenter of ΔPQR is
(a) $2x-y=4$ (b) $2x+y=3$
(c) $x-2y=4$ (d) $x+2y=3$

50. If $P \equiv (6,8)$, then area of ΔQRS is $\frac{192}{25}\sqrt{\lambda}$ sq units. The value of λ is
(a) 2 (b) 3 (c) 5 (d) 6

51. If $P \equiv (3,4)$, then the coordinates of S are
(a) $\left(-\frac{46}{25},\frac{63}{25}\right)$ (b) $\left(-\frac{51}{25},-\frac{68}{25}\right)$
(c) $\left(-\frac{46}{25},\frac{68}{25}\right)$ (d) $\left(-\frac{68}{25},\frac{51}{25}\right)$

Paragraph III

(Q. Nos. 52 to 54)

Equation of the circumcircle of a triangle formed by the lines $L_1=0$, $L_2=0$ and $L_3=0$ can be written as $L_1L_2+\lambda L_2L_3+\mu L_3L_1=0$, where λ and μ are such that coefficient of x^2 = coefficient of y^2 and coefficient of $xy=0$.

52. $L_1{L_2}^2+\lambda L_2L_3^2+\mu L_3L_1^2=0$ represents
(a) a curve passing through point of intersection of $L_1=0$, $L_2=0$ and $L_3=0$
(b) a circle is coefficient of $x^2=$ coefficient of y^2 and coefficient of $xy=0$
(c) a parabola
(d) pair of straight lines

53. $L_1=0$, $L_2=0$ be the distinct parallel lines, $L_3=0$, $L_4=0$ be two other distinct parallel lines which are not parallel to $L_1=0$. The equation of a circle passing through the vertices of the parallelogram formed must be of the form
(a) $\lambda L_1L_4+\mu L_2L_3=0$ (b) $\lambda L_1L_3+\mu L_2L_4=0$
(c) $\lambda L_1L_2+\mu L_3L_4=0$ (d) $\lambda L_1^2L_3+\mu L_2^2L_4=0$

54. If $L_1{L_2}+\lambda L_2L_3+\mu L_3L_1=0$ is such that $\mu=0$ and λ is non-zero, then it represents
(a) a parabola
(b) a pair of straight lines
(c) a circle
(d) an ellipse

Paragraph IV

(Q. Nos. 55 to 57)

Given two circles intersecting orthogonally having the length of common chord $\frac{24}{5}$ unit. The radius of one of the circles is 3 units.

55. If radius of other circle is λ units, then λ is
(a) 2 (b) 4 (c) 5 (d) 6

56. If angle between direct common tangents is 2θ, then $\sin 2\theta$ is
(a) $\frac{4}{5}$ (b) $\frac{4\sqrt{6}}{25}$ (c) $\frac{12}{25}$ (d) $\frac{24}{25}$

57. If length of direct common tangent is λ units, then λ^2 is
(a) 12 (b) 24 (c) 36 (d) 48

Paragraph V

(Q. Nos. 58 to 60)

Consider the two circles $C_1 : x^2 + y^2 = a^2$ and $C_2 : x^2 + y^2 = b^2$ $(a > b)$. Let A be a fixed point on the circle C_1, say $A(a, 0)$ and B be a variable point on the circle C_2. The line BA meets the circle C_2 again at C. 'O' being the origin.

58. If $(OA)^2 + (OB)^2 + (BC)^2 = \lambda$, then $\lambda \in$
(a) $[5b^2 - 3a^2, 5b^2 + a^2]$ (b) $[4b^2, 4b^2 + a^2]$
(c) $[4a^2, 4b^2]$ (d) $[5b^2 - 3a^2, 5b^2 + 3a^2]$

59. The locus of the mid-point of AB is
(a) $\left(x - \frac{a}{2}\right)^2 + y^2 = \frac{b^2}{4}$ (b) $\left(x - \frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4}$
(c) $\left(x - \frac{b}{2}\right)^2 + y^2 = \frac{a^2}{4}$ (d) $\left(x - \frac{b}{2}\right)^2 + y^2 = \frac{b^2}{4}$

60. If $(BC)^2$ is maximum, then the locus of the mid-point of AB is
(a) $x^2 + y^2 = b^2$ (b) $x^2 + y^2 = (a + b)^2$
(c) $x^2 + y^2 = (a - b)^2$ (d) None of these

Paragraph VI

(Q. Nos. 61 to 63)

Two variable chords AB and BC of a circle $x^2 + y^2 = a^2$ are such that $AB = BC = a$, M and N are the mid-points of AB and BC respectively such that line joining MN intersect the circle at P and Q, where P is closer to AB and O is the centre of the circle.

61. $\angle OAB$ is
(a) 15° (b) 30°
(c) 45° (d) 60°

62. Angle between tangents at A and C is
(a) 60° (b) 90°
(c) 120° (d) 150°

63. Locus of point of intersection of tangents at A and C is
(a) $x^2 + y^2 = a^2$ (b) $x^2 + y^2 = 2a^2$
(c) $x^2 + y^2 = 4a^2$ (d) $x^2 + y^2 = 8a^2$

Paragraph VII

(Q. Nos. 64 to 66)

t_1, t_2, t_3 are lengths of tangents drawn from a point (h, k) to the circles $x^2 + y^2 = 4$, $x^2 + y^2 - 4x = 0$ and $x^2 + y^2 - 4y = 0$ respectively further, $t_1^4 = t_2^2\, t_3^2 + 16$. Locus of the point (h, k) consist of a straight line L_1 and a circle C_1 passing through origin. A circle C_2, which is equal to circle C_1 is drawn touching the line L_1 and the circle C_1 externally.

64. Equation of L_1 is
(a) $x + y = 0$ (b) $x - y = 0$
(c) $2x + y = 0$ (d) $x + 2y = 0$

65. Equation of C_1 is
(a) $x^2 + y^2 - x - y = 0$ (b) $x^2 + y^2 - 2x + y = 0$
(c) $x^2 + y^2 - x + 2y = 0$ (d) $x^2 + y^2 - 2x - 2y = 0$

66. The distance between the centres of C_1 and C_2 is
(a) $\sqrt{2}$ (b) 2
(c) $2\sqrt{2}$ (d) 4

Circle Exercise 4 : Single Integer Answer Type Questions

- This section contains **10 questions**. The answer to each question is a **single digit integer**, ranging from 0 to 9 (both inclusive).

67. The point (1, 4) lies inside the circle $x^2 + y^2 - 6x - 10y + \lambda = 0$. If the circle neither touches nor cuts the axes, then the difference between the maximum and the minimum possible values of λ is

68. Consider the family of circles $x^2 + y^2 - 2x - 2\lambda y - 8 = 0$ passing through two fixed points A and B. Then the distance between the points A and B is

69. If $C_1 : x^2 + y^2 = (3+2\sqrt{2})^2$ be a circle and PA and PB are pair of tangents on C_1, where P is any point on the director circle of C_1, then the radius of the smallest circle which touches C_1 externally and also the two tangents PA and PB, is

70. If a circle $S(x, y) = 0$ touches the point $(2,3)$ of the line $x + y = 5$ and $S(1,2) = 0$, then radius of such circle is $\frac{1}{\sqrt{\lambda}}$ units, then the value of λ^2 is.

71. If real numbers x and y satisfy $(x+5)^2 + (y-12)^2 = 196$, then the maximum value of $(x^2 + y^2)^{\frac{1}{3}}$ is

72. If the equation of circle circumscribing the quadrilateral formed by the lines in order are $2x + 3y = 2$, $3x - 2y = 3$, $x + 2y = 3$ and $2x - y = 1$ is given by $x^2 + y^2 + \lambda x + \mu y + \nu = 0$. Then the value of $|\lambda + 2\mu + \nu|$ is

73. A circle $x^2 + y^2 + 4x - 2\sqrt{2}y + c = 0$ is the director circle of the circle C_1 and C_1 is the director circle of circle C_2 and so on. If the sum of radii of all these circles is 2 and if $c = \lambda\sqrt{2}$, then the value of λ is

74. If the area bounded by the circles $x^2 + y^2 = r^2$, $r = 1, 2$ and the rays given by $2x^2 - 3xy - 2y^2 = 0$, $y > 0$ is $\frac{\lambda\pi}{4}$ sq units, then the value of λ is

75. The length of a common internal tangent of two circles is 5 and that of a common external tangent is 13. If the product of the radii of two circles is λ, then the value of $\frac{\lambda}{4}$ is

76. Consider a circles S with centre at the origin and radius 4. Four circles A, B, C and D each with radius unity and centres $(-3,0)$, $(-1,0)$, $(1,0)$ and $(3,0)$ respectively are drawn. A chord PQ of the circle S touches the circle B and passes through the centre of the circle C. If the length of this chord can be expressed as $\sqrt{\lambda}$, then the value of $\frac{\lambda}{9}$ is

Circle Exercise 5 : Matching Type Questions

- This section contains **4 questions**. Questions 77 and 78 have four statements (A, B, C and D) given in **Column I** and four statements (p, q, r and s) in **Column II**, and questions 79 and 80 have three statements (A, B and C) given in **Column I** and five statements (p, q, r, s and t) in **Column II**. Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II**.

77. Consider the circles $S_1 : x^2 + y^2 - 4x - 6y + 12 = 0$ and $S_2 : (x-5)^2 + (y-6)^2 = r^2 > 1$

	Column I		Column II
(A)	S_1 and S_2 touch internally, then $(r-1)^2$ is divisible by	(p)	3
(B)	S_1 and S_2 touch externally, then $r^2 + 2r + 3$ is divisible by	(q)	4
(C)	S_1 and S_2 intersect orthogonally, then $r^2 - 1$ is divisible by	(r)	5
(D)	S_1 and S_2 intersect so that the common chord is longest, then $r^2 + 5$ is divisible by	(s)	6

78. Match the following

	Column I		Column II
(A)	If $ax + by - 5 = 0$ is the equation of the chord of the circle $(x-3)^2 + (y-4)^2 = 4$, which passes through $(2,3)$ and at the greatest distance from the centre of the circle, then	(p)	$a + b = 1$
(B)	Let O be the origin and P be a variable point on the circle $x^2 + y^2 + 2x + 2y = 0$. If the locus of mid-point of OP is $x^2 + y^2 + 2ax + 2by = 0$, then	(q)	$a + b = 2$
(C)	If (a, b) be coordinates of the centre of the smallest circle which cuts the circle $x^2 + y^2 - 2x - 4y - 4 = 0$ and $x^2 + y^2 - 10x + 12y + 52 = 0$ orthogonally, then	(r)	$a^2 + b^2 = 2$
(D)	If a and b are the slope of tangents which are drawn to the circle $x^2 + y^2 - 6\sqrt{3}x - 6y + 27 = 0$ from the origin, then	(s)	$a^2 + b^2 = 3$

79. Match the following

	Column I		Column I
(A)	If the shortest and largest distance from the point (10,7) to the circle $x^2 + y^2 - 4x - 2y - 20 = 0$ are L and M respectively, then	(p)	$M + L = 10$
(B)	If the shortest and largest distance from the point (3, −6) to the circle $x^2 + y^2 - 16x - 12y - 125 = 0$ are L and M respectively, then	(q)	$M + L = 20$
(C)	If the shortest and largest distance from the point (6, −6) to the circle $x^2 + y^2 - 4x + 6y - 12 = 0$ are L and M respectively, then	(r)	$M + L = 30$
		(s)	$M - L = 10$
		(t)	$M - L = 26$

80. Match the following

	Column I		Column II
(A)	If the straight lines $y = a_1x + b$ and $y = a_2x + b$ $(a_1 \neq a_2)$ and $b \in R$ meet the coordinate axes in concyclic points, then	(p)	$a_1^2 + a_2^2 = 4$
(B)	If the chord of contact of the tangents drawn to $x^2 + y^2 = b^2$ and $b \in R$ from any point on $x^2 + y^2 = a_1^2$, touches the circle $x^2 + y^2 = a_2^2$ $(a_1 \neq a_2)$, then	(q)	$a_1 + a_2 = 3$
(C)	If the circle $x^2 + y^2 + 2a_1x + b = 0$ and $x^2 + y^2 + 2a_2x + b = 0 (a_1 \neq a_2)$ and $b \in R$ cuts orthogonally, then	(r)	$a_1a_2 = b$
		(s)	$a_1a_2 = 1$
		(t)	$a_1a_2 = b^2$

Circle Exercise 6 : Statement I and II Type Questions

■ **Directions** (Q. Nos. 81 to 88) are Assertion-Reason type questions. Each of these questions contains two statements:
Statement I (Assertion) and **Statement II** (Reason) Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below :

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

81. **Statement I** Only one tangent can be drawn from the point (1, 3) to the circle $x^2 + y^2 = 1$

Statement II Solving $\frac{|3 - m|}{\sqrt{(1 + m^2)}} = 1$, we get only one real value of m

82. **Statement I** Tangents cannot be drawn from the point $(1, \lambda)$ to the circle $x^2 + y^2 + 2x - 4y = 0$

Statement II $(1+1)^2 + (\lambda + 2)^2 < 1^2 + 2^2$

83. **Statement I** Number of circles passing through (1, 4), (2, 3), (−1, 6) is one

Statement II Every triangle has one circumcircle

84. **Statement I** Two tangents are drawn from a point on the circle $x^2 + y^2 = 50$ to the circle $x^2 + y^2 = 25$, then angle between tangents is $\frac{\pi}{3}$

Statement II $x^2 + y^2 = 50$ is the director circle of $x^2 + y^2 = 25$.

85. **Statement I** Circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 6x + 5 = 0$ intersect each other at two distinct points

Statement II Circles with centres C_1, C_2 and radii r_1, r_2 intersect at two distinct points if $|C_1C_2| < r_1 + r_2$

86. **Statement I** The line $3x - 4y = 7$ is a diameter of the circle $x^2 + y^2 - 2x + 2y - 47 = 0$

Statement II Normal of a circle always pass through centre of circle

87. **Statement I** A ray of light incident at the point (−3, −1) gets reflected from the tangent at $(0, -1)$ to the circle $x^2 + y^2 = 1$. If the reflected ray touches the circle, then equation of the reflected ray is $4y - 3x = 5$

Statement I The angle of incidence = angle of reflection i.e. $\angle i = \angle r$

88. **Statement I** The chord of contact of the circle $x^2 + y^2 = 1$ w.r.t. the points (2,3), (3,5) and (1,1) are concurrent.

Statement II Points (1, 1), (2, 3) and (3,5) are collinear.

Circle Exercise 7 :
Subjective Type Questions

- In this section, there are **16 subjective questions**.

89. Find the equation of the circle passing through $(1, 0)$ and $(0, 1)$ and having the smallest possible radius.

90. Find the equation of the circle which touches the circle $x^2 + y^2 - 6x + 6y + 17 = 0$ externally and to which the lines $x^2 - 3xy - 3x + 9y = 0$ are normals.

91. A line meets the coordinate axes at A and B. A circle is circumscribed about the triangle OAB. If the distance of the points A and B from the tangent at O, the origin, to the circle are m and n respectively, find the equation of the circle.

92. Find the equation of a circle which passes through the point $(2\ ,\ 0)$ and whose centre is the limit of the point of intersection of the lines $3x + 5y = 1$ and $(2 + c)\ x + 5c^2 y = 1$ as $c \to 1$.

93. Tangents are drawn from $P(6, 8)$ to the circle $x^2 + y^2 = r^2$. Find the radius of the circle such that the area of the Δ formed by tangents and chord of contact is maximum.

94. $2x - y + 4 = 0$ is a diameter of the circle which circumscribed a rectangle $ABCD$. If the coordinates of A and B are $A(4, 6)$ and $B(1, 9)$, find the area of rectangle $ABCD$.

95. Find the radius of smaller circle which touches the straight line $3x - y = 6$ at $(1, -3)$ and also touches the line $y = x$.

96. If the circle C_1, $x^2 + y^2 = 16$ intersects another circle C_2 of radius 5 in such a manner that the common chord is of maximum length and has a slope equal to (3/4), find the coordinates of centre C_2.

97. Let $2x^2 + y^2 - 3xy = 0$ be the equation of a pair of tangents drawn from the origin O to a circle of radius 3 with centre in the first quadrant. If A is one of the points of contact, find the length of OA.

98. The circle $x^2 + y^2 = 1$ cuts the X-axis at P and Q. another circle with centre at Q and variable radius intersects the first circle at R above the X-axis and the line segment PQ at S. Find the maximum area of the ΔQSR.

99. If the two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ cut the coordinate axes in concyclic points, prove that $a_1a_2 = b_1b_2$ and find the equation of the circle.

100. The centre of the circle $S = 0$ lie on the line $2x - 2y + 9 = 0$ and $S = 0$ cuts orthogonally the circle $x^2 + y^2 = 4$. Show that circle $S = 0$ passes through two fixed points and find their coordinates.

101. Find the condition on a, b, c such that two chords of the circle $x^2 + y^2 - 2ax - 2by + a^2 + b^2 - c^2 = 0$ passing through the point $(a, b + c)$ are bisected by the line $y = x$.

102. Two straight lines rotate about two fixed points. If they start from their position of coincidence such that one rotates at the rate double that of the other. Prove that the locus of their point of intersection is a circle.

103. The base AB of a triangle is fixed and its vertex C moves such that $\sin A = k \sin B\ (k \neq 1)$. Show that the locus of C is a circle whose centre lies on the line AB and whose radius is equal to $\dfrac{ak}{(1 - k^2)}$, a being the length of the base AB.

104. Consider a curve $ax^2 + 2hxy + by^2 = 1$ and a point P not on the curve. A line drawn from the point P intersects the curve at points Q and R. If the product $PQ \cdot PR$ is independent of the slope of the line, then show that the curve is a circle.

Circle Exercise 8 :
Questions Asked in Previous 13 Year's Exams

■ This section contains questions asked in **IIT-JEE, AIEEE, JEE Main & JEE Advanced** from year **2005** to **2017**.

105. A circle is given by $x^2+(y-1)^2=1$, another circle C touches it externally and also the X-axis, then the locus of its centre is **[IIT-JEE 2005, 3M]**

(a) $\{(x,y):x^2=4y\}\cup\{(x,y):y\le 0\}$

(b) $\{(x,y):x^2+(y-1)^2=4\}\cup\{(x,y):y\le 0\}$

(c) $\{(x,y):x^2=y\}\cup\{(0,y):y\le 0\}$

(d) $\{(x,y):x^2=4y\}\cup\{(0,y):y\le 0\}$

106. If the circles $x^2+y^2+2ax+cy+a=0$ and $x^2+y^2-3ax+dy-1=0$ intersect in two distinct points P and Q, then the line $5x+by-a=0$ passes through P and Q for **[AIEEE 2005, 6M]**

(a) exactly one value of a (b) no value of a

(c) infinitely many values of a (d) exactly two values of a

107. A circle touches the X-axis and also touches the circle with centre at (0, 3) and radius 2. The locus of the centre of the circle is **[AIEEE 2005, 3M]**

(a) an ellipse (b) a circle

(c) a hyperbola (d) a parabola

108. If a circle passes through the point (a,b) and cuts the circle $x^2+y^2=p^2$ orthogonally, then the equation of the locus of its centre is **[AIEEE 2005, 3M]**

(a) $x^2+y^2-3ax-4by+(a^2+b^2-p^2)=0$

(b) $2ax+2by-(a^2-b^2+p^2)=0$

(c) $x^2+y^2-2ax-3by+(a^2-b^2-p^2)=0$

(d) $2ax+2by-(a^2+b^2+p^2)=0$

Paragraph
(Q. Nos. 109 to 111)

ABCD is a square of side length 2 units. C_1 is the circle touching all the sides of the square ABCD and C_2 is the circumcircle of square ABCD. L is a fixed line in the same plane and R is a fixed point.

109. If P is any point of C_1 and Q is another point on C_2, then $\dfrac{PA^2+PB^2+PC^2+PD^2}{QA^2+QB^2+QC^2+QD^2}$ is equal to

(a) 0.75 (b) 1.25 (c) 1 (d) 0.5

110. If a circle is such that it touches the line L and the circle C_1 externally, such that both the circles are on the same side of the line, then the locus of centre of the circle is

(a) ellipse (b) hyperbola

(c) parabola (d) pair of straight line

111. A line L' through A is drawn parallel to BD. Point S moves such that its distances from the line BD and the vertex A are equal. If locus of S cuts L' at T_2 and T_3 and AC at T_1, then area of $\Delta T_1T_2T_3$ is **[IIT-JEE 2006, 5+5+5 M]**

(a) $\frac{1}{2}$ sq units (b) $\frac{2}{3}$ sq units

(c) 1 sq units (d) 2 sq units

112. If the lines $3x-4y-7=0$ and $2x-3y-5=0$ are two diameters of a circle of area 49π square units, the equation of the circle is **[AIEEE 2006, 6M]**

(a) $x^2+y^2+2x-2y-47=0$

(b) $x^2+y^2+2x-2y-62=0$

(c) $x^2+y^2-2x+2y-62=0$

(d) $x^2+y^2-2x+2y-47=0$

113. Let C be the circle with centre (0, 0) and radius 3 units. The equation of the locus of the mid-points of the chords of the circle C that subtend an angle of $\frac{2\pi}{3}$ at its centre is **[AIEEE 2006, 6M]**

(a) $x^2+y^2=\frac{3}{2}$ (b) $x^2+y^2=1$

(c) $x^2+y^2=\frac{27}{4}$ (d) $x^2+y^2=\frac{9}{4}$

114. Tangents are drawn from the point (17, 7) to the circle $x^2+y^2=169$.

Statement I The tangents are mutually perpendicular. because

Statement II The locus of the points from which mutually perpendicular tangents can be drawn to the given circle is $x^2+y^2=338$. **[IIT-JEE 2007, 3M]**

(a) Statement I is True, statement II is True; statement II is a correct explanation for statement I

(b) Statement I is True, statement II is True; statement II is not a correct explanation for statement I

(c) Statement I is True, statement II is False

(d) Statement I is False, statement II is True

115. Consider a family of circles which are passing through the point $(-1,1)$ and are tangent to X-axis. If (h,k) are the coordinate of the centre of the circles, then the set of values of k is given by the interval **[AIEEE 2007, 3M]**

(a) $-\frac{1}{2}\le k\le\frac{1}{2}$ (b) $k\le\frac{1}{2}$

(c) $0\le k\le\frac{1}{2}$ (d) $k\ge\frac{1}{2}$

Paragraph

(Q. Nos. 116 to 118)

A circle C of radius 1 is inscribed in an equilateral triangle PQR. The points of contact of C with the sides PQ, QR, RP are D, E, F, respectively. The line PQ is given by the equation $\sqrt{3}x + y - 6 = 0$ *and the point D is* $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$. *Further, it is given that the origin and the centre of C are on the same side of the line PQ.*

116. The equation of circle C is

(a) $(x-2\sqrt{3})^2 + (y-1)^2 = 1$ (b) $(x-2\sqrt{3})^2 + \left(y+\frac{1}{2}\right)^2 = 1$

(c) $(x-\sqrt{3})^2 + (y+1)^2 = 1$ (d) $(x-\sqrt{3})^2 + (y-1)^2 = 1$

117. Points E and F are given by

(a) $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right), (\sqrt{3}, 0)$ (b) $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), (\sqrt{3}, 0)$

(c) $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right), \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ (d) $\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right), \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

118. Equations of the sides QR, RP are **[IIT-JEE 2008, (4 + 4 + 4) M]**

(a) $y = \frac{2}{\sqrt{3}}x + 1,\ y = -\frac{2}{\sqrt{3}}x - 1$ (b) $y = \frac{1}{\sqrt{3}}x,\ y = 0$

(c) $y = \frac{\sqrt{3}}{2}x + 1,\ y = -\frac{\sqrt{3}}{2}x - 1$ (d) $y = \sqrt{3}x,\ y = 0$

119. Consider $L_1 : 2x + 3y + p - 3 = 0$; $L_2 : 2x + 3y + p + 3 = 0$ where, p is a real number, and $C : x^2 + y^2 + 6x - 10y + 30 = 0$

Statement I If line L_1 is a chord of circle C, then line L_2 is not always a diameter of circle C and

Statement II If line L_1 is a diameter of circle C, then line L_2 is not a chord of circle C. **[IIT-JEE 2008, 3M]**

(a) Statement I is True, statement II is True; statement II is a correct explanation for statement I

(b) Statement I is True, statement II is True; statement II is not a correct explanation for statement I

(c) Statement I is True, statement II is False

(d) Statement I is False, statement II is True

120. The point diametrically opposite to the point $P(1, 0)$ on the circle $x^2 + y^2 + 2x + 4y - 3 = 0$ is **[AIEEE 2008, 3M]**

(a) $(3, -4)$ (b) $(-3, 4)$ (c) $(-3, -4)$ (d) $(3, 4)$

121. Tangents drawn from the point $P(1, 8)$ to the circle $x^2 + y^2 - 6x - 4y - 11 = 0$ touch the circle at the points A and B. The equation of the circumcircle of the triangle PAB is **[IIT-JEE 2009, 3M]**

(a) $x^2 + y^2 + 4x - 6y + 19 = 0$ (b) $x^2 + y^2 - 4x - 10y + 19 = 0$

(c) $x^2 + y^2 - 2x + 6y - 29 = 0$ (d) $x^2 + y^2 - 6x - 4y + 19 = 0$

122. The centres of two circles C_1 and C_2 each of unit radius are at a distance of 6 units from each other. Let P be the mid point of the line segment joining the centres of C_1 and C_2 and C be a circle touching circles C_1 and C_2 externally. If a common tangent to C_1 and C passing through P is also a common tangent to C_2 and C, then the radius of the circle C is **[IIT-JEE 2009, 4M]**

123. If P and Q are the points of intersection of the circles $x^2 + y^2 + 3x + 7y + 2p - 5 = 0$ and $x^2 + y^2 + 2x + 2y - p^2 = 0$ then there is a circle passing through P, Q and (1, 1) for : **[AIEEE 2009, 4M]**

(a) all except one value of p

(b) all except two values of p

(c) exactly one value of p

(d) all values of p

124. The circle $x^2 + y^{2d} = 4x + 8y + 5$ intersects the line $3x - 4y = m$ at two distinct points if **[AIEEE 2010, 4M]**

(a) $-35 < m < 15$ (b) $15 < m < 65$

(c) $35 < m < 85$ (d) $-85 < m < -35$

125. The circle passing through the point $(-1, 0)$ and touching the Y-axis at (0, 2) also passes through the point. **[IIT-JEE 2011, 3M]**

(a) $\left(-\frac{3}{2}, 0\right)$ (b) $\left(-\frac{5}{2}, 2\right)$

(c) $\left(-\frac{3}{2}, \frac{5}{2}\right)$ (d) $(-4, 0)$

126. The straight line $2x - 3y = 1$ divides the circular region $x^2 + y^2 \le 6$ into two parts.

If $S = \left\{\left(2, \frac{3}{4}\right), \left(\frac{5}{2}, \frac{3}{4}\right), \left(\frac{1}{4}, -\frac{1}{4}\right), \left(\frac{1}{8}, \frac{1}{4}\right)\right\}$ then the number of point(s) in S lying inside the smaller part is **[IIT-JEE 2011, 4M]**

127. The two circles $x^2 + y^2 = ax$ and $x^2 + y^2 = c^2 (c > 0)$ touch each other if **[AIEEE 2011, 4M]**

(a) $|a| = c$ (b) $a = 2c$

(c) $|a| = 2c$ (d) $2|a| = c$

128. The locus of the mid-point of the chord of contact of tangents drawn from points lying on the straight line $4x - 5y = 20$ to the circle $x^2 + y^2 = 9$ is **[IIT-JEE 2012, 3M]**

(a) $20(x^2 + y^2) - 36x + 45y = 0$

(b) $20(x^2 + y^2) + 36x - 45y = 0$

(c) $36(x^2 + y^2) - 20x + 45y = 0$

(d) $36(x^2 + y^2) + 20x - 45y = 0$

Paragraph

(Q. Nos. 129 and 130)

A tangent PT is drawn to the circle $x^2 + y^2 = 4$ at the point $P(\sqrt{3}, 1)$. A straight line L, perpendicular to PT is a tangent to the circle $(x-3)^2 + y^2 = 1$.

129. A possible equation of L is

(a) $x - \sqrt{3}y = 1$ (b) $x + \sqrt{3}y = 1$

(c) $x - \sqrt{3}y = -1$ (d) $x + \sqrt{3}y = 5$

130. A common tangent of the two circles is **[IIT-JEE 2012, (3 + 3) M]**

(a) $x = 4$ (b) $y = 2$

(c) $x + \sqrt{3}y = 4$ (d) $x + 2\sqrt{2}y = 6$

131. The length of the diameter of the circle which touches the X -axis at the point (1, 0) and passes through the point (2, 3) is **[AIEEE 2012, 4M]**

(a) $\frac{10}{3}$ (b) $\frac{3}{5}$

(c) $\frac{6}{5}$ (d) $\frac{5}{3}$

132. The circle passing through $(1, -2)$ and touching the axis of x at (3, 0) also passes through the point **[JEE Main 2013, 4M]**

(a) $(-5, 2)$ (b) $(2, -5)$

(c) $(5, -2)$ (d) $(-2, 5)$

133. Circle(s) touching X-axis at a distance 3 from the origin and having an intercept of length $2\sqrt{7}$ on Y-axis is (are) **[JEE Advanced 2013, 3M]**

(a) $x^2 + y^2 - 6x + 8y + 9 = 0$

(b) $x^2 + y^2 - 6x + 7y + 9 = 0$

(c) $x^2 + y^2 - 6x - 8y + 9 = 0$

(d) $x^2 + y^2 - 6x - 7y + 9 = 0$

134. Let C be the circle with centre at $(1, 1)$ and radius = 1. If T is the circle centred at $(0, y)$, passing through origin and touching the circle C externally, then the radius of T is equal to **[JEE Main 2014, 4M]**

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$

(c) $\frac{\sqrt{3}}{\sqrt{2}}$ (d) $\frac{\sqrt{3}}{2}$

135. A circle S passes through the point (0, 1) and is orthogonal to the circles $(x-1)^2 + y^2 = 16$ and $x^2 + y^2 = 1$. Then **[JEE Advanced 2014, 3M]**

(a) radius of S is 8 (b) radius of S is 7

(c) centre of S is $(-7, 1)$ (d) centre of S is $(-8, 1)$

136. Locus of the image of the point (2, 3) in the line $(2x - 3y + 4) + k(x - 2y + 3) = 0, k \in R$, is a **[JEE Main 2015, 4M]**

(a) circle of radius $\sqrt{2}$

(b) circle of radius $\sqrt{3}$

(c) straight line parallel to X-axis

(d) straight line parallel to Y-axis

137. The number of common tangents to the circles $x^2 + y^2 - 4x - 6x - 12 = 0$ and $x^2 + y^2 + 6x + 18y + 26 = 0$, is **[JEE Main 2015, 4M]**

(a) 3 (b) 4

(c) 1 (d) 2

138. The centres of those circles which touch the circle, $x^2 + y^2 - 8x - 8y - 4 = 0$, externally and also touch the X-axis, lie on **[JEE Main 2016, 4M]**

(a) a hyperbola

(b) a parabola

(c) a circle

(d) an ellipse which is not a circle

139. If one of the diameters of the circle, given by the equation, $x^2 + y^2 - 4x + 6y - 12 = 0$, is a chord of a circle S, whose centre is at $(-3, 2)$, then the radius of S is **[JEE Main 2016, 4M]**

(a) 5 (b) 10

(c) $5\sqrt{2}$ (d) $5\sqrt{3}$

140. Let RS be the diameter of the circle $x^2 + y^2 = 1$, where S is the point (1, 0). Let P be a variable point (other than R and S) on the circle and tangents to the circle at S and P meet at the point Q. The normal to the circle at P intersects a line drawn through Q parallel to RS at point E. Then the locus of E passes through the point(s) **[JEE Advanced 2016, 4M]**

(a) $\left(\frac{1}{3}, \frac{1}{\sqrt{3}}\right)$ (b) $\left(\frac{1}{4}, \frac{1}{2}\right)$

(c) $\left(\frac{1}{3}, -\frac{1}{\sqrt{3}}\right)$ (d) $\left(\frac{1}{4}, -\frac{1}{2}\right)$

141. For how many values of p, the circle $x^2 + y^2 + 2x + 4y - p = 0$ and the co-ordinate axes have exactly three common points? **[JEE Advanced 2017, 3M]**

Answers

Exercise for Session 1

1. (d) 2. (b) 3. (b) 4. (b) 5. (a)
6. (c) 7. (d) 8. (a) 9. (c) 10. (c)
11. (b) 12. (a) 13. $\left(\frac{-2}{5}, \frac{4}{5}\right)$; 2
15. $x^2 + y^2 - 2x - 4y - 4 = 0$
16. $x^2 + y^2 - 2x - 8y + 15 = 0$
18. $(x+1)^2 + (y-3)^2 = 4$; $(-1, 3)$; 2

Exercise for Session 2

1. (c) 2. (b) 3. (b) 4. (c) 5. (a)
6. (a) 7. (c) 8. (b) 9. (d) 10. (c)
11. (c) 12. (a) 13. $x^2 + y^2 - y - 16 = 0; \left(0, \frac{1}{2}\right); \frac{\sqrt{65}}{2}$
14. $(-2, -7)$ 15. $x^2 + y^2 - 2x - 3y - 18 = 0$
16. $(x^2 + y^2 - 4x - 2y + 4) = 0$
17. $x^2 + y^2 - 6x + 2y - 15 = 0$

Exercise for Session 3

1. (d) 2. (a) 3. (a) 4. (d) 5. (d)
6. (c) 7. (a) 8. (a) 9. (d) 10. (b)
11. (a) 12. (c) 13. $x^2 + y^2 \pm 10x - 6y + 9 = 0$
15. $\lambda \in (-1, 4)$ 16. $x^2 + y^2 - 4x - 6y = 0$

Exercise for Session 4

1. (c) 2. (c) 3. (c) 4. (d) 5. (b)
6. (b) 7. (b) 8. (a) 9. (d) 10. (b)
11. (d) 12. (a)
14. (i) $3x - 4y + 20 = 0$ and $3x - 4y - 10 = 0$ (ii) $4x + 3y + 5 = 0$ and $4x + 3y - 25 = 0$,
15. centre of the circle $(0, 1, \pm r\sqrt{2})$, where r is radius
16. 15, -35

Exercise for Session 5

1. (c) 2. (b) 3. (c) 4. (b) 5. (d)
6. (d) 7. (b) 8. (c) 9. (b) 10. (a)
11. (c) 12. (d) 14. 8 sq units
16. $3x + 2y - 13 = 0$ 17. $\left(\frac{5}{16}\right)$

Exercise for Session 6

1. (a) 2. (c) 3. (a) 4. (d) 5. (c) 6. (b)
7. (c) 8. (b) 9. (b) 10. (d) 11. (b) 12. (b)
13. $4x^2 + 4y^2 + 30x - 13y - 25 = 0$ 14. $2\sqrt{2}$
16. Direct common tangents are $3x + 4y = 57$, $7x - 24y = 233$,
Transverse common tangents are $4x - 3y = 26$, $24x + 7y = 156$

Exercise for Session 7

1. (d) 2. (c) 3. (a) 4. (b) 5. (c)
6. (a) 7. (b) 8. (d) 9. (b) 10. (a) 11. (d)
12. (a) 14. $x^2 + y^2 - 6x - 6y + 9 = 0$ 16. $\left(\frac{-16}{21}, \frac{-31}{63}\right)$
17. $4x^2 + 4y^2 + 6x + 10y - 1 = 0$ 18. $x + y - 5 = 0$

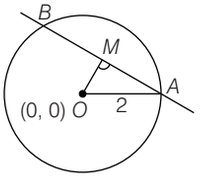
Chapter Exercises

1. (b) 2. (a) 3. (b) 4. (b) 5. (a) 6. (d)
7. (a) 8. (a) 9. (c) 10. (b) 11. (c) 12. (c)
13. (d) 14. (d) 15. (a) 16. (d) 17. (d) 18. (d)
19. (b) 20. (a) 21. (c) 22. (b) 23. (d) 24. (c)
25. (a) 26. (b) 27. (b) 28. (c) 29. (b) 30. (a)
31. (b,d) 32. (b,d) 33. (a,b,c,d) 34. (a,d) 35. (b,d)
36. (b,c) 37. (a,b,c,d) 38. (b,d) 39. (a,d) 40. (a,d)
41. (a,c) 42. (a,c) 43. (a,c) 44. (a,c,d) 45. (a,b) 46. (b)
47. (c) 48. (b) 49. (b) 50. (d) 51. (b) 52. (a)
53. (c) 54. (b) 55. (b) 56. (b) 57. (b) 58. (a)
59. (a) 60. (d) 61. (d) 62. (a) 63. (c) 64. (a)
65. (d) 66. (c) 67. (4) 68. (6) 69. (1) 70. (4)
71. (9) 72. (3) 73. (4) 74. (3) 75. (9) 76. (7)
77. (A) $\rightarrow$ (p, s); (B) $\rightarrow$ (q, r); (C) $\rightarrow$ (q); (D) $\rightarrow$ (p, q, s)
78. (A) $\rightarrow$ (q, r) ; (B) $\rightarrow$ (p); (C) $\rightarrow$ (p) (D) $\rightarrow$ (s)
79. (A) $\rightarrow$ (q, s); (B) $\rightarrow$ (r,t); (C) $\rightarrow$ (p,s)
80. (A) $\rightarrow$ (p, q, s); (B) $\rightarrow$ (p, q, s, t); (C) $\rightarrow$ (p, q, r, s)
81. (d) 82. (a) 83. (d) 84. (d) 85. (c) 86. (b)
87. (b) 88. (a) 89. $(x^2 + y^2 - x - y) = 0$
90. $x^2 + y^2 - 6x - 2y + 1 = 0$
91. $x^2 + y^2 \pm \sqrt{m(m+n)}\, x \pm \sqrt{n(m+n)}\, y = 0$
92. $25x^2 + 25y^2 - 20x + 2y - 60 = 0$
93. (5) 94. 18 sq units
95. 1.5 units 96. $\left(\frac{9}{5}, \frac{-12}{5}\right)$ or $\left(\frac{-9}{5}, \frac{12}{5}\right)$
97. $3(3 + \sqrt{10})$ 98. $\frac{4\sqrt{3}}{9}$ sq units
99. $a_1 a_2 (x^2 + y^2) + (a_1 c_2 + a_2 c_1) x + (b_1 c_2 + b_2 c_1) y = 0$
100. $(-4, 4)$ or $\left(\frac{-1}{2}, \frac{1}{2}\right)$
101. $4a^2 + 4b^2 - c^2 - 8ab + 4bc - 4ca < 0$
105. (d) 106. (b) 107. (d) 108. (d) 109. (a) 110. (b)
111. (c) 112. (d) 113. (d) 114. (a) 115. (d) 116. (d)
117. (a) 118. (d) 119. (c) 120. (c) 121. (b) 122. (8)
123. (a) 124. (a) 125. (d) 126. (2) 127. (a) 128. (a)
129. (a) 130. (d) 131. (a) 132. (c) 133. (a,c) 134. (b)
135. (b,c) 136. (a) 137. (a) 138. (b) 139. (d) 140. (a,c)
141. (2)

Solutions

1. $AB^2 = 4AM^2 = 4\left(4 - \frac{n^2}{2}\right) = 2(8 - n^2)$

Hence, required sum $= 2(8 - 1^2 + 8 - 2^2) = 22$



2. Tangent at $(5\sqrt{2}\cos\theta, 5\sqrt{2}\sin\theta)$ is

$$x\cos\theta + y\sin\theta = 5\sqrt{2}$$

$\therefore$ $OP = 5\sqrt{2}\sec\theta$, $OP, = 5\sqrt{2}\,\text{cosec}\,\theta$

$\therefore$ Area $(\Delta PP_1P_2) = 2 \times$ area of ΔOPP_1

$$= 2 \times \frac{1}{2} \times 5\sqrt{2}\sec\theta \times 5\sqrt{2}\ \text{cosec}\,\theta$$

$$= \frac{100}{\sin 2\theta},$$

Area $(\Delta PP_1P_2)_{\min} = 100$

$\Rightarrow$ $\theta = \frac{\pi}{4} \Rightarrow OP = 10$

$\Rightarrow$ $P \equiv (10, 0), (-10, 0)$

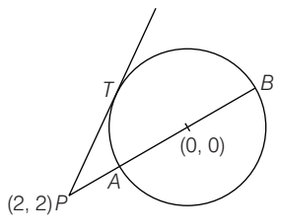
3. Let $S \equiv x^2 + y^2 - 2 = 0$, $P \equiv (2,2)$

$\therefore$ $S_1 = 2^2 + 2^2 - 2 = 6 > 0$

$\Rightarrow P$ lies outside the circle

$$PA.PB = (PT)^2 = S_1 = 6 \quad \ldots(i)$$

given $$\frac{PB}{PA} = 3 \quad \ldots(ii)$$



From Eqs. (i) and (ii), we get

$$PA = \sqrt{2},\ PB = 3\sqrt{2}$$

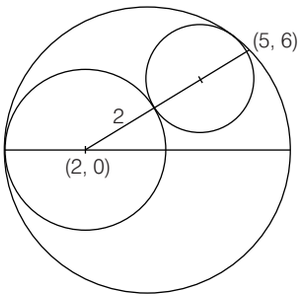
$\Rightarrow$ $AB = PB - PA = 2\sqrt{2} =$ Diameter of circle

Hence, chord AB passes through the centre $(0,0)$, $y = x$

4. Let $S \equiv (x-2)^2 + y^2 - 4 = 0$

Its centre $C \equiv (2, 0)$ and radius $r = 2$

Distance between $C \equiv (2,0)$ and $(5, 6)$ is $\sqrt{(2-5)^2 + (0-6)^2}$



i.e. $3\sqrt{5}$

$\therefore$ $r_1 r_2 = \left(\frac{3\sqrt{5} - 2}{2}\right).\left(\frac{3\sqrt{5} + 2}{2}\right)$

$$= \frac{41}{4}$$

5. Any circle which touches $3x + 4y - 7 = 0$ at $(1, 1)$ will be of the form

$$S(x,y) \equiv (x-1)^2 + (y-1)^2 + \lambda(3x + 4y - 7) = 0$$

Since, $S(2,3) = 16 \Rightarrow 1 + 4 + \lambda(11) = 16$

$\therefore$ $\lambda = 1$

So, required circle will be $x^2 + y^2 + x + 2y - 5 = 0$

6. Let $S \equiv x^2 + y^2 - 2x + 4y - \lambda = 0$

for interval point $P(2,8)$,

$$4 + 64 - 4 + 32 - \lambda < 0$$

$\Rightarrow$ $\lambda > 96$ $\quad \ldots(i)$

and x-intercept $= 2\sqrt{(1+\lambda)}$ and y-intercept $= 2\sqrt{(4+\lambda)}$

given $2\sqrt{(1+\lambda)} < 0 \Rightarrow \lambda < -1$ $\quad \ldots(ii)$

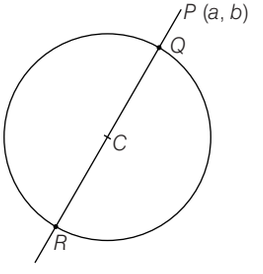
and $2\sqrt{(4+\lambda)} < 0 \Rightarrow \lambda < -4$ $\quad \ldots(iii)$

from Eqs. (i), (ii) and (iii), we get

$\lambda \in \phi$

7. Let $S \equiv x^2 + y^2 + 2x + 4y - 4 = 0$

Its centre $C \equiv (-1, -2)$ and radius $r = 3$. The points on the circle which are nearest and farthest to the point $P(a,b)$ are Q and R respectively. Thus, the circle centred at Q having radius PQ will be the smallest circle while the circle centred at R having radius PR will be the largest required circle.



Hence, the difference between their radii is

$$PR - PQ = QR = 2 \times 3 = 6$$

8. Radius $= \sqrt{(a-\pi)^2 + (b-e)^2}$

$$= \text{irrational} = k$$

$\therefore$ Circle $(x-\pi)^2 + (y-e)^2 = k^2$

9. Given $\lambda L_2L_3 + \mu L_3L_1 + \nu L_1L_2 = 0$

$\Rightarrow \lambda(y - m_2x - c_2)(y - m_3x - c_3) + \mu(y - m_3x - c_3)(y - m_1x - c_1)$
$+ \nu(y - m_1x - c_1)(y - m_2x - c_2) = 0$

for circle coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$,

then, $\lambda(m_2m_3 - 1) + \mu(m_3m_1 - 1) + \nu(m_1m_2 - 1) = 0$

and $(m_2 + m_3)\lambda + (m_3 + m_1)\mu + (m_1 + m_2)\nu = 0$

10. $\because f(x,y) \equiv x^2 + y^2 + 2ax + 2by + c = 0$

$\Rightarrow$ $f(x,0) = 0 \Rightarrow x^2 + 2ax + c = (x-2)^2$

$\therefore$ $a = -2, c = 4$

and $f(0,y) = 0 \Rightarrow y^2 + 2by + c = (y-2)(y-3)$

$\therefore$ $2b = -5, c = 6$

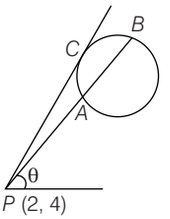
or $b = -\frac{5}{2}, c = 6$

Clearly, that data are not consistent.

11. Given, $(1 + \alpha x)^n = 1 + 8x + 24x^2 + \ldots$

$\Rightarrow$ $1 + n(\alpha x) + \frac{n(n-1)}{1.2}(\alpha x)^2 + \ldots$

$= 1 + 8x + 24x^2 + \ldots$



Equating the coefficients of x and x^2, we get

$n\alpha = 8, \frac{n\alpha(n\alpha - \alpha)}{1.2} = 24$

or $\frac{8(8-\alpha)}{2} = 24 \Rightarrow 8 - \alpha = 6$

$\therefore$ $\alpha = 2$ and $n = 4$

Equation of line is $\frac{x-2}{\cos\theta} = \frac{y-4}{\sin\theta} = r$, then point $(2 + r\cos\theta, 4 + r\sin\theta)$ lies on the circle $x^2 + y^2 = 4$,

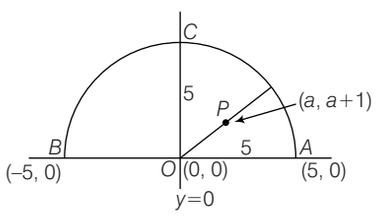
then, $(2 + r\cos\theta)^2 + (4 + r\sin\theta)^2 = 4$

or $r^2 + 4r(\cos\theta + 2\sin\theta) + 16 = 0$

$\therefore$ $PA \cdot PB = r_1r_2 = \frac{16}{1} = 16$

Aliter : $PA \cdot PB = (PC)^2 = 2^2 + 4^2 - 4 = 16$

12. For interior point $OP < 5$



$\Rightarrow$ $\sqrt{a^2 + (a+1)^2} < 5$

$\Rightarrow$ $2a^2 + 2a - 24 < 0$

$\therefore$ $a^2 + a - 12 < 0$

$(a+4)(a-3) < 0$

$\Rightarrow -4 < a < 3$ and for I and II quadrant

$a + 1 > 0$

$\therefore$ $a > -1$

Hence, $-1 < a < 3$

13. Let $S(x,y) \equiv x^2 + y^2 + 2gx + 2fy + c = 0$

$\Rightarrow$ $S(x,2) = 0$

$\Rightarrow$ $x^2 + 4 + 2gx + 4f + c = (x-1)^2$

$\therefore$ $g = -1, 3 + 4f + c = 0$ ...(i)

and $S(1,y) = 0 \Rightarrow 1 + y^2 + 2g + 2fy + c = (y-0)(y-2)$

$\therefore$ $f = -1, 1 + 2g + c = 0$ ...(ii)

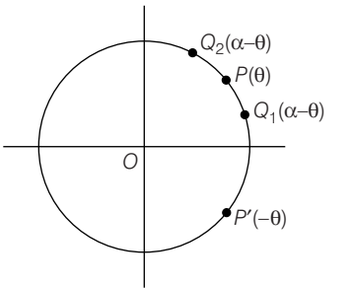
From Eqs. (i) and (ii), we get

$g = -1, f = -1, c = 1$

$\therefore$ Equation of required circle is

$x^2 + y^2 - 2x - 2y + 1 = 0$

14. See the diagram. Since, there is no condition on θ, Q can be placed either at Q_1 or Q_2 for a particular position of P. So option (a) and (b) cannot be definitely true.



Consider a line through origin $y = mx$. If Q and P are reflection of each other with line mirror $y = mx$

(Slope of PQ) $\times m = -1$

or $m\left(\frac{\sin\theta - \sin(\alpha - \theta)}{\cos\theta - \cos(\alpha - \theta)}\right) = -1$

$$m\left(\frac{2\cos\left(\frac{\alpha}{2}\right).\sin\left(\frac{2\theta - \alpha}{2}\right)}{2\sin\left(\frac{\alpha}{2}\right).\sin\left(\frac{\alpha - 2\theta}{2}\right)}\right) = -1$$

or $m\left(-\cot\frac{\alpha}{2}\right) = -1$

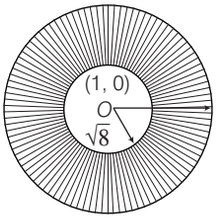
or $m = \tan\left(\frac{\alpha}{2}\right)$

15. Since, $x^2 + y^2 < 25$ and x and y are integers, the possible values of x and $y \in (0, \pm 1, \pm 2, \pm 3, \pm 4)$. Thus, x and y can be chosen in $9 \times 9 = 81$ ways. However, we have to exclude cases (3, 4), (4,3) and (4, 4) i.e. $3 \times 4 = 12$ cases. Hence, the number of permissible values

$= 81 - 12 = 69$

16. ∵ The point $([P+1],[P])$ lies inside the circle $x^2+y^2-2x-15=0$, then

$$[P+1]^2+[P]^2-2[P+1]-15<0$$



$$\Rightarrow \quad ([P]+1)^2+[P]^2-2([P]+1)-15<0$$

$$\Rightarrow \quad 2[P]^2-16<0 \Rightarrow [P]^2<8 \qquad \ldots(i)$$

∵ Circles are concentric

∴ point $([P+1],[P])$ out side the circle

$$x^2+y^2-2x-7=0$$

$$\therefore \quad ([P]+1)^2+[P]^2-2([P+1])-7>0$$

$$\Rightarrow \quad ([P]+1)^2+[P]^2-2([P]+1)-7>0$$

$$\Rightarrow \quad 2[P]^2-8>0$$

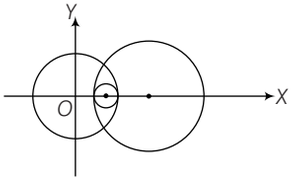
$$\therefore \quad [P]^2>4 \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$4<[P]^2<8$ which is impossible

∴ For no value of 'P' the point will be within the region.

17. The circles are $x^2+y^2=2^2$ and $(x-4)^2+(y-0)^2=3^2$



For the point P to enclose greatest area, the arbitrarily chosen point should be $\left(\frac{3}{2},0\right)$ and P should move in a circle of radius $\frac{1}{2}$

The locus of P is $\left(x-\frac{3}{2}\right)^2+(y-0)^2=\left(\frac{1}{2}\right)^2$

$$\Rightarrow \quad x^2+y^2-3x+2=0.$$

18. Since, $y=|x|+c$ and $x^2+y^2-8|x|-9=0$ both are symmetrical about Y-axis for $x>0$, $y=x+c$.

Equation of tangent to circle $x^2+y^2-8x-9=0$

Parallel to $y=x+c$ is $y=(x-4)+5\sqrt{(1+1)}$

$$\Rightarrow \quad y=x+(5\sqrt{2}-4)$$

for no solution $c>5\sqrt{2}-4$,

$$\therefore \quad c\in(5\sqrt{2}-4,\infty)$$

19. Let $\angle AOL=\theta$

$$\therefore \quad A\equiv(a\cos\theta,a\sin\theta)$$

$$\therefore \quad M\equiv(a+a\cos\theta,a\sin\theta)$$

$$x=a+a\cos\theta$$

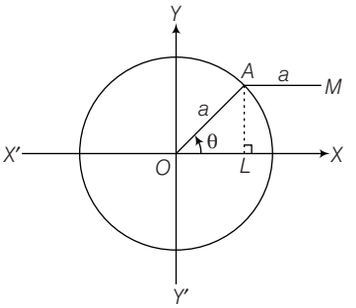
$$\Rightarrow \quad (x-a)=a\cos\theta \qquad \ldots(i)$$

and $\quad y=a\sin\theta$

From Eqs. (i) and (ii), we get

$$(x-a)^2+y^2=a^2$$

$$\Rightarrow \quad x^2+y^2-2ax=0$$



$$\Rightarrow \quad x^2+y^2=2ax$$

20. Equation of circle is $(2x-y+1)(x-2y+3)+\lambda xy=0$

for circle coefficient of $xy=0$

i.e. $\quad -5+\lambda=0,$

$$\therefore \quad \lambda=5$$

∴ Circle is $2x^2+2y^2+7x-5y+3=0$

$$\Rightarrow \quad x^2+y^2+\frac{7}{2}x-\frac{5}{2}y+\frac{3}{2}=0$$

∴ Centre is $\left(-\frac{7}{4},\frac{5}{4}\right)$

21. $\sqrt{\left\{\frac{\lambda^2}{4}+\frac{(1-\lambda)^2}{4}-5\right\}}\le 5$

$$\Rightarrow \quad \lambda^2+(1-\lambda)^2-20\le 100$$

$$\Rightarrow \quad 2\lambda^2-2\lambda-119\le 0$$

$$\therefore \quad \frac{1-\sqrt{239}}{2}\le\lambda\le\frac{1+\sqrt{239}}{2}$$

$$\Rightarrow \quad -7.2\le\lambda\le 8.2 \qquad \text{(approx.)}$$

$$\therefore \quad \lambda=-7,-6,-5,\ldots,7,8$$

22. Let $\phi(x,y)\equiv x^2+y^2+2gx+2fy+c=0$

$$\therefore \quad \phi(0,\lambda)=0+\lambda^2+0+2f\lambda+c=0$$

have equal roots,

Then, $2+2=-\frac{2f}{1}$ and $2.2=\frac{c}{1}$

$$\therefore \quad f=-2 \text{ and } c=4$$

and $\phi(\lambda,0)\equiv\lambda^2+0+2g\lambda+0+c=0$

$$\therefore \quad \lambda^2+2g\lambda+c=0$$

Here, $c=4$

$$\therefore \quad \lambda^2+2g\lambda+4=0$$

have roots 4/5, 5

$$\therefore \quad \frac{4}{5}+5=-2g$$

$$\Rightarrow \quad g=-\frac{29}{10}$$

$$\therefore \quad \text{Centre}\equiv(-g,-f)=\left(\frac{29}{10},2\right)$$

23. Circle is $x^2+y^2=r^2\cos^2\theta+r^2\sin^2\theta$

$$x^2+y^2=r^2$$

Equation of tangent at θ is

$$x\cos\theta+y\sin\theta=r \qquad \ldots(i)$$

and at $\left(\theta+\frac{\pi}{3}\right)$ is $x\cos\left(\theta+\frac{\pi}{3}\right)+y\sin\left(\theta+\frac{\pi}{3}\right)=r$

$$\Rightarrow\ x\left(\frac{1}{2}\cos\theta-\frac{\sqrt{3}}{2}\sin\theta\right)+y\left(\frac{1}{2}\sin\theta+\frac{\sqrt{3}}{2}\cos\theta\right)=r$$

$$\Rightarrow\quad x\cos\theta+y\sin\theta-x\sqrt{3}\sin\theta+y\sqrt{3}\cos\theta=2r$$

$$\Rightarrow\quad r-\sqrt{3}(x\sin\theta-y\cos\theta)=2r$$

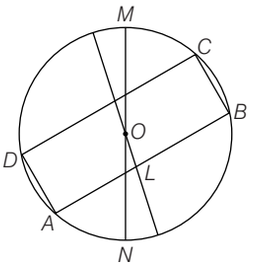
or $$x\sin\theta-y\cos\theta=-\frac{r}{\sqrt{3}} \qquad \ldots(ii)$$

Squaring and adding Eqs. (i) and (ii), then we get

$$x^2+y^2=\frac{4r^2}{3}\Rightarrow 3(x^2+y^2)=4r^2$$

24. Let MN be the diameter of the circle whose equation is

$$4y=x+7 \qquad \ldots(i)$$



and coordinates of A and B are $(-3,4)$ and $(5,4)$ respectively.

Equation of $\perp$ bisector of AB is $[L\equiv(1,4)]$

$$y-4=-\frac{1}{0}(x-1) \qquad (\because \text{slope of } AB=0)$$

$$\therefore\quad x=1 \qquad \ldots(ii)$$

Solving Eqs. (i) and (ii), we get the coordinates of the centre of the circle as (1, 2)

$$\therefore\quad OL=\sqrt{(1-1)^2+(4-2)^2}=2$$

$$\therefore\quad BC=2OL=4 \text{ unit}$$

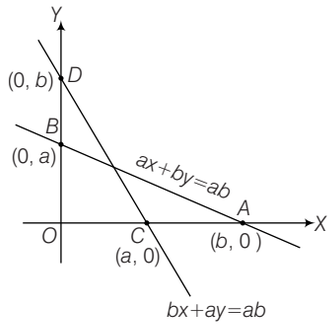
$$AB=8 \text{ unit}$$

$\therefore$ Area of rectangle $ABCD=4\times 8$

$=32$ sq units.

25. $\because OC\cdot OA=ab=OB\cdot OD$

$\therefore A, B, C, D$ are concyclic.



26. Coordinates of P, Q, R are $(\cos\alpha,\sin\alpha),(\cos\beta,\sin\beta)$ and $(\cos\gamma,\sin\gamma)$ respectively.

and $\quad A\equiv(-1,0)$

$$\therefore\quad AP=\sqrt{((1+\cos\alpha)^2+\sin^2\alpha)}=2\cos\left(\frac{\alpha}{2}\right)$$

$$AQ=\sqrt{((1+\cos\beta)^2+\sin^2\beta)}=2\cos\left(\frac{\beta}{2}\right)$$

$$AR=\sqrt{((1+\cos\gamma)^2+\sin^2\gamma)}=2\cos\left(\frac{\gamma}{2}\right)$$

$\because AP, AQ, AR$ are in GP, then

$$\cos\left(\frac{\alpha}{2}\right),\cos\left(\frac{\beta}{2}\right),\cos\left(\frac{\gamma}{2}\right) \text{ are also in GP.}$$

27. Let equation of circle be

$$x^2+y^2+2gx+2fy+c=0$$

It pass through (2, 0) and (0, 4), then $4+0+4g+0+c=0$

$$\Rightarrow\quad g=-\frac{(c+4)}{4} \text{ and } 0+16+0+8f+c=0$$

$$\Rightarrow\quad f=-\frac{(c+16)}{8}$$

$$\because\quad \text{Radius } r=\sqrt{(g^2+f^2-c)}$$

$$=\sqrt{\left\{\frac{(c+4)^2}{16}+\frac{(c+16)^2}{64}-c\right\}}$$

$$=\sqrt{\left\{\frac{4\{c^2+8c+16\}+\{c^2+32c+256\}-64c}{64}\right\}}$$

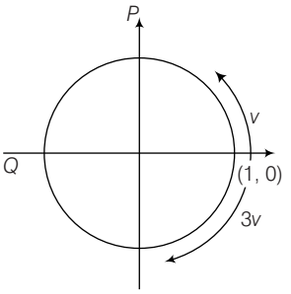
$$=\sqrt{\left(\frac{5c^2+320}{64}\right)}$$

For minimum radius $c=0$

$$\therefore\quad g=-1,\ f=-2$$

Required circle is $x^2+y^2-2x-4y=0$

28. The particle which moves clockwise is moving three times as fast as the particle moving anticlockwise ($\because$ speed in clockwise $3v$ and in anticlockwise v).



This mean the clockwise particle travels $\left(\frac{3}{4}\right)$th of the way around the circle, the anticlockwise particle will travel $\left(\frac{1}{4}\right)$th of the way around the circle. So, the second particle will meet at $P(0,1)$.

Using the same logic, they will meet at $Q(-1,0)$, when they meet the second time.

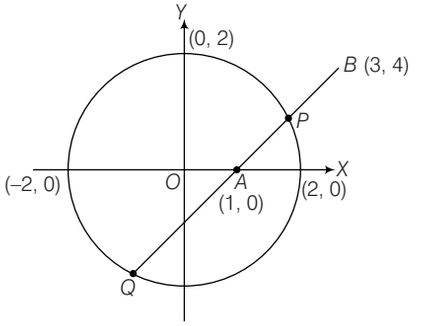
29. $\because \quad \frac{BP}{PA} = \alpha$

$BP : PA = \alpha : 1$

$\therefore$ Coordinates of P is $\left(\frac{3+\alpha}{1+\alpha}, \frac{4}{\alpha+1}\right)$, P lie on $x^2 + y^2 = 4$

$\Rightarrow \quad (\alpha+3)^2 + 16 = 4(\alpha+1)^2$

$\Rightarrow \quad 3\alpha^2 + 2\alpha - 21 = 0 \quad \ldots(i)$



and $\frac{BQ}{QA} = \frac{\beta}{1}$

$BQ : QA = \beta : 1$ or $\frac{BQ}{QA} - 1 = \beta - 1$

$\frac{AB}{QA} = \frac{(\beta-1)}{1}$

$AB : QA = (\beta-1) : 1$

$\therefore$ Coordinates of Q is $\left(\frac{\beta-3}{\beta-1}, \frac{-4}{\beta-1}\right)$

Q lie on $x^2 + y^2 = 4$

$\therefore \quad (\beta-3)^2 + 16 = 4(\beta-1)^2$

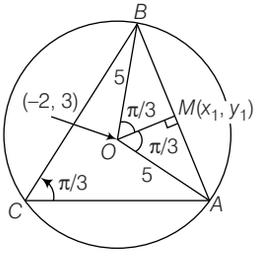
$\Rightarrow \quad 3\beta^2 - 21\beta - 21 = 0$

Hence, α is a root of $3x^2 + 2x - 21 = 0$

and β is a root of $3x^2 - 2x - 21 = 0$

30. $\because \angle ACB = \frac{\pi}{3}$ (given)

$\therefore \angle AOB = \frac{2\pi}{3}$



$\Rightarrow \quad \angle AOM = \angle BOM = \frac{\pi}{3}$

Let mid-point of chord AB is (x_1, y_1)

$\therefore$ In ΔAOM,

$$\cos\frac{\pi}{3} = \frac{OM}{OA} = \frac{\sqrt{(x_1+2)^2 + (y_1-3)^2}}{5}$$

or $\left(\frac{5}{2}\right)^2 = (x_1+2)^2 + (y_1-3)^2$

$\Rightarrow \quad \frac{25}{4} = x_1^2 + y_1^2 + 4x_1 - 6y_1 + 13$

$4x_1^2 + 4y_1^2 + 16x_1 - 24y_1 + 27 = 0$

$\therefore$ Locus of mid-point is

$x^2 + y^2 + 4x - 6y + \frac{27}{4} = 0$

$\Rightarrow \quad (x+2)^2 + (y-3)^2 = 6.25$

31. $\because \quad r = a\cos\theta + b\sin\theta$

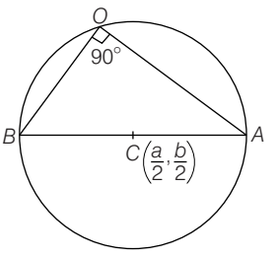
$\Rightarrow \quad r^2 = a(r\cos\theta) + b(r\sin\theta)$

or $x^2 + y^2 = ax + by \quad (\because x = r\cos\theta, y = r\sin\theta)$

or $x^2 + y^2 - ax - by = 0 \quad \ldots(i)$

Circle pass through O (origin)

given $\angle AOB = 90°$



$\therefore AB$ is Diameter of circle Eq. (i), Centre is $\left(\frac{a}{2}, \frac{b}{2}\right)$

$\therefore$ Locus of mid point AB ($\because$ mid-point of AB is C)

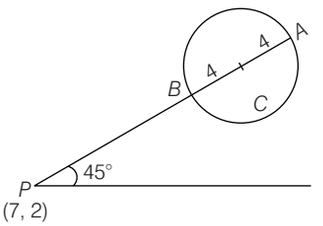
$x = \frac{a}{2}, y = \frac{b}{2}$

32. Slope of $PC = \frac{-3-2}{2-7} = 1$

If $\tan\theta = 1$

$\therefore \quad \theta = 45°$

Equation of PA is $\frac{x-7}{\frac{1}{\sqrt{2}}} = \frac{y-2}{\frac{1}{\sqrt{2}}} = r$



$\therefore \quad \left(7 + \frac{r}{\sqrt{2}}, 2 + \frac{r}{\sqrt{2}}\right)$ lie on circle,

then, $\left(7 + \frac{r}{\sqrt{2}}\right)^2 + \left(2 + \frac{r}{\sqrt{2}}\right)^2 - 4\left(7 + \frac{r}{\sqrt{2}}\right) + 6\left(2 + \frac{r}{\sqrt{2}}\right) - 3 = 0$

$\therefore \quad r^2 = -5\sqrt{2} \pm 4$

$\therefore \quad r = -5\sqrt{2} \pm 4$

$\therefore$ Points $\left(7+\frac{-5\sqrt{2}\pm 4}{\sqrt{2}}, 2+\frac{-5\sqrt{2}\pm 4}{\sqrt{2}}\right)$

$\Rightarrow (2\pm 2\sqrt{2}, -3\pm 2\sqrt{2})$

Taking + ve sign for A, –ve sign for B.

33. Let $S \equiv x^2+y^2+2gx+2fy+c=0$

$S_1 \equiv x^2+y^2-4=0$

$S_2 \equiv x^2+y^2-6x-8y+10=0$

$S_3 \equiv x^2+y^2+2x-4y-2=0$

$\therefore$Common chords are

$S-S_1 \equiv 2gx+2fy+c+4=0$ ...(i)

$S-S_2 \equiv (2g+6)x+(2f+8)y+c-10=0$ ...(ii)

$S-S_3 \equiv (2g-2)x+(2f+4)y+c+2=0$ ...(iii)

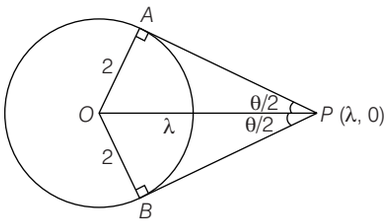
For cutting the extremities of diameter, chords Eqs. (i), (ii) and (iii) pass through the centres of S_1, S_2 and S_3 respectively, then

$\therefore \quad c+4=0, (2g+6)3+(2f+8)4+c-10=0$

and $(2g-2)(-1)+(2f+4)(2)+c+2=0$

after solving $c=-4$, $g=-2$, $f=-3$

34.



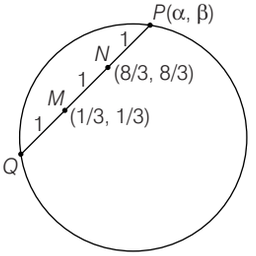
$\because \quad \frac{\pi}{2}<\theta<\frac{2\pi}{3} \Rightarrow \frac{\pi}{4}<\frac{\theta}{2}<\frac{\pi}{3}$

$\Rightarrow \quad \frac{1}{\sqrt{2}}<\sin\left(\frac{\theta}{2}\right)<\frac{\sqrt{3}}{2}$

$\Rightarrow \quad \frac{1}{\sqrt{2}}<\frac{2}{\lambda}<\frac{\sqrt{3}}{2}$ or $\sqrt{2}>\frac{\lambda}{2}>\frac{2}{\sqrt{3}}$

or $\quad 2\sqrt{2}>\lambda>\frac{4}{\sqrt{3}}$ or $\frac{4}{\sqrt{3}}<\lambda<2\sqrt{2}$

35. It is clear that N is the mid-point of M and P



$\therefore \quad \alpha=\frac{16}{3}-\frac{1}{3}=5$ and $\beta=\frac{16}{3}-\frac{1}{3}=5$

$(\alpha,\beta)\equiv(5,5)$ lie on the circle

$\therefore \quad 5^2+5^2-4\times 5-2\times 5-c=0$

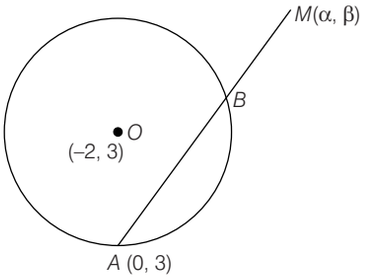
$c=20$

or $\quad c^2-40c+400=0$

36. Let $M(\alpha,\beta)$

$\because \quad AM=2AB \Rightarrow AB+BM=2AB$

$\therefore \quad AB=BM$



$\therefore B$ is the mid point of AM.

$\therefore$Coordinates of B are $\left(\frac{\alpha}{2}, \frac{3+\beta}{2}\right)$ which lie on

$x^2+4x+(y-3)^2=0$

$\frac{\alpha^2}{4}+4\times\frac{\alpha}{2}+\left(\frac{3+\beta}{2}-3\right)^2=0$

$\alpha^2+8\alpha+(\beta-3)^2=0$

$\therefore$Requires locus is

$x^2+8x+(y-3)^2=0$

or $\quad x^2+y^2+8x-6y+9=0$

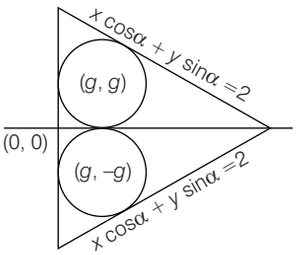
37. $x\cos\alpha+y\sin\alpha=2$ or $\frac{x}{2\sec\alpha}+\frac{y}{2\,\mathrm{cosec}\,\alpha}=1$

$2\sec\alpha =$ + ve in IV

and $\quad 2\,\mathrm{cosec}\,\alpha =$ –ve in IV

value of g in I quadrant

$g=\left(\frac{g\cos\alpha+g\sin\alpha-2}{\sqrt{1}}\right)$



$\therefore \quad g=\frac{2}{\cos\alpha+\sin\alpha\pm 1}$

and value g of in IV quadrant

$g=\left|\frac{g\cos\alpha-g\sin\alpha-2}{1}\right|$

$\Rightarrow \quad \pm g=g(\cos\alpha-\sin\alpha)-2$

$g=\frac{2}{\cos\alpha-\sin\alpha\pm 1}$

Equation of circles in I quadrant

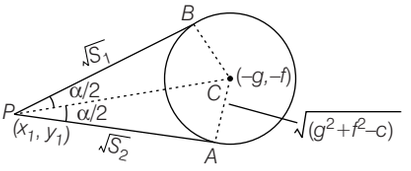
$(x-g)^2+(y-g)^2=g^2$

And in IV quadrant is $(x-g)^2+(y-g)^2=g^2$

38. $\therefore \quad \cot\left(\frac{\alpha}{2}\right) = \frac{\sqrt{S_1}}{\sqrt{(g^2+f^2-c)}}$

$\therefore \quad \tan\left(\frac{\alpha}{2}\right) = \frac{\sqrt{(g^2+f^2-c)}}{\sqrt{S_1}}$

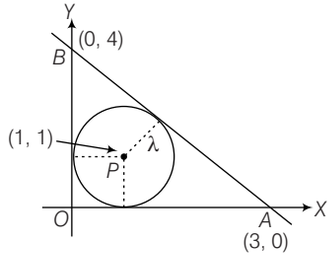
$\therefore \quad \alpha = 2\tan^{-1}\left(\frac{\sqrt{(g^2+f^2-c)}}{\sqrt{S_1}}\right)$



39. For condition of tangency

$$\frac{|4\lambda+3\lambda-12|}{5} = \lambda \Rightarrow \quad 7\lambda-12 = \pm 5\lambda$$

$\therefore \quad \lambda = 1, 6$



40. Let coordinates of $P \equiv (3\cos\alpha, 3\sin\alpha)$

Let x-coordinate of Q is x_1, then

y-coordinate of Q is $-7x_1 - 3$

$\therefore \quad Q \equiv (x_1, -7x_1 - 3)$

$\because x - y + 1 = 0$ is the perpendicular bisector of PQ, then mid-point of PQ lie on $x - y + 1 = 0$

$$\Rightarrow \quad \frac{3\cos\alpha + x_1}{2} - \frac{3\sin\alpha - 7x_1 - 3}{2} + 1 = 0$$

$$\Rightarrow \quad 8x_1 + 3\cos\alpha - 3\sin\alpha + 5 = 0$$

$$\Rightarrow \quad 24x_1 + 9\cos\alpha - 9\sin\alpha + 15 = 0 \qquad \ldots(i)$$

and slope of $(x - y + 1 = 0) \times$ slope of $PQ = -1$

$$\Rightarrow \quad 1 \times \frac{3\sin\alpha + 7x_1 + 3}{3\cos\alpha - x_1} = -1$$

$$\Rightarrow \quad 3\sin\alpha + 7x_1 + 3 = -3\cos\alpha + x_1$$

$$\Rightarrow \quad 6x_1 + 3\sin\alpha + 3\cos\alpha + 3 = 0$$

$$\Rightarrow \quad 24x_1 + 12\sin\alpha + 12\cos\alpha + 12 = 0 \qquad \ldots(ii)$$

Subtracting Eqs. (i) and (ii), we obtain

$$-3\cos\alpha - 21\sin\alpha + 3 = 0$$

$$\Rightarrow \quad (1 - \cos\alpha) = 7\sin\alpha$$

$$\Rightarrow \quad (1-\cos\alpha)^2 = 49(1 - \cos^2\alpha)$$

$$\Rightarrow \quad (1-\cos\alpha)^2 = 49(1+\cos\alpha)(1-\cos\alpha)$$

$$\Rightarrow \quad (1-\cos\alpha)(1 - \cos\alpha - 49 - 49\cos\alpha) = 0$$

$\therefore \quad \cos\alpha = 1$ and $\cos\alpha = -\frac{24}{25}$

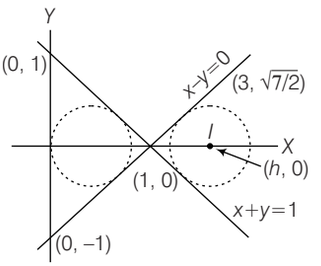
$\therefore$ Coordinates of P are (3, 0) and $\left(-\frac{72}{25}, \frac{21}{25}\right)$.

41. Circle possible in I region and centre of circle on X-axis.

If centre is $(h, 0)$, then

$$\frac{|h-0-1|}{\sqrt{2}} = \sqrt{(h-3)^3 + \frac{7}{2}}$$

$\therefore \quad h = 6, 4$



Then, centres are (6, 0) and (4, 0)

42. $\because C_2$ is the director circle of C_1

$\therefore$ Equation of C_2 is

$$x^2 + y^2 = 2(2)^2 = 8$$

Again C_3 is the director circle of C_2. Hence, the equation of C_3 is

$$x^2 + y^2 = 2(8) = 16$$

43. The equation of tangent in terms of slope of $x^2 + y^2 = 25$ is

$$y = mx \pm 5\sqrt{(1+m^2)} \qquad \ldots(i)$$

Given Eq. (i), pass through $(-2, 11)$, then

$$11 = -2m \pm 5\sqrt{(1+m^2)}$$

squaring both sides, then we get

$$21m^2 - 44m - 96 = 0$$

$$\Rightarrow \quad (7m - 24)(3m + 4) = 0$$

$\therefore \quad m = -\frac{4}{3}, \frac{24}{7}$

There from Eq. (i) we get required tangents are

$$24x - 7y \pm 125 = 0 \text{ and } 4x + 3y = \pm 25$$

Hence, tangents are $24x - 7y + 125 = 0$ and $4x + 3y = 25$

44. $C_1 \equiv x^2 + y^2 - 2x - 4y - 4 = 0 \qquad \ldots(i)$

and $C_2 \equiv x^2 + y^2 + 2x + 4y + 4 = 0 \qquad \ldots(ii)$

$\therefore$ Radical axis is $C_1 - C_2 = 0$

$$\Rightarrow \quad -4x - 8y - 8 = 0$$

or $x + 2y + 2 = 0$ which is $L = 0$

(a) Option is correct.

Centre and radius of $C_1 = 0$ are (1, 2) and 3.

$\because$ Length of $\perp$ from (1, 2) on $L = 0$

is $\frac{|1+4+2|}{\sqrt{1+4}} = \frac{7}{\sqrt{5}} \neq$ radius

$\therefore$(b) Option is wrong.

L is also the common chord of C_1 and C_2.

$\therefore$(c) Option is correct.

$\because$ Centres of $C_1 = 0$ and $C_2 = 0$ are (1, 2) and $(-1, -2)$

$\therefore$ Slope of Line joining centres of circles $C_1 = 0$ and $C_2 = 0$ is

$$\frac{-2-2}{-1-1} = \frac{4}{2} = 2 = m_1 \qquad \text{(say)}$$

and slope of $L=0$ is $-\frac{1}{2}=m_2$ (say)

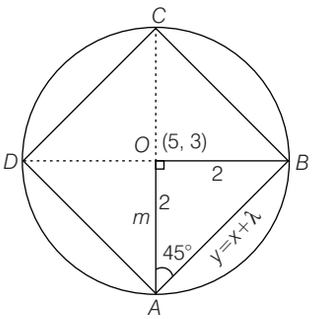
$\therefore \quad m_1m_2=-1$

Hence, L is perpendicular to the line joining centres of C_1 and C_2.

$\therefore$ **(d) Option is correct.**

45. Let slope of OA is m,

Then, $\left|\frac{m-1}{1+m}\right|=\tan 45°$



$\Rightarrow \quad \frac{m-1}{1+m}=\pm 1 \quad$ or $\quad \frac{m-1}{m+1}=-1$

$\Rightarrow \quad m-1=-m-1$

$\therefore \quad m=0$

$\therefore$Equation of OA is $y=3$

Solving $y=3$ and $x^2+y^2-10x-6y+30=0$

$\Rightarrow \quad x^2+9-10x-18+30=0$

$\Rightarrow \quad x^2-10x+21=0$

$\Rightarrow \quad (x-7)(x-3)=0$

or $\quad x=3,7$

$\therefore$Two vertices are (3, 3) and (7, 3) and other diagonal is $\perp$ to $y=3$ and through centre (5, 3) is $x=5$.

Now, solving $x=5$ and $x^2+y^2-10x-6y+30=0$

$\Rightarrow \quad 25+y^2-50-6y+30=0$

$\Rightarrow \quad y^2-6y+5=0$

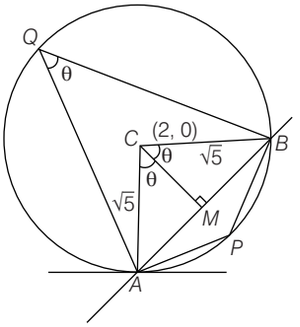
$\Rightarrow \quad (y-1)(y-5)=0$ or $y=1,5$

$\therefore$Other two vertices are (5, 1) and (5, 5).

Sol. (Q. Nos. 46 to 48)

$S: x^2+y^2-4x-1=0$ ...(i)

$L: y=3x-1$ ...(ii)



From Eqs. (i) and (ii), we get $x^2+(3x-1)^2-4x-1=0$

or $\quad 10x^2-10x=0$

$\therefore \quad x=0$ and $x=1$

From Eq. (ii), $y=-1$ and $y=2$

$\therefore \quad A\equiv(0,-1)$ and $B\equiv(1,2)$

46. $AB=\sqrt{(0-1)^2+(-1-2)^2}=\sqrt{10}$

47. Let $\angle ACB=2\theta$

$\therefore \quad CM=\frac{5}{\sqrt{10}}=\sqrt{\frac{5}{2}}$

$\Rightarrow \quad \cos\theta=\frac{CM}{\sqrt{5}}=\frac{1}{\sqrt{2}}$

$\therefore \quad \theta=\frac{\pi}{4}$

Required angle $=\angle APB=\pi-\theta=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$

48. Equation of tangent at $A(0,-1)$ to the circle S is

$0.x+(-1).y-2(x+0)-1=0$

or $\quad 2x+y+1=0$

$\therefore$ Slope $(m_1)=-2$

and slope of line L is $m_2=3$

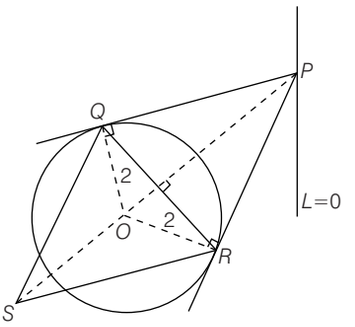
If θ be the angle between L and S, then $\tan\theta=\left|\frac{-2-3}{1+(-2)(3)}\right|=1$

$\therefore \quad \theta=\frac{\pi}{4}$

Sol. (Q. Nos. 49 to 51)

$\because \quad PQ=PR$,

parallelogram $PQRS$ is a rhombus.



$\therefore$Mid-point of QR = mid point of PS and $QR\perp PS$

Therefore, S in the mirror image of P w.r.t. QR.

49. Let $P(\lambda,6-2\lambda)$ be any point on $L=0$

$\because$ Circumcircle of ΔPQR always pass through O

$\therefore OP$ is diameter of circle.

Then, centre is $\left(\frac{\lambda}{2},3-\lambda\right)$ So,

$x=\frac{\lambda}{2}$ and $y=3-\lambda$

or $2x+y=3$ is the locus of the circumcenter of ΔPQR.

50. Area of ΔQRS = Area of $\Delta PQR = \dfrac{RL^3}{R^2+L^2}$ ...(i)

Here, $R = 2,\ L = \sqrt{S_1} = \sqrt{(6^2+8^2-4)} = \sqrt{96}$

From Eq. (i),

Area of $\Delta QRS = \dfrac{2\times 96\times\sqrt{96}}{4+96} = \dfrac{192\sqrt{6}}{25} = \dfrac{192\sqrt{\lambda}}{25}$ (given)

$\therefore \quad \lambda = 6$

51. Let $S \equiv (\alpha, \beta)$

$\because$ S is the mirror image of P w.r.t. QR.

Eq. (i) of QR is $3x + 4y = 4$, then

$$\frac{\alpha-3}{3} = \frac{\beta-4}{4} = \frac{-2(3.3+4.4-4)}{3^2+4^2}$$

$$= -\frac{42}{25}$$

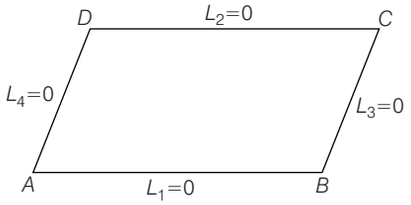
$\therefore \quad \alpha = -\dfrac{51}{25}$ and $\beta = -\dfrac{68}{25}$

Hence, $S \equiv \left(-\dfrac{51}{25}, -\dfrac{68}{25}\right)$

Sol. (Q. Nos. 52 to 54)

52. This is third degree equation which satisfy the point of intersection of $L_1 = 0$, $L_2 = 0$ and $L_3 = 0$.

53. $\lambda L_1L_2 + \mu L_3L_4 = 0$ will always pass through the vertices of the parallelogram for all $\lambda, \mu \in R \sim \{0\}$



54. Since, $\mu = 0$

So, $\quad L_1L_2 + \lambda L_2L_3 = 0$

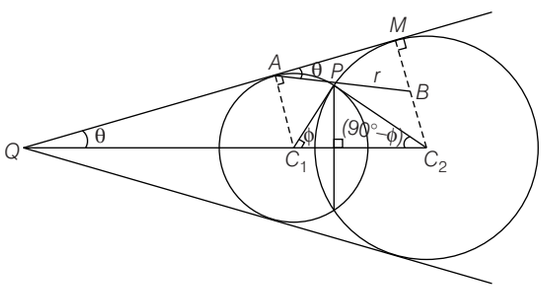
$\Rightarrow \quad L_2(L_1 + \lambda L_3) = 0$

This is the equation of pair of straight lines.

Sol. (Q. Nos. 55 to 57)

Let $2d$ = length of common chord and $\angle PC_1C_2 = \phi$, then

$\sin\phi = \dfrac{d}{3}$



and $\quad \sin(90°-\phi) = \dfrac{d}{r}$ or $\cos\phi = \dfrac{d}{r}$

$$1 = \frac{d^2}{9} + \frac{d^2}{r^2}$$

$\Rightarrow \quad d = \dfrac{3r}{\sqrt{9+r^2}} = \dfrac{1}{2}\left(\dfrac{24}{5}\right)$ (given)

$\Rightarrow \quad \dfrac{r}{\sqrt{(9-r^2)}} = \dfrac{4}{5}$

$\therefore \quad r = 4$

55. $\lambda = r = 4$

56. $\because (C_1C_2)^2 = r^2 + 3^2 = 16 + 9 = 25$

$\therefore \quad C_1C_2 = 5$

In ΔABM, $\quad \sin\theta = \dfrac{BM}{AB} = \dfrac{4-3}{C_1C_2} = \dfrac{1}{5}$

$\therefore \quad \sin 2\theta = 2\sin\theta\cos\theta = 2\times\dfrac{1}{5}\times\sqrt{1-\left(\dfrac{1}{5}\right)^2} = \dfrac{4\sqrt{6}}{25}$

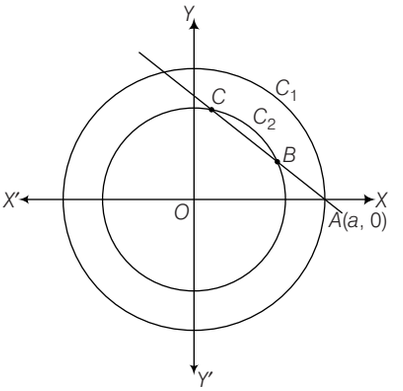
57. Length of direct common tangent $= \sqrt{(C_1C_2)^2 - (r-3)^2}$

$$= \sqrt{(5)^2 - (1)^2}$$

$$= \sqrt{24} = \lambda$$ (given)

$\therefore \quad \lambda^2 = 24$

Sol. (Q. Nos. 58 to 60)



Let equation of line AB be

$$\frac{x-a}{\cos\theta} = \frac{y-0}{\sin\theta} = r$$

Coordinates of any point on the line AB is $(a + r\cos\theta, r\sin\theta)$, then

$$B \equiv (a + r_1\cos\theta, r_1\sin\theta)$$

and $\quad C \equiv (a + r_2\cos\theta, r_2\sin\theta)$

58. $\lambda = (OA)^2 + (OB)^2 + (BC)^2$

$$= a^2 + b^2 + (r_1 - r_2)^2\cos^2\theta + (r_1 - r_2)^2\sin^2\theta$$

$$= (a^2 + b^2) + (r_1 - r_2)^2$$

$$= (a^2 + b^2) + 4a^2\cos^2\theta - 4(a^2 - b^2)$$

($\because$ B and C lie on $x^2 + y^2 = b^2$

$\therefore \quad (a + r\cos\theta)^2 + (r\sin\theta)^2 = b^2$

or $\quad r^2 + 2ar\cos\theta + a^2 - b^2 = 0$

$\therefore \quad r_1 - r_2 = \sqrt{4a^2\cos^2\theta - 4(a^2 - b^2)}$

$\Rightarrow \quad \lambda = 5b^2 - 3a^2 + 4a^2\cos^2\theta$

$\because \quad 0 \le \cos^2\theta \le 1$

$\therefore \quad \lambda \in [5b^2 - 3a^2, 5b^2 + a^2]$

59. Let (h,k) be the mid-point of AB and let (α,β) be the coordinates of B, then

$$\frac{a+\alpha}{2}=h \text{ and } \frac{0+\beta}{2}=k$$

$$\Rightarrow \quad \alpha=2h-a \text{ and } \beta=2k$$

Since, (α,β) lies on $x^2+y^2=b^2$

$$\Rightarrow \quad (2h-a)^2+(2k)^2=b^2$$

or $$\left(h-\frac{a}{2}\right)^2+k^2=\frac{b^2}{4}$$

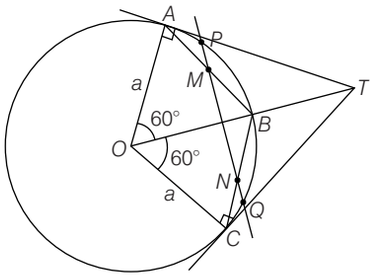
Hence, locus of (h,k) is

$$\left(x-\frac{a}{2}\right)^2+y^2=\frac{b^2}{4}$$

60. The locus of mid-point of AB, when BC is maximum is a fixed point M on X-axis.

Sol. (Q. Nos. 61 to 63)

61. From the figure $OA=OB=AB=a$



$$\therefore \quad \angle OAB=60^\circ$$

62. Let T be the point of intersection of tangents.

$$\because \quad \angle AOC=120^\circ$$

$$\therefore \quad \angle ATC=180^\circ-120^\circ=60^\circ$$

63. Locus of point of intersection of tangents at A and C is

$$x^2+y^2=\frac{a^2}{\sin^2\left(\frac{\alpha}{2}\right)}$$

$$=\frac{a^2}{\sin^2(30^\circ)}=4a^2 \qquad (\text{Here, } \alpha=60^\circ)$$

Hence, $x^2+y^2=4a^2$

Sol. (Q. Nos. 64 to 66)

Here, $$t_1^2=h^2+k^2-4,$$

$$t_2^2=h^2+k^2-4h,$$

$$t_3^2=h^2+k^2-4k$$

given $$t_1^4=t_2^2t_3^2+16$$

or $$(h^2+k^2-4)^2=(h^2+k^2-4h)(h^2+k^2-4k)+16$$

or $$(h^2+k^2)^2-8(h^2+k^2)+16=(h^2+k^2)^2-4(h^2+k^2)(h+k)+16hk+16$$

or $$(h^2+k^2)(h+k)-2(h^2+k^2)-4hk=0$$

or $$(h^2+k^2)(h+k)-2(h+k)^2=0$$

or $$(h+k)(h^2+k^2-2(h+k))=0$$

or $$h+k=0 \qquad \ldots(i)$$

and $$h^2+k^2-2h-2k=0 \qquad \ldots(ii)$$

64. Locus of (h,k) is [From Eq. (i)]

$$L_1: x+y=0$$

65. Locus of (h,k) is [From Eq. (ii)]

$$C_1: x^2+y^2-2x-2y=0$$

66. $\because$ Circle C_2 is equal to circle C_1

$\therefore$ Radius of circle C_2 = radius of circle $C_1=\sqrt{2}$

$\therefore$ Distance between the centres of C_1 and C_2

$$=\sqrt{2}+\sqrt{2}$$

$$=2\sqrt{2} \qquad (\because \text{ circles } C_1 \text{ and } C_2 \text{ touch externally})$$

67. The given circle is

$$S:(x-3)^2+(y-5)^2=(\sqrt{(34-\lambda)})^2$$

Since, point $P(1,4)$ lies inside the circle

$$\therefore \quad S_1<0$$

$$\Rightarrow \quad 1+16-6-40+\lambda<0$$

or $$\lambda<29 \qquad \ldots(i)$$

Also, circle neither touches nor cuts the axes, then

$$3>\sqrt{(34-\lambda)} \text{ or } \lambda>25 \qquad \ldots(ii)$$

and $$5>\sqrt{(34-\lambda)} \text{ or } \lambda>9 \qquad \ldots(iii)$$

From Eqs. (i), (ii) and (iii), we get

$$25<\lambda<29$$

Hence, difference $=29-25=4$

68. $x^2+y^2-2x-2\lambda y-8=0$

or $(x^2+y^2-2x-8)-2\lambda y=0$

which is of the form $S+\lambda L=0$

All the circles pass through the point of intersection of the circle $x^2+y^2-2x-8=0$ and $y=0$

Solving, we get

$$x^2-2x-8=0$$

or $$(x-4)(x+2)=0$$

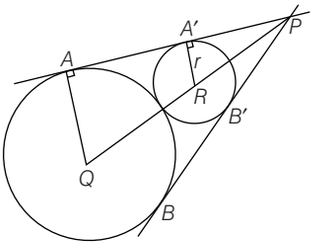
$$\therefore \quad A\equiv(-2,0)$$

and $$B\equiv(4,0)$$

Hence, $|AB|=6$

69. $\because AQ=BQ=3+2\sqrt{2}$

and PQ = radius of director circle of C_1



$$=\sqrt{2}(3+2\sqrt{2})=3\sqrt{2}+4$$

Let 'r' be required radius

$$\therefore \quad 3\sqrt{2}+4=3+2\sqrt{2}+r+r\sqrt{2} \qquad \left(\because \frac{PR}{PQ}=\frac{A'R}{AQ}\right)$$

$$\Rightarrow \quad r=\frac{(\sqrt{2}+1)}{(\sqrt{2}+1)}=1$$

70. Since, **A.** $x+y-5=0$ is a tangent to the circle at $P(2,3)$, so the centre of the circle lies on $x-y+1=0$. Now $Q(1,2)$ lies on the circle.

Equation of right bisector of PQ is $x+y-4=0$

$\therefore$ Centre of circle is point of intersection of $x-y+1=0$ and $x+y-4=0$ which is $\left(\frac{3}{2},\frac{5}{2}\right)$

$\therefore$ Radius $\frac{1}{\sqrt{\lambda}}=\frac{\left|\frac{3}{2}+\frac{5}{2}-5\right|}{\sqrt{2}}=\frac{1}{\sqrt{2}}$

$\therefore \quad \lambda=2$

Hence, $\lambda^2=4$

71. Since, given circle is $(x+5)^2+(y-12)^2=(14)^2$

or $\left(\frac{x+5}{14}\right)^2+\left(\frac{y-12}{14}\right)^2=1$

Let, $\frac{x+5}{14}=\cos\theta$ and $\frac{y-12}{14}=\sin\theta$

or $x=14\cos\theta-5$ and $y=14\sin\theta+12$

$\therefore \quad x^2+y^2=(14\cos\theta-5)^2+(14\sin\theta+12)^2$

$=196+25+144+28(12\sin\theta-5\cos\theta)$

$=365+28(12\sin\theta-5\cos\theta)$

Maximum value of $(x^2+y^2)=365+28\times13=729$

$(\because -13\le 12\sin\theta-5\cos\theta\le 13)$

or maximum value of $(x^2+y^2)^{\frac{1}{3}}=9$

72. Let the given lines are represented by L_1, L_2, L_3 and L_4, then

$L_1\equiv 2x+3y-2=0 \qquad L_2\equiv 3x-2y-3=0$

$L_3\equiv x+2y-3=0 \qquad L_4\equiv 2x-y-1=0$

Equation of second degree conic circumscribing a quadrilateral whose sides are $L_1=0$, $L_2=0$, $L_3=0$ and $L_4=0$ is

$L_1L_3+kL_2L_4=0$

or $(2x+3y-2)(x+2y-3)+k(3x-2y-3)(2x-y-1)=0$ ...(i)

For circle, coefficient of $x^2=$ coefficient of y^2

or $2+6k=6+2k$

$\therefore \quad k=1$

From, Eq. (i), required circle is

$8x^2+8y^2-17x-8y+9=0$

or $x^2+y^2-\left(\frac{17}{8}\right)x-y+\left(\frac{9}{8}\right)=0$

Here, $\lambda=-\frac{17}{8},\mu=-1,\nu=\frac{9}{8}$

then, $|\lambda+2\mu+\nu|=3$

73. Let radius of given circle is a

$\therefore \quad a=\sqrt{(4+2-c)}=\sqrt{(6-c)}$ ...(i)

Now, radius of circle $C_1=\frac{a}{\sqrt{2}}$ and radius of circle $C_2=\frac{a}{(\sqrt{2})^2}$

and so on.

Also, $a+\frac{a}{\sqrt{2}}+\frac{a}{(\sqrt{2})^2}+\ldots\infty=2$ (gjven)

Then, $\frac{a}{1-\frac{1}{\sqrt{2}}}=2$

or $a=2-\sqrt{2}=\sqrt{6-c}$ [From Eq. (i)]

or $4+2-4\sqrt{2}=6-c$

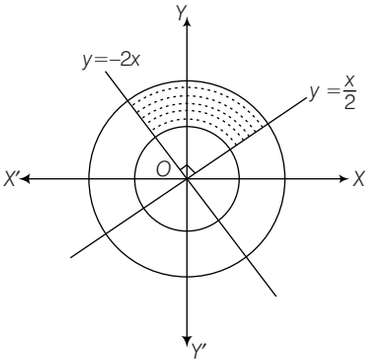
$\therefore \quad c=4\sqrt{2}$

Hence, $\lambda=4$

74. The rays are given as $2x^2-3xy-2y^2=0$ i.e. $(2x+y)(x-2y)=0$

$\Rightarrow \; y=\frac{x}{2}$ or $y=-2x$ which are perpendicular.

$\therefore$ Required area



$=\frac{1}{4}(\pi(2)^2-\pi(1)^2)$

$=\frac{3\pi}{4}=\frac{\lambda\pi}{4}$ (given)

$\therefore \quad \lambda=3$

75. Let radii of circles be r_1 and r_2 and distance between centres is d, then

$\sqrt{d^2-(r_1+r_2)^2}=5$ ...(i)

and $\sqrt{d^2-(r_1-r_2)^2}=13$ ...(ii)

From Eqs. (i) and (ii), we get

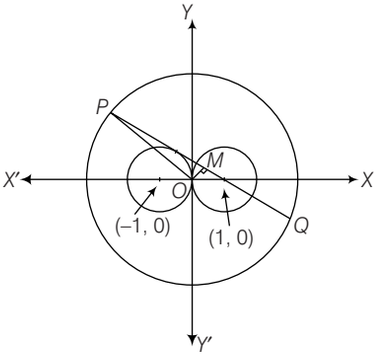
$(r_1+r_2)^2-(r_1-r_2)^2=(13)^2-(5)^2$

$\Rightarrow \quad 4r_1r_2=144$ or $\lambda=36$ $(\because r_1r_2=\lambda)$

$\therefore \quad \frac{\lambda}{4}=9$

76. Let $ax+by=1$ ...(i)

be a chord PQ.



which pass through (1, 0), then $a + 0 = 1$

$\therefore \quad a = 1$

and touch circle B with centre $(-1, 0)$ and radius 1, then

$$\frac{|-a + 0 - 1|}{\sqrt{a^2 + b^2}} = 1$$

or $\quad a^2 + b^2 = (a+1)^2$

or $\quad b^2 = 2a + 1 = 3$

$\therefore \quad b = \sqrt{3}$

From Eq. (i), Equation of chord is $x + \sqrt{3}\, y = 4$

$\therefore \quad OM = \frac{1}{2}$ and $OP = 4$

$\therefore \quad PQ = 2PM = 2\sqrt{(OP)^2 - (OM)^2}$

$$= 2\sqrt{\left(16 - \frac{1}{4}\right)} = \sqrt{63} = \sqrt{\lambda}$$

$\therefore \quad \lambda = 63$

Hence, $\quad \frac{\lambda}{9} = 7$

77. $\because \quad S_1 : (x-2)^2 + (y-3)^2 = 1$

Centre C_1:(2,3) and radius r_1:1 and $S_2 : (x-5)^2 + (y-6)^2 = r^2$

Centre C_2:(5,6) and radius r_2:r

$\Rightarrow \quad C_1C_2 = 3\sqrt{2}$

(A) $\because S_1$ and S_2 touch internally, then

$C_1C_2 = r_2 - r_1$

or $\quad 3\sqrt{2} = r - 1$

or $\quad (r-1)^2 = 18$

(B) $\because S_1$ and S_2 touch externally, then

$C_1C_2 = r_2 + r_1$

or $\quad 3\sqrt{2} = r + 1$

or $\quad r^2 + 2r + 3 = (r+1)^2 + 2 = 18 + 2 = 20$

(C) $\because S_1$ and S_2 intersect orthogonally, then

$(C_1C_2)^2 = r_1^2 + r_2^2$

$\Rightarrow \quad 18 = r^2 + 1$

or $\quad r^2 - 1 = 16$

(D) $\because S_1$ and S_2 intersect, the common chord is $S_1 - S_2 = 0$

i.e. $\quad 6x + 6y + r^2 - 49 = 0$

Given, common chord is longest, then passes through (2, 3)

$\Rightarrow \quad 12 + 18 + r^2 - 49 = 0$

or $\quad r^2 = 19$

$\therefore \quad r^2 + 5 = 24$

78. **(A)** Since (2, 3) lies on $ax + by - 5 = 0$

$\therefore \quad 2a + 3b - 5 = 0 \quad \ldots$(i)

Since, line is at greatest distance from centre

$\Rightarrow \quad \left(\frac{4-3}{3-2}\right)\left(-\frac{a}{b}\right) = -1$ i.e. $a = b$

From Eq. (i), $a = b = 1$

$\therefore \quad a + b = 2$ and $a^2 + b^2 = 2$

(B) Let P be the point (h,k), then $h^2 + k^2 + 2h + 2k = 0$ ...(i)

mid-point of OP is $\left(\frac{h}{2}, \frac{k}{2}\right)$

Let $\quad x = \frac{h}{2}, y = \frac{k}{2}$

or $\quad h = 2x, k = 2y$

From Eq. (i), $4x^2 + 4y^2 + 4x + 4y = 0$

or $\quad x^2 + y^2 + x + y = 0$

On comparing, we get

$2a = 1, 2b = 1$

or $\quad a = b = \frac{1}{2}$

$\therefore \quad a + b = 1$

(C) Centre of circles are C_1:(1,2) and C_2:(5,−6)

Equation of C_1C_2 is $y - 2 = \frac{-6-2}{5-1}(x-1)$

or $\quad 2x + y - 4 = 0 \quad \ldots$(i)

Equation of radical axis is $8x - 16y - 56 = 0$

or $\quad x - 2y - 7 = 0 \quad \ldots$(ii)

Point of intersection of Eqs. (i) and (ii) are (3, −2)

$\therefore \quad a = 3, b = -2$

or $\quad a + b = 1$

(D) Let $S \equiv x^2 + y^2 - 6\sqrt{3}x - 6y + 27 = 0$

$S_1 = 0^2 + 0^2 - 0 - 0 + 27 = 27$

$T = 0.x + 0.y - 3\sqrt{3}(x+0) - 3(y+0) + 27$

$= -3\sqrt{3}x - 3y + 27$

Eq. of the pair of tangents is given by

$SS_1 = T^2$

$\Rightarrow \quad 27(x^2 + y^2 - 6\sqrt{3}x - 6y + 27) = (-3\sqrt{3}x - 3y + 27)^2$

or $\quad 3(x^2 + y^2 - 6\sqrt{3}x - 6y + 27) = (\sqrt{3}x + y - 9)^2$

or $\quad 18y^2 - 18\sqrt{3}xy = 0$

or $\quad 18y(y - \sqrt{3}x) = 0$

the tangents are $y = 0, y = \sqrt{3}x$

$\therefore \quad a = 0, b = \sqrt{3}$

then, $\quad a^2 + b^2 = 3$

79. **(A)** $\because \quad P \equiv (10,7)$

and $\quad S = x^2 + y^2 - 4x - 2y - 20$

$\therefore \quad S_1 = 100 + 49 - 40 - 14 - 20 > 0$

$\therefore P$ outside the circle, radius $r = \sqrt{4 + 1 + 20} = 5$

centre $C \equiv (2,1)$

$\therefore$ Shortest distance $L = CP - r$

$= \sqrt{(10-2)^2 + (7-1)^2} - 5$

$= \sqrt{(64+36)} - 5 = 10 - 5 = 5$

And largest distance $M = CP + r$

$= 10 + 5 = 15$

$M + L = 20, M - L = 10$

(B) $\because \quad P \equiv (3,-6)$

and $\quad S \equiv x^2+y^2-16x-12y-125$

$\therefore \quad S_1 = 9+36-48+72-125<0$

$\therefore$ P inside the circle radius $r=\sqrt{(64+36+125)}=15$

centre $C \equiv (8,6)$

$\therefore$ Shortest distance $L = r-CP$

$$= 15-\sqrt{(8-3)^2+(6+6)^2}$$

$$= 15-13=2$$

And largest distance $M = r+CP$

$$= 15+13=28$$

$$M+L=30,$$

$$M-L=26$$

(C) $P \equiv (6,-6)$

and $\quad S \equiv x^2+y^2-4x+6y-12$

$\therefore \quad S_1 = (6)^2+(-6)^2-24-36-12=0$

P on the circle

$\therefore$ radius $r=\sqrt{(4+9+12)}=5$

$\therefore$ Shortest distance $L=0$

and largest distance $M=2r=10$

$$M+L=10,\ M-L=10$$

80. **(A)** Equation of conic $(y-a_1x-b)(y-a_2x-b)+\lambda xy=0$ represent a circle. If coefficient of x^2 = coefficient of y^2

$\therefore \quad a_1a_2=1$

$\because \quad AM>GM$

$\therefore \quad \dfrac{a_1+a_2}{2} > \sqrt{a_1a_2} = 1$

$\Rightarrow \quad a_1+a_2>2$

and $\quad \dfrac{a_1^2+a_2^2}{2} > \sqrt{a_1^2.a_2^2} = |a_1a_2| = 1$

$\therefore \quad a_1^2+a_2^2>2$

(B) Let $(a_1\cos\theta, a_1\sin\theta)$ be any point on $x^2+y^2=a_1^2$, then chord of contact is

$$x(a_1\cos\theta)+y(a_1\sin\theta)=b^2$$

which is tangent of $x^2+y^2=a_2^2$

$\therefore \quad \dfrac{|0+0-b^2|}{\sqrt{\{(a_1\cos\theta)^2+(a_1\sin\theta)^2\}}} = a_2$ or $b^2=a_1a_2$

for $\quad b=1,\ a_1a_2=1$

$$AM>GM$$

$\Rightarrow \quad \dfrac{a_1+a_2}{2} > \sqrt{a_1a_2} = |b|$

$\therefore \quad a_1+a_2>2|b|,$

for $\quad b=1,\ a_1+a_2>2$

and then, $a_1^2+a_2^2>2$

(C) $2gg_1+2ff_1=c+c_1$

$\Rightarrow \quad 2\times a_1\times a_2+0=b+b$

$\therefore \quad a_1a_2=b$ for $b=1, a_1a_2=1$

and then, $\quad a_1+a_2>2$

Also, $\quad a_1^2+a_2^2>2$

81. Let $S \equiv x^2+y^2-1=0$

$\therefore \quad S(1,3)=1^2+3^2-1=9>0$

$\Rightarrow$ Point (1, 3) outside the circle $x^2+y^2=1$

$\therefore$ Two tangents can be drawn from (1, 3) to circle $x^2+y^2=1$

$\Rightarrow$ Statement I is false

Also, $\quad \dfrac{|3-m|}{\sqrt{(1+m^2)}}=1$

Squaring both sides, we get

$$9+m^2-6m=1+m^2$$

or $\quad 6m=8 \Rightarrow m=\dfrac{4}{3}$

$\Rightarrow$ Statement II is true.

82. $\because x^2+y^2+2x-4y=0$

$\Rightarrow \quad (x+1)^2+(y-2)^2=1^2+2^2$

Let $S \equiv (x+1)^2+(y-2)^2-1^2-2^2$

Value of $S(1,\lambda)=(1+1)^2+(\lambda-2)^2-1^2-2^2<0$

or $\quad (1+1)^2+(\lambda-2)^2<1^2+2^2$

$\therefore$ Points (1, λ) inside the circle.

$\Rightarrow$ No tangents can be drawn from the point (1, λ) to the circle

$$x^2+y^2+2x-4y=0$$

and $\quad (1+1)^2+(\lambda-2)^2<1^2+2^2$

$\Rightarrow \quad (\lambda-2)^2<1$

or $\quad -1<\lambda-2<1$

or $\quad 1<\lambda<3$

$\therefore$ Statement II and Statement II are both true and statement II is correct explanation for statement I.

83. Let $A \equiv (1,4),\ B \equiv (2,3)$ and $C \equiv (-1,6)$

$$\therefore \text{Area of } \Delta ABC = \frac{1}{2}\begin{vmatrix} 1 & 4 & 1 \\ 2 & 3 & 1 \\ -1 & 6 & 1 \end{vmatrix} = 0$$

$\Rightarrow A, B, C$ are collinear

i.e. no circle is drawn.

Hence, Statement I is false and through three non-collinear points in a plane only one circle can be drawn.

$\therefore$ Statement II is true.

84. $\because$ Locus of point of intersection of perpendicular tangents is director circle.

$\because x^2+y^2=50$ is director circle of $x^2+y^2=25$

$\Rightarrow$ Statement I is false and statement II is true.

85. Here, $C_1 \equiv (0,0),\ r_1=2$

and $\quad C_2 \equiv (3,0),\ r_2=2$

$\therefore \quad |C_1C_2|=3$ and $r_1+r_2=4$

and $\quad |r_1-r_2|=0$

$\Rightarrow \quad |r_1-r_2|<|C_1C_2|<r_1+r_2$

Hence, circles $x^2+y^2=4$ and $x^2+y^2-6x+5=0$ intersect each other at two distinct points.

$\Rightarrow$ Statement I is true and Statement II is false.

86. $\because$ Centre of circle $(1, -1)$ lies on $3x - 4y = 7$

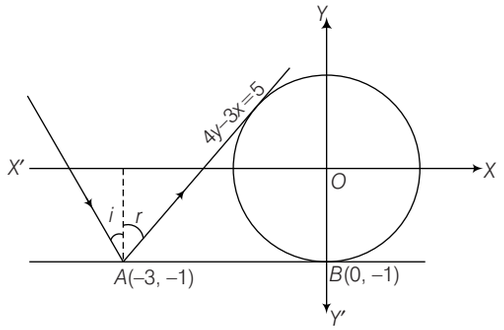
$\therefore$ $3x - 4y = 7$ is a diameter of circle

$$x^2 + y^2 - 2x + 2y - 47 = 0$$

$\therefore$Statement I is true.

Statement II is always true but Statement I is not a correct explanation of Statement I.

87. Statement I is true, line $4y - 3x - 5 = 0$ passes through $A(-3, -1)$ and its distance from $O(0, 0)$ is 1 unit = radius of circle



The tangent at $B(0, -1)$ is $y = -1$ and normal at A to line AB is $x = -3$

$\therefore$Equation of incident ray is $y + 1 = \left(-\frac{4}{3}\right)(x + 3)$

$\Rightarrow$ $4x + 3y + 15 = 0$

Statement II is obviously true.

Hence, both statements are true and but Statement II is not correct explanation of Statement I.

88. The given points are $A(1,1)$, $B(2,3)$ and $C(3,5)$ which are collinear as

Slope of AB = Slope of $BC = 2$

Hence, Statement II is true

The chords of contact are concurrent, then,

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Hence, points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear.

Therefore, Both statements are true and statement II is correct explanation of statement I.

89. Let the equation of circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \ldots(i)$$

Since, circle Eq. (i) passes through (1,0) and (0,1) then

$$1 + 2g + c = 0 \quad \Rightarrow \quad g = -\frac{(1 + c)}{2} \quad \ldots(ii)$$

and $$1 + 2f + c = 0 \quad \Rightarrow \quad f = -\frac{(1 + c)}{2} \quad \ldots(iii)$$

Radius of the circle Eq. (ii) $= \sqrt{(g^2 + f^2 - c)}$

$$= \sqrt{\frac{(1 + c)^2}{4} + \frac{(1 + c)^2}{4} - c} = \sqrt{\left(\frac{1 + c^2}{2}\right)}$$

For minimum radius, c must be equal to zero, then from Eqs. (ii) and (iii),

$$g = -\frac{1}{2} \quad \text{and} \quad f = -\frac{1}{2}$$

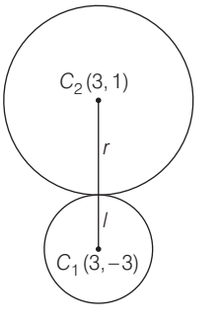
Equation of required circle, from Eq. (i), is

$$x^2 + y^2 - x - y = 0$$

90. The given circle is

$$x^2 + y^2 - 6x + 6y + 17 = 0$$

Centre and radius of this circle are $(3, -3)$ and $\sqrt{9 + 9 - 17} = 1$ respectively. But given the required circle has normals



$$x^2 - 3xy - 3x + 9y = 0$$

or $$(x - 3)(x - 3y) = 0$$

or $$x = 3 \quad \text{and} \quad x - 3y = 0 \quad \ldots(i)$$

but point of intersection of normals is the centre of the circle. Point of intersection of normals represented by Eq. (i) is (3, 1) which is centre of the required circle. Since, given circle and required circle touch each other externally, then (if radius of required circles is r)

Sum of radii = Distance between the centres

$$r + 1 = \sqrt{(3 - 3)^2 + (-3 - 1)^2} = 4$$

$\therefore$ $$r = 3$$

$\therefore$Equation of required circle is

$$(x - 3)^2 + (y - 1)^2 = 3^2$$

or $$x^2 + y^2 - 6x - 2y + 1 = 0$$

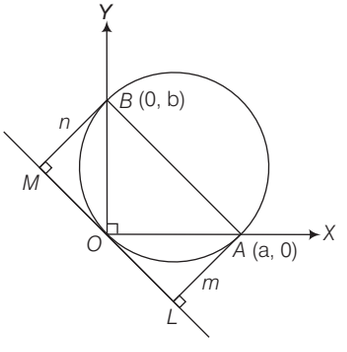
91. Let $OA = a$ and $OB = b$ then the coordinates of A and B are $(a, 0)$ and $(0, b)$ respectively.

Since, $\angle AOB = \pi / 2$

Hence, AB is the diameter of the required circle whose equation is

$$(x - a)(x - 0) + (y - 0)(y - b) = 0$$

or $$x^2 + y^2 - ax - by = 0 \quad \ldots(i)$$



Equation of tangent at (0,0) of Eq. (i) is

$$0 \cdot x + 0 \cdot y - \frac{a}{2}(x + 0) - \frac{b}{2}(0 + y) = 0$$

or $$ax + by = 0$$

$$\therefore \quad m = AL = \frac{a \cdot a + 0}{\sqrt{a^2 + b^2}}$$

or $$m = \frac{a^2}{\sqrt{a^2 + b^2}} \quad \text{...(ii)}$$

and $$n = BM = \frac{a \cdot 0 + b \cdot b}{\sqrt{a^2 + b^2}}$$

$$\therefore \quad n = \frac{b^2}{\sqrt{a^2 + b^2}} \quad \text{...(iii)}$$

Adding Eqs. (ii) and (iii), we get

$$m + n = \sqrt{(a^2 + b^2)}$$

From Eqs. (ii) and (iii), we get

$$a = \pm\sqrt{m\,(m + n)} \quad \text{and} \quad b = \pm\sqrt{n\,(m + n)}$$

From Eq. (i), equation of required circle is

$$x^2 + y^2 \pm \sqrt{m\,(m + n)}\,x \pm \sqrt{n(m + n)}\,y = 0$$

92. Solving the equations

$$(2 + c)\,x + 5c^2 y = 1 \quad \text{and} \quad 3x + 5y = 1$$

then $$(2 + c)\,x + 5c^2\left(\frac{1 - 3x}{5}\right) = 1$$

or $$(2 + c)\,x + c^2\,(1 - 3x) = 1$$

$$\therefore \quad x = \frac{1 - c^2}{2 + c - 3c^2}$$

or $$x = \frac{(1 + c)\,(1 - c)}{(3\,c + 2)\,(1 - c)} = \frac{1 + c}{3c + 2}$$

$$\therefore \quad x = \lim_{c \to 1} \frac{1 + c}{3c + 2}$$

$$x = \frac{2}{5}$$

$$\therefore \quad y = \frac{1 - 3x}{5} = \frac{1 - \frac{6}{5}}{5} = -\frac{1}{25}$$

Therefore, the centre of the required circle is $\left(\frac{2}{5}, -\frac{1}{25}\right)$ but circle passes through (2, 0)

$$\therefore \text{ Radius of the required circle} = \sqrt{\left(\frac{2}{5} - 2\right)^2 + \left(-\frac{1}{25} - 0\right)^2}$$

$$= \sqrt{\frac{64}{25} + \frac{1}{625}} = \sqrt{\frac{1601}{625}}$$

Hence, the required equation of the circle is

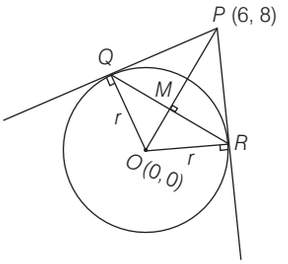
$$\left(x - \frac{2}{5}\right)^2 + \left(y + \frac{1}{25}\right)^2 = \frac{1601}{625}$$

or $$25x^2 + 25y^2 - 20x + 2y - 60 = 0$$

93. Equation of chord of contact (QR) is $6x + 8y - r^2 = 0$

$$\therefore \quad PM = \frac{|\,6 \cdot 6 + 8 \cdot 8 - r^2\,|}{\sqrt{(6^2 + 8^2)}} = \frac{|\,100 - r^2\,|}{10}$$

and $$OM = \frac{|\,0 + 0 - r^2\,|}{\sqrt{(6^2 + 8^2)}} = \frac{r^2}{10}$$



then $$QR = 2 \cdot QM = 2\sqrt{\{(OQ)^2 - (OM)^2\}}$$

$$= 2\sqrt{\left(r^2 - \frac{r^4}{100}\right)}$$

$\therefore$ Area of $\Delta QPR = \frac{1}{2} \cdot QR \cdot PM$

$$\Delta\,(\text{say}) = \frac{1}{2} \cdot 2\sqrt{\left(r^2 - \frac{r^4}{100}\right)} \cdot \frac{|\,100 - r^2\,|}{10}$$

$$\therefore \quad \Delta^2 = \frac{r^2\,(100 - r^2)^3}{1000} = z \quad \text{(say)}$$

$$\therefore \quad \frac{dz}{dr} = \frac{1}{1000}\{r^2 \,.\, 3\,(100 - r^2)^2 \,.\, (-2r) + (100 - r^2)^3 \cdot 2r\}$$

$$= \frac{2r\,(100 - r^2)^2}{1000}\{100 - r^2 - 3r^2\}$$

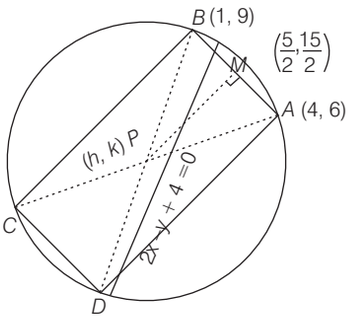
For maximum or minimum $\frac{dz}{dr} = 0$, then we get $r = 5$, ($r \neq 10$ as P is outside the circle)

and $$\left.\frac{d^2 z}{dr^2}\right|_{r=5} = -\text{ve}$$

$\therefore \Delta$ is also maximum at $r = 5$.

94. Since, A (4, 6) and B (1, 9) do not lie on $2x - y + 4 = 0$.

Let M be the mid-point of AB, then coordinates of M is $\left(\frac{5}{2}, \frac{15}{2}\right)$



Slope of $AB = \dfrac{9-6}{1-4} = -1$

$\therefore$ Slope of $PM = 1$

Equation of PM is, $\quad y - \dfrac{15}{2} = 1 \cdot \left(x - \dfrac{5}{2}\right)$

which passes through $P(h, k)$, then

$$k - \frac{15}{2} = h - \frac{5}{2} \quad \text{or} \quad k - h = 5 \qquad \ldots(i)$$

and (h, k) lie on $2x - y + 4 = 0$

$$\therefore \quad 2h - k + 4 = 0 \qquad \ldots(ii)$$

Solving Eqs. (i) and (ii), we get

$$h = 1 \quad \text{and} \quad k = 6$$

Now, $\quad AD = 2PM = 2\sqrt{\left(1 - \dfrac{5}{2}\right)^2 + \left(6 - \dfrac{15}{2}\right)^2}$

$$= 2\sqrt{\frac{9}{4} + \frac{9}{4}} = 3\sqrt{2}$$

and $\quad AB = \sqrt{(4-1)^2 + (6-9)^2} = 3\sqrt{2}$

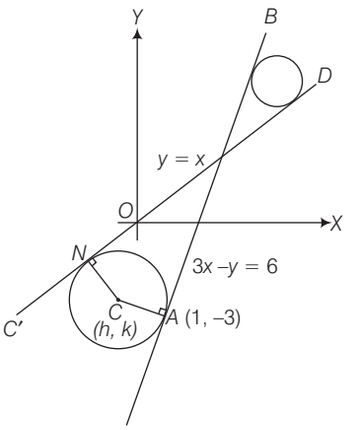
$\therefore$ Area of rectangle

$$ABCD = AB \times AD = 3\sqrt{2} \times 3\sqrt{2} = 18 \text{ sq units.}$$

95. Let $C(h, k)$ be the centre of the circle. Let AB and $C'D$ be the lines represented by

$3x - y = 6$ and $y = x$ respectively.

Clearly, the circle touches AB at $A(1, -3)$.



Equation of line $\perp$ to $3x - y = 6$ is $x + 3y = \lambda$ which passes through $(1, -3)$.

then $\quad 1 - 9 = \lambda$

$$\therefore \quad \lambda = -8$$

$\therefore \perp$ line is $\quad x + 3y + 8 = 0$

which passes through $C(h, k)$

then $\quad h + 3k + 8 = 0 \qquad \ldots(i)$

Now, centre $\quad C(h, k) \equiv C(-3k - 8, k)$

Radius $\quad CN = \dfrac{|-3k - 8 - k|}{\sqrt{1+1}} = CA$

$$= \sqrt{(-3k - 8 - 1)^2 + (k + 3)^2}$$

$$\Rightarrow \quad \frac{4|k+2|}{\sqrt{2}} = |k+3|\sqrt{10} \qquad \ldots(ii)$$

$$\Rightarrow \quad 4|k+2| = 2\sqrt{5}\,|k+3|$$

$$\Rightarrow \quad 2|k+2| = \sqrt{5}\,|k+3|$$

$$\therefore \quad k + 2 = \pm\frac{\sqrt{5}}{2}(k+3)$$

$$\therefore \quad k = -7 - 2\sqrt{5} \quad \text{or} \quad k = -7 + 2\sqrt{5}$$

Since, radius from Eq. (ii),

$$r = \frac{4|k+2|}{\sqrt{2}}$$

$$(\text{radius})_{\text{at } k = -7 - 2\sqrt{5}} = \frac{4|-7 - 2\sqrt{5} + 2|}{\sqrt{2}}$$

$$= 10\sqrt{2} + 4\sqrt{10} = 26.79$$

$$(\text{radius})_{\text{at } k = -7 + 2\sqrt{5}} = \frac{4|-7 + 2\sqrt{5} + 2|}{\sqrt{2}}$$

$$= 10\sqrt{2} - 4\sqrt{10} = 1.5$$

Hence, radius of smaller circle is 1.5 units.

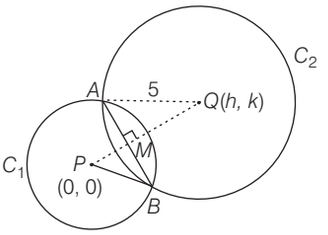
96. Let the centre of the circle C_2 is $Q(h, k)$, equation of the circle C_2 is

$$(x - h)^2 + (y - k)^2 = 5^2$$

or $\quad x^2 + y^2 - 2xh - 2yk + h^2 + k^2 - 25 = 0$

$$C_2 \equiv x^2 + y^2 - 2xh - 2yk + h^2 + k^2 - 25 = 0$$

and equation of circle $C_1 \equiv x^2 + y^2 - 16 = 0$



$\therefore$ Equation of common chord is

$$C_1 - C_2 = 0$$

$$\Rightarrow \quad -2xh - 2yk + h^2 + k^2 - 9 = 0$$

or $\quad 2hx + 2ky - (h^2 + k^2 - 9) = 0 \qquad \ldots(i)$

Slope of this line $= -h/k$

But, it is given that its slope $= 3/4$

$$\therefore \quad -\frac{h}{k} = \frac{3}{4}$$

or $\quad 3k + 4h = 0 \qquad \ldots(ii)$

Let p be the length perpendicular from $P(0, 0)$ on chord (1), then

$$p = \frac{h^2 + k^2 - 9}{\sqrt{(4h^2 + 4k^2)}}$$

or $\quad p = \dfrac{(h^2 + k^2 - 9)}{2\sqrt{(h^2 + k^2)}} \qquad \ldots(iii)$

Length of this chord $AB = 2AM$

$$= 2\sqrt{(16 - p^2)}$$

This chord has maximum length, then $p = 0$, then from Eq. (iii),

$$h^2 + k^2 - 9 = 0 \qquad \ldots(iv)$$

On solving Eqs. (ii) and (iv) we get

$$h = \mp\frac{9}{5}, k = \pm\frac{12}{5}$$

$\therefore$ Centre of C_2 is $\left(\frac{9}{5}, -\frac{12}{5}\right)$ or $\left(-\frac{9}{5}, \frac{12}{5}\right)$

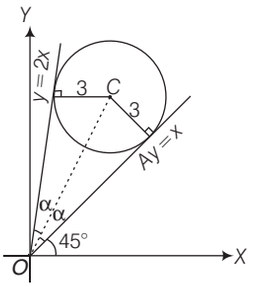
97. $\because\ 2x^2 + y^2 - 3xy = 0$

$\Rightarrow \quad (2x - y)(x - y) = 0$

$\Rightarrow \quad y = 2x, y = x$

are the equations of straight lines passing through origin.

Now, let the angle between tangents is 2α,



then, $\tan(45° + 2\alpha) = 2$

$\Rightarrow \quad \frac{\tan 45° + \tan 2\alpha}{1 - \tan 45° \tan 2\alpha} = 2$

$\Rightarrow \quad \frac{1 + \tan 2\alpha}{1 - \tan 2\alpha} = \frac{2}{1} \Rightarrow \frac{2\tan 2\alpha}{2} = \frac{1}{3}$

(By componendo and dividendo rule)

$\Rightarrow \quad \frac{2\tan\alpha}{1 - \tan^2\alpha} = \frac{1}{3}$

$\Rightarrow \quad \tan^2\alpha + 6\tan\alpha - 1 = 0$

$\therefore \quad \tan\alpha = \frac{-6 \pm \sqrt{(36 + 4)}}{2} = -3 \pm \sqrt{10}$

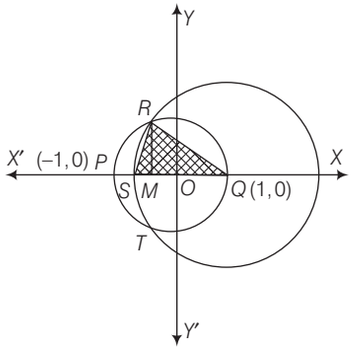
$= -3 + \sqrt{10} \qquad \left(\because 0 < \alpha < \frac{\pi}{4}\right)$

Now, in ΔOAC, $\tan\alpha = \frac{3}{OA} = (\sqrt{10} - 3)$

$\therefore \quad OA = \frac{3}{(\sqrt{10} - 3)} \frac{(\sqrt{10} + 3)}{(\sqrt{10} + 3)} = 3(3 + \sqrt{10})$

98. The given circle is $x^2 + y^2 = 1$ ...(i)

with centre at $O(0,0)$ and radius 1. It cuts X-axis at the points when $y = 0$ then $x = \pm 1$ i.e., *at* $P(-1, 0)$ and $Q(1, 0)$.



Equation of circle with centre at $Q(1, 0)$ and radius r is

$$(x - 1)^2 + (y - 0)^2 = r^2 \qquad \text{...(ii)}$$

$(0 < r < 2)$

Solving Eqs. (i) and (ii), we get

$$x = \frac{2 - r^2}{2} \quad \text{and} \quad y = \pm\frac{r\sqrt{4 - r^2}}{2}$$

but R above the X-axis.

$\therefore \quad R \equiv \left(\frac{2 - r^2}{2}, \frac{r\sqrt{4 - r^2}}{2}\right)$

So, $SQ = r$ and $MR = \frac{r\sqrt{4 - r^2}}{2}$

$\therefore$ Area of $\Delta QSR = \frac{1}{2} \cdot QS \cdot MR$

$$\Delta = \frac{1}{2} \cdot r \cdot \frac{r\sqrt{4 - r^2}}{2}$$

$$\Delta\,(\text{say}) = \frac{r^2\sqrt{(4 - r^2)}}{4}$$

$$\Delta^2 = \frac{r^4}{16}(4 - r^2) = A \qquad \text{(say)}$$

$\therefore \quad A = \frac{1}{16}(4r^4 - r^6)$

$\therefore \quad \frac{dA}{dr} = \frac{1}{16}(16r^3 - 6r^5)$ and $\frac{d^2A}{dr^2} = \frac{1}{16}(48r^2 - 30r^4)$

For maximum and minimum area, $\frac{dA}{dr} = 0$

$$r^2 = \frac{8}{3}$$

$\therefore \quad \left(\frac{d^2A}{dr^2}\right)_{r = \sqrt{\frac{8}{3}}} = \frac{1}{16}\left(48 \times \frac{8}{3} - 30 \times \frac{64}{9}\right) < 0$

$\therefore$ A is maximum. Hence, Δ is also maximum.

$\therefore$ Maximum value of $\Delta = \frac{1}{2} \times \frac{8}{3} \times \frac{1}{2}\sqrt{\left(4 - \frac{8}{3}\right)}$

$$= \frac{2}{3} \times \frac{2}{\sqrt{3}}$$

$$= \frac{4\sqrt{3}}{9} \text{ sq units.}$$

99. The equation of any curve passing through

$$a_1x + b_1y + c_1 = 0$$

$$a_2x + b_2y + c_2 = 0$$

$y = 0$ and $x = 0$ is

$$(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) + \lambda xy = 0 \qquad \text{...(i)}$$

where, λ is a parameter.

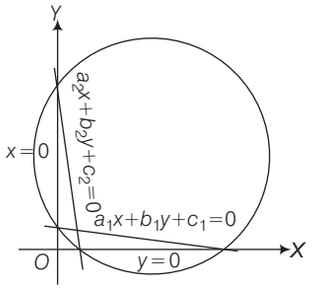
This curve will represent a circle. If the coefficient of x^2 = coefficient of y^2,

i.e. $a_1a_2 = b_1b_2$ ...(ii)

and if the coefficient of $xy = 0$

then $a_1b_2 + a_2b_1 + \lambda = 0$

$\therefore \quad \lambda = -(a_1b_2 + a_2b_1)$



Substituting the value of λ in Eq. (i) then

$$(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) - (a_1b_2 + a_2b_1)\, xy = 0$$

$$\Rightarrow\ a_1a_2x^2 + b_1b_2y^2 + (a_1c_2 + a_2c_1)\, x + (b_1c_2 + b_2c_1)\, y = 0$$

From Eq. (ii), $b_1b_2 = a_1a_2$

$\therefore$ Equation of required circle is

$$a_1a_2(x^2 + y^2) + (a_1c_2 + a_2c_1)\, x + (b_1c_2 + b_2c_1)\, y = 0$$

100. Let circle be

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \ldots(i)$$

Since, centre of this circle $(-g, -f)$ lie on $2x - 2y + 9 = 0$

$$\therefore \qquad -2g + 2f + 9 = 0 \qquad \ldots(ii)$$

and the circle $S = 0$ and $x^2 + y^2 - 4 = 0$ cuts orthogonally.

$$\therefore \qquad 2g \times 0 + 2f \times 0 = c - 4$$

$$\therefore \qquad c = 4 \qquad \ldots(iii)$$

Substituting the values of g and c from Eqs. (ii) and (iii) in Eq. (i), then

$$x^2 + y^2 + (2f + 9)\, x + 2fy + 4 = 0$$

or $$(x^2 + y^2 + 9x + 4) + 2f\,(x + y) = 0$$

Hence, the circle $S = 0$ passes through fixed point

$(\because$ form $S' + \lambda P = 0)$

$\therefore\ \ x^2 + y^2 + 9x + 4 = 0$ and $x + y = 0$

After solving we get $(-4, 4)$ or $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

101. Chords are bisected on the line $y = x$. Let (x_1, x_1) be the mid-point of the chord, then equation of the chord is $T = S_1$

$$\therefore\ \ xx_1 + yx_1 - a\,(x + x_1) - b\,(y + x_1) + a^2 + b^2 - c^2$$

$$= x_1^2 + x_1^2 - 2ax_1 - 2bx_1 + a^2 + b^2 - c^2$$

$$\Rightarrow\ (x_1 - a)\, x\ + (x_1 - b)\, y + ax_1 + bx_1 - 2x_1^2 = 0$$

This chord passes through $(a, b + c)$

$$\Rightarrow\ (x_1 - a)\, a\ + (x_1 - b)\,(b + c) + ax_1 + bx_1 - 2x_1^2 = 0$$

$$\Rightarrow \qquad 2x_1^2 - (2a + 2b + c)\, x_1 + a^2 + b^2 + bc = 0$$

which is quadratic in x_1. Since, it is given that two chords are bisected on the line $y = x$, then x_1 must have two real roots,

$$B^2 - 4AC > 0$$

$$\Rightarrow\ (2a + 2b + c)^2 - 4 \cdot 2\,(a^2 + b^2 + bc) > 0$$

$$\Rightarrow\ 4a^2 + 4b^2 + c^2 + 8ab + 4bc + 4ac - 8a^2 - 8b^2 - 8b^2 - 8bc > 0$$

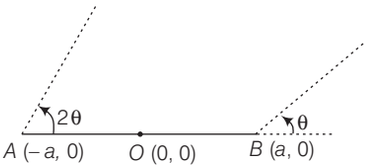
$$\Rightarrow\ 4a^2 + 4b^2 - 8ab + 4bc - 4ac - c^2 < 0$$

Hence, the condition on a, b, c is

$$4a^2 + 4b^2 - c^2 - 8ab + 4bc - 4ca < 0$$

102. Let $A \equiv (-a, 0)$ and $B \equiv (a, 0)$ be two fixed points.

Let one line which rotates about B an angle θ with the X-axis at any time t and at that time the second line which rotates about A make an angle 2θ with X-axis.



Now, equation of lines through B and A are respectively

$$y - 0 = \tan\theta\,(x - a) \qquad \ldots(i)$$

and $$y - 0 = \tan 2\theta\,(x + a) \qquad \ldots(ii)$$

From Eq. (ii), $y = \dfrac{2\tan\theta}{1 - \tan^2\theta}(x + a)$

$$= \left\{\frac{\frac{2y}{(x-a)}}{1 - \frac{y^2}{(x-a)^2}}\right\}(x + a) \qquad \text{[from Eq. (i)]}$$

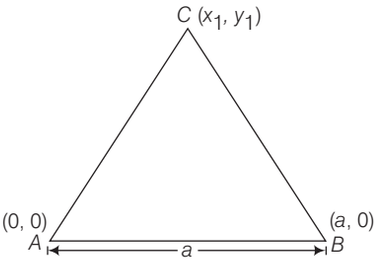
$$\Rightarrow \qquad y = \frac{2y\,(x - a)\,(x + a)}{(x - a)^2 - y^2}$$

$$\Rightarrow\ (x - a)^2 - y^2 = 2\,(x^2 - a^2)$$

or $x^2 + y^2 + 2ax - 3a^2 = 0$ which is the required locus.

103. Let the coordinate of C is (x_1, y_1)

and let the coordinates of A and B are $(0, 0)$ and $(a, 0)$



given $$k = \frac{\sin A}{\sin B} = \frac{BC}{AC}$$

$$\Rightarrow \qquad (BC)^2 = k^2\,(AC)^2$$

$$\Rightarrow\ \ (x_1 - a)^2 + y_1^2 = k^2\,(x_1^2 + y_1^2)$$

$$\Rightarrow (1 - k^2)\, x_1^2 + (1 - k^2)\, y_1^2 - 2ax_1 + a^2 = 0 \qquad (\because k \neq 1)$$

$$\Rightarrow \qquad x_1^2 + y_1^2 - \frac{2ax_1}{1 - k^2} + \frac{a^2}{1 - k^2} = 0$$

Hence, locus of C is

$$x^2 + y^2 - \frac{2ax}{1 - k^2} + \frac{a^2}{1 - k^2} = 0$$

This is a circle whose centre is

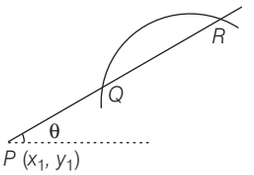
$$\left(\frac{a}{1 - k^2}, 0\right)$$

and $$\text{radius} = \sqrt{\frac{a^2}{(1 - k^2)^2} - \frac{a^2}{(1 - k^2)}} = \frac{ak}{(1 - k^2)}$$

104. Let P be (x_1, y_1) and line through $P(x_1, y_1)$ makes an angle θ with X-axis, then

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$$

Coordinates of any point on the curve is $(x_1 + r\cos\theta, y_1 + r\sin\theta)$. This point must lie on $ax^2 + 2hxy + by^2 = 1$



$$\therefore\ a(x_1 + r\cos\theta)^2 + 2h(x_1 + r\cos\theta)(y_1 + r\sin\theta) + b(y_1 + r\sin\theta)^2 = 1$$

$$\therefore\ (a\cos^2\theta + h\sin 2\theta + b\sin^2\theta)r^2 + 2(ax_1\cos\theta + hx_1\sin\theta + hy_1\cos\theta)r + ax_1^2 + 2hx_1y_1 + by_1^2 = 0$$

It is quadratic equation in r. Let roots of this equation are r_1 and r_2 then

$$r_1r_2 = \frac{ax_1^2 + 2hx_1y_1 + by_1^2}{(a\cos^2\theta + h\sin 2\theta + b\sin^2\theta)}$$

$$\therefore\quad PQ \cdot PR = \frac{ax_1^2 + 2hx_1y_1 + by_1^2}{a\cos^2\theta + h\sin 2\theta + b\sin^2\theta} \text{ for } a = b, h = 0$$

$$\therefore\quad PQ \cdot PR = \frac{ax_1^2 + 0 + ay_1^2}{a\cos^2\theta + 0 + a\sin^2\theta} = x_1^2 + y_1^2$$

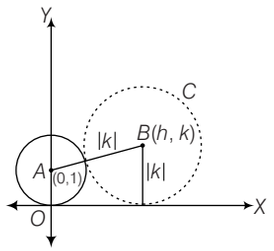
which is independent of θ.

Then curve $ax^2 + 2hxy + by^2 = 1$ becomes $ax^2 + 0 + ay^2 = 1$

$$\Rightarrow\quad x^2 + y^2 = \frac{1}{a}$$

is a circle with centre $(0, 0)$ and radius $\frac{1}{\sqrt{a}}$.

105. Let the centre of circle C be (h,k). Then as this circle touches axis of x, its radius $=|k|$



Also, it touches the given circle $x^2 + (y-1)^2 = 1$, centre $(0, 1)$ radius 1, externally

Therefore, the distance between centres = sum of radii

$$\Rightarrow\quad \sqrt{(h-0)^2 + (k-1)^2} = 1 + |k|$$

$$\Rightarrow\quad h^2 + k^2 - 2k + 1 = (1 + |k|)^2$$

$$\Rightarrow\quad h^2 + k^2 - 2k + 1 = 1 + 2|k| + k^2$$

$$\Rightarrow\quad h^2 = 2k + 2|k|$$

$\therefore$ Locus of (h,k) is, $x^2 = 2y + 2|y|$

Now if $y > 0$, it becomes $x^2 = 4y$

and if $y \le 0$, it becomes $x = 0$

$\therefore$Combining the two, the required locus is

$$\{(x,y) : x^2 = 4y\} \cup \{(0,y) : y \le 0\}$$

106. $s_1 = x^2 + y^2 + 2ax + cy + a = 0$

$$s_2 = x^2 + y^2 - 3ax + dy - 1 = 0$$

Equation of common chord of circles s_1 and s_2 is given by $s_1 - s_2 = 0$

$$\Rightarrow\quad 5ax + (c-d)y + a + 1 = 0$$

Given, that $5x + by - a = 0$ passes through P and Q

$\therefore$ The two equations should represent the same line

$$\Rightarrow\quad \frac{a}{1} = \frac{c-d}{b} = \frac{a+1}{-a}$$

$$\Rightarrow\quad a + 1 = -a^2$$

$$a^2 + a + 1 = 0$$

No real value of a.

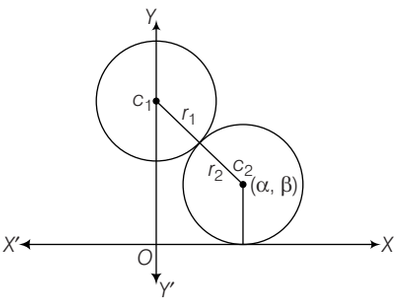
107. Equation of circle with centre $(0, 3)$ and radius 2 is

$$x^2 + (y-3)^2 = 4$$

Let centre of the variable circle is (α,β)

$\because$ It touches X-axis.

$\therefore$It's equation is $(x-\alpha)^2 + (y-\beta)^2 = \beta^2$



Circle touch externally

$$\Rightarrow\quad c_1c_2 = r_1 + r_2$$

$$\therefore\quad \sqrt{\alpha^2 + (\beta-3)^2} = 2 + \beta$$

$$\alpha^2 + (\beta-3)^2 = \beta^2 + 4 + 4\beta$$

$$\Rightarrow\quad \alpha^2 = 10(\beta - 1/2)$$

$\therefore$Locus is $x^2 = 10\left(y - \frac{1}{2}\right)$ which is a parabola.

108. Let the centre be (α,β)

$\because$ It cuts the circle $x^2 + y^2 = p^2$ orthogonally

$\therefore$Using $2g_1g_2 + 2f_1f_2 = c_1 + c_2$, we get

$$2(-\alpha)\times 0 + 2(-\beta)\times 0 = c_1 - p^2$$

$$\Rightarrow\quad c_1 = p^2$$

Let equation of circle is

$$x^2 + y^2 - 2\alpha x - 2\beta y + p^2 = 0$$

It passes through

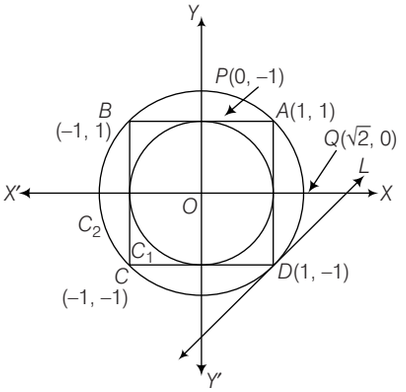
$$(a,b) \Rightarrow a^2 + b^2 - 2\alpha a - 2\beta b + p^2 = 0$$

$\therefore$ Locus of (α,β) is

$\therefore$ $\quad 2ax + 2by - (a^2 + b^2 + p^2) = 0$

109. Without loss of generally it we can assume the square $ABCD$ with its vertices $A(1,1)$, $B(-1,1)$, $C(-1,-1)$, $D(1,-1)$

P to be the point $(0,1)$ and Q as $(\sqrt{2},0)$.



Then, $\dfrac{PA^2 + PB^2 + PC^2 + PD^2}{QA^2 + QB^2 + QC^2 + QD^2}$

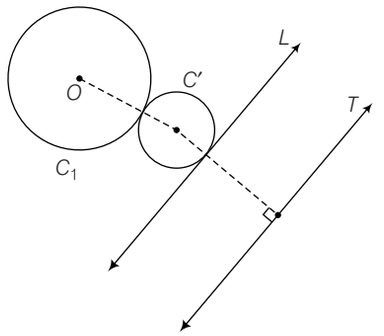
$$= \frac{1+1+5+5}{2[(\sqrt{2}-1)^2+1] + 2((\sqrt{2}+1)^2+1)}$$

$$= \frac{12}{16} = 0.75$$

110. Let C' be the circle touching circle C_1 and L, so that C_1 and C' are on the same side of L. Let us draw a line T parallel to L at a distance equal to the radius of circle C_1, on opposite side of L.

Then, the centre of C' is equidistant from the centre of C_1 and from line T.

$\Rightarrow$ locus of centre of C' is a parabola.

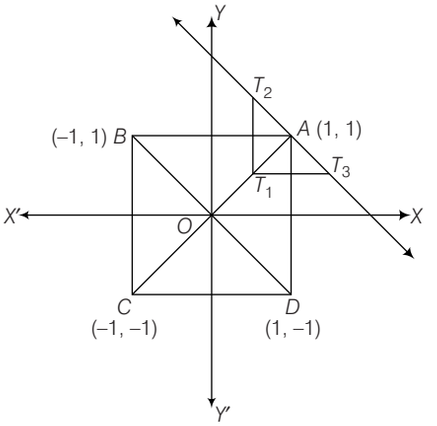


111. Since, S is equidistant from A and line BD, it traces a parabola.

Clearly, AC is the axis, $A(1,1)$ is the focus and $T_1\left(\frac{1}{2},\frac{1}{2}\right)$ is the vertex of parabola.

$$AT_1 = \frac{1}{\sqrt{2}}.$$

T_2T_3 = latusrectum of parabola



$$= 4 \times \frac{1}{\sqrt{2}} = 2\sqrt{2}$$

$\therefore$ $\quad$ Area $(\Delta T_1T_2T_3) = \dfrac{1}{2} \times \dfrac{1}{\sqrt{2}} \times 2\sqrt{2} = \dfrac{1}{2} = 1$ sq units.

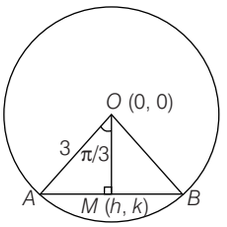
112. Point of intersection of $3x - 47 - 7 = 0$ and $2x - 3y - 5 = 0$ is $(1,-1)$ which is the centre of the circle and radius $= 7$

$\therefore$Equation is $(x-1)^2 + (y+1)^2 = 49$

$\Rightarrow$ $\quad x^2 + y^2 - 2x + 2y - 47 = 0$

113. Let $M(h,k)$ be the mid-point of chord AB where

$$\angle AOB = \frac{2\pi}{3}$$



$\therefore$ $\quad \angle AOM = \dfrac{\pi}{3}.$

Also, $\quad OM = 3\cos\dfrac{\pi}{3} = \dfrac{3}{2}$

$\Rightarrow$ $\quad \sqrt{h^2 + k^2} = \dfrac{3}{2}$

$\Rightarrow$ $\quad h^2 + k^2 = \dfrac{9}{4}$

$\therefore$ Locus of (h,k) is $x^2 + y^2 = \dfrac{9}{4}$

114. Equation of director circle of the given circle $x^2 + y^2 = 169$ is $x^2 + y^2 = 2 \times 169 = 338$.

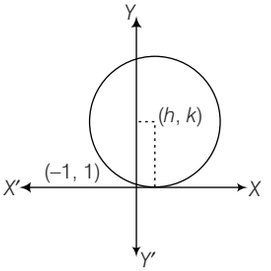
We know from every point on director circle, the tangents drawn to given circle are perpendicular to each other.

Here, (17, 7) lies on director circle.

$\therefore$ The tangent from (17, 7) to given circle are mutually perpendicular.

115. Equation of circle whose centre is (h,k)

$$(x-h)^2+(y-k)^2=k^2$$



(radius of circle = k because circle is tangent to x-axis)

Equation of circle passing through $(-1,+1)$

$\therefore \quad (-1-h)^2+(1-k)^2=k^2$

$\Rightarrow \quad 1+h^2+2h+1+k^2-2k=k^2$

$\Rightarrow \quad h^2+2h-2k+2=0 \quad D\geq 0$

$\therefore \quad (2)^2-4\times 1.(-2K+2)\geq 0$

$\Rightarrow \quad 4-4(-2k+2)\geq 0$

$\Rightarrow \quad 1+2k-2\geq 0 \quad \Rightarrow \quad k\geq \frac{1}{2}$

116. Slope of $CD=\frac{1}{\sqrt{3}}$

$\therefore$ Parametric equation of CD is

$$\frac{x-\frac{3\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}}=\frac{y-\frac{3}{2}}{\frac{1}{2}}=\pm 1$$

$\therefore$ Two possible coordinates of C are

$$\left(\frac{\sqrt{3}}{2}+\frac{3\sqrt{3}}{2},\frac{1}{2}+\frac{3}{2}\right) \text{ or } \left(\frac{-\sqrt{3}}{2}+\frac{3\sqrt{3}}{2},-\frac{1}{2}+\frac{3}{2}\right)$$

i.e. $(2\sqrt{3},2)$ or $(\sqrt{3},1)$

As (0, 0) and C lie on the same side of PQ

$\therefore(\sqrt{3},1)$ should be the coordinates of C.

Remark : Remember (x_1,y_1) and (x_2,y_2) lie on the same or opposite side of a line $ax+by+c=0$ according as $\frac{ax_1+by_1+c}{ax_2+by_2+c}>0$ or <0. $\therefore$ Equation of the circle is $(x-\sqrt{3})^2+(y-1)^2=1$

117. ΔPQR is an equilateral triangle, the incentre C must coincide with centroid of ΔPQR and D, E, F must coincide with the mid points of sides PQ, QR and RP respectively.

Also, $\angle CPD=30^\circ \Rightarrow PD=\sqrt{3}$

Writing the equation of side PQ in symmetric form we get,

$$\frac{x-\frac{3\sqrt{3}}{2}}{-\frac{1}{2}}=\frac{y-\frac{3}{2}}{\frac{\sqrt{3}}{2}}=\mp\sqrt{3}$$

$\therefore$ Coordinates of $P=\left(\frac{\sqrt{3}}{2}+\frac{3\sqrt{3}}{2},\frac{-3}{2}+\frac{3}{2}\right)=(2\sqrt{3},0)$

and coordinates of $Q=\left(\frac{-\sqrt{3}}{2}+\frac{3\sqrt{3}}{2},\frac{3}{2}+\frac{3}{2}\right)=(\sqrt{3},3)$

Let coordinates of R be (α,β), then using the formula for centriod of Δ we get

$$\frac{\sqrt{3}+2\sqrt{3}+\alpha}{3}=\sqrt{3} \text{ and } \frac{3+0+\beta}{3}=1$$

$\Rightarrow \quad \alpha=0$ and $\beta=0$

$\therefore$ Coordinates of $R=(0,0)$

Now, coordinates of $E=$ mid point of $QR=\left(\frac{\sqrt{3}}{2},\frac{3}{2}\right)$ and coordinates of $F=$ mid-point of $PR=(\sqrt{3},0)$

118. Equation of side QR is $y=\sqrt{3}x$ and equation of side RP is $y=0$

119. The given circle is $x^2+y^2+6x-10y+30=0$ Centre $(-3,5)$, radius = 2

$$L_1: 2x+3y+(p-3)=0;$$
$$L_2: 2x+3y+p+3=0$$

Clearly, $L_1 || L_2$

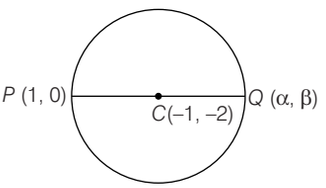
Distance between L_1 and L_2

$$=\left|\frac{p+3-p+3}{\sqrt{2^2+3^2}}\right|=\frac{6}{\sqrt{13}}<2$$

$\Rightarrow$ If one line is a chord of the given circle, other line may or may not the diameter of the circle.

Statement I is true and statement II is false.

120. The given circle is $x^2+y^2+2x+4y-3=0$



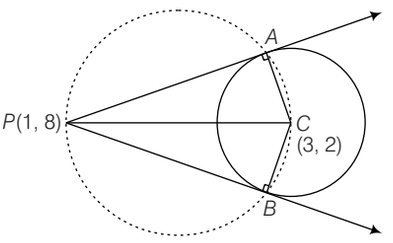
Centre $(-1,-2)$

Let $Q(\alpha,\beta)$ be the point diametrically opposite to the point $P(1,0)$,

then, $\frac{1+\alpha}{2}=-1$ and $\frac{0+\beta}{2}=-2$

$\Rightarrow \quad \alpha=-3, \beta=-4$, So, Q is $(-3,-4)$

121. Tangents PA and PB are drawn from the point $P(1,3)$ to circle $x^2+y^2-6x-4y-11=0$ with centre $C(3,2)$



Clearly the circumcircle of ΔPAB will pass through C and as $\angle A=90^\circ$, PC must be a diameter of the circle.

$\therefore$ Equation of required circle is

$$(x-1)(x-3)+(y-2)=0$$

$\Rightarrow \quad x^2+y^2-4x-10y+19=0$

122. Let r be the radius of required circle.

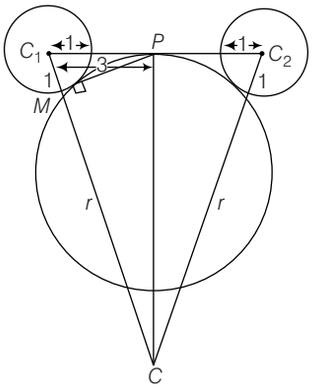
Clearly, in $\Delta C_1CC_2, C_1C = C_2C = r+1$ and P is mid-point of C_1C_2

$\therefore$ $CP \perp C_1C_2$

Also, $PM \perp CC_1$

Now, $\Delta PMC_1 \sim \Delta CPC_1$ (by AA similarity)

$\therefore$ $\dfrac{MC_1}{PC_1} = \dfrac{PC_1}{CC_1}$



$\Rightarrow$ $\dfrac{1}{3} = \dfrac{3}{r+1} \Rightarrow r+1 = 9 \Rightarrow r = 8.$

123. The given circles are

$S_1 \equiv x^2 + y^2 + 3x + 7y + 2p - 5 = 0$ ...(i)

$S_2 \equiv x^2 + y^2 + 2x + 2y - p^2 = 0$ ...(ii)

$\therefore$ Equation of common chord PQ is $S_1 - S_2 = 0$

$\Rightarrow$ $L \equiv x + 5y + p^2 + 2p - 5 = 0$

$\Rightarrow$ Equation of circle passing through P and Q is $S_1 + \lambda L = 0$

$\Rightarrow$ $(x^2 + y^2 + 3x + 7y + 2p - 5) + \lambda(x + 5y + p^2 + 2p - 5) = 0$

As it passes through (1, 1), therefore

$\Rightarrow$ $(7 + 2p) + \lambda(2p + p^2 + 1) = 0$

$\Rightarrow$ $\lambda = -\dfrac{2p+7}{(p+1)^2}$

which does not exist for $p = -1$

124. Circle $x^2 + y^2 - 4x - 8y - 5 = 0$

Centre $= (2, 4)$

Radius $= \sqrt{4 + 16 + 5} = 5$

If circle is intersecting line $3x - 4y = m$, at two distinct points.

$\Rightarrow$ length of perpendicular from centre to the line < radius

$\Rightarrow$ $\dfrac{|6 - 16 - m|}{5} < 5$

$\Rightarrow$ $|10 + m| < 25$

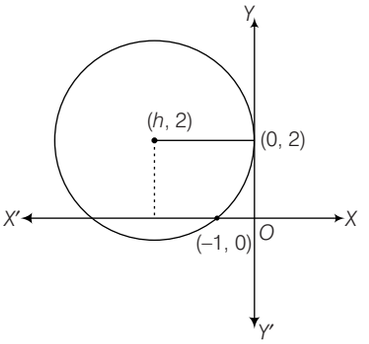
$\Rightarrow$ $-25 < m + 10 < 25$

$\Rightarrow$ $-35 < m < 15$

125. Let centre of the circle be $(h, 2)$ then radius $= |h|$

$\therefore$ Equation of circle becomes $(x-h)^2 + (y-2)^2 = h^2$

As it passes through $(-1, 0)$



$\Rightarrow$ $(-1-h)^2 + 4 = h^2 \Rightarrow h = \dfrac{-5}{2}$

$\therefore$ Centre $\left(\dfrac{-5}{2}, 2\right)$ and $r = \dfrac{5}{2}$

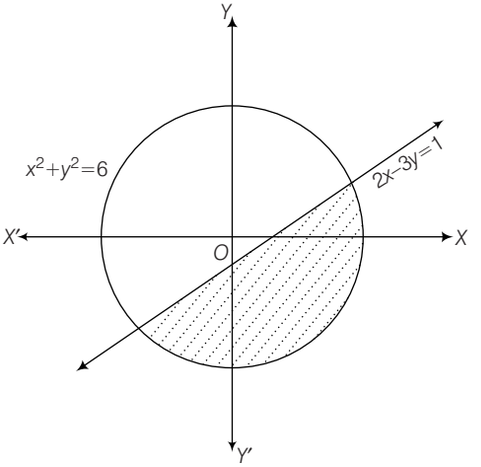
Distance of centre from $(-4, 0)$ is $\dfrac{5}{2}$

$\therefore$ It lies on the circle.

126. The smaller region of circle is the region given by

$x^2 + y^2 \le 6$ ...(i)

and $2x - 3y \ge 1$ ...(ii)



We observe that only two points $\left(2, \dfrac{3}{4}\right)$ and $\left(\dfrac{1}{4}, -\dfrac{1}{4}\right)$ satisfy both the inequations Eqs. (i) and (ii), we get

$\therefore$ 2 points in S lie inside the smaller part.

127. As centre of one circle is (0, 0) and other circle passes through (0, 0), therefore

Aslo, $C_1\left(\dfrac{a}{2}, 0\right) C_2(0, 0)$

$r_1 = \dfrac{|a|}{2}\ r_2 = C$

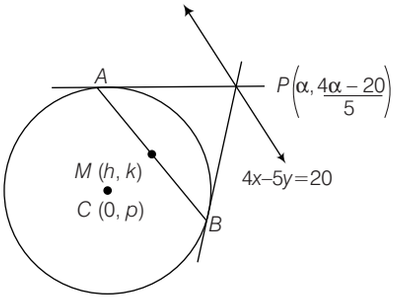
$C_1C_2 = r_1 - r_2 = \dfrac{|a|}{2}$

$\Rightarrow$ $c - \dfrac{|a|}{2} = \dfrac{|a|}{2} \Rightarrow c = |a|$

If the two circles touch each other, then they must touch each other internally.

128. Any point P on line $4x-5y=20$ is $\left(\alpha, \frac{4\alpha-20}{5}\right)$.

Equation of chord of contact AB to the circle $x^2+y^2=9$



drawn from point $P\left(\alpha, \frac{4\alpha-20}{5}\right)$ is

$$x.\alpha + y.\left(\frac{4\alpha-20}{5}\right)=9 \qquad \ldots(i)$$

Also, the equation of chord AB whose mid-point is (h,k) is

$$hx+ky=h^2+k^2 \qquad \ldots(ii)$$

$\because$ Eqs. (i) and (ii) represent the same line, therefore

$$\frac{h}{\alpha}=\frac{k}{\frac{4\alpha-20}{5}}=\frac{h^2+k^2}{9}$$

$\Rightarrow \quad 5k\alpha=4h\alpha-20h$

and $\quad 9h=\alpha(h^2+k^2)$

$\Rightarrow \quad \alpha=\frac{20h}{4h-5k}$ and $\alpha=\frac{9h}{h^2+k^2}$

$\Rightarrow \quad \frac{20h}{4h-5k}=\frac{9h}{h^2+k^2}$

$\Rightarrow \quad 20(h^2+k^2)=9(4h-5k)$

$\therefore$ Locus of (h,k) is $20(x^2+y^2)-36x+45y=0$

Sol. (Q. Nos. 129 to 130)

Equation of tangent PT to the circle $x^2+y^2=4$ at the point $P(\sqrt{3},1)$ is $x\sqrt{3}+y=4$

Let the line L, perpendicular to tangent PT be

$$x-y\sqrt{3}+\lambda=0 \qquad \ldots(i)$$

As it is tangent to the circle $(x-3)^2+y^2=1$

$\therefore$ length of perpendicular from centre of circle to the tangent = radius of circle.

$\Rightarrow \quad \left|\frac{3+\lambda}{2}\right|=1 \Rightarrow \lambda=-1 \text{ or } -5$

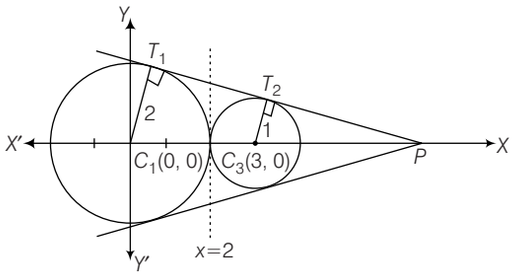
129. From Eq. (i)

Equation of L can be

$$x-\sqrt{3}y=1$$

or $\quad x-\sqrt{3}y=5$

130.



From the figure it is clear that the intersection point of two direct common tangents lies on X-axis.

Aslo $\quad \Delta PT_1C_1 \sim \Delta PT_2C_2$

$\Rightarrow \quad PC_1 : PC_2 = 2:1$

or P divides C_1C_2 in the ratio 2:1 externally

$\therefore$Coordinates of P are $(6,0)$

Let the equation of tangent through P be

$$y=m(x-6)$$

As it touches $x^2+y^2=4$

$\therefore \quad \left|\frac{6m}{\sqrt{m^2+1}}\right|=2$

$\Rightarrow \quad 36m^2=4(m^2+1)$

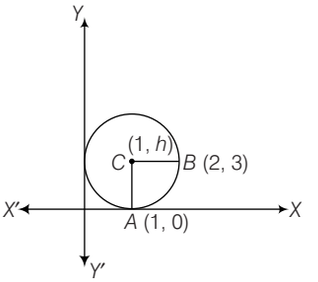
$\Rightarrow \quad m=\pm\frac{1}{2\sqrt{2}}$

$\therefore$Equations of common tangents are $y=\pm\frac{1}{2\sqrt{2}}(x-6)$

Also $x=2$ is the common tangent to the two circles.

131. Let centre of the circle ne $(1,4)$

[$\because$ circle touches x-axis at (1, 0)]



Let the circle passes through the point $B(2,3)$

$\therefore \quad CA=CB$ (radius)

$\Rightarrow \quad CA^2=CB^2$

$\Rightarrow \quad (1-1)^2+(h-0)^2=(1-2)^2+(h-3)^2$

$\Rightarrow \quad h^2=1+h^2+9-6h$

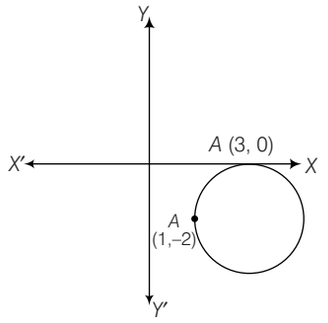
$\Rightarrow \quad h=\frac{10}{6}=\frac{5}{3}$

Thus, diameter is $2h=\frac{10}{3}$.

132. Since, circle touches X-axis at (3, 0)

$\therefore$ The equation of circle be

$$(x-3)^2+(y-0)^2+\lambda y=0$$



As it passes through (1, –2)

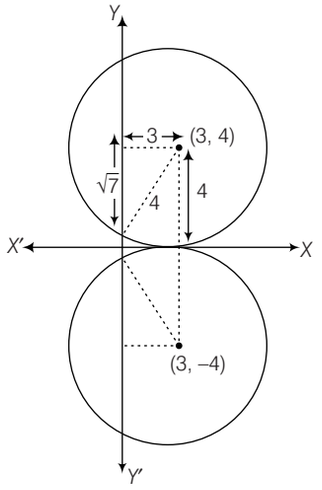
$\therefore$ Put $x=1, y=-2$

$\Rightarrow \quad (1-3)^2+(-2)^2+\lambda(-2)=0 \Rightarrow \lambda=4$

$\therefore$Equation of circle is $(x-3)^2+y^2-8=0$

Now, from the options (5, –2) satisfies equation of circle.

133. There can be two possibilities for the given circle as shown in the figure



$\therefore$ The equations of circle can be

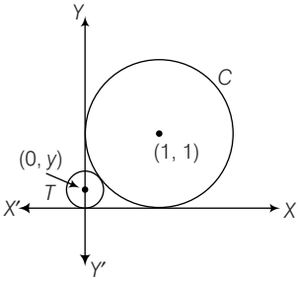
$$(x-3)^2+(y-4)^2=4^2$$

or $\quad (x-3)^2+(y+4)^2=4^2$

i.e. $\quad x^2+y^2-6x+8y+9=0$

or $\quad x^2+y^2-6x+8y+9=0$

134.



Equation of circle $C \equiv (x-1)^2+(y-1)^2=1$

Radius of $T=|y|$

T touches C externally

therefore,

Distance between the centres = sum of their radii

$\Rightarrow \quad \sqrt{(0-1)^2+(y-1)^2}=1+|y|$

$\Rightarrow \quad (0-1)^2+(y-1)^2=(1+|y|)^2$

$\Rightarrow \quad 1+y^2+1-2y=1+y^2+2|y|$

$$2|y|=1-2y$$

If $y>0$ then, $2y=1-2y \Rightarrow y=\frac{1}{4}$

If $y<0$ then, $-2y=1-2y \Rightarrow 0=1$ (not possible)

$\therefore \quad y=\frac{1}{4}$

135. Let the equation of circle be

$$x^2+y^2+2gx+2fy+c=0$$

It passes through (0, 1)

$\therefore \quad 1+2f+c=0$ ...(i)

This circle is orthogonal to $(x-1)^2+y^2=16$

i.e. $\quad x^2+y^2-2x-15=0$

and $\quad x^2+y^2-1=0$

$\therefore$We should have

$$2g(-1)+2f(0)=c-15$$

or $\quad 2g+c-15=0$ ...(ii)

and $\quad 2g(0)+2f(0)=c-1$ or $c=1$

Solving Eqs. (i), (ii) and (iii), we get

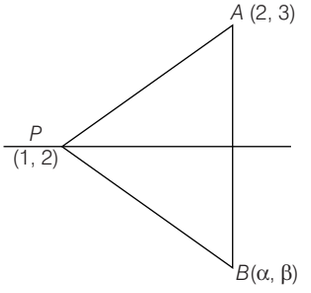
$$c=1, g=7, f=-1$$

$\therefore$ Required circle is

$$x^2+y^2+14x-2y+1=0$$

With centre (–7,1) and radius = 7

136. Intersection point of $2x-3y+4=0$ and $x-2y+3=0$ is (1, 2)



Since, P is the fixed point for given family of lines

So, $PB = PA$

$$(\alpha-1)^2+(\beta-2)^2=(2-1)^2+(3-2)^2$$

$$(\alpha-1)^2+(\beta-2)^2=1+1=2$$

$$(x-1)^2+(y-2)^2=(\sqrt{2})^2$$

$(x-a)^2+(y-b)^2=r^2$

Therefore, given locus is a circle with centre (1,2) and radius $\sqrt{2}$.

137. $x^2+y^2-4x-6y-12=0$ ...(i)

Centre $C_1=(2,3)$ and Radius, $r_1=5$ units

$$x^2+y^2+6x+18y+26=0 \quad \text{...(ii)}$$

Centre, $C_2=(-3,-9)$

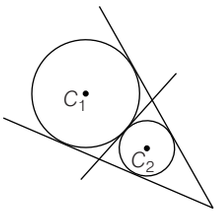
and radius, $r_2=8$ units

$|C_1C_2|=\sqrt{(2+3)^2+(3+9)^2}=13$ units

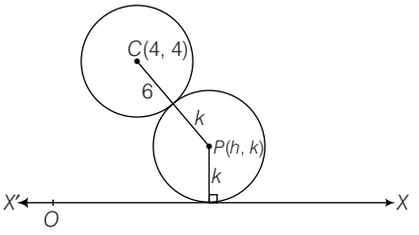
$r_1+r_2=5+8=13$

$\therefore \quad |C_1C_2|=r_1+r_2$

Therefore, there are three common tangents.



138. For the given circle, centre : (4, 4), radius =6

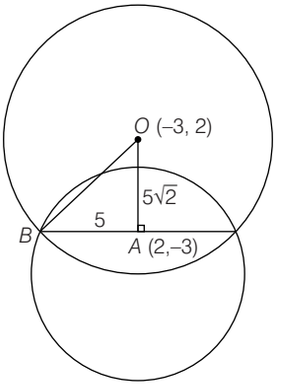


$$6+k=\sqrt{(h-4)^2+(k-4)^2}$$

$$(h-4)^2=20k+20$$

$\therefore$ locus of (h, k) is $(x-4)^2=20(y+1)$, which is parabola.

139. Centre of $S:O(-3,2)$ and centre of given circle is $A(2,-3)$



$\Rightarrow \quad OA=5\sqrt{2}$

Also, $AB=5$ ($\because AB=$ radius of the given circle)

Now, in ΔOAB,

$$(OB)^2=(AB)^2+(OA)^2=25+50=75$$

$\therefore \quad OB=5\sqrt{3}$

140. Circle : $x^2+y^2=1$

Equation of tangent at $P(\cos\theta,\sin\theta)$

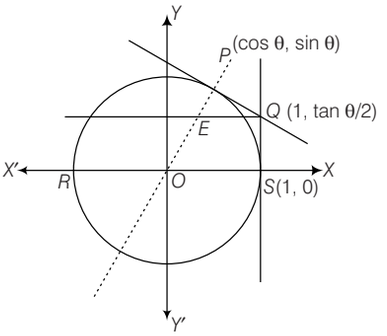
$$x\cos\theta+y\sin\theta=1 \quad \text{...(i)}$$

Equation of normal at P

$$y=x\tan\theta \quad \text{...(ii)}$$

Equation of tangent at S is $x=1$

$$\therefore \quad Q\left(1,\frac{1-\cos\theta}{\sin\theta}\right)=Q\left(1,\tan\frac{\theta}{2}\right)$$



$\therefore$ Equation of line through Q and parallel to RS is $y=\tan\frac{\theta}{2}$

$\therefore$ Intersection point E of normal and $y=\tan\frac{\theta}{2}$

$$\tan\frac{\theta}{2}=x\tan\theta$$

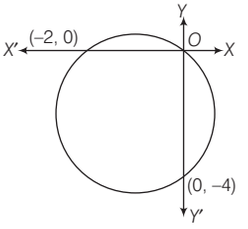
$$\Rightarrow \quad x=\frac{1-\tan^2\frac{\theta}{2}}{2}$$

$$\therefore \quad \text{Locus of } E: x=\frac{1-y^2}{2} \text{ or } y^2=1-2x$$

It is satisfied by the points $\left(\frac{1}{3},\frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{3},\frac{-1}{\sqrt{3}}\right)$.

141. (2) Equation of circle can be written as

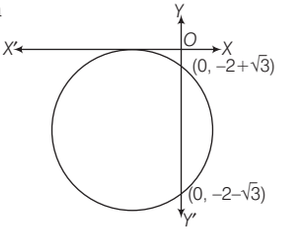
$(x+1)^2+(y+2)^2=p+5$ ...(i)



Case I. For $p=0$, circle passes through origin and cuts x-axis and y-axis at $(-2, 0)$ and $(0, -4)$ respectively.

Case II. If circle touch X-axis, then

$$(1)^2=-p \Rightarrow p=-1$$



From Eq. (i), we get

$(x+1)^2+(y+2)^2=2^2$

Cut off Y-axis at (put $x=0$)

$(y+2)^2=3$

$\Rightarrow \quad y=-2\pm\sqrt{3}$

or $(0, -2\pm\sqrt{3})$

Case III. If circle touch Y-axis, then

$$(2)^2=-p$$

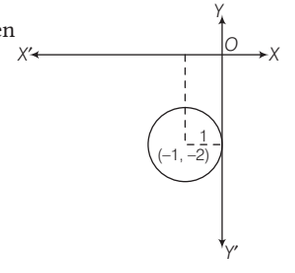
$\Rightarrow \quad p=-4$

From Eq. (i), we get

$(x+1)^2+(y+2)^2=1$

Cut off X-axis at (put $y=0$)

$(x+1)^2=-3$ (impossible)



CHAPTER

05

Parabola

Learning Part

Session 1

- Introduction
- Section of a Right Circular Cone by Different Planes
- Equation of Conic Section
- How to Find the Centre of Conics
- Standard Equation of Parabola
- Other forms of Parabola with Latusrectum $4a$
- General Equation of a Parabola
- The Generalised form $(y-k)^2 = 4a(x-h)$
- Conic Section
- Conic Section : Definition
- Recognisation of Conics
- Parabola : Definition
- Some Terms Related to Parabola
- Smart Table
- Equation of Parabola if Equation of axis, Tangent at Vertex and Latusrectum are given
- Parabolic Curve

Session 2

- Position of a Point (x_1, y_1) with respect to a Parabola $y^2 = 4ax$
- Intersection of a Line and a Parabola
- Point of Intersection of Tangents at any Two Points on the Parabola
- Point of Intersection of Normals at any Two Points on the Parabola
- Co-normal Points
- Parametric Relation between the Coordinates of the Ends of a Focal Chord of a Parabola
- Equation of Tangent in Different Forms
- Equation of Normals in Different Forms
- Relation Between 't_1' and 't_2' if Normal at 't_1' meets the Parabola Again at 't_2'
- Circle Through Co-normal Points

Session 3

- Pair of Tangents $SS_1 = T^2$
- Equation of the Chord Bisected at a Given Point
- Lengths of Tangent, Subtangent, Normal and Subnormal
- Reflection Property of a Parabola
- Chord of Contact
- Diameter
- Some Standard Properties of the Parabola
- Study of Parabola of the Form $(\alpha x + \beta y)^2 + 2gx + 2fy + c = 0$

Practice Part

- JEE Type Examples
- Chapter Exercises

Arihant on Your Mobile !

Exercises with the symbol can be practised on your mobile. See inside cover page to activate for free.

Session 1

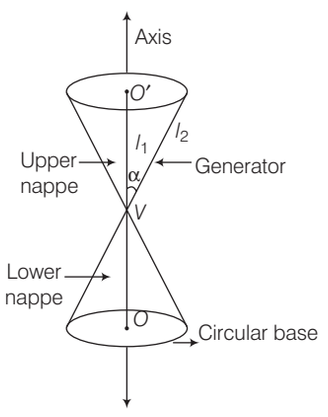
Introduction, Conic Section, Section of a Right Circular Cone by Different Planes, Conic Section : Definition, Equation of Conic Section, Recognisation of Conics, How to Find the Centre of Conics, Parabola : Definition, Other forms of Parabola with Latusrectum 4a, General Equation of a Parabola, The Generalised form $(y-k)^2 = 4a(x-h)$, Parabolic Curve

Introduction

The famous Greek mathematician Euclid, the father of creative Geometry, near about 300 BC considering various plane sections of a right circular cone found many curves, which are called conics or conic sections.

Conic Section

Let l_1 be a fixed vertical line and l_2 be another line intersecting it at a fixed point V and inclined to it at an angle α.

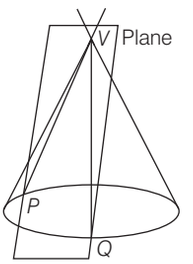


Suppose we rotate the line l_2 around the line l_1 in such a way the angle remains constant then, the surface generated is a double-napped right circular hollow cone.

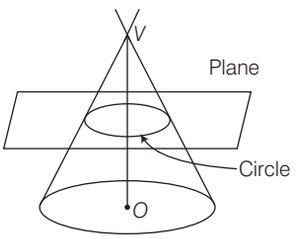
The point V is called the vertex, the line l_1 is the axis of the cone. The rotating line l_2 is called a generator of the cone. The vertex separates the cone into two parts called nappes. The constant angle α is called the semi-vertical angle of the cone.

Section of a Right Circular Cone by Different Planes

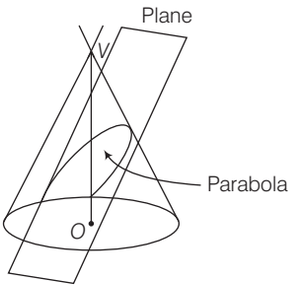
1. Section of a right circular cone by a plane which is passing through its vertex is a pair of straight lines, lines always passes through the vertex of the cone.



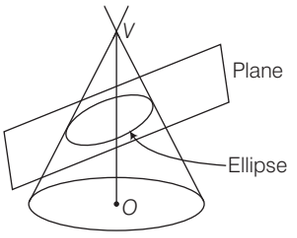
2. Section of a right circular cone by a plane which parallel to its base is a circle.



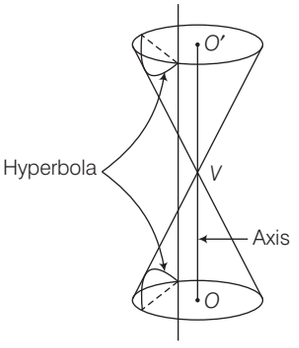
3. Section of a right circular cone by a plane which is parallel to a generator of the cone is a parabola.



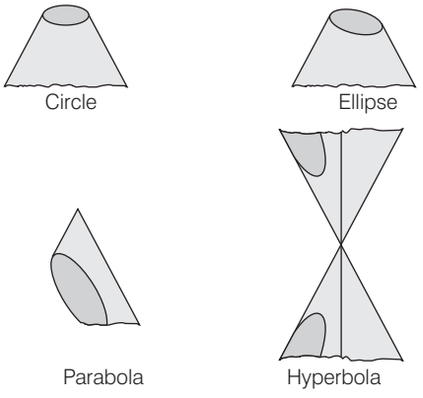
4. Section of a right circular cone by a plane which is not parallel to any generator and not parallel or perpendicular to the axis of the cone is an ellipse.



5. Section of a right circular cone by a plane which is parallel to the axis of the cone is a hyperbola.

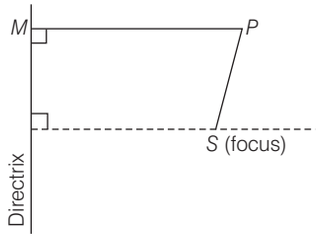


3D View



Conic Section : Definition

The locus of a point which moves in a plane such that the ratio of its distance from a fixed point to its perpendicular distance from a fixed straight line is always constant, is known as a **conic section** or **a conic.**



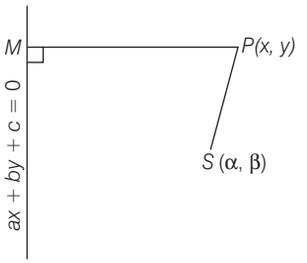
The fixed point is called the **focus** of the conic and this fixed line is called the **directrix** of the conic. Also, this constant ratio is called the **eccentricity** of the conic and is denoted by e.

In the figure, $\frac{SP}{PM} = \text{constant} = e$

$\Rightarrow \quad SP = e\,PM$

Equation of Conic Section

If the focus is (α,β) and the directrix is $ax + by + c = 0$, then the equation of the conic section whose eccentricity $= e$ is $SP = e\,PM$



$$\Rightarrow \sqrt{(x-\alpha)^2 + (y-\beta)^2} = e \cdot \frac{|ax + by + c|}{\sqrt{(a^2 + b^2)}}$$

$$\Rightarrow \quad (x-\alpha)^2 + (y-\beta)^2 = e^2 \cdot \frac{(ax + by + c)^2}{(a^2 + b^2)}.$$

Important Terms

Axis The straight line passing through the focus and perpendicular to the directrix is called the axis of the conic section.

Vertex The points of intersection of the conic section and the axis is (are) called vertex (vertices) of the conic section.

Focal Chord Any chord passing through the focus is called focal chord of the conic section.

Double Ordinate A straight line drawn perpendicular to the axis and terminated at both end of the curve is a double ordinate of the conic section.

Latusrectum The double ordinate passing through the focus is called the latusrectum of the conic section.

Centre The point which bisects every chord of the conic passing through it, is called the centre of the conic section.

Remark

Parabola has no centre but circle, ellipse and hyperbola have centre.

Example 1 Find the locus of a point, which moves such that its distance from the point (0, –1) is twice its distance from the line $3x + 4y + 1 = 0$.

Sol. Let $P(x_1, y_1)$ be the point, whose locus is required.

Its distance from $(0, -1) = 2\times$ its distance from the line $3x + 4y + 1 = 0$.

$$\Rightarrow \quad \sqrt{(x_1 - 0)^2 + (y_1 + 1)^2} = 2 \times \frac{|3x_1 + 4y_1 + 1|}{\sqrt{(3^2 + 4^2)}}$$

$$\Rightarrow \quad 5\sqrt{x_1^2 + (y_1 + 1)^2} = 2\,|3x_1 + 4y_1 + 1|$$

Squaring and simplifying, we have

$$25(x_1^2 + y_1^2 + 2y_1 + 1) = 4(9x_1^2 + 16y_1^2 + 24x_1y_1 + 6x_1 + 8y_1 + 1)$$

or $11x_1^2 + 39y_1^2 + 96x_1y_1 + 24x_1 - 18y_1 - 21 = 0$

Hence, the locus of (x_1, y_1) is

$$11x^2 + 39y^2 + 96xy + 24x - 18y - 21 = 0$$

Example 2 What conic does the equation $25(x^2 + y^2 - 2x + 1) = (4x - 3y + 1)^2$ represent ?

Sol. Given equation is

$$25(x^2 + y^2 - 2x + 1) = (4x - 3y + 1)^2 \qquad \ldots(i)$$

Write the right hand side of this equation, so that it appears in perpendicular distance form, then

$$(4x - 3y + 1)^2 = 25\left(\frac{4x - 3y + 1}{\sqrt{(4^2 + 3^2)}}\right)^2$$

then, Eq. (i) can be re-written as

$$25[(x - 1)^2 + (y - 0)^2] = 25\left[\frac{4x - 3y + 1}{\sqrt{(4^2 + 3^2)}}\right]^2$$

or $$\sqrt{(x - 1)^2 + (y - 0)^2} = \frac{|4x - 3y + 1|}{\sqrt{(4^2 + 3^2)}}$$

Here, $e = 1$

Thus, the given equation represents a parabola. It may noted that (1,0) is the focus and $4x - 3y + 1 = 0$ is the directrix of the parabola.

Recognisation of Conics

The equation of conics represented by the general equation of second degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \ldots(i)$$

can be recognised easily by the condition given in the tabular form. For this, first we have to find discriminant of the equation. We know that the discriminant of above equation is represented by Δ, where

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$$

The nature of the conic section depends upon the position of the focus S with respect to the directrix and also upon the value of the eccentricity e. Two different cases arise.

***Case* I (When the focus lies on the directrix)**

In this case Eq. (i) represents the **Degenerate conic** whose nature is given in the following table :

Condition	Nature of Conic
$e > 1$; $\Delta = 0$, $h^2 > ab$	The lines will be real and distinct intersecting at S.
$e = 1$; $\Delta = 0$, $h^2 = ab$	The lines will coincident
$e < 1$; $\Delta = 0$, $h^2 < ab$	The lines will be imaginary.

***Case* II (When the focus does not lie on the directrix)**

In this case Eq. (i) represents the **Non-degenerate conic** whose nature is given in the following table :

Condition	Nature of Conic
$e = 1$; $\Delta \neq 0$, $h^2 = ab$	a parabola
$0 < e < 1$; $\Delta \neq 0$, $h^2 < ab$	an ellipse
$e > 1$; $\Delta \neq 0$, $h^2 > ab$	a hyperbola
$e > 1$; $\Delta \neq 0$, $h^2 > ab$; $a + b = 0$	rectangular hyperbola

Remark

1. If conic represents an empty set, then $\Delta \neq 0$, $h^2 < ab$.
2. If conic represents a single point, the $\Delta = 0$, $h^2 < ab$.

Example 3 What conic does $13x^2 - 18xy + 37y^2 + 2x + 14y - 2 = 0$ represent?

Sol. Compare the given equation with

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

$\therefore \quad a = 13, h = -9, b = 37, g = 1, f = 7, c = -2$,

then, $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$$= (13)(37)(-2) + 2(7)(1)(-9) - 13(7)^2 - 37(1)^2 + 2(-9)^2$$

$$= -962 - 126 - 637 - 37 + 162 = -1600 \neq 0$$

and also $h^2 = (-9)^2 = 81$ and $ab = 13 \times 37 = 481$

Here, $h^2 < ab$

So, we have $h^2 < ab$ and $\Delta \neq 0$.

Hence, the given equation represents an ellipse.

Example 4 What conic is represented by the equation $\sqrt{ax} + \sqrt{by} = 1$?

Sol. Given conic is $\sqrt{ax} + \sqrt{by} = 1$

On squaring both sides, we get

$$ax + by + 2\sqrt{abxy} = 1$$

$$\Rightarrow \quad ax + by - 1 = -2\sqrt{abxy}$$

Again, on squaring both sides, then

$$(ax + by - 1)^2 = 4abxy$$

$$\Rightarrow \quad a^2x^2 + b^2y^2 + 1 + 2abxy - 2by - 2ax = 4abxy$$

$$\Rightarrow \quad a^2x^2 + b^2y^2 - 2abxy - 2ax - 2by + 1 = 0$$

$$\Rightarrow \quad a^2x^2 - 2abxy + b^2y^2 - 2ax - 2by + 1 = 0 \quad ...(i)$$

Comparing the Eq. (i) with the equation

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0$$

$\therefore \quad A = a^2, H = -ab, B = b^2, G = -a, F = -b, C = 1$

then, $\Delta = ABC + 2FGH - AF^2 - BG^2 - CH^2$

$= a^2b^2 - 2a^2b^2 - a^2b^2 - a^2b^2 - a^2b^2$

$= -4a^2b^2 \neq 0$ and $H^2 = a^2b^2 = AB$

So, we have $\Delta \neq 0$ and $H^2 = AB$.

Hence, the given equation represents a parabola.

Example 5 If the equation $x^2 - y^2 - 2x + 2y + \lambda = 0$ represents a degenerate conic, find the value of λ.

Sol. For degenerate conic $\Delta = 0$

Comparing the given equation of conic with

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

$\therefore \quad a = 1, b = -1, h = 0, g = -1, f = 1, c = \lambda$

$\therefore \quad \Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$\Rightarrow (1)(-1)(\lambda) + 0 - 1 \times (1)^2 + 1 \times (-1)^2 - \lambda(0)^2 = 0$

$\Rightarrow \quad -\lambda - 1 + 1 = 0 \Rightarrow \lambda = 0$

Example 6 If the equation $x^2 + y^2 - 2x - 2y + c = 0$ represents an empty set, then find the value of c.

Sol. For empty set $\Delta \neq 0$ and $h^2 < ab$.

Now, comparing the given equation of conic with

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c' = 0$$

then $a = 1, h = 0, b = 1, g = -1, f = -1, c' = c$

$\because \quad h^2 < ab$

$\therefore \quad 0 < 1$ which is true

and $\Delta = abc' + 2fgh - af^2 - bg^2 - c'h^2 \neq 0$

$\Rightarrow \quad (1)(1)(c) + 0 - 1 \times (-1)^2 - 1 \times (-1)^2 - 0 \neq 0$

$\Rightarrow \quad c - 2 \neq 0$

$\therefore \quad c \neq 2$

Hence, $c \in R \sim (2)$

Example 7 If the equation of conic

$$2x^2 + xy + 3y^2 - 3x + 5y + \lambda = 0$$

represent a single point, then find the value of λ.

Sol. For single point,

$$h^2 < ab \quad \text{and} \quad \Delta = 0$$

Comparing the given equation with

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

then, $a = 2, h = \frac{1}{2}, b = 3, g = -\frac{3}{2}, f = \frac{5}{2}, c = \lambda$.

$\because \quad h^2 = \frac{1}{4}, \; ab = 6$

$\therefore \quad h^2 < ab$

and $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= (2)(3)(\lambda) + 2 \times \frac{5}{2} \times -\frac{3}{2} \times \frac{1}{2}$

$\quad - 2 \times \frac{25}{4} - 3 \times \frac{9}{4} - \lambda \times \frac{1}{4}$

$= 6\lambda - \frac{15}{4} - \frac{25}{2} - \frac{27}{4} - \frac{\lambda}{4}$

$= \frac{23\lambda}{4} - 23 = 0$

$\therefore \quad \lambda = 4$

Example 8 For what value of λ the equation of conic $2xy + 4x - 6y + \lambda = 0$ represents two intersecting straight lines, if $\lambda = 17$, then this equation represents?

Sol. Comparing the given equation of conic with

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

$\therefore \quad a = 0, b = 0, h = 1, g = 2, f = -3, c = \lambda$

For two intersecting lines,

$h^2 > ab, \Delta = 0$

$\because \quad ab = 0, h = 1$

$\therefore \quad h^2 > ab$

and $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= 0 + 2 \times -3 \times 2 \times 1 - 0 - 0 - \lambda(1)^2$

$= -12 - \lambda = 0$

$\therefore \quad \lambda = -12$

For $\lambda = 17$, then the given equation of conic $2xy + 4x - 6y + 17 = 0$ according to the first system but here $c = 17$.

$\therefore \quad a = 0, b = 0, h = 1, g = 2, f = -3, c = 17,$

$\therefore \quad \Delta = abc + 2f\ gh - af^2 - bg^2 - ch^2$

$= 0 + 2 \times -3 \times 2 \times 1 - 0 - 0 - 17 \times (1)^2$

$= -12 - 17 = -29 \neq 0$

$\therefore \quad \Delta \neq 0$ and $h^2 > ab$

So, we have $\Delta \neq 0$ and $h^2 > ab$.

Hence, the given equation represents a hyperbola.

How to Find the Centre of Conics

If $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

Partially differentiating w.r.t. x and y, we get

$$\frac{\partial S}{\partial x} = 2ax + 2hy + 2g; \quad \frac{\partial S}{\partial y} = 2hx + 2by + 2f$$

(Treating y as constant) (Treating x as constant)

For centre, $\frac{\partial S}{\partial x}=0$ and $\frac{\partial S}{\partial y}=0$

$\therefore \quad 2ax+2hy+2g=0 \quad$ and $\quad 2hx+2by+2f=0$

$\Rightarrow \quad ax+hy+g=0 \quad$ and $\quad hx+by+f=0$

Solving these equations we get the centre

$$(x,y)=\left(\frac{hf-bg}{ab-h^2},\frac{gh-af}{ab-h^2}\right).$$

Remembering Method

Since, $\Delta=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

Write first two rows,

i.e. $\begin{matrix} a & h & g & a \\ h & b & f & h \end{matrix}$ (Repeat Ist member)

$\therefore \quad ab-h^2,\ hf-bg,\ gh-af$

or points $\left(\frac{hf-bg}{ab-h^2},\frac{gh-af}{ab-h^2}\right)$ or $\left(\frac{C_{13}}{C_{33}},\frac{C_{23}}{C_{33}}\right)$.

OR

According to first two rows,

$$ax+hy+g=0 \quad \text{and} \quad hx+by+f=0.$$

After solving we get find the centre of conic.

Example 9 *Find the centre of the conic*

$14x^2-4xy+11y^2-44x-58y+71=0$

Sol. Let $f(x,y)\equiv 14x^2-4xy+11y^2-44x-58y+71=0$

Differentiating partially w.r.t. x and y, then

$\frac{\partial f}{\partial x}=28x-4y-44$ and $\frac{\partial f}{\partial y}=-4x+22y-58$

For centre, $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$,

$\therefore \quad 28x-4y-44=0$

or $\quad 7x-y-11=0 \quad \ldots$(i)

and $\quad -4x+22y-58=0$

or $\quad -2x+11y=29 \quad \ldots$ (ii)

On solving Eqs. (i) and (ii) we get,

$$x=2 \text{ and } y=3$$

$\therefore$ Centre is (2, 3).

Aliter : Comparing the given conic with

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

$\therefore \quad a=14, h=-2, b=11, g=-22, f=-29, c=71$

$\therefore \quad$ Centre $\left(\frac{hf-bg}{ab-h^2},\frac{gh-af}{ab-h^2}\right)$

or $\quad =\left(\frac{(-2)(-29)-(11)(-22)}{(14)(11)-(-2)^2},\frac{(-22)(-2)-(14)(-29)}{(14)(11)-(-2)^2}\right)$

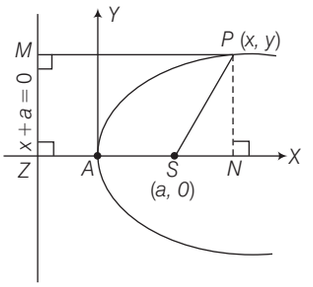
or $\quad =(2,3)$

Parabola : Definition

A parabola is the locus of a point which moves in a plane such that its distance from a fixed point (i.e. focus) is always equal to its distance from a fixed straight line (i.e., directrix).

Standard Equation of Parabola

Let S be the focus and ZM be the directrix of the parabola. Draw SZ perpendicular to ZM, let A be the mid point of SZ, then as $AS=AZ$



So, A lies on the parabola. Take A as the origin and a line AY through A perpendicular to AX as Y-axis.

Let $\quad AS=AZ=a>0$

then, coordinate of S is $(a,0)$ and the equation of ZM is

$$x=-a \text{ or } x+a=0$$

Now, take $P(x,y)$ be any point on the parabola. Join SP and from P draw PM perpendicular to the directrix ZM.

Then, $\quad SP=\sqrt{(x-a)^2+(y-0)^2}=\sqrt{(x-a)^2+y^2}$

and $\quad PM=ZN=AZ+AN=a+x$

Now, for the parabola $\quad SP=PM$

$\Rightarrow \quad (SP)^2=(PM)^2 \Rightarrow (x-a)^2+y^2=(a+x)^2$

$\Rightarrow \quad y^2=(a+x)^2-(x-a)^2=4ax$

$\therefore \quad y^2=4ax,$

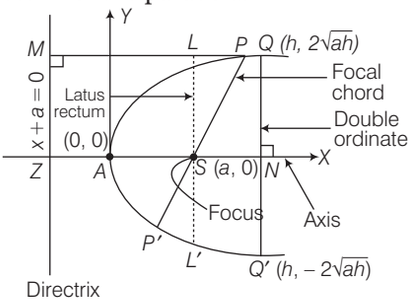
which is required equation of the parabola.

Remark

A parabola has two real foci situated on its axis one of which is the focus S and the other lies at infinity. the corresponding directrix is also at infinity.

Some Terms Related to Parabola

1. **Axis** The axis of the parabola is the straight line which is passes through focus and perpendicular to the directrix of the parabola.



For the parabola $y^2 = 4ax$, X-axis is the **axis**.

Here, all powers of y are even in $y^2 = 4ax$ then, parabola $y^2 = 4ax$ is symmetrical about its axis (i.e. X-axis).

or

If the point (x, y) lie on the parabola $y^2 = 4ax$, then the point $(x, -y)$ also lies on it. Hence, the parabola is symmetrical about X-axis (i.e. axis of parabola).

2. **Vertex** The point of intersection of the parabola and its axis is called the vertex of the parabola. For the parabola $y^2 = 4ax$.

$A(0, 0)$ i.e. the origin is the vertex.

3. **Double ordinate** If Q be the point on the parabola, draw QN perpendicular to the axis of parabola and produced to meet the curve again at Q', then QQ' is called a double ordinate.

If abscissa of Q is h, then ordinate of Q,

$$y^2 = 4ah \quad \text{or } y = 2\sqrt{ah} \qquad \text{(for first quadrant)}$$

and ordinate of Q' is $y = -2\sqrt{ah}$ (for fourth quadrant)

Hence, coordinates of Q and Q' are $(h, 2\sqrt{ah})$ and $(h, -2\sqrt{ah})$, respectively.

4. **Latusrectum** The double ordinate LL' passes through the focus is called the latusrectum of the parabola.

Since focus $S(a, 0)$ the equation of the latusrectum of the parabola is $x = a$, then solving

$$x = a \quad \text{and} \quad y^2 = 4ax$$

then, we get $y = \pm 2a$

Hence, the coordinates of the extremities of the latusrectum are $L(a, 2a)$ and $L'(a, -2a)$, respectively.

Since, $LS = L'S = 2a$

$\therefore$ Length of latusrectum $LL' = 2(LS) = 2(L'S) = 4a$.

5. **Focal chord** A chord of a parabola which is passing through the focus is called a focal chord of the parabola. In the given figure, PP' and LL' are the focal chords.

Remarks

1. In objective questions use LL' as focal chord and in subjective questions use PP' as focal chord.
2. Length of smallest focal chord of the parabola $4a$. Hence, the latusrectum of a parabola is the smallest focal chord.

6. **Focal distance** The focal distance of any point P on the parabola is its distance from the focus S i.e. SP

Also, $SP = PM$ = Distance of P from the directrix.

If $P \equiv (x, y)$

then, $SP = PM = x + a$

7. **Parametric equations** From the equation of the parabola $y^2 = 4ax$, we can write $\frac{y}{2a} = \frac{2x}{y} = t$

where 't' is a parameter.

Then, $y = 2at$ and $x = at^2$

The equations $x = at^2$ and $y = 2at$ are called parametric equations. The point $(at^2, 2at)$ is also referred to as the point 't'.

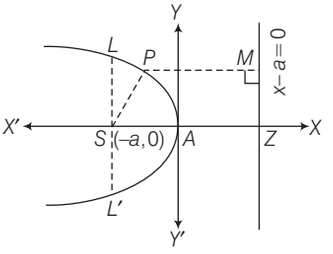
Remarks

1. Coordinates of any point on the parabola $y^2 = 4ax$, may be taken as $(at^2, 2at)$.
2. Equation of chord joining t_1 and t_2 is $2x - (t_1 + t_2)y + 2at_1t_2 = 0$.
3. If the chord joining t_1, t_2 and t_3, t_4 pass through a point $(c, 0)$ on the axis, then $t_1t_2 = t_3t_4 = -\frac{c}{a}$.

Other forms of Parabola with Latusrectum $4a$

(1) Parabola opening to left (i.e. $y^2 = -4ax$) : ($a > 0$)

(i) Vertex is $A(0, 0)$.
(ii) Focus is $S(-a, 0)$.
(iii) Equation of the directrix MZ is $x - a = 0$.
(iv) Equation of the axis is $y = 0$ i.e. X-axis.
(v) Equation of the tangent at the vertex is $x = 0$ i.e. Y-axis.

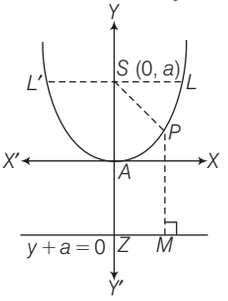


(vi) Length of latusrectum = $LL' = 4a$.

(vii) Ends of latusrectum are $L(-a, 2a)$ and $L'(-a, -2a)$.
(viii) Equation of latusrectum is $x = -a$ i.e. $x + a = 0$.
(ix) Parametric coordinates is $(-at^2, 2at)$.

(2) Parabola opening upwards (i.e. $x^2 = 4ay$) : $(a > 0)$

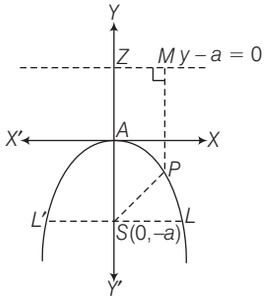
(i) Vertex is $A(0, 0)$.
(ii) Focus is $S(0, a)$.
(iii) Equation of the directrix MZ is $y + a = 0$.



(iv) Equations of the axis is $x = 0$ i.e. Y-axis.
(v) Equation of the tangent at the vertex is $y = 0$ i.e. X-axis.
(vi) Ends of latusrectum are $L(2a, a)$ and $L'(-2a, a)$.
(vii) Length of latusrectum $= LL' = 4a$.
(viii) Equation of latusrectum is $y = a$ i.e. $y - a = 0$.
(ix) Parametric coordinates is $(2at, at^2)$.

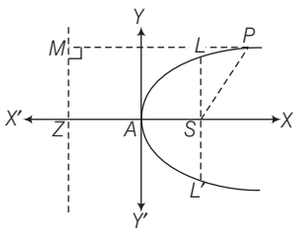
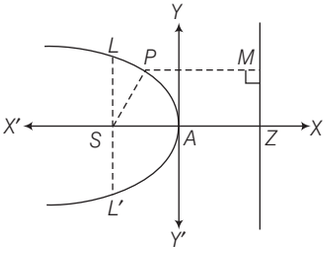
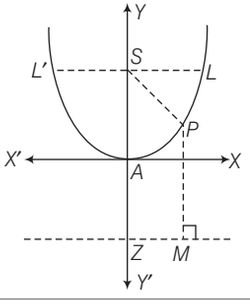
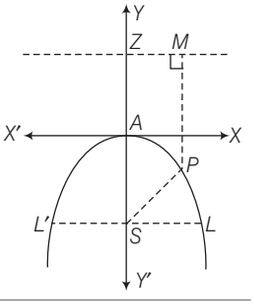
(3) Parabola opening downwards (i.e. $x^2 = -4ay$) : $(a > 0)$

(i) Vertex is $A(0, 0)$. (ii) Focus is $S(0, -a)$.
(iii) Equation of the directrix MZ is $y - a = 0$.
(iv) Equation of the axis is $x = 0$ i.e. Y-axis.
(v) Equation of the tangent at the vertex is $y = 0$ i.e. X-axis.



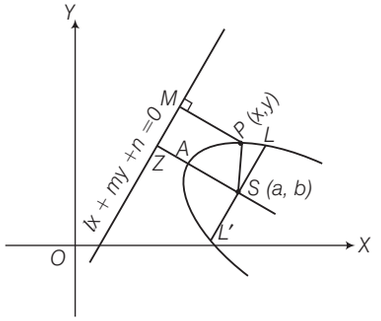
(vi) Length of latusrectum $= LL' = 4a$
(vii) Ends of latusrectum are $L(2a, -a)$ and $L'(-2a, -a)$
(viii) Equation of latusrectum is $y = -a$ i.e. $y + a = 0$.
(ix) Parametric coordinates are $(2at, -at^2)$.

Smart Table : The Study of Standard Parabolas

Equation and Graph of the parabola	$y^2 = 4ax, a > 0$	$y^2 = -4ax, a > 0$	$x^2 = 4ay, a > 0$	$x^2 = -4ay, a > 0$
				
Vertex	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
Focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Equation of the axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Equation of tangent at vertex	$x = 0$	$x = 0$	$y = 0$	$y = 0$
Equation of directrix	$x + a = 0$	$x - a = 0$	$y + a = 0$	$y - a = 0$
Length of latusrectum	$4a$	$4a$	$4a$	$4a$
Ends points of latusrectum	$(a, \pm 2a)$	$(-a, \pm 2a)$	$(\pm 2a, a)$	$(\pm 2a, -a)$
Equation of latusrectum	$x - a = 0$	$x + a = 0$	$y - a = 0$	$y + a = 0$
Focal distance of a point $P(x, y)$	$x + a$	$a - x$	$y + a$	$a - y$
Parametric coordinates	$(at^2, 2at)$	$(-at^2, 2at)$	$(2at, at^2)$	$(2at, -at^2)$
Eccentricity (e)	1	1	1	1

General Equation of a Parabola

Let $S(a,b)$ be the focus, and $lx+my+n=0$ is the equation of the directrix. Let $P(x,y)$ be any point on the parabola. Then by definition $SP = PM$



$$\Rightarrow \sqrt{(x-a)^2+(y-b)^2} = \frac{|lx+my+n|}{\sqrt{(l^2+m^2)}}$$

$$\Rightarrow (x-a)^2+(y-b)^2 = \frac{(lx+my+n)^2}{(l^2+m^2)}$$

$$\Rightarrow m^2x^2+l^2y^2-2lmxy+x \text{ term}+y \text{ term}+\text{constant}=0$$

This is of the form $(mx-ly)^2+2gx+2fy+c=0$.

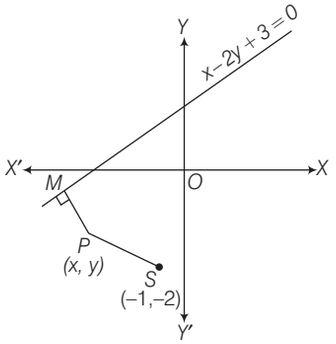
This equation is the general equation of parabola.

Remark

Second degree terms in the general equation of a parabola forms a perfect square.

Example 10 Find the equation of the parabola whose focus is at $(-1,-2)$ and the directrix is the straight line $x-2y+3=0$.

Sol. Let $P(x,y)$ be any point on the parabola whose focus is $S(-1,-2)$ and the directrix $x-2y+3=0$. Draw PM perpendicular from $P(x,y)$ on the directrix $x-2y+3=0$.



Then, by definition

$$SP = PM$$

$$\Rightarrow (SP)^2 = (PM)^2$$

$$\Rightarrow (x+1)^2+(y+2)^2 = \left(\frac{|x-2y+3|}{\sqrt{(1)^2+(-2)^2}}\right)^2$$

$$\Rightarrow 5(x^2+y^2+2x+4y+5) = (x^2+4y^2-4xy+6x-12y+9)$$

$$\therefore 4x^2+y^2+4xy+4x+32y+16=0$$

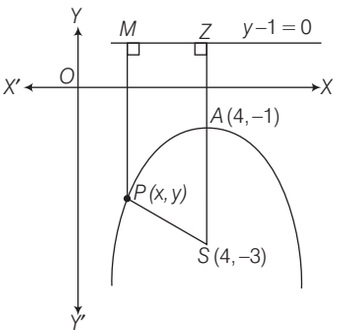
Example 11 Find the equation of the parabola whose focus is $(4,-3)$ and vertex is $(4,-1)$.

Sol. Let $A(4,-1)$ be the vertex and $S(4,-3)$ be the focus.

$$\therefore \text{Slope of } AS = \frac{-3+1}{4-4} = \infty$$

which is parallel to Y-axis.

$\therefore$ Directrix parallel to X-axis.



Let $Z(x_1,y_1)$ be any point on the directrix, then A is the mid-point of SZ.

$$\therefore 4 = \frac{x_1+4}{2} \Rightarrow x_1 = 4$$

and $$-1 = \frac{y_1-3}{2} \Rightarrow y_1 = 1$$

$$\therefore Z = (4,1)$$

Also, directrix is parallel to X-axis and passes through $Z(4,1)$, so equation of directrix is

$$y=1 \quad \text{or} \quad y-1=0$$

Now, let $P(x,y)$ be any point on the parabola. Join SP and draw PM perpendicular to the directrix. Then, by definition

$$SP = PM$$

$$\Rightarrow (SP)^2 = (PM)^2$$

$$\Rightarrow (x-4)^2+(y+3)^2 = \left(\frac{|y-1|}{\sqrt{1^2}}\right)^2$$

$$\Rightarrow (x-4)^2+(y+3)^2 = (y-1)^2$$

$$\therefore x^2-8x+8y+24=0$$

Aliter :

Here $a = AS = 2$

$\therefore$ Length of latusrectum $= 4a = 8$

Equation of parabola with vertex $(0,0)$ and open downward is $x^2 = -8y$.

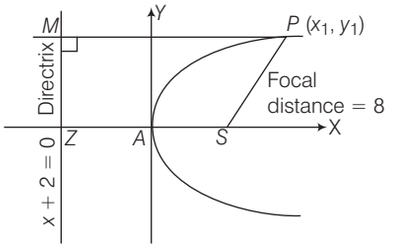
Shifting $(4,-1)$ on $(0,0)$, we get required parabola

$$(x-4)^2 = -8(y+1)$$

$$\therefore x^2-8x+8y+24=0$$

Example 12 The focal distance of a point on a parabola $y^2 = 8x$ is 8. Find it.

Sol. Comparing $y^2 = 8x$ with $y^2 = 4ax$



$\therefore \quad 4a = 8 \implies a = 2$

$\therefore$ Equation of directrix is $x + 2 = 0$.

Let $P(x_1, y_1)$ on the parabola

$$y^2 = 8x$$

$\therefore \quad y_1^2 = 8x_1$ ...(i)

$\because \quad SP = 8$

$\Rightarrow \quad PM = 8 \quad [\because SP = PM]$

$\Rightarrow \quad x_1 + 2 = 8$

or $\quad x_1 = 6$

From Eq. (i), $\quad y_1^2 = 8 \times 6$

$\therefore \quad y_1 = \pm 4\sqrt{3}$

$\therefore$ The required points are $(6, 4\sqrt{3})$ and $(6, -4\sqrt{3})$.

Example 13 QQ' is a double ordinate of a parabola $y^2 = 4ax$. Find the locus of its point of trisection.

Sol. Let the double ordinate QQ' meet the axis of the parabola

$$y^2 = 4ax \quad \text{...(i)}$$

Let coordinates of Q be (x_1, y_1), then coordinates of Q' be $(x_1 - y_1)$ since, Q and Q' lies on Eq. (i), then

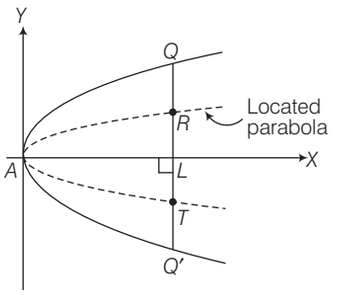
$$y_1^2 = 4ax_1 \quad \text{...(ii)}$$

Let R and T be the points of trisection of QQ'. Then, the coordinates of R and T are

$$\left(\frac{1\cdot x_1 + 2\cdot x_1}{1+2}, \frac{1\cdot(-y_1) + 2\cdot y_1}{1+2}\right) \text{ or } \left(x_1, \frac{y_1}{3}\right)$$

and $\left(\frac{2\cdot x_1 + 1\cdot x_1}{2+1}, \frac{2\cdot(-y_1) + 1\cdot y_1}{2+1}\right)$ or $\left(x_1, -\frac{y_1}{3}\right)$

respectively.



Since, R divide QQ' in 1 : 2 (internally)

and T divide QQ' in 2 : 1 (internally).

For locus, let $R(h,k)$, then

$$x_1 = h \quad \text{and} \quad \frac{y_1}{3} = k \quad \text{or} \quad y_1 = 3k$$

On substituting the values of x_1 and $y_1$in Eq. (ii), then

$$(3k)^2 = 4a(h) \quad \text{or} \quad 9k^2 = 4ah$$

$\therefore$ The required locus is $9y^2 = 4ax$ similarly, let $T(h',k')$

then, $\quad x_1 = h' \quad \text{and} \quad -\frac{y_1}{3} = k'$

or $\quad y_1 = -3k'$

On substituting the values of x_1 and y_1 in Eq. (ii), then

$$(-3k')^2 = 4a(h')$$

or $\quad 9k'^2 = 4ah'$

$\therefore$ The required locus is $9y^2 = 4ax$.

Hence, the locus of point of trisection is

$$9y^2 = 4ax.$$

Aliter : Let R and T be the points of trisection of double ordinates QQ'. Let (h,k) be the coordinates of R,

then, $\quad AL = h \quad \text{and} \quad RL = k$

$$RT = RL + LT = k + k = 2k.$$

Since, $\quad RQ = TR = Q'T = 2k$

$\therefore \quad LQ = LR + RQ = k + 2k = 3k$

Thus, the coordinates of Q are $(h, 3k)$.

Since, $(h, 3k)$ lies on $y^2 = 4ax$

$\Rightarrow \quad 9k^2 = 4ah$

Hence, the locus of (h, k) is $9y^2 = 4ax$.

Example 14 Prove that the area of the triangle inscribed in the parabola $y^2 = 4ax$ is $\frac{1}{8a}(y_1 \sim y_2)(y_2 \sim y_3)(y_3 \sim y_1)$, where y_1, y_2, y_3 are the ordinates of the vertices.

Sol. Let the vertices of the triangle be $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3).

$\because \quad (x_1, y_1)$ is a point on the parabola $y^2 = 4ax$.

$\therefore \quad y_1^2 = 4ax_1$

$\therefore \quad x_1 = \frac{y_1^2}{4a}$

Similarly, $\quad x_2 = \frac{y_2^2}{4a}$

and $\quad x_3 = \frac{y_3^2}{4a}$

Now , vertices of triangle are

$$\left(\frac{y_1^2}{4a}, y_1\right), \left(\frac{y_2^2}{4a}, y_2\right) \text{and} \left(\frac{y_3^2}{4a}, y_3\right).$$

$\therefore$ Required area of the triangle $= \frac{1}{2}\left|\begin{vmatrix} \frac{y_1^2}{4a} & y_1 & 1 \\ \frac{y_2^2}{4a} & y_2 & 1 \\ \frac{y_3^2}{4a} & y_3 & 1 \end{vmatrix}\right|$

$$= \frac{1}{8a}\left|\begin{vmatrix} y_1^2 & y_1 & 1 \\ y_2^2 & y_2 & 1 \\ y_3^2 & y_3 & 1 \end{vmatrix}\right| = \frac{1}{8a}(y_1 \sim y_2)(y_2 \sim y_3)(y_3 \sim y_1)$$

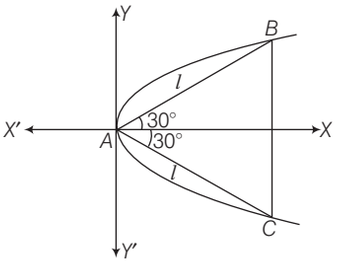
Example 15 Find the length of the side of an equilateral triangle inscribed in the parabola $y^2 = 4ax$, so that one angular point is at the vertex.

Sol. Let ABC be the inscribed equilateral triangle, with one angular point at the vertex A of the parabola

$$y^2 = 4ax \quad \ldots(i)$$

Let the length of the side of equilateral triangle $= l$

$\therefore$ $AB = BC = CA = l$



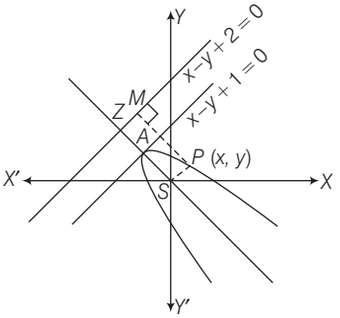
$\therefore$ The coordinates of B is $(l\cos 30°, l\sin 30°)$

i.e., $\left(\frac{l\sqrt{3}}{2}, \frac{l}{2}\right)$.

Since, B lies on Eq. (i), then $\left(\frac{l}{2}\right)^2 = 4a\left(\frac{l\sqrt{3}}{2}\right)$ or $l = 8a\sqrt{3}$

Example 16 Prove that the equation of the parabola whose focus is (0,0) and tangent at the vertex is $x - y + 1 = 0$ is $x^2 + y^2 + 2xy - 4x + 4y - 4 = 0$.

Sol. Let focus is $S(0, 0)$ and A is the vertex of the parabola take any point Z such that $AS = AZ$ given tangent at vertex is $x - y + 1 = 0$, since directrix is parallel to the tangent at the vertex.



$\therefore$ Equation of directrix is $x - y + \lambda = 0$

where, λ is constant.

$\because$ A is the mid-point of SZ.

$\therefore$ $SZ = 2SA$

$\Rightarrow$ $\frac{|0 - 0 + \lambda|}{\sqrt{(1^2 + (-1)^2)}} = 2 \times \frac{|0 - 0 + 1|}{\sqrt{(1^2 + (-1)^2)}}$

$\Rightarrow$ $\frac{|\lambda|}{\sqrt{2}} = \frac{2}{\sqrt{2}}$

$\therefore$ $\lambda = \pm 2$

$\Rightarrow$ $\lambda = 2$

[$\because$ λ is positive since directrix in this case always lies in II quadrant]

$\therefore$ Equation of directrix is $x - y + 2 = 0$.

Now, take $P(x, y)$ be any point on the parabola, draw $PM \perp ZM$, then from definition,

$$SP = PM$$

$\Rightarrow$ $(SP)^2 = (PM)^2$

$\Rightarrow$ $(x - 0)^2 + (y - 0)^2 = \left(\frac{|x - y + 2|}{\sqrt{2}}\right)^2$

$\Rightarrow$ $2(x^2 + y^2) = (x - y + 2)^2$

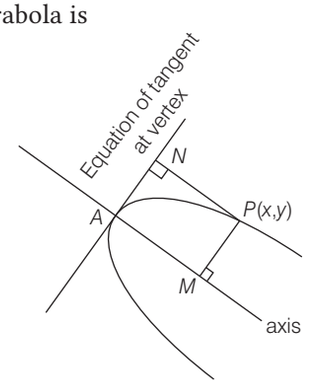
$\Rightarrow$ $2x^2 + 2y^2 = x^2 + y^2 - 2xy + 4x - 4y + 4$

$\therefore$ $x^2 + y^2 + 2xy - 4x + 4y - 4 = 0$

Equation of Parabola if Equation of axis, Tangent at Vertex and Latusrectum are given

Let equation of axis is $ax + by + c = 0$ and equation of tangent at vertex is $bx - ay + d = 0$.

Equation of parabola is

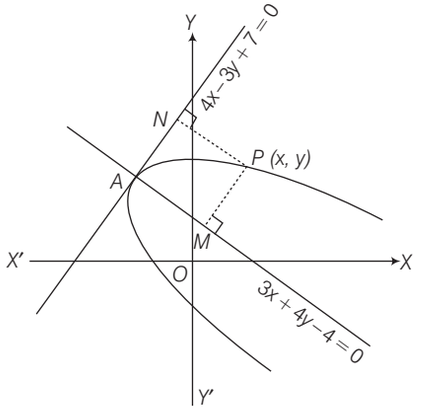


$$(PM)^2 = (\text{Latusrectum})\ (PN)$$

$$\Rightarrow \left(\frac{ax + by + c}{\sqrt{(a^2 + b^2)}}\right)^2 = (\text{Latusrectum})\left(\frac{bx - ay + d}{\sqrt{(b^2 + a^2)}}\right)$$

Example 17 Find the equation of the parabola whose latusrectum is 4 units, axis is the line $3x+4y-4=0$ and the tangent at the vertex is the line $4x-3y+7=0$.

Sol. Let $P(x, y)$ be any point on the parabola and let PM and PN are perpendiculars from P on the axis and tangent at the vertex respectively, then



$$(PM)^2 = (\text{latusrectum})\,(PN)$$

$$\Rightarrow \quad \left(\frac{3x+4y-4}{\sqrt{3^2+4^2}}\right)^2 = 4\left(\frac{4x-3y+7}{\sqrt{4^2+(-3)^2}}\right)$$

$$\therefore \quad (3x+4y-4)^2 = 20\,(4x-3y+7)$$

which is required parabola.

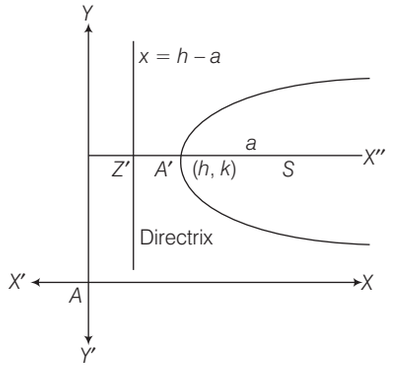
The Generalised form $(y-k)^2 = 4a(x-h)$

The parabola

$$y^2 = 4ax \qquad \ldots(i)$$

can be written as $(y-0)^2 = 4a\,(x-0)$.

The vertex of this parabola is $A(0,0)$



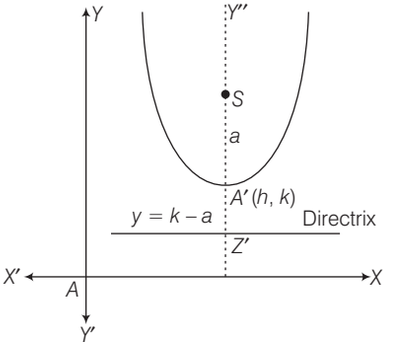
Now, when origin is shifted at $A'(h, k)$ without changing the direction of axes, its equation becomes

$$(y-k)^2 = 4a\,(x-h) \qquad \ldots(ii)$$

This is called generalised form of the parabola Eq. (i) and axis $A'X'' \| AX$ with its vertex at $A'(h, k)$. Its focus is at $(a+h, k)$ and length of latusrectum = $4a$, the equation of the directrix is

$$x = h-a \;\Rightarrow\; x+a-h=0$$

Another form is $(x-h)^2 = 4a(y-k)$ axis parallel to



Y-axis with its vertex (h,k) its focus is at $(h,a+k)$ and length of latusrectum = $4a$, the equation of the directrix is

$$y = k-a \;\Rightarrow\; y+a-k=0.$$

Remark

The parametric equation of $(y-k)^2 = 4a(x-h)$ are $x = h + at^2$ and $y = k + 2at$.

Parabolic Curve

The equations $y = Ax^2 + Bx + C$ and $x = Ay^2 + By + C$ are always represents parabolas generally called parabolic curve.

Now, $\quad y = Ax^2 + Bx + C$

$$= A\left\{x^2 + \frac{B}{A}x + \frac{C}{A}\right\}$$

$$= A\left\{\left(x+\frac{B}{2A}\right)^2 - \frac{B^2}{4A^2} + \frac{C}{A}\right\}$$

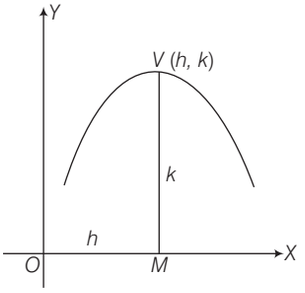
$$= A\left\{\left(x+\frac{B}{2A}\right)^2 - \frac{(B^2-4AC)}{4A^2}\right\}$$

or $\quad \left(x+\frac{B}{2A}\right)^2 = \frac{1}{A}\left(y+\frac{B^2-4AC}{4A}\right)$

Comparing it with $(x-h)^2 = 4a(y-k)$ it represent a parabola with vertex at $(h,k) = \left(-\frac{B}{2A}, -\frac{B^2-4AC}{4A}\right)$

and axis parallel to Y-axis and latusrectum $= \frac{1}{|A|}$

and the curve opening upwards and downwards depending upon the sign of A and B.



The optimum distance of its vertex V from OX is

$$\left| -\frac{B^2 - 4AC}{4A} \right|$$

and $$x = Ay^2 + By + C$$

$$= A\left\{ y^2 + \frac{B}{A}y + \frac{C}{A} \right\}$$

$$= A\left\{ \left(y + \frac{B}{2A} \right)^2 - \frac{B^2}{4A^2} + \frac{C}{A} \right\}$$

$$= A\left\{ \left(y + \frac{B}{2A} \right)^2 - \frac{B^2 - 4AC}{4A^2} \right\}$$

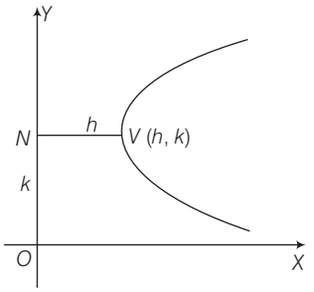
$$\Rightarrow \left(y + \frac{B}{2A} \right)^2 = \frac{1}{A}\left(x + \frac{B^2 - 4AC}{4A} \right)$$

Comparing it with $(y-k)^2 = 4a(x-h)$, it represent a parabola with vertex at

$$(h, k) = \left(-\frac{B^2 - 4AC}{4A}, -\frac{B}{2A} \right)$$

and axis parallel to X-axis and latusrectum $= \frac{1}{|A|}$

and the curve opening left and right depending upon the sign of A and B.



The optimum distance of its vertex V from OY is

$$\left| -\frac{B^2 - 4AC}{4A} \right|.$$

Remarks

1. The optimum distance of vertex from OX or OY can be easily obtained using calculus Method.
2. Equation of the parabola with axis parallel to the X-axis is of the form $x = Ay^2 + By + C$.
3. Equation of the parabola with axis parallel to the Y-axis is of the form $y = Ax^2 + Bx + C$.

Method to Make Perfect Square

If $$x = \alpha y^2 \pm \beta y + \gamma$$

first make the coefficient of y^2 is unity

i.e., $$x = \alpha\left\{ y^2 \pm \frac{\beta}{\alpha}y + \frac{\gamma}{\alpha} \right\}$$

Now, in braces write y and put the sign after y which between y^2 and y i.e. $\pm$ and after this sign write the half the coefficient of y i.e. $\frac{\beta}{2\alpha}$.

Now, write in braces $\left(y \pm \frac{\beta}{2\alpha} \right)^2$

and always subtract $\left(\frac{\beta}{2\alpha} \right)^2 = \frac{\beta^2}{4\alpha^2}$

$$\therefore \quad x = \alpha\left\{ \left(y \pm \frac{\beta}{2\alpha} \right)^2 - \frac{\beta^2}{4\alpha^2} + \frac{\gamma}{\alpha} \right\}$$

$$= \alpha\left\{ \left(y \pm \frac{\beta}{2\alpha} \right)^2 - \frac{(\beta^2 - 4\gamma\alpha)}{4\alpha^2} \right\}$$

Example 18 Find the vertex, focus, latusrectum, axis and the directrix of the parabola $x^2 + 8x + 12y + 4 = 0$.

Sol. The equation of parabola is

$$x^2 + 8x + 12y + 4 = 0 \quad \ldots(i)$$

$$\Rightarrow (x+4)^2 - 16 + 12y + 4 = 0$$

$$\Rightarrow (x+4)^2 - 12 + 12y = 0$$

$$\Rightarrow (x+4)^2 = -12y + 12$$

$$\Rightarrow (x+4)^2 = -12(y-1)$$

Let $$x + 4 = X,\ y - 1 = Y \quad \ldots(ii)$$

$$\therefore \quad X^2 = -12Y \quad \ldots(iii)$$

Comparing it with $X^2 = -4aY$

$$\therefore \quad a = 3$$

$\therefore$ Vertex of Eq. (iii) is (0, 0)

i.e. $$X = 0, Y = 0$$

From Eq. (ii),

$$x + 4 = 0\,,\ y - 1 = 0$$

$\therefore$ $x = -4,\ y = 1$

$\therefore$ Vertex of Eq. (i) is $(-4, 1)$.

Foucs of Eq. (iii) is $(0, -3)$

i.e. $X = 0, Y = -3$

From Eq. (ii),

$$x + 4 = 0,\ y - 1 = -3$$

$\therefore$ $x = -4\,,\ y = -2$

$\therefore$ Focus of Eq. (i) is $(-4, -2)$.

and latusrectum $= 4a = 12$.

Equation of axis of Eq. (iii) is $X = 0$

$\therefore$ Equation of axis of Eq. (i) is $x + 4 = 0$

Equation of directrix of Eq. (iii) is

$$Y = 3 \text{ or } y - 1 = 3$$

$\therefore$ $y - 4 = 0$

$\therefore$ Equation of directrix of Eq. (i) is

$$y - 4 = 0.$$

Example 19 Prove that the equation $y^2 + 2ax + 2by + c = 0$ represents a parabola whose axis is parallel to the axis of x. Find its vertex.

Sol. The equation of parabola is

$$y^2 + 2ax + 2by + c = 0$$

$$(y + b)^2 - b^2 + 2ax + c = 0$$

$$\Rightarrow \quad (y + b)^2 = -2ax + b^2 - c$$

$$\Rightarrow \quad (y + b)^2 = -2a\left(x - \frac{b^2 - c}{2a}\right) \qquad \ldots(i)$$

Let $y + b = Y,\ x - \dfrac{b^2 - c}{2a} = X$

From Eq. (i) ,

$$Y^2 = -2aX \qquad \ldots(ii)$$

axis of its parabola is $Y = 0$

or $y + b = 0,$

which is parallel to X-axis

and vertex of Eq. (ii) is $X = 0, Y = 0$

$$\Rightarrow \quad x - \frac{b^2 - c}{2a} = 0, \quad y + b = 0$$

$$\Rightarrow \quad x = \frac{b^2 - c}{2a}, \quad y = -b$$

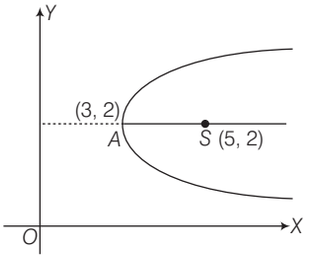
$\therefore$ Vertex of given parabola is $\left(\dfrac{b^2 - c}{2a}, -b\right)$.

Example 20 Find the equation of the parabola with its vertex at (3, 2) and its focus at (5, 2).

Sol. Let Vertex $A(3,2)$ and focus is $S(5,2)$

Slope of $AS = \dfrac{2-2}{5-3} = 0$, which is parallel to X-axis.

Hence, axis of parabola parallel to X-axis.



The equation is of the form

$$(y - k)^2 = 4a\,(x - h)$$

or $(y - 2)^2 = 4a\,(x - 3)$

as (h,k) is the vertex (3,2)

$a =$ distance between the focus and the vertex

$$= \sqrt{(5-3)^2 + (2-2)^2} = 2$$

Hence, the required equation is

$$(y - 2)^2 = 8(x - 3)$$

or $y^2 - 8x - 4y - 28 = 0.$

Example 21 Find the equation of the parabola with latusrectum joining the points (3, 6) and (3, −2).

Sol. Slope of (3,6) and (3, −2) is $\dfrac{-2-6}{3-3} = \infty$, since latusrectum is perpendicular to axis. Hence, axis parallel to X-axis. The equation of the two possible parabolas will be of the form

$$(y - k)^2 = \pm\, 4a\,(x - h) \qquad \ldots(i)$$

Since, latusrectum $= \sqrt{(3-3)^2 + (6+2)^2} = 8$

$\therefore$ $4a = 8$

$\Rightarrow$ $a = 2$

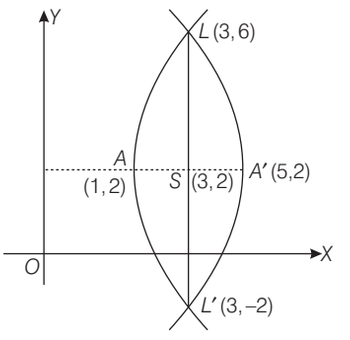
$\therefore$ From Eq. (i),

$$(y - k)^2 = \pm\, 8\,(x - h)$$

Since, (3,6) and (3, −2) lie on the parabola, then

$$(6 - k)^2 = \pm\, 8\,(3 - h) \qquad \ldots(ii)$$

and $(-2 - k)^2 = \pm\, 8\,(3 - h) \qquad \ldots(iii)$



On solving Eqs. (ii) and (iii), we get

$$k = 2$$

From Eq. (ii),

$$16 = \pm 8(3-h),$$

$$\therefore \quad h = 3 \pm 2$$

$$\therefore \quad h = 5,1$$

Hence, values of (h, k) are (5,2) and (1, 2).

The required parabolas are

$$(y-2)^2 = 8(x-5)$$

and $\quad (y-2)^2 = -8(x-1).$

Example 22 Find the equation to the parabola whose axis parallel to the Y-axis and which passes through the points (0, 4) (1, 9) and (4, 5) and determine its latusrectum .

Sol. The equation of parabola parallel to Y-axis is

$$y = Ax^2 + Bx + C \qquad \ldots(i)$$

The points (0,4) , (1,9) and (4,5) lie on Eq. (i), then

$$4 = 0 + 0 + C \quad \Rightarrow \quad C = 4 \qquad \ldots(ii)$$

$$\Rightarrow \qquad 9 = A + B + C$$

$$\Rightarrow \qquad 9 = A + B + 4 \qquad [\because C = 4]$$

$$\therefore \qquad A + B = 5 \qquad \ldots(iii)$$

and $\qquad 5 = 16A + 4B + C \qquad [\because C = 4]$

$$\Rightarrow \qquad 5 = 16A + 4B + 4$$

$$\therefore \qquad 16A + 4B = 1$$

$$\Rightarrow \qquad 4A + B = \frac{1}{4} \qquad \ldots(iv)$$

On solving Eqs. (iii) and (iv), we get

$$A = -\frac{19}{12}, B = \frac{79}{12} \qquad \ldots(v)$$

On substituting the values of A, B and C from Eqs. (ii) and Eq. (v) in Eq. (i), then equation of parabola is

$$y = -\frac{19}{12}x^2 + \frac{79}{12}x + 4$$

Hence, length of latusrectum $= \dfrac{1}{\left|-\dfrac{19}{12}\right|} = \dfrac{12}{19}.$

Exercise for Session 1

1. The vertex of the parabola $y^2 + 6x - 2y + 13 = 0$ is

(a) (–2, 1) (b) (2, – 1)

(c) (1, 1) (d) (1 , –1)

2. If the parabola $y^2 = 4ax$ passes through (3, 2), then the length of latusrectum is

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$

(c) 1 (d) $\frac{4}{3}$

3. The value of p such that the vertex of $y = x^2 + 2px + 13$ is 4 units above the X-axis is

(a) ± 2 (b) 4

(c) ± 3 (d) 5

4. The length of the latusrectum of the parabola whose focus is (3,3) and directrix is $3x - 4y - 2 = 0$, is

(a) 1 (b) 2

(c) 4 (d) 8

5. If the vertex and focus of a parabola are (3,3) and (–3, 3) respectively , then its equation is

(a) $x^2 - 6x + 24y - 63 = 0$ (b) $x^2 - 6x + 24y + 81 = 0$

(c) $y^2 - 6y + 24x - 63 = 0$ (d) $y^2 - 6y - 24x + 81 = 0$

6. If the vertex of the parabola $y = x^2 - 8x + c$ lies on X-axis, then the value of c is

(a) 4 (b) – 4

(c) 16 (d) –16

7. The parabola having its focus at (3,2) and directrix along the Y-axis has its vertex at

(a) $\left(\frac{3}{2}, 1\right)$ (b) $\left(\frac{3}{2}, 2\right)$

(c) $\left(\frac{3}{2}, \frac{1}{2}\right)$ (d) $\left(\frac{3}{2}, -\frac{1}{2}\right)$

8. The directrix of the parabola $x^2 - 4x - 8y + 12 = 0$ is

(a) $y = 0$ (b) $x = 1$

(c) $y = -1$ (d) $x = -1$

9. The equation of the latusrectum of the parabola $x^2 + 4x + 2y = 0$ is

(a) $3y - 2 = 0$ (b) $3y + 2 = 0$

(c) $2y - 3 = 0$ (d) $2y + 3 = 0$

10. The focus of the parabola $x^2 - 8x + 2y + 7 = 0$ is

(a) $\left(0, -\frac{1}{2}\right)$ (b) (4,4)

(c) $\left(4, \frac{9}{2}\right)$ (d) $\left(-4, -\frac{9}{2}\right)$

11. The equation of the parabola with the focus (3,0) and directrix $x + 3 = 0$ is

(a) $y^2 = 2x$ (b) $y^2 = 3x$

(c) $y^2 = 6x$ (d) $y^2 = 12x$

12. Equation of the parabola whose axis is parallel to *Y*-axis and which passes through the points (1,0) , (0,0) and (−2,4), is

(a) $2x^2 + 2x = 3y$ (b) $2x^2 - 2x = 3y$

(c) $2x^2 + 2x = y$ (d) $2x^2 - 2x = y$

13. Find the equation of the parabola whose focus is (5,3) and directrix is the line $3x - 4y + 1 = 0$.

14. Find the equation of the parabola is focus is at (–6, –6) and vertex is at (–2, 2).

15. Find the vertex, focus, axis, directrix and latusrectum of the parabola $4y^2 + 12x - 20y + 67 = 0$.

16. Find the name of the conic represented by $\sqrt{\left(\frac{x}{a}\right)} + \sqrt{\left(\frac{y}{b}\right)} = 1$.

17. Determine the name of the curve described parametrically by the equations

$x = t^2 + t + 1,\ y = t^2 - t + 1$.

18. Prove that the equation of the parabola whose vertex and focus are on the *X*-axis at a distance a and a' from the origin respectively is $y^2 = 4(a' - a)(x - a)$.

19. Find the equation of the parabola whose axis is parallel to *X*-axis and which passes through the points $(0, 4)$, $(1, 9)$ and (–2, 6). Also, find its latusrectum.

20. The equation $ax^2 + 4xy + y^2 + ax + 3y + 2 = 0$ represents a parabola, then find the value of a.

Session 2

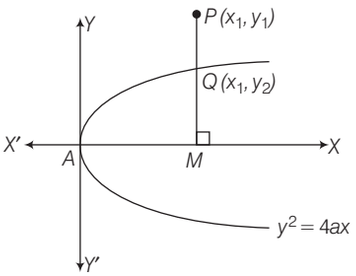
Position of a Point (x_1, y_1) with respect to a Parabola $y^2 = 4ax$, Parametric Relation between the Coordinates of the Ends of a Focal Chord of a Parabola, Intersection of a Line and a Parabola, Equation of Tangent in Different Forms, Point of Intersection of Tangents at any Two Points on the Parabola, Equation of Normals in Different Forms, Point of Intersection of Normals at any Two Points on the Parabola, Circle Through Co-normal Points

Position of a Point (x_1, y_1) with Respect to a Parabola $y^2 = 4ax$

Theorem The point (x_1, y_1) lies outside, on or inside the parabola $y^2 = 4ax$ according as

$$y_1^2 - 4ax_1 >, =, \text{or} < 0.$$

Proof Let $P(x_1, y_1)$ be a point. From P draw $PM \perp AX$ (on the axis of parabola) meeting the parabola $y^2 = 4ax$ at Q let the coordinate of Q be (x_1, y_2).



Since, $Q(x_1, y_2)$ lies on the parabola

$$y_1^2 = 4ax$$

then, $$y_2^2 = 4ax_1 \quad \ldots(i)$$

Now, P will be outside, on or inside the parabola $y^2 = 4ax$ according as

$$PM >, =, \text{or} < QM$$

$$\Rightarrow \quad (PM)^2 >, =, \text{or} < (QM)^2$$

$$\Rightarrow \quad y_1^2 >, =, \text{or} < y_2^2$$

$$\Rightarrow \quad y_1^2 >, =, \text{or} < 4ax_1 \quad \text{[from Eq. (i)]}$$

Hence, $y_1^2 - 4ax_1 >, =, \text{or} < 0$

Remarks

1. The point (x_1, y_1) lies inside, on or outside $y^2 = -4ax$ according as $y_1^2 + 4ax_1 <, =, \text{or} > 0$
2. The point (x_1, y_1) lies inside, on or outside $x^2 = 4ay$ according as $x_1^2 - 4ay_1 <, =, \text{or} > 0$
3. The point (x_1, y_1) lies inside, on or outside $x^2 = -4ay$ according as $x_1^2 + 4ay_1 <, =, \text{or} > 0$

Example 23 Show that the point (2, 3) lies outside the parabola $y^2 = 3x$.

Sol. Let the point $(h, k) = (2, 3)$

We have, $k^2 - 3h = 3^2 - 3.2 = 9 - 6 = 3 > 0$

$\therefore \quad k^2 - 3h > 0$

This shows that (2,3) lies outside the parabola $y^2 = 3x$.

Example 24 ***Find the position of the point*** $(-2,2)$ ***with respect to the parabola*** $y^2 - 4y + 9x + 13 = 0$.

Sol. Let the point $(h,k) = (-2,2)$

We have, $k^2 - 4k + 9h + 13 = (2)^2$

$$- 4(2) + 9(-2) + 13 = 4 - 8 - 18 + 13 = -9 < 0$$

Hence, $k^2 - 4k + 9h + 13 < 0$

Therefore, the point $(-2,2)$ lies inside the parabola

$$y^2 - 4y + 9x + 13 = 0.$$

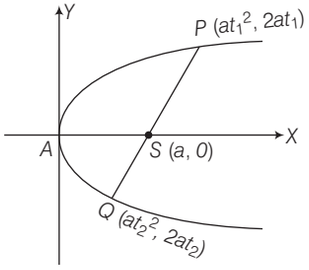
Parameteric Relation between the Coordinates of the Ends of a Focal Chord of a Parabola

Let $y^2 = 4ax$ be a parabola, if PQ be a focal chord.

Then, $P \equiv (at_1^2, 2at_1)$ and $Q \equiv (at_2^2, 2at_2)$

Since, PQ passes through the focus $S(a,0)$.

$\therefore$ Q, S, P are collinear.



$\therefore$ Slope of PS = Slope of QS

$$\Rightarrow \quad \frac{2at_1 - 0}{at_1^2 - a} = \frac{0 - 2at_2}{a - at_2^2} \Rightarrow \frac{2t_1}{t_1^2 - 1} = \frac{2t_2}{t_2^2 - 1}$$

$$\Rightarrow \quad t_1(t_2^2 - 1) = t_2(t_1^2 - 1)$$

$$\Rightarrow \quad t_1 t_2 (t_2 - t_1) + (t_2 - t_1) = 0$$

$$\Rightarrow \quad t_2 - t_1 \neq 0 \quad \text{or} \quad t_1 t_2 + 1 = 0$$

$$\Rightarrow \quad t_1 t_2 = -1 \quad \text{or} \quad t_2 = -\frac{1}{t_1}, \qquad \ldots(i)$$

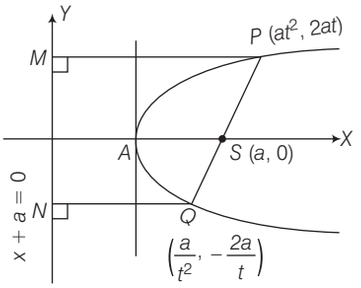
which is required relation.

Remark

If one extremity of a focal chord is $(at_1^2, 2at_1)$ then the other extremity $(at_2^2, 2at_2)$ becomes $\left(\frac{a}{t_1^2}, -\frac{2a}{t_1}\right)$ by virtue of relation Eq. (i).

Example 25 If the point $(at^2, 2at)$ be the extremity of a focal chord of parabola $y^2 = 4ax$ then show that the length of the focal chord is $a\left(t + \frac{1}{t}\right)^2$.

Sol. Since, one extremity of focal chord is $P(at^2, 2at)$, then the other extremity is $Q\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$ [Replacing t by $-1/t$]



$\therefore$ Length of focal chord $= PQ$

$$= SP + SQ \qquad [\because SP = PM \text{ and } SQ = QN]$$

$$= PM + QN$$

$$= at^2 + a + \frac{a}{t^2} + a$$

$$= a\left(t^2 + \frac{1}{t^2} + 2\right) = a\left(t + \frac{1}{t}\right)^2$$

Remark

$\because$ $\left|t + \frac{1}{t}\right| \geq 2$ for all $t \neq 0$ [$\because$ AM $\geq$ GM]

$\therefore$ $a\left(t + \frac{1}{t}\right)^2 \geq 4a$

$\Rightarrow$ Length of focal chord $\geq$ latusrectum i.e. The length of smallest focal chord of the parabola is $4a$. Hence, the latusrectum of a parabola is the smallest focal chord.

Example 26 Prove that the semi-latusrectum of the parabola $y^2 = 4ax$ is the harmonic mean between the segments of any focal chord of the parabola.

Sol. Let parabola be $y^2 = 4ax$

If PQ be the focal chord, if

$$P \equiv (at^2, 2at), \text{ then } Q \equiv \left(\frac{a}{t^2}, \frac{-2a}{t}\right)$$

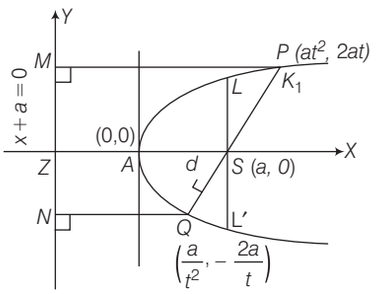
$\therefore$ Length of latusrectum $LL' = 4a$.

$\therefore$ Semi-latusrectum $= \frac{1}{2}(4a) = 2a$.

If sections of focal chord are k_1 and k_2,

then, $k_1 = SP = PM = a + at^2 = a(1 + t^2)$

and $k_2 = SQ = QN = a + \frac{a}{t^2} = \frac{a(1 + t^2)}{t^2}$



$\therefore$ Harmonic mean of k_1

and $k_2 = \frac{2k_1k_2}{k_1 + k_2}$

$$= \frac{2}{\frac{1}{k_2} + \frac{1}{k_1}} = \frac{2}{\frac{t^2}{a(1+t^2)} + \frac{1}{a(1+t^2)}}$$

$$= \frac{2}{\frac{1}{a}} = 2a = \text{Semi-latusrectum.}$$

Remarks

1. The length of focal chord having parameters t_1 and t_2 for its end points is $a(t_2 - t_1)^2$.
2. If l_1 and l_2 are the length of segments of a focal chord of a parabola, then its latusrectum is $\frac{4l_1l_2}{l_1 + l_2}$.

Example 27. ***Show that the focal chord of parabola*** $y^2 = 4ax$ ***makes an angle*** α ***with the X-axis is of length*** $4a\,\text{cosec}^2\alpha$.

Sol. Let $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ be the end points of a focal chord PQ which makes an angle α with the axis of the parabola. Then,

$$PQ = a(t_2 - t_1)^2$$
$$= a[(t_2 + t_1)^2 - 4t_1t_2] \quad \ldots(i)$$
$$= a[(t_2 + t_1)^2 + 4] \quad [\because t_1t_2 = -1]$$

$$\therefore \quad \tan\alpha = \text{slope of } PQ$$
$$= \frac{2at_2 - 2at_1}{at_2^2 - at_1^2}$$

$$\Rightarrow \quad \tan\alpha = \frac{2}{t_2 + t_1}$$

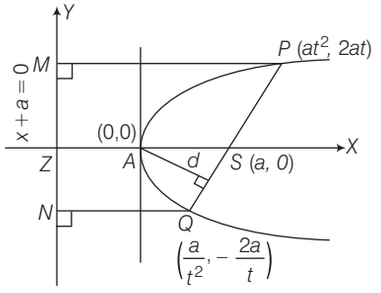
$$\Rightarrow \quad t_2 + t_1 = 2\cot\alpha \quad \ldots(ii)$$

On substituting the value of $t_2 + t_1$ from Eq. (ii) in Eq. (i), then

$$PQ = a(4\cot^2\alpha + 4)$$
$$= 4a\,\text{cosec}^2\alpha.$$

Example 28 ***Prove that the length of a focal chord of a parabola varies inversly as the square of its distance from the vertex.***

Sol. Let $P(at^2, 2at)$ be one end of a focal chord of the parabola $y^2 = 4ax$. The focus of its parabola is $S(a,0)$.



$\therefore$ Equation of focal chord is (i.e. equation of PS)

$$y - 0 = \frac{2at - 0}{at^2 - a}(x - a)$$

$$\Rightarrow \quad y = \frac{2t}{(t^2 - 1)}(x - a)$$

$$\Rightarrow \quad (t^2 - 1)y = 2tx - 2at$$

$$\Rightarrow \quad 2tx - (t^2 - 1)y - 2at = 0$$

If d be the distance of this focal chord from the vertex (0,0) of the parabola $y^2 = 4ax$, then

$$d = \frac{|0 - 0 - 2at|}{\sqrt{(2t)^2 + (t^2 - 1)^2}}$$

$$= \frac{|2at|}{(t^2 + 1)} = \frac{2a}{\left|t + \frac{1}{t}\right|}$$

$$\Rightarrow \quad d^2 = \frac{4a^2}{\left(t + \frac{1}{t}\right)^2} \quad \ldots(i)$$

The other end of the focal chord is $Q\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$

If length of focal chord = $PQ = (l \text{ say})$

$\therefore \quad l = PQ = PS + SQ = PM + QN$

$$\therefore \quad l = at^2 + a + \frac{a}{t^2} + a$$

$$\Rightarrow \quad l = a\left(t^2 + \frac{1}{t^2} + 2\right) \Rightarrow l = a\left(t + \frac{1}{t}\right)^2$$

$$\Rightarrow \quad \frac{l}{a} = \left(t + \frac{1}{t}\right)^2 \quad \ldots(ii)$$

From Eqs. (i) and (ii), $d^2 = \frac{4a^2}{(l/a)} = \frac{4a^3}{l}$

$$\therefore \quad l = \frac{4a^3}{d^2} \Rightarrow l \propto \frac{1}{d^2}$$

i.e. the length of the focal chord varies inversely as the square of its distance from vertex.

Intersection of a Line and a Parabola

Let the parabola be $y^2 = 4ax$ ...(i)

and the given line be $y = mx + c$ ...(ii)

On eliminating x from Eqs. (i) and (ii), then

$$y^2 = 4a\left(\frac{y - c}{m}\right)$$

$$\Rightarrow \quad my^2 - 4ay + 4ac = 0 \quad \ldots(iii)$$

This equation being quadratic in y, gives two values of y, shows that every straight line will cut the parabola in two points may be real, coincident or imaginary according as discriminant of Eq. (iii) $>, =, < 0$

i.e. $(-4a)^2 - 4 \cdot m \cdot 4ac >, =, < 0$ or $a - mc >, =, < 0$

or $a >, =, < mc$ ...(iv)

Condition of tangency

If the line Eq. (ii) touches the parabola Eq. (i), then Eq. (iii) has equal roots

$\therefore$ Discriminant of Eq. (iii) $= 0$

$$\Rightarrow \quad (-4a)^2 - 4m \cdot 4ac = 0$$

$$\Rightarrow \qquad c = \frac{a}{m}, m \neq 0 \qquad \ldots(v)$$

So, the line $y = mx + c$ touches the parabola $y^2 = 4ax$ if $c = \frac{a}{m}$ (which is condition of tangency).

Substituting the value of c from Eq. (v) in Eq. (ii), then

$$y = mx + \frac{a}{m}, m \neq 0$$

Hence, the line $y = mx + \frac{a}{m}$ will always be a tangent to the parabola $y^2 = 4ax$.

The point of contact Substituting $c = \frac{a}{m}$ in Eq. (iii), then

$$my^2 - 4ay + 4a\left(\frac{a}{m}\right) = 0$$

$$\Rightarrow \qquad m^2y^2 - 4amy + 4a^2 = 0$$

$$\Rightarrow \qquad (my - 2a)^2 = 0$$

$$\Rightarrow \qquad my - 2a = 0$$

or $$y = \frac{2a}{m}$$

Substituting this value of y in $y = mx + \frac{a}{m}$

$$\therefore \qquad \frac{2a}{m} = mx + \frac{a}{m}$$

$$\Rightarrow \qquad mx = \frac{a}{m} \quad \text{or} \quad x = \frac{a}{m^2}$$

Hence, the point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right), (m \neq 0)$ this known as ***m*-point** on the parabola.

Remark

If $m = 0$, then Eq. (iii) gives

$$0 - 4ay + 4ac = 0$$

$$\Rightarrow \qquad y = c$$

which gives only one value of y and so every line parallel to X-axis cuts the parabola only in one real point.

Example 29 Prove that the straight line $lx + my + n = 0$ touches the parabola $y^2 = 4ax$, if $\ln = am^2$.

Sol. The given line is $lx + my + n = 0$

or $$y = -\frac{l}{m}x - \frac{n}{m} \qquad \ldots(i)$$

Comparing this line with $y = Mx + c$

$$\therefore \qquad M = -\frac{l}{m} \quad \text{and} \quad c = -\frac{n}{m}$$

The line Eq. (i) will touch the parabola $y^2 = 4ax$, if

$$c = \frac{a}{M} \Rightarrow cM = a$$

$$\Rightarrow \qquad \left(-\frac{n}{m}\right)\left(-\frac{l}{m}\right) = a$$

$$\therefore \qquad \ln = am^2$$

Aliter :

Given line $lx + my + n = 0$ ...(i)

and the parabola $y^2 = 4ax$ ...(ii)

Substituting the value of x from Eq. (i) i.e. $x = -\frac{n + my}{l}$ in Eq. (ii), then

(we should not substituting the value of y from Eq. (i), in Eq. (ii) since y is quadratic, substituting the value of x since x is linear).

$$y^2 = 4a\left(-\frac{n + my}{l}\right)$$

$$\Rightarrow \qquad ly^2 + 4amy + 4an = 0 \qquad \ldots(iii)$$

Since, Eq. (i) touches the parabola Eq. (ii), then roots of Eq. (iii) must be coincident and condition for the same is $B^2 = 4AC$,

i.e., $$(4am)^2 = 4 \cdot l4an$$

$$\Rightarrow \qquad am^2 = ln$$

$$\therefore \qquad ln = am^2$$

Example 30 Show that the line $x\cos\alpha + y\sin\alpha = p$ touches the parabola $y^2 = 4ax$, if $p\cos\alpha + a\sin^2\alpha = 0$ and that the point of contact is $(a\tan^2\alpha, -2a\tan\alpha)$.

Sol. The given line is

$$x\cos\alpha + y\sin\alpha = p$$

$$\Rightarrow \qquad y = -x\cot\alpha + p\,\text{cosec}\alpha$$

Comparing this line with $y = mx + c$.

$$\therefore \qquad m = -\cot\alpha \text{ and } c = p\,\text{cosec}\alpha$$

since, the given line touches the parabola

$$\therefore \qquad c = \frac{a}{m} \text{ or } cm = a$$

$$\Rightarrow \qquad (p\,\text{cosec}\alpha)(-\cot\alpha) = a$$

$$\Rightarrow \qquad a\sin^2\alpha + p\cos\alpha = 0$$

and point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

i.e. $$\left(\frac{a}{\cot^2\alpha}, -\frac{2a}{\cot\alpha}\right)$$

or $$(a\tan^2\alpha, -2a\tan\alpha)$$

Example 31 Prove that the line $\frac{x}{l}+\frac{y}{m}=1$ touches the parabola $y^2=4a(x+b)$, if $m^2(l+b)+al^2=0$.

Sol. The given parabola is

$$y^2 = 4a(x+b) \quad \ldots(i)$$

Vertex of this parabola is $(-b, 0)$.

Now, shifting $(0, 0)$ at $(-b, 0)$,

then, $x = X + (-b)$ and $y = Y + 0$

or $x + b = X$ and $y = Y$ ...(ii)

From Eq. (i), $Y^2 = 4aX$ ...(iii)

and the line $\frac{x}{l}+\frac{y}{m}=1$

reduces to $\frac{X-b}{l}+\frac{Y}{m}=1$

$$Y = m\left(1-\frac{X-b}{l}\right)$$

$$\Rightarrow \quad Y = \left(-\frac{m}{l}\right)X + m\left(1+\frac{b}{l}\right) \quad \ldots(iv)$$

The line Eq. (iv) will touch the parabola Eq. (iii), if

$$m\left(1+\frac{b}{l}\right) = \frac{a}{\left(-\frac{m}{l}\right)}$$

$$\Rightarrow \quad \frac{m^2}{l}\left(1+\frac{b}{l}\right) = -a$$

$$\therefore \quad m^2(l+b)+al^2 = 0$$

Aliter :

The given line and parabola are

$$\frac{x}{l}+\frac{y}{m}=1 \quad \ldots(i)$$

and $$y^2 = 4a(x+b) \quad \ldots(ii)$$

respectively substituting the value of x from Eq. (i)

$$x = l\left(1-\frac{y}{m}\right)$$

in Eq. (ii), then $$y^2 = 4a\left\{l\left(1-\frac{y}{m}\right)+b\right\}$$

or $$y^2 + \frac{4al}{m}y - 4a(l+b) = 0 \quad \ldots(iii)$$

Since, the line Eq. (i) touches the parabola Eq. (ii), then the roots of Eq. (iii) are equal.

$$\therefore \quad \left(\frac{4al}{m}\right)^2 - 4\cdot 1\{-4a(l+b)\} = 0$$

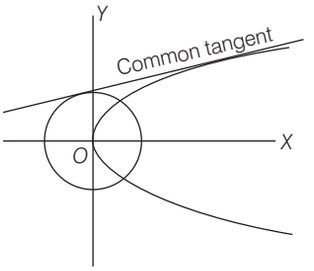
$$\Rightarrow \quad \frac{al^2}{m^2} + (l+b) = 0$$

$$\Rightarrow \quad al^2 + m^2(l+b) = 0$$

$$\Rightarrow \quad m^2(l+b) + al^2 = 0$$

Example 32 Find the equations of the straight lines touching both $x^2+y^2=2a^2$ and $y^2=8ax$.

Sol. The given curves are



$$x^2 + y^2 = 2a^2 \quad \ldots(i)$$

and $$y^2 = 8ax \quad \ldots(ii)$$

The parabola Eq. (ii) is $y^2 = 8ax$

or $$y^2 = 4(2a)x$$

$\therefore$ Equation of tangent of Eq. (ii) is

$$y = mx + \frac{2a}{m}$$

or $$m^2x - my + 2a = 0 \quad \ldots(iii)$$

It is also tangent of Eq. (i), then the length of perpendicular from centre of Eq. (i) i.e. (0, 0) to Eq. (iii) must be equal to the radius of Eq. (i) i.e. $a\sqrt{2}$.

$$\therefore \quad \frac{|0-0+2a|}{\sqrt{(m^2)^2+(-m)^2}} = a\sqrt{2} \quad \Rightarrow \quad \frac{4a^2}{m^4+m^2} = 2a^2$$

$$\Rightarrow \quad m^4 + m^2 - 2 = 0$$

$$\Rightarrow \quad (m^2+2)(m^2-1) = 0$$

$$\because \quad m^2 + 2 \neq 0 \quad \text{[gives the imaginary values]}$$

$$\therefore \quad m^2 - 1 = 0$$

$$\Rightarrow \quad m = \pm 1$$

Hence, from Eq. (iii) the required tangents are

$$x \pm y + 2a = 0.$$

Equation of Tangent in Different Forms

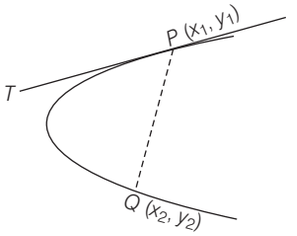
1. Point Form :

To find the equation of the tangent to the parabola $y^2 = 4ax$ at the point (x_1, y_1).

(First Principal Method) Equation of parabola is

$$y^2 = 4ax \quad \ldots(i)$$

Let $P \equiv (x_1, y_1)$ and $Q \equiv (x_2, y_2)$ be any two points on parabola (i), then



$$y_1^2 = 4ax_1 \quad \ldots(ii)$$

and $$y_2^2 = 4ax_2 \quad \ldots(iii)$$

Subtracting Eq. (ii) from Eq. (iii), then

$$y_2^2 - y_1^2 = 4a(x_2 - x_1)$$

$$\Rightarrow \quad \frac{y_2 - y_1}{x_2 - x_1} = \frac{4a}{y_2 + y_1} \quad \ldots(iv)$$

Equation of PQ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \quad \ldots(v)$$

From Eqs. (iv) and (v), then

$$y - y_1 = \frac{4a}{y_2 + y_1}(x - x_1) \quad \ldots(vi)$$

Now, for tangent at $P, Q \to P$, i.e. $x_2 \to x_1$ and $y_2 \to y_1$, then Eq. (vi) becomes

$$y - y_1 = \frac{4a}{2y_1}(x - x_1) \;\Rightarrow\; yy_1 - y_1^2 = 2ax - 2ax_1$$

$$\Rightarrow \quad yy_1 = 2ax + y_1^2 - 2ax_1$$

$$\Rightarrow \quad yy_1 = 2ax + 4ax_1 - 2ax_1 \quad [\text{from Eq. (ii)}]$$

$$\Rightarrow \quad yy_1 = 2ax + 2ax_1$$

$$\therefore \quad yy_1 = 2a(x + x_1),$$

which is the required equation of tangent at (x_1, y_1).

Remarks

1. The equation of tangent at (x_1, y_1) can also be obtained by replacing x^2 by xx_1, y^2 by yy_1, x by $\frac{x + x_1}{2}$, y by $\frac{y + y_1}{2}$ and xy by $\frac{xy_1 + x_1y}{2}$ and without changing the constant (if any) in the equation of curve. This method is apply only when the equations of parabola is polynomial of second degree in x and y.
2. Equation of tangents of all standard parabolas at (x_1, y_1).

Equations of Parabolas	Tangent at(x_1, y_1)
$y^2 = 4ax$	$yy_1 = 2a(x + x_1)$
$y^2 = -4ax$	$yy_1 = -2a(x + x_1)$
$x^2 = 4ay$	$xx_1 = 2a(y + y_1)$
$x^2 = -4ay$	$xx_1 = -2a(y + y_1)$

2. Parametric Form :

To find the equation of tangent to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ or 't'.

Since, the equation of tangent of the parabola $y^2 = 4ax$ at (x_1, y_1) is $yy_1 = 2a(x + x_1)$ ...(i)

replacing x_1 by at^2 and y_1 by $2at$, then Eq. (i) becomes

$$y(2at) = 2a(x + at^2) \;\Rightarrow\; ty = x + at^2$$

Remark

The equations of tangent of all standard parabolas at 't'.

Equations of Parabolas	Parametric coordinates 't'	Tangent at 't'
$y^2 = 4ax$	$(at^2, 2at)$	$ty = x + at^2$
$y^2 = -4ax$	$(-at^2, 2at)$	$ty = -x + at^2$
$x^2 = 4ay$	$(2at, at^2)$	$tx = y + at^2$
$x^2 = -4ay$	$(2at, -at^2)$	$tx = -y + at^2$

3. Slope Form :

To find the equation of tangent and point of contact in terms of m(slope) to the parabola $y^2 = 4ax$.

The equation of tangent to the parabola $y^2 = 4ax$ at (x_1, y_1) is $yy_1 = 2a(x + x_1)$. ...(i)

Since, m is the slope of the tangent, then

$$m = \frac{2a}{y_1} \;\Rightarrow\; y_1 = \frac{2a}{m}$$

Since, (x_1, y_1) lies on $y^2 = 4ax$, therefore

$$y_1^2 = 4ax_1 \Rightarrow \frac{4a^2}{m^2} = 4ax_1$$

$$\therefore \quad x_1 = \frac{a}{m^2}$$

Substituting the values of x_1 and y_1 in Eq. (i), we get

$$y = mx + \frac{a}{m} \quad \ldots(ii)$$

Thus, $y = mx + \frac{a}{m}$ is a tangent to the parabola $y^2 = 4ax$, where, m is the slope of the tangent.

The coordinates of the point of contact are $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

Comparing Eq. (ii) with $y = mx + c$,

$$c = \frac{a}{m}$$

which is condition of tangency.

when, $y = mx + c$ is the tangent of $y^2 = 4ax$.

Remark

The equation of tangent, condition of tangency and point of contact in terms of slope (m) of all standard parabolas.

Equation of parabolas	Point of contact in terms of slope (m)	Equation of tangent in terms of slope (m)	Condition of tangency
$y^2 = 4ax$	$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$	$y = mx + \frac{a}{m}$	$c = \frac{a}{m}$
$y^2 = -4ax$	$\left(-\frac{a}{m^2}, \frac{2a}{m}\right)$	$y = mx - \frac{a}{m}$	$c = -\frac{a}{m}$
$x^2 = 4ay$	$(2am, am^2)$	$y = mx - am^2$	$c = -am^2$
$x^2 = -4ay$	$(2am, -am^2)$	$y = mx + am^2$	$c = am^2$
$(y-k)^2 = 4a(x-h)$	$\left(h + \frac{a}{m^2}, k + \frac{2a}{m}\right)$	$y = mx - mh + k + \frac{a}{m}$	$c + mh = k + \frac{a}{m}$
$(y-k)^2 = -4a(x-h)$	$\left(h - \frac{a}{m^2}, k + \frac{2a}{m}\right)$	$y = mx - mh + k - \frac{a}{m}$	$c + mh = k - \frac{a}{m}$
$(x-h)^2 = 4a(y-k)$	$(h + 2am, k + am^2)$	$y = mx - mh + k - am^2$	$c + mh = k - am^2$
$(x-h)^2 = -4a(y-k)$	$(h + 2am, k - am^2)$	$y = mx - mh + k + am^2$	$c + mh = k + am^2$

Point of Intersection of Tangents at any two Points on the Parabola

Let the parabola be $y^2 = 4ax$

let two points on the parabola are

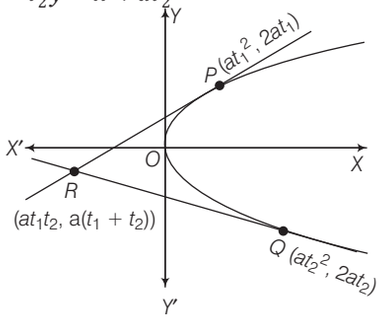
$$P \equiv (at_1^2, 2at_1) \quad \text{and} \quad Q \equiv (at_2^2, 2at_2).$$

Equation of tangents at $P(at_1^2, 2at_1)$

and $Q(at_2^2, 2at_2)$

are $t_1 y = x + at_1^2$ ...(i)

and $t_2 y = x + at_2^2$ ...(ii)



On solving these equations, we get $x = at_1 t_2, y = a(t_1 + t_2)$

Thus, the coordinates of the point of intersection of tangents at

$(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ are $(at_1 t_2, a(t_1 + t_2))$.

Remarks

1. The geometric mean of the x-coordinates of P and Q (i.e. $\sqrt{at_1^2 \times at_2^2} = at_1 t_2$) is the x-coordinate of the point of intersection of tangents at P and Q on the parabola. If P and Q are the ends points of focal chord, then x-coordinate of point of intersection of tangents at P and Q is $(-at_1 t_2)$.
2. The arithmetic mean of the y-coordinates of P and Q $\left(\text{i.e. } \frac{2at_1 + 2at_2}{2} = a(t_1 + t_2)\right)$ is the y-coordinate of the point of intersection of tangents at P and Q on the parabola.

Remembering Method :

G O A [GOA rule]

GM of at_1^2 and at_2^2 , AM of $2at_1$ and $2at_2$

i.e. $at_1 t_2$ i.e. $\frac{2at_1 + 2at_2}{2} = a(t_1 + t_2)$

Example 33 Find the equation of the common tangents to the parabola $y^2 = 4ax$ and $x^2 = 4by$.

Sol. The equation of any tangent in terms of slope (m) to the parabola $y^2 = 4ax$ is

$$y = mx + \frac{a}{m} \quad \text{...(i)}$$

If this line is also tangent to the parabola $x^2 = 4ay$, then Eq. (i) meets $x^2 = 4by$ in two coincident points.

Substituting the value of y from Eq. (i) in $x^2 = 4by$, we get

$$x^2 = 4b\left(mx + \frac{a}{m}\right)$$

$$\Rightarrow \quad x^2 - 4bmx - \frac{4ab}{m} = 0$$

The roots of this quadratic are equal provided

$$B^2 = 4AC$$

i.e., $$(-4bm)^2 = 4 \cdot 1\left(\frac{-4ab}{m}\right)$$

$$\Rightarrow \quad 16b^2m^3 + 16ab = 0,\ m \neq 0$$

or $$m^3 = -a/b$$

$$\therefore \quad m = -a^{1/3}/b^{1/3}$$

Substituting the value of m in Eq. (i) the required equation is

$$y = -\frac{a^{1/3}}{b^{1/3}}x - \frac{ab^{1/3}}{a^{1/3}}$$

$$\Rightarrow \quad y = -\frac{a^{1/3}}{b^{1/3}}x - a^{2/3}b^{1/3}$$

$$\therefore \quad a^{1/3}x + b^{1/3}y + a^{2/3}b^{2/3} = 0$$

Example 34 The tangents to the parabola $y^2 = 4ax$ make angle θ_1 and θ_2 with X-axis. Find the locus of their point of intersection, if $\cot\theta_1 + \cot\theta_2 = c$.

Sol. Let the equation of any tangent to the parabola $y^2 = 4ax$ is

$$y = mx + (a/m) \qquad \ldots(i)$$

Let (x_1, y_1) be the point of intersection of the tangents to $y^2 = 4ax$, then Eq. (i) passes through (x_1, y_1).

$$\therefore \quad y_1 = mx_1 + (a/m)$$

or $$m^2x_1 - my_1 + a = 0$$

Let m_1 and m_2 be the roots of this quadratic equation, then

$$m_1 + m_2 = y_1/x_1 \quad \text{and} \quad m_1m_2 = a/x_1$$

or $$\tan\theta_1 + \tan\theta_2 = y_1/x_1$$

and $$\tan\theta_1\tan\theta_2 = a/x_1 \qquad \ldots(ii)$$

Now, $$\cot\theta_1 + \cot\theta_2 = c \qquad \text{(given)}$$

$$\Rightarrow \quad \frac{1}{\tan\theta_1} + \frac{1}{\tan\theta_2} = c$$

$$\Rightarrow \quad \frac{\tan\theta_1 + \tan\theta_2}{\tan\theta_1\tan\theta_2} = c \qquad \text{[from Eq. (ii)]}$$

$$\Rightarrow \quad \frac{y_1/x_1}{a/x_1} = c$$

$$\Rightarrow \quad y_1 = ac$$

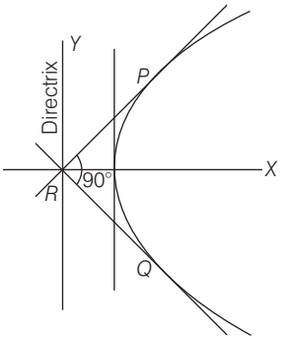
The required locus is $y = ac$,
which is a line parallel to X-axis.

Example 35. Show that the locus of the points of intersection of the mutually perpendicular tangents to a parabola is the directrix of the parabola.

Sol. Let the points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ tangents at P and Q are

$$t_1y = x + at_1^2 \qquad \ldots(i)$$

and $$t_2y = x + at_2^2 \qquad \ldots(ii)$$



$\because$ Point of intersection of these tangents is $(at_1t_2, a(t_1 + t_2))$
Let this point is (h, k),

then, $$h = at_1t_2 \qquad \ldots(iii)$$

and $$k = a(t_1 + t_2) \qquad \ldots(iv)$$

Slope of tangents Eqs. (i) and (ii) are $\frac{1}{t_1}$ and $\frac{1}{t_2}$, respectively.

Since, tangents are perpendicular, then

$$\frac{1}{t_1} \times \frac{1}{t_2} = -1$$

or $$t_1t_2 = -1 \qquad \ldots(v)$$

From Eqs. (iii) and (v), we get

$$h = -a \quad \text{or} \quad h + a = 0$$

$\therefore$ Locus of the point of intersection of tangents is

$$x + a = 0$$

which is directrix of $y^2 = 4ax$.

Aliter :

Let the equation of any tangent to the parabola $y^2 = 4ax$ is

$$y = mx + a/m \qquad \ldots(i)$$

Let the point of intersection of the tangents to $y^2 = 4ax$ then, Eq (i) passes through (x_1, y_1) .

$$\therefore \quad y_1 = mx_1 + a/m$$

or $$m^2x_1 - my_1 + a = 0$$

Let m_1, m_2 be the roots of this quadratic equation then

$$m_1m_2 = a/x_1 = -1$$

[since, tangents are perpendiculars]

$$\Rightarrow \quad a + x_1 = 0$$

$\therefore$ Locus of the point of intersection of tangents is $x + a = 0$ which is directrix of $y^2 = 4ax$.

Remark

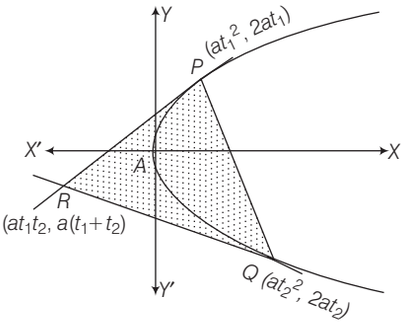
Locus of the point of intersection of the perpendicular tangents to the parabola $y^2 = 4ax$ is called the director circle. Its equation is $x + a = 0$, which is parabola's own directrix.

Example 36 The tangents to the parabola $y^2 = 4ax$ at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ intersect at R. Prove that the area of the ΔPQR is $\frac{1}{2}a^2|(t_1 - t_2)|^3$.

Sol. Equations of tangents at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ are

$$t_1 y = x + at_1^2 \quad \ldots(i)$$

and $$t_2 y = x + at_2^2 \quad \ldots(ii)$$



Since, point of intersect of Eqs. (i) and (ii) is $R(at_1t_2, a(t_1 + t_2))$.

$$\therefore \text{ Area of } \Delta PQR = \frac{1}{2}\begin{vmatrix} at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \\ at_1t_2 & a(t_1+t_2) & 1 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$= \frac{1}{2}\begin{vmatrix} at_1^2 & 2at_1 & 1 \\ a(t_2^2 - t_1^2) & 2a(t_2 - t_1) & 0 \\ at_1(t_2 - t_1) & a(t_2 - t_1) & 0 \end{vmatrix}$$

Expanding with respect to first row

$$= \frac{1}{2}\begin{vmatrix} a(t_2^2 - t_1^2) & 2a(t_2 - t_1) \\ at_1(t_2 - t_1) & a(t_2 - t_1) \end{vmatrix}$$

$$= \frac{1}{2}a^2(t_2 - t_1)^2\begin{vmatrix} t_2 + t_1 & 2 \\ t_1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}a^2(t_1 - t_2)^2|(t_2 - t_1)|$$

$$= \frac{1}{2}a^2(t_1 - t_2)^2|(t_1 - t_2)|$$

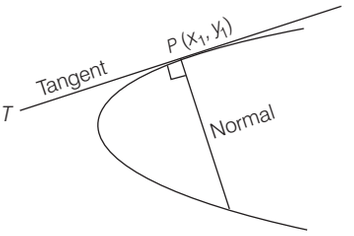
$$= \frac{1}{2}a^2|(t_1 - t_2)^3|.$$

Equations of Normals in Different Forms

1. Point Form: To find the equation of the normal to the parabola $y^2 = 4ax$ at the point (x_1, y_1).

Since, the equation of the tangent to the parabola $y^2 = 4ax$ at (x_1, y_1) is

$$yy_1 = 2a(x + x_1)$$



The slope of the tangent at $(x_1, y_1) = 2a / y_1$

Since, the normal at (x_1, y_1) is perpendicular to the tangent at (x_1, y_1).

$\therefore$ Slope of normal at $(x_1, y_1) = -y_1 / 2a$

Hence, the equation of normal at (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1).$$

Remarks

1. The equation of normal at (x_1, y_1) can also be obtained by this method

$$\frac{x - x_1}{a'x_1 + hy_1 + g} = \frac{y - y_1}{hx_1 + by_1 + f} \quad \ldots(i)$$

a', b, g, f, h are obtained by comparing the given parabola with

$$a'x^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \ldots(ii)$$

and denominators of Eq. (i) can easily remembered by the first two rows of this determinant

i.e. $$\begin{vmatrix} a' & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Since, first row $a'(x_1) + h(y_1) + g(1)$

and second row, $h(x_1) + b(y_1) + f(1)$

Here, parabola $y^2 = 4ax$

or $$y^2 - 4ax = 0 \quad \ldots(iii)$$

Comparing Eqs. (ii) and (iii), then we get

$a' = 0, b = 1, g = -2a, h = 0, f = 0$

From Eq. (i), equation of normal of Eq. (iii) is

$$\frac{x - x_1}{0 + 0 - 2a} = \frac{y - y_1}{0 + y_1 + 0}$$

or $$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

2. Equations of normals of all standard parabolas at (x_1, y_1).

Equations of Parabola	Normal at (x_1, y_1)
$y^2 = 4ax$	$y - y_1 = -\frac{y_1}{2a}(x - x_1)$
$y^2 = -4ax$	$y - y_1 = \frac{y_1}{2a}(x - x_1)$
$x^2 = 4ay$	$y - y_1 = -\frac{2a}{x_1}(x - x_1)$
$x^2 = -4ay$	$y - y_1 = \frac{2a}{x_1}(x - x_1)$

2. Parametric form :

To find the equation of normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ or 't'.

Since, the equation of normal of the parabola $y^2 = 4ax$ at (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1) \quad \ldots(i)$$

Replacing x_1 by at^2 and y_1 by $2at$, then Eq. (i) becomes

$$y - 2at = -t(x - at^2)$$

or $$\boldsymbol{y + tx = 2at + at^3}$$

Remark

The equations of normals of all standard parabolas at 't'

Equations of Parabolas	Parametric coordinates 't'	Normals at 't'
$y^2 = 4ax$	$(at^2, 2at)$	$y + tx = 2at + at^3$
$y^2 = -4ax$	$(-at^2, 2at)$	$y - tx = 2at + at^3$
$x^2 = 4ay$	$(2at, at^2)$	$x + ty = 2at + at^3$
$x^2 = -4ay$	$(2at, -at^2)$	$x - ty = 2at + at^3$

3. Slope form :

To find the Equation of normal, condition for normality and point of contact in terms of m (slope) to the parabola $y^2 = 4ax$

The equation of normal to the parabola $y^2 = 4ax$ at (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1) \quad \ldots(i)$$

Since, m is the slope of the normal,

then, $$m = -\frac{y_1}{2a} \Rightarrow y_1 = -2am$$

Since, (x_1, y_1) lies on $y^2 = 4ax$, therefore

$$y_1^2 = 4ax_1$$

$$\Rightarrow \quad 4a^2m^2 = 4ax_1$$

$$\therefore \quad x_1 = am^2$$

On substituting the values of x_1 and y_1 in Eq. (i) we get

$$y + 2am = m(x - am^2)$$

$$\therefore \quad y = mx - 2am - am^3 \quad \ldots(ii)$$

Thus, $y = mx - 2am - am^3$ is a normal to the parabola $y^2 = 4ax$, where m is the slope of the normal. The coordinates of the point of contact are $(am^2, -2am)$

On comparing Eq. (ii) with

$$y = mx + c$$

$$\therefore \quad c = -2am - am^3$$

which is condition of normality when $y = mx + c$ is the normal of $y^2 = 4ax$.

Remark

The equations of normals, point of contact and condition of normality in terms of slope (m) of all standard parabolas.

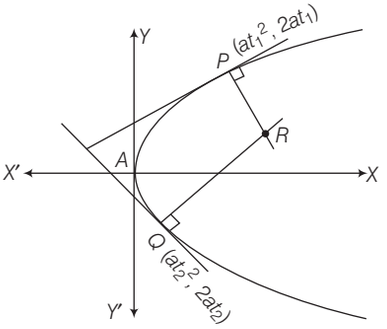
Equation of parabolas	Point of contact in terms of slope (m)	Equation of normals in terms of slope (m)	Condition of normality
$y^2 = 4ax$	$(am^2, -2am)$	$y = mx - 2am - am^3$	$c = -2am - am^3$
$y^2 = -4ax$	$(-am^2, 2am)$	$y = mx + 2am + am^3$	$c = 2am + am^3$
$x^2 = 4ay$	$\left(-\frac{2a}{m}, \frac{a}{m^2}\right)$	$y = mx + 2a + \frac{a}{m^2}$	$c = 2a + \frac{a}{m^2}$
$x^2 = -4ay$	$\left(\frac{2a}{m}, -\frac{a}{m^2}\right)$	$y = mx - 2a - \frac{a}{m^2}$	$c = -2a - \frac{a}{m^2}$
$(y-k)^2 = 4a(x-h)$	$(h + am^2, k - 2am)$	$y - k = m(x-h) - 2am - am^3$	$c = k - mh - 2am - am^3$
$(y-k)^2 = -4a(x-h)$	$(h - am^2, k + 2am)$	$y - k = m(x-h) + 2am + am^3$	$c = k - mh + 2am + am^3$
$(x-h)^2 = 4a(y-k)$	$\left(h - \frac{2a}{m}, k + \frac{a}{m^2}\right)$	$y - k = m(x-h) + 2a + \frac{a}{m^2}$	$c = k - mh + 2a + \frac{a}{m^2}$

Point of Intersection of Normals at any Two Points on the Parabola

Let the parabola be $y^2 = 4ax$.

Let the points on the parabola are

$$P \equiv (at_1^2, 2at_1) \quad \text{and} \quad Q = (at_2^2, 2at_2).$$



Equations of normals at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ are

$$y = -t_1 x + 2at_1 + at_1^3 \quad \ldots(i)$$

and $$y = -t_2 x + 2at_2 + at_2^3 \quad \ldots(ii)$$

On solving Eqs. (i) and (ii), we get

$$x = 2a + a(t_1^2 + t_2^2 + t_1 t_2) \text{ and } y = -at_1 t_2 (t_1 + t_2)$$

If R is the point of intersection, then

$$\boldsymbol{R \equiv [2a + a(t_1^2 + t_2^2 + t_1 t_2), -at_1 t_2 (t_1 + t_2)]}$$

(Remember)

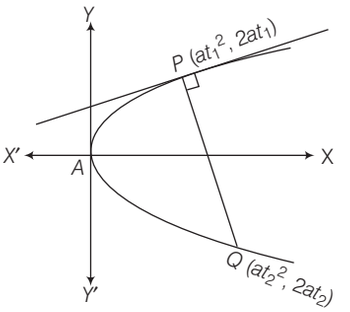
Point of intersection of normals at t_1 and t_2

Parabola	Equation of normal at any point 't'	Point of intersection of normals at t_1 and t_2
$y^2 = 4ax$	$y + tx = 2at + at^3$	$(2a + a(t_1^2 + t_1 t_2 + t_2^2), -at_1 t_2 (t_1 + t_2))$
$y^2 = -4ax$	$y - tx = 2at + at^3$	$(-2a - a(t_1^2 + t_1 t_2 + t_2^2), at_1 t_2 (t_1 + t_2))$
$x^2 = 4ay$	$x + ty = 2at + at^3$	$(-at_1 t_2 (t_1 + t_2), 2a + a(t_1^2 + t_1 t_2 + t_2^2))$
$x^2 = -4ay$	$x - ty = 2at + at^3$	$(at_1 t_2 (t_1 + t_2), -2a - a(t_1^2 + t_1 t_2 + t_2^2))$

Relation between 't_1' and 't_2' if Normal at 't_1' meets the Parabola Again at 't_2'

Let the parabola be $y^2 = 4ax$, equation of normal at $P(at_1^2, 2at_1)$ is

$$y = -t_1 x + 2at_1 + at_1^3 \quad \ldots(i)$$



Since, it meet the parabola again at $Q(at_2^2, 2at_2)$, then Eq. (i) passes through $Q(at_2^2, 2at_2)$.

$$\therefore \quad 2at_2 = -at_1 t_2^2 + 2at_1 + at_1^3$$

$$\Rightarrow \quad 2a(t_2 - t_1) + at_1(t_2^2 - t_1^2) = 0$$

$$\Rightarrow \quad a(t_2 - t_1)[2 + t_1(t_2 + t_1)] = 0$$

$$\because \quad a(t_2 - t_1) \neq 0$$

$[\because t_1 \text{ and } t_2 \text{ are different}]$

$$\therefore \quad 2 + t_1(t_2 + t_1) = 0$$

$$\therefore \quad \boldsymbol{t_2 = -t_1 - \frac{2}{t_1}}$$

Remarks

1. If normals at 't_1' and 't_2' meets the parabola $y^2 = 4ax$ at same point, then $t_1 t_2 = 2$.

   Proof Suppose normals meet at 'T', then

   $$T = -t_1 - \frac{2}{t_1} = -t_2 - \frac{2}{t_2}$$

   $$\Rightarrow \quad (t_1 - t_2) = 2\left(\frac{1}{t_2} - \frac{1}{t_1}\right)$$

   or $$t_1 t_2 = 2 \quad [\because t_1 \neq t_2]$$

2. If the normals to the parabola $y^2 = 4ax$ at the points t_1 and t_2 intersect again on the parabola at the point t_3, then $t_3 = -(t_1 + t_2)$ and the line joining t_1 and t_2 passes through a fixed point $(-2a, 0)$.

Example 37 Show that normal to the parabola $y^2 = 8x$ at the point (2, 4) meets it again at (18, –12). Find also the length of the normal chord.

Sol. Comparing the given parabola (i.e. $y^2 = 8x$) with $y^2 = 4ax$.

$$\therefore \quad 4a = 8 \quad \Rightarrow \quad a = 2$$

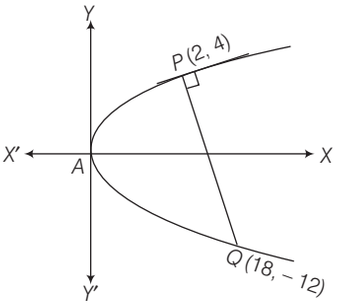
Since, normal at (x_1, y_1) to the parabola $y^2 = 4ax$ is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

Here, $x_1 = 2$ and $y_1 = 4$.

$\therefore$ Equation of normal is

$$y - 4 = -\frac{4}{4}(x - 2)$$



$\Rightarrow \quad y - 4 = -x + 2$

$\Rightarrow \quad x + y - 6 = 0$ ...(i)

On solving Eq. (i) and $y^2 = 8x$,

then, $y^2 = 8(6 - y)$

$\Rightarrow \quad y^2 + 8y - 48 = 0$

$\Rightarrow \quad (y + 12)(y - 4) = 0$

$\therefore \quad y = -12$ and $y = 4$

then, $x = 18$ and $x = 2$.

Hence, the point of intersection of normal and parabola are $(18, -12)$ and $(2, 4)$, therefore normal meets the parabola at $(18, -12)$ and length of normal chord is distance between their points

$$= PQ = \sqrt{(18-2)^2 + (-12-4)^2} = 16\sqrt{2}$$

Example 38 ***Prove that the chord*** $y - x\sqrt{2} + 4a\sqrt{2} = 0$ ***is a normal chord of the parabola*** $y^2 = 4ax$***. Also, find the point on the parabola when the given chord is normal to the parabola.***

Sol. We have, $y - x\sqrt{2} + 4a\sqrt{2} = 0$

i.e., $y = x\sqrt{2} - 4a\sqrt{2}$ ...(i)

Comparing the Eq. (i) with the equation $y = mx + c$, then

$$m = \sqrt{2},\ c = -4a\sqrt{2}$$

Since, $-2am - am^3 = -2a\sqrt{2} - a(\sqrt{2})^3$

$$= -2a\sqrt{2} - 2a\sqrt{2} = -4a\sqrt{2} = c$$

Hence, the given chord is normal to the parabola $y^2 = 4ax$.

The coordinates of the points are $(am^2, -2am)$ i.e. $(2a, -2\sqrt{2}a)$.

Example 39 If the normal to a parabola $y^2 = 4ax$, makes an angle ϕ with the axis. Show that it will cut the curve again at an angle $\tan^{-1}\left(\frac{1}{2}\tan\phi\right)$.

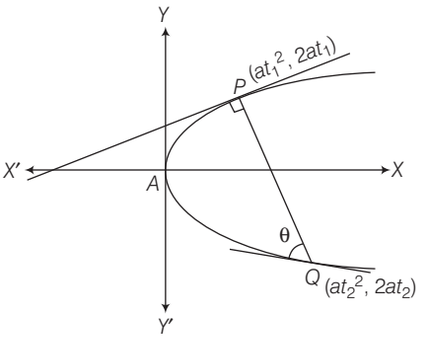
Sol. Let the normal at $P(at_1^2, 2at_1)$ be

$$y = -t_1 x + 2at_1 + at_1^3.$$

$\therefore \quad \tan\phi = -t_1 =$ slope of the normal, ...(i)

it meet the curve again Q say $(at_2^2, 2at_2)$.

$\therefore \quad t_2 = -t_1 - \frac{2}{t_1}$ ...(ii)



Now, angle between the normal and parabola

= Angle between the normal and tangent at Q

(i.e. $t_2 y = x + at_2^2$)

If θ be the angle, then

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{-t_1 - \frac{1}{t_2}}{1 + (-t_1)\left(\frac{1}{t_2}\right)} = -\frac{t_1 t_2 + 1}{t_2 - t_1}$$

$$= -\frac{t_1\left(-t_1 - \frac{2}{t_1}\right) + 1}{-t_1 - \frac{2}{t_1} - t_1} \quad \text{[from Eq. (ii)]}$$

$$= -\frac{-t_1^2 - 1}{-2\left(\frac{1 + t_1^2}{t_1}\right)} = -\frac{t_1}{2}$$

$$= \frac{\tan\phi}{2} \quad \text{[from Eq. (i)]}$$

$\therefore \quad \theta = \tan^{-1}\left(\frac{1}{2}\tan\phi\right)$

Example 40 P***rove*** that the normal chord to a parabola $y^2 = 4ax$ at the point whose ordinate is equal to abscissa subtends a right angle at the focus.

Sol. Let the normal at $P(at_1^2, 2at_1)$ meet the curve at $Q(at_2^2, 2at_2)$.

$\therefore \quad PQ$ is a normal chord

and $t_2 = -t_1 - \frac{2}{t_1}$. ...(i)

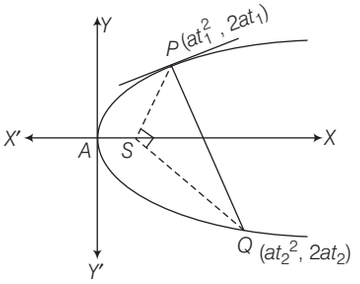
By given condition, $2at_1 = at_1^2$

$\therefore \quad t_1 = 2$ from Eq. (i), $t_2 = -3$

then, $P(4a, 4a)$ and $Q(9a, -6a)$

but focus $S(a, 0)$.

$\therefore \quad$ Slope of $SP = \frac{4a - 0}{4a - a} = \frac{4a}{3a} = \frac{4}{3}$



and slope of $SQ = \dfrac{-6a-0}{9a-a} = -\dfrac{6a}{8a} = -\dfrac{3}{4}$

$\because$ Slope of $SP \times$ Slope of $SQ = \dfrac{4}{3} \times -\dfrac{3}{4} = -1$

$\therefore \quad \angle PSQ = \pi/2$

i.e. PQ subtends a right angle at the focus S.

Example 41 If the normal to the parabola $y^2 = 4ax$ at point t_1 cuts the parabola again at point t_2, prove that $t_2^2 = 8$.

Sol. A normal at point t_1 cuts the parabola again at t_2. Then,

$$t_2 = -t_1 - \frac{2}{t_1} \Rightarrow t_1^2 + t_1 t_2 + 2 = 0$$

Since, t_1 is real , so $(t_2)^2 - 4 \cdot 1 \cdot 2 \geq 0$

$\Rightarrow \quad t_2^2 \geq 8$

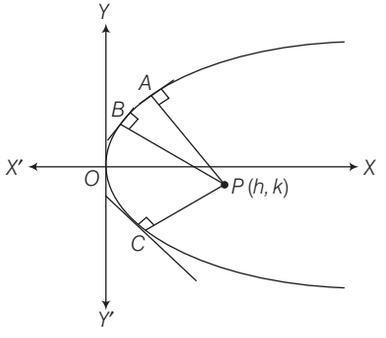
Co-normal Points

In general three normals can be drawn from a point to a parabola and their feet, points where they meet the parabola are called conormal points.

Let $P(h, k)$ be any given point and $y^2 = 4ax$ be a parabola.

The equation of any normal to $y^2 = 4ax$ is

$$y = mx - 2am - am^3$$



If it passes through (h, k), then

$$k = mh - 2am - am^3$$

$\Rightarrow \quad am^3 + m(2a - h) + k = 0 \qquad ...(i)$

This is a cubic equation in m, so it has three roots, say m_1, m_2 and m_3.

$\therefore \quad m_1 + m_2 + m_3 = 0,$

$$m_1 m_2 + m_2 m_3 + m_3 m_1 = \frac{(2a-h)}{a},$$

$$m_1 m_2 m_3 = -\frac{k}{a} \qquad ...(ii)$$

Hence, for any given point $P(h, k)$, Eq. (i) has three real or imaginary roots. Corresponding to each of these three roots, we have one normal passing through $P(h, k)$. Hence, in total, we have three normals PA, PB and PC drawn through P to the parabola.

Points A, B, C in which the three normals from $P(h, k)$ meet the parabola are **called co-normal points.**

Corollary 1 *The algebraic sum of the slopes of three concurrent normals is zero. This follows from Eq.* (ii).

Corollary 2 *The algebraic sum of ordinates of the feets of three normals drawn to a parabola from a given point is zero.*

Let the ordinates of A, B, C be y_1, y_2, y_3 respectively, then

$$y_1 = -2am_1, y_2 = -2am_2 \text{ and } y_3 = -2am_3$$

$\therefore$ Algebraic sum of these ordinates is

$$\begin{aligned} y_1 + y_2 + y_3 &= -2am_1 - 2am_2 - 2am_3 \\ &= -2a(m_1 + m_2 + m_3) \\ &= -2a \times 0 \qquad \text{[from Eq. (ii)]} \\ &= 0 \end{aligned}$$

Corollary 3 *If three normals drawn to any parabola $y^2 = 4ax$ from a given point (h, k) be real then $h > 2a$.*

When normals are real, then all the three roots of Eq. (i) are real and in that case

$$m_1^2 + m_2^2 + m_3^2 > 0 \qquad \text{(for any values of } m_1, m_2, m_3)$$

$\Rightarrow \quad (m_1 + m_2 + m_3)^2 - 2 \qquad (m_1 m_2 + m_2 m_3 + m_3 m_1) > 0$

$\Rightarrow \quad (0)^2 - \dfrac{2(2a-h)}{a} > 0 \Rightarrow h - 2a > 0$

$\therefore \quad h > 2a$

Remark

For $a = 1$ normals drawn to the parabola $y^2 = 4x$ from any point (h, k) are real, if $h > 2$.

Corollary 4 *If three normals drawn to any parabola $y^2 = 4ax$ from a given point (h, k) be real and distinct, then $27ak^2 < 4(h - 2a)^3$*

Let $\quad f(m) = am^3 + m(2a - h) + k$

$\therefore \quad f'(m) = 3am^2 + (2a - h)$

Two distinct roots of $f'(m) = 0$ are

$$\alpha = \sqrt{\left(\frac{h-2a}{3a}\right)} \quad \text{and} \quad \beta = -\sqrt{\left(\frac{h-2a}{3a}\right)},$$

Now, $f(\alpha)\, f(\beta) < 0 \Rightarrow f(\alpha)\, f(-\alpha) < 0$

$\Rightarrow \quad (a\alpha^3 + \alpha(2a-h) + k)(-a\alpha^3 - \alpha(2a-h) + k) < 0$

$\Rightarrow \quad k^2 - (a\alpha^2 + (2a-h))^2 \alpha^2 < 0$

$\Rightarrow \quad k^2 - \left(\frac{h-2a}{3} + (2a-h)\right)^2 \frac{(h-2a)}{3a} < 0$

$\Rightarrow \quad k^2 - \left(\frac{4a-2h}{3}\right)^2 \left(\frac{h-2a}{3a}\right) < 0$

$\Rightarrow \quad k^2 - \frac{4(h-2a)^3}{27a} < 0 \Rightarrow 27ak^2 - 4(h-2a)^3 < 0$

$\therefore \quad 27ak^2 < 4(h-2a)^3$

Corollary 5 *The centroid of the triangle formed by the feet of the three normals lies on the axis of the parabola.*

If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be vertices of ΔABC, then its centroid is

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right) = \left(\frac{x_1 + x_2 + x_3}{3}, 0\right).$$

Since, $y_1 + y_2 + y_3 = 0$ (from corollary 2). Hence, the centroid lies on the X-axis OX, which is the axis of the parabola also.

Now, $\frac{x_1 + x_2 + x_3}{3} = \frac{1}{3}(am_1^2 + am_2^2 + am_3^2)$

$$= \frac{a}{3}(m_1^2 + m_2^2 + m_3^2)$$

$$= \frac{a}{3}\{(m_1 + m_2 + m_3)^2 - 2(m_1m_2 + m_2m_3 + m_3m_1)\}$$

$$= \frac{a}{3}\left[(0)^2 - 2\left\{\frac{2a-h}{a}\right\}\right] = \frac{2h-4a}{3}$$

$\therefore$ Centroid of ΔABC is $\left(\frac{2h-4a}{3}, 0\right)$.

Example 42 *Show that the locus of points such that two of the three normals drawn from them to the parabola* $y^2 = 4ax$ *coincide is* $27ay^2 = 4(x-2a)^3$.

Sol. Let (h, k) be the point of intersection of three normals to the parabola $y^2 = 4ax$. The equation of any normal to $y^2 = 4ax$ is

$$y = mx - 2am - am^3$$

If it passes through (h, k), then

$$k = mh - 2am - am^3$$

$$\Rightarrow \quad am^3 + m(2a-h) + k = 0 \qquad \ldots(i)$$

Let the roots of Eq. (i) be m_1, m_2 and m_3.

Then, from Eq. (i), $m_1 + m_2 + m_3 = 0$ ...(ii)

$$m_1m_2 + m_2m_3 + m_3m_1 = \frac{(2a-h)}{a} \qquad \ldots(iii)$$

and $$m_1m_2m_3 = -\frac{k}{a} \qquad \ldots(iv)$$

But here, two of the three normals are given to be coincident i.e. $m_1 = m_2$.

Putting $m_1 = m_2$ in Eqs. (ii) and (iv), we get

$$2m_1 + m_3 = 0 \qquad \ldots(v)$$

and $$m_1^2 m_3 = -\frac{k}{a} \qquad \ldots(vi)$$

Putting $m_3 = -2m_1$ from Eq. (v) in Eq. (vi), we get

$$-2m_1^3 = -\frac{k}{a}$$

$$\Rightarrow \quad m_1^3 = \frac{k}{2a}$$

Since, m_1 is a root of Eq. (i).

$\therefore \quad am_1^3 + m_1(2a-h) + k = 0$

$$\Rightarrow \quad a\left(\frac{k}{2a}\right) + \left(\frac{k}{2a}\right)^{1/3}(2a-h) + k = 0$$

$$\left[\text{putting } m_1 = \left(\frac{k}{2a}\right)^{1/3}\right]$$

$$\Rightarrow \quad \left(\frac{k}{2a}\right)^{1/3}(2a-h) = -\frac{3k}{2}$$

$$\Rightarrow \quad \frac{k}{2a}(2a-h)^3 = -\frac{27k^3}{8}$$

$$\Rightarrow \quad 27ak^2 = 4(h-2a)^3$$

Hence, the locus of (h, k) is

$$27ay^2 = 4(x-2a)^3.$$

Example 43 *Find the locus of the point through which pass three normals to the parabola* $y^2 = 4ax$ *such that two of them make angles* α *and* β *respectively with the axis such that* $\tan\alpha \tan\beta = 2$.

Sol. Let (h, k) be the point of intersection of three normals to the parabola $y^2 = 4ax$.

The equation of any normal to $y^2 = 4ax$ is

$$y = mx - 2am - am^3$$

If it passes through (h_1, k), then

$$k = mh - 2am - am^3$$

$$\Rightarrow \quad am^3 + m(2a-h) + k = 0 \qquad \ldots(i)$$

Let roots of Eq. (i) be m_1, m_2, m_3 then from Eq. (i)

$$m_1m_2m_3 = -\frac{k}{a} \quad ...(ii)$$

Also $m_1 = \tan\alpha$, $m_2 = \tan\beta$ and $\tan\alpha \tan\beta = 2$

$$\therefore \quad m_1m_2 = 2 \quad ...(iii)$$

From Eqs. (ii) and (iii), $2m_3 = -\frac{k}{a}$

or $$m_3 = -\frac{k}{2a}$$

Which being a root of Eq. (i) must satisfy it

i.e. $$am_3^3 + m_3(2a - h) + k = 0$$

$$\Rightarrow \quad a\left(-\frac{k}{2a}\right)^3 - \frac{k}{2a}(2a - h) + k = 0$$

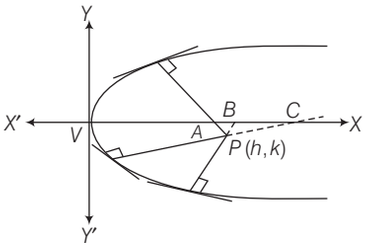
$$\Rightarrow \quad -\frac{k^3}{8a^2} - k + \frac{kh}{2a} + k = 0$$

$$\Rightarrow \quad k^2 - 4ah = 0$$

$\therefore$ Required locus of (h, k) is $y^2 - 4ax = 0$.

Example 44 If the three normals from a point to the parabola $y^2 = 4ax$ cut the axis in points whose distance from the vertex are in AP, show that the point lies on the curve $27ay^2 = 2(x - 2a)^3$.

Sol. Let (h, k) be the point of intersection of three normals to the parabola $y^2 = 4ax$. The equation of any normal to $y^2 = 4ax$ is $y = mx - 2am - am^3$ ...(i)



If it passes through (h, k) then

$$k = mh - 2am - am^3$$

$$\Rightarrow \quad am^3 + m(2a - h) + k = 0 \quad ...(ii)$$

Let roots of Eq. (ii) be m_1, m_2, m_3 then from Eq. (ii)

$$m_1 + m_2 + m_3 = 0 \quad ...(iii)$$

$$m_1m_2 + m_2m_3 + m_3m_1 = \frac{(2a - h)}{a} \quad ...(iv)$$

and $$m_1m_2m_3 = -\frac{k}{a} \quad ...(v)$$

Since, Eq. (i) cuts the axis of parabola *viz.* $y = 0$ at $(2a + am^2, 0)$.

$\therefore$ The normal through (h, k) cut the axis at $A(2a + am_1^2, 0)$, $B(2a + am_2^2, 0)$ and $C(2a + am_3^2, 0)$ and let $V(0, 0)$ be the vertex of the parabola $y^2 = 4ax$.

Then, $VA = 2a + am_1^2$, $VB = 2a + am_2^2$ and $VC = 2a + am_3^2$.

Given, VA, VB and VC are in AP.

$$\therefore \quad 2VB = VA + VC$$

$$\Rightarrow \quad 4a + 2am_2^2 = 2a + am_1^2 + 2a + am_3^2$$

$$\Rightarrow \quad 2m_2^2 = m_1^2 + m_3^2$$

$$\Rightarrow \quad 2m_2^2 = (m_1 + m_3)^2 - 2m_1m_3$$

$$\Rightarrow \quad 2m_2^2 = (m_1 + m_2 + m_3 - m_2)^2 - \frac{2m_1m_2m_3}{m_2}$$

$$\Rightarrow \quad 2m_2^2 = (0 - m_2)^2 - \frac{2}{m_2}\left(-\frac{k}{a}\right)$$

[from Eqs. (iii) and (v)]

$$\Rightarrow \quad m_2^3 = \frac{2k}{a} \quad ...(vi)$$

Now, from Eq. (iv), $m_2(m_1 + m_3) + m_3m_1 = \frac{(2a - h)}{a}$

$$\Rightarrow m_2(m_1 + m_2 + m_3 - m_2) + \frac{m_1m_2m_3}{m_2} = \frac{(2a - h)}{a}$$

$$\Rightarrow \quad m_2(0 - m_2) - \frac{k}{am_2} = \frac{(2a - h)}{a}$$

$$\Rightarrow \quad -am_2^3 - k = m_2(2a - h)$$

$$\Rightarrow \quad (-am_2^3 - k)^3 = m_2^3(2a - h)^3$$

$$\Rightarrow \quad (-2k - k)^3 = \frac{2k}{a}(2a - h)^3 \quad \text{[from Eq. (vi)]}$$

$$\Rightarrow \quad -27k^3 = -\frac{2k}{a}(h - 2a)^3$$

$$\Rightarrow \quad 27ak^2 = 2(h - 2a)^3$$

Hence, locus of (h, k) is

$$27ay^2 = 2(x - 2a)^3.$$

Example 45 The normals at P, Q, R on the parabola $y^2 = 4ax$ meet in a point on the line $y = k$. Prove that the sides of the ΔPQR touch the parabola $x^2 - 2ky = 0$.

Sol. Any normal to the parabola $y^2 = 4ax$ is

$$y = mx - 2am - am^3 \quad ...(i)$$

Also, any point on the line $y = k$ is (x_1, k).

If Eq. (i) passes through (x_1, k) then $k = mx_1 - 2am - am^3$

or $$am^3 + m(2a - x_1) + k = 0$$

If the roots of this equation are m_1, m_2, m_3 then we get

$$m_1 + m_2 + m_3 = 0 \quad ...(ii)$$

$$m_1m_2 + m_2m_3 + m_3m_1 = \frac{(2a - x_1)}{a} \quad ...(iii)$$

and $$m_1m_2m_3 = -\frac{k}{a} \quad ...(iv)$$

Also, coordinates of three points P, Q and R are $(am_1^2, -2am_1)(am_2^2, -2am_2)$ and $(am_3^2, -2am_3)$, respectively.

$\therefore$ The equation of the line PQ is

$$y-(-2am_1)=\frac{(-2am_2)-(-2am_1)}{am_2^2-am_1^2}(x-am_1^2)$$

$$\Rightarrow \quad y+2am_1=-\frac{2}{(m_2+m_1)}(x-am_1^2)$$

$$\Rightarrow \quad y(m_1+m_2)+2am_1(m_1+m_2)=-2x+2am_1^2$$

$$\Rightarrow \quad y(m_1+m_2)+2am_1m_2=-2x$$

$$\Rightarrow \quad y(m_1+m_2+m_3-m_3)+\frac{2am_1m_2m_3}{m_3}=-2x$$

$$\Rightarrow \quad y(0-m_3)-\frac{2k}{m_3}=-2x \qquad \text{[from Eqs. (ii) and (iv)]}$$

$$\Rightarrow \quad -ym_3^2-2k=-2m_3x$$

$$\Rightarrow \quad ym_3^2-2m_3x+2k=0,$$

which is a quadratic in m_3.

Since, PQ will touch it, then

$$B^2-4AC=0$$

$$\Rightarrow \quad (-2x)^2-4\cdot y\cdot 2k=0$$

$$\therefore \quad x^2-2ky=0$$

Example 46 ***Find the point on the axis of the parabola*** $3y^2+4y-6x+8=0$ ***from when three distinct normals can be drawn.***

Sol. Given, parabola is $3y^2+4y-6x+8=0$

$$\Rightarrow \quad 3\left(y^2+\frac{4}{3}y\right)=6x-8$$

$$\Rightarrow \quad 3\left\{\left(y+\frac{2}{3}\right)^2-\frac{4}{9}\right\}=6x-8$$

$$\Rightarrow \quad 3\left(y+\frac{2}{3}\right)^2=\left(6x-8+\frac{4}{3}\right)$$

$$\therefore \quad \left(y+\frac{2}{3}\right)^2=2\left(x-\frac{10}{9}\right)$$

Let $\quad y+\frac{2}{3}=Y,\ x-\frac{10}{9}=X$

Then, $\quad Y^2=2X$

Comparing with $\quad Y^2=4aX$

$$\therefore \quad a=\frac{1}{2}$$

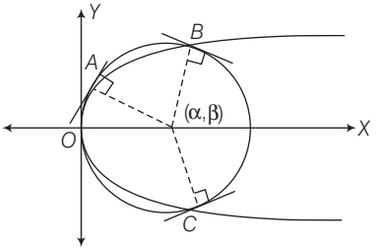
any point on the axis of parabola is $\left(x,-\frac{2}{3}\right)$

and $\quad X>2a \ \Rightarrow \ x-\frac{10}{9}>1$

$$\Rightarrow \quad x>\frac{19}{9}$$

Circle Through Co-normal Points

To find the equation of the circle passing through the three (conormal) points on the parabola, normals at which pass through a given point (α,β).



Let $A(am_1^2,-2am_1)$, $B(am_2^2,-2am_2)$ and $C(am_3^2,-2am_3)$ be the three points on the parabola

$$y^2=4ax$$

Since, point of intersection of normals is (α,β), then

$$am^3+(2a-\alpha)m+\beta=0 \qquad \ldots(E)$$

$$\therefore \quad m_1+m_2+m_3=0 \qquad \ldots(i)$$

$$m_1m_2+m_2m_3+m_3m_1=\frac{(2a-\alpha)}{a} \qquad \ldots(ii)$$

and $$m_1m_2m_3=-\frac{\beta}{a} \qquad \ldots(iii)$$

Let the equation of the circle through A, B, C be

$$x^2+y^2+2gx+2fy+c=0 \qquad \ldots(iv)$$

If the point $(am^2,-2am)$ lies on it, then

$$(am^2)^2+(-2am)^2+2g(am^2)+2f(-2am)+c=0$$

$$\Rightarrow \quad a^2m^4+(4a^2+2ag)m^2-4afm+c=0 \qquad \ldots(v)$$

This is a biquadratic equation in m. Hence, there are four values of m, say m_1, m_2, m_3 and m_4 such that the circle pass through the points.

$A(am_1^2,-2am_1), B(am_2^2,-2am_2), C(am_3^2,-2am_3)$ and $D(am_4^2,-2am_4)$.

$$\therefore \quad m_1+m_2+m_3+m_4=0 \qquad \ldots(F)$$

$$\Rightarrow \quad 0+m_4=0 \qquad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad m_4=0$$

$$\therefore \quad (am_4^2,-2am_4)=(0,0)$$

Thus, the circle passes through the vertex of the parabola $y^2=4ax$ from Eq. (iv),

$$0+0+0+0+c=0$$

$$\therefore \quad c=0$$

From Eq. (v), $a^2m^4+(4a^2+2ag)m^2-4afm=0$

$$\Rightarrow \quad am^3+(4a+2g)m-4f=0 \qquad \ldots(vi)$$

Now, Eqs. (E) and (vi) are identical.

$$\therefore \quad 1 = \frac{4a + 2g}{2a - \alpha} = -\frac{4f}{\beta}$$

$$\therefore \quad 2g = -(2a + \alpha), 2f = -\beta/2$$

$\therefore$ The equation of the required circle is

$$x^2 + y^2 - (2a + \alpha)\, x - \frac{\beta}{2} y = 0$$

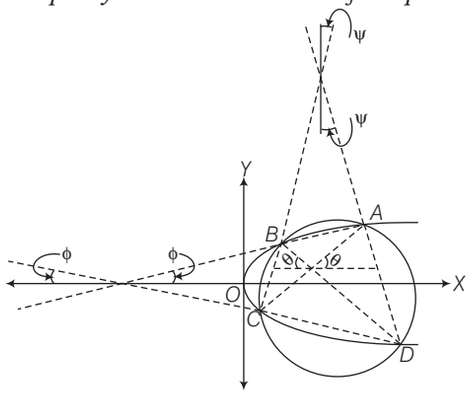
[from Eq. (iv)]

Corollary 1. *The algebraic sum of the ordinates of the four points of intersection of a circle and a parabola is zero.*

Sum of ordinates

$$= -2am_1 - 2am_2 - 2am_3 - 2am_4$$
$$= -2a(m_1 + m_2 + m_3 + m_4) \qquad \text{[from Eq. (F)]}$$
$$= -2a \times 0 = 0$$

Corollary 2. *The common chords of a circle and a parabola are in pairs equally inclined to the axis of the parabola.*



Let A, B, C, D be the points of intersection of the circle and the parabola with $A(am_1^2, -2am_1)$, $B(am_2^2, -2am_2)$ $C(am_3^2, -2am_3)$ and $D(am_4^2, -2am_4)$ then equation of AC and BD are

$$y(m_1 + m_3) = -2x - 2am_1m_3$$

and $\quad y(m_2 + m_4) = -2x - 2am_2m_4$, respectively.

$\therefore$ Slopes of the chords AC and BD are

$$-\frac{2}{m_1 + m_3} \text{ and } -\frac{2}{m_2 + m_4}, \text{ respectively.}$$

$$\therefore \quad \text{Slope of } AC = -\frac{2}{m_1 + m_3}$$
$$= \frac{2}{m_2 + m_4} \qquad [\because m_1 + m_2 + m_3 + m_4 = 0]$$
$$= -\left(-\frac{2}{m_2 + m_4}\right) = -\text{ Slope of } BD$$

$\therefore$ Their slopes are equal in magnitude and opposite in sign.

$\therefore$ The chords of AC and BD are equally inclined to the axis.

Remark

This is likewise true for the pairs of chords AB, CD and AD, BC.

Corollary 3. *The circle through conormal point passes through the vertex $(0, 0)$ of the parabola.*

Corollary 4. *The centroid of four points; in which a circle intersects a parabola, lies on the axis of the parabola.*

$$\text{Centroid} = \left(\frac{\sum_{i=1}^{4} am_i^2}{4}, \frac{\sum_{i=1}^{4}(-2am_i)}{4}\right)$$
$$= \left(\frac{a}{4}\{(\Sigma m_1)^2 - 2\Sigma m_1 m_2\}, -\frac{a}{2}(\Sigma m_1)\right)$$
$$= \left(\frac{a}{4}\left(0 - \frac{2(4a^2 + 2ag)}{a^2}\right), 0\right)$$
$$= (-2a - g, 0)$$

Here $y = 0$, which is axis of the parabola $y^2 = 4ax$.

Example 47 A circle cuts the parabola $y^2 = 4ax$ at right angles and passes through the focus, show that its centre lies on the curve $y^2(a + 2x) = a(a + 3x)^2$.

Sol. Let the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ ...(i)

meet the parabola $y^2 = 4ax$ at any point $P(at^2, 2at)$ cutting it at right angles.

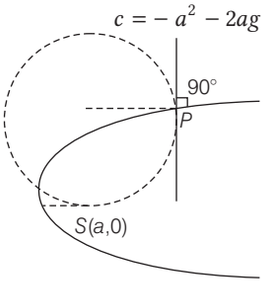
We have to find locus of centre of circle Eq. (i),

i.e. $(-g, -f)$

But given circle Eq. (i) passes through the focus $(a, 0)$, then

$$a^2 + 0 + 2ga + 0 + c = 0$$

$$\Rightarrow \quad c = -a^2 - 2ag \qquad \text{...(ii)}$$



Now, the circle and parabola intersect at $P(at^2, 2at)$ at right angles.

Since, tangent at $P(at^2, 2at)$ to the parabola $y^2 = 4ax$ is

$$ty = x + at^2$$

Hence, this tangent must pass through the centre $(-g, -f)$ of the circle

$$\therefore \quad -ft = -g + at^2$$

$$\Rightarrow \quad ft = g - at^2 \qquad \text{...(iii)}$$

Also the point $P(at^2, 2at)$ lies on the circle (i), then

$$a^2t^4 + 4a^2t^2 + 2agt^2 + 4aft + c = 0$$

Hence, from Eqs. (ii) and (iii) when we put values for c and ft.

$\therefore \quad a^2t^4 + 4a^2t^2 + 2agt^2 + 4ag - 4a^2t^2 - a^2 - 2ag = 0$

$\Rightarrow \quad a^2t^4 + 2agt^2 + 2ag - a^2 = 0$

$\Rightarrow \quad at^4 + 2gt^2 + 2g - a = 0$

$\Rightarrow \quad a(t^4 - 1) + 2g(t^2 + 1) = 0$

$\Rightarrow \quad (t^2 + 1)[a(t^2 - 1) + 2g] = 0 \quad$ But $\quad t^2 + 1 \neq 0$

$\Rightarrow \quad a(t^2 - 1) + 2g = 0 \quad \Rightarrow \quad t^2 = \dfrac{a - 2g}{a}$

Hence, from Eq. (iii),

$$f\sqrt{\left(\frac{a-2g}{a}\right)} = g - a\left(\frac{a-2g}{a}\right)$$

$\Rightarrow \quad f\sqrt{\left(\dfrac{a-2g}{a}\right)} = (3g - a)$

$\Rightarrow \quad f^2(a - 2g) = a(3g - a)^2$

$\Rightarrow \quad (-f)^2[a + 2(-g)] = a(a - 3g)^2$

$\Rightarrow \quad (-f)^2[a + 2(-g)] = a[a + 3(-g)]^2$

Hence, locus of the centre $(-g, -f)$ is the curve

$$y^2(a + 2x) = a(a + 3x)^2.$$

Exercise for Session 2

1. If $2x + y + \lambda = 0$ is a normal to the parabola $y^2 = -8x$, then the value of λ is

(a) -24 (b) -16 (c) -8 (d) 24

2. The slope of a chord of the parabola $y^2 = 4ax$ which is normal at one end and which subtends a right angle at the origin is

(a) $\dfrac{1}{\sqrt{2}}$ (b) $\sqrt{2}$ (c) $-\dfrac{1}{\sqrt{2}}$ (d) $-\sqrt{2}$

3. The common tangent to the parabola $y^2 = 4ax$ and $x^2 = 4ay$ is

(a) $x + y + a = 0$ (b) $x + y - a = 0$ (c) $x - y + a = 0$ (d) $x - y - a = 0$

4. The circle $x^2 + y^2 + 4\lambda x = 0$ which $\lambda \in R$ touches the parabola $y^2 = 8x$. The value of λ is given by

(a) $\lambda \in (0, \infty)$ (b) $\lambda \in (-\infty, 0)$ (c) $\lambda \in (1, \infty)$ (d) $\lambda \in (-\infty, 1)$

5. If the normals at two points P and Q of a parabola $y^2 = 4ax$ intersect at a third point R on the curve, then the product of ordinates of P and Q is

(a) $4a^2$ (b) $2a^2$ (c) $-4a^2$ (d) $8a^2$

6. The normals at three points P, Q, R of the parabola $y^2 = 4ax$ meet in (h, k). The centroid of ΔPQR lies on

(a) $x = 0$ (b) $y = 0$ (c) $x = -a$ (d) $y = a$

7. The set of points on the axis of the parabola $y^2 - 4x - 2y + 5 = 0$ from which all the three normals to the parabola are real, is

(a) $(\lambda, 0)\ ; \lambda > 1$ (b) $(\lambda, 1)\ ; \lambda > 3$ (c) $(\lambda, 2)\ ; \lambda > 6$ (d) $(\lambda, 3)\ ; \lambda > 8$

8. Prove that any three tangents to a parabola whose slopes are in harmonic progression enclose a triangle of constant area.

9. A chord of parabola $y^2 = 4ax$ subtends a right angle at the vertex. Find the locus of the point of intersection of tangents at its extremities.

10. Find the equation of the normal to the parabola $y^2 = 4x$ which is

(a) parallel to the line $y = 2x - 5$. (b) perpendicular to the line $2x + 6y + 5 = 0$.

11. The ordinates of points P and Q on the parabola $y^2 = 12x$ are in the ratio 1 : 2. Find the locus of the point of intersection of the normals to the parabola at P and Q.

12. The normals at P, Q, R on the parabola $y^2 = 4ax$ meet in a point on the line $y = c$. Prove that the sides of the ΔPQR touch the parabola $x^2 = 2cy$.

13. The normals are drawn from $(2\lambda, 0)$ to the parabola $y^2 = 4x$. Show that λ must be greater than 1. One normal is always the X-axis. Find λ for which the other two normals are perpendicular to each other.

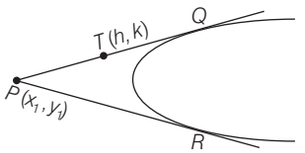
Session 3

Pair of Tangents $SS_1 = T^2$, Chord of Contact, Equation of the Chord Bisected at a Given Point, Diameter, Lengths of Tangent, Subtangent, Normal and Subnormal, Some Standard Properties of the Parabola, Reflection Property of a Parabola, Study of Parabola of the Form $(\alpha x^2 + \beta y)^2 + 2gx + 2fy + c = 0$

Pair of Tangents $SS_1 = T^2$

If $y_1^2 - 4ax_1 > 0$, then any point $P(x_1, y_1)$ lies outside the parabola and a pair of tangents PQ, PR can be drawn to it from P. We find their equation as follows.

Let $T(h, k)$ be any point on the pair of tangents PQ or PR drawn from any external point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$.



Equation of PT is

$$y - y_1 = \frac{k - y_1}{h - x_1}(x - x_1)$$

$$\Rightarrow \quad y = \left(\frac{k - y_1}{h - x_1}\right)x + \left(\frac{hy_1 - kx_1}{h - x_1}\right)$$

which is tangent to the parabola

$$y^2 = 4ax$$

$$\therefore \quad c = \frac{a}{m}$$

$$\Rightarrow \quad cm = a \quad \Rightarrow \quad \left(\frac{hy_1 - kx_1}{h - x_1}\right)\left(\frac{k - y_1}{h - x_1}\right) = a$$

$$\Rightarrow \quad (k - y_1)(hy_1 - kx_1) = a(h - x_1)^2$$

$\therefore$ Locus of (h, k), equation of pair of tangents is

$$(y - y_1)(xy_1 - x_1y) = a(x - x_1)^2$$

$$\Rightarrow \quad (y^2 - 4ax)(y_1^2 - 4ax_1) = \{yy_1 - 2a(x + x_1)\}^2$$

$$\therefore \quad SS_1 = T^2$$

where $S = y^2 - 4ax, S_1 = y_1^2 - 4ax_1$

and $T = yy_1 - 2a(x + x_1)$.

Aliter :

Let the parabola be $y^2 = 4ax$ ...(i)

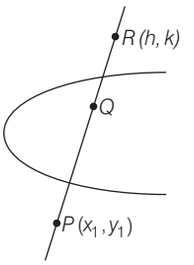
Let $P(x_1, y_1)$ be any point outside the parabola. Let a chord of the parabola through the point $P(x_1, y_1)$ cut the parabola at Q and let $R(h, k)$ be any arbitrary point on the line PQ (R inside or outside).

Let Q divide PR in the ratio $\lambda : 1$, then coordinates of Q are

$$\left(\frac{\lambda h + x_1}{\lambda + 1}, \frac{\lambda k + y_1}{\lambda + 1}\right) \qquad [\because PQ : QR = \lambda : 1]$$

Since, Q lies on parabola Eq. (i), then

$$\left(\frac{\lambda k + y_1}{\lambda + 1}\right)^2 = 4a\left(\frac{\lambda h + x_1}{\lambda + 1}\right)$$



$$\Rightarrow \quad (\lambda k + y_1)^2 - 4a(\lambda h + x_1)(\lambda + 1) = 0$$

$$\Rightarrow \quad (k^2 - 4ah)\lambda^2 + 2[ky_1 - 2a(h + x_1)]\lambda + (y_1^2 - 4ax_1) = 0 \quad ...(ii)$$

Line PR will become tangent to parabola Eq. (i), then roots of Eq. (ii) are equal

$$\therefore \quad 4[ky_1 - 2a(h + x_1)]^2 - 4(k^2 - 4ah)(y_1^2 - 4ax_1) = 0$$

$$\text{or} \quad \{ky_1 - 2a(h + x_1)\}^2 = (k^2 - 4ah)(y_1^2 - 4ax_1)$$

Hence, locus of $R(h, k)$ i.e. equation of pair of tangents from $P(x_1, y_1)$ is

$$\{yy_1 - 2a(x + x_1)\}^2 = (y^2 - 4ax)(y_1^2 - 4ax_1)$$

i.e. $$T^2 = SS_1 \quad \text{or} \quad SS_1 = T^2$$

Remark

$S = 0$ is the equation of the curve, S_1 is obtained from S by replacing x by x_1 and y by y_1 and $T = 0$ is the equation tangent at (x_1, y_1) to $S = 0$.

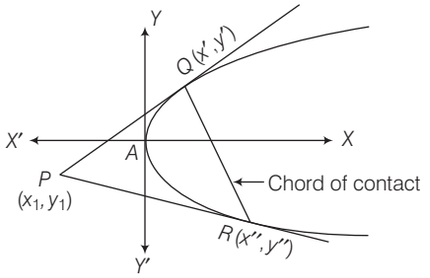
Chord of Contact

The chord joining the points of contact of two tangents drawn from an external point to a parabola is known as the chord of contact of tangents drawn from external point.

Theorem The chord of contact of tangents drawn from a point (x_1, y_1) to the parabola $y^2 = 4ax$ is

$$yy_1 = 2a(x + x_1).$$

Proof Let PQ and PR be tangents to the parabola $y^2 = 4ax$ drawn from any external point $P(x_1, y_1)$, then QR is called **chord of contact** of the parabola $y^2 = 4ax$.



Let $Q \equiv (x', y')$ and $R \equiv (x'', y'')$

Equation of tangent PQ is

$$yy' = 2a(x + x') \quad \ldots(i)$$

and equation of tangent PR is

$$yy'' = 2a(x + x'') \quad \ldots(ii)$$

Since, lines Eqs. (i) and (ii) pass through (x_1, y_1), then

$$y_1y' = 2a(x_1 + x') \text{ and } y_1y'' = 2a(x_1 + x'')$$

Hence, it is clear $Q(x', y')$ and $R(x'', y'')$ lie on

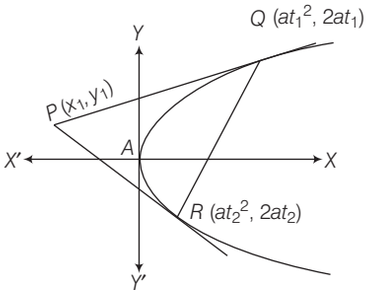
$$yy_1 = 2a(x + x_1)$$

which is **chord of contact** QR.

Example 48 Tangents are drawn from the point (x_1, y_1) to the parabola $y^2 = 4ax$, show that the length of their chord of contact is $\frac{1}{|a|}\sqrt{(y_1^2 - 4ax_1)(y_1^2 + 4a^2)}$.

Sol. Given parabola is

$$y^2 = 4ax \quad \ldots(i)$$



Let $P \equiv (x_1, y_1)$

and the tangents from P touch the parabola at $Q(at_1^2, 2at_1)$ and $R(at_2^2, 2at_2)$ then P is the point of intersection of tangents.

$\therefore \quad x_1 = at_1t_2$ and $y_1 = a(t_1 + t_2)$

$\Rightarrow \quad t_1t_2 = \frac{x_1}{a}$ and $t_1 + t_2 = \frac{y_1}{a}$ ...(ii)

Now, $QR = \sqrt{(at_1^2 - at_2^2)^2 + (2at_1 - 2at_2)^2}$

$= \sqrt{a^2(t_1 - t_2)^2\,[(t_1 + t_2)^2 + 4]}$

$= |a|\,|t_1 - t_2|\sqrt{\{(t_1 + t_2)^2 + 4\}}$

$= |a|\sqrt{\{(t_1 + t_2)^2 - 4t_1t_2\}}\sqrt{\{(t_1 + t_2)^2 + 4\}}$

$= |a|\sqrt{\left(\frac{y_1^2}{a^2} - \frac{4x_1}{a}\right)} \cdot \sqrt{\left(\frac{y_1^2}{a^2} + 4\right)}$

[from Eq. (ii)]

$= |a|\frac{\sqrt{(y_1^2 - 4ax_1)}}{|a|} \cdot \frac{\sqrt{(y_1^2 + 4a^2)}}{|a|}$

$= \frac{1}{|a|}\sqrt{(y_1^2 - 4ax_1)(y_1^2 + 4a^2)}$

Aliter :

Equation of QR is $yy_1 = 2a(x + x_1)$

$\Rightarrow \quad x = \frac{yy_1 - 2ax_1}{2a}$

The ordinates of Q and R are the roots of the equation

$$y^2 = 4a\left(\frac{yy_1 - 2ax_1}{2a}\right)$$

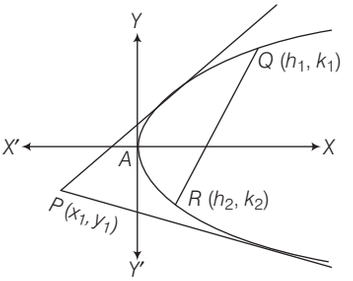
$\Rightarrow \quad y^2 = 2(yy_1 - 2ax_1)$ ... (i)

$\Rightarrow \quad y^2 - 2yy_1 + 4ax_1 = 0$

$\therefore \quad k_1 + k_2 = 2y_1$ and $k_1k = 4ax_1$

$\therefore \quad (k_2 - k_1) = \sqrt{(k_1 + k_2)^2 - 4k_1k_2}$

$= \sqrt{(4y_1^2 - 16ax_1)} = 2\sqrt{(y_1^2 - 4ax_1)}$ ...(ii)



Since, $Q(h_1, k_1)$ and $R(h_2, k_2)$ lie on the parabola $y^2 = 4ax$, therefore

$$k_1^2 = 4ah_1 \quad \text{and} \quad k_2^2 = 4ah_2$$

$\Rightarrow \quad k_2^2 - k_1^2 = 4a(h_2 - h_1)$

$(k_2 + k_1)(k_2 - k_1) = 4a(h_2 - h_1)$

$$\Rightarrow \quad 2y_1(k_2 - k_1) = 4a(h_2 - h_1)$$

$$\Rightarrow \quad (h_2 - h_1) = \frac{y_1(k_2 - k_1)}{2a} \quad \ldots(iii)$$

Now, $QR = \sqrt{(k_2 - k_1)^2 + (h_2 - h_1)^2}$

$$= \sqrt{(k_2 - k_1)^2 + \frac{y_1^2 (k_2 - k_1)^2}{4a^2}} \quad \text{[from Eq. (iii)]}$$

$$= \frac{(k_2 - k_1)}{2|a|}\sqrt{(y_1^2 + 4a^2)}$$

$$= \frac{2\sqrt{(y_1^2 - 4ax_1)}}{2|a|}\sqrt{(y_1^2 + 4a^2)} \quad \text{[from Eq. (ii)]}$$

$$= \frac{1}{|a|}\sqrt{(y_1^2 - 4ax_1)(y_1^2 + 4a^2)}$$

Example 49 Prove that the area of the triangle formed by the tangents drawn from (x_1, y_1) to $y^2 = 4ax$ and their chord of contact is $(y_1^2 - 4ax_1)^{3/2}/2a$.

Sol. Equation of QR (chord of contact) is

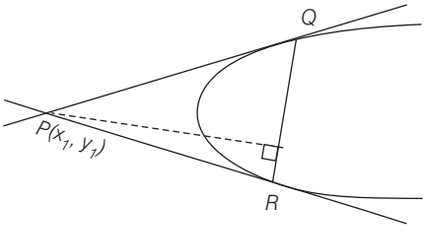
$$yy_1 = 2a(x + x_1)$$

$$\Rightarrow \quad yy_1 - 2a(x + x_1) = 0$$

$\because$ PM = Length of perpendicular from $P(x_1, y_1)$ on QR

$$= \frac{|y_1 y_1 - 2a(x_1 + x_1)|}{\sqrt{(y_1^2 + 4a^2)}} = \frac{|(y_1^2 - 4ax_1)|}{\sqrt{(y_1^2 + 4a^2)}}$$

[Since, $P(x_1, y_1)$ lies outside the parabola. So, $y_1^2 - 4ax_1 > 0$]



Now, area of $\Delta PQR = \frac{1}{2} QR \cdot PM$

$$= \frac{1}{2}\frac{1}{|a|}\sqrt{(y_1^2 - 4ax_1)(y_1^2 + 4a^2)}\,\frac{(y_1^2 - 4ax_1)}{\sqrt{(y_1^2 + 4a^2)}}$$

$$= (y_1^2 - 4ax_1)^{3/2}/2a, \text{ if } a > 0$$

Equation of the Chord Bisected at a Given Point

Theorem The equation of the chord of the parabola $y^2 = 4ax$ which is bisected at (x_1, y_1) is

$$yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$$

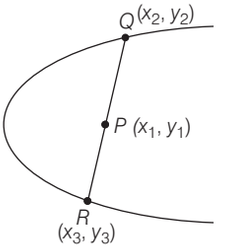
or $$T = S_1,$$

where, $T = yy_1 - 2a(x + x_1)$ and $S_1 = y_1^2 - 4ax_1$.

Proof Since, equation of the parabola is

$$y^2 = 4ax \quad \ldots(i)$$

Let QR be the chord of the parabola whose mid-point is $P(x_1, y_1)$.



Since, Q and R lie on parabola (i),

$$y_2^2 = 4ax_2 \text{ and } y_3^2 = 4ax_3$$

$$\therefore \quad y_3^2 - y_2^2 = 4a(x_3 - x_2)$$

$$\Rightarrow \quad \frac{y_3 - y_2}{x_3 - x_2} = \frac{4a}{y_3 + y_2}$$

$$= \frac{4a}{2y_1} \quad [\because P(x_1, y_1) \text{ is mid-point of } QR]$$

$$\therefore \quad \frac{y_3 - y_2}{x_3 - x_2} = \frac{2a}{y_1} = \text{ Slope of } QR$$

Equation of QR is $y - y_1 = \frac{2a}{y_1}(x - x_1)$

$$\Rightarrow \quad yy_1 - y_1^2 = 2ax - 2ax_1$$

$$\Rightarrow \quad yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$$

[subtracting $2ax_1$ from both sides]

$$\therefore \quad T = S_1,$$

where $T = yy_1 - 2a(x + x_1)$ and $S_1 = y_1^2 - 4ax_1$.

Example 50 Find the locus of the mid-points of the chords of the parabola $y^2 = 4ax$ which subtend a right angle at the vertex of the parabola.

Sol. Let $P(h, k)$ be the mid-point of a chord QR of the parabola $y^2 = 4ax$ then equation of chord QR is

$$T = S_1$$

or $$yk - 2a(x + h) = k^2 - 4ah$$

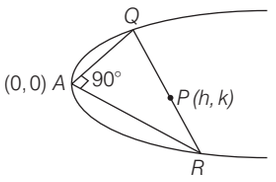
$$\Rightarrow \quad yk - 2ax = k^2 - 2ah \quad \ldots(i)$$

If A is the vertex of the parabola. For combined equation of AQ and AR making homogeneous of $y^2 = 4ax$ with the help of Eq. (i).

$$\therefore \quad y^2 = 4ax$$

$$\Rightarrow \quad y^2 = 4ax\left(\frac{yk - 2ax}{k^2 - 2ah}\right)$$

$$\Rightarrow \quad y^2(k^2 - 2ah) - 4akxy + 8a^2x^2 = 0$$



Since, $\angle QAR = 90°$

$\therefore$ Coefficient of x^2 + Coefficient of $y^2 = 0$

$$k^2 - 2ah + 8a^2 = 0$$

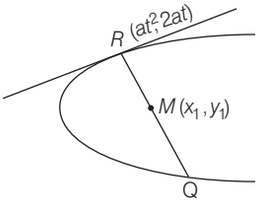
Hence, the locus of $P(h, k)$ is $y^2 - 2ax + 8a^2 = 0$.

Example 51 Show that the locus of the middle points of normal chords of the parabola $y^2 = 4ax$ is

$$y^4 - 2a(x - 2a)y^2 + 8a^4 = 0.$$

Sol. Equation of the normal chord at any point $(at^2, 2at)$ of the parabola $y^2 = 4ax$ is

$$y + tx = 2at + at^3 \qquad \ldots(i)$$



But if $M(x_1, y_1)$ be its middle point its equation must be also

$$T = S_1$$

$$\Rightarrow \quad yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$$

$$\Rightarrow \quad yy_1 - 2ax = y_1^2 - 2ax_1 \qquad \ldots(ii)$$

$\because$ From Eqs. (i) and (ii) are identical, comparing, them

$$\frac{1}{y_1} = \frac{t}{-2a} = \frac{2at + at^3}{y_1^2 - 2ax_1}$$

From first two relations, $t = -\dfrac{2a}{y_1}$ ...(iii)

From last two relations, $\dfrac{t}{-2a} = \dfrac{2at + at^3}{y_1^2 - 2ax_1}$

$$\Rightarrow \quad \frac{y_1^2 - 2ax_1}{-2a} = 2a + at^2$$

$$\Rightarrow \quad \frac{y_1^2 - 2ax_1}{-2a} = 2a + a\left(\frac{-2a}{y_1}\right)^2 \qquad \text{[from Eq. (iii)]}$$

$$\Rightarrow \quad \frac{y_1^2 - 2ax_1}{-2a} = \frac{2ay_1^2 + 4a^3}{y_1^2}$$

$$\Rightarrow \quad y_1^4 - 2ax_1y_1^2 = -4a^2y_1^2 - 8a^4$$

$$\Rightarrow \quad y_1^4 - 2a(x_1 - 2a)y_1^2 + 8a^4 = 0$$

Hence, the locus of middle point (x_1, y_1) is

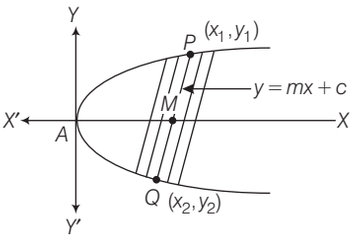
$$y^4 - 2a(x - 2a)y^2 + 8a^4 = 0.$$

Diameter

The locus of the middle points of a system of parallel chords is called a diameter and in case of a parabola this diameter is shown to be a straight line which is parallel to the axis of the parabola.

Theorem The equation of the diameter bisecting chords of slope m of the parabola $y^2 = 4ax$ is $y = \dfrac{2a}{m}$.

Proof Let $y = mx + c$ be system of parallel chords to $y^2 = 4ax$ for different chords c varies, m remains constant. Let the extremities of any chord PQ of the set be $P(x_1, y_1)$ and $Q(x_2, y_2)$ and let its middle point be $M(h, k)$.



On solving equations

$$y^2 = 4ax \quad \text{and} \quad y = mx + c.$$

$$\therefore \quad y^2 = 4a\left(\frac{y - c}{m}\right)$$

$$\therefore \quad my^2 - 4ay + 4ac = 0$$

$$\therefore \quad y_1 + y_2 = \frac{4a}{m} \quad \text{or} \quad \frac{y_1 + y_2}{2} = \frac{2a}{m}$$

[$\because (h, k)$ is the mid-point of PQ]

Hence, locus of $M(h, k)$ is $y = \dfrac{2a}{m}$.

Aliter :

Let (h, k) be the middle point of the chord $y = mx + c$ of the parabola $y^2 = 4ax$ then

$$T = S_1 \Rightarrow ky - 2a(x + h) = k^2 - 4ah$$

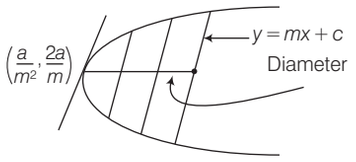
$$\text{slope} = \frac{2a}{k} = m \Rightarrow k = \frac{2a}{m}$$

Hence, locus of the mid-point is $y = \dfrac{2a}{m}$.

Remarks

1. The point in which any diameter meets the curve is called the extremity of the diameter.
2. Any line which is parallel to the axis of the parabola drawn through any point on the parabola is called diameter and its equation is y-coordinate of that point.
   If point on diameter (x_1, y_1), then diameter is $y = y_1$.

Corollary 1. *The tangent at the extremity of a diameter of a parabola is parallel to the system of chords it bisects.*



Let $y = mx + c$ (c variable) represents the system of parallel chords, then the equation of diameter of $y^2 = 4ax$ is $y = \frac{2a}{m}$.

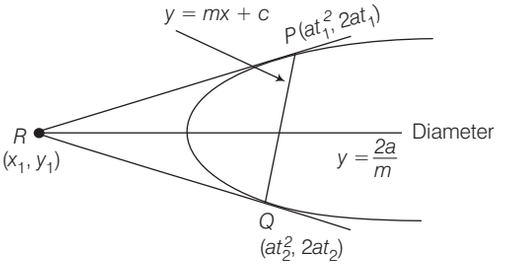
The diameter meets the parabola $y^2 = 4ax$ at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

and tangent is $y = mx + \frac{a}{m}$ which is parallel to $y = mx + c$.

Corollary 2. *Tangents at the end of any chord meet on the diameter which bisects the chords.*

If extremities of the chord be $P(at_1^2, 2at_1)$ *and* $Q(at_2^2, 2at_2)$ *then its slope*

$$m = \frac{2at_2 - 2at_1}{at_2^2 - at_1^2} \implies m = \frac{2}{(t_2 + t_1)}.$$



∴ Equation of diameter is

$$y = 2a/m \implies y = a(t_1 + t_2) \quad \ldots(i)$$

Now, tangents at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ meet at a point $[at_1t_2, a(t_1 + t_2)]$ which lies on Eq. (i).

Aliter

Let equation of any chord PQ be $y = mx + c$.

If tangents at P and Q meet at $R(x_1, y_1)$, then PQ is the chord of contact with respect to $R(x_1, y_1)$.

∴ Equation of PQ is

$$yy_1 = 2x(x + x_1) \quad \text{or} \quad y = \frac{2a}{y_1}x + \frac{2ax_1}{y_1}$$

which is identical to $y = mx + c$

$$m = \frac{2a}{y_1} \quad \text{or} \quad y_1 = \frac{2a}{m}.$$

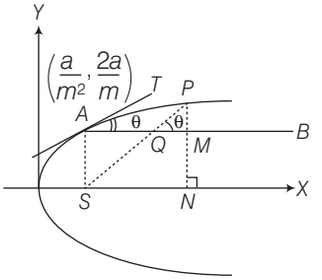
Hence, locus of $R(x_1, y_1)$ is $y = \frac{2a}{m}$, which bisects the chord PQ.

Corollary 3. *To find the equation of a parabola when the axes are any diameter and the tangent to the parabola at the point where this diameter meets the curve.*

Let the equation of the parabola be

$$y^2 = 4ax \quad \ldots(i)$$

Let AB be the diameter of the parabola Eq. (i), then its equation is $y = \frac{2a}{m}$



Since, A is the extremity of the diameter

∴ Coordinates of A is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

where, $m = \tan\theta$

then, the equation of tangent AT at A is

$$y = mx + \frac{a}{m}.$$

Now, let P be any point on the parabola Eq. (i), whose coordinates referred to Vx and Vy are (x, y) and referred to diameter AB and tangent AT are (X, Y).

then $X = AQ$ and $Y = QP$ [since, $PQ \| AT$]

Now, $VN = VL + LN = VL + AM = VL + AQ + QM$

$$= \frac{a}{m^2} + X + QP\cos\theta$$

or $$x = \frac{a}{m^2} + X + Y\cos\theta \quad \ldots(ii)$$

and $PN = PM + MN = PM + AL$

$$= QP\sin\theta + \frac{2a}{m}$$

∴ $$y = Y\sin\theta + \frac{2a}{m} \quad \ldots(iii)$$

From Eqs. (ii) and (iii) coordinates of P are

$$\left(\frac{a}{m^2} + X + Y\cos\theta, Y\sin\theta + \frac{2a}{m}\right)$$

Now, P lies on Eq. (i).

∴ $$\left(Y\sin\theta + \frac{2a}{m}\right)^2 = 4a\left(\frac{a}{m^2} + X + Y\cos\theta\right)$$

$$\Rightarrow \quad Y^2 \sin^2\theta + \frac{4a^2}{m^2} + \frac{4a}{m} Y \sin\theta$$

$$= \frac{4a^2}{m^2} + 4aX + 4aY\cos\theta$$

$$\Rightarrow \quad Y^2 \sin^2\theta + 4a^2 \cot^2\theta + 4a\cos\theta Y$$

$$= 4a^2 \cot^2\theta + 4aX + 4aY\cos\theta \quad [\because m = \tan\theta]$$

$$\Rightarrow \quad Y^2 \sin^2\theta = 4aX$$

$$\therefore \quad Y^2 = (4a\,\text{cosec}^2\theta)\, X,$$

which is the required parabola referred to diameter and tangent at the extremity of the diameter as axes.

Remark

The quantity $4a\,\text{cosec}^2\theta$ is called the parameter of the diameter AQ. It is equal to length of the chord which is parallel to AT and passes through the focus.

i.e. $a\,\text{cosec}^2\theta = a(1 + \cot^2\theta) = a + a\cot^2\theta$

$$= a + \frac{a}{m^2}$$

$$= a + VL = SP$$

But length of focal chord if $P(at^2, 2at)$ is $[\because S(a,0)]\ a\left(t + \frac{1}{t}\right)^2$.

$$\therefore \quad \tan\theta = \frac{2at - 0}{at^2 - a} = \frac{2t}{t^2 - 1} \text{ or } t - \frac{1}{t} = 2\cot\theta$$

$$\therefore \quad a\left(t + \frac{1}{t}\right)^2 = a\left\{\left(t - \frac{1}{t}\right)^2 + 4\right\}$$

$$= a\{(2\cot\theta)^2 + 4\}$$

$$= 4a\,\text{cosec}^2\theta = 4 \cdot SP$$

Example 52 If the diameter through any point P of a parabola meets any chord in A and the tangent at the end of the chord meets the diameter in B and C, then prove that $PA^2 = PB \cdot PC$.

Sol. The equation of the parabola referred to the diameter through P and tangent at P as axes is

$$y^2 = 4\lambda x \quad \ldots(i)$$

where, $\lambda = a\,\text{cosec}^2\theta$ [from previous corollary]

Let QR be any chord of the parabola Eq. (i). Let the extremities Q and R be $(\lambda t_1^2, 2\lambda t_1)$ and $(\lambda t_2^2, 2\lambda t_2)$.

Then, the equation of QR is

$$y(t_1 + t_2) - 2x - 2\lambda t_1 t_2 = 0 \quad \ldots(ii)$$

It meets the diameter through P i.e. X-axis or $y = 0$, then Eq. (ii) reduces

$$0 - 2x - 2\lambda\, t_1 t_2 = 0$$

$$\Rightarrow \quad x = -\lambda t_1 t_2 = PA$$

Now, tangent at Q is $t_1 y = x + \lambda t_1^2 \quad \ldots(iii)$

It meets the diameter through P i.e. X-axis or $y = 0$, then Eq. (iii) reduces

$$0 = x + \lambda t_1^2$$

$$\Rightarrow \quad x = -\lambda t_1^2 = PB \quad \ldots(iv)$$

Similarly, $-\lambda t_2^2 = PC \quad \ldots(v)$

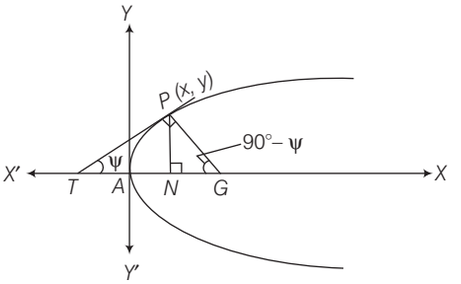
$$\therefore \quad (PA)^2 = (-\lambda t_1 t_2)^2 = \lambda^2 t_1^2 t_2^2$$

$$= (-\lambda t_1^2)(-\lambda t_2^2) \quad \text{[from Eqs. (iv) and (v)]}$$

$$= PB \cdot PC$$

Lengths of Tangent, Subtangent, Normal and Subnormal

Let the parabola $y^2 = 4ax$. Let the tangent and normal at $P(x, y)$ meet the axis of parabola at T and G respectively and let tangent at $P(x, y)$ makes angle ψ with the positive direction of X-axis.



Then, PT = Length of Tangent

PG = Length of Normal

TN = Length of Subtangent

and NG = Length of Subnormal

If $A(0, 0)$ is the vertex of the parabola.

$\because \quad PN = y$

$\therefore \quad PT = PN\ \text{cosec}\ \psi = y\ \text{cosec}\ \psi$

$PG = PN\ \text{cosec}\ (90° - \psi) = y\ \sec\ \psi$

$TN = PN\cot\psi = y\cot\psi$

and $NG = PN\cot(90° - \psi) = y\tan\psi$

where, $\tan\psi = \frac{2a}{y} = m$ [slope of tangent at $P(x, y)$]

Example 53 Find the length of tangent, subtangent, normal and subnormal to $y^2 = 4ax$ at $(at^2, 2at)$.

Sol. $\because$ Equation of tangent of $(at^2, 2at)$ of parabola $y^2 = 4ax$ is

$$ty = x + at^2$$

Slope of this tangent $m = \frac{1}{t}$

Let tangent makes angle ψ with positive direction of X-axis

$$\tan\psi = \frac{1}{t}$$

then $t = \cot\psi$

$\therefore$ Length of tangent at

$$(at^2, 2at) = 2at \operatorname{cosec}\psi$$

$$= 2at\sqrt{(1+\cot^2\psi)} = 2at\sqrt{(1+t^2)}$$

Length of normals at

$$(at^2, 2at) = 2at \sec\psi = 2at\sqrt{(1+\tan^2\psi)}$$

$$= 2a\sqrt{(t^2 + t^2\tan^2\psi)} = 2a\sqrt{(t^2+1)}$$

Length of subtangent at $(at^2, 2at) = 2at \cot\psi$

$$= 2at^2$$

Length of subnormal at $(at^2, 2at) = 2at \tan\psi$

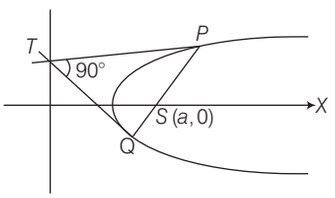
$$= 2a$$

Some Standard Properties of the Parabola

(1) The tangents at the extremities of a focal chord intersect at right angles on the directrix.

The extremities of a focal chord PQ are

$$P \equiv (at^2, 2at) \text{ and } Q \equiv \left(\frac{a}{t^2}, -\frac{2a}{t}\right)$$



Let parabola is $y^2 = 4ax$

$\therefore$ Tangent at $P(at^2, 2at)$ is

$$ty = x + at^2 \quad \ldots(i)$$

Its slope $\frac{1}{t} = m_1$ (say)

and tangent at $Q\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$ is

$$-\frac{y}{t} = x + \frac{a}{t^2} \quad \ldots(ii)$$

$\Rightarrow$ $-ty = xt^2 + a$

Its slope $-t = m_2$ (say)

$\because$ $m_1 m_2 = -1$

$\therefore$ $\angle PTQ = 90°$

and adding Eqs. (i) and (ii), we get

$$(x+a)(1+t^2) = 0 \implies 1 + t^2 \neq 0$$

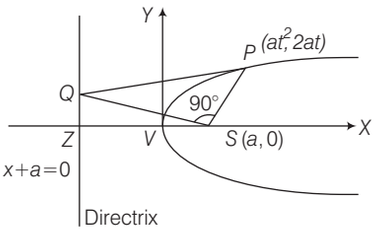
$\therefore$ $x + a = 0$, which is directrix.

(2) The portion of a tangent to a parabola intercepted between the directrix and the curve subtends a right angle at the focus.

The equation of the tangent to the parabola $y^2 = 4ax$ at $P(at^2, 2at)$ is

$$ty = x + at^2 \quad \ldots(i)$$

Let Eq. (i) meet the directrix $x + a = 0$ at Q, then coordinates of Q are $\left(-a, \frac{at^2 - a}{t}\right)$, also focus S is $(a, 0)$.



$\therefore$ Slope of $SP = \frac{2at - 0}{at^2 - a}$

$$= \frac{2t}{t^2 - 1} = m_1 \quad \text{(say)}$$

and slope of $SQ = \frac{\frac{at^2 - a}{t} - 0}{-a - a} = \frac{t^2 - 1}{-2t} = m_2$ (say)

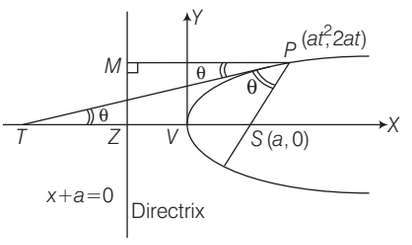
$\therefore$ $m_1 m_2 = -1$

i.e. SP is perpendicular to SQ i.e. $\angle PSQ = 90°$

(3) The tangent at any point P of a parabola bisects the angle between the focal chord through P and the perpendicular from P to the directrix.

Let the tangent at $P(at^2, 2at)$ to the parabola $y^2 = 4ax$ meets the axis of the parabola i.e. X-axis or $y = 0$ at T. The equation of tangent to the parabola $y^2 = 4ax$ at $P(at^2, 2at)$ is

$$ty = x + at^2$$



For coordinates of T solve it with $y = 0$.

$\therefore \quad T(-at^2, 0)$

$\therefore \quad ST = SV + VT = a + at^2 = a(1 + t^2)$

Also, $SP = PM = a + at^2 = a(1 + t^2)$

$\therefore \quad SP = ST$, i.e. $\angle STP = \angle SPT$

But $\angle STP = \angle MPT$ [alternate angles]

$\angle SPT = \angle MPT$

(4) The foot of the perpendicular from the focus on any tangent to a parabola lies on the tangent at the vertex.

Equation of tangent at $P(at^2, 2at)$

On the parabola $y^2 = 4ax$

is $ty = x + at^2$

$\Rightarrow \quad x - ty + at^2 = 0$ ...(i)

Now, the equation of line through $S(a, 0)$ and perpendicular to Eq. (i) is

$tx + y = \lambda$

Since, it passes through $(a, 0)$.

$\therefore \quad ta + 0 = \lambda$

$\therefore$ Equation $tx + y = ta$ or $t^2x + ty - at^2 = 0$ ...(ii)

By adding Eqs. (i) and (ii), we get

$x(1 + t^2) = 0$

$\Rightarrow \quad x = 0 \quad [\because 1 + t^2 \neq 0]$

Hence, the point of intersection of Eq. (i) and (ii) lies on $x = 0$ i.e. on Y-axis (which is tangent at the vertex of a parabola).

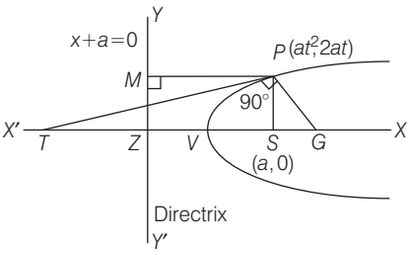
(5) If S be the focus of the parabola and tangent and normal at any point P meet its axis in T and G respectively then $ST = SG = SP$.

Let $P(at^2, 2at)$ be any point on the parabola $y^2 = 4ax$, then equation of tangent and normal at $P(at^2, 2at)$ are

$ty = x + at^2$ and $y = -tx + 2at + at^2$, respectively.

Since, tangent and normal meet its axis in T and G.

$\therefore$ Coordinates of T' and G are $(-at^2, 0)$ and $(2a + at^2, 0)$ respectively.



$\therefore \quad SP = PM = a + at^2$

$SG = VG - VS = 2a + at^2 - a$

$= a + at^2$

and $ST = VS + VT = a + at^2$

Hence, $SP = SG = ST$

(6) If S be the focus and SH be perpendicular to the tangent at P, then H lies on the tangent at the vertex and $SH^2 = OS \cdot SP$, where O is the vertex of the parabola.

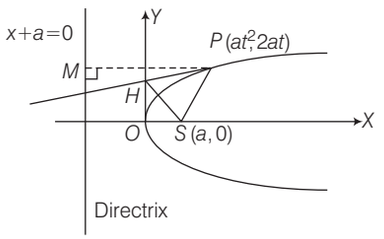
Let $P(at^2, 2at)$ be any point on the parabola

$y^2 = 4ax$ ... (i)

then, tangent at $P(at^2, 2at)$ to the parabola Eq. (i) is

$ty = x + at^2$

It meets the tangent at the vertex i.e. $x = 0$.



$\therefore$ Coordinates of H is $(0, at)$

and $SP = PM = a + at^2 \Rightarrow \quad OS = a$

and $SH = \sqrt{(a-0)^2 + (0-at)^2} = \sqrt{a^2 + a^2t^2}$

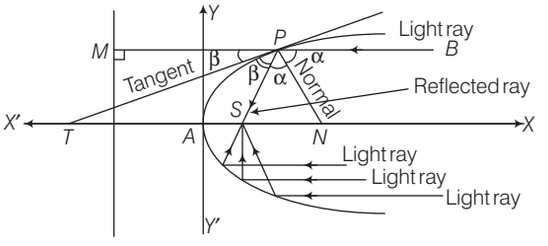
or $(SH)^2 = a\{a(1 + t^2)\} = OS \cdot SP$.

Reflection Property of Parabola

The tangent (PT) and normal (PN) of the parabola

$y^2 = 4ax$

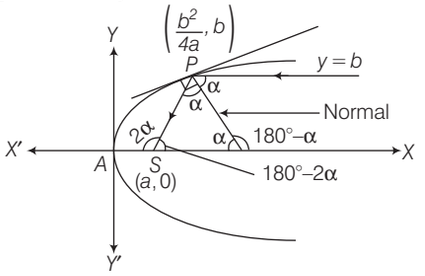
at P are the internal and external bisectors of $\angle SPM$ and BP is parallel to the axis of the parabola and $\angle BPN = \angle SPN$.



All rays of light coming from the positive direction of X-axis and parallel to the axis of the parabola after reflection pass through the focus of the parabola.

Example 54 ***A ray of light is coming along the line $y = b$ from the positive direction of X-axis and strikes a concave mirror whose intersection with the xy-plane is a parabola $y^2 = 4ax$. Find the equation of the reflected ray and show that it passes through the focus of the parabola. Both a and b are positive.***

Sol. Given parabola is $y^2 = 4ax$



Equation of tangent at $P\left(\frac{b^2}{4a}, b\right)$ is $yb = 2a\left(x + \frac{b^2}{4a}\right)$,

Slope of tangent is $\frac{2a}{b}$.

Hence, slope of normal $= -\frac{b}{2a} = \tan(180° - \alpha)$.

$\therefore \quad \tan\alpha = \frac{b}{2a}$

$\therefore$ Slope of reflected ray $= \tan(180° - 2\alpha)$

$$= -\tan 2\alpha$$

$$= -\left\{\frac{2\tan\alpha}{1-\tan^2\alpha}\right\} = -\left\{\frac{2\cdot\frac{b}{2a}}{1-\frac{b^2}{4a^2}}\right\} = -\frac{4ab}{(4a^2-b^2)}$$

Hence, equation of reflected ray is

$$y - b = -\frac{4ab}{(4a^2-b^2)}\left(x - \frac{b^2}{4a}\right)$$

$\Rightarrow \quad (y-b)(4a^2-b^2) = -(4ax - b^2)$

which obviously passes through the focus $S(a, 0)$.

Study of Parabola of the Form $(\alpha x + \beta y)^2 + 2gx + 2fy + c = 0$

Given equation can be written as

$$(\alpha x + \beta y)^2 = -2gx - 2fy - c$$

Now, add an arbitrary constant λ in the square root of the second degree terms. Then the equation will be of the form

i.e. $\quad (\alpha x + \beta y + \lambda)^2 = xf_1(\lambda) + yf_2(\lambda) + f_3(\lambda) \quad$ ...(i)

Now, choose λ such that the lines

$\alpha x + \beta y + \lambda = 0$ and $xf_1(\lambda) + yf_2(\lambda) + f_3(\lambda) = 0$

are perpendicular

i.e. (slope of $\alpha x + \beta y + \lambda = 0$) $\times$ (slope of $xf_1(\lambda) + yf_2(\lambda) + f_3(\lambda) = 0$) $= -1$

$\Rightarrow \quad -\frac{\alpha}{\beta} \times -\frac{f_1(\lambda)}{f_2(\lambda)} = -1$

$\Rightarrow \quad \alpha f_1(\lambda) + \beta f_2(\lambda) = 0 \quad$ ...(ii)

Now, substitute the value of λ in Eq. (i) from Eq. (ii).

Multiply and divide $(\alpha^2 + \beta^2)$ in LHS of Eq. (i)

i.e. $\quad (\alpha x + \beta y + \lambda)^2 = (\alpha^2 + \beta^2)\left(\frac{\alpha x + \beta y + \lambda}{\sqrt{\alpha^2 + \beta^2}}\right)^2$

and RHS of Eq. (i) by $\sqrt{(\alpha^2 + \beta^2)}$

i.e. $\quad xf_1(\lambda) + yf_2(\lambda) + f_3(\lambda)$

$$= \sqrt{(\alpha^2 + \beta^2)}\left(\frac{xf_1(\lambda) + yf_2(\lambda) + f_3(\lambda)}{\sqrt{(\alpha^2 + \beta^2)}}\right)$$

Then, Eq. (i) reduce in the form

$$\left(\frac{\alpha x + \beta y + \lambda}{\sqrt{\alpha^2 + \beta^2}}\right)^2 = 4\rho\left(\frac{\beta x - \alpha y + \mu}{\sqrt{(\alpha^2 + \beta^2)}}\right)$$

which is of the form $Y^2 = 4\rho X$

$$Y = \frac{\alpha x + \beta y + \lambda}{\sqrt{(\alpha^2 + \beta^2)}}, \quad X = \frac{\beta x - \alpha y + \mu}{\sqrt{(\alpha^2 + \beta^2)}} \quad \text{and} \quad 4\rho = \frac{1}{\sqrt{(\alpha^2 + \beta^2)}}.$$

Latusrectum is $4\rho = \frac{1}{\sqrt{(\alpha^2 + \beta^2)}}$.

Axis is $Y = 0$ or $\alpha x + \beta y + \lambda = 0$.

Equation of tangent at vertex is

$$X = 0 \quad \text{or} \quad \beta x - \alpha y + \mu = 0.$$

Vertex is the point of intersection of

$$X = 0 \quad \text{and} \quad Y = 0$$

i.e. $\beta x - \alpha y + \mu = 0$ and $\alpha x + \beta y + \lambda = 0$.

Equation of directrix is $X + \rho = 0$.

Equation of latusrectum is $X - \rho = 0$.

Focus Since, axis and latusrectum intersect at the focus S its coordinates are detained by solving

$$X - \rho = 0 \quad \text{and} \quad Y = 0.$$

Example 55. ***Find the length of latusrectum of the parabola $(a^2 + b^2)(x^2 + y^2) = (bx + ay - ab)^2$.***

Sol. Given equation may be written as

$$a^2x^2 + a^2y^2 + b^2x^2 + b^2y^2 = b^2x^2 + a^2y^2 + a^2b^2 + 2abxy - 2a^2by - 2ab^2x$$

$$\Rightarrow \quad a^2x^2 - 2abxy + b^2y^2 = -2ab^2x - 2a^2by + a^2b^2$$

$$\Rightarrow \quad (ax - by)^2 = -2ab\left(bx + ay - \frac{ab}{2}\right)$$

Since, $ax - by = 0$ and $bx + ay - \frac{ab}{2} = 0$ are perpendicular.

$$\therefore (a^2 + b^2)\left(\frac{ax - by}{\sqrt{a^2 + b^2}}\right)^2 = -2ab\sqrt{(a^2 + b^2)}\left(\frac{bx + ay - \frac{ab}{2}}{\sqrt{(a^2 + b^2)}}\right)$$

$$\Rightarrow \quad \left(\frac{ax - by}{\sqrt{a^2 + b^2}}\right)^2 = \frac{-2ab}{\sqrt{(a^2 + b^2)}}\left(\frac{bx + ay - \frac{ab}{2}}{\sqrt{(a^2 + b^2)}}\right)$$

which is of the form $Y^2 = -4\rho X$.

Therefore, the latusrectum $= 4\rho = \frac{2ab}{\sqrt{(a^2 + b^2)}}$.

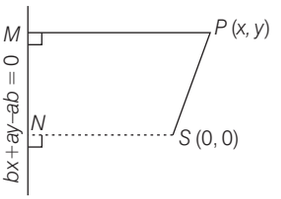
Aliter :

Given equation may be written as

$$x^2 + y^2 = \frac{(bx + ay - ab)^2}{(a^2 + b^2)}$$

$$\Rightarrow \sqrt{(x^2 + y^2)} = \frac{|bx + ay - ab|}{\sqrt{(a^2 + b^2)}}$$

$$\Rightarrow \quad \sqrt{(x - 0)^2 + (y - 0)^2} = \frac{|bx + ay - ab|}{\sqrt{(a^2 + b^2)}}$$

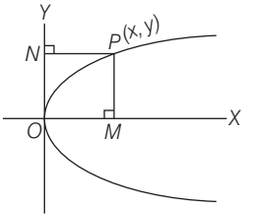


which is of the form $SP = PM$.

Since, distance from focus S to $(bx + ay - ab = 0) = \frac{1}{2}(4\rho)$

$$\Rightarrow \quad \frac{1}{2}(4\rho) = \frac{ab}{\sqrt{a^2 + b^2}}$$

$$\therefore \quad 4\rho = \frac{2ab}{\sqrt{(a^2 + b^2)}}$$



Remark

Consider the equation of parabola is $y^2 = 4ax$.

i.e. $(MP)^2 =$ (Latusrectum) NP.

i.e. if P is any point on the given parabola, then

(the distance of P from its axis)^2 = (Latusrectum)

(The distance of P from the tangent at its vertex).

Exercise for Session 3

1. If m_1, m_2 are slopes of the two tangents that are drawn from (2, 3) to the parabola $y^2 = 4x$, then the value of $\frac{1}{m_1} + \frac{1}{m_2}$ is

(a) –3 (b) 3 (c) $\frac{2}{3}$ (d) $\frac{3}{2}$

2. The angle between the tangents drawn from the origin to the parabola $y^2 = 4a(x - a)$ is

(a) 90° (b) 30° (c) $\tan^{-1}\left(\frac{1}{2}\right)$ (d) 45°

3. If (a, b) is the mid-point of chord passing through the vertex of the parabola $y^2 = 4x$, then

(a) $a = 2b$ (b) $2a = b$ (c) $a^2 = 2b$ (d) $2a = b^2$

4. The diameter of the parabola $y^2 = 6x$ corresponding to the system of parallel chords $3x - y + c = 0$ is

(a) $y - 1 = 0$ (b) $y - 2 = 0$ (c) $y + 1 = 0$ (d) $y + 2 = 0$

5. From the point $(-1, 2)$ tangent lines are drawn to the parabola $y^2 = 4x$, the area of triangle formed by chord of contact and the tangents is given by

(a) 8 (b) $8\sqrt{3}$ (c) $8\sqrt{2}$ (d) None of these

6. For parabola $x^2 + y^2 + 2xy - 6x - 2y + 3 = 0$, the focus is

(a) (1, – 1) (b) (–1, 1) (c) (3, 1) (d) None of these

7. The locus of the mid-point of that chord of parabola which subtends right angle on the vertex will be

(a) $y^2 - 2ax + 8a^2 = 0$ (b) $y^2 = a(x - 4a)$ (c) $y^2 = 4a(x - 4a)$ (d) $y^2 + 3ax + 4a^2 = 0$

8. A ray of light moving parallel to the X-axis gets reflected from a parabolic mirror whose equation is $(y-2)^2 = 4(x+1)$. After reflection, the ray must pass through the point

(a) (–2, 0) (b) (–1, 2) (c) (0, 2) (d) (2, 0)

9. Prove that the locus of the point of intersection of tangents to the parabola $y^2 = 4ax$ which meet at an angle α is $(x+a)^2 \tan^2\alpha = y^2 - 4ax$.

10. Find the locus of the middle points of the chords of the parabola $y^2 = 4ax$ which pass through the focus.

11. From the point $P(-1,2)$ tangents are drawn to the parabola $y^2 = 4x$. Find the equation of the chord of contact. Also, find the area of the triangle formed by the chord of contact and the tangents.

Shortcuts and Important Results to Remember

1 Second degree terms in the equation of a parabola should make perfect squares.

2 If l_1 and l_2 are the lengths of segments of a focal chord then the latusrectum of the parabola is $\frac{4l_1l_2}{l_1+l_2}$.

3 If α be the inclination of a focal chord with axis of the parabola then its length is $4a\text{cosec}^2\alpha$.

4 If tangents of $y^2 = 4ax$ at $P(t_1)$ and $Q(t_2)$ meets at R, then area of ΔPQR is $\frac{1}{2}a^2(t_1-t_2)^3$.

5 The foot of the perpendicular from the focus on any tangent to a parabola lies on the tangent at the vertex.

6 The area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents, at these points.

7 The equation of the common tangent to the parabolas $x^2 = 4ay$ and $y^2 = 4ax$ is $x + y + a = 0$.

8 If the chord joining t_1, t_2 and t_3, t_4 pass through a point $(c, 0)$ on the axis, then

$$t_1t_2 = t_3t_4 = -c/a.$$

9 If the normals to the parabola $y^2 = 4ax$ at the points t_1 and t_2 intersect again on the parabola at the point 't_3', then $t_1t_2 = 2; t_3 = -(t_1+t_2)$ and the line joining t_1 and t_2 passes through a fixed point $(-2a, 0)$.

10 If the length of a focal chord of $y^2 = 4ax$ at a distance b from the vertex is c then $b^2c = 4a^3$.

11 From an external point only one normal can be drawn.

12 If a normal to $y^2 = 4ax$ makes an angle θ with the axis of $y^2 = 4ax$ then it will cut the curve again at an angle of $\tan^{-1}\left(\frac{\tan\theta}{2}\right)$.

13 The orthocentre of any triangle formed by the three tangents to a parabola $y^2 = 4ax$ 't_1', 't_2' and 't'$_3$ lies on the directrix and has the coordinates $(-a, a(t_1+t_2+t_3+t_1t_2t_3))$.

14 Normals at the end points of the latusrectum of a parabola $y^2 = 4ax$ intersect at right angle on the axis of the parabola and their point of intersection is $(3a, 0)$.

15 A line ray parallel to axis of the parabola after reflection passes through the focus.

JEE Type Solved Examples :
Single Option Correct Type Questions

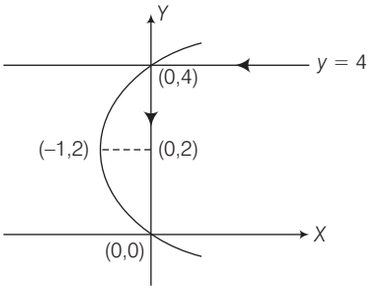
■ This section contains **10 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

● ***Ex. 1*** *A ray of light travels along a line $y = 4$ and strikes the surface of curves $y^2 = 4(x + y)$, then the equation of the line along which the reflected ray travels is*

(a) $x = 0$ (b) $x = 2$

(c) $x + y = 4$ (d) $2x + y = 4$

Sol. (a) The given curve is $(y-2)^2 = 4(x+1)$



The focus is $(0, 2)$.

The point of intersection of the curve and $y = 4$ is $(0, 4)$. From the reflection property of parabola the reflected ray passes through the focus.

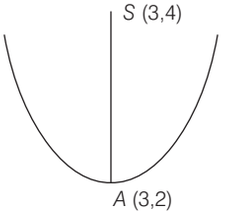
Therefore, $x = 0$ is the reflected ray.

● ***Ex. 2*** *A parabola is drawn with focus at $(3, 4)$ and vertex at the focus of the parabola $y^2 - 12x - 4y + 4 = 0$. The equation of the parabola is*

(a) $x^2 - 6x - 8y + 25 = 0$ (b) $y^2 - 8x - 6y + 25 = 0$

(c) $x^2 - 6x + 8y - 25 = 0$ (d) $x^2 + 6x - 8y - 25 = 0$

Sol. (a) $y^2 - 12x - 4y + 4 = 0 \Rightarrow (y-2)^2 = 12x$



Its vertex is $(0, 2)$ and $a = 3$,

its focus $= (3, 2)$.

Hence, for the required parabola; focus is $(3, 4)$, vertex $= (3, 2)$ and $a = 2$,

Hence, the equation of the parabola is

$$(x-3)^2 = 4(2)(y-2)$$

or $x^2 - 6x - 8y + 25 = 0$

● ***Ex. 3*** *Two parabolas have the same focus. If their directrices are the X-axis and the Y-axis, respectively, then the slope of their common chord is*

(a) ± 1 (b) $\frac{4}{3}$ (c) $\frac{3}{4}$ (d) None of these

Sol. (a) Let the focus be (a, b)

Then, equations are $(x-a)^2 + (y-b)^2 = y^2$ and

$$(x-a)^2 + (y-b)^2 = x^2$$

If $S_1 \equiv (x-a)^2 + (y-b)^2 - y^2 = 0$

and $S_2 \equiv (x-a)^2 + (y-b)^2 - x^2 = 0$

$\therefore$ Equation of common chord $S_1 - S_2 = 0$ gives

$$x^2 - y^2 = 0 \quad \text{or} \quad y = \pm x$$

Hence, slope of common chord is ± 1.

● ***Ex. 4*** *Let us define a region R in xy-plane as a set of points (x, y) satisfying $[x^2] = [y]$ (where $[x]$ denotes greatest integer $\le x$), then the region R defines*

(a) a parabola whose axis is horizontal

(b) a parabola whose axis is vertical

(c) integer point of the parabola $y = x^2$

(d) None of the above

Sol. (d) $\because [x^2] = [y]$

If $0 \le y < 1$,

then, $[y] = 0$

$\therefore$ $[x^2] = 0$

$0 \le x^2 < 1$

$\Rightarrow$ $x \in (-1, 1)$

for $1 \le y < 2$,

then $[y] = 1$

$\therefore$ $[x^2] = 1$

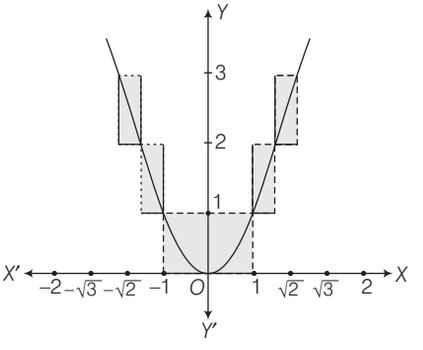
$\Rightarrow$ $1 \le x^2 < 2$

$\Rightarrow$ $x \in (-\sqrt{2}, -1] \cup [1, \sqrt{2})$

for $2 \le y < 3$, then $[y] = 2$

then, $[x^2] = 2 \Rightarrow 2 \le x^2 < 3$

$\therefore$ $x \in (-\sqrt{3}, -\sqrt{2}] \cup [\sqrt{2}, \sqrt{3})$



The graph of the region will not only contain of the parabola $y = x^2$ but $[x^2] = [y]$ contain points within the rectangles of side 1, 2, ; 1, $\sqrt{2} - 1$; 1, $\sqrt{3} - \sqrt{2}$ etc.

Hence, a, b, c are incorrects.

● ***Ex. 5*** *The minimum area of circle which touches the parabolas $y = x^2 + 1$ and $x = y^2 + 1$ is*

(a) $\frac{9\pi}{16}$ sq units (b) $\frac{9\pi}{32}$ sq units

(c) $\frac{9\pi}{8}$ sq units (d) $\frac{9\pi}{4}$ sq units

Sol. (b) The parabolas $y = x^2 + 1$ and $x = y^2 + 1$ are symmetrical about $y = x$.

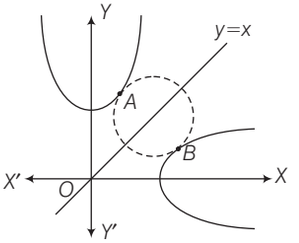
Therefore, the tangent at point A is parallel to $y = x$

then $\frac{dy}{dx} = 2x = 1$

or $x = \frac{1}{2}, y = \frac{5}{4}$

$\therefore$ $A \equiv \left(\frac{1}{2}, \frac{5}{4}\right)$

and $B \equiv \left(\frac{5}{4}, \frac{1}{2}\right)$



Hence, Radius $(r) = \frac{1}{2} AB = \frac{1}{2}\sqrt{\left(\frac{1}{2} - \frac{5}{4}\right)^2 + \left(\frac{5}{4} - \frac{1}{2}\right)^2} = \frac{3\sqrt{2}}{8}$

$\therefore$ Area $= \pi r^2 = \frac{9\pi}{32}$ sq units

● ***Ex. 6*** *Let the line $lx + my = 1$ cut the parabola $y^2 = 4ax$ in the points A and B. Normals at A and B meet at point C. Normal from C other than these two meet the parabola at D, then the coordinate of D are*

(a) $(a, 2a)$ (b) $\left(\frac{4am}{l^2}, \frac{4a}{l}\right)$

(c) $\left(\frac{2am^2}{l^2}, \frac{2a}{l}\right)$ (d) $\left(\frac{4am^2}{l^2}, \frac{4am}{l}\right)$

Sol. (d) Let $A \equiv (am_1^2, -2am_1)$ and $B \equiv (am_2^2, -2am_2)$

Now, A and B lie on $lx + my = 1$

$\Rightarrow$ $l(am_1^2) + m(-2am_1) = 1$ ...(i)

and $l(am_2^2) + m(-2am_2) = 1$ ...(ii)

Subtracting Eq. (ii) from Eq. (i), then

$la(m_1^2 - m_2^2) - 2am(m_1 - m_2) = 0$

$\Rightarrow$ $a(m_1 - m_2) \neq 0$

$\therefore$ $l(m_1 + m_2) - 2m = 0$

$\Rightarrow$ $m_1 + m_2 = \frac{2m}{l}$ ...(iii)

Let $D \equiv (am_3^2, -2am_3)$ and $C \equiv (h, k)$

$\therefore$ Equation of normal in terms of slope

$y = Mx - 2aM - aM^3$

then $aM^3 - (h - 2a)M + k = 0$

$\therefore$ $m_1 + m_2 + m_3 = 0$

$\Rightarrow$ $\frac{2m}{l} + m_3 = 0$

$\therefore$ $m_3 = -\frac{2m}{l}$

$\therefore$ $D \equiv \left(a\left(\frac{-2m}{l}\right)^2, -2a\left(\frac{-2m}{l}\right)\right)$

$\Rightarrow$ $D \equiv \left(\frac{4am^2}{l^2}, \frac{4am}{l}\right)$

● ***Ex. 7*** *If d is the distance between the parallel tangents with positive slope to $y^2 = 4x$ and $x^2 + y^2 - 2x + 4y - 11 = 0$, then*

(a) $10 < d < 20$ (b) $4 < d < 6$

(c) $d < 4$ (d) None of these

Sol. (c) Tangent to the parabola $y^2 = 4x$ having slope m is

$$y = mx + \frac{1}{m}$$

and tangent to the circle $(x-1)^2 + (y+2)^2 = 4^2$ having slope m is

$$y + 2 = m(x - 1) + 4\sqrt{(1 + m^2)}$$

$\therefore$ Distance between tangents $(d) = \left|\frac{4\sqrt{(1+m^2)} - m - 2 - \frac{1}{m}}{\sqrt{(1+m^2)}}\right|$

$$= \left|4 - \frac{2}{\sqrt{(1+m^2)}} - \frac{\sqrt{(1+m^2)}}{m}\right|$$

As $m > 0$ (given)

we get $d < 4$

● ***Ex. 8*** *Two parabolas C and D intersect at two different points, where C is $y = x^2 - 3$ and D is $y = kx^2$. The intersection at which the x value is positive is designated point A , and $x = a$ at this intersection the tangent line l at A to the curve D intersects curve C at point B, other than A. If x-value of point B is 1, then a is equal to*

(a) 1 (b) 2 (c) 3 (d) 4

Sol. (c) $C : y = x^2 - 3$ and $D : y = kx^2$

Solving C and D, then

$kx^2 = x^2 - 3$ ...(i)

or $x^2 = \frac{3}{1-k}$,

then, $y = \frac{3k}{1-k}$

$\therefore$ $A \equiv \left(\sqrt{\frac{3}{1-k}}, \frac{3k}{1-k}\right)$ (given x-value of A is positive)

and $a = \sqrt{\left(\frac{3}{1-k}\right)}$

then $A \equiv (a, ka^2) \equiv (a, a^2 - 3)$ [from Eq. (i)]

tangent 'l' at A to the curve D is

$$\frac{y+a^2-3}{2}=kx\cdot a$$

$$\Rightarrow \quad y+a^2-3=2ax\left(1-\frac{3}{a^2}\right) \qquad \text{[from Eq. (i)]} \quad ...(ii)$$

$$\therefore \quad B\equiv(1,-2)\ (a\neq 1)$$

From Eq. (ii), $\quad -2+a^2-3=2a\left(1-\frac{3}{a^2}\right)$

$$\Rightarrow \quad a^3-5a=2a^2-6$$

$$\Rightarrow \quad a^3-2a^2-5a+6=0$$

$$\Rightarrow \quad (a-1)(a+2)(a-3)=0$$

$$\therefore \quad a=3 \qquad (\because a\neq 1, a\neq -2)$$

● ***Ex. 9*** $\min[(x_1-x_2)^2+(3+\sqrt{(1-x_1^2)}-\sqrt{4x_2})^2]$, $\forall$ $x_1, x_2 \in R$, *is*

(a) $4\sqrt{5}+1$ (b) $3-2\sqrt{2}$ (c) $\sqrt{5}+1$ (d) $\sqrt{5}-1$

Sol. (b) Let $y_1=3+\sqrt{(1-x_1^2)}$ and $y_2=\sqrt{4x_2}$

or $x_1^2+(y_1-3)^2=1$ and $y_2^2=4x_2$

Thus, (x_1, y_1) lies on the circle $x^2+(y-3)^2=1$ and (x_2, y_2) lies on the parabola $y^2=4x$.

Thus, the given expression is the shortest distance between the curve $x^2+(y-3)^2=1$ and $y^2=4x$.

Now, the shortest distance always occurs along the common normal to the curves and normal to the circle passes through the centre of the circle.

Normal to parabola $y^2=4x$ is $y=mx-2m-m^3$. It passes through centre of circle $(0,3)$.

Therefore, $3=-2m-m^3 \Rightarrow m^3+2m+3=0$ which has only one real value $m=-1$.

Hence, the corresponding point on the parabola is $(1,2)$

$$\therefore \sqrt{(x_1-x_2)^2+(3+\sqrt{(1-x_1^2)}-\sqrt{4x_2})^2}=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

= Distance between $(1,2)$ and $(0,3)$ – radius

$$=\sqrt{(1+1)}-1=(\sqrt{2}-1)$$

or $\quad (x_1-x_2)^2+(3+\sqrt{(1-x_1^2)}-\sqrt{4x_2})^2=(\sqrt{2}-1)^2$

$$\therefore \quad \min\cdot[(x_1-x_2)^2+(3+\sqrt{(1-x_1^2)}-\sqrt{4x_2})^2]=3-2\sqrt{2}$$

● ***Ex. 10*** *The condition that the parabolas* $y^2=4c(x-d)$ *and* $y^2=4ax$ *have a common normal other than X-axis* $(a>0, c>0)$ *is*

(a) $2a<2c+d$ (b) $2c<2a+d$

(c) $2d<2a+c$ (d) $2d<2c+a$

Sol. (a) Normals of parabolas $y^2=4ax$ and $y^2=4c(x-d)$ in terms of slope are

$$y=mx-2am-am^3 \qquad ...(i)$$

and $\quad y=m(x-d)-2cm-cm^3 \qquad ...(ii)$

Subtracting Eqs. (ii) from (i), then

$$md-2am-am^3+2cm+cm^3=0$$

$$m\neq 0$$

$$\therefore \quad d-2a-am^2+2c+cm^2=0$$

$$\Rightarrow \quad (a-c)m^2=d-2a+2c$$

$$\Rightarrow \quad m^2=\frac{d-2a+2c}{(a-c)}$$

$$\Rightarrow \quad \frac{d}{a-c}-2>0$$

$$\Rightarrow \quad d>2a-2c$$

$$\Rightarrow \quad 2a<2c+d$$

JEE Type Solved Examples :
More than One Correct Option Type Questions

▪ This section contains **5 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of **which MORE THAN ONE** may be correct.

● ***Ex. 11*** *The locus of the mid-point of the focal radii of a variable point moving on the parabola* $y^2=4ax$ *is a parabola whose*

(a) latusrectum is half the latusrectum of the original parabola

(b) vertex is $\left(\frac{a}{2}, 0\right)$

(c) directrix is Y-axis

(d) focus has the coordinates $(a, 0)$

Sol. (a, b, c, d) Let $P(at^2, 2at)$ be a point on the parabola $y^2=4ax$ with focus $S(a, 0)$

Now, mid-point of focal radii SP is $M\left(\frac{a}{2}(t^2+1), at\right)$.

Let $\quad x=\frac{a}{2}(t^2+1)$ and $y=at$

or $\quad x=\frac{a}{2}\left(\frac{y^2}{a^2}+1\right)$

$$\Rightarrow \quad y^2=2ax-a^2 \qquad \text{or} \quad y^2=2a\left(x-\frac{a}{2}\right)$$

which is a parabola with vertex $\left(\frac{a}{2}, 0\right)$ and latusrectum is $2a$, directrix is $x-\frac{a}{2}=\frac{a}{2}$ i.e. $x=0$ (Y) and focus $\left(\frac{a}{2}+\frac{a}{2}, 0\right)$ i.e. $(a, 0)$.

● ***Ex. 12*** *If P_1P_2 and Q_1Q_2, two focal chords of a parabola $y^2 = 4ax$ at right angles, then*

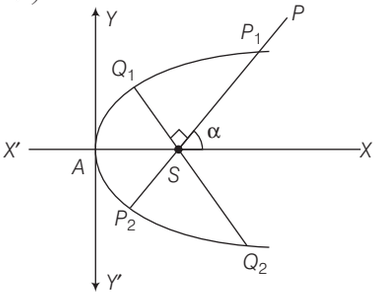
(a) area of the quadrilateral $P_1Q_1P_2Q_2$ is minimum when the chords are inclined at an angle $\pi/4$ to the axis of the parabola.

(b) minimum area is twice the area of the square on the latusrectum of the parabola.

(c) minimum area of quadrilateral $P_1Q_1P_2Q_2$ cannot be found.

(d) minimum area is thrice the area of the square on the latusrectum of the parabola.

Sol. (a, b) Let coordinates of P_1 are $(at^2, 2at)$, then coordinates of P_2 are $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$.



Let focal chord P_1P_2 makes an angle α with X-axis, then

$$\tan\alpha = \frac{2at - \left(-\frac{2a}{t}\right)}{at^2 - \frac{a}{t^2}} = \frac{2}{t - \frac{1}{t}}$$

$$\Rightarrow \quad t - \frac{1}{t} = 2\cot\alpha \qquad \ldots (i)$$

Now, $$P_1P_2 = a\left(t + \frac{1}{t}\right)^2 = a\left\{\left(t - \frac{1}{t}\right)^2 + 4\right\}$$

$$= a\{4\cot^2\alpha + 4\} = 4a\,\text{cosec}^2\alpha \quad [\text{from Eq. (i)}]$$

Similarly, $$Q_1Q_2 = 4a\,\text{cosec}^2(90 - \alpha) = 4a\sec^2\alpha$$

$\therefore$ Area of quadrilateral $P_1Q_1P_2Q_2 = \frac{1}{2}(P_1P_2)(Q_1Q_2)$

$$= 8a^2\sec^2\alpha\,\text{cosec}^2\alpha = 32a^2\,\text{cosec}^2\alpha$$

$\therefore$ Minimum area $= 32a^2 = 2(\text{latusrectum})^2$ and is inclined at

$$\alpha = \pi/4 \qquad (\because \text{cosec}\,2\alpha = 1)$$

● ***Ex. 13*** *The equation of the line that touches the curves $y = x|x|$ and $x^2 + (y-2)^2 = 4$, where $x \neq 0$, is*

(a) $y = 4\sqrt{5}x + 20$ (b) $y = 4\sqrt{3}x - 12$

(c) $y = 0$ (d) $y = -4\sqrt{5}x - 20$

Sol. (a, b, c) $\because y = x|x| = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$

$\because$ Equation of tangent in terms of slope (m) of $x^2 = 4ay$ is

$$y = mx - am^2 \qquad \ldots (i)$$

Also, line (i) touches the circle $x^2 + (y-2)^2 = 4$, then

$$\frac{|2 + am^2|}{\sqrt{(1 + m^2)}} = 2$$

$$\Rightarrow \quad 4 + a^2m^4 + 4am^2 = 4 + 4m^2$$

$$\therefore \quad m^2 = \frac{4 - 4a}{a^2} \text{ and } m^2 = 0$$

Put $4a = 1$ for $y = x^2$, $x \geq 0$, then $m^2 = 48$

and put $4a = -1$ for $y = -x^2$, $x < 0$, then $m^2 = 80$

$\therefore$ Common tangents are

$$y = 0,\ y = 4\sqrt{3}x - 12 \text{ and } y = 4\sqrt{5}x + 20$$

● ***Ex. 14*** *Let V be the vertex and L be the latusrectum of the parabola $x^2 = 2y + 4x - 4$. Then, the equation of the parabola whose vertex is at V, latusrectum $L/2$ and axis is perpendicular to the axis of the given parabola.*

(a) $y^2 = x - 2$ (b) $y^2 = x - 4$

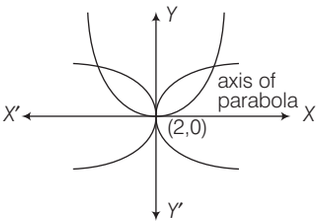
(c) $y^2 = 2 - x$ (d) $y^2 = 4 - x$

Sol. (a, c) Given parabola is $x^2 = 2y + 4x - 4$

$$\Rightarrow \quad (x-2)^2 = 2y$$

Vertex of the parabola is (2, 0) and length of latusrectum $= 2$

$$\therefore \quad V(2, 0) \text{ and } L = 2$$



Length of latusrectum of required parabola $= L/2 = 1$

$\therefore$ Equation of the required parabola is $(y-0)^2 = \pm 1(x-2)$

$$\Rightarrow \quad y^2 = x - 2 \text{ or } y^2 = -x + 2$$

● ***Ex. 15*** *Consider a circle with its centre lying on the focus of the parabola $y^2 = 2ax$ such that it touches the directrix of the parabola. Then a point of intersection of the circle and the parabola is*

(a) $\left(\frac{a}{2}, a\right)$ (b) $\left(\frac{a}{2}, -a\right)$

(c) $\left(-\frac{a}{2}, a\right)$ (d) $\left(-\frac{a}{2}, -a\right)$

Sol. (a, b) Given parabola is $y^2 = 2ax$ ...(i)

$\therefore$ Focus and equation of directrix are $S\left(\frac{a}{2}, 0\right)$ and $x = -\frac{a}{2}$ respectively.

$\therefore$ Equation of circle is $\left(x - \frac{a}{2}\right)^2 + (y-0)^2 = a^2$ ... (ii)

$(\because$ radius = distance between focus and directrix)

From Eqs. (i) and (ii),

$$\left(x-\frac{a}{2}\right)^2+2ax=a^2 \Rightarrow x^2+ax-\frac{3a^2}{4}=0$$

or $x=\frac{a}{2}, x\neq-\frac{3a}{2}$ ($\because$ for $x=-\frac{3a}{2}, y=$ imaginary)

From Eq. (i),

$$y^2=a^2$$

$\therefore$ $y=\pm a$

Hence, required point of intersection are $\left(\frac{a}{2}, \pm a\right)$.

JEE Type Solved Examples : Paragraph Based Questions

- This section contains **2 Solved Paragraphs** based upon each of the paragraph **3 multiple choice question.** Each of these questions has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Paragraph I

(Q. Nos. 16 to 18)

Tangents are drawn to the parabola $y^2=4x$ at the point P which is the upper end of latusrectum.

16. Image of the parabola $y^2=4x$ in the tangent line at the point P is

(a) $(x+4)^2=16y$ (b) $(x+2)^2=8(y-2)$

(c) $(x+1)^2=4(y-1)$ (d) $(x-2)^2=2(y-2)$

17. Radius of the circle touching the parabola $y^2=4x$ at the point P and passing through its focus is

(a) 1 (b) $\sqrt{2}$ (c) $\sqrt{3}$ (d) 2

18. Area enclosed by the tangent line at P, X-axis and the parabola is

(a) $\frac{2}{3}$ sq units (b) $\frac{4}{3}$ sq units

(c) $\frac{14}{3}$ sq units (d) $\frac{16}{3}$ sq units

Sol. Upper end of latusrectum is $P(1,2)$

$\therefore$ The equation of tangent at $P(1,2)$ is

$$y\cdot 2=2(x+1)\Rightarrow x-y+1=0$$

16. (c) Any point on the given parabola is $(t^2,2t)$. The image of (h,k) of the point $(t^2,2t)$ on $x-y+1=0$ is given by

$$\frac{h-t^2}{1}=\frac{k-2t}{-1}=\frac{-2(t^2-2t+1)}{1+1}$$

$\Rightarrow$ $h=2t-1$ and $k=t^2+1$

or $k=\left(\frac{h+1}{2}\right)^2+1 \Rightarrow (h+1)^2=4(k-1)$

The required equation of image is $(x+1)^2=4(y-1)$.

17. (c) $\because$ Focus is $S(1,0)$ and P is $(1,2)$.

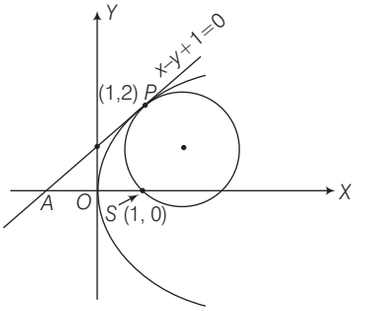
Equation of circle touching the parabola at $(1,2)$ is $(x-1)^2+(y-2)^2+\lambda(x-y+1)=0$ it passes through $(1,0)$.

Therefore, $4+2\lambda=0$ or $\lambda=-2$

Thus, the required circle is

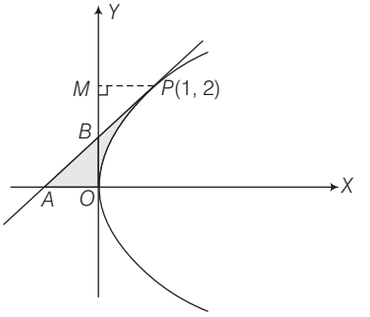
$(x-1)^2+(y-2)^2-2(x-y+1)=0$

or $x^2+y^2-4x-2y+3=0$



Its radius is $\sqrt{(4+1-3)}=\sqrt{2}$.

18. (a) Area bounded by $AOPA$ = Area of ΔAOB + Area of $OPBO$



$$=\frac{1}{2}\times1\times1+\int_0^2\frac{2y^2}{4}dy-\frac{1}{2}\times1\times1=\left[\frac{y^3}{12}\right]_0^2=\frac{8}{12}=\frac{2}{3}\text{ sq unit}$$

Paragraph II

(Q. Nos. 19 to 21)

Let C_1 and C_2 be respectively, the parabolas $x^2=y-1$ and $y^2=x-1$. Let P be any point on C_1 and Q be any point on C_2. Let P_1 and Q_1 be the reflections of P and Q, respectively with respect to the line $y=x$.

19. P_1 and Q_1 lie on

(a) C_1 and C_2 respectively (b) C_2 and C_1 respectively

(c) Cannot be determined (d) None of these

20. If the point $P(\lambda, \lambda^2+1)$ and $Q(\mu^2+1, \mu)$, then P_1 and Q_1 are

(a) (λ^2+1, λ) and (μ^2+1, μ) (b) (λ^2+1, λ) and (μ, μ^2+1)

(c) (λ, λ^2+1) and (μ, μ^2+1) (d) (λ, λ^2+1) and (μ^2+1, μ)

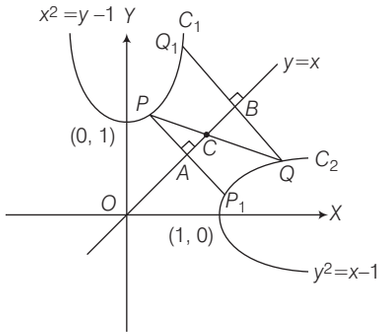
21. Arithmetic mean of PP_1 and QQ_1 is always less than

(a) PQ (b) $\frac{1}{2}PQ$ (c) $2PQ$ (d) $\frac{3}{2}PQ$

Sol. Since, the reflection of a point (p, q) with respect to line $y = x$ is (q, p).

Let $P(\lambda, \lambda^2+1)$ and $Q(\mu^2+1, \mu)$ be points on C_1 and C_2, respectively.

$\therefore$ Reflection of $P(\lambda, \lambda^2+1)$ with respect to line $y = x$ is $P_1(\lambda^2+1, \lambda)$ and reflection of $Q(\mu^2+1, \mu)$ with respect to line $y = x$ is $Q_1(\mu, \mu^2+1)$.



$\therefore$ P_1 and Q_1 are (λ^2+1, λ)

and (μ, μ^2+1) (**Ans.** 20(b))

Also, P_1 and Q_1 lie on $y^2 = x-1$

and $x^2 = y-1$.

Hence, P_1 and Q_1 lie on C_2 and C_1, respectively. (**Ans.** 19(b))

21. (a) $\because$ A is mid-point of PP_1 and B is mid-point of QQ_1.

$$\therefore \quad PA = \frac{1}{2}PP_1$$

$$\text{and} \quad QB = \frac{1}{2}QQ_1 \qquad \text{...(i)}$$

$$\Rightarrow \quad PC \geq PA \qquad \text{...(ii)}$$

$$\text{and} \quad QC \geq QB \qquad \text{...(iii)}$$

On adding Eqs. (ii) and (iii), then

$$PC + QC \geq PA + QB$$

$$= \frac{1}{2}PP_1 + \frac{1}{2}QQ_1 \quad \text{[from Eq. (i)]}$$

$$= \left(\frac{PP_1 + QQ_1}{2}\right)$$

$$\therefore \quad PC + QC \geq \left(\frac{PP_1 + QQ_1}{2}\right)$$

$$\Rightarrow \quad PQ \geq (\text{AM of } PP_1 \text{ and } QQ_1)$$

JEE Type Solved Examples :
Single Integer Answer Type Questions

- This section contains **2 solved examples.** The answer to each example is **a single digit integer,** ranging from 0 to 9 (both inclusive).

● ***Ex. 22*** *Points A, B, C lie on the parabola $y^2 = 4ax$. The tangents to the parabola at A, B, C taken in pairs intersect at points P, Q, R, then, the ratio of the areas of the ΔABC and ΔPQR is*

Sol. (2) Let $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ and $(at_3^2, 2at_3)$ be the points A, B and C respectively.

$$\therefore \text{Area of } \Delta ABC = \frac{1}{2}\left|\begin{vmatrix} at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \\ at_3^2 & 2at_3 & 1 \end{vmatrix}\right| = a^2\left|\begin{vmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \end{vmatrix}\right|$$

$$= a^2|(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)|$$

Coordinates of P, Q and R are $(at_2t_3, a(t_2+t_3))$, $(at_3t_1, a(t_3+t_1))$, $(at_1t_2, a(t_1+t_2))$ respectively, then

$$\text{Area of } \Delta PQR = \frac{1}{2}\left|\begin{vmatrix} at_2t_3 & a(t_2+t_3) & 1 \\ at_3t_1 & a(t_3+t_1) & 1 \\ at_1t_2 & a(t_1+t_2) & 1 \end{vmatrix}\right| = \frac{a^2}{2}\left|\begin{vmatrix} t_2t_3 & t_2+t_3 & 1 \\ t_3t_1 & t_3+t_1 & 1 \\ t_1t_2 & t_1+t_2 & 1 \end{vmatrix}\right|$$

$$= \frac{a^2}{2}|\Sigma t_2t_3(t_3 - t_2)|$$

$$= \frac{a^2}{2}|(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)|$$

$$= \frac{1}{2} \times \text{Area of } \Delta ABC$$

$$\therefore \quad \frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta PQR} = 2$$

● ***Ex. 23*** *If the orthocentre of the triangle formed by the points t_1, t_2, t_3 on the parabola $y^2 = 4ax$ is the focus, the value of $|t_1t_2 + t_2t_3 + t_3t_1|$ is*

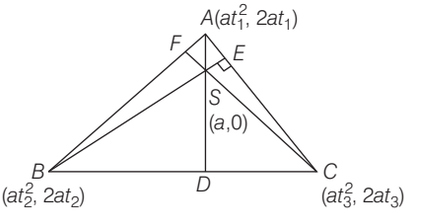
Sol. (5) $\because$ SA is perpendicular to BC[S is focus $(a, 0)$]

$$\Rightarrow \quad \left(\frac{2at_1 - 0}{at_1^2 - a}\right)\left(\frac{2at_3 - 2at_2}{at_3^2 - at_2^2}\right) = -1$$

$$\Rightarrow \quad \left(\frac{2t_1}{t_1^2 - 1}\right)\left(\frac{2}{t_3 + t_2}\right) = -1$$

$$\Rightarrow \quad 4t_1 + t_1^2(t_2 + t_3) = t_2 + t_3 \qquad \ldots(i)$$

Similarly, $\quad 4t_2 + t_2^2(t_3 + t_1) = t_3 + t_1 \qquad \ldots(ii)$



On subtracting Eq. (ii) from Eq. (i), then

$$4 + (t_1t_2 + t_2t_3 + t_3t_1) = -1 \Rightarrow t_1t_2 + t_2t_3 + t_3t_1 = -5$$

$$\therefore \quad |t_1t_2 + t_2t_3 + t_3t_1| = 5$$

JEE Type Solved Examples : Matching Type Questions

- This section contains **one solved example.** Example 24 has three statements (A, B and C) given in **Column I** and five statements (p, q, r, s and t) in **Column II.** Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II.**

● ***Ex. 24*** *Match the following.*

	Column I		Column II
(A)	If PQ is any focal chord of the parabola $y^2 = 32x$ and length of PQ can never be less than λ units, then λ is divisible by	(p)	2
(B)	A tangent is drawn to the parabola $y^2 = 4x$ at the point 'P' whose abscissa lies in the interval $[1, 4]$. If maximum possible area of the triangle formed by the tangent at 'P', ordinate of the point 'P' and the X-axis is λ sq units, then λ is divisible by	(q)	3
(C)	The normal at the ends of the latusrectum of the parabola $y^2 = 4x$ meet the parabola again at A and A'. If length $AA' = \lambda$ unit, then λ is divisible by	(r)	4
		(s)	6
		(t)	8

Sol. (A) $\rightarrow$ (p, r, t); (B) $\rightarrow$ (p, r, t); (C) $\rightarrow$ (p, q, r, s)

(A) Let $P(at^2, 2at)$ be any point on the parabola $y^2 = 4ax$ and $S(a, 0)$ be the focus, then the other end of focal chord through P will be $Q\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$.

Then, length of focal chord $PQ = a\left(t + \frac{1}{t}\right)^2$

$\because$ $\quad t + \frac{1}{t} \geq 2 \Rightarrow a\left(t + \frac{1}{t}\right)^2 \geq 4a$

or $\quad PQ \geq 32 \qquad [\because 4a = 32]$

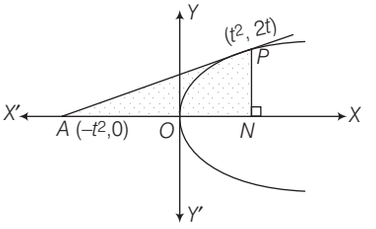
$\therefore \quad \lambda = 32$

(B) Equation of tangent to parabola $y^2 = 4x$ at $P(t^2, 2t)$ is

$$ty = x + t^2$$

Given $\quad t^2 \in [1, 4]$

$\therefore$ Area of $\Delta APN = \frac{1}{2}(AN)(PN)$



$$= \frac{1}{2}(2t^2)(2t)$$

$$\Rightarrow \quad \Delta = 2t^3 = 2(t^2)^{\frac{3}{2}}$$

$\because$ $\quad t^2 \in [1, 4]$

$\Rightarrow$ $\quad \Delta_{\max}$ occurs, when $t^2 = 4$

$$\Rightarrow \quad \Delta_{\max} = 2(4)^{\frac{3}{2}} = 16 \text{ sq units}$$

$$\therefore \quad \lambda = 16$$

(C) Given parabola $y^2 = 4x$. Now, the ends of latusrectum are $P(1, 2)$ and $Q(1, -2)$ or $P(t^2, 2t)$ and $Q(t_1^2, 2t_1)$, where $t = 1$, $t_1 = -1$.

We know that the other end of normal is given by $t_2 = -t - \frac{2}{t}$

$\Rightarrow$ $\quad A(t_2^2, 2t_2)$ and $\quad A'(t_3^2, 2t_3)$,

where $\quad t_2 = -3, t_3 = 3$

or $\quad A(9, -6)$ and $A'(9, 6)$

$\therefore$ $\quad AA' = 12$ units

$\therefore$ $\quad \lambda = 12$

JEE Type Solved Examples :
Statement I and II Type Questions

■ **Directions** (Ex. Nos. 25 and 26) are Assertion-Reason type questions. Each of these question contains two statements :
Statement I (Assertion) and
Statement II (Reason)
Each of these examples also has four alternative choices, only one of which is the correct answer.
You have to select the correct choice

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true, Statement II is false.
(d) Statement I is false, Statement II is true.

● ***Ex. 25*** **Statement I** *Through the point* $(\lambda, \lambda+1)$, $\lambda < 2$, *there cannot be more than one normal to the parabola* $y^2 = 4x$.

Statement II *The point* $(\lambda, \lambda+1)$ *cannot lie inside the parabola* $y^2 = 4x$.

Sol. (b) Let $S = y^2 - 4x$

$\therefore \quad S_1 = (\lambda+1)^2 - 4\lambda = (\lambda-1)^2 \geq 0$

$\therefore$ Point $(\lambda, \lambda+1)$ cannot lie inside the parabola $y^2 = 4x$.

$\therefore$ Statement II is true.

Now, equation of normal at $(t^2, 2t)$ is $y + tx = 2t + t^3$ passes through $(\lambda, \lambda+1)$.

$$\Rightarrow \quad \lambda + 1 + t\lambda = 2t + t^3$$

$$\Rightarrow \quad t^3 + (2-\lambda)t - (\lambda+1) = 0 \qquad \ldots(i)$$

On differentiating w.r.t. t, we get

$$3t^2 + (2-\lambda) > 0, \text{ for } \lambda < 2$$

The cubic Eq. (i) has only one real root.

$\therefore$ Statement I is true.

Hence, both statements are true but Statement II is not correct explanation for Statement I.

● ***Ex. 26*** **Statement I** *If there exist points on the circle* $x^2 + y^2 = \lambda^2$ *from which two perpendicular tangents can be drawn to the parabola* $y^2 = 2x$, *then* $\lambda \geq \frac{1}{2}$.

Statement II *Perpendicular tangents to the parabola meet at the directrix.*

Sol. (a) Statement II is true as it is property of parabola. Equation of directrix of parabola $y^2 = 2x$ is $x = -\frac{1}{2}$.

Any point on directrix is $\left(-\frac{1}{2}, y\right)$, now this point exists on the circle, then

$$\frac{1}{4} + y^2 = \lambda^2$$

$$\Rightarrow \quad y^2 = \lambda^2 - \frac{1}{4} \geq 0$$

$$\therefore \quad \lambda \geq \frac{1}{2}$$

Hence, both statements are true and Statement II is correct explanation for Statement I.

Subjective Type Examples

■ In this section, there are **16 subjective solved examples.**

● ***Ex. 27*** *The two parabolas* $y^2 = 4a(x-l)$ *and* $x^2 = 4a(y-l_1)$ *always touch one another,* l *and* l_1 *being variable parameters. Prove that the point of contact lies on the curve* $xy = 4a^2$.

Sol. Let (h, k) be the common point and if the parabolas touch each other, then the tangents at (h, k) should be identical. Their equations are

$$ky = 2a(x - l + h)$$

$$\Rightarrow \quad 2ax - ky = 2a(l-h) \qquad \ldots(i)$$

and $$hx = 2a(y - l_1 + k)$$

$$\Rightarrow \quad hx - 2ay = 2a(k - l_1) \qquad \ldots(ii)$$

On comparing the coefficients of x and y in Eq. (i) and (ii), we get

$$\frac{2a}{h} = \frac{k}{2a}$$

$$\Rightarrow \quad hk = 4a^2$$

Hence, the locus is $xy = 4a^2$,

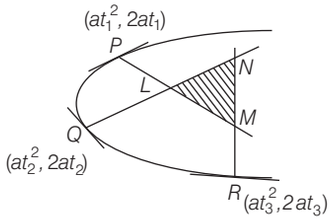
which is independent of l and l_1.

● ***Ex. 28*** *Show that the area formed by the normals to* $y^2 = 4ax$ *at the points* t_1, t_2, t_3 *is* $\frac{1}{2}a^2 |(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)| \lambda^2$, *where* $\lambda = (t_1 + t_2 + t_3)$.

Sol. Equation of normals at P, Q, R are

$$y = -t_1 x + 2at_1 + at_1^3$$

$y = -t_2x + 2at_2 + at_2^3$, $y = -t_3x + 2at_3 + at_3^3$, respectively.



Subtracting them two by two, coordinates of L, M and N are

$$L \equiv \{2a + a(t_1^2 + t_1t_2 + t_2^2), -at_1t_2(t_1 + t_2)\}$$

$$M \equiv \{2a + a(t_1^2 + t_1t_3 + t_3^2), -at_1t_3(t_1 + t_3)\}$$

$$N \equiv \{2a + a(t_2^2 + t_2t_3 + t_3^2), -at_2t_3(t_2 + t_3)\}$$

$\therefore$ Area of ΔLMN

$$= \frac{1}{2}\left|\begin{vmatrix} 2a + a(t_1^2 + t_1t_2 + t_2^2) & -at_1t_2(t_1 + t_2) & 1 \\ 2a + a(t_1^2 + t_1t_3 + t_3^2) & -at_1t_3(t_1 + t_3) & 1 \\ 2a + a(t_2^2 + t_2t_3 + t_3^2) & -at_2t_3(t_2 + t_3) & 1 \end{vmatrix}\right|$$

On applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$= \frac{a^2}{2}\left|\begin{vmatrix} 2 + (t_1^2 + t_1t_2 + t_2^2) & -t_1t_2(t_1 + t_2) & 1 \\ (t_3 - t_2)(t_1 + t_2 + t_3) & t_1(t_3 - t_2)(t_1 + t_2 + t_3) & 0 \\ (t_3 - t_1)(t_1 + t_2 + t_3) & t_2(t_3 - t_1)(t_1 + t_2 + t_3) & 0 \end{vmatrix}\right|$$

Expanding by last column

$$= \frac{a^2}{2}|(t_3 - t_1)(t_3 - t_2)|(t_1 + t_2 + t_3)^2\left|\begin{vmatrix} 1 & t_1 \\ 1 & t_2 \end{vmatrix}\right|$$

$$= \frac{1}{2}a^2|(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)|(t_1 + t_2 + t_3)^2$$

$$= \frac{1}{2}a^2|(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)|\lambda^2$$

Aliter

Equations of sides of ΔLMN formed by the normals are

$$y + t_1x - (2at_1 + at_1^3) = 0$$

$$y + t_2x - (2at_2 + at_2^3) = 0$$

$$y + t_3x - (2at_3 + at_3^3) = 0$$

$\therefore$ Required area $= \dfrac{1}{2|C_1C_2C_3|}\begin{vmatrix} t_1 & 1 & -(2at_1 + at_1^3) \\ t_2 & 1 & -(2at_2 + at_2^3) \\ t_3 & 1 & -(2at_3 + at_3^3) \end{vmatrix}^2$

$$= \frac{a^2}{|2(t_2 - t_3)(t_3 - t_1)(t_1 - t_2)|}\begin{vmatrix} t_1 & 1 & 2t_1 + t_1^3 \\ t_2 & 1 & 2t_2 + t_2^3 \\ t_3 & 1 & 2t_3 + t_3^3 \end{vmatrix}^2$$

$$= \frac{a^2}{2|(t_2 - t_3)(t_3 - t_1)(t_1 - t_2)|}\left\{\begin{vmatrix} t_1 & 1 & 2t_1 \\ t_2 & 1 & 2t_2 \\ t_3 & 1 & 2t_3 \end{vmatrix} + \begin{vmatrix} t_1 & 1 & t_1^3 \\ t_2 & 1 & t_2^3 \\ t_3 & 1 & t_3^3 \end{vmatrix}\right\}^2$$

$$= \frac{a^2}{2|(t_2 - t_3)(t_3 - t_1)(t_1 - t_2)|}$$

$$\{0 - (t_1 - t_2)(t_2 - t_3)(t_3 - t_1)(t_1 + t_2 + t_3)\}^2$$

$$= \frac{1}{2}a^2|(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)|(t_1 + t_2 + t_3)^2$$

$$= \frac{1}{2}a^2|(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)|\lambda^2.$$

● ***Ex. 29*** *Prove that the two parabolas $y^2 = 4ax$ and $y^2 = 4c(x - b)$ cannot have common normal, other than the axis unless $b/(a - c) > 2$.*

Sol. Given parabolas $y^2 = 4ax$ and $y^2 = 4c(x - b)$ have common normals. Then equation of normals in terms of slopes are

$$y = mx - 2am - am^3$$

and $$y = m(x - b) - 2cm - cm^3$$

respectively then normals must be identical, compare the coefficients

$$1 = \frac{2am + am^3}{mb + 2cm + cm^3}$$

$\Rightarrow$ $$m[(c - a)m^2 + (b + 2c - 2a)] = 0,$$

$$m \neq 0 \qquad [\because \text{other than axis}]$$

and $$m^2 = \frac{2a - 2c - b}{c - a},$$

$$m = \pm\sqrt{\frac{2(a - c) - b}{c - a}}$$

$\Rightarrow$ $$m = \pm\sqrt{\left(-2 - \frac{b}{c - a}\right)}$$

$\therefore$ $$-2 - \frac{b}{c - a} > 0$$

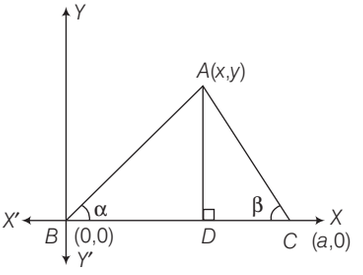
$\Rightarrow$ $$-2 + \frac{b}{a - c} > 0$$

$\Rightarrow$ $$\frac{b}{a - c} > 2$$

● ***Ex. 30*** *If on a given base BC, a triangle is described such that the sum of the tangents of the base angles is m, then prove that the locus of the opposite vertex A is a parabola.*

Sol. Let the given point B and C be $(0, 0)$ and $(a, 0)$, respectively.

Given, $\tan\alpha + \tan\beta = m$ (constant)



$\Rightarrow$ $$\frac{y}{x} + \frac{y}{a - x} = m$$

$\Rightarrow$ $$\frac{ay}{x(a - x)} = m$$

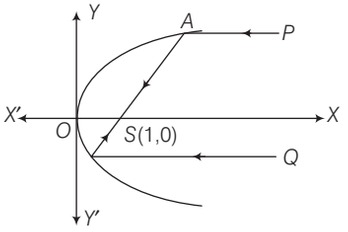
$\therefore$ $$ay = amx - mx^2$$

which is the equation of parabola.

● ***Ex. 31*** *A parabolic mirror is kept along $y^2 = 4x$ and two light rays parallel to its axis are reflected along one straight line. If one of the incident rays is at 3 units distance from the axis, then find the distance of the other incident ray from the axis.*

Sol. Let the incident rays be PA and QB.

After reflection, both rays passes through the focus $S(1, 0)$.



Therefore, AB is a focal chord.

Let A be $(t^2, 2t)$, then B be $\left(\frac{1}{t^2}, \frac{-2}{t}\right)$.

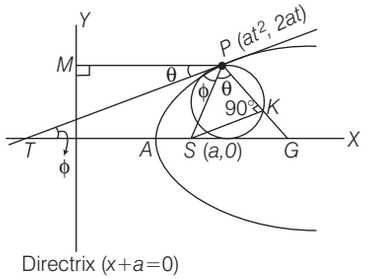
Given, $2t = 3 \ \therefore \ t = 3/2$

Hence, distance of B from the axis of parabola is $\left|-\frac{2}{t}\right| = \frac{4}{3}$ units.

● ***Ex. 32*** *Prove that the length of the intercept on the normal at the point $P(at^2, 2at)$ of a parabola $y^2 = 4ax$ made by the circle on the line joining the focus and point P as diameter is $a\sqrt{(1+t^2)}$.*

Sol. Let the normal at $P(at^2, 2at)$ cut the circle in K and the axis of parabola at G then PK is required intercept.

$$SP = PM = a + at^2$$



Since angle in a semi-circle being right angle.

$\therefore \quad \angle SKP = 90°$

and normal at $P(at^2, 2at)$ is

$$y = -tx + 2at + at^3$$

$$\Rightarrow \quad tx + y - 2at - at^3 = 0 \qquad \ldots(i)$$

$\therefore$ SK is the perpendicular distance from $S(a, 0)$ to the normal (i),

then $$SK = \frac{|at + 0 - 2at - at^3|}{\sqrt{t^2+1}} = a|t|\sqrt{1+t^2}$$

$\therefore$ In ΔSPK, $(Pk)^2 = (SP)^2 - (SK)^2$

$$= a^2(1+t^2)^2 - a^2t^2(1+t^2) = a^2(1+t^2)$$

$$\therefore \quad PK = a\sqrt{(1+t^2)}$$

● ***Ex. 33*** *A parabola of latusrectum $4a$, touches a fixed equal parabola , the axes of the two curves being parallel; prove that the locus of the vertex of the moving curve is a parabola of latusrectum $8a$.*

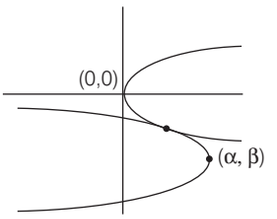
Sol. Let the given parabola is

$$y^2 = 4ax \qquad \ldots(i)$$

If the vertex of moving parabola is (α, β), then equation of moving parabola is

$$(y-\beta)^2 = -4a(x-\alpha) \qquad \ldots(ii)$$

On substituting the value of i.e. $x = \frac{y^2}{4a}$ in Eq. (ii),



then $$(y-\beta)^2 = -4a\left(\frac{y^2}{4a} - \alpha\right)$$

$$\Rightarrow \quad y^2 - 2\beta y + \beta^2 = -y^2 + 4a\alpha$$

$$2y^2 - 2\beta y + \beta^2 - 4a\alpha = 0 \qquad \ldots(iii)$$

Since, two parabolas (i) and (ii) touch each other.

Hence, roots of Eq. (iii) are equal i.e. Discriminant = 0

$$\Rightarrow \quad \text{“}B^2 - 4AC = 0\text{”}$$

$$\therefore \quad (-2\beta)^2 = 4 \cdot 2 \cdot (\beta^2 - 4a\alpha)$$

$$\Rightarrow \quad 4\beta^2 = 32a\alpha$$

or $$\beta^2 = 8a\alpha$$

the required locus is $$y^2 = 8ax$$

which has latusrectum double that of given parabola.

● ***Ex. 34*** *The normal at point P on a given parabola meet the axis of parabola at Q. Then prove that a line through Q and perpendicular to this normal always touches a fixed parabola whose length of latusrectum is same as that of given parabola.*

Sol. Let the equation of parabola is

$$y^2 = 4ax. \qquad \ldots(i)$$

Let $P(at^2, 2at)$

Normal at P is $$y + tx = 2at + at^3$$

$\because$ Normal meet the axis of parabola (i.e. X-axis), then $Q(2a + at^2, 0)$.

Now, equation of the line through Q and perpendicular to the normal is

$$y - 0 = \frac{1}{t}(x - 2a - at^2)$$

$\Rightarrow \quad ty = x - 2a - at^2$

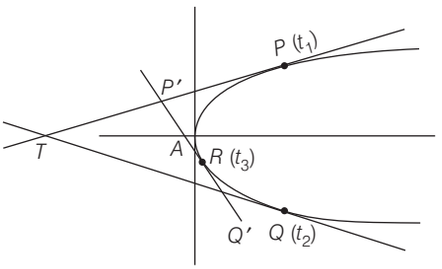
$\Rightarrow \quad ty = (x - 2a) - at^2$

which is clearly tangent to the parabola

$$y^2 = 4a(2a - x).$$

● **Ex. 35** *TP and TQ are any two tangents to a parabola and the tangent at a third point R cuts them in P′ and Q′, prove that* $\frac{TP'}{TP} + \frac{TQ'}{TQ} = 1$.

Sol. Let parabola be $y^2 = 4ax$ and coordinates of P and Q on this parabola are $P \equiv (at_1^2, 2at_1)$ and $Q \equiv (at_2^2, 2at_2)$; T is the point of intersection of tangents at t_1 and t_2.



$\therefore$ Coordinates of $T \equiv \{at_1 t_2, a(t_1 + t_2)\}$

Similarly $\quad P' \equiv \{at_3 t_1, a(t_3 + t_1)\}$

$\quad Q' \equiv \{at_2 t_3, a(t_2 + t_3)\}$

Let $\quad TP' : TP = \lambda : 1$

$\therefore \quad \lambda = \frac{t_3 - t_2}{t_1 - t_2} \quad \Rightarrow \quad \frac{TP'}{TP} = \frac{t_3 - t_2}{t_1 - t_2}$

Similarly, $\quad \frac{TQ'}{TQ} = \frac{t_1 - t_3}{t_1 - t_2}$

$\therefore \quad \frac{TP'}{TP} + \frac{TQ'}{TQ} = 1$

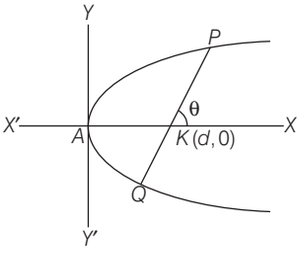
● **Ex. 36** *Prove that on the axis of any parabola there is a certain point 'K' which has the property that, if a chord PQ of parabola be drawn through it, then* $\frac{1}{PK^2} + \frac{1}{QK^2}$ *is the same for all positions of the chord .*

Sol. Any line passing through K is $\frac{x - d}{\cos\theta} = \frac{y - 0}{\sin\theta} = r$

$\therefore$ Coordinates of P and Q are

$$(d + KP\cos\theta, KP\sin\theta)$$

and $\quad (d - KQ\cos\theta, -KQ\sin\theta).$



Points P and Q lie on $y^2 = 4ax$ then

$$(KP)^2 \sin^2\theta = 4a(d + KP\cos\theta) \quad \ldots(i)$$

and $\quad (KQ)^2 \sin^2\theta = 4a(d - KQ\cos\theta) \quad \ldots(ii)$

From Eq. (i), $\quad KP = \frac{4a\cos\theta + \sqrt{(16a^2\cos^2\theta + 16ad\sin^2\theta)}}{2\sin^2\theta}$

From Eq. (ii), $\quad KQ = \frac{-4a\cos\theta + \sqrt{(16a^2\cos^2\theta + 16ad\sin^2\theta)}}{2\sin^2\theta}$

$\therefore \quad \frac{1}{(KP)^2} + \frac{1}{(KQ)^2} = \frac{2a\cos^2\theta + d\sin^2\theta}{2ad^2}$

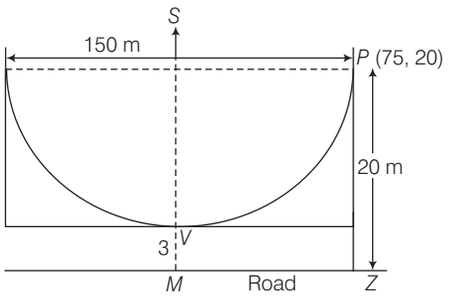
For $d = 2a$

$\therefore \quad \frac{1}{(KP)^2} + \frac{1}{(KQ)^2} = \frac{1}{d^2}$

$= \frac{1}{4a^2}$, which is independent of θ.

● **Ex. 37** *If the distribution of weight is uniform, then the rope of the suspended bridge takes the form of parabola. The height of the supporting towers is* 20 m*, the distance between these towers is* 150 *m and the height of the lowest point of the rope from the road is* 3 m*. Find the equation of the parabolic shape of the rope considering the floor of the bridge as X-axis and the axis of the parabola as Y-axis. Find the height of that tower which supports the rope and is at a distance of* 30 m *from the centre of the road .*

Sol. Here MZ is the road let V is lowest point of the rope given $VZ = 3$ m.



Let MZ is X-axis and let MVS is Y-axis.

Taking X and Y-axes as shown in the given figure, the equation of parabola is of the form

$$x^2 = 4ay$$

if V is origin

Coordinates of P is $\left\{\frac{1}{2}(150), 20 - 3\right\}$ i.e. $(75, 17)$ P lies on Eq. (i), then

$$(75)^2 = 4a(17)$$

or $\quad 4a = \frac{5625}{17}$

From Eq. (i), $\quad x^2 = \frac{5625}{17} y$

Now, shifting the origin (0, 0) to $M(0,-3)$,
then equation of parabola in new form is

$$x^2=\frac{5625}{17}(y-3) \qquad ...(ii)$$

Again, let the required height of supporting tower at a distance of 30 m from the centre of road be h metres, then the coordinates of the top of this tower are $(30, h)$ referred to given axes

from Eq. (ii), $(30)^2=\frac{5625}{17}(h-3)$

$$\therefore \quad h=\left(\frac{900\times17}{5625}\right)+3=5.72 \text{ m}$$

● ***Ex. 38*** *Tangent is drawn at any point (x_1, y_1) on the parabola $y^2=4ax$. Now tangents are drawn from any point on this tangent to the circle $x^2+y^2=a^2$ such that all the chords of contact pass through a fixed point (x_2, y_2). Prove that* $4\left(\frac{x_1}{x_2}\right)+\left(\frac{y_1}{y_2}\right)^2=0.$

Sol. Tangent at (x_1, y_1) on the parabola $y^2=4ax$

$$\therefore \quad yy_1=2a(x+x_1)$$

Any point on this tangent will be $\left(x', \frac{2a(x'+x_1)}{y_1}\right)$

Equation of chord of contact of the point $\left(x', \frac{2a(x'+x_1)}{y_1}\right)$ w.r.t. circle $x^2+y^2=a^2$ is $xx'+y\cdot\frac{2a(x'+x_1)}{y_1}=a^2$

$$\Rightarrow \quad xx'y_1+2ayx'+2ayx_1=a^2y_1$$

$$\Rightarrow \quad (2ayx_1-a^2y_1)+x'(xy_1+2ay)=0$$

which is family of straight lines passing through point of intersection of

$$2ayx_1-a^2y_1=0 \quad \text{and} \quad xy_1+2ay=0$$

$$\therefore \quad y=\frac{ay_1}{2x_1}$$

and $$x=-\frac{2ay}{y_1}=\left(-\frac{2a}{y_1}\right)\left(\frac{ay_1}{2x_1}\right)$$

$$x=-\frac{a^2}{x_1}$$

The fixed point is (x_2, y_2).

$$\therefore \quad x_2=-\frac{a^2}{x_1} \quad \text{and} \quad y_2=\frac{ay_1}{2x_1}$$

$$\therefore \quad 4\left(\frac{x_1}{x_2}\right)+\left(\frac{y_1}{y_2}\right)^2=4\left(\frac{x_1}{-a^2/x_1}\right)+\left(\frac{y_1}{ay_1/2x_1}\right)^2$$

$$=-\frac{4x_1^2}{a^2}+\frac{4x_1^2}{a^2}=0$$

Hence, $4\left(\frac{x_1}{x_2}\right)+\left(\frac{y_1}{y_2}\right)^2=0.$

● ***Ex. 39*** *A variable chord PQ of the parabola $y=4x^2$ subtends a right angle at the vertex. Find the locus of the points of intersection of the normals at P and Q.*

Sol. Parametric point on the parabola

$$y=4x^2 \text{ is } (t, 4t^2)$$

Let $P(t_1, 4t_1^2)$ and $Q(t_2, 4t_2^2)$,
if A is vertex of the parabola, then
Slope of $AP\times$ Slope of $AQ=-1$ $[\because AP\perp AQ]$

$$\therefore \quad 4t_1\times4t_2=-1$$

$$\therefore \quad t_1t_2=-\frac{1}{16}$$

Equations of normals at P and Q are

$$x+4t_1y=t_1+16t_1^3 \qquad ...(i)$$

and $$x+4t_2y=t_2+16t_2^3 \qquad ...(ii)$$

Let the normals (i) and (ii) intersect at (h, k), then on solving Eqs. (i) and (ii), then we get

$$k=\frac{1+16(t_1^2+t_2^2+t_1t_2)}{4}$$

$$\Rightarrow \quad k=\frac{1+16(t_1^2+t_2^2)+16t_1t_2}{4}$$

$$=\frac{1+16((t_1+t_2)^2-2t_1t_2)-1}{4} \qquad [\because 16t_1t_2=-1]$$

$$=4\left\{(t_1+t_2)^2+\frac{1}{8}\right\}$$

$$k=4(t_1+t_2)^2+\frac{1}{2} \qquad ...(iii)$$

Again $$h=-16t_1t_2(t_1+t_2)$$

$$\Rightarrow \quad h=t_1+t_2 \qquad ...(iv)$$

On eliminating t_1 and t_2 from Eqs. (iii) and (iv), we get

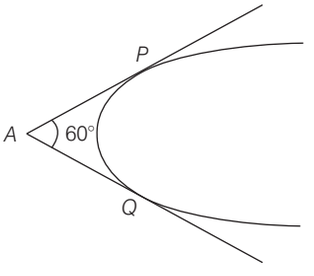
$$k=4h^2+\frac{1}{2}$$

$$\Rightarrow \quad 2k=8h^2+1$$

Hence, locus of (h, k) is $2y=8x^2+1$.

● ***Ex. 40*** *Equilateral triangles are circumscribed to the parabola $y^2=4ax$. Prove that their angular points lie on the conic $(3x+a)(x+3a)=y^2$.*

Sol. Let coordinates of P and Q are $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$, respectively.



Tangents at P and Q are $t_1y = x + at_1^2$ and $t_2y = x + at_2^2$, respectively.

Slope of tangents at P and Q are $\frac{1}{t_1}$ and $\frac{1}{t_2}$, respectively.

Let one angular point of equilateral triangle is A.

$\therefore \quad A \equiv (at_1t_2, a(t_1 + t_2))$

Given, $\tan 60° = \left| \frac{\frac{1}{t_1} - \frac{1}{t_2}}{1 + \frac{1}{t_1} \cdot \frac{1}{t_2}} \right| = \left| \frac{t_2 - t_1}{1 + t_1t_2} \right|$

$\Rightarrow \quad (\sqrt{3})^2 (1 + t_1t_2)^2 = (t_2 - t_1)^2$

$\Rightarrow \quad 3(1 + t_1t_2)^2 = (t_1 + t_2)^2 - 4t_1t_2$

For locus of A, put $at_1t_2 = x$ and $a(t_1 + t_2) = y$, then

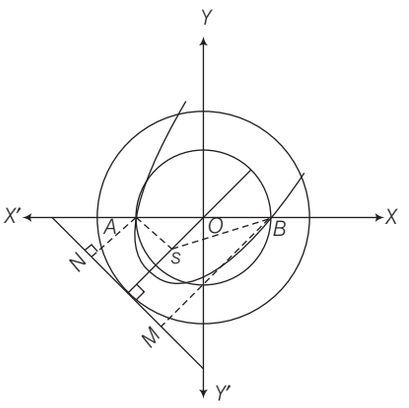
$$3\left(1 + \frac{x}{a}\right)^2 = \left(\frac{y}{a}\right)^2 - 4\frac{x}{a}$$

$\Rightarrow \quad y^2 = 3x^2 + 10ax + 3a^2$

$\therefore \quad (3x + a)(x + 3a) = y^2$

● ***Ex. 41*** *A parabola is drawn to pass through A and B, the ends of diameter of a given circle of radius a and to have a directrix a tangent to a concentric circle of radius b, the axes being AB and the perpendicular diameter. Prove that the locus of the focus of the parabola is* $\frac{x^2}{b^2} + \frac{y^2}{b^2 - a^2} = 1$.

Sol. Let AB be taken along X-axis, it mid-point O as origin and a line through O perpendicular to AB as Y-axis.



Equation of circle on AB as diameter is

$$x^2 + y^2 = a^2 \qquad \ldots(i)$$

The equation of a circle of radius b and concentric to circle (i) is

$$x^2 + y^2 = b^2 \qquad \ldots(ii)$$

Let $S(h, k)$ be the focus of the parabola which passes through $A(-a, 0)$ and $B(a, 0)$.

Let equation of tangent to circle (ii) be

$$x\cos\theta + y\sin\theta = 6. \qquad \ldots(iii)$$

This is taken as directrix, then

$$SP = PM$$

$\Rightarrow \quad BS = BM \Rightarrow (h - a)^2 + (k - 0)^2 = (a\cos\theta - b)^2 \qquad \ldots(iv)$

and $\quad AS = AN$

$\Rightarrow \quad (h + a)^2 + (k - 0)^2 = (a\cos\theta + b)^2 \qquad \ldots(v)$

On adding Eqs. (iv) and (v), then

$$a^2\cos^2\theta + b^2 = h^2 + k^2 + a^2 \qquad \ldots(vi)$$

On subtracting Eq. (iv) from Eq. (v), then

$$4ab\cos\theta = 4ah \text{ or } \cos\theta = \frac{h}{b} \qquad \ldots(vii)$$

From Eqs. (vi) and (vii), we get

$$\frac{a^2h^2}{b^2} + b^2 = h^2 + k^2 + a^2$$

$\Rightarrow \quad h^2\frac{(b^2 - a^2)}{a^2} + k^2 = b^2 - a^2$

$\Rightarrow \quad \frac{h^2}{a^2} + \frac{k^2}{b^2 - a^2} = 1$

Hence, the locus of $S(h, k)$ is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2 - a^2} = 1.$$

● ***Ex. 42*** *Two straight lines are at right angles to one another and one of them touches* $y^2 = 4a(x + a)$ *and the other* $y^2 = 4b(x + b)$ *. Prove that the point of intersection of the lines lies on the line* $x + a + b = 0$.

Sol. We know that any tangent in terms of slope (m) of the parabola $y^2 = 4ax$ is $y = mx + \frac{a}{m}$.

Replacing x by $x + a$, we get

$$y = m(x + a) + \frac{a}{m} \text{ which is tangent to}$$

$$y^2 = 4a(x + a) \qquad \ldots(i)$$

Similarly, tangent in terms of slope of $y^2 = 4b(x + b)$ is

$$y = m_1(x + b) + \frac{b}{m_1} \qquad \ldots(ii)$$

Given tangents are perpendicular, we have

$$mm_1 = -1 \quad \text{or} \quad m_1 = -\frac{1}{m}$$

then Eq. (ii) becomes $y = -\frac{(x + b)}{m} - bm$

On subtracting Eqs. (i) and (ii), then

$$0 = (x + a + b)\left(m + \frac{1}{m}\right)$$

$\Rightarrow \quad m + \frac{1}{m} \neq 0$

Hence, $\quad x + a + b = 0$

Parabola Exercise 1 :
Single Option Correct Type Questions

■ This section contains **30 multiple choice questions.** Each question has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct

1. A common tangent is drawn to the circle $x^2 + y^2 = a^2$ and the parabola $y^2 = 4bx$. If the angle which this tangent makes with the axis of x is $\frac{\pi}{4}$, then the relationship between a and b is $(a, b > 0)$

(a) $b = \sqrt{2}a$ (b) $a = b\sqrt{2}$ (c) $c = 2a$ (d) $a = 2c$

2. The equation of parabola whose vertex and focus lie on the axis of x at distances a and a_1 from the origin respectively, is

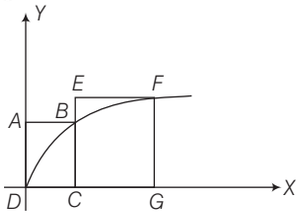
(a) $y^2 = 4(a_1 - a)x$ (b) $y^2 = 4(a_1 - a)(x - a)$

(c) $y^2 = 4(a_1 - a)(x - a_1)$ (d) $y^2 = 4aa_1 x$

3. If parabolas $y^2 = ax$ and $25[(x-3)^2 + (y+2)^2] = (3x - 4y - 2)^2$ are equal, then the value of a is

(a) 3 (b) 6 (c) 7 (d) 9

4. $ABCD$ and $EFGC$ are squares and the curve $y = \lambda\sqrt{x}$ passes through the origin D and the points B and F. The ratio $\frac{FG}{BC}$ is



(a) $\frac{\sqrt{3}+1}{4}$ (b) $\frac{\sqrt{3}+1}{2}$ (c) $\frac{\sqrt{5}+1}{4}$ (d) $\frac{\sqrt{5}+1}{2}$

5. Let A and B be two points on a parabola $y^2 = x$ with vertex V such that VA is perpendicular to VB and θ is the angle between the chord VA and the axis of the parabola. The value of $\frac{|VA|}{|VB|}$ is

(a) $\tan\theta$ (b) $\cot^2\theta$ (c) $\tan^3\theta$ (d) $\cot^3\theta$

6. The vertex of the parabola whose parametric equation is $x = t^2 - t + 1, y = t^2 + t + 1, t \in R$, is

(a) (1, 1) (b) (2, 2) (c) (3, 3) (d) $\left(\frac{1}{2}, \frac{1}{2}\right)$

7. The circle $x^2 + y^2 + 2px = 0$, $p \in R$, touches the parabola $y^2 = 4x$ externally, then

(a) $p > 0$ (b) $p < 0$ (c) $p > 1$ (d) $p > 2$

8. If $a \neq 0$ and the line $2bx + 3cy + 4d = 0$ passes through the points of intersection of the parabolas $y^2 = 4ax$ and $x^2 = 4ay$, then

(a) $d^2 + (2b + 3c)^2 = 0$ (b) $d^2 + (3b + 2c)^2 = a^2$

(c) $d^2 + (2b - 3c)^2 = 0$ (d) $d^2 + (2b + 3c)^2 = a^2$

9. A parabola $y = ax^2 + bx + c$ crosses the X-axis at $(\alpha, 0)$ and $(\beta, 0)$ both to the right of the origin. A circle also passes through these two points. The length of a tangent from the origin to the circle is

(a) $\sqrt{\frac{bc}{a}}$ (b) ac^2 (c) $\frac{b}{a}$ (d) $\sqrt{\frac{c}{a}}$

10. Two mutually perpendicular tangent of the parabola $y^2 = 4ax$ meet the axis at P_1 and P_2. It S is the focus of the parabola, then $\frac{1}{SP_1} + \frac{1}{SP_2}$ is equal to

(a) $\frac{1}{4a}$ (b) $\frac{1}{a}$ (c) $\frac{2}{a}$ (d) $\frac{4}{a}$

11. If the normal to the parabola $y^2 = 4ax$ at P meets the curve again at Q and if PQ and the normal at Q make angles α and β respectively with the X-axis, then $\tan\alpha(\tan\alpha + \tan\beta)$ has the value equal to

(a) -2 (b) -1 (c) $-\frac{1}{2}$ (d) 0

12. If the normals to the parabola $y^2 = 4ax$ at three points P, Q and R meet at A and S is the focus, then $SP \cdot SQ \cdot SR$ is equal to

(a) $(SA)^2$ (b) $(SA)^3$ (c) $a(SA)^2$ (d) $a(SA)^3$

13. The length of the shortest normal chord of the parabola $y^2 = 4ax$ is

(a) $2a\sqrt{27}$ (b) $9a$ (c) $a\sqrt{54}$ (d) $18a$

14. The largest value of a for which the circle $x^2 + y^2 = a^2$ falls totally in the interior of the parabola $y^2 = 4(x + 4)$ is

(a) $4\sqrt{3}$ (b) 4 (c) $\frac{4\sqrt{6}}{7}$ (d) $2\sqrt{3}$

15. From a point $(\sin\theta, \cos\theta)$, if three normals can be drawn to the parabola $y^2 = 4ax$, then the value of a is

(a) $\left(\frac{1}{2}, 1\right)$ (b) $\left[-\frac{1}{2}, 0\right)$

(c) $\left[\frac{1}{2}, 1\right]$ (d) $\left(\frac{-1}{2}, 0\right) \cup \left(0, \frac{1}{2}\right)$

16. If two different tangents of $y^2 = 4x$ are the normals to $x^2 = 4$ by, then

(a) $|b| < \frac{1}{2\sqrt{2}}$ (b) $|b| < \frac{1}{\sqrt{2}}$

(c) $|b| > \frac{1}{2\sqrt{2}}$ (d) $|b| > \frac{1}{\sqrt{2}}$

17. The shortest distance between the parabolas $2y^2 = 2x - 1$ and $2x^2 = 2y - 1$ is

(a) $\frac{1}{2\sqrt{2}}$ (b) $\frac{1}{2}$ (c) $2\sqrt{2}$ (d) 4

18. Normals at two points (x_1, y_1) and (x_2, y_2) of the parabola $y^2 = 4x$ meet again on the parabola, where $x_1 + x_2 = 4$, then $|y_1 + y_2|$ is equal to

(a) $\sqrt{2}$ (b) $2\sqrt{2}$ (c) $4\sqrt{2}$ (d) $8\sqrt{2}$

19. A line is drawn from $A(-2, 0)$ to intersect the curve $y^2 = 4x$ at P and Q in the first quadrant such that $\frac{1}{AP} + \frac{1}{AQ} < \frac{1}{4}$. Then the slope of the line is always

(a) $< \frac{1}{\sqrt{3}}$ (b) $> \frac{1}{\sqrt{3}}$ (c) $> \sqrt{2}$ (d) $> \sqrt{3}$

20. An equilateral triangle SAB is inscribed in the parabola $y^2 = 4ax$ having its focus at S. If chord AB lies towards the left of S, then the side length of this triangle is

(a) $a(2 - \sqrt{3})$ (b) $2a(2 - \sqrt{3})$

(c) $4a(2 - \sqrt{3})$ (d) $8a(2 - \sqrt{3})$

21. Let C be a circle with centre (0, 1) and radius unity. P is the parabola $y = ax^2$. The set of values of a for which they meet at a point other than origin is

(a) $(0, \infty)$ (b) $\left(0, \frac{1}{2}\right)$ (c) $\left(\frac{1}{4}, \frac{1}{2}\right)$ (d) $\left(\frac{1}{2}, \infty\right)$

22. Let S be the focus of $y^2 = 4x$ and a point P be moving on the curve such that its abscissa is increasing at the rate of 4 units/s. Then the rate of increase of the projection of SP on $x + y = 1$ when p is at (4, 4) is

(a) $-\sqrt{2}$ (b) $-\frac{3}{\sqrt{2}}$ (c) -1 (d) $\sqrt{2}$

23. If P is a point on the parabola $y^2 = 3(2x - 3)$ and M is the foot of perpendicular drawn from P on the directrix of the parabola, then the length of each side of the equilateral triangle SMP, where S is the focus of the parabola, is

(a) 2 (b) 4 (c) 6 (d) 8

24. Consider the parabola $y^2 = 4x$. Let $A \equiv (4, -4)$ and $B \equiv (9, 6)$ be two fixed points on the parabola. Let C be a moving point on the parabola between A and B such that the area of the triangle ABC is maximum. Then the coordinates of C are

(a) $\left(\frac{1}{4}, 1\right)$ (b) $(3, -2\sqrt{3})$

(c) $(3, 2\sqrt{3})$ (d) (4, 4)

25. Through the vertex O of the parabola $y^2 = 4ax$, two chords OP and OQ are drawn and the circles on OP and OQ as diameters intersect at R. If θ_1, θ_2 and ϕ are the angles made with the axis by the tangents at P and Q on the parabola and by OR, then the value of $\cot\theta_1 + \cot\theta_2$ is

(a) $-2\tan\phi$ (b) $2\tan\phi$

(c) 0 (d) $2\cot\phi$

26. AB is a double ordinate of the parabola $y^2 = 4ax$. Tangents drawn to the parabola at A and B meet the Y-axis at A_1 and B_1 respectively. If the area of trapezium AA_1B_1B is equal to $24a^2$, then the angle subtended by A_1B_1 at the focus of the parabola is equal to

(a) $\tan^{-1}2$ (b) $\tan^{-1}3$ (c) $2\tan^{-1}2$ (d) $2\tan^{-1}3$

27. If the 4th term in the expansion of $\left(px + \frac{1}{x}\right)^n$, $n \in N$ is $\frac{5}{2}$ and three normals to the parabola $y^2 = x$ are drawn through a point $(q, 0)$, then

(a) $q = p$ (b) $q > p$ (c) $q < p$ (d) $pq = 1$

28. The set of points on the axis of the parabola $y^2 - 4x - 2y + 5 = 0$ from which all the three normals to the parabola are real is

(a) $(k, 0); k > 1$ (b) $(k, 1); k > 3$

(c) $(k, 2); k > 6$ (d) $(k, 3); k > 8$

29. The triangle formed by the tangent to the parabola $y = x^2$ at the point whose abscissa is x_0 ($x_0 \in [1, 2]$), the Y-axis and the straight line $y = x_0^2$ has the greatest area if x_0 is equal to

(a) 0 (b) 1 (c) 2 (d) 3

30. The set of points (x, y) whose distance from the line $y = 2x + 2$ is the same as the distance from (2, 0) is a parabola. This parabola is congruent to the parabola in standard from $y = kx^2$ for some k which is equal to

(a) $\frac{4}{\sqrt{5}}$ (b) $\frac{12}{\sqrt{5}}$ (c) $\frac{\sqrt{5}}{4}$ (d) $\frac{\sqrt{5}}{12}$

Parabola Exercise 2 :
More than One Correct Option Type Questions

This section contains **15 multiple choice questions**. Each question has four choices (a), (b), (c) and (d) out of which **MORE THAN ONE** may be correct.

31. Equation of the common tangent to the circle $x^2 + y^2 = 50$ and the parabola $y^2 = 40x$ can be
(a) $x + y - 10 = 0$ (b) $x - y + 10 = 0$
(c) $x + y + 10 = 0$ (d) $x - y - 10 = 0$

32. Let PQ be a chord of the parabola $y^2 = 4x$. A circle drawn with PQ as a diameter passes through the vertex V of the parabola. If area of $\Delta PVQ = 20$ unit^2, then the coordinates of P are
(a) $(16,8)$ (b) $(16, -8)$
(c) $(-16, 8)$ (d) $(-16, -8)$

33. Let $y^2 = 4ax$ be a parabola and $x^2 + y^2 + 2bx = 0$ be a circle. If parabola and circle touch each other externally, then
(a) $a > 0, b < 0$ (b) $a > 0, b > 0$
(c) $a < 0, b > 0$ (d) $a < 0, b < 0$

34. Tangent is drawn at any point (x_1, y_1) other than the vertex on the parabola $y^2 = 4ax$. If tangents are drawn from any point on this tangent to the circle $x^2 + y^2 = a^2$ such that all the chords of chords of contact pass through a fixed point (x_2, y_2), then
(a) x_1, a, x_2 are in GP (b) $\frac{y_1}{2}, a, y_2$ are in GP
(c) $-4, \frac{y_1}{y_2}, \frac{x_1}{x_2}$ are in GP (d) $x_1x_2 + y_1y_2 = a^2$

35. Let P, Q and R are three co-normal points on the parabola $y^2 = 4ax$. Then the correct statement(s) is/at
(a) algebraic sum of the slopes of the normals at P, Q and R vanishes
(b) algebraic sum of the ordinates of the points P, Q and R vanishes
(c) centroid of the triangle PQR lies on the axis of the parabola
(d) Circle circumscribing the triangle PQR passes through the vertex of the parabola

36. Let P be a point whose coordinates differ by unity and the point does not lie on any of the axes of reference. If the parabola $y^2 = 4x + 1$ passes through P, then the ordinate of P may be
(a) 3 (b) -1
(c) 5 (d) 1

37. If a point P on $y^2 = 4x$, the foot of the perpendicular from P on the directrix and the focus form an equilateral triangle, then the coordinates of P may be
(a) $(3, -2\sqrt{3})$ (b) $(-3, 2\sqrt{3})$
(c) $(3, 2\sqrt{3})$ (d) $(-3, -2\sqrt{3})$

38. The locus of the foot of the perpendicular from the focus on a tangent to the parabola $y^2 = 4ax$ is
(a) the directrix (b) the tangent at the vertex
(c) $x = a$ (d) $x = 0$

39. The extremities of latusrectum of a parabola are (1, 1) and (1, −1). Then the equation of the parabola can be
(a) $y^2 = 2x - 1$ (b) $y^2 = 1 - 2x$
(c) $y^2 = 2x - 3$ (d) $y^2 = 2x - 4$

40. If from the vertex of a parabola $y^2 = 4ax$ a pair of chords be drawn at right angles to one another and with these chords as adjacent sides a rectangle be made, then the locus of the further angle of the rectangle is
(a) an equal parabola
(b) a parabola with focus at $(8a, 0)$
(c) a parabola with directrix as $x - 7a = 0$
(d) not a parabola

41. If two chords drawn from the point (4, 4) to the parabola $x^2 = 4y$ are divided by the line $y = mx$ in the ratio 1 : 2, then
(a) $m \in (-\infty, -\sqrt{3})$ (b) $m \in (-\infty, -\sqrt{3} - 1)$
(c) $m \in (\sqrt{3}, \infty)$ (d) $m \in (\sqrt{3} - 1, \infty)$

42. Through a point $P(-2, 0)$, tangents PQ and PR are drawn to the parabola $y^2 = 8x$. Two circles each passing through the focus of the parabola and one touching at Q and the other at R are drawn. Which of the following point (s) with respect to the triangle PQR lie (s) on the common chord of the two circles?
(a) centroid (b) orthocentre
(c) incentre (d) circumcentre

43. The set of points on the axis of the parabola $(y-2)^2 = 4\left(x - \frac{1}{2}\right)$ from which three distinct normals can be drawn to the parabola are
(a) (3, 2) (b) (1, 2)
(c) (4, 2) (d) (5, 2)

44. Three normals are drawn from the point (14, 7) to the curve $y^2 - 16x - 8y = 0$. Then the coordinates of the feet of the normals are
(a) $(3, -4)$ (b) (8, 16)
(c) (0, 0) (d) (2, 2)

45. A quadrilateral is inscribed in a parabola, then
(a) the quadrilateral may be cyclic
(b) diagonals of the quadrilateral may be equal
(c) all possible pairs of adjacent sides may be perpendicular
(d) None of the above

Parabola Exercise 3 : Paragraph Based Questions

- This section contains **8 paragraphs** based upon each of the paragraph **3 multiple choice questions** have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Paragraph I

(Q. Nos. 46 to 48)

Consider a parabola P touches coordinate axes at $(4, 0)$ *and* $(0, 3)$.

46. If focus of parabola P is (a, b), then the value of $b - a$ is
(a) $\frac{1}{25}$ (b) $\frac{3}{25}$ (c) $\frac{4}{25}$ (d) $\frac{12}{25}$

47. Length of latus rectum of parabola P is
(a) $\frac{72}{125}$ (b) $\frac{144}{125}$
(c) $\frac{288}{125}$ (d) $\frac{576}{125}$

48. Equation of directrix of parabola P is
(a) $4x + 3y = 0$ (b) $3x + 4y = 12$
(c) $3x + 4y = 0$ (d) $4x + 3y = 12$

Paragraph II

(Q. Nos. 49 to 51)

Let C be the locus of the circumcentre of a variable triangle having sides Y-axis, $y = 2$ *and* $ax + by = 1$, *where* (a, b) *lies on the parabola* $y^2 = 4\lambda x$.

49. For $\lambda = 2$, the product of coordinates of the vertex of the curve C is
(a) -8 (b) -6 (c) 6 (d) 8

50. For $\lambda = \frac{1}{32}$, the length of smallest focal chord of the curve C is
(a) $\frac{8}{3}$ (b) 2 (c) 4 (d) 8

51. The curve C is symmetrical about the line
(a) $x = -\frac{3}{2}$ (b) $y = -\frac{3}{2}$
(c) $x = \frac{3}{2}$ (d) $y = \frac{3}{2}$

Paragraph III

(Q. Nos. 52 to 54)

Consider a parabola (P) $x^2 - 4xy + 4y^2 - 32x + 4y + 16 = 0$.

52. The focus of the parabola (P) is
(a) $(2, 1)$ (b) $(-2, 1)$ (c) $(-2, -1)$ (d) $(2, -1)$

53. Length of latusrectum of the parabola (P) is
(a) $\frac{3}{\sqrt{5}}$ (b) $\frac{6}{\sqrt{5}}$ (c) $\frac{12}{\sqrt{5}}$ (d) $\frac{24}{\sqrt{5}}$

54. Equation of directrix of parabola (P) is
(a) $x - 2y - 4 = 0$ (b) $2x + y - 3 = 0$
(c) $x - 2y + 4 = 0$ (d) $2x + y + 3 = 0$

Paragraph IV

(Q. Nos. 55 to 57)

If l and m are variable real numbers such that $5l^2 - 4lm + 6m^2 + 3l = 0$, *then the variable line* $lx + my = 1$ *always touches a fixed parabola, whose axis is parallel to the X-axis.*

55. If (a, b) is the vertex of the parabola, then the value of $|a - b|$ is
(a) 2 (b) 3
(c) 4 (d) 5

56. If (c, d) is the focus of the parabola, then the value of $2^{|d-c|}$ is
(a) 1 (b) 2 (c) 4 (d) 8

57. If $ex + f = 0$ is directrix of the parabola and e, f are prime numbers, then the value of $|e - f|$ is
(a) 2 (b) 4
(c) 6 (d) 8

Paragraph V

(Q. Nos. 58 to 60)

C_1 is a curve $y^2 = 4x$, *C_2 is curve obtained by rotating C_1, 120° in anti-clockwise direction C_3 is reflection of C_2 with respect to* $y = x$ *and S_1, S_2, S_3 are focii of C_1, C_2 and C_3, respectively, where O is origin.*

58. If $(t^2, 2t)$ are parametric form of curve C_1, then the parametric form of curve C_2 is

(a) $\left(\frac{1}{2}(t^2 + 2\sqrt{3}t), \frac{1}{2}(\sqrt{3}t^2 + 2t)\right)$

(b) $\left(\frac{1}{2}(-t^2 + 2\sqrt{3}t), \frac{1}{2}(\sqrt{3}t^2 + 2t)\right)$

(c) $\left(\frac{1}{2}(-t^2 + 2\sqrt{3}t), \frac{1}{2}(-\sqrt{3}t^2 + 2t)\right)$

(d) $\left(\frac{1}{2}(-t^2 + 2\sqrt{3}t), \frac{1}{2}(-\sqrt{3}t^2 - 2t)\right)$

59. Area of ΔOS_2S_3 is

(a) $\frac{1}{8}$ (b) $\frac{1}{4}$

(c) $\frac{1}{2}$ (d) 1

60. If $S_1(x_1, y_1)$, $S_2(x_2, y_2)$ and $S_3(x_3, y_3)$, then the value of $\Sigma x_1^2 + \Sigma y_1^2$ is

(a) 2 (b) 3

(c) 4 (d) 5

Paragraph VI

(Q. Nos. 61 to 63)

Tangent to the parabola $y = x^2 + ax + 1$ at the point of intersection of the Y-axis also touches the circle $x^2 + y^2 = c^2$. It is known that no point of the parabola is below X-axis.

61. The value of $5c^2$ when a attains its maximum value is

(a) 1 (b) 3

(c) 5 (d) 7

62. The slope of the tangent when C is maximum, is

(a) -1 (b) 0 (c) 1 (d) 2

63. Let Δ be the minimum area bounded by the tangent and the coordinate axes, then the value of 8Δ is

(a) 1 (b) 2 (c) 4 (d) 8

Paragraph VII

(Q. Nos. 64 to 66)

A parabola (P) touches the conic

$$x^2 + xy + y^2 - 2x - 2y + 1 = 0$$

at the points when it is cut by the line $x + y + 1 = 0$.

64. If equation of parabola (P) is $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, then the value of $|a + b + c + f + g + h|$ is

(a) 8 (b) 10 (c) 12 (d) 14

65. The length of latusrectum of parabola (P) is

(a) $\sqrt{2}$ (b) $3\sqrt{2}$ (c) $5\sqrt{2}$ (d) $7\sqrt{2}$

66. If (a, b) is the vertex of the parabola (P), then the value of $|a - b|$ is

(a) 0 (b) $\frac{1}{2}$ (c) 1 (d) $\frac{3}{2}$

Paragraph VIII

(Q. Nos. 67 to 69)

$y = 3x$ is tangent to the parabola $2y = ax^2 + b$.

67. The minimum value of $a + b$ is

(a) 2 (b) 4 (c) 6 (d) 8

68. If $(2, 6)$ is the point of contact, then the value of $2a$ is

(a) 2 (b) 3 (c) 4 (d) 5

69. If $b = 18$, then the point of contact is

(a) (1, 3) (b) (2, 6)

(c) (3, 9) (d) (6, 18)

Parabola Exercise 4 : Single Integer Answer Type Questions

- This section contains **10 questions**. The answer to each question is a **single digit integer**, ranging from 0 to 9 (both inclusive).

70. Two tangents are drawn from the point $(-2, -1)$ to the parabola $y^2 = 4x$. If θ is the angle between these tangents, then the value of $\tan\theta$ is

71. If the distances of two points P and Q from the focus of a parabola $y^2 = 4x$ are 4 and 9 respectively, the distance of the point of intersection of tangents at P and Q from the focus is

72. The tangents and normals are drawn at the extremities of the latusrectum of the parabola $y^2 = 4x$. The area of quadrilateral so formed is λ sq units, the value of λ is

73. Three normals are drawn from the point $(a, 0)$ to the parabola $y^2 = x$. One normal is the X-axis. If other two normals are perpendicular to each other, then the value of $4a$ is

74. AB is the chord of the parabola $y^2 = 6x$ with the vertex at A. BC is drawn perpendicular to AB meeting the axis at C. The projection of BC on the axis is λ units, then the value of λ is

75. The parabolas $y = x^2 - 9$ and $y = \lambda x^2$ intersect at points A and B. If length of AB is equal to $2a$ and if $\lambda a^2 + \mu = a^2$, then the value of μ is

76. Let n be the number of integral points lying inside the parabola $y^2 = 8x$ and circle $x^2 + y^2 = 16$, then the sum of the digits of number n is

77. Radius of the largest circle which passes through the focus of the parabola $y^2 = 4x$ and contained in it, is

78. If the circle $(x-6)^2 + y^2 = r^2$ and the parabola $y^2 = 4x$ have maximum number of common chords, then the least integral value of r is

79. The slope of the line which belongs to the family of lines $(1+a)x + (a-1)y + 2(1-a) = 0$ and makes shortest intercept on $x^2 - 4y + 4 = 0$ is

Parabola Exercise 5 : Matching Type Questions

- This section contains **3 questions.** Each question has four statements (A, B, C and D) given in **Column** I and four statements (p, q, r and s) in **Column** II. Any given statement in **Column** I can have correct matching with one or more statement(s) given in **Column** II.

80. Match the following.

	Column I		**Column II**
(A)	The number of common chords of the parabola $x = y^2 - 6y + 11$ and $y = x^2 - 6x + 11$ is	(p)	Prime number
(B)	AB is a chord of the parabola $y^2 = 4x$ with vertex A, BC is drawn perpendicular to AB meeting the axis at C. The projection of BC on the axis of the parabola is	(q)	Composite number
(C)	The maximum number of common normals of $y^2 = 4ax$ and $x^2 = 4by$ is	(r)	Perfect number
(D)	If the locus of the middle of point of contact of tangents drawn to the parabola $y^2 = 8x$ and the foot of perpendicular drawn from its focus to the tangents is a conic, then the length of latusrectum of this conic is	(s)	Even number

81. Match the following.

	Column I		**Column II**
(A)	If the parabola $x^2 = ay$ makes an intercept of length $\sqrt{40}$ on the line $y - 2x = 1$, then the values of a are	(p)	-2
(B)	If the tangents drawn from the point $(0, 2)$ to the parabola $y^2 = 4ax$ are inclined at an angle $\frac{3\pi}{4}$, then the values of a are	(q)	1
(C)	If two distinct chords of a parabola $y^2 = 4ax$ passing through $(a, 2a)$ are bisected on the line $x + y = 1$, then the length of latusrectum can be	(r)	2
(D)	If the focus of the parabola $x^2 - ay + 3 = 0$ is $(0, 2)$ and if two values of a are a_1, a_2 such that $a_1 > a_2$, then the value of $\frac{a_1}{a_2}$ is	(s)	3

82. Match the following.

	Column I		**Column II**
(A)	The common chord of the circle $x^2 + y^2 = 5$ and the parabola $6y = 5x^2 + 7x$ will passes through the point (s)	(p)	$(1, 2)$
(B)	Tangents are drawn from point $(2, 3)$ to the parabola $y^2 = 4x$. Then, the points of contact are	(q)	$(4, 4)$
(C)	From a point P on the circle $x^2 + y^2 = 5$, the equation of chord of contact to the parabola $y^2 = 4x$ is $y = 2(x - 2)$. Then, the coordinates of point P will be	(r)	$(-2, 1)$
(D)	$P(4, -4)$ and Q are points on the parabola $y^2 = 4x$ such that the area of ΔPOQ is 6 sq units, where O is the vertex. Then, the coordinates of Q may be	(s)	$(9, -6)$

Parabola Exercise 6 :
Statement I and II Type Questions

■ **Directions** (Q. Nos. 83 to 90) are Assertion-Reason type questions. Each of these questions contains two statements:
Statement I (Assertion) and
Statement II (Reason)
Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below :

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

83. **Statement I** The equation of the common tangent to the parabolas $y^2 = 4x$ and $x^2 = 4y$ is $x + y + 1 = 0$.

Statement II Both the parabolas are reflected to each other about the line $y = x$.

84. **Statement I** Two perpendicular normals can be drawn from the point $\left(\frac{5}{2}, -2\right)$ to the parabola $(y+2)^2 = 2(x-1)$.

Statement II Two perpendicular normals can be drawn from the point $(3a, 0)$ to the parabola $y^2 = 4ax$.

85. **Statement I** The line $y = mx + \frac{a}{m}$ is tangent to the parabola $y^2 = 4ax$ for all values of m.

Statement II A straight line $y = mx + c$ that intersects the parabola $y^2 = 4ax$ one point is a tangent line.

86. **Statement I** The conic $\sqrt{ax} + \sqrt{by} = 1$ represents a parabola.

Statement II Conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a parabola, if $h^2 = ab$.

87. **Statement I** The lines from the vertex to the two extremities of a focal chord of the parabola $y^2 = 4ax$ are perpendicular to each other.

Statement II If extremities of focal chord of a parabola are $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$, then $t_1 t_2 = -1$.

88. **Statement I** Length of focal chord of a parabola $y^2 = 8x$ making an angle of $60°$ with X-axis is $32/3$.

Statement II Length of focal chord of a parabola $y^2 = 4ax$ making an angle α with X-axis is $4a\sec^2(\alpha/2)$.

89. **Statement I** Straight line $x + y = \lambda$ touch the parabola $y = x - x^2$, if $k = 1$.

Statement II Discriminant of $(x-1)^2 = x - x^2$ is zero.

90. **Statement I** Length of latusrectum of parabola $(3x + 4y + 5)^2 = 4(4x + 3y + 2)$ is 4.

Statement II Length of latusrectum of parabola $y^2 = 4ax$ is $4a$.

Parabola Exercise 7 :
Subjective Type Questions

■ In this section, there are **15 subjective questions**.

91. If the tangent to the parabola $y^2 = 4ax$ meets the axis in T and tangent at the vertex A in Y and the rectangle $TAYG$ is completed, show that the locus of G is $y^2 + ax = 0$.

92. If incident ray from point $(-1, 2)$ parallel to the axis of the parabola $y^2 = 4x$ strikes the parabola, find the equation of the reflected ray.

93. Prove that the normal chord to a parabola at the point whose ordinate is equal to the abscissa subtends a right angle at the focus.

94. Find the shortest distance between the parabola $y^2 = 4x$ and circle $x^2 + y^2 - 24y + 128 = 0$.

95. Show that the locus of a point that divides a chord of slope 2 of the parabola $y^2 = 4x$ internally in the ratio 1 : 2 is a parabola. Find the vertex of this parabola.

96. Show that the locus of the points of intersection of tangents to $y^2 = 4ax$, which intercept a constant length d on the directrix is $(y^2 - 4ax)(x + a)^2 = d^2x^2$.

97. Through the vertex O of the parabola $y^2 = 4ax$ two chords OP and OQ are drawn and the circle on OP and OQ as diameters intersect in R. If θ_1 and θ_2 are the angles made with the axes by the tangents at P and Q to the parabola and ϕ is the angle made by OR with the axis of the parabola, then prove that $\cot\theta_1 + \cot\theta_2 + 2\tan\phi = 0$.

98. Three normals with slopes m_1, m_2 and m_3 are drawn from a point P not on the axis of the parabola $y^2 = 4x$. If $m_1m_2 = \alpha$, results in the locus of P being a part of the parabola, find the value of α.

99. Find the locus of centres of a family of circles passing through the vertex of the parabola $y^2 = 4ax$ and cutting the parabola orthogonally at the other point of intersection.

100. TP and TQ are tangents to the parabola $y^2 = 4ax$. The normals at P and Q intersect at R on the curve. Prove that the circle circumscribing the ΔTPQ lies on the parabola $2y^2 = a(x - a)$.

101. A family of chords of the parabola $y^2 = 4ax$ is drawn so that their projections on a straight line inclined equally to both the axes are all of a constant length c; prove that the locus of their middle points is the curve $(y^2 - 4ax)(y + 2a)^2 + 2a^2c^2 = 0$.

102. The normals at P, Q, R are concurrent and PQ meets the diameter through R on the directrix $x = -a$. Prove that PQ touches [or PQ envelopes] the parabola $y^2 + 16a(x + a) = 0$.

103. If the normals to the parabola $y^2 = 4ax$ at three points P, Q and R meet at A and S be the focus, prove that $SP \cdot SQ \cdot SR = a(SA)^2$.

104. From a point A common tangents are drawn to the circle $x^2 + y^2 = (a^2/2)$ and the parabola $y^2 = 4ax$. Find the area of the quadrilateral formed by the common tangents, the chords of contact of the point A, with respect to the circle and the parabola.

105. Prove that the any three tangents to a parabola whose slopes are in harmonic progression enclose a triangle of constant area.

Parabola Exercise 8 : Questions Asked in Previous 13 Year's Exams

■ This section contains questions asked in **IIT-JEE, AIEEE, JEE Main & JEE Advanced** from year **2005** to **2017**.

106. Tangent to the curve $y = x^2 + 6$ at a point (1, 7) touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at a point Q. Then the coordinates of Q are **[IIT-JEE 2005, 3M]**

(a) $(-6, -11)$ (b) $(-9, -13)$
(c) $(-10, -15)$ (d) $(-6, -7)$

107. Let P be a point (1, 0) and Q a point on the locus $y^2 = 8x$. The locus of mid-point of PQ is **[AIEEE 2005, 3M]**

(a) $x^2 - 4y + 2 = 0$ (b) $x^2 + 4y + 2 = 0$
(c) $y^2 + 4x + 2 = 0$ (d) $y^2 - 4x + 2 = 0$

108. The axis of a parabola is along the line $y = x$ and the distance of its vertex from origin is $\sqrt{2}$ and that from its focus is $2\sqrt{2}$. If vertex and focus both lie in the first quadrant, the equation of the parabola is **[IIT-JEE 2006, 3M]**

(a) $(x + y)^2 = (x - y - 2)$ (b) $(x - y)^2 = (x + y + 2)$
(c) $(x - y)^2 = 4(x + y - 2)$ (d) $(x - y)^2 = 8(x + y - 2)$

109. The equations of the common tangents to the parabolas $y = x^2$ and $y = -(x - 2)^2$ is/are **[IIT-JEE 2006, 5M]**

(a) $y = 4(x - 1)$ (b) $y = 0$
(c) $y = -4(x - 1)$ (d) $y = -30x - 50$

110. The locus of the vertices of the family of parabolas $y = \frac{a^3x^2}{3} + \frac{a^2x}{2} - 2a$ is **[AIEEE 2006, 4.5 M]**

(a) $xy = \frac{105}{64}$ (b) $xy = \frac{3}{4}$
(c) $xy = \frac{35}{16}$ (d) $xy = \frac{64}{105}$

111. Angle between the tangents to the curve $y = x^2 - 5x + 6$ at the points (2, 0) and (3, 0) is **[AIEEE 2006, 4.5 M]**

(a) $\pi/3$ (b) $\pi/2$
(c) $\pi/6$ (d) $\pi/4$

112. Consider the circle $x^2 + y^2 = 9$ and the parabola $y^2 = 8x$. They intersect at P and Q in the first and fourth quadrants, respectively. Tangents to the circle at P and Q intersect the X-axis at R and tangents to the parabola at P and Q intersect the X-axis at S.

(i) The ratio of the areas of the ΔPQS and ΔPQR is

(a) $1:\sqrt{2}$ (b) 1 : 2

(c) 1 : 4 (d) 1 : 8

(ii) The radius of the circumcircle of the ΔPRS is

(a) 5 (b) $3\sqrt{3}$

(c) $3\sqrt{2}$ (d) $2\sqrt{3}$

(iii) The radius of the incircle of the ΔPQR is

[IIT-JEE 2007, (4 + 4 + 4) M]

(a) 4 (b) 3

(c) 8/3 (d) 2

113. **Statement I** The curve $y = -\frac{x^2}{2} + x + 1$ is symmetric with respect to the line $x = 1$ because

Statement II A parabola is symmetric about its axis.

[IIT-JEE 2007, 3M]

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I

(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I

(c) Statement I is true, Statement II is false

(d) Statement I is false, Statement II is true

114. The equation of a tangent to the parabola $y^2 = 8x$ is $y = x + 2$. The point on this line from which the other tangent to the parabola is perpendicular to the given tangent is

[AIEEE 2007, 3M]

(a) $(-1, 1)$ (b) $(0, 2)$

(c) $(2, 4)$ (d) $(-2, 0)$

115. Consider the two curves $C_1: y^2 = 4x, C_2: x^2 + y^2 - 6x + 1 = 0$, then

[IIT-JEE 2008, 3M]

(a) C_1 and C_2 touch each other only at one point

(b) C_1 and C_2 touch each other exactly at two points

(c) C_1 and C_2 intersect (but do not touch) at exactly two points

(d) C_1 and C_2 neither intersect nor touch each other

116. A parabola has the origin as its focus and the line $x = 2$ as the directrix. The vertex of the parabola is at

[AIEEE 2008, 3M]

(a) (0, 2) (b) (1, 0)

(c) (0, 1) (d) (2, 0)

117. The tangent PT and the normal PN to the parabola $y^2 = 4ax$ at a point P on it meet its axis at points T and N, respectively. The locus of the centroid of the ΔPTN is a parabola whose

[IIT-JEE 2009, 4M]

(a) vertex is $\left(\frac{2a}{3}, 0\right)$ (b) directrix is at $x = 0$

(c) latusrectum is $\frac{2a}{3}$ (d) focus is $(a, 0)$

118. Let A and B be two distinct points on the parabola $y^2 = 4x$. If the axis of the parabola touches a circle of radius r having AB as its diameter, The slope of the line joining A and B can be

[IIT-JEE 2010, 3M]

(a) $-\frac{1}{r}$ (b) $\frac{1}{r}$ (c) $\frac{2}{r}$ (d) $-\frac{2}{r}$

119. If two tangents drawn from a point P to the parabola $y^2 = 4x$ are at right angles, the locus of P is

[AIEEE 2010, 4M]

(a) $2x + 1 = 0$ (b) $x = -1$

(c) $2x - 1 = 0$ (d) $x = 1$

120. Consider the parabola $y^2 = 8x$. Let Δ_1 be the area of the triangle formed by the end points of its latusrectum and the point $P\left(\frac{1}{2}, 2\right)$ on the parabola and Δ_2 be the area of the triangle formed by drawing tangent at P and at the end points of the latusrectum. Then, $\frac{\Delta_1}{\Delta_2}$ is

[IIT-JEE 2011, 4M]

121. Let (x, y) be any point on the parabola $y^2 = 4x$. Let P be the point that divides the line segment from $(0, 0)$ to (x, y) in the ratio 1 : 3. Then, the locus of P is

[IIT-JEE 2011, 3M]

(a) $x^2 = y$ (b) $y^2 = 2x$

(c) $y^2 = x$ (d) $x^2 = 2y$

122. Let L be a normal to the parabola $y^2 = 4x$. If L passes through the point (9, 6), then L is given by

[IIT-JEE 2011, 4M]

(a) $y - x + 3 = 0$ (b) $y + 3x - 33 = 0$

(c) $y + x - 15 = 0$ (d) $y - 2x + 12 = 0$

123. The shortest distance between line $y - x = 1$ and curve $x = y^2$ is

[AIEEE 2011, 4M]

(a) $\frac{3\sqrt{2}}{8}$ (b) $\frac{8}{3\sqrt{2}}$

(c) $\frac{4}{\sqrt{3}}$ (d) $\frac{\sqrt{3}}{4}$

124. Let S be the focus of the parabola $y^2 = 8x$ and let PQ be the common chord of the circle $x^2 + y^2 - 2x - 4y = 0$ and the given parabola. The area of the ΔPQS is **[IIT-JEE 2012, 4M]**

Paragraph

(Q. Nos. 125 and 126)

Let PQ be a focal chord of the parabola $y^2 = 4ax$. The tangent to the parabola at P and Q meet at a point lying on the line $y = 2x + a, a > 0$. **[JEE Advanced 2013, 3+3 M]**

125. If chord PQ subtends an angle θ at the vertex of $y^2 = 4ax$, them $\tan\theta$ is equal to

(a) $\frac{2}{3}\sqrt{7}$ (b) $-\frac{2}{3}\sqrt{7}$ (c) $\frac{2}{3}\sqrt{5}$ (d) $-\frac{2}{3}\sqrt{5}$

126. Length of chord PQ is

(a) $7a$ (b) $5a$ (c) $2a$ (d) $3a$

127. The slope of the line touching the parabolas $y^2 = 4x$ and $x^2 = -32y$ is **[JEE Main 2014, 4M]**

(a) 1/8 (b) 2/3
(c) 1/2 (d) 3/2

128. The common tangent to the circle $x^2 + y^2 = 2$ and the parabola $y^2 = 8x$ touch the circle at the points P, Q and the parabola at the points R, S. Then, the area of the quadrilateral $PQRS$ is **[JEE Advanced 2014, 3 M]**

(a) 3 (b) 6 (c) 9 (d) 15

Paragraph

(Q. Nos. 129 and 130)

Let a, r, s and t be non-zero real numbers. Let $P(at^2\ 2at)$, $Q\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$, $R(ar^2, 2ar)$ and $S(as^2, 2as)$ be distinct points on the parabola $y^2 = 4ax$. Suppose that PQ is the focal chord and lines QR and PK are parallel, where K is the point $(2a, 0)$. **[JEE Advanced 2014, (3 + 3) M]**

129. The value of r is

(a) $-\frac{1}{t}$ (b) $\frac{t^2+1}{t}$ (c) $\frac{1}{t}$ (d) $\frac{t^2-1}{t}$

130. If $st = 1$, then the tangent at P and the normal at S to the parabola meet at a point whose ordinate is

(a) $\frac{(t^2+1)^2}{2t^3}$ (b) $\frac{a(t^2+1)^2}{2t^3}$
(c) $\frac{a(t^2+1)^2}{t^3}$ (d) $\frac{a(t^2+2)^2}{t^3}$

131. Let O be the vertex and Q be any point on the parabola $x^2 = 8y$. If the point P divides the line segment OQ internally in the ratio 1 : 3, then the locus of P is **[JEE Main 2015, 4M]**

(a) $x^2 = y$ (b) $y^2 = x$
(c) $y^2 = 2x$ (d) $x^2 = 2y$

132. If the normals of the parabola $y^2 = 4x$ drawn at the end points of its latusrectum are tangents to the circle $(x-3)^2 + (y+2)^2 = r^2$, then the value of r^2 is **[JEE Advanced 2015, 4M]**

133. Let the curve C be the mirror image of the parabola $y^2 = 4x$ with respect to the line $x + y + 4 = 0$. If A and B are the points of intersection of C with the line $y = -5$, the distance between A and B is **[JEE Advanced 2015, 4M]**

134. Let P and Q be distinct points on the parabola $y^2 = 2x$ such that a circle with PQ as diameter passes through the vertex O of the parabola. If P lies in the first quadrant and the area of the ΔOPQ is $3\sqrt{2}$, then which of the following is (are) the coordinates of P? **[JEE Advanced 2015, 4M]**

(a) $(4, 2\sqrt{2})$ (b) $(9, 3\sqrt{2})$
(c) $\left(\frac{1}{4}, \frac{1}{\sqrt{2}}\right)$ (d) $(1, \sqrt{2})$

135. Let P be the point on the parabola $y^2 = 8x$, which is at a minimum distance from the centre C of the circle $x^2 + (y+6)^2 = 1$, the equation of the circle passing through C and having its centre at P, is **[JEE Main 2016, 4M]**

(a) $x^2 + y^2 - 4x + 8y + 12 = 0$
(b) $x^2 + y^2 - x + 4y - 12 = 0$
(c) $x^2 + y^2 - \frac{x}{4} + 2y - 24 = 0$
(d) $x^2 + y^2 - 4x + 9y + 18 = 0$

136. The circle $C_1 : x^2 + y^2 = 3$ with centre at O, intersects the parabola $x^2 = 2y$ at the point P in the first quadrant. Let the tangent to the circle C_1 at P touches other two circles C_2 and C_3 at R_2 and R_3, respectively. Suppose C_2 and C_3 have equal radii $2\sqrt{3}$ and centres Q_2 and Q_3, respectively. If Q_2 and Q_3 lie on the Y-axis, then **[JEE Advanced 2016, 4M]**

(a) $Q_2Q_3 = 12$
(b) $R_2R_3 = 4\sqrt{6}$
(c) area of ΔOR_2R_3 is $6\sqrt{2}$
(d) area of ΔPQ_2Q_3 is $4\sqrt{2}$

137. Let P be the point on the parabola $y^2 = 4x$ which is at the shortest distance from the centre S of the circle $x^2 + y^2 - 4x - 16y + 64 = 0$. Let Q be the point on the circle dividing the line segment SP internally. Then,

(a) $SP = 2\sqrt{5}$ **[JEE Advanced 2016, 4M]**

(b) $SQ : QP = (\sqrt{5} + 1) : 2$

(c) the x-intercept of the normal to the parabola at P is 6

(d) the slope of the tangent to the circle at Q is $\frac{1}{2}$

138. The radius of a circle, having minimum area, which touches the curve $y = 4 - x^2$ and the lines, $y = |x|$ is **[JEE Main 2017, 4M]**

(a) $4(\sqrt{2} + 1)$

(b) $2(\sqrt{2} + 1)$

(c) $2(\sqrt{2} - 1)$

(d) $4(\sqrt{2} - 1)$

139. If a chord, which is not a tangent of the parabola $y^2 = 16x$ has the equation $2x + y = p$, and mid-point (h, k), then which of the following is (are) possible value(s) of p, h and k ? **[JEE Advanced 2017, 4M]**

(a) $p = 2, h = 3, k = -4$

(b) $p = -1, h = 1, k = -3$

(c) $p = -2, h = 2, k = -4$

(d) $p = 5, h = 4, k = -3$

Answers

Exercise for Session 1

1. (a) **2.** (d) **3.** (c) **4.** (b) **5.** (c) **6.** (c)
7. (b) **8.** (c) **9.** (c) **10.** (b) **11.** (d) **12.** (b)
13. $16x^2 + 9y^2 + 24xy - 256x - 142y + 849 = 0$
14. $4x^2 + y^2 - 4xy + 104x + 148y - 124 = 0$
15. $\left(\frac{-7}{2}, \frac{5}{2}\right); \left(\frac{-17}{2}, \frac{5}{2}\right); y = \frac{5}{2}; x + \frac{11}{4} = 0; 3$
16. Parabola **17.** Parabola
19. $x = \frac{2}{5}y^2 - 5y + \frac{68}{5}; \frac{5}{2}$ **20.** 4

Exercise for Session 2

1. (d) **2.** (b, d) **3.** (a) **4.** (a) **5.** (d) **6.** (d)
7. (b) **9.** $x + 4a = 0$ **10.** (a) $y = 2x - 12$ (b) $y = 3x - 33$
11. $y^2 = \frac{12}{343}(x - 6)^2$ **13.** $\lambda = \frac{3}{2}$

Exercise for Session 3

1. (b) **2.** (a) **3.** (d) **4.** (a) **5.** (c) **6.** (c)
7. (a) **8.** (c) **10.** $y^2 = 2a(x - a)$ **11.** $y = x - 1, 8\sqrt{2}$ sq units

Chapter Exercise

1. (a) **2.** (b) **3.** (b) **4**. (d) **5.** (d) **6**. (a)
7. (a) **8.** (a) **9**. (d) **10.** (b) **11.** (a) **12.** (c)
13. (a) **14.** (d) **15.** (d) **16.** (a) **17**. (a) **18.** (c)
19. (d) **20.** (c) **21.** (d) **22.** (a) **23.** (c) **24.** (a)
25. (a) **26.** (c) **27.** (b) **28.** (b) **29.** (c) **30.** (d)
31. (b,c) **32.** (a,b) **33.** (b,d) **34.** (b,c,d) **35.** (a,b,c, d) **36.** (a, c)
37. (a,c) **38.** (b,d) **39.** (a,c) **40**. (a,c) **41.** (b,c,d) **42.** (a,b,c,d)
43. (a,c,d) **44.**(a,b,c) **45.** (a,b) **46.** (d) **47.** (d) **48.** (c)
49. (b) **50.** (c) **51.** (d) **52.** (d) **53.** (c) **54.** (d)
55. (b) **56.** (b) **57.** (d) **58.** (d) **59.** (b) **60.** (b)
61. (a) **62.** (b) **63.** (b) **64.** (c) **65.** (d) **66.** (a)
67. (c) **68.** (b) **69.** (d) **70.** (3) **71.** (6) **72.** (8)
73. (3) **74.** (6) **75**. (9) **76.** (8) **77.** (4) **78.** (5)
79. (0)
80. (A) $\rightarrow$ (q,r,s); (B) $\rightarrow$ (q,s); (C) $\rightarrow$ (p); (D) $\rightarrow$ (q)
81. (A) $\rightarrow$ (p,q); (B) $\rightarrow$ (p,r); (C) $\rightarrow$ (q,r,s); (D) $\rightarrow$ (s)
82. (A) $\rightarrow$ (p,r); (B) $\rightarrow$ (p,q); (C) $\rightarrow$ (r); (D) $\rightarrow$ (p,s)
83. (a) **84.** (a) **85.** (a) **88.** (c) **87.** (d) **88.** (c)
89. (c) **90.** (d) **92.** $x = 1$
94. $4(\sqrt{5} - 1)$ **95.** $\left(\frac{8}{9}, \frac{2}{9}\right)$ **98.** (2)
99. $2y^2(2y^2 + x^2 - 12ax) = ax(3x - 4a)^2$
104. $\frac{15a^2}{4}$ sq units **106.** (d) **107.** (d) **108.** (d) **109.** (a, b)
110.(a) **111.** (b) **112.** [i] (c) [ii] (b) [iii] (d) **113.** (a)
114.(d) **115.** (b) **116.** (b) **117.** (a,d) **118.** (c,d) **119.** (b)
120. (2) **121.** (c) **122.** (a,b,d)**123.** (a) **124.** 4 sq units
125. (d) **126.** (b) **127.** (c) **128.** (d) **129.** (d) **130.** (b)
131. (d) **132.** (2) **133.** (4) **134.** (a,d)
135. (a) **136.** (a,b,c) **137.** (a,c,d)**138.** (d) **139.** (a)

Solutions

1. Equation of tangent of $y^2 = 4bx$ is

$$y = mx + \frac{b}{m} \qquad \ldots(i)$$

Here, $m = \tan\frac{\pi}{4} = 1$

From Eq. (i), $y = x + b$

For common tangent $y = x + b$ is also tangent of circle $x^2 + y^2 = a^2$, then

$$\frac{|0 - 0 + b|}{\sqrt{1+1}} = a$$

$\Rightarrow$ $b = a\sqrt{2}$ $[\because\ a > 0, b > 0]$

2. The coordinates of vertex and focus of required parabola are $(a, 0)$ and $(a_1, 0)$ respectively. Therefore, the distance between the vertex and the focus is $AS = a_1 - a$. So, the length of latusrectum is $4(a_1 - a)$. Thus, the equation of the required parabola is

$$(y-0)^2 = 4(a_1 - a)(x - a)$$

or $y^2 = 4(a_1 - a)(x - a)$.

3. The parabolas are equal if the lengths of their latusrectum are equal.

The length of latusrectum of $y^2 = ax$ is a

The equation of second parabola can be written as

$$\sqrt{(x-3)^2 + (y+2)^2} = \left(\frac{3x - 4y - 2}{5}\right)$$

Here, focus is $(3, -2)$ and the equation of directrix is $3x - 4y - 2 = 0$.

$\therefore$ Length of latusrectum= $2\times$ Distance between focus and directrix

$$= 2\left|\frac{9 - 4\times -2 - 2}{\sqrt{(9+16)}}\right| = 6$$

Thus, the two parabolas are equal if $a = 6$.

4. Let $DC = CB = BA = AD = k$

$\therefore$ Coordinates of B are (k, k),

which lie on $y = \lambda\sqrt{x}$

$\therefore$ $k = \lambda\sqrt{k}$

$\therefore$ $k = \lambda^2$

$\therefore$ $BC = k = \lambda^2$ $\ldots(i)$

Also, let $CG = GF = FE = EC = k_1$

$\therefore$ Coordinates of F are $(\lambda^2 + k_1, k_1)$,

which lie on $y = \lambda\sqrt{x}$

Then $k_1 = \lambda\sqrt{(\lambda^2 + k_1)}$

$\Rightarrow$ $k_1^2 = \lambda^4 + \lambda^2 k_1$

or $k_1^2 - \lambda^2 k_1 - \lambda^4 = 0$

$\therefore$ $k_1 = \dfrac{\lambda^2 \pm \sqrt{(\lambda^4 + 4\lambda^4)}}{2}$ $[\because\ k_1 > 0]$

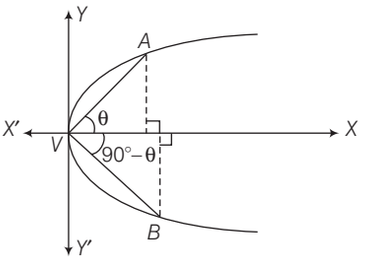
or $\dfrac{k_1}{\lambda^2} = \dfrac{1+\sqrt{5}}{2}$

or $\dfrac{FG}{BC} = \dfrac{\sqrt{5}+1}{2}$

5. Coordinates of A and B are

$(VA\cos\theta, VA\sin\theta)$ and $(VB\sin\theta, -VB\cos\theta)$ respectively

$\because$ A and B lie on $y^2 = x$, then $(VA\sin\theta)^2 = VA\cos\theta$



$\Rightarrow$ $VA = \dfrac{\cos\theta}{\sin^2\theta}$

Similarly, $VB = \dfrac{\sin\theta}{\cos^2\theta}$

$\therefore$ $\dfrac{|VA|}{|VB|} = \cot^3\theta$

6. $\because$ $x = t^2 - t + 1, y = t^2 + t + 1$

$\therefore$ $x + y = 2t^2 + 2$ and $y - x = 2t$

or $2(x + y - 2) = 4t^2 = (2t)^2 = (y - x)^2$

or $(x-y)^2 = 2(x + y - 2)$

or $\left(\dfrac{x-y}{\sqrt{2}}\right)^2 = \sqrt{2}\left(\dfrac{x+y-2}{\sqrt{2}}\right)$

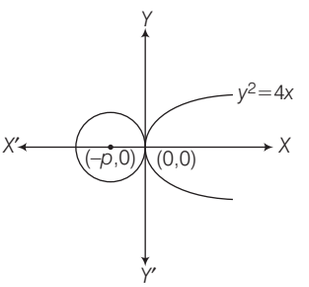
For vertex, $\dfrac{x-y}{\sqrt{2}} = 0$ and $\dfrac{x+y-2}{\sqrt{2}} = 0$

or $x = y$ and $x + y = 2$

$\therefore$ $x = y = 1$

Hence, vertex is $(1, 1)$.

7. It is clear from figure



$-p < 0$ or $p > 0$

8. The point of intersection of the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ are $(0, 0)$ and $(4a, 4a)$ but $a \neq 0$.

Now, $2bx + 3cy + 4d = 0$ passes through $(0, 0)$ and $(4a, 4a)$.

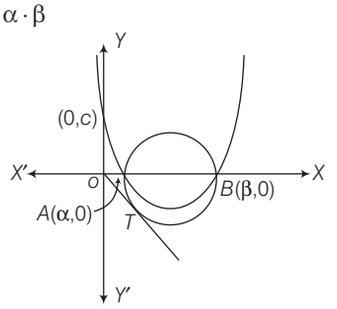
Therefore, $d = 0$ and $2b\,(4a) + 3c\,(4a) = 0$ i.e.,

$$2b + 3c = 0 \qquad [\because\ a \neq 0]$$

or $$d^2 + (2b + 3c)^2 = 0$$

9. $\because (OT)^2 = OA \cdot OB$

$= \alpha \cdot \beta$



or $$OT = \sqrt{\alpha\beta} \qquad \ldots(i)$$

Parabola cuts X-axis at α and β.

$\therefore \alpha, \beta$ are the roots of $ax^2 + bx + c = 0$

$$\therefore \qquad \alpha\beta = \frac{c}{a} \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$OT = \sqrt{\frac{c}{a}}$$

10. $\because$ Two perpendicular tangents meet a point on directrix**.**

Now, equations of tangents at $(at_1^2, 2at_1)$ and $(at_2^2, 2\,at_2)$ are $t_1y = x + at_1^2$ and $t_2y = x + at_2^2$, respectively.

$\therefore \quad P_1 \equiv (-at_1^2, 0), P_2 \equiv (-at_2^2, 0)$

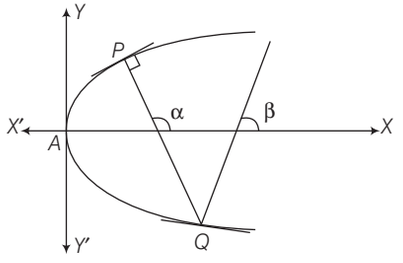
$\Rightarrow \quad SP_1 = a\,(1 + t_1^2), SP_2 = a\,(1 + t_2^2)$ and $t_1t_2 = -1$

$$\therefore \qquad \frac{1}{SP_1} + \frac{1}{SP_2} = \frac{1}{a\,(1 + t_1^2)} + \frac{1}{a\,(1 + t_2^2)}$$

$$= \frac{1}{a\,(1 + t_1^2)} + \frac{1}{a\left(1 + \frac{1}{t_1^2}\right)} \qquad \left[\because t_2 = -\frac{1}{t_1}\right]$$

$$= \frac{1}{a\,(1 + t_1^2)} + \frac{t_1^2}{a\,(1 + t_1^2)} = \frac{1}{a}$$

11. Let $P \equiv (at_1^2, 2at_1)$ and $Q \equiv (at_2^2, 2at_2)$



Equation of normal at P is

$$y + t_1\,x = 2at_1 + at_1^3$$

meet the curve again at Q, then

$$t_2 = -t_1 - \frac{2}{t_1} \qquad \ldots(i)$$

Here $\tan\alpha = -t_1$ and $\tan\beta = -t_2$, from Eq. (i)

$$\therefore \qquad -\tan\beta = \tan\alpha + \frac{2}{\tan\alpha}$$

$$\Rightarrow \qquad \tan\alpha\,(\tan\alpha + \tan\beta) = -2$$

12. Let $A \equiv (\alpha, \beta)$

The equation of normal at $(at^2, 2at)$

$$y + tx = 2at + at^3 \qquad \ldots(i)$$

(α, β) lie on Eq. (i), then

$$at^3 + (2a - \alpha)\,t - \beta = 0 \qquad \ldots(ii)$$

Let t_1, t_2 and t_3 be the roots of Eq. (ii), then

$$at^3 + (2a - \alpha)\,t - \beta = a\,(t - t_1)\,(t - t_2)\,(t - t_3) \qquad \ldots(iii)$$

Let $P \equiv (at_1^2, 2at_1)$, $Q \equiv (at_2^2, 2at_2)$ and $R \equiv (at_3^2, 2at_3)$

Since, the focus is $S\,(a, 0)$

$$\therefore \qquad SP = a\,(1 + t_1^2), SQ = a\,(1 + t_2^2)$$

and $$SR = a\,(1 + t_3^2)$$

On putting $t = i = \sqrt{-1}$ in Eq. (iii), we get

$$-ai + (2a - \alpha)\,i - \beta = a\,(i - t_1)\,(i - t_2)\,(i - t_3)$$

or $|\,(a - \alpha)\,i - \beta\,| = a\,|\,i - t_1\,|\,|\,i - t_2\,|\,i - t_3\,|$

$$\Rightarrow \qquad \sqrt{(a - \alpha)^2 + \beta^2} = a\sqrt{(1 + t_1^2)}\,\sqrt{(1 + t_2^2)}\,\sqrt{(1 + t_3^2)}$$

or $a\,((a - \alpha)^2 + \beta^2) = a\,(1 + t_1^2) \cdot a\,(1 + t_2^2) \cdot a\,(1 + t_3^2)$

$$a\,(SA)^2 = SP \cdot SQ \cdot SR$$

or $$SP \cdot SQ \cdot SR = a\,(SA)^2$$

13. Let AB be a normal chord, where $A \equiv (at_1^2, 2at_1)$ and $B \equiv (at_2^2, 2at_2)$, we have $t_2 = -t_1 - \frac{2}{t_1}$

Now, $$AB = \sqrt{(at_1^2 - at_2^2)^2 + (2at_1 - 2at_2)^2}$$

$$= a\,|\,t_1 - t_2\,|\,\sqrt{(t_1 + t_2)^2 + 4}$$

$$= a\left|\,2\left(t_1 + \frac{1}{t_1}\right)\right|\sqrt{\left(\frac{4}{t_1^2} + 4\right)}$$

$$= 4a\sqrt{\frac{(t_1^2 + 1)^3}{t_1^4}}$$

$$(AB)^2 = 16a^2\,\frac{(t_1^2 + 1)^3}{t_1^4}$$

$$\frac{d\,(AB)^2}{dt_1} = 16a^2\left\{\frac{(1 + t_1^2)^2}{t_1^5}\,(t_1^2 - 2)\right\}$$

For $$\frac{d\,(AB)^2}{dt_1} = 0 \ \Rightarrow\ t_1 = \sqrt{2}$$

For which $(AB)^2$ is minimum, thus

$$AB_{\min} = \sqrt{\frac{16a^2\,(2 + 1)^3}{4}} = 2a\sqrt{27}$$

14. On solving

$$x^2 + y^2 = a^2 \text{ and } y^2 = 4(x + 4)$$

$$\Rightarrow \quad x^2 + 4(x + 4) = a^2$$

or $$x^2 + 4x + 16 - a^2 = 0$$

If the circle and parabola touch each other, then

$$D = 0 \Rightarrow 16 - 4 \cdot 1 \cdot (16 - a^2) = 0$$

$$\Rightarrow \quad a^2 = 12 \text{ or } a = 2\sqrt{3}$$

15. $\sin\theta > |2a|$ $[\because \ h > 2a]$

$$\Rightarrow \quad 0 < |2a| < 1 \quad \text{or} \quad 0 < |a| < \frac{1}{2}$$

$$\therefore \quad a \in \left(-\frac{1}{2}, 0\right) \cup \left(0, \frac{1}{2}\right)$$

16. Tangent to $y^2 = 4x$ in terms of m is

$$y = mx + \frac{1}{m} \qquad \ldots(i)$$

and normal to $x^2 = 4by$ in terms of m is

$$y = mx + 2b + \frac{b}{m^2} \qquad \ldots(ii)$$

$\because$ Eqs. (i) and (ii) are same, then

$$\frac{1}{m} = 2b + \frac{b}{m^2}$$

$$\Rightarrow \quad 2bm^2 - m + b = 0$$

For two different tangents

$$\therefore \quad D > 0 \Rightarrow 1 - 8b^3 > 0$$

or $$|b| < \frac{1}{2\sqrt{2}}$$

17. The given parabolas $2y^2 = 2x - 1$ and $2x^2 = 2y - 1$ are symmetrical about the line $y = x$. The shortest distance occurs along the common normal which is perpendicular to the line $y = x$.

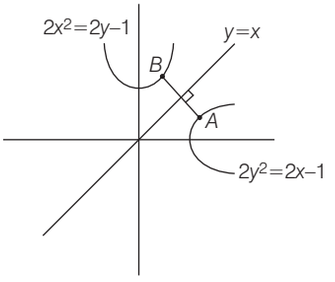
Therefore, the tangent at point A on $2y^2 = 2x - 1$ is parallel to $y = x$. Therefore,

$$4y\frac{dy}{dx} = 2 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{2y} = 1$$

or $$y = \frac{1}{2} \quad \text{and} \quad x = \frac{3}{4}$$

$\therefore$ $A \equiv \left(\frac{3}{4}, \frac{1}{2}\right)$, then coordinates of B on $2x^2 = 2y - 1$ is

$B \equiv \left(\frac{1}{2}, \frac{3}{4}\right)$



$\therefore$ Shortest distance $= AB$

$$= \sqrt{\left(\frac{3}{4} - \frac{1}{2}\right)^2 + \left(\frac{1}{2} - \frac{3}{4}\right)^2}$$

$$= \sqrt{\left(\frac{1}{16} + \frac{1}{16}\right)} = \frac{1}{\sqrt{8}} = \frac{1}{2\sqrt{2}}$$

18. We know that normals at $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ meet again on the parabola. Then $t_1t_2 = 2$

Here, $a = 1$

and $$x_1 = at_1^2 = t_1^2 \text{ and } y_1 = 2at_1 = 2t_1$$

$$x_2 = at_2^2 = t_2^2 \text{ and } y_2 = 2at_2 = 2t_2$$

Given $$x_1 + x_2 = 4 \quad \Rightarrow t_1^2 + t_2^2 = 4$$

or $$(t_1 + t_2)^2 - 2t_1t_2 = 4$$

$$\Rightarrow \quad (t_1 + t_2)^2 = 8 \qquad [\because \ t_1t_2 = 2]$$

or $$|t_1 + t_2| = \sqrt{8}$$

or $$|2t_1 + 2t_2| = 2\sqrt{8}$$

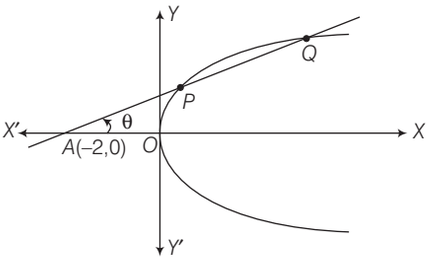
or $$|y_1 + y_2| = 2\sqrt{8} = 4\sqrt{2}$$

19. Let any point at distance r from A on the parabola is

$$(-2 + r\cos\theta, r\sin\theta),$$

then $$r^2\sin^2\theta = 4(-2 + r\cos\theta)$$

or $$r^2\sin^2\theta - 4r\cos\theta + 8 = 0$$



Let P and Q are distances r_1 and r_2 from A, then

$$r_1 + r_2 = \frac{4\cos\theta}{\sin^2\theta}$$

and $$r_1r_2 = \frac{8}{\sin^2\theta}$$

Now, $$\frac{1}{AP} + \frac{1}{AQ} = \frac{1}{r_1} + \frac{1}{r_2} = \frac{r_1 + r_2}{r_1r_2} = \frac{\cos\theta}{2}$$

given that $$\frac{1}{AP} + \frac{1}{AQ} < \frac{1}{4}$$

$$\Rightarrow \quad \frac{\cos\theta}{2} < \frac{1}{4}$$

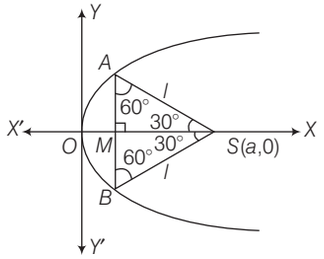
or $$\cos\theta < \frac{1}{2}$$

or $$\tan\theta > \sqrt{3}$$

$$\left[\because \cos\theta \text{ is decreasing and } \tan\theta \text{ is increasing in } \left(0, \frac{\pi}{2}\right)\right]$$

$\therefore$ Slope $(m) > \sqrt{3}$

20. Coordinates of A are $(a - l\cos 30°, l\sin 30°)$



or $$A \equiv \left(a - \frac{l\sqrt{3}}{2}, \frac{l}{2}\right)$$

which lies on $y^2 = 4ax$, then

$$\frac{l^2}{4} = 4a\left(a - \frac{l\sqrt{3}}{2}\right) \Rightarrow l^2 + 8a\sqrt{3}l - 16a^2 = 0$$

$\therefore$ $$l = \frac{-8a\sqrt{3} \pm \sqrt{(192a^2 + 64a^2)}}{2}$$

$$= \frac{-8a\sqrt{3} \pm 16a}{2}$$

$$= 4a(2 - \sqrt{3}) \qquad \text{[taking +ve sign]}$$

Aliter :

Let $A \equiv (at_1^2, 2at_1)$ and $B \equiv (at_1^2 - 2at_1)$.

Slope of $SA = \tan(180° - 30°)$

$$\frac{2at_1}{at_1^2 - a} = -\tan 30°$$

$$\frac{2t_1}{t_1^2 - 1} = -\frac{1}{\sqrt{3}}$$

or $t_1^2 + 2\sqrt{3}t_1 - 1 = 0'$ or $t_1 = -\sqrt{3} \pm 2$

Thus, $t_1 = 2 - \sqrt{3}$ $[\because\ t_1 = -2 - \sqrt{3}$ rejected]

Here, $AB = 4at_1 = 4a(2 - \sqrt{3})$

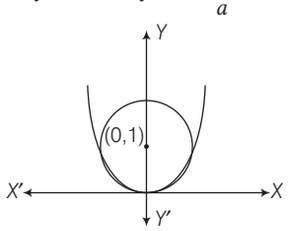
21. $C : x^2 + (y-1)^2 = 1$ and $P : y = ax^2$

Putting $x^2 = \frac{y}{a}$ in $x^2 + (y-1)^2 = 1$

or $$\frac{y}{a} + (y-1)^2 = 1$$

or $$y^2 - 2y + \frac{y}{a} = 0$$

$\therefore$ $$y = 0 \quad \text{or} \quad y = 2 - \frac{1}{a}$$



On substituting $y = 2 - \frac{1}{a}$ in $y = ax^2$, then

$$ax^2 = 2 - \frac{1}{a}$$

or $$x^2 = \frac{2a-1}{a^2} > 0$$

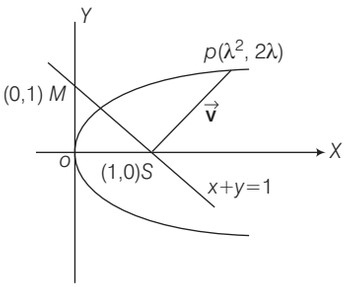
or $$2a - 1 > 0$$

$\therefore$ $$a > \frac{1}{2}$$

or $$a \in \left(\frac{1}{2}, \infty\right)$$

22. $\vec{\mathbf{V}} = (\lambda^2 - 1)\,\hat{\mathbf{i}} + (2\lambda - 0)\,\hat{\mathbf{j}}$

$$\vec{\mathbf{V}} = (\lambda^2 - 1)\,\hat{\mathbf{i}} + (2\lambda)\,\hat{\mathbf{j}} \qquad \text{...(i)}$$



and $$\vec{\mathbf{n}} = (0-1)\,\hat{\mathbf{i}} + (1-0)\,\hat{\mathbf{j}}$$

$$= -\hat{\mathbf{i}} + \hat{\mathbf{j}}$$

The projection of $\vec{\mathbf{V}}$ on $\vec{\mathbf{n}} = y$ [given]

$\therefore$ $$y = \frac{\vec{\mathbf{V}} \cdot \vec{\mathbf{n}}}{|\vec{\mathbf{n}}|} = \frac{-(\lambda^2 - 1) + 2\lambda}{\sqrt{2}} \qquad \text{...(ii)}$$

Given, $$\frac{dx}{dt} = 4$$

$\Rightarrow$ $$\frac{d}{dt}(\lambda^2 - 1) = 4 \quad \Rightarrow \quad 2\lambda\frac{d\lambda}{dt} = 4$$

When $P \equiv (4, 4)$,

We have $\lambda = 2$, therefore

$$\frac{d\lambda}{dt} = 1 \qquad \text{...(iii)}$$

From Eq. (ii),

$$\frac{dy}{dt} = \frac{(2 - 2\lambda)\frac{d\lambda}{dt}}{\sqrt{2}} \qquad \text{[from Eq. (ii)]}$$

$$= \frac{(2-4)\times 1}{\sqrt{2}} = -\sqrt{2}$$

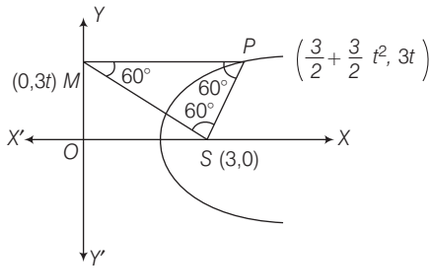
23. Given parabola is

$$y^2 = 6\left(x - \frac{3}{2}\right)$$

The equation of directrix is $x - \frac{3}{2} + \frac{6}{4} = 0$

i.e. $$x = 0$$

Let the coordinates of P be $\left(\frac{3}{2} + \frac{3}{2}t^2, 3t\right)$



Therefore, $MS = MP$

$$\sqrt{(9+9t^2)} = \frac{3}{2} + \frac{3}{2}t^2$$

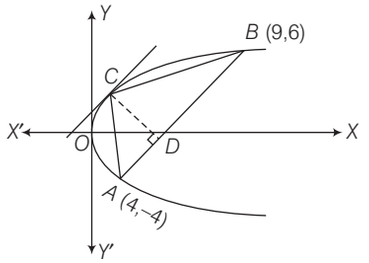
or $$9 + 9t^2 = \frac{9}{4}(1+t^2)^2$$

or $$1 + t^2 = 4$$

$\therefore$ Length of each side $= MS$

$$= \sqrt{9(1+t^2)} = \sqrt{36} = 6$$

24. The area of ΔABC is maximum if CD is maximum, because AB is fixed.



It is clear that tangent drawn to the parabola at C should be parallel to AB.

For $y^2 = 4x$

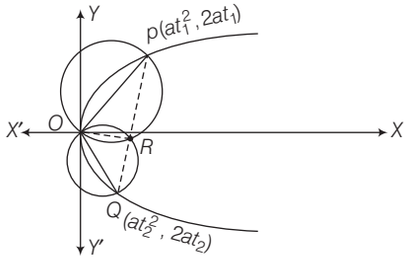
$\therefore$ $$2y\frac{dy}{dx} = 4 \quad \text{or} \quad \frac{dy}{dx} = \frac{2}{y} = \text{slope of } AB$$

$\Rightarrow$ $$\frac{2}{y} = \frac{6+4}{9-4} = 2 \quad \text{or} \quad y = 1, \text{ then } x = \frac{1}{4}$$

Hence, coordinates of C are $\left(\frac{1}{4}, 1\right)$.

25. $\because$ Tangent at $P(at_1^2, 2at_1)$ is $t_1 y = x + at_1^2$.

$\therefore$ $$\tan\theta_1 = \frac{1}{t_1}$$



Similarly, $$\tan\theta_2 = \frac{1}{t_2}$$

$\because$ Angle in a semicircle is 90°.

$\therefore$ $\angle ORP = \angle ORQ = 90°$

$\Rightarrow$ P, R, Q are collinear.

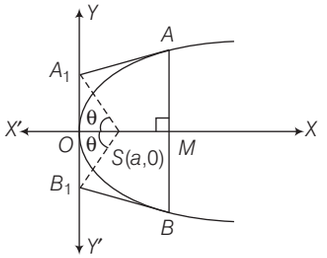
Slope of $PQ = \dfrac{2}{t_1 + t_2}$ $\quad [\because\ OR \perp PQ]$

$\therefore$ Slope of $OR = -\dfrac{(t_1 + t_2)}{2} = \tan\phi$

or $$t_1 + t_2 = -2\tan\phi$$

$\Rightarrow$ $$\cot\theta_1 + \cot\theta_2 = -2\tan\phi$$

26. Let $A \equiv (at^2, 2at)$ and $B \equiv (at^2, -2at)$



The equations of tangents at A and B are

$$ty = x + at^2 \text{ and } -ty = x + at^2.$$

These tangents meet the Y-axis at $A_1(0, at)$ and $B_1(0, -at)$ respectively.

Area of trapezium

$$AA_1B_1B = \frac{1}{2}(AB + A_1B_1) \times OM$$

$$24a^2 = \frac{1}{2}(4at + 2at) \times at^2$$

$$24a^2 = 3a^2t^3 \Rightarrow t^3 = 8$$

$\therefore$ $$t = 2$$

$\therefore$ Coordinates of A_1 and B_1 are $(0, 2a)$ and $(0, -2a)$ respectively.

If $\angle OSA_1 = \angle OSB_1 = \theta$, then

$\therefore$ $$\tan\theta = \frac{2a}{a} = 2$$

$\Rightarrow$ $$\theta = \tan^{-1} 2$$

Hence, subtended angle $= 2\theta = 2\tan^{-1} 2$

27. Given, $$\frac{5}{2} = {}^nC_3(px)^{n-3}\left(\frac{1}{x}\right)^3 = {}^nC_3 \cdot p^{n-3}x^{n-6} \qquad \ldots(i)$$

Since, LHS of Eq. (i) is independent of x

$\therefore$ $$n - 6 = 0 \Rightarrow n = 6$$

From Eq. (i),

$$\frac{5}{2} = {}^6C_3p^3 = 20p^3$$

$\Rightarrow$ $$p^3 = \left(\frac{1}{2}\right)^3 \Rightarrow p = \frac{1}{2}$$

Given, parabola is $y^2 = x$

Here, $$4a = 1$$

$\Rightarrow$ $$a = \frac{1}{4}$$

Since, three normals are drawn from point $(q, 0)$,

$\therefore$ $$q > 2a \text{ or } q > \frac{1}{2} \text{ or } q > p \qquad \left[\because\ p = \frac{1}{2}\right]$$

28. $y^2 - 4x - 2y + 5 = 0$

$\Rightarrow \quad (y-1)^2 - 4x + 4 = 0$

$\Rightarrow \quad (y-1)^2 = 4(x-1)$

Let $y - 1 = Y$ and $x - 1 = X$

then, $y^2 = 4X$ comparing with $Y^2 = 4aX$

$\therefore \quad a = 1$

$\because$ All three normals to the parabola are real and meet on the axis of parabola, then

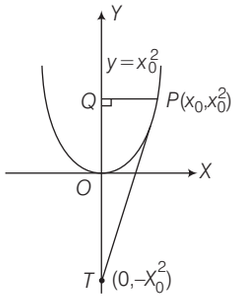
$X > 2a$ and $Y = 0$

i.e., $x - 1 > 2$ and $y - 1 = 0$

$\therefore \quad x > 3$ and $y = 1$

or $(k, 1); k > 3$

29. Let $P(x_0, x_0^2)$ be any point on the parabola $y = x^2$



Equation of tangent at $P(x_0, x_0^2)$ is

$$xx_0 = \frac{1}{2}(y + x_0^2)$$

$\Rightarrow \quad 2xx_0 - y - x_0^2 = 0$

Tangent meets the Y-axis at $T(0, -x_0^2)$.

Hence, the area of the triangle $\Delta PTQ = \frac{1}{2} \times PQ \times QT$

$$= \frac{1}{2} \times x_0 \times 2x_0^2 = x_0^3$$

which increases in the interval $[1, 2]$ and hence is greatest when $x_0 = 2$.

30. $\left(\frac{2x - y + 2}{\sqrt{5}}\right) = \sqrt{(x-2)^2 + (y-0)^2}$ [given]

$\Rightarrow \quad (2x - y + 2)^2 = 5(x^2 + y^2 - 4x + 4)$

or $x^2 + 4y^2 + 4xy = 28x - 4y - 16$

$\Rightarrow \quad (x + 2y)^2 = 4(7x - y - 4)$

$\because \quad x + 2y = 0$ and $7x - y - 4 = 0$ and are not perpendicular.

$\therefore \quad (x + 2y + \lambda)^2 = (2\lambda + 28)x + (4\lambda - 4)y + \lambda^2 - 16$ ...(i)

Now, (slope of $x + 2y + \lambda = 0$) $\times$ (slope of $(2\lambda + 28)x + (4\lambda - 4)y + \lambda^2 - 16 = 0$) $= -1$

$\therefore \quad -\frac{1}{2} \times -\frac{(2\lambda + 28)}{(4\lambda - 4)} = -1$

$\Rightarrow \quad 2\lambda + 28 = -8\lambda + 8$

or $10\lambda = -20$

$\therefore \quad \lambda = -2$

From Eq. (i),

$(x + 2y - 2)^2 = (24x - 12y - 12)$

$\Rightarrow \quad \left(\frac{x + 2y - 2}{\sqrt{5}}\right)^2 = \frac{12}{\sqrt{5}}\left(\frac{2x - y - 1}{\sqrt{5}}\right)$

Let $\frac{x + 2y - 2}{\sqrt{5}} = X, \frac{2x - y - 1}{\sqrt{5}} = Y$

or $X^2 = \frac{12}{\sqrt{5}} Y$ or $Y = \frac{\sqrt{5}}{12} X^2$

$\therefore \quad k = \frac{\sqrt{5}}{12}$

31. Equation of tangent of parabola

$$y^2 = 40x \text{ is } y = mx + \frac{10}{m} \quad \text{... (i)}$$

which is also tangent of circle $x^2 + y^2 = 50$, then

$$\frac{\left|\frac{10}{m}\right|}{\sqrt{(m^2 + 1)}} = 5\sqrt{2}$$

$\Rightarrow \quad m^4 + m^2 - 2 = 0$

$\Rightarrow \quad (m^2 + 2)(m^2 - 1) = 0$

$\therefore \quad m^2 = 1, m^2 + 2 \neq 0$ or $m = \pm 1$

From Eq. (i), common tangents are

$y = x + 10$ and $y = -x - 10$

or $x - y + 10 = 0$ and $x + y + 10 = 0$

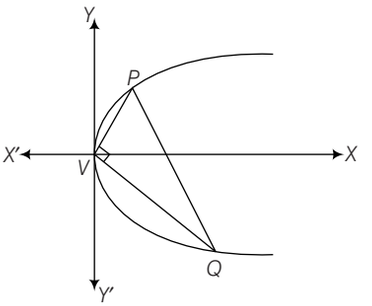
32. Let coordinates of P be $(t^2, 2t)$

$\therefore$ Slope of $PV = \frac{2t - 0}{t^2 - 0} = \frac{2}{t}$

$\Rightarrow$ Slope of QV is $-\frac{t}{2}$

$\therefore$ Equation of QV is $y = \frac{-t}{2}x$

Solving it with $y^2 = 4x$, we get $Q\left(\frac{16}{t^2}, \frac{-8}{t}\right)$



Area of $\Delta PVQ = 20$ (given)

$\Rightarrow \quad \frac{1}{2}\left|\begin{vmatrix} t^2 & 2t \\ \frac{16}{t^2} & \frac{-8}{t} \end{vmatrix}\right| = 20$

$\Rightarrow \quad t + \frac{4}{t} = \pm 5$

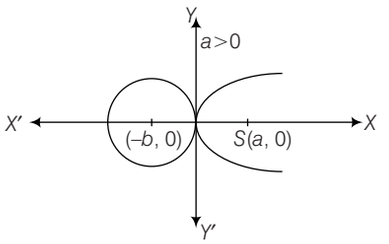
$\Rightarrow \quad t^2 - 5t + 4 = 0$ or $t^2 + 5t + 4 = 0$

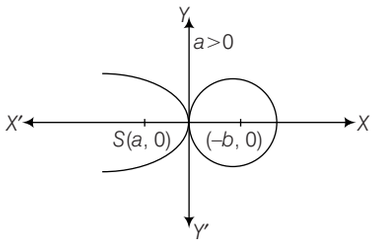
$\therefore \quad t = 1, 4$ or $t = -1, -4$

Hence coordinates of P are $(1, 2), (16, 8), (1 - 2), (16, -8)$

33. Parabola is $y^2 = 4ax$ and circle is $(x + b)^2 + (y - 0)^2 = b^2$

If parabola and circle touch each other externally, then





If $a > 0, -b < 0$ and if $a < 0, -b > 0$

or $a > 0, b > 0$ and $a < 0, b < 0$

34. Let $(x_1, y_1) \equiv (at^2, 2at)$

Equation of tangent at $(at^2, 2at)$ is $ty = x + at^2$

Let any point on this tangent is $\left(\lambda, \dfrac{\lambda + at^2}{t}\right)$

The chord of contact of this point w.r.t the circle $x^2 + y^2 = a^2$ is

$$x \cdot \lambda + y \cdot \left(\frac{\lambda + at^2}{t}\right) = a^2$$

$$\Rightarrow \quad (aty - a^2) + \lambda \cdot \left(x + \frac{y}{t}\right) = 0$$

which is family of straight lines passing through the point of intersection of

$$aty - a^2 = 0 \text{ and } x + \frac{y}{t} = 0$$

So, the fixed point is $\left(\dfrac{-a}{t^2}, \dfrac{a}{t}\right)$, therefore

$$x_2 = -\frac{a}{t^2} \quad , \quad y_2 = \frac{a}{t}$$

Clearly, $x_1x_2 = -a^2 \quad , \quad y_1y_2 = 2a^2$

$$\Rightarrow \quad x_1x_2 + y_1y_2 = a^2$$

Also, $\dfrac{x_1}{x_2} = -t^4 \quad , \quad \dfrac{y_1}{y_2} = 2t^2$

or $$\left(\frac{y_1}{y_2}\right)^2 + 4\left(\frac{x_1}{x_2}\right) = 0$$

$\Rightarrow -4, \dfrac{y_1}{y_2}, \dfrac{x_1}{x_2}$ are in G.P.

Also, $y_1y_2 = 2a^2 \Rightarrow \dfrac{y_1}{2}, a, y_2$ are in G.P.

35. Equation of normal in slope form is

$$y = mx - 2am - am^3,$$

if normals meet at (h, k), then

$$am^3 - (h - 2a)m + k = 0 \qquad \text{...(i)}$$

Let $P \equiv (am_1^2, -2am_1), Q = (am_2^2, -2am_2)$

and $R \equiv (am_3^2, -2am_3)$

$\Rightarrow$ m_1, m_2, m_3 are the roots of Eq. (i), then

$$m_1 + m_2 + m_3 = 0 \qquad \text{...(ii)}$$

i.e. algebraic sum of the slopes of the normals at P, Q and R vanishes.

From Eq. (ii)

$$-2am_1 - 2am_2 - 2am_3 = 0$$

i.e. algebraic sum of the ordinates of the points P, Q and R vanishes.

Also, y-coordinate of centroid of ΔPQR is zero

$\therefore$ centroid lies on X-axis

and circle circumscribing the triangle PQR always passes through the vertex of the parabola.

36. Let $P \equiv (\lambda, \lambda + 1)$, where $\lambda \neq 0, -1$

or $P \equiv (\lambda, \lambda - 1)$, where $\lambda \neq 0, 1$

The point $(\lambda, \lambda + 1)$ is on $y^2 = 4x + 1$, therefore

$$(\lambda + 1)^2 = 4\lambda + 1$$

$$\Rightarrow \quad \lambda^2 - 2\lambda = 0$$

$$\therefore \quad \lambda = 2$$

Therefore, the ordinate of P is 3

and the point $(\lambda, \lambda - 1)$ is on $y^2 = 4x + 1$, therefore

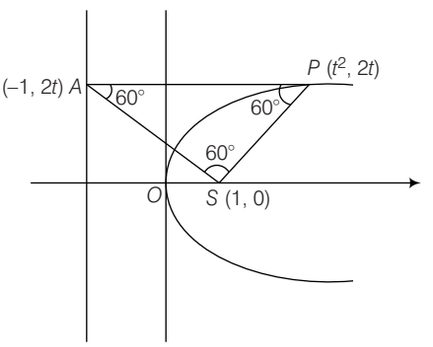
$$(\lambda - 1)^2 = 4(\lambda) + 1$$

$$\Rightarrow \quad \lambda^2 - 6\lambda = 0$$

$$\therefore \quad \lambda = 6$$

Therefore, the ordinate of P is 5,

37. From figure,



Slope of $AP = 0$ and slope of $AS = -t$

$$\therefore \quad \tan 60° = \left|\frac{0 - (-t)}{1 + 0}\right|$$

$$\Rightarrow \quad \sqrt{3} = |t|$$

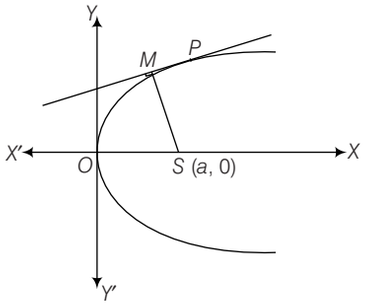
$$\Rightarrow \quad t = \pm\sqrt{3}$$

$\therefore$ Coordinates of P are $(3, \pm 2\sqrt{3})$

38. Let $M(\alpha, \beta)$ be the foot of the perpendicular from the focus $S(a, 0)$ on any tangent to the parabola at $P(at^2, 2at)$.

i.e. $ty + x + at^2$

$\Rightarrow \quad \alpha - t\beta + at^2 = 0 \quad \ldots(i)$



Since, SM is perpendicular to the tangent

$\therefore \quad \frac{1}{t} \times \frac{\beta - 0}{\alpha - a} = -1$

$\Rightarrow \quad \alpha t + \beta - at = 0 \quad \ldots(ii)$

On solving Eqs. (i) and (ii), we get

$$\alpha = 0$$

The locus of $M(\alpha, \beta)$ is the tangent at the vertex.

i.e. $x = 0$

39. Given that the extremities of the latusrectum are (1, 1) and (1, − 1), then $4a = 2$ or $a = \frac{1}{2}$

So, the focus of the parabola is (1, 0).

Hence, the vertex can be $\left(\frac{1}{2}, 0\right)$ or $\left(\frac{3}{2}, 0\right)$.

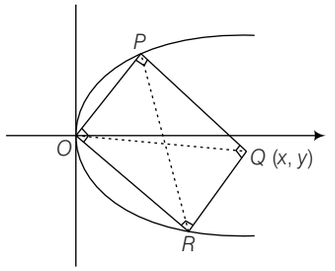
Therefore, the equations of the parabola can be $y^2 = 2\left(x - \frac{1}{2}\right)$

or $y^2 = 2\left(x - \frac{3}{2}\right)$

$\Rightarrow y^2 = 2x - 1$ or $y^2 = 2x - 3$.

40. Let $P \equiv (at^2, 2at)$ and $R \equiv (at_1^2, 2at_1)$

$\because \quad OP \perp OR$



$\therefore$ Slope of $OP \times$ Slope of $OR = -1$

$\Rightarrow \quad \frac{2}{t} \times \frac{2}{t_1} = -1$

$\therefore \quad t_1 = -\frac{4}{t}$

Now, coordinates of R are $\left(\frac{16a}{t^2}, \frac{-8a}{t}\right)$

$\because$ $OPQR$ is a rectangle.

$\therefore$ Mid-point of OQ = mid-point of PR

$\Rightarrow \quad x = at^2 + \frac{16a}{t^2} \quad \Rightarrow \quad \frac{x}{a} = t^2 + \frac{16}{t^2} \quad \ldots$ (i)

and $\quad y = 2at - \frac{8a}{t} \quad \Rightarrow \quad \frac{y}{2a} = t - \frac{4}{t} \quad \ldots$ (ii)

From Eqs. (i) and (ii), we get

$$\left(\frac{y}{2a}\right)^2 = t^2 + \frac{16}{t^2} - 8 = \frac{x}{a} - 8$$

$\Rightarrow \quad y^2 = 4ax - 32a^2$

or $\quad y^2 = 4a(x - 8a)$

which is equal parabola and focus $(9a, 0)$ and directrix

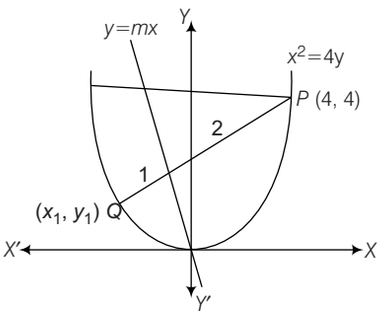
$$x - 8a = -a$$

or $\quad x - 7a = 0.$

41. Let point of intersection of the line $y = mx$ with the chord be $(\lambda, m\lambda)$, then

$$\lambda = \frac{1.4 + 2.x_1}{1 + 2} \quad \Rightarrow \quad x_1 = \frac{3\lambda - 4}{2}$$

and $\quad m\lambda = \frac{1.4 + 2.y_1}{1 + 2} \quad \Rightarrow \quad y_1 = \frac{3m\lambda - 4}{2}$



$\because Q(x_1, y_1)$ lies on the parabola $x^2 = 4y$, then

$$\left(\frac{3\lambda - 4}{2}\right)^2 = 4\left(\frac{3m\lambda - 4}{2}\right)$$

$\Rightarrow \quad 9\lambda^2 - 24\lambda(1 + m) + 48 = 0$

For two distinct chords $\quad D > 0$

$\Rightarrow \quad (24)^2 (1 + m)^2 - 4 \cdot 9 \cdot 48 > 0$

or $\quad (1 + m)^2 > 3$

$\Rightarrow \quad 1 + m < -\sqrt{3}$

or $\quad 1 + m > \sqrt{3}$

$\therefore \quad m < -\sqrt{3} - 1$

or $\quad m > \sqrt{3} - 1$

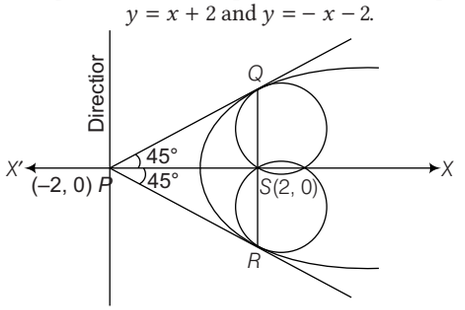
Hence , $m \in (-\infty, -\sqrt{3} - 1) \cup (\sqrt{3} - 1, \infty)$

42. The given parabola is $y^2 = 4(2)x$

$\Rightarrow \quad a = 2$

Since, $P(-2, 0)$ lies on the directrix and the axis.

$\Rightarrow$ The tangents will have slope $m = \pm 1$ and the equations are $y = x + 2$ and $y = -x - 2$.



The chord of contact of tangents is QR as $x = 2$ (i.e. L.R.)

$\because$ Common chord of two circles is X-axis.

$\because$ $\angle QPR = 90°$

$\therefore$ Circumcentre is $S(2, 0)$ on X-axis and orthocentre is $P(-2, 0)$ on X-axis.

Centroid and incentre also lies on X-axis,

($\because$ orthocentre, centroid, circumcentre and incentre are collinear).

43. Given parabola is $(y-2)^2 = 4\left(x - \frac{1}{2}\right)$

Let $x - \frac{1}{2} = X, y - 2 = Y$

$\therefore$ Parabola is $Y^2 = 4X$

Any point on axis of parabola is $(x, 2)$ for three distinct normals $X > 2.1$

$\Rightarrow$ $x - \frac{1}{2} > 2$ or $x > \frac{5}{2}$

$\therefore$ $x = 3, 4, 5$

Hence, points are (3, 2), (4, 2) and (5, 2).

44. The given parabola is $y^2 - 16x - 8y = 0$

$\Rightarrow$ $(y-4)^2 = 16(x+1)$

Shifting the origin to the point $(-1, 4)$ the equation of parabola becomes $y^2 = 16x$

then the coordinates of the point (14, 7) becomes (15, 3).

$\therefore$ Equation of any normal to the parabola is $Y + tX = 8t + 4t^3$.

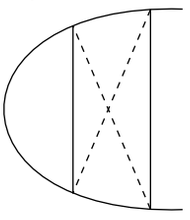
Since, it passes through (15, 3)

$\therefore$ $3 + 15t = 8t + 4t^3 \Rightarrow 4t^3 - 7t - 3 = 0$

or $(t+1)(2t-3)(2t+1) = 0 \Rightarrow t = -1, \frac{3}{2}, -\frac{1}{2}$

$\therefore$ Corresponding points are $(4, -8)$, $(9, 12)$ and $(1, -4)$.
Hence, the coordinates of the feet of the normals w.r.t. the original system of coordinates are $(3, -4)$, $(8, 16)$ and $(0, 0)$.

45. As a circle can intersect a parabola at four points, the quadrilateral may be cyclic.



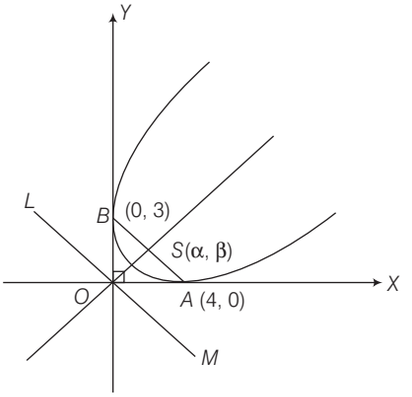
The diagonals of the quadrilateral may be equal as the quadrilateral may be an isosceles trapezium.

A rectangle cannot be inscribed in a parabola.

Sol. (Q. Nos. 46 to 48)

Let $A \equiv (4, 0)$ and $B \equiv (0, 3)$.

$\because$ OA and OB are mutually perpendicular tangents to the parabola. Therefore, O will lie on the directrix of the parabola.



Let $S(\alpha, \beta)$ be the focus of the parabola.

$\because$ Portion of a tangent to a parabola intercepted between the directrix and point of contact subtends a right angle at the focus.

$\therefore$ $\angle OSA = \angle OSB = \frac{\pi}{2}$

Now, $OS \perp SA \Rightarrow$ Slope of $OS \times$ Slope of $SA = -1$

$\Rightarrow$ $\left(\frac{\beta - 0}{\alpha - 0}\right) \times \left(\frac{\beta - 0}{\alpha - 4}\right) = -1$

$\Rightarrow$ $\alpha^2 + \beta^2 - 4\alpha = 0$ ...(i)

Again, $OS \perp SB \Rightarrow$ Slope of $OS \times$ Slope of $SB = -1$

$\Rightarrow$ $\left(\frac{\beta - 0}{\alpha - 0}\right) \times \left(\frac{\beta - 3}{\alpha - 0}\right) = -1$

$\Rightarrow$ $\alpha^2 + \beta^2 - 3\beta = 0$ ...(ii)

From Eqs. (i) and (ii), we get

$4\alpha = 3\beta$ ...(iii)

Since, tangents AO and BO at A and B to parabola are at right angles, therefore AB will be a focal chord of the parabola.

Equation of AB is

$\frac{x}{4} + \frac{y}{3} = 1 \Rightarrow \frac{\alpha}{4} + \frac{\beta}{3} = 1$

$\Rightarrow$ $3\alpha + 4\beta = 12$ ...(iv)

From Eqs. (iii) and (iv), we get

$\alpha = \frac{36}{25}$ and $\beta = \frac{48}{25}$

$\therefore$ Focus of parabola is $\left(\frac{36}{25}, \frac{48}{25}\right)$.

46. $\because$ Focus is (a, b).

$\therefore$ $b - a = \beta - \alpha = \frac{12}{25}$

47. $\because \quad AS = \sqrt{\left(4 - \frac{36}{25}\right)^2 + \left(0 - \frac{48}{25}\right)^2} = \frac{16}{5}$

and $\quad BS = \sqrt{\left(0 - \frac{36}{25}\right)^2 + \left(3 - \frac{48}{25}\right)^2} = \frac{9}{5}$

If l be the semi-latusrectum, then l = HM of AS and BS

$\therefore \quad \frac{2}{l} = \frac{5}{16} + \frac{5}{9} = \frac{125}{144}$

$\Rightarrow \quad l = \frac{288}{125}$

$\therefore \quad 2l = \frac{576}{125}$

48. $\because$ Slope of $OS = \frac{\beta}{\alpha} = \frac{4}{3}$

$\therefore$ Slope of directrix $LM = -\frac{3}{4}$

$\therefore$ Equation of directrix is $y = -\frac{3}{4}x$

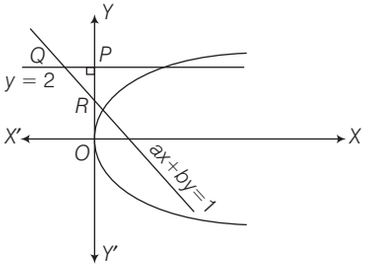
$\Rightarrow \quad 3x + 4y = 0.$

Sol. (Q. Nos. 49 to 51)

Since, (a, b) lies on parabola, $y^2 = 4\lambda x$

$\therefore \quad b^2 = 4a\lambda$ ...(i)

It is clear that ΔPQR is right angled at $P(0, 2)$.



So, its circumcentre is the mid-point of Q and R, where

$$Q \equiv \left(\frac{1-2b}{a}, 2\right) \text{ and } R \equiv \left(0, \frac{1}{b}\right).$$

$\therefore$ Circumcentre $\equiv \left(\frac{1-2b}{2a}, 1 + \frac{1}{2b}\right)$

$\therefore \quad x = \frac{1-2b}{2ab}, y = 1 + \frac{1}{2b}$

$\Rightarrow \quad 2b = \frac{1}{y-1}$ and $2a = \frac{(y-2)}{x(y-1)}$ ...(ii)

From Eqs. (i) and (ii), we get

$$(y-1)(y-2) = \frac{x}{8\lambda}$$

$\Rightarrow \quad \left(y - \frac{3}{2}\right)^2 = \frac{1}{8\lambda}(x + 2\lambda)$

$\therefore$ Vertex is $\left(-2\lambda, \frac{3}{2}\right)$

and length of latusrectum is $\frac{1}{8\lambda}$.

49. Product of coordinates of vertex $= -2\lambda \times \frac{3}{2} = -3\lambda$

$= -3 \times 2 \quad [\because \lambda = 2]$

$= -6$

50. Length of smallest focal chord = Length of latusrectum

$= \frac{1}{8\lambda}$

$= \frac{1}{8 \times \left(\frac{1}{32}\right)} = 4 \quad \left[\because \lambda = \frac{1}{32}\right]$

51. Let $y - \frac{3}{2} = Y$, $x + 2\lambda = X$

$\therefore \quad Y^2 = \frac{1}{8\lambda}X$

Curve is symmetrical about $Y = 0$

$\Rightarrow \quad y - \frac{3}{2} = 0$

$\therefore \quad y = \frac{3}{2}$

Sol. (Q. Nos. 52 to 54)

Given parabola (P) can be written as

$$(x - 2y)^2 = 32x - 4y - 16$$

On adding $2(x - 2y)\lambda + \lambda^2$ both sides, it becomes

$$(x - 2y + \lambda)^2 = 32x - 4y - 16 + 2(x - 2y)\lambda + \lambda^2$$

$$= 2(\lambda + 16)x - 4(\lambda + 1)y + \lambda^2 - 16 \quad ...(i)$$

We choose λ such that lines

$$x - 2y + \lambda = 0 \text{ and } 2(\lambda + 16)x - 4(\lambda + 1)y + \lambda^2 - 16 = 0$$

are perpendicular to each other. It requires

$$\frac{1}{2} \times \frac{2(\lambda + 16)}{4(\lambda + 1)} = -1$$

$\Rightarrow \quad \lambda + 16 = -4\lambda - 4$

$\Rightarrow \quad \lambda = -4$

Hence, Eq. (i) becomes

$$(x - 2y - 4)^2 = 24x + 12y = 12(2x + y)$$

$\Rightarrow \quad \left(\frac{x - 2y - 4}{\sqrt{5}}\right)^2 = \frac{12}{\sqrt{5}}\left(\frac{2x + y}{\sqrt{5}}\right) \Rightarrow Y^2 = 4\rho X$

where, $\quad X = \frac{2x + y}{\sqrt{5}}, Y = \frac{x - 2y - 4}{\sqrt{5}}$

and $\quad 4\rho = \frac{12}{\sqrt{5}} \Rightarrow \rho = \frac{3}{\sqrt{5}}.$

52. $\because$ Equation of axis is $Y = 0$

$\Rightarrow \quad x - 2y - 4 = 0$ ...(i)

and equation of latusrectum is $X = \rho$

$\Rightarrow \quad \frac{2x + y}{\sqrt{5}} = \frac{3}{\sqrt{5}}$

$\Rightarrow \quad 2x + y - 3 = 0$ ...(ii)

On solving Eqs. (i) and (ii),

focus is $\quad (x, y) = (2, -1).$

53. Length of latusrectum $= 4\rho = \frac{12}{\sqrt{5}}$

54. Equation of directrix is $X + \rho = 0$

$\Rightarrow \quad \frac{2x + y}{\sqrt{5}} + \frac{3}{\sqrt{5}} = 0$

$\therefore \quad 2x + y + 3 = 0$

Sol. (Q. Nos. 55 to 57)

Any parabola whose axis is parallel to the X-axis will be of the form

$$(y - q)^2 = 4\lambda(x - p) \quad \text{...(i)}$$

Now, $lx + my = 1$ can be written as

$$(y - q) = -\frac{l}{m}(x - p) + \left(\frac{1 - mq - lp}{m}\right) \quad \text{...(ii)}$$

Eq. (ii) will touch Eq. (i), then

$$\frac{1 - mq - lp}{m} = \frac{\lambda}{-\frac{l}{m}}$$

$\Rightarrow \quad -l + mlq + l^2p = m^2\lambda$

$\Rightarrow \quad pl^2 + qlm - \lambda m^2 - l = 0 \quad \text{...(iii)}$

But given that

$$5l^2 - 4lm + 6m^2 + 3l = 0 \quad \text{...(iv)}$$

On comparing Eqs. (iii) and (iv), we get

$$\frac{p}{5} = \frac{q}{-4} = \frac{-\lambda}{6} = \frac{-1}{3}$$

$\Rightarrow \quad p = -\frac{5}{3}, q = \frac{4}{3}, \lambda = 2$

So, the parabola is

$$\left(y - \frac{4}{3}\right)^2 = 8\left(x + \frac{5}{3}\right).$$

55. $\because$ Vertex is $\left(-\frac{5}{3}, \frac{4}{3}\right)$.

Here, $a = -\frac{5}{3}$ and $b = \frac{4}{3}$

$\therefore \quad a - b = -3$

$\Rightarrow \quad |a - b| = 3$

56. For focus,

$$x + \frac{5}{3} = 2 \text{ and } y - \frac{4}{3} = 0$$

$\therefore$ Coordinates of focus are $\left(\frac{1}{3}, \frac{4}{3}\right)$.

Here, $c = \frac{1}{3}$ and $d = \frac{4}{3}$

$\therefore \quad d - c = 1$

$\Rightarrow \quad 2^{|d-c|} = 2$

57. For directrix $\left(x + \frac{5}{3}\right) + 2 = 0$

$\Rightarrow \; 3x + 11 = 0$ given $ex + f = 0$

$\therefore \quad e = 3, f = 11$

Now, $\quad |e - f| = |3 - 11| = 8$

Sol. (Q. Nos. 58 to 60)

$\because \quad C_1 : y^2 = 4x$

$\therefore \quad S_1 : (1, 0)$

Let $z = x + iy$ and $z_1 = x_1 + iy_1$

If z_1 is obtained by rotating z, 120° in anti-clockwise direction, then

$$z_1 = ze^{2\pi i/3} = (x + iy)\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$$

$$= \left(-\frac{x}{2} - \frac{\sqrt{3}y}{2}\right) + i\left(\frac{\sqrt{3}}{2}x - \frac{y}{2}\right)$$

$\therefore$ Equation of curve C_2 is

$$\left(\frac{\sqrt{3}}{2}x - \frac{y}{2}\right)^2 = 4\left(-\frac{x}{2} - \frac{\sqrt{3}}{2}y\right) \quad \text{...(i)}$$

For focus $-\frac{x}{2} - \frac{\sqrt{3}}{2}y = 1$ and $\frac{\sqrt{3}}{2}x - \frac{y}{2} = 0$.

$\therefore \quad x = -\frac{1}{2}, y = -\frac{\sqrt{3}}{2}$

$\Rightarrow \quad S_2 : \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

$\because \quad C_3$ is reflection of C_2 with respect to $y = x$.

$\therefore \quad S_3 : \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.

58. $\because (t^2, 2t)$ are parametric form of curve C_1.

$\therefore$ From Eq. (i),

$\frac{\sqrt{3}}{2}x - \frac{y}{2} = 2t$ and $-\frac{x}{2} - \frac{\sqrt{3}}{2}y = t^2$,

we get $x = \frac{1}{2}(-t^2 + 2\sqrt{3}t), y = \frac{1}{2}(-\sqrt{3}t^2 - 2t)$

$\therefore$ Parametric coordinates of C_2 are

$$\left(\frac{1}{2}(-t^2 + 2\sqrt{3}t), \frac{1}{2}(-\sqrt{3}t^2 - 2t)\right).$$

59. Area of ΔOS_2S_3

$$= \frac{1}{2}\left|\begin{vmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{vmatrix}\right|$$

$$= \frac{1}{2}\left|\frac{1}{4} - \frac{3}{4}\right| = \frac{1}{4} \text{ sq unit}$$

60. $\Sigma x_1^2 + \Sigma y_1^2 = (1)^2 + \left(-\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2 + (0)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2$

$$= 1 + \frac{1}{4} + \frac{3}{4} + 0 + \frac{3}{4} + \frac{1}{4} = 3$$

Sol. (Q. Nos. 61 to 63)

Since, no point of the parabola $y = x^2 + ax + 1$ is below the X-axis.

$\therefore \quad D \le 0$

$\Rightarrow \quad a^2 - 4 \le 0$

$\Rightarrow \quad -2 \le a \le 2$

61. The maximum value of a is 2.

The equation of the parabola is $y = x^2 + 2x + 1$.

It intersect the Y-axis at $(0, 1)$.

$\therefore$ Equation of tangent at $(0, 1)$ is

$$\frac{y+1}{2} = 0 \cdot x + (x + 0) + 1$$

$$\Rightarrow \quad y = 2x + 1$$

$$\Rightarrow \quad 2x - y + 1 = 0$$

Since, $2x - y + 1 = 0$ touches the circle $x^2 + y^2 = c^2$, then

$$\frac{0 - 0 + 1}{\sqrt{4+1}} = c \qquad [\because c > 0]$$

$$\Rightarrow \quad c = \frac{1}{\sqrt{5}}$$

$$\therefore \quad 5c^2 = 1$$

62. Equation of the tangent at $(0, 1)$ to the parabola

$$y = x^2 + ax + 1 \text{ is}$$

$$\Rightarrow \quad \frac{y+1}{2} = 0 \cdot x + \frac{a}{2}(x + 0) + 1$$

$$\Rightarrow \quad y = ax + 1$$

$$\Rightarrow \quad ax - y + 1 = 0$$

As it touches the circle $x^2 + y^2 = c^2$, then

$$\frac{1}{\sqrt{(a^2+1)}} = c$$

i.e. c is maximum, when $a = 0$.

Therefore, the equation of the tangent is $y = 1$.

$\therefore$ Slope of the tangent is 0.

63. Equation of tangent is

$$y = ax + 1$$

$$\Rightarrow \quad ax - y = -1$$

$$\Rightarrow \quad \frac{x}{\left(-\frac{1}{a}\right)} + \frac{y}{(1)} = 1$$

Therefore, the area of the triangle bounded by the tangent and the axes is

$$\frac{1}{2}\left|-\frac{1}{a}\right||1| = \frac{1}{2|a|}$$

$$\therefore \quad \Delta = \frac{1}{4} \qquad [\text{for minimum area } a = 2]$$

$$\Rightarrow \quad 8\Delta = 2$$

Sol. (Q. Nos. 64 to 66)

The conic is $S \equiv x^2 + xy + y^2 - 2x - 2y + 1 = 0$

and the line is $L \equiv x + y + 1 = 0$

It is required to find equation of the parabola (P) which touches the conic $S = 0$ at those (two) points, where the line $L = 0$ intersect the conic. Obviously at these points the parabola is in double contact with the conic.

$\therefore$ The equation of any such conic is $\phi \equiv S + \lambda L^2 = 0$

$$\Rightarrow (x^2 + xy + y^2 - 2x - 2y + 1) + \lambda(x + y + 1)^2 = 0 \qquad ...(i)$$

$$\Rightarrow (1 + \lambda)x^2 + (1 + 2\lambda)xy + (1 + \lambda)y^2 + 2(\lambda - 1)x + 2(\lambda - 1)y + \lambda + 1 = 0$$

It will be a parabola, if $h^2 = ab$

$$\Rightarrow \quad \frac{1}{4}(1 + 2\lambda)^2 = (1 + \lambda)^2$$

$$\Rightarrow \quad 1 + 4\lambda + 4\lambda^2 = 4 + 8\lambda + 4\lambda^2$$

$$\therefore \quad \lambda = -\frac{3}{4}$$

Hence, from Eq. (i), the required parabola is

$$(x^2 + xy + y^2 - 2x - 2y + 1) - \frac{3}{4}(x + y + 1)^2 = 0$$

$$\Rightarrow \quad x^2 - 2xy + y^2 - 14x - 14y + 1 = 0 \qquad ...(ii)$$

64. Comparing parabola (ii) with

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

$\therefore$ $a = 1, h = -1, b = 1, g = -7, f = -7, c = 1$

Now, $|a + b + c + f + g + h| = |1 + 1 + 1 - 7 - 7 - 1| = 12$

65. The parabola (P) can be written as

$$(x - y)^2 = (14x + 14y - 1)$$

$$\Rightarrow \quad \left(\frac{x-y}{\sqrt{2}}\right)^2 = 7\sqrt{2}\left(\frac{14x + 14y - 1}{\sqrt{(14)^2 + (14)^2}}\right) \qquad ...(iii)$$

$\therefore$ Length of latusrectum is $7\sqrt{2}$.

66. For vertex,

$$\frac{x-y}{\sqrt{2}} = 0, \frac{14x + 14y - 1}{\sqrt{(14)^2 + (14)^2}} = 0 \qquad [\text{from Eq. (iii)}]$$

$$\Rightarrow \quad x = y, 14x + 14y = 1$$

$$\therefore \quad x = y = \frac{1}{28}$$

$$\text{Vertex is } \left(\frac{1}{28}, \frac{1}{28}\right) = (a, b) \qquad [\text{given}]$$

$$\therefore \quad |a - b| = 0$$

Sol. (Q. Nos. 67 to 69)

$\because$ $y = 3x$ is tangent to the parabola

$$2y = ax^2 + b \qquad ...(i)$$

$$\therefore \quad 2(3x) = ax^2 + b$$

[substitute the value of $y = 3x$ in $2y = ax^2 + b$]

$$\Rightarrow \quad ax^2 - 6x + b = 0$$

$$D = 0$$

$[\because y = 3x \text{ is tangent to } 2y = ax^2 + b]$

$$\therefore \quad 36 - 4ab = 0$$

$$\Rightarrow \quad ab = 9 \qquad ...(ii)$$

From Eqs. (i) and (ii), we get

$$2y = ax^2 + \frac{9}{a} \qquad ...(iii)$$

67. $\because$ $\frac{a+b}{2} \geq \sqrt{ab} = 3$ [from Eq. (ii)]

$$\Rightarrow \quad a + b \geq 6$$

$\therefore$ Minimum value of $a + b$ is 6.

68. $\because$ (2, 6) is the point of contact.

From Eq. (iii), we get

$$12 = 4a + \frac{9}{a}$$

$$\Rightarrow \quad 4a^2 - 12a + 9 = 0$$

$$\Rightarrow \quad (2a - 3)^2 = 0$$

$$\therefore \quad 2a = 3$$

69. For $b = 18$

From Eq. (ii), $\quad a = \frac{1}{2}$

From Eq. (iii), $\quad 2y = \frac{x^2}{2} + 18$ ...(iv)

On solving $y = 3x$ and Eq. (iv), we get

$$6x = \frac{x^2}{2} + 18$$

$$\Rightarrow \quad x^2 - 12x + 36 = 0$$

$$\Rightarrow \quad (x - 6)^2 = 0$$

$$\therefore \quad x = 6, \text{ then } y = 3x = 18$$

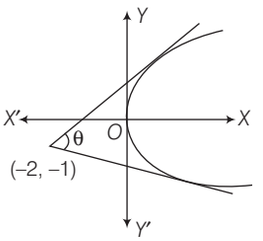
$\therefore$ Point of contact is (6, 18).

70. Equation of tangent in terms of slope (m) of the parabola $y^2 = 4x$ is $y = mx + \frac{1}{m}$.

$\because$ Point of intersection of tangents is $(-2, -1)$, then

$$-1 = -2m + \frac{1}{m}$$

$$\Rightarrow \quad 2m^2 - m - 1 = 0$$



Let m_1, m_2 be slopes of the tangents, then

$$m_1 - m_2 = \frac{\sqrt{D}}{a} = \frac{\sqrt{(1+8)}}{2} = \frac{3}{2}$$

and $\quad m_1 m_2 = -\frac{1}{2}$

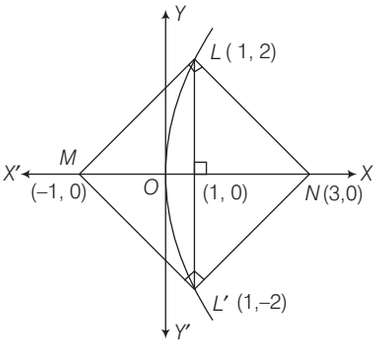
$$\therefore \quad \tan\theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right| = \left| \frac{3/2}{1 - 1/2} \right| = 3$$

71. Let S be the focus and point of intersection of tangents at P and Q is R.

$\because \quad (SR)^2 = SP \cdot SQ = 4 \times 9 = 36$

$\therefore \quad SR = 6$

72. The four lines form a square. The tangents at $L(1, 2)$ and $L'(1, -2)$ are $x - y + 1 = 0$ and $x + y + 1 = 0$. They intersect at $M(-1, 0)$.



The area of the square $MLNL'$ is $(ML)^2 = (1 + 1)^2 + (2 - 0)^2 = 8$ sq units

$\therefore \quad \lambda = 8$

73. Parabola is $y^2 = x$

$\therefore \quad a = \frac{1}{4}$

The normal at t is

$$tx + y = \frac{t^3}{4} + \frac{t}{2}$$

It passes through $(a, 0)$.

$$\therefore \quad ta = \frac{t^3}{4} + \frac{t}{2}$$

$$\Rightarrow \quad t^2 = 4\left(a - \frac{1}{2}\right) \Rightarrow a > \frac{1}{2}$$

$$\therefore \quad t_1 = 2\sqrt{\left(a - \frac{1}{2}\right)}, \; t_2 = -2\sqrt{\left(a - \frac{1}{2}\right)}$$

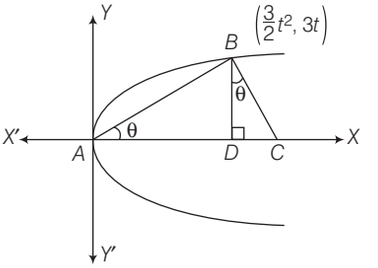
The normals are perpendicular $\Rightarrow t_1 t_2 = -1$

$$\Rightarrow \quad -4\left(a - \frac{1}{2}\right) = -1$$

$$\therefore \quad 4a = 3$$

74. Let B as $\left(\frac{3}{2}t^2, 3t\right)$ $\quad \left[\text{Here, } a = \frac{3}{2}\right]$

and let $BD \perp AC$ and $\angle DAB = \theta$



$$\Rightarrow \quad \tan\theta = \frac{BD}{AD} = \frac{3t}{\frac{3}{2}t^2} = \frac{2}{t}$$

$\therefore$ Projection of BC on the axis $= DC$

$$= BD\tan\theta = 3t\left(\frac{2}{t}\right) = 6 \text{ units}$$

$$\therefore \quad \lambda = 6$$

75. The parabolas are $y = x^2 - 9$ and $y = \lambda x^2$.

$\Rightarrow \quad x^2 - 9 = \lambda x^2 \Rightarrow \quad x^2(1-\lambda) = 9$

$\Rightarrow \quad x^2 = \dfrac{9}{1-\lambda}$

$\therefore \quad x = \pm \dfrac{3}{\sqrt{(1-\lambda)}}$

Now, from the symmetry about Y-axis,

$AB = 2a = \dfrac{6}{\sqrt{(1-\lambda)}} \Rightarrow \quad a = \dfrac{3}{\sqrt{(1-\lambda)}}$

$\Rightarrow \quad a^2(1-\lambda) = 9 \quad \Rightarrow \quad \lambda a^2 + 9 = a^2$

$\Rightarrow \quad a^2 + \mu = a^2$ [given]

$\therefore \quad \mu = 9$

76. Given, $y^2 - 8x < 0$, $x^2 + y^2 < 16$ and $x > 0$.

For $x = 1$,

$y^2 < 8$ and $y^2 < 15 \Rightarrow y^2 < 8$

$\therefore \quad y = 0, \pm 1, \pm 2$

$\therefore \quad$ 5 points.

For $x = 2$,

$y^2 < 16, y^2 < 12 \Rightarrow y^2 < 12$

$\therefore \quad y = 0, \pm 1, \pm 2, \pm 3$

$\therefore \quad$ 7 points

For $x = 3$,

$y^2 < 24, y^2 < 7 \Rightarrow y^2 < 7$

$\therefore \quad y = 0, \pm 1, \pm 2$

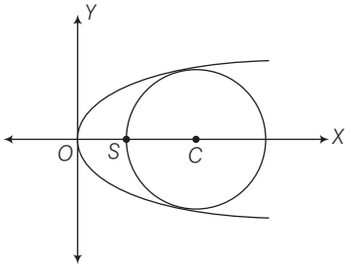
$\therefore \quad$ 5 points

Hence, total points is 17.

$\therefore \quad n = 17$

Sum of digits of n is 8.

77. Focus of the parabola $y^2 = 4x$ is $S(1, 0)$.



Let radius of circle be r.

$\therefore$ Centre of circle is $(1 + r, 0)$.

$\Rightarrow$ Equation of circle is $(x - 1 - r)^2 + y^2 = r^2$

$\Rightarrow \quad (x - 1 - r)^2 + 4x = r^2$ [$\because y^2 = 4x$]

$\Rightarrow \quad x^2 + 2(2 - 1 - r)x + 2r + 1 = 0$

$\Rightarrow \quad x^2 + 2(1 - r)x + 2r + 1 = 0$

It would have same roots due to symmetry.

$\therefore \quad D = 0$

$\Rightarrow \quad 4(1 - r)^2 - 4 \cdot 1 \cdot (2r + 1) = 0$

$\therefore \quad r = 0, 4$

Hence, $r = 4$ [$\because r \neq 0$]

78. For maximum number of common chords, the circle and the parabola must intersect at four points.

Now, solving the given curves, we have

$(x - 6)^2 + 4x = r^2$

$\Rightarrow \quad x^2 - 8x + 36 - r^2 = 0$

The curves touch, if $D = 0$

$\Rightarrow \quad 64 - 4 \cdot 1 \cdot (36 - r^2) = 0$

$\Rightarrow \quad r^2 = 20$

$\therefore \quad r = 2\sqrt{5}$

Hence, the least integral value of r for which the curves intersect is 5.

79. Given parabola is

$x^2 = 4(y - 1)$

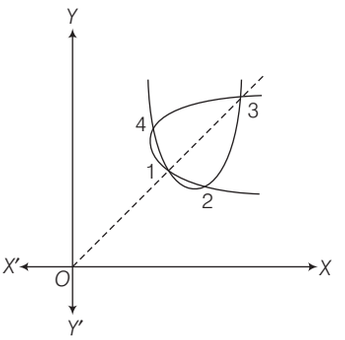
$\therefore$ Focus is $(0, 2)$.

Now, the shortest intercept of the line on the parabola which passes through the focus is latusrectum. The axis of the given parabola is the Y-axis.

Therefore, the latusrectum is parallel to the X-axis.

$\therefore \quad$ Slope = 0

80. (A) The given parabolas are symmetrical about the line $y = x$ as shown in the figure



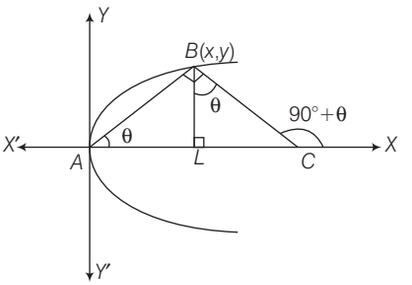
They intersect each other at four distinct points.

Hence, the number of common chords is

$${}^4C_2 = \frac{4 \cdot 3}{1 \cdot 2} = 6$$

Which is perfect number [$\because 1 \times 2 \times 3 = 1 + 2 + 3 = 6$]

(B) $\because \quad \tan\theta = \dfrac{y}{x}$ ...(i)



LC = Projection of BC on X-axis

$= BL \tan\theta = y \tan\theta$

$$= y \times \frac{y}{x} \qquad \text{[from Eq. (i)]}$$

$$= \frac{y^2}{x}$$

$$= 4 \qquad [\because y^2 = 4x]$$

(C) Normals to $y^2 = 4ax$ and $x^2 = 4by$ in terms of m are

$y = mx - 2am - am^3$ and $y = mx + 2b + \frac{b}{m^2}$.

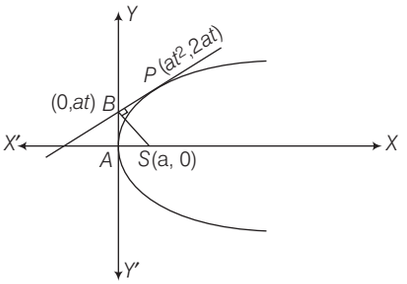
For common normal,

$$2b + \frac{b}{m^2} = -2am - am^3$$

or $$am^5 + 2am^3 + 2bm^2 + b = 0$$

It is clear that at most five common normals.

(D) Let middle point of P and B be (h,k), then $2h = at^2$



and $$2k = 3at$$

or $$2h = a\left(\frac{2k}{3a}\right)^2$$

or $$2k^2 = 9ah$$

∴ Locus of mid-point is

$$2y^2 = 9ax$$

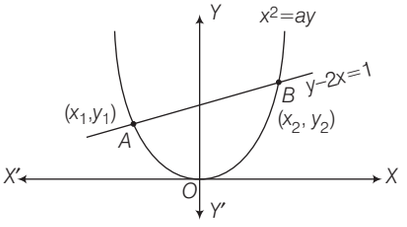
∴ Length of latusrectum $= \frac{9a}{2}$

$$= \frac{9}{2} \times 2 \qquad [\because a = 2]$$

$$= 9$$

81. (A) Given parabola is $x^2 = ay$ ...(i)

and the given line is $y - 2x = 1$ ...(ii)



On solving Eqs. (i) and (ii), we get

$$x^2 = a(2x + 1) \Rightarrow x^2 - 2ax - a = 0$$

Let coordinates of A and B are (x_1, y_1) and (x_2, y_2) respectively, then

$$|x_1 - x_2| = \frac{\sqrt{D}}{a} = \frac{\sqrt{(4a^2 + 4a)}}{1} = 2\sqrt{(a^2 + a)} \qquad \text{...(iii)}$$

Also A, B lie on $y - 2x = 1$

∴ $$y_1 - 2x_1 = 1 \text{ and } y_2 - 2x_2 = 1$$

or $$y_2 - y_1 = 2(x_2 - x_1) \qquad \text{...(iv)}$$

∴ Length of $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$= \sqrt{5}\,|x_1 - x_2| \qquad \text{[from Eq. (iv)]}$$

$$= 2\sqrt{5}\,\sqrt{(a^2 + a)} \qquad \text{[from Eq. (iii)]}$$

Given, $2\sqrt{5}\,\sqrt{(a^2 + a)} = \sqrt{40}$

$\Rightarrow$ $$a^2 + a = 2$$

$\Rightarrow$ $$a^2 + a - 2 = 0$$

$\Rightarrow$ $$(a + 2)(a - 1) = 0$$

$\Rightarrow$ $$a = -2, 1$$

(B) If tangents are drawn from (x_1, y_1) to the parabola $y^2 = 4ax$ and angle between tangents is θ, then

$$|\tan\theta| = \frac{\sqrt{(y_1^2 - 4ax_1)}}{|x_1 + a|}$$

Here, $$x_1 = 0, y_1 = 2, \theta = \frac{3\pi}{4}, \text{ then}$$

$$|-1| = \frac{2}{|0 + a|}$$

$\Rightarrow$ $$|a| = 2$$

∴ $$a = \pm 2$$

Aliter :

Observe that one tangent is the Y-axis, the other tangent is at $\theta = \frac{\pi}{4}$ and its equation is $ty = x + at^2$

for $t = 1$, we get $y = x + a$

$\Rightarrow a = 2$ from the symmetry $a = -2$ is also possible.

(C) Let the other end be at $(at^2, 2at)$

So, mid-point is $\left(\frac{at^2 + a}{2}, \frac{2at + 2a}{2}\right)$

which satisfy $x + y = 1$

or $$\frac{at^2 + a}{2} + at + a = 1$$

$\Rightarrow$ $$at^2 + 2at + 3a - 2 = 0$$

Since, two distinct chords are possible, so $D > 0$.

∴ $$4a^2 - 4a(3a - 2) > 0$$

$\Rightarrow$ $$-8a^2 + 8a > 0$$

$\Rightarrow$ $$8a(a - 1) < 0$$

∴ $$0 < a < 1$$

or $$0 < 4a < 4$$

or $0 <$ Length of latusrectum < 4

∴ Length of latusrectum can be 1 or 2 or 3 from the given values.

(D) The given parabola is

$$x^2 - ay + 3 = 0$$

or $$x^2 = a\left(y - \frac{3}{a}\right)$$

Let $x = X, y - \frac{3}{a} = Y$

Then, the parabola is

$$X^2 = aY$$

For focus $X = 0, Y = \frac{a}{4}$

$\Rightarrow$ $x = 0, y - \frac{3}{a} = \frac{a}{4}$

$\therefore$ Focus is $\left(0, \frac{3}{a} + \frac{a}{4}\right)$ given focus is $(0, 2)$

$\therefore$ $\frac{3}{a} + \frac{a}{4} = 2$

$\Rightarrow$ $a^2 - 8a + 12 = 0$

$\Rightarrow$ $(a - 6)(a - 2) = 0$

$\therefore$ $a = 2, 6$

Here, $a_1 = 6, a_2 = 2$

$\therefore$ $\frac{a_1}{a_2} = 3$

82. (A) Points $(1, 2)$ and $(-2, 1)$ satisfy both the curves.

(B) Equation of tangent at $(t^2, 2t)$ on $y^2 = 4x$ is

$$ty = x + t^2$$

It passes through the point $(2, 3)$, then

$$3t = 2 + t^2$$

$\Rightarrow$ $t^2 - 3t + 2 = 0$

or $(t - 1)(t - 2) = 0$

or $t = 1$ or 2

The point of contact is $(1, 2)$ or $(4, 4)$.

(C) Let $P(\sqrt{5}\cos\theta, \sqrt{5}\sin\theta)$, then the chord of contact of the parabola $y^2 = 4x$ w.r.t. P is

$$y \cdot \sqrt{5}\sin\theta = 2(x + \sqrt{5}\cos\theta)$$

or $y = \frac{2x}{\sqrt{5}\sin\theta} + 2\cot\theta$

On comparing with $y = 2(x - 2)$, then

$$\sqrt{5}\sin\theta = 1 \text{ and } \cot\theta = -2$$

or $\sqrt{5}\sin\theta = 1$ and $\sqrt{5}\cos\theta = -2$

Hence, coordinates of P are $(-2, 1)$.

(D) Let coordinates of Q be $(t^2, 2t)$.

Now, the area of ΔOPQ is

$$\frac{1}{2}\left|\begin{vmatrix} t^2 & 2t \\ 4 & -4 \end{vmatrix}\right| = 6 \qquad \text{[given]}$$

$\Rightarrow$ $2t^2 + 4t = \pm 6$

or $t^2 + 2t \pm 3 = 0$

$\therefore$ $t^2 + 2t - 3 = 0$ $\quad [\because t^2 + 2t + 3 \neq 0]$

$\Rightarrow$ $(t + 3)(t - 1) = 0$

Then, $t = 1$ or -3

Hence, the point Q are $(1, 2)$ or $(9, -6)$.

83. Equation of tangent to $y^2 = 4x$ is

$$y = mx + \frac{1}{m}$$

it is also tangent to $x^2 = 4y$, then

$$x^2 = 4\left(mx + \frac{1}{m}\right)$$

$\Rightarrow$ $x^2 - 4mx - \frac{4}{m} = 0$

It discriminant $= 0$

$\Rightarrow$ $16m^2 + \frac{16}{m} = 0$

$\Rightarrow$ $m^3 = -1$

$\therefore$ $m = -1$

$\therefore$ Equation of common tangent is $x + y + 1 = 0$.

Statement II is also true and it is correct explanation of Statement I.

84. The vertex of $(y + 2)^2 = 2(x - 1)$ is $(1, -2)$ and equation of axis is $y = -2$.

Here, $4a = 2$

$\therefore$ $a = \frac{1}{2} \Rightarrow 3a = \frac{3}{2}$

$\therefore$ Required point is $\left(1 + \frac{3}{2}, -2\right)$, i.e. $\left(\frac{5}{2}, -2\right)$.

Hence, both statements are true and Statement II is correct explanation for Statement I.

85. $\because$ $y = mx + c$ ...(i)

and $y^2 = 4ax$ ...(ii)

From Eqs. (i) and (ii),

$$y^2 = 4a\left(\frac{y - c}{m}\right)$$

$\Rightarrow$ $my^2 - 4ay + 4ac = 0$

If line Eq. (i) intersect the parabola $y^2 = 4ax$ at one point, then

$$B^2 = 4AC$$

$\Rightarrow$ $(-4a)^2 = 4 \cdot m \cdot 4ac$

$\Rightarrow$ $c = \frac{a}{m}$

From Eq. (i), equation of tangent is

$$y = mx + \frac{a}{m}$$

$\therefore$ Statement I and Statement II are individual true and Statement II is correct explanation of Statement I.

86. $\because$ $\sqrt{ax} + \sqrt{by} = 1$ ...(i)

On squaring both sides, then

$$ax + by + 2\sqrt{abxy} = 1$$

$\Rightarrow$ $(ax + by - 1) = -2\sqrt{abxy}$

Again, on squaring both sides, we get

$$a^2x^2 + b^2y^2 + 1 + 2abxy - 2ax - 2by + 1 = 4abxy$$

$\Rightarrow$ $a^2x^2 - 2abxy + b^2y^2 - 2ax - 2by + 1 = 0$

Now, comparing with

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0$$

$\therefore \quad A = a^2, H = -ab, B = b^2, G = -a, F = -b, C = 1$

$\therefore \quad \Delta = ABC + 2FGH - AF^2 - BG^2 - CH^2$

$= a^2b^2 - 2a^2b^2 - a^2b^2 - a^2b^2 - a^2b^2$

$= -4a^2b^2 \neq 0$

and $H^2 = AB$

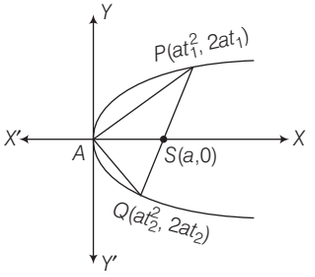
Hence, Eq. (i) represent a parabola.

$\therefore$ Statement I is true and Statement II is false.

87. $\because$ Slope of $AP = m_1 = \dfrac{2at_1 - 0}{at_1^2 - 0} = \dfrac{2}{t_1}$

and slope of $AQ = m_2 = \dfrac{2at_2 - 0}{at_2^2 - 0} = \dfrac{2}{t_2}$

$\because \quad m_1m_2 = \dfrac{4}{t_1t_2} \neq -1$



Also, P, S, Q are collinear, then

$$\frac{2at_1 - 0}{at_1^2 - a} = \frac{0 - 2at_2}{a - at_2^2}$$

$\Rightarrow \quad t_1 - t_1t_2^2 = -t_1^2t_2 + t_2$

$\Rightarrow \quad (t_1 - t_2) + t_1t_2(t_1 - t_2) = 0$

$\because \quad t_1 - t_2 \neq 0, 1 + t_1t_2 = 0$

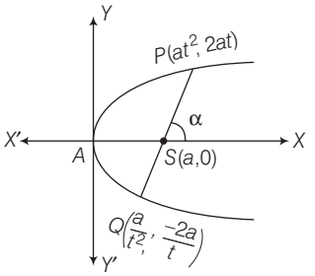
$\therefore \quad t_1t_2 = -1$

Hence, Statement I is false and Statement II is true.

88. Length of focal chord $PQ = a\left(t + \dfrac{1}{t}\right)^2$ ...(i)

and $\tan\alpha = \dfrac{2at + \dfrac{2a}{t}}{at^2 - \dfrac{a}{t^2}} = \dfrac{2t}{t^2 - 1} = \dfrac{2\left(\dfrac{1}{t}\right)}{1 - \left(\dfrac{1}{t}\right)^2}$

$\therefore \quad \dfrac{1}{t} = \tan(\alpha/2)$



From Eq. (i)

$$PQ = a(\cot(\alpha/2) + \tan(\alpha/2))^2$$

$$= a\left(\frac{1}{\sin(\alpha/2)\cos(\alpha/2)}\right)^2$$

$$= a\left(\frac{2}{\sin\alpha}\right)^2 = 4a\ \text{cosec}^2\ \alpha$$

For $\alpha = 60°$, $4a = 8$ and

$$PQ = 8\ \text{cosec}^2\ 60° = 8\left(\frac{2}{\sqrt{3}}\right)^2 = 32/3$$

$\therefore$ Statement I is true and Statement II is false.

89. $\because \quad x + y = \lambda \Rightarrow y = \lambda - x$ ...(i)

and $\quad y = x - x^2$ ...(ii)

From Eqs. (i) and (ii), we get

$$\lambda - x = x - x^2$$

$\Rightarrow \quad x^2 - 2x + \lambda = 0$

$\because$ Eq. (i) touch the parabola Eq. (ii), then

$$B^2 - 4AC = 0$$

$\Rightarrow \quad (-2)^2 - 4 \cdot 1 \cdot \lambda = 0$

$\therefore \quad \lambda = 1$

$\Rightarrow$ Statement I is true.

From Statement II,

$$(x - 1)^2 = x - x^2$$

$\Rightarrow \quad 2x^2 - 3x + 1 = 0$

$\therefore$ Discriminant $= (-3)^2 - 4 \cdot 2 \cdot 1 = 1 \neq 0$

$\therefore$ Statement II is false.

90. $\because \quad 3x + 4y + 5 = 0$ and $4x + 3y + 2 = 0$ are not perpendicular to each other.

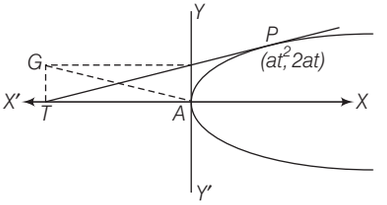
$\therefore$ Latusrectum $\neq 4$

$\therefore$ Statement I is false and Statement II is true.

91. Let $P(at^2, 2at)$ be any point on the parabola $y^2 = 4ax$.

Then, tangent at $P(at^2, 2at)$ is $ty = x + at^2$.

Since, tangent meet the axis of parabola in T and tangent at the vertex in Y



$\therefore$ Coordinates of T and Y are $(-at^2, 0)$ and $(0, at)$ respectively.

Let coordinates of G be (x_1, y_1).

Since, $TAYG$ is rectangle.

$\because$ Mid-points of diagonals TY and GA is same.

$$\Rightarrow \quad \frac{x_1 + 0}{2} = \frac{-at^2 + 0}{2}$$

$$\Rightarrow \quad x_1 = -at^2 \quad \text{...(i)}$$

and $\frac{y_1 + 0}{2} + \frac{0 + at}{2} \quad \Rightarrow \quad y_1 = at$ ...(ii)

Eliminating t from Eqs. (i) and (ii), we get

$$x_1 = -a\left(\frac{y_1}{a}\right)^2$$

$\Rightarrow \quad y_1^2 = -ax_1$

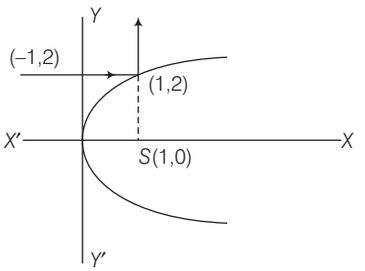
$\Rightarrow \quad y_1^2 + ax_1 = 0$

Hence, the locus of $G(x_1, y_1)$ is $y^2 + ax = 0$.

92. Equation of incident ray parallel to axis of parabola (X-axis) is $y = \lambda$, which pass through $(-1, 2)$, then $2 = \lambda$.

$\therefore$ Equation of incident ray is $y = 2$

Incident ray strikes the parabola $y^2 = 4x$ at $(1, 2)$.



The reflected ray passes through the focus $(1, 0)$.

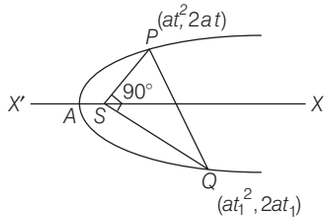
Hence, the equation of the reflected ray is $x = 1$.

93. Let PQ be a normal chord to a parabola at $P(at^2, 2at)$.

Since, the ordinate and abscissa of P are equal.

$\Rightarrow \quad at^2 = 2at, t \neq 0$

$\therefore \quad t = 2$



Since, normal at $P(at^2, 2at)$ meet the parabola at $Q(at_1^2, 2at_1)$.

$\therefore \quad t_1 = -t - \frac{2}{t}$

or $\quad t_1 = -2 - 1$

or $\quad t_1 = -3$

$\therefore$ Coordinates of P and Q are $(4a, 4a)$ and $(9a, -6a)$, respectively.

$\therefore$ Slope of $SP = \frac{4a - 0}{4a - a} = \frac{4}{3} = m_1$ (say)

and slope of $SQ = \frac{-6a - 0}{9a - a} = -\frac{3}{4} = m_2$ (say)

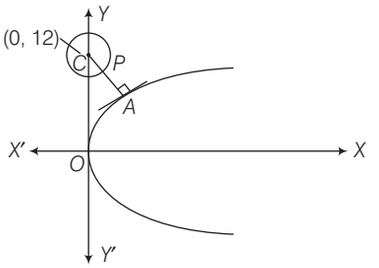
$\therefore \quad m_1 m_2 = -1$

Hence, SP and SQ are perpendicular to each other

i.e. $\quad \angle PSQ = 90°$

94. The centre and radius of the given circle are $(0, 12)$ and 4, respectively.

Now, the shortest distance always occurs along the common normal.



Let $\quad A \equiv (t^2, 2t)$

Equation of normal at A is $y + tx = 2t + t^3$,

which passes through $(0, 12)$, then

$$12 + 0 = 2t + t^3 \Rightarrow t^3 + 2t - 12 = 0$$

or $\quad (t - 2)(t^2 + 2t + 6) = 0$

$\therefore \quad t = 2 \quad [\because t^2 + 2t + 6 \neq 0]$

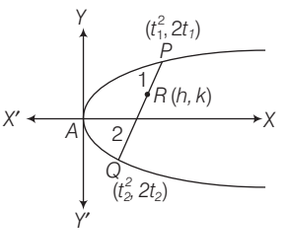
Coordinates of A are $(4, 4)$.

$\therefore$ Shortest distance $= AP = AC - CP = \sqrt{80} - 4 = 4(\sqrt{5} - 1)$

95. Let $P(t_1^2, 2t_1)$ and $Q(t_2^2, 2t_2)$ be extremities of the chord with slope 2.

$\therefore \quad \frac{2t_1 - 2t_2}{t_1^2 - t_2^2} = 2$

$\Rightarrow \quad t_1 + t_2 = 1$ ...(i)



Let $R(h, k)$ be coordinates of the point which divides PQ in the ratio 1 : 2, then

$$h = \frac{2t_1^2 + t_2^2}{3}$$

and $\quad k = \frac{4t_1 + 2t_2}{3}$

$\Rightarrow \quad 3h = 2t_1^2 + (1 - t_1)^2$

and $\quad 3k = 4t_1 + 2(1 - t_1)$ [from Eq. (i)]

or $\quad 3h = 3t_1^2 - 2t_1 + 1$

and $\quad 3k = 2t_1 + 2$

Eliminating t_1

$\therefore \quad 3h = 3\left(\frac{3k - 2}{2}\right)^2 - 2\left(\frac{3k - 2}{2}\right) + 1$

$\Rightarrow \quad 9k^2 - 16k - 4h + 8 = 0$

$\Rightarrow \quad k^2 - \frac{16k}{9} - \frac{4h}{9} + \frac{8}{9} = 0$

$$\Rightarrow \quad \left(k-\frac{8}{9}\right)^2 = \frac{4}{9}\left(h-\frac{2}{9}\right)$$

$\therefore$ Locus of $R(h, k)$ is $\left(y-\frac{8}{9}\right)^2 = \frac{4}{9}\left(x-\frac{2}{9}\right)$,

which is a parabola with vertex $\left(\frac{8}{9}, \frac{2}{9}\right)$.

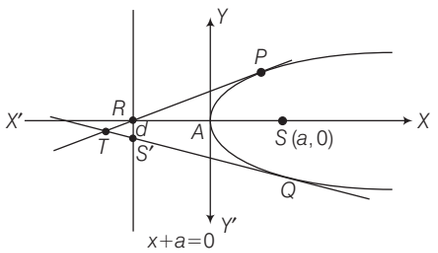
96. Let coordinates of P and Q on the parabola $y^2 = 4ax$ are $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$.

Equation of tangents at P and Q are

$$t_1y = x + at_1^2 \quad \text{and} \quad t_2y = x + at_2^2$$

Let these tangents meet $x + a = 0$ at R and S, then coordinates of R and S' are $\left(-a, \frac{a(t_1^2-1)}{t_1}\right)$ and $\left(-a, \frac{a(t_2^2-1)}{t_2}\right)$,

respectively.



Given, $|RS'| = d$

$$\therefore \quad \left|a\left(\frac{t_1^2-1}{t_1}\right) - a\left(\frac{t_2^2-1}{t_2}\right)\right| = d$$

$$\Rightarrow \quad \left|\frac{a}{t_1t_2}t_1^2t_2 - t_2 - t_1t_2^2 + t_1\right| = d$$

$$\Rightarrow \quad \left|\frac{a}{t_1t_2}((t_1-t_2)(1+t_1t_2))\right| = d$$

$$\Rightarrow \quad \left|\frac{a\sqrt{\{(t_1+t_2)^2 - 4t_1t_2\}}\,(1+t_1t_2)}{t_1t_2}\right| = d$$

Let the point of intersection of tangents at P at Q is T then

$$T = (at_1t_2, a(t_1 + t_2))$$

Now, let $T = (h, k)$

$$\therefore \quad h = at_1t_2 \quad \text{and} \quad k = a(t_1 + t_2)$$

$$\therefore \quad t_1t_2 = \frac{h}{a} \quad \text{and} \quad t_1 + t_2 = \frac{k}{a} \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$\left|\frac{a\sqrt{\left(\frac{k^2}{a^2} - \frac{4h}{a}\right)}\left(1+\frac{h}{a}\right)}{\frac{h}{a}}\right| = d$$

$$\Rightarrow \quad (k^2 - 4ah)(h + a)^2 = h^2d^2$$

Hence, required locus is

$$(y^2 - 4ax)(x + a)^2 = d^2x^2.$$

Aliter :

Let point of intersection of tangents at P and Q is $T(h, k)$ then equation of pair of tangents TP and TQ is

$$SS_1 = T^2$$

$$\Rightarrow \quad (y^2 - 4ax)(k^2 - 4ah) = [ky - 2a(x + h)]^2 \qquad \ldots(i)$$

Let the pair of tangents cut the directrix $x + a = 0$ are in R and S, then substitute $x = -a$ in Eq. (i), then we get

$$hy^2 - k(h - a)y + a\{(h + a)^2 - k^2\} = 0$$

Now let coordinates of R and S be

$$(-a, y_1) \quad \text{and} \quad (-a, y_2).$$

$$\therefore \quad y_1 + y_2 = \frac{k(h-a)}{h}$$

and $$y_1y_2 = \frac{a\{(h+a)^2 - k^2\}}{h}$$

but given $$|RS| = d$$

$$\Rightarrow \quad (RS)^2 = d^2$$

$$\Rightarrow \quad (y_1 - y_2)^2 = d^2$$

$$\Rightarrow \quad (y_1 + y_2)^2 - 4y_1y_2 = d^2$$

$$\Rightarrow \quad \frac{k^2}{h^2}(h-a)^2 - \frac{4a}{h}\{(h+a)^2 - k^2\} = d^2$$

$$\Rightarrow \quad k^2\{(h-a)^2 + 4ah\} - 4ah(h+a)^2 = d^2h^2$$

$$\Rightarrow \quad (k^2 - 4ah)(h + a)^2 = d^2h^2$$

Hence, locus of $T(h, k)$ is

$$(y^2 - 4ax)(x + a)^2 = d^2x^2.$$

97. Let $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ be two points on the parabola $y^2 = 4ax$

The equations of tangents at P and Q are

$$t_1y = x + at_1^2$$

and $$t_2y = x + at_2^2$$

$$\therefore \quad \tan\theta_1 = \frac{1}{t_1}, \tan\theta_2 = \frac{1}{t_2}$$

Equations of circles with OP and OQ as diameters are

$$(x - 0)(x - at_1^2) + (y - 0)(y - 2at_1) = 0$$

and $(x - 0)(x - at_2^2) + (y - 0)(y - 2at_2) = 0$ respectively.

i.e. $$x^2 + y^2 - axt_1^2 - 2at_1y = 0$$

and $$x^2 + y^2 - axt_2^2 - 2at_2y = 0$$

then, point of intersection of circles are $O(0, 0)$

and $$R\left(\frac{-4at_1t_2}{(t_1+t_2)^2+4}, \frac{2at_1t_2(t_1+t_2)}{(t_1+t_2)^2+4}\right)$$

Since, OR makes an angle ϕ with the X-axis. Therefore,

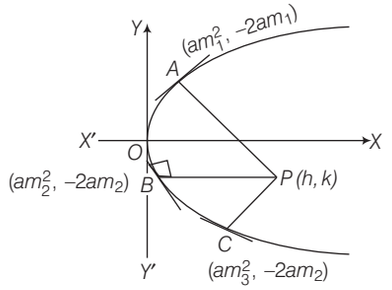
$$\tan\phi = -\left(\frac{t_1+t_2}{2}\right)$$

Now, $\cot\theta_1 + \cot\theta_2 + 2\tan\phi = t_1 + t_2 - (t_1 + t_2)$

$= 0$

98. Any normal of the parabola $y^2 = 4x$ with slope m is

$$y = mx - 2m - m^3$$



It passes through P, then

$$k = mh - 2m - m^3$$

$$\Rightarrow m^3 + (2 - h)m + k = 0 \quad \ldots(i)$$

Thus, $m_1m_2m_3 = -k$

$$\alpha m_3 = -k \quad (\because m_1m_2 = \alpha)$$

$$\Rightarrow m_3 = -\frac{k}{\alpha}$$

$\because$ m_3 is a root of Eq. (i), then

$$-\frac{k^3}{\alpha^3} + (2 - h)\left(-\frac{k}{\alpha}\right) + k = 0$$

$$\Rightarrow k^3 + (2 - h)k\alpha^2 - k\alpha^3 = 0$$

$\therefore$ Locus of $P(h, k)$ is

$$y^3 + (2 - x)y\alpha^2 - y\alpha^3 = 0$$

$$\Rightarrow y^2 + (2 - x)\alpha^2 - \alpha^3 = 0 \quad [\because y \neq 0]$$

(P does not lie on the axis of the parabola)

$$\Rightarrow y^2 = \alpha^2 x - 2\alpha^2 + \alpha^3$$

If it is a part of the parabola $y^2 = 4x$

then, $\alpha^2 = 4$

and $-2\alpha^2 + \alpha^3 = 0$

$$\Rightarrow \alpha^2(\alpha - 2) = 0$$

$$\Rightarrow \alpha - 2 = 0, \alpha \neq 0$$

$$\therefore \alpha = 2$$

99. Let $P(at^2, 2at)$ be any point on $y^2 = 4ax$. Then, vertex $A(0,0)$.

The equation of tangent at P is

$$ty = x + at^2 \quad \ldots(i)$$

Tangent at P will be normal to the circle, AP is a chord whose mid-point is $\left(\frac{at^2}{2}, at\right)$ and slope is $\frac{2}{t}$.

$\therefore$ Equation of the line passing through mid-point of AP and perpendicular to AP is

$$y - at = -\frac{t}{2}\left(x - \frac{at^2}{2}\right)$$

$$\Rightarrow tx + 2y = \frac{at^3}{2} + 2at \quad \ldots(ii)$$

Eqs. (i) and (ii) both pass through (x_1, y_1) which is the centre of the circle

$$ty_1 = x_1 + at^2 \quad \ldots(iii)$$

and $2tx_1 + 4y_1 = at^3 + 4at \quad \ldots(iv)$

Multiplying Eq. (iii) by t and subtracting Eq. (iv), we have

$$t^2y_1 + t(4a - 3x_1) - 4y_1 = 0 \quad \ldots(v)$$

Also, from Eq. (iii), $at^2 - ty_1 + x_1 = 0 \quad \ldots(vi)$

Eliminating t from Eqs. (v) and (vi), we get

$$\frac{t^2}{x_1(4a - 3x_1) - 4y_1^2} = \frac{t}{-4ay_1 - x_1y_1} = \frac{1}{-y_1^2 - a(4a - 3x_1)}$$

On simplyfying, we get

$$2y_1^2(2y_1^2 + x_1^2 - 12ax_1) = ax_1(3x_1 - 4a)^2$$

Hence, required locus is

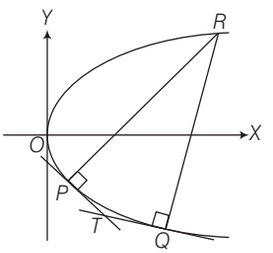
$$2y^2(2y^2 + x^2 - 12ax) = ax(3x - 4a)^2.$$

100. Let $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ be two points on the parabola $y^2 = 4ax$ such that the normals at P and Q intersect at a point $R(aT^2, 2aT)$ on the parabola $y^2 = 4ax$, then

$$T = -t_1 - \frac{2}{t_1} = -t_2 - \frac{2}{t_2}, \text{ then } t_1t_2 = 2$$

$\because$ Tangents at P and Q intersect at $T(at_1t_2, a(t_1 + t_2))$

i.e. $T(2a, a(t_1 + t_2))$.



Also, the coordinates of R being the point of intersection of normals at P and Q are

$$(2a + a(t_1^2 + t_2^2 + t_1t_2), -at_1t_2(t_1 + t_2))$$

$$\Rightarrow (4a + a(t_1^2 + t_2^2), -2a(t_1 + t_2))$$

Now, $\angle TPR = \angle TQR = 90^\circ$

$$\Rightarrow \angle TPR + \angle TQR = 180^\circ$$

$\Rightarrow$ Quadrilateral $PTQR$ is a cyclic quadrilateral, therefore centre of the circle circumscribing the ΔTPQ is at the mid-point of TR.

Let its coordinates be (h, k), then

$$2h = 2a + 4a + a(t_1^2 + t_2^2)$$

$$\Rightarrow \frac{2h - 6a}{a} = (t_1 + t_2)^2 - 2t_1t_2$$

$$\Rightarrow \frac{2h - 2a}{a} = (t_1 + t_2)^2 \quad [\because t_1t_2 = 2] \ldots(i)$$

and $2k = a(t_1 + t_2) - 2a(t_1 + t_2)$

$$\Rightarrow -\frac{2k}{a} = (t_1 + t_2) \quad \ldots(ii)$$

From Eqs. (i) and (ii), then $\left(\frac{2h-2a}{a}\right)=\left(-\frac{2k}{a}\right)^2$

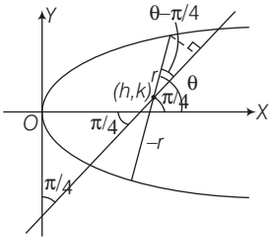
$\Rightarrow \quad 2k^2 = a(h-a)$

Hence, locus of (h, k) is $2y^2 = a(x-a)$.

101. Let the equation of straight line (h, k) as its mid-point,

then, $\frac{x-h}{\cos\theta}=\frac{y-k}{\sin\theta}=r$ ...(i)

Any point on the line Eq. (i) is $(h + r\cos\theta, k + r\sin\theta)$.



Solving with the equation of parabola

$y^2 = 4ax$, we get

$(k + r\sin\theta)^2 = 4a(h + r\cos\theta)$

$\Rightarrow \quad r^2\sin^2\theta + 2r(k\sin\theta - 2a\cos\theta) + k^2 - 4ah = 0$ ...(ii)

which is quadratic in r.

The roots of the quadratic equation will be equal but of opposite sign as (h, k) is the mid-point.

$$\text{Sum of roots} = -\frac{2(k\sin\theta - 2a\cos\theta)}{\sin^2\theta} = 0$$

$\therefore \quad k\sin\theta - 2a\cos\theta = 0$

$\therefore \quad \tan\theta = \frac{2a}{k}$

Now, from Eq. (ii),

$$r^2\sin^2\theta + (k^2 - 4ah) = 0$$

$\Rightarrow \quad r^2 \cdot \frac{4a^2}{(4a^2 + k^2)} + (k^2 - 4ah) = 0$ ...(iii)

Length of the chord will be $2r$. Angle between the two lines will be $(\theta - \pi/4)$ and the projection of the chord on the given line will be $2r\cos(\theta - \pi/4) = c$

$\Rightarrow \quad \frac{2r}{\sqrt{2}}(\cos\theta + \sin\theta) = c$

$\Rightarrow \quad \frac{2r}{\sqrt{2}}\left(\frac{k+2a}{\sqrt{4a^2+k^2}}\right) = c$

$\Rightarrow \quad 2r^2(k + 2a)^2 = c^2(4a^2 + k^2)$

$\Rightarrow \quad \frac{2r^2}{(4a^2 + k^2)} = \frac{c^2}{(k + 2a)^2}$ ...(iv)

From Eqs. (iii) and (iv), we get

$$\frac{2a^2c^2}{(k + 2a)^2} + (k^2 - 4ah) = 0$$

$\Rightarrow \quad (k^2 - 4ah)(k + 2a)^2 + 2a^2c^2 = 0$

Hence, the locus of the middle points is

$$(y^2 - 4ax)(y + 2a)^2 + 2a^2c^2 = 0.$$

102. Equation of normal at $(am^2, -2am)$ is

$$y = mx - 2am - am^3$$

If the three normals of P, Q, R meet at (h, k), then

$$am^3 + m(2a - h) + k = 0$$

$\therefore \quad \Sigma m_1 = 0$

$$\Sigma m_1 m_2 = \frac{(2a - h)}{a},$$

$$m_1 m_2 m_3 = \frac{-k}{a}$$

$P \equiv (am_1^2, -2am_1)$

$Q \equiv (am_2^2, -2am_2)$

and $R \equiv (am_3^2, -2am_3)$

Equation of PQ is

$-y(m_1 + m_2) = 2(x + am_1m_2)$ ...(i)

and equation of diameter through R is

$y = -2am_3$ ...(ii)

Point of intersection of Eqs. (i) and (ii) is on the directrix and hence it must be $(-a, -2am_3)$ and it satisfies Eq. (ii), then

$2am_3(m_1 + m_2) = 2(-a + am_1m_2)$

$\Rightarrow \quad m_3(-m_3) = -1 + m_1m_2$

$\Rightarrow \quad m_1m_2 = 1 - m_3^2$

$\therefore$ Equation of PQ becomes

$$-y(0 - m_3) = 2(x + a - am_3^2)$$

$\Rightarrow \quad 2am_3^2 + m_3y - 2(x + a) = 0 \qquad [m_3 \text{ is parameter}]$

Its envelope is given by the discriminant of this quadratic equated to zero.

$\therefore \quad (y)^2 - 4\cdot 2a\cdot\{-2\cdot(x + a)\} = 0$

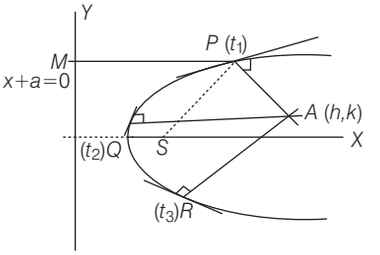
$\Rightarrow \quad y^2 + 16a(x + a) = 0$

103. Equation of normal at 't' is $y = -tx + 2at + at^3$.

Let A be (h, k), then $k = -th + 2at + at^3$

or $at^3 - t(h - 2a) - k = 0$ ...(i)

Let the coordinates of P, Q, R are $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$, $(at_3^2, 2at_3)$ respectively, then from Eq. (i),



$t_1 + t_2 + t_3 = 0$

$\Rightarrow \quad t_1t_2 + t_2t_3 + t_3t_1 = -\left(\frac{h - 2a}{a}\right)$

and $t_1t_2t_3 = \frac{k}{a}$

Since, $SP = PM = a + at_1^2$

Similarly, $SQ = a + at_2^2$

and $SR = a + at_3^2$

$\therefore \quad SP \cdot SQ \cdot SR = a(1 + t_1^2) \cdot a(1 + t_2^2) \cdot a(1 + t_3^2)$

$= a^3\{1 + (t_1^2 + t_2^2 + t_3^2) + (t_1^2t_2^2 + t_2^2t_3^2 + t_3^2 t_1^2) + (t_1^2\, t_2^2\, t_3^2)\}$

$= a^3\{1 + (t_1 + t_2 + t_3)^2 - 2(t_1t_2 + t_2t_3 + t_3t_1)$

$+ (t_1t_2 + t_2t_3 + t_3t_1)^2 - 2t_1t_2t_3(t_1 + t_2 + t_3) + (t_1t_2t_3)^2\}$

$= a^3\left\{1 + 0 + 2\left(\frac{h-2a}{a}\right) + \left(\frac{h-2a}{a}\right)^2 - 0 + \frac{k^2}{a^2}\right\}$

$= a^3\left\{\left(\frac{h-2a}{a} + 1\right)^2 + \frac{k^2}{a^2}\right\}$

$= a\{(h-a)^2 + (k-0)^2\} = a(SA)^2$

104. Equation of tangent at $P(at^2, 2at)$ is

$$ty = x + at^2 \quad \Rightarrow \quad x - ty + at^2 = 0 \qquad \ldots(i)$$

which is also tangent to the circle

$$x^2 + y^2 = a^2/2 \qquad \ldots(ii)$$

then, length of perpendicular from centre of Eq. (ii) to Eq. (i) radius of the circle

$$\Rightarrow \quad \frac{|at^2|}{\sqrt{1+t^2}} = \frac{a}{\sqrt{2}}$$

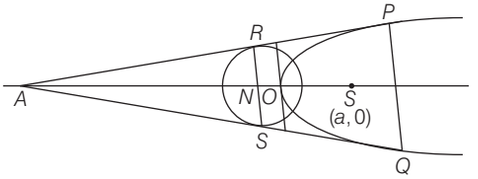
$$\Rightarrow \quad 2t^4 = (1 + t^2)$$

$$\Rightarrow \quad (t^2 - 1)(2t^2 + 1) = 0$$

$$\therefore \quad 2t^2 + 1 \neq 0 \quad \therefore \quad t^2 - 1 = 0$$

then, $t = \pm 1$

then, coordinates of P and Q are $(a, 2a)$ and $(a, -2a)$ respectively.



$$\therefore \quad PQ = 4a$$

$\therefore$ Equation of tangent at $P(a, 2a)$ is

$$x - y + a = 0 \qquad \ldots(iii)$$

[from Eq. (i)]

Let R be (x_1, y_1)

then, equation of tangent at $R(x_1, y_1)$ on Eq. (ii) is

$$xx_1 + yy_1 = \frac{a^2}{2} \qquad \ldots(iv)$$

Hence, Eqs. (iii) and (iv) are identical.

$$\therefore \quad \frac{x_1}{1} = \frac{y_1}{-1} = -\frac{a}{2}$$

$$\therefore \quad (x_1, y_1) = \left(-\frac{a}{2}, \frac{a}{2}\right)$$

Hence coordinates of S are $\left(-\frac{a}{2}, \frac{a}{2}\right)$.

$$\therefore \quad RS' = a$$

Hence, quadrilateral $PQRS'$ is trapezium whose area

$$= \frac{1}{2}(PQ + RS') \times \left(a + \frac{a}{2}\right)$$

$$= \frac{1}{2} \times (4a + a) \times \frac{3a}{2} = \frac{15a^2}{4} \text{ sq units.}$$

Aliter :

Here, centre of the circle is the vertex of the parabola and both circle and parabola are symmetrical about axis of parabola. In this case the point of intersection of common tangents must lie on the directrix and axis of the parabola.

i.e. $A(-a, 0)$.

Chord of contact of circle w.r.t. $A(-a, 0)$ is

$$x(-a) + y \cdot 0 = \frac{a^2}{2}$$

$$\therefore \quad x = -\frac{a}{2}$$

$\therefore$ Coordinates of R are $\left(-\frac{a}{2}, \frac{a}{2}\right)$ and chord of contact of parabola w.r.t. $A(-a, 0)$ is

$$y \cdot 0 = 2a(x - a)$$

i.e. $x = a$

$\therefore$ Coordinates of P is $(a, 2a)$

$\therefore$ Area of quadrilateral

$$PQRS' = 2\{\text{Area of } \Delta PAS - \text{Area of } \Delta RAN\}$$

$$= 2\left\{\frac{1}{2} \cdot 2a \cdot 2a - \frac{1}{2} \cdot \frac{a}{2} \cdot \frac{a}{2}\right\}$$

$$= 4a^2 - \frac{a^2}{4} = \frac{15a^2}{4} \text{ sq units}$$

105. Let parabola be $y^2 = 4ax$ and let $P(at_1^2, 2at_1)$, $Q(at_2^2, 2at_2)$ and $R(at_3^2, 2at_3)$ are three points on the parabola.

$\because$ Tangents at P, Q and R on parabola $y^2 = 4ax$

are $t_1y = x + at_1^2$, $t_2y = x + at_2^2$

and $t_3y = x + at_3^2$

Slopes of these tangents are $\frac{1}{t_1}, \frac{1}{t_2}, \frac{1}{t_3}$

but given slopes are in HP.

$\therefore$ t_1, t_2, t_3 are in AP if d is the common difference. Then,

$$t_2 - t_1 = d, \; t_3 - t_2 = d$$

and $t_3 - t_1 = 2d \qquad \ldots(i)$

Let the tangents at Q and R meet at P', R and P meet at Q', P and Q meet at R'.

$$\therefore \quad P' = \{at_2t_3, a(t_2 + t_3)\}$$

$$Q' = \{at_3t_1, a(t_3 + t_1)\}$$

and $R' = \{at_1t_2, a(t_1 + t_2)\}$

$$\therefore \text{ Area of } \Delta P'Q'R' = \frac{1}{2}\begin{vmatrix} at_2t_3 & a(t_2+t_3) & 1 \\ at_3t_1 & a(t_3+t_1) & 1 \\ at_1t_2 & a(t_1+t_2) & 1 \end{vmatrix}$$

$$= \frac{1}{2}\{at_2t_3\{a(t_3+t_1)-(t_1+t_2)\}+\ldots+\ldots\}$$

$$= \frac{1}{2}a^2\Sigma\, t_2t_3(t_3-t_2)$$

$$= \frac{1}{2}a^2(t_1-t_2)(t_2-t_3)(t_3-t_1)$$

$$= \frac{1}{2}a^2(-d)(-d)(2d) \qquad \text{[by using Eq. (i)]}$$

$$= a^2d^3, \text{ which is constant.}$$

Remark

$\Sigma t_1^2(t_2-t_3) = \Sigma\, t_2t_3(t_3-t_2) = (t_1-t_2)(t_2-t_3)(t_3-t_1)$

Corollary Area of triangle of $P(at_1^2, 2at_1)$, $Q(at_2^2, 2at_2)$ and $R(at_3^2, 2at_3)$ is $a^2(t_1-t_2)(t_2-t_3)(t_3-t_1) = \Delta$ (say)

It is clear from just previous example the area of the triangle inscribed in a parabola is twice the area of the triangle formed by the tangents at the vertices.

106. Equation of tangent at (1, 7) to $y = x^2 + 6$

$$\Rightarrow \quad \frac{1}{2}(y+7) = x\cdot 1 + 6$$

$$\Rightarrow \quad y = 2x + 5 \qquad \ldots(i)$$

This tangent also touches the circle.

$$x^2 + y^2 + 16x + 12y + c = 0 \qquad \ldots(ii)$$

Now, solving Eqs. (i) and (ii), we get

$$x^2 + (2x+5)^2 + 16x + 12(2x+5) + c = 0$$

$$\Rightarrow \quad 5x^2 + 60x + 85 + c = 0$$

Since, roots are equal, so

$$B^2 - 4AC = 0$$

$$\Rightarrow \quad (60)^2 - 4\times 5\times(85+c) = 0$$

$$\Rightarrow \quad 85 + c = 180$$

$$\Rightarrow \quad 5x^2 + 60x + 180 = 0$$

$$\Rightarrow \quad x = -\frac{60}{10} = -6 \quad \Rightarrow \quad y = -7$$

Hence, point of contact is $(-6, -7)$.

107. $\because$ $P \equiv (1,0)$, let $Q \equiv (h, k)$

such that $k^2 = 8h \qquad \ldots(i)$

Let (α, β) be the mid-point of PQ.

$$\therefore \quad \alpha = \frac{h+1}{2}, \beta = \frac{k+0}{2}$$

$$\Rightarrow \quad h = 2\alpha - 1, k = 2\beta$$

From Eq. (i), we get

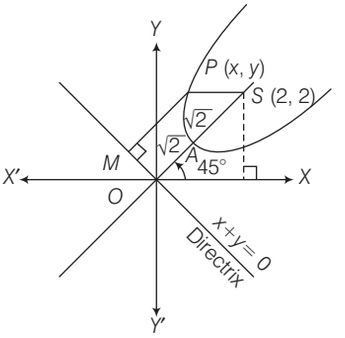
$$(2\beta)^2 = 8(2\alpha - 1)$$

$$\Rightarrow \quad \beta^2 = 4\alpha - 2$$

$$\Rightarrow \quad \beta^2 - 4\alpha + 2 = 0$$

$\therefore$ Required locus is $y^2 - 4x + 2 = 0$.

108. Coordinates of S are $(2\sqrt{2}\cos 45°, 2\sqrt{2}\sin 45°)$ i.e. (2, 2).



$$\therefore \quad SP = PM$$

$$\Rightarrow \quad (SP)^2 = (PM)^2$$

$$\Rightarrow \quad (x-2)^2 + (y-2)^2 = \left[\frac{(x+y)}{\sqrt{2}}\right]^2$$

$$\Rightarrow \quad 2(x^2 + y^2 - 4x - 4y + 8) = x^2 + y^2 + 2xy$$

$$\Rightarrow \quad x^2 + y^2 - 2xy - 8x - 8y + 16 = 0$$

$$\therefore \quad (x-y)^2 = 8(x+y-2)$$

109. Equation of tangent to $y = x^2$ is

$$y = mx - \frac{1}{4}m^2 \qquad \ldots(i)$$

Equation of tangent to $(x-2)^2 = -y$ is

$$y = m(x-2) + \frac{1}{4}m^2 \qquad \ldots(ii)$$

$\because$ Eqs. (i) and (ii) are identical.

$$\Rightarrow \quad m = 0 \text{ or } 4$$

$\therefore$ Common tangents are $y = 0$ and $y = 4x - 4 = 4(x-1)$.

110. Given parabola is

$$y = \frac{a^3x^2}{3} + \frac{a^2x}{2} - 2a \qquad \ldots(i)$$

For vertex $\frac{dy}{dx} = 0 \Rightarrow x = -\frac{3}{4a}$

Substitute $x = -\frac{3}{4a}$ in Eq. (i), we get

$$y = -\frac{35a}{16}$$

$\therefore$ Coordinates of vertex are $\left(-\frac{3}{4a}, -\frac{35a}{16}\right)$.

For locus let $x = -\frac{3}{4a}$ and $y = -\frac{35a}{16}$.

$\therefore \quad xy = \frac{105}{64}$, which is the required locus.

111. $\because$ $y = x^2 - 5x + 6$

$\therefore$ Equation of tangent at (2, 0) is

$$\frac{y+0}{2} = x\cdot 2 - \frac{5}{2}(x+2) + 6$$

$$\Rightarrow \quad y = -x + 2 \qquad \ldots(i)$$

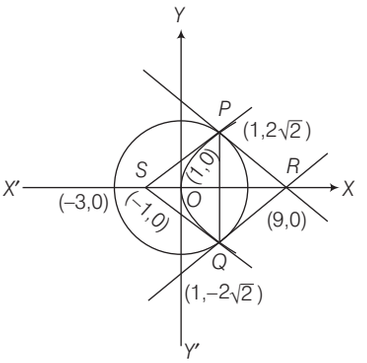
and equation of tangent at (3, 0) is

$\Rightarrow \quad y = x - 3$ ...(ii)

$\because$ Eqs. (i) and (ii) are perpendicular.

$\therefore$ Angle between tangents is $\pi / 2$.

112. (i) Coordinates of P and Q are $(1, 2\sqrt{2})$ and $(1, -2\sqrt{2})$.



Area of $\Delta PQR = \frac{1}{2} \cdot 4\sqrt{2} \cdot 8 = 16\sqrt{2}$

Area of $\Delta PQS = \frac{1}{2} \cdot 4\sqrt{2} \cdot 2 = 4\sqrt{2}$

$\therefore$ Ratio of area of ΔPQS and ΔPQR is 1 : 4.

(ii) Equation of circumcircle of ΔPRS is

$$(x + 1)(x - 9) + y^2 + \lambda y = 0$$

It will pass through $(1, 2\sqrt{2})$, then

$$-16 + 8 + \lambda\, 2\sqrt{2} = 0 \Rightarrow \lambda = \frac{8}{2\sqrt{2}} = 2\sqrt{2}$$

Equation of circumcircle is

$$x^2 + y^2 - 8x + 2\sqrt{2}y - 9 = 0$$

Hence, radius is $3\sqrt{3}$.

Aliter :

Let $\angle PSR = \theta \Rightarrow \sin\theta = \frac{2\sqrt{2}}{2\sqrt{3}}$

$\Rightarrow \quad PR = 6\sqrt{2} = 2R \cdot \sin\theta \Rightarrow R = 3\sqrt{3}$.

(iii) Radius of incircle is $r = \frac{\Delta}{s}$.

As $\quad \Delta = 16\sqrt{2}$

$\therefore \quad s = \frac{6\sqrt{2} + 6\sqrt{2} + 4\sqrt{2}}{2} = 8\sqrt{2}$

$\therefore \quad r = \frac{16\sqrt{2}}{8\sqrt{2}} = 2$

113. $y = -\frac{x^2}{2} + x + 1 \Rightarrow y - \frac{3}{2} = -\frac{1}{2}(x - 1)^2$

$\Rightarrow$ It is symmetric about $x = 1$.

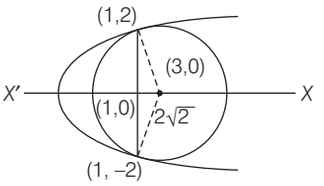
Hence, both statement are true and Statement II is correct explanation of Statement I.

114. $\because$ Point of intersection of two perpendicular tangents to the parabola lies on directrix of the parabola.

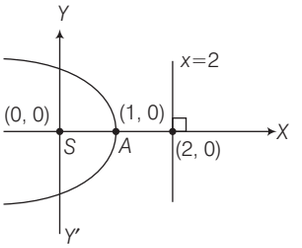
$\therefore$ Equation of directrix is $x + 2 = 0$.

So, point is $(-2, 0)$.

115. The circle and the parabola touch each other at $x = 1$, i.e. at the points (1, 2) and (1, – 2) as shown in figure.

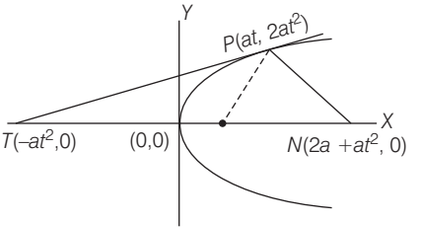


116. Vertex is (1,0).



117. $G \equiv (h, k)$

$\Rightarrow \quad h = \frac{2a + at^2}{3}, k = \frac{2at}{3}$



$\Rightarrow \quad \left(\frac{3h - 2a}{a}\right) = \frac{9k^2}{4a^2}$

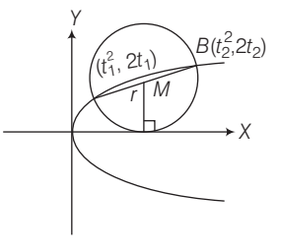
$\therefore$ Required parabola is

$$\frac{9y^2}{4a^2} = \frac{(3x - 2a)}{a} = \frac{3}{a}\left(x - \frac{2a}{3}\right)$$

$\Rightarrow \quad y^2 = \frac{4a}{3}\left(x - \frac{2a}{3}\right)$

$\therefore$ Vertex $\equiv \left(\frac{2a}{3}, 0\right)$; Focus $\equiv (a, 0)$.

118. Slope of $AB = \frac{2t_2 - 2t_1}{(t_2^2 - t_1^2)} = \frac{2}{(t_2 + t_1)}$ ...(i)



$M = \text{Mid-point of } AB = \left(\frac{t_1^2 + t_2^2}{2}, t_1 + t_2\right)$

$\therefore \quad r = |t_1 + t_2| \Rightarrow t_1 + t_2 = \pm r$

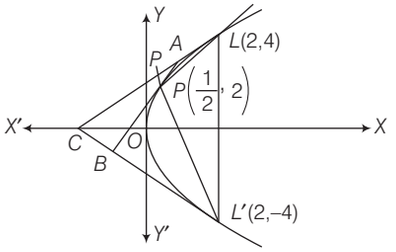
Now, from Eq. (i),

$$\text{slope of } AB = \pm \frac{2}{r}.$$

119. The locus of perpendicular tangent is directrix, i.e., $x + 1 = 0$ or $x = -1$.

120. $\Delta_1 =$ Area of $\Delta PLL'$

$$= \frac{1}{2} \times 8 \times \left(2 - \frac{1}{2}\right) = 6 \text{ sq unitss}$$



Now, equation of AB is $y = 2x + 1$,
equation of AC is $y = x + 2$ and
equation of BC is $y = -x - 2$
On solving above equations, we get
$A(1, 3)$, $B(-1, -1)$ and $C(-2, 0)$

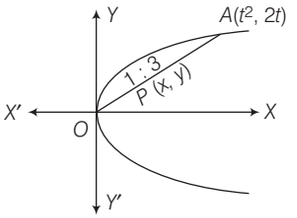
$$\therefore \quad \Delta_2 = \frac{1}{2}\begin{Vmatrix} 1+2 & 3-0 \\ -1+2 & -1-0 \end{Vmatrix} = 3 \text{ sq units}$$

$$\therefore \quad \frac{\Delta_1}{\Delta_2} = 2$$

121. Let $A(x, y) = A(t^2, 2t)$ be any point on the parabola $y^2 = 4x$, then

$$x = \frac{t^2}{4} \quad \ldots(i)$$

and
$$y = \frac{2t}{4} \quad \ldots(ii)$$



From Eqs. (i) and (ii), we get

$$x = y^2$$

122. The equation of normal to

$$y^2 = 4x \text{ is } y = mx - 2m - m^3 \quad \ldots(i)$$

As it passes through (9, 6), then

$$6 = 9m - 2m - m^3$$

$$\Rightarrow \quad m^3 - 7m + 6 = 0$$

$$\Rightarrow (m-1)(m-2)(m+3) = 0$$

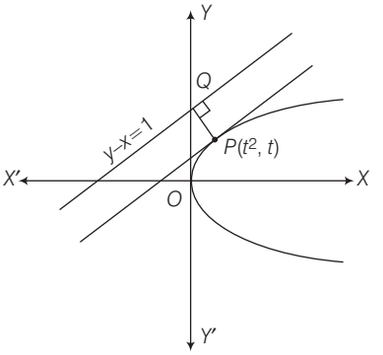
$$\Rightarrow \quad m = 1, 2, -3$$

From Eq. (i), equations of normals are

$y = x - 3, y = 2x - 12, \ y = -3x + 33$

$\Rightarrow \ y - x + 3 = 0, y - 2x + 12 = 0, \ y + 3x - 33 = 0$

123. The shortest distance between $y = x - 1$ and $y^2 = x$ is along the normal of $y^2 = x$.



Let $P(t^2, t)$ be any point on $y^2 = x$.

$\therefore$ Tangent at P is $y = \frac{x}{2t} + \frac{t}{2}$.

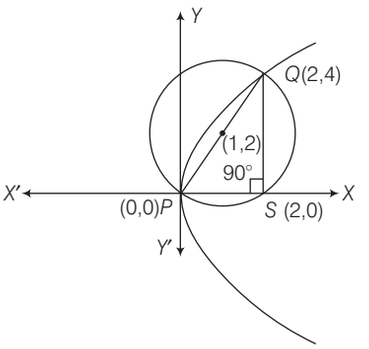
$\therefore$ Slope of tangent $= \frac{1}{2t}$

and tangent at P is parallel to $y - x = 1$

$$\therefore \quad \frac{1}{2t} = 1 \ \Rightarrow \ t = \frac{1}{2} \Rightarrow P\left(\frac{1}{4}, \frac{1}{2}\right)$$

Hence, shortest distance $= PQ = \dfrac{\left|\frac{1}{2} - \frac{1}{4} - 1\right|}{\sqrt{(1+1)}} = \dfrac{3}{4\sqrt{2}} = \dfrac{3\sqrt{2}}{8}$

124. We observe that both parabola $y^2 = 8x$ and circle $x^2 + y^2 - 2x - 4y = 0$ pass through origin say $P(0, 0)$.



Let Q be the point $(2t^2, 4t)$, then it will satisfy the equation of circle.

$$\therefore \quad (2t^2)^2 + (4t)^2 - 2(2t^2) - 4(4t) = 0$$

$$\Rightarrow \quad 4t^4 + 12t^2 - 16t = 0$$

$$\Rightarrow \quad t(t-1)(t^2 + t - 4) = 0 \ \Rightarrow \ t = 0 \text{ or } 1$$

For $t = 0$, we get point P, therefore $t = 1$ gives point Q as (2, 4).
Here, $P(0, 0)$ and $Q(2, 4)$ are end points of diameter of the given circle and focus of the parabola is the point $S(2, 0)$.

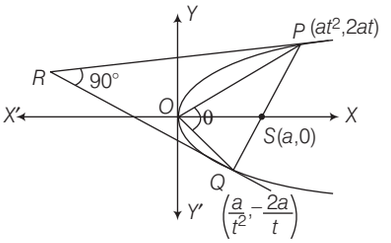
$$\therefore \quad \angle PSQ = 90^\circ$$

Hence, area of $\Delta PQS = \frac{1}{2} \times 2 \times 4 = 4$ sq units.

Sol. (Q. Nos. 125 and 126)

$\because$ PQ is the focal chord of $y^2 = 4ax$.

$\therefore$ Coordinates of P and Q are $(at^2, 2at)$ and $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$.



Tangents at P and Q are

$$ty = x + at^2 \text{ and } ty = xt^2 + a$$

which intersect each other at $R\left(-a, a\left(t - \frac{1}{t}\right)\right)$.

As R lies on the line $y = 2x + a, a > 0$

$$\therefore \quad a\left(t - \frac{1}{t}\right) = -2a + a$$

$$\Rightarrow \quad t - \frac{1}{t} = -1$$

125. $\because$ Slope of $OP = \frac{2}{t}$ and slope of $OQ = -2t$

$$\therefore \quad \tan\theta = \left|\frac{\frac{2}{t} + 2t}{1 - 4}\right| = \frac{2}{3}\left|t + \frac{1}{t}\right|$$

$$= \frac{2}{3}\sqrt{\left(t - \frac{1}{t}\right)^2 + 4} = \frac{2}{3}\sqrt{5} \qquad \left[\because t - \frac{1}{t} = -1\right]$$

$\because$ $\theta > 90°$

$$\therefore \quad \tan\theta = -\frac{2}{3}\sqrt{5}$$

126. $PQ = a\left(t + \frac{1}{t}\right)^2 = a\left\{\left(t - \frac{1}{t}\right)^2 + 4\right\}$

$= a(1 + 4) = 5a \qquad \left[\because t - \frac{1}{t} = -1\right]$

127. Equation of tangent of $y^2 = 4x$ in terms of slope is

$$y = mx + \frac{1}{m} \qquad \ldots(i)$$

$\because$ Line Eq. (i) touches $x^2 = -32y$

$$\Rightarrow \quad x^2 = -32\left(mx + \frac{1}{m}\right)$$

$$\Rightarrow \quad x^2 + 32mx + \frac{32}{m} = 0 \qquad \ldots(ii)$$

For touching roots of Eq. (ii) are equal.

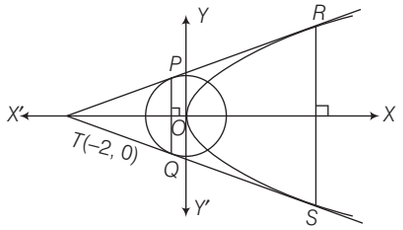
$$\therefore \quad D = 0$$

$$\Rightarrow \quad (32m)^2 = 4 \cdot 1 \cdot \left(\frac{32}{m}\right)$$

$$\Rightarrow \quad m^3 = \frac{1}{8}$$

$$\therefore \quad m = 1/2$$

128. Let the tangent to, $y^2 = 8x$ be $y = mx + \frac{2}{m}$.



If it is common tangent to parabola and circle, then

$y = mx + \frac{2}{m}$ is a tangent to $x^2 + y^2 = 2$.

$$\therefore \quad \frac{\frac{2}{m}}{\sqrt{(1 + m^2)}} = \sqrt{2}$$

$$\Rightarrow \quad \frac{4}{m^2(1 + m^2)} = 2$$

$$\Rightarrow \quad m^4 + m^2 - 2 = 0$$

$$\Rightarrow \quad (m^2 + 2)(m^2 - 1) = 0$$

$$\therefore \quad m = \pm 1$$

$\therefore$ Required tangents are $y = x + 2$ and $y = -x - 2$.

Their common point is $T(-2, 0)$.

Chord of contact PQ to circle is

$$x \cdot (-2) + y \cdot 0 = 2$$

$$\Rightarrow \quad x = -1$$

Hence, coordinates of P and Q are $(-1, 1)$ and $(-1, -1)$ and chord of contact RS to parabola is

$$y \cdot 0 = 4(x - 2)$$

$$\Rightarrow \quad x = 2$$

Hence, coordinates of R and S are $(2, 4)$ and $(2, -4)$.

$\therefore$ Area of trapezium $PQRS = \frac{1}{2}(2 + 8) \times 3 = 15$ sq units

Sol. (Q. Nos. 129 and 130)

$\because$ PQ is a focal chord, then $Q\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$.

Also, $\quad QR \parallel PK \Rightarrow m_{QR} = m_{PK}$

$$\Rightarrow \quad \frac{2ar + \frac{2a}{t}}{ar^2 - \frac{a}{t^2}} = \frac{0 - 2at}{2a - at^2}$$

$$\Rightarrow \quad \frac{2}{r - \frac{1}{t}} = \frac{-2t}{2 - t^2}$$

$[\because r + \frac{1}{t} \neq 0$, otherwise Q will coincide with $R]$

$$\Rightarrow \quad 2 - t^2 = -rt + 1$$

$$\therefore \quad r = \frac{t^2 - 1}{t}$$

129. $r = \frac{t^2 - 1}{t}$

130. Tangent at P is $ty = x + at^2$ ...(i)

Normal at S is $y + sx = 2as + as^3$ ...(ii)

Putting the value of x from Eq. (i) in Eq. (ii), then

$$y + s(ty - at^2) = 2as + as^3$$

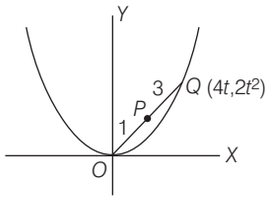
$$\Rightarrow \quad y + (st)y - a(st)t = 2as + as^3$$

$$\Rightarrow \quad y + y - at = \frac{2a}{t} + \frac{a}{t^3} \qquad [\because st = 1]$$

$$\Rightarrow \quad 2y = a\left(t + \frac{2}{t} + \frac{1}{t^3}\right)$$

$$\therefore \quad y = \frac{a(t^2 + 1)^2}{2t^3}$$

131. Let any point Q on $x^2 = 8y$ is $(4t, 2t^2)$ and given $P(h, k)$ divides OQ in the ratio 1:3 (internally).



Then, $h = \frac{4t}{4} = t$ and $k = \frac{2t^2}{4} \Rightarrow 2k = h^2$

$\therefore$ Required locus of P is $x^2 = 2y$.

132. End points of latusrectum of

$$y^2 = 4x \text{ are } (1, \pm 2).$$

Equation of normal to $y^2 = 4x$ at $(1, 2)$ is

$$y - 2 = -\frac{2}{2}(x - 1)$$

$$\Rightarrow \quad x + y - 3 = 0$$

As it is tangent to circle $(x - 3)^2 + (y + 2)^2 = r^2$

$$\therefore \quad \frac{|3 - 2 - 3|}{\sqrt{(1 + 1)}} = r \Rightarrow r^2 = 2$$

133. Let $(t^2, 2t)$ be any point on $y^2 = 4x$. Let (h, k) be image of $(t^2, 2t)$ with respect to the line $x + y + 4 = 0$, then

$$\frac{h - t^2}{1} = \frac{k - 2t}{1} = \frac{-2(t^2 + 2t + 4)}{1 + 1}$$

$$\Rightarrow \quad h = -(2t + 4) \text{ and } k = -(t^2 + 4)$$

$$\Rightarrow \quad (k + 4) = -\left(\frac{h + 4}{-2}\right)^2$$

$$\Rightarrow \quad (h + 4)^2 = -4(k + 4)$$

Locus of (h, k) is $(x + 4)^2 = -4(y + 4)$.

$\therefore$ Curve C is $(x + 4)^2 = -4(y + 4)$

Now, intersection of C with $y = -5$, then

$$(x + 4)^2 = -4(-5 + 4) = 4$$

$$\therefore \quad x + 4 = \pm 2 \Rightarrow x = -6, -2$$

$\therefore$ $A(-6, -5)$ and $B(-2, -5)$

$$\therefore \quad AB = 4$$

134. Let $P\left(\frac{t_1^2}{2}, t_1\right)$ and $Q\left(\frac{t_2^2}{2}, t_2\right)$ such that $t_1 > 0$

$[\because P \text{ lies in first quadrant}]$

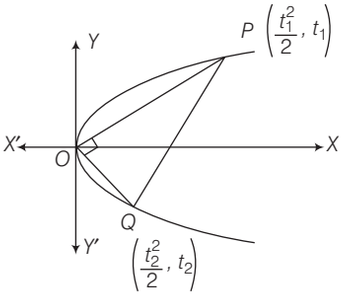
$\because$ Circle with PQ as diameter passes through the vertex $O(0, 0)$ of the parabola.

$$\therefore \quad \angle POQ = 90^\circ$$

$\Rightarrow$ Slope of $OP \times$ Slope of $OQ = -1$

$$\Rightarrow \quad \frac{2}{t_1} \times \frac{2}{t_2} = -1$$

$$\Rightarrow \quad t_1 t_2 = -4 \qquad [\because t_2 < 0]$$



Now, area of $\Delta OPQ = 3\sqrt{2}$

$$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} \frac{t_1^2}{2} & t_1 \\ \frac{t_2^2}{2} & t_2 \end{vmatrix} = 3\sqrt{2}$$

$$\Rightarrow \quad \frac{1}{4}t_1 t_2(t_1 - t_2) = \pm 3\sqrt{2}$$

$$\Rightarrow \quad t_1 - t_2 = \pm 3\sqrt{2} \qquad [\because t_1 t_2 = -4]$$

$$\Rightarrow \quad t_1 + \frac{4}{t_1} = \pm 3\sqrt{2}$$

or $\quad t_1 + \frac{4}{t_1} = 3\sqrt{2} \qquad [\because t_1 > 0]$

$$\Rightarrow \quad t_1^2 - 3\sqrt{2}t_1 + 4 = 0$$

$$\therefore \quad t_1 = \frac{3\sqrt{2} \pm \sqrt{2}}{2} = 2\sqrt{2}, \sqrt{2}.$$

$\therefore$ Point P can be $(4, 2\sqrt{2})$ or $(1, \sqrt{2})$.

135. Let $P(2t^2, 4t)$ and $C(0, -6)$.

$$\therefore \quad (CP)^2 = 4t^4 + (4t + 6)^2 = z \qquad \text{(say)}$$

$$\therefore \quad \frac{dz}{dt} = 0$$

$$\Rightarrow \quad 16t^3 + 2(4t + 6) \cdot 4 = 0$$

$$\Rightarrow \quad t^3 + 2t + 3 = 0$$

$$\Rightarrow \quad (t + 1)(t^2 - t + 3) = 0$$

$$\therefore \quad t = -1$$

$$\Rightarrow \quad P(2, -4)$$

Equation of circle is

$$(x - 2)^2 + (y + 4)^2 = (2 - 0)^2 + (-4 + 6)^2$$

$$\Rightarrow \quad x^2 + y^2 - 4x + 8y + 12 = 0$$

136. $\because C_1 : x^2 + y^2 = 3$ and parabola $x^2 = 2y$, then

$$y^2 + 2y - 3 = 0 \Rightarrow y = 1, -3$$

$\therefore \quad P(\sqrt{2}, 1)$ [$\because P$ lies in first quadrant]

Now, tangent at $P(\sqrt{2}, 1)$ on the circle C_1 is

$$x\sqrt{2} + y = 3$$

Let Q_2 or $Q_3(0, \lambda)$

$$\therefore \quad \frac{|0 + \lambda - 3|}{\sqrt{(2+1)}} = 2\sqrt{3}$$

$$\Rightarrow \quad |\lambda - 3| = 6$$

$$\therefore \quad \lambda = 9 \text{ or } -3$$

$\Rightarrow Q_2(0, -3)$ and $Q_3(0, 9)$.

Alternate (a) $Q_2Q_3 = 12$

Alternate (b) $R_2R_3 =$ Length of external common tangent

$$= \sqrt{(Q_2Q_3)^2 - (2\sqrt{3} + 2\sqrt{3})^2}$$

$$= \sqrt{(144 - 48} = 4\sqrt{6}$$

Alternate (c) Area of $\Delta OR_2R_3 = \frac{1}{2} \times R_2R_3 \times \frac{|0 + 0 - 3|}{\sqrt{(2+1)}}$

$$= \frac{1}{2} \times 4\sqrt{6} \times \frac{3}{\sqrt{3}} = 6\sqrt{2}$$

Alternate (d) Area of $\Delta PQ_2Q_3 = \frac{1}{2} \times Q_2Q_3 \times \sqrt{2}$

$$= \frac{1}{2} \times 12 \times \sqrt{2} = 6\sqrt{2}$$

137. Let $P(t^2, 2t)$, $S(2, 8)$ and $r = \sqrt{(4 + 64 - 64)} = 2$

We know that, shortest distance between two curves lies along their common normal. The common normal will pass through centre of circle.

$\therefore$ Slope of $PS =$ Slope of normal to the parabola $y^2 = 4x$ at $P(t^2, 2t)$

$$\Rightarrow \quad \frac{2t - 8}{t^2 - 2} = -t \text{ or } t^3 = 8 \Rightarrow t = 2$$

$\therefore \quad P(4, 4)$

Alternate (a) $SP = \sqrt{(2-4)^2 + (8-4)^2} = 2\sqrt{5}$

Alternate (b) $SQ = r = 2$

$$\therefore \quad \frac{SQ}{QP} = \frac{SQ}{SP - SQ} = \frac{2}{2\sqrt{5} - 2}$$

$$= \frac{1}{(\sqrt{5} - 1)} \times \frac{(\sqrt{5} + 1)}{(\sqrt{5} + 1)} = \frac{\sqrt{5} + 1}{4}$$

$$\Rightarrow \quad SQ : QP = (\sqrt{5} + 1) : 4$$

Alternate (c) Equation of normal at $P(4, 4)$ is

$$y - 4 = -\frac{4}{2}(x - 4)$$

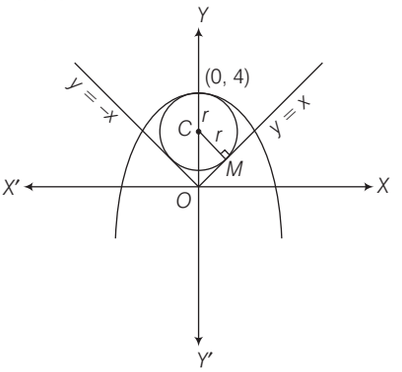
$$\Rightarrow \quad y - 4 = -2x + 8$$

$$\Rightarrow \quad 2x + y = 12$$

$\therefore$ Intercept on X-axis is 6.

Alternate (d) Slope of tangent at $Q =$ Slope of tangent at $P = \frac{1}{2}$

138. Centre of circle



$$C \equiv (0, 4 - r)$$

$$\because \quad CM = r$$

$$\therefore \quad \frac{|10 - (4 - r)|}{\sqrt{2}} = r$$

$$\Rightarrow \quad 4 - r = r\sqrt{2}$$

or $$r = \frac{4}{\sqrt{2} + 1}$$

$$= 4(\sqrt{2} - 1)$$

139. (a) Equation of chord of parabola $y^2 = 16x$ whose mid-point (h, k) is

$$T = S_1$$

or $$ky - 8(x + h) = k^2 - 16h$$

or $$8x - ky = 8h - k^2 \quad \text{...(i)}$$

Now comparing Eq. (i) and $2x + y = p$, then

$$\frac{8}{2} = \frac{-k}{1} = \frac{8h - k^2}{p}$$

$$\Rightarrow \quad k = -4 \text{ and } 4p = 8h - k^2$$

or $$k = -4 \text{ and } p = 2h - 4$$

Hence, $$p = 2, h = 3, k = -4$$

CHAPTER

06

Ellipse

Learning Part

Session 1

- Ellipse Definition
- The Foci and Two Directrices of an Ellipse
- Some Terms Related to an Ellipse
- The Shape of the Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, when $b>a$
- Standard Equation of Ellipse
- Tracing of the Ellipse
- Focal Distances of a Point
- Mechanical Construction of an Ellipse

Session 2

- Position of a Point with Respect to an Ellipse
- Equation of Tangent in Different Forms
- Properties of Eccentric Angles of the Co-normal Points
- Smart Table
- Intersection of a Line and an Ellipse
- Equations of Normals in Different Forms
- Co-normal Points Lie on a Fixed Curve

Session 3

- Pair of Tangents
- Chord Bisected at a Given Point
- Conjugate Diameters
- Director Circle
- Co+ncyclic Points
- Reflection Property of an Ellipse
- Equation of an Ellipse Referred to Two Perpendicular Lines
- Chord of Contact
- Diameter
- Equi-Conjugate Diameters
- Sub-Tangent and Sub-Normal
- Some Standard Properties of the Ellipse

Practice Part

- JEE Types Examples
- Chapter Exercises

Arihant on Your Mobile !

Exercises with the symbol can be practised on your mobile. See inside cover page to activate for free.

Session 1

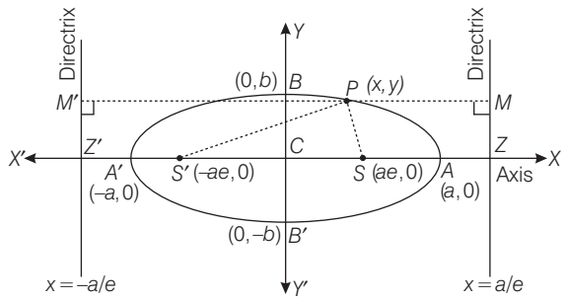
Ellipse Definition, Standard Equation of Ellipse, Tracing of the Ellipse,Focal Distances of a Point, The Shape of the Ellipse $x^2/a^2 + y^2/b^2 = 1$, When $b > a$, Mechanical Construction of an Ellipse,

Ellipse Definition

An ellipse is the locus of a point which moves in a plane such that its distance from a fixed point (i.e. focus) is a constant ratio from a fixed line (i.e. directrix).
This ratio is called eccentricity and is denoted by e.
For an ellipse $0 < e < 1$.

Standard Equation of Ellipse

Let S be the focus and ZM the directrix of the ellipse. Drow $SZ \perp ZM$. Divide SZ internally and externally in the ratio $e:1 (0 < e < 1)$ and let A and A' be these internal and external point of division



Then, $SA = eAZ$ ...(i)

and $SA' = eA'Z$ ...(ii)

Clearly A and A' will lie on the ellipse

Let $AA' = 2a$ and take C the mid point of AA' as origin

$\therefore$ $CA = CA' = a$ ...(iii)

Let $P(x, y)$ by any point on the ellipse referred to CA and CB as co-ordinate axes

Then, adding Eqs. (i) and (ii),

$$SA + SA' = e(AZ + A'Z)$$

$\Rightarrow$ $AA' = e(CZ - CA + CA' + CZ)$ (from figure)

$\Rightarrow$ $AA' = e(2CZ)$ ($\because CA = CA'$)

$\Rightarrow$ $2a = 2eCZ$

$\because$ $CZ = a/e$

$\therefore$ The directrix MZ is $x = CZ = a/e$

or $\dfrac{a}{e} - x = 0$ $\left(\because e < 1, \therefore \dfrac{a}{e} > 1\right)$

and subtracting Eqs. (i) from (ii), then

$$SA' - SA = e(A'Z - AZ)$$

$\Rightarrow$ $(CA' + CS) - (CA - CS) = e(AA')$

$\Rightarrow$ $2CS = e(AA')$ ($\because CA = CA'$)

$\Rightarrow$ $2CS = e(2a)$

$\therefore$ $CS = ae$

$\therefore$ The focus S is $(CS, 0)$ i.e. $(ae, 0)$

Now draw $PM \perp MZ$

$\therefore$ $\dfrac{SP}{PM} = e$ or $(SP)^2 = e^2 (PM)^2$

$$(x - ae)^2 + (y - 0)^2 = e^2 \left(\frac{a}{e} - x\right)^2$$

$\Rightarrow$ $(x - ae)^2 + y^2 = (a - ex)^2$

$\Rightarrow$ $x^2 + a^2e^2 - 2aex + y^2 = a^2 - 2aex + e^2x^2$

$\Rightarrow$ $x^2(1 - e^2) + y^2 = a^2(1 - e^2)$

$\Rightarrow$ $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2(1 - e^2)} = 1$

or $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

where, $$b^2=a^2(1-e^2)$$

This is the standard equation of an ellipse, AA' and BB' are called the major and minor axes of the ellipse. (Here $b<a$) and A and A' are the vertices of the ellipse.

Remark

Two ellipse are said to be similar if they have the same value of eccentricity.

Generally,

The equation to the ellipse, whose focus is the point (h, k) and directrix is $lx+my+n=0$ and whose eccentricity is e, is

$$(x-h)^2+(y-k)^2=e^2\cdot\frac{(lx+my+n)^2}{(l^2+m^2)}$$

The Foci and Two Directrices of an Ellipse

On the negative side of origin take a point S' which is such that

$$CS=CS'=ae$$

and another point Z', then

$$CZ=CZ'=\frac{a}{e}$$

$\therefore$ Coordinates of S' are $(-ae, 0)$ and equation of second directrix (*i.e.*, $Z'M'$) is $x=-\frac{a}{e}$

Let $P(x, y)$ be any point on the ellipse, then

$$S'P=ePM'$$

or $$(S'P)^2=e^2(PM')^2$$

or $$(x+ae)^2+(y-0)^2=e^2\left(x+\frac{a}{e}\right)^2$$

or $$(x+ae)^2+y^2=(ex+a)^2$$

or $$x^2(1-e^2)+y^2=a^2(1-e^2)$$

or $$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1$$

or $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

where, $$b^2=a^2(1-e^2)$$

The equation being the same as that of the ellipse when $S(ae, 0)$ is focus and MZ i.e. $x=a/e$ is directrix.

Hence, coordinates of foci are $(\pm ae, 0)$ and equations of directrices are $x=\pm a/e$.

Remarks

1. Distance between foci $SS'=2ae$ and distance between directrices $ZZ'=2a/e$
2. If $e=0$

   then $$b^2=a^2(1-0)$$

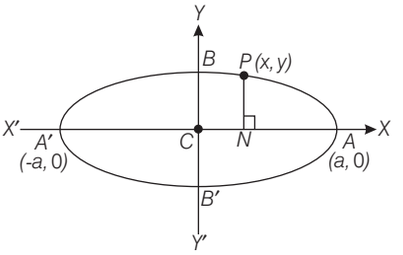
   $\therefore$ $$b^2=a^2$$

   then equation of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ changes in circle

   i.e. $$x^2+y^2=a^2$$
3. Since, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

   $\Rightarrow$ $$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}=\frac{a^2-x^2}{a^2}$$

   $$\frac{y^2}{b^2}=\frac{(a+x)(a-x)}{a^2}$$



   or $$\frac{(PN)^2}{b^2}=\frac{A'N\cdot AN}{a^2}$$

   or $$\frac{(PN)^2}{AN\cdot A'N}=\frac{b^2}{a^2}=\frac{(BC)^2}{(AC)^2}$$

   i.e. $$(PN)^2:AN\cdot A'N::(BC)^2:(AC)^2$$
4. The distance of every focus from the extremity of minor axis is equal to a.

   i.e. $$\sqrt{(a^2e^2+b^2)}=\sqrt{(a^2-b^2+b^2)}=a$$

Tracing of the Ellipse

Equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \ldots(i)$$

1. The ellipse (i) cuts X-axis at $A(a, 0)$ and $A'(-a, 0)$ and cuts Y-axis at $B(0, b)$ and $B'(0, -b)$.
2. The Eq. (i) does not change when y is replaced $by-y$ and x is replaced by $-x$, Hence ellipse (i) is symmetrical about both axes

3. The equation (i), may be written in either of the form

$$y = \pm b\sqrt{\left(1-\frac{x^2}{a^2}\right)} \quad \text{...(ii)}$$

or $$x = \pm a\sqrt{\left(1-\frac{y^2}{b^2}\right)} \quad \text{...(iii)}$$

From Eq. (ii), it follows that (y is real)

if $$1-\frac{x^2}{a^2} \geq 0 \text{ or } a^2 - x^2 \geq 0$$

or $$x^2 \leq a^2 \text{ or } -a \leq x \leq a$$

Also from Eq. (iii), it follows that (x is real)

if $$1-\frac{y^2}{b^2} \geq 0 \text{ or } b^2 - y^2 \geq 0 \text{ or } y^2 \leq b^2$$

$\therefore$ $$-b \leq y \leq b$$

Ellipse (i) is a closed curve lies entirely between the lines $x = a$ and $x = -a$ and the lines $y = b$ and $y = -b$.

Since, when x increases, then y decreases from Eq. (ii) and when y decreases, then x increases from Eq. (iii).

Remark

Area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is πab.

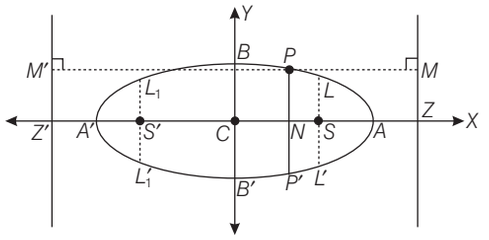
Some Terms Related to an Ellipse

Let the equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

($a > b$)

1. **Centre :** All chords passing through C is bisected at C

   Here $C \equiv (0, 0)$



2. **Foci :** S and S' are two foci of the ellipse and their coordinates are $(ae, 0)$ and $(-ae, 0)$ respectively.
3. **Directrices :** ZM and $Z'M'$ are two directrices of the ellipse and their equation are $x = \frac{a}{e}$ and $x = -\frac{a}{e}$ respectively.
4. **Axes :** The lines AA' and BB' are called the **major** and **minor** axes of the ellipse

   $\because$ $0 < e < 1$

   or $0 < e^2 < 1 \quad (\therefore\ 0 > -e^2 > -1)$

   or $0 < 1 - e^2 < 1 \quad (\text{or } 1 > 1 - e^2 > 1 - 1)$

   or $a^2(1 - e^2) < a^2 \quad (\text{or } 0 < 1 - e^2 < 1)$

   or $b^2 < a^2$

   i.e. $b < a$

Remark

The major and minor axis together are called **principal axes** of the ellipse.

5. **Double ordinates :** If P be a point on the ellipse draw PN perpendicular to the axis of the ellipse and produced to meet the curve again at P'. Then PP' is called a double ordinate.

   If abscissa of P is h, then ordinate of P,

$$\frac{y^2}{b^2} = 1 - \frac{h^2}{a^2}$$

$$y = \frac{b}{a}\sqrt{(a^2 - h^2)} \quad \text{(for first quadrant)}$$

   and ordinate of P' is

$$y = -\frac{b}{a}\sqrt{(a^2 - h^2)} \quad \text{(for fourth quadrant)}$$

   Hence, coordinates of P and P' are

$$\left(h, \frac{b}{a}\sqrt{(a^2 - h^2)}\right) \text{ and } \left(h, -\frac{b}{a}\sqrt{(a^2 - h^2)}\right)$$

   respectively.

6. **Latusrectum** The double ordinates LL' and L_1L_1' are latusrectums of the ellipse. These line are perpendicular to major axis $A'A$ and through the foci S and S', respectively.

   Length of the latusrectum

   Now let $LL' = 2k$

   then $LS = L'S = k$

   Coordinates of L and L' are (ae, k) and $(ae, -k)$ lies on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$\therefore$ $$\frac{a^2e^2}{a^2}+\frac{k^2}{b^2}=1 \text{ or } k^2 = b^2(1 - e^2)$$

$$= b^2\left(\frac{b^2}{a^2}\right) \quad [\because b^2 = a^2(1 - e^2)]$$

$\therefore \quad k = \dfrac{b^2}{a} \qquad (\because k > 0)$

$\therefore \quad 2k = \dfrac{2b^2}{a} = LL'$

$\therefore$ Length of latusrectum $LL' = L_1 L_1' = \dfrac{2b^2}{a}$ and end of points of latusrectum are

$$L \equiv \left(ae, \frac{b^2}{a} \right); \; L' \equiv \left(ae, -\frac{b^2}{a} \right)$$

$$L_1 \equiv \left(-ae, \frac{b^2}{a} \right); \; L_1' \equiv \left(-ae, -\frac{b^2}{a} \right)$$

respectively.

Remark

Latusrectum

$$= LL' = \frac{2b^2}{a} = \frac{(2b)^2}{(2a)} = \frac{(\text{Minor axis})^2}{(\text{Major axis})}$$

$$= 2a(1 - e^2) = 2e\left(\frac{a}{e} - ae \right)$$

$= 2e$ (distance from focus to the corresponding directrix)

7. **Focal chord :** A chord of the ellipse passing through its focus is called a focal chord.

Remark

Semi latusrectum is the harmonic mean of the segments of focal chord or $\dfrac{1}{SP} + \dfrac{1}{SQ} = \dfrac{2a}{b^2}$, $(a > b)$, where PQ is the focal chord and S is the focus.

8. **Vertices :** The vertices of the ellipse are the points where the ellipse meets its major axis.

   Hence, A and A' are the vertices

   $\therefore \quad A \equiv (a, 0)$ and $A' \equiv (-a, 0)$

Example 1 If PSQ is a focal chord of the ellipse $16x^2 + 25y^2 = 400$, such that $SP = 8$, then find the length of SQ.

Sol. The given ellipse is $16x^2 + 25y^2 = 400$ or $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$

$\because \quad \dfrac{1}{SP} + \dfrac{1}{SQ} = \dfrac{2a}{b^2}$

$\Rightarrow \quad \dfrac{1}{8} + \dfrac{1}{SQ} = \dfrac{2(5)}{16} = \dfrac{5}{8}$ or $\dfrac{1}{SQ} = \dfrac{1}{2}$

$\therefore \quad SQ = 2$

Example 2 If the latusrectum of an ellipse is equal to half of its minor-axis, then find its eccentricity.

Sol. $\because$ Latusrectum $= \dfrac{1}{2}$ (minor axis)

$\Rightarrow \quad \dfrac{2b^2}{a} = \dfrac{1}{2}(2b)$

or $\quad 2b = a$

or $\quad 4b^2 = a^2$

$\Rightarrow \quad 4a^2(1 - e^2) = a^2$

or $\quad 4e^2 = 3$

$\therefore \quad e = \dfrac{\sqrt{3}}{2}$

Example 3 If the distance between the directrices is thrice the distance between the foci, then find eccentricity of the ellipse.

Sol. Given, $\dfrac{2a}{e} = 3(2ae) \Rightarrow e^2 = \dfrac{1}{3}$

$\therefore \quad e = \dfrac{1}{\sqrt{3}}$

Example 4 If $P(x, y)$ be any point on the ellipse $16x^2 + 25y^2 = 400$ and $F_1 \equiv (3, 0), F_2 \equiv (-3, 0)$, then find the value of $PF_1 + PF_2$.

Sol. We have, $16x^2 + 25y^2 = 400$

$\Rightarrow \quad \dfrac{x^2}{5^2} + \dfrac{y^2}{4^2} = 1$

$\because$ Coordinates of foci an $(\pm ae, 0)$

or $\quad \left(\pm \sqrt{(a^2 - b^2)}, 0 \right)$

i.e. $\quad (\pm \sqrt{(25 - 16)}, 0)$ or $(\pm 3, 0)$

Thus, F_1 and F_2 are foci of the ellipse.

$\therefore \quad PF_1 + PF_2 =$ Length of major axis

$= 2a = 2 \times 5 = 10$

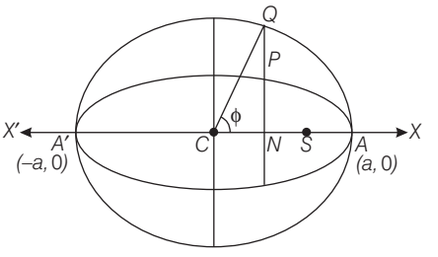
9. **Parametric equation of the ellipse :** The circle described on the major-axis of an ellipse as diameter is called the **auxiliary circle** of the ellipse.

   Let the equation of ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$\therefore$ Equation of its auxiliary circle is

$x^2 + y^2 = a^2$ ($\because AA'$ is diameter of the circle)



Let Q be a point on the auxiliary circle $x^2 + y^2 = a^2$ such that QP produced is perpendicular to the X-axis.

Then P and Q are the **corresponding points** on the ellipse and the auxiliary circle respectively.

Let $\angle QCA = \phi \qquad (0 \le \phi < 2\pi)$

i.e. the eccentric angle of P on an ellipse is the angle which the radius (or **radius vector)** through the corresponding point on the auxiliary circle makes with the major axis.

$\therefore \quad Q \equiv (a\cos\phi, a\sin\phi)$

$\therefore$ Now x-coordinate of P is $a\cos\phi$

Let y-coordinate of P is y, then $(a\cos\phi, y)$ lies on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{a^2\cos^2\phi}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow \quad y^2 = b^2\sin^2\phi$$

$$\therefore \quad y = \pm b\sin\phi$$

$\because P$ is in I quadrant

$$\therefore \quad y = b\sin\phi$$

Coordinates of P are $(a\cos\phi, b\sin\phi)$. We have $x = a\cos\phi, y = b\sin\phi$ are called parametric equations of the ellipse.

This point $(a\cos\phi, b\sin\phi)$ is also called the point 'ϕ'.

Remark

The equation of the chord joining the points

$P \equiv (a\cos\phi_1, b\sin\phi_1)$

and $Q \equiv (a\cos\phi_2, b\sin\phi_2)$ is

$$\frac{x}{a}\cos\left(\frac{\phi_1+\phi_2}{2}\right) + \frac{y}{b}\sin\left(\frac{\phi_1+\phi_2}{2}\right) = \cos\left(\frac{\phi_1-\phi_2}{2}\right)$$

If its focal chord, then it pass through $(ae, 0)$ or $(-ae, 0)$, then

$$\pm e\cos\left(\frac{\phi_1+\phi_2}{2}\right) + 0 = \cos\left(\frac{\phi_1-\phi_2}{2}\right)$$

$$\Rightarrow \quad \frac{\cos\left(\frac{\phi_1-\phi_2}{2}\right)}{\cos\left(\frac{\phi_1+\phi_2}{2}\right)} = \pm\frac{e}{1}$$

$$\Rightarrow \quad \frac{\cos\left(\frac{\phi_1-\phi_2}{2}\right) - \cos\left(\frac{\phi_1+\phi_2}{2}\right)}{\cos\left(\frac{\phi_1-\phi_2}{2}\right) + \cos\left(\frac{\phi_1+\phi_2}{2}\right)} = \left(\frac{\pm e-1}{\pm e+1}\right)$$

$$\Rightarrow \quad \tan\left(\frac{\phi_1}{2}\right)\tan\left(\frac{\phi_2}{2}\right) = \left(\frac{\pm e-1}{\pm e+1}\right)$$

if focal chord pass through $(ae, 0)$, then

$$\tan\left(\frac{\phi_1}{2}\right)\tan\left(\frac{\phi_2}{2}\right) = \left(\frac{e-1}{e+1}\right)$$

and if focal chord pass through $(-ae, 0)$, then

$$\tan\left(\frac{\phi_1}{2}\right)\tan\left(\frac{\phi_2}{2}\right) = \left(\frac{e+1}{e-1}\right)$$

or $$\cot\left(\frac{\phi_1}{2}\right)\cot\left(\frac{\phi_2}{2}\right) = \left(\frac{e-1}{e+1}\right)$$

Remark

Circle described on focal length as diameter always touches the auxiliary circle.

Proof

Consider ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

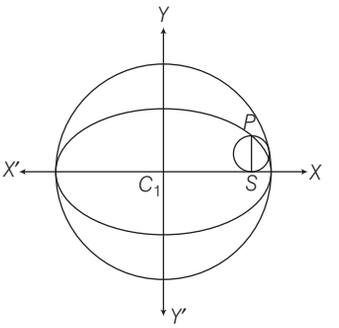
Let P be $(a\cos\theta, b\sin\theta)$

and S be $(ae, 0)$

$$\therefore \quad SP = ePM = e\left(\frac{a}{e} - a\cos\theta\right)$$

$$= (a - ae\cos\theta)$$

The auxiliary circle $x^2 + y^2 = a^2$ having center $C_1(0,0)$ and radius $r_1 = a$



The circle having SP as the diameter has center

$$C_2\left(\frac{ae + a\cos\theta}{2}, \frac{b\sin\theta}{2}\right)$$

and radius $r_2 = \frac{SP}{2} = \frac{a(1 - e\cos\theta)}{2}$

$$\text{Now, } C_1C_2 = \sqrt{\left(\frac{ae + a\cos\theta}{2}\right)^2 + \left(\frac{b\sin\theta}{2}\right)^2}$$

$$= \frac{1}{2}\sqrt{a^2(e + \cos\theta)^2 + a^2(1 - e^2)\sin^2\theta}$$

$$= \frac{a}{2}\sqrt{(e^2 + \cos^2\theta + 2e\cos\theta + \sin^2\theta - e^2\sin^2\theta)}$$

$$= \frac{a}{2}\sqrt{(e^2\cos^2\theta + 2e\cos\theta + 1)}$$

$$= \frac{a}{2}(1 + e\cos\theta)$$

$$= r_1 - r_2$$

Hence, the circle on SP as diameter touches the auxiliary circle internally.

Example 5 Find the eccentric angle of a point on the ellipse $\frac{x^2}{6}+\frac{y^2}{2}=1$. Whose distance from the center of the ellipse is $\sqrt{5}$.

Sol. Any point on the ellipse is $P(\sqrt{6}\cos\theta, \sqrt{2}\sin\theta)$

Where, $0 \le \theta < 2\pi$ and $C(0,0)$ is center, given $CP = \sqrt{5}$

$$\Rightarrow \quad (CP)^2 = 5$$

$$\Rightarrow \quad 6\cos^2\theta + 2\sin^2\theta = 5$$

$$\Rightarrow \quad 6(1-\sin^2\theta) + 2\sin^2\theta = 5$$

$$\text{or} \quad \sin^2\theta = \frac{1}{4}$$

$$\text{or} \quad \sin\theta = \pm\frac{1}{2} = \pm\sin\frac{\pi}{6}$$

$$\therefore \quad \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

Focal Distances of a Point

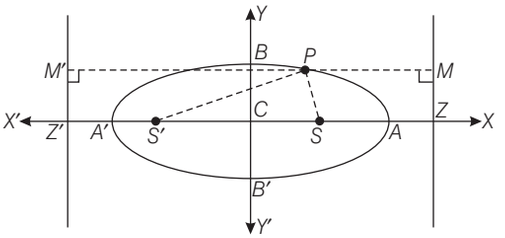
The sum of focal distances of any point on the ellipse is equal to the major axis. The ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \ldots(i)$$

The foci S and S' are $(ae, 0)$ and $(-ae, 0)$.

The equations of its directrices MZ and $M'Z'$ are $x = a/e$ and $x = -a/e$

Let $P(x_1, y_1)$ be any point on Eq. (i)



Now $$SP = ePM = e\left(\frac{a}{e} - x_1\right)$$

$$= a - ex_1$$

and $$S'P = ePM' = e\left(\frac{a}{e} + x_1\right)$$

$$= a + ex_1$$

$$SP + S'P = (a - ex_1) + (a + ex_1)$$

$$= 2a = AA' = \text{major axis}$$

Hence, the sum of the focal distances of a point on the ellipse is constant and is equal to the length of the major axis of the ellipse.

Another definition of ellipse : An ellipse is the locus of a point which moves in a plane such that the sum of its distances from two fixed points in the same plane is always constant.

Remark

SP and *S'P* are also called focal radii of the ellipse

$\therefore \quad SP = a - ex_1$ and $S'P = a + ex_1$

Example 6 An ellipse having foci at (3, 3) and (−4, 4) and passing through the origin, then find eccentricity of the ellipse.

Sol. The ellipse is passing through $O(0,0)$ and has foci $P(3,3)$ and $Q(-4,4)$, then

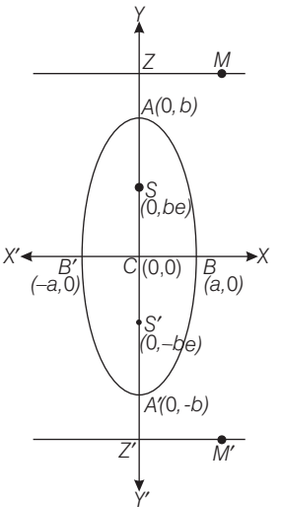
$$OP + OQ = 2a \text{ and } PQ = 2ae$$

$$\therefore \quad e = \frac{PQ}{OP+OQ} = \frac{\sqrt{50}}{3\sqrt{2}+4\sqrt{2}} = \frac{5}{7}$$

The Shape of the Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, when $b > a$

In this case major and minor-axis of the ellipse along Y-axis and X-axis respectively.

then $AA' = 2b$ and $BB' = 2a$



The foci S and S' are $(0, be)$ and $(0, -be)$ respectively

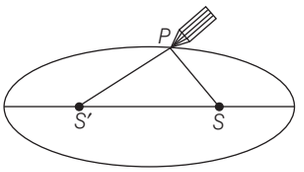
The directrices are MZ and $M'Z'$ are

$$y = \frac{b}{e} \quad \text{and} \quad y = -\frac{b}{e}$$

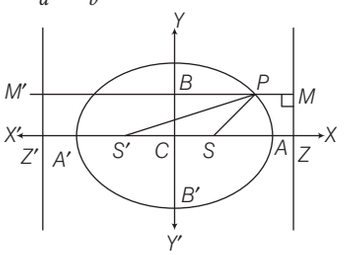
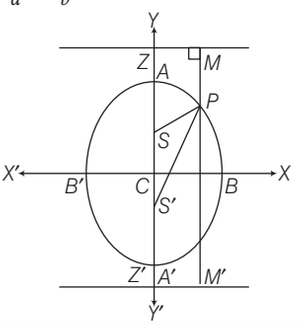
are respectively,

Mechanical Construction of an Ellipse

Let S and S' be two drawing pins and let an inextensible string whose ends at S and S' and length is equal to sum of SP and $S'P$ i.e. $2a$, where P is point of pencil. The point of pencil move on paper and the fixed ends always tight. So as to satisfy these conditions it will trace out the curve on the paper. This curve is an ellipse. Hence the locus of the point of pencil is an ellipse.



Smart Table : Difference between both (Horizontal and Vertical) Ellipses will be clear from the following table

Equation and graph of the ellipse → / Basic fundamentals ↓	**Horizontal ellipse** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,;\,a>b$ 	**Vertical ellipse** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,;\,a<b$ 
Centre	$(0, 0)$	$(0, 0)$
Vertices	$(\pm a, 0)$	$(0, \pm b)$
Length of major axis	$2a$	$2b$
Length of minor axis	$2b$	$2a$
Foci	$(\pm ae, 0)$ or $(\pm \sqrt{(a^2-b^2)}, 0)$	$(0, \pm be)$ or $(0, \pm \sqrt{(b^2-a^2)})$
Distance between foci	$2ae$ or $2\sqrt{(a^2-b^2)}$	$2be$ or $2\sqrt{(b^2-a^2)}$
Equation of directrices	$x = \pm \frac{a}{e}$	$y = \pm \frac{b}{e}$
Distance between directrices	$\frac{2a}{e}$	$\frac{2b}{e}$
Relation between a, b and e	$b^2 = a^2(1-e^2)$	$a^2 = b^2(1-e^2)$
Length of latusrectum	$\frac{2b^2}{a}$ or $2a(1-e^2)$	$\frac{2a^2}{b}$ or $2b(1-e^2)$
End points of latusrectum	$\left(\pm ae, \pm \frac{b^2}{a}\right)$	$\left(\pm \frac{a^2}{b}, \pm be\right)$
Focal radii	$SP = a - ex_1$, $S'P = a + ex_1$ and $SP + S'P = 2a$	$SP = b - ey_1$, $S'P = b + ey_1$ and $SP + S'P = 2b$
Parametric Coordinates	$(a\cos\theta, b\sin\theta)$, $0 \le \theta < 2\pi$	$(a\cos\theta, b\sin\theta)$, $0 \le \theta < 2\pi$
Tangents at the vertices	$x = \pm a$	$y = \pm b$

Example 7 Find the lengths of major and minor axes, the coordinates of foci, vertices and the eccentricity of the ellipse $3x^2 + 2y^2 = 6$. Also, find the equation of the directrices.

Sol. The equation of ellipse is

$$3x^2 + 2y^2 = 6$$

$$\Rightarrow \quad \frac{x^2}{2} + \frac{y^2}{3} = 1$$

$$\Rightarrow \quad \frac{x^2}{(\sqrt{2})^2} + \frac{y^2}{(\sqrt{3})^2} = 1$$

Comparing this equation with

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\therefore \quad a = \sqrt{2},\ b = \sqrt{3} \qquad \text{(Here } b > a\text{)}$$

Length of major axis $= 2b = 2\sqrt{3}$

and Length of minor axis $= 2a = 2\sqrt{2}$

If e be the eccentricity, then $a^2 = b^2(1 - e^2)$

$$\Rightarrow \quad 2 = 3(1 - e^2) \ \Rightarrow \ e^2 = \frac{1}{3}$$

$$\therefore \quad e = \frac{1}{\sqrt{3}}$$

Vertices $= (0, \pm b) = (0, \pm \sqrt{3})$

and foci are $(0, \pm be) = (0, \pm 1)$

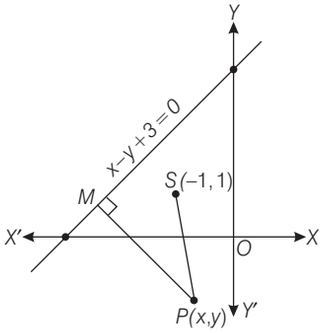
and equation of the directrices are $y = \pm b/e$

$$y = \pm \frac{\sqrt{3}}{(1/\sqrt{3})}$$

$$\therefore \quad y = \pm 3$$

Example 8 Find the equation of an ellipse whose focus is $(-1, 1)$, eccentricity is $\frac{1}{2}$ and the directrix is $x - y + 3 = 0$.

Sol. Let $P(x, y)$ be any point on the ellipse whose focus is $S(-1, 1)$ and the directrix is $x - y + 3 = 0$. Draw PM perpendicular from $P(x, y)$ on the directrix $x - y + 3 = 0$. Then by definition



$SP = ePM$

$$\Rightarrow \quad (SP)^2 = e^2 (PM)^2$$

$$\Rightarrow \quad (x + 1)^2 + (y - 1)^2 = \frac{1}{4}\left\{\frac{x - y + 3}{\sqrt{2}}\right\}^2$$

$$\Rightarrow \quad 8(x^2 + y^2 + 2x - 2y + 2)$$

$$= x^2 + y^2 + 9 - 2xy + 6x - 6y$$

$$\Rightarrow \quad 7x^2 + 7y^2 + 2xy + 10x - 10y + 7 = 0$$

which is the required equation of the ellipse.

Example 9 Show that the line $lx + my + n = 0$ will cut the ellipse $x^2/a^2 + y^2/b^2 = 1$ in points whose eccentric angles differ by $(\pi/2)$, if $a^2l^2 + b^2m^2 = 2n^2$.

Sol. Let eccentric angles are θ and ϕ, then

$$\theta - \phi = \frac{\pi}{2} \qquad \text{(given)}$$

$$\therefore \quad \theta = \frac{\pi}{2} + \phi$$

The line joining the point 'θ' and 'ϕ' is

$$\frac{x}{a}\cos\left(\frac{\theta + \phi}{2}\right) + \frac{y}{b}\sin\left(\frac{\theta + \phi}{2}\right) = \cos\left(\frac{\theta - \phi}{2}\right)$$

or $$\frac{x}{a}\cos\left(\frac{\pi}{4} + \phi\right) + \frac{y}{b}\sin\left(\frac{\pi}{4} + \phi\right) = \cos\frac{\pi}{4} \quad \left(\because \theta = \frac{\pi}{2} + \phi\right)$$

or $$\frac{x}{a}\cos\left(\frac{\pi}{4} + \phi\right) + \frac{y}{b}\sin\left(\frac{\pi}{4} + \phi\right) = \frac{1}{\sqrt{2}} \qquad \text{...(i)}$$

and the given line is $lx + my + n = 0$

or $$lx + my = -n \qquad \text{...(ii)}$$

Now, Eqs. (i) and (ii) represent the same line, so comparing them, we get

$$\frac{\cos\left(\frac{\pi}{4} + \phi\right)}{la} = \frac{\sin\left(\frac{\pi}{4} + \phi\right)}{mb} = -\frac{1}{n\sqrt{2}}$$

$$\therefore \quad \cos\left(\frac{\pi}{4} + \phi\right) = -\frac{la}{n\sqrt{2}} \qquad \text{...(iii)}$$

and $$\sin\left(\frac{\pi}{4} + \phi\right) = -\frac{mb}{n\sqrt{2}} \qquad \text{...(iv)}$$

Squaring and adding Eqs. (iii) and (iv), then

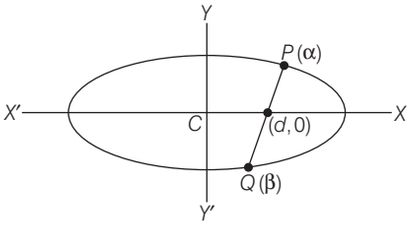
$$\frac{l^2a^2}{2n^2} + \frac{m^2b^2}{2n^2} = 1$$

$$l^2a^2 + m^2b^2 = 2n^2$$

Example 10 If a chord joining two points whose eccentric angles are α, β cut the major axis of an ellipse at a distance d from the centre. Show that $\tan(\alpha/2)\tan(\beta/2) = (d - a)/(d + a)$, where $2a$ is the length of major axis.

Sol. Let the equation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \ldots(i)$$



The equation of the line joining 'α' and 'β' on the ellipse Eq. (i) is

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right) \quad \ldots(ii)$$

Since, it cuts the major axis of the ellipse at a distance d from the centre i.e. passes through the point $(d, 0)$, then

$$\frac{d}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+0=\cos\left(\frac{\alpha-\beta}{2}\right)$$

or $$\frac{d}{a}=\frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}$$

or $$\frac{d-a}{d+a}=\frac{\cos\left(\frac{\alpha-\beta}{2}\right)-\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)+\cos\left(\frac{\alpha+\beta}{2}\right)}$$

(By componendo and dividendo)

$$=\frac{2\sin(\alpha/2)\sin(\beta/2)}{2\cos(\alpha/2)\cos(\beta/2)}$$

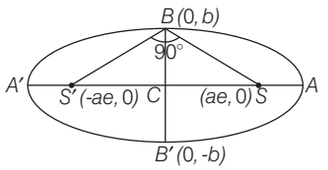
$$=\tan(\alpha/2)\tan(\beta/2)$$

$\therefore$ $$\tan(\alpha/2)\tan(\beta/2)=\frac{d-a}{d+a}$$

Example 11 If the angle between the straight lines joining foci and the ends of the minor axis of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, is 90°. Find its eccentricity.

Sol. The equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Let $a > b$



$\therefore$ The ends of minor axis are $B(0, b)$ and $B'(0, -b)$. If the eccentricity of the ellipse is e, then the foci are $S(ae, 0)$ and $S'(-ae, 0)$

$\therefore$ Slope of BS is $m_1=\frac{b-0}{0-ae}=-\frac{b}{ae}$

and slope of BS' is $m_2=\frac{b-0}{0+ae}=\frac{b}{ae}$

$\because$ The angle between BS and BS' is 90°,

$\therefore$ $m_1m_2=-1$

$\Rightarrow$ $-\frac{b}{ae}\times\frac{b}{ae}=-1$

$\Rightarrow$ $b^2=a^2e^2$

$a^2(1-e^2)=a^2e^2$

$\Rightarrow$ $1-e^2=e^2$

$\Rightarrow$ $2e^2=1$

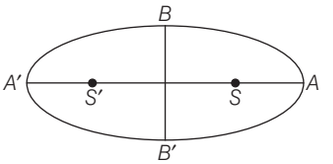
$\therefore$ $e=\frac{1}{\sqrt{2}}$

Example 12 Find the equation of the ellipse referred to its centre whose minor axis is equal to the distance between the foci and whose latusrectum is 10.

Sol. Let the equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad (\text{let } a>b)$$

Then, the foci are $S(ae, 0)$ and $S'(-ae, 0)$, length of minor axis $BB'=2b$ and length of latusrectum $=\frac{2b^2}{a}$



$\therefore$ According to the question

$$BB'=SS' \Rightarrow 2b=2ae$$

$\Rightarrow$ $b=ae$ ...(i)

and $\frac{2b^2}{a}=10$

$\Rightarrow$ $b^2=5a$ ...(ii)

also we have $b^2=a^2(1-e^2)$ ...(iii)

Putting the value of b from Eq. (i) in Eq. (iii), we have

$$a^2e^2=a^2(1-e^2)$$

$\Rightarrow$ $e^2=1-e^2$

$\Rightarrow$ $2e^2=1$

$\therefore$ $e=\frac{1}{\sqrt{2}}$

From Eq. (i), we have

$$b=\frac{a}{\sqrt{2}}$$

$\therefore$ $b^2=\frac{a^2}{2}$

$\Rightarrow \quad 5a = \frac{a^2}{2}$ [from Eq. (ii)]

$\Rightarrow \quad a = 10$

From Eq. (ii)

$$b^2 = 5 \times 10 = 50$$

Putting the values of a and b in $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the equation of required ellipse is

$$\frac{x^2}{100} + \frac{y^2}{50} = 1$$

$$x^2 + 2y^2 = 100$$

Example 13 Prove that the ratio of area of any triangle PQR inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and that of triangle formed by the corresponding points on the auxiliary circle is $\frac{b}{a}$.

Sol. Let the three points on the ellipse be $P(a\cos\alpha, b\sin\alpha)$, $Q(a\cos\beta, b\sin\beta)$ and $R(a\cos\gamma, b\sin\gamma)$.

Then, the corresponding points on the auxiliary circle are $A(a\cos\alpha, a\sin\alpha)$, $B(a\cos\beta, a\sin\beta)$ and $C(a\cos\gamma, a\sin\gamma)$, then

$$\frac{\text{Area of } \Delta PQR}{\text{Area of } \Delta ABC} = \frac{\frac{1}{2}\begin{vmatrix} a\cos\alpha & b\sin\alpha & 1 \\ a\cos\beta & b\sin\beta & 1 \\ a\cos\gamma & b\sin\gamma & 1 \end{vmatrix}}{\frac{1}{2}\begin{vmatrix} a\cos\alpha & a\sin\alpha & 1 \\ a\cos\beta & a\sin\beta & 1 \\ a\cos\gamma & a\sin\gamma & 1 \end{vmatrix}}$$

$$= \frac{ab\begin{vmatrix} \cos\alpha & \sin\alpha & 1 \\ \cos\beta & \sin\beta & 1 \\ \cos\gamma & \sin\gamma & 1 \end{vmatrix}}{a^2\begin{vmatrix} \cos\alpha & \sin\alpha & 1 \\ \cos\beta & \sin\beta & 1 \\ \cos\gamma & \sin\gamma & 1 \end{vmatrix}}$$

$$= \frac{b}{a}$$

Example 14 The extremities of a line segment of length l move in two fixed perpendicular straight lines. Find the locus of that point which divides this line segment in ratio 1 : 2.

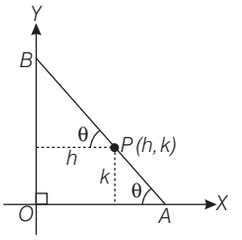
Sol. Let $PA : PB = 1 : 2$

$$PA = \frac{l}{3} \quad \text{and} \quad PB = \frac{2l}{3}$$

$$k = \frac{l}{3}\sin\theta$$

or $\quad 3k = l\sin\theta$ ...(i)

and $\quad h = \frac{2l}{3}\cos\theta$



or $\quad \frac{3h}{2} = l\cos\theta$ ...(ii)

Squaring and adding Eqs. (i) and (ii), then

$$9k^2 + \frac{9h^2}{4} = l^2$$

or $\quad 9h^2 + 36k^2 = 4l^2$

$\therefore$ Locus of $P(h, k)$ is

$$9x^2 + 36y^2 = 4l^2$$

Aliter :

Let $A \equiv (a, 0)$ and $B \equiv (0, b)$

$\because$ $P(h, k)$ divide AB in the ratio 1 : 2 (internally), then

$$h = \frac{1.0 + 2.a}{1 + 2} \Rightarrow a = \frac{3h}{2}$$

and $\quad k = \frac{1.b + 2.0}{1 + 2} \Rightarrow b = 3k$

$\because \quad a^2 + b^2 = l^2$

$\Rightarrow \quad \frac{9h^2}{4} + 9k^2 = l^2$

or $\quad 9h^2 + 36k^2 = 4l^2$

$\therefore$ Locus of $P(h, k)$ is

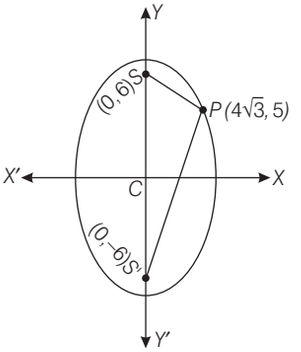
$$9x^2 + 36y^2 = 4l^2$$

Example 15 ***Find the lengths and equations of the focal radii drawn from the point*** $(4\sqrt{3}, 5)$ ***on the ellipse*** $25x^2 + 16y^2 = 1600$.

Sol. The equation of the ellipse is

$$25x^2 + 16y^2 = 1600$$

or $\quad \frac{x^2}{64} + \frac{y^2}{100} = 1$



Here, $b > a$

$$a^2 = 64,\ b^2 = 100$$

$$a^2 = b^2(1 - e^2)$$

$$\therefore \quad 64 = 100(1 - e^2)$$

$$\Rightarrow \quad e = 3/5$$

Let $P(x_1, y_1) \equiv (4\sqrt{3}, 5)$

be a point on the ellipse, then SP and $S'P$ are the focal radii

$$\therefore \quad SP = b - ey_1 \text{ and } S'P = b + ey_1$$

$$\therefore \quad SP = 10 - \frac{3}{5} \times 5 \text{ and } S'P = 10 + \frac{3}{5} \times 5$$

$$\Rightarrow \quad SP = 7 \text{ and } S'P = 13$$

Also, S is $(0, be)$

i.e. $\left(0, 10 \times \frac{3}{5}\right)$ i.e. $(0, 6)$

and S' is $(0, -be)$

i.e. $\left(0, -10 \times \frac{3}{5}\right)$

i.e. $(0, -6)$

$\therefore$ Equation of SP is

$$y - 5 = \frac{6 - 5}{0 - 4\sqrt{3}}(x - 4\sqrt{3})$$

$$-4\sqrt{3y} + 20\sqrt{3} = x - 4\sqrt{3}$$

or $\quad x + 4\sqrt{3}y - 24\sqrt{3} = 0$

and equation of $S'P$ is

$$\therefore \quad y - 5 = \frac{-6 - 5}{0 - 4\sqrt{3}}(x - 4\sqrt{3})$$

$$\Rightarrow \quad -4\sqrt{3}y + 20\sqrt{3} = -11x + 44\sqrt{3}$$

or $\quad 11x - 4\sqrt{3}y - 24\sqrt{3} = 0$

Exercise for Session 1

1. The length of the major axis of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is three times the length of minor axis, it eccentricity is

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{2\sqrt{2}}{3}$ (d) $\frac{2\sqrt{2}}{5}$

2. The equation $\frac{x^2}{10 - a} + \frac{y^2}{4 - a} = 1$, represents an ellipse, if

(a) $a < 4$ (b) $a > 4$ (c) $4 < a < 10$ (d) $a > 10$

3. The eccentricity of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, whose latusrectum is half of its major axis, is

(a) $\frac{1}{\sqrt{3}}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\sqrt{\left(\frac{2}{3}\right)}$

4. If the eccentricity of an ellipse is $\frac{1}{\sqrt{2}}$, then its latusrectum is equal to its

(a) minor axis (b) semi minor axis (c) major axis (d) semi major axis

5. If the distance between the foci of an ellipse is equal to its minor axis, then its eccentricity is

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{1}{3}$ (d) $\frac{1}{\sqrt{3}}$

6. The eccentric angle of a point on the ellipse $\frac{x^2}{6} + \frac{y^2}{2} = 1$, whose distance from the centre of ellipse is 2, is

(a) $-\frac{\pi}{4}$ (b) $\frac{\pi}{4}$ (c) $\frac{3\pi}{2}$ (d) $\frac{5\pi}{3}$

7. If $\tan\alpha \tan\beta = -\frac{a^2}{b^2}$, then the chord joining two points α and β on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, will subtend a right angle at

(a) focus (b) centre (c) end of major axis (d) end of minor axis

8. If the eccentricity of the two ellipses $\frac{x^2}{169}+\frac{y^2}{25}=1$ and $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are equal, then the value of a/b is

(a) $\frac{5}{13}$ (b) $\frac{6}{13}$

(c) $\frac{13}{5}$ (d) $\frac{13}{6}$

9. The ratio of the areas of triangle inscribed in ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, to that of triangle formed by the corresponding points on the auxiliary circle is $\frac{1}{2}$, then the eccentricity of the ellipse is

(a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2}$

(c) $\frac{1}{\sqrt{2}}$ (d) $\frac{1}{\sqrt{3}}$

10. If PSQ is a focal chord of the ellipse $16x^2+25y^2=400$ such that $SP=16$, then the length SQ is

(a) $\frac{2}{9}$ (b) $\frac{4}{9}$

(c) $\frac{8}{9}$ (d) $\frac{16}{9}$

11. Let P be a variable point on the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ with foci at S and S'. If A be the area of triangle PSS', then the maximum value of A is

(a) 12 sq units (b) 24 sq units

(c) 36 sq units (d) 48 sq units

12. If S and S' are the foci of an ellipse of major axis of length 10 units and P is any point on the ellipse such that the perimeter of triangle PSS' is 15 units, then the eccentricity of the ellipse is

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$

(c) $\frac{7}{25}$ (d) $\frac{3}{4}$

13. Find the latusrectum, eccentricity, coordinates of the foci, coordinates of the vertices, the length of the axes and the centre of the ellipse

$4x^2+9y^2-8x-36y+4=0$

14. The distance between the foci of an ellipse is 10 and its latusrectum is 15 ; find its equation referred to its axes as axes of coordinates.

15. Find the equation of the ellipse whose axes are parallel to the coordinate axes having its centre at the point $(2,-3)$ one focus at $(3,-3)$ and one vertex at $(4,-3)$.

16. Find the equation of the ellipse whose foci are $(2,3)$, $(-2,3)$ and whose semi-minor axes is $\sqrt{5}$.

17. Show that the equation $(10x-5)^2+(10y-5)^2=(3x+4y-1)^2$ represents an ellipse, find the eccentricity of the ellipse.

18. Find the locus of the extremities of the latusrectum of the family of ellipses $b^2x^2+y^2=a^2b^2$ having a given major axes.

Session 2

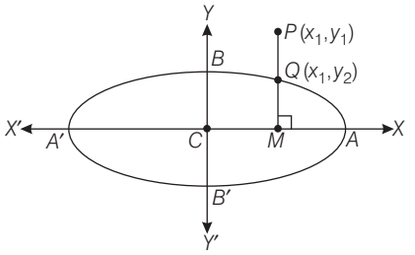
Position of a Point with Respect to an Ellipse, Intersection of a Line and an Ellipse, Equation of Tangent in Different Forms, Equations of Normals in Different Forms, Properties of Eccentric Angles of the Co-normal Points, Co-normal Points Lie on a Fixed Curve

Position of a Point with Respect to an Ellipse

Theorem : Prove that the point $P(x_1, y_1)$ lies outside, on, or inside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ according as

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1>,=\text{or},<0$$

Proof : From point $P(x_1, y_1)$ draw perpendicular PM on AA' to meet the ellipse at $Q(x_1, y_2)$.



Since, $Q(x_1, y_2)$ lies on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

then, $$\frac{x_1^2}{a^2}+\frac{y_2^2}{b^2}=1$$

$$\Rightarrow \quad \frac{y_2^2}{b^2}=1-\frac{x_1^2}{a^2} \qquad ...(i)$$

Now, point P lies outside, on or inside the ellipse according as

$$PM>,=\text{or},<QM$$

$$\Rightarrow \quad y_1>,=\text{or},<y_2$$

$$\Rightarrow \quad \frac{y_1^2}{b^2}>,=\text{or},<\frac{y_2^2}{b^2}$$

$$\Rightarrow \quad \frac{y_1^2}{b^2}>,=\text{or},<1-\frac{x_1^2}{a^2} \qquad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}>,=\text{or},<1,$$

or $$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1>,=\text{or},<0$$

Hence, the point $P(x_1, y_1)$ lies outside, on or inside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ according as

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1>0,=\text{or},<0$$

Remark

Let $S=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$, then $S_1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$

The point (x_1, y_1) lies outside, on or inside the ellipse $S=0$ according as $S_1>,=\text{or}<0$.

Example 16 Find the position of the point $(4,-3)$ relative to the ellipse $5x^2+7y^2=140$.

Sol. The given ellipse can be written as $\frac{x^2}{28}+\frac{y^2}{20}-1=0$

Let $$S=\frac{x^2}{28}+\frac{y^2}{20}-1$$

$\therefore$ $$S_1=\frac{(4)^2}{28}+\frac{(-3)^2}{20}-1=\frac{3}{140}>0$$

So, the point $(4, -3)$ lies outside the ellipse $5x^2 + 7y^2 = 140$.

Example 17 Find the integral value of α for which the point $\left(7 - \frac{5\alpha}{4}, -\alpha\right)$ lies inside the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

Sol. Since, the point $\left(7 - \frac{5\alpha}{4}, -\alpha\right)$ lies inside the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, then $\frac{1}{25}\left(7 - \frac{5\alpha}{4}\right)^2 + \frac{1}{16}(-\alpha)^2 - 1 < 0$

$$\Rightarrow \quad (28 - 5\alpha)^2 + 25\alpha^2 - 400 < 0$$

$$\Rightarrow \quad 50\alpha^2 - 280\alpha + 384 < 0$$

$$\Rightarrow \quad 25\alpha^2 - 140\alpha + 192 < 0$$

$$\Rightarrow \quad (5\alpha - 12)(5\alpha - 16) < 0$$

$$\therefore \quad \frac{12}{5} < \alpha < \frac{16}{5}$$

Hence, integral value of α is 3

Intersection of a Line and an Ellipse

Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ...(i)

and the given line be $y = mx + c$ ...(ii)

Eliminating y from Eqs. (i) and (ii), then

$$\frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} = 1$$

$$\Rightarrow \quad (a^2m^2 + b^2)x^2 + 2mca^2x + c^2a^2 - a^2b^2 = 0 \quad \text{...(iii)}$$

Above equation being a quadratic in x gives two values of x. Shows that every straight line will cut the ellipse in two points may be real, coincident or imaginary according as

Discriminant of Eq. (iii) $>, =, < 0$

i.e. $4m^2c^2a^4 - 4(a^2m^2 + b^2)(c^2a^2 - a^2b^2) >, =, < 0$

or $-a^2b^2c^2 + a^4b^2m^2 + a^2b^4 >, =, < 0$

or $a^2m^2 + b^2 >, =, < c^2$ ...(iv)

Condition of Tangency : If the line Eq. (ii) touches the ellipse Eq. (i), then Eq. (iii) has equal roots.

$\therefore$ Discriminant of Eq. (iii) $= 0$

$$\Rightarrow \quad c^2 = a^2m^2 + b^2 \quad \text{or} \quad c = \pm\sqrt{(a^2m^2 + b^2)} \quad \text{...(v)}$$

so, the line $y = mx + c$ touches the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{if} \quad c^2 = a^2m^2 + b^2$$

(which is condition of tangency)

Substituting the value of c from Eq. (v) in Eq. (ii), then

$$y = mx \pm \sqrt{(a^2m^2 + b^2)}$$

Hence, the lines $y = mx \pm \sqrt{(a^2m^2 + b^2)}$ will always tangents to the ellipse.

Point of contact : Substituting $c = \pm\sqrt{(a^2m^2 + b^2)}$ in Eq. (iii), then

$$(a^2m^2 + b^2)x^2 \pm 2ma^2x\sqrt{(a^2m^2 + b^2)} + (a^2m^2 + b^2)a^2 - a^2b^2 = 0$$

or $(a^2m^2 + b^2)x^2 \pm 2ma^2x\sqrt{(a^2m^2 + b^2)} + a^4m^2 = 0$

or $\left(x\sqrt{(a^2m^2 + b^2)} \pm a^2m\right)^2 = 0$

$$\therefore \quad x = \pm\frac{a^2m}{\sqrt{(a^2m^2 + b^2)}} = \pm\frac{a^2m}{c}$$

From Eq. (i), $\frac{a^4m^2}{c^2} \cdot \frac{1}{a^2} + \frac{y^2}{b^2} = 1$

$$\Rightarrow \quad \frac{y^2}{b^2} = 1 - \frac{a^2m^2}{c^2} = \frac{c^2 - a^2m^2}{c^2} = \frac{b^2}{c^2}$$

$$y = \pm\frac{b^2}{c}$$

Hence, the point of contact is $\left(\pm\frac{a^2m}{c}, \pm\frac{b^2}{c}\right)$ this known as m-point on the ellipse.

Remark

If $m = 0$, then Eq. (iii) gives $b^2x^2 + c^2a^2 - a^2b^2 = 0$

or $b^2x^2 + (a^2m^2 + b^2)a^2 - a^2b^2 = 0$

$$\therefore \quad x = \pm\frac{a^2m}{b}$$

which gives two values of x.

Example 18 Prove that the straight line $lx + my + n = 0$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $a^2l^2 + b^2m^2 = n^2$.

Sol. The given line is

$$lx + my + n = 0$$

or $y = -\frac{l}{m}x - \frac{n}{m}$ ...(i)

Comparing this line with $y = Mx + c$

$$\therefore \quad M = -\frac{l}{m} \quad \text{and} \quad c = -\frac{n}{m}$$

The line Eq. (i) will touch the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if

$$c^2 = a^2M^2 + b^2$$

$$\frac{n^2}{m^2} = \frac{a^2l^2}{m^2} + b^2$$

$$a^2l^2 + b^2m^2 = n^2$$

Example 19 ***Show that the line*** $x\cos\alpha + y\sin\alpha = p$ ***touches the ellipse*** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ***if*** $a^2\cos^2\alpha + b^2\sin^2\alpha = p^2$ ***and that point of contact is*** $\left(\frac{a^2\cos\alpha}{p}, \frac{b^2\sin\alpha}{p}\right)$.

Sol. The given line is $x\cos\alpha + y\sin\alpha = p$

$$y = -x\cot\alpha + p\,\text{cosec}\,\alpha$$

Comparing this line with $y = mx + c$

$\therefore$ $m = -\cot\alpha$ and $c = p\,\text{cosec}\,\alpha$

Hence, the given line touches the ellipse, then

$$c^2 = a^2m^2 + b^2$$

$\Rightarrow$ $p^2\,\text{cosec}^2\alpha = a^2\cot^2\alpha + b^2$

$\Rightarrow$ $p^2 = a^2\cos^2\alpha + b^2\sin^2\alpha$

and point of contact is $\left(-\frac{a^2m}{c}, \frac{b^2}{c}\right)$

i.e. $\left(-\frac{a^2(-\cot\alpha)}{p\,\text{cosec}\,\alpha}, \frac{b^2}{p\,\text{cosec}\,\alpha}\right)$

i.e. $\left(\frac{a^2\cos\alpha}{p}, \frac{b^2\sin\alpha}{p}\right)$

Example 20 For what value of λ does the line $y = x + \lambda$ touches the ellipse $9x^2 + 16y^2 = 144$.

Sol. $\because$ Equation of ellipse is

$$9x^2 + 16y^2 = 144 \quad \text{or} \quad \frac{x^2}{16}+\frac{y^2}{9}=1$$

Comparing this with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

then, we get $a^2 = 16$ and $b^2 = 9$

and comparing the line $y = x + \lambda$ with $y = mx + c$

$\therefore$ $m = 1$

and $c = \lambda$

If the line $y = x + \lambda$ touches the ellipse

$$9x^2 + 16y^2 = 144$$

then $c^2 = a^2m^2 + b^2$

$\Rightarrow$ $\lambda^2 = 16 \times 1^2 + 9$

$\Rightarrow$ $\lambda^2 = 25$

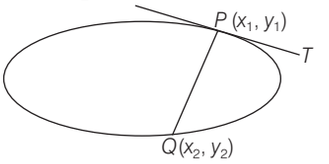
$\therefore$ $\lambda = \pm 5$

Equation of Tangent in Different Forms

1. Point Form

Theorem : The equation of tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_1, y_1) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

Proof : (By first Principal Method)



$\because$ Equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ...(i)

Let $P \equiv (x_1, y_1)$ and $Q \equiv (x_2, y_2)$ be any two point on Eq. (i), then

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1 \quad \text{...(ii)}$$

and $$\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}=1 \quad \text{...(iii)}$$

Subtracting Eqs. (ii) from (iii), then

$$\frac{1}{a^2}(x_2^2 - x_1^2) + \frac{1}{b^2}(y_2^2 - y_1^2) = 0$$

$\Rightarrow$ $$\frac{(x_2 + x_1)(x_2 - x_1)}{a^2} + \frac{(y_2 + y_1)(y_2 - y_1)}{b^2} = 0$$

$\Rightarrow$ $$\frac{y_2 - y_1}{x_2 - x_1} = -\frac{b^2(x_1 + x_2)}{a^2(y_1 + y_2)} \quad \text{...(iv)}$$

Equation of PQ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \quad \text{...(v)}$$

From Eqs. (iv) and (v), then

$$y - y_1 = -\frac{b^2(x_1 + x_2)}{a^2(y_1 + y_2)}(x - x_1) \quad \text{...(vi)}$$

Now, for tangent at $P, Q \to P$ *i.e.*, $x_2 \to x_1$ and $y_2 \to y_1$, then Eq. (vi) becomes

$$y - y_1 = -\frac{b^2(2x_1)}{a^2(2y_1)}(x - x_1)$$

or $$\frac{yy_1 - y_1^2}{b^2} = -\left(\frac{xx_1 - x_1^2}{a^2}\right)$$

or $$\frac{xx_1}{a^2}+\frac{yy_1}{b^2} = \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2} \text{ or } \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \quad [\text{from (ii)}]$$

which is required equation of tangent at (x_1, y_1).

Remark

The equation of tangent at (x_1, y_1) can be obtained by replacing x^2 by xx_1, y^2 by yy_1, x by $\frac{x + x_1}{2}$, y by $\frac{y + y_1}{2}$ and xy by $\frac{xy_1 + x_1y}{2}$.

This method is applicable only when the equation of ellipse is a polynomial of second degree in x and y.

2. Parametric form

Theorem : The equation of tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $(a\cos\phi, b\sin\phi)$ is

$$\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1$$

Proof : The equation of tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ (by point form)

Replacing x_1 by $a\cos\phi$ and y_1 by $b\sin\phi$, then we get

$$\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1$$

Remark

Point of intersection of tangent at 'θ' and 'ϕ' on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\left(\frac{a\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}, \frac{b\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}\right)$

Remembering method : $\because$ Equation of chord joining $(a\cos\theta, b\sin\theta)$ and $(a\cos\phi, b\sin\phi)$ is

$$\frac{x}{a}\cos\left(\frac{\theta+\phi}{2}\right) + \frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right) = \cos\left(\frac{\theta-\phi}{2}\right)$$

$$\Rightarrow \quad \frac{x}{a}\left\{\frac{\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}\right\} + \frac{y}{b}\left\{\frac{\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}\right\} = 1$$

or $$\frac{x}{a^2}\left\{\frac{a\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}\right\} + \frac{y}{b^2}\left\{\frac{b\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}\right\} = 1$$

3. Slope form

Theorem : The equations of tangents of slope m to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are $y = mx \pm \sqrt{(a^2m^2 + b^2)}$ and the coordinates of the points of contact are

$$\left(\mp\frac{a^2m}{\sqrt{(a^2m^2 + b^2)}}, \pm\frac{b^2}{\sqrt{(a^2m^2 + b^2)}}\right)$$

Proof : Let $y = mx + c$ be a tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Then the equation $\frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} = 1$

$$\Rightarrow \quad x^2(a^2m^2 + b^2) + 2a^2mcx + a^2(c^2 - b^2) = 0 \qquad \text{...(i)}$$

must have equal roots

$$4a^4m^2c^2 - 4(a^2m^2 + b^2)a^2(c^2 - b^2) = 0$$

$$\{\because B^2 - 4AC = 0\}$$

$$\Rightarrow \quad a^2m^2c^2 - (a^2m^2 + b^2)(c^2 - b^2) = 0$$

$$\Rightarrow \quad a^2m^2c^2 - a^2m^2c^2 + a^2b^2m^2 - b^2c^2 + b^4 = 0$$

$$\Rightarrow \quad a^2b^2m^2 - b^2c^2 + b^4 = 0$$

$$\Rightarrow \quad c^2 = a^2m^2 + b^2$$

$$\therefore \quad c = \pm\sqrt{(a^2m^2 + b^2)}$$

Substituting this value of c in $y = mx + c$, we get

$$y = mx \pm \sqrt{(a^2m^2 + b^2)}$$

as the required equations of tangent of ellipse in terms of slope, putting $c = \pm\sqrt{(a^2m^2 + b^2)}$ in (i), we get

$$x^2(a^2m^2 + b^2) \pm 2a^2m\sqrt{(a^2m^2 + b^2)}\,x + a^4m^2 = 0$$

$$\Rightarrow \quad (\sqrt{(a^2m^2 + b^2)}\,x \pm a^2m)^2 = 0$$

$$\Rightarrow \quad x = \mp\frac{a^2m}{\sqrt{(a^2m^2 + b^2)}}$$

Substituting this value of x in

$$y = mx \pm \sqrt{a^2m^2 + b^2}$$

we obtained $$y = \pm\frac{b^2}{\sqrt{(a^2m^2 + b^2)}}$$

Thus, the coordinates of the points of contact are

$$\left(\mp\frac{a^2m}{\sqrt{(a^2m^2 + b^2)}}, \pm\frac{b^2}{\sqrt{(a^2m^2 + b^2)}}\right)$$

Example 21 If the line $3x + 4y = \sqrt{7}$ touches the ellipse $3x^2 + 4y^2 = 1$, then find the point of contact.

Sol. Let the given line touches the ellipse at point $P(x_1, y_1)$.

The equation of tangent at P is

$$3xx_1 + 4yy_1 = 1 \qquad \text{...(i)}$$

Comparing Eq. (i) with the given equation of line $3x + 4y = \sqrt{7}$, we get

$$\frac{3x_1}{3} = \frac{4y_1}{4} = \frac{1}{\sqrt{7}}$$

$$\therefore \quad x_1 = y_1 = \frac{1}{\sqrt{7}}$$

Hence, point of contact (x_1, y_1) is $\left(\frac{1}{\sqrt{7}}, \frac{1}{\sqrt{7}}\right)$.

Example 22 Find the equations of the tangents to the ellipse $3x^2 + 4y^2 = 12$ which are perpendicular to the line $y + 2x = 4$.

Sol. Let m be the slope of the tangent, since the tangent is perpendicular to the line $y + 2x = 4$.

$$\therefore \quad m \times -2 = -1$$

$$\Rightarrow \quad m = \frac{1}{2}$$

Since $3x^2 + 4y^2 = 12$

or $\frac{x^2}{4} + \frac{y^2}{3} = 1$

Comparing this with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\therefore \quad a^2 = 4$$

and $b^2 = 3$

So the equations of the tangents are

$$y = \frac{1}{2}x \pm \sqrt{4 \times \frac{1}{4} + 3}$$

$$\Rightarrow \quad y = \frac{1}{2}x \pm 2$$

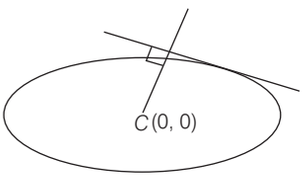
or $x - 2y \pm 4 = 0$

Example 23 Find the locus of the foot of the perpendicular drawn from centre upon any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Sol. Any tangent of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$y = mx + \sqrt{(a^2m^2 + b^2)} \qquad \ldots(i)$$

Equation of the line perpendicular to Eq. (i) and passing through $(0, 0)$ is



$$y = -\frac{1}{m}x \text{ or } m = -\frac{x}{y} \qquad \ldots(ii)$$

Substituting the value of m from Eq. (ii) in Eq. (i), then

$$y = -\frac{x^2}{y} + \sqrt{\left(a^2\frac{x^2}{y^2} + b^2\right)}$$

$$\Rightarrow \quad (x^2 + y^2)^2 = a^2x^2 + b^2y^2$$

or changing to polars by putting $x = r\cos\theta$, $y = r\sin\theta$ it becomes

$$r^2 = a^2\cos^2\theta + b^2\sin^2\theta$$

Example 24 ***Find the point on the ellipse $16x^2 + 11y^2 = 256$, where the common tangent to it and the circle $x^2 + y^2 - 2x = 15$ touch.***

Sol. The given ellipse is $\frac{x^2}{16} + \frac{y^2}{(256/11)} = 1$

Equation of tangent to it at point $\left(4\cos\theta, \frac{16}{\sqrt{11}}\sin\theta\right)$ is

$$\frac{x}{4}\cos\theta + y\frac{\sqrt{11}}{16}\sin\theta = 1$$

It also touch the circle $(x-1)^2 + (y-0)^2 = 4^2$

Therefore,

$$\frac{\left|\frac{1}{4}\cos\theta - 1\right|}{\sqrt{\left(\frac{\cos^2\theta}{16} + \frac{11}{256}\sin^2\theta\right)}} = 4$$

$$\Rightarrow \quad |\cos\theta - 4| = \sqrt{(16\cos^2\theta + 11\sin^2\theta)}$$

or $4\cos^2\theta + 8\cos\theta - 5 = 0$

or $(2\cos\theta - 1)(2\cos\theta + 5) = 0$

or $\cos\theta = \frac{1}{2} \qquad \left(\because \cos\theta \neq -\frac{5}{2}\right)$

$$\therefore \quad \theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

Therefore, points are $\left(2, \pm\frac{8\sqrt{3}}{11}\right)$.

Example 25 Find the maximum area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which touches the line $y = 3x + 2$.

Sol. $\because$ Line $y = 3x + 2$ touches ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Here, $m = 3$ and $c = 2$

Substituting in $c^2 = a^2m^2 + b^2$

or $4 = 9a^2 + b^2 \qquad \ldots(i)$

Now, $AM \geq GM$

$\Rightarrow \quad \frac{9a^2 + b^2}{2} \geq \sqrt{(9a^2)\, b^2} \quad \Rightarrow \quad \frac{9a^2 + b^2}{2} \geq 3ab$

$\Rightarrow \quad 2 \geq 3ab$ [from Eq. (i)]

or $\frac{2\pi}{3} \geq \pi\, ab$

or $\frac{2\pi}{3} \geq$ Area of ellipse

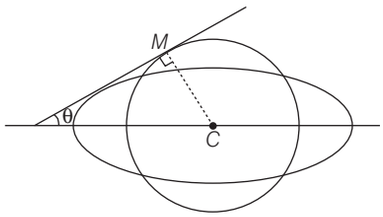
Therefore, the maximum area of the ellipse is $\frac{2\pi}{3}$.

Example 26 A circle of radius r is concentric with the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Prove that the common tangent is inclined to the major axis at an angle $\tan^{-1}\sqrt{\left(\frac{r^2 - b^2}{a^2 - r^2}\right)}$.

Sol. Equation of the circle of radius r and concentric with ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$x^2 + y^2 = r^2 \qquad \ldots(i)$$

any tangent to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is



$$y = mx + \sqrt{(a^2m^2 + b^2)} \qquad (\text{where } m = \tan\theta)$$

If it is a tangent to circle, then perpendicular from $(0, 0)$ is equal to radius r,

$\therefore \quad \frac{\sqrt{(a^2m^2 + b^2)}}{\sqrt{(m^2 + 1)}} = |\, r\, |$ or $a^2m^2 + b^2 = m^2r^2 + r^2$

$$(a^2 - r^2)\, m^2 = r^2 - b^2$$

$\therefore \quad m = \sqrt{\left(\frac{r^2 - b^2}{a^2 - r^2}\right)}$

$$\tan\theta = \sqrt{\left(\frac{r^2 - b^2}{a^2 - r^2}\right)}$$

$$\theta = \tan^{-1}\sqrt{\left(\frac{r^2 - b^2}{a^2 - r^2}\right)}$$

Example 27 Show that the product of the perpendiculars from the foci of any tangent to an ellipse is equal to the square of the semi minor axis, and the feet of a these perpendiculars lie on the auxiliary circle.

Sol. Let equation of ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \ldots(i)$$

Equation of any tangent in term of slope (m) of (i) is

$$y = mx + \sqrt{(a^2m^2 + b^2)}$$

or $y - mx = \sqrt{a^2m^2 + b^2}$ ...(ii)

Equation of a line perpendicular to Eq. (ii) and passing through $S\,(ae, 0)$ is

$$y - 0 = -\frac{1}{m}(x - ae)$$

or $x + my = ae$ ...(iii)

The lines Eq. (ii) and Eq. (iii) will meet at the foot of perpendicular whose locus is obtained by eliminating the variable m between Eq. (ii) and Eq. (iii), then squaring and adding Eq. (ii) and Eq. (iii), we get

$$(y - mx)^2 + (x + my)^2 = a^2m^2 + b^2 + a^2e^2$$

$\Rightarrow \quad (1 + m^2)(x^2 + y^2) = a^2m^2 + b^2 + a^2 - b^2$

$\Rightarrow \quad (1 + m^2)(x^2 + y^2) = a^2(1 + m^2)$

or $x^2 + y^2 = a^2$

which is auxiliary circle of ellipse, similarly we can show that the other foot drawn from second focus also lies on $x^2 + y^2 = a^2$.

Again if p_1 and p_2 be perpendiculars from foci $S(ae, 0)$ and $S'(-ae, 0)$ on (ii), then

$$p_1 = \frac{|\sqrt{(a^2m^2 + b^2)} + mae|}{\sqrt{(1 + m^2)}}$$

and $p_2 = \frac{|\sqrt{(a^2m^2 + b^2)} - mae|}{\sqrt{(1 + m^2)}}$

$\therefore \quad p_1p_2 = \frac{|\, a^2m^2 + b^2 - a^2e^2m^2|}{(1 + m^2)}$

$$= \frac{|\, a^2m^2 + b^2 - (a^2 - b^2)\, m^2|}{(1 + m^2)}$$

$$= \frac{b^2(1 + m^2)}{(1 + m^2)}$$

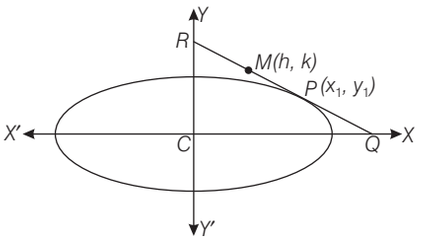
$$= b^2 = (\text{semi minor axis})^2$$

Example 28 ***Prove that the locus of mid-points of the portion of the tangents to the ellipse*** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ***intercepted between the axes is*** $a^2y^2+b^2x^2=4x^2y^2$ **.**

Sol. Let $P(x_1, y_1)$ be any point on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad ...(i)$$

$\therefore$ Equation of tangent at (x_1, y_1) to (i) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$



This meet the coordinate axes at

$$Q\left(\frac{a^2}{x_1}, 0\right) \quad \text{and} \quad R\left(0, \frac{b^2}{y_1}\right)$$

Let $M(h, k)$ be the mid-point of QR then,

$$h=\frac{\frac{a^2}{x_1}+0}{2}, \quad k=\frac{0+\frac{b^2}{y_1}}{2}$$

$$\Rightarrow \qquad x_1=\frac{a^2}{2h}, y_1=\frac{b^2}{2k}$$

Since, (x_1, y_1) lies on Eq. (i)

$$\therefore \qquad \frac{\left(\frac{a^2}{2h}\right)^2}{a^2}+\frac{\left(\frac{b^2}{2k}\right)^2}{b^2}=1$$

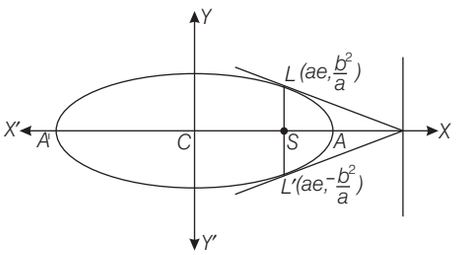
$$\Rightarrow \qquad \frac{a^2}{4h^2}+\frac{b^2}{4k^2}=1$$

$$\Rightarrow \qquad a^2k^2+b^2h^2=4h^2k^2$$

Hence, the locus of $M(h, k)$ is $a^2y^2+b^2x^2=4x^2y^2$

Example 29 Prove that the tangents at the extremities of latusrectum of an ellipse intersect on the corresponding directrix.

Sol. Let LSL' be a latusrectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.



$\therefore$ The coordinates of L and L' are

$$\left(ae, \frac{b^2}{a}\right) \quad \text{and} \quad \left(ae, -\frac{b^2}{a}\right) \text{ respectively}$$

$\therefore$ Equation of tangent at $L\left(ae, \frac{b^2}{a}\right)$ is

$$\Rightarrow \qquad \frac{x(ae)}{a^2}+\frac{y\left(\frac{b^2}{a}\right)}{b^2}=1$$

$$\Rightarrow \qquad xe+y=a \qquad ...(i)$$

The equation of the tangent at $L'\left(ae, -\frac{b^2}{a}\right)$ is

$$\frac{x(ae)}{a^2}+\frac{y\left(-\frac{b^2}{a}\right)}{b^2}=1$$

$$\Rightarrow \qquad ex-y=a \qquad ...(ii)$$

Solving Eqs. (i) and (ii), we get

$$x=\frac{a}{e} \quad \text{and} \quad y=0$$

Thus, the tangents at L and L' intersect at $(a/e, 0)$ which is a point lying on the corresponding directrix i.e. $x=\frac{a}{e}$.

Equations of Normals in Different Forms

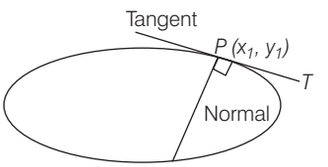
1. Point form

Theorem : The equation of normal at (x_1, y_1) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$$

Proof : Since the equation of tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at (x_1, y_1) is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$$



The slope of the tangent at $(x_1, y_1)=-\frac{b^2x_1}{a^2y_1}$

$\therefore$ Slope of Normals at $(x_1, y_1) = \dfrac{a^2 y_1}{b^2 x_1}$

Hence, the equation of normal at (x_1, y_1) is

$$y - y_1 = \frac{a^2 y_1}{b^2 x_1}(x - x_1)$$

or $$\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$$

Remark

The equation of normal at (x_1, y_1) can also be obtained by this method

$$\frac{x - x_1}{a'x_1 + hy_1 + g} = \frac{y - y_1}{hx_1 + b'y_1 + f} \quad \text{...(i)}$$

a', b', g, f, h are obtained by comparing the given ellipse with

$$a'x^2 + 2hxy + b'y^2 + 2gx + 2fy + c = 0 \quad \text{...(ii)}$$

The denominors of (i) can easily remembered by the first two rows of this determinant

i.e. $$\begin{vmatrix} a' & h & g \\ h & b' & f \end{vmatrix}$$

Since, first row, $a'(x_1) + h(y_1) + g(1)$

and second row, $h(x_1) + b'(y_1) + f(1)$

Here ellipse $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

or $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \quad \text{... (iii)}$$

Comparing Eqs. (ii) and (iii), then we get

$$a' = \frac{1}{a^2}, b' = \frac{1}{b^2}, g = 0, f = 0, h = 0$$

From, Eq. (i), equation of normal of Eq. (iii) at (x_1, y_1) is

$$\frac{x - x_1}{\frac{1}{a^2}x_1 + 0 + 0} = \frac{y - y_1}{0 + \frac{1}{b^2}y_1 + 0}$$

or $$\frac{a^2(x - x_1)}{x_1} = \frac{b^2(y - y_1)}{y_1}$$

or $$\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$$

2. Parametric form

Theorem : The equation of normal to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ at } (a\cos\phi, b\sin\phi) \text{ is}$$

$$ax \sec\phi - by \operatorname{cosec}\phi = a^2 - b^2$$

Proof : Since, the equation of normal of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at (x_1, y_1) is

$$\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2 \quad \text{...(i)}$$

Replacing x_1 by $a\cos\phi$ and y_1 by $b\sin\phi$, then Eq.(i) becomes

$$\frac{a^2 x}{a\cos\phi} - \frac{b^2 y}{b\sin\phi} = a^2 - b^2$$

$$ax \sec\phi - by \operatorname{cosec}\phi = a^2 - b^2$$

is the equation of normal at $(a\cos\phi, b\sin\phi)$

3. Slope form

Theorem : The equations of the normals of slope m to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ are given by

$$y = mx \mp \frac{m(a^2 - b^2)}{\sqrt{(a^2 + b^2m^2)}}$$

at the points $\left(\pm \dfrac{a^2}{\sqrt{(a^2 + b^2m^2)}}, \pm \dfrac{mb^2}{\sqrt{(a^2 + b^2m^2)}} \right)$.

Proof : The equation of normal to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at (x_1, y_1) is

$$\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2 \quad \text{...(i)}$$

Since, 'm' is the slope of the normal, then

$$m = \frac{a^2 y_1}{b^2 x_1}$$

$$y_1 = \frac{b^2 x_1 m}{a^2} \quad \text{...(ii)}$$

Since, (x_1, y_1) lies on $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$\therefore$ $$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$$

or $$\frac{x_1^2}{a^2} + \frac{b^4 x_1^2 m^2}{a^4 b^2} = 1$$

or $$\frac{x_1^2}{a^2} + \frac{b^2 x_1^2 m^2}{a^4} = 1 \text{ or } x_1^2 = \frac{a^4}{(a^2 + b^2m^2)}$$

$\therefore$ $$x_1 = \pm \frac{a^2}{\sqrt{a^2 + b^2m^2}}$$

From Eq. (ii),

$$y_1 = \pm \frac{mb^2}{\sqrt{(a^2 + b^2m^2)}}$$

$\therefore$ Equation of normal in terms of slope is

$$y - \left(\pm \frac{mb^2}{\sqrt{a^2 + b^2m^2}} \right) = m \left(x - \left(\pm \frac{a^2}{\sqrt{a^2 + b^2m^2}} \right) \right)$$

$$\Rightarrow \quad y = mx \mp \frac{m(a^2 - b^2)}{\sqrt{(a^2 + b^2m^2)}} \qquad \text{...(iii)}$$

Thus $y = mx \pm \frac{m(a^2 - b^2)}{\sqrt{(a^2 + b^2m^2)}}$ is a normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where m is the slope of the normal.

The coordinates of the point of contact are

$$\left(\pm \frac{a^2}{\sqrt{a^2 + b^2m^2}}, \pm \frac{mb^2}{\sqrt{a^2 + b^2m^2}} \right)$$

Comparing Eq. (iii) with,

$$y = mx + c$$

$$\therefore \quad c = \mp \frac{m(a^2 - b^2)}{\sqrt{(a^2 + b^2m^2)}}$$

or $$c^2 = \frac{m^2(a^2 - b^2)^2}{(a^2 + b^2m^2)}$$

which is condition of normality, when $y = mx + c$ is the normal of

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

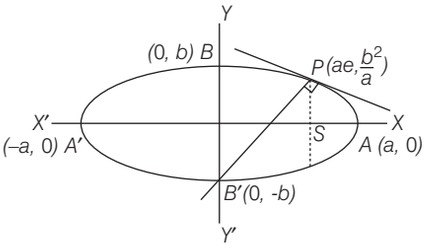
Example 30 *If the normal at an end of a latusrectum of an ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *passes through one extremity of the minor axis, show that the eccentricity of the ellipse is given by*

$$e^4 + e^2 - 1 = 0 \quad \textbf{or} \quad e^2 = \frac{\sqrt{5} - 1}{2}$$

Sol. The coordinates of an end of the latusrectum are $(ae, b^2/a)$. The equation of normal at $P(ae, b^2/a)$ is

$$\frac{a^2x}{ae} - \frac{b^2(y)}{b^2/a} = a^2 - b^2$$

or $$\frac{ax}{e} - ay = a^2 - b^2$$



It passes through one extremity of the minor axis whose coordinates are $(0, -b)$

$$\therefore \quad 0 + ab = a^2 - b^2$$

or $$(a^2b^2) = (a^2 - b^2)^2$$

or $$a^2a^2(1 - e^2) = (a^2e^2)^2$$

or $$1 - e^2 = e^4$$

or $$e^4 + e^2 - 1 = 0$$

or $$(e^2)^2 + e^2 - 1 = 0$$

$$\therefore \quad e^2 = \frac{-1 \pm \sqrt{1 + 4}}{2}$$

$$\Rightarrow \quad e^2 = \frac{\sqrt{5} - 1}{2} \qquad \text{(taking + ve sign)}$$

Example 31 Prove that the straight line $lx + my + n = 0$ is a normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $\frac{a^2}{l^2} + \frac{b^2}{m^2} = \frac{(a^2 - b^2)^2}{n^2}$.

Sol. The equation of any normal to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$ax \sec\phi - by \operatorname{cosec}\phi = a^2 - b^2 \qquad \text{... (i)}$$

The straight line $lx + my + n = 0$ will be a normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Therefore, Eq. (i) and $lx + my + n = 0$ represent the same line

$$\frac{a \sec\phi}{l} = \frac{-b \operatorname{cosec}\phi}{m} = \frac{a^2 - b^2}{-n}$$

$$\cos\phi = \frac{-na}{l(a^2 - b^2)}$$

and $$\sin\phi = \frac{nb}{m(a^2 - b^2)}$$

$\because$ $$\sin^2\phi + \cos^2\phi = 1$$

$$\therefore \quad \frac{n^2b^2}{m^2(a^2 - b^2)^2} + \frac{n^2a^2}{l^2(a^2 - b^2)^2} = 1$$

$$\Rightarrow \quad \frac{a^2}{l^2} + \frac{b^2}{m^2} = \frac{(a^2 - b^2)^2}{n^2}$$

Example 32 A normal inclined at an angle of 45° to x-axis of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is drawn. It meets the major and minor axes in P and Q respectively. If C is the centre of the ellipse, prove that area of ΔCPQ is $\frac{(a^2-b^2)^2}{2(a^2+b^2)}$ sq units.

Sol. Let $R(a\cos\phi, b\sin\phi)$ be any point on the ellipse, then equation of normal at R is

$$ax\sec\phi - by\,\text{cosec}\,\phi = a^2 - b^2$$

or $$\frac{x}{\frac{\cos\phi\,(a^2-b^2)}{a}}+\frac{y}{\frac{-\sin\phi\,(a^2-b^2)}{b}}=1$$

If meets the major and minor axes at $P\left(\frac{(a^2-b^2)}{a}\cos\phi, 0\right)$ and $Q\left(0, -\frac{(a^2-b^2)}{b}\sin\phi\right)$ are respectively

$$\therefore \quad CP=\left(\frac{a^2-b^2}{a}\right)|\cos\phi|$$

and $$CQ=\left(\frac{a^2-b^2}{b}\right)|\sin\phi|$$

$$\therefore \text{ Area of } \Delta CPQ=\frac{1}{2}\times CP\times CQ$$

$$=\frac{(a^2-b^2)^2\,|\sin\phi\cos\phi|}{2ab} \quad \text{... (i)}$$

But slope of normal $=\frac{a}{b}\tan\phi=\tan 45°$ (given)

$$\frac{a}{b}\tan\phi=1$$

$$\tan\phi=\frac{b}{a}$$

$$\therefore \quad \sin 2\phi=\frac{2\tan\phi}{1+\tan^2\phi}=\frac{2ab}{a^2+b^2}$$

$$\therefore \text{ From Eq. (i), Area of } \Delta CPQ=\frac{(a^2-b^2)^2\left|\frac{\sin 2\phi}{2}\right|}{2ab}$$

$$=\frac{(a^2-b^2)^2\,\frac{ab}{(a^2+b^2)}}{2ab}$$

$$=\frac{(a^2-b^2)^2}{2(a^2+b^2)}\text{ sq units.}$$

Example 33 Any ordinate MP of an ellipse meets the auxiliary circle in Q. Prove that the locus of the point of intersection of the normals at P and Q is the circle $x^2+y^2=(a^2+b^2)^2$.

Sol. Let equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ its auxiliary circle is

$$x^2+y^2=a^2$$

Coordinates of P and Q are $(a\cos\phi, b\sin\phi)$ and $(a\cos\phi, a\sin\phi)$ respectively. Equation of normal at P to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$ax\sec\phi - by\,\text{cosec}\,\phi = a^2-b^2 \quad \text{...(i)}$$

and equation of normal at Q to the circle $x^2+y^2=a^2$ is

$$y=x\tan\phi \quad \text{...(ii)}$$

From Eq. (ii), $\tan\phi=\frac{y}{x}$

$$\therefore \quad \sin\phi=\frac{y}{\sqrt{(x^2+y^2)}} \quad \text{and} \quad \cos\phi=\frac{x}{\sqrt{(x^2+y^2)}}$$

or $$\text{cosec}\,\phi=\frac{\sqrt{(x^2+y^2)}}{y}$$

and $$\sec\phi=\frac{\sqrt{(x^2+y^2)}}{x} \quad \text{...(iii)}$$

Substituting the values of $\sec\phi$ and $\text{cosec}\,\phi$ from Eq. (iii) in Eq. (i)

$$\therefore \quad ax\times\frac{\sqrt{(x^2+y^2)}}{x}-by\times\frac{\sqrt{(x^2+y^2)}}{y}=a^2-b^2$$

or $$(a-b)\sqrt{(x^2+y^2)}=(a+b)(a-b)$$

or $$\sqrt{x^2+y^2}=a+b$$

$$\therefore \quad x^2+y^2=(a+b)^2$$

which is required locus.

Properties of Eccentric Angles of the Co-normal Points

1. **In general, four normals can be drawn to an ellipse from any point and if $\alpha, \beta, \gamma, \delta$ the eccentric angles of these four co-normal points, then $\alpha+\beta+\gamma+\delta$ is an odd multiple of π.**

   Let $Q(h, k)$ be any given point and let $P(a\cos\phi, b\sin\phi)$ be any point on the ellipse

   $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

   Equation of normal at $P(a\cos\phi, b\sin\phi)$ is

   $$ax\sec\phi - by\,\text{cosec}\,\phi = a^2-b^2$$

it passes through $Q(h, k)$

$$\therefore \quad ah\sec\phi - bk\,\text{cosec}\,\phi = a^2 - b^2$$

or
$$\frac{ah}{\cos\phi} - \frac{bk}{\sin\phi} = a^2 - b^2 \qquad ...(i)$$

or
$$\frac{ah}{\left(\frac{1-\tan^2(\phi/2)}{1+\tan^2(\phi/2)}\right)} - \frac{bk}{\left(\frac{2\tan(\phi/2)}{1+\tan^2(\phi/2)}\right)} = a^2 - b^2 \qquad ...(ii)$$

Let $\tan\phi/2 = t$

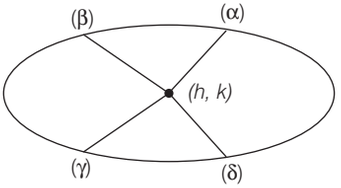
then, Eq. (ii) reduced to

$$bkt^4 + 2\{ah + (a^2 - b^2)\}t^3 + 2\{ah - (a^2 - b^2)\}t - bk = 0 \qquad ... (iii)$$

Which is a fourth degree equation in t, hence four normals can be drawn to an ellipse from any point.

Consequently, it has four values of ϕ say $\alpha, \beta, \gamma, \delta$ ($\because t = \tan\phi/2$).

Now,
$$\tan\left(\frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} + \frac{\delta}{2}\right) = \frac{S_1 - S_3}{1 - S_2 + S_4}$$



$$= \frac{\frac{-2\{ah + (a^2 - b^2)\}}{bk} + \frac{2\{(ah - (a^2 - b^2)\}}{bk}}{1 - 0 - 1}$$

$$= \infty \qquad \text{(From trigonometry)} \qquad (\because a \neq b)$$

or
$$\cot\left(\frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} + \frac{\delta}{2}\right) = 0$$

or
$$\frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} + \frac{\delta}{2} = \text{an odd multiple of } \pi/2$$

$$\Rightarrow \quad \alpha + \beta + \gamma + \delta = \text{an odd multiple of } \pi$$

Aliter :

Let
$$z = e^{i\phi} = \cos\phi + i\sin\phi$$

$$\therefore \quad \frac{1}{z} = e^{-i\phi} = \cos\phi - i\sin\phi$$

$$\therefore \quad \cos\phi = \frac{z + \frac{1}{z}}{2} = \frac{z^2 + 1}{2z}$$

and
$$\sin\phi = \frac{z - \frac{1}{z}}{2i} = \frac{z^2 - 1}{2iz}$$

Now, Eq. (i), reduces to

$$\frac{ah}{\left(\frac{z^2+1}{2z}\right)} - \frac{bk}{\left(\frac{z^2-1}{2iz}\right)} = a^2 - b^2$$

$$\Rightarrow \quad (a^2 - b^2)z^4 - 2(ah - ibk)z^3 + 2(ah + ibk)z - (a^2 - b^2) = 0 \qquad ...(iv)$$

Consequently $z = e^{i\phi}$

gives four values of ϕ, say $\alpha, \beta, \gamma, \delta$ (Here, sum of four angles)

$$\therefore \quad z_1 . z_2 . z_3 . z_4 = -1$$

$$\Rightarrow \quad e^{i\alpha} \cdot e^{i\beta} \cdot e^{i\gamma} \cdot e^{i\delta} = -1$$

$$\Rightarrow \quad e^{i(\alpha+\beta+\gamma+\delta)} = -1$$

$$\cos(\alpha + \beta + \gamma + \delta) + i\sin(\alpha + \beta + \gamma + \delta) = -1$$

or
$$\cos(\alpha + \beta + \gamma + \delta) = -1$$

and
$$\sin(\alpha + \beta + \gamma + \delta) = 0$$

$$\alpha + \beta + \gamma + \delta = (2n + 1)\pi$$

and
$$\alpha + \beta + \gamma + \delta = r\pi$$

where, $n\, r, \in I$

Hence, $\alpha + \beta + \gamma + \delta$ = odd multiple of π

2. If α, β, γ are the eccentric angles of three points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the normals at which are concurrent, then

$\sin(\alpha + \beta) + \sin(\beta + \gamma) + \sin(\gamma + \alpha) = 0$

Here, in each term sum of two eccentric angles

$\therefore$ From Eq. (iv),

$$\Sigma z_1 z_2 = 0$$

or
$$z_1z_2 + z_1z_3 + z_1z_4 + z_2z_3 + z_2z_4 + z_3z_4 = 0$$

$$\Rightarrow \quad e^{i(\alpha+\beta)} + e^{i(\alpha+\gamma)} + e^{i(\alpha+\delta)} + e^{i(\beta+\gamma)} + e^{i(\beta+\delta)} + e^{i(\gamma+\delta)} = 0$$

$$\Rightarrow \quad [\cos(\alpha + \beta) + \cos(\alpha + \gamma) + \cos(\alpha + \delta) + \cos(\beta + \gamma) + \cos(\beta + \delta) + \cos(\gamma + \delta)] + i[(\sin(\alpha + \beta) + \sin(\alpha + \gamma) + \sin(\alpha + \delta) + \sin(\beta + \gamma) + \sin(\beta + \delta) + \sin(\gamma + \delta)] = 0$$

Comparing the imaginary part, then

$$\sin(\alpha + \beta) + \sin(\alpha + \gamma) + \sin(\alpha + \delta) + \sin(\beta + \gamma) + \sin(\beta + \delta) + \sin(\gamma + \delta) = 0 \qquad ...(v)$$

Since, from property Eq. (i)

$$\alpha + \beta + \gamma + \delta = \text{odd multiple of } \pi$$

$$(\alpha + \delta) = \text{odd multiple of } \pi - (\beta + \gamma)$$

$$(\beta + \delta) = \text{odd multiple of } \pi - (\alpha + \gamma)$$

$$(\gamma+\delta)=\text{odd multiple of }\pi-(\alpha+\beta)$$

$$\left.\begin{aligned}\sin(\alpha+\delta)&=\sin(\beta+\gamma)\\ \sin(\beta+\delta)&=\sin(\alpha+\gamma)\\ \sin(\gamma+\delta)&=\sin(\alpha+\beta)\end{aligned}\right\}$$

$\{\because \sin(n\pi-\alpha)=\sin\alpha, \text{ if } n \text{ is integer}\}$ ...(vi)

From Eqs. (v) and (vi), we get

$$2\sin(\alpha+\beta)+2\sin(\beta+\gamma)+2\sin(\gamma+\alpha)=0$$

Hence, $\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)=0$

Aliter :

From Eq. (iii), $\Sigma t_1t_2=0$ ...(vii)

and $t_1t_2t_3t_4=-1$ ...(viii)

Now, $\Sigma t_1t_2=0$

$$\Rightarrow \quad t_1t_2+t_2t_3+t_3t_1=-t_4(t_1+t_2+t_3)$$

$$\Rightarrow \quad t_1t_2+t_2t_3+t_3t_1=\frac{t_1+t_2+t_3}{t_1t_2t_3} \quad \{\text{from (viii)}\}$$

$$\Rightarrow \quad t_1t_2+t_2t_3+t_3t_1=\frac{1}{t_2t_3}+\frac{1}{t_3t_1}+\frac{1}{t_1t_2}$$

$$\Rightarrow \quad \tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\tan\frac{\gamma}{2}\tan\frac{\alpha}{2}$$

$$=\cot\frac{\beta}{2}\cot\frac{\gamma}{2}+\cot\frac{\gamma}{2}\cot\frac{\alpha}{2}+\cot\frac{\alpha}{2}\cot\frac{\beta}{2}$$

$$\Rightarrow \sum\left(\tan\frac{\alpha}{2}\tan\frac{\beta}{2}-\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\right)=0$$

$$\Rightarrow \sum\left(\frac{\sin^2(\alpha/2)\sin^2(\beta/2)-\cos^2(\alpha/2)\cos^2(\beta/2)}{\sin(\alpha/2)\sin(\beta/2)\cos(\alpha/2)\cos(\beta/2)}\right)=0$$

$$\Rightarrow \sum -4\left(\frac{\{\cos(\alpha/2)\cos(\beta/2)+\sin(\alpha/2)\sin(\beta/2)\}\{\cos(\alpha/2)\cos(\beta/2)-\sin(\alpha/2)\sin(\beta/2)\}}{\sin\alpha\sin\beta}\right)=0$$

$$\Rightarrow \quad \sum -4\left(\frac{\cos\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)}{\sin\alpha\sin\beta}\right)=0$$

$$\Rightarrow \quad \sum -2\frac{[\cos\alpha+\cos\beta]}{\sin\alpha\sin\beta}=0$$

$$\Rightarrow \quad \sum\frac{\sin\gamma(\cos\alpha+\cos\beta)}{\sin\alpha\sin\beta\sin\gamma}=0$$

$$\Rightarrow \quad \sum\sin\gamma(\cos\alpha+\cos\beta)=0$$

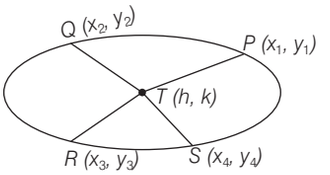
$$\Rightarrow \quad \sin\gamma(\cos\alpha+\cos\beta)+\sin\alpha(\cos\beta+\cos\gamma)+\sin\beta(\cos\gamma+\cos\alpha)=0$$

$$\Rightarrow \quad \sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)=0$$

Co-normal Points Lie on a Fixed Curve

Let $P(x_1,y_1)$, $Q(x_2,y_2)$, $R(x_3,y_3)$ and $T(x_4,y_4)$ be conormal points so that normal drawn from them meet in $T(h,k)$.

Then, equation of normal at $P(x_1,y_1)$ is



$$\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$$

or $(a^2-b^2)x_1y_1+b^2yx_1-a^2xy_1=0$

but the point $T(h,k)$ lies on it

$\therefore \quad (a^2-b^2)x_1y_1+b^2kx_1-a^2hy_1=0$

Similarly, for points Q, R and S are

$$(a^2-b^2)x_2y_2+b^2kx_2-a^2hy_2=0$$

$$(a^2-b^2)x_3y_3+b^2kx_3-a^2hy_3=0$$

and $(a^2-b^2)x_4y_4+b^2kx_4-a^2hy_4=0$

Hence, P, Q, R, S all lie on the curve

$$(a^2-b^2)xy+b^2kx-a^2hy=0$$

This curve is called **Apollonian rectangular hyperbola**.

Remark

The feet of the normals from any fixed point to the ellipse lie at the intersections of the **Apollonian rectangular hyperbola** with the ellipse.

Exercise for Session 2

1. The number of values of c such that the straight line $y = 4x + c$ touches the curve $\frac{x^2}{4} + y^2 = 1$, is

(a) 0 (b) 1 (c) 2 (d) infinite

2. If any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ cuts off intercepts of length h and k on the axes, then $\frac{a^2}{h^2} + \frac{b^2}{k^2}$ is equal to

(a) -1 (b) 0 (c) 1 (d) None of these

3. The equations of the tangents to the ellipse $3x^2 + y^2 = 3$ making equal intercepts on the axes are

(a) $y = \pm x \pm 2$ (b) $y = \pm x \pm 4$ (c) $y = \pm x \pm \sqrt{30}$ (d) $y = \pm x \pm \sqrt{35}$

4. If $\frac{x}{a} + \frac{y}{b} = \sqrt{2}$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then its, eccentric angle θ is equal to

(a) 0 (b) 45° (c) 60° (d) 90°

5. The number of values of $\theta \in [0, 2\pi]$ for which the line $2x \cos\theta + 3y \sin\theta = 6$ touches the ellipse $4x^2 + 9y^2 = 36$ is

(a) 1 (b) 2 (c) 4 (d) infinite

6. The common tangent of $x^2 + y^2 = 4$ and $2x^2 + y^2 = 2$ is

(a) $x + y + 4 = 0$ (b) $x - y + 7 = 0$ (c) $2x + 3y + 8 = 0$ (d) None of these

7. If the normal at any point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the axes in G and g respectively, then $PG \cdot Pg =$

(a) $a : b$ (b) $a^2 : b^2$ (c) $b : a$ (d) $b^2 : a^2$

8. Number of distinct normal lines that can be drawn to the ellipse $\frac{x^2}{169} + \frac{y^2}{25} = 1$, from the point $(0, 6)$ is

(a) one (b) two (c) three (d) four

9. If a tangent of slope 2 of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is normal to the circle $x^2 + y^2 + 4x + 1 = 0$, then the maximum value of ab is

(a) 4 (b) 2 (c) 1 (d) None of these

10. If the normal at the point $P(\theta)$ to the ellipse $\frac{x^2}{14} + \frac{y^2}{5} = 1$, intersect it again at the point $Q(2\theta)$, then $\cos\theta$ is equal to

(a) $\frac{2}{3}$ (b) $-\frac{2}{3}$ (c) $\frac{3}{2}$ (d) $-\frac{3}{2}$

11. The line $5x - 3y = 8\sqrt{2}$ is a normal to the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$. If '$\theta$' be eccentric angle of the foot of this normal, then 'θ' is equal to

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

12. If the tangent drawn at point $(\lambda^2, 2\lambda)$ on the parabola $y^2 = 4x$ is same as the normal drawn at a point $(\sqrt{5}\cos\theta, 2\sin\theta)$ on the ellipse $4x^2 + 5y^2 = 20$. Find the values of λ and θ.

13. If the normal at any point P of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, meets the major and minor axes in G and H respectively and C in the centre of the ellipse, then prove that

$$a^2 (CG)^2 + b^2 (CH)^2 = (a^2 - b^2)^2$$

14. If the normal at the point $P(\theta)$ to the ellipse $5x^2 + 14y^2 = 70$ intersects it again at the point $Q(2\theta)$, show that $\cos\theta = -\frac{2}{3}$.

15. The tangent and normal at any point P of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ cut its major axis in point Q and R respectively. If $QR = a$ prove that the eccentric angle of the point P is given by

$$e^2 \cos^2\phi + \cos\phi - 1 = 0$$

Session 3

Pair of Tangents, Chord of Contact, Chord Bisected at a Given Point, Diameter, Conjugate Diameters, Equi-Conjugate Diameters, Director Circle, Sub-Tangent and Sub-Normal, Concyclic Points, Some Standard Properties of the Ellipse, Reflection Property of an Ellipse, Equation of an Ellipse Referred to Two Perpendicular Lines

Pair of Tangents

Theorem : The combined equation of the pair of tangents drawn from a point (x_1, y_1) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

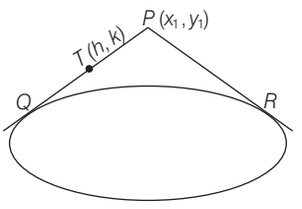
$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2$$

or $$SS_1 = T^2$$

where $$S=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\,;\, S_1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$$

and $$T=\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$$

Proof : Let $T(h, k)$ be any point on the pair of tangents PQ or PR drawn from any external point $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



$\therefore$ Equation of PT is

$$y-y_1=\frac{k-y_1}{h-x_1}(x-x_1)$$

or $$y=\left(\frac{k-y_1}{h-x_1}\right)x+\left(\frac{hy_1-kx_1}{h-x_1}\right)$$

which is the tangent to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$\therefore$ $$c^2=a^2m^2+b^2$$

or $$\left(\frac{hy_1-kx_1}{h-x_1}\right)^2=a^2\left(\frac{k-y_1}{h-x_1}\right)^2+b^2$$

$\Rightarrow$ $$(hy_1-kx_1)^2=a^2(k-y_1)^2+b^2(h-x_1)^2$$

Hence, locus of (h, k) is

$$(xy_1-x_1y)^2=a^2(y-y_1)^2+b^2(x-x_1)^2$$

or $$(xy_1-x_1y)^2=(b^2x^2+a^2y^2)+(b^2x_1^2+a^2y_1^2)-2(b^2xx_1+a^2yy_1)$$

or $$\left(\frac{xy_1-x_1y}{ab}\right)^2=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)+\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right)-2\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}\right)$$

or $$\left(\frac{xy_1-x_1y}{ab}\right)^2-\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)-\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right)+\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}\right)^2$$

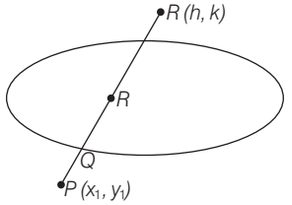
$$= \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2}\right)^2 + 1 - 2\left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2}\right)$$

or $$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right)^2$$

or $$SS_1 = T^2$$

Aliter : Let the ellipse be $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{...(i)}$$

Let $P(x_1, y_1)$ be any point outside the ellipse let a chord of the ellipse through the point $P(x_1, y_1)$ cut the ellipse at Q and let $R(h, k)$ be any arbitrary point on the line PQ (R inside or outside). Let Q divides PR in the ratio $\lambda : 1$ then coordinates of Q is



$$\left(\frac{\lambda h + x_1}{\lambda + 1}, \frac{\lambda k + y_1}{\lambda + 1}\right) \qquad (\because PQ : QR = \lambda : 1)$$

since Q lies on ellipse Eq. (i), then

$$\frac{1}{a^2}\left(\frac{\lambda h + x_1}{\lambda + 1}\right)^2 + \frac{1}{b^2}\left(\frac{\lambda k + y_1}{\lambda + 1}\right)^2 = 1$$

$$\Rightarrow \quad b^2(\lambda h + x_1)^2 + a^2(\lambda k + y_1)^2 = a^2b^2(\lambda + 1)^2$$

$$\Rightarrow \quad (a^2k^2 + b^2h^2 - a^2b^2)\lambda^2 + 2$$

$$(hx_1b^2 + ky_1a^2 - a^2b^2)\lambda + (b^2x_1^2 + a^2y_1^2 - a^2b^2) = 0 \quad \text{...(ii)}$$

Line PR will become tangent to ellipse Eq. (i) then roots of Eq. (ii) are equal

$$\therefore \quad 4(hx_1b^2 + ky_1a^2 - a^2b^2)^2$$

$$- 4(a^2k^2 + b^2h^2 - a^2b^2)(b^2x_1^2 + a^2y_1^2 - a^2b^2) = 0$$

Dividing by $4a^4b^4$

$$\therefore \quad \left(\frac{hx_1}{a^2} + \frac{ky_1}{b^2} - 1\right)^2 = \left(\frac{k^2}{b^2} + \frac{h^2}{a^2} - 1\right)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right)$$

Hence, locus of $R(h, k)$ i.e. equation of pair of tangents from $P(x_1, y_1)$ is

$$\left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right)^2 = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right)$$

i.e. $$T^2 = SS_1 \quad \text{or} \quad SS_1 = T^2$$

Remark

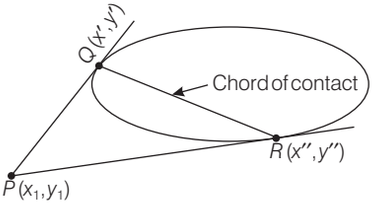
$S = 0$ is the equation of the curve, S_1 is obtained from S by replacing x by x_1 and y by y_1 and $T = 0$ is the equation of tangent at (x_1, y_1) to $S = 0$

Chord of Contact

Theorem : The equation of chord of contact of tangents drawn from a point (x_1, y_1) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

Proof : Let PQ and PR be the tangents drawn from a point $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ such that



$$Q \equiv (x', y') \quad \text{and} \quad R \equiv (x'', y'')$$

are the points of contacts of these tangents the chord QR is called chord of contact of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Equations of tangents at $Q(x', y')$ and $R(x'', y'')$ are

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1$$

and $$\frac{xx''}{a^2} + \frac{yy''}{b^2} = 1, \text{ respectively}$$

These tangents pass through $P(x_1, y_1)$ therefore,

$$\frac{x'x_1}{a^2} + \frac{y'y_1}{b^2} = 1 \quad \text{and} \quad \frac{x''x_1}{a^2} + \frac{y''y_1}{b^2} = 1$$

$$\Rightarrow \quad (x', y') \text{ and } (x'', y'') \text{ lie on } \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

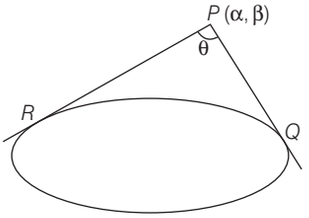
Hence, the equation QR is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

which is same as the equation of tangent but position of point differ.

Example 34 Find the locus of the points of the intersection of tangents to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which make an angle θ .

Sol. Given ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ...(i)

Equation of any tangent to ellipse Eq. (i) in terms of slope (m) is $y=mx+\sqrt{(a^2m^2+b^2)}$



Since, its passes through $P(\alpha,\beta)$ then

$$\beta=m\alpha+\sqrt{(a^2m^2+b^2)}$$

$$\Rightarrow \quad (\beta-m\alpha)=\sqrt{(a^2m^2+b^2)}$$

$$\Rightarrow \quad (\beta-m\alpha)^2=a^2m^2+b^2$$

$$\Rightarrow \quad m^2(a^2-\alpha^2)+2\alpha\beta m+(b^2-\beta^2)=0 \qquad ...(ii)$$

Eq. (ii) being a quadratic equation in m.

Let roots of Eq. (ii) are m_1 and m_2, then

$$\therefore \quad m_1+m_2=-\frac{2\alpha\beta}{(a^2-\alpha^2)},\ m_1m_2=\frac{b^2-\beta^2}{a^2-\alpha^2}$$

$$\therefore \quad (m_1-m_2)=\sqrt{(m_1+m_2)^2-4m_1m_2}$$

$$=\sqrt{\frac{4\alpha^2\beta^2}{(a^2-\alpha^2)^2}-\frac{4(b^2-\beta^2)}{(a^2-\alpha^2)}}$$

$$=\sqrt{\frac{4\alpha^2\beta^2-4(b^2-\beta^2)(a^2-\alpha^2)}{(a^2-\alpha^2)^2}}$$

$$=\sqrt{\frac{4\{\alpha^2\beta^2-a^2b^2+b^2\alpha^2+a^2\beta^2-\alpha^2\beta^2\}}{(a^2-\alpha^2)^2}}$$

$$=\frac{2}{|(a^2-\alpha^2)|}\sqrt{(a^2\beta^2+b^2\alpha^2-a^2b^2)}$$

$\because$ θ be the angle between these two tangents, then

$$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|$$

$$=\left|\frac{\frac{2}{(a^2-\alpha^2)}\sqrt{(a^2\beta^2+b^2\alpha^2-a^2b^2)}}{\left(1+\frac{b^2-\beta^2}{a^2-\alpha^2}\right)}\right|$$

$$\tan\theta=\left|\frac{2\sqrt{(a^2\beta^2+b^2\alpha^2-a^2b^2)}}{a^2+b^2-\alpha^2-\beta^2}\right|$$

or $\quad (a^2+b^2-\alpha^2-\beta^2)^2\tan^2\theta=4\quad(a^2\beta^2+b^2\alpha^2-a^2b^2)$

$\Rightarrow \quad (\alpha^2+\beta^2-a^2-b^2)^2\tan^2\theta=4\quad(b^2\alpha^2+a^2\beta^2-a^2b^2)$

$\therefore$ Locus of $P(\alpha,\beta)$ is

$$(x^2+y^2-a^2-b^2)^2\tan^2\theta=4\quad(b^2x^2+a^2y^2-a^2b^2)$$

Example 35 ***Prove that the chord of contact of tangents drawn from the point*** (h,k) ***to the ellipse*** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ***will subtend a right angle at the centre, if*** $\frac{h^2}{a^4}+\frac{k^2}{b^4}=\frac{1}{a^2}+\frac{1}{b^2}$***. Also, find the locus of*** (h,k) **.**

Sol. The equation of chord of contact of tangents drawn from $P(h,k)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{hx}{a^2}+\frac{ky}{b^2}=1$ ...(i)

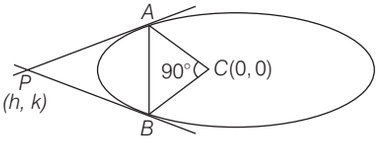
The equation of the straight lines CA and CB is obtained by making homogeneous ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with the help of Eq. (i)

$$\therefore \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\frac{hx}{a^2}+\frac{ky}{b^2}\right)^2$$

$$\Rightarrow \quad \left(\frac{k^2}{a^4}-\frac{1}{a^2}\right)x^2+\left(\frac{k^2}{b^4}-\frac{1}{b^2}\right)y^2+\frac{2hk}{a^2b^2}xy=0 \qquad ...(ii)$$

But given $\angle ACB=90°$

$\therefore$ Coefficient of x^2 + Coefficient of $y^2=0$



$$\Rightarrow \quad \frac{h^2}{a^4}-\frac{1}{a^2}+\frac{k^2}{b^4}-\frac{1}{b^2}=0$$

$$\Rightarrow \quad \frac{h^2}{a^4}+\frac{k^2}{b^4}=\frac{1}{a^2}+\frac{1}{b^2}$$

Hence, locus of (h,k) is

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2}+\frac{1}{b^2}$$

Chord Bisected at a Given Point

Theorem : The equation of a chord of the ellipse

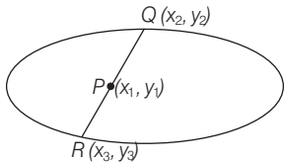
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

bisected at the point (x_1,y_1) is given by

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$$

or $$T=S_1$$

Proof : Let $Q\equiv(x_2,y_2)$ and $R\equiv(x_3,y_3)$ be the end points of a chord QR of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and let $P\equiv(x_1,y_1)$ be its mid point.



Now, $Q \equiv (x_2, y_2)$ and $R \equiv (x_3, y_3)$ lie on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then

$$\frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} = 1 \quad \text{...(i)}$$

and $$\frac{x_3^2}{a^2} + \frac{y_3^2}{b^2} = 1 \quad \text{...(ii)}$$

Subtracting Eq. (ii) from Eq. (i),

$$\therefore \quad \frac{1}{a^2}(x_2^2 - x_3^2) + \frac{1}{b^2}(y_2^2 - y_3^2) = 0$$

$$\Rightarrow \quad \frac{(x_2 + x_3)(x_2 - x_3)}{a^2} + \frac{(y_2 + y_3)(y_2 - y_3)}{b^2} = 0$$

$$\Rightarrow \quad \frac{y_2 - y_3}{x_2 - x_3} = -\frac{b^2(x_2 + x_3)}{a^2(y_2 + y_3)} = -\frac{b^2}{a^2} \cdot \frac{2x_1}{2y_1}$$

$$\left(\because x_1 = \frac{x_2 + x_3}{2} \text{ and } y_1 = \frac{y_2 + y_3}{2} \right)$$

$$= -\frac{b^2 x_1}{a^2 y_1} \quad \text{...(iii)}$$

$\therefore$ Equation of QR is

$$y - y_1 = \frac{y_2 - y_3}{x_2 - x_3}(x - x_1)$$

$$\Rightarrow \quad y - y_1 = -\frac{b^2 x_1}{a^2 y_1}(x - x_1) \quad \text{[from Eq. (iii)]}$$

$$\Rightarrow \quad \frac{yy_1}{b^2} - \frac{y_1^2}{b^2} = -\frac{xx_1}{a^2} + \frac{x_1^2}{a^2}$$

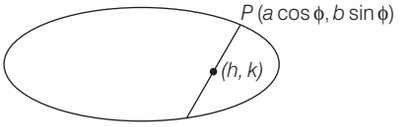
$$\Rightarrow \quad \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 \text{ or } T = S_1$$

where, $$T = \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$$

and $$S_1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$$

Example 36 Prove that the locus of the middle points of normal chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is the curve $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 \left(\frac{a^6}{x^2} + \frac{b^6}{y^2}\right) = (a^2 - b^2)^2$.

Sol. Let (h, k) be the middle point of any chord of an ellipse, then its equation is $T = S_1$



or $$\frac{xh}{a^2} + \frac{yk}{b^2} - 1 = \frac{h^2}{a^2} + \frac{k^2}{b^2} - 1 \text{ or } \frac{xh}{a^2} + \frac{yk}{b^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2} \quad \text{...(i)}$$

If Eq. (i) is a normal chords, then it must be of the form

$$ax \sec \phi - by \operatorname{cosec} \phi = a^2 - b^2 \quad \text{...(ii)}$$

Thus, the Eqs. (i) and (ii) represents the same normal chord of the ellipse with its middle point (h, k).

Hence, they are identical and comparing their co-efficients,

we get $$\frac{h/a^2}{a\sec \phi} = \frac{k/b^2}{-b \operatorname{cosec} \phi} = \frac{\frac{h^2}{a^2} + \frac{k^2}{b^2}}{(a^2 - b^2)}$$

$$\Rightarrow \quad \cos \phi = \frac{a^3}{h} \frac{\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)}{(a^2 - b^2)} \quad \text{...(iii)}$$

and $$\sin \phi = -\frac{b^3}{k} \frac{\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)}{(a^2 - b^2)} \quad \text{...(iv)}$$

Squaring and adding Eqs. (iii) and (iv), then

$$\cos^2 \phi + \sin^2 \phi = \frac{\left(\frac{a^6}{h^2} + \frac{b^6}{k^2}\right)\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)^2}{(a^2 - b^2)^2}$$

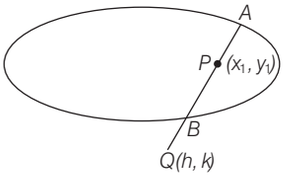
$$\Rightarrow \quad 1 = \frac{\left(\frac{a^6}{h^2} + \frac{b^6}{k^2}\right)\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)^2}{(a^2 - b^2)^2}$$

$$\Rightarrow \quad \left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)^2 \left(\frac{a^6}{h^2} + \frac{b^6}{k^2}\right) = (a^2 - b^2)^2$$

Hence, locus of (h, k) is $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 \left(\frac{a^6}{x^2} + \frac{b^6}{y^2}\right) = (a^2 - b^2)^2$

Example 37 Show that the locus of the middle points of chords of an ellipse which pass through a fixed point, is another ellipse

Sol. Let $P(x_1, y_1)$ be the middle point of any chord AB of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then equation of chord AB is



$$T = S_1$$

$$\Rightarrow \quad \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$$

$$\Rightarrow \quad \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} \qquad \text{... (i)}$$

But it passes through a fixed point $Q(h, k)$ its coordinates must satisfy Eq. (i),

$$\therefore \quad \frac{hx_1}{a^2} + \frac{ky_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$$

This can be re-written as

$$\frac{\left(x_1 - \frac{h}{2}\right)^2}{a^2} + \frac{\left(y_1 - \frac{k}{2}\right)^2}{b^2} = \frac{1}{4}\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)$$

Hence, locus of $P(x_1, y_1)$ is

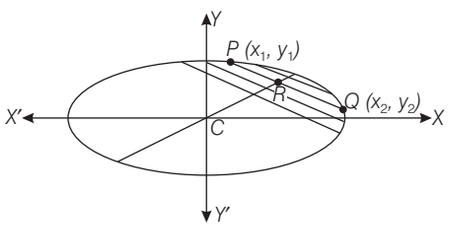
$$\frac{\left(x - \frac{h}{2}\right)^2}{a^2} + \frac{\left(y - \frac{k}{2}\right)^2}{b^2} = \frac{1}{4}\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)$$

Its obviously an ellipse with centre at $\left(\frac{h}{2}, \frac{k}{2}\right)$ and axes parallel to coordinates axes.

Diameter

The locus of the middle points of a system of parallel chords of an ellipse is called a diameter and the point where the diameter intersects the ellipse is called the vertex of the diameter.

Let $y = mx + c$ be system of parallel chords to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for different chords c varies, m remains constant.



Let the extremities of any chord PQ of the set be $P(x_1, y_1)$ and $Q(x_2, y_2)$ and let its middle point be $R(h, k)$, then solving equations.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ and } y = mx + c$$

$$\therefore \quad \frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} = 1$$

$$\Rightarrow \quad (a^2m^2 + b^2)\, x^2 + 2mca^2 x + a^2\,(c^2 - b^2) = 0$$

Since, x_1 and x_2 be the roots of this equation, then

$$x_1 + x_2 = -\frac{2mca^2}{a^2m^2 + b^2} \qquad \text{...(i)}$$

Since, (h, k) be the middle point of QR, then

$$h = \frac{x_1 + x_2}{2}$$

then, from Eq. (i),

$$2h = -\frac{2mca^2}{a^2m^2 + b^2}$$

$$\Rightarrow \quad h = -\frac{mca^2}{a^2m^2 + b^2} \qquad \text{...(ii)}$$

but (h, k) lies on $y = mx + c$

$$\therefore \quad k = mh + c, \;\; c = k - mh \qquad \text{...(iii)}$$

From Eqs. (ii) and (iii), then , $h = -\frac{ma^2\,(k - mh)}{a^2m^2 + b^2}$

$$\Rightarrow \quad a^2m^2h + b^2h = -mka^2 + m^2a^2h$$

$$\Rightarrow \quad b^2h = -mka^2 \text{ or } k = -\frac{b^2h}{a^2m}$$

Hence, locus of $R(h, k)$ is $y = -\frac{b^2x}{a^2m}$

which is diameter of the ellipse passing through $(0, 0)$.

Aliter : Let (h, k) be the middle point of the chord $y = mx + c$ of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then

$$T = S_1$$

$$\Rightarrow \quad \frac{xh}{a^2} + \frac{ky}{b^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2}$$

$$\therefore \quad \text{Slope} = -\frac{b^2h}{a^2k} = m$$

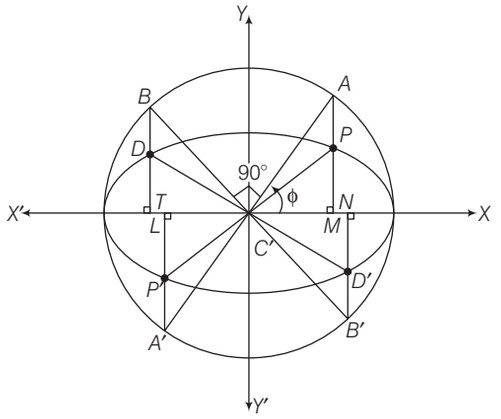
$$\Rightarrow \quad k = -\frac{b^2h}{a^2m}$$

Hence, locus of the mid-point is

$$y = -\frac{b^2x}{a^2m}$$

Conjugate Diameters

Two diameters are said to be *conjugate* when each bisects all chords parallel to the other. If $y = mx$ and $y = m_1x$ be two conjugate diameters of an ellipse, then $mm_1 = \frac{b^2}{a^2}$.



Conjugate diameters of circle i.e. AA' and BB' are perpendicular to each other. Hence, conjugate diameters of ellipse are PP' and DD'.

Hence, angle between conjugate diameters of ellipse $\neq 90°$.

Now the co-ordinates of the four extremities of two conjugate diameters are

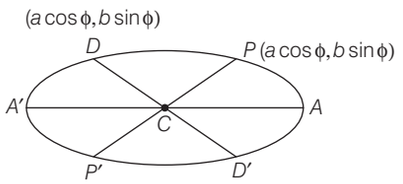
$$\boldsymbol{P\ (a\cos\phi,\ b\sin\phi),\ P'\ (-a\cos\phi, -b\sin\phi),}$$

$$\boldsymbol{D\ (-a\sin\phi,\ b\cos\phi),\ \ D'\ (a\sin\phi, -b\cos\phi)}$$

Properties of Conjugate Diameters

Prop. 1 : *The eccentric angles of the ends of a pair of conjugate diameters of an ellipse differ by a right angle.*

Let PCP' and DCD' be two conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and let the eccentric angles of the



extremities P and D be ϕ and ϕ' respectively. Then, the co-ordinates of P and D are $(a\cos\phi, b\sin\phi)$ and $(a\cos\phi', b\sin\phi')$ respectively.

Now $m_1 =$ slope of $CP = \frac{b}{a}\tan\phi$

and $m_2 =$ slope of $CD = \frac{b}{a}\tan\phi'$

since, the diameters PCP' and DCD' are conjugate diameters.

$$\therefore \qquad m_1 m_2 = -\frac{b^2}{a^2}$$

$$\Rightarrow \qquad \frac{b^2}{a^2}\tan\phi\tan\phi' = -\frac{b^2}{a^2}$$

$$\Rightarrow \qquad \tan\phi\tan\phi' = -1$$

$$\Rightarrow \qquad \tan\phi = -\cot\phi' = \tan\left(\frac{\pi}{2}+\phi'\right)$$

$$\Rightarrow \qquad \phi = \frac{\pi}{2}+\phi' \ \Rightarrow\ \phi-\phi' = \frac{\pi}{2}$$

Prop. 2 : *The sum of the squares of any two conjugate semi diameters of an ellipse is constant and equal to the sum of the squares of the semi-axes of the ellipse* i.e.

$$CP^2 + CD^2 = a^2 + b^2$$

Let CP and CD be two conjugate semi-diameters of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and let eccentric angle of P is ϕ. The eccentric angle of D is $\frac{\pi}{2}+\phi$. So the coordinates of P and D are

$$(a\cos\phi, b\sin\phi) \quad \text{and} \quad \left(a\cos\left(\frac{\pi}{2}+\phi\right), b\sin\left(\frac{\pi}{2}+\phi\right)\right)$$

i.e. $(-a\sin\phi, b\cos\phi)$ respectively

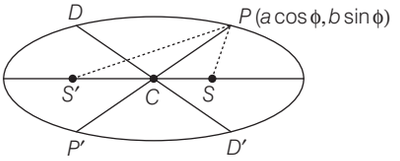
$$\therefore \qquad CP^2 + CD^2 = (a^2\cos^2\phi + b^2\sin^2\phi) + (a^2\sin^2\phi + b^2\cos^2\phi)$$

$$= a^2 + b^2$$

Prop. 3 : *The product of the focal distances of a point on an ellipse is equal to the square of the semi diameter which is conjugate to the diameter through the point.*

Let PCP' and DCD' be the conjugate diameters of an ellipse and let the eccentric angle of P is ϕ then coordinates of P is $(a\cos\phi, b\sin\phi)$

$\therefore$ Coordinates of D is $(-a\sin\phi, b\cos\phi)$



Let S and S' be two foci of the ellipse. Then

$$SP\cdot S'P = (a-ae\cos\phi)\cdot(a+ae\cos\phi)$$

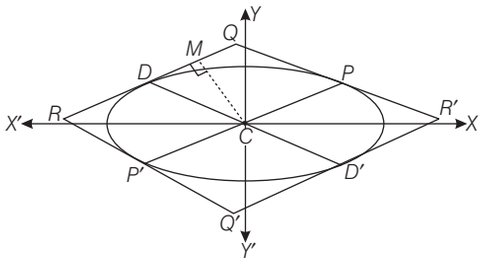
$$= a^2 - a^2e^2\cos^2\phi$$

$$= a^2 - (a^2-b^2)\cos^2\phi \quad \{\because b^2 = a^2(1-e^2)\}$$

$$= a^2\sin^2\phi + b^2\cos^2\phi = CD^2$$

$$\{\therefore a^2 - b^2 = a^2e^2\}$$

Prop. 4 : *The tangents at the extremities of a pair of conjugate diameters form a parallelogram whose area is constant and equal to product of the axes.*



Let PCP' and DCD' be a pair of conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Let the eccentric angle of P be ϕ. Then the eccentric angle of D is $\left(\frac{\pi}{2}+\phi\right)$ so the coordinates of P and D are

$$(a\cos\phi, b\sin\phi) \text{ and } \left(a\cos\left(\frac{\pi}{2}+\phi\right), b\sin\left(\frac{\pi}{2}+\phi\right)\right)$$

i.e. $(-a\sin\phi, b\cos\phi)$

Similarly the coordinates of P' and D' are

$(-a\cos\phi, -b\sin\phi)$ and $(a\sin\phi, -b\cos\phi)$ respectively.

Equation of tangents at P, D, P' and D' are respectively.

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1, -\frac{x}{a}\sin\phi+\frac{y}{b}\cos\phi=1,$$

$$-\frac{x}{a}\cos\phi-\frac{y}{b}\sin\phi=1$$

and $\frac{x}{a}\sin\phi-\frac{y}{b}\cos\phi=1$

Clearly the tangents at P and P' are parallel. Also, the tangents at D and D' are parallel. Hence, the tangents at P, D, P', D' form a parallelogram.

Area of parallelogram $QRQ'R' = 4$ (the area of parallelogram $QDCP$)

$$= 4\cdot|QD|\cdot\{\perp \text{ from } C \text{ on } QD\}$$

$$= 4\cdot|CP|\cdot\{\perp \text{ from } C \text{ on } QD\} \quad \ldots(i)$$

Now $|CP| = \sqrt{(a^2\cos^2\phi + b^2\sin^2\phi)}$

$\because$ tangent at D is $-\frac{x}{a}\sin\phi+\frac{y}{b}\cos\phi=1$

$\therefore$ $\perp$ from C on

$$QD = \frac{1}{\sqrt{\frac{\sin^2\phi}{a^2}+\frac{\cos^2\phi}{b^2}}} = \frac{ab}{\sqrt{a^2\cos^2\phi+b^2\sin^2\phi}}$$

Now from Eq. (i),

Area of parallelogram $QRQ'R'$

$$= 4\times\sqrt{(a^2\cos^2\phi+b^2\sin^2\phi)}\times\frac{ab}{\sqrt{a^2\cos^2\phi+b^2\sin^2\phi)}}$$

$$= 4ab\,(=\text{constant}) = (2a)\,(2b)$$

= Area of rectangle contained under major and minor axes.

Prop. 5 : *The polar of any point with respect to ellipse is parallel to the diameter to the one on which the point lies. Hence obtain the equation of the chord whose mid-point is* (h, k).

Let (h, k) be the point on the diameter $y = m_1 x$

$\therefore$ $m_1 = k/h$

any diameter conjugate to it is $y = m_2 x$

but $m_1 m_2 = -\frac{b^2}{a^2} \Rightarrow \frac{k}{h}m_2 = -\frac{b^2}{a^2}$

$\therefore$ $m_2 = -\frac{b^2 h}{a^2 k}$

Polar of (h, k) is $\frac{hx}{a^2}+\frac{ky}{b^2}=1$ ...(i)

Its slope is $-\frac{b^2 h}{a^2 k} = m_2$ and hence parallel.

Now, equation of chord parallel to the Eq. (i) is

$$\frac{hx}{a^2}+\frac{ky}{b^2}=\lambda \quad \ldots(ii)$$

It is passes through points (h, k)

$\therefore$ $\lambda = \frac{h^2}{a^2}+\frac{k^2}{b^2}$ ... (iii)

From Eqs. (ii) and (iii), $\frac{hx}{a^2}+\frac{ky}{b^2}=\frac{h^2}{a^2}+\frac{k^2}{b^2}$

i.e. $T = S_1$

which is the equation of chord of the ellipse, if mid-points is (h, k).

Equi-Conjugate Diameters

Two conjugate diameters are called *equi-conjugate* if their lengths are equal. In such cases therefore.

$$(CP)^2 = (CD)^2$$

$\therefore$ $a^2\cos^2\phi + b^2\sin^2\phi = a^2\sin^2\phi + b^2\cos^2\phi$

$\Rightarrow$ $(a^2\cos^2\phi - \sin^2\phi) - b^2(\cos^2\phi - \sin^2\phi) = 0$

$$\Rightarrow \quad (a^2 - b^2)\cos 2\phi = 0$$

$$\because \quad (a^2 - b^2) \neq 0$$

$$\therefore \quad \cos 2\phi = 0$$

$$\therefore \quad \phi = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}$$

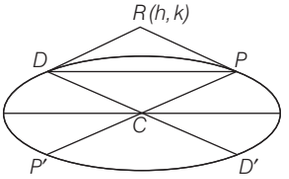
$$\therefore \quad (CP) = (CD) = \sqrt{\frac{(a^2 + b^2)}{2}}$$

Example 38 Show that the tangents at the ends of conjugate diameters of the ellipse $x^2/a^2 + y^2/b^2 = 1$ intersect on the ellipse $x^2/a^2 + y^2/b^2 = 2$.

Or

Prove that the locus of the poles of the line joining the extremities of two conjugate diameters is the ellipse $x^2/a^2 + y^2/b^2 = 2$.

Sol. Let CP and CD be two semi-conjugate diameters, so that if eccentric angle of P is ϕ then eccentric angle of D is $\frac{\pi}{2} + \phi$



$\therefore$ Coordinates of P and D are

$(a\cos\phi, b\sin\phi)$ and $\left(a\cos\left(\frac{\pi}{2} + \phi\right), b\sin\left(\frac{\pi}{2} + \phi\right)\right)$

respectively

$\therefore$ Equation of (PD) is

$$\frac{x}{a}\cos\left(\frac{\phi + \frac{\pi}{2} + \phi}{2}\right) + \frac{y}{b}\sin\left(\frac{\phi + \frac{\pi}{2} + \phi}{2}\right) = \cos\left(\frac{\frac{\pi}{2} + \phi - \phi}{2}\right)$$

$$\Rightarrow \quad \frac{x}{a}\cos\left(\frac{\pi}{4} + \phi\right) + \frac{y}{b}\sin\left(\frac{\pi}{4} + \phi\right) = \frac{1}{\sqrt{2}} \qquad \ldots(i)$$

If its pole or point of intersection of tangents at its extremities be (h, k), then its equation is the same as that of the polar or the chord of contact of (h, k).

i.e. $$\frac{hx}{a^2} + \frac{ky}{b^2} = 1 \qquad \ldots(ii)$$

Since, Eqs. (i) and (ii) are identical, comparing

$$\frac{h}{a\cos\left(\frac{\pi}{4} + \phi\right)} = \frac{k}{b\sin\left(\frac{\pi}{4} + \phi\right)} = \sqrt{2}$$

or $$\sqrt{2}\cos\left(\frac{\pi}{4} + \phi\right) = \frac{h}{a} \qquad \ldots(iii)$$

or $$\sqrt{2}\sin\left(\frac{\pi}{4} + \phi\right) = \frac{k}{b} \qquad \ldots(iv)$$

Squaring and adding Eqs. (iii) and (iv), then

$$\frac{h^2}{a^2} + \frac{k^2}{b^2} = 2\left(\cos^2\left(\frac{\pi}{4} + \phi\right) + \sin^2\left(\frac{\pi}{4} + \phi\right)\right)$$

$$\therefore \quad \frac{h^2}{a^2} + \frac{k^2}{b^2} = 2$$

Hence, locus of (h, k) is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$

Aliter : Equation of tangents at P and D are

$$\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1 \qquad \ldots(i)$$

and $$\frac{x}{a}\cos\left(\frac{\pi}{2} + \phi\right) + \frac{y}{b}\sin\left(\frac{\pi}{2} + \phi\right) = 1$$

i.e. $$-\frac{x}{a}\sin\phi + \frac{y}{b}\cos\phi = 1 \qquad \ldots(ii)$$

Squaring and adding Eqs. (i) and (ii), we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$$

which is required locus.

Example 39 If $x\cos\alpha + y\sin\alpha = p$ is a chord joining the ends P and D of conjugate semi-diameters, of the ellipse then prove that $a^2\cos^2\alpha + b^2\sin^2\alpha = 2p^2$ and hence or otherwise deduce that the line PD always touches a similar ellipse.

Sol. Let equation of ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, eccentric angle of P is ϕ, then eccentric angle of D is $\frac{\pi}{2} + \phi$

$\therefore$ Coordinates of P and D are

$(a\cos\phi, b\sin\phi)$ and $\left(a\cos\left(\frac{\pi}{2} + \phi\right), b\sin\left(\frac{\pi}{2} + \phi\right)\right)$

$\therefore$ Equation of PD is

$$\frac{x}{a}\cos\left(\frac{\phi + \frac{\pi}{2} + \phi}{2}\right) + \frac{y}{b}\sin\left(\frac{\phi + \frac{\pi}{2} + \phi}{2}\right) = \cos\left(\frac{\frac{\pi}{2} + \phi - \phi}{2}\right)$$

$$\Rightarrow \quad \frac{x}{a}\cos\left(\frac{\pi}{4} + \phi\right) + \frac{y}{b}\sin\left(\frac{\pi}{4} + \phi\right) = \frac{1}{\sqrt{2}} \qquad \ldots(i)$$

If it is same as $x\cos\alpha + y\sin\alpha = p$ $\qquad \ldots(ii)$

then on comparing, we get

$$\frac{\cos\left(\frac{\pi}{4} + \phi\right)}{a\cos\alpha} = \frac{\sin\left(\frac{\pi}{4} + \phi\right)}{b\sin\alpha} = \frac{1}{p\sqrt{2}}$$

or $a\cos\alpha = p\sqrt{2}\cos\left(\frac{\pi}{4}+\phi\right)$ ...(iii)

and $b\sin\alpha = p\sqrt{2}\sin\left(\frac{\pi}{4}+\phi\right)$ ...(iv)

Squaring and adding Eq. (iii) and (iv), we get

$$a^2\cos^2\alpha + b^2\sin^2\alpha = (p\sqrt{2})^2 \left\{\cos^2\left(\frac{\pi}{4}+\phi\right)+\sin^2\left(\frac{\pi}{4}+\phi\right)\right\}$$

$\Rightarrow$ $a^2\cos^2\alpha + b^2\sin^2\alpha = 2p^2$

Again, line Eq. (i) can be written as

$$\frac{x}{a/\sqrt{2}}\cos\theta + \frac{y}{b/\sqrt{2}}\sin\theta = 1,$$

where $\theta = \frac{\pi}{4}+\phi$

which is clearly a tangent to the ellipse

$$\frac{x^2}{(a/\sqrt{2})^2}+\frac{y^2}{(b\sqrt{2})^2}=1 \text{ or } \frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{2} \quad ...(v)$$

If e' be its eccentricity, then

$$(b/\sqrt{2})^2 = (a/\sqrt{2})^2(1-e'^2)$$

$\Rightarrow$ $b^2 = a^2(1-e'^2)$

but $b^2 = a^2(1-e^2)$

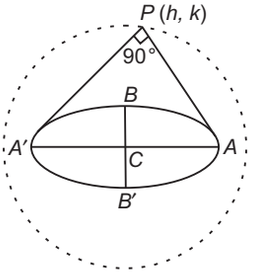
$\therefore$ $e = e'$

Hence, ellipse (v) is a similar ellipse.

Director Circle

The locus of the point of intersection of the tangents to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which are perpendicular to each other is called director circle.

Let any tangent in terms of slope of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is



$$y = mx + \sqrt{(a^2m^2+b^2)}$$

It is passes through (h, k)

$\therefore$ $k = mh + \sqrt{(a^2m^2+b^2)}$

or $(k-mh)^2 = a^2m^2 + b^2$

$\Rightarrow$ $k^2 + m^2h^2 - 2mhk = a^2m^2 + b^2$

$\Rightarrow$ $m^2(h^2-a^2) - 2hkm + k^2 - b^2 = 0$

It is quadratic equation in m let slope of two tangents are m_1 and m_2

$\therefore$ $m_1m_2 = \frac{k^2-b^2}{h^2-a^2}$

$$-1 = \frac{k^2-b^2}{h^2-a^2}$$

($\because$ tangents are perpendicular)

$\Rightarrow$ $-h^2 + a^2 = k^2 - b^2$

or $h^2 + k^2 = a^2 + b^2$

Hence, locus of $P(h, k)$ is

$$\boldsymbol{x^2 + y^2 = a^2 + b^2}$$

Aliter :

If any tangent $y = mx + \sqrt{(a^2m^2+b^2)}$ ...(i)

and $y = -\frac{x}{m} + \sqrt{\left\{a^2\left(-\frac{1}{m}\right)^2 + b^2\right\}}$ ...(ii)

touch the ellipse and intersect at right angles.

From Eq. (i),

$$y - mx = \sqrt{(a^2m^2+b^2)} \quad ...(iii)$$

Eq. (ii) can be re-written as

$$x + my = \sqrt{(a^2+b^2m^2)} \quad ...(iv)$$

Squaring and adding Eqs. (iii) and (iv), then

$$(y-mx)^2 + (x+my)^2 = a^2m^2 + b^2 + a^2 + b^2m^2$$

$\Rightarrow$ $(1+m^2)(x^2+y^2) = (1+m^2)(a^2+b^2)$

Hence, $x^2 + y^2 = a^2 + b^2$ is the **director circle** of the ellipse.

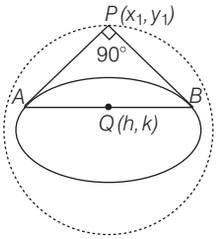
Example 40 ***Tangents at right angles are drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Show that the locus of the middle points of the chord of contact is the curve***

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2 = \frac{x^2+y^2}{a^2+b^2}$$

Sol. Let $Q(h, k)$ be the middle point of the chord of contact

$\therefore$ Equation of chord AB whose mid point $Q(h, k)$ is

$T = S_1$



$$\frac{hx}{a^2} + \frac{ky}{b^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2} \quad \text{...(i)}$$

and equation of chord of contact AB with respect to $P(x_1, y_1)$ is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \quad \text{...(ii)}$$

the Eqs. (i) and (ii) are identical, hence comparing their coefficient, we get

$$\frac{x_1}{h} = \frac{y_1}{k} = \frac{1}{\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)}$$

$$\therefore \quad x_1 = \frac{h}{\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)} \quad \text{...(iii)}$$

and $$y_1 = \frac{k}{\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)} \quad \text{...(iv)}$$

Since tangents are at right angles, then the point (x_1, y_1) must lie on the director circle $x^2 + y^2 = a^2 + b^2$ of the ellipse

$$x_1^2 + y_1^2 = a^2 + b^2$$

$$\frac{h^2}{\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)^2} + \frac{k^2}{\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)^2} = a^2 + b^2$$

[from Eqs. (iii) and (iv)]

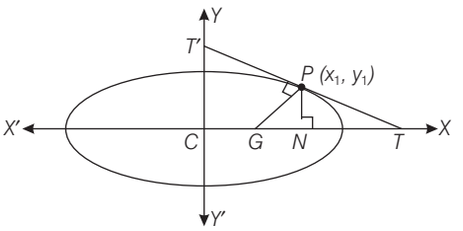
$$\therefore \quad \left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)^2 = \frac{h^2 + k^2}{a^2 + b^2}$$

Hence, locus of mid-point $Q(h, k)$ is $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = \frac{x^2 + y^2}{a^2 + b^2}$

Sub-Tangent and Sub-Normal

Let the tangent and normal at $P(x_1, y_1)$ meet the axes at T and G respectively.

Equation of tangent at $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is



$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \quad \text{...(i)}$$

$\because$ T lies on X-axis.

Put $y = 0$ in Eq. (i) $\Rightarrow x = CT$

$$\therefore \quad CT = \frac{a^2}{x_1} \text{ and } CN = x_1$$

and **length of sub-tangent** $NT = CT - CN = \frac{a^2}{x_1} - x_1$

Equation of normal at $P(x_1, y_1)$ to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is}$$

$$\frac{x - x_1}{x_1 / a^2} = \frac{y - y_1}{y_1 / b^2} \quad \text{...(ii)}$$

$\because$ G lies on X-axis. Put $y = 0$ in Eq. (ii)

$$\Rightarrow \quad x = CG$$

$$\therefore \quad CG = x_1 - \frac{b^2}{a^2} x_1$$

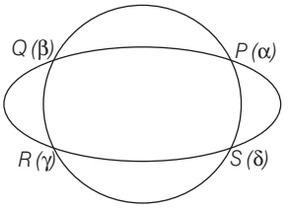
$\therefore$ **Length of sub-normal**

$$GN = CN - CG = x_1 - \left(x_1 - \frac{b^2}{a^2} x_1\right)$$

$$= \frac{b^2}{a^2} x_1 = (1 - e^2)\, x_1$$

Concyclic Points

Any circle intersects an ellipse in two or four real points. They are called concyclic points and the sum of their eccentric angles is an even multiple of π. If $\alpha, \beta, \gamma, \delta$ be the eccentric angles of the four concyclic points on an ellipse, then prove that $\alpha + \beta + \gamma + \delta = 2n\pi$ where n is any integer.



Let the given circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad ...(i)$$

and the given ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad ...(ii)$$

Let $(a\cos\phi, b\sin\phi)$ be a point of intersection of Eqs. (i) and (ii).

As it lies on the circle Eq. (i).

$$\therefore \quad a^2\cos^2\phi + b^2\sin^2\phi + 2ga\cos\phi + 2fb\sin\phi + c = 0 \quad ...(iii)$$

$$\text{or} \quad a^2\left(\frac{1-\tan^2(\phi/2)}{1+\tan^2(\phi/2)}\right)^2 + b^2\left(\frac{2\tan(\phi/2)}{1+\tan^2(\phi/2)}\right)^2 + 2ga\left(\frac{1-\tan^2(\phi/2)}{1+\tan^2(\phi/2)}\right) + 2fb\left(\frac{2\tan(\phi/2)}{1+\tan^2(\phi/2)}\right) + c = 0 \quad ...(iv)$$

Put $\tan(\phi/2) = t$

$\therefore$ Eq. (iv) reduces to

$$a^2\left(\frac{1-t^2}{1+t^2}\right)^2 + b^2\left(\frac{2t}{1+t^2}\right)^2 + 2ga\left(\frac{1-t^2}{1+t^2}\right) + 2fb\left(\frac{2t}{1+t^2}\right) + c = 0$$

$$\text{or} \quad (a^2 - 2ga + c)t^4 + 4bf\,t^3 + (4b^2 - 2a^2 + 2c)\,t^2 + 4bft + (a^2 + 2ga + c) = 0 \quad ...(v)$$

which is biquadratic equation in t.

i.e. it has four values of t

$$t = \tan(\phi/2)$$

Since, four values of eccentric angles are $\alpha, \beta, \gamma, \delta$

$$\therefore \quad \tan\left(\frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} + \frac{\delta}{2}\right) = \frac{S_1 - S_3}{1 - S_2 + S_4} = \frac{\Sigma t_1 - \Sigma t_1 t_2 t_3}{1 - \Sigma t_1 t_2 + t_1 t_2 t_3 t_4} = 0$$

$$\Rightarrow \quad \frac{1}{2}(\alpha + \beta + \gamma + \delta) = n\pi$$

$\therefore \quad \alpha + \beta + \gamma + \delta = 2n\pi$, when n is any integer.

Aliter : Let P, Q, R, S be four concyclic points on an ellipse, whose eccentric angles $\alpha, \beta, \gamma, \delta$ respectively.

Then equation of the chords PQ and RS are (Take any two chords)

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) - \cos\left(\frac{\alpha-\beta}{2}\right) = 0$$

and

$$\frac{x}{a}\cos\left(\frac{\gamma+\delta}{2}\right) + \frac{y}{b}\sin\left(\frac{\gamma+\delta}{2}\right) - \cos\left(\frac{\gamma-\delta}{2}\right) = 0$$

Now, the equation of any curve passing through P, Q, R and S is given by

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) + \lambda \left(\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) - \cos\left(\frac{\alpha-\beta}{2}\right)\right) \times \left(\frac{x}{a}\cos\left(\frac{\gamma+\delta}{2}\right) + \frac{y}{b}\sin\left(\frac{\gamma+\delta}{2}\right) - \cos\left(\frac{\gamma-\delta}{2}\right)\right) = 0$$

But the given points are concyclic. Hence this equation will represent a circle, if co-efficient of x^2 = co-efficient y^2. and co-efficient of $xy = 0$

Now equation of the co-efficient of $xy = 0$

$$\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\gamma+\delta}{2}\right) + \sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\gamma+\delta}{2}\right) = 0$$

$$\text{or} \quad \sin\left(\frac{\alpha+\beta+\gamma+\delta}{2}\right) = 0 = \sin n\pi$$

$$\therefore \quad \frac{1}{2}(\alpha+\beta+\gamma+\delta) = n\pi \quad \text{or} \quad \alpha+\beta+\gamma+\delta = 2n\pi$$

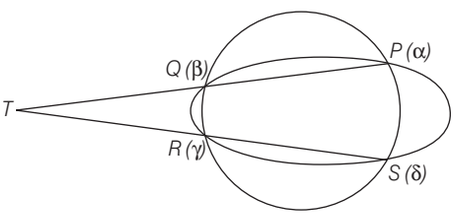
where, n is any integer.

Hence, the sum of eccentric angles of four concylic points on an ellipse is always an even multiple of π

Corollary 1 : *Prove that the common chords of a circle and an ellipse are equally inclined to the axes of the ellipse.*

If the point of intersection of chords PQ and RS is T, then equation of chord PQ is

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$$



$$\therefore \text{ Slope of } PQ = -\frac{b}{a}\cot\left(\frac{\alpha+\beta}{2}\right) = -\frac{b}{a}\cot\left(n\pi - \frac{\gamma+\delta}{2}\right) \quad (\because \alpha+\beta+\gamma+\delta = 2n\pi) = \frac{b}{a}\cot\left(\frac{\gamma+\delta}{2}\right) = -(\text{slope of } RS)$$

Hence, PQ and RS are equally inclined to the axis of x.

Corollary 2 : *Find the centre of the circle passing through the three points on an ellipse whose eccentric angles are* α, β, γ.

Let the point of intersection of ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad ...(i)$$

and circle $x^2+y^2+2gx+2fy+c=0$ ...(ii)

be $\alpha, \beta, \gamma, \delta$

$\therefore \quad \alpha+\beta+\gamma+\delta=2n\pi$ (where n is an integer)

Let ϕ be any point on Eq. (i)

$\therefore \quad x=a\cos\phi, y=b\sin\phi$

This point also lie on Eq. (ii)

$\therefore \quad a^2\cos^2\phi+b^2\sin^2\phi+2ga\cos\phi+2fb\sin\phi+c=0$...(iii)

$\Rightarrow \quad \{(a^2-b^2)\cos^2\phi+2ga\cos\phi+(b^2+c)\}^2=4f^2b^2(1-\cos^2\phi)$

$\Rightarrow (a^2-b^2)^2\cos^4\phi+4ga(a^2-b)\cos^3\phi+\{2(a^2-b^2)(b^2+c)+4g^2a^2+4f^2b^2\}\cos^2\phi+4ga(b^2+c)\cos\phi+\{b^2+c^2-4f^2b^2\}=0$

This is a fourth degree equation in $\cos\phi$.

It has four roots (i.e. $\cos\alpha, \cos\beta, \cos\gamma, \cos\delta$)

$$\therefore \quad \cos\alpha+\cos\beta+\cos\gamma+\cos\delta=-\frac{4ga}{(a^2-b^2)} \qquad ...(iv)$$

Similarly changing Eq. (iii) in $\sin\phi$, we get

$$\sin\alpha+\sin\beta+\sin\gamma+\sin\delta=-\frac{4fb}{b^2-a^2} \qquad ...(v)$$

$\because \quad \alpha+\beta+\gamma+\delta=2n\pi$

$\therefore \quad \delta=2n\pi-(\alpha+\beta+\gamma)$

$\therefore \quad \sin\delta=-\{\sin(\alpha+\beta+\gamma)\}$ and $\cos\delta=\cos(\alpha+\beta+\gamma)$

then, from Eqs. (iv) and (v), we get

$$-g=\left(\frac{a^2-b^2}{4a}\right)\{\cos\alpha+\cos\beta+\cos\gamma+\cos(\alpha+\beta+\gamma)\}$$

and $$-f=\left(\frac{b^2-a^2}{4b}\right)\{\sin\alpha+\sin\beta+\sin\gamma-\sin(\alpha+\beta+\gamma)\}$$

which give co-ordinate of centre of circle through α, β and γ.

Corollary 3 : *$P'CP$ and $D'CD$ are conjugate diameters of an ellipse and α is the eccentric angle of P. Prove that the eccentric angle of the point where the circle through P, P', D again cuts the ellipse is* $\frac{\pi}{2}-3\alpha$.

The eccentric angles of P, P' and D are $\alpha, \pi+\alpha, \frac{\pi}{2}+\alpha$ respectively. Let β be the eccentric angle of the fourth point.

As above

$$\alpha+(\pi+\alpha)+\left(\frac{\pi}{2}+\alpha\right)+\beta=2n\pi$$

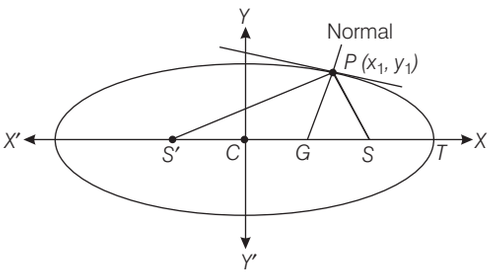
$$\therefore \quad \beta=2n\pi-\left(\frac{3\pi}{2}+3\alpha\right)=\frac{\pi}{2}-3\alpha \qquad (\text{for } n=1)$$

Note

Any other values of n gives the same point on the ellipse.

Some Standard Properties of the Ellipse

(i) If S be the focus and G be the point where the normal at P meets the axis of an ellipse, then $SG=e\cdot SP$ and the tangent and normal at P bisects the external and internal angles between the focal distances of P.



Let P be any point on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$\therefore$ Equation of normals PG is

$$(x-x_1)\frac{a^2}{x_1}=(y-y_1)\frac{b^2}{y_1}$$

Putting $y=0$. For the point G, we have

$$(x-x_1)\frac{a^2}{x_1}=-b^2$$

$$\therefore x=CG=\left(\frac{a^2-b^2}{a^2}\right)x_1=\frac{a^2e^2}{a^2}x_1=e^2x_1$$

$$\therefore \quad SG=CS-CG=ae-e^2x_1=e(a-ex_1)$$

$$=eSP$$

Similarly $S'G=eS'P$

$$\therefore \quad \frac{SG}{S'G}=\frac{eSP}{eS'P}=\frac{SP}{S'P}$$

$\therefore$ The normal PG bisects the internal $\angle SPS'$ between the focal distances but tangent and normal are at right angles, the tangent PT bisects the external angle SPL between them.

(ii) The locus of the feet of the perpendiculars from the foci on any tangent to an ellipse is the auxiliary circle.

The equation of any tangent in terms slope (m) of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$y = mx \pm \sqrt{(a^2m^2 + b^2)}$$

or $$y - mx = \sqrt{(a^2m^2 + b^2)} \quad \text{...(i)}$$

Equation of perpendicular line of Eq. (i) and passes through $(\pm ae, 0)$ is

$$my + x = \pm ae \quad \text{...(ii)}$$

The locus of the point of intersection of the line given by Eqs. (i) and (ii) can be obtained by eliminating m between them, squaring and adding Eqs. (i) and (ii), we get

$$y^2(1+m^2) + x^2(1+m^2) = a^2m^2 + b^2 + a^2e^2$$

$$\Rightarrow \quad (1+m^2)(x^2+y^2) = a^2m^2 + a^2$$

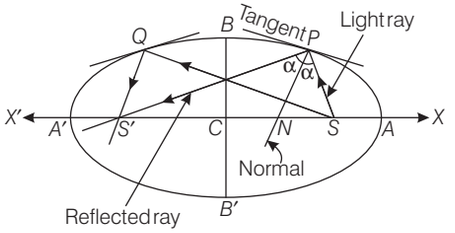
$$= a^2(1+m^2)$$

or $$x^2 + y^2 = a^2$$

which is the equation of the auxiliary circle of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Reflection Property of an Ellipse

If an incoming light ray passes through on focus (S) strike the concave side of the ellipse, then it will get reflected towards other focus (S').



and $$\angle SPS' = \angle SQS'$$

Example 41 A ray emanating from the point $(-3, 0)$ is incident on the ellipse $16x^2 + 25y^2 = 400$ at the point P with ordinate 4. Find the equation of the reflected ray after first reflection.

Sol. For point P, y-coordinate = 4

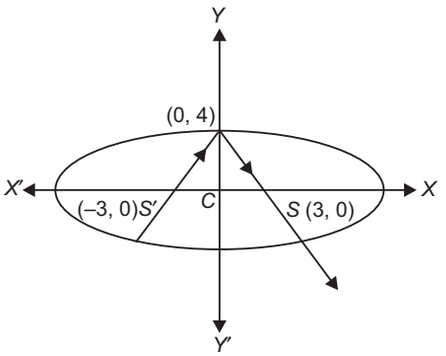
$\because$ Given ellipse is $16x^2 + 25y^2 = 400$

$$16x^2 + 25(4)^2 = 400$$

Coordinate of P is $(0, 4)$

$$e^2 = 1 - \frac{16}{25} = \frac{9}{25}$$

$$e = \frac{3}{5}$$



foci $(\pm ae, 0)$ i.e. $(\pm 3, 0)$

Equation of reflected ray (i.e. PS) is

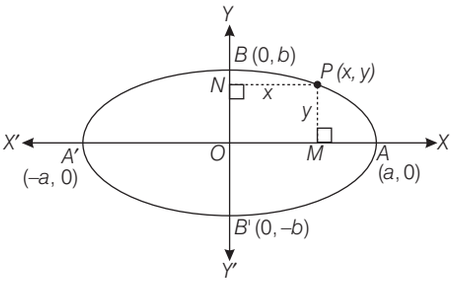
$$\frac{x}{3} + \frac{y}{4} = 1 \text{ or } 4x + 3y = 12.$$

Equation of an Ellipse Referred to Two Perpendicular Lines

Let $P(x, y)$ be any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

then, $$y = PM, x = PN$$

$\therefore$ $$\frac{(PN)^2}{a^2} + \frac{(PM)^2}{b^2} = 1$$

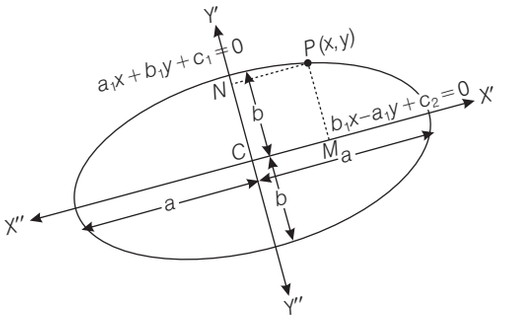


suppose if axes along the lines

$$a_1x + b_1y + c_1 = 0 \quad \text{and} \quad b_1x - a_1y + c_2 = 0$$

then
$$PN = \frac{|a_1x + b_1y + c_1|}{\sqrt{(a_1^2 + b_1^2)}}$$

$$PM = \frac{|b_1x - a_1y + c_2|}{\sqrt{(b_1^2 + a_1^2)}}$$



then, equation of ellipse is

$$\frac{\left\{\dfrac{a_1x + b_1y + c_1}{\sqrt{(a_1^2 + b_1^2)}}\right\}^2}{a^2} + \frac{\left\{\dfrac{b_1x - a_1y + c_2}{\sqrt{(b_1^2 + a_1^2)}}\right\}^2}{b^2} = 1$$

Centre : Is the point of intersection of $a_1x + b_1y + c_1 = 0$ and $b_1x - a_1y + c_2 = 0$

Equations of Major and Minor Axes :

(i) If $a > b$, then major axis lies along $b_1x - a_1y + c_2 = 0$ and minor axis lies along $a_1x + b_1y + c_1 = 0$.

(ii) If $a < b$, then major axis lies along $a_1x + b_1y + c_1 = 0$ and minor axis lies along $b_1x - a_1y + c_2 = 0$

Eccentricity :

(i) If $a > b, b^2 = a^2(1 - e^2)$ (ii) If $a < b, a^2 = b^2(1 - e^2)$

Foci :

(i) If $a > b$

$$\frac{a_1x + b_1y + c_1}{\sqrt{(a_1^2 + b_1^2)}} = \pm ae, \frac{b_1x - a_1y + c_2}{\sqrt{(b_1^2 + a_1^2)}} = 0$$

we get after solving (x, y)

(ii) If $a < b$

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = 0, \frac{b_1x - a_1y + c_2}{\sqrt{b_1^2 + a_1^2}} = \pm be$$

we get after solving (x, y)

Directrices :

(i) If $a > b$,

$$\frac{a_1x + b_1y + c_1}{\sqrt{(a_1^2 + b_1^2)}} = \pm \frac{a}{e}$$

(ii) If $a < b$,

$$\frac{b_1x - a_1y + c_2}{\sqrt{(a_1^2 + b_1^2)}} = \pm \frac{b}{e}$$

Example 42 *Determine the equations of major and minor axes of the ellipse*

$$4(x - 2y + 1)^2 + 9(2x + y + 2)^2 = 25$$

Also, find its centre, length of the latusrectum and eccentricity.

Sol. The equation of the ellipse can be written as

$$4 \times 5\left(\frac{x - 2y + 1}{\sqrt{5}}\right)^2 + 9 \times 5\left(\frac{2x + y + 2}{\sqrt{5}}\right)^2 = 25$$

or
$$\frac{\left(\dfrac{x - 2y + 1}{\sqrt{5}}\right)^2}{(5/4)} + \frac{\left(\dfrac{2x + y + 2}{\sqrt{5}}\right)^2}{(5/9)} = 1$$

or
$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$$

Here $a > b$

$\therefore$ **Equation of major axis is** $Y = 0$

i.e. $2x + y + 2 = 0$

and **Equation of minor axis is** $X = 0$

i.e. $x - 2y + 1 = 0$

Centre : $X = 0, Y = 0$

$\Rightarrow$ $x - 2y + 1 = 0, 2x + y + 2 = 0$

we get $x = -1, y = 0$

$\therefore$ Centre is $(-1, 0)$

Length of latusrectum $= \frac{2b^2}{a} = \frac{2 \times 5/9}{5/4} = \frac{8}{9}$

Eccentricity : $b^2 = a^2(1 - e^2)$

$\Rightarrow$ $\frac{5}{9} = \frac{5}{4}(1 - e^2)$

$\Rightarrow$ $\frac{4}{9} = 1 - e^2$

$\Rightarrow$ $e^2 = \frac{5}{9}$

$\therefore$ $e = \frac{\sqrt{5}}{3}$

Exercise for Session 3

1. The angle between the pair of tangents drawn from the point (1, 2) to the ellipse $3x^2 + 2y^2 = 5$ is

(a) $\tan^{-1}\left(\frac{12}{5}\right)$ (b) $\tan^{-1}\left(\frac{6}{\sqrt{5}}\right)$ (c) $\tan^{-1}\left(\frac{12}{\sqrt{5}}\right)$ (d) $\tan^{-1}(12\sqrt{5})$

2. If chords of contact of tangents from two points (x_1, y_1) and (x_2, y_2) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are at right angles, then $\frac{x_1x_2}{y_1y_2}$ is equal to

(a) $\frac{a^2}{b^2}$ (b) $-\frac{b^2}{a^2}$ (c) $-\frac{a^4}{b^4}$ (d) $-\frac{b^4}{a^4}$

3. From the point $(\lambda, 3)$ tangents are drawn to $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and are perpendicular to each other than λ is

(a) $\pm$ 1 (b) $\pm$ 2 (c) $\pm$ 3 (d) $\pm$ 4

4. The eccentric angle of one end of a diameter of $x^2 + 3y^2 = 3$ is $\pi/6$, then the eccentric angle of the other end will be

(a) $\frac{5\pi}{6}$ (b) $-\frac{5\pi}{6}$ (c) $-\frac{2\pi}{3}$ (d) $\frac{2\pi}{3}$

5. The locus of the mid-points of a focal chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

(a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{ex}{a}$ (b) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{ex}{a}$ (c) $x^2 + y^2 = a^2 + b^2$ (d) $x^2 - y^2 = a^2 + b^2$

6. The centre of the ellipse $\frac{(x + y - 2)^2}{9} + \frac{(x - y)^2}{16} = 1$ is

(a) (0, 0) (b) (1, 0) (c) (0, 1) (d) (1, 1)

7. The locus of the point of intersection of two perpendicular tangents of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ is

(a) $x^2 + y^2 = 4$ (b) $x^2 + y^2 = 9$ (c) $x^2 + y^2 = 13$ (d) $x^2 + y^2 = 5$

8. The area of the parallelogram inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose diagonals are the conjugate diameters of the ellipse is given by

(a) $2ab$ (b) $3ab$ (c) $4ab$ (d) $5ab$

9. Find the locus of the vertices of equilateral triangle circumscribing the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

10. A tangent to the ellipse $x^2 + 4y^2 = 4$ meets the ellipse $x^2 + 2y^2 = 6$ at P and Q. Prove that the tangents at P and Q of the ellipse $x^2 + 2y^2 = 6$ are at right angles.

11. Find the locus of the mid-point of chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 (a > b)$ passing through the point $(2a, 0)$.

12. Find the locus of the point the chord of contact of tangents from which to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, touches the circle $x^2 + y^2 = c^2$.

13. Find the centre and eccentricity of the ellipse

$$4(x - 2y + 1)^2 + 9(2x + y + 2)^2 = 5$$

14. A ray emanating from the point $(0, -\sqrt{5})$ is incident on the ellipse $9x^2 + 4y^2 = 36$ at the point P with abscissa 2. Find the equation of the reflected ray after first reflection.

Shortcuts and Important Results to Remember

1 If S and S' are foci and P be a point, then

(a) If $|SP| + |S'P| > |SS'|$, then the locus of P is an ellipse.

(b) If $|SP| + |S'P| = |SS'|$, then the locus of P is a straight line.

(c) If $|SP| + |S'P| < |SS'|$, then the locus of P is an empty set.

2 If the ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$ and $\frac{x^2}{\theta^2} + \frac{y^2}{\phi^2} = 1$ have a common tangent, then $\begin{vmatrix} a^2 & b^2 & 1 \\ \alpha^2 & \beta^2 & 1 \\ \theta^2 & \phi^2 & 1 \end{vmatrix} = 0$

3 Area of the quadrilateral formed by the common tangents of the circle $x^2 + y^2 = c^2$ and the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; $c \in (a, b)$ is $\frac{2c^2|a^2 - b^2|}{\sqrt{(a^2 - c^2)(c^2 - b^2)}}$

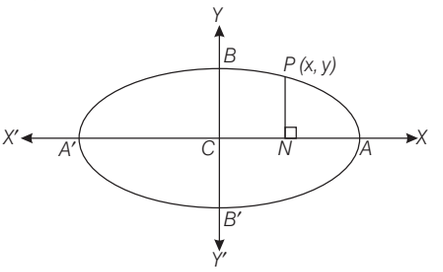
4 The product of the two perpendicular distances from the foci on any tangent of an ellipse is b^2.

5 If the normals at the point $P(x_1, y_1); Q(x_2, y_2)$ and $R(x_3, y_3)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are concurrent, then

$\begin{vmatrix} x_1 & y_1 & x_1y_1 \\ x_2 & y_2 & x_2y_2 \\ x_3 & y_3 & x_3y_3 \end{vmatrix} = 0$ and if points $P(\alpha), Q(\beta)$ and $R(\gamma)$, then

$\begin{vmatrix} \sec\alpha & \operatorname{cosec}\alpha & 1 \\ \sec\beta & \operatorname{cosec}\beta & 1 \\ \sec\gamma & \operatorname{cosec}\gamma & 1 \end{vmatrix} = 0$

6 If ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then

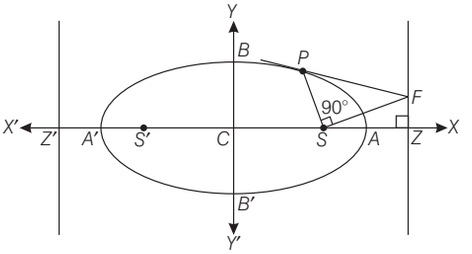


$(PN)^2 : AN \cdot A'N = (BC)^2 : (AC)^2$

7 If α and β are the eccentric angles of extremittes of a focal chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then $\tan(\alpha/2).\tan(\beta/2) = \frac{e-1}{e+1}$ or $\frac{e+1}{e-1}$ according as focus (ae, 0) or $(-ae, 0)$.

8 If the tangent at P on an ellipse meets the directrix in F; then the PF will subtend a right angle at the corresponding focus.

i.e. $\angle PSF = \pi/2$



JEE Type Solved Examples :
Single Option Correct Type Questions

▪ This section contains **10 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

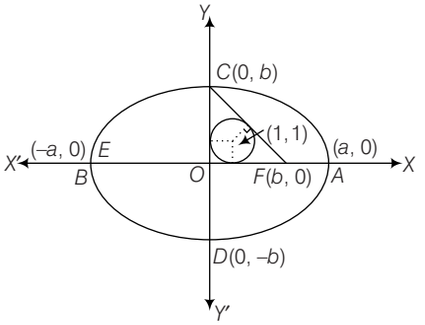
● ***Ex. 1*** *Point 'O' is the centre of the ellipse with major axis AB and minor axis CD. Point F is one focus of the ellipse. If OF = 6 and the diameter of the inscribed circle of triangle OCF is 2, then the product (AB) (CD) is equal to*

(a) 52 (b) 56 (c) 78 (d) None of these

Sol. (b) ∵ Diameter of the inscribed circle of triangle OCF is 2.

$\therefore$ Radius $= 1$

Centre of the circle is (1, 1) and equation of CF is $\frac{x}{6}+\frac{y}{b}=1$



Now, length of perpendicular from (1, 1) on $CF = 1$ (radius)

$$\Rightarrow \quad \frac{\left|\frac{1}{6}+\frac{1}{b}-1\right|}{\sqrt{\left(\frac{1}{36}+\frac{1}{b^2}\right)}}=1 \quad \text{or} \quad -\left(\frac{1}{b}-\frac{5}{6}\right)=\sqrt{\left(\frac{1}{36}+\frac{1}{b^2}\right)}$$

$$\Rightarrow \quad \left(\frac{5}{6}-\frac{1}{b}\right)=\sqrt{\left(\frac{1}{36}+\frac{1}{b^2}\right)}$$

On squaring both sides, then $\left(\frac{5}{6}-\frac{1}{b}\right)^2=\frac{1}{36}+\frac{1}{b^2}$

$$\Rightarrow \quad \frac{25}{36}+\frac{1}{b^2}-\frac{5}{3b}=\frac{1}{36}+\frac{1}{b^2}$$

$$\Rightarrow \quad \frac{5}{3b}=\frac{24}{36}=\frac{2}{3}$$

$$\therefore \quad b=\frac{5}{2}$$

Also, $b^2=a^2(1-e^2)$

$b^2=a^2-(ae)^2$

$$\Rightarrow \quad a^2=b^2+(ae)^2=\frac{25}{4}+36 \qquad [\because ae=6]$$

$$\Rightarrow \quad a=\frac{13}{2}$$

Hence, $(AB)(CD)=(2a)(2b)=13\times 5=65$

● ***Ex. 2*** *Let P_i and P_i' be the feet of the perpendiculars drawn from the foci S and S′ on a tangent T_i to an ellipse whose length of semi-major axis is 20. If* $\sum_{i=1}^{10}(SP_i)(S'P_i')=2560$, *then the value of eccentricity is*

(a) $\frac{1}{5}$ (b) $\frac{2}{5}$

(c) $\frac{3}{5}$ (d) $\frac{4}{5}$

Sol. (c) $\therefore$ Product of length of perpendiculars from foci on a tangent to an ellipse $= b^2$

$$\therefore \quad (SP_1)(S'P_1)=(SP_2)(S'P_2)=\ldots=(SP_{10})(S'P_{10}')=b^2$$

Given $\sum_{i=1}^{10}(SP_i)(S'P_i')=2560$

$$\Rightarrow \quad 10b^2=2560 \quad \text{or} \quad b^2=256$$

$$\text{or} \quad b=16$$

and $b^2=a^2(1-e^2)$

$$\Rightarrow \quad (16)^2=(20)^2(1-e^2)$$

$$\text{or} \quad 1-e^2=\frac{16}{25}$$

$$\text{or} \quad e^2=\frac{9}{25} \quad \text{or} \quad e=\frac{3}{5}$$

● ***Ex. 3*** *Coordinates of the vertices B and C of a ΔABC are (2, 0) and (8, 0) respectively. The vertex A is varing in such a way that* $4\tan\left(\frac{B}{2}\right).\tan\left(\frac{C}{2}\right)=1$. *Then, the locus of A is*

(a) $\frac{(x-5)^2}{25}+\frac{y^2}{16}=1$ (b) $\frac{(x-5)^2}{16}+\frac{y^2}{9}=1$

(c) $\frac{(x-5)^2}{25}+\frac{y^2}{9}=1$ (d) $\frac{(x-5)^2}{16}+\frac{y^2}{25}=1$

Sol. (a) ∵ $4\tan\left(\frac{B}{2}\right)\cdot\tan\left(\frac{C}{2}\right)=1$

$$\Rightarrow \quad 4\times\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}\times\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}=1$$

$$\Rightarrow \quad \frac{s-a}{s}=\frac{1}{4}$$

$$\Rightarrow \quad 3s=4a$$

$$\Rightarrow \quad 3\left(\frac{a+b+c}{2}\right)=4a$$

$$\text{or} \quad b+c=\frac{5a}{3}=10=2a_1 \quad , \qquad [\because a=BC=6]$$

Since, the sum of distances of A from two given fixed points B and C is always 10.

Here, B and C are foci.

$\therefore$ Centre $(5, 0)$ and distance between foci $= 6$

$\Rightarrow \quad 2a_1 e = 6$

$\Rightarrow \quad e = \frac{6}{10} = \frac{3}{5} \quad [\because a_1 = 5]$

and $\quad b^2 = a_1^2(1 - e^2) = 25\left(1 - \frac{9}{25}\right) = 16$ or $b = 4$

$\because$ A lies on the ellipse

$\therefore$ Locus of A is

$$\frac{(x-5)^2}{(5)^2} + \frac{(y-0)^2}{(4)^2} = 1$$

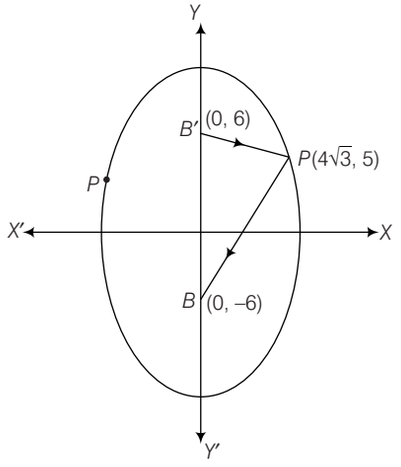
or $$\frac{(x-5)^2}{25} + \frac{y^2}{16} = 1$$

● **Ex. 4** *A ray emanating from the point* $(0, 6)$ *is incident on the ellipse* $25x^2 + 16y^2 = 1600$ *at the point P with ordinate S. After reflection, ray cuts the Y-axis at B. The length of PB is*

(a) 5 (b) 7 (c) 12 (d) 13

Sol. (d) $\because$ Ellipse is $25x^2 + 16y^2 = 1600$

or $$\frac{x^2}{8^2} + \frac{y^2}{10^2} = 1$$



Coordinating foci an $\left(0, \pm\sqrt{(10^2 - 8^2)}\right)$

i.e. $\quad (0, \pm 6)$

Let coordinates of P are $(\lambda, 5)$

$\therefore \quad \frac{\lambda^2}{8^2} + \frac{5^2}{10^2} = 1$

$\Rightarrow \quad \lambda = \pm 4\sqrt{3}$

$\therefore$ $P \equiv (\pm 4\sqrt{3}, 5)$ (lie in I or II quadrants)

According to reflection property, a ray passing through focus $B'(0, 6)$ will passing through $B(0, -6)$ (other focus). If P lies in I quadrant, then

$$PB = \sqrt{48 + (5+6)^2} = \sqrt{169} = 13$$

● **Ex. 5** *If the ellipse* $\frac{x^2}{4} + y^2 = 1$ *meets the ellipse* $x^2 + \frac{y^2}{a^2} = 1$ *in four distinct points and* $a = b^2 - 5b + 7$, *then b belongs to*

(a) $(1, 4)$ (b) $(-\infty, 2) \cup (3, \infty)$

(c) $(2, 3)$ (d) None of these

Sol. (b) The ellipse $\frac{x^2}{1^2} + \frac{y^2}{a^2} = 1$ will intersect in four distinct points with ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$, where $a^2 > 1$

$\Rightarrow \quad a > 1 \quad [\because$ a always positive$]$

Now, $\quad a = b^2 - 5b + 7$

$\Rightarrow \quad b^2 - 5b + 7 > 1$

or $\quad b^2 - 5b + 6 > 0$

or $\quad (b-2)(b-3) > 0$

or $\quad b \in (-\infty, 2) \cup (3, \infty)$

● **Ex. 6** *The normal at a variable point P on an ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *of eccentricity e meets the axes of the ellipse in Q and R, then the locus of the mid-point of QR is a conic with eccentricity e′ such that*

(a) e' is independent of e (b) $e' = 1$

(c) $e' = e$ (d) $e' = \frac{1}{e}$

Sol. (c) Normal at $P(a\cos\theta, b\sin\theta)$ is

$$ax\sec\theta - by\,\mathrm{cosec}\,\theta = a^2 - b^2 \quad (\text{let } a > b)$$

It meets the axes at

$$Q\left(\frac{(a^2-b^2)}{a}\cos\theta,\ 0\right) \text{ and } R\left(0, -\frac{(a^2-b^2)}{b}\sin\theta\right)$$

Let mid-point of QR is $T(x, y)$, then

$$2x = \frac{(a^2-b^2)}{a}\cos\theta$$

or $\quad 2ax = (a^2 - b^2)\cos\theta \quad$ ...(i)

and $\quad 2y = -\frac{(a^2-b^2)}{b}\sin\theta$

or $\quad 2by = -(a^2 - b^2)\sin\theta \quad$ ...(ii)

On squaring and adding Eq. (i) and Eq.(ii), we get

$$4a^2x^2 + 4b^2y^2 = (a^2 - b^2)^2$$

$$\Rightarrow \quad \frac{x^2}{\left(\frac{a^2-b^2}{2a}\right)^2} + \frac{y^2}{\left(\frac{a^2-b^2}{2b}\right)^2} = 1$$

which is an ellipse, having eccentricity e', then

Let $\quad A = \frac{a^2-b^2}{2a},\ B = \frac{a^2-b^2}{2b}$

$$A^2 = B^2(1-e'^2) \qquad [\because B > A]$$

$$\Rightarrow \quad e'^2 = 1 - \frac{A^2}{B^2} = 1 - \frac{b^2}{a^2} = e^2$$

$$\therefore \quad e' = e$$

● ***Ex. 7*** *If the curves* $\frac{x^2}{4} + y^2 = 1$ *and* $\frac{x^2}{a^2} + y^2 = 1$ *for a suitable value of a cut on four concyclic points, the equation of the circle passing through these four points is*

(a) $x^2 + y^2 = 8$ (b) $x^2 + y^2 = 4$

(c) $x^2 + y^2 = 2$ (d) $x^2 + y^2 = 1$

Sol. (d) The equation of conic through the point of intersection of given two ellipses is

$$\left(\frac{x^2}{4} + y^2 - 1\right) + \lambda\left(\frac{x^2}{a^2} + y^2 - 1\right) = 0$$

or $$x^2\left(\frac{1}{4} + \frac{\lambda}{a^2}\right) + y^2(1+\lambda) = (1+\lambda)$$

or $$x^2\left(\frac{a^2 + 4\lambda}{4a^2(1+\lambda)}\right) + y^2 = 1$$

for circle $$\frac{a^2 + 4\lambda}{4a^2(1+\lambda)} = 1$$

$$\Rightarrow \quad \lambda = -\frac{3a^2}{4(a^2-1)}$$

Therefore, the circle is $x^2 + y^2 = 1$

● ***Ex. 8*** *If p is the length of perpendicular drawn from the origin to any normal to the ellipse* $\frac{x^2}{25} + \frac{y^2}{16} = 1$, *then the maximum value of p is*

(a) 5 (b) 4

(c) 2 (d) 1

Sol. (d) The equation of any normal at $(5\cos\theta, 4\sin\theta)$ to the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ is

$$5x\sec\theta - 4y\,\mathrm{cosec}\,\theta = 9$$

$$\therefore \quad p = \frac{|0-0-9|}{\sqrt{(25\sec^2\theta + 16\,\mathrm{cosec}^2\theta)}} = \frac{9}{\sqrt{(25\tan^2\theta + 16\cot^2\theta + 41)}}$$

Now, AM ≥ GM

$$\Rightarrow \quad \frac{25\tan^2\theta + 16\cot^2\theta}{2} \geq \sqrt{25\times 16} = 20$$

$$\therefore \quad 25\tan^2\theta + 16\cot^2\theta \geq 40$$

$$\Rightarrow \quad 25\tan^2\theta + 16\cot^2\theta + 41 \geq 81$$

or $$\frac{9}{\sqrt{(25\tan^2\theta + 16\cot^2\theta + 41)}} \leq 1$$

$$\Rightarrow \quad p \leq 1$$

Thus, maximum value of p is 1.

● ***Ex. 9*** *If* $f(x)$ *is a decreasing function, then the set of values of 'k', for which the major axis of the ellipse* $\frac{x^2}{f(k^2+2k+5)} + \frac{y^2}{f(k+11)} = 1$ *is the X-axis, is*

(a) $k \in (-2, 3)$

(b) $k \in (-3, 2)$

(c) $k \in (-\infty, -3) \cup (2, \infty)$

(d) $k \in (-\infty, -2) \cup (3, \infty)$

Sol. (b) $\because$ $f(x)$ is a decreasing function and for major axis to be X-axis.

$$\therefore \quad f(k^2+2k+5) > (f(k+11)$$

$$\Rightarrow \quad k^2 + 2k + 5 < k + 11$$

or $$k^2 + k - 6 < 0$$

or $$(k+3)(k-2) < 0$$

or $$k \in (-3, 2)$$

● ***Ex. 10*** *If a tangent of slope 2 of the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *is normal to the circle* $x^2 + y^2 + 4x + 1 = 0$, *then the maximum value of ab is*

(a) 1 (b) 2

(c) 4 (d) 8

Sol. (c) Equation of tangent is $y = 2x + \sqrt{(4a^2 + b^{2)}}$

This is normal to the circle

$$x^2 + y^2 + 4x + 1 = 0$$

This tangent passes through $(-2, 0)$, then

$$0 = -4 + \sqrt{(4a^2 + b^2)}$$

$$\Rightarrow \quad 4a^2 + b^2 = 16$$

$$\because \quad AM \geq GM$$

$$\therefore \quad \frac{4a^2 + b^2}{2} \quad \sqrt{(4a^2)(b^2)}$$

$$\Rightarrow \quad \frac{16}{2} \geq 2ab$$

or $$ab \leq 4$$

Hence, maximum value of ab is 4.

JEE Type Solved Examples :
More than One Correct Option Type Questions

■ This section contains **5 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which **MORE THAN ONE** may be correct.

● ***Ex. 11*** *Extremities of the latusrectum of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 (a>b)$ *having a given major axis* $2a$ *lies on*

(a) $x^2 = a(a-y)$ (b) $x^2 = a(a+y)$

(c) $y^2 = a(a+x)$ (d) $y^2 = a(a-x)$

Sol. (a, b) $\because$ Extremities of the latusrectum are $\left(ae, \pm\frac{b^2}{a}\right)$

Let $x = \pm ae$ and $y = \pm\frac{b^2}{a}$

or $x^2 = a^2e^2$ and $b^2 = \pm ay$

or $x^2 = a^2 - b^2$ and $b^2 = \pm ay$

$\therefore$ $x^2 = a^2 \pm ay$ or $x^2 = a(a \pm y)$

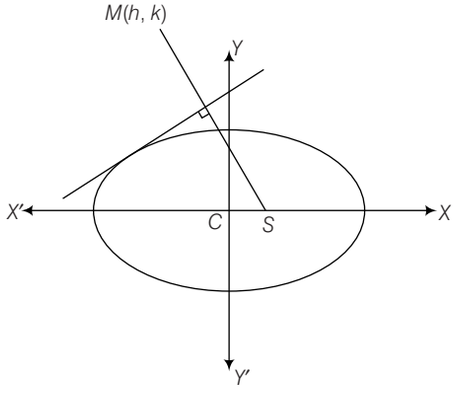
● ***Ex. 12*** *The locus of the image of the focus of the ellipse* $\frac{x^2}{25}+\frac{y^2}{9}=1$, *with respect to any of the tangent to the ellipse is*

(a) $(x+4)^2 + y^2 = 100$ (b) $(x+2)^2 + y^2 = 50$

(c) $(x-4)^2 + y^2 = 100$ (d) $(x-2)^2 + y^2 = 50$

Sol. (a, c) Let $M(h, k)$ be the image SM cuts a tangent at a point which lies on the auxiliary circle of the ellipse, therefore

$\left(\frac{h \pm 4}{2}\right)^2 + \frac{k^2}{4} = (5)^2$ [$\because$ foci of the given ellipse are $(\pm 4, 0)$]



or $(h \pm 4)^2 + k^2 = 100$

Hence, the locus is $(x \pm 4)^2 + y^2 = 100$

● ***Ex. 13*** *A tangent to the ellipse* $4x^2 + 9y^2 = 36$ *is cut by the tangent at the extremities of the major axis at* T *and* T'. *The circle* TT' *as diameter passes through the point*

(a) $(-\sqrt{5}, 0)$ (b) $(\sqrt{5}, 0)$

(c) $(\sqrt{3}, 0)$ (d) $(-\sqrt{3}, 0)$

Sol. (a, b) Given equation of the ellipse is $4x^2 + 9y^2 = 36$

i.e., $\frac{x^2}{9}+\frac{y^2}{4}=1$ ...(i)

The equation of tangent at $(3\cos\theta, 2\sin\theta)$ is

$$\frac{x}{3}\cos\theta + \frac{y}{2}\sin\theta = 1$$

which meets the tangent at $x = 3$ and $x = -3$ at the extremities of major axis

$$T \equiv \left(3, \frac{2(1-\cos\theta)}{\sin\theta}\right)$$

and $$T' \equiv \left(-3, \frac{2(1+\cos\theta)}{\sin\theta}\right)$$

$\therefore$ Equation of circle on TT' as diameter is

$$(x-3)(x+3) + \left(y - \frac{2(1-\cos\theta)}{\sin\theta}\right)\left(y - \frac{2(1+\cos\theta)}{\sin\theta}\right) = 0$$

$\Rightarrow$ $$x^2 + y^2 - \frac{4}{\sin\theta}\cdot y - 5 = 0$$

or $$(x^2 + y^2 - 5) - \frac{4}{\sin\theta}y = 0$$ ...(ii)

Clearly Eq. (ii), passes through point of intersection of

$$x^2 + y^2 - 5 = 0 \text{ and } y = 0 \text{ i.e., } (\pm\sqrt{5}, 0)$$

● ***Ex. 14*** *Consider the ellipse* $\frac{x^2}{\tan^2\alpha}+\frac{y^2}{\sec^2\alpha}=1$, *where* $\alpha \in (0, \pi/2)$. *Which of the following quantities would vary as* α *varies?*

(a) degree of flatness (b) ordinate of the vertex

(c) coordinate of the foci (d) length of latusrectum

Sol. (a, b, d) In $\alpha \in (0, \pi/2)$

$$\sec^2\alpha > \tan^2\alpha$$

$\therefore$ Coordinates of foci $(0, \pm\sqrt{(\sec^2\alpha + \tan^2\alpha)})$ i.e., $(0, \pm 1)$ which is independent of α.

Vertices are $(0, \pm\sec\alpha)$ and latusrectum

$$= \frac{2a^2}{b} = \frac{2\tan^2\alpha}{\sec\alpha}$$

$\because$ $\alpha \in (0, \pi/2)$

$\Rightarrow$ $\tan\alpha \in (0, \infty)$ and $\sec\alpha \in (1, \infty)$

Hence, $\alpha \propto$ degree of flatness.

● ***Ex. 15*** *Let $A(\theta)$ and $B(\phi)$ be the extremities of a chord of an ellipse. If the slope of AB is equal to the slope of the tangent at a point $C(\alpha)$ on the ellipse, then the value of α is*

(a) $\frac{\theta+\phi}{2}$ (b) $\frac{\theta-\phi}{2}$

(c) $\frac{\theta+\phi}{2}+\pi$ (d) $\frac{\theta+\phi}{2}-\pi$

Sol. (a. c) $\because$ Slope of AB = Slope of tangent at C

$$\Rightarrow \left(\frac{b\sin\phi - b\sin\theta}{a\cos\phi - a\cos\theta}\right) = -\frac{b}{a}\frac{\cos\alpha}{\sin\alpha}$$

$$\Rightarrow \frac{b.2\cos\left(\frac{\theta+\phi}{2}\right).\sin\left(\frac{\phi-\theta}{2}\right)}{-a.2\sin\left(\frac{\theta+\phi}{2}\right).\sin\left(\frac{\phi-\theta}{2}\right)} = -\frac{b}{a}\cot\alpha$$

$$\therefore \tan\alpha = \tan\left(\frac{\theta+\phi}{2}\right)$$

$$\Rightarrow \alpha = n\pi + \left(\frac{\theta+\phi}{2}\right), n \in I$$

JEE Type Solved Examples : Paragraph Based Questions

■ This section contains **2 solved Paragraphs** based upon each of the Paragraph **3 multiple choice** questions have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Paragraph I

(Q. Nos. 16 to 18)

A sequence of ellipses $E_1, E_2, E_3, \ldots, E_n$ is constructed as follows : Ellipse E_n is drawn so as to touch ellipse E_{n-1} as the extremities of the major axis of E_{n-1} and to have its foci at the extremities of the mirror axis of E_{n-1}.

16. If E_n is independent of n, then the eccentricity of ellipse E_{n-2} is

(a) $\left(\frac{3-\sqrt{5}}{2}\right)$ (b) $\left(\frac{\sqrt{5}-1}{2}\right)$

(c) $\left(\frac{2-\sqrt{3}}{2}\right)$ (d) $\left(\frac{\sqrt{3}-1}{2}\right)$

17. If eccentricity of ellipse E_n is e_n, then the locus of (e_n^2, e_{n-1}^2) is

(a) a parabola (b) an ellipse

(c) a hyperbola (d) a rectangular hyperbola

18. If equation of ellipse E_1 is $\frac{x^2}{9}+\frac{y^2}{16}=1$, then the equation of ellipse E_3 is

(a) $\frac{x^2}{9}+\frac{y^2}{16}=1$ (b) $\frac{x^2}{25}+\frac{y^2}{49}=1$

(c) $\frac{x^2}{25}+\frac{y^2}{41}=1$ (d) $\frac{x^2}{16}+\frac{y^2}{25}=1$

Sol.

16. (b) If $E_n : \frac{x^2}{a_n^2}+\frac{y^2}{b_n^2}=1$ and eccentricity of E_n is e_n

If $a_n > b_n$

Then, $b_n^2 = a_n^2(1-e_n^2)$ ...(i)

According to the question, $b_n = b_{n-1}$ ...(ii)

and $a_{n-1} = a_n e_n$ ...(iii)

For ellipse E_{n-1}, $a^2_{n-1} = b^2_{n-1}(1-e^2_{n-1})$ ...(iv)

From Eqs. (i) and (ii), we get $b^2_{n-1} = a_n^2(1-e_n^2)$ ...(v)

Substituting the values of a_{n-1} and b^2_{n-1} from Eqs. (iii) and (v) in Eq. (iv), then

$$a_n^2 e_n^2 = a_n^2(1-e_n^2)(1-e^2_{n-1})$$

$$\Rightarrow e_n^2 = (1-e_n^2)(1-e^2_{n-1}) \quad \text{...(vi)}$$

$\because$ E_n is independent of n

$\therefore$ $e_n = e_{n-1} = e$ [say]

From Eq. (vi), we get

$$e^2 = (1-e^2)^2$$

$$\Rightarrow e^4 - 3e^2 + 1 = 0$$

$$\therefore e^2 = \frac{3\pm\sqrt{5}}{2} = \frac{6\pm 2\sqrt{5}}{4} = \left(\frac{\sqrt{5}\pm 1}{2}\right)^2$$

$$\therefore e = \frac{\sqrt{5}-1}{2} \quad [\because 0 < e < 1]$$

17. (d) From Eq. (vi), $e_n^2 = (1-e_n^2)(1-e^2_{n-1})$

Locus of (e_n^2, e^2_{n-1}) is

$$x = (1-x)(1-y)$$

$$\Rightarrow xy - 2x - y + 1 = 0$$

Here, $a=0, b=0, c=1,$

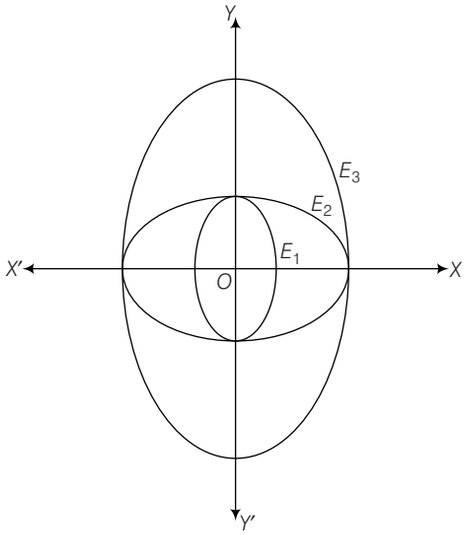
$$f = -\frac{1}{2}, g = -1, h = \frac{1}{2}$$

$$\therefore \Delta = 0 + 2\times -\frac{1}{2}\times -1\times\frac{1}{2} - 0 - 0 - 1\times\frac{1}{4} = \frac{1}{4} \neq 0$$

and $h^2 > ab, a+b=0$

$\Rightarrow$ rectangular hyperbola.

18. (c) From Eq. (vi), $e_n = \sqrt{\left(\frac{1-e_{n-1}^2}{2-e_{n-1}^2}\right)}$ ...(vii)



$\because \quad E_1 \equiv \frac{x^2}{9} + \frac{y^2}{16} = 1$

$\therefore \quad a_1 = 3, b_1 = 4$ and $9 = 16(1 - e_1^2)$

$\therefore \quad e_1 = \frac{\sqrt{7}}{4}$

From Eq. (vii), $e_2 = \sqrt{\left(\frac{1-e_1^2}{2-e_1^2}\right)} = \sqrt{\left(\frac{1-\frac{7}{16}}{2-\frac{7}{16}}\right)} = \frac{3}{5}$

and then, $e_3 = \sqrt{\left(\frac{1-\frac{9}{25}}{2-\frac{9}{25}}\right)} = \frac{4}{\sqrt{41}}$

Also, $a_1 = a_2 e_2$ and $b_1 = b_2$

$\therefore \quad b_2 = 4 = b_3 e_3 \Rightarrow b_3 = \sqrt{41}$

and $a_3^2 = b_3^2(1 - e_3^2) = 41\left(1 - \frac{16}{41}\right) = 25$

$\therefore$ Ellipse E_3 is $\frac{x^2}{25} + \frac{y^2}{41} = 1$

Paragraph II

(Q. Nos. 19 to 21)

Consider an ellipse $E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, *centred at point 'O' and having AB and CD as its major and minor axes respectively if* S_1 *be one of the focus of the ellipse, radius of incircle of* ΔOCS_1 *be 1 unit and* $OS_1 = 6$ *units.*

19. If area of ellipse (E) is Δ sq unit, then the value of 4Δ is

(a) 63π (b) 64π (c) 65π (d) 66π

20. If perimeter of ΔOCS_1 is p units, then the value of p is

(a) 10 (b) 15 (c) 20 (d) 25

21. The equation of the director circle of (E) is

(a) $x^2 + y^2 = 48.5$ (b) $x^2 + y^2 = 97$

(c) $x^2 + y^2 = \sqrt{48.5}$ (d) $x^2 + y^2 = \sqrt{97}$

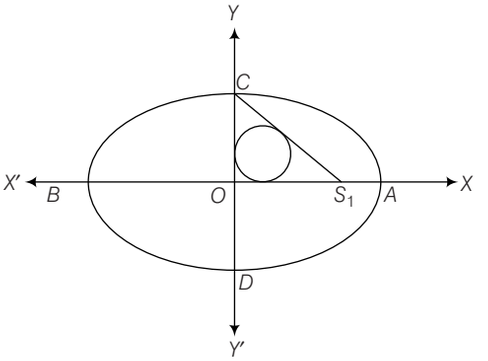
Sol.

$\because \quad OS_1 = ae = 6, OC = b$

$\therefore \quad CS_1 = \sqrt{(OS_1)^2 + (OC)^2} = \sqrt{(a^2e^2 + b^2)} = a$

Area of $\Delta OCS_1 = \frac{1}{2} \times (OS_1) \times (OC) = \frac{1}{2} \times 6 \times b = 3b$

and semi-perimeter of $\Delta OCS_1 = \frac{1}{2}(OS_1 + OC + CS_1) = \frac{1}{2}(6 + a + b)$



$\because$ In radius of $\Delta OCS_1 = 1$

$\therefore \quad \frac{3b}{\frac{1}{2}(6 + a + b)} = 1$

$\Rightarrow \quad b = \frac{1}{5}(6 + a)$ ...(i)

Also, $b^2 = a^2(1 - e^2) = a^2 - 36$ ...(ii)

From Eqs. (i) and (ii), we get

$$\frac{1}{25}(6 + a)^2 = a^2 - 36$$

$\Rightarrow \quad 2a^2 - a - 78 = 0$

or $a = \frac{13}{2}, -6$

19. (c) $\therefore \quad a = \frac{13}{2}$ and $b = \frac{5}{2}$ [from Eq. (i)]

$\because \quad \Delta = \pi ab = \pi \times \frac{13}{2} \times \frac{5}{2}$

$\therefore \quad 4\Delta = 65\pi$

20. (b) $p = (OS_1 + OC + CS_1) = 6 + b + a$

$$= 6 + \frac{5}{2} + \frac{13}{2} = 15$$

21. (a) Equation of director circle of E is

$$x^2 + y^2 = a^2 + b^2$$

$$= \frac{169 + 25}{4} = 48.5$$

JEE Type Solved Examples :
Single Integer Answer Type Questions

■ This section contains **2 examples.** The answer to each example is a **single digit integer**, ranging from 0 to 9 (both inclusive).

● ***Ex. 22*** *If the normals at the four points* $(x_1, y_1), (x_2, y_2)$, (x_3, y_3) *and* (x_4, y_4) *on the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *are concurrent, then the value of* $\left(\sum_{i=1}^{4} x_i\right)\left(\sum_{i=1}^{4}\frac{1}{x_i}\right)$ *is*

Sol. **(4)** Let point of concurrent is (h, k).

Equation of normal at (x', y') is

$$\frac{x-x'}{x'a^2}=\frac{y-y'}{y'b^2}$$

It is passes through (h, k), then

$$y'^2\{a^2(h-x')+b^2x'^2\}=b^4k^2x'^2 \qquad ...(i)$$

But $$\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1 \text{ or } y'^2=\frac{b^2}{a^2}(a^2-x'^2) \qquad ...(ii)$$

Value of y'^2 from Eq. (ii), putting in Eq. (i), we get

$$\frac{b^2}{a^2}(a^2-x'^2)\{a^2h+(b^2-a^2)x'\}^2=b^4k^2x'^2$$

$$\Rightarrow \quad \frac{b^2}{a^2}(a^2-x'^2)\{a^4h^2+(b^2-a^2)^2x'^2+2a^2hx'(b^2-a^2)\}$$

$$=b^4k^2x'^2$$

Arranging above as a fourth degree equation in x', we get

$$\Rightarrow \quad -(a^2-b^2)^2x'^4+2ha^2(a^2-b^2)x'^3+x'^2(\ldots)$$

$$-2a^4h(a^2-b^2)x'+a^6h^2=0$$

above equation being of fourth degree in x', therefore roots of the above equation are x_1, x_2, x_3, x_4, then

$$(x_1+x_2+x_3+x_4)=-\frac{2ha^2(a^2-b^2)}{-(a^2-b^2)^2}=\frac{2ha^2}{(a^2-b^2)} \qquad ...(iii)$$

$$\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)=\frac{\Sigma x_1x_2x_3}{x_1\cdot x_2\cdot x_3\cdot x_4}$$

$$=\frac{\frac{2a^4h(a^2-b^2)}{-(a^2-b^2)^2}}{\frac{a^6h^2}{-(a^2-b^2)^2}}=\frac{2(a^2-b^2)}{a^2h} \qquad ...(iv)$$

Multiplying Eqs. (iii) and (iv), we get

$$(x_1+x_2+x_3+x_4)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)=4$$

or $$\left(\sum_{i=1}^{4}x_i\right)\left(\sum_{i=1}^{4}\frac{1}{x_i}\right)=4$$

● ***Ex. 23*** *If* $x, y \in R$, *satisfies the equation* $\frac{(x-4)^2}{4}+\frac{y^2}{9}=1$, *then the difference between the largest and the smallest value of the expression* $\frac{x^2}{4}+\frac{y^2}{9}$ *is*

Sol. **(8)** Parametric coordinates on $\frac{(x-4)^2}{4}+\frac{y^2}{9}=1$ are

$(4+2\cos\theta, 3\sin\theta)$

Now, let $$E=\frac{x^2}{4}+\frac{y^2}{9}$$

$$=\frac{(4+2\cos\theta)^2}{4}+\frac{(3\sin\theta)^2}{9}$$

$$=(2+\cos\theta)^2+\sin^2\theta$$

$$=4+4\cos\theta+\cos^2\theta+\sin^2\theta$$

$$=5+4\cos\theta$$

$\therefore$ $E_{max}=5+4(1)=9$ $\quad (\because -1\le\cos\theta\le 1)$

and $E_{min}=5+4(-1)=1$

Hence, $E_{max}-E_{min}=9-1=8$

JEE Type Solved Examples : Matching Type Questions

■ This section contains **only one example**. This example has four statements (A, B, C and D) given in **Column I** and four statements (p, q, r and s) in **Column II.** Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II.**

● ***Ex. 24*** *Match the following*

	Column I	Column II
A.	Let $f(x)=\begin{cases} a^x, & x<2 \\ 8, & x=2 \\ \dfrac{b(x^2-b^2)}{(x-2)}, & x>2 \end{cases}$ If f is continuous at $x=2$, then the locus of the pair of perpendicular tangents to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is $x^2+y^2=r^2$, then r^2 is divisible by	(p) 3
B.	If the ellipse $\dfrac{(x-h)^2}{M}+\dfrac{(y-k)^2}{N}=1$ has major axis on the line $y=2$, minor-axis on the line $x=-1$, major axis has length 10 and minor axis has length 4. Then, $h+k+M+N$ is divisible by	(q) 4
C.	If PQ is a focal chord of ellipse $\dfrac{x^2}{25}+\dfrac{y^2}{16}=1$, which passes through $S(3,0)$ and $PS=2$, then length of PQ is divisible by	(r) 5
D.	A tangent to the ellipse $\dfrac{x^2}{27}+\dfrac{y^2}{48}=1$ having slope $\left(-\dfrac{4}{3}\right)$ cuts the x and y-axis at the points A and B respectively. If O is the origin, then area of ΔOAB is divisible by	(s) 6

Sol. (A) → (p, q, s); (B) → (p, r, s); (C) → (r); (D) → (p, q, s)

(A) $\underset{x\to 2^-}{Lt} f(x)=\underset{x\to 2^+}{Lt} f(x)=f(2)$

$\Rightarrow \underset{h\to 0}{Lt}\, a^{2-h}=\underset{h\to 0}{Lt}\dfrac{b((2+h)^2-b^2)}{2+h-2}=8$

$\Rightarrow a^2=b\underset{h\to 0}{Lt}\dfrac{(2+h)^2-b^2}{h}=8$

at $h\to 0,(2+h)^2-b^2\to 0$

$\therefore b^2=4$ and $a^2=8$

$\therefore$ Locus of the pair of perpendicular tangents to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$\therefore$ Required locus is

$$x^2+y^2=a^2+b^2=8+`4=12$$

$\Rightarrow r^2=12$

(B) $\because$ Major axis on the line $y=2$ and minor axis on the line $x=-1$

$\therefore$ Centre of ellipse is $(-1,2)$

$\Rightarrow h=-1, k=2$

Also, $2a=10$ and $2b=4$

$\therefore M=a^2=25$

and $N=b^2=4$

Now, $h+k+M+N=-1+2+25+4=30$

(C) Here, $a=5, b=4$

$\because b^2=a^2(1-e^2) \Rightarrow 16=25(1-e^2)$

$\therefore e=\dfrac{3}{5}$

Foci $(\pm 3, 0)$

Here, $SA=2$ [A and A' are vertices]

Also gives $PS=2$

$\therefore$ P coincides with A and Q coincides with A'

$\therefore PQ=2a=10$

(D) Let $(\sqrt{27}\cos\theta, \sqrt{48}\sin\theta)$ be a point on the ellipse $\dfrac{x^2}{27}+\dfrac{y^2}{48}=1$

$\therefore$ Equation of tangent at $(\sqrt{27}\cos\theta, \sqrt{48}\sin\theta)$ is

$$\frac{x\cos\theta}{\sqrt{27}}+\frac{y\sin\theta}{\sqrt{48}}=1$$

$\therefore$ slope $=-\dfrac{\sqrt{48}}{\sqrt{27}}\cdot\dfrac{\cos\theta}{\sin\theta}=-\dfrac{4}{3}$ [given]

$\therefore \tan\theta=1$

$\Rightarrow \theta=\dfrac{\pi}{4}$

$\therefore$ Equation of tangent is $\dfrac{x}{\sqrt{54}}+\dfrac{y}{\sqrt{96}}=1$

$\therefore$ Area of triangle $=\dfrac{1}{2}\times 3\sqrt{6}\times 4\sqrt{6}=36$

JEE Type Solved Examples :
Statement I and II Type Questions

■ **Directions** (Ex. Nos. 25 and 26) are Assertion. Reason type examples. Each of these examples contains two statements.

Statement I (Assertion) and **Statement II** (Reason)

Each of these example also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below.

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for statement I

(b) Statement I is true, Statement II is true, Statement II is not a correct explanation for Statement I

(c) Statement I is true, Statement II is flase

(d) Statement I is false, Statement II is true

● ***Ex. 25*** **Statement I** *Feet of perpendiculars drawn from foci of an ellipse* $4x^2 + y^2 = 16$ *on the line* $2\sqrt{3}x + y = 8$ *lie on the circle* $x^2 + y^2 = 16$.

Statement II *If perpendiculars are drawn from foci of an ellipse to its any tangent, the feet of these perpendiculars lie on director circle of the ellipse.*

Sol. (c) Simultaneously solving the equations of ellipse and the given line, we get

$$4x^2 + (8 - 2\sqrt{3}x)^2 = 16$$

$$\Rightarrow \quad x^2 + (4 - \sqrt{3}x)^2 = 4$$

$$\Rightarrow \quad 4x^2 - 8\sqrt{3}\,x + 12 = 0$$

$$\text{or} \quad x^2 - 2\sqrt{3}x + 3 = 0$$

$$\text{or} \quad (x - \sqrt{3})^2 = 0$$

$\therefore\ 2\sqrt{3}x + y = 8$ is a tangent to the ellipse, the auxiliary circle is $x^2 + y^2 = 16$.

Hence, Statement I is true and Statement II is false.

● ***Ex. 26*** **Statement II** *The condition on a and b for which two distinct chords of the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$ *passing through* $(a, -b)$ *are bisected by the line* $x + y = b$ *is* $a^2 + 6ab - 7b^2 > 0$.

Statement II *Equation of chord of the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *whose mid-point* (x_1, y_1) *is* $T = S_1$

Sol. (a) Let $(\lambda, b - \lambda)$ is a point on the line $x + y = b$, then equation of chord whose mid-point $(\lambda, b - \lambda)$ is $T = S_1$

$$\text{or} \quad \frac{\lambda x}{2a^2} + \frac{(b-\lambda)y}{2b^2} - 1 = \frac{\lambda^2}{2a^2} + \frac{(b-\lambda)^2}{2b^2} - 1 \qquad \ldots(i)$$

$(a, -b)$ lies on Eq. (i), then

$$\frac{\lambda a}{2a^2} - \frac{b(b-\lambda)}{2b^2} = \frac{\lambda^2}{2a^2} + \frac{(b-\lambda)^2}{2b^2}$$

$$\Rightarrow \quad \lambda^2(a^2 + b^2) - ab\lambda(3a + b) + 2a^2b^2 = 0$$

For two distinct chords $D > 0$

$$\Rightarrow \quad a^2b^2(3a + b)^2 - 8a^2b^2(a^2 + b^2) > 0$$

$$\Rightarrow \quad a^2 + 6ab - 7b^2 > 0$$

Hence, both Statements are true and Statement II is correct explanation for Statement I.

Subjective Type Questions

■ In this section, there are **12 subjective solved examples.**

● ***Ex. 27*** *Find the locus of the centroid of an equilateral triangle inscribed in the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Sol. Let the vertices of the equilateral triangle P, Q and R and whose eccentric angles are α, β and γ.

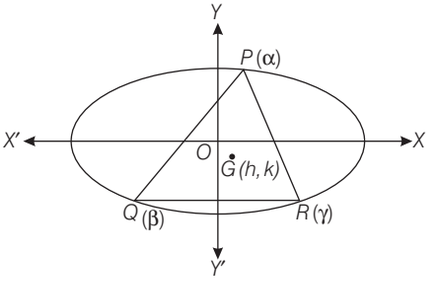
Let the centroid of ΔPQR be (h, k) then

$$h = \frac{a}{3}(\cos\alpha + \cos\beta + \cos\gamma) \qquad \ldots(i)$$

$$\text{and} \quad k = \frac{b}{3}(\sin\alpha + \sin\beta + \sin\gamma) \qquad \ldots(ii)$$

$\because\ \Delta PQR$ is equilateral.

$\therefore$ Centroid of the ΔPQR is same as the circumcentre.



$\because$ Circumcentre of ΔPQR be

$$h=\frac{(a^2-b^2)}{4a}\{\cos\alpha+\cos\beta+\cos\gamma+\cos(\alpha+\beta+\gamma)\}$$

$$\left.\begin{matrix}\text{Concyclic points}\\ \text{Coro : 2}\end{matrix}\right\}$$

$$k=\frac{(b^2-a^2)}{4b}\{\sin\alpha+\sin\beta+\sin\gamma-\sin(\alpha+\beta+\gamma)\}$$

Using Eqs. (i) and (ii), then

$$\cos(\alpha+\beta+\gamma)=h\left(\frac{4a}{a^2-b^2}-\frac{3}{a}\right)=\frac{h(a^2+3b^2)}{a(a^2-b^2)} \quad ...(iii)$$

and $$\sin(\alpha+\beta+\gamma)=\frac{(3a^2+b^2)k}{b(a^2-b^2)} \quad ...(iv)$$

Squaring and adding Eqs. (iii) and (iv), we get

$$\therefore \quad I=\frac{(a^2+3b^2)^2}{a^2(a^2-b^2)^2}h^2+\frac{(3a^2+b^2)^2}{b^2(a^2-b^2)^2}k^2$$

Hence, locus of (h,k) is

$$\frac{(a^2+3b^2)^2}{a^2}x^2+\frac{(3a^2+b^2)^2}{b^2}y^2=(a^2-b^2)^2.$$

● ***Ex. 28*** *If the normals to* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *at the ends of the chords* $lx+my=1$ *and* $l'x+m'y=1$ *be concurrent; show that* $a^2ll'=b^2mm'=-1$.

Sol. Let these normals meet in (h,k), then their feet lie on the curve

$$b^2x(k-y)+a^2y(x-h)=0$$

$$\Rightarrow \quad (a^2-b^2)xy+b^2kx-a^2hy=0$$

$\therefore$ The locus passes through the points where the lines

$$lx+my-1=0 \quad \text{and} \quad l'x+m'y-1=0$$

meet the given ellipse,

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)-\lambda(lx+my-1)(l'x+m'y-1)$$

$$\equiv(a^2-b^2)xy+b^2kx-a^2hy$$

Comparing the co-efficients of x^2, y^2 and constant terms.

$$\frac{1}{a^2}-\lambda ll'=0, \frac{1}{b^2}-\lambda mm'=0, -1-\lambda=0$$

$$\therefore \quad \frac{1}{a^2}+ll'=0 \quad \text{and} \quad \frac{1}{b^2}+mm'=0 \qquad (\because \lambda=-1)$$

$$\therefore \quad a^2ll'=b^2mm'=-1$$

● ***Ex. 29*** *An ellipse slides between two straight lines at right angles to each other. Show that the locus of its centre is a circle.*

Sol. Let the length of major and minor axes of an ellipse are $2a$ and $2b$ and if the centre of the ellipse be $C(h,k)$

If $S(x_1,y_1)$ and $S'(x_2,y_2)$ be two foci of the ellipse, then

$$SS'=2ae$$

$$\Rightarrow (x_1-x_2)^2+(y_1-y_2)^2=4a^2e^2$$

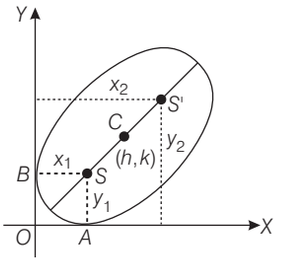
$$\Rightarrow (x_1+x_2)^2-4x_1x_2+(y_1+y_2)^2$$

$$-4y_1y_2=4(a^2-b^2)$$

$$\Rightarrow$$

$$(2h)^2+(2k)^2-4(x_1x_2+y_1y_2)$$

$$=4(a^2-b^2) \quad ...(i)$$



Since, the ellipse always slides between the two fixed lines OX and OY, they are always tangents to it. Therefore y_1, y_2 and x_1, x_2 are perpendicular distances of the foci from their tangents whose product are always b^2.

Hence, $$x_1x_2=y_1y_2=6^2$$

$\Rightarrow$ Eq. (i) becomes

$$4h^2+4k^2-8b^2=4(a^2-b^2)$$

$$\Rightarrow \quad h^2+k^2=a^2+b^2$$

$\therefore$ Locus of the centre (h,k) is the circle

$$x^2+y^2=a^2+b^2$$

● ***Ex. 30*** *Triangles are formed by pairs of tangents drawn from any point on the ellipse* $a^2x^2+b^2y^2=(a^2+b^2)^2$ *to the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, *and the chord of contact. Show that the orthocentre of each such triangles lies on the ellipse.*

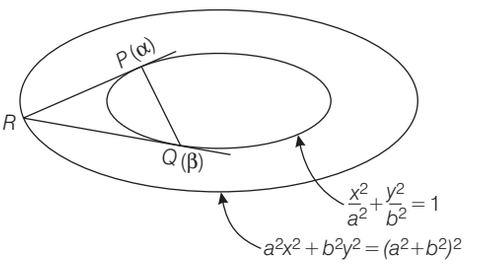
Sol. Let the coordinates of P and Q on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are

$$P\equiv(a\cos\alpha, b\sin\alpha) \quad \text{and} \quad Q=(a\cos\beta, b\sin\beta)$$

Equation of tangents at P and Q are

$$\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha=1 \quad ...(i)$$

and $$\frac{x}{a}\cos\beta+\frac{y}{b}\sin\beta=1 \quad ...(ii)$$



Point of intersection of Eqs. (i) and (ii) are

$$R\left\{a\frac{\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}, b\frac{\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}\right\}$$

This point R lies on the ellipse

$$a^2x^2 + b^2y^2 = (a^2 + b^2)^2$$

$$\Rightarrow \quad a^4\cos^2\left(\frac{\alpha+\beta}{2}\right) + b^4\sin^2\left(\frac{\alpha+\beta}{2}\right) = (a^2+b^2)^2\cos^2\left(\frac{\alpha-\beta}{2}\right) \quad ...(iii)$$

The equation of the line of perpendicular from Q on the tangent at P is

$$a\sin\alpha(x - a\cos\beta) - b\cos\alpha(y - b\sin\beta) = 0 \quad ...(iv)$$

Similarly the equation to other perpendicular will be

$$a\sin\beta(x - a\cos\beta) - b\cos\beta(y - b\sin\alpha) = 0 \quad ...(v)$$

On solving Eqs. (iv) and (v), we get

$$x = \frac{a^3}{a^2+b^2}\cdot\frac{\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)} \quad \text{and} \quad y = \frac{b^3}{a^2+b^2}\cdot\frac{\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}$$

Since, this orthocentre lies on ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

then, $a^4\cos^2\left(\frac{\alpha+\beta}{2}\right) + b^4\sin^2\left(\frac{\alpha+\beta}{2}\right) = (a^2+b^2)^2\cos^2\left(\frac{\alpha-\beta}{2}\right)$ ...(vi)

which is true by Eq. (iii).

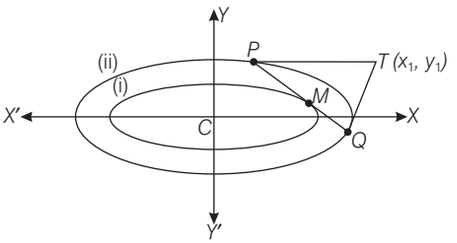
Hence, the orthocentre of the triangle lies on the ellipse.

● ***Ex. 31*** *Show that the tangents drawn at those points of the ellipse* $\frac{x^2}{a} + \frac{y^2}{b} = (a+b)$, *where it is cut by any tangent to* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, *intersect at right angles.*

Sol. The given ellipses, are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad ...(i)$$

$$\frac{x^2}{a(a+b)} + \frac{y^2}{b(a+b)} = 1 \quad ...(ii)$$



Chord of contact of (x_1, y_1) w.r.t ellipse (ii) is

$$\frac{xx_1}{a(a+b)} + \frac{yy_1}{b(a+b)} = 1 \quad ...(iii)$$

or $$lx + my = n \quad \text{(say)}$$

If it touches $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

then $$a^2l^2 + b^2m^2 = n^2 \quad \text{[from corollary]}$$

$$\Rightarrow \quad \frac{a^2x_1^2}{a^2(a+b)^2} + \frac{b^2y_1^2}{b^2(a+b)^2} = 1$$

$$\Rightarrow \quad x_1^2 + y_1^2 = (a+b)^2$$

Locus of $T(x_1, y_1)$ is $x^2 + y^2 = (a+b)^2$

$$= a(a+b) + b(a+b)$$

which is the director circle of (ii).

Hence, PT and TQ are intersect at right angle.

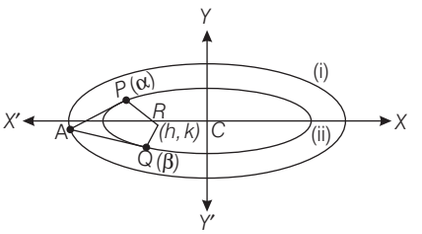
● ***Ex. 32*** *From any point on the conic* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 4$, *tangents are drawn to the conic* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. *Prove that the normals at the points of contact meet on the conic*

$$a^2x^2 + b^2y^2 = \frac{1}{4}(a^2 - b^2)^2.$$

Sol. Equations of concentric ellipses are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 4 \quad ...(i)$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad ...(ii)$$



Let points $P(a\cos\alpha, b\sin\alpha)$ and $Q(a\cos\beta, b\sin\beta)$ on the ellipse (ii). Equation of tangents at P and Q are

$$\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = 1 \quad ...(iii)$$

and $$\frac{x}{a}\cos\beta + \frac{y}{b}\sin\beta = 1 \quad ...(iv)$$

Point of intersection of Eqs. (iii) and (iv) is

$$A\left\{\frac{a\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}, \frac{b\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}\right\}$$

this point A satisfy Eq.(i)

$$\therefore \quad \frac{\cos^2\left(\frac{\alpha+\beta}{2}\right)}{\cos^2\left(\frac{\alpha-\beta}{2}\right)} + \frac{\sin^2\left(\frac{\alpha+\beta}{2}\right)}{\cos^2\left(\frac{\alpha-\beta}{2}\right)} = 4$$

$$\Rightarrow \quad 1 = 4\cos^2\left(\frac{\alpha-\beta}{2}\right), \frac{1}{2}$$

$$= 1 + \cos(\alpha-\beta)$$

$$\therefore \quad \cos(\alpha-\beta) = -\frac{1}{2} \qquad ...(v)$$

Equations of normals PR and QR are

$$ax\sec\alpha - by\,\text{cosec}\,\alpha = a^2 - b^2$$

and $$ax\sec\beta - by\,\text{cosec}\,\beta = (a^2 - b^2)$$

or $$ax\sin\alpha - by\cos\alpha = (a^2-b^2)\sin\alpha\cos\alpha \qquad ...(vi)$$

$$ax\sin\beta - by\cos\beta = (a^2-b^2)\sin\beta\cos\beta \qquad ...(vii)$$

Multiplying Eq. (vi) by $\cos\beta$ and (vii) by $\cos\alpha$ and subtracting, we get

$$ax\sin(\alpha-\beta) = (a^2-b^2)\cos\alpha\cos\beta(\sin\alpha - \sin\beta)$$

$$\frac{ax}{a^2-b^2} = \frac{\cos\alpha\cos\beta\cdot 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}{2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}$$

$$= \frac{\cos\alpha\cos\beta\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}$$

$$= \pm 2\cos\alpha\cos\beta\cos\left(\frac{\alpha+\beta}{2}\right) \qquad \text{[From (v)]}$$

$$= \pm\cos\left(\frac{\alpha+\beta}{2}\right)[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$$

$$\frac{ax}{a^2-b^2} = \pm\cos\left(\frac{\alpha+\beta}{2}\right)\left\{\cos(\alpha+\beta) - \frac{1}{2}\right\} \qquad \text{[Using (v)]}$$

$$\Rightarrow \quad \frac{a^2x^2}{(a^2-b^2)^2} = \cos^2\left(\frac{\alpha+\beta}{2}\right)\left[\cos(\alpha+\beta) - \frac{1}{2}\right]^2$$

Similarly, $$\frac{b^2y^2}{(a^2-b^2)^2} = \sin^2\left(\frac{\alpha+\beta}{2}\right)\left[\cos(\alpha+\beta) + \frac{1}{2}\right]^2$$

$$\therefore \quad \frac{a^2x^2}{(a^2-b^2)^2} + \frac{b^2y^2}{(a^2-b^2)^2}$$

$$= \cos^2(\alpha+\beta) + \frac{1}{4} - \cos^2(\alpha+\beta)$$

Hence, required locus is

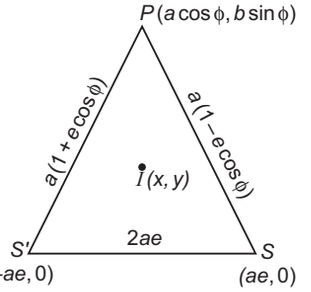
$$a^2x^2 + b^2y^2 = \frac{1}{4}(a^2-b^2)^2$$

● ***Ex. 33*** *A variable point P on an ellipse of eccentricity e, is joined to its foci S, S′ prove that the locus of the incentre of the triangle PSS′ is an ellipse whose eccentricity is* $\sqrt{\left(\frac{2e}{1+e}\right)}$.

Sol. Let the given ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Let the coordinates of P are $(a\cos\phi, b\sin\phi)$



By hypothesis,

$$b^2 = a^2(1-e^2) \text{ and } S(ae, 0), S'(-ae, 0)$$

$\therefore$ SP = focal distance of the point P

$$= a - ae\cos\phi \text{ and } S'P = a + ae\cos\phi$$

Also $SS' = 2ae$

If (x, y) be the incentre of the $\Delta PSS'$, then

$$\therefore \quad x = \frac{(2ae)\,a\cos\phi + a(1-e\cos\phi)(-ae) + 1(1+e\cos\phi)\,ae}{2ae + a(1-e\cos\phi) + a(1+e\cos\phi)}$$

$$x = ae\cos\phi \qquad ...(i)$$

and $$y = \frac{2ae(b\sin\phi) + a(1+e\cos\phi)\cdot 0 + a(1-e\cos\phi)\cdot 0}{2ae + a(1-e\cos\phi) + a(1+e\cos\phi)},$$

$$\Rightarrow \quad y = \frac{eb\sin\phi}{(e+1)} \qquad ...(ii)$$

Eliminating ϕ from Eqs. (i) and (ii), we get

$$\frac{x^2}{a^2e^2} + \frac{y^2}{\left[\frac{be}{e+1}\right]^2} = 1$$

which represents an ellipse.

Let e_1 be its eccentricity.

$$\therefore \quad \frac{b^2e^2}{(e+1)^2} = a^2e^2(1-e_1^2) \;\Rightarrow\; e_1^2 = 1 - \frac{b^2}{a^2(e+1)^2}$$

$$= 1 - \frac{1-e^2}{(e+1)^2} = 1 - \frac{1-e}{1+e} = \frac{2e}{1+e} \;\Rightarrow\; e_1 = \sqrt{\left(\frac{2e}{1+e}\right)}$$

● ***Ex. 34*** *Prove that the eccentricity of the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *is given by* $2\cot\omega = \frac{e^2\sin 2\theta}{\sqrt{(1-e^2)}}$ *where,* ω *is one of the angles between the normals at the points whose eccentric angles are* θ *and* $\frac{\pi}{2} + \theta$.

Sol. The equation of the normals at two given points θ and $\frac{\pi}{2} + \theta$ are

$$ax\sec\theta - by\,\text{cosec}\,\theta = a^2 - b^2 \qquad ...(i)$$

and $$ax\sec\left(\frac{\pi}{2}+\theta\right) - by\,\text{cosec}\left(\frac{\pi}{2}+\theta\right) = a^2 - b^2 \qquad ...(ii)$$

their slopes are $m_1 = \frac{a}{b}\tan\theta$ and $m_2 = -\frac{a}{b}\cot\theta$

$$\therefore \quad \tan\omega = \frac{m_2 - m_1}{1 + m_2 m_1} = \frac{-\frac{a}{b}(\cot\theta + \tan\theta)}{\left(1 - \frac{a^2}{b^2}\right)} = \frac{2ab}{(a^2 - b^2)\sin 2\theta}$$

$$\Rightarrow \quad 2\cot\omega = \frac{(a^2 - b^2)\sin 2\theta}{ab} = \frac{a^2 e^2}{a \cdot a\sqrt{1 - e^2}}\sin 2\theta$$

$$\therefore \quad \frac{2\cot\omega}{\sin 2\theta} = \frac{e^2}{\sqrt{(1 - e^2)}}$$

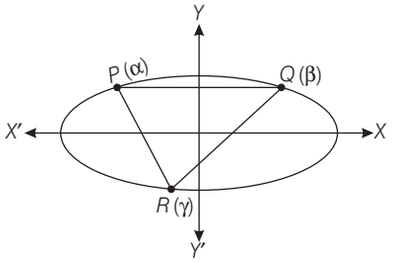
● ***Ex. 35*** *Show that the area of the triangle inscribed in an ellipse is*

$$\left[2ab\sin\left(\frac{\beta - \gamma}{2}\right)\sin\left(\frac{\gamma - \alpha}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)\right],$$

where α, β, γ are the eccentric angles of the vertices and hence find the condition that the area of a triangle inscribed in an ellipse may be maximum.

Sol. The vertices of the triangle PQR are

$$P \equiv (a\cos\alpha, b\sin\alpha)$$
$$Q \equiv (a\cos\beta, b\sin\beta)$$
$$R \equiv (a\cos\gamma, b\sin\gamma)$$



$$\therefore \text{ Area of } \Delta PQR = \frac{1}{2}\begin{vmatrix} a\cos\alpha & b\sin\alpha & 1 \\ a\cos\beta & b\sin\beta & 1 \\ a\cos\gamma & b\sin\gamma & 1 \end{vmatrix} = \frac{ab}{2}\begin{vmatrix} \cos\alpha & \sin\alpha & 1 \\ \cos\beta & \sin\beta & 1 \\ \cos\gamma & \sin\gamma & 1 \end{vmatrix}$$

$$= \frac{ab}{2}[\sin(\gamma - \beta) + \sin(\alpha - \gamma) + \sin(\beta - \alpha)]$$

$$= \frac{ab}{2}\left\{2\sin\left(\frac{\alpha - \beta}{2}\right)\cos\left(\frac{2\gamma - \alpha - \beta}{2}\right) - 2\sin\left(\frac{\alpha - \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)\right\}$$

$$= ab\sin\left(\frac{\alpha - \beta}{2}\right)\left[\cos\left(\frac{2\gamma - \alpha - \beta}{2}\right) - \cos\left(\frac{\alpha - \beta}{2}\right)\right]$$

$$= 2ab\sin\left(\frac{\alpha - \beta}{2}\right)\sin\left(\frac{\gamma - \beta}{2}\right)\sin\left(\frac{\alpha - \gamma}{2}\right)$$

$$\therefore \text{ Area of } \Delta PQR = 2ab\sin\left(\frac{\alpha - \beta}{2}\right)\sin\left(\frac{\beta - \gamma}{2}\right)\sin\left(\frac{\gamma - \alpha}{2}\right) \quad \ldots(i)$$

If P_1, Q_1, R_1 be the points upon the auxiliary circle corresponding to points P, Q, R, then

$$\text{Area of } \Delta P_1Q_1R_1 = 2a^2\sin\left(\frac{\alpha - \beta}{2}\right)\sin\left(\frac{\beta - \gamma}{2}\right)\sin\left(\frac{\gamma - \alpha}{2}\right)$$

{taking $b = a$ in Eq. (i)}

$$\therefore \quad \frac{\text{Area of } \Delta PQR}{\text{Area of } \Delta P_1Q_1R_1} = \frac{b}{a}$$

$$\therefore \quad (\text{Area of } \Delta PQR) = \left(\frac{b}{a}\right)\text{Area of } (\Delta P_1Q_1R_1)$$

$\therefore$ Area of ΔPQR is the greatest when area of $\Delta P_1Q_1R_1$ be the greatest and hence $\Delta P_1Q_1R_1$ must be equilateral

$$\therefore \quad \alpha - \beta = \beta - \gamma = \gamma - \alpha = \frac{2\pi}{3}$$

$\therefore$ The eccentric angles of P, Q, R will be

$$\alpha, \alpha + \frac{2\pi}{3}, \alpha + \frac{4\pi}{3}$$

● ***Ex. 36*** *Let ABC be an equilateral triangle inscribed in the circle $x^2 + y^2 = a^2$ suppose perpendiculars form A, B, C to the major axis of the ellipse*

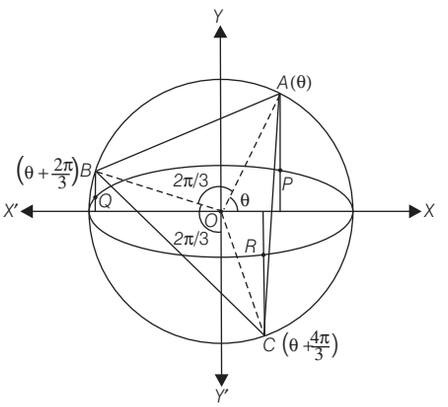
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (a > b)$$

meet the ellipse respectively at P, Q, R so that P, Q, R lie on the same side of the major axis as A, B, C respectively. Prove that the normal to the ellipse drawn at the points P, Q and R are concurrent.

Sol. Equation of normal at $P(a\cos\theta, b\sin\theta)$ is

$$ax\sec\theta - by\,\text{cosec}\,\theta = (a^2 - b^2)$$

or $$ax\sin\theta - by\cos\theta = \frac{1}{2}(a^2 - b^2)\sin 2\theta \quad \ldots(i)$$



Similarly, the other normals

$$ax\sin\left(\theta + \frac{2\pi}{3}\right) - by\cos\left(\theta + \frac{2\pi}{3}\right)$$

$$= \frac{1}{2}(a^2 - b^2)\sin\left(2\theta + \frac{4\pi}{3}\right) \quad \ldots(ii)$$

and $$ax\sin\left(\theta + \frac{4\pi}{3}\right) - by\cos\left(\theta + \frac{4\pi}{3}\right)$$

$$= \frac{1}{2}(a^2 - b^2)\sin\left(2\theta + \frac{8\pi}{3}\right) \quad \ldots(iii)$$

$$\because \quad \sin\left(\theta + \frac{4\pi}{3}\right) = \sin\left(\theta - \frac{2\pi}{3}\right), \cos\left(\theta + \frac{4\pi}{3}\right) = \cos\left(\theta - \frac{2\pi}{3}\right)$$

and $$\sin\left(2\theta + \frac{8\pi}{3}\right) = \sin\left(2\theta - \frac{4\pi}{3}\right)$$

Further the three normals are concurrent, then

$$\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta+\frac{2\pi}{3}\right) & \cos\left(\theta+\frac{2\pi}{3}\right) & \sin\left(2\theta+\frac{4\pi}{3}\right) \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta+\frac{8\pi}{3}\right) \end{vmatrix}=0$$

Applying $R_2 \to R_2 + R_3$, then

$$\text{LHS} = \begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ 2\sin\theta\cos\frac{2\pi}{3} & 2\cos\theta\cos\frac{2\pi}{3} & 2\sin(2\theta+2\pi)\cos\frac{2\pi}{3} \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta+\frac{8\pi}{3}\right) \end{vmatrix}$$

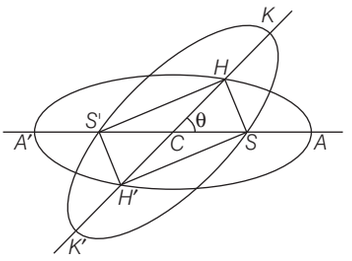
$$= -\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta+\frac{8\pi}{3}\right) \end{vmatrix} = 0 \quad (\because R_1 = R_2)$$

● ***Ex. 37*** *If two concentric ellipses be such that the foci of one be on the other and if e and e′ be their eccentricities. Prove that the angle between their axes is*

$$\cos^{-1}\left\{\frac{\sqrt{(e^2+e'^2-1)}}{ee'}\right\}$$

Sol. Let S and S' be the foci of one ellipse and H and H' be the other, C being their common centre. Then, $SHS'H'$ is a parallelogram and since

$$SH + S'H = HS' + H'S' = 2a$$



Since, the sum of the focal distances of any point on an ellipse is equal to its major axis which is $2a$.

Then $CS = ae, CH = ae'$

Let θ be the angle between their axes.

then $SH^2 = a^2e^2 + a^2e'^2 - 2a^2ee'\cos\theta$

$$HS'^2 = a^2e^2 + a^2e'^2 + 2a^2ee'\cos\theta$$

Now, $2a = SH + S'H$

Squaring both sides, then

$$4a^2 = (SH)^2 + (S'H)^2 + 2(SH)\cdot(S'H)$$

$$\Rightarrow \quad 4a^2 = 2a^2(e^2+e'^2) + 2\sqrt{(a^2e^2+a^2e'^2)^2 - 4a^4e^2e'^2\cos^2\theta}$$

$$\Rightarrow \quad (2-e^2-e'^2)^2 = (e^2+e'^2)^2 - 4e^2e'^2\cos^2\theta$$

$$\Rightarrow \quad 4+(e^2+e'^2)^2 - 4(e^2+e'^2) = (e^2+e'^2)^2 - 4e^2e'^2\cos^2\theta$$

$$\Rightarrow \quad 1-e^2-e'^2 = -e^2e'^2\cos^2\theta$$

$$\therefore \quad \cos\theta = \frac{\sqrt{e^2+e'^2-1}}{ee'}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{\sqrt{(e^2+e'^2-1}}{ee'}\right)$$

● ***Ex. 38*** *If the normals at the four points* $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ *and* (x_4, y_4) *on the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *are concurrent. Prove that*

$$(x_1+x_2+x_3+x_4)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)=4$$

or $(\Sigma\cos\alpha)(\Sigma\sec\alpha)=4$

where, $\alpha, \beta, \gamma, \delta$ *are the eccentric angles of the points.*

Sol. Let point of concurrent is (h, k).

Equation of normal at (x', y') is

$$\frac{x-x'}{x'/a^2} = \frac{y_1-y'}{y'/b^2}$$

It is passes through (h, k) then

$$y'^2\{a^2(h-x')+b^2x'\}^2 = b^4k^2x'^2 \qquad \ldots(i)$$

But $\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1$ or $y'^2 = \frac{b^2}{a^2}(a^2-x'^2)$ ...(ii)

Values of y'^2 from Eq. (ii), putting in Eq. (i) we get

$$\frac{b^2}{a^2}(a^2-x'^2)\{a^2h+(b^2-a^2)x'\}^2 = b^4k^2x'^2$$

$$\Rightarrow \frac{b^2}{a^2}(a^2-x'^2)\{a^4h^2+(b^2-a^2)^2x'^2+2a^2hx'(b^2-a^2)\} = b^4k^2x'^2$$

Arranging above as a fourth degree equation in x', we get

$$\Rightarrow \quad -(a^2-b^2)^2x'^4 + 2ha^2(a^2-b^2)x'^3 + x'^2(\ldots)$$
$$-2a^4h(a^2-b^2)x' + a^6h^2 = 0$$

above equation being of fourth degree in x', therefore roots of the above equation are x_1, x_2, x_3, x_4 then

$$(x_1+x_2+x_3+x_4) = -\frac{2ha^2(a^2-b^2)}{-(a^2-b^2)^2} = \frac{2ha^2}{(a^2-b^2)} \qquad \ldots(iii)$$

$$\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right) = \frac{\Sigma x_1x_2x_3}{x_1\cdot x_2\cdot x_3\cdot x_4}$$

$$= \frac{\frac{2a^4h(a^2-b^2)}{-(a^2-b^2)^2}}{\frac{a^6h^2}{-(a^2-b^2)^2}} = \frac{2(a^2-b^2)}{a^2h} \qquad \ldots(iv)$$

Multiplying Eqs. (iii) and (iv), we get

$$(x_1+x_2+x_3+x_4)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)=4$$

Replacing x_1 by $a\cos\alpha$, x_2 by $a\cos\beta$, ... etc, then, we get the second form

i.e. $\Sigma\cos\alpha\,\Sigma\sec\alpha = 4.$

Ellipse Exercise 1 :
Single Option Correct Type Questions

■ This section contains **30 multiple choice questions.** Each question has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

1. Given f is increasing, the equation $\frac{x^2}{f(2a)}+\frac{y^2}{f(a^2-3)}=1$ represents an ellipse with X-axis as major axis if
(a) $[-1,3]$ (b) $[1,3]$ (c) $(-1,3)$ (d) $(0,5)$

2. If $\frac{x^2}{f(4a)}+\frac{y^2}{f(a^2-5)}$ represents an ellipse with major axis as Y-axis and f is a decreasing fucntion, then
(a) $a\in(-\infty,1)$ (b) $a\in(5,\infty)$ (c) $a\in(1,4)$ (d) $a\in(-1,5)$

3. The curve represented by the equation $\frac{x^2}{\sin\sqrt{2}-\cos\sqrt{3}}+\frac{y^2}{\sin\sqrt{3}-\cos\sqrt{2}}=1$ is
(a) an ellipse with foci on X -axis
(b) an ellipse with foci on Y-axis
(c) a hyperbola with foci on X-axis
(d) a hyperbola with foci on Y-axis

4. The minimum distance of the centre of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ from the chord of contact of mutually perpendicular tangents of the ellipse is
(a) $\frac{144}{5}$ (b) $\frac{16}{5}$ (c) $\frac{9}{5}$ (d) None of these

5. P and Q are the foci of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and B is an end of the minor axis. If PBQ, is an equilateral triangle, then the eccentricity of the ellipse is
(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{3}$
(c) $\frac{1}{2}$ (d) $\frac{\sqrt{3}}{2}$

6. A circle of radius $\frac{5}{\sqrt{2}}$ is concentric with the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$, then the acute angle made by the common tangent with the line $\sqrt{3}x-y+6=0$ is
(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{6}$ (d) $\frac{\pi}{12}$

7. Consider the particle travelling clockwise on the elliptical path $\frac{x^2}{100}+\frac{y^2}{25}=1$. The particle leaves the orbit at the point $(-8,3)$ and travels in a straight line tangent to the ellipse, then the point will the particle cross the Y-axis is
(a) $\left(0,-\frac{25}{3}\right)$ (b) $\left(0,\frac{7}{3}\right)$
(c) $\left(0,\frac{25}{3}\right)$ (d) $(0,9)$

8. C is the centre of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ and A and B are two points on the ellipse such that $\angle ACB=90°$, then $\frac{1}{(CA)^2}+\frac{1}{(CB)^2}=$
(a) $\frac{7}{12}$ (b) $\frac{12}{7}$
(c) $\frac{25}{144}$ (d) $\frac{144}{25}$

9. Let (α,β) be a point from which two perpendicular tangents can be drawn to the ellipse $4x^2+5y^2=20$. If $F=4\alpha+3\beta$, then
(a) $-15\le F\le 15$ (b) $F\ge 0$
(c) $-5\le F\le 20$ (d) $F\le -5\sqrt{5}$ or $F\ge 5\sqrt{5}$

10. If $a=[t^2-3t+4]$ and $b=[3+5t]$, where $[\cdot]$ denotes the greatest integer function, then the latusrectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $t=\frac{3}{2}$ is
(a) 20 (b) 10
(c) $\frac{1}{5}$ (d) $\frac{1}{10}$

11. If the line $x+2y+4=0$ cutting the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in points whose eccentric angles are $30°$ and $60°$ subtends a right angle at the orgin, then its equation is
(a) $\frac{x^2}{8}+\frac{y^2}{4}=1$
(b) $\frac{x^2}{16}+\frac{y^2}{4}=1$
(c) $\frac{x^2}{4}+\frac{y^2}{16}=1$
(d) None of the above

12. An arc of a bridge is semi-elliptical with major axis horizontal. If the length of the base is 9 m and the highest part of the bridge is 3 m from the horizontal; the best approximation of the height of the arch 2 m from the centre of the base is

(a) $\frac{11}{4}$m (b) $\frac{8}{3}$m

(c) $\frac{7}{2}$m (d) 2m

13. A tangent to the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ at any point P meet the line $x=0$ at a point Q. Let R be the image of Q in the line $y=x$, then circle whose extremities of a diameter are Q and R passes through a fixed point. The fixed point is

(a) (3, 0) (b) (4, 0)

(c) (5, 0) (d) (0, 0)

14. If tangents PQ and PR are drawn from a point on the circle $x^2+y^2=25$ to the ellipse $\frac{x^2}{16}+\frac{y^2}{b^2}=1, (b<4)$, so that the fourth vertex S of parallelogram $PQSR$ lies on the circumcircle of triangle PQR, then the eccentricity of the ellipse is

(a) $\frac{\sqrt{5}}{4}$ (b) $\frac{\sqrt{5}}{3}$

(c) $\frac{\sqrt{7}}{4}$ (d) $\frac{\sqrt{7}}{3}$

15. The equation of the chord of contact of the pair of tangents drawn to the ellipse $4x^2+9y^2=36$ from the point (m, n), where $m\cdot n=m+n$, m, n being non-zero positive integers is

(a) $2x+9y=18$ (b) $2x+2y=1$

(c) $4x+9y=18$ (d) $4x+2y=1$

16. $x-2y+4=0$ is a common tangent to $y^2=4x$ and $\frac{x^2}{4}+\frac{y^2}{b^2}=1$. Then, the value of b and the other common tangent are given by

(a) $b=\sqrt{3}$; $x+2y+4=0$

(b) $b=3$; $x+2y+4=0$

(c) $b=\sqrt{3}$; $x+2y-4=0$

(d) $b=\sqrt{3}$, $x-2y-4=0$

17. The point , at shortest distance from the line $x+y=7$ and lying on an ellipse $x^2+2y^2=6$, has coordinates

(a) $(\sqrt{2}, \sqrt{2})$ (b) $(0, \sqrt{3})$

(c) (2, 1) (d) $\left(\sqrt{5}, \frac{1}{\sqrt{2}}\right)$

18. From a point on the axis of x common tangents are drawn to the parabola $y^2=4x$ and the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1(a>b>0)$. If these tangents form an equilateral triangle with their chord of contact w.r.t parabola, then set of exhaustive values of a is

(a) (0, 3) (b) $\left(0, \frac{3}{2}\right)$

(c) $\left(\frac{3}{2}, 3\right)$ (d) $\left(1, \frac{3}{2}\right)$

19. If circumcentre of an equilateral triangle inscribed in $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, with vertices having eccentric angle α, β, γ respectively is (h, k), then $\Sigma\cos(\alpha-\beta)=$

(a) $\frac{9h^2}{a^2}+\frac{9k^2}{b^2}+\frac{3}{2}$ (b) $9h^2-9k^2+a^2b^2$

(c) $\frac{9h^2}{a^2}+\frac{9k^2}{b^2}+3$ (b) $\frac{9h^2}{2a^2}+\frac{9k^2}{2b^2}-\frac{3}{2}$

20. A parabola is drawn with focus at one of the foci of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a>b$ and directrix passing through the other focus and perpendicular to the major axis of the ellipse. If the latusrectum of the ellipse and that of the parabola are some, then the eccentricity of the ellipse is

(a) $\sqrt{2}-1$ (b) $\sqrt{3}-1$

(c) $2\sqrt{2}-2$ (d) $3\sqrt{3}-5$

21. If the maximum distance of any point on the ellipse $x^2+2y^2+2xy=1$ from its center is r, then r is equal to

(a) $\frac{\sqrt{6}+1}{2}$ (b) $\frac{\sqrt{5}+1}{2}$

(c) $\frac{\sqrt{3}+1}{2}$ (d) $\frac{\sqrt{2}+2}{2}$

22. The length of the common chord of the ellipse $\frac{(x-1)^2}{9}+\frac{(y-2)^2}{4}=1$ and the circle $(x-1)^2+(y-2)^2=1$ is

(a) zero (b) one

(c) three (d) eight

23. The eccentricity of the ellipse $ax^2+by^2+2fx+2gy+c=0$ if axis of ellipse parallel to X-axis is

(a) $\sqrt{\left(\frac{b-a}{b}\right)}$ (b) $\sqrt{\left(\frac{a+b}{b}\right)}$

(c) $\sqrt{\left(\frac{a+b}{a}\right)}$ (d) None of these

24. A circle has the same center as an ellipse and passes through the foci F_1 and F_2 of the ellipse, such that two curves intersect at four points. Let P be any one of their point of intersection. If the major axis of the ellipse is 17 and the area of triangle PF_1F_2 is 30, then the distance between the foci is
(a) 13 (b) 11
(c) 9 (d) 7

25. The area of the rectangle formed by the perpendicular from the centre of the standard ellipse to the tangent and normal at its point whose eccentric angle is $\frac{\pi}{4}$, is
(a) $\frac{(a^2-b^2)\,ab}{a^2+b^2}$
(b) $\frac{(a^2-b^2)}{(a^2+b^2)\,ab}$
(c) $\frac{(a^2+b^2)\,ab}{(a^2-b^2)}$
(d) $\frac{(a^2+b^2)}{(a^2-b^2)\,ab}$

26. An ellipse is inscribed in a circle and a point within the circle is chosen at random. If the probability that this point lies outside the ellipse is 2/3, then the eccentricity of the ellipse is
(a) $\frac{2\sqrt{2}}{3}$ (b) $\frac{\sqrt{5}}{3}$
(c) $\frac{8}{9}$ (d) $\frac{2}{3}$

27. An ellipse slides between two perpendicular straight lines.
Then, the locus of its centre is a/an
(a) parabola (b) ellipse
(c) hyperbola (d) circle

28. The length of the chord of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$, where mid-point is $\left(\frac{1}{2},\frac{2}{3}\right)$ is
(a) $\frac{1}{10}$ (b) $\frac{\sqrt{8161}}{10}$
(c) $\frac{\sqrt{8061}}{10}$ (d) None of these

29. The equation of the locus of the middle point of the portion of the tangent to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ include between the coordinate axes is the curve
(a) $9x^2+16y^2=4x^2y^2$ (b) $16x^2+9y^2=4x^2y^2$
(c) $3x^2+4y^2=4x^2y^2$ (d) $9x^2+16y^2=x^2y^2$

30. The tangent at the point 'θ' on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the auxilliary circle in two points which subtends a right angle at the centre, then the eccentricity 'e' of the ellipse is given by the equation
(a) $e^2(1+\cos^2\theta)=1$
(b) $e^2(\operatorname{cosec}^2\theta+1)=1$
(c) $e^2(1+\sin^2\theta)=1$
(d) $e^2(1+\tan^2\theta)=1$

Ellipse Exercise 2 : More than One Correct Option Type Questions

■ This section contains **15 multiple choice questions**. Each question has four choices (a), (b), (c) and (d) out of which **MORE THAN ONE** may be correct.

31. The locus extremities of the latusrectum of the family of ellipse $b^2x^2+y^2=a^2b^2$ is
(a) $x^2-ay=a^2$ (b) $x^2-ay=b^2$
(c) $x^2+ay=a^2$ (d) $x^2+ay=b^2$

32. The distance of a point on the ellipse $\frac{x^2}{6}+\frac{y^2}{2}=1$ from the centre is 2, then the eccentric angles are
(a) $\frac{\pi}{4}$ (b) $\frac{3\pi}{4}$ (c) $\frac{5\pi}{4}$ (d) $\frac{7\pi}{4}$

33. If the equation of family of ellipse is $x^2\sec^2\theta+y^2\operatorname{cosec}^2\theta=1$, where $\frac{\pi}{4}<\theta<\frac{\pi}{2}$, then the locus of extremities of the latusrectum is
(a) $2y^2(1+x^2)=(1-x^2)^2$ (b) $2x^2(1+y^2)=(1-y^2)^2$
(c) $2y(1-x^2)=1+x^2$ (d) $2y^2(1+x^2)=1+y^4-2x^2$

34. Let F_1, F_2 be two focii of the ellipse and PT and PN be the tangent and the normal respectively to the ellipse at point P. Then.
(a) PN bisects $\angle F_1PF_2$
(b) PT bisects $\angle F_1PF_2$
(c) PT bisects angle $(180^\circ-\angle F_1PF_2)$
(d) None of the above

35. $\frac{x^2}{(a^2-a-6)}+\frac{y^2}{(a^2-6a+5)}=1$ will represent an ellipse if a lies in the interval
(a) $(-\infty,-2)$ (b) $(1,\infty)$
(c) $(3,\infty)$ (d) $(5,\infty)$

36. A latusrectum of an ellipse is a line
(a) passing through a focus
(b) passing through the centre
(c) perpendicular to the major axis
(d) parallel to the minor axis

37. If the axes of an ellipse coincide with the coordinate axes and the ellipse passes through the point $(4,-1)$ and touches the line $x+4y-10=0$, then its equation is
(a) $x^2+64y^2=80$ (b) $x^2+4y^2=20$
(c) $x^2+20y^2=100$ (d) $x^2+8y^2=40$

38. If P is a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose foci are S and S'. Let $\angle PSS'=\theta$ and $\angle PS'S=\phi$, then
(a) $SP+S'P=2a$, if $a>b$ (b) $SP+S'P=2b$, if $b>a$
(c) $\tan\left(\frac{\theta}{2}\right)\tan\left(\frac{\phi}{2}\right)=\frac{1-e}{1+e}$ (d) $\tan\left(\frac{\theta}{2}\right)\tan\left(\frac{\phi}{2}\right)=\frac{e-1}{e+1}$

39. If $(5,12)$ and $(24,7)$ are the focii of a conic passing through the origin, then the eccentricity of conic is
(a) $\frac{\sqrt{386}}{38}$ (b) $\frac{\sqrt{386}}{12}$
(c) $\frac{\sqrt{386}}{13}$ (d) $\frac{\sqrt{386}}{25}$

40. If a pair of variable straight lines $x^2+4y^2+axy=0$ (where a is a real parameter) cuts the ellipse $x^2+4y^2=4$ at two points A and B, then the locus of the point of intersection of tangents at A and B is
(a) $x-2y=0$ (b) $2x-y=0$
(c) $x+2y=0$ (d) $2x+y=0$

41. In the ellipse $25x^2+9y^2-150x-90y+225=0$
(a) foci are at (3, 1), (3, 9) (b) $e=\frac{4}{5}$
(c) centre is (5, 3) (d) major axis is 6

42. If the tangent to the ellipse $x^2+4y^2=16$ at the point $P(\theta)$ is a normal to the circle $x^2+y^2-8x-4y=0$, then θ equals
(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$
(c) 0 (d) $-\frac{\pi}{4}$

43. The product of eccentricties of two conics is unity, one of them can be a/an
(a) parabola (b) ellipse
(c) hyperbola (d) circle

44. The parametric angle α, where $-\pi<\alpha\le\pi$ of the point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at which the tangent drawn cuts the intercept of minimum length on the coordinate axes, is/are
(a) $\tan^{-1}\sqrt{\left(\frac{b}{a}\right)}$ (b) $-\tan^{-1}\sqrt{\left(\frac{b}{a}\right)}$
(c) $\pi-\tan^{-1}\sqrt{\left(\frac{b}{a}\right)}$ (d) $\pi+\tan^{-1}\sqrt{\left(\frac{b}{a}\right)}$

45. If latusrectum of the ellipse $x^2\tan^2\alpha+y^2\sec^2\alpha=1$ is 1/2, then $\alpha(0<\alpha<\pi)$ is equal to
(a) $\pi/12$ (b) $\pi/6$
(c) $5\pi/12$ (d) $\pi/2$

Ellipse Exercise 3 : Paragraph Based Questions

- This section contains **5 paragraphs** based upon each paragraph **3 multiple choice question** have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Paragraph I

(Q. Nos. 46 to 48)

A conic is represented by

$$C\equiv 9x^2+4xy+6y^2-22x-16y+9=0$$

46. The centre of the conic C is
(a) (0, 0) (b) (1, 0) (c) (0, 1) (d) (1, 1)

47. The eccentricity of the conic C is
(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$
(c) $\frac{2}{3}$ (d) $\frac{2}{\sqrt{5}}$

48. The lengths of axes of conic C are
(a) $2, 2\sqrt{2}$ (b) $4, 2\sqrt{3}$
(c) $6, 2\sqrt{5}$ (d) $5, 2\sqrt{6}$

Paragraph II

(Q. Nos. 49 to 51)

An ellipse E has its centre $C(3,1)$, focus at $(3,6)$ and passing through the point $P(7,4)$

49. The product of the lengths of the perpendicular segments from the focii on tangent at point P is

(a) 20 (b) 45

(c) 40 (d) 90

50. If F and F' are the foot of perpendiculars from focii S and S' on tangent at point P, then point of intersection of lines SF' and $S'F$ is

(a) $\left(5, \frac{5}{3}\right)$ (b) $\left(3, \frac{4}{3}\right)$

(c) $\left(5, \frac{10}{3}\right)$ (d) $\left(3, \frac{8}{3}\right)$

51. If the normal at a variable point on the ellipse (E) meets its axes in Q and R, then the locus of the mid-point of QR is a conic with eccentricity (e_1), then

(a) $e_1 = \frac{3}{\sqrt{5}}$ (b) $e_1 = \frac{\sqrt{5}}{3}$

(c) $e_1 = \frac{3}{\sqrt{10}}$ (d) $e_1 = \frac{\sqrt{10}}{3}$

Paragraph III

(Q. Nos. 52 to 54)

Curves $C_1 : x^2 + y^2 = r^2$ and $C_2 : \frac{x^2}{16} + \frac{y^2}{9} = 1$ intersect at four distinct points A, B, C and D. Their common tangents form a parallelogram PQRS.

52. If $ABCD$ is a square, then the value of $25r^2$ is

(a) 36 (b) 72

(c) 144 (d) 288

53. If $PQRS$ is a square, then the value of $2r^2$ is

(a) 12 (b) 15

(c) 20 (d) 25

54. If $PQRS$ is a square, then the ratio of the area of curve C_1 to the area of circumcircle of ΔPQR is

(a) 1 : 4 (b) 1 : 2

(c) 3 : 4 (d) 9 : 16

Paragraph IV

(Q. Nos. 55 to 57)

An ellipse whose distance between foci S and S′ is 4 units is inscribed in the ΔABC touching the sides AB, AC and BC at P, Q and R, respectively. If centre of ellipse is at origin and major axis along X-axis, $SP + S'P = 6$.

55. Equation of the ellipse is

(a) $9x^2 + 5y^2 = 45$ (b) $4x^2 + 9y^2 = 36$

(c) $5x^2 + 9y^2 = 45$ (c) $9x^2 + 4y^2 = 36$

56. If $AB = BC$ and $\angle B = \pi / 2$, then locus of A is

(a) $(x^2 + y^2 - 14)^2 = 4(5x^2 + 9y^2 - 45)$

(b) $(x^2 + y^2 - 14)^2 = 4(5x^2 + 9y^2 - 54)$

(c) $(x^2 + y^2 - 14)^2 = 4(9x^2 + 5y^2 - 45)$

(d) $(x^2 + y^2 - 14)^2 = 4(9x^2 + 5y^2 - 54)$

57. If the difference of eccentric angles of P and Q is $\pi / 3$, then locus of A is

(a) $5x^2 + 9y^2 = 15$ (b) $5x^2 + 9y^2 = 60$

(c) $9x^2 + 5y^2 = 14$ (d) $9x^2 + 5y^2 = 144$

Paragraph V

(Q. Nos. 58 to 60)

The line $2px + y\sqrt{(1 - p^2)} = 1$ $(|p| < 1)$ for different values of p, touches a fixed ellipse whose axes are the coordinate axes.

58. The eccentricity of the ellipse is

(a) $\frac{1}{\sqrt{5}}$ (b) $\frac{1}{\sqrt{3}}$

(c) $\frac{\sqrt{3}}{2}$ (d) $\frac{2}{\sqrt{5}}$

59. The foci of the ellipse are

(a) $\left(\pm\frac{\sqrt{3}}{2}, 0\right)$ (b) $\left(0, \pm\frac{\sqrt{3}}{2}\right)$

(c) $(\pm\sqrt{3}, 0)$ (d) $(0, \pm\sqrt{3})$

60. The locus of the point of intersection of perpendicular tangents of ellipse is

(a) $x^2 + y^2 = 2$ (b) $x^2 + y^2 = \frac{3}{2}$

(c) $x^2 + y^2 = \frac{5}{4}$ (d) $x^2 + y^2 = \frac{1}{2}$

Ellipse Exercise 4 :
Single Integer Answer Type Questions

■ This section contains **10 questions**. The answer to each question is a **single digit integer**, ranging from 0 to 9 (both inclusive).

61. Two concentric ellipse be such that the foci of one be on the other and if 3/5 and 4/5 be their eccentricities. If θ be the angle between their axes, then the value of $2(1+\sin^2\theta+\sin^4\theta)$ must be

62. Rectangle $ABCD$ has area 200 unit^2. An ellipse with area 200π unit^2, passes through A and C and has foci at B and D. If the perimeter of the rectangle is P, then the value of $\sqrt{(P+1)}$ is

63. Number of points on the ellipse $2x^2+5y^2=100$ from which pair of perpendicular tangents may be drawn to the ellipse $9x^2+16y^2=144$ is

64. The length of sides of square which can be made by four perpendicular tangents to the ellipse $\frac{x^2}{7}+\frac{2y^2}{11}=1$ is

65. The length of the focal chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which makes an angle θ with the major axis is $\frac{\lambda ab^2}{a^2\sin^2\theta+b^2\cos^2\theta}$, then the value of λ is

66. Number of distinct normal lines that can be drawn to ellipse $\frac{x^2}{169}+\frac{y^2}{25}=1$ from the point $P(0,6)$ is n, then the value of 2^n is

67. If p is the length of the perpendicular from a focus upon the tangent at any point P of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and r is the distance of P from the focus, then the value of $\frac{2a}{r}-\frac{b^2}{p^2}$ is

68. An ellipse passing through the origin has its foci (3, 4) and (6, 8). The length of its semi-minor axis is b, then the value of $b/\sqrt{2}$ is

69. The maximum value of 5λ for which four normals can be drawn to ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ through a point $(\lambda,0)$ is

70. An ellipse with major and minor axis $6\sqrt{3}$ and 6 respectively, slides along the coordinate axes and always remains confined in the first quadrant. If the length of arc described by centre of ellipse is $\frac{\pi\lambda}{6}$, then the value of λ is

Ellipse Exercise 5 : Matching Type Questions

■ This Section Contains **3 questions**. Each question has four statements (A,B,C and D) given in **Column** I and four statements (p,q,r and s) in **Column II**. Any given statement in **Column I** can have correct matching with one or more statement (s) given in **Column II.**

71. Match the following

	Column I		Column II
(A)	For the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ with vertices A and A', tangents drawn at the point P in the first quadrant meets the $y-$axis at Q and the chord $A'P$ meets the y-axis at M. If O is the origin, then OQ^2-MQ^2 is a	(p)	Natural number
(B)	If $y=x$ and $3y+2x=0$ are the equations of a pair of conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and e be the eccentricity, then $4\,(1+e^2+e^4+...+\infty)$ is a	(q)	Composite number
(C)	If the variable line $y=kx+2h$ is tangent to an ellipse $2x^2+3y^2=6$, then the locus of $P(h,k)$ is a conic C whose eccentricity is e, thus $3e^2$ is a	(r)	Prime number
(D)	If extremities of the latusrectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $(a>1)$ having positive ordinates lie on the parabola $x^2=-2(y-2)$, then a is a	(s)	Perfect number

72. Match the following

	Column I		Column II
(A)	An ellipse is sliding along the cordinate axes. If the foci of the ellipse are (1,1) and (3,3) if area of the director circle of the ellipse (in square units) is $\pi\lambda$, then λ is divisible by	(p)	3
(B)	If from a point P (0, λ) two normals other then axes are drawn to the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$, such that $\lvert\lambda\rvert<p$, then $4p$ is divisible by,	(q)	4
(C)	The orbit of the earth is an ellipse with eccentricity $\frac{1}{60}$ with the sun at one focus the major axis being approximately 186×10^6 miles in length. If the shortest and longest distances of the earth from the sun are $\lambda\times10^5$ miles and $\mu\times10^5$ miles, then $\mu-(\lambda+1)$ is divisible by	(r)	5
(D)	If the mid point of a chord of the ellipse $\frac{x^2}{16}+\frac{y^2}{25}=1$ is (0,3) and length of the chord is $\frac{4\lambda}{5}$, then λ is divisible by	(s)	6

73. Match the following

	Column I		Column II
(A)	The minimum and maximum distances of a point (2,6) from the ellipse $9x^2+8y^2-36x-16y-28=0$ are L and G, then	(p)	L + G = 10
(B)	The minimum and maximum distances of a point (1,2) from the ellipse $4x^2+9y^2+8x-36y+4=0$ are L and G, then	(q)	L + G = 6
(C)	The minimum and maximum distances of a point $\left(\frac{9}{5},\frac{12}{5}\right)$ from the ellipse $4(3x+4y)^2+9(4x-3y)^2=900$ are L and G, then	(r)	G – L = 8
(D)	The minimum and maximum distances of a point (0,4) from the ellipse $25x^2+9y^2=225$ are L and G, then	(s)	G – L = 6

Ellipse Exercise 6 :
Statement I and II Type Questions

■ **Directions** (Q. Nos. 74 to 81) are Assertion-Reason type questions. Each of these qustions contains two statements :
Statement I (Assertion) and
Statement II (Reason)
Each of these questions also has four alternative choices, only one of which is the correct answer.

You have to select the correct choice as given below.

(a) Statement I is true, statement II is true; statement II is a correct explanation for statement I

(b) Statement I is true, statement II is true; statement II is not a correct explanation for statement I

(c) Statement I is true, statement II is false

(d) Statement I is false, statement II is true

74. Let the equation of ellipse be $\frac{x^2}{(a^2+1)}=\frac{y^2}{(a^2+2})=1$

Statement I If eccentricity of the ellipse be $\frac{1}{\sqrt{6}}$, then length of latusrectum is $\frac{10}{\sqrt{6}}$.

Statement II Length of latusrectum $=\frac{2(a^2+1)}{\sqrt{(a^2+2)}}$.

75. **Statement I** The area of the ellipse $2x^2+3y^2=6$ is more than the area of the circle $x^2+y^2-2x+4y+4=0$.

Statement II The length of semi-major axis of an ellipse is more than the radius of the circle.

76. **Statement I** The equation of the director circle to the ellipse $4x^2+9y^2=36$ is $x^2+y^2=13$

Statement II The locus of the point of intersection of perpendicular tangents to an ellipse is called the director circle.

77. **Statement I** In an ellipse the distance between foci is always less than the sum of focal distances of any point on it.

Statement II If e be the eccentricity of the ellipse, then $0<e<1$.

78. **Statement I** The sum of the focal distances of a point on the ellipse $4x^2+5y^2-16x-30y+41=0$ is $2\sqrt{5}$.

Statement II The equation $4x^2+5y^2-16x-30y+41=0$ can be expressed as $4(x-2)^2+5(y-3)^2=20$.

79. **Statement I** Locus of centre of a variable circle touching to circles $(x-1)^2+(y-2)^2=25$ and $(x-2)^2+(y-1)^2=16$ is an ellipse.

Statement II If a circle $S_2=0$ lies completely inside the circle $S_1=0$, then the locus of centre of a variable circle $S=0$ which touches both the circles is an ellipse.

80. **Statement I** The tangent and normal at any point P on a ellipse bisect the external and internal angles between the focal distance of P.

Statement II The straight line joining the foci of the ellipse subtends of a right angle at P.

81. **Statement I** In a ΔABC, if base BC is fixed and perimeter of the triangle is also fixed, then vertex moves on an ellipse.

Statement II If sum of distances of a point P from two fixed points is constant, then locus of P is an ellipse.

Ellipse Exercise 7 :
Subjective Type Questions

- In this section, there are **12 subjective questions**.

82. If three of the sides of a quadrilateral inscribed in an ellipse are parallel respectively to three given straight lines. Show that fourth side will also be parallel to a fixed straight line.

83. If S and S' are the focii of an ellipse and P any point on the curve, show that

$$\tan\left(\frac{PSS'}{2}\right)\tan\left(\frac{PS'S}{2}\right)=\frac{1-e}{1+e}$$

Hence, or otherwise show that the locus of the centre of the circle inscribed in the triangle PSS' is another ellipse.

84. If the normal at any point P of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the major and minor axes in G and g respectively, and if CF is the perpendicular upon this normal from the centre C of the ellipse.

Show that $PF\cdot PG=b^2$

and $PF\cdot Pg=a^2$.

85. PQ is double ordinate of the ellipse $x^2/a^2+y^2/b^2=1$ and B an extremity of the minor axis. PB and QB meet the major axis in M and N. Show that M and N are inverse points with respect to the auxiliary circle of the ellipse.

86. Consider the family of circle $x^2+y^2=r^2, 2<r<5$. If in the first quadrant the common tangent to a circle of the family and the ellipse $4x^2+25y^2=100$ meets the coordinate axes at A and B, then find the equation of the locus of the mid-point of AB.

87. Prove that the normals at the four points, where the lines

$$\frac{lx}{a}+\frac{my}{b}=1$$

and $\frac{x}{la}+\frac{y}{mb}=-1$

cut the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are concurrent.

88. The eccentric angle of any point P measured from the semi major axis CA is ϕ. If S be the focus nearest to A, and $\angle ASP=\theta$. Prove that

$$\tan\left(\frac{\theta}{2}\right)=\sqrt{\left(\frac{1+e}{1-e}\right)}\tan\left(\frac{\phi}{2}\right)$$

89. If θ is the difference of the eccentric angles of two points on an ellipse, the tangents at which are at right angles. Prove that $ab\sin\theta=d_1d_2$, where d_1, d_2 are the semi diameters parallel to the tangents at the points and a, b are the semi-axes of the ellipse.

90. A straight line PQ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the circle $x^2+y^2=r^2\ (b<r<a)$. RS is a focal chord of the ellipse. If RS is parallel to PQ and meets the circle at points R and S. Find the length of RS.

91. Let d be the perpendicular distance from the centre of the ellipse to the tangent drawn at a point P on the ellipse. If F_1 and F_2 are two foci of the ellipse. Show that

$$(PF_1-PF_2)^2=4a^2\left(1-\frac{b^2}{d^2}\right).$$

92. TP and TQ are tangents drawn from an external point (x_1, y_1) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Show that

(i) $\frac{ST^2}{SP\cdot SQ}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$

(ii) $ST\cdot S'T\cos\theta=(CT)^2-a^2-b^2$

where, S and S' are the foci, C the centre and θ is the angle between the tangents.

93. If PSQ and PHR are focal chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where S and H are the foci. The tangents at Q and R meet at T. Show that the locus of T as P moves round the ellipse is

$$(1+e^2)^2\frac{x^2}{a^2}+(1-e^2)^2\frac{y^2}{b^2}=(1+e^2)^2$$

where, e is the eccentricity of the ellipse.

Ellipse Exercise 8 :
Questions Asked in Previous 13 Year's Exams

■ This section contains questions asked in **IIT-JEE, AIEEE, JEE Main & JEE Advanced** from year **2005** to **2017**.

94. The minimum area of triangle formed by the tangent to the $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and coordinate axes is **[IIT-JEE 2003, 3M]**

(a) ab sq units (b) $\frac{a^2+b^2}{2}$ sq units

(c) $\frac{(a+b)^2}{2}$ sq units (d) $\frac{a^2+ab+b^2}{3}$ sq units

95. Find the equation of the common tangent in 1st quadrant to the circle $x^2+y^2=16$ and the ellipse $\frac{x^2}{25}+\frac{y^2}{4}=1$. Also find the length of the intercept of the tangent between the coordinate axes. **[IIT-JEE 2005, 4M]**

96. An ellipse has OB as semi minor axis, F and F' its focii and the angle FBF' is a right angle. Then, the eccentricity of the ellipse is **[AIEEE 2005, 3M]**

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) $\frac{1}{\sqrt{3}}$

97. In an ellipse, the distance between its foci is 6 and minor axis is 8. Then, its eccentricity is **[AIEEE 2006, 4.5M]**

(a) $\frac{3}{5}$ (b) $\frac{1}{2}$

(c) $\frac{4}{5}$ (d) $\frac{1}{\sqrt{5}}$

98. Let $P(x_1, y_1)$ and $Q(x_2, y_2), y_1<0, y_2<0$ be the end points of the latusrectum of the, ellipse $x^2+4y^2=4$. The equations of parabolas with latusrectum PQ are **[IIT-JEE 2008, 4M]**

(a) $x^2+2\sqrt{3}y=3+\sqrt{3}$ (b) $x^2-2\sqrt{3}y=3+\sqrt{3}$

(c) $x^2+2\sqrt{3}y=3-\sqrt{3}$ (d) $x^2-2\sqrt{3}y=3-\sqrt{3}$

99. A focus of an ellipse is tat the origin. The directrix is the line $x=4$ and the eccentricity is $\frac{1}{2}$. Then, the length of the semi-major axis is **[AIEEE 2008, 3M]**

(a) $\frac{8}{3}$ (b) $\frac{2}{3}$

(c) $\frac{4}{3}$ (d) $\frac{5}{3}$

100. The line passing through the extermity A of the major axis and extremity B of the minor axis of the ellipse $x^2+9y^2=9$ meets its auxiliary circle at the point M. Then, the area of the triangle with vertices at A, M and the origin O is **[IIT-JEE 2009, 3M]**

(a) $\frac{31}{10}$ (b) $\frac{29}{10}$

(c) $\frac{21}{10}$ (d) $\frac{27}{10}$

101. The normal at a point P on the ellipse $x^2+4y^2=16$ meets the X-axis at Q. If M is the mid-point of the line segment PQ, then the locus of M intersects the latusrectum of the given ellipse at the points **[IIT-JEE 2009, 3M]**

(a) $\left(\pm\frac{3\sqrt{5}}{2},\pm\frac{2}{7}\right)$ (b) $\left(\pm\frac{3\sqrt{5}}{2},\pm\sqrt{\frac{19}{4}}\right)$

(c) $\left(\pm2\sqrt{3},\pm\frac{1}{7}\right)$ (d) $\left(\pm2\sqrt{3},\pm\frac{4\sqrt{3}}{7}\right)$

102. In a triangle ABC with fixed based BC, the vertex A moves such that

$$\cos B+\cos C=4\sin^2\frac{A}{2}$$

If a, b and c denote the lengths of the sides of the triangle opposite to the angles A, B and C, respectively, then **[IIT-JEE 2009, 4M]**

(a) $b+c=4a$

(b) $b+c=2a$

(c) locus of point A is an ellipse

(d) locus of point A is a pair of straight lines

103. The conic having parametric representation

$x=\sqrt{3}\left(\frac{1-t^2}{1+t^2}\right), y=\frac{2t}{1+t^2}$ is **[IIT-JEE 2009, 2M]**

(a) a circle (b) a parabola

(c) an ellipse (d) a hyperbola

104. The ellipse $x^2+4y^2=4$ is inscribed in a rectangle aligned with the coordinate axes, which in turn is inscribed in another ellipse that passes through the point (4, 0). Then, the equation of the ellipse is **[AIEEE 2009, 4M]**

(a) $x^2+12y^2=16$ (b) $4x^2+48y^2=48$

(c) $4x^2+64y^2=48$ (d) $x^2+16y^2=16$

Paragraph

(Q. Nos. 105 to 107)

Tangents are drawn from the point $P(3,4)$ *to the ellipse* $\frac{x^2}{9}+\frac{y^2}{4}=1$ *touching the ellipse at points A and B.*

105. The coordinates of A and B are

(a) (3, 0) and (0, 2)

(b) $\left(-\frac{8}{5},\frac{2\sqrt{161}}{15}\right)$ and $\left(-\frac{9}{5},\frac{8}{5}\right)$

(c) $\left(-\frac{8}{5},\frac{2\sqrt{161}}{15}\right)$ and (0, 2)

(d) (3, 0) and $\left(-\frac{9}{5},\frac{8}{5}\right)$

106. The orthocenter of the triangle PAB is

(a) $\left(5,\frac{8}{7}\right)$ (b) $\left(\frac{7}{5},\frac{25}{8}\right)$

(c) $\left(\frac{11}{5},\frac{8}{5}\right)$ (d) $\left(\frac{8}{25},\frac{7}{5}\right)$

107. The equation of the locus of the point whose distances from the point P and the line AB are equal, is **[IIT-JEE 2010, (3+3+3)M]**

(a) $9x^2+y^2-6xy-54x-62y+241=0$

(b) $x^2+9y^2+6xy-54x+62y-241=0$

(c) $9x^2+9y^2-6xy-54x-62y-241=0$

(d) $x^2+y^2-2xy+27x+31y-120=0$

108. Equation of the ellipse whose axes are the axes of coordinates an which passes through the point (−3, 1) and has eccentricity $\sqrt{\frac{2}{5}}$ is **[AIEEE 2011, 4M]**

(a) $5x^2+3y^2-48=0$ (b) $3x^2+5y^2-15=0$

(c) $5x^2+3y^2-32=0$ (d) $3x^2+5y^2-32=0$

109. The ellipse $E_1:\frac{x^2}{9}+\frac{y^2}{4}=1$ is inscribed in a rectangle R whose sides are parallel to the coordinate axes. Another ellipse E_2 passing through the point (0, 4) circumscribes the rectangle R. the eccentricity of the ellipse E_2 is **[IIT-JEE 2012, 4M]**

(a) $\frac{\sqrt{2}}{2}$ (b) $\frac{\sqrt{3}}{2}$

(c) $\frac{1}{2}$ (d) $\frac{3}{4}$

110. **Statement I :** An equation of a common tangent to the parabola $y^2=16\sqrt{3}x$ and the ellipse $2x^2+y^2=4$ is $y=2x+2\sqrt{3}$

Statement II : If the line $y=mx+\frac{4\sqrt{3}}{m},(m\neq 0)$ is a common tangent to the parabola $y^2=16\sqrt{3}x$ and the ellipse $2x^2+y^2=4$, then m satisfies $m^2+2m^2=24$ **[AIEEE 2012, 4M]**

(a) Statement I is false, statement-II is true

(b) Statement I is true, statementII is true; statement-II is a correct explanation for statement-I

(c) Statement I is true, statement II is true; statement II is not a correct explanation for statement I

(d) Statement I is true, statement II is false

111. An ellipse is drawn by taking a diameter of the circle $(x-1)^2+y^2=1$ as its semi-minor axis and diameter of the circle $x^2+(y-2)^2=4$ is semi-major axis. If the centre of the ellipse is at the origin and its axes are the coordinate axes, then the equation of the ellipse is **[AIEEE 2012, 4M]**

(a) $4x^2+y^2=4$ (b) $x^2+4y^2=8$

(c) $4x^2+y^2=8$ (d) $x^2+4y^2=16$

112. The equation of the circle passing through the foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$, and having centre at (0, 3) is **[JEE Main 2013, 4M]**

(a) $x^2+y^2-6y-7=0$ (b) $x^2+y^2-6y+7=0$

(c) $x^2+y^2-6y-5=0$ (d) $x^2+y^2-6y+5=0$

113. A vertical line passing through the point (h, 0) intersects the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$ at the points P and Q. Let the tangents to the ellipse at P and Q meet at the point R. If $\Delta(h)$ = area of the triangle PQR, $\Delta_1=\frac{1}{2}\le h\le \max\Delta(h)$ and $\Delta_2=\frac{1}{2}\le h\le \min\Delta(h)$, then $\frac{8}{\sqrt{5}}\Delta_1-8\Delta_2=$ **[JEE Advanced 2013, 3M]**

114. The locus of the foot of perpendicular drawn from the centre of the ellipse $x^2+3y^2=6$ on any tangent to it is **[JEE Main 2014, 4M]**

(a) $(x^2+y^2)^2=6x^2+2y^2$ (b) $(x^2+y^2)^2=6x^2-2y^2$

(c) $(x^2-y^2)^2=6x^2+2y^2$ (d) $(x^2-y^2)^2=6x^2-2y^2$

115. The area (in sq units) of the quadrilateral formed by the tangents at the end points of the latus rectum to the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$ is **[JEE Main 2015, 4M]**

(a) $\frac{27}{2}$ (b) 27

(c) $\frac{27}{4}$ (d) 18

116. Let E_1 and E_2 be two ellipses whose centers are at the origin. The major axes of E_1 and E_2 lie along the X-axis and the Y-axis, respectively. Let S be the circle $x^2 + (y-1)^2 = 2$. The straight line $x + y = 3$ touches the curves, S, E_1 and E_2 at P, Q and R respectively. Suppose that $PQ = PR = \frac{2\sqrt{2}}{3}$. If e_1 and e_2 are the eccentricities of E_1 and E_2, respectively, then the correct expression(s) is (are) **[JEE Advanced, 2015 4M]**

(a) $e_1^2 + e_2^2 = \frac{43}{40}$ (b) $e_1 e_2 = \frac{\sqrt{7}}{2\sqrt{10}}$

(c) $|e_1^2 - e_2^2| = \frac{5}{8}$ (d) $e_1 e_2 = \frac{\sqrt{3}}{4}$

117. Suppose that the foci of the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$ are $(f_1, 0)$ and $(f_2\ 0)$ where $f_1 > 0$ and $f_2 > 0$. Let P_1 and P_2 be two parabolas with a common vertex at (0, 0) and with foci at $(f_1, 0)$ and $(2f_2, 0)$, respectively. Let T_1 be a tangent to P_1 which passes through $(2f_2, 0)$ and T_2 be a tangent to P_2 which passes through $(f_1, 0)$. If m_1 is the slope of T_1 and m_2 is the slope of T_2, then the9 value of $\left(\frac{1}{m_1^2} + m_2^2\right)$ is **[JEE Advanced 2015, 4M]**

Paragraph

(Q. Nos. 118 and 119)

Let $F_1(x_1, 0)$ and $F_2(x_2, 0)$ for $x_1 < 0$ and $x_2 > 0$, be the foci of the ellipse $\frac{x^2}{9} + \frac{y^2}{8} = 1$. Suppose a parabola having vertex at the origin and focus at F_2 intersects the ellipse at point M in the first quadrant and at point N in the fourth quadrant.

118. The orthocentre of the triangle F_1MN is

(a) $\left(-\frac{9}{10}, 0\right)$ (b) $\left(\frac{2}{3}, 0\right)$ (c) $\left(\frac{9}{10}, 0\right)$ (d) $\left(\frac{2}{3}, \sqrt{6}\right)$

119. If the tangents to the ellipse at M and N meet at R and the normal to the parabola at M meets the X-axis at Q, then the ratio of area of the triangle MQR to area of the quadrilateral MF_1NF_2 is **[JEE Advanced 2016, (3+3)M]**

(a) 3 : 4 (b) 4 : 5 (c) 5 : 8 (d) 2 : 3

120. The eccentricity of an ellipse whose centre is at the origin is 1/2. If one of its directices is $x = -4$, then the equation of the normal to it at $(1, 3/2)$ is **[JEE Main 2017, 4M]**

(a) $x + 2y = 4$ (b) $2y - x = 2$

(c) $4x - 2y = 1$ (d) $4x + 2y = 7$

Answers

Exercise for Session 1

1. (c) **2.** (a) **3.** (b) **4.** (d) **5.** (b)
6. (a,b) **7.** (b) **8.** (c) **9.** (b) **10.** (d)
11. (a) **12.** (a)
13. $\frac{8}{3}, \frac{\sqrt{5}}{3}$; $(1 \pm \sqrt{5}, 2)$; $(-2, 2)$ and $(4, 2)$; 6 and 4; $(1, 2)$
14. $3x^2 + 4y^2 = 300$ **15.** $\frac{(x-2)^2}{4} + \frac{(y+3)^2}{3} = 1$
16. $5x^2 + 9y^2 - 54y + 36 = 0$ **17.** $\frac{1}{2}$
18. $x^2 \pm ay = a^2$

Exercise for Session 2

1. (c) **2.** (c) **3.** (a) **4.** (b) **5.** (d)
6. (d) **7.** (d) **8.** (c) **9.** (a) **10.** (b)
11. (b) **12.** $\lambda = -\frac{1}{\sqrt{5}}$; $\theta = \cos^{-1}\left(-\frac{1}{\sqrt{5}}\right)$

Exercise for Session 3

1. (c) **2.** (c) **3.** (b) **4.** (b) **5.** (a) **6.** (d)
7. (b) **8.** (a)
9. $3(x^2 + y^2 - a^2 - b^2)^2 = 4(b^2x^2 + a^2y^2 - a^2b^2)$
11. $\frac{(x-a)^2}{a^2} + \frac{y^2}{b^2} = 1$ **12.** $b^4c^2x^2 + a^2c^2y^2 = a^4b^4$
13. $(-1, 0)$; $5\sqrt{3}$ **14.** $x\sqrt{5} + 2y = 2\sqrt{5}$

Chapter Exercises

1. (c) **2.** (d) **3.** (a) **4.** (b) **5.** (c) **6.** (d)
7. (a) **8.** (c) **9.** (a) **10.** (c) **11.** (b) **12.** (b)
13. (d) **14.** (c) **15.** (c) **16.** (a) **17.** (c) **18.** (c)
19. (d) **20.** (a) **21.** (b) **22.** (a) **23.** (a) **24.** (a)
25. (a) **26.** (a) **27.** (d) **28.** (d) **29.** (a) **30.** (c)
31. (a,c) **32.** (a,b,c,d) **33.** (b,d) **34.** (a,c) **35.** (a,d) **36.** (a, c)
37. (a,b) **38.** (a,b,c) **39.** (a,b) **40.** (a,c) **41.** (a,b) **42.** (a,c)
43. (a,b,c) **44.** (a,b,c) **45.** (a,c) **46.** (d) **47.** (b) **48.** (a)
49. (a) **50.** (c) **51.** (b) **52.** (d) **53.** (d) **54.** (b)
55. (c) **56.** (a) **57.** (b) **58.** (c) **59.** (b) **60.** (c)
61. (6) **62.** (9) **63.** (4) **64.** (5) **65.** (2) **66.** (8)
67. (1) **68.** (5) **69.** (9) **70.** (6)
71. (A) → (p,q); (B) → (p,q,s); (C) → (p,r); (D) → (p,r)
72. (A) → (q); (B) → (p); (C) → (p,r,s); (D) → (q)
73. (A) → (p,s); (B) → (q); (C) → (q,s); (D) → (p,r)
74. (a) **75.** (b) **76.** (a) **77.** (a) **78.** (b) **79.** (c)
80. (c) **81.** (a) **86.** $y = \pm\sqrt{\left(\frac{r^2-4}{(25-r^2)}\right)}x$ **90.** (2b)
94. (a) **95.** $\left(\frac{14}{\sqrt{3}}\right)$ **96.** (a) **97.** (a) **98.** (b,c) **99.** (a)
100. (d) **101.** (c) **102.** (b,c) **103.** (c) **104.** (a) **105.** (d)
106. (c) **107.** (a) **108.** (d) **109.** (c) **110.** (b) **111.** (d)
112. (a) **113.** (9) **114.** (a) **115.** (b) **116.** (a,b) **117.** (4)
118. (a) **119.** (c) **120.** (c)

Solutions

1. $\because$ Major axis on X-axis.

$\therefore \quad f(2a) > f(a^2 - 3)$

$\Rightarrow \quad 2a > a^2 - 3 \qquad [\because f \text{ is increasing}]$

$\Rightarrow \quad a^2 - 2a - 3 < 0 \text{ or } (a-3)(a+1) < 0$

or $\quad -1 < a < 3$

$\therefore \quad a \in (-1, 3)$

2. $\because$ Major axis on Y-axis.

$\therefore \quad f(a^2 - 5) > f(4a)$

$\Rightarrow \quad a^2 - 5 < 4a \qquad [\because f \text{ is decreasing}]$

$\Rightarrow \quad a^2 - 4a - 5 < 0$

or $\quad (a-5)(a+1) < 0$

or $\quad -1 < a < 5$

$\therefore \quad a \in (-1, 5)$

3. Let $\quad a^2 = \sin\sqrt{2} - \cos\sqrt{3} = \sin\sqrt{2} - \sin\left(\frac{\pi}{2} - \sqrt{3}\right)$

$$= \sin\sqrt{2} + \sin\left(\sqrt{3} - \frac{\pi}{2}\right)$$

$$= \sin(1.41) + \sin(1.73 - 1.57)$$

$$= \sin(1.41) + \sin(0.16) > 0$$

and $\quad b^2 = \sin\sqrt{3} - \cos\sqrt{2}$

$$= \sin\sqrt{3} - \sin\left(\frac{\pi}{2} - \sqrt{2}\right)$$

$$= \sin(1.73) - \sin(1.57 - 1.41)$$

$$= \sin(1.73) - \sin(0.36) > 0$$

Also, $a^2 > b^2$

Hence, curve represent an ellipse with foci on X-axis.

4. Mutually perpendicular tangents have been drawn by taking any point on the director circle of given ellipse. Any point on the director circle can be taken as $(5\cos\theta, 5\sin\theta)$. Equation of corresponding chord of contact is

$$\frac{5x}{16}\cos\theta + \frac{5y}{9}\sin\theta - 1 = 0$$

It's distance from the origin is equal to

$$\frac{1}{\sqrt{\left(\frac{25\cos^2\theta}{256} + \frac{25\sin^2\theta}{81}\right)}} = \frac{144}{5}$$

$$\Rightarrow \quad \frac{1}{\sqrt{(256 - 175\cos^2\theta)}} \le \frac{144}{5.9} = \frac{16}{5}$$

5. We have, $PQ = BP$

$$\Rightarrow \quad 2ae = \sqrt{(a^2e^2 + b^2)} = \sqrt{a^2} = a$$

$$\therefore \quad e = \frac{1}{2}$$

6. Any tangent to $\frac{x^2}{16} + \frac{y^2}{9} = 1$, is

$$y = mx + \sqrt{(16m^2 + 9)}$$

It is also tangent to the circle $x^2 + y^2 = \left(\frac{5}{\sqrt{2}}\right)^2$, then perpendicular distance from $(0, 0)$ is equal to $\frac{5}{\sqrt{2}}$

$$\therefore \quad \frac{\sqrt{(16m^2 + 9)}}{\sqrt{(m^2 + 1)}} = \frac{5}{\sqrt{2}}$$

$$\Rightarrow \quad 32m^2 + 18 = 25m^2 + 25$$

or $\quad 7m^2 = 7 \Rightarrow m = \pm 1$

for acute $m = 1$.

Now, let θ be the angle between common tangent and the line $\sqrt{3}x - y + 6 = 0$, then

$$\tan\theta = \left|\frac{\sqrt{3} - 1}{1 + \sqrt{3}}\right| = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = \tan\left(\frac{\pi}{12}\right)$$

$$\therefore \quad \theta = \frac{\pi}{12}$$

7. Tangent at $(-8, 3)$ is

$$\frac{x(-8)}{100} + \frac{y(3)}{25} = 1$$

or $\quad 2x - 3y + 25 = 0$

for Y-axis put $x = 0$, then $y = \frac{25}{3}$

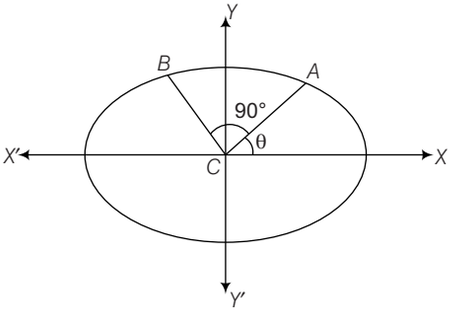
Hence, required point is $\left(0, \frac{25}{3}\right)$.

8. Let $A \equiv (4\cos\theta, 3\sin\theta)$, then

$B \equiv (-4\sin\theta, 3\cos\theta)$

$\therefore \quad (CA)^2 = 16\cos^2\theta + 9\sin^2\theta$

and $(CB)^2 = 16\sin^2\theta + 9\cos^2\theta$



$$\therefore \frac{1}{(CA)^2} + \frac{1}{(CB)^2} = \frac{1}{16\cos^2\theta + 9\sin^2\theta} + \frac{1}{16\sin^2\theta + 9\cos^2\theta}$$

$$= \frac{25}{144(\sin^4\theta + \cos^4\theta) + 337\sin^2\theta\cos^2\theta}$$

$$= \frac{25}{144(1 - 2\sin^2\theta\cos^2\theta) + 337\sin^2\theta\cos^2\theta}$$

$$= \frac{25}{144 + 49\sin^2\theta\cos^2\theta} = \frac{25}{144} \qquad (\text{for } \theta = 0)$$

9. (α, β) lies on the director circle of the ellipse i.e. on $x^2 + y^2 = 9$

So, we can assume

$$\alpha = 3\cos\theta, \beta = \sin\theta$$

$$\therefore \quad F = 12\cos\theta + 9\sin\theta = 3(4\cos\theta + 3\sin\theta)$$

$$\Rightarrow \quad -15 \le F \le 15$$

10. At, $t = \frac{3}{2}$, $\quad b > a$

$$\therefore \text{ Length of latusrectum} = \frac{2a^2}{b} = \frac{2[t^2 - 3t + 4]^2}{[3 + 5t]}$$

$$= \frac{2\left[\frac{9}{4} - \frac{9}{2} + 4\right]^2}{\left[3 + \frac{15}{2}\right]} \qquad [\text{at } t = \frac{3}{2}]$$

$$= \frac{2[1.75]^2}{[10.5]} = \frac{2(1)^2}{10} = \frac{1}{5}$$

11. Slope of line $= \frac{b}{a}\left(\frac{\sin 60° - \sin 30°}{\cos 60° - \cos 30°}\right) = -\frac{b}{a} = -\frac{1}{2}$ or $a = 2b$

Homogenizing the ellipse with $x + 2y + 4 = 0$, then

$$\frac{x^2}{4b^2} + \frac{y^2}{b^2} = \left(\frac{x + 2y}{-4}\right)^2$$

or $\quad x^2\left(\frac{1}{4b^2} - \frac{1}{16}\right) + y^2\left(\frac{1}{b^2} - \frac{1}{4}\right) - \frac{xy}{4} = 0$

Now, coefficients of x^2 + coefficient of $y^2 = 0$

$$\Rightarrow \quad \frac{1}{4b^2} - \frac{1}{16} + \frac{1}{b^2} - \frac{1}{4} = 0$$

or

$$\frac{5}{4b^2} = \frac{5}{16}$$

or $\quad b^2 = 4$ then, $a^2 = 16$

$\therefore$ Ellipse is $\frac{x^2}{16} + \frac{y^2}{4} = 1$

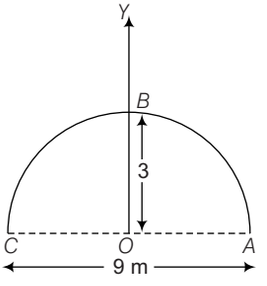
12. Let the equation of the semi elliptical arch be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad [y > 0]$$

Length of the major axis $= 2a = 9$

$$\Rightarrow \quad a = \frac{9}{2}$$

Length of the semi minor axis $b = 3$



So, the equation of the arch becomes

$$\frac{4x^2}{81} + \frac{y^2}{9} = 1$$

If $\quad x = 2$, then $y^2 = \frac{65}{9}$

$$\Rightarrow \quad y = \frac{\sqrt{65}}{3} = \frac{8}{3} \qquad [\text{approximately}]$$

13. Equation of tangent of the ellipse at $P(5\cos\theta, 4\sin\theta)$ is

$$\frac{x}{5}\cos\theta + \frac{y}{4}\sin\theta = 1$$

It meets the line $x = 0$ at $Q(0, 4\ \text{cosec}\ \theta)$ and image of Q in the line $y = x$ is $R(4\ \text{cosec}\ \theta, 0)$.

$\therefore$Equation of the circle is

$$(x - 0)(x - 4\ \text{cosec}\ \theta) + (y - 4\ \text{cosec}\ \theta)(y - 0) = 0$$

i.e. $\quad x^2 + y^2 - 4(x + y)\ \text{cosec}\ \theta = 0$

$\therefore$Each member of the family passes through the intersection of

$$x^2 + y^2 = 0 \text{ and } x + y = 0$$

i.e. the point $(0, 0)$.

14. A cyclic parallelogram will be a rectangle or square.

So, $\quad \angle QPR = 90°$

Therefore, P lies on the director circle of the ellipse

$$\frac{x^2}{16} + \frac{y^2}{b^2} = 1$$

Hence, $x^2 + y^2 = 25$ is the director circle of $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$,

then $\quad 16 + b^2 = 25$ or $b^2 = 9$

$$\Rightarrow \quad a^2(1 - e^2) = 9$$

$$\Rightarrow \quad 16(1 - e^2) = 9 \qquad [\because a^2 = 16]$$

$$\Rightarrow \quad e^2 = 1 - \frac{9}{16} = \frac{7}{16}$$

$$\therefore \quad e = \frac{\sqrt{7}}{4}$$

15. $\because m, n \in N$ and $m.n = m + n$

or $\quad m(n - 1) = n$

n is divisible by $n - 1$. Therefore

$$n - 1 = 1$$

$$\Rightarrow \quad n = 2, m = 2$$

Hence, the chord of contact of tangents drawn from

$(2, 2)$ to $4x^2 + 9y^2 = 36$

is $\quad 4x \cdot 2 + 9y \cdot 2 = 36$

or $\quad 4x + 9y = 18$

16. $\because x - 2y + 4 = 0$ or $y = \frac{x}{2} + 2$

Condition of tangency

$$c^2 = a^2 m^2 + b^2 \Rightarrow 4 = 4 \times \frac{1}{4} + b^2$$

$$\therefore \quad b = \sqrt{3}$$

Now, common tangent meet on X-axis at a point $(-4, 0)$.

According to symmetry other common tangent meet Y-axis at $(0, -2)$.

$\therefore$Equation of other common tangent is

$$\frac{x}{-4} + \frac{y}{-2} = 1 \text{ or } x + 2y + 4 = 0$$

17. The tangent at the point of shortest distance from the line $x + y = 7$ parallel to the given line.

Any point on the given ellipse is

$$(\sqrt{6}\cos\theta, \sqrt{3}\sin\theta).$$

Equation of the tangent is

$$\frac{x\cos\theta}{\sqrt{6}} + \frac{y\sin\theta}{\sqrt{3}} = 1.$$

It is parallel to $x + y = 7$

$$\Rightarrow \quad \frac{\cos\theta}{\sqrt{6}} = \frac{\sin\theta}{\sqrt{3}}$$

$$\Rightarrow \quad \frac{\cos\theta}{\sqrt{2}} = \frac{\sin\theta}{1} = \frac{1}{\sqrt{3}}$$

The required point is (2, 1).

18. Let $y = mx + \frac{1}{m}$ be a tangent to the parabola, then for the required equilateral triangle.

$m = \pm\frac{1}{\sqrt{3}}$ and it is tangent to ellipse also, if

$$\left(\frac{1}{m}\right)^2 = a^2m^2 + b^2$$

$$\Rightarrow \quad 3 = \frac{a^2}{3} + b^2$$

$$\Rightarrow \quad 3 = \frac{a^2}{3} + a^2 - a^2e^2$$

$$\Rightarrow \quad a^2e^2 = \frac{4a^2}{3} - 3$$

$$\text{or} \quad e^2 = \frac{4}{3} - \frac{3}{a^2}$$

Now, $0 < e^2 < 1$

$$\Rightarrow \quad 0 < \frac{4}{3} - \frac{3}{a^2} < 1$$

$$\Rightarrow \quad \frac{3}{2} < a < 3$$

19. For equilateral triangle

Circumcentre = Centroid

$$\therefore \quad h = \frac{a\cos\alpha + a\cos\beta + a\cos\gamma}{3} \text{ or } \frac{3h}{a} = \Sigma\cos\alpha$$

$$\text{or} \quad \left(\frac{3h}{a}\right)^2 = (\Sigma\cos\alpha)^2 = \Sigma\cos^2\alpha + 2\,\Sigma\cos\alpha\cos\beta \qquad \ldots(i)$$

$$\text{and} \quad k = \frac{b\sin\alpha + b\sin\beta + b\sin\gamma}{3}$$

$$\text{or} \quad \frac{3k}{b} = \Sigma\sin\alpha$$

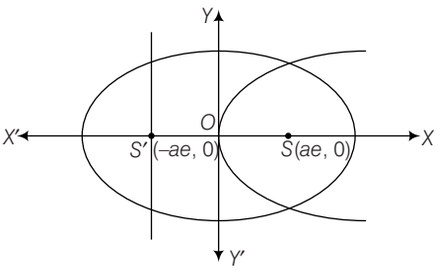
$$\text{or} \quad \left(\frac{3k}{b}\right)^2 = (\Sigma\sin\alpha)^2 = \Sigma\sin^2\alpha + 2\,\Sigma\sin\alpha\sin\beta \qquad \ldots(ii)$$

Adding Eqs. (i) and (ii), we get

$$\frac{9h^2}{a^2} + \frac{9k^2}{b^2} = 3 + 2\,\Sigma\cos(\alpha - \beta)$$

$$\text{or} \quad \Sigma\cos(\alpha - \beta) = \frac{9h^2}{2a^2} + \frac{9k^2}{2b^2} - \frac{3}{2}$$

20. $\because$ Latusrectum and parabola = $4(ae)$ ...(i)



and latusrectum of ellipse $= \frac{2b^2}{a}$ ...(ii)

$\because$ Latusrectum and parabola and ellipse be equal

$$\therefore \quad \frac{2b^2}{a} = 4ae \qquad \text{[from Eqs. (i) and (ii)]}$$

$$\Rightarrow \quad 2b^2 = 4a^2e$$

$$\text{or} \quad 2a^2(1 - e^2) = 4a^2e \quad \text{or} \quad e^2 + 2e - 1 = 0$$

$$\text{Therefore, } e = \frac{-2 \pm \sqrt{(4+4)}}{2} = -1 \pm \sqrt{2}$$

$$\text{Hence,} \quad e = \sqrt{2} - 1$$

21. Here, the center of the ellipse is (0, 0)

Let $P(r\cos\theta, r\sin\theta)$ be any point on the given ellipse, then

$$r^2\cos^2\theta + 2r^2\sin^2\theta + 2r^2\sin\theta\cos\theta = 1$$

$$\text{or} \quad r^2 = \frac{1}{1 + \sin^2\theta + \sin 2\theta}$$

$$= \frac{2}{2 + (1 - \cos 2\theta) + 2\sin 2\theta} = \frac{2}{2\sin 2\theta - \cos 2\theta + 3}$$

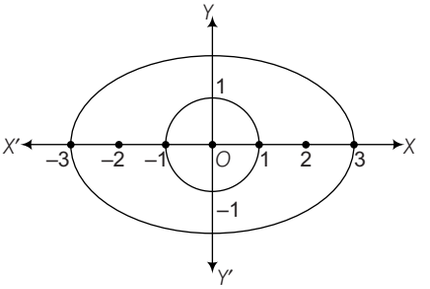
$\because \quad 3 - \sqrt{5} \le 2\sin 2\theta - \cos 2\theta + 3 \le 3 + \sqrt{5}$

For r_{max}, $2\sin 2\theta - \cos 2\theta + 3 = 3 - \sqrt{5}$

$$\therefore \quad r_{max}^2 = \frac{2}{3 - \sqrt{5}} = \frac{2(3 + \sqrt{5})}{4} = \frac{6 + 2\sqrt{5}}{4} = \left(\frac{\sqrt{5} + 1}{2}\right)^2$$

$$\text{Hence,} \quad r_{max} = \frac{\sqrt{5} + 1}{2}$$

22. Centre of the ellipse if (1, 2) and length of major axis and minor axis are 6 and 4 respectively and centre and radius of the circle are (1, 2) and 1 respectively.



Hence, ellipse and circle do not touch or cut.

$\therefore$ Common chord impossible.

$\therefore$ Hence, length of common chord = 0

23. $ax^2 + by^2 + 2fx + 2gy + c = 0$

$$a\left\{x^2 + \frac{2fx}{a}\right\} + b\left\{y^2 + \frac{2gy}{b}\right\} + c = 0$$

$$\Rightarrow \quad a\left(x + \frac{f}{a}\right)^2 + b\left(y + \frac{g}{b}\right)^2 = \left(\frac{f^2}{a} + \frac{g^2}{b} - c\right)$$

$$\Rightarrow \quad \frac{(x + f/a)^2}{\left(\frac{f^2}{a} + \frac{g^2}{b} - c\right)/a} + \frac{(y + g/b)^2}{\left(\frac{f^2}{a} + \frac{g^2}{b} - c\right)/b} = 1$$

If e eccentricity, then

$$\frac{\left(\frac{f^2}{a} + \frac{g^2}{b} - c\right)}{b} = \frac{\left(\frac{f^2}{a} + \frac{g^2}{b} - c\right)}{a}(1 - e^2)$$

$$\Rightarrow \quad 1 - e^2 = \frac{a}{b}$$

$$\Rightarrow \quad e^2 = \frac{b - a}{b}$$

$$\therefore \quad e = \sqrt{\left(\frac{b - a}{b}\right)}$$

24. Let ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the circle be $x^2 + y^2 = a^2e^2$,

point of intersection of the circle and the ellipse is

$$P\left(\frac{a}{e}\sqrt{(2e^2 - 1)}, \frac{a}{e}(1 - e^2)\right)$$

Now, area of $\Delta\, PF_1F_2 = \left|\frac{1}{2} \times 2ae \times \frac{a}{e}(1 - e^2)\right|$

$$= a^2(1 - e^2) = 30 \qquad \text{[given]}$$

or $\quad a^2e^2 = a^2 - 30 = \left(\frac{17}{2}\right)^2 - 30 = \frac{169}{4}$

$$\therefore \quad ae = \frac{13}{2} \text{ or } 2ae = 13$$

25. Let standard ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (a > b)$

Equations of tangent and normal at $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$

are $bx + ay - \sqrt{2}\,ab = 0$ and $ax - by - \frac{(a^2 - b^2)}{\sqrt{2}} = 0$

If sides of rectangles are p and q, then

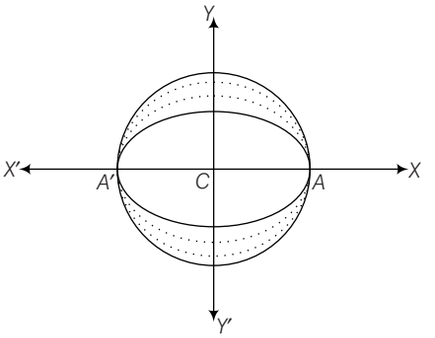
$$p = \frac{|0 + 0 - \sqrt{2}\,ab|}{\sqrt{(a^2 + b^2)}} = \frac{\sqrt{2}\,ab}{\sqrt{(a^2 + b^2)}}$$

and $$q = \left|\frac{0 - 0 - \frac{(a^2 - b^2)}{\sqrt{2}}}{\sqrt{(a^2 + b^2)}}\right|$$

$$= \frac{(a^2 - b^2)}{\sqrt{2}\sqrt{(a^2 + b^2)}}$$

$\therefore$ Area of rectangle $= pq = \frac{(a^2 - b^2)\,ab}{(a^2 + b^2)}$

26. Let equation of ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



and equation of circle is $x^2 + y^2 = a^2$

$\therefore$ Area of ellipse $= \pi ab$

and Area of circle $= \pi a^2$

$$\therefore \quad \text{Probability} = \frac{\pi a^2 - \pi ab}{\pi a^2} = \frac{a - b}{a} = \frac{2}{3} \qquad \text{[given]}$$

$$\Rightarrow \quad 3a - 3b = 2a$$

or $\quad a = 3b$

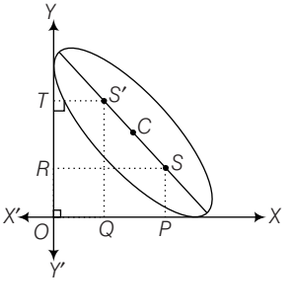
or $\quad a^2 = 9b^2 = 9a^2(1 - e^2)$

$$\Rightarrow \quad 9e^2 = 8$$

$$\therefore \quad e = \frac{2\sqrt{2}}{3}$$

27. Let $S \equiv (x_1, y_1)$, $S' \equiv (x_2, y_2)$

Let $C \equiv (h, k)$



$$\therefore \quad \frac{x_1 + x_2}{2} = h$$

$$\Rightarrow \quad x_1 + x_2 = 2h$$

and $\quad y_1 + y_2 = 2k$

$\because \quad SP\,S'Q = b^2$

$$\Rightarrow \quad y_1\,y_2 = b^2 \text{ and } SR \cdot S'T = b^2$$

$$\Rightarrow \quad x_1\,x_2 = b^2$$

Distance between foci $SS' = 2ae$

$$\Rightarrow \quad \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = (2ae)$$

$$\Rightarrow \quad (x_1 - x_2)^2 + (y_1 - y_2)^2 = 4a^2e^2$$

$$\Rightarrow (x_1 + x_2)^2 - 4x_1x_2 + (y_1 + y_2)^2 - 4\,y_1y_2 = 4a^2e^2$$

$$\Rightarrow \quad 4h^2 - 4b^2 + 4k^2 - 4b^2 = 4(a^2 - b^2)$$

$$\Rightarrow \quad h^2 + k^2 - 2b^2 = a^2 - b^2$$

$$\therefore \quad h^2 + k^2 = a^2 + b^2$$

Locus of centre is $x^2 + y^2 = a^2 + b^2$ which is a circle.

28. If chord AB, then equation of AB, here mid-point of the chord is $\left(\frac{1}{2}, \frac{2}{3}\right)$

$\therefore \quad T = S_1$

$$\frac{x}{50} + \frac{y}{24} = \frac{1}{100} + \frac{1}{36} \quad \text{...(i)}$$

Let $A \equiv (x_1, y_1)$ and $B \equiv (x_2, y_2)$

$$\therefore \quad \frac{x_1}{50} + \frac{y_1}{24} = \frac{1}{100} + \frac{1}{36} \quad \text{...(ii)}$$

and $$\frac{x_2}{50} + \frac{y_2}{24} = \frac{1}{100} + \frac{1}{36} \quad \text{...(iii)}$$

$$\therefore \quad \frac{(x_2 - x_1)}{50} + \frac{(y_2 - y_1)}{24} = 0$$

$$\therefore \quad AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(x_2 - x_1)^2 + \left(\frac{24}{50}\right)^2 (x_2 - x_1)^2}$$

$$= \frac{|x_2 - x_1|}{25}\sqrt{769}$$

Now, solving $\frac{x}{50} + \frac{y}{24} = \frac{1}{100} + \frac{1}{36}$

and $$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

then, $$x^2 - x - \frac{5336}{9} = 0$$

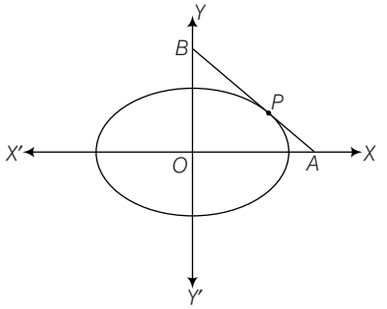
$$\therefore \quad |x_2 - x_1| = \frac{\sqrt{D}}{a} = \frac{\sqrt{21353}}{3}$$

$$\therefore \quad AB = \frac{\sqrt{21353}}{3 \times 25} \times \sqrt{769}$$

29. Equation of tangent at the point $P(4\cos\theta, 3\sin\theta)$ is

$$\frac{x}{4}\cos\theta + \frac{y}{3}\sin\theta = 1$$

or $$\frac{x}{4\sec\theta} + \frac{y}{3\,\text{cosec}\,\theta} = 1$$



Coordinates of A and B are

$$A \equiv (4\sec\theta, 0) \text{ and } B \equiv (0, 3\,\text{cosec}\,\theta)$$

Let middle point (h, k)

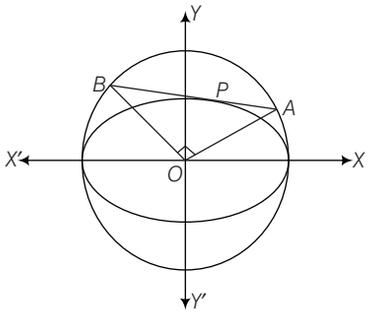
$$\therefore \quad h = 2\sec\theta, k = \frac{3}{2}\,\text{cosec}\,\theta$$

$$\Rightarrow \quad \left(\frac{2}{h}\right)^2 + \left(\frac{3}{2k}\right)^2 = 1$$

$\therefore$ Locus is $9x^2 + 16y^2 = 4x^2y^2$

30. Equation of tangent at $(a\cos\alpha, b\sin\alpha)$ is

$$\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = 1 \quad \text{...(i)}$$



Equation of auxiliary circle is

$$x^2 + y^2 = a^2 \quad \text{...(ii)}$$

For combined equation of OA and OB, making homogeneous Eq. (ii) with the help of Eq. (i), then

$$x^2 + y^2 = a^2\left(\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha\right)^2$$

$$\Rightarrow \quad x^2\sin^2\alpha + y^2\left(1 - \frac{a^2}{b^2}\sin^2\alpha\right) - \frac{2axy}{b}\sin\alpha\cos\alpha = 0$$

$\because \quad \angle AOB = 90°$

$\therefore$ Coefficient of x^2 + Coefficient of $y^2 = 0$

$$\Rightarrow \quad \sin^2\alpha + 1 - \frac{a^2}{b^2}\sin^2\alpha = 0$$

$$\Rightarrow \quad 1 + \sin^2\alpha - \frac{a^2}{a^2(1 - e^2)}\sin^2\alpha = 0$$

$$\Rightarrow \quad (1 - e^2)(1 + \sin^2\alpha) - \sin^2\alpha = 0$$

$$\Rightarrow \quad 1 - e^2(1 + \sin^2\alpha) = 0$$

or $$e^2(1 + \sin^2\alpha) = 1$$

31. $\because \; b^2x^2 + y^2 = a^2b^2 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2b^2} = 1$

Let $a > ab$,

Then, $$(ab)^2 = a^2(1 - e^2)$$

$$\Rightarrow \quad b^2 = 1 - e^2 \quad \text{...(i)}$$

$\therefore$ Extremities of the latusrectum $(\pm ae, \pm ab^2)$

Let $x_1 = \pm ae$ and $y_1 = \pm ab^2 = \pm a(1 - e^2)$

or $$y_1 = \pm(a - ae^2)$$

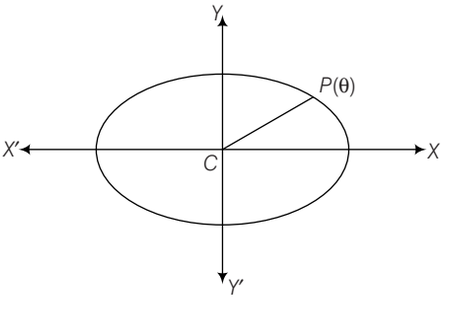
$$\Rightarrow \quad ay_1 = \pm(a^2 - a^2e^2)$$

$$= \pm(a^2 - x_1^2) \quad [\because x_1 = \pm ae]$$

or $$x_1^2 \pm ay_1 = a^2$$

$\therefore$ Required locus is $x^2 \pm ay = a^2$

32. Let $P \equiv (\sqrt{6}\cos\theta, \sqrt{2}\sin\theta)$ be any point on the ellipse



$$\frac{x^2}{6}+\frac{y^2}{2}=1$$

Given, $CP=2 \Rightarrow (CP)^2=4$

$\Rightarrow \quad 6\cos^2\theta+2\sin^2\theta=4$

$\Rightarrow \quad 6\cos^2\theta+2(1-\cos^2\theta)=4$

or $\quad \cos^2\theta=\frac{1}{2}=\left(\frac{1}{\sqrt{2}}\right)^2=\left(\cos\frac{\pi}{4}\right)^2$

$\therefore \quad \theta=n\pi\pm\frac{\pi}{4}, n\in I$

$\therefore \quad \theta=\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}$

33. Here, $a=\cos\theta$ and $b=\sin\theta$

$\therefore \quad \frac{\pi}{4}<\theta<\frac{\pi}{2}$

$\therefore \quad a<b$

then, $\quad a^2=b^2(1-e^2)$

$\Rightarrow \quad \cos^2\theta=\sin^2\theta(1-e^2)$

$\therefore \quad e^2=1-\cot^2\theta$

or $\quad e=\sqrt{(1-\cot^2\theta)} \qquad \left(\because \frac{\pi}{4}<\theta<\frac{\pi}{2}\right)$

$\because$ Extremities of latusrectum are $\left(\pm\frac{a^2}{b},\pm be\right)$

$\Rightarrow \quad x=\pm\frac{a^2}{b},\ y=\pm be$

or $\quad b^2x^2=a^4$ and $y^2=b^2e^2=\sin^2\theta-\cos^2\theta$

or $\quad \sin^2\theta x^2=\cos^4\theta$ and $y^2=1-2\cos^2\theta$

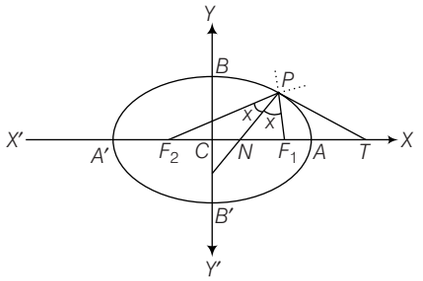
or $\quad (1-\cos^2\theta)x^2=\cos^4\theta$ and $\cos^2\theta=\frac{1-y^2}{2}$

$\therefore \quad \left(1-\frac{1-y^2}{2}\right)x^2=\left(\frac{1-y^2}{2}\right)^2$

or $\quad 2x^2(1+y^2)=(1-y^2)^2$

or $\quad 2y^2(x^2+1)=1+y^4-2x^2$

34. $\because \frac{PF_1}{PF_2}=\frac{NF_1}{NF_2}$



$\therefore$ PN bisects the $\angle F_1PF_2$

$\because$ Bisectors are perpendicular to each other.

$\therefore$ PT bisects the angle $(180^\circ-\angle F_1PF_2)$

35. $\because \ a^2-a-6>0 \Rightarrow (a-3)(a+2)>0$

$\therefore \quad a<-2$ or $a>3$ ...(i)

and $\quad a^2-6a+5>0 \Rightarrow (a-1)(a-5)>0$

$\therefore \quad a<1$ or $a>5$ ...(ii)

Also $\quad a^2-a-6 \neq a^2-6a+5$

or $\quad a\neq\frac{11}{5}$ ...(iii)

From Eqs. (i), (ii) and (iii) we get

$a\in(-\infty,-2)\cup(5,\infty)$

36. $\because$ A latusrectum of an ellipse is a line passing through a focus, perpendicular to major axis i.e. parallel to minor axis.

37. Let the required equation be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad ...(i)$$

Since, Eq. (i) passes through $(4,-1)$, then

$$\frac{16}{a^2}+\frac{1}{b^2}=1 \qquad ...(ii)$$

The equation of given line is $y=-\frac{x}{4}+\frac{10}{4}$

Which touches the ellipse Eq. (i),

$$\left(\frac{10}{4}\right)^2=a^2\left(-\frac{1}{4}\right)^2+b^2$$

$$\Rightarrow \quad \frac{a^2}{16}+b^2=\frac{25}{4} \qquad ...(iii)$$

Solving, Eqs. (ii) and (iii), we get $a^2=20,\ b^2=5$

or $\quad a^2=80,\ b^2=\frac{5}{4}$

Therefore, the required equation of ellipse will be

$$\frac{x^2}{20}+\frac{y^2}{5}=1 \text{ or } \frac{x^2}{80}+\frac{y^2}{5/4}=1$$

i.e. $\quad x^2+4y^2=20$ or $x^2+64y^2=80$

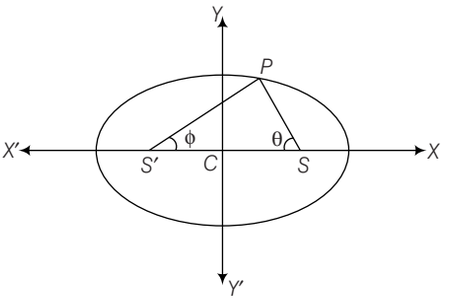
38. From focal property of ellipse

$SP+S'P=2a$, if $a>b$

and $\quad SP+S'P=2b$, if $b>a$

Also, $\quad PS\cos\theta+PS'\cos\phi=2ae$ ...(i)

$PS\sin\theta-PS'\sin\phi=0$ ...(ii)



From Eqs. (i) and (ii), we get

$$PS=\frac{2ae\sin\phi}{\sin(\theta+\phi)},\ PS'=\frac{2ae\sin\theta}{\sin(\theta+\phi)}$$

$\because \quad SP+S'P=2a$

$\therefore \quad e(\sin\theta+\sin\phi)=\sin(\theta+\phi)$

$$\Rightarrow \quad e\cdot 2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)=2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta+\phi}{2}\right)$$

or $$\frac{\cos\left(\frac{\theta-\phi}{2}\right)}{\cos\left(\frac{\theta+\phi}{2}\right)}=\frac{1}{e}$$

or $$\frac{\cos\left(\frac{\theta-\phi}{2}\right)-\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)+\cos\left(\frac{\theta+\phi}{2}\right)}=\frac{1-e}{1+e}$$

or $$\tan\left(\frac{\theta}{2}\right)\tan\left(\frac{\phi}{2}\right)=\frac{1-e}{1+e}$$

39. Let $S\equiv(5,12)$, $S'\equiv(24,7)$ and let $P\equiv(0,0)$

Now, $$SS'=\sqrt{(24-5)^2+(7-12)^2}$$
$$=\sqrt{(361+25)}=\sqrt{(386)}$$

For ellipse $SP+S'P=2a$

$\Rightarrow$ $13+25=2a$

$\therefore$ $2a=38$

$\because$ $SS'=2ae \Rightarrow e=\frac{\sqrt{386}}{38}$

and for hyperbola $S'P-SP=2a$

$\Rightarrow$ $25-13=2a \Rightarrow 2a=12$

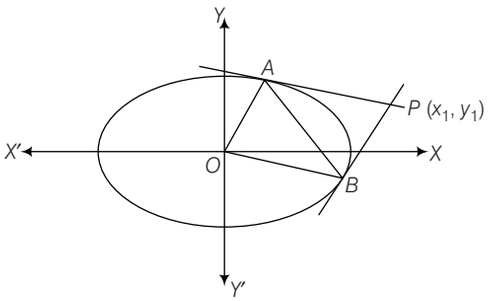
$\therefore$ $SS'=2ae \Rightarrow e=\frac{\sqrt{386}}{12}$

40. Let the point of intersection of tangents at A and B be $P(x_1, y_1)$.

Then, equation of AB is

$$xx_1+4yy_1=4$$

or $$\frac{xx_1}{4}+yy_1=1 \quad \ldots(i)$$



For point of OA and OB, homogenizing the equation of ellipse with the help of Eq. (i), then

$$x^2+4y^2=4\left(\frac{xx_1}{4}+yy_1\right)^2$$

or $$x^2\left(\frac{x_1^2}{4}-1\right)+y^2(4y_1^2-4)+2xyx_1y_1=0 \quad \ldots(ii)$$

But gives equation of OA and OB is

$$x^2+4y^2+axy=0 \quad \ldots(iii)$$

Since, Eqs. (ii) and (iii) an identical, then

$$\frac{\frac{x_1^2}{4}-1}{1}=\frac{4y_1^2-4}{4}=\frac{2x_1y_1}{a}$$

or $$x_1^2-4=4(y_1^2-1)$$

or $$x_1^2-4y_1^2=0$$

Therefore, the locus is $x^2-4y^2=0$

or $$(x+2y)(x-2y)=0$$

or $x+2y=0$ and $x-2y=0$

41. $\because$ $25x^2+9y^2-150x-90y+225=0$

$\Rightarrow$ $25(x^2-6x)+9(y^2-10y)+225=0$

$\Rightarrow$ $25\{(x-3)^2-9\}+9\{(y-5)^2-25\}+225=0$

$\Rightarrow$ $25(x-3)^2+9(y-5)^2=225$

$\Rightarrow$ $\frac{(x-3)^2}{3^2}+\frac{(y-5)^2}{5^2}=1 \quad \ldots(i)$

Let $x-3=X,\ y-5=Y$

Then, Eq. (i) becomes

$$\frac{X^2}{3^2}+\frac{Y^2}{5^2}=1 \quad \ldots(ii)$$

Now, comparing Eq. (ii) with

$$\frac{X^2}{a^2}+\frac{Y^2}{b^2}=1$$

$\therefore$ $a=3, b=5$

$\Rightarrow$ $a^2=b^2(1-e^2)$

$\Rightarrow$ $9=25(1-e^2)$

$\Rightarrow$ $e=\frac{4}{5}$

Centre : $X=0,\ Y=0$

$x-3=0,\ y-5=0$

$\therefore$ Centre $\equiv(3,5)$

Foci : $X=0, Y=\pm be$

$\Rightarrow$ $x-3=0, y-5=\pm 4$

$\Rightarrow$ $y=5\pm 4$

$\therefore$ Foci (3, 1) and (3, 9)

Major axis $2b=10$

42. $P(\theta)$ on the ellipse $x^2+4y^2=16$ is $(4\cos\theta, 2\sin\theta)$

Equation of tangent at P is

$$4\cos\theta\, x+8\sin\theta\, y=16$$

$\Rightarrow$ $x\cos\theta+2y\sin\theta=4$

It passes through centre (4, 2) of the given circle

$\therefore$ $4\cos\theta+4\sin\theta=4$

$\Rightarrow$ $\cos\theta+\sin\theta=1,\ \theta=0, \frac{\pi}{2}$

43. Since, the product of the two eccentricities e and e' is 1. Either $e=e'=1$ in which case both the conics are parabolas or if $e>1, e'<1$ and *vice-versa.*

So, one of them is an ellipse and the other is a hyperbola.

44. Equation of tangent at 'α' is

$$\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha=1$$

$\Rightarrow$ $$\frac{x}{a\sec\alpha}+\frac{y}{b\,\mathrm{cosec}\,\alpha}=1$$

Now, length of intercept,

$$z=\sqrt{(a^2\sec^2\alpha+b^2\,\mathrm{cosec}^2\,\alpha)}$$

or $$z^2=a^2\sec^2\alpha+b^2\,\mathrm{cosec}^2\,\alpha$$

$\therefore$ $$\frac{d(z^2)}{d\alpha}=2\left(\frac{a^2\sin\alpha}{\cos^3\alpha}-\frac{b^2\cos\alpha}{\sin^3\alpha}\right)$$

and $$\frac{d^2(z^2)}{d\alpha^2}>0$$

For minimum of z, $\frac{d(z^2)}{d\alpha} = 0$

$\therefore \quad \tan^4\alpha = \frac{b^2}{a^2}$

or $\quad \tan\alpha = \pm\sqrt{\left(\frac{b}{a}\right)}$

$\Rightarrow \quad \alpha = \tan^{-1}\sqrt{\left(\frac{b}{a}\right)},\ \pi - \tan^{-1}\sqrt{\left(\frac{b}{a}\right)},$

$-\pi + \tan^{-1}\sqrt{\left(\frac{b}{a}\right)}, -\tan^{-1}\sqrt{\left(\frac{b}{a}\right)}$

45. $x^2\tan^2\alpha + y^2\sec^2\alpha = 1$

$\Rightarrow \quad \frac{x^2}{\cot^2\alpha} + \frac{y^2}{\cos^2\alpha} = 1$

$\because \quad \cos^2\alpha = \cot^2\alpha(1-e^2)$

$\Rightarrow \quad \sin^2\alpha = (1-e^2)$

$\therefore \quad e^2 = \cos^2\alpha \qquad (\alpha \neq 90°)$

$e = \cos\alpha$

$\because \quad$ Latusrectum $= \frac{1}{2} = \frac{2b^2}{a}$

$\Rightarrow \quad a = 4b^2 \quad \Rightarrow \cot\alpha = 4\cos^2\alpha$

$\Rightarrow \quad \frac{1}{\sin\alpha} = 4\cos\alpha \quad \Rightarrow \sin 2\alpha = \frac{1}{2}$

$2\alpha = n\pi + (-1)^n\frac{\pi}{6} \quad \Rightarrow \alpha = \frac{n\pi}{2} + (-1)^n\frac{\pi}{12}$

For $n = 0$

$\alpha = \frac{\pi}{12}$ and for $n = 1$

$\alpha = \frac{\pi}{2} - \frac{\pi}{12} = \frac{5\pi}{12}$

Sol. (Q. Nos. 46 to 48)

$\because \quad 9x^2 + 4xy + 6y^2 - 22x - 16y + 9 = 0$

or $\quad 2(2x + y - 3)^2 + (x - 2y + 1)^2 = 10$

or $\quad 2\left(\frac{2x+y-3}{\sqrt{5}}\right)^2 + \left(\frac{x-2y+1}{\sqrt{5}}\right)^2 = 2$

or $\quad \frac{\left(\frac{2x+y-3}{\sqrt{5}}\right)^2}{1} + \frac{\left(\frac{x-2y+1}{\sqrt{5}}\right)^2}{2} = 1$

or $\quad \frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$

Here, $b > a$

46. For centre: $X = 0, Y = 0$

$\Rightarrow \quad 2x + y - 3 = 0$

and $\quad x - 2y + 1 = 0$

we get, $\quad x = 1, y = 1$

$\therefore$ Centre is (1, 1).

47. For eccentricity: $a^2 = b^2(1-e^2)$

$\Rightarrow \quad 1 = 2(1-e^2)$ or $e^2 = \frac{1}{2}$

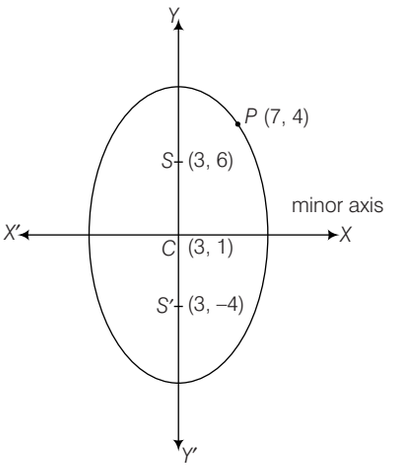
$\therefore \quad e = \frac{1}{\sqrt{2}}$

48. Lengths of axes are $2a$ and $2b$

i.e. 2 and $2\sqrt{2}$.

Sol. (Q. Nos. 49 to 51)

Here, $\quad b > a$



and $\quad CS = 5 = be$ ...(i)

Equation of ellipse is

$\frac{(x-3)^2}{a^2} + \frac{(y-1)^2}{b^2} = 1$ ...(ii)

Which pass through $P(7, 4)$, then

$\frac{16}{a^2} + \frac{9}{b^2} = 1$

$\Rightarrow \quad \frac{16}{b^2-25} + \frac{9}{b^2} = 1 \quad [\because a^2 = b^2(1-e^2) = b^2 - 25]$

$\Rightarrow \quad b^4 - 50b^2 + 225 = 0$

or $\quad (b^2 - 45)(b^2 - 5) = 0$

$\therefore \quad b = 3\sqrt{5} \qquad (b \neq \sqrt{5})$

From Eq. (i),

$e = \frac{\sqrt{5}}{3}$ and $a^2 = 45 - 25 = 20$

From Eq. (ii), ellipse (E) is

$\frac{(x-3)^2}{20} + \frac{(y-1)^2}{45} = 1$

49. Product of the lengths of the perpendicular segments from the foci on tangent at $P(7, 4)$ is equal to a^2 i.e. 20 ($\because b > a$)

50. $\because$ Lines SF' and $S'F$ meet the normal PG and bisects it.

$\therefore$ Required point is mid-point of PG.

$\because$ Equation of ellipse E is

$\frac{(x-3)^2}{20} + \frac{(y-1)^2}{45} = 1$

$\Rightarrow \quad \frac{2(x-3)}{20} + \frac{2(y-1)}{45}\frac{dy}{dx} = 0$

$\therefore \quad \left.\frac{dy}{dx}\right|_{(7,4)} = -3$

$\therefore$ Slope of normal $= \frac{1}{3}$

Equation of normal at $P(7, 4)$ is

$$y-4=\frac{1}{3}(x-7)$$

$$\Rightarrow \quad x-3y+5=0$$

Which meet the major axis in $G\left(3,\frac{8}{3}\right)$.

Required point is mid-point of PG i.e. $\left(5,\frac{10}{3}\right)$.

51. Since locus of mid-point of QR is another ellipse having the same eccentricity as that of ellipse (E).

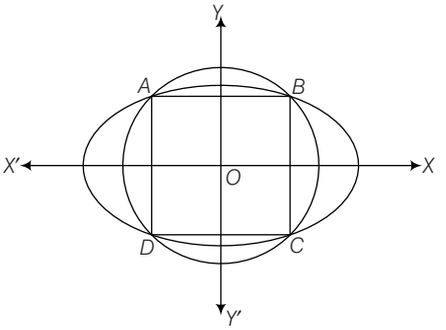
$$\Rightarrow\ e_1=e=\sqrt{\left(1-\frac{a^2}{b^2}\right)}=\sqrt{\left(1-\frac{20}{45}\right)}=\sqrt{\left(1-\frac{4}{9}\right)}=\frac{\sqrt{5}}{3}$$

Sol. (Q. Nos. 52 to 54)

52. Solving curves C_1 and C_2

$$x^2+y^2-r^2=0$$

and $$9x^2+16y^2-144=0$$



then,

$$\begin{array}{cccccc} x^2 & y^2 & 1 & x^2 & y^2 \\ 1 & 1 & -r^2 & 1 & 1 \\ 9 & 16 & -144 & 9 & 16 \end{array}$$

$$\therefore \quad \frac{x^2}{16r^2-144}=\frac{y^2}{144-9r^2}=\frac{1}{7}$$

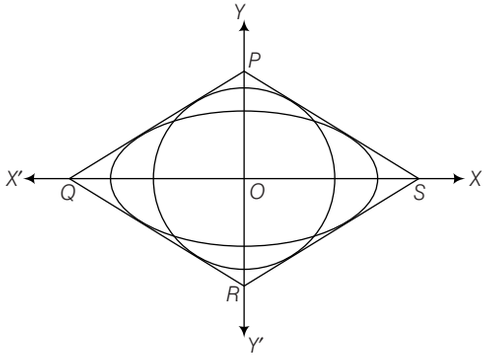
$$\therefore \quad x^2=\frac{16r^2-144}{7} \text{ and } y^2=\frac{144-9r^2}{7}$$

If $ABCD$ is a square, then $x^2=y^2$

or $$\frac{16r^2-144}{7}=\frac{144-9r^2}{7} \Rightarrow 25r^2=288$$

53. Tangents of slope m to the circle and ellipse are respectively.

$$y=mx\pm r\sqrt{(m^2+1)} \text{ and } y=mx\pm\sqrt{(16m^2+9)}$$



For common tangent, $r^2(m^2+1)=16m^2+9$

$$\Rightarrow \quad r=\sqrt{\left(\frac{16m^2+9}{m^2+1}\right)}$$

If $PQRS$ is a square,

then, $\quad m=\pm1$

or $\quad r\sqrt{2}=\sqrt{25}$

or $\quad 2r^2=25$

54. $\because$ $PQRS$ is a square, then $\angle PQR=90°$

$\therefore$ PR is diameter of circle through P, Q, R

$\therefore$ Area of circumcircle of

$$\Delta PQR=\pi(OP)^2=\pi(r\sqrt{2})^2=2\pi r^2$$

and area of circle C_1 is πr^2.

Hence, the required ratio is 1/2.

Sol. (Q. Nos. 55 to 57)

55. $\because$ Distance between foci S and S' is 4 units

$\therefore \quad SS'=4$

$\Rightarrow \quad 2ae=4$

$\therefore \quad ae=2$ … (i)

and given $SP+S'P=6$

$\Rightarrow \quad 2a=6$

$\because \quad a=3$ … (ii)

From Eqs. (i) and (ii), we get

$$e=\frac{2}{3}$$

and $$b^2=a^2(1-e^2)=a^2-(ae)^2=9-4=5$$

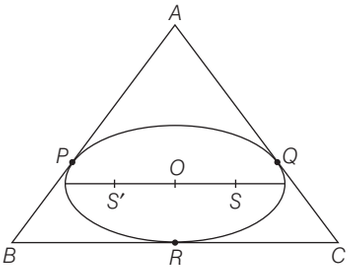
$\therefore$ Equation of ellipse with centre (0, 0) is

$$\frac{x^2}{9}+\frac{y^2}{5}=1 \Rightarrow 5x^2+9y^2=45$$

56. $\because$ $AB=AC$ and $\angle B=\pi/2$

$\because \quad \angle A=\angle C=\pi/4$

Now, if two tangents drawn from (x_1, y_1) on ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



and angle between tangents is α, then

$$\tan\alpha=\frac{2ab\sqrt{\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)}}{|x_1^2+y_1^2-a^2-b^2|}$$

Let $A(x_1, y_1)$

Here, $\alpha=\frac{\pi}{4}$ and $a=3, b=\sqrt{5}$

then, $$1=\frac{6\sqrt{5}\sqrt{\left(\frac{x_1^2}{9}+\frac{y_1^2}{5}-1\right)}}{(x_1^2+y_1^2-9-5)}$$

or $$(x_1^2+y_1^2-14)^2=180\left(\frac{x_1^2}{9}+\frac{y_1^2}{5}-1\right)$$

$\therefore$ Required locus is

$$(x^2+y^2-14)^2=4(5x^2+9y^2-45)$$

57. Let $P=(3\cos\theta, \sqrt{5}\sin\theta)$ and $Q\equiv(3\cos\phi, \sqrt{5}\sin\phi)$

Given, $\theta-\phi=\frac{\pi}{3}$

$\because$ A is the point of intersection of tangents at P and Q

$$\therefore \quad A\equiv\left(\frac{3\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}, \frac{\sqrt{5}\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}\right)$$

Let $$x=\frac{3\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}=\frac{3\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\pi}{6}\right)}=2\sqrt{3}\cos\left(\frac{\theta+\phi}{2}\right) \quad \ldots (i)$$

and $$y=\frac{\sqrt{5}\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}$$

$$=\frac{2\sqrt{5}\sin\left(\frac{\theta+\phi}{2}\right)}{\sqrt{3}} \quad \ldots (ii)$$

From Eqs. (i) and (ii), we get

$$\left(\frac{x}{\sqrt{3}}\right)^2+\left(\frac{\sqrt{3}}{\sqrt{5}}y\right)^2=4 \text{ or } 5x^2+9y^2=60$$

which is required locus of A.

Sol. (Q. Nos. 58 to 60)

Given, line is

$$2px+y\sqrt{(1-p^2)}=1$$

$$\Rightarrow \quad y=-\frac{2p}{\sqrt{(1-p^2)}}x+\frac{1}{\sqrt{(1-p^2)}}$$

$\because$ Equation of tangent in terms of slope (m) of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $y=mx+\sqrt{(a^2m^2+b^2)}$

$$\therefore \quad m=-\frac{2p}{\sqrt{(1-p^2)}}$$

and $$a^2m^2+b^2=\frac{1}{1-p^2}$$

or $$a^2m^2+b^2=\frac{1}{4}m^2+1 \qquad \left(\because p^2=\frac{m^2}{m^2+4}\right)$$

$$\therefore \quad a^2=\frac{1}{4} \text{ and } b^2=1$$

Equation of ellipse is $\frac{x^2}{1/4}+\frac{y^2}{1}=1$

58. Here, $a^2=\frac{1}{4}$, $b^2=1$

$\because$ $b>a$

$$\therefore \quad a^2=b^2(1-e^2) \Rightarrow \frac{1}{4}=1-e^2$$

$$\Rightarrow \quad e^2=\frac{3}{4}$$

$$\therefore \quad e=\frac{\sqrt{3}}{2}$$

59. Foci of ellipse are $(0,\pm\sqrt{(b^2-a^2)})$

i.e. $$\left(0,\pm\frac{\sqrt{3}}{2}\right)$$

60. Required locus in the director circle of ellipse is

$$x^2+y^2=\frac{1}{4}+1$$

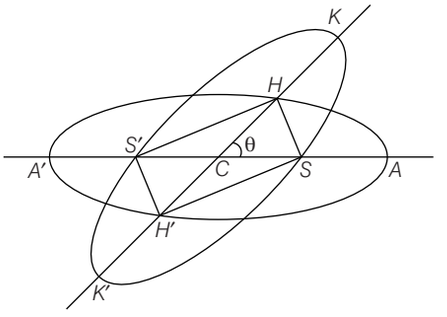
i.e. $$x^2+y^2=5/4$$

61. Let S and S' be the foci of one ellipse and H and H' be the other, C being their common centre. Then, $SHS'H'$ is a parallelogram and since

$$SH+S'H=HS'+H'S'=2a$$

Since, the sum of the focal distances of any point on an ellipse is equal to its major axis which is $2a$.

Then, $$CS=ae, \; CH=ae'$$



Let θ be the angle between their axes.

Then, $$SH^2=a^2e^2+a^2e'^2-2a^2ee'\cos\theta$$

$$HS^2=a^2e^2+a^2e'^2+2a^2ee'\cos\theta$$

Now, $$2a=SH+S'H$$

Squaring both sides, then

$$4a^2=(SH)^2+(S'H)^2+2(SH)\cdot(S'H)$$

$$\Rightarrow \quad 4a^2=2a^2(e^2+e'^2)+2\sqrt{(a^2e^2+a^2e'^2)^2-4a^4e^2e'^2\cos^2\theta}$$

$$\Rightarrow \quad (2-e^2-e'^2)^2=(e^2+e'^2)^2-4e^2e'^2\cos^2\theta$$

$$\Rightarrow \quad 4+(e^2+e'^2)^2-4(e^2+e'^2)=(e^2+e'^2)^2-4e^2e'^2\cos^2\theta$$

$$\Rightarrow \quad 1-e^2-e'^2=-e^2e'^2\cos^2\theta$$

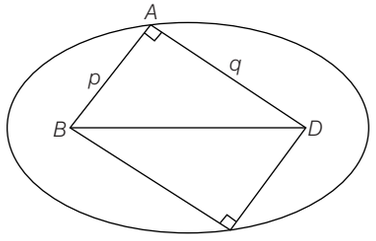
$$\therefore \quad \cos\theta=\frac{\sqrt{e^2+e'^2-1}}{ee'}=\frac{\sqrt{\left(\frac{9}{25}+\frac{16}{25}-1\right)}}{12/25}=0$$

$$\therefore \quad \theta=\frac{\pi}{2}$$

$$\sin\theta=1$$

$$\therefore \quad 2(1+\sin^2\theta+\sin^4\theta)=2(1+1+1)=6$$

62. Let the sides of the rectangle be p and q.



$\therefore$ Area of rectangle $= pq = 200$ ...(i)

and area of ellipse $= \pi ab = 200\pi$

$\therefore$ $ab = 200$ ...(ii)

Perimeter of rectangle $= 2(p + q)$ ...(iii)

In ΔABD,

Distance $BD = \sqrt{(p^2 + q^2)}$ = Distance between foci

$= 2ae$

or $p^2 + q^2 = 4a^2e^2 = 4(a^2 - b^2)$ ...(iv)

Also, $BA + DA = 2a$ [$\because SP + S'P = 2a$]

$\Rightarrow$ $p + q = 2a$ ...(v)

From Eq. (iv),

$(p + q)^2 - 2pq = 4(a^2 - b^2)$

$\Rightarrow$ $(2a)^2 - 2 \times 200 = 4(a^2 - b^2)$ [From Eqs. (i) and (v)]

or $b^2 = 100$ or $b = 10$

From Eq. (ii), $a = 20$

From Eq. (iii),

$P = 2(p + q) = 2(2a) = 4a = 80$

$\therefore$ $P + 1 = 81$

Hence, $\sqrt{(P + 1)} = 9$

63. For ellipse

$$9x^2 + 16y^2 = 144 \text{ or } \frac{x^2}{16} + \frac{y^2}{9} = 1$$

Equation of director circle is

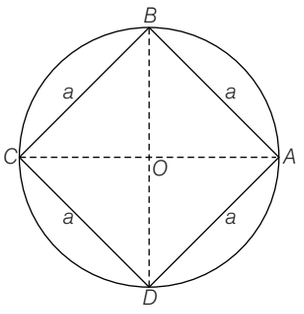
$$x^2 + y^2 = 25$$

This director circle will cut the ellipse $\frac{x^2}{50} + \frac{y^2}{20} = 1$ at four points.

Hence, number of points = 4

64. $\because$ Perpendicular tangents meets on director circle of ellipse $\frac{x^2}{7} + \frac{y^2}{11/2} = 1$ is $x^2 + y^2 = 7 + \frac{11}{2}$

i.e., $$x^2 + y^2 = \left(\frac{5}{\sqrt{2}}\right)^2$$



Let length of sides $= a$

$\therefore$ $OA = OB = OC = OD = \frac{a}{\sqrt{2}}$

Given $\frac{a}{\sqrt{2}} = \frac{5}{\sqrt{2}}$

$\therefore$ $a = 5$

65. We know that, for $\theta = \frac{\pi}{2}$ the focal chord will be latusrectum with length $\frac{2b^2}{a}$.

$$\Rightarrow \frac{\lambda ab^2}{a^2 \sin^2\left(\frac{\pi}{2}\right) + b^2 \cos^2\left(\frac{\pi}{2}\right)} = \frac{2b^2}{a}$$

$$\Rightarrow \frac{\lambda ab^2}{a^2 + 0} = \frac{2b^2}{a}$$

$\therefore$ $\lambda = 2$

66. Equation of normal at the point $(13\cos\theta, 5\sin\theta)$ is

$$y - 5\sin\theta = \frac{13}{5}\frac{\sin\theta}{\cos\theta}(x - 13\cos\theta)$$

or $5y\cos\theta - 25\sin\theta\cos\theta = 13x\sin\theta - 169\sin\theta\cos\theta$

It passes through (0, 6), then

$30\cos\theta - 25\sin\theta\cos\theta = 0 - 169\sin\theta\cos\theta$

$\Rightarrow$ $30\cos\theta + 144\sin\theta\cos\theta = 0$

or $6\cos\theta(5 + 24\sin\theta) = 0$

or $\cos\theta = 0,\ \sin\theta = -\frac{5}{24}$

$\therefore$ $\theta = \frac{\pi}{2},\ \theta = \pi + \sin^{-1}\left(\frac{5}{24}\right),$

$$\theta = 2\pi - \sin^{-1}\left(\frac{5}{24}\right)$$

Hence, three normals can be drawn.

i.e. $n = 3$

$\therefore$ $2^n = 2^3 = 8$

67. The equation of the tangent at $P(a\cos\theta, b\sin\theta)$ on the given ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$

Length of the perpendicular from the focus $(ae, 0)$ on the ellipse is

$$p = \frac{|e\cos\theta - 1|}{\sqrt{\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right)}}$$

$$= \frac{|ab(e\cos\theta - 1)|}{\sqrt{(b^2\cos^2\theta + a^2(1 - \cos^2\theta))}}$$

$$= \frac{ab(1 - e\cos\theta)}{\sqrt{a^2 - a^2e^2\cos^2\theta}} = b\sqrt{\left(\frac{1 - e\cos\theta}{1 + e\cos\theta}\right)}$$

$$\Rightarrow \frac{b^2}{p^2} = \frac{1 + e\cos\theta}{1 - e\cos\theta}$$

Now, $r^2 = (ae - a\cos\theta)^2 + b^2\sin^2\theta$

$= a^2[(e - \cos\theta)^2 + (1 - e^2)\sin^2\theta]$

$$= a^2[e^2 \cos^2\theta - 2e\cos\theta + 1]$$
$$= a^2(1 - e\cos\theta)^2$$
$$\Rightarrow \quad r = a(1 - e\cos\theta)$$
Now, $\frac{2a}{r} - \frac{b^2}{p^2} = \frac{2}{1 - e\cos\theta} - \frac{1 + e\cos\theta}{1 - e\cos\theta} = 1$

68. The points are $A(3, 4)$, $B(6, 8)$ and $O(0, 0)$.

$\because$ $OA + OB = 2a$ (where, a is semi major axis)

$\Rightarrow \quad 5 + 10 = 2a$

$\therefore \quad a = \frac{15}{2}$

Now, $AB = 2ae$

$\Rightarrow \quad \sqrt{(6-3)^2 + (8-4)^2} = 2 \times \frac{15}{2} \times e$

or $5 = 15e$

$\therefore \quad e = \frac{1}{3}$

$\therefore \quad b^2 = a^2(1 - e^2) = \frac{225}{4}\left(1 - \frac{1}{9}\right) = 50$

Hence, $\left(\frac{b}{\sqrt{2}}\right) = 5$

69. For the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, a normal at $P(5\cos\theta, 4\sin\theta)$ is

$$5x\sin\theta - 4y\cos\theta = 9\sin\theta\cos\theta$$

Since, the normal pass through $(\lambda, 0)$, then

$$5\lambda\sin\theta - 0 = 9\sin\theta\cos\theta$$

For, $\sin\theta \neq 0$ (i.e. $\theta \neq 0, \pi$), we get

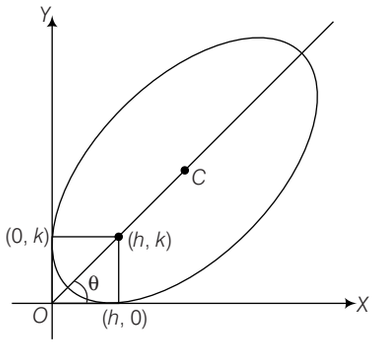
$5\lambda = 9\cos\theta < 9$ $\quad (\because \cos\theta < 1)$

$\therefore \quad 5\lambda < 9$

Hence, maximum value of 5λ is 9.

70. Here, $a = 3\sqrt{3}, b = 3$

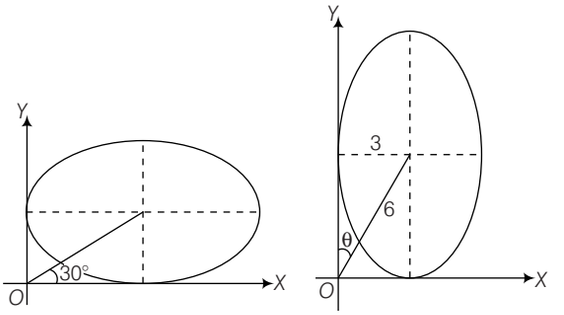
Since, axes are at 90° to each other.



$\therefore$ $O(0, 0)$ being the centre of arc and radius of arc is OC

$\Rightarrow \quad OC = \sqrt{(27 + 9)} = 6$ units

The two extreme positions are horizontal and vertical orientation.



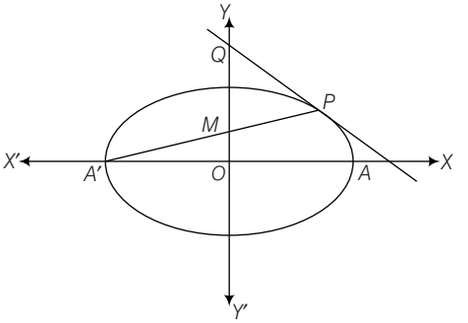
Hence, rotation 30° takes place in totality

$\Rightarrow$ Length of arc $= \frac{2\pi r}{12} = \frac{2\pi}{12} \times 6 = \pi = \frac{\pi\lambda}{6}$ [given]

$\therefore \quad \lambda = 6$

71. (A) Let point P be $(a\cos\theta, b\sin\theta)$. The equation of tangent at point P is $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$

The point Q is $(0, b\,\text{cosec}\,\theta)$.



The equations of chord $A'P$ is

$$y - 0 = \frac{b\sin\theta - 0}{a\cos\theta + a}(x + a)$$

The point M is $\left(0, \frac{b\sin\theta}{1 + \cos\theta}\right)$

Then, $(OQ)^2 - (MQ)^2 = b^2\,\text{cosec}^2\theta - \left(b\,\text{cosec}\,\theta - \frac{b\sin\theta}{1 + \cos\theta}\right)^2$

$$= \frac{2b^2}{1 + \cos\theta} - \frac{b^2\sin^2\theta}{(1 + \cos\theta)^2}$$
$$= \frac{2b^2}{1 + \cos\theta} - \frac{b^2(1 - \cos^2\theta)}{(1 + \cos\theta)^2}$$
$$= \frac{2b^2}{(1 + \cos\theta)} - \frac{b^2(1 - \cos\theta)}{(1 + \cos\theta)}$$
$$= b^2 = 4$$

(B) We know that, two diameters $y = m_1x$, $y = m_2x$ are conjugate diameters of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $m_1m_2 = -\frac{b^2}{a^2}$

$\Rightarrow \quad 1 \times -\frac{2}{3} = \frac{-b^2}{a^2}$

or $b^2 = \frac{2}{3}a^2$

or $a^2(1-e^2) = \frac{2}{3}a^2$

or $1-e^2 = \frac{2}{3}$ or $e^2 = \frac{1}{3}$

Hence, $4\left(1+\frac{1}{3}+\frac{1}{9}+\ldots\infty\right) = 4\left(\frac{1}{1-\frac{1}{3}}\right) = 6$

(C) $\because\ 2x^2+3y^2=6 \Rightarrow \frac{x^2}{3}+\frac{y^2}{2}=1$

By using condition of tangency $(2h)^2 = 3\times(k)^2+2$

Therefore, the locus of $P(h,k)$ is

$4x^2-3y^2=2$ (which is hyperbola)

or $\frac{x^2}{1/2}-\frac{y^2}{2/3}=1$

Hence, $\frac{2}{3}=\frac{1}{2}(e^2-1)$

$\Rightarrow\ e^2 = 1+\frac{4}{3}$ or $3e^2=7$

(D) $\left(\pm ae, \frac{b^2}{a}\right)$ are extremities of the latusrectum having positive ordinates, then,

$$a^2e^2 = -2\left(\frac{b^2}{a}-2\right) \quad \ldots(i)$$

But $b^2 = a^2(1-e^2)$ ...(ii)

From Eqs. (i) and (ii), we get

$$a^2e^2 = -2(a(1-e^2)-2)$$

$\Rightarrow\ a^2e^2-2ae^2+2a-4=0$

$\Rightarrow\ ae^2(a-2)+2(a-2)=0$

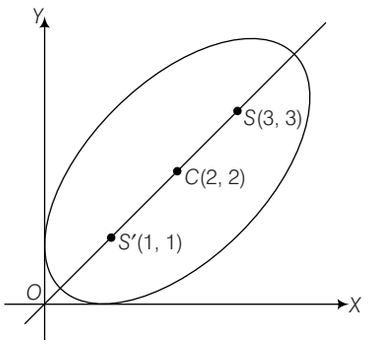
$\therefore\ (ae^2+2)(a-2)=0$

Hence, $a=2$

72. (A) $\because$ (0, 0) lies on the director circle and mid-point of foci (2, 2) is the center of the circle.

Hence, radius of director circle = OC

$r = \sqrt{(4+4)} = 2\sqrt{2}$



Therefore, the area is πr^2 i.e. $8\pi = \pi\lambda$ (given)

$\therefore\ \lambda = 8$

(B) Equation of normal at $(5\cos\theta,\ 4\sin\theta)$ is

$5x\sec\theta - 4y\,\text{cosec}\,\theta = 9$ and it passes through $P(0,\lambda)$.

Then, $0-4\lambda\ \text{cosec}\,\theta = 9$

or $\lambda = -\frac{9}{4}\sin\theta$

or $|\lambda| = \frac{9}{4}|\sin\theta| < \frac{9}{4}$

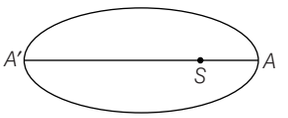
$\therefore\ p = \frac{9}{4} \Rightarrow 4p = 9$

(C) Let the orbit of the earth be the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \ldots(i)$$

Its major axis $= 2a = 186\times10^6$ miles (given)

i.e. $a = 93\times10^6$ miles and $e = 1/60$ (given)



Let the Sun be at the focus $S(ae,0)$. Then, the Earth will be at shortest and longest distance from the Sun when the Earth is at the extremities of the major axis which are respectively nearest and farthest from this focus S.

$\therefore$ Shortest distance of the Earth from the Sun

$= SA$, where S is $(ae, 0)$ and A is $(a,0)$

$= a-ae = a(1-e)$

$= (93\times10^6)(1+(1/60))$

$= 91450000$ miles $= 914.5\times10^5$ miles and longest distance of Earth from the Sun

$= SA'$, where S is $(ae,0)$ and A' is $(-a,0)$

$= a+ae$

$= a(1+e)$

$= (93\times10^6)(1+(1/60))$

$= 94550000$ miles $= 945.5\times10^5$ miles

$\therefore\ \mu-\lambda+1 = 945.5-914.5-1 = 30$

(D) Equation of chord whose mid-point (0,3) is $T = S_1$

$\Rightarrow\ 0+\frac{3.y}{25}-1 = 0+\frac{9}{25}-1$

$\Rightarrow\ y = 3$

It intersects the ellipse $\frac{x^2}{16}+\frac{y^2}{25}=1$

at $\frac{x^2}{16} = 1-\frac{3^2}{25} = \frac{16}{25}$

or $x = \pm\frac{16}{5}$

$\therefore$ Length of chord $= \frac{32}{5} = \frac{4\lambda}{5}$ (given)

$\therefore\ \lambda = 8$

73. (A) Let $S = 9x^2+8y^2-36x-16y-28$

$\therefore$ Value of S at (2,6)

$S_1 = 9(2)^2+8(6)^2-36(2)-16(6)-28$

$= 36+288-72-96-28$

$= 128 > 0$

$\therefore$ Point (2,6) lies outside the ellipse.

The equation of the given ellipse be rewritten as

$$9(x-2)^2+8(y-1)^2=72$$

$$\Rightarrow \quad \frac{(x-2)^2}{8}+\frac{(y-1)^2}{9}=1$$

Centre of ellipse is (2,1) and axis parallel to Y-axis

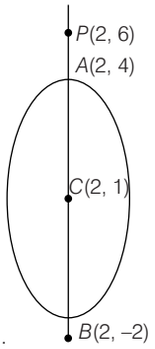
$\therefore$ Vertices are $x-2=0$

and $y-1=\pm 3$

or (2,–2) and (2,4)

$\therefore$ Minimum distance

$$L=PA=2$$



and maximum distance

$$G=PB=8$$

Then, $L+G=10, G-L=6$

(B) Let $S=4x^2+9y^2+8x-36y+4$

$\therefore$ Value of S at (1,2) is

$$S_1=4(1)^2+9(2)^2+8(1)-36(2)+4$$
$$=4+36+8-72+4$$
$$=-20<0$$

$\therefore$ Point (1,2) lies inside the ellipse

The equation of the given ellipse be rewritten as

$$4(x+1)^2+9(y-2)^2=36$$

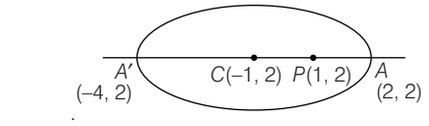
or $$\frac{(x+1)^2}{9}+\frac{(y-2)^2}{4}=1$$

$\therefore$ Centre of ellipse is $(-1, 2)$ and axis parallel to X-axis.

$\therefore$ Vertices are $x+1=\pm 3$

and $y-2=0$

or (-4,2) and (2,2)



$\therefore$ Minimum distance $L=PA=1$

and Maximum distance

$$G=PA'=AA'-PA=6-1=5$$

$\therefore \quad L+G=6, G-L=4,$

(C) Here $3x+4y=0$ and $4x-3y=0$ are mutually perpendicular lines, then substituting

$$\frac{3x+4y}{\sqrt{(3^2+4^2)}}=X \text{ and } \frac{4x-3y}{\sqrt{((4)^2+(-3)^2)}}=Y$$

Then, the given equation can be written as

$$4X^2+9Y^2=36 \Rightarrow \frac{X^2}{9}+\frac{Y^2}{4}=1$$

$\therefore$ Vertices, $X=\pm 3, Y=0$

or $$\frac{3x+4y}{5}=\pm 3, \frac{4x-3y}{5}=0$$

or $$3x+4y=\pm 15, y=\frac{4x}{3}$$

Vertices are $\left(\frac{9}{5}, \frac{12}{5}\right)$ and $\left(-\frac{9}{5}, \frac{-12}{5}\right)$

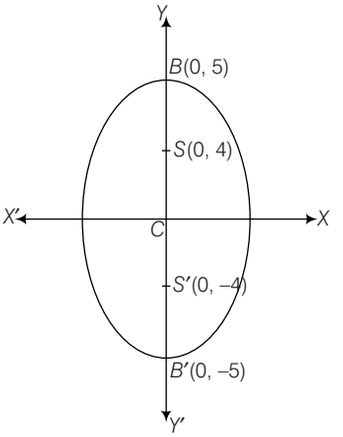
$\because$ Given point is a vertex.

$\therefore$ Minimum distance $L=0$

and maximum distance $G=$ Length of major axis $=2\times 3=6$

Then, $L+G=6, G-L=6$

(D) Given ellipse is $25x^2+9y^2=225$



or $$\frac{x^2}{9}+\frac{y^2}{25}=1$$

$\therefore$ focii $(0, \pm \sqrt{(25-9)})$ or $(0, \pm 4)$

and vertices are $B(0, 5)$ and $B'(0, -5)$

$\therefore$ Minimum distance $L=SB=1$

and maximum distance $G=SB'=4+5=9$

Then, $L+G=10, G-L=8$

74. Here, $(a^2+1)<(a^2+2)$

$\therefore$ Length of latusrectum $=\frac{2(a^2+1)}{\sqrt{(a^2+2)}}=\frac{10}{\sqrt{6}}$

$\Rightarrow \quad 6(a^2+1)^2=25(a^2+2)$

or $6a^4-13a^2-44=0$

or $(a^2-4)(6a^2+11)=0$

$\therefore \quad a^2=4, \ 6a^2+11\neq 0$

If e be the eccentricity, then

$$(a^2+1)=(a^2+2)(1-e^2)$$

$\Rightarrow \quad e^2=1-\frac{5}{6}=\frac{1}{6}$

$\therefore \quad e=\frac{1}{\sqrt{6}}$

Hence, both statements are true, statement II is correct explanation for statement I.

75. The given ellipse is $\frac{x^2}{3}+\frac{y^2}{2}=1$

$\therefore$ Area of ellipse $=\pi\sqrt{3}\sqrt{2}=\pi\sqrt{6}$

The circle is $x^2+y^2-2x+4y+4=0$

or $(x-1)^2+(y+2)^2=1$

Its area is π. Hence, statement I is true.

Also, statement II is true but it is not the correct explanation of statement I.

76. Given, ellipse is $4x^2+9y^2=36$

or $$\frac{x^2}{3^2}+\frac{y^2}{2^2}=1 \quad \ldots(i)$$

Equation of tangents in terms of slope (m) are

$$y=mx\pm\sqrt{(9m^2+4)}$$

$\Rightarrow$ $(y-mx)^2=9m^2+4$

$\Rightarrow$ $m^2(x^2-9)-2mxy+y^2-4=0$

$\therefore$ $m_1m_2=-1 \Rightarrow \frac{y^2-4}{x^2-9}=-1$

$\Rightarrow x^2+y^2=13$, which is director circle of (i).

Hence, both statements are true and statement II is a correct explanation for statement I.

77. Distance between foci $=2ae$ and sum of focal distances from a point $=2a$

$\therefore$ $2ae<2a \Rightarrow e<1$

Both statements are true, statement II is correct explanation for statement I.

78. $4x^2+5y^2-16x-30y+41=0$ ...(i)

$\Rightarrow$ $4(x^2-4x)+5(y^2-6y)+41=0$

$\Rightarrow$ $4\{(x-2)^2-4\}+5\{(y-3)^2-9\}+41=0$

$\Rightarrow$ $4(x-2)^2+5(y-3)^2=20$

$\therefore$ statement II is true.

and $\frac{(x-2)^2}{5}+\frac{(y-3)^2}{4}=1$ or $\frac{(x-2)^2}{(\sqrt{5})^2}+\frac{(y-3)^2}{2^2}=1$

$\therefore$ Sum of focal distances of a point (P) on the ellipse (i) is $2\sqrt{5}$.

$\therefore$ Statement I is true, but statement II is not a correct explanation for statement I.

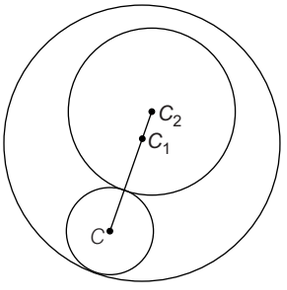
79. Let variable circle is $(x-h)^2+(y-k)^2=r^2$ ...(i)

$\therefore$ $C(h, k)$ and radius r and given circles

$$(x-1)^2+(y-2)^2=5^2 \quad \ldots(ii)$$

$\therefore$ $C_1(1, 2)$ and $r_1=5$

and $$(x-2)^2+(y-1)^2=4^2 \quad \ldots(iii)$$



$\therefore$ $C_2(2, 1)$ and $r_2=4$

$$|C_1C_2|=\sqrt{2}>|r_1-r_2|$$

$\therefore$ Circle (iii) lies completely inside the circle (ii)

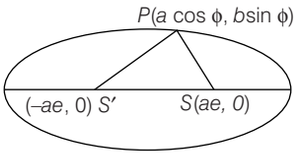
$\because$ $CC_2=r+4$ and $CC_1=5-r$

$\Rightarrow$ $CC_1+CC_2=9=$ constant

$\therefore$ C lies on a ellipse.

Statement I is true and Statement II is false.

80. Statement I is obviously true since it is a theorem

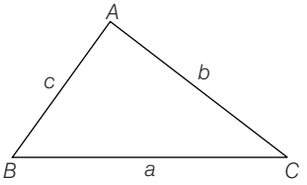


$\because$ (Slope of SP) $\times$ (Slope of $S'P$) $\neq -1$

$\therefore$ Statement II is false.

81. Given, $a=$ constant and $a+b+c=$ constant $=2s$.

Now, $BA+CA=c+b=2s-a=$ constant



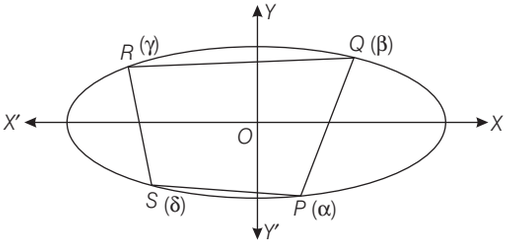
$\therefore$ Locus of A is an ellipse.

$\Rightarrow$ Both statements are true and statement II is a correct explanation for statement I.

82. Let $PQRS$ be a quadrilateral inscribed in the ellipse.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Let PQ, QR and RS be the three sides parallel to the given lines.



Equation of PQ is

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$$

Its slope is $-\frac{b}{a}\cot\left(\frac{\alpha+\beta}{2}\right)$ which is constant by hypothesis

$\therefore$ $\alpha+\beta=$ constant $=2\lambda_1$ (say)

Similarly $\beta+\gamma=$ constant $=2\lambda_2$ (say)

and $\gamma+\delta=$ constant $=2\lambda_3$ (say)

Now the equation of SP is

$$\frac{x}{a}\cos\left(\frac{\alpha+\delta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\delta}{2}\right)$$

$$=\cos\left(\frac{\alpha-\delta}{2}\right)$$

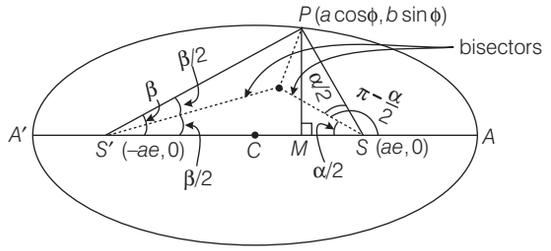
Its slope $m = -\frac{b}{a}\cot\left(\frac{\alpha+\delta}{2}\right)$

But $\alpha + \delta = (\alpha+\beta) - (\beta+\gamma) + (\gamma+\delta)$
$= 2\lambda_1 - 2\lambda_2 + 2\lambda_3$
$=$ constant

Hence, the slope of the fourth side PS is constant. Hence the fourth side is also parallel to a fixed straight line.

83. Let $\angle PSS' = \alpha$ and $\angle PS'S = \beta$

$\therefore \quad SS' = 2ae$



By sine rule in $\Delta PSS'$, then

$$\frac{SP}{\sin\beta} = \frac{S'P}{\sin\alpha} = \frac{SS'}{\sin\{\pi - (\alpha+\beta)\}}$$

$$\Rightarrow \quad \frac{SP}{\sin\beta} = \frac{S'P}{\sin\alpha} = \frac{SS'}{\sin(\alpha+\beta)}$$

or $$\frac{SP + S'P}{\sin\beta + \sin\alpha} = \frac{SS'}{(\sin\alpha+\beta)}$$

or $$\frac{2a}{\sin\beta + \sin\alpha} = \frac{2ae}{\sin(\alpha+\beta)} \qquad [\because SP + S'P = 2a]$$

or $$\frac{2a}{2ae} = \frac{\sin\beta + \sin\alpha}{\sin(\alpha+\beta)}$$

$$\Rightarrow \quad \frac{1}{e} = \frac{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)} \Rightarrow \frac{1}{e} = \frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}$$

$$\frac{1-e}{1+e} = \frac{\cos\left(\frac{\alpha-\beta}{2}\right) - \cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right) + \cos\left(\frac{\alpha+\beta}{2}\right)}$$

{By componendo and dividendo rule}

$$\Rightarrow \quad \frac{1-e}{1+e} = \tan(\alpha/2)\tan(\beta/2) \qquad \ldots(i)$$

or $$\tan\left(\frac{PSS'}{2}\right)\tan\left(\frac{PS'S}{2}\right) = \frac{1-e}{1+e}$$

The centre of the circle inscribed in $\Delta PSS'$ will lie on the bisectors of angles α and β whose equations are

$$y = \tan\left(\frac{\beta}{2}\right)(x + ae)$$

and $y = \tan\left(\pi - \frac{\alpha}{2}\right)(x - ae)$ or $y = -\tan\left(\frac{\alpha}{2}\right)(x - ae)$

By multiplying we get

$$y^2 = -\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)(x^2 - a^2e^2)$$

$$y^2 = -\left(\frac{1-e}{1+e}\right)(x^2 - a^2e^2) \qquad \text{[From Eq. (i)]}$$

or $$y^2 + x^2\left(\frac{1-e}{1+e}\right) = a^2e^2\left(\frac{1-e}{1+e}\right)$$

or $$\frac{x^2}{a^2e^2} + \frac{y^2}{a^2e^2\left(\frac{1-e}{1+e}\right)} = 1$$

which is an ellipse.

84. Let P be $(a\cos\phi, b\sin\phi)$

$\therefore$ Equation of tangent at P is

$$\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1 \qquad \ldots(i)$$

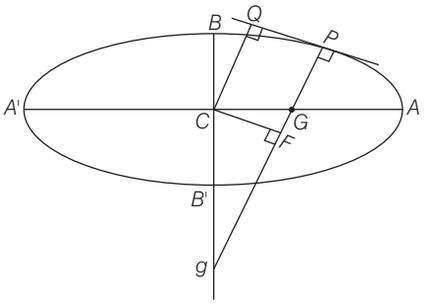
and equation of normal at P is

$$ax\sec\phi - by\,\text{cosec}\,\phi = a^2 - b^2 \qquad \ldots(ii)$$

It is clear from the figure $CFPQ$ is rectangle

$\therefore \quad PF = CQ$

{i.e. the perpendicular from $C(0, 0)$ on Eq. (i)}



$$= \frac{1}{\sqrt{\left(\frac{\cos^2\phi}{a^2} + \frac{\sin^2\phi}{b^2}\right)}} = \frac{ab}{\sqrt{(a^2\sin^2\phi + b^2\cos^2\phi)}} \qquad \ldots(iii)$$

Putting $y = 0$ in Eq. (ii), the point G is $\left(\frac{a^2-b^2}{a}\cos\phi,\ 0\right)$

and Putting $x = 0$ in Eq. (ii), the point g is

$$\left(0, -\frac{(a^2-b^2)}{b}\sin\phi\right)$$

$$\therefore \quad (PG)^2 = \left(a\cos\phi - \left(\frac{a^2-b^2}{2}\right)\cos\phi\right)^2 + (b\sin\phi - 0)^2$$

$$= \frac{b^4}{a^2}\cos^2\phi + b^2\sin^2\phi = \frac{b^2}{a^2}(b^2\cos^2\phi + a^2\sin^2\phi)$$

$$\therefore \quad PG = \frac{b}{a}\sqrt{(a^2\sin^2\phi + b^2\cos^2\phi)} \qquad \ldots(iv)$$

and $$(Pg)^2 = (a\cos\phi - 0)^2 + \left(b\sin\phi + \frac{(a^2-b^2)}{b}\sin\phi\right)^2$$

$$= a^2\cos^2\phi + \frac{a^4}{b^2}\sin^2\phi = \frac{a^2}{b^2}(a^2\sin^2\phi + b^2\cos^2\phi)$$

$$\therefore \quad Pg = \frac{a}{b}\sqrt{(a^2\sin^2\phi + b^2\cos^2\phi)} \qquad \ldots(v)$$

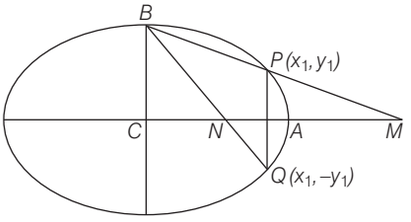
From Eqs. (iii) and (iv), $PF \cdot PG = b^2$
and from Eqs. (iii) and (v), $PF \cdot Pg = a^2$.

85. Let the coordinates of P be (x_1, y_1) then Q is $(x_1, -y_1)$.
Also B is $(0, b)$

$\therefore$ Equation of PB is

$$y - y_1 = \frac{b - y_1}{0 - x_1}(x - x_1)$$

It meets major axis i.e. $y = 0$ in M



$$\therefore \qquad M \text{ is } \left(\frac{bx_1}{(b - y_1)}, 0\right)$$

Also equation of BQ is $y + y_1 = \frac{b + y_1}{0 - x_1}(x - x_1)$

It meets the major axis *i.e.*, $y = 0$ in N.

$$\therefore \qquad N \text{ is } \left(\frac{bx_1}{b + y_1}, 0\right)$$

Now the polar of M with respect to the auxiliary circle $x^2 + y^2 = a^2$ is

$$\frac{xbx_1}{b - y_1} + 0 \cdot y = a^2$$

or $$bxx_1 = a^2(b - y_1) \qquad \ldots(i)$$

If Eq. (i) passes through N, then

$$\frac{b^2x_1^2}{(b + y_1)} = a^2(b - y)$$

$$\Rightarrow \qquad b^2x_1^2 = a^2(b^2 - y_1^2)$$

or $$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1 \text{ which is true.}$$

86. The equation of any tangent to the ellipse

$$4x^2 + 25y^2 = 100 \text{ or } \frac{x^2}{25} + \frac{y^2}{4} = 1 \text{ is}$$

$$y = mx + \sqrt{(25m^2 + 4)} \qquad \ldots(i)$$

It meets the coordinate of axes at

$$A\left(-\frac{\sqrt{(25m^2 + 4)}}{m}, 0\right) \quad \text{and} \quad B\left(0, \sqrt{(25m^2 + 4}\right)$$

Let $P(h, k)$ be the mid-point of AB. Then

$$2h = -\frac{\sqrt{(25m^2 + 4)}}{m} \quad \text{and} \quad 2k = \sqrt{(25m^2 + 4)}$$

$$\Rightarrow \frac{2k}{2h} = -m \quad \Rightarrow \quad m = -\frac{k}{h}$$

Since, Eq. (i) touches the circle $x^2 + y^2 = r^2$

Therefore, $$\frac{\sqrt{(25m^2 + 4)}}{\sqrt{(1 + m^2)}} = |r|$$

or $$\frac{25m^2 + 4}{1 + m^2} = r^2$$

or $$25\left(\frac{k^2}{h^2}\right) + 4 = r^2\left(1 + \frac{k^2}{h^2}\right) \qquad \left(\because m = -\frac{k}{h}\right)$$

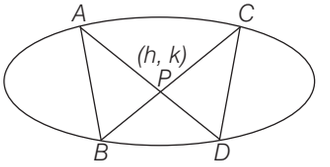
Hence the locus of $P(h, k)$ is

$$25y^2 + 4x^2 = r^2(x^2 + y^2)$$

or $$y = \pm\sqrt{\left(\frac{(r^2 - 4)}{(25 - r^2)}\right)}x$$

87. Let the normals at the points when the line, $\frac{lx}{a} + \frac{my}{b} = 1$ meet in $P(h, k)$

Let the line joining the feet of the other two normals from (h, k) have for its equation, $rx + sy = 1$



Then the general equation of a curve through the intersection of the ellipse and the two lines is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) + \lambda\left(\frac{lx}{a} + \frac{my}{b} - 1\right) \times (rx + sy - 1) = 0 \qquad \ldots(i)$$

But the four points A, B, C, D also lie on the ellipse

$$(a^2 - b^2)\,xy + kb^2x - ha^2y = 0 \qquad \ldots(ii)$$

Curves Eqs. (i) and (ii) must be identical.

This is possible only one coefficient of x^2 and y^2 and the constant terms in Eq. (i) must vanish separately.

$$\frac{1}{a^2} + \frac{\lambda rl}{a} = 0, \quad \text{and} \quad \frac{1}{b^2} + \frac{\lambda ms}{b} = 0, \lambda - 1 = 0$$

$$\therefore \qquad \lambda = 1,\ 1 + arl = 0 = 1 + bms$$

$$\therefore \qquad r = \frac{-1}{al} \quad \text{and} \quad s = \frac{-1}{mb}$$

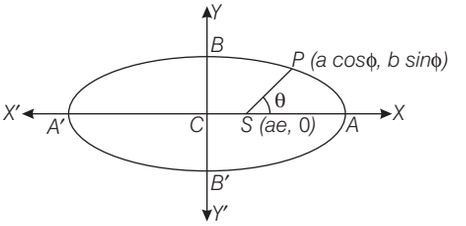
Therefore, the line $rx + sy = 1$ become

$$-\frac{x}{al} - \frac{y}{mb} = 1 \text{ or } \frac{x}{al} + \frac{y}{mb} = -1$$

88. $\therefore \quad \tan\theta = \dfrac{b\sin\phi}{(a\cos\phi - ae)} = \dfrac{\left(\frac{b}{a}\right)\sin\phi}{(\cos\phi - e)}$

$\because \qquad b^2 = a^2(1 - e^2)$

$\therefore \qquad \dfrac{b}{a} = \sqrt{(1 - e^2)}$



$$\Rightarrow \tan\theta = \frac{\sqrt{(1 - e^2)}\sin\phi}{(\cos\phi - e)}$$

$$\Rightarrow \sin\theta\cos\phi - e\sin\theta = \sqrt{(1 - e^2)}\sin\phi\cos\theta$$

$$\Rightarrow \frac{2\tan(\theta/2)}{1+\tan^2(\theta/2)}\cdot\frac{1-\tan^2(\phi/2)}{1+\tan^2(\phi/2)}-\frac{e\,(2\tan(\theta/2))}{(1+\tan^2(\theta/2))}$$

$$=\sqrt{(1-e^2)}\cdot\frac{2\tan(\phi/2)}{1+\tan^2(\phi/2)}\cdot\frac{1-\tan^2(\theta/2)}{1+\tan^2(\theta/2)}$$

$$\Rightarrow 2\tan(\theta/2)(1-\tan^2(\phi/2))-2e\tan(\theta/2)(1+\tan^2(\phi/2))$$

$$=2\sqrt{(1-e^2)}\cdot\tan(\phi/2)(1-\tan^2(\theta/2))$$

$$\Rightarrow (1-e)\tan(\theta/2)-(1+e)\tan(\theta/2)$$

$$\tan^2(\phi/2)-\sqrt{(1-e^2)}\tan(\phi/2)$$

$$+\sqrt{(1-e^2)}\tan^2(\theta/2)\tan(\phi/2)=0$$

$$\Rightarrow (1+e)\tan^2(\phi/2)+\sqrt{(1-e^2)}\tan(\phi/2)$$

$$(\cot(\theta/2)-\tan(\theta/2))-(1-e)=0$$

$$\Rightarrow \sqrt{(1+e)}\tan(\phi/2)\{\sqrt{(1+e)}\tan(\phi/2)-\sqrt{(1-e)}\tan(\theta/2)\}$$

$$+\sqrt{(1-e)}\cot(\theta/2)\{\sqrt{(1+e)}\tan(\phi/2)-\sqrt{(1-e)}\tan(\theta/2)\}=0$$

$$\Rightarrow (\sqrt{(1+e)}\tan(\phi/2)+\sqrt{(1-e)}\cot(\theta/2))$$

$$(\sqrt{(1+e)}\tan(\phi/2)-\sqrt{(1-e)}\tan(\theta/2))=0$$

But ϕ and θ lies between 0 and π

then $\tan(\phi/2)>0$ and $\tan(\theta/2)>0$

then $\sqrt{(1+e)}\tan(\phi/2)+\sqrt{(1-e)}\cot(\theta/2)\neq 0$

$$\therefore \sqrt{(1+e)}\tan(\phi/2)-\sqrt{(1-e)}\tan(\theta/2)=0$$

$$\tan(\theta/2)=\sqrt{\left(\frac{1+e}{1-e}\right)}\tan(\phi/2)$$

89. Let the given ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad ...(i)$$

Let $P(\alpha)$ and $Q(\beta)$ be two points on Eq. (i) such that

$$\theta=\alpha-\beta \quad ...(ii)$$

Given that tangents at P and Q are right angles.

$$\therefore \left(-\frac{b}{a}\cot\alpha\right)\left(-\frac{b}{a}\cot\beta\right)=-1$$

$$\Rightarrow a^2\sin\alpha\sin\beta+b^2\cos\alpha\cos\beta=0 \quad ...(iii)$$

But the diameter parallel to the tangent at $P(\alpha)$ will be conjugate to the diameter CP, then its extremities will be

$$(-a\sin\alpha,\ b\cos\alpha)$$

$$\therefore d_1^2=a^2\sin^2\alpha+b^2\cos^2\alpha$$

Similarly $d_2^2=a^2\sin^2\beta+b^2\cos^2\beta$

$$\Rightarrow d_1^2d_2^2=(a^2\sin^2\alpha+b^2\cos^2\alpha)(a^2\sin^2\beta+b^2\cos^2\beta)$$

$$=(a^2\sin\alpha\sin\beta+b^2\cos\alpha\cos\beta)^2$$

$$+a^2b^2(\sin\alpha\cos\beta-\cos\alpha\sin\beta)^2$$

$$=0+a^2b^2\sin^2(\alpha-\beta) \quad \text{[from Eq. (iii)]}$$

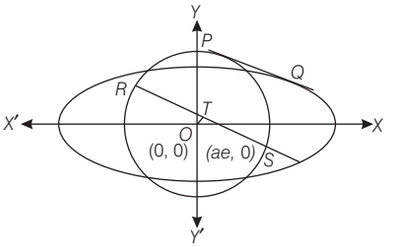
$$=a^2b^2\sin^2\theta \quad \text{[from Eq. (ii)]}$$

$$\therefore ab\sin\theta=d_1d_2$$

90. $y=mx+\sqrt{(a^2m^2+b^2)}$ is a tangent to the ellipse this tangent also touches the circle $x^2+y^2=r^2$

then $$\pm r=\frac{0-0+\sqrt{a^2m^2+b^2}}{\sqrt{1+m^2}}$$

we get $$m=\sqrt{\left(\frac{r^2-b^2}{a^2-r^2}\right)} \quad (\because b<r<a)$$



The line RS passes through $(ae, 0)$ and parallel to PQ is

$$y-0=m(x-ae)$$

$$\Rightarrow mx-y-ame=0$$

Let T be the feet of the perpendicular dropped from the origin on RS

$$\therefore OT=\frac{ame}{\sqrt{1+m^2}}$$

$$(RT)^2=(OR)^2-(OT)^2$$

$$=r^2-\frac{a^2m^2e^2}{1+m^2}=r^2-(r^2-b^2)=b^2$$

$$\therefore RT=b,\ RS=2b$$

91. Equation of the tangent at the point $P(a\cos\phi, b\sin\phi)$ on $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1 \quad ...(i)$$

and foci $F_1\equiv(ae, 0)$ $F_2\equiv(-ae, 0)$

$\therefore$ $d=$ perpendicular distance of Eq. (i) form the centre $(0, 0)$ of the ellipse

$$\therefore d=\frac{1}{\sqrt{\frac{\cos^2\phi}{a^2}+\frac{\sin^2\phi}{b^2}}}=\frac{ab}{\sqrt{(a^2\sin^2\phi+b^2\cos^2\phi)}}$$

$$\therefore 4a^2\left(1-\frac{b^2}{d^2}\right)=4a^2\left(1-\frac{b^2(a^2\sin^2\phi+b^2\cos^2\phi)}{a^2b^2}\right)$$

$$=4a^2\left(1-\sin^2\phi-\frac{b^2}{a^2}\cos^2\phi\right)$$

$$=4a^2\cos^2\phi\,\frac{(a^2-b^2)}{a^2}$$

$$=4\cos^2\phi\,(a^2-b^2)$$

$$=4a^2e^2\cos^2\phi=(2ae\cos\phi)^2$$

$$=[(a-ae\cos\phi)-(a+ae\cos\phi)]^2$$

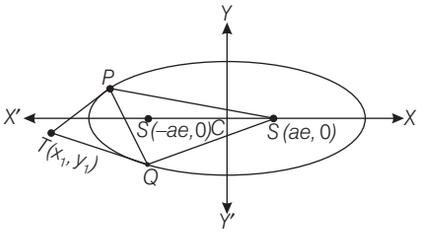
$$=(PF_1-PF_2)^2$$

Hence $(PF_1-PF_2)^2=4a^2\left(1-\frac{b^2}{d^2}\right)$.

92. **(i)** Equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ...(i)

PQ is the chord of contact.

Equation of chord of contact is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ ...(ii)



Eliminate y from Eqs. (i) and (ii), we get

$x^2(a^2y_1^2+b^2x_1^2)-2a^2b^2xx_1+a^4(b^2-y_1^2)=0$

Let x' and x'' be it roots

$\therefore \quad x'+x''=\frac{2a^2b^2x_1}{(a^2y_1^2+b^2x_1^2)}, x'x''=\frac{a^4(b^2-y_1^2)}{(a^2y_1^2+b^2x_1^2)}$

Now, $ST^2=(x_1-ae)^2+y_1^2$

Also, $SP\cdot SQ=(a-ex')(a-ex'')$

$=a^2-ae(x'+x'')+e^2x'x''$

$=a^2-\frac{ae\cdot(2a^2b^2x_1)}{(a^2y_1^2+b^2x_1^2)}+\frac{e^2a^4(b^2-y_1^2)}{(a^2y_1^2+b^2x_1^2)}$

$=\frac{a^2}{(a^2y_1^2+b^2x_1^2)}\{b^2(x_1-ae)^2+b^2y_1^2\}$

$=\frac{a^2b^2(ST)^2}{(a^2y_1^2+b^2x_1^2)}$

$\therefore \quad \frac{(ST)^2}{SP\cdot SQ}=\frac{y_1^2}{b^2}+\frac{x_1^2}{a^2}$

(ii) $ST\cdot S'T\cos\theta=(\sqrt{(x_1-ae)^2+y_1^2})(\sqrt{(x_1+ae)^2+y_1^2})\cos\theta$

$=\sqrt{(x_1^2+y_1^2+a^2e^2)^2-4a^2e^2x_1^2}\cos\theta$ ...(i)

Equation pair of tangents.

$SS_1=T^2$

$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2$

$\therefore \quad \tan\theta=\frac{2ab\sqrt{\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)}}{(x_1^2+y_1^2-a^2-b^2)}$

$\therefore \quad \cos\theta=\frac{(x_1^2+y_1^2-a^2-b^2)}{\sqrt{(x_1^2+y_1^2+a^2e^2)^2-4a^2e^2x_1^2}}$

From Eq. (i), $ST\cdot S'T\cos\theta=x_1^2+y_1^2-a^2-b^2$

Hence $ST\cdot S'T\cos\theta=(CT)^2-a^2-b^2$

93. Let coordinates of T be (x_1, y_1). Let eccentric angles of P, Q, R be ϕ, ϕ_1, ϕ_2

Thus equation of chords PSQ and PHR are

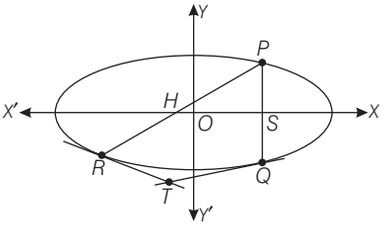
$\frac{x}{a}\cos\left(\frac{\phi+\phi_1}{2}\right)+\frac{y}{b}\sin\left(\frac{\phi+\phi_1}{2}\right)=\cos\left(\frac{\phi-\phi_1}{2}\right)$ ...(i)

and $\frac{x}{a}\cos\left(\frac{\phi+\phi_2}{2}\right)+\frac{y}{b}\sin\left(\frac{\phi+\phi_2}{2}\right)=\cos\left(\frac{\phi-\phi_2}{2}\right)$ ...(ii)

$\because$ Eqs. (i) and (ii) passes through $(ae, 0)$ and $(-ae, 0)$ then Eqs.(i) and (ii) becomes

$e\cos\left(\frac{\phi+\phi_1}{2}\right)=\cos\left(\frac{\phi-\phi_1}{2}\right)$ ...(iii)

and $-e\cos\left(\frac{\phi+\phi_2}{2}\right)=\cos\left(\frac{\phi-\phi_2}{2}\right)$ ...(iv)



By componendo and dividendo

$\frac{1-e}{1+e}=-\frac{2\cos(\phi/2)\cos(\phi_2/2)}{2\sin(\phi/2)\sin(\phi_2/2)}$

or $\frac{1-e}{1+e}=-\cot(\phi/2)\cot(\phi_2/2)$

or $\tan(\phi_2/2)=-\frac{(1+e)}{(1-e)}\cot(\phi/2)$

Similarly, $\tan(\phi_1/2)=-\frac{(1-e)}{(1+e)}\cot(\phi/2)$

the equation of QR is

$\frac{x}{a}\cos\left(\frac{\phi_1+\phi_2}{2}\right)+\frac{y}{b}\sin\left(\frac{\phi_1+\phi_2}{2}\right)=\cos\left(\frac{\phi_1-\phi_2}{2}\right)$

Dividing both sides by $\cos(\phi_1/2)\cos(\phi_2/2)$, we get

$\Rightarrow \frac{x}{a}(1-\tan(\phi_1/2)\tan(\phi_2/2)+\frac{y}{b}(\tan(\phi_1/2)+\tan(\phi_2/2))$

$=1+\tan(\phi_1/2)\tan(\phi_2/2)$ or $\frac{x}{a}(1-\cot^2(\phi/2))+\frac{y}{b}$

$\left\{-2\left(\frac{1+e^2}{1-e^2}\right)\cot(\phi/2)\right\}=1+\cot^2(\phi/2)$

or $\frac{x}{a}\frac{(1-\cot^2(\phi/2))}{(1+\cot^2(\phi/2))}-\frac{1+e^2}{1-e^2}\frac{2\cot\phi/2}{1+\cot^2\phi/2}\cdot\frac{y}{b}=1$

or $-\frac{x}{a}\cos\phi-\left(\frac{1+e^2}{1-e^2}\right)\sin\phi\cdot\frac{y}{b}=1$

or $\frac{x}{a}(1-e^2)\cos\phi+\frac{y}{b}(1+e^2)\sin\phi=-(1-e^2)$ ...(v)

But QR is also chord of contact.

Thus equation of QR is

$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ ...(vi)

Comparing the co-efficients of Eqs. (v) and (vi), we get

$\frac{(1-e^2)\cos\phi}{x_1/a}=\frac{(1+e^2)\sin\phi}{y_1/b}=-(1-e^2)$

$\Rightarrow \quad \cos^2\phi=\frac{x_1^2}{a^2}$ and $\sin^2\phi=\frac{y_1^2}{b^2}\frac{(1-e^2)^2}{(1+e^2)^2}$

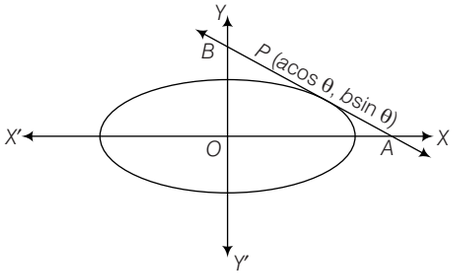
$\because \quad \cos^2\phi+\sin^2\phi=1$

$\therefore \quad \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\cdot\frac{(1-e^2)^2}{(1+e^2)^2}=1$

Locus of (x_1, y_1) is $\frac{x^2}{a^2}(1+e^2)^2+(1-e^2)^2\frac{y^2}{b^2}=(1+e^2)^2$.

94. Any tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $P(a\cos\theta, b\sin\theta)$ is

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$$



It meets coordinate axes at $A(a\sec\theta, 0)$ and $B(0, b\,\text{cosec}\theta)$.

$\therefore$ Area of $\Delta OAB=\frac{1}{2}\times a\sec\theta\times b\,\text{cosec}\theta$

$\Rightarrow \quad \Delta=\frac{ab}{\sin 2\theta}$

For Δ to be min, $\sin 2\theta$ should be max and we know max value of $\sin 2\theta = 1$.

$\therefore \quad \Delta_{max}=ab$ sq units.

95. Let the common tangent to circle $x^2+y^2=16$ and ellipse $x^2/25+y^2/4=1$ be

$$y=mx+\sqrt{25m^2+4} \qquad \text{...(i)}$$

As it is tangent to circle $x^2+y^2=16$, we should have

$$\frac{\sqrt{25m^2+4}}{\sqrt{m^2+1}}=4 \quad \text{[Using: length of perpendicular from (0, 0) to (1) = 4]}$$

$\Rightarrow \quad 25m^2+4=16m^2+16$

$\Rightarrow \quad 9m^2=12$

$\Rightarrow \quad m=\frac{-2}{\sqrt{3}}$

[Leaving + ve sign to consider tangent in I quadrant]

$\therefore$ Equation of common tangent is

$$y=-\frac{2}{\sqrt{3}}x+\sqrt{25\cdot\frac{4}{3}+4}$$

$\Rightarrow \quad y=-\frac{2}{\sqrt{3}}x+4\sqrt{\frac{7}{3}}$

This tangent meets the axes at $A(2\sqrt{7}, 0)$ and $B\left(0, 4\sqrt{\frac{7}{3}}\right)$.

$\therefore$ Length of intercepted portion of tangent between axes

$$AB=\sqrt{(2\sqrt{7})+\left(4\sqrt{\frac{7}{3}}\right)^3}$$

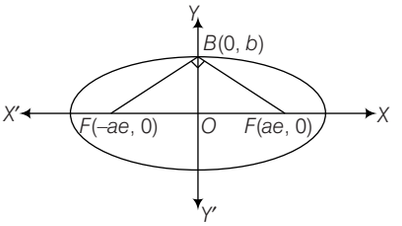
$$=14/\sqrt{3}$$

96. $\because \angle FBF'=90^\circ \Rightarrow FB^2+F'B^2=FF'^2$

$\therefore \quad (\sqrt{a^2e^2+b^2})^2+(\sqrt{a^2e^2+b^2})^2=(2ae)^2$

$\Rightarrow \quad 2(a^2e^2+b^2)=4a^2e^2$

$\Rightarrow \quad e^2=\frac{b^2}{a^2}$



Also, $\quad e^2=1-b^2/a^2=1-e^2$

$\Rightarrow \quad 2e^2=1, e=\frac{1}{\sqrt{2}}$

97. $2ae=6 \Rightarrow ae=3; 2b=8$

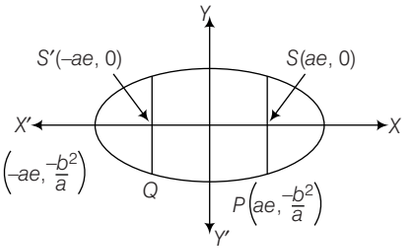
$\Rightarrow \quad b=4$

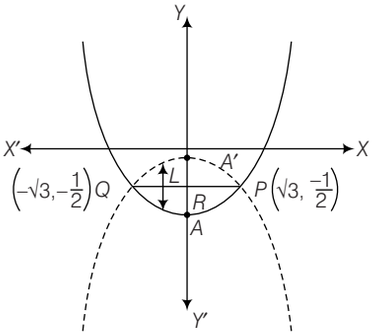
$b^2=a^2(1-e^2); 16=a^2-a^2e^2$

$\Rightarrow \quad a^2=16+9=25 \Rightarrow a=5$

$\therefore \quad e=\frac{3}{a}=\frac{3}{5}$

98.





Given, ellipse is $x^2+4y^2=4$

or $\quad \frac{x^2}{2^2}+\frac{y^2}{1}=1 \Rightarrow a=2, b=1$

$\therefore \quad e=\sqrt{1-\frac{1}{4}}=\frac{\sqrt{3}}{2}$

$\therefore \quad ae=\sqrt{3}$

As per question $P\equiv(ae, -b^2/a)=\left(\sqrt{3}, -\frac{1}{2}\right)$

$$Q\equiv(-ae, -b^2/a)=\left(-\sqrt{3}, -\frac{1}{2}\right)$$

$\therefore \quad PQ=2\sqrt{3}$

Now, if PQ is the length of latusrectum to be found, then

$$PQ=4a=2\sqrt{3} \Rightarrow a=\frac{\sqrt{3}}{2}$$

Also, as PQ is horizontal, parabola with PQ as latusrectum can be upward parabola (with vertex at A) or down ward parabola (with vertex at A').

For upward parabola, $Ar = a = \frac{\sqrt{3}}{2}$

$\therefore$ Coordinates of $A = \left(0, -\left(\frac{\sqrt{3}+1}{2}\right)\right)$

So, Equation of upward parabola is given by

$$x^2 = 2\sqrt{3}\left(y + \frac{\sqrt{3}+1}{2}\right) \text{ or } x^2 - 2\sqrt{3}y = 3 + \sqrt{3} \quad \ldots(i)$$

For downward parabola $A'R = a = \frac{\sqrt{3}}{2}$

$\therefore$ Coordinates of $A' = \left(0, -\left(\frac{1-\sqrt{3}}{2}\right)\right)$

So, equation of downward parabola is given by

$$x^2 = -2\sqrt{3}\left(y + \frac{1-\sqrt{3}}{2}\right)$$

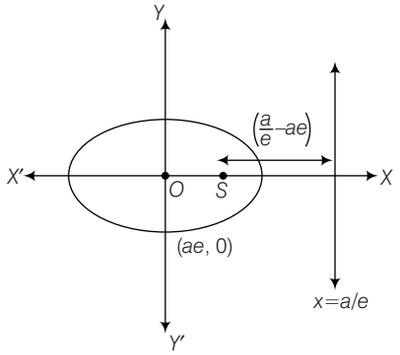
or $$x^2 + 2\sqrt{3}y = 3 - \sqrt{3} \quad \ldots(ii)$$

$\therefore$ Equation of required parabola is given by Eqs. (i) and (ii).

99. Perpendicular distance of directrix from focus

$$= \frac{a}{e} - ae = 4$$

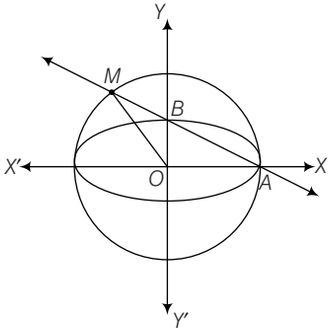
$$\Rightarrow a\left(2 - \frac{1}{2}\right) = 4 \Rightarrow a = \frac{8}{3}$$



$\therefore$ Semi major axis = 8/3

100. The given ellipse is $x^2 + 9y^2 = 9$

or $$\frac{x^2}{3^2} + \frac{y^2}{1^2} = 1$$



So, the $A(3, 0)$ and $B(0, 1)$

$\therefore$ Equation of AB is $\frac{x}{3} + \frac{y}{1} = 1$ or $x + 3y - 3 = 0$ ...(i)

Also, auxillary circle of given ellipse is

$$x^2 + y^2 = 9 \quad \ldots(ii)$$

Solving Eqs. (i) and (ii), we get the point M where line AB meets the auxillary circle.

Putting $x = 3 - 3y$ from Eqs. (i) and (ii)

we get $$(3 - 3y)^2 + y^2 = 9$$

$$\Rightarrow 9 - 18y + 9y^2 + y^2 = 9$$

$$\Rightarrow 10y^2 - 18y = 0$$

$$\Rightarrow y = 0, \frac{9}{5} \Rightarrow x = 3, \frac{-12}{5}$$

Clearly, $M\left(\frac{-12}{5}, \frac{9}{5}\right)$

$$\therefore \text{Area of } \Delta OAM = \frac{1}{2} \times \left| \begin{vmatrix} 0 & 0 & 1 \\ 3 & 0 & 1 \\ \frac{-12}{5} & \frac{9}{5} & 1 \end{vmatrix} \right| = \frac{27}{10}$$

101. The given ellipse is $\frac{x^2}{4^2} + \frac{y^2}{2^2} = 1$

such that $a^2 = 16$ and $b^2 = 4$

$$\therefore e^2 = 1 - \frac{4}{16} = \frac{3}{4}$$

$$\Rightarrow e = \frac{\sqrt{3}}{2}$$

Let $P(4\cos\theta, 2\sin\theta)$ be any point on the ellipse, then equation of normal at P is

$$4x\sin\theta - 2y\cos\theta = 12\sin\theta\cos\theta$$

$$\Rightarrow \frac{x}{3\cos\theta} - \frac{y}{6\sin\theta} = 1$$

$\therefore$ Q, the point where normal at P meets X-axis, has coordinates $(3\cos\theta, 0)$

$\therefore$ Mid-point of PQ is $M\left(\frac{7\cos\theta}{2}, \sin\theta\right)$

For locus of point M we consider

$$x = \frac{7\cos\theta}{2} \text{ and } y = \sin\theta$$

$$\Rightarrow \cos\theta = \frac{2x}{7} \text{ and } \sin\theta = y \Rightarrow \frac{4x^2}{49} + y^2 = 1 \quad \ldots(i)$$

Also, the latusrectum of given ellipse is

$$x = \pm\, ae = \pm\, 4 \times \frac{\sqrt{3}}{2} = \pm 2\sqrt{3}$$

or $$x = \pm 2\sqrt{3} \quad \ldots(ii)$$

Solving Eqs. (i) and (ii), we get

$$\frac{4 \times 12}{49} + y^2 = 1$$

$$\Rightarrow y^2 = \frac{1}{49} \text{ or } y = \pm\frac{1}{7}$$

$\therefore$ The required points are $\left(\pm 2\sqrt{3}, \pm\frac{1}{7}\right)$.

102. In ΔABC, given that $\cos B + \cos C = 4\sin^2\frac{A}{2}$

$$\Rightarrow 2\cos\frac{B+C}{2}\cos\frac{B-C}{2} - 4\sin^2\frac{A}{2} = 0$$

$$\Rightarrow 2\sin\frac{A}{2}\left[\cos\frac{B-C}{2} - 2\sin\frac{A}{2}\right] = 0$$

$$\Rightarrow \sin\frac{A}{2} = 0 \text{ or } \left(\cos\frac{B-C}{2} - 2\cos\frac{B+C}{2}\right) = 0$$

$$\left(\because \frac{A}{2} = 90° - \left(\frac{B+C}{2}\right)\right)$$

But in a triangle $\sin\frac{A}{2} \neq 0$

$$\cos\frac{B-C}{2} - 2\cos\frac{B+C}{2} = 0$$

$$\Rightarrow \frac{\cos\left(\frac{B+C}{2}\right)}{\cos\left(\frac{B-C}{2}\right)} = \frac{1}{2}$$

Applying componendo and dividendo, we get

$$\frac{\cos\left(\frac{B+C}{2}\right) + \cos\left(\frac{B-C}{2}\right)}{\cos\left(\frac{B+C}{2}\right) - \cos\left(\frac{B-C}{2}\right)} = \frac{1+2}{1-2} = -3$$

$$\Rightarrow -\frac{2\cos\frac{B}{2}\cos\frac{C}{2}}{2\sin\frac{B}{2}\sin\frac{C}{2}} = -3$$

$$\Rightarrow \tan\frac{B}{2}\tan\frac{C}{2} = \frac{1}{3}$$

$$\Rightarrow \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}\sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \frac{1}{3}$$

$$\Rightarrow \frac{s-a}{s} = \frac{1}{3}$$

or $2s = 3a$

$\Rightarrow a + b + c = 3a$

or $b + c = 2a$

i.e. $AC + AB =$ constant

[$\because$ base $BC = a$ is given to be constant]

$\Rightarrow A$ moves on an ellipse.

103. $\left(\frac{x}{\sqrt{3}}\right)^2 + (y)^2 = \left(\frac{1-t^2}{1+t^2}\right)^2 + \left(\frac{2t}{1+t^2}\right)^2$

$$= \frac{(1+t^2)^2}{(1+t^2)^2} = 1$$

$$\Rightarrow \frac{x^2}{3} + y^2 = 1$$

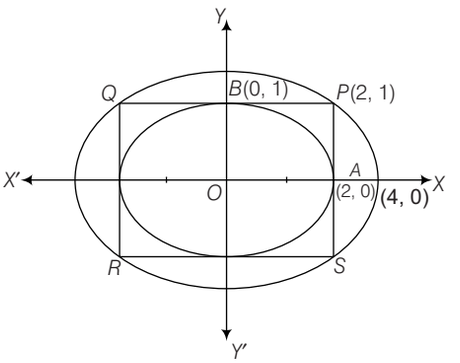
Which is an ellipse.

104. The given ellipse is $\frac{x^2}{4} + \frac{y^2}{1} = 1$

So, $A = (2, 0)$ and $B = (0,1)$

If $PQRS$ is the rectangle in which it is inscribed, then $P = (2, 1)$.

Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be the ellipse circumscribing the rectangle $PQRS$. Then it passes through $P(2, 1)$



$$\therefore \frac{4}{a^2} + \frac{1}{b^2} = 1 \quad \ldots(i)$$

Also, given that, it passes through (4, 0)

$$\therefore \frac{16}{a^2} + 0 = 1$$

$$\Rightarrow a^2 = 16$$

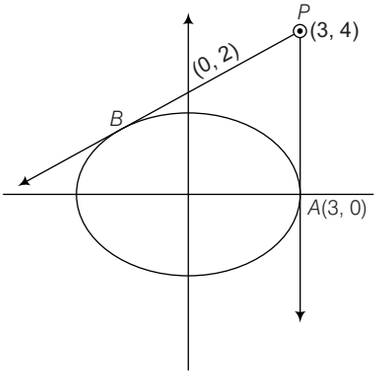
$$\Rightarrow b^2 = 4/3 \quad [\text{substituting } a^2 = 16 \text{ in Eq. (i)}]$$

$\therefore$ The required ellipse is $\frac{x^2}{16} + \frac{y^2}{4/3} = 1$

or $x^2 + 12y^2 = 16$

105. Tangent to $\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$ at the point $(3\cos\theta, 2\sin\theta)$ is

$$\frac{x\cos\theta}{3} + \frac{y\sin\theta}{2} = 1$$



As it passes through (3, 4), we get

$$\cos\theta + 2\sin\theta = 1$$

$$\Rightarrow 4\sin^2\theta = 1 + \cos^2\theta - 2\cos\theta$$

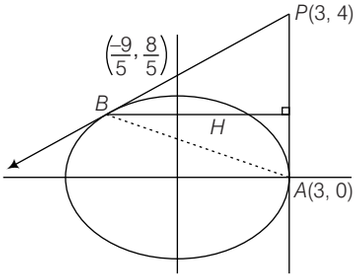
$$\Rightarrow 5\cos^2\theta - 2\cos\theta - 3 = 0$$

$$\Rightarrow \cos\theta = 1, -\frac{3}{5}$$

$$\Rightarrow \sin\theta = 0, \frac{4}{5}$$

$\therefore$ Required points are $A(3, 0)$ and $B\left(\frac{9}{5}, \frac{8}{5}\right)$.

106. Let H be orthocentre of ΔPAB, then as $BH \perp AP$, BH is a horizontal line through B.



$\therefore y$-coordinate of $B = 8/5$

Let H has coordinate $(\alpha, 8/5)$

Then, slope of $PH = \dfrac{\frac{8}{5} - 4}{\alpha - 3} = \dfrac{-12}{5(\alpha - 3)}$

and s lope of $AB = \dfrac{\frac{8-0}{5}}{-\frac{9}{5-3}} = \dfrac{8}{-24} = \dfrac{-1}{3}$

But $PH \perp AB$

$$\Rightarrow \quad \frac{-12}{5(\alpha - 3)} \times \left(\frac{-1}{3}\right) = -1$$

$$\Rightarrow \quad 4 = -5\alpha + 15 \text{ or } \alpha = 11/5$$

Hence $H\left(\dfrac{11}{5}, \dfrac{8}{5}\right)$.

107. Clearly, the moving point traces a parabola with focus at $P(3, 4)$ and directrix as

$$AB: \frac{y-0}{x-3} = \frac{-1}{3}$$

or $\quad x + 3y - 3 = 0$

$\therefore$ Equation of parabola is

$$(x-3)^2 + (y-4)^2 = \frac{(x+3y-3)^2}{10}$$

or $\quad 9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$

108. Let the ellipse be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

It passes through $(-3, 1)$, so

$$\frac{9}{a^2} + \frac{1}{b^2} = 1 \qquad \ldots(i)$$

Also, $\quad b^2 = a^2(1 - 2/5)$

$$\Rightarrow \quad 5b^2 = 3a^2 \qquad \ldots(ii)$$

Solving Eqs. (i) and (ii), we get

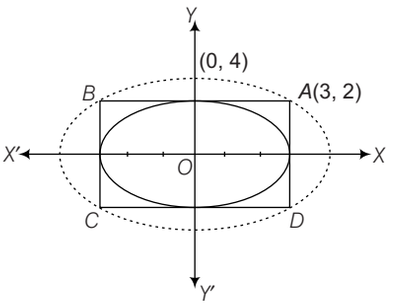
$$a^2 = \frac{32}{3}, b^2 = \frac{32}{5}$$

So, the equation of the ellipse is $3x^2 + 5y^2 = 32$

109. As rectangle $ABCD$ circumscribed the ellipse

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

$\therefore \quad A = (3, 2)$



Let the ellipse circumscribing the rectangle $ABCD$ is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Given that, it passes through (0, 4)

$\therefore \quad b^2 = 16$

Also, it passes through $A(3, 2)$

$$\therefore \quad \frac{9}{a^2} + \frac{4}{16} = 1 \Rightarrow a^2 = 12$$

$$\therefore \quad e = \sqrt{1 - \frac{12}{16}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

110. Given, equation of ellipse is $2x^2 + y^2 = 4$

$$\Rightarrow \quad \frac{2x^2}{4} + \frac{y^2}{4} = 1 \Rightarrow \frac{x^2}{2} + \frac{y^2}{4} = 1$$

Equation of tangent to the ellipse $\dfrac{x^2}{2} + \dfrac{y^2}{4} = 1$ is

$$y = mx \pm \sqrt{2m^2 + 4} \qquad \ldots(i)$$

$\because$ equation of tangent to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

is $y = mx + c$, where $c = \pm\sqrt{a^2m^2 + b^2}$

Now, equation of tangent to the parabola

$\because \quad y^2 = 16\sqrt{3}x$ is $y = mx + \dfrac{4\sqrt{3}}{m} \qquad \ldots(ii)$

($\because$ Equation of tangent to the parabola $y^2 = 4ax$ is $y = mx + \dfrac{a}{m}$)

On comparing Eqs. (i) and (ii), we get

$$\frac{4\sqrt{3}}{m} = \pm\sqrt{2m^2 + 4}$$

Squaring on both the sides, we get

$$16(3) = (2m^2 + 4)m^2$$

$$\Rightarrow \quad 48 = m^2(2m^2 + 4)$$

$$\Rightarrow \quad 2m^2 + 4m^2 - 48 = 0$$

$$\Rightarrow \quad m^4 + 2m^2 - 24 = 0$$

$$\Rightarrow \quad (m^2 + 6)(m^2 - 4) = 0$$

$$\Rightarrow \quad m^2 = 4 \qquad (\because m^2 \neq -6)$$

$$\Rightarrow \quad m = \pm 2$$

$\Rightarrow$ Equation of common tangents are $y = \pm 2x \pm 2\sqrt{3}$

Thus, statement I is true.

statement II is obviously true.

111. Equation of circle is $(x-1)^2 + y^2 = 1$

$\Rightarrow$ Radius $= 1$ and Diameter $= 2$

$\therefore$ Length of semi-minor axis is 2.

Equation of circle is $x^2 + (y-2)^2 = 4 = (2)^2$

$\Rightarrow$ Radius $= 2$ and Diameter $= 4$

$\Rightarrow$ Length of semi-major axis is 4

We know, equation of ellipse is given by

$$\frac{x^2}{(\text{Major-axis})^2} + \frac{y^2}{(\text{Major-axis})^2} = 1$$

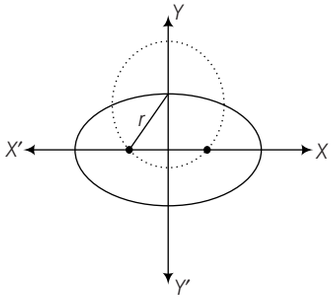
$$\Rightarrow \quad \frac{x^2}{(4)^2} + \frac{y^2}{(2)^2} = 1 \Rightarrow \frac{x^2}{16} + \frac{y^2}{4} = 1$$

$$\Rightarrow \quad x^2 + 4y^2 = 16$$

112. From the given equation of ellipse, we have

$$a = 4, b = 3e = \sqrt{1 - \frac{9}{16}}$$

$$\Rightarrow \quad e = \frac{\sqrt{7}}{4}$$



Now, radius of this circle $= a^2 = 16$

$$\Rightarrow \quad \text{Foci} = (\pm\sqrt{7}, 0)$$

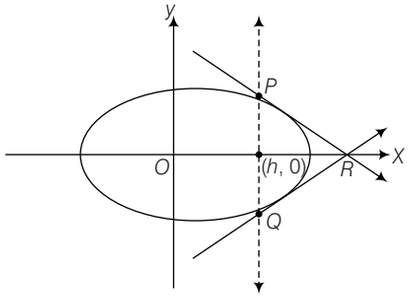
Now, equation of circle is $(x-0)^2 + (y-3)^2 = 16$

$$x^2 + y^2 - 6y - 7 = 0$$

113. Vertical line $x = h$, meets the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$ at

$$P\left(h, \frac{\sqrt{3}}{2}\sqrt{(4-h^2)}\right) \text{ and } Q\left(h, \frac{-\sqrt{3}}{2}\sqrt{(4-h^2)}\right)$$

By symmetry, tangents at P and Q will meet each other at X-axis



Tangent at P is $\frac{xh}{4} + \frac{y\sqrt{3}}{6}\sqrt{(4-h^2)} = 1$

Which meets X-axis at $R\left(\frac{4}{h}, 0\right)$

Area of $\Delta PQR = \frac{1}{2} \times \sqrt{3}\sqrt{(4-h^2)} \times \left(\frac{4}{h} - h\right)$

i.e. $\Delta(h) = \frac{\sqrt{3}}{2}\frac{(4-h^2)^{3/2}}{h}$

$$\frac{d\Delta}{dh} = -\sqrt{3}\left[\frac{\sqrt{(4-h^2)}(h^2+2)}{h^2}\right] < 0$$

$\therefore \Delta(h)$ is a decreasing function.

$$\therefore \quad \frac{1}{2} \le h \le 1$$

$$\Rightarrow \quad \Delta_{\max} = \Delta\left(\frac{1}{2}\right) \text{ and } \Delta_{\max} = \Delta(1)$$

$$\therefore \quad \Delta_1 = \frac{\sqrt{3}}{2}\left(\frac{\left(4 - \frac{1}{4}\right)^{3/2}}{\frac{1}{2}}\right) = \frac{45}{8}\sqrt{5}$$

$$\Delta_2 = \frac{\sqrt{3}}{2} \cdot \frac{3\sqrt{3}}{1} = \frac{9}{2}$$

$$\therefore \quad \frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 = 45 - 36 = 9$$

114. Given, equation of ellipse can be written as

$$\frac{x^2}{6} + \frac{y^2}{2} = 1$$

$$\Rightarrow \quad a^2 = 6, b^2 = 2$$

Now, equation of any variable tangent is

$$y = mx \pm \sqrt{a^2m^2 + b^2} \quad \ldots(i)$$

where m is slope of the tangent

So, equation of perpendicular line drawn from centre to tangent is

$$y = \frac{-x}{m} \quad \ldots(ii)$$

Eliminating m,we get

$$(x^4 + y^4 + 2x^2y^2) = a^2x^2 + b^2y^2$$

$$\Rightarrow \quad (x^2 + y^2)^2 = a^2x^2 + b^2y^2$$

$$\Rightarrow \quad (x^2 + y^2)^2 = 6x^2 + 2y^2$$

115. The end point of latusrectum of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in first quadrant is $\left(ae, \frac{b^2}{a}\right)$ and the tangent at this point intersects X-axis at $\left(\frac{a}{e}, 0\right)$ and Y- axis at $(0, a)$.

The given ellipse is $\frac{x^2}{9} + \frac{y^2}{5} = 1$

Then, $a^2 = 9, b^2 = 5$

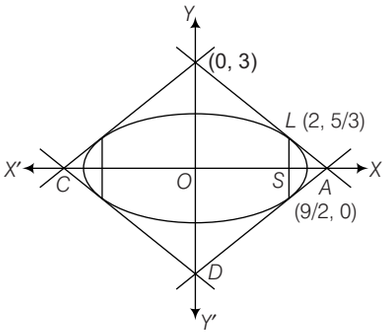
$$\Rightarrow \quad e = \sqrt{\left(1 - \frac{5}{9}\right)} = \frac{2}{3}$$

$\therefore$ End point of latusrectum in first quadrant is $L(2, 5/3)$

Equation of tangent at L is $\frac{2x}{9} + \frac{y}{3} = 1$

It meets X-axis at A (9/2, 0) and Y-axis at B (0, 3)

$\therefore$ Area of $\Delta OAB = \frac{1}{2} \times \frac{9}{2} \times 3 = \frac{27}{4}$



By symmetry area of quadrilateral

$$= 4 \times (\text{Area } \Delta OAB) = 4 \times \frac{27}{4} = 27 \text{ sq units}$$

116. Let $E_1 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$

and $E_2 : \frac{x^2}{c^2} + \frac{y^2}{d^2} = 1$, where $c < d$

Also, $S : x^2 + (y-1)^2 = 2$

Tangent at $P(x_1, y_1)$ to S is $x + y = 3$

To find point of contact put $x = 3 - y$ in S, we get $P(1, 2)$

writing equation of tangent in parametric form

$$\frac{x-1}{\frac{-1}{\sqrt{2}}} = \frac{y-2}{\frac{1}{\sqrt{2}}} = \pm \frac{2\sqrt{2}}{3}$$

$$x = \frac{-2}{3} + 1 \quad \text{or} \quad \frac{2}{3} + 1 \text{ and } y = \frac{2}{3} + 2 \text{ or } \frac{-2}{3} + 2$$

$\Rightarrow \quad x = \frac{1}{3}$ or $\frac{5}{3}$ and $y = \frac{8}{3}$ or $\frac{4}{3}$

$\therefore \quad Q\left(\frac{5}{3}, \frac{4}{3}\right)$ and $R\left(\frac{1}{3}, \frac{8}{3}\right)$

Equation of tangent to E_1 at Q is

$\frac{5x}{3a^2} + \frac{4y}{3b^2} = 1$ which is identical to $\frac{x}{3} + \frac{y}{3} = 1$

$\Rightarrow \; a^2 = 5$ and $b^2 = 4 \; \Rightarrow \; e_1^2 = 1 - \frac{4}{5} = \frac{1}{5}$

Equation of tangent to E_2 at R is

$\frac{x}{3c^2} + \frac{8y}{3d^2} = 1$ identical to $\frac{x}{3} + \frac{y}{3} = 1$

$\Rightarrow \quad c^2 = 1, d^2 = 8 \; \Rightarrow \; e_2^2 = 1 - \frac{1}{8} = \frac{7}{8}$

$\therefore \quad e_1^2 + e_2^2 = \frac{43}{40}, e_1 e_2 = \frac{\sqrt{7}}{2\sqrt{10}}, \left|e_1^2 - e_2^2\right| = \frac{27}{40}$

117. Ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1 \Rightarrow a = 3, b = \sqrt{5}$ and $e = \frac{2}{3}$

$\therefore \quad f_1 = 2$ and $f_2 = -2$

$$P_1 : y^2 = 8x \quad \text{and} \quad P_2 : y^2 = -16x$$

$$T_1 : y = m_1 x + \frac{2}{m_1}$$

It passes through (–4, 0),

$$0 = -4m_1 + \frac{2}{m_1} \; \Rightarrow \; m_1^2 = \frac{1}{2}$$

$$T_2 : y = m_2 x - \frac{4}{m_2}$$

It passes through (2, 0)

$$0 = 2m_2 - \frac{4}{m_2} \; \Rightarrow \; m_2^2 = 2$$

$\therefore \quad \frac{1}{m_1^2} + m_2^2 = 4$

Sol. (Q. Nos. 118 & 119)

118. For ellipse $\frac{x^2}{9} + \frac{y^2}{8} = 1$, $e = \sqrt{\left(1 - \frac{8}{9}\right)} = \frac{1}{3}$

$\therefore \quad F_1(-1, 0)$ and $F_2(1, 0)$

Parabola with vertex at (0, 0) and focus at F_2 (1, 0) is $y^2 = 4x$.

Intersection points of ellipse and parabola are $M\left(\frac{3}{2}, \sqrt{6}\right)$ and $N\left(\frac{3}{2}, -\sqrt{6}\right)$.

For orthocentre of $\Delta F_1 MN$, clearly one altitude is X-axis i.e. $y = 0$ and altitude from M to F_1 N is

$$y - \sqrt{6} = \frac{5}{2\sqrt{6}}\left(x - \frac{3}{2}\right)$$

Putting $y = 0$ in above equation, we get $x = -\frac{9}{10}$

Orthocentre $\left(-\frac{9}{10}, 0\right)$

119. Tangents to ellipse at M and N are

$$\frac{x}{6} + \frac{y\sqrt{6}}{8} = 1 \text{ and } \frac{x}{6} - \frac{y\sqrt{6}}{8} = 1$$

their intersection point is R (6, 0)

Also, normal to parabola at $M\left(\frac{3}{2}, \sqrt{6}\right)$ is

$$y - \sqrt{6} = -\frac{\sqrt{6}}{2}\left(x - \frac{3}{2}\right)$$

Its intersection with x-axis is $Q\left(\frac{7}{2}, 0\right)$

Now, $\quad \text{ar}(\Delta MQR) = \frac{1}{2} \times \frac{5}{2} \times \sqrt{6} = \frac{5\sqrt{6}}{4}$

Also, area $(MF_1NF_2) = 2 \times$ Area of (F_1MF_2)

$$= 2 \times \frac{1}{2} \times 2 \times \sqrt{6} = 2\sqrt{6}$$

$$\frac{\text{area}(\Delta MQR)}{\text{area}(MF_1NF_2)} = \frac{5\sqrt{6}}{4 \times 2\sqrt{6}} = 5 : 8$$

120. Here, $e = \frac{1}{2}$ and $x = -\frac{a}{e} = -4$

$\therefore \quad a = 2$

and $\quad b^2 = a^2(1 - e^2) = 4(1 - 1/4) = 3$

$\therefore$ Equation of ellipse is $\frac{x^2}{4} + \frac{y^2}{3} = 1$

$\Rightarrow$ Equation of normal at (1, 3/2) is

$$\frac{4x}{1} - \frac{3y}{3/2} = 4 - 3$$

or $\quad 4x - 2y = 1$

CHAPTER

07

Hyperbola

Learning Part

Session 1

- Hyperbola : Definition
- The Foci and Two Directrices of a Hyperbola
- Some Terms Related to Hyperbola
- Conjugate Hyperbola
- Intersection of a Line and a Hyperbola
- Standard Equation of Hyperbola
- Tracing of the Hyperbola
- Focal Distances of a Point
- Position of a Point with Respect to a Hyperbola

Session 2

- Equations of Tangents in Different Forms
- Pair of Tangents
- Equation of the Chord Bisected at a Given Point
- Equations of Normals in Different Forms
- Chord of Contact

Session 3

- Diameter
- Properties of Hyperbola
- Director Circle
- Rectangular Hyperbola
- Study of Hyperbola $xy = c^2$
- Reflection Property of a Hyperbola
- Equation of a Hyperbola Referred to Two Perpendicular Lines
- Conjugate Diameters
- Intersection of Conjugate Diameters and Hyperbola
- Asymptotes
- The Rectangular Hyperbola $xy = c^2$
- Properties of Rectangular Hyperbola $xy = c^2$

Practice Part

- JEE Type Examples
- Chapter Exercises

Arihant on Your Mobile !

Exercises with the symbol can be practised on your mobile. See inside cover page to activate for free.

Session 1

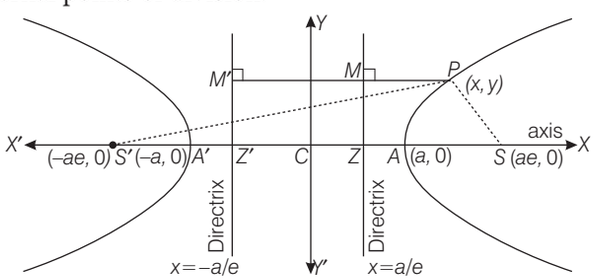
Hyperbola : Definition, Standard Equation of Hyperbola, The Foci and Two Directrices of a Hyperbola, Tracing of the Hyperbola, Focal Distances of a Point, Conjugate Hyperbola, Position of a Point with Respect to a Hyperbola, Intersection of a Line and a Hyperbola

Hyperbola : Definition

The locus of a point which moves in a plane such that its distance from a fixed point (i.e. focus) is e times its distance from a fixed line (i.e. directrix) is known as hyperbola. For hyperbola $e > 1$.

Standard Equation of Hyperbola

Let S be the focus and ZM the directrix of the hyperbola. Draw $SZ \perp ZM$. Divide SZ internally and externally in the ratio $e:1\,(e>1)$ and let A and A' be their internal and external points of division.



then $$SA = e\,AZ \qquad \ldots(i)$$

and $$SA' = e\,A'Z \qquad \ldots(ii)$$

Clearly A and A' will lie on the hyperbola. Let $AA' = 2a$ and take C the mid-point of AA' as origin

$$\therefore \quad CA = CA' = a$$

Let $P(x, y)$ be any point on the hyperbola and CA as X-axis, the line through C perpendicular to CA as Y-axis.

Then adding Eqs. (i) and (ii)

$$\therefore \quad SA + SA' = e\,(AZ + A'Z)$$

$$\Rightarrow \quad CS - CA + CS + CA' = e\,(AA')$$

$$\Rightarrow \quad 2CS = e\,(2a) \qquad (\because CA = CA')$$

$$\therefore \quad CS = ae$$

$\therefore$ The focus S is $(CS, 0)$ i.e. $(ae, 0)$ and subtracting Eq. (i) from Eq. (ii), then

$$SA' - SA = e\,(A'Z - AZ)$$

$$AA' = e\,[(CA' + CZ) - (CA - CZ)]$$

$$\Rightarrow \quad AA' = e\,(2CZ) \qquad (\because CA = CA')$$

$$\Rightarrow \quad 2a = e\,(2CZ)$$

$$\therefore \quad CZ = a/e$$

$\therefore$ The directrix MZ is $x = CZ = a/e$

or $$x - a/e = 0 \qquad \left(\because e > 1, \therefore \ \frac{a}{e} < 1\right)$$

Now, draw $PM \perp MZ$,

$$\therefore \quad \frac{SP}{PM} = e \quad \text{or} \quad (SP)^2 = e^2\,(PM)^2$$

or $$(x - ae)^2 + (y - 0)^2 = e^2\left(x - \frac{a}{e}\right)^2$$

or $$(x - ae)^2 + y^2 = (ex - a)^2$$

$$\Rightarrow \quad x^2 + a^2e^2 - 2aex + y^2 = e^2x^2 - 2aex + a^2$$

$$\Rightarrow \quad x^2(e^2 - 1) - y^2 = a^2\,(e^2 - 1)$$

$$\Rightarrow \quad \frac{x^2}{a^2} - \frac{y^2}{a^2\,(e^2 - 1)} = 1$$

or $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ where, } b^2 = a^2\,(e^2 - 1)$$

This is the standard equation of the hyperbola.

Generally : The equation of the hyperbola whose focus is the point (h, k) and directrix is $lx + my + n = 0$ and whose eccentricity is e, is

$$(x - h)^2 + (y - k)^2 = e^2\,\frac{(lx + my + n)^2}{(l^2 + m^2)}$$

The Foci and Two Directrices of a Hyperbola

On the negative side of origin take a point S' which is such that

$$CS = CS' = ae$$

and another point Z', then $CZ = CZ' = \frac{a}{e}$

$\therefore$ Coordinates of S' are $(-ae, 0)$ and equation of second directrix (i.e. $Z'M'$) is

$$x = -\frac{a}{e}$$

Let $P(x, y)$ be any point on the hyperbola, then

$$S'P = ePM' \text{ or } (S'P)^2 = e^2 (PM')^2$$

or $$(x + ae)^2 + (y - 0)^2 = e^2 \left(x + \frac{a}{e}\right)^2$$

or $$(x + ae)^2 + y^2 = (ex + a)^2$$

or $$x^2 + 2aex + a^2e^2 + y^2 = e^2x^2 + 2aex + a^2$$

or $$x^2 (e^2 - 1) - y^2 = a^2(e^2 - 1)$$

or $$\frac{x^2}{a^2} - \frac{y^2}{a^2 (e^2 - 1)} = 1$$

or $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ where, } b^2 = a^2 (e^2 - 1)$$

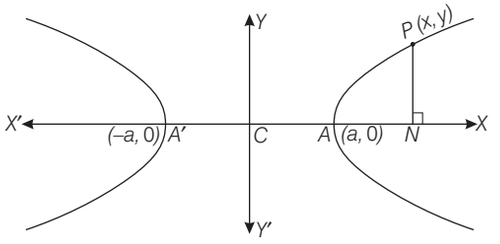
The equation being the same as that of hyperbola when $S(ae, 0)$ is focus and MZ i.e. $x = \frac{a}{e}$ is directrix.

Hence, coordinates of foci are $(\pm ae, 0)$ and equations of directrices are

$$x = \pm a / e$$

Remarks

1. Distance between foci $SS' = 2ae$ and distance between directrices $ZZ' = 2a/e$
2. Two hyperbolas are said to be similar if they have the same value of eccentricity.
3. Since, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



$$\Rightarrow \quad \frac{y^2}{b^2} = \frac{x^2}{a^2} - 1$$

$$\frac{y^2}{b^2} = \frac{(x - a)(x + a)}{a^2} \quad \ldots(i)$$

$\because$ $AN = CN - CA = (x - a)$

and $A'N = CN + CA' = (x + a)$

and $PN = y$

$\therefore$ From Eq. (i), $\frac{(PN)^2}{AN \cdot A'N} = \frac{b^2}{a^2}$

Tracing of the Hyperbola

Equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \ldots(i)$$

(i) Since only even powers of x and y occur in this equation so it (hyperbola) is symmetrical about the both axes.

(ii) The hyperbola (i) does not cut Y-axis in real points where as it cuts X-axis at $(a, 0)$ and $(-a, 0)$.

(iii) The Eq. (i) may be written as

$$y = \pm \frac{b}{a}\sqrt{(x^2 - a^2)}$$

If follows that $x^2 - a^2 \geq 0$

$\therefore$ $x^2 \geq a^2$

$\Rightarrow$ $x \leq -a$ or $x \geq a$

Hence, $x \notin (-a, a)$

The curve does not exist in the region

$$x = -a \text{ to } x = a .$$

(iv) As x increases, y also increases i.e. the curve extends to infinity.

Some Terms Related to Hyperbola

Let the equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(1) Centre : All chords passing through C and bisected at C.

Here $C \equiv (0, 0)$

(2) Eccentricity : For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ we have

$$b^2 = a^2 (e^2 - 1)$$

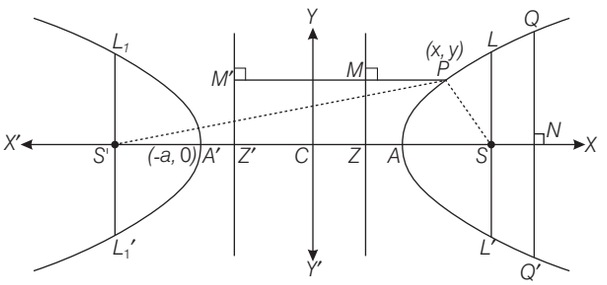
$$\Rightarrow \quad e^2 = \frac{a^2 + b^2}{a^2}$$

$$\Rightarrow \qquad e=\sqrt{1+\left(\frac{b^2}{a^2}\right)} \Rightarrow e=\sqrt{\left\{1+\frac{(2b)^2}{(2a)^2}\right\}}$$

$$\Rightarrow \qquad e=\sqrt{\left\{1+\frac{(\text{conjugate axis})^2}{(\text{transverse axis})^2}\right\}}$$

(3) Foci and directrices : S and S' are the foci of the ellipse and their coordinates are $(ae, 0)$ and $(-ae, 0)$ respectively and ZM and $Z'M'$ are two directrices of the hyperbola and their equations are

$$x=\frac{a}{e} \quad \text{and} \quad x=-\frac{a}{e} \text{ respectively.}$$



(4) Axes : The points $A(a, 0)$ and $A'(-a, 0)$ are called the vertices of the hyperbola and line AA' is called **transverse axis** and the line perpendicular to its through the centre $(0, 0)$ of the hyperbola is called **conjugate axis.**

The length of transverse and conjugate axes are taken as $2a$ and $2b$ respectively.

(5) Double ordinates : If Q be a point on the hyperbola draw QN perpendicular to the axis of the hyperbola and produced to meet the curve again at Q'. Then QQ' is called a double ordinate of Q.

If abscissa of Q is h, then ordinates of Q are

$$\frac{y^2}{b^2}=\frac{h^2}{a^2}-1$$

$$\therefore \qquad y=\pm\frac{b}{a}\sqrt{(h^2-a^2)}$$

$$\therefore \qquad y=\frac{b}{a}\sqrt{(h^2-a^2)} \qquad \text{(for I quadrant)}$$

and ordinate of Q' is

$$y=-\frac{b}{a}\sqrt{(h^2-a^2)} \qquad \text{(for IV quadrant)}$$

Hence, coordinates of Q and Q' are

$$\left(h, \frac{b}{a}\sqrt{(h^2-a^2)}\right) \text{ and } \left(h, -\frac{b}{a}\sqrt{(h^2-a^2)}\right)$$

respectively.

(6) Latusrectum : The double ordinates LL' and L_1L_1' are the latusrectums of the hyperbola. These lines are perpendicular to transverse axis AA' and through the foci S and S' respectively.

Length of latusrectum

Now, let $LL'=2k$, then $LS=L'S=k$

Coordinates of L and L' are (ae, k) and $(ae, -k)$

lie on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$$\therefore \qquad \frac{a^2e^2}{a^2}-\frac{k^2}{b^2}=1$$

or $\quad k^2=b^2(e^2-1)=b^2\left(\frac{b^2}{a^2}\right) \qquad [\because b^2=a^2(e^2-1)]$

$$\therefore \qquad k=\frac{b^2}{a} \qquad (\because k>0)$$

$$\therefore \qquad 2k=\frac{2b^2}{a}=LL'$$

$\therefore$ Length of latusrectum $LL'=L_1L'_1=\frac{2b^2}{a}$ and end points of latusrectums are

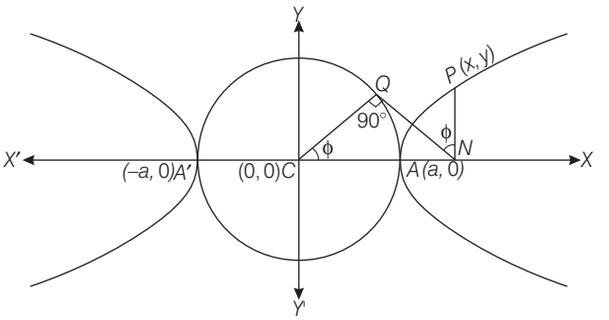
$$L\equiv\left(ae, \frac{b^2}{a}\right); L'\equiv\left(ae, -\frac{b^2}{a}\right);$$

$$L_1\equiv\left(-ae, \frac{b^2}{a}\right); L_1'\equiv\left(-ae, -\frac{b^2}{a}\right) \text{ respectively.}$$

(7) Focal chord : A chord of hyperbola passing through its focus is called a focal chord.

(8) Parametric equations of the hyperbola : Let $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ be the hyperbola with centre C and transverse axis $A'A$. Therefore, circle drawn with centre C and segment $A'A$ as a diameter is called auxiliary circle of the hyperbola.

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$



$\therefore$ Equation of the auxiliary circle is $x^2 + y^2 = a^2$

Let $P(x, y)$ be any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Draw PN perpendicular to X-axis.

Let NQ be a tangent to the auxiliary circle $x^2 + y^2 = a^2$. Join CQ

Let $\angle QCN = \phi$

Here, P and Q are the **corresponding points** of the hyperbola and the auxiliary circle. ϕ is the eccentric angle of $P \cdot (0 \le \phi < 2\pi)$

Since, $Q \equiv (a\cos\phi, a\sin\phi)$

Now, $x = CN = \frac{CN}{CQ} \cdot CQ = \sec\phi \cdot a$

$\therefore$ $x = CN = a\sec\phi$

$\therefore$ $P(x, y) \equiv (a\sec\phi, y)$

$\because$ P lies on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\therefore$ $\frac{a^2\sec^2\phi}{a^2} - \frac{y^2}{b^2} = 1$

or $\frac{y^2}{b^2} = \sec^2\phi - 1 = \tan^2\phi$

$\therefore$ $y = \pm b\tan\phi$

$\therefore$ $y = b\tan\phi$ (P lies in I quadrant)

The equations of $x = a\sec\phi$ and $y = b\tan\phi$ are known as the parametric equations of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Position of points Q an auxiliary circle and the corresponding point P which describes the hyperbola and $0 \le \phi < 2\pi$.

ϕ varies from	$Q(a\cos\phi, a\sin\phi)$	$P(a\sec\phi, b\tan\phi)$
0 to $\frac{\pi}{2}$	I	I
$\frac{\pi}{2}$ to π	II	III
π to $\frac{3\pi}{2}$	III	II
$\frac{3\pi}{2}$ to 2π	IV	IV

Remark

Equation of the chord joining the points $P \equiv (a\sec\phi_1, b\tan\phi_1)$ and $Q \equiv (a\sec\phi_2, b\tan\phi_2)$ is

$$\frac{x}{a}\cos\left(\frac{\phi_1 - \phi_2}{2}\right) - \frac{y}{b}\sin\left(\frac{\phi_1 + \phi_2}{2}\right) = \cos\left(\frac{\phi_1 + \phi_2}{2}\right)$$

If it is focal chord, then pass through $(ae, 0)$ or $(-ae, 0)$. Suppose it pass through $(ae, 0)$, then

$$e\cos\left(\frac{\phi_1 - \phi_2}{2}\right) - 0 = \cos\left(\frac{\phi_1 + \phi_2}{2}\right)$$

$$\Rightarrow \quad \frac{\cos\left(\frac{\phi_1 - \phi_2}{2}\right)}{\cos\left(\frac{\phi_1 + \phi_2}{2}\right)} = \frac{1}{e}$$

$$\Rightarrow \quad \frac{\cos\left(\frac{\phi_1 - \phi_2}{2}\right) - \cos\left(\frac{\phi_1 + \phi_2}{2}\right)}{\cos\left(\frac{\phi_1 - \phi_2}{2}\right) + \cos\left(\frac{\phi_1 + \phi_2}{2}\right)} = \frac{1 - e}{1 + e}$$

or $$\tan\left(\frac{\phi_1}{2}\right)\tan\left(\frac{\phi_2}{2}\right) = \frac{1 - e}{1 + e}$$

Hence, if ϕ_1 and ϕ_2 are the eccentric angles of extremities of a focal chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then

$$\tan\left(\frac{\phi_1}{2}\right)\tan\left(\frac{\phi_2}{2}\right) = \frac{1 - e}{1 + e} \quad \text{or} \quad \frac{1 + e}{1 - e}$$

according as focus $(ae, 0)$ or $(-ae, 0)$.

Focal Distances of a Point

The difference of the focal distances of any point on the hyperbola is constant and equal to length of the transverse axis of the hyperbola.

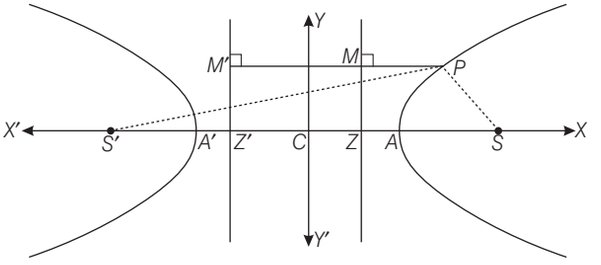
The hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ...(i)

The foci S and S' are $(ae, 0)$ and $(-ae, 0)$.

The equation of its directrices MZ and $M'Z'$ are

$$x = \frac{a}{e} \quad \text{and} \quad x = -\frac{a}{e}$$

respectively.



Let $P(x_1, y_1)$ be any point on Eq. (i).

Now $SP = ePM = e\left(x_1 - \frac{a}{e}\right) = ex_1 - a$

and $S'P' = ePM' = e\left(x_1 + \frac{a}{e}\right) = ex_1 + a$

$\therefore$ $S'P - SP = (ex_1 + a) - (ex_1 - a) = 2a = AA'$

= Transverse axis

A hyperbola is the locus of a point which moves in a plane such that the difference of its distances from two fixed point (foci) is always constant.

Example 1 To find the equation of the hyperbola from the definition that hyperbola is the locus of a point which moves such that the difference of its distances from two fixed points is constant with the fixed point as foci.

Sol. Let two fixed point be $S(ae, 0)$ and $S'(-ae, 0)$. Let $P(x, y)$ be a moving point such that

$$S'P - SP = \text{constant} = 2a \text{ (say)}$$

i.e. $\sqrt{(x+ae)^2 + (y-0)^2} - \sqrt{(x-ae)^2 + (y-0)^2} = 2a$

or $\sqrt{(x^2 + y^2 + 2aex + a^2e^2)}$

$$- \sqrt{(x^2 + y^2 - 2aex + a^2e^2)} = 2a \quad \text{...(i)}$$

Let $l = x^2 + y^2 + 2aex + a^2e^2$ ...(ii)

and $m = x^2 + y^2 - 2aex + a^2e^2$ ...(iii)

Eq. (i) can be re-written as

$$\sqrt{l} - \sqrt{m} = 2a \quad \text{...(iv)}$$

From Eqs. (ii) and (iii),

$$l - m = 4aex$$

$\Rightarrow$ $(\sqrt{l} + \sqrt{m})(\sqrt{l} - \sqrt{m}) = 4aex$

$\Rightarrow$ $2a(\sqrt{l} + \sqrt{m}) = 4aex$ [from Eq. (iv)]

or $\sqrt{l} + \sqrt{m} = 2ex$ ...(v)

Adding Eqs. (iv) and (v), then

$$2\sqrt{l} = 2a + 2ex$$

$\Rightarrow$ $\sqrt{l} = a + ex \Rightarrow l = (a + ex)^2$

$$x^2 + y^2 + 2aex + a^2e^2 = a^2 + 2aex + e^2x^2$$

or $x^2(e^2 - 1) - y^2 = a^2(e^2 - 1)$

or $\dfrac{x^2}{a^2} - \dfrac{y^2}{a^2(e^2-1)} = 1$

or $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, where $b^2 = a^2(e^2 - 1)$.

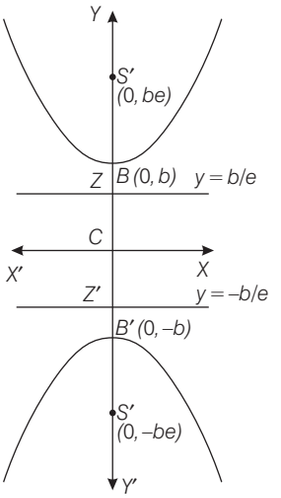
Conjugate Hyperbola

The hyperbola whose transverse and conjugate axes are respectively the conjugate and transverse axes of a given hyperbola is called the conjugate hyperbola of the given hyperbola.

The conjugate hyperbola of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{is} \quad -\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

its shape is shown alongside.



Various results related to hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ and its conjugate

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{or} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

are given in the following table

Basic fundamentals \ Hyperbola	$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$	$-\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ or $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = -1$
Centre	$(0, 0)$	$(0, 0)$
Length of transverse axis	$2a$	$2b$
Length of conjugate axis	$2b$	$2a$
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrices	$x = \pm a/e$	$y = \pm b/e$
Eccentricity	$e = \sqrt{\left(\dfrac{a^2 + b^2}{a^2}\right)}$	$e = \sqrt{\left(\dfrac{a^2 + b^2}{b^2}\right)}$
Length of latusrectum	$\dfrac{2b^2}{a}$	$\dfrac{2a^2}{b}$
Parametric coordinates	$(a \sec \phi, b \tan \phi)$, $0 \le \phi < 2\pi$	$(a \tan \phi, b \sec \phi)$, $0 \le \phi < 2\pi$
Focal radii	$SP = ex_1 - a$ and $S'P = ex_1 + a$	$SP = ey_1 - b$ and $S'P = ey_1 + b$
Difference of focal radii ($S'P - SP$)	$2a$	$2b$
Tangents at the vertices	$x = -a, x = a$	$y = -b, y = b$
Equation of the transverse axis	$y = 0$	$x = 0$
Equation of the conjugate axis	$x = 0$	$y = 0$

Remarks

1. If the centre of hyperbola is (h,k) and axes are parallel to the coordinates axes, then its equation is

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1.$$

By shifting the origin at (h,k) without rotating the coordinate axes, the above equation reduces to

$$\frac{X^2}{a^2}-\frac{Y^2}{b^2}=1$$

where, $X=X+h, y=Y+k$

2. If e_1 and e_2 are the eccentricities of a hyperbola and its conjugate, respectively, then $e_1^2+e_2^2=1$.

Proof For hyperbola $b^2=a^2(e_1^2-1)$

or $$e_1^2=1+\frac{b^2}{a^2}=\frac{a^2+b^2}{a^2} \quad \ldots \text{(i)}$$

and for conjugate hyperbola $a^2=b^2(e_2^2-1)$

or $$e_2^2=1+\frac{a^2}{b^2}=\frac{a^2+b^2}{b^2} \quad \ldots \text{(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{1}{e_1^2}+\frac{1}{e_2^2}=1 \text{ or } e_1^{-2}+e_2^{-2}=1$$

3. The foci of a hyperbola and its conjugate are concyclic and form the vertices of a square.

4. A simple method to find the coordinates of the centre of the hyperbola expressed as a general equation of degree two should be remembered as :

Let $\phi(x, y)=0$ represents a hyperbola.

Find $\frac{\partial\phi}{\partial x}$ (differentiate w.r.t x keeping y as constant) and $\frac{\partial\phi}{\partial y}$ (differentiate w.r.t. y keeping x as constant).

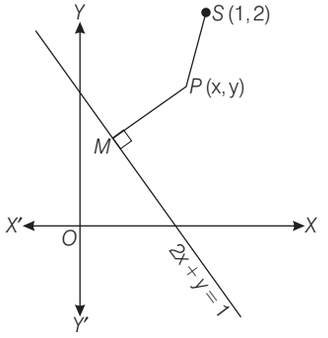
Then the point of intersection of $\frac{\partial\phi}{\partial x}=0$ and $\frac{\partial\phi}{\partial y}=0$ gives the centre of the hyperbola.

Example 2 Find the equation of the hyperbola whose directrix is $2x+y=1$, focus $(1,2)$ and eccentricity $\sqrt{3}$.

Sol. Let $P(x, y)$ be any point on the hyperbola. Draw PM perpendicular from P on the directrix.

Then, by definition $SP = e\ PM$

$\Rightarrow$ $(SP)^2 = e^2\ (PM)^2$



$$\Rightarrow (x-1)^2+(y-2)^2=3\left\{\frac{2x+y-1}{\sqrt{4+1}}\right\}^2$$

$$\Rightarrow 5\{x^2+y^2-2x-4y+5\}$$
$$=3(4x^2+y^2+1+4xy-2y-4x)$$

$$\Rightarrow 7x^2-2y^2+12xy-2x+14y-22=0$$

which is the required hyperbola.

Example 3 Find the lengths of transverse axis and conjugate axis, eccentricity, the coordinates of foci, vertices, lengths of the latusrectum and equations of the directrices of the following hyperbolas

(i) $9x^2-y^2=1$ (ii) $16x^2-9y^2=-144$

Sol. (i) The equation $9x^2-y^2=1$ can be written as

$$\frac{x^2}{(1/9)}-\frac{y^2}{1}=1.$$

This is of the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$\therefore$ $$a^2=\frac{1}{9}, b^2=1$$

$\Rightarrow$ $$a=\frac{1}{3}, b=1$$

Length of transverse axis : The length of transverse, axis $=2a=\frac{2}{3}$

Length of conjugate axis : The length of conjugate axis $=2b=2$

Eccentricity : $e=\sqrt{\left(1+\frac{b^2}{a^2}\right)}=\sqrt{1+\frac{1}{(1/9)}}=\sqrt{10}$

Foci : The coordinates of the foci are $(\pm ae, 0)$ i.e. $\left(\pm\frac{\sqrt{10}}{3}, 0\right)$.

Vertices : The coordinates of the vertices are $(\pm a, 0)$ i.e. $\left(\pm\frac{1}{3}, 0\right)$.

Length of latusrectum : The length of latusrectum

$$=\frac{2b^2}{a}=\frac{2(1)^2}{1/3}=6$$

Equation of the directrices : The equations of the directrices are

$$x=\pm\frac{a}{e}$$

i.e. $$x=\pm\frac{1/3}{\sqrt{10}}$$

or $$x=\pm\frac{1}{3\sqrt{10}}$$

(ii) The equation $16x^2 - 9y^2 = -144$ can be written as

$$\frac{x^2}{9} - \frac{y^2}{16} = -1$$

This is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

$\therefore \quad a^2 = 9, b^2 = 16$

$\Rightarrow \quad a = 3, b = 4$

Length of transverse axis : The length of transverse axis $= 2b = 8$.

Length of conjugate axis : The length of conjugate axis $= 2a = 6$.

Eccentricity : $e = \sqrt{\left(1 + \frac{a^2}{b^2}\right)} = \sqrt{\left(1 + \frac{9}{16}\right)} = \frac{5}{4}$

Foci : The coordinates of the foci are $(0, \pm be)$ i.e. $(0, \pm 5)$

Vertices : The coordinates of the vertices are $(0, \pm b)$ i.e. $(0, \pm 4)$.

Length of latusrectum : The length of latusrectum $= \frac{2a^2}{b}$

$$= \frac{2(3)^2}{4} = \frac{9}{2}$$

Equation of directrices : The equation of directrices are

$$y = \pm \frac{b}{e}$$

$$y = \pm \frac{4}{(5/4)}$$

$$\Rightarrow \quad y = \pm \frac{16}{5}$$

Example 4 Find the eccentricity of the hyperbola whose latusrectum is half of its transverse axis.

Sol. Let the equation of hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Then, transverse axis = 2 a and latusrectum $= \frac{2b^2}{a}$

According to question $\frac{2b^2}{a} = \frac{1}{2}(2a)$

$$\Rightarrow \quad 2b^2 = a^2 \qquad (\because b^2 = a^2(e^2 - 1))$$

$$\Rightarrow \quad 2a^2(e^2 - 1) = a^2$$

$$\Rightarrow \quad 2e^2 - 2 = 1$$

$$\Rightarrow \quad e^2 = \frac{3}{2}$$

$$\therefore \quad e = \sqrt{\frac{3}{2}}$$

Hence, the required eccentricity is $\sqrt{\frac{3}{2}}$.

Example 5 Prove that the point $\left\{\frac{a}{2}\left(t + \frac{1}{t}\right), \frac{b}{2}\left(t - \frac{1}{t}\right)\right\}$ lies on the hyperbola for all values of t $(t \neq 0)$.

Sol. Let $x = \frac{a}{2}\left(t + \frac{1}{t}\right)$

$$\text{or} \quad \frac{2x}{a} = t + \frac{1}{t}$$

$$\text{or} \quad \left(\frac{2x}{a}\right)^2 = t^2 + \frac{1}{t^2} + 2 \qquad \ldots(i)$$

and let $y = \frac{b}{2}\left(t - \frac{1}{t}\right)$

$$\text{or} \quad \frac{2y}{b} = t - \frac{1}{t}$$

$$\text{or} \quad \left(\frac{2y}{b}\right)^2 = t^2 + \frac{1}{t^2} - 2 \qquad \ldots(ii)$$

Subtracting Eqs. (ii) from (i),

$$\frac{4x^2}{a^2} - \frac{4y^2}{b^2} = 4$$

$$\text{or} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

which is hyperbola.

Example 6 Show that the equation $7y^2 - 9x^2 + 54x - 28y - 116 = 0$ represent a hyperbola. Find the coordinates of the centre, lengths of transverse and conjugate axes, eccentricity, latusrectum, coordinate of foci, vertices and equations of the directrices of the hyperbola.

Sol. We have, $7y^2 - 9x^2 + 54x - 28y - 116 = 0$

$$\text{or} \quad 7(y^2 - 4y) - 9(x^2 - 6x) - 116 = 0$$

$$\text{or} \quad 7(y^2 - 4y + 4) - 9(x^2 - 6x + 9) = 116 + 28 - 81$$

$$\text{or} \quad 7(y - 2)^2 - 9(x - 3)^2 = 63$$

$$\text{or} \quad \frac{(y-2)^2}{9} - \frac{(x-3)^2}{7} = 1$$

$$\text{or} \quad \frac{Y^2}{9} - \frac{X^2}{7} = 1$$

where $X = x - 3$ and $Y = y - 2$

This equation represents conjugate hyperbola. Comparing it with

$$\frac{Y^2}{b^2} - \frac{X^2}{a^2} = 1$$

we get, $b^2 = 9$ and $a^2 = 7$

$\therefore \quad b = 3$ and $a = \sqrt{7}$

Centre : $X = 0, Y = 0$

i.e. $x - 3 = 0, y - 2 = 0$

$\therefore$ Centre is $(3, 2)$.

Length of transverse axis : Length of transverse axis
$= 2b = 6.$

Length of conjugate axis : Length of conjugate axis
$= 2a = 2\sqrt{7}.$

Eccentricity : The eccentricity e is given by

$$e = \sqrt{1 + \frac{a^2}{b^2}} = \sqrt{1 + \frac{7}{9}} = \frac{4}{3}$$

Length of latusrectum : The length of latusrectum

$$= \frac{2a^2}{b} = \frac{2(7)}{3} = \frac{14}{3}$$

Foci : The coordinates of foci are $(0, \pm be)$

$\therefore \quad X = 0, Y = \pm be$

$\Rightarrow \quad x - 3 = 0, y - 2 = \pm 3 \times \frac{4}{3}$

or $\quad (3, 2 \pm 4)$

i.e. $\quad (3, -2)$ and $(3, 6)$

Vertices : The coordinates of vertices are $(0, \pm b)$

or $\quad X = 0, Y = \pm b$

or $\quad x - 3 = 0, y - 2 = \pm 3$ or $(3, 2 \pm 3)$

or $\quad$ vertices are $(3, -1)$ and $(3, 5)$

Equation of directrices : The equation of directrices are

$$Y = \pm \frac{b}{e}$$

$$\Rightarrow \quad y - 2 = \pm \frac{3}{4/3}$$

$$\Rightarrow \quad y = \left(2 \pm \frac{9}{4}\right)$$

i.e. $\quad y = \frac{17}{4}$ and $y = -\frac{1}{4}$

Example 7 Find the equation of the hyperbola whose foci are (6, 4) and (–4, 4) and eccentricity is 2.

Sol. The centre of the hyperbola is the mid-point of the line joining the two foci. So, the coordinates of the centre are

$$\left(\frac{6-4}{2}, \frac{4+4}{2}\right) \text{ i.e. } (1, 4)$$

Let $2a$ and $2b$ be the lengths of transverse and conjugate axes and let e be the eccentricity. Then, equation of hyperbola is

$$\frac{(x-1)^2}{a^2} - \frac{(y-4)^2}{b^2} = 1$$

$\because \quad$ Distance between the foci $= 2ae$

$$\sqrt{(6+4)^2 + (4-4)^2} = 2a \times 2$$

$\Rightarrow \quad 10 = 4a$

$\therefore \quad a = 5/2$

$\therefore \quad b^2 = a^2(e^2 - 1) = \frac{25}{4}(4 - 1) = \frac{75}{4}$

Thus, the equation of the hyperbola is

$$\frac{(x-1)^2}{\left(\frac{25}{4}\right)} - \frac{(y-4)^2}{\left(\frac{75}{4}\right)} = 1$$

or $\quad 12(x-1)^2 - 4(y-4)^2 = 75$

or $\quad 12(x^2 - 2x + 1) - 4(y^2 - 8y + 16) = 75$

or $\quad 12x^2 - 4y^2 - 24x + 32y - 127 = 0$

Example 8 *Obtain the equation of a hyperbola with coordinate axes as principal axes given that the distances of one of its vertices from the foci are 9 and 1 units.*

Sol. Let equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \ldots(i)$$

If vertices are $A(a, 0)$ and $A'(-a, 0)$ and foci are $S(ae, 0)$ and $S'(-ae, 0)$.

Given $\quad l(S'A) = 9$ and $l(SA) = 1$

$\Rightarrow \quad a + ae = 9$ and $ae - a = 1$

or $\quad a(1 + e) = 9$ and $a(e - 1) = 1$

$\therefore \quad \frac{a(1+e)}{a(e-1)} = \frac{9}{1}$

$\Rightarrow \quad 1 + e = 9e - 9 \quad \Rightarrow \quad e = \frac{5}{4}$

$\because \quad a(1 + e) = 9$

$\therefore \quad a\left(1 + \frac{5}{4}\right) = 9 \quad \Rightarrow \quad a = 4$

$$b^2 = a^2(e^2 - 1) = 16\left(\frac{25}{16} - 1\right)$$

$\therefore \quad b^2 = 9$

From Eq. (i) equation of hyperbola is

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

Example 9 The foci of a hyperbola coincide with the foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$. Find the equation of the hyperbola if its eccentricity is 2.

Sol. The given ellipse is $\frac{x^2}{25} + \frac{y^2}{9} = 1$

Comparing with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\therefore \quad a^2 = 25$ and $b^2 = 9$

then $\quad$ eccentricity $e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$

$\therefore \quad$ Foci of ellipse are $(\pm ae, 0)$ i.e. $(\pm 4, 0)$

So, the coordinates of foci of the hyperbola are $(\pm 4, 0)$.

Let e' be the eccentricity of the required hyperbola and its equation be

$$\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1 \qquad \ldots(i)$$

the coordinates of foci are $(\pm Ae', 0)$

$\therefore \quad Ae' = 4 \quad \Rightarrow \quad A \times 2 = 4 \quad \Rightarrow \quad A = 2$

Also $\quad B^2 = A^2 (e'^2 - 1) \qquad (\because \text{given } e' = 2)$

$$= 4(4-1) = 12$$

Substituting the values of A and B in Eq. (i), we get

$$\frac{x^2}{4} - \frac{y^2}{12} = 1 \text{ or } 3x^2 - y^2 - 12 = 0$$

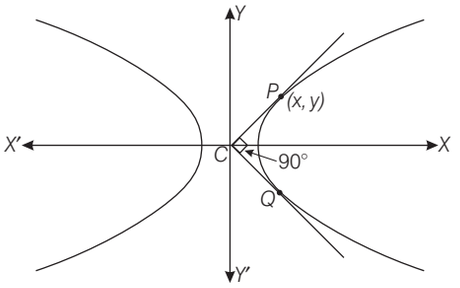
which is required hyperbola.

Example 10 If two points P and Q on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ whose centre is C, are such that CP is perpendicular to CQ, $a < b$, then prove that

$$\frac{1}{(CP)^2} + \frac{1}{(CQ)^2} = \frac{1}{a^2} - \frac{1}{b^2}.$$

Sol. Let $P(x, y)$ be any point on the given hyperbola. Let slope of CP is m, then equation of CP is $y = mx$.

Solving, $\quad y = mx \quad$ and $\quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



$$\frac{x^2}{a^2} - \frac{m^2x^2}{b^2} = 1$$

$$\therefore \qquad x^2 = \frac{a^2b^2}{(b^2 - a^2m^2)}$$

and $\qquad y^2 = m^2x^2 = \frac{a^2m^2b^2}{b^2 - a^2m^2}$

$$\therefore \qquad x^2 + y^2 = \frac{a^2b^2}{(b^2 - a^2m^2)}(1 + m^2)$$

$$\therefore \qquad (CP)^2 = x^2 + y^2 = \frac{a^2b^2(1+m^2)}{(b^2 - a^2m^2)}$$

$$\Rightarrow \qquad \frac{1}{(CP)^2} = \frac{b^2 - a^2m^2}{a^2b^2(1+m^2)} \qquad \ldots(i)$$

and equation of CQ is $\quad y = -\frac{1}{m}x$

replacing m by $-\frac{1}{m}$ in Eq. (i)

then $\quad \frac{1}{(CQ)^2} = \frac{b^2 - a^2\left(-\frac{1}{m}\right)^2}{a^2b^2\left(1 + \left(-\frac{1}{m}\right)^2\right)} = \frac{b^2m^2 - a^2}{a^2b^2(1+m^2)} \qquad \ldots(ii)$

Adding Eqs. (i) and (ii), then

$$\frac{1}{(CP)^2} + \frac{1}{(CQ)^2} = \frac{b^2(1+m^2) - a^2(1+m^2)}{a^2b^2(1+m^2)}$$

$$= \frac{b^2 - a^2}{a^2b^2} = \frac{1}{a^2} - \frac{1}{b^2}$$

Position of a Point with Respect to a Hyperbola

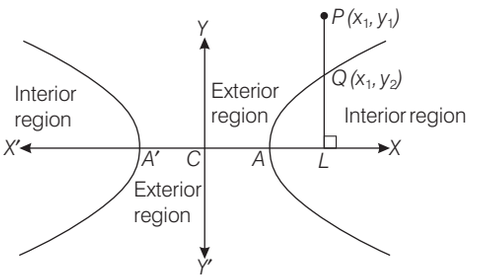
Theorem : The point (x_1, y_1) lies outside, on or inside the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

according as $\quad \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 <, =$ or > 0

Proof : Let $P \equiv (x_1, y_1)$ then $Q \equiv (x_1, y_2)$

Draw PL perpendicular to X-axis



Clearly, $PL > QL$

$$\Rightarrow \qquad y_1 > y_2$$

$$\Rightarrow \qquad \frac{y_1^2}{b^2} > \frac{y_2^2}{b^2}$$

$$\Rightarrow \qquad -\frac{y_1^2}{b^2} < -\frac{y_2^2}{b^2}$$

$$\Rightarrow \qquad \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} < \frac{x_1^2}{a^2} - \frac{y_2^2}{b^2}$$

$$\Rightarrow \qquad \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} < 1 \quad \left(\because Q(x_1, y_2) \text{ lies on } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\right)$$

$$\Rightarrow \qquad \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 < 0$$

Thus, the point $P(x_1, y_1)$ lies outside the hyperbola. Hence the point (x_1, y_1) lies outside, on or inside the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

according as $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 <, =$ or > 0.

Example 11 ***Find the position of the point*** $(5, -4)$ ***relative to the hyperbola*** $9x^2 - y^2 = 1$.

Sol. Since, $9(5)^2 - (-4)^2 - 1 = 225 - 16 - 1 = 208 > 0$,

So, the point $(5, -4)$ inside the hyperbola $9x^2 - y^2 = 1$.

Intersection of a Line and a Hyperbola

Let the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ...(i)

and the given line be $y = mx + c$ ...(ii)

Eliminating y from Eqs. (i) and (ii),

$$\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1 \Rightarrow b^2x^2 - a^2(mx+c)^2 = a^2b^2$$

$$\Rightarrow \quad (a^2m^2 - b^2)x^2 + 2mca^2x + a^2(b^2 + c^2) = 0 \quad \text{...(iii)}$$

Above equation being a quadratic in x gives two values of x. It shows that every straight line will cut the hyperbola in two points, may be real, coincident or imaginary, according as discriminant of Eq. (iii) $>, =, < 0$

i.e. $4m^2c^2a^4 - 4(a^2m^2 - b^2)a^2(b^2 + c^2) >, =, < 0$

or $-a^2m^2 + b^2 + c^2 >, =, < 0$

or $c^2 >, =, < a^2m^2 - b^2$ ...(iv)

Condition of Tangency : If the line (ii) touches the hyperbola (i), then Eq. (iii) has equal roots.

$\therefore$ Discriminant of Eq. (iii) $= 0$

$\Rightarrow \quad c^2 = a^2m^2 - b^2$

or $c = \pm\sqrt{(a^2m^2 - b^2)}$ ...(v)

So, the line $y = mx + c$ touches the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{if} \quad c^2 = a^2m^2 - b^2$$

(which is condition of tangency)

Substituting the value of c from Eq. (v) in Eq. (ii)

$$y = mx \pm \sqrt{(a^2m^2 - b^2)}$$

Hence, the line $y = mx \pm \sqrt{(a^2m^2 - b^2)}$ will always be tangent to the hyperbola.

Point of contact : Substituting $c = \pm\sqrt{(a^2m^2 - b^2)}$ in Eq. (iii)

$$(a^2m^2 - b^2)x^2 \pm 2ma^2\sqrt{(a^2m^2 - b^2)}\,x + a^4m^2 = 0$$

or $(x\sqrt{(a^2m^2 - b^2)} \pm a^2m)^2 = 0$

$\therefore \quad x = \pm\frac{a^2m}{\sqrt{(a^2m^2 - b^2)}} = \pm\frac{a^2m}{c}$

From Eq. (i), $\frac{a^4m^2}{c^2} \cdot \frac{1}{a^2} - \frac{y^2}{b^2} = 1$

$\Rightarrow \quad \frac{y^2}{b^2} = \frac{a^2m^2}{c^2} - 1 = -\frac{(c^2 - a^2m^2)}{c^2} = \frac{b^2}{c^2}$

$\therefore \quad y = \pm\frac{b^2}{c}$

Hence, the point of contact is $\left(\pm\frac{a^2m}{c}, \pm\frac{b^2}{c}\right)$.

This is known as ***m*-point** on the hyperbola.

Remark

If $m = 0$, then Eq. (iii) gives $-b^2x^2 + a^2(b^2 + c^2) = 0$

$$x^2 = \frac{a^2(b^2 + c^2)}{b^2}$$

$\therefore \quad x = \pm\frac{a}{b}\sqrt{(b^2 + c^2)}$

which gives two values of x.

Example 12 Prove that the straight line $lx + my + n = 0$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $a^2l^2 - b^2m^2 = n^2$.

Sol. The given line is $lx + my + n = 0$ or $y = -\frac{l}{m}x - \frac{n}{m}$

Comparing this line with

$y = Mx + c$ ...(i)

$\therefore \quad M = -\frac{l}{m}$ and $c = -\frac{n}{m}$

This line (i) will touch the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ if } \quad c^2 = a^2M^2 - b^2$$

$$\Rightarrow \quad \frac{n^2}{m^2} = \frac{a^2l^2}{m^2} - b^2 \text{ or } a^2l^2 - b^2m^2 = n^2$$

Example 13 Show that the line $x\cos\alpha + y\sin\alpha = p$ touches the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{if} \quad a^2\cos^2\alpha - b^2\sin^2\alpha = p^2.$$

Sol. The given line is

$$x\cos\alpha + y\sin\alpha = p$$

$$\Rightarrow \quad y\sin\alpha = -x\cos\alpha + p$$

$$\Rightarrow \quad y = -x\cot x + p\,\text{cosec}\,\alpha$$

Comparing this line with $y = mx + c$

$$\Rightarrow \quad m = -\cot\alpha,\ c = p\,\text{cosec}\,\alpha$$

Since, the given line touches the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad \text{then} \quad c^2 = a^2m^2 - b^2$$

$$\Rightarrow \quad p^2\text{cosec}^2\alpha = a^2\cot^2\alpha - b^2$$

or $$p^2 = a^2\cos^2\alpha - b^2\sin^2\alpha$$

Example 14 For what value of λ does the line $y = 2x + \lambda$ touches the hyperbola $16x^2 - 9y^2 = 144$?

Sol. Equation of hyperbola is

$$16x^2 - 9y^2 = 144$$

or $$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

Comparing this with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get $a^2 = 9,\ b^2 = 16$

and comparing this line $y = 2x + \lambda$ with $y = mx + c$

$\therefore \quad m = 2$ and $c = \lambda$

If the line $y = 2x + \lambda$ touches the hyperbola

$$16x^2 - 9y^2 = 144$$

then $$c^2 = a^2m^2 - b^2$$

$$\Rightarrow \quad \lambda^2 = 9\,(2)^2 - 16 = 36 - 16 = 20$$

$$\therefore \quad \lambda = \pm 2\sqrt{5}$$

Example 15 If it possible to draw the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ having slope 2, then find its range of eccentricity.

Sol. Tangent having slope m is $y = mx \pm \sqrt{(a^2m^2 - b^2)}$

The tangent having slope 2 is $y = 2x \pm \sqrt{(4a^2 - b^2)}$, which is real if

$$4a^2 - b^2 \geq 0$$

or $$\frac{b^2}{a^2} \leq 4 \quad \text{or} \quad e^2 - 1 \leq 4 \quad \text{or} \quad e^2 \leq 5$$

or $$1 < e \leq \sqrt{5} \qquad \text{(for hyperbola } e > 1\text{)}$$

Exercise for Session 1

1. The eccentricity of the conic represented by $x^2 - y^2 - 4x + 4y + 16 = 0$ is

(a) 1 (b) $\frac{1}{2}$ (c) –1 (d) $\sqrt{2}$

2. If e_1 and e_2 are the eccentricities of the conic sections $16x^2 + 9y^2 = 144$ and $9x^2 - 16y^2 = 144$, then

(a) $e_1^2 - e_2^2 = 1$ (b) $e_1^2 + e_2^2 < 3$ (c) $e_1^2 + e_2^2 = 3$ (d) $e_1^2 + e_2^2 > 3$

3. The transverse axis of a hyperbola is of length $2a$ and a vertex divides the segment of the axis between the centre and the corresponding focus in the ratio 2 : 1, then the equation of the hyperbola is

(a) $4x^2 - 5y^2 = 4a^2$ (b) $4x^2 - 5y^2 = 5a^2$ (c) $5x^2 - 4y^2 = 4a^2$ (d) $5x^2 - 4y^2 = 5a^2$

4. The eccentricity of the hyperbola whose latusrectum is 8 and conjugate axis is equal to half of the distance between the foci, is

(a) $\frac{2}{\sqrt{3}}$ (b) $\frac{3}{\sqrt{3}}$ (c) $\frac{4}{\sqrt{3}}$ (d) $\frac{5}{\sqrt{3}}$

5. The straight line $x + y = \sqrt{2}p$ will touch the hyperbola $4x^2 - 9y^2 = 36$, if

(a) $p^2 = 2$ (b) $p^2 = 5$ (c) $5p^2 = 2$ (d) $2p^2 = 5$

6. The equation of the tangent, parallel to $y - x + 5 = 0$ drawn to $\frac{x^2}{3} - \frac{y^2}{2} = 1$ is

(a) $x - y - 1 = 0$ (b) $x - y + 2 = 0$ (c) $x + y - 1 = 0$ (d) $x + y + 2 = 0$

7. If e and e' are the eccentricities of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{x^2}{b^2} - \frac{y^2}{a^2} = 1$, then the point $\left(\frac{1}{e}, \frac{1}{e'}\right)$ lies on the circle

(a) $x^2 + y^2 = 1$ (b) $x^2 + y^2 = 2$ (c) $x^2 + y^2 = 3$ (d) $x^2 + y^2 = 4$

8. If e and e' are the eccentricities of the ellipse $5x^2 + 9y^2 = 45$ and the hyperbola $5x^2 - 4y^2 = 45$ respectively, then $ee' =$

(a) -1 (b) 1 (c) -4 (d) 9

9. The equation $\frac{x^2}{10 - \lambda} + \frac{y^2}{6 - \lambda} = 1$ represents

(a) a hyperbola if $\lambda < 6$ (b) an ellipse if $\lambda > 6$ (c) a hyperbola if $6 < \lambda < 10$ (d) an ellipse if $0 < \lambda < 6$

10. The eccentricity of the hyperbola conjugate to $x^2 - 3y^2 = 2x + 8$ is

(a) $\frac{2}{\sqrt{3}}$ (b) $\sqrt{3}$ (c) 2 (d) $\sqrt{2}$

11. For hyperbola $\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$ which of the following remains constant with change in 'α'

(a) Abscissae of vertices (b) Abscissae of foci (c) Eccentricity (d) Directrix

12. If the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincide, then the value of b^2 is

(a) 1 (b) 5 (c) 7 (d) 9

13. Find the equation of the hyperbola whose foci are $(0, \pm\sqrt{10})$ and which passes through the point $(2, 3)$.

14. Find the equation of the hyperbola whose foci are $(10, 5)$ and $(-2, 5)$ and eccentricity 3.

15. Prove that the straight lines $\frac{x}{a} - \frac{y}{b} = m$ and $\frac{x}{a} + \frac{y}{b} = \frac{1}{m}$ always meet on a hyperbola.

16. Find the centre, eccentricity and length of axes of the hyperbola $3x^2 - 5y^2 - 6x + 20y - 32 = 0$.

17. Find the eccentricity of the hyperbola conjugate to the hyperbola $x^2 - 3y^2 = 1$.

18. For what value of λ, does the line $y = 3x + \lambda$ touch the hyperbola $9x^2 - 5y^2 = 45$?

19. Find the equation of the tangent to the hyperbola $4x^2 - 9y^2 = 1$ which is parallel to the line $4y = 5x + 7$. Also find the point of contact.

Session 2

Equations of Tangents in Different Forms, Equations of Normals in Different Forms, Pair of Tangents, Chord of Contact, Equation of the Chord Bisected at a Given Point

Equations of Tangents in Different Forms

1. Point form (first principal method) :

Theorem : The equation of the tangent to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \text{ at } (x_1, y_1) \text{ is}$$

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1.$$

Proof : Equation of hyperbola is

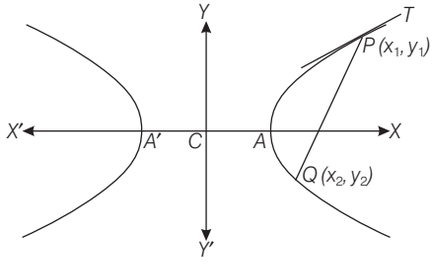
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad \ldots(i)$$

Let $P \equiv (x_1, y_1)$ and $Q \equiv (x_2, y_2)$ be any two points in Eq. (i), then

$$\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1 \qquad \ldots(ii)$$

and

$$\frac{x_2^2}{a^2}-\frac{y_2^2}{b^2}=1 \qquad \ldots(iii)$$



Subtracting Eq. (ii) from Eq. (iii), then

$$\frac{1}{a^2}(x_2^2-x_1^2)-\frac{1}{b^2}(y_2^2-y_1^2)=0$$

$$\Rightarrow \quad \frac{(x_2-x_1)(x_2+x_1)}{a^2}-\frac{(y_2-y_1)(y_2+y_1)}{b^2}=0$$

$$\Rightarrow \quad \frac{y_2-y_1}{x_2-x_1}=\frac{b^2}{a^2}\frac{(x_1+x_2)}{(y_1+y_2)} \qquad \ldots(iv)$$

Equation of PQ is

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1) \qquad \ldots(v)$$

From Eqs. (iv) and (v),

$$y-y_1=\frac{b^2}{a^2}\cdot\frac{(x_1+x_2)}{(y_1+y_2)}(x-x_1) \qquad \ldots(vi)$$

Now, tangent at $P, Q \to P$

i.e. $x_2 \to x_1$ and $y_2 \to y_1$, then Eq. (vi) becomes

$$y-y_1=\frac{b^2}{a^2}\cdot\frac{(2x_1)}{(2y_1)}(x-x_1)$$

or $$\frac{yy_1-y_1^2}{b^2}=\frac{xx_1-x_1^2}{a^2} \text{ or } \frac{xx_1}{a^2}-\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}$$

or $$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1 \qquad \text{[from Eq. (ii)]}$$

which is the required equation of tangent at (x_1, y_1).

Remark

The equation of tangent at (x_1, y_1) can be obtained by replacing x^2 by xx_1, y^2 by yy_1, x by $\frac{x+x_1}{2}$, y by $\frac{y+y_1}{2}$ and xy by $\frac{xy_1+x_1y}{2}$. This method is applied only when the equation of conic is a polynomial of second degree in x and y.

2. Parametric form :

Theorem : The equation of tangent to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \text{ at } (a\sec\phi, b\tan\phi) \text{ is}$$

$$\frac{x}{a}\sec\phi-\frac{y}{b}\tan\phi=1$$

Proof : Since, equation of tangent at (x_1, y_1) is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

Now, replacing x_1 by $a\sec\phi$ and y_1 by $b\tan\phi$, we get

$$\frac{x}{a}\sec\phi - \frac{y}{b}\tan\phi = 1$$

Remark

The point of intersection of tangents at 'θ' and 'ϕ' on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$\left(\frac{a\cos\left(\frac{\theta-\phi}{2}\right)}{\cos\left(\frac{\theta+\phi}{2}\right)}, \frac{b\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta+\phi}{2}\right)}\right)$$

Remembering method :

$\because$ Equations of chord joining 'θ' and 'ϕ' is

$$\frac{x}{a}\cos\left(\frac{\theta-\phi}{2}\right) - \frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right) = \cos\left(\frac{\theta+\phi}{2}\right)$$

or $$\frac{x}{a}\left\{\frac{\cos\left(\frac{\theta-\phi}{2}\right)}{\cos\left(\frac{\theta+\phi}{2}\right)}\right\} - \frac{y}{b}\left\{\frac{\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta+\phi}{2}\right)}\right\} = 1$$

or $$\frac{x}{a^2}\left\{\frac{a\cos\left(\frac{\theta-\phi}{2}\right)}{\cos\left(\frac{\theta+\phi}{2}\right)}\right\} - \frac{y}{b^2}\left\{\frac{b\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta+\phi}{2}\right)}\right\} = 1$$

i.e. $$\left\{\frac{a\cos\left(\frac{\theta-\phi}{2}\right)}{\cos\left(\frac{\theta+\phi}{2}\right)}, \frac{b\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta+\phi}{2}\right)}\right\}$$

3. Slope form :

Theorem : The equations of tangents of slope m to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is given by

$$y = mx \pm \sqrt{(a^2m^2 - b^2)}$$

The coordinates of the points of contact are

$$\left(\pm\frac{a^2m}{\sqrt{(a^2m^2 - b^2}}, \pm\frac{b^2}{\sqrt{(a^2m^2 - b^2)}}\right)$$

Proof : Let $y = mx + c$ be a tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \ldots(i)$$

Substituting the value of $y = mx + c$ in Eq. (i), then

$$\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1$$

or $\quad (a^2m^2 - b^2)\,x^2 + 2mca^2x + a^2(c^2 + b^2) = 0$

must have equal roots

$\therefore \qquad 4a^4m^2c^2 - 4(a^2m^2 - b^2)\,a^2(c^2 + b^2) = 0$

$[\because B^2 - 4AC = 0]$

$\Rightarrow \qquad a^2m^2c^2 - (a^2m^2 - b^2)(b^2 + c^2) = 0$

$\Rightarrow \qquad a^2m^2c^2 - a^2b^2m^2 - a^2m^2c^2 + b^4 + b^2c^2 = 0$

$\Rightarrow \qquad -a^2b^2m^2 + b^4 + b^2c^2 = 0 \Rightarrow c^2 = a^2m^2 - b^2$

$\therefore \qquad c = \pm\sqrt{(a^2m^2 - b^2)}$

Substituting this value of c in $y = mx + c$, we get

$$\boldsymbol{y = mx \pm \sqrt{(a^2m^2 - b^2)}} \qquad \ldots(ii)$$

as the required equation of tangents of hyperbola in terms of slope.

$\because$ Tangent at (x_1, y_1) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1 \qquad \ldots(iii)$$

On comparing Eqs. (ii) and (iii) we get

$$\frac{x_1/a^2}{m} = \frac{y_1/b^2}{1} = \pm\frac{1}{\sqrt{a^2m^2 - b^2}}$$

Thus, the coordinates of the points of contact are

$$\left(\frac{\pm a^2m}{\sqrt{(a^2m - b^2)}}, \pm\frac{b^2}{\sqrt{(a^2m - b^2)}}\right)$$

Remark

The equations of tangents of slope m to the hyperbola $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are given by $y = mx \pm \sqrt{(b^2 - a^2m^2)}$

and the coordinates of the points of contact are

$$\left(\mp\frac{a^2m}{\sqrt{(b^2 - a^2m^2)}}, \mp\frac{b^2}{\sqrt{(b^2 - a^2m^2)}}\right)$$

Example 16 Find the equation of the tangent to the hyperbola $x^2 - 4y^2 = 36$ which is perpendicular to the line $x - y + 4 = 0$.

Sol. Let m be the slope of the tangent. Since, the tangent is perpendicular to the line $x - y + 4 = 0$

$\therefore \qquad m \times 1 = -1$

$\Rightarrow \qquad m = -1$

Since $x^2 - 4y^2 = 36$ or $\frac{x^2}{36} - \frac{y^2}{9} = 1$

Comparing this with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\therefore$ $a^2 = 36$ and $b^2 = 9$

So the equation of tangents are

$$y = (-1)\, x \pm \sqrt{36 \times (-1)^2 - 9}$$

$\Rightarrow$ $y = -x \pm \sqrt{27}$

or $x + y \pm 3\sqrt{3} = 0$

Example 17 Find the equation and the length of the common tangents to hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{and} \quad \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$$

Sol. Tangent at $(a \sec \phi, b \tan \phi)$ on the 1st hyperbola is

$$\frac{x}{a} \sec \phi - \frac{y}{b} \tan \phi = 1 \qquad \ldots(i)$$

Similarly tangent at any point $(b \tan\theta, a \sec \theta)$ on 2nd hyperbola is

$$\frac{y}{a} \sec \theta - \frac{x}{b} \tan\theta = 1 \qquad \ldots(ii)$$

If Eqs. (i) and (ii) are common tangents then they should be identical. Comparing the coefficients of x and y

$\Rightarrow$ $\frac{\sec \theta}{a} = -\frac{\tan \phi}{b}$

or $\sec \theta = -\frac{a}{b} \tan \phi \qquad \ldots(iii)$

and $-\frac{\tan\theta}{b} = \frac{\sec \phi}{a}$

or $\tan\theta = -\frac{b}{a} \sec \phi \qquad \ldots(iv)$

$\because$ $\sec^2\theta - \tan^2\theta = 1$

$\Rightarrow$ $\frac{a^2}{b^2} \tan^2 \phi - \frac{b^2}{a^2} \sec^2 \phi = 1$ [from Eqs. (iii) and (iv)]

or $\frac{a^2}{b^2} \tan^2 \phi - \frac{b^2}{a^2} (1 + \tan^2 \phi) = 1$

or $\left(\frac{a^2}{b^2} - \frac{b^2}{a^2}\right) \tan^2 \phi = 1 + \frac{b^2}{a^2}$

$\therefore$ $\tan^2 \phi = \frac{b^2}{a^2 - b^2}$

and $\sec^2\phi = 1 + \tan^2 \phi = \frac{a^2}{a^2 - b^2}$

Hence, the points of contact are

$$\left\{\pm \frac{a^2}{\sqrt{(a^2 - b^2)}}, \pm \frac{b^2}{\sqrt{(a^2 - b^2)}}\right\}$$

and $\left\{\mp \frac{b^2}{\sqrt{(a^2 - b^2)}}, \mp \frac{a^2}{\sqrt{(a^2 - b^2)}}\right\}$ [from Eqs. (iii) and (iv)]

Length of common tangent i.e. the distance between the above points is $\sqrt{2}\, \frac{(a^2 + b^2)}{\sqrt{(a^2 - b^2)}}$ and equation of common tangent on putting the values of $\sec \phi$ and $\tan \phi$ in Eqs. (i) is

$$\pm \frac{x}{\sqrt{(a^2 - b^2)}} \mp \frac{y}{\sqrt{(a^2 - b^2)}} = 1$$

or $x \mp y = \pm \sqrt{(a^2 - b^2)}$

Aliter : The given two hyperbolas are

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \ldots(i)$$

and $\frac{x^2}{(-b^2)} - \frac{y^2}{(-a^2)} = 1 \qquad \ldots(ii)$

we know that

$$y = mx \pm \sqrt{(a^2m^2 - b^2)} \qquad \ldots(iii)$$

is tangent to Eq. (i) for all m.

Similarly $y = m_1 x \pm \sqrt{(-b^2)\, m_1^2 - (-a^2)}$

$$y = m_1 x \pm \sqrt{(a^2 - b^2 m_1^2)} \qquad \ldots(iv)$$

will be tangent to Eq. (ii)

For common tangents to Eqs. (i) and (ii), the lines (iii) and (iv) must be identical

i.e. $m = m_1$ and $a^2m^2 - b^2 = a^2 - b^2m_1^2$

i.e. $(a^2 + b^2)(m^2 - 1) = 0$

$\Rightarrow$ $m^2 = 1 \Rightarrow m = \pm 1$

$\therefore$ The equation of common tangent lines are

$$\boldsymbol{y = \pm x \pm \sqrt{(a^2 - b^2)}} \qquad \text{[from Eq. (iii)]}$$

or $y \mp x = \pm\sqrt{(a^2 - b^2)} \qquad \ldots(v)$

Equation of tangent to Eq. (i) at (x_1, y_1) is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1 \qquad \ldots(vi)$$

On comparing Eqs.(v) and (vi), then

$$\frac{x_1 / a^2}{\mp 1} = \frac{-y_1 / b^2}{1} = \frac{1}{\pm \sqrt{(a^2 - b^2)}}$$

i.e. $\left(\mp \frac{a^2}{\sqrt{(a^2 - b^2)}}, \mp \frac{b^2}{\sqrt{(a^2 - b^2)}}\right)$

and equation of tangent to Eq. (ii) at (x_2, y_2) is

$$\frac{xx_2}{(-b^2)} - \frac{yy_2}{(-a^2)} = 1 \qquad \ldots(vii)$$

On comparing Eqs. (v) and (vii), then

$$\frac{x_2 / (-b^2)}{\mp 1} = \frac{y_2}{1} = \frac{1}{\pm\sqrt{(a^2 - b^2)}}$$

i.e. $\left(\pm\frac{b^2}{\sqrt{(a^2-b^2)}},\pm\frac{a^2}{\sqrt{(a^2-b^2)}}\right)$

The points of contact are

$$\left(\pm\frac{a^2}{\sqrt{(a^2-b^2)}},\pm\frac{b^2}{\sqrt{(a^2-b^2)}}\right)$$

and $\left(\mp\frac{b^2}{\sqrt{(a^2-b^2)}},\mp\frac{a^2}{\sqrt{(a^2-b^2)}}\right)$

Hence, the length of common tangent is

$$\sqrt{2}\cdot\frac{(a^2+b^2)}{\sqrt{(a^2-b^2)}}$$

Example 18 *PQ* is the chord joining the points ϕ_1 and ϕ_2 on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If $\phi_1-\phi_2=2\alpha$, where α is constant, prove that *PQ* touches the hyperbola

$$\frac{x^2}{a^2}\cos^2\alpha-\frac{y^2}{b^2}=1.$$

Sol. Given hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ ...(i)

Equation of the chord *PQ* to the hyperbola (i) is

$$\frac{x}{a}\cos\left(\frac{\phi_1-\phi_2}{2}\right)-\frac{y}{b}\sin\left(\frac{\phi_1+\phi_2}{2}\right)=\cos\left(\frac{\phi_1+\phi_2}{2}\right)$$

$$\Rightarrow\quad \frac{x}{a}\cos\alpha-\frac{y}{b}\sin\left(\frac{\phi_1+\phi_2}{2}\right)=\cos\left(\frac{\phi_1+\phi_2}{2}\right)$$

(Given $\phi_1-\phi_2=2\alpha$)

i.e. $$y=\frac{b}{a}\frac{\cos\alpha}{\sin\left(\frac{\phi_1+\phi_2}{2}\right)}x+\frac{b\cos\left(\frac{\phi_1+\phi_2}{2}\right)}{\sin\left(\frac{\phi_1+\phi_2}{2}\right)}$$ ...(ii)

Comparing this line with $y=mx+c$

$$\therefore\quad m=\frac{b}{a}\cdot\frac{\cos\alpha}{\sin\left(\frac{\phi_1+\phi_2}{2}\right)}\text{ and }c=\frac{b\cos\left(\frac{\phi_1+\phi_2}{2}\right)}{\sin\left(\frac{\phi_1+\phi_2}{2}\right)}$$

For line $y=mx+c$ to be a tangent on $\frac{x^2}{a^2}\cos^2\alpha-\frac{y^2}{b^2}=1$, we have

$$c^2=\frac{a^2}{\cos^2\alpha}m^2-b^2$$

$$\therefore\quad \text{LHS}=c^2=\frac{b^2\cos^2\left(\frac{\phi_1+\phi_2}{2}\right)}{\sin^2\left(\frac{\phi_1+\phi_2}{2}\right)}$$

and $$\text{RHS}=\frac{a^2}{\cos^2\alpha}m^2-b^2$$

$$=\frac{a^2}{\cos^2\alpha}\times\frac{b^2\cos^2\alpha}{a^2\sin^2\left(\frac{\phi_1+\phi_2}{2}\right)}-b^2$$

$$=\frac{b^2}{\sin^2\left(\frac{\phi_1+\phi_2}{2}\right)}-b^2$$

$$=\frac{b^2\cos^2\left(\frac{\phi_1+\phi_2}{2}\right)}{\sin\left(\frac{\phi_1+\phi_2}{2}\right)}$$

Hence proved.

Example 19 If the line $y=mx+\sqrt{(a^2m^2-b^2)}$ touches the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $(a\sec\theta,b\tan\theta)$, show that $\theta=\sin^{-1}\left(\frac{b}{am}\right)$.

Sol. Since $(a\sec\theta, b\tan\theta)$ lies on

$$y=mx+\sqrt{(a^2m^2-b^2)}$$

$$\therefore\quad b\tan\theta=am\sec\theta+\sqrt{(a^2m^2-b^2)}$$

$$\Rightarrow\quad (b\tan\theta-am\sec\theta)^2=a^2m^2-b^2$$

$$\Rightarrow\quad b^2\tan^2\theta+a^2m^2\sec^2\theta-2abm\tan\theta\sec\theta=a^2m^2-b^2$$

$$\Rightarrow\quad a^2m^2\tan^2\theta-2abm\tan\theta\ \theta+b^2\sec^2\theta=0$$

or $$a^2m^2\sin^2\theta-2abm\sin\theta+b^2=0\qquad(\because\cos\theta\neq0)$$

$$\therefore\quad \sin\theta=\frac{2abm\pm\sqrt{4a^2b^2m^2-4a^2b^2m^2}}{2a^2m^2}=\left(\frac{b}{am}\right)$$

$$\therefore\quad \theta=\sin^{-1}\left(\frac{b}{am}\right)$$

Example 20 If *SY* and *S′Y′* be drawn perpendiculars from foci to any tangent to a hyperbola. Prove that *Y* and *Y′* lie on the auxiliary circle and that product of these perpendiculars is constant.

Sol. Let hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$ ...(i)

Tangent at $P(a\sec\phi, b\tan\phi)$ on Eq. (i) is

$$\frac{x}{a}\sec\phi-\frac{y}{b}\tan\phi=1$$ ...(ii)

Its slope is $\frac{(b\sec\phi)}{(a\tan\phi)}$

Equation of *SY* which is perpendicular to Eq. (i) and passes through focus *S* i.e. $(ae, 0)$ is

$$y - 0 = -\frac{a\tan\phi}{b\sec\phi}(x - ae)$$

or $$\frac{x}{b}\tan\phi + \frac{y}{a}\sec\phi = \frac{ae}{b}\tan\phi \qquad \ldots(iii)$$

$\because$ Lines (ii) and (iii) intersect at Y and in order to find its locus we have to eliminate ϕ between Eqs. (ii) and (iii), for which squaring and adding Eqs. (ii) and (iii) then, we get

$$(x^2 + y^2)\left(\frac{\sec^2\phi}{a^2} + \frac{\tan^2\phi}{b^2}\right) = 1 + \frac{a^2e^2}{b^2}\tan^2\phi$$

$$= 1 + \frac{a^2 + b^2}{b^2}\tan^2\phi = (1 + \tan^2\phi) + \frac{a^2}{b^2}\tan^2\phi$$

$$= \left(\sec^2\phi + \frac{a^2}{b^2}\tan^2\phi\right) = a^2\left(\frac{\sec^2\phi}{a^2} + \frac{\tan^2\phi}{b^2}\right)$$

$\therefore\ x^2 + y^2 = a^2$ is the required locus.

Similarly the point Y' also lies on it. Again, if p_1 and p_2 be the length of perpendiculars from $S(ae, 0)$ and $S'(-ae, 0)$ on the tangent (ii), then

$$p_1p_2 = \frac{(e\sec\phi - 1)\cdot(e\sec\phi + 1)}{\sqrt{\left(\frac{\sec^2\phi}{a^2} + \frac{\tan^2\phi}{b^2}\right)}\sqrt{\left(\frac{\sec^2\phi}{a^2} + \frac{\tan^2\phi}{b^2}\right)}}$$

$$= \frac{a^2b^2(e^2\sec^2\phi - 1)}{b^2\sec^2\phi + a^2\tan^2\phi}$$

$$= \frac{a^2b^2(e^2\sec^2\phi - 1)}{a^2(e^2 - 1)\sec^2\phi + a^2\tan^2\phi}$$

$$= \frac{b^2(e^2\sec^2\phi - 1)}{(e^2\sec^2\phi - 1)} = b^2 \qquad [(\because b^2 = a^2(e^2 - 1)]$$

Equations of Normals in Different Forms

1. Point form :

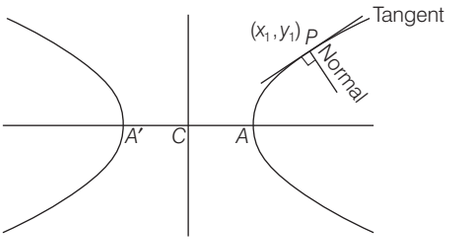
Theorem : The equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$

Proof : Since the equation of tangent to the hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

The slope of tangent at $(x_1, y_1) = \frac{b^2x_1}{a^2y_1}$

$\therefore$ Slope of normal at $(x_1, y_1) = -\frac{a^2y_1}{b^2x_1}$



Hence, the equation of normal at (x_1, y_1)

$$y - y_1 = -\frac{a^2y_1}{b^2x_1}(x - x_1)$$

or $$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$$

Remark

The equation of normal at (x_1, y_1) can also be obtained by this method

$$\frac{x - x_1}{a'x_1 + hy_1 + g} = \frac{y - y_1}{hx_1 + b'y_1 + f} \qquad \ldots(i)$$

a', b', g, f, h are obtained by comparing the given hyperbola with

$$a'x^2 + 2hxy + b'y^2 + 2gx + 2fy + c = 0 \qquad \ldots(ii)$$

The denominator of Eq. (i) can easily be remembered by the first two rows of this determinant

i.e. $$\begin{vmatrix} a' & h & g \\ h & b' & f \end{vmatrix}$$

Since first row is $a'(x_1) + h(y_1) + g(1)$

and second row i $h(x_1) + b'(y_1) + f(1)$

Here, hyperbola $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

or $$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0 \qquad \ldots(iii)$$

On comparing Eqs. (ii) and (iii), we get

$$a' = \frac{1}{a^2}, b' = -\frac{1}{b^2}, g = 0, f = 0, h = 0$$

From Eq. (i), equation of normal of (iii) at (x_1, y_1) is

$$\frac{x - x_1}{\frac{1}{a^2}x_1 + 0 + 0} = \frac{y - y_1}{0 - \frac{y_1}{b^2} + 0}$$

or $$\frac{a^2x}{x_1} - a^2 = -\frac{b^2y}{y_1} + b^2$$

or $$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$$

2. Parametric form :

Theorem : The equation of normal at $(a\sec\phi, b\tan\phi)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $ax\cos\phi + by\cot\phi = a^2 + b^2$

Proof : Since the equation of normals of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at (x_1, y_1) is

$$\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2 \qquad ...(i)$$

Replacing x_1 by $a\sec\phi$ and y_1 by $b\tan\phi$, then Eq. (i) becomes

$$\frac{a^2x}{a\sec\phi}+\frac{b^2y}{b\tan\phi}=a^2+b^2$$

or $$ax\cos\phi+by\cot\phi=a^2+b^2$$

is equation of normal at $(a\sec\phi, b\tan\phi)$.

3. Slope form :

Theorem : The equations of normals of slope m to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are given by

$$y=mx\mp\frac{m(a^2+b^2)}{\sqrt{(a^2-m^2b^2)}}$$

at the points $\left(\pm\frac{a^2}{\sqrt{(a^2-m^2b^2)}}, \mp\frac{mb^2}{\sqrt{(a^2-m^2b^2)}}\right)$

Proof : The equation of normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at (x_1, y_1) is

$$\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2 \qquad ...(i)$$

Since 'm' is the slope of the normal, then

$$m=-\frac{a^2y_1}{b^2x_1}$$

$$y_1=-\frac{b^2x_1m}{a^2} \qquad ...(ii)$$

Since (x_1, y_1) lies on $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$\therefore$ $$\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1$$

or $$\frac{x_1^2}{a^2}-\frac{b^4x_1^2m^2}{a^4b^2}=1$$

or $$x_1^2=\frac{a^4}{a^2-b^2m^2}$$

$\therefore$ $$x_1=\pm\frac{a^2}{\sqrt{(a^2-b^2m^2)}}$$

From Eq. (ii), $$y_1=\mp\frac{mb^2}{\sqrt{(a^2-b^2m^2)}}$$

$\therefore$ Equation of normal in terms of slope is

$$y-\left(\mp\frac{mb^2}{\sqrt{(a^2-m^2b^2)}}\right)=m\left(x-\left(\pm\frac{a^2}{\sqrt{(a^2-m^2b^2)}}\right)\right)$$

$\Rightarrow$ $$\boldsymbol{y=mx\mp\frac{m(a^2+b^2)}{\sqrt{(a^2-m^2b^2)}}} \qquad ...(iii)$$

Thus Eq. (iii) is a normal to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

where m is the slope of the normal.

The coordinates of the point of contact are

$$\left(\pm\frac{a^2}{\sqrt{(a^2-b^2m^2)}}, \mp\frac{mb^2}{\sqrt{(a^2-b^2m^2)}}\right)$$

Comparing Eq. (iii) with $y=mx+c$

$\therefore$ $$c=\mp\frac{m(a^2+b^2)}{\sqrt{(a^2-m^2b^2)}}$$

or $$c^2=\frac{m^2(a^2+b^2)^2}{(a^2-m^2b^2)}$$

which is condition of normality, where $y=mx+c$ is the normal of

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

Remark

Normal other than transverse axis, never passes through the focus.

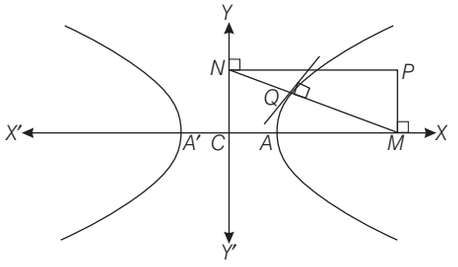
Example 21 ***A normal to the hyperbola*** $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ ***meets the axes in*** M ***and*** N ***and lines*** MP ***and*** NP ***are drawn perpendiculars to the axes meeting at*** P***. Prove that the locus of*** P ***is the hyperbola***

$$a^2x^2-b^2y^2=(a^2+b^2)^2.$$

Sol. The equation of normal at the point $(a\sec\phi, b\tan\phi)$ to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$ax\cos\phi + by\cot\phi = a^2 + b^2 \quad \ldots(i)$$

The normal (i) meets the X-axis in $M\left(\frac{a^2+b^2}{a}\sec\phi, 0\right)$ and Y-axis in $N\left(0, \frac{a^2+b^2}{b}\tan\phi\right)$



$\therefore$ Equation of MP, the line through M and perpendicular to X axis, is

$$x = \left(\frac{a^2+b^2}{a}\right)\sec\phi$$

or $$\sec\phi = \frac{ax}{(a^2+b^2)} \quad \ldots(ii)$$

and the equation of NP, the line through N and perpendicular to the Y-axis, is

$$y = \left(\frac{a^2+b^2}{b}\right)\tan\phi$$

or $$\tan\phi = \frac{by}{(a^2+b^2)} \quad \ldots(iii)$$

The locus of the point of intersection of MP and NP will be obtained by elminating ϕ from Eqs. (ii) and (iii), we have

$$\sec^2\phi - \tan^2\phi = 1$$

$$\Rightarrow \quad \frac{a^2x^2}{(a^2+b^2)^2} - \frac{b^2y^2}{(a^2+b^2)^2} = 1$$

or $$a^2x^2 - b^2y^2 = (a^2+b^2)^2$$

is the required locus of P.

Example 22 Prove that the line $lx + my - n = 0$ will be a normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $\frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2+b^2)^2}{n^2}$.

Sol. The equation of any normal to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$ax\cos\phi + by\cot\phi = a^2 + b^2$$

or $$ax\cos\phi + by\cot\phi - (a^2+b^2) = 0 \quad \ldots(i)$$

The straight line $lx + my - n = 0$ will be a normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then Eq. (i) and $lx + my - n = 0$ represent the same line

$$\frac{a\cos\phi}{l} = \frac{b\cot\phi}{m} = \frac{(a^2+b^2)}{n}$$

or $$\sec\phi = \frac{na}{l(a^2+b^2)} \text{ and } \tan\phi = \frac{nb}{m(a^2+b^2)}$$

$\because$ $$\sec^2\phi - \tan^2\phi = 1$$

$\therefore$ $$\frac{n^2a^2}{l^2(a^2+b^2)^2} - \frac{n^2b^2}{m^2(a^2+b^2)^2} = 1$$

$\Rightarrow$ $$\frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2+b^2)^2}{n^2}$$

Example 23 If the normal at `ϕ' on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meet transverse axis at G, prove that

$$AG \cdot A'G = a^2(e^4\sec^2\phi - 1).$$

where A and A' are the vertices of the hyperbola.

Sol. The equation of normal at $(a\sec\phi, b\tan\phi)$ to the given hyperbola is $ax\cos\phi + by\cot\phi = (a^2+b^2)$

This meets the transverse axis i.e. X-axis at G. So, the coordinates of G are $\left(\left(\frac{a^2+b^2}{a}\right)\sec\phi, 0\right)$ and the coordinates of the vertices A and A' are $A(a, 0)$ and $A'(-a, 0)$ respectively.

$$\therefore AG \cdot A'G = \left(-a + \left(\frac{a^2+b^2}{a}\right)\sec\phi\right)\left(a + \left(\frac{a^2+b^2}{a}\right)\sec\phi\right)$$

$$= \left(\frac{a^2+b^2}{a}\right)^2\sec^2\phi - a^2$$

$$= (ae^2)^2\sec^2\phi - a^2 = a^2(e^4\sec^2\phi - 1)$$

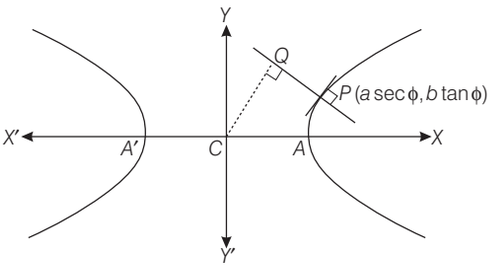
Example 24 Find the locus of the foot of perpendicular from the centre upon any normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Sol. Normal at $P(a\sec\phi, b\tan\phi)$ is

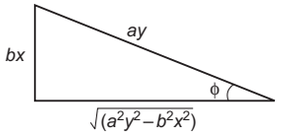
$$ax\cos\phi + by\cot\phi = a^2 + b^2 \quad \ldots(i)$$

and equation of line perpendicular to Eq. (i) and passing through origin is

$$bx - ay\sin\phi = 0 \quad \ldots(ii)$$



Eliminating ϕ from Eqs. (i) and (ii), we will get the equation of locus of Q, as from Eq. (ii),



$$\sin\phi = \frac{bx}{ay}$$

$$\therefore \quad \cos\phi = \frac{\sqrt{(a^2y^2 - b^2x^2)}}{ay}$$

$$\text{and} \quad \cot\phi = \frac{\sqrt{(a^2y^2 - b^2x^2)}}{bx}$$

From Eq. (i),

$$ax \times \frac{\sqrt{a^2y^2 - b^2x^2}}{ay} + by \times \frac{\sqrt{(a^2y^2 - b^2x^2)}}{bx} = a^2 + b^2$$

$$\Rightarrow \quad (x^2 + y^2)\sqrt{(a^2y^2 - b^2x^2)} = (a^2 + b^2)xy$$

$$\text{or} \quad (x^2 + y^2)^2(a^2y^2 - b^2x^2) = (a^2 + b^2)^2 x^2y^2$$

which is required locus.

Pair of Tangents

Theorem : The combined equation of the pair of tangents drawn from a point $P(x_1, y_1)$, lying outside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

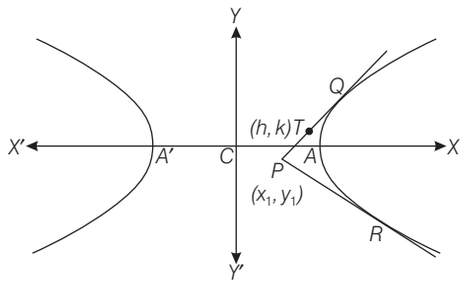
$$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1\right)^2$$

or $$SS_1 = T^2$$

where $$S = \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1;$$

$$S_1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 \text{ and } T = \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$$

Proof : Let $T(h, k)$ be any point on the pair of tangents PQ or PR drawn from any external point $P(x_1, y_1)$ to the hyperbola



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$\therefore$ Equation of PT is $$y - y_1 = \frac{k - y_1}{h - x_1}(x - x_1)$$

or $$y = \left(\frac{k - y_1}{h - x_1}\right)x + \left(\frac{hy_1 - kx_1}{h - x_1}\right)$$

which is the tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$\therefore$ $$c^2 = a^2m^2 - b^2$$

or $$\left(\frac{hy_1 - kx_1}{h - x_1}\right)^2 = a^2\left(\frac{k - y_1}{h - x_1}\right)^2 - b^2$$

$\Rightarrow$ $$(hy_1 - kx_1)^2 = a^2(k - y_1)^2 - b^2(h - x_1)^2$$

Hence, locus of (h, k) is

$$(xy_1 - x_1y)^2 = a^2(y - y_1)^2 - b^2(x - x_1)^2$$

or $$(xy_1 - x_1y)^2 = -(b^2x^2 - a^2y^2) - (b^2x_1^2 - a^2y_1^2) - 2(a^2yy_1 - b^2xx_1)$$

or $$\left(\frac{xy_1 - x_1y}{ab}\right)^2 = -\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) - \left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}\right) + 2\left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2}\right)$$

or $$-\left(\frac{xy_1 - x_1y}{ab}\right)^2 - \left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) - \left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}\right) + 1 + \left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2}\right)^2$$

$$= \left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2}\right)^2 - 2\left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2}\right) + 1$$

or $$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1\right)^2$$

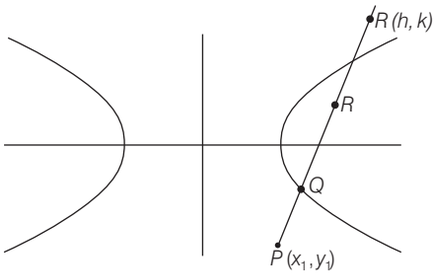
or $$SS_1 = T^2$$

Aliter :

Let the hyperbola be $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \ldots(i)$$

Let $P(x_1, y_1)$ be any point outside the hyperbola.

Let a chord of the hyperbola through the point $P(x_1, y_1)$ cut the hyperbola at Q and R. Let $R(h, k)$ be any arbitrary point on the line PQ (R inside or outside).



Let Q divides PR in the ratio $\lambda : 1$, then coordinates of Q are

$$\left(\frac{\lambda h + x_1}{\lambda + 1}, \frac{\lambda k + y_1}{\lambda + 1}\right) \qquad (\because PQ : QR = \lambda : 1)$$

Since, Q lies on hyperbola (i), then

$$\frac{1}{a^2}\left(\frac{\lambda h + x_1}{\lambda + 1}\right)^2 - \frac{1}{b^2}\left(\frac{\lambda k + y_1}{\lambda + 1}\right)^2 = 1$$

$$\Rightarrow \quad b^2(\lambda h + x_1)^2 - a^2(\lambda k + y_1)^2 = a^2 b^2 (\lambda + 1)^2$$

$$\Rightarrow \quad (b^2h^2 - a^2k^2 - a^2b^2)\lambda^2 + 2(b^2hx_1 - a^2ky_1 - a^2b^2)\lambda + (b^2x_1^2 - a^2y_1^2 - a^2b^2) = 0 \quad \text{...(ii)}$$

Let PR will become tangent to the hyperbola (i), then roots of Eq. (ii) are equal

$$4(b^2hx_1 - a^2ky_1 - a^2b^2)^2 - 4(b^2h^2 - a^2k^2 - a^2b^2) \times (b^2x_1^2 - a^2y_1^2 - a^2b^2) = 0$$

Dividing by $4a^4b^4$

$$\therefore \quad \left(\frac{hx_1}{a^2} - \frac{ky_1}{b^2} - 1\right)^2 = \left(\frac{h^2}{a^2} - \frac{k^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1\right)$$

Hence, locus of $R(h, k)$, i.e. equation of pair of tangents from $P(x_1, y_1)$ is

$$\left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1\right)^2 = \left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} - \frac{y^2}{b^2} - 1\right)$$

i.e. $$T^2 = SS_1 \quad \text{or} \quad SS_1 = T^2$$

Remark

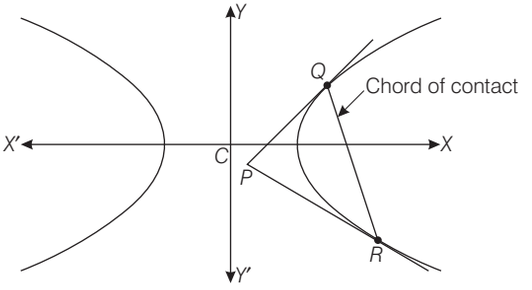
$S = 0$ is the equation of the curve, S_1 is obtained from S by replacing x by x_1 and y by y_1 and $T = 0$ is the equation of the tangent at (x_1, y_1) to $S = 0$.

Chord of Contact

Theorem : If the tangents from a point $P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ touch the hyperbola at Q and R, then the equation of the chord of contact QR is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

Proof : Let $Q \equiv (x', y')$ and $R \equiv (x'', y'')$



Now, equation of tangents PQ and PR are

$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1 \qquad \text{...(i)}$$

and $$\frac{xx''}{a^2} - \frac{yy''}{b^2} = 1 \qquad \text{...(ii)}$$

Since, Eqs. (i) and (ii) pass through $P(x_1, y_1)$, then

$$\frac{x'x_1}{a^2} - \frac{y'y_1}{b^2} = 1 \qquad \text{...(iii)}$$

and $$\frac{x''x_1}{a^2} - \frac{y''y_1}{b^2} = 1 \qquad \text{...(iv)}$$

Hence, it is clear that $Q(x', y')$ and $R(x'', y'')$ lie on

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1 \quad \text{or} \quad T = 0$$

which is **chord of contact** QR.

Equation of the Chord Bisected at a Given Point

Theorem : The equation of the chord of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

bisected at the point (x_1, y_1) is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$$

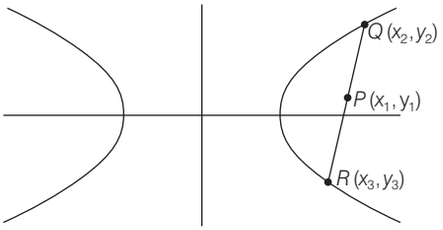
or $$T = S_1, \text{ where } T = \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$$

and $$S_1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1.$$

Proof : Since equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{...(i)}$$

Let QR be the chord of the hyperbola whose mid-point is $P(x_1, y_1)$. Since Q and R lie on the hyperbola (i).



$$\therefore \quad \frac{x_2^2}{a^2} - \frac{y_2^2}{b^2} = 1 \qquad \text{...(ii)}$$

$$\text{and} \quad \frac{x_3^2}{a^2} - \frac{y_3^2}{b^2} = 1 \qquad \text{...(iii)}$$

Subtracting Eq. (iii) from Eq. (ii),

$$\frac{1}{a^2}(x_2^2 - x_3^2) - \frac{1}{b^2}(y_2^2 - y_3^2) = 0$$

$$\Rightarrow \quad \frac{(x_2 + x_3)(x_2 - x_3)}{a^2} - \frac{(y_2 + y_3)(y_2 - y_3)}{b^2} = 0$$

$$\Rightarrow \quad \frac{y_2 - y_3}{x_2 - x_3} = \frac{b^2(x_2 + x_3)}{a^2(y_2 + y_3)}$$

$$= \frac{b^2}{a^2}\frac{(2x_1)}{(2y_1)} \qquad [P \text{ is the mid-point of } QR]$$

$$= \frac{b^2 x_1}{a^2 y_1} \qquad \text{...(iv)}$$

$\therefore$ Equation of QR is

$$y - y_1 = \frac{y_2 - y_3}{x_2 - x_3}(x - x_1)$$

$$\Rightarrow \quad y - y_1 = \frac{b^2 x_1}{a^2 y_1}(x - x_1) \qquad [\text{from Eq. (iv)}]$$

$$\Rightarrow \quad \frac{yy_1}{b^2} - \frac{y_1^2}{b^2} = \frac{xx_1}{a^2} - \frac{x_1^2}{a^2}$$

$$\Rightarrow \quad \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}$$

$$\Rightarrow \quad \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$$

$$\Rightarrow \quad T = S_1$$

Example 25 Find the locus of the mid-points of the chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ which subtend a right angle at the origin.

Sol. Let (h, k) be the mid-point of the chord of the hyperbola. Then its equation is

$$\frac{hx}{a^2} - \frac{ky}{b^2} - 1 = \frac{h^2}{a^2} - \frac{k^2}{b^2} - 1$$

$$\text{or} \quad \frac{hx}{a^2} - \frac{ky}{b^2} = \frac{h^2}{a^2} - \frac{k^2}{b^2} \qquad \text{...(i)}$$

The equation of the lines joining the origin to the points of intersection of the hyperbola and the chord (i) is obtained by making homogeneous hyperbola with the help of Eq. (i)

$$\therefore \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{\left(\frac{hx}{a^2} - \frac{ky}{b^2}\right)^2}{\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2}$$

$$\Rightarrow \quad \frac{1}{a^2}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2 x^2 - \frac{1}{b^2}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2 y^2$$

$$= \frac{h^2}{a^4}x^2 + \frac{k^2}{b^4}y^2 - \frac{2hk}{a^2b^2}xy \qquad \text{...(ii)}$$

The lines represented by Eq. (ii) will be at right angle if

Coefficient of x^2 + Coefficient of $y^2 = 0$

$$\Rightarrow \quad \frac{1}{a^2}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2 - \frac{h^2}{a^4} - \frac{1}{b^2}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2 - \frac{k^2}{b^4} = 0$$

$$\Rightarrow \quad \left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2 \left(\frac{1}{a^2} - \frac{1}{b^2}\right) = \frac{h^2}{a^4} + \frac{k^2}{b^4}$$

Hence, the locus of (h, k) is

$$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2 \left(\frac{1}{a^2} - \frac{1}{b^2}\right) = \frac{x^2}{a^4} + \frac{y^2}{b^4}$$

Example 26 From the points on the circle $x^2 + y^2 = a^2$, tangents are drawn to the hyperbola $x^2 - y^2 = a^2$; prove that the locus of the middle-points of the chords of contact is the curve $(x^2 - y^2)^2 = a^2(x^2 + y^2)$.

Sol. Since any point on the circle $x^2 + y^2 = a^2$ is $(a\cos\theta, a\sin\theta)$ chord of contact of this point w.r.t. hyperbola $x^2 - y^2 = a^2$ is

$$x(a\cos\theta) - y(a\sin\theta) = a^2$$

$$\text{or} \quad x\cos\theta - y\sin\theta = a \qquad \text{...(i)}$$

If its mid-point be (h, k), then it is same as

$$T = S_1$$

$$\text{i.e.} \quad hx - ky - a^2 = h^2 - k^2 - a^2$$

$$\text{or} \quad hx - ky = h^2 - k^2 \qquad \text{...(ii)}$$

On comparing Eqs. (i) and (ii), we get

$$\frac{\cos\theta}{h} = \frac{\sin\theta}{k} = \frac{a}{(h^2 - k^2)}$$

or $(h^2 - k^2)\cos\theta = ah$ ...(iii)

and $(h^2 - k^2)\sin\theta = ak$ ...(iv)

Squaring and adding Eqs. (iii) and (iv), we get

$$(h^2 - k^2)^2 = a^2h^2 + a^2k^2$$

$$\Rightarrow \quad (h^2 - k^2)^2 = a^2(h^2 + k^2)$$

Hence, the required locus is

$$(x^2 - y^2)^2 = a^2(x^2 + y^2).$$

Example 27 Prove that the locus of the middle-points of the chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ which pass through a fixed point (α, β) is a hyperbola whose centre is $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$.

Sol. Let the mid-point of the chord be (h, k). The equation of the chord whose mid-point is (h, k) is

$$\frac{hx}{a^2} - \frac{ky}{b^2} - 1 = \frac{h^2}{a^2} - \frac{k^2}{b^2} - 1 \text{ or } \frac{hx}{a^2} - \frac{ky}{b^2} = \frac{h^2}{a^2} - \frac{k^2}{b^2}$$

It passes through (α, β), then

$$\frac{\alpha h}{a^2} - \frac{\beta k}{b^2} = \frac{h^2}{a^2} - \frac{k^2}{b^2}$$

$\therefore$ locus of (h, k) is

$$\frac{x\alpha}{a^2} - \frac{y\beta}{b^2} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

$$\Rightarrow \quad \frac{1}{a^2}(x^2 - \alpha x) - \frac{1}{b^2}(y^2 - \beta y) = 0$$

$$\Rightarrow \quad \frac{1}{a^2}\left\{\left(x - \frac{\alpha}{2}\right)^2 - \frac{\alpha^2}{4}\right\} - \frac{1}{b^2}\left\{\left(y - \frac{\beta}{2}\right)^2 - \frac{\beta^2}{4}\right\} = 0$$

$$\text{or} \quad \frac{\left(x - \frac{\alpha}{2}\right)^2}{a^2} - \frac{\left(y - \frac{\beta}{2}\right)^2}{b^2} = \frac{1}{4}\left\{\frac{\alpha^2}{a^2} - \frac{\beta^2}{b^2}\right\} = \lambda \text{ (say)}$$

$$\therefore \quad \frac{\left(x - \frac{\alpha}{2}\right)^2}{a^2\lambda} - \frac{\left(y - \frac{\beta}{2}\right)^2}{b^2\lambda} = 1$$

The centre of this hyperbola is $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$.

Exercise for Session 2

1. The tangents from $(1, 2\sqrt{2})$ to the hyperbola $16x^2 - 25y^2 = 400$ include between them an angle equal to

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

2. If $4x^2 + \lambda y^2 = 45$ and $x^2 - 4y^2 = 5$ cut orthogonally, then the value of λ is

(a) $\frac{1}{9}$ (b) $\frac{1}{3}$ (c) 9 (d) 18

3. If the tangent at the point $(2\sec\phi, 3\tan\phi)$ of the hyperbola $\frac{x^2}{4} - \frac{y^2}{9} = 1$ is parallel to $3x - y + 4 = 0$, then the value of ϕ is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{5\pi}{12}$

4. If the line $2x + \sqrt{6}y = 2$ touches the hyperbola $x^2 - 2y^2 = 4$, then the point of contact is

(a) $(-2, \sqrt{6})$ (b) $(-5, 2\sqrt{6})$ (c) $\left(\frac{1}{2}, \frac{1}{\sqrt{6}}\right)$ (d) $(4, -\sqrt{6})$

5. The equation of the chord of hyperbola $25x^2 - 16y^2 = 400$, whose mid-point is (5, 3), is

(a) $115x - 47y = 434$ (b) $125x - 48y = 481$

(c) $127x - 49y = 488$ (d) $155x - 67y = 574$

6. The value of m for which $y = mx + 6$ is a tangent to the hyperbola $\frac{x^2}{100} - \frac{y^2}{49} = 1$ is

(a) $\sqrt{\left(\frac{17}{20}\right)}$ (b) $-\sqrt{\left(\frac{17}{21}\right)}$ (c) $\sqrt{\left(\frac{20}{17}\right)}$ (d) $-\sqrt{\left(\frac{21}{17}\right)}$

7. P is a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, N is the foot of the perpendicular from P on the transverse axis. The tangent to the hyperbola at P meets the transverse axis at T. If O is the centre of the hyperbola, then $OT \cdot ON$ is equal to

(a) a^2 (b) b^2 (c) e^2 (d) b^2/a

8. If $x = 9$ is the chord of contact of the hyperbola $x^2 - y^2 = 9$, then the equation of the corresponding pair of tangents, is

(a) $9x^2 - 8y^2 + 18x - 9 = 0$ (b) $9x^2 - 8y^2 - 18x + 9 = 0$

(c) $9x^2 - 8y^2 - 18x - 9 = 0$ (d) $9x^2 - 8y^2 + 18x + 9 = 0$

9. Let $P(a\sec\theta, b\tan\theta)$ and $Q(a\sec\phi, b\tan\phi)$, when $\theta + \phi = \frac{\pi}{2}$, be two points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If (h,k) is the point of intersection of the normals at P and Q, then k is equal to

(a) $\left(\frac{a^2 + b^2}{a}\right)$ (b) $-\left(\frac{a^2 + b^2}{a}\right)$ (c) $\left(\frac{a^2 + b^2}{b}\right)$ (d) $-\left(\frac{a^2 + b^2}{b}\right)$

10. The tangent at a point P on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ passes through the point $(0, -b)$ and the normal at P passes through $(2a\sqrt{2}, 0)$; then eccentricity of the hyperbola is

(a) $\frac{5}{4}$ (b) $\frac{3}{2}$ (c) $\sqrt{2}$ (d) $2\sqrt{2}$

11. A tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ cuts the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in points P and Q. Find the locus of the mid-point of PQ.

12. A line through the origin meets the circle $x^2 + y^2 = a^2$ at P and the hyperbola $x^2 - y^2 = a^2$ at Q. Prove that the locus of the point of intersection of tangent at P to the circle with the tangent at Q to the hyperbola is the curve.

13. Normals are drawn to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the points $P(a\sec\theta_1, b\tan\theta_1)$ and $Q(a\sec\theta_2, b\tan\theta_2)$ meeting the conjugate axis at G_1 and G_2 respectively. If $\theta_1 + \theta_2 = \pi/2$, prove that

$$CG_1 \cdot CG_2 = \frac{a^2e^4}{(e^2 - 1)}$$

where C is the centre of the hyperbola and e is its eccentricity.

14. Chords of the hyperbola $x^2 - y^2 = a^2$ touch the parabola $y^2 = 4ax$. Prove that the locus of their middle-points is the curve $y^2(x - a) = x^3$.

Session 3

Diameter, Conjugate Diameters, Properties of Hyperbola, Intersection of Conjugate Diameters and Hyperbola, Director Circle, Asymptotes, Rectangular Hyperbola, The Rectangular Hyperbola $xy = c^2$, Reflection Property of a Hyperbola, Equation of a Hyperbola Referred to Two Perpendicular Lines

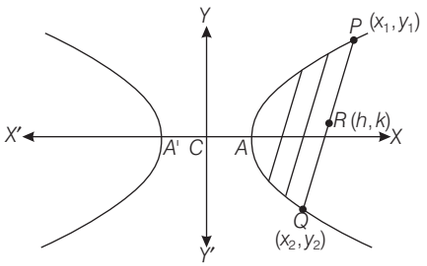
Diameter

The locus of the middle points of a system of parallel chords of a hyperbola is called a diameter and the point where the diameter intersects the hyperbola is called the vertex of the diameter.

Theorem : The equation of a diameter bisecting a system of parallel chords of slope m of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$y = \frac{b^2}{a^2 m} x$$

Proof : Let $y = mx + c$ be a system of a parallel chords to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ for different chords. As c varies, m remain constant.



Let the extremities of any chord PQ of the set be $P(x_1, y_1)$ and $Q(x_2, y_2)$ and let its middle point be $R(h, k)$. Then solving equations

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ and } y = mx + c$$

we get $$\frac{x^2}{a^2} - \frac{(mx + c)^2}{b^2} = 1$$

$$\Rightarrow \quad (a^2m^2 - b^2)\, x^2 + 2mca^2 x + a^2 (c^2 + b^2) = 0$$

Since, x_1 and x_2 be the roots of this equation, then

$$x_1 + x_2 = \frac{-2mca^2}{a^2m^2 - b^2} \quad \text{...(i)}$$

Since, (h, k) be the middle point of QR, then

$$h = \frac{x_1 + x_2}{2}$$

then from Eq. (i), $h = -\frac{mca^2}{a^2m^2 - b^2}$

but (h, k) lies on $y = mx + c$

$$\therefore \quad k = mh + c$$

or $$c = k - mh \quad \text{...(ii)}$$

From Eqs. (i) and (ii),

$$h = -\frac{ma^2 (k - mh)}{a^2m^2 - b^2}$$

$$\Rightarrow \quad a^2m^2h - b^2h = -ma^2k + m^2a^2h$$

$$\Rightarrow \quad -b^2h = -ma^2k \quad \text{or} \quad k = \frac{b^2h}{a^2m}$$

Hence, locus of $R(h, k)$ is $y = \frac{b^2 x}{a^2 m}$

which is the diameter of the hyperbola passing through (0, 0).

Aliter :

Let (h, k) be the middle-point of the chord $y = mx + c$ of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ then

$$T = S_1$$

$$\Rightarrow \quad \frac{xh}{a^2} - \frac{yk}{b^2} = \frac{h^2}{a^2} - \frac{k^2}{b^2}$$

$$\therefore \quad \text{Slope} = \frac{hb^2}{a^2k} = m$$

$$\Rightarrow \quad k = \frac{b^2h}{a^2m}$$

Hence, the locus of mid-point is $y = \frac{b^2x}{a^2m}$.

Conjugate Diameters

Two diameters are said to be conjugate when each bisects all chords parallel to the others.

If $y = mx, y = m_1x$ be conjugate diameters, then $mm_1 = \frac{b^2}{a^2}$.

Let $y = m_1x + c$ be a set of chords parallel to $y = m_1x$, then the diameter $y = \frac{b^2}{a^2m}x$ bisects them all. But being the conjugate diameter $y = mx$ also bisects them.

Hence, these two lines must be identical

$$\therefore \quad m = \frac{b^2}{a^2m_1} \quad \Rightarrow \quad mm_1 = \frac{b^2}{a^2}$$

Properties of Hyperbola

Prop. 1. *If a pair of diameters be conjugate with respect to a hyperbola, they are conjugate with respect to its conjugate hyperbola also.*

Let $y = mx$ and $y = m_1x$ be two conjugate diameters of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then

$$mm_1 = \frac{b^2}{a^2} \qquad \text{...(i)}$$

Now conjugate hyperbola of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

or

$$\frac{x^2}{-a^2} - \frac{y^2}{-b^2} = 1 \qquad \text{...(ii)}$$

If $y = m_2x$ and $y = m_3x$ are the conjugate diameters of Eq. (ii), then

$$m_2m_3 = \frac{(-b^2)}{(-a^2)}$$

$$= \frac{b^2}{a^2} = mm_1 \qquad \text{[from Eq. (i)]}$$

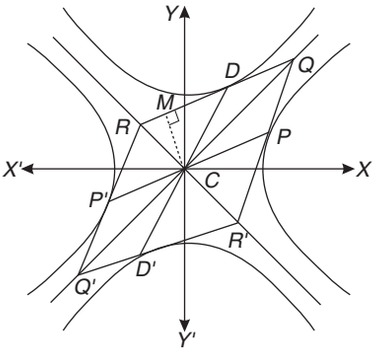
This proves the proposition.

Prop. 2. *The parallelogram formed by the tangents at the extremities of conjugate diameters of a hyperbola has its vertices lying on the asymptotes and is of constant area.*

Since, P and D lie on hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

and its conjugate diameter $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

then coordinates of P and D are $(a \sec \phi, b \tan \phi)$ and $(a \tan \phi, b \sec \phi)$ respectively.



Then $D' \equiv (-a \tan \phi, -b \sec \phi)$

and $P' \equiv (-a \sec \phi, -b \tan \phi)$

Since, equations of asymptotes CQ and CR are $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$ respectively and the equations of tangents at P, P', D and D' are

$$\frac{x}{a}\sec \phi - \frac{y}{b}\tan \phi = 1 \qquad \text{...(i)}$$

$$-\frac{x}{a}\sec \phi + \frac{y}{b}\tan \phi = 1 \qquad \text{...(ii)}$$

$$-\frac{x}{a}\tan \phi + \frac{y}{b}\sec \phi = 1 \qquad \text{...(iii)}$$

and

$$\frac{x}{a}\tan \phi - \frac{y}{b}\sec \phi = 1 \qquad \text{...(iv)}$$

respectively.

Now, the lines (i) and (ii) are parallel and so are Eqs. (iii) and (iv).

Hence these tangents form a parallelogram. Solving Eqs. (i) and (iii), we get the coordinate of Q as $[a(\sec\phi+\tan\phi), b(\sec\phi+\tan\phi)]$ which clearly lies on the asymptote $y=\frac{b}{a}x$ similarly the other points of intersection lies on the asymptotes.

The equations of PCP' and DCD' are

$$y=\frac{b}{a}\frac{\tan\phi}{\sec\phi}x \text{ or } bx\tan\phi-ay\sec\phi=0 \quad ...(v)$$

and $$y=\frac{b}{a}\frac{\sec\phi}{\tan\phi}x \text{ or } bx\sec\phi-ay\tan\phi=0 \quad ...(vi)$$

Hence by symmetry.

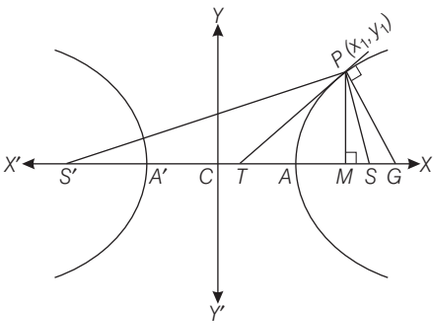
Area of parallelogram $QRQ'R'=4$ (Area of parallelogram $QDCP$)

$$=4\cdot CP\times(\text{Perpendicular length from } C \text{ on } QD)$$

$$=4\cdot\sqrt{(a^2\sec^2\phi+b^2\tan^2\phi)}\times\frac{1}{\sqrt{\frac{\tan^2\phi}{a^2}+\frac{\sec^2\phi}{b^2}}}$$

$$=4ab=\text{ Constant.}$$

Prop. 3. *If the normal at P meets the transverse axis in G, then $SG=e\cdot SP$. Prove also that the tangent and normal bisect the angle between the focal distances of P.*



Let the coordinates of P be (x_1, y_1). The equation of normal at P on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ *is*

$$\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2 \quad ... (i)$$

The normal (i) meets the X-axis i.e. $y=0$ in Eq. (i), then coordinates of G are

$$\left(\frac{(a^2+b^2)}{a^2}x_1, 0\right) \text{ or } (e^2x_1, 0)$$

$\therefore$ $CG=e^2x_1$

Now $SG=CG-CS$

$$=e^2x_1-ae=e(ex_1-a)=e\cdot SP$$

Similarly, $S'G=e\cdot S'P$

$\therefore$ $\frac{SG}{S'G}=\frac{SP}{S'P}$

This relation shows that the normal PG is the external bisector of the angle SPS'. The tangent PT being perpendicular to PG is therefore the internal bisector of the angle SPS'.

Prop. 4. *If a pair of conjugate diameters meet the hyperbola in P, P′ and its conjugate in D, D′, then the asymptotes bisect PD, PD′, P′D and P′D′.*

The coordinates of four points P, D, P', D' are $(a\sec\phi, b\tan\phi)$; $(a\tan\phi, b\sec\phi)$; $(-a\sec\phi, -b\tan\phi)$;$(-a\tan\phi, -b\sec\phi)$ respectively.

If (h, k) be the middle point of PD, then

$$h=\frac{a}{2}(\sec\phi+\tan\phi)$$

and $$k=\frac{b}{2}(\tan\phi+\sec\phi)$$

$\therefore$ $\frac{h}{k}=\frac{a}{b}$

$\therefore$ Locus of mid-point (h, k) is $y=\frac{b}{a}x$ which is equation of asymptote of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Similarly other mid-points lie on the other asymptotes.

Intersection of Conjugate Diameters and Hyperbola

To prove that of a pair of conjugate diameters of a hyperbola, only one meets the curve in real points

Let $y=mx$ ...(i)

and $y=m_1x$ ...(ii)

be a pair of conjugate diameters of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then

$$mm_1=\frac{b^2}{a^2} \quad ...(iii)$$

On solving Eq. (i) and the equation of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we get

$$\frac{x^2}{a^2}-\frac{m^2x^2}{b^2}=1$$

or $$x^2=\frac{a^2b^2}{b^2-a^2m^2} \quad ...(iv)$$

Similarly Eq. (ii) meets the hyperbola at points whose abscissa are given by

$$x^2=\frac{a^2b^2}{b^2-a^2m_1^2} \quad ...(v)$$

The two values of x given by Eq. (iv) will be real if $b^2-a^2m^2>0$ i.e. $m<\left(\frac{b}{a}\right)$

i.e. $$\frac{b^2}{a^2m_1}<\left(\frac{b}{a}\right) \qquad \left\{\because mm_1=\frac{b^2}{a^2}\right\}$$

$\Rightarrow$ $$\frac{b}{a}<m_1 \text{ i.e. if } b^2<a^2m_1^2$$

i.e. if $b^2-a^2m_1^2<0$

Then from Eq. (v) the values of x are imaginary.

Hence, if Eq. (i) meets the hyperbola in real points then Eq. (ii) meets it in imaginary points and *vice-versa.*

Remark

If CD is the conjugate diameter of a diameter CP of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where P is $(a\sec\phi, b\tan\phi)$, then equation of CP, where C is $(0,0)$ is

$$y-0=\frac{b\tan\phi-0}{a\sec\phi-0}(x-0) \Rightarrow y=\frac{b}{a}\sin\phi.x$$

Comparing this with $y=mx$

$$m=\frac{b}{a}\sin\phi$$

$\because$ $$mm'=\frac{b^2}{a^2} \Rightarrow \frac{b}{a}\sin\phi\times m'=\frac{b^2}{a^2}$$

$\therefore$ $$m'=\left(\frac{b}{a}\right)\text{cosec }\phi$$

$\therefore$ Equation of conjugate diameter is $y=m'x=(b/a)\text{cosec }\phi.x$ on solving

$$y=(b/a)\text{cosec }\phi\cdot x \quad \text{and} \quad -\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Then we get coordinates of D

i.e. $D\equiv(a\tan\phi, b\sec\phi)$

Example 28 If a pair of conjugate diameters meets the hyperbola and its conjugate in P and D respectively, then prove that $CP^2-CD^2=a^2-b^2$.

Sol. $\because$ Coordinates of P and D are $(a\sec\phi, b\tan\phi)$ and $(a\tan\phi, b\sec\phi)$ respectively.

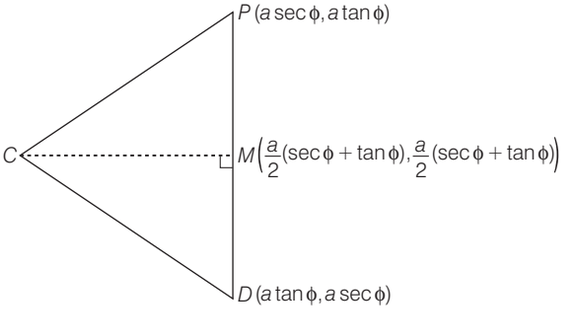
Then $(CP)^2-(CD)^2=a^2\sec^2\phi+b^2\tan^2\phi-a^2\tan^2\phi-b^2\sec^2\phi$

$=a^2(\sec^2\phi-\tan^2\phi)-b^2(\sec^2\phi-\tan^2\phi)$

$=a^2(1)-b^2(1)=a^2-b^2$

Example 29 For the hyperbola $x^2-y^2=a^2$, prove that the triangle CPD is isosceles and has constant area, where CP and CD are a pair of its conjugate diameters.

Sol. Since, CP and CD are the conjugate diameters of $x^2-y^2=a^2$, hence coordinates of P and D are $P\equiv(a\sec\phi, a\tan\phi)$, $D\equiv(a\tan\phi, a\sec\phi)$ respectively.



$\therefore$ $$CP=\sqrt{a^2(\sec^2\phi+\tan^2\phi)}=CD \qquad [\because C=0(0,0)]$$

Hence, ΔCPD is isosceles triangle.

Draw $\perp$ from C on PD, where M is the mid-point of PD

$\therefore$ $$M\equiv\left(\frac{a}{2}(\sec\phi+\tan\phi), \frac{a}{2}(\sec\phi+\tan\phi)\right)$$

$\therefore$ $$CM=\frac{|(\sec\phi+\tan\phi)|}{2}\sqrt{(a^2+a^2)}$$

and $$PD=\sqrt{[a(\sec\phi-\tan\phi)]^2+[a(\sec\phi-\tan\phi)]^2}$$

$$=|\sec\phi-\tan\phi)|\sqrt{(a^2+a^2)}$$

$\therefore$ Area of $\Delta PCD=\frac{1}{2}\cdot PD\cdot CM$

$$=\frac{1}{4}(a^2+a^2)=\frac{a^2}{2}=\text{constant}$$

Example 30 Find the condition for the lines $Ax^2+2Hxy+By^2=0$ to be conjugate diameters of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Sol. Let the lines represented by

$$Ax^2+2Hxy+By^2=0$$

are $y=mx$ and $y=m_1x$

$\therefore$ $$mm_1=\frac{A}{B} \quad ...(i)$$

But $y=mx$ and $y=m_1x$ are the conjugate diameters of

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

then $$mm_1=\frac{b^2}{a^2} \quad \ldots(ii)$$

$\therefore$ From Eqs. (i) and (ii), $\frac{A}{B}=\frac{b^2}{a^2}$

or $$a^2A=b^2B$$

which is the required condition.

Example 31 If the lines $lx+my+n=0$ passes through the extremities of a pair of conjugate diameters of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, show that $a^2l^2-b^2m^2=0$.

Sol. The extremities of a pair of conjugate diameters of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are $(a\sec\phi, b\tan\phi)$ and $(a\tan\phi, b\sec\phi)$ respectively.

According to the question, since extremities of a pair of conjugate diameters lie on $lx+my+n=0$

$\therefore$ $l(a\sec\phi)+m(b\tan\phi)+n=0$ ...(i)

$\Rightarrow$ $l(a\tan\phi)+m(b\sec\phi)+n=0$ ...(ii)

then from Eq. (i), $al\sec\phi+bm\tan\phi=-n$

or $a^2l^2\sec^2\phi+b^2m^2\tan^2\phi+2ablm\sec\phi\tan\phi=n^2$ ...(iii)

and from Eq. (ii), $al\tan\phi+bm\sec\phi=-n$

or $a^2l^2\tan^2\phi+b^2m^2\sec^2\phi+2ablm\sec\phi\tan\phi=n^2$ ...(iv)

then subtracting Eq. (iv) from Eq. (iii),

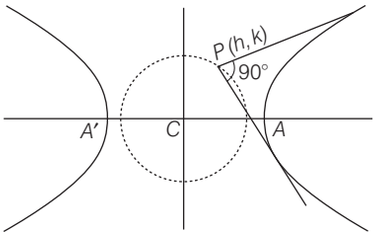
$$a^2l^2(\sec^2\phi-\tan^2\phi)+b^2m^2(\tan^2\phi-\sec^2\phi)=0$$

or $$a^2l^2-b^2m^2=0$$

Director Circle

The locus of the point of intersection of the tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, which are perpendicular to each other is called director circle.

Let any tangent in terms of slope of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is



$$y=mx+\sqrt{(a^2m^2-b^2)}$$

It passes through (h, k)

$\therefore$ $$k=mh+\sqrt{(a^2m^2-b^2)}$$

or $$(k-mh)^2=a^2m^2-b^2$$

$\Rightarrow$ $$k^2+m^2h^2-2mhk=a^2m^2-b^2$$

$\Rightarrow$ $$m^2(h^2-a^2)-2hkm+k^2+b^2=0$$

It is quadratic equation in m. Let slopes of two tangents are m_1 and m_2

$\therefore$ $$m_1m_2=\frac{k^2+b^2}{h^2-a^2}$$

$$-1=\frac{k^2+b^2}{h^2-a^2} \quad (\because \text{tangents are perpendicular})$$

$\Rightarrow$ $$-h^2+a^2=k^2+b^2$$

or $$h^2+k^2=a^2-b^2$$

Hence, locus of $P(h, k)$ is

$$x^2+y^2=a^2-b^2 \qquad (a>b)$$

Aliter :

If tangents $y=mx+\sqrt{(a^2m^2-b^2)}$ ...(i)

and $$y=-\frac{x}{m}+\sqrt{\left\{a^2\left(-\frac{1}{m}\right)^2-b^2\right\}} \quad \ldots(ii)$$

touch the hyperbola and intersects at right angles

$\therefore$ From Eq. (i),

$$y-mx=\sqrt{(a^2m^2-b^2)} \quad \ldots(iii)$$

Eq. (ii) can be rewritten as

$$x+my=\sqrt{(a^2-b^2m^2)} \quad \ldots(iv)$$

Squaring and adding Eqs. (iii) and (iv), then

$$(y-mx)^2+(x+my)^2=a^2m^2-b^2+a^2-b^2m^2$$

$$(1+m^2)(x^2+y^2)=(1+m^2)(a^2-b^2)$$

Hence, $x^2+y^2=a^2-b^2$ is the **director circle** of the hyperbola.

Remarks

1. For director circle of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, a must be greater than b.
   If $a<b$, then director circle $x^2+y^2=a^2-b^2$ does not exist.
2. The equation of director circle of $-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is
   $x^2+y^2=b^2-a^2\,(b>a)$,
   If $b<a$, then director circle does not exist.

Example 32 If any tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with centre C, meets its director circle in P and Q, show that CP and CQ are conjugate semi-diameters of the hyperbola.

Sol. Since, equation of hyperbola is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad ...(i)$$

Equation of tangent at $(a\sec\phi, b\tan\phi)$ on Eq. (i) is

$$\frac{x}{a}\sec\phi-\frac{y}{b}\tan\phi=1 \qquad ...(ii)$$

Equation of director circle of Eq. (i) is

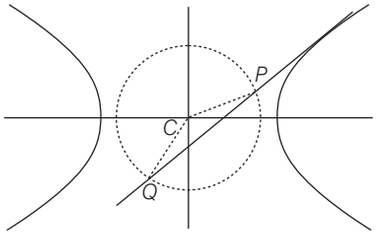
$$x^2+y^2=a^2-b^2 \qquad ...(iii)$$

Equation of lines joining the point of intersection of Eqs. (i) and (ii) to the origin is obtained by making Eq. (iii) homogeneous with the help of Eq. (ii).

$$\therefore \quad x^2+y^2=(a^2-b^2)\left(\frac{x}{a}\sec\phi-\frac{y}{b}\tan\phi\right)^2$$

or $$x^2\left(1-\frac{(a^2-b^2)\sec^2\phi}{a^2}\right)+y^2\left(1-\frac{(a^2-b^2)\tan^2\phi}{b^2}\right)+\frac{2(a^2-b^2)\sec\phi\tan\phi}{ab}xy=0 \qquad ...(iv)$$

Let m and m' represent the slopes of the lines given by Eq. (iv), then



$$mm'=\frac{1-\frac{(a^2-b^2)\sec^2\phi}{a^2}}{1-\frac{(a^2-b^2)\tan^2\phi}{b^2}}=\frac{b^2}{a^2}\cdot\frac{a^2-(a^2-b^2)\sec^2\phi}{b^2-(a^2-b^2)\tan^2\phi}$$

$$=\frac{b^2}{a^2}\cdot\frac{b^2\sec^2\phi-a^2\tan^2\phi}{b^2\sec^2\phi-a^2\tan^2\phi}=\frac{b^2}{a^2}$$

Hence, the lines CP and CQ are conjugate semi-diameters of the hyperbola.

Asymptotes

An asymptotes of any hyperbola or a curve is a straight line which touches in it two points at infinity.

Asymptotes of hyperbola : *The equations of two asymptotes of the hyperbola* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ *are* $y=\pm\frac{b}{a}x$ *or* $\frac{x}{a}\pm\frac{y}{b}=0$.

Proof : Let $y=mx+c$ be an asymptote of the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad ...(i)$$

Substituting the value of y in Eq. (i),

$$\frac{x^2}{a^2}-\frac{(mx+c)^2}{b^2}=1$$

or $$(a^2m^2-b^2)x^2+2a^2mcx+a^2(b^2+c^2)=0 \qquad ...(ii)$$

If the line $y=mx+c$ is an asymptote to the given hyperbola, then it touches the hyperbola at infinity. So both roots of Eq. (ii) must be infinite.

$$\therefore \quad a^2m^2-b^2=0$$

and $$-2a^2mc=0$$

then $$m=\pm\frac{b}{a} \quad \text{and} \quad c=0$$

Substituting the value of m and c in $y=mx+c$, we get

$$y=\pm\frac{b}{a}x \implies \frac{x}{a}\pm\frac{y}{b}=0$$

Aliter :

The difference between the second degree curve and pair of asymptotes is constant.

$\because$ Given hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$\therefore$ Pair of asymptotes is $\frac{x^2}{a^2}-\frac{y^2}{b^2}+\lambda=0$ ... (i)

Eq. (i) represents a pair of lines, then $\Delta=0$

$$\therefore \quad \frac{1}{a^2}\cdot\left(-\frac{1}{b^2}\right)\cdot\lambda+0-0-0-\lambda\cdot0=0$$

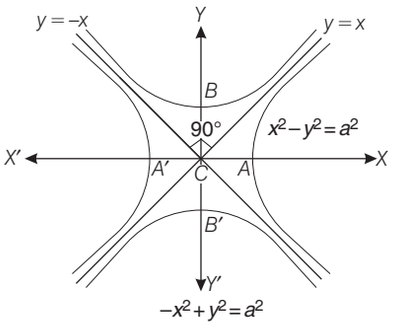
$$\therefore \quad \lambda=0$$

From Eq. (i), pair of asymptotes is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$

or $$y=\pm\frac{b}{a}x \text{ or } \frac{x}{a}\pm\frac{y}{b}=0$$

Remarks

1. If $b=a$, then $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ reduces to $x^2-y^2=a^2$. The asymptotes of rectangular hyperbola $x^2-y^2=a^2$ are $y=\pm x$ which are at right angles.
2. A hyperbola and its conjugate hyperbola have the same asymptotes.
3. The angle between the asymptotes of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $2\tan^{-1}\left(\frac{b}{a}\right)$.



4. If the angle between the asymptotes of a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is 2θ, then $e=\sec\theta$.

5. The asymptotes pass through the centre of the hyperbola.

6. The bisectors of the angles between the asymptotes are the coordinate axes.

7. Let $H\equiv\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0$

$$A\equiv\frac{x^2}{a^2}-\frac{y^2}{b^2}=0 \quad \text{and} \quad C\equiv\frac{x^2}{a^2}-\frac{y^2}{b^2}+1=0$$

be the equation of the hyperbola, asymptotes and the conjugate hyperbola respectively, then clearly $C+H=2A$.

Example 33 Find the asymptotes of the hyperbola $xy-3y-2x=0$.

Sol. Since equation of a hyperbola and its asymptotes differ in constant terms only,

$\therefore$ Pair of asymptotes is given by

$$xy-3y-2x+\lambda=0 \qquad \text{...(i)}$$

where, λ is any constant such that it represents two straight lines.

$\therefore$ $abc+2fgh-af^2-bg^2-ch^2=0$

$\Rightarrow$ $0+2\times-\frac{3}{2}\times-1\times\frac{1}{2}-0-0-\lambda\left(\frac{1}{2}\right)^2=0$

$\therefore$ $\lambda=6$

From Eq. (i), the asymptotes of given hyperbola are given by

$$xy-3y-2x+6=0$$

or $(y-2)(x-3)=0.$

$\therefore$ Asymptotes are $x-3=0$ and $y-2=0$.

Example 34 The asymptotes of a hyperbola having centre at the point $(1,2)$ are parallel to the lines $2x+3y=0$ and $3x+2y=0$. If the hyperbola passes through the point $(5,3)$, show that its equation is $(2x+3y-8)(3x+2y+7)=154$.

Sol. Let the asymptotes be $2x+3y+\lambda=0$ and $3x+2y+\mu=0$. Since asymptotes passes through $(1,2)$, then

$$\lambda=-8 \text{ and } \mu=-7$$

Thus, the equation of asymptotes are

$$2x+3y-8=0 \quad \text{and} \quad 3x+2y-7=0$$

Let the equation of hyperbola be

$$(2x+3y-8)(3x+2y-7)+v=0 \qquad \text{... (i)}$$

It passes through $(5,3)$, then

$$(10+9-8)(15+6-7)+v=0$$

$\Rightarrow$ $11\times14+v=0$

$\therefore$ $\gamma=-154$

Putting the value of v in Eq. (i), we obtain

$$(2x+3y-8)(3x+2y-7)-154=0$$

which is the equation of required hyperbola.

Example 35 Show that the tangent at any point of a hyperbola cuts off a triangle of constant area from the asymptotes and that the portion of it intercepted between the asymptotes is bisected at the point of contact.

Sol. Let $P(a\sec\phi,\ b\tan\phi)$ be any point on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad \text{... (i)}$$

Asymptotes of Eq. (i) are

$$y=\pm\frac{b}{a}x$$

Equation of tangent of Eq. (i) at $P(a\sec\phi, b\tan\phi)$ is

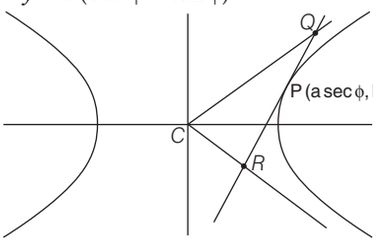
$$\frac{x}{a}\sec\phi-\frac{y}{b}\tan\phi=1 \qquad \text{... (ii)}$$

Solving Eq. (ii) and $y=\frac{b}{a}x$

We get $\frac{x}{a}(\sec\phi-\tan\phi)=(\sec^2\phi-\tan^2\phi)$

$\therefore$ $x=a(\sec\phi+\tan\phi)$

and $y=b(\sec\phi+\tan\phi)$



Let $Q\equiv[a(\sec\phi+\tan\phi), b(\sec\phi+\tan\phi)]$

Now solving Eq. (ii) and $y=-\frac{b}{a}x$

We get, $\frac{x}{a}(\sec\phi+\tan\phi)=(\sec^2\phi-\tan^2\phi)$

or $x=a(\sec\phi-\tan\phi)$

and $y=-b(\sec\phi-\tan\phi)$

Let $R\equiv[a(\sec\phi-\tan\phi), -b(\sec\phi-\tan\phi)]$

Mid-point of QR is $(a\sec\phi, b\tan\phi)$ which is coordinate of P.

Area of $\Delta CQR=\frac{1}{2}|(x_1y_2-x_2y_1)|$

$$=\frac{1}{2}|-ab-ab|=ab=\text{constant.}$$

Rectangular Hyperbola

A hyperbola whose asymptotes include a right angle is said to be rectangular hyperbola.

OR

If the lengths of transverse and conjugate axes of any hyperbola be equal, it is called **rectangular or equilateral hyperbola**.

According to the first definition

$$2\tan^{-1}\left(\frac{b}{a}\right)=\frac{\pi}{2}$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{b}{a}\right)=\frac{\pi}{4} \Rightarrow \frac{b}{a}=1$$

$$a=b$$

then, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ becomes $x^2-y^2=a^2$

According to the second definition

When $a=b, \frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ becomes

$$x^2-y^2=a^2$$

$$\text{Eccentricity, } e=\sqrt{\left(1+\frac{b^2}{a^2}\right)}=\sqrt{2}$$

Hence, $x^2-y^2=a^2$ is the general form of the equation of the rectangular hyperbola.

Remark

All the results of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are applicable to the hyperbola $x^2-y^2=a^2$ after changing b by a.

The Rectangular Hyperbola $xy = c^2$

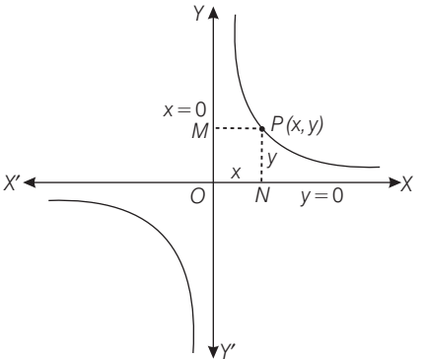
When the centre of any rectangular hyperbola be at the origin and its asymptotes coincide with the coordinate axes its, equation is $xy=c^2$.

Since, asymptotes coincides the coordinate axes.

Hence, asymptotes are $y=0$ and $x=0$.

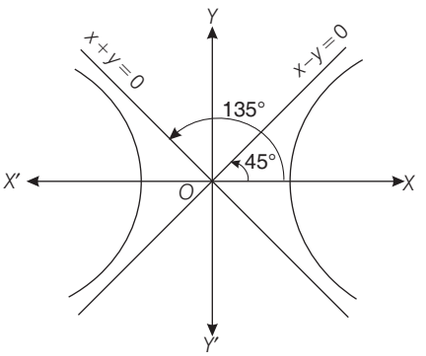
$\therefore$ Combined equation of asymptotes is $xy=0$. Now equation of any hyperbola and its asymptotes differ in respect of constant terms only.

$\therefore$ Equation of rectangular hyperbola is $xy=c^2$ where c is any constant.



Second method (By rotation of axes) :

The equation of rectangular hyperbola is $x^2-y^2=a^2$ and its asymptotes are $x-y=0$ and $x+y=0$. Since, asymptotes are inclined at 45° and 135° to the X-axis respectively.



If we rotate the axes through $\theta=-45°$ without changing the origin. Thus, when we replace (x, y) by

$$[x\cos(-45°)-y\sin(-45°), x\sin(-45)° + y\cos(-45°)]$$

i.e. $\left(\frac{x+y}{\sqrt{2}}, \frac{-x+y}{\sqrt{2}}\right)$

then equation $x^2-y^2=a^2$ reduces to

$$\left(\frac{x+y}{\sqrt{2}}\right)^2-\left(\frac{-x+y}{\sqrt{2}}\right)^2=a^2$$

or $$\frac{1}{2}\{(x+y)^2-(-x+y)^2\}=a^2$$

$$\Rightarrow \quad \frac{1}{2}(2y)(2x)=a^2$$

or $$xy=\frac{a^2}{2}=\left(\frac{a}{\sqrt{2}}\right)^2=c^2 \quad \text{(say)}$$

or $$xy=c^2$$

Study of Hyperbola $xy = c^2$

(i) Vertices : $A(c,c)$ and $A'(-c,-c)$.

(ii) Transverse axis : $y = x$.

(iii) Conjugate axis : $y = -x$.

(iv) Foci : $S(c\sqrt{2}, c\sqrt{2})$ and $S'(-c\sqrt{2}, -c\sqrt{2})$.

(v) Length of latusrectum = Length of $AA' = 2\sqrt{2}\,c$.

(vi) Equation of auxiliary circle : $x^2 + y^2 = 2c^2$.

(vii) Equation of director circle $x^2 + y^2 = 0$.

(viii) Asymptotes : $x = 0, y = 0$.

Remarks

1. The equations of the asymptotes and the conjugate hyperbola of the rectangular hyperbola $xy = c^2$, where the axes are the asymptotes, are $xy = 0$ and $xy = -c^2$ respectively.
2. The equation of a rectangular hyperbola having coordinate axes as its asymptotes is $xy = c^2$. If the asymptotes of a rectangular hyperbola are $x = \alpha$, $y = \beta$ then its equation is $(x - \alpha)(y - \beta) = c^2$ or $xy - \alpha y - \beta x + \lambda = 0$.
3. Since $x = ct$, $y = \frac{c}{t}$ satisfies $xy = c^2$

   $\therefore (x, y) = \left(ct, \frac{c}{t}\right) (t \neq 0)$ is called a 't' point on the rectangular hyperbola. The set $\left\{x = ct, y = \frac{c}{t}\right\}$ represents its parametric equations with parameter 't'.

Properties of Rectangular Hyperbola $xy = c^2$

(i) Equation of the chord joining 't_1' and 't_2' is $x + yt_1t_2 - c(t_1 + t_2) = 0$.

(ii) Equation of tangent at (x_1, y_1) is $xy_1 + x_1y = 2c^2$.

(iii) Equation of tangent at 't' is $\frac{x}{t} + yt = 2c$.

(iv) Point of intersection of tangents at 't_1' and 't_2' is $\left(\frac{2ct_1t_2}{t_1 + t_2}, \frac{2c}{t_1 + t_2}\right)$.

(v) Equation of normal at (x_1, y_1) is $xx_1 - yy_1 = x_1^2 - y_1^2$.

(vi) Equation of normal at 't' is $xt^3 - yt - ct^4 + c = 0$.

(vii) Point of intersection of normals at 't_1' and 't_2' is.

$$\left(\frac{c\{t_1t_2(t_1^2 + t_1t_2 + t_2^2) - 1\}}{t_1t_2(t_1 + t_2)}, \frac{c\{t_1^3t_2^3 + (t_1^2 + t_1t_2 + t_2^2)\}}{t_1t_2(t_1 + t_2)}\right).$$

Example 36 If the normal at the point 't_1' to the rectangular hyperbola $xy = c^2$ meets it again at the point 't_2', prove that $t_1^3t_2 = -1$.

Sol. Since, the equation of normal at $\left(ct_1, \frac{c}{t_1}\right)$ to the hyperbola $xy = c^2$ is

$$xt_1^3 - yt_1 - ct_1^4 + c = 0$$

but this passes through $\left(ct_2, \frac{c}{t_2}\right)$, then

$$ct_2t_1^3 - \frac{c}{t_2}t_1 - ct_1^4 + c = 0$$

$$\Rightarrow \quad t_2^2t_1^3 - t_1 - t_1^4t_2 + t_2 = 0$$

$$\Rightarrow \quad t_2t_1^3(t_2 - t_1) + (t_2 - t_1) = 0$$

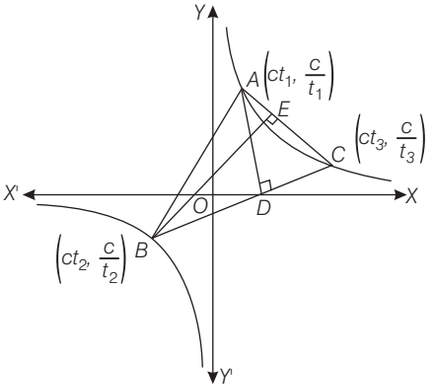
$$\Rightarrow \quad (t_1^3t_2 + 1)(t_2 - t_1) = 0$$

$$\Rightarrow \quad t_1^3t_2 + 1 = 0 \qquad [\because t_2 \neq t_1]$$

$$\therefore \quad t_1^3t_2 = -1$$

Example 37 A triangle has its vertices on a rectangular hyperbola. Prove that the orthocentre of the triangle also lies on the same hyperbola.

Sol. Let "t_1", "t_2" and "t_3" are the vertices of the triangle ABC, described on the rectangular hyperbola $xy = c^2$.



$\therefore$ Coordinates of A, B and C are $\left(ct_1, \frac{c}{t_1}\right)$, $\left(ct_2, \frac{c}{t_2}\right)$ and $\left(ct_3, \frac{c}{t_3}\right)$ respectively.

Now, slope of BC is $\frac{\frac{c}{t_3} - \frac{c}{t_2}}{ct_3 - ct_2} = -\frac{1}{t_2t_3}$

$\therefore$ Slope of AD is t_2t_3

Equation of altitude AD is

$$y - \frac{c}{t_1} = t_2t_3(x - ct_1)$$

or $t_1y - c = x\ t_1t_2t_3 - ct_1^2t_2t_3$ ...(i)

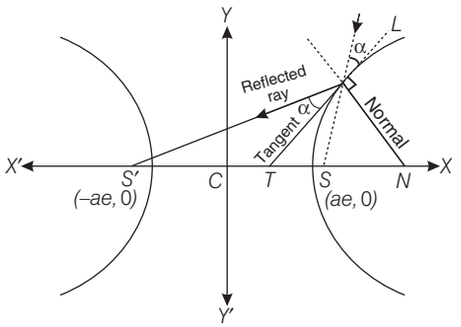
Similarly equation of altitude BE is

$$t_2y - c = x\ t_1t_2t_3 - c\ t_1t_2^2t_3 \quad ...(ii)$$

Solving Eqs. (i) and (ii), we get the orthocentre $\left(-\frac{c}{t_1t_2t_3}, -c\ t_1t_2t_3\right)$ which lies on $xy = c^2$.

Reflection Property of a Hyperbola

If an incoming light ray passing through one focus S strike convex side of the hyperbola then it will get reflected towards other focus S'.

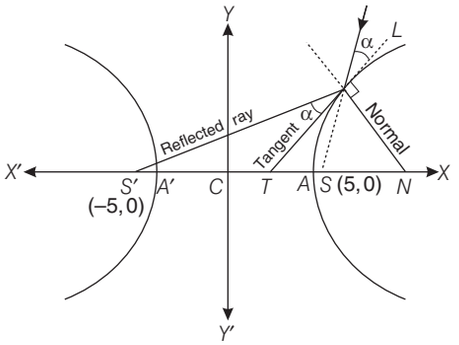


Example 38 A ray emanating from the point $(5, 0)$ is incident on the hyperbola $9x^2 - 16y^2 = 144$ at the point P with abscissa 8. Find the equation of the reflected ray after first reflection and point P lies in first quadrant.

Sol. Given, hyperbola is $9x^2 - 16y^2 = 144$. This equation can be rewritten as

$$\frac{x^2}{16} - \frac{y^2}{9} = 1 \quad ...(i)$$

Coordinates of foci are $(0, \pm\sqrt{16+9})$ i.e., $(0, \pm 5)$



Since x-coordinate of P is 8. Let y-coordinate of P is α.

$\therefore$ $(8, \alpha)$ lies on Eq. (i)

$\therefore$ $\frac{64}{16} - \frac{\alpha^2}{9} = 1$

$\Rightarrow$ $\alpha^2 = 27$

$\Rightarrow$ $\alpha = 3\sqrt{3}$ ($\because$ P lies in first quadrant)

Hence coordinate of point P is $(8, 3\sqrt{3})$.

$\because$ Equation of reflected ray passing through $P(8, 3\sqrt{3})$ and $S'(-5, 0)$.

$\therefore$ Its equation is $y - 3\sqrt{3} = \frac{0 - 3\sqrt{3}}{-5 - 8}(x - 8)$

or $13y - 39\sqrt{3} = 3\sqrt{3}x - 24\sqrt{3}$

or $3\sqrt{3}x - 13y + 15\sqrt{3} = 0.$

Equation of a Hyperbola Referred to Two Perpendicular Lines

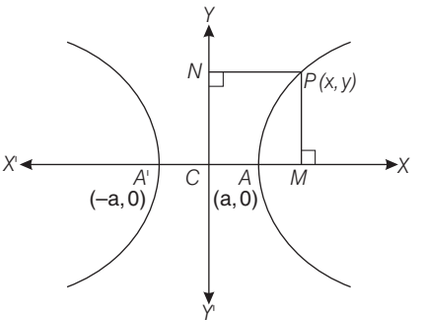
Let $P(x, y)$ be any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,

then

$$PM = y \text{ and } PN = x$$

$\therefore$ $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\Rightarrow$ $\frac{(PN)^2}{a^2} - \frac{(PM)^2}{b^2} = 1$

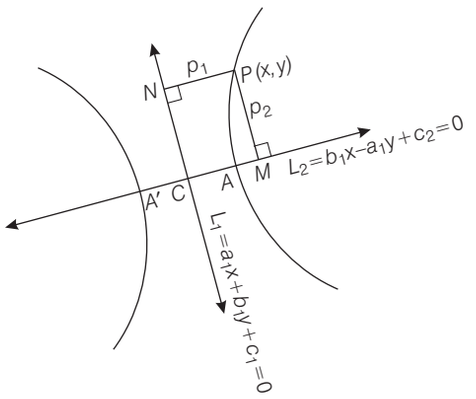


If perpendicular lines represented by

$$L_1 \equiv a_1x + b_1y + c_1 = 0$$

and $L_2 \equiv b_1x - a_1y + c_2 = 0$

then $PN = p_1 = \left(\frac{a_1x + b_1y + c_1}{\sqrt{(a_1^2 + b_1^2)}}\right)$



and $$PM = p_2 = \left(\frac{b_1 x - a_1 y + c_2}{\sqrt{(b_1^2 + a_1^2)}} \right)$$

Now equation of hyperbola becomes

$$\frac{\left(\frac{a_1 x + b_1 y + c_1}{\sqrt{(a_1^2 + b_1^2)}} \right)^2}{a^2} - \frac{\left(\frac{b_1 x - a_1 y + c_2}{\sqrt{(b_1^2 + a_1^2)}} \right)^2}{b^2} = 1$$

Then the point P describes a hyperbola in the plane of the given lines such that :

(i) Centre : The centre of the hyperbola is the point of intersection of the lines $L_1 = 0$ and $L_2 = 0$.

(ii) Transverse axis : The transverse axis lies along $L_2 = 0$.

Conjugate axis : The conjugate axis lies along $L_1 = 0$.

(iii) Length of transverse and conjugate axes : The length of transverse and conjugate axes are $2a$ and $2b$ respectively.

(iv) Foci : The foci of the hyperbola is the point of intersection of the lines

$$\frac{a_1 x + b_1 y + c_1}{\sqrt{(a_1^2 + b_1^2)}} = \pm ae \text{ and } L_2 = 0.$$

(v) Directrix : The directrices of the hyperbola are

$$\frac{a_1 x + b_1 y + c_1}{\sqrt{(a_1^2 + b_1^2)}} = \pm \frac{a}{e}$$

(vi) Latusrectum : The latusrectum of the hyperbola is $\frac{2b^2}{a}$.

Example 39 The equations of the transverse and conjugate axes of a hyperbola are respectively $3x + 4y - 7 = 0$, $4x - 3y + 8 = 0$ and their respective lengths are 4 and 6. Find the equation of the hyperbola.

Sol. The equation of the required hyperbola is

$$\frac{\left(\frac{3x + 4y - 7}{\sqrt{(3^2 + 4^2)}} \right)^2}{\left(\frac{4}{2} \right)^2} - \frac{\left(\frac{4x - 3y + 8}{\sqrt{4^2 + (-3)^2}} \right)^2}{\left(\frac{6}{2} \right)^2} = 1$$

$$\Rightarrow \quad \frac{1}{100}(3x + 4y - 7)^2 - \frac{1}{225}(4x - 3y + 8)^2 = 1$$

$$\Rightarrow \quad 9(3x + 4y - 7)^2 - 4(4x - 3y + 8)^2 = 900$$

$$\Rightarrow \quad 9(9x^2 + 16y^2 + 49 + 24xy - 42x - 56y) - 4(16x^2 + 9y^2 + 64 - 24xy + 64x - 48y) = 900$$

$$\Rightarrow \quad 17x^2 + 312xy + 108y^2 - 634x - 312y - 715 = 0.$$

Exercise for Session 3

1. The diameter of $16x^2 - 9y^2 = 144$ which is conjugate to $x = 2y$ is

(a) $y = \frac{16}{9}x$ (b) $y = \frac{32}{9}x$ (c) $x = \frac{16}{9}y$ (d) $x = \frac{32}{9}y$

2. Tangents drawn from a point on the circle $x^2 + y^2 = 9$ to the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$, then tangents are at angle

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

3. If $H \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$, $C \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} + 1 = 0$ and $A \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ then H, A and C are in

(a) AP (b) GP (c) HP (d) AGP

4. The angle between the asymptotes of $\frac{x^2}{4} - \frac{y^2}{9} = 1$ is equal to

(a) $\tan^{-1}\left(\frac{2}{3}\right)$ (b) $\tan^{-1}\left(\frac{3}{2}\right)$ (c) $2\tan^{-1}\left(\frac{2}{3}\right)$ (d) $2\tan^{-1}\left(\frac{3}{2}\right)$

5. If e and e_1 are the eccentricities of the hyperbolas $xy = c^2$ and $x^2 - y^2 = a^2$, then $(e + e_1)^2$ is equal to

(a) 2 (b) 4 (c) 6 (d) 8

6. The product of the lengths of perpendiculars drawn from any point on the hyperbola $\frac{x^2}{2} - y^2 = 1$ to its asymptotes is

(a) $\frac{1}{2}$ (b) 2 (c) $\frac{2}{3}$ (d) $\frac{3}{2}$

7. The number of points on hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 3$ from which mutually perpendicular tangents can be drawn to the circle $x^2 + y^2 = a^2$ is/are

(a) 0 (b) 2 (c) 3 (d) 4

8. If the sum of the slopes of the normal from a point P to the hyperbola $xy = c^2$ is equal to $\lambda (\lambda \in R^+)$, then the locus of point P is

(a) $x^2 = \lambda c^2$ (b) $y^2 = \lambda c^2$ (c) $xy = \lambda c^2$ (d) None of these

9. If $S \equiv x^2 + 4xy + 3y^2 - 4x + 2y + 1 = 0$, then the value of λ for which $S + \lambda = 0$ represents its asymptotes is

(a) 20 (b) 18 (c) -16 (d) -22

10. A ray emanating from the point $(-\sqrt{41}, 0)$ is incident on the hyperbola $16x^2 - 25y^2 = 400$ at the point P with abscissa 10. Then the equation of the reflected ray after first reflection and point P lies in second quadrant is

(a) $4\sqrt{3}x - (10 - \sqrt{41})\, y + 4\sqrt{123} = 0$ (b) $4\sqrt{3}x + (10 - \sqrt{41})\, y - 4\sqrt{123} = 0$

(c) $4\sqrt{3}x + (10 - \sqrt{41})\, y + 4\sqrt{123} = 0$ (d) $4\sqrt{3}x - (10 - \sqrt{41})\, y - 4\sqrt{123} = 0$

11. A ray of light incident along the line $3x + (5 - 4\sqrt{2})y = 15$ gets reflected from the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$, then its reflected ray goes along the line

(a) $x\sqrt{2} - y + 5 = 0$ (b) $y\sqrt{2} - x + 5 = 0$ (c) $y\sqrt{2} - x - 5 = 0$ (d) None of these

12. The equations of the transverse and conjugate axes of a hyperbola are $x + 2y - 3 = 0$ and $2x - y + 4 = 0$ respectively and their respective lengths are $\sqrt{2}$ and $\frac{2}{\sqrt{3}}$. The equation of the hyperbola is

(a) $2(x + 2y - 3)^2 - 3(2x - y + 4)^2 = 5$ (b) $2(2x - y + 4)^2 - 3(x + 2y - 3)^2 = 5$

(c) $2(x + 2y - 3)^2 - 3(2x - y + 4)^2 = 1$ (d) $2(2x - y + 4)^2 - 3(x + 2y - 3)^2 = 1$

13. Find the equation of that diameter which bisects the chord $7x + y - 2 = 0$ of the hyperbola $\frac{x^2}{3} - \frac{y^2}{7} = 1$.

14. Find the equation of the hyperbola which has $3x - 4y + 7 = 0$ and $4x + 3y + 1 = 0$ for its asymptotes and which passes through the origin.

15. The asymptotes of a hyperbola are parallel to lines $2x + 3y = 0$ and $3x + 2y = 0$. The hyperbola has its centre at $(1, 2)$ and it passes through $(5, 3)$, find its equation.

16. If the pair of straight lines $Ax^2 + 2Hxy + By^2 = 0$ be conjugate diameters of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then prove that $Aa^2 = Bb^2$.

17. A circle cuts the rectangular hyperbola $xy = 1$ in points $(x_r, y_r), r = 1, 2, 3, 4$ then prove that $x_1x_2x_3x_4 = y_1y_2y_3y_4 = 1$.

Shortcuts and Important Results to Remember

1 If P be any point and F_1 and F_2 are any other two points then :
(a) If $|PF_1 - PF_2| < |F_1F_2|$, then the locus of P is a hyperbola.
(b) If $|PF_1 - PF_2| = |F_1F_2|$, then the locus of P is a straight line.
(c) If $|PF_1 - PF_2| > |F_1F_2|$, then the locus of P is an empty set.

2 The orthocentre of triangle inscribed in the hyperbola $xy = c^2$ lies on it.

3 Length of the chord of the rectangular hyperbola $xy = c^2$ whose middle-point is (h, k) is $2\sqrt{\left(\frac{(h^2 + k^2)(hk - c^2)}{hk}\right)}$.

4 The product of length of perpendicular drawn from any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ to its asymptotes is $\frac{a^2b^2}{(a^2 + b^2)}$.

5 Asymptotes are the tangents from the centre of a hyperbola.

6 If the angle between the asymptotes is 2α, then eccentricity of the hyperbola is $\sec\alpha$.

7 If the tangent and normal to a rectangular hyperbola $xy = c^2$ at a point cuts off intercepts a_1 and a_2 on one axis and b_1, b_2 on the other axis, then $a_1a_2 + b_1b_2 = 0$.

8 The equation of common tangents to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are $y = \pm x \pm \sqrt{(a^2 - b^2)}$.

9 The director circle of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ will be imaginary if $a < b$ and will become a circle, if $a > b$ (for $a = b$, point circle).

10 The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{a^2 - k^2} - \frac{y^2}{k^2 - b^2} = 1$ $(a > k > b)$ are confocal and therefore orthogonal.

11 If four normals can be drawn to a hyperbola from any point and if $\alpha, \beta, \gamma, \delta$ be eccentric angles of these four co-normal points, then $\alpha + \beta + \gamma + \delta$ = odd multiple of π.

12 If α, β, γ are the eccentric angles of three points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the normals at which are concurrent, then $\sin(\alpha + \beta) + \sin(\beta + \gamma) + \sin(\gamma + \alpha) = 0$.

13 The locus of the foot of the perpendiculars drawn from the focus of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ upon any tangent is its auxiliary circle i.e. $x^2 + y^2 = a^2$ and product of the perpendiculars is b^2.

14 The portion of tangent between the point of the contact and the directrix subtends a right angle at the corresponding focus.

15 The equation of the pair of asymptotes differ the hyperbola and the conjugate hyperbola by the same constant only.

16 The asymptotes pass through the centre of the hyperbola and the bisectors of the angles between the asymptotes are the axes of the hyperbola.

17 The asymptotes of a hyperbola are the diagonals of the rectangle formed by the lines drawn through the extremities of each axis parallel to the other axis.

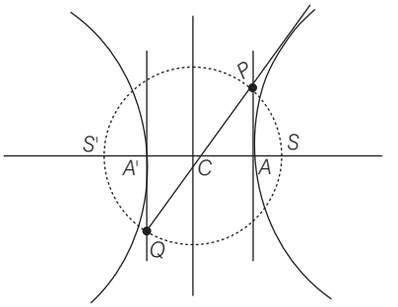
18 Perpendicular from the foci on either asymptote meet it in the same points as the corresponding directrix and the common points of intersection lie on the auxiliary circle.

19 If from any point on the asymptote a straight line be drawn perpendicular to the transverse axis, the product of the segments of this line, intercepted between the point and the curve is always equal to the square of the semi conjugate axis.

20 The tangent at any point P on a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with centre C, meets the asymptotes in Q and R and cuts off a ΔCQR of constant area equal to ab from the asymptotes and the portion of the tangent intercepted between the asymptotes is bisected at the point of the contact. This implies that locus of the centre of the circle circumscribing the ΔCQR in case of rectangular hyperbola is the hyperbola itself and for a standard hyperbola the locus would be the curve $4(a^2x^2 - b^2y^2) = (a^2 + b^2)^2$.

21 If a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ intersects a rectangular hyperbola $xy = \lambda^2$ or $x^2 - y^2 = a^2$ at four points then the Arithmetic mean of the points of intersection lies on the middle of the line joining the centres of the circle and hyperbola.

22 The points (two) in which any tangent meets the tangents at the vertices and the foci of the hyperbola are concyclic i.e. S, P, S' and Q are lie on circle.



JEE Type Solved Examples :
Single Option Correct Type Questions

▪ This section contains 10 **multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

● ***Ex. 1*** *If the eccentricity of the hyperbola* $x^2 - y^2 \sec^2\alpha = 5$ *is* $\sqrt{3}$ *times the eccentricity of the ellipse* $x^2 \sec^2\alpha + y^2 = 25$, *then a value of* α *is*

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

Sol. (b) For the hyperbola $\frac{x^2}{5} - \frac{y^2}{5\cos^2\alpha} = 1$

we have, $e_1^2 = 1 + \frac{5\cos^2\alpha}{5} = 1 + \cos^2\alpha$ ... (i)

For the ellipse $\frac{x^2}{25\cos^2\alpha} + \frac{y^2}{25} = 1$

we have, $e_2^2 = 1 - \frac{25\cos^2\alpha}{25} = 1 - \cos^2\alpha = \sin^2\alpha$ ... (ii)

Given that, $e_1 = \sqrt{3}\, e_2$ or $e_1^2 = 3e_2^2$

$\Rightarrow$ $1 + \cos^2\alpha = 3\sin^2\alpha$ [from Eqs. (i) and (ii)]

$\Rightarrow$ $2 = 4\sin^2\alpha$

or $\sin\alpha = \frac{1}{\sqrt{2}}$

$\Rightarrow$ $\alpha = \frac{\pi}{4}$

● ***Ex. 2*** *The asymptote of the hyperbola* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ *form with any tangent to the hyperbola a triangle whose area is* $a^2 \tan\lambda$ *in magnitude, then its eccentricity is*

(a) $\sec\lambda$ (b) $\operatorname{cosec}\lambda$

(c) $\sec^2\lambda$ (d) $\operatorname{cosec}^2\lambda$

Sol. (a) Any tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ forms a triangle with the asymptotes which has constant area *ab*.

Given, $ab = a^2\tan\lambda$

or $b = a\tan\lambda$

or $b^2 = a^2\tan^2\lambda$

or $a^2(e^2 - 1) = a^2\tan^2\lambda$

or $e^2 = 1 + \tan^2\lambda = \sec^2\lambda$

$\therefore$ $e = \sec\lambda$

● ***Ex. 3*** *The equation of the chord joining two points* (x_1, y_1) *and* (x_2, y_2) *on the rectangular hyperbola* $xy = c^2$ *is*

(a) $\frac{x}{x_1 + x_2} + \frac{y}{y_1 + y_2} = 1$ (b) $\frac{x}{x_1 - x_2} + \frac{y}{y_1 - y_2} = 1$

(c) $\frac{x}{y_1 + y_2} + \frac{y}{x_1 + x_2} = 1$ (d) $\frac{x}{y_1 - y_2} + \frac{y}{x_1 - x_2} = 1$

Sol. (a) The mid-point of the chord is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

$\therefore$ The equation of chord whose mid-point $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ is $T = S_1$

or $\frac{1}{2}\left(x\left(\frac{y_1 + y_2}{2}\right) + y\left(\frac{x_1 + x_2}{2}\right)\right) - c^2$

$= \left(\frac{x_1 + x_2}{2}\right)\left(\frac{y_1 + y_2}{2}\right) - c^2$

$\Rightarrow$ $x(y_1 + y_2) + y(x_1 + x_2) = (x_1 + x_2)(y_1 + y_2)$

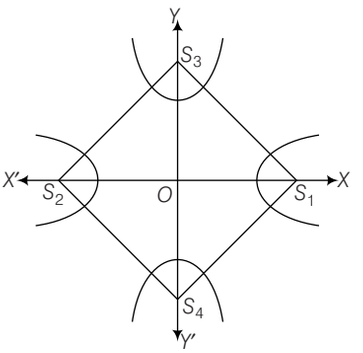
or $\frac{x}{x_1 + x_2} + \frac{y}{y_1 + y_2} = 1$

● ***Ex. 4*** *Area of quadrilateral formed with the foci of the hyperbola* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ *and* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ *is*

(a) $4(a^2 + b^2)$ (b) $2(a^2 + b^2)$

(c) $(a^2 + b^2)$ (d) $\frac{1}{2}(a^2 + b^2)$

Sol. (b) Required area $= 4 \times$ Area of $\Delta S_1 O S_3$



$= 4 \times \frac{1}{2} ae \times be_1 = (2ab)(ee_1)$ ...(i)

$\because$ $b^2 = a^2(e^2 - 1)$

$\therefore$ $e = \frac{\sqrt{(a^2 + b^2)}}{a}$ ...(ii)

and $a^2 = b^2(e_1^2 - 1)$

$$\therefore \quad e_1 = \frac{\sqrt{(a^2+b^2)}}{b} \qquad \text{...(iii)}$$

Substituting the values of e and e_1 from Eqs. (ii) and (iii) in Eq. (i), then

$$\text{Required area} = 2ab \times \frac{\sqrt{(a^2+b^2)}}{a} \times \frac{\sqrt{(a^2+b^2)}}{b}$$
$$= 2(a^2+b^2)$$

● ***Ex. 5*** *Let $P(a\sec\theta, b\tan\theta)$ and $Q(a\sec\phi, b\tan\phi)$, where $\theta+\phi=\frac{\pi}{2}$, be two points on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If (h,k) is the point of intersection of normals at P and Q, then k is equal to*

(a) $\left(\frac{a^2+b^2}{a}\right)$ (b) $-\left(\frac{a^2+b^2}{a}\right)$

(c) $\left(\frac{a^2+b^2}{b}\right)$ (d) $-\left(\frac{a^2+b^2}{b}\right)$

Sol. (d) Equations of the normals at $P(\theta)$ and $Q(\phi)$ are

$$ax\cos\theta + by\cot\theta = a^2+b^2 \qquad \text{... (i)}$$

and $$ax\cos\phi + by\cot\phi = a^2+b^2 \qquad \text{... (ii)}$$

Now, dividing by $\cos\theta$ and $\cos\phi$ in Eqs. (i) and (ii) respectively, then

$$ax + by\,\text{cosec}\,\theta = (a^2+b^2)\sec\theta \qquad \text{... (iii)}$$

and $$ax + by\,\text{cosec}\,\phi = (a^2+b^2)\sec\phi \qquad \text{... (iv)}$$

Subtracting Eq. (iv) from Eq. (iii), we get

$$y = \left(\frac{a^2+b^2}{b}\right).\left(\frac{\sec\theta-\sec\phi}{\text{cosec}\,\theta-\text{cosec}\,\phi}\right)$$

$$\therefore \quad k = y$$

$$= \left(\frac{a^2+b^2}{b}\right).\left(\frac{\sec\theta-\sec\left(\frac{\pi}{2}-\theta\right)}{\text{cosec}\,\theta-\text{cosec}\left(\frac{\pi}{2}-\theta\right)}\right) \qquad \left(\because \theta+\phi=\frac{\pi}{2}\right)$$

$$= \left(\frac{a^2+b^2}{b}\right).\left(\frac{\sec\theta-\text{cosec}\,\theta}{\text{cosec}\,\theta-\sec\theta}\right) = -\left(\frac{a^2+b^2}{b}\right)$$

● ***Ex. 6*** *Let the major axis of a standard ellipse equals the transverse axis of a standard hyperbola and their director circles have radius equal to 2R and R respectively. If e_1 and e_2 are the eccentricities of the ellipse and hyperbola, then the correct relation is*

(a) $4e_1^2 - e_2^2 = 6$ (b) $e_1^2 - 4e_2^2 = 2$

(c) $4e_2^2 - e_1^2 = 6$ (d) $e_2^2 - 4e_1^2 = 2$

Sol. (c) Let equation of ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,(a>b)$$

$\therefore$ Director circle is $x^2+y^2=a^2+b^2$

and $$b^2 = a^2(1-e_1^2)$$

given $$a^2+b^2=(2R)^2$$

$$\Rightarrow \quad a^2+a^2(1-e_1^2)=4R^2$$

$$\Rightarrow \quad 2-e_1^2=\frac{4R^2}{a^2} \qquad \text{... (i)}$$

and equation of hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b_1^2}=1$

$\therefore$ Director circle is $x^2+y^2=a^2-b_1^2$

and $$b_1^2 = a^2(e_2^2-1)$$

Given $$a^2-b_1^2=R^2$$

$$\Rightarrow \quad a^2-a^2(e_2^2-1)=R^2$$

$$\Rightarrow \quad 2-e_2^2=\frac{R^2}{a^2} \qquad \text{... (ii)}$$

Dividing Eq. (i) by Eq. (ii), then

$$\frac{2-e_1^2}{2-e_2^2}=4 \;\Rightarrow\; 4e_2^2-e_1^2=6$$

● ***Ex. 7*** *The tangent to the hyperbola $xy=c^2$ at point $P(t)$ intersects the X-axis at T and the Y-axis. at T′. The normal to the hyperbola at $P(t)$ intersects the X-axis at N and the Y-axis at N′. The areas of the triangles PNT and PN′T′ are Δ and Δ′ respectively, then $\frac{1}{\Delta}+\frac{1}{\Delta'}$ is*

(a) equal to 1 (b) depends on t

(c) depends on c (d) equal to 2

Sol. (c) Equation of tangent at $P\left(ct, \frac{c}{t}\right)$ is $\frac{x}{t}+ty=2c$

or $$\frac{x}{2ct}+\frac{y}{(2c/t)}=1$$

$$\therefore \quad T \equiv (2ct, 0), T' \equiv \left(0, \frac{2c}{t}\right)$$

and equation of normal at $P\left(ct, \frac{c}{t}\right)$ is

$$xt^3 - yt - ct^4 + c = 0$$

or $$\frac{x}{\left(\frac{c(t^4-1)}{t^3}\right)}+\frac{y}{\left(\frac{c(t^4-1)}{-t}\right)}=1$$

$$\therefore \quad N \equiv \left(\frac{c(t^4-1)}{t^3}, 0\right), N' \equiv \left(0, -\frac{c(t^4-1)}{t}\right)$$

$\because$ Area of triangle $PNT = \Delta$

$$\therefore \quad \Delta = \frac{1}{2}\left|\begin{vmatrix} ct-2ct & \frac{c}{t}-0 \\ \frac{c(t^4-1)}{t^3}-2ct & 0-0 \end{vmatrix}\right|$$

$$= \frac{1}{2}\left| -\frac{c}{t}\left(\frac{c(t^4-1)}{t^3} - 2ct \right) \right| = \frac{c^2}{2}\left(\frac{t^4+1}{t^4} \right)$$

and Area of $\Delta PN'T' = \Delta'$

$$\therefore \quad \Delta' = \frac{1}{2}\left| \begin{vmatrix} ct-0 & \frac{c}{t} - \frac{2c}{t} \\ 0-0 & -\frac{c(t^4-1)}{t} - \frac{2c}{t} \end{vmatrix} \right|$$

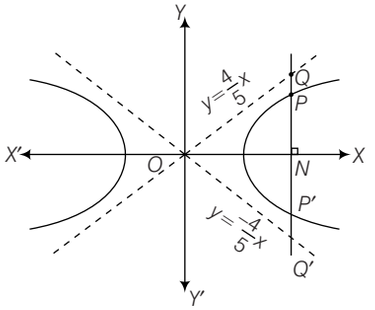
$$= \frac{1}{2}\left| ct\left(-\frac{c(t^4-1)}{t} - \frac{2c}{t} \right) \right| = \frac{c^2}{2}(t^4+1)$$

$$\therefore \quad \frac{1}{\Delta} + \frac{1}{\Delta'} = \frac{2t^4}{c^2(t^4+1)} + \frac{2}{c^2(t^4+1)} = \frac{2}{c^2}$$

● ***Ex. 8*** *Let any double ordinate PNP' of the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ be produced both sides to meet the asymptotes is Q and Q', then $(PQ)(P'Q)$ is equal to*

(a) 9 (b) 16 (c) 25 (d) 41

Sol. (b) Let $Q \equiv (x_1, y_1)$



We have $NP = \frac{4}{5}\sqrt{(x_1^2 - 25)}$

and Q is on $y = \frac{4}{5}x$

$\therefore \quad NQ = \frac{4}{5}x_1$

Now, $PQ = NQ - NP$

$$= \frac{4}{5}(x_1 - \sqrt{(x_1^2 - 25)})$$

and $P'Q = NP' + NQ = NP + NQ \quad (\because NP' = NP)$

$$= \frac{4}{5}(x_1 + \sqrt{(x_1^2 - 25)})$$

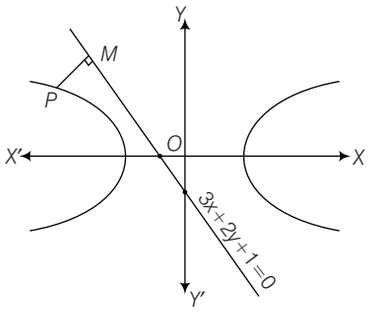
$$\therefore \quad (PQ)(P'Q) = \frac{16}{25}(x_1^2 - (x_1^2 - 25))$$

$$= \frac{16}{25} \times 25 = 16$$

● ***Ex. 9*** *The coordinates of a point on the hyperbola $\frac{x^2}{24} - \frac{y^2}{18} = 1$. Which is nearest to the line $3x + 2y + 1 = 0$ are*

(a) (6, 3) (b) (−6,−3)
(c) (6,−3) (d) (−6,3)

Sol. (d)



Let point P is nearest to the given line if the tangent at P is parallel to the given line.

Now, equation of tangent at $P(x_1, y_1)$ is

$$\frac{xx_1}{24} - \frac{yy_1}{18} = 1$$

$\therefore$ Slope of tangent at $P(x_1, y_1)$ is

$$\frac{3}{4}\frac{x_1}{y_1}$$

which must be equal to $-\frac{3}{2}$

Therefore,

$$\frac{3}{4}.\frac{x_1}{y_1} = -\frac{3}{2} \text{ or } x_1 = -2y_1 \qquad \ldots(i)$$

Also, $P(x_1, y_1)$ lies on the curve

Hence, $$\frac{x_1^2}{24} - \frac{y_1^2}{18} = 1 \qquad \ldots(ii)$$

Solving Eqs. (i) and (ii), we get two points $(6,-3)$ and $(-6,3)$ of which $(-6,3)$ is the nearest.

Aliter :

The equation of normal at $P(2\sqrt{6}\sec\theta, 3\sqrt{2}\tan\theta)$ is

$$(2\sqrt{6}\cos\theta)x + (3\sqrt{2}\cot\theta)y = 42$$

Now, this line will be perpendicular to

$$3x + 2y + 1 = 0$$

then, $(2\sqrt{6}\cos\theta) \times 3 + (3\sqrt{2}\cot\theta) \times 2 = 0$

or $$\sin\theta = -\frac{1}{\sqrt{3}}$$

$\therefore \quad \tan\theta = \frac{1}{\sqrt{2}}$ and $\sec\theta = -\frac{\sqrt{3}}{\sqrt{2}}$

Hence, $P \equiv (-6,3)$

● ***Ex. 10*** *For each positive integer n, consider the point P with abscissa n on the curve $y^2 - x^2 = 1$. If d_n represents the shortest distance from the point P to the line $y = x$, then $\lim_{n\to\infty}(n.d_n)$ has the value equal to*

(a) $\frac{1}{2\sqrt{2}}$ (b) $\frac{1}{2}$

(c) $\frac{1}{\sqrt{2}}$ (d) 0

Sol. (a) Let $P \equiv (n, \sqrt{(n^2+1)})$

$\therefore \quad d_n = \left(\frac{\sqrt{(n^2+1)} - n}{\sqrt{2}}\right)$

or $\quad n \,.\, d_n = \frac{n}{\sqrt{2}}(\sqrt{(n^2+1)} - n)$

or $\quad \lim_{n\to\infty}(n.d_n) = \frac{1}{\sqrt{2}} \lim_{n\to\infty} n(\sqrt{(n^2+1)} - n)$

$$= \frac{1}{\sqrt{2}} \lim_{n\to\infty} \frac{n(\sqrt{(n^2+1)} - n)(\sqrt{(n^2+1)} + n)}{(\sqrt{(n^2+1)} + n)}$$

$$= \frac{1}{\sqrt{2}} \lim_{n\to\infty} \frac{n(1)}{n\left(\sqrt{\left(1+\frac{1}{n^2}\right)} + 1\right)}$$

$$= \frac{1}{\sqrt{2}} \lim_{n\to\infty} \left(\frac{1}{\sqrt{\left(1+\frac{1}{n^2}\right)} + 1}\right)$$

$$= \frac{1}{\sqrt{2}} \cdot \left(\frac{1}{1+1}\right) = \frac{1}{2\sqrt{2}}$$

JEE Type Solved Examples :
More than One Correct Option Type Questions

- This section contains 5 **multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which **MORE THAN ONE** may be correct.

● ***Ex. 11*** *If two tangents can be drawn to the different branches of hyperbola* $\frac{x^2}{1} - \frac{y^2}{4} = 1$ *from the point* (α, α^2), *then*

(a) $\alpha \in (-\infty, -2)$ (b) $\alpha \in (-2, 0)$

(c) $\alpha \in (0, 2)$ (d) $\alpha \in (2, \infty)$

Sol. (a, d) $\because (\alpha, \alpha^2)$ lie on the parabola $y = x^2$

$\therefore (\alpha, \alpha^2)$ must lie between the asymptotes of hyperbola

$\frac{x^2}{1} - \frac{y^2}{4} = 1$ in I and II quadrants.

$\because$ Asymptotes of $\frac{x^2}{1} - \frac{y^2}{4} = 1$ are $y = \pm 2x$

then $\quad 2\alpha < \alpha^2$ and $-2\alpha < \alpha^2$

$\Rightarrow \quad \alpha(\alpha - 2) > 0$ and $\alpha(\alpha + 2) > 0$

$\therefore \quad \alpha < 0$ or $\alpha > 2$ and $\alpha < -2$ or $\alpha > 0$

$\therefore \quad \alpha \in (-\infty, -2) \cup (2, \infty)$

● ***Ex. 12*** *If the ellipse* $x^2 + \lambda^2 y^2 = \lambda^2 a^2; \lambda^2 > 1$ *is confocal with the hyperbola* $x^2 - y^2 = a^2$, *then*

(a) ratio of eccentricities of ellipse and hyperbola is $1:\sqrt{3}$

(b) ratio of major axis of ellipse and transverse axis of hyperbola is $\sqrt{3}:1$

(c) The ellipse and hyperbola cuts each other orthogonally

(d) ratio of length of latusrectum of ellipse and hyperbola is 1 : 3

Sol. (a, b, c) Given ellipse is $\frac{x^2}{\lambda^2 a^2} + \frac{y^2}{a^2} = 1$; $\lambda^2 a^2 > a^2$ and let e_1 and e_2 be the eccentricities of ellipse and hyperbola, then

$$a^2 = \lambda^2 a^2 (1 - e_1^2)$$

or $\quad e_1 = \sqrt{\left(1 - \frac{1}{\lambda^2}\right)}$

and $\quad e_2 = \sqrt{2}$

Now, $\quad \lambda a e_1 = a e_2$

$\Rightarrow \quad \lambda\sqrt{\left(1 - \frac{1}{\lambda^2}\right)} = \sqrt{2}$

$\Rightarrow \quad \lambda^2 - 1 = 2$ or $\lambda = \pm\sqrt{3}$

$\therefore \quad \lambda = \sqrt{3}$ $\quad (\because \lambda > 0)$

Alternate (a) : $\frac{e_1}{e_2} = \frac{1}{\lambda} = \frac{1}{\sqrt{3}}$

Alternate (b) : $= \frac{\text{Major axis of ellipse}}{\text{Transverse axis of hyperbola}}$

$$= \frac{2\lambda a}{2a} = \lambda = \sqrt{3}$$

Alternate (c) : Equations of tangents of ellipse and hyperbola at (x_1, y_1) are

$$\frac{xx_1}{\lambda^2 a^2} + \frac{yy_1}{a^2} = 1$$

and $\quad xx_1 - yy_1 = a^2$

i.e., $\quad$ slopes are $= -\frac{x_1}{\lambda^2 y_1}$ and $\frac{x_1}{y_1}$ $\quad$ (say m_1 and m_2)

According to alternate,

$$m_1 m_2 = -\frac{x_1}{\lambda^2 y_1} \times \frac{x_1}{y_1} = -\frac{x_1^2}{\lambda^2 y_1^2}$$

$$= -\frac{\frac{2a^2}{\lambda^2+1}}{\frac{a^2(\lambda^2-1)}{(\lambda^2+1)}} = -\frac{2}{\lambda^2-1} \quad \left(\because \begin{matrix} x_1^2 = \frac{2a^2\lambda^2}{\lambda^2+1} \\ \text{and } y_1^2 = \frac{a^2(\lambda^2-1)}{(\lambda^2+1)} \end{matrix}\right)$$

$$= -\frac{2}{3-1} = -1 \qquad (\because \lambda = \sqrt{3})$$

Alternate (d) : $= \dfrac{\text{Latus reaction of ellipse}}{\text{Latus reaction of hyperbola}}$

$$= \frac{\dfrac{2a^2}{\lambda a}}{\dfrac{2a^2}{a}} = \frac{1}{\lambda} = \frac{1}{\sqrt{3}}$$

● ***Ex. 13*** *If the circle $x^2 + y^2 = a^2$ intersects the hyperbola $xy = c^2$ at four points $P(x_1, y_1), Q(x_2, y_2), R(x_3, y_3)$ and $S(x_4, y_4)$, then*

(a) $\Sigma x_1 = 0$ (b) $\Sigma y_1 = 0$

(c) $\Pi x_1 = 0$ (d) $\Pi y_1 = 0$

Sol. (a, b, c, d) Solving $x^2 + y^2 = a^2$ and $xy = c^2$,

we have, $x^2 + \dfrac{c^4}{x^2} = a^2$

or $x^4 - a^2x^2 + c^4 = 0$

or $\Sigma x_1 = 0$ and $\Pi x_1 = c^4$

Similarly, if we eliminate x, then

$$y^4 - a^2y^2 + c^4 = 0$$

or $\Sigma y_1 = 0$ and $\Pi y_1 = c^4$

● ***Ex. 14*** *A straight line touches the rectangular hyperbola $9x^2 - 9y^2 = 8$ and the parabola $y^2 = 32x$, the equation of the line is*

(a) $9x + 3y - 8 = \infty$ (b) $9x - 3y + 8 = 0$

(c) $9x + 3y + 8 = 0$ (d) $9x - 3y - 8 = 0$

Sol. (b, c) Equation of tangent to the parabola $y^2 = 32x$ is

$$y = mx + \frac{8}{m} \qquad \text{... (i)}$$

Which is also touches the hyperbola $9x^2 - 9y^2 = 8$

i.e. $x^2 - y^2 = \dfrac{8}{9}$

So that, $\left(\dfrac{8}{m}\right)^2 = \dfrac{8}{9} \times m^2 - \dfrac{8}{9}$

$\Rightarrow$ $\dfrac{8}{m^2} = \dfrac{m^2}{9} - \dfrac{1}{9}$

$\Rightarrow$ $m^4 - m^2 - 72 = 0$

$\Rightarrow$ $(m^2 - 9)(m^2 + 8) = 0$

$\therefore$ $m^2 - 9 = 0$

but $m^2 + 8 \neq 0$

$\therefore$ $m = \pm 3$

Hence, from Eq. (i), the equation of tangents are

$$y = 3x + \frac{8}{3}$$

and $y = -3x - \dfrac{8}{3}$

or $9x - 3y + 8 = 0$

and $9x + 3y + 8 = 0$

● ***Ex. 15*** *The differential equation $\dfrac{dx}{dy} = \dfrac{3y}{2x}$ represents a family of hyperbolas (except when it represents a pair of lines) with eccentricity*

(a) $\sqrt{\dfrac{7}{3}}$ (b) $\sqrt{\dfrac{5}{3}}$

(c) $\sqrt{\dfrac{3}{2}}$ (d) $\sqrt{\dfrac{5}{2}}$

Sol. (b, d) $\because \dfrac{dx}{dy} = \dfrac{3y}{2x}$

or $\int 2x\,dx = \int 3y\,dy$

or $x^2 = \dfrac{3y^2}{2} + c$

or $\dfrac{x^2}{3} - \dfrac{y^2}{2} = \dfrac{c}{3}$

or $\dfrac{x^2}{3} - \dfrac{y^2}{2} = c_1$

Case I If $c_1 > 0$, then $e = \sqrt{\left(1 + \dfrac{2}{3}\right)} = \sqrt{\dfrac{5}{3}}$

Case II If $c_1 < 0$, then $e = \sqrt{\left(1 + \dfrac{3}{2}\right)} = \sqrt{\dfrac{5}{2}}$

JEE Type Solved Examples : Paragraph Based Questions

- This section contains 2 **solved paragraphs** based upon each of the paragraph 3 **multiple choice** questions have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Paragraph I

(Q. Nos. 16 to 18)

A conic C satisfies the differential equation.
$(1+y^2)\,dx - xy\,dy = 0$ and passes through the point (1, 0).
An ellipse E which is confocal with C having its eccentricity equal to $\sqrt{\left(\frac{2}{3}\right)}$.

16. Length of latusrectum of the conic C is
(a) 1 (b) 2
(c) 3 (d) 4

17. Equations of the ellipse E is
(a) $\frac{x^2}{3}+\frac{y^2}{1}=1$ (b) $\frac{x^2}{1}+\frac{y^2}{3}=1$
(c) $\frac{x^2}{4}+\frac{y^2}{9}=1$ (d) $\frac{x^2}{9}+\frac{y^2}{4}=1$

18. Locus of the point of intersection of the perpendicular tangents to the ellipse E, is
(a) $x^2+y^2=4$ (b) $x^2+y^2=8$
(c) $x^2+y^2=10$ (d) $x^2+y^2=13$

Sol. Given differential equation is $(1+y^2)\,dx - xy\,dy = 0$

$\Rightarrow \quad \frac{dx}{x} - \frac{ydy}{(1+y^2)} = 0$ or $\frac{2dx}{x} - \frac{2y\,dy}{1+y^2} = 0$

On integrating, we have $2\ln x - \ln(1+y^2) = \ln c$

or $\ln x^2 = \ln(c(1+y^2))$ or $x^2 = c(1+y^2)$

or $\frac{x^2}{c} - \frac{y^2}{1} = 1$ (Hyperbola)

It passes through (1, 0), then $\frac{1}{c} - 0 = 1$

$\therefore \quad c = 1$

$\therefore C$ is rectangular hyperbola $x^2 - y^2 = 1$, having its eccentricity $= \sqrt{2}$ and its foci are $(\pm\sqrt{2}, 0)$.

If a is the length of semi major axis of ellipse E, then $ae = \sqrt{2}$

or $a\sqrt{\frac{2}{3}} = \sqrt{2}$ or $a = \sqrt{3}$

Now, $b^2 = a^2(1-e^2) = 3\left(1-\frac{2}{3}\right) = 1$

16. (b) Length of latusrectum of the conic C is $\frac{2(1)^2}{1} = 2$

17. (a) Equation of the ellipse E is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ i.e, } \frac{x^2}{3}+\frac{y^2}{1}=1$$

18. (a) Required locus is the director circle of E which is $x^2+y^2=3+1$ or $x^2+y^2=4$

Paragraph II

(Q. Nos. 19 to 21)

For the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, the normal at point P meets the transverse axis AA' in G and the conjugate axis BB' in g and CF be perpendicular to the normal from the centre.

19. The value of $\frac{PF.PG}{(CB)^2}$ is equal to
(a) 4 (b) 3
(c) 2 (d) 1

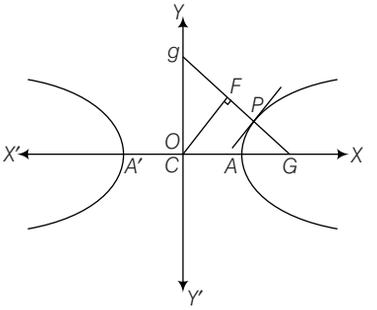
20. The value of $PF.Pg$ is equal to
(a) $(CA)^2$ (b) $(CF)^2$
(c) $(CB)^2$ (d) $CA\,.\,CB$

21. Locus of middle-point of G and g is a hyperbola of eccentricity
(a) $\frac{1}{\sqrt{(e^2-1)}}$ (b) $\frac{e}{\sqrt{(e^2-1)}}$
(c) $2\sqrt{(e^2-1)}$ (d) $\frac{e}{2}$

Sol. $C \equiv (0,0)$

$AA' = 2a,\ BB' = 2b$

$\therefore \quad CA = CA' = a$ and $CB = CB' = b$



Let $P \equiv (a\sec\theta,\ b\tan\theta)$

Equation of normals at P is

$$ax\cos\theta + by\cot\theta = a^2 + b^2 \quad \text{... (i)}$$

$\therefore \quad G \equiv \left(\frac{(a^2+b^2)}{a}\sec\theta,\ 0\right),\ g \equiv \left(0,\ \frac{(a^2+b^2)}{b}\tan\theta\right)$

Equation of CF which is perpendicular to Eq. (i) and through origin is $bx\cot\theta - ay\cos\theta = 0$

$$\therefore \quad PF = \frac{ab}{\sqrt{(b^2\sec^2\theta + a^2\tan^2\theta)}}$$

and $$PG = \sqrt{\left(a\sec\theta - \frac{a^2+b^2}{a}\sec\theta\right)^2 + (b\tan\theta - 0)^2}$$

$$= \frac{b}{a}\sqrt{(b^2\sec^2\theta + a^2\tan^2\theta)}$$

19. (d) $\because PF \,.\, PG = \frac{ab}{\sqrt{(b^2\sec^2\theta + a^2\tan^2\theta)}} \cdot \frac{b}{a}\sqrt{(b^2\sec^2\theta + a^2\tan^2\theta)}$

$$= b^2 = (CB)^2$$

$$\therefore \quad \frac{PF \,.\, PG}{(CB)^2} = 1$$

20. (a) $\because \quad Pg = \sqrt{(a\sec\theta - 0)^2 + \left(b\tan\theta - \frac{(a^2+b^2)}{b}\tan\theta\right)^2}$

$$= \sqrt{\left(a^2\sec^2\theta + \frac{a^4}{b^2}\tan^2\theta\right)}$$

$$= \frac{a}{b}\sqrt{(b^2\sec^2\theta + a^2\tan^2\theta)}$$

$$\therefore \; PF \,.\, Pg = \frac{ab}{\sqrt{(b^2\sec^2\theta + a^2\tan^2\theta)}} \times \frac{a}{b}\sqrt{(b^2\sec^2\theta + a^2\tan^2\theta)}$$

$$= a^2 = (CA)^2$$

21. (b) $\because$ Mid-point of G and g is

$$\left(\frac{(a^2+b^2)}{2a}\sec\theta,\ \frac{(a^2+b^2)}{2b}\tan\theta\right)$$

$$\therefore \quad x = \left(\frac{a^2+b^2}{2a}\right)\sec\theta,\ y = \left(\frac{a^2+b^2}{2a}\right)\tan\theta$$

$$\therefore \quad (2ax)^2 - (2by)^2 = (a^2+b^2)^2$$

or $$\frac{x^2}{\left(\frac{a^2+b^2}{2a}\right)^2} - \frac{y^2}{\left(\frac{a^2+b^2}{2b}\right)^2} = 1$$

$$\therefore \quad e_1 = \sqrt{\frac{\left(\frac{a^2+b^2}{2a}\right)^2 + \left(\frac{a^2+b^2}{2b}\right)^2}{\left(\frac{a^2+b^2}{2a}\right)^2}} = \sqrt{\left(\frac{\frac{1}{a^2}+\frac{1}{b^2}}{\frac{1}{a^2}}\right)}$$

$$= \sqrt{\frac{a^2+b^2}{2b^2}} = \sqrt{\frac{a^2+a^2(e^2-1)}{a^2(e^2-1)}} = \frac{e}{\sqrt{(e^2-1)}}$$

JEE Type Solved Examples :
Single Integer Answer Type Questions

▪ This section contains 2 **examples.** The answer to each example in **a single digit integer,** ranging from 0 to 9 (both inclusive).

● ***Ex. 22*** *The equation of transverse axis of hyperbola (passing through origin) having asymptotes* $3x - 4y - 1 = 0$ *and* $4x - 3y - 6 = 0$ *is* $ax + by - c = 0$, $a, b, c \in N$ *and g.c.d* $(a, b, c) = 1$, *then the value of* $a + b + c$ *is*

Sol. (7) Since the equation of asymptotes are

$$3x - 4y - 1 = 0 \ \text{ and } \ 4x - 3y - 6 = 0$$

$\therefore$ Equation of transverse axis is given by

$$\frac{|(3x-4y-1)|}{\sqrt{(3)^2+(-4)^2}} = \frac{|4x-3y-6|}{\sqrt{(4)^2+(-3)^2}}$$

or $$|3x - 4y - 1| = |4x - 3y - 6|$$

or $$(3x - 4y - 1) = \pm(4x - 3y - 6) \Rightarrow x + y - 5 = 0$$

and $$x - y - 7 = 0$$

$\therefore$ Transverse axis is given by $ax + by - c = 0$; $a, b, c \in N$

$\Rightarrow \quad ax + by - c \equiv x + y - 5 = 0$

$\Rightarrow \quad a = 1, b = 1, c = 5$

Hence, $\quad a + b + c = 1 + 1 + 5 = 7$

● ***Ex. 23*** *If a variable line has its intercepts on the coordinate axes are* e *and* e', *where* $\frac{e}{2}$ *and* $\frac{e'}{2}$ *are the eccentricities of a hyperbola and its conjugate hyperbola, then the line always touches the circle* $x^2 + y^2 = r^2$, *where* r *is*

Sol. (2) Since $\frac{e}{2}$ and $\frac{e'}{2}$ are the eccentricities of a hyperbola and its conjugate, we have

$$\left(\frac{e}{2}\right)^{-2} + \left(\frac{e'}{2}\right)^{-2} = 1 \ \text{ or } \ \frac{4}{e^2} + \frac{4}{(e')^2} = 1$$

or $$4 = \frac{e^2e'^2}{e^2+e'^2} \qquad \ldots (i)$$

Equations of variable line is $\frac{x}{e} + \frac{y}{e'} = 1$

i.e. $$xe' + ye - ee' = 0.$$

It is tangent to the circle $x^2 + y^2 = r^2$

$$\therefore \quad \frac{|0 - ee'|}{\sqrt{e^2+e'^2}} = r$$

$$\therefore \quad r = \frac{ee'}{\sqrt{(e^2+e'^2)}} = 2 \qquad \text{[from Eq. (i)]}$$

JEE Type Solved Examples : Matching Type Questions

- This section contain **only one example.** This example has three statements (A, B and C) given in **Column I** and four statements (p, q, r, and s) in **Column II**. Any given statment in **Column I** can have correct matching with one or more statements (s) given in **Column II.**

● ***Ex. 24*** *Match the following.*

	Column I		Column II
(A)	The locus of the point of intersection of the lines $\sqrt{3}x - y - 4\sqrt{3}t = 0$ and $\sqrt{3}tx + ty - 4\sqrt{3} = 0$ (where t is a parameter) is a hyperbola whose eccentricity is	(p)	a natural number
(B)	If the product of the perpendicular distances from any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ of eccentricity $e = \sqrt{3}$ from its asymptotes is equal to 6, then the length of the transverse axis of the hyperbola is	(q)	a prime number
(C)	The area of the triangle that a tangent at a point of the hyperbola $\frac{x^2}{16} - \frac{y^2}{a} = 1$ makes with its asymptotes is	(r)	a composite number
		(s)	a perfect number

Sol. (A) → (p, q); (B) → (p, r, s); (C)→ (p, r)

(A) The given lines are

$$\sqrt{3}x - y - 4\sqrt{3}t = 0 \quad \text{... (i)}$$

and $$\sqrt{3}tx + ty - 4\sqrt{3} = 0 \quad \text{... (ii)}$$

Eliminate 't' from Eqs. (i) and (ii), then

$$\frac{\sqrt{3}x - y}{4\sqrt{3}} = \frac{4\sqrt{3}}{\sqrt{3}x + y}$$

$$\Rightarrow \quad 3x^2 - y^2 = 48$$

or $$\frac{x^2}{16} - \frac{y^2}{48} = 1$$

or $$48 = 16(e^2 - 1)$$

$$\Rightarrow \quad e^2 = 4$$

$$\therefore \quad e = 2$$

(B) Here, $e = \sqrt{3}$

$$\therefore \quad b^2 = a^2(3-1) = 2a^2 \quad \text{... (i)}$$

Now, hyperbola convert in the form

$$\frac{x^2}{a^2} - \frac{y^2}{2a^2} = 1$$

Let $P(a\sec\theta, a\sqrt{2}\tan\theta)$ be any point on the hyperbola.

∵ Asymptotes of hyperbola are

$$\frac{x^2}{a^2} - \frac{y^2}{2a^2} = 0$$

or $$2x^2 - y^2 = 0$$

or $$x\sqrt{2} + y = 0$$

and $$x\sqrt{2} - y = 0$$

∵ Product of the perpendiculars from P on asymptotes =6

$$\therefore \quad \frac{|a\sqrt{2}\sec\theta + a\sqrt{2}\tan\theta|}{\sqrt{(2+1)}} \cdot \frac{|a\sqrt{2}\sec\theta - a\sqrt{2}\tan\theta|}{\sqrt{2+1}} = 6$$

or $$\frac{2a^2}{3} = 6$$

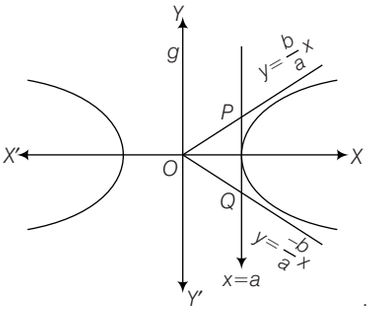
$$\therefore \quad a = 3$$

Hence, length of transverse axis $= 2a = 6$

(C) Equation of tangent at $(a, 0)$ is $x = a$

Equation of asymptotes are $y = \pm\frac{b}{a}x$

$$\therefore \quad P \equiv (a, b), Q \equiv (a, -b)$$



$\therefore$ Required area $= \frac{1}{2} \times a \times 2b$

$$= ab$$

$$= (4)(3) = 12 \quad \text{(Here } a = 4, b = 3\text{)}$$

JEE Type Solved Examples :
Statement I and II Type Questions

■ **Directions** (Ex. Nos. 25 and 26) are Assertion-Reason type examples. Each of these examples contains two statements. **Statement I** (Assertion) and **Statement II** (Reason) Each of these also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below :

(a) Statement I is true, statement II is true; statement II is a correct explanation for statement I

(b) Statement I is true, statement II is true; statement II is not a correct explanation for statement I

(c) Statement I is true, statement II is false

(d) Statement I is false, statement II is true

● ***Ex. 25*** **Statement I** *Director circle of hyperbola* $\frac{x^2}{a^2}-\frac{y^2}{b^2}+1=0$ *is defined only when* $b \geq a$

Statement II *Director circle of hyperbola* $\frac{x^2}{25}-\frac{y^2}{9}=1$ *is* $x^2+y^2=16$.

Sol. (b) Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}+1=0$ can be re-written as

$$\frac{x^2}{(-a)^2}-\frac{y^2}{(-b^2)}=1$$

So, the director circle will be

$$x^2+y^2=(-a^2)-(-b^2)=b^2-a^2;$$

Which will be defined only where $b \geq a$ (i.e. $b^2-a^2 \geq 0$)

$\therefore$ Statement I is true and director circle of hyperbola $\frac{x^2}{25}-\frac{y^2}{9}=1$ is $x^2+y^2=25-9=16$

$\therefore$ Statement II is true.

Hence, both statements are true but statement II is not a correct explanation of statement I.

● ***Ex. 26*** **Statement I** *If a circle* $S \equiv 0$ *intersect a hyperbola* $xy = 4$ *at four points, three of them being (2, 2), (4, 1) and (6, 2/3), then the coordinates of the fourth point are* $\left(\frac{1}{4}, 16\right)$.

Statement II *If a circle* $S \equiv 0$ *intersects a hyperbola* $xy = c^2$ *at* t_1, t_2, t_3 *and* t_4, *then* $t_1 t_2 t_3 t_4 = 1$.

Sol. (d) Let circle $S \equiv x^2 + y^2 - a^2 = 0$

and given hyperbola $xy = c^2$

Then, $x^2+\frac{c^4}{x^2}=a^2$ or $x^4-a^2x^2+c^4=0$

If four intersecting points are $\left(ct_1, \frac{c}{t_1}\right), \left(ct_2\frac{c}{t_2}\right), \left(ct_3, \frac{c}{t_3}\right)$ and $\left(ct_4, \frac{c}{t_4}\right)$, then

$$(ct_1)(ct_2)(ct_3)(ct_4)=c^4$$

$\therefore$ $t_1 t_2 t_3 t_4 = 1$

$\therefore$ Statement II is true

For the point (2, 2); $t_1 = 1$

For the point (4, 1); $t_2 = 2$

For the point (6, 2/3); $t_3 = 3$

For the point $\left(\frac{1}{4}, 16\right)$; $t_4 = \frac{1}{8}$

Now, $t_1 t_2 t_3 t_4 = \frac{3}{4} \neq 1$

$\therefore$ Statement I is false.

Subjective Type Questions

■ In this section there are 11 **subjective examples.**

● ***Ex. 27*** *PP′ is a diameter of the rectangular hyperbola* $xy = c^2$. *Show that the intersection of the tangent of P with the straight line through P′ parallel to either asymptote is the locus* $xy + 3c^2 = 0$.

Sol. The given rectangular hyperbola is

$$xy = c^2 \quad ...(i)$$

Given that PP' is a diameter of Eq. (i). Therefore, if P is the point $(ct, c/t)$ then P' is $(-ct, -c/t)$. Equation of tangent at P is

$$x + yt^2 = 2ct \quad ...(ii)$$

Asymptotes of Eq. (i) are $x = 0, y = 0$.

$\therefore$ The equation of the line through P' and parallel to the asymptote $y = 0$ is

$$y = -c/t \quad ...(iii)$$

Putting the value of t from Eq. (iii) in Eq. (ii), then

$$x + y \times \frac{c^2}{y^2} = 2c \times -\frac{c}{y}$$

or $$xy + c^2 = -2c^2$$

or $$xy + 3c^2 = 0$$

which is the required locus of the point of intersection of Eqs. (ii) and (iii).

Again the equation of the line through P' and parallel to the asymptote $x = 0$ is

$$x = -ct \qquad \text{...(iv)}$$

Putting the value of t from Eq. (iv) in Eq. (ii), we get

$$x + y\left(\frac{x^2}{c^2}\right) = 2c \times -\frac{x}{c} \quad \text{or} \quad xy + 3c^2 = 0$$

which is the required locus of the point of intersection of Eqs. (ii) and (iv).

● ***Ex. 28*** *The tangent at a point P of a rectangular hyperbola meets the asymptotes at L and M and C is the centre of the hyperbola. Prove that*

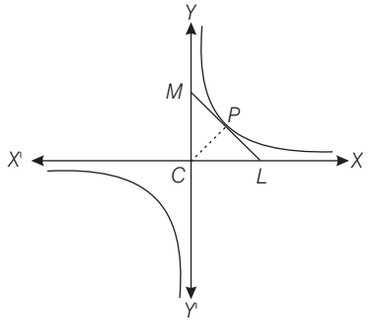
$$PL = PM = CP.$$

Sol. Since hyperbola is

$$xy = c^2 \qquad \text{....(i)}$$

Let $P\left(ct, \frac{c}{t}\right)$ be any point on Eq. (i), then equation of tangent at P on Eq. (i) is

$$x + t^2 y = 2ct \qquad \text{...(ii)}$$



Since, asymptotes of $xy = c^2$ are $x = 0$ and $y = 0$

i.e. Y-axis and X-axis.

$\because$ Tangents meet X-axis at L and Y-axis at M.

$\therefore$ Coordinates of L and M are $(2ct, 0)$ and $\left(0, \frac{2c}{t}\right)$ respectively.

The middle-point of LM is $\left(\frac{2ct+0}{2}, \frac{0+\frac{2c}{t}}{2}\right)$ i.e. $\left(ct, \frac{c}{t}\right)$.

i.e. the point of P.

$\therefore \quad PL = PM$

Also, $\quad CP = \sqrt{\left(c^2t^2 + \frac{c^2}{t^2}\right)}$

and $\quad ML = \sqrt{\left(4c^2t^2 + \frac{4c^2}{t^2}\right)} = 2CP$

$\therefore \quad CP = \frac{1}{2} ML = PL = PM$

Hence, $\quad PL = PM = CP.$

● ***Ex. 29*** *Prove that the perpendicular focal chords of a rectangular hyperbola are equal.*

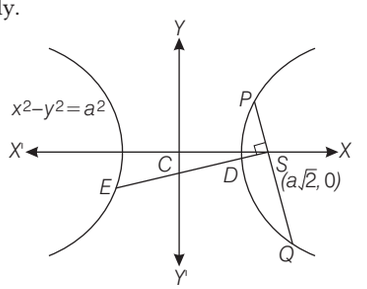
Sol. Let rectangular hyperbola is $x^2 - y^2 = a^2$.

Let equations of PQ and DE are

$$y = mx + c \qquad \text{...(i)}$$

and $\quad y = m_1 x + c_1 \qquad \text{...(ii)}$

respectively.



Be any two focal chords of any rectangular hyperbola $x^2 - y^2 = a^2$ through its focus. We have to prove $PQ = DE$.

Since, $PQ \perp DE$.

$\therefore \quad mm_1 = -1 \qquad \text{...(iii)}$

Also PQ passes through $S(a\sqrt{2}, 0)$ then from Eq. (i),

$$0 = ma\sqrt{2} + c$$

or $\quad c^2 = 2a^2m^2 \qquad \text{...(iv)}$

Let (x_1, y_1) and (x_2, y_2) be the coordinates of P and Q then

$$(PQ)^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 \qquad \text{...(v)}$$

Since, (x_1, y_1) and (x_2, y_2) lie on Eq. (i)

$\therefore \quad y_1 = mx_1 + c$ and $y_2 = mx_2 + c$

$\therefore \quad (y_1 - y_2) = m(x_1 - x_2) \qquad \text{...(vi)}$

From Eqs. (v) and (vi),

$$(PQ)^2 = (x_1 - x_2)^2 (1 + m^2) \qquad \text{...(vii)}$$

Now, solving $y = mx + c$ and $x^2 - y^2 = a^2$ then

$$x^2 - (mx + c)^2 = a^2$$

or $\quad (m^2 - 1)\, x^2 + 2mcx + (a^2 + c^2) = 0$

$\therefore \quad x_1 + x_2 = \frac{2mc}{m^2 - 1}$ and $x_1 x_2 = \frac{a^2 + c^2}{m^2 - 1}$

$\Rightarrow \quad (x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2$

$$= \frac{4m^2c^2}{(m^2-1)^2} - \frac{4(a^2 + c^2)}{(m^2 - 1)}$$

$$= \frac{4\{a^2 + c^2 - a^2m^2\}}{(m^2-1)^2}$$

$$= \frac{4a^2(m^2 + 1)}{(m^2-1)^2} \qquad [\because c^2 = 2a^2m^2]$$

From Eq. (vii), $\quad (PQ)^2 = 4a^2\left(\frac{m^2+1}{m^2-1}\right)^2$

Similarly, $\quad (DE)^2 = 4a^2\left(\frac{m_1^2+1}{m_1^2-1}\right)^2$

$$= 4a^2\left(\frac{\left(-\frac{1}{m}\right)^2+1}{\left(-\frac{1}{m}\right)^2-1}\right)^2 \qquad (\because mm_1=-1)$$

$$= 4a^2\left(\frac{m^2+1}{m^2-1}\right)^2$$

$$= (PQ)^2$$

Thus, $(PQ)^2 = (DE)^2 \Rightarrow PQ = DE.$

Hence, perpendicular focal chords of a rectangular hyperbola are equal.

● ***Ex. 30*** *The normals at three points P, Q, R on a rectangular hyperbola intersect at a point T on the curve. Prove that the centre of the hyperbola is the centroid of the triangle PQR.*

Sol. Equation of the normal at point $\left(ct, \frac{c}{t}\right)$ on $xy = c^2$ is

$$xt^3 - yt - ct^4 + c = 0$$

It will pass through the point (h, k)

$$\therefore \qquad ht^3 - kt - ct^4 + c = 0 \qquad \text{...(i)}$$

Also $T(h, k)$ lies on $xy = c^2$

$$\therefore \qquad hk = c^2$$

Therefore, $h = cq$ and $k = \frac{c}{q}$

From, $$cqt^3 - \frac{c}{q}t - ct^4 + c = 0$$

or $$qt^3 - \frac{t}{q} - t^4 + 1 = 0$$

or $$q^2t^3 - t - qt^4 + q = 0$$

or $$q(qt^3+1) - t(qt^3+1) = 0$$

or $$(q-t)(qt^3+1) = 0$$

$$q \neq t$$

$$\therefore \qquad qt^3 + 1 = 0$$

The three points other than T are given by

$$qt^3 + 1 = 0 \qquad \text{...(ii)}$$

If coordinates of $P\left(ct_1, \frac{c}{t_1}\right), Q\left(ct_2, \frac{c}{t_2}\right)$ and $R\left(ct_3, \frac{c}{t_3}\right)$

then from Eq. (ii),

$$t_1 + t_2 + t_3 = 0 \qquad \text{...(iii)}$$

$$t_1t_2 + t_2t_3 + t_3t_1 = 0 \qquad \text{...(iv)}$$

and $$t_1t_2t_3 = -1 \qquad \text{...(v)}$$

From Eq. (iii),

$$c(t_1 + t_2 + t_3) = 0$$

and from Eq. (iv),

$$t_1t_2t_3\left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3}\right) = 0$$

or $$(-1)\left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3}\right) = 0 \qquad \text{[from Eq. (v)]}$$

$$\therefore \qquad \frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} = 0$$

or $$c\left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3}\right) = 0$$

Hence, $c(t_1 + t_2 + t_3) = 0, c\left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3}\right) = 0$

$\Rightarrow$ Centroid of ΔPQR is the origin.

● ***Ex. 31*** *Find the equation of the hyperbola whose asymptotes are $x + 2y + 3 = 0$ and $3x + 4y + 5 = 0$ and which passes through the point $(1, -1)$. Find also the equation of the conjugate hyperbola.*

Sol. Combined equation of asymptotes is

$$(x + 2y + 3)(3x + 4y + 5) = 0$$

or $$3x^2 + 10xy + 8y^2 + 14x + 22y + 15 = 0 \qquad \text{...(i)}$$

Also, we know that the equation of the hyperbola differs from that of asymptotes by a constant.

Let the equation of the hyperbola be

$$3x^2 + 10xy + 8y^2 + 14x + 22y + \lambda = 0 \qquad \text{...(ii)}$$

Since it passes through $(1, -1)$ then

$$3(1)^2 + 10(1)(-1) + 8(-1)^2 + 14(1) + 22(-1) + \lambda = 0$$

$\Rightarrow$ $$3 - 10 + 8 + 14 - 22 + \lambda = 0$$

$\therefore$ $$\lambda = 7$$

From Eq. (ii), equation of hyperbola is

$$3x^2 + 10xy + 8y^2 + 14x + 22y + 7 = 0 \qquad \text{...(iii)}$$

But we know that equation of conjugate hyperbola

= 2 (Combined equation of asymptotes) − (Equation of hyperbola)

$\Rightarrow$ $$6x^2 + 20xy + 16y^2 + 28x + 44y + 30$$

$$-3x^2 - 10xy - 8y^2 - 14x - 22y - 7 = 0$$

or $$3x^2 + 10xy + 8y^2 + 14x + 22y + 23 = 0.$$

● ***Ex. 32*** *A triangle is inscribed in $xy = c^2$ and two of its sides are parallel to $y = m_1x$ and $y = m_2x$ prove that the third side envelopes the hyperbola $4m_1m_2xy = c^2(m_1 + m_2)^2$.*

Sol. Let a triangle PQR be inscribed in $xy = c^2$.

Let the coordinates of the vertices of the triangle be

$$P\left(ct_1, \frac{c}{t_1}\right), Q\left(ct_2, \frac{c}{t_2}\right) \text{ and } R\left(ct_3, \frac{c}{t_3}\right)$$

Now, the equation of chord joining P and Q is

$$x + yt_1t_2 = c(t_1 + t_2) \qquad \text{...(i)}$$

and the equation of chord joining Q and R is

$$x + yt_2t_3 = c(t_2 + t_3) \qquad \text{...(ii)}$$

Let Eq. (i) be parallel to $y = m_1x$ and Eq. (ii) be parallel to $y = m_2x$.

$$\therefore \qquad m_1 = -\frac{1}{t_1 t_2} \quad \text{and} \quad m_2 = -\frac{1}{t_2 t_3}$$

$$\therefore \qquad \frac{m_1}{m_2} = \frac{t_3}{t_1} \text{ i.e. } t_3 = \left(\frac{m_1}{m_2}\right) t_1 \qquad \ldots(iii)$$

Again the equation to the third side RP is

$$x + y t_3 t_1 = c(t_3 + t_1)$$

$$x + y\left(\frac{m_1}{m_2}\right) t_1^2 = c\left(\frac{m_1}{m_2} t_1 + t_1\right)$$

or $\qquad y m_1 t_1^2 - c t_1 (m_1 + m_2) + x m_2 = 0 \qquad \ldots(iv)$

t_1 being a parameter. Since t_1 is real the envelope of Eq. (iv) is given by the discriminant of Eq. (iv) $= 0$

i.e. $\qquad c^2 (m_1 + m_2)^2 - 4 y m_1 \cdot x m_2 = 0$

or $\qquad 4 m_1 m_2 xy = c^2 (m_1 + m_2)^2.$

● ***Ex. 33*** *In both an ellipse and a hyperbola, prove that the focal distance of any point and the perpendicular from the centre upon the tangent at it meet on a circle whose centre is the focus and whose radius is the semi-transverse axis.*

Sol. Let the hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \ldots(i)$$

Let there be any point $P(a \sec \phi, b \tan \phi)$ on the hyperbola Eq. (i).

$\therefore$ Equation of tangent at 'P' is

$$\frac{x}{a} \sec \phi - \frac{y}{b} \tan \phi = 1 \qquad \ldots(ii)$$

Equation of focal chord SP is [if $S \equiv (-ae, 0)$]

$$y - 0 = \frac{b \tan \phi - 0}{a \sec \phi + ae}(x + ae)$$

$$\Rightarrow \qquad y = \frac{b \sin \phi (x + ae)}{(a + ae \cos \phi)} \qquad \ldots(iii)$$

Equation of line passing through (0, 0) and perpendicular to Eq. (ii) is

$$y = -\frac{a}{b} \sin \phi \cdot x \qquad \ldots(iv)$$

Let Eqs. (iii) and (iv) are passing through (h, k), then

$$k = \frac{b(h + ae) \sin \phi}{(a + ae \cos \phi)} \qquad \ldots(v)$$

$$k = -\frac{a}{b} \sin \phi \cdot h \qquad \ldots(vi)$$

From Eq. (iv), $\sin \phi = -\dfrac{bk}{ah}$

and from Eq. (v), $\cos \phi = -\left\{\dfrac{a^2 h + (h + ae) b^2}{a^2 eh}\right\}$

$$= -\left\{\frac{(a^2 + b^2)h + aeb^2}{a^2 eh}\right\}$$

$$= -\left\{\frac{a^2 e^2 h + aeb^2}{a^2 eh}\right\} = -\left\{\frac{aeh + b^2}{ah}\right\}$$

$\therefore \qquad \sin^2 \phi + \cos^2 \phi = 1$

$$\Rightarrow \qquad \frac{b^2 k^2}{a^2 h^2} + \frac{(aeh + b^2)^2}{a^2 h^2} = 1$$

$\therefore$ Locus of (h, k) is

$$b^2 y^2 + (aex + b^2)^2 = a^2 x^2$$

$$\Rightarrow \qquad b^2 y^2 + a^2 e^2 x^2 + b^4 + 2aeb^2 x = a^2 x^2$$

$$\Rightarrow \qquad b^2 y^2 + (a^2 + b^2) x^2 + b^4 + 2aeb^2 x = a^2 x^2$$

$$\Rightarrow \qquad y^2 b^2 + b^2 (x^2 + 2aex) + b^4 = 0$$

$$\Rightarrow \qquad b^2 y^2 + b^2 \{(x + ae)^2 - a^2 e^2\} + b^4 = 0$$

$$\Rightarrow \qquad b^2 y^2 + b^2 (x + ae)^2 - b^2 (a^2 + b^2) + b^4 = 0$$

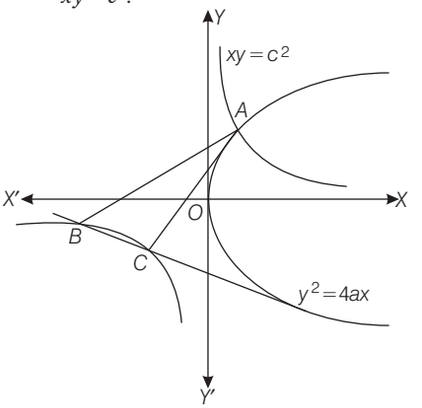
$$\Rightarrow \qquad b^2 y^2 + b^2 (x + ae)^2 - a^2 b^2 = 0$$

$$\Rightarrow \qquad (x + ae)^2 + y^2 = a^2$$

● ***Ex. 34*** *Show that an infinite number of triangles can be inscribed in the rectangular hyperbola $xy = c^2$, whose sides all touch the parabola $y^2 = 4ax$.*

Sol. From any point $A\left(ct_1, \dfrac{c}{t_1}\right)$ on the rectangular hyperbola

$$xy = c^2. \qquad \ldots(i)$$



Tangent AB and AC are drawn to the parabola $y^2 = 4ax$,

then $\qquad B \equiv \left(ct_2, \dfrac{c}{t_2}\right)$ and $\quad C \equiv \left(ct_3, \dfrac{c}{t_3}\right)$.

Then ΔABC is inscribed in the rectangular hyperbola and its sides AB and AC touch the parabola.

$\because$ BC is also a tangent to it

Equations of chords AB, BC and CA are

$$AB \equiv x + y t_1 t_2 = c(t_1 + t_2)$$

$$BC \equiv x + y t_2 t_3 = c(t_2 + t_3)$$

$$CA \equiv x + y t_3 t_1 = c(t_3 + t_1)$$

$\because$ AB is tangent to parabola $y^2 = 4ax$, its equation

$$y = -\frac{1}{t_1 t_2} x + \frac{c(t_1 + t_2)}{t_1 t_2}$$

Touching condition $c = \dfrac{a}{m}$ or $cm = a$

$$\frac{c(t_1+t_2)}{t_1t_2}\cdot\left(\frac{-1}{t_1t_2}\right)=a$$

$$\Rightarrow \quad -at_1^2t_2^2=c(t_1+t_2)$$

$$(at_1^2)\,t_2^2+ct_2+ct_1=0$$

Similarly, $(at_2^2)\,t_3^2+ct_3+ct_1=0$

Hence, $(at_1^2)\,t^2+ct+ct_1=0$ ...(ii)

and t_2 and t_3 are the roots of Eq. (ii),

$$\therefore \quad t_2+t_3=-\frac{c}{at_1^2} \quad \text{and} \quad t_2t_3=\frac{c}{at_1}$$

$$\Rightarrow \quad c=at_1t_2t_3$$

$$\therefore \quad c(t_2+t_3)=-\frac{c^2}{at_1^2}=-\frac{(at_1t_2t_3)^2}{at_1^2}=-at_2^2t_3^2$$

$$\Rightarrow \quad -at_2t_3=\frac{c(t_2+t_3)}{t_2t_3} \text{ is the condition.}$$

This proves BC to be also a tangent to the parabola. But any point A can be taken in infinite ways anywhere on the curve representing the rectangular hyperbola. Hence such triangles exist in infinite number.

● ***Ex. 35*** *A circle with centre* $(3\alpha, 3\beta)$ *and of variable radius cuts the rectangular hyperbola* $x^2-y^2=9a^2$ *at the points P, Q, R, S. Prove that the locus of the centroid of the triangle PQR is* $(x-2\alpha)^2-(y-2\beta)^2=a^2$.

Sol. The equation of the circle with centre $(3\alpha, 3\beta)$ is

$$x^2+y^2-6\alpha x-6\beta y+k=0 \quad ...(i)$$

where k is variable.

Given hyperbola is $x^2-y^2=9a^2$ ...(ii)

Eliminating y from Eq. (i) with the help of Eq. (ii), we get

$$(x^2+x^2-9a^2-6\alpha x+k)^2=36\beta^2(x^2-9a^2)$$

$$\Rightarrow \quad 4x^4-24\alpha x^3+....=0$$

this is a biquadratic equation.

Let the abscissas of four points P, Q, R and S are x_1, x_2, x_3 and x_4, then

$$x_1+x_2+x_3+x_4=6\alpha \quad ...(iii)$$

Similarly, $y_1+y_2+y_3+y_4=6\beta$ ...(iv)

If (x', y') be the centroid of the triangle PQR, then

$$x'=\frac{x_1+x_2+x_3}{3}=\frac{6\alpha-x_4}{3}$$

$$\therefore \quad -3(x'-2\alpha)=x_4$$

and $$y'=\frac{y_1+y_2+y_3}{3}=\frac{6\beta-y_4}{3}$$

$$\therefore \quad -3(y'-2\beta)=y_4$$

But $x_4^2-y_4^2=9a^2$

$$\Rightarrow \quad 9(x'-2\alpha)^2-9(y'-2\beta)^2=9a^2$$

$$\Rightarrow \quad (x'-2\alpha)^2-(y'-2\beta)^2=a^2$$

Hence, locus of centroid (x', y') is $(x-2\alpha)^2-(y-2\beta)^2=a^2$.

● ***Ex. 36*** *If the normals at* (x_i, y_i), $i=1,2,3,4$ *on the rectangular hyperbola,* $xy=c^2$, *meet at the point* (α, β) *show that*

(i) $\Sigma x_i=\alpha$ *(ii)* $\Sigma y_i=\beta$ *(iii)* $\Pi x_i=\Pi y_i=-c^4$

(iv) $\Sigma x_i^2=\alpha^2$ *(v)* $\Sigma y_i^2=\beta^2$

Sol. Let $(x_i, y_i)=\left(ct_i, \frac{c}{t_i}\right)$, $i=1,2,3,4$ are the points on the rectangular hyperbola $xy=c^2$.

Equation of normal to the hyperbola

$$xy=c^2 \text{ at } \left(ct, \frac{c}{t}\right) \text{ is}$$

$$ct^4-t^3x+ty-c=0$$

It passes through (α, β), then

$$ct^4-t^3\alpha+t\beta-c=0$$

its biquadratic equation in t. Let the roots of this equation are t_1, t_2, t_3, t_4 then

$$\Sigma t_1=\frac{\alpha}{c},$$

$$\Sigma t_1t_2=0, \Sigma t_1t_2t_3=-\beta/c, t_1t_2t_3t_4=-1$$

Now, (i) $\Sigma x_i=c\,\Sigma t_1=\alpha$

(ii) $$\Sigma y_i=c\left(\Sigma\frac{1}{t_1}\right)=c\left(\frac{\Sigma t_1t_2t_3}{t_1t_2t_3t_4}\right)=\beta$$

(iii) $\Pi x_i=c^4\Pi t_i=-c^4$

and $$\Pi y_i=c^4\left(\frac{1}{\Pi t_i}\right)=-c^4$$

(iv) $$\Sigma x_i^2=c^2(\Sigma t_1^2)=c^2\{(\Sigma t_1)^2-2\Sigma t_1t_2\}$$
$$=\alpha^2$$

(v) $$\Sigma y_i^2=(\Sigma y_1)^2-2\Sigma y_1y_2$$
$$=\beta^2-2c^2\Sigma\left(\frac{1}{t_1t_2}\right)=\beta^2-2c^2\cdot\frac{\Sigma t_1t_2}{t_1t_2t_3t_4}=\beta^2$$

● ***Ex. 37*** *If a circle cuts a rectangular hyperbola* $xy=c^2$ *in A, B, C and D and the parameters of these four points be* t_1, t_2, t_3 *and* t_4 *respectively prove the following :*

(i) $t_1t_2t_3t_4=1$, *find the value of* $\Sigma\frac{1}{t_1}$.

(ii) If H be the orthocentre of the triangle ABC, then H and D are extremities of a diameter of the rectangular hyperbola.

(iii) The centre of mean position of the four points bisects the distance between the centres of the two curves.

(iv) The centre of the circle through A, B and C is

$$\left[\frac{c}{2}\left(t_1+t_2+t_3+\frac{1}{t_1t_2t_3}\right), \frac{c}{2}\left(\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}+t_1t_2t_3\right)\right].$$

Sol. (i) Let the equation of the circle is

$$x^2 + y^2 + 2gx + 2fy + k = 0 \quad \text{...(i)}$$

and the equation of the rectangular hyperbola is

$$xy = c^2 \quad \text{...(ii)}$$

Put $x = ct$ and $y = \frac{c}{t}$ in Eq. (i)

then, $$c^2t^2 + \frac{c^2}{t^2} + 2gct + \frac{2fc}{t} + k = 0$$

$\Rightarrow$ $$c^2t^4 + 2gct^3 + kt^2 + 2fct + c^2 = 0$$

this equation being fourth degree in t. Let roots be t_1, t_2, t_3, t_4 then

$$\Sigma t_1 = -\frac{2g}{c}, \Sigma t_1t_2 = \frac{k}{c^2},$$

$$\Sigma t_1t_2t_3 = -\frac{2f}{c} \text{ and } t_1t_2t_3t_4 = 1$$

$\therefore$ $$\Sigma \frac{1}{t_1} = \frac{\Sigma t_1t_2t_3}{t_1t_2t_3t_4} = \frac{\frac{-2f}{c}}{1} = -\frac{-2f}{c}$$

(ii) Orthocentre of the ΔABC is $\left(\frac{-c}{t_1t_2t_3}, -ct_1t_2t_3\right)$

Hence orthocentre is $\left(\frac{-ct_4}{t_1t_2t_3t_4}, \frac{-ct_1t_2t_3t_4}{t_4}\right)$

i.e. $$\left(-ct_4, \frac{-c}{t_4}\right)$$

But D is $$\left(ct_4, \frac{c}{t_4}\right).$$

Hence H and D are the extremities of a diagonal of rectangular hyperbola.

(iii) The centre of the mean position of the four points is

$$\left(\frac{c}{4}(t_1 + t_2 + t_3 + t_4), \frac{c}{4}\left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4}\right)\right)$$

i.e. $$\left(\frac{c}{4}\Sigma t_1, \frac{c}{4}\Sigma\left(\frac{1}{t_1}\right)\right)$$

i.e. $$\left(-\frac{g}{2}, -\frac{f}{2}\right)$$ [from relation Eq. (i)]

$\because$ Centres of the circle and rectangular hyperbola are $(-g, -f)$ and $(0, 0)$; mid point of centres of circle and rectangular hyperbola is $\left(\frac{-g}{2}, \frac{-f}{2}\right)$.

Hence the centre of the mean position of the four points bisects the distance between the centres of the two curves (circle and rectangular hyperbola).

(iv) Let the circle passing through ABC meet the hyperbola in fourth point D; then centre of circle is

$$(-g, -f), (-g, -f)$$

$\Rightarrow$ $$\left(\frac{c}{2}\left(\frac{-2g}{c}\right), \frac{c}{2}\left(\frac{-2f}{2}\right)\right)$$ [from Eq. (i) relation]

$\Rightarrow$ $$\left(\frac{c}{2}(t_1 + t_2 + t_3 + t_4), \frac{c}{2}\left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4}\right)\right)$$

But $$t_1t_2t_3t_4 = 1$$

$\therefore$ $$t_4 = \frac{1}{t_1t_2t_3}$$

Hence, centre of circle is

$$\left(\frac{c}{2}\left(t_1 + t_2 + t_3 + \frac{1}{t_1t_2t_3}\right), \frac{c}{2}\left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + t_1t_2t_3\right)\right).$$

Hyperbola Exercise 1 :
Single Option Correct Type Questions

■ This section contains **30 multiple choice questions.** Each question has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

1. P is any point on the hyperbola $x^2 - y^2 = a^2$. If F_1 and F_2 are the foci of the hyperbola and $PF_1 \cdot PF_2 = \lambda(OP)^2$, where O is the origin, then λ is equal to
(a) 1 (b) $\sqrt{2}$ (c) 2 (d) 3

2. If the sum of the slopes of the normals from a point P to the hyperbola $xy = c^2$ is equal to $\lambda\,(\lambda \in R^+)$, then locus of point P is
(a) $x^2 - y^2 = \lambda c^2$ (b) $y^2 = \lambda c^2$
(c) $xy = \lambda c^2$ (d) $x^2 = \lambda c^2$

3. If $xy = \lambda^2 - 9$ be a rectangular hyperbola whose branches lie only in the second and fourth quadrant, then
(a) $|\lambda| \geq 3$ (b) $|\lambda| < 3$
(c) $\lambda \in R - \{-3, 3\}$ (d) None of these

4. If there are two points A and B on rectangular hyperbola $xy = c^2$ such that abscissa of A = ordinate of B, then the locus of point of intersection of tangents at A and B is
(a) $y^2 = x^2 + 2c^2$ (b) $y^2 = x^2 + \frac{c^2}{2}$
(c) $y = x$ (d) $y = 3x$

5. A series of hyperbola is drawn having a common transverse axis of length $2a$. Then the locus of a point P on each hyperbola, such that its distance from the transverse axis is equal to its distance from an asymptote, is
(a) $(x^2 - y^2)^2 = 4x^2(x^2 - a^2)$
(b) $(x^2 - y^2)^2 = x^2(x^2 - a^2)$
(c) $(x^2 - y^2)^2 = 4y^2(x^2 - a^2)$
(d) $(x^2 - y^2)^2 = y^2(x^2 - a^2)$

6. If a rectangular hyperbola $(x-1)(y-2) = 4$ cuts a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at points (3, 4) (5, 3), (2, 6) and $(-1,0)$, then the value of $(g + f)$ is equal to
(a) -8 (b) -9 (c) 8 (d) 9

7. If $f(x) = ax^3 + bx^2 + cx + d, (a, b, c, d$ are rational numbers) and roots of $f(x) = 0$ are eccentricities of a parabola and a rectangular hyperbola, then $a + b + c + d$ equals
(a) -1 (b) 0 (c) 1 (d) data inadequate

8. From a point on the line $y = x + c, c$ (parameter), tangents are drawn to the hyperbola $\frac{x^2}{2} - \frac{y^2}{1} = 1$ such that chords of contact pass through a fixed point (x_1, y_1). Then, $\frac{x_1}{y_1}$ is equal to
(a) 2 (b) 3
(c) 4 (d) None of these

9. Two conics $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $x^2 = -\frac{a}{b}y$ intersect, if
(a) $0 < b \leq \frac{1}{2}$ (b) $0 < a < \frac{1}{2}$
(c) $a^2 < b^2$ (d) $a^2 > b^2$

10. The number of points outside the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$ from where two perpendicular tangents can be drawn to the hyperbola are
(a) 0 (b) 1
(c) 2 (d) None of these

11. Let $A \equiv (-3, 4)$ and $B \equiv (2, -1)$ be two fixed points. A point C moves such that
$$\tan\left(\frac{1}{2}\angle ABC\right) : \tan\left(\frac{1}{2}\angle BAC\right) = 3:1$$
Thus, locus of C is a hyperbola, distance between whose foci is
(a) 5 (b) $5\sqrt{2}$ (c) $\frac{5}{2}$ (d) $\frac{5}{\sqrt{2}}$

12. A point P is taken on the right half of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ having its foci as S_1 and S_2. If the internal angle bisector of the angle $\angle S_1PS_2$ cuts the x-axis at point $Q(\alpha, 0)$, then range of α is
(a) $[-a, a]$ (b) $[0, a]$
(c) $(0, a]$ (d) $[-a, 0)$

13. If angle between asymptotes of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is 120° and product of perpendiculars drawn from foci upon its any tangent is 9, then locus of point of intersection of perpendicular tangents of the hyperbola can be
(a) $x^2 + y^2 = 3$ (b) $x^2 + y^2 = 6$
(c) $x^2 + y^2 = 9$ (d) $x^2 + y^2 = 18$

14. If $\alpha+\beta=3\pi$, then the chord joining the points α and β for the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ passes through
(a) focus
(b) centre
(c) one of the end point of the transverse axis
(d) one of the end points of the conjugate axis

15. If the curves $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 (a>b)$ and $x^2-y^2=c^2$ cut at right angles, then
(a) $a^2+b^2=2c^2$ (b) $b^2-a^2=2c^2$
(c) $a^2-b^2=2c^2$ (d) $a^2b^2=2c^2$

16. If chords of the hyperbola $x^2-y^2=a^2$ touch the parabola $y^2=4ax$, then the locus of the middle points of these chords is the curve
(a) $y^2(x+a)=x^3$ (b) $y^2(x-a)=x^3$
(c) $y^2(x+2a)=3x^3$ (d) $y^2(x-2a)=2x^3$

17. An ellipse has eccentricity 1/2 and one focus at the point $P(1/2,1)$. Its one directrix is the common tangent nearer to the point P, to the circle $x^2+y^2=1$ and the hyperbola $x^2-y^2=1$. The equation of the ellipse is standard form is
(a) $9x^2+12y^2=108$
(b) $9(x-1/3)^2+12(y-1)^2=1$
(c) $9(x-1/3)^2+4(y-1)^2=36$
(d) None of the above

18. The equation of the line passing through the centre of a rectangular hyperbola is $x-y-1=0$. If one of its asymptote is $3x-4y-6=0$, the equation of the other asymptote is
(a) $4x-3y+8=0$ (b) $4x+3y+17=0$
(c) $3x-2y+15=0$ (d) None of these

19. The condition that a straight line with slope m will be normal to parabola $y^2=4ax$ as well as a tangent to rectangular hyperbola $x^2-y^2=a^2$ is
(a) $m^6-4m^2+2m-1=0$ (b) $m^4+3m^3+2m+1=0$
(c) $m^6-2m=0$ (d) $m^6+4m^4+3m^2+1=0$

20. The locus of the middle points of chords of hyperbola $3x^2-2y^2+4x-6y=0$ parallel to $y=2x$ is
(a) $3x-4y=4$ (b) $3y-4x+4=0$
(c) $4x-4y=3$ (d) $3x-4y=2$

21. The coordinates of the centre of the hyperbola $x^2+3xy+2y^2+2x+3y+2=0$ is
(a) $(-1,0)$ (b) $(1,0)$ (c) $(-1,1)$ (d) $(1,-1)$

22. Let F_1, F_2 are foci of the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ and F_3, F_4 are the foci of its conjugate hyperbola. If e_H and e_C are their eccentricities respectively, then the statement which holds true is
(a) their equations of their asymptotes are different
(b) $e_H>e_C$
(c) area of the quadrilateral formed by their foci is 50 sq units
(d) their auxiliary circles will have the same equation

23. Locus of the point of intersection of the tangents at the points with eccentric angles ϕ and $\frac{\pi}{2}-\phi$ on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is
(a) $x=a$ (b) $y=b$
(c) $x=ab$ (d) $y=ab$

24. Latusrectum of the conic satisfying the differential equation $xdy+ydx=0$ and passing through the point $(2,8)$ is
(a) $4\sqrt{2}$ (b) 8
(c) $8\sqrt{2}$ (d) 16

25. The points of the intersection of the curves whose parametric equations are $x=t^2+1, y=2t$ and $x=2s, y=\frac{2}{s}$ is given by
(a) $(1,-3)$ (b) $(2,2)$
(c) $(-2,4)$ (d) $(1,2)$

26. If the tangent and normal to a rectangular hyperbola cut off intercepts x_1 and x_2 on one axis and y_1 and y_2 on the other axis, then
(a) $x_1y_1+x_2y_2=0$ (b) $x_1y_2+x_2y_1=0$
(c) $x_1x_2+y_1y_2=0$ (d) None of these

27. The focus of rectangular hyperbola $(x-h)(y-k)=p^2$ is
(a) $(h-p,k-p)$
(b) $(h-p,k+p)$
(c) $(h+p,k-p)$
(d) None of the above

28. The equation of a hyperbola, conjugate to the hyperbola $x^2+3xy+2y^2+2x+3y=0$ is
(a) $x^2+3xy+2y^2+2x+3y+1=0$
(b) $x^2+3xy+2y^2+2x+3y+2=0$
(c) $x^2+3xy+2y^2+2x+3y+3=0$
(d) $x^2+3xy+2y^2+2x+3y+4=0$

29. If the values of m for which the line $y = mx + 2\sqrt{5}$ touches the hyperbola $16x^2 - 9y^2 = 144$ are the roots of the equation $x^2 - (a+b)x - 4 = 0$, then the value of $a + b$ is

(a) -2 (b) 0
(c) 2 (d) 4

30. Let C be a curve which is the locus of the point of intersection of lines $x = 2 + m$ and $my = 4 - m$. A circle $s = (x-2)^2 + (y+1)^2 = 25$ intersects the curve C at four points P, Q, R and S. If O is the centre of the curve C, then $(OP)^2 + (OQ)^2 + (OR)^2 + (OS)^2$ is

(a) 25 (b) 50 (c) 100 (d) 200

Hyperbola Exercise 2 : More than One Correct Option Type Questions

- This section contains **15 multiple choice questions.** Each question has four choices (a), (b), (c) and (d) out of which **MORE THAN ONE** may be correct.

31. Equation of common tangent to the parabola $y^2 = 8x$ and hyperbola $x^2 - \frac{y^2}{3} = 1$ is

(a) $2x - y + 1 = 0$ (b) $2x - y - 1 = 0$
(c) $2x + y + 1 = 0$ (d) $2x + y - 1 = 0$

32. If the foci of the ellipse $\frac{x^2}{k^2a^2} + \frac{y^2}{a^2} = 1$ and the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ coincide, then k is equal to

(a) $-\sqrt{2}$ (b) $\sqrt{2}$ (c) $-\sqrt{3}$ (d) $\sqrt{3}$

33. If $(a\sec\theta, b\tan\theta)$ and $(a\sec\phi, b\tan\phi)$ are the ends of a focal chord of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $\tan\left(\frac{\theta}{2}\right)\tan\left(\frac{\phi}{2}\right)$ is equal to

(a) $\frac{e-1}{e+1}$ (b) $\frac{1-e}{1+e}$ (c) $\frac{1+e}{1-e}$ (d) $\frac{e+1}{e-1}$

34. If foci of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ coincide with the foci of $\frac{x^2}{25} + \frac{y^2}{16} = 1$ and eccentricity of the hyperbola is 3, then

(a) $a^2 + b^2 = 9$
(b) there is no director circle to the hyperbola
(c) centre of the director circle is $(0, 0)$
(d) length of the latusrectum of the hyperbola = 16

35. The equation $16x^2 - 3y^2 - 32x - 12y - 44 = 0$ represents a hyperbola with

(a) length of the transverse axis = $2\sqrt{3}$
(b) length of the conjugate axis = 8
(c) centre at $(1, -2)$
(d) eccentricity = $\sqrt{19}$

36. If the line $ax + by + c = 0$ is normal to the hyperbola $xy = 1$, then

(a) $a > 0, b > 0$ (b) $a > 0, b < 0$
(c) $a < 0, b > 0$ (d) $a < 0, b < 0$

37. If $P(x_1, y_1), Q(x_2, y_2), R(x_3, y_3)$ and $S(x_4, y_4)$ are 4 concyclic points on the rectangular hyperbola $xy = c^2$, the coordinates of the orthocentre of the ΔPQR are

(a) $(x_4, -y_4)$ (b) (x_4, y_4)
(c) $(-x_4, -y_4)$ (d) $(-x_4, y_4)$

38. The line $y = x + 5$ touches

(a) the parabola $y^2 = 20x$
(b) the ellipse $9x^2 + 16y^2 = 144$
(c) the hyperbola $\frac{x^2}{29} - \frac{y^2}{4} = 1$
(d) the circle $x^2 + y^2 = 25$

39. The coordinates of a point common to a directrix and an asymptote of the hyperbola $x^2/25 - y^2/16 = 1$ are

(a) $(25/\sqrt{41}, 20/\sqrt{41})$ (b) $(-25/\sqrt{41}, -20/\sqrt{41})$
(c) $(25/3, 20/3)$ (d) $(-25/3, -20/3)$

40. If (5, 12) and (24, 7) are the foci of a hyperbola passing through the origin, then

(a) $e = \frac{\sqrt{386}}{12}$ (b) $e = \frac{\sqrt{386}}{13}$
(c) latusrectum = $\frac{121}{3}$ (d) latusrectum = $\frac{121}{6}$

41. For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, let n be the number of points on the plane through which perpendicular tangents are drawn

(a) if $n = 1$, then $e = \sqrt{2}$
(b) if $n > 1$, then $0 < e < \sqrt{2}$
(c) if $n = 0$, then $e > \sqrt{2}$
(d) None of the above

42. Which of the following equations in parametric form can represent a hyperbola, where 't' is a parameter?

(a) $x = \frac{a}{2}\left(t + \frac{1}{t}\right)$ and $y = \frac{b}{2}\left(t - \frac{1}{t}\right)$

(b) $\frac{tx}{a} - \frac{y}{b} + t = 0$ and $\frac{x}{a} + \frac{ty}{b} - 1 = 0$

(c) $x = e^t + e^{-t}$ and $y = e^t - e^{-t}$

(d) $x^2 - 6 = 2\cos t$ and $y^2 + 2 = 4\cos^2\left(\frac{t}{2}\right)$

43. Equation of common tangent to the two hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ is

(a) $y = x + \sqrt{(a^2 - b^2)}$

(b) $y = x - \sqrt{(a^2 - b^2)}$

(c) $y = -x + \sqrt{(a^2 - b^2)}$

(d) $y = -x - \sqrt{(a^2 - b^2)}$

44. Given ellipse $\frac{x^2}{16} + \frac{y^2}{7} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$, if the ordinate of one of the points of intersection is produced to cut an asymptote at P, then which of the following is true?

(a) They have the same foci

(b) Square of the ordinate of point of intersection is $\frac{63}{25}$

(c) Sum of the squares of coordinate of P is 16

(d) P lies on the auxiliary circle formed by ellipse

45. Solution of the differential equation $(1 - x^2)\frac{dy}{dx} + xy = ax$, where $a \in R$, is

(a) a conic which is an ellipse

(b) centre of the conic is $(0, a)$

(c) length of one of the principal axes is 1

(d) length of one of the principal axes is equal to 2

Hyperbola Exercise 3 : Paragraph Based Questions

- This section contains **5 paragraphs** based upon each of the paragraph 3 **multiple choice questions** have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Paragraph I

(Q. Nos. 46 to 48)

The graph of the conic $x^2 - (y-1)^2 = 1$ has one tangent line with positive slope that passes through the origin. The point of tangency being (a, b)`.

46. The value of $\sin^{-1}\left(\frac{a}{b}\right)$ is

(a) $\frac{5\pi}{12}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$

47. Length of the latusrectum of the conic is

(a) 1 (b) $\sqrt{2}$ (c) 2 (d) 4

48. If e be the eccentricity of the conic, then the value of $(1 + e^2 + e^4)$ is

(a) 3 (b) 7 (c) $\frac{7}{4}$ (d) 21

Paragraph II

(Q. Nos. 49 to 51)

A point P moves such that the sum of the slopes of the normals drawn from it to the hyperbola $xy = 4$ is equal to the sum of the ordinates of feet of normals. The locus of P is a curve C.

49. The equation of the curve C is

(a) $x^2 = 2y$ (b) $x^2 = 4y$

(c) $x^2 = 6y$ (d) $x^2 = 8y$

50. If the tangent to the curve C cuts the coordinate axes at A and B, then, the locus of the middle-point of AB is

(a) $x^2 + 2y = 0$ (b) $x^2 = y$

(c) $2x^2 + y = 0$ (d) $x^2 = 2y$

51. The area of the equilateral triangle inscribed in the curve C having one vertex as the vertex of curve C is

(a) $8\sqrt{3}$ sq units (b) $12\sqrt{3}$ sq units

(c) $27\sqrt{3}$ sq units (d) $48\sqrt{3}$ sq units

Paragraph III

(Q. Nos. 52 to 54)

Let $P(x, y)$ be a variable point such that

$|\sqrt{(x-1)^2 + (y-2)^2} - \sqrt{(x-5)^2 + (y-5)^2}| = 4$

which represents a hyperbola.

52. The eccentricity of the corresponding conjugate hyperbola is

(a) $\frac{5}{4}$ (b) $\frac{4}{3}$

(c) $\frac{5}{3}$ (d) $\frac{3}{2}$

53. Locus of point of intersection of two perpendicular tangents to the hyperbola is

(a) $(x-3)^2+(y-7/2)^2=\frac{1}{4}$

(b) $(x-3)^2+(y-7/2)^2=\frac{3}{4}$

(c) $(x-3)^2+(y-7/2)^2=\frac{5}{4}$

(d) $(x-3)^2+(y-7/2)^2=\frac{7}{4}$

54. If origin is shifted to point (3, 7/2) and axes are rotated in anticlockwise sense through an angle θ, so that the equation of hyperbola reduces to its standard form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then θ equals

(a) $\tan^{-1}\left(\frac{4}{3}\right)$ (b) $\tan^{-1}\left(\frac{3}{4}\right)$

(c) $\tan^{-1}\left(\frac{5}{4}\right)$ (d) $\tan^{-1}\left(\frac{4}{5}\right)$

Paragraph IV

(Q. Nos. 55 to 57)

Let $P(\theta_1)$ and $Q(\theta_2)$ are the extremities of any focal chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ whose eccentricity is e. Let θ be the angle between its asymptotes. Tangents are drawn to the hyperbola at some arbitrary point R. These tangent meet the coordinate axes at the points A and B respectively. The rectangle OACB (O being the origin) is completed, then

55. Locus of point *C* is

(a) $\frac{b^2}{x^2}-\frac{a^2}{y^2}=1$ (b) $\frac{b^2}{x^2}+\frac{a^2}{y^2}=1$

(c) $\frac{a^2}{x^2}+\frac{b^2}{y^2}=1$ (d) $\frac{a^2}{x^2}-\frac{b^2}{y^2}=1$

56. If $\cos^2\left(\frac{\theta_1+\theta_2}{2}\right)=\lambda\cos^2\left(\frac{\theta_1-\theta_2}{2}\right)$, then λ is equal to

(a) $\left(\frac{a^2+b^2}{a^2}\right)$ (b) $\left(\frac{a^2+b^2}{b^2}\right)$

(c) $\left(\frac{a^2+b^2}{ab}\right)$ (d) $\left(\frac{a^2+b^2}{2ab}\right)$

57. The value of $\cos\left(\frac{\theta}{2}\right)$ is

(a) $\frac{1}{2e}$ (b) $\frac{1}{e}$

(c) $\frac{1}{e^2}$ (d) $\frac{1}{2e^2}$

Paragraph V

(Q. Nos. 58 to 60)

The vertices of ΔABC lie on a rectangular hyperbola such that the orthocentre of the triangle is (2, 3) and the asymptotes of the rectangular hyperbola are parallel to the coordinate axes. The two perpendicular tangents of the hyperbola intersect at the point (1, 1).

58. The equation of the asymptotes is

(a) $xy-1=y-x$ (b) $xy+1=x+y$

(c) $xy-1=x-y$ (d) $xy+1=-x-y$

59. The equation of the rectangular hyperbola is

(a) $xy-5=y-x$

(b) $xy-1=x+y$

(c) $xy=x+y+1$

(d) $xy-11=-x-y$

60. The number of real tangents that can be drawn from the point (1, 1) to the rectangular hyperbola is

(a) 0 (b) 2

(c) 3 (d) 4

Hyperbola Exercise 4 :
Single Integer Answer Type Questions

■ This section contains **10 questions.** The answer to each question is **a single digit integer** ranging from 0 to 9 (both inclusive)

61. The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the hyperbola $\frac{x^2}{A^2}-\frac{y^2}{B^2}=1$ are given to be confocal and length of minor axis of the ellipse is same as the conjugate axis of the hyperbola. If e_1 and e_2 represents the eccentricities of ellipse and hyperbola respectively, then the value of $e_1^{-2}+e_2^{-2}$ is

62. If abscissa of orthocentre of a triangle inscribed in a rectangular hyperbola $xy=4$ is $\frac{1}{2}$, then the ordinate of orthocentre of triangle is

63. Normal drawn to the hyperbola $xy=2$ at the point $P(t_1)$ meets the hyperbola again at $Q(t_2)$, then minimum distance between the point P and Q is

64. The normal at P to a hyperbola of eccentricity $\frac{3}{2\sqrt{2}}$ intersects the transverse and conjugate axes at M and N respectively. The locus of mid-point of MN is a hyperbola, then its eccentricity

65. If radii of director circle of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b_1^2}=1$ are in the ratio 1 : 3 and $4e_1^2-e_2^2=\lambda$, where e_1 and e_2 are the eccentricities of ellipse and hyperbola respectively, then the value of λ is

66. The shortest distance between the curves $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $4x^2+4y^2=a^2\,(b>a)$ is $f(a,b)$, then the value of $f(4,6)+f(2,3)$ is

67. ABC is a triangle such that $\angle ABC=2\angle BAC$. If AB is fixed and locus of C is a hyperbola, then the eccentricity of the hyperbola is

68. Point P lie on $2xy=1$. A triangle is constructed by P, S and S' (where S and S' are foci). The locus of ex-centre opposite S (S and P lie in first quadrant) is $(x+py)^2=(\sqrt{2}-1)^2(x-y)^2+q$, then the value of $p+q$ is

69. Chords of the circle $x^2+y^2=4$, touch the hyperbola $\frac{x^2}{4}-\frac{y^2}{16}=1$. The locus of their middle-points is the curve $(x^2+y^2)^2=\lambda x^2-16y^2$, then the value of λ is

70. Tangents are drawn from the point (α,β) to the hyperbola $3x^2-2y^2=6$ and are inclined at angles θ and ϕ to the X-axis. If $\tan\theta\,.\,\tan\phi=2$, then the value of $2\alpha^2-\beta^2$ is

Hyperbola Exercise 5 : Matching Type Questions

■ This section contains 3 **questions.** Each question has four statements (A), (B), (C) and (D) given in **Column I** and four statements (p, q, r and s) in **Column II.** Any given statement in **Column I** can have correct matching with one or move statements (*s*) given in **Column II.**

71. Match the following.

	Column I		Column II
(A)	If λ be the length of the latusrectum of the hyperbola $16x^2 - 9y^2 + 32x + 36y - 164 = 0$, then 3λ is divisible by	(p)	4
(B)	If the chord $x\cos\alpha + y\sin\alpha = p$ of the hyperbola $\frac{x^2}{16} - \frac{y^2}{18} = 1$ subtends a right angle at the centre, a circle touches the given chord and concentric with hyperbola, then the diameter of circle is divisible by	(q)	6
(C)	For the hyperbola $xy = 8$ any tangent of it at P meets coordinate axes at Q and R, then the area of triangle CQR is divisible by (where 'C' is centre of the hyperbola)	(r)	8
(D)	For the hyperbola $x^2 - 3y^2 = 9$, acute angle between its asymptotes is $\frac{\pi\lambda}{24}$, then λ is divisible by	(s)	16

72. Match the following.

	Column I		Column II
A.	If the vertex of a hyperbola bisects the distance between its centre and the corresponding focus, then the ratio of the square of its conjugate axis to the square of its transverse axis is	(p)	A Natural number
B.	With one focus of the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$ as the centre, a circle is drawn which is tangent to the hyperbola with no part of the circle being outside the hyperbola. The radius of the circle is	(q)	A Prime number
C.	If S_1 and S_2 are the foci of the hyperbola whose length of the transverse axis is 4 and that of the conjugate axis is 6, and S_3 and S_4 are the foci of the conjugate hyperbola, then the area of quadrilateral $S_1S_2S_3S_4$ is	(r)	A Composite number
D.	If equation of hyperbola whose conjugate axis is 5 and distance between its foci is 13, is $ax^2 - by^2 = c$, where a and b are co-prime, then $\frac{3ab}{2c}$ is	(s)	A Perfect number

73. If e_1 and e_2 are the roots of the equation $x^2 - \lambda x + 2 = 0$

	Column I		Column II
A.	If e_1 and e_2 are the eccentricities of ellipse and hyperbola respectively, then the values of λ are	(p)	$2\sqrt{2}$
B.	If both e_1 and e_2 are the eccentricities of the hyperbolas, then the values of λ are	(q)	$2\sqrt{3}$
C.	If e_1 and e_2 are the eccentricities of the hyperbola and conjugate hyperbola, then the values of λ are	(r)	$2\sqrt{5}$
D.	If e_1 is the eccentricity of the hyperbola for which there exist infinite points from which perpendicular tangents can be drawn and e_2 is the eccentricity of the hyperbola in which no such points exist, then the values of λ are	(s)	$2\sqrt{6}$

Hyperbola Exercise 6 :
Statement I and II Type Questions

■ **Directions** (Q. Nos. 74 to 81) are Assertion-Reason type questions. Each of these questions contains two statements :
Statement I (Assertion) and **Statement II** (Reason)
Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below :

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I

(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I

(c) Statement I is true, Statement II is false

(d) Statement I is false, Statement II is true

74. **Statement I** $\frac{5}{3}$ and $\frac{5}{4}$ are the eccentricities of two conjugate hyperbolas.

Statement II If e_1 and e_2 are the eccentricities of two conjugate hyperbolas, then $e_1 e_2 > 1$.

75. **Statement I** A hyperbola and its conjugate hyperbola have the same asymptotes.

Statement II The difference between the second degree curve and pair of asymptotes is constant.

76. **Statement I** The equation of the director circle to the hyperbola $5x^2 - 4y^2 = 20$ is $x^2 + y^2 = 1$.

Statement II Director circle is the locus of the point of intersection of perpendicular tangents.

77. **Statement I** Two tangents are drawn from a point on the circle $x^2 + y^2 = 9$ to the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$, then angle between tangents is $\pi / 2$.

Statement II $x^2 + y^2 = 9$ is the director circle of $\frac{x^2}{25} - \frac{y^2}{16} = 1$.

78. **Statement I** If eccentricity of a hyperbola is 2, then eccentricity of its conjugate hyperbola is $2/\sqrt{3}$.

Statement II If e and e_1 are the eccentricities of two conjugate hyperbolas, then $ee_1 > 1$.

79. **Statement I** The line $4x - 5y = 0$ will not meet the hyperbola $16x^2 - 25y^2 = 400$.

Statement II The line $4x - 5y = 0$ is an asymptote to the hyperbola.

80. **Statement I** The point $(5, -3)$ inside the hyperbola $3y^2 - 5x^2 + 1 = 0$.

Statement II The point (x_1, y_1) inside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 < 0$.

81. **Statement I** A hyperbola whose asymptotes include $\pi / 3$ is said to be equilateral hyperbola.

Statement II The eccentricity of an equilateral hyperbola is $\sqrt{2}$.

Hyperbola Exercise 7 :
Subjective Type Questions

■ In this section, there are **12 Subjective questions.**

82. Given the base of a triangle and the ratio of the tangent of half the base angles. Show that the vertex moves on a hyperbola whose foci are the extremities of the base.

83. A, B, C are three points on the rectangular hyperbola $xy = c^2$, find

(i) The area of the triangle ABC.

(ii) The area of the triangle formed by the tangents at A, B and C.

84. If a hyperbola be rectangular and its equation be $xy = c^2$, prove that the locus of the middle points of chords of constant length $2d$ is $(x^2 + y^2)(xy - c^2) = d^2 xy$.

85. If four points be taken on a rectangular hyperbola such that the chord joining any two is perpendicular to the chord joining the other two, and if $\alpha, \beta, \gamma, \delta$ be the inclinations to either asymptote of the straight line joining these points to the centre, prove that $\tan\alpha \tan\beta \tan\gamma \tan\delta = 1$.

86. P and Q are two variable points on the rectangular hyperbola $xy = c^2$ such that tangent at Q passes through the foot of the ordinate of P. Show that the locus of the intersection of tangents at P and Q is a hyperbola with the same asymptotes as of the given hyperbola.

87. A circle cuts two perpendicular lines so that each intercept is of given length. Prove that the locus of the centre of the circle is a rectangular hyperbola.

88. (a) Prove that any line parallel to either of the asymptotes of a hyperbola shall meet it in one point at infinity.

(b) Prove that the asymptotes of a hyperbola are the diagonals of the rectangle formed by the lines drawn parallel to the axes at the vertices of the hyperbola [i.e. at $(\pm a, 0)$ and $(0, \pm b)$].

89. Let the tangent at a point P on the ellipse meet the major axis at B and the ordinate from *it* meet the major axis at A. If Q is a point on the line AP such that $AQ = AB$, prove that the locus of Q is a hyperbola. Find the asymptotes of this hyperbola.

90. From the point (x_1, y_1) and (x_2, y_2), tangents are drawn to the rectangular hyperbola $xy = c^2$. If the conic passing through the two given points and the four points of contact is a circle, then show that $x_1 x_2 = y_1 y_2$ and $x_1 y_2 + x_2 y_1 = 4c^2$.

91. A rectangular hyperbola passes through two fixed points and its asymptotes are in given directions. Prove that its vertices lie on an ellipse and hyperbola which intersect orthogonally.

92. Let normals are drawn from (α, β) to the hyperbola $xy = 1$, and (x_i, y_i), $i = 1, 2, 3, 4$ be the feet of the co-normal points. If the algebraic sum of the perpendicular distances drawn from (x_i, y_i), $i = 1, 2, 3, 4$ onto a variable line vanishes, show that the variable line passes through the point $(\alpha/4, \beta/4)$.

93. A series of hyperbolas is drawn having a common transverse axis of length $2a$. Prove that the locus of a point P on each hyperbola such that its distance from the transverse axis is equal to its distance from on asymptote, is the curve $(x^2 - y^2)^2 = 4x^2 (x^2 - a^2)$.

Hyperbola Exercise 8 : Questions Asked in Previous 13 Year's Exams

- This section contains questions asked in **IIT-JEE, AIEEE, JEE Main & JEE Advanced** from year **2005** to **2017**.

94. The locus of a point $P(\alpha, \beta)$ moving under the condition that the line $y = \alpha x + \beta$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is **[AIEEE 2005, 3M]**

(a) an ellipse
(b) a circle
(c) a parabola
(d) a hyperbola

95. Let a hyperbola passes through the focus of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$. The transverse and conjugate axes of this hyperbola coincide with the major and minor axes of the given ellipse, also the product of eccentricities of given ellipse and hyperbola is 1, then **[IIT- JEE 2006, 5M]**

(a) the equation of hyperbola is $\frac{x^2}{9} - \frac{y^2}{16} = 1$
(b) the equations of hyperbola is $\frac{x^2}{9} - \frac{y^2}{25} = 1$
(c) focus of hyperbola is $(5, 0)$
(d) vertex of hyperbola is $(5\sqrt{3}, 0)$

96. A hyperbola, having the transverse axis of length $2\sin\theta$, is confocal with the ellipse $3x^2 + 4y^2 = 12$. Then, its equation is **[IIT- JEE 2007, 3M]**

(a) $x^2 \operatorname{cosec}^2\theta - y^2 \sec^2\theta = 1$
(b) $x^2 \sec^2\theta - y^2 \operatorname{cosec}^2\theta = 1$
(c) $x^2 \sin^2\theta - y^2 \cos^2\theta = 1$
(d) $x^2 \cos^2\theta - y^2 \sin^2\theta = 1$

97. Two branches of a hyperbola **[IIT- JEE 2007, 1.5M]**

(a) have a common tangent
(b) have a common normal
(c) do not have a common tangent
(d) do not have a common normal

98. For the hyperbola $\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$, which of the following remains constant when α varies **[AIEEE 2007, 3M]**

(a) abscissae of vertices (b) abscissae of foci
(c) eccentricity (d) directrix

99. Consider a branch of the hyperbola $x^2 - 2y^2 - 2\sqrt{2}x - 4\sqrt{2}y - 6 = 0$ with vertex at the point A. Let B be one of the end points of its latusrectum. If C is the focus of the hyperbola nearest to the point A, then the area of the triangle ABC is **[IIT- JEE 2008, 3M]**

(a) $1-\sqrt{\frac{2}{3}}$ (b) $\sqrt{\frac{3}{2}}-1$ (c) $1+\sqrt{\frac{2}{3}}$ (d) $\sqrt{\frac{3}{2}}+1$

100. An ellipse intersects the hyperbola $2x^2 - 2y^2 = 1$ orthogonally. The eccentricity of the ellipse is reciprocal of that of the hyperbola. If the axes of the ellipse are along the coordinate axes, then **[IIT- JEE 2009, 4M]**

(a) equation of ellipse is $x^2 + 2y^2 = 2$
(b) the foci of ellipse are $(\pm 1, 0)$
(c) equation of ellipse is $x^2 + 2y^2 = 4$
(d) the foci of ellipse are $(\pm\sqrt{2}, 0)$

Paragraph

(Q. Nos. 100 and 102)

The circle $x^2 + y^2 - 8x = 0$ *and hyperbola* $\frac{x^2}{9} - \frac{y^2}{4} = 1$ *intersect at the points A and B.*

101. Equation of a common tangent with positive slope to the circle as well as to the hyperbola is

(a) $2x - \sqrt{5}y - 20 = 0$ (b) $2x - \sqrt{5}y + 4 = 0$
(c) $3x - 4y + 8 = 0$ (d) $4x - 3y + 4 = 0$

102. Equation of the circle with AB as its diameter is

(a) $x^2 + y^2 - 12x + 24 = 0$ (b) $x^2 + y^2 + 12x + 24 = 0$
(c) $x^2 + y^2 + 24x - 12 = 0$ (d) $x^2 + y^2 - 24x - 12 = 0$ **[IIT- JEE 2010, 3 + 3M]**

103. The line $2x + y = 1$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If this line passes through the point of intersection of the nearest directrix and the X-axis, then the eccentricity of the hyperbola is **[IIT- JEE 2010, 3M]**

104. Let $P(6,3)$ be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If the normal at the point P intersects the X-axis at $(9, 0)$, then the eccentricity of the hyperbola is **[IIT- JEE 2011, 3M]**

(a) $\sqrt{\frac{5}{2}}$ (b) $\sqrt{\frac{3}{2}}$
(c) $\sqrt{2}$ (d) $\sqrt{3}$

105. Let the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be reciprocal to that of the ellipse $x^2 + 4y^2 = 4$. If the hyperbola passes through a focus of the ellipse, then **[IIT- JEE 2011, 4M]**

(a) the equation of the hyperbola is $\frac{x^2}{3} - \frac{y^2}{2} = 1$
(b) a focus of the hyperbola is $(2,0)$
(c) the eccentricity of the hyperbola is $\sqrt{\frac{5}{3}}$
(d) the equation of the hyperbola is $x^2 - 3y^2 = 3$

106. Tangents are drawn to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$, parallel to the straight line $2x - y = 1$. The points of contact of the tangents on the hyperbola are **[IIT- JEE 2012, 4M]**

(a) $\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ (b) $\left(-\frac{9}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$
(c) $(3\sqrt{3}, -2\sqrt{2})$ (d) $(-3\sqrt{3}, 2\sqrt{2})$

107. Consider the hyperbola $H: x^2 - y^2 = 1$ and a circle S with centre $N(x_2, 0)$. Suppose that H and S touch each other at a point $P(x_1, y_1)$ with $x_1 > 1$ and $y_1 > 0$. The common tangent to H and S at P intersects the X-axis at point M. If (l, m) is the centroid of the triangle PMN, then the correct expression(s) is(are) **[JEE Advanced 2015, 4M]**

(a) $\frac{dl}{dx_1} = 1 - \frac{1}{3x_1^2}$ for $x_1 > 1$ (b) $\frac{dm}{dx_1} = \frac{x_1}{3(\sqrt{(x_1^2 - 1)})}$ for $x_1 > 1$
(c) $\frac{dl}{dx_1} = 1 + \frac{1}{3x_1^2}$ for $x_1 > 1$ (d) $\frac{dm}{dy_1} = \frac{1}{3}$ for $y_1 > 0$

108. The eccentricity of the hyperbola whose length of the latusrectum is equal to 8 and the length of its conjugate axis is equal to half of the distance between its foci, is **[JEE Main 2016, 4M]**

(a) $\frac{2}{\sqrt{3}}$ (b) $\sqrt{3}$
(c) $\frac{4}{3}$ (d) $\frac{4}{\sqrt{3}}$

109. A hyperbola passes through the point $P(\sqrt{2}, \sqrt{3})$ and has foci at $(\pm 2, 0)$. Then the taught to this hyperbola at P also passes through the point **[JEE Main 2017, 4M]**

(a) $(-\sqrt{2}, -\sqrt{3})$ (b) $(3\sqrt{2}, 2\sqrt{3})$
(c) $(2\sqrt{2}, 3\sqrt{3})$ (d) $(\sqrt{3}, \sqrt{2})$

110. If $2x - y + 1 = 0$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{16} = 1$, then which of the following cannot be sides of a right angled triangle? **[JEE Advanced 2017, 4M]**

(a) $2a, 8, 1$ (b) $a, 4, 1$
(c) $a, 4, 2$ (d) $2a, 4, 1$

■ **Direction** (Q. No. 111 to 113) Matching the information given in the three columns of the following table. Columns 1, 2 and 3 contain conics, equations of tangents to the conics and points of contact, respectively.

[JEE Advanced 2017, (3 + 3 + 3) M]

Column 1		Column 2		Column 3	
(I)	$x^2 + y^2 = a^2$	(i)	$my = m^2x + a$	(P)	$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$
(II)	$x^2 + a^2y^2 = a^2$	(ii)	$y = mx + a\sqrt{m^2+1}$	(Q)	$\left(\frac{-ma}{\sqrt{m^2+1}}, \frac{a}{\sqrt{m^2+1}}\right)$
(III)	$y^2 = 4ax$	(iii)	$y = mx + \sqrt{a^2m^2-1}$	(R)	$\left(\frac{-a^2m}{\sqrt{a^2m^2+1}}, \frac{1}{\sqrt{a^2m^2+1}}\right)$
(IV)	$x^2 - a^2y^2 = a^2$	(iv)	$y = mx + \sqrt{a^2m^2+1}$	(S)	$\left(\frac{-a^2m}{\sqrt{a^2m^2-1}}, \frac{-1}{\sqrt{a^2m^2-1}}\right)$

111. The tangent to a suitable conic (Column 1) at $(\sqrt{3}, \frac{1}{2})$ is found to be $\sqrt{3}x + 2y = 4$, then which of the following options is the only correct combination?

(a) (IV) (iii) (S)
(b) (II) (iv) (R)
(c) (IV) (iv) (S)
(d) (II) (iii) (R)

112. For $a = \sqrt{2}$, if a tangent is drawn to a suitable conic (Column 1) at the point of contact $(-1, 1)$, then which of the following options is the only correct combination for obtaining its equation?

(a) (III) (i) (P) (b) (I) (i) (P)
(c) (II) (ii) (Q) (d) (I) (ii) (Q)

113. If a tangent of a suitable conic (Column 1) is found to be $y = x + 8$ and its point of contact is (8, 16), then which of the following options is the only correct combination?

(a) (III) (i) (P) (b) (III) (ii) (Q)
(c) (II) (iv) (R) (d) (I) (ii) (Q)

Answers

Exercise for Session 1

1. (d) **2.** (b) **3.** (d) **4.** (a) **5.** (d)
6. (a) **7.** (a) **8.** (b) **9.** (c,d) **10.** (c)
11. (b) **12.** (c) **13.** $y^2 - x^2 = 5$
14. $8x^2 - y^2 - 64x + 10y + 71 = 0$
16. (1, 2); $2\sqrt{\left(\frac{2}{5}\right)}$; $2\sqrt{5}$; $2\sqrt{3}$ **17.** 2 **18.** $\lambda = \pm 6$
19. $30x - 24y \pm \sqrt{(161)} = 0; \left(\pm\frac{15}{2\sqrt{161}}, \pm\frac{8}{3\sqrt{161}}\right)$

Exercise for Session 2

1. (d) **2.** (c) **3.** (a) **4.** (d) **5.** (b)
6. (a) **7.** (a) **8.** (b) **9.** (d) **10.** (c)
11. $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = \frac{x^2}{a^2} - \frac{y^2}{b^2}$

Exercise for Session 3

1. (b) **2.** (d) **3.** (a) **4.** (d) **5.** (d)
6. (c) **7.** (a) **8.** (a) **9.** (d) **10.** (b)
11. (d) **12.** (b) **13.** $x + 3y = 0$
14. $12x^2 - 7xy - 12y^2 + 31x + 17y = 0$
15. $6x^2 + 13xy + 6y^2 - 38x - 37y - 98 = 0$

Chapter Exercises

1. (a) **2.** (d) **3.** (b) **4.** (c) **5.** (a) **6.** (a)
7. (b) **8.** (a) **9.** (b) **10.** (a) **11.** (b) **12.** (c)
13. (d) **14.** (b) **15.** (c) **16.** (b) **17.** (b) **18.** (b)
19. (d) **20.** (a) **21.** (a) **22.** (c) **23.** (b) **24.** (c)
25. (b) **26.** (c) **27.** (a) **28.** (b) **29.** (b) **30.** (c)
31. (a,c) **32.** (c,d) **33.** (b,c) **34.** (a,b,d) **35.** (a,b,c) **36.** (b,c)
37. (b,c) **38.** (a,b,c) **39.** (a,b) **40.** (a,d) **41.** (a,b,c) **42.** (a,c,d)
43. (a,b,c,d) **44.** (a,b,c,d) **45.** (a,b,d)
46. (c) **47.** (c) **48.** (b) **49.** (b) **50.** (c) **51.** (d)
52. (c) **53.** (d) **54.** (b) **55.** (d) **56.** (a) **57.** (b)
58. (b) **59.** (c) **60.** (b) **61.** (2) **62.** (8) **63.** (4)
64. (3) **65.** (7) **66.** (3) **67.** (2) **68.** (5) **69.** (4)
70. (7)
71. (A) → (p,r,s); (B) → (p,q,r); (C) → (p,r,s); (D) → (p,r)
72. (A) → (p,q); (B) → (p,q); (C) → (p,r); (D) → (p,r,s)
73. (A) → (q,r,s); (B) → (p); (C) → (p); (D) → (q,r,s)
74. (b) **75.** (a) **76.** (d) **77.** (a) **78.** (b) **79.** (a)
80. (c) **81.** (d)
83. (i) $\frac{c^2}{2t_1t_2t_3}|(t_1 - t_2)((t_2 - t_3)(t_3 - t_1)|$

(ii) $2c^2\left|\frac{(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)}{(t_1 + t_2)(t_2 + t_3)(t_3 + t_1)}\right|$

89. $x = 0$ and $x + y = 0$
94. (d) **95.** (a,c) **96.** (a) **97.** (b,c) **98.** (b) **99.** (b)
100. (a,b) **101.** (b) **102.** (a) **103.** (2) **104.** (b) **105.** (b,d)
106. (a,b) **107.** (a,b,d) **108.** (a) **109.** (c) **110.** (a,b,c)
111. (b) **112.** (d) **113.** (a)

Solutions

1. $\because PF_1 \cdot PF_2 = e\left(x_1 - \frac{a}{e}\right).e\left(x_1 + \frac{a}{e}\right)$

$= e^2x_1^2 - a^2$

$= 2x_1^2 - a^2$ (for rectangular hyperbola $e = \sqrt{2}$)

$= x_1^2 + x_1^2 - a^2$ ($\because P(x_1, y_1)$, $x_1^2 - y_1^2 = a^2$)

$= x_1^2 + y_1^2$

$= (OP)^2$

$\therefore \quad \lambda = 1$

2. Let the point of contact of normal from point $P(h, k)$, be $R(t)$ for the hyperbola $xy = c^2$

$\Rightarrow$ The equation of normal is

$ct^4 - xt^3 + yt - c = 0$ which gives

$ct^4 - ht^3 + kt - c = 0$ [$\because$ It passes through (h,k)]...(i)

$\Rightarrow \quad t_1^2 + t_2^2 + t_3^2 + t_4^2 = \lambda$ (given) ... (ii)

From Eq. (i),

$\Sigma t_1 = \frac{h}{c}, \Sigma t_1 t_2 = 0$

$\therefore \quad (\Sigma t_1)^2 - 2\Sigma t_1 t_2 = t_1^2 + t_2^2 + t_3^2 + t_4^2$

$\Rightarrow \quad \left(\frac{h}{c}\right)^2 - 0 = \lambda$

or $h^2 = \lambda c^2$

$\therefore$ Locus of P is $x^2 = \lambda c^2$

3. As branches lies in the second and fourth quadrant.

$\therefore$ We have $xy < 0$

$\Rightarrow \quad m^2 - 9 < 0 \Rightarrow |m| < 3$

4. Let A is (α, β), the B is (β, α)

$\because$ A and B an symmetrical about the line $y = x$.

So, tangents at A and B will be mirror images of each other about $y = x$. Thus, point of intersection will lie on $y = x$.

5. Let $P(h,k)$ be any point on any one member of hyperbola family, having equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$; b is any arbitrary constant, then its asymptotes are given by $y = \pm \frac{b}{a}x$, then according to question

$$|k| = \frac{\left|\pm \frac{b}{a}h - k\right|}{\sqrt{\left(\frac{b^2}{a^2} + 1\right)}}$$

$$\Rightarrow \quad k^2 = \left(\pm \frac{b}{a}h - k\right)^2 \left(\frac{a^2}{a^2 + b^2}\right)$$

or $$k^2 = \frac{(\pm bh - ak)^2}{(a^2 + b^2)} \quad \text{...(i)}$$

Further (h,k) lies on hyperbola

$$\frac{h^2}{a^2} - \frac{k^2}{b^2} = 1 \Rightarrow b = \sqrt{\left(\frac{a^2k^2}{h^2 - a^2}\right)} \quad \text{...(ii)}$$

From Eq. (i) and (ii), we get

$$(k^2 - h^2) = \pm 2h\sqrt{(h^2 - a^2)}$$

$$\Rightarrow \quad (k^2 - h^2)^2 = 4h^2(h^2 - a^2)$$

$\therefore$ Required locus is

$$(x^2 - y^2)^2 = 4x^2(x^2 - a^2)$$

6. We know that if a circle cuts a rectangular hyperbola, then arithmetic mean of points of intersections is the mid-point of centre of hyperbola and circle.

So, $$\frac{3 + 5 + 2 + (-1)}{4} = \frac{-g + 1}{2}$$

and $$\frac{4 + 3 + 6 + 0}{4} = \frac{-f + 2}{2}$$

$\therefore \quad g + f = -8$

7. We know that eccentricity of a parabola and rectangular hyperbola are 1 and $\sqrt{2}$ respectively. Also, irrational roots occur in conjugate pair, thus roots of $f(x) = 0$ are 1, $\sqrt{2}$ and $-\sqrt{2}$

$\therefore \quad f(x) = (x - 1)(x - \sqrt{2})(x + \sqrt{2})$

$= x^3 - x^2 - 2x + 2$

$\therefore \quad a + b + c + d = 1 - 1 - 2 + 2 = 0$

8. Let the point be (α, β)

$\Rightarrow \quad \beta = \alpha + c$

Chord of contact of hyperbola $T = 0$.

$\therefore \quad \frac{x\alpha}{2} - \frac{y\beta}{1} = 1$

$\Rightarrow \quad \frac{x\alpha}{2} - y(\alpha + c) = 1$

$\Rightarrow \quad \left(\frac{x}{2} - y\right)\alpha - (yc + 1) = 0$

Since, this passes through point (x_1, y_1)

$\therefore \quad x_1 = 2y_1$ and $y_1c + 1 = 0$

$\therefore \quad y_1 = \frac{x_1}{2}$

Hence, $$\frac{x_1}{y_1} = 2$$

9. Eliminating x, we have

$$\frac{y^2}{b^2} + \frac{y}{ab} + 1 = 0$$

This equation has real and distinct roots

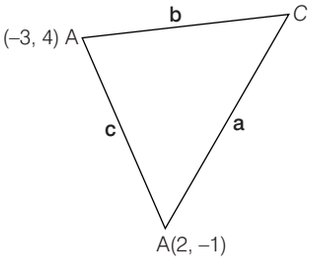
$\therefore \quad \frac{1}{a^2b^2} - \frac{4}{b^2} > 0$

i.e. $$\frac{1}{a^2} > 4 \text{ or } a^2 < \frac{1}{4}$$

$\Rightarrow \quad a < \frac{1}{2}$ and hence the conics intersect if $0 < a < \frac{1}{2}$.

10. Points from where perpendicular tangents can be drawn to the give hyperbola lie on the director circle $x^2 + y^2 = 9 - 16 = -7$ which is an imaginary circle. Hence, no point exists.

11. $\because \dfrac{\tan\left(\frac{1}{2}\angle ABC\right)}{\tan\left(\frac{1}{2}\angle BAC\right)} = \dfrac{3}{1}$



or $\dfrac{\tan\left(\frac{B}{2}\right)}{\tan\left(\frac{A}{2}\right)} = 3 \quad \text{or} \quad \dfrac{\frac{\Delta}{s(s-b)}}{\frac{\Delta}{s(s-a)}} = 3$

or $\dfrac{s-a}{s-b} = 3$

$\Rightarrow \quad s - a = 3s - 3b$

or $2s = 3b - a$

or $a + b + c = 3b - a$

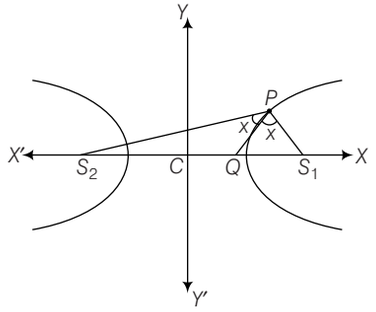
or $b - a = \dfrac{c}{2}$

or $AC - BC = \text{constant}$

$\therefore$ Locus of C is a hyperbola, whose foci are A and B.

$\therefore$ Distance between foci $= |AB| = 5\sqrt{2}$.

12.



Let $P \equiv (x_1, y_1)$

It is clear from figure

$\alpha > 0$ ... (i)

and $Q(\alpha, 0)$ out side on the hyperbola, then

$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - 1 \le 0$

$\Rightarrow \quad \dfrac{\alpha^2}{a^2} - 1 \le 0$

or $\alpha^2 \le a^2$

$\therefore \quad -a \le \alpha \le a$ ... (ii)

From Eqs. (i) and (ii), we get

$0 < \alpha \le a$

$\alpha \in (0, a]$

13. $\because$ Angle between asymptotes $= 2\tan^{-1}\left(\dfrac{b}{a}\right) = 120°$

or $\tan^{-1}\left(\dfrac{b}{a}\right) = 60°$

or $\dfrac{b}{a} = \sqrt{3}$

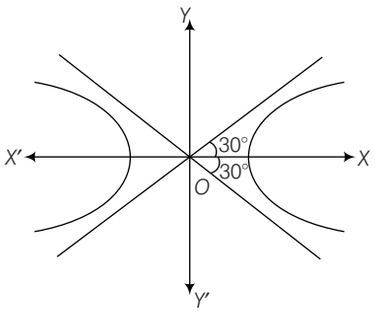
$\therefore \quad b = a\sqrt{3}$ or $b^2 = 3a^2$

$\Rightarrow \quad a^2 = 3 \qquad (\because b^2 = 9)$

Required locus is director circle of the hyperbola and which is $x^2 + y^2 = a^2 - b^2 = 3 - 9 = -6$

which is not possible.

Now, angle between asymptotes $= 2\tan^{-1}\left(\dfrac{b}{a}\right) = 60°$.



or $\tan^{-1}\left(\dfrac{b}{a}\right) = 30°$

or $\dfrac{b}{a} = \dfrac{1}{\sqrt{3}}$

or $a^2 = 3b^2 = 27 \qquad (\because b^2 = 9)$

$\therefore$ Required locus is

$x^2 + y^2 = a^2 - b^2 = 27 - 9 = 18$

14. Equation of chord joining α and β is

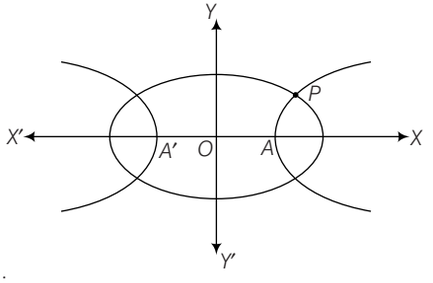
$\dfrac{x}{a}\cos\left(\dfrac{\alpha - \beta}{2}\right) - \dfrac{y}{b}\sin\left(\dfrac{\alpha + \beta}{2}\right) = \cos\left(\dfrac{\alpha + \beta}{2}\right)$

Put $\alpha + \beta = 3\pi$, then

$\dfrac{x}{a}\cos\left(\dfrac{\alpha - \beta}{2}\right) + \dfrac{y}{b} = 0$

It passes through the centre $(0, 0)$.

15. Let $P(a\cos\theta, b\sin\theta)$ on the ellipse.



$\therefore$ Equation of tangent at P on ellipse is

$\dfrac{x}{a}\cos\theta + \dfrac{y}{b}\sin\theta = 1$ ... (i)

and equation of tangent at P on $x^2 - y^2 = c^2$ is

$$x(a\cos\theta) - y(b\sin\theta) = c^2 \quad \ldots(ii)$$

Since, curves intersect at right angle, then

$$-\frac{b}{a}\cot\theta \times \frac{a}{b}\cot\theta = -1$$

$$\therefore \quad \tan^2\theta = 1 \quad \ldots(iii)$$

Since, $P(a\cos\theta, b\sin\theta)$ also lies on hyperbola

$$\therefore \quad (a\cos\theta)^2 - (b\sin\theta)^2 = c^2$$

Dividing both sides by $\cos^2\theta$, then

$$\text{or} \quad a^2 - b^2\tan^2\theta = c^2(1 + \tan^2\theta)$$

$$a^2 - b^2 = 2c^2 \quad \text{[from Eq. (iii)]}$$

16. Let mid-point of the chord is (h,k).

$\therefore$ Equation of chord of $x^2 - y^2 = a^2$ is

$$T = S_1$$

$$hx - ky = h^2 - k^2$$

$$\text{or} \quad y = \frac{h}{k}x - \frac{(h^2 - k^2)}{k}$$

which is tangent of $y^2 = 4ax$

$$\therefore \quad -\frac{(h^2 - k^2)}{k} = \frac{a}{\frac{h}{k}}$$

$$\text{or} \quad -(h^2 - k^2) = \frac{ak^2}{h}$$

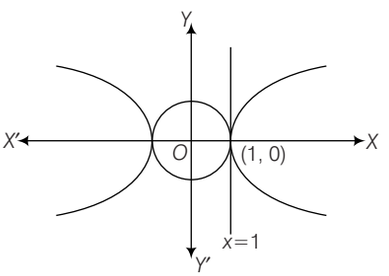
$$\text{or} \quad -h^3 + hk^2 = ak^2$$

$$\text{or} \quad k^2(h - a) = h^3$$

Hence, locus of mid-point is

$$y^2(x - a) = x^3$$

17. It is clear from the figure the common tangent to the circle $x^2 + y^2 = 1$ and hyperbola $x^2 - y^2 = 1$ is $x = 1$ (which is nearer to P (1/2, 1) and given one focus at P (1/2, 1), so the equation of the directrix is $x = 1$. Hence, the equation of the ellipse is



$$\sqrt{(x - 1/2)^2 + (y - 1)^2} = \frac{1}{2}|x - 1|$$

$$\Rightarrow \quad 9(x - 1/3)^2 + 12(y - 1)^2 = 1$$

18. Since, asymptotes of rectangular hyperbola are perpendicular to each other.

$\because$ Given asymptote is $3x - 4y - 6 = 0$

$\therefore$ Other asymptote is $4x + 3y + \lambda = 0 \quad \ldots(i)$

Given, centre of hyperbola lies on $x - y - 1 = 0$

Since, asymptotes pass through the centre of hyperbola

$\therefore$ Centre is the point of intersection of $x - y - 1 = 0$

and $3x - 4y - 6 = 0$

$\therefore$ Centre is $(-2, -3)$, also $(-2, -3)$ lies on Eq. (i)

then $\quad -8 - 9 + \lambda = 0$

$$\therefore \quad \lambda = 17$$

Hence, other asymptote is

$$4x + 3y + 17 = 0 \quad \text{[from Eq. (i)]}$$

19. Equation of normal of $y^2 = 4ax$ is

$$y = mx - 2am - am^3$$

which is tangent of $x^2 - y^2 = a^2$

$$\therefore \quad (-2am - am^3)^2 = a^2m^2 - a^2$$

$$\text{or} \quad 4m^2 + m^6 + 4m^4 = m^2 - 1$$

$$\text{or} \quad m^6 + 4m^4 + 3m^2 + 1 = 0$$

20. Let the middle-point of the chord is (h,k)

$$\therefore \quad T = S_1$$

$$3xh - 2yk + 2(x + h) - 3(y + k) = 3h^2 - 2k^2 + 4h - 6k$$

$$\text{Slope of this chord} = \frac{3h + 2}{2k + 3} = 2 \quad \text{(given)}$$

$$\text{or} \quad 3h + 2 = 4k + 6$$

$$\Rightarrow \quad 3h - 4k = 4$$

Hence, locus of middle-point is

$$3x - 4y = 4$$

21. Let $f(x, y) \equiv x^2 + 3xy + 2y^2 + 2x + 3y + 2 = 0$

$$\therefore \frac{\partial f}{\partial x} = 2x + 3y + 2 \text{ and } \frac{\partial f}{\partial y} = 3x + 4y + 3$$

For centre, $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$$\therefore \quad 2x + 3y + 2 = 0$$

$$3x + 4y + 3 = 0$$

After solving, we get

$$x = -1, y = 0$$

$\therefore$ Coordinates of centre are $(-1, 0)$.

22. $\because$ Hyperbola is $\frac{x^2}{16} - \frac{y^2}{9} = 1 \quad \ldots(i)$

Foci are $(\pm 5, 0)$

$$F_1 \equiv (5, 0), F_2 \equiv (-5, 0)$$

Also, $\quad 4e_H = 5$

$$\therefore \quad e_H = \frac{5}{4}$$

Conjugate hyperbola of Eq. (i) is

$$-\frac{x^2}{16} + \frac{y^2}{9} = 1 \quad \ldots(ii)$$

Foci are $(0, \pm 5)$

$$F_3 \equiv (0, 5), F_4 \equiv (0, -5)$$

Also, $\quad 3e_c = 5$

$$\therefore \quad e_c = \frac{5}{3}$$

Equation of asymptotes of Eqs. (i) and (ii) are same

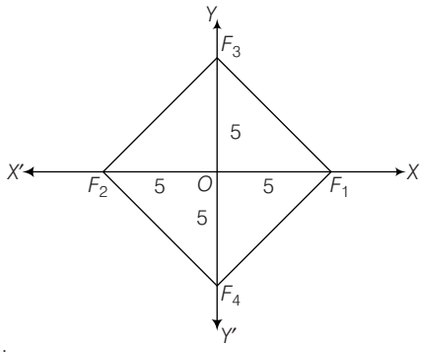
$$y = \pm\frac{3}{4}x \text{ and } e_H < e_c$$

Auxiliary circle of Eq. (i) is $x^2 + y^2 = 7$ and Eq. (ii) is

$$x^2 + y^2 = -7$$

and area of quadrilateral formed by their foci $= 4 \times \frac{1}{2} \times 5 \times 5$

$= 50$ sq units



23. Point of intersection of tangents at $(a\sec\alpha, b\tan\alpha)$ and $(a\sec\beta, b\tan\beta)$ is

$$\left(\frac{a\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}, \frac{b\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}\right)$$

Here, $\alpha = \phi, \beta = \frac{\pi}{2} - \phi$

then, point of intersection is $\left(\frac{a\cos\left(\frac{\pi}{4}-\phi\right)}{\frac{1}{\sqrt{2}}}, b\right)$

$$x = a\sqrt{2}\cos\left(\frac{\pi}{4} - \phi\right)$$

and $y = b$

$\therefore$ Required locus is $y = b$

24. $xdy + ydx = 0$

$\Rightarrow \quad d(xy) = 0$

$\therefore \quad xy = c$

which pass through (2, 8), then $2 \times 8 = c$

$\therefore \quad c = 16$

Equation of conic is $xy = 16$

$\therefore$ Length of latusrectum $= 2\sqrt{2}\,(4) = 8\sqrt{2}$

25. $x = t^2 + 1, y = 2t$

$$\Rightarrow \quad x = \left(\frac{y}{2}\right)^2 + 1$$

$$\Rightarrow \quad x = \frac{y^2}{4} + 1$$

$$\Rightarrow \quad y^2 = 4(x-1) \qquad \ldots (i)$$

Also, $x = 2s, y = \frac{2}{s}$

$$\therefore \quad xy = 4 \qquad \ldots (ii)$$

From Eqs. (i) and (ii),

$$\left(\frac{4}{x}\right)^2 = 4(x-1)$$

$$\Rightarrow \quad \frac{16}{x^2} = 4(x-1)$$

$$\Rightarrow \quad 4 = x^3 - x^2$$

$$\Rightarrow \quad x^3 - x^2 - 4 = 0$$

$$\Rightarrow \quad (x-2)(x^2 + x + 2) = 0$$

$$\therefore \quad x = 2,\ x^2 + x + 2 \neq 0$$

From Eq. (ii),

$$y = \frac{4}{2} = 2$$

Point of intersection is (2, 2)

26. Let rectangular hyperbola $xy = c^2$

Equation of tangent at 't' is

$$\frac{x}{t} + yt = 2c$$

$$\Rightarrow \quad \frac{x}{2ct} + \frac{y}{\left(\frac{2c}{t}\right)} = 1 \qquad \ldots (i)$$

and equation of normal at 't' is

$$xt^3 - yt - ct^4 + c = 0$$

$$\Rightarrow \quad xt^3 - ty = ct^4 - c$$

$$\Rightarrow \quad \frac{x}{\left(\frac{ct^4 - c}{t^3}\right)} + \frac{x}{\left(\frac{-ct^4 + c}{t}\right)} = 1 \qquad \ldots (ii)$$

From Eqs. (i) and (ii) it is clear that

$$x_1 = 2ct,\ x_2 = \left(\frac{ct^4 - c}{t^3}\right)$$

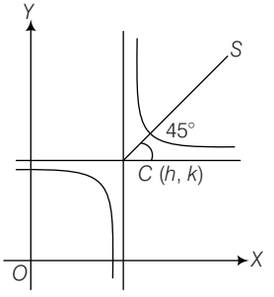
and $\quad y_1 = \frac{2c}{t}, y_2 = \left(\frac{-ct^4 + c}{t}\right)$

$\therefore \quad x_1x_2 + y_1y_2$

$$= \frac{2ct(ct^4 - c)}{t^3} + \frac{2c}{t} \cdot \frac{(-ct^4 + c)}{t}$$

$$= \frac{2c^2(t^4 - 1)}{t^2} + \frac{2c^2(t^4 - 1)}{t^2} = 0$$

27. $CS = p\sqrt{2}$



$\therefore$ Coordinate of S is either

$(h + p, k + p)$ or $(h - p, k - p)$

28. $\because H : x^2 + 3xy + 2y^2 + 2x + 3y = 0$

Let pair of asymptotes is

$$x^2 + 3xy + 2y^2 + 2x + 3y + \lambda = 0$$

$\therefore \quad \Delta = 0$

$$1\times2\times\lambda+2\times\frac{3}{2}\times1\times\frac{3}{2}-1\times\frac{9}{4}-2\times1-\lambda\times\frac{9}{4}=0$$

$$\Rightarrow\quad -\frac{\lambda}{4}+\frac{9}{2}-\frac{9}{4}-2=0$$

$$\Rightarrow\quad \frac{\lambda}{4}=\frac{9}{4}-2=\frac{1}{4}$$

$$\therefore\quad \lambda=1$$

$$A: x^2+3xy+2y^2+2x+3y+1=0$$

$$\because\quad H+C=2A$$

$$\therefore\quad C=2A-H$$

$$=x^2+3xy+2y^2+2x+3y+2$$

$\therefore$ Conjugate hyperbola is $x^2+3xy+2y^2+2x+3y+2=0$

29. The equation of the hyperbola is $\frac{x^2}{9}-\frac{y^2}{16}=1$

The equation of the tangent is

$$y=mx+\sqrt{(9m^2-16)}$$

or $\sqrt{(9m^2-16)}=2\sqrt{5}$

or $9m^2-16=20$

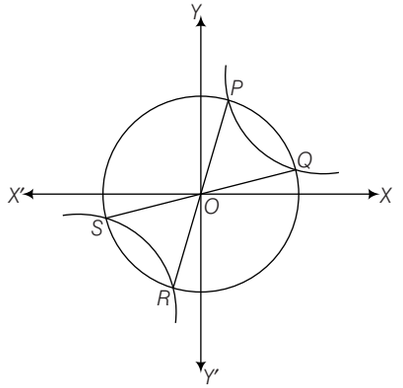
or $m^2=4$

$\therefore\quad m=\pm2$

or $a+b=$ sum of roots $=0$

30. $\because\ x-2=m$ and $y+1=\frac{4}{m}$

$\therefore\quad (x-2)(y+1)=4$



Let $x-2=X,\ y+1=Y$

$\therefore\quad XY=4$ ... (i)

and $s=(x-2)^2+(y+1)^2=25$

or $X^2+Y^2=25$

Curve C and circle are concentric, therefore,

$$(OP)^2+(OQ)^2+(OR)^2+(OS)^2=4r^2$$

$$=4(5)^2$$

$$=100$$

31. Equation of any tangent to the parabola $y^2=8x$

is $y=mx+\frac{2}{m}$ ... (i)

which is also touches the hyperbola

$\frac{x^2}{1}-\frac{y^2}{3}=1$, then $\left(\frac{2}{m}\right)^2=1\times m^2-3$

or $m^4-3m^2-4=0$

or $(m^2-4)(m^2+1)=0$

$\therefore\quad m^2-4=0$

or $m=\pm2$

From Eq. (i), common tangents are

$$y=2x+1 \text{ and } y=-2x-1$$

i.e. $2x-y+1=0$ and $2x+y+1=0$

32. Foci of the ellipse $\frac{x^2}{k^2a^2}+\frac{y^2}{a^2}=1$ are $(\pm a\sqrt{(k^2-1)},\ 0)$ and foci of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are $(\pm ae, 0)$, $e>1$

$\because$ Foci are coincide, then

$a\sqrt{(k^2-1)}=ae$ or $\sqrt{(k^2-1)}>1$

or $k^2-1>1$ or $k^2>2$

$\therefore\quad k<-\sqrt{2}$ or $k>\sqrt{2}$

33. Equation of chord joining θ and ϕ is

$$\frac{x}{a}\cos\left(\frac{\theta-\phi}{2}\right)-\frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right)=\cos\left(\frac{\theta+\phi}{2}\right)$$

It passes through $(kae,\ 0)$, where $k=\pm1$, then

$$\frac{\cos\left(\frac{\theta-\phi}{2}\right)}{\cos\left(\frac{\theta+\phi}{2}\right)}=\frac{1}{ke}$$

$$\Rightarrow\quad \frac{\cos\left(\frac{\theta-\phi}{2}\right)-\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)+\cos\left(\frac{\theta+\phi}{2}\right)}=\frac{1-ke}{1+ke}$$

$$\therefore\quad \tan\left(\frac{\theta}{2}\right)\tan\left(\frac{\phi}{2}\right)=\frac{1-ke}{1+ke}$$

$$=\begin{cases}\frac{1-e}{1+e}, & \text{for } k=1\\ \frac{1+e}{1-e}, & \text{for } k=-1\end{cases}$$

34. $\because$ Foci of hyperbola are $(\pm3a, 0)$ and foci of ellipse are $(\pm\sqrt{(25-16)}, 0)$ i.e. $(\pm3, 0)$ according to question.

$$a=1$$

and $b^2=a^2(e^2-1)=1(9-1)=8$

Now, hyperbola is $\frac{x^2}{1}-\frac{y^2}{8}=1$

Alternate (a) $a^2+b^2=1+8=9$

Alternate (b) Director circle is $x^2+y^2=1-8=-7$

i.e. there is no director circle.

Alternate (d) Length of latusrectum $=\frac{2b^2}{a}=\frac{2(8)}{1}=16$

35. $16x^2-3y^2-32x-12y-44=0$

$$\Rightarrow\quad 16(x^2-2x)-3(y^2+4y)-44=0$$

$$\Rightarrow 16\{(x-1)^2-1\}-3\{(y+2)^2-4\}-44=0$$

$$\Rightarrow\quad 16(x-1)^2-3(y+2)^2=48$$

$$\frac{(x-1)^2}{(\sqrt{3})^2}-\frac{(y+2)^2}{4^2}=1$$

Alternate (a) : Length of transverse axis
$= 2 \times \sqrt{3} = 2\sqrt{3}$

Alternate (b) : Length of conjugate axis
$= 2 \times 4 = 8$

Alternate (c) : Centre $x - 1 = 0$

and $y + 2 = 0$, i.e. $(1, -2)$

Alternate (d) : $4^2 = (\sqrt{3})^2 (e^2 - 1)$

or $e^2 - 1 = \frac{16}{3}$

$\therefore$ $e^2 = \frac{19}{3}$

then $e = \sqrt{\frac{19}{3}}$

36. $xy = 1$

Equation of normal at 't' i.e. $\left(t, \frac{1}{t}\right)$

is $xt^3 - yt - t^4 + 1 = 0$

Slope is $t^2 = -\frac{a}{b}$,

$-\frac{a}{b} > 0 \Rightarrow \frac{a}{b} < 0$

$\therefore$ $a > 0, b < 0$

or $a < 0, b > 0$

37. P, Q, R, S lies on the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{... (i)}$$

and also lies on

$$xy = c^2 \quad \text{... (ii)}$$

Solving Eqs. (i) and (ii), then

$$x^2 + \left(\frac{c^2}{x}\right)^2 + 2gx + \frac{2fc^2}{x} + c = 0$$

$\Rightarrow$ $x^4 + 2gx^3 + cx^2 + 2fc^2x + c^4 = 0$

$\therefore$ $x_1 x_2 x_3 x_4 = c^4$ ... (iii)

and $P \equiv (x_1, y_1) \equiv \left(x_1, \frac{c^2}{x_1}\right)$

$Q \equiv \left(x_2, \frac{c^2}{x_2}\right)$

and $R \equiv \left(x_3, \frac{c^2}{x_3}\right)$

Let orthocentre $O \equiv (h, k)$

Then, slope of $QR \times$ slope of $OP = -1$

$$\left(\frac{\frac{c^2}{x_3} - \frac{c^2}{x_2}}{x_3 - x_2}\right) \times \left(\frac{k - \frac{c^2}{x_1}}{h - x_1}\right) = -1$$

$\Rightarrow$ $$-\frac{c^2}{x_2 x_3} \times \left(\frac{k - \frac{c^2}{x_1}}{h - x_1}\right) = -1$$

$\Rightarrow$ $$k - \frac{c^2}{x_1} = \frac{hx_2x_3}{c^2} - \frac{x_1x_2x_3}{c^2} \quad \text{... (iv)}$$

Also, slope of $PQ \times$ slope of $OR = -1$

$$k - \frac{c^2}{x_3} = \frac{h\,x_1x_2}{c^2} - \frac{x_1x_2x_3}{c^2} \quad \text{...(v)}$$

From Eqs. (iii) and (iv),

$\therefore$ $h = -\frac{c^4}{x_1x_2x_3}$ and $k = -\frac{x_1x_2x_3}{c^2}$

From Eq. (iii),

$h = -x_4$ and $k = -\frac{c^2}{x_4}$

$\therefore$ Orthocentre lies on $xy = c^2$

i.e. (x_4, y_4) and $(-x_4, -y_4)$

38. (c) $y = x + 5$

Comparing with $y = mx + c$

$\therefore$ $m = 1, c = 5$

Alternate (a) : Condition of tangency

$c = \frac{a}{m}$

$5 = \frac{5}{1}$

which is true.

Alternate (b) :

$9x^2 + 16y^2 = 144$

$\Rightarrow$ $\frac{x^2}{16} + \frac{y^2}{9} = 1$

$\because$ Condition of tangency

$c^2 = a^2m^2 + b^2$

$\Rightarrow$ $25 = 16 \times 1 + 9 = 25$

Which is true

Alternate (c) :

$\frac{x^2}{29} - \frac{y^2}{4} = 1$

$\because$ Condition of tangency

$c^2 = a^2m^2 - b^2$

$25 = 29 \times 1 - 4 = 25$

Which is true.

Alternate (d) : Now length of perpendicular from centre (0, 0) to the line $y = x + 5$ is $\frac{|5|}{\sqrt{2}}$ i.e. $\frac{5}{\sqrt{2}} \neq$ radius (5).

39. Equation of the directrices of the given hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ are

$$x = \pm \frac{5 \times 5}{\sqrt{(25 + 16)}} = \pm \frac{25}{\sqrt{41}} \quad \text{... (i)}$$

Equations of the asymptotes of the given hyperbola are

$$\frac{x^2}{15} - \frac{y^2}{16} = 0 \quad \text{...(ii)}$$

The points of intersection of Eqs. (i) and (ii) are
$(\pm 25/\sqrt{41}, \pm 20/\sqrt{41})$

40. $\because |SP - S'P| = 2a$

$$\Rightarrow \quad 2a = \sqrt{(24-0)^2 + (7-0)^2} - \sqrt{(5-0)^2 + (12-0)^2}$$

$$= 25 - 13 = 12$$

$$\therefore \quad a = 6$$

and $$2ae = \sqrt{(24-5)^2 + (7-12)^2} = \sqrt{(386)}$$

$$\therefore \quad e = \frac{\sqrt{386}}{12}$$

and $$b^2 = a^2(e^2 - 1) = 36\left(\frac{386}{144} - 1\right) = \frac{121}{2}$$

$$\therefore \quad \text{Latusrectum} = \frac{2b^2}{a} = \frac{121}{6}$$

41. The locus of the point of intersection of perpendicular tangents is director circle

$$x^2 + y^2 = a^2 - b^2$$

Now, $$e^2 = 1 + \frac{b^2}{a^2}$$

if $a^2 = b^2$, there is exactly one point (centre of the hyperbola)

i.e, $$e = \sqrt{2}$$

if $$a^2 > b^2 \text{ or } \frac{b^2}{a^2} < 1$$

i.e. $$e^2 < 2$$

or $0 < e < \sqrt{2}$, there are infinite (or more than one) points on the circle.

if $$a^2 < b^2 \text{ or } \frac{b^2}{a^2} > 1$$

i.e. $$e^2 > 2$$

or $e > \sqrt{2}$, there does not exist any point on the plane.

42. **Alternate (a) :**

$$\left(\frac{2x}{a}\right)^2 - \left(\frac{2y}{b}\right)^2 = \left(t + \frac{1}{t}\right)^2 - \left(t - \frac{1}{t}\right)^2 = 4$$

$$\Rightarrow \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Alternate (b) :

$$\because \quad \frac{tx}{a} - \frac{y}{b} + t = 0 \Rightarrow t = \frac{ay}{b(x+a)} \qquad \text{... (i)}$$

and $$\frac{x}{a} + \frac{ty}{b} - 1 = 0 \Rightarrow t = \frac{(a-x)b}{ay} \qquad \text{... (ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{ay}{b(x+a)} = \frac{(a-x)b}{ay}$$

$$\Rightarrow \quad a^2y^2 = b^2(a^2 - x^2)$$

$$\Rightarrow \quad b^2x^2 + a^2y^2 = a^2b^2$$

or $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Alternate (c) : $x^2 - y^2 = (e^t + e^{-t})^2 - (e^t - e^{-t})^2 = 4$

Alternate (d) : $\because x^2 - 6 = 2\cos t$

$$\Rightarrow \quad x^2 - 4 = 2(1 + \cos t) = 4\cos^2\left(\frac{t}{2}\right) \qquad \text{... (i)}$$

and $$y^2 + 2 = 4\cos^2\left(\frac{t}{2}\right) \qquad \text{... (ii)}$$

From Eqs. (i) and (ii), then

$$x^2 - 4 = y^2 + 2$$

or $$x^2 - y^2 = 6$$

43. Equation of any tangent to hyperbola is

$$y = mx \pm \sqrt{(a^2m^2 - b^2)} \qquad \text{... (i)}$$

which is tangent of $-\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

then $$a^2m^2 - b^2 = a^2 - b^2m^2$$

$$\Rightarrow \quad (a^2 + b^2)m^2 = (a^2 + b^2)$$

$$\therefore \quad m^2 = 1$$

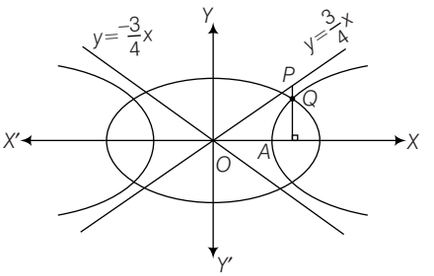
or $$m = \pm 1$$

From Eq. (i), common tangent are

$$y = \pm x \pm \sqrt{(a^2 - b^2)}$$

44. Vertices and foci of ellipse $\frac{x^2}{16} + \frac{y^2}{7} = 1$ are $(\pm 4, 0)$ and $(\pm 3, 0)$ respectively.

Vertices and foci of the hyperbola $\frac{x^2}{\left(\frac{144}{25}\right)} - \frac{y^1}{\left(\frac{81}{25}\right)} = 1$ are $\left(\pm \frac{12}{5}, 0\right)$ and $(\pm 3, 0)$, respectively.



For point of intersection

$$7\left(1 - \frac{x^2}{16}\right) = 81\left(\frac{x^2}{144} - \frac{1}{25}\right)$$

$$\Rightarrow \quad x^2 = \frac{256}{25}$$

and $$y^2 = 7\left(1 - \frac{16}{25}\right) = \frac{63}{25}$$

= square of the ordinate of point of intersection

$$y^2 \text{ for } P = \frac{9}{16}x^2 = \frac{9}{16} \times \frac{256}{25} = \frac{144}{25}$$

$\therefore$ Sum of the squares of coordinates of P is $x^2 + y^2 = 16$ which is auxiliary circle formed by ellipse.

45. $(1 - x^2)\frac{dy}{dx} + xy = ax$

or $$\frac{2dy}{(a-y)} = \frac{2x\,dx}{(1-x^2)}$$

$\Rightarrow \quad -2\ln(a-y) = -\ln(1-x^2) - \ln c$

or $\quad (a-y)^2 = c(1-x^2)$

$\Rightarrow \quad cx^2 + (y-a)^2 = c$

or $\quad \dfrac{x^2}{1} + \dfrac{(y-a)^2}{c} = 1$

for $c > 0$, ellipse and for $c < 0$, hyperbola centre of the conic is $(0, a)$ and length of one of the principal axes $= 2a = 2$

Sol. (Q. Nos. 46 to 48)

Give conic is

$$x^2 - y^2 + 2y - 2 = 0 \qquad \ldots (i)$$

Equation of tangent at (a,b) is

$$ax - by + (y + b) - 2 = 0$$

$\because$ Tangent line pass through $(0, 0)$, then

$$b = 2$$

Also, (a, b) lies on Eq. (i), then

$$a^2 - b^2 + 2b - 2 = 0$$

or $\quad a^2 - 4 + 4 - 2 = 0$

$\Rightarrow \quad a^2 = 2$

$\therefore \quad a = \pm\sqrt{2}$

46. $\because$ Slope of tangent $= \dfrac{a}{b}$

$$= \frac{\sqrt{2}}{2} \qquad \text{(for positive slope)}$$

$$= \frac{1}{\sqrt{2}}$$

$$\therefore \quad \sin^{-1}\left(\frac{a}{b}\right) = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{\pi}{4}$$

47. $\because$ Conic is $x^2 - (y-1)^2 = 1$

$\therefore$ Length of latusrectum $= \dfrac{2(1)^2}{1} = 2$

48. $\because$ Given curve is rectangular hyperbola

$\therefore \quad e = \sqrt{2}$

Thus, $\quad 1 + e^2 + e^4 = 1 + 2 + 4 = 7$

Sol. (Q. Nos. 49 to 51)

Any point on the hyperbola is $(2t, 2/t)$ normal at this point is

$$xt^3 - yt - 2t^4 + 2 = 0$$

If the normal passes through $P(h,k)$, then

$$2t^4 - ht^3 + kt - 2 = 0$$

The equation has roots t_1, t_2, t_3, t_4, then

$$\Sigma t_1 = \frac{h}{2}, \Sigma t_1 t_2 = 0,$$

$$\Sigma t_1 t_2 t_3 = -\frac{k}{2}, t_1 t_2 t_3 t_4 = -1$$

$\because$ Feet of normals are $\left(2t_1, \dfrac{2}{t_1}\right), \left(2t_2, \dfrac{2}{t_2}\right),$

$$\left(2t_3, \frac{2}{t_3}\right) \text{ and } \left(2t_4, \frac{2}{t_4}\right)$$

$\because$ Sum of slopes of normals = Sum of ordinates of feet of normals

$$\therefore \quad t_1^2 + t_2^2 + t_3^2 + t_4^2 = \frac{2}{t_1} + \frac{2}{t_2} + \frac{2}{t_3} + \frac{2}{t_4}$$

$$\Rightarrow \quad \Sigma t_1^2 = 2\Sigma\left(\frac{1}{t_1}\right)$$

49. $(\Sigma t_1)^2 - 2\Sigma t_1 t_2 = 2\left(\dfrac{\Sigma t_1 t_2 t_3}{t_1 t_2 t_3 t_4}\right)$

$$\Rightarrow \quad \left(\frac{h}{2}\right)^2 - 0 = 2\left(\frac{-k/2}{-1}\right) \text{ or } h^2 = 4k$$

Hence, the locus of (h, k) is $x^2 = 4y$

50. $\because$ Curve C is $x^2 = 4y$

$\therefore$ Equation of tangent at $(2t, t^2)$ is $tx = y + t^2$

$\Rightarrow \quad A \equiv (t, 0)$ and $B \equiv (0, -t^2)$

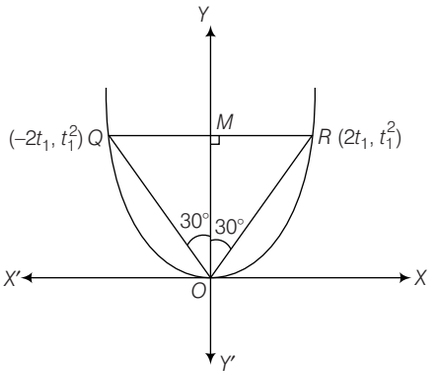
Let mid-point of AB is (h, k).

$\therefore \quad 2h = t$ and $2k = -t^2$

$\Rightarrow \quad 2k = -(2h)^2$ or $2h^2 + k = 0$

$\therefore$ Required locus is $2x^2 + y = 0$

51.



$$\therefore \quad \tan 30° = \frac{2t_1}{t_1^2} \text{ or } \frac{1}{\sqrt{3}} = \frac{2}{t_1}$$

$$\therefore \quad t_1 = 2\sqrt{3}$$

$$\Rightarrow \quad QR = 4t_1 = 8\sqrt{3}$$

Area of equilateral ΔOQR

$$= \frac{\sqrt{3}}{4}(8\sqrt{3})^2 = 48\sqrt{3} \text{ sq units}$$

Sol. (Q. Nos. 52 to 54)

$\because S_1(1,2)$ and $S_2(5, 5)$ are the foci

then $\quad S_1S_2 = 5 = 2ae \qquad \ldots (i)$

and $\quad |S'P - SP| = 2a$

Here, $\quad 2a = 4 \qquad \ldots (ii)$

From Eqs. (i) and (ii)

$$e = \frac{5}{4}$$

52. Let e' be the eccentricity of conjugate hyperbola.

$$\therefore \quad \frac{1}{e^2} + \frac{1}{e'^2} = 1$$

$$\Rightarrow \quad \frac{16}{25} + \frac{1}{e'^2} = 1 \Rightarrow \frac{1}{e'^2} = \frac{9}{25}$$

$$\therefore \quad e' = \frac{5}{3}$$

53. Locus of point of intersection of two mutually perpendicular tangents is the director circle given by

$$(x-h)^2+(y-k)^2=(a^2-b^2);$$

where $(h,\ k)$ is the centre of hyperbola given by

$$(h,k)=\left(\frac{1+5}{2},\frac{2+5}{2}\right)\equiv\left(3,\frac{7}{2}\right)$$

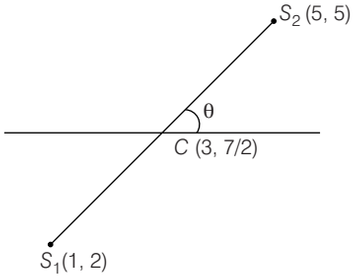
$\therefore$ $a^2=4,\ b^2=a^2(e^2-1)$

$$=4\left(\frac{25}{16}-1\right)=\frac{9}{4}$$

$\therefore$Required locus (director circle) is

$$(x-3)^2+\left(y-\frac{7}{2}\right)^2=4-\frac{9}{4}=\frac{7}{4}$$

54.



$\because$ θ should be the angle between the transverse axis and X-axis, given by

$$\tan\theta=\frac{5-2}{5-1}=\frac{3}{4}$$

$\therefore$ $\theta=\tan^{-1}\left(\frac{3}{4}\right)$

Sol. (Q. Nos. 55 to 57)

$\because$ $P(a\sec\theta_1,\ b\tan\theta_1)$ and $Q(a\sec\theta_2,\ b\tan\theta_2)$ are the extremities of focal chord, then

$$\tan\left(\frac{\theta_1}{2}\right)\tan\left(\frac{\theta_2}{2}\right)=\frac{1-e}{1+e}\text{ for focus }S(ae,0)\quad\ldots(i)$$

and $\theta=2\tan^{-1}\left(\frac{b}{a}\right)$ ... (ii)

Let $R\equiv(a\sec\phi,\ b\tan\phi)$

$\therefore$Equation of tangent at R is

$$\frac{x}{a}\sec\phi-\frac{y}{b}\tan\phi=1$$

$\therefore$ $A\equiv(a\cos\phi,0)$

and $B\equiv(0,-b\cot\phi)$

55. Let $C=(h,k)$

$\because$ $OACB$ is rectangle

$\therefore$ Mid-point of OC = mid-point of AB

$\Rightarrow$ $\left(\frac{h}{2},\frac{k}{2}\right)\equiv\left(\frac{a\cos\phi}{2},\frac{-b\cot\phi}{2}\right)$

or $h=a\cos\phi\ \Rightarrow\ \sec\phi=\frac{a}{h}$ ... (iii)

and $k=-b\cot\phi\ \Rightarrow\ \tan\phi=-\frac{b}{k}$ ... (iv)

From Eqs. (iii) and (iv), we get

$$\frac{a^2}{h^2}-\frac{b^2}{k^2}=1$$

Hence, required locus is

$$\frac{a^2}{x^2}-\frac{b^2}{y^2}=1$$

56. From Eq. (i)

$$1+\tan\left(\frac{\theta_1}{2}\right)\tan\left(\frac{\theta_2}{2}\right)=\frac{2}{1+e}$$

$\Rightarrow$ $\frac{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}{\cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right)}=\frac{2}{1+e}$ ... (v)

and $1-\tan\left(\frac{\theta_1}{2}\right)\tan\left(\frac{\theta_2}{2}\right)=\frac{2e}{1+e}$

$\Rightarrow$ $\frac{\cos\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right)}=\frac{2e}{1+e}$ ... (vi)

From Eqs. (v) and (vi), we get

$$\frac{\cos\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}=e\ \text{ or }\ \frac{\cos^2\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos^2\left(\frac{\theta_1-\theta_2}{2}\right)}=e^2$$

or $\lambda=e^2$ (given)

$$=\frac{a^2+b^2}{a^2}$$

57. From Eq. (ii),

$$\theta=2\tan^{-1}\left(\frac{b}{a}\right)$$

$\therefore$ $\tan\left(\frac{\theta}{2}\right)=\frac{b}{a}$

$\Rightarrow$ $\cos\left(\frac{\theta}{2}\right)=\frac{1}{\sec\left(\frac{\theta}{2}\right)}=\frac{1}{\sqrt{1+\tan^2\left(\frac{\theta}{2}\right)}}$

$$=\frac{1}{\sqrt{\left(1+\frac{b^2}{a^2}\right)}}$$

$$=\frac{1}{\sqrt{1+e^2-1}}=\frac{1}{e}$$

Sol. (Q. Nos. 58 to 60)

58. $\because$ Perpendicular tangents intersect at the centre of rectangular hyperbola. Hence, centre of the hyperbola is (1, 1) and the equations of asymptotes are

$$x-1=0\text{ and }y-1=0$$

$\therefore$Pair of asymptotes is

$$(x-1)(y-1)=0$$

or $xy-x-y+1=0$

or $xy+1=x+y$

59. Let equation of the hyperbola be

$$xy - x - y + \lambda = 0$$

It passes through (2, 3), then

$$6 - 2 - 3 + \lambda = 0$$

$$\therefore \quad \lambda = -1$$

So, equation of hyperbola is

$$xy = x + y + 1$$

60. From the centre of the hyperbola, we can draw two real tangents to the rectangular hyperbola.

61. For ellipse (if $a > b$)

$$ae_1 = \sqrt{(a^2 - b^2)}$$

or $$a^2e_1^2 = a^2 - b^2$$

and for hyperbola

$$Ae_2 = \sqrt{(A^2 + B^2)}$$

$$\Rightarrow \quad = \sqrt{(A^2 + b^2)} \qquad [\because 2b = 2B \text{ (given)}]$$

or $$A^2e_2^2 = A^2 + b^2$$

$$\Rightarrow \quad \frac{1}{e_1^2} + \frac{1}{e_2^2} = \frac{a^2}{(a^2 - b^2)} + \frac{A^2}{(A^2 + b^2)} \qquad \text{... (i)}$$

Since, both the curves are confocal $\Rightarrow ae_1 = Ae_2$

or $$a^2e_1^2 = A^2e_2^2$$

$$\Rightarrow \quad a^2 - b^2 = A^2 + b^2$$

or $$A^2 = a^2 - 2b^2 \qquad \text{... (ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{1}{e_1^2} + \frac{1}{e_2^2} = \frac{a^2}{a^2 - b^2} + \frac{a^2 - 2b^2}{a^2 - b^2}$$

$$= \frac{2(a^2 - b^2)}{(a^2 - b^2)} = 2$$

62. Othrocentre of triangle formed by the points $\left(ct_1, \frac{c}{t_1}\right)$, $\left(ct_2, \frac{c}{t_2}\right)$ and $\left(ct_3, \frac{c}{t_3}\right)$ on the rectangular hyperbola.

$$xy = c^2 \text{ is } \left(-\frac{c}{t_1t_2t_3}, -c\,t_1t_2t_3\right)$$

Here, $$c = 2 \text{ and } \frac{-c}{t_1t_2t_3} = \frac{1}{2} \quad \text{or} \quad t_1t_2t_3 = -2c$$

$\therefore$ Ordinate of orthocentre $= -c\,t_1t_2t_3$

$$= -c \times -2c$$

$$= 2c^2 = 2(2)^2$$

$$= 8$$

63. Since the normal drawn at $P(t_1)$ meets the hyperbola $xy = 2$ again at $Q(t_2)$, then

$$t_1^3\,t_2 = -1 \qquad \text{... (i)}$$

$\because$ $$P \equiv \left(\sqrt{2}\,t_1, \frac{\sqrt{2}}{t_1}\right) \text{ and } Q \equiv \left(\sqrt{2}\,t_2, \frac{\sqrt{2}}{t_2}\right)$$

$\therefore$ Distance $$PQ = \sqrt{(\sqrt{2}t_1 - \sqrt{2}t_2)^2 + \left(\frac{\sqrt{2}}{t_1} - \frac{\sqrt{2}}{t_2}\right)^2}$$

$$= \sqrt{2}\,|t_1 - t_2|\sqrt{\left(1 + \frac{1}{t_1^2\,t_2^2}\right)}$$

$$= \sqrt{2}\,\frac{(t_1^4 + 1)^{3/2}}{|t_1^3|} \qquad \text{[from Eq. (i)]}$$

$$= \sqrt{2}\left(t_1^2 + \frac{1}{t_1^2}\right)^{3/2} \geq \sqrt{2}\,(2)^{3/2} = 4 \quad (\because AM \geq GM)$$

$\therefore$ Distance $PQ \geq 4$

Hence, minimum distance between the points P and Q is 4.

64. Let $P(a\sec\theta, b\tan\theta)$ be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with eccentricity $e = \frac{3}{2\sqrt{2}}$

$\therefore$ $$a^2 = 8b^2 \qquad (\because b^2 = a^2(e^2 - 1)) \quad \text{...(i)}$$

$\because$ Equation of normal at P is

$$ax\cos\theta + by\cot\theta = a^2 + b^2$$

$\therefore$ $$M \equiv \left(\left(\frac{a^2 + b^2}{a}\right)\sec\theta, 0\right)$$

$$N \equiv \left(0, \left(\frac{a^2 + b^2}{b}\right)\tan\theta\right)$$

Let mid-point of MN is $R(h,k)$

$\therefore$ $$2h = \left(\frac{a^2 + b^2}{a}\right)\sec\theta \qquad \text{... (ii)}$$

and $$2k = \left(\frac{a^2 + b^2}{b}\right)\tan\theta \qquad \text{... (iii)}$$

From Eqs. (ii) and (iii), we get

$$4a^2h^2 - 4b^2k^2 = (a^2 + b^2)^2$$

$\therefore$Locus of R is

$$\frac{x^2}{\left(\frac{a^2 + b^2}{2a}\right)^2} - \frac{y^2}{\left(\frac{a^2 + b^2}{2b}\right)^2} = 1$$

which have eccentricity

$$e_1 = \sqrt{\left(1 + \frac{a^2}{b^2}\right)} = \sqrt{(1 + 8)} = 3 \qquad \text{[from Eq. (i)]}$$

65. Equations of director circle of ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is } x^2 + y^2 = a^2 + b^2$$

$\therefore$ Radius $(r_1) = \sqrt{(a^2 + b^2)}$

$$= \sqrt{a^2 + a^2(1 - e_1^2)}$$

$$= a\sqrt{(2 - e_1^2)} \qquad \text{... (i)}$$

and director circle of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b_1^2} = 1$ is

$$x^2 + y^2 = a^2 - b_1^2$$

$\therefore$ Radius $(r_2) = \sqrt{(a^2 - b_1^2)}$

$$= \sqrt{a^2 - a^2(e_2^2 - 1)} = a\sqrt{(2 - e_2^2)} \qquad \text{... (ii)}$$

Given, $\frac{r_1}{r_2} = \frac{1}{3}$

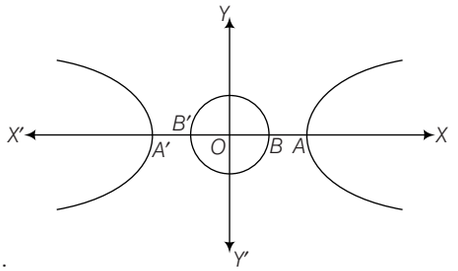
$\Rightarrow \quad \frac{a\sqrt{(2-e_1^2)}}{a\sqrt{(2-e_2^2)}} = \frac{1}{3}$ [from Eqs. (i) and (ii)]

$\Rightarrow \quad 9 - 9e_1^2 = 2 - e_2^2$

$\therefore \quad 9e_1^2 - e_2^2 = 7 = \lambda$

$\therefore \quad \lambda = 7$

66. $\because \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $x^2 + y^2 = \frac{a^2}{4}$ having origin as their common centre.



So, vertex is the nearest point. Hence, shortest distance = BA

$= a - \frac{a}{2} = \frac{a}{2}$

$\therefore \quad f(a, b) = \frac{a}{2}$

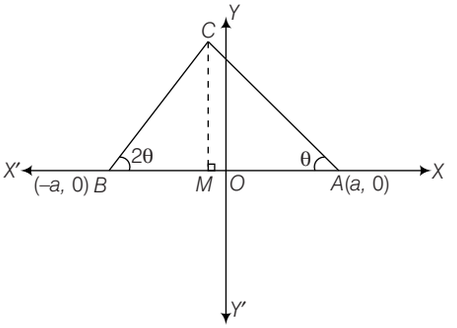
Hence, $f(4,6) + f(2,3) = \frac{4}{2} + \frac{2}{2} = 3$

67. Let $A \equiv (a, 0)$ and $B \equiv (-a, 0)$

If $C \equiv (x, y)$

$\therefore \quad \tan\theta = \frac{y}{a + x}$ ... (i)

and $\tan 2\theta = \frac{y}{a - x}$ ... (ii)



From Eq. (ii),

$\frac{2\tan\theta}{1 - \tan^2\theta} = \frac{y}{a - x}$

$\Rightarrow \quad \frac{\frac{2y}{a + x}}{1 - \left(\frac{y}{a + x}\right)^2} = \frac{y}{a - x}$ [from Eq. (i)]

or $\frac{2y(a + x)}{a^2 + x^2 + 2ax - y^2} = \frac{y}{a - x}$

$\Rightarrow \quad 2(a^2 - x^2) = a^2 + x^2 + 2ax - y^2$

or $3x^2 + 2ax - y^2 - a^2 = 0$

or $3\left(x + \frac{a}{3}\right)^2 - y^2 = \frac{2a^2}{3}$

or $\frac{\left(x + \frac{a}{3}\right)^2}{\left(\frac{2a^2}{9}\right)} - \frac{y^2}{\left(\frac{2a^2}{3}\right)} = 1$

$\therefore \quad$ Eccentricity $(e) = \sqrt{\left(\frac{\frac{2a^2}{9} + \frac{2a^2}{3}}{\frac{2a^2}{9}}\right)} = 2$

68. Let $P\left(\frac{t}{\sqrt{2}}, \frac{1}{t\sqrt{2}}\right)$ be any point on $2xy = 1$ and $S \equiv (1,1)$

and $S' \equiv (-1,-1)$

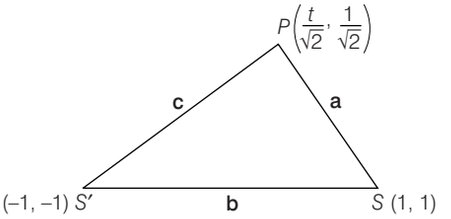
Here $t > 0$

Let $R(h, k)$ be the ex-centre of $\Delta PSS'$ opposite to vertex S.

$\Rightarrow \quad h = \frac{-a - c + \frac{tb}{\sqrt{2}}}{a + b - c}$

and $k = \frac{-a - c + \frac{b}{t\sqrt{2}}}{a + b - c}$

Here, $b = 2\sqrt{2}, a = ePM$



$= \sqrt{2}\,\frac{\left(\frac{t}{\sqrt{2}} + \frac{1}{t\sqrt{2}} - 1\right)}{\sqrt{2}} = \frac{t}{\sqrt{2}} + \frac{1}{t\sqrt{2}} - 1$

and $c = \sqrt{2}\,\frac{\left(\frac{t}{\sqrt{2}} + \frac{1}{t\sqrt{2}} + 1\right)}{\sqrt{2}}$

$= \frac{t}{\sqrt{2}} + \frac{1}{t\sqrt{2}} + 1$ ($\because$ Directrices are $x + y = \pm 1$)

$\therefore \quad h = \frac{-\sqrt{2}\left(t + \frac{1}{t}\right) + 2t}{2\sqrt{2} - 2}$

and $k = \frac{-\sqrt{2}\left(t + \frac{1}{t}\right) + \frac{2}{t}}{2\sqrt{2} - 2}$

$\therefore \quad h + k = -\left(t + \frac{1}{t}\right)$ ... (i)

and $h-k=\dfrac{t-\dfrac{1}{t}}{\sqrt{2}-1}$ ... (ii)

$\because \quad \left(t+\dfrac{1}{t}\right)^2=\left(t-\dfrac{1}{t}\right)^2+4$

$\Rightarrow \quad (h+k)^2=(\sqrt{2}-1)^2(h-k)^2+4$

$\therefore$ Locus of R is $(x+y)^2=(\sqrt{2}-1)^2(x-y)^2+4$

On comparing, we get $p=1$ and $q=4$

$\therefore \quad p+q=5$

69. Let $P(h,k)$ be the middle-point, then equation of chord whose mid-point $P(h,k)$ is

$$T=S_1$$

$\Rightarrow \quad hx+ky-4=h^2+k^2-4$

or $\quad y=-\dfrac{h}{k}x+\dfrac{(h^2+k^2)}{k}$

It will touch the hyperbola $\dfrac{x^2}{4}-\dfrac{y^2}{16}=1$, then

$$\left(\frac{h^2+k^2}{k}\right)^2=4\times\frac{h^2}{k^2}-16$$

$\Rightarrow \quad (h^2+k^2)^2=4h^2-16k^2$

$\therefore$ Locus of $P(h,k)$ is

$$(x^2+y^2)^2=4x^2-16y^2$$

$\therefore \quad \lambda=4$

70. The given hyperbola is $\dfrac{x^2}{2}-\dfrac{y^2}{3}=1$

Equation of tangents is $y=mx\pm\sqrt{(2m^2-3)}$

$\because$ Tangents from the point (α,β) will be

$$(\beta-m\alpha)^2=2m^2-3$$

or $\quad m^2(\alpha^2-2)-2m\alpha\beta+\beta^2+3=0$

$\because \quad m_1m_2=\dfrac{\beta^2+3}{\alpha^2-2} \Rightarrow \tan\theta\tan\phi=\dfrac{\beta^2+3}{\alpha^2-2}$

$\Rightarrow \quad 2=\dfrac{\beta^2+3}{\alpha^2-2}$

or $\quad 2\alpha^2-\beta^2=7$

71. **(A)** $16x^2-9y^2+32x+36y-164=0$

$\Rightarrow \quad 16(x^2+2x+1)-9(y^2-4y+4)-144=0$

or $\quad 16(x+1)^2-9(y-2)^2=144$

or $\quad \dfrac{(x+1)^2}{9}-\dfrac{(y-2)^2}{16}=1$

$\therefore \quad \lambda=$ length latusrectum $=\dfrac{2\times16}{3}=\dfrac{32}{3}$

or $\quad 3\lambda=32$

(B) Making $\dfrac{x^2}{16}-\dfrac{y^2}{18}=1$ homogeneous with the help of $x\cos\alpha+y\sin\alpha=p$, then

$$\frac{x^2}{16}-\frac{y^2}{18}=\left(\frac{x\cos\alpha+y\sin\alpha}{p}\right)^2$$

Since, these lines are perpendicular to each other, then

$$\left(\frac{1}{16}-\frac{\cos^2\alpha}{p^2}\right)+\left(-\frac{1}{18}-\frac{\sin^2\alpha}{p^2}\right)=0$$

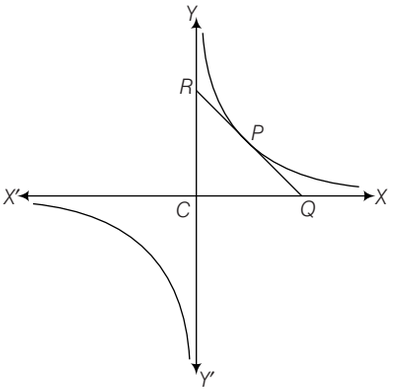
or $\quad \dfrac{1}{16}-\dfrac{1}{18}=\dfrac{1}{p^2}$

or $\quad p=\pm12$

$\therefore$ Radius of circle $=|p|=2$

Hence, diameter of the circle $=24$

(C) Any point on $xy=8$ is $P\left(\sqrt{8}\,t,\dfrac{\sqrt{8}}{t}\right)$



Equation of tangent at P

$$x\times\frac{\sqrt{8}}{t}+y\times\sqrt{8}\,t=16 \text{ or } \frac{x}{\left(\frac{16t}{\sqrt{8}}\right)}+\frac{x}{\left(\frac{16}{\sqrt{8}\,t}\right)}=1$$

$\therefore \quad CQ=\dfrac{16t}{\sqrt{8}}$ and $CR=\dfrac{16}{\sqrt{8}t}$

Hence, area of $\Delta CQR=\dfrac{1}{2}\times CQ\times CR$

$$=\frac{1}{2}\times\frac{16t}{\sqrt{8}}\times\frac{16}{\sqrt{8}t}=16$$

(D) Hyperbola is $x^2-3y^2=9$

or $\quad \dfrac{x^2}{9}-\dfrac{y^2}{3}=1$

$\because$ Angle between asymptotes

$$=2\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)=2\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$=2\times\frac{\pi}{6}=\frac{\pi\lambda}{24} \qquad \text{(given)}$$

$\therefore \quad \lambda=8$

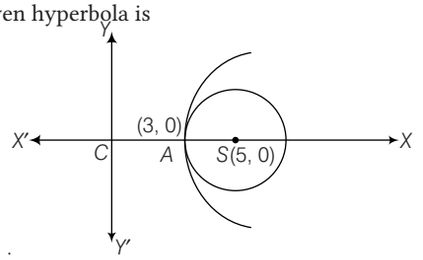
72. **(A)** Let the hyperbola be $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$

then $\quad 2a=ae$

$\Rightarrow \quad e=2$

$\therefore \quad \dfrac{(2b)^2}{(2a)^2}=\dfrac{b^2}{a^2}=e^2-1=(2)^2-1=3$

(B) Given hyperbola is



$$\frac{x^2}{9}-\frac{y^2}{16}=1 \qquad \text{... (i)}$$

or $$e^2=1+\frac{16}{9}=\frac{25}{9}$$

or $$e=\frac{5}{3}$$

Hence, its foci are $(\pm 5, 0)$

The equation of the circle with (5, 0) as centre is

$$(x-5)^2+(y-0)^2=r^2 \qquad \text{... (ii)}$$

Solving Eqs. (i) and (ii), we have

$$16x^2-9(r^2-(x-5)^2)=144$$

or $$25x^2-90x-9r^2+81=0$$

Since, the circle touches the hyperbola, the above equation must have equal roots.

Hence, $$(90)^2-4(25)(81-9r^2)=0$$

or $$r=0$$

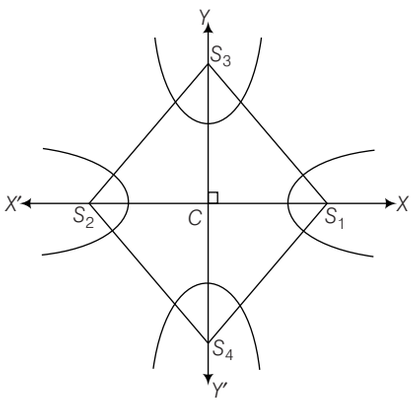
which is not possible.

Hence, the circle can not touch at two points. It can only be tangent at the vertex.

Hence, $$r=5-3=2$$

(C) Equation of hyperbola is

$$\frac{x^2}{2^2}-\frac{y^2}{3^2}=1$$



Its foci $(\pm\sqrt{(2^2+3^2)}, 0)$

or $S_1 \equiv (\sqrt{13}, 0)$ and $S_2 \equiv (-\sqrt{13}, 0)$

and equation of conjugate hyperbola is

$$-\frac{x^2}{2^2}+\frac{y^2}{3^2}=1$$

Its foci $(0, \pm\sqrt{(3^2+2^2)})$

or $S_3 \equiv (0, \sqrt{13})$ and $S_4 \equiv (0, -\sqrt{13})$

Hence, area of quadrilateral $S_1S_2S_3S_4$

$$=4\times \text{Area of } \Delta S_1CS_3$$

$$=4\times\frac{1}{2}\times CS_1\times CS_3$$

$$=2\times\sqrt{13}\times\sqrt{13}=26$$

(D) Given hyperbola is

$$ax^2-by^2=c$$

or $$\frac{x^2}{(c/a)}-\frac{y^2}{(c/b)}=1$$

$\because$ $$\sqrt{\frac{c}{b}}=\frac{5}{2} \qquad \text{...(i)}$$

and $$2\sqrt{\left(\frac{c}{a}+\frac{c}{b}\right)}=13 \qquad \text{...(ii)}$$

From Eqs. (i) and (ii)

$$2\sqrt{\left(\frac{c}{a}+\frac{25}{4}\right)}=13$$

or $$\frac{c}{a}=\frac{169}{4}-\frac{25}{4}=36 \qquad \text{... (iii)}$$

and $$\frac{c}{b}=\frac{25}{4} \qquad \text{... (iv)}$$

$\therefore$ $$\frac{a}{b}=\frac{\frac{c}{b}}{\frac{c}{a}}=\frac{25}{144}$$

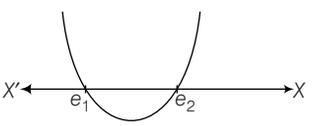
$\because$ a and b are co-prime

$\therefore$ $a=25, b=144$

and from Eq. (iii), $c=900$

Hence, $$\frac{3ab}{2c}=\frac{3\times 25\times 144}{2\times 900}=6$$

73. Let $f(x)=x^2-\lambda x+2$



$\therefore$ $$D=\lambda^2-8$$

x-coordinate of vertex $=\frac{\lambda}{2}$ and $f(1)=3-\lambda$

(A) We must have

$$e_1<1<e_2$$

then $D>0$ and $f(1)<0$

$\Rightarrow$ $\lambda^2>8$ and $3-\lambda<0$

$\Rightarrow$ $\lambda\in(-\infty, -2\sqrt{2})\cup(2\sqrt{2}, \infty)$ and $\lambda>3$

$\therefore$ $\lambda>3$

(B) $\because$ $e_1>1, e_2>1$

then $D\geq 0$ and $f(1)>0$ and $\frac{\lambda}{2}>1$

$\Rightarrow$ $\lambda\in(-\infty, -2\sqrt{2})]\cup[2\sqrt{2}, \infty)$

and $\lambda<3$ and $\lambda>2$

$\therefore$ $\lambda\in[2\sqrt{2}, 3)$

(C) We must have $\frac{1}{e_1^2}+\frac{1}{e_2^2}=1$

or $e_1^2+e_2^2=(e_1e_2)^2$

$\Rightarrow (e_1+e_2)^2-2e_1e_2=(e_1e_2)^2$

$\Rightarrow \lambda^2-4=(2)^2$ or $\lambda^2=8$

$\therefore \lambda=\pm 2\sqrt{2}$

(D) We must have $e_1<\sqrt{2}<e_2$

then $D>0$ and $f(\sqrt{2})<0$

$\Rightarrow \lambda^2>8$ and $2-\lambda\sqrt{2}+2<0$

or $\lambda\in(-\infty,-2\sqrt{2})\cup(2\sqrt{2},\infty)$

and $\lambda>2\sqrt{2}$

$\therefore \lambda>2\sqrt{2}$

74. Since, e_1 and e_2 are the eccentricities of two conjugate hyperbolas, so $e_1>1$ and $e_2>1$.

$\Rightarrow e_2e_2>1$

$\therefore$ Statement II is true.

As for e_1 and e_2 for hyperbola and its conjugate,

$$\frac{1}{e_1^2}+\frac{1}{e_2^2}=1$$

Let $e_1=\frac{5}{3}$ and $e_2=\frac{5}{4}$

Now, $\frac{1}{e_1^2}+\frac{1}{e_2^2}=\frac{9}{25}+\frac{16}{15}=1$

$\therefore$Statement I is true, but Statement II is not correct explanation of Statement I.

75. By definition, if $H(x,y)=0$

then $A_1(x,y)=H(x,y)+\lambda$

$\therefore$ Statement II is true

Since, $C(x,y)=H(x,y)+2$

$=A_1(x,y)+(2-\lambda)$

Since, for $A_1(x,y)=0$, we have $\Delta=0$ and also for $A_2(x,y)$, $\Delta=0$, So $A_1=A_2$

$\therefore$Statement I is true and Statement II is correct explanation of Statement I.

76. $\because$ Director circle is the locus of the point of intersection of perpendicular tangents.

$\therefore$ Statement II is true.

$\because$ Director circle of $5x^2-4y^2=20$

or $\frac{x^2}{4}-\frac{y^2}{5}=1$

is $x^2+y^2=4-5$ or $x^2+y^2=-1$

$\therefore$ Statement I is false.

77. Director circle of $\frac{x^2}{25}-\frac{y^2}{16}=1$ is

$$x^2+y^2=25-16=9$$

Hence, angle between tangents is $\pi/2$.

$\therefore$ Statement I and Statement II are true and Statement II is a correct explanation for Statement I.

78. The conjugate hyperbolas are

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \text{ and } \frac{y^2}{b^2}-\frac{x^2}{a^2}=1$$

then $e^2=\frac{a^2+b^2}{a^2}$ and $e_1^2=\frac{a^2+b^2}{b^2}$

$\therefore \frac{1}{e^2}+\frac{1}{e_1^2}=1$

$\because \frac{1}{2^2}+\frac{1}{\left(\frac{2}{\sqrt{3}}\right)^2}=\frac{1}{4}+\frac{3}{4}=1$ and $ee_1=\frac{4}{\sqrt{3}}>1$

$\therefore$ Statement I and Statement II are both true and Statement II is not a correct explanation of Statement I.

79. If we solve $4x-5y=0$

or $y=\frac{4}{5}x$

and $16x^2-25y^2=400$

or $\frac{x^2}{25}-\frac{y^2}{16}=1$

we get, $0=1\Rightarrow$ No solution

$\Rightarrow$ The line $y=\frac{4}{5}x$ does not meet but the line satisfies the condition of being a tangent.

$$c^2-a^2m^2+b^2=0-25\times\frac{16}{25}+16=0$$

$\Rightarrow$ It must touch the curve at infinity.

$\therefore$Statement I and Statement II are both true but Statement II is a correct explanation for Statement I.

80. The point (x_1,y_1) inside the hyperbola.

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1>0$

$\therefore$ Statement II is false.

Also, $3y^2-5x^2+1=0$

or $5x^2-3y^2-1=0$

$\therefore$ Value of $5x^2-3y^2-1$ at $(5,-3)$ is

$5(5)^2-3(-3)^2-1$

$=125-27-1=97>0$

$\therefore$ Statement I is true.

81. If hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$\therefore$Angle between asymptotes $=2\tan^{-1}\left(\frac{b}{a}\right)$

For equilateral hyperbola $a=b$

then, angle between asymptotes

$$=2\tan^{-1}(1)=2\cdot\frac{\pi}{4}=\frac{\pi}{2}$$

$\therefore$Statement I is false.

and eccentricity $e=\sqrt{\left(\frac{a^2+b^2}{a^2}\right)}=\sqrt{\left(\frac{a^2+a^2}{a^2}\right)}=\sqrt{2}$

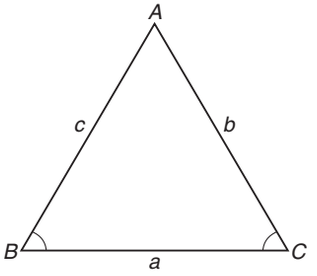
$\therefore$Statement 2 is true.

82. BC = base of the triangle = a (constant) and A is the vertex.

$$\frac{\tan(B/2)}{\tan(C/2)} = \frac{\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}}{\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}}$$

$$= \frac{s-c}{s-b} = \frac{2s-2c}{2s-2b}$$

$$= \frac{a+b-c}{a-b+c} = k \qquad \text{(let)}$$



By componendo and dividendo

$$\Rightarrow \qquad \frac{k-1}{k+1} = \frac{b-c}{a}$$

$$\Rightarrow \qquad b - c = a\left(\frac{k-1}{k+1}\right) = \text{ constant}$$

$$\Rightarrow \qquad CA - BA = \text{constant}$$

By the focal property the locus of A is a hyperbola whose foci are B and C.

83. Let coordinates of A, B and C on the hyperbola $xy = c^2$ are $\left(ct_1, \frac{c}{t_1}\right), \left(ct_2, \frac{c}{t_2}\right)$ and $\left(ct_3, \frac{c}{t_3}\right)$ respectively.

(i) $\therefore$ Area of triangle $ABC = \frac{1}{2}\left|\begin{vmatrix} ct_1 & \frac{c}{t_1} \\ ct_2 & \frac{c}{t_2} \end{vmatrix} + \begin{vmatrix} ct_2 & \frac{c}{t_2} \\ ct_3 & \frac{c}{t_3} \end{vmatrix} + \begin{vmatrix} ct_3 & \frac{c}{t_3} \\ ct_1 & \frac{c}{t_1} \end{vmatrix}\right|$

$$= \frac{c^2}{2}\left|\frac{t_1}{t_2} - \frac{t_2}{t_1} + \frac{t_2}{t_3} - \frac{t_3}{t_2} + \frac{t_3}{t_1} - \frac{t_1}{t_3}\right|$$

$$= \frac{c^2}{2t_1t_2t_3}\left|t_2^2t_3 - t_2^2t_3 + t_1t_2^2 - t_3^2t_1 + t_2t_3^2 - t_1^2t_2\right|$$

$$= \frac{c^2}{2t_1t_2t_3}\left|(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)\right|$$

(ii) Equations of tangents at A, B, C are

$$x + yt_1^2 - 2ct_1 = 0$$

$$x + yt_2^2 - 2ct_2 = 0$$

and $$x + yt_3^2 - 2ct_3 = 0$$

$\therefore$ Required area $= \frac{1}{2|C_1C_2C_3|}\begin{vmatrix} 1 & t_1^2 & -2ct_1 \\ 1 & t_2^2 & -2ct_2 \\ 1 & t_3^2 & -2ct_3 \end{vmatrix}^2$ ... (i)

where $C_1 = \begin{vmatrix} 1 & t_2^2 \\ 1 & t_3^2 \end{vmatrix}$, $C_2 = -\begin{vmatrix} 1 & t_1^2 \\ 1 & t_3^2 \end{vmatrix}$ and $C_3 = \begin{vmatrix} 1 & t_1^2 \\ 1 & t_2^2 \end{vmatrix}$

$\therefore$ $C_1 = t_3^2 - t_2^2$, $C_2 = t_1^2 - t_3^2$ and $C_3 = t_2^2 - t_1^2$

From Eq. (i)

$$= \frac{1}{2|(t_3^2 - t_2^2)(t_1^2 - t_3^2)(t_2^2 - t_1^2)|}$$

$$4c^2 \cdot (t_1 - t_2)^2 (t_2 - t_3)^2 (t_3 - t_1)^2$$

$$= 2c^2\left|\frac{(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)}{(t_1 + t_2)(t_2 + t_3)(t_3 + t_1)}\right|$$

$\therefore$ Required area is, $2c^2\left|\frac{(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)}{(t_1 + t_2)(t_2 + t_3)(t_3 + t_1)}\right|$

84. Let chord be PQ and coordinates of P and Q are $\left(ct_1, \frac{c}{t_1}\right)$ and $\left(ct_2, \frac{c}{t_2}\right)$ respectively.

Now let mid-point of PQ is $M(h, k)$

$\therefore$ $$h = \frac{1}{2}(ct_1 + ct_2)$$

$\therefore$ $$t_1 + t_2 = \frac{2h}{c} \quad \text{and} \quad k = \frac{1}{2}\left\{\frac{c}{t_1} + \frac{c}{t_2}\right\}$$

or $$\frac{2k}{c} = \frac{t_1 + t_2}{t_1t_2}$$

or $$\frac{2k}{c} = \frac{\frac{2h}{c}}{t_1t_2} \qquad \left(\because t_1 + t_2 = \frac{2h}{c}\right)$$

or $$t_1t_2 = \frac{h}{k}$$

$\therefore$ $$(t_1 - t_2)^2 = (t_1 + t_2)^2 - 4t_1t_2 = \frac{4h^2}{c^2} - \frac{4h}{k}$$

$$= \frac{4h}{c^2k}(hk - c^2) \qquad \text{...(i)}$$

Since, $|PQ| = 2d$

$\therefore$ $$\sqrt{(ct_1 - ct_2)^2 + \left(\frac{c}{t_1} - \frac{c}{t_2}\right)^2} = 2d$$

or $$c^2(t_1 - t_2)^2\left\{1 + \frac{1}{t_1^2t_2^2}\right\} = (2d)^2$$

or $$c^2 \times \frac{4h}{c^2k}(hk - c^2)\left\{1 + \frac{k^2}{h^2}\right\} = 4d^2 \qquad \text{[from Eq. (i)]}$$

or $$(h^2 + k^2)(hk - c^2) = d^2hk$$

Hence, locus of the middle-point of PQ is

$$(x^2 + y^2)(xy - c^2) = d^2xy.$$

85. Let the rectangular hyperbola is

$$xy = c^2 \qquad \text{...(i)}$$

Since, the centre of hyperbola (i) is origin (0, 0) and equation of asymptotes are $x = 0$ and $y = 0$.

The equation of a line through (0, 0) and makes an angle θ with asymptote (X-axis) is $y = x \tan\theta$.

It will meet the hyperbola, where $x(x\tan\theta) = c^2$

i.e. $x = c\sqrt{\cot\theta}$

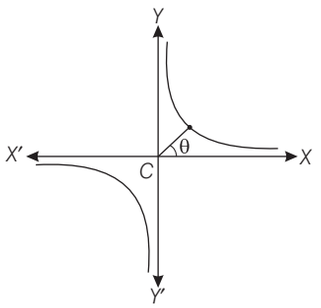
Putting $x = c\sqrt{\cot\theta}$ in (1)

then $y = c\sqrt{(\tan\theta)}$

$\therefore$ The four points are $(c\sqrt{\cot\theta}, c\sqrt{\tan\theta})$ where

$Q = \alpha, \beta, \gamma, \delta.$

The line joining the points α and β is perpendicular to the line joining the points γ and δ.



Therefore, the product of their slopes $= -1$

i.e. $$\frac{c\sqrt{\tan\beta} - c\sqrt{\tan\alpha}}{c\sqrt{\cot\beta} - c\sqrt{\cot\alpha}} \times \frac{c\sqrt{\tan\delta} - c\sqrt{\tan\gamma}}{c\sqrt{\cot\delta} - c\sqrt{\cot\gamma}} = -1$$

$\Rightarrow$ $$(-\sqrt{\tan\alpha}\sqrt{\tan\beta}) \times (-\sqrt{\tan\gamma}\sqrt{\tan\delta}) = -1$$

or $$\tan\alpha\tan\beta\tan\gamma\tan\delta = 1$$

86. Let $P\left(ct_1, \frac{c}{t_1}\right)$ and $Q\left(ct_2, \frac{c}{t_2}\right)$ be any two points on $xy = c^2$.

Then tangents at P and Q are

$$x + yt_1^2 = 2ct_1 \quad \text{...(i)}$$

and $$x + yt_2^2 = 2ct_2 \quad \text{...(ii)}$$

On solving Eqs. (i) and (ii), the point of intersection, say (h, k), is given by

$$h = \frac{2ct_1t_2}{t_1 + t_2} \quad \text{and} \quad k = \frac{2c}{t_1 + t_2} \quad \text{...(iii)}$$

The foot of the ordinate of P is $(ct_1, 0)$ and it lies on Eq. (ii) then

$$ct_1 + 0 = 2ct_2$$

$\therefore$ $$t_1 = 2t_2 \quad \text{...(iv)}$$

Then, from Eqs. (iii) and (iv),

$$h = \frac{2c \cdot 2t_2 \cdot t_2}{2t_2 + t_2} \quad \text{and} \quad k = \frac{2c}{2t_2 + t_2}$$

$\therefore$ $$h = \frac{4c}{3}t_2 \quad \text{and} \quad k = \frac{2c}{3t_2}$$

$\therefore$ $$h \cdot k = \frac{4c}{3}t_2 \times \frac{2c}{3t_2}$$

$\therefore$ $$hk = \frac{8}{9}c^2$$

$\therefore$ Locus of (h, k) is $xy = \frac{8}{9}c^2$, which is a rectangular hyperbola with the same asymptotes $x = 0$ and $y = 0$ as those of $xy = c^2$.

87. Take two given perpendicular straight lines as the coordinate axes and let the equation of variable circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...(i)}$$

Suppose circle (i) make an intercept of length a on X-axis and an intercept of length b on Y-axis.

$\therefore$ $$a = 2\sqrt{(g^2 - c)}$$

and $$b = 2\sqrt{(f^2 - c)}$$

Squaring and subtracting these, we get

$$4(g^2 - c) - 4(f^2 - c) = a^2 - b^2$$

or $$g^2 - f^2 = \frac{1}{4}(a^2 - b^2)$$

or $$(-g)^2 - (-f)^2 = \frac{1}{4}(a^2 - b^2)$$

Hence locus of the centre of circle $(-g, -f)$ is

$$x^2 - y^2 = \frac{1}{4}(a^2 - b^2)$$

which is a rectangular hyperbola.

88. Let the hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{...(i)}$$

Its asymptotes are

$$\frac{x}{a} - \frac{y}{b} = 0 \quad \text{and} \quad \frac{x}{a} + \frac{y}{b} = 0$$

(a) Let us take the asymptote

$$\frac{x}{a} - \frac{y}{b} = 0 \Rightarrow y = \frac{b}{a}x \quad \text{...(ii)}$$

Any line parallel to Eq. (ii) is

$$y = \frac{b}{a}x + c \quad \text{...(iii)}$$

Eliminate y from Eqs. (i) and (iii), then

$$\frac{x^2}{a^2} - \frac{1}{b^2}\left(\frac{b}{a}x + c\right)^2 = 1$$

$\Rightarrow$ $$\frac{x^2}{a^2} - \frac{1}{b^2}\left(\frac{b^2x^2}{a^2} + c^2 + \frac{2bc}{a}x\right) = 1$$

$\Rightarrow$ $$0 \cdot x^2 - \frac{2c}{ab}x - \frac{(c^2 + b^2)}{b^2} = 0$$

One root of its equation is infinite since coefficient of x^2 in it is zero. Also from Eq. (iii), when $y \to \infty$ as $x \to \infty$.

$\therefore$ Eqs. (i) and (iii) meet in one point at infinity.

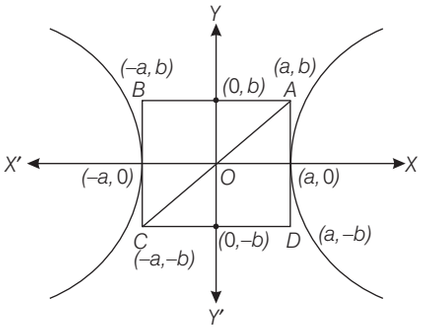
(b) Lines through $(a, 0); (-a, 0); (0, b)$ and $(0, -b)$ are parallel to the principal axes, enclose a rectangle whose vertices are $A(a, b)$, $B(-a, b)$, $C(-a, -b)$ and $D(a, -b)$. Now equation of diagonal AC is

$$y - b = \frac{b - (-b)}{a - (-a)}(x - a)$$

$\Rightarrow$ $$y - b = \frac{b}{a}(x - a)$$

$\Rightarrow$ $$y = \frac{b}{a}x$$

$\Rightarrow$ $$\frac{x}{a} - \frac{y}{b} = 0$$



i.e. The diagonal AC is one of the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Similarly the diagonal BD is the other asymptote.

89. Let the equation of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad ...(i)$$

Tangent to Eq. (i) at $P(a\cos\phi, b\sin\phi)$ is

$$\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1$$

Coordinate of point B is

$$B \equiv (a\sec\phi, 0) \text{ and } AB = a\{\sec\phi - \cos\phi\}$$

Let coordinate of Q be (x_1, y_1) then

$$x_1 = a\cos\phi \text{ and } y_1 = a(\sec\phi - \cos\phi)$$

So $$y_1 = a\left(\frac{a}{x_1} - \frac{x_1}{a}\right)$$

$$\Rightarrow \quad x_1 y_1 = a^2 - x_1^2$$

Hence, locus of Q is $xy = a^2 - x^2$ ...(ii)

which is clearly a hyperbola.

Since, the equation of a hyperbola and its asymptotes differ in constant terms only, asymptotes of Eq. (ii) are given by $x^2 + xy - a^2 + k = 0$, k is any constant.

It represents two straight lines. The required condition for this is

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

then $$k = a^2$$

$\therefore$ Asymptotes are

$$x = 0 \quad \text{and} \quad x + y = 0$$

90. The given rectangular hyperbola is

$$xy = c^2 \qquad ...(i)$$

Equations of chords of contact of (x_1, y_1) and (x_2, y_2) w.r.t., $xy = c^2$ are

$$xy_1 + x_1 y = 2c^2 \qquad ...(ii)$$

and $$xy_2 + x_2 y = 2c^2 \qquad ...(iii)$$

The equation of the conic passing through Eq. (i) with Eqs. (ii) and (iii) is

$$(xy_1 + x_1 y - 2c^2)(xy_2 + x_2 y - 2c^2) + \lambda (xy - c^2) = 0 \qquad ...(iv)$$

Now, Eq. (iv) represents a circle

$\therefore$ coefficient of x^2 = coefficient of y^2

$\therefore \quad y_1 y_2 = x_1 x_2$

and coefficient of $xy = 0$

$$\Rightarrow \quad x_1 y_2 + x_2 y_1 + \lambda = 0 \qquad ...(v)$$

Again the conic (iv) passes through (x_1, y_1) and (x_2, y_2) then

$$(2x_1 y_1 - 2c^2)(x_1 y_2 + x_2 y_1 - 2c^2) + \lambda (x_1 y_1 - c^2) = 0$$

or $$2(x_1 y_2 + x_2 y_1 - 2c^2) + \lambda = 0 \qquad (\because x_1 y_1 \neq c^2) \quad ...(vi)$$

and $$(x_2 y_1 + y_2 x_1 - 2c^2)(y_2 x_2 + x_2 y_2 - 2c^2) + \lambda (x_2 y_2 - c^2) = 0$$

$$\Rightarrow \quad 2(x_2 y_1 + x_1 y_2 - 2c^2) + \lambda = 0 \qquad (\because x_2 y_2 \neq c^2) \quad ... \text{ (vii)}$$

From Eqs. (v) and (vi), we get

$$2(-\lambda - 2c^2) + \lambda = 0$$

$$-\lambda - 4c^2 = 0$$

$\therefore \quad \lambda = -4c^2$

Then from Eq. (v)

$$x_1 y_2 + x_2 y_1 = 4c^2$$

which is the other condition.

91. Let two perpendicular asymptotes of a rectangular hyperbola are CD and CE.

Let the coordinates of A, B and C are $(a, 0)$, $(0, b)$, (α, β). OX and OY are parallel to CD and CE.

Then equations of CD and CE are

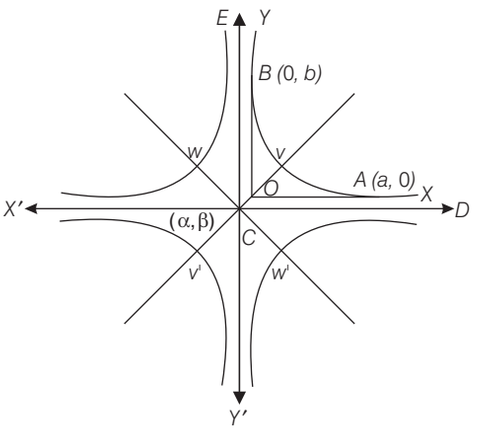
$$y - \beta = 0 \quad \text{and} \quad x - \alpha = 0$$

Thus combined equation of CD and CE is

$$(x - \alpha)(y - \beta) = 0$$

or $$xy - \alpha y - \beta x + \alpha\beta = 0$$

is the equation of asymptotes of rectangular hyperbola.



Hence, equation of rectangular hyperbola is

$$xy - \alpha y - \beta x + k = 0,$$

where k is any constant.

It passes through $A(a, 0)$ and $B(0, b)$

$\therefore \quad 0 - 0 - \beta a + k = 0$

and $$0 - \alpha b - 0 + k = 0$$

$\therefore \quad \beta = \frac{k}{a}$ and $\alpha = \frac{k}{b}$

Hence equation of rectangular hyperbola becomes

$$xy - k\left(\frac{x}{a} + \frac{y}{b} - 1\right) = 0 \qquad ...(i)$$

Eq. (i) represents two sets of rectangular hyperbolas whose vertices lie on the straight lines VCV' and WCW' both passing through (α, β) inclined at 45° and 135° to OX.

Then $\alpha = \frac{k}{b}, \beta = \frac{k}{a}$ and their equations are

$$y - \beta = \pm (x - \alpha)$$

$$\Rightarrow \quad y - \frac{k}{a} = \pm \left(x - \frac{k}{b} \right)$$

or $$x + y = k \left(\frac{1}{a} + \frac{1}{b} \right)$$

and $$x - y = k \left(\frac{1}{b} - \frac{1}{a} \right) \quad \text{...(ii)}$$

or $$k = \frac{ab\,(x + y)}{(a + b)}$$

and $$k = \frac{ab\,(x - y)}{(a - b)}$$

Putting in Eq. (i) then

$$\frac{x^2}{a} + \frac{y^2}{b} - x - y = 0$$

and $$\frac{x^2}{b} - \frac{y^2}{a} - x + y = 0$$

Differentiating these equations, we get

$$\frac{2x}{a} + \frac{2y}{b}\frac{dy}{dx} - 1 - \frac{dy}{dx} = 0$$

and $$\frac{2x}{b} - \frac{2y}{a}\frac{dy}{dx} - 1 + \frac{dy}{dx} = 0$$

$\therefore$ Slopes of lines are

$$m_1 = \frac{dy}{dx} = \frac{b\,(a - 2x)}{a\,(2y - b)}$$

and $$m_2 = \frac{dy}{dx} = \frac{a\,(2x - b)}{b\,(2y - a)}$$

At $(0, 0)$ $\quad m_1 = -1 \quad$ and $\quad m_2 = 1$

$\therefore \quad m_1 m_2 = -1$

which shows that curves intersect orthogonally.

92. Equation of the normal to the hyperbola $xy = 1$ at t is

$$xt^3 - yt - t^4 + 1 = 0 \quad \text{...(i)}$$

$\because$ It passes through (α, β)

$\therefore \quad t^4 - \alpha t^3 + \beta t - 1 = 0$

If foot of the co-normal points are

$$\left(t_1, \frac{1}{t_1} \right), \left(t_2, \frac{1}{t_2} \right), \left(t_3, \frac{1}{t_3} \right) \text{and} \left(t_4, \frac{1}{t_4} \right)$$

$\therefore \quad t_1 + t_2 + t_3 + t_4 = \alpha$

$\Rightarrow \quad x_1 + x_2 + x_3 + x_4 = \alpha$

and $$\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} = \beta$$

$\Rightarrow \quad y_1 + y_2 + y_3 + y_4 = \beta$

Let the variable line be

$$px + qy + r = 0 \quad \text{...(ii)}$$

Since, algebraic sum of perpendicular distances from $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ and (x_4, y_4) to Eq. (ii) is zero.

$$\left(\frac{px_1 + qy_1 + r}{\sqrt{(p^2 + q^2)}} \right) + \left(\frac{px_2 + qy_2 + r}{\sqrt{p^2 + q^2}} \right) + \left(\frac{px_3 + qy_3 + r}{\sqrt{(p^2 + q^2)}} \right) + \left(\frac{px_4 + qy_4 + r}{\sqrt{(p^2 + q^2)}} \right) = 0$$

$$\Rightarrow \quad p\,(x_1 + x_2 + x_3 + x_4) + q\,(y_1 + y_2 + y_3 + y_4) + 4r = 0$$

$$\Rightarrow \quad p \left(\frac{x_1 + x_2 + x_3 + x_4}{4} \right) + q \left(\frac{y_1 + y_2 + y_3 + y_4}{4} \right) + r = 0$$

$$\Rightarrow \quad p \cdot \frac{\alpha}{4} + q \cdot \frac{\beta}{4} + r = 0$$

$\therefore$ Eq. (ii) passes through $\left(\frac{\alpha}{4}, \frac{\beta}{4} \right)$.

93. Let the equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

whose asymptotes are

$$\frac{x}{a} + \frac{y}{b} = 0$$

and $$\frac{x}{a} - \frac{y}{b} = 0$$

Let there be any point $P\,(h, k)$ on the hyperbola

$k =$ distance of P from transverse axis

$h =$ distance of P from any asymptotes

$$\text{say} \left(\frac{x}{a} - \frac{y}{b} = 0 \right)$$

$\therefore \quad k = \frac{bh - ak}{\sqrt{(a^2 + b^2)}}$

Squaring $(a^2 + b^2)\,k^2 = b^2h^2 + a^2k^2 - 2ab\,hk$

$\Rightarrow \quad b\,(h^2 - k^2) = 2ahk$

$\Rightarrow \quad (h^2 - k^2) = \frac{2ahk}{b}$

Squaring $\quad (h^2 - k^2)^2 = 4a^2h^2 \left(\frac{k^2}{b^2} \right)$

$$= 4a^2h^2 \left(\frac{h^2}{a^2} - 1 \right) \qquad \left\{ \because \frac{h^2}{a^2} - \frac{k^2}{b^2} = 1 \right\}$$

$\Rightarrow \quad (h^2 - k^2)^2 = 4h^2(h^2 - a^2)$

Hence locus of $P\,(h, k)$ is

$$(x^2 - y^2)^2 = 4x^2(x^2 - a^2).$$

94. Tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

Given that $y = \alpha x + \beta$ is the tangent of hyperbola

$\Rightarrow \quad m = \alpha$ and $a^2m^2 - b^2 = \beta^2$

$\therefore \quad a^2\alpha^2 - b^2 = \beta^2$

Locus is $a^2x^2 - y^2 = b^2$ which is hyperbola.

95. For the given ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$

$$\Rightarrow \qquad e=\sqrt{\left(1-\frac{16}{25}\right)}=\frac{3}{5}$$

$\Rightarrow$ Eccentricity of hyperbola $=\frac{5}{3}$

Let the hyperbola be $\frac{x^2}{A^2}-\frac{y^2}{B^2}=1$ then

$$B^2=A^2\left(\frac{25}{9}-1\right)=\frac{16}{9}A^2$$

$\therefore \frac{x^2}{A^2}-\frac{9y^2}{16A^2}=1$, As it passes through focus of ellipse i.e. $(3,0)$

$\therefore$ We get $A^2=9\Rightarrow B^2=16$

$\therefore$ Equation of hyperbola is $\frac{x^2}{9}-\frac{y^2}{16}=1$, focus of hyperbola is $(5, 0)$, vertex of hyperbola is $(3, 0)$.

96. The length of transverse axis $=2\sin\theta=2a$

$$\Rightarrow \qquad a=\sin\theta$$

Also for ellipse $3x^2+4y^2=12$

or $$\frac{x^2}{4}+\frac{y^2}{3}=1,\ a^2=4,\ b^2=3$$

$$e=\sqrt{1-\frac{b^2}{a^2}}=\sqrt{1-\frac{3}{4}}=\frac{1}{2}$$

$\therefore$ Focus of ellipse $=\left(2\times\frac{1}{2},0\right)\Rightarrow(1,0)$

As hyperbola is confocal with ellipse, focus of hyperbola $=(1,0)$

$$\Rightarrow \qquad ae=1\ \Rightarrow \sin\theta\times e=1$$

$$\Rightarrow \qquad e=\operatorname{cosec}\theta$$

$$\therefore \qquad b^2=a^2(e^2-1)$$

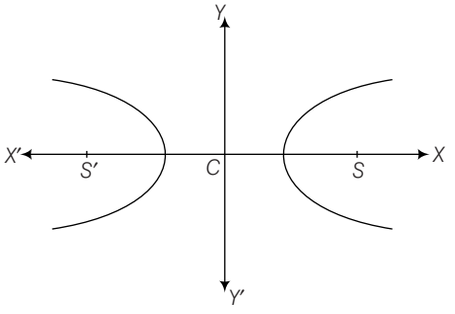
$$=\sin^2\theta(\operatorname{cosec}^2\theta-1)=\cos^2\theta$$

$\therefore$Equation of hyperbola is

$$\frac{x^2}{\sin^2\theta}-\frac{y^2}{\cos^2\theta}=1$$

or, $$x^2\operatorname{cosec}^2\theta-y^2\sec^2\theta=1$$

97. Two branches of hyperbola have no common tangent but have a common normal joining SS'.



98. Given, equation of hyperbola is

$$\frac{x^2}{\cos^2\alpha}-\frac{y^2}{\sin^2\alpha}=1$$

Here, $a^2=\cos^2\alpha$ and $b^2=\sin^2\alpha$

$$\because \qquad b^2=a^2(e^2-1)$$

$$\therefore \qquad \sin^2\alpha=\cos^2\alpha\,(e^2-1)$$

or $\sin^2\alpha+\cos^2\alpha=\cos^2\alpha\,.\,e^2$

or $e^2=1+\tan^2\alpha=\sec^2\alpha\Rightarrow e=\sec\alpha$

$$\therefore \qquad ae=\cos\alpha\,.\,\frac{1}{\cos\alpha}=1$$

Coordinates of foci are $(\pm ae, 0)$ i.e. $(\pm 1,0)$

Hence, abscissae of foci remain constant, when α varies.

99. The given hyperbola is

$$x^2-2y^2-2\sqrt{2}x-4\sqrt{2}y-6=0$$

$$\Rightarrow\ (x^2-2\sqrt{2}x+2)-2(y^2+2\sqrt{2}y+2)=6+2-4$$

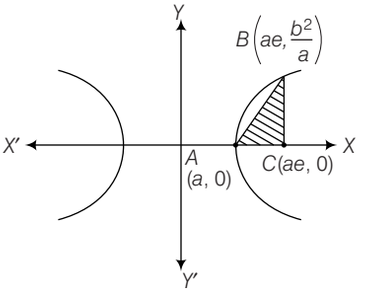
$$\Rightarrow \qquad (x-\sqrt{2})^2-2(y+\sqrt{2})^2=4$$

$$\Rightarrow \qquad \frac{(x-\sqrt{2})^2}{2^2}-\frac{(y+\sqrt{2})^2}{(\sqrt{2})^2}=1$$

$$\therefore \qquad a=2,\ b=\sqrt{2}\ \Rightarrow e=\sqrt{\left(1+\frac{2}{4}\right)}=\sqrt{\frac{3}{2}}$$

Clearly, ΔABC is a right triangle.

$$B\left(ae,\frac{b^2}{a}\right)$$



$$\therefore \quad \text{Area of } (\Delta ABC)=\frac{1}{2}\times AC\times BC$$

$$=\frac{1}{2}\times(ae-a)\times\frac{b^2}{a}$$

$$=\frac{1}{2}(e-1)\,b^2=\frac{1}{2}\left(\sqrt{\frac{3}{2}}-1\right).2$$

$$=\left(\sqrt{\frac{3}{2}}-1\right)$$

100. The given hyperbola is

$$x^2-y^2=\frac{1}{2} \qquad \ldots (i)$$

which is rectangular hyperbola

$$\therefore \qquad e=\sqrt{2}$$

Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Its eccentricity $=\frac{1}{\sqrt{2}}$

$$\therefore \qquad b^2=a^2(1-e^2)=a^2\left(1-\frac{1}{2}\right)=\frac{a^2}{2}$$

So, the equation of ellipse becomes

$$x^2 + 2y^2 = a^2 \quad \text{... (ii)}$$

Let the hyperbola (i) and ellipse (ii) intersect each other at $P(x_1, y_1)$.

$\therefore$ (Slope of hyperbola (i) at (x_1, y_1)) $\times$ (Slope of ellipse (ii) at $(x_1, y_1) = -1$

$$\Rightarrow \quad \frac{x_1}{y_1} \times \frac{-x_1}{2y_1} = -1$$

or $$x_1^2 = 2y_1^2 \quad \text{... (iii)}$$

Also (x_1, y_1) lies on $x^2 - y^2 = \frac{1}{2}$

$$\therefore \quad x_1^2 - y_1^2 = \frac{1}{2} \quad \text{... (iv)}$$

From Eqs. (iii) and (iv), we get $y_1^2 = \frac{1}{2}$ and $x_1^2 = 1$

and (x_1, y_1) lies on ellipse $x^2 + 2y^2 = a^2$

$$\therefore \quad x_1^2 + 2y_1^2 = a^2$$

$$\Rightarrow \quad 1 + 1 = a^2 \text{ or } a^2 = 2$$

$\therefore$ Equation of ellipse is $x^2 + 2y^2 = 2$

whose foci $(\pm 1, 0)$.

Sol. (Q. Nos. 101 and 102)

The intersection points of given circle

$$x^2 + y^2 - 8x = 0 \quad \text{... (i)}$$

and hyperbola

$$4x^2 - 9y^2 = 36 \quad \text{... (ii)}$$

can be obtained by solving these equations substituting value of y^2 from Eq. (i) in Eq. (ii), we get

$$4x^2 - 9(8x - x^2) = 36$$

$$\Rightarrow \quad 13x^2 - 72x - 36 = 0$$

$$\Rightarrow \quad x = 6, -\frac{6}{13}$$

$$\Rightarrow \quad y^2 = 12, -\frac{660}{169} \quad \text{(not possible)}$$

$\therefore A(6, 2\sqrt{3})$ and $B(6, -2\sqrt{3})$ are points of intersection.

101. Equation of tangent to hyperbola having slope m is

$$y = mx + \sqrt{(9m^2 - 4)} \quad \text{... (i)}$$

Equation of tangent to circle is

$$y = m(x - 4) + \sqrt{(16m^2 + 16)} \quad \text{... (ii)}$$

Eqs. (i) and (ii) will be identical, then

$$m = \frac{2}{\sqrt{5}}$$

So, equation of common tangents

$$y = \frac{2x}{\sqrt{5}} + \sqrt{\left(\frac{36}{5} - 4\right)}$$

$$\Rightarrow \quad y = \frac{2x}{\sqrt{5}} + \frac{4}{\sqrt{5}}$$

or $2x - \sqrt{5}y + 4 = 0$

102. Equation of circle with AB as its diameter is

$$(x - 6)(x - 6) + (y - 2\sqrt{3})(y + 2\sqrt{3}) = 0$$

$$\Rightarrow \quad x^2 + y^2 - 12x + 24 = 0$$

103. $\because$ line $2x + y = 1$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$\therefore \quad (1)^2 = a^2(-2)^2 - b^2$$

$$\Rightarrow \quad 4a^2 - b^2 = 1 \quad \text{... (i)}$$

Intersection point of nearest directrix $x = \frac{a}{e}$ and X-axis is $\left(\frac{a}{e}, 0\right)$

As, $2x + y = 1$ passes through $\left(\frac{a}{e}, 0\right)$

$$\therefore \quad \frac{2a}{e} + 0 = 1 \Rightarrow a = \frac{e}{2} \quad \text{... (ii)}$$

and $$b^2 = a^2(e^2 - 1) = \frac{e^2}{4}(e^2 - 1) \quad \text{... (iii)}$$

Substituting the values of a and b from Eqs. (ii) and (iii) in Eq. (i), then

$$e^2 - \frac{e^2}{4}(e^2 - 1) = 1$$

$$\Rightarrow \quad (e^2 - 4)(e^2 - 1) = 0$$

$$\therefore \quad e^2 = 4, e^2 \neq 1 \quad (\because e > 1)$$

Hence, $$e = 2$$

104. Equation of normal at P (6, 3) is

$$\frac{a^2 x}{6} + \frac{b^2 y}{3} = a^2 + b^2$$

$\because$ Normal intersects the X-axis at (9, 0), then

$$\frac{9a^2}{6} + 0 = a^2 + b^2 \Rightarrow 3a^2 = 2a^2 + 2b^2$$

or $$a^2 = 2b^2$$

or $$a^2 = 2a^2(e^2 - 1)$$

$$\therefore \quad e^2 = \frac{3}{2}$$

Hence, $$e = \sqrt{\frac{3}{2}}$$

105. Given, ellipse is $x^2 + 4y^2 = 4$

or $$\frac{x^2}{4} + \frac{y^2}{1} = 1$$

$$\therefore \quad e = \sqrt{\left(1 - \frac{1}{4}\right)} = \frac{\sqrt{3}}{2}$$

and foci are $(\pm\sqrt{3}, 0)$

$\therefore$ Eccentricity of hyperbola $= \frac{2}{\sqrt{3}} = e_1$

and $$b^2 = a^2(e_1^2 - 1) = a^2\left(\frac{4}{3} - 1\right) = \frac{a^2}{3}$$

then equation of hyperbola becomes

$$x^2 - 3y^2 = a^2$$

which pass through $(\pm\sqrt{3}, 0)$

$$3 - 0 = a^2$$

$$\Rightarrow \quad a^2 = 3$$

$\therefore$Equation of hyperbola is

$$x^2 - 3y^2 = 3$$

and foci of hyperbola are $\left(\pm\sqrt{3} \times \frac{2}{\sqrt{3}}, 0\right)$ i.e., $(\pm 2, 0)$

106. Equation of tangent at (x_1, y_1) is

$$\frac{xx_1}{9} - \frac{yy_1}{4} = 1 \quad \text{... (i)}$$

Equation of line parallel to

$$2x - y = \lambda \quad \text{... (ii)}$$

$\because$ Line (ii) is tangent of $\frac{x^2}{9} - \frac{y^2}{4} = 1$, then

$$\lambda^2 = 9 \times 2^2 - 4$$

$$\therefore \quad \lambda = \pm 4\sqrt{2}$$

From Eq. (ii), Equation of tangent is

$$2x - y = \pm 4\sqrt{2} \quad \text{... (iii)}$$

Comparing Eqs. (i) and (iii), we get $\frac{x_1}{18} = \frac{y_1}{4} = \pm \frac{1}{4\sqrt{2}}$

or $\quad x_1 = \pm \frac{9}{2\sqrt{2}}$ and $y_1 = \pm \frac{1}{\sqrt{2}}$

Hence, points of contact of the tangents on the hyperbola are $\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{9}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

107. $H : x^2 - y^2 = 1$

S : Circle with centre $N(x_2, 0)$

Common tangent to H and S at $P(x_1, y_1)$ is

$$xx_1 - yy_1 = 1 \Rightarrow m_1 = \frac{x_1}{y_1}$$

Also radius of circle S with centre $N(x_2, 0)$ through point of contact (x_1, y_1) is perpendicular to tangent.

$$\therefore \quad m_1 m_2 = -1 \Rightarrow \frac{x_1}{y_1} \times \frac{0 - y_1}{x_2 - x_1} = -1$$

$$\Rightarrow \quad x_1 = x_2 - x_1 \text{ or } x_2 = 2x_1$$

M is the point of intersection of tangent at P and X-axis

$$\therefore \quad M\left(\frac{1}{x_1}, 0\right)$$

$\because$ Centroid of ΔPMN is (ℓ, m)

$$\therefore \quad x_1 + \frac{1}{x_1} + x_2 = 3\ell \text{ and } y_1 = 3 \text{ m}$$

Using $x_2 = 2x_1$,

$$\Rightarrow \quad \frac{1}{3}\left(3x_1 + \frac{1}{x_1}\right) = l \text{ and } \frac{y_1}{3} = m$$

$$\therefore \quad \frac{dl}{dx_1} = 1 - \frac{1}{3x_1^2}, \frac{dm}{dy_1} = \frac{1}{3}$$

Also, (x_1, y_1) lies on H,

$$\therefore \quad x_1^2 - y_1^2 = 1 \text{ or } y_1 = \sqrt{(x_1^2 - 1)}$$

$$\therefore \quad m = \frac{1}{3}\sqrt{(x_1^2 - 1)}$$

$$\therefore \quad \frac{dm}{dx_1} = \frac{x_1}{3\sqrt{(x_1^2 - 1)}}$$

108. $\because \frac{2b^2}{a} = 8$ and $2b = \frac{1}{2}(2ae)$

$$\Rightarrow \quad b^2 = 4a \text{ and } b = \frac{1}{2}ae \text{ or } b^2 = \frac{1}{4}a^2e^2$$

$$\Rightarrow \quad a^2(e^2 - 1) = \frac{1}{4}a^2e^2 \Rightarrow 4e^2 - 4 = e^2 \text{ or } e^2 = \frac{4}{3}$$

$$\therefore \quad e = \frac{2}{\sqrt{3}}$$

$$\Rightarrow \quad b^2 = 4a \text{ and } b = \frac{1}{2}ae \text{ or } b^2 = \frac{1}{4}a^2e^2$$

109. Let Equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

When it passes through $P(\sqrt{2}, \sqrt{3})$, then $\frac{2}{a^2} - \frac{3}{b^2} = 1$

$$\Rightarrow \quad 2b^2 - 3a^2 = a^2b^2 \quad \text{...(i)}$$

and $ae = 2 \Rightarrow a^2 + b^2 = 4$ ...(ii)

From Eqs. (i) and (ii), we get $2(4 - a^2) - 3a^2 = a^2(4 - a^2)$

$$\Rightarrow \quad a^2 - 9a^2 + 8 = 0 \Rightarrow (a^2 - 1)(a^2 - 8) = 0$$

$$\therefore \quad a^2 = 1 \qquad (\because a^2 \neq 8)$$

From Eq. (ii), $b^2 = 3$

$\therefore$ Equation of Hyperbola is $\frac{x^2}{1} - \frac{y^2}{3} = 1$

Equation of tangent at $P(\sqrt{2}, \sqrt{3})$ is $\frac{x\sqrt{2}}{1} - \frac{y\sqrt{3}}{3} = 1$

Which is passes through $(2\sqrt{2}, 3\sqrt{2})$

110. (a,b,c) $\because 2x - y + 1 = 0$ is a tangent of $\frac{x^2}{a^2} - \frac{y^2}{16} = 1$, then

$$(1)^2 = a^2(2)^2 - 16 \text{ or } (2a)^2 = (1)^2 + (4)^2$$

$\therefore 2a$, 4, 1 are the sides of a right angled triangle.

Sol. (Q. Nos. 111-113)

I. $x^2 + y^2 = a^2$

Equations of tangents in terms of slopes are

$$y = mx + a\sqrt{(m^2 + 1)} \text{ (ii)}$$

and points of contact in terms of slopes are

$$\left(\frac{-ma}{\sqrt{(m^2 + 1)}}, \frac{a}{\sqrt{(m^2 + 1)}}\right) \text{(Q)}$$

$\therefore$ I (ii) (Q) (Ans. Q.No. 112) (Here, $a = \sqrt{2}$, $m = \pm 1$)

II. $x^2 + a^2y^2 = a^2$ or $\frac{x^2}{a^2} + \frac{y^2}{1^2} = 1$

Equations of tangents in terms of slopes are

$$y = mx + \sqrt{(a^2m^2 + 1)} \text{ (iv)}$$

and points of contact in terms of slopes are

$$\left(\frac{-a^2m}{\sqrt{(a^2m^2 + 1)}}, \frac{1}{\sqrt{(a^2m^2 + 1)}}\right) \text{ (R)}$$

$\therefore$ II (iv) (R) (Ans. Q.No. 111) (Here, $a = 2$, $m = \frac{-\sqrt{3}}{2}$)

III. $y^2 = 4ax$

Equation of tangent in terms of slopes are

$$y = mx + \frac{a}{m} \text{ or } my = m^2x + a \quad \text{(i)}$$

and points of contact in terms of slopes are

$$\left(\frac{a}{m^2}, \frac{2a}{m}\right) \text{ (P)}$$

$\therefore$ III (i) (P) (Ans. Q.No. 113) (Here, $a = 8$, $m = 1$)

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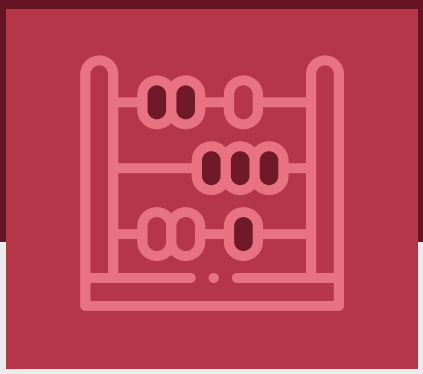

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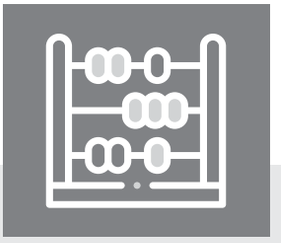

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Differential Calculus

With Sessionwise Theory & Exercises

Amit M. Agarwal

arihant

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PREFACE

"YOU CAN DO ANYTHING IF YOU SET YOUR MIND TO IT, I TEACH CALCULUS TO JEE ASPIRANTS BUT BELIEVE THE MOST IMPORTANT FORMULA IS
COURAGE + DREAMS = SUCCESS"

It is a matter of great pride and honour for me to have received such an overwhelming response to the previous editions of this book from the readers. In a way, this has inspired me to revise this book thoroughly as per the changed pattern of JEE Main & Advanced. I have tried to make the contents more relevant as per the needs of students, many topics have been re-written, a lot of new problems of new types have been added in etcetc. All possible efforts are made to remove all the printing errors that had crept in previous editions. The book is now in such a shape that the students would feel at ease while going through the problems, which will in turn clear their concepts too.

A Summary of changes that have been made in Revised & Enlarged Edition

- Theory has been completely updated so as to accommodate all the changes made in JEE Syllabus & Pattern in recent years.
- The most important point about this new edition is, now the whole text matter of each chapter has been divided into small sessions with exercise in each session. In this way the reader will be able to go through the whole chapter in a systematic way.
- Just after completion of theory, Solved Examples of all JEE types have been given, providing the students a complete understanding of all the formats of JEE questions & the level of difficulty of questions generally asked in JEE.
- Along with exercises given with each session, a complete cumulative exercises have been given at the end of each chapter so as to give the students complete practice for JEE along with the assessment of knowledge that they have gained with the study of the chapter.
- Last 10 Years questions asked in JEE Main & Adv, IIT-JEE & AIEEE have been covered in all the chapters.

However I have made the best efforts and put my all calculus teaching experience in revising this book. Still I am looking forward to get the valuable suggestions and criticism from my own fraternity i.e. the fraternity of JEE teachers.

I would also like to motivate the students to send their suggestions or the changes that they want to be incorporated in this book. All the suggestions given by you all will be kept in prime focus at the time of next revision of the book.

Amit M. Agarwal

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- To draw $y = f(|x|)$, when $f(x)$ is given
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PRACTICE PART
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- Chapter Exercises

SYLLABUS

JEE MAIN

Limit, Continuity and Differentiability
Real valued functions, algebra of functions, polynomials, rational, trigonometric, logarithmic and exponential functions, inverse functions. Graphs of simple functions. Limits, continuity and differentiability.

Differentiation of the sum, difference, product and quotient of two functions. Differentiation of trigonometric, inverse trigonometric, logarithmic exponential, composite and implicit functions derivatives of order upto two. Rolle's and Lagrange's Mean Value Theorems. Applications of derivatives: Rate of change of quantities, monotonic - increasing and decreasing functions, Maxima and minima of functions of one variable, tangents and normals.

JEE ADVANCED

Differential Calculus
Real valued functions of a real variable, into, onto and one-to-one functions, sum, difference, product and quotient of two functions, composite functions, absolute value, polynomial, rational, trigonometric, exponential and logarithmic functions.

Limit and continuity of a function, limit and continuity of the sum, difference, product and quotient of two functions, l'Hospital rule of evaluation of limits of functions.

Even and odd functions, inverse of a function, continuity of composite functions, intermediate value property of continuous functions.

Derivative of a function, derivative of the sum, difference, product and quotient of two functions, chain rule, derivatives of polynomial, rational, trigonometric, inverse trigonometric, exponential and logarithmic functions.

Derivatives of implicit functions, derivatives up to order two, geometrical interpretation of the derivative, tangents and normals, increasing and decreasing functions, maximum and minimum values of a function, applications of Rolle's Theorem and Lagrange's Mean Value Theorem.

CHAPTER

01

Essential Mathematical Tools

Learning Part

Session 1

- Basic Definitions

Session 2

- Intervals
- Modulus or Absolute Value Function

Session 3

- Number Line Rule
- Wavy Curve Method

Session 4

- Quadratic Expression
- Non-negative Functions

Practice Part

- Chapter Exercises

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Session 1

Basic Definitions

(i) **Natural Numbers** The set of numbers $\{1, 2, 3, 4, \ldots\ldots\}$ is called natural numbers and it is **denoted by N**.
i.e. $N = \{1, 2, 3, 4, \ldots\ldots\}$

(ii) **Whole Numbers** The set of numbers $\{0, 1, 2, 3, 4, ...\}$ is called whole numbers and it is **denoted by W.**
i.e. $W = \{0, 1, 2, 3, 4, \ldots\ldots\}$

(iii) **Integers** The set of numbers $\{\ldots\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\ldots\}$ is called integers and it is **denoted by I or Z,**
i.e. I (or Z) $= \{\ldots\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\ldots\}$
where we represent;

(a) **Positive integers** by $I^+ = \{1, 2, 3, 4, \ldots\ldots\}$ = natural numbers.

(b) **Negative integers** by $I^- = \{\ldots\ldots, -4, -3, -2, -1\}$

(c) **Non-negative integers** $\{0, 1, 2, 3, 4, \ldots\ldots\}$ = whole numbers

(d) **Non-positive integers** $\{\ldots\ldots, -3, -2, -1, 0\}$

(iv) **Rational Numbers** All the numbers of the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$ are called rational numbers and their set is **denoted by Q**.
i.e. $Q = \left\{\frac{a}{b} : a, b \in I \text{ and } b \neq 0 \text{ and HCF of } a,b \text{ is } 1\right\}$.

Remarks

(a) Every integer is a rational number as it can be written as $\frac{a}{b}$ (where $b = 1$)

(b) All recurring decimals are rational numbers.
For example, $\frac{1}{3} = 0.3333\ldots\ldots$

(v) **Irrational Numbers** Those values which neither terminate nor could be expressed as recurring decimals are called irrational numbers, i.e. they can't be expressed as $\frac{a}{b}$ form and are **denoted by Q^C** (i.e. complement of Q).
e.g. $\sqrt{2}, 1+\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{-3}{\sqrt{2}}, \frac{2}{\sqrt{2}}, \sqrt{3}, 1+\sqrt{3}, \pm\frac{1}{\sqrt{3}}, \pi, \ldots$ etc.

Remark

The set of rational and irrational numbers cannot be expressed in roster form.

(vi) **Prime Numbers** A counting number is called a prime number when it has exactly two factors, 1 and itself.
e.g. 2, 3, 5, 7, 11, 13, 17, … etc

Remarks

(a) 2 is the only even number which is prime.
(b) A prime is always greater than 1.

How to test a number is prime or not

Let given number be p, then
Find whole number x such that $x > \sqrt{p}$.
Take all prime numbers less than or equal to x.
If none of these divides with p exactly, then p is a prime otherwise p is non-prime.
e.g. Let $p = 193$; clearly, $14 > \sqrt{193}$
Prime numbers upto 14 are 2, 3, 5, 7, 11, 13.
No one of these divides 193 exactly.
Hence, 193 is a prime number.

(vii) **Co-prime Numbers** Two natural numbers are said to be co-primes, if their HCF is 1.
e.g. (7, 9), (15, 16) are called co-prime numbers.

Remark

Co-prime numbers may or may not be prime.

(viii) **Twin Prime** A prime number that is either 2 less or 2 more than another prime number, i.e. a twin prime is a prime that has a prime gap of two.
e.g. (3, 5), (5, 7), (11,13), (17, 19), (29, 31),… are called twin prime pairs.

(ix) **Composite Numbers** Composite numbers are non-prime natural numbers. They must have atleast one factor apart from 1 and itself.
e.g. 4, 6, 8, 9,… are called composite numbers.

Remarks

(a) Composite numbers can be both even or odd.
(b) 1 is neither prime nor composite number.

(x) **Real Numbers** The set which contains both rational and irrational numbers is called the real numbers and is **denoted by R**.
i.e. $R = Q \cup Q^C$
$R = \{x : x \in Q \text{ or } x \in Q^C\}$

$$\therefore \quad R = \left\{ \ldots\ldots, -2, -1, 0, 1, 2, 3, \ldots\ldots, \frac{5}{6}, \frac{3}{4}, \frac{7}{9}, \frac{1}{3}, \frac{1}{7}, \frac{1}{5}, \ldots\ldots, \sqrt{2}, \sqrt{3}, \pi, \ldots\ldots \right\}$$

Remarks

(i) As from above definitions;

$$N \subset W \subset I \subset Q \subset R,$$

It could be shown that real numbers can be expressed on number line with respect to origin as

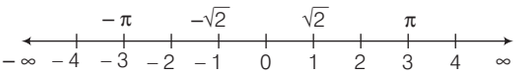


Figure 1.1

(ii) The sum, difference, product and quotient of two rational numbers is a rational number.

(iii) The sum, difference, product and quotient of two irrational numbers need not be irrational.

(iv) The sum, difference, product and quotient of a non-zero rational and an irrational number is always irrational.

Exercise for Session 1

■ **Directions** (Q. Nos. 1 to 6) *State whether following statements are True/False.*

1. All the rational numbers are irrational also.

2. All the integers are irrational also.

3. Irrational numbers are real numbers also.

4. Zero is a natural number.

5. Sum of two natural numbers is a rational number.

6. A positive integer is a natural number also.

■ **Directions** (Q. Nos. 7 to 10) *These questions have only one option correct.*

7. Sum of two rational numbers is

(a) rational (b) irrational
(c) Both (a) and (b) (d) None of these

8. Sum of two irrational numbers is

(a) rational (b) irrational
(c) real (d) None of these

9. Product of two rational numbers is

(a) always rational (b) rational or irrational
(c) always irrational (d) None of these

10. If *a* is an irrational number which is divisible by *b*, then the number *b*

(a) must be rational (b) must be irrational
(c) may be rational or irrational (d) None of these

Session 2

Intervals, Modulus or Absolute Value Function

Intervals

The set of numbers between any two real numbers is called intervals. The types of interval discussed below.

(i) **Open-Open Interval** If a and b are real numbers and $a < b$, then the set of all real numbers x such that $a < x < b$ is called open-open interval. It is denoted by (a, b) or $]a,\ b[$.

i.e. If $a < x < b \Rightarrow x \in]a, b[$ or $x \in (a, b)$

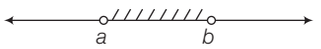


Figure 1.2

e.g. $1 < x < 2$

$\Rightarrow \quad x \in (1, 2)$ or $x \in]1, 2[$ (using above definition)

Note At $\pm\infty$ brackets are always open.

(ii) **Open-Closed Interval** If a and b are real numbers and $a < b$, then the set of all real numbers x such that $a < x \leq b$ is called open-closed interval. Here, a is excluded and b is included.

i.e. If $a < x \leq b$

$\Rightarrow \quad x \in]a, b]$ or $x \in (a, b]$

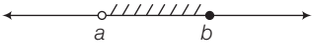


Figure 1.3

(iii) **Closed-Open Interval** If a and b are real numbers and $a < b$, then the set of all real 'x' such that $a \leq x < b$ is called closed-open interval.

Here, a is included and b is excluded.

i.e. If $a \leq x < b$

$\Rightarrow \quad x \in [a, b[$

or $x \in [a, b)$

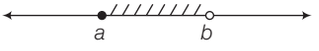


Figure 1.4

(iv) **Closed-Closed Interval** If a and b are real numbers and $a < b$, then the set of all real 'x' such that $a \leq x \leq b$ is called closed-closed interval. Here, a and b both are included.

i.e. If $a \leq x \leq b \Rightarrow x \in [a, b]$

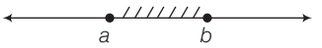
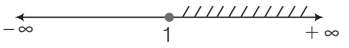


Figure 1.5

Example 1 Solve $2x + 1 > 3$.

Sol. Here, $2x + 1 > 3$ or $2x > 2$ or $x > 1$, i.e. $x \in (1, \infty)$

This solution can be graphed on a real line as;



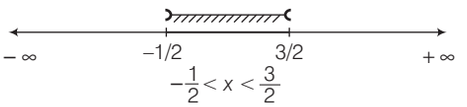
Example 2 Solve $-2 < 2x - 1 < 2$.

Sol. Here, $-2 < 2x - 1 < 2$

or $-1 < 2x < 3$

or $-\frac{1}{2} < x < \frac{3}{2}$, i.e. $x \in \left(-\frac{1}{2}, \frac{3}{2}\right)$

This solution can be graphed on a real line as;



Example 3 Solve the following inequations.

(i) $\frac{3(x-2)}{5} \geq \frac{5(2-x)}{3}$ (ii) $\frac{2x-3}{4} + 9 \geq 3 + \frac{4x}{3}$

Sol. (i) We have, $\frac{3(x-2)}{5} \geq \frac{5(2-x)}{3}$

$\Rightarrow \quad 3(3x - 6) \geq 5(10 - 5x)$

$\Rightarrow \quad 9x - 18 \geq 50 - 25x$

$\Rightarrow \quad 34x \geq 68 \Rightarrow x \geq 2$

Hence, the solution set is $x \in [2, \infty)$ and graphically it could be shown as

(ii) We have, $\frac{2x-3}{4} + 9 \geq 3 + \frac{4x}{3}$

$\Rightarrow \quad \frac{2x-3}{4} - \frac{4x}{3} \geq 3 - 9$

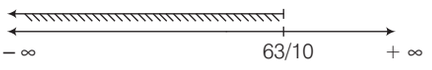
$$\Rightarrow \quad \frac{3(2x-3)-16x}{12} \geq -6 \quad \Rightarrow \quad \frac{-10x-9}{12} \geq -6$$

$$\Rightarrow \quad -10x-9 \geq -72 \quad \Rightarrow \quad -10x \geq -63$$

As we know the inequality sign changes, if multiplied by (–ve).

$$\therefore \quad 10x \leq 63 \quad \text{or} \quad x \leq \frac{63}{10}$$

Hence, the solution set of the given inequation is $\left(-\infty, \frac{63}{10}\right]$. Graphically it could be shown as



Example 4 Solve for x.

$$\frac{4}{x+1} \leq 3 \leq \frac{6}{x+1}, (x>0)$$

Sol. Consider the first inequality

$$\frac{4}{x+1} \leq 3 \Rightarrow 4 \leq 3(x+1)$$

$$\Rightarrow \quad 4 \leq 3x+3$$

$$\Rightarrow \quad 4-3 \leq 3x \Rightarrow 3x \geq 1$$

$$\Rightarrow \quad x \geq \frac{1}{3} \qquad \text{...(i)}$$

Again, $3 \leq \frac{6}{x+1} \Rightarrow 3(x+1) \leq 6$

$$\Rightarrow \quad 3x+3 \leq 6 \quad \Rightarrow \quad 3x \leq 6-3$$

$$\Rightarrow \quad 3x \leq 3$$

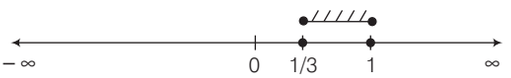
$$\Rightarrow \quad x \leq 1 \qquad \text{...(ii)}$$

Combining the results (i) and (ii), we have

$$\frac{1}{3} \leq x \leq 1, \quad \text{i.e.} \quad x \in \left[\frac{1}{3}, 1\right]$$

Hence, the solution set of the given in equations is $\left[\frac{1}{3}, 1\right]$.

Graphically, it could be shown as



Modulus or Absolute Value Function

The function defined by $f(x) = |x| = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$ is called modulus function. This always gives a positive result.

As we know modulus means numerical value,

i.e. $|3| = |-3| = 3,$

$$|-2| = 2, |-1.3291| = 1.3291\ldots\ldots \text{ etc.}$$

or it is the distance defined with respect to origin, as $|x| = 1$ means distance covered is one unit on right hand side or left hand side of origin shown as:

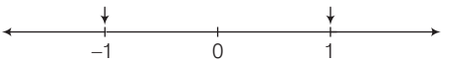


Figure 1.6

$$\therefore \quad |x| = 1 \Rightarrow x = \pm 1$$

Again, $|x| < 1$ means distance covered is less than one unit on right hand side or left hand side of origin shown as:

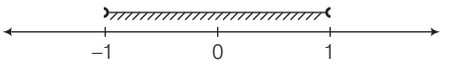


Figure 1.7

Similarly, $|x| > 1$ means distance covered is more than one unit on right hand side or left hand side of origin shown as:

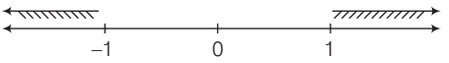


Figure 1.8

Graphical Representation of Modulus

As we define,

$$y = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

This behaviour is due to two straight lines represented by modulus.

For plotting the graph of the modulus function, we put

x	0	1	2	–1	–2
y	0	1	2	1	2

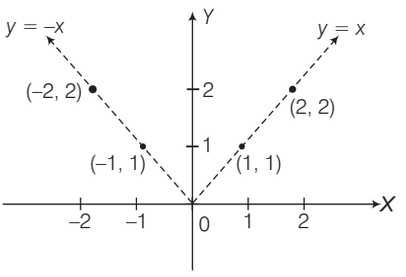


Figure 1.9

Now, we consider the following cases

***Case* I** When $x \geq 0$

Equation of the straight line, passing through (0, 0) and (1, 1) is

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1},\ \text{i.e.}\ \frac{y-0}{x-0}=\frac{1-0}{1-0}$$

$\Rightarrow \quad y = x$, when $x \geq 0$

***Case* II** When $x \leq 0$

Equation of the straight line passing through $(-1, 1)$ and $(-2, 2)$

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$

i.e. $\quad \frac{y-1}{x+1}=\frac{2-1}{-2+1} \Rightarrow -y+1=x+1 \quad \text{or} \quad y=-x$

when $x \leq 0$.

Thus for every modulus function it exhibits two values, which could be shown graphically.

Similarly, $\quad y=|x-1|=\begin{cases}(x-1), & x \geq 1\\ -(x-1), & x \leq 1\end{cases}$

Remarks

(a) Every modulus function exhibits two values which are represented by +ve and –ve, signs but it only gives the positive outcomes. So, students shouldn't get confused by + ve or –ve, signs as these signs are in different intervals, but the outcomes are positive.

(b) Modulus function is never negative, thus $|x| \geq 0$ for any real x and $|x| \not< 0$.

Example 5 Explain the following :

(i) $|x|=5$ (ii) $|x|=-5$

(iii) $|x|<5$ (iv) $|x|<-5$

(v) $|x|>-5$ (vi) $|x|>5$

Sol. (i) If $|x|=5$

$\Rightarrow x = \pm 5$, which means, x is at a distance of 5 units from 0, which is certainly 5 and − 5.

Aliter

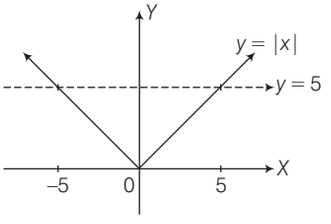
$|x| = 5$.

Here, students are advised to consider two different functions, as

$$y = |x| \ \text{ and } \ y = 5.$$

Now, we plot graph of these two equations.

which intersect at two points, i.e. $x = 5$ and $x = -5$.



$\therefore \quad |x| = 5$ possess two solutions $x = 5$ and $x = -5$.

(ii) If $|x|=-5$

$\Rightarrow \quad x$ has no solution.

As $|x|$ is always positive or zero, it can never be negative.

$\therefore \quad$ RHS < LHS

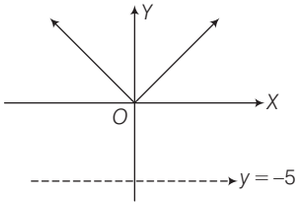
or given relation has no solution.

Aliter

Same as in (i), plotting the graph of $y = |x|$ and $y = -5$.

The two graphs do not intersect.

$\therefore$ No solution.

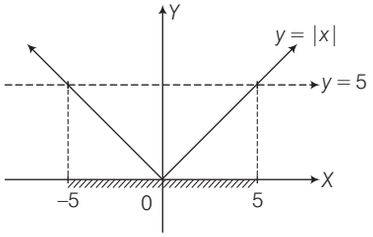


(iii) $|x|< 5$

It means that x is the number, which is at distance less than 5 from 0.

Hence, $\quad -5 < x < 5$

Aliter $|x| < 5$. Plotting the graph of we have $y = |x|$ and $y = 5$.



We see that, $|x| < 5$, when $-5 < x < 5$

(iv) $|x|< -5$

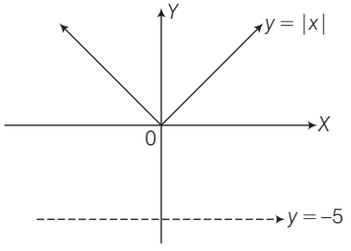
which shows no solution.

As LHS is non-negative and RHS is negative or $|x|<-5$ does not possess any solution.

Aliter $|x| < -5$.

Plotting the graph of $y = |x|$ and $y = -5$.

We see that, $|x| < -5$ is not possible as $|x| > -5$, for all $x \in R$.



$\therefore \quad |x| < -5 \Rightarrow$ No solution.

(v) $|x| > -5$ We know, here LHS ≥ 0 and RHS < 0

So, LHS > RHS

i.e. above statement is true for all real x.

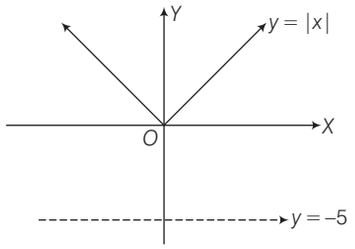
(as we know that non-negative number is always greater than negative).

Aliter

Here, $|x| > -5$

Plotting the graph of $y = |x|$ and $y = -5$.

We see that, $|x| > -5$, for all $x \in$ real number.



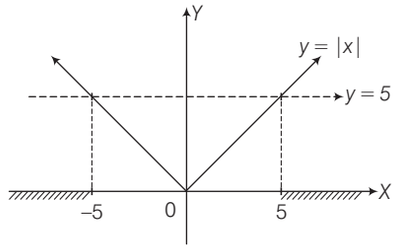
(vi) $|x| > 5$

It means that x is the number which is at distance greater than 5 from 0. Hence, $x < -5$ or $x > 5$

Aliter

$|x| > 5$

Plotting the graph of $y = |x|$ and $y = 5$. We have



We see that, $x < -5$ or $x > 5$.

Generalized Results

(i) For any real number x, we have $x^2 = |x|^2$

(ii) For any real number x, we have $\sqrt{x^2} = |x|$

(iii) If $a < 0$, then

(a) $x^2 \leq a^2 \Leftrightarrow |x| \leq a \Leftrightarrow -a \leq x \leq a$

(b) $x^2 < a^2 \Leftrightarrow |x| < a \Leftrightarrow -a < x < a$

(c) $x^2 \geq a^2 \Leftrightarrow |x| \geq a \Leftrightarrow x \leq -a$ or $x \geq a$

(d) $x^2 > a^2 \Leftrightarrow |x| > a \Leftrightarrow x < -a$ or $x > a$

(e) $a^2 \leq x^2 \leq b^2 \Leftrightarrow a \leq |x| \leq b$

$\Leftrightarrow x \in [-b, -a] \cup [a, b]$

(f) $a^2 < x^2 < b^2 \Leftrightarrow a < |x| < b$

$\Leftrightarrow x \in (-b, -a) \cup (a, b)$

(iv) If $a < 0$, then

$|x| \leq a \Rightarrow$ No solution.

$|x| \geq a \Rightarrow$ All real numbers solutions.

(v) $|x + y| = |x| + |y| \Leftrightarrow (x \geq 0$ and $y \geq 0)$ or $(x \leq 0$ and $y \leq 0) \Leftrightarrow xy \geq 0$.

(vi) $|x - y| = |x| - |y| \Leftrightarrow (x \geq 0, y \geq 0$ and $|x| \geq |y|)$ or $(x \leq 0, y \leq 0$ and $|x| \geq |y|)$

(vii) $|x \pm y| \leq |x| + |y|$ (viii) $|x \pm y| \geq \big||x| - |y|\big|$

Example 6 Solve for x, where

(i) $f(x) = |x| \geq 0$ (ii) $f(x) = |x| > 0$

Sol. (i) $f(x) = |x| \geq 0$

As we know modulus is non-negative quantity.

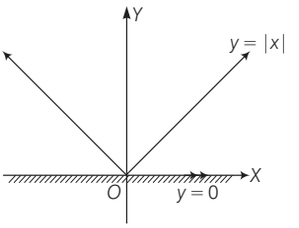
(i.e. It is always greater than equal to zero)

$\therefore$ $x \in R$ is the required solution.

Aliter $f(x) = |x| \geq 0$.

Plotting two graphs $y = |x|$ and $y = 0$.

From graph $|x| \geq 0$, for all $x \in R$.



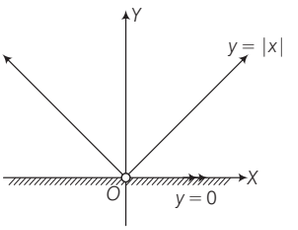
(ii) $f(x) = |x| > 0$

Here equal sign is absent, so we have to exclude those value of x for which $|x| = 0$.

$\therefore$ $x \in R$ except $x = 0$ or $x \in R - \{0\}$ is the required solution.

Aliter $f(x) = |x| > 0$.

Plotting two graphs $y = |x|$ and $y = 0$.



From graph $|x| > 0$, for all $x \in R$ except 0.

$\therefore$ $x \in R - \{0\}$.

Example 7 *Solve* $|x - 3| < 5$.

Sol. $|x - 3| < 5$

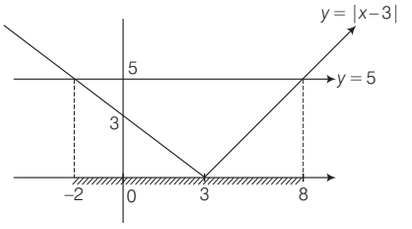
$\Rightarrow -5 < x - 3 < 5$

$\Rightarrow -5 + 3 < x < 5 + 3$

$\Rightarrow -2 < x < 8$ or $x \in]-2, 8[$ or $x \in (-2, 8)$

Aliter For $|x-3|<5$.

Plotting two graphs $y=|x-3|$ and $y=5$



From graph, $-2<x<8$; $\therefore\ x\in(-2,8)$.

Example 8 Solve $|x-1|\le 2$.

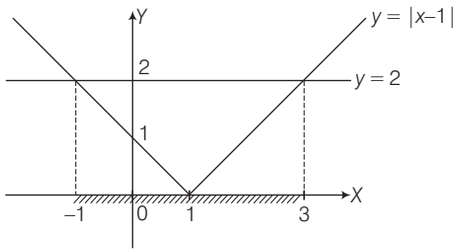
Sol. $|x-1|\le 2$

$\Rightarrow\ -2\le x-1\le 2\ \Rightarrow\ -2+1\le x\le 2+1$

$\Rightarrow\ -1\le x\le 3$ or $x\in[-1,3]$

Aliter For $|x-1|\le 2$.

Plotting two graphs $y=|x-1|$ and $y=2$



From graph, $-1\le x\le 3$

$\therefore\ x\in[-1,3]$

Example 9 Solve $1\le|x-1|\le 3$.

Sol. Here, $1\le|x-1|\le 3$

$\Rightarrow\ -3\le(x-1)\le -1$ or $1\le(x-1)\le 3$

$[\because a\le|x|\le b\Rightarrow x\in[-b,-a]\cup[b,a]]$

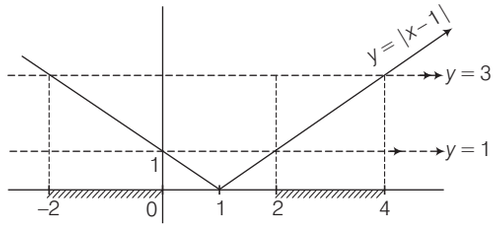
i.e. the distance covered between 1 unit to 3 units.

$\Rightarrow\ -2\le x\le 0$ or $2\le x\le 4$

Hence, the solution set of the given inequation is $x\in[-2,0]\cup[2,4]$.

Aliter Here, $1\le|x-1|\le 3$

Plotting three graphs $y=1$, $y=|x-1|$ and $y=3$.



From graph, $-2\le x\le 0$ or $2\le x\le 4$.

$\therefore\ x\in[-2,0]\cup[2,4]$

Example 10 Solve $|x-1|\le 5, |x|\ge 2$.

Sol. Here, $|x-1|\le 5$ and $|x|\ge 2$

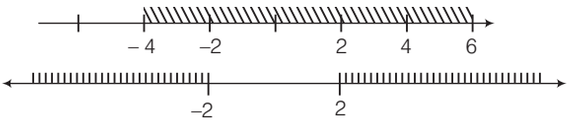
i.e. $(-5\le x-1\le 5)$

$\Rightarrow\ (-4\le x\le 6)$ ...(i)

Similarly, $(x\le -2$ or $x\ge 2)$

$\Rightarrow\ (x\le -2$ or $x\ge 2)$ ...(ii)

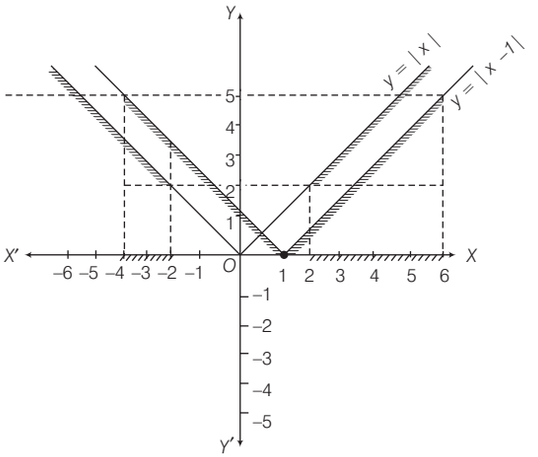
From Eqs. (i) and (ii) could be graphically shown as



Thus, the shaded portion, i.e. common to both Eqs. (i) and (ii) is the required region $x\in[-4,-2]\cup[2,6]$.

Aliter Here, $|x-1|\le 5$ and $|x|\ge 2$.

Take $(y=|x-1|, y=5)$ and $(y=|x|, y=2)$ on graph and take common.



From the graph, $-4\le x\le -2$

or $2\le x\le 6$

$\therefore\ x\in[-4,-2]\cup[2,6]$

Example 11 *Solve* $\left|\frac{2}{x-4}\right|>1,\ x\ne 4$.

Sol. We have, $\left|\frac{2}{x-4}\right|>1$, where $x\ne 4$ ...(i)

$\Rightarrow\ \frac{2}{|x-4|}>1$ $\left[\because\left|\frac{a}{b}\right|=\frac{|a|}{|b|}\text{ and }|2|=2\right]$

$\Rightarrow\ 2>|x-4|$

$\Rightarrow\ |x-4|<2$

$\Rightarrow\ -2<x-4<2$

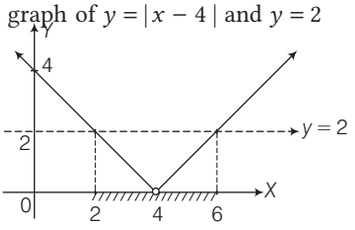
$\Rightarrow\ 2<x<6$

$\therefore\ x\in(2,6)$, but $x\ne 4$ [from Eq. (i)]

Hence, the solution set of the given inequation is $x \in (2, 4) \cup (4, 6)$.

Aliter $\left|\frac{2}{x-4}\right| > 1,\ x \neq 4 \implies |x-4| < 2$

Plotting the graph of $y = |x-4|$ and $y = 2$



From graph, $2 < x < 6, x \neq 4$

i.e. $x \in (2, 6) - \{4\}$

Note

Students should always remember that they have to compare the solution set with the initial condition.

Example 12 *Solve* $|x-1|+|x-2| \geq 4$.

Sol. On the LHS of the given inequation, we have two modulus, so we should define each modulus i.e. by equating it to zero.

i.e. $|x-1| = \begin{cases} (x-1), & x \geq 1 \\ -(x-1), & x < 1 \end{cases}$

and $|x-2| = \begin{cases} (x-2), & x \geq 2 \\ -(x-2), & x < 2 \end{cases}$

Thus, it gives three cases :

***Case* I** When $-\infty < x < 1$

i.e. $|x-1|+|x-2| \geq 4$

$\Rightarrow$ $-(x-1)-(x-2) \geq 4$

$\Rightarrow$ $-2x+3 \geq 4 \implies -2x \geq 1$

$\Rightarrow$ $x \leq -\frac{1}{2}$ ...(i)

But $-\infty < x < 1$

$\therefore$ Solution set is $x \in \left(-\infty, -\frac{1}{2}\right]$.

***Case* II** When $1 \leq x \leq 2$

i.e. $|x-1|+|x-2| \geq 4$

$\Rightarrow$ $(x-1)-(x-2) \geq 4$

$\Rightarrow$ $1 \geq 4$, which is meaningless.

$\therefore$ No solution for $x \in [1, 2]$. ...(ii)

***Case* III** When $2 < x < \infty$

i.e. $|x-1|+|x-2| \geq 4$

$\Rightarrow$ $(x-1)+(x-2) \geq 4$

$\Rightarrow$ $2x-3 \geq 4$

$\Rightarrow$ $x \geq 7/2$

But $2 < x < \infty$

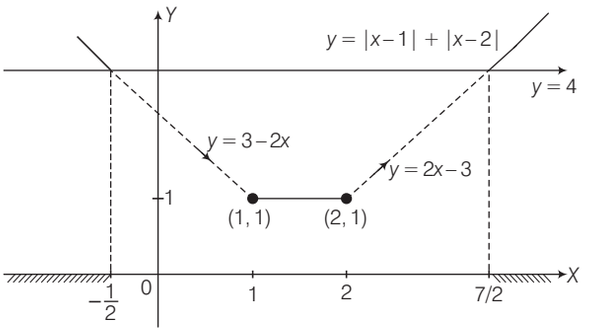
$\therefore$ Solution set is $\left[\frac{7}{2}, \infty\right)$. ...(iii)

From Eqs. (i), (ii) and (iii), we get

$$x \in \left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{7}{2}, \infty\right)$$

Aliter

Plotting two graphs $y = |x-1|+|x-2|$ and $y = 4$ shown below.



From graph, $x \leq -\frac{1}{2}$ or $x \geq \frac{7}{2}$

$\therefore$ $x \in \left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{7}{2}, \infty\right)$

Exercise for Session 2

■ **Directions** (Q. Nos. 1 to 5) *Solve the following inequalities for real values of x.*

1. $|x-1| < 2$
2. $|x-3| > 5$
3. $0 < |x-1| < 3$
4. $|x-1| + |2x-3| = |3x-4|$
5. $\left|\frac{x-3}{x^2-4}\right| \leq 1$

Session 3

Number Line Rule, Wavy Curve Method

Number Line Rule

It is used to solve algebraic inequalities using following steps:

(i) Factorize numerator as well as denominator.

(ii) Now, check the coefficients of x and make them positive.

(iii) Put only odd power factors in numerator and denominator and put them equal to zero separately and find the value of x.

(*As for polynomial function only numerator* $= 0$, *denominator* $\neq 0$).

(iv) Plot these points on number line in increasing order.

(v) Start number line from right to left taking sign of $f(x)$.

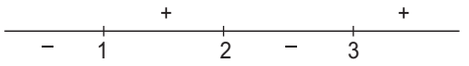
(vi) Check your answer so that it should not contain a point for which $f(x)$ doesn't exist.

Example 13 Find the interval in which $f(x)$ is positive or negative: $f(x) = (x-1)(x-2)(x-3)$.

Sol. Here, $f(x) = (x-1)(x-2)(x-3)$ has all factors with odd powers, so put them as zero.

i.e. $x-1=0, x-2=0, x-3=0$, we get $x = 1, 2, 3$ [using step (iii)]

Using steps (iv) and (v), plotting on number line, we get



$f(x) > 0$ when $1 < x < 2$ and $x > 3$

$f(x) < 0$ when $x < 1$ and $2 < x < 3$

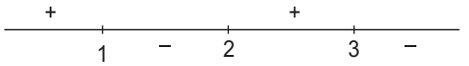
Example 14 Solve $f(x) = \dfrac{(x-1)(2-x)}{(x-3)} \geq 0$.

Sol. Here, $f(x) = \dfrac{(x-1)(2-x)}{(x-3)} \geq 0$

or $f(x) = -\dfrac{(x-1)(x-2)}{(x-3)}$, which gives

$x - 3 \neq 0$ or $x \neq 3$ ...(i)

Using number line rule as shown in the figure,



which shows $f(x) \geq 0$ when $x \leq 1$ or $2 \leq x < 3$

i.e. $x \in (-\infty, 1] \cup [2, 3)$. (as $x \neq 3$)

Example 15 ***Find the values of*** x ***for which***

$$f(x) = \frac{(2x-1)(x-1)^2(x-2)^3}{(x-4)^4} > 0.$$

Sol. $f(x) = \dfrac{(2x-1)(x-1)^2(x-2)^3}{(x-4)^4}$, which gives $x \neq 4$ ...(i)

As denominator $\neq 0$ and $x \neq 1$, as at $x = 1$, $f(x)$ has even powers.

Putting zero to $(2x-1)$ and $(x-2)^3$ as they have odd powers and neglecting $(x-1)^2$ and $(x-4)^4$ on number line as shown in figure.

which shows $f(x) > 0$ when $x < 1/2$ or $x > 2$ but except for 4 and 1.

$\therefore$ $x \in (-\infty, 1/2) \cup (2, \infty) - \{4\}$

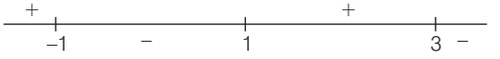
Example 16 Find the value of x for which

$$f(x) = \frac{(x-2)^2(1-x)(x-3)^3(x-4)^2}{(x+1)} \leq 0.$$

Sol. Here, $f(x) = \dfrac{(x-2)^2(1-x)(x-3)^3(x-4)^2}{(x+1)}$

or $f(x) = -\dfrac{(x-2)^2(x-1)(x-3)^3(x-4)^2}{(x+1)}, (x \neq -1)$ ...(i)

Putting zero to $(x-1), (x-3)^3, (x+1)$ as having odd powers and neglecting $(x-2)^2, (x-4)^2$, we get



$f(x) \leq 0$ when $-1 < x \leq 1$ or $3 \leq x < \infty$ or $f(x) = 0$ at $x = 2$

or $x \in (-1, 1] \cup [3, \infty) \cup \{2\}$ [using Eq. (i) as $x \neq -1$]

Example 17 Solve $\dfrac{|x|-1}{|x|-2} \geq 0$; $x \in R$, $x \neq \pm 2$

Sol. We have, $\dfrac{|x|-1}{|x|-2} \geq 0 \Rightarrow \dfrac{y-1}{y-2} \geq 0$; where $y = |x|$

$\Rightarrow$ $y \leq 1$ or $y > 2$ using number line rule

$\Rightarrow$ $|x| \leq 1$ or $|x| > 2$

$\Rightarrow$ $(-1 \leq x \leq 1)$ or $(x < -2$ or $x > 2)$

$\Rightarrow \quad x \in [-1, 1] \cup (-\infty, -2) \cup (2, \infty)$

Hence, the solution set is

$x \in (-\infty, -2) \cup [-1, 1] \cup (2, \infty)$.

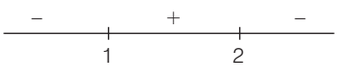
Example 18 Solve $\dfrac{-1}{|x|-2} \geq 1$, where $x \in R$, $x \neq \pm 2$.

Sol. We have, $\dfrac{-1}{|x|-2} \geq 1$

$\Rightarrow \quad \dfrac{-1}{|x|-2} - 1 \geq 0 \Rightarrow \dfrac{-1-(|x|-2)}{|x|-2} \geq 0$

$\Rightarrow \quad \dfrac{1-|x|}{|x|-2} \geq 0 \Rightarrow \dfrac{-(|x|-1)}{(|x|-2)} \geq 0$

Using number line rule,



$\Rightarrow \quad 1 \leq |x| < 2 \Rightarrow x \in (-2, -1] \cup [1, 2)$

$\{\because a \leq |x| < b \Leftrightarrow x \in (-b, -a] \cup [a, b)\}$

Hence, the solution set is $(-2, -1] \cup [1, 2)$.

Example 19 Solve $\dfrac{|x+3|+x}{x+2} > 1$.

Sol. Here, $\dfrac{|x+3|+x}{x+2} - 1 > 0 \Rightarrow \dfrac{|x+3|+x-x-2}{x+2} > 0$

$\Rightarrow \quad \dfrac{|x+3|-2}{x+2} > 0$ ...(i)

Now, two cases arise :

***Case* I** When $x + 3 \geq 0$, i.e. $x \geq -3$ ...(ii)

$\Rightarrow \quad \dfrac{x+3-2}{x+2} > 0 \Rightarrow \dfrac{x+1}{x+2} > 0$

$\Rightarrow \quad x \in (-\infty, -2) \cup (-1, \infty)$ using number line rule as shown in the figure.

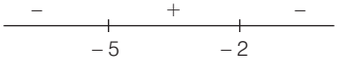
But $x \geq -3$ [from Eq. (ii)]

$\Rightarrow \quad x \in [-3, -2) \cup (-1, \infty)$ ...(a)

***Case* II** When $x + 3 < 0$, i.e. $x < -3$ ...(iii)

$\Rightarrow \quad \dfrac{-(x+3)-2}{x+2} > 0 \Rightarrow \dfrac{-(x+5)}{(x+2)} > 0$

$\Rightarrow x \in (-5, -2)$ using number line rule as shown in the figure.



But $x < -3$ [from Eq. (iii)]

$\therefore \quad x \in (-5, -3)$ ...(b)

Thus, from Eqs. (a) and (b), we have

$x \in [-3, -2) \cup (-1, \infty) \cup (-5, -3)$

$\Rightarrow \quad x \in (-5, -2) \cup (-1, \infty)$

Wavy Curve Method

The method of intervals (or wavy curve) is used for solving inequalities of the form :

$$f(x) = \frac{(x-a_1)^{n_1}(x-a_2)^{n_2} \ldots (x-a_k)^{n_k}}{(x-b_1)^{m_1}(x-b_2)^{m_2} \ldots (x-b_p)^{m_p}} > 0$$

$(< 0, \leq 0 \text{ or } \geq 0)$

where $n_1, n_2, \ldots\ldots, n_k$ and $m_1, m_2, \ldots\ldots, m_p$ are natural numbers.

$a_1, a_2, \ldots\ldots, a_k$; $b_1, b_2, \ldots\ldots, b_p$ are any real numbers such that $a_i \neq b_j$, where $i = 1, 2, 3, \ldots, k$ and $j = 1, 2, 3, \ldots, p$

It consists of the following statements :

1. All zeros of the function $f(x)$ (the values of x corresponding to numerator = 0) contained on the left hand side of the inequality should be marked on the number line with inked (black) circles.
2. All points of discontinuities of the function $f(x)$ (the values of x corresponding to denominator = 0) contained on the left hand side of the inequality should be marked on the number line with uninked (white) circles.
3. Check the value of $f(x)$ for any real number greater than the rightmost marked number on the number line.
4. From right to left, beginning above the number line (in case of value of $f(x)$ is positive in step (iii) otherwise from below the number line), a wavy curve should be drawn to pass through all the marked points so that when it passes through a simple point, the curve intersects the number line and when passing through a double point, the curve remains located on one side of the number line.
5. The appropriate intervals are chosen in accordance with the sign of inequality (the function $f(x)$ is positive whenever the curve is situated above the number line, it is negative if the curve is found below the number line). Their union just represents the solution of the inequality.

Remarks

(a) Points of discontinuity will never be include in the answers.

(b) If asked to find the intervals where $f(x)$ is non-negative or non-positive, then make the intervals closed, corresponding to the roots of the numerator and let it remain open corresponding to the roots of denominator.

Example 20 *Let* $f(x) = \dfrac{(x-1)^3(x+2)^4(x-3)^5(x+6)}{x^2(x-7)^3}$.

Solve the following inequalities :

(i) $f(x) > 0$ (ii) $f(x) \geq 0$

(iii) $f(x) < 0$ (iv) $f(x) \leq 0$

Sol. Given, $f(x)=\dfrac{(x-1)^3(x+2)^4(x-3)^5(x+6)}{x^2(x-7)^3}$

Put numerator = 0

$\Rightarrow \quad (x-1)^3(x+2)^4(x-3)^5(x+6)=0$

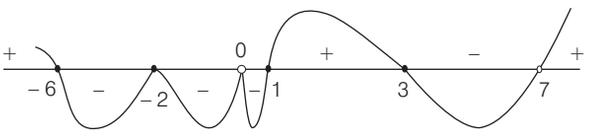
$\Rightarrow \quad x=1,-2,3,-6$

Again, put denominator = 0

$\Rightarrow \quad x^2(x-7)^3=0$

$\Rightarrow \quad x=0,\ 7$

We mark on the number line zeros of the function: $1,-2,3$ and -6 (with black circles) and the points of discontinuities 0 and 7 (with white circles). Isolate the double points: -2 and 0 and draw the curve of signs.



From the graph, we get

(i) if $f(x)>0$, then $x\in(-\infty,-6)\cup(1,3)\cup(7,\infty)$

(ii) if $f(x)\geq 0$, then $x\in(-\infty,-6]\cup\{-2\}\cup[1,3]\cup(7,\infty)$

(iii) if $f(x)<0$, then $x\in(-6,-2)\cup(-2,0)\cup(0,1)\cup(3,7)$

(iv) if $f(x)\leq 0$, then $x\in[-6,0)\cup(0,1]\cup[3,7)$

Example 21 *Let*

$$f(x)=\frac{\left(\sin x-\frac{1}{2}\right)(\ln x-1)^2(x-2)(\tan x-\sqrt{3})}{(e^x-e^2)(x-3)^2\cdot\cos x}$$

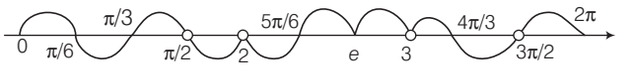
Solve the following inequalities for $x\in[0,2\pi]$ **:**

(i) $f(x)>0$ (ii) $f(x)\geq 0$

(iii) $f(x)<0$ (iv) $f(x)\leq 0$

Sol. Clearly, $x\neq 2,3,\dfrac{\pi}{2},\dfrac{3\pi}{2}$ and $f(x)=0$

for $x=\dfrac{\pi}{6},\dfrac{\pi}{3},\dfrac{5\pi}{6},e,\dfrac{4\pi}{3}$



Now, sign of $f(x)$ will not change around $x=2,e,3$.

Then, for $f(x)>0$

$$\left(\sin x-\frac{1}{2}\right)(\tan x-\sqrt{3})>0$$

$$\Rightarrow \quad x\in\left(0,\frac{\pi}{6}\right)\cup\left(\frac{\pi}{3},\frac{\pi}{2}\right)\cup\left(\frac{5\pi}{6},\frac{4\pi}{3}\right)\cup\left(\frac{3\pi}{2},2\pi\right)-\{e,3\}$$

Hence, solution of

(i) $f(x)>0$ is $x\in\left(0,\dfrac{\pi}{6}\right)\cup\left(\dfrac{\pi}{3},\dfrac{\pi}{2}\right)\cup\left(\dfrac{5\pi}{6},\dfrac{4\pi}{3}\right)\cup\left(\dfrac{3\pi}{2},2\pi\right)-\{e,3\}$

(ii) $f(x)\geq 0$ is $x\in\left(0,\dfrac{\pi}{6}\right]\cup\left[\dfrac{\pi}{3},\dfrac{\pi}{2}\right)\cup\left[\dfrac{5\pi}{6},\dfrac{4\pi}{3}\right]\cup\left(\dfrac{3\pi}{2},2\pi\right)-\{3\}$

(iii) $f(x)<0$ is $x\in\left(\dfrac{\pi}{6},\dfrac{\pi}{3}\right)\cup\left(\dfrac{\pi}{2},\dfrac{5\pi}{6}\right)\cup\left(\dfrac{4\pi}{3},\dfrac{3\pi}{2}\right)-\{2\}$

(iv) $f(x)\leq 0$ is $x\in\left[\dfrac{\pi}{6},\dfrac{\pi}{3}\right]\cup\left(\dfrac{\pi}{2},\dfrac{5\pi}{6}\right]\cup\left[\dfrac{4\pi}{3},\dfrac{3\pi}{2}\right)\cup\{e\}-\{2\}$

Example 22 Let

$$f(x)=\frac{(\cos x+|\cos x|)\left(\sin x-\frac{3}{2}\right)^3(\tan x-1)^5}{(\cos x-2)^2(\tan x-\sqrt{3})^3}$$

Find the interval of $x\in\left(\dfrac{-\pi}{2},\dfrac{\pi}{2}\right)$ for which

(i) $f(x)>0$ (ii) $f(x)<0$

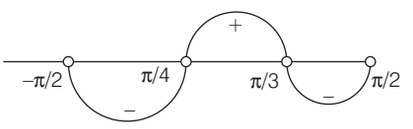
Sol. For $x\in\left(\dfrac{-\pi}{2},\dfrac{\pi}{2}\right)$

$\cos x>0$ for all $x \ \Rightarrow\ |\cos x|=\cos x$

$\Rightarrow \quad \cos x+|\cos x|=2\cos x$

$\therefore \quad f(x)=\dfrac{2\cos x\left(\sin x-\frac{3}{2}\right)^3(\tan x-1)^5}{(\cos x-2)^2(\tan x-\sqrt{3})^3}$

Critical points $\cos x=0\Rightarrow x=\dfrac{\pi}{2}$ (not in domain)



$\sin x=\dfrac{3}{2}$ (not possible)

$\tan x=1\Rightarrow x=\dfrac{\pi}{4}\ \Rightarrow \cos x=2$ (not possible)

$\tan x=\sqrt{3}\Rightarrow x=\dfrac{\pi}{3}\cdot$

Then, $f(x)>0\Rightarrow x\in\left(\dfrac{\pi}{4},\dfrac{\pi}{3}\right)$

and $f(x)<0\Rightarrow x\in\left(\dfrac{-\pi}{2},\dfrac{\pi}{4}\right)\cup\left(\dfrac{\pi}{3},\dfrac{\pi}{2}\right)$

Exercise for Session 3

■ **Directions** (Q. Nos. 1 to 5) *Solve the following inequalities.*

1. $\frac{x-2}{x+2} > \frac{2x-3}{4x-1}$

2. $\frac{2x-1}{2x^3+3x^2+x} > 0$

3. $\frac{4x}{x^2+3} \geq 1$

4. $\frac{(x-1)(x+1)(x+4)(x+6)}{7x^2+8x+4} > 0$

5. $\frac{x}{x^2-5x+9} \leq 1$

6. Solution of inequality $|x-1| < 0$ is

(a) $x = 0$
(b) $x = 1$
(c) $x \neq 1$
(d) No solution

7. Solution of inequality $x^2 + x + |x| + 1 \leq 0$ is

(a) (1, 2)
(b) (0, 1)
(c) No solution
(d) None of these

8. Solution of inequality $|x+3| > |2x-1|$ is

(a) $\left(-\frac{2}{3}, 4\right)$
(b) $(4, \infty)$
(c) $\left(-\frac{2}{3}, 1\right)$
(d) None of these

9. Solution of inequality $\left|x + \frac{1}{x}\right| < 4$ is

(a) $(2-\sqrt{3}, 2+\sqrt{3}) \cup (-2-\sqrt{3}, -2+\sqrt{3})$
(b) $R - (2-\sqrt{3}, 2+\sqrt{3})$
(c) $R - (-2-\sqrt{3}, 2+\sqrt{3})$
(d) None of these

10. The solution of $|x^2 + 3x| + x^2 - 2 \geq 0$ is

(a) $(-\infty, 1)$
(b) (0, 1)
(c) $\left(-\infty, -\frac{2}{3}\right] \cup \left[\frac{1}{2}, \infty\right)$
(d) None of these

11. The solution of $||x|-1| < |1-x|, x \in R$ is

(a) (–1, 1)
(b) $(0, \infty)$
(c) $(-1, \infty)$
(d) None of these

12. The solution of $2^x + 2^{|x|} \geq 2\sqrt{2}$ is

(a) $(-\infty, \log_2(\sqrt{2}+1))$
(b) $(0, \infty)$
(c) $\left(\frac{1}{2}, \log_2(\sqrt{2}-1)\right)$
(d) $(-\infty, \log_2(\sqrt{2}-1)] \cup \left[\frac{1}{2}, \infty\right)$

Session 4

Quadratic Expression, Non-negative Functions

Quadratic Expression

The expression $ax^2 + bx + c$ is said to be a real quadratic expression in x, where a, b, c are real and $a \neq 0$. Let $f(x) = ax^2 + bx + c$, where $a, b, c \in R\,(a \neq 0)$. $f(x)$ can be rewritten as

$$f(x) = a\left\{\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right\} = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a^2}\right]$$

where $D = b^2 - 4ac$ is the **discriminant** of the quadratic expression.

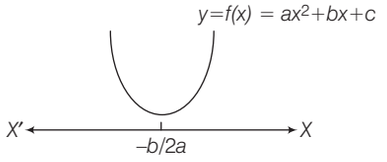
Therefore, $y = f(x)$ represents a parabola whose axis is parallel to the Y-axis, with vertex at $A\left(-\frac{b}{2a}, \frac{-D}{4a}\right)$.

Graph for Quadratic Expressions

That if $a > 0$, the parabola will be concave upwards and if $a < 0$, the parabola will be concave downwards and it depends on the sign of $b^2 - 4ac$ that the parabola cuts the X-axis at two points $(b^2 - 4ac > 0)$, touches the X-axis $(b^2 - 4ac = 0)$ or never intersects with the X-axis $(b^2 - 4ac < 0)$.

This gives rise to the following cases:

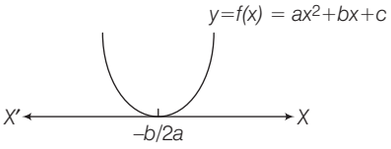
(i) If $a > 0$ and $b^2 - 4ac < 0$.



$\Leftrightarrow \quad f(x) > 0, \forall\ x \in R$

In this case the parabola always remains concave upwards and above the X-axis.

(ii) If $a > 0$ and $b^2 - 4ac = 0$.



$\Leftrightarrow \quad f(x) \geq 0, \forall\ x \in R$

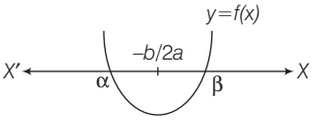
In this case the parabola touches the X-axis and remains concave upwards.

(iii) If $a > 0$ and $b^2 - 4ac > 0$.

Let $f(x) = 0$ has two real roots α and $\beta\,(\alpha < \beta)$.

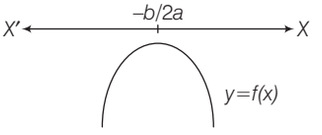
Then, $f(x) > 0, \forall\ x \in (-\infty, \alpha) \cup (\beta, \infty)$,

$f(x) < 0, \forall\ x \in (\alpha, \beta)$ and $f(x) = 0$ for $x \in \{\alpha, \beta\}$.



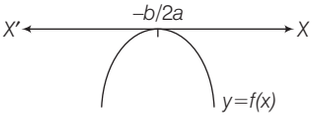
In this case the parabola cuts the X-axis at two points α and β and remains concave upwards.

(iv) If $a < 0$ and $b^2 - 4ac < 0 \ \Leftrightarrow\ f(x) < 0, \forall\ x \in R$



In this case the parabola remains concave downwards and always below the X-axis.

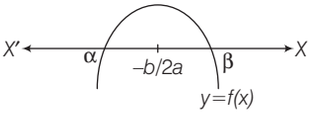
(v) If $a < 0$ and $b^2 - 4ac = 0$.



$\Leftrightarrow \quad f(x) \leq 0, \forall\ x \in R$

In this case the parabola touches the X-axis and remains concave downwards.

(vi) If $a < 0$ and $b^2 - 4ac > 0$.



Let $f(x) = 0$ have two real roots α and $\beta\,(\alpha < \beta)$.

Then, $f(x) < 0, \forall\ x \in (-\infty, \alpha) \cup (\beta, \infty)$,

$f(x) > 0, \forall\ x \in (\alpha, \beta)$ and $f(x) = 0$ for $x \in \{\alpha, \beta\}$.

In this case the parabola cuts the X-axis at two points α and β and remains concave downwards.

Quick and Important Results for Characteristic Expression

1. The expression $ax^2 + bx + c$ will be at same sign for all real values of x, iff $D < 0$.
2. $ax^2 + bx + c$ will always be positive iff $D < 0$ and $a > 0$.
3. $ax^2 + bx + c$ will always be negative iff $D < 0$ and $a < 0$.
4. If $D > 0$, then sign of the expression between the roots will be opposite to that of a.
5. If $a > 0$, then minima of $f(x)$ occurs at $x = \frac{-b}{2a}$ and if $a < 0$, then maxima of $f(x)$ occurs at $x = \frac{-b}{2a}$ and maximum or minimum value of $f(x)$ will be given by $\left(\frac{-D}{4a}\right)$.
6. If $f(x) = 0$ has two distinct real roots, then $a \cdot f(d) < 0$ if and only if d lies between the roots and $a \cdot f(d) > 0$ if and only if d lies outside the roots.

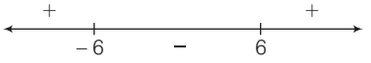
Example 23 ***Find*** a ***for which*** $3x^2 + ax + 3 > 0$, $\forall\, x \in R$.

Sol. Here, $3x^2 + ax + 3 > 0, \forall\, x \in R$

$\Rightarrow \quad D < 0 \quad [\because \text{if } f(x) > 0 \text{ and } a > 0, \text{ then } D < 0]$

$\Rightarrow \quad (a)^2 - 4(3)(3) < 0 \quad \Rightarrow \quad a^2 - 36 < 0$

$\Rightarrow (a-6)(a+6) < 0$, using number line rule as shown in figure,



which shows $-6 < a < 6$ or $a \in (-6, 6)$

Example 24 ***Find*** a ***for which*** $ax^2 + x - 1 < 0$, $\forall\, x \in R$ ***?***

Sol. Here, $ax^2 + x - 1 < 0, \forall\, x \in R$

$\Rightarrow \quad a < 0 \quad \text{and} \quad D < 0 \quad [\because a < 0, D < 0, \text{ so } f(x) < 0]$

$\Rightarrow \quad a < 0 \quad \text{and} \quad 1 + 4a < 0$

$\Rightarrow \quad a < 0 \quad \text{and} \quad a < -1/4$

$\therefore \quad a \in (-\infty, -1/4)$

Non-negative Functions

The sum of several non-negative terms is zero if and only if each term is zero.

i.e. $a^2 + b^2 + c^2 = 0$

if $a = b = c$

Example 25 Solve $(x+1)^2 + (x^2 + 3x + 2)^2 = 0$.

Sol. Here, $(x+1)^2 + (x^2 + 3x + 2)^2 = 0$ if and only if each term is zero simultaneously,

$(x+1) = 0$ and $(x^2 + 3x + 2) = 0 \Rightarrow (x+1)(x+2) = 0$

i.e. $x = -1$ and $x = -1, -2$

$\therefore$ The common solution is $x = -1$.

Hence, solution of the above equation is $x = -1$.

Example 26 Solve $|x+1| + \sqrt{x-1} = 0$.

Sol. Here, $|x+1| + \sqrt{x-1} = 0$, where each term is non-negative.

$\therefore \quad |x+1| = 0$ and $\sqrt{x-1} = 0$

should be zero simultaneously.

i.e. $x = -1$ and $x = 1$, which is not possible.

$\therefore$ There is no x for which each term is zero simultaneously. Hence, there is no solution.

Example 27 Solve

$$|x^2 - 1| + (x-1)^2 + \sqrt{x^2 - 3x + 2} = 0.$$

Sol. Here, each of the term is non-negative, thus each term must be zero simultaneously.

i.e. $(x^2 - 1) = 0, (x-1)^2 = 0$ and $x^2 - 3x + 2 = 0$

$\Rightarrow \quad x = \pm 1, \; x = 1$ and $x = 1, 2$

The common solution is $x = 1$.

Therefore, $\boldsymbol{x = 1}$ is the solution of above equation.

Example 28 Let $f(x) = x$ and $g(x) = |x|$ be two real-valued functions, $\phi(x)$ be a function satisfying the condition;

$[\phi(x) - f(x)]^2 + [\phi(x) - g(x)]^2 = 0$. Then, find $\phi(x)$.

Sol. Here, $[\phi(x) - f(x)]^2 + [\phi(x) - g(x)]^2 = 0$ is only possible, if $\phi(x) - f(x) = 0$ and $\phi(x) - g(x) = 0$

$\Rightarrow \quad \phi(x) = f(x) = g(x)$ or $\phi(x) = x = |x|$,

which is only possible, if x is non-negative.

Therefore, $\phi(x) = x, \forall\, x \in [0, \infty)$

Exercise for Session 4

1. Find all values of 'm' for which $(2m-3)x^2 + 2mx + 4 < 0$ for all real x.
2. If $ax^2 - bx + 5 = 0$ does not have two distinct real roots, then find the minimum value of $5a + b$.
3. If $a, b, c \in R, a \neq 0$ and the quadratic equation, $ax^2 + bx + c = 0$ has no real root, then show that $a(a + b + c) > 0$.
4. If $x, y \in [0, 10]$, then find the number of solutions (x, y) of the inequation $3^{\sec^2 x - 1} \cdot \sqrt{9y^2 - 6y + 2} \leq 1$.

Essential Mathematical Tools Exercise 1 : Subjective Questions

■ **Directions** (Q. Nos. 1 to 11) *Solve each of the following system of equations :*

1. For $a<0$, determine all solutions of the equation $x^2-2a|x-a|-3a^2=0$.

2. Solve $|x^2+4x+3|+2x+5=0$.

3. Solve $|x^2-3x-4|=9-|x^2-1|$.

4. Solve $2^{|x+1|}-2^x=|2^x-1|+1$.

5. Find the set of all real 'a' such that $5a^2-3a-2, a^2+a-2$ and $2a^2+a-1$ are the lengths of the sides of a triangle?

6. Solve $(x+3)^5-(x-1)^5\geq 244$.

7. Solve $||x-2|-1|\geq 3$.

8. Solve $1\leq\dfrac{3x^2-7x+8}{x^2+1}\leq 2$.

9. Let $f(x)=\dfrac{2x}{2x^2+5x+2}$ and $g(x)=\dfrac{1}{x+1}$.
Find the set of real values of x for which $f(x)>g(x)$.

10. For $x\in R, ||x||$ is defined as follows;

$$||x||=\begin{cases}x+1, & 0\leq x<2\\ |x-4|, & 2\leq x\end{cases}$$

Then, solve the equation, $||x||^2+x=||x||+x^2$.

11. Solve the inequality $|x-1|+|2-x|>3+x$.

12. Solve the equation

$$\sqrt{x^2+12y}+\sqrt{y^2+12x}=33,\ x+y=23.$$

13. Solve the equation

$$\sqrt{2x-1}+\sqrt{3x-2}=\sqrt{4x-3}+\sqrt{5x-4}.$$

14. If x, y and z are three real numbers such that $x+y+z=4$ and $x^2+y^2+z^2=6$, then show that each of x, y and z lies in the closed interval $\left[\dfrac{2}{3},2\right]$.

15. If $\{(\alpha+1)(\beta-1)+(\beta+1)(\alpha-1)\}a+(\alpha-1)(\beta-1)=0$ and $a(\alpha+1)(\beta+1)-(\alpha-1)(\beta-1)=0$

Also, let $A=\left\{\dfrac{\alpha+1}{\alpha-1},\dfrac{\beta+1}{\beta-1}\right\}$

and $B=\left\{\dfrac{2\alpha}{\alpha+1},\dfrac{2\beta}{\beta+1}\right\}$. If $A\cap B\neq\phi$, then find all the permissible values of the parameter 'a'.

16. Solve $\left|\dfrac{x-1}{3+2x-8x^2}\right|+|1-x|=\dfrac{(x-1)^2}{|3+2x-8x^2|}+1$

17. Let $f(x)=(x^2-2|x|)(2|x|-2)-9\dfrac{(2|x|-2)}{x^2-2|x|}$.
Solve the following inequalities

(i) $f(x)>0$ (ii) $f(x)\geq 0$

(iii) $f(x)<0$ (iv) $f(x)\leq 0$

18. Solve the inequality $\left|1-\dfrac{|x|}{1+|x|}\right|\geq\dfrac{1}{2}$.

Essential Mathematical Tools Exercise 2 : More Than One Correct Option Type Questions

19. If $\cos x-y^2-\sqrt{y-x^2-1}\geq 0$, then

(a) $y\geq 1$ (b) $x\in R$ (c) $y=1$ (d) $x=0$

20. If $(\sin\alpha)x^2-2x+b\geq 2$ for all real values of $x\leq 1$ and $\alpha\in\left(0,\dfrac{\pi}{2}\right)\cup\left(\dfrac{\pi}{2},\pi\right)$, then the possible real values of b is/are

(a) 2 (b) 3 (c) 4 (d) 5

21. If $|ax^2+bx+c|\leq 1$ for all x is [0, 1], then

(a) $|a|\leq 8$

(b) $|b|\leq 8$

(c) $|c|\leq 1$

(d) $|a|+|b|+|c|\leq 17$

Essential Mathematical Tools Exercise 3 : Passage Based Questions

Passage I

(Q. Nos. 22 to 24)

Let $f(x) = ax^2 + bx + c$; $a, b, c \in R$

It is given $|f(x)| \le 1, \forall\ |x| \le 1$.

Now, answer the following questions.

22. The possible value of $|a + c|$, if $\frac{8}{3}a^2 + 2b^2$ is maximum, is given by
(a) 1 (b) 0 (c) 2 (d) 3

23. The possible value of $|a + b|$, if $\frac{8}{3}a^2 + 2b^2$ is maximum, is given by
(a) 1 (b) 0 (c) 2 (d) 3

24. The possible maximum value of $\frac{8}{3}a^2 + 2b^2$ is given by
(a) 32 (b) $\frac{32}{3}$ (c) $\frac{2}{3}$ (d) $\frac{16}{3}$

Passage II

(Q. Nos. 25 to 27)

Consider the equation $|2x| - |x - 4| = x + 4$.

25. The least integer satisfying the equation, is
(a) − 4 (b) 4 (c) 5 (d) − 5

26. Total number of prime numbers less than 20 satisfying the equation, is
(a) 3 (b) 4 (c) 5 (d) 6

27. If P = greatest composite number less than 34 satisfying the given equation, then P^{2007} has the digit on its units place as
(a) 8 (b) 1
(c) 7 (d) 0

Passage III

(Q. Nos. 28 to 30)

Consider a number $N = 2\,1\,P\,5\,3\,Q\,4$.

28. The number of ordered pairs (P, Q) so that the number 'N' is divisible by 9, is
(a) 11 (b) 12
(c) 10 (d) 8

29. The number of values of Q so that the number 'N' is divisible by 8, is
(a) 4 (b) 3
(c) 2 (d) 6

30. The number of ordered pairs (P, Q) so that the number 'N' is divisible by 44, is
(a) 2 (b) 3 (c) 4 (d) 5

Passage IV

(Q. Nos. 31 to 35)

Consider the nine digit number $n = 7\,3\,\alpha\,4\,9\,6\,1\,\beta\,0$.

31. If p is the number of all possible distinct values of $(\alpha - \beta)$, then p is equal to
(a) 17 (b) 18 (c) 19 (d) 20

32. If q is the number of all possible values of β for which the given number is divisible by 8, then q is equal to
(a) 2 (b) 3 (c) 4 (d) 5

33. The number of ordered pairs (α, β) for which the given number is divisible by 88, is
(a) 1 (b) 2 (c) 3 (d) 4

34. The number of possible values of $(\alpha + \beta)$ for which the given number is divisible by 6, is
(a) 3 (b) 4 (c) 6 (d) 7

35. The number of possible values of β for which $i^N = 1$ (where $i = \sqrt{-1}$), is
(a) 2 (b) 3 (c) 4 (d) 5

Passage V

(Q. Nos. 36 to 38)

The set of integers can be classified into k classes, according to the remainder obtained when they are divided by k (where k is a fixed natural number). The classification enables is solving even some more difficult problems of number theory e.g.

(i) *even, odd classification is based on whether remainder is* 0 *or* 1 *when divided by* 2.

(ii) *when divided by* 3, *the remainder may be* 0, 1, 2. *Thus, there are three classes.*

36. The remainder obtained, when the square of an integer is divided by 3, is
(a) 0, 1 (b) 1, 2 (c) 0, 2 (d) 0, 1, 2

37. $n^2 + n + 1$ is never divisible by
(a) 2 (b) 3
(c) 111 (d) None of these

38. If n is odd, $n^5 - n$ is not divisible by
(a) 16 (b) 15 (c) 240 (d) 720

Essential Mathematical Tools Exercise 4 : Single Integer Answer Type Questions

39. The number of solutions of the equation $|x-1|-|2x-5| \geq 4$

40. The number of integral solution of the equation $|x^2-7| \leq 9$ are

41. The number of solutions of the system of equation $x+2y=6$ and $|x-3|=y$ is/are

Answers

Exercise for Session 1

1. False **2.** False **3.** True **4.** False **5.** True
6. True **7.** (a) **8.** (c) **9.** (a) **10.** (b)

Exercise for Session 2

1. $x \in (-1, 3)$

2. $x \in (-\infty, -2) \cup (8, \infty)$

3. $x \in (-2, 1) \cup (1, 4)$

4. $x \in (-\infty, 1] \cup \left[\frac{3}{2}, \infty\right)$

5. $x \in \left(-\infty, \frac{-1-\sqrt{29}}{2}\right] \cup \left[\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right] \cup \left[\frac{-1+\sqrt{29}}{2}, \infty\right)$

Exercise for Session 3

1. $x \in (-\infty, -2) \cup \left(\frac{1}{4}, 1\right) \cup (4, \infty)$

2. $x \in (-\infty, -1) \cup \left(\frac{-1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right)$

3. $x \in [1, 3]$

4. $x \in [-\infty, -6) \cup (-4, -1) \cup (1, 0)$ **5.** $x \in R$

6. (d) **7.** (c) **8.** (a) **9.** (a) **10.** (c) **11.** (d)
12. (d)

Exercise for Session 4

1. $m \in \phi$ **2.** (−1) **4.** (4)

Chapter Exercises

1. $\{(1-\sqrt{2})a, (-1+\sqrt{6})a\}$ **2.** $\{-4, -1-\sqrt{3}\}$

3. $\{-2, 2\}$ **4.** $\{-2\} \cup [0, \infty)$

5. $\left(\frac{3+\sqrt{57}}{8} < a < \frac{5+\sqrt{17}}{4}\right)$ **6.** $(-\infty, -2] \cup [0, \infty)$

7. $(-\infty, -2] \cup [6, \infty)$ **8.** $[1, 6]$ **9.** $x \in (-2, -1) \cup \left(-\frac{2}{3}, -\frac{1}{2}\right)$

10. $x = \{0, 2\}$ **11.** $(-\infty, 0) \cup (6, \infty)$

12. (13, 10) and (10, 13) **13.** $x = 1$

14. $x, y, z \in \left[\frac{2}{3}, 2\right]$

15. $a \in \{-1, 1+i, 1-i\}$

16. $x \in \left\{0, 2, \frac{1 \pm \sqrt{129}}{16}, \frac{3 \pm \sqrt{73}}{16}\right\}$

17. (i) $x \in (-\infty, -3) \cup (-2, -1) \cup (1, 2) \cup (3, \infty)$
(ii) $x \in (-\infty, -3] \cup (-2, -1] \cup [1, 2) \cup [3, \infty)$
(iii) $x \in (-3, -2) \cup (-1, 0) \cup (0, 1) \cup (2, 3)$
(iv) $x \in [-3, -2) \cup [-1, 0) \cup (0, 1] \cup (2, 3]$

18. $[-1, 1]$

19. (c,d)

20. (b,c,d)

21. (a,b,c,d) **22.** (a) **23.** (c) **24.** (b)
25. (a) **26.** (d) **27.** (c) **28.** (a) **29.** (b) **30.** (c)
31. (c) **32.** (a) **33.** (b) **34.** (d) **35.** (d) **36.** (a)
37. (c) **38.** (d) **39.** (0) **40.** (9) **41.** (2)

Solutions

1. For $a < 0, |x - a| = \begin{cases} (x-a), & x \geq a \\ (a-x), & x \leq a \end{cases}$.

***Case* I** $x \geq a$

$$x^2 - 2a(x-a) - 3a^2 = 0$$

$\Rightarrow$ $x^2 - 2ax - a^2 = 0$

$\Rightarrow$ $x^2 - 2ax + a^2 = 2a^2$

$\Rightarrow$ $(x-a)^2 = 2a^2 \Rightarrow x = \pm\sqrt{2}a + a$

$\Rightarrow$ $x = a(1+\sqrt{2}), a(1-\sqrt{2})$

Since, $x \geq a$ and $a < 0$

$\therefore$ Neglecting $\{a(1+\sqrt{2})\} \Rightarrow x = a(1-\sqrt{2})$

***Case* II** $x \leq a$

$\Rightarrow$ $x^2 + 2a(x-a) - 3a^2 = 0 \Rightarrow x^2 + 2ax = 5a^2$

$\Rightarrow$ $(x+a)^2 = 6a^2 \Rightarrow x = -a \pm \sqrt{6}a$

$\therefore$ $x = a(\sqrt{6}-1), -a(\sqrt{6}+1)$

Since, $x \leq a$ and $a < 0$

$\therefore$ Neglecting $\{-a(\sqrt{6}+1)\} \Rightarrow x = a(\sqrt{6}-1)$

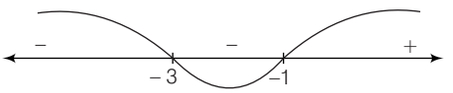
Hence, $x \in \{a(1-\sqrt{2}), a(\sqrt{6}-1)\}$.

2. $|x^2 + 4x + 3| + 2x + 5 = 0$

Take two cases, i.e. $|x^2 + 4x + 3| = \pm(x^2 + 4x + 3)$

***Case* I** $x^2 + 4x + 3 \geq 0$

$$(x+3)(x+1) \geq 0$$



$\Rightarrow$ $x \notin (-3, -1)$

or $x \in (-\infty, -3] \cup [-1, \infty)$

$\Rightarrow$ $x^2 + 4x + 3 + 2x + 5 = 0$

$\Rightarrow$ $x^2 + 6x + 8 = 0$

$\Rightarrow$ $(x+4)(x+2) = 0$

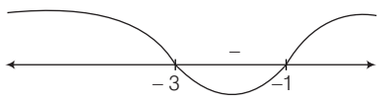
$\Rightarrow$ $x = -2, -4$.

But $x \notin (-3, -1)$

$\therefore$ Neglecting $x = -2$

$\Rightarrow$ $x = -4$ ...(i)

***Case* II** $x^2 + 4x + 3 \leq 0$



$$(x+3)(x+1) \leq 0$$

$\Rightarrow$ $x \in [-3, -1]$

$\Rightarrow$ $x^2 + 4x + 3 - 2x - 5 = 0$

$\Rightarrow$ $x^2 + 2x - 2 = 0$

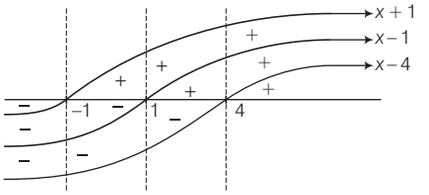
$\Rightarrow$ $x = -1 \pm \sqrt{3}$, as $x \in [-3, -1]$

$\therefore$ Neglecting $x = -1 + \sqrt{3} \Rightarrow x = -1 - \sqrt{3}$ ...(ii)

$\therefore$ $x \in \{-4, -1 - \sqrt{3}\}$.

3. $|x^2 - 3x - 4| + |x^2 - 1| = 9$

$\Rightarrow$ $|x+1|\{|x-4| + |x-1|\} = 9$



***Case* I** $x \geq 4$

$$(x+1)\{x - 4 + x - 1\} = 9$$

$\Rightarrow$ $(x+1)(2x-5) = 9$

$\Rightarrow$ $2x^2 - 3x - 14 = 0$

$\Rightarrow$ $x = \frac{7}{2}, -2$

[neglecting both as $x \geq 4$]

***Case* II** $1 < x < 4$

$\Rightarrow$ $(x+1)\{4 - x + x - 1\} = 9$

$\Rightarrow$ $(x+1) \cdot 3 = 9$

$\Rightarrow$ $x = 2 \in (1, 4)$ ...(i)

***Case* III** $-1 < x < 1$

$$(x+1)\{4 - x + 1 - x\} = 9$$

$\Rightarrow$ $(x+1)(5-2x) = 9$

$\Rightarrow$ $2x^2 - 3x + 4 = 0$, having imaginary roots.

$\therefore$ No solution.

***Case* IV** $x \leq -1$

$$-(x+1)\{4 - x + 1 - x\} = 9$$

$$(x+1)(2x-5) = 9 \Rightarrow x = -2, \frac{7}{2}\cdot$$

As $x \leq -1$,

$\therefore$ $x = -2$, neglecting $x = \frac{7}{2}$. ...(ii)

Thus, from Eqs. (i) and (ii), we get $x \in \{-2, 2\}$.

4. $2^{|x+1|} - 2^x = |2^x - 1| + 1$

***Case* I** $x \geq 0$

$$2^{x+1} - 2^x = 2^x - 1 + 1 \Rightarrow 2^x \cdot 2 = 2 \cdot 2^x$$

i.e. true for all $x \geq 0$ ...(i)

***Case* II** $-1 \leq x \leq 0$

$$2^{+x+1} - 2^x = 1 - 2^x + 1$$

$\Rightarrow$ $2^{x+1} = 2$

$\Rightarrow$ $x = 0$ ...(ii)

***Case* III** $x \leq -1$

$\Rightarrow$ $2^{-x-1} - 2^x = 1 - 2^x + 1$

$\Rightarrow$ $2^{-x-1} = 2$

$\Rightarrow$ $x = -2$ ...(iii)

Thus, from Eqs. (i), (ii) and (iii), we get $x \in \{-2\} \cup [0, \infty)$.

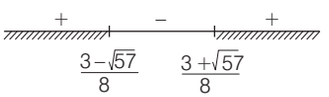
5. If $5a^2-3a-2, a^2+a-2, 2a^2+a-1$ are lengths of sides of triangle, then sum of any two sides must be greater than third side.

***Case* I** $(5a^2-3a-2)+(a^2+a-2)>2a^2+a-1$

$\Rightarrow \quad 6a^2-2a-4>2a^2+a-1$

$\Rightarrow \quad 4a^2-3a-3>0$

$\Rightarrow \left\{a-\left(\frac{3+\sqrt{57}}{8}\right)\right\}\left\{a-\left(\frac{3-\sqrt{57}}{8}\right)\right\}>0$



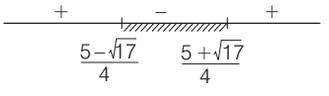
$\Rightarrow \quad a\in\left(-\infty, \frac{3-\sqrt{57}}{8}\right)\cup\left(\frac{3+\sqrt{57}}{8}, \infty\right)$ ...(i)

***Case* II** $(a^2+a-2)+(2a^2+a-1)>5a^2-3a-2$

$\Rightarrow \quad 3a^2+2a-3>5a^2-3a-2$

$\Rightarrow \quad 2a^2-5a+1<0$

$\Rightarrow \left\{a-\left(\frac{5+\sqrt{17}}{4}\right)\right\}\left\{a-\left(\frac{5-\sqrt{17}}{4}\right)\right\}<0$



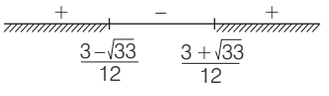
$\Rightarrow \quad a\in\left(\frac{5-\sqrt{17}}{4}, \frac{5+\sqrt{17}}{4}\right)$ ...(ii)

***Case* III** $(5a^2-3a-2)+(2a^2+a-1)>(a^2+a-2)$

$\Rightarrow \quad 7a^2-2a-3>a^2+a-2$

$\Rightarrow \quad 6a^2-3a-1>0$

$\Rightarrow \quad \left\{a-\left(\frac{3+\sqrt{33}}{12}\right)\right\}\left\{a-\left(\frac{3-\sqrt{33}}{12}\right)\right\}>0$



$\Rightarrow \quad a\in\left(-\infty, \frac{3-\sqrt{33}}{12}\right)\cup\left(\frac{3+\sqrt{33}}{12}, \infty\right)$ ...(iii)

From Eqs. (i), (ii) and (iii), we get

$$a\in\left(\frac{3+\sqrt{57}}{8}, \frac{5+\sqrt{17}}{4}\right)$$

6. $(x+3)^5-(x-1)^5\geq 244$

Let $y=\frac{(x+3)+(x-1)}{2}\Rightarrow y=x+1$

$\therefore \quad (y+2)^5-(y-2)^5\geq 244$

$\Rightarrow \quad 2\{{}^5C_1\cdot y^4\cdot 2+{}^5C_3\cdot y^2\cdot 2^3+{}^5C_5\cdot 2^5\}\geq 244$

$\Rightarrow \quad 2\{10y^4+80y^2+32\}\geq 244$

$\Rightarrow \quad 4\{5y^4+40y^2+16\}\geq 244$

$\Rightarrow \quad 5y^4+40y^2+16\geq 61$

$\Rightarrow \quad y^4+8y^2-9\geq 0$

$\Rightarrow \quad (y^2+9)(y^2-1)\geq 0$

$\Rightarrow \quad y^2\geq 1$

i.e. $\quad y\leq -1$ or $y\geq 1$

$\Rightarrow \quad x+1\leq -1$ or $x+1\geq 1$

$\Rightarrow \quad x\in(-\infty,-2]\cup[0,\infty)$

7. $||x-2|-1|\geq 3$

$\Rightarrow \quad |x-2|-1\leq -3$ or $|x-2|-1\geq 3$

$\Rightarrow \quad |x-2|\leq -2$ or $|x-2|\geq 4$

$\Rightarrow$ No solution or $x-2\leq -4$ or $x-2\geq 4$

$\Rightarrow \quad x\leq -2$ or $x\geq 6$

$\Rightarrow \quad x\in(-\infty,-2]\cup[6,\infty)$

8. $1\leq\frac{3x^2-7x+8}{x^2+1}\leq 2$

$\Rightarrow \quad 2x^2-7x+7\geq 0$ and $x^2-7x+6\leq 0$

$\Rightarrow \quad x\in R$ and $x\in[1,6]$

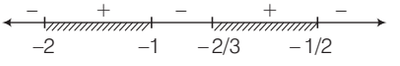
$\therefore \quad x\in[1,6]$

9. $f(x)>g(x)\Rightarrow \frac{2x}{2x^2+5x+2}>\frac{1}{x+1}$

$\Rightarrow \quad \frac{2x}{(2x+1)(x+2)}-\frac{1}{(x+1)}>0$

$\Rightarrow \quad \frac{2x^2+2x-(2x^2+5x+2)}{(2x+1)(x+2)(x+1)}>0$

$\Rightarrow \quad \frac{-(3x+2)}{(2x+1)(x+2)(x+1)}>0$



$\Rightarrow \quad x\in(-2,-1)\cup(-2/3,-1/2)$

10. $||x||^2+x=||x||+x^2$.

***Case* I** When $0\leq x<2$

$\Rightarrow \quad (x+1)^2+x=(x+1)+x^2$

$\Rightarrow \quad x^2+2x+1+x=x^2+x+1$

$\Rightarrow \quad 2x=0$

$\Rightarrow \quad x=0$ ...(i)

***Case* II** When $x\geq 2$

$\Rightarrow \quad (x-4)^2+x=|x-4|+x^2$

Now, if $2\leq x\leq 4$

$\Rightarrow \quad (x^2-8x+16)+x=4-x+x^2$

$\Rightarrow \quad 6x=12$

$\Rightarrow \quad x=2$ ...(ii)

Again, if $x\geq 4$

$\Rightarrow \quad (x-4)^2+x=x-4+x^2$

$\Rightarrow \quad x^2-8x+16+x=x^2+x-4$

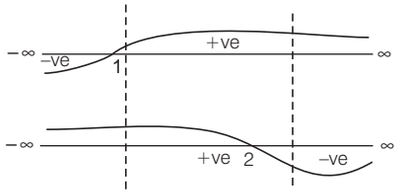
$\Rightarrow \quad 8x=20$

$\Rightarrow \quad x=\frac{5}{2}$,

but $x\geq 4$.

$\therefore$ Only two solutions $x=0$ and 2.

11. The points $x = 1$ and $x = 2$ divide the number axis into three intervals as follows :



We solve the inequality on each intervals.

If $x < 1$, then $x - 1 < 0$ and $2 - x > 0$.

$\therefore \quad |x-1| + |2-x| > 3 - x$

$\Rightarrow \quad 1 - x + 2 - x > 3 + x$

$\Rightarrow \quad 3 - 2x > 3 + x$

$\Rightarrow \quad x < 0 \quad ...(i)$

If $1 \le x \le 2$, then $x - 1 \ge 0$ and $2 - x \ge 0$, we have

$x - 1 + 2 - x > 3 + x$

$\Rightarrow \quad 1 > 3 + x \Rightarrow x < -2$

The system of inequalities obtained has no solution, for $1 \le x \le 2$. If $x > 2$, then $x - 1 > 0$ and $2 - x < 0$, we have

$x - 1 + x - 2 > 3 + x$

$\Rightarrow \quad 2x - 3 > 3 + x \Rightarrow x > 6$

Combining the solutions obtained on all parts of the domain of admissible values of the given inequality, we get the solution set $(-\infty, 0) \in (6, \infty)$.

12. Here, $\sqrt{x^2 + 12(23 - x)} + \sqrt{(23 - x)^2 + 12x} = 33 \ (\because y = 23 - x)$

$\Rightarrow \quad \sqrt{x^2 - 12x + 276} + \sqrt{x^2 - 34x + 529} = 33$

Let $\quad a = \sqrt{x^2 - 12x + 276}$

and $\quad b = \sqrt{x^2 - 34x + 529} \quad ...(i)$

$\Rightarrow \quad a + b = 33$ and $a^2 - b^2 = 11(2x - 23)$

$a - b = \dfrac{2x - 23}{3}$

$\therefore \quad a = \dfrac{x + 38}{3} \quad ...(ii)$

$\Rightarrow \quad x^2 - 12x + 276 = \left(\dfrac{x+38}{3}\right)^2$ [using Eqs. (i) and (ii)]

$\Rightarrow \quad 8x^2 - 184x + 1040 = 0 \Rightarrow x = 13, 10$

$\Rightarrow \quad y = 10, 13$

$\therefore$ (13, 10) and (10, 13) are the required solutions.

13. Here, $u + v = p + q \quad ...(i)$

Where $u = \sqrt{2x - 1}$, $v = \sqrt{3x - 2}$, $p = \sqrt{4x - 3}$,

$q = \sqrt{5x - 4}$

$\therefore \quad u^2 - v^2 = 1 - x$

$p^2 - q^2 = 1 - x \Rightarrow u^2 - v^2 = p^2 - q^2$

$\Rightarrow \quad u - v = p - q \quad ...(ii)$ {using Eq. (i), $u + v = p + q$}

$\therefore$ From Eqs. (i) and (ii), we get

$2u = 2p \Rightarrow 2x - 1 = 4x - 3$

or $\quad x = 1$, which clearly satisfies

$\sqrt{2x - 1} + \sqrt{3x - 2} = \sqrt{4x - 3} + \sqrt{5x - 4}$

$\therefore \quad x = 1$ is the required solution.

14. Here, $x^2 + y^2 + (4 - x - y)^2 = 6 \quad (\because z = 4 - x - y)$

$x^2 + (y - 4)x + (y^2 + 5 - 4y) = 0$

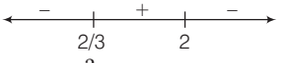
Since, x is real.

$\Rightarrow \quad D \ge 0$

$\Rightarrow \quad (y - 4)^2 - 4(y^2 - 4y + 5) \ge 0$

$\Rightarrow \quad -3y^2 + 8y - 4 \ge 0$

$\Rightarrow \quad -(3y - 2)(y - 2) \ge 0$



$\Rightarrow \quad \dfrac{2}{3} \le y \le 2$

Similarly, we can show that x, y and $z \in \left[\dfrac{2}{3}, 2\right]$.

15. Here, $\dfrac{\alpha + 1}{\alpha - 1} + \dfrac{\beta + 1}{\beta - 1} = -\dfrac{1}{a}$ and $\dfrac{(\alpha + 1)(\beta + 1)}{(\alpha - 1)(\beta - 1)} = \dfrac{1}{a}$

Now, the quadratic $x^2 - \left(-\dfrac{1}{a}\right)x + \dfrac{1}{a} = 0$ has roots $\left(\dfrac{\alpha + 1}{\alpha - 1} \text{ and } \dfrac{\beta + 1}{\beta - 1}\right)$

$\Rightarrow \quad ax^2 + x + 1 = 0 \quad ...(i)$

Let $\quad x = \dfrac{2\alpha}{\alpha + 1} \Rightarrow \alpha = \dfrac{x}{2 - x} \Rightarrow \dfrac{\alpha + 1}{\alpha - 1} = \dfrac{1}{x - 1}$

Now, replacing x by $\dfrac{1}{x - 1}$ in Eq. (i), we get a quadratic whose roots are $\dfrac{2\alpha}{\alpha + 1}$ and $\dfrac{2\beta}{\beta + 1}$.

$\dfrac{a}{(x - 1)^2} + \dfrac{1}{(x - 1)} + 1 = 0$

$\Rightarrow \quad x^2 - x + a = 0 \quad ...(ii)$

Hence, Eqs. (i) and (ii) are the two quadratics whose roots are elements of set 'A' and set 'B', respectively.

$\therefore$ They must have a root in common as $A \cap B \ne \phi$

***Case* I** Both roots are common.

$\Rightarrow \quad \dfrac{a}{1} = \dfrac{1}{-1} = \dfrac{1}{a}$

$\Rightarrow \quad a = -1$ and $a^2 = 1 \Rightarrow a = -1$

***Case* II** For a common root,

$\dfrac{x^2}{a + 1} = \dfrac{x}{1 - a^2} = \dfrac{1}{-(a + 1)}$

$\Rightarrow \quad a = 1 \pm i$

$\therefore \quad a \in \{-1, 1 + i, 1 - i\}$

16. Here, $\left|\dfrac{x - 1}{3 + 2x - 8x^2}\right| + |1 - x| = \dfrac{(x - 1)^2}{|3 + 2x - 8x^2|} + 1$

Let $\quad a = \dfrac{x - 1}{3 + 2x - 8x^2}; \ b = (x - 1)$,

then $\quad |a| + |b| = |ab| + 1$

$\Rightarrow \quad |ab| - |a| - |b| + 1 = 0$

$\Rightarrow \quad (|a| - 1)(|b| - 1) = 0$

$\therefore \quad |a| = 1$ or $|b| = 1$

When $|a|=1$

$\Rightarrow \quad \left|\dfrac{x-1}{3+2x-8x^2}\right|=1$

$\dfrac{x-1}{3+2x-8x^2}=1 \quad \text{or} \quad \dfrac{x-1}{3+2x-8x^2}=-1$

$\Rightarrow \quad x-1=3+2x-8x^2$

or $\quad x-1=-3-2x+8x^2$

$\Rightarrow \quad 8x^2-x-4=0 \text{ or } 8x^2-3x-2=0$

$\Rightarrow \quad x=\dfrac{1\pm\sqrt{1+128}}{16} \text{ or } x=\dfrac{3\pm\sqrt{9+64}}{16}$

$\Rightarrow \quad x=\dfrac{1\pm\sqrt{129}}{16} \quad \text{or} \quad x=\dfrac{3\pm\sqrt{73}}{16} \qquad ...(i)$

Again, when $|b|=1, |x-1|=1$

$\Rightarrow \quad x-1=1 \text{ or } x-1=-1$

$\Rightarrow \quad x=0 \text{ or } x=2 \qquad ...(ii)$

From Eqs. (i) and (ii), solution set is

$$x\in\left\{0,2,\ \frac{1\pm\sqrt{129}}{16},\frac{3\pm\sqrt{73}}{16}\right\}$$

17. We have, $f(x)=(x^2-2|x|)(2|x|-2)-9\cdot\dfrac{(2|x|-2)}{x^2-2|x|}$

$$=(2|x|-2)\left\{x^2-2|x|-\frac{9}{x^2-2|x|}\right\}$$

$$=(2|x|-2)\left\{\frac{(x^2-2|x|)^2-9}{x^2-2|x|}\right\}$$

$$=\frac{(2|x|-2)(x^2-2|x|+3)(x^2-2|x|-3)}{x^2-2|x|}$$

$$=\frac{(2|x|-2)\{(|x|-1)^2+2\}\{(|x|+1)(|x|-3)\}}{x^2-2|x|}$$

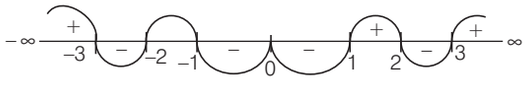
Taking, $\quad N^r=0 \Rightarrow |x|=1,3$

$\Rightarrow \quad x=\pm1,\pm3 \text{ and } D^r=0$

$\Rightarrow \quad |x|(|x|-2)=0$

$\Rightarrow \quad 0,\pm2$

We mark these roots on a number line :



From the wavy curve method, we have

(i) $f(x)>0 \Rightarrow x\in(-\infty,-3)\cup(-2,-1)\cup(1,2)\cup(3,\infty)$

(ii) $f(x)\geq0 \Rightarrow x\in(-\infty,-3]\cup(-2,-1]\cup[1,2)\cup[3,\infty)$

(iii) $f(x)<0 \Rightarrow x\in(-3,-2)\cup(-1,0)\cup(0,1)\cup(2,3)$

(iv) $f(x)\leq0 \Rightarrow x\in[-3,-2)\cup[-1,0)\cup(0,1]\cup(2,3]$

18. The domain of admissible values of this inequality consists of all the real numbers. The inequality is equivalent to the collection of two systems :

$$\begin{cases}\left|1-\dfrac{x}{1+x}\right| \geq \dfrac{1}{2}\\ x \geq 0\end{cases}, \quad \begin{cases}\left|1-\dfrac{-x}{1-x}\right| \geq \dfrac{1}{2}\\ x \leq 0\end{cases}$$

We solve the first system :

$$\begin{cases}\left|1-\dfrac{x}{1+x}\right| \geq \dfrac{1}{2}\\ x \geq 0\end{cases} \Leftrightarrow \begin{cases}\dfrac{1}{1+x} \geq \dfrac{1}{2}\\ x \geq 0\end{cases}$$

$$\Leftrightarrow \quad \begin{cases}\dfrac{2-1-x}{1+x} \geq 0\\ x \geq 0\end{cases} \Leftrightarrow \begin{cases}\dfrac{1-x}{1+x}\\ x\geq0\end{cases} \Leftrightarrow 0\leq x\leq1$$

Again, we solve the second system. Its first inequality is equivalent to the inequality $\left|\dfrac{1}{1-x}\right|\geq\dfrac{1}{2}$.

If $x<0$, then $1-x>0$ and consequently the second system is equivalent to the system

$$\begin{cases}\dfrac{1}{1-x} \geq \dfrac{1}{2}\\ x \leq 0\end{cases} \Leftrightarrow \begin{cases}\dfrac{2-1+x}{1-x} \geq 0\\ x \leq 0\end{cases}$$

$$\Leftrightarrow \quad \begin{cases}\dfrac{1+x}{1-x} \geq 0\\ x \leq 0\end{cases} \Leftrightarrow -1\leq x<0$$

Thus, the set of all solutions of the original inequality consists of the numbers belonging to the interval $[-1,1]$.

19. Given, $\cos x-y^2-\sqrt{y-x^2-1}\geq0 \qquad ...(i)$

Clearly, $\sqrt{y-x^2-1}$ is defined when $y-x^2-1\geq0$

or $y\geq x^2+1$. So, minimum value of y is 1.

From Eq. (i), we have $\cos x-y^2\geq\sqrt{y-x^2-1}$

where $\cos x-y^2\leq0$ [as when $\cos x$ is maximum $(=1)$ and y^2 is minimum $(=1)$. So, $\cos x-y^2$ is maximum.]

Also, $\quad \sqrt{y-x^2-1}\geq0$

Hence, $\quad \cos x-y^2=\sqrt{y-x^2-1}=0$

$\Rightarrow \quad y=1 \text{ and } \cos x=1,\ y=x^2+1$

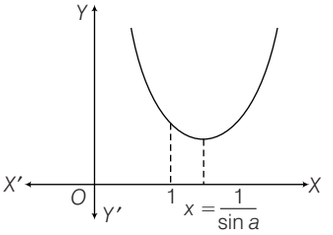
$\Rightarrow \quad x=0,\ y=1$

20. Given, $(\sin\alpha)x^2-2x+b\geq2$

Let $f(x)=(\sin\alpha)x^2-2x+b-2$

Abscissa of the vertex is given by $x=\dfrac{1}{\sin\alpha}>1$

The graph of $f(x)=(\sin\alpha)x^2-2x+b-2, \forall x\leq1$ is shown in the figure.



Therefore, minimum of $f(x)=(\sin\alpha)x^2-2x+b-2$ must be greater than zero but minimum is at $x=1$.

i.e. $\quad \sin\alpha-2+b-2\geq0,$

$\Rightarrow \quad b\geq4-\sin\alpha,\ \alpha\in(0,\pi)-\left\{\dfrac{\pi}{2}\right\}$

21. Given, $|ax^2 + bx + c| \le 1, \forall x \in [0,1]$ ...(i)

Putting successively $x = 0,\ 1$ and $\frac{1}{2}$ in Eq. (i), we get

$$-1 \le c \le 1 \quad \text{...(ii)}$$

$$-1 \le a + b + c \le 1 \quad \text{...(iii)}$$

$$-4 \le a + 2b + 4c \le 4 \quad \text{...(iv)}$$

From Eqs. (ii), (iii) and (iv), we get

$$|b| \le 8 \text{ and } |a| \le 8$$

$$\Rightarrow \quad |a| + |b| + |c| \le 17$$

Sol. (Q. Nos. 22-24)

$$|f(1) - f(0)| \le 2 \qquad (\because \text{if } |u| \le 1, |v| \le 1, \text{ then } |u - v| \le 2)$$

$$\Rightarrow \quad |a + b| \le 2 \qquad \{\because f(1) = a + b + c \text{ and } f(0) = c\}$$

$$\Rightarrow \quad (a + b)^2 \le 4$$

Also, $|f(-1) - f(0)| \le 2$

$$\Rightarrow \quad |a - b| \le 2 \qquad [\because f(-1) = a - b + c \text{ and } f(0) = c]$$

$$\Rightarrow \quad (a - b)^2 \le 4$$

Now, $4a^2 + 3b^2 = 2(a + b)^2 + 2(a - b)^2 - b^2$

$$\le 2 \times 4 + 2 \times 4 - b^2 \le 16 - b^2 \le 16$$

$\Rightarrow$ Maximum value of $4a^2 + 3b^2 = 16$, when $b = 0$

$$\Rightarrow \quad |a + b| + |a - b| = 4 \Rightarrow |a| = 2$$

Also, $|f(1) - f(0)| = |a + c - c| = |a| = 2$

$$\Rightarrow \quad |a + c| = |c| = 1$$

$\therefore$ The possible values of (a, b, c) are $(2, 0, -1)$ or $(-2, 0, 1)$

Also, $\frac{8}{3}a^2 + 2b^2 = \frac{2}{3}(4a^2 + 3b^2) \le \frac{2}{3}(16)$

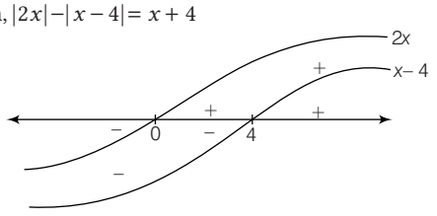
22. $|a + c| = 1$, when $\frac{8a^2}{3} + 2b^2$ is maximum.

23. $|a + b| = 2$, when $\frac{8a^2}{3} + 2b^2$ is maximum.

24. The maximum value of $\frac{8a^2}{3} + 2b^2$ is $\frac{32}{3}$.

Sol. (Q. Nos. 25-27)

25. Given, $|2x| - |x - 4| = x + 4$



***Case* I** When, $-\infty < x \le 0$

$$-2x - 4 + x = x + 4$$

$$\Rightarrow \quad -2x = 8 \Rightarrow x = -4$$

***Case* II** When, $0 < x < 4$

$$2x - (4 - x) = x + 4$$

$$\Rightarrow \quad 2x = 8 \Rightarrow x = 4$$

***Case* III** When, $4 \le x < \infty$

$$2x - x + 4 = x + 4$$

$$\Rightarrow \quad x + 4 = x + 4 \Rightarrow x \in R$$

Solution of the given equation is $\{-4\} \cup [4, \infty)$.

$\therefore$ Least integer $= -4$.

26. Prime numbers less than 20 satisfying the equation are 5, 7, 11, 13, 17, 19 i.e. 6 prime number.

27. $P = 33,\ P^{2007} = (33)^{2007}$ = number has 7 at its units place.

Sol. (Q. Nos. 28-30)

28. Sum of digits $= P + Q + 15$

N is divisible by 9, if $P + Q + 15 = 18, 27$

$$\Rightarrow \quad P + Q = 3 \quad \text{...(i)}$$

or $\quad P + Q = 12 \quad \text{...(ii)}$

$$P = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$$

From Eq. (i), we get $\left.\begin{matrix} P = 0, Q = 3 \\ P = 1, Q = 2 \\ P = 2, Q = 1 \\ P = 3, Q = 0 \end{matrix}\right\}$ Number of ordered pairs is 4.

From Eq. (ii), we get

$$\left.\begin{matrix} P = 3, Q = 9 \\ P = 4, Q = 8 \\ \ldots\ \ \ldots \\ P = 8, Q = 4 \\ P = 9, Q = 3 \end{matrix}\right\} \text{ Number of ordered pairs is 7.}$$

Total number of ordered pairs is 11.

29. N is divisible by 8, if $Q = 0, 4, 8$

Number of values of Q is 3.

30. $S_O = P + 9$

$S_E = Q + 6$

$$S_O - S_E = P - Q + 3$$

N is divisible is 11, if

$$P - Q + 3 = 0, 11$$

$$P - Q = -3 \quad \text{...(i)}$$

or $\quad P - Q = 8 \quad \text{...(ii)}$

N is divisible by 4, if

$$Q = 0, 2, 4, 6, 8$$

From Eq. (i), $\quad Q = 0, \quad P = -3$ (not possible)

$$Q = 2, \quad P = -1 \quad \text{(not possible)}$$

$$Q = 4, \quad P = 1$$

$$Q = 6, \quad P = 1$$

$$Q = 8, \quad P = 5$$

$\therefore$ Number of ordered pairs is 3.

From Eq. (ii), $\quad Q = 0 \quad P = 8$

$$Q = 2 \quad P = 10 \quad \text{(not possible)}$$

Similarly, $\quad Q \ne 4, 6, 8$

$\therefore$ Number of ordered pairs is 1.

$\therefore$ Total number of ordered pairs, so that the number N is divisible by 44 is 4.

Sol. (Q. Nos. 31-35)

31. If $\alpha = 0$, then the possible values of $\alpha - \beta$ are

$$\{0, -1, -2, -3, \ldots, -9\}$$

If $\alpha = 1$, then the possible values of $\alpha - \beta$ are

$$\{1, 0, -1, -2, -3, \ldots, -8\}$$

If $\alpha = 2$, then the possible values of $\alpha - \beta$ are

$$\{2, 1, 0, -1, -2, -3, \ldots, -7\} \text{ and so on}$$

If $\alpha = 9$, then the possible values of $\alpha - \beta$ are

$\{9, 8, 7, 6, 5, 4, 3, 2, 1, 0\}$

Thus, in all the values of $\alpha - \beta$ are

$\{-9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, \ldots, 9\}$

$\therefore$ Total number of values of $\alpha - \beta$ is 19.

$\therefore \quad p = 19$

32. Number N is divisible by 8, if $100 + 10\beta$ is divisible by 8.

$\therefore$ Possible values of β are 2, 6.

$\therefore \quad q = 2$

33. $N = 7\ 3\ \alpha\ 4\ 9\ 6\ 1\ \beta\ 0$

N is divisible by 88, if N is divisible by 8 as well as 11.

N is divisible by 8, then $\beta = 2, 6$

For divisibility by 11,

$S_O = 17 + \alpha,\ S_E = 13 + \beta$

$S_O - S_E = (17 + \alpha) - (13 + \beta) = \alpha - \beta + 4$

$\alpha - \beta + 4$ be the integer lying between -5 and 13.

But, $\alpha - \beta + 4$ is divisible by 11, so $\alpha - \beta + 4 = 0,\ 11$

***Case* I** $\alpha - \beta + 4 = 0$

when $\beta = 2, \alpha = -2$ (rejected)

when $\beta = 6, \alpha = 2$

***Case* II** $\alpha - \beta + 4 = 11$ when $\alpha = 9, \beta = 2$

when $\beta = 6, \alpha = 13$ (rejected)

$\therefore \quad (\alpha, \beta) = (9, 2)\ (2, 6)$

$\therefore$ Number of ordered pairs are 2.

34. N is divisible by 6, if it is divisible is 2 and 3.

$N = 7\ 3\ \alpha\ 4\ 9\ 6\ 1\ \beta\ 0$ is divisible by 2 for all values of α and β.

$30 \le (30 + \alpha + \beta) \in I \le 48$

But $30 + \alpha + \beta$ is divisible by 3, if

$30 + \alpha + \beta = 30, 33, 36, 39, 42, 45, 48$

$\alpha + \beta = 0, 3, 6, 9, 12, 15, 18$

Number of possible values of $(\alpha + \beta)$ is equal to 7.

35. $i^N = 1 \Rightarrow N$ must be divisible by 4.

$\Rightarrow \beta = 0, 2, 4, 6, 8$

Number of possible values of β is 5.

Sol. (Q. Nos. 36-38)

36. Let $n = 3k + 4,\ r = 0, 1, 2$

$n^2 = 9k^2 + 6kr + r^2 = 3m + r^2$

$r^2 = 0, 1, 4$

$\therefore$ Remainder = 0 or 1.

37. (a) $n^2 + n + 1$ is an odd number.

(b) when $n = 1$, $n^2 + n + 1$ is divisible by 3.

(c) $111 = 10^2 + 10 + 1 \quad \therefore$ when $n = 10$

$n^2 + n + 1$ is divisible by 111.

38. n = odd

Let $n = 2k + 1$

$n^5 - n = n(n^4 - 1) = n(n-1)(n+1)(n^2 + 1)$

$= (2k + 1)(2k)(2k + 2)(4k^2 + 4k + 2)$

$= 8(k)(k + 1)(2k + 1)(2k^2 + 2k + 1)$

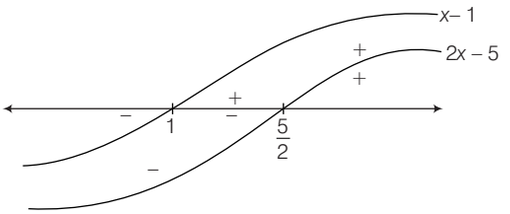
which is divisible by 15, 16 and 240 but not divisible by 720

When $n = 5 \Rightarrow n(n^4 - 1)$

$= 5(5^4 - 1) = 5(625 - 1) = 5 \times 624 = 3120,$

which is not divisible by 720.

39. Given, $|x-1| - |2x-5| \ge 4$



***Case* I** When, $x < 1$

$1 - x - (5 - 2x) \ge 4 \Rightarrow x - 4 \ge 4 \Rightarrow x \ge 8$

but $x < 1$

$\Rightarrow$ No solution

***Case* II** $1 \le x \le 5/2$

$x - 1 + 2x - 5 \ge 4$

$\Rightarrow \quad 3x - 6 \ge 4 \Rightarrow 3x \ge 10$

$\Rightarrow \quad x \ge \frac{10}{3}$ but $1 \le x \le 5/2$

$\Rightarrow$ No such x exist.

***Case* III** $x \ge 5/2$

$\Rightarrow \quad x - 1 - (2x - 5) \ge 4 \Rightarrow x - 1 - 2x + 5 \ge 4$

$\Rightarrow \quad -x + 4 \ge 4 \Rightarrow x \le 0$

$\Rightarrow$ No such x exist.

$\therefore$ Number of solution is 0.

40. Given, $|x^2 - 7| \le 9$

$\Rightarrow \quad -9 \le x^2 - 7 \le 9 \Rightarrow -2 \le x^2 \le 16$

$\Rightarrow \quad 0 \le x^2 \le 16 \Rightarrow 0 \le |x| \le 4$

$\Rightarrow \quad x \in [-4, 4]$

Hence, number of integral solution 9.

41. System of equations are $x + 2y = 6$ and $|x - 3| = y$

***Case* I** When $x \ge 3$, $x + 2y = 6$ and $x - 3 = y$

Solving these equations, we get $x = 4, y = 1$

***Case* II** When $x < 3$, $x + 2y = 6$ and $3 - x = y$

Solving these equations, we get $x = 0, y = 3$

Hence, two solutions are (0,3) and (4,1).

CHAPTER

02

Differentiation

Learning Part

Session 1

- Geometrical Meaning of the Derivative
- Derivative of $f(x)$ from the First Principle or *ab*-Initio Method
- Rules of Differentiation

Session 2

- Chain Rule

Session 3

- Differentiation of Implicit Functions

Session 4

- Differentiation of Inverse Trigonometric Functions
- Graphical Approach for Differentiation of Inverse Trigonometric Functions

Session 5

- Differentiation of a Function in Parametric Form

Session 6

- Logarithmic Differentiation
- Differentiation of Infinite Series

Session 7

- Differentiation of a Function w.r.t. Another Function

Session 8

- Higher Derivatives of a Function

Session 9

- Differentiation of a Function given in the form of a Determinant

Session 10

- Derivative of an Inverse Function

Practice Part

- JEE Type Examples
- Chapter Exercises

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Session 1

Geometrical Meaning of the Derivative, Derivative of *f*(*x*) from the First Principle or *ab*-Initio Method, Rules of Differentiation

Derivative is the instantaneous rate of change of a function with respect to dependent variable. Also, the derivative is the slope of tangent to a curve at a point.

Geometrical Meaning of the Derivative

Let us consider a function $y = f(x)$ in a rectangular coordinate system. We also consider a point $P(x, y)$ on the curve. If a point corresponding to an increased value of the argument $x + \Delta x$ is considered, its ordinate value is given by $y + \Delta y = f(x + \Delta x)$.

In figure, the point $(x + \Delta x, y + \Delta y)$ is represented by A.

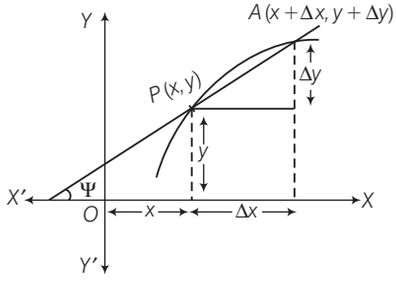


Figure 2.1

Here, PA is the secant to the curve.

Now, as $\Delta x \to 0$, $\Delta y \to 0 \Rightarrow PA \to 0$

(i.e. the distance PA tends to zero or to a single point P)

$\Rightarrow \lim\limits_{\Delta x \to 0}$ (slope of chord PA) $\to$ (slope of tangent at P).

or $$\tan\psi = \lim_{\Delta x \to 0} \tan\theta$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} \quad \text{or} \quad f'(x)$$

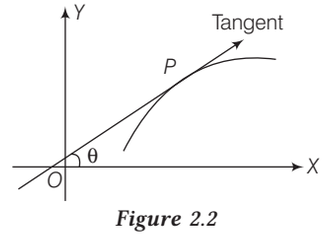


Figure 2.2

which means that **the value of the derivative $f'(x)$ for a given value of x is equal to the tangent of the angle formed by the tangent line to the graph of the function $y = f(x)$ at the point $P(x, y)$ with the positive X-axis.**

Relation between *dy*/*dx* and *dx*/*dy*

Let x and y be two variables connected by a relation of the form $f(x, y) = 0$.

Let Δx be a small change in x and Δy be a small change in y, then

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \quad \text{and} \quad \frac{dx}{dy} = \lim_{\Delta y \to 0} \frac{\Delta x}{\Delta y}$$

Now, $$\frac{\Delta y}{\Delta x} \cdot \frac{\Delta x}{\Delta y} = 1$$

$$\Rightarrow \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x} \cdot \frac{\Delta x}{\Delta y} \right) = 1$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \cdot \lim_{\Delta y \to 0} \frac{\Delta x}{\Delta y} = 1 \qquad [\because \Delta x \to 0 \Leftrightarrow \Delta y \to 0]$$

$$\Rightarrow \frac{dy}{dx} \cdot \frac{dx}{dy} = 1$$

So, $$\frac{dy}{dx} = \frac{1}{dx/dy}$$

Derivatives of *f*(*x*) from the First Principle

or ab-Initio Method or Delta Method or By Definition

If $f(x)$ is differentiable function, then

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x) = \frac{dy}{dx}$$

Simply, $$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

Example 1 Differentiate the following functions w.r.t. x using first principle.

(i) $f(x)=\tan x$ (ii) $f(x)=e^{2x}$

(iii) e^{x^2} (iv) $\sqrt{\sin x}$

Sol. (i) Let $f(x)=\tan x$

Then, $f(x+h)=\tan(x+h)$

$$\therefore \quad \frac{d}{dx}(\tan x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\tan(x+h)-\tan(x)}{h}$$

$$=\lim_{h\to 0}\frac{\tan(x+h-x)\cdot[1+\tan x\cdot\tan(x+h)]}{h}$$

$[\because \tan A-\tan B=\tan(A-B)\cdot(1+\tan A\tan B)]$

$$=\lim_{h\to 0}\frac{\tan h}{h}\cdot(1+\tan^2 x) \quad \left[\because \lim_{h\to 0}\frac{\tan h}{h}=1\right]$$

$$=\sec^2 x$$

$$\therefore \quad \frac{d}{dx}(\tan x)=\sec^2 x$$

(ii) Let $f(x)=e^{2x} \Rightarrow f(x+h)=e^{2(x+h)}$

$$\therefore \quad \frac{d}{dx}(f(x))=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \frac{d}{dx}(e^{2x})=\lim_{h\to 0}\frac{e^{2(x+h)}-e^{2x}}{h}=\lim_{h\to 0}e^{2x}\cdot\frac{(e^{2h}-1)}{h}$$

$$=e^{2x}\cdot\left(\lim_{h\to 0}\frac{e^{2h}-1}{2h}\right)\times 2=2e^{2x} \quad \left[\because \lim_{x\to 0}\frac{e^x-1}{x}=1\right]$$

$$\therefore \quad \frac{d}{dx}(e^{2x})=2e^{2x}$$

(iii) Let $f(x)=e^{x^2} \Rightarrow f(x+h)=e^{(x+h)^2}$

$$\therefore \quad \frac{d}{dx}f(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{e^{(x+h)^2}-e^{x^2}}{h}$$

$$=\lim_{h\to 0}\frac{e^{x^2+h^2+2hx}-e^{x^2}}{h}$$

$$=\lim_{h\to 0}\frac{e^{x^2}\cdot e^{h^2+2hx}-e^{x^2}}{h}=\lim_{h\to 0}e^{x^2}\left(\frac{e^{h^2+2hx}-1}{h}\right)$$

$$=e^{x^2}\lim_{h\to 0}\left(\frac{e^{h^2+2hx}-1}{h^2+2hx}\right)\times\left(\frac{h^2+2hx}{h}\right)$$

$$=e^{x^2}\lim_{t\to 0}\left(\frac{e^t-1}{t}\right)\lim_{h\to 0}(h+2x), \text{ where } t=h^2+2hx$$

$$=e^{x^2}\times 1\times\lim_{h\to 0}(h+2x) \quad \left[\because \lim_{x\to 0}\frac{e^x-1}{x}=1\right]$$

$$=e^{x^2}\times 2x=2xe^{x^2}$$

(iv) Let $f(x)=\sqrt{\sin x}$

$\Rightarrow f(x+h)=\sqrt{\sin(x+h)}$

$$\therefore \quad \frac{d}{dx}f(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{\sin(x+h)}-\sqrt{\sin x}}{h}$$

$$=\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h\,(\sqrt{\sin(x+h)})+\sqrt{\sin x}}$$

[rationalising the numerator]

$$=\lim_{h\to 0}\frac{2\sin\left(\frac{h}{2}\right)\cos\left(\frac{2x+h}{2}\right)}{h\,(\sqrt{\sin(x+h}+\sqrt{\sin x})}$$

$$\left[\because \sin C-\sin D=2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$$

$$=\lim_{h/2\to 0}\frac{\sin h/2}{h/2}\lim_{h\to 0}\frac{\cos(x+h/2)}{\sqrt{\sin(x+h)}+\sqrt{\sin x}}$$

[as $h\to 0\Rightarrow h/2\to 0$]

$$=1\cdot\frac{\cos x}{2\sqrt{\sin x}}=\frac{\cos x}{2\sqrt{\sin x}}$$

Differential Coefficients of Standard Functions

(i) $\frac{d}{dx}(\text{constant})=0$ (ii) $\frac{d}{dx}(x^n)=nx^{n-1}$

(iii) $\frac{d}{dx}(|x|)=\frac{x}{|x|}$ (iv) $\frac{d}{dx}(e^x)=e^x$

(v) $\frac{d}{dx}(a^x)=a^x\log a$ (vi) $\frac{d}{dx}(\log_e|x|)=\frac{1}{x}$

(vii) $\frac{d}{dx}(\log_a|x|)=\frac{1}{x\log_e a}$ (viii) $\frac{d}{dx}(\sin x)=\cos x$

(ix) $\frac{d}{dx}(\cos x)=-\sin x$ (x) $\frac{d}{dx}(\tan x)=\sec^2 x$

(xi) $\frac{d}{dx}(\cot x)=-\text{cosec}^2 x$ (xii) $\frac{d}{dx}(\sec x)=\sec x\tan x$

(xiii) $\frac{d}{dx}(\text{cosec } x)=-\text{cosec } x\cot x$

(xiv) $\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}$, $-1<x<1$ or $|x|<1$

(xv) $\frac{d}{dx}(\cos^{-1}x)=\frac{-1}{\sqrt{1-x^2}}$, $-1<x<1$ or $|x|<1$

(xvi) $\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$, $-\infty<x<\infty$ or $x\in R$

(xvii) $\frac{d}{dx}(\sec^{-1}x)=\frac{1}{|x|\sqrt{x^2-1}}$, $|x|>1$ or $x\in R-[-1,1]$

(xviii) $\frac{d}{dx}(\cot^{-1}x)=\frac{-1}{1+x^2}$, $-\infty<x<\infty$ or $x\in R$

(xix) $\frac{d}{dx}(\text{cosec}^{-1}x)=\frac{-1}{|x|\sqrt{x^2-1}}$, $|x|>1$ or $x\in R-[-1,1]$

Rules for Differentiation

Here, we will discuss some rules which are related to differentiation.

Differentiation of Sum and Difference of Functions

When a function is expressed as sum or difference of two or more functions then, to find the derivative of function, we need to differentiate each term seperately. i.e.

If $y = u(x) \pm v(x) \pm w(x) \pm \ldots$, then

$$\frac{dy}{dx} = \frac{du(x)}{dx} \pm \frac{dv(x)}{dx} \pm \frac{dw(x)}{dx} \pm \ldots$$

is known as term-by-term differentiation.

Example 2 *If* $y = \sin x + e^x$, *then find* $\frac{dy}{dx}$.

Sol. Here, $y = \sin x + e^x$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x + e^x)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{d}{dx}\sin x + \frac{d}{dx}e^x$$

$$\Rightarrow \quad \frac{dy}{dx} = \cos x + e^x$$

Example 3 *If* $y = x^2 + \sin^{-1} x + \log_e x$, *then find* $\frac{dy}{dx}$.

Sol. Here, $y = x^2 + \sin^{-1} x + \log_e x$

On differentiating, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin^{-1} x) + \frac{d}{dx}(\log_e x)$$

$$\Rightarrow \quad \frac{dy}{dx} = 2(x)^{2-1} + \frac{1}{\sqrt{1-x^2}} + \frac{1}{x}$$

$$\therefore \quad \frac{dy}{dx} = 2x + \frac{1}{\sqrt{1-x^2}} + \frac{1}{x}$$

Example 4 If $y = \log_{10} x + \log_e 10 + \log_x x + \log_{10} 10$, then find $\frac{dy}{dx}$.

Sol. Here, $y = \log_{10} x + \log_e 10 + \log_x x + \log_{10} 10$

$$\Rightarrow \quad y = \log_{10} x + \log_e 10 + 1 + 1 \qquad [\because \log_a a = 1]$$

$$\Rightarrow \quad y = \log_{10} x + \log_e 10 + 2$$

On differentiating, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\log_{10} x) + \frac{d}{dx}(\log_e 10) + \frac{d}{dx}(2)$$

$$= \frac{1}{x \log_e 10} + 0 + 0$$

$$= \frac{1}{x \log_e 10}$$

Example 5 *If* $y = x^{-1/2} + \log_5 x + \frac{\sin x}{\cos x} + 2^x$, *then find* $\frac{dy}{dx}$.

Sol. Here, $y = x^{-1/2} + \log_5 x + \tan x + 2^x$

On differentiating, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x)^{-1/2} + \frac{d}{dx}(\log_5 x) + \frac{d}{dx}\tan x + \frac{d}{dx}(2^x)$$

$$= -\frac{1}{2}(x)^{-1/2-1} + \frac{1}{x \log_e 5} + \sec^2 x + 2^x \log 2$$

$$= -\frac{1}{2}x^{-3/2} + \frac{1}{x \log_e 5} + \sec^2 x + 2^x \log 2$$

Differentiation of a Function Multiplied with a Constant

If a function multiplied by a constant, then the derivative of constant times a function is the constant times the derivative of the function. i.e.

If $y = k f(x)$, then on differentiating, we get

$$\frac{dy}{dx} = k \cdot \frac{d}{dx} f(x)$$

Example 6 If $y = m^2 \sec^{-1} x$, then find $\frac{dy}{dx}$.

Sol. Here, $y = m^2 \sec^{-1} x$

On differentiating, we get

$$\frac{dy}{dx} = m^2 \frac{d}{dx}\sec^{-1} x$$

$$= m^2 \times \frac{1}{|x|\sqrt{x^2-1}}$$

$$= \frac{m^2}{|x|\sqrt{x^2-1}}$$

Example 7 If $y = \log x^3 + 3\sin^{-1} x + kx^2$, then find $\frac{dy}{dx}$.

Sol. Here, $y = \log x^3 + 3\sin^{-1} x + kx^2$

On differentiating, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\log x^3] + \frac{d}{dx}[3\sin^{-1} x] + \frac{d}{dx}[kx^2]$$

$$= 3\frac{d}{dx}[\log x] + 3\frac{d}{dx}(\sin^{-1} x) + k\frac{d}{dx}(x^2)$$

$$[\because \log x^n = n\log x]$$

$$= 3\cdot\frac{1}{x} + 3\cdot\frac{1}{\sqrt{1-x^2}} + k(2x) = \frac{3}{x} + \frac{3}{\sqrt{1-x^2}} + 2kx$$

Product Rule

If $u(x)$ and $v(x)$ are two differentiable functions, then $u(x)\cdot v(x)$ is also differentiable.

If $y = u(x)\cdot v(x)$, then

$$\frac{dy}{dx} = \left\{\frac{d}{dx}u(x)\right\}\cdot v(x) + u(x)\cdot\left\{\frac{d}{dx}v(x)\right\}.$$

which is known as product rule.

i.e. **derivative of product of two functions**

= (first function) × (derivative of second function) + (second function) × (derivative of first function)

Example 8 ***If*** $y = e^x \sin x$, ***then find*** $\frac{dy}{dx}$.

Sol. Here, $y = e^x \sin x$

On differentiating, we get

$$\frac{dy}{dx} = \left\{\frac{d}{dx}(e^x)\right\}\cdot\sin x + e^x\cdot\left\{\frac{d}{dx}(\sin x)\right\}$$

$$= e^x\cdot\sin x + e^x\cdot\cos x = e^x(\sin x + \cos x)$$

Example 9 ***If*** $y = e^x \tan x + x\cdot\log_e x$, ***then find*** $\frac{dy}{dx}$.

Sol. Here, $y = e^x \tan x + x\cdot\log_e x$

On differentiating, we get

$$\frac{dy}{dx} = \frac{d}{dx}(e^x \tan x) + \frac{d}{dx}(x\log x)$$

$$= \left\{\frac{d}{dx}e^x\right\}\cdot\tan x + e^x\cdot\left\{\frac{d}{dx}\tan x\right\} + \left\{\frac{d}{dx}x\right\}\cdot\log x + x\cdot\left\{\frac{d}{dx}\log x\right\}$$

$$= e^x\cdot\tan x + e^x\cdot\sec^2 x + 1\cdot\log x + x\cdot\frac{1}{x}$$

Hence, $\frac{dy}{dx} = e^x(\tan x + \sec^2 x) + (\log x + 1)$

Differentiation of Product of More than Two Functions

(i) If 3 functions are involved, then remember

$$D\{f(x)\cdot g(x)\cdot h(x)\} = f(x)\cdot g(x)\cdot h'(x) + f(x)\cdot g'(x)\cdot h(x) + f'(x)\cdot g(x)\cdot h(x)$$

(ii) The result can be generalised to product of n terms as

$$D\{f_1(x) f_2(x) f_3(x)\ldots f_n(x)\} = [f_1'(x)\cdot f_2(x)\cdot f_3(x)\ldots f_n(x)] + [f_1(x)\cdot f_2'(x) f_3(x)\ldots f_n(x)] + [f_1(x) f_2(x) f_3'(x)\ldots f_n(x)] + \ldots + [f_1(x)\ f_2(x) f_3(x)\ldots f_n'(x)]$$

Example 10 ***Let*** f, g ***and*** h ***be differentiable functions. If*** $f(0) = 1, g(0) = 2, h(0) = 3$ ***and the derivative of their pairwise product at*** $x = 0$ ***are*** $(fg)'(0) = 6$; $(gh)'(0) = 4$ ***and*** $(hf)'(0) = 5$***, then compute the value of*** $(fgh)'(0)$.

Sol. We know that, $(fgh)' = \frac{(fg)'h + (gh)'f + (hf)'g}{2}$

$$\therefore\quad (fgh)'(0) = \frac{\left[\begin{array}{r}(fg)'(0)\cdot h(0) + (gh)'(0)\cdot f(0)\\ + (hf)'(0)\cdot g(0)\end{array}\right]}{2}$$

$$= \frac{(6)\cdot(3) + (4)\cdot(1) + (5)\cdot(2)}{2} = 16$$

Example 11 ***If*** $y = (1+x)(1+x^2)(1+x^4)\ldots(1+x^{2n})$, ***then find*** $\frac{dy}{dx}$ ***at*** $x = 0$.

Sol. Here, $y = (1+x)(1+x^2)(1+x^4)\ldots(1+x^{2n})$

On differentiating, we get

$$\frac{dy}{dx} = (1)(1+x^2)(1+x^4)\ldots(1+x^{2n}) + (1+x)(2x)(1+x^4)\ldots(1+x^{2n}) + (1+x)(1+x^2)(4x^3)\ldots(1+x^{2n}) + \ldots + (1+x)(1+x^2)(1+x^4)(1+x^8)\ldots(1+x^{2n-1})\times 2n\cdot x^{2n-1}$$

Now, $\left(\frac{dy}{dx}\right)_{x=0} = 1$

Quotient Rule

If $u(x)$ and $v(x)$ are two differentiable functions such that $v(x) \neq 0$, then $\frac{u(x)}{v(x)}$ is also differentiable.

If $y = \frac{u(x)}{v(x)}$, then

$$\frac{dy}{dx} = \frac{\left\{\frac{d}{dx}u(x)\right\}\cdot v(x) - \left\{\frac{d}{dx}v(x)\right\}\cdot u(x)}{\{v(x)\}^2}$$

is known as the quotient rule of differentiation.

Reader's can learn the given formula in following ways

Derivative of quotient of two functions

$$= \frac{\textbf{(denominator)} \times \textbf{(derivative of numerator)} - \textbf{(numerator)} \times \textbf{derivative of denominator}}{\textbf{(denominator)}^2}$$

Example 12 *If* $y = \frac{x}{x^2+1}$, *then find* $\frac{dy}{dx}$.

Sol. Here, $y = \frac{x}{x^2+1}$

On differentiating, we get

$$\frac{dy}{dx} = \frac{(x^2+1)\frac{d}{dx}(x) - x\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$= \frac{(x^2+1)\cdot 1 - x\cdot(2x)}{(x^2+1)^2} = \frac{1-x^2}{(1+x^2)^2}$$

Example 13 If $y = \frac{x\sin x}{\log_e x}$, then find $\frac{dy}{dx}$.

Sol. Here, $y = \frac{x\sin x}{\log_e x}$

On differentiating, we get

$$\frac{dy}{dx} = \frac{\log_e x\frac{d}{dx}(x\sin x) - x\sin x\frac{d}{dx}(\log_e x)}{(\log_e x)^2}$$

$$= \frac{\log_e x(x\cos x + \sin x) - x\sin x\left(\frac{1}{x}\right)}{(\log_e x)^2}$$

$$= \frac{\log_e x\,(x\cos x + \sin x) - \sin x}{(\log_e x)^2}$$

Example 14 If $y = \frac{3a^2x - x^3}{a^3 - 3ax^2}$, then find $\frac{dy}{dx}$.

Sol. Here, $y = \frac{3a^2x - x^3}{a^3 - 3ax^2}$

On differentiating, we get

$$\frac{dy}{dx} = \frac{\left[(a^3-3ax^2)\frac{d}{dx}(3a^2x - x^3) - (3a^2x - x^3)\frac{d}{dx}(a^3 - 3ax^2)\right]}{(a^3-3ax^2)^2}$$

$$= \frac{(a^3 - 3ax^2)(3a^2 - 3x^2) - (3a^2\,x - x^3)(-6ax)}{(a^3 - 3ax^2)^2}$$

$$= \frac{3ax^4 + 6a^3x^2 + 3a^5}{(a^3 - 3ax^2)^2}$$

Example 15 *If* $y = \frac{e^x - \tan x}{x^n + \cot x}$, *then find* $\frac{dy}{dx}$.

Sol. Here, $y = \frac{e^x - \tan x}{x^n + \cot x}$

On differentiating, we get

$$\frac{dy}{dx} = \frac{\left[(x^n + \cot x)\frac{d}{dx}(e^x - \tan x) - (e^x - \tan x)\frac{d}{dx}(x^n + \cot x)\right]}{(x^n + \cot x)^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\left[(e^x - \sec^2 x)(x^n + \cot x) - (e^x - \tan x)(nx^{n-1} - \operatorname{cosec}^2 x)\right]}{(x^n + \cot x)^2}$$

Example 16 *If* $y = \frac{\log x}{x} + e^x \sin x + \log_5 x$, *then find* $\frac{dy}{dx}$.

Sol. Here, $y = \frac{\log x}{x} + e^x \sin x + \log_5 x$

On differentiating, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\log x}{x}\right) + \frac{d}{dx}(e^x \sin x) + \frac{d}{dx}(\log_5 x)$$

$$= \frac{\left\{\frac{d}{dx}(\log x)\right\}\cdot x - \log x\left\{\frac{d}{dx}x\right\}}{x^2} + \left\{\frac{d}{dx}e^x\right\}\cdot\sin x + e^x\cdot\left\{\frac{d}{dx}\sin x\right\} + \frac{1}{x\log_e 5}$$

$$= \frac{\frac{1}{x}\cdot x - \log x\cdot 1}{x^2} + e^x\sin x + e^x\cdot\cos x + \frac{1}{x\log_e 5}$$

Hence, $\frac{dy}{dx} = \left(\frac{1 - \log x}{x^2}\right) + e^x(\sin x + \cos x) + \frac{1}{x\log_e 5}$

Remark

While applying the quotient rule, think twice and check whether your function could be simplified prior to differentiation. (See Examples 17 and 18).

Example 17 If $y=\dfrac{x^4+x^2+1}{x^2+x+1}$ and $\dfrac{dy}{dx}=ax+b$, then find a and b.

Sol. Here, $y=\dfrac{x^4+x^2+1}{x^2+x+1}$

$$\Rightarrow \quad y=\frac{x^4+2x^2+1-x^2}{x^2+x+1}=\frac{(x^2+1)^2-x^2}{(x^2+x+1)}$$

$$\therefore \quad y=\frac{(x^2+x+1)(x^2-x+1)}{(x^2+x+1)}$$

$$[\because a^2-b^2=(a+b)(a-b)]$$

$$\Rightarrow \quad y=x^2-x+1 \Rightarrow \frac{dy}{dx}=2x-1$$

$$\Rightarrow \quad ax+b=2x-1 \qquad \left[\because \frac{dy}{dx}=ax+b\right]$$

$$\therefore \quad a=2, b=-1$$

Example 18 ***If*** $y=\dfrac{\sec x+\tan x-1}{(-\sec x)+\tan x+1}$, ***then find*** $\dfrac{dy}{dx}$ ***at*** $x=\dfrac{\pi}{4}$.

Sol. Here, $y=\dfrac{\sec x+\tan x-1}{-\sec x+\tan x+1}$

$$\Rightarrow \quad y=\frac{(\sec x+\tan x)-(\sec^2 x-\tan^2 x)}{-\sec x+\tan x+1}$$

$$(\because \sec^2 x-\tan^2 x=1)$$

$$\Rightarrow \quad y=\frac{(\sec x+\tan x)\{1-\sec x+\tan x\}}{\tan x-\sec x+1}$$

$$\Rightarrow \quad y=\sec x+\tan x \Rightarrow \frac{dy}{dx}=\sec x\tan x+\sec^2 x$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{\text{at } x=\frac{\pi}{4}}=\sqrt{2}+2$$

Example 19 If $y=\dfrac{\tan^{-1}x-\cot^{-1}x}{\tan^{-1}x+\cot^{-1}x}$, then find $\left(\dfrac{dy}{dx}\right)$ at $x=-1$.

Sol. Here, $y=\dfrac{\tan^{-1}x-\cot^{-1}x}{\tan^{-1}x+\cot^{-1}x}$, as $\tan^{-1}x+\cot^{-1}x=\dfrac{\pi}{2}$

$$\therefore \quad y=\frac{\tan^{-1}x-(\pi/2-\tan^{-1}x)}{(\pi/2)}$$

$$\Rightarrow \quad y=\frac{2}{\pi}\left[2\tan^{-1}x-\frac{\pi}{2}\right] \Rightarrow \frac{dy}{dx}=\frac{2}{\pi}\left[\frac{2}{1+x^2}\right]$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{x=-1}=\frac{2}{\pi}$$

Exercise for Session 1

■ **Directions** (Q. Nos. 1 to 9) *Differentiate the following functions w.r.t. x.*

1. $e^{x\log a}+e^{a\log x}+e^{a\log a}$

2. $\sqrt{\dfrac{1-\cos 2x}{1+\cos 2x}}$

3. $\log_3 x+3\log_e x+2\tan x$

4. $|x|+a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n$

5. $\sec^{-1}\left(\dfrac{x+1}{x-1}\right)+\sin^{-1}\left(\dfrac{x-1}{x+1}\right)$

6. $x^n\log_a x\,e^x$

7. $\dfrac{2^x\cot x}{\sqrt{x}}$

8. $\dfrac{\sin x-x\cos x}{x\sin x+\cos x}$

9. If $y=1+\dfrac{x}{1!}+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\dfrac{x^4}{4!}+\ldots\infty$, then find the value of $\dfrac{dy}{dx}$.

10. Find the values of 'x' for which the rate of change of $\dfrac{x^4}{4}+\dfrac{x^3}{3}-x$ is more than the rate of change of $\dfrac{x^4}{4}$.

Session 2

Chain Rule (Derivative of Composite Function)

Chain Rule

(Derivative of Composite Function)

If $u(x)$ and $v(x)$ are differentiable functions, then $uov(x)$ or $u[v(x)]$ is also differentiable.

If $y\,[uov(x)] = [u\{v(x)\}]$, then

$$\frac{dy}{dx} = \frac{du\{v(x)\}}{d\{v(x)\}} \times \frac{d}{dx} v(x)$$

is known as chain rule. Or

If $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

The chain rule can be extended as follows

If $y\,[uovow(x)] = u\,[v\{w(x)\}]$, then

$$\frac{dy}{dx} = \frac{du[v\{w(x)\}]}{dv\{w(x)\}} \times \frac{dv\{w(x)\}}{dw(x)} \times \frac{dw(x)}{dx}$$

Example 20 *If* $y = \log(\sin x)$*, then find* $\frac{dy}{dx}$.

Sol. Here, $y = \log(\sin x)$

On differentiating, we get

$$\frac{dy}{dx} = \frac{d[\log(\sin x)]}{d(\sin x)} \times \frac{d}{dx}(\sin x) = \frac{1}{\sin x} \times \cos x$$

[by chain rule]

Hence, $\frac{dy}{dx} = \cot x$

Aliter

Here, $y = \log(\sin x)$

Put $\sin x = u$, we have, $y = \log u$ where $u = \sin x$

On differentiating, we get

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$ [by chain rule]

$$\frac{dy}{dx} = \frac{d}{dx}(\log u) \cdot \frac{d}{dx}(\sin x)$$

$$= \frac{1}{u} \times \cos x$$

$$\frac{dy}{dx} = \frac{1}{\sin x} \times \cos x = \cot x \qquad [\because u = \sin x]$$

Example 21 If $y = e^{(\tan^{-1} x)^3}$, then find $\frac{dy}{dx}$.

Sol. Here, $y = e^{(\tan^{-1} x)^3}$

On differentiating, we get

$$\frac{dy}{dx} = \frac{d\{e^{(\tan^{-1} x)^3}\}}{d\{(\tan^{-1} x)^3\}} \cdot \frac{d\{(\tan^{-1} x)^3\}}{d(\tan^{-1} x)} \cdot \frac{d(\tan^{-1} x)}{dx}$$

$$= e^{(\tan^{-1} x)^3} \cdot 3(\tan^{-1} x)^2 \cdot \frac{1}{1+x^2}$$ [by chain rule]

Aliter

Here, $y = e^{(\tan^{-1} x)^3}$

Put $(\tan^{-1} x) = v$ and $(\tan^{-1} x)^3 = u$

$\Rightarrow$ $u = v^3$

$\therefore$ $y = e^u$, $u = v^3$ and $v = \tan^{-1} x$

On differentiating, we get

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{d}{du} e^u \times \frac{d(v)^3}{dv} \times \frac{d}{dx} \tan^{-1} x = e^u \cdot 3v^2 \cdot \frac{1}{1+x^2}$$

$$= e^{(\tan^{-1} x)^3} \cdot (3\tan^{-1} x)^2 \cdot \frac{1}{1+x^2}$$

Example 22 If $y = \log_e(\tan^{-1}\sqrt{1+x^2})$, then find $\frac{dy}{dx}$.

Sol. Here, $y = \log_e(\tan^{-1}\sqrt{1+x^2})$

On differentiating, we get

$$\frac{dy}{dx} = \frac{d[\log_e(\tan^{-1}\sqrt{1+x^2})]}{d(\tan^{-1}\sqrt{1+x^2})} \cdot \frac{d(\tan^{-1}\sqrt{1+x^2})}{d(\sqrt{1+x^2})} \cdot \frac{d(\sqrt{1+x^2})}{d(1+x^2)} \cdot \frac{d(1+x^2)}{dx}$$

$$= \frac{1}{\tan^{-1}(\sqrt{1+x^2})} \cdot \frac{1}{1+(\sqrt{1+x^2})^2} \cdot \frac{1}{2\sqrt{1+x^2}} \cdot 2x$$

$$= \frac{x}{\tan^{-1}(\sqrt{1+x^2})(2+x^2)\sqrt{1+x^2}}$$

Aliter

Here, $y = \log_e(\tan^{-1}\sqrt{1+x^2})$

Put $1+x^2=w, \sqrt{1+x^2}=v$

$\Rightarrow \quad v=\sqrt{w}$ and $\tan^{-1}\sqrt{1+x^2}=u$

$\Rightarrow \quad u=\tan^{-1}v$

$\therefore \quad y=\log u, u=\tan^{-1}v$

$v=\sqrt{w}$ and $w=1+x^2$

On differentiating, we get

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dw}\cdot\frac{dw}{dx}$$

$$=\frac{d}{du}(\log u)\times\frac{d}{dv}\tan^{-1}v\cdot\frac{d}{dw}\sqrt{x}\cdot\frac{d}{dx}(1+x^2)$$

$$=\frac{1}{u}\times\frac{1}{1+v^2}\times\frac{1}{2\sqrt{w}}\times(2x)$$

$$=\frac{1}{\tan^{-1}\sqrt{1+x^2}}\times\frac{1}{2+x^2}\times\frac{1}{2\sqrt{1+x^2}}\times(2x)$$

$$=\frac{1}{\tan^{-1}\sqrt{1+x^2}}\times\frac{1}{2+x^2}\times\frac{x}{\sqrt{1+x^2}}$$

$$=\frac{x}{(\tan^{-1}\sqrt{1+x^2})(2+x^2)\sqrt{1+x^2}}$$

Example 23 *If* $y=e^{ax}\cdot\cos(bx+c)$, *then find* $\frac{dy}{dx}$.

Sol. Here, $y=e^{ax}\cdot\cos(bx+c)$

On differentiating, we get

$$\frac{dy}{dx}=\left\{\frac{d}{dx}(e^{ax})\right\}\cdot\cos(bx+c)+e^{ax}\cdot\left\{\frac{d}{dx}\{\cos(bx+c)\}\right\}$$

$$=\frac{d(e^{ax})}{d(ax)}\cdot\frac{d(ax)}{dx}\cdot\cos(bx+c)+e^{ax}\cdot\frac{d\cos(bx+c)}{d(bx+c)}\cdot\frac{d(bx+c)}{dx}$$

$$=e^{ax}\cdot a\cdot\cos(bx+c)+e^{ax}\cdot\{-\sin(bx+c)\}\cdot b$$

$$=ae^{ax}\cos(bx+c)-be^{ax}\sin(bx+c)$$

Remark

From now and onwards, we apply chain rule directly.

Example 24 *Differentiate the following w.r.t.* x.

(i) $\sqrt{\log\left\{\sin\left(\frac{x^2}{3}-1\right)\right\}}$ (ii) $\log_e\left(\frac{x+\sqrt{x^2-a^2}}{x-\sqrt{x^2-a^2}}\right)$

Sol. (i) Let $y=\sqrt{\log\left\{\sin\left(\frac{x^2}{3}-1\right)\right\}}$

On differentiating, we get

$$\frac{dy}{dx}=\frac{1}{2\sqrt{\log\left\{\sin\left(\frac{x^2}{3}-1\right)\right\}}}\times\frac{1}{\sin\left(\frac{x^2}{3}-1\right)}\times\cos\left(\frac{x^2}{3}-1\right)\times\frac{2x}{3}$$

$$=\frac{x\cot\left(\frac{x^2}{3}-1\right)}{3\log\left\{\sin\left(\frac{x^2}{3}-1\right)\right\}}$$

(ii) Let $y=\log_e\left(\frac{x+\sqrt{x^2-a^2}}{x-\sqrt{x^2-a^2}}\right)$

$$\Rightarrow y=\log(x+\sqrt{x^2-a^2})-\log(x-\sqrt{x^2-a^2})$$

$$\left[\because \log\frac{a}{b}=\log a-\log b\right]$$

On differentiating, we get

$$\frac{dy}{dx}=\left(\frac{1}{x+\sqrt{x^2-a^2}}\right)\left(1+\frac{2x}{2\sqrt{x^2-a^2}}\right)-\left[\left(\frac{1}{x-\sqrt{x^2-a^2}}\right)\left(1-\frac{2x}{2\sqrt{x^2-a^2}}\right)\right]$$

$$=\frac{1}{\sqrt{x^2-a^2}}+\frac{1}{\sqrt{x^2-a^2}}=\frac{2}{\sqrt{x^2-a^2}}$$

Exercise for Session 2

■ **Directions** (Q. Nos. 1 to 19) *Differentiate the following w.r.t. x.*

1. $(x^2+x+1)^4$
2. $\sqrt{x^2+x+1}$
3. $\sin^3 x$
4. $\frac{1}{\sqrt{a^2-x^2}}$
5. $e^{x\sin x}$
6. $\sin^{-1}\left(\frac{a+b\cos x}{b+a\cos x}\right), b>a>1$
7. e^{e^x}
8. $\log(x+\sqrt{a^2+x^2})$

9. $\log\left(\frac{a + b \sin x}{a - b \sin x}\right)$

10. $\log\sqrt{\frac{1+\sin x}{1-\sin x}}$

11. $\frac{e^x + \log x}{\sin 3x}$

12. $\sin(m \sin^{-1} x), |x| < 1$

13. $a^{(\sin^{-1} x)^2}, |x| < 1$

14. $e^{\cos^{-1}(\sqrt{1-x^2})}, |x| < 1$

15. $\frac{x \sin^{-1} x}{\sqrt{1-x^2}} + \log\sqrt{1-x^2}, |x| < 1$

16. $\log_{10} x + \log_x 10 + \log_x x + \log_{10} 10$

17. $5^{3-x^2} + (3 - x^2)^5$

18. $\frac{\sqrt{a^2+x^2} + \sqrt{a^2-x^2}}{\sqrt{a^2+x^2} - \sqrt{a^2-x^2}}$

19. $\sqrt{4 + \sqrt{4 + \sqrt{4 + x^2}}}$

Multiple Choice Questions

20. The differential coefficient of $f(\log_e x)$ w.r.t. x, where $f(x) = \log_e x$, is

(a) $\frac{x}{\log_e x}$ (b) $\frac{1}{x}\log_e x$ (c) $\frac{1}{x \log_e x}$ (d) None of these

21. If $f(x) = \log_e |x|, x \neq 0$, then $f'(x)$ is equal to

(a) $\frac{1}{|x|}$ (b) $\frac{1}{x}$ (c) $-\frac{1}{x}$ (d) None of these

22. Let $f(x) = \sin x$, $g(x) = x^2$ and $h(x) = \log x$. If $F(x) = h(f(g(x)))$, then $F'(x)$ is

(a) $2x \cot x^2$ (b) $2 \operatorname{cosec}^3 x$ (c) $-2 \operatorname{cosec}^2 x$ (d) None of these

23. If $f(x) = \cos x \cdot \cos 2x \cdot \cos 4x \cdot \cos 8x \cdot \cos 16x$, then $f'\left(\frac{\pi}{4}\right)$ is

(a) $\sqrt{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) 1 (d) None of these

24. If $y = f\left(\frac{3x+4}{5x+6}\right)$ and $f'(x) = \tan x^2$, then $\frac{dy}{dx}$ is equal to

(a) $-2\tan\left(\frac{3x+4}{5x+6}\right)^2 \cdot \frac{1}{(5x+6)^2}$ (b) $f\left(\frac{3\tan x^2 + 3}{5\tan x^2 + 6}\right)\tan x^2$

(c) $2x \tan\left(\frac{3x+4}{5x+6}\right)$ (d) $\tan x^2$

25. If $y = |\cos x| + |\sin x|$, then $\frac{dy}{dx}$ at $x = \frac{2\pi}{3}$ is

(a) $\frac{1}{2}(\sqrt{3}+1)$ (b) $2(\sqrt{3}-1)$ (c) $\frac{1}{2}(\sqrt{3}-1)$ (d) None of these

26. If $f'(x) = \sin x + \sin 4x \cdot \cos x$, then $f'\left(2x^2 + \frac{\pi}{2}\right)$ is

(a) $4x\{\cos(2x^2) - \sin 8x^2 \cdot \sin 2x^2\}$ (b) $4x\{\cos(2x^2) + \sin 8x^2 \cdot \cos 2x^2\}$

(c) $\{\cos(2x^2) - \sin 8x^2 \cdot \sin 2x^2\}$ (d) None of these

27. If $f'(x) = \sqrt{2x^2 - 1}$ and $y = f(x^2)$, then $\frac{dy}{dx}$ at $x = 1$, is

(a) 1 (b) –1 (c) –2 (d) 2

Session 3

Differentiation of Implicit Functions

Differentiation of Implicit Functions

In previous rules, we have studied functions in which y was only expressed in terms of x without complication called **explicit functions**.

But, if the relation between the variables x and y are given by an equation and this equation is not immediately solvable for y, then y is called an **implicit function** of x.

Implicit functions are given by $\phi(x, y) = 0$.

In this case, to find $\frac{dy}{dx}$, we differentiate both sides of the given relation with respect to x and collect the terms containing $\frac{dy}{dx}$ to the left hand side and transpose the other terms to the right hand side. Now, divide both sides by coefficient of $\frac{dy}{dx}$.

Shortcut for Differentiation of Implicit Functions

(Only to be Applied while Solving Objective MCQs)

For implicit function put; $\frac{d}{dx}\{f(x, y)\} = \frac{-\partial f / \partial x}{\partial f / \partial y}$,

where $\frac{\partial f}{\partial x}$ is a **partial differentiation** of given function w.r.t. x (i.e. differentiating f w.r.t. x keeping y as constant) and $\frac{\partial f}{\partial y}$ means partial differentiation of given function w.r.t. y (i.e. differentiating f w.r.t. y keeping x as constant). In simple words, we write this method as for implicit functions

$$\frac{dy}{dx} = \frac{\text{Differentiation of function w.r.t. } x \text{ keeping } y \text{ as constant}}{\text{Differentiation of function w.r.t. } y \text{ keeping x as constant}}$$

Example 25 *If* $x^2 + y^2 + xy = 2$, *then find* $\frac{dy}{dx}$.

Sol. Here, $x^2 + y^2 + xy = 2$

Differentiating both sides, we get

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) + \frac{d}{dx}(xy) = \frac{d}{dx}(2)$$

$$\Rightarrow \quad 2x + 2y\frac{dy}{dx} + \left\{\frac{dx}{dx}\right\}y + x\left\{\frac{d}{dx}y\right\} = 0$$

$$\Rightarrow \quad 2x + 2y\frac{dy}{dx} + 1 \cdot y + x \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \quad (2y + x)\frac{dy}{dx} = -(2x + y)$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{(2x + y)}{(2y + x)}$$

Aliter

Put $f = x^2 + y^2 + xy - 2$

Now, $\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$ ...(i)

where, $\frac{\partial f}{\partial x} = 2x + 0 + y - 0 = 2x + y$

and $\frac{\partial f}{\partial y} = 2y + x$

Substituting in Eq. (i), we get

$$\frac{dy}{dx} = -\frac{(2x + y)}{(2y + x)}$$

Example 26 If $y = x \cos y + y \cos x$, then find $\frac{dy}{dx}$.

Sol. Here, $y = x \cos y + y \cos x$

Differentiating both the sides, we get

$$\frac{dy}{dx} = \left\{\frac{d}{dx}(x)\right\}\cos y + x\left\{\frac{d}{dx}(\cos y)\right\} + y\left\{\frac{d}{dx}(\cos x)\right\} + \left\{\frac{d}{dx}(y)\right\} \cdot \cos x$$

$$\Rightarrow \quad \frac{dy}{dx} = 1 \cdot \cos y + x(-\sin y)\frac{dy}{dx} + y(-\sin x) + \frac{dy}{dx}(\cos x)$$

$$\Rightarrow \quad \frac{dy}{dx}(1 + x \sin y - \cos x) = \cos y - y \sin x$$

$$\therefore \quad \frac{dy}{dx} = \frac{\cos y - y \sin x}{1 + x \sin y - \cos x}$$

Aliter

Let $f = x\cos y + y\cos x - y \Rightarrow \dfrac{\partial f}{\partial x} = \cos y - y\sin x$

and $\dfrac{\partial f}{\partial y} = -x\sin y + \cos x - 1$

$$\therefore \quad \frac{dy}{dx} = \frac{-\left(\dfrac{\partial f}{\partial x}\right)}{\left(\dfrac{\partial f}{\partial y}\right)} = \frac{-(\cos y - y\sin x)}{(-x\sin y + \cos x - 1)}$$

or $$\frac{dy}{dx} = \frac{\cos y - y\sin x}{1 + x\sin y - \cos x}$$

Example 27 ***If*** $y\sqrt{1-x^2} + x\sqrt{1-y^2} = 1$, ***when*** $|x| < 1$ ***and*** $|y| < 1$, ***then find*** $\dfrac{dy}{dx}$.

Sol. Here, $y\sqrt{1-x^2} + x\sqrt{1-y^2} = 1$, $|x| < 1$ and $|y| < 1$

Let $x = \sin\theta$, $y = \sin\phi$, then

$$\sin\phi\sqrt{1-\sin^2\theta} + \sin\theta\sqrt{1-\sin^2\phi} = 1$$

$$\Rightarrow \quad \sin\phi\cos\theta + \sin\theta\cos\phi = 1$$

$$\Rightarrow \quad \sin(\theta + \phi) = 1$$

$$\Rightarrow \quad \theta + \phi = \sin^{-1}(1)$$

$$\Rightarrow \quad \sin^{-1}x + \sin^{-1}y = \sin^{-1}(1)$$

$$\left[\because \sin\theta = x \Rightarrow \theta = \sin^{-1}x \text{ and } \sin\phi = y \Rightarrow \phi = \sin^{-1}y\right]$$

Differentiating both the sides, we get

$$\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-y^2}}\cdot\frac{dy}{dx} = 0$$

Hence, $$\frac{dy}{dx} = -\sqrt{\frac{1-y^2}{1-x^2}}$$

Aliter

Let $f = y\sqrt{1-x^2} + x\sqrt{1-y^2} - 1$

$$\Rightarrow \quad \frac{\partial f}{\partial x} = \frac{-xy}{\sqrt{1-x^2}} + \sqrt{1-y^2} \text{ and } \frac{\partial f}{\partial y} = \sqrt{1-x^2} - \frac{xy}{\sqrt{1-y^2}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{-\left(\dfrac{\partial f}{\partial x}\right)}{\left(\dfrac{\partial f}{\partial y}\right)} = -\frac{[\sqrt{1-x^2}\sqrt{1-y^2} - xy]/\sqrt{1-x^2}}{[\sqrt{1-x^2}\sqrt{1-y^2} - xy]/\sqrt{1-y^2}}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

Example 28 If $\sqrt{1-x^6} + \sqrt{1-y^6} = a^3(x^3 - y^3)$, then prove that $\dfrac{dy}{dx} = \dfrac{x^2}{y^2}\sqrt{\dfrac{1-y^6}{1-x^6}}$.

Sol. Here, $\sqrt{1-x^6} + \sqrt{1-y^6} = a^3(x^3 - y^3)$

Let $x^3 = \sin\theta$, $y^3 = \sin\phi$, then we get

$$\sqrt{1-\sin^2\theta} + \sqrt{1-\sin^2\phi} = a^3(\sin\theta - \sin\phi)$$

$$\Rightarrow \quad \cos\theta + \cos\phi = a^3(\sin\theta - \sin\phi)$$

$$\Rightarrow 2\cos\left(\frac{\theta+\phi}{2}\right)\cdot\cos\left(\frac{\theta-\phi}{2}\right) = a^3\left[2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)\right]$$

$$\Rightarrow \quad \cos\left(\frac{\theta-\phi}{2}\right) = a^3\sin\left(\frac{\theta-\phi}{2}\right) \quad \Rightarrow \quad \cot\left(\frac{\theta-\phi}{2}\right) = a^3$$

$$\Rightarrow \quad \theta - \phi = 2\cot^{-1}(a^3)$$

or $$\sin^{-1}x^3 - \sin^{-1}y^3 = 2\cot^{-1}(a^3)$$

$$\left[\because \cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\right.$$

$$\left.\text{and } \sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$$

Differentiating both the sides, we get

$$\frac{1}{\sqrt{1-x^6}}\cdot 3x^2 - \frac{1}{\sqrt{1-y^6}}\cdot 3y^2\cdot\frac{dy}{dx} = 0$$

Hence, $$\frac{dy}{dx} = \frac{x^2\sqrt{1-y^6}}{y^2\sqrt{1-x^6}}$$

Example 29 If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots\infty}}}$, where $\sin x > 0$, then find $\dfrac{dy}{dx}$.

Sol. We have, $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots\infty}}}$

$$\Rightarrow \quad y = \sqrt{(\sin x) + y}$$

$$\Rightarrow \quad y^2 = \sin x + y$$

Differentiating both the sides, we get

$$2y\frac{dy}{dx} = \cos x + \frac{dy}{dx}$$

$$\Rightarrow (2y-1)\frac{dy}{dx} = \cos x \Rightarrow \frac{dy}{dx} = \frac{\cos x}{2y-1}$$

Remark

In above examples, it is advised to use $\dfrac{dy}{dx} = -\left(\dfrac{\partial f/\partial x}{\partial f/\partial y}\right)$

Exercise for Session 3

1. If $\log(x^2 + y^2) = 2\tan^{-1}\left(\frac{y}{x}\right)$, then show that $\frac{dy}{dx} = \frac{x + y}{x - y}$.

2. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, then prove that $\frac{dy}{dx} = \frac{-1}{(x+1)^2}$.

3. If $\cos^{-1}\left(\frac{x^2 - y^2}{x^2 + y^2}\right) = \tan^{-1} a$, then prove that $\frac{dy}{dx} = \frac{y}{x}$.

4. If $\sin y = x \sin(a + y)$, then prove that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$.

5. If $x^2 + y^2 = t - \frac{1}{t}$ and $x^4 + y^4 = t^2 + \frac{1}{t^2}$, then prove that $\frac{dy}{dx} = \frac{1}{x^3 y}$.

Multiple Choice Questions

6. If $\sin(xy) + \cos(xy) = 0$, then $\frac{dy}{dx}$ is

(a) $\frac{y}{x}$ (b) $-\frac{y}{x}$

(c) $-\frac{x}{y}$ (d) $\frac{x}{y}$

7. If $ax^2 + 2hxy + by^2 = 0$, then $\frac{dy}{dx}$ is

(a) $\frac{y}{x}$ (b) $\frac{x}{y}$

(c) $-\frac{x}{y}$ (d) None of these

8. If $x^2 \cdot e^y + 2xye^x + 13 = 0$, then $\frac{dy}{dx}$ is

(a) $\frac{-2xe^{y-x} - 2y(x+1)}{x(xe^{y-x} + 2)}$ (b) $\frac{2xe^{x-y} + 2y(x+1)}{x(xe^{y-x} + 2)}$

(c) $\frac{-2xe^{x-y} + 2y(x+1)}{x(xe^{y-x} + 2)}$ (d) None of these

9. If $\log(x + y) = 2xy$, then $y'(0)$ is

(a) 1 (b) -1 (c) 2 (d) 0

10. If $x\log_e y + y\log_e x = 5$, then $\frac{dy}{dx}$ is

(a) $-\frac{y}{x}\left(\frac{x\log y + y}{x + y\log x}\right)$ (b) $-\frac{x}{y}\left(\frac{x\log y + y}{x - y\log x}\right)$

(c) $-\frac{y}{x}\left(\frac{x\log y - y}{x + y\log x}\right)$ (d) None of these

Session 4

Differentiation of Inverse Trigonometric Functions

Differentiation of Inverse Trigonometric Functions

Inverse trigonometric functions are continuous functions, their domain and range are shown in the table given below.

S. No.	Function	Domain	Range	Principal value branch
1.	$\sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, where $y = \sin^{-1} x$
2.	$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$	$0 \le y \le \pi$, where $y = \cos^{-1} x$
3.	$\tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$, where $y = \tan^{-1} x$
4.	$\text{cosec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, where $y = \text{cosec}^{-1} x$, $y \ne 0$
5.	$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$	$0 \le y \le \pi$, where $y = \sec^{-1} x$, $y \ne \frac{\pi}{2}$
6.	$\cot^{-1} x$	R	$(0, \pi)$	$0 < y < \pi$, where $y = \cot^{-1} x$

To differentiate inverse trigonometric functions, we need to know some basic trigonometric identities and substitution, they can help you finding the differentiation of inverse trigonometric functions.

Useful Trigonometric Identities

(i) $\sin 2x = 2 \sin x \cos x$

(ii) $\cos 2x = 2\cos^2 x - 1, \cos 2x = 1 - 2\sin^2 x$

(iii) $\sin 2x = \dfrac{2 \tan x}{1 + \tan^2 x}$

(iv) $\cos 2x = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$

(v) $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$, where $x \ne (2n + 1)\dfrac{\pi}{4}$

(vi) $\sin 3x = 3 \sin x - 4 \sin^3 x$

$= 4 \sin (60° - A) \cdot \sin A \cdot \sin (60° + A)$

(vii) $\cos 3x = 4 \cos^3 x - 3 \cos x$

$= 4 \cos (60° - A) \cdot \cos A \cdot \cos (60° + A)$

(viii) $\tan 3x = \dfrac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}$

$= \tan (60° - A) \cdot \tan A \cdot \tan (60° + A)$

(ix) $\cos A \cdot \cos 2A \cdot \cos 2^2 A \ldots \cos 2^{n-1} A = \dfrac{\sin 2^n A}{2^n \sin A}$

Remark

If no branch of an inverse trigonometric function is mentioned, then it means that the principal value branch of the function have to be taken.

Substitutions for Inverse Trigonometric Functions

S.No.	Expression	Substitutions
1.	$\sqrt{a^2 - x^2}$	$x = a\sin\theta$ or $a\cos\theta$
2.	$\sqrt{a^2 + x^2}$	$x = a\tan\theta$ or $a\cot\theta$
3.	$\sqrt{x^2 - a^2}$	$x = a\sec\theta$ or $a\text{cosec}\,\theta$
4.	$\sqrt{\frac{a+x}{a-x}}$ or $\sqrt{\frac{a-x}{a+x}}$	$x = a\cos\theta$ or $a\cos 2\theta$
5.	$\sqrt{(a-x)(x-b)}$ or $\sqrt{\frac{a-x}{x-b}}$ or $\sqrt{\frac{x-b}{a-x}}$	$x = a\cos^2\theta - b\sin^2\theta$
6.	$\sqrt{(x-a)(x-b)}$ or $\sqrt{\frac{x-a}{x-b}}$ or $\sqrt{\frac{x-b}{a-x}}$	$x = a\sec^2\theta + b\tan^2\theta$
7.	$\sqrt{2ax - x^2}$	$x = a(1 - \cos\theta)$

Example 30 If $y=\sec^{-1}(\sqrt{1+x^2})$, when $-1<x<1$, then find $\frac{dy}{dx}$.

Sol. Here, $y=\sec^{-1}(\sqrt{1+x^2})$,

put $x=\tan\theta$

$\Rightarrow$ $y=\sec^{-1}(\sqrt{\sec^2\theta})$

$=\sec^{-1}(\sec\theta)=\theta=\tan^{-1}x$

$\therefore$ $\frac{dy}{dx}=\frac{1}{1+x^2}$

Example 31 If $y=\tan^{-1}\sqrt{\frac{1-x}{1+x}}$, then find $\frac{dy}{dx}$, when $-1<x<1$.

Sol. Here, $y=\tan^{-1}\sqrt{\frac{1-x}{1+x}}$, put $x=\cos\theta$

$\Rightarrow$ $y=\tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$

$\Rightarrow$ $y=\tan^{-1}\sqrt{\frac{2\sin^2\theta/2}{2\cos^2\theta/2}}$

$\Rightarrow$ $y=\tan^{-1}|\tan\theta/2|$ $\left[\because -1<\cos\theta<1\Rightarrow 0<\theta<\pi,\ \therefore 0<\frac{\theta}{2}<\frac{\pi}{2}\right]$

$\therefore$ $y=\tan^{-1}(\tan\theta/2)$

$[\because \tan^{-1}(\tan x)=x,\ -\pi/2<x<\pi/2]$

$\Rightarrow$ $y=\frac{\theta}{2}\Rightarrow y=\frac{1}{2}\cos^{-1}x$

$\therefore$ $\frac{dy}{dx}=-\frac{1}{2\sqrt{1-x^2}}$

Example 32 If $y=\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$, then find $\frac{dy}{dx}$, when $0<x<\frac{\pi}{2}$.

Sol. Here, $y=\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$

$\Rightarrow$ $y=\tan^{-1}\sqrt{\frac{2\sin^2 x/2}{2\cos^2 x/2}}=\tan^{-1}\sqrt{\tan^2 x/2}$

$\Rightarrow$ $y=\tan^{-1}(\tan x/2)$

$\Rightarrow$ $y=\frac{x}{2}$ $\left[\because \tan^{-1}(\tan x)=x,\ -\frac{\pi}{2}<x<\frac{\pi}{2}\right]$

Hence, $\frac{dy}{dx}=\frac{1}{2}$

Example 33 ***If*** $y=\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$, ***then find*** $\frac{dy}{dx}$ ***when*** $-1\le x\le 1$.

Sol. Here, $y=\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$, let $x=\tan\theta$

$\Rightarrow$ $y=\tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$

$\Rightarrow$ $y=\tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$

$\Rightarrow$ $y=\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)=\tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$

$\Rightarrow$ $y=\tan^{-1}\left(\tan\frac{\theta}{2}\right)=\frac{\theta}{2}$

$\Rightarrow$ $y=\frac{1}{2}\tan^{-1}x\ \Rightarrow\ \frac{dy}{dx}=\frac{1}{2(1+x^2)}$

Example 34 ***Find*** $\frac{dy}{dx}$, ***if*** $y=\tan^{-1}\left(\frac{a\cos x-b\sin x}{b\cos x+a\sin x}\right)$, ***where*** $-\frac{\pi}{2}<x<\frac{\pi}{2}$ ***and*** $\frac{a}{b}\tan x>-1$.

Sol. Here, $y=\tan^{-1}\left(\frac{a\cos x-b\sin x}{b\cos x+a\sin x}\right)$

$\Rightarrow$ $y=\tan^{-1}\left(\frac{\frac{a}{b}-\tan x}{1+\frac{a}{b}\tan x}\right)$

$=\tan^{-1}\left(\frac{a}{b}\right)-\tan^{-1}(\tan x)=\tan^{-1}\left(\frac{a}{b}\right)-x$

$\Rightarrow$ $\frac{dy}{dx}=0-1=-1$

Example 35 ***Find*** $\frac{dy}{dx}$ ***for the function***

$$y=\sin^{-1}\left(\frac{5x+12\sqrt{1-x^2}}{13}\right).$$

Sol. Here, $y=\sin^{-1}\left(\frac{5x+12\sqrt{1-x^2}}{13}\right)$

Put $x=\sin\theta, 5=r\cos\phi$ and $12=r\sin\phi$

$\Rightarrow$ $r=\sqrt{(r\cos\phi)^2+(r\sin\phi)^2}$

$=\sqrt{(5)^2+(12)^2}=\sqrt{25+144}=\sqrt{169}=13$

$\phi=\tan^{-1}\left(\frac{12}{5}\right)$

$$\therefore \quad y = \sin^{-1}\left[\frac{r\cos\phi\sin\theta + r\sin\phi\cos\theta}{13}\right]$$

$$\Rightarrow \quad y = \sin^{-1}(\sin(\theta+\phi)) = \theta + \phi$$

$$\Rightarrow \quad y = \sin^{-1}x + \tan^{-1}\left(\frac{12}{5}\right)$$

$$\left[\because \theta = \sin^{-1}x \text{ and } \phi = \tan^{-1}\left(\frac{12}{5}\right)\right]$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Example 36 Find $\frac{dy}{dx}$, for $y = \tan^{-1}\sqrt{\frac{a-x}{a+x}}$, where $-a < x < a$.

Sol. Here, $y = \tan^{-1}\sqrt{\frac{a-x}{a+x}}$, where $-a < x < a$

Put $x = a\cos\theta \Rightarrow \theta = \cos^{-1}\frac{x}{a}$

$$\therefore \quad y = \tan^{-1}\left\{\sqrt{\frac{a-a\cos\theta}{a+a\cos\theta}}\right\}$$

$$\Rightarrow \quad y = \tan^{-1}\left\{\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right\}$$

$$\Rightarrow \quad y = \tan^{-1}\left\{\sqrt{\tan^2\frac{\theta}{2}}\right\}$$

$$\Rightarrow \quad y = \tan^{-1}\left|\tan\frac{\theta}{2}\right|$$

Also, $-a < x < a \Rightarrow -1 < \cos\theta < 1$

$$\Rightarrow \quad \theta \in (0,\pi) \Rightarrow \frac{\theta}{2} \in \left(0, \frac{\pi}{2}\right)$$

$$\therefore \quad y = \tan^{-1}\left|\tan\frac{\theta}{2}\right| = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2}$$

$$\Rightarrow \quad y = \frac{1}{2}\cos^{-1}\frac{x}{a}$$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{2}\times\frac{1}{\sqrt{1-\frac{x^2}{a^2}}}\frac{d}{dx}\left(\frac{x}{a}\right) = -\frac{1}{2\sqrt{a^2-x^2}}$$

Graphical Approach for Differentiation of Inverse Trigonometric Functions

In this section, we are going to introduce, graphs of some important inverse trigonometric functions which are highly useful in differential calculus.

Example 37 Sketch the graph for $y = \sin^{-1}(\sin x)$ and hence find $\frac{dy}{dx}$.

Sol. As, $y = \sin^{-1}(\sin x)$ is periodic with period 2π.

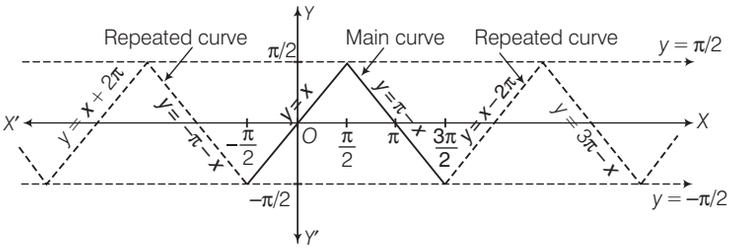
$\therefore$ To draw this graph we should draw the graph for one interval of length 2π and repeat it for entire values of x.

As we know,

$$\sin^{-1}(\sin x) = \begin{cases} x, & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ (\pi - x), & -\frac{\pi}{2} \le \pi - x < \frac{\pi}{2} \end{cases} \quad \left(\text{i.e. } \frac{\pi}{2} \le x \le \frac{3\pi}{2}\right)$$

$$\text{or} \quad \sin^{-1}(\sin x) = \begin{cases} x, & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} \le x \le \frac{3\pi}{2} \end{cases}$$

which is defined for the interval of length 2π, plotted as;



Thus, the graph for $y = \sin^{-1}(\sin x)$, is a straight line up and a straight line down with slopes 1 and −1 respectively lying between $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$\text{Thus, } \frac{dy}{dx} = \begin{cases} 1, & \left(2n-\frac{1}{2}\right)\pi < x < \left(2n+\frac{1}{2}\right)\pi \\ -1, & \left(2n+\frac{1}{2}\right)\pi < x < \left(2n+\frac{3}{2}\right)\pi \\ \text{does not exist}, & x = \left\{(2n+1)\frac{\pi}{2}; n \in I\right\} \end{cases}$$

Remark

Students are advised to learn the definition of $\sin^{-1}(\sin x)$ as

$$y = \sin^{-1}(\sin x) = \begin{cases} x + 2\pi, & -\frac{5\pi}{2} \le x \le -\frac{3\pi}{2} \\ -\pi - x, & -\frac{3\pi}{2} \le x \le -\frac{\pi}{2} \\ x, & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} \le x \le \frac{3\pi}{2} \\ x - 2\pi & \frac{3\pi}{2} \le x \le \frac{5\pi}{2} \end{cases} \text{... and so on}$$

Example 38 Sketch the graph for $y = \cos^{-1}(\cos x)$ and hence find $\frac{dy}{dx}$.

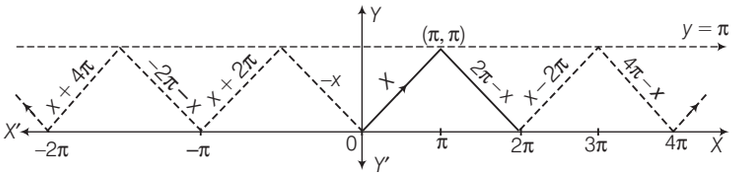
Sol. As, $y = \cos^{-1}(\cos x)$ is periodic with a period 2π.

$\therefore$ To draw this graph, we should draw the graph for one interval of length 2π and repeat it for entire values of x of length 2π.

As we know, $\cos^{-1}(\cos x) = \begin{cases} x, & 0 \le x \le \pi \\ 2\pi - x, & 0 \le 2\pi - x \le \pi \end{cases}$

or $\cos^{-1}(\cos x) = \begin{cases} x, & 0 \le x \le \pi \\ 2\pi - x, & \pi \le x \le 2\pi \end{cases}$

Thus, it has been defined for $0 < x < 2\pi$ that has length 2π. So, its graph could be plotted as



Thus, the curve $y = \cos^{-1}(\cos x)$ and hence

$$\frac{dy}{dx} = \begin{cases} \text{does not exist}, & x = \{n\pi, n \in I\} \\ 1, & 2n\pi < x < (2n+1)\pi \\ -1, & (2n+1)\pi < x < (2n+2)\pi \end{cases}$$

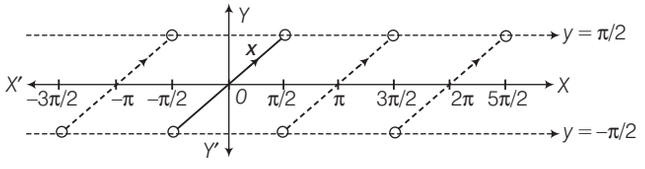
Example 39 Sketch the graph for $y = \tan^{-1}(\tan x)$ and hence find $\frac{dy}{dx}$.

Sol. As, $y = \tan^{-1}(\tan x)$ is periodic with period π.

$\therefore$ To draw this graph we should draw the graph for one interval of length π and repeat it for entire values of x.

As we know, $\tan^{-1}(\tan x) = \left\{ x, -\frac{\pi}{2} < x < \frac{\pi}{2} \right\}$

Thus, it has been defined for $-\frac{\pi}{2} < x < \frac{\pi}{2}$ that has length π. So, its graph could be plotted as



Thus, the curve for $y = \tan^{-1}(\tan x)$, where y is not defined for $x \in (2n+1)\frac{\pi}{2}$.

and hence $\frac{dy}{dx} = \begin{cases} 1, & \left(2n - \frac{1}{2}\right)\pi < x < \left(2n + \frac{1}{2}\right)\pi \\ \text{does not exit}, & x = (2n+1)\frac{\pi}{2} \end{cases}$

Example 40 ***Sketch the graphs for***

(i) $y = \sin(\sin^{-1} x)$ *(ii)* $y = \cos(\cos^{-1} x)$

(iii) $y = \tan(\tan^{-1} x)$ *(iv)* $y = \text{cosec}(\text{cosec}^{-1} x)$

(v) $y = \sec(\sec^{-1} x)$

(vi) $y = \cot(\cot^{-1} x)$ and hence find $\frac{dy}{dx}$

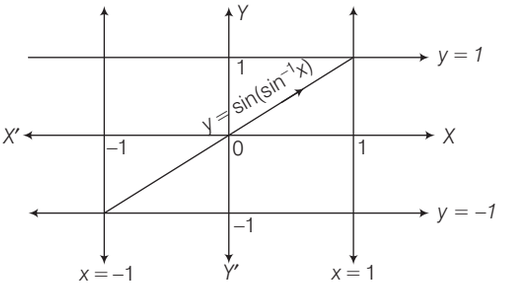
Sol. As we know, all the above mentioned six curves are non-periodic, but have restricted domains and ranges.

So, we shall first define each curve for its domain and range and then sketch the curves.

(i) **Sketch for the curve $y = \sin(\sin^{-1} x)$**

We know that, domain $x \in [-1, 1]$, i.e. $-1 \le x \le 1$ and range $y = x \Rightarrow y \in [-1, 1]$.

Hence, we should sketch $y = \sin(\sin^{-1} x)$ only when $x \in [-1, 1]$ and $y = x$. So, its graph could be plotted as shown in the figure.



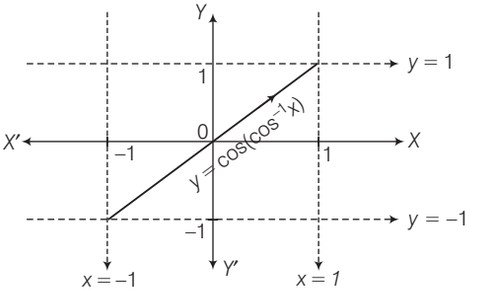
Thus, $y = \sin(\sin^{-1} x) = x, -1 \le x \le 1$.

$\Rightarrow \quad \frac{dy}{dx} = 1, \ -1 < x < 1.$

(ii) **Sketch for the curve $y = \cos(\cos^{-1} x)$**

We know that, domain $x \in [-1, 1]$, i.e. $-1 \le x \le 1$ and range $y = x \Rightarrow y \in [-1, 1]$.

Hence, we should sketch $y = \cos(\cos^{-1} x) = x$ only when $x \in [-1, 1]$. So, its graph could be plotted as shown in the figure.



Thus, $y = \cos(\cos^{-1} x) = x, -1 \le x \le 1$

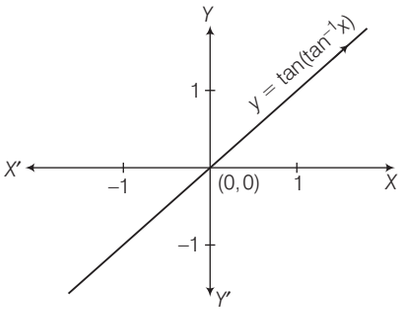
$\Rightarrow \quad \frac{dy}{dx} = 1, \ -1 < x < 1.$

(iii) **Sketch for the curve $y = \tan(\tan^{-1} x)$**

We know that, domain $x \in R$, i.e. $-\infty < x < \infty$
and range $y = x \Rightarrow y \in R$.

Hence, we should sketch $y = \tan(\tan^{-1} x) = x$,
$\forall\ x \in R$.

So, its graph could be plotted as shown in the figure.

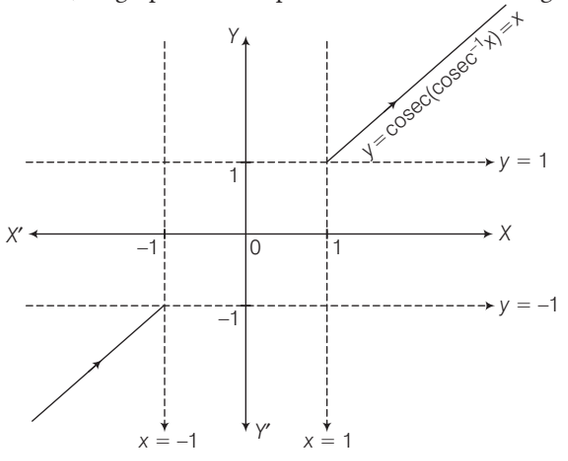


Thus, $y = \tan(\tan^{-1} x) = x,\ x \in R$

$\Rightarrow \quad \dfrac{dy}{dx} = 1,\ x \in R$

(iv) **Sketch for the curve $y = \text{cosec}(\text{cosec}^{-1} x)$**

We know that, domain $\in R - (-1, 1)$
(i.e. $-\infty < x \le -1$ or $1 \le x < \infty$)
and range $y = x \Rightarrow y \in R - (-1, 1)$.
Hence, we should sketch
$y = \text{cosec}(\text{cosec}^{-1} x) = x$ only when
$x \in (-\infty, -1] \cup [1, \infty)$.
So, its graph could be plotted as shown in the figure.



Thus, $y = \text{cosec}(\text{cosec}^{-1} x) = x,\ |x| \ge 1$

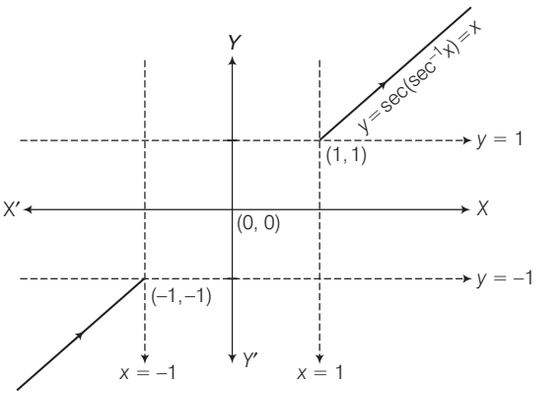
$\Rightarrow \quad \dfrac{dy}{dx} = 1,\ |x| > 1$

(v) **Sketch for the curve $y = \sec(\sec^{-1} x)$**

We know that, domain $\in R - (-1, 1)$
(i.e. $-\infty < x \le -1$ or $1 \le x < \infty$)
and range $y = x$
$\Rightarrow \quad y \in R - (-1, 1)$
Hence, we should sketch $y = \sec(\sec^{-1} x) = x$, only when

$$x \in (-\infty, -1] \cup [1, \infty)$$

So, its graph could be plotted as shown in the figure.

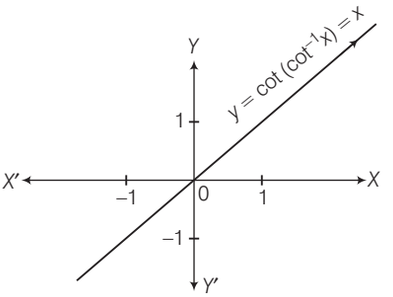


Thus, $y = \sec(\sec^{-1} x) = x,\ |x| \ge 1$

$\Rightarrow \quad \dfrac{dy}{dx} = 1,\ |x| > 1$

(vi) **Sketch for the curve $y = \cot(\cot^{-1} x)$**

We know that, domain $\in R$ (i.e. $-\infty < x < \infty$)
and range $y = x$
$\Rightarrow \quad y = R$.
Hence, we should sketch
$y = \cot(\cot^{-1} x) = x,\ \forall\ x \in R$.
So, its graph could be plotted as shown in the figure.



Thus, $y = \cos(\cot^{-1} x) = x,\ x \in R$

$\Rightarrow \quad \dfrac{dy}{dx} = 1,\ x \in R$.

Exercise for Session 4

1. If $y = \tan^{-1}\left(\frac{1-\cos x}{\sin x}\right)$, then $\frac{dy}{dx}$ is

(a) $\begin{cases} \frac{1}{2}, & x \in R - \{n\pi\}; \quad n \in I \\ -\frac{1}{2}, & x = \{n\pi\}; \qquad n \in I \end{cases}$

(b) $\begin{cases} \frac{1}{2}, & x \in R - \{n\pi\}; \quad n \in I \\ \text{does not exist}, & x = \{n\pi\}; \qquad n \in I \end{cases}$

(c) $\begin{cases} -\frac{1}{2}, & x \in R - \{n\pi\}; \quad n \in I \\ \text{does not exist}, & x = \{n\pi\}; \qquad n \in I \end{cases}$

(d) None of these

2. If $y = \cos^{-1}\left(\frac{x - x^{-1}}{x + x^{-1}}\right)$, then $\frac{dy}{dx}$ is

(a) $\begin{cases} \frac{2}{1+x^2}, & x > 0 \\ -\frac{2}{1+x^2}, & x < 0 \end{cases}$

(b) $\begin{cases} \frac{2}{1+x^2}, & x > 0 \\ \text{does not exist}, & x = 0 \\ \frac{-2}{1+x^2}, & x < 0 \end{cases}$

(c) $\begin{cases} \frac{2}{1+x^2}, & x < 0 \\ \text{does not exist}, & x = 0 \\ \frac{-2}{1+x^2}, & x > 0 \end{cases}$

(d) None of these

3. If $y = \tan^{-1}\left\{\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right\}$, then $\frac{dy}{dx}$ is

(a) $\begin{cases} \frac{1}{2}, & \cos\frac{x}{2} > \sin\frac{x}{2} \\ -\frac{1}{2}, & \cos\frac{x}{2} < \sin\frac{x}{2} \\ \text{does not exist}, & x = \{n\pi\}; n \in \text{integer} \end{cases}$

(b) $\begin{cases} -\frac{1}{2}, & \cos\frac{x}{2} > \sin\frac{x}{2} \\ \frac{1}{2}, & \cos\frac{x}{2} < \sin\frac{x}{2} \\ \text{does not exist}, & x = \{n\pi\}; n \in \text{integer} \end{cases}$

(c) $\begin{cases} -\frac{1}{2}, & \cos\frac{x}{2} \geq \sin\frac{x}{2} \\ \frac{1}{2}, & \cos\frac{x}{2} < \sin\frac{x}{2} \end{cases}$

(d) None of these

4. If $y = \cot^{-1}(\cot x)$, then $\frac{dy}{dx}$ is

(a) $1, x \in R$

(b) $1, x \in R - \{n\pi\}$

(c) $\begin{cases} 1, & x \in R - \{n\pi\} \\ \text{does not exist}, & x \in \{n\pi\}; n \in \text{integer} \end{cases}$

(d) None of these

5. Sketch the graph for

(i) $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

(ii) $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

(iii) $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

(iv) $y = \tan^{-1}\left(\frac{3x - x^3}{1-3x^2}\right)$

(v) $y = \sin^{-1}(3x - 4x^3)$

(vi) $y = \cos^{-1}(4x^3 - 3x)$

Session 5

Differentiation of a Function in Parametric Form

Differentiation of a Function in Parametric Form

Until now we have differentiate the variable y w.r.t. x, but what can we do if the variable x and y are depend on third variable 't' (say) i.e. $x = g(t), y = f(t)$. In such case x and y are called parametric functions and t is called parameter.

In this case to find $\frac{dy}{dx}$, we first obtain the relationship between x and y by eliminating the parameter t, then we differentiate w.r.t. x. But always, it is not possible to eliminate t, then we use the following formula

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{f'(t)}{g'(t)}$$

Example 41 If $x = e^{-t^2}$ and $y = \tan^{-1}(2t+1)$, then find $\frac{dy}{dx}$.

Sol. Here, $x = e^{-t^2}$

On differentiating both sides, we get

$$\frac{dx}{dt} = e^{-t^2} \cdot (-2t) \text{ and } y = \tan^{-1}(2t+1)$$

On differentiating both sides, we get

$$\frac{dy}{dt} = \frac{1}{1+(2t+1)^2}(2)$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{2}{1+(4t^2+4t+1)}}{-\frac{2t}{e^{t^2}}}$$

Hence, $$\frac{dy}{dx} = \frac{-e^{t^2}}{2t(2t^2+2t+1)}$$

Example 42 If $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$, then find $\frac{dy}{dx}$.

Sol. Here, $x = a(\theta - \sin\theta)$

On differentiating both the sides, we get

$$\frac{dx}{d\theta} = a(1-\cos\theta)$$

and $y = a(1-\cos\theta)$

On differentiating both sides, we get

$$\frac{dy}{d\theta} = a(\sin\theta)$$

So, $$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \cot\frac{\theta}{2}$$

Example 43 If $x = \sqrt{a^{\sin^{-1}t}}$, $y = \sqrt{a^{\cos^{-1}t}}$, $a > 0$ and $-1 < t < 1$. Show that $\frac{dy}{dx} = -\frac{y}{x}$.

Sol. Given, $x = \sqrt{a^{\sin^{-1}t}}$ and $y = \sqrt{a^{\cos^{-1}t}}$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{1}{2}(a^{\sin^{-1}t})^{-\frac{1}{2}} \frac{d}{dt}(a^{\sin^{-1}t})$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{1}{2}(a^{\sin^{-1}t})^{-\frac{1}{2}} \cdot a^{\sin^{-1}t} \log_e a \times \frac{1}{\sqrt{1-t^2}} = \frac{x \log_e a}{2\sqrt{1-t^2}}$$

Again, $$\frac{dy}{dt} = \frac{1}{2}(a^{\cos^{-1}t})^{-1/2} \frac{d}{dt}(a^{\cos^{-1}t})$$

$$= \frac{1}{2}(a^{\cos^{-1}t})^{-1/2} \cdot a^{\cos^{-1}t} \log_e a \times \left(\frac{1}{\sqrt{1-t^2}}\right) = \frac{-y \log_e a}{2\sqrt{1-t^2}}$$

So, $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{y}{x}$$

Aliter

We have, $xy = \sqrt{a^{\sin^{-1}t}} \cdot \sqrt{a^{\cos^{-1}t}} = \sqrt{a^{\sin^{-1}t + \cos^{-1}t}} = \sqrt{a^{\pi/2}}$

$$\Rightarrow \quad x^2y^2 = a^{\pi/2} \qquad \left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$$

On differentiating, we get

$$2xy^2 + 2x^2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$$

Example 44 If $x = a\left(\cos t + \frac{1}{2}\log \tan^2 t\right)$ and $y = a \sin t$, then find $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$.

Sol. Here, $x = a\left(\cos t + \frac{1}{2}\log \tan^2 t\right)$ and $y = a \sin t$

On differentiating both equations w.r.t. t, we get

$$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan \frac{t}{2}} \sec^2 \frac{t}{2} \times \frac{1}{2}\right)$$

$$\Rightarrow \quad \frac{dx}{dt} = a\left(-\sin t + \frac{1}{2 \sin \frac{t}{2} \cos \frac{t}{2}}\right)$$

$$\Rightarrow \quad \frac{dx}{dt} = a\left(-\sin t + \frac{1}{\sin t}\right) = \frac{a \cos^2 t}{\sin t} \text{ and } \frac{dy}{dt} = a \cos t$$

So, $\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = \frac{a \cos t}{\frac{a \cos^2 t}{\sin t}} = \tan t$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{t = \frac{\pi}{4}} = \tan \frac{\pi}{4} = 1$$

Exercise for Session 5

1. If $x = 2\cos \theta - \cos 2\theta$ and $y = 2 \sin \theta - \sin 2\theta$, prove that $\frac{dy}{dx} = \tan\left(\frac{3\theta}{2}\right)$.
2. If $x = e^{\cos 2t}$ and $y = e^{\sin 2t}$, then prove that $\frac{dy}{dx} = \frac{-y \log x}{x \log y}$.
3. If $x = \cos t$ and $y = \sin t$, then prove that $\frac{dy}{dx} = \frac{1}{\sqrt{3}}$, at $t = \frac{2\pi}{3}$.
4. If $x = a\left(t + \frac{1}{t}\right)$ and $y = a\left(t - \frac{1}{t}\right)$, then prove that $\frac{dy}{dx} = \frac{x}{y}$.
5. If $x = \sin^{-1}\left(\frac{2t}{1+t^2}\right)$ and $y = \tan^{-1}\left(\frac{2t}{1-t^2}\right)$, then prove that $\frac{dy}{dx} = 1$.
6. If $x = a \sec^3 \theta$ and $y = a \tan^3 \theta$, then find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{3}$.
7. Let $y = x^3 - 8x + 7$ and $x = f(t)$. If $\frac{dy}{dt} = 2$ and $x = 3$ at $t = 0$, then find the value of $\frac{dx}{dt}$ at $t = 0$.

Session 6

Logarithmic Differentiation, Differentiation of Infinite Series

Logarithmic Differentiation

So far, we have discussed derivatives of the functions of the form $(f(x))^n$, $n^{f(x)}$ and n^n, where $f(x)$ is a function of x and n is a constant. In this section, we will mainly be discussing derivatives of the functions of the form $(f(x))^{g(x)}$, where $f(x)$ and $g(x)$ are functions of x or $f(x)s(x)h(x)\ldots$ or $\dfrac{f_1(x)f_2(x)\ldots}{g_1(x)g_2(x)\ldots}$

To find the derivatives of these types of functions, we proceed as follows

Let $\boldsymbol{y = (f(x))^{g(x)}}$

Taking logarithm on both sides, we have

$$\log y = g(x)\log\{f(x)\}$$

Differentiating w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = g(x)\cdot\frac{1}{f(x)}\cdot\frac{d\,f(x)}{dx} + \log\{f(x)\}\cdot\frac{d\,g(x)}{dx}$$

$$\therefore\quad \frac{dy}{dx} = y\left[\frac{g(x)}{f(x)}\cdot\frac{d\,f(x)}{dx} + \log\{f(x)\}\cdot\frac{d\,g(x)}{dx}\right]$$

or $$\boldsymbol{\frac{dy}{dx} = (f(x))^{g(x)}\left\{\frac{g(x)}{f(x)}\cdot\frac{d\,f(x)}{dx} + \log\{f(x)\}\cdot\frac{d\,g(x)}{dx}\right\}}$$

Logarithmic Differentiation

(For Quickly Solving Objective MCQs)

If $y = \{f(x)\}^{\phi(x)}$, then

$$\boldsymbol{\frac{dy}{dx}} = \textbf{Differentiation of } y \textbf{ taking } f(x) \textbf{ as constant} + \textbf{Differentiation of } y \textbf{ taking } \phi(x) \textbf{ as constant.}$$

or when we have to differentiate the function of the form $y = (\text{Variable})^{\text{variable}}$, take log on both sides and then differentiate.

Example 45 *If* $y = x^{\sin x}$, *then find* $\dfrac{dy}{dx}$.

Sol. $y = x^{\sin x}$, taking log on both sides, we get

$$\log y = \sin x \log x$$

On differentiating both sides, we get

$$\frac{1}{y}\frac{dy}{dx} = \sin x\cdot\left(\frac{1}{x}\right) + \log x\cdot(\cos x)$$

$$\Rightarrow\quad \frac{dy}{dx} = y\left[\frac{\sin x}{x} + (\cos x)\log x\right]$$

$$\therefore\quad \frac{dy}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + (\cos x)\log x\right]$$

Aliter

Here, $y = x^{\sin x}$ could also be differentiated by using definition;

$$\frac{d}{dx}(\text{variable})^{\text{variable}} = \frac{d}{dx}(\text{variable})^{\text{constant}} + \frac{d}{dx}(\text{constant})^{\text{variable}}$$

i.e. $$\frac{dy}{dx} = \underset{\text{variable}\quad\text{constant}}{\frac{d}{dx}(x)^{\sin x}} + \underset{\text{constant}\qquad\text{variable}}{\frac{d}{dx}(x)^{\sin x}}$$

$$= \sin x\,(x)^{\sin x - 1} + (x)^{\sin x}\cdot(\log x)\cdot\cos x$$

$$= (\sin x)\frac{(x)^{\sin x}}{x} + (x)^{\sin x}\cdot\log x\cdot\cos x$$

$$= x^{\sin x}\left\{\frac{\sin x}{x} + \cos x\cdot\log x\right\}$$

Example 46 *If* $x^y\cdot y^x = 1$, *then find* $\dfrac{dy}{dx}$.

Sol. Taking log on both the sides, $y\log x + x\log y = \log 1$

Differentiating both the sides, we get

$$y\cdot\frac{d}{dx}(\log x) + \left\{\frac{d}{dx}y\right\}\cdot\log x + \left\{\frac{d}{dx}x\right\}\cdot\log y + x\left\{\frac{d}{dx}\log y\right\} = 0$$

$$\Rightarrow\quad y\cdot\frac{1}{x} + \log x\cdot\frac{dy}{dx} + 1\cdot\log y + x\cdot\frac{1}{y}\cdot\frac{dy}{dx} = 0$$

$$\Rightarrow\quad \left[\log x + \frac{x}{y}\right]\frac{dy}{dx} = -\left[\frac{y}{x} + \log y\right]$$

$$\Rightarrow\quad \frac{dy}{dx} = -\frac{(y + x\log y)}{(x + y\log x)}\cdot\frac{y}{x}$$

Example 47 If $y^{\cot x} + (\tan^{-1} x)^y = 1$, then find $\frac{dy}{dx}$.

Sol. Let $u = y^{\cot x}$ and $v = (\tan^{-1} x)^y$, we get

$$u + v = 1 \quad \text{...(i)}$$

Taking log on both sides for u and v, we get

$$\log u = \cot x \log y \text{ and } \log v = y \log (\tan^{-1} x)$$

On differentiating both sides, we get

$$\frac{1}{u} \cdot \frac{du}{dx} = \cot x \cdot \frac{1}{y}\frac{dy}{dx} + \log y \cdot (-\text{cosec}^2 x)$$

$$\Rightarrow \quad \frac{du}{dx} = y^{\cot x}\left[\frac{\cot x}{y} \cdot \frac{dy}{dx} - (\text{cosec}^2 x) \log y\right] \quad \text{...(ii)}$$

Again, differentiating $\log v = y \log (\tan^{-1} x)$, we get

$$\frac{1}{v} \cdot \frac{dv}{dx} = y \cdot \frac{1}{\tan^{-1} x} \cdot \frac{1}{1 + x^2} + \log (\tan^{-1} x) \cdot \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dv}{dx} = (\tan^{-1} x)^y \left[\frac{y}{(1 + x^2) \tan^{-1} x} + \log (\tan^{-1} x) \cdot \frac{dy}{dx}\right] \quad \text{...(iii)}$$

From Eq. (i), we get $\frac{du}{dx} + \frac{dv}{dx} = 0$

$$\Rightarrow y^{\cot x}\left\{\frac{\cot x}{y} \cdot \frac{dy}{dx} - (\text{cosec}^2 x) \log y\right\} + (\tan^{-1} x)^y \left[\frac{y}{(1 + x^2) \tan^{-1} x} + \log (\tan^{-1} x) \cdot \frac{dy}{dx}\right] = 0$$

$$\Rightarrow \quad \left\{\frac{y^{\cot x} \cdot \cot x}{y} + (\tan^{-1} x)^y \cdot \log (\tan^{-1} x)\right\}\frac{dy}{dx} = \left\{y^{\cot x} \cdot (\text{cosec}^2 x) \log y - \frac{(\tan^{-1} x)^y \cdot y}{(1 + x^2) \tan^{-1} x}\right\}$$

Hence, $\frac{dy}{dx} = \frac{(y^{\cot x} \cdot \text{cosec}^2 x \cdot \log y) - \left\{\frac{(\tan^{-1} x)^{y-1} \cdot y}{1 + x^2}\right\}}{(y^{\cot x - 1} \cdot \cot x) + \{(\tan^{-1} x)^y \cdot \log (\tan^{-1} x)\}}$

Derivative of the Product of Finite Number of Functions

Example 48 If $f(x) = (x + 1)(x + 2)(x + 3) \ldots (x + n)$, then find $f'(0)$.

Sol. Here $f(x) = (x + 1)(x + 2)(x + 3) \ldots (x + n)$

$$\log f(x) = \log \{(x + 1)(x + 2) \ldots (x + n)\}$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{f(x)} f'(x) = \frac{1}{x + 1} + \frac{1}{x + 2} + \frac{1}{x + 3} + \ldots + \frac{1}{x + n}$$

$$\therefore \quad f'(0) = f(0)\left\{1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right\}$$

$$= (1 \cdot 2 \cdot 3 \cdot \ldots \cdot n)\left\{1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right\}$$

$$= (n)!\left\{1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right\}$$

Example 49 If $f(x) = \prod_{n=1}^{100} (x - n)^{n(101-n)}$, then find $\frac{f(101)}{f'(101)}$.

Sol. $f(x) = \prod_{n=1}^{100} (x - n)^{n(101-n)}$

$$\Rightarrow \quad \log f(x) = \left\{n(101 - n) \log\left\{\prod_{n=1}^{100} (x - n)\right\}\right\}$$

{Here, Π changes to Σ when taken log}

$$\Rightarrow \quad \log f(x) = \sum_{n=1}^{100} n(101 - n) \log (x - n)$$

Differentiating both sides, we get

$$\frac{f'(x)}{f(x)} = \sum_{n=1}^{100} n(101 - n) \cdot \frac{1}{x - n}$$

$$\therefore \quad \frac{f'(101)}{f(101)} = \sum_{n=1}^{100} \frac{n(101 - n)}{(101 - n)} = \sum_{n=1}^{100} n = 5050$$

$$\Rightarrow \quad \frac{f(101)}{f'(101)} = \frac{1}{5050}$$

Differentiation of Infinite Series

When the value of y is given as an infinite series, then we use the fact that, if one term is deleted from an infinite series, it remains unaffected, to replace all terms except first term by y. Thus, we convert it into a finite series or function.

Example 50 If $y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \ldots \infty}}}$ then find $\frac{dy}{dx}$.

Sol. We have, $y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \ldots \infty}}}$

Then $\quad y = \sqrt{\log x + y}$

$$\Rightarrow \quad y^2 = \log x + y$$

Differentiating both sides, we get

$$2y \cdot \frac{dy}{dx} = \frac{1}{x} + \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}(2y - 1) = \frac{1}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x(2y - 1)}$$

Exercise for Session 6

■ **Directions** (Q. Nos. 1 to 10) *Differentiate the following functions w.r.t. x.*

1. x^x **2.** $x^{\sqrt{x}}$ **3.** x^{x^x}

4. $(x^x)^x$ **5.** $(x^x)\sqrt{x}$ **6.** $(\cos x)^x$

7. $(\sin x)^{\cos x}$ **8.** $(\sin x)^{\cos^{-1} x}$ **9.** $\cos(x^x)$

10. $\log(x^x + \operatorname{cosec}^2 x)$

11. If $y = (\sin x)^{\tan x} + (\cos x)^{\sec x}$, then find $\frac{dy}{dx}$.

12. If $x^y = e^{x-y}$, then prove that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$

13. If $x^y + y^x = 2$, then find $\frac{dy}{dx}$.

14. If $(\cos x)^y = (\sin y)^x$, then find $\frac{dy}{dx}$.

15. If $x \sin(a+y) + \sin a \cdot \cos(a+y) = 0$. Prove that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$.

16. If $y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \ldots \infty}}}$, then prove that $\frac{dy}{dx} = \frac{\sin x}{1-2y}$.

17. If $y = (\tan x)^{(\tan x)^{(\tan x)^{\cdots \infty}}}$, then prove that $\frac{dy}{dx} = 2$ at $x = \frac{\pi}{4}$.

18. If $y = e^{x^{e^x}} + x^{e^{e^x}} + e^{x^{x^e}}$, then prove that

$$\frac{dy}{dx} = e^{x^{e^x}} \cdot x^{e^x} \cdot \left\{\frac{e^x}{x} + e^x \cdot \log x\right\} + x^{e^{e^x}} \cdot e^{e^x} \cdot \left\{\frac{1}{x} + e^x \cdot \log x\right\} + e^{x^{x^e}} \cdot x^{x^e} \cdot x^{e-1}\{1 + e \log x\}.$$

19. If $y = \tan^{-1}\left(\frac{a_1 x - \alpha}{a_1 \alpha + x}\right) + \tan^{-1}\left(\frac{a_2 - a_1}{1 + a_1 a_2}\right) + \tan^{-1}\left(\frac{a_3 - a_2}{1 + a_2 a_3}\right) + \ldots + \tan^{-1}\left(\frac{a_n - a_{n-1}}{1 + a_n a_{n-1}}\right) - \tan^{-1}(a_n)$, then find $\frac{dy}{dx}$.

Session 7

Differentiation of a Function w.r.t. Another Function

Differentiation of a Function w.r.t. Another Function

Suppose it is required to differentiate a function $f(x)$ with respect to another function $g(x)$. Let $y = f(x)$ and $z = g(x)$. Here, both functions are different but both are in same variable x.

Then, to find the derivative of $f(x)$ with respect to $g(x)$ or derivative of $g(x)$ with respect to $f(x)$, i.e. to find $\frac{dy}{dz}$ or $\frac{dz}{dy}$. We firstly differentiate both functions $f(x)$ and $g(x)$ with respect to x separately and then put these values in the following formulae:

$$\frac{dy}{dz} = \frac{dy/dx}{dz/dx} \quad \text{or} \quad \frac{dz}{dy} = \frac{dz/dx}{dy/dx}$$

Example 51 ***Differentiate*** $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ ***w.r.t.*** $\tan^{-1} x$, $-1 < x < 1$.

Sol. Let $u = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ and $v = \tan^{-1} x$

Putting $x = \tan\theta \Rightarrow \theta = \tan^{-1} x$

$$\Rightarrow \quad u = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$\Rightarrow \quad u = \sin^{-1}(\sin 2\theta) \qquad \left[\because \sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}\right]$$

$$\Rightarrow \quad u = 2\theta \Rightarrow u = 2\tan^{-1} x$$

$$\Rightarrow \quad \frac{du}{dx} = \frac{2}{1+x^2} \qquad \text{...(i)}$$

Again, $v = \tan^{-1} x \Rightarrow \frac{dv}{dx} = \frac{1}{1+x^2}$ ...(ii)

Now, $$\frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{\frac{2}{1+x^2}}{\frac{1}{1+x^2}} = 2$$

Example 52 ***If*** $x \in \left(\frac{1}{\sqrt{2}}, 1\right)$. ***Differentiate*** $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ ***w.r.t.*** $\cos^{-1}(2x\sqrt{1-x^2})$.

Sol. Let $u = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

and $v = \cos^{-1}(2x\sqrt{1-x^2})$

Putting $x = \sin\theta$, then $x \in \left(\frac{1}{\sqrt{2}}, 1\right)$

$$\Rightarrow \quad \frac{1}{\sqrt{2}} < \sin\theta < 1 \Rightarrow \frac{\pi}{4} < \theta < \frac{\pi}{2}$$

Now, $u = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) \Rightarrow u = \tan^{-1}\left(\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}\right)$

$$\Rightarrow \quad u = \tan^{-1}\left(\frac{\cos\theta}{\sin\theta}\right) \Rightarrow u = \tan^{-1}(\cot\theta)$$

$$\Rightarrow \quad u = \tan^{-1}\left\{\tan\left(\frac{\pi}{2} - \theta\right)\right\}$$

$$\left[\because \frac{\pi}{4} < \theta < \frac{\pi}{2} \Rightarrow 0 < \frac{\pi}{2} - \theta < \frac{\pi}{4}\right]$$

$$\Rightarrow \quad u = \frac{\pi}{2} - \theta \Rightarrow u = \frac{\pi}{2} - \sin^{-1} x$$

$$\Rightarrow \quad \frac{du}{dx} = -\frac{1}{\sqrt{1-x^2}} \qquad \text{...(i)}$$

Again, $v = \cos^{-1}(2x\sqrt{1-x^2})$

$$\Rightarrow \quad v = \frac{\pi}{2} - \sin^{-1}(2x\sqrt{1-x^2})$$

$$\Rightarrow \quad v = \frac{\pi}{2} - \sin^{-1}(\sin 2\theta), \text{ where } x = \sin\theta$$

$$\Rightarrow \quad v = \frac{\pi}{2} - \sin^{-1}(\sin(\pi - 2\theta)) \Rightarrow v = \frac{\pi}{2} - (\pi - 2\theta)$$

$$\left[\because \frac{\pi}{4} < \theta < \frac{\pi}{2} \Rightarrow 0 < \pi - 2\theta > \frac{\pi}{2}\right]$$

$$\Rightarrow \quad v = -\frac{\pi}{2} + 2\theta \Rightarrow v = -\frac{\pi}{2} + 2\sin^{-1} x$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{2}{\sqrt{1-x^2}} \qquad \text{...(ii)}$$

$$\therefore \quad \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{-\frac{1}{\sqrt{1-x^2}}}{\frac{2}{\sqrt{1-x^2}}} = -\frac{1}{2}$$

Example 53 *Find the derivative of* $\sec^{-1}\left(\frac{1}{2x^2-1}\right)$ *w.r.t.* $\sqrt{1-x^2}$ *at* $x=\frac{1}{2}$.

Sol. Let $u=\sec^{-1}\left(\frac{1}{2x^2-1}\right)$ and $v=\sqrt{1-x^2}$

Put $x=\cos\theta$

$\therefore$ $u=\sec^{-1}(\sec 2\theta)=2\theta$

and $v=\sqrt{1-\cos^2\theta}=\sin\theta$

$\Rightarrow$ $\frac{du}{d\theta}=2$

and $\frac{dv}{d\theta}=\cos\theta$

$\Rightarrow$ $\frac{du}{dv}=\frac{du/d\theta}{dv/d\theta}=\frac{2}{\cos\theta}=\frac{2}{x}$

Now, $\left(\frac{du}{dv}\right)_{\text{at } x=\frac{1}{2}}=\frac{2}{\frac{1}{2}}=4$

Example 54 If $y=f(x^3)$, $z=g(x^5)$, $f'(x)=\tan x$ and $g'(x)=\sec x$, then find the value of $\lim_{x\to 0}\frac{dy/dz}{x}$.

Sol. Here, $y=f(x^3)\Rightarrow \frac{dy}{dx}=f'(x^3)\cdot 3x^2$

$\frac{dy}{dx}=3x^2\tan x^3$ $[\because f'(x)=\tan x]$

and $z=g(x^5)\Rightarrow \frac{dz}{dx}=g'(x^5)\cdot 5x^4$

$\frac{dz}{dx}=5x^4\cdot\sec x^5$ $[\because g'(x)=\sec x]$

Now, $\frac{dy}{dz}=\frac{3x^2\tan x^3}{5x^4\sec x^5}=\frac{3}{5x^2}\times\frac{\tan x^3}{\sec x^5}$

$\therefore$ $\lim_{x\to 0}\frac{\frac{dy}{dz}}{x}=\lim_{x\to 0}\frac{3\tan x^3}{5x^3\sec x^5}=\frac{3}{5}$

Example 55 Find the derivative of $f(\tan x)$ w.r.t. $g(\sec x)$ at $x=\frac{\pi}{4}$, where $f'(1)=2$ and $g'(\sqrt{2})=4$.

Sol. Let $u=f(\tan x)$ and $v=g(\sec x)$

$\Rightarrow$ $\frac{du}{dx}=f'(\tan x)\cdot\sec^2 x$

and $\frac{dv}{dx}=g'(\sec x)\cdot\sec x\cdot\tan x$

$\Rightarrow$ $\frac{du}{dv}=\frac{du/dx}{dv/dx}=\frac{f'(\tan x)\sec^2 x}{g'(\sec x)\sec x\tan x}$

$\Rightarrow$ $\left(\frac{du}{dv}\right)_{x=\frac{\pi}{4}}=\frac{f'\left(\tan\frac{\pi}{4}\right)}{g'\left(\sec\frac{\pi}{4}\right)\sin\frac{\pi}{4}}=\frac{f'(1)\sqrt{2}}{g'(\sqrt{2})}=\frac{2\sqrt{2}}{4}=\frac{1}{\sqrt{2}}$

Exercise for Session 7

1. Find the derivative of $\tan^{-1}\frac{2x}{1-x^2}$ w.r.t. $\sin^{-1}\frac{2x}{1+x^2}$, if

   (i) $x\in(-1,1)$ (ii) $x\in(1,\infty)$

   (iii) $x\in(-\infty,-1)$

2. Differentiate $x^{\sin^{-1}x}$ w.r.t. $\sin^{-1}x$.

3. Differentiate $\sin^{-1}(2ax\sqrt{1-a^2x^2})$ w.r.t. $\sqrt{1-a^2x^2}$.

4. Differentiate $\log\sin x$ w.r.t. $\sqrt{\cos x}$

5. Differentiate $\tan^{-1}\left\{\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right\}$ w.r.t. $\cos^{-1}x^2$.

6. Differentiate x^x w.r.t. $x\log x$.

7. Differentiate $\sin^{-1}(4x\sqrt{1-4x^2})$ w.r.t. $\sqrt{1-4x^2}$, if

   (i) $x\in\left(-\frac{1}{2\sqrt{2}},\frac{1}{2\sqrt{2}}\right)$ (ii) $x\in\left(\frac{1}{2\sqrt{2}},\frac{1}{2}\right)$

   (iii) $x\in\left(-\frac{1}{2},-\frac{1}{2\sqrt{2}}\right)$

Session 8

Higher Derivatives of a Function

Higher Derivatives of a Function

If $y = f(x)$, then the derivative of $\frac{dy}{dx}$ w.r.t. x is called the second derivative of y w.r.t. x and it is denoted by $\frac{d^2y}{dx^2}$.

Also, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$; $\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$ and so on.

The nth order derivative of y w.r.t. x is denoted by $\frac{d^ny}{dx^n}$. If $y = f(x)$, then other notations for $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots, \frac{d^ny}{dx^n}$ are $y_1, y_2, y_3, \ldots, y_n$ or $y', y'', y''', \ldots, y^n$ or $f'(x), f''(x), f'''(x), \ldots, f^n(x)$.

Remark

If y is a function of x, given parametrically by $x = \phi(t)$ $y = \psi(t)$, then

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{\psi'(t)}{\phi'(t)}\right) = \frac{d}{dt}\cdot\left(\frac{\psi'(t)}{\phi'(t)}\right) \times \frac{dt}{dx}$$

Example 56 If $y = x^3 \log_e x$, then find y'' and y'''.

Sol. Here, $y = x^3 \log_e x \Rightarrow y' = x^3 \cdot \frac{1}{x} + 3x^2 \cdot \log_e x$

$$y' = x^2 + 3x^2 \cdot \log_e x$$

Again differentiating both sides, we get

$$y'' = 2x + 6x \cdot \log_e x + 3x^2 \cdot \frac{1}{x}$$

$$y'' = 5x + 6x \cdot \log_e x$$

Differentiating again,

$$y''' = 5 + 6 \cdot \log x + 6x \cdot \frac{1}{x}$$

$$y''' = 11 + 6 \log x$$

Example 57 If $y = \sin(\sin x)$, then prove that $y'' + (\tan x)\, y' + y \cos^2 x = 0$.

Sol. Here, $y = \sin(\sin x)$ ...(i)

$\Rightarrow$ $y' = \cos(\sin x) \cdot \cos x$

$\Rightarrow$ $(\sec x)\, y' = \cos(\sin x)$ ...(ii)

Again differentiating w.r.t. x, we get

$(\sec x) y'' + (\sec x \tan x) y' = -\sin(\sin x) \cdot \cos x$

$\Rightarrow$ $y'' + (\tan x) y' = -\sin(\sin x) \cdot \cos^2 x$

Using Eq. (i), we get $y = \sin(\sin x)$

$\Rightarrow$ $y'' + (\tan x) y' + (\cos^2 x) y = 0$

Example 58 If $(a + bx)e^{y/x} = x$, show that $x^3 y'' = (xy' - y)^2$.

Sol. Here, $(a + bx)e^{y/x} = x \Rightarrow e^{y/x} = \frac{x}{a + bx}$

Taking logarithm on both sides, we have

$$\frac{y}{x} = \log\left(\frac{x}{a + bx}\right) \quad \text{...(i)}$$

$\Rightarrow$ $\frac{y}{x} = \log x - \log(a + bx)$

Differentiating both sides, we get

$$\frac{xy' - y \cdot 1}{x^2} = \frac{1}{x} - \frac{b}{a + bx}$$

$\Rightarrow$ $xy' - y = \frac{ax}{(a + bx)} = ae^{y/x}$ [from Eq. (i)]

Again taking logarithm on both sides, we get

$$\log(xy' - y) = \log a + \frac{y}{x} \cdot \log e$$

Again differentiating both sides w.r.t. x, we get

$$\frac{xy'' + y' - y'}{xy' - y} = 0 + \frac{(xy' - y \cdot 1)}{x^2}$$

$\Rightarrow$ $x^3 y'' = (xy' - y)^2$.

Example 59 ***If*** $x = a(t + \sin t)$ ***and*** $y = a(1 - \cos t)$, ***then find*** $\frac{d^2y}{dx^2}$.

Sol. Here, $x = a(t + \sin t)$ and $y = a(1 - \cos t)$

Differentiating both sides w.r.t. t, we get

$$\frac{dx}{dt} = a(1 + \cos t) \quad \text{and} \quad \frac{dy}{dt} = a(\sin t)$$

$$\therefore \quad \frac{dy}{dx} = \frac{a \sin t}{a(1 + \cos t)} = \frac{2 \sin \frac{t}{2} \cdot \cos \frac{t}{2}}{2 \cos^2 \frac{t}{2}} = \tan\left(\frac{t}{2}\right)$$

Again, differentiating both sides, we get

$$\frac{d^2y}{dx^2} = \sec^2\left(\frac{t}{2}\right)\cdot\frac{1}{2}\cdot\frac{dt}{dx} = \frac{1}{2}\sec^2\left(\frac{t}{2}\right)\cdot\frac{1}{a(1+\cos t)}$$

$$= \frac{1}{2a}\cdot\frac{\sec^2\left(\frac{t}{2}\right)}{2\left(\cos^2\frac{t}{2}\right)}$$

Hence, $\frac{d^2y}{dx^2} = \frac{1}{4a}\cdot\sec^4\left(\frac{t}{2}\right)$

Example 60 Show that the function $y = f(x)$ defined by the parametric equations $x = e^t \cdot \sin t$, $y = e^t \cdot \cos t$, satisfies the relation $y''(x+y)^2 = 2(xy' - y)$.

Sol. Here, $x = e^t \cdot \sin t$

On differentiating w.r.t. t, we get

$$\frac{dx}{dt} = e^t \cdot \cos t + e^t \cdot \sin t = e^t(\sin t + \cos t)$$

and $y = e^t \cdot \cos t$

On differentiating w.r.t. t, we get

$$\frac{dy}{dt} = e^t \cdot(-\sin t) + e^t \cdot(\cos t) = e^t(\cos t - \sin t)$$

So, $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{e^t(\cos t - \sin t)}{e^t(\cos t + \sin t)} = \frac{y-x}{y+x}$$

or $$\frac{dy}{dx} = \frac{y-x}{y+x} \quad \text{...(i)}$$

On differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{(y+x)\cdot\left\{\frac{dy}{dx}-1\right\} - (y-x)\left\{\frac{dy}{dx}+1\right\}}{(y+x)^2}$$

or $$y'' = \frac{(y+x)\cdot(y'-1) - (y-x)(y'+1)}{(y+x)^2}$$

Therefore, $(x+y)^2 \cdot y'' = 2(xy' - y)$

Exercise for Session 8

1. If $y = x^x$, then prove that $\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0$.
2. If $y = A\cos(\log x) + B\sin(\log x)$, then prove that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$.
3. If $y = x\log\left(\frac{x}{a+bx}\right)$, then prove that $x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$.
4. If $y = \log(x + \sqrt{x^2+a^2})$, then prove that $(x^2+a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$.
5. If $y = (x + \sqrt{x^2+1})^m$, then prove that $(x^2+1)y_2 + xy_1 - m^2y = 0$.
6. If $x = at^2$, $y = 2at$, then find $\frac{d^2y}{dx^2}$.
7. If $x = a\cos^3\theta$, $y = a\sin^3\theta$, then find $\frac{d^2y}{dx^2}$.
8. If $x = \tan\left(\frac{1}{a}\log y\right)$, then show that $(1+x^2)\frac{d^2y}{dx^2} = (a-2x)\frac{dy}{dx}$.
9. If $x = a\cos\theta + b\sin\theta$ and $y = a\sin\theta - b\cos\theta$, then prove that $y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$.
10. If $y = \frac{ax+b}{cx+d}$, then prove that $2y_1y_3 = 3(y_2)^2$.
11. If $x = f(t)$ and $y = \phi(t)$, prove that $\frac{d^2y}{dx^2} = \frac{f_1\phi_2 - f_2\phi_1}{f_1^3}$, where suffixes denote differentiation w.r.t. 't'.
12. If $x = \sin t$, $y = \sin Kt$, then show that $(1-x^2)y_2 - xy_1 + K^2y = 0$.
13. If $\frac{x+b}{2} = a\tan^{-1}(a\log_e y), a > 0$, then prove that $yy'' - yy'\log y = (y')^2$.
14. Let $f(x)$ be a polynomial function of degree 2 and $f(x) > 0$ for all $x \in R$. If $g(x) = f(x) + f'(x) + f''(x)$, then for any x show that $g(x) > 0$.

Session 9

Differentiation of a Function Given in the Form of a Determinant

Differentiation of a Function Given in the Form of a Determinant

To find the derivative of a determinant, we differentiate the row (or columns) keeping other rows (or columns) unchanged, i.e.

$$\text{If } y=\begin{vmatrix} u(x) & v(x) & w(x) \\ p(x) & q(x) & r(x) \\ \lambda(x) & \mu(x) & \gamma(x) \end{vmatrix}, \text{ then}$$

$$\frac{dy}{dx}=\begin{vmatrix} u'(x) & v'(x) & w'(x) \\ p(x) & q(x) & r(x) \\ \lambda(x) & \mu(x) & \gamma(x) \end{vmatrix}+\begin{vmatrix} u(x) & v(x) & w(x) \\ p'(x) & q'(x) & r'(x) \\ \lambda(x) & \mu(x) & \gamma(x) \end{vmatrix}+\begin{vmatrix} u(x) & v(x) & w(x) \\ p(x) & q(x) & r(x) \\ \lambda'(x) & \mu'(x) & \gamma'(x) \end{vmatrix}$$

The differentiation can also be done column-wise.

Example 61 *If* $f(x)=\begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$, *then find* $f'(x)$.

Sol. Here, $f(x)=\begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$

On differentiating, we get

$$f'(x)=\begin{vmatrix} \frac{d}{dx}(x) & \frac{d}{dx}(x^2) & \frac{d}{dx}(x^3) \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}+\begin{vmatrix} x & x^2 & x^3 \\ \frac{d}{dx}(1) & \frac{d}{dx}(2x) & \frac{d}{dx}(3x^2) \\ 0 & 2 & 6x \end{vmatrix}+\begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ \frac{d}{dx}(0) & \frac{d}{dx}(2) & \frac{d}{dx}(6x) \end{vmatrix}$$

$$\text{or } f'(x)=\begin{vmatrix} 1 & 2x & 3x^2 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}+\begin{vmatrix} x & x^2 & x^3 \\ 0 & 2 & 6x \\ 0 & 2 & 6x \end{vmatrix}+\begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 0 & 6 \end{vmatrix}$$

As we know, if any two rows or columns are equal, then value of the determinant is zero.

$$\therefore \quad f'(x)=0+0+\begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 0 & 6 \end{vmatrix}=6(2x^2-x^2)$$

Therefore, $f'(x)=6x^2$

Example 62 *If* $f(x)=\begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ ac & bc & x+c^2 \end{vmatrix}$, *then find* $f'(x)$.

Sol. We have, $f(x)=\begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ ac & bc & x+c^2 \end{vmatrix}$

$$\therefore \quad f'(x)=\begin{vmatrix} 1 & 0 & 0 \\ ab & x+b^2 & bc \\ ac & bc & x+c^2 \end{vmatrix}+\begin{vmatrix} x+a^2 & ab & ac \\ 0 & 1 & 0 \\ ac & bc & x+c^2 \end{vmatrix}+\begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ 0 & 0 & 1 \end{vmatrix}$$

$$=\{(x+b^2)(x+c^2)-b^2c^2\}+\{(x+a^2)(x+c^2)-a^2c^2\}+\{(x+a^2)(x+b^2)-a^2b^2\}$$

$$=3x^2+2x(a^2+b^2+c^2)$$

Example 63 Let $f(x)=\begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$, where p is constant. Then, find $\frac{d^3}{dx^3}[f(x)]$ at $x=0$. ***[IIT JEE 1997]***

Sol. Given, $f(x)=\begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$

On differentiating, we get

$$\frac{d}{dx}[f(x)]=\begin{vmatrix} \frac{d}{dx}(x^3) & \frac{d}{dx}(\sin x) & \frac{d}{dx}(\cos x) \\ 6 & (-1) & 0 \\ p & p^2 & p^3 \end{vmatrix} + \begin{vmatrix} x^3 & \sin x & \cos x \\ 0 & 0 & 0 \\ p & p^2 & p^3 \end{vmatrix} + \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

So, $\frac{d}{dx}[f(x)]=\begin{vmatrix} 3x^2 & \cos x & -\sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$

Again differentiating, we get

$$\frac{d^2}{dx^2}[f(x)]=\begin{vmatrix} 6x & -\sin x & -\cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix} + \text{Remaining two determinants as zero.}$$

Again differentiating, we get

$$\frac{d^3}{dx^3}[f(x)]=\begin{vmatrix} 6 & -\cos x & \sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix} + \text{Remaining two determinants as zero}$$

At $x=0$, $\frac{d^3}{dx^3}[f(x)]=\begin{vmatrix} 6 & -1 & 0 \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}=0$

$\therefore \quad \left(\frac{d^3}{dx^3}[f(x)]\right)_{\text{at } x=0} = 0$ (i.e. independent of p)

Example 64 *If* $y=\cos ax$, *prove that* $\begin{vmatrix} y & y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & y_8 \end{vmatrix}=0$, *where* $y_r=\frac{d^r}{dx^r}y$.

Sol. Given, $y=\cos(ax)$

Then, $y_1=-a\sin(ax)=a\cos\left(\frac{\pi}{2}+ax\right)$

$y_2=-a^2\cos(ax)=a^2\cos\left(\frac{2\pi}{2}+ax\right)$

$y_3=+a^3\sin(ax)=a^3\cos\left(\frac{3\pi}{2}+ax\right)$

.....

.....

$$y_n=a^n\cos\left(\frac{n\pi}{2}+ax\right)$$

$\therefore$ The determinant $\Delta(x)=\begin{vmatrix} y & y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & y_8 \end{vmatrix}$

$$\Delta(x)=\begin{vmatrix} \cos ax & a\cos\left(\frac{\pi}{2}+ax\right) & a^2\cos\left(\frac{2\pi}{2}+ax\right) \\ a^3\cos\left(\frac{3\pi}{2}+ax\right) & a^4\cos\left(\frac{4\pi}{2}+ax\right) & a^5\cos\left(\frac{5\pi}{2}+ax\right) \\ a^6\cos\left(\frac{6\pi}{2}+ax\right) & a^7\cos\left(\frac{7\pi}{2}+ax\right) & a^8\cos\left(\frac{8\pi}{2}+ax\right) \end{vmatrix}$$

$$\Delta(x)=a^3\times a^6\begin{vmatrix} \cos(ax) & -a\sin(ax) & -a^2\cos(ax) \\ \sin(ax) & a\cos(ax) & -a^2\sin(ax) \\ -\cos(ax) & a\sin(ax) & a^2\cos(ax) \end{vmatrix}$$

$$\Delta(x)=a^9\times(0)\{R_1\to -R_3\}$$

Hence, $\Delta(x)=0$

Example 65 If $f(x)=\begin{vmatrix} (1+x)^a & (1+2x)^b & 1 \\ 1 & (1+x)^a & (1+2x)^b \\ (1+2x)^b & 1 & (1+x)^a \end{vmatrix}$,

then find

(i) constant term (ii) coefficient of x.

Sol. Here, $f(x)=\begin{vmatrix} (1+x)^a & (1+2x)^b & 1 \\ 1 & (1+x)^a & (1+2x)^b \\ (1+2x)^b & 1 & (1+x)^a \end{vmatrix}$

$$=A+Bx+Cx^2+\ldots \qquad \ldots(i)$$

Putting $x=0$, we get

$$f(0)=\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}=A+B(0)+C(0)^2+\ldots$$

$\Rightarrow \quad A=0$

Again, differentiating Eq. (i) w.r.t. x, we get

$$f'(x)=\begin{vmatrix} a(1+x)^{a-1} & 2b(1+2x)^{b-1} & 0 \\ 1 & (1+x)^a & (1+2x)^b \\ (1+2x)^b & 1 & (1+x)^a \end{vmatrix} + \begin{vmatrix} (1+x)^a & (1+2x)^b & 1 \\ 0 & a(1+x)^{a-1} & 2b(1+2x)^{b-1} \\ (1+2x)^b & 1 & (1+x)^a \end{vmatrix} + \begin{vmatrix} (1+x)^a & (1+2x)^b & 1 \\ 1 & (1+x)^a & (1+2x)^b \\ 2b(1+2x)^{b-1} & 0 & a(1+x)^{a-1} \end{vmatrix} = B+2Cx+\ldots$$

$$f'(0)=\begin{vmatrix} a & 2b & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}+\begin{vmatrix} 1 & 1 & 1 \\ 0 & a & 2b \\ 1 & 1 & 1 \end{vmatrix}+\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2b & 0 & a \end{vmatrix}=B$$

$\Rightarrow \quad B=0$ ...(ii)

$\therefore$ Coefficient of constant term = coefficient of $x=0$.

Example 66 If $a_i, b_i \in N$ for $i=1,2,3$, then coefficient of x in the determinant;

$$\begin{vmatrix} (1+x)^{a_1b_1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3} \\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3} \\ (1+x)^{a_3b_1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3} \end{vmatrix}$$

Sol. Here,

$$\begin{vmatrix} (1+x)^{a_1b_1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3} \\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3} \\ (1+x)^{a_3b_1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3} \end{vmatrix}=A+Bx+Cx^2+\ldots$$

Differentiating both the sides w.r.t. x, we get

$$\begin{vmatrix} a_1b_1(1+x)^{a_1b_1-1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3} \\ a_2b_1(1+x)^{a_2b_1-1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3} \\ a_3b_1(1+x)^{a_3b_1-1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3} \end{vmatrix}$$

$$+\begin{vmatrix} (1+x)^{a_1b_1} & a_1b_2(1+x)^{a_1b_2-1} & (1+x)^{a_1b_3} \\ (1+x)^{a_2b_1} & a_2b_2(1+x)^{a_2b_2-1} & (1+x)^{a_2b_3} \\ (1+x)^{a_3b_1} & a_3b_2(1+x)^{a_3b_2-1} & (1+x)^{a_3b_3} \end{vmatrix}$$

$$+\begin{vmatrix} (1+x)^{a_1b_1} & (1+x)^{a_1b_2} & a_1b_3(1+x)^{a_1b_3-1} \\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & a_2b_3(1+x)^{a_2b_3-1} \\ (1+x)^{a_3b_1} & (1+x)^{a_3b_2} & a_3b_3(1+x)^{a_3b_3-1} \end{vmatrix}$$

$$=B+2Cx+\ldots$$

Putting $x=0$,

$$B=\begin{vmatrix} a_1b_1 & 1 & 1 \\ a_2b_1 & 1 & 1 \\ a_3b_1 & 1 & 1 \end{vmatrix}+\begin{vmatrix} 1 & a_1b_2 & 1 \\ 1 & a_2b_2 & 1 \\ 1 & a_3b_2 & 1 \end{vmatrix}+\begin{vmatrix} 1 & 1 & a_1b_3 \\ 1 & 1 & a_2b_3 \\ 1 & 1 & a_3b_3 \end{vmatrix}$$

$\Rightarrow \quad B=0$

Exercise for Session 9

1. If $f(x)$, $g(x)$ and $h(x)$ are three polynomials of degree 2, then prove that $\phi(x)=\begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}$ is a constant polynomial.

2. If f, g and h are differentiable functions of x and $\Delta=\begin{vmatrix} f & g & h \\ (xf)' & (xg)' & (xh)' \\ (x^2f)'' & (x^2g)'' & (x^2h)'' \end{vmatrix}$.

Prove that $\Delta'=\begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3f'')' & (x^3g'')' & (x^3h'')' \end{vmatrix}$.

3. Find $\frac{dy}{dx}$ at $x=-1$, when $(\sin y)^{\sin\frac{\pi x}{2}}+\frac{\sqrt{3}}{2}\sec^{-1}(2x)+2^x\tan\{\log_e(x+2)\}=0$.

Session 10

Derivative of an Inverse Function

Derivative of an Inverse Function

Theorem If the inverse functions f and g are defined by $y = f(x)$ and $x = g(y)$ and if $f'(x)$ exists and $f'(x) \neq 0$, then $g'(y) = \dfrac{1}{f'(x)}$. This result can also be written as, if $\dfrac{dy}{dx}$ exists and $\dfrac{dy}{dx} \neq 0$, then

$$\frac{dx}{dy} = \frac{1}{\dfrac{dy}{dx}} \quad \text{or} \quad \frac{dy}{dx}\cdot\frac{dx}{dy} = 1 \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}} \qquad \left[\because \frac{dx}{dy} \neq 0\right]$$

Example 67 If $y = f(x) = x^3 + x^5$ and g is the inverse of f, then find $g'(2)$ (means dx/dy when $y = 2$).

Sol. Here, $y = x^3 + x^5$

On differentiation, we get

$$\frac{dy}{dx} = 3x^2 + 5x^4 \Rightarrow g'(y) = \frac{dx}{dy} = \frac{1}{3x^2 + 5x^4}$$

when $y = 2$, then $2 = x^3 + x^5 \Rightarrow x = 1$

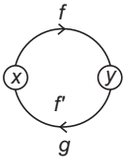
$$\therefore \quad g'(2) = \left.\frac{dx}{dy}\right]_{\substack{x=1\\y=2}} = \frac{1}{3+5} = \frac{1}{8}$$

Aliter $y = f(x),\ x = g(y)$

$$(gof)(x) = x$$

$$g'[f(x)]\, f'(x) = 1$$

when $f(x) = 2$, then $x = 1$



$$g'(2)\cdot f'(1) = 1 \qquad [\text{but } f'(1) = 8]$$

$$\therefore \quad g'(2) = 1/8$$

Example 68 Let $f(x) = \exp(x^3 + x^2 + x)$ for any real number x and let g be the inverse function for f. The value of $g'(e^3)$ is

(a) $\dfrac{1}{6e^3}$ (b) $\dfrac{1}{6}$ (c) $\dfrac{1}{34e^{39}}$ (d) 6

Sol. Let $y = e^{x^3 + x^2 + x}$,

On differentiating, $\dfrac{dy}{dx} = e^{x^3 + x^2 + x}\cdot(3x^2 + 2x + 1)$

$$\Rightarrow \quad g'(y) = \frac{dx}{dy} = \frac{1}{e^{x^3 + x^2 + x}\cdot[3x^2 + 2x + 1]}$$

$$y = e^3$$

$$\Rightarrow \quad e^3 = e^{x^3 + x^2 + x}$$

$$\Rightarrow \quad x^3 + x^2 + x - 3 = 0$$

$$\Rightarrow \quad (x-1)(x^2 + 2x + 3) = 0$$

$$\Rightarrow \quad x = 1 \qquad [\because x^2 + 2x + 3 > 0]$$

$$\therefore \quad g'(e^3) = \frac{1}{6e^3}$$

Hence, (*a*) is the correct option.

Exercise for Session 10

1. The function $f(x) = e^x + x$, being differentiable and one to one, has a differentiable increase $f^{-1}(x)$. Find $\dfrac{d}{dx}(f^{-1}(x))$ at point $f(\log 2)$.
2. Let $g(x)$ be the inverse of an invertible function $f(x)$ which is differentiable for all real x, then find $g''(f(x))$.
3. Let $g(x)$ be the inverse of an invertible function $f(x)$ which is differentiable at $x = c$, then $g'(f(c))$ equals

   (a) $f'(c)$ (b) $\dfrac{1}{f'(c)}$ (c) $f(c)$ (d) None of these

4. If $f(x) = x + \tan x$ and f is inverse of g, then $g'(x)$ equals

   (a) $\dfrac{1}{1+[g(x)-x]^2}$ (b) $\dfrac{1}{2-[g(x)-x]^2}$ (c) $\dfrac{1}{2+[g(x)-x]^2}$ (d) None of these

JEE Type Solved Examples :
Single Option Correct Type Questions

● **Ex. 1** *Let f be a twice differentiable function such that $f''(x) = -f(x)$ and $f'(x) = g(x)$. If $h(x) = \{f(x)\}^2 + \{g(x)\}^2$, where $h(5) = 11$. Find $h(10)$.*

(a) 0 (b) 9
(c) 11 (d) None of these

Sol. (*c*) Given, $h(x) = \{f(x)\}^2 + \{g(x)\}^2$

On differentiating both the sides w.r.t. x, we get

$h'(x) = 2f(x) \cdot f'(x) + 2g(x) \cdot g'(x)$ ...(i)

Now, $f'(x) = g(x)$.

Then, $f''(x) = g'(x)$

$\Rightarrow$ $-f(x) = g'(x)$ ...(ii)

$[\because f''(x) = -f(x),$ given$]$

From Eqs. (i) and (ii), we get

$h'(x) = 2f(x) \cdot g(x) + 2g(x) \cdot \{-f(x)\}$

[using $f'(x) = g(x)$ and $g'(x) = -f(x)$]

$\therefore$ $h'(x) = 0$

So, $h(x)$ must be constant. $\left[\because \frac{d}{dx}(\text{constant}) = 0\right]$

But $h(5) = 11$

$h(x) = 11$

So, $h(10) = 11$

● **Ex. 2** *Let $f(x)$ be a real valued differentiable function not identically zero such that $f(x + y^{2n+1}) = f(x) + \{f(y)\}^{2n+1}, n \in N$ and x, y are any real numbers and $f'(0) \geq 0$. Find the values of $f(5)$ and $f'(10)$.*

(a) 3, 2 (b) 0, 1
(c) 1, 5 (d) 5, 1

Sol. (*d*) Here, $f(x + y^{2n+1}) = f(x) + \{f(y)\}^{2n+1}$ ...(i)

Putting $x = 0$, $y = 0$, we get

$f(0) = f(0) + \{f(0)\}^{2n+1} \Rightarrow f(0) = 0$

$f'(0) \geq 0$ [given]

$\Rightarrow$ $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} \geq 0 \Rightarrow \lim_{x \to 0} \frac{f(x)}{x} \geq 0$

Now, if $x > 0 \Rightarrow f(x) \geq 0$ ...(ii)

Putting $x = 0$, $y = 1$ in Eq. (i), we get

$f(1) = f(0) + \{f(1)\}^{2n+1}$ or $f(1)[1 - \{f(1)\}^{2n}] = 0$

$\therefore$ $f(1) = 0$ or 1 [using Eq. (ii)]

Putting $y = 1$ in Eq. (i), for all real x.

$f(x + 1) = f(x) + \{f(1)\}^{2n+1}$ ...(iii)

Now, two cases arise either $f(1) = 0$ or 1

***Case* I** If $f(1) = 0$

$\Rightarrow$ $f(x + 1) = f(x)$ [using Eq. (iii)]

$\Rightarrow$ $f(1) = f(2) = f(3) = ... = 0$

$\therefore$ $f(x)$ is identically zero. (which is not possible)

***Case* II** If $f(1) = 1$

$\Rightarrow$ $f(x + 1) = f(x) + 1$ [using Eq. (iii)]

$\therefore$ $f(2) = f(1) + 1 = 1 + 1 = 2$

$f(3) = f(2) + 1 = 2 + 1 = 3$

$f(4) = f(3) + 1 = 3 + 1 = 4$

$f(5) = f(4) + 1 = 4 + 1 = 5$

Proceeding in same way, we get

$f(x) = x$ and $f'(x) = 1 \Rightarrow f'(10) = 1$

So, $f(5) = 5$ and $f'(10) = 1$

● **Ex. 3** *Let $f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}$ and $f'(0) = a$ and $f(0) = b$. Find $f''(x)$ (where y is independent of x), when $f(x)$ is differentiable.*

(a) 0 (b) 1
(c) a (d) None of these

Sol. (*a*) Here $f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}$, this holds for any real x, y and y is independent of x. i.e. $\frac{dy}{dx} = 0$

On differentiating w.r.t. x, we get

$$f'\left(\frac{x+y}{2}\right) \cdot \frac{1}{2}\left(1 + \frac{dy}{dx}\right) = \frac{1}{2}\left\{f'(x) + f'(y) \cdot \frac{dy}{dx}\right\}$$

$\therefore$ $\frac{1}{2} f'\left(\frac{x+y}{2}\right) = \frac{1}{2} f'(x)$ $\left[\text{as } \frac{dy}{dx} = 0\right]$

$\Rightarrow$ $f'\left(\frac{x+y}{2}\right) = f'(x)$ ...(i)

Taking $x = 0$ and $y = x$ in Eq. (i), we get

$f'\left(\frac{0+x}{2}\right) = f'(0) \Rightarrow f'\left(\frac{x}{2}\right) = a$ $[\because f'(0) = a,$ given$]$

which shows $f\left(\frac{x}{2}\right) = a\left(\frac{x}{2}\right) + c$ [using integration]

$\therefore$ $f(x) = ax + c$

Let $x = 0$

$\Rightarrow$ $f(0) = b = c$

$\therefore$ $f(x) = ax + b$

On differentiating,

$f'(x) = a$ and $f''(x) = 0$

● **Ex. 4** *If* $y=\dfrac{(a-x)\sqrt{a-x}-(b-x)\sqrt{x-b}}{\sqrt{a-x}+\sqrt{x-b}}$, *then find* $\dfrac{dy}{dx}$ *whenever defined.*

(a) $\dfrac{x-(a+b)}{\sqrt{(a-x)(x-b)}}$ (b) $\dfrac{2x+(a+b)}{\sqrt{a-x}(x-b)}$

(c) $\dfrac{2x-(a+b)}{2\sqrt{(a-x)(x-b)}}$ (d) None of these

Sol. (*c*) Here, $y=\dfrac{(a-x)^{3/2}+(x-b)^{3/2}}{(a-x)^{1/2}+(x-b)^{1/2}}$, use

$a^3+b^3=(a+b)(a^2-ab+b^2)$

$$\therefore \quad y=\frac{\left[\begin{array}{l}\{(a-x)^{1/2}+(x-b)^{1/2}\}\{(a-x)\\ \quad -\sqrt{(a-x)(x-b)}+(x-b)\}\end{array}\right]}{\{(a-x)^{1/2}+(x-b)^{1/2}\}}$$

$$\Rightarrow \quad y=(a-b)-\sqrt{(ax+bx-ab-x^2)}$$

$$\therefore \quad \frac{dy}{dx}=-\frac{(a+b-2x)}{2\sqrt{(a-x)(x-b)}}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{2x-(a+b)}{2\sqrt{(a-x)(x-b)}}$$

● **Ex. 5** *If* $x^2+y^2=R^2\ (R>0)$, *then* $K=\dfrac{y''}{\sqrt{(1+(y')^2)^3}}$. *Find* K *in terms of* R.

(a) $\dfrac{1}{R}$ (b) $-\dfrac{1}{R}$

(c) R (d) $\dfrac{1}{2R}$

Sol. (*b*) Here, $x^2+y^2=R^2$

On differentiating w.r.t. x, we get

$$2x+2yy'=0 \Rightarrow x+yy'=0$$

$$\Rightarrow \quad y'=-\frac{x}{y} \qquad \ldots(i)$$

Again, differentiating both the sides,

$$1+yy''+(y')^2=0$$

$$\therefore \quad y''=-\frac{(1+(y')^2)}{y} \qquad \ldots(ii)$$

Now, $K=\dfrac{y''}{\sqrt{(1+(y')^2)^3}}=-\dfrac{(1+(y')^2)}{y\cdot(1+(y')^2)^{3/2}}$

$$=-\frac{1}{y\sqrt{1+(y')^2}}=-\frac{1}{y\sqrt{1+\dfrac{x^2}{y^2}}} \qquad \text{[from Eq. (i)]}$$

$$=-\frac{1}{\sqrt{x^2+y^2}}=-\frac{1}{R}$$

$$\therefore \quad K=-\frac{1}{R}$$

● **Ex. 6** *Let* $f(x)=x+\sin x$, *suppose* g *denotes the inverse function of* f. *Then, find the value of* $g'\left(\dfrac{\pi}{4}+\dfrac{1}{\sqrt{2}}\right)$.

(a) $2+\sqrt{2}$ (b) $\sqrt{2}-2$

(c) $2-\sqrt{2}$ (d) $2\sqrt{2}$

Sol. (*c*) Here, $f(x)=x+\sin x$

$$\therefore \quad \frac{dy}{dx}=1+\cos x \Rightarrow g'(y)=\frac{dx}{dy}=\frac{1}{1+\cos x}$$

where, $y=\dfrac{\pi}{4}+\dfrac{1}{\sqrt{2}}=x+\sin x \Rightarrow x=\dfrac{\pi}{4}$

$$\therefore \quad g'\left(\frac{\pi}{4}+\frac{1}{\sqrt{2}}\right)=\frac{1}{1+\dfrac{1}{\sqrt{2}}}=\frac{\sqrt{2}}{\sqrt{2}+1}=2-\sqrt{2}$$

● **Ex. 7** *Let* $e^{f(x)}=\log x$. *If* $g(x)$ *is the inverse of* $f(x)$, *then find* $g'(x)$.

(a) e^{e^x} (b) e^x (c) e^{e^x+x} (d) None of these

Sol. (*c*) Let $f(x)=y \Rightarrow x=f^{-1}(y)=g(y)$

$$\Rightarrow \quad x=e^{e^y} \Rightarrow \frac{dx}{dy}=e^{e^y}\cdot e^y=e^{e^y+y}=g'(y)$$

$$\therefore \quad g'(x)=e^{e^x+x}$$

● **Ex. 8** *If* $f(x)=4x^3-6x^2\cos 2a+3x\cdot\sin 2a\cdot\sin 6a+\sqrt{\log(2a-a^2)}$, *then*

(a) $f'\left(\dfrac{1}{2}\right)<0$ (b) $f'\left(\dfrac{1}{2}\right)\le 0$

(c) $f'\left(\dfrac{1}{2}\right)>0$ (d) None of these

Sol. (*c*) Here, $f(x)=4x^3-6x^2\cos 2a+3x\sin 2a\cdot\sin 6a+\sqrt{\log(2a-a^2)}$ for $f(x)$ to exists.

$$\log(2a-a^2)\ge 0 \Rightarrow (2a-a^2)\ge e^0$$

i.e. $2a-a^2\ge 1$ or $a^2-2a+1\le 0 \Rightarrow (a-1)^2\le 0$

which is only possible, if $(a-1)^2=0$, i.e. $a=1$.

$$\therefore \quad f(x)=4x^3-6x^2\cos 2+3x\sin 2\cdot\sin 6$$

$$\Rightarrow \quad f'(x)=12x^2-12x\cos 2+3\sin 2\sin 6$$

$$\Rightarrow \quad f'\left(\frac{1}{2}\right)=3-6\cos 2+3\sin 2\sin 6$$

$$=3(1+\sin 2\sin 6)-6\cos 2 \Rightarrow f'\left(\frac{1}{2}\right)>0$$

Note

$\cos 2<0$ and $1+\sin 2\sin 6>0$

● **Ex. 9** *Suppose,* $f(x) = e^{ax} + e^{bx}$, *where* $a \neq b$ *and* $f''(x) - 2f'(x) - 15f(x) = 0$ *for all* $x \in R$. *Then, find* ab.

(a) 15 (b) −15 (c) 10 (d) 16

Sol. (*b*) $f(x) = e^{ax} + e^{bx}$

$\Rightarrow$ $f'(x) = ae^{ax} + be^{bx}$, $f''(x) = a^2e^{ax} + b^2e^{bx}$

$\therefore$ $f''(x) - 2f'(x) - 15f(x) = 0$

$\Rightarrow$ $\{a^2e^{ax} + b^2e^{bx}\} - 2\{ae^{ax} + be^{bx}\} - 15\{e^{ax} + e^{bx}\} = 0$, for all x.

$\Rightarrow$ $(a^2 - 2a - 15)e^{ax} + (b^2 - 2b - 15)e^{bx} = 0$, for all x.

$\Rightarrow$ $a^2 - 2a - 15 = 0$ and $b^2 - 2b - 15 = 0$

$\Rightarrow$ $(a - 5)(a + 3) = 0$

and $(b - 5)(b + 3) = 0$

$\Rightarrow$ $a = 5$ or -3

and $b = 5$ or -3

But $a \neq b$, hence, $a = 5, b = -3$

or $a = -3, b = 5 \Rightarrow ab = -15$

JEE Type Solved Examples :
More than One Correct Option Type Questions

● **Ex. 10** *The functions* $u = e^x \cdot \sin x$ *and* $v = e^x \cdot \cos x$ *satisfy the equation*

(a) $v\frac{du}{dx} - u\frac{dv}{dx} = u^2 + v^2$ (b) $v\frac{du}{dx} + u\frac{dv}{dx} = u^2 + v^2$

(c) $\frac{du}{dx} + \frac{dv}{dx} = 2v$ (d) $\frac{du}{dx} + \frac{dv}{dx} = 2u$

Sol. (*a, c*) We have, $u = e^x \sin x$, $v = e^x \cos x$

Differentiating both the equations w.r.t. x, we get

$$\frac{du}{dx} = e^x(\sin x + \cos x) \quad ...(i)$$

and $$\frac{dv}{dx} = e^x(\cos x - \sin x) \quad ...(ii)$$

Now, $\frac{du}{dx} + \frac{dv}{dx} = 2e^x \cos x$ [using Eqs. (i) and (ii)]

$\Rightarrow$ $\frac{du}{dx} + \frac{dv}{dx} = 2v$

Also, $\frac{du}{dx} = u + v$ and $\frac{dv}{dx} = v - u$

$\therefore$ $v\frac{du}{dx} = uv + v^2$ and $u\frac{dv}{dx} = uv - u^2$

$\Rightarrow$ $v\frac{du}{dx} - u\frac{dv}{dx} = u^2 + v^2$

● **Ex. 11** *If* $y + \log(1 + x) = 0$, *then which of the following is true?*

(a) $e^y = xy' + 1$ (b) $y' = -\frac{1}{(x + 1)}$

(c) $y' + e^y = 0$ (d) $y' = e^y$

Sol. (*a, b, c*) We have, $y + \log_e(1 + x) = 0$

$\Rightarrow$ $y = -\log(1 + x) \Rightarrow y = \log(1 + x)^{-1}$

$\Rightarrow$ $e^y = \frac{1}{1 + x}$ ...(i)

$\Rightarrow$ $xe^y + e^y = 1$

On differentiating w.r.t. x, we get

$xe^y y' + e^y + e^y y' = 0$

$xy' + y' + 1 = 0$ [$\because e^y \neq 0$] ...(ii)

$\Rightarrow$ $y' = -\frac{1}{x + 1}$

$\Rightarrow$ $y' = -e^y$ [from Eq. (i)]

$\Rightarrow$ $y' + e^y = 0$

From Eq. (ii), we have

$xy' + 1 = -y' \Rightarrow xy' + 1 = e^y$

● **Ex. 12** *If* $x^p \cdot y^q = (x + y)^{p+q}$, *then* $\frac{dy}{dx}$ *is*

(a) independent of p

(b) independent of q

(c) dependent both p and q

(d) $\frac{y}{x}$

Sol. (*a, b, d*) We have, $x^p \cdot y^q = (x + y)^{p+q}$

On taking log both sides, we get

$p\log x + q\log y = (p + q)\log(x + y)$

On differentiating w.r.t. x, we get

$$\frac{p}{x} + \frac{q}{y}\frac{dy}{dx} = \frac{p + q}{x + y}\left(1 + \frac{dy}{dx}\right)$$

$\Rightarrow$ $\frac{dy}{dx}\left(\frac{q}{y} - \frac{p + q}{x + y}\right) = \frac{p + q}{x + y} - \frac{p}{x}$

$\Rightarrow$ $\frac{dy}{dx} = \frac{y}{x}$

● **Ex. 13** *Two functions* f *and* g *have first and second derivatives at* $x = 0$ *and satisfy the relations* $f(0) = \frac{2}{g(0)}$, $f'(0) = 2g'(0) = 4g(0)$, $g''(0) = 5f''(0) = 6f(0) = 3$ *Then,*

(a) If $h(x)=\dfrac{f(x)}{g(x)}$, then $h'(0)=\dfrac{15}{4}$

(b) If $k(x)=f(x)\cdot g(x)\cdot \sin x$, then $k'(0)=2$

(c) $\lim\limits_{x\to 0}\dfrac{g'(x)}{f'(x)}=\dfrac{1}{2}$

(d) None of the above

Sol. (a, b, c) We have, $f(0)=\dfrac{2}{g(0)}$,

$$f'(0)=2g'(0)=4g(0)$$
$$g''(0)=5f''(0)=6f(0)=3$$

Now, on solving these equations, we get

$$f(0)=\frac{1}{2},\ g(0)=4,\ f'(0)=16,\ g'(0)=8$$
$$f''(0)=\frac{3}{5},\ g''(0)=3$$

(a) We have, $h(x)=\dfrac{f(x)}{g(x)}$,

$$h'(x)=\frac{g(x)f'(x)-g'(x)f'(x)}{[g(x)]^2}$$
$$\therefore\quad h'(0)=\frac{g(0)f'(0)-g'(0)f'(0)}{[g(0)]^2}$$
$$=\frac{4\times 16-8\times\frac{1}{2}}{(4)^2}=\frac{15}{4}$$

(b) $k(x)=f(x)\cdot g(x)\cdot\sin x$

$k'(x)=f(x)g(x)\cos x+f(x)\sin x\cdot g'(x)+g(x)\sin x f'(x)$

$k'(0)=f(0)g(0)\cos 0+f(0)\sin 0\,g'(0)+g(0)\sin 0\,f'(0)$

$$k'(0)=\frac{1}{2}\times 4\times 1+0+0=2$$

(c) $\lim\limits_{x\to 0}\dfrac{g'(x)}{f'(x)}=\dfrac{g'(0)}{f'(0)}=\dfrac{1}{2}$

● ***Ex. 14*** *If $f(x)=|x^2-3|x|+2|$, then which of the following is/are true?*

(a) $f'(x)=2x-3$ for $x\in(0,1)\cup(2,\infty)$

(b) $f'(x)=2x+3$ for $x\in(-\infty,-2)\cup(-1,0)$

(c) $f'(x)=-2x-3$ for $x\in(-2,-1)$

(d) None of the above

Sol. $(a,\ b,\ c)$ We have,

$$f(x)=|x^2-3|x|+2|=\begin{cases}|x^2-3x+2|, & x\ge 0\\ |x^2+3x+2|, & x<0\end{cases}$$

$$\Rightarrow\quad f(x)=\begin{cases}x^2-3x+2, & x^2-3x+2\ge 0,\quad x\ge 0\\ -x^2+3x-2, & x^2-3x+2<0,\quad x\ge 0\\ x^2+3x+2, & x^2+3x+2\ge 0,\quad x<0\\ -x^2-3x-2, & x^2+3x+2<0,\quad x<0\end{cases}$$

$$\Rightarrow\quad f(x)=\begin{cases}x^2-3x+2, & x\in[0,1]\cup(2,\infty)\\ -x^2+3x+2, & x\in(1,2)\\ x^2+3x+2, & x\in(-\infty,-2]\cup[-1,0)\\ -x^2-3x+2, & x\in(-2,-1)\end{cases}$$

$$\Rightarrow\quad f'(x)=\begin{cases}2x-3, & x\in(0,1)\cup(2,\infty)\\ -2x+3, & x\in(1,2)\\ 2x+3, & x\in(-\infty,-2)\cup(-1,0)\\ -2x-3, & x\in(-2,-1)\end{cases}$$

JEE Type Solved Examples :
Statements I and II Type Questions

■ **Directions** (Ex. Nos. 15-17) *This section is based on Statement I and Statement II. Select the correct answer from the codes given below.*

(a) Both Statement I and Statement II are correct and Statement II is the correct explanation of Statement I

(b) Both Statement I and Statement II are correct but Statement II is not the correct explanation of Statement I

(c) Statement I is correct but Statement II is incorrect

(d) Statement II is correct but Statement I is incorrect

● ***Ex. 15*** *Consider* $f(x)=\dfrac{x}{x^2-1}$ *and* $g(x)=f''(x)$

Statement I *Graph of $g(x)$ is concave up for $x>1$.*

Statement II

$$\frac{d^n}{dx^n}f(x)=\frac{(-1)^n n!}{2}\left\{\frac{1}{(x+1)^{n+1}}+\frac{1}{(x-1)^{n+1}}\right\}\ n\in N$$

Sol. (b) We have, $f(x)=\dfrac{x}{x^2-1}\ \Rightarrow\ f(x)=\dfrac{x}{(x+1)(x-1)}$

$$\Rightarrow\quad f(x)=\frac{1}{2}\left[\frac{1}{x+1}+\frac{1}{x-1}\right]$$
$$\Rightarrow\quad f'(x)=\frac{1}{2}\left[-\frac{1}{(x+1)^2}-\frac{1}{(x-1)^2}\right]$$
$$\Rightarrow\quad f''(x)=\frac{(-1)(-2)}{2}\left[\frac{1}{(x+1)^3}+\frac{1}{(x-1)^3}\right]$$
$$\vdots\qquad\qquad\vdots$$
$$\Rightarrow\quad f^n(x)=\frac{(-1)^n n!}{2}\left[\frac{1}{(x+1)^{n+1}}+\frac{1}{(x-1)^{n+1}}\right]$$

Now, $g(x)=f''(x)$

$g'(x)=f'''(x)$

$\Rightarrow\quad g''(x)=f''''(x)$

$$\Rightarrow \quad g''(x) = \frac{(-1)^4 4!}{2}\left[\frac{1}{(x+1)^5} + \frac{1}{(x-1)^5}\right]$$

$$\Rightarrow \quad g''(x) = \frac{4!}{2}\left[\frac{1}{(x+1)^5} + \frac{1}{(x-1)^5}\right]$$

$$\therefore \quad g''(1) > 0$$

$\therefore$ Graph of $g(x)$ is concave up for $x > 1$.

Hence, both statements are correct but Statement II is not the correct explanation of Statement I.

● ***Ex. 16*** **Statement I** *If differentiable function $f(x)$ satisfies the relation $f(x) + f(x-2) = 0, \forall\ x \in R$, and if $\left(\frac{d}{dx} f(x)\right)_{x=a} = 6$, then $\left(\frac{d}{dx} f(x)\right)_{a+4000} = 6$*

Statement II *$f(x)$ is a periodic function with period 4.*

Sol. (a) We have, $f(x) + f(x-2) = 0$ ...(i)

Replace x by $x-2$ in Eq. (i), we have

$$f(x-2) + f(x-4) = 0 \quad \text{...(ii)}$$

From Eqs. (i) and (ii),

$$f(x) - f(x-4) = 0 \quad \text{...(iii)}$$

Replace x by $x+4$ in Eq. (iii), we get

$$f(x+4) - f(x) = 0$$

$$\Rightarrow \quad f(x+4) = f(x)$$

$$\Rightarrow \quad f(x) = f(x+4) = f(x+8) \ldots f(x+4000)$$

$$\Rightarrow \quad f'(x) = f'(x+4000)$$

Hence, both the statements are true and Statement II is the correct explanation of Statement I.

● ***Ex. 17*** **Statement I** *Let $f(x) = x[x]$ and $[\cdot]$ denotes greatest integer function, when x is not an integral, then rule for $f'(x)$ is given by $[x]$.*

Statement II *$f'(x)$ does not exist for any $x \in Z$.*

Sol. (a) $$f(x) = x[x] = \begin{cases} -x, & -1 \le x < 0 \\ 0, & 0 \le x < 1 \\ x, & 1 \le x < 2 \\ 2x, & 2 \le x < 3 \\ \vdots & \vdots \end{cases}$$

$$\Rightarrow \quad f'(x) = \begin{cases} -1, & -1 < x < 0 \\ 0, & 0 < x < 1 \\ 1, & 1 < x < 2 \\ 2, & 2 < x < 3 \\ \vdots & \vdots \end{cases}$$

$$\Rightarrow \quad f'(x) = [x]$$

Hence, both statements are true and Statement II is the correct explanation of Statement I.

JEE Type Solved Examples : Passage Based Questions

Passage I

(Ex. Nos. 18 to 20)

A non-zero polynomial with real coefficients has the property that $f''(x) \cdot f'(x) = f(x)$. Then,

● ***Ex. 18*** *The leading coefficient of $f(x)$, is*

(a) 1/6 (b) 1/9
(c) 1/12 (d) 1/18

● ***Ex. 19*** *The degree of $f(x)$, is*

(a) 2 (b) 3
(c) 4 (d) 5

● ***Ex. 20*** *The value of $f'''(x)$, is*

(a) 1/18 (b) 1/3
(c) 1/9 (d) 1/6

Sol. (Ex. Nos. 18 to 20)

Let degree of $f(x) = n$

$\therefore$ Degree of $f'(x) = n-1$, degree of $f''(x) = n-2$

Since, $f''(x) \cdot f'(x) = f(x)$

$$\Rightarrow \quad (n-1) + (n-2) = n$$

$$\Rightarrow \quad 2n - 3 = n \quad \text{or} \quad n = 3$$

$\therefore$ Degree of $f(x) = 3$

Hence, $f(x) = ax^3 + bx^2 + cx + d$

$$\Rightarrow \quad f'(x) = 3ax^2 + 2bx + c$$

$$\Rightarrow \quad f''(x) = 6ax + 2b$$

$$f'''(x) = 6a$$

As, $f''(x) \cdot f'(x) = f(x)$

$$\Rightarrow \quad (6ax + 2b)(3ax^2 + 2bx + c) = (ax^3 + bx^2 + cx + d)$$

On comparing coefficients of x^3, we get

$$18a^2 = a \ \Rightarrow\ a = 0, \frac{1}{18}$$

$$\therefore \quad a = \frac{1}{18} \qquad [\because a \neq 0]$$

The leading coefficient of $f(x) = \frac{1}{18}$ and $f'''(x) = 6a = \frac{1}{3}$.

18. (d) **19.** (b) **20.** (b)

Passage II

(Ex. Nos. 21 to 22)

The ends A and B of a rod of length $\sqrt{5}$ are sliding along the curve $y = 2x^2$. Let x_A and x_B be the x-coordinate of the ends. Then,

● ***Ex. 21*** *The moment when A is at $(0, 0)$ and B is at $(1, 2)$. The derivative $\frac{dx_B}{dx_A}$, is*

(a) $\frac{1}{9}$ (b) $\frac{1}{7}$

(c) $\frac{1}{5}$ (d) None of these

● ***Ex. 22*** *The moment when A is at $(1, 2)$ and B is at $(0, 0)$. The derivative $\frac{dx_B}{dx_A}$, is*

(a) 16 (b) 8 (c) 9 (d) 2

Sol. (Ex. Nos. 21 to 22)

We have, $y = 2x^2$

$$(AB)^2 = (x_B - x_A)^2 + (2x_B^2 - 2x_A^2)^2 = 5$$

$$\Rightarrow \quad (x_B - x_A)^2 + 4(x_B^2 - x_A^2)^2 = 5$$

Differentiating w.r.t. x_A and denoting $\frac{dx_B}{dx_A} = D$

$$2(x_B - x_A)(D - 1) + 8(x_B^2 - x_A^2) \cdot \{2x_B D - 2x_A\} = 0 \quad \ldots(i)$$

Now, when $A(0, 0)$ and $B(1, 2) \Rightarrow x_A = 0,\ x_B = 1$

Then, Eq. (i) reduces to

$$2(1 - 0)(D - 1) + 8(1 - 0)\{2D - 0\} = 0$$

$$\Rightarrow \quad 2D - 2 + 16D = 0$$

$$\Rightarrow \quad D = \frac{1}{9} \Rightarrow \frac{dx_B}{dx_A} = \frac{1}{9}$$

Again, when $A(1, 2)$ and $B(0, 0) \Rightarrow x_A = 1,\ x_B = 0$

Eq. (i) reduces to

$$2(0 - 1)(D - 1) + 8(0 - 1)\{0 - 2\} = 0$$

$$\Rightarrow \quad -2D + 2 + 16 = 0 \Rightarrow D = 9$$

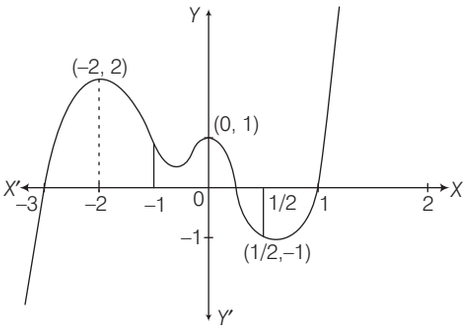
$$\therefore \quad \frac{dx_B}{dx_A} = 9$$

21. (a) **22.** (c)

Passage III

(Ex. Nos. 23 to 24)

The figure of $y = P(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$ is given below



● ***Ex. 23*** *If $P''(x)$ has real roots α, β, γ. Then, $[\alpha] + [\beta] + [\gamma]$ is*

(a) -2 (b) -3

(c) -1 (d) 0

Sol. (*b*) Here, $P'(x) = 0$ at $x = -2, -1, 0, \frac{1}{2}$.

$\Rightarrow$ $P''(x)$ will have real roots belongs to $(-2,-1), (-1,0), \left(0, \frac{1}{2}\right)$.

[$\because$ between any two roots of a function, atleast one root of its derivative exist]

$\therefore \quad [\alpha] + [\beta] + [\gamma] = -3$

● ***Ex. 24*** *The minimum number of real roots of equation $(P''(x))^2 + P'(x) \cdot P'''(x) = 0$*

(a) 5

(b) 7

(c) 6

(d) 4

Sol. (*c*) Let $f(x) = (P''(x))^2 + P'(x) \cdot P'''(x)$

$$\Rightarrow \quad f(x) = \frac{d}{dx}(P'(x) \cdot P''(x))$$

Since, $P'(x)$ has 4 real roots.

So, $P''(x)$ has 3 real roots.

$\Rightarrow f(x)$ has 6 real roots.

JEE Type Solved Examples :
Matching Type Questions

● ***Ex. 25*** *Match the Column I with Column II and mark the correct option from the codes given below.*

	Column I	Column II	
(i)	If $f'(x) = \sqrt{3x^2 + 6}$ and $y = f(x^3)$ then at $x = 1$, $\frac{dy}{dx}$ is	p.	-2
(ii)	If f is a differentiable function such that $f(xy) = f(x) + f(y) : x, y \in R$, then $f(e) + f(1/e)$ is	q.	-1
(iii)	If f is a twice differentiable function such that $f''(x) = -f(x)$ and $f'(x) = g(x)$. If $h(x) = [f(x)]^2 + [g(x)]^2$ and $h(5) = 9$, then $h(10)$ is	r.	0
(iv)	$y = \tan^{-1}(\cot x) + \cot^{-1}(\tan x)$, $\frac{\pi}{2} < x < \pi$, then $\frac{dy}{dx}$ is	s.	9

Codes

	i	ii	iii	iv		i	ii	iii	iv
(a)	s	r	s	p	(b)	p	q	r	s
(c)	q	p	r	r	(d)	s	p	q	q

Sol. (*a*) (i) $\because$ $y = f(x^3)$

$\therefore$ $\frac{dy}{dx} = f'(x^3) \cdot 3x^2$

$\left(\frac{dy}{dx}\right)_{x=1} = f'(1) \cdot 3 = 9$

(ii) $\because$ $f(xy) = f(x) + f(y)$

Put $x = y = 1$,

$f(1) = f(1) + f(1)$

$\therefore$ $f(1) = 0$

Also, $f(1) = f(e) + f(1/e)$

i.e. $x = e, y = \frac{1}{e}$

$\therefore$ $f(e) + f(1/e) = 0$

(iii) $\because$ $f''(x) = -f(x)$ and $g(x) = f'(x)$

$\therefore$ $g'(x) = f''(x) = -f(x)$ ...(i)

Also, $h(x) = [f(x)]^2 + [g(x)]^2$

$\therefore$ $h'(x) = 2f(x)f'(x) + 2g(x)g'(x)$

$= 2f(x)g(x) - 2g(x)f(x) = 0$

$\therefore$ $h(x) = c, \forall\ x \in R$

Here, $h(5) = 9 \implies h(10) = 9$

(iv) $\because$ $y = \tan^{-1}(\cot x) + \cot^{-1}(\tan x), \frac{\pi}{2} < x < \pi$

$\therefore$ $\frac{dy}{dx} = \frac{-\operatorname{cosec}^2 x}{1 + \cot^2 x} + \frac{(-1)\sec^2 x}{1 + \tan^2 x} = -1 - 1 = -2$

Hence, (i) $\to$ s; (ii) $\to$ r; (iii) $\to$ s; (iv) $\to$ p

● ***Ex. 26*** *Match the Column I with Column II and mark the correct option from the codes given below.*

	Column I	Column II	
(i)	If $y = \cos^{-1}(\cos x)$, then y' at $x = 5$ is equal to	p.	-1
(ii)	For the function $f(x) = \log_e \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$, if $\frac{dy}{dx} = \sec x + p$, then p is equal to	q.	0
(iii)	The derivative of $\tan^{-1}\left(\frac{1+x}{1-x}\right)$ at $x = -1$ is	r.	1/2
(iv)	The derivative of $\frac{\log \lvert x \rvert}{x}$ at $x = -1$ is	s.	1

Codes

	i	ii	iii	iv		i	ii	iii	iv
(a)	p	q	r	s	(b)	q	p	r	s
(c)	s	r	q	p	(d)	r	s	p	q

Sol. (*a*) (i) We have, $y = \cos^{-1}(\cos x)$

$$y = \begin{cases} x, & 0 \le x \le \pi \\ 2\pi - x, & \pi \le x \le 2\pi \end{cases} \implies \frac{dy}{dx} = \begin{cases} 1, & 0 \le x \le \pi \\ -1, & \pi \le x \le 2\pi \end{cases}$$

$\because$ $5 \in [\pi, 2\pi]$ $\therefore$ $\left(\frac{dy}{dx}\right)_{x=5} = -1$

(ii) We have, $f(x) = \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$

$$f'(x) = \frac{1}{\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)} \times \sec^2\left(\frac{\pi}{4} + \frac{x}{2}\right) \times \frac{1}{2}$$

$$f'(x) = \frac{1}{2\sin\left(\frac{\pi}{4} + \frac{x}{2}\right)\cos\left(\frac{\pi}{4} + \frac{x}{2}\right)}$$

$$f'(x) = \frac{1}{\sin\left(\frac{\pi}{2} + x\right)} = \frac{1}{\cos x} = \sec x$$

$\therefore$ $p = 0$

(iii) Let $y = \tan^{-1}\left(\frac{1+x}{1-x}\right) \implies y = \tan^{-1}(1) + \tan^{-1} x$

$\therefore$ $\frac{dy}{dx} = \frac{1}{1 + x^2} \implies \left(\frac{dy}{dx}\right)_{x=-1} = \frac{1}{2}$

(iv) Let $y = \frac{\log |x|}{x} \implies \frac{dy}{dx} = \frac{-\log |x|}{x^2} + \frac{1}{x^2}$

$\left(\frac{dy}{dx}\right)_{x=-1} = \frac{-\log 1}{1} + \frac{1}{1} = 1$

Hence, (i) $\to$ p; (ii) $\to$ q; (iii) $\to$ r; (iv) $\to$ s

JEE Type Solved Examples :
Single Integer Answer Type Questions

● **Ex. 27** *If* $y=\sqrt{(x-\sin x)+\sqrt{(x-\sin x)+\ldots}}$, *then* $\left|\left|\frac{dx}{dy}\right|^2_{x=\frac{\pi}{2}}-2\pi\right|=$ ..... .

Sol. (3) Here, $y=\sqrt{(x-\sin x)+\sqrt{(x-\sin x)+....\infty}}$

So, $y=\sqrt{(x-\sin x)+y}$

$\therefore$ $y^2=x-\sin x+y$

Differentiating, we get

$$2y\frac{dy}{dx}=1-\cos x+\frac{dy}{dx} \qquad ...(i)$$

At $x=\frac{\pi}{2}$, $y^2-y=\frac{\pi}{2}-1 \Rightarrow y^2-y+\frac{1}{4}=\frac{\pi}{2}-\frac{3}{4}$

$\Rightarrow$ $\left(y-\frac{1}{2}\right)^2=\frac{2\pi-3}{4} \Rightarrow (2y-1)=\pm\sqrt{2\pi-3}$

From Eq. (i), we get $(2y-1)\frac{dy}{dx}=1-\cos x$

$\therefore$ $\left|\frac{dy}{dx}\right|_{x=\frac{\pi}{2}}=\frac{1}{\sqrt{2\pi-3}} \Rightarrow \left|\left|\frac{dx}{dy}\right|^2_{x=\frac{\pi}{2}}-2\pi\right|=3$

● **Ex. 28** *If* $f(x)=(x-a)(x-b)$ *for* $a,b\in R$. *Then, the minimum number of roots of equation* $\pi(f'(x))^2\cos(\pi f(x))+\sin(\pi f(x))f''(x)=0$ *in* (α,β), *where* $f(\alpha)=3=f(\beta)$, *is* ..... (*where* $\alpha<a<b<\beta$).

Sol. (4) Let $g(x)=f'(x)\sin(\pi f(x))=2\left[x-\left(\frac{a+b}{2}\right)\right]\sin(\pi\ f(x))$

$g'(x)=\pi(f'(x))^2\cos(\pi\ f(x))+\sin(\pi\ f(x))\ f''(x)=0$

$g(a)=0=g(b)=g\left(\frac{a+b}{2}\right)=g(\alpha)=g(\beta)$

$\therefore$ $g'(x)$ must have atleast one root in the intervals $(\alpha,\ a)$, $\left(a,\frac{a+b}{2}\right),\left(\frac{a+b}{2},\ b\right)$ and (b,β). i.e. minimum 4 roots.

● **Ex. 29** *Let* $f(x)=\int_{-2}^{x}e^{(1+t)^2}\ dt$ *and* $g(x)=f(h(x))$, *where* $h(x)$ *is defined for all* $x\in R$. *If* $g'(2)=e^4$ *and* $h'(2)=1$, *then absolute value of sum of all possible values of* $h(2)$, *is* ..... .

Sol. (2) Here, $f(x)=\int_{-2}^{x}e^{(1+t)^2}dt$

$$g(x)=\int_{-2}^{h(x)}e^{(1+t)^2}\ dt$$

$\Rightarrow$ $g'(x)=h'(x)\cdot e^{(1+h(x))^2}$ [by Leibnitz Rule]

$\Rightarrow$ $g'(2)=h'(2)\cdot e^{(1+h(2))^2}$

$\Rightarrow$ $e^4=1\cdot e^{(1+h(2))^2}$ [given $g'(2)=e^4$ and $h'(2)=1$]

$\Rightarrow$ $(1+h(2))^2=4 \Rightarrow h(2)=-3,1$

$\therefore$ Absolute value of sum of all possible values of $h(2)=|-3+1|=2$.

Subjective Type Questions

● **Ex. 30** *If* $f(x)=\cos\left\{\frac{\pi}{2}[x]-x^3\right\}, 1<x<2$ *and* $[x]$ = *the greatest integer* $\le x$, *then find* $f'\left(\sqrt[3]{\frac{\pi}{2}}\right)$.

Sol. As we know, $1<\sqrt[3]{\frac{\pi}{2}}<2$

$\therefore$ If $x=\sqrt[3]{\frac{\pi}{2}} \Rightarrow [x]=1$, so $f(x)=\cos\left\{\frac{\pi}{2}-x^3\right\}$

$f(x)=\sin x^3$ $\left[\text{at } x=\sqrt[3]{\frac{\pi}{2}}\in(1,2)\right]$

$\Rightarrow$ $f'(x)=\cos x^3\cdot 3x^2$

$\therefore$ $f'\left(\sqrt[3]{\frac{\pi}{2}}\right)=3\left(\frac{\pi}{2}\right)^{2/3}\cdot\cos\frac{\pi}{2}=0 \Rightarrow f'\left(\sqrt[3]{\frac{\pi}{2}}\right)=0$

● **Ex. 31** *If* $u=f(x^3), v=g(x^2)$, $f'(x)=\cos x$ *and* $g'(x)=\sin x$, *then find* $\frac{du}{dv}$.

Sol. Here, $u=f(x^3)$

$\Rightarrow$ $\frac{du}{dx}=f'(x^3)\cdot\frac{d}{dx}(x^3)=\{\cos(x^3)\}\cdot 3x^2=3x^2\cdot\cos x^3$

and $v=g(x^2)$

$\Rightarrow$ $\frac{dv}{dx}=g'(x^2)\cdot\frac{d}{dx}(x^2)=\{\sin x^2\}\cdot(2x)=2x\cdot\sin x^2$

$$\therefore \quad \frac{du}{dv}=\frac{\frac{du}{dx}}{\frac{dv}{dx}}=\frac{3x^2\cdot\cos x^3}{2x\cdot\sin x^2}$$

$$\Rightarrow \quad \frac{du}{dv}=\frac{3}{2}x\cdot\cos x^3\cdot\operatorname{cosec} x^2$$

● **Ex. 32** *Find a, b, c and d, where $f(x)=(ax+b)\cos x+(cx+d)\sin x$ and $f'(x)=x\cos x$ is identity in x.*

Sol. Here, $f'(x)=x\cos x$

$\Rightarrow a\cos x-(ax+b)\sin x+c\sin x+(cx+d)\cos x\equiv x\cos x$

or $(a+cx+d)\cos x+(-ax-b+c)\sin x\equiv x\cos x+0\cdot\sin x$

$\Rightarrow \quad a+d+cx=x$ and $-ax-b+c=0$

which is again identity in x.

$\Rightarrow \quad a+d=0, c=1, -a=0, -b+c=0$

$\Rightarrow \quad a=0, b=1, c=1, d=0$

● **Ex. 33** *If $f(x)=x^3+x^2f'(1)+xf''(2)+f'''(3)$ for all $x\in R$. Then, find $f(x)$ independent of $f'(1)$, $f''(2)$ and $f'''(3)$.*

Sol. Here, $f(x)=x^3+x^2f'(1)+xf''(2)+f'''(3)$

Put $f'(1)=a,\ f''(2)=b,\ f'''(3)=c$ ...(i)

$\therefore \quad f(x)=x^3+ax^2+bx+c$

$\Rightarrow \quad f'(x)=3x^2+2ax+b$ or $f'(1)=3+2a+b$ ...(ii)

$\Rightarrow \quad f''(x)=6x+2a$ or $f''(2)=12+2a$ ...(iii)

$\Rightarrow \quad f'''(x)=6$ or $f'''(3)=6$ ...(iv)

From Eqs. (i) and (iv), $c=6$

From Eqs. (i), (ii) and (iii), we have $a=-5, \quad b=2$

$\therefore \quad f(x)=x^3-5x^2+2x+6$

● **Ex. 34** *Let $f(x)=x^2+xg'(1)+g''(2)$ and $g(x)=f(1)\cdot x^2+xf'(x)+f''(x)$, then find $f(x)$ and $g(x)$.*

Sol. Here, put $g'(1)=a,\ g''(2)=b$ ...(i)

Then, $f(x)=x^2+ax+b, f(1)=1+a+b$

$\Rightarrow \quad f'(x)=2x+a, \quad f''(x)=2$

$\therefore \quad g(x)=(1+a+b)x^2+(2x+a)\cdot x+2$

$=x^2(3+a+b)+ax+2$

$\Rightarrow \quad g'(x)=2x(3+a+b)+a$

Hence, $g'(1)=2(3+a+b)+a$ ...(ii)

$g''(2)=2(3+a+b)$ ...(iii)

From Eqs. (i), (ii) and (iii), we have

$a=2(3+a+b)+a$ and $b=2(3+a+b)$

i.e. $3+a+b=0$ and $b+2a+6=0$

Hence, $b=0$ and $a=-3$

So, $f(x)=x^2-3x$ and $g(x)=-3x+2$

● **Ex. 35** *If $y=\dfrac{\sin x}{1+\dfrac{\cos x}{1+\dfrac{\sin x}{1+\cos x\ \ldots\infty}}}$, prove that $\dfrac{dy}{dx}=\dfrac{(1+y)\cos x+\sin x}{1+2y+\cos x-\sin x}$.*

Sol. Given function is $y=\dfrac{\sin x}{1+\dfrac{\cos x}{1+y}}=\dfrac{(1+y)\sin x}{1+y+\cos x}$

$\Rightarrow y+y^2+y\cos x=(1+y)\sin x$

On differentiating both the sides w.r.t. x, we get

$$\frac{dy}{dx}+2y\frac{dy}{dx}+y(-\sin x)+\cos x\cdot\frac{dy}{dx}$$

$$=(1+y)\cos x+\frac{dy}{dx}\cdot\sin x$$

$$\Rightarrow \frac{dy}{dx}\{1+2y+\cos x-\sin x\}=(1+y)\cos x+y\sin x$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{(1+y)\cos x+y\sin x}{(1+2y+\cos x-\sin x)}$$

● **Ex. 36** *If $y=\dfrac{x}{x+\dfrac{\sqrt[3]{x}}{x+\dfrac{\sqrt[3]{x}}{x+\sqrt[3]{x}+\ldots.\infty}}}$, then find $\dfrac{dy}{dx}$.*

Sol. $$y=\frac{x}{x+\dfrac{\sqrt[3]{x}}{x+\dfrac{\sqrt[3]{x}}{x+\sqrt[3]{x}+\ldots.\infty}}}$$

$$\Rightarrow \quad y=\frac{x}{x+\dfrac{x}{x+\dfrac{\sqrt[3]{x}}{x+\sqrt[3]{x}+\ldots.\infty}}}\cdot x^{-2/3}$$

$$\Rightarrow \quad y=\frac{x}{x+y\cdot x^{-2/3}}$$

$$\Rightarrow \quad y\{x^{5/3}+y\}=x^{5/3}$$

or $$x^{5/3}y+y^2=x^{5/3}$$

Differentiating both the sides w.r.t. x, we get

$$x^{5/3}\cdot\frac{dy}{dx}+\frac{5}{3}\cdot x^{2/3}\cdot y+2y\frac{dy}{dx}=\frac{5}{3}\cdot x^{2/3}$$

$$\Rightarrow \quad (x^{5/3}+2y)\frac{dy}{dx}=\frac{5}{3}x^{2/3}-\frac{5}{3}x^{2/3}y$$

or $$\frac{dy}{dx}=\frac{\frac{5}{3}x^{2/3}(1-y)}{(x^{5/3}+2y)}$$

● **Ex. 37** *If* $y = \tan^{-1}\dfrac{1}{x^2+x+1} + \tan^{-1}\dfrac{1}{x^2+3x+3} + \tan^{-1}\dfrac{1}{x^2+5x+7}$ *+... to n terms, show that* $\dfrac{dy}{dx} = \dfrac{1}{(x+n)^2+1} - \dfrac{1}{x^2+1}$.

Sol. Given, $y = \tan^{-1}\dfrac{1}{x^2+x+1} + \tan^{-1}\dfrac{1}{x^2+3x+3} + \tan^{-1}\dfrac{1}{x^2+5x+7} + \ldots$ to n terms

$$= \tan^{-1}\left\{\frac{1}{1+x(x+1)}\right\} + \tan^{-1}\left\{\frac{1}{1+(x+1)(x+2)}\right\} + \tan^{-1}\left\{\frac{1}{1+(x+2)(x+3)}\right\} + \ldots + \tan^{-1}\left\{\frac{1}{1+\{x+(n-1)\}\{x+n\}}\right\}$$

$$= \tan^{-1}\left\{\frac{(x+1)-x}{1+(x+1)x}\right\} + \tan^{-1}\left\{\frac{(x+2)-(x+1)}{1+(x+2)(x+1)}\right\} + \tan^{-1}\left\{\frac{(x+3)-(x+2)}{1+(x+3)(x+2)}\right\} + \ldots + \tan^{-1}\left\{\frac{(x+n)-(x+n-1)}{1+(x+n)(x+n-1)}\right\}$$

$\therefore\quad y = \{\tan^{-1}(x+1) - \tan^{-1}(x)\} + \{\tan^{-1}(x+2) - \tan^{-1}(x+1)\} + \{\tan^{-1}(x+3) - \tan^{-1}(x+2)\} + \ldots + \{\tan^{-1}(x+n) - \tan^{-1}(x+n-1)\}$

$$y = \tan^{-1}(x+n) - \tan^{-1}(x)$$

On differentiating both the sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{1+(x+n)^2} - \frac{1}{1+x^2}$$

● **Ex. 38** *If* $f(\theta) = \cos\theta_1 \cdot \cos\theta_2 \cdot \cos\theta_3 \ldots \cos\theta_n$, *show that* $\{\tan\theta_1 + \tan\theta_2 + \tan\theta_3 + \ldots + \tan\theta_n\} = -\left\{\dfrac{f'(\theta)}{f(\theta)}\right\}$, *where* $\dfrac{d\theta_1}{d\theta} = \dfrac{d\theta_2}{d\theta} = \ldots = \dfrac{d\theta_n}{d\theta} = 1$.

Sol. $f(\theta) = \cos\theta_1 \cdot \cos\theta_2 \cdot \cos\theta_3 \ldots \cos\theta_n$

Taking log on both the sides,

$$\log f(\theta) = \log(\cos\theta_1) + \log(\cos\theta_2) + \ldots + \log(\cos\theta_n)$$

On differentiating both the sides w.r.t. θ, we get

$$\frac{1}{f(\theta)} \cdot f'(\theta) = \frac{1}{\cos\theta_1}\cdot(-\sin\theta_1) + \frac{1}{\cos\theta_2}\cdot(-\sin\theta_2) + \ldots + \frac{1}{\cos\theta_n}\cdot(-\sin\theta_n)$$

Hence, $(\tan\theta_1 + \tan\theta_2 + \ldots + \tan\theta_n) = -\left\{\dfrac{f'(\theta)}{f(\theta)}\right\}$

● **Ex. 39** *Find the sum of the series* $\sin x + 3\sin 3x + 5\sin 5x + \ldots + (2k-1)\sin(2k-1)x$ *(using calculus).*

Sol. Let $S = \cos x + \cos 3x + \cos 5x + \ldots\ldots + \cos(2k-1)x$

Here, the angles are in AP whose first term is x and common difference is $2x$.

$$\therefore\quad S = \frac{\sin\left(\frac{k\cdot 2x}{2}\right)}{\sin\left(\frac{2x}{2}\right)} \cdot \cos\left\{\frac{x}{1} + \frac{(k-1)2x}{2}\right\} = \frac{\sin kx}{\sin x}\cdot\cos kx$$

or $\{\cos x + \cos 3x + \cos 5x + \ldots + \cos(2k-1)x\} = \dfrac{\sin 2kx}{2\sin x}$ ...(i)

On differentiating Eq. (i) w.r.t. x, we get

$$-\{\sin x + 3\sin 3x + 5\sin 5x + \ldots + (2k-1)\sin(2k-1)x\} = \frac{1}{2}\left\{\frac{2k(\cos 2kx)\cdot\sin x - (\sin 2kx)\cdot\cos x}{\sin^2 x}\right\}$$

$$\therefore\quad [\sin x + 3\sin 3x + 5\sin 5x + \ldots + (2k-1)\sin(2k-1)x] = -\frac{1}{2\sin^2 x}\left[k\{\sin(2k+1)x - \sin(2k-1)x\} - \frac{1}{2}\{\sin(2k+1)x + \sin(2k-1)x\}\right]$$

$$= \frac{1}{4\sin^2 x}[(2k+1)\sin(2k-1)x - (2k-1)\sin(2k+1)x]$$

● **Ex. 40** *Find the sum of series* $\sum_{r=1}^{n} r\,x^{r-1}$, *using calculus.*

Sol. Let $S = 1 + x + x^2 + x^3 + x^4 + \ldots + x^n$

which is a geometric progression.

$$\therefore\quad S = 1 + x + x^2 + x^3 + \ldots + x^n = \frac{1(1-x^{n+1})}{1-x}$$

On differentiating both the sides, we get

$$0 + 1 + 2x + 3x^2 + 4x^3 + \ldots + nx^{n-1} = \frac{(1-x)\cdot[-(n+1)x^n] - (1-x^{n+1})\cdot(-1)}{(1-x)^2}$$

$$\therefore\quad \sum_{r=1}^{n} r\,x^{r-1} = \frac{1}{(1-x)^2}\cdot\{1-(n+1)x^n + n\cdot x^{n+1}\}$$

● **Ex. 41** *Use calculus to find the sum of* $\left[\dfrac{1}{x+1} + \dfrac{2}{x^2+1} + \dfrac{4}{x^4+1} + \ldots + \dfrac{2^n}{x^{2^n}+1}\right]$.

Sol. We know that, $(1-x)(1+x) = (1-x^2)$

$$(1-x)(1+x)(1+x^2) = (1-x^4) = (1-x^{2^2})$$

$$(1-x)(1+x)(1+x^2)(1+x^4) = (1-x^8) = (1-x^{2^3})$$

$$(1-x)(1+x)(1+x^2)(1+x^4)\ldots(1+x^{2^n}) = (1-x^{2^{n+1}}) \quad \ldots(i)$$

Taking log on both the sides of Eq. (i), we get

$$\log\{(1-x)(1+x)(1+x^2)(1+x^4)\ldots(1+x^{2^n})\} = \log(1-x^{2^{n+1}})$$

or $\log(1-x)+\log(1+x)+\log(1+x^2)+\log(1+x^4)$

$$+\ldots+\log(1+x^{2^n}) = \log(1-x^{2^{n+1}})$$

On differentiating the above equation, we get

$$\frac{1(-1)}{1-x}+\frac{1\cdot 1}{1+x}+\frac{1\,(2x)}{1+x^2}+\frac{1\cdot(4x^3)}{1+x^4}+\ldots+\frac{1\cdot 2^n\cdot x^{2^n-1}}{1+x^{2^n}}$$

$$=\frac{-1\cdot 2^{n+1}\cdot x^{2^{n+1}-1}}{1-x^{2^{n+1}}}$$

or $$\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\ldots+\frac{2^n\cdot x^{2^n-1}}{1+x^{2^n}}$$

$$=\frac{1}{1-x}-\frac{2^{n+1}\cdot x^{2^{n+1}-1}}{1-x^{2^{n+1}}}$$

Multiplying both the sides by x,

$$\left[\frac{x}{1+x}+\frac{2x^2}{1+x^2}+\frac{4x^4}{1+x^4}+\ldots+\frac{2^n\cdot x^{2^n}}{1+x^{2^n}}\right]$$

$$=\frac{x}{1-x}-\frac{2^{n+1}\cdot x^{2^{n+1}}}{1-x^{2n+1}}$$

or $$\frac{(x+1)-1}{1+x}+\frac{2\,(x^2+1)-2}{1+x^2}+\frac{4\,(x^4+1)-4}{1+x^4}$$

$$+\ldots+\frac{2^n\,(x^{2^n}+1)-2^n}{1+x^{2^n}}=\frac{x}{1-x}-2^{n+1}\cdot\frac{x^{2^{n+1}}}{1-x^{2^{n+1}}}$$

or $$[(1+2+2^2+\ldots+2^n)-P]=\frac{x}{1-x}-\frac{2^{n+1}\cdot x^{2^{n+1}}}{1-x^{2^{n+1}}}$$

where, $P=\dfrac{1}{1+x}+\dfrac{2}{1+x^2}+\dfrac{4}{1+x^4}+\ldots+\dfrac{2^n}{1+x^{2^n}}$

Then, $$P=\frac{2^{n+1}-1}{2-1}-\frac{x}{1-x}+\frac{2^{n+1}\cdot x^{2^{n+1}}}{1-x^{2^{n+1}}}$$

$\Rightarrow$ $$P=2^{n+1}-1-\frac{x}{1-x}+\frac{2^{n+1}\cdot x^{2^{n+1}}}{1-x^{2^{n+1}}}$$

$\Rightarrow$ $$P=2^{n+1}\left[1+\frac{x^{2^{n+1}}}{1-x^{2^{n+1}}}\right]-\left[1+\frac{x}{1-x}\right]$$

$\Rightarrow$ $$P=2^{n+1}\left[\frac{1}{1-x^{2^{n+1}}}\right]-\left[\frac{1}{1-x}\right]$$

Hence, $$\left\{\frac{1}{1+x}+\frac{2}{1+x^2}+\frac{4}{1+x^4}+\ldots+\frac{2^n}{1+x^{2^n}}\right\}$$

$$=\left\{\frac{2^{n+1}}{1-x^{2^{n+1}}}\right\}-\left\{\frac{1}{1-x}\right\}$$

● ***Ex. 42*** *If $f_n(x)=e^{f_{n-1}(x)}$ for all $n\in N$ and $f_0(x)=x$, then find $\dfrac{d}{dx}\{f_n(x)\}$.*

Sol. Here, $$f_n(x)=e^{f_{n-1}(x)} \quad \ldots(i)$$

$\Rightarrow$ $$\frac{d}{dx}\{f_n(x)\}=e^{f_{n-1}(x)}\cdot\frac{d}{dx}\{f_{n-1}(x)\}$$

or $$\frac{d}{dx}\{f_n(x)\}=f_n(x)\frac{d}{dx}\{f_{n-1}(x)\}$$

[using Eq. (i)] ...(ii)

Replacing n by $n-1$ in Eq. (ii), we have

$$\frac{d}{dx}\{f_{n-1}(x)\}=f_{n-1}(x)\cdot\frac{d}{dx}\{f_{n-2}(x)\} \quad \ldots(iii)$$

$\therefore$ From Eqs. (ii) and (iii), we get

$$\frac{d}{dx}\{f_n(x)\}=f_n(x)\cdot f_{n-1}(x)\cdot\frac{d}{dx}\{f_{n-2}(x)\} \quad \ldots(iv)$$

Similarly, $$\frac{d}{dx}\{f_{n-2}(x)\}=f_{n-2}(x)\cdot\frac{d}{dx}\{f_{n-3}(x)\}$$

$$\frac{d}{dx}\{f_1(x)\}=f_1(x)\frac{d}{dx}\{f_0(x)\} \quad \ldots(v)$$

From Eqs. (iv) and (v), we get

$$\frac{d}{dx}\{f_n(x)\}=f_n(x)f_{n-1}(x)\cdot f_{n-2}(x)\ldots f_2(x)\cdot f_1(x)\left\{\frac{d}{dx}f_0(x)\right\}$$

$\Rightarrow$ $$\frac{d}{dx}\{f_n(x)\}=f_n(x)f_{n-1}(x)\cdot f_{n-2}(x)\ldots f_2(x)\cdot f_1(x)\cdot 1$$

$$\left[\because\ f_0(x)=x\ \Rightarrow\frac{d}{dx}\{f_0(x)\}=1\right]$$

● ***Ex. 43*** *If $y^3-y=2x$, then prove that $\dfrac{d^2y}{dx^2}=-\dfrac{24y}{(3y^2-1)^3}$.*

Hence, show that $\left(x^2-\dfrac{1}{27}\right)\dfrac{d^2y}{dx^2}+x\cdot\dfrac{dy}{dx}=\dfrac{1}{9}y$.

Sol. Given, $$y^3-y=2x \quad \ldots(i)$$

On differentiating both the sides w.r.t. x, we get

$$(3y^2-1)\frac{dy}{dx}=2 \quad \text{or} \quad \frac{dy}{dx}=\frac{2}{3y^2-1} \quad \ldots(ii)$$

Again, differentiating w.r.t. x,

$$\frac{d^2y}{dx^2}=\frac{-2(6y)\cdot\frac{dy}{dx}}{(3y^2-1)^2}=\frac{-12y}{(3y^2-1)^2}\cdot\frac{2}{(3y^2-1)} \quad \text{[from Eq. (ii)]}$$

$\therefore$ $$\frac{d^2y}{dx^2}=-\frac{24y}{(3y^2-1)^3} \quad \ldots(iii)$$

Now, LHS $=\left(x^2-\dfrac{1}{27}\right)\dfrac{d^2y}{dx^2}+x\cdot\dfrac{dy}{dx}$

$$=\left(\frac{y^2(y^2-1)^2}{4}-\frac{1}{27}\right)\left(\frac{-24y}{(3y^2-1)^3}\right)+\frac{y\,(y^2-1)}{2}\cdot\frac{2}{(3y^2-1)}$$

[from Eqs. (i), (ii) and (iii)]

$$= \left(\frac{27y^2(y^2-1)^2-4}{108}\right)\frac{-24y}{(3y^2-1)^3} + \frac{y(y^2-1)}{(3y^2-1)}$$

$$= \frac{y}{9}\left[\frac{-54y^2(y^2-1)^2+8}{(3y^2-1)^3} + \frac{9(y^2-1)}{(3y^2-1)}\right]$$

Let $(3y^2-1) = \alpha$

Then, LHS $= \frac{y}{9}\left[\frac{-2(1+\alpha)(\alpha-2)^2+8}{\alpha^3} + \frac{3(\alpha-1)}{\alpha}\right]$

$$= \frac{y}{9}\left[\frac{-2\alpha^3+6\alpha^2+3\alpha^3-6\alpha^2}{\alpha^3}\right]$$

$$= \frac{y}{9} = \text{RHS}$$

Hence, $\left(x^2 - \frac{1}{27}\right)\frac{d^2y}{dx^2} + x\cdot\frac{dy}{dx} = \frac{y}{9}$

● ***Ex. 44*** *If $2x = y^{1/5} + y^{-1/5}$, then express y as an explicit function of x and prove that $(x^2-1)\frac{d^2y}{dx^2} + x\cdot\frac{dy}{dx} = 25y$.*

Sol. We have, $(y^{1/5} + y^{-1/5})^2 = (y^{1/5} - y^{-1/5})^2 + 4$

Then, $(y^{1/5} - y^{-1/5})^2 = 4x^2 - 4$ $[\because\ 2x = y^{1/5} + y^{-1/5}]$

$\therefore$ $y^{1/5} - y^{-1/5} = 2\sqrt{x^2-1}$ ...(i)

and $y^{1/5} + y^{-1/5} = 2x$ [given]...(ii)

Adding Eqs. (i) and (ii), we get

$$2y^{1/5} = 2x + 2\sqrt{x^2-1}$$

$\therefore$ $y^{1/5} = x + \sqrt{x^2-1}$

or $y = (x + \sqrt{x^2-1})^5$ ...(iii)

On differentiating both the sides w.r.t. x, we get

$$\frac{dy}{dx} = 5(x+\sqrt{x^2-1})^4\cdot\left\{1 + \frac{1\cdot 2x}{2\sqrt{x^2-1}}\right\}$$

$\Rightarrow$ $\frac{dy}{dx} = \frac{5(x+\sqrt{x^2-1})^5}{\sqrt{x^2-1}}$

$\Rightarrow$ $\frac{dy}{dx} = \frac{5y}{\sqrt{x^2-1}}$ [using Eq. (iii)]

$\Rightarrow$ $(x^2-1)\left(\frac{dy}{dx}\right)^2 = 25y^2$ ...(iv)

Again, differentiating both the sides w.r.t. x, we get

$$2x\left(\frac{dy}{dx}\right)^2 + (x^2-1)\cdot 2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} = 50y\cdot\frac{dy}{dx}$$

On dividing by $2\frac{dy}{dx}$ on both the sides,

$$x\cdot\frac{dy}{dx} + (x^2-1)\frac{d^2y}{dx^2} = 25y \qquad \left[\because\ \frac{dy}{dx} \neq 0\right]$$

● ***Ex. 45*** *If $y = 1 + \frac{a_1}{(x-a_1)} + \frac{a_2x}{(x-a_1)(x-a_2)} + \frac{a_3x^2}{(x-a_1)(x-a_2)(x-a_3)} + \ldots$ to $(n+1)$ terms, show that $\frac{dy}{dx} = \frac{y}{x}\left[n - \frac{x}{x-a_1} - \frac{x}{x-a_2} - \frac{x}{x-a_3} - \ldots - \frac{x}{x-a_n}\right]$.*

Sol. Here, $y = 1 + \frac{a_1}{(x-a_1)} + \frac{a_2x}{(x-a_1)(x-a_2)} + \frac{a_3x^2}{(x-a_1)(x-a_2)(x-a_3)} + \ldots$ to $(n+1)$ terms

$\Rightarrow$ $y = \frac{x}{(x-a_1)} + \frac{a_2x}{(x-a_1)(x-a_2)} + \frac{a_3x^2}{(x-a_1)(x-a_2)(x-a_3)} + \ldots$ to (n) terms

[adding first two terms]

$\Rightarrow$ $y = \frac{x^2}{(x-a_1)(x-a_2)} + \frac{a_3x^2}{(x-a_1)(x-a_2)(x-a_3)} + \ldots$ to $(n-1)$ terms

[again adding first two terms]

Proceeding in the same way, we get

$$y = \frac{x^n}{(x-a_1)(x-a_2)(x-a_3)\ldots(x-a_n)} \quad \text{...(i)}$$

Taking logarithm on both the sides of Eq. (i), we have

$$\log y = n\log x - \log(x-a_1) - \log(x-a_2) - \log(x-a_3) - \ldots - \log(x-a_n)$$

On differentiating both the sides w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{n}{x} - \frac{1}{(x-a_1)} - \frac{1}{(x-a_2)} - \ldots - \frac{1}{(x-a_n)}$$

or $\frac{dy}{dx} = \frac{y}{x}\left[n - \frac{x}{(x-a_1)} - \frac{x}{(x-a_2)} - \ldots - \frac{x}{(x-a_n)}\right]$

● ***Ex. 46*** *If for all x, y the function f is defined by $f(x) + f(y) + f(x)\cdot f(y) = 1$ and $f(x) > 0$. Then, show $f'(x) = 0$, when $f(x)$ is differentiable.*

Sol. Here, $f(x) + f(y) + f(x)\cdot f(y) = 1$ ...(i)

Put $x = y = 0$, we get

$$2f(0) + \{f(0)\}^2 = 1$$

$\Rightarrow$ $\{f(0)\}^2 + 2f(0) - 1 = 0$

$$f(0) = \frac{-2 \pm \sqrt{4+4}}{2} = -1 - \sqrt{2} \quad \text{and} \quad -1 + \sqrt{2}$$

As $f(0) > 0 \Rightarrow f(0) = \sqrt{2} - 1$

[neglecting $f(0) = -1 - \sqrt{2}$ as $f(0)$ is positive]

Again, putting $y = x$ in Eq. (i),

$$2f(x) + \{f(x)\}^2 = 1$$

On differentiating w.r.t. x, we have

$$2f'(x) + 2f(x)f'(x) = 0$$

$$2f'(x)\{1 + f(x)\} = 0$$

$$\Rightarrow \quad f'(x) = 0$$

Because $f(x) > 0$

Therefore, $f'(x) = 0$ when $f(x) > 0$

● ***Ex. 47*** *If* $y = f\left(\dfrac{2x-1}{x^2+1}\right)$ *and* $f'(x) = \sin x^2$, *then find* $\dfrac{dy}{dx}$.

Sol. Here, $\dfrac{dy}{dx} = \dfrac{d}{dx} f\left(\dfrac{2x-1}{x^2+1}\right)$

$$= \frac{df\left(\dfrac{2x-1}{x^2+1}\right)}{d\left(\dfrac{2x-1}{x^2+1}\right)} \cdot \frac{d\left(\dfrac{2x-1}{x^2+1}\right)}{dx}$$

$$= f'\left(\frac{2x-1}{x^2+1}\right) \cdot \left\{\frac{(x^2+1)\cdot(2) - (2x-1)\cdot(2x)}{(x^2+1)^2}\right\}$$

$$= \sin\left(\frac{2x-1}{x^2+1}\right)^2 \cdot \frac{2\{x^2+1-2x^2+x\}}{(x^2+1)^2}$$

$$= \sin\left(\frac{2x-1}{x^2+1}\right)^2 \cdot \frac{2(1+x-x^2)}{(x^2+1)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2(1+x-x^2)}{(x^2+1)^2} \cdot \sin\left(\frac{2x-1}{x^2+1}\right)^2$$

● ***Ex 48*** *Let* $f(x)$ *be a polynomial function of second degree. If* $f(1) = f(-1)$ *and* a_1, a_2, a_3 *are in AP, then show that* $f'(a_1), f'(a_2), f'(a_3)$ *are in AP.*

Sol. Let $f(x) = \lambda x^2 + \mu x + \nu$

Then, $f'(x) = 2\lambda x + \mu$

Also, $f(1) = f(-1)$

$\Rightarrow \quad \lambda + \mu + \nu = \lambda - \mu + \nu$

$\Rightarrow \quad \mu = 0$

$\therefore \quad f'(x) = 2\lambda x$

$\Rightarrow \quad f'(a_1) = 2\lambda a_1,\ f'(a_2) = 2\lambda a_2$

and $f'(a_3) = 2\lambda a_3$

As, a_1, a_2, a_3 are in AP. Therefore, $f'(a_1), f'(a_2), f'(a_3)$ are in AP.

● ***Ex. 49*** *If* $5f(x) + 3f\left(\dfrac{1}{x}\right) = x + 2$ *and* $y = x\ f(x)$, *then find* $\dfrac{dy}{dx}$ *at* $x = 1$.

Sol. Here, $5f(x) + 3f\left(\dfrac{1}{x}\right) = x + 2$ ...(i)

Put $x = \dfrac{1}{x}$, we get $5f\left(\dfrac{1}{x}\right) + 3f(x) = \dfrac{1}{x} + 2$ ...(ii)

Solving Eqs. (i) and (ii), we get

$$16f(x) = 5x - \frac{3}{x} + 4 \qquad \text{...(iii)}$$

$$\Rightarrow \quad y = x f(x)$$

$$\Rightarrow \quad y = x \cdot \frac{1}{16}\left\{5x - \frac{3}{x} + 4\right\}$$

$$\Rightarrow \quad y = \frac{1}{16}\{5x^2 - 3 + 4x\}$$

or $\dfrac{dy}{dx} = \dfrac{1}{16}\{10x + 4\}$

Now, $\dfrac{dy}{dx}$ at $x = 1$,

$$\left(\frac{dy}{dx}\right)_{\text{at } x=1} = \frac{10+4}{16} = \frac{14}{16} = \frac{7}{8}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{\text{at } x=1} = \frac{7}{8}$$

Differentiation Exercise 1 :
Single Option Correct Type Questions

1. If $y=\dfrac{\sec x-\tan x}{\sec x+\tan x}$, then $\dfrac{dy}{dx}$ equals

(a) $2\sec x(\sec x-\tan x)$ (b) $-2\sec x(\sec x-\tan x)^2$

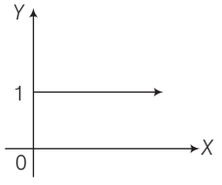
(c) $2\sec x(\sec x+\tan x)^2$ (d) $-2\sec x(\sec x+\tan x)^2$

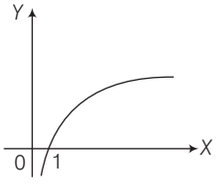
2. If $y=\dfrac{1+x^2+x^4}{1+x+x^2}$ and $\dfrac{dy}{dx}=ax+b$, then

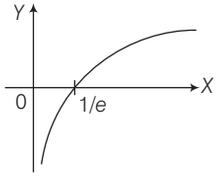
(a) $a=2,\ b=1$ (b) $a=-2,\ b=1$

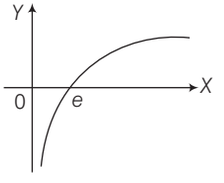
(c) $a=2,\ b=-1$ (d) $a=-2,\ b=-1$

3. Which of the following could be the sketch of graph of $y=\dfrac{d}{dx}(x\log x)$?

(a) 

(b) 

(c) 

(d) 

4. Let $f(x)=x+3\log(x-2)$ and $g(x)=x+5\log(x-1)$, then the set of x satisfying the inequality $f'(x)<g'(x)$ is

(a) $(2,7/2)$ (b) $(1,2)\cup(7/2,\infty)$

(c) $(2,\infty)$ (d) $(7/2,\infty)$

5. If $\cos^{-1}\left(\dfrac{x^2-y^2}{x^2+y^2}\right)=\log a$, then $\dfrac{dy}{dx}$ is

(a) $-\dfrac{x}{y}$ (b) $-\dfrac{y}{x}$ (c) $\dfrac{y}{x}$ (d) $\dfrac{x}{y}$

6. If $f(x)=(|x|)^{|\sin x|}$, then $f'(-\pi/4)$ is equal to

(a) $\left(\dfrac{\pi}{4}\right)^{1/\sqrt{2}}\cdot\left(\dfrac{\sqrt{2}}{2}\log\dfrac{4}{\pi}-\dfrac{2\sqrt{2}}{\pi}\right)$

(b) $\left(\dfrac{\pi}{4}\right)^{1/\sqrt{2}}\cdot\left(\dfrac{\sqrt{2}}{2}\log\dfrac{4}{\pi}+\dfrac{2\sqrt{2}}{\pi}\right)$

(c) $\left(\dfrac{\pi}{4}\right)^{1/\sqrt{2}}\cdot\left(\dfrac{\sqrt{2}}{2}\log\dfrac{\pi}{4}-\dfrac{2\sqrt{2}}{\pi}\right)$

(d) $\left(\dfrac{\pi}{4}\right)^{1/\sqrt{2}}\cdot\left(\dfrac{\sqrt{2}}{2}\log\dfrac{\pi}{4}+\dfrac{2\sqrt{2}}{\pi}\right)$

7. If $y=\cfrac{x}{a+\cfrac{x}{b+\cfrac{x}{a+\cfrac{x}{b+\ldots\infty}}}}$, then $\dfrac{dy}{dx}$ equals

(a) $\dfrac{a}{ab+2ay}$ (b) $\dfrac{b}{ab+2by}$

(c) $\dfrac{a}{ab+2by}$ (d) $\dfrac{b}{ab+2ay}$

8. If $y=x^{x^2}$, then $\dfrac{dy}{dx}$ equals

(a) $2\log x\cdot x^2$

(b) $(2\log x+1)\cdot x^{x^2}$

(c) $(\log x+1)\cdot x^{x^2+1}$

(d) $x^{x^2+1}\cdot(\log(ex^2))$

9. If $x\sqrt{1+y}+y\sqrt{1+x}=0$, then $\dfrac{dy}{dx}$ equals

(a) $\dfrac{1}{(1+x)^2}$ (b) $\dfrac{-1}{(1+x)^2}$

(c) $\dfrac{-1}{(1+x)}+\dfrac{1}{(1+x)^2}$ (d) None of these

10. If $x^2e^y+2xye^x+13=0$, then $\dfrac{dy}{dx}$ equals

(a) $\dfrac{-2xe^{y-x}+2y(x-1)}{x(xe^{y-x}+2)}$ (b) $-\left[\dfrac{2xe^{y-x}+2y(x+1)}{x(xe^{y-x}+2)}\right]$

(c) $\dfrac{2xe^{x-y}+2y(x-1)}{x(xe^{y-x}+2)}$ (d) None of these

11. If $x=e^{y+e^{y+\ldots\infty}}$; $x>0$, then $\dfrac{dy}{dx}$ equals

(a) $\dfrac{x}{1+x}$ (b) $\dfrac{1+x}{x}$

(c) $\dfrac{1-x}{x}$ (d) $\dfrac{1}{x}$

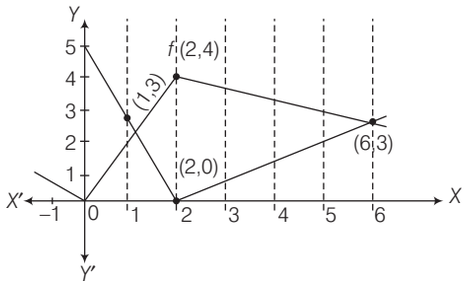
12. Let g be the inverse function of f and $f'(x)=\dfrac{x^{10}}{1+x^2}$.

If $g(2)=a$, then $g'(2)$ is equal to

(a) $\dfrac{5}{2^{10}}$ (b) $\dfrac{1+a^2}{a^{10}}$

(c) $\dfrac{a^{10}}{1+a^2}$ (d) $\dfrac{1+a^{10}}{a^2}$

13. If f and g are the functions whose graphs are as shown, let $u(x) = f(g(x)); w(x) = g(g(x))$



Then, the value of $u'(1) + w'(1)$ is

(a) $-\frac{1}{2}$ (b) $-\frac{3}{2}$ (c) $-\frac{5}{4}$ (d) does not exist

14. If $f'(x) = g(x)$ and $g'(x) = -f(x)$ for all real x and $f(5) = 2 = f'(5)$, then $f^2(10) + g^2(10)$ equals

(a) 2 (b) 4
(c) 8 (d) None of these

15. If $f(x) = x + \frac{x^2}{1!} + \frac{x^3}{2!} + \ldots + \frac{x^n}{(n-1)!}$, then $f(0) + f'(0) + f''(0) + \ldots + f^n(0)$ is equal to

(a) $\frac{n(n+1)}{2}$ (b) $\frac{(n^2+1)}{2}$
(c) $\left(\frac{n(n+1)}{2}\right)^2$ (d) $\frac{n(n+1)(2n+1)}{6}$

16. If $y = (fofof)(x)$ and $f(0) = 0, f'(0) = 2$, then $y'(0)$ is equal to

(a) 6 (b) 7 (c) 8 (d) 9

17. If $y^2 = p(x)$ is a polynomial of degree 3, then $2\frac{d}{dx}\left(y^3 \cdot \frac{d^2y}{dx^2}\right)$ is equal to

(a) $p'''(x) \cdot p'(x)$ (b) $p''(x) \cdot p'''(x)$
(c) $p(x) \cdot p'''(x)$ (d) None of these

18. If $y = f(x)$ and $x = g(y)$ are inverse functions of each other, then

(a) $g''(y) = \frac{1}{f'(x)}$ (b) $g''(y) = -\frac{f''(x)}{(f'(x))^3}$
(c) $g''(y) = -\frac{f'(x)}{(f''(x))^3}$ (d) None of these

19. If y is a function of x, then $\frac{d^2y}{dx^2} + y \cdot \frac{dy}{dx} = 0$. If x is a function of y, then the equation becomes

(a) $\frac{d^2x}{dy^2} - x \cdot \frac{dx}{dy} = 0$ (b) $\frac{d^2x}{dy^2} + y\left(\frac{dx}{dy}\right)^3 = 0$
(c) $\frac{d^2x}{dy^2} - y\left(\frac{dx}{dy}\right)^2 = 0$ (d) $\frac{d^2x}{dy^2} - x\left(\frac{dx}{dy}\right)^2 = 0$

20. Let $g(x) = \log_e f(x)$, where $f(x)$ is twice differentiable positive function on $(0, \infty)$ such that $f(x+1) = x f(x)$, for $N = 1, 2, 3, \ldots$, then

$$g''\left(N + \frac{1}{2}\right) - g''\left(\frac{1}{2}\right) \text{ equals}$$

(a) $-4\left\{1 + \frac{1}{9} + \frac{1}{25} + \ldots + \frac{1}{(2N-1)^2}\right\}$

(b) $4\left\{1 + \frac{1}{9} + \frac{1}{25} + \ldots + \frac{1}{(2N-1)^2}\right\}$

(c) $-4\left\{1 + \frac{1}{9} + \frac{1}{25} + \ldots + \frac{1}{(2N+1)^2}\right\}$

(d) $4\left\{1 + \frac{1}{9} + \frac{1}{25} + \ldots + \frac{1}{(2N+1)^2}\right\}$

21. If the function $f(x) = x^3 + e^{x/2}$ and $g(x) = f^{-1}(x)$, then the value of $g'(1)$ is

(a) 1 (b) 2
(c) 3 (d) e

22. Let $f(\theta) = \sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$, where $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$.

Then, the value of $\frac{d}{d(\tan\theta)} \cdot f(\theta)$, is

(a) 1 (b) 2
(c) 3 (d) 4

23. If $y = \log_{\sin x}(\tan x)$, then $\frac{dy}{dx}$ at $x = \frac{\pi}{4}$ is equal to

(a) $\frac{4}{\log 2}$ (b) $-4\log 2$
(c) $\frac{-4}{\log 2}$ (d) None of these

24. If $y = \sum_{r=1}^{x} \tan^{-1}\left(\frac{1}{1 + r + r^2}\right)$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{1}{1 + x^2}$ (b) $\frac{1}{1 + (1 + x)^2}$
(c) 0 (d) None of these

25. If $y = \sin^{-1}\left(\frac{\sin\alpha \sin x}{1 - \cos\alpha \sin x}\right)$, then $y'(0)$ is equal to

(a) 1 (b) $2\tan\alpha$
(c) $\left(\frac{1}{2}\right)\tan\alpha$ (d) $\sin\alpha$

26. If $f(x) = \cot^{-1}\left(\frac{x^x - x^{-x}}{2}\right)$, then $f'(1)$ equals

(a) -1 (b) 1
(c) $\log_e 2$ (d) $-\log_e 2$

27. The function $f(x) = e^x + x$, being differentiable and one-one, has a differentiable inverse $f^{-1}(x)$. The value of $\frac{d}{dx}(f^{-1})$ at the point $f(\log 2)$ is

(a) $\frac{1}{\ln 2}$ (b) $\frac{1}{3}$

(c) $\frac{1}{4}$ (d) None of these

28. If $f''(x) = -f(x)$, $g(x) = f'(x)$, $F(x) = \left(f\left(\frac{x}{2}\right)\right)^2 + \left(g\left(\frac{x}{2}\right)\right)^2$ and $F(5) = 5$, then $F(10)$ is equal to

(a) 5 (b) 10 (c) 0 (d) 15

29. If $x = \sec\theta - \cos\theta$ and $y = \sec^n\theta - \cos^n\theta$, then $(x^2 + 4)\left(\frac{dy}{dx}\right)^2$ is equal to

(a) $n^2(y^2 - 4)$ (b) $n^2(4 - y^2)$

(c) $n^2(y^2 + 4)$ (d) None of these

30. If $x = f(t)\cos t - f'(t)\sin t$ and $y = f(t)\sin t + f'(t)\cos t$, then $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$ is equal to

(a) $f(t) - f''(t)$ (b) $[f(t) - f''(t)]^2$

(c) $[f(t) + f''(t)]^2$ (d) None of these

31. If $y = at^2 + 2bt + c$ and $t = ax^2 + 2bx + c$, then $\frac{d^3y}{dx^3}$ is equal to

(a) $24a^2(at + b)$ (b) $24a(ax + b)^2$

(c) $24a(at + b)^2$ (d) $24a^2(ax + b)$

32. Differential coefficient of $\left(x^{\frac{l+m}{m-n}}\right)^{\frac{1}{n-l}} \cdot \left(x^{\frac{m+n}{n-l}}\right)^{\frac{1}{l-m}} \cdot \left(x^{\frac{n+l}{l-m}}\right)^{\frac{1}{m-n}}$ w.r.t. x, is

(a) 1 (b) 0 (c) -1 (d) x^{lmn}

33. If $y = (A + Bx)e^{mx} + (m-1)^{-2}e^x$, then $\frac{d^2y}{dx^2} - 2m\frac{dy}{dx} + m^2y$ is equal to

(a) e^x (b) e^{mx} (c) e^{-mx} (d) $e^{(1-m)x}$

34. If $f(x) = -\frac{x^3}{3} + x^2\sin 1.5a - x\sin a\cdot\sin 2a - 5\sin^{-1}(a^2 - 8a + 17)$, then

(a) $f(x)$ is not defined at $x = \sin 8$

(b) $f'(\sin 8) > 0$

(c) $f'(x)$ is not defined at $x = \sin 8$

(d) $f'(\sin 8) < 0$

35. If f and g are differentiable functions such that $g'(a) = 2$ and $g(a) = b$ and if fog is an identity function, then $f'(b)$ has the value equal to

(a) 2/3 (b) 1

(c) 0 (d) 1/2

36. The derivative of the function, $f(x) = \cos^{-1}\left\{\frac{1}{\sqrt{13}}(2\cos x - 3\sin x)\right\} + \sin^{-1}\left\{\frac{1}{\sqrt{13}}(2\cos x + 3\sin x)\right\}$ w.r.t. $\sqrt{1+x^2}$ at $x = \frac{3}{4}$ is

(a) $\frac{3}{2}$ (b) $\frac{5}{2}$

(c) $\frac{10}{3}$ (d) 0

37. If $f(x) = \sqrt{x + 2\sqrt{2x-4}} + \sqrt{x - 2\sqrt{2x-4}}$, then the value of $10f'(102^+)$, is

(a) -1 (b) 0

(c) 1 (d) does not exist

38. Let $y = \ln(1 + \cos x)^2$, then the value of $\frac{d^2y}{dx^2} + \frac{2}{e^{y/2}}$ equals

(a) 0 (b) $\frac{2}{1+\cos x}$

(c) $\frac{4}{(1+\cos x)}$ (d) $\frac{-4}{(1+\cos x)^2}$

39. If $f(x) = \frac{a + \sqrt{a^2 - x^2} + x}{\sqrt{a^2 - x^2} + a - x}$, where $a > 0$ and $x < a$, then $f'(0)$ has the value equal to

(a) $\sqrt{a}$ (b) a

(c) $\frac{1}{\sqrt{a}}$ (d) $\frac{1}{a}$

40. Let $u(x)$ and $v(x)$ be differentiable functions such that $\frac{u(x)}{v(x)} = 7$. If $\frac{u'(x)}{v'(x)} = p$ and $\left(\frac{u(x)}{v(x)}\right)' = q$, then $\frac{p+q}{p-q}$ has the value equal to

(a) 1 (b) 0

(c) 7 (d) -7

41. If $f(x) = |\log_e|x||$, then $f'(x)$ equals

(a) $\frac{1}{|x|}$, $x \neq 0$

(b) $\frac{1}{x}$ for $|x| > 1$ and $-\frac{1}{x}$ for $|x| < 1$

(c) $-\frac{1}{x}$ for $|x| > 1$ and $\frac{1}{x}$ for $|x| < 1$

(d) $\frac{1}{x}$ for $x > 0$ and $-\frac{1}{x}$ for $x < 0$

42. If $f(x)$ is given by, $f(x) = (\cos x + i \sin x)$ $(\cos 3x + i \sin 3x) \ldots (\cos (2n-1)\, x + i \sin (2n-1)\, x)$, then $f''(x)$ is equal to

(a) $n^3 f(x)$ (b) $-n^4 f(x)$

(c) $-n^2 f(x)$ (d) $n^4 f(x)$

43. Let $f(x) = x^n$, n being a non-negative integer. The value of n for which the equality $f'(x+y) = f'(x) + f'(y)$ is valid for all $x, y > 0$, is

(a) 0, 1 (b) 1, 2

(c) 2, 4 (d) None of these

44. If $f(x) = \sin\left\{\frac{\pi}{3}[x] - x^2\right\}$ for $2 < x < 3$ and $[x]$ denotes the greatest integer less than or equal to x, then $f'(\sqrt{\pi/3})$ is equal to

(a) $\sqrt{\pi/3}$ (b) $-\sqrt{\pi/3}$

(c) $-\sqrt{\pi}$ (d) None of these

45. The functions $u = e^x \sin x, v = e^x \cos x$ satisfy the equation

(a) $v\frac{du}{dx} - u\frac{dv}{dx} = u^2 + v^2$ (b) $\frac{d^2u}{dx^2} = 2v$

(c) $\frac{d^2v}{dx^2} = -2u$ (d) All of these

46. If $f(x) = \log_x \{\ln(x)\}$, then $f'(x)$ at $x = e$, is

(a) e (b) $-e$

(c) e^2 (d) e^{-1}

47. Let f be a differentiable function satisfying

$$[f(x)]^n = f(nx) \text{ for all } x \in R.$$

Then, $f'(x)f(nx)$ equals

(a) $f(x)$ (b) 0

(c) $f(x)f'(nx)$ (d) None of these

48. If $y = f(x)$ is an odd differentiable function defined on $(-\infty, \infty)$ such that $f'(3) = -2$, then $f'(-3)$ equals

(a) 4 (b) 2

(c) -2 (d) 0

49. If $y = \sqrt{x + \sqrt{y + \sqrt{x + \sqrt{y + \ldots \infty}}}}$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{y+x}{y^2 - 2x}$ (b) $\frac{y^3 - x}{2y^2 - 2xy - 1}$

(c) $\frac{y^3 + x}{2y^2 - x}$ (d) None of these

50. If $f(x) = |\cos x - \sin x|$, then $f'\left(\frac{\pi}{4}\right)$ is equal to

(a) $\sqrt{2}$ (b) $-\sqrt{2}$

(c) 0 (d) None of these

51. Let $f(x) = x^2 + xg'(1) + g''(2)$ and $g(x) = x^2 + xf'(2) + f''(3)$. Then,

(a) $f'(1) = 4 + f'(2)$

(b) $g'(2) = 8 + g'(1)$

(c) $g''(2) + f''(3) = 4$

(d) All of the above

52. If $f(x) = x^n$, then the value of

$$f(1) - \frac{f'(1)}{1!} + \frac{f''(1)}{2!} - \frac{f'''(1)}{3!} + \frac{f''''(1)}{4!} - \ldots + \frac{(-1)^n f^n(1)}{n!} \text{ is}$$

(a) 1 (b) 2^n

(c) 2^{n-1} (d) 0

Differentiation Exercise 2 : More than One Option Correct Type Questions

53. If $y + \log(1+x) = 0$, which of the following is true?

(a) $e^y = xy' - 1$ (b) $y' = -\frac{1}{(x+1)}$

(c) $y' + e^y = 0$ (d) $y' = e^y$

54. If $y = 2^{3^x}$, then y' equals

(a) $3^x \cdot \log 3 \cdot \log 2$

(b) $y \cdot (\log_2 y) \cdot \log 3 \cdot \log 2$

(c) $2^{3^x} \cdot 3^x \cdot \log 6$

(d) $2^{3^x} \cdot 3^x \cdot \log 3 \cdot \log 2$

55. If g is the inverse of f and $f(x) = x^2 + 3x - 3, (x > 0)$, then $g'(1)$ equals

(a) $\frac{1}{2g(1)+3}$ (b) -1 (c) $\frac{1}{5}$ (d) $-\frac{f'(1)}{(f(1))^2}$

56. If $x^3 - 2x^2y^2 + 5x + y - 5 = 0$ and $y(1) = 1$, then

(a) $y'(1) = \frac{4}{3}$ (b) $y''(1) = -\frac{1}{3}$

(c) $y''(1) = -8\frac{22}{27}$ (d) $y'(1) = \frac{2}{3}$

57. Let $y=\sqrt{x+\sqrt{x+\sqrt{x+\ldots\infty}}}$, then dy/dx equals

(a) $\frac{1}{2y-1}$ (b) $\frac{x}{x+2y}$

(c) $\frac{1}{\sqrt{1-4x}}$ (d) $\frac{y}{2x+y}$

58. If $y=x^{(\ln x)^{\ln(\ln x)}}$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{y}{x}(\ln x^{\ln x-1}+2\ln x\ln(\ln x))$

(b) $\frac{y}{x}(\ln x)^{\ln(\ln x)}(2\ln(\ln x)+1)$

(c) $\frac{y}{x\ln x}((\ln x)^2+2\ln x(\ln x))$

(d) $\frac{y}{x}\cdot\frac{\ln y}{\ln x}(2\ln(\ln x)+1)$

59. Which of the following functions are not derivable at $x=0$?

(a) $f(x)=\sin^{-1}2x\sqrt{1-x^2}$ (b) $g(x)=\sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$

(c) $h(x)=\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ (d) $k(x)=\sin^{-1}(\cos x)$

60. Let $f(x)=\frac{\sqrt{x-2\sqrt{x-1}}}{\sqrt{x-1}-1}\cdot x$, then

(a) $f'(10)=1$ (b) $f'(3/2)=-1$

(c) domain of $f(x)$ is $x\geq 1$ (d) None of these

61. If $2^x+2^y=2^{x+y}$, then $\frac{dy}{dx}$ is equal to

(a) $-\frac{2^y}{2^x}$ (b) $\frac{1}{1-2^x}$

(c) $1-2^y$ (d) $\frac{2^x(1-2^y)}{2^y(2^x-1)}$

62. For the function $y=f(x)=(x^2+bx+c)e^x$, which of the following holds?

(a) If $f(x)>0$ for all real $x\not\Rightarrow f'(x)>0$

(b) If $f(x)>0$ for all real $x\Rightarrow f'(x)>0$

(c) If $f'(x)>0$ for all real $x\Rightarrow f(x)>0$

(d) If $f'(x)>0$ for all real $x\not\Rightarrow f(x)>0$

63. If $\sqrt{y+x}+\sqrt{y-x}=c$, where $c\neq 0$, then $\frac{dy}{dx}$ has the value equal to

(a) $\frac{2x}{c^2}$ (b) $\frac{x}{y+\sqrt{y^2-x^2}}$

(c) $\frac{y-\sqrt{y^2-x^2}}{x}$ (d) $\frac{c^2}{2y}$

64. If $y=\tan x\tan 2x\tan 3x$, then $\frac{dy}{dx}$ has the value equal to

(a) $3\sec^2 3x\tan x\tan 2x+\sec^2 x\tan 2x\tan 3x+2\sec^2 2x\tan 3x\tan x$

(b) $2y(\operatorname{cosec}2x+2\operatorname{cosec}4x+3\operatorname{cosec}6x)$

(c) $3\sec^2 3x-2\sec^2 2x-\sec^2 x$

(d) $\sec^2 x+2\sec^2 2x+3\sec^2 3x$

Differentiation Exercise 3 : Statements I and II Type Questions

■ **Directions** (Q. Nos. 65 to 74) *This section is based on Statement* I *and Statement* II. *Select the correct answer from the codes given below.*

(a) Both Statement I and Statement II are correct and Statement II is the correct explanation of Statement I

(b) Both Statement I and Statement II are correct but Statement II is not the correct explanation of Statement I

(c) Statement I is correct but Statement II is incorrect

(d) Statement II is correct but Statement I is incorrect

65. Consider $f(x)=\frac{x}{x^2-1}$

Statement I Graph of $f(x)$ is concave up for $x>1$.

Statement II If $f(x)$ is concave up, then $f''(x)>0$.

66. If $f(x)=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$, then

Statement I The value of $f'(2)=-\frac{2}{5}$.

Statement II $f(x)=\sin^{-1}\left(\frac{2x}{1+x^2}\right)=\frac{-2}{1+x^2}$, for $x>1$.

67. **Statement I** If $f(0)=a$, $f'(0)=b$, $g(0)=0$, $(fog)'(0)=c$, then $g'(0)=\frac{c}{b}$.

Statement II $(f(g(x))'=f'(g(x))\cdot g'(x)$, for all n.

68. Let f and g be real valued functions defined on interval $(-1,1)$ such that $g''(x)$ is continuous, $g(0)=0$, $g'(0)=0$, $g''(0)=0$ and $f(x)=g(x)\sin x$.

Statement I $\lim_{x\to 0}(g(x)\cot x-g(0)\operatorname{cosec}x)=f''(0)$.

Statement II $f'(0)=g'(0)$.

69. **Statement I** If $y = \sin^{-1}(3x - 4x^3)$, then $\frac{dy}{dx} = \frac{3}{\sqrt{1-x^2}}$ only when $\frac{-1}{2} \le x < \frac{1}{2}$.

Statement II $\sin^{-1}(3x - 4x^3)$

$$= \begin{cases} -\pi - 3\sin^{-1} x, & -1 \le x \le -\frac{1}{2} \\ 3\sin^{-1} x, & -\frac{1}{2} \le x \le \frac{1}{2} \\ \pi - 3\sin^{-1} x, & \frac{1}{2} \le x \le 1. \end{cases}$$

70. If $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, then

Statement I $\frac{dy}{dx} = \frac{2}{1+x^2}$ for $x \in R$.

Statement II $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \begin{cases} 2\tan^{-1} x, & x \ge 0 \\ -2\tan^{-1} x, & x < 0. \end{cases}$

71. If $y = x + [x]$, then

Statement I $\frac{dy}{dx} = 1$ for all $x \in R$.

Statement II $\frac{d([x])}{dx} = \begin{cases} 0, & x \notin \text{Integer} \\ \text{does not exist}, & x \in \text{Integer}. \end{cases}$

72. **Statement I** If $f(x)$ is a continuous function defined from R to Q and $f(5) = 3$, then differential coefficient of $f(x)$ w.r.t. x will be 0.

Statement II Differentiation of constant function is always zero.

73. **Statement I** Derivative of $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ w.r.t. $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ is 1 for $0 < x < 1$.

Statement II $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ for $-1 \le x \le 1$.

74. Consider function $f(x)$ satisfies the relation $f(x + y^3) = f(x) + f(y^3), \forall\ x, y \in R$ and differentiable for all x.

Statement I If $f'(2) = a$, then $f'(-2) = a$.

Statement II $f(x)$ is an odd function.

Differentiation Exercise 4 : Passage Based Questions

Passage I (Q. Nos. 75 to 77)

Let $\frac{f(x+y) - f(x)}{2} = \frac{f(y) - 1}{2} + xy$, *for all* $x, y \in R$, $f(x)$ *is differentiable and* $f'(0) = 1$. *Let* $g(x)$ *be a derivable function at* $x = 0$ *and follows the functional rule*

$g\left(\frac{x+y}{k}\right) = \frac{g(x) + g(y)}{k}$; $k \in R$, $k \ne 0, 2$

and $g'(0) - \lambda g'(0) \ne 0$.

75. Domain of $\log(f(x))$, is

(a) R^+ (b) $R - \{0\}$
(c) R (d) R^-

76. Range of $y = \log_{3/4}(f(x))$ is

(a) $(-\infty, 1]$ (b) $[3/4, \infty)$
(c) $(-\infty, \infty)$ (d) R

77. If the graphs of $y = f(x)$ and $y = g(x)$ intersect in coincident points then λ can take values

(a) -3 (b) 1
(c) -1 (d) 4

Passage II (Q. Nos. 78 to 80)

Left hand derivative and right hand derivative of a function $f(x)$ *at a point* $x = a$ *are defined as*

$$f'(a^-) = \lim_{h \to 0^+} \frac{f(a) - f(a-h)}{h} = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

and $$f'(a^+) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0^+} \frac{f(a) - f(a+h)}{h} = \lim_{h \to 0^+} \frac{f(a) - f(x)}{a - x}$$ *respectively.*

Let f *be a twice differentiable function. We also know that derivative of an even function is odd function and derivative of an odd function is even function.*

78. If f is odd, then which of the following is left hand derivative of f at $x = -a$?

(a) $\lim_{h \to 0^-} \frac{f(a-h) - f(a)}{-h}$

(b) $\lim_{h \to 0^-} \frac{f(h-a) - f(a)}{h}$

(c) $\lim_{h \to 0^+} \frac{f(a) + f(a-h)}{-h}$

(d) $\lim_{h \to 0^-} \frac{f(a) - f(a-h)}{-h}$

79. If f is even function, then which of the following is right hand derivative of f' at $x = a$?

(a) $\lim_{h \to 0^-} \frac{f'(a) + f'(-a+h)}{h}$

(b) $\lim_{h \to 0^+} \frac{f'(a) + f'(-a-h)}{h}$

(c) $\lim_{h \to 0^-} \frac{-f'(-a) + f'(-a-h)}{-h}$

(d) $\lim_{h \to 0^+} \frac{f'(a) + f'(-a+h)}{-h}$

80. The statement $\lim_{h \to 0} \frac{f(-x) - f(-x-h)}{h}$ $= \lim_{h \to 0} \frac{f(x) - f(x-h)}{-h}$ implies that for all $x \in R$,

(a) f is odd
(b) f is even
(c) f is neither even nor odd
(d) Nothing can be concluded

Passage III

(Q. Nos. 81 to 82)

If $f(x) = \sin^{-1}(3x - 4x^3)$. *Then answer the following*

81. The value of $f'(0)$ is

(a) -3 (b) 3
(c) $\sqrt{2}$ (d) $-\sqrt{2}$

82. The value of $f'\left(\frac{1}{\sqrt{2}}\right)$ is

(a) -3 (b) 3
(c) $-3\sqrt{2}$ (d) $3\sqrt{2}$

Passage IV (Q. Nos. 83 to 84)

Let the derivative of $f(x)$ *be defined as*

$D^* f(x) = \lim_{h \to 0} \frac{f^2(x+h) - f^2(x)}{h}$, *where* $f^2(x) = \{f(x)\}^2$.

83. If $u = f(x)$, $v = g(x)$, then the value of $D^*(u \cdot v)$ is

(a) $(D^* u)v + (D^* v)\, u$ (b) $u^2(D^* v) + v^2(D^* u)$
(c) $D^* u + D^* v$ (d) $uv\, D^*(u + v)$

84. If $u = f(x)$, $v = g(x)$, then the value of $D^*\left(\frac{u}{v}\right)$ is

(a) $\frac{u^2 (D^* v) - v^2 (D^* u)}{v^4}$ (b) $\frac{u (D^* v) - v (D^* u)}{v^2}$
(c) $\frac{v^2 (D^* u) - u^2 (D^* v)}{v^4}$ (d) $\frac{v (D^* u) - u (D^* v)}{v^2}$

Passage V

(Q. Nos. 85 to 87)

A curve is represented parametrically by the equations $x = e^t \cos t$ *and* $y = e^t \sin t$, *where t is a parameter. Then*,

85. The relation between the parameter 't' and the angle α between the tangent to the given curve and the X-axis is given by, 't' equals

(a) $\frac{\pi}{2} - \alpha$ (b) $\frac{\pi}{4} + \alpha$
(c) $\alpha - \frac{\pi}{4}$ (d) $\frac{\pi}{4} - \alpha$

86. The value of $\frac{d^2 y}{dx^2}$ at the point, where $t = 0$, is

(a) 1 (b) 2
(c) -2 (d) 3

87. If $F(t) = \int (x + y)\, dt$, then the value of $F\left(\frac{\pi}{2}\right) - F(0)$ is

(a) 1 (b) -1
(c) $e^{\pi/2}$ (d) 0

Passage VI

(Q. Nos. 88 to 90)

Equation $x^n - 1 = 0$, $n > 1$, $n \in N$ *has roots* $1, a_1, a_2, \ldots, a_n$.

88. The value of $(1 - a_1)(1 - a_2) \ldots (1 - a_n)$, is

(a) $\frac{n^2}{2}$ (b) n
(c) $(-1)^n n$ (d) None of these

89. The value of $\sum_{r=1}^{n} \frac{1}{2 - a_r}$, is

(a) $\frac{2^{n-1}(n-2) + 1}{2^n - 1}$ (b) $\frac{2^n (n-2) + 1}{2^n - 1}$
(c) $\frac{2^{n-1}(n-1) - 1}{2^n - 1}$ (d) None of these

90. The value of $\sum_{r=1}^{n} \frac{1}{1 - a_r}$, is

(a) $\frac{n}{4}$ (b) $\frac{n(n-1)}{2}$
(c) $\frac{n-1}{2}$ (d) None of these

Differentiation Exercise 5 : Matching Type Questions

91. Match the entries between following two columns.

	Column I		Column II
(A)	$y = f(x)$ be given by $x = t^5 - 5t^3 - 20t + 7$ and $y = 4t^3 - 3t^2 - 18t + 3$, then $-5 \times \frac{dy}{dx}$ at $t = 1$	(p)	0
(B)	$P(x)$ be a polynomial of degree 4 with $P(2) = -1$, $P'(2) = 0$, $P''(2) = 2$, $P'''(2) = -12$ and $P''''(2) = 24$, then $P''(3)$ is equal to	(q)	–2
(C)	$y = \frac{1}{x}$, then $\frac{\frac{dy}{\sqrt{1+y^4}}}{\frac{dx}{\sqrt{1+x^4}}}$	(r)	2
(D)	$f\left(\frac{2x+3y}{5}\right) = \frac{2f(x)+3f(y)}{5}$ and $f'(0) = p$ and $f(0) = q$, then $f''(0)$	(s)	–1

92. Match the following.

	Column I		Column II
(A)	$y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, then $\frac{dy}{dx} = -\frac{2}{1+x^2}$	(p)	for $x < 0$
(B)	$y = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$, then $\frac{dy}{dx} = -\frac{1}{1+x^2}$	(q)	for $x > 1$
(C)	$y = \lvert e^{\lvert x \rvert} - e \rvert$, then $\frac{dy}{dx} > 0$	(r)	for $x < -1$
(D)	$u = \log \lvert 2x \rvert$, $v = \lvert \tan^{-1} x \rvert$, then $\frac{du}{dv} > 2$	(s)	for $-1 < x < 0$

Differentiation Exercise 6 : Single Integer Answer Type Questions

93. If $f(x) = x^3 + x^2 f'(1) + xf''(2) + f'''(3)$ for all $x \in R$. Then, $f(1)$ is ..... .

94. Let $f(x) = \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)$, then the value of $\frac{d(\tan f(x))}{d(\tan^{-1} x)}$, when $x = \frac{1}{2}$, is ..... .

95. Let $x, x_1, x_2, x_3, x_4, \ldots, x_8$ be 9 real zeroes, of the polynomial $P(x) = x^{10} + ax^2 + bx + c$, where $a, b, c \in R$. If the value of $Q(x_1) = \frac{p}{q}$, where p and q are coprime to each other. If $Q(x) = (x - x_2)(x - x_3)\ldots(x - x_8)$ and $x_1 = \frac{1}{2}$, then the value of $q - p$ is ..... .

96. If $f(x) = \begin{vmatrix} (x-a)^4 & (x-a)^3 & 1 \\ (x-b)^4 & (x-b)^3 & 1 \\ (x-c)^4 & (x-c)^3 & 1 \end{vmatrix}$ and $f'(x) = \lambda \begin{vmatrix} (x-a)^4 & (x-a)^2 & 1 \\ (x-b)^4 & (x-b)^2 & 1 \\ (x-c)^4 & (x-c)^2 & 1 \end{vmatrix}$, then value of λ is .... .

97. Let $P(x)$ be a polynomial of degree 4 such that $P(1) = P(3) = P(5) = P(7) = 0$. If the real number $x \neq 1, 3, 5$ is such that $P(x) = 0$ can be expressed as $x = \frac{p}{q}$, where p and q are relatively prime, then $(p - 8q)$ is ..... .

98. If $x^2 + y^2 = t - \frac{1}{t}$ and $x^4 + y^4 = t^2 + \frac{1}{t^2}$, then $\left(\frac{dy}{dx}\right)_{(1,1)}$ is ..... .

99. If $x^2 + y^2 + z^2 - 2xyz = 1$, then the value of $\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} + \frac{dz}{\sqrt{1-z^2}}$ is equal to ..... .

100. If y is twice differentiable function of x, then the expression $(1-x^2)\cdot\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y$ by means of the transformation $x = \sin t$ in terms of t is $\frac{d^2y}{dt^2} + \lambda y$.
Thus, λ is ..... .

101. The derivative of $f(x) = \cos^{-1}\left(\frac{1}{\sqrt{3}}(2\cos x - 3\sin x)\right) + \left\{\sin^{-1}\left(\frac{1}{\sqrt{3}}(2\cos x + 3\sin x)\right)\right\}$ w.r.t. $\sqrt{1+x^2}$ at $x = \frac{1}{\sqrt{3}}$ is ..... .

102. Suppose $f(x) = e^{ax} + e^{bx}$, where $a \neq b$ and $f''(x) - 2f'(x) - 15f(x) = 0$ for all x. Then, the value of $|a+b|$ is equal to ..... .

103. Suppose, $A = \frac{dy}{dx}$ of $x^2 + y^2 = 4$ at $(\sqrt{2}, \sqrt{2})$, $B = \frac{dy}{dx}$ of $\sin y + \sin x = \sin x \cdot \sin y$ at (π, π) and $C = \frac{dy}{dx}$ of $2e^{xy} + e^x e^y - e^x - e^y = e^{xy+1}$ at (1, 1), then $(A - B - C)$ has the value equal to ..... .

104. A function is represented parametrically by the equations $x = \frac{1+t}{t^3}$; $y = \frac{3}{2t^2} + \frac{2}{t}$, then $\frac{dy}{dx} - x\cdot\left(\frac{dy}{dx}\right)^3$ has the absolute value equal to ..... .

105. Suppose, the function $f(x) - f(2x)$ has the derivative 5 at $x = 1$ and derivative 7 at $x = 2$. The derivative of the function $f(x) - f(4x)$ at $x = 1$, has the value $10 + \lambda$, then the value of λ is equal to ..... .

106. If $x + y = 3e^2$, then $D(x^y)$ vanishes when x equals to λe^2, then the value of λ is equal to ..... .

107. Let $h(x)$ be differentiable for all x and let $f(x) = (kx + e^x)h(x)$, where k is some constant. If $h(0) = 5$, $h'(0) = -2$ and $f'(0) = 18$, then the value of k is equal to ..... .

Complex Number Exercise 7 :
Questions Asked in Previous 10 Years' Exams

(i) JEE Advanced & IIT-JEE

108. For $x \in R$, $f(x) = |\log 2 - \sin x|$ and $g(x) = f(f(x))$, then **[One Correct Option, 2016 Adv.]**

(a) g is not differentiable at $x = 0$ (b) $g'(0) = \cos(\log 2)$
(c) $g'(0) = -\cos(\log 2)$ (d) g is differentiable at $x = 0$ and $g'(0) = -\sin(\log 2)$

109. Let $f: R \to R$, $g: R \to R$ and $h: R \to R$ be differentiable functions such that $f(x) = x^3 + 3x + 2$, $g(f(x)) = x$ and $h(g(g(x))) = x$ for all $x \in R$. Then, **[More than One Correct Option, 2016 Adv.]**

(a) $g'(2) = \frac{1}{15}$ (b) $h'(1) = 666$ (c) $h(0) = 16$ (d) $h(g(3)) = 36$

(ii) JEE Main & AIEEE

110. If for $x \in \left(0, \frac{1}{4}\right)$, the derivative of $\tan^{-1}\left(\frac{6x\sqrt{x}}{1-9x^3}\right)$ is $\sqrt{x}\cdot g(x)$, then $g(x)$ equals **[2017 JEE Main]**

(a) $\frac{3}{1+9x^3}$ (b) $\frac{9}{1+9x^3}$ (c) $\frac{3x\sqrt{x}}{1-9x^3}$ (d) $\frac{3x}{1-9x^3}$

111. Let $g(x) = \log f(x)$, where $f(x)$ is a twice differentiable positive function on $(0, \infty)$ such that $f(x+1) = x f(x)$. Then, for $N = 1, 2, 3, \ldots$, $g''\left(N + \frac{1}{2}\right) - g''\left(\frac{1}{2}\right)$ is equal to **[2008 AIEEE]**

(a) $-4\left\{1 + \frac{1}{9} + \frac{1}{25} + \ldots + \frac{1}{(2N-1)^2}\right\}$ (b) $4\left\{1 + \frac{1}{9} + \frac{1}{25} + \ldots + \frac{1}{(2N-1)^2}\right\}$
(c) $-4\left\{1 + \frac{1}{9} + \frac{1}{25} + \ldots + \frac{1}{(2N+1)^2}\right\}$ (d) $4\left\{1 + \frac{1}{9} + \frac{1}{25} + \ldots + \frac{1}{(2N+1)^2}\right\}$

112. $\dfrac{d^2x}{dy^2}$ equals **[2007 AIEEE]**

(a) $\left(\dfrac{d^2y}{dx^2}\right)^{-1}$ (b) $-\left(\dfrac{d^2y}{dx^2}\right)^{-1}\left(\dfrac{dy}{dx}\right)^{-3}$

(c) $\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)^{-2}$ (d) $-\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)^{-3}$

113. If $f''(x) = -f(x)$, where $f(x)$ is a continuous double differentiable function and $g(x) = f'(x)$.

If $F(x) = \left\{f\left(\dfrac{x}{2}\right)\right\}^2 + \left\{g\left(\dfrac{x}{2}\right)\right\}^2$ and $F(5) = 5$, then $F(10)$ is **[2006 AIEEE]**

(a) 0 (b) 5 (c) 10 (d) 25

114. If y is a function of x and $\log(x + y) = 2xy$, then the value of $y'(0)$ is **[2004 AIEEE]**

(a) 1
(b) −1
(c) 2
(d) 0

115. If $x^2 + y^2 = 1$, then **[2000 AIEEE]**

(a) $yy'' - 2(y')^2 + 1 = 0$
(b) $yy'' + (y')^2 + 1 = 0$
(c) $yy'' + (y')^2 - 1 = 0$
(d) $yy'' + 2(y')^2 + 1 = 0$

Answers

Exercise for Session 1

1. $a^x \log a + ax^{a-1}$ **2.** $\dfrac{\tan x}{|\tan x|}\sec^2 x$

3. $\dfrac{1}{x \log_e 3} + \dfrac{3}{x} + 2\sec^2 x$

4. $\dfrac{x}{|x|} + na_0x^{n-1} + (n-1)a_1x^{n-2} + (n-2)a_2x^{n-3} + \ldots + a_{n-1}1$

5. 0

6. $e^x x^{n-1}\left\{n \log_a x + \dfrac{1}{\log_e a} + x \log_a x\right\}$

7. $\dfrac{2^x}{\sqrt{x}}\left\{\log 2 \cot x - \operatorname{cosec}^2 x - \dfrac{\cot x}{2x}\right\}$

8. $\dfrac{x^2}{(x \sin x + \cos x)^2}$ **9.** y

10. $x \in (-\infty, -1) \cup (1, \infty)$

Exercise for Session 2

1. $4(x^2 + x + 1)^3 \cdot (2x + 1)$

2. $\dfrac{1}{2\sqrt{x^2 + x + 1}} \cdot (2x + 1)$

3. $3 \sin^2 x \cos x$ **4.** $\dfrac{x}{(a^2 - x^2)^{3/2}}$

5. $e^{x \sin x}(x \cos x + \sin x)$ **6.** $-\dfrac{\sqrt{b^2 - a^2}}{b + a \cos x}$

7. $e^{e^x} \cdot e^x$ **8.** $\dfrac{1}{\sqrt{a^2 + x^2}}$

9. $\dfrac{2ab \cos x}{a^2 - b^2 \sin^2 x}$

10. $\sec x$

11. $\dfrac{(\sin 3x)\left(e^x + \dfrac{1}{x}\right) - 3(e^x + \log x)\cos 3x}{\sin^2 3x}$

12. $\dfrac{m}{\sqrt{1 - x^2}} \cos(m \sin^{-1} x)$

13. $\dfrac{2 \log a \cdot \sin^{-1} x}{\sqrt{1 - x^2}} \cdot a^{(\sin^{-1} x)^2}$

14. $e^{\cos^{-1}\sqrt{1 - x^2}} \cdot \dfrac{1}{\sqrt{1 - x^2}}$ **15.** $\dfrac{\sin^{-1} x}{(1 - x^2)^{3/2}}$

16. $\dfrac{1}{x \log_e 10} - \dfrac{\log_e 10}{x(\log_e x)^2}$

17. $-2x\{5^{3 - x^2} \cdot \log_e 5 + 5(3 - x^2)^4\}$

18. $\dfrac{-2a^2}{x^3}\left\{1 + \dfrac{a^2}{\sqrt{a^4 - x^4}}\right\}$

19. $\dfrac{x}{4\sqrt{4 + \sqrt{4 + \sqrt{4 + x^2}}}\sqrt{4 + \sqrt{4 + x^2}}\sqrt{4 + x^2}}$

20. (c) **21.** (b) **22.** (a) **23.** (a)
24. (a) **25.** (c) **26.** (a) **27.** (d)

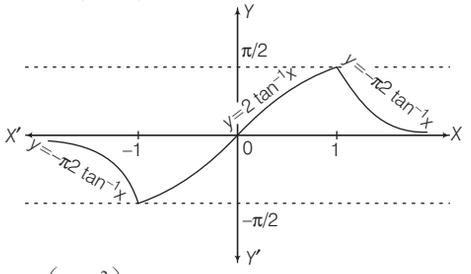
Exercise for Session 3

6. (b) **7.** (a) **8.** (a) **9.** (a) **10.** (a)

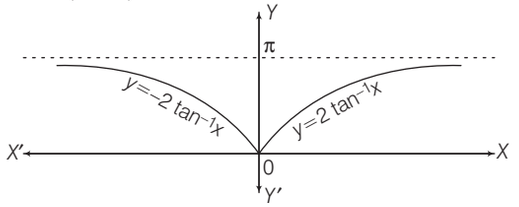
Exercise for Session 4

1. (b) **2.** (c) **3.** (b) **4.** (c)

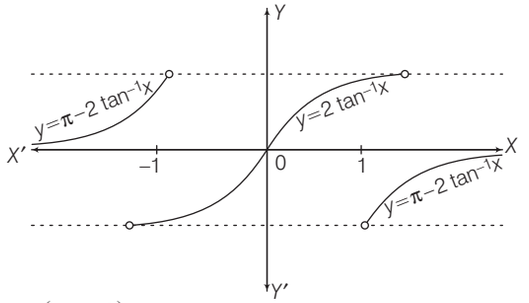
5. (i) $y = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$



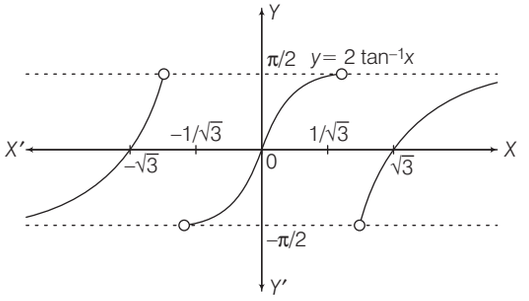
(ii) $y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$



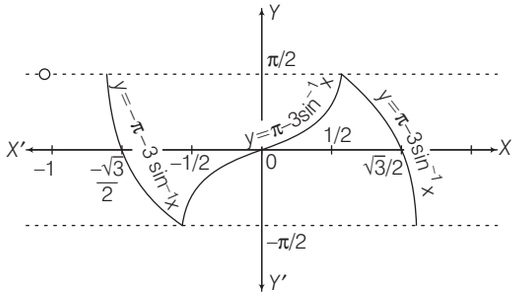
(iii) $y = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$



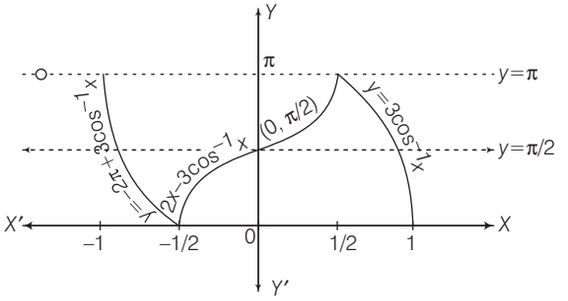
(iv) $y = \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)$



(v) $y = \sin^{-1}(3x - 4x^3)$



(vi) $y = \cos^{-1}(4x^3 - 3x)$



Exercise for Session 5

6. $\dfrac{\sqrt{3}}{2}$ **7.** $\dfrac{2}{19}$

Exercise for Session 6

1. $x^x(1 + \log x)$

2. $x^{\sqrt{x}}\left(\dfrac{\log x}{2\sqrt{x}} + \dfrac{1}{\sqrt{x}}\right)$

3. $x^{x^x} \cdot x^x \cdot \left\{(1 + \log x) \cdot \log x + \dfrac{1}{x}\right\}$

4. $x \cdot x^{x^2} \cdot (2 \log x + 1)$

5. $x^{x+\frac{1}{2}}\left\{\left(\dfrac{2x+1}{2x}\right) + \log x\right\}$

6. $(\cos x)^x\ (\log \cos x - x \tan x)$

7. $(\sin x)^{\cos x}\left(-\sin x \log \sin x + \dfrac{\cos^2 x}{\sin x}\right)$

8. $(\sin x)^{\cos^{-1} x}\left(\cos^{-1} x \cot x - \dfrac{\log (\sin x)}{\sqrt{1-x^2}}\right)$

9. $-x^x\ (\sin x^x) \cdot \{1 + \log x\}$

10. $\dfrac{1}{x^x + \operatorname{cosec}^2 x}\{x^x(1 + \log x) - 2\ \operatorname{cosec}^2 x \cot x\}$

11. $(\sin x)^{\tan x} \cdot \{\sec^2 x (\log \sin x) + 1\}$
$+ (\cos x)^{\sec x} \cdot \{\sec x \tan x \cdot \log (\cos x) - \sec x \tan x\}$

13. $-\left(\dfrac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}\right)$

14. $\dfrac{\log \sin y + y \tan x}{\log \cos x - x \cot y}$

19. $\dfrac{\alpha}{\alpha^2 + x^2}$

Exercise for Session 7

1. (i) 1 (ii) – 1 (iii) – 1

2. $x^{\sin^{-1} x}\left(\log x + \dfrac{\sqrt{1-x^2}}{x}\sin^{-1} x\right)$

3. $-\dfrac{2}{ax}$ **4.** $-2\sqrt{\cos x} \cdot \cot x \operatorname{cosec} x$

5. $-\dfrac{1}{2}$ **6.** x^x **7.** (i) $-\dfrac{1}{x}$ (ii) $\dfrac{1}{x}$ (iii) $\dfrac{1}{x}$

Exercise for Session 8

6. $-\frac{1}{2at^3}$ **7.** $\frac{1}{3a}\sec^4\theta\,\text{cosec}\,\theta$

Exercise for Session 9

3. $\frac{3}{\pi\sqrt{\pi^2-3}}$

Exercise for Session 10

1. $\frac{1}{3}$ **2.** $-\frac{f''(x)}{(f'(x))^3}$ **3.** (b) **4.** (c)

Chapter Exercises

1. (b) **2.** (c) **3.** (c) **4.** (d) **5.** (c)
6. (a) **7.** (d) **8.** (d) **9.** (b)
10. (b) **11.** (c) **12.** (b) **13.** (b) **14.** (c)
15. (a) **16.** (c) **17.** (c) **18.** (b) **19.** (c)
20. (a) **21.** (b) **22.** (a) **23.** (c) **24.** (b)
25. (d) **26.** (a) **27.** (b) **28.** (a)
29. (c) **30.** (c) **31.** (d) **32.** (b)
33. (a) **34.** (d) **35.** (d) **36.** (c) **37.** (c)
38. (a) **39.** (d) **40.** (a) **41.** (b) **42.** (b)
43. (b) **44.** (b) **45.** (d) **46.** (d) **47.** (c)
48. (c) **49.** (d) **50.** (d) **51.** (d) **52.** (d)
53. (b, c) **54.** (b, d) **55.** (a, c) **56.** (a, c) **57.** (a, d)
58. (b, d) **59.** (b, c, d) **60.** (a, b) **61.** (a, b, c, d)
62. (a, c) **63.** (a, b) **64.** (a, b, c) **65.** (a)
66. (a) **67.** (a) **68.** (b) **69.** (a)
70. (d) **71.** (d) **72.** (a) **73.** (c)
74. (a) **75.** (c) **76.** (a) **77.** (c)
78. (a) **79.** (a) **80.** (b) **81.** (b)
82. (c) **83.** (b) **84.** (c) **85.** (c)
86. (b) **87.** (c) **88.** (b) **89.** (a)
90. (c)
91. (A) $\rightarrow$ (q), (B) $\rightarrow$ (r), (C) $\rightarrow$ (s), (D) $\rightarrow$ (p)
92. (A) $\rightarrow$ (q, r), (B) $\rightarrow$ (p), (C) $\rightarrow$ (q, s), (D) $\rightarrow$ (q, r)
93. (4) **94.** (1) **95.** (1) **96.** (3) **97.** (1) **98.** (1)
99. (0) **100.** (1) **101.** (2) **102.** (2) **103.** (1) **104.** (−l)
105. (9) **106.** (1) **107.** (3) **108.** (b) **109.** (b, c) **110.** (c)
111. (a) **112.** (d) **113.** (b) **114.** (a) **115.** (b)

Solutions

1. Here, $y = \dfrac{\sec x - \tan x}{\sec x + \tan x} \cdot \dfrac{\sec x - \tan x}{\sec x - \tan x}$

$\therefore \quad y = (\sec x - \tan x)^2$

$\Rightarrow \quad \dfrac{dy}{dx} = 2(\sec x - \tan x)\cdot(\sec x \tan x - \sec^2 x)$

$= -2\sec x(\sec x - \tan x)^2$

2. Here, $y = \dfrac{1 + x^2 + x^4}{1 + x + x^2} = \dfrac{(1 + x^2)^2 - x^2}{(1 + x + x^2)}$

$= \dfrac{(1 + x + x^2)(1 - x + x^2)}{(1 + x + x^2)} = (1 - x + x^2)$

$\therefore \quad \dfrac{dy}{dx} = 2x - 1 = ax + b$

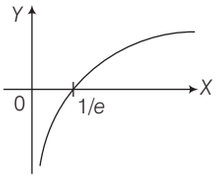
$\Rightarrow \quad a = 2, \; b = -1$

3. $y = \dfrac{d}{dx}(x \log x) = x \cdot \dfrac{1}{x} + \log x \cdot 1$

$\Rightarrow \quad y = 1 + \log_e x$

$\therefore y = 0$, when $\log_e x = -1 \;\Rightarrow\; x = \dfrac{1}{e}$

Thus, $y = 1 + \log_e x$ can be shown as



4. Here, $f'(x) = 1 + \dfrac{3}{x-2} = \dfrac{x+1}{x-2}$ and $g'(x) = 1 + \dfrac{5}{x-1} = \dfrac{x+4}{x-1}$

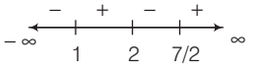
Since, $f'(x) < g'(x)$

$\Rightarrow \quad \dfrac{x+1}{x-2} < \dfrac{x+4}{x-1}$

or $\dfrac{x+1}{x-2} - \dfrac{x+4}{x-1} < 0$

$\Rightarrow \quad \dfrac{(x^2-1)-(x^2+2x-8)}{(x-1)(x-2)} < 0$

$\Rightarrow \quad \dfrac{-2x+7}{(x-1)(x-2)} < 0 \;\Rightarrow\; \dfrac{2x-7}{(x-1)(x-2)} > 0$



$\therefore \quad x \in (1,2) \cup (7/2, \infty)$, since $\log(x-2)$ exists, when $x > 2$.

$\Rightarrow \quad x \in \left(\dfrac{7}{2}, \infty\right)$

5. Here, $\cos^{-1}\left(\dfrac{x^2 - y^2}{x^2 + y^2}\right) = \log a$

$\Rightarrow \quad \cos^{-1}\left(\dfrac{1 - (y/x)^2}{1 + (y/x)}\right) = \log a$

Put $\dfrac{y}{x} = \tan\theta$

$\Rightarrow \quad \cos^{-1}\left(\dfrac{1 - \tan^2\theta}{1 + \tan^2\theta}\right) = \log a$

$\Rightarrow \quad \cos^{-1}(\cos 2\theta) = \log a$

$\Rightarrow \quad 2\theta = \log_e a$

$\Rightarrow \quad 2\tan^{-1}\left(\dfrac{y}{x}\right) = \log a$

$\Rightarrow \quad \dfrac{y}{x} = \tan\left(\dfrac{\log a}{2}\right)$

On differentiating both sides, we get

$$\frac{x\dfrac{dy}{dx} - y \cdot 1}{x^2} = 0$$

$\Rightarrow \quad x\dfrac{dy}{dx} - y = 0$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{y}{x}$

6. Here, $f(x) = (|x|)^{|\sin x|}$ as $x \to \dfrac{-\pi}{4}$

$\Rightarrow \quad f(x) = (-x)^{-\sin x}$

Taking logarithm on both sides, we get

$$\log(f(x)) = (-\sin x)\log(-x)$$

On differentiating both sides, we get

$$\frac{1}{f(x)} \cdot f'(x) = (-\sin x)\frac{1}{x} + \log(-x)\cdot(-\cos x)$$

At $x = -\dfrac{\pi}{4}$,

$$f'\left(-\frac{\pi}{4}\right) = f\left(-\frac{\pi}{4}\right)\left[\frac{-\dfrac{1}{\sqrt{2}}}{\dfrac{\pi}{4}} + \log\left(\frac{\pi}{4}\right)\cdot\left(-\frac{1}{\sqrt{2}}\right)\right]$$

$$= \left(\frac{\pi}{4}\right)^{\frac{1}{\sqrt{2}}}\left[\frac{\sqrt{2}}{2}\log\frac{4}{\pi} - \frac{2\sqrt{2}}{\pi}\right]$$

7. Here, $y = \dfrac{x}{a + \dfrac{x}{b+y}} = \dfrac{x(b+y)}{ab + ay + x}$

$\Rightarrow \quad aby + ay^2 + xy = bx + xy$

$\Rightarrow \quad aby + ay^2 = bx$

On differentiating w.r.t. x, we get

$$ab\frac{dy}{dx} + 2ay\frac{dy}{dx} = b$$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{b}{a(b+2y)}$

8. Taking logarithm on both sides, we get

$$\log y = x^2 \log x$$

On differentiating both sides, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = x^2\cdot\frac{1}{x} + 2x\cdot(\log x)$$

$$\Rightarrow \quad \frac{dy}{dx} = x^{x^2}\cdot(x + 2x\cdot\log x)$$

$$= x^{x^2+1}\cdot(1+2\log x) = x^{x^2+1}\cdot(\log(e\,x^2))$$

9. Here, $x\sqrt{1+y} = -y\sqrt{1+x}$

On squaring both sides, we get

$$x^2(1+y) = y^2(1+x)$$

$$\Rightarrow \quad x^2 + x^2y = y^2 + y^2x$$

$$\Rightarrow \quad (x-y)(x+y) + xy(x-y) = 0$$

$$\Rightarrow \quad (x-y)(x+y+xy) = 0 \Rightarrow \; x+y+xy = 0$$

$$\Rightarrow \quad y = \frac{-x}{(1+x)}$$

$$\therefore \quad \frac{dy}{dx} = -\left[\frac{(1+x)\cdot 1 - x\cdot 1}{(1+x)^2}\right] \Rightarrow \; \frac{dy}{dx} = \frac{-1}{(1+x)^2}$$

10. Here, $x^2e^y + 2xye^x + 13 = 0$

On differentiating both sides, we get

$$x^2\cdot e^y\cdot\frac{dy}{dx} + 2x\cdot e^y + 2\left[xe^x\cdot\frac{dy}{dx} + ye^x + yx\cdot e^x\right] = 0$$

$$\Rightarrow \quad \frac{dy}{dx}\cdot x(xe^y + 2e^x) + 2(xe^y + ye^x + xye^x) = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{2[xe^y + ye^x(x+1)]}{x(xe^y + 2e^x)}$$

or $$\frac{dy}{dx} = -2\left[\frac{(xe^{y-x} + y(x+1))}{x(xe^{y-x} + 2)}\right]$$

11. Here, $x = e^{y+x}$. Taking log on both sides, we get

$$\log x = (y+x)\log e$$

On differentiating w.r.t. x, we get

$$\frac{1}{x} = \frac{dy}{dx} + 1 \Rightarrow \frac{dy}{dx} = \frac{1}{x} - 1 = \frac{1-x}{x}$$

12. $g(f(x)) = x$, as g is the inverse of f.

$$\Rightarrow \quad g'(f(x))\cdot f'(x) = 1$$

$$\Rightarrow \quad g'(f(x)) = \frac{1+x^2}{x^{10}}, \text{ as } g(2) = a$$

Put $x = a$

$$\Rightarrow \quad g'(f(a)) = \frac{1+a^2}{a^{10}} \quad \Rightarrow f(a) = 2$$

$$\therefore \quad g'(2) = \frac{1+a^2}{a^{10}}$$

13. Here, $u'(x) = f'(g(x))\cdot g'(x)$

$$\Rightarrow \quad u'(1) = f'(g(1))\cdot g'(1) = f'(3)\cdot g'(1)$$

$$= \left(-\frac{1}{4}\right)\cdot(-3) = \frac{3}{4}$$

and $w'(x) = g'(g(x))\cdot g'(x)$

$$\Rightarrow \quad w'(1) = g'(g(1))\cdot g'(1) = g'(3)\cdot g'(1)$$

$$= \frac{3}{4}\cdot(-3) = -\frac{9}{4}$$

$$\therefore \quad u'(1) + w'(1) = \frac{3}{4} - \frac{9}{4} = -\frac{6}{4} = -\frac{3}{2}$$

14. Let $h(x) = f^2(x) + g^2(x)$

On differentiating both sides, we get

$$h'(x) = 2f(x)\cdot f'(x) + 2g(x)\cdot g'(x)$$

$$= -2g'(x)\cdot f'(x) + 2f'(x)\cdot g'(x) = 0$$

$\therefore$ $h(x)$ is constant.

Thus, $h(10) = f^2(10) + g^2(10) = h(5)$

$$\therefore \quad f^2(10) + g^2(10) = f^2(5) + g^2(5) = f^2(5) + (f'(5))^2 = 4 + 4 = 8$$

15. Here, $f(0) = 0$

$f'(0) = 1$

$f''(0) = 2$

$f'''(0) = 3$

$\vdots \quad \vdots$

$f^n(0) = n$

$$\therefore \; f(0) + f'(0) + f''(0) + f'''(0) + \ldots + f^n(0) = 1 + 2 + 3 + \ldots + n$$

$$= \frac{n(n+1)}{2}$$

16. Here, $\frac{dy}{dx} = f'(f(f(x)))\cdot f'(f(x))\cdot f'(x)$

$$\therefore \quad y'(0) = f'(f(f(0)))\cdot f'(f(0))\cdot f'(0)$$

$$= f'(f(0))\cdot f'(0)\cdot 2 = f'(0)\cdot f'(0)\cdot 2 = 2\times 2\times 2 = 8$$

17. Here, $y^2 = p(x)$

Differentiating both sides, we get $2y\cdot y' = p(x)$

Again, differentiating both sides, we get

$$2yy'' + 2(y')^2 = p''(x)$$

On multiplying by y^2, we get

$$2y^3\cdot y'' + 2(y\cdot y')^2 = y^2p''(x)$$

$$\because \quad y\cdot y' = \frac{p'(x)}{2} \qquad \left[\because\; y\cdot y' = \frac{p'(x)}{2}\right]$$

$$\Rightarrow \quad 2y^3\cdot y'' + 2\cdot\frac{(p'(x))^2}{4} = y^2p''(x)$$

$$\Rightarrow \quad 2y^3\cdot y'' + \frac{1}{2}\cdot(p'(x))^2 = p''(x)\cdot y^2$$

Again, differentiating both sides, we get

$$2\cdot\frac{d}{dx}\left(y^3\cdot\frac{d^2y}{dx^2}\right) + \frac{1}{2}\cdot 2p'(x)\cdot p''(x) = p'''(x)y^2 + 2y\cdot y'p''(x)$$

$$\Rightarrow 2\cdot\frac{d}{dx}\left(y^3\cdot\frac{d^2y}{dx^2}\right) = p'''(x)\cdot p(x) + 2y\cdot y'\cdot p''(x) - p'(x)\cdot p''(x)$$

$$[\because 2y\cdot y' = p'(x)]$$

$$= p'''(x)\cdot p(x) + p'(x)\cdot p''(x) - p'(x)\cdot p''(x)$$

$$= p'''(x)\cdot p(x)$$

18. As, $\frac{dy}{dx} = f'(x)$ and $\frac{dx}{dy} = g'(y)$

$$\Rightarrow \quad g'(y) = \frac{1}{f'(x)} \qquad \ldots(\text{i})$$

On differentiating w.r.t. y, we get

$$g''(y)=\frac{d}{dy}\left(\frac{1}{f'(x)}\right)=\frac{d}{dx}\left(\frac{1}{f'(x)}\right)\cdot\frac{dx}{dy}$$
$$=-\frac{f''(x)}{(f'(x))^2}\cdot\frac{dx}{dy}$$
$$=-\frac{f''(x)}{(f'(x))^2}\cdot\frac{1}{f'(x)}=-\frac{f''(x)}{(f'(x))^3}$$

$\therefore \quad g''(y)=-\dfrac{f''(x)}{(f'(x))^3}$

19. As discussed in Q.18,

$$\frac{d^2x}{dy^2}=-\frac{d^2y/dx^2}{\left(\frac{dy}{dx}\right)^3}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}=-\frac{d^2x}{dy^2}\cdot\left(\frac{dy}{dx}\right)^3 \qquad ...(i)$$

Now, $\dfrac{d^2y}{dx^2}+y\cdot\dfrac{dy}{dx}=0$

$$\Rightarrow -\frac{d^2x}{dy^2}\cdot\left(\frac{dy}{dx}\right)^3+y\left(\frac{dy}{dx}\right)=0$$

$$\Rightarrow \quad \frac{d^2x}{dy^2}-y\left(\frac{dx}{dy}\right)^2=0$$

20. $g(x+1)=\log(f(x+1))=\log x+\log(f(x))$

$\Rightarrow \; g(x+1)-g(x)=\log x$

On differentiating both sides, we get

$$g'(x+1)-g'(x)=\frac{1}{x}$$

Again, differentiating both sides, we get

$$g''(x+1)-g''(x)=-\frac{1}{x^2} \qquad ...(i)$$

$\therefore$
$$g''\left(1+\frac{1}{2}\right)-g''\left(\frac{1}{2}\right)=-4$$
$$g''\left(2+\frac{1}{2}\right)-g''\left(1+\frac{1}{2}\right)=-\frac{4}{9}$$
$$g''\left(3+\frac{1}{2}\right)-g''\left(2+\frac{1}{2}\right)=\frac{-4}{25}$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$g''\left(N+\frac{1}{2}\right)-g''\left(N-\frac{1}{2}\right)=-\frac{4}{(2N-1)^2}$$

On adding, we get

$$g''\left(N+\frac{1}{2}\right)-g''\left(\frac{1}{2}\right)=-4\left(1+\frac{1}{9}+\frac{1}{25}+\ldots+\frac{1}{(2N-1)^2}\right)$$

21. Here, $f(x)=x^3+e^{x/2}$ and $g(x)=f^{-1}(x)$

$$\Rightarrow \quad g'(f(x))\cdot f'(x)=1$$

Putting $f(x)=1\Rightarrow x^3+e^{x/2}=1 \;\Rightarrow\; x=0$

$$\therefore \quad g'(1)\cdot f'(0)=1,\; f'(x)=3x^2+\frac{1}{2}e^{x/2}$$

$$\Rightarrow \quad g'(1)\cdot\frac{1}{2}=1 \;\Rightarrow\; g'(1)=2$$

22. Let $f(\theta)=\sin\alpha$, where $\alpha=\tan^{-1}\left(\dfrac{\sin\theta}{\sqrt{\cos 2\theta}}\right)$

$$\Rightarrow \quad \tan\alpha=\frac{\sin\theta}{\sqrt{\cos 2\theta}}$$

$$\therefore \quad \sin\alpha=\frac{\sin\theta}{\cos\theta}=\tan\theta \qquad [\because\ \theta\in(-\pi/4,\pi/4)]$$

$$\Rightarrow \quad f(\theta)=\tan\theta \qquad \therefore \quad \frac{d(f(\theta))}{d(\tan\theta)}=1$$

23. We have, $y=\dfrac{\log\tan x}{\log\sin x}$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{(\log\sin x)\left(\frac{\sec^2 x}{\tan x}\right)-(\log\tan x)(\cot x)}{(\log\sin x)^2}$$

$$\therefore \quad \left[\frac{dy}{dx}\right]_{x=\pi/4}=\frac{-4}{\log 2} \qquad \text{[on simplification]}$$

24. We have, $y=\sum_{r=1}^{x}\tan^{-1}\left(\dfrac{1}{1+r+r^2}\right)=\sum_{r=1}^{x}\tan^{-1}\left[\dfrac{(r+1)-r}{1+(r+1)r}\right]$

$$=\sum_{r=1}^{x}[\tan^{-1}(r+1)-\tan^{-1}r]$$
$$=[\tan^{-1}2-\tan^{-1}1+\tan^{-1}3-\tan^{-1}2+\ldots+\tan^{-1}x$$
$$-\tan^{-1}(x-1)+\tan^{-1}(x+1)-\tan^{-1}x]$$
$$=[\tan^{-1}(x+1)-\tan^{-1}1]$$

$$\therefore \quad \frac{dy}{dx}=\frac{1}{1+(x+1)^2}$$

25. We have, $y=\sin^{-1}\left(\dfrac{\sin\alpha\sin x}{1-\cos\alpha\sin x}\right)$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{\sqrt{1-\frac{\sin^2\alpha\sin^2 x}{(1-\cos\alpha\sin x)^2}}}\cdot\frac{d}{dx}\left(\frac{\sin\alpha\sin x}{1-\cos\alpha\sin x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{(1-\cos\alpha\sin x)}{\sqrt{1+\cos 2\alpha\sin^2 x-2\cos\alpha\sin x}}\cdot\frac{\sin\alpha}{\cos\alpha}$$

$$\Rightarrow \quad \frac{dy}{dx}=-\frac{\frac{d}{dx}\left(-\frac{1}{1-\cos\alpha\sin x}\right)\tan\alpha\,(1-\cos\alpha\sin x)}{\{\sqrt{1+\cos 2\alpha\sin^2 x-2\sin x\cos\alpha}\}}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{\left[\frac{-\cos\alpha\cos x}{(1-\cos\alpha\sin x)^2}\right]\sin\alpha\cos x}{(1-\cos\alpha\sin x)\sqrt{[1+\cos 2\alpha\sin^2 x-2\cos\alpha\sin x]}}$$

$$\therefore \quad \left[\frac{dy}{dx}\right]_{x=0}=\sin\alpha \;\Rightarrow\; y'(0)=\sin\alpha$$

26. We have, $f(x)=\cot^{-1}\left(\dfrac{x^x-x^{-x}}{2}\right)$

$$\Rightarrow \quad f'(x)=-\frac{1}{1+\left(\frac{x^x-x^{-x}}{2}\right)^2}\cdot\frac{d}{dx}\left(\frac{x^x-x^{-x}}{2}\right)$$

$$\Rightarrow \quad f'(x)=\frac{-2}{4+(x^x-x^{-x})^2}\cdot\frac{d}{dx}(x^x-x^{-x})$$

$$\Rightarrow \quad f'(x)=\frac{-2}{(x^x+x^{-x})^2}\cdot\frac{d}{dx}(e^{x\log x}-e^{-x\log x})$$

$$\Rightarrow \quad f'(x)=\frac{-2}{(x^x+x^{-x})^2}\left\{e^{x\log x}\cdot\frac{d}{dx}(x\log x)-e^{-x\log x}\cdot\frac{d}{dx}(-x\log x)\right\}$$

$$\Rightarrow \quad f'(x)=\frac{-2}{(x^x+x^{-x})^2}\{x^x(1+\log x)+x^{-x}(1+\log x)\}$$

$$\Rightarrow \quad f'(x)=\frac{-2(1+\log x)}{(x^x+x^{-x})^2}\cdot(x^x+x^{-x})=\frac{-2(1+\log x)}{x^x+x^{-x}}$$

$$\therefore \quad f'(1)=\frac{-2}{(1+1)}=-1$$

27. $f(g(x))=x \Rightarrow f'(g(x))\,g'(x)=1$

$$\Rightarrow (e^{g(x)}+1)\,g'(x)=1$$

$$\Rightarrow (e^{g(f(\log 2))}+1)\,g'(f(\log 2))=1$$

$$\Rightarrow (e^{\log 2}+1)\,g'(f(\log 2))=1$$

$$\Rightarrow g'(f(\log 2))=\frac{1}{3}$$

28. $F'(x)=\left[f\left(\frac{x}{2}\right)\cdot f'\left(\frac{x}{2}\right)+g\left(\frac{x}{2}\right)g'\left(\frac{x}{2}\right)\right]$

Here, $g(x)=f'(x)$ and $g'(x)=f''(x)=-f(x)$

So, $F'(x)=f\left(\frac{x}{2}\right)g\left(\frac{x}{2}\right)-f\left(\frac{x}{2}\right)g\left(\frac{x}{2}\right)=0$

$\therefore$ $F(x)$ is a constant function. Hence, $F(10)=5$

29. We have, $x=\sec\theta-\cos\theta$ and $y=\sec^n\theta-\cos^n\theta$

$$\Rightarrow \quad \frac{dx}{d\theta}=\sec\theta\tan\theta+\sin\theta=\tan\theta(\sec\theta+\cos\theta)$$

and $$\frac{dy}{d\theta}=n[\sec^n\theta\tan\theta+\cos^{n-1}\theta\sin\theta]$$

$$=n\tan\theta(\sec^n\theta+\cos^n\theta)$$

$$\therefore \quad \frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{n\tan\theta(\sec^n\theta+\cos^n\theta)}{\tan\theta(\sec\theta+\cos\theta)}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2=n^2\cdot\frac{(\sec^n\theta-\cos^n\theta)^2+4}{(\sec\theta-\cos\theta)^2+4}=\frac{n^2(y^2+4)}{(x^2+4)}$$

$$\Rightarrow \quad (x^2+4)\left(\frac{dy}{dx}\right)^2=n^2(y^2+4)$$

30. We have, $x=f(t)\cos t-f'(t)\sin t$

and $y=f(t)\sin t+f'(t)\cos t$

$$\Rightarrow \quad \frac{dx}{dt}=-f(t)\sin t+f'(t)\cos t-f'(t)\cos t-f''(t)\sin t$$

$$=-[f(t)+f''(t)]\sin t$$

and $$\frac{dy}{dt}=f(t)\cos t+f'(t)\sin t-f'(t)\sin t+f''(t)\cos t$$

$$=[f(t)+f''(t)]\cos t$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=[f(t)+f''(t)]^2$$

31. Note that in y highest degree of x is 4 and therefore $\frac{d^3y}{dx^3}$ is a linear function of x, which is satisfied only in (d).

32. Exponent of $x=\frac{[l^2-m^2+m^2-n^2+n^2-l^2]}{(l-m)(m-n)(n-l)}$

$$\Rightarrow \quad y=x^0=1 \Rightarrow y=1 \Rightarrow y'=0$$

33. $y=(A+Bx)e^{mx}+(m-1)^{-2}\cdot e^x$

$$\Rightarrow y\cdot e^{-mx}=(A+Bx)+(m-1)^{-2}\cdot e^{(1-m)x}$$

$$\Rightarrow e^{-mx}\cdot y_1-mye^{-mx}=B-(m-1)^{-1}\cdot e^{-(m-1)x}$$

$$\Rightarrow e^{-mx}\cdot y_2-y_1e^{-mx}\cdot m-m[e^{-mx}\cdot y_1-ye^{-mx}\cdot m]=e^{-(m-1)x}$$

$$\Rightarrow e^{-mx}\cdot y_2-2my_1e^{-mx}+m^2y\cdot e^{-mx}=e^{-(m-1)x}$$

$$\therefore \quad y_2-2my_1+m^2y=e^x$$

34. We have, $f(x)=-\frac{x^3}{3}+x^2\sin 6-x\sin 4\cdot\sin 8$

$$-5\sin^{-1}((a-4)^2+1)\,(a=4)$$

$$\Rightarrow f'(x)=-x^2+2x\sin 6-\sin 4\sin 8$$

$$\Rightarrow f'(\sin 8)=-\sin^2 8+2\sin 6\sin 8-\sin 4\sin 8$$

$$=\sin 8\,[-\sin 8+2\sin 6-\sin 4]$$

$$=-\sin 8\,[\sin 8+\sin 4-2\sin 6]$$

$$=-\sin 8\,[2\sin 6\cos 2-2\sin 6]$$

$$=2\sin 8\sin 6\,[1-\cos 2]<0$$

sin 8	
sin 4	sin 6

35. $f(g(x))=x \Rightarrow f'(g(x))\cdot g'(x)=1;\ f'(g(a))\cdot g'(a)=1;\ f'(b)\cdot 2=1$

$$\Rightarrow \quad f'(b)=\frac{1}{2}$$

36. Put $\cos\phi=\frac{2}{\sqrt{13}};\sin\phi=\frac{3}{\sqrt{13}};\tan\phi=\frac{3}{2}$

$$y=\cos^{-1}\{\cos(x+\phi)\}+\sin^{-1}\{\cos(x-\phi)\}$$

$$=\cos^{-1}\{\cos(x+\phi)\}+\frac{\pi}{2}-\cos^{-1}\{\cos(\phi-x)\}$$

$$=x+\phi+\frac{\pi}{2}-\phi+x$$

$$y=2x+\frac{\pi}{2};z=\sqrt{1+x^2}$$

Now, $$\frac{dy}{dz}=\frac{\frac{dy}{dx}}{\frac{dz}{dx}}=\frac{2\sqrt{1+x^2}}{x}$$

$$\therefore \quad \left(\frac{dy}{dz}\right)_{x=\frac{3}{4}}=\frac{10}{3}$$

37. $f(x)=\sqrt{x+2\sqrt{2x-4}}+\sqrt{x-2\sqrt{2x-4}}$

$$\therefore \ f(x)=\sqrt{(\sqrt{x-2}+\sqrt{2})^2}+\sqrt{(\sqrt{x-2}-\sqrt{2})^2}$$

$$=|\sqrt{x-2}+\sqrt{2}|+|\sqrt{x-2}-\sqrt{2}|$$

For $\sqrt{x-2}$ to exist, $x\geq 2$

Also, $\sqrt{x-2}+\sqrt{2}>0$ [true]

But $\sqrt{x-2}-\sqrt{2}\geq 0$ only, if $x\geq 4 < 0$ only, if $x<4$

Now, $f(x)$ becomes

$$f(x)=\begin{cases}\sqrt{x-2}+\sqrt{2}-\sqrt{x-2}+\sqrt{2}, & \text{for } 2\le x<4\\ \sqrt{x-2}+\sqrt{2}+\sqrt{x-2}-\sqrt{2}, & \text{for } x\ge 4\end{cases}$$

$$\therefore\quad f(x)=\begin{cases}2\sqrt{2}, & \text{for } 2\le x<4\\ 2\sqrt{x-2}, & \text{for } 4\le x<\infty\end{cases}$$

$\because$ f is continuous in $[2, 4)\cup[4, \infty)$.

$$\therefore\quad f'(x)=\begin{cases}0, & 2\le x<4\\ \dfrac{1}{\sqrt{x-2}}, & 4\le x<\infty\end{cases}$$

$$\therefore\quad f'(102^+)=\frac{1}{\sqrt{102-2}}=\frac{1}{10}$$

$$\therefore\quad 10f'(102^+)=1$$

38. Here, $y=2\ln(1+\cos x)\Rightarrow y_1=\dfrac{-2\sin x}{1+\cos x}$

$$y_2=-2\left[\frac{(1+\cos x)\cos x-\sin x(-\sin x)}{(1+\cos x)^2}\right]$$

$$=-2\left[\frac{\cos x+1}{(1+\cos x)^2}\right]=\frac{-2}{(1+\cos x)}$$

$$\therefore\ 2e^{-y/2}=2\cdot e^{\frac{-\ln(1+\cos x)^2}{2}}=\frac{2}{(1+\cos x)}$$

$$\therefore\quad y_2+\frac{2}{e^{y/2}}=0$$

39. $y=\dfrac{(a+x)+\sqrt{a-x}\cdot\sqrt{a+x}}{(a-x)+\sqrt{a-x}\cdot\sqrt{a+x}}$

$$y=\frac{\sqrt{(a+x)}\,(\sqrt{a+x}+\sqrt{a-x})}{\sqrt{a-x}\ \ (\sqrt{a+x}+\sqrt{a-x})}=\left(\frac{a+x}{a-x}\right)^{1/2}$$

$$\frac{dy}{dx}=\frac{1}{2}\sqrt{\frac{a-x}{a+x}}\left(\frac{(a-x)+(a+x)}{(a-x)^2}\right)=\frac{1}{2}\frac{\sqrt{a-x}}{\sqrt{a+x}}\times\frac{2a}{(a-x)^2}$$

$$\therefore\quad \left(\frac{dy}{dx}\right)_{x=0}=\frac{1}{a}$$

40. We have, $u(x)=7v(x)$

$$\Rightarrow\quad u'(x)=7v'(x)\ \Rightarrow\ p=7$$

Again, $$\frac{u(x)}{v(x)}=7\ \Rightarrow\left(\frac{u(x)}{v(x)}\right)'=0$$

$$\Rightarrow\quad q=0$$

Now, $$\frac{p+q}{p-q}=\frac{7+0}{7-0}=1$$

41. For $x>1$, we have $f(x)=|\log|x||=\log x$

$$\Rightarrow\quad f'(x)=\frac{1}{x}$$

For $x<-1$, we have

$$f(x)=|\log|x||=\log(-x)$$

$$\Rightarrow\quad f'(x)=\frac{1}{x}$$

For $0<x<1$, we have

$$f(x)=|\log|x||=-\log x$$

$$f'(x)=-\frac{1}{x}$$

For $-1<x<0$, we have

$$f(x)=-\log(-x)$$

$$f'(x)=-\frac{1}{x}$$

Hence, $$f'(x)=\begin{cases}\dfrac{1}{x}, & |x|>1\\ -\dfrac{1}{x}, & |x|<1\end{cases}$$

42. We have, $f(x)=\cos\{x+3x+\ldots+(2n-1)x\}$

$$+i\sin\{x+3x+5x+\ldots+(2n-1)x\}$$

$$\Rightarrow\quad f(x)=\cos n^2x+i\sin n^2x$$

$$\Rightarrow\quad f'(x)=-n^2(\sin n^2x)+n^2(i\cos n^2x)$$

$$\Rightarrow\quad f''(x)=-n^4\cos n^2x-n^4i\sin n^2x$$

$$\Rightarrow\quad f''(x)=-n^4(\cos n^2x+i\sin n^2x)$$

$$\Rightarrow\quad f''(x)=-n^4f(x)$$

43. We have, $f(x)=x^n$

$$\Rightarrow\quad f(x+y)=(x+y)^n\Rightarrow f'(x+y)=n(x+y)^{n-1}$$

Also, $f'(x)=nx^{n-1}$ and $f'(y)=ny^{n-1}$

$$\therefore\quad f'(x+y)=f'(x)+f'(y)$$

$$\Rightarrow\ n(x+y)^{n-1}=n\cdot x^{n-1}+n\cdot y^{n-1}$$

$$\Rightarrow\quad (x+y)^{n-1}=x^{n-1}+y^{n-1}\qquad\ldots(i)$$

For $n-1>1$, we find that LHS of Eq. (i) is greater than the RHS. So, we must have $n-1\le 1$, i.e. $n-1=0$ or $n-1=1$.

Hence, $n=1$ or $n=2$

44. For $2<x<3$, we have $[x]=2$

$$\therefore\quad f(x)=\sin\left(\frac{2\pi}{3}-x^2\right)$$

$$\Rightarrow\quad f'(x)=-2x\cos\left(\frac{2\pi}{3}-x^2\right)$$

$$\Rightarrow\quad f'(\sqrt{\pi/3})=-2\sqrt{\pi/3}\cos\pi/3=-\sqrt{\pi/3}$$

45. We have, $u=e^x\sin x$

$$\Rightarrow\quad \frac{du}{dx}=e^x\sin x+e^x\cos x=u+v$$

and $$v=e^x\cos x$$

$$\Rightarrow\quad \frac{dv}{dx}=e^x\cos x+e^x\sin x=v-u$$

$$\therefore\quad v\frac{du}{dx}-u\frac{dv}{dx}=v(u+v)-u(v-u)=u^2+v^2$$

$$\frac{d^2u}{dx^2}=\frac{du}{dx}+\frac{dv}{dx}=u+v+v-u=2v$$

and $$\frac{d^2v}{dx^2}=\frac{dv}{dx}-\frac{du}{dx}=(v-u)-(v+u)=-2u$$

46. We have, $f(x) = \log_x \{\ln(x)\} = \dfrac{\ln\{\ln(x)\}}{\ln(x)}$

$$\therefore \quad f'(x) = \frac{\ln(x)\cdot\frac{1}{\ln(x)}\cdot\frac{1}{x} - \ln\{\ln(x)\}\frac{1}{x}}{\{\ln(x)\}^2} = \frac{1-\ln\{\ln(x)\}}{x\{\ln(x)\}^2}$$

$$\Rightarrow \quad f'(e) = \frac{1-\ln\{\ln(e)\}}{e\{\ln(e)\}^2} = \frac{1-\ln(1)}{e} = \frac{1}{e} \text{ or } e^{-1}$$

47. We have, $[f(x)]^n = f(nx)$ for all x

$$\Rightarrow \quad n[f(x)]^{n-1}f'(x) = nf'(nx)$$

$$\Rightarrow \quad n[f(x)]^n f'(x) = nf(x)f'(nx)$$

[multiplying both sides by $f(x)$]

$$\Rightarrow \quad nf(nx)f'(x) = nf(x)f'(nx) \qquad [\because [f(x)]^n = f(nx)]$$

$$\Rightarrow \quad f(nx)f'(x) = f(x)f'(nx)$$

48. Since, $f(x)$ is an odd differentiable function defined on R.

Therefore, $f(-x) = -f(x)$ for all $x \in R$

Differentiating both the sides w.r.t. x, we get

$$-f'(-x) = -f'(x) \text{ for all } x \in R$$

$$\Rightarrow \quad f'(-x) - f'(x) = 0 \text{ for all } x \in R$$

$$\Rightarrow \quad f'(-3) = f'(3) = -2$$

Aliter

We know that the derivative of a differentiable odd function is an even function. Therefore, $f'(x)$ is an even function.

Hence, $f'(-3) = f'(3) = -2$

49. We have, $y = \sqrt{x+\sqrt{y+\sqrt{x+\sqrt{y+\ldots\infty}}}}$

$$\Rightarrow \quad y^2 = x + \sqrt{y+\sqrt{x+\sqrt{y+\ldots\infty}}}$$

$$\Rightarrow \quad y^2 = x + \sqrt{y+y} \;\Rightarrow\; (y^2-x)^2 = 2y$$

Differentiating both the sides w.r.t. x, we get

$$2(y^2-x)\left(2y\frac{dy}{dx} - 1\right) = 2\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y^2-x}{2y^3-2xy-1}$$

50. We have, $f(x) = |\cos x - \sin x|$

$$\Rightarrow \quad f(x) = \begin{cases} \cos x - \sin x, & \text{for } 0 < x \le \dfrac{\pi}{x} \\ \sin x - \cos x, & \text{for } \dfrac{\pi}{4} < x < \dfrac{\pi}{2} \end{cases}$$

Clearly, $\left(\text{LHD at } x = \dfrac{\pi}{4}\right) = \left\{\dfrac{d}{dx}(\cos x - \sin x)\right\}_{\text{at } x=\frac{\pi}{4}}$

$$= (-\sin x - \cos x)_{x=\frac{\pi}{4}} = -\sqrt{2}$$

and $\left(\text{RHD at } x = \dfrac{\pi}{4}\right) = \left\{\dfrac{d}{dx}(\sin x - \cos x)\right\}_{\text{at } x=\frac{\pi}{4}}$

$$= (\cos x + \sin x)_{x=\frac{\pi}{4}} = \sqrt{2}$$

$$\therefore \quad \left(\text{LHD at } x = \frac{\pi}{4}\right) \neq \left(\text{RHD at } x = \frac{\pi}{4}\right)$$

Thus, $f'\left(\dfrac{\pi}{4}\right)$ doesn't exist.

51. We have, $f(x) = x^2 + xg'(1) + g''(2)$

and $g(x) = x^2 + xf'(2) + f''(3)$

$$\Rightarrow \quad f'(x) = 2x + g'(1)$$

and $g'(x) = 2x + f'(2)$ ...(i)

Putting $x = 1$ in Eq (i), we get

$$f'(1) = 2 + g'(1)$$

and $g'(1) = 2 + f'(2)$

$$\Rightarrow \quad f'(1) = 4 + f'(2)$$

Putting $x = 2$ in Eq. (i), we get

$$f'(2) = 4 + g'(1)$$

and $g'(2) = 4 + f'(2)$

$$\Rightarrow \quad g'(2) = 4 + 4 + g'(1) = 8 + g'(1)$$

Differentiating Eq. (i) w.r.t. x, we get

$$f''(x) = 2$$

and $g''(x) = 2$ for all x

$$\Rightarrow \quad f''(3) = 2$$

and $g''(2) = 2$

$$\Rightarrow g''(2) + f''(3) = 2 + 2 = 4$$

52. We have, $f(x) = x^n$

$$f^r(x) = n(n-1)(n-2)\ldots\{n-(r-1)\}x^{n-r}$$

$$\Rightarrow \quad f^r(x) = \frac{n!}{(n-r)!}x^{n-r}$$

$$\Rightarrow \quad f^r(1) = \frac{n!}{(n-r)!}$$

$$\therefore \; f(1) - \frac{f'(1)}{1!} + \frac{f''(1)}{2!} - \frac{f'''(1)}{3!} + \ldots + \frac{(-1)^n f^n(1)}{n!}$$

$$= \sum_{r=0}^{n}(-1)^r \frac{f^r(1)}{r!}, \text{ where } f^0(1) = f(1)$$

$$= \sum_{r=0}^{n}(-1)^r \frac{n!}{(n-r)!\,r!} = \sum_{r=0}^{n}(-1)^r {}^nC_r = 0$$

53. We have, $y + \log(1+x) = 0$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{1}{(1+x)} = 0$$

$$\Rightarrow \quad y' = -\frac{1}{(x+1)} \qquad \ldots(i)$$

Also, $\log(1+x) = -y$

$$\Rightarrow \quad 1 + x = e^{-y} \;\Rightarrow\; 1 = e^{-y}\cdot(-y')$$

$$\Rightarrow \quad e^y = -y'$$

or $y' + e^y = 0$

54. As, $y = 2^{3^x}$

$$\Rightarrow \quad \frac{dy}{dx} = 2^{3^x} \cdot \log 2 \cdot \left(\frac{d}{dx}(3^x)\right)$$

$$= 2^{3^x} \cdot \log 2 \cdot 3^x \cdot \log 3$$

$$= 2^{3^x} \cdot 3^x \cdot \log 2 \cdot \log 3 \quad \ldots(i)$$

Also, $y = 2^{3^x}$ and $\log_2 y = 3^x$

$\therefore$ Eq. (i) becomes,

$$\frac{dy}{dx} = y \cdot (\log_2 y) \cdot \log 3 \cdot \log 2$$

55. Here, $f(x) = x^2 + 3x - 3$

$$g(f(x)) = x$$

$$\Rightarrow \quad g'(f(x)) \cdot f'(x) = 1,\ f'(x) = 2x + 3$$

$$\Rightarrow \quad g'(f(x)) = \frac{1}{2x+3}$$

$$\therefore \quad g'(1) = \frac{1}{2g(1)+3} \quad \ldots(i)$$

and $g^{-1}(x) = f(x)$

When $f(x) = 1$,

then $g^{-1}(x) = 1 \Rightarrow x = g(1)$

Also, $f(x) = 1$

$$\Rightarrow \quad 1 = x^2 + 3x - 3$$

$$\Rightarrow \quad x^2 + 3x - 4 = 0$$

$$\Rightarrow \quad (x+4)(x-1) = 0$$

$$\Rightarrow \quad x = 1 \quad [\text{as } x > 0]$$

$$\therefore \quad g'(1) = \frac{1}{2(1)+3} = \frac{1}{5}$$

56. Here, $x^3 - 2x^2y^2 + 5x + y - 5 = 0$

On differentiating, we get

$3x^2 - 4xy^2 - 4x^2y\,dy/dx + 5 + dy/dx = 0$

$$\Rightarrow \quad y' = \frac{3x^2 - 4xy^2 + 5}{4x^2y - 1}$$

$$\therefore \quad y'(1) = \frac{3-4+5}{4-1} = \frac{4}{3}$$

Also, $y''(x)$

$$= \frac{[(6x - 4y^2 - 8xyy')](4x^2y - 1) - (8xy + 4x^2y')(3x^2 - 4xy^2 + 5)]}{(4x^2y - 1)^2}$$

$$\Rightarrow \quad y''(1) = \frac{\left(6 - 4 - 8 \cdot \frac{4}{3}\right) \cdot (4-1) - \left(8 + 4 \cdot \frac{4}{3}\right) \cdot (3 - 4 + 5)}{(4-1)^2}$$

$$= -8\frac{22}{27}$$

57. Here, $y = \sqrt{x+y}$

$$\Rightarrow \quad y^2 = x + y$$

On differentiating w.r.t. x, we get

$$2y \cdot \frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2y-1}$$

Also, $y = \frac{x}{y} + 1$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y \cdot 1 - x \cdot \frac{dy}{dx}}{y^2}$$

$$\Rightarrow \quad (y^2 + x)\frac{dy}{dx} = y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{y^2 + x} \quad [\because y^2 = x + y]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{(x+y)+x} = \frac{y}{2x+y}$$

58. We have, $y = x^{(\ln x)^{\ln(\ln x)}}$

$$\Rightarrow \quad \ln y = (\ln x)^{\ln(\ln x)} \ln x \quad \ldots(i)$$

$$\Rightarrow \ln(\ln y) = \ln(\ln x) \cdot \ln(\ln x) + \ln(\ln x)$$

$$\Rightarrow \quad \frac{1}{\ln y} \cdot \frac{1}{y} \cdot \frac{dy}{dx} = \frac{2\ln(\ln x)}{\ln x} \cdot \frac{1}{x} + \frac{1}{x \ln x}$$

$$= \frac{2\ln(\ln x) + 1}{x \ln x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{y}{x} \cdot \frac{\ln y}{\ln x}(2\ln(\ln x) + 1)$$

Substituting the value of $\ln y$ from Eq. (i), we get

$$\frac{dy}{dx} = \frac{y}{x}(\ln x)^{\ln(\ln x)}(2\ln(\ln x) + 1)$$

59. (a) We have, $y = \sin^{-1} 2x\sqrt{1-x^2}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2 - 4x^2}{\sqrt{1 - 4x^2 + 4x^4}}$$

Clearly, it is not defined at $x = \frac{1}{\sqrt{2}}$.

Not derivable at $x = \frac{1}{\sqrt{2}}$, check with $x = \sin\theta$ or direct differentiation.

(b) $g'(x) = \frac{1}{\sqrt{1 - \left(\frac{2 \cdot 2^x}{1 + 2^{2x}}\right)^2}} \cdot \frac{(1+4^x)(2^{x+1})\ln 2 - 2^{x+1} \cdot 4^x \cdot \ln 4}{(1+4^x)^2}$

$$= \frac{1 + 2^{2x}}{\sqrt{1 + 4^{2x} - 2 \cdot 2^{2x}}} \times \frac{2^{x+1}\ln 2\,(1 - 4^x)}{(1+4^x)^2}$$

$$= \frac{2^{x+1}\ln 2(1 - 4^x)}{\left|1 - 2^{2x}\right|(1 + 4^x)}$$

Clearly, it is not defined, when $1 - 2^x = 0$

i.e. at $x = 0$

$\Rightarrow$ Not derivable at $x = 0$

(c) $h(x) = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \frac{\pi}{2} - \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

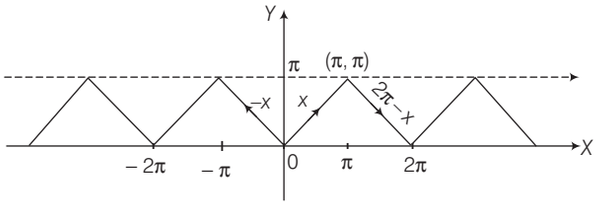
$$= \begin{cases} \frac{\pi}{2} - 2\tan^{-1}x, & x \geq 0 \\ \frac{\pi}{2} + 2\tan^{-1}x, & x < 0 \end{cases}$$

$$\therefore \quad h'(x) = \begin{cases} \frac{-2}{1+x^2}, & x > 0 \\ \text{not derivable}, & x = 0 \\ \frac{2}{1+x^2}, & x < 0 \end{cases}$$

$\therefore$ Not differentiable at $x = 0$.

(d) $k(x) = \sin^{-1}(\cos x) = \frac{\pi}{2} - \cos^{-1}(\cos x)$

and graph for $\cos^{-1}(\cos x)$, is shown as



$\therefore$ $k(x)$ is not differentiable at $x = 0$.

60. $f(x) = \frac{\sqrt{(\sqrt{x-1})^2 + 1 - 2\sqrt{x-1}}}{\sqrt{x-1}-1} \cdot x$

$$= \frac{|\sqrt{x-1}-1|}{\sqrt{x-1}-1} \cdot x = \begin{cases} -x, & \text{if } x \in [1, 2) \\ x, & \text{if } x \in (2, \infty) \end{cases}$$

$$\Rightarrow \quad f'(x) = \begin{cases} -1, & \text{if } x \in [1, 2) \\ 1, & \text{if } x \in (2, \infty) \end{cases}$$

$\therefore$ $f'(10) = 1$ and $f'\left(\frac{3}{2}\right) = -1$

61. $2^x + 2^y = 2^{x+y}$

Differentiating both sides, we get

$$2^x \log 2 + 2^y \log 2 \frac{dy}{dx} = 2^{x+y} \cdot \log 2 \left(1 + \frac{dy}{dx}\right)$$

$$\Rightarrow \quad \log 2 \cdot 2^y (2^x - 1) \frac{dy}{dx} = 2^x (1 - 2^y) \log 2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2^x (1-2^y)}{2^y (2^x - 1)} \qquad \ldots(i)$$

Also, $2^x = 2^y (2^x - 1)$

$$\therefore \quad \frac{dy}{dx} = (1 - 2^y) \qquad \ldots(ii)$$

when $2^y = 2^x (2^y - 1)$ [from Eq. (i)]

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-1}{2^x - 1} = \frac{1}{1 - 2^x} \qquad \ldots(iii)$$

As, $2^x = 2^y (2^x - 1)$ and $2^y = 2^x (2^y - 1)$

$$\therefore \quad -\frac{2^y}{2^x} = \frac{2^x (1-2^y)}{2^y (2^x - 1)}$$

Substituting this in Eq. (i), we get

$$\frac{dy}{dx} = -\frac{2^y}{2^x}$$

62. $f(x) = (x^2 + bx + c)e^x$

$$\therefore \quad f'(x) = \{x^2 + (b+2)x + (b+c)\}e^x$$

$f(x) > 0$ iff $D = b^2 - 4c < 0$

Now, $f'(x) > 0$ iff $D' = (b+2)^2 - 4(b+c)$

$$= D + 4 < 0$$

Thus, for $f'(x) > 0$, $D + 4 < 0$ holds.

$$\Rightarrow \quad D < 0$$

$$\Rightarrow \quad f(x) > 0$$

63. We have, $\sqrt{y+x} + \sqrt{y-x} = c$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx}\left(\frac{1}{\sqrt{y+x}} + \frac{1}{\sqrt{y-x}}\right) = \frac{1}{\sqrt{y-x}} - \frac{1}{\sqrt{y+x}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sqrt{y+x} - \sqrt{y-x}}{(\sqrt{y+x} + \sqrt{y-x})}$$

$$\frac{dy}{dx} = \frac{(y+x)-(y-x)}{(\sqrt{y+x} + \sqrt{y-x})^2} = \frac{2x}{c^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{x}{y + \sqrt{y^2 - x^2}}$$

Now, $\frac{dy}{dx} = \frac{(\sqrt{y+x} - \sqrt{y-x})^2}{(y+x)-(y-x)} = \frac{y - \sqrt{y^2 - x^2}}{x}$

64. $y = \tan x \cdot \tan 2x \cdot \tan 3x$ ...(i)

Differentiating both sides, we get

$$\frac{dy}{dx} = 3\sec^2 3x \cdot \tan x \cdot \tan 2x + \sec^2 x \cdot \tan 2x \cdot \tan 3x + 2\sec^2 2x \cdot \tan x \cdot \tan 3x \qquad \ldots(ii)$$

Taking log on both sides of Eq. (i), we get

$$\log y = \log \tan x + \log \tan 2x + \log \tan 3x$$

Differentiating both sides, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2 \operatorname{cosec} 2x + 4 \operatorname{cosec} 4x + 6 \operatorname{cosec} 6x$$

$$\therefore \quad \frac{dy}{dx} = 2y (\operatorname{cosec} 2x + 2 \operatorname{cosec} 4x + 3 \operatorname{cosec} 6x) \qquad \ldots(iii)$$

$$\tan(3x) = \tan(2x + x) = \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$$

$$\Rightarrow \quad \tan 3x \cdot \tan 2x \cdot \tan x = \tan 3x - \tan 2x - \tan x$$

Differentiating both sides, we get

$$\frac{dy}{dx} = 3\sec^2 3x - 2\sec^2 2x - \sec^2 x$$

65. Here, $f(x) = \frac{x}{(x-1)(x+1)} = \frac{1}{2}\left[\frac{(x+1)+(x-1)}{(x-1)(x+1)}\right]$

$$f(x) = \frac{1}{2}\left[\frac{1}{x-1} + \frac{1}{x+1}\right]$$

$$\Rightarrow \quad f'(x) = \frac{1}{2}\left[-\frac{1}{(x-1)^2} - \frac{1}{(x+1)^2}\right]$$

$$f''(x)=\frac{1}{2}\left[\frac{2}{(x-1)^3}+\frac{2}{(x+1)^3}\right]>0, \text{ for all } x>1$$

$\therefore f(x)$ is concave up.

66. Let $x=\tan\theta$, where $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$.

$$\Rightarrow \quad f(x)=\sin^{-1}(\sin 2\theta)$$

$$\therefore f(x)=\begin{cases}\pi-2\theta, & \frac{\pi}{2}<2\theta<\pi\\ 2\theta, & -\frac{\pi}{2}\le 2\theta\le\frac{\pi}{2}\\ -\pi-2\theta, & -\pi<2\theta<-\frac{\pi}{2}\end{cases}=\begin{cases}\pi-2\tan^{-1}x, & x>1\\ 2\tan^{-1}x, & -1\le x\le 1\\ -\pi-2\tan^{-1}x, & x<-1\end{cases}$$

$$\therefore f'(x)=\begin{cases}\frac{-2}{1+x^2}, & x>1\\ \frac{2}{1+x^2}, & -1\le x\le 1\\ \frac{-2}{1+x^2}, & x<-1\end{cases} \Rightarrow f'(2)=\frac{-2}{1+4}=\frac{-2}{5}$$

67. As, $(f(g(x))'=f'(g(x))\cdot g'(x)$

$$(f(g(0)))'=f'(g(0))\cdot g'(0)=c$$

$$\Rightarrow \quad f'(0)\cdot g'(0)=c \Rightarrow g'(0)=\frac{c}{b}$$

68. $\lim\limits_{x\to 0}\dfrac{g(x)\cos x-g(0)}{\sin x}=\lim\limits_{x\to 0}\dfrac{g'(x)\cos x-g(x)\sin x}{\cos x}=0$

Now, $f(x)=g(x)\sin x$

$$f'(x)=g(x)\cos x+g'(x)\sin x$$

$\therefore \quad f'(0)=0$

On differentiating both sides, we get

$$f''(x)=g'(x)\cos x-g(x)\sin x+g''(x)\sin x+g'(x)\cos x$$
$$=g''(x)\sin x+2g'(x)\cos x-g(x)\sin x$$

Also, $f'(0)=g'(0)$

$\therefore$ Both statements are correct but Statement II is not the correct explanation of Statement I.

69. Statement II is correct.

$$\therefore \quad y=\sin^{-1}(3x-4x^3)=3\sin^{-1}x, \text{ when } -\frac{1}{2}\le x\le\frac{1}{2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{3}{\sqrt{1-x^2}} \text{ only when } \frac{-1}{2}\le x\le\frac{1}{2}$$

$\therefore$ Both statements are true and Statement II is the correct explanation of Statement I.

70. Clearly, Statement I is false.

$$\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)=\begin{cases}2\tan^{-1}x, & x\ge 0\\ -2\tan^{-1}x, & x<0\end{cases}$$

$\therefore$ Statement II is correct but Statement I is incorrect.

71. Clearly, Statement II is true.

$$\therefore \quad \frac{dy}{dx}=\begin{cases}1, & x\notin \text{Integer}\\ \text{does not exist}, & x\in \text{Integer}\end{cases}$$

Thus, Statement I is incorrect but Statement II is correct.

72. We have, $f(x)$ is a continuous function $f: R\to Q$.

$f: R\to 0$, iff $f(x)$ is constant function.

$$f(5)=3$$
$$f(x)=3$$
$$f'(x)=0$$

So, Statements I and II both are correct and Statement II is the correct explanation of Statement I.

73. Let $f(x)=\sin^{-1}\dfrac{2x}{1+x^2}$, $g(x)=\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

$$f(x)=\begin{cases}2\tan^{-1}x, & -1\le x\le 1\\ \pi-2\tan^{-1}x, & x>1\\ -\pi-2\tan^{-1}x, & x<-1\end{cases}$$

$$g(x)=\begin{cases}2\tan^{-1}x, & x\ge 0\\ -2\tan^{-1}x, & x<0\end{cases}$$

$$f'(x)=\begin{cases}\frac{2}{1+x^2}, & -1\le x\le 1\\ -\frac{2}{1+x^2}, & x>1\\ -\frac{2}{1+x^2}, & x<-1\end{cases}$$

$$g'(x)=\begin{cases}\frac{2}{1+x^2}, & x\ge 0\\ -\frac{2}{1+x^2}, & x<0\end{cases}$$

$$\frac{f'(x)}{g'(x)}=1,\ 0<x<1$$

$\therefore \quad f'(x)\ne g'(x), -1\le x\le 1$

Statement I is correct and Statement II is incorrect.

74. Given, $f(x+y^3)=f(x)+f(y^3), \forall\ x, y\in R$

Put $x=y=0$, we get

$$f(0+0)=f(0)+f(0) \Rightarrow f(0)=0$$

Now, put $y=-x^{1/3}$, we get

$$f(0)=f(x)+f(-x)$$

$$\Rightarrow \quad f(x)+f(-x)=0$$

$\Rightarrow f(x)$ is an odd function.

$\Rightarrow f'(x)$ is an even function.

$$\Rightarrow \quad f'(2)=f'(-2)=a$$

Sol. (Q. Nos. 75 to 77)

Consider, $f(x+y)-f(x)=f(y)-1+2xy$

$$\Rightarrow \quad f(0+0)-f(0)=f(0)-1+0$$

$$\Rightarrow \quad f(0)=1 \text{ and } f'(0)=1 \qquad \text{[given]}$$

Also, $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=\lim\limits_{h\to 0}\left[\dfrac{f(h)-1}{h}+\dfrac{2xh}{h}\right]$

$$f'(x)=f'(0)+2x$$

$\therefore \quad f'(x)=2x+1$

Integrating both sides, $f(x)=x^2+x+c$

As, $\quad f(0)=1 \Rightarrow c=1$

$\therefore \quad f(x)=x^2+x+1$

Thus, domain of $\log(f(x)) = \log(x^2 + x + 1)$ is $x \in R$

Range of $x^2 + x + 1 \geq 3/4$

$\therefore$ Range of $\log_{3/4}(x^2 + x + 1) \leq 1$

$\Rightarrow$ Range $\in (-\infty, 1]$

and $\quad g(0) = \dfrac{g(0) + g(0)}{k} \Rightarrow 2g(0) = kg(0)$

$\Rightarrow \quad g(0) = 0 \qquad [\because k \neq 2]$

$$g'(x) = \lim_{h \to 0} \frac{g\left(\frac{x+h}{1}\right) - g(x)}{h} = \lim_{h \to 0} \frac{g(x) + g(h) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{g(h) - g(0)}{h} = g'(0) = \lambda$$

$\therefore \quad g(x) = \lambda x + c$

As $\quad g(0) = 0 \Rightarrow c = 0$

$\Rightarrow \quad g(x) = \lambda x$

$\Rightarrow \quad \lambda x = x^2 + x + 1 \Rightarrow x^2 + x(1 - \lambda) + 1 = 0$

$\because \quad D = 0$

$\therefore \quad (1 - \lambda)^2 - 4 = 0 \Rightarrow \lambda = 3, -1$

75. (c) ***76.*** (a) ***77.*** (c)

Sol. (Q. Nos. 78 to 80)

$$\text{LHD} = \lim_{h \to 0^-} \frac{f(a-h) - f(a)}{h}$$

$$= \lim_{h \to 0^-} \frac{-f(a-h) + f(a)}{h} = \lim_{h \to 0^-} \frac{f(a-h) - f(a)}{-h}$$

$$\text{RHD} = \lim_{h \to 0^-} \frac{f'(a) + f'(-a+h)}{h} = \lim_{h \to 0^-} \frac{f'(a) + f'(h-a)}{h}$$

$$\text{Also, } \lim_{h \to 0^-} \frac{f(-x) - f(-x-h)}{h} = \lim_{h \to 0^-} \frac{f(-x-h) - f(-x)}{-h}$$

$$= f'(-x) \qquad \ldots(\text{i})$$

and $\quad \lim_{h \to 0} \dfrac{f(x) - f(x-h)}{-h} = -f'(x) \qquad \ldots(\text{ii})$

From Eqs. (i) and (ii), $f'(x)$ is odd function and hence $f(x)$ is even function.

78. (a) ***79.*** (a) ***80.*** (b)

Sol. (Q. Nos. 81 to 82)

Since, $\quad f(x) = \sin^{-1}(3x - 4x^3)$

$$= \begin{cases} -\pi - 3\sin^{-1} x, & -1 \leq x \leq -\dfrac{1}{\sqrt{2}} \\ 3\sin^{-1} x, & -\dfrac{1}{2} \leq x \leq \dfrac{1}{2} \\ \pi - 3\sin^{-1} x, & \dfrac{1}{2} \leq x \leq 1 \end{cases}$$

$$\therefore \quad f'(x) = \begin{cases} \dfrac{3}{\sqrt{1-x^2}}, & -\dfrac{1}{2} \leq x \leq \dfrac{1}{2} \\ -\dfrac{3}{\sqrt{1-x^2}}, & \dfrac{1}{2} \leq |x| \leq 1 \end{cases}$$

$\Rightarrow \quad f'(0) = 3$ and $f'\left(\dfrac{1}{\sqrt{2}}\right) = -3\sqrt{2}$

81. (b) ***82.*** (c)

Sol. (Q. Nos. 83 to 84)

$$D^*(u \cdot v) = \lim_{h \to 0} \frac{f^2(x+h) \cdot g^2(x+h) - f^2(x) \cdot g^2(x)}{h}$$

$$= \lim_{h \to 0} \frac{\begin{matrix} f^2(x+h) \cdot g^2(x+h) - f^2(x+h)\, g^2(x) \\ + f^2(x+h)\, g^2(x) - f^2(x)\, g^2(x) \end{matrix}}{h}$$

$$= \lim_{h \to 0} f^2(x+h) \cdot \left(\frac{g^2(x+h) - g^2(x)}{h}\right) + g^2(x) \left(\lim_{h \to 0} \frac{f^2(x+h) - f^2(x)}{h}\right)$$

$$= f^2(x) \cdot (D^* v) + g^2(x) \cdot (D^* u)$$

$$= u^2 (D^* v) + v^2 (D^* u)$$

$$\text{Also, } D^*\left(\frac{u}{v}\right) = \lim_{h \to 0} \frac{\dfrac{f^2(x+h)}{g^2(x+h)} - \dfrac{f^2(x)}{g^2(x)}}{h}$$

$$= \lim_{h \to 0} \frac{f^2(x+h) \cdot g^2(x) - g^2(x+h) \cdot f^2(x)}{g^2(x+h) \cdot g^2(x) \cdot h}$$

$$= \lim_{h \to 0} \frac{\begin{bmatrix} f^2(x+h)\, g^2(x) - f^2(x)\, g^2(x) \\ + f^2(x)\, g^2(x) - g^2(x+h)\, f^2(x) \end{bmatrix}}{g^2(x+h) \cdot g^2(x) \cdot h}$$

$$= \lim_{h \to 0} \frac{\begin{bmatrix} \left(\dfrac{f^2(x+h) - f^2(x)}{h}\right) \cdot g^2(x) \\ - f^2(x) \cdot \left(\lim\limits_{h \to 0} \dfrac{g^2(x+h) - g^2(x)}{h}\right) \end{bmatrix}}{g^2(x+h) \cdot g^2(x)}$$

$$= \frac{g^2(x) \cdot (D^* u) - f^2(x) \cdot (D^* v)}{v^2 \cdot v^2}$$

$$= \frac{v^2 \cdot (D^* u) - u^2 (D^* v)}{v^4}$$

83. (b) ***84.*** (c)

85. $y = e^t \sin t \Rightarrow \dfrac{dy}{dt} = e^t [\cos t + \sin t]$

$x = e^t \cos t \Rightarrow \dfrac{dx}{dt} = e^t [\cos t - \sin t]$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{\cos t + \sin t}{\cos t - \sin t} = \tan \alpha$

$\Rightarrow \quad \tan\left(\dfrac{\pi}{4} + t\right) = \tan \alpha$

$\Rightarrow \quad \left(\dfrac{\pi}{4} + t\right) = \alpha \Rightarrow t = \alpha - \dfrac{\pi}{4}$

86. $\dfrac{d^2y}{dx^2} = \dfrac{\sec^2\left(\dfrac{\pi}{4} + t\right)}{e^t(\cos t - \sin t)}$

$\therefore \quad \left(\dfrac{d^2y}{dx^2}\right)_{t=0} = 2$

87. $F(t) = \int e^t(\cos t + \sin t)\, dt = e^t \sin t + C$

$$F\left(\frac{\pi}{2}\right) - F(0) = (e^{\pi/2} + C) - 0 = e^{\pi/2}$$

Sol. (Q. Nos. 88 to 90)

88. Since, 1, a_1, a_2, ..., a_n are roots of $x^n - 1 = 0$.

$$\therefore \quad x^n - 1 = (x-1)(x-a_1)(x-a_2)\ldots(x-a_n) \qquad \ldots(i)$$

$$\Rightarrow \quad \frac{x^n - 1}{x-1} = (x-a_1)(x-a_2)\ldots(x-a_n)$$

$$\Rightarrow \quad \lim_{x\to 1}\frac{x^n-1}{x-1} = \lim_{x\to 1}[(x-a_1)(x-a_2)\ldots(x-a_n)]$$

$$\Rightarrow \quad (1-a_1)(1-a_2)\ldots(1-a_n) = n$$

89. From Eq. (i), $\log(x^n - 1) = \log(x-1) + \log(x-a_1) + \ldots + \log(x-a_n)$

Differentiating w.r.t. x, we get

$$\frac{nx^{n-1}}{x^n-1} = \frac{1}{x-1} + \frac{1}{x-a_1} + \frac{1}{x-a_2} + \ldots + \frac{1}{x-a_n} \qquad \ldots(ii)$$

Putting $x = 2$ in Eq. (ii), we get

$$\frac{n2^{n-1}}{x^n-1} = 1 + \frac{1}{2-a_1} + \frac{1}{2-a_2} + \ldots + \frac{1}{2-a_n}$$

$$\Rightarrow \quad \frac{1}{2-a_1} + \frac{1}{2-a_2} + \ldots + \frac{1}{2-a_n} = \frac{n2^{n-1}}{2^n-1} - 1$$

$$= \frac{n2^{n-1} - 2^n + 1}{2^n - 1} = \frac{2^{n-1}(n-2)+1}{2^n-1}$$

90. From Eq. (ii), $\frac{nx^{n-1}}{x^n-1} - \frac{1}{x-1} = \frac{1}{x-a_1} + \frac{1}{x-a_2} + \ldots + \frac{1}{x-a_n}$

$$\Rightarrow \quad \frac{nx^{n-1} - 1(1 + x + x^2 + \ldots + x^{n-1})}{x^n - 1}$$

$$= \frac{1}{x-a_1} + \frac{1}{x-a_2} + \ldots + \frac{1}{x-a_n}$$

$$\Rightarrow \quad \lim_{x\to 1}\frac{nx^{n-1} - 1(1 + x + x^2 + \ldots + x^{n-1})}{x^n - 1}$$

$$= \lim_{x\to 1}\left(\frac{1}{x-a_1} + \frac{1}{x-a_2} + \ldots + \frac{1}{x-a_n}\right)$$

$$\Rightarrow \quad \lim_{x\to 1}\frac{n(n-1)x^{n-2} - \{1 + 2x + \ldots + (n-1)x^{n-2}\}}{nx^{n-1}}$$

$$= \frac{1}{1-a_1} + \frac{1}{1-a_2} + \ldots + \frac{1}{1-a_n}$$

[applying L'Hospital's rule on LHS]

$$\Rightarrow \quad \frac{n(n-1) - \{1 + 2 + \ldots + (n-1)\}}{n}$$

$$= \frac{1}{1-a_1} + \frac{1}{1-a_2} + \ldots + \frac{1}{1-a_n}$$

$$\Rightarrow \quad \frac{1}{1-a_1} + \frac{1}{1-a_2} + \ldots + \frac{1}{1-a_n} = \frac{n-1}{2}$$

91. (A) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{12t^2 - 6t - 18}{5t^4 - 15t^2 - 20}$

$$\Rightarrow \left[\frac{dy}{dx}\right]_{t=1} = \frac{12 - 6 - 18}{5 - 15 - 20} = \frac{2}{5} \Rightarrow \left[-5\frac{dy}{dx}\right]_{t=1} = -2$$

(B) Let us take

$$P(x) = a(x-2)^4 + b(x-2)^3 + c(x-2)^2 + d(x-2) + e$$

$$-1 = P(2) = e$$

$$0 = P'(2) = d$$

$$2 = P''(2) = 2c$$

$$\Rightarrow \quad c = 1$$

$$-12 = P'''(2) = 6b$$

$$\Rightarrow \quad b = -2$$

$$P^{iv}(2) = 24\, a = 24 \Rightarrow a = 1$$

Thus, $P''(x) = 12(x-2)^2 - 12(x-2) + 2$

$$\Rightarrow \quad P''(3) = 12 - 12(1) + 2 = 2$$

(C) Here, $\sqrt{(1+y^4)} = \sqrt{\left(1 + \frac{1}{x^4}\right)} = \frac{\sqrt{1+x^4}}{x^2} \qquad \left[\because y = \frac{1}{x}\right]$

$$\Rightarrow \quad \frac{\sqrt{1+y^4}}{\sqrt{1+x^4}} = \frac{1}{x^2} \qquad \ldots(i)$$

But $\quad y = \frac{1}{x}$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{x^2} \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$\frac{\sqrt{1+y^4}}{\sqrt{1+x^4}} = -\frac{dy}{dx} \quad \Rightarrow \frac{dy}{\sqrt{(1+y^4)}} + \frac{dx}{\sqrt{(1+x^4)}} = 0$$

$$\Rightarrow \quad \frac{\frac{dy}{\sqrt{1+y^4}}}{\frac{dx}{\sqrt{1+x^4}}} = -1$$

(D) Obviously, $f(x)$ is a linear function.

Also, from $f'(0) = p$ and $f(0) = q$,

$$f(x) = px + q \Rightarrow f''(0) = 0$$

92. (A) We know that,

$$2\tan^{-1}x = \begin{cases} \sin^{-1}\left(\frac{2x}{1+x^2}\right), & \text{if } -1 \le x \le 1 \\ \pi - \sin^{-1}\left(\frac{2x}{1+x^2}\right), & \text{if } x > 1 \\ -\pi - \sin^{-1}\left(\frac{2x}{1+x^2}\right), & \text{if } x < -1 \end{cases}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{2}{1+x^2}, \text{ if } x < -1 \text{ or } x > 1$$

(B) $\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \begin{cases} \tan^{-1}x, & x \ge 0 \\ -\tan^{-1}x, & x < 0 \end{cases}$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{1+x^2}, \text{ if } x < 0$$

(C) $y = |e^{|x|} - e| = \begin{cases} |e^x - e|, & x \geq 0 \\ |e^{-x} - e|, & x < 0 \end{cases}$

$$= \begin{cases} e^x - e, & x \geq 1 \\ e - e^x, & 0 \leq x < 1 \\ e - e^{-x}, & -1 \leq x < 0 \\ e^{-x} - e, & x < -1 \end{cases}$$

$\Rightarrow \quad \frac{dy}{dx} > 0$, if $x > 1$ or $-1 < x < 0$

(D) $\frac{du}{dx} = \frac{1}{x}$ and $\frac{dv}{dx} = \begin{cases} \frac{1}{1+x^2}, & x > 0 \\ -\frac{1}{1+x^2}, & x < 0 \end{cases}$

$\Rightarrow \quad \frac{du}{dv} = \begin{cases} \frac{1+x^2}{x}, & x > 0 \\ -\frac{1+x^2}{x}, & x < 0 \end{cases}$

Now, we know that $\frac{1+x^2}{x} = x + \frac{1}{x} > 2$, if $x > 1$ and < -2, if $x < -1 \Rightarrow \frac{du}{dv} > 2$, if $x < -1$ or $x > 1$

93. Here, $f(x) = x^3 + x^2 f'(1) + xf''(2) + f'''(3)$

Put $f'(1) = a, f''(2) = b, f'''(3) = c$

$\Rightarrow \quad f(x) = x^3 + ax^2 + bx + c$

$f'(x) = 3x^2 + 2ax + b$

$\Rightarrow \quad f'(1) = 3 + 2a + b$ or $a = 3 + 2a + b$

$\Rightarrow \quad a + b = -3$ ...(i)

$f'' = 6x + 2a \Rightarrow f''(2) = 12 + 2a$

$\therefore \quad b = 12 + 2a \Rightarrow 2a - b = -12$ ...(ii)

$f'''(x) = 6 \Rightarrow c = 6$ ...(iii)

From Eqs. (i) and (ii), we get

$a = -5, \ b = 2$

$\therefore \quad f(1) = 1 + a + b + c$

$= 1 - 5 + 2 + 6 = 4$

94. Let $y = \tan\left(\sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)\right)$ and $z = \tan^{-1} x$.

$\Rightarrow \quad y = \tan\left(\sin^{-1}\left(\frac{2x+2}{\sqrt{(2x+2)^2+9}}\right)\right)$

$= \tan\left(\tan^{-1}\left(\frac{2x+3}{3}\right)\right) = \frac{2x+3}{3}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{2}{3}$ and $\frac{dz}{dx} = \frac{1}{1+x^2}$

$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{2}{3}(1 + x^2)$

$\Rightarrow \quad \left(\frac{dy}{dz}\right)_{x=\frac{1}{\sqrt{2}}} = \frac{2}{3}\left(1 + \frac{1}{2}\right) = \frac{2}{3} \times \frac{3}{2} = 1$

95. $P(x) = (x - x_1)^3 (x - x_2)(x - x_3) \ldots (x - x_8)$

$\Rightarrow \quad P(x) = (x - x_1)^3 \cdot Q(x)$

$\therefore \quad Q(x) = \frac{P(x)}{(x - x_1)^3}$

As, $Q(x)$ is polynomial.

$\therefore \quad Q(x)$ must be continuous at $x = x_1$.

$\therefore \quad Q(x_1) = \lim_{x \to x_1} Q(x)$

$= \lim_{x \to x_1} \frac{x^{10} + ax^2 + bx + c}{(x - x_1)^3}$

By L' Hospital's rule, we have

$Q(x_1) = \frac{10 \times 9 \times 8 \times x_1^7}{3 \times 2 \times 1}$

$\therefore \quad Q\left(\frac{1}{2}\right) = \frac{10 \times 9 \times 8 \times \left(\frac{1}{2}\right)^7}{3 \times 2 \times 1} = \frac{15}{16} = \frac{p}{q}$

$\Rightarrow \quad q - p = 1$

96. Here, $f'(x) = \begin{vmatrix} 4(x-a)^3 & (x-a)^3 & 1 \\ 4(x-b)^3 & (x-b)^3 & 1 \\ 4(x-c)^3 & (x-c)^3 & 1 \end{vmatrix}$

$$+ \begin{vmatrix} (x-a)^4 & 3(x-a)^2 & 1 \\ (x-b)^4 & 3(x-b)^2 & 1 \\ (x-c)^4 & 3(x-c)^2 & 1 \end{vmatrix} + \begin{vmatrix} (x-a)^4 & (x-a)^3 & 0 \\ (x-b)^4 & (x-b)^3 & 0 \\ (x-c)^4 & (x-c)^3 & 0 \end{vmatrix}$$

$$= 0 + 3\begin{vmatrix} (x-a)^4 & (x-a)^2 & 1 \\ (x-b)^4 & (x-b)^2 & 1 \\ (x-c)^4 & (x-c)^2 & 1 \end{vmatrix} + 0$$

$\Rightarrow \quad \lambda = 3$

97. Let $P(x) = a(x-1)(x-3)(x-5)(x-\lambda)$

$\Rightarrow \log P(x) = \log a + \log(x-1) + \log(x-3) + \log(x-5) + \log(x-7)$

Differentiating both sides, we get

$\frac{1}{P(x)} \cdot P'(x) = \frac{1}{x-1} + \frac{1}{x-3} + \frac{1}{x-5} + \frac{1}{x-\lambda}$

As, $P'(7) = 0$ [$\because P(x) = 0 \Rightarrow P'(x) = 0 \Rightarrow P'(7) = 0$]

$\Rightarrow \quad 0 = \frac{1}{6} + \frac{1}{4} + \frac{1}{2} + \frac{1}{7-\lambda}$

$\Rightarrow \quad \lambda = \frac{89}{11} = \frac{p}{q}$

$\therefore \quad p - 8q = 1$

98. We have, $x^2 + y^2 = t - \frac{1}{t}$

$\Rightarrow \quad (x^2 + y^2)^2 = \left(t - \frac{1}{t}\right)^2 = t^2 + \frac{1}{t^2} - 2$

$\Rightarrow \quad x^4 + y^4 + 2x^2y^2 = x^4 + y^4 - 2$

$\Rightarrow \quad 2x^2y^2 = -2$

$\Rightarrow \quad y^2 = -x^{-2}$

Differentiating w.r.t. x, we get

$$2y \cdot \frac{dy}{dx} = +2x^{-3}$$

$$\therefore \quad \frac{dy}{dx} = +\frac{1}{x^3 y}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(1,1)} = 1$$

99. We have, $x^2 + y^2 + z^2 - 2xyz = 1$ ...(i)

$$\Rightarrow \quad d(x^2 + y^2 + z^2 - 2xyz) = 0$$

$$\Rightarrow \quad 2x\,dx + 2y\,dy + 2z\,dz - 2(xy\,dz + yz\,dx + zx\,dy) = 0$$

$$\Rightarrow \quad (x - yz)\,dx + (y - zx)\,dy + (z - xy)\,dz = 0$$

$$\Rightarrow \quad \frac{dx}{(y - zx)(z - xy)} + \frac{dy}{(x - yz)(z - xy)} + \frac{dz}{(x - yz)(y - zx)} = 0$$

Now, $(y - zx)^2 (z - xy)^2 = (y^2 - 2xyz + z^2x^2)(z^2 - 2xyz + x^2y^2)$

$= (1 - x^2 - z^2 + z^2x^2)(1 - x^2 - y^2 + x^2y^2)$ [using Eq. (i)]

$= (1 - x^2)(1 - z^2)(1 - x^2)(1 - y^2)$

$= (1 - x^2)^2 (1 - y^2)(1 - z^2)$

$\therefore (y - zx)(z - xy) = (1 - x^2) \cdot \sqrt{(1 - y^2)(1 - z^2)}$

Similarly, $(x - yz)(z - xy) = (1 - y^2) \cdot \sqrt{(1 - x^2)(1 - z^2)}$

and $(x - yz)(y - zx) = (1 - z^2) \cdot \sqrt{(1 - x^2((1 - y^2)}$

Substituting these values in Eq. (i), we get

$$\frac{dx}{\sqrt{1 - x^2}} + \frac{dy}{\sqrt{1 - y^2}} + \frac{dz}{\sqrt{1 - z^2}} = 0$$

100. We have, $x = \sin t$ or $t = \sin^{-1} x$

Now, $$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{\sqrt{1 - x^2}}$$

$$\Rightarrow \quad \sqrt{1 - x^2}\,\frac{dy}{dx} = \frac{dy}{dt}$$

Differentiating w.r.t. x, we get

$$\sqrt{1 - x^2} \cdot \frac{d^2y}{dx^2} + \frac{1(-2x)}{2\sqrt{1 - x^2}} \cdot \frac{dy}{dx} = \frac{d}{dx}\left(\frac{dy}{dt}\right)$$

$$\Rightarrow \quad \sqrt{1 - x^2} \cdot \frac{d^2y}{dx^2} - \frac{x}{\sqrt{1 - x^2}}\frac{dy}{dx} = \frac{d^2y}{dt^2} \cdot \frac{dt}{dx}$$

$$\Rightarrow \quad (1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = \sqrt{1 - x^2} \cdot \frac{d^2y}{dt^2} \cdot \frac{1}{\sqrt{1` - x^2}}$$

$$\therefore \quad (1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = \frac{d^2y}{dt^2}$$

Adding y on both sides, we get

$$(1 - x^{2)}\frac{d^2y}{dx^2} - x\frac{dx}{dy} + y = \frac{d^2y}{dx^2} + y$$

$$\therefore \quad \lambda = 1$$

101. Put $\cos\phi = \frac{2}{\sqrt{13}}$

and $\sin\phi = \frac{3}{\sqrt{13}}$

$$\Rightarrow \quad y = \cos^{-1}\{\cos(x + \phi)\} + \sin^{-1}\{\cos(x - \phi)\}$$

$$= x + \phi + \frac{\pi}{2} - \phi + x$$

$$y = 2x + \frac{\pi}{2}$$

and $z = \sqrt{1 + x^2}$

$$\therefore \quad \frac{dy}{dz} = \frac{2\sqrt{1 + x^2}}{x} = \left(\frac{2\sqrt{1 + x^2}}{x}\right)_{x = \frac{1}{\sqrt{3}}} = \frac{2\sqrt{1 + \frac{1}{3}}}{\frac{1}{\sqrt{3}}} = 4$$

102. Here, $f''(x) - 2f'(x) - 15f(x) = 0$

$$\Rightarrow \quad (a^2 - 2a - 15)\,e^{ax} + (b^2 - 2b - 15)\,e^{bx} = 0$$

$$\Rightarrow \quad a^2 - 2a - 15 = 0 \quad \text{and} \quad b^2 - 2b - 15 = 0$$

$$\Rightarrow \quad a = (5 \text{ or } -3) \text{ and } (b = 5 \text{ or } -3)$$

Since, $a \neq b$

$$\therefore \quad a + b = 2 \text{ or } -2$$

$$\Rightarrow \quad |a + b| = 2$$

103. $A : 2x + 2yy' = 0 \Rightarrow y' = -\frac{x}{y}$

$$\therefore \quad y'(\sqrt{2}) = -1$$

$B : \cos y \cdot y' + \cos x = \sin x \cdot \cos y \cdot y' + \sin y \cdot \cos x$

when $x = y = \pi$

$$\Rightarrow \quad -y' - 1 = 0 + 0$$

$$\Rightarrow \quad y'(\pi) = -1$$

$C : 2e^{xy}(xy' + y) + e^x e^y y' + e^y e^x - e^x - e^y y'$

$$= e \cdot e^{xy}(xy' + y)$$

At $x = 1, y = 1$

$2e(y' + 1) + e^2 y' + e^2 - e - ey' = e^2(y' + 1)$

$$\Rightarrow \quad ey' + e = 0 \Rightarrow y' = -1$$

Hence, $A - B - C = 1$

104. $\frac{dx}{dt} = -\frac{3}{t^4} - \frac{2}{t^3} = -\left(\frac{3 + 2t}{t^4}\right)$

$$\frac{dy}{dt} = -\left(\frac{3}{t^3} + \frac{2}{t^2}\right) = -\left(\frac{3 + 2t}{t^3}\right)$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = t$$

$$= t - \left(\frac{1 + t}{t^3}\right) \cdot t^3 = -1$$

Now, $\frac{du}{dx} - x \cdot \left(\frac{dy}{dx}\right)^3$

105. We have, $y = f(x) - f(2x)$

$\Rightarrow \quad y' = f'(x) - 2f'(2x)$

$y'(1) = f'(1) - 2f'(2) = 5 \quad ...(i)$

and $\quad y'(2) = f'(2) - 2f'(4) = 7 \quad ...(ii)$

Now, let $\quad y = f(x) - f(4x)$

$\Rightarrow \quad y' = f'(x) - 4f'(4x)$

$\Rightarrow \quad y'(1) = f'(1) - 4f'(4) \quad ...(iii)$

Substituting the value of $f'(2) = 7 + 2f'(4)$ in Eq. (i),

$f'(1) - 2[7 + 2f'(4)] = 5$

$\Rightarrow \quad f'(1) - 4f'(4) = 19$

Hence, $\quad 19 = 10 + \lambda$

$\Rightarrow \quad \lambda = 9$

106. $y = 3e^2 - x$

Let $\quad x^y = x^{3e^2 - x}$

$\Rightarrow \quad f(x) = x^{3e^2 - x}$

$\Rightarrow \quad \ln(f(x)) = (3e^2 - x)\ln x$

$\Rightarrow \quad \frac{1}{f(x)} \cdot f'(x) = \frac{3e^2 - x}{x} - \ln x$

$\therefore \quad f'(x) = 0$

$\Rightarrow \quad 3e^2 - x = x \ln x$

$\Rightarrow \quad 3e^2 = x(1 + \ln x)$

$\Rightarrow \quad x = e^2 \quad$ [by verification]

Hence, $\quad \lambda = 1$

107. $f'(x) = (kx + e^x)h'(x) + h(x)(k + e^x)$

$\therefore \quad f'(0) = h'(0) + h(0)(k + 1)$

$\Rightarrow \quad 18 = -2 + 5(k + 1)$

$\Rightarrow \quad k = 3$

108. We have, $f(x) = |\log 2 - \sin x|$

and $\quad g(x) = f(f(x)),\ x \in R$

Note that, for $x \to 0$, $\log 2 > \sin x$

$\therefore \quad f(x) = \log 2 - \sin x$

$\Rightarrow \quad g(x) = \log 2 - \sin(f(x))$

$= \log 2 - \sin(\log 2 - \sin x)$

Clearly, $g(x)$ is differentiable at $x = 0$ as $\sin x$ is differentiable.

Now, $\quad g'(x) = -\cos(\log 2 - \sin x)(-\cos x)$

$= \cos x \cdot \cos(\log 2 - \sin x)$

$\therefore \quad g'(0) = 1 \cdot \cos(\log 2)$

109. As, $g(f(x)) = x$

Thus, $g(x)$ is inverse of $f(x)$.

$\Rightarrow \quad g(f(x)) = x$

$\Rightarrow \quad g'(f(x)) \cdot f'(x) = 1$

$\therefore \quad g'(f(x)) = \frac{1}{f'(x)} \quad$ [where, $f'(x) = 3x^2 + 3$] $...(i)$

When $f(x) = 2$, then

$x^3 + 3x + 2 = 2$

$\Rightarrow \quad x = 0$

i.e. when $x = 0$, then $f(x) = 2$

$\therefore \quad g'(f(x)) = \frac{1}{3x^2 + 3}$ at $(0, 2)$

$\Rightarrow \quad g'(2) = \frac{1}{3}$

$\therefore$ Option (a) is incorrect.

Now, $\quad h(g(g(x))) = x$

$\Rightarrow \quad h(g(g(f(x)))) = f(x)$

$\Rightarrow \quad h(g(x)) = f(x) \quad ...(ii)$

As $\quad g(f(x)) = x$

$\therefore \quad h(g(3)) = f(3)$

$= 3^3 + 3(3) + 2 = 38$

$\therefore$ Option (d) is incorrect.

From Eq. (ii),

$h(g(x)) = f(x)$

$\Rightarrow \quad h(g(f(x))) = f(f(x))$

$\Rightarrow \quad h(x) = f(f(x)) \quad ...(iii)$

[using $g(f(x)) = x$]

$\Rightarrow \quad h'(x) = f'(f(x)) \cdot f'(x) \quad ...(iv)$

Putting $x = 1$, we get

$h'(1) = f'(f(1)) \cdot f'(1)$

$= (3 \times 36 + 3) \times (6) = 111 \times 6 = 666$

$\therefore$ Option (b) is correct.

Putting $x = 0$ in Eq. (iii), we get

$h(0) = f(f(0))$

$= f(2) = 8 + 6 + 2 = 16$

$\therefore$ Option (c) is correct.

110. Let $y = \tan^{-1}\left(\frac{6x\sqrt{x}}{1 - 9x^3}\right)$, where $x \in \left(0, \frac{1}{4}\right)$

$= \tan^{-1}\left(\frac{2 \cdot (3x^{3/2})}{1 - (3x^{3/2})^2}\right) = 2\tan^{-1}(3x^{3/2})$

As $3x^{3/2} \in \left(0, \frac{3}{8}\right)$

$\therefore \quad \frac{dy}{dx} = 2 \times \frac{1}{1 + 9x^3} \times 3 \times \frac{3}{2} \times x^{1/2}$

$= \frac{9}{1 + 9x^3}\sqrt{x}$

$\therefore \quad g(x) = \frac{9}{1 + 9x^3}$

111. Since, $f(x) = e^{g(x)}$

$\Rightarrow e^{g(x+1)} = f(x+1) = xf(x) = xe^{g(x)}$

and $\quad g(x+1) = \log x + g(x)$

i.e. $\quad g(x+1) - g(x) = \log x \quad ...(i)$

Replacing x by $x-\frac{1}{2}$, we get

$$g\left(x+\frac{1}{2}\right)-g\left(x-\frac{1}{2}\right)=\log\left(x-\frac{1}{2}\right)$$

$$=\log(2x-1)-\log 2$$

$$\therefore \quad g''\left(x+\frac{1}{2}\right)-g''\left(x-\frac{1}{2}\right)=\frac{-4}{(2x-1)^2} \qquad ...(ii)$$

On substituting $x = 1,2,3,...,N$ in Eq. (ii) and adding, we get

$$g''\left(N+\frac{1}{2}\right)-g''\left(\frac{1}{2}\right)=-4\left\{1+\frac{1}{9}+\frac{1}{25}+...+\frac{1}{(2N-1)^2}\right\}.$$

112. Since, $\frac{dx}{dy}=\frac{1}{dy/dx}=\left(\frac{dy}{dx}\right)^{-1}$

$$\Rightarrow \quad \frac{d}{dy}\left(\frac{dx}{dy}\right)=\frac{d}{dx}\left(\frac{dy}{dx}\right)^{-1}\frac{dx}{dy}$$

$$\Rightarrow \quad \frac{d^2x}{dy^2}=-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}\left(\frac{dx}{dy}\right)$$

$$=-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$$

113. Since, $f''(x)=-f(x)$

$$\Rightarrow \quad \frac{d}{dx}\{f'(x)\}=-f(x)$$

$$\Rightarrow \quad g'(x)=-f(x) \qquad [\because g(x)=f'(x), \text{ given}] \ ...(i)$$

Also, $F(x)=\left\{f\left(\frac{x}{2}\right)\right\}^2+\left\{g\left(\frac{x}{2}\right)\right\}^2$

$$\Rightarrow \quad F'(x)=2\left(f\left(\frac{x}{2}\right)\right)\cdot f'\left(\frac{x}{2}\right)\cdot\frac{1}{2}$$

$$+2\left(g\left(\frac{x}{2}\right)\right)\cdot g'\left(\frac{x}{2}\right)\cdot\frac{1}{2}=0 \text{ [from Eq.(i)]}$$

$\therefore$ $F(x)$ is constant $\Rightarrow F(10)=F(5)=5$

114. Given that, $\log(x+y)=2xy$ ...(i)

$\therefore$ At $x = 0$

$\Rightarrow \log(y)=0$

$\Rightarrow \quad y=1$

$\therefore$ To find $\frac{dy}{dx}$ at $(0, 1)$

On differentiating Eq. (i) w.r.t. x, we get

$$\frac{1}{x+y}\left(1+\frac{dy}{dx}\right)=2x\frac{dy}{dx}+2y\cdot 1$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{2y(x+y)-1}{1-2(x+y)x}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(0,1)}=1$$

115. Given, $x^2+y^2=1$

On differentiating w.r.t. x, we get

$$2x+2yy'=0$$

$$\Rightarrow \quad x+yy'=0$$

Again, differentiating w.r.t. x, we get

$$1+y'y'+yy''=0$$

$$\therefore \quad 1+(y')^2+yy''=0$$

CHAPTER

03

Functions

Learning Part

Session 1
- Function

Session 2
- Domain
- Algebraic Functions

Session 3
- Transcendental Functions

Session 4
- Piecewise Functions

Session 5
- Range

Session 6
- Odd and Even Functions

Session 7
- Periodic Functions

Session 8
- Mapping of Functions

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- Identical Functions

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Session 11
- Inverse of a Function

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- Miscellaneous Problems on Functions

Practice Part

- JEE Type Examples
- Chapter Exercises

Arihant on Your Mobile !
Exercises with this symbol can be practised on your mobile. See title inside to activate for free.

Session 1

Function

Function

Let A and B be two non-empty sets and f be a relation which associates each element of set A with unique element of set B, f is called a function from A to B.

Or

If for each element in a set A there is assigned a unique element of set B, we call such an assignment a function $f : A \to B$. "f is a function from A to B".

If $a \in A$, the element in B which is assigned to 'a' is called the image of 'a' and denoted by $f(a)$.

Here, set A is called the domain of f and B is called the codomain of f. The set of elements of B, which are the images of the elements of set A is called the range of f.

e.g. Let $A = \{a, b, c, d\}$, $B = \{1, 2, 3, 4, 5\}$. Here, $f(a) = 2$, $f(b) = 3$, $f(c) = 5$, $f(d) = 1$ given by

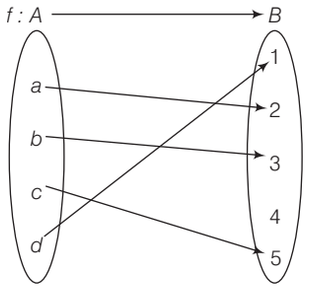


Figure 3.1

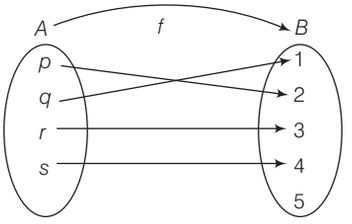
i.e. $A \to$ Domain of $f = \{a, b, c, d\}$

$B \to$ Codomain of $f = \{1, 2, 3, 4, 5\}$

Range of $f = \{1, 2, 3, 5\}$

An element of A (i.e. Domain) cannot associate with more than one element in B.

Example 1 In the given figure, find the domain, codomain and range.



Sol. From the given figure, we conclude that

$A \to$ Domain of $f = \{p, q, r, s\}$

$B \to$ Codomain of $f = \{1, 2, 3, 4, 5\}$

Range of $f = \{1, 2, 3, 4\}$

Remark

We should always remember that "the range is a subset of codomain".

Number of Functions (or Mapping) from *A* to *B*

Let $A = \{x_1, x_2, x_3, \ldots, x_m\}$ [i.e. m elements]

and $B = \{y_1, y_2, y_3, \ldots, y_n\}$ [i.e. n elements]

Then, each element x_i's $(i = 1, 2, \ldots, m)$ of domain corresponds n images.

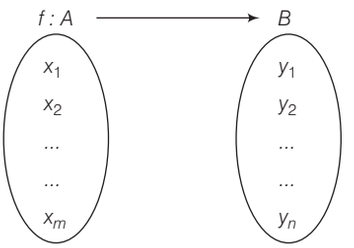


Figure 3.2

i.e. x_1 can take n images.

x_2 can take n images.

.....

.....

x_m can take n images.

Thus, **total number of functions from *A* to *B***

$$= n \times n \times \ldots m \text{ times} = n^m$$

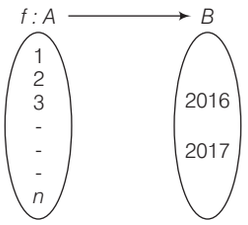
i.e. (Number of elements in codomain) $^{\text{number of elements in domain}}$.

Example 2 ***The number of functions*** $f : \{1, 2, 3, \ldots, n\} \to \{2016, 2017\}$, ***where*** $n \in N$, ***which satisfy the condition*** $f(1) + f(2) + \ldots + f(n)$ ***is an odd number are***

(a) 2^n (b) $n \cdot 2^{n-1}$

(c) 2^{n-1} (d) $n!$

Sol.



We can send 1, 2, 3, ..., $(n-1)$ any where in 2^{n-1} ways, the value $f(n)$ can uniquely determined according to sum of $f(1)+f(2)+...+f(n-1)$ is even or odd.

i.e. if $\{f(1)+f(2)+...+f(n-1)\} \in$ even, $f(n)=2017$

and if $\{f(1)+f(2)+...+f(n-1)\} \in$ odd, $f(n)=2016$

$\therefore$ Number of functions $= 2^{n-1}$

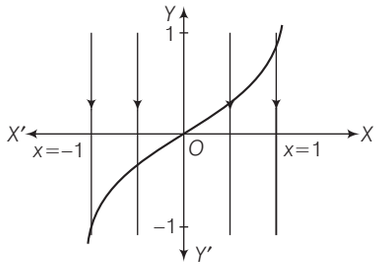
Graphical Representation of Functions

The graph of $y=f(x)$, where f is a function, consists of points whose coordinates (x, y) satisfy $y=f(x)$. Also, it is such that lines drawn parallel to the Y-axis **do not intersect the curve at more than one point**.

Vertical line test for f to be a function If graph of a function is cut a line parallel to Y-axis at more than one point, then it does not form a function.

Example 3 Find whether $f(x)=x^3$ forms a mapping or not.

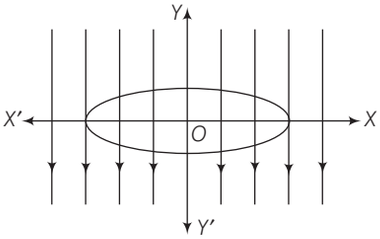
Sol. We have, $y=f(x)=x^3, \forall\, x \in R$.



Here, all the straight lines drawn parallel to Y-axis cut $y=x^3$ only at one point. Thus, $y=f(x)$ forms a mapping.

Example 4 ***Find whether*** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a>b$, ***forms a mapping or not.***

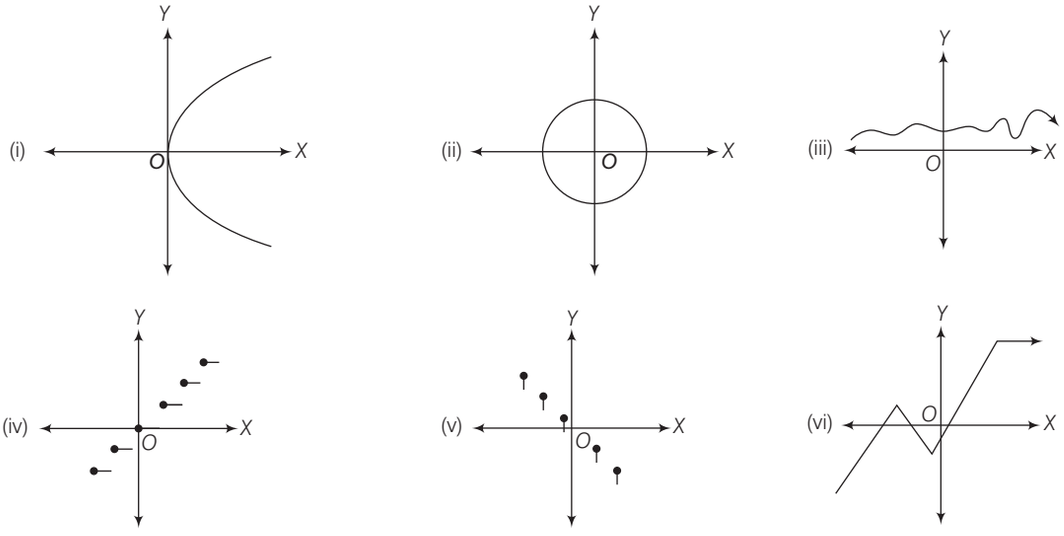
Sol. We have, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, i.e. $y=\pm\frac{b}{a}\sqrt{a^2-x^2}$, which represents a horizontal ellipse.

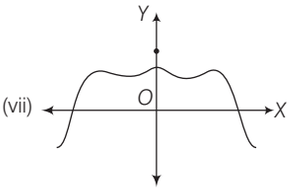


Here, straight lines drawn parallel to Y-axis meet the curve at more than one points. Thus, $f(x)=\pm\frac{b}{a}\sqrt{a^2-x^2}$ does not form a mapping.

Exercise for Session 1

1. Which of the following graphs are graphs of a function?



(vii)


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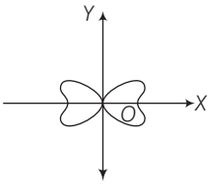
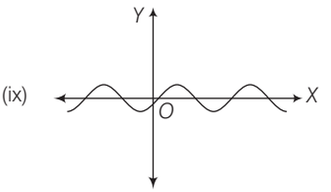
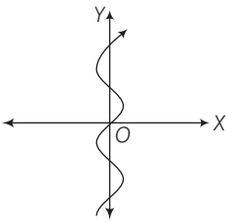


(ix)


(x)


2. For which of the following, y can be a function of x, $(x \in R, y \in R)$?

(i) $(x-h)^2 + (y-k)^2 = r^2$ (ii) $y^2 = 4ax$

(iii) $x^4 = y^2$ (iv) $x^6 = y^3$

(v) $3y = (\log x)^2$

3. Let $g(x)$ be a function defined on $[-1, 1]$. If the area of equilateral triangle with two of its vertices at $(0, 0)$ and $(x, g(x))$, is $\frac{\sqrt{3}}{4}$ sq unit, the function $g(x)$ may be

(a) $g(x) = \pm\sqrt{1-x^2}$ (b) $g(x) = \sqrt{1-x^2}$

(c) $g(x) = -\sqrt{1-x^2}$ (d) $g(x) = \sqrt{1+x^2}$

4. Represent diagramatically all possible functions from $\{\alpha, \beta\}$ to $\{1, 2\}$.

5. The number of functions from $f : \{a_1, a_2, \ldots, a_{10}\} \to \{b_1, b_2, \ldots, b_5\}$ is

(a) 10^5 (b) 5^{10}

(c) $\frac{10!}{5!}$ (d) $5!$

Session 2

Domain, Algebraic Functions

Domain

The domain of $y = f(x)$ is the set of all real x for which $f(x)$ is defined (real).

Rules for Finding Domain

(i) If domain of $y = f(x)$ and $y = g(x)$ are D_1 and D_2 respectively, then the domain of $f(x) \pm g(x)$ or $f(x) \cdot g(x)$ is $D_1 \cap D_2$.

(ii) Domain of $\dfrac{f(x)}{g(x)}$ is $D_1 \cap D_2 - \{x : g(x) = 0\}$.

Example 5 Find the domain of the following functions.

(i) $y = \sqrt{5x-3}$ (ii) $y = \sqrt[3]{5x-3}$

(iii) $y = \dfrac{1}{(x-1)(x-2)}$ (iv) $y = \dfrac{1}{\sqrt[3]{(x-1)}}$

Sol. (i) Here, $y = \sqrt{5x-3}$ is defined, if

$$5x - 3 \geq 0$$

$$\Rightarrow \quad x \geq \frac{3}{5} \quad \Rightarrow \quad x \in \left[\frac{3}{5}, \infty\right).$$

Thus, domain is $\left[\dfrac{3}{5}, \infty\right)$.

(ii) Here, $y = \sqrt[3]{5x-3}$ is defined, if

$$5x - 3 \in R \Rightarrow x \in R$$

Thus, domain is R.

(iii) Here, $y = \dfrac{1}{(x-1)(x-2)}$ is defined, if

$$(x-1)(x-2) \neq 0$$

i.e. if $x \neq 1, 2$

Thus, domain is $R - \{1, 2\}$.

(iv) Here, $y = \dfrac{1}{\sqrt[3]{x-1}}$ is defined, if

$$x - 1 \neq 0 \Rightarrow x \neq 1$$

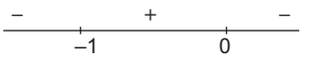
Thus, domain is $R - \{1\}$.

Example 6 Find the domain of $f(x) = \sqrt{\dfrac{1-5^x}{7^{-x}-7}}$.

Sol. For $f(x)$ to be defined, $\dfrac{1-5^x}{7^{-x}-7} \geq 0$ and $7^{-x} - 7 \neq 0$

$$\Rightarrow \quad \frac{5^x - 1}{7^{-x} - 7} \leq 0 \text{ and } 7^{-x} - 7 \neq 0$$

Now, $5^x - 1 = 0 \Rightarrow x = 0$ and $7^{-x} - 7 = 0 \Rightarrow x = -1$



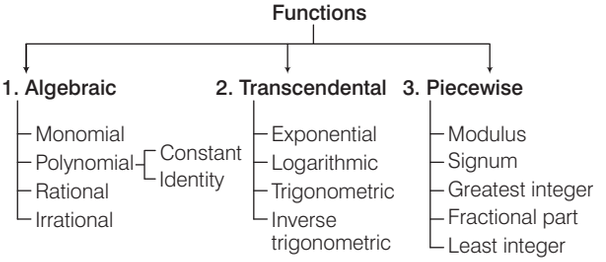
The sign of $\dfrac{5^x - 1}{7^{-x} - 7}$

Hence, from the sign scheme $x \in (-\infty, -1) \cup [0, \infty)$.

$\therefore$ Domain of $f(x)$ is $(-\infty, -1) \cup [0, \infty)$.

Classification of Functions

Functions are classified by the type of mathematical equation which represent their relationship.



Algebraic Functions

A function f is called an algebraic function, if it can be constructed using algebraic operations such as addition, subtraction, multiplication, division and taking roots, starting with polynomials.

e.g. $f(x) = \sqrt{1+x}$,

$$g(x) = \frac{x^3 - 16x}{x + \sqrt{x}} + (x-2)\sqrt[3]{x-1}$$

Different types of algebraic functions are described below

(i) Monomial Function

A function of the form $y = ax^n$, where a is constant and n is a non-negative integer, called monomial function.

e.g. $y = x^2, y = 2x, y = -x, \ldots\ldots$.

(ii) Polynomial Function

A function $y = f(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_n$, where $a_0, a_1, a_2, \ldots, a_n$ are real constants and n is a non-negative integer, then $f(x)$ is called a polynomial function. If $a_0 \neq 0$, then n is the degree of polynomial function. e.g.

Expression	Degree
$f(x) = x^{1920} + 5x^{1919} + 6x$	polynomial of degree 1920
$g(x) = x^2 + 3x + 3$	polynomial of degree 2
$h(x) = 7 = 7x^0$	polynomial of degree 0

A polynomial of degree one with no constant term is called an **odd linear function**, i.e. $f(x) = ax, a \neq 0$

In case $f(x) = 0$, it is constant function for which degree is not defined. A polynomial of odd degree has its range $(-\infty, \infty)$ but a polynomial of even degree has a range which is always subset of R.

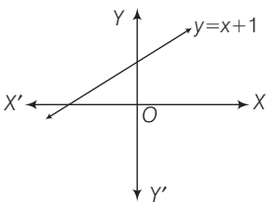
Note: All polynomial functions are algebraic functions but not the converse.

Example 7 Draw the graph of polynomial functions and find their domain and range.

(i) $y = x + 1$ (ii) $y = x^2$

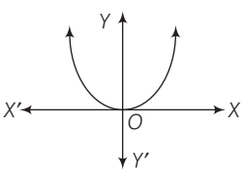
(iii) $y = x^3 + 1$ (iv) $y = x(x-1)(x-2)$

Sol. (i) Graph of $y = (x + 1)$ is shown as



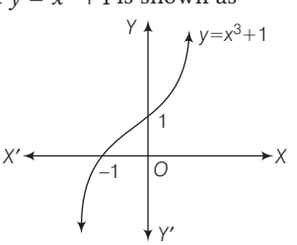
From graph domain is $x \in R$ and range is $y \in R$.

(ii) Graph of $y = x^2$ is shown as



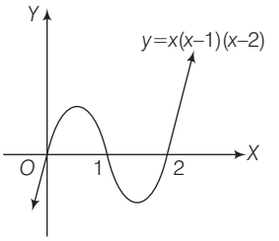
From graph domain is $x \in R$ and range is $y \in [0, \infty)$.

(iii) Graph of $y = x^3 + 1$ is shown as



∴ Domain; $x \in R$ and Range; $y \in R$.

(iv) Graph of $y = x(x-1)(x-2)$ is shown as



∴ Domain; $x \in R$ and Range; $y \in R$.

Constant Function

If the range of a function f consists of only one number, then f is called a constant function. The graph of a constant function is as shown below.

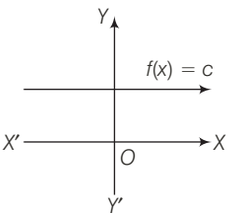


Figure 3.3

Thus, domain is R and range is $\{c\}$.

Identity Function

The function $y = f(x) = x$ for all $x \in R$ is called an identity function on R.

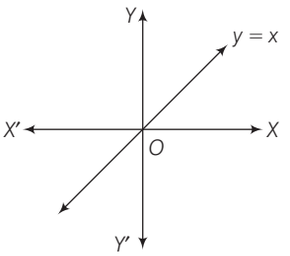


Figure 3.4

Here, domain of identity function is the set of real number. (As here for all values of x, $f(x)$ exists.)

Thus, domain is R and range is R.

(iii) Rational Function

It is defined as the ratio of two polynomials.

Let $P(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_n$

and $Q(x) = b_0 x^m + b_1 x^{m-1} + \ldots + b_m$.

Then, $f(x) = \dfrac{P(x)}{Q(x)}$ is a rational function provided $Q(x) \neq 0$.

Here, we can say that domain of $f(x)$ is all real numbers except the numbers where denominator is zero
[i.e. Domain $(f) = R - \{x : Q(x) = 0\}$].

e.g. $f(x) = \dfrac{x^2 - 3x + 5}{(x-1)(x-2)}$; Domain is $R - \{1, 2\}$

e.g. $f(x) = \dfrac{1}{2 - \cos 3x}$; As $2 - \cos 3x \neq 0$. $[\because -1 \leq \cos 3x \leq 1]$

$\therefore$ Domain of $f(x)$ is R.

(iv) Irrational Function

An algebraic function having **non-integral** rational exponent is called irrational function.

e.g. $f(x) = x^{1/2}$,

e.g. $f(x) = \dfrac{x^2 + \sqrt{x}}{\sqrt{1 + 3x^2}}$, etc.

An Important Result

There are two polynomial functions, satisfying the relation;

$f(x) \cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$.

They are
(i) $f(x) = 1 + x^n$ and (ii) $f(x) = 1 - x^n$,
where n is a positive integer.
For proof, refer Example 38 in Jee Type Solved Examples.

Exercise for Session 2

■ **Directions** (Q. Nos. 1 to 6) *Find the domain of the following.*

1. $f(x) = \sqrt{x^2 - 5x + 6}$
2. $f(x) = \sqrt{\dfrac{2x + 1}{x^3 - 3x^2 + 2x}}$
3. $f(x) = \sqrt{1 - \sqrt{1 - \sqrt{1 - x^2}}}$
4. $f(x) = \sqrt{x^{12} - x^9 + x^4 - x + 1}$
5. $f(x) = {}^{16-x}C_{2x-1} + {}^{20-3x}P_{4x-5}$
6. $f(x) = \sqrt{{}^{x^2+4x}C_{2x^2+3}}$

Session 3

Transcendental Functions

Transcendental Functions

The functions which are not algebraic are called transcendental functions. Exponential, logarithmic, trigonometric and inverse trigonometric functions are transcendental functions.

(i) Exponential Functions

The function $f(x) = a^x, a > 0, a \neq 1, a \in$ constant is said to be an exponential function.

If $0 < a < 1$ If $a > 1$

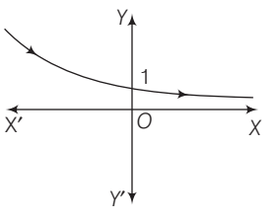


Figure 3.5

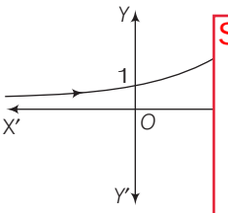


Figure 3.6

In the given graphs, none of them intersect the X-axis. Thus, we can say that they exist for all $x \in R$ and their corresponding values of y are greater than zero.

Thus, domain is R and range is $]\,0, \infty[$.

(ii) Logarithmic Functions

The function $f(x) = \log_a x;\ (x, a > 0)$ and $a \neq 1$ is a logarithmic function. Thus, domain of a logarithmic function is the set of all positive numbers and their range is the set R of real numbers.

If $0 < a < 1$ If $a > 1$

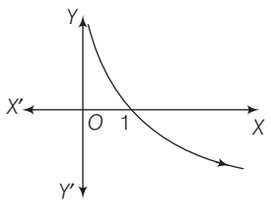


Figure 3.7

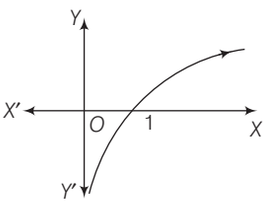


Figure 3.8

Properties of Logarithmic Functions

1. $\log_e(ab) = \log_e a + \log_e b \quad [a, b > 0]$

2. $\log_e\left(\frac{a}{b}\right) = \log_e a - \log_e b \quad [a, b > 0]$

3. $\log_e a^m = m \log_e a \quad [a > 0, m \in R]$

4. $\log_a a = 1 \quad [a > 0, a \neq 1]$

5. $\log_{b^m} a = \frac{1}{m} \log_b a \quad [a, b > 0, b \neq 1 \text{ and } m \in R - \{0\}]$

6. $\log_b a = \frac{1}{\log_a b} \quad [a, b > 0 \text{ and } a, b \neq 1$

$\log_b a = \frac{\log_m a}{\log_m b} \quad [a, b, m > 0 \text{ and } b, m \neq 1]$

$^{\text{og}_a m} = m \quad [a, m > 0 \text{ and } a \neq 1]$

$^{\text{og}_c b} = b^{\log_c a} \quad [a, b, c > 0 \text{ and } c \neq 1]$

$\log_m x > \log_m y \Rightarrow \begin{cases} x > y, & \text{if } m > 1 \\ x < y, & \text{if } 0 < m < 1 \end{cases}$

$[m, x, y > 0, m \neq 1]$

11. $\log_m a = b \Rightarrow a = m^b$

$[m, a > 0, m \neq 1,\ b \in R]$

12. $\log_m a > b \Rightarrow \begin{cases} a > m^b, & \text{if} \quad m > 1 \\ a < m^b, & \text{if} \quad 0 < m < 1 \end{cases}$

13. $\log_m a < b \Rightarrow \begin{cases} a < m^b, & \text{if} \quad m > 1 \\ a > m^b, & \text{if} \quad 0 < m < 1 \end{cases}$

Example 8 Find the domain of a single-valued function $y = f(x)$ given by the equation $10^x + 10^y = 10$.

Sol. Given, $10^x + 10^y = 10$

$\Rightarrow \quad 10^y = 10 - 10^x$

$\Rightarrow \quad y = \log_{10}(10 - 10^x) \quad [\text{as } a^m = b \Rightarrow m = \log_a b]$

Now, $10^1 - 10^x > 0$

$\Rightarrow \quad 10^1 > 10^x \quad \Rightarrow \ 1 > x$

Therefore, domain of the single-valued function

$y = f(x)$ is $x < 1$ or $x \in (-\infty, 1)$

Example 9 Find the domain of

$$f(x)=\frac{1}{\sqrt{\log_{1/2}(x^2-7x+13)}}.$$

Sol. $f(x)=\frac{1}{\sqrt{\log_{1/2}(x^2-7x+13)}}$ exists, if

$$\log_{1/2}(x^2-7x+13)>0$$

$\Rightarrow$ $(x^2-7x+13)<1$ ...(i)

and $x^2-7x+13>0$ ...(ii)

Considering Eq. (ii), $x^2-7x+13>0$, we get

$$\left(x^2-7x+\frac{49}{4}\right)+13-\frac{49}{4}>0$$

$\Rightarrow$ $\left(x-\frac{7}{2}\right)^2+\frac{3}{4}>0$

which is true for all $x\in R$. [As $\left(x-\frac{7}{2}\right)^2\geq 0$ for all x.] ...(iii)

Now, considering Eq. (i), we get

$$x^2-7x+12<0$$

$\Rightarrow$ $(x-3)(x-4)<0$

$\Rightarrow$ $3<x<4$ ...(iv)

Combining Eqs. (iii) and (iv), we get

Domain of $f(x)$ is $]3, 4[$.

Example 10 ***Find the domain of function***

$$f(x)=\frac{1}{\log_{10}(1-x)}+\sqrt{(x+2)}.$$

Sol. $f(x)=\frac{1}{\log_{10}(1-x)}+\sqrt{x+2}$

(As we know, $\log_a x$ is defined when $x, a>0$ and $a\neq 1$, also $\log_a 1=0$)

Thus, $\log_{10}(1-x)$ exists when

$1-x>0$ ...(i)

Also, $\frac{1}{\log_{10}(1-x)}$ exists when

$1-x>0$ and $1-x\neq 1$ ...(ii)

$\Rightarrow$ $x<1$ and $x\neq 0$ ...(iii)

Now, we have $\sqrt{x+2}$, which exists when $x+2\geq 0$

or $x\geq -2$ ...(iv)

Thus, $f(x)=\frac{1}{\log_{10}(1-x)}+\sqrt{x+2}$ exists when Eqs. (iii) and (iv) both hold true.

$\Rightarrow$ $-2\leq x<1$ and $x\neq 0$

$\Rightarrow$ $x\in[-2,0)\cup(0,1)$

Thus, domain of $f(x)$ is $[-2,0)\cup(0,1)$.

Example 11 Find the domain of $f(x)=\log_{10}(1+x^3)$.

Sol. $f(x)=\log_{10}(1+x^3)$ exists,

if $1+x^3>0$

$\Rightarrow$ $(1+x)(1-x+x^2)>0$

[where $(1-x+x^2)$ is always positive as $D<0$ and $a>0$]

So, $1+x>0$ or $x>-1$ or $x\in\,]-1,\infty[$

Thus, domain of above function $f(x)$ is $]-1,\infty[$.

Example 12 Find the domain of $f(x)=\log_{10}\log_{10}(1+x^3)$.

Sol. $f(x)=\log_{10}\log_{10}(1+x^3)$ exists,

if $\log_{10}(1+x^3)>0$ and $1+x^3>0$

$\Rightarrow$ $1+x^3>10^0$ and $1+x^3>0$

$\Rightarrow$ $1+x^3>1$ and $1+x^3>0$

$\Rightarrow$ $1+x^3>1\Rightarrow x^3>0$

$\Rightarrow$ $x\in(0,\infty)$

Thus, domain of above function $f(x)$ is $(0,\infty)$.

Example 13 ***Find the domain of*** $f(x)=\log_{10}\{\log_{10}\log_{10}\log_{10}x\}$.

Sol. $f(x)$ exists, if $\{\log_{10}\log_{10}\log_{10}x\}>0$ and $x>0$

$\Rightarrow$ $\log_{10}\log_{10}x>10^0$ and $x>0$

$\Rightarrow$ $\log_{10}\log_{10}x>1$ and $x>0$

$\Rightarrow$ $\log_{10}x>10^1$ and $x>0$

$\Rightarrow$ $x>10^{10}$ and $x>0$

Thus, $f(x)$ exists, if $x\in(10^{10},\infty)$.

Hence, domain of above function $f(x)$ is $(10^{10},\infty)$.

Example 14 ***Find the domain for***

$$f(x)=\sqrt{\log_{0.4}\left(\frac{x-1}{x+5}\right)}.$$

Sol. $f(x)=\sqrt{\log_{0.4}\left(\frac{x-1}{x+5}\right)}$ exists, if

$$\log_{0.4}\left(\frac{x-1}{x+5}\right)\geq 0 \quad \text{and} \quad \left(\frac{x-1}{x+5}\right)>0$$

$\Rightarrow$ $\frac{x-1}{x+5}\leq(0.4)^0$ and $\frac{x-1}{x+5}>0$

$\Rightarrow$ $\frac{x-1}{x+5}\leq 1$ and $\frac{x-1}{x+5}>0$

$\Rightarrow$ $\frac{x-1}{x+5}-1\leq 0$ and $\frac{x-1}{x+5}>0$

$\Rightarrow$ $\frac{-6}{x+5}\leq 0$ and $\frac{x-1}{x+5}>0$

$\Rightarrow \quad x+5>0 \quad$ and $\quad \dfrac{x-1}{x+5}>0$

$\Rightarrow \quad x>-5 \quad$ and $\quad x-1>0 \quad$ [as $x+5>0$]

$\Rightarrow \quad x>-5 \quad$ and $\quad x>1$

Thus, domain $f(x)$ is $(1, \infty)$.

Example 15 Find the domain of

$$f(x)=\log_{100x}\left(\frac{2\log_{10} x+1}{-x}\right)$$

Sol. $f(x)$ exists, if $\quad 100x>0 \quad$ ...(i)

$$100x \neq 1 \quad \text{...(ii)}$$

$$\frac{2\log_{10} x+1}{-x}>0 \quad \text{...(iii)}$$

$$2\log_{10} x+1<0 \quad [\text{as } x>0 \text{ from Eq. (i)}]$$

$$\log_{10} x<-\frac{1}{2}$$

$$x<(10)^{-1/2} \quad \text{...(iv)}$$

Now, from Eqs. (i), (ii) and (iv), we get

$$x \in (0,10^{-2}) \cup (10^{-2},10^{-1/2}).$$

Thus, domain of $f(x)$ is $(0, 10^{-2}) \cup (10^{-2}, 10^{-½})$.

Example 16 ***Find the domain of definition of***

$$f(x)=\frac{\log_2(x+3)}{x^2+3x+2}.$$

[IIT JEE 2001]

Sol. Here, $f(x)=\dfrac{\log_2(x+3)}{x^2+3x+2}=\dfrac{\log_2(x+3)}{(x+1)(x+2)}$ exists, if

Numerator $\quad x+3>0 \Rightarrow x>-3 \quad$ ...(i)

and denominator $\quad (x+1)(x+2)\neq 0$

$\Rightarrow \quad x\neq -1,-2 \quad$ ...(ii)

Thus, from Eqs. (i) and (ii), we get domain of $f(x)$ is $(-3, \infty) - \{-1,-2\}$.

(iii) Trigonometric Functions

Sine Function $f(x) = \sin x$

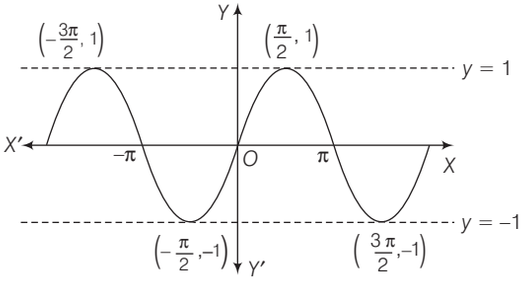


Figure 3.9

Here, domain is R and range is $[-1, 1]$.

Cosine Function $f(x) = \cos x$

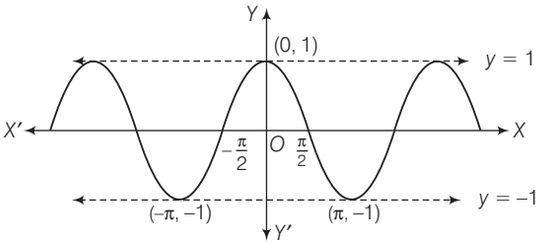


Figure 3.10

Here, domain is R and range is $[-1, 1]$.

Tangent Function $f(x) = \tan x$

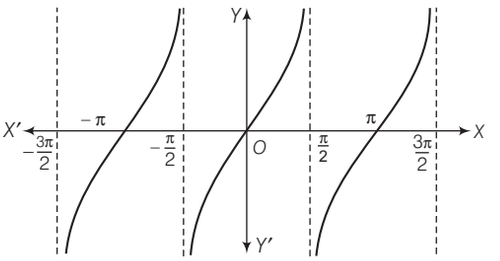


Figure 3.11

Here, domain is $R-\left\{\dfrac{(2n+1)\pi}{2}, n\in Z\right\}$ and range is R.

Cosecant Function $f(x) = \operatorname{cosec} x$

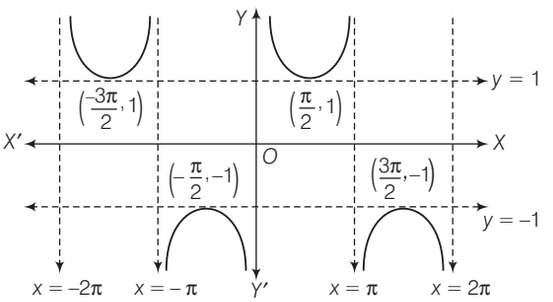


Figure 3.12

Here, domain is $R - \{n\pi, n \in Z\}$ and range is $R - (-1, 1)$.

Secant Function $f(x) = \sec x$

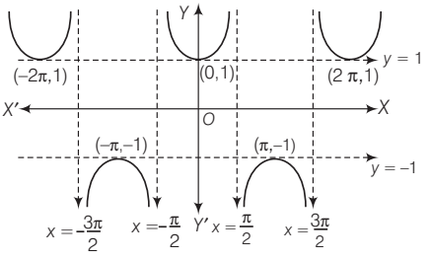


Figure 3.13

Here, domain is $R-\{(2n+1)\dfrac{\pi}{2}, n\in Z\}$ and range is $R-(-1, 1)$.

Cotangent Function $f(x) = \cot x$

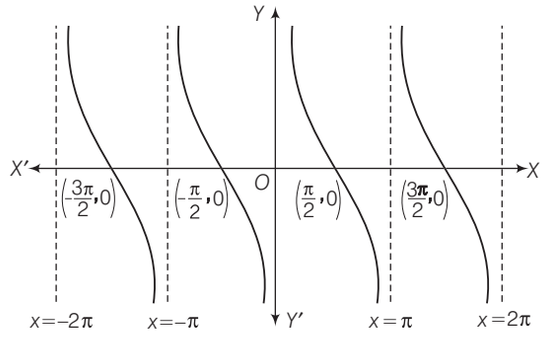


Figure 3.14

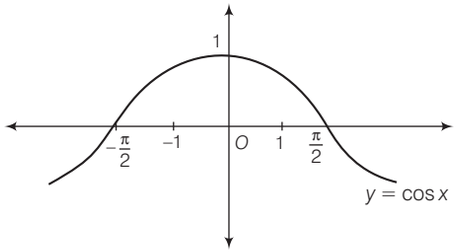
Here, domain is $R - \{n\pi, n \in Z\}$ and range is R.

Example 17 Find domain for $f(x) = \sqrt{\cos(\sin x)}$.

Sol. $f(x) = \sqrt{\cos(\sin x)}$ is defined, if

$$\cos(\sin x) \geq 0 \qquad \ldots(i)$$

As, we know $-1 \leq \sin x \leq 1$ for all x



$\therefore$ $\cos(\sin x) \geq 0$, for all x (using graph of $y = \cos x$)

i.e. $x \in R$

Thus, domain $f(x)$ is R.

Example 18 The function $f(x)$ is defined in $[0, 1]$. Find the domain of $f(\tan x)$.

Sol. Here, $f(x)$ is defined in $[0, 1]$.

$\Rightarrow x \in [0, 1]$, i.e. we can substitute only those values of x, which lie in $[0, 1]$.

For $f(\tan x)$ to be defined, we must have

$$0 \leq \tan x \leq 1 \qquad [\text{as } x \text{ is replaced by } \tan x]$$

i.e. $$n\pi \leq x \leq n\pi + \frac{\pi}{4} \qquad [\text{in general}]$$

or $$0 \leq x \leq \frac{\pi}{4} \qquad [\text{in particular}]$$

Thus, domain for $f(\tan x)$ is $\left[n\pi, n\pi + \frac{\pi}{4}\right]$.

Inverse Trigonometric Functions

We know that trigonometric functions are many-one in their actual domain. Hence, for inverse functions to get defined, the actual domain of trigonometric functions must be restricted to make the function one-one.

$y = \sin^{-1} x$

Since, $y = \sin x$ is not one-one. Therefore, it is not invertible. Let us consider the function $y = \sin x, x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ and $y \in [-1, 1]$ whose graph is the portion of the $\sin x$ curve in the interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, which is strictly increasing. Hence, one-one and its inverse is $y = \sin^{-1} x$.

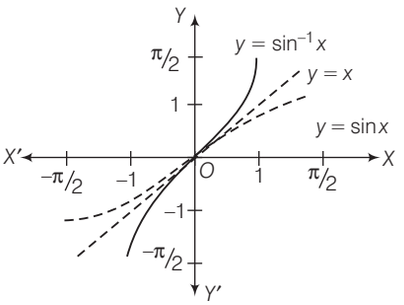


Figure 3.15

Here, domain is $[-1, 1]$; range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$y = \cos^{-1} x$

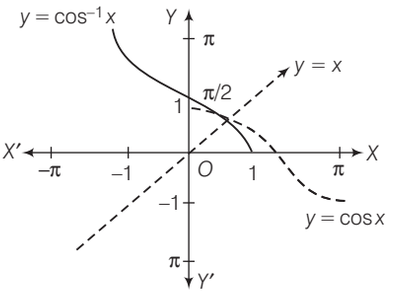


Figure 3.16

Here, domain is $[-1, 1]$; range is $[0, \pi]$.

$y = \tan^{-1} x$

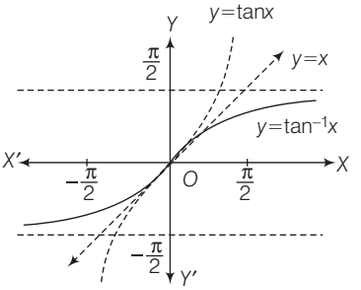


Figure 3.17

Here, domain is R; range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$y = \cot^{-1} x$

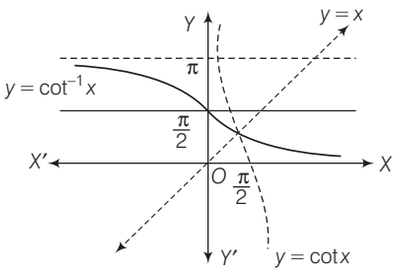


Figure 3.18

Here, domain is R ; range is $(0, \pi)$.

$y = \sec^{-1} x$

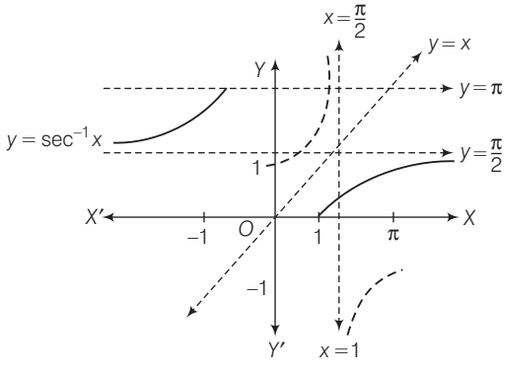


Figure 3.19

Here, domain is $R - (-1, 1)$; range is $[0, \pi] - \{\pi/2\}$.

$y = \text{cosec}^{-1} x$

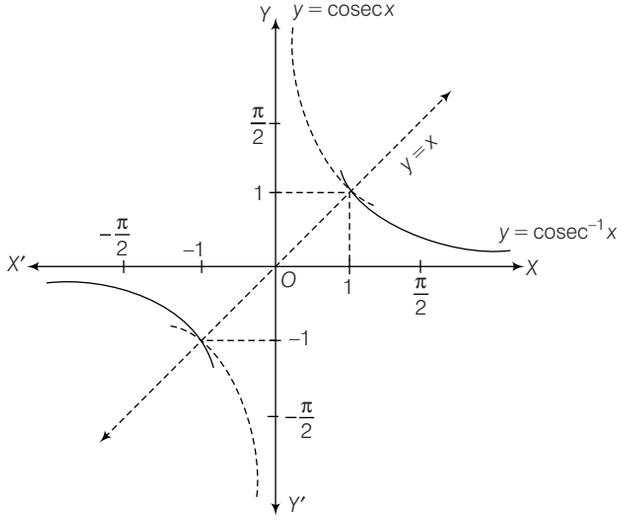


Figure 3.20

Here, domain is $R - (-1, 1)$; range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

Example 19 Find the domain for $f(x) = \sin^{-1}\left(\frac{x^2}{2}\right)$.

Sol. $f(x) = \sin^{-1}\left(\frac{x^2}{2}\right)$ is defined, if

$$-1 \le \frac{x^2}{2} \le 1 \text{ or } -2 \le x^2 \le 2$$

$\Rightarrow \quad 0 \le x^2 \le 2$ [as x^2 cannot be negative]

$\Rightarrow \quad -\sqrt{2} \le x \le \sqrt{2}$

Therefore, domain of $f(x)$ is $[-\sqrt{2}, \sqrt{2}]$.

Example 20 Find the domain for

$$y = \sin^{-1}\left(\log_2\left(\frac{x^2}{2}\right)\right).$$

Sol. For y to be defined, $\frac{x^2}{2} > 0$ ...(i)

and $\quad -1 \le \log_2\left(\frac{x^2}{2}\right) \le 1$ ...(ii)

From Eq. (i), we have $x \in R - \{0\}$ ...(iii)

From Eq. (ii), we have

$$2^{-1} \le \frac{x^2}{2} \le 2^1 \Rightarrow 1 \le x^2 \le 4$$

$\Rightarrow \quad -2 \le x \le -1$ or $1 \le x \le 2$ ...(iv)

Thus, from Eqs. (iii) and (iv), we get

$$x \in [-2, -1] \cup [1, 2]$$

Example 21 Find the domain for $f(x) = \sin^{-1}\left(\frac{1+x^2}{2x}\right)$.

Sol. $f(x) = \sin^{-1}\left(\frac{1+x^2}{2x}\right)$ is defined, if

$$-1 \le \frac{1+x^2}{2x} \le 1 \text{ or } \left|\frac{1+x^2}{2x}\right| \le 1$$

$\Rightarrow \quad |1+x^2| \le |2x|$

$\Rightarrow \quad 1+x^2 \le |2x|$ [as $1+x^2 > 0$]

$\Rightarrow \quad x^2 - 2|x| + 1 \le 0$

$\Rightarrow \quad |x|^2 - 2|x| + 1 \le 0$ [as $x^2 = |x|^2$]

$\Rightarrow \quad (|x|-1)^2 \le 0$

But $(|x|-1)^2$ is either always positive or zero.

Thus, $\quad (|x|-1)^2 = 0$

$\Rightarrow \quad |x| = 1 \Rightarrow x = \pm 1$

Thus, domain for $f(x)$ is $\{-1, 1\}$.

Example 22 Find the domain for $y = \sqrt{\sin^{-1}(\log_2 x)}$.

Sol. y is defined, if $x > 0$, ...(i)

$$-1 \le \log_2 x \le 1$$

$$\Rightarrow \quad 2^{-1} \le x \le 2$$

$$\Rightarrow \quad \frac{1}{2} \le x \le 2 \qquad \text{...(ii)}$$

and $\sin^{-1}(\log_2 x) \ge 0$ ...(iii)

$$\Rightarrow \quad \log_2 x \ge 0$$

$$\Rightarrow \quad x \ge 2^0$$

$$\Rightarrow \quad x \ge 1 \qquad \text{...(iv)}$$

From Eqs. (i), (ii) and (iv), we get

$$1 \le x \le 2$$

Hence, domain is $1 \le x \le 2$ or $x \in [1, 2]$.

Example 23 Find the domain of the function,

$$f(x) = \log\left\{\log_{|\sin x|}(x^2 - 8x + 23) - \frac{3}{\log_2|\sin x|}\right\}.$$

Sol. Here, $f(x)$ is defined, if

$$\log_{|\sin x|}(x^2 - 8x + 23) - \frac{3}{\log_2|\sin x|} > 0$$

$$\Rightarrow \quad \log_{|\sin x|}(x^2 - 8x + 23) - 3\log_{|\sin x|} 2 > 0$$

$$\Rightarrow \quad \log_{|\sin x|}\left(\frac{x^2 - 8x + 23}{8}\right) > 0$$

$$\Rightarrow \quad \frac{x^2 - 8x + 23}{8} < |\sin x|^0$$

and $|\sin x| \ne 0, 1$

$$\Rightarrow \quad x^2 - 8x + 23 < 8 \text{ and } |\sin x| \ne 0, 1$$

$$\Rightarrow \quad x^2 - 8x + 15 < 0 \text{ and } \sin x \ne 0, \pm 1$$

$$\Rightarrow \quad (x - 3)(x - 5) < 0 \text{ and } x \ne n\pi, (2n + 1)\frac{\pi}{2}$$

$$\Rightarrow \quad x \in (3, 5) \text{ and } x \ne \pi, \frac{3\pi}{2} \quad \text{[using number line rule]}$$

$$\Rightarrow \quad x \in (3, \pi) \cup \left(\pi, \frac{3\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 5\right)$$

Thus, domain of $f(x)$ is $(3, \pi) \cup \left(\pi, \frac{3\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 5\right)$.

Exercise for Session 3

■ **Directions** (Q. Nos. 1 to 10) *Find the domain of the following.*

1. $f(x) = \log_{10}(\sqrt{x - 4} + \sqrt{6 - x})$
2. $f(x) = \sqrt{\log_{1/2}\left(\frac{5x - x^2}{4}\right)}$
3. $f(x) = \sqrt{\log_{0.3}\left(\frac{3x - x^2}{x - 1}\right)}$
4. $f(x) = \sqrt{\frac{\log_{0.3}(x - 1)}{x^2 - 2x - 8}}$
5. $f(x) = \log_{10}(1 - \log_{10}(x^2 - 5x + 16))$
6. $f(x) = \sin|x| + \sin^{-1}(\tan x) + \sin(\sin^{-1} x)$
7. $f(x) = \sqrt{e^{\cos^{-1}(\log_4 x^2)}}$
8. $f(x) = \sin^{-1}\left(\frac{3 - 2x}{5}\right) + \sqrt{3 - x}$
9. $f(x) = \frac{\log_{2x} 3}{\cos^{-1}(2x - 1)}$
10. $f(x) = \log_{10} \log_2 \log_{2/\pi}(\tan^{-1} x)^{-1}$

Session 4

Piecewise Functions

Piecewise Functions

(i) Absolute Value Function (or Modulus Function)

The modulus function is defined as

$$y = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}.$$

It is the numerical value of x.

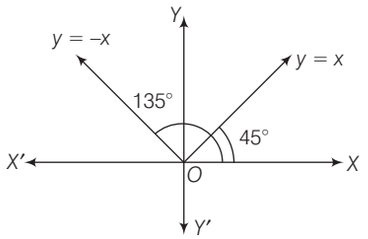


Figure 3.21

Geometrical Interpretation of Modulus of a Function

Geometrically, $|x|$ represents the distance of the point $P(x, 0)$ or $Q(-x, 0)$ from origin.

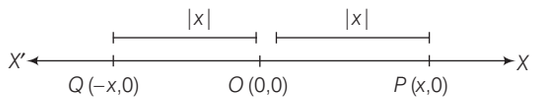


Figure 3.22

i.e. $$d(O, P) = \sqrt{(x-0)^2 + (0-0)^2} = \sqrt{x^2} = |x|$$

$$d(O, Q) = \sqrt{(-x-0)^2 + (0-0)^2} = \sqrt{x^2} = |x|$$

Properties of Modulus

(i) $|x|^2 = x^2$ (ii) $\sqrt{x^2} = |x|$

(iii) $||x|| = |-x| = |x|$ (iv) $|x| = \max\{-x, x\}$

(v) $-|x| = \min\{-x, x\}$ (vi) $\max(a, b) = \dfrac{a+b}{2} + \left|\dfrac{a-b}{2}\right|$

(vii) $\min(a, b) = \dfrac{a+b}{2} - \left|\dfrac{a-b}{2}\right|$

(viii) $|x + y| \leq |x| + |y|$

(ix) $|x + y| = |x| + |y|$, iff $xy \geq 0$

(x) $|x - y| = |x| + |y|$, iff $xy \leq 0$

(xi) $|x| \leq a$, (a is +ve) $\Rightarrow -a \leq x \leq a$ or $x \in [-a, a]$

(xii) $|x| \geq a$, (a is +ve) $\Rightarrow x \leq -a$ or $x \geq a$
or $x \in (-\infty, -a] \cup [a, \infty)$

(xiii) $|x| \leq a$, (a is −ve) $\Rightarrow$ No solution or $x \in \phi$

(xiv) $|x| \geq a$, (a is −ve) $\Rightarrow x \in$ Real number or $x \in (-\infty, \infty)$

(xv) $a \leq |x| \leq b$, where a and b are +ve.

$\Rightarrow -b \leq x \leq -a$ or $a \leq x \leq b$ or $x \in [-b, -a] \cup [a, b]$

Example 24 Solve for x.

(i) $(|x| - 1)(|x| - 2) \leq 0$ (ii) $\dfrac{|x| - 1}{|x| - 2} \geq 0$

(iii) $|x - 3| + |4 - x| = 1$ (iv) $|x| + |x + 4| = 4$

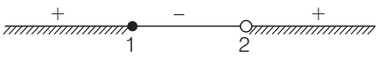
Sol. (i) $(|x| - 1)(|x| - 2) \leq 0$

Using number line rule for $|x|$, we get

$\Rightarrow \quad 1 \leq |x| \leq 2$

i.e. $x \in [-2, -1] \cup [1, 2]$

(ii) $\dfrac{|x| - 1}{|x| - 2} \geq 0$. Using number line rule for $|x|$, we get



$\Rightarrow \quad |x| \leq 1$ or $|x| > 2$

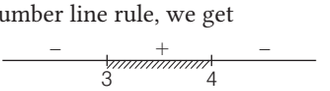
i.e. $x \in [-1, 1] \cup (-\infty, -2) \cup (2, \infty)$

(iii) $|x - 3| + |4 - x| = 1$

As we know, $|x| + |y| = |x + y|$, iff $xy \geq 0$

$\therefore \quad (x - 3)(4 - x) \geq 0$ or $-(x - 3)(x - 4) \geq 0$

Using number line rule, we get



$\Rightarrow \quad x \in [3, 4]$

(iv) $|x| + |x + 4| = 4$

As we know, $|x| + |y| = |x - y|$, iff $xy \leq 0$

$\therefore \quad x(x + 4) \leq 0$

Using number line rule, we get

$\Rightarrow \quad x \in [-4, 0]$

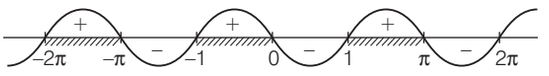
Example 25 ***Solve*** $|x^2-1+\sin x|=|x^2-1|+|\sin x|$, ***where*** $x\in[-2\pi,2\pi]$.

Sol. Let $f(x)=x^2-1$ and $g(x)=\sin x$

We know, $|f(x)+g(x)|=|f(x)|+|g(x)|$ iff $f(x)\cdot g(x)\geq 0$.

$\therefore$ $(x^2-1)\cdot\sin x\geq 0$ on $[-2\pi,2\pi]$.

Using number line rule, we get



$\Rightarrow$ $x\in[-2\pi,-\pi]\cup[-1,0]\cup[1,\pi]$

Example 26 Solve $\left|\frac{x}{|x-1|}\right|+|x|=\frac{x^2}{|x-1|}$.

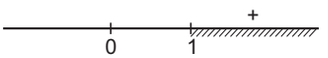
Sol. Let $f(x)=\frac{x}{x-1}$ and $g(x)=x$

Then, $f(x)+g(x)=\frac{x}{x-1}+x=\frac{x^2}{x-1}$

We know, $|f(x)|+|g(x)|=|f(x)+g(x)|$,

iff $f(x)\cdot g(x)\geq 0$

$\therefore$ $\frac{x}{x-1}\cdot x\geq 0 \Rightarrow \frac{x^2}{x-1}\geq 0$



$\Rightarrow$ $x\in\{0\}\cup(1,\infty)$

Example 27 Find domain for $y=\frac{1}{\sqrt{|x|-x}}$.

Sol. y is defined, if $(|x|-x)>0 \Rightarrow |x|>x$, which is true for negative x only.

Hence, domain $\in(-\infty,0)$.

Example 28 ***Find domain for***

$$y=\cos^{-1}\left(\frac{1-2|x|}{3}\right)+\log_{|x-1|}x.$$

Sol. Here, $\cos^{-1}\left(\frac{1-2|x|}{3}\right)$ exists, if $-1\leq\frac{1-2|x|}{3}\leq 1$

$\Rightarrow$ $-3\leq 1-2|x|\leq 3 \Rightarrow -4\leq -2|x|\leq 2$

$\Rightarrow$ $2\geq|x|\geq -1 \Rightarrow -2\leq x\leq 2$ ...(i)

Also, $\log_{|x-1|}x$ exists, if

$x>0$, $|x-1|>0$ and $|x-1|\neq 1$

$x>0$, $x\in R-\{1\}$ and $x\neq 0,\ 2$ ...(ii)

Hence, from Eqs. (i) and (ii), we get

$x\in(0,1)\cup(1,2)$

Example 29 ***Domain of the function*** $f(x)=\frac{1}{\sqrt{4x-|x^2-10x+9|}}$ ***is***

(a) $(7-\sqrt{40},7+\sqrt{40})$ (b) $(0,7+\sqrt{40})$

(c) $(7-\sqrt{40},\infty)$ (d) None of these

Sol. Here, $f(x)=\frac{1}{\sqrt{4x-|x^2-10x+9|}}$ would exists, if

$4x-|x^2-10x+9|>0$

i.e. $|x^2-10x+9|<4x$,

where

$$|x^2-10x+9|=\begin{cases}x^2-10x+9, & x\leq 1 \text{ or } x\geq 9\\ -(x^2-10x+9), & 1<x<9\end{cases}$$

$\Rightarrow$ Two cases

Case **I** When $x\leq 1$ or $x\geq 9$

$\therefore$ $x^2-10x+9<4x$

$\Rightarrow$ $x^2-14x+9<0 \Rightarrow (x-7)^2<40$

$\Rightarrow$ $x\in(7-\sqrt{40},7+\sqrt{40})$ [but $x\leq 1$ or $x\geq 9$]

$\Rightarrow$ $x\in(7-\sqrt{40},1]\cup[9,7+\sqrt{40})$ ...(i)

Case **II** When $1<x<9$

$-x^2+10x-9<4x \Rightarrow x^2-6x+9>0$

$\Rightarrow$ $(x-3)^2>0$, which is always true except at $x=3$

$\therefore$ $x\in(1,9)-\{3\}$ ...(ii)

From Eqs. (i) and (ii), domain of $f(x)$ is

$(7-\sqrt{40},7+\sqrt{40})-\{3\}$

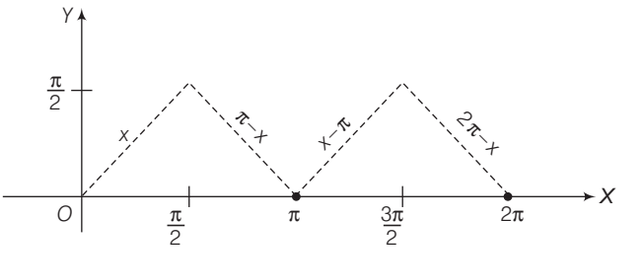
Hence, (d) is the correct answer.

Example 30 ***The domain of the function*** $f(x)=\sqrt{|\sin^{-1}(\sin x)|-\cos^{-1}(\cos x)}$ ***in*** $[0,2\pi]$ ***is***

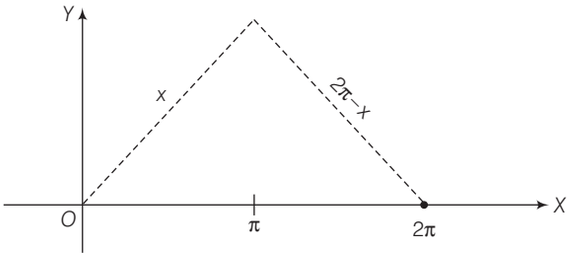
(a) $\left[0,\frac{\pi}{2}\right]\cup\left[\frac{3\pi}{2},2\pi\right]$ (b) $[\pi,2\pi]$

(c) $[0,\pi]-\left\{\frac{\pi}{2}\right\}$ (d) $[0,2\pi]-\left\{\frac{\pi}{2},\frac{3\pi}{2}\right\}$

Sol. As, $|\sin^{-1}(\sin x)|$ could be sketched, as

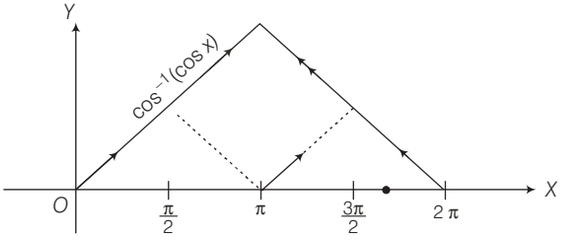


and $\cos^{-1}(\cos x)$ could be sketched as



$\therefore |\sin^{-1}(\sin x)| > \cos^{-1}(\cos x)$ is not possible.

Only equality holds as



Thus, $|\sin^{-1}(\sin x)| = \cos^{-1}(\cos x)$

When $x \in \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$

So, domain for $f(x)$ is $\left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$.

Hence, (a) is the correct answer.

(ii) Signum Function

Signum function is denoted by sgn (x) and it is defined by

$$y = \text{sgn}(x) = \begin{cases} \frac{|x|}{x} \text{ or } \frac{x}{|x|}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \end{cases}$$

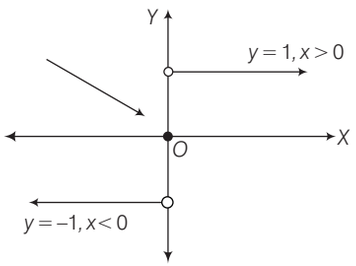


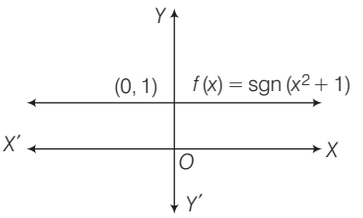
Figure 3.23

Example 31 Sketch the graph of

(i) $f(x) = \text{sgn}(x^2 + 1)$. (ii) $f(x) = \text{sgn}(\log_e x)$.

(iii) $f(x) = \text{sgn}(\sin x)$. (iv) $f(x) = \text{sgn}(\cos x)$.

Sol. (i) $f(x) = \text{sgn}(x^2 + 1) = \begin{cases} -1, & \text{if } x^2 + 1 < 0 \\ 0, & \text{if } x^2 + 1 = 0 \\ 1, & \text{if } x^2 + 1 > 0 \end{cases}$

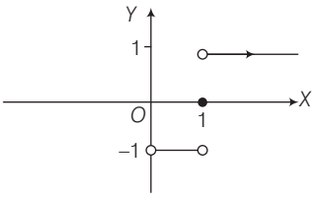


$f(x) = 1, x \in R$

(ii) $f(x) = \text{sgn}(\log_e x)$

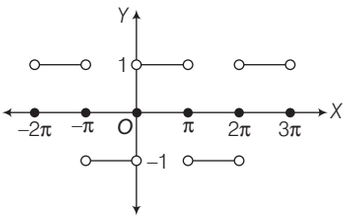
$$= \begin{cases} 1, & \log_e x > 0 \\ -1, & \log_e x < 0 \\ 0, & \log_e x = 0 \end{cases} \quad \therefore f(x) = \begin{cases} 1, & x > 1 \\ -1, & 0 < x < 1 \\ 0, & x = 1 \end{cases}$$

Graph for, $f(x) = \text{sgn}(\log_e x)$ is



(iii) $f(x) = \text{sgn}(\sin x)$

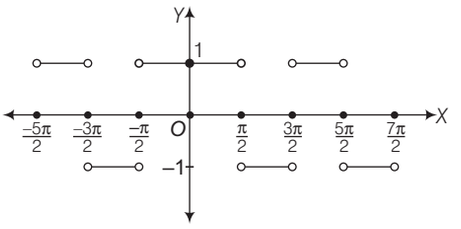
$$f(x) = \begin{cases} 1, & \sin x > 0 \\ -1, & \sin x < 0 \\ 0, & \sin x = 0 \end{cases} = \begin{cases} 1, & 2n\pi < x < (2n+1)\pi \\ -1, & (2n+1)\pi < x < (2n+2)\pi \\ 0, & x = n\pi \end{cases}$$



(iv) $f(x) = \text{sgn}(\cos x)$

$$f(x) = \begin{cases} 1, & \cos x > 0 \\ -1, & \cos x < 0 \\ 0, & \cos x = 0 \end{cases}$$

$$= \begin{cases} 1, & 2n\pi - \pi/2 < x < 2n\pi + \pi/2 \\ -1, & 2n\pi + \pi/2 < x < 2n\pi + 3\pi/2 \\ 0, & x = (2n+1)\pi/2 \end{cases}$$



(iii) Greatest Integer Function or Step Function

$[x]$ indicates the integral part of x, which is nearest and smaller to x. It is also known as floor of x.

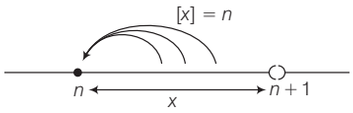


Figure 3.24

Thus, $[2.3202]=2, [0.23]=0, [5]=5,$
$[-8.0725]=-9, [-0.6]=-1$

In general, $n \le x < n+1 \quad [n \in \text{Integer}]$

$\Rightarrow \quad [x]=n$

Here, $f(x)=[x]$ could be expressed graphically as

x	$[x]$
$0 \le x < 1$	0
$1 \le x < 2$	1
$2 \le x < 3$	2
$-1 \le x < 0$	−1
$-2 \le x < -1$	−2

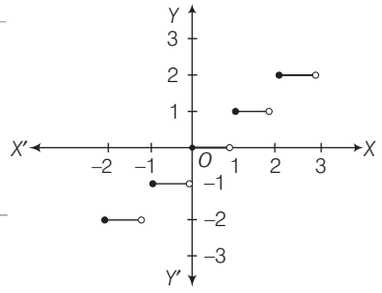


Figure 3.25

(•) darkened circle represents value is taken, (◦) represents value is neglected.

Properties of the Greatest Integer Functions

(i) $[x] \le x < [x]+1$

(ii) $x-1<[x]<x$

(iii) $I \le x < I+1 \Rightarrow [x]=I$, where $I \in$ Integer

(iv) $[[x]]=[x]$

(v) $[x]+[-x]=\begin{cases} 0, & \text{if } x \in \text{Integer} \\ -1, & \text{if } x \notin \text{Integer} \end{cases}$

i.e. $[-x]=\begin{cases} -x, & \text{if } x \in \text{Integer} \\ -1-[x], & \text{if } x \notin \text{Integer} \end{cases}$

(vi) $[x]-[-x]=\begin{cases} 2x, & \text{if } x \in \text{Integer} \\ 2[x]+1, & \text{if } x \notin \text{Integer} \end{cases}$

(vii) $[x \pm n]=[x] \pm n, n \in$ Integer

(viii) $[x] \ge n \Leftrightarrow x \ge n, n \in$ Integer

(ix) $[x] > n \Leftrightarrow x \ge n+1, n \in$ Integer

(x) $[x] \le n \Leftrightarrow x < n+1, n \in$ Integer

(xi) $[x] < n \Leftrightarrow x < n, n \in$ Integer

(xii) $[x]=\left[\frac{x}{2}\right]+\left[\frac{x+1}{2}\right]$

(xiii) $\left[\frac{n+1}{2}\right]+\left[\frac{n+2}{4}\right]+\left[\frac{n+4}{8}\right]+\ldots = n,$
$n \in$ Natural number

(xiv) $[x]+[y] \le [x+y] \le [x]+[y]+1$

(xv) $[x]+\left[x+\frac{1}{n}\right]+\left[x+\frac{2}{n}\right]+$
$\ldots+\left[x+\frac{n-1}{n}\right]=[nx], n \in N$

Example 32 Find domain for, $f(x)=\cos^{-1}[x]$.

Sol. As, $y=\cos^{-1}x$ exists, when $-1 \le x \le 1$.

$\therefore\ f(x)=\cos^{-1}[x]$ exists, when $-1 \le [x] \le 1 \Rightarrow -1 \le x < 2$

or $\quad x \in [-1, 2)$

Example 33 ***Find the value of***

$$\left[\frac{3}{4}\right]+\left[\frac{3}{4}+\frac{1}{100}\right]+\left[\frac{3}{4}+\frac{2}{100}\right]+\ldots+\left[\frac{3}{4}+\frac{99}{100}\right].$$

Sol. $\left[\frac{3}{4}\right]+\left[\frac{3}{4}+\frac{1}{100}\right]+\ldots+\left[\frac{3}{4}+\frac{24}{100}\right]+\left[\frac{3}{4}+\frac{25}{100}\right]$
$+\ldots+\left[\frac{3}{4}+\frac{99}{100}\right]$

$\Rightarrow \underbrace{[0.75]+\ldots+[0.99]}_{\substack{25 \text{ terms are each} \\ \text{equal to zero}}} + \underbrace{[1.0]+\ldots+[1.74]}_{\substack{75 \text{ terms are each} \\ \text{equal to 1}}} \Rightarrow 75$

Aliter Apply property (xv)

Here, $x=\frac{3}{4}$ and $n=100$

$\therefore \left[\frac{3}{4}\right]+\left[\frac{3}{4}+\frac{1}{100}\right]+\ldots+\left[\frac{3}{4}+\frac{99}{100}\right]=\left[\frac{3}{4}\times 100\right]=75$

Example 34 ***Given*** $y=2[x]+3$ ***and*** $y=3[x-2]+5$, ***find the value of*** $[x+y]$.

Sol. We have, $\quad 2[x]+3=3[x-2]+5$

$\Rightarrow \quad 2[x]+3=3[x]-6+5$

$\Rightarrow \quad [x]=4 \Rightarrow 4 \le x < 5$

or $\quad x=4+f \quad [f \to \text{fraction}]$

and $\quad y=2[x]+3=11$

Hence, $\quad [x+y]=[4+f+11]=[15+f]=15$

Example 35 ***Find domain for*** $f(x)=[\sin x]\cos\left(\frac{\pi}{[x-1]}\right)$.

Sol. $[\sin x]$ is always defined. $\cos\left(\frac{\pi}{[x-1]}\right)$ is also defined except when $[x-1]=0 \Rightarrow 0\le x-1<1$

$\Rightarrow \quad 1\le x<2$

Hence, domain of $f(x)$ is $R-[1,2)$.

Example 36 ***The domain of the function*** $f(x)=\frac{\log_4(5-[x-1]-[x]^2)}{x^2+x-2}$ ***is***

(where $[x]$ ***denotes greatest integer function)***

Sol. For domain of $f(x)$,

$$5-[x-1]-[x]^2>0$$

and $\quad x^2+x-2\ne 0$

$\Rightarrow \quad (x+2)(x-1)\ne 0$

$\Rightarrow \quad x\ne 1,-2$

Now, $\quad 5-[x]+1-[x]^2>0$

$\Rightarrow \quad [x]^2+[x]-6<0$

$\Rightarrow \quad ([x]+3)([x]-2)<0$

$\Rightarrow \quad -3<[x]<2$

$\Rightarrow \quad -2\le x<2$

$\because \quad x\ne 1,-2$

$\therefore$ Domain $\in(-2,1)\cup(1,2)$

Example 37 Let $[\sqrt{n^2+1}]=[\sqrt{n^2+\lambda}]$, where $n,\lambda\in N$. Show that λ can have $2n$ different values.

Sol. We have, $n^2+1=(n+1)^2-2n<(n+1)^2; n\in N$

i.e. $\quad \sqrt{n^2+1}<n+1$

$\Rightarrow \quad n<\sqrt{n^2+1}<n+1$

$\Rightarrow \quad [\sqrt{n^2+1}]=n$

$\therefore \quad [\sqrt{n^2+\lambda}]=n$

$\Rightarrow \quad n<\sqrt{n^2+\lambda}<n+1$

or $\quad n^2<(n^2+\lambda)<(n+1)^2$

$\Rightarrow \quad 0<\lambda<2n+1$

Thus, λ can take $2n$ different values.

Example 38 $f(x)=\frac{1}{\sqrt{[x]-x}}$, where $[\cdot]$ denotes the greatest integer function less than or equals to x. Then, find the domain of $f(x)$.

Sol. $f(x)=\frac{1}{\sqrt{[x]-x}}$ exists, if

$$[x]-x>0$$

i.e. $\quad [x]>x$

But from definition of greatest integral function, we know

$$[x]\le x \qquad [\text{as } x=[x]+\{x\}]$$

Thus, it is not possible that $[x]>x$.

Hence, $\quad$ domain $f(x)=\phi$

Example 39 ***If domain for*** $y=f(x)$ ***is*** $[-3,2]$ ***, find the domain of*** $g(x)=f\{|[x]|\}$.

Sol. Here, $f(x)$ is defined in $[-3,2]$.

$\Rightarrow \quad x\in[-3,2]$.

(i.e. we can substitute only those values of x, which lie between $[-3,2]$).

For $g(x)=f\{|[x]|\}$ to be defined, we must have

$-3\le|[x]|\le 2$

$\Rightarrow \quad 0\le|[x]|\le 2 \qquad [\text{as } |x|\ge 0 \text{ for all } x]$

$\Rightarrow \quad -2\le[x]\le 2 \qquad [\text{as } |x|\le a\Rightarrow -a\le x\le a]$

$\Rightarrow \quad -2\le x<3$

[by definition of greatest integral function]

Hence, domain of $g(x)$ is $[-2,3[$ or $[-2,3)$.

Example 40 Find the domain of function $f(x)=\frac{1}{[|x-1|]+[|7-x|]-6}$, where $[\cdot]$ denotes the greatest integer function.

Sol. $f(x)$ is defined when

$$[|x-1|]+[|7-x|]-6\ne 0$$

$$\Rightarrow \begin{cases}[1-x]+[7-x]\ne 6, & \text{when } x\le 1 \quad ...(i)\\ [x-1]+[7-x]\ne 6, & \text{when } 1\le x\le 7 \quad ...(ii)\\ [x-1]+[x-7]\ne 6, & \text{when } x\ge 7 \quad ...(iii)\end{cases}$$

$$[1-x]+[7-x]\ne 6$$

$1+[-x]+7+[-x]\ne 6 \Rightarrow 2[-x]\ne -2$

$\Rightarrow \quad [-x]\ne -1$

$\Rightarrow \quad x\notin(0,1] \qquad ...(a)$

From Eq. (ii), we have

$$[x-1]+[7-x]\ne 6$$

$\Rightarrow \quad [x]-1+7+[-x]\ne 6$

$\Rightarrow \quad [x]+[-x]\ne 0$

$\Rightarrow \quad x\notin$ Integer

$\Rightarrow \quad x\notin\{1,2,3,4,5,6,7\} \qquad ...(b)$

From Eq. (iii), we have

$$[x-1]+[x-7]\ne 6$$

$\Rightarrow \quad [x]-1+[x]-7\ne 6$

$\Rightarrow \quad 2[x]\ne 14 \Rightarrow [x]\ne 7$

$\Rightarrow \quad x\notin[7,8) \qquad ...(c)$

Hence, froms (a), (b) and (c), we get

Domain $f(x)$ is $R-\{(0,1)\cup\{1,2,3,4,5,6,7\}\cup(7,8)\}$.

Example 41 If the function $f(x)=[3.5+b\sin x]$ (where $[\cdot]$ denotes the greatest integer function) is an even function, the complete set of values of b is

(a) $(-0.5, 0.5)$ (b) $[-0.5, 0.5]$

(c) $(0, 1)$ (d) $[-1, 1]$

Sol. We have $f(x)=[3.5+b\sin x]$ For $f(x)$ to be an even funcion.

$$3<3.5+b\sin x<4,\ \forall x\in R$$

$$\Rightarrow\quad -0.5<b<0.5\ \Rightarrow\ b\in(-0.5,0.5)$$

Hence, (a) is the correct answer.

Example 42 The domain of the function

$$f(x)=\log_3\ \log_{1/3}(x^2+10x+25)+\frac{1}{[x]+5}$$

(where $[\cdot]$ denotes the greatest integer function) is

(a) $(-4,-3)$ (b) $(-6,-5)$

(c) $(-6,-4)$ (d) None of these

Sol. Here, $\log_3\log_{1/3}(x^2+10x+25)$ is defined, when

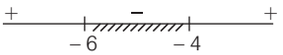
I. $x^2+10x+25>0$, i.e. $(x+5)^2>0\Rightarrow x\neq -5$ ...(i)

II. $\log_{1/3}(x^2+10x+25)>0$

$$\Rightarrow\quad x^2+10x+25<1$$

$$\text{or}\quad x^2+10x+24<0$$

$$\text{or}\quad (x+6)(x+4)<0$$



$$\Rightarrow\quad x\in(-6,-4)\quad ...(ii)$$

From Eqs. (i) and (ii), we get

$$x\in(-6,-5)\cup(-5,-4)\quad ...(iii)$$

III. $[x]+5\neq 0$

$$\Rightarrow\quad [x]\neq -5$$

$$\Rightarrow\quad x\notin[-5,-4)\quad ...(iv)$$

From Eqs. (iii) and (iv), we get

Domain of $f(x)$ is $(-6,-5)$.

Hence, (b) is the correct answer.

Example 43 Let $[x]$ be the greatest integer less than or equal to x.

Then, the equation $\sin x=[1+\sin x]+[1-\cos x]$ has

(a) one solution in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ (b) one solution in $\left[\frac{\pi}{2},\pi\right]$

(c) one solution in R (d) no solution in R

Sol. Given that, $[1+\sin x]+[1-\cos x]=\sin x$

$$\Rightarrow\quad 1+[\sin x]+1+[-\cos x]=\sin x$$

$$\Rightarrow\quad 2+[\sin x]+[-\cos x]=\sin x$$

$$\Rightarrow\quad 2+[-\cos x]=\{\sin x\}$$

Here, LHS is 1, 2 or 3, but RHS $\in[0,1)$

$\therefore$ No solution. Hence, (d) is the correct answer.

(iv) Fractional Part Function

$y=\{x\}$, $\{x\}$ indicates fractional part of x.

Let $x=I+f$, where $I=[x]$

and $f=\{x\}$

Then, $y=\{x\}$

$$=x-[x]$$

Here, $y=\{x\}$ could be expressed graphically, as

x	$\{x\}$
$0\le x<1$	x
$1\le x<2$	$x-1$
$2\le x<3$	$x-2$
$-1\le x<0$	$x+1$
$-2\le x<-1$	$x+2$

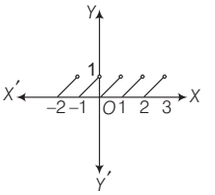


Figure 3.26

Properties of Fractional Part of x

(i) $\{x\}=x$, if $0\le x<1$.

(ii) $\{x\}=0$, if $x\in$ integer.

(iii) $\{-x\}=1-\{x\}$, if $x\notin$ integer.

(iv) $\{x\pm\text{integer}\}=\{x\}$.

Example 44 ***If*** $\{x\}$ ***and*** $[x]$ ***represent fractional and integral parts of*** x ***respectively, then find the value of*** $[x]+\sum_{r=1}^{2000}\frac{\{x+r\}}{2000}$.

Sol. Clearly, $[x]+\sum_{r=1}^{2000}\frac{\{x+r\}}{2000}=[x]+\sum_{r=1}^{2000}\frac{\{x\}}{2000}$

[$\because$ we know that $\{x+r\}=\{x\}$, if $r\in$ Integer]

$$=[x]+\left[\frac{\{x\}}{2000}+\frac{\{x\}}{2000}+\ldots+\text{upto 2000 times}\right]$$

$$=[x]+\frac{2000\{x\}}{2000}$$

$$=[x]+\{x\}$$

$$=x$$

Thus, $[x]+\sum_{r=1}^{2000}\frac{\{x+r\}}{2000}=x$

Example 45 Solve $4\{x\} = x + [x]$.

Sol. We know, $x = [x] + \{x\}$

$\therefore \quad 4\{x\} = [x] + \{x\} + [x] \Rightarrow \{x\} = \dfrac{2[x]}{3}$ ...(i)

But $0 \le \{x\} < 1$

So, $0 \le \dfrac{2[x]}{3} < 1$

$\Rightarrow \quad 0 \le [x] < \dfrac{3}{2}$, therefore $[x] = 0$ or 1

If $[x] = 1$, then $\{x\} = \dfrac{2}{3}$ [from Eq. (i)]

Thus, $x = \dfrac{5}{3}$

If $[x] = 0, \{x\} = 0$ so, $x = 0$

Thus, solutions of $4\{x\} = x + [x]$ are $x \in \left\{0, \dfrac{5}{3}\right\}$.

Example 46 Prove that $[x] + [y] \le [x + y]$, where $x = [x] + \{x\}$ and $y = [y] + \{y\}$, $[\cdot]$ represents greatest integer function and $\{\cdot\}$ represents fractional part function.

Sol. Here, $x + y = [x] + [y] + \{x\} + \{y\}$

$\therefore \quad [x + y] = [[x] + [y] + \{x\} + \{y\}]$

$= [x] + [y] + [\{x\} + \{y\}]$

[using $[x + I] = [x] + I, I \in$ Integer]

$\Rightarrow \quad [x + y] \ge [x] + [y]$

Remark

This could be generalised for n terms as

$[x_1] + [x_2] + \ldots + [x_n] \le [x_1 + x_2 + \ldots + x_n]$.

Example 47 ***Find the number of solutions of*** $|[x] - 2x| = 4$, ***where*** $[x]$ ***is the greatest integer*** $\le x$.

Sol. Let $x = I + f$, $I \in$ integer, $f \in$ fractional part

[i.e. $0 \le f < 1$]

Then, $|[x] - 2x| = 4$

$\Rightarrow \quad |[I + f] - 2(I + f)| = 4$

$\Rightarrow \quad |I - 2I - 2f| = 4 \Rightarrow |I + 2f| = 4$,

which is possible, only if $\left(f = \dfrac{1}{2} \text{ or } 0\right)$

If $f = \dfrac{1}{2}$,

$|I + 1| = 4 \Rightarrow I + 1 = \pm 4$

So, $I = 3, -5$ and $f = \dfrac{1}{2}$

If $f = 0, |I| = 4$

So, $I = \pm 4$ and $f = 0$

Thus, number of solutions are $x = \left\{\pm 4, \dfrac{7}{2}, -\dfrac{9}{2}\right\}$,

i.e. 4 solutions.

(v) Least Integer Function

$y = (x) = [x], [x]$ or (x) indicates the integral part of x which is the nearest and greater to x.

It is known as **ceiling of x**.

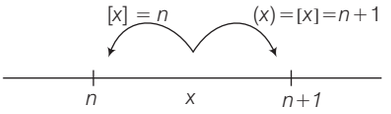


Figure 3.27

Thus, $[2.3203] = 3, \quad (0.23) = 1, (-8.0725) = -8, (-0.6) = 0$

In general, $n < x \le n + 1 \quad (n \in$ Integer)

$\Rightarrow \quad (x) = n + 1 = [x]$

Here, $f(x) = (x) = [x]$ can be expressed graphically as

x	$[x] = (x)$
$-1 < x \le 0$	0
$0 < x \le 1$	1
$1 < x \le 2$	2
$2 < x \le 3$	3
$-2 < x \le -1$	-1

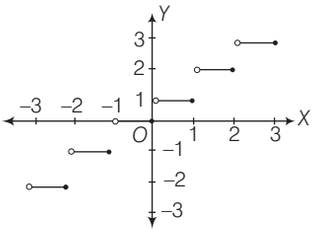


Figure 3.28

(•) represents value is taken.
(∘) represents value is neglected.

Properties of Least Integer Function

(i) $(x) = x = [x]$ holds, if x is Integer.

(ii) $(x + I) = [x + I] = (x) + I, I \in$ Integer.

(iii) Greatest integer function $[x]$ converts $x = I + f$ into I, while (x) converts to $I + 1$.

(iv) $x = (x) + \{x\} - 1$,

(v) $(-x) = \begin{cases} -(x), & \text{if } x \in I \\ -(x) + 1, & \text{if } x \notin I \end{cases}$

(vi) $(x) - (-x) = \begin{cases} 2(x), & \text{if } x \in I \\ 2(x) - 1, & \text{if } x \notin I \end{cases}$

(vii) $(x) \ge n \Leftrightarrow x > n - 1, n \in I$

(viii) $(x) > n \Leftrightarrow x > n, n \in I$

(ix) $(x) \le n \Leftrightarrow x \le n, n \in I$

(x) $(x) < n \Leftrightarrow x < n - 1, n \in I$

(xi) $\left(\dfrac{(x)}{n}\right) = \left(\dfrac{x}{n}\right), n \in N$

(xii) $\left(\dfrac{n+1}{2}\right) + \left(\dfrac{n+2}{4}\right) + \left(\dfrac{n+4}{8}\right) + \ldots = 2(n-1), n \in N$

(xiii) $(x) = \begin{cases} [x], & x \in I \\ [x] + 1, & x \notin I \end{cases}$

(xiv) $(x) + \left(x + \dfrac{1}{n}\right) + \left(x + \dfrac{2}{n}\right) + \ldots + \left(x + \dfrac{n-1}{n}\right) = (nx) + n - 1, n \in N$

Example 48 ***Find the solution set of*** $(x)^2 + (x+1)^2 = 25$, ***where*** (x) ***is the least integer greater than equals to*** x.

Sol. Let $x = I + f$, where I (integer) and f (fractional part) such that $0 < f < 1$.

Then, $(I+f)^2 + (I+f+1)^2 = 25$

$\Rightarrow \quad \{I+1\}^2 + \{I+2\}^2 = 25$

$\Rightarrow \quad I^2 + 2I + 1 + I^2 + 4I + 4 = 25$

$\Rightarrow \quad 2I^2 + 6I + 5 - 25 = 0$

$\Rightarrow \quad 2I^2 + 6I - 20 = 0$

So, $I = 2, -5$

Thus, $x = 2 + f, -5 + f$

where $0 < f < 1$

$\Rightarrow \quad 2 < 2 + f < 3, -5 < -5 + f < -4$ ...(i)

Again, let $x = I$

Then, $x^2 + (x+1)^2 = 25$

$\Rightarrow \quad x = 3, -4$ ...(ii)

From Eqs. (i) and (ii), we get $x \in (-5, -4] \cup (2, 3]$

Example 49 ***If*** $[x]$ = ***the greatest integer less than or equal to*** x ***and*** (x) = ***the least integer greater than or equal to*** x ***and*** $[x]^2 + (x)^2 > 25$, ***then*** x ***belongs to ..... .***

Sol. Let $x = I + f$, where $I \in$ integer, $f \in$ fractional part such that $0 \le f < 1$.

$\therefore \quad [x]^2 + (x)^2 > 25$

$\Rightarrow \quad [I+f]^2 + (I+f)^2 > 25$

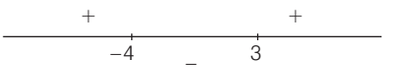
$\Rightarrow \quad I^2 + \{I+1\}^2 > 25$

$\Rightarrow \quad I^2 + I^2 + 2I + 1 > 25$

$\Rightarrow \quad 2I^2 + 2I - 24 > 0$

$\Rightarrow \quad I^2 + I - 12 > 0$

$\Rightarrow \quad (I+4)(I-3) > 0$



$\therefore \quad I < -4$

or $I > 3$

Here, $x = I + f$

So, $x < -4 + f$

or $x > 3 + f$ ...(i)

Now, let $x = I$, then $x^2 + x^2 > 25$

$\Rightarrow \quad x^2 > 12.5$

$\Rightarrow \quad x \le -4$ or $x \ge 4$

Form Eqs. (i) and (ii), we get $x \in (-\infty, -4] \cup [4, \infty)$

Exercise for Session 4

- **Directions** (Q. Nos. 1 to 18) *Find the domain of the following.*

1. $f(x) = \sqrt{x^2 - |x| - 2}$
2. $f(x) = \sqrt{2 - |x|} + \sqrt{1 + |x|}$
3. $f(x) = \log_e |\log_e x|$
4. $f(x) = \sin^{-1}\left(\dfrac{2 - 3[x]}{4}\right)$, which $[\cdot]$ denotes the greatest integer function.
5. $f(x) = \log(x - [x])$, where $[\cdot]$ denotes the greatest integer function.
6. $f(x) = \dfrac{1}{\sqrt{[x]^2 - [x] - 6}}$, where $[\cdot]$ denotes the greatest integer function.
7. $f(x) = \text{cosec}^{-1}[1 + \sin^2 x]$, where $[\cdot]$ denotes the greatest integer function.

8. $f(x) = \cos^{-1}\sqrt{\log_{[x]}\frac{|x|}{x}}$, where $[\cdot]$ denotes the greatest integer function.

9. $f(x) = \sqrt{\frac{x-1}{x-2\{x\}}}$, where $\{\cdot\}$ denotes the fractional part function.

10. $f(x) = \sin^{-1}\left(\frac{[x]}{\{x\}}\right)$, where $[\cdot]$ and $\{\cdot\}$ denotes the greatest integer and fractional part function.

11. $f(x) = \sin^{-1}[2x^2 - 3]$, where $[\cdot]$ denotes the greatest integer function.

12. $f(x) = \sin^{-1}\left[\log_2\left(\frac{x^2}{2}\right)\right]$, where $[\cdot]$ denotes the greatest integer function.

13. $f(x) = \sqrt{2\{x\}^2 - 3\{x\} + 1}$, $x \in [-1, 1]$, where $\{\cdot\}$ denotes the fractional part function.

14. $f(x) = \frac{1}{[|x-2|] + [|x-10|] - 8}$, where $[\cdot]$ denotes the greatest integer function.

15. If a function is defined, as $g(x) = |\sin x| + \sin x$, $\phi(x) = \sin x + \cos x, 0 \le x \le \pi$, then find the domain for $f(x) = \sqrt{\log_{\phi(x)} g(x)}$.

16. Solve the equations, $y = \frac{1}{3}[\sin x + [\sin x + [\sin x]]]$ and $[y + [y]] = 2\cos x$, where $[\cdot]$ denotes the greatest integer function.

17. If $[x] = \left[\frac{x}{2}\right] + \left[\frac{x+1}{2}\right]$, where $[\cdot]$ denotes the greatest integer function and n be positive integer, then show that $\left[\frac{n+1}{2}\right] + \left[\frac{n+2}{4}\right] + \left[\frac{n+4}{8}\right] + \left[\frac{n+8}{16}\right] + \ldots = n$.

18. Find the integral solutions to the equation $[x][y] = x + y$. Show that all the non-integral solutions lie on exactly two lines. Determine these lines.

Session 5

Range

Range

The range of a function is the set of all possible values that can be produced by that function.

e.g. Let $f(x) = x^2$

No matter what value we substitute for x, $f(x)$ will always be positive. Therefore, we would say that the range of this function is the set of all positive real numbers.

Rules for Finding the Range

First of all find the domain of given function $y = f(x)$

(i) If domain contains finite number of points, range is the set of corresponding $f(x)$ values.

(ii) If domain is R or R – {some finite points},
express x in terms of y. From this, find y for x to be defined.
(i.e. Find the values of y for which x exists.)

(iii) If domain is a finite interval, then find the least and the greatest values for range, using monotonicity.
e.g. If $f(x)$ is defined on $[a, b]$, then
Step 1 Put, $f'(x) = 0 \Rightarrow x = \alpha_1, \alpha_2, \ldots, \alpha_n \in (a, b)$
Step 2 Find, $\{f(a), f(\alpha_1), f(\alpha_2), \ldots, f(\alpha_n), f(b)\}$
The greatest and least values gives the range of $f(x)$.

Example 50 ***Find the range for*** $y = \dfrac{x - [x]}{1 - [x] + x}$.

Sol. Here, $y = \dfrac{x - [x]}{1 + x - [x]} = \dfrac{\{x\}}{1 + \{x\}}$

Thus, domain is R.

Now, from $y = \dfrac{\{x\}}{1 + \{x\}}$,

we have, $y + y\{x\} = \{x\} \Rightarrow \{x\} = \dfrac{y}{1 - y}$

Here, $0 \le \{x\} < 1$, so $0 \le \dfrac{y}{1 - y} < 1 \Rightarrow 0 \le y < \dfrac{1}{2}$

Hence, range $= \left[0, \dfrac{1}{2}\right)$.

Example 51 ***Find the range for*** $f(x) = \dfrac{e^x}{1 + [x]}$ ***when*** $x \ge 0$.

Sol. Here, $f(x)$ is defined for all $x \ge 0$. Also, $f(x)$ is an increasing function in $[0, \infty)$.

Thus, range $= [f(0), f(\infty))$

Hence, range $= [1, \infty)$

Example 52 ***Find the domain and range of the function*** $y = \log_e (3x^2 - 4x + 5)$.

Sol. y is defined, if $3x^2 - 4x + 5 > 0$

Where $D = 16 - 4(3)(5) = -44 < 0$

and coefficient of $x^2 = 3 > 0$

Hence, $(3x^2 - 4x + 5) > 0, \ \forall x \in R$

Thus, domain $\in R$

Now, from $y = \log_e (3x^2 - 4x + 5)$,

we have, $3x^2 - 4x + 5 = e^y$ or $3x^2 - 4x + (5 - e^y) = 0$

Since, x is real, therefore discriminant ≥ 0.

i.e. $(-4)^2 - 4(3)(5 - e^y) \ge 0 \Rightarrow 12e^y \ge 44$

So, $e^y \ge \dfrac{11}{3}$. Thus, $y \ge \log\left(\dfrac{11}{3}\right)$

Hence, range is $\left[\log\left(\dfrac{11}{3}\right), \infty\right)$.

Aliter Since, log is an increasing function and $3x^2 - 4x + 5$ is minimum at $x = \dfrac{2}{3}$, i.e. $\log_e(3x^2 - 4x + 5)$ is minimum at $x = \dfrac{2}{3}$ and minimum value of $y = \log_e \dfrac{11}{3}$, i.e. $\left(\log_e \dfrac{11}{3} \le y < \infty\right)$.

Thus, range is $\left[\log_e\left(\dfrac{11}{3}\right), \infty\right)$.

Example 53 Find the range of $y = \sqrt{x - 1} + \sqrt{5 - x}$.

Sol. Here, domain $x \ge 1$ and $x \le 5$, i.e. $x \in [1, 5]$.

Now, check monotonicity, $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x - 1}} - \dfrac{1}{2\sqrt{5 - x}}$

When $\dfrac{dy}{dx} = 0$, we get $x = 3$

Thus, $y_{\min}$ at $x = 1$ or $x = 5$

$y_{\min}$ at $x = 1$ is 2;

$y_{\min}$ at $x = 5$ is 2.

Thus, minimum value of $y = 2$.

Also, $y_{\max}$ at $x = 3$ is $2\sqrt{2}$.

Hence, range $= [2, 2\sqrt{2}]$

Remarks

(i) When y is minimum at two or more points, the smallest value amongst them is taken.

(ii) When y is maximum at two or more points, the largest value amongst them is taken.

Example 54 *Find the range of* $\log_3\{\log_{1/2}(x^2+4x+4)\}$.

Sol. Firstly, finding domain

$\log_3\{\log_{1/2}(x^2+4x+4)\}$ exists, if

$\log_{1/2}(x^2+4x+4) > 0$

$\Rightarrow \quad x^2+4x+4 < \left(\frac{1}{2}\right)^0$

[using $\log_a x > b \Rightarrow x < a^b$, if $0 < a < 1$]

$\Rightarrow \quad x^2+4x+4 < 1$

$\Rightarrow \quad x^2+4x+3 < 0$

$\Rightarrow \quad (x+1)(x+3) < 0 \quad \Rightarrow -3 < x < -1$ ...(i)

and $x^2+4x+4 > 0 \quad \Rightarrow (x+2)^2 > 0$,

which is always true except for $x = -2$. ...(ii)

From Eqs. (i) and (ii), we have

$x \in (-3,-2) \cup (-2,-1)$

Thus, domain is $(-3,-2) \cup (-2,-1)$

Now, we find out the range.

Since, $0 < \log_{1/2}(x^2+4x+4) < \infty, \forall\ x \in$ domain y

$\Rightarrow -\infty < \log_3\{\log_{1/2}(x^2+4x+4)\} < \infty$

Thus, range (y) is R.

Example 55 *Range of the function* $f(x) = (\cos^{-1}|1-x^2|)$ *is*

(a) $\left[0, \frac{\pi}{2}\right]$ (b) $\left[0, \frac{\pi}{3}\right]$ (c) $(0, \pi)$ (d) $\left(\frac{\pi}{2}, \pi\right)$

Sol. Here, $f(x) = \cos^{-1}|1-x^2|$ would exists, only when $|1-x^2| \le 1$

$\Rightarrow \quad -1 \le 1-x^2 \le 1 \quad \Rightarrow \quad 2 \ge x^2 \ge 0$

or $x \in [-\sqrt{2}, \sqrt{2}]$

$\because \quad 0 \le |1-x^2| \le 1, \ \forall\ x \in [-\sqrt{2}, \sqrt{2}]$

$\therefore \quad 0 \le \cos^{-1}|1-x^2| \le \frac{\pi}{2}$

Hence, (a) is the correct answer.

Example 56 If x, y and z are real such that $x+y+z=4$, $x^2+y^2+z^2=6$, x belongs to

(a) $(-1, 1)$ (b) $[0, 2]$

(c) $[2, 3]$ (d) $\left[\frac{2}{3}, 2\right]$

Sol. Eliminating z,

$x^2+y^2+(4-(x+y))^2 = 6$

$\Rightarrow \quad y^2+(x-4)y+x^2-4x+5 = 0$

$\Rightarrow \quad D \ge 0$

$\Rightarrow \quad (x-4)^2-4(x^2-4x+5) \ge 0$

$\Rightarrow \quad 3x^2-8x+4 \le 0$

$\Rightarrow \quad (x-2)(3x-2) \le 0$

$\therefore \quad x \in [2/3, 2]$

Hence, (d) is the correct answer.

Example 57 The range of the function $f(x) = \frac{1}{|\sin x|} + \frac{1}{|\cos x|}$ is

(a) $[2\sqrt{2}, \infty)$ (b) $(\sqrt{2}, 2\sqrt{2})$ (c) $(0, 2\sqrt{2})$ (d) $(2\sqrt{2}, 4)$

Sol. $f(x) = \frac{1}{|\sin x|} + \frac{1}{|\cos x|}$

Using AM $\ge$ GM, we get

$$\frac{\frac{1}{|\sin x|} + \frac{1}{|\cos x|}}{2} \ge \left(\frac{1}{|\sin x||\cos x|}\right)^{1/2}$$

$\Rightarrow \quad \frac{1}{|\sin x|} + \frac{1}{|\cos x|} \ge 2\,(2\,|\operatorname{cosec} 2x|)^{1/2}$

[where, $|\operatorname{cosec} 2x| \ge 1$]

$\Rightarrow \quad \frac{1}{|\sin x|} + \frac{1}{|\cos x|} \ge 2\sqrt{2}$

$\therefore$ Range of $f(x)$ is $[2\sqrt{2}, \infty)$.

Hence, (a) is the correct answer.

Example 58 If $z = x+iy$ and $x^2+y^2=16$, then the range of $||x|-|y||$ is

(a) $[0, 4]$ (b) $[0, 2]$

(c) $[2, 4]$ (d) None of these

Sol. Let $x = 4\cos\theta, y = 4\sin\theta$, then

$$|4|\cos\theta| - 4|\sin\theta|| = 4||\cos\theta| - |\sin\theta||$$
$$= 4\sqrt{1 - 2|\cos\theta||\sin\theta|}$$
$$= 4\sqrt{1 - |\sin 2\theta|}$$

$\therefore$ Range is $[0, 4]$. Hence, (a) is the correct answer.

Example 59 The range of $f(x) = \frac{1}{\pi}(\sin^{-1}x + \tan^{-1}x) + \frac{x+1}{x^2+2x+5}$ is

(a) $\left[-\frac{3}{4}, \frac{1}{5}\right]$ (b) $\left[-\frac{5}{4}, \frac{3}{4}\right]$

(c) $\left[-\frac{3}{4}, \frac{5}{4}\right]$ (d) $\left[-\frac{3}{4}, 1\right]$

Sol. Here, $f(x) = \frac{1}{\pi}(\sin^{-1}x + \tan^{-1}x) + \frac{1}{(x+1) + \frac{4}{(x+1)}}$

$= g(x) + h(x)$,

where domain of $g(x)$ is $[-1, 1]$

$\therefore$ Maximum value of $g(x) = g(1) = \frac{3}{4}$

and minimum value of $g(x) = g(-1) = -\frac{3}{4}$

Also, maximum value of $h(x)$ occurs, when $(x+1) + \frac{4}{(x+1)}$ is minimum at $x = 1$.

$\Rightarrow$ Range of $f(x)$ is $\left[-\frac{3}{4}, 1\right]$.

Hence, (d) is the correct answer.

Example 60 The range of the function $f(x) = \sin^2 x - 5\sin x - 6$ is

(a) $[-10, 0]$ (b) $[-1, 1]$ (c) $[0, \pi]$ (d) $\left[-\frac{49}{4}, 0\right]$

Sol. Here, $f(x) = \sin^2 x - 5\sin x - 6$

$$= \left(\sin^2 x - 5\sin x + \frac{25}{4}\right) - 6 - \frac{25}{4}$$
$$= \left(\sin x - \frac{5}{2}\right)^2 - \frac{49}{4} \quad \ldots(i)$$

where $\frac{9}{4} \le \left(\sin x - \frac{5}{2}\right)^2 \le \frac{49}{4}$ ...(ii)

From Eqs. (i) and (ii), we get $-10 \le f(x) \le 0$

$\Rightarrow$ Range of $f(x)$ is $[-10, 0]$.

Hence, (a) is the correct answer.

Example 61 If $f(x) = [x^2] - [x]^2$, where $[\cdot]$ denotes the greatest integer function and $x \in [0, n], n \in N$, then the number of elements in the range of $f(x)$ is

(a) $(2n+1)$ (b) $4n-3$ (c) $3n-3$ (d) $2n-1$

Sol. When $x = n-1$, $f(x) = (n-1)^2 - (n-1)^2 = 0$

When $n - 1 < x < n$,

$[x] = n - 1$

and $(n-1)^2 \le [x^2] \le n^2 - 1$

$\therefore$ $0 \le [x^2] - [x]^2 \le n^2 - 1 - (n-1)^2$

$\Rightarrow$ $0 \le f(x) \le 2n - 2$, but $f(x)$ has to be an integer.

The set of values of $f(x)$ is $\{0, 1, 2, \ldots, 2n-2\}$.

Hence, (d) is the correct answer.

Example 62 ***Range of the function*** $f(x) = \sqrt{|\sin^{-1}|\sin x| - \cos^{-1}|\cos x|}$, ***is***

(a) $\{0\}$ (b) $\left[0, \sqrt{\frac{\pi}{2}}\right]$ (c) $[0, \sqrt{\pi}]$ (d) None of these

Sol. We know that, $|\sin^{-1}|\sin x|| = \cos^{-1}|\cos x|$,

$\forall\ x \in$ domain

$\therefore f(x) = \sqrt{|\sin^{-1}|\sin x| - \cos^{-1}|\cos x|} = 0, \forall\ x \in$ domain

$\therefore$ Range of $f(x)$ is $\{0\}$.

Hence, (a) is the correct answer.

Example 63 The number of values of y in $[-2\pi, 2\pi]$ satisfying the equation $|\sin 2x| + |\cos 2x| = |\sin y|$ is

(a) 3 (b) 4 (c) 5 (d) 6

Sol. Here, $1 \le |\sin 2x| + |\cos 2x| \le \sqrt{2}$ and $|\sin y| \le 1$.

So, solution is possible only when $|\sin y| = 1$.

$\Rightarrow$ $\sin y = \pm 1 \quad \Rightarrow \quad y = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$

$\therefore$ Number of values of y is 4.

Hence, (b) is the correct answer.

Example 64 Let $f(x) = \cot^{-1}(x^2 - 4x + 5)$, then range of $f(x)$ is equal to

(a) $\left(0, \frac{\pi}{2}\right)$ (b) $\left(0, \frac{\pi}{4}\right]$

(c) $\left[0, \frac{\pi}{4}\right)$ (d) None of these

Sol. Here, $x^2 - 4x + 5 \Rightarrow (x-2)^2 + 1 \ge 1$

$\therefore$ $1 \le x^2 - 4x + 5 < \infty$

$\Rightarrow$ $0 < \cot^{-1}(x^2 - 4x + 5) \le \frac{\pi}{4}$

[using, $x_1 < x_2 \Rightarrow \cot^{-1}x_1 > \cot^{-1}x_2$, since $\cot^{-1}x$ is decreasing]

$\Rightarrow$ Range of $f(x)$ is $\left(0, \frac{\pi}{4}\right]$.

Hence, (b) is the correct answer.

To Find Range for Rational Expressions

Let $f(x)=\dfrac{ax^2+bx+c}{px^2+qx+r}$

***Step* 1** Write the given function as a equation

$$y=\frac{ax^2+bx+c}{px^2+qx+r}$$

***Step* 2** Rewrite the above equation for x in standard form

$$x^2(py-a)+x(qy-b)+ry-c=0$$

***Step* 3** Find the discriminant to the above equation

$$D=(qy-b)^2-4(py-a)(ry-c)$$

$$D=q^2y^2-2qyb+b^2-4\ [pry^2-(pc+ar)y+ac]$$

$$D=y^2(q^2-4pr)+(4pc+4ar-2bq)y+b^2-4ac$$

***Step* 4** Find $y_1, y_2=\dfrac{-B\pm\sqrt{B^2-4AC}}{2A}$

where, $A=q^2-4pr,\ B=4pc+4ar-2bq, C=b^2-4ac.$

***Step* 5** Let $y_1<y_2$, then range of $f(x)$ is $[y_1, y_2]$ if $A<0$ and $A>0$, then range of $f(x)$ is $R-(y_1, y_2)$.

Example 65 *Find the range of* $f(x)=\dfrac{x^2+14x+9}{x^2+2x+3}$, *where* $x\in R$.

Sol. Here, $A=4-12=-8,\ B=12+36-56=-8,\ C=160$

Now, $\dfrac{-B\pm\sqrt{B^2-4AC}}{2A}=\dfrac{-1\pm\sqrt{1+80}}{-2}=-5, 4$ and here $A<0$

$\therefore$ Range is $[-5, 4]$

Example 66 *For what real values of a does the range of* $f(x)=\dfrac{x+1}{a+x^2}$ *contains the interval* $[0,1]$ *?*

Sol. Let $y=\dfrac{x+1}{a+x^2}$

$\Rightarrow\quad y(a+x^2)=x+1$

$\Rightarrow yx^2-x+(ay-1)=0$ has real roots for every $y\in[0,1]$.

$\therefore\quad 1-4y(ay-1)\geq 0$

$\Rightarrow\quad 1-4ay^2+4y\geq 0$ holds for $0\leq y\leq 1$ ...(i)

***Case* I** If $y=0$

Here, $y=0$ is assumed at $x=-1$ for any $a\neq -1$.

For $a=-1, y=\dfrac{x+1}{x^2-1}$ is undefined for $x=-1$.

***Case* II** If $0<y\leq 1$. Put $z=\dfrac{1}{y}$

$\Rightarrow\quad 1\leq z<\infty$

Now, from Eq. (i), we get

$z^2+4z-4a\geq 0$ holds for $1\leq z<\infty$

$\Rightarrow\quad (z+2)^2-4-4a\geq 0$ holds for $1\leq z<\infty$

Consider, $(z+2)^2-4-4a\geq 0$

$\Rightarrow\quad (z+2)^2\geq 4+4a$

$\Rightarrow\quad z+2\geq\sqrt{4+4a};\ 4+4a\geq 0$

$a\geq -1$ ...(ii)

but $z\geq 1$

$\therefore\quad 9\geq 4+4a$

$\Rightarrow\quad 5\geq 4a\quad [\text{as } (u+2)^2\geq 9 \text{ for } u=1]$

$\Rightarrow\quad a\leq\dfrac{5}{4}, a\neq -1$ ...(iii)

$\therefore$ From Eqs. (i), (ii) and (iii), we get $a\in\left(-1,\dfrac{5}{4}\right]$.

Example 67 Find the range of the function $f(x)=\dfrac{\sin^2 x+\sin x-1}{\sin^2 x-\sin x+2}$.

Sol. Let $y=\dfrac{\sin^2 x+\sin x-1}{\sin^2 x-\sin x+2}$

Let $t=\sin x\ \Rightarrow\ -1\leq t\leq 1$ and $y=\dfrac{t^2+t-1}{t^2-t+2}$

$\Rightarrow\quad (y-1)t^2-(y+1)t+(2y+1)=0$ ...(i)

Since t is real, $\dfrac{3-2\sqrt{11}}{7}\leq y\leq\dfrac{3+2\sqrt{11}}{7}$

***Case* I** If both roots of Eq. (i) are greater than 1, i.e. $t_1>1$ and $t_2>1$.

$\Rightarrow\quad t_1+t_2>2$

and $(t_1-1)(t_2-1)>0$

$\Rightarrow\quad \dfrac{y+1}{y-1}>2$

and $\dfrac{2y+1}{y-1}-\dfrac{y+1}{y-1}+1>0$

$\Rightarrow\quad 1<y<3$

and $y>1$ or $y<\dfrac{1}{2}$

$\therefore\quad y\in(1,3)$ ...(ii)

***Case* II** If $t_1<-1$ and $t_2<-1$

$\Rightarrow\quad t_1+t_2<-2$ and $(t_1+1)(t_2+1)>0$

$\Rightarrow\quad \dfrac{y+1}{y-1}+2<0$ and $\dfrac{2y+1}{y-1}+\dfrac{y+1}{y-1}+1>0$

$\Rightarrow \quad \frac{1}{3} < y < 1 \text{ and } y > 1 \text{ or } y < -\frac{1}{4}$

$\Rightarrow \quad y \in \phi \qquad \ldots(iii)$

***Case* III** If $t_1 < -1$ and $t_2 > -1, t_1 < 1$ and $t_2 > 1$

$\Rightarrow \quad (t_1 + 1)(t_2 + 1) < 0 \text{ and } (t_1 - 1)(t_2 - 1) < 0$

$\Rightarrow \quad \frac{2y+1}{y-1} + \frac{y+1}{y-1} + 1 < 0 \text{ and } \frac{2y+1}{y-1} - \frac{y+1}{y-1} + 1 < 0$

$\Rightarrow \quad -\frac{1}{4} < y < 1 \text{ and } \frac{1}{2} < y < 1 \Rightarrow \frac{1}{2} < y < 1$

$\therefore \quad R_f = \left[\frac{3 - 2\sqrt{11}}{7}, \frac{1}{2}\right] \cup \{1\}$

Exercise for Session 5

■ **Directions** (Q. Nos. 1 to 25) *Find the range of the following.*

1. $f(x) = \sqrt{9 - x^2}$

2. $f(x) = \frac{x}{1 + x^2}$

3. $f(x) = \sin x + \cos x + 3$

4. $f(x) = |x - 1| + |x - 2|, \ -1 \le x \le 3$

5. $f(x) = \log_3 (5 + 4x - x^2)$

6. $f(x) = \frac{x^2 - 2}{x^2 - 3}$

7. $f(x) = \frac{x^2 + 2x + 3}{x}$

8. $f(x) = |x - 1| + |x - 2| + |x - 3|$

9. $f(x) = \log_{[x-1]} \sin x$, where $[\cdot]$ denotes the greatest integer function.

10. $f(x) = \cos^{-1} \sqrt{\log_{[x]} \frac{|x|}{x}}$, where $[\cdot]$ denotes the greatest integer function.

11. $f(x) = \sqrt{[\sin 2x] - [\cos 2x]}$, where $[\cdot]$ denotes the greatest integer function.

12. $f(x) = \sin^{-1}\left[x^2 + \frac{1}{2}\right] + \cos^{-1}\left[x^2 - \frac{1}{2}\right]$, where $[\cdot]$ denotes the greatest integer function.

13. $f(x) = \sin^{-1}(\sqrt{x^2 + x + 1})$

14. $f(x) = \cos^{-1}\left(\frac{x^2}{\sqrt{1 + x^2}}\right)$

15. $f(x) = \sqrt{\log(\cos(\sin x))}$

16. $f(x) = \frac{x - 1}{x^2 - 2x + 3}$

17. $f(x) = \frac{\sin x}{\sqrt{1 + \tan^2 x}} - \frac{\cos x}{\sqrt{1 + \cot^2 x}}$

18. $f(x) = \frac{\tan(\pi [x^2 - x])}{1 + \sin(\cos x)}$

19. $f(x) = \frac{e^x}{[x + 1]}, x \ge 0$

20. $f(x) = [|\sin x| + |\cos x|]$, where $[\cdot]$ denotes the greatest integer function.

21. $f(x) = \sqrt{-x^2 + 4x - 3} + \sqrt{\sin \frac{\pi}{2}\left(\sin \frac{\pi}{2}(x - 1)\right)}$

22. Find the image of the following sets under the mapping $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 10$ (i) $(-\infty, 1)$ (ii) $[1, 2]$.

23. Find the domain and range of $f(x) = \log\left[\cos |x| + \frac{1}{2}\right]$, where $[\cdot]$ denotes the greatest integer function.

24. Find the domain and range of $f(x) = \sin^{-1}(\log [x]) + \log(\sin^{-1} [x])$, where $[\cdot]$ denotes the greatest integer function.

25. Find the domain and range of $f(x) = [\log(\sin^{-1} \sqrt{x^2 + 3x + 2})]$, where $[\cdot]$ denotes the greatest integer function.

Session 6

Odd and Even Functions

Odd and Even Functions

Odd Functions

A function $f(x)$ is said to be an odd function, if $f(-x) = -f(x)$ for all x. Graph of an odd function is **symmetrical in opposite quadrants**, i.e. the curve in first quadrant is identical to the curve in the third quadrant and the curve in second quadrant is identical to the curve in fourth quadrant. Some graphs which are symmetrical in opposite quadrants (or about origin) are

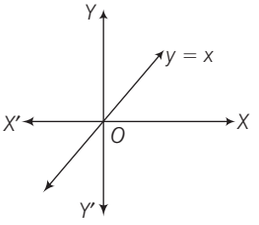


Figure 3.29

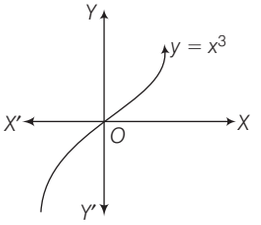


Figure 3.30

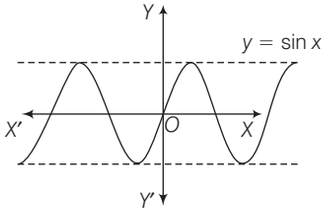


Figure 3.31

Even Functions

A function $f(x)$ is said to be an even, if $f(-x) = f(x)$ for all x. The graph is always **symmetrical about Y-axis**, i.e. the graph on left hand side of Y-axis is the mirror image of the curve on its right hand side.

Some graphs which are symmetrical about ***Y*-axis** are

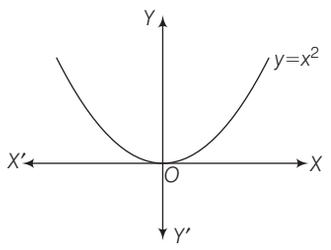


Figure 3.32

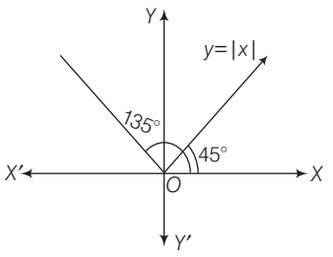


Figure 3.33

Properties of Odd and Even Functions

(i) Product of two odd functions or two even functions is an even function.

(ii) Product of odd and even function is an odd function.

(iii) Every function $y = f(x)$ can be expressed as the sum of an even and odd function.

(iv) The derivative of an odd function is an even function and derivative of an even function is an odd function.

(v) A function which is even or odd, when squared becomes an even function.

(vi) The only function which is both even and odd is $f(x) = 0$, i.e. zero function.

Example 68 If f is an even function, then find the real values of x satisfying the equation

$$f(x) = f\left(\frac{x+1}{x+2}\right).$$ ***[IIT JEE 1996, 2001]***

Sol. Since, $f(x)$ is even, so $f(-x) = f(x)$

Thus, $$x = \frac{x+1}{x+2}$$

or $$-x = \frac{x+1}{x+2}$$

$\Rightarrow$ $$x^2 + 2x = x + 1$$

or $$-x^2 - 2x = x + 1$$

$\Rightarrow$ $$x^2 + x - 1 = 0$$

or $$-x^2 - 3x - 1 = 0$$

$\Rightarrow$ $$x = \frac{-1 \pm \sqrt{5}}{2}$$

or $$x = \frac{-3 \pm \sqrt{5}}{2}$$

Thus, $x \in \left\{\frac{-1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}, \frac{-3+\sqrt{5}}{2}, \frac{-3-\sqrt{5}}{2}\right\}$

Example 69 Find out whether the given function is even, odd or neither even nor odd,

where $f(x) = \begin{cases} x|x| & , \quad x \le -1 \\ [1+x] + [1-x] & , \quad -1 < x < 1 \\ -x|x| & , \quad x \ge 1 \end{cases}$

where $|\,|$ and $[\,]$ represent the modulus and greatest integer functions.

Sol. The given function can be written as

$$f(x)=\begin{cases}-x^2 & , \quad x\le -1\\ 2+[x]+[-x] & , \quad -1<x<1\\ -x^2 & , \quad x\ge 1\end{cases}$$

[using definition of modulus function and the properties of greatest integer functions]

$$\Rightarrow \quad f(x)=\begin{cases}-x^2 & , \quad x\le -1\\ 2-1+0 & , \quad -1<x<0\\ 2 & , \quad x=0\\ 2+0-1 & , \quad 0<x<1\\ -x^2 & , \quad x\ge 1\end{cases}$$

$$f(x)=\begin{cases}-x^2 & , \quad x\le -1\\ 1 & , \quad -1<x<0\\ 2 & , \quad x=0\\ 1 & , \quad 0<x<1\\ -x^2 & , \quad x\ge 1\end{cases}$$

which is clearly even as

if $f(-x)=f(x)$.

Thus, $f(x)$ is even.

Example 70 Find whether the given function is even or odd function, where $f(x)=\dfrac{x(\sin x+\tan x)}{\left[\dfrac{x+\pi}{\pi}\right]-\dfrac{1}{2}}$, when $x\neq n\pi$, where $[\cdot]$ denotes the greatest integer function.

Sol. $f(x)=\dfrac{x(\sin x+\tan x)}{\left[\dfrac{x+\pi}{\pi}\right]-\dfrac{1}{2}}=\dfrac{x(\sin x+\tan x)}{\left[\dfrac{x}{\pi}\right]+1-\dfrac{1}{2}}$

$$f(x)=\frac{x(\sin x+\tan x)}{\left[\dfrac{x}{\pi}\right]+0.5}$$

Now, $f(-x)=\dfrac{-x(\sin(-x)+\tan(-x))}{\left[-\dfrac{x}{\pi}\right]+0.5}$

$$\Rightarrow \quad f(-x)=\begin{cases}\dfrac{x(\sin x+\tan x)}{-1-\left[\dfrac{x}{\pi}\right]+0.5} & , \quad x\neq n\pi\\ 0 & , \quad x=n\pi\end{cases}$$

So, $f(-x)=-\dfrac{x(\sin x+\tan x)}{\left[\dfrac{x}{\pi}\right]+0.5}$ or $[\because x\neq n\pi]$

$\therefore f(-x)=-f(x)$. Hence, $f(x)$ is an odd function (if $x\neq n\pi$).

Exercise for Session 6

1. Determine whether the following functions are even or odd.

(i) $f(x)=\log(x+\sqrt{1+x^2})$ (ii) $f(x)=x\left(\dfrac{a^x+1}{a^x-1}\right)$

(iii) $f(x)=\sin x+\cos x$ (iv) $f(x)=x^2-|x|$

(v) $f(x)=\log\left(\dfrac{1-x}{1+x}\right)$ (vi) $f(x)=\{(\text{sgn } x)^{sgn\,x}\}^n$; n is an odd integer.

(vii) $f(x)=\text{sgn}(x)+x^2$ (viii) $f(x+y)+f(x-y)=2f(x)\cdot f(y)$; where $f(0)\neq 0$ and $x,y\in R$.

2. Determine whether function; $f(x)=(-1)^{[x]}$ is even, odd or neither of two (where $[\cdot]$ denotes the greatest integer function).

3. A function defined for all real numbers is defined for $x\ge 0$ as follows $f(x)=\begin{cases}x|x|, & 0\le x<1\\ 2x & , \quad x\ge 1\end{cases}$.

How is f defined for $x\le 0$, if (i) f is even? (ii) f is odd?

4. Show that the function $f(x)=\dfrac{2x(\sin x+\tan x)}{2\left[\dfrac{x+21\pi}{\pi}\right]-41}$ is symmetric about origin.

5. If $f:[-20,20]\to R$ defined by $f(x)=\left[\dfrac{x^2}{a}\right]\sin x+\cos x$ is an even function, find the set of values of 'a' (where $[\cdot]$ denotes the greatest integer function).

Session 7

Periodic Functions

Periodic Functions

Definition A function $f(x)$ is said to be periodic function, if there exists a positive real number, T such that $f(x+T)=f(x)$, $\forall\ x \in \text{Dom}\,(f)$. Then, $f(x)$ is periodic with period T, where T is least positive value.

Graphically If the graph repeats at fixed interval, the function is said to be periodic and its period is the width of that interval.

Example 71 Prove that $\sin x$ is periodic and find its period.

Sol. Let $f(x)=\sin x$ and $T>0$, then $f(x)$ is periodic, if $f(x+T)=f(x)$.

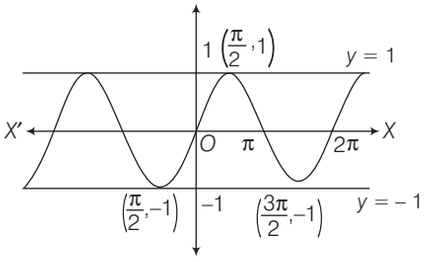
$\Rightarrow \quad \sin(x+T)=\sin(x), \forall\ x \in R$

$\Rightarrow \quad T = 2\pi, 4\pi, 6\pi, \ldots.$

But period of $f(x)$ is smallest positive real number.

Thus, period of $f(x)$ is 2π.

Aliter $f(x)=\sin x$ could be expressed graphically as shown in figure.



Here, graph repeats at an interval of 2π.

Thus, $f(x)$ is periodic with period 2π.

Example 72 Prove that $f(x)=x-[x]$ is periodic function. Also, find its period.

Sol. Let $T>0$.

Then, $\quad f(x+T)=f(x), \forall\ x \in R$

$\Rightarrow \quad (x+T)-[x+T]=x-[x], \forall\ x \in R$

$\Rightarrow \quad [x+T]-[x]=T, \forall\ x \in R$

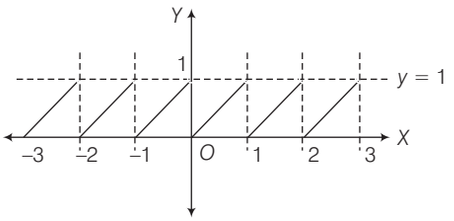
$\Rightarrow \quad T = 1, 2, 3, 4, \ldots$

[since, subtraction of two integers]

The smallest value of T satisfying $f(x+T)=f(x)$ is 1.

Thus, it is periodic with period 1.

Graphically $f(x)=x-[x]=\{x\}$



Clearly, from the given graph, the function repeats itself at an interval of 1 unit. Thus, period of $f(x)=1$.

Remark

For those functions whose periods are not deducted by graphs, they can be judged by inspection method.

Example 73 Let $f(x)$ be periodic and k be a positive real number such that $f(x+k)+f(x)=0$ for all $x \in R$. Prove that $f(x)$ is a periodic with period $2k$.

Sol. We have,

$f(x+k)+f(x)=0, \forall\ x \in R$

$\Rightarrow \quad f(x+k)=-f(x), \forall\ x \in R$, put $x=x+k$

$\Rightarrow \quad f(x+2k)=-f(x+k), \forall\ x \in R$

[as $f(x+k)=-f(x)$]

$\Rightarrow \quad f(x+2k)=f(x), \forall\ x \in R$

which clearly shows that $f(x)$ is periodic with period $2k$.

Some Standard Results on Periodic Functions

Functions	Periods
(i) $\sin^n x, \cos^n x$	π, if n is even.
$\sec^n x, \text{cosec}^n x$	2π, if n is odd or fraction.
(ii) $\tan^n x, \cot^n x$	π, n is even or odd.
(iii) $\lvert\sin x\rvert, \lvert\cos x\rvert, \lvert\tan x\rvert$ $\lvert\cot x\rvert, \lvert\sec x\rvert, \lvert\text{cosec } x\rvert$	π
(iv) $x-[x]$	1
(v) Algebraic functions e.g. $\sqrt{x}, x^2, x^3+5$, ... etc.	Period doesn't exist.
(vi) $f(x)$ = constant	Periodic with no fundamental period.

Example 74 Find periods for

(i) $\cos^4 x$. (ii) $\sin^3 x$. (iii) $\cos\sqrt{x}$. (iv) $\sqrt{\cos x}$.

Sol. (i) $\cos^4 x$ has a period π as n is even.

(ii) $\sin^3 x$ has a period 2π as n is odd.

(iii) $\cos\sqrt{x}$ is not periodic, as for no value of T,

$$f(x+T)=f(x) \Rightarrow \cos\sqrt{x+T}=\cos(\sqrt{x})$$

Thus, there exists no value of T for which $f(x+T)=f(x)$.

Hence, $\cos\sqrt{x}$ is not periodic.

(iv) $f(x)=\sqrt{\cos x}$ has the period 2π as n is in fraction.

Aliter $f(x+T)=f(x) \Rightarrow \sqrt{\cos(x+T)}=\sqrt{\cos(x)}$

$\Rightarrow \quad T=2\pi, 4\pi, \ldots$

But T is the least positive value, hence $f(x)$ is periodic with period 2π.

Properties of Periodic Functions

(i) If $f(x)$ is periodic with period T, then

(a) $c \cdot f(x)$ is periodic with period T

(b) $f(x+c)$ is periodic with period T

(c) $f(x) \pm c$ is periodic with period T, where c is any constant.

We know, $\sin x$ has period 2π.

Then, $f(x)=5(\sin x)+4$ is also periodic with period 2π.

i.e. "If constant is added, subtracted, multiplied or divided in a periodic function, its period remains the same."

(ii) If $f(x)$ is periodic with period T, then $kf(cx+d)$ has period $\frac{T}{|c|}$,

i.e. period is only affected by coefficient of x,

where, $k, c, d \in$ constant.

We know, $f(x)=\left\{7\sin\left(2x+\frac{\pi}{9}\right)\right\}-12$ has the period $\frac{2\pi}{|2|}=\pi$; as $\sin x$ is periodic with period 2π.

(iii) If $f_1(x)$, $f_2(x)$ are periodic functions with periods T_1, T_2 respectively, then $h(x)=f_1(x)+f_2(x)$ has period

$$=\begin{cases}\text{LCM of } \{T_1, T_2\}, \text{ if } h(x) \text{ is not an even function.}\\ \qquad\qquad OR\\ \frac{1}{2}\text{ LCM of } \{T_1, T_2\}, \text{ if } f_1(x) \text{ and } f_2(x) \text{ are complementary}\\ \text{pair-wise comparable functions.}\end{cases}$$

While taking LCM we should always remember

(a) LCM of $\left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}\right)=\frac{\text{LCM of } (a,c,e)}{\text{HCF of } (b,d,f)}$

e.g. LCM of $=\frac{\text{LCM of } (2\pi, \pi, \pi)}{\text{HCF of } (3, 6, 12)}=\frac{2\pi}{3}$

$\therefore$ LCM of $\left(\frac{2\pi}{3}, \frac{\pi}{6}, \frac{\pi}{12}\right)=\frac{2\pi}{3}$

(b) LCM of rational with rational is possible.

LCM of irrational with irrational is possible.

But LCM of rational and irrational is not possible.

e.g. LCM of $(2\pi, 1, 6\pi)$ is not possible, as $2\pi, 6\pi \in$ irrational and $1 \in$ rational.

Periodicity of Constant Function

The LCM rule is not applicable, if function reduces to constant.

e.g. $\quad f(x)=\sin^2 x+\cos^2 x.$

Since, period of $\sin^2 x$ and $\cos^2 x$ are π.

$\therefore$ Period of $f(x)=\frac{1}{2}\text{LCM}\{\pi, \pi\}=\frac{\pi}{2}$, which is not correct.

Whereas, $\sin^2 x+\cos^2 x=1$ is a constant function and period is undetermined.

Example 75 Find the period, if $f(x)=\sin x+\{x\}$, where $\{x\}$ is fractional part of x .

Sol. Here, $\sin x$ is periodic with period 2π and $\{x\}$ is periodic with period 1. Thus, LCM of 2π and $1 \Rightarrow$ Does not exist.

Hence, $f(x)$ is not periodic.

Example 76 Find period of $f(x)=\tan 3x+\sin\left(\frac{x}{3}\right)$.

Sol. Period for $\tan 3x$ is $\left|\frac{\pi}{3}\right|$.

Period for $\sin\frac{x}{3}$ is $\left|2\pi\times\frac{3}{1}\right|=|6\pi|$

Thus, LCM of $\frac{\pi}{3}$

and $\quad \frac{6\pi}{1} \Rightarrow \frac{6\pi}{1}$

Hence, $f(x)$ is periodic with period 6π.

Example 77 Find the period of

$$f(x)=\sin x+\tan\frac{x}{2}+\sin\frac{x}{2^2}+\tan\frac{x}{2^3}+\ldots+\sin\frac{x}{2^{n-1}}+\tan\frac{x}{2^n}.$$

Sol. We have, Period of $\left(\sin x+\tan\frac{x}{2}\right)$ is 2π,

$\left(\sin\frac{x}{2^2}+\tan\frac{x}{2^3}\right)$ is $2^3\pi$,

.....

.....

$\left(\sin\frac{x}{2^{n-1}}+\tan\frac{x}{2^n}\right)$ is $2^n\pi$.

Thus, LCM of $\{2\pi, 2^3\pi, \ldots, 2^n\pi\}=2^n\pi$.

Hence, the period of $f(x)$ is $2^n\pi$.

Example 78 Find the period of $f(x) = |\sin x| + |\cos x|$.

Sol. $|\sin x|$ has period π and $|\cos x|$ has period π.

Here, $f(x)$ is an even function and $\sin x$, $\cos x$ are complementary.

Thus, period of $f(x) = \frac{1}{2}\{\text{LCM of } \pi \text{ and } \pi\} = \frac{\pi}{2}$

Thus, period for $f(x)$ is $\frac{\pi}{2}$.

Example 79 ***Find the period of*** $f(x) = \sin^4 x + \cos^4 x$.

Sol. $\sin^4 x$ and $\cos^4 x$ both has a period π. [as n is even]

But $f(x)$ is an even function and $\sin x$ and $\cos x$ are complementary.

Hence, $f(x)$ has period $= \frac{1}{2}\{\text{LCM of } \pi, \pi\} = \frac{\pi}{2}$

Thus, period of $f(x)$ is $\frac{\pi}{2}$.

Example 80 Find the period of $f(x) = \cos(\cos x) + \cos(\sin x)$.

Sol. Here, $\cos(\cos x)$ has period π; as it is even, also $\cos(\sin x)$ has period π; as it is even.

Thus, period of $f(x) = \frac{1}{2}\{\text{LCM of } \pi \text{ and } \pi\}$

Hence, period of $f(x) = \frac{\pi}{2}$

Example 81 Find the period of $f(x) = \cos^{-1}(\cos x)$.

Sol. Here, $f(x) = \cos^{-1}(\cos x)$; $f(x + T) = f(x)$

$\Rightarrow \cos^{-1}\{\cos(x + T)\} = \cos^{-1}\{\cos(x)\}$

$\Rightarrow T = 2\pi, 4\pi, 6\pi, \ldots$

But, T is the least positive value. Hence, $T = 2\pi$ or period is 2π.

Aliter $f(x) = \cos^{-1}(\cos x)$ has period 2π, since $\cos x$ has period 2π.

(i.e. In composition of function (fog) or (gof) are periodic, if $g(x)$ and $f(x)$ are periodic, respectively.)

Example 82 The period of $f(x) = \cos(|\sin x| - |\cos x|)$ is

(a) π (b) 2π

(c) $\frac{\pi}{2}$ (d) None of these

Sol. As, $\cos\theta$ is even and $|\sin x| - |\cos x|$ has the period π.

$\therefore$ $\cos(|\sin x| - |\cos x|)$ has period $\frac{\pi}{2}$.

(i.e. Half the period of $g(x)$, if $f(x)$ is even in fog).

Hence, (c) is the correct answer.

Example 83 Period of the function $f(x) = \sin(\sin(\pi x)) + e^{\{3x\}}$, where $\{.\}$ denotes the fractional part of x is

(a) 1 (b) 2

(c) 3 (d) None of these

Sol. As, $\sin(\pi x)$ has period $= \left|\frac{2\pi}{\pi}\right| = 2$

$\therefore$ $\sin(\sin(\pi x))$ has period 2

and $e^{\{3x\}}$ has period $\frac{1}{3}$.

$\therefore$ Period of $f(x) = \sin(\sin(\pi x)) + e^{\{3x\}}$ is LCM of $\left\{2, \frac{1}{3}\right\} = 2$

Hence, (b) is the correct answer.

Example 84 $\sin ax + \cos ax$ and $|\cos x| + |\sin x|$ are periodic functions of same fundamental period, if 'a' equals

(a) 0 (b) 1 (c) 2 (d) 4

Sol. Fundamental period of $|\sin x| + |\cos x|$ is $\frac{\pi}{2}$.

Fundamental period of $(\sin ax + \cos ax)$ is $\frac{2\pi}{a}$.

$\therefore$ $a = 4$

Hence, (d) is the correct answer.

Example 85 ***Let*** $f(x) = \sin x + \cos(\sqrt{4 - a^2})\,x$. ***Then, the integral values of 'a' for which*** $f(x)$ ***is a periodic function, are given by***

(a) $\{2, -2\}$ (b) $(-2, 2]$

(c) $[-2, 2]$ (d) None of these

Sol. $f(x)$ will be periodic, if $\sqrt{4 - a^2}$ is a rational which is only possible when $(4 - a^2)$ is a perfect square.

$\Rightarrow$ $a = 0, 2, -2$ or $a \in \{-2, 0, 2\}$

Hence, (d) is the correct answer.

Example 86 ***Let*** $f(x) = \begin{cases} -1 + \sin K_1\pi x, & x \text{ is rational.} \\ 1 + \cos K_2\pi x, & x \text{ is irrational.} \end{cases}$

If $f(x)$ ***is a periodic function, then***

(a) either $K_1, K_2 \in$ rational or $K_1, K_2 \in$ irrational

(b) $K_1, K_2 \in$ rational only

(c) $K_1, K_2 \in$ irrational only

(d) $K_1, K_2 \in$ irrational such that $\frac{K_1}{K_2}$ is rational

Sol. Range of $-1 + \sin K_1\pi x$ is $[-2, 0]$ and range of $1 + \cos K_2\pi x$ is $[0, 2]$.

$\Rightarrow$ If $g(x) = -1 + \sin K_1\pi x$

$\Rightarrow g(x+T_1) = -1 + \sin K_1\pi x$, where T_1 is a period of $1 + \sin K_1 \pi x$

$\Rightarrow T_1$ is rational

$\Rightarrow$ Period of $f(x)$ is rational.

$\Rightarrow K_1$ and K_2 are rational.

Hence, (b) is the correct answer.

Example 87 If $f(x) = \tan^2 \dfrac{\pi x}{n^2 - 5n + 8} + \cot(n+m)\pi x;\ (n \in N, m \in Q)$ is a periodic function with 2 as its fundamental period, then m can't belong to

(a) $(-\infty, -2) \cup (-1, \infty)$ (b) $(-\infty, -3) \cup (-2, \infty)$

(c) $(-2, -1) \cup (-3, -2)$ (d) $\left(-3, -\dfrac{5}{2}\right) \cup \left(-\dfrac{5}{2}, -2\right)$

Sol. Period is LCM of $n^2 - 5n + 8$ and $\dfrac{1}{n+m}$.

$\Rightarrow \quad n^2 - 5n + 8 = 1 \quad \text{or} \quad n^2 - 5n + 8 = 2$

$\Rightarrow \quad n^2 - 5n + 7 = 0 \quad \text{or} \quad n^2 - 5n + 6 = 0$

Since, $n \in N \therefore n = 2, 3$

and $\left(\dfrac{1}{n+m}\right) K_1 = 2,\ (K_1 \in I) \Rightarrow \dfrac{1}{n+m} \nmid 1$

$\therefore \quad \dfrac{1}{2+m} \nmid 1 \quad \text{or} \quad \dfrac{1}{3+m} \nmid 1$

$\Rightarrow \quad m \notin (-2, -1) \cup (-3, -2)$

Hence, (c) is the correct answer.

Example 88 Let $f(x)$ be a periodic function with period 3 and $f\left(-\dfrac{2}{3}\right) = 7$ and $g(x) = \int_0^x f(t+n)\,dt$, where $n = 3K, K \in N$. Then, $g'\left(\dfrac{7}{3}\right)$ is equal to

(a) $-\dfrac{2}{3}$ (b) 7

(c) -7 (d) $\dfrac{7}{3}$

Sol. $g'(x) = f(x+n) = f(x + (n-3) + 3)$

$= f(x + n - 3)$

.....

.....

$= f(x+3) = f(x)$

$\Rightarrow \quad g'(x) = f(x)$

$\Rightarrow g'\left(\dfrac{7}{3}\right) = f\left(\dfrac{7}{3}\right) = f\left(-\dfrac{2}{3} + 3\right) = f\left(-\dfrac{2}{3}\right) = 7$

Hence, (b) is the correct answer.

Exercise for Session 7

1. Find the periods of following functions.

(i) $f(x) = [\sin 3x] + |\cos 6x|$

(ii) $f(x) = \dfrac{1}{2}\left\{\dfrac{|\sin x|}{\cos x} + \dfrac{|\cos x|}{\sin x}\right\}$

(iii) $f(x) = e^{\cos^4 \pi x + x - [x] + \cos^2 \pi x}$

(iv) $f(x) = 3\sin\dfrac{\pi x}{3} + 4\cos\dfrac{\pi x}{4}$

(v) $f(x) = \cos 3x + \sin\sqrt{3}\pi x$

(vi) $f(x) = \sin\dfrac{\pi x}{n!} - \cos\dfrac{\pi x}{(n+1)!}$

(vii) $f(x) = x - [x - b]$

(viii) $f(x) = e^{\ln(\sin x)} + \tan^3 x - \operatorname{cosec}(3x - 5)$

2. Find the period of the real-valued function satisfying $f(x) + f(x+4) = f(x+2) + f(x+6)$.

3. Check whether the function defined by $f(x+\lambda) = 1 + \sqrt{2f(x) - f^2(x)},\ \forall\ x \in R$ is periodic. If yes, find its period.

4. Let $f(x+p) = 1 + \{2 - 3f(x) + 3(f(x))^2 - (f(x))^3\}^{1/3},\ \forall\ x \in R$, where $p > 0$, prove that $f(x)$ is periodic.

5. Let $f(x)$ be a function such that ; $f(x-1) + f(x+1) = \sqrt{3}\,f(x),\ \forall\ x \in R$. If $f(5) = 100$, find $\sum_{r=0}^{99} f(5 + 12r)$.

Session 8

Mapping of Functions

Mapping of Functions

As discussed earlier, a function exists only if, "to every element in domain there exists unique image in the codomain".

i.e. To every element of A there exists one and only one element of B.

This is written as $f: A \to B$ and is read as f maps from A to B and this correspondence is denoted by $y = f(x)$.

From definition, it follows that there may exist some elements in B, which may not have any corresponding element in set A.

But there should not be any x left (element of A) for which there is no element in set B.

There are four types of mappings defined as

1. One-one Mapping or Injective or Monomorphic

A function $f: A \to B$ is said to be one-one mapping or injective, if different elements of A have different images in B.

Thus, no two elements of set A can have the same f image.

Verbal Description Let us consider set $A = \{1, 3, 5\}$ and $B = \{3, 7, 11, 15\}$, where $f: A \to B$ and $f(x) = 2x + 1$, then here every element in domain possess distinct images in codomain.

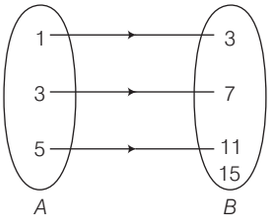


Figure 3.34

Thus, $f(x)$ is one-one or injective.

From above definition, following mappings are not one-one.

(i) $f: A \to B$ (ii) $f: A \to B$

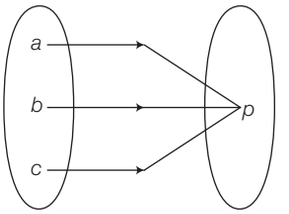


Figure 3.35

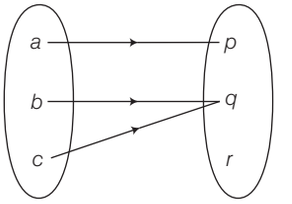


Figure 3.36

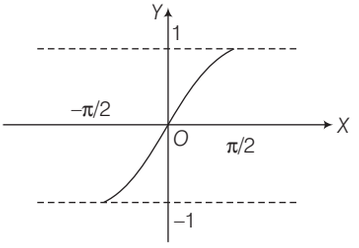
Method to Check One-one Mapping

***Method* 1. Theoretically** If $f(x) = f(y) \Rightarrow x = y$, then $f(x)$ is one-one.

***Method* 2. Graphically** A function is one-one iff no line parallel to X-axis meets the graph of function at more than one point.

Example 89 Let $f: [-\pi/2, \pi/2] \to [-1, 1]$ where $f(x) = \sin x$. Find whether $f(x)$ is one-one or not.

Sol. Here, $f: [-\pi/2, \pi/2] \to [-1, 1]$ indicates that domain $\in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and codomain $\in [-1, 1]$.



Thus, the graph of $f(x) = \sin x$ should be plotted in $[-\pi/2, \pi/2]$.

Which is clearly not intersected at more than one point by any straight line parallel to X-axis.

Thus, $f(x)$ is one-one.

***Method* 3. By Calculus** For checking whether $f(x)$ is one-one, find whether function is only increasing or only decreasing in its domain. If yes, then one-one.

i.e. if $f'(x) \geq 0, \forall\ x \in$ domain

or if $f'(x) \leq 0, \forall\ x \in$ domain, then one-one.

Remark

Students are advised to use the graphical or calculus method for finding one-one.

Number of One-one Mapping

If A and B are finite sets having m and n elements, then number of one-one function from A to B.

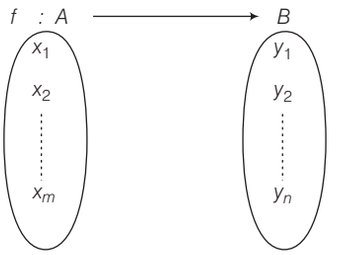


Figure 3.37

Here, x_1 can take n images,

x_2 can take $(n-1)$ images,

x_3 can take $(n-2)$ images,

.....

.....

x_m can take $(n-m+1)$ images.

Thus, number of mapping $\Rightarrow n(n-1)(n-2)\ldots(n-m+1)$

$$\Rightarrow \quad \begin{cases} {}^{n}P_{m}, & \text{if } n \geq m \\ 0, & \text{if } n < m \end{cases}$$

Example 90 $f(x) = x^3 + 3x^2 + 4x + b \sin x + c \cos x$, $\forall\, x \in R$ is a one-one function, the value of $b^2 + c^2$ is

(a) ≥ 1 (b) ≥ 2 (c) ≤ 1 (d) None of these

Sol. Here, $f(x) = x^3 + 3x^2 + 4x + b \sin x + c \cos x$

$$\Rightarrow \quad f'(x) = 3x^2 + 6x + 4 + b \cos x - c \sin x$$

Now, for $f(x)$ to be one-one, the only possibility is

$$f'(x) \geq 0, \ \forall\, x \in R$$

$$\Rightarrow \quad 3x^2 + 6x + 4 + b \cos x - c \sin x \geq 0, \forall\, x \in R$$

$$\Rightarrow \quad 3x^2 + 6x + 4 \geq c \sin x - b \cos x, \ \forall\, x \in R$$

$$\Rightarrow \quad 3x^2 + 6x + 4 \geq \sqrt{b^2 + c^2}, \ \forall\, x \in R$$

$$\Rightarrow \quad \sqrt{b^2 + c^2} \leq 3(x^2 + 2x + 1) + 1, \forall\, x \in R$$

$$\Rightarrow \quad \sqrt{b^2 + c^2} \leq 3(x+1)^2 + 1, \forall\, x \in R$$

$$\Rightarrow \quad \sqrt{b^2 + c^2} \leq 1, \forall\, x \in R \Rightarrow b^2 + c^2 \leq 1, \forall\, x \in R$$

Hence, (c) is the correct answer.

2. Many-one Mapping

A mapping $f : A \to B$ is said to be many-one function, if two or more elements of set A have the same image in B.

In other words; $f : A \to B$ is a many-one function, if it is not one-one function.

Verbal Description Let $f : A \to B$ and $g : X \to Y$ be two functions represented by

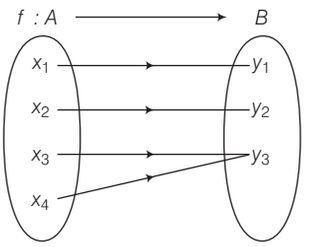


Figure 3.38

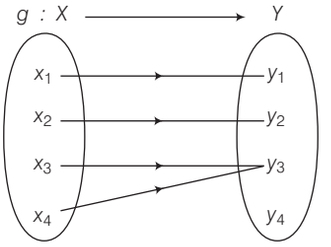


Figure 3.39

Clearly, f and g both are many-one as there are two elements x_3, x_4 which correspond to the same image.

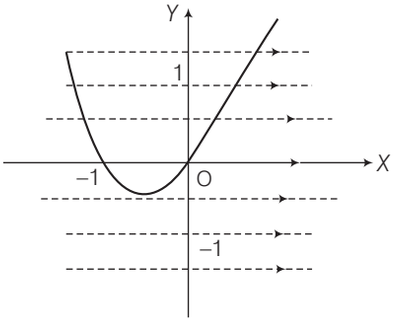
i.e. $f(x_3) = f(x_4) = y_3$. Thus, many-one.

Method to Check Many-one

They are same as for one-one because, if mapping is not one-one it is many-one.

Example 91 Show $f : R \to R$ defined by $f(x) = x^2 + x$ for all $x \in R$ is many-one.

Sol. By graph



$f(x) = x^2 + x$ can be represented graphically, as shown in figure,

where the straight line parallel to X-axis meets the curve at two points (i.e. more than one point).

Thus, it is not one-one it is many-one.

Aliter By calculus,

$$f(x) = x^2 + x$$

$$\Rightarrow \quad f'(x) = 2x + 1$$

where $f'(x) > 0$

if $x > -\frac{1}{2}$

and $f'(x) < 0$

if $x < -\frac{1}{2}$

which shows $f(x)$ is neither increasing nor decreasing i.e. not monotonic. Hence, many-one.

3. Onto Mapping or Surjective

A function $f: A \to B$ is an onto function, if such that each element of B is the f image of atleast one element in A. It is expressed as $f: A \to B$.

Here, range of f = codomain.

i.e. $f(A) = B$

Method to show onto or surjective

Find the range of $y = f(x)$ and show range of $f(x)$ = codomain of $f(x)$.

Remark

If range = codomain, then $f(x)$ is onto. Any polynomial of odd degree has range all real numbers and is onto for $f: R \to R$.

Example 92 Show $f: R \to R$ defined by $f(x) = (x-1)(x-2)(x-3)$ is surjective but not injective

Sol. We have,

$$f(x) = (x-1)(x-2)(x-3)$$
$$f(1) = f(2) = f(3) = 0$$

Hence, it is not injective, it is many-one function.

Now, $f(x)$ is a polynomial of degree 3, i.e. odd.

Hence, $f(x)$ is surjective.

Number of Onto Functions

If A and B are two sets having m and n elements respectively, such that $1 \le n \le m$, then number of onto functions from A to B is

Coefficient of x^m in $m!(e^x - 1)^n$

$\Rightarrow$ Coefficient of x^m in

$$m!\{{}^nC_0 e^{nx} - {}^nC_1 e^{(n-1)x} + \ldots + (-1)^r {}^nC_r e^{rx} + \ldots + (-1)^n {}^nC_n\}$$

$$= \sum_{r=1}^{n} (-1)^{n-r} {}^nC_r \, r^m$$

Or

Consider the set of all possible functions from A to B,

i.e. $\{f: A \to B\}$.

Now, let us define a subset Q such that

$$A_i = \{f \in Q \mid i \notin \text{Range of } f\}, i \in n$$

i.e. $\bigcup_{i=1}^{n} A_i = \{f \text{ is injective}\}$

To find number of onto functions

= Total number of functions

− Number of injective functions

$= n^m - \{\text{Number of injective functions}\}$

$$= n^m - \left| \bigcup_{i=1}^{n} A_i \right|, \quad \ldots(i)$$

where $\bigcup_{i=1}^{n} A_i = \sum_{i=1}^{n} |A_i| - \sum_{i,j} A_i \cap A_j + \ldots$

$$= n \cdot (n-1)^m - {}^nC_2 (n-2)^m + \ldots + (-1)^{n-1} \cdot {}^nC_{n-1} (1)^m \quad \ldots(ii)$$

$\therefore$ Number of onto functions

$$= n^m - \{n \cdot (n-1)^m - {}^nC_2 (n-2)^m + \ldots + (-1)^{n-1} \cdot {}^nC_{n-1} (1)^m\}$$

$$= \sum_{r=1}^{n} (-1)^{n-r} \, {}^nC_r \cdot r^m$$

Example 93 ***If*** $f: R \to \left[\frac{\pi}{6}, \frac{\pi}{2}\right)$, $f(x) = \sin^{-1}\left(\frac{x^2 - a}{x^2 + 1}\right)$ ***is an onto function, the set of values of 'a' is***

(a) $\left\{-\frac{1}{2}\right\}$ (b) $\left[-\frac{1}{2}, -1\right)$

(c) $(-1, \infty)$ (d) None of these

Sol. Here, $f(x)$ is onto.

$$\therefore \quad \frac{\pi}{6} \le \sin^{-1}\left(\frac{x^2 - a}{x^2 + 1}\right) < \frac{\pi}{2} \Rightarrow \frac{1}{2} \le \frac{x^2 - a}{x^2 + 1} < 1$$

$$\Rightarrow \frac{1}{2} \le 1 - \frac{(a+1)}{x^2 + 1} < 1, \forall \; x \in R \Rightarrow a + 1 > 0 \Rightarrow a \in (-1, \infty)$$

Hence, (c) is the correct answer.

4. Into Mapping

A function $f: A \to B$ is an into function, if there exists an element in B having no pre-image in A.

In other words, $f: A \to B$ is into function, if it is not onto function (mapping).

Example 94 ***Show*** $f: R \to R$ ***defined by*** $f(x) = x^2 + 4x + 5$ ***is into.***

Sol. We have, $f(x) = x^2 + 4x + 5$

$$f(x) = (x+2)^2 + 1$$

Since, the codomain of f is R but the range of f is $[1, \infty)$.

Hence, f is into.

One-one Onto Mapping or Bijective

A function is one-one onto or bijective, if it is both one-one and onto. A function is bijective if and only if every possible image is mapped to by exactly one argument.

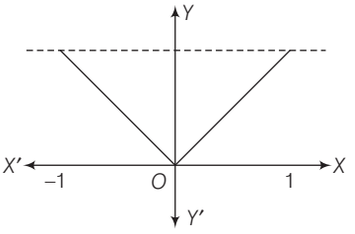
The function $f: A \to B$ is bijective iff for all $y \in B$, there is a unique $x \in A$ such that $f(x) = y$.

Example 95 Let $A=\{x:-1\le x\le 1\}=B$ and a mapping $f:A\to B$. For each of the following functions from A to B, find whether it is surjective or bijective.

(i) $f(x)=|x|$ (ii) $f(x)=x|x|$ (iii) $f(x)=x^3$

(iv) $f(x)=[x]$ (v) $f(x)=\sin\frac{\pi x}{2}$

Sol. (i) $f(x)=|x|$



Which shows many-one, as the straight line is parallel to X-axis cuts at two points. Here, range for $f(x)\in[0,1]$. Which is clearly a subset of codomain.

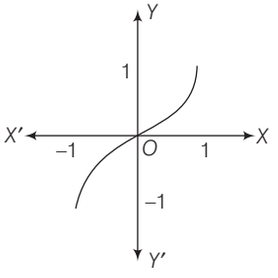
i.e. $[0,1]\subseteq[-1,1]$

Thus, into. Hence, function is many-one-into.

$\therefore$ Neither injective nor surjective.

(ii) $f(x)=x|x|$

The graph shows that $f(x)$ is one-one, as the straight line parallel to X-axis cuts only at one point.



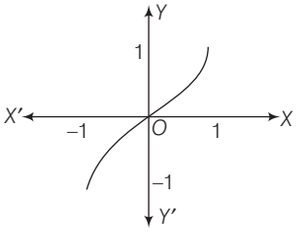
Here, range $f(x)\in[-1, 1]$

Thus, range = codomain

Hence, onto. Therefore, $f(x)$ is one-one onto or (bijective).

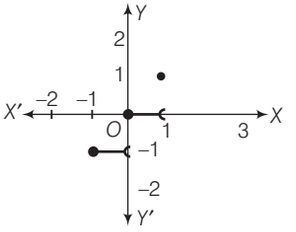
(iii) $f(x)=x^3$

Graph shows $f(x)$ is one-one onto (i.e. bijective). (as explained above)



(iv) $f(x)=[x]$

Graph shows that $f(x)$ is many-one, as the straight line parallel to X-axis meets at more than one points.



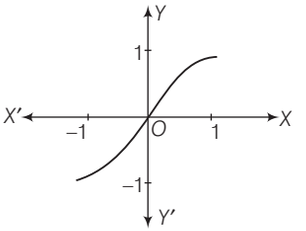
Here, range $f(x)\in\{-1, 0, 1\}$

which shows into as range $\subseteq$ codomain.

Hence, many-one-into.

(v) $f(x)=\sin\frac{\pi x}{2}$

Graph shows $f(x)$ is one-one and onto as range = codomain.



Therefore, $f(x)$ is bijective.

Example 96 The function $f:R\to R$ defined as $f(x)=\log_e(\sqrt{\sqrt{x^2+1}+x}+\sqrt{\sqrt{x^2+1}-x})$ is

(a) one-one and onto both

(b) one-one but not onto

(c) onto but not one-one

(d) Neither one-one nor onto

Sol. (d) Here, $f(x)=\frac{2}{2}\log\left(\sqrt{\sqrt{x^2+1}+x}+\sqrt{\sqrt{x^2+1}-x}\right)$

$$=\frac{1}{2}\log\left(\sqrt{\sqrt{x^2+1}+x}+\sqrt{\sqrt{x^2+1}-x}\right)^2$$

$$=\frac{1}{2}\log\left[\sqrt{x^2+1}+x+\sqrt{x^2+1}-x+2\sqrt{(\sqrt{x^2+1})^2-x^2}\right]$$

$$=\frac{1}{2}\log\left[2\sqrt{x^2+1}+2\sqrt{x^2+1-x^2}\right]$$

$$=\frac{1}{2}\log[2\sqrt{x^2+1}+2]$$

Now, $f(-x)=f(x)$

$\therefore$ $f(x)$ is even function.

Hence, neither one-one nor onto.

Quick Review

Let $X = \{x_1, x_2, x_3, \ldots, x_n\}$ [i.e. n elements]

and $Y = \{y_1, y_2, y_3, \ldots, y_r\}$ [i.e. r elements]

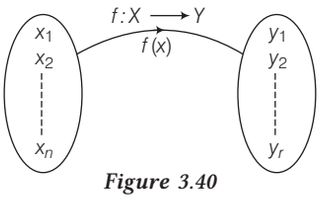


Figure 3.40

(a) Total number of functions $= r^n$
$=$ (Number of elements in codomain)$^{\text{number of elements in domain}}$

(b) Total number of one to one functions $= \begin{cases} {}^rC_n \cdot n!, & r \geq n \\ 0, & r < n \end{cases}$

(c) Total number of many-one functions $= \begin{cases} r^n - {}^rC_n \cdot n!, & r \geq n \\ r^n, & r < n \end{cases}$

(d) Total number of constant functions $= r$

(e) Total number of onto functions

$$= \begin{cases} {}^rC_1(r-1)^n + {}^rC_2(r-2)^n - {}^rC_3(r-3)^n + \ldots, & r < n \\ r!, & r = n \\ 0, & r > n \end{cases}$$

(f) Total number of into functions

$$= \begin{cases} {}^rC_1(r-1)^n - {}^rC_2(r-2)^n + {}^rC_3(r-3)^n - \ldots, & r \leq n \\ r^n, & r > n \end{cases}$$

Example 97 If $X = \{1, 2, 3, 4, 5\}$ and $Y = \{a, b, c, d, e, f\}$ and $f : X \to Y$, find the total number of

(i) functions (ii) one to one functions
(iii) many-one functions (iv) constant functions
(v) onto functions (vi) into functions

Sol. (i) Total number of functions $= 6^5 = 7776$

(ii) Total number of one to one functions
$= {}^6C_5 \cdot 5! = 6! = 720$

(iii) Total number of many-one functions $= 6^5 - 6! = 7056$

(iv) Total number of constant functions $= 6$

(v) Total number of onto functions $= 0$ (as $r > n$)

(vi) Total number of into functions $= 6^5 = 7776$

Example 98 Find the number of surjections from A to B, where $A = \{1, 2, 3, 4\}$, $B = \{a, b\}$. ***[IIT JEE 2000]***

Sol. Number of surjections from A to $B = \sum_{r=1}^{2} (-1)^{2-r} {}^2C_r (r)^4$

$$= (-1)^{2-1} {}^2C_1 (1)^4 + (-1)^{2-2} {}^2C_2 (2)^4 = -2 + 16 = 14$$

Therefore, number of onto mappings from A to $B = 14$.

Aliter Total number of mapping from A to B is 2^4 of which two functions $f(x) = a$ for all $x \in A$ and $g(x) = b$ for all $x \in A$ are not surjective.

Thus, total number of surjections from

$$A \text{ to } B = 2^4 - 2 = 14.$$

Exercise for Session 8

1. There are exactly two distinct linear functions, which map $[-1, 1]$ onto $[0, 3]$. Find the point of intersection of the two functions.

2. Let f be one-one function with domain $\{x, y, z\}$ and range $\{1, 2, 3\}$. It is given that exactly one of the following statement is true and remaining two are false.
$f(x) = 1, f(y) \neq 1, f(z) \neq 2$. Determine $f^{-1}(1)$.

3. Let $A = R - \{3\}$, $B = R - \{1\}$ and $f : A \to B$ defined by $f(x) = \dfrac{x-2}{x-3}$. Is 'f' bijective? Give reasons.

4. Let $f : R \to R$ defined by $f(x) = \dfrac{x^2}{1+x^2}$. Prove that f is neither injective nor surjective.

5. If the function $f : R \to A$, given by $f(x) = \dfrac{x^2}{x^2+1}$ is surjection, find A.

6. If the function $f : R \to A$, given by $f(x) = \dfrac{e^x - e^{-|x|}}{e^x + e^{|x|}}$ is surjection, find A.

7. Let $f(x) = ax^3 + bx^2 + cx + d \sin x$. Find the condition that $f(x)$ is always one-one function.

8. Let $f : X \to Y$ be a function defined by $f(x) = a \sin\left(x + \dfrac{\pi}{4}\right) + b \cos x + c$. If f is both one-one and onto, find the sets X and Y.

Session 9

Identical (or Equal) Functions

Identical (or Equal) Functions

Two functions f and g are said to be identical (or equal) functions, if

(i) the domain of f = the domain of g,

(ii) the range of f = the range of g, and

(iii) $f(x) = g(x), \forall\ x \in$ domain.

Examples of Equal or Identical Functions

(i) $f(x) = \ln x^2, g(x) = 2 \ln x$ (NI)

$f(x) = \text{cosec } x, g(x) = \dfrac{1}{\sin x}$ (I)

(ii) $f(x) = \cot(\cot^{-1} x), g(x) = x$ (I)

$f(x) = \tan x, g(x) = \dfrac{1}{\cot x}$ (NI)

(iii) $f(x) = \sin^{-1}(3x - 4x^3), g(x) = 3 \sin^{-1} x$ (NI)

(iv) $f(x) = \text{sgn}(x^2 + 1), g(x) = \sin^2 x + \cos^2 x$ (I)

(v) $f(x) = \tan^2 x \cdot \sin^2 x, g(x) = \tan^2 x - \sin^2 x$ (I)

(vi) $f(x) = \sec^2 x - \tan^2 x, g(x) = 1$ (NI)

(vii) $f(x) = \log_x e, g(x) = \dfrac{1}{\log_e x}$ (I)

(viii) $f(x) = \tan(\cot^{-1} x), g(x) = \cot(\tan^{-1} x)$ (I)

(ix) $f(x) = \sqrt{x^2 - 1}, g(x) = \sqrt{x - 1} \cdot \sqrt{x + 1}$ (NI)

(x) $f(x) = \tan x \cdot \cot x, g(x) = \sin x \cdot \text{cosec } x$ (NI)

(xi) $f(x) = e^{\ln e^x}, g(x) = e^x$ (I)

(xii) $f(x) = \sqrt{\dfrac{1 - \cos 2x}{2}}, g(x) = \sin x$ (NI)

(xiii) $f(x) = \sqrt{x^2}, \ g(x) = (\sqrt{x})^2$ (NI)

(xiv) $f(x) = \log(x + 2) + \log(x - 3)$,
$g(x) = \log(x^2 - x - 6)$ (NI)

(xv) $f(x) = x \mid x \mid, g(x) = x^2 \text{ sgn } x$ (I)

(xvi) $f(x) = \lim\limits_{n \to \infty} \dfrac{x^{2n} - 1}{x^{2n} + 1}, g(x) = \text{sgn}(\mid x \mid - 1)$ (I)

(xvii) $f(x) = \sin(\sin^{-1} x), g(x) = \cos(\cos^{-1} x)$ (I)

(xviii) $f(x) = \dfrac{1}{1 + \dfrac{1}{x}}, g(x) = \dfrac{x}{1 + x}$ (NI)

(xix) $f(x) = e^{\ln \sec^{-1} x}, g(x) = \sec^{-1} x$ (NI)

Identical, if $x \in (-\infty, -1] \cup [1, \infty)$

(xx) $F(x) = (fog)(x), G(x) = (gof)(x)$, where $f(x) = e^x$, $g(x) = \ln x$ (NI)

Example 99 ***If*** $f(x) = \log_{x^2} 25$ ***and*** $g(x) = \log_x 5$***, then*** $f(x) = g(x)$ ***holds, now find the interval for*** x**.**

Sol. Domain of $f \in R - \{\pm 1, 0\}$

Domain of $g \in (0, \infty) - \{1\}$

For $f(x) = g(x)$,

domain of f = domain of g.

i.e., $f(x) = g(x)$, if $x \in (0, \infty) - \{1\}$

Example 100 Let $A = \{1, 2\}$, $B = \{3, 6\}$ and $f : A \to B$ given by $f(x) = x^2 + 2$ and $g : A \to B$ given by $g(x) = 3x$. Find whether they equal or not.

Sol. $f(1) = 3, f(2) = 6$

$g(1) = 3, g(2) = 6$

which shows $f(x)$ and $g(x)$ have same domains and range, thus $f = g$.

Example 101 Which pair of functions is identical?

(a) $\sin^{-1}(\sin x)$ and $\sin(\sin^{-1} x)$

(b) $\log_e e^x$, $e^{\log_e x}$

(c) $\log_e x^2, 2 \log_e x$

(d) None of the above

Sol. Here, (a) $\sin^{-1}(\sin x)$ is defined for $x \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$,

while $\sin(\sin^{-1} x)$ is defined only for $x \in [-1, 1]$.

(b) $\log_e e^x$ is defined for all x,

while $e^{\log_e x}$ is defined for $x > 0$.

(c) $\log_e x^2$ is defined for all $x \in R - \{0\}$, while $2 \log_e x$ is defined for $x > 0$.

$\therefore$ None is identical.

Hence, (d) is the correct answer.

Exercise for Session 9

- **Directions** (Q. Nos. 1 to 10) *Which of the following are identical (or equal) functions?*

1. $f(x) = \ln e^{x^2}, g(x) = e^{\ln x}$
2. $f(x) = \sec x, g(x) = \dfrac{1}{\cos x}$
3. $f(x) = \sec^{-1} x + \operatorname{cosec}^{-1} x,\ g(x) = \dfrac{\pi}{2}$
4. $f(x) = \cot^2 x \cdot \cos^2 x,\ g(x) = \cot^2 x - \cos^2 x$
5. $f(x) = \operatorname{sgn}(\cot^{-1} x), g(x) = \operatorname{sgn}(x^2 - 4x + 5)$
6. $f(x) = \log_e x,\ g(x) = \dfrac{1}{\log_x e}$
7. $f(x) = \sqrt{1 - x^2}, g(x) = \sqrt{1 - x} \cdot \sqrt{1 + x}$
8. $f(x) = \dfrac{1}{|x|}, g(x) = \sqrt{x^{-2}}$
9. $f(x) = [\{x\}], g(x) = \{[x]\}$ [Note that $f(x)$ and $g(x)$ are constant functions]
10. $f(x) = e^{\ln \cot^{-1} x}, g(x) = \cot^{-1} x$

Session 10

Composite Functions

Composite Functions

Let us consider two functions, $f : X \to Y_1$ and $g : Y_1 \to Y$.
We define function $h : X \to Y$, such that,
$h(x) = g(f(x)) = (gof)(x)$.

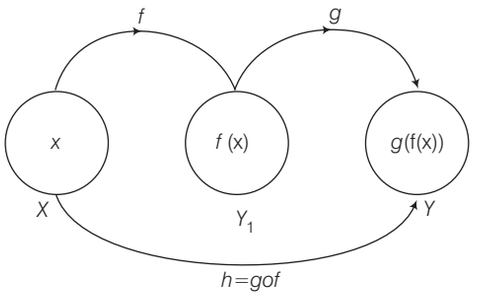


Figure 3.41

To obtain $h(x)$, we first take f-image of an element $x \in X$ so that $f(x) \in Y_1$, which is the domain of $g(x)$. Then, we take g-image of $f(x)$, i.e. $g(f(x))$ which would be an element of Y.

The diagram below shows the steps to be taken

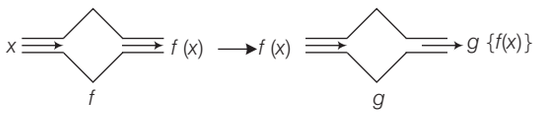


Figure 3.42

The function h defined in the diagram is called composition of f and g, and is denoted by (gof). Clearly,
domain $(gof) = \{X : x \in \text{domain}(f), f(x) \in \text{domain}(g)\}$.
Similarly, we can write, $(fog)(x) = f\{g(x)\}$
and domain $(fog) = \{x : x \in \text{domain}(g), g(x) \in \text{domain } f\}$
In general, $fog \neq gof$.

Properties of Composite Functions

(i) It should be noted that *gof* exists, iff the range of $f \subseteq$ domain of g. Similarly, *fog* exists; iff the range of $g \subseteq$ domain of f.

(ii) Composite of functions is not commutative, i.e. $gof \neq fog$.

(iii) Composite of functions is associative, i.e. if f, g and h are three functions such that $fo(foh)$ and $(fog)oh$ are defined, then $fo(goh) = (fog)oh$.
e.g. Associativity $f : (N) \to I_0 \qquad f(x) = 2x$

$q : I_0 \to Q \qquad g(x) = \frac{1}{x}$

$h : Q \to R \qquad h(x) = e^{1/x}$

$(hog)\,of = ho\,(gof) = e^{2x}$

(iv) The composite of two bijections is a bijection, i.e. if f and g are two bijections such that *gof* is defined, then *gof* is also a bijection.

Proof $f : A \to B$ and $g : B \to C$ be two bijections. Then, *gof* exists such that $gof : A \to C$.

We have to prove that *gof* is one-one and onto.

One-One Let $a_1, a_2 \in A$ such that $(gof)(a_1) = (gof)(a_2)$, then

$(gof)(a_1) = (gof)(a_2) \Rightarrow g[f(a_1)] = g[f(a_2)]$

$\Rightarrow f(a_1) = f(a_2)$ [$\because g$ is one-one]

$\Rightarrow a_1 = a_2$ [$\because f$ is one-one]

$\therefore$ *gof* is also one-one function.

Onto Let $c \in C$, then $c \in C$

$\Rightarrow \exists\, b \in B$ such that $g(b) = c$ [$\because g$ is onto]

and $b \in B \Rightarrow \exists\, a \in A$ such that $f(a) = b$ [$\because f$ is onto]

Therefore, we see that

$c \in C \Rightarrow \exists\, a \in A$ such that $(gof)(a) = g[f(a)] = g(g) = c$

i.e. Every element of C is the *gof* image of some element of A. As such *gof* is an onto function. Hence, *gof* being one-one and onto is a bijection.

(v) f is even, g is even $\Rightarrow$ *fog* is even function.

(vi) f is odd, g is odd $\Rightarrow$ *fog* is odd function.

(vii) f is even, g is odd $\Rightarrow$ *fog* is even function.

(viii) f is odd, g is even $\Rightarrow$ *fog* is even function.

Example 102 Let $f : A \to B$ and $g : B \to C$ be functions and $gof : A \to C$. Which of the following statements is true?

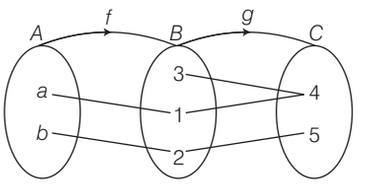
(a) If *gof* is one-one, then f and g both are one-one

(b) If *gof* is one-one, then f is one-one

(c) If *gof* is a bijection, then f is one-one and g is onto

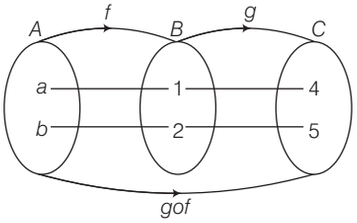
(d) If f and g are both one-one, then *gof* is one-one

Sol. (a) As shown *gof* is one-one, but g is many-one.



$\Rightarrow$ (a) is not correct.

(b) If gof is one-one, then f is also one-one,
if f is many-one, then gof cannot be one-one.



$\Rightarrow$ (c) and (d) are obviously true.
Hence, (b), (c) and (d) are correct answers.

Example 103 If $f:R\to R, f(x)=x^2$ and $g:R\to R$; $g(x)=2x+1$. Find fog and gof, also show $fog\neq gof$.

Sol. $\because$ $(gof)(x)=g\{f(x)\}=g\{x^2\}$

$$(gof)(x)=2x^2+1 \text{ and}$$

$$(fog)(x)=f\{g(x)\}=f(2x+1)$$

$$(fog)(x)=(2x+1)^2$$

where $(2x^2+1)\neq(2x+1)^2$. Therefore, $(gof)\neq(fog)$.

Example 104 Let $g(x)=1+x-[x]$

and $$f(x)=\begin{cases}-1, & x<0\\ 0, & x=0.\\ 1, & x>0\end{cases}$$

Then, for all x, find $f(g(x))$. ***[IIT JEE 2001]***

Sol. Here, $g(x)=1+x-[x]$

$\Rightarrow$ $g(x)=1+\{x\}$ [as $x-[x]=\{x\}$]

i.e. $g(x)\geq 1$.

So, $f(g(x))=1$. Since, $f(x)=1$ for all $x>0$.

Thus, $f(g(x))=1$, for all $x\in R$.

Example 105 Let $f(x)=\begin{cases}1+x, & 0\leq x\leq 2\\ 3-x, & 2<x\leq 3\end{cases}$,

find $(fof)(x)$.

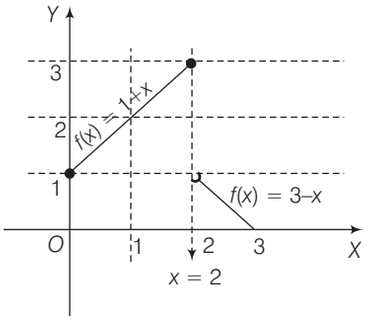
Sol. Clearly, $fof(x)=f\{f(x)\}=\begin{cases}f(1+x), & 0\leq x\leq 2\\ f(3-x), & 2<x\leq 3\end{cases}$

$$=\begin{cases}f(1+x), & 0\leq x\leq 1\\ f(1+x), & 1\leq x\leq 2\\ f(3-x), & 2<x\leq 3\end{cases}$$

$$=\begin{cases}1+(1+x), & 0\leq x\leq 1\\ 3-(1+x), & 1<x\leq 2\\ 1+(3-x), & 2<x\leq 3\end{cases}$$

$$=\begin{cases}2+x, & 0\leq x\leq 1\\ 2-x, & 1<x\leq 2\\ 4-x, & 2<x\leq 3\end{cases}$$

Aliter $f(x)$ can be expressed graphically as shown in figure below;



when

$0\leq f(x)<1; 2<x\leq 3$ where $f(x)=3-x$.

$1\leq f(x)\leq 2; 0\leq x\leq 1$ where $f(x)=1+x$.

$2<f(x)\leq 3; 1\leq x\leq 2$ where $f(x)=1+x$.

Thus, $(fof)(x)=\begin{cases}1+f(x), & 0\leq f(x)\leq 2\\ 3-f(x), & 2<f(x)\leq 3\end{cases}$

$$(fof)(x)=\begin{cases}1+f(x), & 0\leq f(x)<1\\ 1+f(x), & 1\leq f(x)\leq 2\\ 3-f(x), & 2<f(x)\leq 3\end{cases}$$

$$(fof)(x)=\begin{cases}1+(3-x), & 2<x\leq 3\\ 1+(1+x), & 0\leq x\leq 1\\ 3-(1+x), & 1<x\leq 2\end{cases}$$

$$(fof)(x)=\begin{cases}4-x, & 2<x\leq 3\\ 2+x, & 0\leq x\leq 1\\ 2-x, & 1<x\leq 2\end{cases}$$

$$(fof)(x)=\begin{cases}2+x, & 0\leq x\leq 1\\ 2-x, & 1<x\leq 2\\ 4-x, & 2<x\leq 3\end{cases}$$

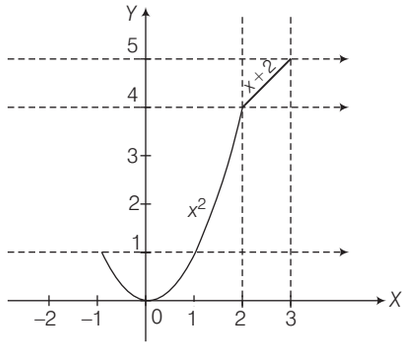
Example 106 Let $f(x)=\begin{cases}x+1, & x\leq 1\\ 2x+1, & 1<x\leq 2\end{cases}$

and $g(x)=\begin{cases}x^2, & -1\leq x<2\\ x+2, & 2\leq x\leq 3\end{cases}$.

Find (fog).

Sol. $f(g(x))=\begin{cases}g(x)+1, & g(x)\leq 1\\ 2g(x)+1, & 1<g(x)\leq 2\end{cases}$

Here, $g(x)$ becomes the variable that means we should draw the graph. It is clear that $g(x) \le 1;\ \forall\ x \in [-1, 1]$



and $1 < g(x) \le 2;\ \forall\ x \in (1, \sqrt{2}]$.

$$\therefore \quad f(g(x)) = \begin{cases} x^2 + 1, & -1 \le x < 1 \\ 2x^2 + 1, & 1 < x \le \sqrt{2} \end{cases}$$

Example 107 If $f(x) = 2x + |x|$, $g(x) = \frac{1}{3}(2x - |x|)$ and $h(x) = f(g(x))$, domain of $\underbrace{\sin^{-1}(h(h(h(h \ldots h(x) \ldots))))}_{n \text{ times}}$ is

(a) $[-1, 1]$ (b) $\left[-1, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, 1\right]$

(c) $\left[-1, -\frac{1}{2}\right]$ (d) $\left[\frac{1}{2}, 1\right]$

Sol. Since, $f(x) = \begin{cases} 2x + x, & x \ge 0 \\ 2x - x, & x < 0 \end{cases} = \begin{cases} 3x, & x \ge 0 \\ x, & x < 0 \end{cases}$

and $g(x) = \frac{1}{3}\begin{cases} 2x - x, & x \ge 0 \\ 2x + x, & x < 0 \end{cases} = \begin{cases} \frac{x}{3}, & x \ge 0 \\ x, & x < 0 \end{cases}$

$$\therefore \quad f(g(x)) = \begin{cases} 3\left(\frac{x}{3}\right), & x \ge 0 \\ x, & x < 0 \end{cases}$$

$\Rightarrow \quad f(g(x)) = x, \forall\ x \in R$

$\therefore \quad h(x) = x$

$\Rightarrow \sin^{-1}(h(h(h \ldots h(x) \ldots))) = \sin^{-1} x$

$\therefore$ Domain of $\sin^{-1}(h(h(h(h \ldots h(x) \ldots))))$ is $[-1, 1]$.

Hence, (a) is the correct answer.

Example 108 ***A function*** $f : R \to R$ ***satisfies***

$$\sin x \cos y\,(f(2x + 2y) - f(2x - 2y)) = \cos x \sin y\,(f(2x + 2y) + f(2x - 2y)).$$

If $f'(0) = \frac{1}{2}$, ***then***

(a) $f''(x) = f(x) = 0$ (b) $4f''(x) + f(x) = 0$

(c) $f''(x) + f(x) = 0$ (d) $4f''(x) - f(x) = 0$

Sol. We have, $\frac{f(2x + 2y)}{f(2x - 2y)} = \frac{\sin(x + y)}{\sin(x - y)}$

$$\Rightarrow \quad \frac{f(\alpha)}{\sin\frac{\alpha}{2}} = \frac{f(\beta)}{\sin\frac{\beta}{2}} = K$$

$$\Rightarrow \quad f(x) = K \sin\frac{x}{2}$$

$$\Rightarrow \quad f'(x) = \frac{K}{2}\cos\frac{x}{2}$$

and $f''(x) = \frac{-K}{4}\sin\frac{x}{2}$

$\Rightarrow \quad 4f''(x) + f(x) = 0$

Hence, (b) is the correct answer.

Exercise for Session 10

1. Consider the real-valued function satisfying $2f(\sin x) + f(\cos x) = x$. Find the domain and range of $f(x)$.
2. If $f(x)$ is defined in $[-3, 2]$, find the domain of definition of $f([|x|])$ and $f([2x + 3])$.
3. $f(x) = \begin{cases} x - 1, & -1 \le x \le 0 \\ x^2, & 0 < x \le 1 \end{cases}$ and $g(x) = \sin x$. Find $h(x) = f(|g(x)|) + |f(g(x))|$.
4. Let $f(x)$ be defined on $[-2, 2]$ and is given by $f(x) = \begin{cases} -1, & -2 \le x \le 0 \\ x - 1, & 0 \le x \le 2 \end{cases}$ and $g(x) = f(|x|) + |f(x)|$, find $g(x)$.
5. Let two functions are defined as $g(x) = \begin{cases} x^2, & -1 \le x < 2 \\ x + 2, & 2 \le x \le 3 \end{cases}$ and $f(x) = \begin{cases} x + 1, & x \le 1 \\ 2x + 1, & 1 < x \le 2 \end{cases}$, find gof.

Session 11

Inverse of a Function

Inverse of a Function

Let $f: A \to B$ be a one-one and onto function, then there exists a unique function, $g: B \to A$ such that $f(x) = y \Leftrightarrow g(y) = x, \ \forall \ x \in A$ and $y \in B$.

Then, g is said to be inverse of f.

Thus, $g = f^{-1}: B \to A = \{(f(x), x) | (x, f(x)) \in f\}$

Let us consider a one-one function with domain A and range B.

Where, $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6, 8\}$ and $f: A \to B$ is given by $f(x) = 2x$, then write f and f^{-1} as a set of ordered pairs.

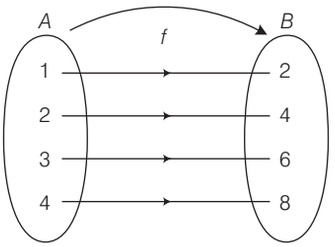


Figure 3.43

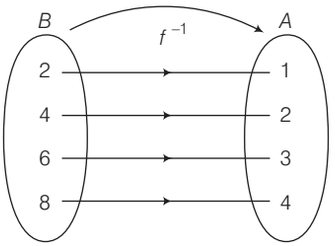


Figure 3.44

Here, member $y \in B$ arises from one and only one member $x \in A$.

So, $f = \{(1, 2)(2, 4)(3, 6)(4, 8)\}$

and $f^{-1} = \{(2, 1)(4, 2)(6, 3)(8, 4)\}$

In above function,

Domain of $f = \{1, 2, 3, 4\} =$ Range of f^{-1}.

Range of $f = \{2, 4, 6, 8\} =$ Domain of f^{-1}.

which represents for a function to have its inverse, it must be **one-one onto** or **bijective**.

Example 109 If $f(x) = 3x - 5$, find $f^{-1}(x)$.

Sol. Here, $f(x) = 3x - 5$

which is clearly bijective as it is linear in x.

Now, let $f(x) = y \Rightarrow y = 3x - 5$

$\Rightarrow \quad x = \dfrac{y+5}{3}$

$\Rightarrow \quad f^{-1}(y) = \dfrac{y+5}{3}$ [as $f(x) = y \Rightarrow x = f^{-1}(y)$]

Therefore, $f^{-1}(x) = \dfrac{x+5}{3}$

Example 110 If $f: [1, \infty) \to [2, \infty)$ is given by $f(x) = x + \dfrac{1}{x}$, find $f^{-1}(x)$, (assume bijection). ***[IIT JEE 2001, 2002]***

Sol. Let $y = f(x)$

$\therefore \quad y = \dfrac{x^2+1}{x} \quad \Rightarrow \quad x^2 - xy + 1 = 0$

$\Rightarrow \quad x = \dfrac{y \pm \sqrt{y^2-4}}{2}$

$\Rightarrow \quad f^{-1}(y) = \dfrac{y \pm \sqrt{y^2-4}}{2}$ [as $f(x) = y \Rightarrow x = f^{-1}(y)$]

$\Rightarrow \quad f^{-1}(x) = \dfrac{x \pm \sqrt{x^2-4}}{2}$

Since, range of inverse function is $[1, \infty)$, therefore neglecting the negative sign, we have

$$f^{-1}(x) = \frac{x + \sqrt{x^2-4}}{2}$$

Example 111 ***Let $f(x) = x^3 + 3$ be bijective, then find its inverse.***

Sol. Let $y = x^3 + 3$ [i.e. $y = f(x)$]

$\Rightarrow \quad x^3 = y - 3$

$\Rightarrow \quad x = (y-3)^{1/3}$

$\Rightarrow \quad f^{-1}(y) = (y-3)^{1/3}$ [$y = f(x) \Rightarrow f^{-1}(y) = x$]

$\Rightarrow \quad f^{-1}(x) = (x-3)^{1/3}$

Thus, $f^{-1}(x) = (x-3)^{1/3}$

when $f(x) = x^3 + 3$ is bijective.

Example 112 Find the inverse of the function, (assuming onto).

$$y = \log_a (x + \sqrt{x^2 + 1}),\ (a > 1).$$

Sol. We have, $y = \log_a(x + \sqrt{x^2 + 1})$

Since, $\sqrt{x^2 + 1} > |x|$

$\therefore$ It is defined for all x.

Now, $y = \log_a(x + \sqrt{x^2 + 1})$,

which is strictly increasing when $a > 1$.

Thus, one-one. Also, given that $f(x)$ is onto.

[where $y = f(x)$]

Hence, the given function is invertible.

Now, $y = \log_a (x + \sqrt{x^2 + 1})$

$\Rightarrow$ $a^y = x + \sqrt{x^2 + 1}$ and $a^{-y} = \sqrt{x^2 + 1} - x$

$\Rightarrow$ $x = \frac{1}{2}(a^y - a^{-y})$

Hence, the inverse in the form $y = f^{-1}(x)$ is,

$$y = \frac{1}{2}(a^x - a^{-x})$$

Graphical Representation of Invertible Functions

Let (h, k) be a point on the graph of the function f. Then, (k, h) is the corresponding point on the graph of inverse of f, i.e. g.

The line segment joining the points (h,k) and (k,h) is bisected at right angle by the line $y = x$.

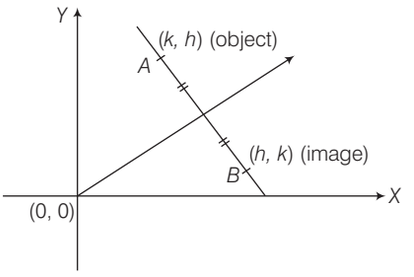


Figure 3.45

So, that the two points play object-image role in the line $y = x$ as plane mirror.

It follows that the graph of $y = f(x)$ and its inverse written in form $y = g(x)$ are symmetrical about the line $y = x$.

The graphs $y = f(x)$ and $y = f^{-1}(x)$, if they intersect then they meet on the line $y = x$ only. Hence, the solutions of $f(x) = f^{-1}(x)$ are also the solutions of $f(x) = x$.

Example 113 Let $f: R \to R$ be defined by $f(x) = \frac{e^x - e^{-x}}{2}$. Is $f(x)$ invertible? If so, find its inverse.

Sol. Let us check for invertibility of $f(x)$.

(a) **One-one** Here, $f'(x) = \frac{e^x + e^{-x}}{2}$

$\Rightarrow$ $f'(x) = \frac{e^{2x} + 1}{2e^x}$, which is strictly increasing as $e^{2x} > 0$ for all x. Thus, it is one-one.

(b) **Onto** Let $y = f(x)$

$\Rightarrow$ $y = \frac{e^x - e^{-x}}{2}$, where y is strictly monotonic.

Hence, range of $f(x) = (f(-\infty), f(\infty))$

$\Rightarrow$ Range of $f(x) = (-\infty, \infty)$

So, range of $f(x) =$ codomain

Hence, $f(x)$ is one-one and onto.

(c) **To find f^{-1}** $y = \frac{e^{2x} - 1}{2e^x}$

$\Rightarrow$ $e^{2x} - 2e^x y - 1 = 0 \Rightarrow e^x = \frac{2y \pm \sqrt{4y^2 + 4}}{2}$

$\Rightarrow$ $x = \log (y \pm \sqrt{y^2 + 1})$

$\Rightarrow$ $f^{-1}(y) = \log (y \pm \sqrt{y^2 + 1})$

[as $f(x) = y \Rightarrow x = f^{-1}(y)$]

Since, $e^{f^{-1}(x)}$ is always positive.

So, neglecting the negative sign.

Hence, $f^{-1}(x) = \log (x + \sqrt{x^2 + 1})$

Example 114 Let $f: [1/2, \infty) \to [3/4, \infty)$, where $f(x) = x^2 - x + 1$. Find the inverse of $f(x)$. Hence, solve the equation $x^2 - x + 1 = \frac{1}{2} + \sqrt{x - \frac{3}{4}}$.

Sol. (a) $f(x) = x^2 - x + 1$

$\Rightarrow f(x) = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$, which is clearly one-one and onto in given domain and codomain.

Thus, its inverse can be obtained.

Let $y = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$

$\Rightarrow$ $x - \frac{1}{2} = \pm\sqrt{y - \frac{3}{4}}$ $\Rightarrow$ $x = \frac{1}{2} \pm \sqrt{y - \frac{3}{4}}$

$\Rightarrow$ $f^{-1}(y) = \frac{1}{2} + \sqrt{y - \frac{3}{4}}$ $[f(x) = y \Rightarrow x = f^{-1}(y)]$

[neglecting – ve sign as x is always +ve]

$$\Rightarrow \quad f^{-1}(x) = \frac{1}{2} + \sqrt{x - \frac{3}{4}}$$

(b) To solve $x^2 - x + 1 = \frac{1}{2} + \sqrt{x - \frac{3}{4}}$, since solution of $f(x) = f^{-1}(x)$ are solutions of $f(x) = x$.

i.e. $\quad f(x) = x \Rightarrow x^2 - x + 1 = x$

$\Rightarrow \quad x^2 - 2x + 1 = 0 \Rightarrow (x-1)^2 = 0$

$\therefore\ x = 1$ is the required solution.

Properties of Inverse Functions

(i) *The inverse of a bijection is unique.*

Proof Let $f : A \to B$ be a bijection. If possible let $g : B \to A$ and $h : B \to A$ be two inverse functions of f. Also, let $a_1, a_2 \in A$ and $b \in B$ such that $g(b) = a_1$ and $h(b) = a_2$, then $g(b) = a_1 \Rightarrow f(a_1) = b$

$$h(b) = a_2 \Rightarrow f(a_2) = b$$

But, since f is one-one, so $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$

$$\Rightarrow \quad g(b) = h(b), \forall\ b \in B$$

(ii) *If $f : A \to B$ is a bijection and $g : B \to A$ is the inverse of f, then $fog = I_B$ and $gof = I_A$, where I_A and I_B are identity functions on the sets A and B, respectively.*

Remarks

(a) The graphs of f and g are the mirror images of each other in the line $y = x$. As shown, in the figure given below a point (x', y') corresponding to $y = x^2 (x \geq 0)$ changes to (y', x') corresponding to $y = +\sqrt{x}$, the changed form of $x = \sqrt{y}$.

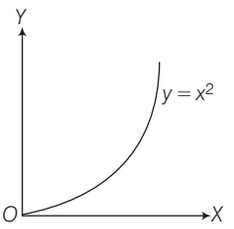


Figure 3.46

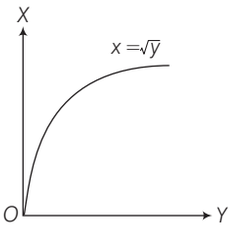


Figure 3.47

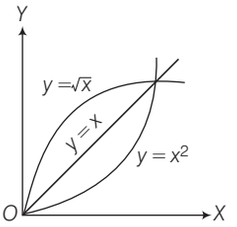


Figure 3.48

(b) If $f(x)$ has its own inverse as in $f(x) = \frac{1}{x}$, then $f(x) = f^{-1}(x)$ will have infinite solutions but $f(x) = f^{-1}(x) = x$ will have only one solution.

(iii) *The inverse of a bijection is also a bijection.*

Proof Let $f : A \to B$ be a bijection and $g : B \to A$ be its inverse. We have to show that g is one-one and onto.

One-one Let $g(b_1) = a_1$ and

$g(b_2) = a_2 : a_1, a_2 \in A$ and $b_1, b_2 \in B$

Then, $\quad g(b_1) = g(b_2)$

$\Rightarrow \quad a_1 = a_2$

$\Rightarrow \quad f(a_1) = f(a_2)$

$\Rightarrow \quad b_1 = b_2 \qquad [\because f$ is a bijection$]$

$[\because g(b_1) = a_1 \Rightarrow b_1 = f(a_1), g(b_2) = a_2 \Rightarrow\ b_2 = f(a_2)]$

which proves that g is one-one.

Onto Again, if $a \in A$, then

$a \in A \Rightarrow \exists\, b \in B$ such that $f(a) = b$ [by definition of f]

$\Rightarrow \exists\, b \in B$ such that $g(b) = a \ [\because f(a) = b \Rightarrow a = g(b)]$

which proves that g is onto.

Hence, g is also a bijection.

(iv) If f *and* g are two bijections $f : A \to B, g : B \to C$, then the inverse of gof exists and $(gof)^{-1} = f^{-1}og^{-1}$.

Proof Since, $f : A \to B$ and $g : B \to C$ are two bijections.

$\therefore\ gof : A \to C$ is also a bijection.

[by theorem, the composite of two bijections is a bijection]

As such gof has an inverse function $(gof)^{-1} : C \to A$.

We have to show that $(gof)^{-1} = f^{-1}og^{-1}$.

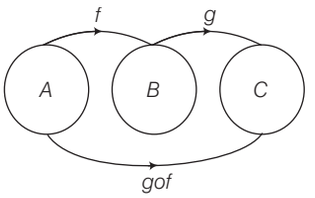


Figure 3.49

Now, let $a \in A, b \in B, c \in C$ such that

$$f(a) = b \text{ and } g(b) = c$$

So, $\quad (gof)(a) = g[f(a)] = g(b) = c$

Now, $\quad f(a) = b \Rightarrow\ a = f^{-1}(b) \qquad \ldots(i)$

$$g(b) = c \Rightarrow\ b = g^{-1}(c) \qquad \ldots(ii)$$

$$(gof)(a) = c \Rightarrow\ a = (gof)^{-1}(c) \qquad \ldots(iii)$$

Also, $(f^{-1}og^{-1})(c) = f^{-1}[g^{-1}(c)]$ [by definition]

$= f^{-1}(b)$ [by Eq. (ii)]

$= a$ [by Eq. (i)]

$= (gof)^{-1}(c)$ [by Eq. (iii)]

$\therefore \quad (gof)^{-1} = f^{-1}og^{-1}$,

which proves the theorem.

Example 115 Let $g(x)$ be the inverse of $f(x)$ and $f'(x) = \frac{1}{1+x^3}$. Find $g'(x)$ in terms of $g(x)$.

Sol. We know, if $g(x)$ is inverse of $f(x)$.

$\Rightarrow \quad g\{f(x)\} = x \Rightarrow g'\{f(x)\} \cdot f'(x) = 1$

$\Rightarrow g'\{f(x)\} = \frac{1}{f'(x)} = 1 + x^3 \Rightarrow g'\{f(g(x))\} = 1 + (g(x))^3$

$\Rightarrow \quad g'(x) = 1 + (g(x))^3 \qquad [\because f(g(x)) = x]$

Example 116 ***If*** $f : R \to R$ ***is defined by*** $f(x) = x^2 + 1$, ***find the value of*** $f^{-1}(17)$ ***and*** $f^{-1}(-3)$***.***

Sol. $f(x) = x^2 + 1$; $f^{-1}(17) \Rightarrow f(x) = 17 \Rightarrow x^2 + 1 = 17$

$\Rightarrow \quad x = \pm 4$ and $f^{-1}(-3)$

$\Rightarrow \quad f(x) = -3 \Rightarrow x^2 + 1 = -3$

$\Rightarrow \quad x^2 = -4$ [which is not possible]

Hence, $f^{-1}(17) = \pm 4$ and $f^{-1}(-3) = \phi$

Example 117 If the function f and g are defined as $f(x) = e^x$ and $g(x) = 3x - 2$, where $f : R \to R$ and $g : R \to R$, find the function fog and gof. A***lso,*** find the domain of $(fog)^{-1}$ and $(gof)^{-1}$.

Sol. $(fog)(x) = f\{g(x)\}$

$\Rightarrow \quad f\{g(x)\} = f(3x - 2)$

$\Rightarrow \quad f\{g(x)\} = e^{3x-2}$ ...(i)

and $(gof)(x) = g\{f(x)\}$

$\Rightarrow \quad g\{f(x)\} = g\{e^x\}$

$\Rightarrow \quad g\{f(x)\} = 3e^x - 2$ ...(ii)

For finding $(fog)^{-1}$ and $(gof)^{-1}$.

Let $(fog)(x) = y = e^{3x-2}$

$\Rightarrow \quad 3x - 2 = \log y \quad \Rightarrow \quad x = \frac{\log y + 2}{3}$

$\Rightarrow \quad (fog)^{-1} y = \frac{\log y + 2}{3}$ and $(fog)^{-1} x = \frac{\log x + 2}{3}$

and domain of $(fog)^{-1}$ is $x > 0$, i.e. $x \in (0, \infty)$.

Again, let $(gof)x = y = 3e^x - 2 \Rightarrow e^x = \frac{y+2}{3}$

$\Rightarrow \quad x = \log\left(\frac{y+2}{3}\right)$

$\Rightarrow \quad (gof)^{-1} y = \log\left(\frac{y+2}{3}\right)$

$\Rightarrow \quad (gof)^{-1} x = \log\left(\frac{x+2}{3}\right)$

and domain of $(gof)^{-1}$ is $\frac{x+2}{3} > 0$.

Hence, domain of $(gof)^{-1}$ is $x > -2$, i.e. $x \in (-2, \infty)$.

Example 118 If $f(x) = ax + b$ and the equation $f(x) = f^{-1}(x)$ be satisfied by every real value of x, then

(a) $a = 2, b = -1$ (b) $a = -1, b \in R$

(c) $a = 1, b \in R$ (d) $a = 1, b = -1$

Sol. If $f(x) = ax + b$

$\Rightarrow \quad f^{-1}(x) = \frac{x}{a} - \frac{b}{a}$

Since, $f(x) = f^{-1}(x), \forall \ x \in R$

$\Rightarrow \quad \frac{1}{a} = a$ and $b = -\frac{b}{a} \Rightarrow a = -1$ and $b \in R$

Hence, (b) is the correct answer.

Example 119 If $g(x)$ is the inverse of $f(x)$ and $f'(x) = \sin x$, then $g'(x)$ is equal to

(a) $\sin(g(x))$ (b) $\text{cosec}(g(x))$

(c) $\tan(g(x))$ (d) None of these

Sol. Given, $g(x) = f^{-1}(x)$

So, $x = f(g(x))$

On differentiating w.r.t. 'x', we get $1 = f'(g(x)) \cdot g'(x)$

Therefore, $g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\sin(g(x))}$

$\therefore \quad g'(x) = \text{cosec}(g(x))$

Hence, (b) is the correct answer.

Example 120 If A and B are the points of intersection of $y = f(x)$ and $y = f^{-1}(x)$, then

(a) A and B necessarily lie on the line $y = x$

(b) A and B must be coincident

(c) slope of line AB may be -1

(d) None of the above

Sol. If solution of $f(x) = f^{-1}(x)$ doesn't lie on $y = x$, then they must be of the form (α, β) and (β, α).

$\therefore$ Slope of line AB may be -1. Hence, (c) is the correct answer.

General Results

If x, y are independent variables, then

(i) $f(xy) = f(x) + f(y) \Rightarrow f(x) = k \ln x$ or $f(x) = 0$.

(ii) $f(xy) = f(x) \cdot f(y) \Rightarrow f(x) = x^n, n \in R$.

(iii) $f(x + y) = f(x) \cdot f(y) \Rightarrow f(x) = a^{kx}$.

(iv) $f(x + y) = f(x) = f(y) \in f(x) = k$, where k is constant.

(v) $f(x)$ takes rational values for all $x \Rightarrow f(x)$ is a constant function.

Exercise for Session 11

1. Find the inverse of the following functions

(i) $f(x) = \sin^{-1}\left(\frac{x}{3}\right), x \in [-3, 3]$

(ii) $f(x) = \log_e (x^2 + 3x + 1), x \in [1, 3]$

(iii) $f(x) = 5^{\log_e x}, x > 0$

(iv) $f(x) = \log_e (x + \sqrt{x^2 + 1})$

(v) $f(x) = \begin{cases} x, & x < 1 \\ x^2, & 1 \le x \le 4 \\ 8\sqrt{x}, & x > 4 \end{cases}$

2. If the function $f : [1, \infty) \to [1, \infty)$ is defined by $f(x) = 2^{x(x-1)}$, then find $f^{-1}(x)$.

Session 12

Miscellaneous Problems on Functions

Miscellaneous Problems On Functions

We should consider certain examples to make the concept clear.

Example 121 For $x \in R$, the function $f(x)$ satisfies $2f(x)+f(1-x)=x^2$. The value of $f(4)$ is equal to

(a) $\frac{13}{3}$ (b) $\frac{43}{3}$ (c) $\frac{23}{3}$ (d) None of these

Sol. We have, $2f(x)+f(1-x)=x^2$ ...(i)

In Eq. (i) x is replaced by $(1-x)$, we get

$$2f(1-x)+f(x)=(1-x)^2 \quad ...(ii)$$

On solving Eqs. (i) and (ii), we get

$$\therefore \quad 2f(x)+f(1-x)=x^2 \} \times 2$$

$$\Rightarrow \quad \underline{-2f(1-x) \pm f(x)=(1-x)^2}$$

$$\Rightarrow \quad 3f(x)=2x^2-(1-x)^2$$

$$\Rightarrow \quad f(x)=\frac{1}{3}\{x^2+2x-1\}$$

$$\therefore \quad f(4)=\frac{1}{3}\{16+8-1\}=\frac{23}{3}$$

Example 122 ***Let*** $f(x)=ax^7+bx^3+cx-5$***, where*** a, b and c ***are constants. If*** $f(-7)=7$, ***find*** $f(7)$.

Sol. As, $f(x)=ax^7+bx^3+cx-5$

$$\therefore \quad f(-x)=-ax^7-bx^3-cx-5$$

$$\Rightarrow \quad f(x)+f(-x)=-10$$

Put $x=7$, $f(7)+f(-7)=-10 \Rightarrow f(7)=-17$

Example 123 If $f(x)+f\left(1-\frac{1}{x}\right)=1+x$ for $x \in R-\{0,1\}$. The value of $4f(2)$ is equal to

Sol. Here, $f(x)+f\left(1-\frac{1}{x}\right)=1+x$ ...(i)

$$\Rightarrow \quad f(y)+f\left(1-\frac{1}{y}\right)=1+y, \text{ let } y=1-\frac{1}{x}$$

$$\Rightarrow \quad f\left(1-\frac{1}{x}\right)+f\left(1-\frac{x}{x-1}\right)=1+\left(1-\frac{1}{x}\right)$$

$$\Rightarrow \quad f\left(1-\frac{1}{x}\right)+f\left(\frac{-1}{x-1}\right)=2-\frac{1}{x}$$

or $$f\left(1-\frac{1}{x}\right)+f\left(\frac{1}{1-x}\right)=2-\frac{1}{x} \quad ...(ii)$$

Also, $f(z)+f\left(1-\frac{1}{z}\right)=1+z$, put $z=\frac{1}{1-x}$

$$f\left(\frac{1}{1-x}\right)+f(1-(1-x)=1+\frac{1}{1-x}$$

$$\Rightarrow \quad f\left(\frac{1}{1-x}\right)+f(x)=1+\frac{1}{1-x} \quad ...(iii)$$

On subtracting Eq. (ii) from Eq. (iii), we get

$$f(x)-f\left(1-\frac{1}{x}\right)=\frac{1}{1-x}+\frac{1}{x}-1 \quad ...(iv)$$

From Eqs. (i) and (iv), we get

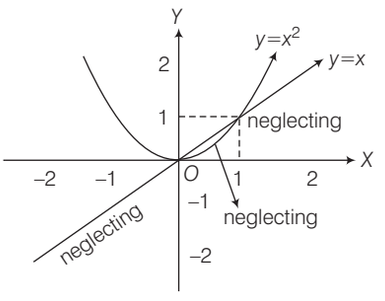
$$2f(x)=\frac{1}{1-x}+\frac{1}{x}+x$$

$$\therefore \quad 2f(2)=\frac{1}{-1}+\frac{1}{2}+2=\frac{3}{2}$$

$$\Rightarrow \quad f(2)=\frac{3}{4} \Rightarrow 4f(2)=3$$

Example 124 Let $f(x)=\max\{x, x^2\}$. Then, find equivalent definition of $f(x)$.

Sol. **Note** *These type of questions, where $f(x)$ are either maximum or minimum should be solved graphically for better representation.*



Let $f_1(x)=x$ and $f_2(x)=x^2$

Now, draw graph for $f_1(x) = x$ and $f_2(x) = x^2$.

Here, neglecting the graph, below point of intersection. Since, we want to find the maximum of two functions $f_1(x)$ and $f_2(x)$.

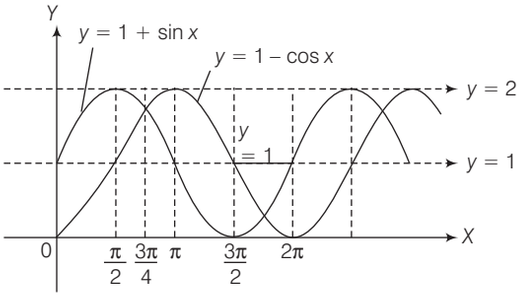
$$\therefore \quad f(x) = \begin{cases} x^2, & x \le 0 \text{ or } x \ge 1 \\ x, & 0 \le x \le 1 \end{cases}$$

Example 125 *Let*

$f(x) = \max\{1+\sin x, 1-\cos x, 1\}, \forall\ x \in [0, 2\pi]$ ***and*** $g(x) = \max\{1, |x-1|\}, \forall\ x \in R$. ***Determine*** $f\{g(x)\}$ ***and*** $g\{f(x)\}$ ***in terms of x.***

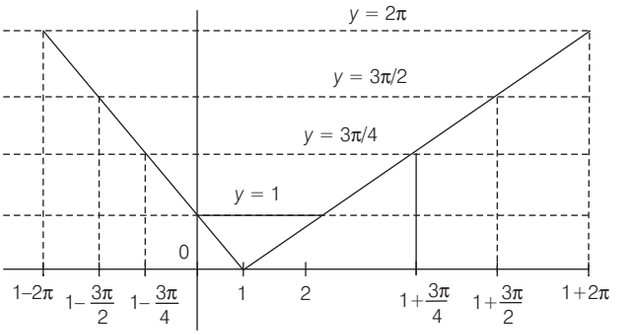
Sol. Here, $f(x) = \max\{1+\sin x, 1-\cos x, 1\}$.

Graphically it can be shown as



$$f(x) = \begin{cases} 1+\sin x, & 0 \le x \le \dfrac{3\pi}{4} \\ 1-\cos x, & \dfrac{3\pi}{4} < x \le \dfrac{3\pi}{2} \\ 1, & \dfrac{3\pi}{2} < x \le 2\pi \end{cases} \qquad \text{[using above graph]}$$

Again, $g(x) = \max\{1, |x-1|\}$, graphically it can be shown as



$$\therefore \quad g(x) = \begin{cases} 1-x, & x \le 0 \\ 1, & 0 < x \le 2 \\ x-1, & x > 2 \end{cases}$$

$$\text{Now,} \quad g\{f(x)\} = \begin{cases} 1-f(x), & f(x) \le 0 \\ 1, & 0 < f(x) \le 2 \\ f(x)-1, & f(x) > 2 \end{cases}$$

Since, $f(x) \in [1, 2], \forall\ x \in [0, 2\pi]$; $g(f(x)) = 1, \forall\ x \in [0, 2\pi]$

$$\text{Also,} \quad f\{g(x)\} = \begin{cases} 1+\sin\{g(x)\}, & 0 \le g(x) \le \dfrac{3\pi}{4} \\ 1-\cos\{g(x)\}, & \dfrac{3\pi}{4} < g(x) \le \dfrac{3\pi}{2} \\ 1, & \dfrac{3\pi}{2} < g(x) \le 2\pi \end{cases}$$

$$\Rightarrow f(g(x)) = \begin{cases} 1, & 1-2\pi \le x < 1-3\pi/2 \\ 1-\cos(1-x), & 1-3\pi/2 \le x < 1-3\pi/4 \\ 1+\sin(1-x), & 1-3\pi/4 \le x \le 0 \\ 1+\sin 1, & 0 < x \le 2 \\ 1+\sin(x-1), & 2 < x \le 1+3\pi/4 \\ 1-\cos(x-1), & 1+3\pi/4 < x \le 1+3\pi/2 \\ 1, & 1+3\pi/2 < x \le 2\pi+1 \end{cases}$$

Exercise for Session 12

1. For $x \in R - \{1\}$, the function $f(x)$ satisfies $f(x) + 2f\left(\dfrac{1}{1-x}\right) = x$. Find $f(2)$.
2. Let $f(x)$ and $g(x)$ be functions which take integers as arguments. Let $f(x+y) = f(x) + g(y) + 8$ for all integers x and y. Let $f(x) = x$ for all negative integers x and let $g(8) = 17$. Find $f(0)$.
3. The function $f : R \to R$ satisfies the condition $mf(x-1) + nf(-x) = 2|x| + 1$. If $f(-2) = 5$ and $f(1) = 1$, find $(m+n)$.
4. The equivalent definition of $f(x) = \max\{x^2, (1-x)^2, 2x(1-x)\}$, where $0 \le x \le 1$.

JEE Type Solved Examples :
Single Option Correct Type Questions

● **Ex. 1** *Let* $f(x)=\dfrac{a_{2k}x^{2k}+a_{2k-1}x^{2k-1}+\ldots+a_1x+a_0}{b_{2k}x^{2k}+b_{2k-1}x^{2k-1}+\ldots+b_1x+b_0}$, *where k is a positive integer,* $a_i, b_i \in R$ *and* $a_{2k}\neq 0$, $b_{2k}\neq 0$ *such that* $b_{2k}x^{2k}+b_{2k-1}x^{2k-1}+\ldots+b_1x+b_0=0$ *has no real roots, then*

(a) $f(x)$ must be one to one

(b) $a_{2k}x^{2k}+a_{2k-1}x^{2k-1}+\ldots+a_1x+a_0=0$ must have real roots

(c) $f(x)$ must be many to one

(d) Nothing can be said about the above options

Sol. (*c*) $f(x)$ is continuous, $\forall\ x\in R$.

$$\lim_{x\to-\infty} f(x)=\lim_{x\to\infty} f(x)=\frac{a_{2k}}{b_{2k}}$$

$\therefore f(x)$ is many to one.

● **Ex. 2** *If* $\log_{10}\left(\sin\left(x+\dfrac{\pi}{4}\right)\right)=\dfrac{\log_{10}6-1}{2}$, *the value of* $\log_{10}(\sin x)+\log_{10}(\cos x)$ *is*

(a) −1 (b) −2 (c) 2 (d) 1

Sol. (*a*) $2\log_{10}\left(\dfrac{\sin x+\cos x}{\sqrt{2}}\right)=\log_{10}\left(\dfrac{6}{10}\right)$

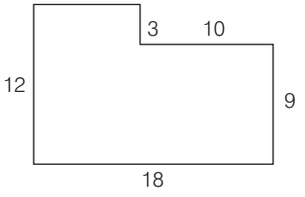
$$\Rightarrow \log_{10}\left(\frac{1+2\sin x\cos x}{2}\right)=\log_{10}\left(\frac{6}{10}\right)$$

$$\Rightarrow \frac{1}{2}+\sin x\cos x=\frac{3}{5}$$

$$\Rightarrow \sin x\cos x=\frac{1}{10}$$

$$\therefore \log_{10}(\sin x)+\log_{10}(\cos x)=-1$$

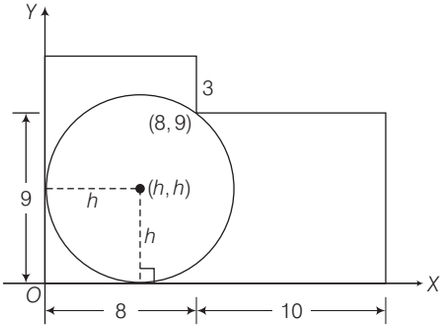
● **Ex. 3** *The diagram shows the dimensions of the floor of an L-shaped room. (All the angles are right angles). The area of the largest circle that can be drawn on the floor of this room is*



(a) 16π (b) 25π

(c) $\dfrac{81\pi}{4}$ (d) $\dfrac{145\pi}{4}$

Sol. (*b*)



Here, $(x-h)^2+(y-h)^2=h^2$ passes through (8, 9).

$$(8-h)^2+(9-h)^2=h^2$$

$$\Rightarrow h^2-34h+145=0$$

$$\Rightarrow (h-5)(h-29)=0$$

$$\Rightarrow h=5, \text{ neglecting } h=29$$

$$\therefore r=5$$

Area of the largest circle $=\pi(5)^2=25\pi$.

● **Ex. 4** *Suppose that the temperature T at every point* (x, y) *in the cartesian plane is given by the formula* $T=1-x^2+2y^2$. *The correct statement about the maximum and minimum temperature along the line* $x+y=1$ *is*

(a) Minimum is −1. There is no maximum

(b) Maximum is −1. There is no minimum

(c) Maximum is 0. Minimum is − 1

(d) There is neither a maximum nor a minimum along the line

Sol. (*a*) $T=1-x^2+2y^2$, where $x+y=1$

$$T=1-x^2+2(1-x)^2$$

$$=1-x^2+2(1+x^2-2x)$$

$$T=x^2-4x+3$$

$$=(x-2)^2-1$$

$$\therefore T_{\max}=\text{doesn't exist}$$

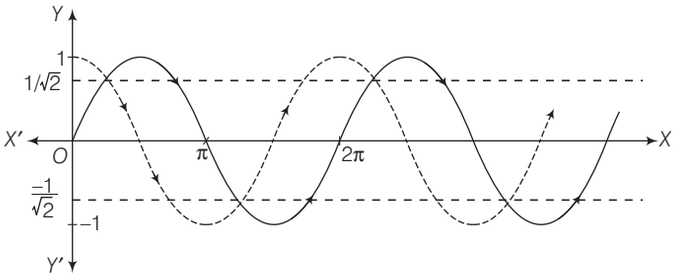
and $T_{\min}=-1$

● **Ex. 5** *The domain of the function* $f(x)=\max\{\sin x, \cos x\}$ *is* $(-\infty,\infty)$. *The range of* $f(x)$ *is*

(a) $\left[-\dfrac{1}{\sqrt{2}}, 1\right]$ (b) $\left[-\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right]$

(c) [0, 1] (d) [− 1, 1]

Sol. (*a*) Here, $f(x) = \max\{\sin x, \cos x\}$



From graph, range of $f(x) \in \left[-\frac{1}{\sqrt{2}}, 1\right]$.

● **Ex. 6** *Let a function f be defined as* $f: \{1,2,3,4\} \to \{1,2,3,4\}$. *If f satisfy* $f(f(x)) = f(x)$, $\forall\ x \in \{1,2,3,4\}$, *the number of such functions is*

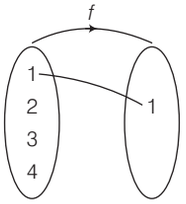
(a) 10 (b) 40 (c) 41 (d) 31

Sol. (*c*) Here, $f(f(x)) = f(x)$, let $f(x) = y$

$\Rightarrow$ $f(y) = y$

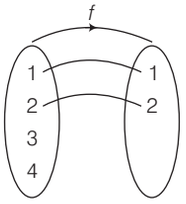
***Case* I** Range contains exactly one element.

It can be done in ${}^4C_1$ ways.



Say, $f(1) = 1$ remaining 3 elements 2, 3, 4 can be mapped only in one way $\Rightarrow$ Total $= {}^4C_1 \cdot 1 = 4$ ...(i)

***Case* II** Range contains two elements, this can be done in ${}^4C_2$ ways.

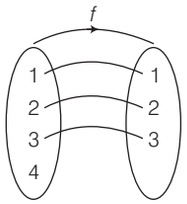


Say, $f(1) = 1, f(2) = 2$

Remaining 2 elements, i.e. 3 and 4 each can be mapped in 2 ways.

$\therefore$ Total $= {}^4C_2 \times 2 \times 2 = 24$ ...(ii)

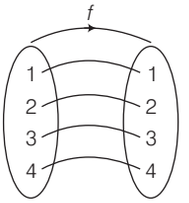
***Case* III** Range contains exactly 3 elements which can be done in ${}^4C_3 = 4$ ways. Say, $f(1) = 1, f(2) = 2, f(3) = 3$.



Now, remaining element can be mapped only in 3 ways.

$\therefore$ Total $= {}^4C_3 \cdot 3 = 12$ ways ...(iii)

***Case* IV** Range contains all 4 elements $f(1) = 1, f(2) = 2$, $f(3) = 3, f(4) = 4$.



$\therefore$ Only 1 way.

$\therefore$ Total ways $= 4 + 24 + 12 + 1 = 41$

● **Ex. 7** *Area bounded by the relation* $[2x] + [y] = 5$, $x, y > 0$ *is* (*where* $[\cdot]$ *represent greatest integer function*)

(a) 2 (b) 3 (c) 4 (d) 5

Sol. (*b*) Here, $[2x] + [y] = 5$

Let us consider $[2x] = 0 \Rightarrow 0 \le 2x < 1$ and $[y] = 5$

i.e. $x \in [0, 1/2)$ and $y \in [5, 6)$.

Similarly, we can consider $[2x] = 1, 2, 3, 4$ and 5.

when, $[2x] = 1 \Rightarrow 1/2 \le x < 1$ and $y \in [4, 5)$

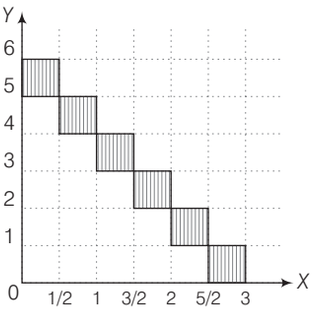
$[2x] = 2 \Rightarrow 1 \le x < 3/2$ and $y \in [3, 4)$

$[2x] = 3 \Rightarrow 3/2 \le x < 2$ and $y \in [2, 3)$

$[2x] = 4 \Rightarrow 2 \le x < \frac{5}{2}$ and $y \in [1, 2)$

$[2x] = 5 \Rightarrow 5/2 \le x < 3$ and $y \in [0, 1)$

which can be shown as



$\therefore$ From the graph, area is 3 sq units.

● **Ex. 8** *If the integers a, b, c, d are in arithmetic progression and* $a < b < c < d$ *and* $d = a^2 + b^2 + c^2$, *the value of* $(a + 10b + 100c + 1000d)$ *is*

(a) 2008 (b) 2010 (c) 2099 (d) 2016

Sol. (*c*) Let $a = x - t$, $b = x$, $c = x + t$ and $d = x + 2t$.

Where, $x \in I$ and $t > 0$ and t is an integer (as $a < b < c < d$).

Now, $d = a^2 + b^2 + c^2$

$\Rightarrow$ $x + 2t = (x - t)^2 + x^2 + (x + t)^2$

$\Rightarrow$ $x + 2t = 3x^2 + 2t^2$

$\Rightarrow$ $x(1 - 3x) = 2t(t - 1)$ ...(i)

as $t > 0$ and $t \in I$, hence $t \ge 1 \Rightarrow$ RHS ≥ 0

$\therefore$ LHS $= x(1-3x) \geq 0 \Rightarrow x \in [0, 1/3]$

The only integer is zero $\Rightarrow x = 0$

From Eq. (i), $t = 0$ or $t = 1$

But $t > 0 \Rightarrow t = 1$

$\therefore$ $a = -1, b = 0, c = 1, d = 2$

$\Rightarrow$ $a + 10b + 100c + 1000d = 2099$

● ***Ex. 9*** *Let $f(n)$ denotes the square of the sum of the digits of natural number n, where $f^2(n)$ denotes $f(f(n))$, $f^3(n)$ denotes $f(f(f(n)))$ and so on. Then, the value of* $\dfrac{f^{2017}(2011) - f^{2016}(2011)}{f^{2017}(2011) - f^{2018}(2011)}$*, is*

(a) 1 (b) 3 (c) 5 (d) 7

Sol. (*a*) Here, $f(n) =$ (sum of digits of natural number $n)^2$

$\therefore$ $f(2011) = (2+0+1+1)^2 = 16$

$f^2(2011) = f(f(2011)) = f(16) = (1+6)^2 = 49$

$f^3(2011) = f(49) = (4+9)^2 = 169$

$f^4(2011) = f(169) = (1+6+9)^2 = 256$

$f^5(2011) = f(256) = (2+5+6)^2 = 169$

$f^6(2011) = f(169) = (1+6+9)^2 = 256$

Thus, $f^{2n}(2011) = 256$ and $f^{2n+1}(2011) = 169$ $n \geq 2$

$\Rightarrow$ $\dfrac{f^{2017}(2011) - f^{2016}(2011)}{f^{2017}(2011) - f^{2018}(2011)} = \dfrac{169 - 256}{169 - 256} = 1$

JEE Type Solved Examples :
More Than One Correct Option Type Questions

● ***Ex. 10*** *If $\sum_{i=1}^{4} a_i^2 x^2 - 2\sum_{i=1}^{4} a_i a_{i+1} x + \sum_{i=1}^{4} a_{i+1}^2 \leq 0$, where $a_i > 0$ and all are distinct. Then,*

(a) $a_1 + a_5 > 2a_3$ (b) $\sqrt{a_1 a_5} = a_3$

(c) $\dfrac{2}{\sqrt{a_1 a_4}} > \dfrac{1}{a_1} + \dfrac{1}{a_4}$ (d) $\prod_{i=1}^{5} a_i = a_3^5$

Sol. (*a, b, d*) $(a_1^2 + a_2^2 + a_3^2 + a_4^2)x^2 - 2(a_1a_2 + a_2a_3 + a_3a_4 + a_4a_5)x + (a_2^2 + a_3^2 + a_4^2 + a_5^2) \leq 0$

$\Rightarrow (a_1^2 x^2 - 2a_1a_2 x + a_2^2) + (a_2^2 x^2 - 2a_2a_3 x + a_3^2) + \ldots + (a_4^2 x^2 - 2a_4a_5 x + a_5^2) \leq 0$

$\Rightarrow (a_1x - a_2)^2 + (a_2x - a_3)^2 + \ldots + (a_4x - a_5)^2 \leq 0$

$\therefore$ $(a_1x - a_2)^2 + (a_2x - a_3)^2 + \ldots + (a_4x - a_5)^2 = 0$

$\therefore$ $x = \dfrac{a_2}{a_1} = \dfrac{a_3}{a_2} = \dfrac{a_4}{a_3} = \dfrac{a_5}{a_4}$

$\Rightarrow$ a_1, a_2, a_3, a_4, a_5 are in GP.

$\therefore$ $\dfrac{a_1 + a_5}{2} > \sqrt{a_1 a_5} \Rightarrow a_1 + a_5 > 2a_3.$ $(\because \sqrt{a_1 a_5} = a_3)$

HM of a_1 and a_4 < GM of a_1 and $a_4 \Rightarrow \dfrac{2}{\sqrt{a_1 a_4}} < \dfrac{1}{a_1} + \dfrac{1}{a_4}$

$a_1 \cdot a_2 \cdot a_3 \cdot a_4 \cdot a_5 = a_3^5.$ $[\because a_1 a_5 = a_3^2, a_2 \cdot a_4 = a_3^2]$

● ***Ex. 11*** *If $f(x-y)$, $f(x) \cdot f(y)$, $f(x+y)$ are in AP for all $x, y \in R$ and $f(0) \neq 0$, then*

(a) $f'(x)$ is an even function

(b) $f'(1) + f'(-1) = 0$

(c) $f'(2) - f'(-2) = 0$

(d) $f'(3) + f'(-3) = 0$

Sol. (*b, d*) Given, $2f(x) \cdot f(y) = f(x+y) + f(x-y)$

At $x = y = 0$,

$2f(0) \cdot f(0) = 2f(0)$

$\Rightarrow$ $2f(0) \cdot (f(0) - 1) = 0$

$\Rightarrow$ $f(0) = 1$ [as $f(0) \neq 0$] ...(i)

At $x = 0$,

$2f(0) \cdot f(y) = f(y) + f(-y)$ [using Eq. (i), $f(0) = 1$]

$\Rightarrow$ $2f(y) = f(y) + f(-y)$

$\therefore$ $f(y) = f(-y) \Rightarrow f(x)$ is even function.

We know, if $f(x)$ is even, then $f'(x)$ is odd.

$\Rightarrow$ $f'(-x) = -f'(x)$

$\therefore$ $f'(1) = -f'(-1)$ and $f'(3) = -f'(-3)$

● ***Ex. 12*** *If the equation $x^2 + 4 + 3\cos(ax + b) = 2x$ has atleast one solution where $a, b \in [0, 5]$, the value of $(a + b)$ equal to*

(a) 5π (b) 3π (c) 2π (d) π

Sol. (*b, d*) $x^2 - 2x + 4 = -3\cos(ax + b)$

$(x-1)^2 + 3 = -3\cos(ax + b)$

Here, LHS ≥ 3 and RHS ≤ 3. Thus, the above equation have solution if LHS = RHS = 3

$\Rightarrow$ $x = 1$ and $\cos(ax + b) = -1$

Hence, $\cos(a + b) = -1 \therefore a + b = \pi, 3\pi, 5\pi$

But, $0 \leq a + b \leq 10$ $\Rightarrow$ $a + b = \pi$ or 3π

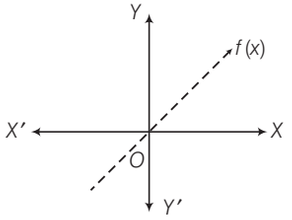
Hence, (b) and (d) are the correct answers.

● ***Ex. 13*** *Which of the following functions have the same graph?*

(a) $f(x) = \log_e e^x$ (b) $g(x) = |x| \operatorname{sgn} x$

(c) $h(x) = \cot^{-1}(\cot x)$ (d) $k(x) = \lim_{n \to \infty} \frac{2|x|}{\pi} \tan^{-1}(nx)$

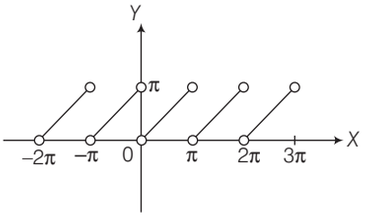
Sol. (a,b,d) (a) $f(x) = \log_e e^x = x \cdot \log_e e = x$ shown as,



(b) $g(x) = |x| \operatorname{sgn} x = \begin{cases} |x| \cdot \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases} = \begin{cases} x, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$\therefore \quad g(x) = x$, which is same as $f(x)$.

(c) $h(x) = \cot^{-1}(\cot x)$ is shown as,



which is not same as $f(x)$ and $g(x)$.

(d) $k(x) = \lim_{n \to \infty} \frac{2|x|}{\pi} \cdot \tan^{-1}(nx)$

$$= \begin{cases} \frac{2x}{\pi} \cdot \frac{\pi}{2}, & x > 0 \\ -\frac{2x}{\pi} \cdot -\frac{\pi}{2}, & x < 0 \\ 0, & x = 0 \end{cases} = \begin{cases} x, & x > 0 \\ x, & x < 0 \\ 0, & x = 0 \end{cases}$$

$\therefore k(x) = x, \forall x.$

Hence, (a), (b) and (d) are the correct answers.

JEE Type Solved Examples :
Statements I and II Type Questions

■ **Directions** (Q. Nos. 14 to 15) *For the following questions, choose the correct answers from the codes* (a), (b), (c) *and* (d) *defined as follows*

(a) Statement I is true, Statement II is also true; Statement II is the correct explanation of Statement I

(b) Statement I is true, Statement II is also true; Statement II is not the correct explanation of Statement I

(c) Statement I is true, Statement II is false

(d) Statement I is false, Statement II is true

● ***Ex. 14*** *Let* $f(\theta) = \frac{\sin^2 \theta \cos \theta}{(\sin \theta + \cos \theta)} - \frac{1}{4} \tan\left(\frac{\pi}{4} - \theta\right)$, $\forall \theta \in R - \left\{m\pi - \frac{\pi}{4}\right\}, n \in I.$

Statement I *The largest and smallest value of* $f(\theta)$ *differ by* $\frac{1}{\sqrt{2}}$.

Statement II $a \sin x + b \cos x + c \in [c - \sqrt{a^2 + b^2}, c + \sqrt{a^2 + b^2}], \forall x \in R$, *where* $a, b, c \in R$.

Sol. *(a)* We have, $f(\theta) = \frac{4 \sin^2 \theta \cos \theta - \cos \theta + \sin \theta}{4(\cos \theta + \sin \theta)}$

$$= \frac{2 \sin \theta ((\sin \theta + \cos \theta)^2 - 1) - (\cos \theta + \sin \theta) + 2 \sin \theta}{4(\cos \theta + \sin \theta)}$$

$$= \frac{1}{2} \sin \theta (\sin \theta + \cos \theta) - \frac{1}{4}$$

$$= \frac{1}{4}(\sin 2\theta - \cos 2\theta) \in \left[-\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}\right]$$

● ***Ex. 15*** *Consider two functions* $f(x) = 1 + e^{\cot^2 x}$ *and* $g(x) = \sqrt{2|\sin x| - 1} + \frac{1 - \cos 2x}{1 + \sin^4 x}$.

Statement I *The solutions of the equation* $f(x) = g(x)$ *is given by* $x = (2n + 1)\frac{\pi}{2}, \forall n \in I.$

Statement II *If* $f(x) \geq k$ *and* $g(x) \leq k$ *(where* $k \in R$*), then solutions of the equation* $f(x) = g(x)$ *is the solution corresponding to the equation* $f(x) = k$.

Sol. *(c)* LHS $= 1 + e^{\cot^2 x} \geq 2$

As, $\sqrt{2|\sin x| - 1} \leq 1$

and $$\frac{1-\cos 2x}{1+\sin^4 x}=\frac{2\sin^2 x}{1+\sin^4 x}=\frac{2}{\frac{1}{\sin^2 x}+\sin^2 x}\le 1$$

$\therefore$ RHS $=\sqrt{2|\sin x|-1}+\frac{1-\cos 2x}{1+\sin^4 x}\le 2$

Equation will satisfy, if LHS = RHS = 2 which is possible when $\cot^2 x=0$ and $|\sin x|=1$.

$\Rightarrow$ $x=(2n+1)\frac{\pi}{2}, n\in I$

$\therefore$ Statement I is correct.

Statement II is not always correct because solution of the equation $f(x)=g(x)$ will be solutions corresponding to $f(x)=g(x)=k$ in the domain of $f(x)$ and $g(x)$ both. Hence, (c) is the correct answer.

JEE Type Solved Examples : Passage Based Questions

Passage I

(Ex. Nos. 16 to 18)

Let $a_m\,(m=1,2,\ldots,p)$ be the possible integral values of a for which the graphs of

$$f(x)=ax^2+2bx+b$$

and $$g(x)=5x^2-3bx-a$$

meet at some points for all real values of b.

Let $t_r=\prod_{m=1}^{p}(r-a_m)$ *and* $S_n=\sum_{r=1}^{n}t_r, n\in N.$

● ***Ex. 16*** *The minimum possible value of a is*

(a) $\frac{1}{5}$ (b) $\frac{5}{26}$ (c) $\frac{3}{28}$ (d) $\frac{2}{43}$

● ***Ex.17*** *The sum of values of n for which S_n vanishes is*

(a) 8 (b) 9
(c) 10 (d) 15

● ***Ex. 18*** *The value of* $\sum_{r=5}^{\infty}\frac{1}{t_r}$ *is equal to*

(a) $\frac{1}{3}$ (b) $\frac{1}{6}$ (c) $\frac{1}{15}$ (d) $\frac{1}{18}$

■ ***Sol.*** (Q. Nos. 16 to 18)

16. (a) **17.** (c) **18.** (d)

$$ax^2+2bx+b=5x^2-3bx-a$$

$\Rightarrow$ $(a-5)x^2+5bx+(b+a)=0$

If $a\ne 5$, then since $x\in R$,

$$D=25b^2-4(b+a)(a-5)\ge 0,\forall\ b\in R$$

$\Rightarrow$ $25b^2-4(a-5)b-4a(a-5)\ge 0,\forall\ b\in R$

$\therefore$ $16(a-5)^2+16(25)a(a-5)\le 0$

$\Rightarrow$ $16(a-5)(a-5+25a)\le 0$

$\Rightarrow$ $(a-5)(26a-5)\le 0$ $\therefore$ $a\in\left[\frac{5}{26},5\right)$

If $a=5$, $5bx+(b+5)=0$ is not satisfied for $b=0$.

$\therefore$ $a_m\in\{1,2,3,4\};\ t_r=(r-1)(r-2)(r-3)(r-4)$

$$S_n=\frac{1}{5}\sum_{r=1}^{n}(r-4)(r-3)(r-2)(r-1)(r-(r-5))$$

$$=\frac{1}{5}\sum_{r=1}^{n}[(r-4)(r-3)(r-2)(r-1)r-(r-5)(r-4)(r-3)(r-2)(r-1)]$$

$$=\frac{1}{5}n(n-1)(n-2)(n-3)(n-4)$$

$S_n=0$

$\Rightarrow$ $n=1,2,3,4$

$\therefore$ Sum of values of n for which S_n vanishes is

$1+2+3+4=10$ [$n=0$ rejected]

$$\sum_{r=5}^{\infty}\frac{1}{t_r}=\frac{1}{3}\lim_{n\to\infty}\sum_{r=5}^{n}\frac{(r-1)-(r-4)}{(r-4)(r-3)(r-2)(r-1)}$$

$$=\frac{1}{3}\lim_{n\to\infty}\sum_{r=5}^{n}\left[\frac{1}{(r-4)(r-3)(r-2)}-\frac{1}{(r-3)(r-2)(r-1)}\right]$$

$$=\lim_{n\to\infty}\frac{1}{3}\left[\frac{1}{6}-\frac{1}{(n-3)(n-2)(n-1)}\right]=\frac{1}{18}$$

Passage II

(Ex. Nos. 19 to 21)

Let w be non-real fifth root of 3 and $x=w^3+w^4$. If $x^5=f(x)$, where $f(x)$ is real quadratic polynomial, with roots α and β, $(\alpha,\beta\in C)$, then determine $f(x)$ and answer the following questions.

● ***Ex. 19*** *Every term of the sequence $\{f(n)\}, n\in N$ is divisible by*

(a) 12 (b) 18
(c) 24 (d) 27

● ***Ex. 20*** *Which of the following is not true?*

(a) $\alpha+\beta=-3$ (b) $\alpha\beta=12/5$
(c) $|\alpha-\beta|=\sqrt{3/5}$ (d) $|\alpha|+|\beta|=2\sqrt{3/5}$

● ***Ex. 21*** *If α and β are represented by points A and B in argand plane, then circumradius of ΔOAB, where O is origin, is*

(a) 4 / 5 (b) 8 / 5 (c) 16 / 5 (d) 32 / 5

■ ***Sol.*** (Q. Nos. 19 to 21)

19. (b) **20.** (d) **21.** (a)

Here, $x=w^3+w^4$

$\Rightarrow \quad x^5=(w^2)^5(w+w^2)^5$

$=(w^5)^2\ [w^5+5w^4\cdot w^2+10w^3\cdot w^4+10w^2\cdot w^6+5w\cdot w^8+w^{10}]$

Put $w^5=3$ and $w^{10}=9$

Now, $x^5=9\ [3+5\cdot(3w)+10\cdot(3w^2)+10\cdot 3w^3+5\cdot(3w^4)+3^2]$

$=9[12+15(w+w^2)+15(w^2+w^3)+15(w^3+w^4)]$

$=9\left[12+15\left(\frac{x}{w^2}\right)+15\cdot\left(\frac{x}{w}\right)+15x\right]$

$=9\left[12+15x\left(\frac{1}{w^2}+\frac{1}{w}\right)+15x\right]$

$=9\left[12+15x\cdot\frac{w+w^2}{w^3}\cdot\frac{w^2}{w^2}+15x\right]$

$=9\left[12+15x\cdot\frac{(w^3+w^4)}{w^5}+15x\right]$

$=9\left[12+15x\cdot\frac{x}{3}+15x\right]$

$=9[12+15x+5x^2]=f(x)$

Now, $f(x)=9(5x^2+15x+12)$

$\therefore \quad f(x)=0$

$\Rightarrow \quad x=\frac{-15\pm\sqrt{-15}}{10}=\frac{-15\pm i\sqrt{15}}{10}=\alpha,\beta$

(i) For sequence $\{f(n)\}$,

$f(n)=9(5n^2+15n+12)=9[5n(n+1)+10n+12]$

$=9\times$ an even number

$\therefore$ Every term of $\{f(n)\}$ is always divisible by 18.

(ii) Clearly, $\alpha+\beta=-3, \alpha\beta=\frac{12}{5}, \ |\alpha-\beta|=\left|\frac{2i\sqrt{15}}{10}\right|=\sqrt{\frac{3}{5}}$

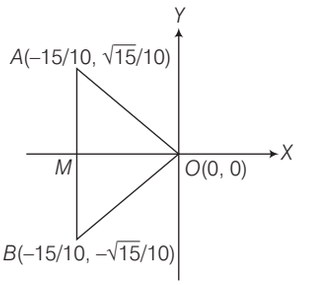
But $|\alpha|+|\beta|=2\left(\sqrt{\left(\frac{15}{10}\right)^2+\left(\frac{\sqrt{15}}{10}\right)^2}\right)=4\sqrt{\frac{3}{5}}$

(iii) In argand plane,

$OA=OB=\sqrt{\frac{12}{5}}$ and $AB=2\frac{\sqrt{15}}{10}=\sqrt{\frac{3}{5}}$

and area of ΔOAB,

$\Delta=\frac{1}{2}AB(OM)=\frac{1}{2}\cdot\sqrt{\frac{3}{5}}\cdot\left(\frac{15}{10}\right)$



$\therefore$ Circumradius, $R=\frac{OA\cdot AB\cdot OB}{4\Delta}=\frac{\sqrt{\frac{12}{5}}\sqrt{\frac{3}{5}}\sqrt{\frac{12}{5}}}{4\cdot\frac{1}{2}\cdot\sqrt{\frac{3}{5}}\left(\frac{15}{10}\right)}=\frac{4}{5}$

Passage III
(Ex. Nos. 22 to 24)

Let $A=\{1,2,3,4,5\}$ and $B=\{-2,-1,0,1,2,3,4,5\}$.

● ***Ex. 22*** *Increasing function from A to B is*

(a) 120 (b) 72 (c) 60 (d) 56

● ***Ex. 23*** *Non-decreasing functions from A to B is*

(a) 216 (b) 540 (c) 792 (d) 840

● ***Ex. 24*** *Onto functions from A to A such that $f(i)\neq i$ for all i, is*

(a) 44 (b) 120 (c) 56 (d) 76

■ ***Sol.*** (Q. Nos. 22 to 24)

22. (*d*) The desired number is ${}^8C_5=56$

23. (*c*) Here, for non-decreasing functions from A to B, is

${}^8C_1\cdot 1+{}^8C_2\cdot 4+{}^8C_3\cdot 6+{}^8C_4\cdot 4+{}^8C_5\cdot 1=792$

Explanation for case ${}^8C_2$, say two elements of set B are selected in ${}^8C_2$ is $\{-1,0\}$.

Now, $x_1+x_2=5 \qquad [x_1\geq 1, x_2\geq 1]$

where, x_1 denotes number of elements of set A maps to -1 and x_2 denotes number of elements of set A maps to 0.

$\therefore$ Total number of solutions is ${}^4C_1$.

Similarly, explanation for case ${}^8C_3$ say three elements selected in ${}^8C_3$ is $\{-1,0,1\}$.

Now, $x_1+x_2+x_3=5 \qquad (x_1\geq 1, x_2\geq 1, x_3\geq 1)$

$\therefore$ Total number of solutions is ${}^4C_2$ etc.

24. (*a*) Onto functions from A to A such that $f(i)\neq i$ for all i.

$\Rightarrow \quad 5!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}\right)=44$

JEE Type Solved Examples :
Matching Type Questions

● ***Ex. 25*** *Match the statements of Column I with values of Column II.*

	Column I	Column II
(A)	Let $f(x)$ be a function on $(-\infty, \infty)$ and $f(x+2) = f(x-2)$. If $f(x) = 0$ has only three real roots in $[0, 4]$ and one of them is 4, then the number of real roots of $f(x) = 0$ in $(-8, 10]$ is	(p) 4
(B)	Let $r_1, r_2, r_3, \ldots, r_n$ be n positive integers, not necessarily distinct, such that $(x + r_1)(x + r_2) \ldots (x + r_n) = x^n + 56x^{n-1} + \ldots + 2009$. The possible value of n is	(q) 5
(C)	If x and y are positive integers and $2xy = 2009 - 3y$, then the number of ordered pairs of (x, y) is	(r) 8
(D)	If $x, y \in R$, satisfying the equation $\frac{(x-4)^2}{4} + \frac{y^2}{9} = 1$, then the difference between the largest and smallest value of the expression $\frac{x^2}{4} + \frac{y^2}{9}$ is	(s) 9

Sol. (A) → (s), (B) → (p), (C) → (q), (D) → (r)

(A) Given, $f(x+2) = f(x-2)$, domain is R $x \to x+2$

$f(x+4) = f(x)$

$\Rightarrow f$ is periodic with period 4.

Put $x = 0, f(4) = f(0)$

But $f(4) = 0$

$\Rightarrow \quad f(0) = 0$

$\Rightarrow \quad x = 0$, is also the root.

Since, in $[0, 4]$, f has only 3 roots.

So, $f(x)$ is the form of $\sin\left(\frac{\pi x}{2}\right)$ fulfilling all conditions given in the question, hence the 3rd root in $[0, 4]$ is 2 as $f(2) = 0$.

$\therefore$ The number of real roots in $(-8, 10]$ is 9 and the roots are $(-6, -4, -2, 0, 2, 4, 6, 8, 10)$.

(B) $(r_1 + r_2 + \ldots + r_n) = 56$

and $\quad r_1 r_2 \ldots r_n = 2009 = 7^2 \cdot 41$

[using theory of equation]

As $\quad 2009 = 7^2 \cdot 41$

Therefore, no value of r_i can be 49 or larger factor of 2009 otherwise their sum is larger than 56.

As $\quad 56 = 41 + 7 + 7 + 1$

Therefore, $\quad r_1 = 41; r_2 = r_3 = 7$

and $\quad r_4 = 1$

Hence, $\quad n = 4$

(C) $2xy = 2009 - 3y$ yields $2xy + 3y = 2009$

$\Rightarrow \quad y = \frac{2009}{2x+3}$

So, $2x + 3$ must be the factors of 2009.

Since, $\quad 2009 = 7^2 \cdot 41$

Thus, $2x + 3$ can be one of the factors 7, 49, 41, 287 and 2009.

or $\quad 2x + 3 = 7; 2x + 3 = 49; 2x + 3 = 41;$
$2x + 3 = 287$

and $\quad 2x + 3 = 2009.$

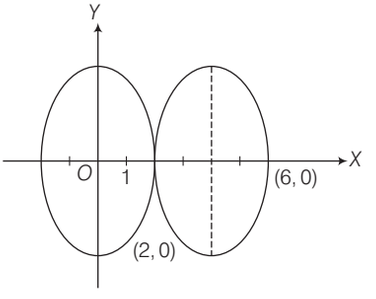
Hence, $x = 2, 23, 19, 142$ and 1003

$\Rightarrow$ Ordered pairs are (2, 287), (23, 41), (19, 49), (142, 7), (1003, 1).

(D) Let $x - 4 = 2\cos\theta$

$\Rightarrow \quad x = 2\cos\theta + 4$

and $\quad y = 3\sin\theta$



Now, $\quad z = \frac{x^2}{4} + \frac{y^2}{9} = \frac{(2\cos\theta + 4)^2}{4} + \sin^2\theta$

$= \frac{4\cos^2\theta + 16 + 16\cos\theta + 4\sin^2\theta}{4} = \frac{20 + 16\cos\theta}{4}$

$z = 5 + 4\cos\theta$

Hence, $\quad z_{\max} - z_{\min} = (9 - 1) = 8$

JEE Type Solved Examples :
Single Integer Answer Type Questions

● **Ex. 26** *Let* $f(x)=\sin^{23}x-\cos^{22}x$ *and*

$$g(x)=1+\frac{1}{2}\tan^{-1}|x|.$$

The number of values of x *in interval* $[-10\pi, 20\pi]$ *satisfying the equation* $f(x)=sgn(g(x))$, *is* $5a$. *Then,* a *is equal to ..... .*

Sol. (3) $g(x)=\frac{1}{2}\tan^{-1}|x|+1$

$\Rightarrow \quad \text{sgn}\,(g(x))=1$

$\Rightarrow \quad \sin^{23}x-\cos^{22}x=1$

$\Rightarrow \quad \sin^{23}x=1+\cos^{22}x$

which is possible, if $\sin x=1$ and $\cos x=0$

$$x=2n\pi+\frac{\pi}{2}$$

Hence, $\quad -10\pi\le 2n\pi+\frac{\pi}{2}\le 20\pi$

$\Rightarrow \quad -\frac{21}{2}\le 2n\le\frac{39}{2}$

$\Rightarrow \quad -\frac{21}{4}\le n\le\frac{39}{4}$

$\Rightarrow \quad -5\le n\le 9$

Hence, number of values of $x=15\Rightarrow a=3$.

● **Ex. 27** *Consider the function* $g(x)$ *defined as*

$g(x)\cdot(x^{(2^{2008}-1)}-1)=(x+1)(x^2+1)(x^4+1)$

$\ldots(x^{2^{2007}}+1)-1$ *the value of* $g(2)$ *equals ..... .*

Sol. (2) RHS $=\dfrac{(x-1)(x+1)(x^2+1)(x^4+1)\ldots(x^{2^{2007}}+1)}{(x-1)}-1$

$$=\frac{(x^2-1)(x^2+1)\ldots(x^{2^{2007}}+1)}{(x-1)}-1$$

$$=\frac{(x^{2^2}-1)(x^{2^2}+1)\ldots(x^{2^{2007}}+1)}{(x-1)}-1$$

Hence, $g(x)\cdot(x^{(2^{2008}-1)}-1)=\dfrac{(x^{2^{2008}}-1)-(x-1)}{(x-1)}$

$$=\frac{x(x^{2^{2008}-1}-1)}{(x-1)}$$

$\Rightarrow \quad g(x)=\dfrac{x}{x-1}$

$\therefore \quad g(2)=2$

● **Ex. 28** *If* $f(x)=\sqrt[3]{\dfrac{9}{\log_2(3-2x)}-1}$, *the value of 'a'*

which satisfies $f^{-1}(2a-4)=\dfrac{1}{2}$, *is*

Sol. (3) Given, $f^{-1}(2a-4)=\frac{1}{2}$

$\therefore \quad f\left(\frac{1}{2}\right)=2a-4$

Put $x=\frac{1}{2}$ in $f(x)=\sqrt[3]{\dfrac{9}{\log_2(3-2x)}-1}$

$\therefore \quad 2a-4=\sqrt[3]{\dfrac{9}{\log_2 2}-1}$

$\Rightarrow \quad 2a-4=\sqrt[3]{9-1}$

$\Rightarrow \quad 2a-4=2$

$\Rightarrow \quad 2a=6$

$\therefore \quad a=3$

● **Ex. 29** *Let* f *be defined on the natural numbers as follow:*

$f(1)=1$ *and for* $n>1$, $f(n)=f[f(n-1)]+f[n-f(n-1)]$,

the value of $\dfrac{1}{30}\sum_{r=1}^{20}f(r)$ *is*

Sol. (7) Here, $\quad f(1)=1$

$f(2)=f[f(1)]+f[2-f(1)]$, using $f(1)=1$

$f(2)=f(1)+f(1)=2.$

$f(3)=f[f(z)]+f[3-f(2)]$

$=f(2)+f(1)=2+1=3$

Thus, $\quad f(n)=n$

$\therefore \quad \dfrac{1}{30}\sum_{r=1}^{20}f(r)=\dfrac{1}{30}[1+2+3+\ldots+20]$

$$=\frac{1}{30}\times\frac{20(20+1)}{2}=7$$

● **Ex. 30** *If* a,b,c *are real roots of the cubic equation* $f(x)=0$ *such that* $(x-1)^2$ *is a factor of* $f(x)+2$ *and* $(x+1)^2$ *is a factor of* $f(x)-2$, *then* $|ab+bc+ca|$ *is equal to*

Sol. (3) Let $f(x)=Ax^3+Bx^2+Cx+D$

Since, $(x-1)^2$ is a factor of $f(x)+2$.

$\Rightarrow \quad f(1)+2=0$

$\Rightarrow \quad A+B+C+D+2=0$ ...(i)

Also, $(x+1)^2$ is a factor of $f(x)-2$.

$\Rightarrow \quad f(-1)-2=0$

$\Rightarrow \quad -A+B-C+D-2=0$ ...(ii)

From Eq. (i), $D=-A-B-C-2$

$\therefore \quad f(x)=Ax^3+Bx^2+Cx-A-B-C-2$

$\Rightarrow \quad f(x)+2=A(x^3-1)+B(x^2-1)+C(x-1)$

Since, $(x-1)^2$ is factor of $f(x)+2$.

$\Rightarrow \quad f'(1)=0 \Rightarrow 3A+2B+C=0$ ...(iii)

Similarly, $(x+1)^2$ is factor of $f(x)-2$.

$\Rightarrow \quad f'(-1)=0 \Rightarrow 3A-2B+C=0$ ...(iv)

From Eqs. (i), (ii), (iii) and (iv), $A=1, C=-3, B=D=0$.

$\therefore \quad f(x)=x^3-3x$

Hence, $|ab+bc+ca|=3$

● ***Ex. 31*** *The minimum value of $k(k\in I)$ for which the equation $e^x=kx^2$ has exactly three real solutions, is*

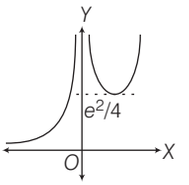
Sol. (2) Here, $e^x=kx^2 \Rightarrow \dfrac{e^x}{x^2}=k$ ...(i)

Let $f(x)=\dfrac{e^x}{x^2}$

$$f'(x)=\frac{(x-2)e^x}{x^3}$$

$\therefore f(x)$ is increasing on $(-\infty,0)\cup[2,\infty)$ and decreasing on $(0,2)$.

$\therefore f(x)$ can be plotted as



So, clearly from graph , $k>\dfrac{e^2}{4}, k\in I$

Hence, $k_{\min}=2$

● ***Ex. 32*** *The value of expression*

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{\ldots\infty}}}}}$$ *is*

Sol. (3) As, we can write

$$n=\sqrt{n^2}=\sqrt{1+(n^2-1)}=\sqrt{1+(n-1)(n+1)}$$

$$=\sqrt{1+(n-1)\cdot\sqrt{(n+1)^2}}.$$

$$=\sqrt{1+(n-1)\cdot\sqrt{1+(n+1)^2-1}}$$

$$=\sqrt{1+(n-1)\cdot\sqrt{1+(n)\cdot(n+2)}}\cdot$$

$$=\sqrt{1+(n-1)\cdot\sqrt{1+n\cdot\sqrt{(n+2)^2}}}$$

$$=\sqrt{1+(n-1)\cdot\sqrt{1+n\cdot\sqrt{1+(n+2)^2-1}}}$$

$$=\sqrt{1+(n-1)\cdot\sqrt{1+n\cdot\sqrt{1+(n+1)(n+3)}}}$$

$$=\sqrt{1+(n-1)\cdot\sqrt{1+n\cdot\sqrt{1+(n+1)\cdot\sqrt{(n+3)^2}}}}$$ ... and so on.

$\therefore \quad \sqrt{1+2\cdot\sqrt{1+3\cdot\sqrt{1+4\cdot\sqrt{1+5\sqrt{.....\infty}}}}}=3.$

Subjective Type Questions

● ***Ex. 33*** *Let a sequence $x_1, x_2, x_3, \ldots$ of complex numbers be defined by $x_1=0$, $x_{n+1}=x_n^2-i$ for all $n>1$, where $i^2=-1$. Find the distance of x_{2000} from x_{1997} in the complex plane.*

Sol. Given, $x_1=0$

Then, $x_2=0^2-i=-i$

$$x_3=(-i)^2-i=-1-i=-(1+i)$$

$$x_4=[-(1+i)]^2-i=2i-i=i$$

$$x_5=(i)^2-i=-1-i=x_3$$

$\therefore \quad x_6=x_4$ and hence $x_7=x_5$ and so on.

$\therefore \quad x_{2n}=i,\ x_{2n+1}=-(1+i)$

$\therefore \quad x_{2000}=i=(0,1)$ and

$x_{1997}=-1-i=(-1,-1)$ in the complex plane.

So, the distance between x_{2000} and x_{1997} is $\sqrt{1+4}=\sqrt{5}$.

● ***Ex. 34*** *If a, b, c, d, e are positive real numbers, such that $a+b+c+d+e=8$ and $a^2+b^2+c^2+d^2+e^2=16$, find the range of e.*

Sol. As we know,

$$\left(\frac{a+b+c+d}{4}\right)^2 \le \frac{a^2+b^2+c^2+d^2}{4}$$ ...(i)

[using Tchebycheff's Inequality]

where $a+b+c+d+e=8$

and $a^2+b^2+c^2+d^2+e^2=16$

From Eq (i), reduces to

$$\left(\frac{8-e}{4}\right)^2 \le \frac{16-e^2}{4}$$

$\Rightarrow \quad 64+e^2-16e \le 4(16-e^2)$

$\Rightarrow \quad 5e^2 - 16e \le 0$

$\Rightarrow \quad e(5e - 16) \le 0$

$\Rightarrow \quad 0 \le e \le \frac{16}{5}$ [using number line rule]

Thus, $e \in \left[0, \frac{16}{5}\right]$.

● **Ex. 35** *Find the set of all solutions of the equation* $2^{|y|} - \left|2^{y-1} - 1\right| = 2^{y-1} + 1$. ***[IIT JEE 1997]***

Sol. Here, $2^{|y|} - \left|2^{y-1} - 1\right| = 2^{y-1} + 1$

We know, how to define modulus, we have three cases as

***Case* I** $y < 0$

$$\Rightarrow \quad 2^{-y} + (2^{y-1} - 1) = 2^{y-1} + 1; \begin{cases} \text{as when } y < 0 \\ |y| = -y \\ \text{and } \left|2^{y-1} - 1\right| = -(2^{y-1} - 1) \end{cases}$$

$$\Rightarrow \quad 2^{-y} = 2^1$$

Hence, $y = -1$, which is true when $y < 0$. ...(i)

***Case* II** $0 \le y < 1$

$$\Rightarrow \quad 2^{y} + (2^{y-1} - 1) = 2^{y-1} + 1; \begin{cases} \text{as when } 0 \le y < 1 \\ |y| = y \\ \text{and } \left|2^{y-1} - 1\right| = -(2^{y-1} - 1) \end{cases}$$

$$\Rightarrow \quad 2^{y} = 2$$

$\Rightarrow \quad y = 1$, which shows no solution as,

$0 \le y < 1$ ...(ii)

***Case* III** $y \ge 1$

$$\Rightarrow \quad 2^{y} - (2^{y-1} - 1) = 2^{y-1} + 1 \qquad \begin{cases} \text{as when } y \ge 0 \\ |y| = y \\ \text{and } \left|2^{y-1} - 1\right| = (2^{y-1} - 1) \end{cases}$$

$$\Rightarrow \quad 2^{y} = 2^{y-1} + 2^{y-1};$$

$$\Rightarrow \quad 2^{y} = 2 \cdot 2^{y-1}$$

$\Rightarrow 2^y = 2^y$, which is an identity, therefore it is true, $\forall\ y \ge 1$. ...(iii)

Hence, from Eqs. (i), (ii) and (iii) the solution set is $\{y : y \ge 1 \cup y = -1\}$.

● **Ex. 36** *Solve the equation* $[x]\{x\} = x$, *where* [ ] *and* { } *denote the greatest integer function and fractional part, respectively.*

Sol. We know that, $x = [x] + \{x\}$ ...(i)

Thus, we have $[x]\{x\} = [x] + \{x\}$

$$\Rightarrow \quad \{x\} = \frac{[x]}{[x] - 1} \qquad \text{...(ii)}$$

Here, in Eq. (ii), $[x] \ne 1$

But, if $[x] = 1$, then given equation

$\{x\} = x$ ...(iii)

which is true only when $x \in [0, 1)$

and $[x] = 1 \Rightarrow x \in [1, 2)$ ...(iv)

From Eqs. (iii) and (iv) no value of x,

when $[x] = 1$ ...(v)

Now, from Eq. (ii), $\{x\} = \frac{[x]}{[x] - 1}$,

when $[x] \ne 1$

Again, as we know $\{x\} \in [0, 1)$

$\therefore \quad 0 \le \frac{[x]}{[x] - 1} < 1$

i.e. $\frac{[x]}{[x] - 1} < 1$ and $\frac{[x]}{[x] - 1} \ge 0$

i.e. $\frac{1}{[x] - 1} < 0$ and $\{[x] \le 0$ or $[x] > 1\}$

i.e. $[x] < 1$ and $\{[x] \le 0$ or $[x] > 1\}$

[using number line rule]

i.e. $[x] \le 0$

$\therefore \quad x = [x] + \{x\} \Rightarrow x = [x] + \frac{[x]}{[x] - 1}$

$\Rightarrow \quad x = \frac{[x]^2}{[x] - 1}$, [where x takes values less than 1]

$\therefore \quad x = \frac{[x]^2}{[x] - 1}$, [where $x < 1$]

$\Rightarrow \quad x = \frac{[x]^2}{[x] - 1}$, [where $[x]$ is any non-positive integer]

● **Ex. 37** *Find all possible values of x satisfying*

$$\frac{[x]}{[x-2]} - \frac{[x-2]}{[x]} = \frac{8\{x\} + 12}{[x-2][x]}$$

(*where* [ ] *denotes the greatest integer function and* { } *is the fractional part*).

Sol. Here, $\frac{[x]}{[x-2]} - \frac{[x-2]}{[x]} = \frac{8\{x\} + 12}{[x-2][x]}$

$$\Rightarrow \quad \frac{[x]^2 - [x-2]^2}{[x-2][x]} = \frac{8\{x\} + 12}{[x][x-2]}$$

$$\Rightarrow \frac{([x] - [x-2])([x] + [x-2])}{[x-2][x]} = \frac{8\{x\} + 12}{[x][x-2]}$$

$\Rightarrow ([x] - [x-2])([x] + [x-2]) = 8\{x\} + 12$

$\Rightarrow ([x] - [x] + 2)([x] + [x] - 2) = 8\{x\} + 12$

$[\because [x + I] = [x] + I]$

$\Rightarrow \quad 4([x] - 1) = 8\{x\} + 12$

$\Rightarrow \quad [x] - 4 = 2\{x\}$ ...(i)

Now, as we know $0 \le \{x\} < 1 \Rightarrow 0 \le 2\{x\} < 2$

$\Rightarrow \quad 0 \le [x]-4 < 2 \;\Rightarrow\; 4 \le [x] < 6$

$\Rightarrow \quad [x] = 4, 5$

$$\left.\begin{array}{c}\text{if } [x]=4 \;\Rightarrow\; 2\{x\}=[x]-4 \\ \text{becomes } \{x\}=0\end{array}\right\} \quad \ldots(\text{ii})$$

$$\left.\begin{array}{c}\text{and if } [x]=5 \Rightarrow 2\{x\}=[x]-4 \\ \text{becomes } \{x\}=\dfrac{1}{2}\end{array}\right\} \quad \ldots(\text{iii})$$

Thus, from Eqs. (ii) and (iii), we have

$$x = [x] + \{x\}$$

i.e. $x = 4 + 0 = 4$ [using Eq. (ii)]

and $x = 5 + \dfrac{1}{2} = \dfrac{11}{2}$ [using Eq. (iii)]

$\Rightarrow \quad x \in \left\{4, \dfrac{11}{2}\right\}$

● ***Ex. 38*** *If $f(x)$ is a polynomial function satisfying $f(x)\cdot f\left(\dfrac{1}{x}\right) = f(x) + f\left(\dfrac{1}{x}\right)$ and $f(4) = 65$. Find $f(6)$.*

Sol. Here, $f(x)\cdot f\left(\dfrac{1}{x}\right) = f(x) + f\left(\dfrac{1}{x}\right)$

$\Rightarrow \quad f(x)\cdot f\left(\dfrac{1}{x}\right) - f(x) = f\left(\dfrac{1}{x}\right)$

$\Rightarrow \quad f(x) = \dfrac{f(1/x)}{f(1/x)-1} \quad \ldots(\text{i})$

Also, $f(x)\cdot f\left(\dfrac{1}{x}\right) = f(x) + f\left(\dfrac{1}{x}\right)$

$\Rightarrow \quad f(x)\cdot f\left(\dfrac{1}{x}\right) - f\left(\dfrac{1}{x}\right) = f(x)$

$\Rightarrow \quad f\left(\dfrac{1}{x}\right) = \dfrac{f(x)}{f(x)-1} \quad \ldots(\text{ii})$

On multiplying Eqs. (i) and (ii), we get

$$f(x)\cdot f\left(\frac{1}{x}\right) = \frac{f(1/x)\cdot f(x)}{\{f(1/x)-1\}\{f(x)-1\}}$$

$\Rightarrow \quad \left[f\left(\dfrac{1}{x}\right)-1\right][f(x)-1] = 1 \quad \ldots(\text{iii})$

Since, $f(x)$ is a polynomial function, so $\{f(x)-1\}$ and $\left\{f\left(\dfrac{1}{x}\right)-1\right\}$ are reciprocals of each other. Also, x and $\dfrac{1}{x}$ are reciprocals of each other.

Thus, Eq. (iii) can hold only when

$f(x) - 1 = \pm x^n$, where $n \in R$

$\therefore \quad f(x) = \pm x^n + 1$, but $f(4) = 65$

$\Rightarrow \quad \pm 4^n + 1 = 65 \;\Rightarrow\; 4^n = 64$

$\Rightarrow \quad 4^n = 4^3 \;\Rightarrow\; n = 3 \quad [\because 4^n > 0]$

So, $f(x) = x^3 + 1$

Hence, $f(6) = 6^3 + 1 = 217$

Aliter Let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n$

Then, $f(x)\cdot f\left(\dfrac{1}{x}\right) = f(x) + f\left(\dfrac{1}{x}\right)$

$$\Rightarrow (a_0x^n + a_1x^{n-1} + \ldots + a_n)\left(\frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + \ldots + a_n\right)$$

$$= (a_0x^n + a_1x^{n-1} + \ldots + a_n) + \left(\frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + \ldots + a_n\right)$$

On comparing the coefficient of x^n, we have

$a_0a_n = a_0 \;\Rightarrow\; a_n = 1$ [as $a_0 \neq 0$]

Comparing the coefficient of x^{n-1}, we have

$a_0a_{n-1} + a_na_1 = a_1$

$\Rightarrow \quad a_0a_{n-1} + a_1 = a_1$ [as $a_n = 1$]

$\Rightarrow \quad a_0a_{n-1} = 0$

$\Rightarrow \quad a_{n-1} = 0$ [as $a_0 \neq 0$]

Similarly, $a_{n-1} = a_{n-2} = \ldots\ldots = a_1 = 0$

and $a_0 = \pm 1$

$\therefore \quad f(x) = \pm x^n + 1$

$\Rightarrow \quad f(4) = \pm 4^n + 1$

$\Rightarrow \quad 4^n + 1 = 65 \;\Rightarrow\; 4^n = 64$

$\Rightarrow \quad n = 3$

So, $f(x) = x^3 + 1$

Hence, $f(6) = 6^3 + 1 = 217$

● ***Ex. 39*** *If $f(x)$ satisfies the relation, $f(x+y) = f(x) + f(y)$ for all $x, y \in R$ and $f(1) = 5$, then find $\sum_{n=1}^{m} f(n)$. Also, prove that $f(x)$ is odd.*

Sol. Here, $f(x+y) = f(x) + f(y)$, put $x = r-1, y = 1$

$f(r) = f(r-1) + f(1)$ ...(i)

$\therefore \quad f(r) = f(r-1) + 5$

$\Rightarrow \quad f(r) = \{f(r-2) + 5\} + 5$ [using definition]

$\Rightarrow \quad f(r) = f(r-2) + 2\cdot(5)$

$\Rightarrow \quad f(r) = f(r-3) + 3\cdot(5)$

.....

.....

$\Rightarrow \quad f(r) = f\{r-(r-1)\} + (r-1)\cdot 5$

$\Rightarrow \quad f(r) = f(1) + (r-1)\cdot 5$

$\Rightarrow \quad f(r) = 5 + (r-1)\cdot 5$

$\Rightarrow \quad f(r) = 5r$

$\therefore \quad \sum_{n=1}^{m} f(n) = \sum_{n=1}^{m} (5n) = 5[1 + 2 + 3 + \ldots + m]$

$$= \frac{5m(m+1)}{2}$$

Hence, $\sum_{n=1}^{m} f(n) = \frac{5m(m+1)}{2}$ ...(ii)

Now, putting $x = 0, y = 0$ in the given function, we have

$f(0+0) = f(0) + f(0)$

$\therefore \quad f(0) = 0$

Also, putting $(-x)$ for (y) in the given function,

$f(x-x) = f(x) + f(-x)$

$\Rightarrow \quad f(0) = f(x) + f(-x)$

$\Rightarrow \quad 0 = f(x) + f(-x)$

$\Rightarrow \quad f(-x) = -f(x)$ ...(iii)

Thus, $\sum_{n=1}^{m} f(n) = \frac{5m(m+1)}{2}$ and $f(x)$ is odd.

● ***Ex. 40*** *Let* $f(x) = \frac{9^x}{9^x + 3}$. *Show that* $f(x) + f(1-x) = 1$ *and hence evaluate* $f\left(\frac{1}{1996}\right) + f\left(\frac{2}{1996}\right) + f\left(\frac{3}{1996}\right) + \ldots + f\left(\frac{1995}{1996}\right)$.

Sol. Given, $f(x) = \frac{9^x}{9^x + 3}$ ...(i)

$\Rightarrow \quad f(1-x) = \frac{9^{1-x}}{9^{1-x} + 3}$

$\Rightarrow \quad f(1-x) = \frac{\frac{9}{9^x}}{\frac{9}{9^x} + 3} = \frac{9}{9 + 3 \cdot 9^x}$

$f(1-x) = \frac{9}{3(3 + 9^x)}$ ...(ii)

On adding Eqs. (i) and (ii), we get

$f(x) + f(1-x) = \frac{9^x}{9^x + 3} + \frac{9}{3(3 + 9^x)}$

$= \frac{3 \cdot 9^x + 9}{3(9^x + 3)} = \frac{3(9^x + 3)}{3(9^x + 3)}$

$\therefore \quad f(x) + f(1-x) = 1$ ...(iii)

Now, putting $x = \frac{1}{1996}, \frac{2}{1996}, \frac{3}{1996}, \ldots, \frac{998}{1996}$ in Eq. (iii), we get

$f\left(\frac{1}{1996}\right) + f\left(\frac{1995}{1996}\right) = 1$

$\Rightarrow \quad f\left(\frac{2}{1996}\right) + f\left(\frac{1994}{1996}\right) = 1$

$\Rightarrow \quad f\left(\frac{3}{1996}\right) + f\left(\frac{1993}{1996}\right) = 1$

.....

.....

$\Rightarrow \quad f\left(\frac{997}{1996}\right) + f\left(\frac{999}{1996}\right) = 1$

$\Rightarrow \quad f\left(\frac{998}{1996}\right) + f\left(\frac{998}{1996}\right) = 1$

or $f\left(\frac{998}{1996}\right) = \frac{1}{2}$

On adding all the above expressions, we get

$f\left(\frac{1}{1996}\right) + f\left(\frac{2}{1996}\right) + \ldots + f\left(\frac{1995}{1996}\right)$

$= (1 + 1 + 1 + \ldots + \text{to 997 times}) + \frac{1}{2}$

$= 997 + \frac{1}{2}$

$= 997.5$

● ***Ex. 41*** *ABCD is a square of side a. A line parallel to the diagonal BD at a distance x from the vertex A cuts the two adjacent sides. Express the area of the segment of the square with A at a vertex, as a function of x.*

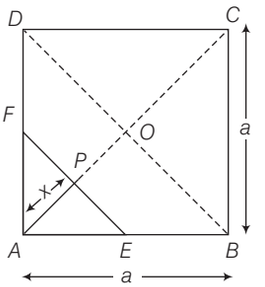
Sol. There are two different situations,

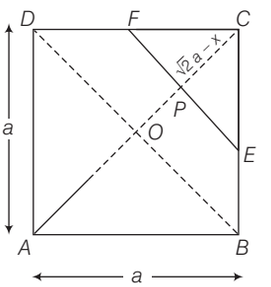
(i) when $x = AP \le OA$,

i.e. $x \le \frac{a}{\sqrt{2}}$

(ii) when $x = AP > OA$,

i.e. $x > \frac{a}{\sqrt{2}}$ but $x \le \sqrt{2}a$





***Case* I** $\text{ar}(\Delta AEF) = \frac{1}{2} x \cdot 2x = x^2$

***Case* II** $\text{ar}(ABEFDA) = \text{ar}(ABCD) - \text{ar}(\Delta CEF)$

$= a^2 - \frac{1}{2}(\sqrt{2}a - x)2(\sqrt{2}a - x)$

$\text{ar}(ABEFDA) = 2\sqrt{2}\,ax - x^2 - a^2$

$\therefore \quad f(x) = \begin{cases} x^2, & 0 \le x \le \frac{a}{\sqrt{2}} \\ 2\sqrt{2}\,ax - x^2 - a^2, & \frac{a}{\sqrt{2}} < x \le \sqrt{2}a \end{cases}$

Ex. 42 *A function $R \to R$ is defined by $f(x) = \dfrac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$. Find the interval of values of α for which f is onto. Is the function one-one for $\alpha = 3$? Justify your answer.* ***[IIT JEE 1996]***

Sol. Since, $f : R \longrightarrow R$ is an onto mapping.

$\therefore$ Range of $f = R$

$\Rightarrow$ $\dfrac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$ assumes all real values of x.

Let $y = \dfrac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$

Then, y assumes all real values for real values of x.

$\Rightarrow \alpha y + 6xy - 8x^2 y = \alpha x^2 + 6x - 8, \ \forall\ y \in R$

$\Rightarrow x^2(\alpha + 8y) + 6x(1 - y) - (8 + \alpha y) = 0, \ \forall\ y \in R$

We know, above equation assumes all real values.

So, $D \geq 0$

$\Rightarrow 36(1 - y)^2 + 4(\alpha + 8y)(8 + \alpha y) \geq 0$

$\Rightarrow 4[9(1 - 2y + y^2) + (8\alpha + \alpha^2 y + 64y + 8\alpha y^2)] \geq 0$

$\Rightarrow [9 - 18y + 9y^2 + 8\alpha + \alpha^2 y + 64y + 8\alpha y^2] \geq 0$

$\Rightarrow [y^2(8\alpha + 9) + y(\alpha^2 + 46) + (8\alpha + 9)] \geq 0$

We know, if $ax^2 + bx + c \geq 0, \forall\ x$ and $a > 0 \ \Rightarrow \ D \leq 0$

So, $(\alpha^2 + 46)^2 - 4(8\alpha + 9)(8\alpha + 9) \leq 0$ and $(8\alpha + 9) > 0$

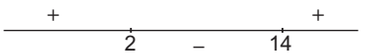
$\Rightarrow (\alpha^2 + 46)^2 - [2(8\alpha + 9)]^2 \leq 0$ and $\alpha > -9/8$

$\Rightarrow (\alpha^2 + 46 - 16\alpha - 18)(\alpha^2 + 46 + 16\alpha + 18) \leq 0$

and $\alpha > -9/8$

$(\alpha^2 - 16\alpha + 28)(\alpha^2 + 16\alpha + 64) \leq 0$ and $\alpha > -9/8$

$\Rightarrow (\alpha - 14)(\alpha - 2)(\alpha + 8)^2 \leq 0$ and $\alpha > -9/8$



$\alpha \in [2, 14] \cup \{-8\}$ and $\alpha > -9/8$

Thus, $\alpha \in [2, 14]$

Hence, f is onto when $\alpha \in [2, 14]$.

Again, if $\alpha = 3$, we have, $f(x) = \dfrac{3x^2 + 6x - 8}{3 + 6x - 8x^2}$.

Clearly, $f(1) = f(-1)$, which shows $f(x)$ is not one-one.

Ex. 43 *Let $f(x, y)$ be a periodic function satisfying the condition $f(x, y) = f((2x + 2y), (2y - 2x)), \forall\ x, y \in R$. Now, define a function g by $g(x, 0) = f(2^x, 0)$. Then, prove that $g(x)$ is a periodic function and find its period.*

Sol. We have $f(x, y) = f(2x + 2y, 2y - 2x)$ ...(i)

$\Rightarrow f(2x + 2y, 2y - 2x) = f\{2(2x + 2y) + 2(2y - 2x), 2(2y - 2x) - 2(2x + 2y)\}$ [using Eq. (i)]

$\Rightarrow f(x, y) = f(2x + 2y, 2y - 2x) = f(8y, -8x)$

$\Rightarrow f(x, y) = f(8y, -8x)$ ...(ii)

$\Rightarrow f(8y, -8x) = f\{8(-8x), -8(8y)\}$ [using Eq. (ii)]

$\Rightarrow f(x, y) = f(2x + 2y, 2y - 2x) = f(8y, -8x)$

$= f(-64x, -64y)$

$\Rightarrow f(x, y) = f(-64x, -64y)$ ...(iii)

$\Rightarrow f(-64x, -64y) = f(64 \times 64x, 64 \times 64y)$

$= f(2^{12} x, 2^{12} y)$

$\Rightarrow f(x, y) = f(2^{12} x, 2^{12} y)$ [using Eq. (iii)]

$\Rightarrow f(2^x, 0) = f(2^{12} \cdot 2^x, 0) = f(2^{12 + x}, 0)$ ...(iv)

Given, $g(x, 0) = f(2^x, 0)$

$\Rightarrow g(x, 0) = f(2^x, 0) = f(2^{12 + x}, 0)$ [using Eq. (iv)]

$\Rightarrow g(x, 0) = g(x + 12, 0)$

Hence, $g(x)$ is periodic with period 12.

Ex. 44 *Solve the equation $10^{(x+1)(3x+4)} - 2 \cdot 10^{(x+1)(x+2)} = 10^{1 - x - x^2}$.*

Sol. Given equation is

$$10^{3x^2 + 7x + 4} - 2 \cdot 10^{x^2 + 3x + 2} = \frac{10}{10^{x + x^2}}$$

$\Rightarrow 10^{4x^2 + 8x + 4} - 2 \cdot 10^{2x^2 + 4x + 2} = 10$

$\Rightarrow 10^{4(x^2 + 2x + 1)} - 2 \cdot 10^{2(x^2 + 2x + 1)} = 10$

$\Rightarrow 10^{4(x + 1)^2} - 2 \cdot 10^{2(x + 1)^2} = 10$

$\Rightarrow \{10^{2(x + 1)^2}\}^2 - 2 \cdot \{10^{2(x + 1)^2}\} = 10$ ...(i)

Let $10^{2(x + 1)^2} = y$ ...(ii)

$\Rightarrow y^2 - 2y = 10 \ \Rightarrow \ y^2 - 2y - 10 = 0$

$\Rightarrow y = \dfrac{2 \pm \sqrt{4 + 40}}{2}$

$\Rightarrow y = 1 \pm \sqrt{11}$

$\Rightarrow y = 1 + \sqrt{11}$ [neglecting –ve sign as $y > 0$]

$\Rightarrow 10^{2(x + 1)^2} = 1 + \sqrt{11}$

$\Rightarrow 2(x + 1)^2 = \log_{10}(1 + \sqrt{11})$

$\Rightarrow (x + 1)^2 = \dfrac{1}{2}\log_{10}(1 + \sqrt{11})$

$\Rightarrow x + 1 = \pm\sqrt{\dfrac{1}{2}\log_{10}(1 + \sqrt{11})}$

Hence, $x = -1 \pm \sqrt{\dfrac{1}{2}\log_{10}(1 + \sqrt{11})}$

Ex. 45 *Find the real solution of $[x] + [5x] + [10x] + [20x] = 36K + 35, K \in$ integer, where $[\cdot]$ denotes the greatest integral function.*

Sol. Let $x = a + r$, where $a \in$ integer and $r \in [0, 1)$. Then, there are four cases

(i) $0 \le r < \frac{1}{20}$ (ii) $\frac{1}{20} \le r < \frac{1}{10}$

(iii) $\frac{1}{10} \le r < \frac{1}{5}$ (iv) $\frac{1}{5} \le r < 1$

***Case* I** If $0 \le r < \frac{1}{20}$.

$\therefore\ [x] + [5x] + [10x] + [20x] = 36K + 35$

$\Rightarrow [a + r] + [5a + 5r] + [10a + 10r] + [20a + 20r] = 36K + 35$

$\Rightarrow a + 5a + 10a + 20a = 36K + 35$

[by definition of the greatest integer function]

$\Rightarrow \quad 36a = 36K + 35,$

which is not possible for any values of a and K as they belongs to integers. ...(i)

***Case* II** If $\frac{1}{20} \le r < \frac{1}{10}$.

$\therefore \quad [x] + [5x] + [10x] + [20x] = 36K + 35$

$\Rightarrow \quad [a + r] + [5a + 5r] + [10a + 10r] + [20a + 20r] = 36K + 35$

$\Rightarrow \quad a + 5a + 10a + 20a + 1 = 36K + 35$

$\Rightarrow \quad 36a + 1 = 36K + 35$

$\Rightarrow \quad 36a = 36K + 34,$

which is again not possible, $\forall\ a, K \in$ integer. ...(ii)

***Case* III** If $\frac{1}{10} \le r < \frac{1}{5}$.

$\therefore\ [x] + [5x] + [10x] + [20x] = 36K + 35$

$\Rightarrow [a + r] + [5a + 5r] + [10a + 10r] + [20a + 20r] = 36K + 35$

$\Rightarrow\ a + 5a + 10a + 1 + 20a + 2 = 36K + 35$

$\Rightarrow \quad 36a + 3 = 36K + 35$

$\Rightarrow \quad 36a = 36K + 32,$

which shows no value of a and K can be given. ...(iii)

***Case* IV** If $\frac{1}{5} \le r < 1$.

$\therefore\ [x] + [5x] + [10x] + [20x] = 36K + 35$

$\Rightarrow [a + r] + [5a + 5r] + [10a + 10r] + [20a + 20r] = 36K + 35$

$\Rightarrow \quad a + 5a + 1 + 10a + 2 + 20a + 4 = 36K + 35$

$\Rightarrow \quad 36a + 7 = 36K + 35$

$\Rightarrow \quad 36a = 36K + 28$

Again, no solution is possible. ...(iv)

Hence, from Eqs. (i), (ii), (iii), (iv) there exists no real value which possesses solution.

● ***Ex. 46*** *Let $f : N \to N$ be a function such that*

(i) $x - f(x) = 19\left[\frac{x}{19}\right] - 90\left[\frac{f(x)}{90}\right], \forall\ x \in N$, *where* $[\cdot]$ *denotes the greatest integer function.*

(ii) $1900 < f(1990) < 2000$.

Find all the possible values of $f(1990)$.

Sol. Since, $1900 < f(1990) < 2000$

$\Rightarrow \quad \frac{1900}{90} < \frac{f(1990)}{90} < \frac{2000}{90} \Rightarrow 21.1 < \frac{f(1990)}{90} < 22.2$

$\therefore \quad \left[\frac{f(1990)}{90}\right] = 21,22$ ...(i)

Given, $x - f(x) = 19\left[\frac{x}{19}\right] - 90\left[\frac{f(x)}{90}\right]$.

Now, there are two cases arise.

***Case* I** $\left[\frac{f(1990)}{90}\right] = 21$

We have, $1990 - f(1990) = 19\left[\frac{1990}{19}\right] - 90\left[\frac{f(1990)}{90}\right]$

$1990 - f(1990) = 19 \cdot (104) - 90 \cdot (21)$

$\Rightarrow \quad f(1990) = 1904$ ... (ii)

***Case* II** $\left[\frac{f(1990)}{90}\right] = 22$

We have, $1990 - f(1990) = 19\left[\frac{1990}{19}\right] - 90\left[\frac{f(1990)}{90}\right]$

$\Rightarrow \quad f(1990) = 1994$ ... (iii)

From Eqs. (ii) and (iii), we have $f(1990) = 1904$ or 1994.

● ***Ex. 47*** *Solve the system of equations;*

$$|x^2 - 2x| + y = 1, x^2 + |y| = 1.$$

Sol. Here, $|x^2 - 2x| + y = 1$ and $x^2 + |y| = 1$ gives four cases as

***Case* I** $x^2 - 2x \ge 0$ and $y \ge 0$ ...(i)

Equations are $x^2 - 2x + y = 1$ and $x^2 + y = 1$

$\Rightarrow \quad 1 - 2x = 1 \Rightarrow x = 0$ and $y = 1$ ...(ii)

From Eq. (i), $x^2 - 2x \ge 0 \Rightarrow x \le 0 \quad x \ge 2$ and $y \ge 0$

Thus, $x = 0$ and $y = 1$ is the only solution when

$x^2 - 2x \ge 0$ and $y \ge 0$

***Case* II** $x^2 - 2x \ge 0$ and $y < 0$

$\Rightarrow \quad (x \le 0$ or $x \ge 2)$ and $(y < 0)$ ...(iii)

$\therefore$ Equations are $x^2 - 2x + y = 1$ and $x^2 - y = 1$.

Adding these equations, we get

$2x^2 - 2x = 2$

or $x = \frac{1 \pm \sqrt{5}}{2}$ and $y = \frac{1 \pm \sqrt{5}}{2}$ ...(iv)

From Eqs. (iii) and (iv), $x = \frac{1 - \sqrt{5}}{2}$ and $y = \frac{1 - \sqrt{5}}{2}$

***Case* III** $x^2 - 2x \le 0$ and $y \ge 0$.

$\Rightarrow \quad (0 \le x \le 2)$ and $(y \ge 0)$ ...(v)

Equations are $-x^2 + 2x + y = 1$ and

$x^2 + y = 1$, subtracting, we get

$-2x^2 + 2x = 0 \Rightarrow x = 0, 1$

i.e. $x = 0, y = 1$ or $x = 1, y = 0$ ...(vi)

which satisfy Eq. (v).

***Case* IV** $x^2 - 2x \le 0$ and $y \le 0$.

$\Rightarrow \quad (0 \le x \le 2)$ and $(y \le 0)$ ...(vii)

$\therefore$ Equations are $-x^2 + 2x + y = 1$ and $x^2 - y = 1$,

adding, we get $2x = 2 \Rightarrow x = 1$

and $y = 0$ which satisfies Eq. (vii).

Thus, solutions are

$(x = 0, y = 1)(x = 1, y = 0)$ and $\left(x = y = \frac{1-\sqrt{5}}{2}\right)$.

● ***Ex. 48*** *Let f and g be real-valued functions such that $f(x+y) + f(x-y) = 2f(x) \cdot g(y), \forall\ x, y \in R$. Prove that, if $f(x)$ is not identically zero and $|f(x)| \le 1, \forall\ x \in R$, then $|g(y)| \le 1, \forall\ y \in R$.*

Sol. Let maximum value of $f(x)$ be M.

$\Rightarrow \quad \max |f(x)| = M$, where $0 < M \le 1$ ...(i)

[since, f is not identically zero and $|f(x)| \le 1, \forall\ x \in R$]

Now, $f(x+y) + f(x-y) = 2f(x) \cdot g(y)$

$\Rightarrow \quad |2f(x)| \cdot |g(y)| = |f(x+y) + f(x-y)|$

$\Rightarrow \quad 2|f(x)| \cdot |g(y)| \le |f(x+y)| + |f(x-y)|$

[as $|a+b| \le |a| + |b|$]

$\Rightarrow \quad 2|f(x)| \cdot |g(y)| \le M + M$

[using Eq. (i), i.e. $\max |f(x)| = M$]

$\Rightarrow \quad |g(y)| \le 1, \forall\ y \in R$

Thus, if $f(x)$ is not identically zero and $|f(x)| \le 1, \forall\ x \in R$, then $|g(y)| \le 1, \forall\ y \in R$.

● ***Ex. 49*** *If p and q are positive integers, f is a function defined for positive numbers and attains only positive values such that $f(x\ f(y)) = x^p y^q$, then prove that $q = p^2$.*

Sol. For $x = \frac{1}{f(y)}$, we have

$$f\left(x \cdot \frac{1}{x}\right) = \frac{1}{f(y)^p} \cdot y^q$$

$\Rightarrow \quad f(1) = \frac{y^q}{\{f(y)\}^p}$

$\Rightarrow \quad f(y) = \frac{y^{q/p}}{\{f(1)\}^{1/p}}$

For $y = 1$, we have $f(1) = 1$

$\therefore \quad f(y) = y^{q/p}$

or $\quad f(x) = x^{q/p}$ ...(i)

Hence, $\quad f(x \cdot y^{q/p}) = x^p \cdot y^q$

Let $\quad y^{q/p} = z$

$\Rightarrow \quad y = z^{p/q}$

$\Rightarrow \quad f(x \cdot z) = x^p \cdot z^p$

or $\quad f(x) = x^p$ ...(ii)

From Eqs. (i) and (ii), we have $x^{q/p} = x^p$

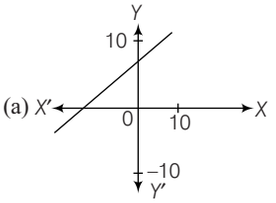
$\Rightarrow \quad \frac{q}{p} = p$ or $q = p^2$

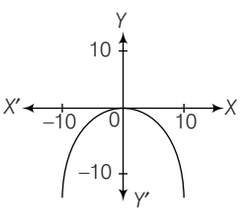
Functions Exercise 1 :
Single Option Correct Type Questions

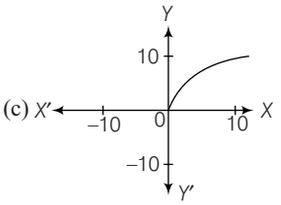
1. $f_1(x) = \begin{cases} x, \text{ for } 0 \le x \le 1 \\ 1, \text{ for } x > 1 \\ 0, \text{ otherwise} \end{cases}$ $f_2(x) = f_1(-x)$, for all x and $f_3(x) = -f_2(x)$, for all x. $f_4(x) = f_3(-x)$, for all x

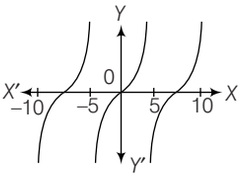
Which of the following is necessarily true?
(a) $f_4(x) = f_1(x)$, for all x (b) $f_1(x) = -f_3(-x)$, for all x
(c) $f_2(-x) = f_4(x)$, for all x (d) $f_1(x) + f_3(x) = 0$, for all x

2. Which one of the following is an odd function?

(a) 

(b) 

(c) 

(d) 

3. If $f(x) = \sqrt{\frac{8}{1-x} + \frac{8}{1+x}}$ and $g(x) = \frac{4}{f(\sin x)} + \frac{4}{f(\cos x)}$, the $g(x)$ is periodic with period
(a) $\frac{\pi}{2}$ (b) π (c) $\frac{3\pi}{2}$ (d) 2π

4. Let f be a function defined by $f(xy) = \frac{f(x)}{y}$ for all positive real numbers x and y. If $f(30) = 20$, the value of $f(40)$ is
(a) 15 (b) 20 (c) 40 (d) 60

5. Let $f(x) = e^{\{e^{|x|} \text{ sgn } x\}}$ and $g(x) = e^{[e^{|x|} \text{ sgn } x]}$, $x \in R$, where $\{x\}$ and $[x]$ denote fractional part and greatest integer function, respectively. Also, $h(x) = \log(f(x)) + \log(g(x))$, then for all real x, $h(x)$ is
(a) an odd function
(b) an even function
(c) neither odd nor even function
(d) both odd as well as even function

6. Which of the following function is surjective but not injective?
(a) $f : R \to R$, $f(x) = x^4 + 2x^3 - x^2 + 1$
(b) $f : R \to R$, $f(x) = x^3 + x + 1$
(c) $f : R \to R^+$, $f(x) = \sqrt{1 + x^2}$
(d) $f : R \to R$, $f(x) = x^3 + 2x^2 - x + 1$

7. If $f(x) = 2x^3 + 7x - 9$, then $f^{-1}(4)$ is
(a) 1 (b) 2 (c) 1/3 (d) non-existent

8. The range of the function

$$f(x) = \frac{e^x \cdot \log x \cdot 5^{x^2+2} \cdot (x^2 - 7x + 10)}{2x^2 - 11x + 12} \text{ is}$$

(a) $(-\infty, \infty)$ (b) $[0, \infty)$ (c) $\left(\frac{3}{2}, \infty\right)$ (d) $\left(\frac{3}{2}, 4\right)$

9. If $x = \cos^{-1}(\cos 4)$ and $y = \sin^{-1}(\sin 3)$, then which of the following holds?
(a) $x - y = 1$ (b) $x + y + 1 = 0$
(c) $x + 2y = 2$ (d) $x + y = 3\pi - 7$

10. Let $f(x) = \left(\frac{2\sin x + \sin 2x}{2\cos x + \sin 2x} \cdot \frac{1 - \cos x}{1 - \sin x}\right)^{2/3}$; $x \in R$.

Consider the following statements.
I. Domain of f is R.
II. Range of f is R.
III. Domain of f is $R - (4n-1)\frac{\pi}{2}$, $n \in I$.
IV. Domain of f is $R - (4n+1)\frac{\pi}{2}$, $n \in I$.

Which one of the following is correct?
(a) I and II (b) II and III
(c) III and IV (d) II, III and IV

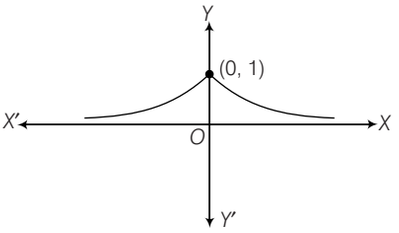
11. If $f(x) = e^{\sin(x - [x])\cos \pi x}$, where $[x]$ denotes the greatest integer function, then $f(x)$ is
(a) non-periodic
(b) periodic with no fundamental period
(c) periodic with period 2
(d) periodic with period π

12. The range of the function, $f(x) = \cot^{-1}(\log_{0.5}(x^4 - 2x^2 + 3))$ is
(a) $(0, \pi)$ (b) $\left(0, \frac{3\pi}{4}\right]$ (c) $\left[\frac{3\pi}{4}, \pi\right)$ (d) $\left[\frac{\pi}{2}, \frac{3\pi}{4}\right]$

13. Range of $f(x) = \left[\frac{1}{\log(x^2 + e)}\right] + \frac{1}{\sqrt{1 + x^2}}$, where $[\cdot]$ denotes greatest integer function, is
(a) $\left(0, \frac{e+1}{e}\right) \cup \{2\}$ (b) $(0, 1)$
(c) $(0, 1] \cup \{2\}$ (d) $[0, 1) \cup \{2\}$

14. The period of the function $f(x) = \sin(x + 3 - [x + 3])$, where $[\cdot]$ denotes greatest integer function, is
(a) $2\pi + 3$ (b) 2π (c) 1 (d) 4

15. Which one of the following functions best represent the graph as shown below?



(a) $f(x)=\dfrac{1}{1+x^2}$ (b) $f(x)=\dfrac{1}{\sqrt{1+|x|}}$

(c) $f(x)=e^{-|x|}$ (d) $f(x)=a^{|x|}, a>1$

16. The solution set for $[x]\{x\}=1$, where $\{x\}$ and $[x]$ denote fractional part and greatest integer functions, is

(a) $R^+-(0,1)$ (b) $R^+-\{1\}$

(c) $\left\{m+\dfrac{1}{m}; m\in I-\{0\}\right\}$ (d) $\left\{m+\dfrac{1}{m}; m\in N-\{1\}\right\}$

17. The domain of definition of function $f(x)=\log(\sqrt{x^2-5x-24}-x-2)$, is

(a) $(-\infty,-3]$ (b) $(-\infty-3]\cup[8,\infty)$

(c) $\left(-\infty,\dfrac{-28}{9}\right)$ (d) None of these

18. If $f(x)$ is a function $f:R\to R$, we say $f(x)$ has property

I. If $f(f(x))=x$ for all real numbers x.

II. If $f(-f(x))=-x$ for all real numbers x.

How many linear functions, have both property I and II?

(a) 0 (b) 2

(c) 3 (d) Infinite

19. Let $f(x)=\dfrac{x}{1+x}$ and $g(x)=\dfrac{rx}{1-x}$. Let S be the set of all real numbers r, such that $f(g(x))=g(f(x))$ for infinitely many real numbers x. The number of elements in set S is

(a) 1 (b) 2

(c) 3 (d) 5

20. Let $f(x)$ be linear functions with the properties that $f(1)\le f(2)$, $f(3)\ge f(4)$ and $f(5)=5$. Which one of the following statements is true?

(a) $f(0)<0$ (b) $f(0)=0$

(c) $f(1)<f(0)<f(-1)$ (d) $f(0)=5$

21. Suppose R is relation whose graph is symmetric to both X-axis and Y-axis and that the point $(1,2)$ is on the graph of R. Which one of the following is not necessarily on the graph of R?

(a) $(-1,2)$ (b) $(1,-2)$

(c) $(-1,-2)$ (d) $(2,1)$

22. The area between the curve $2\{y\}=[x]+1, 0\le y<1$, where $\{\cdot\}$ and $[\cdot]$ are the fractional part and greatest integer functions, respectively and the X-axis, is

(a) $\dfrac{1}{2}$ (b) 1 (c) 0 (d) $\dfrac{3}{2}$

23. If $f(x)=\sin^{-1}x$ and $g(x)=[\sin(\cos x)]+[\cos(\sin x)]$, then range of $f(g(x))$ is (where, $[\cdot]$ denotes greatest integer function)

(a) $\left\{\dfrac{-\pi}{2},\dfrac{\pi}{2}\right\}$ (b) $\left\{\dfrac{-\pi}{2},0\right\}$

(c) $\left\{0,\dfrac{\pi}{2}\right\}$ (d) $\left\{-\dfrac{\pi}{2},0,\dfrac{\pi}{2}\right\}$

24. The number of solutions of the equation $e^{2x}+e^x-2=[\{x^2+10x+11\}]$ is

(where, $\{x\}$ denotes fractional part of x and $[x]$ denotes greatest integer function)

(a) 0 (b) 1 (c) 2 (d) 3

25. Total number of values of x, of the form $\dfrac{1}{n}, n\in N$ in the interval $x\in\left[\dfrac{1}{25},\dfrac{1}{10}\right]$, which satisfy the equation $\{x\}+\{2x\}+\ldots+\{12x\}=78x$ (where, $\{\cdot\}$ represents fractional part function), is

(a) 12 (b) 13

(c) 14 (d) 15

26. The sum of the maximum and minimum values of the function $f(x)=\dfrac{1}{1+(2\cos x-4\sin x)^2}$ is

(a) $\dfrac{22}{21}$ (b) $\dfrac{21}{20}$ (c) $\dfrac{22}{20}$ (d) $\dfrac{21}{11}$

27. Let $f:X\to Y$, $f(x)=\sin x+\cos x+2\sqrt{2}$ be invertible, then $X\to Y$ is/are

(a) $\left[\dfrac{\pi}{4},\dfrac{5\pi}{4}\right]\to[\sqrt{2},3\sqrt{2}]$

(b) $\left[-\dfrac{\pi}{4},\dfrac{3\pi}{4}\right]\to[\sqrt{2},3\sqrt{2}]$

(c) $\left[-\dfrac{3\pi}{4},\dfrac{3\pi}{4}\right]\to[\sqrt{2},-3\sqrt{2}]$

(d) $\left[-\dfrac{3\pi}{4},-\dfrac{\pi}{4}\right]\to[\sqrt{2},3\sqrt{2}]$

28. The range of values of a, so that all the roots of the equation $2x^3-3x^2-12x+a=0$ are real and distinct, belongs to

(a) (7, 20) (b) (−7, 20)

(c) (−20, 7) (d) (−7, 7)

29. If $f(x)$ is continuous such that $|f(x)| \le 1, \forall\ x \in R$ and $g(x) = \dfrac{e^{f(x)} - e^{|f(x)|}}{e^{f(x)} + e^{|f(x)|}}$, then range of $g(x)$ is

(a) $[0, 1]$ (b) $\left[0, \dfrac{e^2+1}{e^2-1}\right]$

(c) $\left[0, \dfrac{e^2-1}{e^2+1}\right]$ (d) $\left[\dfrac{1-e^2}{1+e^2}, 0\right]$

30. Let $f(x) = \sqrt{|x| - \{x\}}$, where $\{\}$ denotes the fractional part of x and X, Y and its domain and range respectively, then

(a) $f : X \to Y : y = f(x)$ is one-one function

(b) $X \in \left(-\infty, -\dfrac{1}{2}\right] \cup [0, \infty)$ and $Y \in \left[\dfrac{1}{2}, \infty\right)$

(c) $X \in \left(-\infty, -\dfrac{1}{2}\right] \cup [0, \infty)$ and $Y \in [0, \infty)$

(d) None of the above

31. If the graphs of the functions $y = \ln x$ and $y = ax$ intersect at exactly two points, then a must be

(a) $(0, e)$ (b) $\left(\dfrac{1}{e}, 0\right)$

(c) $\left(0, \dfrac{1}{e}\right)$ (d) None of these

32. A quadratic polynomial maps from $[-2, 3]$ onto $[0, 3]$ and touches X-axis at $x = 3$, then the polynomial is

(a) $\dfrac{3}{16}(x^2 - 6x + 16)$ (b) $\dfrac{3}{25}(x^2 - 6x + 9)$

(c) $\dfrac{3}{25}(x^2 - 6x + 16)$ (d) $\dfrac{3}{16}(x^2 - 6x + 9)$

33. The range of the function $y = \sqrt{2\{x\} - \{x\}^2 - \dfrac{3}{4}}$ (where, $\{\cdot\}$ denotes the fractional part) is

(a) $\left[-\dfrac{1}{4}, \dfrac{1}{4}\right]$ (b) $\left[0, \dfrac{1}{2}\right)$

(c) $\left[0, \dfrac{1}{4}\right]$ (d) $\left[\dfrac{1}{4}, \dfrac{1}{2}\right]$

34. Let $f(x)$ be a fourth differentiable function such that $f(2x^2 - 1) = 2xf(x), \forall\ x \in R$, then $f^{iv}(0)$ is equal to (where, $f^{iv}(0)$ represents fourth derivative of $f(x)$ at $x = 0$)

(a) 0 (b) 1

(c) -1 (d) Data insufficient

35. Number of solutions of the equation $[y + [y]] = 2 \cos x$ is (where, $y = \dfrac{1}{3}[\sin x + [\sin x + [\sin x]]]$ and $[\cdot]$ denotes the greatest integer function)

(a) 1 (b) 2

(c) 3 (d) None of these

36. If a function satisfies $f(x+1) + f(x-1) = \sqrt{2} f(x)$, then period of $f(x)$ can be

(a) 2 (b) 4 (c) 6 (d) 8

37. If x and α are real, then the inequation $\log_2 x + \log_x 2 + 2 \cos \alpha \le 0$

(a) has no solution

(b) has exactly two solutions

(c) is satisfied for any real α and any real x in $(0, 1)$

(d) is satisfied for any real α and any real x in $(1, \infty)$

38. The range of values of 'a' such that $\left(\dfrac{1}{2}\right)^{|x|} = x^2 - a$ is satisfied for maximum number of values of 'x'

(a) $(-\infty, -1)$ (b) $(-\infty, \infty)$ (c) $(-1, 1)$ (d) $(-1, \infty)$

39. Let $f : R \to R$ be a function defined by $f(x) = \{|\cos x|\}$, where $\{x\}$ represents fractional part of x. Let S be the set containing all real values x lying in the interval $[0, 2\pi]$ for which $f(x) \ne |\cos x|$. The number of elements in the set S is

(a) 0 (b) 1 (c) 3 (d) infinite

40. The domain of the function $f(x) = \sqrt{\log_{\sin x + \cos x}(|\cos x| + \cos x)}$, $0 \le x \le \pi$ is

(a) $(0, \pi)$ (b) $\left(0, \dfrac{\pi}{2}\right)$

(c) $\left(0, \dfrac{\pi}{3}\right)$ (d) None of these

41. If $f(x) = (x^2 + 2\alpha x + \alpha^2 - 1)^{1/4}$ has its domain and range such that their union is set of real numbers, then α satisfies

(a) $-1 < \alpha < 1$ (b) $\alpha \le -1$

(c) $\alpha \ge 1$ (d) $\alpha \le 1$

42. Let $f : (e, \infty) \to R$ be a function defined by $f(x) = \log(\log(\log x))$, the base of the logarithm being e. Then,

(a) f is one-one and onto

(b) f is one-one but not onto

(c) f is onto but not one-one

(d) the range of f is equal to its domain

43. The expression $x^2 - 4px + q^2 > 0$ for all real x and also $r^2 + p^2 < qr$, the range of $f(x) = \dfrac{x + r}{x^2 + qx + p^2}$ is

(a) $\left[\dfrac{p}{2r}, \dfrac{q}{2r}\right]$ (b) $(0, \infty)$

(c) $(-\infty, 0)$ (d) $(-\infty, \infty)$

44. Let $f(x) = \dfrac{x^4 - \lambda x^3 - 3x^2 + 3\lambda x}{x - \lambda}$. If range of $f(x)$ is the set of entire real numbers, the true set in which λ lies is

(a) $[-2,2]$ (b) $[0,4]$

(c) $(1,3)$ (d) None of these

45. Let $a = 3^{1/224} + 1$ and for all $n \geq 3$,

let $f(n) = {}^{n}C_0\, a^{n-1} - {}^{n}C_1 a^{n-2} + {}^{n}C_2 a^{n-3} + \ldots + (-1)^{n-1} \cdot {}^{n}C_{n-1} \cdot a^0$.

If the value of $f(2016) + f(2017) = 3^K$, the value of K is

(a) 6 (b) 8
(c) 9 (d) 10

46. The area bounded by $f(x) = \sin^{-1}(\sin x)$ and

$g(x) = \dfrac{\pi}{2} - \sqrt{\dfrac{\pi^2}{2} - \left(x - \dfrac{\pi}{2}\right)^2}$ is

(a) $\dfrac{\pi^3}{8}$ sq units (b) $\dfrac{\pi^2}{8}$ sq units
(c) $\dfrac{\pi^3}{2}$ sq units (d) $\dfrac{\pi^2}{2}$ sq units

47. If $f : R \to R$, $f(x) = \dfrac{x^2 + bx + 1}{x^2 + 2x + b}, (b > 1)$ and $f(x), \dfrac{1}{f(x)}$ have the same bounded set as their range, the value of b is

(a) $2\sqrt{3} - 2$ (b) $2\sqrt{3} + 2$
(c) $2\sqrt{2} - 2$ (d) $2\sqrt{2} + 2$

48. The period of $\sin \dfrac{\pi [x]}{12} + \cos \dfrac{\pi [x]}{4} + \tan \dfrac{\pi [x]}{3}$, where $[x]$ represents the greatest integer less than or equal to x is

(a) 12 (b) 4
(c) 3 (d) 24

49. If $f(2x + 3y, 2x - 7y) = 20x$, then $f(x, y)$ equals

(a) $7x - 3y$ (b) $7x + 3y$
(c) $3x - 7y$ (d) $x - y$

50. The range of the function $f(x) = \sqrt{x - 1} + 2\sqrt{3 - x}$ is

(a) $[\sqrt{2}, 2\sqrt{2}]$ (b) $[\sqrt{2}, \sqrt{10}]$
(c) $[2\sqrt{2}, \sqrt{10}]$ (d) $[1, 3]$

51. The domain of the function

$f(x) = \cos^{-1}(\sec(\cos^{-1} x)) + \sin^{-1}(\operatorname{cosec}(\sin^{-1} x))$ is

(a) $x \in R$ (b) $x = 1, -1$
(c) $-1 \leq x \leq 1$ (d) $x \in \phi$

52. Let $f(x)$ be a polynomial one-one function such that $f(x)f(y) + 2 = f(x) + f(y) + f(xy), \forall\ x, y \in R - \{0\}$, $f(1) \neq 1, f'(1) = 3$.

Let $g(x) = \dfrac{x}{4}(f(x) + 3) - \int_0^x f(x)\, dx$, then

(a) $g(x) = 0$ has exactly one root for $x \in (0, 1)$
(b) $g(x) = 0$ has exactly two roots for $x \in (0, 1)$
(c) $g(x) \neq 0, \ \forall\ x \in R - \{0\}$
(d) $g(x) = 0, \ \forall\ x \in R - \{0\}$

53. Let $f(x)$ be a polynomial with real coefficients such that $f(x) = f'(x) \times f'''(x)$. If $f(x) = 0$ is satisfied $x = 1, 2, 3$ only, then the value of $f'(1)\, f'(2)\, f'(3)$ is

(a) positive (b) negative
(c) 0 (d) Inadequate data

54. Let $A = \{1, 2, 3, 4, 5\}$ and $f : A \to A$ be an into function such that $f(i) \neq i, \forall\ i \in A$, then number of such functions f are

(a) 1024 (b) 904
(c) 980 (d) None of these

55. If functions $f : \{1, 2, \ldots, n\} \to \{1995, 1996\}$ satisfying $f(1) + f(2) + \ldots + f(1996) =$ odd integer are formed, the number of such functions can be

(a) 2^n (b) $2^{n/2}$ (c) n^2 (d) 2^{n-1}

56. The range of $y = \sin^3 x - 6\sin^2 x + 11 \sin x - 6$ is

(a) $[-24, 2]$ (b) $[-24, 0]$
(c) $[0, 24]$ (d) None of these

57. Let $f(x) = x^2 - 2x$ and $g(x) = f(f(x) - 1) + f(5 - f(x))$, then

(a) $g(x) < 0, \forall x \in R$
(b) $g(x) < 0$, for some $x \in R$
(c) $g(x) \geq 0$, for some $x \in R$
(d) $g(x) \geq 0, \forall x \in R$

58. If $f(x)$ and $g(x)$ are non-periodic functions, then $h(x) = f(g(x))$ is

(a) non-periodic
(b) periodic
(c) may be periodic
(d) always periodic, if domain of $h(x)$ is a proper subset of real numbers

59. If $f(x)$ is a real-valued function discontinuous at all integral points lying in $[0, n]$ and if $(f(x))^2 = 1$, $\forall\ x \in [0, n]$, then number of functions $f(x)$ are

(a) 2^{n+1} (b) 6×3^n (c) $2 \times 3^{n-1}$ (d) 3^{n+1}

60. A function f from integers to integers is defined as

$$f(x) = \begin{cases} n + 3, & n \in \text{odd} \\ n/2, & n \in \text{even} \end{cases}$$

Suppose $k \in$ odd and $f(f(f(k))) = 27$, then the sum of digits of k is

(a) 3 (b) 6 (c) 9 (d) 12

61. If $f : R \to R$ and $f(x) = \dfrac{\sin(\pi\{x\})}{x^4 + 3x^2 + 7}$, where $\{\}$ is a fractional part of x, then

(a) f is injective
(b) f is not one-one and non-constant
(c) f is a surjective
(d) f is a zero function

62. Let $f: R \to R$ and $g: R \to R$ be two one-one and onto functions, such that they are the mirror images of each other about the line $y = a$. If $h(x) = f(x) + g(x)$, then $h(x)$ is
(a) one-one and onto
(b) only one-one and not onto
(c) only onto but not one-one
(d) None of the above

63. Domain of the function $f(x)$, if $3^x + 3^{f(x)}$ = minimum of $\phi(t)$, where $\phi(t) = \min\{2t^3 - 15t^2 + 36t - 25, 2 + |\sin t|\}$ is
(a) $(-\infty, 1)$
(b) $(-\infty, \log_3 e)$
(c) $(0, \log_3 2)$
(d) $(-\infty, \log_3 2)$

64. Let x be the elements of the set $A = \{1,2,3,4,5,6,8,10,12,15,20,24,30,40,60,120\}$ and x_1, x_2, x_3 be positive integers and d be the number of integral solutions of $x_1 x_2 x_3 = x$, then d is
(a) 100 (b) 150
(c) 320 (d) 250

65. If $A > 0, c, d, u, v$ are non-zero constants and the graph of $f(x) = |Ax + c| + d$ and $g(x) = -|Ax + u| + v$ intersect exactly at two points $(1, 4)$ and $(3, 1)$, then the value of $\frac{u+c}{A}$ equals
(a) 4 (b) -4
(c) 2 (d) -2

66. If $f(x) = x^3 + 3x^2 + 4x + a\sin x + b\cos x, \forall x \in R$ is a one-one function, then the greatest value of $(a^2 + b^2)$ is
(a) 1 (b) 2
(c) $\sqrt{2}$ (d) None of these

67. If two roots of the equation $(p-1)(x^2 + x + 1)^2 - (p+1)(x^4 + x^2 + 1) = 0$ are real and distinct and $f(x) = \frac{1-x}{1+x}$, then $f(f(x)) + f\left(f\left(\frac{1}{x}\right)\right)$ is equal to
(a) p (b) $-p$
(c) $2p$ (d) $-2p$

68. Let $f(x) = x^{13} + 2x^{12} + 3x^{11} + \ldots + 13x + 14$ and $\alpha = \cos\frac{2\pi}{15} + i\sin\frac{2\pi}{15}$. If $N = f(\alpha) f(\alpha^2) \ldots f(\alpha^{14})$, then
(a) number of divisors of N is 144
(b) number of divisors of N is 196
(c) number of divisors of N which are perfect squares of 49
(d) number of divisors of N which are perfect square of 12

69. The sum of the maximum and minimum values of function $f(x) = \sin^{-1} 2x + \cos^{-1} 2x + \sec^{-1} 2x$ is
(a) π (b) $\frac{\pi}{2}$
(c) 2π (d) $\frac{3\pi}{2}$

70. The complete set of values of 'a' for which the function $f(x) = \tan^{-1}(x^2 - 18x + a) > 0, \forall\ x \in R$, is
(a) $(81, \infty)$ (b) $[81, \infty)$
(c) $(-\infty, 81)$ (d) $(-\infty, 81]$

71. The domain of the function $f(x) = \sin^{-1}\frac{1}{|x^2 - 1|} + \frac{1}{\sqrt{\sin^2 x + \sin x + 1}}$ is
(a) $(-\infty, \infty)$
(b) $(-\infty, -\sqrt{2}] \cup [\sqrt{2}, \infty)$
(c) $(-\infty, -\sqrt{2}] \cup [\sqrt{2}, \infty) \cup \{0\}$
(d) None of the above

72. The domain of $f(x) = \frac{\log(\sin^{-1}\sqrt{x^2 + x + 1})}{\log(x^2 - x + 1)}$ is
(a) $(-1, 1)$ (b) $(-1, 0) \cup (0, 1)$
(c) $(-1, 0) \cup \{1\}$ (d) None of these

73. The domain of $f(x) = \sqrt{\sin^{-1}(3x - 4x^3)} + \sqrt{\cos^{-1} x}$ is equal to
(a) $\left[-1, -\frac{\sqrt{3}}{2}\right] \cup \left[0, \frac{\sqrt{3}}{2}\right]$ (b) $\left[-1, -\frac{1}{2}\right] \cup \left[0, \frac{1}{2}\right]$
(c) $\left[0, \frac{1}{2}\right]$ (d) None of these

74. The domain of the function $f(x) = \sqrt[6]{4^x + 8^{2/3(x-2)} - 52 - 2^{2(x-1)}}$ is
(a) (0, 1) (b) [3, ∞)
(c) [1, 0) (d) None of these

75. The domain of derivative of the function $f(x) = |\sin^{-1}(2x^2 - 1)|$ is
(a) $(-1, 1)$ (b) $(-1, 1) \sim \left\{0, \pm\frac{1}{\sqrt{2}}\right\}$
(c) $(-1, 1) \sim \{0\}$ (d) $(-1, 1) \sim \left\{\pm\frac{1}{\sqrt{2}}\right\}$

76. The range of a function $f(x) = \tan^{-1}\{\log_{5/4}(5x^2 - 8x + 4)\}$ is
(a) $\left(\frac{-\pi}{4}, \frac{\pi}{2}\right)$ (b) $\left[\frac{-\pi}{4}, \frac{\pi}{2}\right)$
(c) $\left(\frac{-\pi}{4}, \frac{\pi}{2}\right]$ (d) $\left[\frac{-\pi}{4}, \frac{\pi}{2}\right]$

Functions Exercise 2 :
More Than One Option Correct Type Questions

77. Which of the following function(s) is/are transcendal?

(a) $f(x) = 5\sin(\sqrt{x})$ (b) $f(x) = \dfrac{2\sin 3x}{x^2 + 2x - 1}$

(c) $f(x) = \sqrt{x^2 + 2x + 1}$ (d) $f(x) = (x^2 + 3)\cdot 2^x$

78. Let $f(x) = \dfrac{\sqrt{x - 2\sqrt{x-1}}}{\sqrt{x-1} - 1}\cdot x$, then

(a) domain of $f(x)$ is $x \geq 1$ (b) domain of $f(x)$ is $[1, \infty) - \{2\}$

(c) $f'(10) = 1$ (d) $f'\left(\dfrac{3}{2}\right) = -1$

79. $f(x) = \cos^2 x + \cos^2\left(\dfrac{\pi}{3} + x\right) - \cos x \cdot \cos\left(x + \dfrac{\pi}{3}\right)$ is

(a) an odd function (b) an even function

(c) a periodic function (d) $f(0) = f(1)$

80. If the following functions are defined from $[-1, 1]$ to $[-1, 1]$, identify these which are into.

(a) $\sin(\sin^{-1} x)$ (b) $\dfrac{2}{\pi}\cdot\sin^{-1}(\sin x)$

(c) $\text{sgn}(x)\cdot\log(e^x)$ (d) $x^3\text{sgn}(x)$

81. Let $f(x) = \begin{cases} x^2 - 4x + 3, & x < 3 \\ x - 4, & x \geq 3 \end{cases}$

and $g(x) = \begin{cases} x - 3, & x < 4 \\ x^2 + 2x + 2, & x \geq 4 \end{cases}$, which one of the following is/are true?

(a) $(f + g)(3.5) = 0$ (b) $f(g(3)) = 3$

(c) $f(g(2)) = 1$ (d) $(f - g)(4) = 0$

82. If $f(x) = x^2 - 2ax + a(a+1)$, $f : [a, \infty) \to [a, \infty)$. If one of the solutions of the equation $f(x) = f^{-1}(x)$ is 5049, the other may be

(a) 5051 (b) 5048 (c) 5052 (d) 5050

83. The function g defined by $g(x) = \sin\alpha + \cos\alpha - 1; \alpha = \sin^{-1}\sqrt{\{x\}}$, where $\{\}$ denotes fractional part function, is

(a) an even function (b) periodic function

(c) odd function (d) neither even nor odd

84. The graph of $f : R \to R$ defined by $y = f(x)$ is symmetric with respect to $x = a$ and $x = b$. Which of the following is true?

(a) $f(2a - x) = f(x)$ (b) $f(2a + x) = f(-x)$

(c) $f(2b + x) = f(-x)$ (d) f is periodic

85. Let f be the continuous and differentiable function such that $f(x) = f(2 - x)$, $\forall\, x \in R$ and $g(x) = f(1 + x)$, then

(a) $g(x)$ is an odd function

(b) $g(x)$ is an even function

(c) $f(x)$ is symmetric about $x = 1$

(d) None of the above

86. Let $f(x) = |x-1| + |x-2| + |x-3| + |x-4|$, then

(a) least value of $f(x)$ is 4

(b) least value is not attained at unique point

(c) the number of integral solution of $f(x) = 4$ is 2

(d) the value of $\dfrac{f(\pi - 1) + f(e)}{2f\left(\dfrac{12}{5}\right)}$ is 1

87. Let $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 2, 3, 4\}$ and $f : A \to B$ is a function, the

(a) number of onto functions, if $n(f(A)) = 4$ is 240

(b) number of onto functions, if $n(f(A)) = 3$ is 600

(c) number of onto functions, if $n(f(A)) = 2$ is 180

(d) number of onto functions, if $n(f(A)) = 1$ is 4

88. If $2f(x) + x\, f\left(\dfrac{1}{x}\right) - 2f\left(\left|\sqrt{2}\sin\pi\left(x + \dfrac{1}{4}\right)\right|\right)$

$= 4\cos^2\left(\dfrac{\pi x}{2}\right) + x\cos\left(\dfrac{\pi}{x}\right)$, $\forall\, x \in R - \{0\}$, which of the following statement(s) is/are true?

(a) $f(2) + f\left(\dfrac{1}{2}\right) = 1$ (b) $f(2) + f(1) = 0$

(c) $f(2) + f(1) = f\left(\dfrac{1}{2}\right)$ (d) $f(1)\cdot f\left(\dfrac{1}{2}\right)\cdot f(2) = 1$

89. If $f(x)$ is a differentiable function satisfying the condition $f(100x) = x + f(100x - 100)$, $\forall\, x \in R$ and $f(100) = 1$, then $f(10^4)$ is

(a) 5049 (b) $\sum_{r=1}^{100} r$ (c) $\sum_{r=2}^{100} r$ (d) 5050

90. If $[x]$ denotes the greatest integer function then the extreme values of the function $f(x) = [1 + \sin x] + [1 + \sin 2x] + \ldots + [1 + \sin nx]$, $n \in I^+$, $x \in (0, \pi)$ are

(a) $(n - 1)$ (b) n (c) $(n + 1)$ (d) $(n + 2)$

91. Which of the following is/are periodic?

(a) $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$

(b) $f(x) = \begin{cases} x - [x], & 2n \leq x < 2n + 1 \\ \dfrac{1}{2}, & 2n + 1 \leq x < 2n + 2 \end{cases}$, where $[\cdot]$ denotes the greatest integer function

(c) $f(x) = (-1)^{\left[\frac{2x}{\pi}\right]}$, where $[\cdot]$ denotes the greatest integer function

(d) $f(x) = ax - [ax + a] + \tan\left(\dfrac{\pi x}{2}\right)$, where $[\cdot]$ denotes the greatest integer function

92. If $f(x)$ is a polynomial of degree n, such that $f(0)=0$, $f(1)=1/2, \ldots, f(n)=\frac{n}{n+1}$, then the value of $f(n+1)$ is

(a) 1, when n is even (b) $\frac{n}{n+2}$, when n is odd

(c) 1, when n is odd (d) $\frac{n}{n+2}$, when n is even

93. Let $f: R \to R$ be a function defined by $f(x+1)=\frac{f(x)-5}{f(x)-3}, \forall\ x \in R$. Then, which of the following statement(s) is/are true?

(a) $f(2008)=f(2004)$ (b) $f(2006)=f(2010)$

(c) $f(2006)=f(2002)$ (d) $f(2006)=f(2018)$

94. Let $f(x)=1-x-x^3$. Then, the real values of x satisfying the inequality, $1-f(x)-f^3(x)>f(1-5x)$, are

(a) $(-2, 0)$ (b) $(0, 2)$

(c) $(2, \infty)$ (d) $(-\infty, -2)$

95. If a function satisfies $(x-y)f(x+y)-(x+y)f(x-y)=2(x^2y-y^3)$, $\forall\ x, y \in R$ and $f(1)=2$, then

(a) $f(x)$ must be polynomial function

(b) $f(3)=12$

(c) $f(0)=0$

(d) $f(x)$ may not be differentiable

96. If the fundamental period of function $f(x)=\sin x+\cos\left(\sqrt{4-a^2}\right)x$ is 4π, then the value of a is/are

(a) $\frac{\sqrt{15}}{2}$ (b) $-\frac{\sqrt{15}}{2}$ (c) $\frac{\sqrt{7}}{2}$ (d) $-\frac{\sqrt{7}}{2}$

97. Let $f(x)$ be a real valued function such that $f(0)=\frac{1}{2}$ and $f(x+y)=f(x)f(a-y)+f(y)f(a-x)$, $\forall\ x, y \in R$, then for some real a,

(a) $f(x)$ is a periodic function

(b) $f(x)$ is a constant function

(c) $f(x)=\frac{1}{2}$ (d) $f(x)=\frac{\cos x}{2}$

98. If $f(g(x))$ is one-one function, then

(a) $g(x)$ must be one-one (b) $f(x)$ must be one-one

(c) $f(x)$ may not be one-one (d) $g(x)$ may not be one-one

99. Which of the following functions have their range equal to R (the set of real numbers)?

(a) $x \sin x$

(b) $\frac{[x]}{\tan 2x} \cdot x \in \left(-\frac{\pi}{4}\cdot\frac{\pi}{4}\right)-\{0\}$, where $[\cdot]$ denotes the greatest integer function

(c) $\frac{x}{\sin x}$

(d) $[x]+\sqrt{\{x\}}$, where $[\cdot]$ and $\{\cdot\}$, respectively denote the greatest integer and fractional part functions

100. Which of the following pairs of function are identical?

(a) $f(x)=e^{\ln \sec^{-1} x}$ and $g(x)=\sec^{-1} x$

(b) $f(x)=\tan(\tan^{-1} x)$ and $g(x)=\cot(\cot^{-1} x)$

(c) $f(x)=\text{sgn}(x)$ and $g(x)=\text{sgn}(\text{sgn}(x))$

(d) $f(x)=\cot^2 x \cdot \cos^2 x$ and $g(x)=\cot^2 x-\cos^2 x$

101. Let $f: R \to R$ defined by $f(x)=\cos^{-1}(-\{-x\})$, where $\{x\}$ denotes fractional part of x. Then, which of the following is/are correct?

(a) f is many one but not even function

(b) Range of f contains two prime numbers

(c) f is non-periodic

(d) Graph of f does not lie below X-axis

Functions Exercise 3 : Statements I and II Type Questions

■ **Directions** (Q. Nos. 102 to 112) *For the following questions, choose the correct answers from the codes (a), (b), (c) and (d) defined as follows :*

(a) Statement I is true, Statement II is also true; Statement II is the correct explanation of Statement I

(b) Statement I is true, Statement II is also true; Statement II is not the correct explanation of Statement I

(c) Statement I is true, Statement II is false

(d) Statement I is false, Statement II is true

102. **Statement I** The function $f(x)=x \sin x$ and $f'(x)=x \cos x+\sin x$ are both non-periodic.

Statement II The derivative of differentiable function (non-periodic) is non-periodic function.

103. **Statement I** The maximum value of $\sin\sqrt{2}x+\sin ax$ cannot be 2 (where a is positive rational number).

Statement II $\frac{\sqrt{2}}{a}$ is irrational.

104. Let $f: R \to R$ be a function such that $f(x)=\frac{e^{|x|}-e^{-x}}{e^x+e^{-x}}$.

Statement I $f(x)$ is into function.

Statement II $f(x)$ is many-one function and the many-one function is not onto.

105. **Statement I** The range of
$f(x)=\sin\left(\frac{\pi}{5}+x\right)-\sin\left(\frac{\pi}{5}-x\right)-\sin\left(\frac{2\pi}{5}+x\right)+\sin\left(\frac{2\pi}{5}-x\right)$ is $[-1, 1]$.

Statement II $\cos\frac{\pi}{5}-\cos\frac{2\pi}{5}=\frac{1}{2}$

106. **Statement I** The period of
$f(x)=2\cos\frac{1}{3}(x-\pi)+4\sin\frac{1}{3}(x-\pi)$ is 3π.

Statement II If T is the period of $f(x)$, then the period of $f(ax+b)$ is $\frac{T}{|a|}$.

107. f is a function defined on the interval $[-1, 1]$ such that $f(\sin 2x)=\sin x+\cos x$.

Statement I If $x\in\left[-\frac{\pi}{4},\frac{\pi}{4}\right]$, then $f(\tan^2 x)=\sec x$

Statement II $f(x)=\sqrt{1+x}, \forall\ x\in[-1, 1]$

108. **Statement I** The equation $f(x)=4x^5+20x-9=0$ has only one real root.

Statement II $f'(x)=20x^4+20=0$ has no real root.

109. **Statement I** The range of $\log\left(\frac{1}{1+x^2}\right)$ is $(-\infty,\infty)$.

Statement II When $0<x\le 1$, $\log x\in(-\infty, 0]$.

110. Let $f: X\to Y$ be a function defined by
$f(x)=2\sin\left(x+\frac{\pi}{4}\right)-\sqrt{2}\cos x+c$.

Statement I For set X, $x\in\left[0,\frac{\pi}{2}\right]\cup\left[\pi,\frac{3\pi}{2}\right]$, $f(x)$ is one-one function.

Statement II $f'(x)\ge 0, x\in\left[0,\frac{\pi}{2}\right]$

111. Let $f(x)=\sin x$

Statement I f is not a polynomial function.

Statement II nth derivative of $f(x)$, w.r.t. x, is not a zero function for any positive integer n.

112. **Statement I** The function $f: R\to R$, given $f(x)=\log_a(x+\sqrt{x^2+1}), a>0, a\ne 1$ is invertible.

Statement II f is many-one and into.

Functions Exercise 4 : Passage Based Questions

Passage I (Q. Nos. 113 to 115)

Let $f: R\to R$ be a continuous function such that $f(x)-2f\left(\frac{x}{2}\right)+f\left(\frac{x}{4}\right)=x^2$.

113. $f(3)$ is equal to
(a) $f(0)$ (b) $4+f(0)$ (c) $9+f(0)$ (d) $16+f(0)$

114. The equation $f(x)-x-f(0)=0$ have exactly
(a) no solution (b) one solution
(c) two solutions (d) infinite solutions

115. $f'(0)$ is equal to
(a) 0 (b) 1 (c) $f(0)$ (d) $-f(0)$

Passage II (Q. Nos. 116 to 117)

Consider the equation $x+y-[x][y]=0$, where $[\cdot]$ is the greatest integer function.

116. The number of integral solutions to the equation is
(a) 0 (b) 1
(c) 2 (d) None of these

117. Equation of one of the lines on which the non-integral solution of given equation lies, is
(a) $x+y=-1$ (b) $x+y=0$
(c) $x+y=1$ (d) $x+y=5$

Passage III (Q. Nos. 118 to 120)

Let $f(x)=\frac{1}{2}\left[f(xy)+f\left(\frac{x}{y}\right)\right]$ *for $x, y\in R^+$ such that $f(1)=0$; $f'(1)=2$.*

118. $f(x)-f(y)$ is equal to
(a) $f\left(\frac{y}{x}\right)$ (b) $f\left(\frac{x}{y}\right)$ (c) $f(2x)$ (d) $f(2y)$

119. $f'(3)$ is equal to
(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{1}{2}$ (d) $\frac{1}{4}$

120. $f(e)$ is equal to
(a) 2 (b) 1 (c) 3 (d) 4

Passage IV (Q. Nos. 121 to 123)

If $f: R\to R$ and $f(x)=g(x)+h(x)$, where $g(x)$ is a polynomial and $h(x)$ is a continuous and differentiable bounded function on both sides, then $f(x)$ is one-one, we need to differentiate $f(x)$. If $f'(x)$ changes sign in domain of f, then f, if many-one else one-one.

121. If $f: R\to R$ and $f(x)=a_1x+a_3x^3+a_5x^5+\ldots+a_{2n+1}x^{2n+1}-\cot^{-1}x$
where $0<a_1<a_3<\ldots<a_{2n+1}$, then the function $f(x)$ is
(a) one-one into (b) many-one onto
(c) one-one onto (d) many-one into

122. If $f: R \to R$ and $f(x) = \dfrac{x(x^4+1)(x+1)+x^4+2}{x^2+x+1}$, then $f(x)$ is

(a) one-one into (b) many-one onto
(c) one-one onto (d) many-one into

123. If $f: R \to R$ and $f(x) = 2ax + \sin 2x$, then the set of values of a for which $f(x)$ is one-one and onto is

(a) $a \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ (b) $a \in (-1, 1)$
(c) $a \in R - \left(-\frac{1}{2}, \frac{1}{2}\right)$ (d) $a \in R - (-1, 1)$

Passage V (Q. Nos. 124 to 126)

Let $g(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ and $f(x) = \sqrt{g(x)}$, $f(x)$ has its non-zero local minimum and maximum values at -3 and 3, respectively. If $a_3 \in$ the domain of the function

$h(x) = \sin^{-1}\left(\dfrac{1+x^2}{2x}\right)$.

124. The value of $a_1 + a_2$ is

(a) 30 (b) -30 (c) 27 (d) -27

125. The value of a_0 is

(a) equal to 50 (b) greater than 54
(c) less than 54 (d) less than 50

126. $f(10)$ is defined for

(a) $a_0 > 830$ (b) $a_0 < 830$
(c) $a_0 = 830$ (d) None of these

Passage VI (Q. Nos. 127 to 129)

Let $f: [2, \infty) \to [1, \infty)$ defined by $f(x) = 2^{x^4 - 4x^2}$ and $g: \left[\frac{\pi}{2}, \pi\right] \to A$ defined by $g(x) = \dfrac{\sin x + 4}{\sin x - 2}$ be two invertible functions.

127. $f^{-1}(x)$ is equal to

(a) $\sqrt{2+\sqrt{4-\log_2 x}}$ (b) $\sqrt{2+\sqrt{4+\log_2 x}}$
(c) $\sqrt{2-\sqrt{4+\log_2 x}}$ (d) None of these

128. The set A is equal to

(a) $[-5, -2]$ (b) $[2, 5]$ (c) $[-5, 2]$ (d) $[-3, -2]$

129. The domain of $f^{-1}g^{-1}(x)$ is

(a) $[-5, \sin 1]$ (b) $\left[-5, \dfrac{\sin 1}{2 - \sin 1}\right]$
(c) $\left[-5, -\dfrac{(4+\sin 1)}{2-\sin 1}\right]$ (d) $\left[-\dfrac{(4+\sin 1)}{2-\sin 1}, -2\right]$

Passage VII (Q. Nos. 130 to 132)

Let $P(x)$ be polynomial of degree atmost 5 which leaves remainders -1 and 1 upon division by $(x-1)^3$ and $(x+1)^3$, respectively.

130. Numbers of real roots of $P(x) = 0$ is

(a) 1 (b) 3 (c) 5 (d) 2

131. The maximum value of $y = P''(x)$ can be obtained at x is equal to

(a) $-\dfrac{1}{\sqrt{3}}$ (b) 0 (c) $\dfrac{1}{\sqrt{3}}$ (d) 1

132. The sum of pairwise product of all roots (real and complex) of $P(x) = 0$ is

(a) $-\dfrac{5}{3}$ (b) $-\dfrac{10}{3}$ (c) 2 (d) -5

Passage VIII (Q. Nos. 133 to 135)

Consider $\alpha > 1$ and $f: \left[\frac{1}{\alpha}, \alpha\right] \to \left[\frac{1}{\alpha}, \alpha\right]$ be bijective function.

Suppose that $f^{-1}(x) = \dfrac{1}{f(x)}$; for all $x \in \left[\frac{1}{\alpha}, \alpha\right]$.

133. $f(1)$ is equal to

(a) 1 (b) 0
(c) -1 (d) does'nt attain a unique value

134. Which of the following statements can be concluded about $(f(x))$?

(a) $f(x)$ is discontinuous in $\left[\frac{1}{\alpha}, \alpha\right]$
(b) $f(x)$ is increasing in $\left[\frac{1}{\alpha}, \alpha\right]$
(c) $f(x)$ is decreasing in $\left[\frac{1}{\alpha}, \alpha\right]$
(d) None of the above

135. Which of the following statements can be concluded about $f(f(x))$?

(a) $f(f(x))$ is continuous in $\left[\frac{1}{\alpha}, \alpha\right]$
(b) $f(f(x))$ is increasing in $\left[\frac{1}{\alpha}, \alpha\right]$
(c) $f(f(x))$ is decreasing in $\left[\frac{1}{\alpha}, \alpha\right]$
(d) None of the above

Passage IX (Q. Nos. 136 to 137)

Let f be a real valued function from N to N satisfying. The relation $f(m+n) = f(m) + f(n)$ for all $m, n \in N$.

136. The range of f contains all the even numbers, the value of $f(1)$ is

(a) 1 (b) 2 (c) 1 or 2 (d) 4

137. If domain of f is first $3m$ natural numbers and if the number of elements common in domain and range is m, then the value of $f(1)$ is

(a) 2 (b) 3
(c) 6 (d) Can't say

Functions Exercise 5 : Matching Type Questions

138. Match the statements of Column I with values of Column II.

	Column I		Column II
(A)	$\sqrt{\sin(\cos x)}$ has domain	(p)	$x \in R$
(B)	$(\sqrt{\cos(\sin x)})^{-1}$ has domain	(q)	$R - \left\{n\pi \pm \frac{\pi}{6}\right\}$
(C)	$\tan(\pi \sin x)$ has domain	(r)	$x \in \left(n\pi, n\pi + \frac{\pi}{2}\right)$
(D)	$\ln(\tan x)$ has domain	(s)	$x \in \left[2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right]$

139. Match the statements of Column I with values of Column II.

	Column I	Column II
(A)	$\lvert 4\sin x - 1\rvert < \sqrt{5}$, $x \in [0, \pi]$, the domain is	(p) $\left[0, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \pi\right]$
(B)	$\frac{4\sin^2 x - 8}{\sin x + 3} \le 0$, $[0, 2\pi]$, the domain is	(q) $\left[\frac{3\pi}{2}, 2\pi\right] \cup \{0\}$
(C)	$\lvert\tan x\rvert \le 1$ and $x \in [0, \pi]$, the domain is	(r) $\left[0, \frac{3\pi}{10}\right)$
(D)	$\cos x - \sin x \ge 1$ and $[0, 2\pi]$, the domain is	(s) $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$

140. Match the statements of Column I with values of Column II.

	Column I	Column II
(A)	If $f(x) = \begin{cases} x+1, & \text{when } x < 0 \\ x^2 - 1, & \text{when } x \ge 0 \end{cases}$, the $fof(x)$ for $-1 \le x < 0$ is	(p) $\frac{x-3}{2}$
(B)	If $f\left(\frac{2\tan x}{1+\tan^2 x}\right) = \frac{(\cos 2x + 1)(\sec^2 x + 2\tan x)}{2}$, then $f(x)$ is	(q) $x^2 + 2x$
(C)	If $f(x+y+1) = (\sqrt{f(x)} + \sqrt{f(y)})^2$ for all $x, y \in R$ and $f(0) = 1$, then $f(x)$ is	(r) $1 + x$
(D)	If $4 < x < 5$ and $f(x) = \left[\frac{x}{4}\right] + 2x + 2$, where $[y]$ is the greatest integer $\le y$, then $f^{-1}(x)$ is	(s) $(x+1)^2$

141. Match the statements of Column I with values of Column II.

	Column I	Column II
(A)	$f(x) = \sin^2 x + \sin^2\left(x + \frac{\pi}{3}\right) + \cos x \cos\left(x + \frac{\pi}{3}\right)$ then $f(x)$ is	(p) Defined for all real 'x'
(B)	$g(x) = \tan(e^{\{x\}}) + [x + \alpha] - 5 - x$, where $[\cdot]$ denotes the greatest integer less than or equal to x, then $g(x)$ is	(q) Even function
(C)	$h(x) = \frac{x}{5^x - 1} + \frac{x}{2} + 5$, then $h(x)$ is	(r) Odd function
(D)	$k(x) = 2\sin^2 x - \cos 2\alpha + 4\sin\alpha \cdot \sin x \cos(x + \alpha) + \cos 2(x + \alpha)$, $\alpha \in R$, then $k(x)$ is	(s) Periodic function

Functions Exercise 6 : Single Integer Answer Type Questions

142. A function $f : R \to R$ is defined by $f(x+y) - k\,xy = f(x) + 2y^2$, $\forall\ x, y \in R$ and $f(1) = 2$; $f(2) = 8$, where k is some real constant, then $f(x+y) \cdot f\left(\frac{1}{x+y}\right)$ is ..... .

143. If $f : R \to R$ satisfying $f(x - f(y)) = f(f(y)) + x\,f(y) + f(x) - 1$, for all $x, y \in R$, then $\frac{-f(10)}{7}$ is ..... .

144. Let $f : N \to R$ be such that $f(1) = 1$ and $f(1) + 2\,f(2) + 3f(3) + \ldots + nf(n) = n(n+1)f(n)$, for $n \ge 2$, then $\frac{1}{2010\,f(2010)}$ is ..... .

145. If $f(x) = \frac{2010x + 163}{165x - 2010}$, $x > 0$ and $x \ne \frac{2010}{165}$, the least value of $f(f(x)) + f\left(f\left(\frac{4}{x}\right)\right)$ is ..... .

146. If $\alpha, \beta, \gamma \in R; \alpha + \beta + \gamma = 4$ and $\alpha^2 + \beta^2 + \gamma^2 = 6$, the number of integers lie in the exhaustive range of α is ..... .

147. The number of linear functions satisfying $f(x + f(x)) = x + f(x), \forall\ x$ is ..... .

148. If $A = \{1, 2, 3\}$, $B = \{1, 3, 5, 7, 9\}$, the ratio of number of one-one functions to the number of strictly monotonic functions is ..... .

149. If $n(A) = 4, n(B) = 5$ and number of functions from A to B such that range contains exactly 3 elements is k, $\frac{k}{60}$ is ..... .

150. If a and b are constants, such that $f(x) = a\sin x + bx\cos x + 2x^2$ and $f(2) = 15$, $f(-2)$ is ..... .

151. If the functions $f(x) = x^5 + e^{x/3}$ and $g(x) = f^{-1}(x)$, the value of $g'(1)$ is ..... .

152. If $f(x) = x^3 - 12x + p$; $p \in \{1, 2, 3, ..., 15\}$ and for each 'p', the number of real roots of equation $f(x) = 0$ is denoted by θ, the $\frac{1}{5}\Sigma\theta$ is equal to ..... .

153. Let $f(x)$ denotes the number of zeroes in $f'(x)$. If $f(m) - f(n) = 3$, the value of $\frac{(m-n)_{\max} - (m-n)_{\min}}{2}$ is ..... .

154. If $x^2 + y^2 = 4$, the maximum value of $\left(\frac{x^3 + y^3}{x + y}\right)$ is

155. Let $f(n)$ denotes the square of the sum of the digits of natural number n, where $f^2(n)$ denotes $f(f(n))$, $f^3(n)$ denotes $f(f(f(n)))$ and so on. The value of $\frac{f^{2011}(2011) - f^{2010}(2011)}{f^{2013}(2011) - f^{2012}(2011)}$ is ..... .

156. If $[\sin x] + \left[\frac{x}{2\pi}\right] + \left[\frac{2x}{5\pi}\right] = \frac{9x}{10\pi}$, where $[\cdot]$ denotes the greatest integer function, the number of solutions in the interval $(30, 40)$ is ..... .

157. The number of integral solutions of $\frac{1}{x} + \frac{1}{y} = \frac{1}{6}$ with $x \le y$ is 'α'. The value of '$\alpha - 6$' is ..... .

158. If $f(x)$ is a polynomial of degree 4 with leading coefficient '1' satisfying $f(1) = 10$, $f(2) = 20$ and $f(3) = 30$, then $\left(\frac{f(12) + f(-8)}{19840}\right)$ is ..... .

159. If $a + b = 3 - \cos 4\theta$ and $a - b = 4\sin 2\theta$, then ab is always less than or equal to ..... .

160. Let 'n' be the number of elements in the domain set of the function $f(x) = \left|\ \ln \sqrt{{}^{x^2+4x}C_{2x^2+3}}\right|$ and 'Y' be the global maximum value of $f(x)$, then $[n + [Y]]$ is ..... (where $[\cdot]$ = greatest integer function).

161. If $f(x)$ is a function such that $f(x-1) + f(x+1) = \sqrt{3}f(x)$ and $f(5) = 10$, then the sum of digits of the value of $\sum_{r=0}^{19} f(5 + 12r)$ is ..... .

162. If $2f(x) = f(xy) + f\left(\frac{x}{y}\right)$ for all positive values of x and y, $f(1) = 0$ and $f'(1) = 1$, then $f(e)$ is ..... .

163. Let f be a function from the set of positive integers to the set of real number such that $f(1) = 1$ and $\sum_{r=1}^{n} r\,f(r) = n(n+1)f(n),\ \forall\ n \ge 2$, the value of $2126\ f(1063)$ is ..... .

164. If $f(x) = \frac{x^4 + x^2 + 1}{x^2 - x + 1}$, the value of $f(\omega^n)$ (where 'ω' is the non-real root of the equation $z^3 = 1$ and 'n' is a multiple of 3), is ..... .

165. If $f^2(x) \cdot f\left(\frac{1-x}{1+x}\right) = x^3, [x \ne -1, 1$ and $f(x) \ne 0]$, the value of $|[f(-2)]|$ (where $[\cdot]$ is the greatest integer function), is ..... .

166. An odd function is symmetric about the vertical line $x = a (a > 0)$ and if $\sum_{r=0}^{\infty} [f(1 + 4r)]^r = 8$, find the numerical value of $8f(1)$.

167. Let $\frac{e^x - e^{-x}}{e^x + e^{-x}} = \ln\sqrt{\frac{1+x}{1-x}}$, then find x.

168. If the maximum value of $f(x) = \frac{3x^2 + 9x + 17}{3x^2 + 9x + 7}$ is $5k + 1$, the value of k is ..... .

169. The period of the function $f(x)$ which satisfies the relation $f(x) + f(x+4) = f(x+2) + f(x+6)$ is ..... .

170. If a non-zero function $f(x)$ is symmetrical about $y = x$, then the value of p (constant) such that $f^2(x) = (f^{-1}(x))^2 - px \cdot f(x) \cdot f^{-1}(x) + 2x^2 f(x)$ for all $x \in R^+$ is ..... .

171. Let $f:R\to R$ and $f(x)=\dfrac{3x^2+mx+n}{x^2+1}$. If the range of this function is $[-4,3]$, then the value of $\dfrac{m^2+n^2}{4}$ is....

172. Let $f(x)$ be a monotonic polynomial of $(2m-1)$ degree where $m\in N$, then the equation $f(x)+f(3x)+f(5x)+\ldots+f(2m-1)\,x=(2m-1)$ has ..... . roots.

Functions Exercise 7 : Subjective Type Questions

173. Let x be a real number. $[x]$ denotes the greatest integer function, $\{x\}$ denotes the fractional part and (x) denotes the least integer function, then solve the following :

(i) $(x)^2=[x]^2+2x$

(ii) $[2x]-2x=[x+1]$

(iii) $[x^2]+2[x]=3x, 0\le x\le 2$

(iv) $y=4-[x]^2$ and $[y]+y=6$

(v) $[x]+|x-2|\le 0$ and $-1\le x\le 3$

174. Let n be a positive integer and define $f(n)=1!+2!+3!+\ldots+n!$. Find polynomials $P(x)$ and $Q(x)$ such that $f(n+2)=Q(n)\,f(n)+P(n)\,f(n+1)$ for all $n\ge 1$.

175. If $f(x)=\dfrac{a^x}{a^x+\sqrt{a}}(a>0)$, evaluate $\displaystyle\sum_{r=1}^{2n-1}2f\left(\frac{r}{2n}\right)$.

176. Find the domain of the function $f(x)=\log\left\{\log_{|\sin x|}(x^2-8x+23)-\dfrac{3}{\log_2|\sin x|}\right\}$.

177. Let $S(n)$ denotes the number of ordered pairs (x, y) satisfying $\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{1}{n}$, where $n>1$ and $x, y, n\in N$.

(i) Find the value of $S(6)$.

(ii) Show that, if n is prime, then $S(n)=3$, always.

178. Solve $\dfrac{1}{[x]}+\dfrac{1}{[2x]}=\{x\}+\dfrac{1}{3}$, where $[\cdot]$ denotes the greatest integer function and $\{\}$ denotes fractional part of x.

179. Let $f(x)=x^2+3x-3, x\ge 0$. n points $x_1, x_2, \ldots, x_n$ are so chosen on the X-axis that

(i) $\dfrac{1}{n}\displaystyle\sum_{i=1}^{n}f^{-1}(x_i)=f\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)$

(ii) $\displaystyle\sum_{i=1}^{n}f^{-1}(x_i)=\sum_{i=1}^{n}x_i$, where f^{-1} denotes the inverse of f. Find the AM of x_i's.

180. Let $f(x)=x^2-2x, x\in R$ and $g(x)=f(f(x)-1)+f(5-f(x))$, show that $g(x)\ge 0$, $\forall\ x\in R$.

181. If f is a polynomial function satisfying $2+f(x)\cdot f(y)=f(x)+f(y)+f(xy), \forall\ x, y\in R$ and if $f(2)=5$, find $f(f(2))$.

182. If $a+b+c=abc$, a, b and $c\in R^+$, prove that $a+b+c\ge 3\sqrt{3}$.

183. Consider the function $f(x)=\begin{cases}x-[x]-\dfrac{1}{2}, & \text{if } x\notin I\\ 0, & \text{if } x\in I\end{cases}$, where $[\cdot]$ denotes the greatest integral function and I is the set of integers. Find $g(x)=\max\{x^2, f(x), |x|\}$; $-2\le x\le 2$.

184. If $f(x)$ is continuous function in $[0, 2\pi]$ and $f(0)=f(2\pi)$, then prove that there exists a point $c\in(0,\pi)$ such that $f(c)=f(c+\pi)$.

185. Let $g(t)=|t-1|-|t|+|t+1|, \forall\ t\in R$. Find $f(x)=\max\{g(t): -\dfrac{3}{2}\le t\le x\}, \forall\ x\in\left(\dfrac{-3}{2},\infty\right)$.

186. Find the integral solution for $n_1n_2=2n_1-n_2$, where $n_1, n_2\in$ integer.

Functions Exercise 8 : Questions Asked in Previous 10 Year's Exams

(i) JEE Advanced & IIT-JEE

187. Let $f_1 : R \to R, f_2 : [0, \infty) \to R, f_3 : R \to R$ and $f_4 : R \to [0, \infty)$ be defined by

[Match Type Question, 2014 Adv.]

$$f_1(x) = \begin{cases} |x|, & \text{if } x < 0 \\ e^x, & \text{if } x \geq 0 \end{cases}; f_2(x) = x^2; f_3(x) = \begin{cases} \sin x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

and $f_4(x) = \begin{cases} f_2[f_1(x)], & \text{if } x < 0 \\ f_2[f_1(x)] - 1, & \text{if } x \geq 0 \end{cases}$

Column I		Column II	
A.	f_4 is	p.	onto but not one-one
B.	f_3 is	q.	neither continuous nor one-one
C.	$f_2 of_1$, is	r.	differentiable but not one-one
D.	f_2 is	s.	continuous and one-one

Codes

	A	B	C	D
(a)	r	p	s	q
(c)	r	p	q	s

	A	B	C	D
(b)	p	r	s	q
(d)	p	r	q	s

188. If function $f(x) = x^2 + e^{x/2}$ and $g(x) = f^{-1}(x)$, then the value of $g'(1)$ is **[Integer Type Question, 2009]**

189. For the following questions, choose the correct answer from the codes (a), (b), (c) and (d) defined as follows.

(a) Statement I is true, Statement II is also true; Statement II is the correct explanation of Statement I.

(b) Statement I is true, Statement II is also true; Statement II is not the correct explanation of Statement I.

(c) Statement I is true; Statement II is false.

(d) Statement I is false; Statement II is true.

Let $F(x)$ be an indefinite integral of $\sin^2 x$.

Statement I The function $F(x)$ satisfies $F(x + \pi) = F(x)$ for all real x. Because

Statement II $\sin^2(x + \pi) = \sin^2 x$, for all real x.

[Assertion and Reason Type Question, 2007]

190. Match the conditons/expressions in Column I with statement in Column II.

Let $f(x) = \dfrac{x^2 - 6x + 5}{x^2 - 5x + 6}$. **[Match Type Question, 2007]**

Column I		Column II	
A.	If $-1 < x < 1$, then $f(x)$ satisfies	p.	$0 < f(x) < 1$
B.	If $1 < x < 2$, then $f(x)$ satisfies	q.	$f(x) < 0$
C.	If $3 < x < 5$, then $f(x)$ satisfies	r.	$f(x) > 0$
D.	If $x > 5$, then $f(x)$ satisfies	s.	$f(x) < 1$

191. Find the range of values of t for which

$$2 \sin t = \frac{1 - 2x + 5x^2}{3x^2 - 2x - 1}, \; t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

[Subjective Type Question, 2005]

(ii) JEE Main & AIEEE

192. If $f_k(x) = \dfrac{1}{k}(\sin^k x + \cos^k x)$, where $x \in R$ and $k \geq 1$, then $f_4(x) - f_6(x)$ is equal to **[2014 JEE Main]**

(a) 1/6 (b) 1/3 (c) 1/4 (d) 1/12

193. The function $f : [0, 3] \to [1, 29]$, defined by $f(x) = 2x^3 - 15x^2 + 36x + 1$, is **[2012 AIEEE]**

(a) one-one and onto

(b) onto but not one-one

(c) one-one but not onto

(d) neither one-one nor onto

194. Let $f(x) = x^2$ and $g(x) = \sin x$ for all $x \in R$. Then, the set of all x satisfying $(fogogof)(x) = (gogof)(x)$, where $(fog)(x) = f(g(x))$, is **[2011 AIEEE]**

(a) $\pm\sqrt{n\pi}, n \in \{0, 1, 2, \ldots\}$

(b) $\pm\sqrt{n\pi}, n \in \{1, 2, \ldots\}$

(c) $\pi/2 + 2n\pi, n \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

(d) $2n\pi, n \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

195. Let $f : (0, 1) \to R$ be defined by $f(x) = \dfrac{b - x}{1 - bx}$, where b is a constant such that $0 < b < 1$. Then, **[2011 AIEEE]**

(a) f is not invertible on (0, 1)

(b) $f \neq f^{-1}$ on (0, 1) and $f'(b) = \dfrac{1}{f'(0)}$

(c) $f = f^{-1}$ on (0, 1) and $f'(b) = \dfrac{1}{f'(0)}$

(d) f^{-1} is differentiable on (0, 1)

196. Let f be a real-valued function defined on the interval $(-1, 1)$ such that $e^{-x} f(x) = 2 + \int_0^x \sqrt{t^4 + 1}\cdot dt, \forall x \in (-1, 1)$ and let f^{-1} be the inverse function of f. Then, $[f^{-1}(2)]'$ is equal to **[2010 AIEEE]**

(a) 1 (b) 1/3 (c) 1/2 (d) $1/e$

197. If X and Y are two non-empty sets, where $f : X \to Y$, is function defined such that

$f(C) = \{f(x) : x \in C\}$ for $C \subseteq X$ and

$f^{-1}(D) = \{x : f(x) \in D\}$ for $D \subseteq Y$,

for any $A \subseteq Y$ and $B \subseteq Y$, then **[2005 AIEEE]**

(a) $f^{-1}\{f(A)\} = A$

(b) $f^{-1}\{f(A)\} = A$, only if $f(X) = Y$

(c) $f\{f^{-1}(B)\} = B$, only if $B \subseteq f(x)$

(d) $f\{f^{-1}(B)\} = B$

198. $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$, $g(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ x, & \text{if } x \text{ is irrational.} \end{cases}$

Then, $f - g$ is **[2005 AIEEE]**

(a) one-one and into (b) neither one-one nor onto

(c) many one and onto (d) one-one and onto

199. If $f(x) = \sin x + \cos x$, $g(x) = x^2 - 1$, then $g\{f(x)\}$ is invertible in the domain **[2004 AIEEE]**

(a) $\left[0, \frac{\pi}{2}\right]$ (b) $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ (c) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (d) $[0, \pi]$

200. Domain of definition of the function $f(x) = \sqrt{\sin^{-1}(2x) + \frac{\pi}{6}}$ for real valued x, is **[2003 AIEEE]**

(a) $\left[-\frac{1}{4}, \frac{1}{2}\right]$ (b) $\left[-\frac{1}{2}, \frac{1}{2}\right]$ (c) $\left(-\frac{1}{2}, \frac{1}{9}\right)$ (d) $\left[-\frac{1}{4}, \frac{1}{4}\right]$

201. Range of the function $f(x) = \frac{x^2 + x + 2}{x^2 + x + 1}; x \in R$ is **[2003 AIEEE]**

(a) $(1, \infty)$ (b) (1, 11/7)

(c) (1, 7/3] (d) (1, 7/5)

202. If $f : [0, \infty) \to [0, \infty)$ and $f(x) = \frac{x}{1 + x}$, then f is **[2003 AIEEE]**

(a) one-one and onto (b) one-one but not onto

(c) onto but not one-one (d) neither one-one nor onto

203. Let function $f : R \to R$ be defined by $f(x) = 2x + \sin x$ for $x \in R$. Then, f is **[2002 AIEEE]**

(a) one-to-one and onto

(b) one-to-one but not onto

(c) onto but not one-to-one

(d) neither one-to-one nor onto

204. Let $E = \{1, 2, 3, 4\}$ and $F = \{1, 2\}$. Then, the number of onto functions from E to F is **[2001 AIEEE]**

(a) 14 (b) 16 (c) 12 (d) 8

205. Suppose $f(x) = (x + 1)^2$ for $x \geq -1$. If $g(x)$ is the function whose graph is reflection of the graph of $f(x)$ with respect to the line $y = x$, then $g(x)$ equals **[2002 AIEEE]**

(a) $-\sqrt{x} - 1, x \geq 0$ (b) $\frac{1}{(x + 1)^2}, x > -1$

(c) $\sqrt{x + 1}, x \geq -1$ (d) $\sqrt{x} - 1, x \geq 0$

206. If $f : [1, \infty) \to [2, \infty)$ is given by $f(x) = x + \frac{1}{x}$, then $f^{-1}(x)$ equals **[2001 AIEEE]**

(a) $\frac{x + \sqrt{x^2 - 4}}{2}$ (b) $\frac{x}{1 + x^2}$

(c) $\frac{x - \sqrt{x^2 - 4}}{2}$ (d) $1 + \sqrt{x^2 - 4}$

207. Let $f(x) = (1 + b^2)x^2 + 2bx + 1$ and let $m(b)$ be the minimum value of $f(x)$. As b varies, the range of $m(b)$ is **[2001 AIEEE]**

(a) [0, 1] (b) $\left[0, \frac{1}{2}\right]$

(c) $\left[\frac{1}{2}, 1\right]$ (d) (0, 1]

208. The domain of definition of $f(x) = \frac{\log_2(x + 3)}{(x^2 + 3x + 2)}$ is **[2001 AIEEE]**

(a) $R / \{-1, -2\}$ (b) $(-2, \infty)$

(c) $R / \{-1, -2, -3\}$ (d) $(-3, \infty) / \{-1, -2\}$

209. Let $f(x) = \frac{\alpha x}{x + 1}, x \neq -1$. Then, for what value of α is $f[f(x)] = x$? **[2001 AIEEE]**

(a) $\sqrt{2}$ (b) $-\sqrt{2}$

(c) 1 (d) -1

210. Let $g(x) = 1 + x - [x]$ and $f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$, then for all x, $f[g(x)]$ is equal to **[2001 AIEEE]**

(a) x (b) 1

(c) $f(x)$ (d) $g(x)$

211. The domain of definition of the function $y(x)$ is given by the equation $2^x + 2^y = 2$, is **[2000 AIEEE]**

(a) $0 < x \leq 1$

(b) $0 \leq x \leq 1$

(c) $-\infty < x \leq 0$

(d) $-\infty < x < 1$

212. Let $f(\theta) = \sin\theta(\sin\theta + \sin 3\theta)$. Then, $f(\theta)$ **[2000 AIEEE]**

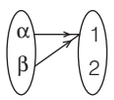
(a) ≥ 0, only when $\theta \geq 0$ (b) ≤ 0, for all real θ

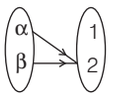
(c) ≥ 0, for all real θ (d) ≤ 0, only when $\theta \leq 0$

Answers

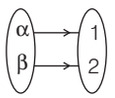
Exercise for Session 1

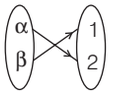
1. (i) not a function (ii) not a function (iii) function (iv) function (v) not a function (vi) function (vii) function (viii) not a function (ix) function (x) not a function.
2. (i) not a function (ii) not a function (iii) not a function (iv) is a function (v) not a function
3. (a)
4. (a) $f:A \to B$ (b) $f:A \to B$





(c) $f:A \to B$ (d) $f:A \to B$





5. (b)

Exercise for Session 2

1. $(-\infty, 2] \cup [3, \infty)$
2. $(-\infty, -1/2] \cup (0, 1) \cup (2, \infty)$
3. $[-1, 1]$
4. $(-\infty, \infty)$
5. $\{2, 3\}$
6. $\{1, 2, 3\}$

Exercise for Session 3

1. $[4, 6]$
2. $(0, 1] \cup [4, 5)$
3. $[1-\sqrt{2}, 0) \cup [1+\sqrt{2}, 3)$
4. $x \in [2, 4]$
5. $(2, 3)$
6. $[-\pi/4, \pi/4]$
7. $[-2, -1/2] \cup [1/2, 2]$
8. $[-1, 3]$
9. $\left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$
10. ϕ

Exercise for Session 4

1. $R - (-2, 2)$
2. $[-2, 2]$
3. $(0, 1) \cup (1, \infty)$
4. $[0, 3)$
5. $R - I$
6. $(-\infty, -2) \cup (4, \infty)$
7. R
8. $(2, \infty)$
9. $(-\infty, 0) \cup (0, 1] \cup [2, \infty)$
10. $(0, 1)$
11. $\left(-\sqrt{\frac{5}{2}}, -1\right] \cup \left[1, \sqrt{\frac{5}{2}}\right)$
12. $(-\sqrt{8}, -1] \cup [1, \sqrt{8})$
13. $\left[-1, -\frac{1}{2}\right] \cup \left[0, \frac{1}{2}\right] \cup \{1\}$
14. $R - (1, 2) \cup \{2, 3, 4, 5, 6, 7, 8, 9, 10\} \cup (10, 11)$
15. $\left[\frac{\pi}{6}, \frac{\pi}{2}\right)$
16. No solution
18. Integral solutions are (2, 2) and (0, 0), all non-integral solutions lie on exactly two lines $x + y = 0$ and $x + y = 6$.

Exercise for Session 5

1. $[0, 3]$
2. $[-1/2, 1/2]$
3. $[3-\sqrt{2}, 3+\sqrt{2}]$
4. $[1, 5]$
5. $(-\infty, \log_3 9)$
6. $\left(-\infty, \frac{2}{3}\right] \cup (1, \infty)$
7. $R - (2-2\sqrt{3}, 2+2\sqrt{3})$
8. $[2, \infty)$
9. $(-\infty, 0]$
10. $\left\{\frac{\pi}{2}\right\}$
11. $\{0, 1\}$
12. $\{\pi\}$
13. $\left[\frac{\pi}{3}, \frac{\pi}{2}\right]$
14. $\left(0, \frac{\pi}{2}\right]$
15. $\{0\}$
16. $\left[\frac{-1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right]$
17. $[-1, 1]$
18. $\{0\}$
19. $[1, \infty)$
20. $\{1\}$
21. $[0, 2]$
22. Image of $(-\infty, 1)$ under f is $(1, \infty)$, Image of $[1, 2]$ under f is $[1, 2]$
23. Range of $f(x)$ is $\{0\}$ and domain is $\cup \left[\left(2n\pi - \frac{\pi}{3}\right), \left(2n\pi + \frac{\pi}{3}\right)\right]$
24. Range of $f(x)$ is $\left\{\log \frac{\pi}{2}\right\}$ and domain is $[1, 2)$
25. Range of $f(x)$ is the set of all non-positive integers and domain is $\left[\frac{-3-\sqrt{5}}{2}, -2\right) \cup \left(-1, \frac{-3+\sqrt{5}}{2}\right]$.

Exercise for Session 6

1. (i) odd (ii) even (iii) neither even nor odd (iv) even (v) odd (vi) odd (vii) neither even nor odd (viii) even
2. f is an even function when $x \in$ integer f is an odd function when $x \notin$ integer.
3. (i) $f(x) = \begin{cases} -2x, & x \le -1 \\ -x|x|, & -1 < x \le 0 \end{cases}$ (ii) $f(x) = \begin{cases} 2x, & x \le -1 \\ x|x|, & -1 < x \le 0 \end{cases}$
5. $a \in (400, \infty)$

Exercise for Session 7

1. (i) $\pi/3$ (ii) 2π (iii) 1 (iv) 24 (v) Does not exist (vi) $2(n+1)!$ (vii) 1 (viii) 2π
2. Period is 8.
3. $f(x)$ is periodic with period 2λ.
4. $f(x)$ is periodic with period $2p$.
5. $f(x)$ is periodic with period 12 and $\sum_{r=0}^{99} f(5+12r) = 10000$

Exercise for Session 8

1. $x = 0$ and $y = \frac{3}{2}$
2. $f^{-1}(1) = y$
3. As monotonic and range = Codomain $\Rightarrow$ Bijective
5. $A \in [0, 1)$
6. $A \in \left[0, \frac{1}{2}\right)$
7. $b^2 < 3a(c - |d|)$
8. $X \in \left[-\frac{\pi}{2} - \alpha, \frac{\pi}{2} - \alpha\right]$ and $Y \in [c-r, c+r]$, where $\alpha = \tan^{-1}\left(\frac{a + b\sqrt{2}}{a}\right)$ and $r = \sqrt{a^2 + \sqrt{2}ab + b^2}$

Exercise for Session 9

1. Not Identical
2. Identical
3. Not Identical
4. Identical
5. Identical
6. Not Identical
7. Identical
8. Identical
9. Identical
10. Identical

Exercise for Session 10

1. Domain $\in [-1, 1]$ and range $\in \left[\frac{-2\pi}{3}, \frac{\pi}{3}\right]$

2. Domain for $f([|x|]) \in (-3, 3)$

Domain for $f([2x + 3]) \in [-3, 0)$

3. $h(x) = \begin{cases} \sin^2 x - \sin x + 1, & -1 < x < 0 \\ 2\sin^2 x, & 0 < x \le 1 \end{cases}$

4. $g(x) = \begin{cases} -x, & -2 \le x \le 0 \\ 0, & 0 \le x \le 1 \\ 2(x-1), & 1 \le x \le 2 \end{cases}$ **5.** $gof = \{(x+1)^2, -2 \le x \le 1\}$

Exercise for Session 11

1. (i) $f^{-1}(x) = 3 \sin x$ (ii) $f^{-1}(x) = \dfrac{-3 + \sqrt{5 + 4e^x}}{2}$

(iii) $f^{-1}(x) = x^{\log_5 e}$, $x > 0$ (iv) $f^{-1}(x) = \dfrac{1}{2}(e^x - e^{-x})$

(v) $f^{-1}(x) = \begin{cases} x, & x < 1 \\ \sqrt{x}, & 1 \le x \le 16 \\ \dfrac{x^2}{64}, & x > 16 \end{cases}$

2. $f^{-1}(x) = \dfrac{1 + \sqrt{1 + 4\log_2 x}}{2}$, $x > 0$

Exercise for Session 12

1. $\frac{2}{3}$ **2.** 17 **3.** $\frac{4}{3}$ **4.** $f(x) = \begin{cases} (1-x)^2, & 0 \le x \le \frac{1}{3} \\ 2x(1-x), & \frac{1}{3} \le x \le \frac{2}{3} \\ x^2, & \frac{2}{3} \le x \le 1 \end{cases}$

Chapter Exercises

1. (b)	**2.** (d)	**3.** (a)	**4.** (a)	**5.** (a)	**6.** (d)	**7.** (a)	**8.** (a)	**9.** (d)	**10.** (c)
11. (c)	**12.** (c)	**13.** (d)	**14.** (c)	**15.** (c)	**16.** (d)	**17.** (a)	**18.** (b)	**19.** (b)	**20.** (d)
21. (d)	**22.** (a)	**23.** (d)	**24.** (b)	**25.** (b)	**26.** (a)	**27.** (a)	**28.** (b)	**29.** (d)	**30.** (c)
31. (c)	**32.** (b)	**33.** (c)	**34.** (a)	**35.** (d)	**36.** (d)	**37.** (c)	**38.** (d)	**39.** (c)	**40.** (d)
41. (b)	**42.** (a)	**43.** (d)	**44.** (a)	**45.** (c)	**46.** (a)	**47.** (a)	**48.** (d)	**49.** (b)	**50.** (b)
51. (b)	**52.** (d)	**53.** (c)	**54.** (c)	**55.** (d)	**56.** (b)	**57.** (d)	**58.** (c)	**59.** (c)	**60.** (b)
61. (b)	**62.** (d)	**63.** (d)	**64.** (c)	**65.** (b)	**66.** (a)	**67.** (a)	**68.** (b)	**69.** (c)	**70.** (a)
71. (c)	**72.** (d)	**73.** (a)	**74.** (b)	**75.** (b)	**76.** (b)	**77.** (a,b,d)	**78.** (b,c,d)	**79.** (b,c,d)	**80.** (b,c,d)
81. (a, b)		**82.** (b, d)	**83.** (a,b)	**84.** (a,b, c, d)		**85.** (b, c)	**86.** (a, b, c, d)		
87. (a, b, c, d)		**88.** (a,b,c)	**89.** (b, d)	**90.** (b, c)		**91.** (a, b, c, d)	**92.** (c, d)	**93.** (a,b, c, d)	
94. (a, c)		**95.** (a, b, c)		**96.** (a, b, c, d)		**97.** (a, b, c)		**98.** (a, c)	
99. (a, d)		**100.** (b, c, d)		**101.** (a,b,d)		**102.** (c)	**103.** (b)	**104.** (c)	**105.** (a)
106. (d)	**107.** (a)	**108.** (a)	**109.** (d)	**110.** (d)	**111.** (a)	**112.** (c)	**113.** (d)	**114.** (c)	**115.** (a)
116. (c)	**117.** (b)	**118.** (b)	**119.** (b)	**120.** (a)	**121.** (c)	**122.** (d)	**123.** (d)	**124.** (c)	**125.** (b)
126. (d)	**127.** (b)	**128.** (a)	**129.** (c)	**130.** (a)	**131.** (c)	**132.** (b)	**133.** (a)	**134.** (b)	**135.** (b)
136. (c)		**137.** (a)							

138. (A) $\to$ (s), (B) $\to$ (p), (C) $\to$ (q), (D) $\to$ (r)

139. (A) $\to$ (r), (B) $\to$ (s), (C) $\to$ (p), (D) $\to$ (q)

140. (A) $\to$ (q), (B) $\to$ (r), (C) $\to$ (s), (D) $\to$ (p)

141. (A) $\to$ (p,q,s), (B) $\to$ (s), (C) $\to$ (q), (D) $\to$ (p,q,r,s)

142. (4)	**143.** (7)	**144.** (2)	**145.** (4)	**146.** (2)	**147.** (2)	**148.** (3)	**149.** (6)	**150.** (1)	**151.** (3)
152. (9)	**153.** (9)	**154.** (6)	**155.** (1)	**156.** (1)	**157.** (4)	**158.** (1)	**159.** (1)	**160.** (5)	**161.** (2)
162. (1)	**163.** (2)	**164.** (3)	**165.** (2)	**166.** (7)	**167.** (0)	**168.** (8)	**169.** (8)	**170.** (2)	**171.** (4)
172. (1)									

173. (i) $0, n + \dfrac{1}{2}$, where $n \in Z$

(ii) $\left\{-1, -\dfrac{1}{2}\right\}$ (iii) $\left\{0, 1, \dfrac{4}{3}, \dfrac{5}{3}\right\}$

(iv) $\{1, -1, \pm 1 + k$, where k is any positive proper fraction$\}$

(v) no solution

174. $P(x) = x + 3$ and $Q(x) = -x - 2$

175. $(2n - 1)$

176. $x \in (3, \pi) \cup \left(\pi, \dfrac{3\pi}{2}\right) \cup \left(\dfrac{3\pi}{2}, 5\right)$

177. (i) $S(6) = 9$

178. Possible solutions are $\dfrac{29}{12}, \dfrac{19}{6}, \dfrac{97}{24}$.

179. $\dfrac{1}{n}\sum_{i=1}^{n} x_i = 1$ **181.** $f(f(2)) = 26$

183. $g(x) = \begin{cases} x^2, & -2 \le x \le -1 \\ -x, & -1 \le x \le -1/4 \\ x + \frac{1}{2}, & -\frac{1}{4} \le x \le 0 \\ x, & 0 \le x \le 1 \\ x^2, & 1 \le x \le 2 \end{cases}$ **185.** $f(x) = \begin{cases} 3/2, & -3/2 < x \le -1/2 \\ 2 + x, & -1/2 \le x < 0 \\ 2, & 0 \le x \le 2 \\ x, & 2 \le x \end{cases}$

186. $(-3, 3), (-2, 4), (0, 0)$ and $(1, 1)$ **187.** (d)

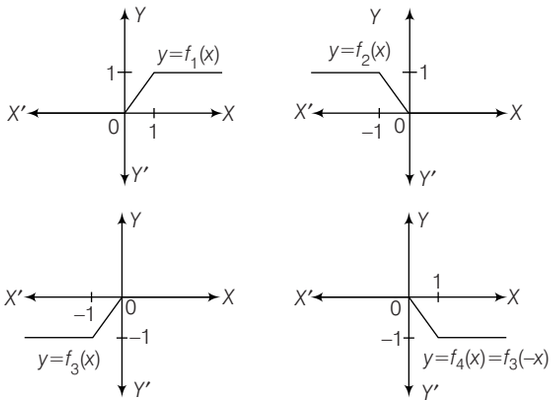
188. (2) **189.** (d) **190.** A $\to$ p, B $\to$ q, C $\to$ q, D $\to$ p

191. $\left[-\dfrac{\pi}{2}, -\dfrac{\pi}{10}\right] \cup \left[\dfrac{3\pi}{10}, \dfrac{\pi}{2}\right]$ **192.** (d) **193.** (d)

194. (a)	**195.** (a)	**196.** (b)	**197.** (c)	**198.** (d)	**199.** (b)
200. (a)	**201.** (c)	**202.** (b)	**203.** (a)	**204.** (a)	**205.** (d)
206. (a)	**207.** (d)	**208.** (d)	**209.** (d)		
210. (b)	**211.** (d)	**212.** (c)			

Solutions

1. Here,



$\Rightarrow \qquad f_1(x) = -f_3(-x)$

2. Only in option (d), the graph has a symmetry w.r.t. origin.

3. Here, $\qquad f(x) = \dfrac{4}{\sqrt{1-x^2}}$

$\Rightarrow \qquad f(\sin x) = \dfrac{4}{|\cos x|}$

and $\qquad f(\cos x) = \dfrac{4}{|\sin x|}$

$\therefore \qquad g(x) = |\sin x| + |\cos x|$

$\therefore \qquad$ Period of $g(x) = \dfrac{\pi}{2}$

4. If $x = 1$, we see $f(y) = \dfrac{f(1)}{y}$ for all y

Put $\qquad y = 30$

$\Rightarrow \qquad f(1) = 30 \cdot f(30) = 30(20)$

$= 600 \qquad$ ...(i)

Now, $\qquad$ let $y = 40$

$\Rightarrow \qquad f(40) = \dfrac{f(1)}{40} = \dfrac{600}{40} = 15$

5. $h(x) = \log(f(x) \cdot g(x)) = \log e^{\{e^{|x|} \text{ sgn } x\}} \cdot e^{[e^{|x|} \text{ sgn } x]}$

$= \log e^{e^{|x|} \text{ sgn } x} = e^{|x|} \text{ sgn } x$

$\therefore \qquad h(x) = e^{|x|} \text{ sgn } x$

$$= \begin{cases} e^x, & x > 0 \\ 0, & x = 0 \\ -e^{-x}, & x < 0 \end{cases}$$

$\Rightarrow \qquad h(x) + h(-x) = 0$ for all x

$\therefore \quad h(x)$ is odd.

6. (a) $f(x) = x^4 + 2x^3 - x^2 + 1$

A polynomial of degree even will always be into.

(b) $f(x) = x^3 + x + 1$

$\Rightarrow f'(x) = 3x^2 + 1$, i.e. injective as well as surjective.

(c) $f(x) = \sqrt{1 + x^2}$, neither injective nor surjective as range $\in [1, \infty)$.

(d) $f(x) = x^3 + 2x^2 - x + 1$

$\Rightarrow \quad f'(x) = 3x^2 + 4x - 1$

$\Rightarrow \qquad D > 0$

$\therefore \quad f(x)$ is surjective but not injective.

7. Here, $f(x)$ is bijective, hence $f^{-1}(4)$ exists when $y = 4$.

$\therefore \qquad 2x^3 + 7x - 9 = 0$

$\Rightarrow \qquad (2x^2 + 2x + 9)(x - 1) = 0$

$\Rightarrow x = 1$ only, as $2x^2 + 2x + 9 = 0$ has no other root.

8. Here, $f(x) = \dfrac{e^x \cdot \log x \cdot 5^{x^2+2} \cdot (x^2 - 7x + 10)}{2x^2 - 11x + 12}$

$$= \frac{e^x \cdot \log x \cdot 5^{x^2+2} \cdot (x-2)(x-5)}{(2x-3)(x-4)}$$

Note that at $x = \dfrac{3}{2}$ and $x = 4$, function is not defined and in open interval $\left(\dfrac{3}{2}, 4\right)$, function is continuous.

$\therefore \quad \lim\limits_{x \to \frac{3}{2}^+} \dfrac{e^x \cdot \log x \cdot 5^{x^2+2} \cdot (x-2)(x-5)}{(2x-3)(x-4)}$

$$= \frac{(+\text{ ve})(+\text{ve})(-\text{ ve})(-\text{ ve})}{(+\text{ ve})(-\text{ ve})} = -\infty$$

and $\lim\limits_{x \to 4^-} \dfrac{e^x \cdot \log x \cdot 5^{x^2+2} \cdot (x-2)(x-5)}{(2x-3)(x-4)}$

$$= \frac{(+\text{ve})(+\text{ve})(-\text{ve})}{(+\text{ve})(-\text{ve})} = \infty$$

In the open interval $\left(\dfrac{3}{2}, 4\right)$, the function is continuous and takes up all real values from $(-\infty, \infty)$.
Hence, range of the function is $(-\infty, \infty)$.

9. As, $x = \cos^{-1}(\cos 4) = \cos^{-1}(\cos(2\pi - 4)) = 2\pi - 4$

and $\qquad y = \sin^{-1}(\sin 3)$

$= \sin^{-1}(\sin(\pi - 3)) = \pi - 3$

$\therefore \qquad x + y = 3\pi - 7$

10. $f(x) = \left(\dfrac{2\sin x + \sin 2x}{2\cos x + \sin 2x} \cdot \dfrac{1 - \cos x}{1 - \sin x}\right)^{2/3}$

$$= \left(\frac{2\sin x \cdot (1 + \cos x)}{2\cos x \cdot (1 + \sin x)} \cdot \frac{(1 - \cos x)}{(1 - \sin x)}\right)^{2/3}$$

For domain, $\sin x \neq \pm 1$

$\Rightarrow \qquad x \in R - \left\{(4n-1)\dfrac{\pi}{2}, (4n+1)\dfrac{\pi}{2}\right\}$

$\therefore \quad f(x) = \left(\tan x \cdot \dfrac{(1-\cos^2 x)}{(1-\sin^2 x)} \right)^{2/3} = \tan^2 x$

$\Rightarrow \quad$ Range $\in [0, \infty)$

11. As, $f(x+2) = e^{\sin\{x+2\} \cdot \cos \pi(x+2)} = e^{\sin\{x\} \cdot \cos \pi x} = f(x)$

$\therefore$ Periodic with period 2.

12. Here, $y = x^4 - 2x^2 + 3 = (x^2-1)^2 + 2 \Rightarrow y_{\min} = 2$

$\Rightarrow \quad \log_{0.5}(x^4 - 2x^2 + 3) \le \log_{1/2} 2 = -1$

$\therefore \quad f(x) = \cot^{-1}(\log_{0.5}(x^4 - 2x^2 + 3)) \ge \cot^{-1}(-1)$

$\Rightarrow$ Range $\in \left[\dfrac{3\pi}{4}, \pi \right)$

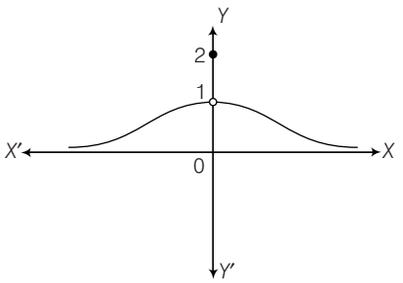
13. Here, $\log_e(x^2 + e) \ge 1$, for all $x \in R$

$\therefore \quad 0 < \dfrac{1}{\log(x^2+e)} \le 1$

$\Rightarrow \quad \left[\dfrac{1}{\log(x^2+e)} \right] = \begin{cases} 0, & x \ne 0 \\ 1, & x = 0 \end{cases}$

$\Rightarrow \quad f(x) = \begin{cases} \dfrac{1}{\sqrt{1+x^2}}, & x \ne 0 \\ 2, & x = 0 \end{cases}$

Which is shown in figure as

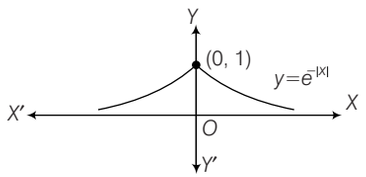


$\therefore$ Range of $f(x) \in [0, 1) \cup \{2\}$.

14. Let $x + 3 = t$

$\therefore \quad f(t) = \sin(\{t\}) \Rightarrow$ Period $= 1$

15. As, $e^{-|x|}$ can be shown by graphical transformation.



16. Here, $[x]\{x\} = 1$

$\Rightarrow \quad \{x\} = \dfrac{1}{[x]}$ or $x - [x] = \dfrac{1}{[x]} \Rightarrow x = [x] + \dfrac{1}{[x]}$

Obviously, $x > 2$

$\therefore \quad x = m + \dfrac{1}{m}, m \in N - \{1\}$

17. Here, $\sqrt{x^2 - 5x - 24} > x + 2$ is equivalent to the collection of two system of inequations.

***Case* I** $x^2 - 5x - 24 \ge 0$ and $x + 2 < 0$

$\Rightarrow \quad (x-8)(x+3) \ge 0$ and $x < -2$

$\Rightarrow \quad x \le -3 \quad \ldots(i)$

***Case* II** $x^2 - 5x - 24 \ge 0, x + 2 \ge 0$

and $\quad x^2 - 5x - 24 > (x+2)^2$

$\Rightarrow \quad x \in (-\infty, -3] \cup [8, \infty), x \in [-2, \infty)$

and $\quad x^2 - 5x - 24 > x^2 + 4x + 4$

$\Rightarrow \quad 9x < -28$

$\Rightarrow \quad x < \dfrac{-28}{9}$

$\therefore$ No solution, i.e. $x \in \phi$. $\quad \ldots(ii)$

From Eqs. (i) and (ii), we get

$x \in (-\infty, -3]$

18. Here, $\quad f(f(x)) = x$

$\Rightarrow \quad m(mx + b) + b = x$

$\Rightarrow \quad m^2 x + b(m+1) = x$

$\Rightarrow \quad m = \pm 1, b = 0$

If $f(-f(x)) = -x \Rightarrow -m(mx + b) + b = -x$

$\Rightarrow \quad -m^2 x + b(-m+1) = -x$

$\Rightarrow \quad m = \pm 1$ and $b = 0$

$\therefore$ Only 2 straight lines, i.e. $y = \pm x$.

19. Here, $f(g(x)) = \dfrac{rx}{1 + (r-1)x}$ and $g(f(x)) = rx$

If $f(g(x)) = g(f(x))$, then $\dfrac{rx}{1 + (r-1)x} = rx$

$\Rightarrow \quad rx = rx(1 + (r-1)x)$

$\Rightarrow \quad r(r-1)\, x^2 = 0$

If this is to be true for infinitely many x, then

$r(r-1) = 0 \Rightarrow r = 0, 1$

20. Since, f is a linear function, so it has the form $f(x) = mx + b$ because $f(1) \le f(2)$, we have $m \ge 0$. Similarly, $f(3) \le f(4)$ $\Rightarrow m \le 0$.

Hence, $m = 0$ and f is constant function. Thus, $f(0) = f(5) = 5$

21. Suppose R is just a rectangle whose four vertices are $(1, 2), (1, -2), (-1, -2), (-1, 2)$.

The X-axis and Y-axis symmetries in the problem are satisfied, but the point $(2, 1)$ is not contained in R.

22. Here, $2\{y\} = [x] + 1$

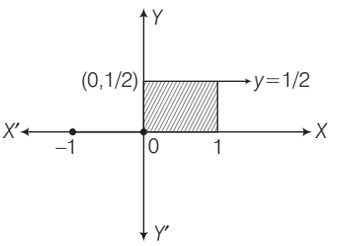
Since, $0 \le y < 1$

$\therefore \quad \{y\} = y$

and $\quad 0 \le [x] + 1 < 2$, i.e. $-1 \le [x] < 1$

When $-1 \le x < 0 \Rightarrow y = 0$

When $\quad 0 \le x < 1 \Rightarrow y = \dfrac{1}{2}$



$\therefore \quad$ The required area $= \dfrac{1}{2}$.

23. Here, $-1 \le \cos x \le 1$

$\Rightarrow \quad -\sin 1 \le \sin(\cos x) \le \sin 1$

$\Rightarrow \quad [\sin(\cos x)] = 0, -1$

Also, $\quad -1 \le \sin x \le 1$

$\Rightarrow \quad \cos 1 \le \cos(\sin x) \le 1$

$\therefore \quad [\cos(\sin x)] = 0, 1$

$\therefore \quad f(g(x))$ has range $\left\{\frac{-\pi}{2}, 0, \frac{\pi}{2}\right\}$.

24. As we know, $\quad \{f(x)\} \in [0, 1)$

$\Rightarrow \quad [\{f(x)\}] = 0$

$\therefore \quad e^{2x} + e^x - 2 = [\{x^2 + 10x + 11\}]$

$\Rightarrow \quad e^{2x} + e^x - 2 = 0$

$\Rightarrow \quad (e^x + 2)(e^x - 1) = 0$

$\therefore \quad e^x = 1$ is the only solution.

$\Rightarrow \quad x = 0$

$\Rightarrow$ Number of solutions is 1.

25. We know that, $\{x\} = x - [x]$.

$\therefore \quad \{x\} + \{2x\} + \{3x\} + \ldots + \{12x\} = 78x.$

$\Rightarrow \quad x - [x] + 2x - [2x] + 3x - [3x] + \ldots + 12x - [12x] = 78x$

$\Rightarrow \quad (x + 2x + 3x + \ldots + 12x) - ([x] + [2x] + \ldots + [12x]) = 78x$

$\Rightarrow \quad [x] + [2x] + \ldots + [12x] = 0$

$\therefore \quad 0 \le 12x < 1$

$\Rightarrow \quad 0 \le x < \frac{1}{12}, \ \forall \ \frac{1}{25} \le x \le \frac{1}{10}$

$\therefore \quad$ Common values of $x \in \left[\frac{1}{25}, \frac{1}{12}\right)$.

Since, x is of the form $\frac{1}{n}$.

$\therefore \quad x = \frac{1}{25}, \frac{1}{24}, \frac{1}{23}, \ldots, \frac{1}{14}, \frac{1}{13}$

i.e. 13 solutions.

26. $-\sqrt{20} \le 2\cos x - 4\sin x \le \sqrt{20}$

$\Rightarrow \quad 0 \le (2\cos x - 4\sin x)^2 \le 20$

$\min = \frac{1}{1 + 20} = \frac{1}{21}; \ \max = 1$

$\Rightarrow \quad M + m = \frac{22}{21}$

27. $f(x) = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right) + 2\sqrt{2}$

or $\quad f(x) = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right) + 2\sqrt{2}$

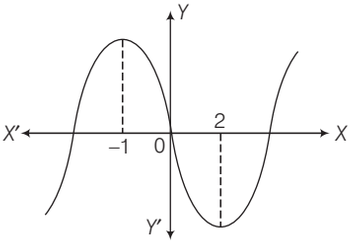
$\Rightarrow \quad Y = [\sqrt{2}, 3\sqrt{2}]$

and $\quad X = \left[-\frac{3\pi}{4}, \frac{\pi}{4}\right]$ or $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

28. Let $f(x) = 2x^3 - 3x^2 - 12x + a$, then

$f'(x) = 6(x^2 - x - 2) = 6(x + 1)(x - 2)$

So, the roots of $f'(x) = 0$ are $x = -1, 2$.



Now, $f(x) = 0$ will have all real roots, if $f(-1) > 0$ and $f(2) < 0$.

$\Rightarrow \quad -2 - 3 + 12 + a > 0$ and $16 - 12 - 24 + a < 0$

$\Rightarrow \quad -7 < a < 20$

29. $g(x) = \frac{e^{f(x)} - e^{|f(x)|}}{e^{f(x)} + e^{|f(x)|}}, \ -1 \le f(x) \le 1$

For $\quad 0 \le f(x) \le 1,$

$g(x) = 0$

$-1 \le f(x) < 0$

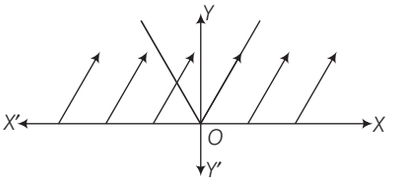
$g(x) = \frac{e^{f(x)} - e^{-f(x)}}{e^{f(x)} + e^{-f(x)}} = \frac{e^{2f(x)} - 1}{e^{2f(x)} + 1} = 1 - \frac{2}{e^{2f(x)} + 1}$

For $\quad -1 \le f(x) < 0,$

$$g(x) \in \left[\frac{1 - e^2}{1 + e^2}, 0\right)$$

For $\quad -1 \le f(x) < 1$, $g(x) \in \left[\frac{1 - e^2}{1 + e^2}, 0\right]$

30. $f(x) = \sqrt{|x| - \{x\}}; \ |x| \ge \{x\}$



$\Rightarrow \quad X \in \left(-\infty, -\frac{1}{2}\right] \cup [0, \infty)$

$\Rightarrow \quad Y \in [0, \infty)$ and $f(x)$ is many-one.

31. Given, curves are $y = \ln x$ and $y = ax$.

$\Rightarrow \ln x = ax$ has exactly two solutions.

$\Rightarrow \frac{\ln x}{x} = a$ has exactly two solutions to find the range of $\frac{\ln x}{x}$.

Let $y = \frac{\ln x}{x}, \ x > 0; \ \ \frac{dy}{dx} = \frac{x \cdot \frac{1}{x} - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$

y is increasing, if $1 - \ln x > 0$ or $\ln x < 1 \Rightarrow 0 < x < e$

Range of $y \in \left(-\infty, \frac{1}{e}\right]$ graph of $y = \frac{\ln x}{x}$

For exactly two solutions of $\frac{\ln x}{x} = a$

$\Rightarrow \quad a \in \left(0, \frac{1}{e}\right)$

32. Let $f(x) = ax^2 + bx + c$ as it touches X-axis at $x = 3$

$$\Rightarrow \quad \frac{-b}{2a} = 3$$

$$\Rightarrow \quad b = -6a \qquad \ldots(i)$$

Also, $\quad 9a + 3b + c = 0 \qquad \ldots(ii)$

$$4a - 2b + c = 3 \qquad \ldots(iii)$$

$$\Rightarrow \quad a = \frac{3}{25}, b = -\frac{18}{25}, c = \frac{27}{25}$$

$$\Rightarrow \quad f(x) = \frac{3}{25}(x^2 - 6x + 9)$$

33. $y = \sqrt{2\{x\} - \{x\}^2 - \frac{3}{4}}$

$$\Rightarrow \quad 2\{x\} - \{x\}^2 - \frac{3}{4} \geq 0 \Rightarrow \frac{1}{2} \leq \{x\} \leq \frac{3}{2}$$

$$\Rightarrow \quad \frac{1}{2} \leq \{x\} < 1 \qquad (\because 0 \leq \{x\} < 1)$$

$\therefore \quad 2\{x\} - \{x\}^2 - \frac{3}{4}$ is increasing for $\frac{1}{2} \leq \{x\} < 1$.

$\Rightarrow$ Range $= \left[0, \frac{1}{4}\right]$

34. Replace x by $-x$

$$\Rightarrow \quad x[f(x) + f(-x)] = 0$$

$\Rightarrow \quad f(x)$ is an odd function.

$\Rightarrow \quad f^{iv}(x)$ is also odd $\Rightarrow f^{iv}(0) = 0$

35. $[y + [y]] = 2\cos x \quad \Rightarrow \quad [y] = \cos x$

$$\Rightarrow \quad y = \frac{1}{3}[\sin x + [\sin x + [\sin x]]] = [\sin x]$$

$\Rightarrow [\sin x] = \cos x$

Number of solutions in $[0, 2\pi]$ is 0.

Hence, total solution is 0.

$\therefore$ Both are periodic with period 2π.

36. By replacing $x = x + 1$ and $x = x - 1$, we get

$$f(x + 2) + f(x) = \sqrt{2} f(x + 1) \qquad \ldots(i)$$

$$f(x) + f(x - 2) = \sqrt{2} f(x - 1) \qquad \ldots(ii)$$

From Eqs. (i) and (ii), gives

$$f(x + 2) + f(x - 2) + 2f(x)$$
$$= \sqrt{2}\,[f(x + 1) + f(x - 1)]$$
$$= \sqrt{2}\sqrt{2} f(x)$$

$\therefore \quad f(x + 2) + f(x - 2) = 0$

On replacing x by $x + 2$, we get

$$f(x + 4) + f(x) = 0$$
$$f(x + 8) = -f(x + 4) = f(x), \forall\ x$$

$\therefore \quad f(x)$ is periodic with period 8.

37. The equation has meaning, if $x > 0, x \neq 1$.

$\therefore \quad$ Domain $= (0, 1) \cup (1, \infty)$

If $x \in (0, 1)$, then $\log_2 x < 0$

and $\quad \log_2 x + \log_x 2 = \frac{\log x}{\log 2} + \frac{\log 2}{\log x}$

$$= \text{sum of a negative number} \leq -2$$

In this case any α will satisfy, since $2\cos\alpha$ can never be more than 2.

Thus, the inequation is satisfied for any x in (0, 1) and for any α.

If $x \in (1, \infty)$, then $\log_2 x > 0 \Rightarrow \frac{\log x}{\log 2} + \frac{\log 2}{\log x} > 0$

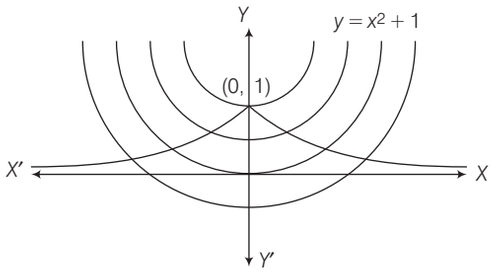
The inequation cannot be satisfied unless

$$\cos\alpha = -1 \text{ and } x = 2$$

i.e. $\quad \log_2 x = 1$

Option (d) is wrong, since in the last case there are infinite solution.

38. If we draw the graph of $\left(\frac{1}{2}\right)^{|x|}$ and $x^2 - a$, then the range of value of a will be $(-1, \infty)$.



Maximum possible solution for 'x' is '2'.

39. $f(x) \neq |\cos x|$ is true only when

$$|\cos x| = 1 \quad \Rightarrow \quad x = 0, \pi, 2\pi$$

40. $|\cos x| + \cos x > 0 \rightarrow x \in \left[0, \frac{\pi}{2}\right)$

For $\quad x \in \left[0, \frac{\pi}{2}\right), 1 \leq \sin x + \cos x \leq \sqrt{2}$

But $\quad \sin x + \cos x \neq 1 \Rightarrow x \in \left(0, \frac{\pi}{2}\right)$

Now, for $x \in \left(0, \frac{\pi}{2}\right)$,

$$\log_{\sin x + \cos x}(|\cos x| + \cos x) \geq 0$$

$$\Rightarrow \quad \cos x \geq \frac{1}{2} \Rightarrow x \in \left(0, \frac{\pi}{3}\right]$$

41. $y = ((x + \alpha)^2 - 1)^{1/4} = [(x + \alpha - 1)(x + \alpha + 1)]^{1/4}$

$$(x + \alpha - 1)(x + \alpha + 1) \geq 0$$

$$x \geq 1 - \alpha \text{ and } x \leq -1 - \alpha \text{ for } \alpha > 0$$

For $\quad \alpha < 0, x \leq 1 - \alpha, x \geq -1 - \alpha$

$$(x + \alpha) \leq 1 \text{ and } (x + \alpha) \geq -1$$

For $\alpha \leq -1, x \leq 0$ and range is $[0, \infty)$.

42. $f(x) = \log(\log(\log x))$

$$\log x > 1 \text{ when } x \in (e, \infty)$$

$\therefore \log(\log x) > 0$ and hence, $\log(\log(\log x))$ is well defined and uniquely.

It is evidently one-one. Since, the range of $\log x = R$, $f(x)$ is one-one and onto.

43. $\quad x^2 - 4px + q^2 > 0, \forall\ x \in R$

$$\Rightarrow \quad 4p^2 - q^2 < 0 \qquad \ldots(i)$$

$$r^2 + p^2 < qr \qquad \ldots(ii)$$

Let $y = \dfrac{x + r}{x^2 + qx + p^2}$

$\Rightarrow$ $x^2y + x(qy - 1) + p^2y - r = 0$ ...(iii)

x is real,

$\Rightarrow$ $(q^2 - 4p^2)y^2 + y(-2q + 4r) + 1 > 0$

From Eq. (i) $\Rightarrow$ Coefficient of y^2 is a positive discriminant.

$= (4r - 2q)^2 - 4(q^2 - 4p^2)$

$= 16(r^2 + p^2 - qr) < 0$ [from Eq. (ii)]

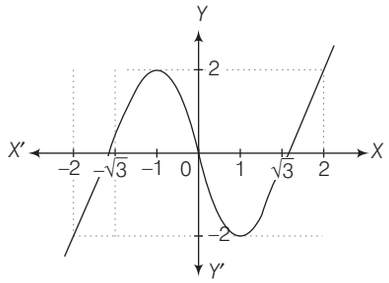
Hence, Eq. (iii) is true for all real y or $y \in (-\infty, \infty)$.

44. Here,

$$f(x) = \frac{x^4 - \lambda x^3 - 3x^2 + 3\lambda x}{x - \lambda}$$

$\therefore$ $f(x) = \dfrac{(x^3 - 3x)(x - \lambda)}{(x - \lambda)}$

Consider, $g(x) = x^3 - 3x$ that can be shown as



Now, the range of $f(x)$ is the set of entire real numbers, if $\lambda \in [-2,2]$.

Because, if $\lambda > 2$ or $\lambda < -2$, then range of $f(x) \notin R$.

45. We know that,

$$(a-1)^n = {}^nC_0a^n - {}^nC_1a^{n-1} + {}^nC_2a^{n-2} - \ldots + (-1)^{n-1} \cdot {}^nC_{n-1} \cdot a + (-1)^n \cdot {}^nC_n$$

$\therefore$ $\dfrac{(a-1)^n}{a} = {}^nC_0a^{n-1} - {}^nC_1 \cdot a^{n-2} + {}^nC_2 \cdot a^{n-3} + \ldots + (-1)^{n-1} \cdot {}^nC_{n-1} + (-1)^n \cdot \dfrac{{}^nC_n}{a}$

Hence, $f(n) = \dfrac{(a-1)^n - (-1)^n}{a}$

Now, $f(2016) + f(2017) = \dfrac{(a-1)^{2016} - 1}{a} + \dfrac{(a-1)^{2017} + 1}{a}$

$= \dfrac{(a-1)^{2016}[1 + a - 1]}{a} = (a-1)^{2016}$, where $a = 3^{\frac{1}{224}} + 1$

$\therefore$ $f(2016) + f(2017) = (3^{\frac{1}{224}})^{2016} = 3^9 = 3^K$

$\Rightarrow$ $K = 9$

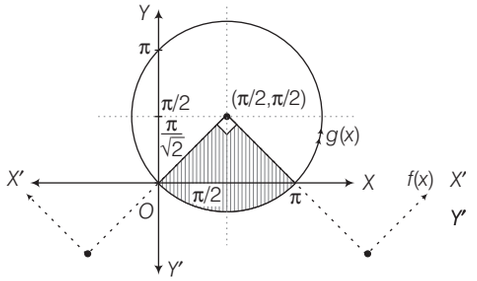
46. Here,

$$g(x) = \frac{\pi}{2} - \sqrt{\frac{\pi^2}{2} - \left(x - \frac{\pi}{2}\right)^2}$$

or $\left(y - \dfrac{\pi}{2}\right)^2 = \dfrac{\pi^2}{2} - \left(x - \dfrac{\pi}{2}\right)^2$

$\Rightarrow$ $\left(x - \dfrac{\pi}{2}\right)^2 + \left(y - \dfrac{\pi}{2}\right)^2 = \dfrac{\pi^2}{2}$, i.e. circle

and $f(x) = \sin^{-1}(\sin x)$ is shown as



$\therefore$ Area of shaded part $= \dfrac{\theta}{360^\circ} \times \pi r^2$

$= \dfrac{90^\circ}{360^\circ} \times \pi \times \dfrac{\pi^2}{2} = \dfrac{\pi^3}{8}$ sq units

47. Let $y = \dfrac{x^2 + bx + 1}{x^2 + 2x + b}\ (b > 1)$

$\Rightarrow$ $(y-1)x^2 + (2y - b)x + (by - 1) = 0$

$\Rightarrow$ $D \geq 0 \Rightarrow (4 - 4b)y^2 + 4y + (b^2 - 4) \geq 0$ ...(i)

Since, $f(x)$ and $\dfrac{1}{f(x)}$ have the same bounded set as their range.

Thus, $(4 - 4b)y^2 + 4y + (b^2 - 4) \geq 0$ have roots α and $\dfrac{1}{\alpha}$.

$\therefore$ Product of roots $= 1$

$\Rightarrow$ $\dfrac{b^2 - 4}{4(1 - b)} = 1 \Rightarrow b^2 - 4 = 4 - 4b$

or $b^2 + 4b - 8 = 0$

$$b = \frac{-4 \pm \sqrt{16 + 32}}{2} = \frac{-4 \pm 4\sqrt{3}}{2} = 2\sqrt{3} - 2$$

48. Since, $\sin\dfrac{\pi[x + 24]}{12} = \sin\dfrac{\pi}{12}(24 + [x])$

$= \sin\left(2\pi + \dfrac{\pi[x]}{2}\right) = \sin\dfrac{\pi[x]}{12}$

The period of $\sin\dfrac{\pi[x]}{12}$ is 24.

Similarly, period of $\cos\dfrac{\pi[x]}{4}$ is 8 and period of $\tan\dfrac{\pi[x]}{3} = 3$.

Hence, the period of the given function = LCM of 24, 8, 3 = 24.

49. Let $f(x, y) = ax + by$

Then, $f(2x + 3y, 2x - 7y)$

$= a(2x + 3y) + b(2x - 7y) = 20x$ [given]

$\therefore$ $2a + 2b = 20$ and $3a - 7b = 0$

$\therefore$ $a = 7$ and $b = 3$

$\therefore$ $f(x, y) = 7x + 3y$

50. Domain of $f(x)$ is [1, 3] and the function is continuous.

$$f'(x) = \frac{1}{2\sqrt{x-1}} - \frac{1}{\sqrt{3-x}} = 0$$

$\Rightarrow$ $\sqrt{3 - x} = 2\sqrt{x - 1}$

$\Rightarrow \quad 3 - x = 4x - 4 \Rightarrow x = \frac{7}{5}$

$\therefore$ Critical points in $[1, 3]$ are $1, \frac{7}{5}$ and 3.

$f(1) = 2\sqrt{2},\ f(3) = \sqrt{2}$

and $f\left(\frac{7}{5}\right) = \sqrt{\frac{2}{5}} + 2\sqrt{\frac{8}{5}} = \sqrt{10}, 2\sqrt{2}$ being $< \sqrt{10}$, the range $= [\sqrt{2}, \sqrt{10}]$.

51. $f(x) = \cos^{-1}(\sec(\cos^{-1} x)) + \sin^{-1}(\operatorname{cosec}(\sin^{-1} x))$

$\Rightarrow \quad -1 \le \sec(\cos^{-1} x) \le 1$

and $\quad -1 \le \operatorname{cosec}(\sin^{-1} x) \le 1$

$\Rightarrow \quad \sec(\cos^{-1} x) = \pm 1$

and $\quad \operatorname{cosec}(\sin^{-1} x) = \pm 1$

$\Rightarrow \quad \cos^{-1} x = 0, \pi \quad \text{and} \quad \sin^{-1} x = \frac{\pi}{2}, -\frac{\pi}{2}$

$\Rightarrow \quad x = \pm 1$ and $x = \pm 1$

$\therefore$ Domain is $x = \pm 1$.

52. Put $x = y = 1 \Rightarrow f(1) = 2$

Put $\quad y = \frac{1}{x} \Rightarrow f(x) + f\left(\frac{1}{x}\right) = f(x)\, f\left(\frac{1}{x}\right)$

$\Rightarrow \quad f(x) = x^3 + 1$

$\Rightarrow \quad g(x) = 0, \forall\ x \in R - \{0\}$

53. $f(x) = f'(x) \times f''(x)$ is satisfied by only the polynomial of degree 4.

Since, $f(x) = 0$ satisfies $x = 1, 2, 3$ only. It is clear one of the roots is twice repeated.

$\therefore \quad f'(1) f'(2) f'(3) = 0$

54. Total number of functions for which $f(i) \ne i = 4^5$ and number of onto functions in which $f(i) \ne i = 44$.

$\therefore$ Required number of functions $= 980$

55. We can send $1, 2, \ldots, n-1$ anywhere and the value of $f(n)$ will then be uniquely determined.

56. Put $\sin x = t$,

$y = t^3 - 6t^2 + 11t - 6, -1 \le t \le 1$

$f(-1) = -24,\ f(1) = 0$

57. $g(x) = f(x^2 - 2x - 1) + f(5 - x^2 + 2x)$

$= 2x^4 - 8x^3 - 4x^2 + 24x + 18$

$g'(x) = 8x^3 - 24x^2 - 8x + 24$

$g'(x) = 0 \quad \Rightarrow \quad x = -1, 1, 3$

We observe that,

$g(x) \ge \min\{g(-1), g(1), g(3)\} = 0$

$\therefore \quad g(x) \ge 0, \forall\ x \in R$

58. Let $f(x) = [x],\ g(x) = \frac{e^{-|x|}}{2}$

$\Rightarrow \quad h(x) = \left[\frac{e^{-|x|}}{2}\right] = 0, \forall\ x \in R$

59. There are four possible functions defined in $0 \le x < 1$, of them 2 are continuous and two are discontinuous, now for each of the points $(1, 2, \ldots, n-1)$, keep functions fixed from left of the point, so there are 4 possible functions defined in between next two consecutive integral points of them only one is continuous and at last for $x = n$, there is only one possibility of discontinuity of the function. So, total number of functions

$= 2 \times 3^{n-1} \times 1$

60. $\because \quad k \in$ odd

$f(x) = k + 3$ [even]

$f(f(k)) = \frac{k+3}{2}$

If $\frac{k+3}{2} \in$ odd $\Rightarrow 27 = \frac{k+3}{2} + 3$

$\Rightarrow \quad k = 45$ not possible.

Now, let $\frac{k+3}{2} \in$ even

$\therefore \quad 27 = f(f(f(k))) = f\left(\frac{k+3}{2}\right) = \frac{k+3}{4}$

$\therefore \quad k = 105$

Verifying $f(f(f(105))) = f(f(108)) = f(54) = 27$

$\therefore \quad k = 105$

Hence, sum of digits of $k = 1 + 0 + 5 = 6$

61. $f(x) = \frac{\sin(\pi\{x\})}{x^4 + 3x^2 + 7}$

Here, $\quad f(1/2) = f(-1/2)$

Clearly, $f(x)$ is not one-one and also it is dependent on x.

62. Since, $f(x)$ and $g(x)$ are one-one and onto and are also the mirror images of each other which respect to the line $y = a$. It clearly indicates that $h(x) = f(x) + g(x)$ will be a constant function and will always be equal to $2a$.

63. Let $g(t) = 2t^3 - 15t^2 + 36t - 25$

$g'(t) = 6t^2 - 30t + 36 = 6(t^2 - 5t + 6)$

$= 6(t-2)(t-3) = 0 \quad \Rightarrow \quad t = 2, 3$

For $\quad 2 \le t \le 3$,

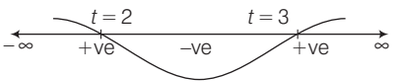
$g(t)_{\min} = g(3) = 2 \times 27 - 15 \times 9 + 36 \times 3 - 25 = 2$

Also, $\quad 2 + |\sin t| \ge 2$

Hence, minimum $\phi(t) = 2$

$\therefore \quad 3^x + 3^{f(x)} = 2$

$\Rightarrow \quad 3^{f(x)} = 2 - 3^x \quad \because \quad 3^{f(x)} > 0$



$\Rightarrow\ 2 - 3^x > 0 \quad \Rightarrow \quad 3^x < 2 \quad \Rightarrow \quad x < \log_3 2$

$\therefore \quad x \in (-\infty, \log_3 2)$

64. Here, $x_1 x_2 x_3 = x$ and x be the element of A.

$\therefore \quad x_1 x_2 x_3 = \frac{120}{x_4}$

$\Rightarrow\ x_1 x_2 x_3 x_4 = 120 = 2^3 \times 3^1 \times 5^1$.

Thus, to find number of positive integer solutions of

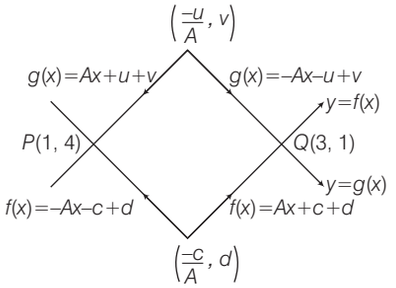
$$x_1 x_2 x_3 x_4 = 2^3 \times 3^1 \times 5^1.$$

i.e. $\alpha+\beta+\gamma+\delta=3, \alpha+\beta+\gamma+\delta=1, \alpha+\beta+\gamma+\delta=1$

$$\Rightarrow \quad {}^{3+4-1}C_{4-1} \cdot {}^{1+4-1}C_{4-1} \cdot {}^{1+4-1}C_{4-1} = {}^6C_3 \cdot {}^4C_3 \cdot {}^4C_3$$

$$= 20 \times 4 \times 4 = 320 \quad \therefore \quad d = 320$$

65.



From above figure,

$4 = A + u + v, \quad 1 = -3A - u + v.$ ...(i)

and $1 = 3A + c + d, 4 = -A - c + d$ ...(ii)

From Eq. (i), $3 = 4A + 2u$ ...(iii)

From Eq. (ii), $-3 = 4A + 2c$ ...(iv)

On adding Eqs. (iii) and (iv), we get

$$8A + 2u + 2c = 0$$

$$\therefore \quad \frac{u+c}{A} = -4$$

66. As, $f(x)$ is one-one, if $f'(x) \geq 0, \forall x \in R$.

$\therefore \quad 3x^2 + 2x + 4 + a\cos x - b\sin x \geq 0, \forall x \in R.$

$\Rightarrow \quad 3x^2 + 2x + 4 \geq b\sin x - a\cos x, \forall x \in R$

$\Rightarrow \quad 3x^2 + 6x + 4 \geq \sqrt{a^2+b^2}, \forall x \in R$

$[\text{as}, |b\sin x - a\cos x| \leq \sqrt{a^2+b^2}]$

$\Rightarrow \quad 3x^2 + 6x + 3 + 1 \geq \sqrt{a^2+b^2}, \forall x \in R$

$\Rightarrow \quad 3(x+1)^2 + 1 \geq \sqrt{a^2+b^2}, \forall x \in R$

$\therefore \quad \sqrt{a^2+b^2} \leq 1 + 3(x+1)^2$, since $1 + 3(x+1)^2 \geq 1$

$\Rightarrow \quad \sqrt{a^2+b^2} \leq 1$

$\therefore$ Greatest value of $(a^2+b^2) = 1$

67. Here, $\frac{p-1}{p+1} = \frac{x^4+x^2+1}{(x^2+x+1)^2} = \frac{x^4+2x^2+1-x^2}{(x^2+x+1)^2} = \frac{(x^2+1)^2-x^2}{(x^2+x+1)^2}$

$$= \frac{(x^2+x+1)(x^2-x+1)}{(x^2+x+1)^2}$$

$\therefore \quad \frac{p-1}{p+1} = \frac{x^2-x+1}{x^2+x+1}$, using componendo and dividendo

$\Rightarrow \quad \frac{2p}{2} = \frac{2(x^2+1)}{2x}$

$\Rightarrow \quad p = x + \frac{1}{x}$ ...(i)

As, $f(x) = \frac{1-x}{1+x} \Rightarrow f(f(x)) + f\left(f\left(\frac{1}{x}\right)\right) = x + \frac{1}{x}$ ...(ii)

From Eqs. (i) and (ii), we get $f(f(x)) + f\left(f\left(\frac{1}{x}\right)\right) = p$

68. As, α^{15} and α is the root of $x^{14} + x^{13} + \ldots + x + 1 = 0$

Now, $f(x) = x^{13} + 2x^{12} + 3x^{11} + \ldots + 13x + 14$

$\therefore \quad \frac{f(x)}{x} = x^{12} + 2x^{11} + 3x^{10} + \ldots + \frac{14}{x}$

On subtracting, we get

$$\left(1 - \frac{1}{x}\right) f(x) = x^{13} + x^{12} + x^{11} + \ldots + x + 1 - \frac{14}{x}$$

Put $x = \alpha, \left(\frac{\alpha-1}{\alpha}\right) f(\alpha) = -\alpha^{14} - \frac{14}{\alpha} = \frac{-\alpha^{15}-14}{\alpha} = -\frac{15}{\alpha}$

$\therefore \quad f(\alpha) = \frac{15}{1-\alpha}.$

Hence, $N = \frac{15^{14}}{(1-\alpha)(1-\alpha^2)\ldots(1-\alpha^{14})}$

and we know that

$x^{15} - 1 = (x-1)(x-\alpha)(x-\alpha^2)(x-\alpha^3)\ldots(x-\alpha^{14})$

$\therefore \quad (x-\alpha)(x-\alpha^2)(x-\alpha^3)\ldots(x-\alpha^{14}) = \frac{x^{15}-1}{x-1}$

As, $x \to 1$.

$$(1-\alpha)(1-\alpha^2)(1-\alpha^3)\ldots(1-\alpha^{14}) = \lim_{x\to 1} \frac{x^{15}-1}{x-1} = 15$$

$\therefore \quad N = 15^3 = 3^{13} \cdot 5^{13}$

Thus, number of divisors $= (13+1)(13+1) = 14^2 = 196.$

69. Here, domain of $f(x) \Rightarrow x \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$ only

$\therefore \quad f(x)$ is minimum when $x = \frac{1}{2}$, i.e. $f_{\min}\left(\frac{1}{2}\right) = \frac{\pi}{2}$

and $f(x)$ is maximum when $x = -\frac{1}{2}$, i.e. $f_{\max}\left(-\frac{1}{2}\right) = \frac{3\pi}{2}$

$\therefore$ Sum of maximum and minimum value of function is 2π.

70. Here, $\tan^{-1}(x^2 - 18x + a) > 0, \forall \ x \in R$

$\Rightarrow \quad x^2 - 18x + a > 0, \ \forall \ x \in R$

$\Rightarrow \quad (18)^2 - 4a < 0$

$\Rightarrow \quad a > 81$

$\Rightarrow \quad a \in (81, \infty)$

71. As, $\sin^2 x + \sin x + 1 > 0, \forall x \in R.$

$\therefore \quad \frac{1}{\sqrt{\sin^2 x + \sin x + 1}}$ is always exists.

For $\sin^{-1}\left(\frac{1}{|x^2-1|}\right)$ to exists,

$$0 < \frac{1}{|x^2-1|} \leq 1$$

$\Rightarrow \quad |x^2 - 1| \geq 1$

$\Rightarrow \quad x^2 - 1 \leq -1 \quad \text{or} \quad x^2 - 1 \geq 1$

$\Rightarrow \quad x^2 \leq 0 \quad \text{or} \quad x^2 \geq 2$

$\Rightarrow \quad x = 0 \quad \text{or} \quad (x \leq -\sqrt{2} \text{ or } x \geq \sqrt{2})$

$\therefore \quad x \in (-\infty, -\sqrt{2}] \cup [\sqrt{2}, \infty) \cup \{0\}.$

72. Here, $\log(x^2 - x + 1)$ is defined, when

$$x^2 - x + 1 > 0 \quad \text{and} \quad x^2 - x + 1 \neq 1$$

$$\Rightarrow \qquad x \in R \quad \text{and } x \neq 0, 1 \Rightarrow x \in R - \{0, 1\} \qquad \ldots(i)$$

Again, $\log(\sin^{-1}\sqrt{x^2 + x + 1})$ exists, when

$$0 < x^2 + x + 1 \leq 1$$

$$\Rightarrow \qquad x^2 + x \leq 0 \quad \Rightarrow \quad x \in [-1, 0] \qquad \ldots(ii)$$

$$x \in [-1, 0)$$

73. Here, $\sqrt{\cos^{-1} x}$ is defined for $x \in [-1, 1]$ for $\sqrt{\sin^{-1}(3x - 4x^3)}$ for x defined $0 \leq 3x - 4x^3 \leq 1$

$$\therefore \qquad 3x - 4x^3 \geq 0 \text{ and } 3x - 4x^3 \leq 1$$

$$\Rightarrow \quad x \in \left(-\infty, -\frac{\sqrt{3}}{2}\right] \cup \left[0, \frac{\sqrt{3}}{2}\right] \text{ and } x \in [-1, \infty)$$

$$\Rightarrow \quad x \in \left[-1, -\frac{\sqrt{3}}{2}\right] \cup \left[0, \frac{\sqrt{3}}{2}\right]$$

74. Here, $f(x)$ exists only, if

$$4^x + 8^{\frac{2}{3}(x-2)} - 52 - 2^{2(x-1)} \geq 0.$$

$$\Rightarrow \qquad 2^{2x} + 2^{2(x-2)} - 2^{2(x-1)} \geq 52$$

$$\Rightarrow \qquad 2^{2x} \geq 64 \Rightarrow x \geq 3$$

75. $y = |\sin^{-1}(2x^2 - 1)|$

$$\therefore \quad \frac{dy}{dx} = \frac{|\sin^{-1}(2x^2 - 1)|}{\sin^{-1}(2x^2 - 1)} \cdot \frac{4x}{|2x|\sqrt{1 - x^2}}$$

which would exist, if

$$|2x| \neq 0, \ \sin^{-1}(2x^2 - 1) \neq 0 \quad \text{and} \quad 1 - x^2 > 0$$

$$\Rightarrow \qquad x \neq 0, \ 2x^2 - 1 \neq 0$$

and $\quad |x| < 1 \Rightarrow x \neq 0, \pm\frac{1}{\sqrt{2}}$ and $|x| < 1$

$$\Rightarrow \qquad x \in (-1, 1) \sim \left\{0, \pm\frac{1}{\sqrt{2}}\right\}$$

76. The given function is defined for $5x^2 - 8x + 4 > 0$ which is true, $\forall\ x \in R$.

Since, coefficient of $x^2 = 5 > 0$ and $D = 64 - 80 = -16 < 0$

Let $\qquad g(x) = 5x^2 - 8x + 4$

Here, $\qquad a = 5 > 0$

$$\therefore \text{Range of } g(x) = \left[-\frac{D}{4a}, \infty\right) = \left[-\frac{-16}{4 \times 5}, \infty\right) = \left[\frac{4}{5}, \infty\right)$$

As, $\frac{4}{5} \leq 5x^2 - 8x + 4 < \infty$

$$\therefore \qquad \log_{5/4}\left(\frac{4}{5}\right) \leq \log_{5/4}(5x^2 - 8x + 4) < \log_{5/4} \infty$$

$$\Rightarrow \qquad -1 \leq \log_{5/4}(5x^2 - 8x + 4) < \infty$$

$$\Rightarrow \quad \tan^{-1}(-1) \leq \tan^{-1}\{\log_{5/4}(5x^2 - 8x + 4)\} < \tan^{-1}(\infty)$$

$$\Rightarrow \qquad -\frac{\pi}{4} \leq f(x) < \frac{\pi}{2}$$

$$\therefore \qquad R_f = \left[\frac{-\pi}{4}, \frac{\pi}{2}\right)$$

77. Functions which are not algebraic, are known as transcendal functions.

78. $f(x) = \dfrac{\sqrt{(\sqrt{x-1})^2 + 1 - 2\sqrt{x-1}}}{\sqrt{x-1} - 1} \cdot x$

$$= \frac{\left|\sqrt{x-1} - 1\right|}{\sqrt{x-1} - 1} \cdot x = \begin{cases} x, & \text{when } x \in [2, \infty) \\ -x, & \text{when } x \in [1, 2) \end{cases}$$

$$f'(x) = \begin{cases} 1, & x \in [2, \infty) \\ -1, & x \in [1, 2) \end{cases}$$

$$\therefore \ f'(10) = 1, \ f'\left(\frac{3}{2}\right) = -1$$

79. $f(x) = \dfrac{1 + \cos 2x}{2} + \dfrac{1 + \cos\left(\frac{2\pi}{3} + 2x\right)}{2} - \dfrac{\left[\cos\left(2x + \frac{\pi}{3}\right) + \cos\left(\frac{\pi}{3}\right)\right.}{2}$

$$= \frac{1}{2}\left[2 + \cos 2x + \cos\left(\frac{2\pi}{3} + 2x\right) - \cos\left(2x + \frac{\pi}{3}\right) - \frac{1}{2}\right]$$

$$= \frac{1}{2}\left[\frac{3}{2} + 2\cos\left(2x + \frac{\pi}{3}\right) \cdot \cos\frac{\pi}{3} - \cos\left(2x + \frac{\pi}{3}\right)\right] = \frac{3}{4}$$

$\therefore\ f(x)$ is even function, periodic function and

$$f(0) = f(1) = \frac{3}{4}.$$

80. (a) Let $f(x) = \sin(\sin^{-1} x) = x, \forall\ x \in [-1, 1]$

So, $f(x)$ is one-one and onto.

(b) Let $f(x) = \dfrac{2}{\pi}\sin^{-1}(\sin x) = \dfrac{2}{\pi}x$

The range is $\left[-\frac{2}{\pi}, \frac{2}{\pi}\right]$.

So, $f(x)$ is one-one and into.

(c) Let $f(x) = \text{sgn}(x) \cdot \log e^x = \begin{cases} x, & x > 0 \\ -x, & x < 0 \\ 0, & x = 0 \end{cases}$

$\therefore$ Range is $[0, 1]$ for $x \in [-1, 1]$

So, $f(x)$ is many-one and into.

(d) Let $f(x) = x^3 \cdot \text{sgn}(x) = \begin{cases} x^3, & x > 0 \\ -x^3, & x < 0 \\ 0, & x = 0 \end{cases}$

So, $f(x)$ is many-one and into.

81. (a) $(f + g)(3.5) = f(3.5) + g(3.5) = (-0.5) + (0.5) = 0$

(b) $f(g(3)) = f(0) = 3$

(c) $f(g(2)) = f(-1) = 8$

(d) $(f - g)(4) = f(4) - g(4) = 0 - 26 = -26$

82. Here, $f(x) = x^2 - 2ax + a(a + 1)$

$$\Rightarrow \qquad f(x) = (x - a)^2 + a, \ x \in [a, \infty)$$

Let $\qquad y = f(x) = (x - a)^2 + a$

Clearly, $y \geq a$

$\Rightarrow \quad (x-a)^2 = (y-a) \Rightarrow x = a + \sqrt{y-a}$

$\therefore \quad f^{-1}(x) = a + \sqrt{x-a}$

Now, $\quad f(x) = f^{-1}(x)$

$\Rightarrow \quad (x-a)^2 + a = a + \sqrt{x-a}$

$\Rightarrow \quad (x-a)^2 = \sqrt{x-a}$

or $\quad (x-a)\left((x-a)^3 - 1\right) = 0$

$\Rightarrow \quad x = a \text{ or } a+1$

If $a = 5049$, then $a + 1 = 5050$

If $a + 1 = 5049$, then $a = 5048$

83. Here, $g(x) = \sin(\sin^{-1}\sqrt{\{x\}}) + \cos(\sin^{-1}\sqrt{\{x\}}) - 1$

$= \sqrt{\{x\}} + \cos(\cos^{-1}\sqrt{1-\{x\}}) - 1$

$= \sqrt{x} + \sqrt{1-\{x\}} - 1$

If $x \in I$, then $\{x\} = 0 \Rightarrow g(x) = 0$

Also, $g(-x) = g(x) \Rightarrow g(x)$ is even.

If $x \notin I$, then $\{-x\} = 1 - \{x\}$

$\Rightarrow \quad g(-x) = \sqrt{1-\{x\}} + \sqrt{\{x^2\}-1} = g(x)$

$\Rightarrow g(x)$ is even function.

$\therefore g(x) = 0, x \in I$ and $g(x) = g(-x), x \notin I$

$\Rightarrow g(x)$ is periodic function.

84. Given, $f(a + x) = f(a - x)$

(a) $f(2a - x) = f(a + (a - x)) = f(a - (a - x)) = f(x)$

$\therefore \quad f(2a - x) = f(x)$ ...(i)

(b) $f(2a + x) = f(a + (a + x)) = f(a - (a + x)) = f(-x)$

$\therefore \quad f(2a + x) = f(-x)$ ...(ii)

(c) $f(2b + x) = f(-x)$ [from Eq. (ii)] ...(iii)

(d) From Eqs. (ii) and (iii), we get

$f(2a + x) = f(2b + x)$

$\therefore$ Period is $(2b - 2a)$.

85. Here, $f(1 + x) = f(1 - x)$

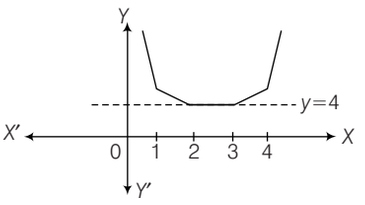
$\Rightarrow f(x)$ is symmetric about $x = 1$.

$\therefore \quad g(x) = f(1 + x) \Rightarrow g(-x) = f(1 - x) = f(1 + x) = g(x)$

$\Rightarrow g(x)$ is an even function.

86. $f(x) = \begin{cases} 10 - 4x, & \text{if } -\infty < x < 1 \\ 8 - 2x, & \text{if } 1 < x \leq 2 \\ 4, & \text{if } 2 < x \leq 3 \\ 2x - 2, & \text{if } 3 < x \leq 4 \\ 4x - 10, & \text{if } 4 < x < \infty \end{cases}$

Could be shown as



Clearly, the least value of $f(x)$ is 4.

The number of integral solutions of $f(x) = 4$ are two, i.e. $\{2, 3\}$.

Also, $\quad \pi - 1, e, \frac{12}{5} \in \{2, 3\}$

$\therefore \quad \dfrac{f(\pi - 1) + f(e)}{2\, f\left(\dfrac{12}{5}\right)} = 1$

87. (a) Number of onto functions

$= 4^5 - 4 \cdot 3^5 + 6 \cdot 2^5 - 4 = 240$

(b) Number of onto functions, whose range is 3 elements

$= {}^4C_3(3^5 - 3 \cdot 2^5 + 3) = 4 \times 150 = 600$

(c) Number of onto functions, whose range is 2 elements

$= {}^4C_2(2^5 - 2) = 6 \times 30 = 180$

(d) Number of onto functions, whose range is 1 element

$= {}^4C_1 = 4$

88. Replacing x by 2,

$\Rightarrow \quad 2f(2) + 2f\left(\frac{1}{2}\right) - 2f(1) = 4$

$\Rightarrow \quad f(2) + f\left(\frac{1}{2}\right) = 2 + f(1)$ ...(i)

Replacing x by 1,

$f(1) = -1$ ...(ii)

Replacing x by $\frac{1}{2}$, $2f\left(\frac{1}{2}\right) + \frac{1}{2}f(2) + 2 = \frac{5}{2}$

$\therefore \quad 2f(2) + \frac{1}{2}f\left(\frac{1}{2}\right) = \frac{1}{2}$ ...(iii)

From Eqs. (i) and (iii), we get

$f(2) = 1, f\left(\frac{1}{2}\right) = 0$

89. Let $g(x) = f(100x)$, $g(1) = 1$

$\Rightarrow \quad g(x + 1) - g(x) = x + 1$

Putting $x = 1, 2, 3, \ldots, 99$ and adding, we get

$$g(100) = 5050 = \sum_{r=1}^{100} r = \frac{100}{2}(100 + 1) = 5050$$

90. We have, $f(x) = [1 + \sin x] + [1 + \sin 2x] + \ldots + [1 + \sin nx]$

$= n + [\sin x] + [\sin 2x] + \ldots + [\sin nx]$

$\because \quad x \in (0, \pi)$

$\Rightarrow \quad 0 < \sin x \leq 1$

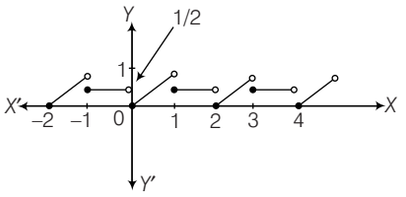
$\Rightarrow \quad f(x) = \begin{cases} n, & \text{if } n \text{ is even} \\ n + 1, & \text{if } n \text{ is odd} \end{cases}$

91. (a) $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$

$\Rightarrow f(x + k) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$

$\Rightarrow f(x)$ is periodic but period cannot be determined.

(b) $f(x)=\begin{cases} x-[x], & 2n\le x<2n+1 \\ \frac{1}{2}, & 2n+1\le x<2n+2 \end{cases}$



$\therefore f(x)$ is periodic with period 2.

(c) $f(x)=(-1)^{\left[\frac{2x}{\pi}\right]}$

$\Rightarrow f(x+\pi)=(-1)^{\left[\frac{2(x+\pi)}{\pi}\right]}=(-1)^{\left[\frac{2x}{\pi}\right]}$

$\therefore\ f(x)$ is periodic with period π.

(d) $f(x)=(ax+a)-[ax+a]+\tan\left(\frac{\pi x}{2}\right)$

$=\{ax+a\}+\tan\left(\frac{\pi x}{2}\right)$

$\therefore$ Period of $f(x)$ is LCM of $\left\{\frac{1}{a},\frac{2}{1}\right\}=2$

92. Here, $(x+1)\,f(x)=x$

$\Rightarrow\ (x+1)\,f(x)-x=0$ is $(n+1)$th degree

$\Rightarrow\ (x+1)f(x)-x=(x-0)\,(x-1)(x-2)\ldots(x-n)\cdot k$

Put $x=n+1\Rightarrow(n+2)f(n+1)-(n+1)=(n+1)!\,(k)$ ...(i)

Put $x=-1\Rightarrow 1=(-1)^{n+1}\cdot(n+1)!(k)$ ...(ii)

From Eqs. (i) and (ii), we get

$$f(n+1)=\frac{(n+1)+(-1)^{n+1}}{(n+2)}=\begin{cases}1, & \text{when } n \text{ is odd}\\ \frac{n}{n+2}, & \text{when } n \text{ is even}\end{cases}$$

93. We have, $f(x+1)=\frac{f(x)-5}{f(x)-3}$ ...(i)

$f(x)\cdot f(x+1)-3f(x+1)=f(x)-5$

$\Rightarrow\quad f(x)=\frac{3f(x+1)-5}{f(x+1)-1}$

On replacing x by $(x-1)$, we have $f(x-1)=\frac{3f(x)-5}{f(x)-1}$ ...(ii)

Using Eq. (i), we get $f(x+2)=\frac{f(x+1)-5}{f(x+1)-3}$

$$=\frac{\frac{f(x)-5}{f(x)-3}-5}{\frac{f(x)-5}{f(x)-3}-3}=\frac{2f(x)-5}{f(x)-2} \quad \text{...(iii)}$$

Using Eq. (ii), we get $f(x-2)=\frac{3f(x-1)-5}{f(x-1)-1}$

$$=\frac{3\left(\frac{3f(x)-5}{f(x)-1}\right)-5}{\frac{3f(x)-5}{f(x)-1}}=\frac{2f(x)-5}{f(x)-2} \quad \text{...(iv)}$$

Using Eqs. (iii) and (iv), we have $f(x+2)=f(x-2)$

$\Rightarrow\quad f(x+4)=f(x)$

$\therefore\ f(x)$ is periodic with period 4.

94. $f(x)=1-x-x^3$

Replacing x by $f(x)$, $f(f(x))=1-f(x)-f^3(x)$

Hence, the given equation is

$f(f(x))>f(1-5x),\ f(x)<1-5x$

$f(x)=1-x-x^3$

$1-x-x^3<1-5x$

$x^3-4x>0$

$x(x-2)(x+2)>0$

So, $x\in(-2,0)\cup(2,\infty)$

95. $(x-y)f(x+y)-(x+y)f(x-y)=2y\,((x-y)(x+y))$

Let $x-y=u,\ x+y=v$

$uf(v)-vf(u)=uv\,(v-u)$

$\frac{f(v)}{v}-\frac{f(u)}{u}=v-u$

$\Rightarrow\quad \left(\frac{f(v)}{v}-v\right)=\left(\frac{f(u)}{u}-u\right)=\text{constant}$

Let $\frac{f(x)}{x}-x=\lambda\Rightarrow\ f(x)=(\lambda x+x^2)$

$f(1)=2$

$\lambda+1=2\quad\Rightarrow\quad\lambda=1$

$f(x)=x^2+x$

96. Period of $\sin x=2\pi$

and period of $\cos(\sqrt{4-a^2}x)=\frac{2\pi}{\sqrt{4-a^2}}$.

$\Rightarrow\quad \text{LCM}\left(2\pi,\frac{2\pi}{\sqrt{4-a^2}}\right)=4\pi$ [given]

i.e. $\sqrt{4-a^2}=\frac{p}{2}$, where $p=1,3$

Hence, $a^2=\frac{15}{4},\frac{7}{4};\ a=\pm\frac{\sqrt{15}}{2},\pm\frac{\sqrt{7}}{2}$

97. $f(x+y)=f(x)f(a-y)+f(y)\,f(a-x)$ ...(i)

Put $x=y=0$, we get $f(a)=\frac{1}{2}$

Let $y=0$

$\Rightarrow\quad f(x)=f(x)f(a)+f(0)\cdot f(a-x)$

$\Rightarrow\quad f(x)=\frac{1}{2}f(x)+\frac{1}{2}f(a-x)$

$\Rightarrow\quad f(x)=f(a-x)$

Put $y=a-x$ in Eq. (i),

$f(a)=(f(x))^2+(f(a-x))^2$

$\Rightarrow\quad (f(x))^2=\frac{1}{4}$

$\Rightarrow\quad f(x)=\pm\frac{1}{2}\qquad\left[\because f(x)\neq-\frac{1}{2}\right]$

Hence, $f(x)=\frac{1}{2}$

98. Since, $f(g(x))$ is one-one function.

$\Rightarrow$ $f(g(x_1)) = f(g(x_2))$ whenever $x_1 \neq x_2$

$\Rightarrow$ $g(x_1) = g(x_2)$ whenever $x_1 \neq x_2$

$\Rightarrow$ $g(x)$ must be one-one.

Let $f(x) = y$ is satisfied by $x = x_1$ and x_2.

If $g(x)$ is such that its range has only one of x_1 and x_2, then $f(g(x))$ can be one-one even, if $f(x)$ is many-one.

99. $x \sin x$ is a continuous function value of $|x \sin x|$ can be made as large as we like for sufficiently large values of x.

Therefore, range of $x \sin x = R \Rightarrow \dfrac{[x]}{\tan 2x} = 0$

For $x \in \left(0, \dfrac{\pi}{4}\right)$, $[x] = 0$

For $x \in \left(-\dfrac{\pi}{4}, 0\right)$, $\dfrac{[x]}{\tan 2x} > 0$.

Therefore, values of $\dfrac{[x]}{\tan 2x}$ are never negative.

Thus, range of $\dfrac{[x]}{\tan 2x} \neq R$.

$\left|\dfrac{x}{\sin x}\right| > 1$, whenever defined.

Thus, range of $\dfrac{x}{\sin x}$ is not R.

$|x| + \sqrt{\{x\}}$ is a continuous function and

$$\lim_{x \to \infty} ([x] + \sqrt{\{x\}}) = \infty,$$

$$\lim_{x \to -\infty} ([x] - \sqrt{\{x\}}) = -\infty$$

Thus, range of $[x] + \sqrt{\{x\}} = R$.

100. (a) $f(x) = e^{\ln \sec^{-1} x}$. $g(1) = 0$ but $f(1)$ is not defined.

Thus, f and g are not identical.

(b) $f(x) = \tan(\tan^{-1} x) = x, \forall\ x \in R$

and $g(x) = \cot(\cot^{-1} x) = x, \forall\ x \in R$

Thus, f and g are identical.

(c) $f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$

$$g(x) = \text{sgn}(\text{sgn } x) = \text{sgn} \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

$$g(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

Thus, f and g are identical.

(d) $g(x) = \cot^2 x - \cos^2 x = \dfrac{\cos^2 x}{\sin^2 x} - \cos^2 x$

$= \cot^2 x \cos^2 x = f(x), \forall\ x \neq \pi$

Thus, f and g are identical.

101. We have, $f(x) = \cos^{-1}(-\{-x\})$

Domain of $f(x) \in R$.

As, $0 \leq \{-x\} < 1$ for all $x \in R$

$\Rightarrow$ $-1 < -\{-x\} \leq 0$

So, range of $f(x) \in \left[\dfrac{\pi}{2}, \pi\right)$.

$\therefore$ Range of $f(x)$ contains two prime numbers 2 and 3. Graph of $f(x)$ does not lie below X-axis. Clearly, f is neither even nor odd but many-one.

$\therefore$ $f(x+1) = f(x) \Rightarrow f(x)$ is periodic.

102. If we take example of $x + \sin x$, it is non-periodic whereas its derivative $1 + \cos x$ is periodic.

103. The value of $\sin \sqrt{2}x + \sin ax$ can be equal to 2, if $\sin \sqrt{2}x$ and $\sin ax$ both are equal to one but both are not equal one for any common value of x.

104. Clearly, $f(x)$ is many-one and into function.

105. $f(x) = 2 \cos \dfrac{\pi}{5} \sin x - 2 \cos \dfrac{2\pi}{5} \sin x$

$= 2 \sin x \left[\cos \dfrac{\pi}{5} - \cos \dfrac{2\pi}{5}\right]$

$= 2 \sin x [\cos 36^\circ - \cos 72^\circ]$

$= 2 \sin x [\cos 36^\circ - \sin 18^\circ]$

$= 2 \sin x \left[\dfrac{\sqrt{5}+1}{4} - \left(\dfrac{\sqrt{5}-1}{4}\right)\right] = \sin x$

$\therefore$ Range is $[-1, 1]$.

106. Period of $2 \cos \dfrac{1}{3}(x - \pi)$ and $4 \sin \dfrac{1}{3}(x - \pi)$ are $\dfrac{2\pi}{\frac{1}{3}}, \dfrac{2\pi}{\frac{1}{3}}$

or $6\pi, 6\pi$.

$\therefore$ Period of their sums $= 6\pi$

107. $(f(\sin 2x))^2 = \sin^2 x + \cos^2 x + 2 \sin x \cos x$

$= 1 + \sin 2x$

$\Rightarrow$ $f(x) = \sqrt{1 + x}, \forall\ x \in [-1, 1]$

$\Rightarrow$ If $x \in \left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$, then

$f(\tan^2 x) = \sqrt{1 + \tan^2 x} = \sec x$

108. Fifth degree equation must have atleast one real root. If it had two real roots, $f'(x) = 0$ must have one real root.

109. Range of $\dfrac{1}{1 + x^2} \in (0, 1]$

For domain R, $\log\left(\dfrac{1}{1 + x^2}\right) \in (-\infty, 0]$

110. $f(x) = \sqrt{2} \sin x + \sqrt{2} \cos x - \sqrt{2} \cos x + c = \sqrt{2} \sin x + c$

$\Rightarrow$ $f(0) = f(\pi) = c$

Hence, many-one function.

111. If $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$, $a_n \neq 0$.

Then, $f^{(n+1)}(x) = 0$, for all x while any derivative of $\sin x$ is never a zero function.

Hence, $\sin x$ is not polynomial function.

112. f is injective, since $x \neq y$ $(x, y \in R)$.

$\Rightarrow \log_a \{x + \sqrt{x^2+1}\} \neq \log_a \{y + \sqrt{y^2+1}\}$

$\Rightarrow f(x) \neq f(y)$

f is onto because $\log_a (x + \sqrt{x^2+1}) = y$

$\Rightarrow x = \dfrac{a^y - a^{-y}}{2}$

113. $f(x) - 2f\left(\dfrac{x}{2}\right) + f\left(\dfrac{x}{4}\right) = x^2$

$\therefore \quad f\left(\dfrac{x}{2}\right) - 2f\left(\dfrac{x}{4}\right) + f\left(\dfrac{x}{8}\right) = \left(\dfrac{x}{2}\right)^2$

$f\left(\dfrac{x}{4}\right) - 2f\left(\dfrac{x}{8}\right) + f\left(\dfrac{x}{16}\right) = \left(\dfrac{x}{4}\right)^2$

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$

$f\left(\dfrac{x}{2^n}\right) - 2f\left(\dfrac{x}{2^{n+1}}\right) + f\left(\dfrac{x}{2^{n+2}}\right) = \left(\dfrac{x}{2^n}\right)^2$

On adding, we get

$$f(x) - f\left(\frac{x}{2}\right) - f\left(\frac{x}{2^{n+1}}\right) + f\left(\frac{x}{2^{n+2}}\right) = x^2\left(1 + \frac{1}{2^2} + \ldots + \frac{1}{2^{2n}}\right)$$

As $n \to \infty$, we get $\quad f(x) - f\left(\dfrac{x}{2}\right) = \dfrac{4x^2}{3}$

Repeating the same procedure again, we get

$$f(x) - f(0) = \frac{16x^2}{9}$$

$\therefore \qquad f(3) = 16 + f(0)$

114. $f(x) - f(0) - x = 0$

$\Rightarrow \quad \dfrac{16x^2}{9} - x = 0 \quad \Rightarrow \quad x = 0, \dfrac{9}{16}$

$\therefore$ Two solutions.

115. $f'(x) = \dfrac{32x}{9} \quad \therefore \; f'(0) = 0$

116. For integral solution,

$x + y - [x][y] = 0 \quad \Rightarrow \quad x + y - xy = 0$

$\Rightarrow \quad -1 + x + y - xy = -1 \quad \Rightarrow (x-1)(y-1) = 1$

i.e. only possible, if

$(x - 1 = 1, y - 1 = 1)$ or $(x - 1 = -1, y - 1 = -1)$

$\Rightarrow \quad x = 2, y = 2 \quad$ or $\quad x = 0, y = 0$

$\therefore$ Solutions are (0, 0) or (2, 2).

117. For non-integral solution,

Let $\qquad x = [x] + f_1$

and $\qquad y = [y] + f_2$

$\therefore \qquad [x][y] = [x] + f_1 + [y] + f_2$

$([x] - 1)([y] - 1) = f_1 + f_2 + 1$

Now, $\qquad 0 \leq f_1 + f_2 < 2$

$\therefore \qquad f_1 + f_2 = 1$

$\Rightarrow ([x] - 1)([y] - 1) = 2$

Which is possible for

$\therefore \qquad [x] = 3$ and $[y] = 2$

$\Rightarrow \qquad [x] = 2$ and $[y] = 3$

$\Rightarrow \qquad [x] = -1$ and $[y] = 0$

$\Rightarrow \qquad [x] = 0$ and $[y] = -1$

$\therefore$ The $x + y = [x][y]$ becomes $x + y = 6 \Rightarrow x + y = 0$

$\therefore$ Non-integer solution lies on $x + y = 6$

$\Rightarrow \qquad x + y = 0$

118. We have, $f(x) = \dfrac{1}{2}\left[f(xy) + f\left(\dfrac{x}{y}\right)\right]$ ...(i)

Interchanging x and y, we get

$$f(y) = \frac{1}{2}\left[f(xy) + f\left(\frac{y}{x}\right)\right] \quad \text{...(ii)}$$

On subtracting, we get

$$f(x) - f(y) = \frac{1}{2}\left\{f\left(\frac{x}{y}\right) - f\left(\frac{y}{x}\right)\right\} \quad \text{...(iii)}$$

From Eq. (i), put $x = 1$, we get

$$f(1) = \frac{1}{2}\left[f(y) + f\left(\frac{1}{y}\right)\right] \quad \text{...(iv)}$$

$$f(1) = 0 \Rightarrow f(y) = -f\left(\frac{1}{y}\right)$$

Hence, $\qquad f\left(\dfrac{x}{y}\right) = -f\left(\dfrac{y}{x}\right)$

$\therefore$ From Eq. (iii) becomes $f(x) - f(y) = f\left(\dfrac{x}{y}\right)$

119. $f'(x) = \lim\limits_{h \to 0} = \dfrac{f(x+h) - f(x)}{h}$

Using the result in Q. No. 118.

$$\lim_{h \to 0} \frac{f\left(\dfrac{x+h}{x}\right)}{h} = \lim_{h \to 0} \frac{f\left(1 + \dfrac{h}{x}\right)}{h}$$

$$= \lim_{h \to 0} \frac{f\left(1 + \dfrac{h}{x}\right)}{\dfrac{h}{x} \cdot x} = \frac{1}{x} f'(1) = \frac{1}{x} \cdot 2 = \frac{2}{x}$$

$\therefore \qquad f'(3) = \dfrac{2}{3}$

120. From Q. No. 119, $f'(x) = \dfrac{2}{x}$

$f(x) = 2 \log x + C$

$\Rightarrow \qquad f(1) = 0 \Rightarrow C = 0$

$\therefore \qquad f(x) = 2 \log x$

$\therefore \qquad f(e) = 2 \log e = 2$

121. $f(x)$ = odd degree polynomial + bounded function

$\cot^{-1} x \in (0, \pi)$

Also, $\qquad f'(x) > 0$

$\therefore \; f(x)$ is one-one and range of $f(x) \in R$

$\therefore \; f(x)$ is onto.

$\Rightarrow \; f(x)$ is one-one onto.

122. We have, $f(x) = x^4 + 1 + \dfrac{1}{x^2 + x + 1}$

= even degree polynomial + bounded function

$$\frac{1}{x^2 + x + 1} \in \left(0, \frac{4}{3}\right)$$

$$f'(x) = \frac{4x^3(x^2 + x + 1)^2 - 2x - 1}{(x^2 + x + 1)^2}$$

$\Rightarrow f'(x) = 0$ has at least one root which is repeated odd number of times or it has one root which is not repeated, since numerator of $f'(x)$ is a polynomial of degree 7.

$\Rightarrow$ $f(x) = 0$ has a point of extreme.

$\therefore$ $f(x)$ is many-one into.

123. $f(x)$ = odd degree polynomial + bounded function $\sin 2x \Rightarrow$ $f(x)$ is onto.

$f(x)$ is one-one, if $f'(x) \geq 0$ or $f'(x) \leq 0, \forall\, x$

$\Rightarrow$ $a \geq 1 \cup a \leq -1 \Rightarrow a \in R - (-1, 1)$

124. $D_h = \{-1, 1\}$, as minimum occurs before maxima for $f(x)$.

$\therefore$ $a_3 = -1$

Now, $g(x) = a_0 + a_1x + a_2x^2 - x^3$

$$g'(x) = a_1 + 2a_2x - 3x^2$$

$$= -3(x - 3)(x + 3) = -3x^2 + 27$$

$\therefore$ $a_1 = 27, a_2 = 0$

$\therefore$ $a_1 + a_2 = 27$

125. Also, $g(-3) > 0$ and $g(3) > 0$

$\Rightarrow$ $a_0 > 54$ and $a_0 < -54$

$\therefore$ $a_0 > 54$

126. Now, $g(x) = a_0 + 27x - x^3$

$$f(x) = \sqrt{a_0 + 27x - x^3}$$

$$f(10) = \sqrt{a_0 + 270 - 1000}$$

Clearly, $f(10)$ is defined for $a_0 > 730$.

127. $\because ff^{-1}(x) = x$

$$2^{(f^{-1}(x))^4 - 4(f^{-1}(x))^2} = x$$

$\Rightarrow (f^{-1}(x))^4 - 4(f^{-1}(x))^2 - \log_2 x = 0$

$\therefore$ $(f^{-1}(x))^2 = 2 + \sqrt{4 + \log_2 x}$

$\therefore$ Range of $f^{-1}(x)$ is $[2, \infty)$.

$\therefore$ $f^{-1}(x) = \sqrt{2 + \sqrt{4 + \log_2 x}}$

$\therefore$ $f^{-1}(x) > 0$

128. $g(x) = \dfrac{\sin x + 4}{\sin x - 2}$

$\Rightarrow$ $g'(x) = \dfrac{-6\cos x}{(\sin x - 2)^2} \geq 0 \qquad \left[\because x \in \left[\frac{\pi}{2}, \pi\right]\right]$

$\Rightarrow g(x)$ is an increasing function, hence one-one function.

$\therefore$ Range is $\left[g\left(\dfrac{\pi}{2}\right), g(\pi)\right]$ and lies in $[-5, -2]$.

129. $x : x$ in domain of $g^{-1}(x)$,

$g^{-1}(x)$ in domain of $f^{-1}(x)$

$$g^{-1}(x) = \sin^{-1}\frac{2(x + 2)}{x - 1}, \forall\, x \in [-5, -2] \qquad \ldots(i)$$

$\Rightarrow$ $\sin^{-1}\dfrac{2(x + 2)}{x - 1} \geq 1$

i.e. $1 \leq \sin^{-1}\dfrac{2(x + 2)}{x - 1} \leq \dfrac{\pi}{2}$

$\Rightarrow$ $\sin 1 \leq \dfrac{2(x + 2)}{x - 1} \leq 1$

Solving this, we get

$$x \leq -\frac{(4 - \sin 1)}{2 - \sin 1} \text{ or } x > 1 \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$x \in \left[-5, -\frac{(4 + \sin 1)}{2 - \sin 1}\right]$$

Sol. (Q. Nos. 130 to 132)

$P(x) + 1 = 0$ has a thrice repeated root at $x = 1$.

$P'(x) = 0$ has a twice repeated root at $x = 1$.

Similarly, $P'(x) = 0$ has a twice repeated root at $x = -1$.

$\Rightarrow P'(x)$ is divisible by $(x - 1)^2(x + 1)^2$

$\because$ $P'(x)$ is degree at most 4.

$\Rightarrow$ $P'(x) = \alpha(x - 1)^2(x + 1)^2$

$\therefore$ $P(x) = \alpha\left(\dfrac{x^5}{5} - \dfrac{2}{3}x^3 + x\right) + c$

Now, $P(1) = -1$ and $P(-1) = 1 \Rightarrow \alpha = -\dfrac{15}{8}$ and $c = 0$

$\therefore$ $P(x) = -\dfrac{15}{8}\left(\dfrac{x^5}{5} - \dfrac{2}{3}x^3 + x\right)$

$$= -\frac{15}{8}x\left(\frac{x^4}{5} - \frac{2x^2}{3} + 1\right)$$

has only one real root $x = 0$, as $\dfrac{x^4}{5} - \dfrac{2x^2}{3} + 1$ has imaginary roots.

Also, sum of pairwise product of all roots $= -\dfrac{10}{3}$.

Also, $P''(x) = -\dfrac{15}{2}(x^3 - x)$

Let $f(x) = -\dfrac{15}{2}(x^3 - x)$

$$f'(x) = -\frac{15}{2}(3x^2 - 1) = -\frac{15}{2}(\sqrt{3}x - 1)(\sqrt{3}x + 1)$$

$$\underset{-}{\qquad} \; \underset{+}{\qquad} \; \underset{-}{\qquad}$$

$\Rightarrow$ Maximum at $x = \dfrac{1}{\sqrt{3}}$.

130. (a) ***131.*** (c) ***132.*** (b)

Sol. (Q. Nos. 133 to 135)

Given, $f^{-1}(x) = \dfrac{1}{f(x)}$ ...(i)

Replace x by $f(x)$.

$\Rightarrow$ $f(f(x)) = \dfrac{1}{x} \Rightarrow f^{-1}\left(\dfrac{1}{x}\right) = f(x)$

$\Rightarrow$ $f^{-1}(x) = f\left(\dfrac{1}{x}\right)$ ...(ii)

From Eqs. (i) and (ii), we get,

$$\frac{1}{f(x)} = f\left(\frac{1}{x}\right) \Rightarrow \quad f(x)\cdot f\left(\frac{1}{x}\right) = 1 \qquad \text{...(iii)}$$

Putting $x = 1$, we get

$$(f(1))^2 = 1$$

133. $f(1) = 1 \qquad [\text{as } f(1) > 0]$

134. If $f(x)$ is continuous, then being bijective, it will be monotonic.

135. $f(f(x))$ would be increasing but $f(f(x)) = \frac{1}{x}$, i.e. decreasing.

Thus, contradicts the facts of $f(x)$ being continuous.

Sol. (Q. Nos. 136 to 137)

$$f(m+n) = f(m) + f(n)$$
$$n = N-1$$
$$\therefore \quad f(N) = f(1) + f(N-1)$$
$$f(N-1) = f(1) + f(N-2)$$
$$f(N-2) = f(1) + f(N-3)$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$f(2) = f(1) + f(1)$$

On adding all, we get

$$f(N) = Nf(1)$$

Now, if $f(1) = 1$

$\Rightarrow f(N) = N$, which contains even numbers

If $f(1) = 2 \Rightarrow f(N) = 2N$, which contains even numbers

$\therefore$ $f(1)$ is 1 or 2.

136. (c) **137.** (a)

138. (A) $\sqrt{\sin(\cos x)}$, $\sin(\cos x) \geq 0$

$$2n\pi \leq \cos x \leq (2n+1)\pi$$
$$0 \leq \cos x \leq 1$$
$$x \in \left[2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right]$$

(B) $\left(\sqrt{\cos(\sin x)}\right)^{-1}$

$$\cos(\sin x) > 0$$
$$-\frac{\pi}{2} < \sin x < \frac{\pi}{2} \quad \Rightarrow x \in R$$

(C) $\tan(\pi \sin x)$

$$\pi \sin x \neq \pm\frac{\pi}{2}$$
$$\sin x \neq \pm\frac{1}{2}$$
$$\therefore \quad x \neq n\pi \pm \frac{\pi}{6} \Rightarrow \quad x \in R - \left\{n\pi \pm \frac{\pi}{6}\right\}$$

(D) $\ln(\tan x)$

$$\tan x > 0 \quad \Rightarrow \quad 0 < x < \frac{\pi}{2}$$
$$n\pi < x < n\pi + \frac{\pi}{2} \Rightarrow x \in \left(n\pi, n\pi + \frac{\pi}{2}\right)$$

139. (A) $|4\sin x - 1| < \sqrt{5} \quad \Rightarrow \quad -\sqrt{5} < 4\sin x - 1 < \sqrt{5}$

$$\Rightarrow \quad 1 - \sqrt{5} < 4\sin x < 1 + \sqrt{5}$$
$$\Rightarrow \quad \frac{1-\sqrt{5}}{4} < \sin x < \frac{1+\sqrt{5}}{4}$$
$$\Rightarrow \quad -\frac{\pi}{10} < x < \frac{3\pi}{10}$$

But $x \in [0, \pi]$ domain $x \in [0, 3\pi/10)$.

(B) $4\sin^2 x - 8\sin x + 3 \leq 0, \quad 0 \leq x \leq 2\pi$

$\Rightarrow (2\sin x - 1)(2\sin x - 3) \leq 0 \Rightarrow 2\sin x - 1 \geq 0$

$$\Rightarrow \quad \sin x \geq \frac{1}{2} \quad \Rightarrow \quad x \in \left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$$

(C) $|\tan x| \leq 1; x \in [-\pi, \pi], \quad -1 \leq \tan x \leq 1$

$$n\pi - \frac{\pi}{4} \leq x \leq n\pi + \frac{\pi}{4}$$

Put $n = 0, \qquad -\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$

Put $n = 1, \qquad \pi - \frac{\pi}{4} \leq x \leq \pi + \frac{\pi}{4}$

$$\frac{3\pi}{4} \leq x \leq \frac{5\pi}{4}$$

But, from domain $\frac{3\pi}{4} \leq x \leq \pi$

Put $x = -1, \; -\pi - \frac{\pi}{4} \leq x \leq -\pi + \frac{\pi}{4} \Rightarrow -\frac{5\pi}{4} \leq x \leq -\frac{3\pi}{4}$

But from domain $-\pi \leq x \leq -\frac{3\pi}{4}$

Finally, $\left[-\pi, -\frac{3\pi}{4}\right] \cup \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \pi\right]$

But, $x \in [0, \pi] \;\; \therefore \; x \in \left[0, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \pi\right]$

(D) $\cos x - \sin x \geq 1, 0 \leq x \leq 2\pi \Rightarrow \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x - \frac{\sin x}{\sqrt{2}}\right) \geq 1$

$$\Rightarrow \quad \left(\cos x \cdot \cos\frac{\pi}{4} - \sin x \cdot \sin\frac{\pi}{4}\right) \geq \frac{1}{\sqrt{2}}$$
$$\Rightarrow \quad \cos\left(x + \frac{\pi}{4}\right) \geq \frac{1}{\sqrt{2}}$$
$$\Rightarrow \quad 2n\pi - \frac{\pi}{4} \leq x + \frac{\pi}{4} \leq 2n\pi + \frac{\pi}{4}$$
$$\Rightarrow \quad 2n\pi - \frac{\pi}{2} \leq x \leq 2n\pi$$

On substituting suitable values of n, according to domain

$$x \in \left[\frac{3\pi}{2}, 2\pi\right] \cup \{0\}$$

140. (A) When $-1 \leq x < 0$,

$$f(x) \in [0, 1) \text{ and } f(x) = x + 1$$
$$\therefore \quad f(f(x)) = (x+1)^2 - 1 = x^2 + 2x$$

(B) $\frac{(\cos 2x + 1)(\sec^2 x + 2\tan x)}{2}$

$$= \left(\frac{1 - \tan^2 x}{1 + \tan^2 x} + 1\right)\left(\frac{1 + \tan^2 x + 2\tan x}{2}\right)$$
$$= \frac{1 + \tan^2 x + 2\tan x}{1 + \tan^2 x}$$

i.e. $\quad f\left(\frac{2\tan x}{1 + \tan^2 x}\right) = 1 + \frac{2\tan x}{1 + \tan^2 x}$

$\therefore \; f(x) = 1 + x$

(C) Put $x = 0, y = 0, f(1) = (1 + 1)^2 = 2^2$

Put $x = 0, y = 1,\ f(2) = (1 + 2)^2 = 3^2$

By induction, $f(x) = (x + 1)^2$

(D) When $4 < x < 5, \left[\frac{x}{4}\right] = 1$

$\Rightarrow \quad f(x) = 2x + 3$

$\therefore \quad y = 2x + 3 \Rightarrow x = \frac{y - 3}{2}$

$\therefore \quad f^{-1}(x) = \frac{x - 3}{2}$

141. (A) Simplifying the expression, we get $f(x) = \frac{5}{4}$.

(B) Period of $\tan(e^{\{x\}}) = 1$, the period of

$[x + \alpha] - (x + \alpha) + \alpha - 5$ is also 1, hence 1 is the period.

(C) $h(x) - h(-x) = 0$, hence even function.

(D) Simplifying the expression, $k(x) = 0$.

142. Here, $f(x + y) - kxy = f(x) + 2y^2$

Put $x = 0 \Rightarrow f(y) = f(0) + 2y^2$

$\Rightarrow \quad f(x) = 2x^2 + f(0)$

Now, $f(1) = 2 + f(0) \Rightarrow f(0) = 0$

$\therefore \quad f(x) = 2x^2$

Hence, $f(x+y) \cdot f\left(\frac{1}{x+y}\right) = 4$

143. Given, $f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$ ...(i)

On putting $x = 0$ and $f(y) = 0$, we get

$f(0) = f(0) + f(0) - 1$

$\Rightarrow \quad f(0) = 1$ ...(ii)

On putting $x = a$ and $f(y) = a$, we get

$f(0) = f(a) + a^2 + f(a) - 1$

$\Rightarrow \quad f(a) = 1 - \frac{a^2}{2} \therefore f(x) = 1 - \frac{x^2}{2}$

$\Rightarrow \quad -f(10) = -1 + \frac{(10)^2}{2} = 49$

Hence, $\frac{-f(10)}{7} = 7$

144. Given,

$f(1) + 2f(2) + 3f(3) + \ldots + nf(n) = n(n + 1)f(n)$ ...(i)

On putting $(n + 1)$ in place of n, we get

$f(1) + 2f(2) + \ldots + nf(n) + (n + 1) f(n + 1)$

$= (n + 1)(n + 2) f(n + 1)$ ...(ii)

On subtracting Eq. (i) from Eq. (ii), we get

$(n + 1) f(n + 1) = (n + 1)(n + 2) f(n + 1) - n(n + 1) f(n)$

$\Rightarrow (n + 1) f(n + 1) - nf(n) = 0$

$\Rightarrow \quad nf(n) = (n + 1) f(n + 1)$

$\Rightarrow \quad 2f(2) = 3f(3) = \ldots = nf(n)$

From Eq. (i),

$f(1) + [nf(n) + nf(n) + \ldots \text{ to } (n - 1) \text{ terms}] = n(n + 1) f(n)$

$\Rightarrow \quad f(1) + (n - 1) nf(n) = n(n + 1) f(n)$

$\Rightarrow \quad f(1) = 2n f(n) \Rightarrow 1 = 2n f(n)$

$\Rightarrow \quad \frac{1}{f(n)} = 2n$

$\Rightarrow \quad \frac{1}{f(2010)} = 4020 \Rightarrow \frac{1}{(2010) f(2010)} = 2$

145. If $f(x) = \frac{ax + b}{cx - a}, x \neq \frac{a}{c}$

Then, $f(f(x)) = x$

$\therefore \quad f(f(x)) = x$ and $f\left(f\left(\frac{4}{x}\right)\right) = \frac{4}{x}$

$\therefore \quad f(f(x)) + f\left(f\left(\frac{4}{x}\right)\right) = x + \frac{4}{x} \geq 4$

146. Here, $\gamma = 4 - (\alpha + \beta)$

$\Rightarrow \quad \alpha^2 + \beta^2 + \gamma^2 = \alpha^2 + \beta^2 + (4 - (\alpha + \beta))^2 = 6$

$\Rightarrow \quad 2\beta^2 + 2\beta(\alpha - 4) + (2\alpha^2 - 8\alpha + 10) = 0$

But, $\beta \in R \Rightarrow D \geq 0$

$\Rightarrow \quad 3\alpha^2 - 8\alpha + 4 \geq 0$

$\Rightarrow \quad \alpha \in [2/3, 2]$

$\therefore$ Integer values of α are $\{1, 2\}$.

147. Here, $f(x) = ax + b$

$\Rightarrow \quad a(x + f(x)) + b = x + ax + b$

$\Rightarrow \quad a f(x) = x$

$\Rightarrow \quad f(x) = \frac{x}{a}$

$\Rightarrow \quad \frac{x}{a} = ax + b$

$\Rightarrow \quad b = 0 \Rightarrow a^2 = 1$

$\Rightarrow \quad a = \pm 1$

$\Rightarrow \quad f(x) = \pm x$, i.e. 2 functions.

148. Number of one-one functions $= {}^5P_3 = 60$

Number of strictly monotonic functions $= 10 + 10 = 20$

$\therefore \quad \frac{\text{Number of one - one functions}}{\text{Number of strictly monotonic functions}} = \frac{60}{20} = 3$

149. The number of different sets contains exactly 3 elements of $B = {}^5C_3 = 10$

The number of onto functions from A to the set contains 3 elements $= 10[3^4 - 3(2)^4 + 3] = 360$

$\therefore \quad \frac{k}{60} = 6$

150. Here, $f(2) = a \sin(2) + 2b \cos(2) + 8$

and $f(-2) = -a \sin(2) - 2b \cos(2) + 8$

On adding, we get

$f(2) + f(-2) = 16$

$\therefore \quad f(-2) = 1$

151. Let $y = f(x) = x^5 + e^{x/3}$

Then, $y(0) = 1$

Also, $g'(y) = \frac{1}{f'(x)}$

$\therefore \quad g'(1) = \frac{1}{f'(0)} = 3$

152. $f(x) = x^3 - 12x + p$

$\Rightarrow \quad f'(x) = 3x^2 - 12 = 0$

$\Rightarrow \quad x = \pm 2$

$\therefore \quad f(-2) = p + 16$ and $f(2) = p - 16$ for all $p \in \{1, 2, ..., 15\}$

$f(-2) > 0$, $f(2) < 0$. It has 3 roots in each case.

$\therefore \quad \Sigma\theta = 3 \times 15 = 45 \Rightarrow \frac{1}{5} \cdot \Sigma\theta = 9$

153. Given, $f(m) - f(n) = 3$

If $m = 125, n = 124 \Rightarrow (m-n)_{\min} = 1$

$m = 24, n = 5$

$\Rightarrow \quad f(m) - f(n) = 3 \Rightarrow (m-n)_{\max} = 19$

$\therefore \quad \frac{(m-n)_{\max} - (m-n)_{\min}}{2} = \frac{19-1}{2} = 9$

154. As, $x^2 + y^2 = 4$

$\Rightarrow \quad x = 2\cos\theta, y = 2\sin\theta$

$\therefore \quad \frac{x^3 + y^3}{x + y} = x^2 + y^2 - xy = 4 - 2\sin 2\theta$

$\Rightarrow \quad \left(\frac{x^3 + y^3}{x + y}\right)_{\max} = (4 - 2(\sin 2\theta))_{\max} = 4 + 2 = 6$

155. $f(2011) = (2 + 0 + 1 + 1)^2 = 16$

$f^2(2011) = (1 + 6)^2 = 49$

$f^3(2011) = (4 + 9)^2 = 169$

$f^4(2011) = (1 + 6 + 9)^2 = 256$

$f^5(2011) = (2 + 5 + 6) = 169$

$\vdots \qquad \vdots$

$f^{2n}(2011) = 256$

$\therefore \quad f^{2n+1}(2011) = 169$

$\Rightarrow \quad \frac{f^{2011}(2011) - f^{2010}(2011)}{f^{2013}(2011) - f^{2012}(2011)} = \frac{169 - 256}{169 - 256} = 1$

156. As, $[\sin x] = \frac{x}{2\pi} - \left[\frac{x}{2\pi}\right] + \frac{2x}{5\pi} - \left[\frac{2x}{5\pi}\right]$

$[\sin x] = \left\{\frac{x}{2\pi}\right\} + \left\{\frac{2x}{5\pi}\right\}$

Thus, only one solution, i.e. $x = 10\pi$.

157. As, $xy - 6(x + y) = 0 \Rightarrow (x-6)(y-6) = 36$

$\therefore (x-6, y-6) = (1, 36), (2, 18), (3, 12), (4, 9), (6, 6),$

$(-36, -1), (-18, -2), (-12, -3), (-4, -9), (-6, -6),$

$\because \quad x \le y \Rightarrow \alpha = 10$

$\therefore \quad (\alpha - 6) = 4$

158. Here, $f(x) - 10x = (x-1)(x-2)(x-3)(x-\alpha)$

$\therefore \quad f(12) + f(-8) = 19840$

$\Rightarrow \quad \frac{f(12) + f(-8)}{19840} = 1$

159. On adding, $a = \frac{3 - \cos 4\theta + 4\sin 2\theta}{2} = (1 + \sin 2\theta)^2$

On subtracting, $b = (1 - \sin 2\theta)^2$

$\Rightarrow \quad ab = \cos^4 2\theta \le 1$

160. Clearly, $x^2 + 4x \ge 0$

$2x^2 + 3 \ge 0 \Rightarrow x^2 + 4x \ge 2x^2 + 3$

and x is an integer.

$\therefore \quad x \in \{1, 2, 3\}$

$\therefore \quad n = 3$

Now, maximum value ${}^{x^2+4x}C_{2x^2+3} = 12$

$\therefore \quad Y = |\ln 12|$

$\therefore \quad [Y] = 2 \ (\ln e^2 < \ln 12 < \ln e^3)$

$\therefore \quad [n + [Y]] = [3 + 2] = 5$

161. $f(x-1) + f(x+1) = \sqrt{3} f(x)$

$\Rightarrow \quad f(x) + f(x+2) = \sqrt{3} f(x+1)$

Putting $x = x + 2$,

$f(x+1) + f(x+3) = \sqrt{3} f(x+2)$

$f(x-1) + 2f(x+2) + f(x+3) = \sqrt{3}[\sqrt{3} f(x+1)]$

$f(x-1) + f(x+3) = f(x+1)$

Again, putting $x = x + 2$,

$f(x+1) + f(x+5) = f(x+3)$

$f(x-1) + f(x+5) = 0$

$f(x+5) = -f(x-1)$

$f(x) = -f(x+6)$

$f(x+12) = f(x)$

$\Rightarrow \sum_{r=0}^{19} f(5 + 12r) = 20 f(5) = 20 \times 10 = 200$

Hence, the sum of digits $= 2 + 0 + 0 = 2$

162. Put $x = 1$,

$$2f(x) = f(xy) + f\left(\frac{x}{y}\right) \qquad \text{...(i)}$$

$$2f(1) = f(y) + f\left(\frac{1}{y}\right) \qquad \text{...(ii)}$$

$$\Rightarrow \quad f(y) = -f\left(\frac{1}{y}\right)$$

Replacing x by y and y by x in Eq. (i), we get

$$2f(y) = f(yx) + f\left(\frac{y}{x}\right) \qquad \text{...(iii)}$$

From Eqs. (i) and (iii), we get

$$2\{f(x) - f(y)\} = f\left(\frac{x}{y}\right) - \left\{-f\left(\frac{x}{y}\right)\right\} = 2f\left(\frac{x}{y}\right) \qquad \text{...(iv)}$$

$$f(x) - f(y) = f\left(\frac{x}{y}\right) \qquad \text{...(v)}$$

$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = f'(1) = 1$

$\lim_{h \to 0} \frac{f(1+h)}{h} = 1, \quad \text{as} \quad f(1) = 0$

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{f\left(1 + \frac{h}{x}\right)}{h} = \frac{1}{x}$

$f(x) = \log |x| + c$

$f(1) = 0 \Rightarrow c = 0$

$f(e) = 1$

163. $f(1) + 2f(2) + 3f(3) + \ldots + n(f(n)) = n(n+1)f(n)$

$\Rightarrow f(1) + 2f(2) + 3f(3) + \ldots + (n+1)f(n+1)$
$= (n+1)(n+2)f(n+1)$

$\Rightarrow n(n+1)f(n) = (n+1)^2 f(n+1)$

$\Rightarrow nf(n) = (n+1)f(n+1)$

i.e. $2f(2) = 3f(3) = \ldots = n(f(n))$

i.e. $f(n) = \dfrac{1}{n}$

$\therefore \quad 2126\ f(1063) = 2126 \times \dfrac{1}{1063} = 2$

164. Now, $f(x) = \dfrac{x^4 + x^2 + 1}{x^2 - x + 1} \Rightarrow f(x) = x^2 + x + 1$

Now, $f(\omega^n) = \omega^{2n} + \omega^n + 1 = 3$

$\because \quad \omega^n = 1$, when n is a multiple of 3.

165. $f^2(x) \cdot f\left(\dfrac{1-x}{1+x}\right) = x^3$ ...(i)

Replacing x by $\dfrac{1-x}{1+x}$, we get

$$f^2\left(\frac{1-x}{1+x}\right) f(x) = \left(\frac{1-x}{1+x}\right)^3 \quad \text{...(ii)}$$

By using Eqs. (i) and (ii), we get $f^3(x) = x^6 \left(\dfrac{1+x}{1-x}\right)^3$

$\Rightarrow \quad f(x) = x^3 \left(\dfrac{1+x}{1-x}\right) \Rightarrow f(-2) = \dfrac{8}{3}$

$\Rightarrow \quad [f(-2)] = 2 \Rightarrow |[f(-2)]| = 2$

166. $f(2a - x) = f(x)$

$\Rightarrow \quad f(2a - x) = -f(x)$

$\because \quad f$ is odd $\Rightarrow f(x + 4a) = f(x)$

$\Rightarrow f$ is periodic with period $4a$.

$\Rightarrow \quad f(1 + 4r) = f(1)$

Now, $\sum_{r=0}^{\infty} [f(1)]^r = 8 \Rightarrow \dfrac{1}{1 - f(1)} = 8$

$\Rightarrow \quad f(1) = \dfrac{7}{8} \Rightarrow 8f(1) = 7$

167. Consider, $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{e^{2x} - 1}{e^{2x} + 1} = 1 - \dfrac{2}{e^{2x} + 1}$

$\Rightarrow \quad f'(x) = \dfrac{4e^{2x}}{(e^{2x} + 1)^2} > 0, \ \forall\ x \in R$

$\Rightarrow f(x)$ is an increasing function.

$\Rightarrow$ Domain : R, Range : $(-1, 1)$

For $f : R \to (-1, 1)$,

$$f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}},\ f^{-x} : (-1, 1) \to R$$

$\Rightarrow \quad x = \dfrac{e^y - e^{-y}}{e^y + e^{-y}}$

$\Rightarrow \quad e^y = \sqrt{\dfrac{1+x}{1-x}} \Rightarrow f^{-1}(x) = \ln \sqrt{\dfrac{1+x}{1-x}}$

Hence, given equation is equivalent to $f(x) = f^{-1}(x)$.

$\Leftrightarrow \quad f(x) = x$ [as f is an increasing function]

$\Rightarrow \quad \ln \sqrt{\dfrac{1+x}{1-x}} = x \Rightarrow \dfrac{1+x}{1-x} = e^{2x}$

Now, draw the graph of $y = \dfrac{1+x}{1-x}$ and $y = e^{2x}$. They intersect each other at $x = 0$.

168. Let $y = \dfrac{3x^2 + 9x + 17}{3x^2 + 9x + 7} = 1 + \dfrac{10}{3x^2 + 9x - 7}$

Now, $3x^2 + 9x + 7 = 3(x^2 + 3x) + 7$

$= 3\left(x + \dfrac{3}{2}\right)^2 + \dfrac{1}{4} \geq \dfrac{1}{4}$ for all $x \in R$

Maximum value of $\dfrac{10}{3x^2 + 9x + 7}$ is 40.

Maximum value of y is $1 + 40 = 41$

$\therefore \quad 5k + 1 = 41 \Rightarrow k = 8$

169. The period of the function is found as follows

Given, $f(x) + f(x+4) = f(x+2) + f(x+6)$ ...(i)

$\therefore$ Replacing x by $x + 2$, we get

$f(x+2) + f(x+6) = f(x+4) + f(x+8)$ ...(ii)

From Eqs. (i) and (ii), we get

$f(x) + f(x+4) = f(x+4) + f(x+8)$

$\Rightarrow \quad f(x) = f(x+8) \Rightarrow$ Thus, 8 is the period.

170. Since, $f(x)$ is symmetric about $y = x$.

$\Rightarrow \quad f^{-1}(x) = f(x)$

$\therefore \quad f^2(x) = (f^{-1}(x))^2 - px \cdot f(x) f^{-1}(x) + 2x^2 f(x)$

$\Rightarrow \quad f^2(x) = f^2(x) - px \cdot f(x) \cdot f(x) + 2x^2 f(x)$

$\Rightarrow \quad f(x) \cdot \{2x^2 - px \cdot f(x)\} = 0$

$\Rightarrow \quad f(x) = \dfrac{2x^2}{px}$ [as $f(x) \neq 0$]

$\Rightarrow \quad f(x) = \dfrac{2x}{p} \Rightarrow p = 2$

171. Here, $f(x) = \dfrac{3x^2 + mx + n}{x^2 + 1} = 3 + \dfrac{mx + n - 3}{x^2 + 1}$

$\therefore \quad y = 3 + \dfrac{mx + n - 3}{1 + x^2}$

For y to lie in $[-4, 3)$.

$mx + n - 3 < 0, \forall x \in R.$

This is possible only if $m = 0$.

When $m = 0$, then $y = 3 + \dfrac{n-3}{1+x^2}$.

If $x \to \infty \Rightarrow y_{\max} \to 3^-$

Now, $y_{\min}$ occurs at $x = 0$. [as $1 + x^2$ is maximum]

$$y_{\min} = 3 + \frac{n-3}{1} = n \quad \text{...(i)}$$

Since, $-4 \leq y < 3$

$\therefore \quad y_{\min} = -4$ ...(ii)

From Eqs. (i) and (ii), $n = -4$

$\therefore \quad \dfrac{m^2 + n^2}{4} = \dfrac{16}{4} = 4$

172. Since, $f(x)$ is monotonic.

$\Rightarrow f'(x) < 0$ or $f'(x) > 0, \forall x \in R.$

$\Rightarrow f'(px) < 0$ or $f'(px) > 0, \forall x \in R.$

$\Rightarrow f(px)$ is monotonic

$\Rightarrow f(x) + f(3x) + f(5x) + ... + f((2m-1)x)$

is a monotonic polynomial of odd degree $(2m-1)$, so it will attains all real values only once.

$\therefore \quad f(x) + f(3x) + f(5x) + ... + f((2m-1)x) = (2m-1)$ has only one root.

173. (i) We have,

$$(x)^2 = [x]^2 + 2x \qquad ...(i)$$

$$\Rightarrow (i+f)^2 = [i+f]^2 = [i+f]^2 + 2i + 2f$$

$[\because x = i + f$, where i is integer and f is fraction$]$

$$\Rightarrow \{i+f\}^2 = (i)^2 + 2i + 2f$$

$$\Rightarrow i^2 + 2i + 1 = i^2 + 2i + 2f$$

$$\Rightarrow f = \frac{1}{2} \Rightarrow [x] = 0, (x) = 1$$

Put the value of (x) and $[x]$ in Eq. (i), we get

$$x = 0, \ n + \frac{1}{2}, n \in Z$$

(ii) We have,

$$[2x] - 2x = [x+1]$$

$$\Rightarrow -\{2x\} = [x] + 1 \Rightarrow [x] + \{2x\} = -1$$

$$\therefore \quad x = -1, -\frac{1}{2}$$

(iii) We have, $[x]^2 + 2(x) = 0, [x]^2 = 3x, 0 \le x \le 2$

For $0 \le x < 1, [x] = 0, [x^2] = 0 \Rightarrow x = 0$

$1 \le x < \sqrt{2}, [x] = 1, [x^2] = 1 \Rightarrow x = 1$

$\sqrt{2} \le x < \sqrt{2}, [x] = 1, [x^2] = 2 \Rightarrow x = \frac{4}{3}$

$\sqrt{3} \le x < 2, [x] = 1, [x^2] = 3 \Rightarrow x = \frac{5}{3}$

$x = 2, [x] = 2, [x^2] = 4 \Rightarrow x = \frac{8}{3}$

$\therefore x \in \left\{0, 1, \frac{4}{3}, \frac{5}{3}\right\} \qquad \left[\because \frac{8}{3} \notin [0, 2]\right]$

(iv) We have, $y = 4 - [x]^2$ and $[y] + y = 6$

Now, $[y] + y = 6, \Rightarrow y = 3$

On substituting the value of y in $y = 4 - [x]^2$

$\Rightarrow [x]^2 = 1, x = \pm 1, \pm 1 + k$, where k is fraction.

(v) We have,

$[x] + |x-2| \le 0, -1 \le x \le 3$

For $-1 \le x < 0, [x] = - \Rightarrow -1 - x + 2 \le 0 \quad x \ge 1$, Not possible.

$0 \le x \le 3, [x]$ and $|x-2|$ are positive.

Hence, no solution.

174. Given, $f(n) = 1! + 2! + 3! + ... + n!$

$$\therefore \quad f(n+2) = Q(n)\ f(n) + P(n) f(n+1)$$

$$\Rightarrow \{1! + 2! + ... + (n+2)!\} = Q(n)\{1! + 2! + ... + n!\} + P(n)\{1! + 2! + ... + (n+1)!\}$$

$$\Rightarrow (1! + 2! + \ldots + n!) + \{(n+1)! + (n+2)!\} = \{Q(n) + P(n)\}$$

$$\{1! + 2! + 3! + \ldots + n!\} + P(n).(n+1)!$$

Equating coefficients of $\{1! + 2! + ... + n!\}$ and $(n+1)!$ on both sides, we get

$$Q(n) + P(n) = 1 \text{ and } P(n) = (n+3)$$

So, $\quad P(n) = n + 3$

or $\quad P(x) = x + 3 \quad$ and $\quad Q(x) = 1 - P(x)$

Hence, $\quad P(x) = x + 3$ and $Q(x) = -x - 2.$

175. Given, $f(x) = \dfrac{a^x}{a^x + \sqrt{a}}$ ...(i)

Now, $f(1-x) = \dfrac{a^{1-x}}{a^{1-x} + \sqrt{a}} = \dfrac{\sqrt{a}}{\sqrt{a} + a^x}$ ...(ii)

From Eqs. (i) and (ii), we have $f(x) + f(1-x) = 1$ ...(iii)

$$\Rightarrow f\left(\frac{r}{2n}\right) + f\left(\frac{2n-r}{2n}\right) = 1$$

$$\Rightarrow \sum_{r=1}^{2n-1} f\left(\frac{r}{2n}\right) + \sum_{r=1}^{2n-1} f\left(\frac{2n-r}{2n}\right) = 2n-1$$

$$\Rightarrow \sum_{r=1}^{2n-1} f\left(\frac{r}{2n}\right) + \sum_{t=1}^{2n-1} f\left(\frac{t}{2n}\right) = 2n-1 \qquad [\text{putting } 2n - r = t]$$

Hence, $\quad \sum_{r=1}^{2n-1} 2f\left(\frac{r}{2n}\right) = 2n-1$

176. $f(x)$ is defined if $\left(\log_{|\sin x|}(x^2 - 8x + 23) - \dfrac{3}{\log_2 |\sin x|}\right) > 0$

$$\Rightarrow \log_{|\sin x|}\left(\frac{x^2 - 8x + 23}{8}\right) > 0$$

$$\left[\text{as } \frac{3}{\log_2 |\sin x|} = \frac{\log_2 8}{\log_2 |\sin x|} = \log_{|\sin x|} 8\right]$$

This is true, if

$$|\sin x| \ne 0, 1 \quad \text{and} \quad \frac{x^2 - 8x + 23}{8} < 1$$

$$[\text{as } |\sin x| < 1 \Rightarrow \log_{|\sin x|} a > 0 \Rightarrow a < 1]$$

Now, $\quad \dfrac{x^2 - 8x + 23}{8} < 1$

$$\Rightarrow x^2 - 8x + 15 < 0$$

$$\Rightarrow x \in (3, 5) - \left\{\pi, \frac{3\pi}{2}\right\}$$

Hence, domain $= (3, \pi) \cup \left(\pi, \dfrac{3\pi}{2}\right) \cup \left(\dfrac{3\pi}{2}, 5\right).$

177. Given, $\quad nx + ny = xy$

$\Rightarrow \ xy - nx - ny + n^2 = n^2 \ \Rightarrow (x-n)(y-n) = n^2$

$\Rightarrow (x-n)$ and $(y-n)$ are two integral factors of n^2.

[as $x, y, n \in N$]

Obviously, if d is one divisor of n^2, then for each such divisor, there will be an ordered pair (x, y).

So, $S(n) =$ number of divisors of n^2.

(i) For $n = 6$, we have $d = 1, 2, 3, 6, 9, 12, 18, 36.$

Thus, $\quad S(6) = 9$

(ii) If n is prime, then $d = 1, n$ and n^2.

Hence, $S(n) = 3$, whenever n is prime.

178. We have, $\frac{1}{[x]} + \frac{1}{[2x]} = \{x\} + \frac{1}{3}$

Case I $x \in I$, $[x] = x$, $[2x] = 2x$ $\{x\} = 0$

$$\therefore \quad \frac{1}{x} + \frac{1}{2x} = \frac{1}{3} \Rightarrow \frac{3}{2x} = \frac{1}{3} \Rightarrow x = \frac{9}{2}$$

$\frac{9}{2} \notin I$, Not possible.

Case II, $x \in I + f$, where I is an integer and f is fraction

$$[x] = I, [2x] = 2I \ \ \{x\} = f, 0 \le f < \frac{1}{2}$$

$$\Rightarrow \quad \frac{1}{I} + \frac{1}{2I} = f + \frac{1}{3} \Rightarrow f = \frac{3}{2I} - \frac{1}{3}$$

$$\Rightarrow \quad 0 \le \frac{3}{2I} = -\frac{1}{3} < \frac{1}{2} \Rightarrow \frac{1}{3} \le \frac{3}{2I} < \frac{5}{6}$$

$$\Rightarrow \quad \frac{9}{5} < I \le \frac{9}{2} \Rightarrow I = 2, 3, 4$$

When $I = 2$, $f = \frac{5}{12}$; $I = 3$, $f = \frac{1}{6}$; $I = 4$, $f = \frac{1}{24}$

$$\therefore \quad x \in \left\{\frac{29}{12}, \frac{19}{6}, \frac{97}{24}\right\}.$$

179. The condition, $\sum_{i=1}^{n} f^{-1}(x_i) = \sum_{i=1}^{n} x_i$ can be written as;

$$\frac{1}{n}\sum_{i=1}^{n} f^{-1}(x_i) = \frac{1}{n}\sum_{i=1}^{n} x_i$$

i.e. AM of y-coordinates of f^{-1} = AM of x-coordinates of f

The given two conditions hold if and only if

$$x^2 + 3x - 3 = x$$

$$\Rightarrow \quad x^2 + 2x - 3 = 0$$

So, $x = -3, +1$

But $x \ge 0 \Rightarrow x = 1$ [neglecting $x = -3$]

Hence, we can write $\frac{1}{n}\sum_{i=1}^{n} x_i = 1$, which is the required result.

180. We have, $f(x) = x^2 - 2x$ ...(i)

and $g(x) = f(f(x) - 1) + f(5 - f(x))$ ...(ii)

$$\therefore \quad g(x) = f(x^2 - 2x - 1) + f(5 - x^2 + 2x)$$

$$= [\{x^2 - 2x - 1\}^2 - 2\{x^2 - 2x - 1\}] + [(5 - x^2 + 2x)^2 - 2\{5 - x^2 + 2x\}]$$

$$\Rightarrow \quad g(x) = 2x^4 - 8x^3 - 4x^2 + 24x + 18$$

To show $g(x) \ge 0$, we find its range.

i.e. $g'(x) = 8x^3 - 24x^2 - 8x + 24$ let $g'(x) = 0$

$$\Rightarrow \quad x = -1, 1 \text{ and } 3$$

$$\Rightarrow \quad g(x) \ge \min \{ g(-1), g(1), g(3)\} = 0$$

Hence, $g(x) \ge 0, \forall x \in R$.

181. Given, $2 + f(x)f(y) = f(x) + f(y) + f(xy)$

or $1 - f(x) - f(y) + f(x)f(y) = f(xy) - 1$

or $(1 - f(x))(1 - f(y)) = f(xy) - 1$

The above result holds if and only if,

$$f(x) = 1 + x^n$$

If $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_0$

Then, consider $(1 + f(x))(1 - f(y)) = f(xy) - 1$

Compare constant term on either side, we have

$1 - a_0 = a_0 - 1 \Rightarrow a_0 = 1$

Comparing coefficient of $x^n y^n$, we get

$a_n^2 = a_n \Rightarrow a_n = 1$ or otherwise polynomial would not be of n degree.

Comparing coefficient of x, x^1, ..., x^{n-1} on either sides, we have

$$a_1 = a_2 = ... = a_{n-1} = 0$$

$$\Rightarrow \quad a_n = 1 \text{ and } f(x) = x^n + 1$$

Given, $f(2) = 5$, i.e. $2^n + 1 = 5$

$$\Rightarrow \quad n = 2$$

Thus, $f(x) = x^2 + 1$

$$\Rightarrow \quad f(f(2)) = f(5) = 5^2 + 1 = 26$$

182. Using AM $\ge$ GM, we have $\frac{a + b + c}{3} \ge (abc)^{1/3}$

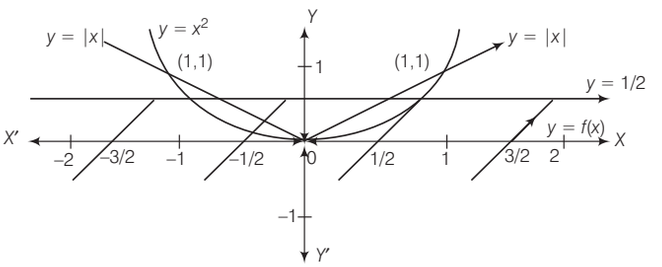
$$\Rightarrow \quad \frac{a + b + c}{3} \ge (a + b + c)^{1/3} \qquad [\because a + b + c = abc]$$

$$\Rightarrow \quad (a + b + c)^{2/3} \ge 3$$

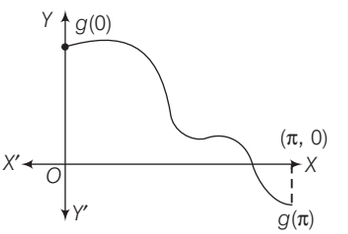
$$\Rightarrow \quad (a + b + c) \ge 3\sqrt{3}$$

183. Clearly, $g(x) =$

$$\begin{cases} x^2, -2 \le x \le -1 \\ -x, -1 \le x \le -\frac{1}{4} \\ x + \frac{1}{2}, -\frac{1}{4} \le x < 0 \text{ as graphically if,} \\ x, 0 \le x \le 1 \\ x^2, 1 \le x \le 2 \text{ can be expressed as shown in the following figure} \end{cases}$$



184. Let $g(x) = f(x) - f(x + \pi)$ ...(i)



At $x = \pi$, $g(\pi) = f(\pi) - f(2\pi)$ ...(ii)

At $x = 0$, $g(0) = f(0) - f(\pi)$ ...(iii)

Adding Eqs. (ii) and (iii), we have

$$g(0) + g(\pi) = f(0) - f(2\pi) = 0$$

$$\Rightarrow \quad g(0) = -g(\pi)$$

$\Rightarrow g(0)$ and $g(\pi)$ are of opposite sign.

$\Rightarrow$ There is a point c between 0 and π such that

$$g(c) = 0 \quad \text{...(iv)}$$

From Eq. (i) putting $x = c$, we have

$$g(c) = f(c) - f(\pi + c) \quad \text{...(v)}$$

From Eqs. (iv) and (v), we have

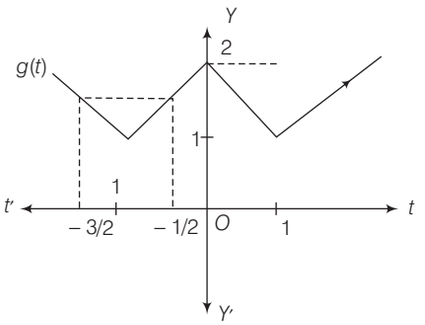
$$f(c) - f(\pi + c) = 0$$

Hence, $f(c) = f(\pi + c)$

185. We have, $g(t) = |t-1| - |t| + |t+1|$

$$\Rightarrow \quad g(t) = \begin{cases} -t, & t < -1 \\ t+2, & -1 \le t < 0 \\ 2-t, & 0 \le t < 1 \\ t, & t \ge 1 \end{cases}$$

Graph of $g(t)$ is



$$f(x) = \max\left\{ g(t) : \frac{-3}{2} \le t \le x \right\} \quad x \in \left(\frac{-3}{2}, \infty \right)$$

Clearly from the graph,

$$f(x) = \begin{cases} 3/2, & \frac{-3}{2} < x \le \frac{-1}{2} \\ x+2, & \frac{-1}{2} \le x < 0 \\ 2, & 0 \le x \le 2 \\ x, & x \ge 2 \end{cases}$$

186. $n_1 n_2 = 2n_1 - n_2$ [given]

$$\Rightarrow \quad n_2(n_1 + 1) = 2n_1 \quad \Rightarrow \quad n_2 = \frac{2n_1}{n_1 + 1}$$

$$\Rightarrow \quad n_2 = 2 - \frac{2}{n_1 + 1}$$

Since, n_1, n_2 are integers.

$$\therefore \quad \frac{2}{n_1 + 1} \in \text{Integer}$$

$$\Rightarrow \quad n_1 + 1 = -2, -1, 1, 2$$

$$\Rightarrow \quad n_1 = -3, -2, 0, 1$$

$$\Rightarrow \quad n_2 = 3, 4, 0, 1$$

$\Rightarrow$ Integral solutions of the form (n_1, n_2) are $(-3, 3), (-2, 4), (0, 0), (1, 1)$.

187. **PLAN**

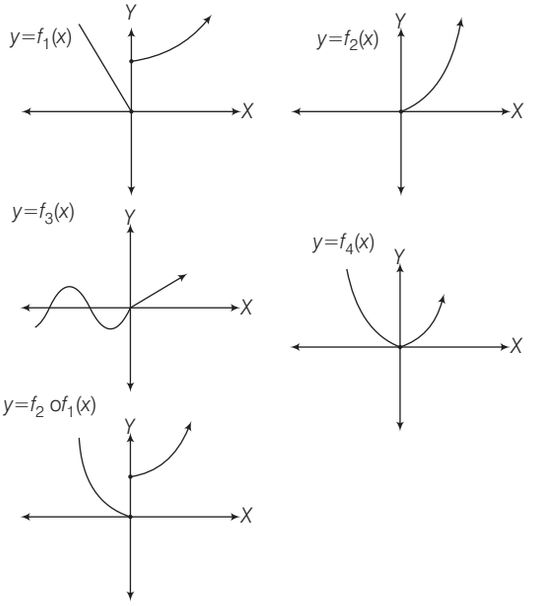
(i) For such questions, we need to properly define the functions and then we draw their graphs.

(ii) From the graphs, we can examine the function for continuity, differentiability, one-one and onto.

$$f_1(x) = \begin{cases} -x, & x < 0 \\ e^x, & x \ge 0 \end{cases} \quad \Rightarrow \quad f_2(x) = x^2,\ x \ge 0$$

$$f_3(x) = \begin{cases} \sin x, & x < 0 \\ x, & x \ge 0 \end{cases} \Rightarrow f_4(x) = \begin{cases} f_2(f_1(x)), & x < 0 \\ f_2(f_1(x)) - 1, & x \ge 0 \end{cases}$$

Now, $f_2(f_1(x)) = \begin{cases} x^2, & x < 0 \\ e^{2x}, & x \ge 0 \end{cases} \Rightarrow f_4 = \begin{cases} x^2, & x < 0 \\ e^{2x} - 1, & x \ge 0 \end{cases}$



As, $f_4(x)$ is continuous, $f'_4(x) = \begin{cases} 2x, & x < 0 \\ 2e^{2x}, & x > 0 \end{cases}$

$f_4'(0)$ is not defined. Its range is $[0, \infty)$, thus f_4 is onto.

Also, horizontal line (drawn parallel to X-axis) meets the curve more than once, thus function is not one-one.

188. Given, $g\{f(x)\} = x \Rightarrow g'\{f(x)\} f'(x) = 1$

If $f(x) = 1 \Rightarrow x = 0,\ f(0) = 1$

Substitute $x = 0$ in Eq. (i), we get

$$g'(1) = \frac{1}{f'(0)}$$

$$\Rightarrow \quad g'(1) = 2 \qquad \left[\because f'(x) = 3x^2 + \frac{1}{2} e^{x/2} \Rightarrow f'(0) = \frac{1}{2} \right]$$

Alternate Solution

Given, $f(x) = x^2 + e^{x/2}$

$$\Rightarrow \quad f'(x) = 3x^2 + \frac{1}{2} e^{x/2}$$

For $x = 0,\ f(0) = 1,\ f'(0) = \frac{1}{2}$ and $g(x) = f^{-1}(x)$

Replacing x by $f(x)$, we have

$$g(f(x)) = x \ \Rightarrow \ g'(f(x)) \cdot f'(x) = 1$$

Put $x = 0$, we get $g'(1) = \frac{1}{f'(0)} = 2$

189. Given, $F(x)=\int \sin^2 x\,dx=\int \frac{1-\cos 2x}{2}dx$

$$F(x)=\frac{1}{4}(2x-\sin 2x)+C$$

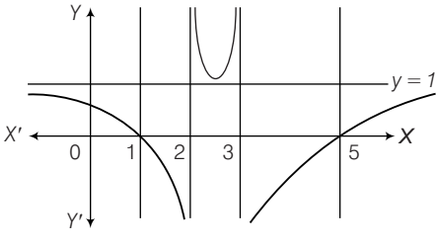
Since, $F(x+\pi)\neq F(x)$

Hence, Statement I is false.

But Statement II is true, as $\sin^2 x$ is periodic with period π.

190. Given, $f(x)=\frac{(x-1)(x-5)}{(x-2)(x-3)}$

The graph of $f(x)$ is shown as



A. If $-1<x<1 \Rightarrow \quad 0<f(x)<1$

B. If $1<x<2 \Rightarrow f(x)<0$

C. If $3<x<5 \Rightarrow f(x)<0$

D. If $x>5 \Rightarrow 0<f(x)<1$

191. Given, $\quad 2\sin t=\frac{1-2x+5x^2}{3x^2-2x-1},\ t\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

Put $\quad 2\sin t=y \Rightarrow -2\le y\le 2$

$$\therefore \quad y=\frac{1-2x+5x^2}{3x^2-2x-1}$$

$$\Rightarrow \quad (3y-5)x^2-2x(y-1)-(y+1)=0$$

Since, $x\in R-\{1,-1/3\}$

$$[\text{as}, 3x^2-2x-1\neq 0 \Rightarrow (x-1)(x+1/3)\neq 0]$$

$$\therefore \quad D\ge 0$$

$$\Rightarrow \quad 4(y-1)^2+4(3y-5)(y+1)\ge 0$$

$$\Rightarrow \quad y^2-y-1\ge 0$$

$$\Rightarrow \quad \left(y-\frac{1}{2}\right)^2-\frac{5}{4}\ge 0$$

$$\Rightarrow \quad \left(y-\frac{1}{2}-\frac{\sqrt{5}}{2}\right)\left(y-\frac{1}{2}+\frac{\sqrt{5}}{2}\right)\ge 0$$

$$\Rightarrow \quad y\le\frac{1-\sqrt{5}}{2} \quad \text{or} \quad y\ge\frac{1+\sqrt{5}}{2}$$

$$\Rightarrow \quad 2\sin t\le\frac{1-\sqrt{5}}{2} \quad \text{or} \quad 2\sin t\ge\frac{1+\sqrt{5}}{2}$$

$$\Rightarrow \quad \sin t\le\sin\left(-\frac{\pi}{10}\right) \quad \text{or} \quad \sin t\ge\sin\left(\frac{3\pi}{10}\right)$$

$$\Rightarrow \quad t\le-\frac{\pi}{10} \quad \text{or} \quad t\ge\frac{3\pi}{10}$$

Hence, range of t is $\left[-\frac{\pi}{2},-\frac{\pi}{10}\right]\cup\left[\frac{3\pi}{10},\frac{\pi}{2}\right]$.

192. Given, $f_k(x)=\frac{1}{k}(\sin^k x+\cos^k x)$,

where $x\in R$ and $k>1$

$$f_4(x)-f_6(x)=\frac{1}{4}(\sin^4 x+\cos^4 x)-\frac{1}{6}(\sin^6 x+\cos^6 x)$$

$$=\frac{1}{4}(1-2\sin^2 x\cdot\cos^2 x)-\frac{1}{6}(1-3\sin^2 x\cdot\cos^2 x)$$

$$=\frac{1}{4}-\frac{1}{6}=\frac{1}{12}$$

193. **Plan** To check nature of function.

(i) **One-one** To check one-one, we must check whether $f'(x)>0$ or $f'(x)<0$ in given domain.

(ii) **Onto** To check onto, we must check

$$\text{Range}=\text{Codomain}$$

Description of Situation To find range in given domain $[a,b]$, put $f'(x)=0$ and find $x=\alpha_1,\alpha_2,\ldots,\alpha_n\in[a,b]$

Now, find $\{f(a),f(\alpha_1),f(\alpha_2),\ldots,f(\alpha_n),f(b)\}$

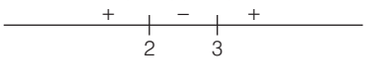
its greatest and least values gives you range.

Now, $f:[0,3]\to[1,29]$

$$f(x)=2x^3-15x^2+36x+1$$

$$\therefore \quad f'(x)=6x^2-30x+36=6(x^2-5x+6)$$

$$=6(x-2)(x-3)$$



For given domain [0, 3], $f(x)$ is increasing as well as decreasing $\Rightarrow$ many-one

Now, put $f'(x)=0$

$$\Rightarrow \quad x=2,3$$

Thus, for range $\quad f(0)=1,\ f(2)=29,\ f(3)=28$

$\Rightarrow$ Range $\in[1,29]$

$\therefore$ Hence, the function is into.

194. $f(x)=x^2,\ g(x)=\sin x$

$$(gof)(x)=\sin x^2$$

$$go(gof)(x)=\sin(\sin x^2)$$

$$(fogogof)(x)=(\sin(\sin x^2))^2$$

Given, $(fogogof)(x)=(gogof)(x)$

$$\Rightarrow \quad (\sin(\sin x^2))^2=\sin(\sin x^2)$$

$$\Rightarrow \quad \sin(\sin x^2)\{\sin(\sin x^2)-1\}=0$$

$$\Rightarrow \sin(\sin x^2)=0 \ \text{ or } \ \sin(\sin x^2)=1$$

$$\Rightarrow \quad \sin x^2=0 \quad \text{or} \quad \sin x^2=\frac{\pi}{2}$$

$$\therefore \quad x^2=n\pi \qquad [\text{i.e. not possible as } -1\le\sin\theta\le 1]$$

$$x=\pm\sqrt{n\pi}$$

195. Here, $f(x)=\frac{b-x}{1-bx}$, where $0<b<1,\ 0<x<1$

For function to be invertible, it should be one-one onto.

Check Range

Let $\quad f(x)=y \Rightarrow y=\frac{b-x}{1-bx}$

$$\Rightarrow \quad y-bxy=b-x \Rightarrow x(1-by)=b-y$$

$\Rightarrow \quad x = \dfrac{b-y}{1-by}$, where $0 < x < 1$

$\therefore \quad 0 < \dfrac{b-y}{1-by} < 1$

$\Rightarrow \quad \dfrac{b-y}{1-by} > 0$ and $\dfrac{b-y}{1-by} < 1$

$\Rightarrow \quad y < b$ or $y > \dfrac{1}{b}$

and $\dfrac{(b-1)(y+1)}{1-by} < 0 \Rightarrow -1 < y < \dfrac{1}{b}$ ...(i)

From Eqs. (i) and (ii), we get ...(ii)

$$y \in (-1, b) \subset \text{codomain}$$

196. We have, $e^{-x} f(x) = 2 + \int_0^x \sqrt{t^4+1}\, dt, \forall\ x \in (-1, 1)$

On differentiating w.r.t. x, we get

$$e^{-x}[f'(x) - f(x)] = \sqrt{x^4+1}$$

$\Rightarrow \quad f'(x) = f(x) + \sqrt{x^4+1} \cdot e^x$

$\because$ f^{-1} is the inverse of f.

$\therefore \quad f^{-1}\{f(x)\} = x$

$\Rightarrow \quad [f^{-1}\{f(x)\}]'\, f'(x) = 1$

$\Rightarrow \quad [f^{-1}\{f(x)\}]' = \dfrac{1}{f'(x)}$

$\Rightarrow \quad [f^{-1}\{f(x)\}]' = \dfrac{1}{f(x) + \sqrt{x^4+1} \cdot e^x}$

At $x = 0$, $f(x) = 2$

$\Rightarrow \quad \{f^{-1}(2)\}' = \dfrac{1}{2+1} = \dfrac{1}{3}$

197. Since, only (c) satisfy given definition,

i.e. $f\{f^{-1}(B)\} = B$

Only, if $B \subseteq f(x)$

198. Let $\phi(x) = f(x) - g(x) = \begin{cases} x, & x \in Q \\ -x, & x \notin Q \end{cases}$

Now, to check one-one.

Take any straight line parallel to X-axis which will intersect $\phi(x)$ only at one point.

$\Rightarrow$ $\phi(x)$ is one-one.

To check onto

As, $f(x) = \begin{cases} x, & x \in Q \\ -x, & x \notin Q \end{cases}$, which shows

$y = x$ and $y = -x$ for rational and irrational values

$\Rightarrow$ $y \in$ real numbers.

$\therefore$ Range = Codomain $\Rightarrow$ onto

Thus, $f - g$ is one-one and onto.

199. By definition of composition of function,

$g(f(x)) = (\sin x + \cos x)^2 - 1$, is invertible (i.e. bijective)

$\Rightarrow \quad g\{f(x)\} = \sin 2x$ is bijective.

We know, $\sin x$ is bijective, only when $x \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

Thus, $g\{f(x)\}$ is bijective, if $-\dfrac{\pi}{2} \le 2x \le \dfrac{\pi}{2}$

$\Rightarrow \quad -\dfrac{\pi}{4} \le x \le \dfrac{\pi}{4}$

200. Here, $f(x) = \sqrt{\sin^{-1}(2x) + \dfrac{\pi}{6}}$, to find domain we must have,

$$\sin^{-1}(2x) + \frac{\pi}{6} \ge 0 \qquad \left[\text{but } -\frac{\pi}{2} \le \sin^{-1}\theta \le \frac{\pi}{2}\right]$$

$$-\frac{\pi}{6} \le \sin^{-1}(2x) \le \frac{\pi}{2} \Rightarrow \sin\left(\frac{-\pi}{6}\right) \le 2x \le \sin\frac{\pi}{2}$$

$$\frac{-1}{2} \le 2x \le 1$$

$\Rightarrow \quad \dfrac{-1}{4} \le x \le \dfrac{1}{2}$

$\because \quad x \in \left[\dfrac{-1}{4}, \dfrac{1}{2}\right]$

201. Let $y = f(x) = \dfrac{x^2+x+2}{x^2+x+1}$, $x \in R$

$\therefore \quad y = \dfrac{x^2+x+2}{x^2+x+1}$

$$y = 1 + \frac{1}{x^2+x+1} \qquad [\text{i.e. } y > 1] \ldots(i)$$

$\Rightarrow \quad yx^2 + yx + y = x^2 + x + 2$

$\Rightarrow \quad x^2(y-1) + x(y-1) + (y-2) = 0, \forall\, x \in R$

Since, x is real, $D \ge 0$

$\Rightarrow \quad (y-1)^2 - 4(y-1)(y-2) \ge 0$

$\Rightarrow \quad (y-1)\{(y-1) - 4(y-2)\} \ge 0$

$\Rightarrow \quad (y-1)(-3y+7) \ge 0$

$\Rightarrow \quad 1 \le y \le \dfrac{7}{3}$ ...(ii)

From Eqs. (i) and (ii), Range $\in \left(1, \dfrac{7}{3}\right]$

202. Given, $f : [0, \infty) \to [0, \infty)$

Here, domain is $[0, \infty)$ and codomain is $[0, \infty)$. Thus, to check one-one

Since, $f(x) = \dfrac{x}{1+x}$

$\Rightarrow \quad f'(x) = \dfrac{1}{(1+x)^2} > 0, \forall\ x \in [0, \infty)$

$\therefore$ $f(x)$ is increasing in its domain. Thus, $f(x)$ is one-one in its domain. To check onto (we find range)

Again, $y = f(x) = \dfrac{x}{1+x}$

$\Rightarrow \quad y + yx = x$

$\Rightarrow \quad x = \dfrac{y}{1-y} \Rightarrow \dfrac{y}{1-y} \ge 0$

Since, $x \ge 0$, therefore $0 \le y < 1$

i.e. Range $\neq$ Codomain

$\therefore$ $f(x)$ is one-one but not onto.

203. Given, $f(x) = 2x + \sin x$

$\Rightarrow$ $f'(x) = 2 + \cos x$

$\Rightarrow$ $f'(x) > 0, \forall x \in R$

which shows $f(x)$ is one-one, as $f(x)$ is strictly increasing.

Since, $f(x)$ is increasing for every $x \in R$.

$\therefore$ $f(x)$ takes all intermediate values between $(-\infty, \infty)$.

Range of $f(x) \in R$.

Hence, $f(x)$ is one-to-one and onto.

204. The number of onto functions from

$E = \{1, 2, 3, 4\}$ to $F = \{1, 2\}$

= Total number of functions which map E to F

– Number of functions for which map $f(x) = 1$ and $f(x) = 2$ for all $x \in E = 2^4 - 2 = 14$

205. It is only to find the inverse.

Let $y = f(x) = (x + 1)^2$ for $x \geq -1$

$\Rightarrow \pm\sqrt{y} = x + 1,\ x \geq -1$

$\Rightarrow \sqrt{y} = x + 1 \Rightarrow y \geq 0,\ x + 1 \geq 0$

$\Rightarrow x = \sqrt{y} - 1$

$\Rightarrow f^{-1}(y) = \sqrt{y} - 1$

$\Rightarrow f^{-1}(x) = \sqrt{x} - 1$

$\Rightarrow x \geq 0$

206. Let $y = x + \frac{1}{x} \Rightarrow y = \frac{x^2 + 1}{x}$

$\Rightarrow xy = x^2 + 1$

$\Rightarrow x^2 - xy + 1 = 0$

$\Rightarrow x = \frac{y \pm \sqrt{y^2 - 4}}{2}$

$\Rightarrow f^{-1}(y) = \frac{y \pm \sqrt{y^2 - 4}}{2}$

$\Rightarrow f^{-1}(x) = \frac{x \pm \sqrt{x^2 - 4}}{2}$

Since, the range of the inverse function is $[1, \infty)$, then

we take $f^{-1}(x) = \frac{x + \sqrt{x^2 - 4}}{2}$

If we consider $f^{-1}(x) = \frac{x - \sqrt{x^2 - 4}}{2}$, then $f^{-1}(x) > 1$

This is possible only if $(x - 2)^2 > x^2 - 4$

$\Rightarrow x^2 + 4 - 4x > x^2 - 4$

$\Rightarrow 8 > 4x$

$\Rightarrow x < 2$, where $x > 2$

Therefore, (a) is the correct answer.

207. Given, $f(x) = (1 + b^2)\, x^2 + 2bx + 1$

$$= (1 + b^2)\left(x + \frac{b}{1 + b^2}\right)^2 + 1 - \frac{b^2}{1 + b^2}$$

$m(b)$ = minimum value of $f(x) = \frac{1}{1 + b^2}$ is positive and $m(b)$ varies from 1 to 0, so range = $(0, 1]$

208. Given, $f(x) = \frac{\log_2 (x + 3)}{(x^2 + 3x + 2)} = \frac{\log_2 (x + 3)}{(x + 1)(x + 2)}$

For numerator, $x + 3 > 0$

$\Rightarrow x > -3$ ...(i)

and for denominator, $(x + 1)(x + 2) \neq 0$

$\Rightarrow x \neq -1, -2$ ...(ii)

From Eqs. (i) and (ii), we get

Domain is $(-3, \infty) / \{-1, -2\}$

209. Given, $f(x) = \frac{\alpha x}{x + 1}$

$$f[f(x)] = f\left(\frac{\alpha x}{x + 1}\right) = \frac{\alpha\left(\frac{\alpha x}{x + 1}\right)}{\frac{\alpha x}{x + 1} + 1}$$

$$= \frac{\frac{\alpha^2 x}{x + 1}}{\frac{\alpha x + (x + 1)}{x + 1}} = \frac{\alpha^2 x}{(\alpha + 1)\, x + 1} = x \text{ [given]} \quad \text{...(i)}$$

$\Rightarrow \alpha^2 x = (\alpha + 1)\, x^2 + x$

$\Rightarrow x[\alpha^2 - (\alpha + 1)\, x - 1] = 0$

$\Rightarrow x(\alpha + 1)(\alpha - 1 - x) = 0$

$\Rightarrow \alpha - 1 = 0$ and $\alpha + 1 = 0 \Rightarrow \alpha = -1$

But $\alpha = 1$ does not satisfy the Eq. (i).

210. $g(x) = 1 + x - [x] \geq 1$

since $x - [x] \geq 0$

$f[g(x)] = 1$, since $f(x) = 1$ for all $x > 0$

211. Given, $2^x + 2^y = 2,\ \forall\ x, y \in R$

But $2^x, 2^y > 0, \forall\ x, y \in R$

Therefore, $2^x = 2 - 2^y < 2 \Rightarrow 0 < 2^x < 2$

Taking log on both sides with base 2, we get

$$\log_2 0 < \log_2 2^x < \log_2 2 \Rightarrow -\infty < x < 1$$

212. It is given,

$f(\theta) = \sin\theta(\sin\theta + \sin 3\theta)$

$= (\sin\theta + 3\sin\theta - 4\sin^3\theta)\sin\theta$

$= (4\sin\theta - 4\sin^3\theta)\sin\theta = \sin^2\theta(4 - 4\sin^2\theta)$

$= 4\sin^2\theta\cos^2\theta = (2\sin\theta\cos\theta)^2 = (\sin 2\theta)^2 \geq 0$

which is true for all θ.

CHAPTER

04

Graphical Transformations

Learning Part

Session 1

- When $f(x)$ vertically transforms to $f(x) \pm a$

Session 2

- When $f(x)$ horizontally transforms to $f(x \pm a)$, where a is any positive constant
- When $f(x)$ transforms to $a\, f(x)$ or $\frac{1}{a} f(x)$
- When $f(x)$ transforms to $f(ax)$ or $f(x/a)$
- When $f(x)$ transforms to $f(-x)$
- When $f(x)$ transforms to $-f(x)$

Session 3

- To draw $y = |f(x)|$, when $y = f(x)$ is given
- To draw $y = f(|x|)$, when $f(x)$ is given
- $f(x)$ transforms to $|f(|x|)|$, i.e. $f(x) \rightarrow |f(|x|)|$
- To draw the graph of $|y| = f(x)$ when $y = f(x)$ is given

Session 4

- To draw $y = [f(x)]$ when the graph of $y = f(x)$ is given and $[\cdot]$ denotes greatest integer or integral function
- To draw $y = f([x])$ when the graph of $y = f(x)$ is given
- When $f(x)$ transforms to $y = f(\{x\})$, where $\{\cdot\}$ denotes fractional part of x, i.e. $\{x\} = x - [x]$ or $f(x) \rightarrow f(\{x\})$
- When $f(x)$ and $g(x)$ are two functions and are transformed to their sum
- When $f(x)$ transforms to $f(x) \rightarrow \frac{1}{f(x)} = h(x)$
- To draw the graph for $y = f(x) \cdot \sin x$ when graph of $y = f(x)$ is given
- When $f(x)$ and $g(x)$ are given, then find $h(x) = \max(f(x), g(x))$ or $h(x) = \min(f(x), g(x))$

Practice Part

- JEE Type Solved Examples
- Chapter Exercises

Arihant on Your Mobile !
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Graph and diagram provide a simple and powerful approach to a variety of problems, from which we can solve them in a very short time. In this chapter, we will discuss how the graph of a function may be transformed either by shifting, stretching or compressing or reflection.

Graphical Transformations

A graphical transformation takes whatever its basic function $f(x)$. The graph of new function is the graph of the original function shifted vertically or horizontally.

Session 1

When $f(x)$ Vertically Transforms to $f(x) \pm a$

(Where a is Any Positive Constant)

For $f(x) \to f(x) + a$,

First, draw the graph of $f(x)$, then shift the graph of $f(x)$ upwards through 'a' units such that the distance between graphs of $f(x)$ and $f(x) + a$ at every point is same and is equal to 'a' units.

For $f(x) \to f(x) - a$,

First, draw the graph of $f(x)$, then shift the graph of $f(x)$ downwards through 'a' units such that the distance between the graphs of $f(x)$ and $f(x) - a$ at every point is same and is equal to 'a' units.

Graphically, it could be stated as shown in figure.

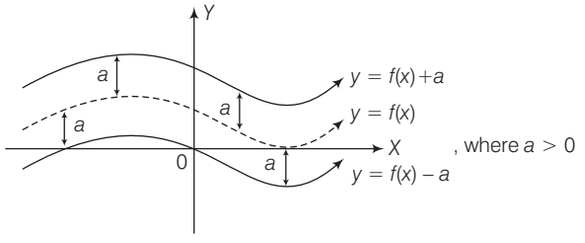


Figure 4.1

Example 1 Plot $y = |x|$ and $y = |x| + 2$.

Sol. We know, $y = |x|$ (modulus function) is define as

$$y = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

So, we have to plot two straight lines $y = x$ and $y = -x$.

For $y = x$, $x \geq 0$

x	1	2	3	4
y	1	2	3	4

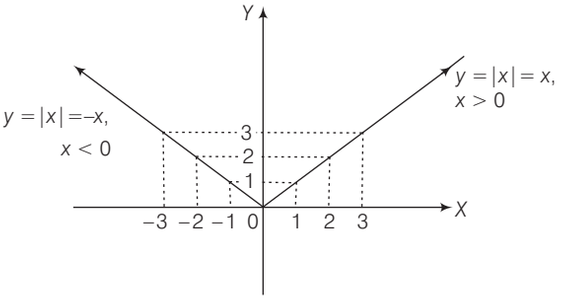
$\therefore$ Locate the points (1, 1), (2, 2), (3, 3), (4, 4) and join them.

For $y = -x$, $x < 0$

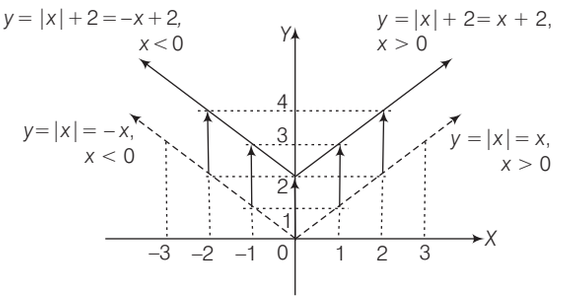
x	–1	–2	–3
y	1	2	3

$\therefore$ Locate the points (–1, 1), (–2, 2), (–3, 3) and join them.

So, we obtained graph of $y = |x|$.

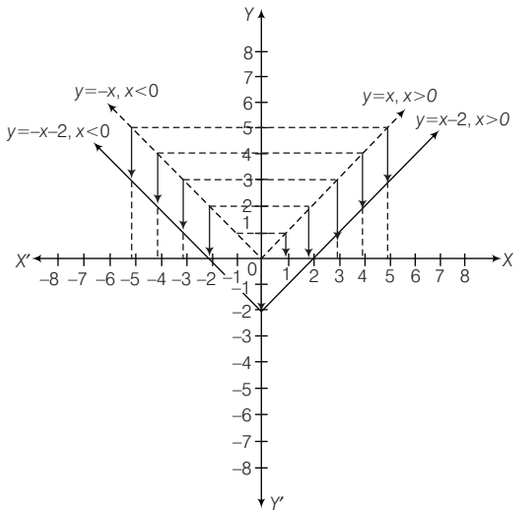


Now, to draw the graph of $y = |x| + 2$ and the graph of $|x|$ is shifted upwards by 2 units as shown below.



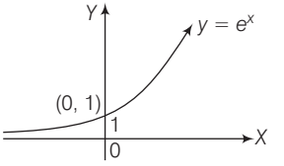
Example 2 Plot $y = |x|$ and $y = |x| - 2$.

Sol. First, we plot the graph of $y = |x|$ as explained in Example 1, then shift the graph of $|x|$ through 2 units downwards as shown below.



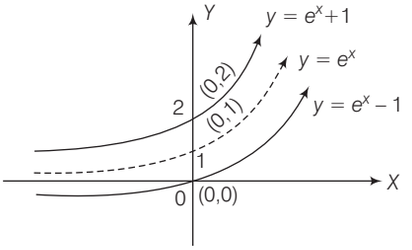
Example 3 Plot $y = e^x$, $y = e^x + 1$ and $y = e^x - 1$.

Sol. We know, $y = e^x$ (exponential function) could be plotted as shown below which cuts the Y-axis at $(0,1)$.



$\Rightarrow y = e^x + 1$ is shifted upward by 1 unit such that it intersects Y-axis at (0, 2) and $y = e^x - 1$ shifted downwards by 1 unit. Such that it intersects Y-axis at $(0,0)$.

The graph of the given functions is shown below.



Exercise for Session 1

■ **Directions** (Q. Nos. 1 to 8) *Plot the following functions.*

1. $y = x^2 + 1$
2. $y = x^2 - 1$
3. $y = x^3 + 1$
4. $y = x^3 - 1$
5. $y = \sin x + 1$
6. $y = \sin x - 1$
7. $y = (\log_e x) + 1$
8. $y = (\log_e x) - 1$

Session 2

When $f(x)$ Horizontally Transforms to $f(x \pm a)$, where a is any Positive Constant, When $f(x)$ Transforms to $af(x)$ or $\frac{1}{a} f(x)$, When $f(x)$ Transforms to $f(ax)$ or $f(x/a)$, When $f(x)$ Transforms to $f(-x)$, When $f(x)$ Transforms to $-f(x)$

For $f(x) \to f(x \pm a)$

For $f(x) \to f(x-a)$, First, draw the graph of $f(x)$, then shift the graph of $f(x)$ through 'a' units towards right such that distance between the graphs of $f(x)$ and $f(x-a)$ is same and is equal to 'a' units.

For $f(x) \to f(x+a)$, First, draw the graph of $f(x)$, then shift the graph of $f(x)$ through 'a' units towards left such that the distance between the graphs of $f(x)$ and $f(x+a)$ is same and is equal to 'a' units.

Graphically, it could be stated as shown in figure.

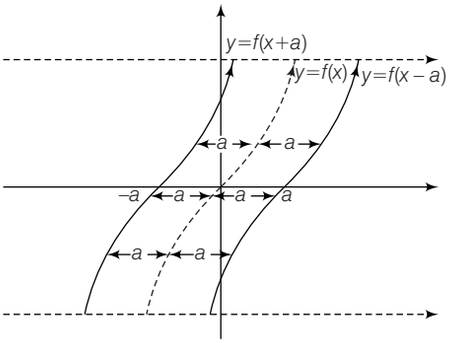
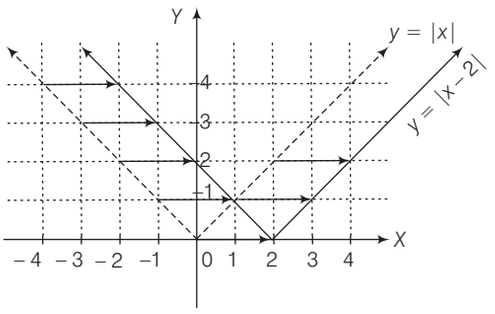


Figure 4.2

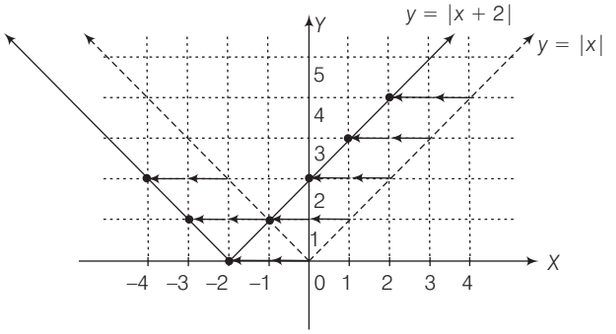
Example 4 ***Plot*** $y = |x|$ ***and*** $y = |x-2|$ **.**

Sol. First we plot the graph of $y = |x|$ as explained in Example 1, then shift the graph of $|x|$ through '2' units towards right as shown below.



Example 5 Plot $y = |x|$ and $y = |x+2|$.

Sol. First we plot the graph of $y = |x|$ as explained in Example 1, then shift the graph of $|x|$ through '2' units towards left.

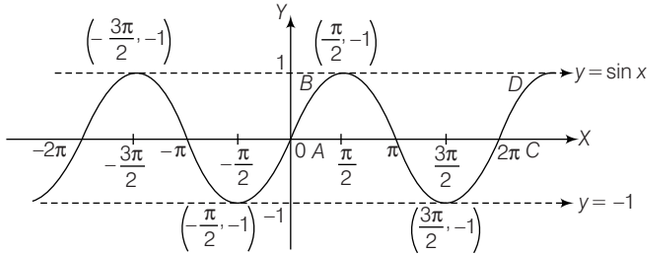


Example 6 ***Plot*** $y = \sin\left(x+\frac{\pi}{4}\right)$ ***and*** $y = \sin\left(x-\frac{\pi}{4}\right)$ **.**

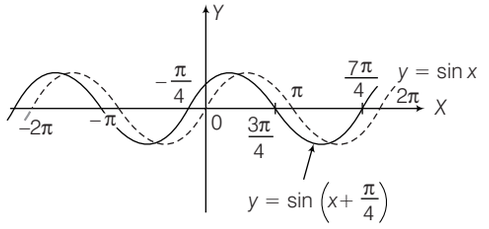
Sol. Let us consider $y = \sin x$, domain of sine functions is R and range is $[-1, 1]$. sin x increases strictly from -1 to 1 as x increases from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, decreases strictly from 1 to -1 as x increases from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$. Also, for $y = \sin x$.

x	-2π	$-\frac{3\pi}{2}$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y	0	1	0	-1	0	1	0	-1	0

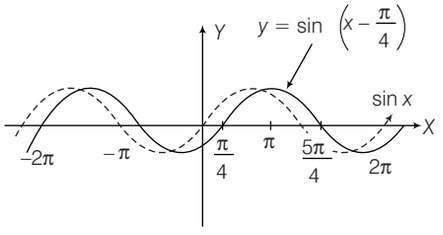
Locate the points $(-2\pi,0)$, $\left(-\frac{3\pi}{2},1\right)$, $(-\pi,0)$, $\left(-\frac{\pi}{2},-1\right)$, $(0,0)$, $\left(\frac{\pi}{2},1\right)$, $(\pi,0)$, $\left(\frac{3\pi}{2},-1\right)$ and $(2\pi,0)$ on the graph paper, then we get the graph of $\sin x$ as shown below.



Now, for the graph of $y = \sin\left(x + \frac{\pi}{4}\right)$ shift the graph of $y = \sin x$ through $\frac{\pi}{4}$ towards left as shown below.

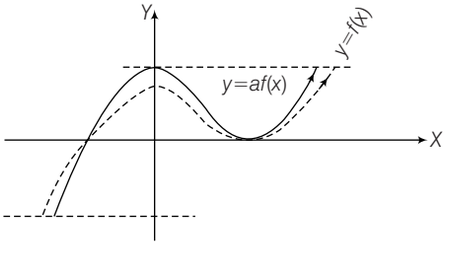


And for the graph of $y = \sin\left(x - \frac{\pi}{4}\right)$, shift the graph of $y = \sin x$ through $\frac{\pi}{4}$ towards right as shown below.



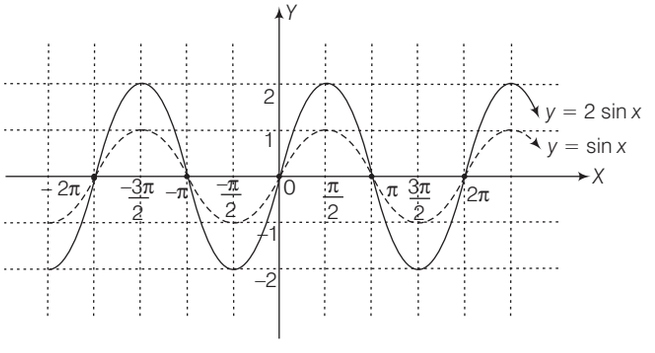
$f(x) \to \{af(x)\}, a > 1$

First draw the graph of $f(x)$, then stretch the graph of $f(x)$, 'a' times along Y-axis.



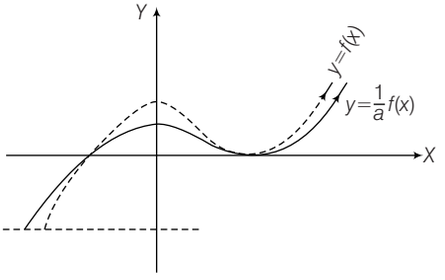
Example 7 Plot $y = \sin x$ and $y = 2\sin x$.

Sol. First we draw the graph of $\sin x$ as explained in Example 6, then stretch the graph of $\sin x$ by 2 times along the Y-axis, we get the graph of $2\sin x$ as shown below.



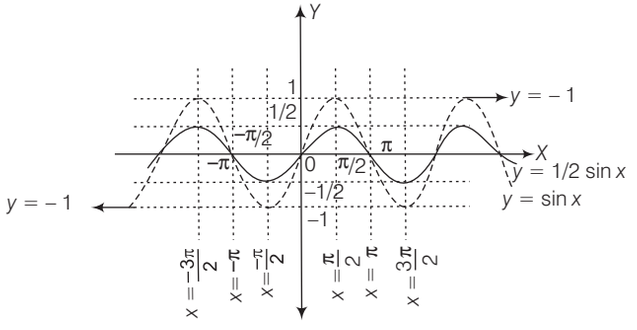
$f(x) \to \frac{1}{a} f(x), a > 1$

First, draw the graph of $f(x)$, then shrink the graph of $f(x)$, 'a' times along Y-axis. As shown in the figure.



Example 8 Plot $y = \sin x$ and $y = \frac{1}{2}\sin x$.

Sol. First we draw the graph of $y = \sin x$ as explained in Example 6, then shrink the graph of $\sin x$ by 2 times along Y-axis, we get the graph of $\frac{1}{2}\sin x$ as shown below.



$f(x) \to f(ax), a > 1$

First draw the graph of $f(x)$, then shrink the graph of $f(x)$, 'a' times along X-axis as shown in figure.

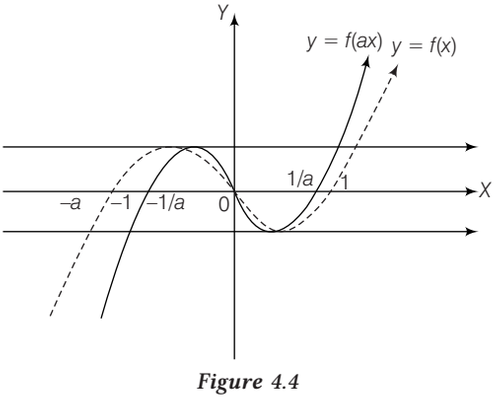
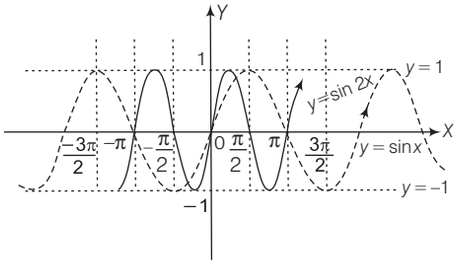


Figure 4.4

Example 9 Plot $y = \sin x$ and $y = \sin 2x$.

Sol. First draw the graph of $\sin x$ as explained in Example 6, then shrink the graph of $\sin x$, 2 times along X-axis, we get the graph of $\sin 2x$ as shown below.



$f(x) \to f\left(\frac{x}{a}\right), a > 1$

First draw the graph of $f(x)$, then
stretch the graph of $f(x)$, 'a' times along X-axis, as shown in figure.

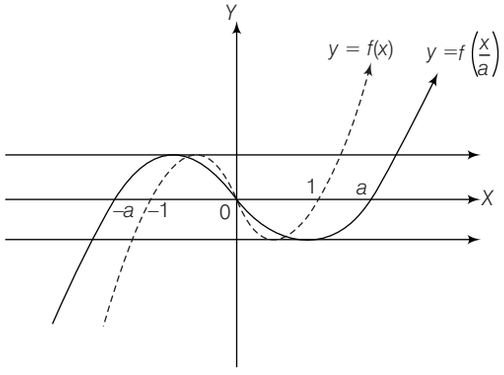
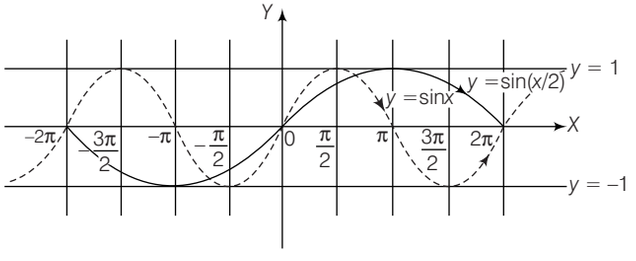


Figure 4.5

Example 10 Plot $y = \sin x$ and $y = \sin \frac{x}{2}$.

Sol. First we draw the graph of $\sin x$ as explained in Example 6, then stretch the graph of $\sin x$, 2 times along X-axis, we get the graph of $\sin \frac{x}{2}$ as shown below.



$f(x) \to f(-x)$

To draw $y = f(-x)$, take the image of the curve $y = f(x)$ in the Y-axis as plane mirror or turn the graph of $f(x)$ by 180° about Y-axis.

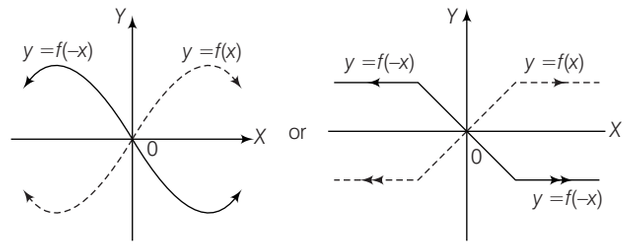
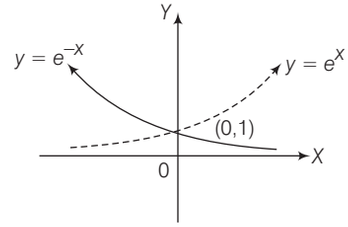


Figure 4.6

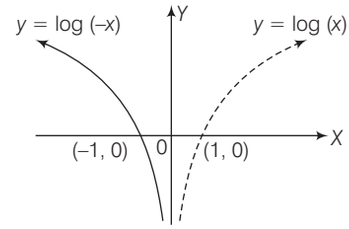
Example 11 Draw the graph of $y = e^{-x}$, when the graph of $y = e^{x}$ is known.

Sol. As $y = e^{x}$ is given, then $y = e^{-x}$ is the image in Y-axis as plane mirror. Thus, it can be drawn as shown in the figure.



Example 12 Draw graph of $y = \log(-x)$, when the graph of $y = \log(x)$ is given.

Sol. As $y = \log(x)$ is given and $y = \log(-x)$ is the image in Y-axis as plane mirror. Thus, it can be drawn as shown in figure.



$f(x) \to -f(x)$

To draw $y = -f(x)$ take image of $f(x)$ in the X-axis as plane mirror or turn the graph of $f(x)$ by 180° about X-axis.

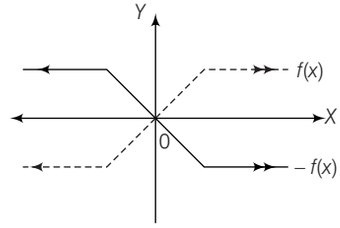
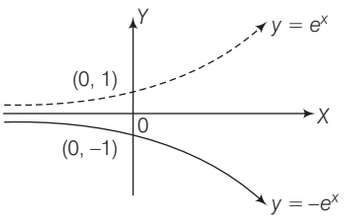


Figure 4.7

Example 13 Draw the graph of $y = -e^x$, when the graph of $y = e^x$ is known.

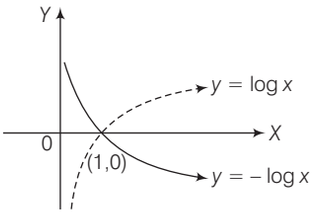
Sol. As $y = e^x$ is given, then $y = -e^x$ is the image of e^x in the X-axis as plane mirror.

Thus, it can be plotted as shown in figure



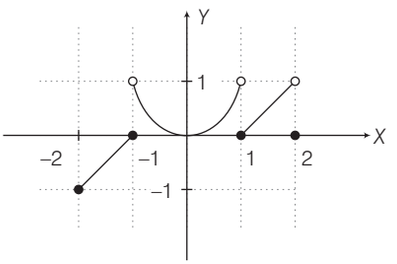
Example 14 Draw the graph of $y = -\log(x)$, when the graph of $y = \log x$ is known.

Sol. As $y = \log x$ is given, then $y = -\log x$ is the image of $\log x$ in the X-axis as plane mirror. Thus, it can be drawn as shown in figure.



Exercise for Session 2

1. Consider the following function f, whose graph is given below.



Draw the graph of the following functions.

(i) $f(x+1)$ (ii) $f(x-1)$ (iii) $-f(x)$ (iv) $f(-x)$

(v) $2f(x)$ (vi) $\frac{1}{2}f(x)$ (vii) $f(2x)$ (viii) $f\left(\frac{x}{2}\right)$

Session 3

To Draw $y = f(|x|)$, When $f(x)$ is Given, $f(x)$ Transforms to Draw $|f(|x|)|$, i.e. $f(x) \to |f(|x|)|$ To Draw $|y| = f(x)$

$f(x) \to |f(x)|$

When $y = f(x)$ is given

We know that $|f(x)| = \begin{cases} f(x), & \text{if } f(x) \geq 0 \\ -f(x), & \text{if } f(x) \leq 0 \end{cases}$

Hence, $y = |f(x)|$ is drawn in two steps

(i) In the step I, leave the positive part of $f(x)$, (i.e. the part of $f(x)$ above X-axis) as it is.

(ii) In the step II, take the image of negative part of $f(x)$, (i.e. the part of $f(x)$ below X-axis) in the X-axis as plane mirror.

Or

Take the mirror image (in X-axis) of the portion of the graph of $f(x)$ which lies below X-axis.

Or

Turn the portion of the graph of $f(x)$ lying below X-axis by 180° about X-axis.

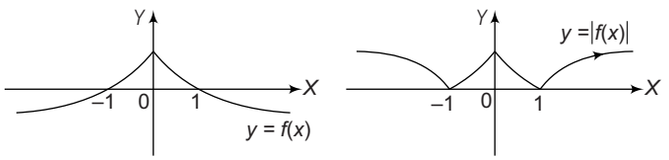
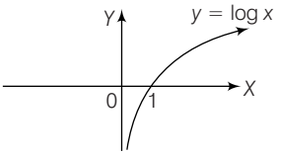


Figure 4.8

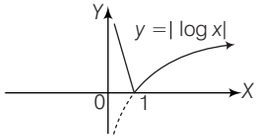
Example 15 Draw the graph of $y = |\log x|$, when the graph of $y = \log(x)$ is known.

Sol. To draw the graph of $y = |\log x|$, we have to proceed in two steps.

(i) In the first step, leave the positive part of $y = \log x$, i.e. the part of $y = \log x$ above X-axis as it is.



(ii) In the second step, take the image of negative part of $y = \log x$, i.e. the part below X-axis in the X-axis as plane mirror. Thus, the graph can be



Remark

The above transformation of graph is very important in explaining the differentiability of $f(x)$.

From above example we could say, $y = \log x$ is differentiable for $(0, \infty)$.

But $y = |\log x|$ is differentiable for $(0, \infty) - \{1\}$. As, we have a sharp edge at $x = 1$ (which indicates that it is not differentiable at $x = 1$).

$f(x) \to f(|x|)$

When $f(x)$ is given

We know that, $y = f|x| = \begin{cases} f(x), & \text{if } x \geq 0 \\ f(-x), & \text{if } x \leq 0 \end{cases}$.

Hence, again $f(|x|)$ could be drawn in two steps

(i) In the step I, leave the graph lying right side of the Y-axis, as it is.

(ii) In the step II, take the image of $f(x)$ in the Y-axis as plane mirror.

The part of $f(x)$ lying left side of the Y-axis (if it exists) is omitted.

e.g. Let us assume the graph of $f(x)$ as shown

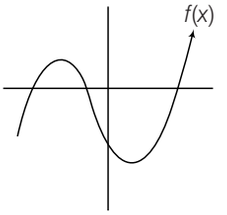


Figure 4.9

Now, **On the right of Y-axis** Plot the graph of $f(x)$ as such

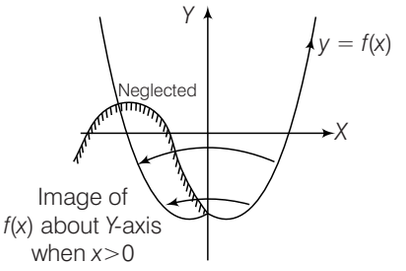


Figure 4.10

On the left of *Y*-axis Take the mirror image of the graph of $f(x)$ (of the portion lying on right of *Y*-axis).

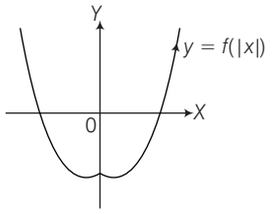


Figure 4.11

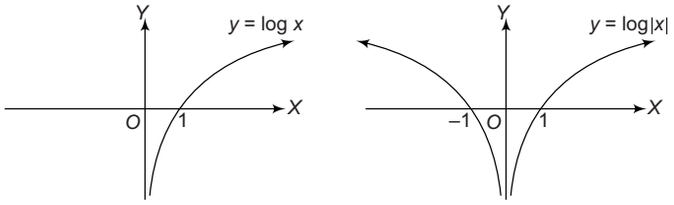
Example 16 Draw the graph of $y = \log|x|$, when graph of $y = \log(x)$ is given.

Sol. We know, $y = \log|x|$ could be drawn in two steps

(i) In step I, leave the graph lying right side of *Y*-axis as it is.

(ii) In step II, take the image of $f(x)$ in the *Y*-axis as plane mirror.

Thus, it can be plotted as shown in figure.



Note $f(|x|)$ is always an even function.

$f(x) \to |f(|x|)|$

Here, we plot $y = |f(|x|)|$ when graph of $f(|x|)$ is given, in two steps

(i) In step I, $f(x)$ transforms to $|f(x)|$.

(ii) In step II, $|f(x)|$ transforms to $|f(|x|)|$.

or

(i) $f(x) \to |f(x)|$ (ii) $f(x) \to |f(|x|)|$

e.g. Let graph for $f(x)$

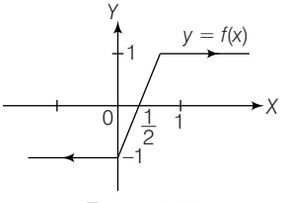


Figure 4.12

Now, $f(x) \to |f(x)|$ can be shown as,

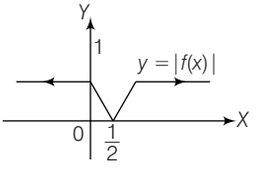


Figure 4.13

Again, $|f(x)| \to |f(|x|)|$ can be shown as,

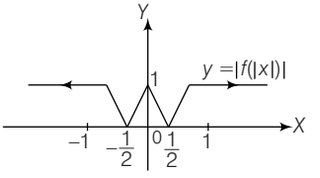
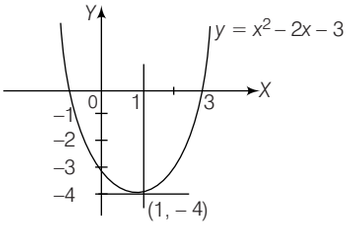


Figure 4.14

Example 17 Draw the graph for $y = \left||x|^2 - 2|x| - 3\right|$, if the graph for $y = x^2 - 2x - 3$ is given.

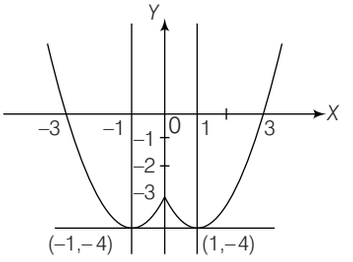
Sol. First we draw the graph for $y = x^2 - 2x - 3$

$\Rightarrow \quad (y + 4) = (x - 1)^2$



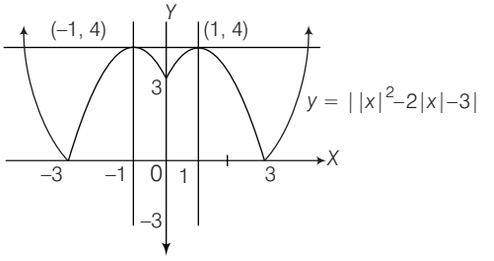
Now, we form $f(x) \to f(|x|)$,

i.e. $y = x^2 - 2x - 3 \Rightarrow y = |x|^2 - 2|x| - 3$ as shown below



In next step, we transform $f(x) = |f(|x|)|$,

i.e. $y = |x|^2 - 2|x| - 3 \to ||x|^2 - 2|x| - 3|$ as shown below



Remarks

Above mentioned transformations are very useful in explaining continuity and differentiability of functions as, if $y = \log(x)$, then

(i) $y = \log|x|$ can be plotted as shown in figure (4.15).

(ii) $y = |\log|x||$ can be plotted as shown in figure (4.16).

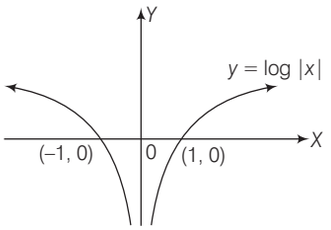


Figure 4.15

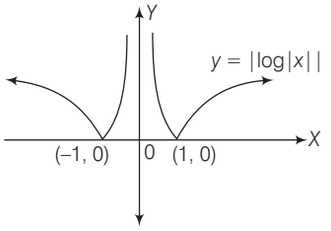


Figure 4.16

While discussing the case (i), we could say that it is continuous for all $x \in R - \{0\}$ and differentiable for all $x \in R - \{0\}$. Again, while discussing the case (ii), we could say that it is continuous for all $x \in R - \{0\}$ and differentiable at all $x \in R - \{-1,0,1\}$.
(As now it has sharp edges at $\{-1,1\}$ and is broken at $\{0\}$).

$|y| = f(x) \rightarrow y = f(x)$

When $y = f(x)$ is given

Clearly, $|y| > 0$, if $f(x) < 0$, graph of $|y| = f(x)$ would not exist.

And if $f(x) \geq 0$, $|y| = f(x)$ would be given as $y = \pm f(x)$.

Hence, the graph of $|y| = f(x)$ exists only in the regions, where $f(x)$ is non-negative and will be reflected about X-axis only when $f(x) \geq 0$. Region where $f(x) < 0$ will be neglected.

Or

To draw the graph of $|y| = f(x)$, we use following steps

(i) Remove the portion of the graph, which lies below X-axis.
[$\because$ the equation $|y| = f(x)$ is not satisfied when $f(x)$ is negative.]

(ii) Plot the remaining portion of the graph and also take its mirror image in the X-axis.
[$\because$ when $f(x) > 0$, then $y = \pm f(x)$]

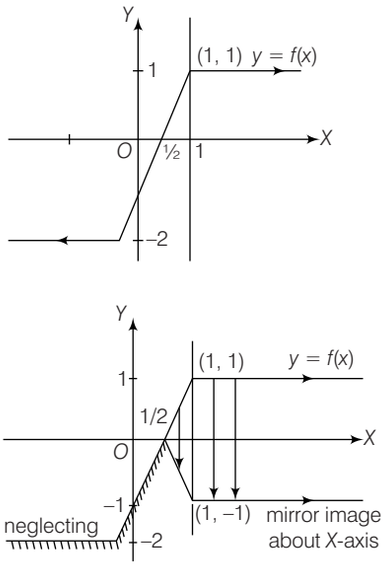
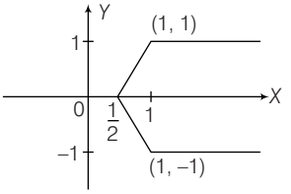


Figure 4.17

Finally, it is as shown

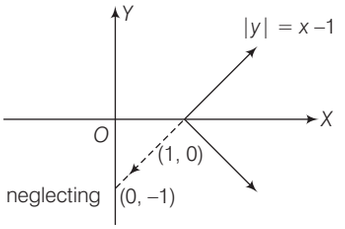


***Graph,* $|y| = f(x)$**
Figure 4.18

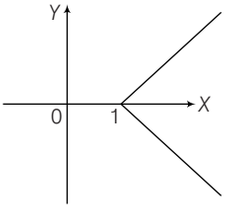
Example 18 Draw the graph for $|y| = (x - 1)$.

Sol. We know, here $|y| = (x - 1)$ exists only when $(x-1)$ is positive. Clearly $y = x - 1$ represent straight line the passing through (0, 1) and (0, –1).

$\therefore$ Neglecting negative values and will be reflected about X-axis. Thus, it can be drawn as shown.



Finally, it is shown as



Graph for $|y| = x - 1$

Example 19 Draw the graph for $|y| = (x - 1)(x - 2)$.

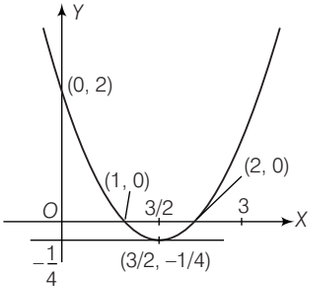
Sol. First we plot $y = (x - 1)(x - 2)$

$$\Rightarrow \quad y = x^2 - 3x + 2 \Rightarrow \left(x - \frac{3}{2}\right)^2 = \left(y + \frac{1}{4}\right)$$

It represent a parabola opening upwards having vertex $\left(\frac{3}{2}, -\frac{1}{4}\right)$.

It intersect X-axis at (0, 1) and (2, 0) and Y-axis at (0, 2).

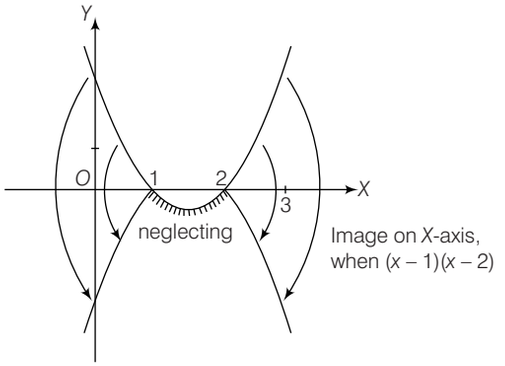
Thus, we can plot the figure as shown below.



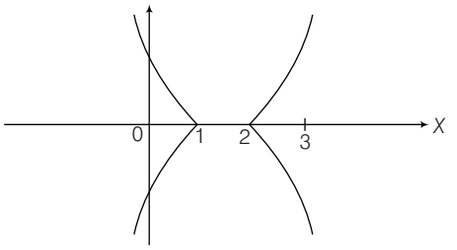
We know, $|y| = (x-1)(x-2)$ exists only, when $(x-1)(x-2) \geq 0$ and neglecting, if $(x-1)(x-2) < 0$.

Now, $f(x)$ will be reflected about X-axis, when $(x-1)(x-2) > 0$.

Thus, it can be plotted as shown.



Finally, $|y| = (x-1)(x-2)$ can be shown as

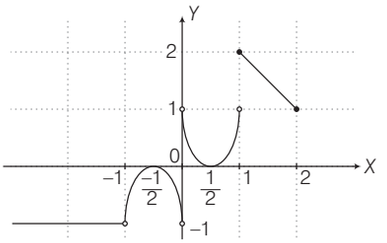


Summary

- $y = f(x)$ to $y = -f(x)$ — *Flip about the X-axis.*
- $y = f(x)$ to $y = f(-x)$ — *Flip about the Y-axis.*
- $y = f(x)$ to $y = |f(x)|$ — *Reflect the parts of the graph that lie in the lower half (negative parts) into the upper half of the axes.*
- $y = f(x)$ to $y = f(|x|)$ — *Discard the left part of the graph (for $x < 0$) and take a reflection of the right part of the graph into the left half of the axes.*
- $y = f(x)$ to $|y| = f(x)$ — *Discard the lower part of the graph ($f(x) < 0$) and take a reflection of the upper part of the graph into the lower half of the axes.*
- $y = f(x)$ to $y = f(x) + k$ — *Shift the graph $|k|$ units upwards or downwards depending on whether k is positive or negative respectively.*
- $y = f(x)$ to $y = f(x + k)$ — *Advance (shift left) or delay (shift right) the graph by $|k|$ units depending on whether k is positive or negative, respectively.*
- $y = f(x)$ to $y = kf(x)$ — *Stretch or compress the graph along the Y-axis depending on whether $|k| > 1$ or $|k| < 1$, respectively. Also, flip it about the X-axis, if k is negative (this latter statement follows from part 1).*
- $y = f(x)$ to $y = f(kx)$ — *Stretch or compress the graph along the X-axis depending on whether $|k| < 1$ or $|k| > 1$, respectively. Also, flip it about the Y-axis, if k is negative.*

Exercise for Session 3

1. Consider the following function f, whose graph is given below.



Draw the graph of the following functions.

(i) $|f(x)|$ (ii) $f(|x|)$ (iii) $|f(|x|) - 1|$

2. Plot the following functions.

(i) $y = |x^2 - 2x - 3|$ (ii) $y = x^2 - 2|x| - 3$ (iii) $y = |\log_2 x|$ (iv) $y = |\log_2 |x||$

(v) $y = \log_2 |1 - x|$ (vi) $y = \log_2 (2-x)^2$ (vii) $y = |\cos |x||$ (viii) $y = |2 - 2^x|$

(ix) $y = \sin(|x|)$ (x) $y = \cos(|x|)$

3. Plot the following functions.

(i) $|f(x)| = \log_2 x$ (ii) $|f(x)| = \log_2 (-x)$

4. Find the number of solutions of $\sin \pi x = |\log_e |x||$.

5. Find the number of solutions of

(i) $2^{|x|} = \sin x^2$ (ii) $\sin x = x^2 + x + 1$

Session 4

To draw $y = [f(x)]$, when the graph of $y = f(x)$ is given and $[\cdot]$ denotes greatest integer fuctions; To draw $y = f([x])$ when the graph of $y = f(x)$ is given; When $f(x)$ transforms to $y = f(\{x\})$; where $\{\cdot\}$ denotes fractional part of x, i.e. $\{x\} = x - [x]$; When $f(x)$ and $g(x)$ are Two Functions and are transformed to their Sum; To draw $f(x) \to 1/f(x) = h(x)$, To draw $y = f(x) \cdot \sin x$, When $f(x)$ & $g(x)$ are given, then find $h(x) = \max(f(x), g(x))$ or $h(x) = \min(f(x), g(x))$

To Draw $y = [f(x)]$

Where, $[\cdot]$ denotes greatest integral function {i.e. now y cannot be fraction}.

Here, in order to draw $y = [f(x)]$ mark the integer on Y-axis. Draw the horizontal lines through integers till they intersect the graph. Draw vertical dotted lines from these intersection points.

Finally, draw horizontal lines parallel to X-axis from any intersection point to the nearest vertical dotted line with blank dot at right end in case $f(x)$ increases.

Or

To plot $y = [f(x)]$, we use the following steps

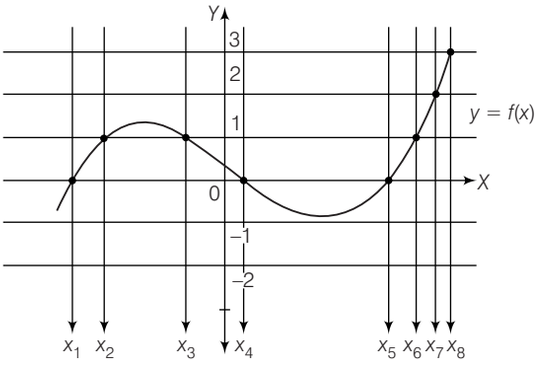
***Step* I** Plot $f(x)$.

***Step* II** Mark the intervals of unit length with integers as end points on Y-axis.

***Step* III** Mark the corresponding intervals {with the help of graph of $f(x)$} on X-axis.

***Step* IV** Plot the value of $[f(x)]$ for each of the marked intervals.

Graphically, it could be shown as



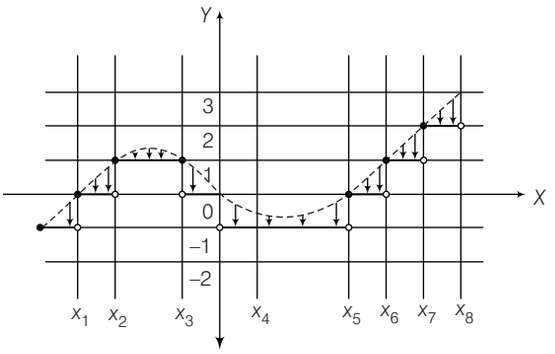


Figure 4.19

Example 20 Draw the graph of $y = [x^3]$, when $-2^{1/3} \le x \le 2^{1/3}$.

Sol. Here, in order to draw the graph of $y = [x^3]$.

***Step* I** First we draw the graph at $y = x^3$.

The function $f(x) = x^3$ is called the cube function. The domain and range of cube function is R. Since, $y = x^3$ is an odd function, so its graph is symmetrical in opposite quardrant.

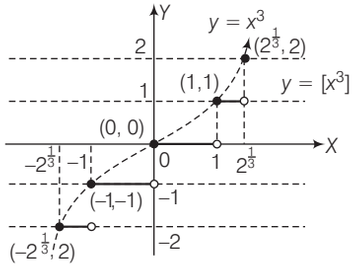
For $y = x^3$, we have following table.

x	0	1	$2^{1/3}$	-1	$-2^{1/3}$
y	0	1	2	-1	-2

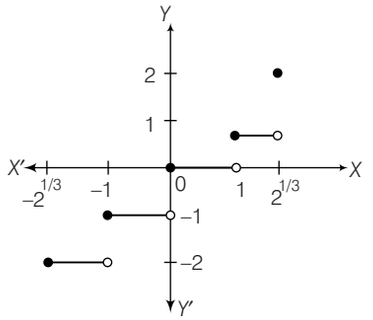
Now, plot the points (0,0) (1,1), $(2^{1/3},2)$, $(-1,-1)$ and $(-2^{1/3},2)$ and join them by free hand, we get the graph of $y = x^3$.

***Step* II** Draw horizontal lines through integers located on Y-axis till they intersect the graph.

***Step* III** Draw vertical dotted lines from these intersection points to intersect X-axis.



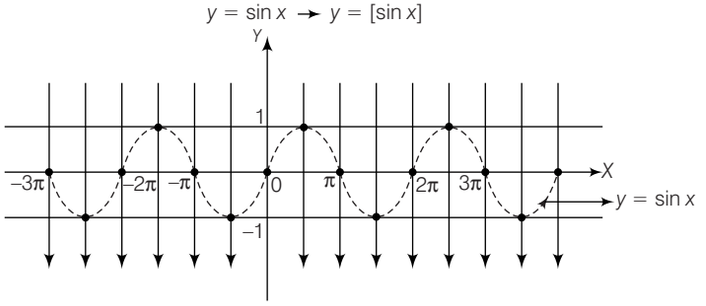
***Step* IV** Finally, draw horizontal lines parallel to X-axis from any intersection point to the nearest vertical dotted line with blank dot at right end.



***Graph for* $y = [x^3]$, $x \in [-2^{1/3}, 2^{1/3}]$**

Example 21 *Draw the graph of* $y = [\sin x]$.

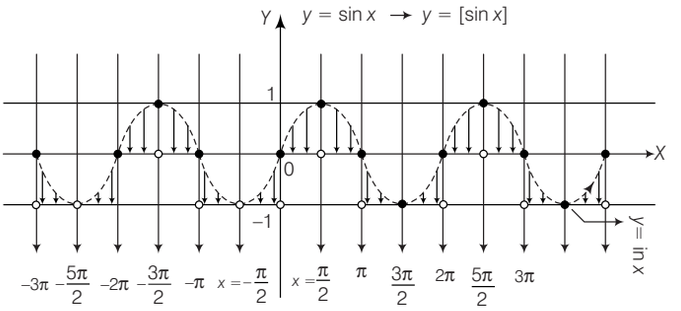
***Sol Step* I** First we draw the graph of $y = \sin x$ as (explained in Example 6) shown below.

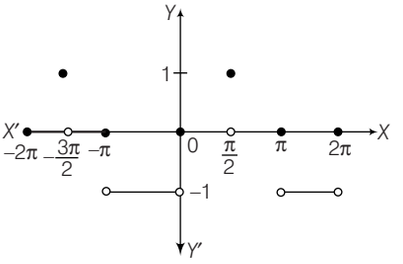


***Step* II** Now, draw horizontal lines through integers located as Y-axis till they intersect the graph.

***Step* III** Draw vertical lines from the intersection points to intersect X-axis.

***Step* IV** Finally, draw horizontal lines parallel to X-axis from any intersection point to the nearest vertical dotted line with blank dot at right end as shown below.





***Graph for* $y = [\sin x]$**

To draw $y = f[x]$

when the graph of $y = f(x)$ is given, following steps are used.

***Step* I** Plot the straight lines parallel to Y-axis for integral value of x, i.e. $n = \ldots -3, -2, -1, 0, 1, 2, 3$.

***Step* II** Now, mark the points of which $x = -3, x = -2, x = -1, x = 0, x = 1, \ldots$ on the curve.

***Step* III** Take the lower marked point for x (say) if $n < x < n + 1$, then take the point at $x = n$ and draw a horizontal line to the nearest vertical line formed by $x = n + 1$, proceeding in this way, we get required curve.

Mark the integers on the X-axis. Draw vertical lines till they intersect the graph of $f(x)$. From these intersection points, draw horizontal lines (parallel to X-axis) to meet the nearest right vertical line with a blank dot on each nearest right vertical line which can be shown as in the figure.

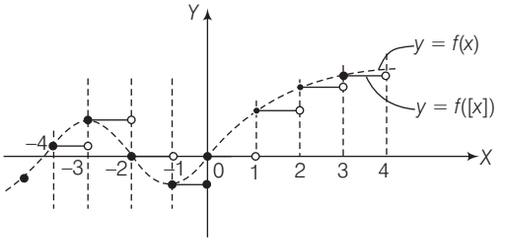
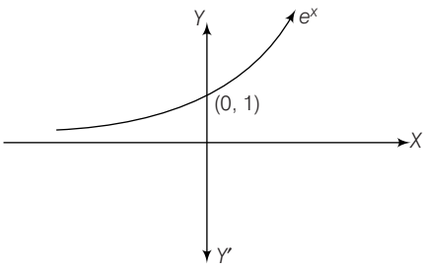


Figure 4.20

Example 22. Draw the curve $y = e^{[x]}$.

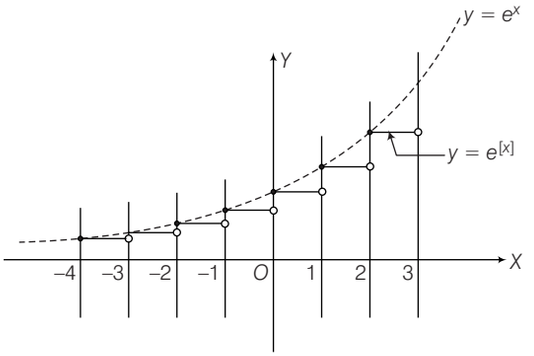
Sol. ***Step* I.** First we draw $y = e^x$ as shown below by figure.



***Step* II.** Draw straight lines parallel Y-axis for integer values of x.

***Step* III.** Now, mark the points as the curve for $n = -4, -3, -2, -1, 0, 1, 2, 3, \ldots$

***Step* IV.** Now, take the lower value of x $(= -4)$ and draw a horizontal line to the nearest vertical line with a blank dot at left end, as shown in figure given below.



To Plot $y = f(\{x\})$

Graph of $f(\{x\})$ can be obtained from the graph of $f(x)$ by following rule. ''Retain the graph of $f(x)$ for values of x lying between interval $[0, 1)$. Now, it can be repeated for rest of the points. (taking periodicity 1).

Now, obtained function is graph for $y = f(\{x\})$.

Graphically, it could be stated as

(i) Graph for $y = f(x)$ (ii) Graph for $y = f(\{x\})$

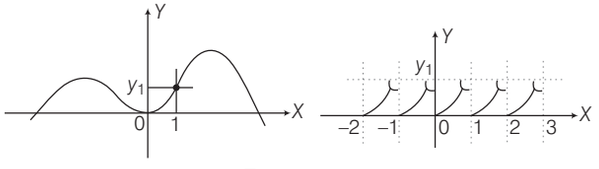
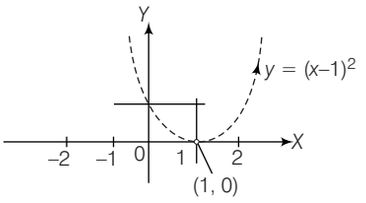


Figure 4.21

Example 23. ***Draw the graph for*** $y = (\{x\} - 1)^2$.

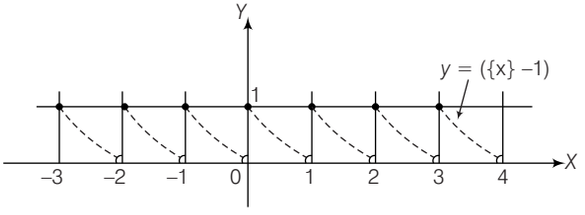
Sol. First we will draw the graph for $y = (x-1)^2$, clearly it represent a parabola opening upward and having vertex (1, 0) which could be ploted as shown below.



Graph for $y = (x-1)^2$

Now, to plot $y = (\{x\} - 1)^2$ retain the graph for the interval $x \in [0, 1)$ and repeat for length one as shown below.

Graph for $y = (\{x\} - 1)^2$



To Plot $y = f(x) + g(x)$, when $a \le g(x) \le b$

There is no direct approach, but we can use the following steps :

If maximum or minimum value of any one is known.

***Step* I** Find the maximum and minimum value of $g(x)$ say $a \le g(x) \le b$.

***Step* II** Plot the curve $f(x)$ between $f(x) + a$ to $f(x) + b$, i.e. $f(x) + a \le h(x) \le f(x) + b$.

***Step* III**

When $g(x) = 0$

$\Rightarrow \quad h(x) = f(x)$

***Step* IV**

When $g(x) > 0$, then $h(x) > f(x)$, i.e. the graph of $h(x)$ lies above the graph of $f(x)$.

***Step* V** When $g(x) < 0$, then $h(x) < f(x)$, i.e. the graph of $h(x)$ lies below the graph of $f(x)$.

The procedure is illustrated by following example.

Example 24 Plot $y = x + \sin x$.

Sol. Here, $y = f(x) + g(x) = x + \sin x$

where $f(x) = x$ and $g(x) = \sin x$.

As we know, $g(x) = \sin x \in [-1, 1]$

$\therefore \quad x - 1 \le y \le x + 1$ ...(i)

To sketch the curve between two parallel line $y = x + 1$ and $y = x - 1$.

Here, $g(x) = 0 \Rightarrow y = x$ ...(ii)

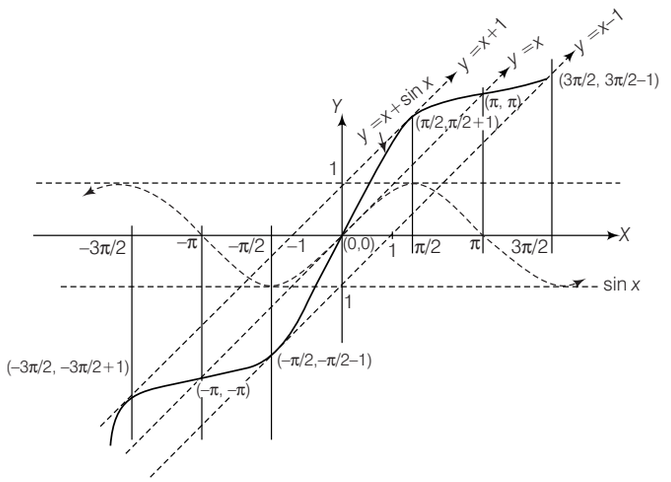
Also, when $g(x) > 0$

$\Rightarrow \quad x + \sin x > x$ ...(iii)

and when $g(x) < 0$

$\Rightarrow \quad x + \sin x < x \quad \ldots(iv)$

Using Eqs. (i), (ii), (iii) and (iv), we have the following graph.



When $f(x)$ Transforms to $\dfrac{1}{f(x)} = h(x)$

(i) when $f(x)$ increases, then $h(x)$ decreases.

(ii) when $f(x)$ decreases, then $h(x)$ increases.

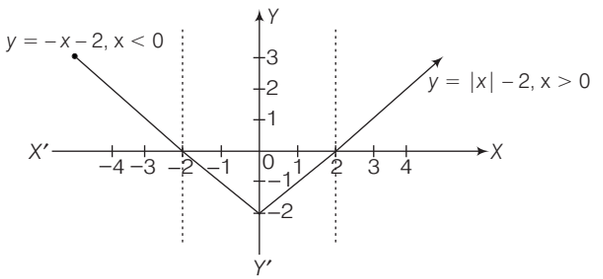
(iii) as $f(x) \to 0, h(x) \to +\infty$.

(iv) as $f(x) \to \infty, h(x) \to 0$.

(v) when $f(x) = \pm 1$, then $h(x)$ is also equal to ± 1.

Example 25 Plot $y = |x| - 2$ and hence $f(x) = \dfrac{1}{|x| - 2}$.

Sol. First we plot $y = |x| - 2$ as explained in Example 2. The graph is shown below.



Here, (i) $y = |x| - 2$ increases when $x > 0$

$\Rightarrow \quad f(x) = \dfrac{1}{|x| - 2}$ decreases when $x > 0$

and (ii) $y = |x| - 2$ decreases when $x < 0$

$\Rightarrow \quad f(x) = \dfrac{1}{|x| - 2}$ increases when $x < 0$

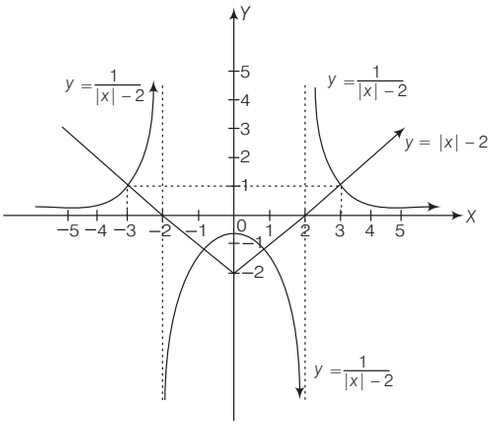
(iii) As, $y \to 0 \Rightarrow x = \pm 2 \Rightarrow f(x) \to \infty$ as $x = \pm 2$

(iv) As, $f(x) \to 0$, i.e. $\dfrac{1}{|x| - 2} \to 0$ as $x \to \infty$

and when $x \to \infty \Rightarrow y = |x| - 2 \to \infty$

(v) $y = \pm 1 \Rightarrow f(x) = \pm 1$

$\therefore$ $f(x)$ could be plotted as,



To Plot the Graph of $y = f(x) \cdot \sin x$, when Graph of $y = f(x)$ is given

Clearly, $-f(x) \le f(x) \cdot \sin x \le f(x) \quad [\text{as } -1 \le \sin x \le 1]$

Hence, graph for $y = f(x) \cdot \sin x$ would be lying between the graph of $y = f(x)$ and $y = -f(x)$. It amounts to just drawing graph of $\sin x$ in between the graphs of $y = \pm f(x)$.

The procedure is illustrated by following example.

Example 26 Let $f(x) = \begin{cases} x \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

and $g(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$.

Discuss the graph for $f(x)$ and $g(x)$, and evaluate the continuity and differentiability of $f(x)$ and $g(x)$.

Sol. We will first try to graphically understand the behaviour of these two functions and then verify our results analytically. Notice that no matter what the argument of the sine function is, its magnitude will always remain between -1 and 1.

Therefore, $\left| x \sin \dfrac{1}{x} \right| \le |x|$ and $\left| x^2 \sin \dfrac{1}{x} \right| \le |x|^2$

This means that the graph of $x \sin \dfrac{1}{x}$ will always lie between the lines $y = \pm x$ and the graph of $x^2 \sin \dfrac{1}{x}$ will always lie between the two curves $y = \pm x^2$. Also, notice that as $|x|$ increases, $\dfrac{1}{x}$ decreases in a progressively slower manner while when $|x|$ is close to 0, the increase in $\dfrac{1}{x}$ is very fast (as $|x|$ decreases visualise the graph of $y = 1/x$).

This means that near the origin, the variation in the graph of $\sin\frac{1}{x}$ will be extremely rapid because the successive zeroes of the graph will become closer and closer. As we keep on increasing x, the variation will become slower and slower and the graph will 'spread out'.

For example, for $x > \frac{1}{\pi}$ there will not be finite root of the function. Only when $x \to \infty$, $\sin\frac{1}{x}$ will again approach 0.

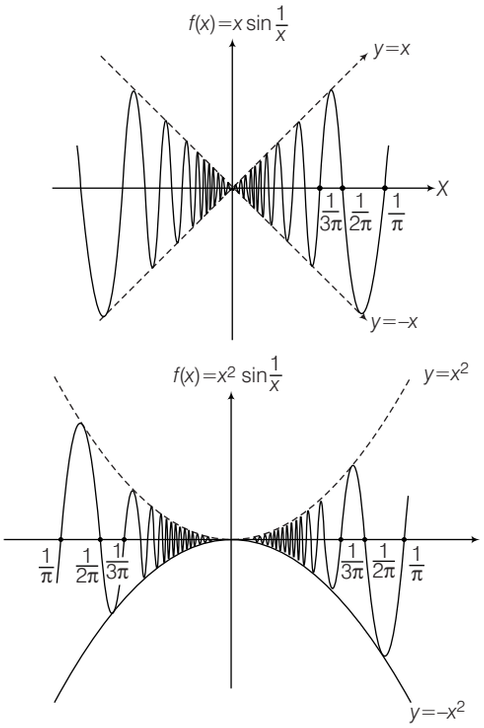


Figure 4.22

Remarks

Notice how the lines $y = \pm x$ 'envelope' the graph of the function in the first case and the curves $y = \pm x^2$ 'envelope' the graph of the function in the second case.

The envelopes shrink to zero vertical width at the origin in both cases. Therefore, we must have

$$\lim_{x \to 0} x \sin\frac{1}{x} = 0 \text{ and } \lim_{x \to 0} x^2 \sin\frac{1}{x} = 0.$$

(This is also analytically obvious; $\sin\frac{1}{x}$ is a finite number between -1 and 1; when it gets multiplied by x (where $x \to 0$) the whole product gets infinitesimally small).

Now lets try to get a 'feel' on what will happen to the derivatives of these two functions at the origin.

For $f(x) = x \sin\frac{1}{x}$, the slope of the envelope is constant (± 1).

Thus, the sinusoidal function inside the envelope will keep on oscillating as we approach the origin, while shrinking in width due to the shrinking envelope. The slope of the curve also keeps on changing and does not approach a fixed value.

However, for $g(x) = x^2 \sin\frac{1}{x}$, the slope of the envelope is itself decreasing as we approach the origin, apart from shrinking in width. This envelope will 'compress' or 'harmmer out' or 'flatten' the sin oscillations near the origin. What should therefore happen to the derivative? It should become 0 at the origin!

Let us 'zoom in' on the graphs of both the functions around the origin, to see what is happening.

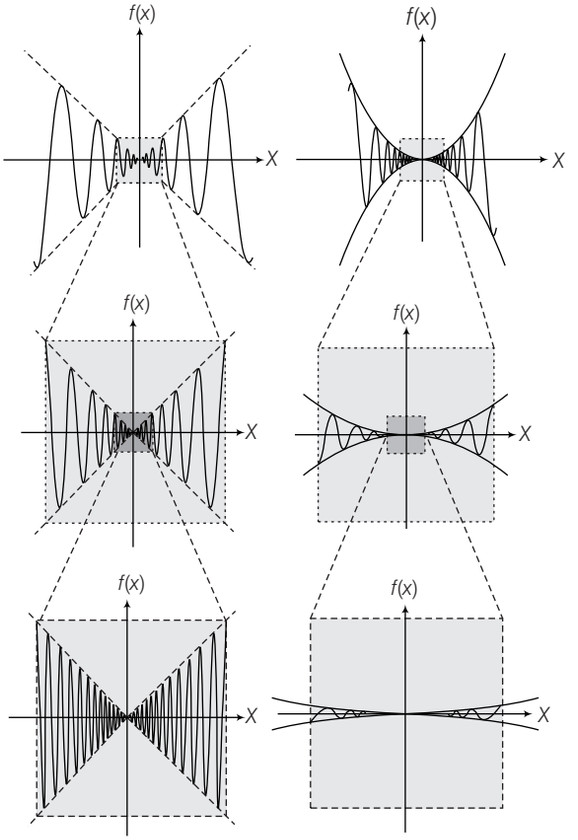


Figure 4.23

These graphs are not very accurate and are only of an approximate nature; but they do give us some feeling on the behaviour of these two functions near the origin.

The $x \sin\frac{1}{x}$ graph keeps 'continuing' in the same manner no matter how much we zoom in; however, in the $x^2 \sin\frac{1}{x}$ graph, the decreasing slope of the envelope itself tends to flatten out the curve and make its slope tend to 0. Hence, the derivative of $x \sin\frac{1}{x}$ at the origin will not have any definite value, while the derivative of $x^2 \sin\frac{1}{x}$ will be 0 at the origin.

Lets verify this analytically

(i) $\boldsymbol{f(x) = x \sin\frac{1}{x}}$ **at** $\boldsymbol{x = 0}$.

$$\text{LHD} = \text{RHD} = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h \sin\frac{1}{h}}{h} = \lim_{h \to 0} \left(\sin\frac{1}{h}\right)$$

This limit, as we know does not exist; hence, the derivative for $f(x)$ does not exist at $x = 0$

(ii) $f(x) = x^2 \sin\frac{1}{x}$ at $x = 0$

$$\text{LHD} = \text{RHD} = \lim_{h\to 0}\frac{h^2\sin\frac{1}{h} - 0}{h} = \lim_{h\to 0} h\sin\frac{1}{h} = 0.$$

To Find $h(x) = \max(f(x), g(x))$ or $h(x) = \min(f(x), g(x))$ when $f(x)$ and $g(x)$ are given

(i) $h(x) = \max\{f(x), g(x)\}$

$$\because \quad h(x) = \begin{cases} f(x), & \text{when } f(x) > g(x) \\ g(x), & \text{when } f(x) < g(x) \end{cases}$$

To draw the graph of $y = \max\{f(x), g(x)\}$, first we draw the graphs of both the functions $f(x)$ and $g(x)$ and find their point of intersection.

Then, we find any two consecutive points of intersection. In between these points $f(x) > g(x)$, then in order to draw the graph of $\max\{f(x), g(x)\}$, we take those segments of $f(x)$ for which $f(x) > g(x)$ between any two consecutive points of intersection of $f(x)$ and $g(x)$.

(ii) $h(x) = \min\{f(x), g(x)\}$

$$\because \quad h(x) = \begin{cases} f(x), & \text{when } f(x) < g(x) \\ g(x), & \text{when } g(x) < f(x) \end{cases}$$

Similarly, in order to draw the graph of $\min\{f(x), g(x)\}$ we take those segments of $f(x)$ for which $f(x) < g(x)$, between any two consecutive points of intersection of $f(x)$ and $g(x)$.

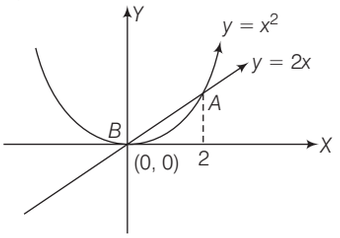
Example 27 *Draw graph for* $y = \max\{2x, x^2\}$ *and discuss the continuity and differentiability.*

Sol. Here, to draw, $y = \max\{2x, x^2\}$.

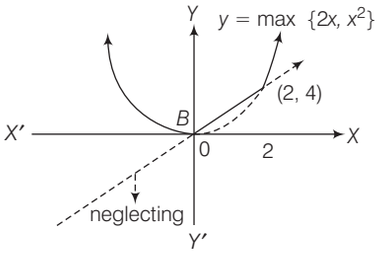
First, plot $y = 2x$ and $y = x^2$ on graph and put $2x = x^2$

$\Rightarrow x = 0, 2$ (i.e. their point of intersection).

Clearly, $y = x^2$ represent a parabola open upward with vertex (0, 0) and $y = 2x$ represent a straight line pointing through (0, 0).



Now, since $y = \max\{2x, x^2\}$ we have to neglect the curve below the point of intersections thus, the required graph is, as shown below.



Thus, from the given graph $y = \max\{2x, x^2\}$, we can say $y = \max\{2x, x^2\}$ is continuous for all $x \in R$.

But $y = \max\{2x, x^2\}$ is differentiable for all $x \in R - \{0,2\}$. As at $x = 0, 2$ curve has corner point.

Remarks

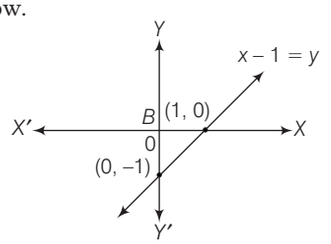
One must remember the formula, we can write

$$\max\{f(x), g(x)\} = \frac{f(x) + g(x)}{2} + \left|\frac{f(x) - g(x)}{2}\right|$$

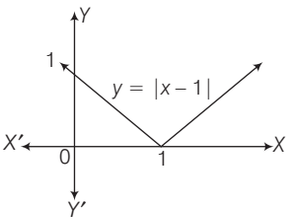
$$\text{and } \min\{f(x), g(x)\} = \frac{f(x) + g(x)}{2} - \left|\frac{f(x) - g(x)}{2}\right|.$$

Example 28 *Draw the graph for* $y = |2 - |x - 1||$.

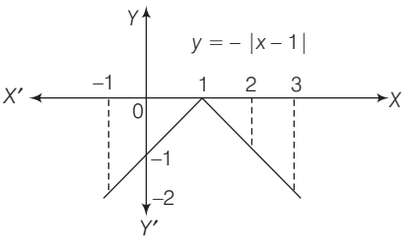
Sol. First we plot $y = x - 1$, it is a straight line intersecting X and Y-axes at (1, 0) and (0, −1), respectively. The graph is shown below.



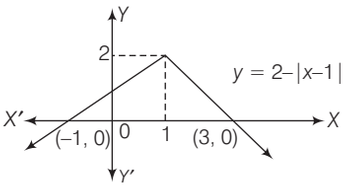
$\Rightarrow$ Now, we plot $y = |x - 1|$, For this, the portion of the graph remain same which lines above the X-axis and reflect the negative portion of the graph about X-axis.



$\Rightarrow$ Now, to plot $y = -|x - 1|$ take the mirror image of the graph of $|x - 1|$ in the X-axis as plane mirror.

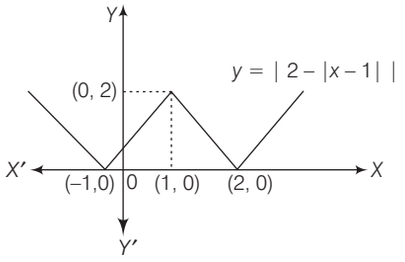


To find the graph of $2-|x-1|$. Shift the graph of $y=-|x-1|$ upward through 2 units.



To find the graph of $|2-|x-1||$, reflect the negative portion of the graph of $2-|x-1|$ in X-axis.

Thus, $y=|2-|x-1||$ can be plotted as shown below.



Remark

From above figure we could say $y=|2-|x-1||$ is not differentiable at $x=\{-1, 1, 3\}$ as sharp edges at $x=-1, 1, 3$.

Example 29 Let $h(x)=\min\{x; x^2\}$ for every real number of x. Then, which one of the following is true?

[IIT JEE 1998]

(a) h is not continuous for all x

(b) h is differentiable for all x

(c) $h'(x)=1$, for all x

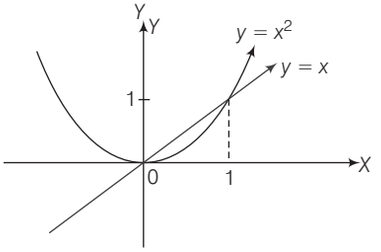
(d) h is not differentiable at two values of x

Sol. Here, $h(x)=\min\{x; x^2\}$ can be drawn on graph in two steps.

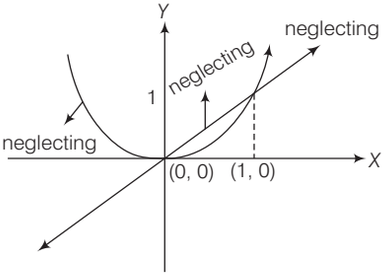
(a) Draw the graph of $y=x$ and $y=x^2$, also find their point of intersection.

i.e. $x=x^2 \Rightarrow \quad x=0, 1$

Thus, points of intersection are (0, 0) and (1,1).



(b) To find $h(x)=\min\{x; x^2\}$ neglecting the graph above the point of intersection, we get



Thus, from the given graph,

$$h(x)=\begin{cases} x, & x\le 0 \text{ or } x\ge 1 \\ x^2, & 0\le x\le 1 \end{cases}$$

which shows $h(x)$ is continuous for all x. But not differentiable at $x=\{0, 1\}$.

Thus, $h(x)$ is not differentiable at two values of x.

Hence, (d) is the correct answer.

Exercise for Session 4

1. Plot the following, where [.] denotes greatest integer function.

(i) $f(x)=[x^2]$, when $-2\le x\le 2$ (ii) $f(x)=[|x|]$ (iii) $f(x)=[|x-2|]$ (iv) $f(x)=[|x|-2]$

(v) $f(x)=\sin^{-1}(\sin|x|)$ (vi) $f(x)=[\cos^{-1}x]$ (vii) $f(x)=\cos(x-[x])$ (viii) $f(x)=[\sin^{-1}(\sin x)]$

2. Plot the graph for $f(x)=\min(x-[x], -x-[-x])$.

3. Find the area enclosed by the curves

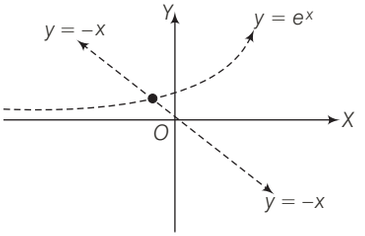
(i) $\max(|x|, |y|)=1$ (ii) $\max(2|x|, 2|y|)=1$ (iii) $\max(|x+y|, |x-y|)=1$

JEE Type Solved Examples :
Single Option Correct Type Questions

● **Ex. 1** *The number of real solutions of the equation* $e^x + x = 0$ *is*

(a) 0 (b) 1
(c) 2 (d) None of these

Sol. (*b*) It is evident from figure that the curves $y = e^x$ and $y = -x$ intersect exactly at one point. So, the equation $e^x = -x$ or $e^x + x = 0$ has one real solution.

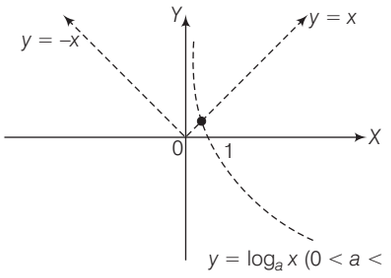


● **Ex. 2** *The number of real solutions of the equation* $\log_a x = |x|, 0 < a < 1$, *is*

(a) 0 (b) 1
(c) 2 (d) None of these

Sol. (*b*) The number of real solutions of the equation $\log_a x = |x|$ is equal to the number of points of intersection of the curves.

$y = \log_a x$ and $y = |x| \quad (0 < a < 1)$



It is evident from the graph that the two curves intersect at one point only.

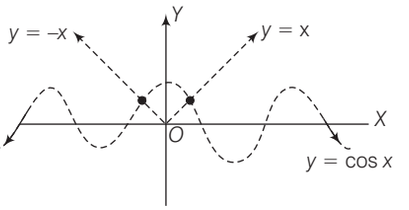
∴ One real solution lying in (0, 1).

● **Ex. 3** *The number of solutions of the equation* $|x| = \cos x$ *is*

(a) 0 (b) 1
(c) 2 (d) 3

Sol. (*c*) The number of real solutions of the equation $|x| = \cos x$ is equal to the number of point of intersection of the curves.

$y = |x|$ and $y = \cos x$ shown as;



It is evident from the graph that the two curves intersect at two points only.

∴ Two real solutions lying in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

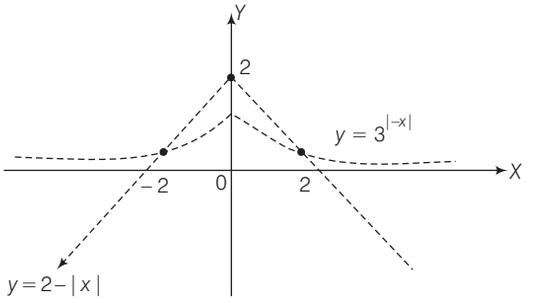
● **Ex. 4** *How many roots does the following equation possess* $3^{|x|}\{|2 - |x||\} = 1$?

(a) 1 (b) 2 (c) 3 (d) 4

Sol. (*b*) Here, $3^{|x|}\{|2 - |x||\} = 1$

$\Rightarrow \quad 2 - |x| = 3^{-|x|}$

In order to determine the number of roots, it is sufficient to find the points of intersection of the curves $y = 2 - |x|$ and $y = 3^{-|x|}$, shown as;



We observe the two curves intersect at two points.
∴ Two real solutions ∈ (–2, 2).

● **Ex. 5** *The number of real solutions of the equation* $x^2 = 1 - |x - 5|$ *is*

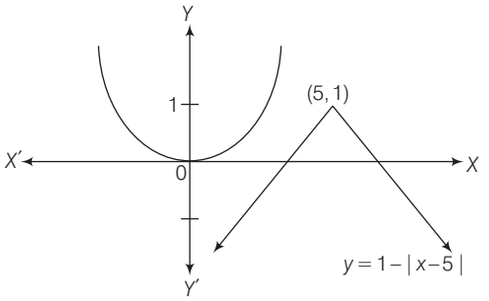
(a) 1 (b) 2
(c) 4 (d) None of these

Sol. (*d*) The number of real solutions of the equation

$$x^2 = 1 - |x - 5|$$

is equal to the number of points of intersection of the curves.

$y = x^2$ and $y = 1 - |x - 5|$, shown as;



It is evident from the graph that the two curves do not intersect.

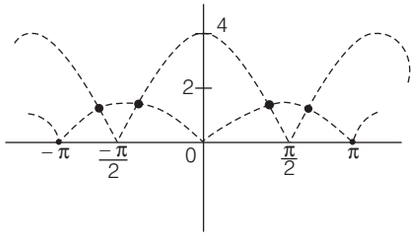
∴ No solution.

● ***Ex. 6*** *Number of solutions for* $2^{\sin|x|} = 4^{|\cos x|}$ *in* $[-\pi, \pi]$ *is equal to*

(a) 2 (b) 4
(c) 6 (d) 8

Sol. (*b*) The total number of solutions for the given equation is equal to the number of points of intersection of curves $y = 4^{|\cos x|}$ and $y = 2^{\sin|x|}$.

Clearly, the two curves intersect at four points. So, there are four solutions of the given equation.



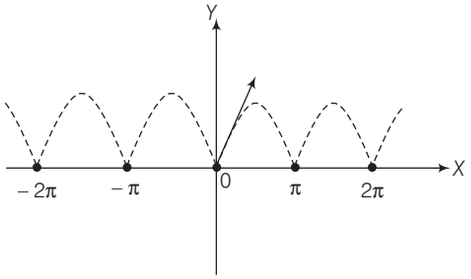
● ***Ex. 7*** *Number of roots of* $|\sin|x|| = x + |x|$ *in* $[-2\pi, 2\pi]$ *is*

(a) 2 (b) 3
(c) 4 (d) 6

Sol. (*b*) The total number of solutions for the given equation is equal to the number of points of intersection of the curves.

$$y = |\sin|x||$$

and $$y = x + |x| = \begin{cases} 2x, & x \geq 0 \\ 0, & x < 0 \end{cases}$$



Clearly, the two curves intersect at three points.

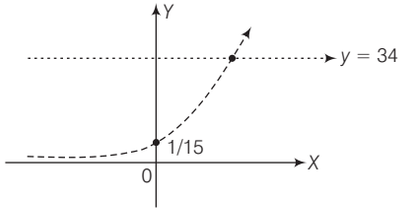
∴ There are three solutions.

● ***Ex. 8*** *The equation* $3^{x-1} + 5^{x-1} = 34$ *has*

(a) one solution (b) two solutions
(c) three solutions (d) four solutions

Sol. (*a*) The total number of solutions is same as the number of points of intersection of the curves.

$$y = 3^{x-1} + 5^{x-1} \text{ and } y = 34$$



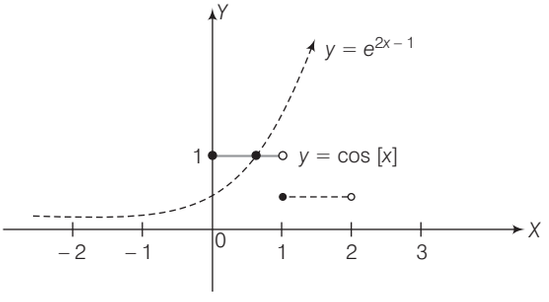
It is evident that these two curves intersect at exactly one point.

∴ Exactly one solution.

● ***Ex. 9*** *The number of solutions of the equation* $\cos[x] = e^{2x-1}$, $x \in [0, 2\pi]$, *where* [.] *denotes the greatest integer function is*

(a) 1 (b) 2
(c) 3 (d) 4

Sol. (*a*) It is evident from the graph the two curves intersect at only one point.



JEE Type Solved Examples :
More than One Correct Option Type Questions

● ***Ex. 10*** *Let* $g(x)=\sqrt{x-2k}, \forall\ 2k \le x < 2(k+1)$ *where,* $k \in I$, *then*

(a) $g(x)=\sqrt{x+2}, -2 \le x < 0$

(b) $g(x)=\sqrt{x-2}, 2 \le x < 4$

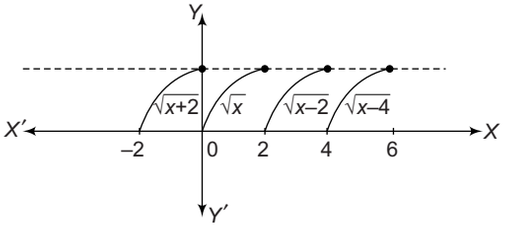
(c) $g(x)=\sqrt{x}, 0 \le x < 2$

(d) period of $g(x)$ is 2

Sol. (*a,b,c,d*) Given, $g(x)=\sqrt{x-2k}, \forall\ 2k \le x < 2(k+1), \forall\ k \in I$

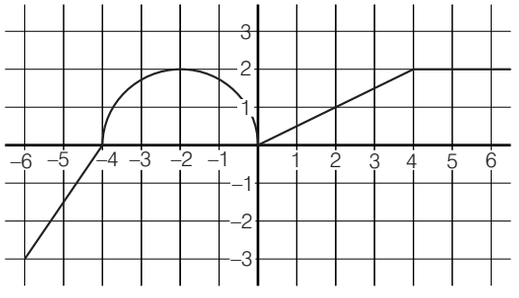
$$= \begin{cases} \ldots\ldots \\ \ldots\ldots \\ \sqrt{x+2}, & -2 \le x < 0 \\ \sqrt{x}, & 0 \le x < 2 \\ \sqrt{x-2}, & 2 \le x < 4 \\ \sqrt{x-4}, & 4 \le x < 6 \\ \ldots\ldots \\ \ldots\ldots \end{cases}$$

The graph of $g(x)$ is shown below

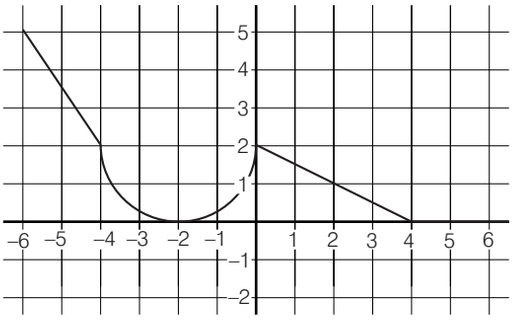


Clearly, $g(x)$ is periodic with period 2.

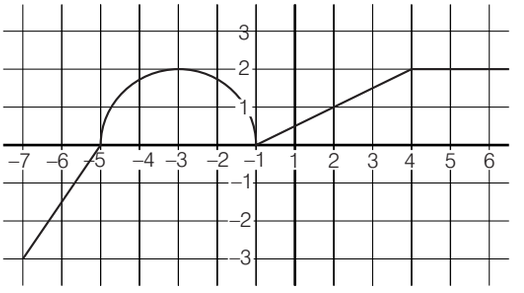
● ***Ex. 11*** *The graph of* $f(x)$ *is given below.*



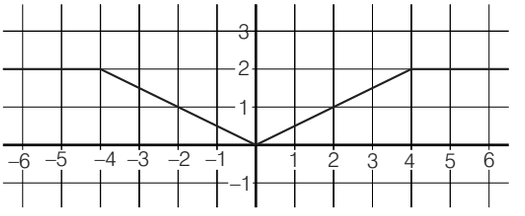
Then, (a) Graph of $-f(x)+2$ is



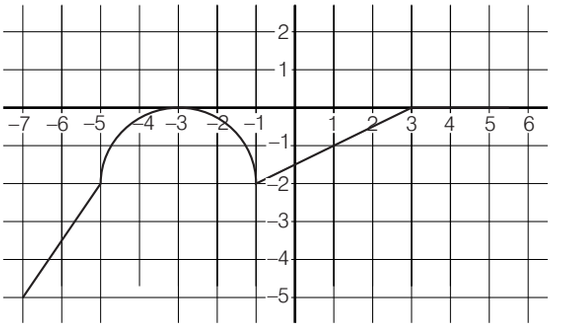
(b) Graph of $-f(x-1)$ is



(c) Graph of $f(|x|)$ is

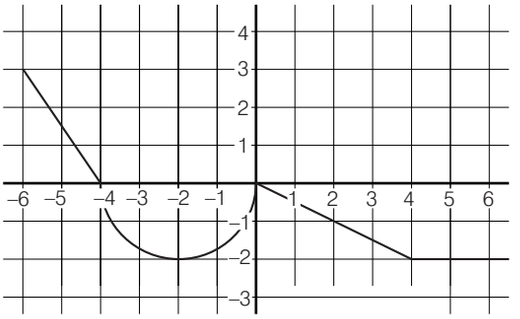


(d) Graph of $f(x+1)-2$ is

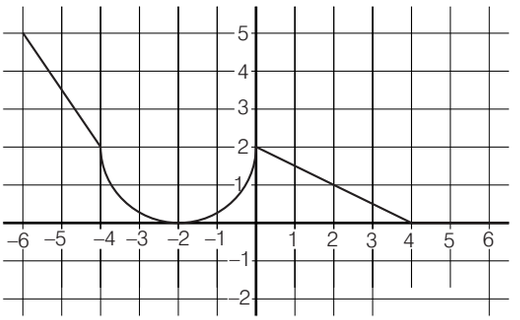


Sol. (*a,c,d*)

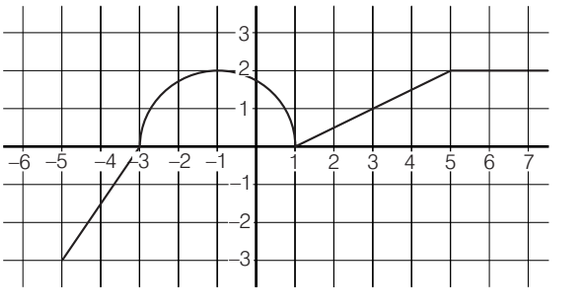
(a) For the graph of $-f(x)+2$, first flip the graph of $y=f(x)$ about, X-axis to get $y=-f(x)$



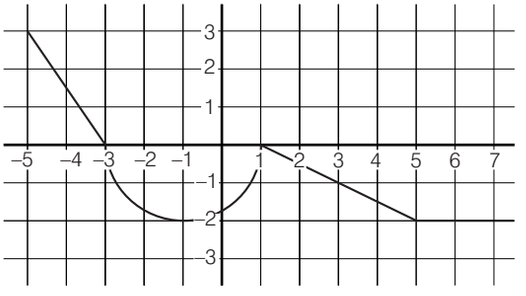
Now, shift the above graph 2 units upward to get $y=2-f(x)$



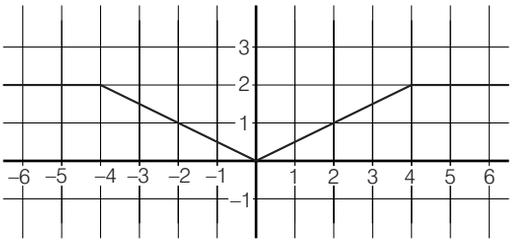
(b) First shift the graph of $f(x)$, through 1 unit right to get $y=f(x-1)$



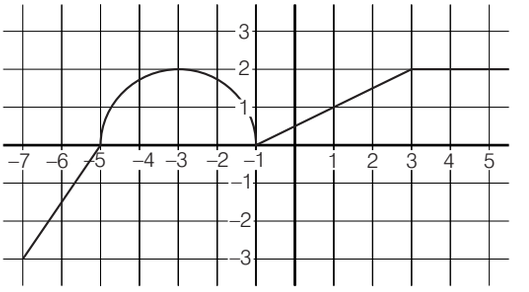
Now, flip the above graph about X-axis to get $y=-f(x-1)$



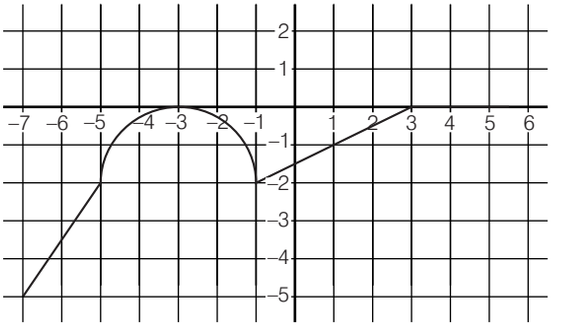
(c) For $y=f(|x|)$, neglect the graph of $y=f(x)$ for $x<0$, and take the mirror image of $y=f(x)$ for $x>0$ about Y-axis, keeping $y=f(x)$ for $x>0$



(d) For $y=f(x+1)-2$, shift the graph of $y=f(x)$, 1 unit left to get $y=f(x+1)$



Now, shift the above graph 2 units downwards to get $f(x+1)-2$



JEE Type Solved Examples :
Single Integer Answer Type Questions

● ***Ex. 12*** *The number of solutions of the equation* $[y+[y]]=2\cos x$, *where* $y=\frac{1}{3}[\sin x+[\sin x+[\sin x]]]$, *where* $[\cdot]$ *denotes the greatest integer function, is*

Sol. Here, $[y+[y]]=2\cos x$

or $[y]+[y]=2\cos x \quad [\because [x+h]=[x]+h, \forall\ x\in I]$

or $2[y]=2\cos x$

or $[y]=\cos x$ ...(i)

Also, $y=\frac{1}{3}[\sin x+[\sin x+[\sin x]]]$

$y=\frac{1}{3}(3[\sin x])$

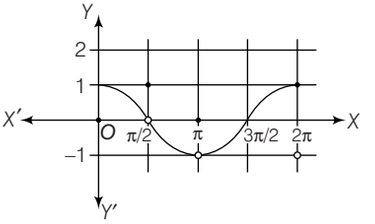
$y=[\sin x]$ ...(ii)

From Eqs. (i) and (ii), we have

$[[\sin x]]=\cos x$

or $[\sin x]=\cos x$

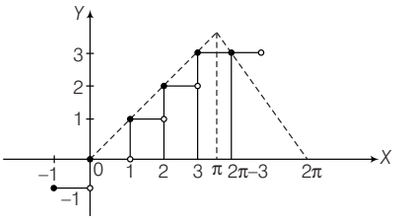
The graph of $y=[\sin x]$ and $y=\cos x$ is shown below.



∴ The number of solution zero.

● ***Ex. 13*** *The sum of the roots of the equation* $\cos^{-1}(\cos x)=[x]$, *where* $[x]$ *denotes greatest integer function, is*

Sol. The graph of $\cos^{-1}(\cos x)=[x]$ is shown below.

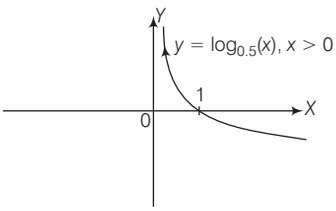


Solutions are clearly 0, 1, 2, 3 and $2\pi-3$.
Hence, there are 5 solutions.

Subjective Type Questions

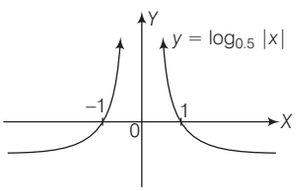
● ***Ex. 14*** *Sketch the graph of* $y=\log_{0.5}|x|$.

Sol. As we know, $y=\log_{0.5}x$ is a decreasing graph given as;



Since, $f(x)\to f(|x|)$, then taking the images about Y-axis for $x>0$

∴ $y=\log_{0.5}|x|$ could be plotted as;



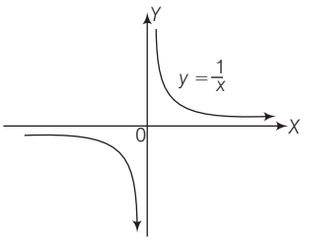
● ***Ex. 15*** *Sketch the graph for* $y=\left|\left|\frac{1}{x}\right|-3\right|$.

Sol. Here, we follow certain steps to plot

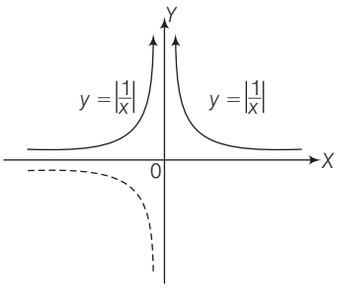
$y=\left|\left|\frac{1}{x}\right|-3\right|$, as first we plot $\frac{1}{x}$

and then successively $\left|\frac{1}{x}\right|$, $\left|\frac{1}{x}\right|-3$, $\left|\left|\frac{1}{x}\right|-3\right|$.

(i) The graph of $\frac{1}{x}$ is shown below

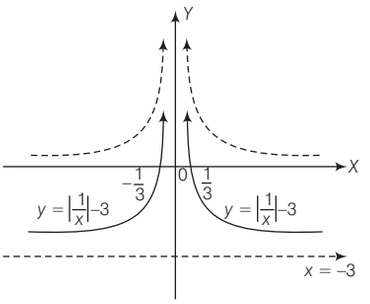


(ii) $\frac{1}{x} \to \left|\frac{1}{x}\right|$ To draw the graph of $\left|\frac{1}{x}\right|$ taking mirror images about X-axis for negative values of $\frac{1}{x}$ in the graph of $\frac{1}{x}$.



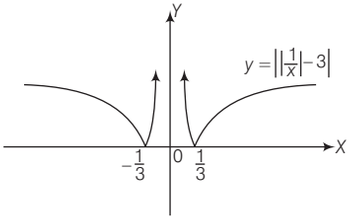
(iii) $\left|\frac{1}{x}\right| \to \left|\frac{1}{x}\right| - 3$

Shift the graph of $\left|\frac{1}{x}\right|$ through 3 units downward.



(iv) $\left|\frac{1}{x}\right| - 3 \to \left|\left|\frac{1}{x}\right| - 3\right|$

To draw the graph of $\left|\left|\frac{1}{x}\right| - 3\right|$, taking mirror images about X-axis for negative value of $\left|\frac{1}{x}\right| - 3$ in the graph of $\left|\frac{1}{x}\right| - 3$.

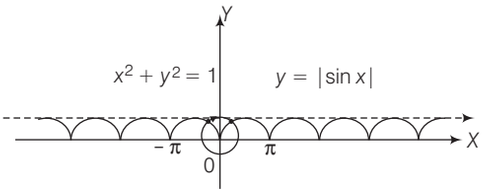


● ***Ex. 16*** *Find the number of solutions of the equations* $y = |\sin x|$ *and* $x^2 + y^2 = 1$.

Sol. To find the number of solutions of two curves we should find the point of intersection of two curves.

As we know, $x^2 + y^2 = 1$ is a circle and $y = |\sin x|$ is the image of negative values of $y = \sin x$ about X-axis. Thus, we can plot them as; which shows the two curves intersects at two points.

∴ Number of solutions is 2.



● ***Ex. 17*** *Find the number of solutions of* $4\{x\} = x + [x]$.

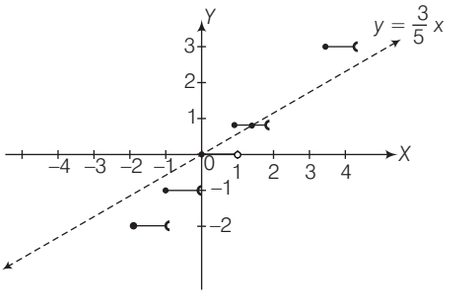
Sol. Here, $4\{x\} = x + [x]$

$\Rightarrow$ $4(x - [x]) = x + [x]$ $[\because \{x\} = x - [x]]$

$\Rightarrow$ $3x = 5[x]$

$\Rightarrow$ $[x] = \frac{3}{5}x$

To find their solution we plot the graph of both $y = [x]$ and $y = \frac{3}{5}x$,



i.e. The two graphs intersects when $[x] = 0$ and 1

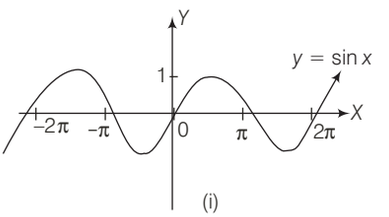
$\Rightarrow$ $x = 0$ and $x = \frac{5}{3}$

● ***Ex. 18*** *Sketch the graph of* $\left|\sin x + \frac{1}{2}\right|$.

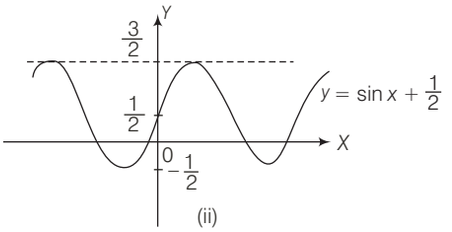
Sol. We follow certain steps to plot $\left|\sin x + \frac{1}{2}\right|$,

i.e. $\sin x \to \sin x + \frac{1}{2} \to \left|\sin x + \frac{1}{2}\right|$

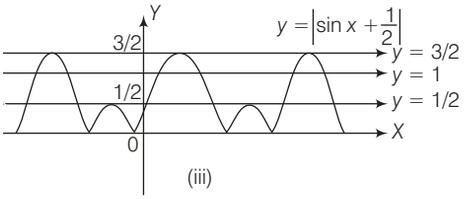
(i) (ii) (iii)



(i)



(ii)



(iii)

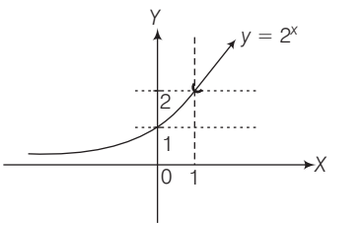
Remarks

Plotting graph of $f(x - [x])$:

Graph of $f(x - [x])$ can be obtained from the graph of $f(x)$ by the following rule.

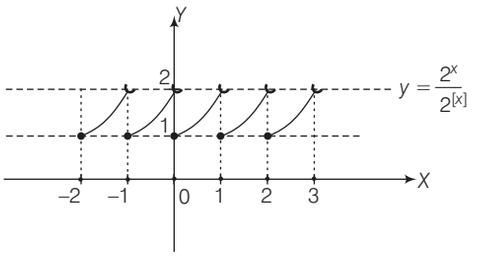
"Retain the graph of $f(x)$ for values of x lying between interval [0, 1). Now, it can be repeated for rest of points. Now, obtained function is periodic with period 1."

● ***Ex. 19*** *Sketch the graph of* $y = \dfrac{2^x}{2^{[x]}}$.



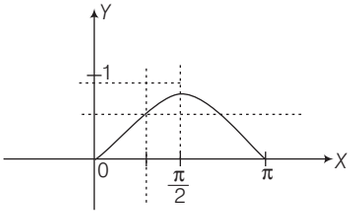
Sol. As we know, 2^x is exponential function and we want to transform it to $2^{x-[x]}$, it retains, the graph for $x \in [0, 1)$ and repeat for rest points.

Here, to retain graph between $x \in [0, 1)$, so we get

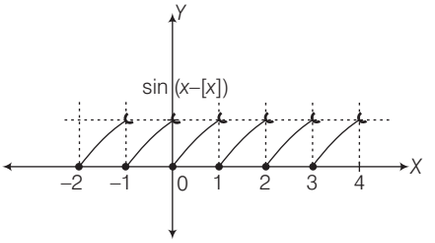


● ***Ex. 20*** *Sketch the region for* $y = \sin(x - [x])$.

Sol. We know, $y = \sin x$ is the periodic function. So, to plot the graph between $x \in [0, 1)$ and repeat for all values of x.

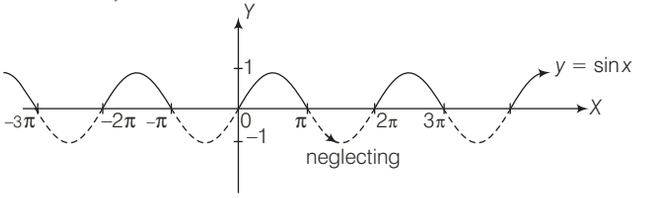


i.e. To retain the graph between $x \in [0, 1)$.

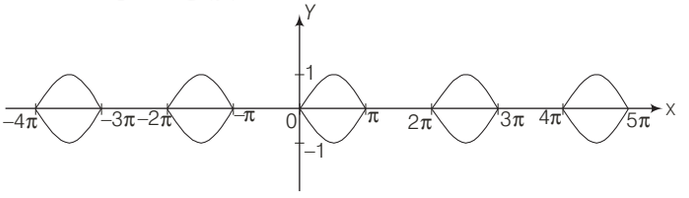


● ***Ex. 21*** *Sketch the region for* $|y| = \sin x$.

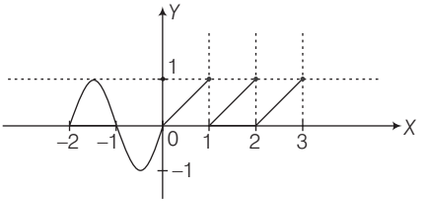
Sol. We know, $y = \sin x$ is a periodic function. So, to plot $|y| = \sin x$ neglect all the points for which y is negative and take the image for positive values of y about X-axis shown as;



Now, plotting, $|y| = \sin x$.



● ***Ex. 22*** *Consider the following function f whose graph is given below.*



Draw the graph of following functions.

(a) $f(x)+1$ (b) $f(x)-1$ (c) $-f(x)$

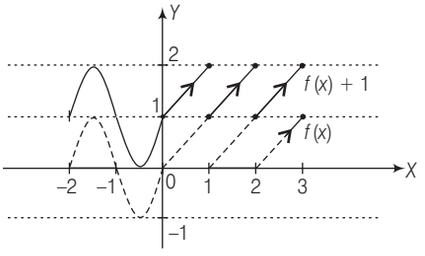
(d) $|f(x)|$ (e) $f(-x)$ (f) $f(|x|)$

(g) $2f(x)$ (h) $f(2x)$ (i) $[f(x)]$

(j) $f(x-[x])$

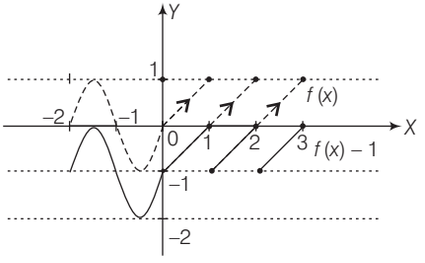
Sol. (a) $f(x) \to f(x)+1$

To draw the graph of $f(x)+1$, shift the graph of $f(x)$ through 1 unit upward.



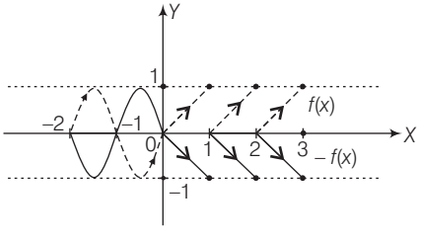
(b) $f(x) \to f(x)-1$

To draw the graph of $f(x)-1$, shift the graph of $f(x)$ through 1 unit downward.

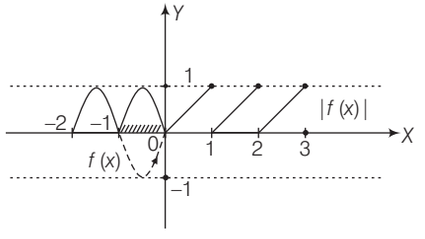


(c) $f(x) \to -f(x)$

To draw the graph of $-f(x)$. Take the image of $f(x)$ in the X-axis as plane mirror.

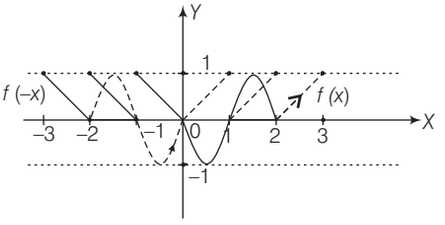


(d) $f(x) \to |f(x)|$, To draw the graph of $|f(x)|$ taking image for negative values of $f(x)$ about X-axis.



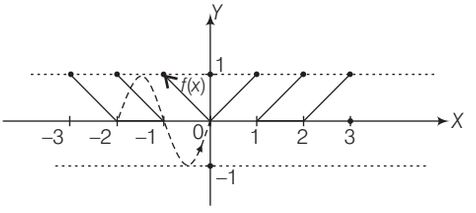
(e) $f(x) \to f(-x)$

To draw the graph of $f(-x)$ take the image of the curve $f(x)$ in the Y-axis as plane mirror.



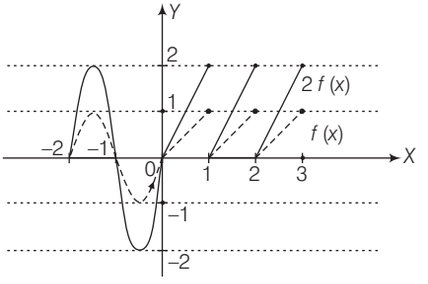
(f) $f(x) \to f(|x|)$

To draw the graph of $f|x|$ neglecting the graph for negative values of $f(x)$ and taking image for positive values of x about Y-axis.



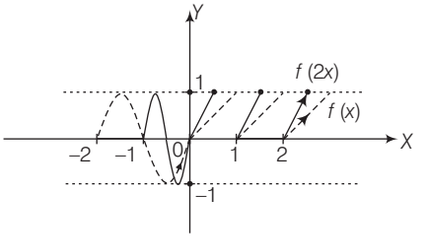
(g) $f(x) \to 2f(x)$

To draw the graph of $2f(x)$ stretch the graph of $f(x)$, 2 times along Y-axis.



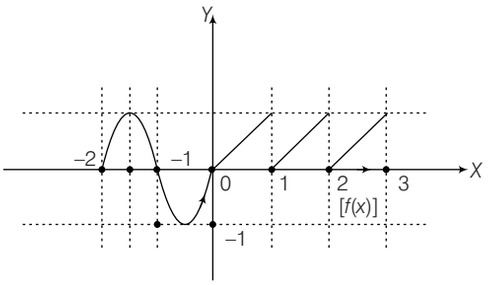
(h) $f(x) \to f(2x)$

To draw the graph of $f(2x)$, stretch the graph of $f(x)$, 2 times along X-axis.



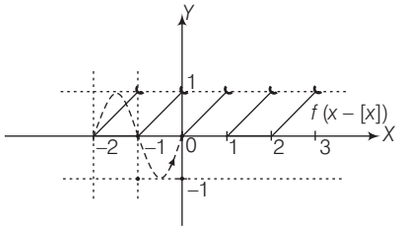
(i) $f(x) \to [f(x)]$

To draw $[f(x)]$ mark the integer on Y-axis. Draw the horizontal lines till they intersect the graph, now draw vertical dotted lines from these intersection point. Finally, draw horizontal lines parallel to X-axis from any intersection point to the nearest vertical dotted line with blank dot at right end.



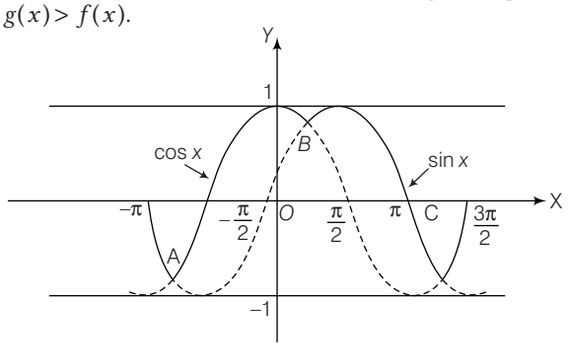
(j) $f(x) \to f(x-[x])$

Retain the graph of $f(x)$ for values of x lying between interval [0,1). Now, it can be repeated for rest of the point taking periodicity '1'.

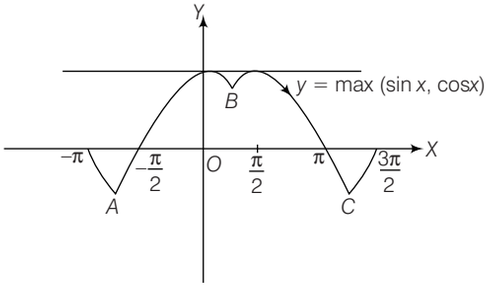


● ***Ex. 23*** *Sketch the graph of* $y = \max(\sin x, \cos x)$, $\forall\ x \in \left(-\pi, \frac{3\pi}{2}\right)$.

Sol. First plot both $y = \sin x$ and $y = \cos x$ by a dotted curve as can be seen from the graph in the interval $\left(-\pi, \frac{3\pi}{2}\right)$ and then darken those dotted line for which $f(x) > g(x)$ or $g(x) > f(x)$.



∴ **Graph of max $\{\sin x, \cos x\}$**

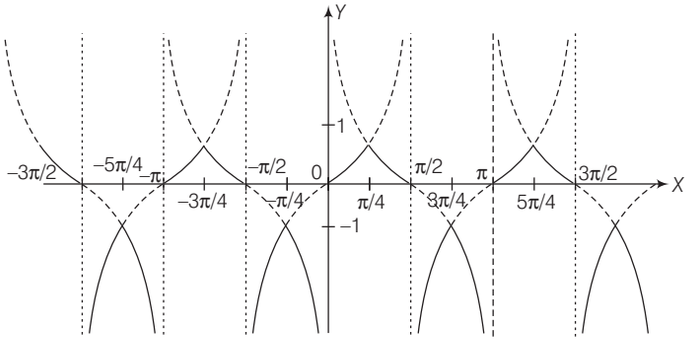


As from the above graph,

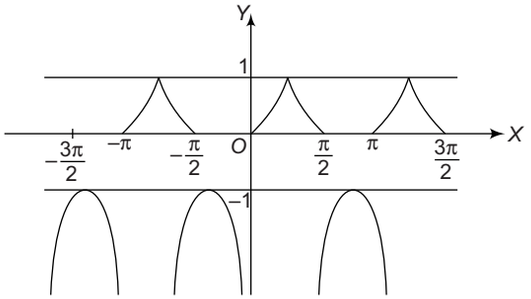
$$\max(\sin x, \cos x) = \begin{cases} \sin x, & -\pi < x \le -\frac{3\pi}{4} \\ \cos x, & -\frac{3\pi}{4} \le x \le \frac{\pi}{4} \\ \sin x, & \frac{\pi}{4} \le x \le \frac{5\pi}{4} \\ \cos x, & \frac{5\pi}{4} \le x \le \frac{3\pi}{2} \end{cases}$$

● ***Ex. 24*** *Sketch the graph for* $y = \min\{\tan x, \cot x\}$.

Sol. First plot both $f(x) = \tan x$ and $g(x) = \cot x$ by a dotted curve as can be seen from the graph and then darken those dotted lines for which $f(x) < g(x)$ and $g(x) < f(x)$.



∴ **Graph of min $\{\tan x, \cot x\}$**

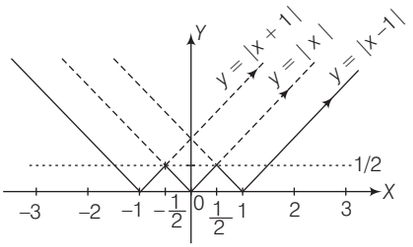


As from the graph, we have

$$\min\{\tan x, \cot x\} = \begin{cases} \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \tan x, & -\frac{\pi}{2} < x \le -\frac{\pi}{4} \\ \cot x, & -\frac{\pi}{4} \le x < 0 \\ \tan x, & 0 \le x \le \frac{\pi}{4} \\ \cot x, & \frac{\pi}{4} \le x \le \frac{\pi}{2} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

● ***Ex. 25*** *Sketch the graph of* $y = \min\{|x|, |x-1|, |x+1|\}$.

Sol. First plot the graph for, $y = |x|$, $y = |x-1|$ and $y = |x+1|$ by a dotted curve as can be seen from the graph and then darken those dotted lines for $|x| < \{|x-1|, |x+1|\}$, $|x-1| < \{|x|, |x+1|\}$ and $|x+1| < \{|x|, |x-1|\}$.



As from the above graph,

$$\min\{|x|, |x-1|, |x+1|\} = \begin{cases} -(x+1), & x \le -1 \\ (x+1), & -1 \le x \le -\frac{1}{2} \\ -(x), & -\frac{1}{2} \le x \le 0 \\ x, & 0 \le x \le \frac{1}{2} \\ -(x-1), & \frac{1}{2} \le x \le 1 \\ (x-1), & 1 \le x \end{cases}$$

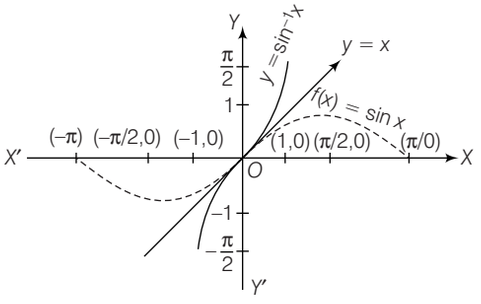
Remark

To plot the graph of $f^{-1}(x)$, take reflection of $f(x)$ in $y = x$ line as a mirror.

● ***Ex. 26*** *Sketch the graph of* $y = \sin^{-1} x, \forall\, x \in [-1, 1]$.

Sol. As, $-1 \le x \le 1 \Rightarrow -\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2} \Rightarrow -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$\sin^{-1} x$ is the reflection of $\sin x$ about $y = x$ (as a mirror) when $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

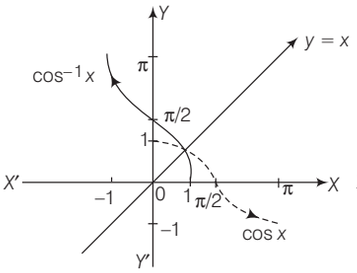


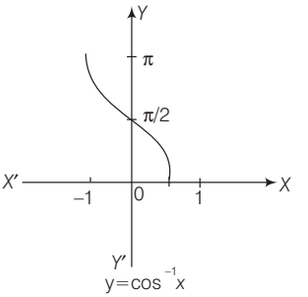
● ***Ex. 27*** *Sketch the graph for* $y = \cos^{-1} x, \forall\, x \in [-1, 1]$.

Sol. As, $-1 \le x \le 1$

$\Rightarrow \quad 0 \le \cos^{-1} x \le \pi \quad \Rightarrow \quad 0 \le y \le \pi$

$\therefore\ \cos^{-1} x$ is reflection of the graph of $\cos x, 0 \le x \le \pi$ in $y = x$ line as a mirror.

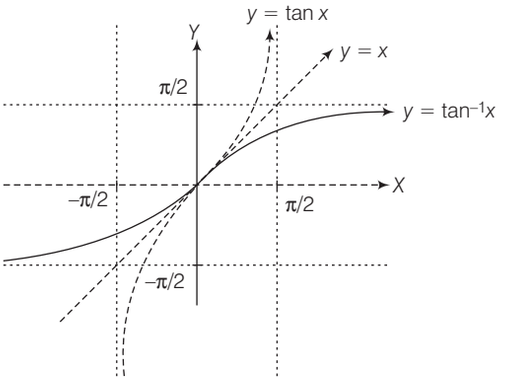




● ***Ex. 28*** *Sketch the graph for* $y = \tan^{-1} x, \forall\, x \in R$.

Sol. As, $x \in R \Rightarrow \frac{-\pi}{2} < \tan^{-1} x < \frac{\pi}{2} \Rightarrow \frac{-\pi}{2} < y < \frac{\pi}{2}$

$\therefore\ \tan^{-1} x$ is the reflection of the graph of $\tan x$, $\frac{-\pi}{2} < x < \frac{\pi}{2}$ in $y = x$ as a mirror.



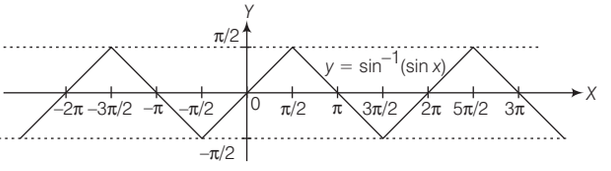
● ***Ex. 29*** *Sketch the graph for* $y = \sin^{-1}(\sin x)$.

Sol. As, $y = \sin^{-1}(\sin x)$ is periodic with period 2π.

$\therefore$ To draw this graph we should draw the graph for one interval of length 2π and repeat it for entire values of x. As we know,

$$\sin^{-1}(\sin x) = \begin{cases} (x), & \frac{-\pi}{2} \le x \le \frac{\pi}{2} \\ (\pi - x), & \frac{\pi}{2} \le x \le \frac{3\pi}{2} \text{ or } \pi - \frac{\pi}{2} \ge \pi - x \ge \pi - \frac{3\pi}{2} \\ & \text{or } -\frac{\pi}{2} \le \pi - x \le \frac{\pi}{2} \end{cases}$$

Thus, it has been defined for $\frac{-\pi}{2} \le x \le \frac{3\pi}{2}$, that has length 2π. So, its graph could be plotted as;



● ***Ex. 30*** *Sketch the graph for* $y = \cos^{-1}(\cos x)$.

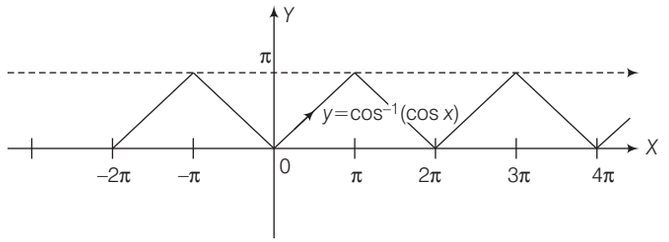
Sol. As, we have that $y = \cos^{-1}(\cos x)$ is periodic with period 2π.

$\therefore$ To draw this graph we should draw the graph for one interval of length 2π and repeat it for entire values of x of length 2π.

As we know,

$$\cos^{-1}(\cos x) = \begin{cases} x, & 0 \le x \le \pi \\ (2\pi - x), & \pi < x \le 2\pi \text{ or } 0 < 2\pi - x < \pi \end{cases}$$

Thus, it has been defined for $0 < x < 2\pi$, that has length 2π.
So, its graph could be plotted as;



● ***Ex. 31*** *Sketch the graph for* $y = \tan^{-1}(\tan x)$.

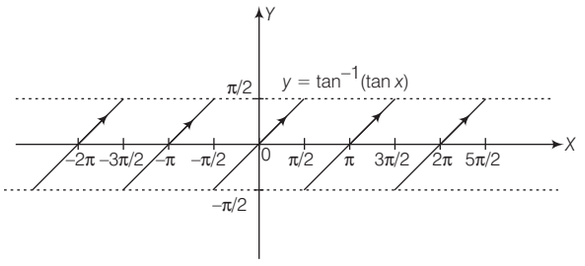
Sol. As, we have that $y = \tan^{-1}(\tan x)$ is periodic with period π.

$\therefore$ To draw this graph we should draw the graph for one interval of length π and repeat for entire values of x.

As we know, $\tan^{-1}(\tan x) = \left\{ x, \quad \frac{-\pi}{2} < x < \frac{\pi}{2} \right.$

Thus, it has been defined for $\frac{-\pi}{2} < x < \frac{\pi}{2}$ that has length π.

So, its graph could be plotted as;



● ***Ex. 32*** *Find the values of x graphically which satisfy* $\left|\frac{x^2}{x-1}\right| \le 1$.

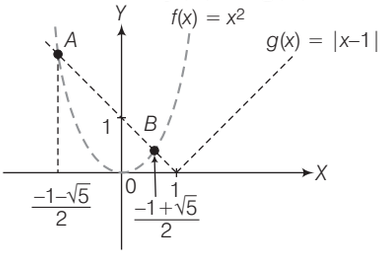
Sol. We have, $\left|\frac{x^2}{x-1}\right| \le 1$

$\Rightarrow$ $|x^2| \le |x-1|, \forall\ x \in R - \{1\}$

or $x^2 \le |x-1|, \forall\ x \in R - \{1\}$

Thus, to find the points for which $f(x) = x^2$ is less than or equal to $g(x) = |x-1|$.

Where the two functions $f(x)$ and $g(x)$ could be plotted as;



Thus, from the graph $f(x) \le g(x)$, when $x \in [A, B]$. So that, A and B are point of intersection of x^2 and $1 - x$.

$\therefore$ Solving $x^2 = 1 - x$, we get $x = \frac{-1-\sqrt{5}}{2} = A$

and $x = \frac{-1+\sqrt{5}}{2} = B.$

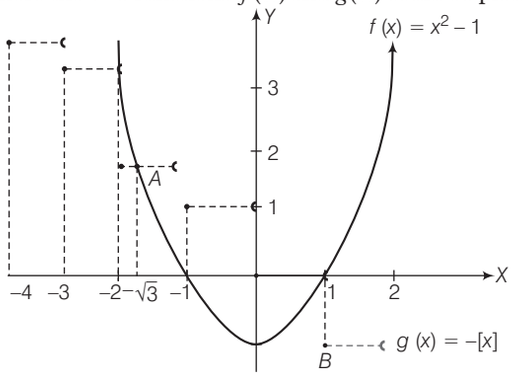
$\therefore$ $\left|\frac{x^2}{x-1}\right| \le 1$ is satisfied, $\forall\ x \in \left[\frac{-1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right]$.

● ***Ex. 33*** *Find the values of x graphically satisfying* $[x] - 1 + x^2 \ge 0$ *; where [·] denotes the greatest integer function.*

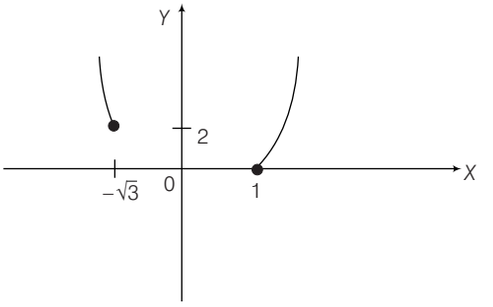
Sol. We have, $[x] - 1 + x^2 \ge 0 \Rightarrow x^2 - 1 \ge -[x]$

Thus, to find the points for which $f(x) = x^2 - 1$ is greater than or equal to $g(x) = -[x]$.

Where the two functions $f(x)$ and $g(x)$ could be plotted as;



Finally.



Thus, from the above graph $f(x) \geq g(x)$ when $x \in (-\infty, A] \cup [B, \infty)$, where A is point of intersection of $x^2 - 1$ and $-[x]$ when $-[x] = +2$.

$\therefore\ x^2 - 1 = +2$, i.e. $x = -\sqrt{3} = A$ and $B = 1$

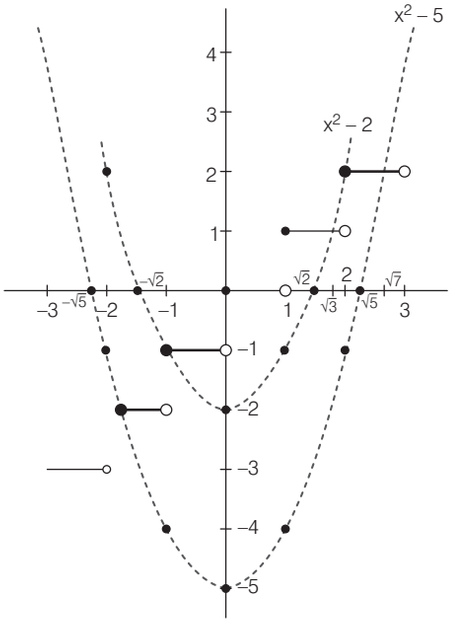
$\therefore\ [x] - 1 + x^2 \geq 0$ is satisfied, $\forall\ x \in (-\infty, -\sqrt{3}] \cup [1, \infty)$.

● ***Ex. 34*** *Find the values of x graphically which satisfy;* $-1 \leq [x] - x^2 + 4 \leq 2$, *where* $[.]$ *denotes the greatest integer function.*

Sol. We have, $-1 \leq [x] - x^2 + 4 \leq 2 \Rightarrow x^2 - 5 \leq [x] \leq x^2 - 2$

Thus, to find the points for which $f(x) = x^2 - 5$ is less than or equal to $g(x) = [x]$ and $g(x) = [x]$ is less than or equal to $h(x) = x^2 - 2$.

Where the three functions $f(x)$, $g(x)$ and $h(x)$ could be plotted as;



Thus, from the above graph; $x^2 - 5 \leq [x] \leq x^2 - 2$, when $x \in [A, B] \cup [C, D]$.

Where A and D is point of intersection of $x^2 - 5 = \pm 2$

$\Rightarrow x = -\sqrt{3}, \sqrt{7}$ B and C is point of intersection of $x^2 - 2 = \pm 1$

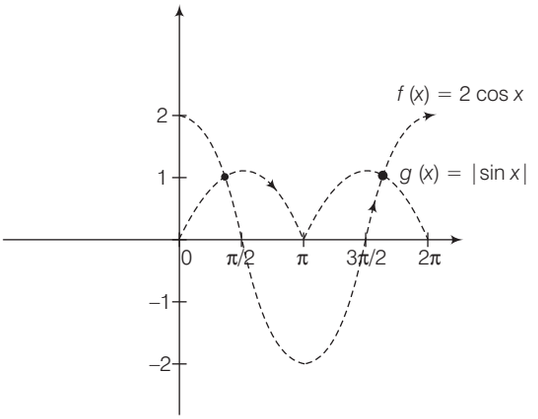
$\Rightarrow x = \sqrt{3}$

$\therefore\ -1 \leq [x] - x^2 + 4 \leq 2$ is satisfied, $x \in [-\sqrt{3}, -1] \cup [\sqrt{3}, \sqrt{7}]$.

● ***Ex. 35*** *Find the number of solutions of* $2\cos x = |\sin x|$ *when* $x \in [0, 4\pi]$.

Sol. As we know, the graph of both $f(x) = 2\cos x$ and $g(x) = |\sin x|$

$\therefore$ Their point of intersection are number of solutions.



Thus, from the above graph the two functions intersect at two points between $[0, 2\pi]$ and we know that $\cos x$ is periodic with period 2π, so it has same number of solutions for the interval $[2\pi, 4\pi]$.

$\therefore$ Total number of solutions of $2\cos x = |\sin x|$ when $x \in [0, 4\pi]$ is 4.

● ***Ex. 36*** *Sketch the curves*

(i) $y = \sqrt{x - [x]}$ (ii) $y = [x] + \sqrt{x - [x]}$

(iii) $y = |[x] + \sqrt{x - [x]}|$

(where $[\cdot]$ denotes the greatest integer function).

Sol. (i) Here, $0 \leq x - [x] < 1, \forall\ x \in R$ and

$x^2 \leq x \leq \sqrt{x}, \forall x \in [0,1)$

$\therefore\quad x - [x] \leq \sqrt{x - [x]}$

$$\Rightarrow\quad y = \sqrt{x - [x]} = \begin{cases} \sqrt{x-1}, & 1 \leq x < 2 \\ \sqrt{x}, & 0 \leq x < 1 \\ \sqrt{x+1}, & -1 \leq x < 0 \end{cases}$$

and so on.

In general, $y = \sqrt{x - [x]} = \sqrt{x}$, when $0 \leq x < 1$

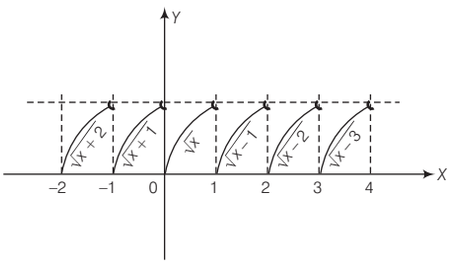
$y = \sqrt{x - [x]} = \sqrt{x - 1}$, when $1 \leq x < 2$

i.e. Shifting $\sqrt{x}$ by 1 unit on right side of X-axis

$y = \sqrt{x - [x]} = \sqrt{x - 2}$, when $2 \leq x < 3$,

then graph of $y = \sqrt{x - [x]}$.

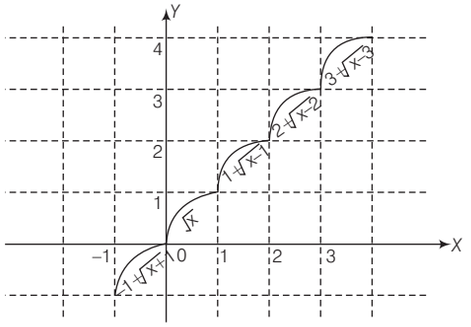
i.e. Shifting $\sqrt{x-1}$ by 1 unit on right side of X-axis and so on. Thus, the graph for $y = \sqrt{x - [x]}$ is shown as in figure, which is periodic with period 1.



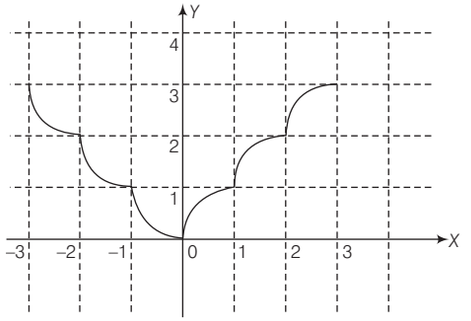
(ii) Again, $y = [x] + \sqrt{x - [x]} \Rightarrow y = k + \sqrt{x - k}$,
$k \le x < k + 1$; $k \in$ integer $-1 + \sqrt{x+1}, -1 \le x < 0$

$$\Rightarrow \qquad y = \begin{cases} \sqrt{x}, & 0 \le x < 1 \\ 1 + \sqrt{x-1}, & 1 \le x < 2 \\ 2 + \sqrt{x-2}, & 2 \le x < 3 \quad \text{and so on.} \end{cases}$$

Thus, the graph for $y = [x] + \sqrt{x - [x]}$ is obtained by the graph of $y = \sqrt{x - [x]}$ by translating it by $[x]$ units in upward or downward directions according as $[x] > 0$ or $[x] < 0$. The graph is shown as



(iii) Graph for $y = |[x] + \sqrt{x - [x]}|$ is obtained by reflecting the portion lying below X-axis of the graph of $y = [x] + \sqrt{x - [x]}$. About X-axis and keeping the portion lying above the X-axis (as it is).
Thus, the graph for $y = \left|[x] + \sqrt{x - [x]}\right|$ is shown below.



Graphical Transformations Exercise 1 :
Single Option Correct Type Questions

1. The number of real solutions of the equation $e^{|x|} - |x| = 0$ is
(a) 0 (b) 1 (c) 2 (d) None of these

2. The number of real solutions of the equation $3^{-|x|} - 2^{|x|} = 0$ is
(a) 0 (b) 1 (c) 2 (d) 3

3. The number of solutions of $3^{|x|} = |2 - |x||$ is
(a) 0 (b) 2 (c) 4 (d) infinite

4. The total number of solutions of the equation $|x - x^2 - 1| = |2x - 3 - x^2|$ is
(a) 0 (b) 1
(c) 2 (d) infinitly many

5. The equation $e^x = m(m+1), m < -1$ has
(a) no real root
(b) exactly one real root
(c) two real roots
(d) None of the above

6. The number of real solutions of the equation $1 - x = [\cos x]$ is
(a) 1 (b) 2
(c) 3 (d) 4

7. The number of roots of the equation $1 + 3^{x/2} = 2^x$ is
(a) 0
(b) 1
(c) 2
(d) None of the above

Graphical Transformations Exercise 2 :
More than One Option Correct Type Questions

8. The equation $x^2 - 2 = [\sin x]$, where $[\cdot]$ denotes the greatest integer function, has
(a) infinity many roots
(b) exactly one integer root
(c) exactly one irrational root
(d) exactly two roots

9. Consider the function $f(x) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \notin I \\ 0, & \text{if } x \in I \end{cases}$,
where $[\cdot]$ denotes greatest integer function and I is the set of integers, then $g(x) = \max\{x^2, f(x), |x|\}, -2 \le x \le 2$ is defined as
(a) $x^2, -2 \le x \le -1$
(b) $1 - x, -1 < x \le -\frac{1}{4}$
(c) $\frac{1}{2} + x, -\frac{1}{4} < x < 0$
(d) $1 + x, 0 \le x < 1$

10. If $f(x)$ is defined on $[-2, 2]$ and is given by $f(x) = \begin{cases} -1, & -2 \le x < 0 \\ x - 1, & 0 < x \le 2 \end{cases}$ and $g(x) = f|x| + |f(x)|$, then $g(x)$ is defined as
(a) $-x, -2 \le x \le 0$ (b) $x, -2 \le x \le 0$
(c) $0, 0 < x \le 1$ (d) $2(x-1), 1 < x \le 2$

Graphical Transformations Exercise 3 :
Statements I and II Type Questions

■ **Directions** (Q. Nos. 11 to 12) *This section is based on Statement* I *and Statement* II. *Select the correct answer from the code given below.*
(a) Both Statement I and Statement II are correct and Statement II is the correct explanation of Statement I
(b) Both Statement I and Statement II are correct but Statement II is not the correct explanation of Statement I
(c) Statement I is correct but Statement II is incorrect
(d) Statement II is correct but Statement I is incorrect

11. **Statement I** The graph of $y = \sec^2 x$ is symmetrical about the Y-axis.

Statement II The graph of $y = \tan x$ is symmetrical about the origin.

12. **Statement I** The equation $|(x-2) + a| = 4$ can have four distinct real solutions for x if a belongs to the interval $(-\infty, -4)$.

Statement II The number of point of intersection of the curve represent the solution of the equation.

Graphical Transformations Exercise 4 : Passage Based Questions

Passage I (Q. Nos. 13 to 14)

Let $f(x) = f_1(x) - 2f_2(x)$, *where*

$$f_1(x) = \begin{cases} \min\{x^2, |x|\}, & |x| \le 1 \\ \max\{x^2, |x|\}, & |x| > 1 \end{cases}$$

$$f_2(x) = \begin{cases} \min\{x^2, |x|\}, & |x| > 1 \\ \max\{x^2, |x|\}, & |x| \le 1 \end{cases},$$

and $$\text{let } g(x) = \begin{cases} \min\{f(t), & -3 \le t \le x, -3 \le x < 0\} \\ \max\{f(t), & 0 \le t < x, 0 \le x \le 3\}. \end{cases}$$

13. For $x \in (-1, 0)$, $f(x) + g(x)$ is

(a) $x^2 - 2x + 1$ (b) $x^2 + 2x - 1$

(c) $x^2 + 2x + 1$ (d) $x^2 - 2x - 1$

14. The graph of $y = g(x)$ in its domain is broken at

(a) 1 point (b) 2 points (c) 3 points (d) None of these

Passage II (Q. Nos. 15 to 16)

Consider the functions $f(x) = \begin{cases} x - 1, & -1 \le x \le 0 \\ x^2, & 0 \le x \le 1 \end{cases}$

and $g(x) = \sin x$.

If $h_1(x) = f(|g(x)|)$

and $h_2(x) = |f(g(x))|$.

15. Which of the following is not true about $h_1(x)$?

(a) It is a periodic function with period π

(b) The range is $[0, 1]$

(c) Domain R

(d) None of the above

16. Which of the following is not true about $h_2(x)$?

(a) The domain is R (b) It is periodic with period 2π

(c) The range is $[0, 1]$ (d) None of these

Graphical Transformations Exercise 5 : Matching Type Questions

■ **Directions** (Q.No. 17) *Choices for the correct combination of elements from Column I and Column II are given as option (a), (b), (c) and (d) out of which are correct.*

17.

	Column I (Equation)	Column II (Number of Roots)	
A.	$x^2 \tan x = 1, x \in [0, 2\pi]$	p.	5
B.	$2^{\cos x} = \lvert\sin x\rvert, x \in [0, 2\pi]$	q.	2
C.	If $f(x)$ is a polynomial of degree 5 with real coefficient such that $f(x) = 0$ has 8 real roots, then the number of roots of $f(x) = 0$	r.	3
D.	$7^{\lvert x\rvert}(5 - \lvert x\rvert) = 1$	s.	4

Codes

	A	B	C	D		A	B	C	D
(a)	p	q	r	s	(b)	q	s	p	s
(c)	q	p	s	r	(d)	s	p	q	r

Graphical Transformations Exercise 6 :
Single Integer Answer Type Questions

18. Let $f(x) = x + 2|x + 1| + 2|x - 1|$. If $f(x) = k$ has exactly one real solution, then the value of k is

19. The number of roots of the equation $x \sin x = 1$, $x \in [-2\pi, 0] \cup (0, 2\pi)$ is

20. The number of solutions of $\tan x - mx = 0, m > 1$ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is

Graphical Transformations Exercise 7 :
Subjective Type Questions

21. Find the number of solutions of the equation $\frac{x^2}{1-|x-2|} = 1$, graphically.

22. Find the number of solutions for $\tan 4x = \cos x$, when $x \in (0, \pi)$.

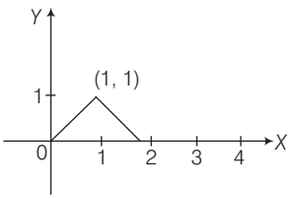
23. Find number of solutions for equation $[\sin^{-1} x] = x - [x]$, where $[\cdot]$ denotes the greatest integer function.

24. If x and y satisfy the equations $\max(|x+y|, |x-y|) = 1$ and $|y| = x - [x]$, the number of ordered pairs (x, y).

25. Find the area enclosed by $|x + y - 1| + |2x + y + 1| = 1$.

26. Find $f(x)$ when it is given by $f(x) = \max\left\{x^3, x^2, \frac{1}{64}\right\}, \forall x \in [0, \infty)$.

27. Find a formula for the function f graphed as



28. Find the domain for $f(x) = \frac{1}{[|x-1|] + [|5-x|] - 4}$, graphically.

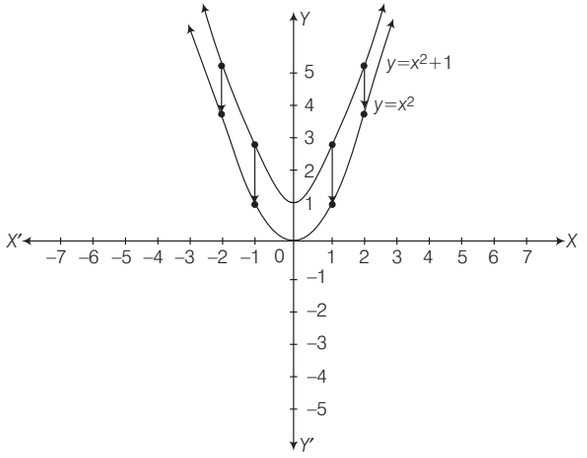
29. Draw the graph for $y = \sqrt{\{x\}}$ and $|y| = \sqrt{\{x\}}$.

30. Draw the graph for $y = -[x] + \sqrt{\{x\}}$.

Answers

Exercise for Session 1

1.



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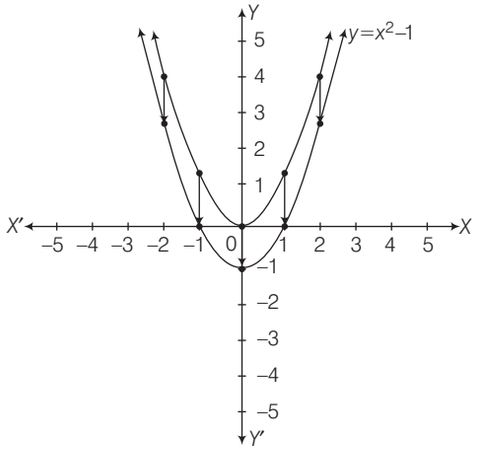

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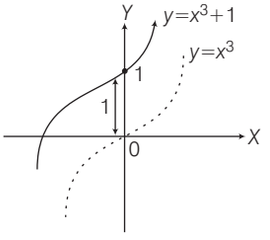

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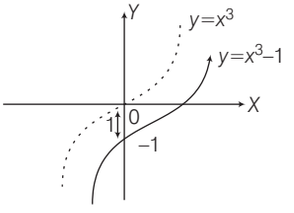

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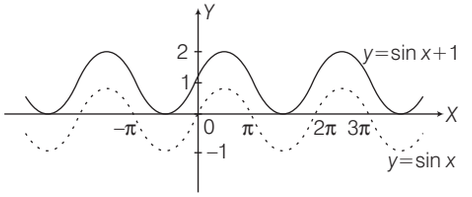
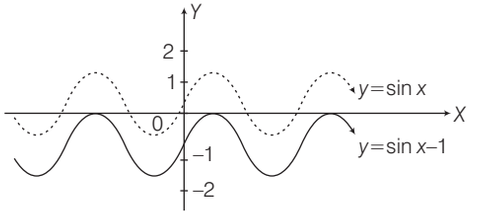
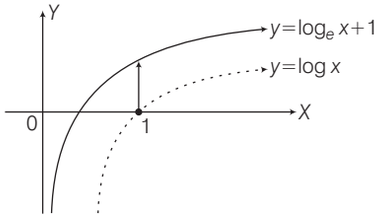


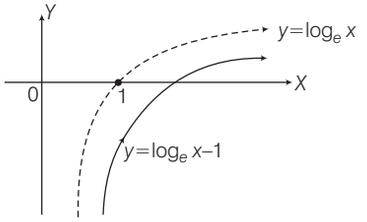
6.



7.

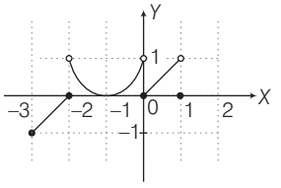


8.

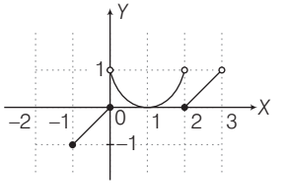


Exercise for Session 2

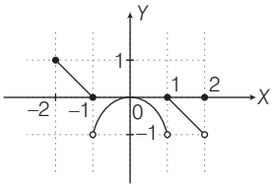
1. (i) $f(x+1)$ is shown as



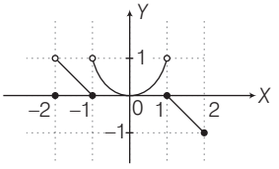
(ii) $f(x-1)$ is shown as



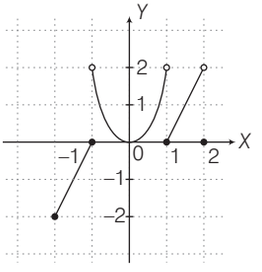
(iii) $-f(x)$ is shown as



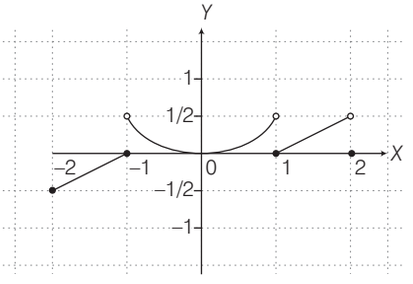
(iv) $f(-x)$ is shown as



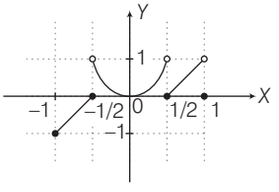
(v) $2f(x)$ is shown as



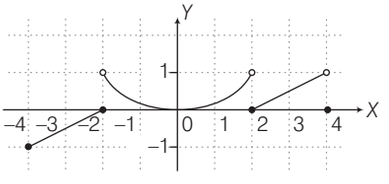
(vi) $\frac{1}{2}f(x)$ is shown as



(vii) $f(2x)$ is shown as

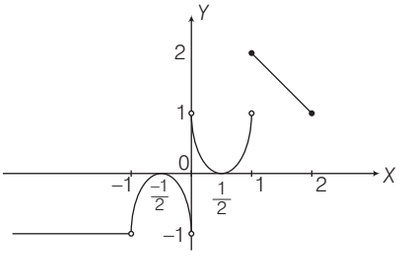


(viii) $f\left(\frac{x}{2}\right)$ is shown as

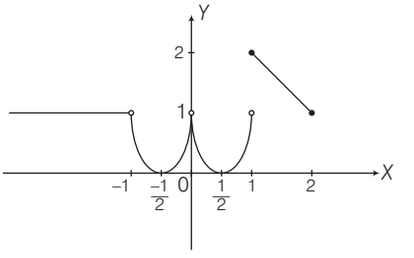


Exercise for Session 3

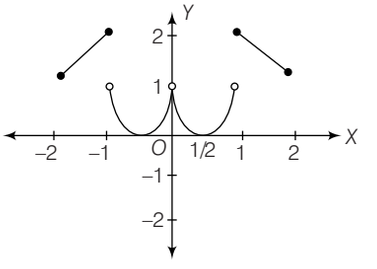
1. Graph for $f(x)$



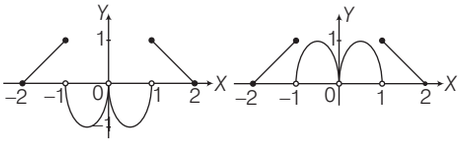
(i) $|f(x)|$ is shown as



(ii) $f(|x|)$ is shown as

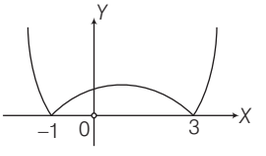


(iii)

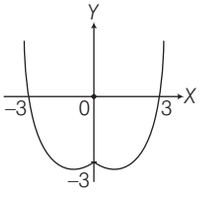


Graph for $f(|x|)-1$ Graph for $|f(|x|)-1|$

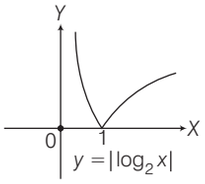
2. (i) $y=|x^2-2x-3|=|(x-3)(x+1)|$



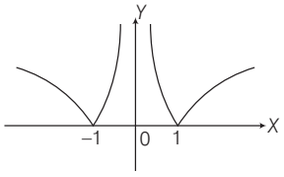
(ii) $y=(x+1)(x-3)=x^2-2x-3,\ y=x^2-2|x|-3$



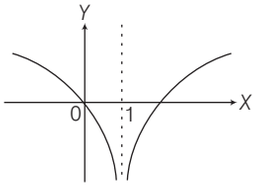
(iii) $y = |\log_2 x|$



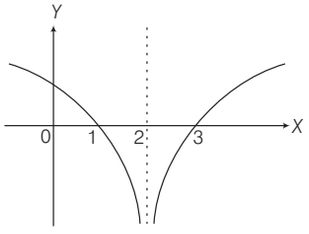
(iv) $y = |\log_2 |x||$



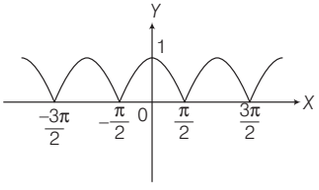
(v) $y = \log_2(1-x) \Rightarrow y = \log_2|1-x|$



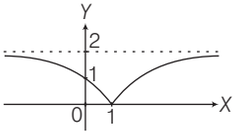
(vi) $y = \log_2(2-x)^2 = 2\log_2|2-x|$



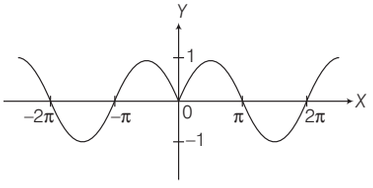
(vii) $y = |\cos|x||$



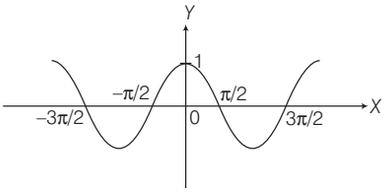
(viii) $y = 2 - 2^x \Rightarrow y = |2 - 2^x|$



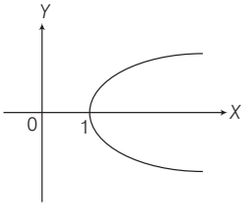
(ix) $y = \sin|x|$



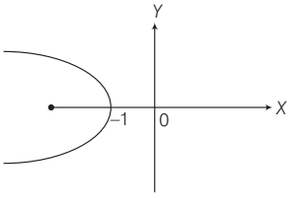
(x) $y = \cos|x|$



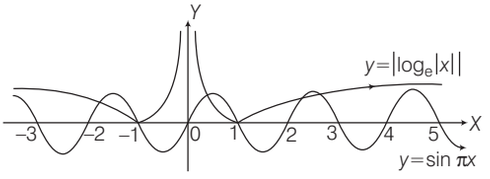
3. (i) $|f(x)| = \log_2 x$



(ii) $|f(x)| = \log_2(-x)$

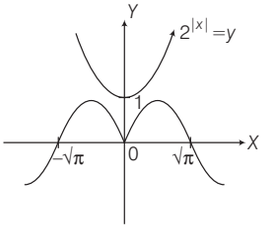


4. $\sin \pi x = |\log_e|x||$



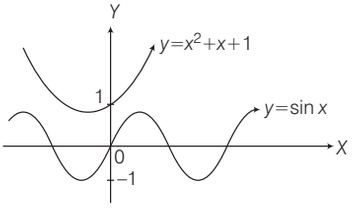
∴ Number of solutions is 6.

5. (i) $2^{|x|} = \sin x^2$



∴ No solution

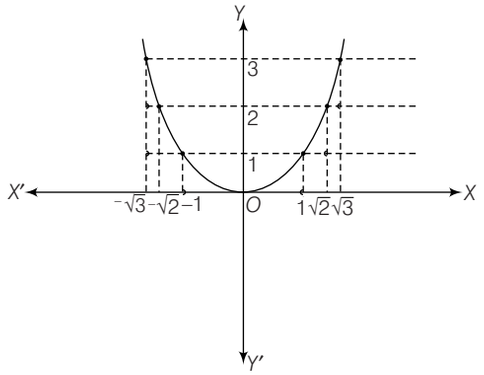
(ii) $\sin x = x^2 + x + 1$



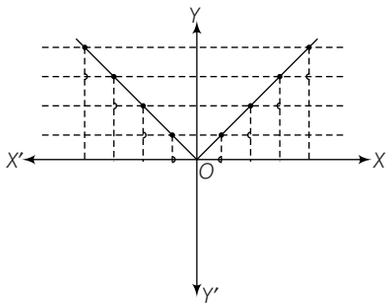
∴ No solution

Exercise for Session 4

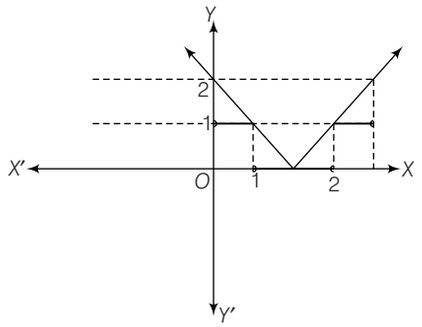
1. (i) Here, $f(x) = [x^2]$



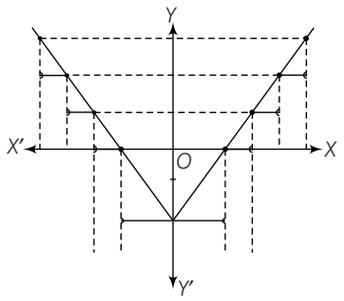
(ii) Here, $f(x) = [|x|]$



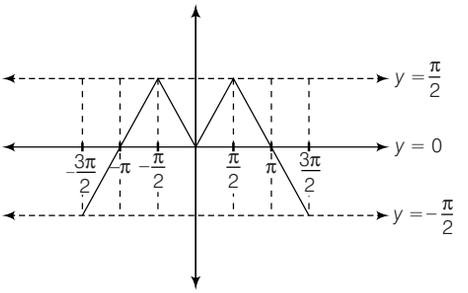
(iii) $f(x) = [(x-2)]$



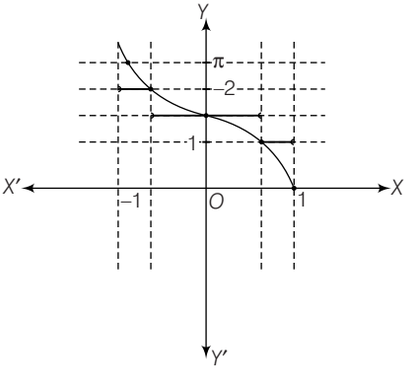
(iv) $f(x) = [|x|-2]$



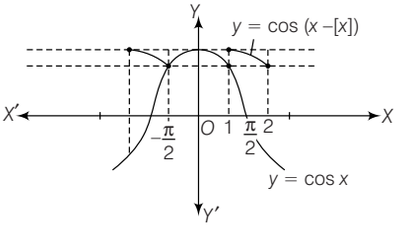
(v) $y = \sin^{-1}(\sin|x|)$



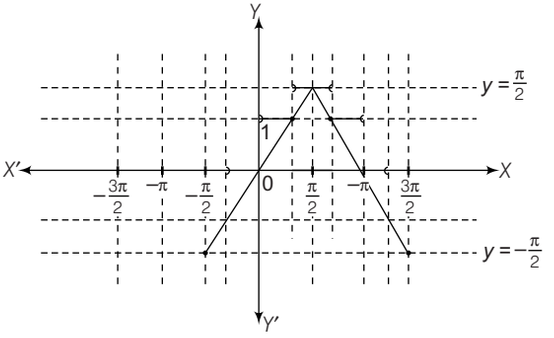
(vi) $y = [\cos^{-1} x]$



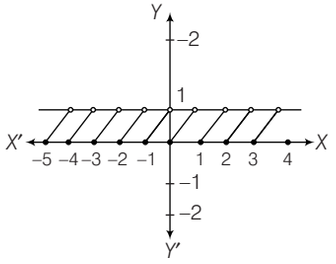
(vii) $f(x) = \cos(x - [x])$



(viii) $f(x) = [\sin^{-1}(\sin x)]$



2. Here, $f(x) = \min(x - [x], -x - [-x])$



3. (i) 4 sq units (ii) 1 sq unit (iii) 2 sq units

Chapter Exercises

1. (a) **2.** (b) **3.** (b) **4.** (b) **5.** (b) **6.** (b) **7.** (b) **8.** (b, c, d) **9.** (a, b, c, d)

10. (a, c, d) **11.** (a) **12.** (a) **13.** (b)

14. (a) **15.** (d) **16.** (c) **17.** (b) **18.** (3)

19. (4) **20.** (3) **21.** 0 **22.** 5 **23.** 1

24. 0 **25.** 4 sq units **26.** $f(x) = \begin{cases} \frac{1}{64}, & 0 < x \le \frac{1}{8} \\ x^2, & \frac{1}{8} < x \le 1 \\ x^3, & x > 1 \end{cases}$

27. $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \\ 0, & x > 2 \end{cases}$

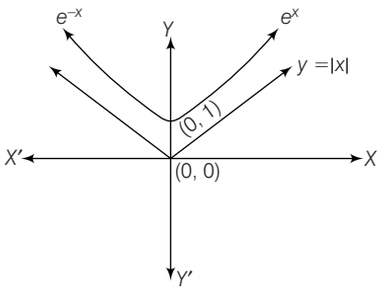
28. $D = R - \{(0,1], 2, 3, 4, [5, 6\}$.

Solutions

1. Given, $e^{|x|} - |x| = 0 \Rightarrow e^{|x|} = |x|$

Now, the number of solution will be the point of intersection of $y = e^{|x|}$ and $y = |x|$.

Both of the curves shown below



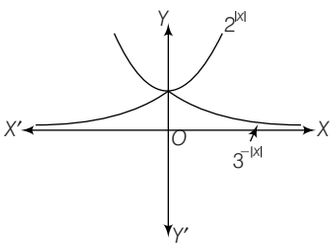
Clearly, from the graph we see that the both curves do not intersect.

∴ No solution.

2. Given, $3^{-|x|} - 2^{|x|} = 0 \Rightarrow 3^{-|x|} = 2^{|x|}$

Now, we plot the graph $y = 3^{-|x|}$

and $y = 2^{|x|}$ as shown below

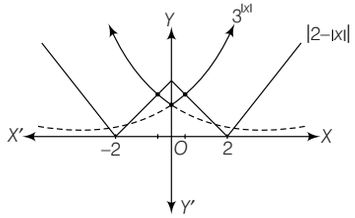


Clearly, the two curves intersect at one points.

Hence, the number of solutions is 1.

3. Given, $3^{|x|} = |2 - |x||$

The graph of $y = 3^{|x|}$ and $y = |2 - |x||$ is shown below



Clearly, both the curve intersect of two points.

∴ Number of solutions is 2.

4. Given, $\left| x - x^2 - 1 \right| = \left| 2x - 3 - x^2 \right|$

Let $f(x) = x - x^2 - 1$ and $g(x) = 2x - 3 - x^2$

First we draw the graph of $f(x)$ and $g(x)$

then, transforms it into $|f(x)|$ and $|g(x)|$.

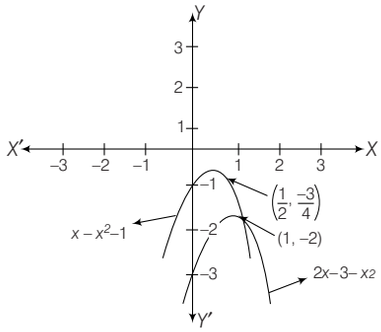
Here, $f(x) = x - x^2 - 1 = y \Rightarrow \left(x - \frac{1}{2}\right)^2 = -\left(y + \frac{3}{4}\right)$

Clearly, it represents a parabola having vertex $\left(-\frac{1}{2}, -\frac{3}{4}\right)$.

Also, $g(x) = 2x - 3 - x^2 = y \Rightarrow (x - 1)^2 = -(y + 2)$

It also represent a parabola having vertex $(1, -2)$.

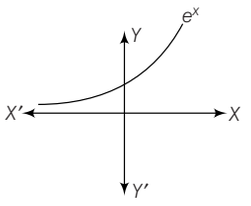
The graph of both curve is given below



Clearly, both the curves intersect at only one point.

∴ Number of solutions is 1.

5. Given, $e^x = m(m+1)$

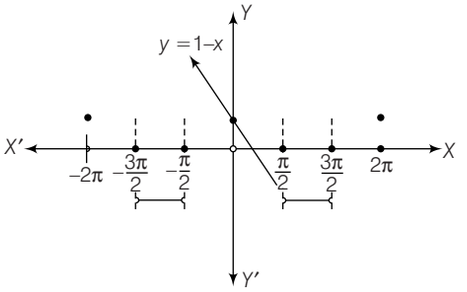


Clearly, for $m < -1$, it will represent a straight line parallel to X-axis which intersect the e^x at only one point.

Hence, exactly one real root.

6. Given, $1 - x = [\cos x]$

The graph of both the curves is given below

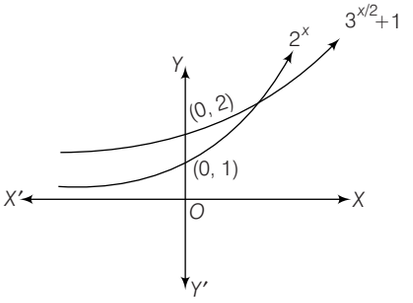


$y = 1 - x$ intersect the graph of $y = [\cos x]$ at exactly two points.

$\therefore$ Number of solutions is 2.

7. Given, $1 + 3^{x/2} = 2^x$

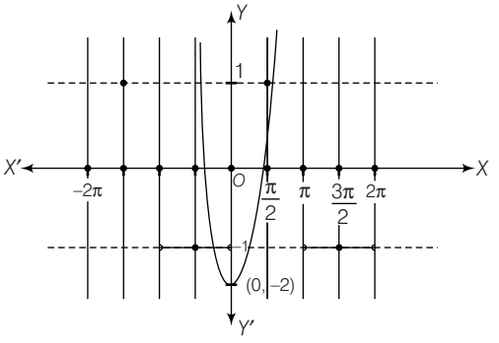
The graph is shown below



Clearly, both the curves intersect at one point.

8. Given, $x^2 - 2 = [\sin x]$

The graph of two curves is given below



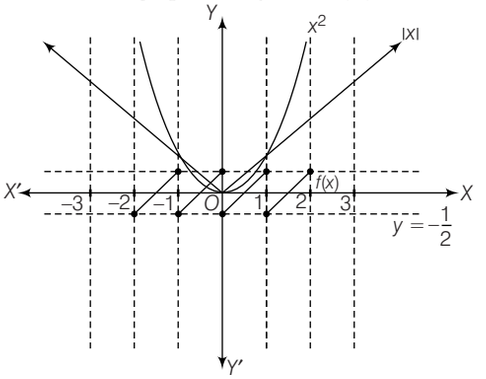
Clearly, the curve $y = x^2 - 2$ intersect the curves $y = [\sin x]$ at two points $(\sqrt{2}, 0)$ and $(-1, -1)$.

$\therefore$ Therefore, the solution has exactly one integer root and one irrational root.

9. Given, $f(x) = \begin{cases} x - [x] - \frac{1}{2}, & x \notin I \\ 0, & x \in I \end{cases}$

and $g(x) = \max\{x^2, f(x), |x|\}$

First we draw the graph of x^2, $f(x)$ and $|x|$ as shown below



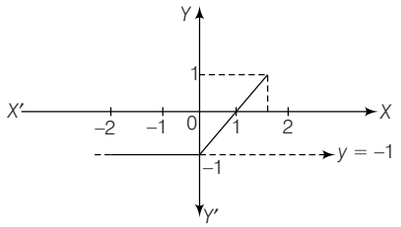
Clearly, from the graph

$$g(x) = \begin{cases} x^2, & -2 \le x \le -1 \\ 1 - x, & -1 < x \le -\frac{1}{4} \\ \frac{1}{2} + x, & -\frac{1}{4} < x < 0 \\ 1 + x, & 0 \le x < 1 \\ x^2, & 1 \le x \le 2 \end{cases}$$

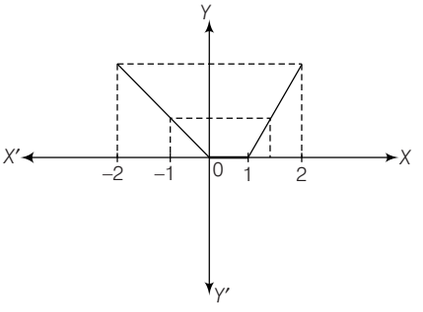
10. $f(x) = \begin{cases} -1 & -2 \le x < 0 \\ x - 1, & 0 < x \le 2 \end{cases}$

$g(x) = f(|x|) + |f(x)|$

The graph of $f(x)$ is shown below



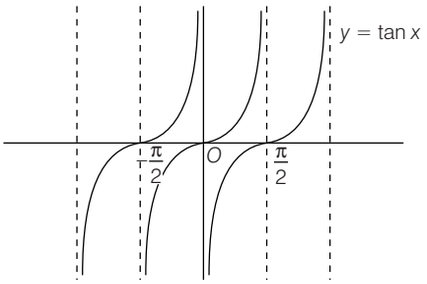
$\therefore$ $g(x) = |f(x)| + f(|x|)$ is shown below



Clearly, from the graph

$$g(x)=\begin{cases}-x, & -2\le x\le 0\\ 0, & 0<x<1\\ 2(x-1) & 1<x\le 2\end{cases}$$

11. We have the graph of $y=\tan x$ as shown below



Clearly, from the graph it is symmetrical about the origin.

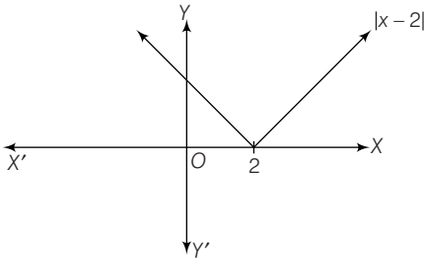
Also, derivative of an odd function is an even function and $\sec^2 x$ is derivative of $\tan x$.

Hence, both the statements are true and Statement II is a correct explanation of Statement I.

12. We know that, the number of point of intersection of two curve is the real solutions of the equation involving these functions.

Also, $||x-2|+a|=4$

$\Rightarrow$ $|x-2|=\pm\,4-a$



For four real roots $4-a>0$ and $-4-a>0$

$\Rightarrow$ $a\in(-\infty,-4)$

Both Statement I and Statement II are correct and Statement II is the correct explanation of Statement I

Sol. (Q.Nos. 13 to 14)

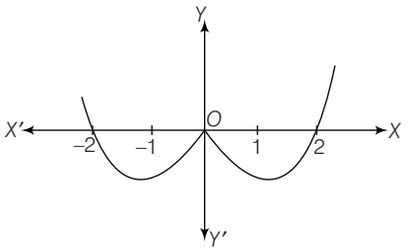
Here, $f_1(x)=x^2$

$f_2(x)=|x|$

or $f(x)=f_1(x)-2f_2(x)$

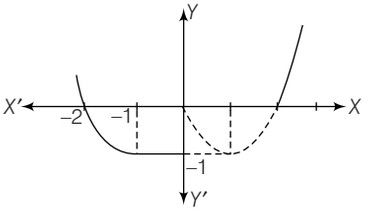
$=x^2-2|x|$

Graph for $f(x)$



Now, $g(x)=\begin{cases}f(x), & -3\le x<-1\\ -1, & -1\le x<0\\ 0, & 0\le x\le 2\\ f(x), & 2<x\le 3\end{cases}$

The graph of $g(x)$ is shown below



$$g(x)=\begin{cases}x^2+2x, & -3\le x<-1\\ -1, & -1\le x<0\\ 0, & 0\le x\le 2\\ x^2-2x, & 2<x<3\end{cases}$$

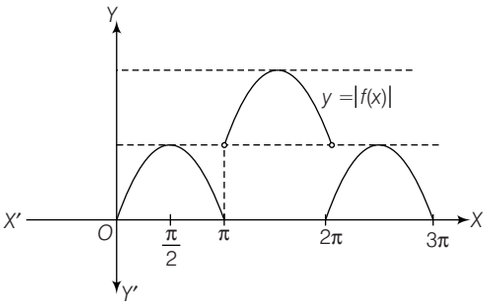
13. For $x\in(-1,0)$, $f(x)+g(x)=x^2+2x-1$

14. Obviously, the graph is broken at $x=0$, i.e. in one point.

15. Clearly, $h_1(x)=f(|g(x)|)=\sin^2 x$ has period π, range $[0,1]$ and domain R.

16. $h_2(x)=|f(g(x))|$

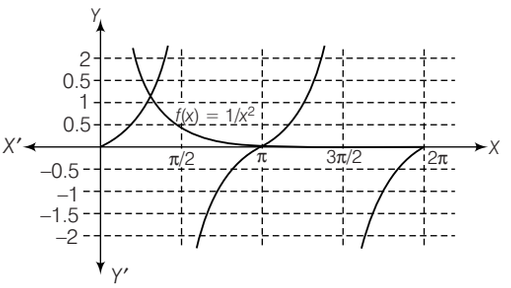
The graph for $h_2(x)$ is



It is clear from the graph that the function is periodic with period 2π and has range $[0,2]$.

17. Let $y=\tan x=\dfrac{1}{x^2}$

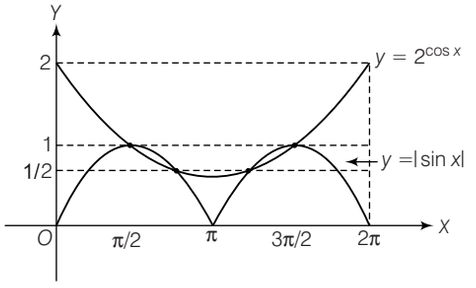
We plot the graph for $\tan x$ and $\dfrac{1}{x^2}$ as shown below



From the graph, it is clear that, it will have two real roots.

(b) $2^{\cos x}=|\sin x|$, $x\in[0,2\pi]$

Plotting the graph of $y = 2^{\cos x}$ and $y = |\sin x|$, as shown below



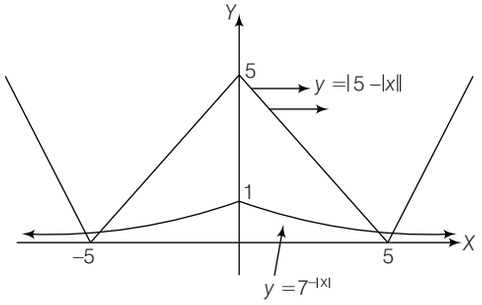
From the graph we see that the two curve meet at four points for $x \in [0, 2\pi]$.

So, the equation $2^{\cos x} = |\sin x|$ has four solutions.

(c) Given, $f(|x|) = 0$ has a real root or $f(x) = 0$ has four positive roots. Since, $f(x)$ is a polynomial of degree 5, $f(x)$ cannot have even number of real roots. Hence, $f(x)$ has all the five roots real and one root is negative.

(d) Here, $7^{|x|}(|5 - |x||) = 1$ or $|5 - |x|| = 7^{-|x|}$

Draw the graph of $y = 7^{-|x|}$ and $y = |5 - |x||$



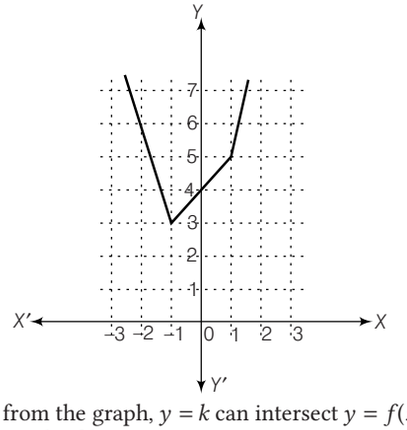
From the graph, the number of real roots is 4.

Hence, a → q; b → s; c → p; d → s.

18. Let $f(x) = x + 2|x + 1| + 2|x - 1|$

$$= \begin{cases} x - 2(x+1) - 2(x-1), & x < -1 \\ x + 2(x+1) - 2(x-1), & -1 \le x \le 1 \\ x + 2(x+1) + 2(x-1), & x > 1 \end{cases} = \begin{cases} -3x, & x < -1 \\ x + 4, & -1 \le x \le 1 \\ 5x, & x > 1 \end{cases}$$

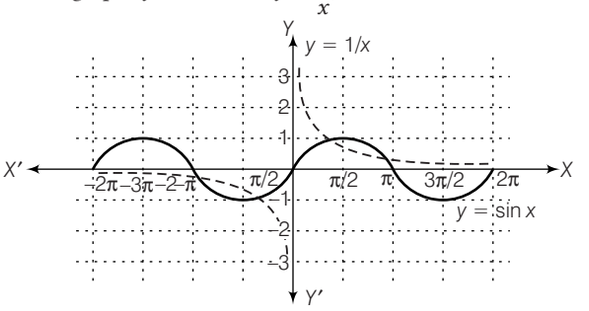
The graph of $f(x)$ is shown below



Clearly, from the graph, $y = k$ can intersect $y = f(x)$ at exactly one point only if $k = 3$.

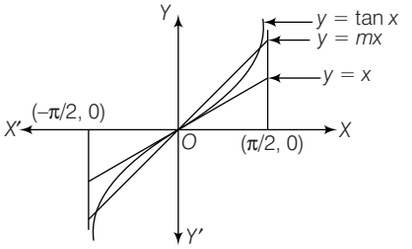
19. Here, $x \sin x = 1$ or $y = \sin x = \dfrac{1}{x}$

The roots of Eq. (i) will be given by the points of intersection of the graphs $y = \sin x$ and $y = \dfrac{1}{x}$, which is shown below.



Graphically, it is clear that the given equation has four roots.

20. In $\left(-\dfrac{\pi}{2}, 0\right)$, the graph of $y = \tan x$ lies below the line $y = x$, which is the tangent at $x = 0$ and in $\left(0, \dfrac{\pi}{2}\right)$, it lies above the line $y = x$.

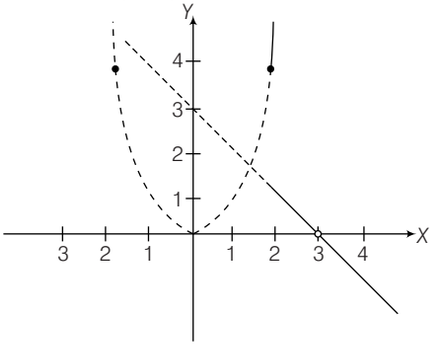


For $m > 1$, the line $y = mx$ lie, below $y = x$ in $\left(-\dfrac{\pi}{2}, 0\right)$ and above $y = x$ in $\left(0, \dfrac{\pi}{2}\right)$.

Thus, graph of $y = \tan x$ and $y = mx$, $m > 1$, meet at three points including $x = 0$ in $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ independent of m.

21. We have, $\dfrac{x^2}{1 - |x-2|} = 1$

(i) When $x \ge 2$, $\dfrac{x^2}{1-(x-2)} = 1 \Rightarrow \dfrac{x^2}{3-x} = 1$

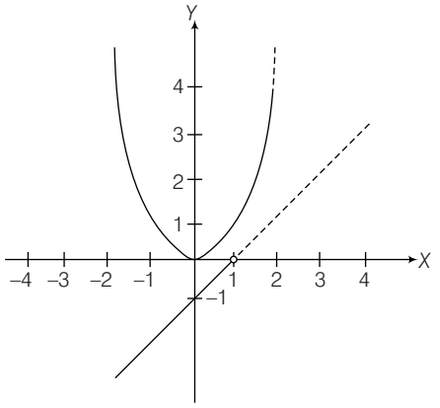


$\Rightarrow \quad x^2 = 3 - x, x \ne 3$

No point of intersection.

(ii) When $x < 2, \dfrac{x^2}{1-(2-x)} = 1 \Rightarrow \dfrac{x^2}{-1+x} = 1$

$\Rightarrow \quad x^2 = -1 + x, x \neq 1$



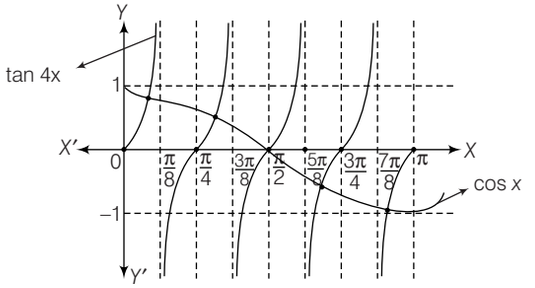
No point of intersection.

Here, the number of solution is zero.

22. Given, $\tan 4x = \cos x$

The number of solutions will be the point of intersection of curves $y = \tan 4x$ and $\cos x$

The graph of $y = \tan 4x$ and $y = \cos x$ is given below.



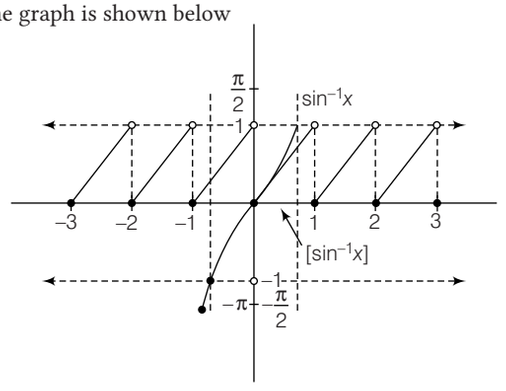
From the graph, it is clear that both curve intersect at five points.

∴Number of solutions is 5.

23. Given, $[\sin^{-1} x] = x - [x]$

The number of solution will be the point of intersection of $y = [\sin^{-1} x]$ and $y = x - [x] = \{x\}$

The graph is shown below



Clearly, the two graphs intersect at only one point.

∴ Number of solutions is 1.

24. Now, $|x + y| = 1$

$\Rightarrow \quad x + y = \pm 1$ ...(i)

and $\quad |x - y| = 1$

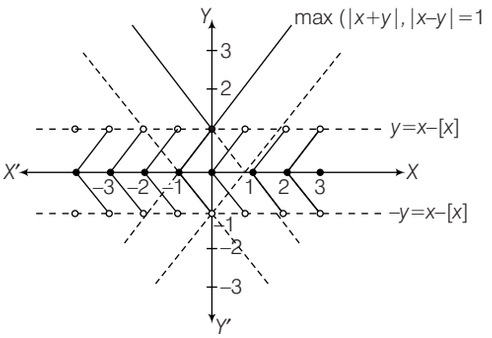
$\Rightarrow \quad x - y = \pm 1$ ...(ii)

Also, $\quad |y| = x - [x]$

$\Rightarrow \quad \pm y = x - [x]$

Here, we see that there is no point of intersection between the two curves.

Here, no ordered pair of (x, y) satisfy the given equations.



25. We have, $\quad |x + y - 1| + |2x + y + 1| = 1$

$\Rightarrow \pm (x + y - 1) \pm (2x + y + 1) = 1$

(i) $(x + y - 1) + (2x + y + 1) = 1$

$\Rightarrow \quad 3x + 2y = 1$ ...(i)

(ii) $+(x + y - 1) - (2x + y + 1) = 1$

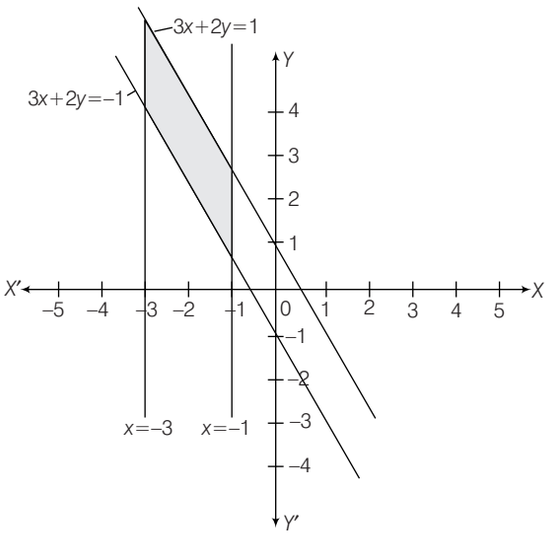
$\Rightarrow \quad -x = 3$ ...(ii)

(iii) $-(x + y - 1) + (2x + y + 1) = 1$

$\Rightarrow \quad x = -1$ ...(iii)

(iv) $-(x + y - 1) - (+2x + y + 1) = 1$

$\Rightarrow \quad -3x - 2y = 1$ ...(iv)

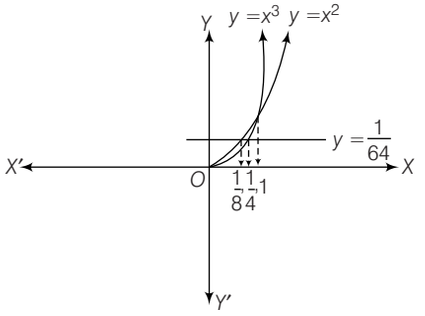


∴ Area of shaded region $= \int_{-1}^{-3} [(3x + 2y - 1) - (3x + 2y + 1)]dx$

$= \int_{-1}^{-3} -2\,dx = -2[x]_{-1}^{-3} = -2[-3 + 1] = 4$ sq units

26. Given, $f(x) = \max\left\{x^3, x^2, \frac{1}{64}\right\}, \forall\ x \in (0,\infty)$

Now, we draw the graph for the curves $y = x^3$, $y = x^2$ and $y = \frac{1}{64}$



Clearly, from the graph $f(x) = \begin{cases} \frac{1}{64}, & 0 < x \le \frac{1}{8} \\ x^2, & \frac{1}{8} < x < 1 \\ x^3, & x > 1 \end{cases}$

27. The line through (0, 0) and (1, 1) has slope $m = 1$ and y-intercept $b = 0$, so its equation is $y = x$. Thus, for the part of the graph f that joins (0, 0) to (1,1), we have $f(x) = x$, if $0 \le x \le 1$

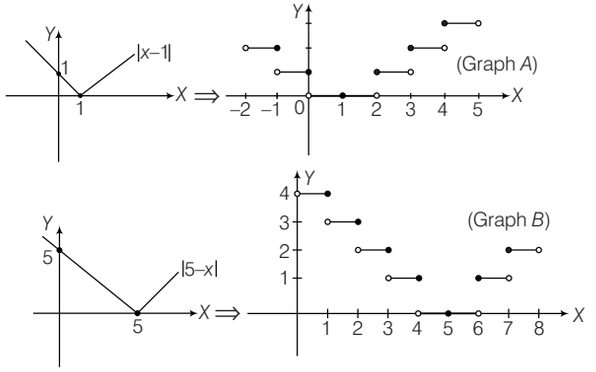
The line through (1, 1) and (2, 0) has slope $m = -1$, so its point slope form is $y - 0 = (-1)(x - 2)$ or $y = 2 - x$.

So, we have $f(x) = 2 - x$, if $1 < x \le 2$

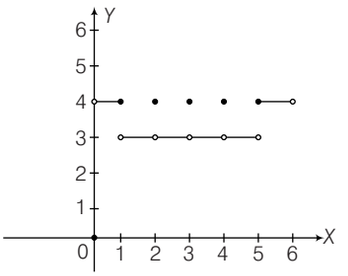
We also see that the graph of f coincides with the X-axis for $x > 2$. Putting this information together, we have the following three-piece formula for $f(x) = \begin{cases} x, & \text{if } 0 \le x \le 1 \\ 2 - x, & \text{if } 1 < x \le 2 \\ 0, & \text{if } x > 2 \end{cases}$

28. The denominator is a bit complicated and we need to analyse it in detail to determine, where it can become zero.

The fastest and easiest way would be to visualise the graph. Draw the graphs for $|x - 1|$ and $|5 - x|$, apply the greatest integer function of these graphs separately, then add them and find the values of x for which this sum becomes 4. For these values of x, the denominator of $f(x)$ becomes 0.

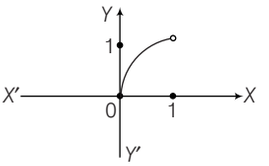


Now, add the graphs of A and B point by point

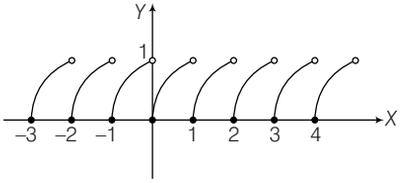


We see that the value of (graph A + graph B) is 4 for the following values of x : (0, 1], {2, 3, 4}, [5, 6).
Hence, domain $D = R / \{(0, 1], 2, 3, 4, [5, 6)\}$.

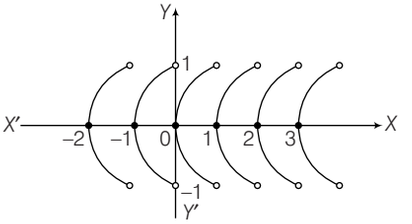
29. Let us first draw the graph for $y = \sqrt{\{x\}}$. Now, $\{x\}$ is the same as x for $0 \le x < 1$. In this interval, $\sqrt{\{x\}}$ will be the same as $\sqrt{x}$.



We can easily see that this same curve will be repeated in every previous and subsequent unit interval, since $\{x\}$ is the same in all such intervals. Hence, we obtain the graph of $y = \sqrt{\{x\}}$.

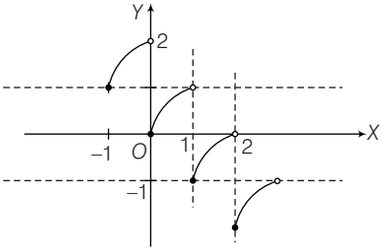


Now, we can easily draw $|y| = \sqrt{\{x\}}$ by taking a reflection in the X-axis.



30. In any interval $n \le x < n + 1$, (where n is an integer) $[x]$ has the value n. In any interval, therefore the graph of $y = -[x] + \sqrt{\{x\}}$ will be the graph of $\sqrt{\{x\}}$ – integer n.

e.g. $0 \le x < 1 \Rightarrow y = \sqrt{\{x\}}$; $1 \le x < 2 \Rightarrow y = -1 + \sqrt{\{x\}}$; $-1 \le x < 0 \Rightarrow y = 1 + \sqrt{\{x\}}$. The graph is drawn below



CHAPTER

05

Limits

Learning Part

Session 1

- Definition of Limits
- Indeterminate Forms
- L'Hospital's Rule
- Evaluation of Limits

Session 2

- Trigonometric Limits

Session 3

- Logarithmic Limits
- Exponential Limits

Session 4

- Miscellaneous Forms

Session 5

- Left Hand and Right Hand Limits

Session 6

- Use of Standard Theorems/Results
- Use of Newton-Leibnitz's Formula in Evaluating the Limits
- Summation of Series Using Definite Integral as the Limit

Practice Part

- JEE Type Examples
- Chapter Exercises

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Session 1

Definition of Limits, Indeterminate Forms, L'Hospital's Rule, Evaluation of Limits

Definition of Limits

Let $\lim_{x \to a} f(x) = l$. It would mean that when we approach the point $x = a$ from the values which are just greater than or smaller than $x = a$, $f(x)$ would have a tendency to move closer to the value 'l'.

This is same as saying 'difference between $f(x)$ and l can be made as small as we feel like by suitably choosing x in the neighbourhood of $x = a$'.

Mathematically, we write this, as $\lim_{x \to a} f(x) = l$, which is equivalent of saying that, $|\boldsymbol{f(x) - l| < \varepsilon, \forall\ x}$ **whenever** $\boldsymbol{0 < |x - a| < \delta}$ **and** ε **and** δ **sufficiently small +ve numbers**.

It is clear from the above discussion that, if we are interested in finding the limit of $f(x)$ at $x = a$, the first thing we have to make sure that $f(x)$ is well defined in the neighbourhood of $x = a$ and not necessarily at $x = a$ (that means $x = a$ may or may not be in the domain of $f(x)$), because we have to examine its behaviour or tendency in the neighbourhood of $x = a$.

Following possibilities may arise :

(a) **Left tendency is same as its right tendency** As shown in figure 5.1, when we approach $x = a$ from the values which are just less than a, $f(x)$ has a tendency to move towards the value l (**left tendency**).

Similarly, when we approach $x = a$ from the values which are just greater than a, $f(x)$ has a tendency to move towards the value l (**right tendency**).

In this case, we say $f(x)$ has limit l at $x = a$,
i.e. $\boldsymbol{\lim_{x \to a} f(x) = l.}$

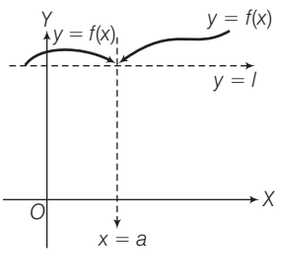


Figure 5.1

(b) **When the left tendency is not the same as its right tendency** Here, left tendency is l_1 and right tendency is l_2, clearly left tendency (l_1) is not same as right tendency (l_2). In this case, we say that the limit of $f(x)$ at $x = a$ will not exist.

i.e. $\boldsymbol{\lim_{x \to a} f(x) =}$ **Doesn't exist.**

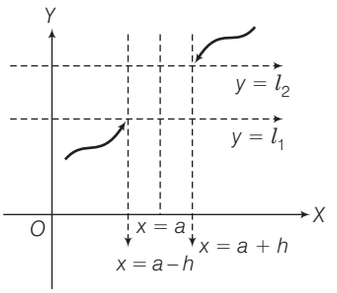


Figure 5.2

(c) **When the left tendency and/or right tendency is not fixed** As shown in the figure 5.3, it is clear that in this case, the function has **erratic** behaviour in the neighbourhood of $x = a$ and it will not be possible to talk about the left and right tendencies of the function in the neighbourhood of $x = a$.

In this case, we conclude that the limit of $f(x)$ at $x = a$ will not exist.

i.e. $\boldsymbol{\lim_{x \to a} f(x) =}$ **Doesn't exist.**

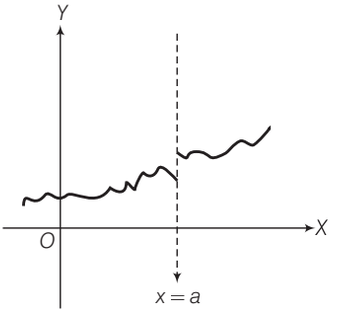


Figure 5.3

Remarks

1. Normally, students have the perception that limit should be a finite number. But, it is not always so. It is quite possible that $f(x)$ has infinite limit at $x = a$.
   If $\lim_{x \to a} f(x) = \infty$, it would simply mean that function has tendency to assume very large positive values in neighbourhood of $x = a$ (as shown in figure 5.4).

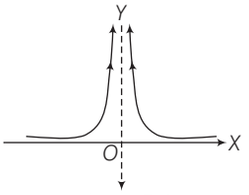


Figure 5.4

   For example $\lim_{x \to 0} \frac{1}{|x|} = \infty$

   which indicates the left tendency as well as right tendency are the same.
   Again, if $\lim_{x \to a} f(x) = -\infty$, it would simply mean that the function has tendency to assume very large negative values in the neighbourhood of $x = a$ as shown in figure 5.5.

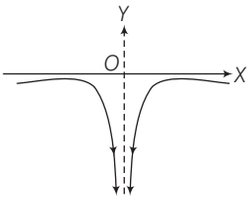


Figure 5.5

   For example (as shown in figure 5.5) $\lim_{x \to 0} \frac{-1}{|x|} = -\infty$

   At the end we discuss the case when left tendency is $(-\infty)$ and right tendency is $(+\infty)$ (i.e. $f(x)$ does not have unique tendency).
   Thus, in this case limit does not exist.

   For example $\lim_{x \to 0} \frac{1}{x}$ does not exist, since left tendency is $(-\infty)$ and right tendency is $(+\infty)$ as shown in figure 5.6.

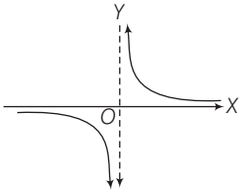


Figure 5.6

2. If $f(x)$ is well defined at $x = a$, it doesn't imply that $\lim_{x \to a} f(x) = f(a)$. Because, it is quite possible that $f(x)$ is well defined at $x = a$ but not in the neighbourhood of $x = a$ or $f(x)$ is well defined in the neighbourhood of $x = a$, but doesn't have a unique tendency.

Indeterminate Forms

If direct substitution of $x = a$ while evaluating $\lim_{x \to a}(x)$ leads to one of the following forms

$\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 1^{\infty}, 0^{0}, \infty^{0}, \infty \times 0$ then it is called indeterminate form.

e.g. $\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \frac{0}{0}$ indeterminate form.

$\lim_{x \to a} \frac{x^n - a^n}{x - a} = \frac{0}{0}$ indeterminate form.

$\lim_{x \to 0} \frac{\sin x}{x} = \frac{0}{0}$ indeterminate form.

Infinity (∞) is a symbol and not a number. It is a symbol for the behaviour of a variable which continuously increases and passes through all limits. Thus, the statement $x = \infty$ is meaningless, we should write $x \to \infty$.

Similarly, $-\infty$ is a symbol for the behaviour of a variable which continuously decreases and passes through all limits. Thus, the statement $x = -\infty$ is meaningless, we should write $x \to -\infty$.

Also, $\frac{1}{x} \to 0$, if $x \to +\infty$ and $\frac{1}{x} \to 0$, if $x \to -\infty$.

We cannot plot ∞ on paper. Infinity does not obey laws of elementary algebra.

(i) $\infty + \infty = \infty$ is indeterminate

(ii) $\infty - \infty$ is indeterminate.

L'Hospital's Rule

This rule states that, if $\lim_{x \to a} \frac{f(x)}{g(x)}$, reduces to $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Then, differentiate numerator and denominator till this form is removed.

i.e. $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$, provided the later limit exists.

But, if it again take form $\left(\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right)$,

then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f''(x)}{g''(x)}$

and this process is continued till $\left(\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right)$ form is removed.

Remark

L'Hospital's rule is applicable to only two indeterminate forms $\left(\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right)$.

e.g. Evaluate $\lim_{x \to 2} \frac{x^6 - 24x - 16}{x^3 + 2x - 12}$.

Sol. We have, $\lim_{x \to 2} \frac{x^6 - 24x - 16}{x^3 + 2x - 12}$ $\left[\frac{0}{0} \text{ form}\right]$

$= \lim_{x \to 2} \frac{6x^5 - 24}{3x^2 + 2}$ [by L'Hospital's rule]

$= \frac{6(2)^5 - 24}{3(2)^2 + 2} = \frac{168}{14} = 12$

Frequently Used Series Expansions

1. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

2. $a^x = 1 + \frac{x \cdot \log a}{1!} + \frac{(\log a)^2 x^2}{2!} + \ldots$

[where, $a \in R^+$]

3. $(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \ldots n \in R$ and $|x| < 1$

4. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

[where $-1 \le x \le 1$]

5. $\frac{x^n - a^n}{x - a} = x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + a^{n-1}$

6. $(1+x)^{1/x} = e\left(1 - \frac{x}{2} + \frac{11x^2}{24} + \ldots\right)$

7. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

8. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

9. $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \ldots$

10. $\sin^{-1} x = x + \frac{1^2}{3!}x^3 + \frac{1^2 \cdot 3^2}{5!}x^5 + \frac{1^2 \cdot 3^2 \cdot 5^2}{7!}x^7 + \ldots$

11. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots$

12. $\sec^{-1} x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \ldots$

13. $(\sin^{-1} x)^2 = \frac{2}{2!}x^2 + \frac{2 \cdot 2^2}{4!}x^4 + \frac{2 \cdot 2^2 \cdot 4^2}{6!}x^6 + \ldots$

14. $x \cot x = 1 - \frac{x^3}{3} + \frac{x^4}{45} - \frac{2x^6}{945} + \ldots$

15. $\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \ldots$

16. $x \operatorname{cosec} x = 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \frac{31x^6}{15120} + \ldots$

Example 1 Evaluate $\lim_{x \to 0} \frac{x \tan 2x - 2x \tan x}{(1 - \cos 2x)^2}$.

[IIT JEE 1999]

Sol. We have, $\lim_{x \to 0} \frac{x \tan 2x - 2x \tan x}{(1 - \cos 2x)^2}$

$$= \lim_{x \to 0} \frac{x\left[2x + \frac{2^3x^3}{3} + 2\frac{2^5x^5}{15} + \ldots\right] - 2x\left[x + \frac{x^3}{3} + 2\frac{x^5}{15} + \ldots\right]}{(2\sin^2 x)^2}$$

$$= \lim_{x \to 0} \frac{x^4\left(\frac{8}{3} - \frac{2}{3}\right) + x^6\left(\frac{64}{15} - \frac{4}{15}\right) + \ldots}{4\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)^4}$$

$$= \lim_{x \to 0} \frac{2 + 4x^2 + \ldots}{4\left(1 - \frac{x^2}{3!} + \ldots\right)^4} = \frac{2}{4} = \frac{1}{2}$$

Evaluation of Limits

Now, according to our plan, first of all we shall learn the evaluation of limits of different forms and then learn the existence of limits.

There are eight indeterminate or meaningless forms, which are

(i) $\frac{0}{0}$ (ii) $\frac{\infty}{\infty}$ (iii) $\infty - \infty$ (iv) $\infty \times \infty$

(v) $\infty \cdot 0$ (vi) 0^0 (vii) ∞^0 (viii) 1^∞

We will divide the problems of evaluation of limits in five categories, which are

1. Limit of algebraic functions
2. Trigonometric limits
3. Logarithmic limits
4. Exponential limits
5. Miscellaneous forms

Now, we discuss one by one in details.

Limit of Algebric Functions

In this section, we evaluate limit of algebraic functions when variable tends to a finite or infinite value. While evaluating algebraic limits the form $\frac{0}{0}, \frac{\infty}{\infty}$ and $\infty - \infty$ arise, which we will discussed here.

(i) $\frac{0}{0}$ Form

This form can be resolved by factorisation method, rationalisation method or by using the formula $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$, which are discussed below.

(a) Factorisation Method

In this method, numerators and denominators are factorised. The common factors are cancelled and the rest output is the result.

Example 2 Evaluate $\lim_{x\to 1}\dfrac{x^2-3x+2}{x-1}$.

Sol. **Method I** We have, $\lim_{x\to 1}\dfrac{x^2-3x+2}{x-1}$ $\left[\dfrac{0}{0}\text{ form}\right]$

$$=\lim_{x\to 1}\frac{(x-1)(x-2)}{(x-1)}$$

[as $x^2-3x+2=(x-1)(x-2)$]

$$=\lim_{x\to 1}(x-2) \qquad [\text{as } x-1\neq 0]$$

$$=1-2=-1$$

Method II We have, $L=\lim_{x\to 1}\dfrac{x^2-3x+2}{x-1}$ $\left[\dfrac{0}{0}\text{ form}\right]$

So, applying L'Hospital's rule,

$$L=\lim_{x\to 1}\frac{2x-3}{1}=\frac{2-3}{1}=-1$$

[i.e. differentiating numerator and denominator separately]

Example 3 ***Evaluate*** $\lim_{x\to 1}\dfrac{x^3-x^2\log x+\log x-1}{x^2-1}$.

Sol. We have, $\lim_{x\to 1}\dfrac{x^3-x^2\log x+\log x-1}{x^2-1}$

$$=\lim_{x\to 1}\frac{(x^3-1)-(x^2-1)\log x}{(x^2-1)} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$=\lim_{x\to 1}\frac{(x-1)\{x^2+x+1-(x+1)\log x\}}{(x-1)(x+1)}$$

$$\left[\because x^3-1=(x-1)(x^2+x+1),\ x^2-1=(x-1)(x+1)\right]$$

$$=\lim_{x\to 1}\frac{x^2+x+1-(x+1)\log x}{(x+1)}$$

$$=\frac{1^2+1+1-(1+1)\log 1}{1+1}=\frac{3}{2} \qquad [\text{as } \log 1=0]$$

(b) Rationalisation Method

Rationalisation is followed when we have fractional powers (like $\frac{1}{2},\frac{1}{3}$ etc.) on expressions in numerator or denominator or in both. After rationalisation the terms are factorised which on cancellation gives the result.

Example 4 Evaluate $\lim_{h\to 0}\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$.

Sol. **Method I** We have, $\lim_{h\to 0}\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$

$$=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\times\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$=\lim_{h\to 0}\frac{(x+h)-(x)}{h(\sqrt{x+h}+\sqrt{x})}=\lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})}$$

$$=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}=\frac{1}{2\sqrt{x}}$$

Method II (L'Hospital's rule) We have,

$$L=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$\therefore$ Applying L'Hospital's rule,

$$L=\lim_{h\to 0}\frac{\frac{1}{2\sqrt{x+h}}-0}{1}$$

[differentiating numerator and denominator w.r.t. h]

$$=\frac{1}{2\sqrt{x}}$$

Example 5 ***Evaluate*** $\lim_{x\to 2a}\dfrac{\sqrt{x-2a}+\sqrt{x}-\sqrt{2a}}{\sqrt{x^2-4a^2}}$.

Sol. We have, $\lim_{x\to 2a}\dfrac{\sqrt{x-2a}+\sqrt{x}-\sqrt{2a}}{\sqrt{x^2-4a^2}}$ $\left[\dfrac{0}{0}\text{ form}\right]$

$$=\lim_{x\to 2a}\frac{\sqrt{x-2a}+\dfrac{x-2a}{\sqrt{x}+\sqrt{2a}}}{\sqrt{x-2a}\sqrt{x+2a}}$$

$$=\lim_{x\to 2a}\frac{1}{\sqrt{x+2a}}+\frac{\sqrt{x-2a}}{\sqrt{x+2a}(\sqrt{x}+\sqrt{2a})}$$

$$=\frac{1}{\sqrt{4a}}=\frac{1}{2\sqrt{a}}$$

(c) Based on Standard Formula

$\lim_{x\to a}\dfrac{x^n-a^n}{x-a}=na^{n-1}$, where n is a rational number.

Proof Let $f(x)=\dfrac{x^n-a^n}{x-a}$

$$=x^{n-1}+ax^{n-2}+a^2x^{n-3}+\ldots+a^{n-1}$$

$$\therefore\ \lim_{x\to a}f(x)=\lim_{x\to a}(x^{n-1}+ax^{n-2}+a^2x^{n-3}+\ldots+a^{n-1})$$

$$=\underbrace{a^{n-1}+a\cdot a^{n-2}+a^2\cdot a^{n-3}+\ldots+a^{n-1}}_{n\text{ terms}}$$

$$=a^{n-1}+a^{n-1}+a^{n-1}+\ldots\text{ upto }n\text{ terms}=n\cdot a^{n-1}$$

Example 6 Evaluate $\lim_{x \to 2} \frac{x^3 - 2^3}{x - 2}$.

Sol. We have, $\lim_{x \to 2} \frac{x^3 - 2^3}{x - 2}$ $\left[\frac{0}{0} \text{ form}\right]$

$= 3(2)^{3-1}$ $\left[\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = n \cdot a^{n-1}\right]$

$= 3(2)^2 = 12$

Example 7 Evaluate $\lim_{x \to 1} \frac{x + x^2 + \ldots + x^n - n}{x - 1}$.

Sol. We have, $\lim_{x \to 1} \frac{x + x^2 + x^3 + \ldots + x^n - n}{x - 1}$

$= \lim_{x \to 1} \frac{(x-1) + (x^2 - 1^2) + (x^3 - 1^3) + \ldots + (x^n - 1^n)}{(x-1)}$

$= \lim_{x \to 1} \left\{\frac{x-1}{x-1} + \frac{x^2 - 1^2}{x-1} + \frac{x^3 - 1^3}{x-1} + \ldots + \frac{x^n - 1^n}{x-1}\right\}$

$= 1 + 2(1)^{2-1} + 3(1)^{3-1} + \ldots + n(1)^{n-1}$

$= 1 + 2 + 3 + \ldots + n$

$= \frac{n(n+1)}{2}$

Example 8 The value of $\lim_{\substack{x \to 1 \\ y \to 0}} \frac{y^3}{x^3 - y^2 - 1}$ as $(x, y) \to (1, 0)$ along the line $y = x - 1$ is

(a) 1

(b) –1

(c) 0

(d) Doesn't exist

Sol. As, $y \to x - 1$ or $x \to y + 1$

$\therefore$ $\lim_{y \to 0} \frac{(y)^3}{(y+1)^3 - y^2 - 1}$

Using L, Hospital's rule,

$\lim_{y \to 0} \frac{3y^2}{3(y+1)^2 - 2y} = \lim_{y \to 0} \frac{6y}{6(y+1) - 2}$

$= \frac{0}{6} = 0$

Hence, (c) is the correct answer.

(ii) Algebraic Function of ∞ Type

(a) $\frac{\infty}{\infty}$ Form

First we should know the limiting values of a^x $(a > 0)$ as $x \to \infty$. *See the graphs of this function.*

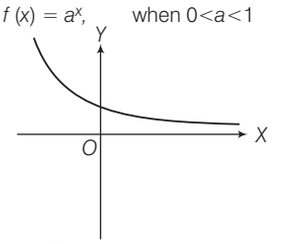


Figure 5.7

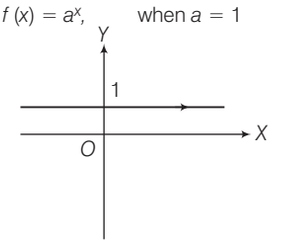


Figure 5.8

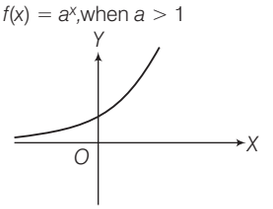


Figure 5.9

Now, see the graph for a^x, when $a > 1$. This graph appears to touch X-axis in the negative side of X-axis and thereafter it increases rapidly. This is why because $\lim_{x \to -\infty} a^x \to 0$, again you will also find the result,

$$\lim_{x \to \infty} a^x \to \infty$$

Thus, we have $\lim_{x \to \infty} a^x = \begin{cases} \infty, & \text{if } a > 1 \\ 1, & \text{if } a = 1 \\ 0, & \text{if } 0 \le a < 1 \end{cases}$

This type of problems are solved by taking the highest power of the terms tending to infinity as common from numerator and denominator. That is after they are cancelled and the rest output is the result (or apply L'Hospital's rule).

Example 9 ***Evaluate*** $\lim_{x \to \infty} \frac{x^2 + 5}{x^2 + 4x + 3}$.

Sol. Let $L = \lim_{x \to \infty} \frac{x^2 + 5}{x^2 + 4x + 3}$

Dividing numerator and denominator by x^2, we get

$$L = \lim_{x \to \infty} \frac{1 + \frac{5}{x^2}}{1 + \frac{4}{x} + \frac{3}{x^2}} = \frac{1 + 0}{1 + 0 + 0} = 1$$

[because $\frac{K}{x} \to 0$, when $x \to \infty$, where K is any constant]

Aliter We have, $L = \lim_{x \to \infty} \frac{x^2 + 5}{x^2 + 4x + 3}$ $\left[\frac{\infty}{\infty} \text{ form}\right]$

Applying L'Hospital's rule, $L = \lim_{x \to \infty} \frac{2x}{2x + 4}$ $\left[\frac{\infty}{\infty} \text{ form}\right]$

Again, applying L'Hospital's rule, $L = \lim_{x \to \infty} \frac{2}{2} = 1$

Example 10 ***Evaluate*** $\lim_{n\to\infty}\dfrac{(n+2)!+(n+1)!}{(n+2)!-(n+1)!}$.

Sol. We have, $\lim_{n\to\infty}\dfrac{(n+2)(n+1)!+(n+1)!}{(n+2)(n+1)!-(n+1)!}$

$$=\lim_{n\to\infty}\frac{(n+1)![n+2+1]}{(n+1)![n+2-1]}=\lim_{n\to\infty}\frac{(n+3)}{(n+1)}\quad\left[\frac{\infty}{\infty}\text{ form}\right]$$

$$=\lim_{n\to\infty}\frac{1+3/n}{1+1/n}=\frac{1+0}{1+0}=1\qquad\left[\because\frac{1}{n}\to 0,\ \text{as } n\to\infty\right]$$

(b) $\infty-\infty$ Form

Such problems are simplified (generally rationalised) first, thereafter they generally acquire $\left(\dfrac{\infty}{\infty}\right)$ form.

Example 11 ***Evaluate*** $\lim_{x\to\infty}(x-\sqrt{x^2+x})$.

Sol. We have, $\lim_{x\to\infty}(x-\sqrt{x^2+x})$

$$=\lim_{x\to\infty}\frac{x-\sqrt{x^2+x}}{1}\times\frac{x+\sqrt{x^2+x}}{x+\sqrt{x^2+x}}\qquad[\infty-\infty\text{ form}]$$

$$=\lim_{x\to\infty}\frac{x^2-(x^2+x)}{x+\sqrt{x^2+x}}=\lim_{x\to\infty}\frac{-x}{x+\sqrt{x^2+x}}$$

$$=\lim_{x\to\infty}\frac{-x}{x\left\{1+\sqrt{1+\dfrac{1}{x}}\right\}}$$

$$=\lim_{x\to\infty}\frac{-1}{1+\sqrt{1+\dfrac{1}{x}}}=-\frac{1}{2}\qquad\left[\because\frac{1}{x}\to 0,\ \text{as } x\to\infty\right]$$

Example 12 ***Evaluate*** $\lim_{x\to\infty}(\sqrt{x^2+x+1}-\sqrt{x^2+1})$.

Sol. We have, $\lim_{x\to\infty}(\sqrt{x^2+x+1}-\sqrt{x^2+1})\qquad[\infty-\infty\text{ form}]$

$$=\lim_{x\to\infty}\frac{(\sqrt{x^2+x+1}-\sqrt{x^2+1})}{1}\times\frac{(\sqrt{x^2+x+1}+\sqrt{x^2+1})}{(\sqrt{x^2+x+1}+\sqrt{x^2+1})}$$

$$=\lim_{x\to\infty}\frac{(x^2+x+1)-(x^2+1)}{\sqrt{x^2+x+1}+\sqrt{x^2+1}}$$

$$=\lim_{x\to\infty}\frac{x}{\sqrt{x^2+x+1}+\sqrt{x^2+1}}$$

$$=\lim_{x\to\infty}\frac{1}{\sqrt{1+\dfrac{1}{x}+\dfrac{1}{x^2}}+\sqrt{1+\dfrac{1}{x^2}}}=\frac{1}{1+1}=\frac{1}{2}$$

$$\left[\because\frac{1}{x}\to 0,\ \text{as } x\to\infty\right]$$

An Important Result

If m, n are positive integers and $a_0, b_0\neq 0$ and non-zero real numbers, then

$$\lim_{x\to\infty}\frac{a_0x^m+a_1x^{m-1}+\ldots+a_{m-1}x+a_m}{b_0x^n+b_1x^{n-1}+\ldots+b_{n-1}x+b_n}$$

$$=\begin{cases}0, & m<n\\ \dfrac{a_0}{b_0}, & m=n\\ \infty, & m>n\ \text{ when } a_0b_0>0\\ -\infty, & m>n\ \text{ when } a_0b_0<0\end{cases}$$

Example 13 ***Evaluate*** $\lim_{x\to\infty}\dfrac{ax^2+b}{x+1}$, ***when*** $a\geq 0$.

Sol. Here, if $a\neq 0$; $\lim_{x\to\infty}\dfrac{ax^2+b}{x+1}$

[as degree of numerator > degree of denominator]

$$=\lim_{x\to\infty}\frac{ax^2+b}{x+1}=\infty\qquad[\text{as } a>0]$$

Again, if $a=0$; $\lim_{x\to\infty}\dfrac{0\cdot x^2+b}{x+1}=\lim_{x\to\infty}\dfrac{b}{x+1}=0$

[as degree of numerator < degree of denominator]

$$\therefore\quad\lim_{x\to\infty}\frac{ax^2+b}{x+1}=\begin{cases}\infty, & a>0\\ 0, & a=0\end{cases}$$

Example 14 ***If*** $\lim_{x\to\infty}\left(\dfrac{x^2+1}{x+1}-ax-b\right)=0$, ***find the values of*** a ***and*** b.

Sol. Given, $\lim_{x\to\infty}\left(\dfrac{x^2+1}{x+1}-ax-b\right)=0$

$$\Rightarrow\lim_{x\to\infty}\frac{x^2+1-ax^2-ax-bx-b}{x+1}=0$$

$$\Rightarrow\lim_{x\to\infty}\frac{x^2(1-a)-x(a+b)+(1-b)}{x+1}=0$$

Since, the limit of above expression is zero.

$\therefore$ Degree of numerator < Degree of denominator.
So, numerator must be a constant, i.e. a zero degree polynomial.

$\therefore\qquad 1-a=0$ and $a+b=0$

Hence, $\qquad a=1$ and $b=-1$

Example 15 ***If*** $\lim_{x\to\infty}\left(\dfrac{x^2+1}{x+1}-ax-b\right)=2$, ***find the values of*** a ***and*** b.

Sol. We have, $\lim_{x\to\infty}\left(\dfrac{x^2+1}{x+1}-ax-b\right)$

$$=\lim_{x\to\infty}\frac{x^2(1-a)-x(a+b)+(1-b)}{(x+1)}=2$$

Since, limit of above expression is a finite non-zero number.

$\therefore$ Degree of numerator = Degree of denominator

$\Rightarrow \quad 1-a=0 \quad \Rightarrow \quad a=1$

Putting $a=1$ in above limit, we get

$$\lim_{x\to\infty}\frac{-x\ (1+b)+(1+b)}{x+1}=2$$

$\Rightarrow \quad -(1+b)=2 \Rightarrow \ b=-3$

Hence, $a=1$ and $b=-3$.

Example 16 *If* $\lim_{x\to\infty}\left(\frac{x^2+1}{x+1}-ax-b\right)=\infty$, *find* a *and* b.

Sol. Given, $\lim_{x\to\infty}\left(\frac{x^2+1}{x+1}-ax-b\right)=\infty$

$$\Rightarrow \quad \lim_{x\to\infty}\frac{x^2(1-a)-x(a+b)+(1-b)}{x+1}=\infty$$

The limit of above expression is infinity.

$\therefore$ Degree of numerator > Degree of denominator

$\Rightarrow \quad 1-a>0\Rightarrow a\neq 1$

Hence, $a<1$ and b can assume any real value.

Example 17 Let $S_n=1+2+3+\cdots+n$ and $P_n=\frac{S_2}{S_2-1}\cdot\frac{S_3}{S_3-1}\cdot\frac{S_4}{S_4-1}\cdots\frac{S_n}{S_n-1}$, where $n\in N\ (n\geq 2)$. Find $\lim_{n\to\infty}P_n$.

Sol. As, $S_n=\frac{n(n+1)}{2}$

$$\Rightarrow \quad S_n-1=\frac{n(n+1)}{2}-1=\frac{n^2+n-2}{2}=\frac{(n+2)(n-1)}{2}$$

$$\therefore \quad \frac{S_n}{S_n-1}=\left(\frac{n}{n-1}\right)\left(\frac{n+1}{n+2}\right)$$

$$\Rightarrow \quad P_n=\left(\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdots\frac{n}{n-1}\right)\left(\frac{3}{4}\cdot\frac{4}{5}\cdot\frac{5}{6}\cdots\frac{n+1}{n+2}\right)$$

$$=\left(\frac{n}{1}\right)\left(\frac{3}{n+2}\right)$$

$$\therefore \quad \lim_{n\to\infty}P_n=\lim_{n\to\infty}\frac{3n}{n+2}=3$$

Example 18 If $\lim_{n\to\infty}\frac{1}{(\sin^{-1}x)^n+1}=1$, x lies in the interval

(a) $(-\sin 1, \sin 1)$ (b) $(-1, 1)$

(c) $(0, 1)$ (d) $(-1, 0)$

Sol. Here, $\lim_{n\to\infty}\frac{1}{(\sin^{-1}x)^n+1}=1$ is possible only, if

$$-1<\sin^{-1}x<1$$

$$\Rightarrow \quad x\in(-\sin 1, \sin 1)$$

Hence, (a) is the correct answer.

Exercise for Session 1

1. If $f(a)=2, f'(a)=1, g(a)=-1, g'(a)=-2$, then $\lim_{x\to a}\frac{g(x)f(a)-g(a)f(x)}{x-a}$ is

(a) -5 (b) 3 (c) -3 (d) 5

2. The value of $\lim_{x\to 0}\frac{x\cos x-\log(1+x)}{x^2}$ is

(a) 1 (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) None of these

3. The value of $\lim_{x\to 0}\frac{e^{x^2}-\cos x}{x^2}$ is

(a) $\frac{3}{2}$ (b) $-\frac{3}{2}$ (c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

4. The value of $\lim_{x\to a}\frac{\cos x-\cos a}{\cot x-\cot a}$ is

(a) $-\sin^3 a$ (b) $\cos^3 a$ (c) $\sin^3 a$ (d) $\cot a$

5. The value of $\lim_{x \to 0} \left(\frac{1}{x^2} - \cot x \right)$ is

(a) 0 (b) 1

(c) $\frac{1}{4}$ (d) None of these

6. The value of $\lim_{x \to \infty} (\sqrt{a^2x^2 + ax + 1} - \sqrt{a^2x^2 + 1}), (a > 0)$ is

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$

(c) Doesn't exist (d) None of these

7. The value of $\lim_{n \to \infty} \frac{1^3 + 2^3 + 3^3 + \ldots + n^3}{(n^2 + 1)^2}$ is

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$

(c) $\frac{1}{2\sqrt{2}}$ (d) None of these

8. The value of $\lim_{n \to \infty} \frac{1 \cdot n + 2 \cdot (n-1) + 3 \cdot (n-2) + \ldots + n \cdot 1}{1^2 + 2^2 + \ldots + n^2}$ is

(a) 1 (b) −1

(c) $\frac{1}{\sqrt{2}}$ (d) $\frac{1}{2}$

9. The value of $\lim_{n \to \infty} \frac{a^n + b^n}{a^n - b^n}$, (where $a > b > 1$) is

(a) 1 (b) −1

(c) $\frac{1}{2}$ (d) $\frac{1}{\sqrt{2}}$

Session 2

Trigonometric Limits

Trigonometric Limits

To evaluate trigonometric limits the following results are given below

(i) $\lim_{x\to 0}\frac{\sin x}{x}=1$ **(ii)** $\lim_{x\to 0}\frac{\tan x}{x}=1$

(iii) $\lim_{x\to 0}\frac{\sin^{-1} x}{x}=1$ **(iv)** $\lim_{x\to 0}\frac{\tan^{-1} x}{x}=1$

(v) $\lim_{x\to 0}\frac{\sin x^{0}}{x}=\frac{\pi}{180^{\circ}}$ **(vi)** $\lim_{x\to 0}\cos x=1$

(vii) $\lim_{x\to a}\frac{\sin(x-a)}{x-a}=1$ **(viii)** $\lim_{x\to a}\frac{\tan(x-a)}{x-a}=1$

Example 19 Evaluate $\lim_{x\to 0}\frac{1-\cos x}{x^2}$.

Sol. We have, $\lim_{x\to 0}\frac{1-\cos x}{x^2}$

$$=\lim_{x\to 0}\frac{2\sin^2 x/2}{x^2}=\lim_{x\to 0}\frac{2}{4}\frac{\sin^2 x/2}{x^2/4}\qquad\left[\frac{0}{0}\text{ form}\right]$$

$$=\lim_{x\to 0}\frac{1}{2}\left(\frac{\sin x/2}{x/2}\right)^2=\frac{1}{2}(1)^2=\frac{1}{2}$$

Example 20 Solve $\lim_{x\to 0}\frac{1-\cos(1-\cos x)}{\sin^4 x}$

Sol. $\lim_{x\to 0}\frac{1-\cos(1-\cos x)}{\sin^4 x}=\lim_{x\to 0}\frac{1-\cos\left(2\sin^2\frac{x}{2}\right)}{x^4}\Big/\frac{\sin^4 x}{x^4}$

$$=\lim_{x\to 0}\frac{1-\cos\left(2\sin^2\frac{x}{2}\right)}{x^4}\Big/\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^4$$

$$=\lim_{x\to 0}\frac{2\sin^2\left(\sin^2\frac{x}{2}\right)}{x^4}\Big/1$$

$$=\lim_{x\to 0}2\cdot\left(\frac{\sin\left(\sin^2\frac{x}{2}\right)}{\sin^2\frac{x}{2}}\cdot\frac{\sin^2\frac{x}{2}}{4\cdot\frac{x^2}{4}}\right)^2$$

$$=2\times\frac{1}{4^2}=\frac{1}{8}$$

Example 21 ***Evaluate*** $\lim_{x\to\infty}2^{-x}\sin(2^x)$.

Sol. Since, $2^{-x}=\frac{1}{2^x}$. We know that, as $x\to\infty, 2^x\to\infty$

$\therefore$ The given limit $= 0\times$

[A finite number between -1 and $+1$] $=0$

Hence, $\lim_{x\to\infty}\frac{\sin(2^x)}{(2^x)}=0$

Example 22 Evaluate $\lim_{x\to\infty}e^x\sin(d/e^x)$.

Sol. When $x\to\infty, e^x\to\infty$

But, angle of sine $=\frac{d}{e^x}=\frac{\text{finite}}{\infty}=0$

$\therefore$ The given limit $=\lim_{x\to\infty}\frac{\sin(d/e^x)}{1/e^x}=\lim_{\frac{d}{e^x}\to 0}\frac{\sin d/e^x}{d/e^x}\times d$

$$=1\times d=d$$

Example 23 Evaluate $\lim_{x\to\infty}\sqrt{\frac{x-\sin x}{x+\cos^2 x}}$.

Sol. We have, $\lim_{x\to\infty}\sqrt{\frac{x-\sin x}{x+\cos^2 x}}=\lim_{x\to\infty}\sqrt{\frac{x\left(1-\frac{\sin x}{x}\right)}{x\left(1+\frac{\cos^2 x}{x}\right)}}$

$$=\lim_{x\to\infty}\sqrt{\frac{1-\frac{\sin x}{x}}{1+\frac{\cos^2 x}{x}}}=\sqrt{\frac{1-0}{1+0}}=1$$

Example 24 ***Evaluate*** $\lim_{x\to y}\frac{\sin^2 x-\sin^2 y}{x^2-y^2}$.

Sol. $\lim_{x\to y}\frac{\sin^2 x-\sin^2 y}{x^2-y^2}=\lim_{x\to y}\frac{\sin(x+y)\sin(x-y)}{(x+y)(x-y)}$

$$=\lim_{x\to y}\frac{\sin(x+y)}{(x+y)}\times\lim_{x\to y}\frac{\sin(x-y)}{(x-y)}=\frac{\sin(2y)}{2y}\times 1$$

[as $x\to y\ \Rightarrow (x-y)\to 0$, but $x+y\to 2y$]

$$=\frac{\sin 2y}{2y}$$

Example 25

$\lim_{x\to\infty}[(x+5)\tan^{-1}(x+5)-(x+1)\tan^{-1}(x+1)]$ ***is equal to***

(a) π (b) 2π

(c) $\frac{\pi}{2}$ (d) None of these

Sol. Here, $\lim_{x\to\infty}((x+5)\tan^{-1}(x+5)-(x+1)\tan^{-1}(x+1))$

$$=\lim_{x\to\infty}(x+5)\left\{\frac{\pi}{2}-\cot^{-1}(x+5)\right\}-(x+1)\left\{\frac{\pi}{2}-\cot^{-1}(x+1)\right\}$$

$$=\lim_{x\to\infty}2\pi-\frac{\tan^{-1}\left(\frac{1}{x+5}\right)}{\left(\frac{1}{x+5}\right)}+\frac{\tan^{-1}\left(\frac{1}{x+1}\right)}{\left(\frac{1}{x+1}\right)}=2\pi$$

Hence, (b) is the correct answer.

Example 26 Evaluate $\lim_{x\to 0}\frac{\sin(\pi\cos^2 x)}{x^2}$.

[IIT JEE 2001]

Sol. $\lim_{x\to 0}\frac{\sin(\pi\cos^2 x)}{x^2}=\lim_{x\to 0}\frac{\sin\{\pi(1-\sin^2 x)\}}{x^2}$

$$=\lim_{x\to 0}\frac{\sin(\pi-\pi\sin^2 x)}{x^2}$$

$$=\lim_{x\to 0}\frac{\sin(\pi\sin^2 x)}{x^2}=\lim_{x\to 0}\left\{\frac{\sin(\pi\sin^2 x)}{\pi\sin^2 x}\times\frac{\pi}{1}\times\frac{\sin^2 x}{x^2}\right\}$$

$$=\lim_{x\to 0}\frac{\sin(\pi\sin^2 x)}{\pi\sin^2 x}\times\pi\times\lim_{x\to 0}\frac{\sin^2 x}{x^2}$$

$$=1\times\pi\times 1=\pi$$

Example 27 Let $a=\min\{x^2+2x+3, x\in R\}$ and $b=\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2}$. The value of $\sum_{r=0}^{n}a^r\cdot b^{n-r}$ is

Sol. Here, $a=\min\{x^2+2x+3, x\in R\}$

i.e. $x^2+2x+3=x^2+2x+1+2$

$$=(x+1)^2+2$$

$\therefore$ a is minimum, when $x=-1$, i.e. $a=2$

Again, $b=\lim_{\theta\to 0}\frac{2\sin^2\theta/2}{\theta^2}=\lim_{\theta\to 0}\frac{2}{4}\cdot\frac{\sin^2\theta/2}{\theta^2/4}$

$$=\lim_{\theta\to 0}\frac{2}{4}\cdot\frac{\sin^2\theta/2}{(\theta/2)^2}=\frac{1}{2}$$

Hence, $\sum_{r=0}^{n}a^r\,b^{n-r}=\sum_{r=0}^{n}2^r\cdot\left(\frac{1}{2}\right)^{n-r}$

$\Rightarrow$ $\frac{1}{2^n}\{2^0+2^2+2^4+\ldots+2^{2n}\}$

$\Rightarrow$ $\frac{1}{2^n}\left\{\frac{1(4^{n+1}-1)}{4-1}\right\}=\frac{4^{n+1}-1}{2^n\times 3}$

[i.e. sum of $(n+1)$ terms of GP]

$\therefore$ $\sum_{r=0}^{n}a^r\,b^{n-r}=\frac{4^{n+1}-1}{2^n\times 3}$

Example 28 Evaluate

$$\lim_{n\to\infty}\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{8}\right)\ldots\cos\left(\frac{x}{2^n}\right).$$

Sol. Here, $\lim_{n\to\infty}\cos\left(\frac{x}{2^n}\right)\cos\left(\frac{x}{2^{n-1}}\right)\ldots\cos\left(\frac{x}{8}\right)\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{2}\right)$

We know, $\cos A\cos 2A\cos 2^2A\ldots\cos 2^{n-1}A=\frac{\sin 2^nA}{2^n\sin A}$

Thus, $\lim_{n\to\infty}\cos\left(\frac{x}{2^n}\right)\cos\left(\frac{x}{2^{n-1}}\right)\ldots\cos\left(\frac{x}{8}\right)\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{2}\right)$

$$=\lim_{n\to\infty}\frac{\sin 2^n\left(\frac{x}{2^n}\right)}{2^n\sin\left(\frac{x}{2^n}\right)}=\lim_{n\to\infty}\frac{\sin x}{2^n\sin\left(\frac{x}{2^n}\right)}$$

$$=\sin x\cdot\lim_{n\to\infty}\frac{1}{2^n\sin\left(\frac{x}{2^n}\right)}\cdot\frac{x}{x}$$

$$=\frac{\sin x}{x}\cdot\lim_{n\to\infty}\frac{1}{2^n\sin\left(\frac{x}{2^n}\right)}\cdot x$$

$$=\frac{\sin x}{x}\cdot 1=\frac{\sin x}{x}\left[n\to\infty,\frac{x}{2^n}\to 0\text{ and }\lim_{\theta\to 0}\frac{\theta}{\sin\theta}=1\right]$$

Exercise for Session 2

1. If $\lim_{x \to 0} (x^{-3} \sin 3x + ax^{-2} + b)$ exists and is equal to zero, then

(a) $a = -3,\ b = \frac{9}{2}$ (b) $a = 3, b = \frac{9}{2}$

(c) $a = -3, b = \frac{-9}{2}$ (d) None of these

2. The value of $\lim_{x \to a} \frac{x \sin a - a \sin x}{x - a}$ is

(a) $a \sin a - \cos a$ (b) $\sin a - a \cos a$

(c) $\cos a + a \sin a$ (d) $\sin a + a \cos a$

3. The value of $\lim_{x \to \pi} \frac{\sqrt{2 + \cos x} - 1}{(\pi - x)^2}$ is

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$

(c) 2 (d) Doesn't exist

4. The value of $\lim_{\theta \to \pi/4} \frac{(\sqrt{2} - \cos\theta - \sin\theta)}{(4\theta - \pi)^2}$ is

(a) $\frac{1}{16\sqrt{2}}$ (b) $\frac{1}{16}$

(c) $\frac{1}{8\sqrt{2}}$ (d) $\frac{1}{2\sqrt{2}}$

5. The value of $\lim_{x \to \pi/4} \frac{(\cos x + \sin x)^3 - 2\sqrt{2}}{1 - \sin 2x}$ is

(a) $\frac{\sqrt{3}}{2}$ (b) $-\frac{3}{\sqrt{2}}$

(c) $\frac{1}{2}$ (d) $-\frac{1}{\sqrt{2}}$

Session 3

Logarithmic Limits, Exponential Limits

Logarithmic Limits

In this section, we will deal with the problems based on expansion of logarithmic series, which is given below

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots \infty$$

where, $-1 \le x \le 1$ and it should be noted that the expansion is true only if the base is e. To evaluate the logarithmic limit, we use $\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$.

Example 29 Evaluate $\lim_{x \to a} \frac{\log\{1+(x-a)\}}{(x-a)}$.

Sol. We have, $\lim_{x \to a} \frac{\log\{1+(x-a)\}}{(x-a)}$

Let $x - a = y$, when $x \to a$; $y \to 0$

$\therefore$ The given limit $= \lim_{y \to 0} \frac{\log\{1+y\}}{y} = 1$

Example 30 ***Evaluate*** $\lim_{h \to 0} \frac{\log_{10}(1+h)}{h}$.

Sol. We have, $\lim_{h \to 0} \frac{\log_{10}(1+h)}{h} = \lim_{h \to 0} \frac{\log_e(1+h) \times \log_{10} e}{h}$

$$= \lim_{h \to 0} \frac{\log_e(1+h)}{h} \times \log_{10} e$$

$$= \log_{10} e \times 1 = \log_{10} e \qquad \left[\because \lim_{x \to 0} \frac{\log(1+x)}{x} = 1\right]$$

Example 31 ***Evaluate*** $\lim_{x \to 0} \frac{\log(5+x) - \log(5-x)}{x}$.

Sol. We have, $\lim_{x \to 0} \frac{\log\left\{5\left(1+\frac{x}{5}\right)\right\} - \log\left\{5\left(1-\frac{x}{5}\right)\right\}}{x}$

$$= \lim_{x \to 0} \frac{\log 5 + \log\left(1+\frac{x}{5}\right) - \log 5 - \log\left(1-\frac{x}{5}\right)}{x}$$

$$= \lim_{x \to 0} \frac{\log\left(1+\frac{x}{5}\right)}{5\left(\frac{x}{5}\right)} - \frac{\log\left(1-\frac{x}{5}\right)}{-5\left(-\frac{x}{5}\right)} = \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$$

$$\left[\because \lim_{x \to 0} \frac{\log(1+x)}{x} = 1\right]$$

Example 32 Evaluate $\lim_{h \to 0} \frac{\log_e(1+2h) - 2\log_e(1+h)}{h^2}$.

[IIT JEE 1999]

Sol. We have, $\lim_{h \to 0} \frac{\log_e(1+2h) - 2\log_e(1+h)}{h^2}$

$$= \lim_{h \to 0} \frac{\left[(2h) - \frac{(2h)^2}{2} + \frac{(2h)^3}{3} - \ldots \infty\right] - 2\left(h - \frac{h^2}{2} + \frac{h^3}{3} - \ldots\right)}{h^2}$$

$$= \lim_{h \to 0} \frac{-h^2 + 2h^3 - \ldots}{h^2}$$

$$= \lim_{h \to 0} \frac{h^2\{-1 + 2h - \ldots\}}{h^2}$$

$$= \lim_{h \to 0} \{-1 + 2h - \ldots\} = -1$$

Example 33 ***Solve*** $\lim_{x \to \infty} \left\{x - x^2 \cdot \log\left(1+\frac{1}{x}\right)\right\}$.

Sol. Here, put $x = \frac{1}{y}$ in $\lim_{x \to \infty} \left\{x - x^2 \cdot \log\left(1+\frac{1}{x}\right)\right\}$

$$= \lim_{y \to 0} \left\{\frac{1}{y} - \frac{\log(1+y)}{y^2}\right\}$$

$$= \lim_{y \to 0} \frac{y - \log(1+y)}{y^2}$$

$$= \lim_{y \to 0} \frac{y - \left\{y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \ldots\right\}}{y^2}$$

$$= \lim_{y \to 0} \frac{y^2\left\{\frac{1}{2} - \frac{y}{3} + \ldots\right\}}{y^2}$$

$$= \lim_{y \to 0} \left\{\frac{1}{2} - \frac{y}{3} + \ldots\right\} = \frac{1}{2}$$

Remark

If we solve above question as

$$\lim_{y \to 0} \left\{\frac{1}{y} - \frac{\log(1+y)}{y^2}\right\} = \lim_{y \to 0} \left\{\frac{1}{y} - \frac{\log(1+y)}{y} \cdot \frac{1}{y}\right\}$$

$$\left[\text{as } \lim_{y \to 0} \frac{\log(1+y)}{y} = 1\right]$$

$$= \lim_{y \to 0} \left(\frac{1}{y} - \frac{1}{y}\right) = 0, \text{ then it is not correct.}$$

Exponential Limits

There are two types of exponential limits discussed below

(i) Based on Series Expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \infty$$

To evaluate the exponential limit, we use the following results.

(a) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$ **(b)** $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$

Example 34 Evaluate $\lim_{x \to 0} \frac{a^x - b^x}{x}$.

Sol. We have, $\lim_{x \to 0} \frac{a^x - b^x}{x} = \lim_{x \to 0} \frac{(a^x - 1) - (b^x - 1)}{x}$

$= \lim_{x \to 0} \frac{a^x - 1}{x} - \lim_{x \to 0} \frac{b^x - 1}{x} = \log a - \log b = \log (a / b)$

Example 35 Evaluate $\lim_{x \to 0} \frac{(ab)^x - a^x - b^x + 1}{x^2}$.

Sol. We have, $\lim_{x \to 0} \frac{(ab)^x - a^x - b^x + 1}{x^2} = \lim_{x \to 0} \frac{a^x b^x - a^x - b^x + 1}{x^2}$

$= \lim_{x \to 0} \frac{a^x (b^x - 1) - (b^x - 1)}{x^2}$

$= \lim_{x \to 0} \frac{(a^x - 1)}{x} \times \lim_{x \to 0} \frac{(b^x - 1)}{x} = \log a \times \log b$

$\left[\because \lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a \right]$

Example 36 Evaluate $\lim_{x \to 0} \frac{e^{\tan x} - e^x}{\tan x - x}$.

Sol. We have, $\lim_{x \to 0} \frac{e^{\tan x} - e^x}{\tan x - x} = \lim_{x \to 0} \frac{e^x \times e^{(\tan x - x)} - e^x}{(\tan x - x)}$

$= \lim_{x \to 0} \frac{e^x \{e^{\tan x - x} - 1\}}{(\tan x - x)}$

$= e^0 \times 1$ [as $x \to 0, \tan x - x \to 0$]

$= 1 \times 1 = 1$

Example 37 Evaluate $\lim_{x \to 0} \frac{ae^x - b}{x} = 2$. Find a and b.

Sol. Given, $\lim_{x \to 0} \frac{ae^x - b}{x} = \lim_{x \to 0} \frac{a\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots \infty\right) - b}{x} = 2$

$\Rightarrow \lim_{x \to 0} \frac{(a - b) + xa + \frac{ax^2}{2!} + \ldots \infty}{x} = 2$

Since, limit is finite, so $(a - b) = 0 \Rightarrow b = a$

$\therefore \lim_{x \to 0} \frac{xa + \frac{ax^2}{2!} + \ldots \infty}{x} = 2$

$\Rightarrow \lim_{x \to 0} a + \frac{ax}{2!} + \ldots \infty = 2 \Rightarrow a = 2$

$\therefore b = 2$

Example 38 Solve $\lim_{x \to 0} \frac{a \sin x - bx + cx^2 + x^3}{2x^2 \log (1 + x) - 2x^3 + x^4}$, if it exists and is finite, also find a, b and c.

Sol. Let $L = \lim_{x \to 0} \frac{a \sin x - bx + cx^2 + x^3}{2x^2 \log (1 + x) - 2x^3 + x^4}$

$= \lim_{x \to 0} \frac{a\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right) - bx + cx^2 + x^3}{2x^2\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right) - 2x^3 + x^4}$

$= \lim_{x \to 0} \frac{(a - b)x + cx^2 + \left(1 - \frac{a}{3!}\right)x^3 + \frac{a}{5!}x^5 + \ldots}{\frac{2}{3}x^5 - \frac{1}{2}x^6 + \ldots}$

For finite limit, $a - b = 0, c = 0, 1 - \frac{a}{3!} = 0$, i.e. $a = b = 6, c = 0$]

$\therefore L = \lim_{x \to 0} \frac{\frac{a}{5!} + \text{higher powers of } x}{\frac{2}{3} - \frac{1}{2}x + \cdots}$

$= \frac{a}{5!} \cdot \frac{3}{2} = \frac{a}{80} = \frac{6}{80} = \frac{3}{40}$ [$\because a = 6$]

So, $\lim_{x \to 0} \frac{a \sin x - bx + cx^2 + x^3}{2x^2 \log (1 + x) - 2x^3 + x^4} = \frac{3}{40}$

Where, $a = 6 = b, c = 0$

Example 39 Find the values of a, b and c such that

$$\lim_{x \to 0} \frac{axe^x - b \log (1 + x) + cxe^{-x}}{x^2 \sin x} = 2.$$

Sol. We have, $\lim_{x \to 0} \frac{axe^x - b \log (1 + x) + c\, xe^{-x}}{x^2 \sin x} = 2$

Using the expansion, we have

$$\lim_{x \to 0} \frac{\left[ax\left(1 + x + \frac{x^2}{2!} + \ldots\right) - b\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\right) + cx\left(1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\right) \right]}{x^2\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)} = 2$$

$$\lim_{x\to 0}\frac{x(a-b+c)-x^2\left(a+\frac{b}{2}-c\right)+x^3\left(\frac{a}{2}-\frac{b}{3}+\frac{c}{2}\right)+\ldots}{x^2\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots\right)}=2$$

Now, above limit would exist, if least power in numerator is greater than or equal to least power in denominator, i.e. Coefficient of x and x^2 must be zero and coefficient of x^3 should be 2.

i.e. $a-b+c=0, a+\frac{b}{2}-c=0, \quad \frac{a}{2}-\frac{b}{3}+\frac{c}{2}=2$

On solving, we get $a=3, b=12, c=9$.

(ii) Evaluation of Exponential Limits of the Form 1^∞

To evaluate the exponential limits of the form 1^∞, we use the following results.

If $\lim_{x\to a} f(x)=\lim_{x\to a} g(x)=0$,

then $\lim_{x\to a}\{1+f(x)\}^{1/g(x)}=e^{\lim_{x\to a}\frac{f(x)}{g(x)}}$

or when $\lim_{x\to a} f(x)=1$ and $\lim_{x\to a} g(x)=\infty$

Then, $\lim_{x\to a}\{f(x)\}^{g(x)}=\lim_{x\to a}[1+f(x)-1]^{g(x)}$

$$=e^{\lim_{x\to a}(f(x)-1)\,g(x)}$$

Particular Cases

(i) $\lim_{x\to 0}(1+x)^{1/x}=e$ (ii) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$

(iii) $\lim_{x\to 0}(1+\lambda x)^{1/x}=e^\lambda$ (iv) $\lim_{x\to\infty}\left(1+\frac{\lambda}{x}\right)^x=e^\lambda$

Example 40 Evaluate $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$.

Sol. We have, $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x=e^{\lim_{x\to\infty}\frac{2}{x}x}=e^2$ [1^∞ form]

Example 41 Evaluate $\lim_{x\to 1}(\log_3 3x)^{\log_x 3}$.

Sol. We have, $\lim_{x\to 1}(\log_3 3x)^{\log_x 3}=\lim_{x\to 1}(\log_3 3+\log_3 x)^{\log_x 3}$

$$=\lim_{x\to 1}(1+\log_3 x)^{1/\log_3 x}\qquad\left[\text{as }\log_b a=\frac{1}{\log_a b}\right]$$

$$=e^{\lim_{x\to 1}\log_3 x\times\frac{1}{\log_3 x}}$$

$$=e^1$$

Example 42 ***Evaluate*** $\lim_{x\to a}\left(2-\frac{a}{x}\right)^{\tan\frac{\pi x}{2a}}$.

Sol. We have, $\lim_{x\to a}\left(2-\frac{a}{x}\right)^{\tan\frac{\pi x}{2a}}$

$$=\lim_{x\to a}\left\{1+\left(1-\frac{a}{x}\right)\right\}^{\tan\frac{\pi x}{2a}}=e^{\lim_{x\to a}\left(1-\frac{a}{x}\right)\cdot\tan\frac{\pi x}{2a}}\qquad[1^\infty\text{ form}]$$

$$=e^{\lim_{x\to a}\left(\frac{x-a}{x}\right)\cdot\tan\frac{\pi x}{2a}}=e^{\lim_{x\to a}f(x)}\qquad\ldots(i)$$

Let $x-a=h$, we get

$$e^{\lim_{h\to 0}\left(\frac{h}{a+h}\right)\cdot\tan\frac{\pi}{2a}(a+h)}=e^{\lim_{h\to 0}\frac{h}{a+h}\cdot\tan\left(\frac{\pi}{2}+\frac{\pi h}{2a}\right)}$$

$$=e^{\lim_{h\to 0}\frac{h}{a+h}\left[-\cot\left(\frac{\pi h}{2a}\right)\right]}=e^{\lim_{h\to 0}\frac{-h}{(a+h)\tan(\pi h/2a)}\cdot\frac{\pi}{2a}\cdot\frac{1}{\pi/2a}}$$

$$=e^{\lim_{h\to 0}\frac{-2a}{\pi(a+h)}\cdot\frac{\pi h/2a}{\tan(\pi h/2a)}}=e^{\frac{-2a}{\pi(a)}}=e^{-2/\pi}\qquad\left[\text{as }\lim_{x\to 0}\frac{x}{\tan x}=1\right]$$

Example 43 Evaluate $\lim_{x\to\infty}\left(\frac{x+6}{x+1}\right)^{x+4}$.

Sol. We have, $\lim_{x\to\infty}\left(\frac{x+6}{x+1}\right)^{x+4}$

As $x\to\infty$, $\lim_{x\to\infty}\frac{x+6}{x+1}=1$ and $(x+4)\to\infty$ [1^∞ form]

$$\therefore\ \lim_{x\to\infty}\left(\frac{x+6}{x+1}\right)^{x+4}=\lim_{x\to\infty}\left[1+\left(\frac{x+6}{x+1}-1\right)\right]^{x+4}$$

$$=\lim_{x\to\infty}\left[1+\frac{5}{x+1}\right]^{x+4}=e^{\lim_{x\to\infty}\left(\frac{5}{x+1}\right)\cdot(x+4)}$$

$$=e^{\lim_{x\to\infty}5\cdot\frac{x+4}{x+1}}=e^{5\cdot(1)}=e^5\qquad\left[\text{as }x\to\infty;\ \lim_{x\to\infty}\frac{x+4}{x+1}=1\right]$$

Example 44 Evaluate $\lim_{x\to 0}(1+\tan^2\sqrt{x})^{\frac{1}{2x}}$.

Sol. We have, $\lim_{x\to 0}(1+\tan^2\sqrt{x})^{\frac{1}{2x}}=e^{\lim_{x\to 0}\tan^2\sqrt{x}\cdot\frac{1}{2x}}$

$$=e^{\frac{1}{2}\lim_{x\to 0}\left(\frac{\tan\sqrt{x}}{\sqrt{x}}\right)^2}=e^{1/2}\qquad\left[\text{as }\lim_{x\to 0}\frac{\tan x}{x}=1\right]$$

Example 45 ***Evaluate*** $\lim_{x\to 0}\left\{\tan\left(\frac{\pi}{4}+x\right)\right\}^{\frac{1}{x}}$.

[IIT JEE 1993]

Sol. We have, $\lim_{x\to 0}\left\{\tan\left(\frac{\pi}{4}+x\right)\right\}^{\frac{1}{x}}=\lim_{x\to 0}\left\{\frac{\tan\frac{\pi}{4}+\tan x}{1-\tan\frac{\pi}{4}\tan x}\right\}^{\frac{1}{x}}$

$$= \lim_{x\to 0}\left\{\frac{1+\tan x}{1-\tan x}\right\}^{\frac{1}{x}} = \lim_{x\to 0}\left\{1+\frac{2\tan x}{1-\tan x}\right\}^{\frac{1}{x}}$$

$$= e^{\lim_{x\to 0}\left(\frac{2\tan x}{1-\tan x}\right)\cdot\frac{1}{x}} = e^{\lim_{x\to 0} 2\cdot\frac{\tan x}{x}\cdot\frac{1}{1-\tan x}} = e^2$$

Example 46 Evaluate $\lim_{x\to 0}\left(\frac{1+5x^2}{1+3x^2}\right)^{1/x^2}$. *[IIT JEE 1996]*

Sol. We have, $\lim_{x\to 0}\left(\frac{1+5x^2}{1+3x^2}\right)^{1/x^2} = \lim_{x\to 0}\left(1+\frac{2x^2}{1+3x^2}\right)^{1/x^2}$ [1^∞ form]

$$= e^{\lim_{x\to 0}\left(\frac{2x^2}{1+3x^2}\right)\cdot\frac{1}{x^2}} = e^{\frac{2}{1+0}} = e^2$$

Example 47 ***Evaluate*** $\lim_{x\to\infty}\left(\frac{x-3}{x+2}\right)^x$. *[IIT JEE 2000]*

Sol. We have, $\lim_{x\to\infty}\left(\frac{x-3}{x+2}\right)^x = \lim_{x\to\infty}\left(1+\frac{(-5)}{x+2}\right)^x$

$$= e^{\lim_{x\to\infty}\left(\frac{-5}{x+2}\right)\cdot x} \qquad = e^{\lim_{x\to\infty}\frac{-5}{1+\frac{2}{x}}} = e^{-5}$$

Example 48 The value of

$$\lim_{x\to 0}\left\{\sin^2\left(\frac{\pi}{2-ax}\right)\right\}^{\sec^2\frac{\pi}{2-bx}}$$ is equal to

(a) $e^{-a/b}$ (b) e^{-a^2/b^2} (c) $a^{2a/b}$ (d) $e^{4a/b}$

Sol. Here, $\lim_{x\to 0}\left(\sin^2\left(\frac{\pi}{2-ax}\right)\right)^{\sec^2\left(\frac{\pi}{2-bx}\right)}$

$$= \lim_{x\to 0}\left\{1-\cos^2\left(\frac{\pi}{2-ax}\right)\right\}^{\sec^2\left(\frac{\pi}{2-bx}\right)}$$ [1^∞ form]

$$= e^{\lim_{x\to 0}-\left\{\cos^2\left(\frac{\pi}{2-ax}\right)\cdot\frac{1}{\cos^2\left(\frac{\pi}{2-bx}\right)}\right\}}$$

$$= e^{\lim_{x\to 0}-\left\{\frac{2\sin\left(\frac{\pi}{2-ax}\right)\cos\left(\frac{\pi}{2-ax}\right)}{2\sin\left(\frac{\pi}{2-bx}\right)\cos\left(\frac{\pi}{2-bx}\right)}\times\frac{\pi a}{(2-ax)^2}\cdot\frac{}{}\times\frac{\pi b}{(2-bx)^2}\right\}}$$

[using L'Hospital's rule]

$$= e^{-\lim_{x\to 0}\frac{\sin\left(\frac{2\pi}{2-ax}\right)}{\sin\left(\frac{2\pi}{2-bx}\right)}\cdot\frac{a}{b}\cdot\frac{(2-bx)^2}{(2-ax)^2}} = e^{-\lim_{x\to 0}\frac{a}{b}\cdot\frac{(2-bx)^3}{(2-ax)^3}} = e^{-\frac{a}{b}}$$

Hence, (a) is the correct answer.

Example 49 ***The value of*** $\lim_{x\to 7/2}(2x^2-9x+8)^{\cot(2x-7)}$ ***is equal to***

(a) $e^{5/2}$ (b) $e^{-5/2}$
(c) $e^{7/2}$ (d) $e^{3/2}$

Sol. Here, $\lim_{x\to 7/2}(2x^2-9x+8)^{\cot(2x-7)}$ [1^∞ form]

$$= \lim_{x\to 7/2}\{1+(2x^2-9x+7)\}^{\cot(2x-7)}$$

$$= e^{\lim_{x\to 7/2}(2x^2-9x+7)\cdot\cot(2x-7)}$$

$$= e^{\lim_{x\to 7/2}\frac{4x-9}{\sec^2(2x-7)\cdot 2}} = e^{5/2}$$

Hence, (a) is the correct answer.

Example 50 ***The value of*** $\lim_{x\to 1}\left(\tan\left(\frac{\pi}{4}+\log x\right)\right)^{\frac{1}{\log x}}$ ***is equal to***

(a) e (b) e^{-1}
(c) e^2 (d) e^{-2}

Sol. Here, $\lim_{x\to 1}\left(\tan\left(\frac{\pi}{4}+\log x\right)\right)^{\frac{1}{\log x}}$ [1^∞ form]

$$= \lim_{x\to 1}\left(1+\frac{2\tan(\log x)}{1-\tan(\log x)}\right)^{\frac{1}{\log x}} = e^{\lim_{x\to 1}\frac{2\tan(\log x)}{\{1-\tan(\log x)\}}\cdot\frac{1}{\log x}}$$

$$= e^{2\lim_{x\to 1}\frac{\tan(\log x)}{\log x}\cdot\frac{1}{1-\tan(\log x)}} = e^{2\cdot(1)} = e^2$$

[as $x\to 1,\ \log x\to 0$]

Hence, (c) is the correct answer.

Example 51 ***The value of*** $\lim_{x\to 0}\left(\sin\frac{x}{m}+\cos\frac{3x}{m}\right)^{\frac{2m}{x}}$ ***is equal to***

(a) e (b) 1 (c) e^{-1} (d) e^2

Sol. Here, $\lim_{x\to 0}\left(\sin\frac{x}{m}+\cos\frac{3x}{m}\right)^{\frac{2m}{x}}$

$$= \lim_{x\to 0}\left\{1+\left(\sin\frac{x}{m}+\cos\frac{3x}{m}-1\right)\right\}^{\frac{2m}{x}}$$ [1^∞ form]

$$= e^{\lim_{x\to 0}\frac{2m}{x}\left(\sin\frac{x}{m}+\cos\frac{3x}{m}-1\right)}$$

$$= e^{\lim_{x\to 0}\frac{2m\left(\frac{1}{m}\cos\left(\frac{x}{m}\right)-\frac{3}{m}\sin\frac{3x}{m}\right)}{1}} = e^{2m\left(\frac{1}{m}\right)} = e^2$$

Hence, (d) is the correct answer.

Example 52 The value of $\lim_{n\to\infty}\left(\frac{a-1+\sqrt[n]{b}}{a}\right)^n$ $(a>0, b>0)$ is equal to

(a) $\sqrt[q]{b}$ (b) $\sqrt[b]{a}$ (c) $\sqrt{b}$ (d) $\sqrt{a}$

Sol. Here, $\lim_{n\to\infty}\left(\frac{a-1+\sqrt[n]{b}}{a}\right)^n = \lim_{n\to\infty}\left(1+\frac{\sqrt[n]{b}-1}{a}\right)^n$

$= e^{\lim_{n\to\infty}\left(\frac{\sqrt[n]{b}-1}{a}\right)\cdot n}$ [1^∞ form]

$= e^{\lim_{n\to\infty}\frac{1}{a}\cdot\frac{b^{1/n}-1}{1/n}} = e^{\lim_{n\to\infty}\frac{1}{a}\cdot\frac{b^{1/n}\log b\left(\frac{-1}{n^2}\right)}{-1/n^2}}$

$= e^{\frac{1}{a}\log_e b} = e^{\log_e b^{1/a}} = b^{1/a}$

Hence, (a) is the correct answer.

Exercise for Session 3

1. The value of $\lim_{x\to 0}\frac{e^x - e^{x\cos x}}{x+\sin x}$ is

(a) 0 (b) 1 (c) −1 (d) None of these

2. The value of $\lim_{x\to y}\frac{x^y - y^x}{x^x - y^y}$ is

(a) $\frac{1-\log x}{1+\log x}$ (b) $\frac{1-\log y}{1+\log y}$ (c) $\frac{\log x - \log y}{\log x + \log y}$ (d) None of these

3. The value of $\lim_{x\to 0}\frac{p^x - q^x}{r^x - s^x}$ is

(a) $\frac{1-\log p}{1+\log p}$ (b) $\frac{\log p - \log q}{\log r - \log s}$ (c) $\frac{\log p\cdot\log q}{\log r\cdot\log s}$ (d) None of these

4. The value of $\lim_{x\to\infty}(x+2)\tan^{-1}(x+2) - (x\tan^{-1}x)$ is

(a) $\frac{\pi}{2}$ (b) Doesn't exist (c) $\frac{\pi}{4}$ (d) None of these

5. The value of $\lim_{x\to 4}\frac{(\cos\alpha)^x - (\sin\alpha)^x - \cos 2\alpha}{x-4}$, $\alpha\in\left(0,\frac{\pi}{2}\right)$ is

(a) $\log(\cos\alpha) + (\sin\alpha)^4\log(\sin\alpha)$ (b) $(\cos^4\alpha)\log(\cos\alpha) - (\sin\alpha)^4\log(\sin\alpha)$

(c) $(\cos^4\alpha)\log(\cos\alpha)$ (d) None of these

6. The value of $\lim_{x\to 0}\left(\frac{1^x + 2^x + 3^x + \ldots + n^x}{n}\right)^{a/x}$ is

(a) $(n!)^{a/n}$ (b) $n!$ (c) $a^{n!}$ (d) Doesn't exist

7. If $\lim_{x\to 0}(1+ax+bx^2)^{2/x} = e^3$, the values of a and b are

(a) $a=\frac{3}{2}, b\in R$ (b) $a=\frac{1}{2}, b\in R$ (c) $a\in R, b\in R$ (d) None of these

8. If α and β are roots of $ax^2+bx+c=0$, the value of $\lim_{x\to\alpha}(1+ax^2+bx+c)^{2/x-\alpha}$ is

(a) $e^{2a(\alpha-\beta)}$ (b) $e^{a(\alpha-\beta)}$ (c) $e^{\frac{2a}{3}(\alpha-\beta)}$ (d) None of these

9. The value of $\lim_{n\to\infty}((15)^n + [(1+0.0001)^{10000}]^n)^{1/n}$, where [·] denotes the greatest integer function, is

(a) 1 (b) $\frac{1}{2}$ (c) Doesn't exist (d) 2

10. The value of $\lim_{x\to 0}|x|^{[\cos x]}$, where $[\cdot]$ denotes the greatest integer function, is

(a) 0 (b) Doesn't exist (c) 1 (d) None of these

Session 4

Miscellaneous Forms

Miscellaneous Forms

(i) 0^0 Form

When $\lim_{x \to a} f(x) \neq 1$ but $f(x)$ is positive in the neighbourhood of $x = a$.

In this case, we write $\{f(x)\}^{g(x)} = e^{\log_e \{f(x)\}^{g(x)}}$

$$\Rightarrow \quad \lim_{x \to a} [f(x)]^{g(x)} = e^{\lim_{x \to a} g(x) \log_e f(x)}$$

Example 53 Evaluate $\lim_{x \to 0} |x|^{\sin x}$.

Sol. $\lim_{x \to 0} |x|^{\sin x} = \lim_{x \to 0} e^{\sin x \log_e |x|} = e^{\lim_{x \to 0} \frac{\log_e |x|}{\operatorname{cosec} x}}$ [0° form]

$= e^{\lim_{x \to 0} \frac{1/x}{-\operatorname{cosec} x \cot x}}$ [by L' Hospital's rule]

$= e^{\lim_{x \to 0} -\frac{\sin^2 x}{x \cos x}} = e^{\lim_{x \to 0} -\left(\frac{\sin x}{x}\right)^2 \cdot \left(\frac{x}{\cos x}\right)}$

$= e^{-(1)^2 \cdot (0)} = e^0 = 1$

Example 54 Evaluate $\lim_{n \to \infty} (\pi n)^{2/n}$.

Sol. Let $A = \lim_{n \to \infty} (\pi n)^{2/n}$ [∞^0 form]

$\therefore \quad \log A = \lim_{n \to \infty} \frac{2 \log (\pi n)}{n}$ $\left[\frac{\infty}{\infty} \text{ form}\right]$

$= \lim_{n \to \infty} \frac{2 \cdot \frac{1}{\pi n} \cdot \pi}{1}$ [by L'Hospital's rule]

$= \lim_{n \to \infty} \frac{2}{n} = 0$

$\therefore \quad \log_e A = 0 \Rightarrow A = 1$

Example 55 Evaluate $\lim_{n \to \infty} \left(\frac{e^n}{\pi}\right)^{1/n}$.

Sol. Let $A = \lim_{n \to \infty} \left(\frac{e^n}{\pi}\right)^{1/n}$ [∞^0 form]

$\therefore \quad \log A = \lim_{n \to \infty} \frac{1}{n} \log \left(\frac{e^n}{\pi}\right)$

$= \lim_{n \to \infty} \frac{n \log e - \log \pi}{n}$ $\left[\frac{\infty}{\infty} \text{ form}\right]$

$= \lim_{n \to \infty} \frac{\log e - 0}{1}$ [by L'Hospital's rule]

$\log A = 1$

$\Rightarrow \quad A = e^1$ or $\lim_{n \to \infty} \left(\frac{e^n}{\pi}\right)^{1/n} = e$

Example 56 ***Evaluate*** $\lim_{x \to 0} (\operatorname{cosec} x)^x$.

Sol. Let $A = \lim_{x \to 0} (\operatorname{cosec} x)^x$ [∞^0 form]

$\Rightarrow \quad \log A = \lim_{x \to 0} x \log (\operatorname{cosec} x)$

$= \lim_{x \to 0} \frac{\log (\operatorname{cosec} x)}{\frac{1}{x}}$ $\left[\frac{\infty}{\infty} \text{ form}\right]$

$= \lim_{x \to 0} \frac{\frac{1}{\operatorname{cosec} x} \cdot (-\operatorname{cosec} x \cot x)}{-\frac{1}{x^2}}$ [by L'Hospital's rule]

$= \lim_{x \to 0} \frac{x^2}{\tan x} = 0$

$\therefore \quad \log A = 0$ or $A = 1 \Rightarrow \lim_{x \to 0} (\operatorname{cosec} x)^x = 1$

(ii) 1^0 Form

Example 57 Solve $\lim_{x \to 0} \frac{(1+x)^{1/x} - e}{x}$.

Sol. Let $L = \lim_{x \to 0} \frac{(1+x)^{1/x} - e}{x}$

$= \lim_{x \to 0} \frac{e^{\log (1+x)^{1/x}} - e}{x} = \lim_{x \to 0} \frac{e\left\{e^{\frac{\log (1+x)}{x} - 1}\right\}}{x}$

$= \lim_{x \to 0} \frac{e\{e^M - 1\}}{M} \cdot \lim_{x \to 0} \frac{\log(1+x) - x}{x^2}$,

where $M = \frac{\log(1+x)}{x} - 1$

$\therefore \quad L = \lim_{M \to 0} \frac{e \cdot \{e^M - 1\}}{M} \cdot \lim_{x \to 0} \frac{\frac{1}{1+x} - 1}{2x}$

$= e \times 1 \times \lim_{x \to 0} \frac{-x}{2x(1+x)} = -\frac{e}{2}$ [as $x \to 0 \Rightarrow M \to 0$]

Exercise for Session 4

1. Evaluate $\lim_{x \to 0^+} (\sin x)^x$.

2. Evaluate $\lim_{x \to 0^+} (\sin x)^{\tan x}$.

3. Evaluate $\lim_{n \to \infty} \left(\frac{e^n}{\pi}\right)^{1/n}$.

4. The value of $\lim_{x \to 0} \frac{e - (1+x)^{1/x}}{\tan x}$ is

(a) e (b) $\frac{11e}{24}$ (c) $\frac{e}{2}$ (d) None of these

Session 5

Left Hand and Right Hand Limits

Left Hand and Right Hand Limits

Let $y = f(x)$ be a given function and $x = a$ is the point under consideration. Left tendency of $f(x)$ at $x = a$ is called its **left hand limit** and right tendency is called its **right hand limit.**

Left tendency (left hand limit) is denoted by $f(a-0)$ or $f(a-)$ and right tendency (right hand limit) is denoted by $f(a+0)$ or $f(a+)$ and are written as

$$\left.\begin{aligned} f(a-0) &= \lim_{h \to 0} f(a-h) \\ f(a+0) &= \lim_{h \to 0} f(a+h) \end{aligned}\right\}$$

where, h is a small positive number.

Thus, for the existence of the limit of $f(x)$ at $x = a$, it is necessary and sufficient that

$f(a-0) = f(a+0)$, if these are finite or $f(a-0)$ and $f(a+0)$ both should be either $+\infty$ or $-\infty$.

Example 58 ***Evaluate the right hand limit and left hand limit of the function***

$$f(x) = \begin{cases} \dfrac{|x-4|}{x-4}, & x \neq 4 \\ 0, & x = 4 \end{cases}.$$

Sol. Given, $f(x) = \begin{cases} \dfrac{|x-4|}{x-4}, & x \neq 4 \\ 0, & x = 4 \end{cases}$

Now, RHL of $f(x)$ at $x = 4$

$$= \lim_{x \to 4^+} f(x) = \lim_{h \to 0} f(4+h)$$

$$= \lim_{h \to 0} \frac{|4+h-4|}{4+h-4} = \lim_{h \to 0} \frac{|h|}{h} = 1$$

LHL of $f(x)$ at $x = 4$

$$= \lim_{x \to 4^-} f(x) = \lim_{h \to 0} f(4-h)$$

$$= \lim_{h \to 0} \frac{|4-h-4|}{4-h-4} = \lim_{h \to 0} \frac{|h|}{-h} = \lim_{h \to 0} \frac{h}{-h} = -1$$

Thus, RHL $\neq$ LHL. So, $\lim_{x \to 4} f(x)$ does not exist.

Example 59 If $f(x) = \begin{cases} 5x-4, & 0 < x \leq 1 \\ 4x^3 - 3x, & 1 < x < 2 \end{cases}$, show that $\lim_{x \to 1} f(x)$ exists.

Sol. We have, $f(x) = \begin{cases} 5x-4, & 0 < x \leq 1 \\ 4x^3 - 3x, & 1 < x < 2 \end{cases}$

LHL of $f(x)$ at $x = 1 = \lim_{x \to 1^-} f(x) = \lim_{h \to 0} f(1-h)$

$$= \lim_{h \to 0} 5(1-h) - 4 = \lim_{h \to 0} 1 - 5h = 1$$

RHL of $f(x)$ at $x = 1 = \lim_{x \to 1^+} f(x) = \lim_{h \to 0} f(1+h)$

$$= \lim_{h \to 0} 4(1+h)^3 - 3(1+h) = 4(1)^3 - 3(1) = 1$$

Thus, RHL = LHL = 1. So, $\lim_{x \to 1} f(x)$ exists and is equal to 1.

Example 60 ***Show that*** $\lim_{x\to 0}\dfrac{e^{1/x}-1}{e^{1/x}+1}$ ***does not exist.***

Sol. Let $f(x)=\dfrac{e^{1/x}-1}{e^{1/x}+1}$.

Then, $\text{LHL}=\lim_{x\to 0^-} f(x)=\lim_{h\to 0} f(0-h)$

$$=\lim_{h\to 0}\frac{e^{-1/h}-1}{e^{-1/h}+1}$$

$$=\lim_{h\to 0}\frac{(1/e^{1/h}-1)}{(1/e^{1/h}+1)}=\frac{0-1}{0+1}=-1$$

$[\text{as } h\to 0 \Rightarrow \frac{1}{h}\to\infty\Rightarrow e^{1/h}\to\infty \Rightarrow 1/e^{1/h}\to 0]$ ...(i)

$\text{RHL}=\lim_{x\to 0^+} f(x)=\lim_{h\to 0} f(0+h)$

$$=\lim_{h\to 0}\frac{e^{1/h}-1}{e^{1/h}+1}=\lim_{h\to 0}\frac{(1-1/e^{1/h})}{(1+1/e^{1/h})}$$

[dividing numerator and denominator both by $e^{1/h}$]

$$=\frac{1-0}{1+0}=1 \qquad \text{[using Eq. (i)]}$$

Clearly, $\lim_{x\to 0^-} f(x)\neq\lim_{x\to 0^+} f(x)$

Hence, $\lim_{x\to 0} f(x)$ doesn't exist.

Example 61 Evaluate $\lim_{x\to 1}\dfrac{\sqrt{1-\cos 2(x-1)}}{(x-1)}$.

[IIT JEE 1998]

Sol. We have, $\lim_{x\to 1}\dfrac{\sqrt{1-\cos 2(x-1)}}{(x-1)}$

$$=\lim_{x\to 1}\frac{\sqrt{2\sin^2(x-1)}}{(x-1)}=\lim_{x\to 1}\frac{\sqrt{2}|\sin(x-1)|}{(x-1)}$$

$\therefore\ \text{LHL}=\lim_{x\to 1^-}\dfrac{\sqrt{2}|\sin(x-1)|}{(x-1)}$

$$=\lim_{h\to 0}\frac{\sqrt{2}|\sin(-h)|}{(-h)}=\lim_{h\to 0}\frac{\sqrt{2}\sin h}{-h}=-\sqrt{2}$$

Again, $\text{RHL}=\lim_{x\to 1^+}\dfrac{\sqrt{2}|\sin(x-1)|}{(x-1)}$

$$=\lim_{h\to 0}\frac{\sqrt{2}|\sin(h)|}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{2}\sin h}{h}=\sqrt{2}$$

Clearly, $\lim_{x\to 1^+} f(x)\neq\lim_{x\to 1^-} f(x)$

Hence, $\lim_{x\to 1} f(x)$ doesn't exist.

Example 62 Solve (i) $\lim_{x\to 1}[\sin^{-1}x]$ (ii) $\lim_{x\to 0^+}\left[\dfrac{\sin x}{x}\right]$

(iii) $\lim_{x\to 0^-}\left[\dfrac{\sin x}{x}\right]$

(where $[\cdot]$ denotes greatest integer function.)

Sol. (i) Here, $L=\lim_{x\to 1}[\sin^{-1}x]$

Put $\sin^{-1}x=t$

$\therefore\quad x=\sin t$ and $t\to\dfrac{\pi}{2}$ as $x\to 1$

$\Rightarrow\quad L=\lim_{t\to\pi/2}[t]=\left[\dfrac{\pi}{2}\right]=1$

$\therefore\quad \lim_{x\to 1}[\sin^{-1}x]=1$

(ii) Here, $L=\lim_{x\to 0^+}\left[\dfrac{\sin x}{x}\right]$

Put $\dfrac{\sin x}{x}=t\Rightarrow t\to 1^-$ as $x\to 0^+$ and $[t]=0$

$\Rightarrow\quad L=\lim_{t\to 1^-}[t]=[1-h]=0$

$[\text{as } x\to 0+h\Rightarrow t\to 1-h]$

(iii) Here, $L=\lim_{x\to 0^-}\left[\dfrac{\sin x}{x}\right]$

Put $\dfrac{\sin x}{x}=t$

$\Rightarrow\ t\to 1^-$ as $x\to 0^-$

$\therefore\quad L=\lim_{t\to 1^-}[t]=[1-h]=0$

$[\text{as } x\to 0-h\Rightarrow t\to 1-h\ \therefore\ [t]=0]$

Example 63 Solve (i) $\lim_{x\to\infty}[\tan^{-1}x]$

(ii) $\lim_{x\to-\infty}[\tan^{-1}x]$

(where $[\cdot]$ denotes greatest integer function.)

Sol. (i) Here, $\lim_{x\to\infty}[\tan^{-1}x]$

Put $\tan^{-1}x=t\ \Rightarrow\ t\to\dfrac{\pi}{2}$ as $x\to\infty$

$\therefore\quad \lim_{t\to\pi/2}[t]=\left[\dfrac{\pi}{2}\right]=1$

(ii) Here, $\lim_{x\to-\infty}[\tan^{-1}x]$

Put $\tan^{-1}x=t\ \Rightarrow\ t\to-\dfrac{\pi}{2}$ as $x\to-\infty$

$\therefore\ \lim_{t\to-\pi/2}[t]=\left[-\dfrac{\pi}{2}\right]=[-1.57]=-2$

Example 64 Solve (i) $\lim_{x \to 0^+} \left[\frac{\tan x}{x}\right]$

(ii) $\lim_{x \to 0^-} \left[\frac{\tan x}{x}\right]$

(where $[\cdot]$ denotes greatest integer function.)

Sol. (i) Here, $\lim_{x \to 0^+} \left[\frac{\tan x}{x}\right]$

Put $\frac{\tan x}{x} = t \Rightarrow t \to 1^+$ as $x \to 0^+$

$\therefore \lim_{t \to 1^+} [t] = [1 + h] = 1 \qquad [\text{as } x \to 0 + h \Rightarrow t \to 1 + h]$

(ii) Here, $\lim_{x \to 0^-} \left[\frac{\tan x}{x}\right]$

Put $\frac{\tan x}{x} = t \Rightarrow t \to 1^+$ as $x \to 0^-$

$\therefore \lim_{t \to 1^+} [t] = [1 + h] = 1 \qquad [\text{as } x \to 0 - h \Rightarrow t \to 1 + h]$

Example 65 ***Solve***

(i) $\lim_{x \to 1^-} [\sin(\sin^{-1} x)]$ (ii) $\lim_{x \to \pi/2} [\sin^{-1}(\sin x)]$

(where $[\cdot]$ denotes greatest integer function.)

Sol. (i) Here, $\lim_{x \to 1^-} [\sin(\sin^{-1} x)]$

$= \lim_{x \to 1^-} [x] \quad [\text{as } \sin(\sin^{-1} x) = x, \text{if } -1 \le x \le 1]$

$= \lim_{h \to 0} [1 - h] \qquad [\text{as } x \to 1^- \Rightarrow x = 1 - h]$

$= 0$

$\therefore \lim_{x \to 1} [\sin(\sin^{-1} x)] = \lim_{x \to 1^-} [\sin(\sin^{-1} x)]$

and no need to check for $\lim_{x \to 1^+} [\sin(\sin^{-1} x)]$

(ii) $\lim_{x \to \frac{\pi}{2}} [\sin^{-1}(\sin x)] = \lim_{x \to \frac{\pi^-}{2}} [\sin^{-1}(\sin x)]$

$= \lim_{x \to \frac{\pi^-}{2}} [x] = \lim_{h \to 0} \left[\frac{\pi}{2} - h\right] = 1$

Remark

If $\lim_{x \to 1} [\sin(\sin^{-1} x)]$, it means you have to calculate only left hand limit and not right hand limit as for $x > 1$, $\sin(\sin^{-1} x)$ is not defined.

Example 66 Solve (i) $\lim_{x \to 0} [\cot x]$

(ii) $\lim_{x \to +\infty} [\cot^{-1} x]$

(where $[\cdot]$ denotes greatest integer function.)

Sol. (i) Here, $\lim_{x \to 0} [\cot x]$

Put $\cot x = t$, now as $x \to 0$; $\cot x$ exhibits two values for $x \to 0^+$ and $x \to 0^-$. i.e. $\cot x \to +\infty$ and $\cot x \to -\infty$, respectively.

$\therefore$ We should apply right hand and left hand limit;

i.e. $\lim_{x \to 0^+} [\cot x] = \lim_{t \to +\infty} [t] = \infty$

$[\because \cot x = t \Rightarrow t \to +\infty \text{ as } x \to 0^+]$

and $\lim_{x \to 0^-} [\cot x] = \lim_{t \to -\infty} [t] = -\infty$

$[\because \cot x = t \Rightarrow t \to -\infty \text{ as } x \to 0^-]$

$\therefore$ Limit doesn't exist.

(ii) Here, $\lim_{x \to +\infty} [\cot^{-1} x] = \lim_{t \to 0^+} [t]$

$[\because \cot^{-1} x = t \Rightarrow t \to 0^+ \text{ as } x \to +\infty]$

$= \lim_{h \to 0} [0 + h] = \lim_{h \to 0} 0 = 0$

Example 67 ***Solve*** $\lim_{x \to 0} \left[\sin \frac{|x|}{x}\right]$***, where*** $[\cdot]$ ***denotes greatest integer function.***

Sol. Here, $\lim_{x \to 0} \left[\sin \frac{|x|}{x}\right]$, since we have greatest integer function, we must define function.

Now, RHL (put $x = 0 + h$) $= \lim_{h \to 0} \left[\frac{\sin|0+h|}{0+h}\right]$

We know that, $\frac{\sin h}{h} \to 1$ as $h \to 0$ but less than 1.

$\therefore$ RHL $= \lim_{h \to 0} 0 = 0 \qquad \left[\because \left(\frac{\sin h}{h}\right) = 0 \text{ as } h \to 0\right]$

Again, LHL (put $x = 0 - h$) $= \lim_{h \to 0} \left[\sin \frac{|0-h|}{0-h}\right]$,

we know $\frac{\sin h}{-h} \to -1$ as $h \to 0$ but greater than -1.

$\therefore$ LHL $= \lim_{h \to 0} -1 = -1 \qquad \left[\because \left(\frac{\sin h}{h}\right) = -1 \text{ as } h \to 0\right]$

Thus, Limit doesn't exist, as RHL = 0 and LHL = -1.

Example 68 Solve $\lim_{x \to 0} \left[\frac{\sin|x|}{|x|}\right]$, where $[\cdot]$ denotes greatest integer function.

Sol. Here, $\lim_{x \to 0} \left[\frac{\sin|x|}{|x|}\right]$ ...(i)

We know that, $\frac{\sin x}{x} \to 1$ as $x \to 0$ or $\frac{\sin|x|}{|x|} \to 1$ as $x \to 0$ from right or left,

i.e. at $x = 0 + h$ or $x = 0 - h$, $\frac{\sin|x|}{|x|} \to 1^-$

$\therefore \quad \left[\frac{\sin|x|}{|x|}\right]=0 \text{ as } \frac{\sin|x|}{|x|}<1$

From Eq. (i) whether we find RHL or LHL

$\lim_{x\to 0}\left[\frac{\sin|x|}{|x|}\right] = \lim_{x\to 0} 0 = 0 \Rightarrow \lim_{x\to 0}\left[\frac{\sin|x|}{|x|}\right]$,

exists and is 0.

Example 69 $\lim_{x\to 0}\left[\frac{-2x}{\tan x}\right]$, where $[\cdot]$ denotes greatest integer function is

(a) -1 (b) 4
(c) 5 (d) None of these

Sol. We know, when $x \to 0$

$\Rightarrow \quad \frac{x}{\tan x}<1 \Rightarrow \frac{-x}{\tan x}>-1$

$\Rightarrow \quad \frac{-2x}{\tan x}>-2$

So, $\lim_{x\to 0}\left[\frac{-2x}{\tan x}\right]=-2$

Hence, (d) is the correct answer.

Exercise for Session 5

1. The value of $\lim_{x\to 1}\{1-x+[x-1]+[1-x]\}$ (where $[\cdot]$ denotes the greatest integral function) is

(a) -1 (b) Doesn't exist
(c) 1 (d) None of these

2. The value of $\lim_{x\to 0}\frac{\sin[x]}{[x]}$ (where $[\cdot]$ denotes the greatest integer function) is

(a) 1 (b) sin 1
(c) Doesn't exist (d) None of these

3. The value of $\lim_{x\to 0}\sin^{-1}\{x\}$ (where $\{\cdot\}$ denotes fractional part of x) is

(a) 0 (b) $\frac{\pi}{2}$
(c) Doesn't exist (d) None of these

4. The value of $\lim_{x\to 0}\left[\frac{x^2}{\sin x \tan x}\right]$ (where $[\cdot]$ denotes the greatest integer function) is

(a) 0 (b) 1
(c) Doesn't exist (d) None of these

Session 6

Use of Standard Theorems/Results, Use of Newton-Leibnitz's Formula in Evaluating the Limits, Summation of Series Using Definite Integral as the Limit

Use of Standard Theorems/Results

Theorem 1 : Sandwich/Squeeze Play Theorem

General The Squeeze principle is used on limit problems where the usual algebraic methods, factorisation or algebraic manipulation etc., are not effective. However, it requires to "squeeze" our problem in between two other simpler function, whose limits can be easily computed and equal. Use of Squeeze principle requires accurate analysis, indepth algebra skills and careful use of inequalities.

Statement If f, g and h are three functions such that $f(x) \le g(x) \le h(x)$ for all x in some interval containing the point $x = c$ and if

$$\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$$

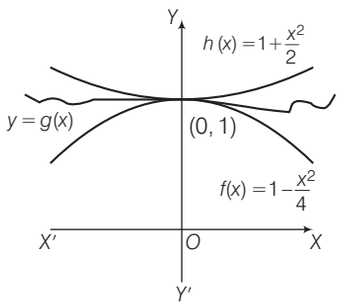


Figure 5.10

Then, $\lim_{x \to c} g(x) = L$

From the figure, note that $\lim_{x \to 0} g(x) = 1$.

Remark

The quantity c may be a finite number, $+\infty$ or $-\infty$. Similarly, L may also be finite number, $+\infty$ or $-\infty$.

Theorem 2 : Limits of Trigonometric Functions

If x is small and is measured in radians, then

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 = \lim_{x \to 0} \frac{x}{\sin x}$$

$$= \lim_{x \to 0} x \operatorname{cosec} x = \lim_{x \to 0} \frac{\sin^{-1} x}{x} = \lim_{x \to 0} \frac{x}{\sin^{-1} x}$$

Proof Consider a circle with unit radius.

Area of $\Delta OAP <$ Area of sector $\overset{\frown}{OAP} <$ Area of ΔOAT

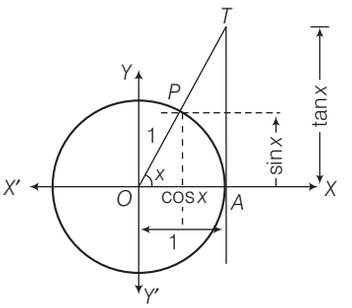


Figure 5.11

$$\frac{\sin x}{2} < \frac{x}{2} < \frac{\tan x}{2} \Rightarrow 1 < \frac{x}{\sin x} < \frac{1}{\cos x} \quad [\because 0 < x < \pi/2]$$

$$\Rightarrow \quad \cos x < \frac{\sin x}{x} < 1$$

Now, using Sandwich theorem,

$$\lim_{x \to 0} \cos x < \lim_{x \to 0^+} \frac{\sin x}{x} < 1$$

Obviously, we have $\lim_{x \to 0^+} \frac{\sin x}{x} = 1$

Put $x = -y$, $\lim_{y \to 0^-} \frac{\sin y}{y} = 1$

Hence, $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Similarly, $\lim_{x\to 0}\frac{\sin x}{x}=1=\lim_{x\to 0}\frac{x}{\sin x}=\lim_{x\to 0}x\operatorname{cosec}x$

$$=\lim_{x\to 0}\frac{\sin^{-1}x}{x}=\lim_{x\to 0}\frac{x}{\sin^{-1}x} \quad ...(i)$$

Using Eq. (i), we can deduce

$$\lim_{x\to 0}\frac{\tan x}{x}=1=\lim_{x\to 0}\frac{x}{\tan x}=\lim_{x\to 0}x\cot x$$

$$=\lim_{x\to 0}\frac{\tan^{-1}x}{x}=\lim_{x\to 0}\frac{x}{\tan^{-1}x}$$

Important Results

The $\lim_{x\to 0}\frac{\sin x}{x}$ always approaches 1 from its left hand,

i.e. 0.9999....

$\Rightarrow \lim_{x\to 0}\left[\frac{\sin x}{x}\right]=0$, where $[\cdot]$ denotes step up function.

$$\left[\text{Note that } \left(\lim_{x\to 0}\frac{\sin x}{x}\right)=1\right]$$

Note that the $\lim_{x\to 0}\frac{\tan x}{x}$ approaches 1 from RHS.

$\Rightarrow \lim_{x\to 0}\left[\frac{\tan x}{x}\right]=1$, where $[\cdot]$ denotes step up function.

Example 70 Evaluate $\lim_{x\to 0}x^3\cos\frac{2}{x}$.

Sol. Here, $\lim_{x\to 0}x^3\cos\frac{2}{x}$

As, $-1\le\cos\frac{2}{x}\le 1\Rightarrow -x^3\le x^3\cos\frac{2}{x}\le x^3$ for $x>0$

and $x^3\le x^3\cos\frac{2}{x}\le -x^3$ for $x<0$,

Thus, $\lim_{x\to 0}x^3\cos\frac{2}{x}=0$, as in both the cases limit is zero.

Example 71 Evaluate $\lim_{x\to\infty}\frac{x^2(2+\sin^2 x)}{x+100}$.

Sol. We have, $\lim_{x\to\infty}\frac{x^2(2+\sin^2 x)}{x+100}$

Now, $\frac{2x^2}{x+100}\le\frac{x^2(2+\sin^2 x)}{x+100}\le\frac{3x^2}{x+100}$, as $0\le\sin^2 x\le 1$

$$\therefore \lim_{x\to\infty}\frac{2x^2}{x+100}\le\lim_{x\to 0}\frac{x^2(2+\sin^2 x)}{x+100}\le\lim_{x\to\infty}\frac{3x^2}{x+100}$$

$$\Rightarrow \lim_{x\to\infty}\frac{x^2(2+\sin^2 x)}{x+100}=\infty$$

$$\left[\because \lim_{x\to\infty}\frac{2x^2}{x+100}=\lim_{x\to\infty}\frac{3x^2}{x+100}=\infty\right]$$

Example 72 *Evaluate*

$$\lim_{n\to\infty}\left(\frac{n}{n^2+1}+\frac{n}{n^2+2}+\frac{n}{n^2+3}+\ldots+\frac{n}{n^2+n}\right).$$

Sol. Let $f(n)=\frac{n}{n^2+1}+\frac{n}{n^2+2}+\frac{n}{n^2+3}+\ldots+\frac{n}{n^2+n}$

Note that $f(n)$ has n terms which are decreasing.

Suppose

$$h(n)=\left(\frac{n}{n^2+1}+\frac{n}{n^2+1}+\frac{n}{n^2+1}+\ldots+\frac{n}{n^2+1}\right),\ n \text{ terms}$$

$$=\frac{n^2}{n^2+1} \qquad [\text{obviously } f(n)<h(n)]$$

and

$$g(n)=\left(\frac{n}{n^2+n}+\frac{n}{n^2+n}+\frac{n}{n^2+n}+\ldots+\frac{n}{n^2+n}\right),\ n \text{ terms}$$

$$=\frac{n^2}{n^2+n} \qquad [\text{obviously } g(n)<f(n)]$$

Hence, $\lim_{n\to\infty}g(n)<\lim_{n\to\infty}f(n)<\lim_{n\to\infty}h(n)$

Since, $\lim_{n\to\infty}g(n)=1=\lim_{n\to\infty}h(n)$

Hence, using Sandwich theorem, $\lim_{n\to\infty}f(n)=1$.

Example 73 The value of the $\lim_{x\to 0}\frac{x}{a}\left[\frac{b}{x}\right]$ $(a\ne 0)$

(where $[\cdot]$ denotes the greatest integer function) is

(a) a (b) b

(c) $\frac{b}{a}$ (d) $1-\frac{b}{a}$

Sol. Since, $\frac{b}{x}-1<\left[\frac{b}{x}\right]\le\frac{b}{x}$

Now, we have two cases depending upon the value of $\frac{x}{a}$.

***Case* I** For $\frac{x}{a}>0$

$$\Rightarrow \lim_{x\to 0}\left(\frac{b}{x}-1\right)\frac{x}{a}<\lim_{x\to 0}\left[\frac{b}{x}\right]\frac{x}{a}\le\lim_{x\to 0}\frac{b}{x}\cdot\frac{x}{a}$$

Using Squeeze play theorem, we have $=\lim_{x\to 0}\frac{x}{a}\left[\frac{b}{x}\right]=\frac{b}{a}$

***Case* II** For $\frac{x}{a}<0$

$$\Rightarrow \lim_{x\to 0}\left(\frac{b}{x}-1\right)\frac{x}{a}>\lim_{x\to 0}\left[\frac{b}{x}\right]\frac{x}{a}\ge\lim_{x\to 0}\frac{b}{x}\cdot\frac{x}{a}$$

Using Squeeze play theorem, we have

$$\lim_{x\to 0}\frac{x}{a}\left[\frac{b}{x}\right]=\frac{b}{a}$$

Hence, (c) is the correct answer.

Example 74 *Evaluate*

$$\lim_{n\to\infty}\frac{[x]+[2x]+[3x]+\ldots+[nx]}{n^2},$$

where $[\cdot]$ denotes the greatest integer function.

Sol. We know that, $x-1<[x]\le x$

$\Rightarrow$ $2x-1<[2x]\le 2x$

$\Rightarrow$ $3x-1<[3x]\le 3x$

.....

.....

$\Rightarrow$ $nx-1<[nx]\le nx$

$\therefore\ (x+2x+3x+\ldots+nx)-n<[x]+[2x]+\ldots+[nx]\le(x+2x+\ldots+nx)$

$$\Rightarrow\quad \frac{xn(n+1)}{2}-n<\sum_{r=1}^{n}[rx]\le\frac{x\cdot n(n+1)}{2}$$

Thus, $\lim_{n\to\infty}\dfrac{[x]+[2x]+\ldots+[nx]}{n^2}$

$$\Rightarrow\quad \lim_{n\to\infty}\frac{x}{2}\left(1+\frac{1}{n}\right)-\frac{1}{n}<\lim_{n\to\infty}\frac{[x]+[2x]+\ldots+[nx]}{n^2}\le\lim_{n\to\infty}\frac{x}{2}\left(1+\frac{1}{n}\right)$$

$$\Rightarrow\quad \frac{x}{2}<\lim_{n\to\infty}\frac{[x]+[2x]+\ldots+[nx]}{n^2}\le\frac{x}{2}$$

$$\therefore\quad \lim_{n\to\infty}\frac{[x]+[2x]+\ldots+[nx]}{n^2}=\frac{x}{2}$$

Aliter We know that, $[x]=x-\{x\}$

$$\sum_{r=1}^{n}rx=[x]+[2x]+\ldots+[nx]$$
$$=x-\{x\}+2x-\{2x\}+\ldots+nx-\{nx\}$$
$$=(x+2x+3x+\ldots+nx)-(\{x\}+\{2x\}+\ldots+\{nx\})$$
$$=\frac{xn(n+1)}{2}-(\{x\}+\{2x\}+\ldots+\{nx\})$$

$$\therefore\frac{1}{n^2}\sum_{r=1}^{n}[rn]=\frac{x}{2}\left(1+\frac{1}{n}\right)-\frac{\{x\}+\{2x\}+\ldots+\{nx\}}{n^2}$$

Since, $0\le\{rx\}<1$

$$\therefore\quad 0\le\sum_{r=1}^{n}\{rx\}<n$$

$$\Rightarrow\quad \lim_{n\to\infty}\frac{\sum_{r=1}^{n}\{rx\}}{n^2}=0$$

$$\therefore\quad \lim_{n\to\infty}\frac{\sum_{r=1}^{n}[rx]}{n^2}=\lim_{n\to\infty}\frac{x}{2}\left(1+\frac{1}{n}\right)-\lim_{n\to\infty}\frac{\sum_{r=1}^{n}\{rx\}}{n^2}$$

$$\Rightarrow\quad \lim_{n\to\infty}\frac{\sum_{r=1}^{n}[rx]}{n^2}=\frac{x}{2}$$

Use of Newton-Leibnitz's Formula in Evaluating the Limits

Let us consider the definite integral,

$$I(x)=\int_{\phi(x)}^{\psi(x)}f(t)\,dt$$

Newton-Leibnitz's formula states that,

$$\frac{d}{dx}\{I(x)\}=f\{\psi(x)\}\cdot\left\{\frac{d}{dx}\psi(x)\right\}-f\{\phi(x)\}\left\{\frac{d}{dx}\phi(x)\right\}$$

Example 75 *Evaluate*

$$\lim_{x\to0}\left(\frac{1}{x^5}\int_0^x e^{-t^2}\,dt-\frac{1}{x^4}+\frac{1}{3x^2}\right).$$

Sol We have, $\lim_{x\to0}\left(\dfrac{1}{x^5}\int_0^x e^{-t^2}dt-\dfrac{1}{x^4}+\dfrac{1}{3x^2}\right)$

$$=\lim_{x\to0}\frac{3\int_0^x e^{-t^2}\,dt-3x+x^3}{3x^5}\qquad\left[\frac{0}{0}\text{ form}\right]$$

$$=\lim_{x\to0}\frac{3\frac{d}{dx}\int_0^x e^{-t^2}\,dt-3+3x^2}{15x^4}\text{ [by L'Hospital's rule]}$$

Applying Newton-Leibnitz's formula,

$$\frac{d}{dx}\int_0^x e^{-t^2}\,dt=e^{-x^2}\cdot\frac{d}{dx}(x)-e^{-0}\frac{d}{dx}(0)=e^{-x^2}$$

$$\therefore\lim_{x\to0}\frac{3\frac{d}{dx}\int_0^x e^{-t^2}\,dt-3+3x^2}{15x^4}=\lim_{x\to0}\frac{3e^{-x^2}-3+3x^2}{15x^4}\qquad\left[\frac{0}{0}\text{ form}\right]$$

$$=\lim_{x\to0}\frac{-3(2x)e^{-x^2}+6x}{60x^3}\qquad\text{[again, apply L'Hospital's rule]}$$

$$=\lim_{x\to0}\frac{-6x(e^{-x^2}-1)}{60x^3}=\lim_{x\to0}\frac{-(e^{-x^2}-1)}{10x^2}=\frac{1}{10}\lim_{x\to0}\left(\frac{e^{-x^2}-1}{-x^2}\right)$$

$$=\frac{1}{10}\times1=\frac{1}{10}\qquad\left[\because\lim_{x\to0}\left(\frac{e^x-1}{x}\right)=1\right]$$

Example 76 Evaluate $\lim_{x\to0}\dfrac{x-\int_0^x\cos t^2\,dt}{x^3-6x}$.

Sol. Let $L=\lim_{x\to0}\dfrac{x-\int_0^x\cos t^2\,dt}{x^3-6x}\qquad\left[\dfrac{0}{0}\text{ form}\right]$

Applying L'Hospital's rule, we get

$$L=\lim_{x\to0}\frac{1-\frac{d}{dx}\int_0^x\cos t^2\,dt}{3x^2-6}$$

Applying Newton-Leibnitz's rule,

$$\frac{d}{dx}\int_0^x (\cos t^2)\,dt = \cos(x^2)\cdot 1 - 0 = \cos(x^2)$$

$$\therefore \quad L = \lim_{x\to 0}\frac{1-\frac{d}{dx}\int_0^x \cos t^2\,dt}{3x^2-6} = \lim_{x\to 0}\frac{1-\cos(x^2)}{3(x^2-2)}$$

$$= \frac{1-\cos 0}{3(0-2)} = \frac{1-1}{3(-2)} = \frac{0}{-6} = 0$$

Example 77 ***Evaluate*** $\lim_{x\to 0}\dfrac{\int_0^{x^2}\cos t^2\,dt}{x\sin x}$.

[IIT JEE 1997]

Sol. Applying Newton-Leibnitz's rule, followed by L' Hospital's rule, we get

$$\lim_{x\to 0}\frac{\cos(x^2)^2\cdot\{2x\}-0}{x\cos x+\sin x} = \lim_{x\to 0}\frac{2\cos x^4}{\cos x+\left(\frac{\sin x}{x}\right)}$$

$$= \frac{2\cos 0}{\cos 0+1} = \frac{2}{1+1} = 1$$

Summation of Series Using Definite Integral as the Limit

The expression of the form,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{r=\phi(x)}^{\psi(x)} f\left(\frac{r}{n}\right) = \int_a^b f(x)\,dx$$

where, (i) Σ is replaced by $\int$,

(ii) $\dfrac{r}{n}$ is replaced by x,

(iii) $\dfrac{1}{n}$ is replaced by dx,

(iv) To obtain, $a = \lim_{n\to\infty}\dfrac{\phi(x)}{n}$ and $b = \lim_{n\to\infty}\dfrac{\psi(x)}{n}$

The value so obtained is the required sum of the given series.

Example 78 ***Evaluate*** $\lim_{n\to\infty}\dfrac{1}{n}\sum_{r=1}^{2n}\dfrac{r}{\sqrt{n^2+r^2}}$.

[IIT JEE 1997]

Sol. Let $L = \lim_{n\to\infty}\dfrac{1}{n}\sum_{r=1}^{2n}\dfrac{r}{\sqrt{n^2+r^2}}$ dividing numerator and denominator both by n, we get

$$L = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{2n}\frac{r/n}{\sqrt{1+(r/n)^2}}$$

Put $\dfrac{r}{n} = x; \dfrac{1}{n} = dx;\ \lim_{n\to\infty}\sum_{r=1}^{2n} = \int_a^b$

where, $a = \lim_{n\to\infty}\dfrac{1}{n} = 0$ and $b = \lim_{n\to\infty}\dfrac{2n}{n} = 2$

$$\therefore \quad L = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{2n}\frac{r/n}{\sqrt{1+(r/n)^2}} = \int_0^2\frac{x}{\sqrt{1+x^2}}\,dx$$

$$= \left(\sqrt{1+x^2}\right)_0^2 = \sqrt{5}-1$$

Example 79 Evaluate

$$\lim_{n\to\infty}\left(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\ldots+\frac{n}{n^2+n^2}\right).$$

Sol. Let $S = \lim_{n\to\infty}\left(\dfrac{n}{n^2+1^2}+\dfrac{n}{n^2+2^2}+\ldots+\dfrac{n}{n^2+n^2}\right)$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{n}{n^2+r^2} = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{n^2}{n^2+r^2}$$

[dividing numerator and denominator both by n]

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+(r/n)^2}$$

Replace $\dfrac{r}{n} = x; \dfrac{1}{n} = dx;\ \lim_{n\to\infty}\sum_{r=1}^{n} = \int_a^b$

where, $a = \lim_{n\to\infty}\dfrac{1}{n} = 0$ and $b = \lim_{x\to\infty}\dfrac{n}{n} = 1$

we get, $S = \lim_{n\to\infty}\dfrac{1}{n}\sum_{r=1}^{n}\dfrac{1}{1+(r/n)^2}$

$$= \int_0^1\frac{1}{1+x^2}\,dx = [\tan^{-1}x]_0^1 = \tan^{-1}(1)-\tan^{-1}(0) = \frac{\pi}{4}$$

Example 80 The value of $\lim_{n\to\infty}\left(\dfrac{n!}{n^n}\right)^{1/n}$ is equal to

(a) $\dfrac{1}{e}$ (b) e (c) e^2 (d) $\dfrac{1}{e^2}$

Sol. Let $A = \lim_{n\to\infty}\left(\dfrac{n!}{n^n}\right)^{1/n}$

$$\therefore \log A = \lim_{n\to\infty}\frac{1}{n}\log\left(\frac{n}{n}\cdot\frac{(n-1)}{n}\cdot\frac{(n-2)}{n}\cdots\frac{3}{n}\cdot\frac{2}{n}\cdot\frac{1}{n}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\log\left(\frac{n-r}{n}\right)$$

$$= \int_0^1 1\cdot\log(1-x)\,dx = -1$$

[replace $\dfrac{r}{n}$ by n and $\dfrac{1}{n}$ by dx $\lim_{n\to\infty}\sum_{r=0}^{n-1} = \int_a^b$, where $\dfrac{0}{n} = 0$, $a = \lim_{n\to\infty}, b = \lim_{n\to\infty}\dfrac{n-1}{n} = 1$]

$$\Rightarrow \quad \log_e A = -1 \quad \Rightarrow \quad A = e^{-1}$$

Hence, (a) is the correct answer.

Exercise for Session 6

1. The value of $\lim_{n\to\infty}\frac{1}{n^3}([1^2x+1^2]+[2^2x+2^2]+\ldots+[n^2x+n^2])$ (where $[\cdot]$ denotes the greatest integer function) is

(a) $\frac{x}{3}$ (b) $x+\frac{1}{3}$

(c) $\frac{x}{3}+\frac{1}{3}$ (d) None of these

2. The value of $\lim_{x\to\infty}\left\{\lim_{n\to\infty}\frac{[1^2(\sin x)^x]+[2^2(\sin x)^x]+\ldots+[n^2(\sin x)^x]}{n^3}\right\}$ (where $[\cdot]$ denotes the greatest integer function) is

(a) $\frac{x}{3}+\frac{\sin x}{3}$ (b) $\frac{x}{3}+(\sin x)^x$

(c) $\frac{1}{3}$ (d) 0

3. The value of $\lim_{x\to 1^+}\frac{\int_1^x |t-1|\,dt}{\sin(x-1)}$ is

(a) 0 (b) 1

(c) Doesn't exist (d) None of these

4. The value of $\lim_{n\to\infty}\sum_{K=1}^{n}\log\left(1+\frac{K}{n}\right)^{1/n}$ is

(a) $\log_e\left(\frac{e}{4}\right)$ (b) $\log_e\left(\frac{4}{e}\right)$

(c) $\log_e 4$ (d) None of these

5. The value of $\lim_{n\to\infty}\left(\frac{1}{na}+\frac{1}{na+1}+\frac{1}{na+2}+\ldots+\frac{1}{nb}\right)$ is

(a) $\log\left(\frac{a}{b}\right)$ (b) $\log\left(\frac{b}{a}\right)$

(c) $\log(ab)$ (d) None of these

JEE Type Solved Examples :
Single Option Correct Type Questions

● **Ex. 1** *The value of* $\lim_{x\to 0}\dfrac{x\sin(\sin x)-\sin^2 x}{x^6}$ *equals*

(a) $\frac{1}{6}$ (b) $\frac{1}{12}$

(c) $\frac{1}{18}$ (d) $\frac{1}{24}$

Sol. *(c)* Here, $\lim_{x\to 0}\dfrac{x\sin(\sin x)-\sin^2 x}{x^6}$

$= \lim_{t\to 0}\dfrac{\sin^{-1}(t)\cdot\sin t - t^2}{(\sin^{-1}t)^6}$ [put $\sin x = t$]

$= \lim_{t\to 0}\dfrac{\sin^{-1}(t)\cdot\sin t - t^2}{t^6}\cdot\dfrac{t^6}{(\sin^{-1}t)^6}$

$= \lim_{t\to 0}\dfrac{\left\{t+\frac{t^3}{6}+\frac{9t^5}{120}+\frac{5t^7}{112}+\ldots\right\}\cdot\left\{t-\frac{t^3}{3!}+\frac{t^5}{5!}-\ldots\right\}-t^2}{t^6}\times 1$

$= \lim_{t\to 0}\dfrac{t^6\left(\frac{1}{5!}-\frac{1}{6\cdot(3!)}+\frac{9}{120}\right)+\text{higher powers to } t}{t^6}$

$= \frac{1}{120}+\frac{9}{120}-\frac{1}{36}=\frac{1}{18}$

● **Ex. 2** *If* $\lim_{x\to 0}\dfrac{\log_e\cot\left(\frac{\pi}{4}-K_1x\right)}{\tan K_2x}=1$, *then*

(a) $K_1 = K_2$ (b) $2K_1 = K_2$

(c) $K_1 = 2K_2$ (d) $K_1 = 4K_2$

Sol. *(b)* Here, $\lim_{x\to 0}\dfrac{\log_e\left[\cot\left(\frac{\pi}{4}-K_1x\right)\right]}{\tan K_2x}=1$

$\Rightarrow \lim_{x\to 0}\dfrac{\log\left[\cot\left(\frac{\pi}{4}-K_1x\right)-1+1\right]}{\tan K_2x}=1$

$\Rightarrow \lim_{x\to 0}\dfrac{\log\left(1+\frac{2\tan K_1x}{1-\tan K_1x}\right)}{\tan K_2x}=1$

$\Rightarrow \lim_{x\to 0}\dfrac{\log\left(1+\frac{2\tan K_1x}{1-\tan K_1x}\right)}{\frac{2\tan K_1x}{1-\tan K_1x}}\cdot\dfrac{\frac{2\tan K_1x}{1-\tan K_1x}}{\tan K_2x}=1$

$\Rightarrow \lim_{x\to 0}\dfrac{2\tan K_1x}{\tan K_2x}=1 \Rightarrow \lim_{x\to 0}\dfrac{\frac{2\tan K_1x}{K_1x}\cdot K_1x}{\frac{\tan K_2x}{K_2x}\cdot K_2x}=1$

$\Rightarrow \dfrac{2K_1}{K_2}=1 \Rightarrow 2K_1 = K_2$

● **Ex. 3** *Let* $f(\theta)=\dfrac{1}{\tan^2\theta}\{(1+\tan\theta)^3+(2+\tan\theta)^3+\ldots+(10+\tan\theta)^3\}-10\tan\theta$.

Then, $\lim_{\theta\to\frac{\pi}{2}^-} f(\theta)$ *is equal to*

(a) 170 (b) 166 (c) 165 (d) None of these

Sol. *(c)* Here,

$\lim_{\theta\to\frac{\pi}{2}^-}\dfrac{(1+\tan\theta)^3}{\tan^2\theta}=\lim_{\theta\to\frac{\pi}{2}^-}\dfrac{1+3\tan\theta+3\tan^2\theta+\tan^3\theta}{\tan^2\theta}$

$[(a+b)^3=a^3+b^3+3a^2b+3ab^2]$

$=\lim_{\theta\to\frac{\pi}{2}^-}3+\tan\theta+\left(\dfrac{3\tan\theta+1}{\tan^2\theta}\right)=\lim_{\theta\to\frac{\pi}{2}^-}(3+\tan\theta)$

$\left[\text{as }\lim_{0\to\frac{\pi}{2}^-}\left(\dfrac{3\tan\theta+1}{\tan^2\theta}\right)=0\right]$

$\Rightarrow \lim_{\theta\to\frac{\pi}{2}^-} f(\theta)=\lim_{\theta\to\frac{\pi}{2}^-}[(3+\tan\theta)+(3(2)+\tan\theta)+\ldots+(3(10)+\tan\theta)]-10\lim_{\theta\to\frac{\pi}{2}^-}\tan\theta$

$=\lim_{\theta\to\frac{\pi}{2}^-}3(1+2+3+\ldots+10)+10\cdot\lim_{\theta\to\frac{\pi}{2}^-}(\tan\theta)-10\cdot\lim_{\theta\to\frac{\pi}{2}^-}(\tan\theta)$

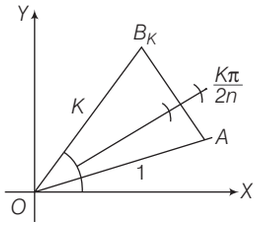
$=\dfrac{3\times 10\times 11}{2}=165$

● **Ex. 4** *For positive integers* $K=1,2,3,\ldots,n$. *Let* S_k *denotes the area of* ΔOAB_K *(where 'O' is the origin) such that* $\angle AOB_K=\dfrac{K\pi}{2n}$, $OA=1$ *and* $OB_K=K$. *The value of the* $\lim_{n\to\infty}\dfrac{1}{n^2}\cdot\sum_{K=1}^{n}S_K$ *is*

(a) $\frac{2}{\pi^2}$ (b) $\frac{4}{\pi^2}$ (c) $\frac{8}{\pi^2}$ (d) $\frac{1}{2\pi^2}$

Sol. *(a)* Here, $OB_K = K$ and $\angle AOB_K = \dfrac{K\pi}{2n}$

$\therefore \quad S_K = \dfrac{1}{2}\cdot(1)(K)\sin\left(\dfrac{K\pi}{2n}\right)$ $\left[\text{using, } \Delta = \dfrac{1}{2}ab\sin\theta\right]$



Then, $L = \lim\limits_{n\to\infty}\dfrac{1}{n^2}\cdot\dfrac{K}{2}\cdot\sin\left(\dfrac{K\pi}{2n}\right)$

$= \dfrac{1}{2n}\cdot\lim\limits_{n\to\infty}\dfrac{K}{n}\cdot\sin\left(\dfrac{\pi}{2}\cdot\dfrac{K}{n}\right) = \dfrac{1}{2}\cdot\int_0^1 x\cdot\sin\left(\dfrac{\pi}{2}x\right)dx$

$= \dfrac{1}{2}\left[\left(\underbrace{\dfrac{-2}{\pi}\cdot x\cdot\cos\dfrac{\pi x}{2}}_{0}\right)_0^1 + \dfrac{2}{\pi}\int_0^1\cos\dfrac{\pi x}{2}\cdot dx\right]$

$= \dfrac{1}{2}\left[\dfrac{2}{\pi}\cdot\dfrac{2}{\pi}\cdot\left(\sin\dfrac{\pi x}{2}\right)_0^1\right] = \dfrac{2}{\pi^2}$

● ***Ex. 5*** *If*

$S_n = \left(1-\tan^4\dfrac{\pi}{2^3}\right)\left(1-\tan^4\dfrac{\pi}{2^4}\right)\ldots\left(1-\tan^4\dfrac{\pi}{2^n}\right).$

The value of $\lim\limits_{n\to\infty} S_n$, *is*

(a) $\dfrac{\pi^3}{4}$ (b) $\dfrac{\pi^3}{16}$

(c) $\dfrac{\pi^3}{32}$ (d) $\dfrac{\pi^3}{256}$

Sol. *(c)* Here,

$$S_n = \left(1-\tan^4\frac{\pi}{2^3}\right)\left(1-\tan^4\frac{\pi}{2^4}\right)\ldots\left(1-\tan^4\frac{\pi}{2^n}\right)$$

$$\left(\cos^2\frac{\pi}{2^3}-\sin^2\frac{\pi}{2^3}\right)\left(\cos^2\frac{\pi}{2^4}-\sin^2\frac{\pi}{2^4}\right)$$

$$= \frac{\ldots\left(\cos^2\dfrac{\pi}{2^n}-\sin^2\dfrac{\pi}{2^n}\right)}{\left(\cos^4\dfrac{\pi}{2^3}\cdot\cos^4\dfrac{\pi}{2^4}\ldots\cos^4\dfrac{\pi}{2^n}\right)}$$

$$= \frac{\left(\cos\dfrac{\pi}{2^2}\right)\cdot\left(\cos\dfrac{\pi}{2^3}\right)\ldots\left(\cos\dfrac{\pi}{2^{n-1}}\right)}{\left(\cos^4\dfrac{\pi}{2^3}\cdot\cos^4\dfrac{\pi}{2^4}\ldots\cos^4\dfrac{\pi}{2^n}\right)}$$

$$= \frac{\dfrac{1}{\sqrt{2}}}{\left(\cos\dfrac{\pi}{2^3}\cdot\cos\dfrac{\pi}{2^3}\ldots\cos\dfrac{\pi}{2^{n-1}}\right)^3}\cdot\frac{1}{\cos^4\dfrac{\pi}{2^n}} \qquad \ldots(i)$$

Let $M = \cos\dfrac{\pi}{2^3}\cdot\cos\dfrac{\pi}{2^4}\ldots\cos\dfrac{\pi}{2^{n-1}}$

$$= \frac{\sin\left(2^{n-3}\cdot\dfrac{\pi}{2^{n-1}}\right)}{2^{n-3}\cdot\sin\left(\dfrac{\pi}{2^{n-1}}\right)} = \frac{\sin\left(\dfrac{\pi}{2^2}\right)}{2^{n-3}\cdot\sin\left(\dfrac{\pi}{2^{n-1}}\right)}$$

$\therefore \quad S_n = \lim\limits_{n\to\infty}\dfrac{\dfrac{1}{\sqrt{2}}}{\left(\dfrac{1}{\sqrt{2}}\right)^3}\times\left\{2^{n-3}\cdot\sin\dfrac{\pi}{2^{n-1}}\right\}^3\times\dfrac{1}{\cos^n\left(\dfrac{\pi}{2^n}\right)}$

$$= \lim_{n\to\infty} 2\cdot\left(\frac{\sin\dfrac{\pi}{2^{n-1}}}{\dfrac{\pi}{2^{n-1}}}\right)^3\cdot\left(\frac{\pi}{4}\right)^3\times\frac{1}{\cos^n\left(\dfrac{\pi}{2^n}\right)}$$

$$= 2\left(\frac{\pi}{4}\right)^3\cdot\frac{1}{1} = \frac{\pi^3}{32}$$

● ***Ex. 6*** *The value of* $\lim\limits_{x\to 2}\dfrac{\sqrt{x+7}-3\sqrt{2x-3}}{\sqrt[3]{x+6}-2\sqrt[3]{3x-5}}$ *is*

(a) $\dfrac{33}{23}$ (b) $\dfrac{34}{23}$ (c) $\dfrac{54}{25}$ (d) None of these

Sol. *(b)* We have, $L = \lim\limits_{x\to 2}\dfrac{\sqrt{x+7}-3\sqrt{2x-3}}{\sqrt[3]{x+6}-2\sqrt[3]{3x-5}}$ $\left[\dfrac{0}{0}\text{ form}\right]$

Let $x-2 = t$, such that $x\to 2 \implies t\to 0$

$\therefore \quad L = \lim\limits_{t\to 0}\dfrac{(t+9)^{1/2}-3(2t+1)^{1/2}}{(t+8)^{1/3}-2(3t+1)^{1/3}}$

$$= \frac{3}{2}\cdot\lim_{t\to 0}\frac{(1+t/9)^{1/2}-(2t+1)^{1/2}}{\left(1+\dfrac{t}{8}\right)^{1/3}-(1+3t)^{1/3}} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \frac{3}{2}\cdot\lim_{t\to 0}\frac{\left(1+\dfrac{1}{2}\cdot\dfrac{t}{9}\right)-\left(1+\dfrac{1}{2}\cdot(2t)\right)}{\left(1+\dfrac{1}{3}\cdot\dfrac{t}{8}\right)-\left(1+\dfrac{1}{3}\cdot(3t)\right)};$$

$[\text{using } (1+x)^n = 1+nx, \text{ as } x\to 0]$

$$= \frac{3}{2}\cdot\lim_{t\to 0}\frac{\dfrac{t}{18}-t}{\dfrac{t}{24}-t} = \frac{3}{2}\cdot\frac{\left(\dfrac{1}{18}-1\right)}{\left(\dfrac{1}{24}-1\right)} = \frac{34}{23}$$

● ***Ex. 7*** *Let* $\Delta(x) = \begin{vmatrix} f(x+\alpha) & f(x+2\alpha) & f(x+3\alpha) \\ f(\alpha) & f(2\alpha) & f(3\alpha) \\ f'(\alpha) & f'(2\alpha) & f'(3\alpha) \end{vmatrix}$ *for some real values differential function* f *and constant* α, $\lim\limits_{x\to 0}\dfrac{\Delta(x)}{x}$ *is equal to*

(a) 0 (b) 1 (c) 2 (d) 3

Sol. *(a)* Here, $\Delta(0)=\begin{vmatrix} f(\alpha) & f(2\alpha) & f(3\alpha) \\ f(\alpha) & f(2\alpha) & f(3\alpha) \\ f'(\alpha) & f'(2\alpha) & f'(3\alpha) \end{vmatrix}$

Thus, $\lim_{x\to 0}\frac{\Delta(x)}{x}$ $\left[\frac{0}{0}\text{ form}\right]$

$=\lim_{x\to 0}\frac{\Delta'(x)}{1}=\Delta'(0)$ [applying L'Hospital's rule]

$\therefore \lim_{x\to 0}\frac{\Delta(x)}{x}=\Delta'(0)$ ...(i)

Given, $\Delta(x)=\begin{vmatrix} f(x+\alpha) & f(x+2\alpha) & f(x+3\alpha) \\ f(\alpha) & f(2\alpha) & f(3\alpha) \\ f'(\alpha) & f'(2\alpha) & f'(3\alpha) \end{vmatrix}$

Using definition of differentiation of determinant,

$$\Delta'(x)=\begin{vmatrix} f'(x+\alpha) & f'(x+2\alpha) & f'(x+3\alpha) \\ f(\alpha) & f(2\alpha) & f(3\alpha) \\ f'(\alpha) & f'(2\alpha) & f'(3\alpha) \end{vmatrix}+\begin{vmatrix} f(x+\alpha) & f(x+2\alpha) & f(x+3\alpha) \\ 0 & 0 & 0 \\ f'(\alpha) & f'(2\alpha) & f'(3\alpha) \end{vmatrix}+\begin{vmatrix} f(x+\alpha) & f(x+2\alpha) & f(x+3\alpha) \\ f(\alpha) & f(2\alpha) & f(3\alpha) \\ 0 & 0 & 0 \end{vmatrix}$$

[as α is constant $\Rightarrow \frac{d}{dx}(\alpha)=0$]

$\therefore\ \Delta'(x)=\begin{vmatrix} f'(x+\alpha) & f'(x+2\alpha) & f'(x+3\alpha) \\ f(\alpha) & f(2\alpha) & f(3\alpha) \\ f'(\alpha) & f'(2\alpha) & f'(3\alpha) \end{vmatrix}$

or $\Delta'(0)=\begin{vmatrix} f'(\alpha) & f'(2\alpha) & f'(3\alpha) \\ f(\alpha) & f(2\alpha) & f(3\alpha) \\ f'(\alpha) & f'(2\alpha) & f'(3\alpha) \end{vmatrix}=0$

[$\because R_1$ and R_3 are identical]

Hence, $\lim_{x\to 0}\frac{\Delta(x)}{x}=\Delta'(0)=0$

● ***Ex. 8*** *Let* $\lim_{x\to 1}\frac{x^a-ax+a-1}{(x-1)^2}=f(a)$. *The value of* $f(101)$ *equals*

(a) 5050
(b) 5151
(c) 4950
(d) 101

Sol. *(a)* We have, $\lim_{x\to 1}\frac{x^a-ax+a-1}{(x-1)^2}$

Put $x=1+h$, we have

$$\lim_{h\to 0}\frac{(1+h)^a-a(1+h)+a-1}{h^2}$$

$$=\lim_{h\to 0}\frac{\left(1+ah+\frac{a(a-1)}{2!}h^2+\ldots\right)-a-ah+a-1}{h^2}$$

$\therefore f(a)=\frac{a(a-1)}{2}$; $f(101)=5050$

● ***Ex. 9*** $\lim_{x\to 1}\frac{nx^{n+1}-(n+1)x^n+1}{(e^x-e)\sin\pi x}$, *where* $n=100$ *is equal to*

(a) $\frac{5050}{\pi e}$ (b) $\frac{100}{\pi e}$ (c) $-\frac{5050}{\pi e}$ (d) $-\frac{4950}{\pi e}$

Sol. *(c)* Let $l=\lim_{x\to 1}\frac{nx^n(x-1)-(x^n-1)}{(e^x-e)\sin\pi x}$

Put $x=1+h$, so that as $x\to 1, h\to 0$

$\therefore\ l=-\lim_{h\to 0}\frac{h\cdot n(1+h)^n-\{(1+h)^n-1\}}{e\,(e^h-1)\sin\pi h}$

$$\Rightarrow\ l=-\lim_{x\to 1}\frac{n\cdot h(1+{}^nC_1h+{}^nC_2h^2+{}^nC_3h^3+\ldots)-(1+{}^nC_1h+{}^nC_2h^2+{}^nC_3h^3+\ldots-1)}{\pi e(h^2)\left(\frac{e^h-1}{h}\right)\left(\frac{\sin\pi h}{\pi h}\right)}$$

$$=-\frac{n^2-{}^nC_2}{\pi e}=-\left[\frac{2n^2-n(n-1)}{2\pi e}\right]=-\frac{n^2+n}{2(\pi e)}=-\frac{n(n+1)}{2(\pi e)}$$

If $n=100\ \Rightarrow\ l=-\left(\frac{5050}{\pi e}\right)$

● ***Ex. 10*** $\lim_{n\to\infty}\frac{1^2n+2^2(n-1)+3^2(n-2)+\ldots+n^2\cdot 1}{1^3+2^3+3^3+\ldots+n^3}$ *is equal to*

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{1}{2}$ (d) $\frac{1}{6}$

Sol. *(a)* We have,

$$\lim_{n\to\infty}\frac{1^2n+2^2(n-1)+3^2(n-2)+\ldots+n^2\{n-(n-1)\}}{\Sigma n^3}$$

Numerator $= n(1^2+2^2+\ldots+n^2)-\{1\cdot 2^2+2\cdot 3^2+3\cdot 4^2+\ldots+(n-1)n^2\}$

$$=n\Sigma n^2-\sum_{r=2}^{n}(r-1)\cdot r^2$$

$$=n\Sigma n^2-\sum_{r=1}^{n}(r^3-r^2)$$

$$=n\Sigma n^2-[\Sigma n^3-\Sigma n^2]=(n+1)\Sigma n^2-\Sigma n^3$$

$$\therefore\ l=\lim_{n\to\infty}\frac{(n+1)\Sigma n^2-\Sigma n^3}{\Sigma n^3}$$

$$=\lim_{n\to\infty}\frac{4(n+1)\,n(n+1)(2n+1)}{6n\,(n+1)\,n\,(n+1)}-1=\frac{4}{3}-1=\frac{1}{3}$$

Aliter $l=\dfrac{1^2n+2^2\,(n-1)+\ldots+n^2\cdot 1}{\underbrace{1^3+2^3+3^3+\ldots+n^3}}+1-1$

$$l=\frac{1^2(n+1)+2^2\,(n+1)+\ldots+n^2\,(n+1)}{\Sigma n^3}-1$$

$$l=\frac{(n+1)\,\Sigma n^2}{\Sigma n^3}-1$$

$$\Rightarrow\quad\lim_{n\to\infty}\frac{(n+1)\cdot n(n+1)(2n+1)}{6\cdot\dfrac{n^2\,(n+1)^2}{4}}-1=\frac{4}{3}-1=\frac{1}{3}$$

● ***Ex. 11*** *The value of* $\lim\limits_{x\to\infty}\dfrac{\cot^{-1}(x^{-a}\log_a x)}{\sec^{-1}(a^x\log_x a)}\,(a>1)$ *is equal to*

(a) 1 (b) 0 (c) $\dfrac{\pi}{2}$ (d) Doesn't exist

Sol. *(a)* We have, $\lim\limits_{x\to\infty}\dfrac{\cot^{-1}\left(\dfrac{\log_a x}{x^a}\right)}{\sec^{-1}\left(\dfrac{a^x}{\log_a x}\right)}$;

as $\lim\limits_{x\to\infty}\left(\dfrac{\log_a x}{x^a}\right)\to 0$ and $\left(\dfrac{a^x}{\log_a x}\right)\to\infty$

[using L' Hospital's rule]

$$\therefore\qquad l=\frac{\frac{\pi}{2}}{\frac{\pi}{2}}=1$$

● ***Ex. 12*** *Suppose that* a *and* $b\,(b\neq a)$ *are real positive numbers, thene the value of* $\lim\limits_{t\to 0}\left(\dfrac{b^{t+1}-a^{t+1}}{b-a}\right)^{1/t}$ *is equal to*

(a) $\dfrac{a\ln b-b\ln a}{b-a}$ (b) $\dfrac{b\ln b-a\ln a}{b-a}$

(c) $b\ln b-a\ln a$ (d) $\left(\dfrac{b^b}{a^a}\right)^{\frac{1}{b-a}}$

Sol. *(d)* Here, $\lim\limits_{t\to 0}\left(\dfrac{b^{t+1}-a^{t+1}}{b-a}\right)^{\frac{1}{t}}$.

Obviously, limit is of the form 1^∞.

Hence, $l=e^{\lim\limits_{t\to 0}\frac{1}{t}\left[\frac{b^{t+1}-a^{t+1}}{b-a}-1\right]}=e^{\lim\limits_{t\to 0}\left(\frac{b^{t+1}-a^{t+1}-b+a}{t(b-a)}\right)}$

$$=e^{\lim\limits_{t\to 0}\left(\frac{b(b^t-1)-a(a^t-1)}{t(b-a)}\right)}=e^{\frac{b\ln b-a\ln a}{b-a}}$$

$$=e^{\frac{\ln b^b-\ln a^a}{b-a}}=e^{\frac{\ln\frac{b^b}{a^a}}{b-a}}=e^{\ln\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}}=\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$$

● ***Ex. 13*** $\lim\limits_{x\to\infty}\dfrac{\cot^{-1}(\sqrt{x+1}-\sqrt{x})}{\sec^{-1}\left\{\left(\dfrac{2x+1}{x-1}\right)^x\right\}}$ *is equal to*

(a) 1 (b) 0 (c) $\dfrac{\pi}{2}$ (d) None of these

Sol. *(a)* As, $\lim\limits_{x\to\infty}\sqrt{x+1}-\sqrt{x}=0\Rightarrow\cot^{-1}(0)=\dfrac{\pi}{2}$

$$\lim_{x\to\infty}\left(\frac{2x+1}{x-1}\right)^x\to\infty\quad\Rightarrow\ \sec^{-1}(\infty)=\frac{\pi}{2}$$

$$\therefore\qquad l=1$$

● ***Ex. 14*** $\lim\limits_{n\to\infty}\cos(\pi\sqrt{n^2+n})$ *(when* n *is an integer) is equal to*

(a) 1 (b) -1 (c) 0 (d) Doesn't exist

Sol. *(c)* Let $l=\lim\limits_{n\to\infty}\pm\cos(n\pi-\pi\sqrt{n^2+n})$

$$=\lim_{n\to\infty}\pm\cos(\pi(n-\sqrt{n^2+n}))$$

$$=\lim_{n\to\infty}\pm\cos\left(\frac{(\pi(+n))}{n+\sqrt{n^2+n}}\right)$$

$$=\lim_{n\to\infty}\pm\cos\left(\frac{n\pi}{n+n\sqrt{1+\frac{1}{n}}}\right)$$

$$=\lim_{n\to\infty}\pm\cos\left(\frac{\pi}{1+\sqrt{1+\frac{1}{n}}}\right)=\cos\frac{\pi}{2}\to 0$$

Aliter

We have, $\pi\sqrt{n^2+n}$

$$=\pi n\left(1+\frac{1}{n}\right)^{1/2}=n\pi\left[1+\frac{1}{2n}+\frac{1}{2}\left(\frac{1}{2}-1\right)\frac{1}{2!}\frac{1}{n^2}+\ldots\right]$$

$$=\pi\left[n+\frac{1}{2}+\frac{1}{2}\left(\frac{1}{2}-1\right)\frac{1}{2!}\frac{1}{n}+\ldots\right]$$

As $n\to\infty;\ \dfrac{\pi}{2}\cdot\left[2n+1+\left(\dfrac{1}{2}-1\right)\dfrac{1}{2!}\dfrac{1}{n}+\ldots\right]=(2n+1)\dfrac{\pi}{2}$

$$\therefore\qquad\lim_{n\to\infty}\cos\left((2n+1)\frac{\pi}{2}\right)=0$$

● **Ex. 15** *The value of* $\lim_{x\to 0}\dfrac{(\tan(\{x\}-1))\sin\{x\}}{\{x\}(\{x\}-1)}$, *where* $\{x\}$ *denotes the fractional part function is*

(a) 1 (b) tan 1 (c) sin 1 (d) Doesn't exist

Sol. *(d)* Let $f(x)=\lim_{x\to 0}\dfrac{(\tan(\{x\}-1))\sin\{x\}}{\{x\}(\{x\}-1)}$

$$\lim_{x\to 0^+} f(x)=\lim_{h\to 0^+}\frac{\tan(h-1)\cdot\sin h}{h(h-1)}=\frac{\tan(-1)}{-1}=\tan 1$$

$$\text{LHL}=\lim_{h\to 0}\frac{\tan((1-h)-1)\sin(1-h)}{(1-h)(1-h-1)}=\frac{\sin 1}{1}=\sin 1$$

Hence, $\lim_{x\to 0} f(x)$ doesn't exist.

● **Ex. 16** $\lim_{x\to 0^-}(-\ln(\{x\}+|[x]|))^{\{x\}}$ *is equal to*

(a) 0 (b) 1 (c) $\ln 2$ (d) $\ln\dfrac{1}{2}$

Sol. *(d)* We have, $\lim_{x\to 0^-}=(-\ln(\{x\}+|[x]|))^{\{x\}}$

$$=\lim_{x\to 0^-}(-\ln(\{0-h\}+|[-h]|))^{\{-h\}}$$

$$=\lim_{x\to 0^-}(-\ln(1-h+1))^{1-h}=-\ln 2=\ln\left(\frac{1}{2}\right)$$

● **Ex. 17** $\lim_{x\to\infty}\dfrac{2+2x+\sin 2x}{(2x+\sin 2x)\,e^{\sin x}}$ *is equal to*

(a) 0 (b) 1 (c) -1 (d) Non-existent

Sol. *(d)* We have, $\lim_{x\to\infty}\dfrac{2+2x+\sin 2x}{(2x+\sin 2x)e^{\sin x}}$

$$=\lim_{x\to\infty}\frac{\frac{2}{x}+2+\frac{\sin 2x}{x}}{\left(2+\frac{\sin 2x}{x}\right)e^{\sin x}},\ \text{as } x\to\infty$$

$\Rightarrow l=\lim_{x\to\infty}\dfrac{2}{2\cdot e^{\sin x}}=$ oscillatory between $\dfrac{1}{e}$ and $\dfrac{1}{e^{-1}}\Rightarrow$ Non-existent

● **Ex. 18** *The value of* $\lim_{x\to 0}(\cos ax)^{\operatorname{cosec}^2 bx}$ *is*

(a) $e^{\left(-\frac{8b^2}{a^2}\right)}$ (b) $e^{\left(-\frac{8a^2}{b^2}\right)}$

(c) $e^{\left(-\frac{a^2}{2b^2}\right)}$ (d) $e^{\left(-\frac{b^2}{2a^2}\right)}$

Sol. *(c)* Let $l=\lim_{x\to 0}(\cos ax)^{\operatorname{cosec}^2 bx}$

$$\Rightarrow\quad l=e^{\lim_{x\to\infty}\operatorname{cosec}^2 bx\,(\cos ax-1)}$$

Now, $-\lim_{x\to 0}\dfrac{1-\cos ax}{\sin^2 bx}=-\lim_{x\to 0}\dfrac{\sin^2 ax}{\sin^2 bx}\cdot\dfrac{1}{1+\cos ax}$

$$=-\frac{a^2}{2b^2},\quad l=e^{-\frac{a^2}{2b^2}}$$

● **Ex. 19** $\lim_{n\to\infty}\sum_{r=1}^{n}\dfrac{r}{n^2+n+r}$ *equals*

(a) 0 (b) 1/3 (c) 1/2 (d) 1

Sol. *(c)* Let $f(n)=\dfrac{1}{n^2+n+1}+\dfrac{2}{n^2+n+2}+\ldots+\dfrac{n}{n^2+n+n}$

Consider $g(n)=\dfrac{1}{n^2+n+n}+\dfrac{2}{n^2+n+n}+\ldots+\dfrac{n}{n^2+n+n}$

$$=\frac{1+2+3+\ldots+n}{n^2+2n}=\frac{n(n+1)}{2(n^2+2n)}$$

$$g(n)<f(n) \qquad \ldots(\text{i})$$

Similarly, $h(n)=\dfrac{1}{n^2+n+1}+\dfrac{2}{n^2+n+1}+\ldots+\dfrac{n}{n^2+n+1}$

$$=\frac{n(n+1)}{2(n^2+n+1)}$$

$$\therefore\quad f(n)<h(n) \qquad \ldots(\text{ii})$$

From Eqs. (i) and (ii), $g(n)<f(n)<h(n)$

But $\lim_{n\to\infty} g(n)=\lim_{n\to\infty} h(n)=\dfrac{1}{2}$.

Hence, using Sandwich theorem, $\lim_{n\to\infty} f(n)=\dfrac{1}{2}$.

● **Ex. 20** *The value of* $\lim_{n\to\infty}(\sqrt{n^2+n+1}-[\sqrt{n^2+n+1}])$, *where* $[\cdot]$ *denotes the greatest integer function is*

(a) 0 (b) 1/2 (c) 2/3 (d) 1/4

Sol. *(b)* We know that, $n<\sqrt{n^2+n+1}<n+1$.

Hence, $[\sqrt{n^2+n+1}]=n$

$$\therefore\ l=\lim_{n\to\infty}(\sqrt{n^2+n+1}-n)=\lim_{n\to\infty}\frac{n+1}{\sqrt{n^2+n+1}+n}=\frac{1}{2}$$

● **Ex. 21** $\lim_{x\to 1}\dfrac{\sin^2(x^3+x^2+x-3)}{1-\cos(x^2-4x+3)}$ *has the value equal to*

(a) 18 (b) 9/2 (c) 9 (d) None of these

Sol. *(a)* We have, $\lim_{x\to 1}\dfrac{\sin^2(x^3+x^2+x-3)}{1-\cos(x^2-4x+3)}$

$$=\lim_{x\to 1}\frac{\sin^2(x^3+x^2+x-3)}{(x^3+x^2+x-3)^2}\cdot\frac{(x^3+x^2+x-3)^2}{1-\cos(x^2-4x+3)}$$

$$=(1)\lim_{x\to 1}\frac{(x^2-4x+3)^2}{1-\cos(x^2-4x+3)}\cdot\frac{(x^3+x^2+x-3)^2}{(x^2-4x+3)^2}$$

$$= (1)(2)\lim_{x\to 1}\left(\frac{x^3+x^2+x-3}{x^2-4x+3}\right)^2 = 2l^2$$

where, $l = \lim_{x\to 1}\frac{3x^2+2x+1}{2x-4}$ [using L' Hospital's rule]

$$= \frac{6}{-2} = -3 \Rightarrow l = 2(-3)^2 = 18$$

● ***Ex. 22*** *The graph of function $y = f(x)$ has a unique tangent at $(e^a, 0)$ through which the graph passes, then* $\lim_{x\to e^a}\frac{\log(1+7f(x)) - \sin(f(x))}{3f(x)}$ *equals*

(a) 1 (b) 2
(c) 7 (d) None of these

Sol. *(b)* Here, $\lim_{x\to e^a}\frac{\log(1+7f(x)) - \sin(f(x))}{3f(x)}$ $\left[\frac{0}{0}\text{ form}\right]$

$$= \lim_{x\to e^a}\frac{7f'(x) - \{\cos(f(x))\cdot f'(x)\}\{1+7f(x)\}}{3f'(x)\cdot\{1+7f(x)\}}$$

[using L'Hospital's rule]

$$= \lim_{x\to e^a}\frac{7-\cos(f(x))\{1+7f(x)\}}{3\{1+7f(x)\}} = \frac{7-1}{3} = 2$$

JEE Type Solved Examples :
More than One Correct Option Type Questions

● ***Ex. 23*** *If $f(x) = \frac{3x^2+ax+a+1}{x^2+x-2}$, which of the following can be correct?*

(a) $\lim_{x\to 1} f(x)$ exists $\Rightarrow a = -2$

(b) $\lim_{x\to -2} f(x)$ exists $\Rightarrow a = 13$

(c) $\lim_{x\to 1} f(x) = 4/3$

(d) $\lim_{x\to -2} f(x) = -1/3$

Sol. *(a, b, c, d)* Given, $f(x) = \frac{3x^2+ax+a+1}{(x+2)(x-1)}$

As $x\to 1$, Dr. $\to 0$, hence as $x\to 1$, Nr. $\to 0$

$\therefore \quad 3+2a+1=0 \Rightarrow a=-2$

As $x\to -2$, Dr. $\to 0$, hence as $x\to -2$, Nr. $\to 0$

$\therefore \quad 12-2a+a+1=0 \Rightarrow a=13$

Now, $\lim_{x\to 1} f(x) = \lim_{x\to 1}\frac{3x^2-2x-1}{(x+2)(x-1)}$

$$= \lim_{x\to 1}\frac{(3x+1)(x-1)}{(x+2)(x-1)} = \frac{4}{3}$$

and $\lim_{x\to -2}\frac{3x^2+13x+14}{(x+2)(x-1)} = \lim_{x\to -2}\frac{(3x+7)(x+2)}{(x+2)(x-1)} = -\frac{1}{3}$

● ***Ex. 24*** *Let* $f(x) = \begin{cases} \frac{\tan^2\{x\}}{x^2-[x]^2}, & \text{for } x>0 \\ 1, & \text{for } x=0 \\ \sqrt{\{x\}\cot\{x\}}, & \text{for } x<0 \end{cases}$ *where $[x]$ is the step up function and $\{x\}$ is the fractional part function of x, then*

(a) $\lim_{x\to 0^+} f(x) = 1$ (b) $\lim_{x\to 0^-} f(x) = 1$

(c) $\cot^{-1}\left(\lim_{x\to 0^-} f(x)\right)^2 = 1$ (d) None of these

Sol. *(a, c)* We have, $f(x) = \begin{cases} \frac{\tan^2\{x\}}{x^2-[x]^2}, & x>0 \\ 1, & x=0 \\ \sqrt{\{x\}\cot\{x\}}, & x<0 \end{cases}$

$$\text{RHL} = \lim_{x\to 0^+} f(x) = \lim_{h\to 0}\frac{\tan^2\{h\}}{h^2-[h]^2} = \lim_{h\to 0}\frac{\tan^2 h}{h^2} = 1$$

$$\text{LHL} = \lim_{x\to 0^-} f(x) = \lim_{h\to 0}\sqrt{\{-h\}\cot\{-h\}}$$

$$= \lim_{h\to 0}\sqrt{(1-h)\cot(1-h)} = \sqrt{\cot 1}$$

$$\therefore \quad \cot^{-1}\left(\lim_{x\to 0^-} f(x)\right)^2 = \cot^{-1}(\sqrt{\cot 1})^2 = 1$$

● ***Ex. 25*** *Given that the derivative $f'(a)$ exists. Indicate which of the following statement(s) is/are always true?*

(a) $f'(a) = \lim_{h\to a}\frac{f(h)-f(a)}{h-a}$

(b) $f'(a) = \lim_{h\to 0}\frac{f(a)-f(a-h)}{h}$

(c) $f'(a) = \lim_{t\to 0}\frac{f(a+2t)-f(a)}{t}$

(d) $f'(a) = \lim_{t\to 0}\frac{f(a+2t)-f(a+t)}{2t}$

Sol. *(a, b)* Here, options (a) and (b) are true by definition.
Option (c) is false, as

$$\lim_{t\to 0}\frac{f(a+2t)-f(a)}{t}=2f'(a)$$

and $\lim_{t\to 0}\frac{f(a+2t)-f(a+t)}{2t}=\frac{1}{2}[2f'(a)-f'(a)]=\frac{1}{2}f'(a)$

Hence, option (d) is false.

● **Ex. 26** *Let* $f(x)=\lim_{n\to\infty}\sum_{r=0}^{n}\frac{x}{(rx+1)\{(r+1)x+1\}}$. *Then,*

(a) $f(0)=0$ (b) $f(0)=x$
(c) $f(0^+)=1$ (d) $f(0^-)=1$

Sol. *(a, c, d)* Given, $f(x)=\lim_{n\to\infty}\sum_{r=0}^{n}\frac{x}{(rx+1)\{(r+1)x+1\}}$

$$=\lim_{n\to\infty}\sum_{r=0}^{n}\frac{[(r+1)x+1)]-(rx+1)}{(rx+1)[(r+1)x+1]}$$

$$=\lim_{n\to\infty}\sum_{r=0}^{n}\left(\frac{1}{rx+1}-\frac{1}{(r+1)x+1}\right)$$

$$=\lim_{n\to\infty}\left[\left(\frac{1}{1}-\frac{1}{x+1}\right)+\left(\frac{1}{x+1}-\frac{1}{2x+1}\right)+\ldots+\left(\frac{1}{nx+1}-\frac{1}{(n+1)x+1}\right)\right]$$

$$=\lim_{n\to\infty}\left[1-\frac{1}{(n+1)x+1}\right]=\lim_{n\to\infty}\frac{(n+1)x}{(n+1)x+1}$$

$\therefore\quad f(x)=\begin{cases}0, & x=0\\ 1, & x\neq 0\end{cases}\Rightarrow f(0)=0,\ f(0^+)=f(0^-)=1.$

JEE Type Solved Examples : Passage Based Questions

Passage I

(Q. Nos. 27 to 29)

Let $f(x)=\begin{cases}e^{\{x^2\}}-1, & x>0\\ \dfrac{\sin x-\tan x+\cos x-1}{2x^2+\ln(2+x)+\tan x}, & x<0,\\ 0, & x=0\end{cases}$

where $\{\}$ *represents fractional part function. Suppose lines* L_1 *and* L_2 *represent tangent and normal to curve* $y=f(x)$ *at* $x=0$. *Consider the family of circles touching both the lines* L_1 *and* L_2.

● **Ex.27** *Ratio of radii of two circles belonging to this family cutting each other orthogonally is*

(a) $2+\sqrt{3}$ (b) $\sqrt{3}$
(c) $2+\sqrt{2}$ (d) $2-\sqrt{2}$

● **Ex.28** *A circle having radius unity is inscribed in the triangle formed by* L_1 *and* L_2 *and a tangent to it. Then, the minimum area of the triangle possible is*

(a) $3+\sqrt{2}$ (b) $2+\sqrt{3}$
(c) $3+2\sqrt{2}$ (d) $3-2\sqrt{2}$

● **Ex.29** *If centres of circles belonging to family having equal radii r are joined, the area of figure formed is*

(a) $2r^2$ (b) $4r^2$
(c) $8r^2$ (d) r^2

■ **Sol.** (Q. Nos. 27 to 29)

$$\text{LHD}=\lim_{h\to 0}\frac{\dfrac{-\sin h+\tan h+\cos h-1}{2h^2+\ln(2-h)-\tan h}-0}{-h}$$

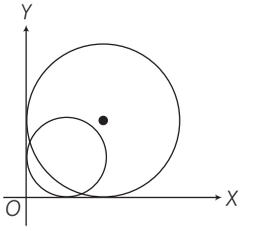
$$=\lim_{h\to 0}\frac{\dfrac{\sin h}{h}-\dfrac{\tan h}{h}+\dfrac{1-\cos h}{h^2}\times h}{2h^2+\ln(2-h)-\tan h}=0$$

$$f'(0^+)=\text{RHD}=\lim_{h\to 0}\frac{e^{h^2}-1-0}{h}=h\times\frac{e^{h^2}-1}{h^2}=0$$

$L_1\equiv y=0$ and $L_2\equiv x=0$

27. *(a)* $(x-r)^2+(y-r)^2=r^2$ [family of circle]

$x^2+y^2-2rx-2ry+r^2=0$



$\because\quad 2(r_1r_2+r_1r_2)=r_1^2+r_2^2$

or $\quad 4r_1r_2=r_1^2+r_2^2$

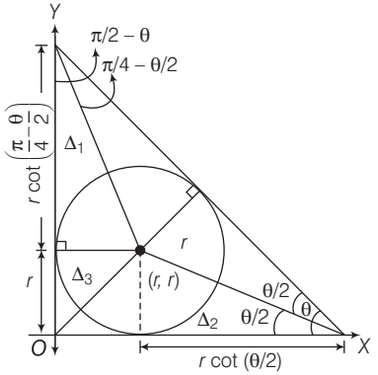
$$\left(\frac{r_2}{r_1}\right)^2-4\left(\frac{r_2}{r_1}\right)+1=0$$

$$\Rightarrow\quad \frac{r_2}{r_1}=\frac{4\pm\sqrt{12}}{2}=2\pm\sqrt{3}$$

28. *(c)* $2[\Delta_1 + \Delta_2 + \Delta_3]$

$$\Delta = 2 \times \frac{1}{2}\left(\cot\left(\frac{\pi}{4} - \frac{\theta}{2}\right) + \cot\frac{\theta}{2} + 1\right) \quad \left[\text{using, } \frac{1}{2}ab\right]$$

$$\Delta = \frac{\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right)}{\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)} + \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} + 1$$



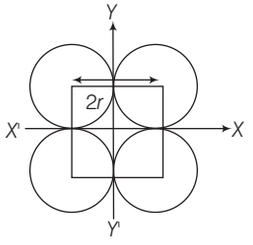
$$\Rightarrow \quad \Delta = 1 + \frac{2\sin\frac{\pi}{4}}{2\sin\frac{\theta}{2}\cdot\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)}$$

$$\Delta = 1 + \frac{\sqrt{2}}{\cos\left(\theta - \frac{\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right)}$$

Δ is minimum, if denominators is maximum when $\theta = \frac{\pi}{4}$,

$$\Delta_{\min} = 1 + \frac{\sqrt{2}}{1 - \frac{1}{\sqrt{2}}} = 1 + \frac{2}{\sqrt{2} - 1} = 1 + 2(\sqrt{2} + 1) = 3 + 2\sqrt{2}$$

29. *(b)* Area $= (2r)^2 = 4r^2$



Passage II

(Q. Nos. 30 to 32)

Let A be $n \times n$ matrix given by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \ldots a_{1n} \\ a_{21} & a_{22} & a_{23} \ldots a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} \ldots a_{nn} \end{bmatrix}$$

Such that each horizontal row is arithmetic progression and each vertical column is a geometrical progression. It is known that each column in geometric progression have the same common ratio. Given that $a_{24} = 1$, $a_{42} = \frac{1}{8}$ and $a_{43} = \frac{3}{16}$.

● ***Ex.30*** *Let $S_n = \sum_{j=1}^{n} a_{4j}$, $\lim_{n\to\infty} \frac{S_n}{n^2}$ is equal to*

(a) $\frac{1}{4}$ (b) $\frac{1}{8}$ (c) $\frac{1}{16}$ (d) $\frac{1}{32}$

Sol. (*d*) Here, $a_{43} = a_{42} + d$

$$\Rightarrow \quad \frac{3}{16} = \frac{1}{8} + d \;\Rightarrow\; d = \frac{1}{16}$$

[common difference of 4th row]

$$\therefore \quad a_{41} = a_{42} - d = \frac{1}{8} - \frac{1}{16} = \frac{1}{16}$$

$$\therefore \quad a_{41} = \frac{1}{16}, a_{42} = \frac{2}{16}, a_{43} = \frac{3}{16}, \ldots, a_{4n} = \frac{n}{16}$$

Now, all elements of 4th row are known

$$S_n = \sum_{j=1}^{n} a_{4j} = \frac{n(n+1)}{2(16)}; \lim_{n\to\infty} \frac{S_n}{n^2} = \frac{1}{32}$$

● ***Ex.31*** *Let d_i be the common difference of the elements in ith row, then $\sum_{i=1}^{n} d_i$ is*

(a) n (b) $\frac{1}{2} - \frac{1}{2^{n+1}}$ (c) $1 - \frac{1}{2^n}$ (d) $\frac{n+1}{2^n}$

Sol. (*c*) Also, $a_{24}r^2 = a_{44} = \frac{4}{16}$

$$\Rightarrow \quad r^2 = \frac{4}{16} \;\Rightarrow\; r = \frac{1}{2} \quad \left[\text{common ratio of all GP is } \frac{1}{2}\right]$$

$$\therefore \quad A = \begin{bmatrix} \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \ldots & \frac{n}{2} \\ \frac{1}{2^2} & \frac{2}{2^2} & \frac{3}{2^2} & \ldots & \frac{n}{2^2} \\ \frac{1}{2^3} & \frac{2}{2^3} & \frac{3}{2^3} & \ldots & \frac{n}{2^3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{2^n} & \frac{2}{2^n} & \frac{3}{2^n} & \ldots & \frac{n}{2^n} \end{bmatrix}_{n\times n}$$

Now, d_i = common difference in ith row $= \frac{1}{2^i}$

$$\therefore \quad \sum_{i=1}^{n} d_i = \sum_{i=1}^{n} \frac{1}{2^i} = \frac{\frac{1}{2}\left(1 - \frac{1}{2^n}\right)}{1 - \frac{1}{2}} = 1 - \frac{1}{2^n}$$

● **Ex.32** *The value of* $\lim_{n\to\infty}\sum_{i=1}^{n} a_{ii}$ *is equal to*

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) 1 (d) 2

Sol. *(d)* Given, $\sum_{i=1}^{n} a_{ii} = S = \frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\ldots+\frac{n}{2^n}$ ...(i)

$\Rightarrow \quad \frac{1}{2}S_n = \frac{1}{2^2}+\frac{2}{2^3}+\frac{3}{2^4}+\ldots+\frac{n-1}{2^n}+\frac{n}{2^{n+1}}$ ...(ii)

On subtracting Eq. (ii) from Eq. (i), we get

$$\frac{1}{2}S_n = \left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^n}\right)-\frac{n}{2^{n+1}}$$

$$\Rightarrow \frac{1}{2}S_n = \frac{\frac{1}{2}\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}}-\frac{n}{2^{n+1}} \Rightarrow S_n = 2\left(1-\frac{1}{2^n}\right)-\frac{n}{2^n}$$

$$\therefore \quad \lim_{n\to\infty}\sum_{i=1}^{n} a_{ii} = \lim_{n\to\infty} 2-\frac{2}{2^n}-\frac{2n}{2^n} = 2-0-0 = 2$$

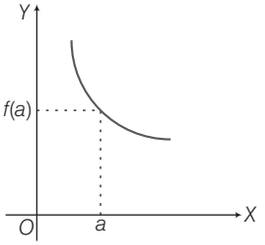
$$\Rightarrow \lim_{n\to\infty}\sum_{i=1}^{n} a_{ii} = 2$$

Passage III

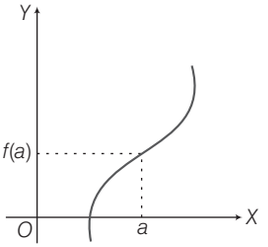
(Q. Nos. 33 to 35)

To evaluate $\lim_{x\to a}[f(x)]$, *we must analyse the* $f(x)$ *in right hand neighbourhood as well as in left hand neighbourhood of* $x = a$. e.g. *In case of continuous function, we may come across following cases.*

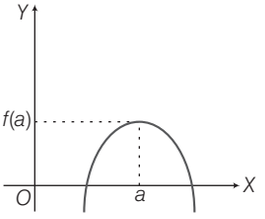
***Case* I**



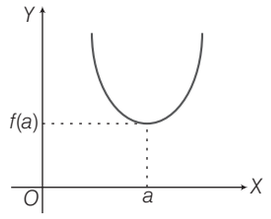
***Case* II**



***Case* III**



***Case* IV**



If $f(a)$ *is an integer, the limit will exist in Case* III *and Case* IV *but not in Case* I *and Case* II. *If* $f(a)$ *is not an integer, the limit exists in all the cases.*

● **Ex.33** *If* $f'(1) = -3$ *and* $\lim_{x\to 1}\left[f(x)-\frac{1}{2}\right]$ *does not exist, (where* $[\cdot]$ *denotes the greatest integer function), then*

(a) $f(1)$ may be integer

(b) $\{f(x)\} = \frac{1}{2}, \forall\ x \in R$ ($\{\cdot\}$ fractional part of x)

(c) $f(1)$ is not an integer

(d) None of the above

Sol. *(c)* $\lim_{x\to 1}\left[f(x)-\frac{1}{2}\right]$ does not exist, when $f(1)$ is not an integer, as $f'(1) = -3$, i.e. decreasing in the neighbourhood at $x = 1$.

● **Ex.34** $\lim_{x\to 0}\left[(1-e^x)\cdot\frac{\sin x}{|x|}\right]$ *(where* $[\cdot]$ *denotes the greatest integer function), equals*

(a) 0 (b) 1 (c) -1 (d) Doesn't exist

Sol. *(c)* Let $f(x) = \begin{cases}(1-e^x)\cdot\frac{\sin x}{x}, & x>0\\ (e^x-1)\cdot\frac{\sin x}{x}, & x<0\end{cases}$.

$$\therefore \quad f(x) = \begin{cases}<0, & \text{for } x>0\\ <0, & \text{for } x<0\end{cases} \Rightarrow \lim_{x\to 0}[f(x)] = -1$$

● **Ex.35** $\lim_{x\to 1}\left[\text{cosec}\frac{\pi x}{2}\right]^{-1/(1-x)}$ *is equal to (where* $[\cdot]$ *denotes the greatest integer function).*

(a) 0 (b) 1 (c) ∞ (d) Doesn't exist

Sol. *(b)* Let $f(x) = \text{cosec}\left(\frac{\pi x}{2}\right)$

$$\Rightarrow \quad f(1^+) = \lim_{h\to 0}\text{cosec}\left(\frac{\pi}{2}(1+h)\right) > 1$$

and $\quad f(1^-) = \lim_{h\to 0}\text{cosec}\left(\frac{\pi}{2}(1-h)\right) > 1$

$$\therefore \lim_{x\to 1}\left[\text{cosec}\frac{\pi x}{2}\right] = 1 \Rightarrow \lim_{x\to 1}\left[\text{cosec}\frac{\pi x}{2}\right]^{\frac{-1}{1-x}} = 1$$

JEE Type Solved Examples :
Statements I and II Type Questions

■ **Directions** (Ex. Nos. 36 to 38) This section is based on Statements I and II. Select the correct answer from the codes given below.

(a) Statement I is true, Statement II is true; Statement II is correct explanation for Statement I

(b) Statement I is true, Statement II is true; Statement II is not the correct explanation for Statement I

(c) Statement I is true, Statement II is false

(d) Statement I is false, Statement II is true

● ***Ex. 36***

Statement I $\lim\limits_{x \to \pi/2} \dfrac{\sin(\cot^2 x)}{(\pi - 2x)^2} = \dfrac{1}{2}$.

Statement II $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$ *and* $\lim\limits_{\theta \to 0} \dfrac{\tan\theta}{\theta} = 1$, *where* θ *is measured in radians.*

Sol. *(d)* $\lim\limits_{x \to \pi/2} \dfrac{\sin(\cot^2 x)}{\cot^2 x} \cdot \dfrac{\cot^2 x}{(\pi - 2x)^2}$; put $x = \dfrac{\pi}{2} - h$

$\Rightarrow \lim\limits_{h \to 0} \dfrac{\tan^2 h}{4h^2} = \dfrac{1}{4}$

$\therefore$ Statement I is false and Statement II is true.

● ***Ex. 37*** **Statement I** $\lim\limits_{m \to \infty} \lim\limits_{n \to \infty} \{\sin^{2m} n!\pi n\} = 0$ $m, n \in N$, when x is rational.

Statement II When $n \to \infty$ and x is rational $n!x$, is integer.

Sol. *(a)* When $n \to \infty$ and x is rational $x = \dfrac{b}{q}$.

Where p and q are integers and $q \neq 0$ $n!x = n! \times \dfrac{p}{q}$ is integer as $n!$ has factor q, when $n \to \infty$

Also, when $n\,!\,x$ is integer, $\sin(n!\pi x) = 0$.

Therefore, the given limit is zero.

● ***Ex. 38*** **Statement I** *If* $\lim\limits_{x \to 0} \left\{ f(x) + \dfrac{\sin x}{x} \right\}$ *does not exist, then* $\lim\limits_{x \to 0} f(x)$ *does not exist.*

Statement II $\lim\limits_{x \to 0} \left(\dfrac{e^{1/x} - 1}{e^{1/x} + 1} \right)$ does not exist.

Sol. *(b)* If $\lim\limits_{x \to 0} f(x)$ exist, then $\lim\limits_{x \to 0} \left(f(x) + \dfrac{\sin x}{x} \right)$ always exist as $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$ exists finitely.

Hence, $\lim\limits_{x \to 0} f(x)$ must not exist.

JEE Type Solved Examples :
Matching Type Questions

● ***Ex. 39*** *Match the following.*

	Column I	Column II
(A)	$\lim\limits_{x \to \infty} \left(\dfrac{x}{1+x} \right)^x$ equals	(p) e^2
(B)	$\lim\limits_{x \to \infty} \left(\sin\dfrac{1}{x} + \cos\dfrac{1}{x} \right)^x$ equals	(q) $e^{-1/2}$
(C)	$\lim\limits_{x \to 0} (\cos x)^{\cot^2 x}$ equals	(r) e
(D)	$\lim\limits_{x \to 0} \left[\tan\left(\dfrac{\pi}{4} + x \right) \right]^{1/x}$ equals	(s) e^{-1}

Sol. (A) $\rightarrow$ (s); (B) $\rightarrow$ (r); (C) $\rightarrow$ (q); (D) $\rightarrow$ (p)

(A) Put $x = \dfrac{1}{y}$, $\lim\limits_{y \to 0} \left(\dfrac{1}{1+y} \right)^{1/y} = e^{\lim\limits_{y \to 0} \frac{1 - y - 1}{y(1+y)}} = e^{-1}$

(B) $\lim\limits_{y \to 0} (\sin y + \cos y)^{1/y} = e^{\lim\limits_{y \to 0} \frac{\sin y + \cos y - 1}{y}} = e$

(C) $e^{\lim\limits_{x \to 0} \frac{\cos x - 1}{\frac{\tan^2 x}{x^2} x^2}} = e^{-\frac{1}{2}}$

(D) $e^{\lim\limits_{x \to 0} \frac{\tan((\pi/4) + x) - \tan(\pi/4)}{x}}$

$= e^{\lim\limits_{x \to 0} \frac{\tan x [1 + \tan((\pi/4) + x) \cdot 1]}{x}} = e^2$

● ***Ex. 40*** *Match the following.*

	Column I	Column II
(A)	$\lim_{x\to\infty}(\sqrt{x+\sqrt{x}}-\sqrt{x-\sqrt{x}})$ equals	(p) -2
(B)	The value of the $\lim_{x\to 0}\frac{\sin 2x-2\tan x}{\ln(1+x^3)}$ is	(q) -1
(C)	$\lim_{x\to 0^+}(\ln\sin^3 x-\ln(x^4+ex^3))$ equals	(r) 0
(D)	Let $\tan(2\pi\mid\sin\theta\mid)=\cot(2\pi\mid\cos\theta\mid)$, where $\theta\in R$ and $f(x)=(\mid\sin\theta\mid+\mid\cos\theta\mid)^x$. The value of $\lim_{x\to\infty}\left[\frac{2}{f(x)}\right]$ equals (here, $[\cdot]$ represents greatest integer function)	(s) 1

Sol. (A) $\to$ (s); (B) $\to$ (p); (C) $\to$ (q); (D) $\to$ (r)

(A) $\lim_{x\to\infty}\dfrac{(x+\sqrt{x})-(x-\sqrt{x})}{\sqrt{x+\sqrt{x}}+\sqrt{x-\sqrt{x}}}$

$$=\lim_{x\to\infty}\frac{2\sqrt{x}}{\sqrt{x+\sqrt{x}}+\sqrt{x-\sqrt{x}}}=1$$

(B) $\lim_{x\to 0}\dfrac{\sin 2x-2\tan x}{x^3}\quad\lim_{x\to 0}\dfrac{x^3}{\log(1+x^3)}$

$$=\lim_{x\to 0}\frac{2\sin x(\cos^2 x-1)}{\cos x\cdot x^3}\cdot 1=\lim_{x\to 0}-\frac{2\tan^3 x}{x^3}=-2$$

(C) $\lim_{x\to 0^+}[\ln\sin^3 x-\ln(x^4+ex^3)]=\lim_{x\to 0}\ln\left(\dfrac{\sin^3 x}{x^3(x+e)}\right)$

$$=\ln\left(\frac{1}{e}\right)=-1$$

(D) $\tan(2\pi\mid\sin\theta\mid)=\tan\left(\dfrac{\pi}{2}-\mid\cos\theta\mid 2\pi\right)$

$$\Rightarrow\quad 2\pi\mid\sin\theta\mid=n\pi+\frac{\pi}{2}-\mid\cos\theta\mid 2\pi$$

$$\Rightarrow\quad 2\pi(\mid\sin\theta\mid+\mid\cos\theta\mid)=n\pi+\frac{\pi}{2}$$

$$\Rightarrow\quad \mid\sin\theta\mid+\mid\cos\theta\mid=\frac{n}{2}+\frac{1}{4}\qquad\ldots(i)$$

Since, $1\le\mid\sin\theta\mid+\mid\cos\theta\mid\le\sqrt{2}$; $1\le\dfrac{n}{2}+\dfrac{1}{4}\le\sqrt{2}$

$$4\le 2n+1\le 4\sqrt{2}$$

$$\Rightarrow\quad\frac{3}{2}\le n\le\frac{4\sqrt{2}-1}{2}$$

Thus, $n=2$ is only possible value. Putting in Eq. (i),

$$\mid\sin\theta\mid+\mid\cos\theta\mid=\frac{5}{4},$$

$$g(x)=\lim_{x\to\infty}\left[2\left(\frac{4}{5}\right)^x\right]=0$$

JEE Type Solved Examples :
Single Integer Answer Type Questions

● ***Ex. 41*** *If* $\lim_{x\to 0}(x^{-3}\sin 3x+ax^{-2}+b)$ *exists and is equal to zero, the value of* $a+2b$ *is ..... .*

Sol. (*6*) We have, $\lim_{x\to 0}\dfrac{\sin 3x}{x^3}+\dfrac{a}{x^2}+b=\lim_{x\to 0}\dfrac{\sin 3x+ax+bx^3}{x^3}$

$$=\lim_{x\to 0}\frac{3\dfrac{\sin 3x}{3x}+a+bx^2}{x^2}\ \text{for}$$

existence of limit $3+a=0\Rightarrow a=-3$

$$\therefore\qquad l=\lim_{x\to 0}\frac{\sin 3x-3x+bx^3}{x^3}$$

$$=27\cdot\frac{\sin t-t}{t^3}+b=0\qquad[\because 3x=t]$$

$$=-\frac{27}{6}+b=0\quad\Rightarrow\quad b=\frac{9}{2}$$

Hence, $a+2b=-3+2\times\dfrac{9}{2}=6$

[using L' Hospital's rule]

● ***Ex. 42*** *For a certain value of* c, $\lim_{x\to-\infty}[(x^5+7x^4+2)^c-x]=\lambda$, *is finite and non-zero. The value of* $3c+\lambda$ *is ..... .*

Sol. (*2*) We have, $\lim_{x\to-\infty}[(x^5+7x^4+2)^c-x]$

$$=\lim_{x\to-\infty}\left(x^{5c}\left(1+\frac{7}{x}+\frac{2}{x^5}\right)^c-x\right)$$

$$=\lim_{x\to-\infty}x\left(x^{5c-1}\left(1+\frac{7}{x}+\frac{2}{x^5}\right)^c-1\right)$$

This will be of the form $\infty\times 0$ only, if

$5c-1=0\quad\Rightarrow\quad c=\dfrac{1}{5}$ substituting $c=\dfrac{1}{5}$, λ becomes

$$\lambda=\lim_{x\to-\infty}x[(1+x)^{1/5}-1],\ \text{where}\ x=\frac{7}{x}+\frac{2}{x^5}$$

$$=\lim_{x\to-\infty}x\left[1+\frac{x}{5}+\ldots.-1\right]=\lim_{x\to-\infty}x\left(\frac{7}{x}+\frac{2}{x^5}\right)\cdot\frac{1}{5}=\frac{7}{5}$$

Hence, $c=\dfrac{1}{5}$ and $\lambda=\dfrac{7}{5}\Rightarrow 3c+\lambda=2$

● ***Ex. 43*** *Consider the curve* $y = \tan^{-1} x$ *and a point* $A\left(1, \frac{\pi}{4}\right)$ *on it. If the variable point* $P_i(x_i, y_i)$ *moves on the curve for* $i = 1, 2, 3, \ldots, n\ (n \in N)$ *such that* $y_r = \sum_{m=1}^{r} \tan^{-1}\left(\frac{1}{2m^2}\right)$ *and* $B(x, y)$ *be the limiting position of variable point* P_n *as* $n \to \infty$*, the value of reciprocal of the slope of AB will be ..... .*

Sol. (*2*) Here, $y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} \sum_{m=1}^{n} \tan^{-1}\left(\frac{1}{2m^2}\right)$

$$= \lim_{n\to\infty} \sum_{m=1}^{n} \tan^{-1}\left(\frac{2}{1+(2m+1)(2m-1)}\right)$$

$$= \lim_{n\to\infty} \sum_{m=1}^{n} \tan^{-1}\left[\frac{(2m+1)-(2m-1)}{1+(2m+1)(2m-1)}\right]$$

$$= \lim_{n\to\infty} \sum_{m=1}^{n} (\tan^{-1}(2m+1) - \tan^{-1}(2m-1))$$

$$= \lim_{n\to\infty} (\tan^{-1}3 - \tan^{-1}1) + (\tan^{-1}5 - \tan^{-1}3) + \ldots + (\tan^{-1}(2n+1) - \tan^{-1}(2n-1))$$

$$= \lim_{n\to\infty} (\tan^{-1}(2n+1) - \tan^{-1}(1))$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

$\because B \to \left[1, \frac{\pi}{4}\right)$, i.e. coordinates of B approaches towards those of A.

Chord AB approaches to the tangent to $y = f(x)$ at A.

$\therefore$ Slope of $AB = \left(\frac{d}{dx}\tan^{-1} x\right)_{\text{at } x=1}$

$$= \left(\frac{1}{1+x^2}\right)_{\text{at } x=1} = \frac{1}{2}$$

$\Rightarrow$ (Slope of $AB)^{-1} = 2$

● ***Ex. 44*** *If* $\lim_{x\to 0} \frac{\int_0^x \frac{2t\,dt}{\sqrt{\frac{2a}{3} - \frac{t}{2} + 104}}}{(e^x - 1 - x)} = \frac{1}{19}$, *then* $\frac{a}{2010}$ *equals ..... .*

Sol. (*1*) Here, $\lim_{x\to 0} \frac{\int_0^x \frac{2t}{\sqrt{\frac{2a}{3} - \frac{t}{2} + 104}}dt}{e^x - 1 - x} = \frac{1}{19}$

[using L'Hospital's rule]

$$\Rightarrow \lim_{x\to 0} \frac{2x}{\sqrt{\frac{2a}{3} - \frac{x}{2} + 104}(e^x - 1)} = \frac{1}{19}$$

$$\Rightarrow 2\lim_{x\to 0} \frac{1}{\sqrt{\frac{2a}{3} - \frac{x}{2} + 104}} \cdot \lim_{x\to 0} \frac{x}{e^x - 1} = \frac{1}{19}$$

$$\Rightarrow 2 \cdot \frac{1}{\sqrt{\frac{2a}{3} + 104}} \cdot 1 = \frac{1}{19}$$

$$\Rightarrow \frac{2a}{3} + 104 = 1444 \Rightarrow 2a = 4020 \Rightarrow \frac{a}{2010} = 1$$

● ***Ex. 45*** *Evaluate* $\lim_{x\to 0^+} \log_{\sin x} \sin 2x$.

Sol. (*1*) Here, $\lim_{x\to 0^+} \log_{\sin x} \sin 2x = \lim_{x\to 0^+} \frac{\log \sin 2x}{\log \sin x}$ $\left[\frac{-\infty}{-\infty}\text{form}\right]$

$$= \lim_{x\to 0^+} \frac{\frac{1}{\sin 2x} \cdot 2\cos 2x}{\frac{1}{\sin x} \cdot \cos x}$$ [using L'Hospital's rule]

$$= \lim_{x\to 0^+} \frac{\left[\frac{(2x)}{\sin(2x)}\right]\cos 2x}{\left(\frac{x}{\sin x}\right)\cos x} = \lim_{x\to 0^+} \frac{\cos 2x}{\cos x} = 1$$

$\left[\because \lim_{x\to 0} \frac{\sin x}{x} = 1\right]$

Subjective Type Questions

● **Ex. 46** *Evaluate a, b, c and d, if*

$$\lim_{x\to\infty}\left(\sqrt{x^4+ax^3+3x^2+bx+2}-\sqrt{x^4+2x^3-cx^2+3x-d}\right)=4.$$

Sol. Here, $\lim_{x\to\infty}\left(\sqrt{x^4+ax^3+3x^2+bx+2}-\sqrt{x^4+2x^3-cx^2+3x-d}\right)=4$ [$\infty-\infty$ form]

Rationalising

$$\lim_{x\to\infty}\frac{(a-2)x^3+(3+c)x^2+(b-3)x+(2+d)}{\left[\sqrt{x^4+ax^3+3x^2+bx+2}+\sqrt{x^4+2x^3-cx^2+3x-d}\right]}=4$$

Since, limit is finite, the degree of the numerator must be 2. So, $a-2=0$, i.e. $a=2$.

$$\Rightarrow \lim_{x\to\infty}\frac{(3+c)x^2+(b-3)x+(2+d)}{\left[\sqrt{x^4+ax^3+3x^2+bx+2}+\sqrt{x^4+2x^3-cx^2+3x-d}\right]}=4$$

On dividing numerator and denominator by x^2, we get

$$\lim_{x\to\infty}\frac{(3+c)+(b-3)/x+(2+d)/x^2}{\sqrt{1+\frac{a}{x}+\frac{3}{x^2}+\frac{b}{x^3}+\frac{2}{x^4}}+\sqrt{1+\frac{2}{x}-\frac{c}{x^2}+\frac{3}{x^3}-\frac{d}{x^4}}}$$

$$=4 \Rightarrow \frac{3+c}{2}=4$$

$\Rightarrow \quad c=5$

$\therefore \quad c=5,\ a=2$

Hence, $a=2,\ c=5$ and b, d are any real numbers.

● **Ex. 47** *If x is a real number in $[0, 1]$. Find the value of*

$$\lim_{m\to\infty}\lim_{n\to\infty}[1+\cos^{2m}(n!\pi x)].$$

Sol. If x is a real number in $[0, 1]$, then we have two cases either.

***Case* I** $x\in Q$ [rational number]

As $x\in Q$, we have $x=0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, 1$

which shows $(n!\pi x)$ is integral multiple of π for large values of n.

$\therefore \quad \cos(n!\pi x)=\pm 1$

Thus, $\lim_{m\to\infty}\lim_{n\to\infty}[1+\cos^{2m}(n!\pi x)]=\lim_{m\to\infty}\lim_{n\to\infty}(1+1)=2$

***Case* II** $x\notin Q$ [irrational number]

As $x\notin Q$, we have $x=\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{5}}, \ldots$

which shows $(n!\pi x)$ is not an integral multiple of π and so, $\cos(n!\pi x)$ will lie between -1 and $+1$.

Thus, $\lim_{m\to\infty}\lim_{n\to\infty}[1+\cos^{2m}(n!\pi x)]$

$=\lim_{m\to\infty}\lim_{n\to\infty}[1+(\text{a value between } -1 \text{ and } +1)^{2m}]$

$=\lim_{n\to\infty}[1+0]=1$ [as $0<x<1 \Rightarrow x^{\infty}=0$]

Thus, $\lim_{m\to\infty}\lim_{n\to\infty}[1+\cos^{2m}(n!\pi x)]=\begin{cases}2, & x\in Q\\ 1, & x\notin Q\end{cases}$

● **Ex. 48** *If $a_1=1$ and $a_{n+1}=\frac{4+3a_n}{3+2a_n}$, $n\geq 1$, show that $a_{n+2}>a_{n+1}$ and if a_n has a limit l as $n\to\infty$, then evaluate $\lim_{n\to\infty}a_n$.*

Sol Here, $a_1=1$

$\therefore \quad a_2=\frac{4+3}{3+2}=\frac{7}{5}>1$

$\therefore \quad a_2>a_1$

Assuming $a_{n+1}>a_n$

$\therefore\ a_{n+2}-a_{n+1}=\frac{4+3a_{n+1}}{3+2a_{n+1}}-\frac{4+3a_n}{3+2a_n}$

$$=\frac{(4+3a_{n+1})(3+2a_n)-(4+3a_n)(3+2a_{n+1})}{(3+2a_{n+1})(3+2a_n)}$$

$$=\frac{a_{n+1}-a_n}{(3+2a_{n+1})(3+2a_n)}>0 \qquad [\because a_{n+1}>a_n]$$

$\therefore \quad a_{n+2}-a_{n+1}>0$

$\Rightarrow \quad a_{n+2}>a_{n+1}$

whenever $a_{n+1}>a_n$.

$\therefore$ The sequence of values a_n is increasing and since $a_1=1$, $a_n>0$, for all n.

Now, let $l=\lim_{n\to\infty}a_n=\lim_{n\to\infty}a_{n+1}$

$\therefore \quad l=\lim_{n\to\infty}\frac{4+3a_n}{3+2a_n}$

$\Rightarrow \quad l=\frac{4+3l}{3+2l}$ $\left[\because \lim_{n\to\infty}a_n=l\right]$

$\Rightarrow \quad 3l+2l^2=4+3l$

$\Rightarrow \quad l^2=2$

$\therefore \quad l=\sqrt{2}$ [neglecting $l=-\sqrt{2}$, as $l>0$]

● ***Ex. 49*** *Let $a_1, a_2, \ldots, a_n$ be sequence of real numbers with $a_{n+1} = a_n + \sqrt{1 + a_n^2}$ and $a_0 = 0$. Prove that* $\lim_{n \to \infty} \left(\frac{a_n}{2^{n-1}} \right) = \frac{4}{\pi}$.

Sol. Here, $a_{n+1} = a_n + \sqrt{1 + a_n^2}$, let $a_n = \cot(\alpha_n)$

$$\Rightarrow \quad a_{n+1} = \cot(\alpha_n) + \operatorname{cosec}(\alpha_n)$$

$$\Rightarrow \quad a_{n+1} = \frac{\cos(\alpha_n) + 1}{\sin(\alpha_n)}$$

$$= \frac{2\cos^2(\alpha_n/2)}{2\sin(\alpha_n/2)\cos(\alpha_n/2)} = \cot\left(\frac{\alpha_n}{2}\right)$$

Putting $n = 1, a_1 = \cot(\alpha_1)$

and $a_1 = a_0 + \sqrt{1 + a_0^2} = 1$

$$\Rightarrow \quad \cot(\alpha_1) = 1 \text{ or } \alpha_1 = \frac{\pi}{4}$$

Again, $a_2 = \cot\left(\frac{\alpha_1}{2}\right) = \cot\left(\frac{\pi}{8}\right)$

$$a_3 = \cot\left(\frac{\alpha_2}{2}\right) = \cot\left(\frac{\pi}{4 \cdot 2^2}\right)$$

$$a_4 = \cot\left(\frac{\alpha_3}{2}\right) = \cot\left(\frac{\pi}{4 \cdot 2^3}\right)$$

.....

.....

$$a_n = \cot\left(\frac{\pi}{4 \cdot 2^{n-1}}\right);$$

Hence,

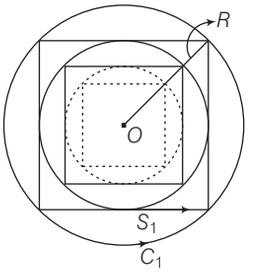
$$\lim_{n \to \infty} \left(\frac{a_n}{2^{n-1}} \right) = \lim_{n \to \infty} \frac{\cot\left(\frac{\pi}{4 \cdot 2^{n-1}}\right)}{2^{n-1}} = \lim_{x \to 0} \frac{1}{\frac{\tan\left(\frac{\pi}{4}x\right)}{x}}$$

$$\left[\text{put} \frac{1}{2^{n-1}} = x \right]$$

$$\therefore \quad \lim_{n \to \infty} \left(\frac{a_n}{2^{n-1}} \right) = \frac{4}{\pi}$$

● ***Ex. 50*** *A square is inscribed in a circle of radius R, a circle is inscribed in this square then a square in this circle and so on, n times. Find the limit of the sum of areas of all the squares as $n \to \infty$.*

Sol. Let the side of a square, S_1 be 'a' units.



Then, $a\sqrt{2} = 2R \Rightarrow R = \frac{a}{\sqrt{2}}$ is radius of circle C_1.

If a_1 be the side of another square, then

$$a_1\sqrt{2} = a \quad \Rightarrow \quad a_1 = \frac{a}{\sqrt{2}}$$

$$a_2\sqrt{2} = a_1 \Rightarrow \quad a_2 = \frac{a_1}{\sqrt{2}} = \frac{a}{2}.$$

.....

.....

So, sum of areas of all the squares,

$S_n = a^2 + a_1^2 + a_2^2 + \ldots$ upto n terms

$$= a^2 + \frac{a^2}{2} + \frac{a^2}{4} + \ldots \text{ upto } n \text{ terms}$$

$$= a^2 \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \text{ upto } n \text{ terms}\right) = a^2 \left\{ \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} \right\}$$

$$= 2a^2 \left(1 - \frac{1}{2^n}\right)$$

$$\therefore \quad \lim_{n \to \infty} S_n = \lim_{n \to \infty} 2a^2 \left(1 - \frac{1}{2^n}\right) = 2a^2 = 4R^2$$

$$\left[\text{as } n \to \infty \Rightarrow \frac{1}{2^n} \to 0 \right]$$

Limits Exercise 1 : Single Option Correct Type Questions

1. $\lim_{x \to 0} \frac{\sin(\pi \cos^2(\tan(\sin x)))}{x^2}$ is equal to

(a) π (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{2}$ (d) None of these

2. $\lim_{t \to 0} \frac{1-(1+t)^t}{\ln(1+t)-t}$ is equal to

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$

(c) 2 (d) -2

3. If $I_1 = \lim_{x \to \infty} (\tan^{-1} \pi x - \tan^{-1} x) \cos x$ and $I_2 = \lim_{x \to 0} (\tan^{-1} \pi x - \tan^{-1} x) \cos x$, then (I_1, I_2) is

(a) (0, 0)
(b) (0, 1)
(c) (1, 0)
(d) None of the above

4. If $f(x) = 0$ is a quadratic equation such that $f(-\pi) = f(\pi) = 0$ and $f\left(\frac{\pi}{2}\right) = -\frac{3\pi^2}{4}$, then $\lim_{x \to -\pi} \frac{f(x)}{\sin(\sin x)}$ is equal to

(a) 0 (b) π

(c) 2π (d) None of these

5. $\lim_{x \to 0} \frac{\sqrt[3]{1+\sin^2 x} - \sqrt[4]{1-2\tan x}}{\sin x + \tan^2 x}$ is equal to

(a) -1 (b) 1

(c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

6. If $x_{n+1} = \sqrt{\frac{1+x_n}{2}}$ and $|x_0| < 1, n \geq 0$, then $\lim_{n \to \infty} \left(\frac{\sqrt{1-x_0^2}}{x_1 x_2 x_3 \ldots x_{n+1}} \right)$ is equal to

(a) -1
(b) 1
(c) $\cos^{-1}(x_0 + 1)$
(d) $\cos^{-1}(x_0)$

7. For $n \in N$, let $f_n(x) = \tan\frac{x}{2}(1+\sec x)(1+\sec 2x)$ $(1+\sec 4x)\ldots(1+\sec 2^n x)$. Then, $\lim_{x \to 0} \frac{f_n(x)}{2x}$ is

(a) 0 (b) 2^n

(c) 2^{n-1} (d) 2^{n+1}

8. Let $f(x)$ be a real valued function defined for all $x \geq 1$, satisfying $f(1) = 1$ and $f'(x) = \frac{1}{x^2 + (f(x))}$; then $\lim_{x \to \infty} f(x)$

(a) doesn't exist
(b) exists and less than $\frac{\pi}{4}$
(c) exists and less than $1 + \frac{\pi}{4}$
(d) exists and equal to 0

9. The quadratic equation whose roots are the minimum value of $\sin^2\theta - \sin\theta + \frac{1}{2}$ and $\lim_{x \to \infty} \sqrt{(x+1)(x+2)} - x$ is

(a) $3x^2 - 7x + 3 = 0$ (b) $8x^2 - 14x + 3 = 0$

(c) $x^2 - 7x + 3 = 0$ (d) $2x^2 - 7x + 3 = 0$

10. If $x_1 = \sqrt{3}$ and $x_{n+1} = \frac{x_n}{1+\sqrt{1+x_n^2}}, \forall\, n \in N$, then $\lim_{x \to \infty} 2^n x_n$ equal to

(a) $\frac{3}{2\pi}$ (b) $\frac{2}{3\pi}$ (c) $\frac{2\pi}{3}$ (d) $\frac{3\pi}{2}$

11. $\lim_{x \to a^-} \frac{\sqrt{x-b} - \sqrt{a-b}}{(x^2 - a^2)}, (a > b)$ is

(a) $\frac{1}{4a}$ (b) $\frac{1}{a\sqrt{a-b}}$

(c) $\frac{1}{2a\sqrt{a-b}}$ (d) $\frac{1}{4a\sqrt{a-b}}$

12. $\lim_{n \to \infty} (\sin^n 1 + \cos^n 1)^n$ is equal to

(a) cot 1 (b) tan 1 (c) cos 1 (d) sin 1

13. The value of $\lim_{x \to 0} \left(\frac{2}{x^3}(\sin^{-1} x - \tan^{-1} x) \right)^{2/x^2}$ equals

(a) e (b) $\sqrt{e}$ (c) $\frac{1}{e}$ (d) $\frac{1}{\sqrt{e}}$

14. The value of $\lim_{n \to \infty} \sum_{k=1}^{n} \frac{n-K}{n^2} \cos\left(\frac{4K}{n}\right)$ equals

(a) $\frac{1}{4}\sin 4 + \frac{1}{16}\cos 4 - \frac{1}{16}$
(b) $\frac{1}{4}\sin 4 - \frac{1}{16}\cos 4 + \frac{1}{16}$
(c) $\frac{1}{16}(1 - \sin 4)$
(d) $\frac{1}{16}(1 - \cos 4)$

15. If $\lim_{x\to 0}\dfrac{1-\sqrt{\cos 2x}\cdot\sqrt[3]{\cos 3x}\cdot\sqrt[4]{\cos 4x}\ldots\sqrt[n]{\cos nx}}{x^2}$ has the value equal to 10, the value of n is

(a) 6 (b) 7 (c) 8 (d) 9

16. $\lim_{z\to\infty}\dfrac{\int_{1/2}^{z}[\cot^{-1}x]\,dx}{\int_{1/2}^{z}\left[1+\dfrac{1}{x}\right]dx}$, where $[\cdot]$ denotes the greatest integer function, equals

(a) 0 (b) 1
(c) cot 1 (d) not defined

17. If α and β are roots of $x^2-(\sqrt{1-\cos 2\theta})x+\theta=0$, where $0<\theta<\dfrac{\pi}{2}$. Then, $\lim_{\theta\to 0^+}\left(\dfrac{1}{\alpha}+\dfrac{1}{\beta}\right)$ is

(a) $\dfrac{1}{\sqrt{2}}$ (b) $-\sqrt{2}$
(c) $\sqrt{2}$ (d) None of these

18. If $f(x)=\dfrac{1}{3}\left(f(x+1)+\dfrac{5}{f(x+2)}\right)$ and $f(x)>0,\ \forall\ x\in R$, then $\lim_{x\to\infty}f(x)$ is

(a) 0 (b) $\sqrt{\dfrac{2}{5}}$ (c) $\sqrt{\dfrac{5}{2}}$ (d) ∞

19. Let $f:(1,2)\to R$ satisfies the inequality $\dfrac{\cos(2x+4)-33}{2}<f(x)<\dfrac{x^2|4x-8|}{x-2},\ \forall\ x\in(1,2)$. Then, $\lim_{x\to 2^-}f(x)$ is

(a) 16 (b) −16
(c) 8 (d) doesn't exist

20. Let $f(x)$ be polynomial of degree 4 with roots 1, 2, 3, 4 and leading coefficient 1 and $g(x)$ be the polynomial of degree 4 with roots 1, $\dfrac{1}{2}$, $\dfrac{1}{3}$ and $\dfrac{1}{4}$ with leading coefficient 1. Then, $\lim_{x\to 1}\dfrac{f(x)}{g(x)}$ equals

(a) $\dfrac{1}{24}$ (b) −24 (c) $\dfrac{1}{12}$ (d) $-\dfrac{1}{12}$

21. The value of $\lim_{x\to\frac{\pi}{4}}\dfrac{4\sqrt{2}-(\cos x+\sin x)^5}{1-\sin 2x}$ is

(a) $\sqrt{2}$ (b) $3\sqrt{5}$
(c) $5\sqrt{2}$ (d) $-5\sqrt{2}$

22. If $\lim_{n\to\infty}\dfrac{n\cdot 3^n}{n(x-2)^n+n\cdot 3^{n+1}-3^n}=\dfrac{1}{3}$, where $n\in N$, the number of integer(s) in the range of 'x' is

(a) 3 (b) 4
(c) 5 (d) infinite

23. If $f(x)=\lim_{n\to\infty}\dfrac{1}{\left(\dfrac{3}{\pi}\tan^{-1}2x\right)^{2n}+5}$, then the set of values of x for which $f(x)=0$, is

(a) $|2x|>\sqrt{3}$ (b) $|2x|<\sqrt{3}$
(c) $|2x|\geq\sqrt{3}$ (d) $|2x|\leq\sqrt{3}$

24. The integer 'n' for which the $\lim_{x\to 0}\dfrac{\cos^2 x-\cos x-e^x\cos x+e^x-\dfrac{x^3}{2}}{x^n}$ is a finite non-zero number, is

(a) 2 (b) 3
(c) 4 (d) None of these

25. If $I_1=\lim_{x\to 0}\sqrt{\dfrac{\tan^{-1}x}{x}-\dfrac{\sin^{-1}x}{x}}$ and $I_2=\lim_{x\to 0}\sqrt{\dfrac{\sin^{-1}x}{x}-\dfrac{\tan^{-1}x}{x}}$, where $|x|<1$, which of the following statement is true?

(a) Neither I_1 nor I_2 exist
(b) I_1 exists and I_2 doesn't exist
(c) I_1 doesn't exist and I_2 exists
(d) None of the above

26. The value of $\lim_{x\to\pi/2}\dfrac{\left[\dfrac{x}{2}\right]}{\log(\sin x)}$ is equal to

(a) 0 (b) 1
(c) −1 (d) doesn't exist

27. The value of $\lim_{n\to\infty}\left(1+\dfrac{1}{a_1}\right)\left(1+\dfrac{1}{a_2}\right)\ldots\left(1+\dfrac{1}{a_n}\right)$, where $a_1=1$ and $a_n=n(1+a_{n-1}),\forall\ n\geq 2$, is

(a) e (b) $\dfrac{e}{2}$
(c) $2e$ (d) $3e$

28. If $f(x+y)=f(x)+f(y)$ for all $x,y\in R$ and $f(1)=1$, then $\lim_{x\to 0}\dfrac{2^{f(\tan x)}-2^{f(\sin x)}}{x^2\,f(\sin x)}$ equals

(a) log 2 (b) log 4 (c) log $\sqrt{2}$ (d) log 8

29. The value of $\lim_{n\to\infty}n^{-n^2}\left[(n+1)\left(n+\dfrac{1}{2}\right)\left(n+\dfrac{1}{2^2}\right)\ldots\left(n+\dfrac{1}{2^{n-1}}\right)\right]^n$ is

(a) e
(b) e^2
(c) e^3
(d) e^4

30. If $f(x)=\begin{cases}\dfrac{\sin[x]}{[x]}, & \text{for } [x]\neq 0\\ 0, & \text{for } [x]=0\end{cases}$, where $[x]$ denotes the greatest integer less than or equal to x, then $\lim\limits_{x\to 0} f(x)$ equals

(a) 1 (b) 0
(c) -1 (d) doesn't exist

31. Let $f(x)=\dfrac{|x^3-6x^2+11x-6|}{x^3-6x^2+11x-6}$, then the number of solutions of 'a', where $\lim\limits_{x\to a} f(x)$ doesn't exist is
(a) 3 (b) 2 (c) 1 (d) 4

32. Let the rth term, t_r of a series is given by $t_r=\dfrac{r}{1+r^2+r^4}$. The value of $\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n} t_r$ is

(a) 2 (b) $\dfrac{1}{2}$ (c) 1 (d) $\dfrac{1}{4}$

33. The value of $\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\cot^{-1}\left(\dfrac{r^3-r+\frac{1}{r}}{2}\right)$ is

(a) π (b) $\dfrac{\pi}{4}$
(c) $\dfrac{\pi}{2}$ (d) π

34. Let $(\tan\alpha)x+(\sin\alpha)y=\alpha$ and $(\alpha\operatorname{cosec}\alpha)x+(\cos\alpha)y=1$ be two variable straight lines, α being the parameter. Let P be the point of intersection of the lines. In the limiting position, when $\alpha\to 0$, the point of intersection of straight lines is
(a) $(2,-1)$ (b) $(2,1)$
(c) $(-2,1)$ (d) $(-2,-1)$

35. The polynomial of least degree, such that $\lim\limits_{x\to 0}\left(1+\dfrac{x^2+f(x)}{x^2}\right)^{1/x}=e^2$ is

(a) x^2 (b) x^3+2x^2
(c) $-x^2+2x^3$ (d) None of these

36. Let [ ] represents the greatest integer function less than or equal to x. The value of $\lim\limits_{x\to 0}\left(\left[\dfrac{n\sin x}{x}\right]+\left[\dfrac{n\tan x}{x}\right]\right)$ is
(a) $n+1$ (b) $2n$
(c) $n-1$ (d) $2n-1$

37. The value of $\lim\limits_{x\to a}[\sqrt{2-x}+\sqrt{1+x}]$, where $a\in\left[0,\dfrac{1}{2}\right]$ and $[\cdot]$ denotes the greatest integer function is
(a) 1 (b) 2
(c) 3 (d) 4

38. The value of

$$\lim_{x\to 0^+}\frac{-1+\sqrt{(\tan x-\sin x)+\sqrt{(\tan x-\sin x)+\sqrt{(\tan x-\sin x)}+\ldots\infty}}}{-1+\sqrt{x^3+\sqrt{x^3+\sqrt{x^3+\ldots\infty}}}}$$ is

(a) $\dfrac{1}{2}$ (b) $\dfrac{1}{4}$
(c) $\dfrac{1}{8}$ (d) 1

39. The value of

$$\lim_{\theta\to 0}\frac{\cos^2(1-\cos^2(1-\cos^2(1\ldots\cos^2\theta))\ldots)}{\sin\left(\dfrac{\pi(\sqrt{\theta+4}-2)}{\theta}\right)}$$ is

(a) 2 (b) $\sqrt{2}$
(c) $\dfrac{1}{2}$ (d) $\dfrac{1}{\sqrt{2}}$

40. The value of $\lim\limits_{n\to\infty} a_n$ when $a_{n+1}=\sqrt{2+a_n}$, $n=1,2,3,\ldots$ is
(a) 1 (b) 2
(c) 3 (d) 4

Limits Exercise 2 : More than One Option Correct Type Questions

41. If $\lim\limits_{x\to\infty} 4x\left(\dfrac{\pi}{4}-\tan^{-1}\dfrac{x+1}{x+2}\right)=y^2+4y+5$, then y can be equal to
(a) 1 (b) -1
(c) -4 (d) -3

42. $\lim\limits_{x\to 0}\dfrac{1-\cos(x^2)}{x^3(4^x-1)}$ is equal to
(a) $\dfrac{1}{2}\ln 2$ (b) $\ln 2$
(c) $\ln 4$ (d) $1-\dfrac{1}{2}\ln\left(\dfrac{e^2}{4}\right)$

43. If $f(x)=e^{[\cot x]}$, where $[y]$ represents the greatest integer less than or equal to y, then

(a) $\lim\limits_{x\to\frac{\pi^+}{2}} f(x)=1$ (b) $\lim\limits_{x\to\frac{\pi^+}{2}} f(x)=\frac{1}{e}$

(c) $\lim\limits_{x\to\frac{\pi^-}{2}} f(x)=\frac{1}{e}$ (d) $\lim\limits_{x\to\frac{\pi^-}{2}} f(x)=1$

44. $\lim\limits_{x\to 0}\left[m\dfrac{\sin x}{x}\right]$ is equal to (where $m\in I$ and $[\cdot]$ denotes greatest integer function)

(a) m, if $m<0$

(b) $m-1$, if $m>0$

(c) $m-1$, if $m<0$

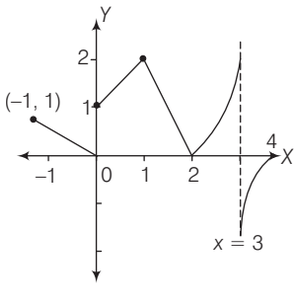
(d) m, if $m>0$

45. If $\lim\limits_{x\to 0}(1+ax+bx^2)^{2/x}=e^3$, then

(a) $a=3, b=0$ (b) $a=\frac{3}{2}, b=1$

(c) $a=\frac{3}{2}, b=4$ (d) $a=2, b=3$

46. The graph of the function $y=f(x)$ is shown in the adjacent figure, then correct statement is



(a) $\lim\limits_{x\to 0^+} f(x)=1$ (b) $\lim\limits_{x\to 1} f(x)=2$

(c) $\lim\limits_{x\to 3} f(x)=$ does not exist (d) $\lim\limits_{x\to 4} f(x)=4$

47. For $\lim\limits_{x\to 0}\dfrac{\cot^{-1}\left(\frac{1}{x}\right)}{x}$

(a) RHL exists (b) LHL does not exist

(c) limit does not exist as RHL is 1 and LHL is -1

(d) limit does not exist as RHL and LHL both are non-existent

48. If $l=\lim\limits_{x\to\infty}\left(\dfrac{x+1}{x-1}\right)^x$, the value of $\{l\}$ and $[l]$ are

(a) 7 (b) $7-e^2$ (c) -7 (d) e^2-7

49. If $f(x)=\begin{cases}\dfrac{\sin[x]}{[x]}, & [x]\neq 0\\ 0, & [x]=0\end{cases}$,

where $[\cdot]$ denotes the greatest integer less than or equal to x, then

(a) $\lim\limits_{x\to 0^-} f(x)=\sin 1$

(b) $\lim\limits_{x\to 0^+} f(x)=0$

(c) limit does not exist at $x=0$

(d) limit exists at $x=0$

50. $\lim\limits_{x\to c} f(x)$ does not exist when

(a) $f(x)=[[x]]-[2x-1]$, $c=3$ (b) $f(x)=[x]-x$, $c=1$

(c) $f(x)=\{x\}^2-\{-x\}^2$, $c=0$ (d) $f(x)=\dfrac{\tan(\operatorname{sgn} x)}{(\operatorname{sgn} x)}$, $c=0$

(where $[\cdot]$ and $\{\cdot\}$ denotes greatest integer and fractional part of x)

51. Identify the correct statement.

(a) $\lim\limits_{n\to\infty}\left[\sum\limits_{r=1}^{n}\dfrac{1}{2^r}\right]=1$

(b) If $f(x)=(x-1)\{x\}$, where $[\cdot]$ and $\{\cdot\}$ denotes greatest integer function and fractional part of x respectively, the limit of $f(x)$ does not exist at $x=1$

(c) $\lim\limits_{x\to 0^+}\left[\dfrac{\tan x}{x}\right]=1$

(d) $\left[\lim\limits_{x\to 0^+}\dfrac{\tan x}{x}\right]=1$

52. For $a>0$, let $l=\lim\limits_{x\to\frac{\pi}{2}}\dfrac{a^{\cot x}-a^{\cos x}}{\cot x-\cos x}$ and

$m=\lim\limits_{x\to-\infty}(\sqrt{x^2+ax}-\sqrt{x^2-ax})$, then

(a) $l>m$, for all $a>0$ (b) $l>m$, only when $a\geq 1$

(c) $l>m$, for all $a>e^{-a}$ (d) $e^l+m=0$

53. Consider the function $f(x)=\left(\dfrac{ax+1}{bx+2}\right)^x$, where $a, b>0$, the $\lim\limits_{x\to\infty} f(x)$ is

(a) exists for all values of a and b

(b) zero for $a<b$

(c) non-existent for $a>b$

(d) $e^{-(1/a)}$ or $e^{-(1/b)}$, if $a=b$

54. If $f(x)=\dfrac{x\cdot 2^x-x}{1-\cos x}$ and $g(x)=2^x\cdot\sin\left(\dfrac{\log 2}{2^x}\right)$, then

(a) $\lim\limits_{x\to 0} f(x)=\log 2$ (b) $\lim\limits_{x\to\infty} g(x)=\log 4$

(c) $\lim\limits_{x\to 0} f(x)=\log 4$ (d) $\lim\limits_{x\to\infty} g(x)=\log 2$

55. If $\lim\limits_{x\to 3}\dfrac{x^3+cx^2+5x+12}{x^2-7x+12}=l$ (finite real number), then

(a) $l=4$ (b) $c=-6$

(c) $c=4$ (d) $x\in R$

Limits Exercise 3 : Passage Based Questions

Passage I (Q. Nos. 56 to 58)

If $\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = \infty$, *then*

$\lim_{x \to a} \{f(x)\}^{g(x)} = e^{\lim_{x \to a} (f(x) - 1) \times g(x)}$.

56. $\lim_{x \to 0} \left(\frac{\sin x}{x} \right)^{\frac{\sin x}{x - \sin x}}$ is equal to

(a) $1/e$ (b) $-1/e$
(c) e (d) $-e$

57. $\lim_{x \to 0} \left(\frac{x - 1 + \cos x}{x} \right)^{\frac{1}{x}}$ is equal to

(a) $e^{1/2}$ (b) $e^{-1/2}$ (c) e^{1} (d) $\frac{1}{e}$

58. $\lim_{x \to 0} \left(\frac{a^x + b^x + c^x}{3} \right)^{\frac{2}{x}}$ is equal to

(a) $a^{2/3} + b^{2/3} + c^{2/3}$ (b) abc
(c) $(abc)^{2/3}$ (d) 1

Passage II (Q. Nos. 59 to 61)

Let $f(x) = \lim_{n \to \infty} \left(\cos \sqrt{\frac{x}{n}} \right)^n$, $g(x) = \lim_{n \to \infty} (1 - x + x \sqrt[n]{e})^n$.

Now, consider the function $y = h(x)$, *where* $h(x) = \tan^{-1}(g^{-1} f^{-1}(x))$.

59. $\lim_{x \to 0} \frac{\ln(f(x))}{\ln(g(x))}$ is equal to

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$
(c) 0 (d) 1

60. Domain of the function $y = h(x)$ is

(a) $(0, \infty)$ (b) R
(c) $(0, 1)$ (d) $[0, 1]$

61. Range of the function $y = h(x)$ is

(a) $\left(0, \frac{\pi}{2}\right)$ (b) $\left(-\frac{\pi}{2}, 0\right)$
(c) R (d) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Passage III (Q. Nos. 62 to 63)

Let K_1 = *Total number of ways of selecting of ball from a bag which contains* n *balls of first colour,* $(n + 1)$ *balls of second colour,* $(n + 2)$ *balls of third colour, ...,* $(2n - 1)$ *balls of* n *colour.*

K_2 = *number of* n*-digit numbers using the digits* 1, 2, 3, ..., n *and*
K_3 = *number of ways of arranging* $(n + 1)$ *objects on a circle.*

62. The value of $\lim_{n \to \infty} \left(\frac{K_1}{K_2} \right)^{1/n}$ is

(a) e^2 (b) $\log 4 - 1$ (c) $\frac{4}{e}$ (d) $\frac{1}{e}$

63. The value of $\lim_{n \to \infty} \left(\frac{K_1}{K_2 + K_3} \right)$ is

(a) e (b) 1
(c) 0 (d) does not exist

Passage IV (Q. Nos. 64 to 65)

Let $f : N \to R$ *and* $g : N \to R$ *be two functions and* $f(1) = 0.8$, $g(1) = 0.6$; $f(n + 1) = f(n)\cos(g(n)) - g(n)\sin(g(n))$ *and* $g(n + 1) = f(n)\sin(g(n)) + g(n)\cos(g(n))$ *for* $n \geq 1$.

64. $\lim_{n \to \infty} f(n)$ is equal to

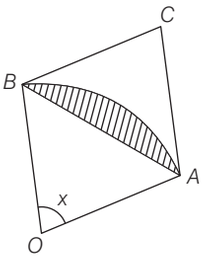
(a) -1 (b) 0
(c) 1 (d) does not exist

65. $\lim_{n \to \infty} g(n)$ is equal to

(a) -1 (b) 0
(c) 1 (d) does not exist

Passage V (Q. Nos. 66 to 68)

A circular arc of radius 1 subtends an angle of x *radian as shown in figure. The centre of the circle is O and the point C is the intersection of two tangent lines at A and B. Let* $T(x)$ *be the area of* ΔABC *and* $S(x)$ *be the area of shaded region.*



66. $\lim_{x \to 0} \frac{T(x)}{x^3}$ is

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) $\frac{1}{8}$

67. $\lim_{x \to 0} \frac{S(x)}{x}$ is

(a) 0 (b) $\frac{1}{2}$ (c) 1 (d) None of these

68. $\lim_{x \to 0} \frac{T(x)}{S(x)}$ is

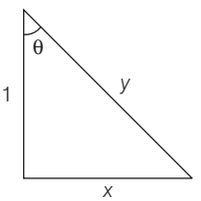
(a) $\frac{1}{4}$ (b) $\frac{3}{4}$ (c) $\frac{3}{2}$ (d) 0

Limits Exercise 4 : Matching Type Questions

69. Match the statements of Column I with values of Column II.

	Column I		Column II
(A)	$\lim\limits_{x \to \frac{\pi^+}{2}} \tan^{-1}(\tan x)$	(p)	0
(B)	$\lim\limits_{n \to \infty}\left[\sum\limits_{r=1}^{n} \frac{1}{2^r}\right]$ ([·] denotes the greatest integer function)	(q)	Doesn't exist
(C)	$\lim\limits_{x \to \infty} \sec^{-1}\left(\frac{x}{x+1}\right)$	(r)	$-\frac{\pi}{2}$
(D)	$\lim\limits_{x \to \frac{\pi}{2}} \frac{\cos x}{(1-\sin x)^{2/3}}$	(s)	$\frac{\pi}{2}$

70. A right angled triangle has legs 1 and x. The hypotenuse is y and the angle opposite to the side x is θ. Shown as



Match the column.

	Column I		Column II
(A)	$\lim\limits_{\theta \to \frac{\pi}{2}} (\sqrt{y} - \sqrt{x})$	(p)	0
(B)	$\lim\limits_{\theta \to \frac{\pi}{2}} (y - x)$	(q)	$\frac{1}{2}$
(C)	$\lim\limits_{\theta \to \frac{\pi}{2}} (y^2 - x^2)$	(r)	1
(D)	$\lim\limits_{\theta \to \frac{\pi}{2}} (y^3 - x^3)$	(s)	∞

71. Match the column.

	Column I		Column II
(A)	$\lim\limits_{x \to 0} \frac{1-\cos 2x}{e^{x^2} - e^x + x}$ equals	(p)	1
(B)	If the value of $\lim\limits_{x \to 0^+}\left(\frac{(3/x)+1}{(3/x)-1}\right)^{1/x}$ can be expressed in the form of $e^{p/q}$, where p and q are relative prime, then $(p+q)$ is equal to	(q)	2
(C)	$\lim\limits_{x \to 0} \frac{\tan^3 x - \tan x^3}{x^5}$ equals	(r)	4
(D)	$\lim\limits_{x \to 0} \frac{x + 2\sin x}{\sqrt{x^2 + 2\sin x + 1} - \sqrt{\sin^2 x - x + 1}}$ equals	(s)	5

72. Match the column.

	Column I		Column II
(A)	$\lim\limits_{n \to \infty} \cos^2(\pi(\sqrt[3]{n^3 + n^2 + 2n} - n))$, where n is an integer, equals	(p)	$\frac{1}{2}$
(B)	$\lim\limits_{n \to \infty} n \sin(2\pi\sqrt{1+n^2})\ (n \in N)$ equals	(q)	$\frac{1}{4}$
(C)	$\lim\limits_{n \to \infty} (-1)^n \sin(\pi\sqrt{n^2 + 0.5n + 1})$ $\left(\sin\frac{(n+1)\pi}{4n}\right)$ is (where $n \in N$)	(r)	π
(D)	If $\lim\limits_{x \to \infty}\left(\frac{x+a}{x-a}\right)^x = e$, where a is some real constant, the value of a is equal to	(s)	non -existent

Limits Exercise 5 : Single Integer Answer Type Questions

73. Let $L = \lim\limits_{x \to \infty}\left(x \log x + 2x \cdot \log \sin\left(\frac{1}{\sqrt{x}}\right)\right)$, then the value of $\left(-\frac{2}{L}\right)$ is ..... .

74. For $n \in N$, let x_n be defined as $\left(1 + \frac{1}{n}\right)^{(n+x_n)} = e$, then $2\lim\limits_{n \to \infty}(x_n)$ equals ..... .

75. Let $f(n) = \left[\sqrt{n} + \frac{1}{2}\right]$, where $[\cdot]$ denotes greatest integer function, $\forall\ n \in N$.

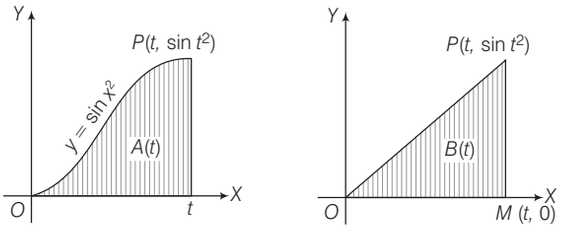
Then, $\sum\limits_{n=1}^{\infty} \frac{2^{f(n)} + 2^{-f(n)}}{2^n}$ is equal to ..... .

76. If the arithmetic mean of the product of all distinct pairs of positive integers whose sum is n, is S_n, then $\lim_{n\to\infty}\frac{n^2}{S_n}$ must equal to ..... .

77. If $k=\sum_{n=1}^{\infty}\frac{6^n}{(3^n-2^n)(3^{n+1}-2^{n+1})}$, the value of k is ..... .

78. The value of $\lim_{x\to\frac{\pi}{2}}\sqrt{\frac{\tan x-\sin\{\tan^{-1}(\tan x)\}}{\tan x+\cos^2(\tan x)}}$ is ..... .

79. The figure shows two regions in the first quadrant. $A(t)$ is the area under the curve $y=\sin x^2$ from 0 to t and $B(t)$ is the area of the triangle with vertices 0, P and $M(t,0)$. If $\lim_{t\to 0}\frac{A(t)}{B(t)}=\frac{m}{n}$, then $(m+n)$ is ..... .



80. If the two lines $AB:\left(\int_0^{2t}\left(\frac{\sin x}{x}+1\right)dx\right)x+y=3t$ and $AC:2tx+y=0$ intersect at a point A, the x-coordinate of a point A as $t\to 0$, is equal to $\frac{p}{q}$ (p and q are in their lowest form), the $(p+q)$ is ..... .

81. Consider a parabola $y=\frac{x^2}{4}$ and the point $F(0,1)$. Let $A_1(x_1,y_1), A_2(x_2,y_2), A_3(x_3,y_3),\ldots, A_n(x_n,y_n)$ are 'n' points on the parabola such that $x_k>0$ and $\angle OFA_k=\frac{k\pi}{2n}(k=1,2,\ldots,n)$. If the value of $\lim_{n\to\infty}\frac{1}{n}\cdot\sum_{k=1}^{n}FA_k=\frac{m}{\pi}$, then m is ..... .

82. If $L=\lim_{x\to\frac{\pi^+}{2}}\frac{\cos(\tan^{-1}(\tan x))}{x-\pi/2}$, then $\cos(2\pi L)$ is ..... .

83. Number of solutions of the equation $\operatorname{cosec}\theta=k$ in $[0,\pi]$, where $k=\lim_{n\to\infty}\prod_{r=2}^{n}\left(\frac{r^3-1}{r^3+1}\right)$, is ..... .

84. If C satisfies the equation $\lim_{x\to\infty}\left(\frac{x+c}{x-c}\right)^x=4$, then $\left|\frac{e^c}{2}\right|$ is ..... .

85. If $\lim_{x\to-\infty}\frac{(3x^4+2x^2)\cdot\sin\left(\frac{1}{x}\right)+|x|^3+5}{|x|^3+|x|^2+|x|+1}=k$, then $\left|\frac{k}{2}\right|$ is ..... .

86. If $f(x)=\lim_{t\to 0}\left[\frac{2x}{\pi}\cdot\tan^{-1}\left(\frac{x}{t^2}\right)\right]$, then $f(1)$ is ..... .

87. The value of $\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2}}+\frac{1}{\sqrt{n^2+1}}+\ldots+\frac{1}{\sqrt{n^2+2n}}\right)$ is

88. If $l=\lim_{x\to 1^+}2^{-2^{\frac{1}{1-x}}}$ and $m=\lim_{x\to 1^+}\frac{x\sin(x-[x])}{x-1}$ (where $[\cdot]$ denotes greatest integer function). Then, $(l+m)$ is ..... .

89. The value of $\lim_{x\to 0}\left[\frac{\sin x\cdot\tan x}{x^2}\right]$ is .....
(where $[\cdot]$ denotes greatest integer function).

90. If $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{kr}{1\times3\times5\times\ldots\times(2r-1)\times(2r+1)}=1$, then k^2 is ..... .

91. If $f(x)=\lim_{n\to\infty}\sin^4 x+\frac{1}{4}\sin^4 2x+\ldots+\frac{1}{4^n}\cdot\sin^4(2^n x)$ and $g(x)$ is a differentiable function satisfying $g(x)+f(x)=1$, then the maximum value of $(\sqrt{f(x)}+\sqrt{g(x)})^4$ is ..... .

92. If $f(x+y+z)=f(x)+f(y)+f(z)$ with $f(1)=1$ and $f(2)=2$ and $x, y, z\in R$, the value of $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{(4r)f(3r)}{n^3}$ is ....

93. If $\lim_{n\to\infty}\frac{n\cdot3^n}{n\cdot(x-1)^n+n\cdot3^{n+1}-3^n}=\frac{1}{3}$, the number of the integral values of x is ..... .

94. The value of $\lim_{x\to\infty}(((x-1)(x-2)(x+3)(x+10)(x+15))^{1/5}-x)$ is ..... .

95. If $\lim_{x\to\infty}([f(x)]+x^2)\{f(x)\}=k$, where $f(x)=\frac{\tan x}{x}$ and $[\cdot]$, $\{\cdot\}$ denotes greatest integer and fractional part of x respectively, the value of $[k/e]$ is ..... .

96. The value of $\lim_{n\to\infty}\{(\sqrt{3}+1)^{2n}\}$ is ..... (where $\{\cdot\}$ denotes fractional part of x).

97. If $f(x)$ is a polynomial of degree 4, having extremum at $x=1,2$ and $\lim_{x\to 0}\left(1+\frac{f(x)}{x^2}\right)=2$.
Then, $f(2)$ is ..... .

98. If α is the number of solutions of $|x| = \log(x - [x])$, (where $[\cdot]$ denotes greatest integer function) and $\lim\limits_{x \to \alpha} \dfrac{x\, e^{ax} - b \sin x}{x^3}$ is finite, the value of $(a + b)$ is

99. Suppose $x_1 = \tan^{-1} 2 > x_2 > x_3 > \ldots$ are the real numbers satisfying $\sin(x_{n+1} - x_n) + 2^{-(n+1)} \cdot \sin x_n \cdot \sin x_{n+1} = 0$ for all $n > 1$ and $l = \lim\limits_{n \to \infty} x_n$, the value of $[4l]$ is ..... (where $[\cdot]$ denotes greatest integer function).

Limits Exercise 6 : Questions Asked in Previous 10 Years' Exams

(i) JEE Advanced & IIT-JEE

100. Let $\alpha, \beta \in R$ be such that $\lim\limits_{x \to 0} \dfrac{x^2 \sin(\beta x)}{\alpha x - \sin x} = 1$. Then, $6(\alpha + \beta)$ equals **[Integer Answer Type, 2016 Adv]**

101. Let m and n be two positive integers greater than 1. If $\lim\limits_{\alpha \to 0} \left(\dfrac{e^{\cos(\alpha^n)} - e}{\alpha^m} \right) = -\left(\dfrac{e}{2} \right)$, then the value of $\dfrac{m}{n}$ is **[Integer Answer Type, 2015 Adv]**

102. Let $f : R \to R$ be a continuous odd function, which vanishes exactly at one point and $f(1) = \dfrac{1}{2}$. Suppose that

$$F(x) = \int_{-1}^{x} f(t)dt \text{ for all } x \in [-1, 2]$$

and $G(x) = \int_{-1}^{x} t\,|f\{f(t)\}|\, dt$ for all $x \in [-1, 2]$.

If $\lim\limits_{x \to 1} \dfrac{F(x)}{G(x)} = \dfrac{1}{14}$, then the value of $f\left(\dfrac{1}{2}\right)$ is **[Integer Answer Type, 2015 Adv]**

103. The largest value of the non-negative integer a for which $\lim\limits_{x \to 1} \left\{ \dfrac{-ax + \sin(x-1) + a}{x + \sin(x-1) - 1} \right\}^{\frac{1-x}{1-\sqrt{x}}} = \dfrac{1}{4}$, is **[Integer Answer Type , 2014 Adv]**

104. Let $L = \lim\limits_{x \to 0} \dfrac{a - \sqrt{a^2 - x^2} - \dfrac{x^2}{4}}{x^4}$, $a > 0$. If L is finite, then **[More than One Correct Option, 2009]**

(a) $a = 2$ (b) $a = 1$

(c) $L = \dfrac{1}{64}$ (d) $L = \dfrac{1}{32}$

(ii) JEE Main & AIEEE

105. The $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\cot x - \cos x}{(\pi - 2x)^3}$ equals **[2017 JEE Main]**

(a) $\dfrac{1}{4}$ (b) $\dfrac{1}{24}$ (c) $\dfrac{1}{16}$ (d) $\dfrac{1}{8}$

106. Let $p = \lim\limits_{x \to 0^+} (1 + \tan^2 \sqrt{x})^{1/2x}$, then $\log p$ is equal to **[2016 JEE Main]**

(a) 2 (b) 1 (c) $\dfrac{1}{2}$ (d) $\dfrac{1}{4}$

107. $\lim\limits_{n \to \infty} \left[\dfrac{(n+1)(n+2)\ldots 3n}{n^{2n}} \right]^{1/n}$ is equal to **[2016 JEE Main]**

(a) $\dfrac{18}{e^4}$ (b) $\dfrac{27}{e^2}$ (c) $\dfrac{9}{e^2}$ (d) $3\log 3 - 2$

108. $\lim\limits_{x \to 0} \dfrac{\sin(\pi \cos^2 x)}{x^2}$ is equal to **[2014 JEE Main]**

(a) $\dfrac{\pi}{2}$ (b) 1 (c) $-\pi$ (d) π

109. $\lim\limits_{x \to 0} \dfrac{(1 - \cos 2x)(3 + \cos x)}{x \tan 4x}$ is equal to **[2013 JEE Main]**

(a) 4 (b) 3 (c) 2 (d) $\dfrac{1}{2}$

110. If $\lim\limits_{x \to \infty} \left(\dfrac{x^2 + x + 1}{x + 1} - ax - b \right) = 4$, then **[2012 AIEEE]**

(a) $a = 1, b = 4$ (b) $a = 1, b = -4$

(c) $a = 2, b = -3$ (d) $a = 2, b = 3$

111. Let $\alpha(a)$ and $\beta(a)$ be the roots of the equation $(\sqrt[3]{1+a} - 1)x^2 - (\sqrt{1+a} - 1)x + (\sqrt[6]{1+a} - 1) = 0$, where $a > -1$. Then, $\lim\limits_{a \to 0^+} \alpha(a)$ and $\lim\limits_{a \to 0^+} \beta(a)$ are **[2012 AIEEE]**

(a) $-\dfrac{5}{2}$ and 1 (b) $-\dfrac{1}{2}$ and -1

(c) $-\dfrac{7}{2}$ and 2 (d) $-\dfrac{9}{2}$ and 3

112. If $\lim\limits_{x \to 0} [1 + x \log(1 + b^2)]^{\frac{1}{x}} = 2b \sin^2 \theta$, $b > 0$ and $\theta \in (-\pi, \pi]$, then the value of θ is **[2011 AIEEE]**

(a) $\pm \dfrac{\pi}{4}$ (b) $\pm \dfrac{\pi}{3}$

(c) $\pm \dfrac{\pi}{6}$ (d) $\pm \dfrac{\pi}{2}$

113. For $x > 0$, $\lim_{x \to 0} \left[(\sin x)^{1/x} + \left(\frac{1}{x}\right)^{\sin x} \right]$ is **[2006 AIEEE]**

(a) 0 (b) − 1 (c) 1 (d) 2

114. $\lim_{h \to 0} \frac{f(2h + 2 + h^2) - f(2)}{f(h - h^2 + 1) - f(1)}$, given that $f'(2) = 6$ and $f'(1) = 4$, **[2003 AIEEE]**

(a) does not exist (b) is equal to –3/2
(c) is equal to 3/2 (d) is equal to 3

115. If $\lim_{x \to 0} \frac{\{(a - n)nx - \tan x\} \sin nx}{x^2} = 0$, where n is non-zero real number, then a is equal to **[2003 AIEEE]**

(a) 0 (b) $\frac{n+1}{n}$ (c) n (d) $n + \frac{1}{n}$

116. The integer n for which $\lim_{x \to 0} \frac{(\cos x - 1)(\cos x - e^x)}{x^n}$ is a finite non-zero number, is **[2002 AIEEE]**

(a) 1 (b) 2 (c) 3 (d) 4

117. Let $f : R \to R$ be such that $f(1) = 3$ and $f'(1) = 6$. Then, $\lim_{x \to 0} \left[\frac{f(1+x)}{f(1)} \right]^{1/x}$ equals **[2002 AIEEE]**

(a) 1 (b) $e^{\frac{1}{2}}$ (c) e^2 (d) e^3

118. For $x \in R$, $\lim_{x \to \infty} \left(\frac{x-3}{x+2} \right)^x$ is equal to **[2000 AIEEE]**

(a) e (b) e^{-1}
(c) e^{-5} (d) e^5

Answers

Exercise for Session 1

1. (c) **2.** (c) **3.** (a) **4.** (c) **5.** (d) **6.** (a)
7. (a) **8.** (d) **9.** (a)

Exercise for Session 2

1. (a) **2.** (b) **3.** (a) **4.** (a) **5.** (b)

Exercise for Session 3

1. (a) **2.** (b) **3.** (b) **4.** (d) **5.** (b) **6.** (a)
7. (a) **8.** (a) **9.** (d) **10.** (c)

Exercise for Session 4

1. (1) **2.** (1) **3.** (e) **4.** (c)

Exercise for Session 5

1. (a) **2.** (c) **3.** (c) **4.** (a)

Exercise for Session 6

1. (c) **2.** (c) **3.** (a) **4.** (b) **5.** (b)

Chapter Exercises

1. (a) **2.** (c) **3.** (a) **4.** (c) **5.** (c) **6.** (d) **7.** (c) **8.** (c) **9.** (b) **10.** (c)
11. (d) **12.** (d) **13.** (d) **14.** (d) **15.** (a) **16.** (a) **17.** (c) **18.** (c) **19.** (b) **20.** (b)
21. (c) **22.** (c) **23.** (a) **24.** (c) **25.** (c) **26.** (a) **27.** (a) **28.** (c) **29.** (b) **30.** (d)
31. (a) **32.** (b) **33.** (c) **34.** (a) **35.** (c) **36.** (d) **37.** (b) **38.** (a) **39.** (b) **40.** (b)
41. (b, d) **42.** (b, d) **43.** (b, d) **44.** (a, b) **45.** (b, c) **46.** (a, b, c) **47.** (a, b) **48.** (a, d)
49. (a,b) **50.** (b, c) **51.** (c, d) **52.** (b, c, d) **53.** (b, c,d) **54.** (c,d) **55.** (a, b) **56.** (a)
57. (b) **58.** (c) **59.** (b) **60.** (c) **61.** (d) **62.** (c) **63.** (b) **64.** (a) **65.** (b) **66.** (d)
67. (a) **68.** (c)
69. (A) → (r); (B) → (p); (C) → (q); (D) → (q)
70. (A) → (p); (B) → (p); (C) → (r); (D) → (s)
71. (A) → (r); (B) → (s); (C) → (p); (D) → (q)
72. (A) → (q); (B) → (r); (C) → (p); (D) → (p)
73. (6) **74.** (1) **75.** (3) **76.** (6) **77.** (2) **78.** (1) **79.** (5) **80.** (5) **81.** (4) **82.** (1)
83. (0) **84.** (1) **85.** (1) **86.** (1) **87.** (2) **88.** (2) **89.** (1) **90.** (4) **91.** (4) **92.** (4)
93. (3) **94.** (5) **95.** (7) **96.** (1) **97.** (0) **98.** (1) **99.** (3) **100.** (7) **101.** (2) **102.** (7)
103. (2) **104.** (c) **105.** (c) **106.** (b) **107.** (d) **108.** (c) **109.** (b) **110.** (d) **111.** (b) **112.** (d)
113. (c) **114.** (d) **115.** (d) **116.** (c) **117.** (c) **118.** (c)

Solutions

1. $\lim\limits_{x\to 0}\dfrac{\sin(\pi-\pi\sin^2\tan(\sin x))}{x^2}$

$$=\lim_{x\to 0}\frac{\sin(\pi\sin^2\tan(\sin x))}{\pi\sin^2\tan(\sin x)}\left(\frac{\pi\sin^2\tan(\sin x)}{\tan^2(\sin x)}\right)\left(\frac{\tan^2(\sin x)}{\sin^2 x}\right)\left(\frac{\sin^2 x}{x^2}\right)=\pi$$

2. $\lim\limits_{t\to 0}\dfrac{1-(1+t)^t}{\ln(1+t)-t}=\lim\limits_{t\to 0}\dfrac{\frac{1-(1+t)^t}{t^2}}{\frac{\ln(1+t)-t}{t^2}}$

$$=\lim_{t\to 0}\frac{1-e^{t(\ln(1+t))}}{\frac{t^2}{-1/2}}=2$$

3. $I_1=\lim\limits_{x\to\infty}(\tan^{-1}\pi x-\tan^{-1}x)$

$$=\lim_{x\to\infty}\tan^{-1}\left(\frac{(\pi-1)x}{1+\pi x^2}\right)=0\Rightarrow I_1=0 \text{ and } I_2=0$$

4. From given $f(x)=x^2-\pi^2$,

$$\lim_{x\to-\pi}\frac{x^2-\pi^2}{\sin(\sin x)}=\lim_{x\to-\pi}\frac{(-\pi+h)^2-\pi^2}{\sin\sin(-\pi+h)}$$

$$=\lim_{h\to 0}\frac{-2h\pi+h^2}{-\sin(\sin h)}=\lim_{h\to 0}\frac{h-2\pi}{\frac{-\sin(\sin h)}{\sin h}\times\frac{\sin h}{h}}=2\pi$$

5. Using approximations

$$L=\lim_{x\to 0}\frac{\frac{\sin^2 x}{3}+\frac{2\tan x}{4}}{\sin x+\tan^2 x}=\lim_{x\to 0}\frac{\frac{x^2}{3}+\frac{2x}{4}}{x+x^2}$$

$$=\lim_{x\to 0}\frac{x\left(\frac{x}{3}+\frac{1}{2}\right)}{x(1+x)}=\frac{1}{2}$$

6. Let $x_0=\cos\theta$, $x_1=\cos\frac{\theta}{2}$, $x_2=\cos\frac{\theta}{2^2}$

$$=\lim_{n\to\infty}\left(\frac{\sqrt{1-x_0^2}}{x_1x_2x_3\ldots x_{n+1}}\right)=\lim_{n\to\infty}\left(\frac{\sin\theta}{\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\ldots\cos\frac{\theta}{2^{n+1}}}\right)$$

$$=\lim_{n\to\infty}\left(\frac{2^n\cdot\sin\frac{\theta}{2^n}}{\cos\frac{\theta}{2^{n+1}}}\right)=\lim_{n\to\infty}\left(\frac{\sin\frac{\theta}{2^n}}{\frac{\theta}{2^n}}\right)\left(\frac{\theta}{\cos\frac{\theta}{2^{n+1}}}\right)$$

$$=\theta=\cos^{-1}(x_0)$$

7. $f_n(x)=\tan\dfrac{x}{2}\left(\dfrac{1+\cos x}{\cos x}\right)(1+\sec 2x)(1+\sec 4x)\ldots(1+\sec 2^n x)$

$$=\tan x\left(\frac{1+\cos 2x}{\cos 2x}\right)(1+\sec 4x)\ldots(1+\sec 2^n x)$$

$$=\tan 2x(1+\sec 2^2 x)\ldots(1+\sec 2^n x)$$

$$=\tan 2^{n-1}x(1+\sec 2^n x)=\tan 2^n x$$

Now, $\lim\limits_{x\to 0}\dfrac{f_n(x)}{2x}=\lim\limits_{x\to 0}\dfrac{\tan 2^n x}{2^n x}\cdot 2^{n-1}=2^{n-1}$

8. As, $f'(x)>0\Rightarrow f(x)$ is increasing.

So, for $t>1$; $f(t)>1$

Now, $f'(t)=\dfrac{1}{t^2+f(t)}<\dfrac{1}{1+t^2}$

$\therefore$ $f(x)=1+\int_1^x f'(t)\,dt<1+\int_1^x\dfrac{dt}{1+t^2}$

$\Rightarrow$ $\lim\limits_{x\to\infty}f(x)<1+\int_1^\infty\dfrac{dt}{1+t^2}\Rightarrow\lim\limits_{x\to\infty}f(x)<1+\dfrac{\pi}{4}$

9. $E=\sin^2\theta-\sin\theta+\dfrac{1}{2}=\left(\sin\theta-\dfrac{1}{2}\right)^2+\dfrac{1}{4}$

$\Rightarrow$ Minimum value is $\dfrac{1}{4}$.

Let $K=\sqrt{(x+1)(x+2)}$, then $K-x=\dfrac{K^2-x^2}{K+x}=\dfrac{3x+2}{K+x}$

$$\lim_{x\to\infty}(K-x)=\lim_{x\to\infty}\frac{3+\frac{2}{x}}{\frac{K}{x}+1}\Rightarrow\lim_{x\to\infty}\frac{K}{x}=1$$

$\therefore$ $\lim\limits_{x\to\infty}(K-x)=\dfrac{3+0}{1+1}=\dfrac{3}{2}$ $\left[\lim\limits_{x\to\infty}\dfrac{K}{x}=1\right]$

The required equation is $x^2-\dfrac{7}{4}x+\dfrac{3}{8}=0$,

i.e. $8x^2-14x+3=0$.

10. Let $x_n=\tan\theta_n$

Now, $\tan\theta_{n+1}=x_{n+1}=\dfrac{x_n}{1+\sqrt{1+x_n^2}}=\dfrac{\tan\theta}{1+\sqrt{1+\tan^2\theta_n}}$

$\Rightarrow$ $\tan\theta_{n+1}=\dfrac{\tan\theta_n}{1+\sec\theta_n}=\dfrac{\sin\theta_n}{1+\cos\theta_n}=\tan\dfrac{\theta_n}{2}$

$\Rightarrow$ $\theta_{n+1}=\dfrac{\theta_n}{2}\Rightarrow\theta_n=\dfrac{\theta_{n-1}}{2}$

Now, $\theta_1=\dfrac{\pi}{3}\Rightarrow\theta_n=\dfrac{\pi}{3\cdot 2^{n-1}}\Rightarrow x_n=\tan\left(\dfrac{2\pi}{3\cdot 2^n}\right)$

$\Rightarrow$ $\lim\limits_{n\to\infty}2^n x_n=\lim\limits_{n\to\infty}\dfrac{\tan\left(\frac{2\pi}{3\cdot 2^n}\right)}{\left(\frac{1}{2^n}\right)}=\dfrac{2\pi}{3}$

11. $L=\lim\limits_{x\to a^-}=\dfrac{(x-b)-(a-b)}{\sqrt{x-b}+\sqrt{a-b}}\times\dfrac{1}{(x^2-a^2)}$

$$=\lim_{x\to a^-}\frac{1}{(x+a)\{\sqrt{x-b}+\sqrt{a-b}\}}=\frac{1}{4a\sqrt{a-b}}$$

12. $\lim\limits_{n\to\infty}(\sin^n 1+\cos^n 1)^n=\sin 1\lim\limits_{n\to\infty}(1+\cot^n 1)^n$

$$=\sin 1\cdot e^{\lim\limits_{n\to\infty}\frac{n}{\tan^n 1}}=\sin 1\cdot e^{\lim\limits_{n\to\infty}\frac{1}{\tan^n 1\cdot\ln(\tan 1)}}=\sin 1$$

13. As, $\sin^{-1}x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5}{112}x^7 + \cdots$

and $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

$$\therefore \lim_{x\to 0}\frac{2}{x^3}(\sin^{-1}x - \tan^{-1}x)$$

$$= \lim_{x\to 0}\frac{2}{x^3}\left[\left(x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots\right) - \left(x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)\right]$$

$$= \lim_{x\to 0}\frac{2}{x^3}\left[x^3\left(\frac{1}{2}\right) + x^5\left(\frac{3}{40} - \frac{1}{5}\right) + \cdots\right] = 1$$

$$\therefore \lim_{x\to 0}\left[\frac{2}{x^3}(\sin^{-1}x - \tan^{-1}x)\right]^{2/x^2} \qquad [1^\infty \text{ form}]$$

$$= e^{\lim_{x\to 0}\left[\frac{2}{x^3}(\sin^{-1}x - \tan^{-1}x) - 1\right]\frac{2}{x^2}}$$

$$= e^{\lim_{x\to 0}\frac{2}{x^2}\cdot\frac{2\left[\left(x + \frac{x^3}{6} + \frac{3x^5}{5} + \cdots\right) - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\right)\right] - x^3}{x^3}}$$

$$= e^{\lim_{x\to 0}\frac{2}{x^5}\left[-\frac{1}{4}x^5 + \text{higher powers of } x\right]} = e^{-1/2} = \frac{1}{\sqrt{e}}$$

14. We have, $S = \frac{1}{n}\cdot\sum_{k=1}^{n}\left(1 - \frac{k}{n}\right)\cdot\cos 4\left(\frac{k}{n}\right)$

$$S = \int_0^1 (1-x)\cos 4x\cdot dx = \left((1-x)\cdot\frac{\sin 4x}{4}\right)_0^1 + \frac{1}{4}\int_0^1 \sin 4x\, dx$$

$$= 0 + \frac{1}{4}\cdot\frac{1}{(-4)}(\cos 4x)_0^1 = -\frac{1}{16}(\cos 4 - \cos 0) = \frac{1}{16}(1 - \cos 4)$$

15. Let $y = \sqrt{\cos 2x}\cdot\sqrt[3]{\cos 3x}\cdot\sqrt[4]{\cos 4x}\ \ldots\ \sqrt[n]{\cos nx}$

$$\log_e y = \frac{1}{2}\log\cos 2x + \frac{1}{3}\log\cos 3x + \frac{1}{4}\log\cos 4x + \ldots + \frac{1}{n}\log\cos nx$$

On differentiating both sides, we get

$$\frac{1}{y}\cdot dy/dx = \frac{1}{2}\cdot\frac{(-2\sin 2x)}{\cos 2x} + \frac{1}{3}\cdot\left(\frac{-3\sin 3x}{\cos 3x}\right) + \ldots + \frac{1}{n}\left[\frac{(-n\sin nx}{\cos nx}\right]$$

$$\therefore \frac{dy}{dx} = \prod_{r=2}^{n}(\cos rx)^{1/r}\cdot\sum_{r=2}^{n}[-\tan(rx)] \qquad \ldots(i)$$

$$\therefore L = \lim_{x\to 0}\frac{1 - \prod_{r=2}^{n}[\cos(rx)]^{1/r}}{x^2}$$

$$= \lim_{x\to 0}\frac{-\prod_{r=2}^{n}(\cos rx)^{1/2}\sum_{r=2}^{n}[-\tan(rx)]}{2x} \qquad \text{[using L' Hospital's rule]}$$

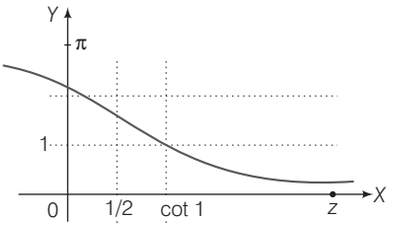
$$= \lim_{x\to 0}\frac{1}{2}\cdot\left[\frac{\tan 2x + \tan 3x + \cdots + \tan nx}{x}\right]$$

$$= \frac{1}{2}[2 + 3 + \cdots + n] = \frac{n^2 + n - 2}{4} = 10 \qquad \text{[given]}$$

$$\Rightarrow (n+7)(n-6) = 0 \quad \Rightarrow \quad n = 6$$

16. Here, $\lim_{z\to\infty}\frac{\int_{1/2}^{z}[\cot^{-1}x]\,dx}{\int_{1/2}^{z}\left[1 + \frac{1}{x}\right]dx} = \frac{\int_{1/2}^{\cot 1}1\cdot dx + \int_{\cot 1}^{z}0\cdot dx}{\int_{1/2}^{1}2\cdot dx + \int_{1}^{z}1\cdot dx}$

As,



$\because$ $[\cot^{-1}x] = 1$, when $1/2 < x < \cot 1$

and $[\cos^{-1}x] = 0$, when $\cot 1 < x < z$

$$\Rightarrow \lim_{z\to\infty}\frac{\int_{1/2}^{\cot 1}1\cdot dx}{\int_{1/2}^{1}2\cdot dx + \int_{1}^{z}1\cdot dx} = \lim_{z\to\infty}\frac{(\cot 1 - 1/2)}{2(1-1/2) + (z-1)} = 0$$

17. $\lim_{\theta\to 0^+}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) = \lim_{\theta\to 0^+}\frac{\alpha+\beta}{\alpha\beta} = \lim_{\theta\to 0^+}\frac{\sqrt{2}|\sin\theta|}{\theta} = \sqrt{2}$

18. Here, $\lim_{x\to\infty}f(x) = \lim_{x\to\infty}\frac{1}{3}\left[f(x+1) + \frac{5}{f(x+2)}\right]$

$$\Rightarrow \qquad l = \frac{1}{3}\cdot\left(l + \frac{5}{l}\right) \Rightarrow l^2 = \frac{1}{3}(l^2 + 5)$$

$$\Rightarrow \qquad 2l^2 = 5 \quad \Rightarrow l = \sqrt{\frac{5}{2}}$$

19. $\lim_{x\to 2^-}\frac{\cos(2x+4) - 33}{2} = -16$ and $\lim_{x\to 2^-}\frac{x^2|4x-8|}{2} = -4(4) = -16$

By Sandwich theorem, $\lim_{x\to 2^-}f(x) = -16$

20. As, $f(x)$ being polynomial having roots 1, 2, 3, 4 and leading coefficient 1.

$$\therefore \qquad f(x) = (x-1)(x-2)(x-3)(x-4)$$

Similarly, $\qquad g(x) = (x-1)\left(x - \frac{1}{2}\right)\left(x - \frac{1}{3}\right)\left(x - \frac{1}{4}\right)$

$$\therefore \qquad \lim_{x\to 1}\frac{f(x)}{g(x)} = \lim_{x\to 1}\frac{(x-1)(x-2)(x-3)(x-4)}{(x-1)\left(x - \frac{1}{2}\right)\left(x - \frac{1}{3}\right)\left(x - \frac{1}{4}\right)}$$

$$= \frac{(-1)(-2)(-3)}{\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)} = -24$$

21. We have, $\lim_{x\to\frac{\pi}{4}}\frac{4\sqrt{2} - (\cos x + \sin x)^5}{1 - \sin 2x}$

$$= \lim_{x\to\frac{\pi}{4}}\frac{[(\cos x + \sin x)^2]^{5/2} - (2)^{5/2}}{(1 + \sin 2x) - 2} = \lim_{x\to\frac{\pi}{4}}\frac{(1 + \sin 2x)^{5/2} - (2)^{5/2}}{(1 + \sin 2x) - 2}$$

Put $y = 1 + \sin 2x$

$$= \lim_{x\to\frac{\pi}{4}}\frac{y^{5/2} - 2^{5/2}}{y - 2} = \frac{5}{2}(2)^{\frac{5}{2}-1} = 5\sqrt{2}\left[\text{as } x\to\frac{\pi}{4} \Rightarrow y\to 2\right]$$

22. We have, $\lim_{n\to\infty}\frac{n\cdot 3^n}{n(x-2)^n + n\cdot 3^{n+1} - 3^n} = \frac{1}{3}$

$$\Rightarrow \lim_{n\to\infty}\frac{1}{\left(\frac{x-2}{3}\right)^n + 3 - \frac{1}{n}} = \frac{1}{3}$$

For, limit to exists and is equal to $\frac{1}{3}$.

We must have, $-1 < \frac{x-2}{3} < 1$.

$\Rightarrow\ -3 < x-2 < 3\ \Rightarrow\ -1 < x < 5$, as $x \in$ integer

$\therefore\quad x \in \{0,\ 1,\ 2,\ 3,\ 4\}$

$\Rightarrow$ Number of integer solutions = 5.

23. Given, $f(x) = \lim\limits_{n\to\infty} \dfrac{1}{\left(\frac{3}{\pi}\tan^{-1}2x\right)^{2n} + 5}$

Now, $f(x) = 0$, if and only if $\left(\frac{3}{\pi}\tan^{-1}2x\right)^2 > 1$

$\Rightarrow\ \tan^{-1}2x > \frac{\pi}{3}$ or $\tan^{-1}2x < -\frac{\pi}{3}$

$\Rightarrow\quad 2x > \sqrt{3}\quad$ or $\quad 2x < -\sqrt{3}\ \Rightarrow\ |2x| > \sqrt{3}$

24. We have, $\lim\limits_{x\to 0} \dfrac{\cos^2 x - \cos x - e^x\cos x + e^x - \frac{x^3}{2}}{x^n}$

$$= \lim_{x\to 0} \frac{(\cos x - 1)(\cos x - e^x) - \frac{x^3}{2}}{x^n}$$

$$= \lim_{x\to 0} \frac{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots - 1\right)\left[\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) - \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots\right)\right] - \frac{x^3}{2}}{x^n}$$

$$= \lim_{x\to 0} \frac{\left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right)\left(-x - x^2 - \frac{x^3}{3!} - \frac{2x^5}{5!} - \ldots\right) - \frac{x^3}{2}}{x^n}$$

$$= \lim_{x\to 0} \frac{\left(\frac{x^3}{2} + \frac{x^4}{2} + \frac{x^5}{12} - \ldots\right) - \frac{x^3}{2}}{x^n} = \lim_{x\to 0} \frac{\frac{x^4}{2} + \frac{x^5}{12} - \frac{x^5}{24}}{x^n}$$

= a non-zero finit number, if $n = 4$.

25. We know that,

$\frac{\tan^{-1}x}{x} < 1$ and $\frac{\sin^{-1}x}{x} > 1,\ \forall\ x \in R$

$\therefore\quad \frac{\tan^{-1}x}{x} - \frac{\sin^{-1}x}{x} < 0$ and $\frac{\sin^{-1}x}{x} - \frac{\tan^{-1}x}{x} > 0$

$\Rightarrow\ I_1$ doesn't exist and I_2 exists.

26. Here, $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{[x/2]}{\log(\sin x)}$

As, $x \to \frac{\pi}{2}\ \Rightarrow\ \frac{x}{2} \to \frac{\pi}{4}$, where $\frac{x}{2} \to \frac{\pi}{4} < 1$

for $\quad x \to \frac{\pi^+}{2}$ or $x \to \frac{\pi^-}{2}$

$\therefore\quad \left[\frac{x}{2}\right] = 0$, since $0 < \left(\frac{x}{2} \to \frac{\pi}{4}\right) < 1$

$\Rightarrow\quad \lim\limits_{x\to\pi/2} \dfrac{\left[\frac{x}{2}\right]}{\log\sin x} = \lim\limits_{x\to\pi/2} 0 = 0$

27. We have, $\lim\limits_{n\to\infty} \left(\frac{a_1+1}{a_1}\right)\left(\frac{a_2+1}{a_2}\right)\ldots\left(\frac{a_n+1}{a_n}\right)$

and $\quad a_{n-1} + 1 = \frac{a_n}{n}$ ...(i)

$$\therefore \lim_{n\to\infty} \left(\frac{a_2}{2}\right)\left(\frac{a_3}{3}\right)\left(\frac{a_4}{4}\right)\ldots\left(\frac{a_{n+1}}{n+1}\right)\cdot\frac{1}{a_1\cdot a_2\ldots a_n}$$

$$= \lim_{n\to\infty} \frac{a_{n+1}}{(n+1)!} = \lim_{n\to\infty} \frac{1+a_n}{n!} \quad \text{[using Eq. (i)]}$$

$$= \lim_{n\to\infty} \left(\frac{1}{n!} + \frac{a_n}{n!}\right) = \lim_{n\to\infty} \left(\frac{1}{n!} + \frac{1}{(n-1)!} + \frac{a_{n-1}}{(n-1)!}\right) \text{[using Eq. (i)]}$$

$$= \lim_{n\to\infty} \left(\frac{1}{n!} + \frac{1}{(n-1)!} + \frac{1}{(n-2)!} + \ldots + \frac{1}{(2)!} + \frac{1}{1!} + \frac{a_1}{1!}\right)$$

$[a_1 = 1,\ \text{given}]$

$$= \lim_{n\to\infty} \left(\frac{1}{n!} + \frac{1}{(n-1)!} + \frac{1}{(n-2)!} + \ldots + \frac{1}{(2)!} + \frac{1}{1!} + \frac{1}{1}\right)$$

$$= e \qquad \left[\text{as } e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\infty\right]$$

28. Here, $\quad f(x+y) = f(x) + f(y)$

$$f(2) = f(1) + f(1) = 2$$

$$f(3) = f(1+2) = f(1) + f(2) = 3$$

$$f(4) = f(1+3) = f(1) + f(3) = 4$$

$$\vdots\quad\vdots\quad\vdots\quad\vdots\quad\vdots\quad\vdots\quad\vdots\quad\vdots$$

$$f(x) = x,\ \text{for all } x \in R$$

$\Rightarrow\quad f(\tan x) = \tan x,\ \ f(\sin x) = \sin x$

$$\therefore \lim_{x\to 0} \frac{2^{f(\tan x)} - 2^{f(\sin x)}}{x^2\, f(\sin x)} = \lim_{x\to 0} \frac{2^{\tan x} - 2^{\sin x}}{x^2 \sin x}$$

$$= \lim_{x\to 0} \frac{2^{\sin x}\cdot\{2^{\tan x - \sin x} - 1\}}{\sin x\cdot x^2} \times \frac{\tan x - \sin x}{\tan x - \sin x}$$

$$= \lim_{x\to 0} \frac{\{2^{\tan x - \sin x} - 1\}}{\tan x - \sin x} \times \left\{\frac{\tan x - \sin x}{x^2 \sin x}\right\} \times 2^{\sin x}$$

$$= \lim_{x\to 0} \frac{2^{\tan x - \sin x} - 1}{\tan x - \sin x} \times \frac{1 - \cos x}{x^2\cos x} \times 2^{\sin x}$$

$$= \lim_{x\to 0} \frac{2^{\tan x - \sin x} - 1}{\tan x - \sin x} \times \frac{2\sin^2 x/2}{4(x/2)^2} \times 2^{\sin x} \times \frac{1}{\cos x}$$

$$= \log 2 \times \frac{1}{2} \times 1 = \frac{1}{2}\log 2$$

29. We have, $\lim\limits_{n\to\infty} n^{-n^2}\left[(n+1)\left(n+\frac{1}{2}\right)\ldots\left(n+\frac{1}{2^{n-1}}\right)\right]^n$

$$= \lim_{n\to\infty} \left[\frac{(n+1)\left(n+\frac{1}{2}\right)\ldots\left(n+\frac{1}{2^{n-1}}\right)}{n^n}\right]^n$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n \cdot \left(\frac{n+\frac{1}{2}}{n}\right)^n \ldots \left(\frac{n+\frac{1}{2^{n-1}}}{n}\right)^n$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \cdot \left(1+\frac{1}{2n}\right)^n \ldots \left(1+\frac{1}{2^{n-1}n}\right)^n \qquad [1^\infty \text{ form}]$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \cdot \left(1+\frac{1}{2n}\right)^{\frac{2n}{2}} \ldots \left(1+\frac{1}{2^{n-1}n}\right)^{\frac{2^{n-1}\cdot n}{2^{n-1}}}$$

$$= e^1 \cdot e^{1/2} \cdot e^{1/4} \ldots e^{1/2^{n-1}} \ldots \infty \qquad \left[\text{using } \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{an} = e^a\right]$$

$$= e^{(1+1/2+1/4+\ldots)} = e^{\frac{1}{1-\frac{1}{2}}} = e^2$$

30. Here, limit can be calculated only after removing greatest integral function (i.e. $[x]$).

$\therefore$ LHL at $x = 0$

$$= \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin[x]}{[x]} = \lim_{h\to 0} \frac{\sin[0-h]}{[0-h]}$$

$$= \lim_{h\to 0} \frac{\sin(-1)}{(-1)} = \sin 1$$

$[\text{as } -1 \le [0-h] < 0 \ \therefore [0-h] = -1, \text{ i.e. } [x] \ne 0]$

Again, RHL at $x = 0$

$$= \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 0 = 0 \qquad \left[\text{as } f(x) = \begin{cases} \dfrac{\sin[x]}{[x]}, & [x] \ne 0 \\ 0, & [x] = 0 \end{cases}\right]$$

$$\therefore \qquad \lim_{x\to 0^+} f(x) = 0$$

Here, $0 \le [0+h] < 1$

$$\therefore \qquad [x] = 0 \Rightarrow f(x) = 0$$

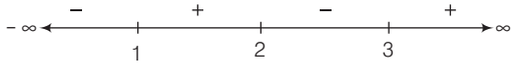
So, $\lim\limits_{x\to 0^-} f(x) \ne \lim\limits_{x\to 0^+} f(x)$. Thus, limit doesn't exist.

31. We write, $f(x) = \dfrac{|(x-1)(x-2)(x-3)|}{(x-1)(x-2)(x-3)}$

$$f(x) = \begin{cases} -1, & x < 1 \\ 1, & 1 < x < 2 \\ -1, & 2 < x < 3 \\ 1, & x > 3 \end{cases}$$

[leaving $x = 1, 2, 3$ as denominator $\ne 0$]

Using Wavy-curve method, as shown in figure



When $x > 3$, then

$|(x-1)(x-2)(x-3)| = +(x-1)(x-2)(x-3)$

When $2 < x < 3$, then

$|(x-1)(x-2)(x-3)| = -(x-1)(x-2)(x-3)$

When $1 < x < 2$, then

$|(x-1)(x-2)(x-3)| = +(x-1)(x-2)(x-3)$

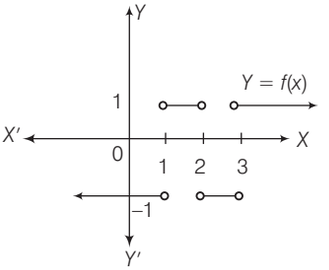
When $x < 1$, then

$|(x-1)(x-2)(x-3)| = -(x-1)(x-2)(x-3)$

Thus,
$$f(x) = \begin{cases} -1, & x < 1 \\ 1, & 1 < x < 2 \\ -1, & 2 < x < 3 \\ 1, & x > 3 \end{cases}$$

Shows limit exists at all points except at $x = 1, 2, 3$.

Graphically, this is shown in given figure.



which shows limit doesn't exist at $x = 1, 2, 3$.

32. Here, $t_r = \dfrac{r}{1+r^2+r^4}$

$$= \frac{1}{2}\left(\frac{2r}{1+2r^2+r^4-r^2}\right) = \frac{1}{2}\left[\frac{2r}{(1+r^2)^2-(r^2)}\right]$$

$$= \frac{1}{2}\left[\frac{2r}{(1+r^2+r)(1+r^2-r)}\right]$$

$$= \frac{1}{2}\left[\frac{(r^2+r+1)-(r^2-r+1)}{(1+r^2+r)(1+r^2-r)}\right] \qquad [\because 2r = (r^2+r+1)-(r^2-r+1)]$$

$$= \frac{1}{2}\left[\frac{1}{1+r^2-r} - \frac{1}{1+r^2+r}\right] = \frac{1}{2}\left[\frac{1}{1+r(r-1)} - \frac{1}{1+r(r+1)}\right] \ldots(\text{i})$$

Thus, $t_r = \dfrac{1}{2}\left(\dfrac{1}{1+r(r-1)} - \dfrac{1}{1+r(r+1)}\right)$

$$t_1 = \frac{1}{2}\left(\frac{1}{1+0} - \frac{1}{1+2}\right) = \frac{1}{2}\left(1-\frac{1}{3}\right)$$

$$t_2 = \frac{1}{2}\left(\frac{1}{1+2} - \frac{1}{1+2\cdot 3}\right) = \frac{1}{2}\left(\frac{1}{3}-\frac{1}{7}\right)$$

$$t_3 = \frac{1}{2}\left(\frac{1}{7}-\frac{1}{13}\right) = \frac{1}{2}\left(\frac{1}{7}-\frac{1}{13}\right)$$

.....

.....

$$t_n = \frac{1}{2}\left(\frac{1}{1+n(n-1)} - \frac{1}{1+n(n+1)}\right)$$

$$\therefore \qquad \lim_{n\to\infty} \sum_{r=1}^{n} t_r = \frac{1}{2}\left(1 - \frac{1}{1+n(n+1)}\right) \qquad \ldots(\text{ii})$$

Hence, $\lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} t_r = \lim\limits_{n\to\infty} \dfrac{1}{2}\left(1 - \dfrac{1}{1+n(n+1)}\right) = \dfrac{1}{2}\left(1-\dfrac{1}{\infty}\right)$

$$\therefore \qquad \lim_{n\to\infty} \sum_{r=1}^{n} t_r = \frac{1}{2}$$

33. Here, $\lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \cot^{-1}\left(\dfrac{r^3 - r + \frac{1}{r}}{2}\right) = \lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \tan^{-1}\left(\dfrac{2r}{1-r^2+r^4}\right)$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \tan^{-1}\left(\frac{2r}{1+(r^2-r)(r^2+r)}\right)$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\tan^{-1}\left(\frac{(r^2+r)-(r^2-r)}{1+(r^2+r)(r^2-r)}\right)$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\{\tan^{-1}(r^2+r)-\tan^{-1}(r^2-r)\}$$

$$= \lim_{n\to\infty}[(\tan^{-1}2-\tan^{-1}0)+(\tan^{-1}6-\tan^{-1}2)+(\tan^{-1}12-\tan^{-1}6)$$
$$+\ldots+\{(\tan^{-1}(n^2+n)-\tan^{-1}(n^2-n)\}]$$

$$= \lim_{n\to\infty}\{\tan^{-1}(n^2+n)-\tan^{-1}(0)\}$$

$$= \tan^{-1}(\infty)-\tan^{-1}(0) = \frac{\pi}{2}$$

$$\therefore \lim_{n\to\infty}\sum_{r=1}^{n}\cot^{-1}\left(\frac{r^3-r+\frac{1}{r}}{2}\right)=\frac{\pi}{2}$$

34. Here, two straight lines $(\tan\alpha)\,x+(\sin\alpha)\,y=\alpha$ and $(\alpha\,\text{cosec}\,\alpha)\,x+(\cos\alpha)\,y=1$ have their point of intersection, as

$$x=\frac{\alpha\cos\alpha-\sin\alpha}{\sin\alpha-\alpha}\quad\text{and}\quad y=\frac{\alpha-x\tan\alpha}{\sin\alpha}$$

$\therefore$ When $\alpha\to 0$, we obtain the point P.

i.e. $\lim\limits_{\alpha\to 0}x=\lim\limits_{\alpha\to 0}\dfrac{\alpha\cos\alpha-\sin\alpha}{\sin\alpha-\alpha}$ $\left[\dfrac{0}{0}\text{ form}\right]$

$$= \lim_{\alpha\to 0}\frac{-\alpha\sin\alpha+\cos\alpha-\cos\alpha}{\cos\alpha-1}\quad[\text{applying L'Hospital's rule}]$$

$$= \lim_{\alpha\to 0}\frac{\alpha\left(2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right)}{2\sin^2\frac{\alpha}{2}} = \lim_{\alpha\to 0}\frac{\alpha}{\tan\frac{\alpha}{2}} = \lim_{\alpha\to 0}\frac{\frac{2\alpha}{2}}{\tan\frac{\alpha}{2}} = 2$$

Again, as $\lim\limits_{\alpha\to 0}y=\lim\limits_{\alpha\to 0}\dfrac{\alpha-x\tan\alpha}{\sin\alpha}$

$$= \lim_{\alpha\to 0}\left(\frac{\alpha}{\sin\alpha}-\frac{x}{\cos\alpha}\right) = \lim_{\alpha\to 0}\frac{\alpha}{\sin\alpha}-\lim_{\alpha\to 0}\frac{x}{\cos\alpha}$$

$= 1-2$ $\left[\text{as }\lim\limits_{\alpha\to 0}x=2\right]$

$\therefore \lim\limits_{\alpha\to 0}y=-1$

Hence, in limiting position $P(2,-1)$.

35. Here, $\lim\limits_{x\to 0}\left(1+\dfrac{x^2+f(x)}{x^2}\right)^{1/x}$

$\Rightarrow \lim\limits_{x\to 0}\left(1+\dfrac{x^2+f(x)}{x^2}\right)^{1/x}=L$ (say)

exists only when

$\lim\limits_{x\to 0}\dfrac{x^2+f(x)}{x^2}=0$ [it converts to 1^∞ form]

So, the least degree in $f(x)$ is of degree 2.

i.e. $f(x)=a_2x^2+a_3x^3+\ldots$

Now, $\lim\limits_{x\to 0}\left(1+\dfrac{x^2+f(x)}{x^2}\right)^{1/x}=e^2$

$$\Rightarrow e^{\lim\limits_{x\to 0}\left(\frac{x^2+f(x)}{x^2}\right)\frac{1}{x}}=e^2\Rightarrow e^{\lim\limits_{x\to 0}\frac{x^2+f(x)}{x^3}}=e^2$$

$$\therefore \lim_{x\to 0}\frac{x^2+f(x)}{x^3}=2\Rightarrow\lim_{x\to 0}\frac{x^2+a_2x^2+a_3x^3+\ldots}{x^3}=2$$

$\Rightarrow a_2=-1, a_3=2$ and a_4, a_5 are any arbitrary constants. Since, we want polynomial of least degree.

Hence, $f(x)=-x^2+2x^3$

36. We know, $n\le[x]<n+1\Rightarrow[x]=n$

Here, $\dfrac{n\sin x}{x}\to n$, as $x\to 0$ but less than n.

Also, $\dfrac{n\tan x}{x}\to n$, as $x\to 0$ but more than n.

Thus, $n-1<\left[\dfrac{n\sin x}{x}\right]<n$, as $x\to 0$

$\Rightarrow \left[\dfrac{n\sin x}{x}\right]=n-1$

Again, $n\le\left[\dfrac{n\tan x}{x}\right]<n+1$, as $x\to 0$

$\Rightarrow \left[\dfrac{n\tan x}{x}\right]=n$

Thus, $\lim\limits_{x\to 0}\left(\left[\dfrac{n\sin x}{x}\right]+\left[\dfrac{n\tan x}{x}\right]\right)=(n-1)+(n)=(2n-1)$

$\therefore \lim\limits_{x\to 0}\left(\left[\dfrac{n\sin x}{x}\right]+\left[\dfrac{n\tan x}{x}\right]\right)=(2n-1)$

37. Here, $\lim\limits_{x\to a}[\sqrt{2-x}+\sqrt{1+x}]$ and we know, we could only apply limit after defining greatest integral function.

Thus, finding range of $[\sqrt{2-x}+\sqrt{1+x}]$, when $x\in\left[0,\dfrac{1}{2}\right]$.

i.e. Let $f(x)=\sqrt{2-x}+\sqrt{1+x}$

For range $f'(x)=\dfrac{1}{2}\left(-\dfrac{1}{\sqrt{2-x}}+\dfrac{1}{\sqrt{1+x}}\right)$

$$= \frac{1}{2}\left(\frac{\sqrt{2-x}-\sqrt{1+x}}{\sqrt{2-x}\cdot\sqrt{1+x}}\right)$$

$\Rightarrow f'(x)$ will be the +ve for $\sqrt{2-x}>\sqrt{1+x}$.

$\Rightarrow f'(x)$ will be the +ve for $2-x>1+x$.

$\Rightarrow f'(x)$ will be the +ve for $2x<1$.

$\Rightarrow x<\dfrac{1}{2}$

$\therefore f(x)$ will be increasing for $x<\dfrac{1}{2}$.

$\therefore f(0)=\sqrt{2}+1$

$\Rightarrow f(1/2)=\sqrt{6}$

which shows range of $f(x)$ is $[1+\sqrt{2},\sqrt{6}]$, when $x\in\left[0,\dfrac{1}{2}\right]$.

$\therefore [\sqrt{2-x}+\sqrt{1+x}]=2$

$\Rightarrow \lim\limits_{x\to a}[\sqrt{2-x}+\sqrt{1+x}]=\lim\limits_{x\to a}2=2$

38. Let $y=\sqrt{(\tan x-\sin x)+\sqrt{(\tan x-\sin x)+\sqrt{(\tan x-\sin x)+\ldots\infty}}}$

$\Rightarrow \quad y=\sqrt{(\tan x-\sin x)+y}$

$\Rightarrow \quad y^2-y-(\tan x-\sin x)=0$

$\Rightarrow \quad y=\dfrac{1+\sqrt{1+(\tan x-\sin x)\,4}}{2}$ [as $y>0$] ...(i)

Again, let $Z=\sqrt{x^3+\sqrt{x^3+\sqrt{x^3+\ldots\infty}}}$

$\Rightarrow \quad Z=\sqrt{x^3+Z}$

$\Rightarrow \quad Z^2-Z-x^3=0$

$\Rightarrow \quad Z=\dfrac{1+\sqrt{(1+4x^3)}}{2}$ [as $Z>0$] ...(ii)

$$\therefore \lim_{x\to 0^+}\frac{-1+\sqrt{(\tan x-\sin x)+\sqrt{(\tan x-\sin x)+\ldots\infty}}}{-1+\sqrt{x^3+\sqrt{x^3+\sqrt{x^3+\ldots\infty}}}}$$

$$=\lim_{x\to 0^+}\frac{-1+\left(\dfrac{1+\sqrt{1+4(\tan x-\sin x)}}{2}\right)}{-1+\left(\dfrac{1+\sqrt{1+4x^3}}{2}\right)}$$ [from Eqs. (i) and (ii)]

$$=\lim_{x\to 0^+}\frac{-1+\sqrt{1+4(\tan x-\sin x)}}{-1+\sqrt{1+4x^3}}$$

Rationalising numerator and denominator, we get

$$\lim_{x\to 0^+}\frac{4(\tan x-\sin x)(1+\sqrt{1+4x^3})}{4x^3(1+\sqrt{1+4(\tan x-\sin x)})}$$

$$=\lim_{x\to 0^+}\frac{\left(\dfrac{\sin x}{\cos x}-\dfrac{\sin x}{1}\right)(1+\sqrt{1+4x^3})}{x^3(1+\sqrt{1+4(\tan x-\sin x)})}$$

$$=\lim_{x\to 0^+}\frac{\sin x(1-\cos x)}{x^3\cos x}\cdot\frac{(1+\sqrt{1+4x^3})}{(1+\sqrt{1+4(\tan x-\sin x)})}$$

$$=\lim_{x\to 0^+}\frac{\sin x}{x}\cdot\frac{2\sin^2\dfrac{x}{2}}{\dfrac{4x^2}{4}}\cdot\frac{1}{\cos x}\cdot\frac{(1+\sqrt{1+4x^3})}{(1+\sqrt{1+4(\tan x-\sin x)})}$$

$$=1\cdot\frac{1}{2}\cdot 1\cdot\frac{(1+1)}{(1+1)}=\frac{1}{2}$$

$$\therefore \lim_{x\to 0^+}\frac{-1+\sqrt{(\tan x-\sin x)+\sqrt{(\tan x-\sin x)+\sqrt{(\tan x-\sin x)}}+\ldots\infty}}{-1+\sqrt{x^3+\sqrt{x^3+\sqrt{x^3+\ldots\infty}}}}=\frac{1}{2}$$

39. Here, $\lim\limits_{\theta\to 0}\dfrac{\cos^2(1-\cos^2(1-\cos^2(1\ldots\cos^2\theta))\ldots)}{\sin\left(\dfrac{\pi(\sqrt{\theta+4}-2)}{\theta}\right)}$

$$=\lim_{\theta\to 0}\frac{\cos^2(\sin^2(\sin^2\ldots(\sin^2\theta))\ldots)}{\sin\left(\dfrac{\pi(\sqrt{\theta+4}-2)}{\theta}\right)}$$

$$=\lim_{\theta\to 0}\frac{\cos^2(\sin^2(\sin^2\ldots(\sin^2\theta))\ldots)}{\sin\left(\pi\lim\limits_{\theta\to 0}\dfrac{\theta}{\theta(\sqrt{\theta+4}+2)}\right)}=\frac{\cos^2(0)}{\sin\left(\dfrac{\pi}{4}\right)}=\sqrt{2}$$

40. Here, $\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}a_{n+1}$.

So, let $l=\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}a_{n+1}$

$\therefore \quad l=\lim\limits_{n\to\infty}a_{n+1}=\lim\limits_{n\to\infty}\sqrt{2+a_n}$

or $l=\sqrt{2+l}$ $\left[\because \lim\limits_{n\to\infty}a_n=l\right]$

$\Rightarrow \quad l^2=2+l \quad\Rightarrow\quad l^2-l-2=0$

$\Rightarrow \quad (l-2)(l+1)=0 \quad\Rightarrow\quad l=2,-1$

$\therefore \quad \lim\limits_{n\to\infty}a_n=2 \text{ or } -1$

$\Rightarrow \quad \lim\limits_{n\to\infty}a_n=2 \quad\left[\text{neglecting } \lim\limits_{n\to\infty}a_n=-1, \text{ as } a_n>0\right]$

41. $\lim\limits_{x\to\infty}4x\dfrac{\tan^{-1}\left(\dfrac{1}{2x+3}\right)}{\left(\dfrac{1}{2x+3}\right)}\times\dfrac{1}{2x+3}=2$

$\Rightarrow \quad y^2+4y+5=2$

$\Rightarrow \quad y=-1,-3.$

42. $\lim\limits_{x\to 0}\dfrac{2\sin^2\left(\dfrac{x^2}{2}\right)}{2}\cdot\dfrac{\left(\dfrac{x^2}{2}\right)^2}{(4^x-1)}\cdot\dfrac{x}{x^4}=\dfrac{1}{2}\ln 4=\ln 2=1-\dfrac{1}{2}\ln\left(\dfrac{e^2}{4}\right)$

43. $f(x)=e^{[\cot x]}$, as $\cot x$ is negative in the II quadrant and

$\cot\dfrac{\pi}{2}=0\,[\cot x]=-1 \quad\Rightarrow\quad \lim\limits_{x\to\frac{\pi^+}{2}}f(x)=e^{-1}=\dfrac{1}{e}$

As $x\to\dfrac{\pi^-}{2}$, $\cot x$ is positive (being in I quadrant) and hence

$[\cot x]=0 \quad\Rightarrow\quad \lim\limits_{x\to\frac{\pi^-}{2}}f(x)=e^0=1$

44. If $m<0$, then for values of x sufficiently close to 0.

$$1+\frac{1}{m}<\frac{\sin x}{x}<1 \quad\Rightarrow\quad m+1>m\frac{\sin x}{x}>m$$

$$\therefore \quad \left[m\frac{\sin x}{x}\right]=m \quad\Rightarrow\quad \lim_{x\to 0}\left[m\frac{\sin x}{x}\right]=m$$

If $m>0$, then for values of x sufficiently close to 0, we can have

$$1-\frac{1}{m}<\frac{\sin x}{x}<1$$

$$\therefore \quad m-1<m\frac{\sin x}{x}<m$$

$$\therefore \quad \lim_{x\to 0}\left[m\frac{\sin x}{x}\right]=m-1$$

45. $\lim_{x\to 0}(1+ax+bx^2)^{\frac{2}{x}}=e^3 \Rightarrow e^{\lim_{x\to\infty}\frac{2}{x}(ax+bx^2)}=e^3$

$\Rightarrow \quad e^{2a}=e^3$

Since, limit value does not involve b.

$\therefore$ b can have any value. Thus, $a=\frac{3}{2}$, $b\in R$

46. (a) $\lim_{x\to 0^+} f(x)=1$

(b) $\lim_{x\to 1} f(x)=2$, as RHL = LHL = 2 (at $x=1$)

(c) $\lim_{x\to 3^-} f(x)\to\infty$ and $\lim_{x\to 3^+} f(x)\to-\infty$

$\therefore$ $\lim_{x\to 3} f(x)$ does not exists.

(d) $\lim_{x\to 4} f(x)=0$

47. RHL $=\lim_{x\to 0^+}\frac{\cot^{-1}(1/x)}{x}=\lim_{x\to 0^+}\frac{\tan^{-1}}{x}=1$

LHL $=\lim_{x\to 0}\frac{\cot^{-1}(1/x)}{x}=\lim_{x\to 0}\frac{\pi+\tan^{-1}x}{x}=\infty$

$\therefore$ RHL exists and LHL does not exists.

48. As, $l=\lim_{x\to\infty}\left(\frac{x+1}{x-1}\right)^x=\lim_{x\to\infty}\left(1+\frac{2}{x-1}\right)^x=e^{\lim_{x\to\infty}\frac{2x}{x-1}}=e^2$

$\therefore$ $[l]=7$ and $\{l\}=e^2-7$.

49. (a) $\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\frac{\sin[x]}{[x]}=\lim_{h\to 0}\frac{\sin(-1)}{(-1)}=\sin 1$

(b) $\lim_{x\to 0^+} f(x)=\lim_{h\to 0}[0+h]=0$

$\therefore$ Limit does not exist at $x=0$.

50. (a) $\lim_{x\to 3}[[x]]-[2x-1]$

RHL $=\lim_{h\to 0}[[3+h]]-[6+2h-1]=\lim_{h\to 0}3-5=-2$

LHL $=\lim_{h\to 0}[[3-h]]-[6-2h-1]=\lim_{h\to 0}2-4=-2$

$\therefore$ Limit exists.

(b) $\lim_{x\to 1}[x]-x$

RHL $=\lim_{h\to 0}[1+h]-(1+h)=\lim_{h\to 0}(1-1-h)=0$

LHL $=\lim_{h\to 0}[1-h]-(1+h)=\lim_{h\to 0}0-1+h=-1$

$\therefore$ Limit does not exist.

(c) $\lim_{x\to 0}\{x\}^2-\{-x\}^2$

RHL $=\lim_{h\to 0}\{h\}^2-\{-h\}^2=\lim_{h\to 0}h^2-(1-h)^2=-1$

LHL $=\lim_{h\to 0}\{-h\}^2-\{h\}^2=\lim_{h\to 0}(1-h)^2-h^2=1$

$\therefore$ Limit does not exist.

(d) $\lim_{x\to 0}\frac{\tan(\operatorname{sgn}x)}{\operatorname{sgn}x}$; RHL $=\lim_{h\to 0}\frac{\tan(1)}{1}=\tan 1$

LHL $=\lim_{h\to 0}\frac{\tan(-1)}{-1}=\tan 1$

$\therefore$ Limit exists.

51. (a) $\lim_{n\to\infty}\left[\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^n}\right]$

$=\lim_{n\to\infty}\left[\frac{\frac{1}{2}\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}}\right]=\lim_{n\to\infty}\left[1-\frac{1}{2^n}\right]=0$

(b) $\lim_{n\to\infty}(x-1)\{x\}$

RHL $=\lim_{h\to 0}(1+h-1)\{1+h\}=\lim_{h\to 0}h(h)=0$

LHL $=\lim_{h\to 0}(1-h-1)\{1-h\}=\lim_{h\to 0}-h(1-h)=0$

$\therefore$ $\lim_{x\to 1}(x-1)\{x\}=0$

(c) As, $\frac{\tan x}{x}>1$

$\therefore$ $\left[\frac{\tan x}{x}\right]\to 1$, as $x\to 0^+ \Rightarrow \lim_{x\to 0^+}\left[\frac{\tan x}{x}\right]=1$

(d) $\left[\lim_{x\to 0^+}\frac{\tan x}{x}\right]=[1+h]=1$

52. Put $x=\frac{\pi}{2}-h$

$l=\lim_{h\to 0}\frac{a^{\tan h}-a^{\sin h}}{\tan h-\sin h}=\lim_{h\to\infty}\frac{a^{\sin h}(a^{\tan h-\sin h}-1)}{\tan h-\sin h}=\log_e a$

and $m=\lim_{x\to-\infty}(\sqrt{x^2+ax}-\sqrt{x^2-ax})$

$=\lim_{x\to-\infty}\frac{(x^2+ax)-(x^2-ax)}{\sqrt{x^2+ax}+\sqrt{x^2-ax}}$

$=\lim_{x\to-\infty}\frac{2ax}{|x|\left(\sqrt{1+\frac{a}{x}}+\sqrt{1-\frac{a}{x}}\right)}=-a$

53. $f(x)=\lim_{x\to\infty}\left[1+\frac{ax+1}{bx+2}-1\right]^x$

$=\lim_{x\to\infty}\left[1+\frac{(a-b)x-1}{bx+2}\right]^x=e^{\lim_{x\to\infty}\left[\frac{(a-b)x-1}{bx+2}\right]^x}$

When $a=b$, $f(x)=e^{\lim_{x\to\infty}-\frac{x}{bx+2}}=e^{-\frac{1}{b}}$ or $e^{-\frac{1}{a}}$.

When $a>b$, $f(x)=e^{\infty}=\infty$, does not exist.

When $a<b$, $f(x)=e^{-\infty}=0$.

54. $\lim_{x\to 0}f(x)=\lim_{x\to 0}\frac{x\cdot(2^x-1)}{2\sin^2 x/2}=\lim_{x\to 0}\frac{2\cdot x^2\frac{1}{4}}{\sin^2\frac{x}{2}}\cdot\frac{2^x-1}{x}=2\cdot\log_e 2=\log_e 4$

and $\lim_{x\to\infty}2^x\cdot\sin(\log 2/2^x)=\lim_{x\to\infty}\frac{\sin(\log 2/2^x)}{(\log 2/2^x)}\cdot\log_e 2=\log_e 2$

55. As, $\lim_{x\to 3}\frac{x^3+cx^2+5x+12}{x^2-7x+12}=l$. [finite]

$\therefore$ $27+9c+15+12=0$

$\Rightarrow \quad 9c+54=0\Rightarrow c=-6$

$\therefore$ $\lim_{x\to 3}\frac{x^3+cx^2+5x+12}{x^2-7x+12}=\lim_{x\to 3}\frac{3x^2+2cx+5}{2x-7}$

$$= \frac{27+6c+5}{9-7} = \frac{27-36+5}{-1}$$

$\therefore \quad c = -6$ and $l = 4$

56. $\lim\limits_{x\to 0} \dfrac{\sin x}{x - \sin x} = \lim\limits_{x\to 0} \dfrac{\frac{\sin x}{x}}{1 - \frac{\sin x}{x}} = \infty$

Hence, $L = e^{\lim\limits_{x\to 0}\left(\frac{\sin x}{x} - 1\right)\frac{\sin x}{x - \sin x}} = e^{\lim\limits_{x\to 0}\frac{-\sin x}{x}} = e^{-1} = \dfrac{1}{e}$

57. $\lim\limits_{x\to 0} \dfrac{x - 1 + \cos x}{x} = \lim\limits_{x\to 0} 1 - \dfrac{2\sin^2\frac{x}{2}}{x} = 1$

Hence, $L = e^{\lim\limits_{x\to 0}\left(\frac{x-1+\cos x}{x} - 1\right)\times\frac{1}{x}}$

$$= e^{\lim\limits_{x\to 0}\left(\frac{-2\sin^2\frac{x}{2}}{x^2}\right)} = e^{-\frac{2}{4}} = e^{-1/2}$$

58. $L = e^{\lim\limits_{x\to 0}\left(\frac{a^x + b^x + c^x}{3} - 1\right)\frac{2}{x}} = e^{\lim\limits_{x\to 0}\left(\frac{a^x - 1}{x} + \frac{b^x - 1}{x} + \frac{c^x - 1}{x}\right)\frac{2}{3}}$

$= e^{\frac{2}{3}(\log a + \log b + \log c)} = e^{\frac{2}{3}\log abc} = (abc)^{2/3}$

Sol. (Q. Nos. 59 to 61)

$$f(x) = \lim_{n\to\infty}\left(\cos\sqrt{\frac{x}{n}}\right)^n = \lim_{n\to\infty}\left(1 + \left(\cos\sqrt{\frac{x}{n}} - 1\right)\right)^n$$

$$= e^{\lim\limits_{n\to\infty}\left(\cos\sqrt{\frac{x}{n}} - 1\right)n} = e^{-\lim\limits_{n\to\infty} 2\frac{\sin^2\left(\frac{1}{2}\sqrt{\frac{x}{n}}\right)n}{\left(\frac{1}{2}\sqrt{\frac{x}{n}}\right)^2}\times\left(\frac{1}{2}\sqrt{\frac{x}{n}}\right)^2}$$

$$= e^{-2\lim\limits_{n\to\infty}\frac{\left(\frac{1}{2}\sqrt{\frac{x}{n}}\right)^2}{\frac{1}{n}}} = e^{-2\lim\limits_{n\to\infty}\frac{1}{4}\frac{x/n}{1/n}} = e^{-\frac{x}{2}}$$

$y = f(x) = e^{-x/2}$, $x \ge 0$, range $= (0, 1]$

$\Rightarrow \quad g(x) = \lim\limits_{n\to\infty}(1 - x + x\sqrt[n]{e})^n$

$$= e^{\lim\limits_{n\to\infty} x\frac{(e^{1/n} - 1)}{1/n}} = e^x, \forall\ x \in R$$

$\Rightarrow \quad h(x) = \tan^{-1}(g^{-1}(f^{-1}(x)))$

Let $\quad y^2 = f(x) \Rightarrow y = e^{-x/2}$

$$-\frac{x}{2} = \ln y \Rightarrow x = 2\ln\frac{1}{y}$$

$\Rightarrow f^{-1}(x) = 2\ln\dfrac{1}{x}$, for $0 < x \le 1$

$y = g(x) = e^x \Rightarrow x = \ln y$, $g^{-1}(x) = \ln x$

$\therefore\ g^{-1}\left(2\ln\dfrac{1}{x}\right) = \ln\left(2\ln\left(\dfrac{1}{x}\right)\right)$, for $0 < x < 1$

$\therefore \quad h(x) = \tan^{-1}\left(\ln\left(\ln\dfrac{1}{x^2}\right)\right)$, for $0 < x < 1$

59. $\lim\limits_{x\to 0^+} \dfrac{\ln(f(x))}{\ln(g(x))} = \lim\limits_{x\to 0} \dfrac{-x/2}{x} = -\dfrac{1}{2}$

60. Domain of $h(x)$ is $(0, 1)$.

61. $h(x) = \tan^{-1}(\ln(\ln 1/x^2))$, for $0 < x < 1$

$$1 < \frac{1}{x^2} < \infty \quad \Rightarrow \quad 0 < \ln\frac{1}{x^2} < \infty$$

$\Rightarrow \quad -\infty < \ln(\ln(1/x^2)) < \infty$

$\therefore$ Range of $h(x)$ is $(-\pi/2, \pi/2)$.

Sol. (Q. Nos. 62 to 63)

Here, $K_1 = (n+1)(n+2)(n+3)\ldots(2n)$

$K_2 = n\times n\times n\times \ldots \times n$ times $= n^n$ and $K_3 = (n)!$

62. $L = \lim\limits_{n\to\infty}\left(\dfrac{K_1}{K_2}\right)^{1/n} = \lim\limits_{n\to\infty}\left(\dfrac{(n+1)(n+2)\ldots(2n)}{n\times n\times\ldots\times n \text{ times}}\right)^{1/n}$

Taking log on both sides, we get

$$\log L = \lim_{n\to\infty}\frac{1}{n}\cdot\sum_{r=1}^{n}\log\left(1 + \frac{r}{n}\right) = \int_0^1 \log(1+x)\,dx = \log(4/e)$$

$\therefore \quad L = 4/e$

63. Also, $M = \lim\limits_{n\to\infty}\dfrac{K_1}{K_2 + K_3} = \lim\limits_{n\to\infty}\dfrac{(n+1)(n+2)\ldots(n+n)}{n^n + n!}$

$$= \lim_{n\to\infty}\frac{n^n((1+1/n)(1+2/n)\ldots(1+n/n))}{n^n\left[1 + \frac{1\cdot 2\cdot 3\ldots n}{n\cdot n\ldots n}\right]} = 1$$

Sol. (Q. Nos. 64 to 65)

Here, $f(1) = 0\cdot 8$ and $g(1) = 0\cdot 6 \Rightarrow (f(1))^2 + (g(1))^2 = 1$

$\therefore$ Let $f(1) = \cos\alpha$ and $g(1) = \sin\alpha$, then

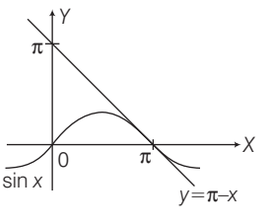
$$f(2) = f(1)\cos(g(1)) - g(1)\sin(g(1))$$

$$= \cos\alpha\cdot\cos(g(1)) - \sin\alpha\cdot\sin(g(1)) = \cos(\alpha + g(1))$$

Similarly, $f(n+1) = \cos(\alpha + g(1) + g(2) + \ldots + g(n))$ ...(i)

Here, for all $x \in (0, \pi)$

$$\sin x < \pi - x \Rightarrow x + \sin x < \pi$$



$\Rightarrow \quad \alpha + g(1) + g(2) + \ldots + g(n) < \pi$

$\therefore$ As $n \to \infty$, $\alpha + g(1) + g(2) + \ldots + g(n) \to \pi$ ...(ii)

64. $\lim\limits_{n\to\infty} f(n) = \lim\limits_{n\to\infty}\cos(\alpha + g(1) + g(2) + \ldots + g(n))$

$$= \cos\pi = -1$$

65. Similarly, $\quad g(n) = \sin(\alpha + g(1) + g(2) + \ldots + g(n-1))$

Again, $\quad g(2) = \sin(\alpha + g(1))$

As, $\quad \lim\limits_{n\to\infty} g(n) = \lim\limits_{n\to\infty}\sin(\alpha + g(1) + g(2) + \ldots + g(n-1))$

$$= \sin(\pi) = 0 \qquad \text{[using Eq. (ii)]}$$

Sol. (Q. Nos. 66 to 68)

Here, $\quad T(x) = \dfrac{1}{2}\left(\sin\left(\dfrac{x}{2}\right)\cdot\tan\left(\dfrac{x}{2}\right)\cdot\cos\left(\dfrac{\pi}{2} - \dfrac{x}{2}\right)\right)\times 2$

$$= \sin^2\left(\frac{x}{2}\right)\cdot\tan\left(\frac{x}{2}\right)$$

$$\therefore \quad S(x)=\frac{1}{2}(1)^2\cdot(x-\sin x)$$

66. $\lim\limits_{x\to 0}\dfrac{T(x)}{x^3}=\lim\limits_{x\to 0}\dfrac{\sin^2\frac{x}{2}\cdot\tan\frac{x}{2}}{x^3}=\dfrac{1}{8}$

67. $\lim\limits_{x\to 0}\dfrac{S(x)}{x}=\lim\limits_{x\to 0}\dfrac{1}{2}\dfrac{(x-\sin x)}{x}=0$

68. $\lim\limits_{x\to 0}\dfrac{T(x)}{S(x)}=\lim\limits_{x\to 0}\dfrac{\sin^2(x/2)\cdot\tan(x/2)}{\frac{1}{2}(x-\sin x)}=\lim\limits_{x\to 0}\dfrac{1}{4}\cdot\dfrac{x^3}{(x-\sin x)}=\dfrac{3}{2}$

69. (A) $\lim\limits_{x\to\frac{\pi^+}{2}}\tan^{-1}(\tan x)=-\dfrac{\pi}{2}$, as $\lim\limits_{x\to\frac{\pi^+}{2}}(\tan x)=-\infty$

(B) $\sum\limits_{r=1}^{n}\dfrac{1}{2^r}=\left(1-\dfrac{1}{2^n}\right)<1$, for all $n\in N$. Thus, $\lim\limits_{n\to 0}\left[\sum\limits_{r=1}^{n}\dfrac{1}{2^r}\right]=0$

(C) As $x\to\infty$, $\dfrac{x}{x+1}\to 1^-$

and hence $\lim\limits_{x\to\infty}\sec^{-1}\left(\dfrac{x}{x+1}\right)$ doesn't exist.

(D) Put $\dfrac{\pi}{2}-x=\theta$, then the given limit is

$$\lim_{\theta\to 0}\frac{\sin\theta}{(1-\cos\theta)^{2/3}}=\lim_{\theta\to 0}\frac{\sin\theta}{2^{2/3}\cdot\sin^{4/3}\frac{\theta}{2}}$$

$$=\lim_{\theta\to 0}\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2^{2/3}\cdot\sin^{4/3}\frac{\theta}{2}}=2^{1/2}\lim_{\theta\to 0}\frac{\cos\frac{\theta}{2}}{\sin^{1/3}\frac{\theta}{2}},$$

which doesn't exist as for $\lim\limits_{\theta\to 0^+}$ limit is ∞ and for $\lim\limits_{\theta\to 0^+}$ limit is $-\infty$.

70. Here, $y=\sec\theta$ and $x=\tan\theta$

(A) $\lim\limits_{\theta\to\frac{\pi}{2}}(\sqrt{y}-\sqrt{x})=\lim\limits_{\theta\to\frac{\pi}{2}}(\sqrt{\sec\theta}-\sqrt{\tan\theta})$

$$=\lim_{\theta\to\frac{\pi}{2}}\frac{1-\sqrt{\sin\theta}}{\sqrt{\cos\theta}}$$

$$=\lim_{\theta\to\frac{\pi}{2}}\left(\frac{\cos\theta}{\sin\theta}\right)^{3/2}=0 \qquad \text{[using L'Hospital's rule]}$$

(B) $\lim\limits_{\theta\to\frac{\pi}{2}}(y-x)=\lim\limits_{\theta\to\frac{\pi}{2}}(\sec\theta-\tan\theta)=\lim\limits_{\theta\to\frac{\pi}{2}}\dfrac{1-\sin\theta}{\cos\theta}$

$$=\lim_{\theta\to\frac{\pi}{2}}\frac{\cos\theta}{\sin\theta}=0 \qquad \text{[using L'Hospital's rule]}$$

(C) $\lim\limits_{\theta\to\frac{\pi}{2}}(y^2-x^2)=\lim\limits_{\theta\to\frac{\pi}{2}}(\sec^2\theta-\tan^2\theta)=1$

(D) $\lim\limits_{\theta\to\frac{\pi}{2}}(y^3-x^3)=\lim\limits_{\theta\to\frac{\pi}{2}}(\sec^3\theta-\tan^3\theta)$

$$=\lim_{\theta\to\frac{\pi}{2}}\frac{1-\sin^3\theta}{\cos^3\theta}$$

$$=\lim_{\theta\to\frac{\pi}{2}}\frac{-3\sin^2\theta\cos\theta}{-3\cos^2\theta\sin\theta}=\frac{\sin\theta}{\cos\theta} \qquad \text{[using L' Hospital's rule]}$$

$$=\lim_{\theta\to\frac{\pi}{2}}\tan\theta=\infty$$

71. (A) $l=\lim\limits_{x\to 0}\dfrac{e^{x^2}-1+x-e^x+1}{2\frac{\sin^2 x}{x^2}\cdot x^2}$

$$=\frac{1}{2}\left[\lim_{x\to 0}\frac{e^{x^2}-1}{x^2}+\lim_{x\to 0}\frac{x-e^x+1}{x^2}\right]$$

$$=\frac{1}{2}\left[1-\lim_{x\to 0}\frac{e^x-x-1}{x^2}\right]$$

$$=\frac{1}{2}\left[1-\frac{1}{2}\right]=\frac{1}{4}\Rightarrow\frac{1}{l}=4$$

(B) $l=\lim\limits_{x\to 0}\left(\dfrac{3+x}{3-x}\right)^{1/x}=e^{\lim\limits_{x\to 0}\frac{1}{x}\left(\frac{3+x}{3-x}-1\right)}$

$$=e^{\lim\limits_{x\to 0}\frac{2x}{x(3-x)}}=e^{2/3}\Rightarrow 2+3=5$$

(C) $\lim\limits_{x\to 0}\dfrac{(\tan^3 x-x^3)-(\tan x^3-x^3)}{x^5}$

$$=\lim_{x\to 0}\frac{\tan^3 x-x^3}{x^5}-\underbrace{\lim_{x\to 0}\frac{\tan x^3-x^3}{x^5}}_{\text{zero (by expansion)}}$$

$$=\lim_{x\to 0}\frac{(\tan x-x)}{x^3}\cdot\frac{(\tan^2 x+x\tan x+x^2)}{x^2}=\frac{1}{3}\times 3=1$$

(D) Rationalising gives

$$\lim_{x\to 0}\frac{(x+2\sin x)\left[\sqrt{(x^2+2\sin x+1)}+\sqrt{\sin^2 x-x+1}\right]}{(x^2+2\sin x+1)-(\sin^2 x-x+1)}$$

$$=2\cdot\lim_{x\to 0}\frac{x+\sin 2x}{x^2-\sin^2 x+2\sin x+x}$$

$$=2\cdot\lim_{x\to 0}\frac{1+\frac{\sin 2x}{x}}{x-\frac{\sin^2 x}{x}+2+1}=2\left(\frac{1+2}{3}\right)=2$$

72. (A) $l=\lim\limits_{n\to\infty}\cos^2(\pi(\sqrt[3]{n^3+n^2+2n}-n))$

Consider $\lim\limits_{n\to\infty}[(n^3+n^2+2n)^{1/3}-n]$

$$=\lim_{n\to\infty}\left[n\left(1+\left(\frac{1}{n}+\frac{2}{n^2}\right)\right)^{1/3}-n\right]$$

$$=\lim_{n\to\infty}n\left[\left\{1+\left(\frac{1}{n}+\frac{2}{n^2}\right)\right\}^{1/3}-1\right]$$

$$= \lim_{n\to\infty} n\left[1+\frac{1}{3}\left(\frac{1}{n}+\frac{2}{n^2}\right)+\ldots-1\right]$$

$$= \frac{1}{3} + \text{terms containing } \frac{1}{n}, \frac{1}{n^2}, \frac{1}{n^3}, \ldots = \frac{1}{3}$$

$$\therefore \quad l = \cos^2\left(\frac{\pi}{3}\right) = \frac{1}{4}$$

(B) $l = \lim_{n\to\infty} n\sin\left(2\pi\sqrt{1+n^2} - 2n\pi\right)$

$$= \lim_{n\to\infty} n\sin\left[\frac{2\pi\left(\sqrt{1+n^2}-n\right)}{\left(\sqrt{1+n^2}+n\right)}\left(\sqrt{1+n^2}+n\right)\right]$$

$$= \lim_{n\to\infty}\left[\frac{n\sin\left(\frac{2\pi}{\sqrt{1+n^2}+n}\right)}{\left(\frac{2\pi}{\sqrt{1+n^2}+n}\right)}\left(\frac{2\pi}{\sqrt{1+n^2}+n}\right)\right]$$

$$= \lim_{n\to\infty} \frac{2n\pi}{n\left(\sqrt{1+\frac{1}{n^2}}+1\right)} = \frac{2\pi}{2} = \pi$$

(C) $l = \lim_{n\to\infty} (-1)^n (-1)^{n-1} \sin\left(n\pi - \pi\sqrt{n^2+\frac{n}{2}+1}\right)$

$$= \lim_{n\to\infty} (-1)^{2n-1}\sin\pi \left[\frac{\left(n-\sqrt{n^2+\frac{n}{2}+1}\right)\left(n+\sqrt{n^2+\frac{n}{2}+1}\right)}{n+\sqrt{n^2+\frac{n}{2}+1}}\right]$$

$$= \lim_{n\to\infty} (-1)^{2n-1}\sin\pi\left(\frac{n^2-n^2-\frac{n}{2}-1}{n\left(1+\sqrt{1+\frac{1}{2n}+\frac{1}{n^2}}\right)}\right)$$

$$= \lim_{n\to\infty} (-1)^{2n}\sin\pi\left(\frac{\frac{1}{2}+\frac{1}{n}}{1+\sqrt{1+\frac{1}{2n}+\frac{1}{n^2}}}\right)$$

$$= (1)\ \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

Also, as $n\to\infty, \sin\dfrac{(n+1)\pi}{4n} \to \dfrac{1}{\sqrt{2}}$

$\therefore$ Final answer is $\dfrac{1}{2}$.

(D) $l = e^{\lim\limits_{x\to\infty} x\left(\frac{x+a}{x-a}-1\right)} = e^{\lim\limits_{x\to\infty} x\left(\frac{2a}{x-a}\right)}$

$$= e^{\lim\limits_{x\to\infty} x\left(\frac{2a}{1-(a/x)}\right)} = e^{2a}$$

$$\therefore \quad e^{2a} = e \Rightarrow a = 1/2$$

73. Here, $L = \lim_{x\to\infty} x\log x + 2x\cdot\log\sin\left(\frac{1}{\sqrt{x}}\right)$,

$$= \lim_{\theta\to 0} \frac{1}{\theta^2}\cdot\log\left(\frac{1}{\theta^2}\right) + \frac{2}{\theta^2}\cdot\log(\sin\theta) \qquad [\text{put } x = \frac{1}{\theta^2}]$$

$$= \lim_{\theta\to 0} \frac{2\log(\sin\theta) - 2\log\theta}{\theta^2}$$

$$= \lim_{\theta\to 0} \frac{2\left[\cot\theta - \frac{1}{\theta}\right]}{2\theta} \qquad [\text{using L' Hospital's rule}]$$

$$= \lim_{\theta\to 0} \frac{2\left[\theta\cot\theta - 1\right]}{2\theta^2} = \frac{\theta-\tan\theta}{\theta^2\tan\theta}$$

$$= \lim_{\theta\to 0} \frac{\theta-\tan\theta}{\theta^3}\cdot\frac{\theta}{\tan\theta} = \lim_{\theta\to 0}\frac{\theta-\tan\theta}{\theta^3}$$

$$L = \lim_{\theta\to 0}\frac{\theta-(\theta+\theta^3/3+\cdots)}{\theta^3} = -1/3$$

$$\therefore \quad \frac{-2}{L} = 6$$

74. We have, $\left(1+\frac{1}{n}\right)^{n+x_n} = e$ ...(i)

On taking log both sides of Eq. (i), we get

$$(n+x_n)\log_e\left(1+\frac{1}{n}\right) = 1$$

$$\Rightarrow n + x_n = \frac{1}{\log\left(1+\frac{1}{n}\right)} \Rightarrow x_n = \frac{1}{\log\left(1+\frac{1}{n}\right)} - n$$

Let $\dfrac{n+1}{n} = \mu \Rightarrow n\mu = n+1$

$$\Rightarrow \qquad n = \frac{1}{\mu-1}$$

$$\therefore \quad \lim_{n\to\infty} x_n = \lim_{\mu\to 1}\left(\frac{1}{\log\mu} - \frac{1}{\mu-1}\right)$$

$$= \lim_{\mu\to 1}\frac{(\mu-1)-\log\mu}{(\mu-1)\log\mu} \qquad \left[\frac{0}{0}\text{form}\right]$$

$$= \lim_{\mu\to 1}\frac{1-\frac{1}{\mu}}{(\mu-1)\cdot\frac{1}{\mu}+\log\mu} \qquad [\text{using L' Hospital's rule}]$$

$$= \lim_{\mu\to 1}\frac{1/\mu^2}{1/\mu^2+1/\mu} = \frac{1}{2}$$

$\therefore 2\lim\limits_{n\to\infty} x_n = 1$

75. Let $\left[\sqrt{n}+\frac{1}{2}\right] = K \Rightarrow K \le \sqrt{n}+\frac{1}{2} < (K+1)$

$$\Rightarrow (K-1/2)^2 \le n < (K+1/2)^2$$

$$\Rightarrow K^2 - K + 1/4 \le n < K^2 + K + 1/4,\ K\in N, n\in N$$

$$\Rightarrow K^2 - K + 1 \le n \le K^2 + K$$

So, $S = \sum_{n=1}^{\infty}\frac{2^{f(n)}+2^{-f(n)}}{2^n} = \sum_{n=1}^{\infty}\frac{2^K+2^{-K}}{2^n}$

$$= \left(\frac{2+2^{-1}}{2} + \frac{2^1+2^{-1}}{2^2}\right) + \left(\frac{2^2+2^{-2}}{2^3} + \frac{2^2+2^{-2}}{2^4} + \frac{2^2+2^{-2}}{2^5} + \frac{2^2+2^{-2}}{2^6}\right)$$

$$\text{As,}\begin{cases} K=1 & \Rightarrow \quad 1\le n\le 2 \\ K=2 & \Rightarrow \quad 3\le n\le 6 \\ K=3 & \Rightarrow \quad 7\le n\le 12 \\ \quad\vdots & \quad\vdots \end{cases}$$

$$\therefore\ S=\sum_{K=1}^{\infty}\sum_{n=K^2-K+1}^{K^2+K}\frac{2^K+2^{-K}}{2^n}=\sum_{K=1}^{\infty}(2^K+2^{-K})\cdot\sum_{n=K^2-K+1}^{K^2+K}\frac{1}{2^n}$$

$$=\sum_{K=1}^{\infty}(2^K+2^{-K})\cdot\left[\frac{1}{2^{K^2-K+1}}+\frac{1}{2^{K^2-K+2}}+\cdots+\frac{1}{2^{K^2+K}}\right]$$

$$=\sum_{K=1}^{\infty}(2^K+2^{-K})\frac{\frac{1}{2^{K^2-K+1}}\left\{1-\frac{1}{2^{2K}}\right\}}{1-1/2}$$

$$=\sum_{K=1}^{\infty}(2^K+2^{-K})\cdot(2^{-K^2+K}-2^{-K^2-K})$$

$$=\sum_{K=1}^{\infty}(2^{-K^2+2K}-2^{-K^2}+2^{-K^2}-2^{-K^2-2K})$$

$$=\sum_{K=1}^{\infty}(2^{-K^2+2K}-2^{-K^2-2K})=\sum_{K=1}^{\infty}(2^{-K(K-2)}-2^{-K(K+2)})$$

$$=(2^1-2^{-3})+(2^0-2^{-8})+(2^{-3}-2^{-15})+(2^{-8}-2^{-24})+\ldots\infty$$

$$=2+1=3$$

76. Arithmetic mean $=\dfrac{1\cdot(n-1)+2\cdot(n-2)+\ldots+(n-1)\cdot 1}{(n-1)}$

$$S_n=\frac{\sum_{r=1}^{n-1}r(n-r)}{(n-1)}=\frac{n\cdot\sum_{r=1}^{n-1}r-\sum_{r=1}^{n-1}r^2}{(n-1)}$$

$$=\frac{n\cdot\frac{n(n-1)}{2!}-\frac{n(n-1)(2n-1)}{6}}{(n-1)}\ \Rightarrow\ S_n=\frac{n^2}{2}-\frac{n(2n-1)}{6}$$

$$\therefore\quad\lim_{n\to\infty}\frac{n^2}{S_n}=\frac{n^2}{\frac{n^2}{2}-\frac{n(2n-1)}{6}}=\frac{1}{1/2-2/6}=6$$

77. Here, $k=\sum_{n=1}^{\infty}\dfrac{6^n}{(3^n-2^n)(3^{n+1}-2^{n+1})}$

$$=\sum_{n=1}^{\infty}\frac{1}{\left[1-\left(\frac{2}{3}\right)^n\right]\left[3\cdot\left(\frac{3}{2}\right)^n-2\right]}$$

$$=\sum_{n=1}^{\infty}\frac{1}{\left[1-\left(\frac{2}{3}\right)^n\right]\cdot\left(\frac{3}{2}\right)^n\cdot\left[3-2\cdot\left(\frac{2}{3}\right)^n\right]}$$

$$=\sum_{n=1}^{\infty}\frac{(2/3)^n\cdot(3-2)}{\left[1-\left(\frac{2}{3}\right)^n\right]\left[3-2\cdot\left(\frac{2}{3}\right)^n\right]}$$

$$=\sum_{n=1}^{\infty}\frac{\left[3-2\left(\frac{2}{3}\right)^n\right]-3\left[1-\left(\frac{2}{3}\right)^n\right]}{\left[1-\left(\frac{2}{3}\right)^n\right]\left[3-2\cdot\left(\frac{2}{3}\right)^n\right]}$$

$$=\lim_{n\to\infty}\sum_{n=1}^{n}\frac{1}{1-\left(\frac{2}{3}\right)^n}-\frac{3}{3-2\cdot\left(\frac{2}{3}\right)^n}$$

$$=\lim_{n\to\infty}\left[\frac{1}{1-(2/3)}-\frac{3}{3-2\cdot(2/3)}\right]+\left[\frac{1}{1-(2/3)^2}-\frac{3}{3-2\cdot(2/3)^2}\right]+\ldots$$

$$+\left[\frac{1}{1-(2/3)^n}-\frac{3}{3-2\cdot(2/3)^n}\right]$$

$$=\lim_{n\to\infty}\left[\frac{1}{1-2/3}-\frac{1}{1-(2/3)^{n+1}}\right]=3-1=2$$

78. We have, $\lim\limits_{x\to\frac{\pi}{2}}\sqrt{\dfrac{\tan x-\sin\{\tan^{-1}(\tan x)\}}{\tan x+\cos^2(\tan x)}}$

Now, RHL at $x=\dfrac{\pi}{2}$,

$$=\lim_{x\to\frac{\pi^+}{2}}\sqrt{\frac{\tan x-\sin\{\tan^{-1}(\tan x)\}}{\tan x+\cos^2(\tan x)}}$$

$$=\lim_{x\to\frac{\pi^+}{2}}\sqrt{\frac{\tan x-\sin(x-\pi)}{\tan x+\cos^2(\tan x)}}$$

$$\left[\because\ \tan^{-1}(\tan x)=x-\pi,\ \text{ when } x>\frac{\pi}{2}\right]$$

$$=\lim_{x\to\frac{\pi^+}{2}}\sqrt{\frac{1+\frac{\sin x}{\tan x}}{1+\frac{\cos^2(\tan x)}{\tan x}}}=\sqrt{\frac{1+0}{1+0}}=1$$

Again, LHL at $x=\dfrac{\pi}{2}$,

$$=\lim_{x\to\frac{\pi^-}{2}}\sqrt{\frac{\tan x-\sin\{\tan^{-1}(\tan x)\}}{\tan x+\cos^2(\tan x)}}$$

$$=\lim_{x\to\frac{\pi^-}{2}}\sqrt{\frac{\tan x-\sin(x)}{\tan x+\cos^2(\tan x)}}\quad\left[\because\tan^{-1}(\tan x)=x,\ \text{when}\ x<\frac{\pi}{2}\right]$$

$$=\lim_{x\to\frac{\pi^-}{2}}\sqrt{\frac{1-\frac{\sin x}{\tan x}}{1+\frac{\cos^2(\tan x)}{\tan x}}}=\sqrt{\frac{1+0}{1+0}}=1$$

$$\therefore\quad\lim_{x\to\frac{\pi}{2}}\sqrt{\frac{\tan x-\sin\{\tan^{-1}(\tan x)\}}{\tan x+\cos^2(\tan x)}}=1$$

79. We have, $A(t)=\int_0^t\sin x^2\cdot dx\ ;\ B(t)=\dfrac{t\sin t^2}{2}\cdot$

$$\therefore\quad\lim_{t\to 0}\frac{A(t)}{B(t)}=\lim_{t\to 0}\frac{2\int_0^t\sin x^2\cdot dx}{t\sin t^2}=\lim_{t\to 0}\frac{2\int_0^t\sin x^2dx}{t^3\cdot\frac{\sin^2 t}{t^2}}$$

$$=\lim_{t\to 0}\frac{2\int_0^t\sin x^2\cdot dx}{t^3}=\lim_{t\to 0}\frac{2\cdot\sin(t^2)}{3t^2}$$

[applying Newton-Leibnitz's rule followed by L' Hospital's rule]

$\therefore \quad \lim_{t\to 0}\frac{A(t)}{B(t)}=\frac{2}{3}$

Then, $m+n=2+3=5$

80. Here, $AB:\left[\int_0^{2t}\left(\frac{\sin x}{x}+1\right)dx\right]x+y=3t$

and $AC: 2tx+y=0$

On solving, we get

$AB:\left[\int_0^{2t}\left(\frac{\sin x}{x}+1\right)dx-2t\right]x=3t \qquad [\because y=-2tx]$

$\therefore \quad x_A=\frac{3t}{\int_0^{2t}\left(\frac{\sin x}{x}+1\right)dx-2t}$

$=\frac{3}{\left(\frac{\sin 2t}{2t}+1\right)2-2}=\frac{3}{2\cdot\frac{\sin 2t}{2t}}$

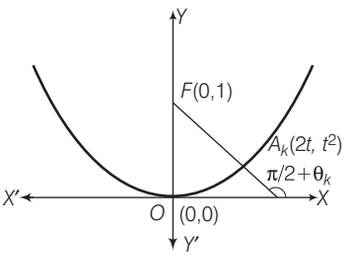
Now, $\lim_{t\to 0}\frac{3}{2\cdot\left(\frac{\sin 2t}{2t}\right)}=\frac{3}{2}$ [using L'Hospital's rule]

$\therefore \quad \frac{p}{q}=\frac{3}{2} \quad \Rightarrow \quad p+q=5$

81. Let $A_k=(2t, t^2)$

$\therefore$ Slope of $FA_k=\frac{t^2-1}{2t-0}$

$=\tan\left(\frac{\pi}{2}+\theta_k\right)$



$\tan(\theta_k)=\frac{2t}{1-t^2}=\tan(2\phi)$, where $t=\tan\phi$

$\therefore \quad \phi=\frac{\theta_k}{2}=\frac{k\pi}{4n}$, where $\tan\phi=t$.

Also, $FA_K=\sqrt{(t^2-1)^2+(2t)^2}$

$=t^2+1=1+\tan^2\phi$

$=\sec^2\left(\frac{k\pi}{4n}\right)$

$\therefore \lim_{n\to\infty}\frac{1}{n}\cdot\sum_{k=1}^{n}FA_k=\lim_{n\to\infty}\frac{1}{n}\cdot\sum_{k=1}^{n}\sec^2\left(\frac{k\pi}{4n}\right)$

$=\int_0^1\sec^2\left(\frac{\pi x}{4}\right)dx$

$=\frac{4}{\pi}\cdot\left[\tan\left(\frac{\pi x}{4}\right)\right]_0^1=\frac{4}{\pi}$

Hence, $m=4$.

82. Here, $L=\lim_{h\to 0}\frac{\cos(\tan^{-1}(\tan(\pi/2+h)))}{h}$

$=\lim_{h\to 0}\frac{\cos(\tan^{-1}(-\cot h))}{h}=\lim_{h\to 0}\frac{\cos\left(-\frac{\pi}{2}+h\right)}{h}=\frac{\sin h}{h}=1$

$\therefore \quad \cos(2\pi L)=\cos(2\pi)=1$

83. Here, $k=\lim_{n\to\infty}\prod_{r=2}^{n}\frac{(r-1)(r^2+r+1)}{(r+1)(r^2-r+1)}$

$=\lim_{n\to\infty}\frac{2}{n(n+1)}\times\frac{n^2+n+1}{3}=\frac{2}{3}.$

So, $\operatorname{cosec}\theta=\frac{2}{3} \Rightarrow$ Number of solution is zero.

84. Here, $\lim_{x\to\infty}\left(\frac{x+c}{x-c}\right)^x=4 \Rightarrow e^{\lim_{x\to\infty}\left(\frac{2c}{x-c}\right)}=4$

$\Rightarrow \quad e^{2c}=4 \Rightarrow e^c=2$ [only positive value]

$\therefore \quad \frac{e^c}{2}=1$

85. $\lim_{x\to-\infty}\frac{(3x^4+2x^2)\cdot\sin\left(\frac{1}{x}\right)-x^3+5}{-x^3+x^2-x+1}=k$

$\Rightarrow \quad \lim_{x\to-\infty}\frac{\left(3+\frac{2}{x^2}\right)\cdot\frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}-1+\frac{5}{x^3}}{-1+\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}}$

$\Rightarrow \quad \frac{(3-1)}{-1}=k \Rightarrow \left|\frac{k}{2}\right|=1$

86. $f(x)=\lim_{t\to 0}\frac{2x}{\pi}\tan^{-1}\left(\frac{x}{t^2}\right)$

***Case* I** When $x>0$,

$f(x)=\lim_{t\to 0}\frac{2x}{\pi}\cdot\tan^{-1}\left(\frac{x}{t^2}\right)=\frac{2x}{\pi}\times\frac{\pi}{2}=x$

***Case* II** When $x<0$,

$f(x)=\lim_{t\to 0}\frac{2x}{\pi}\cdot\tan^{-1}\left(\frac{x}{t^2}\right)=\frac{2x}{\pi}\cdot\left(\frac{-\pi}{2}\right)=-x$

***Case* III** When $x=0 \Rightarrow f(x)=0$

$\therefore \quad f(x)=(x) \Rightarrow f(1)=1$

87. Let $f(x)=\left(\frac{1}{\sqrt{n^2}}+\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\ldots+\frac{1}{\sqrt{n^2+2n}}\right)$

As, $\frac{1}{\sqrt{n^2+2n}}<\frac{1}{\sqrt{n^2}}=\frac{1}{\sqrt{n^2}}$

$\frac{1}{\sqrt{n^2+2n}}<\frac{1}{\sqrt{n^2+1}}<\frac{1}{\sqrt{n^2}}$

$\vdots$

On adding, we get

$\frac{2n+1}{\sqrt{n^2+2n}}<\frac{1}{\sqrt{n^2}}+\frac{1}{\sqrt{n^2+1}}+\ldots+\frac{1}{\sqrt{n^2+2n}}<\frac{2n+1}{\sqrt{n^2}}$

On applying, limit $n\to\infty$

$$2<\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2}}+\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\ldots+\frac{1}{\sqrt{n^2+2n}}\right)<2$$

$$\therefore\quad \lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2}}+\frac{1}{\sqrt{n^2+1}}+\ldots+\frac{1}{\sqrt{n^2+2n}}\right)=2$$

88. $l=\lim\limits_{x\to1^+}2^{-2^{\frac{1}{1-x}}}=\lim\limits_{h\to0}2^{-2^{-\frac{1}{h}}}=2^0=1$

and $m=\lim\limits_{h\to0}\dfrac{(1+h)\cdot\sin h}{h}=1$

$\therefore\quad l+m=2$

89. As, $\dfrac{\sin x\tan x}{x^2}=\dfrac{4\cdot\tan^4\frac{x}{2}}{x^2\cdot(1-\tan^4 x/2)}$

Hence, when $x\to0$, $\dfrac{\sin x\cdot\tan x}{x^2}\to1$. But $\dfrac{\sin x\cdot\tan x}{x^2}>1$

$\therefore\quad \lim\limits_{x\to0}\left[\dfrac{\sin x\tan x}{x^2}\right]=1$

90. Let $S_n=\dfrac{k}{2}\sum\limits_{r=1}^{n}\dfrac{2r}{1\times3\times5\times\ldots\times(2r-1)\times(2r+1)}$

$$=\frac{k}{2}\sum_{r=1}^{n}\left[\frac{1}{1\times3\times5\times\ldots\times(2r-1)}-\frac{1}{1\times3\times5\times\ldots\times(2r+1)}\right]$$

$$=\frac{k}{2}\left[1-\frac{1}{1\times3\times5\times\ldots\times(2n+1)}\right],\text{ as } n\to\infty.$$

$\lim\limits_{n\to\infty}S_n=\dfrac{k}{2}=1\ \Rightarrow\ k^2=4$

91. Here, $\sin^4 x=\sin^2 x-\dfrac{1}{4}\cdot\sin^2 2x$

$$\therefore\ S_n=\left(\sin^2 x-\frac{1}{4}\cdot\sin^2 2x\right)+\left(\frac{1}{4}\cdot\sin^2 2x-\frac{1}{4^2}\cdot\sin^2 2^2x\right)$$

$$+\ldots+\left(\frac{1}{4^n}\cdot\sin^2 2^n x-\frac{1}{4^{n+1}}\cdot\sin^2 2^{n+1}x\right)$$

$$=\sin^2 x-\frac{1}{4^{n+1}}\cdot\sin^2 2^{n+1}x\ \Rightarrow\ f(x)=\sin^2 x,\ g(x)=\cos^2 x$$

$$\therefore\quad [\sqrt{f(x)}+\sqrt{g(x)}]^4=4$$

92. Here, $f(3)=3f(1)=1$

$f(4)=f(2+1+1)=f(2)+f(1)+f(1)=2+1+1=4$ and so on.

In general, $f(x)=x$, for $x\in N$

$$\therefore\quad \lim_{n\to\infty}\sum_{r=1}^{n}\frac{(4r)f(3r)}{n^3}=\frac{12n(n+1)(2n+1)}{6n^3}=4$$

93. As, $0\le\dfrac{x-1}{3}<1\ \Rightarrow 1\le x\le4$

Then, $\lim\limits_{n\to\infty}\dfrac{n}{n\cdot\left(\frac{x-1}{3}\right)^n+n\cdot3-1}=\dfrac{1}{3}$, when $0\le\dfrac{x-1}{3}<1$

$\Rightarrow\ x=1,2,3$

$\therefore$ Number of integral values = 3.

94. As, $x^5-y^5=(x-y)(x^4+x^3y+x^2y^2+xy^3+y^4)$

and the coefficient of x^n in

$$(x-1)(x-2)(x+3)(x+10)(x+15)-x^5$$

$$=-1-2+3+10+15=25$$

$$\therefore\lim_{x\to\infty}((x-1)(x-2)(x+3)(x+10)(x+15)^{1/3}-x)$$

$$=\frac{25}{5}=5$$

95. Here, $k=\lim\limits_{x\to\infty}(1+[f(x)]+x^2-1)\{f(x)\}$

$$=e^{\lim\limits_{x\to\infty}\frac{\left[\frac{\tan x}{x}\right]+x^2-1}{\left\{\frac{\tan x}{x}\right\}}}=e^{\lim\limits_{x\to\infty}\frac{1+x^2-1}{\frac{\tan x}{x}-\left[\frac{\tan x}{x}\right]}}$$

$$=e^{\lim\limits_{x\to\infty}\frac{x^2}{\left(1+\frac{x^2}{3}+\frac{x^4}{15}-1\right)}}=e^3$$

$\therefore\quad [k/e]=[e^2]=7$

96. Here, $(\sqrt{3}+1)^{2n}=I+f$ ...(i)

Let $f=(\sqrt{3}-1)^{2n}$ ...(ii)

$\therefore\quad I+f+F=$ even integer, $0<f,\ f<1$

$\Rightarrow\quad f+F=1\ \Rightarrow\ 0<f+F<2$

$\Rightarrow\quad f+F=1$

$\Rightarrow\quad \{(\sqrt{3}+1)\}^{2n}=f=1-f$

$\therefore\quad \lim\limits_{n\to\infty}f=\lim\limits_{n\to\infty}1-f=1-\lim\limits_{n\to\infty}(\sqrt{3}-1)^{2n}=1-0=1$

97. Here, $f(x)=a_0x^4+a_1x^3+a_2x^2+a_3x+a_4$.

As, $\lim\limits_{x\to0}1+\dfrac{f(x)}{x^2}=2\ \Rightarrow\ \lim\limits_{x\to0}\dfrac{f(x)}{x^2}=1$

So, $a_3=a_4=0$ and $a_2=1$

$f(x)=a_0x^4+a_1x^3+x^2$

$\Rightarrow f'(x)=4a_0x^3+3a_1x^2+2x$

As, $f(x)$ has extremum at $x=1$ and $x=2$.

$f'(1)=4a_0+3a_1+2$

and $f'(2)=32a_0+12a_1+4$

So, $a_0=\dfrac{1}{4}$ and $a_1=-1$

$\Rightarrow\quad f(2)=0$

98. Here, $|x|=\log(\{x\})$ has no solution.

Hence, $\alpha=0$

$$\therefore\lim_{x\to0}\frac{x\,e^{ax}-b\sin x}{x^3}=\lim_{x\to0}\frac{(1-b)x+ax^2+x^3\left(\frac{a^2}{2}-\frac{b}{6}\right)}{x^3}$$

$\Rightarrow\quad a=0,\ b=1$

$\Rightarrow\quad a+b=1$

99. Here, $\sin(x_{n+1}-x_n)+2^{-(n+1)}\sin x_n\cdot\sin x_{n+1}=0$

$\Rightarrow\quad \cot x_{n+1}-\cot x_n=2^{-(n+1)}$

$\Rightarrow\quad \cot x_n-\cot x_{n-1}=2^{-n}$

$\Rightarrow\quad \cot x_{n+1}-\cot x_1=\dfrac{1}{2^2}+\dfrac{1}{2^3}+\ldots+\dfrac{1}{2^{n+1}}$

$\Rightarrow\quad \cot x_{n+1}=\dfrac{1}{2}+\dfrac{1}{2^2}+\ldots\dfrac{1}{2^{n+1}}$

$$\Rightarrow \quad \lim_{n\to\infty} \cot x_{n+1} = 1$$

$$\therefore \quad \lim_{n\to\infty} x_{n+1} = \frac{\pi}{4}$$

$$l = \frac{\pi}{4} \quad \Rightarrow 4l = \pi$$

$$\therefore \quad [4l] = 3$$

100. Here, $\lim_{x\to 0} \frac{x^2 \sin(\beta x)}{\alpha x - \sin x} = 1$

$$\Rightarrow \lim_{x\to 0} \frac{x^2\left(\beta x - \frac{(\beta x)^3}{3!} + \frac{(\beta x)^5}{5!} - \ldots\right)}{\alpha x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)} = 1$$

$$\Rightarrow \lim_{x\to 0} \frac{x^3\left(\beta - \frac{\beta^3 x^2}{3!} + \frac{\beta^5 x^4}{5!} - \ldots\right)}{(\alpha - 1)x + \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots} = 1$$

Limit exists only, when $\alpha - 1 = 0$

$$\Rightarrow \quad \alpha = 1 \qquad \ldots(i)$$

$$\therefore \lim_{x\to 0} \frac{x^3\left(\beta - \frac{\beta^3 x^2}{3!} + \frac{\beta^5 x^4}{5!} - \ldots\right)}{x^3\left(\frac{1}{3!} - \frac{x^2}{5!} - \ldots\right)} = 1$$

$$\Rightarrow \quad 6\beta = 1 \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$6(\alpha + \beta) = 6\,\alpha + 6\beta = 6 + 1 = 7$$

101. Given, $\lim_{\alpha\to 0}\left[\frac{e^{\cos(\alpha^n)} - e}{\alpha^m}\right] = -\frac{e}{2}$

$$\Rightarrow \lim_{\alpha\to 0} \frac{e\{e^{\cos(\alpha^n)-1} - 1\}}{\cos(\alpha^n) - 1} \cdot \frac{\cos(\alpha^n) - 1}{\alpha^m} = \frac{-e}{2}$$

$$\Rightarrow \lim_{\alpha\to 0} e\left\{\frac{e^{\cos(\alpha^n)-1} - 1}{\cos(\alpha^n) - 1}\right\} \cdot \lim_{\alpha\to 0} \frac{-2\sin^2\frac{\alpha^n}{2}}{\alpha^m} = -\frac{e}{2}$$

$$\Rightarrow \quad e \times 1 \times (-2) \lim_{\alpha\to 0} \frac{\sin^2\left(\frac{\alpha^n}{2}\right)}{\frac{\alpha^{2n}}{4}} \cdot \frac{\alpha^{2n}}{4\alpha^m} = \frac{-e}{2}$$

$$\Rightarrow \quad e \times 1 \times -2 \times 1 \times \lim_{\alpha\to 0} \frac{\alpha^{2n-m}}{4} = \frac{-e}{2}$$

For this to be exists, $2n - m = 0 \;\Rightarrow\; \frac{m}{n} = 2$

102. Here, $\lim_{x\to 1} \frac{F(x)}{G(x)} = \frac{1}{14}$

$$\Rightarrow \quad \lim_{x\to 1} \frac{F'(x)}{G'(x)} = \frac{1}{14} \qquad \text{[using L'Hospital's rule]} \ldots(i)$$

As, $F(x) = \int_{-1}^{x} f(t)\,dt$

$$\Rightarrow \quad F'(x) = f(x) \qquad \ldots(ii)$$

and $G(x) = \int_{-1}^{x} t\left|f\{f(t)\}\right| dt$

$$\Rightarrow \quad G'(x) = x\left|f\{f(x)\}\right| \qquad \ldots(iii)$$

$$\therefore \quad \lim_{x\to 1} \frac{F(x)}{G(x)} = \lim_{x\to 1} \frac{F'(x)}{G'(x)} = \lim_{x\to 1} \frac{f(x)}{x\left|f\{f(x)\}\right|}$$

$$= \frac{f(1)}{1\left|f\{f(1)\}\right|} = \frac{\frac{1}{2}}{\left|f\left(\frac{1}{2}\right)\right|} \qquad \ldots(iv)$$

Given, $\lim_{x\to 1} \frac{F(x)}{G(x)} = \frac{1}{14}$

$$\therefore \quad \frac{\frac{1}{2}}{\left|f\left(\frac{1}{2}\right)\right|} = \frac{1}{14} \;\Rightarrow\; \left|f\left(\frac{1}{2}\right)\right| = 7$$

103. Given, $\lim_{x\to 1}\left\{\frac{\sin(x-1) + a(1-x)}{(x-1) + \sin(x-1)}\right\}^{\frac{(1+\sqrt{x})(1-\sqrt{x})}{1-\sqrt{x}}} = \frac{1}{4}$

$$\lim_{x\to 1}\left\{\frac{\frac{\sin(x-1)}{(x-1)} - a}{1 + \frac{\sin(x-1)}{(x-1)}}\right\}^{1+\sqrt{x}} = \frac{1}{4}$$

$$\Rightarrow \quad \left(\frac{1-a}{2}\right)^2 = \frac{1}{4}$$

$$\Rightarrow \quad (a-1)^2 = 1$$

$$\Rightarrow \quad a = 2 \text{ or } 0$$

Hence, the maximum value of a is 2.

104. $L = \lim_{x\to 0} \frac{a - \sqrt{a^2 - x^2} - \frac{x^2}{4}}{x^4}, a > 0$

$$= \lim_{x\to 0} \frac{a - a\cdot\left[1 - \frac{1}{2}\cdot\frac{x^2}{a^2} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2}\cdot\frac{x^4}{a^4} - \ldots\right] - \frac{x^2}{4}}{x^4}$$

$$= \lim_{x\to 0} \frac{\frac{x^2}{2a} + \frac{1}{8}\cdot\frac{x^4}{a^3} + \ldots - \frac{x^2}{4}}{x^4}$$

Since, L is finite $\Rightarrow 2a = 4 \;\Rightarrow\; a = 2$

$$\therefore \quad L = \lim_{x\to 0} \frac{1}{8\cdot a^3} = \frac{1}{64}$$

105. $\lim_{x\to\frac{\pi}{2}} \frac{\cot x(1 - \sin x)}{-8\left(x - \frac{\pi}{2}\right)^3}$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\tan\left(\frac{\pi}{2} - x\right)}{8\left(\frac{\pi}{2} - x\right)} \frac{\left(1 - \cos\left(\frac{\pi}{2} - x\right)\right)}{\left(\frac{\pi}{2} - x\right)^2} = \frac{1}{8}\cdot 1\cdot\frac{1}{2} = \frac{1}{16}$$

106. Given, $p = \lim_{x \to 0^+} (1 + \tan^2 \sqrt{x})^{\frac{1}{2x}}$ [1^∞ form]

$$= e^{\lim_{x \to 0^+} \frac{\tan^2 \sqrt{x}}{2x}} = e^{\frac{1}{2} \lim_{x \to 0^+} \left(\frac{\tan \sqrt{x}}{\sqrt{x}}\right)^2} = e^{\frac{1}{2}}$$

$\therefore \log p = \log e^{\frac{1}{2}} = \frac{1}{2}$

107. Let $l = \lim_{n \to \infty} \left[\frac{(n+1) \cdot (n+2) \ldots (3n)}{n^{2n}}\right]^{\frac{1}{n}}$

$$= \lim_{n \to \infty} \left[\frac{(n+1) \cdot (n+2) \ldots (n+2n)}{n^{2n}}\right]^{\frac{1}{n}}$$

$$= \lim_{n \to \infty} \left[\left(\frac{n+1}{n}\right)\left(\frac{n+2}{n}\right) \ldots \left(\frac{n+2n}{n}\right)\right]^{\frac{1}{n}}$$

Taking log on both sides, we get

$$\log l = \lim_{n \to \infty} \frac{1}{n} \left[\log \left\{\left(1 + \frac{1}{n}\right)\left(1 + \frac{2}{n}\right) \ldots \left(1 + \frac{2n}{n}\right)\right\}\right]$$

$$\Rightarrow \log l = \lim_{n \to \infty} \frac{1}{n} \left[\log\left(1 + \frac{1}{n}\right) + \log\left(1 + \frac{2}{n}\right) + \ldots + \log\left(1 + \frac{2n}{n}\right)\right]$$

$$\Rightarrow \log l = \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{2n} \log\left(1 + \frac{r}{n}\right)$$

$$\Rightarrow \log l = \int_0^2 \log (1 + x)\, dx$$

$$\Rightarrow \log l = \left[\log (1 + x) \cdot x - \int \frac{1}{1 + x} \cdot x\, dx\right]_0^2$$

$$\Rightarrow \log l = [\log (1 + x) \cdot x]_0^2 - \int_0^2 \frac{x + 1 - 1}{1 + x}\, dx$$

$$\Rightarrow \log l = 2 \cdot \log 3 - \int_0^2 \left(1 - \frac{1}{1 + x}\right) dx$$

$$\Rightarrow \log l = 2 \cdot \log 3 - [x - \log |1 + x|]_0^2$$

$$\Rightarrow \log l = 2 \cdot \log 3 - [2 - \log 3]$$

$$\Rightarrow \log l = 3 \cdot \log 3 - 2$$

$$\Rightarrow \log l = \log 27 - 2$$

$$\therefore \quad l = e^{\log 27 - 2}$$

$$= 27 \cdot e^{-2} = \frac{27}{e^2}$$

108. $\lim_{x \to 0} \frac{\sin(\pi \cos^2 x)}{x^2} = \lim_{x \to 0} \frac{\sin \pi (1 - \sin^2 x)}{x^2}$

$$= \lim_{x \to 0} \frac{\sin(\pi - \pi \sin^2 x)}{x^2}$$

$$= \lim_{x \to 0} \frac{\sin(\pi \sin^2 x)}{x^2} \qquad [\because \sin(\pi - \theta) = \sin \theta]$$

$$= \lim_{x \to 0} \frac{\sin \pi \sin^2 x}{\pi \sin^2 x} \times (\pi)\left(\frac{\sin^2 x}{x^2}\right) = \pi \qquad \left[\because \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1\right]$$

109. We have, $\lim_{x \to 0} \frac{(1 - \cos 2x)(3 + \cos x)}{x \tan 4x} = \lim_{x \to 0} \frac{2 \sin^2 x\, (3 + \cos x)}{x \times \frac{\tan 4x}{4x} \times 4x}$

$$= \lim_{x \to 0} \frac{2 \sin^2 x}{x^2} \times \lim_{x \to 0} \frac{(3 + \cos x)}{4} \times \frac{1}{\lim_{x \to 0} \frac{\tan 4x}{4x}}$$

$$= 2 \times \frac{4}{4} \times 1 = 2 \qquad \left[\because \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \text{ and } \lim_{\theta \to 0} \frac{\tan \theta}{\theta} = 1\right]$$

110. **Plan** $\left(\frac{\infty}{\infty}\right)$ form

$$\lim_{x \to \infty} \frac{a_0 x^n + a_1 x^{n-1} + \ldots + a_n}{b_0 x^m + b_1 x^{m-1} + \ldots + b_m} = \begin{cases} 0, & \text{if } n < m \\ \frac{a_0}{b_0}, & \text{if } n = m \\ +\infty, & \text{if } n > m \text{ and } a_0 b_0 > 0 \\ -\infty, & \text{if } n < m \text{ and } a_0 b_0 < 0 \end{cases}$$

Description of Situation As to make degree of numerator equal to degree of denominator.

$$\therefore \quad \lim_{x \to \infty} \left(\frac{x^2 + x + 1}{x + 1} - ax - b\right) = 4$$

$$\Rightarrow \quad \lim_{x \to \infty} \frac{x^2 + x + 1 - ax^2 - ax - bx - b}{x + 1} = 4$$

$$\Rightarrow \quad \lim_{x \to \infty} \frac{x^2(1 - a) + x(1 - a - b) + (1 - b)}{x + 1} = 4$$

Here, we make degree of numerator = degree of denominator

$$\therefore \quad 1 - a = 0$$

$$\Rightarrow \quad a = 1$$

and $\lim_{x \to \infty} \frac{x(1 - a - b) + (1 - b)}{x + 1} = 4$

$$\Rightarrow \quad 1 - a - b = 4$$

$$\Rightarrow \quad b = -4 \qquad [\because (1 - a) = 0]$$

111. **Plan** To make the quadratic into simple form, we should eliminate radical sign.

Description of Situation As for given equation, when $a \to 0$ the equation reduces to identity in x.

i.e. $ax^2 + bx + c = 0, \forall\, x \in R$ or $a = b = c \to 0$

Thus, first we should make above equation independent from coefficients as 0.

Let $a + 1 = t^6$. Thus, when $a \to 0, t \to 1$.

$$\therefore \quad (t^2 - 1)\, x^2 + (t^3 - 1)x + (t - 1) = 0$$

$$\Rightarrow \quad (t - 1)\{(t + 1)x^2 + (t^2 + t + 1)\, x + 1\} = 0, \text{ as } t \to 1$$

$$2x^2 + 3x + 1 = 0$$

$$\Rightarrow \quad 2x^2 + 2x + x + 1 = 0$$

$$\Rightarrow \quad (2x + 1)(x + 1) = 0$$

Thus, $x = -1, -1/2$

or $\lim_{a \to 0^+} \alpha(a) = -1/2$

and $\lim_{a \to 0^+} \beta(a) = -1$

112. Here, $\lim_{x \to 0} \{1 + x \log (1 + b^2)\}^{1/x}$ [1^∞ form]

$$= e^{\lim_{x \to 0} \{x \log (1 + b^2)\} \cdot \frac{1}{x}}$$

$$= e^{\log (1 + b^2)} = (1 + b^2) \quad \ldots(i)$$

Given, $\lim_{x \to 0} \{1 + x \log (1 + b^2)\}^{1/x} = 2b \sin^2 \theta$

$$\Rightarrow \quad (1 + b^2) = 2b \sin^2 \theta$$

$$\therefore \quad \sin^2 \theta = \frac{1 + b^2}{2b} \quad \ldots(ii)$$

By AM $\geq$ GM, $\quad \frac{b + \frac{1}{b}}{2} \geq \left(b \cdot \frac{1}{b}\right)^{1/2}$

$$\Rightarrow \quad \frac{b^2 + 1}{2b} \geq 1 \quad \ldots(iii)$$

From Eqs. (ii) and (iii), we get

$$\sin^2 \theta = 1$$

$$\Rightarrow \quad \theta = \pm \frac{\pi}{2}, \text{ as } \theta \in (-\pi, \pi]$$

113. Here, $\lim_{x \to 0} (\sin x)^{1/x} + \lim_{x \to 0} \left(\frac{1}{x}\right)^{\sin x}$

$$= 0 + \lim_{x \to 0} e^{\log\left(\frac{1}{x}\right)^{\sin x}} = e^{\lim_{x \to 0} \frac{\log (1/x)}{\operatorname{cosec} x}} \qquad \left[\begin{matrix} \lim_{x \to 0} (\sin x)^{1/x} \to 0 \\ \text{as, } (\text{decimal})^\infty \to 0 \end{matrix}\right]$$

Applying L' Hospital's rule, we get

$$e^{\lim_{x \to 0} \frac{x\left(-\frac{1}{x^2}\right)}{-\operatorname{cosec} x \cot x}} = e^{\lim_{x \to 0} \frac{\sin x}{x} \tan x} = e^0 = 1$$

114. Here, $\lim_{h \to 0} \frac{f(2h + 2 + h^2) - f(2)}{f(h - h^2 + 1) - f(1)}$

[$\because f'(2) = 6$ and $f'(1) = 4$, given]

Applying L' Hospital's rule,

$$= \lim_{h \to 0} \frac{\{f'(2h + 2 + h^2)\} \cdot (2 + 2h) - 0}{\{f'(h - h^2 + 1)\} \cdot (1 - 2h) - 0}$$

$$= \frac{f'(2) \cdot 2}{f'(1) \cdot 1}$$

$$= \frac{6 \cdot 2}{4 \cdot 1} = 3 \qquad [\text{using } f'(2)=6 \text{ and } f'(1) = 4]$$

115. Given, $\lim_{x \to 0} \frac{\{(a - n)\, nx - \tan x\} \sin nx}{x^2} = 0$

$$\Rightarrow \lim_{x \to 0} \left\{(a - n)\, n - \frac{\tan x}{x}\right\} \frac{\sin nx}{nx} \times n = 0$$

$$\Rightarrow \quad \{(a - n)\, n - 1\}\, n = 0$$

$$\Rightarrow \quad (a - n)\, n = 1$$

$$\Rightarrow \quad a = n + \frac{1}{n}$$

116. $\lim_{x \to 0} \frac{(\cos x - 1)(\cos x - e^x)}{x^n}$

$$= \lim_{x \to 0} \frac{\left(-2 \sin^2 \frac{x}{2}\right)\left\{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) - \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)\right\}}{x^n}$$

$$= \lim_{x \to 0} \frac{\left(-2 \sin^2 \frac{x}{2}\right)\left(-x - \frac{2x^2}{2!} - \frac{x^3}{3!} - \ldots\right)}{4\left(\frac{x}{2}\right)^2 x^{n-2}}$$

$$= \lim_{x \to 0} \frac{\sin^2 \frac{x}{2}\left(1 + x + \frac{x^2}{3!} + \ldots\right)}{2\left(\frac{x}{2}\right)^2 x^{n-3}}$$

Above limit is finite, if $n - 3 = 0$, i.e. $n = 3$.

117. Let $y = \left[\frac{f(1 + x)}{f(1)}\right]^{1/x}$

$$\Rightarrow \log y = \frac{1}{x} [\log f(1 + x) - \log f(1)]$$

$$\Rightarrow \lim_{x \to 0} \log y = \lim_{x \to 0} \left[\frac{1}{f(1 + x)} \cdot f'(1 + x)\right]$$

[using L'Hospital's rule]

$$= \frac{f(1)}{f(1)} = \frac{6}{3}$$

$$\Rightarrow \log\left(\lim_{x \to 0} y\right) = 2 \Rightarrow \lim_{x \to 0} y = e^2$$

118. For $x \in R$, $\lim_{x \to \infty} \left(\frac{x - 3}{x + 2}\right)^x = \lim_{x \to \infty} \frac{(1 - 3/x)^x}{(1 + 2/x)^x} = \frac{e^{-3}}{e^2} = e^{-5}$

CHAPTER

06

Continuity and Differentiability

Learning Part

Session 1

- Continuous Function
- Continuity of a Function at a Point

Session 2

- Continuity in an Interval or Continuity at End Points

Session 3

- Discontinuity of a Function

Session 4

- Theorems Based on Continuity
- Continuity of Composite Function

Session 5

- Intermediate Value Theorem

Session 6

- Differentiability : Meaning of Derivative

Session 7

- Differentiability in an Interval

Practice Part

- JEE Type Examples
- Chapter Exercises

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Session 1

Continuous Function, Continuity of a Function at a Point

Introduction to Continuity

A real function is continuous at a fixed point if we can draw the graph of the function around that point without lifting the pen from the plane of the paper. In case one has to lift the pen at a point, the graph of the function is said to have a break or discontinuous at that point, say $x = a$.

Different types of situations, which may come up at $x = a$ along the graph, can be shown as below

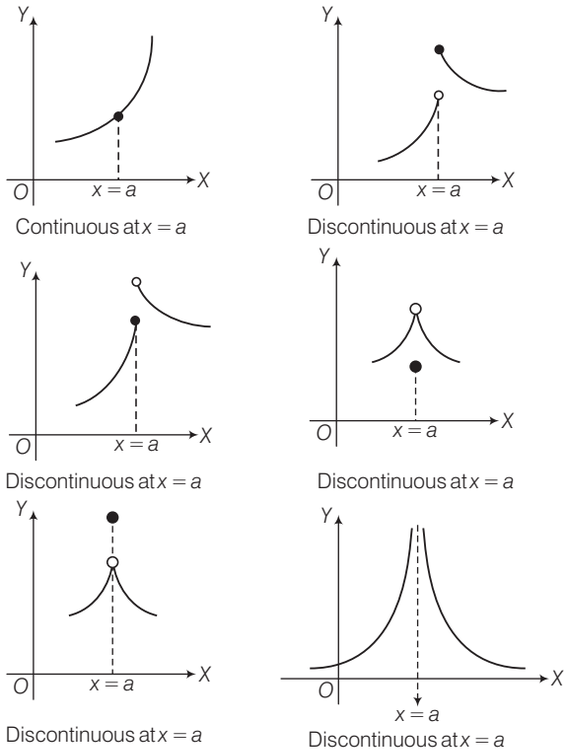


Figure 6.1

- There should be heading ''continuity of a function at point''
- A function $f(x)$ is said to be continuous at $x = a$; where $a \in$ domain of $f(x)$, if $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x) = f(a)$

  i.e. LHL = RHL = Value of a function at $x = a$

  or $\lim\limits_{x \to a} f(x) = f(a)$
- If $f(x)$ is not continuous at $x = a$, we say that $f(x)$ is discontinuous at $x = a$, $f(x)$ will be discontinuous at $x = a$ in any of the following cases

(i) $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$ exist but are not equal.

(ii) $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$ exist and are equal but not equal to $f(a)$.

(iii) $f(a)$ is not defined.

(iv) Atleast one of the limit doesn't exist.

Now, we will discuss different conditions for a function to be discontinuous in detail

(i) $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$ exists but are not equal.

Here, $\lim\limits_{x \to a^-} f(x) = l_1$

and $\lim\limits_{x \to a^+} f(x) = l_2$

$\therefore$ $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$ exist but are not equal.

Thus, $f(x)$ is discontinuous at $x = a$. It does not matter whether $f(a)$ exist or not.

Graphically This conditions can be illustrated as

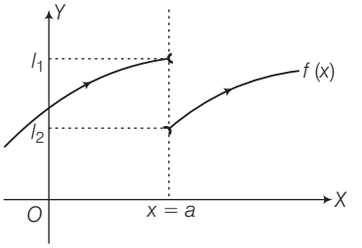


Figure 6.2

Example 1 ***If*** $f(x) = \dfrac{|x|}{x}$***. Discuss the continuity at*** $x = 0$.

Sol. Here, $f(x) = \dfrac{|x|}{x}$

$\therefore$ RHL at $x = 0$

Let $x = 0 + h$

i.e. $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{h \to 0} f(0 + h) = \lim\limits_{h \to 0} \dfrac{|0 + h|}{0 + h} = \lim\limits_{h \to 0} \dfrac{h}{h} = 1$

$\Rightarrow$ $\lim\limits_{x \to 0^+} f(x) = 1$

Again, LHL at $x = 0$

Let $x = 0 - h$

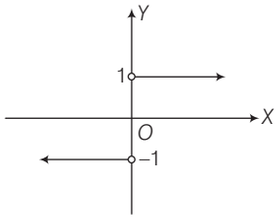
i.e. $\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h)$

$= \lim_{h \to 0} \frac{|0-h|}{0-h} = \lim_{h \to 0} \frac{h}{-h} = -1$

$\Rightarrow \lim_{x \to 0^-} f(x) = -1 \Rightarrow \lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x).$

Thus, $f(x)$ is discontinuous at $x = 0$.

Graphically Here,



$$f(x) = \frac{|x|}{x} = \begin{cases} \frac{x}{x}, & x > 0 \\ -\frac{x}{x}, & x < 0 \end{cases} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

and $f(0) = \frac{0}{0}$ [indeterminate form]

$\Rightarrow$ It is not defined.

Which shows, the graph is broken at $x = 0$.

Where, $\lim_{x \to 0^-} f(x) = -1$ and $\lim_{x \to 0^+} f(x) = -1$

Thus, $\lim_{x \to 0} f(x)$ doesn't exist and hence function is discontinuous.

Remark

Here, $f(x)$ is not defined at $x = 0$, as $f(0) = \frac{0}{0}$. [indeterminate form]

So, we could say directly that the function is discontinuous at $x = 0$

(ii) $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist and are equal but not equal to $f(a)$.

Here, $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$

$f(a)$ is also defined but $f(a) \neq L$.

So, limit of $f(x)$ exists at $x = a$. But, it is not continuous at $x = a$.

Graphically This conditions can be illustrated as

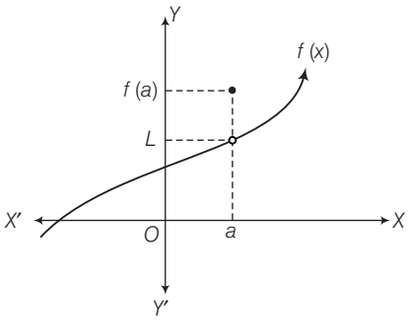


Figure 6.3

Example 2 If $f(x) = \begin{cases} 2x + 3, & \text{when } x < 0 \\ 0, & \text{when } x = 0. \\ x^2 + 3, & \text{when } x > 0 \end{cases}$

Discuss the continuity at $x = 0$.

Sol. Here, $f(x) = \begin{cases} 2x + 3, & \text{when } x < 0 \\ 0, & \text{when } x = 0 \\ x^2 + 3, & \text{when } x > 0 \end{cases}$

$\therefore$ RHL at $x = 0$, let $x = 0 + h$

i.e. $\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h)$

$= \lim_{h \to 0} \{(0+h)^2 + 3\} = 3$

$\Rightarrow \lim_{x \to 0^+} f(x) = 3$

Again, LHL at $x = 0$, let $x = 0 - h$

i.e. $\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h)$

$= \lim_{h \to 0} \{2(0-h) + 3\} = 3$

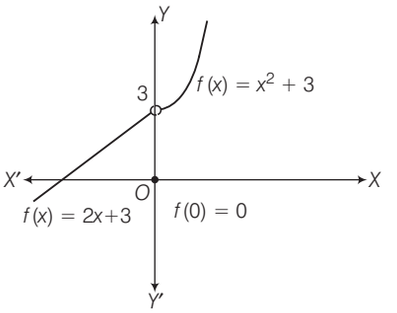
$\Rightarrow \lim_{x \to 0^-} f(x) = 3$

But $f(0) = 0$

Therefore, $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = 3 \neq f(0)$

Thus, $f(x)$ is discontinuous at $x = 0$.

Graphically



Here, $\lim_{x \to 0^-} f(x) = 3$

$\lim_{x \to 0^+} f(x) = 3$

$f(0) = 0$

Thus, $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 3 \neq f(0)$

Hence, $f(x)$ is discontinuous at $x = 0$.

(iii) $f(a)$ is not defined.

Here, $\lim_{x \to a^+} f(x) = L$

and $\lim_{x \to a^-} f(x) = L$

But, $f(a)$ is not defined. So, $f(x)$ is discontinuous at $x = a$.

Graphically This conditions can be illustrated as

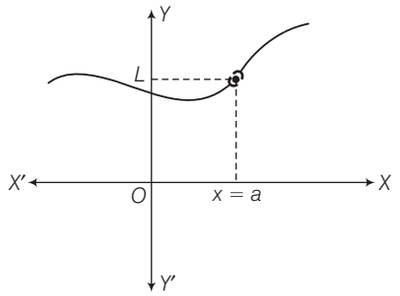


Figure 6.4

Example 3 *If* $f(x)=\dfrac{x^2-1}{x-1}$.

Discuss the continuity at $x=1$.

Sol. Here, $f(x)=\dfrac{x^2-1}{x-1}$

$$\lim_{x\to 1} f(x)=\lim_{x\to 1}\frac{x^2-1}{x-1}$$

$$=\lim_{x\to 1}\frac{(x-1)(x+1)}{(x-1)}$$

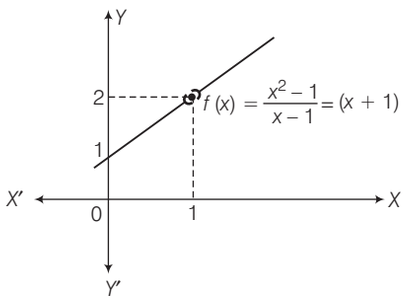
$$=\lim_{x\to 1}(x+1)=2$$

But $f(1)=\dfrac{0}{0}$ [indeterminate form]

$\therefore$ $f(1)$ is not defined at $x=1$.

Hence, $f(x)$ is discontinuous at $x=1$.

Graphically



Here, $\lim_{x\to 1} f(x)=2$ but $f(1)$ is not defined.

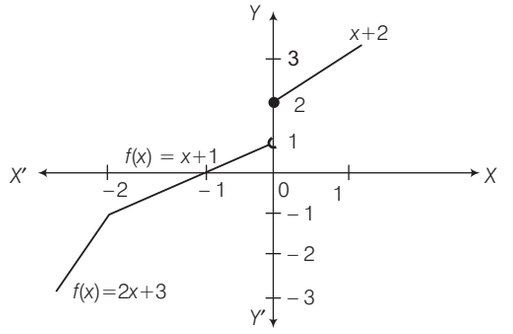
So, $f(x)$ is discontinuous at $x=1$.

Example 4 *Show that the function*

$$f(x)=\begin{cases}2x+3, & -3\le x<-2\\ x+1, & -2\le x<0\\ x+2, & 0\le x\le 1\end{cases}$$

is discontinuous at $x=0$ *and continuous at every other point in interval* $[-3,1]$.

Sol. **Graphically** $f(x)=\begin{cases}2x+3, & -3\le x<-2\\ x+1, & -2\le x<0\\ x+2, & 0\le x\le 1\end{cases}$ is plotted as shown



Here, if we observe the graph we could conclude that at $x=0$, $\lim_{x\to 0^-} f(x)=1$ and $\lim_{x\to 0^+} f(x)=2$, which shows that the function is discontinuous at $x=0$ and continuous at every other point in $[-3,1]$.

Example 5 Examine the function,

$$f(x)=\begin{cases}\dfrac{\cos x}{\pi/2-x}, & x\ne \pi/2\\ 1, & x=\pi/2\end{cases}$$ for continuity *at* $x=\pi/2$.

Sol. We have, $\lim_{x\to \pi/2} f(x)=\lim_{x\to \pi/2}\dfrac{\cos x}{\pi/2-x}$

[as $x\ne\pi/2$ but $x\to\pi/2$]

$$=\lim_{x\to \pi/2}\frac{\cos x}{\pi/2-x}\qquad\left[\frac{0}{0}\text{ form}\right]$$

$$=\lim_{x\to \pi/2}\frac{-\sin x}{0-1}\quad\text{[applying L'Hospital's rule]}$$

$\Rightarrow$ $\lim_{x\to \pi/2} f(x)=\sin\dfrac{\pi}{2}=1$

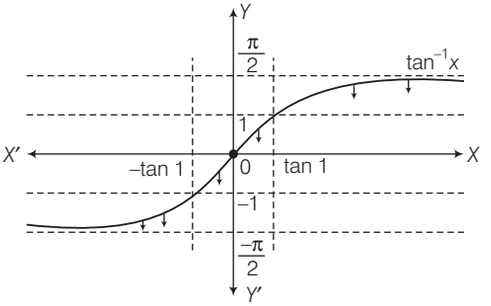
Also, $f(\pi/2)=1$

$\therefore$ $\lim_{x\to \pi/2} f(x)=f(\pi/2)$

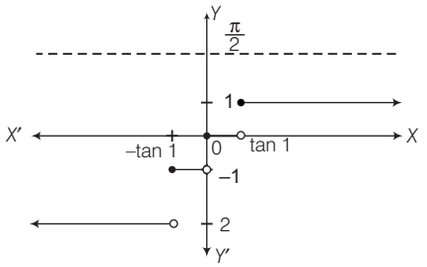
Thus, $f(x)$ is continuous at $x=\pi/2$.

Example 6 *Discuss the continuity of* $f(x)=[\tan^{-1}x]$.

Sol. We know, $y=\tan^{-1}x$ could be plotted as



Thus, $f(x) = [\tan^{-1} x]$ could be plotted as



Which clearly shows the graph is broken at $\{-\tan 1, 0, \tan 1\}$.

$\therefore$ $f(x)$ is not continuous when $x \in \{-\tan 1, 0, \tan 1\}$.

Example 7 Let $y = f(x)$ be defined parametrically as $y = t^2 + t|t|$, $x = 2t - |t|, t \in R$. Then, at $x = 0$, find $f(x)$ and discuss continuity.

Sol. As, $y = t^2 + t|t|$ and $x = 2t - |t|$

Thus, when $t \geq 0$

$\Rightarrow$ $x = 2t - t = t, y = t^2 + t^2 = 2t^2$

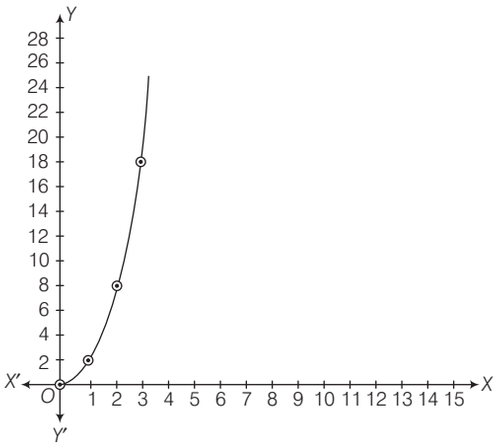
$\therefore$ $x = t$ and $y = 2t^2 \Rightarrow y = 2x^2, \forall\ x \geq 0$

Again, when $t < 0$

$\Rightarrow$ $x = 2t + t = 3t$ and $y = t^2 - t^2 = 0 \Rightarrow y = 0, \forall\ x < 0$

Hence, $f(x) = \begin{cases} 2x^2, & x \geq 0 \\ 0, & x < 0 \end{cases}$

which is clearly continuous for all x as shown graphically.



Example 8 Let $f(x) = \dfrac{e^{\tan x} - e^x + \log(\sec x + \tan x) - x}{\tan x - x}$ be a continuous function at $x = 0$. The value of $f(0)$ equals

(a) $\frac{1}{2}$ (b) $\frac{2}{3}$

(c) $\frac{3}{2}$ (d) 2

Sol. For continuity at $x = 0$, we have

$$f(0) = \lim_{x \to 0} f(x)$$

$$= \lim_{x \to 0} \frac{e^{\tan x} - e^x}{\tan x - x} + \lim_{x \to 0} \frac{\log(\sec x + \tan x) - x}{\tan x - x}$$

$$= \lim_{x \to 0} e^x \frac{(e^{\tan x - x} - 1)}{(\tan x - x)} + \lim_{x \to 0} \frac{\sec x - 1}{\sec^2 x - 1}$$

[using L'Hospital's rule]

$$= 1 + \lim_{x \to 0} \frac{1}{\sec x + 1} = 1 + \frac{1}{2} = \frac{3}{2}$$

Hence, (c) is the correct answer.

Example 9 ***If*** $f(x) = \sqrt{\dfrac{1}{\tan^{-1}(x^2 - 4x + 3)}}$, ***then*** $f(x)$ ***is continuous for***

(a) (1, 3) (b) $(-\infty, 0)$

(c) $(-\infty, 1) \cup (3, \infty)$ (d) None of these

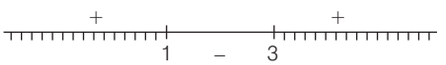
Sol. Here,

$$f(x) = \sqrt{\frac{1}{\tan^{-1}(x^2 - 4x + 3)}}$$

For domain, $\tan^{-1}(x^2 - 4x + 3) > 0$

$\Rightarrow$ $x^2 - 4x + 3 > 0$

$\Rightarrow$ $(x - 1)(x - 3) > 0$



$\therefore$ $x \in (-\infty, 1) \cup (3, \infty)$

Since, every general function is continuous in its domain.

$\therefore f(x)$ is continuous for $x \in (-\infty, 1) \cup (3, \infty)$.

Hence, (c) is correct answer.

Exercise for Session 1

1. If function $f(x)=\dfrac{\sqrt{1+x}-\sqrt[3]{1+x}}{x}$ is continuous function at $x=0$, then $f(0)$ is equal to

(a) 2 (b) $\frac{1}{4}$ (c) $\frac{1}{6}$ (d) $\frac{1}{3}$

2. If $f(x)=\begin{cases}\dfrac{1}{e^{1/x}+1}, & x\neq 0\\ 0, & x=0\end{cases}$, then

(a) $\lim\limits_{x\to 0^-} f(x)=0$ (b) $\lim\limits_{x\to 0^+} f(x)=1$

(c) $f(x)$ is discontinuous at $x=0$ (d) $f(x)$ is continuous at $x=0$

3. If $f(x)=\begin{cases}\dfrac{x^2-(a+2)x+2a}{x-2}, & x\neq 2\\ 2, & x=2\end{cases}$ is continuous at $x=2$, then a is equal to

(a) 0 (b) 1 (c) -1 (d) 2

4. If $f(x)=\begin{cases}\dfrac{\log(1+2ax)-\log(1-bx)}{x}, & x\neq 0\\ k, & x=0\end{cases}$ is continuous at $x=0$, then k is equal to

(a) $2a+b$ (b) $2a-b$ (c) $b-2a$ (d) $a+b$

5. If $f(x)=\begin{cases}[x]+[-x], & x\neq 2\\ \lambda, & x=2\end{cases}$ and f is continuous at $x=2$, where $[\cdot]$ denotes greatest integer function, then λ is

(a) -1 (b) 0 (c) 1 (d) 2

Session 2

Continuity in an Interval or Continuity at End Points

Let a function $y = f(x)$ be defined on $[a, b]$. Then,

the function $f(x)$ is said to be continuous at the left end $x = a$

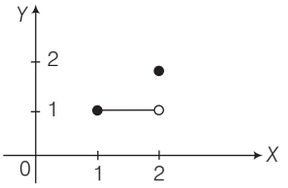
If $f(a) = \lim_{x \to a^+} f(x)$ [need not check LHL]

The function $f(x)$ is said to be continuous at the right end $x = b$

If $f(b) = \lim_{x \to b^-} f(x)$ [need not check RHL]

Example 10 ***If*** $f(x) = [x]$ ***, where*** $[\cdot]$ ***denotes greatest integral function. Then, check the continuity on*** $[1, 2]$**.**

Sol. **Graphically** $f(x) = [x]$ could be plotted as



Here, in the graph $f(x)$ is continuous at all points, where $1 < x < 2$.

To check continuity at $x = 1$ and $x = 2$

(i) **To check continuity at** $x = 1$

Here, $f(1) = 1$ and RHL at $x = 1$

$\Rightarrow \lim_{x \to 1^+} f(x) = 1$, i.e. $f(1) = \lim_{x \to 1^+} f(x)$

Thus, $f(x)$ is continuous at $x = 1$.

(ii) **To check continuity at** $x = 2$

Here, $f(2) = 2$

But LHL at $x = 2 \Rightarrow \lim_{x \to 2^-} f(x) = 1$

which shows $f(2) \neq \lim_{x \to 2^-} f(x)$

Thus, $f(x)$ is discontinuous at $x = 2$.

From the above information, it becomes clear that $f(x)$ is continuous at all points on $[1, 2]$ except at $x = 2$,

i.e. $f(x)$ is continuous for $x \in [1, 2)$.

Exercise for Session 2

1. Let $f(x) = \begin{cases} -2\sin x & \text{for } -\pi \le x \le -\frac{\pi}{2} \\ a\sin x + b & \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \cos x & \text{for } \frac{\pi}{2} \le x \le \pi \end{cases}$. If f is continuous on $[-\pi, \pi)$, then find the values of a and b.

2. Draw the graph of the function $f(x) = x - |x - x^2|, -1 \le x \le 1$ and discuss the continuity or discontinuity of f in the interval $-1 \le x \le 1$.

3. Discuss the continuity of 'f' in $[0, 2]$, where $f(x) = \begin{cases} |4x - 5|[x] & \text{for } x > 1 \\ [\cos \pi x] & \text{for } x \le 1 \end{cases}$; where $[x]$ is greatest integer not greater than x.

4. Let $f(x) = \begin{cases} Ax - B, & x \le -1 \\ 2x^2 + 3Ax + B, & x \in (-1, 1] \\ 4, & x > 1 \end{cases}$

Statement I $f(x)$ is continuous at all x, if $A = \frac{3}{4}$ and $B = -\frac{1}{4}$.

Statement II Polynomial function is always continuous.

A. Both Statement I and Statement II are correct and Statement II is the correct explanation of Statement I

B. Both Statement I and Statement II are correct but Statement II is not the correct explanation of Statement I

C. Statement I is correct but Statement II is incorrect

D. Statement II is correct but Statement I is incorrect

(a) A (b) B (c) C (d) D

Session 3

Discontinuity of a Function

A function f which is not continuous, said to be discontinuous functions, i.e. if there is a break in the function of any kind, then it is discontinuous functions.

There are two types of discontinuity

1. Removable discontinuity
2. Non-removable discontinuity

Removable Discontinuity

In this type of discontinuity, $\lim_{x \to a} f(x)$ necessarily exists, but is either not equal to $f(a)$ or $f(a)$ is not defined. Such function is said to have a removable discontinuity of the first kind.

In this case, it is possible to redefine the function in such a manner that $\lim_{x \to a} f(x) = f(a)$ and thus making function continuous.

These discontinuities can be further classified as

(i) Missing point discontinuity

(ii) Isolated point discontinuity

Examples of Missing Point Discontinuity

Here, $\lim_{x \to a} f(x)$ exists finitely but $f(a)$ is not defined. e.g.

(i) Let $f(x) = \dfrac{(x-1)(9-x^2)}{(x-1)}$,

clearly $f(1) \to \dfrac{0}{0}$ form

$\therefore$ $f(x)$ has missing point discontinuity. Shown as,

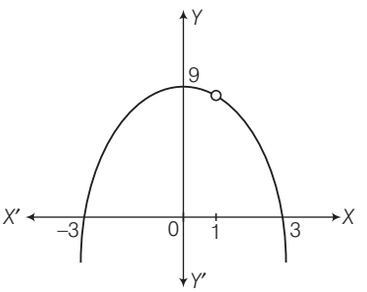


Figure 6.5

(ii) $f(x) = \dfrac{x^2-4}{x-2}$ has missing point discontinuity at $x = 2$.

Shown as,

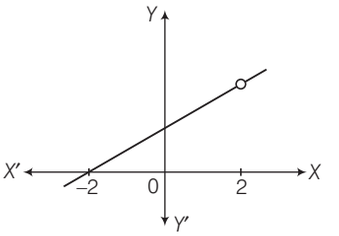


Figure 6.6

(iii) $f(x) = \dfrac{\sin x}{x}$ has missing point discontinuity at $x = 0$.

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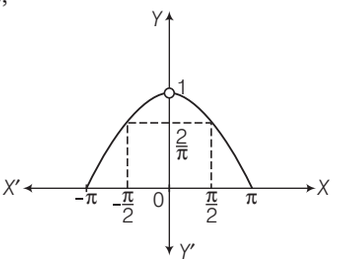


Figure 6.7

Examples of Isolated Point Discontinuity

Here, $\lim_{x \to a} f(x)$ exists and $f(a)$ also exists but $\lim_{x \to a} f(x) \neq f(a)$. e.g.

(i) Let $f(x) = [x] + [-x] \Rightarrow f(x) = \begin{cases} 0, \text{ if } x \in I \\ -1, \text{ if } x \notin I \end{cases}$

where x = Integer, has isolated point discontinuity, can be shown as

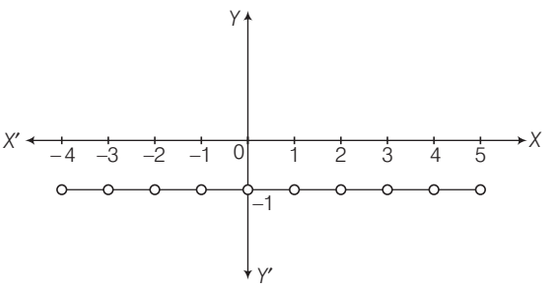


Figure 6.8

(ii) Let $f(x) = \text{sgn}(\cos 2x - 2\sin x + 3)$

$\Rightarrow$ $f(x) = \text{sgn}(1 - 2\sin^2 x - 2\sin x + 3)$

$= \text{sgn}(2(2 + \sin x)(1 - \sin x))$

$$= \begin{cases} 0, \text{ if } x = 2n\pi + \dfrac{\pi}{2} \\ 1, \text{ if } x \neq 2n\pi + \dfrac{\pi}{2} \end{cases}$$

$\therefore f(x)$ has an isolated point discontinuity at $x = 2n\pi + \dfrac{\pi}{2}$.

Shown as,

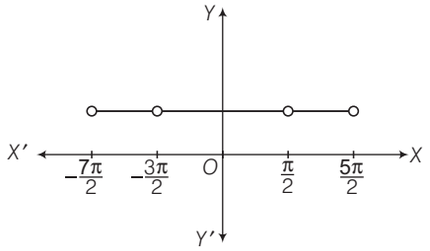


Figure 6.9

Non-removable Discontinuity

In this type of discontinuity $\lim\limits_{x \to a} f(x)$ doesn't exist and therefore, it is not possible to redefine the function in any manner and make it continuous. Such function is said to have non-removable discontinuity or discontinuity of second kind.

Such discontinuities can further be classified into three types.

(i) Finite Discontinuity

If for a function $f(x)$: $\lim\limits_{x \to a^-} f(x) = L_1$ and $\lim\limits_{x \to a^+} f(x) = L_2$ and $L_1 \neq L_2$, then such function is said to have a finite discontinuity or a jump discontinuity.

In this case, the non-negative difference between the two limits is called the **Jump of Discontinuity**. A function having a finite number of jumps in a given interval is called a **Piecewise Continuous** or **Sectionally Continuous Function.**

Examples of Finite Discontinuity

(i) $f(x) \tan^{-1}\left(\dfrac{1}{x}\right)$ at $x = 0$

$$\left.\begin{array}{ll} \text{RHL} & \text{i.e. } f(0^+) = \dfrac{\pi}{2} \\ \text{LHL} & \text{i.e. } f(0^-) = -\dfrac{\pi}{2} \end{array}\right\} \text{jump} = \pi$$

(ii) $f(x) = \dfrac{|\sin x|}{x}$ at $x = 0$

$$\left.\begin{array}{ll} \text{RHL} & \text{i.e. } f(0^+) = 1 \\ \text{LHL} & \text{i.e. } f(0^-) = -1 \end{array}\right\} \text{jump} = 2$$

(ii) Infinite Discontinuity

If for a function $f(x)$: $\lim\limits_{x \to a^-} f(x) = L_1$ and $\lim\limits_{x \to a^+} f(x) = L_2$ and either L_1 or L_2 is infinity, then such function is said to have infinite discontinuity.

In other words, If $x = a$ is a vertical asymptote for the graph of $y = f(x)$, then f is said to have infinite discontinuity at a.

Examples of Infinite Discontinuity

(i) $f(x) = \dfrac{x}{1 - x}$, at $x = 1$

RHL i.e. $f(1^+) = -\infty$

LHL i.e. $f(1^-) = \infty$

(ii) $f(x) = \dfrac{1}{x^2}$, at $x = 0$

RHL i.e. $f(0^+) = \infty$

LHL i.e. $f(0^-) = \infty$

(iii) Oscillatory Discontinuity

If for a function $f(x)$: $\lim\limits_{x \to a} f(x)$ doesn't exist but oscillate between two finite quantities, then such function is said to have oscillatory discontinuity.

Examples of Oscillatory Discontinuity

(i) $f(x) = \sin\left(\dfrac{1}{x}\right)$

$\Rightarrow \lim\limits_{x \to 0} f(x) =$ a value between -1 to 1.

$\therefore$ Limit doesn't exist, as it oscillates between -1 and 1 as $x \to 0$.

(ii) $f(x) = \cos\left(\dfrac{1}{x}\right)$

$\Rightarrow \lim\limits_{x \to 0} \cos\left(\dfrac{1}{x}\right) =$ a value between -1 to 1.

$\therefore$ Limit doesn't exist, as it oscillates between -1 to 1 at $x \to 0$.

List of Continuous Functions

	Function $f(x)$	Interval in which $f(x)$ is continuous
1.	constant c	$(-\infty, \infty)$
2.	x^n, n is an integer ≥ 0	$(-\infty, \infty)$
3.	x^{-n}, n is a positive integer	$(-\infty, \infty)-\{0\}$
4.	$\lvert x-a\rvert$	$(-\infty, \infty)$
5.	$P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$	$(-\infty, \infty)$
6.	$\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial in x	$(-\infty, \infty)-\{x : q(x) = 0\}$
7.	$\sin x$	$(-\infty, \infty)$
8.	$\cos x$	$(-\infty, \infty)$
9.	$\tan x$	$(-\infty, \infty)-\left\{(2n+1)\frac{\pi}{2} : n \in I\right\}$
10.	$\cot x$	$(-\infty, \infty)-\{n\pi : n \in I\}$
11.	$\sec x$	$(-\infty, \infty)-\{(2n+1)\pi/2 : n \in I\}$
12.	$\operatorname{cosec} x$	$(-\infty, \infty)-\{n\pi : n \in I\}$
13.	e^x	$(-\infty, \infty)$
14.	$\log_e x$	$(0, \infty)$

Example 11 ***Examine the function,*** $f(x)=\begin{cases} x-1, & x<0 \\ 1/4, & x=0 \\ x^2-1, & x>0 \end{cases}$

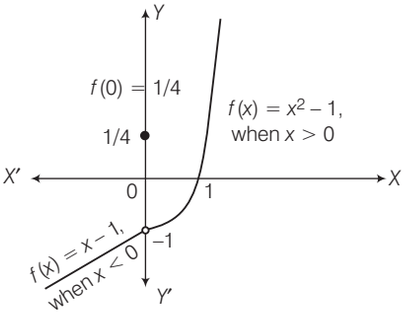
Discuss the continuity and if discontinuous remove the discontinuity.

Sol. $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x) = -1$

But $f(0) = 1/4$. Thus, $f(x)$ has removable discontinuity and $f(x)$ could be made continuous by taking

$$f(0) = -1 \Rightarrow f(x) = \begin{cases} x-1, & x<0 \\ -1, & x=0 \\ x^2-1, & x>0 \end{cases}$$

Graphically $f(x)$ could be plotted as showing



Example 12 ***Show the function,*** $f(x)=\begin{cases} \frac{e^{1/x}-1}{e^{1/x}+1}, & \text{when } x \neq 0 \\ 0, & \text{when } x=0 \end{cases}$

has non-removable discontinuity at $x = 0$.

Sol. We have, $f(x)=\begin{cases} \frac{e^{1/x}-1}{e^{1/x}+1}, & \text{when } x \neq 0 \\ 0, & \text{when } x=0 \end{cases}$

$\therefore$ RHL at $x = 0$, let $x = 0 + h$

$$\Rightarrow \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} \frac{e^{\frac{1}{0+h}}-1}{e^{\frac{1}{0+h}}+1} = \lim_{h \to 0} \frac{e^{\frac{1}{h}}-1}{e^{\frac{1}{h}}+1}$$

$$\Rightarrow \lim_{x \to 0^+} f(x) = \lim_{h \to 0} \frac{1-\frac{1}{e^{1/h}}}{1+\frac{1}{e^{1/h}}} \quad \Rightarrow \lim_{x \to 0^+} f(x) = \frac{1-0}{1+0} = 1$$

$$[\text{as } h \to 0; \frac{1}{h} \to \infty \Rightarrow e^{1/h} \to \infty; 1/e^{1/h} \to 0]$$

$\therefore \quad \lim\limits_{x \to 0^+} f(x) = 1$

Again, LHL at $x = 0$, let $x = 0 - h$

$$\Rightarrow \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h)$$

$$= \lim_{h \to 0} \frac{e^{-1/h}-1}{e^{-1/h}+1} = \frac{0-1}{0+1} = -1$$

$$[\text{as } h \to 0; e^{-1/h} \to 0]$$

$$\lim_{x \to 0^-} f(x) = -1$$

$$\Rightarrow \lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x).$$

Thus, $f(x)$ has non-removable discontinuity.

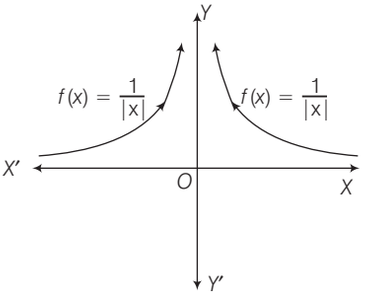
Example 13 ***Show*** $f(x)=\frac{1}{|x|}$ ***has discontinuity of second kind at*** $x = 0$.

Sol. Here, $f(0)=\frac{1}{|0|}$, which shows function has discontinuity of second kind.

Graphically Here, the graph is broken at $x = 0$ as

$$x \to 0$$

$$\Rightarrow \quad f(x) \to \infty$$



Therefore, $f(x)$ has discontinuity of second kind.

Exercise for Session 3

1. Which of the following function(s) has/have removable discontinuity at the origin?

(a) $f(x)=\dfrac{1}{1+2^{\cot x}}$

(b) $f(x)=\cos\left(\dfrac{|\sin x|}{x}\right)$

(c) $f(x)=x\sin\dfrac{\pi}{x}$

(d) $f(x)=\dfrac{1}{\ln|x|}$

2. Function whose jump (non-negative difference of LHL and RHL) of discontinuity is greater than or equal to one. is/are

(a) $f(x)=\begin{cases}\dfrac{(e^{1/x}+1)}{(e^{1/x}-1)}; & x<0\\ \dfrac{(1-\cos x)}{x}; & x>0\end{cases}$

(b) $g(x)=\begin{cases}\dfrac{(x^{1/3}-1)}{(x^{1/2}-1)}; & x>0\\ \dfrac{\ln x}{(x-1)}; & \dfrac{1}{2}<x<1\end{cases}$

(c) $u(x)=\begin{cases}\dfrac{\sin^{-1}2x}{\tan^{-1}3x}; & x\in\left(0,\dfrac{1}{2}\right]\\ \dfrac{|\sin x|}{x}; & x<0\end{cases}$

(d) $v(x)=\begin{cases}\log_3(x+2); & x>2\\ \log_{1/2}(x^2+5); & x<2\end{cases}$

3. Consider the piecewise defined function $f(x)=\begin{cases}\sqrt{-x}, & \text{if } x<0\\ 0, & \text{if } 0\le x\le 4\\ x-4, & \text{if } x>4\end{cases}$, choose the answer which best describes the continuity of this function.

(a) the function is unbounded and therefore cannot be continuous.

(b) the function is right continuous at $x=0$.

(c) the function has a removable discontinuity at 0 and 4, but is continuous on the rest of the real line.

(d) the function is continuous on the entire real line.

4. If $f(x)=\text{sgn}(\cos 2x-2\sin x+3)$, where sgn () is the signum function, then $f(x)$

(a) is continuous over its domain.

(b) has a missing point discontinuity.

(c) has isolated point discontinuity.

(d) has irremovable discontinuity

5. $f(x)=\dfrac{2\cos x-\sin 2x}{(\pi-2x)^2}$; $g(x)=\dfrac{e^{-\cos x}-1}{8x-4\pi}$

$h(x)=f(x)$ for $x<\pi/2=g(x)$ for $x>\pi/2$, then which of the followings does not hold?

(a) h is continuous at $x=\pi/2$

(b) h has an irremovable discontinuity at $x=\pi/2$

(c) h has a removable discontinuity at $x=\pi/2$

(d) $f\left(\dfrac{\pi^+}{2}\right)=g\left(\dfrac{\pi^-}{2}\right)$

Session 4

Theorems Based on Continuity; Continuity of Composite Function

Theorem 1 Sum, difference, product and quotient of two continuous functions is always a continuous function.

However, $h(x)=\dfrac{f(x)}{g(x)}$ is continuous at $x=a$ only if $g(a)\neq 0$.

Theorem 2 If $f(x)$ is continuous and $g(x)$ is discontinuous at $x=a$, then the product function $\phi(x)=f(x)\cdot g(x)$ is not necessarily be discontinuous at $x=a$.

e.g. (i) $f(x)=x$ and $g(x)=\begin{cases}\sin\dfrac{1}{x}, & x\neq 0\\ 0, & x=0\end{cases}$

where $f(x)$ is continuous and $g(x)$ is discontinuous at $x=0$.

But $\phi(x)=f(x)\cdot g(x)=\begin{cases}x\cdot\sin\dfrac{1}{x}, & x\neq 0\\ 0, & x=0\end{cases}$

is continuous at $x=0$.

(ii) $f(x)=\cos(2x-1)\dfrac{\pi}{2}$ is continuous at $x=1$ and $g(x)=[x]$ is discontinuous at $x=1$.

But $\phi(x)=f(x)\cdot g(x)=[x]\cos\left(\dfrac{2x-1}{2}\right)\pi$ is continuous at $x=1$.

Theorem 3 If $f(x)$ and $g(x)$ both are discontinuous at $x=a$, then the product function $\phi(x)=f(x)\cdot g(x)$ is not necessarily be discontinuous at $x=a$.

e.g. $f(x)=\begin{cases}1, & x\geq 0\\ -1, & x<0\end{cases}$ and $g(x)=\begin{cases}-1, & x\geq 0\\ 1, & x<0\end{cases}$

$\therefore\quad \phi(x)=f(x)\cdot g(x)=-1,\forall\, x\in R$

$\phi(x)$ is continuous, where $f(x)$ and $g(x)$ are discontinuous at $x=0$.

Example 14 $f(x)=\begin{cases}\left(\tan\left(\dfrac{\pi}{4}+x\right)\right)^{1/x}, & x\neq 0\\ k, & x=0\end{cases}$.

For what value of k, $f(x)$ ***is continuous at*** $x=0$***?***

Sol. Here, $\lim\limits_{x\to 0} f(x)=\lim\limits_{x\to 0}\left\{\tan\left(\dfrac{\pi}{4}+x\right)\right\}^{1/x}$

$\Rightarrow \lim\limits_{x\to 0} f(x)=\lim\limits_{x\to 0}\left[\dfrac{1+\tan x}{1-\tan x}\right]^{1/x}$ [1^{∞} form]

$\Rightarrow \lim\limits_{x\to 0} f(x)=\lim\limits_{x\to 0}\left[1+\left(\dfrac{1+\tan x}{1-\tan x}-1\right)\right]^{1/x}$

$\Rightarrow \lim\limits_{x\to 0} f(x)=e^{\lim\limits_{x\to 0}\left(\frac{2\tan x}{1-\tan x}\right)\cdot\frac{1}{x}}$

$\Rightarrow \lim\limits_{x\to 0} f(x)=e^{2\lim\limits_{x\to 0}\frac{\tan x}{x(1-\tan x)}}=e^2$

Here, $f(x)$ is continuous at $x=0$, when

$\lim\limits_{x\to 0} f(x)=f(0)$

$\Rightarrow \quad k=e^2$

Example 15 ***A function*** $f(x)$ ***is defined by,***

$$f(x)=\begin{cases}\dfrac{[x^2]-1}{x^2-1}, & \text{for } x^2\neq 1\\ 0, & \text{for } x^2=1\end{cases}$$

Discuss the continuity of $f(x)$ at $x=1$.

Sol. We have, $f(x)=\begin{cases}\dfrac{[x^2]-1}{x^2-1}, & \text{for } x^2\neq 1\\ 0, & \text{for } x^2=1\end{cases}$

$\Rightarrow f(x)=\begin{cases}\dfrac{-1}{x^2-1}, & \text{for } 0<x^2<1\\ 0, & \text{for } x^2=1\\ \dfrac{1-1}{x^2-1}, & \text{for } 1<x^2<2\end{cases}$

$\Rightarrow f(x)=\begin{cases}\dfrac{-1}{x^2-1}, & \text{for } 0<x^2<1\\ 0, & \text{for } x^2=1\\ 0, & \text{for } 1<x^2<2\end{cases}$

$\therefore$ RHL at $x=1$

$\Rightarrow \quad \lim_{x \to 1^+} f(x) = 0$

Also, LHL at $x^2 = 1$

$\Rightarrow \quad \lim_{x \to 1^-} f(x) = \lim_{h \to 0} f(1-h)$

$= \lim_{h \to 0} \frac{-1}{(1-h)^2 - 1} = -\infty$

$\therefore \lim_{x \to 1} f(x)$ doesn't exist. [as RHL $\neq$ LHL]

Hence, $f(x)$ is not continuous at $x = 1$.

Example 16 Discuss the continuity of the function

$$f(x) = \lim_{n \to \infty} \frac{\log(2+x) - x^{2n} \sin x}{1 + x^{2n}} \text{ at } x = 1.$$

Sol. We have, $f(1) = \lim_{n \to \infty} \frac{\log 3 - \sin 1}{2} = \frac{1}{2}(\log 3 - \sin 1)$ ...(i)

We know that, $\lim_{n \to \infty} x^{2n} = \begin{cases} 0, & \text{if } x^2 < 1 \\ \infty, & \text{if } x^2 > 1 \end{cases}$

$\therefore$ For $x^2 < 1$, we have

$$f(x) = \lim_{n \to \infty} \frac{\log(2+x) - x^{2n} \sin x}{1 + x^{2n}} = \log(2+x)$$

Again, for $x^2 > 1$, we have

$$f(x) = \lim_{n \to \infty} \frac{\frac{1}{x^{2n}} \log(2+x) - \sin x}{1 + \frac{1}{x^{2n}}} = -\sin(x)$$

Here, as $x \to 1$

$$\lim_{x \to 1^-} f(x) = \log(3)$$

and $$\lim_{x \to 1^+} f(x) = -\sin(1)$$

So, $$\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$$

Therefore, $f(x)$ is not continuous at $x = 1$.

Example 17 Discuss the continuity of $f(x)$, where

$$f(x) = \lim_{n \to \infty} \left(\sin \frac{\pi x}{2} \right)^{2n}.$$

Sol. Since, $\lim_{n \to \infty} x^{2n} = \begin{cases} 0, & |x| < 1 \\ 1, & |x| = 1 \end{cases}$

$\therefore \quad f(x) = \lim_{n \to \infty} \left(\sin\left(\frac{\pi x}{2} \right) \right)^{2n} = \begin{cases} 0, & \left| \sin \frac{\pi x}{2} \right| < 1 \\ 1, & \left| \sin \frac{\pi x}{2} \right| = 1 \end{cases}$

Thus, $f(x)$ is continuous for all x, except for those values of x for which $\left| \sin \frac{\pi x}{2} \right| = 1$, i.e. x is an odd integer.

$\Rightarrow \quad x = (2n+1)$, where $n \in I$

Check continuity at $x = (2n+1)$

LHL $= \lim_{x \to (2n+1)^-} f(x) = 0$ and $f(2n+1) = 1$

Thus, LHL $\neq f(2n+1)$

$\Rightarrow f(x)$ is discontinuous at $x = (2n+1)$ [i.e. at odd integers]

Hence, $f(x)$ is discontinuous at $x = (2n+1)$; $n \in$ integer.

Example 18 ***Let***

$$f(x) = \begin{cases} \{1 + |\sin x|\}^{a/|\sin x|}, & -\pi/6 < x < 0 \\ b, & x = 0 \\ e^{\tan 2x / \tan 3x}, & 0 < x < \pi/6 \end{cases}$$

Determine a and b such that $f(x)$ is continuous at $x = 0$. ***[IIT JEE 1994]***

Sol. Since, f is continuous at $x = 0$.

Therefore, RHL = LHL = $f(0)$

RHL at $x = 0$

$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} e^{\tan 2h / \tan 3h}$$

$$= \lim_{h \to 0} e^{\frac{\tan 2h}{2h} \cdot \frac{3h}{\tan 3h} \cdot \frac{2}{3}} = e^{2/3} \quad \text{...(i)}$$

Again, LHL at $x = 0$

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h)$$

$$= \lim_{h \to 0} \{1 + |\sin(0-h)|\}^{a/|\sin(0-h)|}$$

$$= \lim_{h \to 0} \{1 + |\sin h|\}^{\frac{a}{|\sin h|}}$$

$$= e^{\lim_{h \to 0} |\sin h| \cdot \frac{a}{|\sin h|}} = e^a \quad \text{...(ii)}$$

and $f(0) = b$ ...(iii)

Thus, $e^{2/3} = e^a = b \Rightarrow a = 2/3$ and $b = e^{2/3}$

Example 19 Fill in the blanks so that the resulting statement is correct.

Let $f(x) = [x] \sin\left(\frac{\pi}{[x+1]} \right)$, where $[\cdot]$ denotes greatest integral function. The domain of f is ..... and the points of discontinuity of f in the domain are ..... . ***[IIT JEE 1996]***

Sol. Let $f(x) = [x] \sin\left(\frac{\pi}{[x+1]} \right)$

Domain of $f(x)$ is $x \in R$ excluding the points where $[x+1] = 0$ [$\because$ denominator can't be zero]

$\Rightarrow \quad 0 \leq x+1 < 1 \Rightarrow -1 \leq x < 0$

i.e. for all $x \in [-1, 0)$, denominator is zero.

So, domain is $x \in R - [-1, 0)$.

$\Rightarrow$ Domain is $x \in (-\infty, -1) \cup [0, \infty)$

and the internal point discontinuity $\in [-1, 0)$.

Example 20 Let $f(x+y)=f(x)+f(y)$ for all x and y. If the function $f(x)$ is continuous at $x=0$, show that $f(x)$ is continuous for all x.

Sol. As, the function is continuous at $x=0$, we have

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=f(0)$$

$$\Rightarrow \lim_{h\to 0} f(0-h)=\lim_{h\to 0} f(0+h)=f(0)$$

$$\Rightarrow \lim_{h\to 0}\{f(0)+f(-h)\}=\lim_{h\to 0}\{f(0)+f(h)\}=f(0)$$

[using $f(x+y)=f(x)+f(y)$]

$$\Rightarrow \lim_{h\to 0} f(-h)=\lim_{h\to 0} f(h)=0 \quad \ldots(i)$$

Now, consider some arbitrary point $x=a$

$$\text{LHL}=\lim_{h\to 0} f(a-h)=\lim_{h\to 0} f(a)+f(-h)$$

[using $f(x+y)=f(x)+f(y)$]

$$=f(a)+\lim_{h\to 0} f(-h)$$

$$\text{LHL}=f(a)+0=f(a)$$

[as $\lim_{h\to 0} f(-h)=0$, using Eq. (i)]

$$\text{RHL}=\lim_{h\to 0} f(a+h)=\lim_{h\to 0} f(a)+f(h)$$

$$=f(a)+\lim_{h\to 0} f(h)$$

$$\text{RHL}=f(a)+0=f(a)$$

[as $\lim_{h\to 0} f(h)=0$, using Eq. (i)]

At any arbitrary point $(x=a)$,

$$\text{LHL}=\text{RHL}=f(a)$$

Therefore, function is continuous for all values of x, if it is continuous at 0.

Example 21 ***Let $f(x)$ be a continuous function defined for $1\le x\le 3$. If $f(x)$ takes rational values for all x and $f(2)=10$, then find the value of $f(1.5)$.***

[IIT JEE 1997]

Sol. As, $f(x)$ is continuous in [1, 3], $f(x)$ will attain all values between $f(1)$ and $f(3)$. As, $f(x)$ takes rational values for all x and there are innumerable irrational values between $f(1)$ and $f(3)$ which implies that $f(x)$ can take rational values for all x, if $f(x)$ has a constant rational value at all points between $x=1$ and $x=3$.

So, $f(2)=f(1.5)=10$

Continuity of composite function

If the function $u=f(x)$ is continuous at the point $x=a$ and the function $y=g(u)$ is continuous at the point $u=f(a)$, then the composite function $y=(gof)(x)=g(f(x))$ is continuous at the point $x=a$.

Example 22 ***Discuss the continuity for*** $f(x)=\dfrac{1-u^2}{2+u^2}$, where $u=\tan x$.

Sol. Here, $u=\tan x$ is discontinuous at $n\pi\pm\dfrac{\pi}{2}$, $n\in I$

and $f(x)=\dfrac{1-u^2}{2+u^2}$ is continuous at every $u\in R$.

Hence, $f(x)$ is continuous on; $x\in R-\left\{n\pi\pm\dfrac{\pi}{2}, n\in I\right\}$.

Also, $\lim\limits_{x\to n\pi\pm\frac{\pi}{2}} f(x)=\lim\limits_{u^2\to\infty}\dfrac{1-u^2}{2+u^2}$

$$=\lim_{u\to\infty}\frac{\frac{1}{u^2}-1}{\frac{2}{u^2}+1}=-1$$

Hence, the points $n\pi\pm\dfrac{\pi}{2}$, $n\in I$ have removable discontinuity.

i.e. If $f(x)$ is defined as

$$f(x)=\begin{cases}\dfrac{1-u^2}{2+u^2}, & x\ne n\pi\pm\dfrac{\pi}{2} \text{ and } u=\tan x\\ -1, & x=n\pi\pm\dfrac{\pi}{2}\end{cases}$$, then

$f(x)$ is continuous for all $x\in R$.

Example 23 Find the points of discontinuity of $y=\dfrac{1}{u^2+u-2}$, where $u=\dfrac{1}{x-1}$.

Sol. The function $u=f(x)=\dfrac{1}{x-1}$ is discontinuous at the point $x=1$. ...(i)

The function $y=g(x)=\dfrac{1}{u^2+u-2}=\dfrac{1}{(u+2)(u-1)}$ is discontinuous at $u=-2$ and $u=1$.

when $u=-2$, $\dfrac{1}{x-1}=u=-2$

$$\Rightarrow x-1=-\frac{1}{2}$$

$$\Rightarrow x=1/2 \quad \ldots(ii)$$

when $u=1$, $\dfrac{1}{x-1}=u=1$

$$\Rightarrow x-1=1$$

$$\Rightarrow x=2 \quad \ldots(iii)$$

Hence, the composite function $y=g(f(x))$ is discontinuous at three points $x=\dfrac{1}{2}$, 1 and 2.

Exercise for Session 4

1. If $f(x)=\dfrac{1}{x^2-17x+66}$, then $f\left(\dfrac{2}{x-2}\right)$ is discontinuous at $x=$

(a) 2 (b) $\dfrac{7}{3}$ (c) $\dfrac{24}{11}$ (d) 6, 11

2. Let f be a continuous function on R.

If $f(1/4^n)=(\sin e^n)\,e^{-n^2}+\dfrac{n^2}{n^2+1}$, then $f(0)$ is

(a) not unique (b) 1

(c) data sufficient to find $f(0)$ (d) data insufficient to find $f(0)$

3. $f(x)$ is continuous at $x=0$, then which of the following are always true?

(a) $\lim\limits_{x\to 0} f(x)=0$

(b) $f(x)$ is non continuous at $x=1$

(c) $g(x)=x^2f(x)$ is continuous at $x=0$

(d) $\lim\limits_{x\to 0^+}(f(x)-f(0))=0$

4. If $f(x)=\cos\left[\dfrac{\pi}{x}\right]\cos\left(\dfrac{\pi}{2}(x-1)\right)$; where $[x]$ is the greatest integer function of x, then $f(x)$ is continuous at

(a) $x=0$ (b) $x=1$

(c) $x=2$ (d) None of these

5. Let $f(x)=[x]$ and $g(x)=\begin{cases}0, & x\in Z\\ x^2, & x\in R-Z\end{cases}$, then (where [.] denotes greatest integer function)

(a) $\lim\limits_{x\to 1} g(x)$ exists, but $g(x)$ is not continuous at $x=1$

(b) $\lim\limits_{x\to 1} f(x)$ does not exist and $f(x)$ is not continuous at $x=1$

(c) *gof* is continuous for all x

(d) *fog* is continuous for all x

6. Let $f(x)=\begin{cases}a\sin^{2n}x, & \text{for } x\geq 0 \text{ and } n\to\infty\\ b\cos^{2m}x-1, & \text{for } x<0 \text{ and } m\to\infty\end{cases}$, then

(a) $f(0^-)\neq f(0^+)$ (b) $f(0^+)\neq f(0)$

(c) $f(0^-)=f(0)$ (d) f is continuous at $x=0$

7. Consider $f(x)=\lim\limits_{n\to\infty}\dfrac{x^{n`}-\sin x^n}{x^n+\sin x^n}$ for $x>0, x\neq 1, f(1)=0$, then

(a) f is continuous at $x=1$

(b) f has a finite discontinuity at $x=1$

(c) f has an infinite or oscillatory discontinuity at $x=1$

(d) f has a removal type of discontinuity at $x=1$

Session 5

Intermediate Value Theorem

If $f(x)$ is continuous in $[a, b]$ and $f(a) \neq f(b)$, then for any value $c \in (f(a), f(b))$, there is atleast one number x_0 in (a, b) for which $f(x_0) = c$.

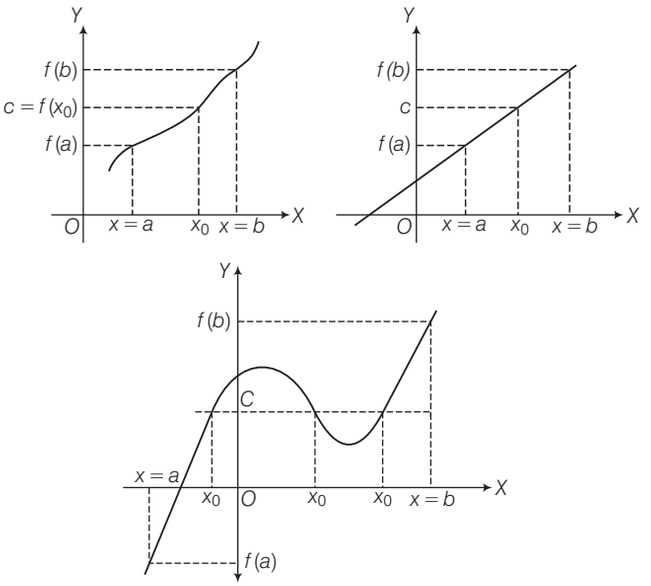


Figure 6.10

Example 24 Show that the function $f(x) = (x-a)^2 (x-b)^2 + x$ takes the value $\frac{a+b}{2}$ for some value of $x \in [a, b]$.

Sol. Here, $f(a) = a$ and $f(b) = b$

Also, $f(x)$ is continuous in $[a, b]$ and $\frac{a+b}{2} \in [a, b]$.

Therefore, using intermediate value theorem, there exists some $c \in [a, b]$, such that

$$f(c) = \frac{a+b}{2}$$

Example 25 ***Suppose that*** $f(x)$ ***is continuous in [0, 1] and*** $f(0) = 0, f(1) = 0$. ***Prove that*** $f(c) = 1 - 2c^2$ ***for some*** $c \in (0, 1)$.

Sol. Let $g(x) = f(x) + 2x^2 - 1$ in $[0, 1]$, $g(0) = -1, g(1) = 1$

$\therefore$ By intermediate value theorem, there exists some

$$c \in (0, 1); g(c) = 0 \Rightarrow f(c) = 1 - 2c^2$$

Continuity for Rational and Irrational Function

Functions should be continuous only at one point and to be defined everywhere. [Single Point Continuity]

e.g.

(i) $f(x) = \begin{cases} x, & \text{if } x \in Q \\ 0, & \text{if } x \notin Q \end{cases}$, is continuous only at $x = 0$.

(ii) $f(x) = \begin{cases} x, & \text{if } x \in Q \\ -x, & \text{if } x \notin Q \end{cases}$, is continuous only at $x = 0$ and defined everywhere.

(iii) $f(x) = \begin{cases} x, & \text{if } x \in Q \\ 1-x, & \text{if } x \notin Q \end{cases}$, is continuous only at $x = \frac{1}{2}$ and defined everywhere.

(iv) $f(x) = \begin{cases} x^2, & \text{if } x \in Q \\ 1, & \text{if } x \notin Q \end{cases}$, is continuous only at $x = 1$ or $x = -1$ and defined everywhere.

Exercise for Session 5

1. Examine the continuity at $x = 0$ of the sum function of the infinite series $\frac{x}{x+1} + \frac{x}{(x+1)(2x+1)} + \frac{x}{(2x+1)(3x+1)} + \ldots \infty$

2. If $g : [a, b]$ onto $[a, b]$ is continuous, then show that there is some $c \in [a, b]$ such that $g(c) = c$.

3. Show that (a) a polynomial of an odd degree has atleast one real root.

(b) a polynomial of an even degree has atleast two real roots, if it attains atleast one value opposite in sign to the coefficient of its highest-degree term.

4. Let $y_n(x) = x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \ldots + \frac{x^2}{(1+x^2)^{n-1}}$ and $y(x) = \lim_{n \to \infty} y_n(x)$. Discuss the continuity of $y_n(x)\,(n = 1, 2, 3, \ldots n)$ and $y(x)$ at $x = 0$.

Session 6

Differentiability : Meaning of Derivative

The instantaneous rate of change of function with respect to the independent variable is called derivative. Let $f(x)$ be a given function of one variable and Δx denotes a number (positive or negative) to be added to the number x.

Let Δf denotes the corresponding change of f, then $\Delta f = f(x+\Delta x) - f(x)$.

$$\frac{\Delta f}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

If $\frac{\Delta f}{\Delta x}$ approaches a limit as Δx approaches to zero, this limit is the derivative of f at the point x. The derivative of a function f is denoted by symbols such as $f'(x)$, $\frac{df}{dx}$, $\frac{d}{dx} f(x)$ or $\frac{d\,f(x)}{dx}$.

$$\Rightarrow \quad \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

The derivative evaluated at a point a is written,

$$f'(a), \left(\frac{d\,f(x)}{dx}\right)_{x=a}, (f'(x))_{x=a} \text{ etc.}$$

Existence of Derivative at $x = a$

The derivative of a function $f(x)$ exists at $x = a$, if

$$\lim_{x \to a} \frac{f(x) - f(a)}{(x-a)} \text{ exists finitely.}$$

or $$\lim_{x \to a^-} \frac{f(x) - f(a)}{(x-a)} = \lim_{x \to a^+} \frac{f(x) - f(a)}{(x-a)}$$

or $$\lim_{h \to 0} \frac{f(a-h) - f(a)}{-h} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

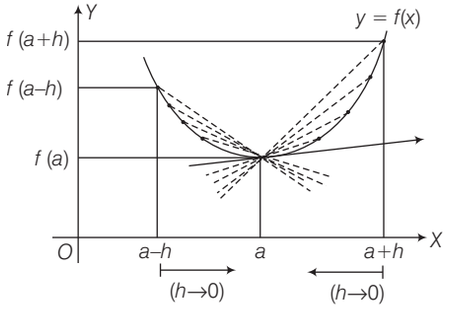


Figure 6.11

(a) **Right hand derivative** The right hand derivative of $f(x)$ at $x = a$ denoted by $f(a^+)$ is defined as

$f'(a^+) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$, provided the limit exists and is finite $(h > 0)$.

(b) **Left hand derivative** The left hand derivative of $f(x)$ at $x = a$ denoted by $f(a^-)$ is defined as

$f'(a^-) = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}$, provided the limit exists and is finite $(h > 0)$.

Hence, $f(x)$ is said to be derivable or differentiable at $x = a$.

If $f'(a^+) = f'(a^-) =$ finite quantity and it is denoted by $f'(a)$; where $f'(a) = f'(a^-) = f'(a^+)$ and it is called derivative or differential coefficient of $f(x)$ at $x = a$.

Relation between Continuity and Differentiability

If a function is differentiable at a point, then it should be continuous at that point as well and a discontinuous function cannot be differentiable. This fact is proved in the following theorem.

Theorem If a function is differentiable at a point, it is necessarily continuous at that point. But the converse is not necessarily true.

Or

$f(x)$ is differentiable at $x = c \Rightarrow f(x)$ is continuous at $x = c$.

Proof Let a function $f(x)$ be differentiable at $x = c$.

Then, $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists finitely.

Let $$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c) \qquad \ldots(i)$$

In order to prove that $f(x)$ is continuous at $x = c$, it is sufficient to show that $\lim_{x \to c} f(x) = f(c)$.

Now, $$\lim_{x \to c} f(x) = \lim_{x \to c} \left[\left(\frac{f(x) - f(c)}{x - c}\right)(x - c) + f(c)\right]$$

$$= \lim_{x \to c} \left[\left\{ \frac{f(x) - f(c)}{x - c} \right\} \cdot (x - c) \right] + f(c)$$

$$= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x \to c} (x - c) + f(c)$$

$$= f'(c) \times 0 + f(c)$$

$$= f(c)$$

Hence, $f(x)$ is continuous at $x = c$.

Converse The converse of above theorem is not necessarily true, i.e. a function may be continuous at a point but may not be differentiable at that point.
e.g. The function $f(x) = |x|$ is continuous at $x = 0$ but it is not differentiable at $x = 0$, as shown by figure.

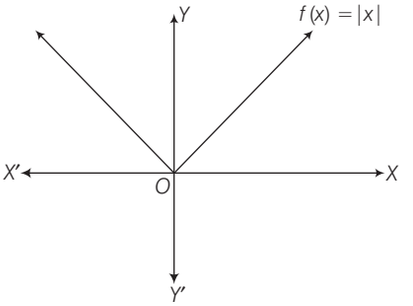


Figure 6.12

Which shows we have sharp edge at $x = 0$, hence not differentiable but continuous at $x = 0$.

Example 26 ***Show that*** $f(x) = \begin{cases} x \sin \dfrac{1}{x}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$

is continuous but not differentiable at $x = 0$.

Sol. (a) **To check continuity at $x = 0$**

Here, $f(0) = 0$

Also, $\lim_{x \to 0} f(x) = \lim_{x \to 0} x \sin \dfrac{1}{x}$

$= 0 \times$ (A finite quantity that lies between −1 to +1)

$=0$ [as $n \to 0$, $\sin \dfrac{1}{x} \to \sin \infty$, which is a finite quantity between −1 to +1]

$\therefore \lim_{x \to 0} f(x) = f(0)$ and hence, $f(x)$ is continuous.

Now,

(b) **To check differentiability at $x = 0$**

(LHD at $x = 0$)

$$= \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{h \to 0} \frac{f(0 - h) - f(0)}{(0 - h) - (0)}$$

$$= \lim_{h \to 0} \frac{-h \sin(-1/h)}{(-h)}$$

$$= -\lim_{h \to 0} \sin\left(\frac{1}{h}\right)$$

[a number which oscillates between −1 and +1]

$\therefore$ (LHD at $x = 0$) doesn't exist.

Similarly, it could be shown that RHD at $x = 0$ doesn't exist.

Hence, $f(x)$ is continuous but not differentiable.

Example 27 ***Let*** $f(x) = \begin{cases} x \exp\left[-\left(\dfrac{1}{|x|} + \dfrac{1}{x} \right) \right], & x \neq 0 \\ 0, & x = 0 \end{cases}$

Test whether (a) $f(x)$ ***is continuous at*** $x = 0$

(b) $f(x)$ ***is differentiable at*** $x = 0$.

[IIT JEE 1997]

Sol. Here, $f(x) = \begin{cases} x \exp\left[-\left(\dfrac{1}{|x|} + \dfrac{1}{x} \right) \right], & x \neq 0 \\ 0, & x = 0 \end{cases}$

$$\Rightarrow \quad f(x) = \begin{cases} xe^{-\left\{\frac{1}{x} + \frac{1}{x}\right\}}, & x > 0 \\ xe^{-\left\{\frac{1}{-x} + \frac{1}{x}\right\}}, & x < 0 \\ 0, & x = 0 \end{cases} \qquad \left[\because |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases} \right]$$

$$\Rightarrow \quad f(x) = \begin{cases} xe^{-2/x}, & x > 0 \\ x, & x < 0 \\ 0, & x = 0 \end{cases} \qquad \ldots(i)$$

(a) **To check continuity of $f(x)$ at $x = 0$**

$$\text{LHL} = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x$$

$$= \lim_{h \to 0} (0 - h) = 0$$

$$\text{RHL} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} xe^{-2/x}$$

$$= \lim_{h \to 0} \frac{h}{e^{2/h}} = 0, \; f(0) = 0$$

$\therefore f(x)$ is continuous at $x = 0$.

(b) **To check differentiability at $x = 0$**

$$\text{LHD} = Lf'(0) = \lim_{h \to 0} \frac{f(0 - h) - f(0)}{-h}, h > 0$$

$$= \lim_{h \to 0} \frac{(-h) - 0}{-h} = 1$$

$$\text{RHD} = Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{he^{\frac{-2}{h}} - 0}{h} = \lim_{h \to 0} e^{\frac{-2}{h}}$$

$$= e^{-\infty} = 0$$

$\therefore \quad Lf'(0) \neq Rf'(0)$

Therefore, $f(x)$ is not differentiable at $x = 0$.

Some Standard Results on Differentiability

Functions $f(x)$	Intervals in which $f(x)$ is differentiable
1. Polynomial	$(-\infty$ to $\infty)$
2. Exponential $(a^x, a > 0)$	$(-\infty$ to $\infty)$
3. Constant	$(-\infty$ to $\infty)$
4. Logarithmic	Each point in its domain
5. Trigonometric	Each point in its domain
6. Inverse trigonometric	Each point in its domain

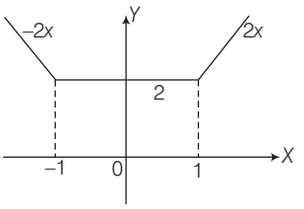
Example 28 Let $f(x) = |x-1| + |x+1|$. Discuss the continuity and differentiability of the function.

Sol. Here, $f(x) = |x-1| + |x+1|$

$$\Rightarrow \quad f(x) = \begin{cases} (x-1)+(x+1), & \text{when} \quad x > 1 \\ -(x-1)+(x+1), & \text{when} \quad -1 \le x \le 1 \\ -(x-1)-(x+1), & \text{when} \quad x < -1 \end{cases}$$

$$\Rightarrow \quad f(x) = \begin{cases} 2x, & \text{when} \quad x > 1 \\ 2, & \text{when} \quad -1 \le x \le 1 \\ -2x, & \text{when} \quad x < -1 \end{cases}$$

Graphically The graph of the function is shown below.



From the graph, it is clear that the function is continuous at all real x, also differentiable at all real x except at $x = \pm 1$. Since, it has sharp edges at $x = -1$ and $x = 1$.

At $x = 1$ we see that the slope from the right, i.e. RHD = 2, while slope from the left, i.e. LHD = 0.

Similarly, at $x = -1$ it is clear that RHD = 0, while LHD = -2

Aliter In this method, first of all we differentiate the function and from the derivative equality sign should be removed from doubtful points.

Here, $f'(x) = \begin{cases} -2, & x < -1 \\ 0, & -1 < x < 1 \\ 2, & x > 1 \end{cases}$ [no equality on -1 and $+1$]

Now, at $x = 1$, $f'(1^+) = 2$, while $f'(1^-) = 0$ and

at $x = -1$, $f'(-1^+) = 0$, while $f'(-1^-) = -2$

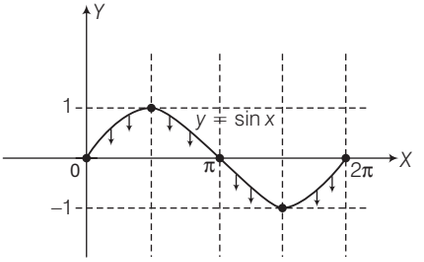
Thus, $f(x)$ is not differentiable at $x = \pm 1$.

Remark

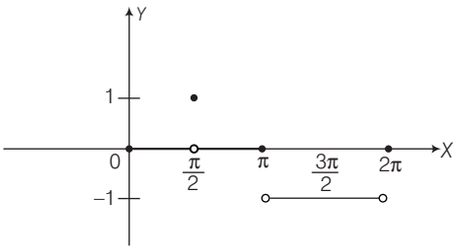
This method is not applicable when function is discontinuous.

Example 29 ***Discuss the continuity and differentiability for*** $f(x) = [\sin x]$ ***when*** $x \in [0, 2\pi]$***; where*** $[\cdot]$ ***denotes the greatest integer function*** x***.***

Sol. In last chapter, we have discussed the plotting of curves, so $y = [\sin x]$ could be plotted as



Or



which shows $f(x) = [\sin x]$ is discontinuous for $x = \frac{\pi}{2}, \pi, 2\pi$, when $x \in [0, 2\pi]$ and $f(x)$ is not differentiable at $x = \frac{\pi}{2}, \pi, 2\pi$.

As we know, function is neither differentiable nor continuous at those points for which graph is broken.

Example 30 *If* $f(x)=\{|x|-|x-1|\}^2$, *draw the graph of* $f(x)$ *and discuss its continuity and differentiability of* $f(x)$.

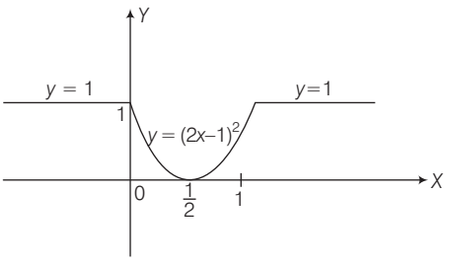
Sol. We know, $|x|-|x-1|=\begin{cases}-x+x-1, & x<0\\ x+x-1, & 0\le x<1\\ x-(x-1), & 1\le x\end{cases}$

$$\Rightarrow \quad |x|-|x-1|=\begin{cases}-1, & x<0\\ 2x-1, & 0\le x<1\\ 1, & 1\le x\end{cases}$$

$$\therefore \quad f(x)=\{|x|-|x-1|\}^2$$

$$=\begin{cases}1, \text{ when } x<0 \text{ or } x\ge 1\\ (2x-1)^2, \text{ when } 0\le x<1\end{cases}$$

Graphically $f(x)$ could be shown as



From above figure, it is clear that $f(x)$ is continuous for $x\in R$; but $f(x)$ is not differentiable at $x=0, 1$.

$\Rightarrow$ $f(x)$ is continuous for all $x\in R$.

$\Rightarrow$ $f(x)$ is differentiable for all $x\in R-\{0,1\}$.

Example 31 *If* $f(x)=\begin{cases}x-3, & x<0\\ x^2-3x+2, & x\ge 0\end{cases}$ *and let* $g(x)=f(|x|)+|f(x)|$. *Discuss the differentiability of* $g(x)$.

Sol. $f(|x|)=\begin{cases}|x|-3, & |x|<0\\ |x|^2-3|x|+2, & |x|\ge 0\end{cases}$

Where, $|x|<0$ is not possible, thus neglecting, we get

$$f(|x|)=\{|x|^2-3|x|+2, \quad |x|\ge 0$$

$$f(|x|)=\begin{cases}x^2+3x+2, & x<0\\ x^2-3x+2, & x\ge 0\end{cases} \qquad \ldots(i)$$

Again, $|f(x)|=\begin{cases}|x-3|, & x<0\\ |x^2-3x+2|, & x\ge 0\end{cases}$

$$|f(x)|=\begin{cases}(3-x), & x<0\\ (x^2-3x+2), & 0\le x<1\\ -(x^2-3x+2), & 1\le x<2\\ (x^2-3x+2), & 2\le x\end{cases} \qquad \ldots(ii)$$

Now, from Eqs. (i) and (ii), $g(x)=f(|x|)+|f(x)|$

$$g(x)=\begin{cases}x^2+2x+5, & x<0\\ 2x^2-6x+4, & 0\le x<1\\ 0, & 1\le x<2\\ 2x^2-6x+4, & x\ge 2\end{cases}$$

and $g'(x)=\begin{cases}2x+2, & x<0\\ 4x-6, & 0<x<1\\ 0, & 1<x<2\\ 4x-6, & x>2\end{cases}$

Therefore, $g(x)$ is continuous in $R-\{0\}$ and $g(x)$ is differentiable in $R-\{0,1,2\}$.

Example 32 *Let* $f(x)=[n+p\sin x], x\in(0,\pi), n\in Z$ *and* p *is a prime number, where* $[\cdot]$ *denotes the greatest integer function. Then, find the number of points, where* $f(x)$ *is not differentiable.*

Sol. Here, $f(x)=[n+p\sin x]$ is not differentiable at those points where $n+p\sin x$ is an integer.

As, p is a prime number.

$\Rightarrow$ $n+p\sin x$ is an integer if $\sin x=1, -1, r/p$

i.e. $x=\frac{\pi}{2}, -\frac{\pi}{2}, \sin^{-1}\frac{r}{p}, \pi-\sin^{-1}\frac{r}{p}$

where $0\le r\le p-1$

But $x\ne -\frac{\pi}{2}, 0$

$\therefore$ Function is not differentiable at

$$x=\frac{\pi}{2}, \sin^{-1}\frac{r}{p}, \pi-\sin^{-1}\frac{r}{p}$$

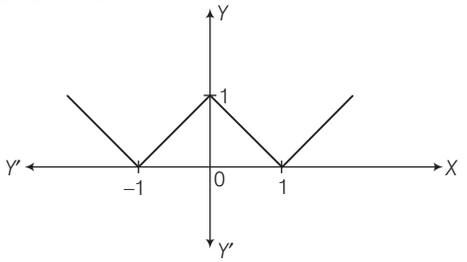
where $0<r\le p-1$

So, the required number of points are

$$=1+2(p-1)=2p-1.$$

Example 33 *If* $f(x)=||x|-1|$, *then draw the graph of* $f(x)$ *and* $fof(x)$ *and also discuss their continuity and differentiability. Also, find derivative of* $(fof)^2$ *at* $x=\frac{3}{2}$.

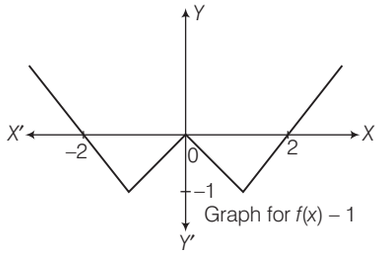
Sol. The graph of $f(x)$ is shown as



It is clear from the graph that, $f(x)$ is continuous for all x, but $f(x)$ is not differentiable at $x = \{-1, 0, 1\}$.

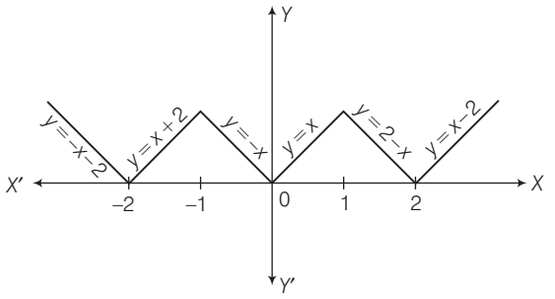
Now, $fof(x) = ||f(x)| - 1| = |f(x) - 1|$ [as $f(x) \geq 0$ for all x]

Now, if $f(x) \to f(x) - 1$, shift the graph one unit below the X-axis, i.e., as shown;



Thus, for graph of $fof(x) = |f(x) - 1|$ is taking image of the graph of $f(x) - 1$ below X-axis and leaving the portion above Y-axis as it is.

$\therefore$ Graph for $fof(x)$ is shown as



which is clearly continuous for all $x \in R$, but not differentiable at $x = \{-2, -1, 0, 1, 2\}$.

Also, $fof(x) = 2 - x, 1 \leq x \leq 2$

$\therefore$ $(fof)^2 = (2 - x)^2$, when $1 \leq x \leq 2$

$\Rightarrow$ $\dfrac{d}{dx}(fof)^2 = 2(2 - x)(-1)$, when $1 \leq x \leq 2$

$\therefore$ $\dfrac{d}{dx}(fof)^2$ (when $x = 3/2$) $= -2(2 - 3/2) = -1$

$\Rightarrow$ $\dfrac{d}{dx}\{(fof)^2\}_{\text{at } x = 3/2} = -1$

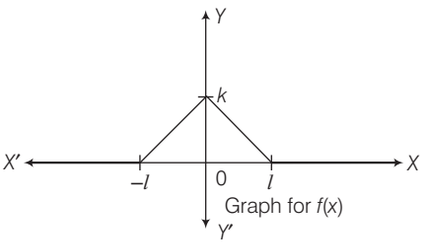
Example 34 Draw the graph of the function $g(x) = f(x + l) + f(x - l)$, where

$$f(x) = \begin{cases} k\left\{1 - \dfrac{|x|}{l}\right\}, & \text{for } |x| \leq l \\ 0, & \text{for } |x| > l \end{cases}$$

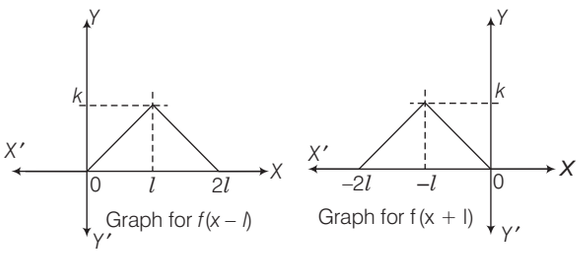
Also, discuss the continuity and differentiability of the function $g(x)$.

Sol. We have, $f(x) = \begin{cases} k\left(1 - \dfrac{|x|}{l}\right), & |x| \leq l \\ 0, & |x| > l \end{cases}$

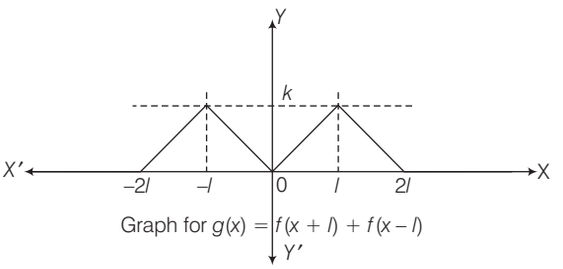
The graph of $f(x)$ is shown as



Now, to plot $g(x) = f(x + l) + f(x - l)$, we shall first plot the graphs of $f(x + l)$ and $f(x - l)$ which are given as



Thus, the graph of $g(x) = f(x + l) + f(x - l)$ is obtained by adding graph of $f(x + l)$ and $f(x - l)$ as



From the above figure, $g(x)$ is continuous for all $x \in R$ and $g(x)$ is differentiable for all $x \in R - \{\pm 2l, \pm l, 0\}$.

Example 35 Let $f(x)=\begin{cases}\int_0^x\{5+|1-t|\}dt, & \text{if } x>2\\ 5x+1, & \text{if } x\le 2\end{cases}$.

Test $f(x)$ for continuity and differentiability for all real x.

Sol. Here, $f(x)$ for $x>2$.

$$f(x)=\int_0^x\{5+|1-t|\}dt$$

$$=\int_0^1(5+1-t)dt+\int_1^x(5+t-1)dt \quad [\text{since } x>2]$$

$$=\left(6t-\frac{t^2}{2}\right)_0^1+\left(4t+\frac{t^2}{2}\right)_1^x.$$

$$=6-\frac{1}{2}+4x+\frac{x^2}{2}-4-\frac{1}{2}$$

$$=1+4x+\frac{x^2}{2}$$

$$\therefore \quad f(x)=\begin{cases}1+4x+x^2/2, & \text{if } x>2\\ 5x+1, & \text{if } x\le 2\end{cases}$$

$\therefore$ RHL (at $x=2$)

$$=\lim_{h\to 0}\left\{1+4(2+h)+\frac{(2+h)^2}{2}\right\}$$

$$=1+8+2=11$$

and LHL (at $x=2$) $=\lim_{h\to 0}\{5(2-h)+1\}$

$$=10+1=11$$

$\therefore$ $f(x)$ is continuous at $x=2$. [as $f(2)=11$]

Also, RHD (at $x=2$)

$$Rf'(2)=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

$$=\lim_{h\to 0}\frac{1+4(2+h)+\frac{(2+h)^2}{2}-11}{h}$$

$$=\lim_{h\to 0}\frac{11+6h+\frac{h^2}{2}-11}{h}=6$$

LHD (at $x=2$)

$$Lf'(2)=\lim_{h\to 0}\frac{f(2-h)-f(2)}{-h}$$

$$=\lim_{h\to 0}\frac{5(2-h)+1-11}{-h}$$

$$=\lim_{h\to 0}\frac{11-5h-11}{-h}=5$$

$\therefore$ $f(x)$ is not differentiable at $x=2$.

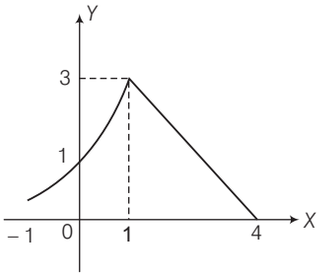
Thus, $f(x)$ is continuous for all $x\in R$ and differentiable for all $x\in R-\{2\}$.

Example 36 Draw the graph of the function and discuss the continuity and differentiability at $x=1$ for,

$$f(x)=\begin{cases}3^x, & \text{when } -1\le x\le 1\\ 4-x, & \text{when } 1<x<4\end{cases}$$

Sol. Here, $f(x)=\begin{cases}3^x, & \text{when } -1\le x\le 1\\ 4-x, & \text{when } 1<x<4\end{cases}$

is shown below graphically.



From the graph, it is clear that it is continuous for all x in $[-1, 4)$ and not differentiable at $x=1$.

Because at $x=1$, LHD >0, while RHD <0

Mathematically,

$$\text{RHD}=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$=\lim_{h\to 0}\frac{4-(1+h)-3}{h}=-1$$

$$\text{LHD}=\lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$$

$$=\lim_{h\to 0}\frac{3^{(1-h)}-3}{-h}=\lim_{h\to 0}3\left(\frac{3^{-h}-1}{-h}\right)$$

$$=3\log_e 3$$

Since, LHD $\neq$ RHD, $f(x)$ is not differentiable at $x=1$.

But $f(x)$ is continuous at $x=1$, because the derivatives from both the sides are finite and definite.

Aliter

The given function is continuous.

Hence, $$f'(x)=\begin{cases}3^x\log 3, & -1\le x<1\\ -1, & 1<x<4\end{cases}$$

Here, $$f'(1^+)=-1$$

and $$f'(1^-)=\lim_{x\to 1^-}3^x\log 3=3\log 3$$

which shows, $$f'(1^+)\neq f'(1^-)$$

Therefore, $f(x)$ is not differentiable at $x=1$.

Example 37 Match the column I with column II.

	Column I		Column II
(i)	$\sin(\pi[x])$	(A)	differentiable everywhere
(ii)	$\sin\{(x-[x])\pi\}$	(B)	nowhere differentiable
		(C)	not differentiable at -1 and $+1$

where [ ] denotes greatest integral function. ***[IIT JEE 1992]***

Sol. (i) We know, $[x] \in I, \forall\ x \in R$

$\therefore\ \sin(\pi[x]) = \sin(I\pi) = 0, \forall\ x \in R$

By theory, we know that every constant function is differentiable in its domain.

Thus, $\sin(\pi[x])$ is differentiable everywhere.

Hence, (i) $\leftrightarrow$ (A)

(ii) Again, $f(x) = \sin\{(x-[x])\pi\}$

Here, we know $x - [x] = \{x\}$,

then $\pi(x-[x]) = \pi\{x\}$

which is not differentiable at integral points.

$\therefore\ f(x) = \sin\{\pi(x-[x])\}$ is not differentiable at $x \in I$. [integers]

Hence, (ii) $\leftrightarrow$ (C)

Example 38 Fill in the blank, in the statement given below.

Let $f(x) = x|x|$. The set of points, where $f(x)$ is twice differentiable is ..... . ***[IIT JEE 1992]***

Sol. The function $f(x) = x|x|$ can be written as,

$$f(x) = \begin{cases} x(x), & \text{if } x \geq 0 \\ x(-x), & \text{if } x < 0 \end{cases}$$

$$\Rightarrow \quad f(x) = \begin{cases} x^2, & \text{if } x \geq 0 \\ -(x)^2, & \text{if } x < 0 \end{cases}$$

$f(x)$ is differentiable, $\forall\ x \in R$.

Again, $$f'(x) = \begin{cases} 2x, & \text{if } x > 0 \\ -2x, & \text{if } x < 0 \end{cases}$$

$$f''(x) = \begin{cases} 2, & \text{if } x > 0 \\ -2, & \text{if } x < 0 \end{cases}$$

Here, $f''(x)$ is discontinuous at $x = 0$

$\therefore\ f''(x)$ is not differentiable at $x = 0$.

Hence, $f(x)$ is twice differentiable, $\forall\ x \in R - \{0\}$.

Example 39 The function $f(x) = (x^2-1)|x^2-3x+2| + \cos(|x|)$ is not differentiable at

(a) -1 (b) 0 (c) 1 (d) 2

[IIT JEE 1992]

Sol. The function $f(x) = (x^2-1)|x^2-3x+2| + \cos(|x|)$ ...(i)

Here, $|x|$ is not differentiable at $x = 0$, but

$$\cos(|x|) = \begin{cases} \cos(-x), & x < 0 \\ \cos(x), & x \geq 0 \end{cases}$$

$$\Rightarrow \quad \cos(|x|) = \begin{cases} \cos(x), & x \geq 0 \\ \cos(x), & x < 0 \end{cases}$$

$\therefore\ \cos(|x|)$ is differentiable at $x = 0$ ...(ii)

Again, $|x^2-3x+2| = |(x-1)(x-2)|$

$$= \begin{cases} (x-1)(x-2), & \text{if } x < 1 \\ -(x-1)(x-2), & \text{if } 1 \leq x < 2 \\ (x-1)(x-2), & \text{if } x \geq 2 \end{cases} \quad \text{...(iii)}$$

So, $$f(x) = \begin{cases} (x^2-1)(x-1)(x-2)+\cos x, & \text{if } -\infty < x < 1 \\ -(x^2-1)(x-1)(x-2)+\cos x, & \text{if } 1 \leq x < 2 \\ (x^2-1)(x-1)(x-2)+\cos x, & \text{if } 2 \leq x < \infty \end{cases}$$

Now, to check differentiability at $x = 1, 2$ [using shortcut method]

$$f'(x) = \begin{cases} (x^2-1)(2x-3)+(2x)(x^2-3x+2)-\sin x, & -\infty < x < 1 \\ -(x^2-1)(2x-3)-(2x)(x^2-3x+2)-\sin x, & 1 \leq x < 2 \\ (x^2-1)(2x-3)+(2x)(x^2-3x+2)-\sin x, & 2 \leq x < \infty \end{cases}$$

Thus, for $f'(1)$, we have

$$f'(1) = \begin{cases} -\sin 1, & x < 1 \\ -\sin 1, & x > 1 \end{cases}$$

Thus, $f(x)$ is differentiable at $x = 1$

Also, $$f'(2) = \begin{cases} -3-\sin 2, & x < 2 \\ 3-\sin 2, & x > 2 \end{cases}$$

Thus, $f(x)$ is not differentiable at $x = 2$.

Hence, (d) is the correct answer.

Example 40 ***If*** $f(x) = \sum_{r=1}^{n} a_r |x|^r$, ***where*** a_i's ***are real constants, then*** $f(x)$ ***is***

(a) continuous at $x = 0$, for all a_i

(b) differentiable at $x = 0$, for all $a_i \in R$

(c) differentiable at $x = 0$, for all $a_{2k+1} = 0$

(d) None of the above

Sol. We know that, $|x|^r, r = 0, 1, 2, \ldots$ are all continuous everywhere.

$\therefore\ f(x) = \sum_{r=1}^{n} a_r |x|^r$ is everywhere continuous.

Since, $|x|, |x|^3, |x|^5, \ldots$ are not differentiable at $x = 0$, whereas $|x|^2, |x|^4, \ldots$ are everywhere differentiable.

$\therefore\ f(x)=\sum_{r=1}^{n} a_r |x|^r$ is not differentiable at $x=0$, if anyone of $a_1, a_3, a_5, \ldots$ is non-zero.

Thus, for $f(x)$ to be differentiable at $x=0$, we must have $a_1 = a_3 = a_5 \ldots = 0$

i.e. $a_{2k+1} = 0$

Hence, (a) and (c) are the correct answers.

Example 41 ***Let f and g be differentiable functions satisfying*** $g'(a)=2$, $g(a)=b$ ***and*** $fog = I$ ***(Identity function***). ***Then,*** $f'(b)$ ***is equal to***

(a) 2 (b) $\frac{2}{3}$ (c) $\frac{1}{2}$ (d) None of these

Sol. We have, $fog = I$

$\Rightarrow \quad f\{g(x)\} = x$, for all $x \in R$

$\therefore \quad f'\{g(x)\}\cdot g'(x) = 1,$ for all $x \in R$

$\Rightarrow \quad f'(g(a)) = \frac{1}{g'(a)} \Rightarrow f'(b) = \frac{1}{2}$

Hence, (c) is the correct answer.

Example 42 If $f(x)=\frac{x}{1+(\log x)(\log x)\ldots\infty}, \forall\ x \in [1, 3]$ is non-differentiable at $x = k$. Then, the value of $[k^2]$, is (where $[\cdot]$ denotes greatest integer function).

(a) 5 (b) 6 (c) 7 (d) 8

Sol. Let $g(x) = (\log x)\cdot(\log x)\ldots\infty$

$$\therefore \quad g(x)=\begin{cases}1, & x=e\\ 0, & 1\le x<e\\ \infty, & x>e\end{cases}$$

$$\therefore \quad f(x)=\begin{cases}x, & 1\le x<e\\ x/2, & x=e\\ 0, & e<x\le 3\end{cases}$$

$\Rightarrow f(x)$ is neither continuous nor differentiable at $x = e = k$.

$\therefore \quad [k^2] = 7$

Hence, (c) is the correct answer.

Example 43 If $f(x)=|1-x|$, then the points where $\sin^{-1}(f(|x|))$ is non-differentiable, are

(a) $\{0, 1\}$ (b) $\{0, -1\}$

(c) $\{0, 1, -1\}$ (d) None of these

Sol. Here, $f(x) = |1-x|$

$$f(|x|) = |1-|x||=\begin{cases}x-1, & x>1\\ 1-x, & 0<x<1\\ 1+x, & -1\le x\le 0\\ -x-1, & x<-1\end{cases}$$

Here, for domain of $\sin^{-1}(f(|x|))$

$-1 \le f(|x|) \le 1$

$\Rightarrow \quad |1-|x||\le 1$

$\Rightarrow \quad -1 \le 1-|x|\le 1$

$\Rightarrow \quad -2 \le -|x|\le 0$

$\Rightarrow \quad 2 \ge |x| \ge 0$

$\therefore \quad x \in [-2, 2]$

Then, $\sin^{-1}(f(|x|))$, for $x \in [-2, 2]$ is not differentiable at $x = \{0, 1, -1\}$.

Hence, (c) is the correct answer.

Exercise for Session 6

1. If a function $f(x)$ is defined as $f(x)=\begin{cases}-x, & x<0\\ x^2, & 0\le x\le 1\\ x^2-x+1, & x>1\end{cases}$, then

(a) $f(x)$ is differentiable at $x=0$ and $x=1$

(b) $f(x)$ is differentiable at $x=0$ but not at $x=1$

(c) $f(x)$ is not differentiable at $x=1$ but not at $x=0$

(d) $f(x)$ is not differentiable at $x=0$ and $x=1$

2. If $f(x)=x^3\operatorname{sgn}(x)$, then

(a) f is differentiable at $x=0$

(b) f is continuous but not differentiable at $x=0$

(c) $f'(0^-)=1$

(d) None of these

3. Which of the following is continuous everywhere in its domain but has atleast one point where it is not differentiable?

(a) $f(x)=x^{1/3}$

(b) $f(x)=\dfrac{|x|}{x}$

(c) $f(x)=e^{-x}$

(d) $f(x)=\tan x$

4. If $f(x)=\begin{cases}x+\{x\}+x\sin\{x\}, & \text{for } x\neq 0\\ 0, & \text{for } x=0\end{cases}$, where $\{x\}$ denotes the fractional part function, then

(a) f is continuous and differentiable at $x=0$

(b) f is continuous but not differentiable at $x=0$

(c) f is continuous and differentiable at $x=2$

(d) None of these

5. If $f(x)=\begin{cases}x\left(\dfrac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{1/x}}\right), & x\neq 0\\ 0, & x=0\end{cases}$, then at $x=0$, $f(x)$ is

(a) differentiable

(b) not differentiable

(c) $f'(0^+)=-1$

(d) $f'(0^-)=1$

Session 7

Differentiability in an Interval

(i) A function $f(x)$ defined in an open interval (a, b) is said to be differentiable or derivable in open interval (a, b), if it is differentiable at each point of (a, b).

(ii) A function $f(x)$ defined in a close interval $[a, b]$ is said to be differentiable or derivable at the end points a and b, if it is differentiable from the right at a and from the left at b. In other words, $\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}$ and $\lim_{x \to b^-} \frac{f(x) - f(b)}{x - b}$ both exist.

Example 44 Discuss the differentiability of $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

Sol. We have, $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$$\Rightarrow f'(x) = \frac{1}{\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}} \times \frac{d}{dx}\left(\frac{2x}{1+x^2}\right)$$

$$= \frac{(1+x^2)}{\sqrt{(1+x^2)^2 - 4x^2}} \times \left[\frac{(1+x^2)(2) - 2x(2x)}{(1+x^2)^2}\right]$$

$$= \frac{(1+x^2)}{\sqrt{1+2x^2+x^4-4x^2}} \times \frac{(2+2x^2-4x^2)}{(1+x^2)^2}$$

$$= \frac{(1+x^2)}{\sqrt{1-2x^2+x^4}} \times \frac{(2-2x^2)}{(1+x^2)^2}$$

$$= \frac{(1+x^2)}{\sqrt{(1-x^2)^2}} \times \frac{(2-2x^2)}{(1+x^2)^2} = \frac{(1+x^2)}{|1-x^2|} \times \frac{2(1-x^2)}{(1+x^2)^2}$$

[since $1 + x^2 \neq 0$]

$$\Rightarrow f'(x) = \frac{1}{|(1-x^2)|} \times \frac{2(1-x^2)}{(1+x^2)} \quad \ldots(i)$$

Here, in Eq. (i), $f'(x)$ exists only if, $|1-x^2| \neq 0$

$$\Rightarrow \quad 1 - x^2 \neq 0$$

$$\Rightarrow \quad x^2 \neq 1 \Rightarrow x \neq \pm 1$$

Thus, $f'(x)$ exists only, if $x \in R - \{-1, 1\}$.

$\therefore$ $f(x)$ is differentiable for all $x \in R - \{1, -1\}$.

Remark

The above example, can also be solved as follows

$$y = f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right), \text{ let } x = \tan\theta$$

$$\therefore \quad y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) \Rightarrow \quad y = \sin^{-1}(\sin 2\theta)$$

$$\therefore \quad y = 2\theta \quad \text{or} \quad y = 2\tan^{-1}x$$

$\frac{dy}{dx} = \frac{2}{1+x^2}$, which states $f'(x)$ exists for all $x \in R$. "Which is wrong as we have not checked the domain of $f(x)$." So, students are advised to solve these problems carefully, while applying this method.

Example 45 ***Let [ ] denotes the greatest integer function and $f(x) = [\tan^2 x]$, then*** ***[IIT JEE 1993]***

(a) $\lim_{x \to 0} f(x)$ doesn't exist (b) $f(x)$ is continuous at $x = 0$

(c) $f(x)$ is not differentiable at $x = 0$

(d) $f'(0) = 1$

Sol. Here, [ ] denotes the greatest integral function.

Thus, $-45° < x < 45°$

$$\Rightarrow \quad \tan(-45°) < \tan x < \tan(45°)$$

$$\Rightarrow \quad -1 < \tan x < 1 \Rightarrow 0 < \tan^2 x < 1$$

Since, $f(x) = [\tan^2 x] = 0$

Therefore, $f(x)$ is zero for all values of x from $(-45°)$ to $(45°)$. Thus, $f(x)$ exists when $x \to 0$ and also it is continuous at $x = 0$, $f(x)$ is differentiable at $x = 0$ and has a value 0. (i.e. $f(0) = 0$).

Hence, (b) is the correct answer.

Theorems of Differentiability

Theorem 1 If $f(x)$ and $g(x)$ are both derivable at $x = a$, $f(x) \pm g(x)$, $f(x) \cdot g(x)$ and $\frac{f(x)}{g(x)}$ will also be derivable at $x = a$ $\left\{\text{only if } g(a) \neq 0 \text{ for } \frac{f(x)}{g(x)}\right\}$.

Theorem 2 If $f(x)$ is derivable at $x = a$ and $g(x)$ is not differentiable at $x = a$, then $f(x) \pm g(x)$ will not be derivable at $x = a$.

e.g. $f(x) = \cos|x|$ is derivable at $x = 0$ and $g(x) = |x|$ is not derivable at $x = 0$.

Then, $\cos|x|+|x|$ is not derivable at $x=0$.

However, nothing can be said about the product function, as in this case

$f(x)=x$ is derivable at $x=0$

$g(x)=|x|$ is not derivable at $x=0$

But, $f(x)\cdot g(x)=\begin{cases} x^2, & \text{if } x\geq 0 \\ -x^2, & \text{if } x<0 \end{cases}$

which is derivable at $x=0$.

Theorem 3 If both $f(x)$ and $g(x)$ are non-derivable, then nothing can be said about the sum/difference/product function.

e.g. $f(x)=\sin|x|$, not derivable at $x=0$

$g(x)=|x|$, not derivable at $x=0$

Then, the function

$F(x)=\sin|x|+|x|$, not derivable at $x=0$

$G(x)=\sin|x|-|x|$, derivable at $x=0$

Theorem 4 If $f(x)$ is derivable at $x=a$ and $f(a)=0$ and $g(x)$ is continuous at $x=a$.

Then, the product function $F(x)=f(x)\cdot g(x)$ will be derivable at $x=a$.

Proof $F'(a^+)=\lim\limits_{h\to 0}\dfrac{f(a+h)\cdot g(a+h)-0}{h}=f'(a)\cdot g(a)$

$$F'(a^-)=\lim_{h\to 0}\frac{f(a-h)\cdot g(a-h)-0}{-h}=f'(a)\cdot g(a)$$

$\therefore$ Derivable at $x=a$.

Theorem 5 Derivative of a continuous function need not be a continuous function.

e.g. $$f(x)=\begin{cases} x^2\cdot\sin\left(\dfrac{1}{x}\right), & \text{if } x\neq 0 \\ 0, & \text{if } x=0 \end{cases}$$

Here, $f(0^+)=0$ and $f(0^-)=0$

$\therefore$ Continuous at $x=0$.

and $$f'(x)=\begin{cases} 2x\cdot\sin\dfrac{1}{x}-x^2\cdot\cos\left(\dfrac{1}{x}\right)\cdot\dfrac{1}{x^2}, & x\neq 0 \\ 0, & x=0 \end{cases}$$

$\Rightarrow$ $f'(x)$ is not continuous at $x=0$.

$\left[\text{as } \lim\limits_{x\to 0} f'(x) \text{ doesn't exist}\right]$

Remark

One must remember the formula which we can write as

$$\max\{f(x),g(x)\}=\frac{f(x)+g(x)}{2}+\left|\frac{f(x)-g(x)}{2}\right|$$

$$\min\{f(x),g(x)\}=\frac{f(x)+g(x)}{2}-\left|\frac{f(x)-g(x)}{2}\right|$$

Example 46 ***Let*** $h(x)=\min\{x,x^2\}$ ***for every real number of*** x***. Then,*** ***[IIT JEE 1998]***

(a) h is not continuous for all x

(b) h is differentiable for all x

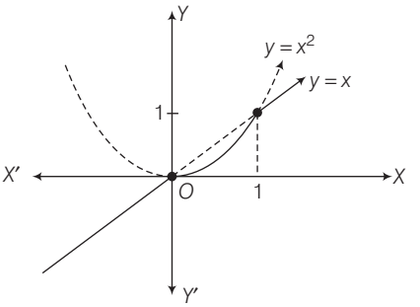
(c) $h'(x)=1$ for all x

(d) h is not differentiable at two values of x

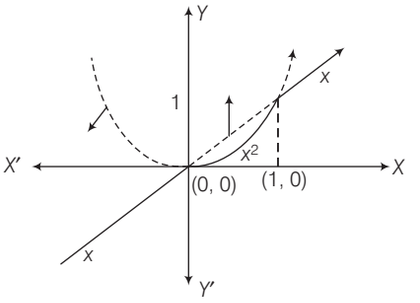
Sol. Here, $h(x)=\min\{x,x^2\}$ can be drawn on graph in two steps.

(a) Draw the graph of $y=x$ and $y=x^2$ also find their point of intersection.

Clearly, $x=x^2 \Rightarrow x=0,1$



(b) To find $h(x)=\min\{x,x^2\}$ neglecting the graph above the point of intersection, we get



Thus, from the above graph,

$$h(x)=\begin{cases} x, & x\leq 0 \text{ or } x\geq 1 \\ x^2, & 0\leq x\leq 1 \end{cases}$$

which shows $h(x)$ is continuous for all x. But not differentiable at $x=\{0,1\}$.

Thus, $h(x)$ is not differentiable at two values of x.

Hence, (d) is the correct answer.

Example 47 Let $f:R\to R$ be a function defined by $f(x)=\max\{x,x^3\}$. The set of all points where $f(x)$ is not differentiable, is ***[IIT JEE 2001]***

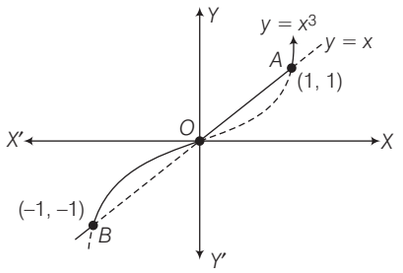
(a) $\{-1,1\}$ (b) $\{-1,0\}$

(c) $\{0,1\}$ (d) $\{-1,0,1\}$

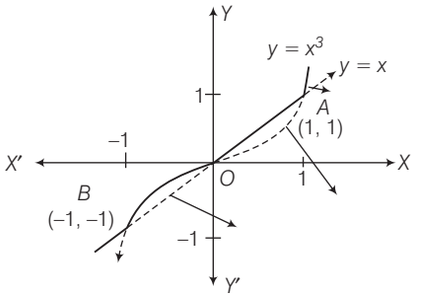
Sol. $f(x) = \max\{x, x^3\}$. Consider the graph separately of $y = x^3$ and $y = x$ and find their point of intersection;

Clearly, $x^3 = x$

$\Rightarrow$ $x = 0, 1, -1$



Now, to find $f(x) = \max\{x, x^3\}$ neglecting the graph below the point of intersection, we get the required graph of $f(x) = \max\{x, x^3\}$.



Thus, from above graph, $f(x) = \begin{cases} x, & \text{if } x \in (-\infty, -1] \cup [0,1] \\ x^3, & \text{if } x \in [-1, 0] \cup [1, \infty) \end{cases}$

which shows $f(x)$ is not differentiable at 3 points, i.e. $x = \{-1, 0, 1\}$. (Due to sharp edges)

Hence, (d) is the correct answer.

Example 48 *Let $f(x)$ be a continuous function, $\forall\ x \in R, f(0) = 1$ and $f(x) \neq x$ for any $x \in R$, then show $f(f(x)) > x, \forall\ x \in R^+$.*

Sol. Let $g(x) = f(x) - x$

So, $g(x)$ is continuous and $g(0) = f(0) - 0$.

$\Rightarrow$ $g(0) = 1$

Now, it is given that $g(x) \neq 0$ for any $x \in R$

[as $f(x) \neq x$ for any $x \in R$]

So, $g(x) > 0, \forall\ x \in R^+$

i.e. $f(x) > x, \forall\ x \in R^+$

$\Rightarrow$ $f(f(x)) > f(x) > x, \ \forall\ x \in R^+$

or $f(f(x)) > x, \forall\ x \in R^+$

Example 49 *The total number of points of non-differentiability of $f(x) = \max\left\{\sin^2 x, \cos^2 x, \frac{3}{4}\right\}$ in [0, 10π], is*

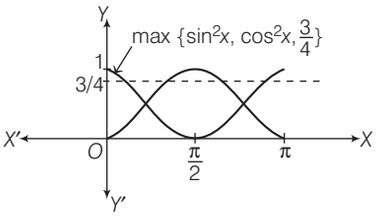
(a) 40

(b) 30

(c) 20

(d) 10

Sol. Here, $f(x) = \max\left\{\sin^2 x, \cos^2 x, \frac{3}{4}\right\}$



Since, $\sin^2 x$ and $\cos^2 x$ are periodic with period π and in $[0, \pi]$, there are four points of non-differentiability of $f(x)$.

$\therefore$ In $[0, 10\pi]$, there are 40 points of non-differentiability.

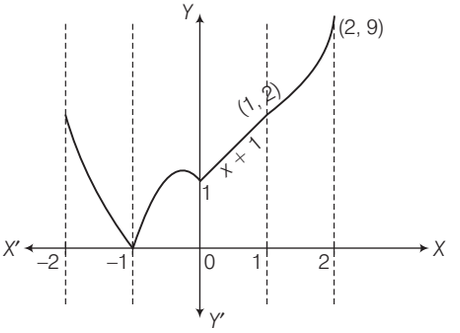
Hence, (a) is the correct answer.

Example 50 If $f(x) = |x + 1|\{|x| + |x - 1|\}$, then draw the graph of $f(x)$ in the interval $[-2, 2]$ and discuss the continuity and differentiability in $[-2, 2]$.

Sol. Here, $f(x) = |x + 1|\{|x| + |x - 1|\}$

$$f(x) = \begin{cases} (x+1)(2x-1), & -2 \le x < -1 \\ -(x+1)(2x-1), & -1 \le x < 0 \\ (x+1), & 0 \le x < 1 \\ (x+1)(2x-1), & 1 \le x \le 2 \end{cases}$$

Thus, the graph of $f(x)$ is



Clearly, continuous for $x \in R$ and has differentiability for $x \in R - \{-1, 0, 1\}$

Example 51 If the function

$$f(x)=\left[\frac{(x-2)^3}{a}\right]\sin(x-2)+a\cos(x-2),$$

(where [ ] denotes the greatest integer function) is continuous and differentiable in $(4,6)$*, then*

(a) $a\in[8,64]$ (b) $a\in(0,8]$

(c) $a\in[64,\infty)$ (d) None of these

Sol. We have, $x\in(4,6)\Rightarrow 2<x-2<4$

$$\Rightarrow \quad \frac{8}{a}<\frac{(x-2)^3}{a}<\frac{64}{a} \qquad [\because a>0]$$

For $f(x)$ to be continuous and differentiable in $(4,6)$, $\left[\frac{(x-2)^3}{a}\right]$ must attain a constant value for $x\in(4,6)$.

Clearly, this is possible only when $a\geq 64$.

In that case, we have

$f(x)=a\cos(x-2)$, which is continuous and differentiable

$\therefore \quad a\in[64,\infty)$

Hence, (c) is the correct answer.

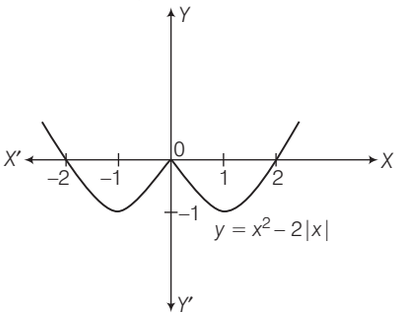
Example 52 If $f(x)=x^2-2|x|$ and

$$g(x)=\begin{cases}\min\{f(t):-2\leq t\leq x, & -2\leq x\leq 0\}\\ \max\{f(t):0\leq t\leq x, & 0\leq x\leq 3\}\end{cases}$$

(i) Draw the graph of $f(x)$ and discuss its continuity and differentiability.

(ii) Find and draw the graph of $g(x)$. Also, discuss the continuity.

Sol. (i) Graph of $f(x)=\begin{cases}x^2-2x, & x\geq 0\\ x^2+2x, & x<0\end{cases}$ is shown as



which shows $f(x)$ is continuous for all $x\in R$ and differentiable for all $x\in R-\{0\}$.

(ii) We know that,

If $f(x)$ is an increasing function on $[a,b]$, then

$\max\{f(t);a\leq t\leq x,\ a\leq x\leq b\}=f(x)$

$\min\{f(t);a\leq t\leq x,\ a\leq x\leq b\}=f(a)$

If $f(x)$ is decreasing function on $[a,b]$, then

$\max\{f(t);a\leq t\leq x,\ a\leq x\leq b\}=f(a)$

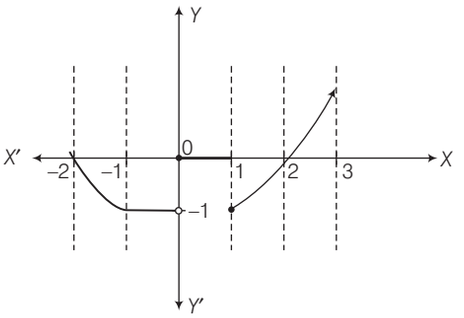
$\min\{f(t);a\leq t\leq x,\ a\leq x\leq b\}=f(x)$

From graph of $f(x)$,

$$g(x)=\begin{cases}f(x), & \text{for } -2\leq x<-1\\ -1, & \text{for } -1\leq x<0\\ 0, & \text{for } 0\leq x<1\\ f(x), & \text{for } x\geq 1\end{cases}$$

$$\Rightarrow\ g(x)=\begin{cases}x^2+2x, & \text{for } -2\leq x<-1\\ -1, & \text{for } -1\leq x<0\\ 0, & \text{for } 0\leq x<1\\ x^2-2x, & \text{for } x\geq 1\end{cases}$$

Thus, graph of $g(x)$ is



From above figure, it is clear that $g(x)$ is not continuous at $x=0,1$.

Example 53 ***Let*** $f(x)=\phi(x)+\psi(x)$ ***and*** $\phi'(a),\psi'(a)$ ***are finite and definite. Then,***

(a) $f(x)$ is continuous at $x=a$

(b) $f(x)$ is differentiable at $x=a$

(c) $f'(x)$ is continuous at $x=a$

(d) $f'(x)$ is differentiable at $x=a$

Sol. We know that the sum of two continuous (differentiable) functions is continuous (differentiable).

$\therefore$ $f(x)$ is continuous and differentiable at $x=a$.

Hence, (a) and (b) are the correct answers.

Example 54 If $f(x)=x+\tan x$ and $g(x)$ is the inverse of $f(x)$, then $g'(x)$ is equal to

(a) $\dfrac{1}{1+(g(x)-x)^2}$ (b) $\dfrac{1}{2+(g(x)+x)^2}$

(c) $\dfrac{1}{2+(g(x)-x)^2}$ (d) None of these

Sol. We have, $f(x)=x+\tan x$

$$\Rightarrow \quad f(f^{-1}(x))=f^{-1}(x)+\tan(f^{-1}(x))$$

$$\Rightarrow \quad x=g(x)+\tan(g(x)) \qquad \text{...(i)}$$

$$[\because g(x)=f^{-1}(x)]$$

$$1=g'(x)+\sec^2(g(x))\cdot g'(x)$$

$$\Rightarrow \quad g'(x)=\frac{1}{1+\sec^2(g(x))}$$

$$\Rightarrow \quad g'(x) = \frac{1}{2 + \tan^2(g(x))}$$

$$\Rightarrow \quad g'(x) = \frac{1}{2 + (x - g(x))^2} \qquad \text{[from Eq. (i)]}$$

Hence, (c) is the correct answer.

Example 55 If $f(x)$ is differentiable function and $(f(x)\cdot g(x))$ is differentiable at $x = a$, then

(a) $g(x)$ must be differentiable at $x = a$

(b) $g(x)$ is discontinuous, then $f(a) = 0$

(c) $f(a) \neq 0$, then $g(x)$ must be differentiable

(d) None of the above

Sol. $$\left[\frac{d}{dx}(f(x)\cdot g(x))\right]_{x=a} = f'(a)g(a) + \lim_{h\to 0}\frac{g(a+h)-g(a)}{h}\cdot f(a)$$

If $f(a) \neq 0 \Rightarrow g'(a)$ must exist.

Also, if $g(x)$ is discontinuous, $f(a)$ must be 0 for $f(x)\cdot g(x)$ to be differentiable.

Hence, (b) and (c) are the correct answers.

Example 56 If $f(x) = [x^{-2}[x^2]]$, (where $[\cdot]$ denotes the greatest integer function) $x \neq 0$, then incorrect statement

(a) $f(x)$ is continuous everywhere

(b) $f(x)$ is discontinuous at $x = \sqrt{2}$

(c) $f(x)$ is non-differentiable at $x = 1$

(d) $f(x)$ is discontinuous at infinitely many points

Sol. Here, $0 \le [x^2] \le x^2$

$$\Rightarrow \quad 0 \le x^{-2}[x^2] \le 1 \Rightarrow [x^{-2}[x^2]] = 0 \text{ or } 1$$

$f(x)$ is discontinuous at $x^2 = n, n \in N \Rightarrow x = \sqrt{n}$

$\therefore f(x)$ is neither continuous nor differentiable at $x = \sqrt{n}, n \in N$.

Hence, (b), (c) and (d) are the correct answers.

Example 57 ***If*** $f(x) = \begin{cases} x^2(\text{sgn}[x]) + \{x\}, & 0 \le x \le 2 \\ \sin x + |x-3|, & 2 \le x \le 4 \end{cases}$, ***where*** $[\ ]$ ***and*** $\{\ \}$ ***represents greatest integer and fractional part function respectively, then***

(a) $f(x)$ is differentiable at $x = 1$

(b) $f(x)$ is continuous but non-differentiable at $x = 2$

(c) $f(x)$ is non-differentiable at $x = 2$

(d) $f(x)$ is discontinuous at $x = 2$

Sol. For continuity at $x = 1$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^2 \text{sgn}[x] + \{x\} = 1 + 0 = 1$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 \text{sgn}[x] + \{x\} = 0 + 1 = 1$$

Also, $f(1) = 1$

$\therefore f(x)$ is Continuous at $x = 1$

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ x^2 + x - 1, & 1 \le x < 2 \end{cases}, \text{ non-differentiable at } x = 1$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2 \text{sgn}[x] + \{x\}$$

$$= 4 \times 1 + 1 = 5$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \sin x + |x-3| = 1 + \sin 2$$

Thus, $f(x)$ is neither continuous nor differentiable at $x = 2$.

Hence, (c) and (d) are the correct answers.

Example 58 ***A real valued function*** $f(x)$ ***is given as***

$$f(x) = \begin{cases} \int_0^x 2\{x\}dx, & x + \{x\} \in I \\ x^2 - x + \frac{1}{2}, & \frac{1}{2} < x < \frac{3}{2} \text{ and } x \neq 1, \\ x^2 - x + \frac{1}{6}, & \text{otherwise} \end{cases}$$

where $[\ ]$ ***denotes greatest integer less than or equals to*** x ***and*** $\{\ \}$ ***denotes fractional part function of*** x***. Then,***

(a) $f(x)$ is continuous and differentiable in $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$

(b) $f(x)$ is continuous and differentiable in $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

(c) $f(x)$ is continuous and differentiable in $x \in \left[\frac{1}{2}, \frac{3}{2}\right]$

(d) $f(x)$ is continuous but not differentiable in $x \in (0, 1)$

Sol. Here, $x + \{x\} \in I \Rightarrow x + x - [x] \in I$

$$\Rightarrow \quad 2x - [x] \in I, \text{ possible for } x = \frac{n}{2}, n \in I$$

$$\therefore \quad f\left(\frac{1}{2}\right) = \int_0^{1/2} 2\{x\}dx = \frac{1}{4}, f\left(\frac{3}{2}\right) = \int_0^{3/2} 2\{x\}dx = \frac{5}{4}$$

and $$f\left(\frac{-1}{2}\right) = \int_0^{-1/2} 2\{x\}dx = \frac{-3}{4}, \ f(1) = 1$$

Then, $$f(x) = \begin{cases} \frac{1}{4}, & x = \frac{1}{2} \\ \frac{5}{4}, & x = \frac{3}{2} \\ \frac{-3}{4}, & x = \frac{-1}{2} \\ 1, & x = 1 \\ x^2 - x + \frac{1}{2}, & \frac{1}{2} < x < \frac{3}{2} \text{ and } x \neq 1 \\ x^2 - x + \frac{1}{6}, & \text{otherwise} \end{cases}$$

Clearly, continuous for $x \in (0, 1)$ but not differentiable.

Hence, (d) is the correct answer.

Exercise for Session 7

1. If $f(x)=\sin(\pi\,(x-[x]))$, $\forall\ x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, where $[\cdot]$ denotes the greatest integer function, then

(a) $f(x)$ is discontinuous at $x=\{-1, 0, 1\}$ (b) $f(x)$ is differentiable for $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)-\{0\}$

(c) $f(x)$ is differentiable for $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)-\{-1, 0, 1\}$ (d) None of these

2. Let $f(x)=\begin{cases}x-1, & -1\le x<0\\ x^2, & 0\le x\le 1\end{cases}$, $g(x)=\sin x$ and $h(x)=f(|g(x)|)+|f(g(x))|$. Then,

(a) $h(x)$ is continuous for $x\in[-1, 1]$ (b) $h(x)$ is differentiable for $x\in[-1, 1]$

(c) $h(x)$ is differentiable for $x\in[-1, 1]-\{0\}$ (d) $h(x)$ is differentiable for $x\in(-1, 1)-\{0\}$

3. If $f(x)=\begin{cases}|1-4x^2|, & 0\le x<1\\ [x^2-2x], & 1\le x<2\end{cases}$, where [ ] denotes the greatest integer function, then

(a) $f(x)$ is continuous for all $x\in[0, 2)$ (b) $f(x)$ is differentiable for all $x\in[0, 2)-\{1\}$

(c) $f(x)$ is differentiable for all $x\in[0, 2)-\left\{\frac{1}{2}, 1\right\}$ (d) None of these

4. Let $f(x)=\int_0^1 |x-t|\,t\,dt$, then

(a) $f(x)$ is continuous but not differentiable for all $x\in R$ (b) $f(x)$ is continuous and differentiable for all $x\in R$

(c) $f(x)$ is continuous for $x\in R-\left\{\frac{1}{2}\right\}$ and $f(x)$ is differentiable for $x\in R-\left\{\frac{1}{4}, \frac{1}{2}\right\}$

(d) None of these

5. Let $f(x)$ be a function such that $f(x+y)=f(x)+f(y)$ for all x and y and $f(x)=(2x^2+3x)\cdot g(x)$ for all x, where $g(x)$ is continuous and $g(0)=3$. Then, $f'(x)$ is equal to

(a) 6 (b) 9 (c) 8 (d) None of these

6. If a function $g(x)$ which has derivatives $g'(x)$ for every real x and which satisfies the following equation $g(x+y)=e^y g(x)+e^x g(y)$ for all x and y and $g'(0)=2$, then the value of $\{g'(x)-g(x)\}$ is equal to

(a) e^x (b) $\frac{2}{3}e^x$ (c) $\frac{1}{2}e^x$ (d) $2e^x$

7. Let $f:R\to R$ be a function satisfying $f\left(\frac{xy}{2}\right)=\frac{f(x)\cdot f(y)}{2}$, $\forall\ x, y\in R$

and $f(1)=f'(1)\ne 0$. Then, $f(x)+f(1-x)$ is (for all non-zero real values of x)

(a) constant (b) can't be discussed (c) x (d) $\frac{1}{x}$

8. Let $f(x)$ be a derivable function at $x=0$ and $f\left(\frac{x+y}{K}\right)=\frac{f(x)+f(y)}{K}$ $(K\in R, K\ne 0, 2)$. Then, $f(x)$ is

(a) even function (b) neither even nor odd function

(c) either zero or odd function (d) either zero or even function

9. Let $f:R-(-\pi,\pi)$ be a differentiable function such that $f(x)+f(y)=f\left(\frac{x+y}{1-xy}\right)$.

If $f(1)=\frac{\pi}{2}$ and $\lim_{x\to 0}\frac{f(x)}{x}=2$. Then, $f(x)$ is equal to

(a) $2\tan^{-1}x$ (b) $\frac{1}{2}\tan^{-1}x$ (c) $\frac{\pi}{2}\tan^{-1}x$ (d) $2\pi\ \tan^{-1}x$

10. Let $f(x)=\sin x$ and $g(x)=\begin{cases}\max\{f(t), 0\le t\le x\}, & \text{for } 0\le x\le\pi\\ \frac{1-\cos x}{2}, & \text{for } x>\pi\end{cases}$. Then, $g(x)$ is

(a) differentiable for all $x\in R$ (b) differentiable for all $x\in R-\{\pi\}$

(c) differentiable for all $x\in(0, \infty)$ (d) differentiable for all $x\in(0, \infty)-\{\pi\}$

JEE Type Solved Examples :
Single Option Correct Type Questions

● **Ex. 1** *The values of a and b so that the function*

$$f(x)=\begin{cases} x+a\sqrt{2}\sin x, & 0\le x<\pi/4 \\ 2x\cot x+b, & \pi/4\le x\le\pi/2 \\ a\cos 2x-b\sin x, & \pi/2<x\le\pi \end{cases}$$ *is continuous for*

$x\in[0,\pi]$, *are*

(a) $a=\dfrac{\pi}{6}, b=-\dfrac{\pi}{6}$ (b) $a=-\dfrac{\pi}{6}, b=\dfrac{\pi}{12}$

(c) $a=\dfrac{\pi}{6}, b=-\dfrac{\pi}{12}$ (d) None of these

Sol. (c) Since, $f(x)$ is continuous for $x\in[0,\pi]$.

$\therefore$ $f(x)$ is continuous at $x=\pi/4$ and $x=\pi/2$ and hence to discuss continuity at $x=\pi/4$ and $x=\pi/2$.

Now, at $x=\pi/4$

Left hand limit (at $x=\pi/4$)

$$=\lim_{x\to\frac{\pi}{4}^-} f(x)=\lim_{x\to\frac{\pi}{4}^-}(x+a\sqrt{2}\sin x)$$

$$=\lim_{h\to 0}\{(\pi/4-h)+a\sqrt{2}\sin(\pi/4-h)\}$$

$\therefore$ $\text{LHL}=\lim\limits_{x\to\frac{\pi}{4}^-} f(x)=\dfrac{\pi}{4}+a$

Right hand limit at $x=\pi/4$

$$=\lim_{x\to\frac{\pi}{4}^+} f(x)=\lim_{x\to\frac{\pi}{4}^+}(2x\cot x+b)$$

$$=\lim_{h\to 0} 2\left(\frac{\pi}{4}+h\right)\cot\left(\frac{\pi}{4}+h\right)+b$$

$\Rightarrow$ $\text{RHL}=\lim\limits_{x\to\frac{\pi}{4}^+} f(x)=\dfrac{\pi}{2}+b$

Also, $f\left(\dfrac{\pi}{4}\right)=\dfrac{\pi}{2}+b$

For continuity, LHL (at $x=\pi/4$) = RHL (at $x=\pi/4$) $=f\left(\dfrac{\pi}{4}\right)$

$\Rightarrow$ $\dfrac{\pi}{4}+a=\dfrac{\pi}{2}+b$

$\Rightarrow$ $a-b=\dfrac{\pi}{4}$ ...(i)

At $x=\pi/2$

Left hand limit (at $x=\pi/2$)

$$=\lim_{x\to\frac{\pi}{2}^-}(2x\cot x+b)=\lim_{h\to 0} 2\left(\frac{\pi}{2}-h\right)\cot\left(\frac{\pi}{2}-h\right)+b$$

$$\text{LHL}=b$$

Right hand limit (at $x=\pi/2$)

$$=\lim_{x\to\frac{\pi}{2}^+}(a\cos 2x-b\sin x)$$

$$=\lim_{h\to 0}\left[a\cos 2\left(\frac{\pi}{2}+h\right)-b\sin\left(\frac{\pi}{2}+h\right)\right]$$

$$\text{RHL}=-a-b$$

and $f(\pi/2)=2\dfrac{\pi}{2}\cot\dfrac{\pi}{2}+b=b$

For continuity, LHL (at $x=\pi/2$) = RHL (at $x=\pi/2$) $=f(\pi/2)$

$\Rightarrow$ $b=-a-b \Rightarrow a+2b=0$ ...(ii)

On solving Eqs. (i) and (ii), we get

$$a=\pi/6,\ b=-\pi/12$$

● **Ex. 2** *Let f be an even function and $f'(0)$ exists, then $f'(0)$ is*

(a) 1 (b) 0

(c) −1 (d) −2

Sol. (b) Since, f is an even function.

So, $f(-x)=f(x)$, for all x ...(i)

Also, $f'(0)$ exists.

So, $Rf'(0)=Lf'(0)$

$\Rightarrow$ $\lim\limits_{h\to 0}\dfrac{f(0+h)-f(0)}{h}=\lim\limits_{h\to 0}\dfrac{f(0-h)-f(0)}{-h}$

$\Rightarrow$ $\lim\limits_{h\to 0}\dfrac{f(h)-f(0)}{h}=\lim\limits_{h\to 0}\dfrac{f(-h)-f(0)}{-h}$

$\Rightarrow$ $\lim\limits_{h\to 0}\dfrac{f(h)-f(0)}{h}=\lim\limits_{h\to 0}\dfrac{f(h)-f(0)}{-h}$

[using Eq. (i), $f(-h)=f(h)$]

$\Rightarrow$ $\lim\limits_{h\to 0}\dfrac{f(h)-f(0)}{h}=-\lim\limits_{h\to 0}\dfrac{f(h)-f(0)}{h}$

$\Rightarrow$ $2\lim\limits_{h\to 0}\dfrac{f(h)-f(0)}{h}=0$

$\Rightarrow$ $2f'(0)=0 \Rightarrow f'(0)=0$

● **Ex. 3** *The set of points where $x^2|x|$ is thrice differentiable, is*

(a) R (b) $R-\{0,1\}$

(c) $[0,\infty)$ (d) $R-\{0\}$

Sol. (d) Let $f(x)=x^2|x|$ which could be expressed as

$$f(x)=\begin{cases} -x^3, & x<0 \\ 0, & x=0 \\ x^3, & x>0 \end{cases}$$

This gives, $f'(x)=\begin{cases}-3x^2, & x<0\\ 0, & x=0\\ 3x^2, & x>0\end{cases}$

So, $f'(x)$ exists for all real x.

$$f''(x)=\begin{cases}-6x, & x<0\\ 0, & x=0\\ 6x, & x>0\end{cases}$$

So, $f''(x)$ exists for all real x.

$$f'''(x)=\begin{cases}-6, & x<0\\ 0, & x=0\\ 6, & x>0\end{cases}$$

However, $f'''(0)$ doesn't exist, since $f'''(0^-)=-6$ and $f'''(0^+)=6$ which are not equal.

Thus, the set of points where $f(x)$ is thrice differentiable is $R-\{0\}$.

● ***Ex. 4*** *The function* $f(x)=\dfrac{|x+2|}{\tan^{-1}(x+2)}$, *is continuous for*

(a) $x\in R$ (b) $x\in R-\{0\}$
(c) $x\in R-\{-2\}$ (d) None of these

Sol. (*c*) Clearly, f is continuous except possibility at $x=-2$

$$\text{RHL (at } x=-2)=\lim_{x\to -2^+} f(x)$$
$$=\lim_{x\to -2^+}\frac{(x+2)}{\tan^{-1}(x+2)}=1$$
$$\text{LHL (at } x=-2)=\lim_{x\to -2^-} f(x)$$
$$=\lim_{x\to -2^-}\frac{-(x+2)}{\tan^{-1}(x+2)}=-1$$

So, f is not continuous at $x=-2$.

● ***Ex. 5*** *If* $f(x)=\begin{cases}\dfrac{\sin[x^2]\pi}{x^2-3x-18}+ax^3+b, & \text{for } 0\le x\le 1\\ 2\cos\pi x+\tan^{-1}x, & \text{for } 1<x\le 2\end{cases}$

differentiable function in $[0, 2]$, *where* $[\cdot]$ *denotes the greatest integer function, then*

(a) $a=\dfrac{1}{6}, b=\dfrac{\pi}{4}-\dfrac{13}{6}$ (b) $a=-\dfrac{1}{6}, b=\dfrac{\pi}{4}$
(c) $a=-\dfrac{1}{6}, b=\dfrac{\pi}{4}-\dfrac{13}{6}$ (d) None of these

Sol. (*a*) Here, $[x^2]=0$, for all $0\le x<1$

and $[x^2]=1$, for $x=1$

$\therefore\ \sin[x^2]\pi=0$, for $0\le x\le 1$

Hence, $f(x)=\begin{cases}ax^3+b, & 0\le x\le 1\\ 2\cos\pi x+\tan^{-1}x, & 1<x\le 2\end{cases}$

As, $f(x)$ is differentiable in $[0, 2]$.

$\Rightarrow$ $f(x)$ is continuous and differentiable at $x=1$

$\Rightarrow$ $\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^+} f(x)=f(1)$

$$a+b=-2+\frac{\pi}{4}=a+b$$

$\Rightarrow$ $a+b=-2+\dfrac{\pi}{4}$ ...(i)

Again, since $f(x)$ is differentiable at $x=1$

(LHD at $x=1$) = (RHD at $x=1$)

$\Rightarrow$ $\lim\limits_{x\to 1^-}\dfrac{f(x)-f(1)}{x-1}=\lim\limits_{x\to 1^+}\dfrac{f(x)-f(1)}{x-1}$

$\Rightarrow$ $\lim\limits_{x\to 1^-}\dfrac{(ax^3+b)-(a+b)}{x-1}$

$$=\lim_{x\to 1^+}\frac{(2\cos\pi x+\tan^{-1}x)-(a+b)}{x-1}$$

$\Rightarrow$ $3a=\lim\limits_{x\to 1^+}\dfrac{-2\pi\sin\pi x+\dfrac{1}{1+x^2}}{1}$

$\Rightarrow$ $3a=\dfrac{1}{2}$ or $a=\dfrac{1}{6}$ ...(ii)

From Eqs. (i) and (ii), we get

$$a=\frac{1}{6}\quad\text{and}\quad b=\frac{\pi}{4}-\frac{13}{6}$$

● ***Ex. 6*** *If* $g(x)=\lim\limits_{m\to\infty}\dfrac{x^m f(1)+h(x)+1}{2x^m+3x+3}$ *is continuous at* $x=1$ *and* $g(1)=\lim\limits_{x\to 1}\{\log_e(ex)\}^{2/\log_e x}$, *then the value of* $2g(1)+2f(1)-h(1)$ *when* $f(x)$ *and* $h(x)$ *are continuous at* $x=1$, *is*

(a) 0 (b) 1 (c) 2 (d) 3

Sol. (*b*) Here, $g(1)=\lim\limits_{x\to 1}\{\log e+\log_e x\}^{2/\log_e x}$

$$=\lim_{x\to 1}\{1+\log_e x\}^{2/\log_e x}$$
$$=e^{\lim\limits_{x\to 1}\log_e x\frac{2}{\log_e x}}$$
$$g(1)=e^2 \quad ...(i)$$

Also, $\lim\limits_{x\to 1^-} g(x)=\lim\limits_{x\to 1^-}\lim\limits_{m\to\infty}\left\{\dfrac{x^m f(1)+h(x)+1}{2x^m+3x+3}\right\}$

$$=\lim_{m\to\infty}\left\{\lim_{x\to 1^-}\frac{x^m f(1)+h(x)+1}{2x^m+3x+3}\right\}$$
$$=\frac{h(1)+1}{3+3}\qquad[\text{as } x<1\Rightarrow\lim_{m\to\infty}x^m=0]$$

$$\lim_{x\to 1^-} g(x)=\frac{h(1)+1}{6} \quad ...(ii)$$

$$\text{Again, } \lim_{x\to 1^+} g(x) = \lim_{x\to 1^+}\lim_{m\to\infty}\left\{\frac{x^m f(1)+h(x)+1}{2x^m+3x+3}\right\}$$

$$= \lim_{m\to\infty}\left\{\lim_{x\to 1^+}\frac{x^m f(1)+h(x)+1}{2x^m+3x+3}\right\}$$

$$= \lim_{m\to\infty}\lim_{x\to 1^+}\frac{f(1)+h(x)/x^m+1/x^m}{2+3/x^{m-1}+3/x^m} = \frac{f(1)}{2}$$

$$\therefore \quad \lim_{x\to 1^+} g(x) = \frac{f(1)}{2} \qquad \text{...(iii)}$$

From Eqs. (i), (ii) and (iii), we get

$$e^2 = \frac{h(1)+1}{6} = \frac{f(1)}{2} \quad [\text{as } g(x) \text{ is continuous at } x=1]$$

$$\Rightarrow \quad h(1) = 6e^2 - 1 \quad \text{and} \quad f(1) = 2e^2$$

$$\therefore \quad 2g(1) + 2f(1) - h(1) = 2e^2 + 4e^2 - 6e^2 + 1 = 1$$

$$\Rightarrow 2g(1) + 2f(1) - h(1) = 1$$

● ***Ex. 7*** *Let $g(x) = \log f(x)$, where $f(x)$ is a twice differentiable positive function on $(0, \infty)$ such that $f(x+1) = xf(x)$. Then, for $N = 1, 2, 3, \ldots, g''\left(N+\frac{1}{2}\right) - g''\left(\frac{1}{2}\right)$ is equal to*

[IIT JEE 2008]

(a) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+\ldots+\frac{1}{(2N-1)^2}\right\}$

(b) $4\left\{1+\frac{1}{9}+\frac{1}{25}+\ldots+\frac{1}{(2N-1)^2}\right\}$

(c) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+\ldots+\frac{1}{(2N+1)^2}\right\}$

(d) $4\left\{1+\frac{1}{9}+\frac{1}{25}+\ldots+\frac{1}{(2N+1)^2}\right\}$

Sol. (*a*) We have, $g(x) = \log f(x)$ ...(i)

$$\Rightarrow \quad f(x) = e^{g(x)}$$

$$\Rightarrow \quad f(x+1) = e^{g(x+1)}$$

$$\Rightarrow \quad x\,f(x) = e^{g(x+1)} \quad [\text{given } f(x+1) = x\,f(x)]$$

$$\Rightarrow \log x + \log f(x) = \log e^{g(x+1)} \quad [\text{taking log both sides}]$$

$$\Rightarrow \quad \log x + g(x) = g(x+1) \quad [\text{from Eq. (i)}]$$

$$\text{i.e.} \quad g(x+1) - g(x) = \log x \qquad \text{...(ii)}$$

On replacing x by $x - \frac{1}{2}$ in Eq. (ii), we get

$$g\left(x+\frac{1}{2}\right) - g\left(x-\frac{1}{2}\right) = \log\left(x-\frac{1}{2}\right)$$

$$= \log(2x-1) - \log 2$$

$$\therefore \quad g''\left(x+\frac{1}{2}\right) - g''\left(x-\frac{1}{2}\right) = \frac{-4}{(2x-1)^2} \qquad \text{...(iii)}$$

On substituting $x = 1, 2, 3, \ldots, N$ in Eq. (iii) and adding

$$g''\left(N+\frac{1}{2}\right) - g''\left(\frac{1}{2}\right) = -4\left\{1+\frac{1}{9}+\frac{1}{25}+\ldots+\frac{1}{(2N-1)^2}\right\}$$

● ***Ex. 8*** *Let $y = f(x)$ be a differentiable function, $\forall\ x \in R$ and satisfies;*

$$f(x) = x + \int_0^1 x^2 z\, f(z)\,dz + \int_0^1 x\,z^2\, f(z)\,dz, \text{ then}$$

(a) $f(x) = \frac{20x}{119}(2+9x)$ (b) $f(x) = \frac{20x}{119}(4+9x)$

(c) $f(x) = \frac{10x}{119}(4+9x)$ (d) $f(x) = \frac{5x}{119}(4+9x)$

Sol. (*b*) We have, $f(x) = x + \int_0^1 x^2 z\, f(z)\,dz + \int_0^1 x z^2 f(z)dz$

$$\text{or} \quad f(x) = x + x^2\int_0^1 z\, f(z)dz + x\int_0^1 z^2 f(z)dz$$

$$\text{Let} \quad \lambda_1 = \int_0^1 z\, f(z)dz \text{ and } \lambda_2 = \int_0^1 z^2\, f(z)dz$$

$$\therefore \quad f(x) = x + x^2\lambda_1 + x\lambda_2 \qquad \text{...(i)}$$

$$\text{Now,} \quad \lambda_1 = \int_0^1 z\, f(z)dz$$

$$\lambda_1 = \int_0^1 z\,(z + z^2\lambda_1 + z\lambda_2)dz$$

[using Eq. (i), as $f(z) = z + z^2\lambda_1 + z\lambda_2$]

$$\Rightarrow \quad \lambda_1 = (1+\lambda_2)\int_0^1 z^2\,dz + \lambda_1\int_0^1 z^3 dz$$

$$\Rightarrow \quad \lambda_1 = (1+\lambda_2)\left(\frac{z^3}{3}\right)_0^1 + \lambda_1\left(\frac{z^4}{4}\right)_0^1$$

$$\Rightarrow \quad \lambda_1 = \frac{1+\lambda_2}{3} + \frac{\lambda_1}{4}$$

$$\Rightarrow \quad 9\lambda_1 - 4\lambda_2 = 4 \qquad \text{...(ii)}$$

$$\text{Also,} \quad \lambda_2 = \int_0^1 z^2\, f(z)\,dz \text{ or } \lambda_2 = \int_0^1 z^2\,\{z + z^2\lambda_1 + z\lambda_2\}dz$$

[using Eq. (i), as $f(z) = z + z^2\lambda_1 + z\lambda_2$]

$$\Rightarrow \quad \lambda_2 = \int_0^1 z^3(1+\lambda_2)dz + \lambda_1\int_0^1 z^4\,dz$$

$$\Rightarrow \quad \lambda_2 = \frac{(1+\lambda_2)}{4} + \frac{\lambda_1}{5}$$

$$\text{or} \quad 15\lambda_2 - 4\lambda_1 = 5 \qquad \text{...(iii)}$$

From Eqs. (ii) and (iii), we get

$$9\lambda_1 - 4\lambda_2 = 4 \quad \text{and} \quad 15\lambda_2 - 4\lambda_1 = 5$$

$$\lambda_1 = \frac{80}{119}, \lambda_2 = \frac{61}{119} \qquad \text{...(iv)}$$

Thus, Eq. (i) becomes

$$f(x) = x + \frac{80}{119}x^2 + \frac{61}{119}x \qquad [\text{from Eq. (iv)}]$$

$$\text{Hence,} \quad f(x) = \frac{20x}{119}(4+9x)$$

● ***Ex. 9*** *A function $f: R \to R$ satisfies the equation $f(x+y) = f(x) \cdot f(y)$ for all , $f(x) \neq 0$. Suppose that the function is differentiable at $x = 0$ and $f'(0) = 2$. Then,*

(a) $f'(x) = 2f(x)$ (b) $f'(x) = f(x)$
(c) $f'(x) = f(x) + 2$ (d) $f'(x) = 2f(x) + x$

Sol. (*a*) We are given that

$$f(x+y) = f(x) \cdot f(y) \quad \text{...(i)}$$

and $$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = 2 \quad \text{...(ii)}$$

Now, $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$\Rightarrow$ $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x+0)}{h}$$

$\Rightarrow$ $$f'(x) = \lim_{h \to 0} \frac{f(x) \cdot f(h) - f(x) \cdot f(0)}{h}$$

[using Eq. (i)]

$\Rightarrow$ $$f'(x) = f(x) \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$\Rightarrow$ $f'(x) = 2f(x)$ [using Eq. (ii)]

● ***Ex. 10*** *Let f be a function such that $f(x + f(y)) = f(x) + y, \forall\ x, y \in R$, then find $f(0)$. If it is given that there exists a positive real δ, such that $f(h) = h$ for $0 < h < \delta$, then find $f'(x)$,*

(a) 0, 1 (b) −1, 0 (c) 2, 1 (d) −2, 0

Sol. (*a*) Let $x = 0, y = 0$ in $f(x + f(y)) = f(x) + y$

$$f(0 + f(0)) = f(0) + 0$$

$\Rightarrow$ $f(f(0)) = f(0)$

$\Rightarrow$ $f(a) = a$ or $f(0) = 0$

Given, $f(h) = h$

Then, $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \text{ for } 0 < h < \delta$$

$\Rightarrow$ $$f'(x) = \lim_{h \to 0} \frac{f(x + f(h)) - f(x)}{h}$$

[given $f(h) = h$]

$\Rightarrow$ $$f'(x) = \lim_{h \to 0} \frac{f(x) + h - f(x)}{h}$$

$\Rightarrow$ $$f'(x) = \lim_{h \to 0} \frac{h}{h}$$

$\Rightarrow$ $f'(x) = 1$

● ***Ex. 11*** *If the function of $f(x) = \left[\frac{(x-5)^2}{A}\right] \sin(x-5) + a\cos(x-2)$, where $[\cdot]$ denotes the greatest integer function, is continuous and differentiable in $(7, 9)$, then*

(a) $A \in [8, 64]$ (b) $A \in [0, 8)$
(c) $A \in [16, \infty)$ (d) $A \in [8, 16]$

Sol. (*c*) As we know, $[x]$ is neither continuous nor differentiable at integer values.

So, $f(x)$ is continuous and differentiable in (7, 9).

If $\left[\frac{(x-5)^2}{A}\right] = 0$

$\Rightarrow$ $A > (x-5)^2$ ...(i)

As, $x \in (7, 9) \Rightarrow x - 5 \in (2, 4)$

$\Rightarrow$ $(x-5)^2 \in (4, 16)$ ...(ii)

From Eqs. (i) and (ii), we get

$A \geq 16$

● ***Ex. 12*** *If $f(x) = [2 + 5|n| \sin x]$, where $n \in I$ has exactly 9 points of non-derivability in $(0, \pi)$, then possible values of n are (where $[x]$ denotes greatest integer function)*

(a) ± 3 (b) ± 2
(c) ± 1 (d) None of these

Sol. (*c*) We have, $[2 + 5|n|\sin x] = 2 + [5|n|\sin x]$, let $y = 5|n|\sin x$

∴ Number of points of non-derivability

$$= 2(5|n| - 1) + 1 = 10|n| - 1$$

According to the question, $10|n| - 1 = 9$

$\Rightarrow$ $10|n| = 10$

$\Rightarrow$ $|n| = 1$

Here, $n = \pm 1$

● ***Ex. 13*** *The number of points of discontinuity of $f(x) = [2x]^2 - \{2x\}^2$ (where $[\cdot]$ denotes the greatest integer function and $\{\cdot\}$ denotes the fractional part of x) in the interval $(-2, 2)$ are*

(a) 6 (b) 8
(c) 4 (d) 3

Sol. (*a*) Here, $f(x) = ([2x] + \{2x\})([2x] - \{2x\}) = 2x(2x - 2\{2x\})$

$$= 4x^2 - 4x\{2x\} \quad \text{...(i)}$$

We know, $\{2x\}$ is discontinuous at integers.

Here, $-2 < x < 2$

$\Rightarrow$ $-4 < 2x < 4$

∴ Integer values of $2x = \{-3, -2, -1, 0, 1, 2, 3\}$, but $f(x) = 4x(x - \{2x\})$ is continuous at $x = 0$

∴ Total number of points of discontinuity are 6.

● ***Ex. 14*** *If $x \in R^+$ and $n \in N$, we can uniquely write $x = mn + r$, where $m \in W$ and $0 \leq r < n$. We define x mod n = r. e.g. 10.3 mod 3 = 1.3. The number of points of discontinuity of the function, $f(x) = (x \bmod 2)^2 + (x \bmod 4)$ in the interval $0 < x < 9$ is*

(a) 0 (b) 2
(c) 4 (d) None of these

Sol. (*c*) Here, $f(x)=(x \bmod 2)^2+(x \bmod 4)$

$$=\begin{cases} x^2+x, 0<x<2 \\ (x-2)^2+x, 2\le x<4 \\ (x-4)^2+(x-4), 4\le x<6 \\ (x-6)^2+(x-4), 6\le x<8 \\ (x-8)^2+(x-8), 8\le x<9 \end{cases}$$

Clearly, $f(2^-)=6, f(2^+)=2 \Rightarrow$ discontinuous at $x=2$.

$f(4^-)=8, f(4^+)=0 \Rightarrow$ discontinuous at $x=4$.

$f(6^-)=6, f(6^+)=2 \Rightarrow$ discontinuous at $x=6$.

$f(8^-)=8, f(8^+)=0 \Rightarrow$ discontinuous $x=8$.

$\therefore$ Number of points of discontinuity = 4.

● ***Ex. 15*** *Let $f: R \to R$ be a differentiable function at $x=0$ satisfying $f(0)=0$ and $f'(0)=1$, then the value of* $\lim_{x\to 0}\frac{1}{x}\cdot\sum_{n=1}^{\infty}(-1)^n\cdot f\left(\frac{x}{n}\right)$, *is*

(a) 0 (b) $-\log 2$

(c) 1 (d) e

Sol. (*b*) Let $L=\lim_{x\to 0}\frac{f(x/n)}{x}=\lim_{x\to 0}\frac{f(x/n)-f(0)}{n\cdot x/n}$,

[using $f(0)=0$]

$\therefore \quad L=\frac{1}{n}\lim_{x\to 0}\frac{f(x/n)-f(0)}{x/n}$

$\Rightarrow \quad L=\frac{1}{n}\cdot f'(0)=\frac{1}{n}$ [as $f'(0)=1$]

$\therefore \quad \lim_{x\to 0}\frac{1}{x}\cdot\sum_{n=1}^{\infty}(-1)^n f\left(\frac{x}{n}\right)=\sum_{n=1}^{\infty}(-1)^n\cdot\frac{1}{n}$

$$=-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\ldots$$

$$=-\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots\right)=-\log 2$$

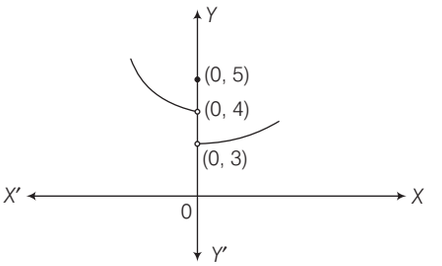
● ***Ex. 16*** *Let $f(x)$ is a function continuous for all $x\in R$ except at $x=0$ such that $f'(x)<0, \forall x\in(-\infty,0)$ and $f'(x)>0, \forall x\in(0,\infty)$. If $\lim_{x\to 0^+} f(x)=3$, $\lim_{x\to 0^-} f(x)=4$ and $f(0)=5$, then the image of the point $(0,1)$ about the line, $y\cdot\lim_{x\to 0} f(\cos^3 x-\cos^2 x)=x\cdot\lim_{x\to 0} f(\sin^2 x-\sin^3 x)$, is*

(a) $\left(\frac{12}{25},\frac{-9}{25}\right)$ (b) $\left(\frac{12}{25},\frac{9}{25}\right)$

(c) $\left(\frac{16}{25},\frac{-8}{25}\right)$ (d) $\left(\frac{24}{25},\frac{-7}{25}\right)$

Sol. (*d*) We have,

$$y\cdot\lim_{x\to 0} f(\cos^3 x-\cos^2 x)=x\cdot\lim_{x\to 0} f(\sin^2 x-\sin^3 x)$$



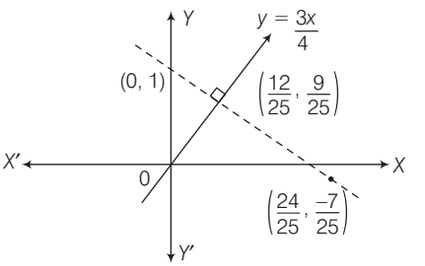
Here, $\cos^3 x-\cos x\to 0$ from LHS

$\therefore \quad \lim_{x\to 0} f(\cos^3 x-\cos x)=4$

$\sin^2 x-\sin^3 x\to 0$ from RHS

$\therefore \quad \lim_{x\to 0} f(\sin^2 x-\sin^3 x)\to 3$

$\therefore$ Equation of straight line is $4y=3x$ ...(i)



Equation of line perpendicular to $y=\frac{3}{4}x$ and passing through (0, 1), is $(y-1)=\frac{-4}{3}(x-0)$

$\Rightarrow \quad y-1=\frac{-4}{3}x$ ...(ii)

On solving Eqs. (i) and (ii), we get

$$x=\frac{12}{25}, y=\frac{9}{25}$$

Hence, image point is $\left(\frac{24}{25},\frac{-7}{25}\right)$.

JEE Type Solved Examples :
More than One Correct Option Type Questions

● ***Ex. 17*** *If $f(x)$ be such that $f(x) = max\,(|3-x|, 3-x^3)$, then*

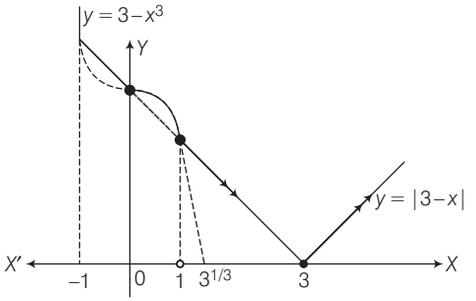
(a) $f(x)$ is continuous $\forall\ x \in R$

(b) $f(x)$ is differentiable $\forall\ x \in R$

(c) $f(x)$ is non-differentiable at three points only

(d) $f(x)$ is non-differentiable at four points only

Sol. (*a,d*) From the graph of $f(x)$, $f(x)$ is continuous, $\forall\ x \in R$ and $f(x)$ is not differentiable at $x = -1, 0, 1, 3$.



● ***Ex. 18*** *Let $f(x) = |x-1|([x]-[-x])$, then which of the following statement(s) is/are correct. (where $[\cdot]$ denotes greatest integer function.)*

(a) $f(x)$ is continuous at $x = 1$

(b) $f(x)$ is derivable at $x = 1$

(c) $f(x)$ is non-derivable at $x = 1$

(d) $f(x)$ is discontinuous at $x = 1$

Sol. (*a, c*) We have, $f(x) = |x-1|([x]-[-x])$

Here, $[x] = 0$, when $0 < x < 1$

$[-x] = -1$, when $0 < x < 1$

$[x] = 1$, when $1 < x < 2$

$[-x] = -2$, when $1 < x < 2$

$[x] = 1, [-x] = -1$, when $x = 1$

$$\therefore\quad f(x) = \begin{cases} (x-1)(1+2), \text{ when } 1 < x < 2 \\ 0, \text{ when } x = 1 \\ (1-x), \text{ when } 0 < x < 1 \end{cases}$$

$$\Rightarrow\quad f'(1^+) = \lim_{h\to 0}\frac{3(h)-0}{h} = 3$$

and $$f'(1^-) = \lim_{h\to 0}\frac{(1-(1-h))-0}{-h} = -1$$

$\Rightarrow$ $f(x)$ is continuous at $x = 1$ and non-derivable at $x = 1$.

● ***Ex. 19*** *If $y = f(x)$ defined parametrically by $x = 2t - |t-1|$ and $y = 2t^2 + t|t|$, then*

(a) $f(x)$ is continuous for all $x \in R$

(b) $f(x)$ is continuous for all $x \in R - \{2\}$

(c) $f(x)$ is differentiable for all $x \in R$

(d) $f(x)$ is differentiable for all $x \in R - \{2\}$

Sol. (*a,d*) Here, $x = 2t - |t-1|$ and $y = 2t^2 + t|t|$

Now, when $t < 0$,

$$x = 2t - \{-(t-1)\} = 3t - 1$$

and $$y = 2t^2 - t^2 = t^2 \Rightarrow y = \frac{1}{9}(x+1)^2$$

When $0 \le t < 1$,

$$x = 2t - (-(t-1)) = 3t - 1$$

and $$y = 2t^2 + t^2 = 3t^2 \Rightarrow y = \frac{1}{3}(x+1)^2$$

When $t \ge 1$,

$$x = 2t - (t-1) = t + 1$$

and $$y = 2t^2 + t^2 = 3t^2 \Rightarrow y = 3(x-1)^2$$

Thus, $$y = f(x) = \begin{cases} \frac{1}{9}(x+1)^2, & x < -1 \\ \frac{1}{3}(x+1)^2, & -1 \le x < 2 \\ 3(x-1)^2, & x \ge 2 \end{cases}$$

Now, to check continuity at $x = -1$ and 2.

Continuity at $x = -1$,

$$\text{LHL} = \lim_{h\to 0} f(-1-h) = \lim_{h\to 0}\frac{1}{9}(-1-h+1)^2 = 0$$

$$\text{RHL} = \lim_{h\to 0} f(-1+h) = \lim_{h\to 0}\frac{1}{3}(-1+h+1)^2 = 0$$

$$f(-1) = 0$$

$\therefore$ $f(x)$ is continuous at $x = -1$.

Now, to check continuity at $x = 2$,

$$\text{LHL} = \lim_{h\to 0} f(2-h) = \lim_{h\to 0}\frac{1}{3}(2-h+1)^2 = 3$$

$$\text{RHL} = \lim_{h\to 0} f(2+h) = \lim_{h\to 0} 3(2+h-1)^2 = 3$$

$$f(2) = 3$$

Thus, $f(x)$ is continuous at $x = 2$.

Now, to check differentiability at $x = -1$ and 2.

Differentiability at $x = -1$,

$$\text{LHD} = Lf'(-1) = \lim_{h\to 0}\frac{f(-1-h)-f(-1)}{-h}$$

$$= \lim_{h\to 0}\frac{\frac{1}{9}(-1-h+1)^2 - 0}{-h} = 0$$

$$\text{RHD} = Rf'(-1) = \lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{3}(-1+h+1)^2 - 0}{h} = 0$$

Hence, $f(x)$ is differentiable at $x = -1$.

Differentiability at $x = 2$,

$$\text{LHD} = Lf'(2) = \lim_{h \to 0} \frac{f(2-h) - f(2)}{-h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{3}(2-h+1)^2 - 3}{-h} = 2$$

$$\text{RHD} = Rf'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{3(2+h-1)^2 - 3}{h} = 6$$

Hence, $f(x)$ is not differentiable at $x = 2$.

$\therefore f(x)$ is continuous for all x and differentiable for all x, except $x = 2$.

● **Ex. 20** *$f(x) = \sin^{-1}[e^x] + \sin^{-1}[e^{-x}]$, where $[\cdot]$ greatest integer function, then*

(a) domain of $f(x) = (-\ln 2, \ln 2)$

(b) range of $f(x) = \{\pi\}$

(c) $f(x)$ has removable discontinuity at $x = 0$

(d) $f(x) = \cos^{-1} x$ has only solution

Sol. (a, c) $0 < e^x < 2$ and $0 < e^{-x} < 2$

$\Rightarrow \quad -\infty < x < \log_e 2$ and $-\infty < -x < \log_e 2$

$\Rightarrow \quad (-\infty < x < \log_e 2)$

and $\quad (-\log_e 2 < x < \infty)$

$\Rightarrow \quad -\log_e 2 < x < \log_e 2$

$$\therefore \quad f(x) = \begin{cases} \pi, & x = 0 \\ \frac{\pi}{2}, & x \in (-\log_e 2, 0) \cup (0, \log_e 2) \end{cases}$$

● **Ex. 21** *$f : R \to R$ is one-one, onto and differentiable function and graph of $y = f(x)$ is symmetrical about the point (4, 0), then*

(a) $f^{-1}(2010) + f^{-1}(-2010) = 8$

(b) $\int_{-2010}^{2018} f(x)\,dx = 0$

(c) if $f'(-100) > 0$, then roots of $x^2 - f'(10)\,x - f'(10) = 0$ may be non-real

(d) if $f'(10) = 20$, then $f'(-2) = 20$

Sol. (a, b, d) Graph is symmetrical about (4, 0).

$\Rightarrow \quad f(4+x) = -f(4-x)$

$\Rightarrow \quad f(x) = -f(8-x)$ ...(i)

Now, let $f(x) = 2010$, then $f(8-x) = -2010$

$\Rightarrow \quad f^{-1}(2010) + f^{-1}(-2010) = 8$

$\Rightarrow$ Option (a) is true.

and $\quad \int_{-2010}^{4} f(x)dx = -\int_{4}^{2018} f(x)\,dx$

$\Rightarrow$ Option (b) is true.

Also, $\quad D = (f'(10))^2 + 4f'(10) > 0$

As, $\quad f'(-100) > 0 \Rightarrow f'(10) \geq 0$

$\Rightarrow \quad x^2 - f'(10)x - f'(10) = 0$ has real roots.

$\Rightarrow$ Option (c) is false.

As, $\quad f'(4+x) = f'(4-x)$

$f'(x)$ is symmetric about $x = 4$

$\Rightarrow \quad f'(10) = f'(-2) = 20$

$\Rightarrow$ Option (d) is true.

● **Ex. 22** *Let f be a real valued function defined on the interval $(0, \infty)$, by $f(x) = \ln x + \int_0^x \sqrt{1+\sin t}\,dt$. Then, which of the following statement(s) is/are true?* **[IIT JEE 2010]**

(a) $f''(x)$ exists for all $x \in (0, \infty)$

(b) $f'(x)$ exists for all $x \in (0, \infty)$ and f' is continuous on $(0, \infty)$ but not differentiable on $(0, \infty)$

(c) There exists $\alpha > 1$ such that $|f'(x)| < |f(x)|$ for all $x \in (\infty, \infty)$

(d) There exists $\beta > 0$ such that $|f(x)| + |f'(x)| \leq \beta$ for all $x \in (0, \infty)$

Sol. (b, c) Here, $f'(x) = \frac{1}{x} + \sqrt{1+\sin x}$, $x > 0$ but $f(x)$ is not differentiable in $(0, \infty)$ as $\sin x$ may be -1 and then $f''(x) = -\frac{1}{x^2} + \frac{1}{2}\frac{\cos x}{\sqrt{1+\sin x}}$ will not exist.

$\Rightarrow f'(x)$ is continuous for all $x \in (0, \infty)$ but $f'(x)$ is not differentiable on $(0, \infty)$.

$\therefore$ Option (b) is true.

Also, $\quad f'(x) \leq 3$, if $x > 1$

and $\quad f(x) > 3$, if $x > e^3$

$\therefore$ Let $\quad \alpha = e^3$

$\Rightarrow$ Option (c) is true.

(d) is not possible, as $f(x) \to \infty$ when $x \to \infty$.

● **Ex. 23** *If $f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$ for all $x, y \in R$ $(xy \neq 1)$ and $\lim_{x \to 0} \frac{f(x)}{x} = 2$, then*

(a) $f\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{3}$ (b) $f\left(\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{3}$

(c) $f'(1) = 1$ (d) $f'(1) = -1$

Sol. (a, c) $f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$ ...(i)

On putting $x = y = 0$, we get $f(0) = 0$

On putting $y = -x$, we get $f(x) + f(-x) = f(0)$

$\Rightarrow \quad f(-x) = -f(x)$ ...(ii)

Also, $\lim_{x\to 0}\frac{f(x)}{x}=2$

Now, $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ ...(iii)

$=\lim_{h\to 0}\frac{f(x+h)+f(-x)}{h}$

[using Eq. (ii), $-f(x)=f(-x)$]

$f'(x)=\lim_{h\to 0}\frac{f\left(\frac{x+h-x}{1-(x+h)(-x)}\right)}{h}$ [using Eq. (i)]

$\Rightarrow \quad f'(x)=\lim_{h\to 0}\left[\frac{f\left(\frac{h}{1+x(x+h)}\right)}{h}\right]$

$\Rightarrow \quad f'(x)=\lim_{h\to 0}\frac{f\left(\frac{h}{1+xh+x^2}\right)}{\left(\frac{h}{1+xh+x^2}\right)}\times\left(\frac{1}{1+xh+x^2}\right)$

$\Rightarrow \quad f'(x)=\lim_{h\to 0}\frac{f\left(\frac{h}{1+xh+x^2}\right)}{\left(\frac{h}{1+xh+x^2}\right)}\times\lim_{h\to 0}\frac{1}{1+xh+x^2}$

$\left[\text{using } \lim_{x\to 0}\frac{f(x)}{x}=2\right]$

$\Rightarrow \quad f'(x)=2\times\frac{1}{1+x^2} \quad \Rightarrow \quad f'(x)=\frac{2}{1+x^2}$

On integrating both sides, we get

$f(x)=2\tan^{-1}(x)+C$

Where $f(0)=0 \Rightarrow C=0$

Thus, $f(x)=2\tan^{-1}x$

Hence, $f\left(\frac{1}{\sqrt{3}}\right)=2\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=2\cdot\frac{\pi}{6}=\frac{\pi}{3}$

and $f'(1)=\frac{2}{1+1^2}=\frac{2}{2}=1$

● ***Ex. 24*** *Let $f:R\to R$ is a function which satisfies condition $f(x+y^3)=f(x)+[f(y)]^3$ for all $x,y\in R$. If $f'(0)\geq 0$, then*

(a) $f(x)=0$ only (b) $f(x)=x$ only

(c) $f(x)=0$ or x only (d) $f(10)=10$

Sol. (*c, d*) Given, $f(x+y^3)=f(x)+[f(y)]^3$ ...(i)

and $f'(0)\geq 0$ ...(ii)

On replacing x, y by 0,

$f(0)=f(0)+f(0)^3 \Rightarrow f(0)=0$ ...(iii)

Also, $f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{f(h)}{h}$ ...(iv)

Let $I=f'(0)=\lim_{h\to 0}\frac{f(0+(h^{1/3})^3)-f(0)}{(h^{1/3})^3}$

$=\lim_{h\to 0}\frac{(f(h^{1/3}))^3}{(h^{1/3})^3}=\lim_{h\to 0}\left(\frac{f(h^{1/3})}{(h^{1/3})}\right)^3=I^3$

$\Rightarrow \quad I=I^3$ or $I=0,1,-1$ as $f'(0)\geq 0$

$\therefore \quad f'(0)=0,1$ ...(v)

Thus, $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$=\lim_{h\to 0}\frac{f(x+(h^{1/3})^3)-f(x)}{(h^{1/3})^3}$

$f'(x)=\lim_{h\to 0}\frac{f(x)+(f(h^{1/3}))^3-f(x)}{(h^{1/3})^3}$ [using Eq. (i)]

$\Rightarrow \quad f'(x)=\lim_{h\to 0}\left(\frac{f(h^{1/3})}{(h^{1/3})}\right)^3=(f'(0))^3$

$\Rightarrow \quad f'(x)=0$ or 1 [as $f'(0)=0$ or 1, using Eq. (v)]

On integrating both sides, we get

$f(x)=C$ or $x+C$ as $f(0)=0$

$\Rightarrow \quad f(x)=0$ or x

Thus, $f(10)=0$ or 10

● ***Ex. 25*** *Let $f(x)=x^3-x^2+x+1$*

and $g(x)=\begin{cases}\max f(t), & 0\leq t\leq x \text{ for } 0\leq x\leq 1\\ 3-x, & 1<x\leq 2\end{cases}$

Then, $g(x)$ in $[0,2]$ is

(a) continuous for $x\in[0,2]-\{1\}$

(b) continuous for $x\in[0,2]$

(c) differentiable for all $x\in[0,2]$

(d) differentiable for all $x\in[0,2]-\{1\}$

Sol. (*b, d*) Here, $f(x)=x^3-x^2+x+1$

$\Rightarrow f'(x)=3x^2-2x+1$, which is strictly increasing in (0, 2).

$\therefore \quad g(x)=\begin{cases}f(x), & 0\leq x\leq 1\\ 3-x, & 1<x\leq 2\end{cases}$

[as $f(x)$ is increasing, so $f(x)$ is maximum when $0\leq t\leq x$]

So, $g(x)=\begin{cases}x^3-x^2+x+1, & 0\leq x\leq 1\\ 3-x, & 1<x\leq 2\end{cases}$

Also, $g'(x)=\begin{cases}3x^2-2x+1, & 0\leq x\leq 1\\ -1, & 1<x\leq 2\end{cases}$

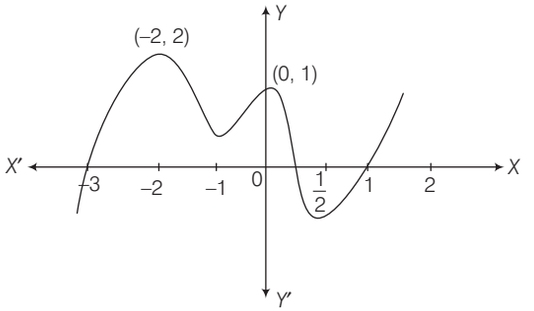
Which clearly shows $g(x)$ is continuous for all $x\in[0,2]$ but $g(x)$ is not differentiable at $x=1$.

JEE Type Solved Examples : Passage Based Questions

Passage I

(Ex. Nos. 26 to 27)

In the given figure graph of $y = P(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$, *is given.*



26. If $P''(x)$ has real roots α, β, γ, then $[\alpha]+[\beta]+[\gamma]$, is

(a) -2 (b) -3 (c) -1 (d) 0

27. The minimum number of real roots of equation $(P''(x))^2 + P'(x)\cdot P'''(x) = 0$, is

(a) 5 (b) 7 (c) 6 (d) 4

Sol. (Ex. Nos. 26 to 27) Here, $P'(x) = 0$ at $x = \left\{-2, -1, 0, \frac{1}{2}\right\}$

$\Rightarrow$ $P''(x)$ will have real roots $\in (-2, -1), (-1, 0), \left(0, \frac{1}{2}\right)$.

$\Rightarrow$ $[\alpha]+[\beta]+[\gamma] = -2-1+0 = -3$

Also, let $f(x) = (P''(x))^2 + P'(x)\cdot P'''(x)$

$\therefore$ $f(x) = \frac{d}{dx}(P'(x)\cdot P''(x))$

Since, $P'(x)$ has 4 real roots.

$\Rightarrow$ $P''(x)$ has 3 real roots.

$\Rightarrow$ $f(x)$ has 6 real roots.

26. (b) **27.** (c)

Passage II

(Ex. Nos. 28 to 30)

If α, β (*where* $\alpha < \beta$) *are the points of discontinuity of the function* $g(x) = f(f(f(x)))$, *where* $f(x) = \frac{1}{1-x}$ *and* $P(a, a^2)$ *is any point on XY-plane. Then,*

28. The points of discontinuity of $g(x)$ is

(a) $x = 0, -1$ (b) $x = 1$ only (c) $x = 0$ only (d) $x = 0, 1$

Sol. (*d*) Here, $f(x) = \frac{1}{1-x}$

$\Rightarrow$ $x \neq 1$

$$f(f(x)) = \frac{1}{1-f(x)} = \frac{1}{1-\frac{1}{1-x}} = \frac{x-1}{x}$$

$\Rightarrow$ $x \neq 0, 1$

and $f(f(f(x))) = \frac{f(x)-1}{f(x)} = \frac{\frac{1}{1-x}-1}{\frac{1}{x-1}} = x$, where $x \neq 0, 1$.

Here, point of discontinuity are $x = 0, 1$.

29. The domain of $f(g(x))$, is

(a) $x \in R$
(b) $x \in R - \{1\}$
(c) $x \in R - \{0, 1\}$
(d) $x \in R - \{0, 1, -1\}$

Sol. (*c*) Here, $g(x) = f(f(f(x))) = x$, $x \in R - \{0, 1\}$

$$f(g(x)) = f(x) = \frac{1}{1-x}$$

$\Rightarrow$ $x \neq 0, 1$

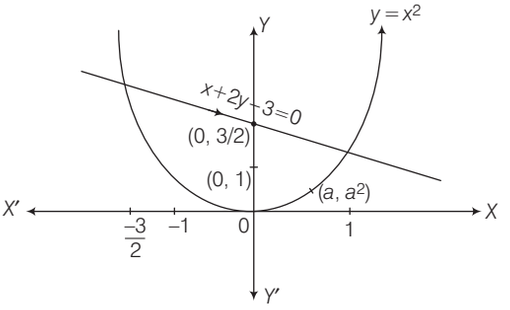
$\therefore$ Domain of $f(g(x)) \in R - \{0, 1\}$

30. If point $P(a, a^2)$ lies on the same side as that of (α, β) with respect to line $x + 2y - 3 = 0$, then

(a) $a \in \left(-\frac{3}{2}, 1\right)$ (b) $a \in R$

(c) $a \in \left(-\frac{3}{2}, 0\right)$ (d) $a \in (0, 1)$

Sol. (*a*) From graph, $a \in \left(-\frac{3}{2}, 1\right)$, as $(\alpha, \beta) = (0, 1)$.



JEE Type Solved Examples :
Matching Type Questions

● ***Ex. 31*** *Let* $f(x)=\begin{cases}[x], & -2\le x<0\\ |x|, & 0\le x\le 2\end{cases}$
(*where* $[\cdot]$ *denotes the greatest integer function*)
$g(x)=\sec x, x\in R-(2n+1)\pi/2$.
Match the following statements in Column I *with their values in Column* II *in the interval* $\left(-\frac{3\pi}{2},\frac{3\pi}{2}\right)$.

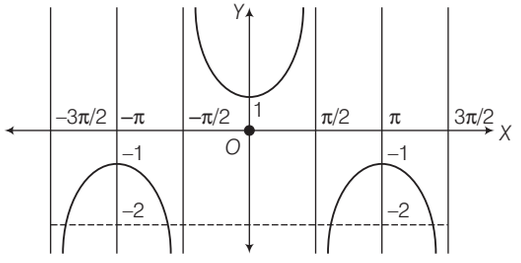
	Column I		Column II
(A)	Limit of *fog* exists at	(p)	-1
(B)	Limit of *gof* doesn't exist at	(q)	π
(C)	Points of discontinuity of *fog* is/are	(r)	$\frac{5\pi}{6}$
(D)	Points of differentiability of *fog* is/are	(s)	$-\pi$

Sol. (A) → (p, q, s), (B) → (p), (C) → (q, s), (D) → (p, r)

$$f(x)=\begin{cases}[x], & -2\le x<0\\ |x|, & 0\le x\le 2\end{cases}\Rightarrow f(x)=\begin{cases}-2, & -2\le x<-1\\ -1, & -1\le x<0\\ x, & 0\le x\le 2\end{cases}$$

$$g(x)=\sec x,\ x\in R-(2n+1)\frac{\pi}{2}$$

$$\Rightarrow\quad fog(x)=\begin{cases}-2, & -2\le \sec x<-1\\ -1, & -1\le \sec x<0\\ \sec x, & 0\le \sec x\le 2\end{cases}$$



$$\therefore\quad fog=\begin{cases}-2, & x\in\left[-\frac{4\pi}{3},-\frac{2\pi}{3}\right]\cup\left[\frac{2\pi}{3},\frac{4\pi}{3}\right]-\{-\pi,\pi\}\\ -1, & x=-\pi,\pi\\ \sec x, & x\in\left[-\frac{\pi}{3},\frac{\pi}{3}\right]\end{cases}$$

Limit of *fog* exist at $x=-\pi,\pi,-1$ points of discontinuity of *fog* are $-\pi,\pi$ and points of differentiability of *fog* are $-1,\frac{5\pi}{6}$.

$$gof=\begin{cases}\sec(-2), & x\in[-2,-1)-\left\{-\frac{\pi}{2}\right\}\\ \sec(-1), & x\in[-1,0)\\ \sec x, & x\in[0,2]-\left\{\frac{\pi}{2}\right\}\end{cases}$$

Limit of *gof* doesn't exist at $x=-1$.

● ***Ex. 32*** *Match the functions in Column* I *with the properties Column* II.

	Column I		Column II
(A)	$g:R\to Q$ (Rational number), $f:R\to Q$ (Rational number); f and g are continuous functions such that $\sqrt{3}\,f(x)+g(x)=3$, then $(1-f(x))^3+(g(x)-3)^3$ is	(p)	1
(B)	If $f(x)$, $g(x)$ and $h(x)$ are continuous and positive functions such that $f(x)+g(x)+h(x)=\sqrt{f(x)g(x)}+\sqrt{g(x)h(x)}+\sqrt{h(x)f(x)}$, then $f(x)+g(x)-2h(x)$ is	(q)	0
(C)	$y=f(x)$ satisfies the equation $y^3-2y^2(x+1)+4xy+(x^2-1)(y-2)=0$, then $y'(1)+y(1)$ would be equal to	(r)	2
(D)	If $y=f(x)$ satisfies $(xf(x))^{99}+(xf(x))^{98}+\ldots+(xf(x))+1=0$, then $(1+f(1))$ is	(s)	3
		(t)	-1

Sol. (A) → (p), (B) → (q), (C) → (p, r,s), (D) → (q)

(A) On comparing, $f(x)=0, g(x)=3$

$\therefore\quad (1-f(x))^3+(g(x)-3)^3=1$

(B) Here, $f(x)+g(x)+h(x)$

$=\sqrt{f(x)\cdot g(x)}+\sqrt{g(x)\cdot h(x)}+\sqrt{h(x)\cdot f(x)}$

$$\Rightarrow\ \frac{1}{2}\{(\sqrt{f(x)}-\sqrt{g(x)})^2+(\sqrt{g(x)}-\sqrt{h(x)})^2+(\sqrt{h(x)}-\sqrt{f(x)})^2\}=0$$

$\Rightarrow\quad f(x)=g(x)=h(x)$

$\therefore\quad f(x)+g(x)-2h(x)=0$

(C) $y^3 - 2y^2(x+1) + 4xy + (x^2-1)(y-2) = 0$

Put $y = 2 \Rightarrow 8 - 8(x+1) + 8x = 0$

$\therefore$ $y = 2$ is solution.

Put $y = (x+1)$

$\Rightarrow (x+1)^3 - 2(x+1)^3 + 4x(x+1) + (x^2-1)(x-1)$

$= -(x+1)^3 + 4x(x+1) + (x+1)(x-1)^2$

$= -(x+1)^3 + (x+1)\{4x + (x-1)^2\}$

$= -(x+1)^3 + (x+1)^3 = 0$

$\therefore$ $y = (x+1)$ is the solution.

Similarly, $y = (x-1)$ is the solution

$\Rightarrow \quad y = 2, x+1, x-1$

$\therefore \quad \frac{dy}{dx} = 0, 1$

$\Rightarrow$ $\begin{cases} y'(1) + y(1) = 0 + 2 = 2, \text{ when } y = 2 \\ y'(1) + y(1) = 1 + 2 = 3, \text{ when } y = x+1 \\ y'(1) + y(1) = 1 + 0 = 1, \text{ when } y = x-1 \end{cases}$

(D) $\frac{(xf(x))^{100} - 1}{xf(x) - 1} = 0 \Rightarrow xf(x) = -1$

As, $xf(x) \neq 1$

$\therefore$ $xf(x) = -1$

$\Rightarrow$ $f(1) = -1$ or $1 + f(1) = 0$

● ***Ex. 33*** *Suppose a function $f(x)$ satisfies the following conditions*

$f(x+y) = \frac{f(x) + f(y)}{1 + f(x)\,f(y)}, \forall\, x, y$ *and* $f'(0) = 1$. *Also,*

$-1 < f(x) < 1, \forall\, x \in R$.

Match the entries of the following two columns.

	Column I		Column II
(A)	$f(x)$ is differentiable over the set	(p)	$R - (-1, 0, 1)$
(B)	$f(x)$ increases in the interval	(q)	R
(C)	Number of the solutions of $f(x) = 0$ is	(r)	0
(D)	The value of the limit $\lim_{x \to \infty} [f(x)]^x$ is	(s)	1

Sol. (A) $\to$ (q), (B) $\to$ (q), (C) $\to$ (s), D $\to$ (s)

Put $x = y = 0 \Rightarrow f(0) = 0$

Now, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{\frac{f(x) + f(h)}{1 + f(x)\,f(h)} - f(x)}{h}$

$= \lim_{h \to 0} \frac{f(h)[1 - \{f(x)\}^2]}{h[1 + f(x)\,f(h)]}$

$= \lim_{h \to 0} \left\{\frac{f(h) - f(0)}{h - 0}\right\} \left[\frac{1 - \{f(x)\}^2}{1 + f(x)\,f(h)}\right]$

$= f'(0)[1 - \{f(x)\}^2] = 1 - \{f^2(x)\}$

$\therefore \quad f'(x) = 1 - \{f^2(x)\}$ ...(i)

On integrating both sides, we get

$\frac{1}{2} \ln\left[\frac{1 + f(x)}{1 - f(x)}\right] = x + c$

or $\frac{1 + f(x)}{1 - f(x)} = ke^{2x}$

Now, $f(0) = 0 \Rightarrow k = 1$

$\therefore \quad f(x) = \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

Clearly, $f(x)$ is differentiable for all $x \in R$ and from Eq. (i), $f'(x) > 0$ for all $x \in R$. Again, $f(x)$ is an odd function, $f(x) = 0 \Rightarrow x = 0$.

Now, $\lim_{x \to \infty} [f(x)]^x = \lim_{x \to \infty} \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^x$

$= e^{\lim_{x \to \infty} \left(\frac{e^x - e^{-x}}{e^x + e^{-x}} - 1\right) x} = e^{-2 \lim_{x \to \infty} \left(\frac{xe^{-x}}{e^x + e^{-x}}\right)}$

$= e^{-2 \lim_{x \to \infty} \left(\frac{x}{e^{2x} + 1}\right)}$

$= e^{-2 \lim_{x \to \infty} \left(\frac{1}{2e^{2x}}\right)} = e^0 = 1$

JEE Type Solved Examples :
Single Integer Answer Type Questions

● **Ex. 34** *Let* $f(x)=\begin{cases}\dfrac{1-\cos 4x}{x^2}, & x<0\\ a, & x=0\\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}, & x>0\end{cases}$

Then, the value of a if possible, so that the function is continuous at $x=0$ *, is ..... .*

Sol. (8) As, $f(x)$ is continuous at $x=0$.

$\therefore$ We must have,

$$\text{RHL (at } x=0)=\text{LHL (at } x=0)=f(0)$$

$$\text{RHL (at } x=0)=\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\frac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}$$

$$=\lim_{h\to 0}\frac{\sqrt{0+h}}{\sqrt{16+\sqrt{0+h}}-4}\times\frac{\sqrt{16+\sqrt{h}}+4}{\sqrt{16+\sqrt{h}}+4} \qquad [\text{putting } x=0+h]$$

$$=\lim_{h\to 0}\frac{\sqrt{h}\{\sqrt{16+\sqrt{h}}+4\}}{16+\sqrt{h}-16}$$

$$=\lim_{h\to 0}\{\sqrt{16+\sqrt{h}}+4\}=8$$

$$\text{Also, LHL (at } x=0)=\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\frac{1-\cos 4x}{x^2}$$

$$=\lim_{h\to 0}\frac{1-\cos 4(0-h)}{(0-h)^2} \qquad [\text{putting } x=0-h]$$

$$=\lim_{h\to 0}\frac{1-\cos 4h}{h^2}$$

$$=\lim_{h\to 0}\frac{2\sin^2 2h}{h^2}$$

$$=\lim_{h\to 0} 8\left(\frac{\sin 2h}{2h}\right)^2=8$$

And $f(0)=a$

Since, $f(x)$ is continuous at $x=0$

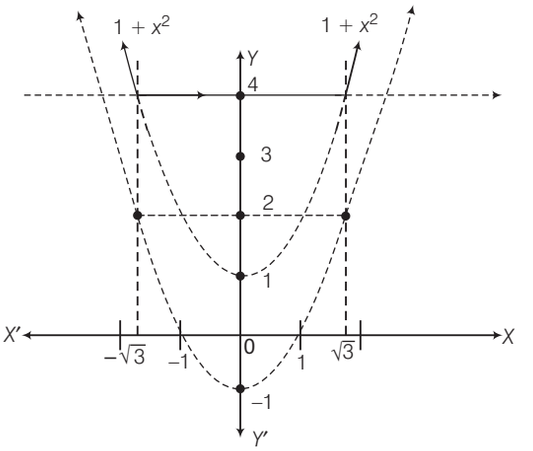
$\Rightarrow$ $f(0)=\text{RHL}=\text{LHL}$

or $f(0)=8$ or $a=8$

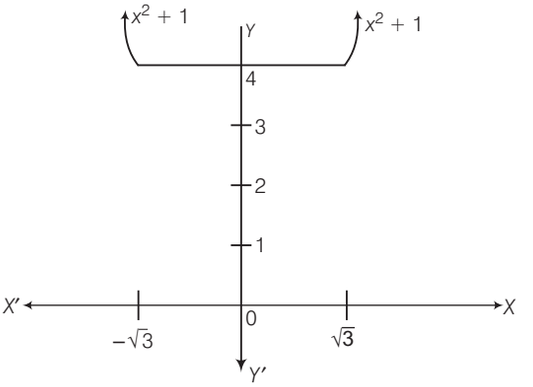
● **Ex. 35** *Let* $f(x)=\text{maximum}\{4,1+x^2,x^2-1\},\forall\, x\in R$.

Then, the total number of points, where $f(x)$ *is not differentiable, ...*

Sol. (2) We have, discussed in the last chapter for sketching maximum $\{4, 1+x^2, x^2-1\}$ as



Or



Thus, from above graph, we can simply say

$f(x)$ is not differentiable at $x=\pm\sqrt{3}$.

$\therefore$ Not differentiable at 2 points.

● **Ex. 36** *Let* $f(x)=x^n$*, n being non-negative integer. Then, the number of value of n for which the equality* $f'(a+b)=f'(a)+f'(b)$ *is valid for all* $a,b>0$*, is*

Sol. (2) Since, $f(x)=x^n$, n being non-negative integer.

Then, $f'(x)=nx^{n-1}$

$$f'(a)=na^{n-1},\ f'(b)=nb^{n-1},$$

$$f'(a+b)=n(a+b)^{n-1}$$

Now, the equality $f'(a+b)=f'(a)+f'(b)$ holds, if

$$n(a+b)^{n-1}=na^{n-1}+nb^{n-1}$$

or $(a+b)^{n-1}=a^{n-1}+b^{n-1}$ ...(i)

Above statement is true, only if $(n-1)=1 \Rightarrow n=2$

i.e. $(a+b)^{n-1}=a^{n-1}+b^{n-1}$ [if $n=2$]

or $(a+b)^1 = a^1 + b^1$

Also, when $n = 1$, then LHS < RHS

and when $n = 3, 4, 5, \ldots$, then LHS > RHS

Again, when $n = 0$, $f'(x) = 0$ for all x.

So, the equality is true for $n = 0$ and 2.

∴ Number of values is 2.

● ***Ex. 37*** *Find the number of points where* $f(x) = [\sin x + \cos x]$ *(where* $[\cdot]$ *denotes greatest integral function),* $x \in [0, 2\pi]$ *is not continuous, is/are..... .*

Sol. (5) We know $[\cdot]$ is not continuous at integral points.

Thus, $f(x) = [\sin x + \cos x]$ will be discontinuous at those points, where $\sin x + \cos x$ is an integer, i.e.

$$x \in \left\{\frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{3\pi}{2}, \frac{7\pi}{4}\right\}$$

Thus, the number of points at which $f(x)$ is discontinuous is 5.

● ***Ex. 38*** *The number of points where* $|xf(x)| + ||x-2|-1||$ *is non-differentiable in* $x \in (0, 3\pi)$*, where*

$$f(x) = \prod_{k=1}^{\infty}\left(\frac{1+2\cos\left(\frac{2x}{3^k}\right)}{3}\right), \text{ is ..... .}$$

Sol. (5) We know,

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta = \sin\theta(1+2\cos 2\theta)$$

$$\therefore \quad \sin 3\theta = \sin\theta\,(1+2\cos 2\theta) \qquad \ldots(i)$$

Now, on putting $3\theta = x, \frac{x}{3}, \frac{x}{3^2}, \ldots$ one by one in Eq. (i), we get

$$\Rightarrow \quad \sin x = \sin\left(\frac{x}{3}\right)\cdot\left(1+2\cos\frac{2x}{3}\right)$$

and $$\sin\frac{x}{3} = \sin\left(\frac{x}{3^2}\right)\cdot\left(1+2\cos\frac{2x}{3^2}\right)$$

$$\therefore \quad \sin\frac{x}{3^2} = \sin\left(\frac{x}{3^3}\right)\cdot\left(1+2\cos\frac{2x}{3^3}\right) \ldots \text{and so on}$$

$$\Rightarrow \quad \sin x = \sin\left(\frac{x}{3^n}\right)\cdot\prod_{k=1}^{n}\left(1+2\cos\frac{2x}{3^k}\right)$$

$$\Rightarrow \quad \frac{\sin x}{x} = \lim_{n\to\infty}\frac{\sin x/3^n}{x/3^n}\prod_{k=1}^{n}\frac{\left(1+2\cos\left(\frac{2x}{3^k}\right)\right)}{3^n}$$

$$\Rightarrow \quad \frac{\sin x}{x} = \prod_{k=1}^{\infty}\left(\frac{1+2\cos\left(\frac{2x}{3^k}\right)}{3}\right) = f(x) \qquad \ldots(ii)$$

Thus, $|xf(x)| + ||x-2|-1| = \frac{|x||\sin x|}{|x|} + ||x-2|-1|$ is not differentiable at $\{\pi, 2\pi, 0, 1, 3\}$.

∴ Number of points are 5.

● ***Ex. 39*** *If* $f\left(\frac{xy}{2}\right) = \frac{f(x)\cdot f(y)}{2}$; $x, y \in R$, $f(1) = f'(1)$.

Then, $\frac{f(3)}{f'(3)}$ *is ..... .*

Sol. (3) Here, $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0}\frac{f\left(\frac{2x+2h}{2}\right)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f\left(\frac{2x(1+h/x)}{2}\right)-f\left(\frac{2x\cdot 1}{2}\right)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{f(2x)\cdot f(1+h/x)}{2}-\frac{f(2x)\cdot f(1)}{2}}{h}$$

$$= \frac{f(2x)}{2x}\cdot\lim_{h\to 0}\frac{f(1+h/x)-f(1)}{h/x}$$

$$= \frac{f(2x)}{2x}\cdot f'(1)$$

$$= \frac{f(2x)}{2x}\cdot f(1), \text{ given } f(1) = f'(1)$$

$$= \frac{f\left(\frac{2x\cdot 1}{2}\right)}{x}, \text{ using } f\left(\frac{xy}{2}\right) = \frac{f(x)\cdot f(y)}{2}$$

$$\therefore \quad f'(x) = \frac{f(x)}{x}$$

$$\Rightarrow \quad \frac{f(x)}{f'(x)} = x \Rightarrow \frac{f(3)}{f'(3)} = 3$$

● ***Ex. 40*** *Let* $f: R \to R$ *be a differentiable function satisfying* $f(x) = f(y)\,f(x-y), \forall x, y \in R$ *and* $f'(0) = \int_0^4 \{2x\}\,dx$*, where* $\{\cdot\}$ *denotes the fractional part function and* $f'(-3) = \alpha e^{\beta}$*. Then,* $|\alpha+\beta|$ *is equal to ..... .*

Sol. (4) Given, $f(x) = f(y)\,f(x-y)$

On replacing x by $(x+y)$,

$$f(x+y) = f(y)\cdot f(x)$$

$$\Rightarrow \quad f(x) = e^{kx} \Rightarrow f'(x) = ke^{kx}$$

But, $$f'(0) = \int_0^4 \{2x\}dx = 2$$

$$\Rightarrow \quad f'(0) = k = 2 \Rightarrow f'(x) = 2e^{2x}$$

$$\Rightarrow \quad f'(-3) = 2e^{-6} \Rightarrow |\alpha+\beta| = 4$$

● **Ex. 41** *Let $f(x)$ is a polynomial function and $(f(\alpha))^2 + (f'(\alpha))^2 = 0$, then find $\lim_{x\to\alpha} \frac{f(x)}{f'(x)}\left[\frac{f'(x)}{f(x)}\right]$, where $[\cdot]$ denotes greatest integer function, is .....*

Sol. (1) Here, $(f(\alpha))^2 + (f'(\alpha))^2 = 0 \Rightarrow f(\alpha) = f'(\alpha) = 0$

$\therefore$ α is repeated root of $f(x)$.

Now, $\lim_{x\to\alpha} \frac{f(x)}{f'(x)}\left[\frac{f'(x)}{f(x)}\right] = \lim_{x\to\alpha} \frac{f(x)}{f'(x)}\left(\frac{f'(x)}{f(x)} - \left\{\frac{f'(x)}{f(x)}\right\}\right)$

$= \lim_{x\to\alpha}\left(1 - \frac{f(x)}{f'(x)} \cdot \left\{\frac{f'(x)}{f(x)}\right\}\right)$

$\left[\text{as, } x = \alpha \text{ is the repeated root and } \left\{\frac{f(x)}{f'(x)}\right\} \text{ is bounded; } \therefore \lim_{x\to\alpha} \frac{f(x)}{f'(x)} = 0\right]$

$= 1 - 0 = 1$

● **Ex. 42** *Let $f : R \to R$ is a function satisfying $f(10 - x) = f(x)$ and $f(2 - x) = f(2 + x), \forall\, x \in R$. If $f(0) = 101$. Then, the minimum possible number of values of x satisfying $f(x) = 101$, $x \in [0, 25]$ is ..... .*

Sol. (9) Since, $f(10 - x) = f(x) = f(4 - x)$

$\Rightarrow$ $f(10 - x) = f(4 - x)$

Say, $4 - x = t \Rightarrow f(6 + t) = f(t)$

$\Rightarrow f(x)$ is periodic function with period 6.

So, for $x \in [0, 25]$

$f(x) = 101$ at $x = 0, 6, 12, 18, 24$

Total numbers = 5

Since, $f(2 - x) = f(2 + x)$

$\Rightarrow f(x)$ is symmetric about $x = 2$ line.

Due to symmetry in one period length.

$f(x) = 101$ has one solution at $x = 4$ other than 0 and 6.

Now, $f(x) = 101$ at $x = 4, 10, 16, 22$

Total numbers = 4

Hence, atleast minimum possible number of values of $x = 9$.

● **Ex. 43** *If $f(x)$ is a differentiable function for all $x \in R$ such that $f(x)$ has fundamental period 2. $f(x) = 0$ has exactly two solutions in $[0, 2]$, also $f(0) \neq 0$. If minimum number of zeros of $h(x) = f'(x)\cos x - f(x)\sin x$ in $(0, 99)$ is $120 + k$, then k is ..... .*

Sol. (7) $h(x) = \frac{d}{dx}(f(x) \cdot \cos x)$

First find the minimum number of zeros of $(f(x) \cdot \cos x) = 0$.

$f(x) = 0$ has minimum 98 roots in $[0, 99)$ and $\cos x = 0$ has 31 roots is $[0, 99)$.

Maximum common possible root is only 1.

Hence, minimum number of roots of $f(x)\cos x = 0$ is 128.

Thus, $\frac{d}{dx}(f(x)\cos x) = 0$ has minimum 127 roots.

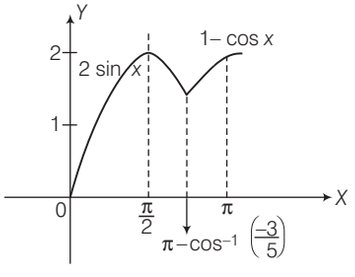
$127 = 120 + 7$

Hence, value of k is 7.

Subjective Type Questions

● **Ex. 44** *Let $f(x) = \max\{2\sin x, 1 - \cos x\}, \forall\, x \in (0, \pi)$, then discuss differentiability of $f(x)$ in $(0, \pi)$.*

Sol. We know that, $f(x) =$ maximum $\{2\sin x, 1 - \cos x\}$ can be plotted as



Thus, $f(x) =$ maximum $\{2\sin x, 1 - \cos x\}$ is not differentiable.

When, $2\sin x = 1 - \cos x$

or $4\sin^2 x = (1 - \cos x)^2$

or $4(1 + \cos x) = (1 - \cos x)$

or $4 + 4\cos x = 1 - \cos x$

or $\cos x = -3/5$

$\Rightarrow$ $x = \cos^{-1}(-3/5)$

$\therefore$ $f(x)$ is not differentiable at

$x = \pi - \cos^{-1}(3/5)$.

● **Ex. 45** *Discuss the continuity of the function $g(x) = [x] + [-x]$ at integral values of x.*

Sol. Here, x can assume two values

(a) integers (b) non-integers

(a) **If x is an integer**

$[x] = x$ and $[-x] = -x$

$\Rightarrow$ $g(x) = x - x = 0$

(b) **If x is not an integer**

Let $x = n + f$, where n is an integer and $f \in (0, 1)$

$\Rightarrow \quad [x] = [n + f] = n$

$\Rightarrow [-x] = [-n - f] = [(-n-1) + (1-f)] = (-n-1)$

$[\because 0 < f < 1 \Rightarrow (1-f) < 1]$

Hence, $g(x) = [x] + [-x] = n + (-n-1) = -1$

$\therefore \quad g(x) = \begin{cases} 0, & \text{if } x \text{ is an integer} \\ -1, & \text{if } x \text{ is not an integer} \end{cases}$

Let us discuss the continuity of $g(x)$ at a point $x = a$ [where $a \in$ integer]

$\text{LHL} = \lim_{x \to a^-} g(x) = -1$

$[\because \text{as } x \to a^-, x \text{ is not an integer}]$

$\text{RHL} = \lim_{x \to a^+} g(x) = -1$

$[\because \text{as } x \to a^+, x \text{ is not an integer}]$

But $g(a) = 0$ because a is an integer.

Hence, $g(x)$ has a removable discontinuity at integral values of x.

● ***Ex. 46*** *If* $f(x) = \dfrac{\sin 2x + A \sin x + B \cos x}{x^3}$ *is continuous at* $x = 0$. *Find the values of A and B. Also, find* $f(0)$.

Sol. As, $f(x)$ is continuous at $x = 0$.

$f(0) = \lim_{x \to 0} f(x)$ and both $f(0)$ and $\lim_{x \to 0} f(x)$ are finite.

$\Rightarrow \quad f(0) = \lim_{x \to 0} \dfrac{\sin 2x + A \sin x + B \cos x}{x^3}$

As, denominator $\to 0$ as $x \to 0$,

Numerator should also $\to 0$ as $x \to 0$

Which is possible only if (for $f(0)$ to be finite).

$\Rightarrow \quad \sin 2(0) + A \sin (0) + B \cos (0) = 0$

$\Rightarrow \quad B = 0$

$\therefore \quad f(0) = \lim_{x \to 0} \dfrac{\sin 2x + A \sin x}{x^3}$

$\Rightarrow \quad f(0) = \lim_{x \to 0} \left(\dfrac{\sin x}{x}\right)\left(\dfrac{2\cos x + A}{x^2}\right)$

$= \lim_{x \to 0} \left(\dfrac{2\cos x + A}{x^2}\right)$

Again, we can see that denominator $\to 0$ as $x \to 0$.

$\therefore$ Numerator should also approach to 0 as $x \to 0$ [$\because$ for $f(0)$ to be finite]

$\Rightarrow \quad 2 + A = 0$

$\Rightarrow \quad A = -2$

$\Rightarrow \quad f(0) = \lim_{x \to 0} \left(\dfrac{2\cos x - 2}{x^2}\right) = \lim_{x \to 0} \left(\dfrac{-4\sin^2 x/2}{x^2}\right)$

$= \lim_{x \to 0} -\left(\dfrac{-\sin^2 x/2}{x^2/4}\right) = -1$

So, we get $A = -2, B = 0$

and $f(0) = -1$

● ***Ex. 47*** *Let* $f : R \to R$ *satisfying* $|f(x)| \le x^2, \forall x \in R$, *then show that* $f(x)$ *is differentiable at* $x = 0$.

Sol. Since, $|f(x)| \le x^2, \forall x \in R$

$\therefore$ At $x = 0, |f(0)| \le 0$

$\Rightarrow \quad f(0) = 0 \quad \ldots(i)$

$\therefore \quad f'(0) = \lim_{h \to 0} \dfrac{f(h) - f(0)}{h} = \lim_{h \to 0} \dfrac{f(h)}{h} \quad \ldots(ii)$

$[f(0) = 0, \text{ from Eq. (i)}]$

Now, $\left|\dfrac{f(h)}{h}\right| \le |h|$

$\Rightarrow \quad -|h| \le \dfrac{f(h)}{h} \le |h|$

$\Rightarrow \quad \lim_{h \to 0} \dfrac{f(h)}{h} \to 0$

[using Cauchy-Squeeze theorem] ...(iii)

$\therefore$ From Eqs. (ii) and (iii), we get $f'(0) = 0$

i.e. $f(x)$ is differentiable at $x = 0$

● ***Ex. 48*** *Show that the function defined by*

$$f(x) = \begin{cases} x^2 \sin 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is differentiable for every value of x, but the derivative is not continuous for $x = 0$.

Sol. For $x \neq 0$,

$$f'(x) = 2x \sin (1/x) + x^2 \left(-\frac{1}{x^2}\right) \cos \left(\frac{1}{x}\right)$$

$$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$$

For $x = 0$,

$$f'(x) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h^2 \sin \frac{1}{h} - 0}{h}$$

$\Rightarrow \quad f'(x) = \lim_{h \to 0} h \sin \dfrac{1}{h} = 0$

Thus, $f'(x) = \begin{cases} 2x \sin \dfrac{1}{x} - \cos \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Now, $f'(x)$ is continuous at $x = 0$, if

(i) $\lim_{x \to 0} f'(x)$ exists (ii) $\lim_{x \to 0} f'(x) = f'(0)$

Again, $\lim_{x \to 0} f'(x) = \lim_{x \to 0} \left(2x \sin \dfrac{1}{x} - \cos \dfrac{1}{x}\right)$, doesn't exist

Since, $\lim_{x \to 0} \cos \dfrac{1}{x}$ doesn't exist.

Hence, $f'(x)$ is not continuous at $x = 0$.

● ***Ex. 49*** *If* $f(x)=\begin{cases} x-[x], & x\notin I \\ 1, & x\in I \end{cases}$

where I is an integer and $[\cdot]$ *represents the greatest integer function and*

$$g(x)=\lim_{n\to\infty}\frac{\{f(x)\}^{2n}-1}{\{f(x)\}^{2n}+1},\ \textit{then}$$

(a) Draw graphs of $f(2x)$, $g(x)$ and $g\{g(x)\}$.

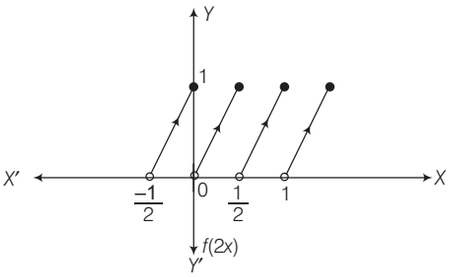
(b) Find the domain and range of these functions.

(c) Are these functions periodic? If yes, find their periods.

Sol. Here, $f(x)=\begin{cases} x-[x], & x\notin I \\ 1, & x\in I \end{cases}$

(a) **Graph of $f(2x)$**

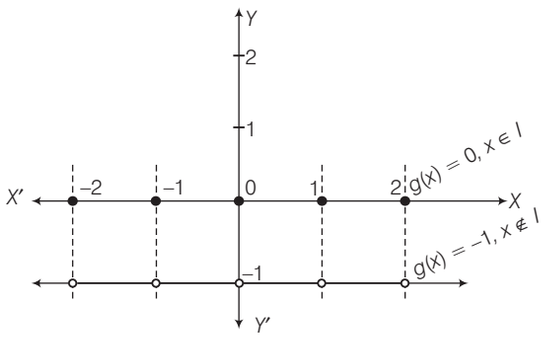
As, $f(2x)=\begin{cases} 2x-[2x], & 2x\notin I \\ 1, & 2x\in I \end{cases}$



Graph of $g(x)$

Here, $g(x)=\lim_{n\to\infty}\frac{\{f(x)\}^{2n}-1}{\{f(x)\}^{2n}+1}$

$$g(x)=\begin{cases} 0, & x\in I \\ -1, & x\notin I, \text{ as } \lim_{n\to\infty}\{f(x)\}^{2n}=0 \end{cases}$$



Graph of $g\{g(x)\}$

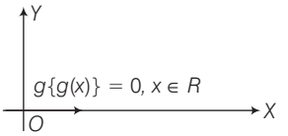
We have, $g(x)=\begin{cases} 0, & x\in I \\ -1, & x\notin I \end{cases}$

$\Rightarrow$ $g\{g(x)\}=\begin{cases} 0, & g(x)\in I \\ -1, & g(x)\notin I \end{cases}$

where $g(x)\in\{0,-1\}$

and thus $g(x)\notin I$, should be neglected.

$\Rightarrow$ $g\{g(x)\}=0,\ x\in R$ and could be plotted as,



(b) From the above three graphs, we have

Domain of $f(2x)\in R$

and Range of $f(2x)\in(0,1]$

Domain of $g(x)\in R$

and Range of $g(x)\in\{0,-1\}$

Domain of $g(g(x))\in R$

and Range of $g(g(x))\in\{0\}$

(c) Here, $f(2x)$ and $g(x)$ are periodic with period 1/2 and 1. Also, $g\{g(x)\}$ is constant function. Therefore, periodic and its period is undetermined.

● ***Ex. 50*** *Prove that* $f(x)=[\tan x]+\sqrt{\tan x-[\tan x]}$. *(where* $[\cdot]$ *denotes greatest integer function) is continuous in* $\left[0,\frac{\pi}{2}\right)$.

Sol. Here, $f(x)=[\tan x]+\sqrt{\tan x-[\tan x]}$

or $g(x)=[x]+\sqrt{x-[x]}$, where $x=\tan x\geq 0$

Then, for $a\in N$. We discuss continuity of $f(x)$ as

LHL (at $x=a$)

$$=\lim_{x\to a^-}g(x)=\lim_{h\to 0}([a-h]+\sqrt{a-h-[a-h]})$$

$$=\lim_{h\to 0}((a-1)+\sqrt{a-h-a+1})=a-1+1=a$$

$$\begin{cases} \text{where}, & a-1<a-h<a \\ \therefore [a-h]=a-1 \end{cases}$$

Now, RHL (at $x=a$)

$$=\lim_{x\to a^+}g(x)=\lim_{h\to 0}([a+h]+\sqrt{a+h-[a+h]})$$

$$=\lim_{h\to 0}(a+\sqrt{a+h-a})=a$$

$$[\because a<a+h<a+1\ \Rightarrow\ [a+h]=a]$$

and $g(a)=[a]+\sqrt{a-[a]}=a$

as $a\in N$

So, $g(x)$ is continuous, $\forall\ a\in N$, now $g(x)$ is clearly continuous in $(a-1,\ a)$, $\forall\ a\in N$.

Hence, $g(x)$ is continuous in $[0,\infty)$.

Now, let $\phi(x)=\tan x$ which is continuous in $[0,\pi/2]$.

So, $g\{\phi(x)\}$ is continuous in $[0,\pi/2)$.

Hence, $f(x)=[\tan x]+\sqrt{\tan x-[\tan x]}$ is continuous in $[0,\pi/2)$.

● **Ex. 51** *Determine the values of x for which the following functions fails to be continuous or differentiable*

$$f(x)=\begin{cases}(1-x), & x<1\\ (1-x)(2-x), & 1\le x\le 2\\ (3-x), & x>2\end{cases}$$

Justify your answer. ***[IIT JEE 1995]***

Sol. By the given definition, it is clear that the function f is continuous and differentiable at all points except possibility at $x=1$ and $x=2$.

Continuity at $x=1$

$$\text{LHL}=\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(1-x)=\lim_{h\to 0}(1-(1-h))=0$$

$$\text{RHL}=\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(1-x)(2-x)$$

$$=\lim_{h\to 0}\{1-(1+h)\}\{2-(1+h)\}=0$$

Also, $f(1)=0$.

Hence, LHL = RHL = $f(1)$

Therefore, $f(x)$ is continuous at $x=1$.

Differentiability at $x=1$

$$Lf'(1)=\lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}=\lim_{h\to 0}\frac{1-(1-h)-0}{-h}=-1$$

$$Rf'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$=\lim_{h\to 0}\frac{\{1-(1+h)\}\{2-(1+h)\}-0}{h}=-1$$

Since, $L\{f'(1)\}=R\{f'(1)\}$ so, we get $f(x)$ is differentiable at $x=1$.

Continuity at $x=2$

$$\text{LHL}=\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}(1-x)(2-x)$$

$$=\lim_{h\to 0}(1-(2-h))(2-(2-h))=0$$

$$\text{RHL}=\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}(3-x)=\lim_{h\to 0}3-(2+h)=1$$

Since, RHL ≠ LHL

Therefore, $f(x)$ is not continuous at $x=2$.

As such $f(x)$ cannot be differentiable at $x=2$.

Hence, $f(x)$ is continuous and differentiable at all points except at $x=2$.

● **Ex. 52** *If $g(x)$ is continuous function in $[0,\infty)$ satisfying $g(1)=1$. If $\int_0^x 2x\cdot g^2(t)\,dt=\left(\int_0^x 2g(x-t)\,dt\right)^2$, find $g(x)$.*

Sol. Here, $\int_0^x g(t)\,dt=\int_0^x g(x-t)\,dt$, the given equation could be written as

$$x\int_0^x g^2(t)\,dt=2\left(\int_0^x g(t)\,dt\right)^2 \qquad ...(i)$$

On differentiating both sides w.r.t. 'x', we get

$$\int_0^x g^2(t)\,dt+xg^2(x)=4g(x)\left\{\int_0^x g(t)\,dt\right\}$$

or $$x\int_0^x g^2(t)\,dt+x^2g^2(x)=4xg(x)\int_0^x g(t)\,dt \qquad ...(ii)$$

Using Eq. (i),

$$2\left(\int_0^x g(t)\,dt\right)^2+x^2g^2(x)=4xg(x)\int_0^x g(t)\,dt$$

$$\Rightarrow\quad 2\left(\int_0^x g(t)\,dt\right)^2-4xg(x)\int_0^x g(t)\,dt+x^2g^2(x)=0$$

which is quadratic in $\int_0^x g(t)\,dt$,

$$\therefore\qquad \int_0^x g(t)\,dt=\frac{2\pm\sqrt{2}}{2}xg(x)$$

On differentiating both sides w.r.t. x, we get

$$\frac{g'(x)}{g(x)}=\frac{1\pm\sqrt{2}}{x}$$

$$\Rightarrow\qquad g(x)=kx^{1\pm\sqrt{2}},\ \text{since } g(1)=1$$

$$\Rightarrow\qquad k=1$$

$$\therefore\qquad g(x)=x^{1\pm\sqrt{2}}$$

which is continuous in $[0,\infty)$.

● **Ex. 53** *Let $f(x)=\begin{cases}x+a, & \text{if } x<0\\ |x-1|, & \text{if } x\ge 0\end{cases}$ and $g(x)=\begin{cases}x+1, & \text{if } x<0\\ (x-1)^2+b, & \text{if } x\ge 0\end{cases}$ where a and b are non-negative real numbers. Determine the composite function gof. If $(gof)(x)$ is continuous for all real x. Determine the values of a and b. Further for these values of a and b, is gof differentiable at $x=0$. Justify your answer.* ***[IIT JEE 2002]***

Sol. Here, $$f(x)=\begin{cases}x+a, & \text{if } x<0\\ |x-1|, & \text{if } x\ge 0\end{cases}$$

and $$g(x)=\begin{cases}x+1, & \text{if } x<0\\ (x-1)^2+b, & \text{if } x\ge 0\end{cases}$$

$$\therefore\qquad gof(x)=g\{f(x)\}=\begin{cases}g(x+a), & x<0\\ g(|x-1|), & x\ge 0\end{cases}$$

$$=\begin{cases}x+a+1, & x+a<0\\ (x+a-1)^2+b, & x+a\ge 0\\ \{|x-1|-1\}^2+b, & x\ge 0\end{cases}$$

$$=\begin{cases}x+a+1, & x<-a\\ (x+a-1)^2+b, & 0>x\ge -a\\ x^2+b, & 0\le x<1\\ (x-2)^2+b, & x\ge 1\end{cases}$$

$\Rightarrow gof(x)$ is continuous for all real x.

$\therefore$ $gof(x)$ is continuous at $x=-a, 0, 1$.

Since, $gof(x)$ is continuous at $x=-a$.

$$\Rightarrow \lim_{x \to -a^-} gof(x) = \lim_{x \to -a^+} gof(x) = gof(-a)$$

$$\Rightarrow \lim_{x \to -a^-} (x + a + 1) = \lim_{x \to -a^+} (x + a - 1)^2 + b$$

$$= (-a + a - 1)^2 + b$$

$$\Rightarrow -a + a + 1 = (-a + a - 1)^2 + b = (-a + a - 1)^2 + b$$

$$\Rightarrow b = 0$$

and $gof(x)$ is continuous at $x = 0$.

$$\Rightarrow \lim_{x \to 0^-} gof(x) = \lim_{x \to 0^+} gof(x) = gof(0)$$

$$\Rightarrow (a-1)^2 + b = b \Rightarrow a = 1$$

Now, $\text{(LHD at } x = 0) = \dfrac{d}{dx}\{(x + a - 1)^2 + b\}_{\text{at } x = 0}$

$$= 2(a - 1) = 0 \qquad [\text{as } a = 1]$$

Again, $\text{(RHD at } x = 0) = \dfrac{d}{dx}\{(x^2 + b)\}_{\text{at } x = 0} = 0$

$\therefore$ $gof(x)$ is differentiable at $x = 0$.

● ***Ex. 54*** *If a function $f : [-2a, 2a] \to R$ is an odd function such that $f(x) = f(2a - x)$ for $x \in [a, 2a]$ and the left hand derivative at $x = a$ is 0, then find the left hand derivative at $x = -a$* ***[IIT JEE 2003]***

Sol. It is given that, LHD (at $x = a$) $= 0$

$$\therefore \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h} = 0$$

$$\Rightarrow \lim_{h \to 0} \frac{f(a-h) - f(a)}{h} = 0 \qquad \text{...(i)}$$

Now, LHD (at $x = -a$)

$$= \lim_{h \to 0} \frac{f(-a-h) - f(-a)}{-h}$$

$$= \lim_{h \to 0} \frac{-f(a+h) + f(a)}{-h} \quad [\text{as } f(-x) = -f(x), \text{ given}]$$

$$= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{f[2a - (a-h)] - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{f(a-h) - f(a)}{h} = 0 \qquad [\text{as } f(x) = f(2a - x)]$$

[using Eq. (i)]

$\therefore$ LHD (at $x = -a$) $= 0$

● ***Ex. 55*** *Discuss the continuity of $f(x)$ in $[0, 2]$, where*

$$f(x) = \begin{cases} [\cos \pi x], & x \le 1 \\ |2x - 3|[x - 2], & x > 1 \end{cases}$$

where $[\cdot]$ denotes the greatest integral function.

Sol. First we shall find the points where $f(x)$ may be discontinuous.

Consider $x \in [0, 1]$

$f(x) = [\cos \pi x]$ is discontinuous where $\cos \pi x \in I$.

In $[0, 1]$, $\cos \pi x$ is an integer at $x = 0, 1/2, 1$.

$\Rightarrow x = 0, x = \dfrac{1}{2}$ and $x = 1$ may be the points at which $f(x)$ may be discontinuous ...(i)

Now, consider $x \in (1, 2]$

$$f(x) = [x - 2]|2x - 3|$$

If $x \in (1, 2)$, then $[x - 2] = -1$ and if $x = 2$, then $[x - 2] = 0$

Also, $|2x - 3| = 0 \Rightarrow x = 3/2$

$\Rightarrow x = 3/2$ and 2 may be the points at which $f(x)$ may be discontinuous ...(ii)

On combining Eqs. (i) and (ii), we have

$$x = 0, 1/2, 1, 3/2, 2$$

On dividing $f(x)$ by about the 5 critical points, we get

$$f(x) = \begin{cases} 1, & x = 0 \quad \because \cos(\pi\, 0) = 1 \\ 0, & 0 < x \le \dfrac{1}{2} \quad \because 0 \le \cos \pi x < 1 \Rightarrow [\cos \pi x] = 0 \\ -1, & \dfrac{1}{2} < x \le 1 \quad \because -1 \le \cos \pi x < 0 \Rightarrow [\cos \pi x] = -1 \\ -(3 - 2x), & 1 < x \le 3/2 \because |2x - 3| = 3 - 2x \\ & \text{and } [x - 2] = -1 \\ -(2x - 3), & 3/2 < x < 2 \because |2x - 3| = 2x - 3 \\ & \text{and } [x - 2] = -1 \\ 0, & x = 2 \quad \because [x - 2] = 0 \end{cases}$$

Checking continuity at $x = 0$

$$\text{RHL} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 0 = 0$$

And $f(0) = 1$

As $\text{RHL} \ne f(0)$

Thus, $f(x)$ is discontinuous at $x = 0$

Checking continuity at $x = 1/2$

$$\text{LHL} = \lim_{x \to \frac{1}{2}^-} f(x) = \lim_{x \to \frac{1}{2}^-} 0 = 0$$

$$\text{RHL} = \lim_{x \to \frac{1}{2}^+} f(x) = \lim_{x \to \frac{1}{2}^+} (-1) = -1$$

As $\text{RHL} \ne \text{LHL}$

Therefore, $f(x)$ is discontinuous at $x = 1/2$

Checking continuity at $x = 1$

$$\text{LHL} = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (-1) = -1$$

$$\text{RHL} = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} -(3 - 2x) = -1$$

And $f(1) = -1$

As $\text{RHL} = \text{LHL} = f(1)$

Therefore, $f(x)$ is continuous at $x = 1$

Checking continuity at $x = 3/2$

$$\text{LHL} = \lim_{x \to 3/2^-} (2x - 3) = 0$$

$$\text{RHL} = \lim_{x \to 3/2^+} (3 - 2x) = 0$$

And $f(3/2) = 0$

As $\text{RHL} = \text{LHL} = f(3/2)$

Hence, $f(x)$ is continuous at $x = 3/2$

Checking continuity at $x = 2$

$$\text{LHL} = \lim_{x \to 2^-} (3 - 2x) = -1$$

And $f(2) = 0$

As $\text{LHL} \neq f(2)$

Hence, $f(x)$ is discontinuous at $x = 2$

Thus, $f(x)$ is continuous when $x \in [0, 2] - \{0, 1/2, 2\}$.

● ***Ex. 56*** *Let f is a differentiable function such that $f(x) = x^2 + \int_0^x e^{-t} f(x - t)\, dt$. Find $f(x)$.*

Sol. We have, $f(x) = x^2 + \int_0^x e^{-t}\, f(x - t)\, dt$

$$\Rightarrow \quad f(x) = x^2 + \int_0^x e^{-(x-t)}\, f(x-(x-t))\, dt$$

$$\left[\text{using} \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$\Rightarrow \quad f(x) = x^2 + e^{-x} \int_0^x e^t\, f(t)\, dt \qquad ...(i)$$

On differentiating both the sides, we get

$$f'(x) = 2x + e^{-x}[e^x\, f(x)] - e^{-x} \int_0^x e^t\, f(t)\, dt \qquad ...(ii)$$

[using Leibnitz rule]

$$\Rightarrow \quad f(x) + f'(x) = x^2 + 2x + f(x)$$

$$\Rightarrow \quad f'(x) = x^2 + 2x \qquad ...(iii)$$

On integrating both the sides of Eq. (iii), we get

$$f(x) = \frac{x^3}{3} + x^2 + C$$

But $f(0) = 0 \Rightarrow C = 0$

Hence, $f(x) = \frac{x^3}{3} + x^2$

● ***Ex. 57*** *Let $f : R^+ \to R$ satisfies the functional equation $f(xy) = e^{xy - x - y}\, \{e^y f(x) + e^x f(y)\}, \forall\, x, y \in R^+$. If $f'(1) = e$, determine $f(x)$.*

Sol. Given that,

$$f(xy) = e^{xy - x - y}\, \{e^y f(x) + e^x f(y)\}, \forall\, x, y \in R^+ \qquad ...(i)$$

On putting $x = y = 1$, we get

$$f(1) = e^{-1}\, \{e^1 f(1) + e^1 f(1)\} \Rightarrow f(1) = 0 \qquad ...(ii)$$

Now, $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{f\left\{x\left(1 + \frac{h}{x}\right)\right\} - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{e^{x\left(1+\frac{h}{x}\right) - x - \left(1+\frac{h}{x}\right)} \left\{e^{1+\frac{h}{x}} f(x) + e^x f\left(1 + \frac{h}{x}\right)\right\} - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{e^h\, f(x) + e^{h - 1 - \frac{h}{x} + x} f\left(1 + \frac{h}{x}\right) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x)(e^h - 1) + e^{h - 1 - \frac{h}{x} + x} \left\{f\left(1 + \frac{h}{x}\right) - f(1)\right\}}{h}$$

$$[\because f(1) = 0]$$

$$= \lim_{h \to 0} \frac{f(x)(e^h - 1)}{h} + \lim_{h \to 0} \frac{e^{h - 1 - \frac{h}{x} + x} \left\{f\left(1 + \frac{h}{x}\right) - f(1)\right\}}{\frac{h}{x} \cdot x}$$

$$= f(x) + \frac{e^{x-1} \cdot f'(1)}{x}$$

$$\left[\because \lim_{h \to 0} \frac{f\left(1 + \frac{h}{x}\right) - f(1)}{\frac{h}{x}} = f'(1) \text{ and } \lim_{h \to 0} \frac{e^h - 1}{h} = 1\right]$$

$$= f(x) + \frac{e^x}{ex} \cdot f'(1)$$

$$\therefore f'(x) = f(x) + \frac{e^x}{x} \Rightarrow \frac{e^x}{x} = f'(x) - f(x) \quad [\because f'(1) = e]$$

$$\Rightarrow \quad \frac{1}{x} = \frac{e^x f'(x) - f(x) \cdot e^x}{e^{2x}}$$

$$\left[\text{as by quotient rule, we can write } \frac{e^x f'(x) - f(x) \cdot e^x}{(e^x)^2} = \frac{d}{dx}\left\{\frac{f(x)}{e^x}\right\}\right]$$

$$\therefore \quad \frac{1}{x} = \frac{d}{dx}\left\{\frac{f(x)}{e^x}\right\}$$

On integrating both the sides w.r.t. 'x', we get

$$\log|x| + C = \frac{f(x)}{e^x}$$

$$f(x) = e^x \{\log|x| + C\}$$

Since, $f(1) = 0 \Rightarrow C = 0$

Thus, $f(x) = e^x \log|x|$

● ***Ex. 58*** *Let f is a differentiable function such that $f'(x) = f(x) + \int_0^2 f(x)\, dx$, $f(0) = \frac{4 - e^2}{3}$, find $f(x)$.*

Sol. It is given that

$$f'(x) = f(x) + \int_0^2 f(x)\, dx$$

$$\Rightarrow \quad f'(x) = f(x) + c \qquad \left[\therefore c = \int_0^2 f(x)\, dx\right]$$

$$\Rightarrow \quad \frac{f'(x)}{f(x) + c} = 1$$

On integrating both the sides of above expression,

$$\log_e\{f(x) + c\} = x + c \Rightarrow f(x) + c = ke^x \qquad ...(i)$$

[k being constant of integration]

Since, $f(0)=\dfrac{4-e^2}{3}$ and $f(0)+c=k$

$$\therefore \quad k=\frac{4-e^2}{3}+c \Rightarrow c=k-\left(\frac{4-e^2}{3}\right) \qquad ...(ii)$$

From Eqs. (i) and (ii), we get

$$f(x)+k-\left(\frac{4-e^2}{3}\right)=k(e^x)$$

$$\Rightarrow \quad f(x)=k(e^x-1)+\left(\frac{4-e^2}{3}\right) \qquad ...(iii)$$

Now, to find constant of integration k.

On integrating both the sides from 0 to 2, we get

$$\int_0^2 f(x)\,dx=k\int_0^2(e^x-1)\,dx+\frac{4-e^2}{3}\int_0^2 dx$$

$$\Rightarrow \quad c=k\,(e^2-2-1)+\frac{4-e^2}{3}(2)$$

$$\Rightarrow \quad c=\frac{2}{3}(4-e^2)+k\,(e^2-3) \qquad ...(iv)$$

From Eqs. (ii) and (iv), we get

$$k-\left(\frac{4-e^2}{3}\right)=\frac{2}{3}(4-e^2)+k(e^2-3) \Rightarrow k=1$$

On putting $k=1$ in Eq. (iii), we get

$$f(x)=(e^x-1)+\frac{4-e^2}{3}$$

Hence, $$f(x)=e^x-\frac{(e^2-1)}{3}$$

● ***Ex. 59*** *A function $f(x)$ satisfies the following property $f(x+y)=f(x)\cdot f(y)$.*
Show that the function is continuous for all values of x, if it is continuous at $x=1$.

Sol. As the function is continuous at $x=1$, we have

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)=f(1)$$

$$\Rightarrow \quad \lim_{h\to 0} f(1-h)=\lim_{h\to 0} f(1+h)=f(1)$$

[using $f(x+y)=f(x)\cdot f(y)$]

$$\Rightarrow \quad \lim_{h\to 0} f(1)\,f(-h)=\lim_{h\to 0} f(1)\cdot f(h)=f(1)$$

$$\Rightarrow \quad \lim_{h\to 0} f(-h)=\lim_{h\to 0} f(h)=1 \qquad ...(i)$$

Now, consider some arbitrary points $x=a$.

Left hand limit

$$\Rightarrow \quad \lim_{h\to 0} f(a-h)=\lim_{h\to 0} f(a)\cdot f(-h)=f(a)\lim_{h\to 0} f(-h)$$

$$\text{LHL} = f(a) \text{ [as } \lim_{h\to 0} f(-h)=1, \text{ using Eq. (i)]}$$

Right hand limit

$$\Rightarrow \quad \lim_{h\to 0} f(a+h)=\lim_{h\to 0} f(a)\cdot f(h)=f(a)\lim_{h\to 0} f(h)$$

$$\text{RHL} = f(a) \quad \text{[as } \lim_{h\to 0} f(h)=1, \text{ using Eq. (i)]}$$

Hence, at any arbitrary point $(x=a)$

$$\text{LHL} = \text{RHL} = f(a)$$

Therefore, function is continuous for all values of x, if it is continuous at 1.

● ***Ex. 60*** *Let $f\left(\dfrac{x+y}{2}\right)=\dfrac{f(x)+f(y)}{2}$ for all real x and y. If $f'(0)$ exists and equals to -1 and $f(0)=1$, then find $f'(x)$.* ***[IIT JEE 1995]***

Sol. Given equation is $f\left(\dfrac{x+y}{2}\right)=\dfrac{f(x)+f(y)}{2}$ ...(i)

On putting $y=0$ and $f(0)=1$ in Eq. (i), we have

$$f\left(\frac{x}{2}\right)=\frac{1}{2}[f(x)+1] \qquad [\because f(0)=1]$$

$$\Rightarrow \quad f(x)=2f\left(\frac{x}{2}\right)-1 \qquad ...(ii)$$

Now, $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{f\left(\dfrac{2x+2h}{2}\right)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\dfrac{f(2x)+f(2h)}{2}-f(x)}{h} \qquad \text{[using Eq. (i)]}$$

$$=\lim_{h\to 0}\frac{f(2x)+f(2h)-2f(x)}{2h}$$

$$=\lim_{h\to 0}\frac{2f(x)-1+f(2h)-2f(x)}{2h} \qquad \text{[using Eq. (ii)]}$$

$$\lim_{h\to 0}\frac{f(2h)-1}{2h}=f'(0)$$

Therefore, $$f'(x)=-1$$

● ***Ex. 61*** *Let $f(x)=1+4x-x^2, \forall\ x\in R$*

$$g(x)=\max\{f(t);\ x\le t\le(x+1);\ 0\le x<3\}$$
$$=\min\{(x+3);\ 3\le x\le 5\}$$

Verify continuity of $g(x)$, for all $x\in[0,5]$.

Sol. Here, $$f(t)=1+4t-t^2$$

$$f'(t)=4-2t$$

When $f'(t)=0$, then $t=2$

At $t=2$, $f(t)$ has a maxima.

Since, $g(x)=\max\{f(t) \text{ for } t\in[x,x+1], 0\le x<3\}$

$$\therefore \quad g(x)=\begin{cases} f(x+1), & \text{if } [x,x+1]<2 \\ f(2), & \text{if } x\le 2\le x+1 \\ f(x), & \text{if } 2<[x,x+1] \end{cases}$$

$$g(x)=\begin{cases}4+2x-x^2, & \text{if} \quad 0\le x<1\\ 5, & \text{if} \quad 1\le x\le 2\\ 1+4x-x^2, & \text{if} \quad 2<x<3\\ 6, & \text{if} \quad 3\le x\le 5\end{cases}$$

Which is clearly continuous for all $x\in[0,5]$ except $x=3$.

$\therefore$ $g(x)$ is continuous for $x=[0,3)\cup(3,5]$

● ***Ex. 62*** *Let* $f(x)=x^4-8x^3+22x^2-24x$ *and*

$$g(x)=\begin{cases}\min f(x);\ x\le t\le x+1,-1\le x\le 1\\ x-10;\quad x>1\end{cases}$$

Discuss the continuity and differentiability of $g(x)$ *in* $[-1,\infty)$.

Sol. Here, $f(x)=x^4-8x^3+22x^2-24x$

$\Rightarrow$ $f'(x)=4x^3-24x^2+44x-24$

or $f'(x)=4(x-1)(x-2)(x-3)$

Which shows $f(x)$ is increasing in $[1,2]\cup[3,\infty)$ and decreasing in $(-\infty,1]\cup[2,3]$.

Thus, minimum $f(x);\ x\le t\le x+1,-1\le x\le 1$

$\Rightarrow$ Minimum $f(x)=\begin{cases}f(x+1),-1\le x\le 0\\ f(1), \quad 0<x\le 1\end{cases}$

Thus, $g(x)=\begin{cases}f(x+1),-1\le x\le 0\\ f(1), \quad 0<x\le 1\\ x-10, \quad x>1\end{cases}$

$$=\begin{cases}(x+1)^4-8(x+1)^3+22(x+1)^2-24(x+1), & -1\le x\le 0\\ 1-8+22-24, & 0<x\le 1\\ x-10, & x>1\end{cases}$$

$$g(x)=\begin{cases}x^4-4x^3+4x^2-9, & -1\le x\le 0\\ -9, & 0<x\le 1\\ x-10, & x>1\end{cases}$$

Also, $g'(x)=\begin{cases}4x^3-12x^2+8x, & -1\le x\le 0\\ 0, & 0<x\le 1\\ 1, & x>1\end{cases}$

Which clearly shows $g(x)$ is continuous in $[-1,\infty)$ but not differentiable at $x=1$.

● ***Ex. 63*** *Let* $g(x)=\int_0^x f(t)\,dt$, *where* f *is such that*

$1/2\le f(t)\le 1$ *for* $t\in[0,1]$ *and* $0\le f(t)\le 1/2$ *for* $t\in[1,2]$. *Then, find the interval in which* $g(2)$ *lies.* ***[IIT JEE 2000]***

Sol. Here, $g(x)=\int_0^x f(t)dt \Rightarrow g(2)=\int_0^2 f(t)dt$

$\Rightarrow$ $g(2)=\int_0^1 f(t)dt+\int_1^2 f(t)dt$ ...(i)

Now, $1/2\le f(t)\le 1$ for $t\in[0,1]$

We get, $\int_0^1\frac{1}{2}dt\le\int_0^1 f(t)dt\le\int_0^1 1\,dt$

$\Rightarrow$ $\frac{1}{2}\le\int_0^1 f(t)dt\le 1$ ...(ii)

Again, $0\le f(t)\le\frac{1}{2}$, for $t\in[1,2]$

$\Rightarrow$ $\int_1^2 0\,dt\le\int_1^2 f(t)dt\le\int_1^2\frac{1}{2}dt$

$\Rightarrow$ $0\le\int_1^2 f(t)dt\le\frac{1}{2}$ ...(iii)

On adding Eq. (ii) and (iii), we get

$$0+\frac{1}{2}\le\int_0^1 f(t)dt+\int_1^2 f(t)dt\le 1+\frac{1}{2}$$

or $\frac{1}{2}\le g(2)\le\frac{3}{2}$ $\Rightarrow$ $g(2)\in\left[\frac{1}{2},\frac{3}{2}\right]$

● ***Ex. 64*** *Let* f *be a one-one function such that* $f(x)\cdot f(y)+2=f(x)+f(y)+f(xy),\ \forall\ x,y\in R-\{0\}$ *and* $f(0)=1$, $f'(1)=2$. *Prove that* $3\left(\int f(x)\,dx\right)-x(f(x)+2)$ *is constant.*

Sol. We have, $f(x)\cdot f(y)+2=f(x)+f(y)+f(xy)$ ...(i)

On replacing $x, y\to 1$, we get

$$(f(1))^2+2=3f(1)$$

$\Rightarrow$ $f^2(1)-3f(1)+2=0$

$\Rightarrow$ $f(1)=2$ or 1

But $f(1)$ cannot be equal to one as $f(0)=1$

$\Rightarrow$ $f(1)=2$ ...(ii)

On replacing y by $1/x$ in Eq. (i), we get

$f(x)\cdot f(1/x)+2=f(x)+f(1/x)+f(1)$

$\Rightarrow$ $f(x)\cdot f(1/x)+2=f(x)+f(1/x)+2$ [using Eq. (ii)]

$\Rightarrow$ $f(x)\cdot f(1/x)=f(x)+f(1/x)$

$\Rightarrow$ $f(x)=\frac{f(1/x)}{f(1/x)-1}$

and $f(1/x)=\frac{f(x)}{f(x)-1}$ ...(iii)

Now, $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to 0}\frac{f(x+h)+\frac{f(1/x)}{1-f(1/x)}}{h}$$

$$=\lim_{h\to 0}\frac{f(x+h)-f(x+h)\cdot f(1/x)+f(1/x)}{h\{1-f(1/x)\}}$$

$$=\lim_{h\to 0}\frac{f(x+h)-f(x+h)-f(1/x)-f\left(\frac{x+h}{x}\right)+2+f(1/x)}{h\{1-f(1/x)\}}$$

[using Eq. (i)]

$$= \lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)-2}{h\{f(1/x)-1\}} = \lim_{h\to 0}\frac{f(1+h/x)-f(1)}{\frac{h}{x}\cdot x\,\{f(1/x)-1\}}$$

[using Eq. (ii)]

$$= \frac{f'(1)}{x\,\{f(1/x)-1\}}$$

$$\left[\because \text{from Eq. (iii)}, f(x)\cdot f\left(\frac{1}{x}\right) = \frac{f(x)f\left(\frac{1}{x}\right)}{\left\{f\left(\frac{1}{x}\right)-1\right\}\{f(x)-1\}}\right]$$

$$f'(x) = \frac{2\{f(x)-1\}}{x} \Rightarrow xf'(x) = 2\{f(x)-1\}$$

On integrating both the sides of the above expression, we get

$$x\,f(x) - \int f(x)\,dx = 2\int f(x)\,dx - 2x + \lambda,$$

where λ is the constant of integral

$$\Rightarrow \quad 3\int f(x)\,dx = x\{2+f(x)\} - \lambda$$

Hence, $3\int f(x)\,dx - x\{2+f(x)\} = \lambda$ (constant)

● ***Ex. 65*** *Let $f: R \to R$, such that $f'(0)=1$ and $f(x+2y) = f(x)+f(2y)+e^{x+2y}(x+2y) - x\cdot e^x - 2y\cdot e^{2y} + 4xy,\ \forall\ x, y \in R$. Find $f(x)$.*

Sol. We have,

$$f(x+2y) = f(x)+f(2y)+e^{x+2y}(x+2y) - x\cdot e^x - 2y\cdot e^{2y} + 4xy$$

On replacing $x, y \to 0$, we get

$$f(0) = f(0)+f(0)+0-0-0+0 \Rightarrow f(0)=0$$

On replacing $2y \to -x$, we get

$$f(0) = f(x)+f(-x)-x\cdot e^x + xe^{-x} - 2x^2$$

$$\Rightarrow \quad -f(x) = f(-x) - x\cdot e^x + xe^{-x} - 2x^2 \qquad ...(i)$$

Now, $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0}\frac{f(x+h)+f(-x)-xe^x+x\cdot e^{-x}-2x^2}{h}$$

$$\Rightarrow f'(x) = \lim_{h\to 0}\frac{\left[\begin{array}{l} f(h)-e^h\cdot h+(x+h)e^{(x+h)}-xe^{-x}+ \\ \qquad 2(x+h)x - xe^x + xe^{-x} - 2x^2\end{array}\right]}{h}$$

$$= \lim_{h\to 0}\frac{f(h)-e^h\cdot h + x\cdot e^x\cdot e^h + h\cdot e^x\cdot e^h + 2hx - xe^x}{h}$$

$$= \lim_{h\to 0}\frac{f(h)-f(0)}{h} + \frac{h(e^x-1)}{h}e^h + \frac{xe^x(e^h-1)}{h} + \frac{2hx}{h}$$

$$= f'(0) + (e^x-1) + x\cdot e^x + 2x$$

$$\therefore \quad f'(x) = 1 + e^x - 1 + xe^x + 2x$$

$$\text{or} \quad f'(x) = e^x(x+1) + 2x \qquad ...(ii)$$

On integrating Eq. (ii) both sides, we get

$$f(x) = \int e^x(x+1)\,dx + 2\int x\,dx$$

$$= (x+1)e^x - \int 1\cdot e^x dx + 2\frac{x^2}{2} + C$$

$$= (x+1)e^x - e^x + x^2 + C$$

$$\Rightarrow \quad f(x) = x^2 + xe^x + C$$

But $f(0)=0 \Rightarrow C=0$

So, $f(x) = x^2 + x\cdot e^x$

● ***Ex. 66*** *Let f be a function such that $f(xy) = f(x)\cdot f(y), \forall\ y \in R$ and $f(1+x) = 1 + x(1+g(x))$, where $\lim_{x\to 0} g(x) = 0$. Find the value of $\int_1^2 \frac{f(x)}{f'(x)}\cdot\frac{1}{1+x^2}dx$.*

Sol. We know, $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0}\frac{f\left(x\left(1+\frac{h}{x}\right)\right)-f(x)}{h}$$

$$\Rightarrow \quad f'(x) = \lim_{h\to 0}\frac{f(x)\cdot f\left(1+\frac{h}{x}\right)-f(x)}{h}$$

[given $f(xy) = f(x)\cdot f(y)$]

$$\Rightarrow \quad f'(x) = \lim_{h\to 0}\frac{f(x)\cdot\left\{1+\frac{h}{x}(1+g(h/x))\right\}-f(x)}{h}$$

[given $f(1+x) = 1 + x(1+g(x))$]

$$\Rightarrow \quad f'(x) = \lim_{h\to 0}\frac{f(x)\cdot\left\{1+\frac{h}{x}(1+g(h/x))-1\right\}}{h}$$

$$\therefore \quad f'(x) = \frac{f(x)}{x} \qquad [\text{as } \lim_{x\to 0} g(x) = 0] \quad ...(i)$$

To find, $\int_1^2 \frac{f(x)}{f'(x)}\cdot\frac{1}{1+x^2}dx = \int_1^2 \frac{x}{1+x^2}dx$ [using Eq. (i)]

$$= \frac{1}{2}[\log|1+x^2|]_1^2 = \frac{1}{2}[\log(5/2)]$$

● ***Ex. 67*** *If $f(x) = ax^2 + bx + c$ is such that $|f(0)| \le 1$, $|f(1)| \le 1$ and $|f(-1)| \le 1$, prove that $|f(x)| \le 5/4, \forall\ x \in [-1,1]$.*

Sol. We have, $f(x) = ax^2 + bx + c$ ...(i)

$$\Rightarrow \quad f(-1) = a - b + c \qquad ...(ii)$$

$$\Rightarrow \quad f(0) = c \qquad ...(iii)$$

$$\Rightarrow \quad f(1) = a + b + c \qquad ...(iv)$$

From Eqs. (ii), (iii) and (iv), we get

$$a=\frac{1}{2}[f(-1)+f(1)-2f(0)],$$

$$b=\frac{1}{2}[f(1)-f(-1)] \text{ and } c=f(0)$$

On substituting the values of a, b and c in Eq. (i), we get

$$\Rightarrow \quad f(x)=\frac{1}{2}[f(-1)+f(1)-2f(0)]x^2 + \frac{1}{2}[f(1)-f(-1)]x+f(0)$$

$$f(x)=\frac{x(x-1)}{2}f(-1)-(x+1)(x-1)f(0) + \frac{(x+1)x}{2}f(1) \quad ...(v)$$

Since, $|f(-1)|, |f(0)|$ and $|f(1)|$ are ≤ 1, we have

$$2|f(x)| \le |x(x-1)| + 2|x^2-1| + |x(x+1)| \quad ...(vi)$$

In the interval, $x \in [-1, 1]$

$$0 \le 1+x \le 2, \quad 0 \le 1-x \le 2 \text{ and } 0 \le 2(1-x^2) \le 2$$

$$\Rightarrow \quad 2|f(x)| \le |x|(1-x+1+x)+2(1-x^2)$$

$$2|f(x)| \le 2(|x|+1-x^2)$$

Therefore, $$|f(x)| \le \left(|x|-\frac{1}{2}\right)^2 + 5/4 \le 5/4$$

$$\Rightarrow \quad |f(x)| \le \frac{5}{4}, \forall\ x \in [-1, 1]$$

● ***Ex. 68*** *Let $\alpha+\beta=1$, $2\alpha^2+2\beta^2=1$ and $f(x)$ be a continuous function such that $f(2+x)+f(x)=2$ for all $x \in [0, 2]$ and $p=\int_0^4 f(x)\,dx-4$, $q=\frac{\alpha}{\beta}$. Then, find the least positive integral value of 'a' for which the equation $ax^2-bx+c=0$ has both roots lying between p and q, where $a, b, c \in N$.*

Sol. Given, $$\alpha+\beta=1 \quad ...(i)$$

$$2\alpha^2+2\beta^2=1 \quad ...(ii)$$

On solving Eqs. (i) and (ii), we get

$$\alpha=\beta=\frac{1}{2} \Rightarrow q=\frac{\alpha}{\beta}=1 \quad ...(iii)$$

and given $f(2+x)+f(x)=2, \forall\ x \in [0, 2]$ ...(iv)

Now, $$p=\int_0^4 f(x)\,dx-4=\int_0^2 f(x)\,dx+\int_2^4 f(x)\,dx-4$$

$$=\int_0^2 f(x)\,dx+\int_0^2 f(t+2)\,dt-4$$

[let $x=t+2$ for second integration]

$$=\int_0^2 f(x)\,dx+\int_0^2 \{2-f(x)\}\,dx-4$$

[given, $f(2+x)+f(x)=2$]

$$\Rightarrow \quad p=\int_0^2 f(x)\,dx+2\int_0^2 dx-\int_0^2 f(x)\,dx-4=0 \quad ...(v)$$

Then, from Eqs. (v) and (iii), $p=0, q=1$

Let the roots of equation $ax^2-bx+c=0$ be α and β.

$$\therefore \quad f(x)=ax^2-bx+c=a(x-\alpha)(x-\beta) \quad ...(vi)$$

Since, equation $f(x)=0$ has both roots lying between 0 and 1.

$$\therefore \quad f(0)\cdot f(1)>0 \quad ...(vii)$$

But $$f(0)\cdot f(1)=c(a-b+c)=\text{an integer} \quad ...(viii)$$

$\therefore$ Least value of $f(0)\cdot f(1)=1$ ...(ix)

Now, from Eq. (vi),

$$f(0)\cdot f(1)=a\alpha\beta\, a(1-\alpha)(1-\beta)$$

$$=a^2\alpha\beta(1-\alpha)(1-\beta) \quad ...(x)$$

As we know,

$\alpha(1-\alpha)$ has greatest value $\frac{1}{4}$ at $\alpha=\frac{1}{2}$ and $\beta(1-\beta)$ has greatest value $\frac{1}{4}$ at $\beta=\frac{1}{2}$. But $\alpha \ne \beta$

Thus, from Eq. (x) greatest value of $f(0)\cdot f(1)<\frac{a^2}{16}$ ...(xi)

$\therefore$ From Eqs. (ix) and (xi), we get $1<\frac{a^2}{16}$

$$\Rightarrow \quad a^2-16>0$$

$$\Rightarrow \quad a<-4 \text{ or } a>4 \quad [\because a \in N]$$

$\Rightarrow$ Least value of $a=5$ [as $a \in$ natural number]

● ***Ex. 69*** *Prove that the function $f(x)=a\sqrt{x-1}+b\sqrt{2x-1}-\sqrt{2x^2-3x+1}$, where $a+2b=2$ and $a, b \in R$ always has a root in $(1, 5)$, $\forall\ b \in R$.*

Sol. Let $b>0$, then $f(1)=b>0$

and $$f(5)=2a+3b-6=2(a+2b)-b-6$$

$$=4-b-6=-(2+b)<0$$

Hence, by IVT, $\exists$ some $c \in (1, 5)$ such that

$$\Rightarrow \quad f(c)=0$$

If $b=0$, then $a=2$

$$f(x)=2\sqrt{x-1}-\sqrt{2x^2-3x+1}=0$$

$$\Rightarrow \quad 4(x-1)=2x^2-3x+1=(2x-1)(x-1)$$

$$\Rightarrow \quad (x-1)(2x-5)=0$$

$$\Rightarrow \quad x=\frac{5}{2}$$

Hence, $f(x)=0$, if $x=\frac{5}{2}$, which lies in (1, 5).

If $b < 0$, $f(1) = b < 0$ and $f(2) = a + b\sqrt{3} - \sqrt{3}$

$= (a + 2b) + (\sqrt{3} - 2)b - \sqrt{3}$

$= (2 - \sqrt{3}) - (2 - \sqrt{3})b$

$= (2 - \sqrt{3})(1 - b) > 0$ [as $b < 0$]

Hence, $f(1)$ and $f(2)$ have opposite signs

$\exists$ some $c \in (1, 2) \subset (1, 5)$ for which $f(c) = 0$.

● ***Ex. 70*** *Let $\alpha \in R$. Prove that a function $f : R \to R$ is differentiable at α, if and only if there is a function $g : R \to R$ which is continuous at $x = \alpha$ and satisfies*
$f(x) - f(\alpha) = g(x)(x - \alpha)$ for all $x \in R$. ***[IIT JEE 2001]***

Sol. If $g(x)$ is continuous at $x = \alpha$.

$\Rightarrow$ $\lim\limits_{x \to \alpha} g(x) = g(\alpha)$ ...(i)

and $f(x) - f(\alpha) = g(x)(x - \alpha), \forall\, x \in R$

$\Rightarrow$ $\dfrac{f(x) - f(\alpha)}{(x - \alpha)} = g(x)$

$\Rightarrow$ $\lim\limits_{x \to \alpha} \dfrac{f(x) - f(\alpha)}{(x - \alpha)} = \lim\limits_{x \to \alpha} g(x) \Rightarrow f'(\alpha) = g(\alpha)$

Therefore, $f(x)$ is differentiable at $x = \alpha$ ...(ii)

Conversely, f is differentiable at $x = \alpha$, then $\lim\limits_{x \to \alpha} \dfrac{f(x) - f(\alpha)}{(x - \alpha)}$ exists finitely.

Let $g(x) = \begin{cases} \dfrac{f(x) - f(\alpha)}{x - \alpha}, & x \neq \alpha \\ f'(\alpha), & x = \alpha \end{cases}$

Clearly, $\lim\limits_{x \to \alpha} g(x) = f'(\alpha)$

Thus, $g(x)$ is continuous at $x = \alpha$.

Hence, $f(x)$ is differentiable at $x = \alpha$, iff $g(x)$ is continuous at $x = \alpha$.

Continuity and Differentiability Exercise 1 :
Single Option Correct Type Questions

1. If $f(x)=\begin{cases}\sin\frac{\pi x}{2}, & x<1\\ [x], & x\geq 1\end{cases}$, where $[x]$ denotes the greatest integer function, then

(a) $f(x)$ is continuous at $x=1$

(b) $f(x)$ is discontinuous at $x=1$

(c) $f(1^+)=0$

(d) $f(1^-)=-1$

2. Consider $f(x)=\begin{cases}\frac{8^x-4^x-2^x+1}{x^2}, & x>0\\ e^x\sin x+\pi x+k\log 4, & x<0\end{cases}$

Then, $f(0)$ so that $f(x)$ is continuous at $x=0$, is

(a) $\log 4$ (b) $\log 2$

(c) $(\log 4)(\log 2)$ (d) None of these

3. Let $f(x)=\begin{cases}\frac{a(1-x\sin x)+b\cos x+5}{x^2}, & x<0\\ 3, & x=0\\ \left[1+\left(\frac{cx+dx^3}{x^2}\right)\right]^{1/x}, & x>0\end{cases}$

If f is continuous at $x=0$, then $(a+b+c+d)$ is

(a) 5 (b) -5 (c) $\log_e 3-5$ (d) $5-\log_e 3$

4. If $f(x)=\begin{cases}\cos^{-1}\{\cot x\}, & x<\pi/2\\ \pi[x]-1, & x\geq\pi/2\end{cases}$, then the jump of discontinuity, (where $[\cdot]$ denotes greatest integer and $\{\}$ denotes fractional part function) is

(a) 1 (b) $\pi/2$ (c) $\frac{\pi}{2}-1$ (d) 2

5. Let $f:[0,1]\xrightarrow{\text{onto}}[0,1]$ be a continuous function, then $f(x)=x$ holds for

(a) at least one $x\in[0,1]$ (b) at least one $x\in[1,2]$

(c) at least one $x\in[-1,0]$ (d) can't be discussed

6. If $f(x)=\frac{x+1}{x-1}$ and $g(x)=\frac{1}{x-2}$, then $(fog)(x)$ is discontinuous at

(a) $x=3$ only (b) $x=2$ only

(c) $x=2$ and 3 (d) $x=1$ only

7. Let $y_n(x)=x^2+\frac{x^2}{(1+x^2)}+\frac{x^2}{(1+x^2)^2}+\ldots+\frac{x^2}{(1+x^2)^{n-1}}$ and $y(x)=\lim_{n\to\infty}y_n(x)$, then $y_n(x)$, $n=1, 2, 3, \ldots, n$ and $y(x)$ is

(a) continuous for $x\in R$ (b) continuous for $x\in R-\{0\}$

(c) continuous for $x\in R-\{1\}$ (d) data unsufficient

8. If $g(x)=\begin{cases}\frac{1+a^x\cdot x\cdot\log a-a^x}{a^x\cdot x^2}, & \text{for } x\leq 0\\ \frac{2^x\cdot a^x-x\log 2-x\log a-1}{x^2}, & \text{for } x>0\end{cases}$

where $a>0$, then a for which $g(x)$ is continuous, is

(a) $-\frac{1}{\sqrt{2}}$ (b) $\frac{1}{\sqrt{2}}$ (c) 2 (d) -2

9. Let $f(x)=\begin{cases}\frac{\left(\frac{\pi}{2}-\sin^{-1}(1-\{x\}^2)\right)\cdot\sin^{-1}(1-\{x\})}{\sqrt{2}(\{x\}-\{x\}^3)}, & x\neq 0,\\ \frac{\pi}{2}, & x=0,\end{cases}$ where $\{\cdot\}$ is fractional part of x, then

(a) $f(0^+)=-\frac{\pi}{2}$

(b) $f(0^-)=\frac{\pi}{4\sqrt{2}}$

(c) $f(x)$ is continuous at $x=0$

(d) None of the above

10. Let $f(x)=\begin{cases}\text{sgn}\,(x)+x, & -\infty<x<0\\ -1+\sin x, & 0\leq x\leq\pi/2\\ \cos x, & \pi/2\leq x<\infty\end{cases}$, then the number of points, where $f(x)$ is not differentiable, is/are

(a) 0 (b) 1

(c) 2 (d) 3

11. Let $f(x)=\begin{cases}\frac{1}{|x|}, & |x|\geq 1\\ ax^2+b, & |x|<1\end{cases}$ be continuous and differentiable for all x. Then, a and b are

(a) $-\frac{1}{2},\frac{3}{2}$ (b) $\frac{1}{2},-\frac{3}{2}$

(c) $\frac{1}{2},\frac{3}{2}$ (d) None of these

12. If $f(x)=\begin{cases}A+Bx^2, & x<1\\ 3Ax-B+2, & x\geq 1\end{cases}$, then A and B, so that $f(x)$ is differentiable at $x=1$, are

(a) $-2, 3$ (b) $2, -3$

(c) $2, 3$ (d) $-2, -3$

13. If $f(x)=\begin{cases}|x-1|([x]-x), & x\neq 1\\ 0, & x=1\end{cases}$, then

(a) $f'(1^+)=0$ (b) $f'(1^-)=0$

(c) $f'(1^-)=-1$ (d) $f(x)$ is differentiable at $x=1$

14. If $f(x)=\begin{cases}[\cos \pi x], & x \le 1 \\ 2\{x\}-1, & x>1\end{cases}$, where $[\cdot]$ and $\{\cdot\}$ denote greatest integer and fractional part of x, then

(a) $f'(1^-)=2$ (b) $f'(1^+)=2$

(c) $f'(1^-)=-2$ (d) $f'(1^+)=0$

15. If $f(x)=\begin{cases}x-3, & x<0 \\ x^2-3x+2, & x \ge 0\end{cases}$, then $g(x)=f(|x|)$ is

(a) $g'(0^+)=-3$

(b) $g'(0^-)=-3$

(c) $g'(0^+)=g'(0^-)$

(d) $g(x)$ is not continuous at $x=0$

16. If $f(x)=\begin{cases}\left\{x+\frac{1}{3}\right\}[\sin \pi x], & 0 \le x<1 \\ [2x]\operatorname{sgn}\left(x-\frac{4}{3}\right), & 1 \le x \le 2\end{cases}$, where $[\cdot]$ and $\{\cdot\}$ denote greatest integer and fractional part of x respectively, then the number of points of non-differentiability, is

(a) 3 (b) 4 (c) 5 (d) 6

17. Let f be differentiable function satisfying $f\left(\frac{x}{y}\right)=f(x)-f(y)$ for all $x, y>0$. If $f'(1)=1$, then $f(x)$ is

(a) $2\log_e x$ (b) $3\log_e x$ (c) $\log_e x$ (d) $\frac{1}{2}\log_e x$

18. Let $f(x+y)=f(x)+f(y)-2xy-1$ for all x and y. If $f'(0)$ exists and $f'(0)=-\sin\alpha$, then $f\{f'(0)\}$ is

(a) -1 (b) 0

(c) 1 (d) 2

19. A derivable function $f: R^+ \to R$ satisfies the condition

$$f(x)-f(y) \ge \log\left(\frac{x}{y}\right)+x-y, \forall\ x, y \in R^+.$$

If g denotes the derivative of f, then the value of the sum $\sum_{n=1}^{100} g\left(\frac{1}{n}\right)$ is

(a) 5050 (b) 5510 (c) 5150 (d) 1550

20. If $\frac{d(f(x))}{dx}=e^{-x}f(x)+e^{x}f(-x)$, then $f(x)$ is, (given $f(0)=0$)

(a) an even function

(b) an odd function

(c) neither even nor odd function

(d) can't say

21. Let $f:(0,\infty) \to R$ be a continuous function such that

$$f(x)=\int_0^x t\ f(t)\,dt.$$

If $f(x^2)=x^4+x^5$, then $\sum_{r=1}^{12} f(r^2)$ is equal to

(a) 216 (b) 219

(c) 222 (d) 225

22. For $x>0$, let $h(x)=\begin{cases}\frac{1}{q}, & \text{if } x=\frac{p}{q} \\ 0, & \text{if } x \text{ is irrational}\end{cases}$, where p and $q>0$ are relatively prime integers, then which one of the following does not hold good?

(a) $h(x)$ is discontinuous for all x in $(0, \infty)$

(b) $h(x)$ is continuous for each irrational in $(0, \infty)$

(c) $h(x)$ is discontinuous for each rational in $(0, \infty)$

(d) $h(x)$ is not derivable for all x in $(0, \infty)$

23. Let $f(x)=\frac{g(x)}{h(x)}$, where g and h are continuous functions on the open interval (a, b). Which of the following statements is true for $a<x<b$?

(a) f is continuous at all x for which $x \ne 0$

(b) f is continuous at all x for which $g(x)=0$

(c) f is continuous at all x for which $g(x) \ne 0$

(d) f is continuous at all x for which $h(x) \ne 0$

24. If $f(x)=\frac{2\cos x-\sin 2x}{(\pi-2x)^2}$, $g(x)=\frac{e^{-\cos x}-1}{8x-4\pi}$ and $h(x)=\begin{cases}f(x), & \text{for } x<\pi/2 \\ g(x), & \text{for } x>\pi/2\end{cases}$,

then which of the following holds?

(a) h is continuous at $x=\pi/2$

(b) h has an irremovable discontinuity at $x=\pi/2$

(c) h has a removable discontinuity at $x=\pi/2$

(d) $f\left(\frac{\pi^+}{2}\right)=g\left(\frac{\pi^-}{2}\right)$

25. If $f(x)=\frac{x-e^x+\cos 2x}{x^2}$, $x \ne 0$ is continuous at $x=0$, then

(a) $f(0)=\frac{5}{2}$ (b) $[f(0)]=-2$

(c) $\{f(0)\}=-0.5$ (d) $[f(0)]\cdot\{f(0)\}=-1.5$

where, $[x]$ and $\{x\}$ denote the greatest integer and fractional part function.

26. Consider the function $f(x)=\begin{cases}x\{x\}+1, & \text{if } 0 \le x<1 \\ 2-\{x\}, & \text{if } 1 \le x \le 2\end{cases}$, where $\{x\}$ denotes the fractional part function. Which one of the following statements is not correct?

(a) $\lim_{x \to 1} f(x)$ exists

(b) $f(0) \ne f(2)$

(c) $f(x)$ is continuous in $[0, 2]$

(d) Rolle's theorem is not applicable to $f(x)$ in $[0, 2]$

27. Let $f(x)=\begin{cases}\dfrac{2^x+2^{3-x}-6}{\sqrt{2^{-x}}-2^{1-x}}, & \text{if } x>2\\ \dfrac{x^2-4}{x-\sqrt{3x-2}}, & \text{if } x<2\end{cases}$, then

(a) $f(2)=8\Rightarrow f$ is continuous at $x=2$
(b) $f(2)=16\Rightarrow f$ is continuous at $x=2$
(c) $f(2^-)\neq f(2^+)\Rightarrow f$ is discontinuous
(d) f has a removable discontinuity at $x=2$

28. Let $[x]$ denotes the integral part of $g(x)=x-[x], x\in R$. Let $f(x)$ be any continuous function with $f(0)=f(1)$, then the function $h(x)=f(g(x))$
(a) has finitely many discontinuities
(b) is discontinuous at some $x=c$
(c) is continuous on R
(d) is a constant function

29. Let f be a differentiable function on the open interval (a, b).

I. f is continuous on the closed interval $[a, b]$.
II. f is bounded on the open interval (a, b).
III. If $a<a_1<b_1<b$ and $f(a_1)<0<f(b_1)$, then there exists a number c such that $a_1<c<b_1$ and $f(c)=0$.

Which of the above statements must be true?
(a) I and II
(b) I and III
(c) II and III
(d) Only III

30. Number of points, where the function $f(x)=(x^2-1)|x^2-x-2|+\sin(|x|)$ is not differentiable, is
(a) 0 (b) 1
(c) 2 (d) 3

31. Consider function $f: R-\{-1, 1\}\to R$. $f(x)=\dfrac{x}{1-|x|}$.

Then, which of the statements is incorrect?
(a) It is continuous at the origin
(b) It is not derivable at the origin
(c) The range of the function is R
(d) f is continuous and derivable in its domain

32. If the functions $f: R\to R$ and $g: R\to R$ are such that $f(x)$ is continuous at $x=\alpha$ and $f(\alpha)=a$ and $g(x)$ is discontinuous at $x=a$ but $g(f(x))$ is continuous at $x=\alpha$, where $f(x)$ and $g(x)$ are non-constant functions.
(a) $x=\alpha$ is a extremum of $f(x)$ and $x=a$ is an extremum of $g(x)$
(b) $x=\alpha$ may not be an extremum of $f(x)$ and $x=a$ is an extremum of $g(x)$
(c) $x=\alpha$ is an extremum of $f(x)$ and $x=a$ may not be an extremum of $g(x)$
(d) None of the above

33. The total number of points of non-differentiability of $f(x)=\min\left[|\sin x|, |\cos x|, \dfrac{1}{4}\right]$ in $(0, 2\pi)$ is
(a) 8 (b) 9 (c) 10 (d) 11

34. The function $f(x)=[x]^2-[x^2]$, where $[y]$ is the greatest integer less than or equal to y, is discontinuous at
(a) all integers
(b) all integers except 0 and 1
(c) all integers except 0
(d) all integers except 1

35. The function $f(x)=(x^2-1)|x^2-6x+5|+\cos|x|$ is not differentiable at
(a) –1 (b) 0 (c) 1 (d) 5

36. Let $f(x)=\begin{cases}e^{\frac{-1}{x^2}}, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$

Then, $f'(0)$ is equal to
(a) 0 (b) 1
(c) – 1 (d) doesn't exist

37. Given $f(x)=\dfrac{e^x-\cos 2x-x}{x^2}$, for $x\in R-\{0\}$

$$g(x)=\begin{cases}f(\{x\}), & \text{for } n<x<n+\dfrac{1}{2}\\ f(1-\{x\}), & \text{for } n+\dfrac{1}{2}\le x<n+1, n\in I\\ \dfrac{5}{2}, & \text{otherwise}\end{cases}$$

$\begin{cases}\text{where } \{x\} \text{ denotes}\\ \text{fractional part function,}\end{cases}$ then $g(x)$ is
(a) discontinuous at all integral values of x only
(b) continuous everywhere except for $x=0$
(c) discontinuous at $x=n+\dfrac{1}{2}, n\in I$ and at some $x\in I$
(d) continuous everywhere

38. The function $g(x)=\begin{cases}x+b, & x<0\\ \cos x, & x\ge 0\end{cases}$ cannot be made differentiable at $x=0$,
(a) if b is equal to zero (b) if b is not equal to zero
(c) if b takes any real value (d) for no value of b

39. The graph of function f contains the point $P(1, 2)$ and $Q(s, r)$. The equation of the secant line through P and Q is $y=\left(\dfrac{s^2+2s-3}{s-1}\right)x-1-s$. The value of $f'(1)$ is
(a) 2 (b) 3
(c) 4 (d) non-existent

40. Consider
$f(x)=\left[\dfrac{2(\sin x-\sin^3 x)+|\sin x-\sin^3 x|}{2(\sin x-\sin^3 x)-|\sin x-\sin^3 x|}\right], x\neq\dfrac{\pi}{2}$ for $x\in(0,\pi)$, $f(\pi/2)=3$, where [ ] denotes the greatest integer function, then
(a) f is continuous and differentiable at $x=\pi/2$
(b) f is continuous but not differentiable at $x=\pi/2$
(c) f is neither continuous nor differentiable at $x=\pi/2$
(d) None of the above

41. If $f(x+y)=f(x)+f(y)+|x|y+xy^2, \forall\ x, y\in R$ and $f'(0)=0$, then
(a) f need not be differentiable at every non-zero x
(b) f is differentiable for all $x\in R$
(c) f is twice differentiable at $x=0$
(d) None of the above

42. Let $f(x)=\max\{|x^2-2|x||, |x|\}$ and $g(x)=\min\{|x^2-2|x||, |x|\}$, then
(a) both $f(x)$ and $g(x)$ are non-differentiable at 5 points
(b) $f(x)$ is not differentiable at 5 points whether $g(x)$ is non-differentiable at 7 points
(c) number of points of non-differentiability for $f(x)$ and $g(x)$ are 7 and 5 points, respectively
(d) both $f(x)$ and $g(x)$ are non-differentiable at 3 and 5 points, respectively

43. Let $g(x)=\begin{cases}3x^2-4\sqrt{x}+1, & \text{for } x<1\\ ax+b, & \text{for } x\geq 1\end{cases}$. If $g(x)$ is the continuous and differentiable for all numbers in its domain, then
(a) $a=b=4$ (b) $a=b=-4$
(c) $a=4$ and $b=-4$ (d) $a=-4$ and $b=4$

44. Let $f(x)$ be continuous and differentiable function for all reals and $f(x+y)=f(x)-3xy+f(y)$.
If $\lim\limits_{h\to 0}\dfrac{f(h)}{h}=7$, then the value of $f'(x)$ is
(a) $-3x$ (b) 7
(c) $-3x+7$ (d) $2f(x)+7$

45. Let $[x]$ be the greatest integer function and $f(x)=\dfrac{\sin\frac{1}{4}\pi[x]}{[x]}$. Then, which one of the following does not hold good?
(a) Not continuous at any point
(b) Continuous at 3/2
(c) Discontinuous at 2
(d) Differentiable at 4/3

46. If $f(x)=\begin{cases}b([x]^2+[x])+1, \text{ for } x\geq -1\\ \sin(\pi(x+a)), \quad \text{for } x<-1\end{cases}$, where $[x]$ denotes the integral part of x, then for what values of a, b the function is continuous at $x=-1$?
(a) $a=2n+(3/2); b\in R; n\in I$
(b) $a=4n+2; b\in R; n\in I$
(c) $a=4n+(3/2); b\in R^+; n\in I$
(d) $a=4n+1; b\in R^+; n\in I$

47. If both $f(x)$ and $g(x)$ are differentiable functions at $x=x_0$, then the function defined as, $h(x)=\max\{f(x), g(x)\}$
(a) is always differentiable at $x=x_0$
(b) is never differentiable at $x=x_0$
(c) is differentiable at $x=x_0$ when $f(x_0)\neq g(x_0)$
(d) cannot be differentiable at $x=x_0$, if $f(x_0)=g(x_0)$

48. Number of points of non-differentiability of the function
$g(x)=[x^2]\{\cos^2 4x\}+\{x^2\}[\cos^2 4x]+x^2\sin^2 4x$
$+[x^2][\cos^2 4x]+\{x^2\}\{\cos^2 4x\}$ in $(-50, 50)$,
where $[\cdot]$ and $\{\cdot\}$ are greatest integer function and fractional part of x, is equal to
(a) 98
(b) 99
(c) 100
(d) 0

49. If $f(x)=\dfrac{\{x\}g(x)}{\{x\}g(x)}$ is a periodic function with period $\dfrac{1}{4}$, where $g(x)$ is a differentiable function, then (where $\{\cdot\}$ denotes fractional part of x).
(a) $g'(x)$ has exactly three roots in $\left(\dfrac{1}{4}, \dfrac{5}{4}\right)$
(b) $g(x)=0$ at $x=\dfrac{k}{4}$, where $k\in I$
(c) $g(x)$ must be non-zero function
(d) $g(x)$ must be periodic function

50. If $f\left(\dfrac{x}{y}\right)=\dfrac{f(x)}{f(y)}$ for all $x, y\in R, y\neq 0$ and $f'(x)$ exists for all x, $f(2)=4$. Then, $f(5)$ is
(a) 3 (b) 5
(c) 25 (d) None of these

Continuity and Differentiability Exercise 2 : More than One Option Correct Type Questions

51. Indicate the correct alternative, if $f(x)=\frac{x}{2}-1$, then on the interval $[0,\pi]$,

(a) $\tan(f(x))$ and $\frac{1}{f(x)}$ are both continuous

(b) $\tan(f(x))$ and $\frac{1}{f(x)}$ are both discontinuous

(c) $\tan(f(x))$ and $f^{-1}(x)$ are both continuous

(d) $\tan(f(x))$ is continuous but $\frac{1}{f(x)}$ is not continuous

52. On the interval $I=[-2,2]$, if the function

$$f(x)=\begin{cases}(x+1)\,e^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}, & x\neq 0\\ 0, & x=0\end{cases}$$ then which of the following hold good?

(a) $f(x)$ is continuous for all values of $x\in I$

(b) $f(x)$ is continuous for $x\in I-\{0\}$

(c) $f(x)$ assumes all intermediate values from $f(-2)$ to $f(2)$

(d) $f(x)$ has a maximum value equal to $3/e$

53. If $f(x)=\begin{cases}3-\left[\cot^{-1}\left(\frac{2x^3-3}{x^2}\right)\right], & \text{for } x>0\\ \{x^2\}\cos(e^{1/x}), & \text{for } x<0\end{cases}$, where $\{x\}$ and $[x]$ denote fractional part and the greatest integer function respectively, then which of the following statements does not hold good?

(a) $f(0^-)=0$

(b) $f(0^+)=0$

(c) $f(0)=0\Rightarrow$ Continuous at $x=0$

(d) Irremovable discontinuity at $x=0$

54. If $f(x)=\begin{cases}b([x]^2+[x])+1, & \text{for } x>-1\\ \sin(\pi(x+a)), & \text{for } x<-1\end{cases}$, where $[x]$ denotes the integral part of x, then for what values of a and b, the function is continuous at $x=-1$?

(a) $a=2n+\frac{3}{2}; b\in R, n\in I$

(b) $a=4n+2; b\in R, n\in I$

(c) $a=4n+\frac{3}{2}; b\in R^+, n\in I$

(d) $a=4n+1; b\in R^+, n\in I$

55. Let $[x]$ be the greatest integer function, then $f(x)=\frac{\sin\frac{1}{4}\pi[x]}{[x]}$ is

(a) not continuous at any point (b) continuous at $x=\frac{3}{2}$

(c) discontinuous at $x=2$ (d) differentiable at $x=\frac{4}{3}$

56. If $f(x)=\begin{cases}(\sin^{-1}x)^2\cos\left(\frac{1}{x}\right), & x\neq 0\\ 0, & x=0\end{cases}$, then $f(x)$ is

(a) continuous nowhere in $-1\le x\le 1$

(b) continuous everywhere in $-1\le x\le 1$

(c) differentiable nowhere in $-1\le x\le 1$

(d) differentiable everywhere in $-1\le x\le 1$

57. Let $f(x)=\cos x$ and $H(x)=\begin{cases}\min(f(t):0\le t<x), & \text{for } 0\le x\le \pi/2\\ \frac{\pi}{2}-x, & \text{for } \frac{\pi}{2}<x\le 3\end{cases}$, then

(a) $H(x)$ is continuous and derivable in $[0,3]$

(b) $H(x)$ is continuous but not derivable at $x=\frac{\pi}{2}$

(c) $H(x)$ is neither continuous nor derivable at $x=\frac{\pi}{2}$

(d) maximum value of $H(x)$ in $[0,3]$ is 1

58. If $f(x)=3(2x+3)^{2/3}+2x+3$, then

(a) $f(x)$ is continuous but not differentiable at $x=-\frac{3}{2}$

(b) $f(x)$ is differentiable at $x=0$

(c) $f(x)$ is continuous at $x=0$

(d) $f(x)$ is differentiable but not continuous at $x=-\frac{3}{2}$

59. If $f(x)=\begin{cases}\frac{x\cdot\ln(\cos x)}{\ln(1+x^2)}, & x\neq 0\\ 0, & x=0\end{cases}$, then

(a) f is continuous at $x=0$

(b) f is continuous at $x=0$ but not differentiable at $x=0$

(c) f is differentiable at $x=0$

(d) f is not continuous at $x=0$

60. Let $[x]$ denotes the greatest integer less than or equal to x. If $f(x) = [x \sin \pi x]$, then $f(x)$ is
(a) continuous at $x = 0$
(b) continuous in $(-1, 0)$
(c) differentiable at $x = 1$
(d) differentiable in $(-1, 1)$

61. The function $f(x) = [|x|] - |[x]|$, where $[x]$ denotes greatest integer function
(a) is continuous for all positive integers
(b) is discontinuous for all non-positive integers
(c) has finite number of elements in its range
(d) is such that its graph does not lie above the X-axis

62. The function $f(x) = \sqrt{1 - \sqrt{1 - x^2}}$
(a) has its domain $-1 \le x \le 1$
(b) has finite one sided derivates at the point $x = 0$
(c) is continuous and differentiable at $x = 0$
(d) is continuous but not differentiable at $x = 0$

63. Consider the function $f(x) = |x^3 + 1|$. Then,
(a) domain of $f(x) \in R$
(b) range of f is R^+
(c) f has no inverse
(d) f is continuous and differentiable for every $x \in R$

64. f is a continuous function in $[a, b]$, g is a continuous function in $[b, c]$. A function $h(x)$ is defined as
$h(x) = \begin{cases} f(x) \text{ for } x \in [a, b) \\ g(x) \text{ for } x \in (b, c] \end{cases}$. If $f(b) = g(b)$, then
(a) $h(x)$ has a removable discontinuity at $x = b$
(b) $h(x)$ may or may not be continuous in $[a, c]$
(c) $h(b^-) = g(b^+)$ and $h(b^+) = f(b^-)$
(d) $h(b^+) = g(b^-)$ and $h(b^-) = f(b^+)$

65. Which of the following function(s) has/have the same range?
(a) $f(x) = \dfrac{1}{1 + x}$
(b) $f(x) = \dfrac{1}{1 + x^2}$
(c) $f(x) = \dfrac{1}{1 + \sqrt{x}}$
(d) $f(x) = \dfrac{1}{\sqrt{3 - x}}$

66. If $f(x) = \sec 2x + \operatorname{cosec} 2x$, then $f(x)$ is discontinuous at all points in
(a) $\{n\pi, n \in N\}$
(b) $\left\{(2n \pm 1)\dfrac{\pi}{4}, n \in I\right\}$
(c) $\left\{\dfrac{n\pi}{4}, n \in I\right\}$
(d) $\left\{(2n \pm 1)\dfrac{\pi}{8}, n \in I\right\}$

67. Let $f(x) = \begin{cases} x^n \sin\left(\dfrac{1}{x^2}\right), & x \ne 0 \\ 0, & x = 0 \end{cases}, (n \in I)$, then
(a) $\lim\limits_{x \to 0} f(x)$ exists for every $n > 1$
(b) f is continuous at $x = 0$ for $n > 1$
(c) f is differentiable at $x = 0$ for every $n > 1$
(d) None of the above

68. A function is defined as $f(x) = \begin{cases} e^x, & x \le 0 \\ |x - 1|, & x > 0 \end{cases}$, then $f(x)$ is
(a) continuous at $x = 0$ (b) continuous at $x = 1$
(c) differentiable at $x = 0$ (d) differentiable at $x = 1$

69. Let $f(x) = \int_{-2}^{x} |t + 1|\, dt$, then
(a) $f(x)$ is continuous in $[-1, 1]$
(b) $f(x)$ is differentiable in $[-1, 1]$
(c) $f'(x)$ is continuous in $[-1, 1]$
(d) $f'(x)$ is differentiable in $[-1, 1]$

70. A function $f(x)$ satisfies the relation $f(x + y) = f(x) + f(y) + xy(x + y), \forall\ x, y \in R$. If $f'(0) = -1$, then
(a) $f(x)$ is a polynomial function
(b) $f(x)$ is an exponential function
(c) $f(x)$ is twice differentiable for all $x \in R$
(d) $f'(3) = 8$

71. If $f(x) = \begin{cases} 3x^2 + 12x - 1, & -1 \le x \le 2 \\ 37 - x, & 2 < x \le 3 \end{cases}$, then
(a) $f(x)$ is increasing on $[-1, 2]$
(b) $f(x)$ is continuous on $[-1, 3]$
(c) $f'(2)$ doesn't exist
(d) $f(x)$ has the maximum value at $x = 2$

72. If $f(x) = 0$ for $x < 0$ and $f(x)$ is differentiable at $x = 0$, then for $x > 0$, $f(x)$ may be
(a) x^2 (b) x (c) $-x$ (d) $-x^{3/2}$

Continuity and Differentiability Exercise 3 : Statements I and II Type Questions

■ **Directions** (Q. Nos. 73 to 82) *For the following questions, choose the correct answer from the codes* (a), (b), (c) *and* (d) *defined as follows :*

(a) Statement I is correct, Statement II is also correct; Statement II is the correct explanation of Statement I

(b) Statement I is correct, Statement II is also correct; Statement II is not the correct explanation of Statement I

(c) Statement I is correct, Statement II is incorrect

(d) Statement I is incorrect, Statement II is correct

73. **Statement I** $f(x) = \sin x + [x]$ is discontinuous at $x = 0$.

Statement II If $g(x)$ is continuous and $f(x)$ is discontinuous, then $g(x) + f(x)$ will necessarily be discontinuous at $x = a$.

74. Consider $f(x) = \begin{cases} 2\sin(a\cos^{-1} x), & \text{if } x \in (0, 1) \\ \sqrt{3}, & \text{if } x = 0 \\ ax + b, & \text{if } x < 0 \end{cases}$

Statement I If $b = \sqrt{3}$ and $a = \dfrac{2}{3}$, then $f(x)$ is continuous in $(-\infty, 1)$.

Statement II If a function is defined on an interval I and limit exists at every point of interval I, then function is continuous in I.

75. Let $f(x) = \begin{cases} \dfrac{\cos x - e^{x^2/2}}{x^3}, & x \neq 0 \\ 0, & x = 0, \text{ then} \end{cases}$

Statement I $f(x)$ is continuous at $x = 0$.

Statement II $\lim\limits_{x \to 0} \dfrac{\cos x - e^{-x^2/2}}{x^3} = -\dfrac{1}{12}$

76. **Statement I** The equation $\dfrac{x^3}{4} - \sin \pi x + \dfrac{2}{3} = 0$ has atleast one solution in $[-2, 2]$.

Statement II Let $f : [a, b] \to R$ be a function and c be a number such that $f(a) < c < f(b)$, then there is at least one number $n \in (a, b)$ such that $f(n) = c$.

77. **Statement I** Range of $f(x) = x\left(\dfrac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\right) + x^2 + x^4$ is not R.

Statement II Range of a continuous even function cannot be R.

78. Let $h(x) = f_1(x) + f_2(x) + f_3(x) + \ldots + f_n(x)$, where $f_1(x), f_2(x), f_3(x), \ldots, f_n(x)$ are real valued functions of x.

Statement I $f(x) = |\cos|x|| + \cos^{-1}(\text{sgn } x) + |\ln x|$ is not differentiable at 3 points in $(0, 2\pi)$.

Statement II Exactly one function , is $f_i(x)$, $i = 1, 2, \ldots, n$ is not differentiable and the rest of the function is differentiable at $x = a$ makes $h(x)$ not differentiable at $x = a$.

79. **Statement I** $f(x) = |x| \sin x$ is differentiable at $x = 0$.

Statement II If $g(x)$ is not differentiable at $x = a$ and $h(x)$ is differentiable at $x = a$, then $g(x) \cdot h(x)$ cannot be differentiable at $x = a$.

80. **Statement I** $f(x) = |\cos x|$ is not derivable at $x = \dfrac{\pi}{2}$.

Statement II If $g(x)$ is differentiable at $x = a$ and $g(a) = 0$, then $|g(x)|$ is non-derivable at $x = a$.

81. Let $f(x) = x - x^2$ and $g(x) = \{x\}, \forall\ x \in R$, where $\{\}$ denotes fractional part function.

Statement I $f(g(x))$ will be continuous, $\forall\ x \in R$.

Statement II $f(0) = f(1)$ and $g(x)$ is periodic with period 1.

82. Let $f(x) = \begin{cases} -ax^2 - b|x| - c, & -\alpha \le x < 0 \\ ax^2 + b|x| + c, & 0 \le x \le \alpha \end{cases}$, where a, b, c are positive and $\alpha > 0$, then

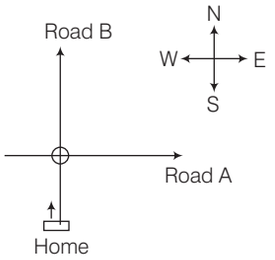
Statement I The equation $f(x) = 0$ has at least one real root for $x \in [-\alpha, \alpha]$.

Statement II Values of $f(-\alpha)$ and $f(\alpha)$ are opposite in sign.

Continuity and Differentiability Exercise 4 : Passage Based Questions

Passage I (Q. Nos. 83 to 85)

A man leaves his home early in the morning to have a walk. He arrives at a junction of roads A and B as shown in figure. He takes the following steps in later journies :



(i) 1 *km in North direction.*

(ii) *Changes direction and moves in North-East direction for* $2\sqrt{2}$ *km.*

(iii) *Changes direction and moves Southwards for distance of* 2 *km.*

(iv) *Finally, he changes the direction and moves in South-East direction to reach road A again.*

Visible/invisible path *The path traced by the man in the direction parallel to road A and road B is called invisible path, the remaining path is called visible.*

Visible points *The point about which the man changes direction are called visible points, except the point from where he changes direction last time.*

Now, if roads A and B are taken as X-axis and Y-axis, then visible point representing the graph of $y = f(x)$.

83 The value of x at which the function is discontinuous, is

(a) 2 (b) 0 (c) 1 (d) 3

84. The value of x for which fof is discontinuous, is

(a) 0 (b) 1 (c) 2 (d) 3

85. If $f(x)$ is periodic with period 3, then $f(19)$ is

(a) 2 (b) 3
(c) 19 (d) None of these

Passage II (Q. Nos. 86 to 89)

Let f *be a function that is differentiable everywhere and that has the following properties :*

(i) $f(x) > 0$ *(ii)* $f'(0) = -1$

(iii) $f(-x) = \dfrac{1}{f(x)}$ *and* $f(x+h) = f(x) \cdot f(h)$

A standard result is $\displaystyle\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + C.$

86. Range of $f(x)$ is

(a) R (b) $R - \{0\}$
(c) R^+ (d) $(0, e)$

87. The range of the function $\Delta = f(|x|)$ is

(a) $[0, 1]$ (b) $[0, 1)$
(c) $(0, 1]$ (d) None of these

88. The function $y = f(x)$ is

(a) odd (b) even
(c) increasing (d) decreasing

89. If $h(x) = f'(x)$, then $h(x)$ is given by

(a) $-f(x)$ (b) $\dfrac{1}{f(x)}$ (c) $f(x)$ (d) $e^{f(x)}$

Passage III (Q. Nos. 90 to 92)

Let $y = f(x)$ *be defined in* $[a, b]$, *then*

(i) *Test of continuity at* $x = c$, $a < c < b$

(ii) *Test of continuity at* $x = a$

(iii) *Test of continuity at* $x = b$

***Case* I** *Test of continuity at* $x = c$, $a < c < b$

If $y = f(x)$ *be defined at* $x = c$ *and its value* $f(c)$ *be equal to limit of* $f(x)$ *as* $x \to c$, i.e. $f(c) = \lim\limits_{x \to c} f(x)$

or $\lim\limits_{x \to c^-} f(x) = f(c) = \lim\limits_{x \to c^+} f(x)$

or LHL $= f(c) =$ RHL

Then, $y = f(x)$ *is continuous at* $x = c$.

***Case* II** *Test of continuity at* $x = a$

If $RHL = f(a)$

Then, $f(x)$ *is said to be continuous at the end point* $x = a$.

***Case* III** *Test of continuity at* $x = b$, *if* $LHL = f(b)$

Then, $f(x)$ *is continuous at right end* $x = b$.

90. If $f(x) = \begin{cases} \sin x, & x \le 0 \\ \tan x, & 0 < x < 2\pi \\ \cos x, & 2\pi \le x < 3\pi \\ 3\pi, & x = 3\pi \end{cases}$, then $f(x)$ is discontinuous at

(a) $\dfrac{\pi}{2}, \dfrac{3\pi}{2}, 2\pi$ (b) $0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}, 3\pi$

(c) $\dfrac{\pi}{2}, 2\pi$ (d) None of these

91. Number of points of discontinuity of $[2x^3 - 5]$ in $[1, 2)$ is (where $[\cdot]$ denotes the greatest integral function)

(a) 14 (b) 13
(c) 10 (d) None of these

92. Max $([x], |x|)$ is discontinuous at

(a) $x = 0$
(b) ϕ
(c) $x = n, n \in I$
(d) None of the above

Passage IV (Q. Nos. 93 to 95)

$f(x) = \cos x$ *and* $H_1(x) = min\{f(t), 0 \le t < x\}$;

$0 \le x \le \frac{\pi}{2} = \frac{\pi}{2} - x; \frac{\pi}{2} < x \le \pi$

$f(x) = \cos x$ *and* $H_2(x) = max\{f(t), 0 \le t \le x\}$;

$0 \le x \le \frac{\pi}{2} = \frac{\pi}{2} - x; \frac{\pi}{2} < x \le \pi$

$g(x) = \sin x$ *and* $H_3(x) = min\{g(t), 0 \le t \le x\}$;

$0 \le x \le \frac{\pi}{2} = \frac{\pi}{2} - x; \frac{\pi}{2} < x \le \pi$

$g(x) = \sin x$ *and* $H_4(x) = max\{g(t), 0 \le t \le x\}$;

$0 \le x \le \frac{\pi}{2} = \frac{\pi}{2} - x; \frac{\pi}{2} < x \le \pi$

93. Which of the following is true for $H_2(x)$?

(a) Continuous and derivable in $[0, \pi]$

(b) Continuous but not derivable at $x = \frac{\pi}{2}$

(c) Neither continuous nor derivable at $x = \frac{\pi}{2}$

(d) None of the above

94. Which of the following is true for $H_3(x)$?

(a) Continuous and derivable in $[0, \pi]$

(b) Continuous but not derivable at $x = \frac{\pi}{2}$

(c) Neither continuous nor derivable at $x = \frac{\pi}{2}$

(d) None of the above

95. Which of the following is true for $H_4(x)$?

(a) Continuous and derivable in $[0, \pi]$

(b) Continuous but not derivable at $x = \frac{\pi}{2}$

(c) Neither continuous nor derivable at $x = \frac{\pi}{2}$

(d) None of the above

Passage V (Q. Nos. 96 to 99)

Let $f(x)$ be a real valued function not identically zero, which satisfied the following conditions

I. $f(x + y^{2n+1}) = f(x) + (f(y))^{2n+1}, n \in N, x, y$ *are any real numbers.*

II. $f'(0) \ge 0$

96. The value of $f(1)$ is

(a) 0 (b) 1
(c) 2 (d) Not defined

97. The value of $f(x)$ is

(a) $2x$ (b) $x^2 + x + 1$
(c) x (d) None of these

98. The value of $f'(10)$ is

(a) 10 (b) 0 (c) $2n + 1$ (d) 1

99. The function $f(x)$ is

(a) odd
(b) even
(c) neither even nor odd
(d) both even as well as odd

Passage VI (Q. Nos. 100 to 101)

If $f : R \to (0, \infty)$ is a differentiable function $f(x)$ satisfying $f(x + y) - f(x - y) = f(x) \cdot \{f(y) - f(-y)\}, \forall\ x, y \in R$, $(f(y) \ne f(-y)$ *for all* $y \in R)$ *and* $f'(0) = 2010$. *Now, answer the following questions*

100. Which of the following is true for $f(x)$?

(a) $f(x)$ is one-one and into
(b) $\{f(x)\}$ is non-periodic, where $\{\cdot\}$ denotes fractional part of x
(c) $f(x) = 4$ has only two solutions
(d) $f(x) = f'(x)$ has only one solution

101. The value of $\frac{f'(x)}{f(x)}$ is

(a) 2016 (b) 2014 (c) 2012 (d) 2010

Continuity and Differentiability Exercise 5 : Matching Type Questions

102. Match the column.

	Column I		Column II
(A)	If $f(x)=\begin{cases}\sin\{x\}; & x<1\\ \cos x+a; & x\geq 1\end{cases}$, where $\{x\}$ denotes the fractional part function, such that $f(x)$ is continuous at $x=1$. If $\lvert K\rvert=\dfrac{a}{\sqrt{2}\sin\dfrac{(4-\pi)}{4}}$, then K is	(p)	1
(B)	If the function $f(x)=\dfrac{1-\cos(\sin x)}{x^2}$ is continuous at $x=0$, then $f(0)$ is	(q)	0
(C)	If $f(x)=\begin{cases}x, & x\in\theta\\ 1-x, & x\notin\theta\end{cases}$, then the values of x at which $f(x)$ is continuous, is	(r)	-1
(D)	If $f(x)=x+\{-x\}+[x]$, where $[x]$ and $\{x\}$ represent greatest integer and fractional part of x, then the values of x at which $f(x)$ is discontinuous, is	(s)	$\frac{1}{2}$

103. Match the column.

	Column I		Column II
(A)	Number of points where the function $f(x)=\begin{cases}1+\left[\cos\dfrac{\pi x}{2}\right], & 1<x\leq 2\\ 1-\{x\}, & 0\leq x<1 \text{ and } f(1)=0\\ \lvert\sin\pi x\rvert, & -1\leq x<0\end{cases}$ is continuous but non-differentiable, where $[\cdot]$ denotes greatest integer function and $\{\cdot\}$ denotes fractional part of x, is	(p)	0
(B)	If $f(x)=\begin{cases}x^2e^{1/x}, & x\neq 0\\ 0, & x=0\end{cases}$, then $f(0^-)$ is	(q)	1
(C)	The number of points at which $f(x)=\dfrac{1}{1+2/f(x)}$ is not differentiable, where $f(x)=\dfrac{1}{1+1/x}$, is	(r)	2
(D)	Number of points where tangent does not exist for the curve $y=\text{sgn}(x^2-1)$, is	(s)	3

104. Match the column.

	Column I		Column II
(A)	The number of values of x in $(0, 2\pi)$, where the function $f(x)=\dfrac{\tan x+\cot x}{2}-\left\lvert\dfrac{\tan x-\cot x}{2}\right\rvert$ is continuous but not differentiable, is	(p)	2
(B)	The number of points where the function $f(x)=\min\{1, 1+x^3, x^2-3x+3\}$ is non-derivable, is	(q)	0
(C)	The number of points where $f(x)=(x+4)^{1/3}$ is non-differentiable, is	(r)	4
(D)	Consider $f(x)=\begin{cases}\dfrac{-\pi}{2}\log\left(\dfrac{2x}{\pi}\right)+\dfrac{\pi}{2}, & 0<x\leq\dfrac{\pi}{2}\\ \sin^{-1}(\sin x), & \dfrac{\pi}{2}<x\leq\dfrac{3\pi}{2}\end{cases}$ Number of points in $\left(0,\dfrac{3\pi}{2}\right)$, where $f(x)$ is non-differentiable, is	(s)	1

105. Match the entries of the following two columns.

	Column I		Column II
(A)	$f(x)=\begin{cases}x+1, & \text{if } x<0\\ \cos x, & \text{if } x\geq 0\end{cases}$ at $x=0$ is	(p)	continuous
(B)	For every $x\in R$, the function $g(x)=\dfrac{\sin(\pi[x-\pi])}{1+[x]^2}$, where $[x]$ denotes the greatest integer function, is	(q)	differentiability
(C)	$h(x)=\sqrt{\{x\}^2}$, where $\{x\}$ denotes fractional part function for all $x\in I$, is	(r)	discontinuous
(D)	$k(x)=\begin{cases}x^{\frac{1}{\ln x}}, & \text{if } x\neq 1\\ e, & \text{if } x=1\end{cases}$ at $x=1$ is	(s)	non-derivable

106. Match the entries of the following two columns.

	Column I		Column II
(A)	$\lim\limits_{x\to\infty}\left(e^{\sqrt{x^4+1}}-e^{(x^2+1)}\right)$ is	(p)	e
(B)	For $a>0$, let $f(x)=\begin{cases}\dfrac{a^x+a^{-x}-2}{x^2}, & \text{if } x>0\\ 3\ln(a-x)-2, & \text{if } x\leq 0\end{cases}$. If f is continuous at $x=0$, then a equals	(q)	e^2
(C)	Let $L=\lim\limits_{x\to a}\dfrac{x^x-a^a}{x-a}$ and $M=\lim\limits_{x\to a}\dfrac{x^x-a^x}{x-a}(a>0)$. If $L=2M$, then the value of a is equal to	(r)	non-existent

Continuity and Differentiability Exercise 6 :
Single Integer Answer Type Questions

107. Number of points of discontinuity of $f(x) = \tan^2 x - \sec^2 x$ in $(0, 2\pi)$ is ..... .

108. Number of point(s) of discontinuity of the function $f(x) = [x^{1/x}]$, $x > 0$, (where [ ] denotes the greatest integral function) is ..... .

109. Let $f(x) = x + \cos x + 2$ and $g(x)$ be the inverse function of $f(x)$, then $g'(3)$ equals to ..... .

110. Let $f(x) = x \tan^{-1}(x^2) + x^4$. Let $f^k(x)$ denotes kth derivative of $f(x)$ w. r. t. x, $k \in N$.

If $f^{2m}(0) \neq 0$, $m \in N$, then m equals to ..... .

111. Let $f_1(x)$ and $f_2(x)$ be twice differentiable functions, where $F(x) = f_1(x) + f_2(x)$ and $G(x) = f_1(x) - f_2(x), \forall\ x \in R, f_1(0) = 2$ and $f_2(0) = 1$. If $f_1'(x) = f_2(x)$ and $f_2'(x) = f_1(x), \forall\ x \in R$, then the number of solutions of the equation $(F(x))^2 = \dfrac{9x^4}{G(x)}$ is ..... .

112. Suppose the function $f(x) - f(2x)$ has the derivative 5 at $x = 1$ and derivative 7 at $x = 2$. The derivative of the function $f(x) - f(4x) - 10x$ at $x = 1$ is equal to ..... .

113. In a ΔABC, angles A, B, C are in AP.

If $f(x) = \lim\limits_{A \to C} \dfrac{\sqrt{3 - 4\sin A \sin C}}{|A - C|}$, then $f'(x)$ is equal to ..... .

114. Let $f(x) = \begin{cases} x\dfrac{\left(\frac{3}{4}\right)^{1/x} - \left(\frac{3}{4}\right)^{-1/x}}{\left(\frac{3}{4}\right)^{1/x} + \left(\frac{3}{4}\right)^{-1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

If $P = f'(0^-) - f'(0^+)$, then

$$4\left(\lim_{x \to P-} \frac{(\exp((x+2)\log 4))^{\frac{[x+1]}{4}} - 16}{4^x - 16}\right) \text{ is ..... .}$$

(where $[x]$ denotes greatest integer function.)

115. Let $f(x) = -x^3 + x^2 - x + 1$ and

$$g(x) = \begin{cases} \min(f(t)), 0 \le t \le x \text{ and } 0 \le x \le 1 \\ x - 1, \quad 1 < x \le 2 \end{cases}$$

Then, the value of $\lim\limits_{x \to 1} g(g(x))$ is..... .

116. The number of points at which the function $f(x) = (x - |x|)^2 (1 - x + |x|)^2$ is not differentiable in the interval $(-3, 4)$ is ..... .

117. If $f(x) = \begin{cases} \dfrac{\frac{\pi}{2} - \sin^{-1}(1 - \{x\}^2)\sin^{-1}(1 - \{x\})}{\sqrt{2}(\{x\} - \{x\}^3)}, & x > 0 \\ k, & x = 0 \\ \dfrac{A\sin^{-1}(1 - \{x\})\cos^{-1}(1 - \{x\})}{\sqrt{2\{x\}}(1 - \{x\})}, & x < 0 \end{cases}$

is continuous at $x = 0$, then the value of $\sin^2 k + \cos^2\left(\dfrac{A\pi}{\sqrt{2}}\right)$, is..... (where $\{\cdot\}$ denotes fractional part of x).

Continuity and Differentiability Exercise 7 :
Subjective Type Questions

118. Discuss the continuity of the function $f(x) = [[x]] - [x - 1]$, where [ ] denotes the greatest integral function.

119. Examine the continuity or discontinuity of the following :

(i) $f(x) = [x] + [-x]$ (ii) $g(x) = \lim\limits_{n \to \infty} \dfrac{x^{2n} - 1}{x^{2n} + 1}$

120. Examine the continuity and differentiability at points $x = 1$ and $x = 2$.

The function f defined by

$f(x) = \begin{cases} x[x], & 0 \le x < 2 \\ (x - 1)[x], & 2 \le x \le 3 \end{cases}$ (where $[\cdot]$ denotes the greatest integral function less than or equal to x).

121. Let f be twice differentiable function, such that $f''(x) = -f(x)$ and $f'(x) = g(x)$, $h(x) = [f(x)]^2 + [g(x)]^2$. Find $h(10)$, if $h(5) = 11$.

122. A function $f : R \to R$ satisfies the equation $f(x+y) = f(x) \cdot f(y)$ for all x, y in R and $f(x) \neq 0$ for any x in R. Let the function be differentiable at $x = 0$ and $f'(0) = 2$. Show that $f'(x) = 2f(x)$ for all x in R. Hence, determine $f(x)$.

123. A function $f : R \to R$, where R is a set of real numbers, satisfying the equation $f\left(\frac{x+y}{3}\right) = \frac{f(x)+f(y)+f(0)}{3}$ for all x, y in R. If the function is differentiable at $x = 0$, then show that it is differentiable for all x in R.

124. Let $f(x+y) = f(x) + f(y) + 2xy - 1$ for all real x, y and $f(x)$ be differentiable functions. If $f'(0) = \cos\alpha$, then prove that $f(x) > 0,\ \forall\ x \in R$.

Continuity and Differentiability Exercise 8 : Questions Asked in Previous 10 Years' Exams

(i) JEE Advanced & IIT-JEE

125. For every pair of continuous function $f,\ g : [0,1] \to R$ such that $\max\{f(x): x \in [0,1]\} = \max\{g(x): x \in [0,1]\}$. The correct statement(s) is/are **[More than One Correct Option, 2014]**

(a) $[f(c)]^2 + 3f(c) = [g(c)]^2 + 3g(c)$ for some $c \in [0,1]$

(b) $[f(c)]^2 + f(c) = [g(c)]^2 + 3g(c)$ for some $c \in [0,1]$

(c) $[f(c)]^2 + 3f(c) = [g(c)]^2 + g(c)$ for some $c \in [0,1]$

(d) $[f(c)]^2 = [g(c)]^2$ for some $c \in [0,1]$

126. Let $f : R \to R$ and $g : R \to R$ be respectively given by $f(x) = |x| + 1$ and $g(x) = x^2 + 1$. Define $h : R \to R$ by

$$h(x) = \begin{cases} \max\{f(x), g(x)\}, & \text{if } x \leq 0 \\ \min\{f(x), g(x)\}, & \text{if } x > 0 \end{cases}.$$

The number of points at which $h(x)$ is not differentiable is **[Integer Answer Type, 2014]**

127. Let $f(x) = \begin{cases} x^2 \left|\cos\frac{\pi}{x}\right|, & x \neq 0\ x \in R, \\ 0, & x = 0, \text{then } f \text{ is} \end{cases}$ **[One Correct Option, 2012]**

(a) differentiable both at $x = 0$ and at $x = 2$

(b) differentiable at $x = 0$ but not differentiable at $x = 2$

(c) not differentiable at $x = 0$ but differentiable at $x = 2$

(d) differentiable neither at $x = 0$ nor at $x = 2$

128. For every integer n, let a_n and b_n be real numbers. Let function $f : R \to R$ be given by

$$f(x) = \begin{cases} a_n + \sin\pi x, & \text{for } x \in [2n, 2n+1] \\ b_n + \cos\pi x, & \text{for } x \in (2n-1, 2n) \end{cases},$$

for all integers n.

If f is continuous, then which of the following hold(s) for all n? **[More than One Correct Option, 2012]**

(a) $a_{n-1} - b_{n-1} = 0$ (b) $a_n - b_n = 1$

(c) $a_n - b_{n+1} = 1$ (d) $a_{n-1} - b_n = -1$

129. Let $f : R \to R$ be a function such that $f(x+y) = f(x) + f(y),\ \forall x, y \in R$. If $f(x)$ is differentiable at $x = 0$, then **[More than One Correct Option, 2011]**

(a) $f(x)$ is differentiable only in a finite interval containing zero

(b) $f(x)$ is continuous for all $x \in R$

(c) $f'(x)$ is constant for all $x \in R$

(d) $f(x)$ is differentiable except at finitely many points

130. If $f(x) = \begin{cases} -x - \frac{\pi}{2}, & x \leq -\frac{\pi}{2} \\ -\cos x, & -\frac{\pi}{2} < x \leq 0, \text{ then} \\ x - 1, & 0 < x \leq 1 \\ \ln x, & x > 1 \end{cases}$ **[More than One Correct Option, 2011]**

(a) $f(x)$ is continuous at $x = -\frac{\pi}{2}$

(b) $f(x)$ is not differentiable at $x = 0$

(c) $f(x)$ is differentiable at $x = 1$

(d) $f(x)$ is differentiable at $x = -\frac{3}{2}$

131. For the function $f(x) = x\cos\frac{1}{x},\ x \geq 1$, **[More than One Correct Option, 2009]**

(a) for at least one x in the interval $[1, \infty)$, $f(x+2) - f(x) < 2$

(b) $\lim_{x \to \infty} f'(x) = 1$

(c) for all x in the interval $[1, \infty)$, $f(x+2) - f(x) > 2$

(d) $f'(x)$ is strictly decreasing in the interval $[1, \infty)$

132. Let $g(x) = \frac{(x-1)^n}{\log\cos^m(x-1)}; 0 < x < 2,\ m$ and n are integers, $m \neq 0,\ n > 0$ and let p be the left hand derivative of $|x-1|$ at $x = 1$. If $\lim_{x \to 1^+} g(x) = p$, then **[One Correct Option, 2008]**

(a) $n = 1, m = 1$ (b) $n = 1, m = -1$

(c) $n = 2, m = 2$ (d) $n > 2, m = n$

133. Let f and g be real valued functions defined on interval $(-1, 1)$ such that $g''(x)$ is continuous, $g(0) \neq 0$, $g'(0) = 0$, $g''(0) \neq 0$ and $f(x) = g(x)\sin x$.

Statement I $\lim_{x \to 0} [g(x)\cos x - g(0)]\operatorname{cosec} x = f''(0)$.

Statement II $f'(0) = g(0)$.

For the above question, choose the correct answer from the codes (a), (b), (c) and (d) defined as follows.

[Assertion and Reason, 2008]

(a) Statement I is true, Statement II is also true; Statement II is the correct explanation of Statement I

(b) Statement I is true, Statement II is also true; Statement II is not the correct explanation of Statement I

(c) Statement I is true; Statement II is false

(d) Statement I is false; Statement II is true

134. In the following, $[x]$ denotes the greatest integer less than or equal to x. **[Matching type Question, 2007]**

	Column I		Column II
A.	$x\|x\|$	p.	continuous in $(-1, 1)$
B.	$\sqrt{\|x\|}$	q.	differentiable in $(-1, 1)$
C.	$x + [x]$	r.	strictly increasing in $(-1, 1)$
D.	$\|x-1\| + \|x+1\|$	s.	not differentiable atleast at one point in $(-1, 1)$

135. If $f(x) = \min\{1, x^2, x^3\}$, then **[More than One Correct, 2006]**

(a) $f(x)$ is continuous everywhere

(b) $f(x)$ is continuous and differentiable everywhere

(c) $f(x)$ is not differentiable at two points

(d) $f(x)$ is not differentiable at one point

136. Let $f(x) = ||x| - 1|$, then points where, $f(x)$ is not differentiable is/are **[One Correct Option, 2005]**

(a) $0, \pm 1$ (b) ± 1

(c) 0 (d) 1

137. If f is a differentiable function satisfying $f\left(\frac{1}{n}\right) = 0, \forall$ $n \geq 1, n \in I$, then **[One Correct Option, 2005]**

(a) $f(x) = 0, x \in (0, 1]$

(b) $f'(0) = 0 = f(0)$

(c) $f(0) = 0$ but $f'(0)$ not necessarily zero

(d) $|f(x)| \leq 1, x \in (0, 1]$

138. The domain of the derivative of the functions

$$f(x) = \begin{cases} \tan^{-1} x, & \text{if } |x| \leq 1 \\ \frac{1}{2}(|x| - 1), & \text{if } |x| > 1 \end{cases}$$ is

[One Correct Option, 2002]

(a) $R - \{0\}$ (b) $R - \{1\}$

(c) $R - \{-1\}$ (d) $R - \{-1, 1\}$

139. The left hand derivative of $f(x) = [x]\sin(\pi x)$ at $x = k, k$ is an integer, is **[One Correct Option, 2001]**

(a) $(-1)^k (k-1)\pi$

(b) $(-1)^{k-1}(k-1)\pi$

(c) $(-1)^k k\pi$

(d) $(-1)^{k-1} k\pi$

140. Which of the following functions is differentiable at $x = 0$? **[One Correct Option, 2001]**

(a) $\cos(|x|) + |x|$ (b) $\cos(|x|) - |x|$

(c) $\sin(|x|) + |x|$ (d) $\sin(|x|) - |x|$

(ii) JEE Main & AIEEE

141. For $x \in R$, $f(x) = |\log 2 - \sin x|$ and $g(x) = f(f(x))$, then

(a) g is not differentiable at $x = 0$ **[2016, JEE Main]**

(b) $g'(0) = \cos(\log 2)$

(c) $g'(0) = -\cos(\log 2)$

(d) g is differentiable at $x = 0$ and $g'(0) = -\sin(\log 2)$

142. If the function $g(x) = \begin{cases} k\sqrt{x+1}, & 0 \leq x \leq 3 \\ mx + 2, & 3 < x \leq 5 \end{cases}$ is differentiable, then the value of $k + m$ is **[2015, JEE Main]**

(a) 2 (b) $\frac{16}{5}$

(c) $\frac{10}{3}$ (d) 4

143. If f and g are differentiable functions in $(0, 1)$ satisfying $f(0) = 2 = g(1)$, $g(0) = 0$ and $f(1) = 6$, then for some $c \in]0, 1[$ **[2014, JEE Main]**

(a) $2f'(c) = g'(c)$ (b) $2f'(c) = 3g'(c)$

(c) $f'(c) = g'(c)$ (d) $f'(c) = 2g'(c)$

144. If $f : R \to R$ is a function defined by $f(x) = [x]\cos\left(\frac{2x-1}{2}\right)\pi$, where $[x]$ denotes the greatest integer function, then f is **[2012 AIEEE]**

(a) continuous for every real x

(b) discontinuous only at $x = 0$

(c) discontinuous only at non-zero integral values of x

(d) continuous only at $x = 0$

Answers

Exercise for Session 1

1. (c) **2.** (c) **3.** (a) **4.** (a) **5.** (a)

Exercise for Session 2

1. $a = -1, b = 1$ **2.** f is continuous in $-1 \le x \le 1$.

3. $f(x)$ is continuous everywhere in $[0, 2]$ except for $x = \frac{1}{2}$, 1 and 2.

4. (d)

Exercise for Session 3

1. (b,c,d) **2.** (a,c,d) **3.** (d) **4.** (c) **5.** (a,c,d)

Exercise for Session 4

1. (a,b,c) **2.** (b,c) **3.** (c,d) **4.** (b,c) **5.** (a,b,c) **6.** (a) **7.** (b)

Exercise for Session 5

1. discontinuity at $x = 0$

4. $y_n(x)$ is continuous at $x = 0$ for all n and $y(x)$ is discontinuous at $x = 0$.

Exercise for Session 6

1. (d) **2.** (a) **3.** (a) **4.** (d) **5.** (b)

Exercise for Session 7

1. (c) **2.** (c) **3.** (c) **4.** (b) **5.** (b) **6.** (d)
7. (a) **8.** (c) **9.** (a) **10.** (c)

Chapter Exercises

1. (a) **2.** (c) **3.** (c) **4.** (c) **5.** (a) **6.** (c)
7. (b) **8.** (b) **9.** (b) **10.** (b) **11.** (a) **12.** (c)
13. (a) **14.** (b) **15.** (a) **16.** (c) **17.** (c) **18.** (c)
19. (c) **20.** (b) **21.** (b) **22.** (b) **23.** (d) **24.** (b)
25. (d) **26.** (c) **27.** (c) **28.** (c) **29.** (d) **30.** (c)
31. (b) **32.** (c) **33.** (d) **34.** (d) **35.** (d) **36.** (a)
37. (d) **38.** (d) **39.** (c) **40.** (a) **41.** (b) **42.** (b)
43. (c) **44.** (c) **45.** (c) **46.** (a) **47.** (c) **48.** (d)
49. (b) **50.** (c) **51.** (c,d) **52.** (b,c,d) **53.** (a,b,c) **54.** (a,c)
55. (b,c,d) **56.** (b,d) **57.** (a,d) **58.** (a,b,c) **59.** (a,c) **60.** (a, b, c, d)
61. (a, b, c, d) **62.** (a, b, d) **63.** (a, c) **64.** (a,c)
65. (b, c) **66.** (a, b, c) **67.** (a, b, c)
68. (a, b) **69.** (a, b, c, d) **70.** (a, c, d)
71. (a, b, c, d) **72.** (a,d) **73.** (a) **74.** (c)
75. (a) **76.** (c) **77.** (a) **78.** (a) **79.** (c) **80.** (c)
81. (a) **82.** (d) **83.** (a) **84.** (b,c) **85.** (a) **86.** (c)
87. (a) **88.** (d) **89.** (a) **90.** (a) **91.** (b) **92.** (b)
93. (c) **94.** (b) **95.** (c) **96.** (b) **97.** (c) **98.** (d)
99. (a) **100.** (b) **101.** (d)

102. (A) $\rightarrow$ (p,r); (B) $\rightarrow$ (s); (C) $\rightarrow$ (s); (D) $\rightarrow$ (p,q,r)

103. (A) $\rightarrow$ (q); (B) $\rightarrow$ (p); (C) $\rightarrow$ (s); (D) $\rightarrow$ (p)

104. (A) $\rightarrow$ (r); (B) $\rightarrow$ (p); (C) $\rightarrow$ (s); (D) $\rightarrow$ (q)

105. (A) $\rightarrow$ (p, s); (B) $\rightarrow$ (p, q); (C) $\rightarrow$ (r, s); (D) $\rightarrow$ (p, q)

106. (A) $\rightarrow$ (r); (B) $\rightarrow$ (p, q); (C) $\rightarrow$ (p)

107. (2) **108.** (1) **109.** (1) **110.** (2) **111.** (2)
112. (9) **113.** (0) **114.** (2) **115.** (1) **116.** (0) **117.** (2)
125. (a,d) **126.** (3) **127.** (b) **128.** (b, d) **129.** (b,c) **130.** (a, b, c, d)
131. (b, c, d) **132.** (c) **133.** (b)

134. (A) $\rightarrow$ (p, q, r); (B) $\rightarrow$ (p, s); (C) $\rightarrow$ (r, s); (D) $\rightarrow$ (p, q)

135. (a,d) **136.** (a) **137.** (b) **138.** (d) **139.** (a) **140.** (d)
141. (b) **142.** (a) **143.** (d) **144.** (a)

Solutions

1. $f(x)=\begin{cases}\sin\dfrac{\pi x}{2}, & x<1\\ [x], & x\geq 1\end{cases}$

$f(1)=1,\ f(1^+)=\lim\limits_{h\to 0}[1+h]=1$

$f(1^-)=\lim\limits_{h\to 0}\sin\dfrac{\pi}{2}(1-h)=\lim\limits_{h\to 0}\sin\left(\dfrac{\pi}{2}-\dfrac{\pi h}{2}\right)=\lim\limits_{h\to 0}\cos\dfrac{\pi h}{2}=1$

$\therefore$ $f(x)$ is continuous at $x=1$.

2. $f(0^+)=\lim\limits_{h\to 0}\dfrac{8^h-4^h-2^h+1}{h^2}=\lim\limits_{h\to 0}\dfrac{(4^h-1)}{h}\dfrac{(2^h-1)}{h}$

$=(\log 4)(\log 2)$

$f(0^-)=\lim\limits_{h\to 0}(e^{-h}\sin(-h)+\pi(-h)+k\log 4)=k\log 4$

Since, $f(x)$ is continuous at $x=0$

$\Rightarrow$ $k\log 4=(\log 4)(\log 2)=f(0)$.

$\therefore$ $f(0)=(\log 4)(\log 2)$, when $k=\log 2$

3. $f(0^-)=\lim\limits_{x\to 0^-}\dfrac{a(1-x\sin x)+b\cos x+5}{x^2}=f(0)$

$=\lim\limits_{x\to 0^-}\dfrac{(a+b+5)+\left(-a-\dfrac{b}{2}\right)x^2+\ldots}{x^2}=3$

$\Rightarrow$ $a+b+5=0$ and $-a-\dfrac{b}{2}=3$

$\Rightarrow$ $a=-1,\ b=-4$

$\Rightarrow$ $f(0^+)=\lim\limits_{x\to 0^+}\left[1+\left(\dfrac{cx+dx^3}{x^2}\right)\right]^{1/x}$ exists.

$\Rightarrow$ $\lim\limits_{x\to 0^+}\dfrac{cx+dx^3}{x^2}=0\Rightarrow c=0$

Now, $\lim\limits_{x\to 0^+}(1+dx)^{1/x}=\lim\limits_{x\to 0^+}((1+dx)^{1/dx})^d=e^d$

So, $e^d=3\Rightarrow d=\log_e 3$

$\therefore$ $a+b+c+d=\log_e 3-5$

4. $f\left(\dfrac{\pi^+}{2}\right)=\lim\limits_{h\to 0}\pi\left[\dfrac{\pi}{2}+h\right]-1=(\pi-1)$

$f\left(\dfrac{\pi^-}{2}\right)=\lim\limits_{h\to 0}\cos^{-1}\left\{\cot\left(\dfrac{\pi}{2}-h\right)\right\}=\lim\limits_{h\to 0}\cos^{-1}\{\tan h\}=\dfrac{\pi}{2}$

$\therefore$ Jump of discontinuity $=(\pi-1)-\dfrac{\pi}{2}=\dfrac{\pi}{2}-1$

5. Consider $g(x)=f(x)-x$

$g(0)=f(0)-0=f(0)\geq 0$ $\quad[\because 0\leq f(x)\leq 1]$

$g(1)=f(1)-1\leq 0$

$\Rightarrow$ $g(0)\cdot g(1)\leq 0$

$\Rightarrow$ $g(x)=0$ has at least one root in [0, 1].

$\Rightarrow$ $f(x)=x$ for at least one root in [0, 1].

6. Here, $f(x)=\dfrac{x+1}{x-1}$, discontinuous at $x=1$

$g(x)=\dfrac{1}{x-2}$, discontinuous at $x=2$

$f(g(x))=\dfrac{g(x)+1}{g(x)-1}$, discontinuous at $g(x)=1$

$\Rightarrow$ $\dfrac{1}{x-2}=1$

$\Rightarrow$ $x=3$

$\therefore$ $(fog)(x)$ is discontinuous at $x=2$ and 3.

7. Here, $y_n(x)=\begin{cases}\dfrac{x^2\left[\dfrac{1}{(1+x^2)^n}-1\right]}{\dfrac{1}{1+x^2}-1}=(1+x^2)\cdot\left[1-\dfrac{1}{(1+x^2)^n}\right], & \text{when } x\neq 0,\ n\in N\\ 0, & \text{when } x=0, n\in N\end{cases}$

$\therefore$ $y(x)=\lim\limits_{n\to\infty}y_n(x)=\begin{cases}(1+x^2), & x\neq 0\\ 0, & x=0\end{cases}$

$\therefore$ $y(x)$ is discontinuous at $x=0$.

8. $g(0^-)=\lim\limits_{h\to 0}\dfrac{a^h-1-h\log a}{h^2}$

$=\lim\limits_{h\to 0}\dfrac{\left[1+h\log a+\dfrac{h^2}{2!}(\log a)^2+\ldots\right]-1-h\log a}{h^2}$

$=\lim\limits_{h\to 0}\left[\dfrac{(\log a)^2}{2!}\cdot 1+\dfrac{(\log a)^3}{3!}\cdot h\ldots\right]=\dfrac{(\log a)^2}{2}$

$g(0^+)=\lim\limits_{h\to 0}\dfrac{2^h a^h-h\log 2-h\log a-1}{h^2}$

$=\lim\limits_{h\to 0}\dfrac{\left[1+h\log(2a)+\dfrac{h^2}{2!}(\log(2a))^2\ldots\right]-h\log 2-h\log a-1}{h^2}$

$=\dfrac{(\log(2a))^2}{2}$

Since, $g(x)$ is continuous.

$\Rightarrow$ $\dfrac{(\log a)^2}{2}=\dfrac{(\log(2a))^2}{2}\Rightarrow(\log a)^2=(\log 2+\log a)^2$

$\Rightarrow$ $\log a=-\dfrac{1}{2}\log 2=\log 2^{-1/2}$

$\Rightarrow$ $a=\dfrac{1}{\sqrt{2}}$

9. $f(0^+)=\lim\limits_{h\to 0}\dfrac{\left(\dfrac{\pi}{2}-\sin^{-1}(1-h^2)\right)\cdot\sin^{-1}(1-h)}{\sqrt{2}(h-h^3)}$

$=\lim\limits_{h\to 0}\dfrac{(\cos^{-1}(1-h^2))\cdot\sin^{-1}(1-h)}{\sqrt{2}(1-h^2)\cdot h}=\dfrac{\pi}{2}$

$$f(0^-)=\lim_{h\to 0}\frac{\left(\frac{\pi}{2}-\sin^{-1}(1-(1-h^2))\right)\cdot\sin^{-1}(1-(1-h))}{\sqrt{2}((1-h)-(1-h)^3)}$$

$$=\lim_{h\to 0}\frac{\frac{\pi}{2}\sin^{-1}h}{\sqrt{2}(1-h)(2-h)h}=\frac{\pi}{4\sqrt{2}}$$

10. $f(x)=\begin{cases}-1+x, & -\infty<x<0\\ -1+\sin x, & 0\le x<\pi/2\\ \cos x, & \pi/2\le x<\infty\end{cases}$

$$f'(x)=\begin{cases}1, & -\infty<x<0\\ \cos x, & 0<x<\pi/2\\ -\sin x, & \pi/2<x<\infty\end{cases}$$

$f'(0^-)=1,\ f'(0^+)=1$

$\Rightarrow f(x)$ is differentiable at $x=0$

$$f'\left(\frac{\pi^-}{2}\right)=0,\quad f'(\pi/2^+)=-1$$

$\Rightarrow f(x)$ is not differentiable at $x=\frac{\pi}{2}$.

$\therefore$ Number of points of non-differentiability is 1.

11. $f(x)=\begin{cases}-\frac{1}{x}, & x\le -1\\ ax^2+b, & -1<x<1\\ \frac{1}{x}, & x\ge 1\end{cases}$

Since, function is continuous everywhere.

$\therefore$ LHL = RHL at $x=-1$

$\Rightarrow f(-1^-)=1,\ f(-1^+)=a+b$

$\Rightarrow a+b=1$ ...(i)

$f(x)$ is differentiable at $x=-1$

$$f'(x)=\begin{cases}\frac{1}{x^2}, & x<-1\\ 2ax, & -1<x<1\\ -\frac{1}{x^2}, & x>1\end{cases}$$

$f'(-1^-)=1,\ f'(-1^+)=-2a\Rightarrow -2a=1,\ a=-1/2$ ...(ii)

From Eqs. (i) and (ii), we get

$$a=-\frac{1}{2},\ b=\frac{3}{2}$$

12. $f(1^-)=A+B,\ f(1^+)=3A-B+2$

Since, continuous $A+B=3A-B+2$

$\Rightarrow 2A-2B=-2\Rightarrow A-B=-1$ ...(i)

$$f'(x)=\begin{cases}2Bx, & x<1\\ 3A, & x>1\end{cases}$$

$f'(1^+)=3A,\ f'(1^-)=2B$

$\Rightarrow 3A=2B$ ...(ii)

On solving Eqs. (i) and (ii), we get

$A=2,\ B=3$

13. $f'(1^+)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$

$$=\lim_{h\to 0}\frac{|1+h-1|([1+h]-(1+h))}{h}=0=\lim_{h\to 0}\frac{h(1-1-h)}{h}=0$$

$$f'(1^-)=\lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}=\lim_{h\to 0}\frac{|-h|([1-h]-(1-h))}{-h}$$

$$=\lim_{h\to 0}\frac{h(0-1+h)}{-h}=1$$

$\Rightarrow f'(1^+)=0,\ f'(1^-)=1$

$\therefore f(x)$ is not differentiable at $x=1$.

14. $f'(1^-)=\lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}=\lim_{h\to 0}\frac{[\cos\pi(1-h)]+1}{-h}$

$$=\lim_{h\to 0}\frac{-1+1}{-h}=0$$

$$f'(1^+)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$=\lim_{h\to 0}\frac{2\{1+h\}-1+1}{h}$$

$$=\lim_{h\to 0}\frac{2h}{h}=2$$

$\therefore\ f'(1^-)=0,\ f'(1^+)=2$

15. $f(x)=\begin{cases}x-3, & x<0\\ x^2-3x+2, & x\ge 0\end{cases}$

$f(|x|)=\begin{cases}|x|-3, & |x|<0 \text{ not possible}\\ |x|^2-3|x|+2, & |x|\ge 0\end{cases}$

$\Rightarrow\ g(x)=\begin{cases}x^2+3x+2, & x\le 0\\ x^2-3x+2, & x\ge 0\end{cases}$

$\Rightarrow\ g(x)$ is continuous at $x=0$.

$$g'(x)=\begin{cases}2x+3, & x\le 0\\ 2x-3, & x\ge 0\end{cases}$$

$g'(0^+)=-3,\ g'(0^-)=3$

16. $f(x)=\begin{cases}0, & 0\le x<1/2\\ 5/6, & x=1/2\\ 0, & 1/2<x<1\\ -2, & 1\le x<4/3\\ 0, & x=4/3\\ 2, & 4/3<x<3/2\\ 3, & 3/2\le x<2\\ 4, & x=2\end{cases}$

Hence, $f(x)$ is neither continuous nor differentiable at $x=\frac{1}{2}, 1, \frac{4}{3}, \frac{3}{2}, 2$.

$\therefore$ Number of points is 5.

17. Put $x=y=1$ in given rule

$\Rightarrow\ f(1)=f(1)-f(1)=0$

$\therefore\quad f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{f\left(\frac{x+h}{x}\right)}{h}$

$$= \lim_{h \to 0} \left(\frac{f\left(1 + \frac{h}{x}\right) - f(1)}{\frac{h}{x}} \right) \frac{1}{x} = \frac{f'(1)}{x} = \frac{1}{x}$$

[given, $f'(1) = 1$]

On integrating both sides, $f(x) = \log x + C$

On putting $x = 1$, we get $C = 0 \Rightarrow f(x) = \log_e x$

18. $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{f(x) + f(h) - 2xh - 1 - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \left(-2x + \frac{f(h) - 1}{h} \right) = -2x + f'(0)$$

On integrating both sides, we get

$\Rightarrow \quad f(x) = -x^2 + f'(0)x + c$

$\Rightarrow \quad f(x) = -x^2 - (\sin\alpha)x + c \qquad [\text{given } f'(0) = -\sin\alpha]$

$\Rightarrow \quad f(0) = c \Rightarrow c = 1$

So, $f\{f'(0)\} = f\{-\sin\alpha\} = -\sin^2\alpha + \sin^2\alpha + 1$

$= 1$

19. $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \geq \lim_{h \to 0} \frac{\log\left(\frac{x+h}{x}\right) + x + h - x}{h}$

$$= \lim_{h \to 0} \frac{\log\left(1 + \frac{h}{x}\right)}{\frac{h}{x}} \frac{1}{x} + 1 \geq \frac{1}{x} + 1 \quad \ldots(\text{i})$$

$$f'(x) = \lim_{h \to 0} \frac{f(x-h) - f(x)}{-h} \leq \lim_{h \to 0} \frac{\log\left(\frac{x-h}{x}\right) + x - h - x}{-h}$$

$$= \lim_{h \to 0} \frac{\log\left(1 - \frac{h}{x}\right)}{-h} + 1 \leq \frac{1}{x} + 1 \quad \ldots(\text{ii})$$

$\therefore$ From Eqs. (i) and (ii), we get $f'(x) = \frac{1}{x} + 1 = g(x)$

$$\Rightarrow \quad \sum_{n=1}^{100} g\left(\frac{1}{n}\right) = g\left(\frac{1}{1}\right) + g\left(\frac{1}{2}\right) + \ldots + g\left(\frac{1}{100}\right)$$

$$= (1+1) + (2+1) + (3+1) + \ldots + (100+1)$$

$$= (1 + 2 + \ldots + 100) + 100$$

$$= 5050 + 100 = 5150$$

20. Here, $\frac{d(f(x))}{dx} = e^{-x} f(x) + e^x f(-x)$

Let $\frac{d(f(x))}{dx} = g(x)$, where $g(x) = e^{-x} f(x) + e^x f(-x)$

$\therefore \quad g(-x) = e^x f(-x) + e^{-x} f(x) = g(x)$

$\Rightarrow$ $g(x)$ is even function.

Hence, $f(x)$ should be an odd function as $f(0) = 0$.

21. We have, $f(x^2) = \int_0^{x^2} t\, f(t)\, dt = x^4 + x^5 \qquad \ldots(\text{i})$

On differentiating both the sides w.r.t. x, we get

$$x^2 f(x^2) \cdot 2x = 4x^3 + 5x^4$$

$$\Rightarrow \quad f(x^2) = 2 + \frac{5}{2}x \qquad \ldots(\text{ii})$$

$$\therefore \sum_{r=1}^{12} f(r^2) = \sum_{r=1}^{12} \left(2 + \frac{5}{2}r\right) = 24 + \frac{5}{2} \cdot \frac{(12)(13)}{2} = 24 + 195 = 219$$

Hence, $\sum_{r=1}^{12} f(r^2) = 219$

22. Let $x = \frac{2}{3}$, which is rational.

$$\Rightarrow \quad h\left(\frac{2}{3}\right) = \frac{1}{3}$$

$\lim_{t \to 0} h\left(\frac{2}{3} + t\right) = 0 \Rightarrow$ Discontinuous at $x \in Q$

Let $x = \sqrt{2} \notin Q$

$h(\sqrt{2}) = 0$, consider $\sqrt{2} = 1.4142135624$

$$h(\sqrt{2}) = h\left(\frac{14142135624}{10^{10}}\right) = \frac{1}{10^{10}} \to 0$$

Hence, h is continuous for all irrational.

23. By theorem, if g and h are continuous functions on the open interval (a, b), then g/h is also continuous at all x in the open interval (a, b), where $h(x) \neq 0$.

24. $h(x) = \begin{cases} \frac{2\cos x - \sin 2x}{(\pi - 2x)^2}, & x < \frac{\pi}{2} \\ \frac{e^{-\cos x} - 1}{8x - 4\pi}, & x > \frac{\pi}{2} \end{cases}$

LHL at $x = \pi/2$

$$\text{LHL} = \lim_{h \to 0} \frac{2\sin h - \sin 2h}{4h^2} = \lim_{h \to 0} \frac{2\sin h\,(1 - \cos h)}{4h^2} = 0$$

$$\text{RHL} = \lim_{h \to 0} \frac{e^{\sin h} - 1}{8((\pi/2) + h) - 4\pi}$$

$$= \lim_{h \to 0} \frac{e^{\sin h} - 1}{8h} \cdot \frac{\sin h}{\sin h} = \frac{1}{8}$$

$\Rightarrow h(x)$ is discontinuous at $x = \pi/2$.

Irremovable discontinuity at $x = \pi/2$.

$$f\left(\frac{\pi^+}{2}\right) = 0 \text{ and } g\left(\frac{\pi^-}{2}\right) = \frac{1}{8}$$

$$\Rightarrow \quad f\left(\frac{\pi^+}{2}\right) \neq g\left(\frac{\pi^-}{2}\right)$$

25. $\lim_{x \to 0} \frac{x - e^x + 1 - (1 - \cos 2x)}{x^2} = -\frac{1}{2} - 2 = -\frac{5}{2}$

Hence, for continuity, $f(0) = -\frac{5}{2}$

$\therefore [f(0)] = -3$; $\{f(0)\} = \left\{-\frac{5}{2}\right\} = \frac{1}{2}$

Hence, $[f(0)] \cdot \{f(0)\} = -\frac{3}{2} = -1.5$

26. $f(1^+) = f(1^-) = f(1) = 2$

$f(0) = 1, \qquad f(2) = 2$

$f(2^-) = 1,$

$\Rightarrow$ f is not continuous at $x = 2$.

27. $f(2^+) = 8,\ f(2^-) = 16$

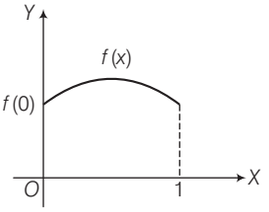
28. $g(x) = x - [x] = \{x\}$

f is continuous with $f(0) = f(1)$

$h(x) = f(g(x)) = f(\{x\})$

Let the graph of f is as shown in the figure satisfying

$$f(0) = f(1)$$



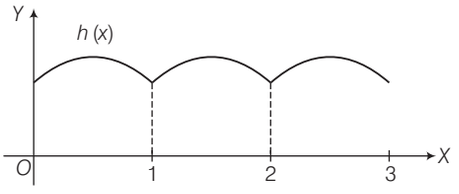
Now,

$h(0) = f(\{0\}) = f(0) = f(1)$

$h(0.2) = f(\{0.2\}) = f(0.2)$

$h(1.5) = f(\{1.5\}) = f(0.5)$ etc.

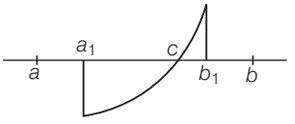
Hence, the graph of $h(x)$ will be a periodic graph as shown



$\Rightarrow h$ is continuous in R.

29. Statements I and II are false. The function $f(x) = 1/x,\ 0 < x < 1$ is a counter example.

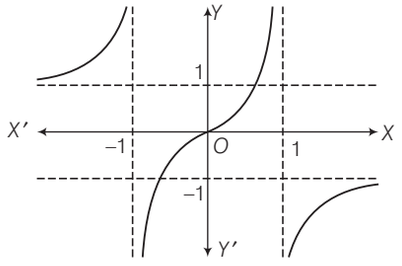
Statement III is true. Apply the intermediate value theorem to f on the closed interval $[a_1, b_1]$.



30. Since, the given function is not differentiable, at $x = 0$ and 2.

Hence, the number of points is 2.

31. $f(x) = \begin{cases} \dfrac{x}{1-x}, & \text{if } x \geq 0,\ x \neq 1 \\ \dfrac{x}{1+x}, & \text{if } x < 0,\ x \neq -1 \end{cases}$



and $f'(x) = \begin{cases} \dfrac{1}{(1-x)^2}, & \text{if } x > 0, x \neq 1 \\ \dfrac{1}{(1+x)^2}, & \text{if } x < 0,\ x \neq -1 \end{cases}$

32. $g[f(x)]$ is continuous at $x = \alpha$

$\Rightarrow \quad g[f(\alpha)] = g(a)$

Also, $\lim\limits_{x \to \alpha} g(f(x)) = g(a)$

$\Rightarrow \quad g[f(\alpha^-)] = g[f(\alpha^+)] = g(a)$

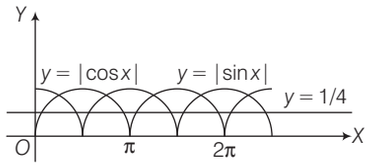
$\Rightarrow g(x)$ takes same limiting values at

$f(\alpha^-),\ f(\alpha^+)$ and $f(\alpha)$.

$\Rightarrow\ f(\alpha^-) = f(\alpha^+) \Rightarrow x = \alpha$ is an extremum of $f(x)$

and $x = a$ may not be an extremum of $g(x)$.

33. From the graph, number of points of non-differentiability = 11



34. Let n be any integer other than 1.

$$\lim_{x \to n^-} f(x) = \lim_{h \to 0} ([n-h]^2 - [(n-h)^2])$$

$$= (n-1)^2 - (n^2 - 1) = 2 - 2n$$

$$\lim_{x \to n^+} f(x) = \lim_{h \to 0} [n+h]^2 - [(n+h)^2] = n^2 - n^2 = 0$$

$\therefore$ LHL $\neq$ RHL unless $n = 1$.

Hence, $f(x)$ is discontinuous at all integral values except 1.

35. At $x = 5,\ f'(x) = \lim\limits_{x \to 5} \dfrac{\{(x-1)^2 (x+1)\,|x-5| + \cos|x|\} - \cos 5}{x-5}$

$$= \lim_{x \to 5} \frac{96\,|x-5|}{x-5} = +96, \text{ if } x > 5 \text{ and } -96, \text{ if } x < 5$$

Hence, $f'(5)$ doesn't exist.

This ambiguity doesn't occur at other points.

$\therefore f(x)$ is not differentiable at $x = 5$.

36. $f'(0^+) = \lim\limits_{x \to 0^+} \dfrac{f(x) - f(0)}{x} = \lim\limits_{x \to 0^+} \dfrac{e^{\frac{-1}{x^2}} - 0}{x} = \lim\limits_{x \to 0^+} \dfrac{1/x}{e^{1/x^2}}$

$$= \lim_{x \to 0^+} \frac{-1/x^2}{e^{1/x^2} \cdot \left(-\frac{2}{x^3}\right)} = \lim_{x \to 0^+} \frac{x}{2e^{1/x^2}} = 0$$

As, f is even, so $f'(0^-) = f'(0^+) = 0$. Thus, $f'(0) = 0$

37. $\lim\limits_{h \to 0} g(n+h) = \lim\limits_{h \to 0} \dfrac{e^h - \cos 2h - h}{h^2}$

$$= \lim_{h \to 0} \frac{e^h - h - 1}{h^2} + \lim_{h \to 0} \frac{(1 - \cos 2h)}{4h^2} \cdot 4 = \frac{1}{2} + 2 = \frac{5}{2}$$

$$\lim_{h \to 0} g(n-h) = \frac{e^{1-\{n-h\}} - \cos 2\,(1 - \{n-h\}) - (1 - \{n-h\})}{(1 - \{n-h\})^2}$$

$$= \lim_{h \to 0} \frac{e^h - \cos 2h - h}{h^2} \left[\{n-h\} = \{-h\} = 1 - h\right] = \frac{5}{2}$$

$g(n) = \dfrac{5}{2}$. Hence, $g(x)$ is continuous, $\forall\ x \in I$.

Hence, $g(x)$ is continuous, $\forall\ x \in R$.

38. $g'(0^+) = \lim\limits_{h \to 0} \dfrac{\cos h - 1}{h} = 0$

$g'(0^-) = \lim\limits_{h \to 0} \dfrac{-h + b - 1}{-h}$ for existence of limit $b = 1$

Thus, $g'(0^-) = 1$

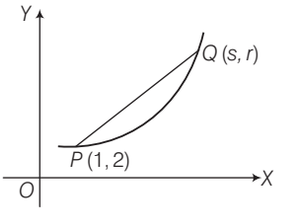
Hence, g cannot be made differentiable for no value of b.

39. By definition $f'(1)$ is the limit of the slope of the secant line when $s \to 1$.

Thus, $f'(1) = \lim_{s \to 1} \dfrac{s^2 + 2s - 3}{s - 1}$

$= \lim_{s \to 1} \dfrac{(s-1)(s+3)}{s-1}$

$= \lim_{s \to 1} (s + 3) = 4$



Aliter

By substituting $x = s$ into the equation of the secant line and cancelling by $s - 1$. Again, we get $y = s^2 + 2s - 1$.

This is $f(s)$ and its derivative is $f'(s) = 2s + 2$, so $f'(1) = 4$.

40. In the immediate neighbourhood of $x = \pi/2$, $\sin x > \sin^3 x \Rightarrow |\sin x - \sin^3 x| = \sin x - \sin^3 x$

Hence, for $x \neq \pi / 2$,

$$f(x) = \left[\frac{2(\sin x - \sin^3 x) + \sin x - \sin^3 x}{2(\sin x - \sin^3 x) - \sin x + \sin^3 x} \right]$$

$$= \frac{3\sin x - 3\sin^3 x}{\sin x - \sin^3 x} = 3$$

Hence, f is continuous and differentiable at $x = \pi/2$.

41. $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \dfrac{f(h) + |x| h + xh^2}{h}$

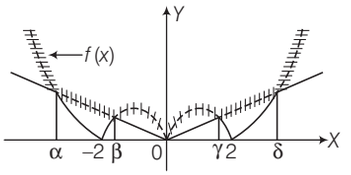
$\because \quad f(0) = 0$

Hence, $f'(x) = \lim_{h \to 0} \left[\dfrac{f(h) - f(0)}{h} + |x| + xh \right]$

$\Rightarrow \quad f'(x) = f'(0) + |x| = |x|$

42. $f(x)$ is non-differentiable at $x = \alpha, \beta, 0, \gamma, \delta$

and $g(x)$ is non-differentiable at $x = \alpha, -2, \beta, 0, \gamma, 2, \delta$.



43. We have, $g(x) = \begin{cases} 3x^2 - 4\sqrt{x} + 1, & \text{for } x < 1 \\ ax + b, & \text{for } x \geq 1 \end{cases}$

For differentiability at $x = 1$, $g'(1^+) = g'(1^-)$

$$a = 6 - \frac{4}{2\sqrt{1}}$$

$\Rightarrow \quad a = 6 - 2 = 4$

For continuity at $x = 1$, $g(1^+) = g(1^-)$

$$a + b = 3 - 4 + 1 \Rightarrow a + b = 0$$

$\Rightarrow \quad b = -4$

Hence, $a = 4$ and $b = -4$

44. $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \dfrac{-3xh + f(h)}{h}$

$= \lim_{h \to 0} \left\{ -3x + \dfrac{f(h)}{h} \right\} = -3x + \lim_{h \to 0} \dfrac{f(h)}{h}$

$= -3x + 7$

45. $f(x) = \dfrac{\sin \dfrac{\pi[x]}{4}}{[x]}$

Obviously, continuity at $x = 3/2$

$$f(2^-) = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

At $x = 2$,

$$f(2) = \frac{\sin \dfrac{\pi}{2}}{2} = \frac{1}{2}$$

Hence, $f(x)$ is discontinuous at $x = 2$.

46. $f(-1) = b(1 - 1) + 1 = 1$

and $\lim_{h \to 0} f(-1 + h) = 1$

$\lim_{h \to 0} f(-1 - h) = \lim_{h \to 0} \sin((-1 - h + a)\pi) = -\sin \pi a$

For continuity, $\sin \pi a = -1 = \sin\left(2n\pi + \dfrac{3\pi}{2}\right)$

$\Rightarrow \quad \pi a = 2n\pi + \dfrac{3\pi}{2}$

$\Rightarrow \quad a = 2n + \dfrac{3}{2}$

Hence, $a = 2n + \dfrac{3}{2}$, $n \in I$ and $b \in R$

47. Consider the graph of $h(x) = \max(x, x^2)$ at $x = 0$ and $x = 1$

For $(d) : h(x) = \max(x^2, -x^2)$

48. Here, $g(x) = [x^2]\{\cos^2 4x\} + \{x^2\}[\cos^2 4x] + x^2 \sin^2 4x$

$$+ [x^2][\cos^2 4x] + \{x^2\}\{\cos^2 4x\}$$

$$= [x^2](\{\cos^2 4x\} + [\cos^2 4x]) + \{x^2\}([\cos^2 4x] + \{\cos^2 4x\}) + x^2 \sin^2 4x$$

$$= ([x^2] + \{x^2\})(\{\cos^2 4x\} + [\cos^2 4x]) + x^2 \sin^2 4x$$

$$= x^2 \cos^2 4x + x^2 \sin^2 4x$$

$$= x^2$$

Clearly, $g(x)$ is always differentiable.

$\therefore$ Number of points of non-differentiability is 0.

49. Here, $f(x) = \dfrac{\{x\} g(x)}{\{x\} g(x)} = 1$, when $\{x\} g(x) \neq 0$

If $f(x)$ is periodic with period $\dfrac{1}{4}$, then $\{x\} g(x) \neq 0$ with period $\dfrac{1}{4}$.

$\Rightarrow \quad g(x) = 0$ at $x = \dfrac{k}{4}$, where $k \in I$.

50. Here, $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{f\left(x\left(1 + \dfrac{h}{x}\right)\right) - f\left(\dfrac{1}{1/x}\right)}{h}$$

$$= \lim_{h \to 0} \frac{f\left(\dfrac{1 + h/x}{1/x}\right) - f\left(\dfrac{1}{1/x}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{f(1+h/x)}{f(1/x)} - \dfrac{f(1)}{f(1/x)}}{h}$$

$$= \lim_{h\to 0} \frac{f(1+h/x)-f(1)}{h/x} \cdot \frac{1}{xf\left(\dfrac{1}{x}\right)} = f'(1)\cdot\frac{1}{x}\cdot\frac{f(x)}{f(1)}$$

$\therefore \quad \dfrac{dy}{dx} = k\cdot\dfrac{y}{x}$, where $\dfrac{f'(1)}{f(1)} = k$, $f(x) = y$.

$\Rightarrow \quad \int \dfrac{dy}{y} = \int k\dfrac{dx}{x}$

$\Rightarrow \quad \log y = k\log x + \log C$

$\Rightarrow \quad \log y = \log(x^k \cdot C)$

$\Rightarrow \quad y = C\cdot x^k$

$\therefore \quad f(x) = C\cdot x^k$

Put $x = 2, y = 1$ in $f\left(\dfrac{x}{y}\right) = \dfrac{f(x)}{f(y)}$

$\Rightarrow \quad f(2) = \dfrac{f(2)}{f(1)} \Rightarrow f(1) = 1$

Also, $f(x) = C\cdot x^k$, put $x = 1$

$\Rightarrow \quad f(1) = C \Rightarrow C = 1$

$\therefore \quad f(x) = x^k$, now $f(2) = 4$

$\Rightarrow \quad f(2) = 2^k$

$4 = 2^k \Rightarrow k = 2$

$\therefore \quad f(x) = x^2 \Rightarrow f(5) = 25$

51. $\tan(f(x)) = \tan\left(\dfrac{x}{2} - 1\right), \quad x \in [0, \pi]$

$$0 \le x \le \pi \Rightarrow -1 \le \frac{x}{2} - 1 \le \frac{\pi}{2} - 1$$

$\therefore \tan(f(x))$ is continuous in $[0, \pi]$.

$\dfrac{1}{f(x)} = \dfrac{2}{x-2}$ is not defined at $x = 2 \in [0, \pi]$.

$y = \dfrac{x-2}{2} \Rightarrow f^{-1}(x) = 2x + 2$ is continuous in R.

52. $\lim\limits_{x\to 0^+} (x+1)\, e^{-\frac{2}{x}} = \lim\limits_{x\to 0^+} \dfrac{(x+1)}{e^{2/x}} = 0$

$\lim\limits_{x\to 0^-} (x+1)\, e^0 = 1$

Hence, continuous for $x \in I - \{0\}$, assumes all intermediate values from $f(-2)$ to $f(2)$ and maximum value $\dfrac{3}{e}$ at $x = 2$.

53. RHL $= \lim\limits_{x\to 0^+} \left(3 - \left[\cot^{-1}\left(\dfrac{2x^3-3}{x^2}\right)\right]\right)$

$= 3 - [\cot^{-1}(-\infty)] = 3 - 3 = 0$

LHL $= \lim\limits_{x\to 0^-} \{x^2\} \cos\left(e^{-\frac{1}{x}}\right)$

$= \lim\limits_{h\to 0} h^2 \cos\left(e^{\frac{1}{h}}\right) = 0$

and $f(0) = 0$

Hence, $f(x)$ is continuous at $x = 0$.

54. $\lim\limits_{x\to -1^+} f(x) = \lim\limits_{x\to -1^+} (b([x]^2 + [x]) + 1)$

$= \lim\limits_{h\to 0} (b([-1+h]^2 + [-1+h]) + 1)$

$= \lim\limits_{h\to 0} (b((-1)^2 - 1) + 1) = 1$

$\Rightarrow \quad b \in R$

$\lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to -1^-} \sin(\pi(x+a))$

$= \lim\limits_{h\to 0} \sin(\pi((-1-h)+a))$

$\therefore \quad \sin\pi a = -1$

$\Rightarrow \quad \pi a = 2n\pi + \dfrac{3\pi}{2} \Rightarrow a = 2n + \dfrac{3}{2}$

55. $f(x) = \begin{cases} \dfrac{\sin\frac{\pi}{4}}{1} = \dfrac{1}{\sqrt{2}}, & 1 \le x < 2 \\ \dfrac{\sin\frac{\pi}{2}}{1} = \dfrac{1}{2}, & 2 \le x < 3 \end{cases}$

Hence, $f(x)$ is continuous at $\dfrac{3}{2}$, differentiable at $\dfrac{4}{3}$ and discontinuous at 2.

56. Since, $\sin^{-1} x$ and $\cos(1/x)$ are continuous and differentiable in $x \in [-1, 1] - \{0\}$.

Now, at $x = 0$

$$f'(0^-) = \lim_{h\to 0} \frac{f(-h) - f(0)}{-h} = 0$$

$$= \lim_{h\to 0} \frac{(\sin^{-1}(-h))^2 \cos\left(-\dfrac{1}{h}\right) - 0}{-h} = 0$$

$$f'(0^+) = \lim_{h\to 0} \frac{(\sin^{-1} h)^2 \cos\left(\dfrac{1}{h}\right) - 0}{h} = 0$$

Hence, LHD = RHD. So, $f(x)$ is continuous and differentiable everywhere in $-1 \le x \le 1$.

57. $H(x) = \begin{cases} \cos x, & 0 \le x < \dfrac{\pi}{2} \\ \dfrac{\pi}{2} - x, & \dfrac{\pi}{2} < x \le 3 \end{cases}$

$$H'\left(\frac{\pi}{2}^-\right) = -\sin x = -1$$

$$H'\left(\frac{\pi}{2}^+\right) = -1$$

Hence, $H(x)$ is continuous and derivable in $[0, 3]$ and has maximum value 1 in $[0, 3]$.

58. Here, $f(x) = 3(2x+3)^{2/3} + 2x + 3$

$\Rightarrow \quad f'(x) = \dfrac{4}{(2x+3)^{1/3}} + 2$

Now, $\quad 2x + 3 \neq 0 \;\Rightarrow\; x \neq -\dfrac{3}{2}$

Hence, $f(x)$ is continuous but not differential at $x = -3/2$.

Also, $f(x)$ is differentiable and continuous at $x = 0$.

59. $f'(0^+) = \lim\limits_{h\to 0} \dfrac{h\ln(\cos h)}{h\ln(1+h^2)} = \lim\limits_{h\to 0} \dfrac{\ln(\cos h)^{1/h^2}}{\dfrac{\ln(1+h^2)}{h^2}}$

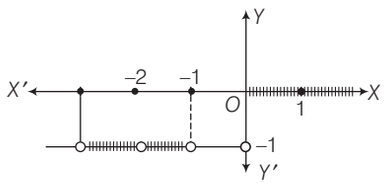
$= \lim\limits_{h\to 0} \dfrac{1}{h^2}(\cos h - 1) = -\dfrac{1}{2}$; similarly $f'(0^-) = -\dfrac{1}{2}$

Hence, f is continuous and derivable at $x = 0$.

60. $f(x) = \begin{cases} 0, & 0 < x < 1 \\ 0, & x = 0 \text{ or } 1 \text{ or } -1 \\ 0, & -1 < x < 0 \end{cases} \;\Rightarrow\; f(x) = 0$ for all in $[-1, 1]$

61. $[|x|] - |[x]| = \begin{cases} 0, & x = -1 \\ -1, & -1 < x < 0 \\ 0, & 0 \le x \le 1 \\ 0, & 1 < x \le 2 \end{cases}$

$\Rightarrow$ Range is $\{0, -1\}$. The graph is



62. $f'(0^+) = \dfrac{1}{\sqrt{2}}, f'(0^-) = -\dfrac{1}{\sqrt{2}}$;

$f(x) = \dfrac{\sqrt{x^2}}{\sqrt{1+\sqrt{1-x^2}}} = \dfrac{|x|}{\sqrt{1+\sqrt{1-x^2}}}$

63. Range is $R^+ \cup \{0\} \Rightarrow$ Option (b) is not correct.

f is not differentiable at $x = -1$

As, $\quad f(x) = \begin{cases} x^3 + 1, & \text{if } x \ge -1 \\ -(x^3+1), & \text{if } x < -1 \end{cases}$

$\Rightarrow \quad f'(x) = \begin{cases} 3x^2, & \text{if } x > -1 \\ -3x^2, & \text{if } x < -1 \end{cases}$

$f'(-1^+) = 3, f'(-1^-) = -3 \Rightarrow f$ is not differentiable at $x = -1$.

Also, f is not bijective, hence it has no inverse.

64. Given, f is continuous in $[a, b]$ ...(i)

$\Rightarrow$ g is continuous in $[b, c]$ ...(ii)

$\Rightarrow$ $f(b) = g(b)$ ...(iii)

$\Rightarrow$ $\left.\begin{matrix} h(x) = f(x) \text{ for } x \in [a, b) \\ = g(x) \text{ for } x \in (b, c] \end{matrix}\right\}$ ...(iv)

$\Rightarrow h(x)$ is continuous in $[a, b) \cup (b, c]$ [using Eqs. (i) and (ii)]

Also, $\quad f(b^-) = f(b),\ g(b^+) = g(b)$

[using Eqs. (i) and (ii)] ...(iv)

$\therefore \quad h(b^-) = f(b^-) = f(b) = g(b) = g(b^+) = h(b^+)$

[using Eqs. (iv) and (v)]

Now, verify each alternative. Of course! $g(b^-)$ and $f(b^+)$ are undefined.

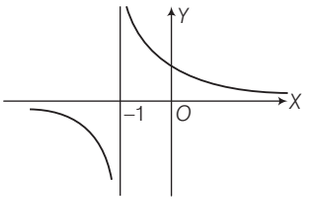
$h(b^-) = f(b^-) = f(b) = g(b) = g(b^+)$

and $\quad h(b^+) = g(b^+) = g(b) = f(b) = f(b^-)$

Hence, $\quad h(b^-) = h(b^+) = f(b) = g(b)$

and $h(b)$ is not defined.

65. (a) Domain is $R - \{-1\}$; Range = $R - \{0\}$



(b) Domain is $x \in R$; Range = $(0, 1]$

(c) Domain is $[0, \infty)$; Range = $(0, 1]$

(d) Domain is $(-\infty, 3)$; Range = $(0, \infty)$

66. $f(x) = \sec 2x + \operatorname{cosec} 2x = \dfrac{2(\sin 2x + \cos 2x)}{2\cos 2x \sin 2x}$

$= \dfrac{2(\sin 2x + \cos 2x)}{\sin 4x}$

is discontinuous, where $4x = n\pi$, $n \in I$ or $x = \dfrac{n\pi}{4}$.

Options (a) and (b) also satisfy the condition, since they are subsets of option (c).

67. $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} x^n \cdot \sin\left(\dfrac{1}{x^2}\right) = 0$, if $n > 0$

and hence true for $n > 1$.

Since, $f(0) = 0$, $f(x)$ is continuous at $x = 0$.

Now, $\lim\limits_{x\to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x\to 0} \dfrac{x^n \sin\dfrac{1}{x^2}}{x}$

$= \lim\limits_{x\to 0} x^{n-1} \sin\left(\dfrac{1}{x^2}\right) = 0$, if $n > 1$.

Hence, $f(x)$ is differentiable at $x = 0$, if $n > 1$.

68. $f(x) = \begin{cases} e^x, & x \le 0 \\ 1 - x, & 0 < x < 1 \\ x - 1, & x \ge 1 \end{cases}$

$\Rightarrow \lim\limits_{x\to 0^-} f(x) = 1, \lim\limits_{x\to 0^+} f(x) = 1$

and $\lim\limits_{x\to 1^-} f(x) = 0, \lim\limits_{x\to 1^+} f(x) = 0$

Hence, $f(x)$ is continuous at $x = 0$ and 1.

$f'(0^-) = 1$ and $f'(0^+) = -1$.

Hence, $f(x)$ is not differentiable at $x = 1$.

69. $f(x) = \int_{-2}^{x} |t+1|\, dt = -\int_{-2}^{-1}(t+1)\, dt + \int_{-1}^{x}(t+1)\, dt$

$$= \frac{1}{2} + \left(\frac{t^2}{2} + t\right)_{-1}^{x} = \frac{x^2}{2} + x + 1, \text{ for } x \geq -1$$

$f(x)$ is a quadratic polynomial.

$\therefore$ $f(x)$ is continuous as well as differentiable in $[-1, 1]$.

Also, $f'(x)$ is continuous as well as differentiable in $[-1, 1]$.

70. We have, $f(x+y) = f(x) + f(y) + xy(x+y)$

$$f(0) = 0$$

$\therefore$ $\lim_{h \to 0} \frac{f(h)}{h} = -1$

$\therefore$ $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{f(x) + f(h) + xh(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(h)}{h} + \lim_{h \to 0} x(x+h) = -1 + x^2$$

$\therefore$ $f'(x) = -1 + x^2$

$\therefore$ $f(x) = \frac{x^3}{3} - x + c$

$\therefore$ $f(x)$ is a polynomial function,

$f(x)$ is twice differentiable for all $x \in R$ and

$f'(3) = 3^2 - 1 = 8$.

71. We have, $f'(x) = 6x + 12$

For $f(x)$ is increasing, $f'(x) \geq 0 \Rightarrow x \geq -2$

Hence, $f(x)$ is increasing in $[-1, 2]$

$\lim_{x \to 2^+} f(x) = 35$, $\lim_{x \to 2^-} f(x) = 35$ and $f(2) = 35$

Hence, $f(x)$ is continuous on $[-1, 3]$, $f'(2^-) = 24$ and $f'(2^+) = -1$.

Hence, $f'(2)$ doesn't exist for maximum, $f(2) = 35$

$$f(-1) = -10, \qquad f(3) = 34$$

Hence, $f(x)$ has maximum value at $x = 2$.

72. Since, $f(x) = 0$, $x < 0$ and differentiable at $x = 0$, LHD $= 0$

(function is on X-axis for $x < 0$). If $f(x)$,

(a) x^2, $x > 0$

RHD at $x = 0$,

$f'(0) = 2 \times 0 = 0$ (possible)

(d) $-x^{3/2}$, $x > 0$

RHD at $x = 0$,

$f'(0) = -3/2\, x^{1/2} = -3/2 \times 0 = 0$ (possible)

73. We know that, $\sin x$ is periodic function in $[0, 2\pi]$.

$\therefore$ $\sin x$ is continuous at $x = 0$

Now, $\lim_{x \to 0} [x]$

RHL $= \lim_{x \to 0^+} [x] = \lim_{h \to 0} [x+h] = x = 0$ $\quad [\because [x+h] = x]$

LHL $= \lim_{x \to 0^-} [x] = \lim_{h \to 0} [x-h] = (x-1)$ $\quad [\because [x-h] = (x-1)]$

$= 0 - 1 = -1$

$\therefore$ RHL $\neq$ LHL

So, $[x]$ is discontinuous at $x = 0$.

We know that, sum of continuous and discontinuous functions is discontinuous.

74. We have,

$$f(x) = \begin{cases} 2\sin(a\cos^{-1} x), & \text{if } x \in (0, 1) \\ \sqrt{3}, & \text{if } x = 0 \\ ax + b, & \text{if } x < 0 \end{cases}$$

Continuity at $x = 0$

(LHL at $x = 0$) $= \lim_{x \to 0^-} ax + b$

$$= \lim_{h \to 0} a(0-h) + b = \lim_{h \to 0} -ah + b$$

$$= b$$

(RHL at $x = 0$) $= \lim_{x \to 0^+} 2\sin(a\cos^{-1} x)$

$$= \lim_{h \to 0} 2\sin(a\cos^{-1}(0+h))$$

$$= \lim_{h \to 0} 2\sin a\cos^{-1} h$$

$$= 2\sin a\cos^{-1} h = 2\sin a\cos^{-1} 0$$

$$= 2\sin\frac{a\pi}{2}$$

$$f(0) = \sqrt{3}$$

For $f(x)$ to be continuous at $x = 0$,

LHL $=$ RHL $= f(0)$

$\therefore$ $b = 2\sin\frac{a\pi}{a} = \sqrt{3}$

$\therefore$ $b = \sqrt{3}$ and $\sin\frac{a\pi}{2} = \frac{\sqrt{3}}{2}$

$\Rightarrow$ $b = \sqrt{3}$ and $\frac{a\pi}{2} = \frac{\pi}{3}$

$\Rightarrow$ $b = \sqrt{3}$ and $a = \frac{2}{3}$

So, Statement I is correct.

Since, for $x < 0$, $f(x) = ax + b$

which is a polynomial function and will be continuous for $(-\infty, 0)$.

Again, for $x \in (0, 1)$,

$f(x) = 2\sin(a\cos^{-1} x)$, which is trigonometric function will be continuous for $x \in (0, 1)$.

$\because$ $f(x)$ is continuous in $(-\infty, 1)$.

$\therefore$ Statement II is also correct.

75. We have, $f(x) = \begin{cases} \dfrac{\cos x - e^{-\frac{x^2}{2}}}{x^3}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Clearly, $f(0) = 0$

Now consider, $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^3}$ $\quad \left[\frac{0}{0} \text{ form}\right]$

$$= \lim_{x \to 0} \frac{-\sin x + e^{-\frac{x^2}{2}} \cdot x}{3x^2} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{-\cos x + e^{-\frac{x^2}{2}} - e^{-\frac{x^2}{2}} \cdot x^2}{6x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{\sin x - e^{-\frac{x^2}{2}} \cdot x - 2xe^{-\frac{x^2}{2}} + e^{-\frac{x^2}{2}}}{6}$$

$$= 0$$

Thus, $\lim_{x \to 0} f(x) = f(0)$

$\Rightarrow f$ is continuous at $x = 0$

Hence, option (c) is correct.

76. Let $f(x) = \frac{x^3}{4} - \sin \pi x + \frac{2}{3}$. Then, $f(x)$ will be continuous function. ($\because$ Sum and difference of two continuous function is continuous)

Here, $f(2) = \frac{8}{3}$ and $f(-2) = -\frac{4}{3}$ $\qquad [\because \sin n\pi = 0, \forall\ n \in Z]$

Now, as $f(-2) < 0 < f(2)$, therefore by intermediate value theorem we can say that there exists atleast one point $n \in [-2, 2]$. Such that $f(n) = 0$

Hence, $f(x) = 0$, i.e. $\frac{x^3}{4} - \sin \pi x + \frac{2}{3} = 0$ has atleast one solution in $[-2, 2]$.

Clearly, Statement II is wrong. Because for this to be true, $f(x)$ should be a continuous function (by intermediate value theorem).

Hence, option (c) is correct.

77. We have,

$$f(x) = x\left(\frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\right) + x^2 + x^4$$

$$\therefore f(-x) = (-x)\left(\frac{e^{2(-x)} - e^{-2(-x)}}{e^{2(-x)} + e^{-2(-x)}}\right) + (-x)^2 + (-x)^4$$

$$= -x\left(\frac{e^{-2x} - e^{+2x}}{e^{-2x} + e^{+2x}}\right) + x^2 + x^4$$

$$= -x\left[-\left(\frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\right)\right] + x^2 + x^4$$

$$= x\left(\frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\right) + x^2 + x^4$$

$$= f(x)$$

$\therefore f(x)$ is even function and even function can't has range equal to R.

78. $y = |\ln x|$ not differentiable at $x = 1$.

$y = |\cos|x||$ is not differentiable at $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

$y = \cos^{-1}(\text{sgn } x) = \cos^{-1}(1) = 0$ differentiable, $\forall\ x \in (0, 2\pi)$.

79. $f'(0^+) = \lim_{h \to 0} \frac{h \sin h - 0}{h} = 0$

$f'(0^-) = \lim_{h \to 0} \frac{h \sin(-h) - 0}{-h} = 0$

$f(x)$ is differentiable at $x = 0$.

e.g. $x|x|$ is derivable at $x = 0$.

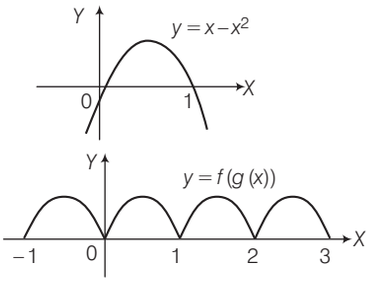
80. Consider $g(x) = x^3$ at $x = 0$, $g(0) = 0$

$|g(x)|$ is derivable at $x = 0$.

Actually nothing definite can be said. Also, for $g(x) = x - 1$ with $g(1) = 0$.

Then, $|g(x)|$ is not derivable at $x = 1$.

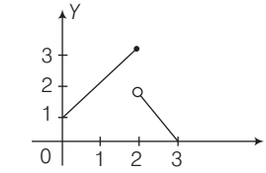
81.



82. $f(x)$ is discontinuous at $x = 0$ and $f(x) < 0, \forall\ x \in [-\alpha, 0)$ and $f(x) > 0, \forall\ x \in [0, \alpha]$.

Sol. (Q. Nos. 83 to 85)

$$f(x) = \begin{cases} x + 1, & 0 \le x \le 2 \\ -x + 3, & 2 < x < 3 \end{cases}$$



$\therefore f(x)$ is discontinuous at $x = 2$.

$$(fof)(x) = \begin{cases} x + 2, & 0 \le x \le 1 \\ -x + 2, & 1 < x \le 2 \\ -x + 4, & 2 < x < 3 \end{cases}$$

$\Rightarrow (fof)(x)$ is discontinuous at $x = 1, 2$

and $\qquad f(19) = f(3 \times 6 + 1) = f(1) = 2$

83. (a) ***84.*** (b, c) ***85.*** (a)

Sol. (Q. Nos. 86 to 89)

Since, $f(-x) = \frac{1}{f(x)}$

$\therefore$ At $x = 0$

$\Rightarrow f(0) = \frac{1}{f(0)} \Rightarrow f^2(0) = 1 \Rightarrow f(0) = +1$, as $f(x) > 0$.

$$\Rightarrow \qquad f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x) \cdot f(h) - f(x)}{h}$$

$$\therefore \qquad f'(x) = f(x) \cdot \left(\lim_{h \to 0} \frac{f(h) - 1}{h}\right) = f(x) \cdot f'(0)$$

$$\Rightarrow \qquad \int \frac{f'(x)}{f(x)} = \int -1\, dx \Rightarrow \log f(x) = -x + c$$

$$f(x) = e^{-x} \cdot \lambda \text{ at } x = 0, \quad \lambda = 1 \Rightarrow f(x) = e^{-x}$$

$\Rightarrow$ Range of $f(x) \in R^+$. $\Rightarrow$ Range of $f(|x|)$ is $[0, 1]$.

$\Rightarrow f(x)$ is decreasing function

and $\qquad f'(x) = -e^{-x} = -f(x)$.

86. (c) ***87.*** (a) ***88.*** (d) ***89.*** (a)

90. $f(x)=\begin{cases}\sin x, & x\le 0\\ \tan x, & 0<x<2\pi\\ \cos x, & 2\pi\le x<3\pi\\ 3\pi, & x=3\pi\end{cases}$

$f(x)$ is discontinuous at $\frac{\pi}{2}, \frac{3\pi}{2}, 2\pi$.

91. $f(x)=[2x^3-5]$

Since $1\le x<2 \Rightarrow 1\le x^3<8$

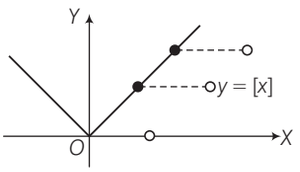
$-3\le 2x^3-5<11$

Now, $2x^3-5$ is discontinuous at integer points.

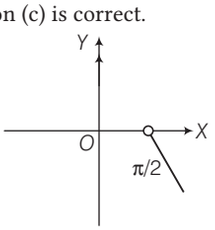
$\therefore$ $-2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$.

Hence, number of points of discontinuity = 13

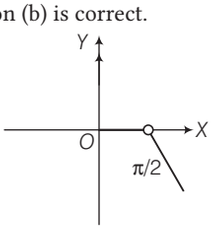
92. Max $([x], |x|)$, hence discontinuity at $x=\phi$.



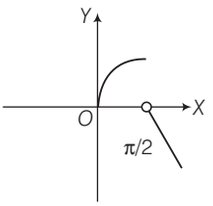
93. From figure, option (c) is correct.



94. From figure, option (b) is correct.



95. From figure, option (c) is correct.



Sol. (Q. Nos. 96 to 99)

Here, $f(x+y^{2n+1})=f(x)+(f(y))^{2n+1}$

On differentiating, we get

$f'(x+y^{2n+1})=f'(x)$

$\left[\because\ x \text{ and } y \text{ are independent, so } \frac{dy}{dx}=0\right]$

$\Rightarrow$ $f'(x)$ is constant, say $f'(x)=k$.

On integrating, we get $f(x)=kx+c$

Now, $f(0)=0 \Rightarrow c=0$ and $f(1)=1 \Rightarrow k=1$

$\therefore$ $f(x)=x \Rightarrow f'(x)=1$, for all $x\in R$

$\therefore$ $f(x)=x$ is an odd function.

96. (b) ***97.*** (c) ***98.*** (d) ***99.*** (a)

We know, $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

and $f'(x)=\lim_{h\to 0}\frac{f(x-h)-f(x)}{-h}$

On adding, we get

$$2f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{f(x-h)-f(x)}{-h} \quad ...(i)$$

$$\Rightarrow \quad 2f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x-h)}{h}$$

$$\Rightarrow \quad 2f'(x)=\lim_{h\to 0} f(x)\cdot\frac{\{f(h)-f(-h)\}}{h}$$

[using $f(x+y)-f(x-y)=f(x)\{f(y)-f(-y)\}$]...(ii)

Also, $2f'(0)=\lim_{h\to 0}\left(\frac{f(h)-f(0)}{h}+\frac{f(-h)-f(0)}{-h}\right)$

[using Eq. (i)]

$$=\lim_{h\to 0}\frac{f(h)-f(-h)}{h} \quad ...(iii)$$

From Eqs. (ii) and (iii), we get

$$2f'(x)=f(x)\cdot 2f'(0) \Rightarrow \frac{f'(x)}{f(x)}=f'(0)$$

$\therefore$ $$\frac{f'(x)}{f(x)}=2010 \quad ...(iv)$$

On integrating both sides, we get

$$\log(f(x))=2010x+c, \text{ as } f(0)=1$$

$\therefore$ $$c=0 \Rightarrow f(x)=e^{2010x}$$

Thus, $\{f(x)\}$ is non-periodic. ...(v)

100. (b) ***101.*** (d)

102. (A) $\lim_{h\to 0}\sin\{1-h\}=\cos 1+a$

$\Rightarrow \lim_{h\to 0}\sin(1-h)-\cos 1=a$

$\Rightarrow a=\sin 1-\cos 1$

Now, $|K|=\frac{\sin 1-\cos 1}{\sqrt{2}\left(\sin 1\cdot\frac{1}{\sqrt{2}}-\cos 1\cdot\frac{1}{\sqrt{2}}\right)}=1 \Rightarrow K=\pm 1$

(B) $f(0)=\lim_{x\to 0}\frac{2\sin^2\left(\frac{\sin x}{2}\right)}{x^2\cdot\left(\frac{\sin x}{2}\right)^2}\cdot\left(\frac{\sin x}{2}\right)^2=\frac{1}{2}$

(C) Function should have same rule for θ and $\theta' \Rightarrow x=1-x$

$\Rightarrow$ $$x=\frac{1}{2}$$

(D) $f(x)=x+\{-x\}+[x]$

x is continuous at $x\in R$.

Check at $x=I$ (where I is integer)

$f(I^+)=2I+1$ or $f(I^-)=2I-1$

So, $f(x)$ is discontinuous at every integer, i.e. $\{1, 0, -1\}$.

103. (A) $f(x)\begin{cases} 1-1, & 1<x\le 2 \\ 0, & x=1 \\ 1-x, & 0\le x<1 \\ -\sin\pi x, & -1\le x<0 \end{cases}$

At $x=0$, $f(x)$ is not continuous and not differentiable. At $x=1$, $f(x)$ is continuous and non-differentiable. At $x=2$ and -1, $f(x)$ is continuous and differentiable.

(B) $f(0^-)=\lim\limits_{h\to 0} h^2 e^{-\frac{1}{h}}=\lim\limits_{h\to 0}\dfrac{h^2}{e^{1/h}}=0$

(C) $f(x)=\dfrac{x}{x+1}$, not defined at $x=-1$.

$$g(x)=\frac{f(x)}{f(x)+2}$$

$g(x)$ is not defined at $f(x)=-2$

$$\Rightarrow \quad \frac{x}{x+1}=-2\Rightarrow x=-\frac{2}{3}$$

Also, $x=0$ is not in domain.

$\therefore$ $f(x)$ is not differentiable at 3 points.

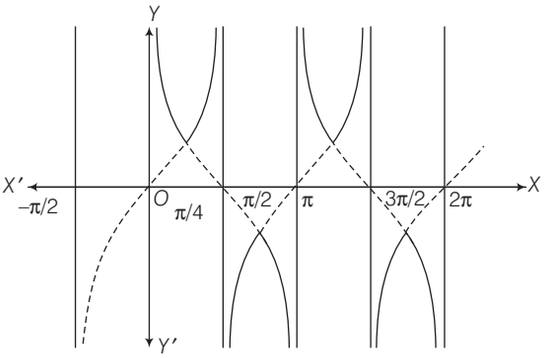
(D) $y=\operatorname{sgn}(x^2-1)=\begin{cases} 1, & x^2-1>0 \\ 0, & x^2-1=0 \\ -1, & x^2-1<0 \end{cases}=\begin{cases} 1, & |x|>1 \\ 0, & |x|=1 \\ -1, & |x|<1 \end{cases}$

$\therefore$ Tangent exists for all x.

$\therefore$ Number of points where tangent does not exists is 0.

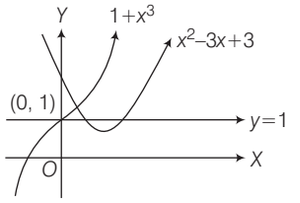
104. (A) $f(x)=\dfrac{\tan x+\cot x}{2}-\left|\dfrac{\tan x-\cot x}{2}\right|$

$$=\begin{cases}\cot x, & \tan x\ge\cot x \\ \tan x, & \cot x>\tan x\end{cases}$$



There are 4 points where the function is continuous but not differentiable in $(0, 2\pi)$.

(B) $f(x)=\min\{1, 1+x^3, x^2-3x+3\}$ can be shown as



$$\therefore \quad f(x)=\begin{cases} 1+x^3, & x\le 0 \\ 1, & 0\le x\le 1 \text{ or } x\ge 2 \\ x^2-3x+3, & 1\le x\le 2\end{cases}$$

Clearly, $f(x)$ is not differentiable at 2 points.

(C) $f(x)=(x+4)^{\frac{1}{3}}$

$$\Rightarrow f'(x)=\frac{1}{3(x+4)^{\frac{2}{3}}}$$

$\therefore$ Functions non-derivable at $x=-4$, i.e., at one point.

(D) $f(x)=\begin{cases} -\dfrac{\pi}{2}\cdot\log\left(\dfrac{2x}{\pi}\right)+\dfrac{\pi}{2}, & 0<x\le\dfrac{\pi}{2} \\ \pi-x, & \dfrac{\pi}{2}<x<\dfrac{3\pi}{2}\end{cases}$

$$f'(x)=\begin{cases}-\dfrac{\pi}{2x}, & 0<x<\dfrac{\pi}{2} \\ -1, & \dfrac{\pi}{2}<x<\dfrac{3\pi}{2}\end{cases}$$

$$f'\left(\frac{\pi^-}{2}\right)=f'\left(\frac{\pi^+}{2}\right)=-1$$

$\therefore$ $f(x)$ is differentiable for all $x\in\left(0,\dfrac{3\pi}{2}\right)$.

$\therefore$ Number of points of non-differentiable is 0.

105. (A) $Rf'(0)=\lim\limits_{h\to 0}\dfrac{\cos h-1}{h}=0$ and $Lf'(0)=\lim\limits_{h\to 0}\dfrac{-h+1-1}{-h}=-1$

Obviously,

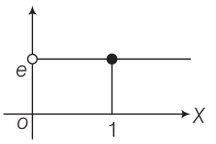
$$f(0)=f(0^-)=f(0^+)=1$$

Hence, continuous and not derivable.

(B) $g(x)=0$ for all x, hence continuous and derivable.

(C) As $0\le\{x\}<1$, hence $h(x)=\sqrt{\{x\}^2}=\{x\}$ which is discontinuous, hence non-derivable all $x\in I$.

(D) $\lim\limits_{x\to 1} x^{\frac{1}{\ln x}}=\lim\limits_{x\to 1} x^{\log_x e}=e=f(1)$



Hence, $k(x)$ is constant for all $x>0$, hence continuous and differentiable at $x=1$.

106. (A) $l=\lim\limits_{x\to\infty} e^{x^2+1}\left[e^{\sqrt{x^4+1}-(x^2+1)}-1\right]$

Consider, $\lim\limits_{x\to\infty}[\sqrt{x^4+1}-(x^2+1)]$

$$=\lim_{x\to\infty}\frac{x^4+1-(x^4+1+2x^2)}{\sqrt{x^4+1}+(x^2+1)}$$

$$=\lim_{x\to\infty}\frac{-2x^2}{x^2\left(\sqrt{1+(1/x^4)}+1+1/x^2\right)}=-1$$

Now, as $x \to \infty, \sqrt{x^4+1} - (x^2+1) \to -1$

$\infty \times \left(\frac{1}{e} - 1\right) \to -\infty$ and hence limit doesn't exist.

(B) $f(0^+) = \lim\limits_{h \to 0} \dfrac{a^{2h} - 2a^h + 1}{h^2} = \lim\limits_{h \to 0} \left(\dfrac{a^h - 1}{h}\right)^2 = \ln^2 a$

$f(0^-) = \lim\limits_{h \to 0} 3 \ln (a+h) - 2 = 3 \ln a - 2 = f(0)$

For continuous

$\ln^2 a = 3 \ln a - 2 \quad \ln^2 a - 3 \ln a + 2 = 0$

$\Rightarrow \quad (\ln a - 2)(\ln a - 1) = 0\ ;\ a = e^2$ or $a = e$

(C) $L = a^a \ln ae$

$\Rightarrow \quad M = a^a$

$\therefore \quad a^a \ln ae = 2a^a$

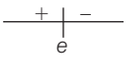
$\therefore \ \ln ae = 2 \Rightarrow ae = e^2 \Rightarrow a = e$

107. $\tan^2 x$ is discontinuous at $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

$\sec^2 x$ is discontinuous at $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

$\Rightarrow$ Number of discontinuities = 2

108. Let $g(x) = x^{1/x}, g'(x) = x^{1/x} \dfrac{1 - \ln x}{x^2}$



$\Rightarrow \quad g_{\max} = e^{1/e} \in (1, 2)$

$\Rightarrow \quad \lim\limits_{x \to 0} x^{1/x} = 0, \ \lim\limits_{x \to \infty} x^{1/x} = 1$

So, $\quad f(x) = \begin{cases} 0, & 0 < x < 1 \\ 1, & 1 \le x < \infty \end{cases}$

109. $f(x) = x + \cos x + 2, f(0) = 3 \Rightarrow g(3) = 0$

$g(f(x)) = x \Rightarrow g'(f(x)) \cdot f'(x) = 1$, putting $x = 0$

$g'(3) \cdot f'(0) = 1$

Now, $\quad f'(x) = 1 - \sin x \Rightarrow f'(0) = 1 \Rightarrow \quad g'(3) = 1$

110. Let $g(x) = x \tan^{-1}(x^2)$. It is an odd function.

So, $\quad g^{2m}(0) = 0.$

Let $\quad h(x) = x^4$

So, $\quad f(x) = g(x) + h(x)$

$\Rightarrow \quad f^{2m}(0) = g^{2m}(0) + h^{2m}(0)$

$= h^{2m}(0) \neq 0$

It happens when $\ 2m = 4 \Rightarrow m = 2$

111. $F(x) = 3e^x$ and $G(x) = e^{-x}$

The equation $9x^4 = (F(x))^2 G(x)$ becomes $x^4 = e^x$

Hence, number of solutions = 2

112. $y' = f'(x) - 2f'(2x)$

$y'(1) = f'(1) - 2f'(2) = 5 \quad$ ...(i)

and $\quad y'(2) = f'(2) - 2f'(4) = 7 \quad$ ...(ii)

Now, let $\quad y = f(x) - f(4x) - 10x$

$y' = f'(x) - 4f'(4x) - 10$

$y'(1) = f'(1) - 4f'(4) - 10 \quad$ ...(iii)

On substituting the value of $f'(2) = 7 + 2f'(4)$ in Eq. (i), we get

$f'(1) - 2[7 + 2f'(4)] = 5$

$f'(1) - 4f'(4) = 19$

$\Rightarrow \quad f'(1) - 4f'(4) - 10 = 9$

113. A, B, C are in AP.

$\therefore \quad 2B = A + C$ and $A + B + C = 180°$

$\therefore \quad B = 60°$

$\therefore \quad \cos B = \dfrac{1}{2} = \dfrac{a^2 + c^2 - b^2}{2ac}$

$\Rightarrow \quad a^2 + c^2 = b^2 + ac$

$\Rightarrow \quad (a - c)^2 = b^2 - ac$

or $\quad |\sin A - \sin C| = \sqrt{\sin^2 B - \sin A \sin C}$

$\Rightarrow \ 2\cos\left(\dfrac{A+C}{2}\right)\left|\sin\left(\dfrac{A-C}{2}\right)\right| = \sqrt{\dfrac{3}{4} - \sin A \sin C}$

$\Rightarrow \quad 2\left|\sin\left(\dfrac{A-C}{2}\right)\right| = \sqrt{3 - 4\sin A \sin C}$

$\therefore \ \lim\limits_{A \to C} \dfrac{\sqrt{3 - 4 \sin A \sin C}}{|A - C|} = \lim\limits_{A \to C} \dfrac{2\left|\sin\left(\dfrac{A-C}{2}\right)\right|}{|A - C|}$

$= \lim\limits_{A \to C} \left| \dfrac{\sin\left(\dfrac{A-C}{2}\right)}{\left(\dfrac{A-C}{2}\right)} \right|$

$= |1| = 1 \Rightarrow f(x) = 1$

$\therefore \quad f'(x) = 0$

114. $\text{RHD} = \lim\limits_{h \to 0} h \dfrac{\left(\left(\frac{3}{4}\right)^{1/h} - \left(\frac{3}{4}\right)^{-1/h}\right) - 0}{\left(\left(\frac{3}{4}\right)^{1/h} + \left(\frac{3}{4}\right)^{-1/h}\right) \cdot h} = -1$

$\text{LHD} = \lim\limits_{h \to 0} (-h) \dfrac{\left(\left(\frac{3}{4}\right)^{-1/h} - \left(\frac{3}{4}\right)^{1/h}\right) - 0}{\left(\left(\frac{3}{4}\right)^{-1/h} + \left(\frac{3}{4}\right)^{1/h}\right) \cdot (-h)} = 1$

$\therefore \quad P = f'(0^-) - f'(0^+) = 1 - (-1) = 2$

Now, $4 \cdot \left(\lim\limits_{x \to 2^-} \dfrac{(\exp((x+2)\log 4))^{\frac{[x+1]}{4}} - 16}{4^x - 16} \right)$

$= 4 \cdot \left[\lim\limits_{x \to 2^-} \dfrac{(4^{x+2})^{\frac{[x+1]}{4}} - 16}{4^x - 16} \right] = 4 \times \dfrac{1}{2} = 2$

115. Here, $f(x) = -x^3 + x^2 - x + 1 \Rightarrow f'(x) = -3x^2 + 2x - 1$

$\Rightarrow \quad D = 4 - 12 = -8 < 0$

$\therefore \quad f(x)$ is decreasing.

Thus, min $f(t) = f(x)$, as $f(x)$ is decreasing and $0 \le t \le x$.

$$\therefore \quad g(x)=\begin{cases}-x^3+x^2-x+1, & 0<x\le 1\\ x-1, & 1<x\le 2\end{cases}$$

$$\Rightarrow \quad \lim_{x\to 1^+} g(g(x))=\lim_{x\to 1^+} g(0^+)=1$$

$$\Rightarrow \quad \lim_{x\to 1^-} g(g(x))=\lim_{x\to 1^-} g(0^+)=1$$

$$\therefore \quad \lim_{x\to 1} g(g(x))=1$$

116. We have, $f(x)=(x-|x|)^2(1-x+|x|)^2$

$$\Rightarrow \quad f(x)=\begin{cases}(2x)^2(1-2x)^2, & x<0\\ 0, & x\ge 0\end{cases}$$

$$=\begin{cases}16x^4-16x^3+4x^2, & x<0\\ 0, & x\ge 0\end{cases}$$

Clearly, $f(x)$ is continuous as well as derivable for all $x\in R$.

$\therefore$ Number of points of non-differentiable = 0.

117. $$\text{RHL}=\lim_{h\to 0}\frac{\left(\frac{\pi}{2}-\sin^{-1}(1-h^2)\right)\sin^{-1}(1-h)}{\sqrt{2}(h-h^3)}$$

$$=\lim_{h\to 0}\frac{\cos^{-1}(1-h^2)\sin^{-1}(1-h)}{\sqrt{2}h(1-h^2)}$$

$$=\lim_{h\to 0}\frac{\sin^{-1}\sqrt{2h^2-h^4}\cdot\sin^{-1}(1-h)}{\sqrt{2}h(1+h)(1-h)}$$

$$=\lim_{h\to 0}\frac{\sin^{-1}(h\sqrt{2-h^2})\sin^{-1}(1-h)}{\sqrt{2}h(1+h)(1-h)}$$

$$=\lim_{h\to 0}\frac{1}{\sqrt{2}}\cdot\frac{\sin^{-1}(h\sqrt{2-h^2})}{h\cdot\sqrt{2-h^2}}\cdot\frac{\sqrt{2-h^2}}{1+h}\cdot\frac{\sin^{-1}(1-h)}{(1-h)}$$

$$=\frac{1}{\sqrt{2}}\cdot\sqrt{2}\cdot\frac{\sin^{-1}(1)}{1}=\frac{\pi}{2}\Rightarrow k=\frac{\pi}{2} \qquad \ldots(i)$$

$$\text{LHL}=\lim_{h\to 0}\frac{A\sin^{-1}(1-(1-h))\cdot\cos^{-1}(1-(1-h))}{\sqrt{2(1-h)}\cdot(1-(1-h))}$$

$$=\lim_{h\to 0}\frac{A\cdot\sin^{-1}(h)\cdot\cos^{-1}(h)}{\sqrt{2(1-h)}\cdot h}=\frac{A\cdot\pi/2}{\sqrt{2}}=\frac{A\pi}{2\sqrt{2}}$$

$$\Rightarrow \quad \frac{A\pi}{2\sqrt{2}}=\frac{\pi}{2}\Rightarrow A=\sqrt{2}$$

Hence, $\sin^2 k+\cos^2\left(\frac{A\pi}{\sqrt{2}}\right)=\sin^2\left(\frac{\pi}{2}\right)+\cos^2(\pi)=1+1=2$.

118. $f(x)=[[x]]-[x-1]$

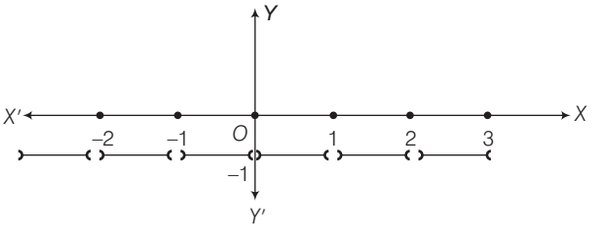
or $f(x)=[x]-[x]+1$

or $f(x)=1$, which is constant function and which is continuous for all real numbers.

119. (i) $f(x)=[x]+[-x]=\begin{cases}x-x, & x\in \text{integers}\\ [x]-1-[x], & x\notin \text{integers}\end{cases}$

$$\therefore\ f(x)=\begin{cases}0, & x\in \text{integers}\\ -1, & x\notin \text{integers}\end{cases}$$

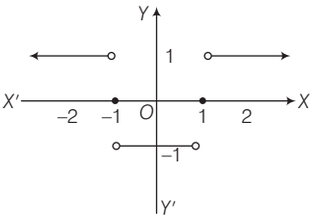
which shows the graph of $f(x)$ as shown in figure.



Thus, $f(x)$ is discontinuous at $x\in$ integers.

(ii) $g(x)=\lim_{n\to\infty}\frac{x^{2n}-1}{x^{2n}+1}=\begin{cases}-1, & |x|<1\\ 0, & |x|=1\\ 1, & |x|>1\end{cases}$

which can be shown as in the figure.



Thus, $g(x)$ is discontinuous at $x=\pm 1$.

120. $f(x)=\begin{cases}x[x], & 0\le x<2\\ (x-1)[x], & 2\le x\le 3\end{cases}$

To check continuity at $x=1$

RHL (at $x=1$) $=\lim_{h\to 0}(1+h)[1+h]=1$

LHL (at $x=1$) $=\lim_{h\to 0}(1-h)[1-h]=0$

Hence, the function is discontinuous at $x=1$.

To check continuity at $x=2$

RHL (at $x=2$) $=\lim_{h\to 0}(2+h-1)[2+h]=2$

LHL (at $x=2$) $=\lim_{h\to 0}(2-h)[2-h]=2\Rightarrow f(2)=(2-1)[2]=2$

Hence, the function is continuous at $x=2$.

To check differentiability at $x=2$

$$\text{RHD (at } x=2)=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

$$=\lim_{h\to 0}\frac{(2+h-1)[2+h]-2}{h}$$

$$=\lim_{h\to 0}\frac{(1+h)2-2}{h}=2$$

$$\text{LHD (at } x=2)=\lim_{h\to 0}\frac{(2-h)[2-h]-2}{-h}$$

$$\Rightarrow \lim_{h\to 0}\frac{2-h-2}{-h}=1$$

which shows $f(x)$ is not differentiable at $x=2$.

Also, $f(x)$ is not differentiable at $x=1$, as $f(x)$ at $x=1$ is not continuous.

121. Differentiating both the sides, we get

$$h'(x) = 2f(x)\cdot f'(x) + 2g(x)\cdot g'(x)$$

$$h'(x) = 2f(x)\cdot g(x) + 2g(x)\cdot f''(x)$$

[as $g(x) = f'(x)$, $g'(x) = f''(x)$]

$$h'(x) = 2f(x)\cdot g(x) - 2g(x)\cdot f(x) \qquad [\because f''(x) = -f(x)]$$

$$h'(x) = 0$$

$\therefore$ $h(x)$ must be constant function.

Given, $h(5) = 11$

Hence, $h(10) = 11$

122. Given that,

$$f(x+y) = f(x)\cdot f(y) \text{ for all } x \in R \qquad \text{...(i)}$$

On putting $x = y = 0$ in Eq. (i), we get

$$f(0)\{f(0) - 1\} = 0$$

$\Rightarrow$ $f(0) = 0$ or $f(0) = 1$

If $f(0) = 0$, then

$$f(x) = f(x+0) = f(x)\cdot f(0)$$

$\Rightarrow$ $f(x) = 0$ for all $x \in R$

which is not true, so $f(0) = 1$

$$\therefore \quad f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x)\cdot f(h) - f(x)}{h} \qquad \text{[from Eq. (i)]}$$

$$= \lim_{h\to 0} f(x)\left\{\frac{f(h)-1}{h}\right\}$$

$$= f(x)\cdot f'(0)$$

$$= 2f(x) \qquad \text{[given } f'(0) = 2\text{]}$$

or $\dfrac{f'(x)}{f(x)} = 2$

On integrating both the sides, we get $\log_e f(x) = 2x + C$

At $x = 0$, $f(0) = 1$

Hence, $\log f(1) = 2(0) + C$

$\Rightarrow$ $\log 1 = 0 + C$

$\Rightarrow$ $C = 0$

$\Rightarrow$ $\log_e f(x) = 2x + 0$

$\therefore$ $f(x) = e^{2x}$

123. Since, $f(x)$ is differentiable at $x = 0$.

$$\Rightarrow \quad \lim_{h\to 0}\frac{f(0+h) - f(0)}{h} = p \text{ (say)} \qquad \text{...(i)}$$

Then, $f'(x) = \lim_{h\to 0}\dfrac{f(x+h) - f(x)}{h}$

or $f'(x) = \lim_{h\to 0}\dfrac{f\left\{\frac{3x+3h}{3}\right\} - f\left(\frac{3x+3\cdot 0}{3}\right)}{h}$

or $f'(x) = \lim_{h\to 0}\dfrac{f(3x) + f(3h) + f(0) - f(3x) - f(0) - f(0)}{3h}$

or $f'(x) = \lim_{h\to 0}\dfrac{f(3h) - f(0)}{3h}$

$$\left[\text{using } f\left(\frac{x+y}{3}\right) = \frac{f(x) + f(y) + f(0)}{3}\right]$$

or $f'(x) = f'(0)$ or $f'(x) = p$ (let) [from Eq. (i)]

$\therefore$ $f(x) = px + q$

which shows $f(x)$ is differentiable for all x in R.

124. We have,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x) + f(h) + 2hx - 1 - f(x)}{h}$$

[using given definition]

$$= \lim_{h\to 0}\left\{2x + \frac{f(h) - 1}{h}\right\}$$

Now, substituting $x = y = 0$ in the given functional relation, we get

$$f(0) = f(0) + f(0) + 0 - 1$$

$\Rightarrow$ $f(0) = 1$

$$\therefore \quad f'(x) = 2x + \lim_{h\to 0}\frac{f(h) - f(0)}{h}$$

$$= 2x + f'(0)$$

$\Rightarrow$ $f'(x) = 2x + \cos\alpha$

On integrating, $f(x) = x^2 + x\cos\alpha + C$

Here, at $x = 0$, $f(0) = 1$

$\therefore$ $1 = C \Rightarrow f(x) = x^2 + x\cos\alpha + 1$

It is a quadratic in x with discriminant,

$$D = \cos^2\alpha - 4 < 0$$

and coefficient of $x^2 = 1 > 0$

$\therefore$ $f(x) > 0, \ \forall\ x \in R$

125. **Plan** If a continuous function has values of opposite sign inside an interval, then it has a root in that interval.

$$f, g : [0, 1] \to R$$

We take two cases.

***Case* I** Let f and g attain their common maximum value at p.

$\Rightarrow$ $f(p) = g(p)$,

where $p \in [0,1]$

***Case* II** Let f and g attain their common maximum value at different points.

$\Rightarrow$ $f(a) = M$ and $g(b) = M$

$\Rightarrow$ $f(a) - g(a) > 0$ and $f(b) - g(b) < 0$

$\Rightarrow f(c) - g(c) = 0$ for some $c \in [0,1]$ as f and g are continuous functions.

$\Rightarrow f(c) - g(c) = 0$ for some $c \in [0,1]$ for all cases. ...(i)

Option (a) $\Rightarrow f^2(c) - g^2(c) + 3[f(c) - g(c)] = 0$

which is true from Eq. (i).

Option (d) $\Rightarrow f^2(c) - g^2(c) = 0$, which is true from Eq. (i).

Now, if we take $f(x) = 1$ and $g(x) = 1$, $\forall x \in [0,1]$

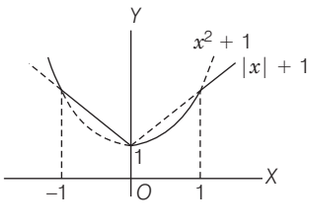
Options (b) and (c) does not hold.

Hence, options (a) and (d) are correct.

126. **Plan**

(i) In these type of questions, we draw the graph of the function.

(ii) The points at which the curve taken a sharp turn, are the points of non-differentiability. Curve of $f(x)$ and $g(x)$ are



$h(x)$ is not differentiable at $x = \pm 1$ and 0.
As, $h(x)$ take sharp turns at $x = \pm 1$ and 0.

Hence, number of points of non-differentiability of $h(x)$ is 3.

127. **Plan** To check differentiability at a point we use RHD and LHD at a point and if RHD =LHD, then $f(x)$ is differentiable at the point.

Description of Situation

As, $$R\{f'(x)\}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

and $$L\{f'(x)\}=\lim_{h\to 0}\frac{f(x-h)-f(x)}{-h}$$

To check differentiable at $x=0$,

$$R\{f'(0)\}=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$=\lim_{h\to 0}\frac{h^2\left|\cos\frac{\pi}{h}\right|-0}{h}=\lim_{h\to 0}h\cdot\left|\cos\frac{\pi}{h}\right|=0$$

$$L\{f'(0)\}=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\frac{h^2\left|\cos\left(-\frac{\pi}{h}\right)\right|-0}{-h}=0$$

So, $f(x)$ is differentiable at $x=0$.

To check differentiability at $x=2$,

$$R\{f'(2)\}=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

$$=\lim_{h\to 0}\frac{(2+h)^2\left|\cos\left(\frac{\pi}{2+h}\right)\right|-0}{h}=\lim_{h\to 0}\frac{(2+h)^2\cdot\cos\left(\frac{\pi}{2+h}\right)}{h}$$

$$=\lim_{h\to 0}\frac{(2+h)^2\cdot\sin\left(\frac{\pi}{2}-\frac{\pi}{2+h}\right)}{h}$$

$$=\lim_{h\to 0}\frac{(2+h)^2\cdot\sin\left(\frac{\pi h}{2(2+h)}\right)}{h\cdot\frac{\pi}{2(2+h)}}\cdot\frac{\pi}{2(2+h)}=\pi$$

and $$L\{f'(2)\}=\lim_{h\to 0}\frac{f(2-h)-f(2)}{-h}$$

$$=\lim_{h\to 0}\frac{(2-h)^2\cdot\left|\cos\frac{\pi}{2-h}\right|-2^2\cdot\left|\cos\frac{\pi}{2}\right|}{-h}$$

$$=\lim_{h\to 0}\frac{(2-h)^2\left(-\cos\frac{\pi}{2-h}\right)-0}{-h}$$

$$=\lim_{h\to 0}\frac{-(2-h)^2\cdot\sin\left(\frac{\pi}{2}-\frac{\pi}{2-h}\right)}{h}$$

$$=\lim_{h\to 0}\frac{(2-h)^2\cdot\sin\left(-\frac{\pi h}{2(2-h)}\right)}{h\times\frac{-\pi}{2(2-h)}}\times\frac{-\pi}{2(2-h)}=-\pi$$

Thus, $f(x)$ is differentiable at $x=0$ but not differentiable at $x=2$.

128. $f(2n)=a_n$, $f(2n^+)=a_n$

$$f(2n^-)=b_n+1 \Rightarrow a_n-b_n=1$$

$$f(2n+1)=a_n \Rightarrow f\{(2n+1)^-\}=a_n$$

$$f\{(2n+1)^+\}=b_{n+1}-1$$

$\Rightarrow$ $a_n=b_{n+1}-1$ or $a_n-b_{n+1}=-1$

or $a_{n-1}-b_n=-1$

129. $f(x+y)=f(x)+f(y)$, as $f(x)$ is differentiable at $x=0$.

$\Rightarrow$ $f'(0)=k$ ...(i)

Now, $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(x)+f(h)-f(x)}{h}=\lim_{h\to 0}\frac{f(h)}{h}\quad\left[\frac{0}{0}\text{form}\right]$$

$$\left[\begin{array}{l}\text{given, } f(x+y)=f(x)+f(y),\ \forall\ x, y\\ \therefore\quad f(0)=f(0)+f(0),\\ \text{when}\quad x=y=0\ \Rightarrow\ f(0)=0\end{array}\right]$$

Using L'Hospital's rule,

$$\lim_{h\to 0}\frac{f'(h)}{1}=f'(0)=k \quad ...(ii)$$

$\Rightarrow$ $f'(x)=k$, integrating both sides, we get

$f(x)=kx+C$, as $f(0)=0 \Rightarrow C=0$

$\therefore$ $f(x)=kx$

$\therefore f(x)$ is continuous for all $x\in R$ and $f'(x)=k$, i.e. constant for all $x\in R$. Hence, (b) and (c) are correct.

130. We have, $$f(x)=\begin{cases}-x-\frac{\pi}{2}, & x\le-\frac{\pi}{2}\\ -\cos x, & -\frac{\pi}{2}<x\le 0\\ x-1, & 0<x\le 1\\ \log x, & x>1\end{cases}$$

Continuity at $x=-\frac{\pi}{2}$, $f\left(-\frac{\pi}{2}\right)=-\left(-\frac{\pi}{2}\right)-\frac{\pi}{2}=0$

$$\text{RHL}=\lim_{h\to 0}-\cos\left(-\frac{\pi}{2}+h\right)=0$$

$\therefore$ Continuous at $x=\left(-\frac{\pi}{2}\right)$.

Continuity at $x=0 \Rightarrow f(0)=-1$

$$\text{RHL}=\lim_{h\to 0}(0+h)-1=-1$$

$\therefore$ Continuous at $x=0$. Continuity at $x=1$, $f(1)=0$

$$\text{RHL}=\lim_{h\to 0}\log(1+h)=0$$

$\therefore$ Continuous at $x=1$

$$f'(x)=\begin{cases}-1, & x\le-\frac{\pi}{2}\\ \sin x, & -\frac{\pi}{2}<x\le 0\\ 1, & 0<x\le 1\\ \frac{1}{x}, & x>1\end{cases}$$

Differentiable at $x = 0$, LHD = 0, RHD = 1
$\therefore$ Not differentiable at $x = 0$
Differentiable at $x = 1$, LHD = 1, RHD = 1
$\therefore$ Differentiable at $x = 1$.
Also, for $x = -\frac{3}{2} \Rightarrow f(x) = -\cos x$
$\therefore$ Differentiable at $x = -\frac{3}{2}$

131. Given, $f(x) = x\cos\frac{1}{x}$, $x \geq 1$

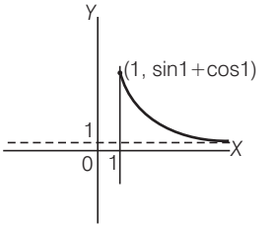
$\Rightarrow \quad f'(x) = \frac{1}{x}\sin\frac{1}{x} + \cos\frac{1}{x}$

$\Rightarrow \quad f''(x) = -\frac{1}{x^3}\cos\left(\frac{1}{x}\right)$

Now $\lim\limits_{x\to\infty} f'(x) = 0 + 1 = 1 \Rightarrow$ Option (b) is correct.

Now, $x \in [1, \infty) \Rightarrow \frac{1}{x} \in (0, 1] \Rightarrow f''(x) < 0$

Option (d) is correct.
As, $f'(1) = \sin 1 + \cos 1 > 1$
$f'(x)$ is strictly decreasing and $\lim\limits_{x\to\infty} f'(x) = 1$
So, graph of $f'(x)$ is shown as below



Now, in $[x, x+2]$, $x \in [1, \infty)$, $f(x)$ is continuous and differentiable so by LMVT, $f'(x) = \frac{f(x+2) - f(x)}{2}$
As, $f'(x) > 1$
For all $x \in [1, \infty)$

$\Rightarrow \quad \frac{f(x+2) - f(x)}{2} > 1 \quad \Rightarrow \quad f(x+2) - f(x) > 2$

For all $x \in [1, \infty)$.

132. Given, $g(x) = \frac{(x-1)^n}{\log\cos^m(x-1)}$; $0 < x < 2$, $m \neq 0$, n are integers
and $|x-1| = \begin{cases} x-1, & x \geq 1 \\ 1-x, & x < 1 \end{cases}$

The left hand derivative of $|x-1|$ at $x = 1$ is $p = -1$.
Also, $\lim\limits_{x\to 1^+} g(x) = p = -1$

$\Rightarrow \quad \lim\limits_{h\to 0} \frac{(1+h-1)^n}{\log\cos^m(1+h-1)} = -1$

$\Rightarrow \quad \lim\limits_{h\to 0} \frac{h^n}{\log\cos^m h} = -1 \Rightarrow \lim\limits_{h\to 0} \frac{h^n}{m\log\cos h} = -1$

$\Rightarrow \quad \lim\limits_{h\to 0} \frac{n\cdot h^{n-1}}{m\frac{1}{\cos h}(-\sin h)} = -1$ [using L'Hospital's rule]

$\Rightarrow \quad \lim\limits_{h\to 0}\left(-\frac{n}{m}\right)\cdot\frac{h^{n-2}}{\left(\frac{\tan h}{h}\right)} = -1 \Rightarrow \left(\frac{n}{m}\right)\lim\limits_{h\to 0}\frac{h^{n-2}}{\left(\frac{\tan h}{h}\right)} = 1$

$\Rightarrow \quad n = 2$ and $\frac{n}{m} = 1$

$\therefore \quad m = n = 2$

133. We have, $\lim\limits_{x\to 0}\frac{g(x)\cos x - g(0)}{\sin x}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim\limits_{x\to 0}\frac{g'(x)\cos x - g(x)\sin x}{\cos x} = 0$

Statement I
Since, $f(x) = g(x)\sin x$
$f'(x) = g'(x)\sin x + g(x)\cos x$
and $f''(x) = g''(x)\sin x + 2g'(x)\cos x - g(x)\sin x$
$\Rightarrow \quad f''(0) = 0$
Thus, $\lim\limits_{x\to 0}[g(x)\cos x - g(0)]\,\text{cosec}\, x = 0 = f''(0)$
$\Rightarrow$ Statement I is true.
Statement II $f'(x) = g'(x)\sin x + g(x)\cos x \Rightarrow f'(0) = g(0)$
Statement II is not a correct explanation of Statement I.

134. Match the Columns

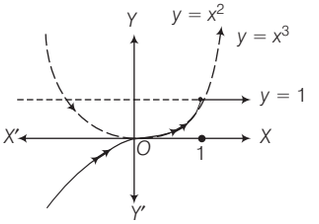
A. $x|x|$ is continuous, differentiable and strictly increasing in $(-1, 1)$.

B. $\sqrt{|x|}$ is continuous in $(-1, 1)$ and not differentiable at $x = 0$.

C. $x + [x]$ is strictly increasing in $(-1, 1)$ and discontinuous at $x = 0$
$\Rightarrow$ not differentiable at $x = 0$.

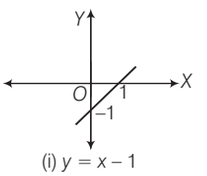
D. $|x-1| + |x+1| = 2$ in $(-1, 1)$
$\Rightarrow$ The function is continuous and differentiable in $(-1, 1)$.

135. Here, $f(x) = \min\{1, x^2, x^3\}$ which could be graphically shown as

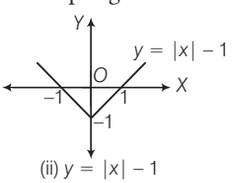


$\Rightarrow f(x)$ is continuous for $x \in R$ and not differentiable at $x = 1$ due to sharp edge. Hence, (a) and (d) are correct answers.

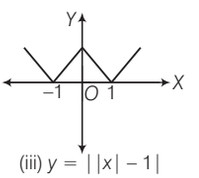
136. Using graphical transformation. As, we know that, the function is not differentiable at sharp edges.



(i) $y = x - 1$



(ii) $y = |x| - 1$



(iii) $y = ||x| - 1|$

In function, $y = ||x| - 1|$, we have 3 sharp edges at $x = -1, 0, 1$.
Hence, $f(x)$ is not differentiable at $\{0, \pm 1\}$.

137. Given, $f(1) = f\left(\frac{1}{2}\right) = f\left(\frac{1}{3}\right) = \ldots = \lim_{n \to \infty} f\left(\frac{1}{n}\right) = 0$

as $f\left(\frac{1}{n}\right) = 0$; $n \in$ integers and $n \geq 1$.

$\Rightarrow \lim_{n \to \infty} f\left(\frac{1}{n}\right) = 0$

$\Rightarrow f(0) = 0$

Since, there are infinitely many points in neighbourhood of $x = 0$.

$\therefore f(x) = 0$

$\Rightarrow f'(x) = 0 \Rightarrow f'(0) = 0$

Hence, $f(0) = f'(0) = 0$

138. Given, $f(x) = \begin{cases} \frac{1}{2}(-x-1), & \text{if } x < -1 \\ \tan^{-1} x, & \text{if } -1 \leq x \leq 1 \\ \frac{1}{2}(x-1), & \text{if } x > 1 \end{cases}$

$f(x)$ is discontinuous at $x = -1$ and $x = 1$.

$\therefore$ Domain of $f'(x) \in R - \{-1, 1\}$

139. Given, $f(x) = [x] \sin \pi x$

If x is just less than k, $[x] = k - 1$

$\Rightarrow f(x) = (k-1) \sin \pi x$.

LHD of $f(x) = \lim_{x \to k} \frac{(k-1)\sin \pi x - k \sin \pi k}{x - k}$

$= \lim_{x \to k} \frac{(k-1)\sin \pi x}{x-k}$,

$= \lim_{h \to 0} \frac{(k-1)\sin \pi (k-h)}{-h}$ [where $x = k - h$]

$= \lim_{h \to 0} \frac{(k-1)(-1)^{k-1} \cdot \sin h\pi}{-h} = (-1)^k (k-1)\pi$

140. RHD of $\sin(|x|) - |x| = \lim_{h \to 0} \frac{\sin h - h}{h} = 1 - 1 = 0 [\because f(0) = 0]$

LHD of $\sin(|x|) - |x|$

$= \lim_{h \to 0} \frac{\sin|-h| - |-h|}{-h}$

$= \lim_{h \to 0} \frac{\sin h - h}{-h} = 0$

Therefore, (d) is the answer.

141. We have, $f(x) = |\log 2 - \sin x|$ and $g(x) = f(f(x)),\ x \in R$

Note that, for $x \to 0$, $\log 2 > \sin x$

$\therefore f(x) = \log 2 - \sin x$

$\Rightarrow g(x) = \log 2 - \sin(f(x))$

$= \log 2 - \sin(\log 2 - \sin x)$

Clearly, $g(x)$ is differentiable at $x = 0$ as $\sin x$ is diferentiable.

Now, $g'(x) = -\cos(\log 2 - \sin x)(-\cos x)$

$= \cos x \cdot \cos(\log 2 - \sin x)$

$\Rightarrow g'(0) = 1 \cdot \cos(\log 2)$

142. Since, $g(x)$ is differentiable $\Rightarrow g(x)$ must be continuous.

$\therefore g(x) = \begin{cases} k\sqrt{x+1}, & 0 \leq x \leq 3 \\ mx + 2, & 3 < x \leq 5 \end{cases}$

At $x = 3$, RHL $= 3m + 2$

and at $x = 3$, LHL $= 2k$

$\therefore 2k = 3m + 2$ ...(i)

Also, $g'(x) = \begin{cases} \frac{k}{2\sqrt{x+1}}, & 0 \leq x < 3 \\ m, & 3 < x \leq 5 \end{cases}$

$\therefore L\{g'(3)\} = \frac{k}{4}$ and $R\{g'(3)\} = m$

$\Rightarrow \frac{k}{4} = m$, i.e. $k = 4m$ ...(ii)

On solving Eqs. (i) and (ii), we get

$k = \frac{8}{5}, m = \frac{2}{5}$

$\Rightarrow k + m = 2$

143. Given, $f(0) = 2 = g(1)$, $g(0) = 0$ and $f(1) = 6$

f and g are differentiable in (0, 1).

Let $h(x) = f(x) - 2g(x)$ ...(i)

$h(0) = f(0) - 2g(0) = 2 - 0 = 2$

and $h(1) = f(1) - 2g(1) = 6 - 2(2) = 2$

$\Rightarrow h(0) = h(1) = 2$

Hence, using Rolle's theorem,

$h'(c) = 0$, such that $c \in (0, 1)$

Differentiating Eq. (i) at c, we get

$\Rightarrow f'(c) - 2g'(c) = 0$

$\Rightarrow f'(c) = 2g'(c)$

144. **Given** A function $f : R \to R$ defined by

$f(x) = [x] \cos \pi\left(x - \frac{1}{2}\right)$, where [] denotes the greatest integer function.

To discuss The continuity of function f.

Now, $\cos x$ is continuous, $\forall x \in R$.

$\Rightarrow \cos \pi\left(x - \frac{1}{2}\right)$ is also continuous, $\forall x \in R$.

Hence, the continuity of f depends upon the continuity of $[x]$.

Since, $[x]$ is discontinuous, $\forall x \in I$.

So, we should check the conitnuity of f at $x = n, \forall n \in I$.

LHL at $x = n$ is given by

$f(n^-) = \lim_{x \to n^-} f(x)$

$= \lim_{x \to n^-} [x] \cos \pi\left(x - \frac{1}{2}\right) = (n-1) \cos \frac{(2n-1)\pi}{2} = 0$

RHL at $x = n$ is given by

$f(n^+) = \lim_{x \to n^+} f(x) = \lim_{x \to n^+} [x] \cos \pi\left(x - \frac{1}{2}\right)$

$= (n) \cos \frac{(2n-1)\pi}{2} = 0$

Also, value of the function at $x = n$ is

$f(n) = [n] \cos \pi\left(n - \frac{1}{2}\right) = (n) \cos \frac{(2n-1)\pi}{2} = 0$

$\therefore f(n^+) = f(n^-) = f(n)$

Hence, f is continuous at $x = n, \forall n \in I$.

CHAPTER

07

dy/dx as a Rate Measurer and Tangents, Normals

Learning Part

Session 1
- Derivative as the Rate of Change
- Velocity and Acceleration in Rectilinear Motion

Session 2
- Differential and Approximation
- Geometrical Meaning of Δx, Δy, dx and dy

Session 3
- Slope of Tangent and Normal
- Equation of Tangent
- Equation of Normal

Session 4
- Angle of Intersection of Two Curves
- Length of Tangent, Subtangent, Normal and Subnormal

Session 5
- Rolle's Theorem
- Lagrange's Mean Value Theorem

Session 6
- Application of Cubic Functions

Practice Part
- JEE Type Examples
- Chapter Exercises

Arihant on Your Mobile !
Exercises with this symbol can be practised on your mobile. See title inside to activate for free.

Session 1

Derivative as the Rate of Change, Velocity and Acceleration in Rectilinear Motion

Derivative as the Rate of Change

If a variable quantity y is some function of time t, i.e. $y = f(t)$, then small change in time Δt have a corresponding change Δy in y.

Thus, the average rate of change $= \dfrac{\Delta y}{\Delta t}$

When limit $\Delta t \to 0$ is applied, the rate of change becomes instantaneous and we get the rate of change of y with respect to the instant t.

i.e. $$\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \frac{dy}{dt}$$

Hence, it is clear that **the rate of change of any variable with respect to some other variable is the derivative of first variable with respect to other variable.**

The differential coefficient of y with respect to x i.e. $\dfrac{dy}{dx}$ is nothing but the rate of increase of y relative to x.

Example 1 ***If the radius of a circle is increasing at a uniform rate of 2 cm/s, then find the rate of increase of area of circle, at the instant when the radius is 20 cm.***

Sol. Given, $\dfrac{dr}{dt} = 2$ cm/s [where, r is radius and t is time]

Now, area of circle is given by $A = \pi r^2$

Differentiating it with respect to time t, we get

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} \Rightarrow \frac{dA}{dt} = 2\pi \cdot 20 \cdot 2 \text{ cm}^2/\text{s}$$

$$\Rightarrow \frac{dA}{dt} = 80\,\pi \text{ cm}^2/\text{s}$$

Thus, the rate of change of area of circle with respect to time is 80π cm^2/s.

Example 2 On the curve $x^3 = 12y$, find the interval at which the abscissa changes at a faster rate than the ordinate.

Sol. Given, $x^3 = 12y$

Differentiating it with respect to y, we get

$$3x^2 \frac{dx}{dy} = 12 \Rightarrow \frac{dx}{dy} = \frac{12}{3x^2}$$

In the interval, at which the abscissa changes at a faster rate than the ordinate, we must have

$$\left|\frac{dx}{dy}\right| > 1 \quad \text{or} \quad \left|\frac{12}{3x^2}\right| > 1$$

or $$\frac{4}{x^2} > 1 \Rightarrow \frac{4 - x^2}{x^2} > 0, \text{ when } x \neq 0$$

$\Rightarrow$ $$4 - x^2 > 0\,;\, x \neq 0 \Rightarrow x^2 - 4 < 0;\, x \neq 0$$

$\Rightarrow$ $$(x-2)(x+2) < 0\,;\, x \neq 0$$

Using number line rule, we have $-2 < x < 2; x \neq 0$

Thus, $x \in (-2, 2) - \{0\}$ is the required interval at which abscissa changes at a faster rate than the ordinate.

Velocity and Acceleration in Rectilinear Motion

The velocity of a moving particle is defined as the rate of change of its displacement with respect to time and the acceleration is defined as the rate of change of its velocity with respect to time.

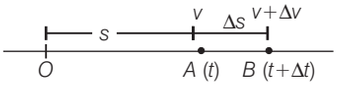


Figure 7.1

Let a particle A move rectilinearly as shown in figure. Let s be the displacement from a fixed point O along the path at time t ; s is considered to be positive on right of the point O and negative on the left of it.

Also, Δs is positive when s increases, i.e. when the particle moves towards right.

Thus, if Δs be the increment in s in time Δt, then average velocity in this interval is

$$\text{Average velocity} = \frac{\Delta s}{\Delta t}$$

and the instantaneous velocity, i.e. velocity at time t is

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

If the velocity varies, then there is change of velocity Δv in time Δt. Hence, the acceleration at time t is

$$\text{Acceleration } (a) = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$

Example 3 If the displacement of a particle is given by $s = \left(\frac{1}{2}t^2 + 4\sqrt{t}\right)$ m. Find the velocity and acceleration at $t = 4$ seconds.

Sol. We have, the displacement of particle is given by,

$$s = \left(\frac{1}{2}t^2 + 4\sqrt{t}\right) \text{ m}$$

As, we know velocity $v = \frac{ds}{dt} = \left(t + \frac{2}{\sqrt{t}}\right)$ m/s ...(i)

and acceleration $a = \frac{d^2s}{dt^2} = \frac{dv}{dt} = \left(1 - \frac{1}{t^{3/2}}\right)$ m/s ...(ii)

Now, velocity when $t = 4$, is $v = \left(4 + \frac{2}{\sqrt{4}}\right)$ m/s [from Eq. (i)]

$\Rightarrow$ $v = 5$ m/s

and acceleration when $t = 4$, is $a = \left(1 - \frac{1}{4^{3/2}}\right)$ m/s^2 [from Eq. (i)]

$\Rightarrow$ $a = \left(1 - \frac{1}{8}\right)$ m/s^2 $\Rightarrow$ $a = \frac{7}{8}$ m/s^2

Example 4 ***If*** $s = \frac{1}{2}t^3 - 6t$, ***then find the acceleration at time when the velocity tends to zero.***

Sol. We have the displacement s is given by, $s = \frac{1}{2}t^3 - 6t$

Now, velocity $v = \frac{ds}{dt} = \left(\frac{3t^2}{2} - 6\right)$ ...(i)

and acceleration, $a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = (3t)$ ...(ii)

To find acceleration, when velocity $\to 0$

Now, when velocit y $\to 0$, then $\frac{3t^2}{2} - 6 = 0$ [from Eq. (i)]

$\Rightarrow$ $t^2 = 4 \Rightarrow t = 2$

Thus, acceleration when velocity tends to zero, is

$a = 3t = 6$ [from Eq. (ii)]

Example 5 ***If*** r ***is the radius,*** S ***the surface area and*** V ***the volume of a spherical bubble, prove that***

(i) $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$ (ii) $\frac{dV}{dS} \propto r$.

Sol. (i) Since, $V = \frac{4}{3}\pi r^3$

$\therefore$ $\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$ ...(i)

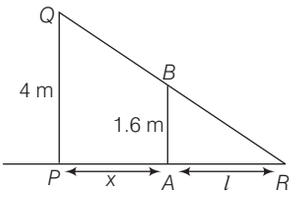
(ii) We know, $S = 4\pi r^2$ $\Rightarrow$ $\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$ ...(ii)

Thus, $\frac{dV}{dS} = \frac{dV/dt}{dS/dt} = \frac{1}{2} \cdot r$ [from Eqs. (i) and (ii)]

$\Rightarrow$ $\frac{dV}{dS} = \frac{1}{2}r$ or $\frac{dV}{dS} \propto r$

Example 6 A man who is 1.6 m tall walks away from a lamp which is 4 m above ground at the rate of 30 m/min. How fast is the man's shadow lengthening?

Sol. Let $PQ = 4$ m be the height of lamp and $AB = 1.6$ m be height of man. Let the end of shadow is R and it is at a distance of l from A when the man is at a distance of x from PQ, at some instance.



Since, ΔPQR and ΔABR are similar, we have

$$\frac{PQ}{AB} = \frac{PR}{AR} \Rightarrow \frac{4}{1.6} = \frac{x + l}{l}$$

$\Rightarrow$ $2x = 3l$ $\Rightarrow$ $2\frac{dx}{dt} = 3\frac{dl}{dt}$ $\left[\text{given} \frac{dx}{dt} = 30 \text{ m/min}\right]$

$\therefore$ $\frac{dl}{dt} = \frac{2}{3} \cdot 30$ m/min $= 20$ m/min (lengthening)

Example 7 If x and y are the sides of two squares such that $y = x - x^2$, find the rate of change of the area of the second square with respect to the first square.

Sol. Given, x and y are sides of two squares, thus the area of two squares are x^2 and y^2.

We have to obtain $\frac{d(y^2)}{d(x^2)} = \frac{2y\frac{dy}{dx}}{2x} = \frac{y}{x} \cdot \frac{dy}{dx}$ ...(i)

The given curve is $y = x - x^2$

$\Rightarrow$ $\frac{dy}{dx} = 1 - 2x$ ...(ii)

Thus, $\frac{d(y^2)}{d(x^2)} = \frac{y}{x}(1 - 2x)$ [from Eqs. (i) and (ii)]

or $\frac{d(y^2)}{d(x^2)} = \frac{(x - x^2)(1 - 2x)}{x}$

$\Rightarrow$ $\frac{d(y^2)}{d(x^2)} = 2x^2 - 3x + 1$

Exercise for Session 1

1. The surface area of a spherical bubble is increasing at the rate of 2 cm^2/s. Find the rate at which the volume of the bubble is increasing at the instant, if its radius is 6 cm.

2. A particle moves along the curve $y = \frac{2}{3}x^3 + 1$. Find the point on the curve at which y-coordinate is changing twice as fast as x-coordinate.

3. The area of an expanding rectangle is increasing at the rate of 48 cm^2/s. The length of rectangle is always equal to square of the breadth. At which rate the length is increasing at the instant when the breadth is 4 cm?

4. An edge of a variable cube is increasing at the rate of 10 cm/s. How fast the volume of the cube is increasing when the edge is 5 cm long?

5. An air-force plane is ascending vertically at the rate of 100 km/h. If the radius of the earth is r km, how fast is the area of the earth, visible from the plane, increasing at 3 min after it started ascending?

   Note Visible area $A = \frac{2\pi r^2 h}{r + h}$, where h is the height of the plane above the earth.

6. Sand is pouring from the pipe at the rate of 12 cm^3/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?

7. Water is dripping out from a conical funnel, at the uniform rate of 2 cm^3/s, through a tiny hole at the vertex of the bottom. When the slant height of the water is 4 cm, find the rate of decrease of the slant height of the water, if the vertical angle of funnel is 120°.

8. From a cylindrical drum containing oil and kept vertical, the oil leaking at the rate of 10 cm^3/s. If the radius of the drum is 10 cm and height is 50 cm, then find the rate at which level of oil is changing when oil level is 20 cm.

9. A kite is 120 m high and 130 m of string is out. If the kite is moving away horizontally at the rate of 52 m/s, find the rate at which the string is being paid out.

10. A ladder 13 m long leans against a wall. The foot of the ladder is pulled along the ground away from the wall, at the rate of 1.5 m/s. How fast is the angle θ between the ladder and the ground changing when the foot of the ladder is 12 m away from the wall?

11. Water is running into a conical vessel, 15 cm deep and 5 cm in radius, at the rate of 0.1 cm^3/s. When the water is 6 cm deep, find at what rate is
    (a) the water level rising?
    (b) the water surface area increasing?
    (c) the wetted surface of the vessel increasing?

12. Height of a tank in the form of an inverted cone is 10 m and radius of its circular base is 2 m. The tank contains water and it is leaking through a hole at its vertex at the rate of 0.02 m^3/s. Find the rate at which the water level changes and the rate at which the radius of water surface changes when height of water level is 5 m.

13. At what points of the ellipse $16x^2 + 9y^2 = 400$ does the ordinate decreases at the same rate at which the abscissa increases?

14. A satellite travels in a circular orbit of radius R. If its x-coordinate decreases at the rate of 2 units/s at the point (a, b), how fast is the y-coordinate changing?

15. The ends of a rod AB which is 5 m long moves along two grooves OX, OY which are at right angles. If A moves at a constant speed of $\frac{1}{2}$ m/s, what is the speed of B, when it is 4 m from O?

Session 2

Differential and Approximation, Geometrical Meaning of Δx, Δy, dx and dy

Differential and Approximation

We are familiar with the symbol dy/dx representing derivative of y w.r.t. x.

Now, we shall give meaning to the symbols dx and dy such that the symbol dy/dx represents the **quotient of dy and dx**.

Let $y = f(x)$ be a function of x.

Let Δx denotes small change in x and let the corresponding change in y be Δy.

Then, $$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} \Rightarrow \frac{\Delta y}{\Delta x} = \frac{dy}{dx} \text{ (approx)}$$

$$\Rightarrow \frac{\Delta y}{\Delta x} = \frac{dy}{dx} + \varepsilon, \text{ where } \varepsilon \to 0 \text{ as } \Delta x \to 0$$

$$\Rightarrow \Delta y = \frac{dy}{dx} \cdot \Delta x + \varepsilon \cdot \Delta x \quad \ldots(i)$$

Now, Δy consists of two parts. The part $\varepsilon \cdot \Delta x$ is very small and hence negligible. The part $\frac{dy}{dx} \cdot \Delta x$ is called **principal part of Δy** and is denoted by dy and is called **differential of y.**

Thus, $$dy = \frac{dy}{dx} \cdot \Delta x \quad \ldots(ii)$$

$$\Rightarrow dx = \frac{dy}{dy} \cdot \Delta x$$

$$= (1)\Delta x = \Delta x \quad \ldots(iii)$$

From Eqs. (ii) and (iii), we get

$$dy = \frac{dy}{dx} \cdot dx$$

$\Rightarrow$ (Differential of y) = (Differential coefficient of y w.r.t. x) $\cdot$ (Differential of x)

$$\Rightarrow \frac{\text{Differential of } y}{\text{Differential of } x} = \text{(Differential coefficient of } y \text{ w.r.t. } x)$$

From Eq. (i), $$\Delta y = \frac{dy}{dx} \cdot \Delta x \text{ (approx)}$$

$$\Rightarrow \Delta y = dy \text{ (approx)}$$

Hence, approximately

$$dy = \Delta y = (y + \Delta y) - y = f(x + \Delta x) - f(x)$$

$$\Rightarrow \frac{dy}{dx} \cdot dx = f(x + \Delta x) - f(x) \text{ (approx)}$$

$$\boxed{f(x + \Delta x) - f(x) = f'(x) \cdot \Delta x \text{ (approx)}}$$

Errors

Definition of Absolute Error

Δx or dx is called absolute error in x.

Definition of Relative Error

$\frac{\Delta x}{x}$ or $\frac{dx}{x}$ is called relative error in x.

Definition of Percentage Error

$\left(\frac{\Delta x}{x}\right) \cdot 100$ or $\left(\frac{dx}{x}\right) \cdot 100$ is called percentage error in x.

Geometrical Meaning of Δx, Δy, dx and dy

Let us take a point $A(x, y)$ on the curve $y = f(x)$, where $f(x)$ is differentiable real function.

Let $B(x + \Delta x, y + \Delta y)$ be a neighbouring point on the curve, where Δx denotes a small change in x and Δy is the corresponding change in y.

From the figure, it is clear that if Δx and Δy are sufficiently small quantities, then

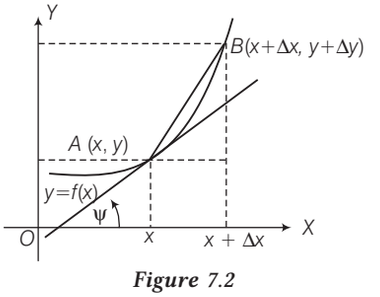


Figure 7.2

$$\frac{\Delta y}{\Delta x} = \tan \psi \cong \frac{dy}{dx} = f'(x)$$

Hence, approximate change in the value of y, called its differential, is given by $\Delta y = f'(x) \cdot \Delta x$

Example 8 Use differential to approximate $\sqrt{101}$.

Sol. Let $f(x)=\sqrt{x}$, then $f'(x)=\dfrac{1}{2\sqrt{x}}$.

Now, let $x=100$ and $\Delta x=1$, so that $x+\Delta x=101$

Then, by definition $f(x+\Delta x)-f(x)=f'(x)\cdot\Delta x$, we get

$$\sqrt{101}-\sqrt{100}=f'(\sqrt{100})\cdot 1$$

$$\sqrt{101}=\sqrt{100}+\frac{1}{2\sqrt{100}}\cdot 1$$

$$=10+\frac{1}{20}=10.05$$

Example 9 ***Use differential to approximate*** $(66)^{1/3}$.

Sol. Let $f(x)=(x)^{1/3}$

Then, $f'(x)=\dfrac{1}{3x^{2/3}}$

Now, let $x=64$, $\Delta x=2$, so that $x+\Delta x=66$

Then, by definition $f(x+\Delta x)-f(x)=f'(x)\cdot\Delta x$

$$\Rightarrow\quad f(66)-f(64)=f'(64)\cdot(2)$$

$$\Rightarrow\quad (66)^{1/3}-(64)^{1/3}=\frac{1}{3(64)^{2/3}}\times 2$$

$$\Rightarrow\quad (66)^{1/3}=4+\frac{2}{48}=\frac{97}{24}=4.0416$$

Example 10 If the radius of a circle increases from 5 cm to 5.1 cm, find the increase in area.

Sol. Let r be the radius of the circle. Then, $r=5$ cm and $r+dr=5.1$ cm, so that $dr=0.1$ cm.

Let A denotes the areas of the circle.

Then, $A=\pi r^2\ \therefore\ \dfrac{dA}{dr}=2\pi r$

$\therefore$ At $r=5$, $\dfrac{dA}{dr}=10\pi$

Now, $dA=\dfrac{dA}{dr}\cdot dr=10\pi\times(0.1)=\pi\ \text{cm}^2$

$\therefore$ Increase in area $=\pi\ \text{cm}^2$

Example 11 Find the approximate value of $\tan^{-1}(0.999)$ using differential.

Sol. Let $f(x)=\tan^{-1}x$

Then, $f'(x)=\dfrac{1}{1+x^2}$

Now, let $x=1$ and $\Delta x=-0.001$. So that $x+\Delta x=0.999$

Then, by definition,

$f(x+\Delta x)-f(x)=f'(x)\cdot\Delta x$, we get

$$f(0.999)-f(1)=f'(1)\cdot(-0.001)$$

$$\Rightarrow\quad \tan^{-1}(0.999)=\tan^{-1}(1)-(0.001)\cdot\frac{1}{1+1^2}$$

$$=\frac{\pi}{4}-0.0005=\frac{1}{4}\times\frac{22}{7}-0.0005=0.7852$$

Example 12 ***The time*** T ***of oscillation of a simple pendulum of length*** l ***is given by*** $T=2\pi\cdot\sqrt{\dfrac{l}{g}}$.

Find percentage error in T ***corresponding to***

(i) an increase of 2% in the value of l.

(ii) decrease of 2% in the value of l.

Sol. (i) We have, $T=2\pi\cdot\sqrt{\dfrac{l}{g}}$

$$\Rightarrow\quad \log T=\log 2+\log\pi+\frac{1}{2}\log l-\frac{1}{2}\log g$$

Differentiating w.r.t. l, we get

$$\frac{1}{T}\frac{dT}{dl}=0+0+\frac{1}{2l}-0\ \Rightarrow\ \frac{dT}{T}=\frac{1}{2}\frac{dl}{l}$$

$$\therefore\quad \frac{dT}{T}\times 100=\frac{1}{2}\left(\frac{dl}{l}\times 100\right)=\frac{1}{2}(2)\%=1\%$$

$$\left[\because \frac{dl}{l}\times 100=2\%\right]$$

Percentage error in $T=1\%$.

(ii) We have, $\dfrac{dT}{T}=\dfrac{1}{2}\dfrac{dl}{l}$

$$\Rightarrow\ \frac{dT}{T}\times 100=\frac{1}{2}\left(\frac{dl}{l}\times 100\right)=\frac{1}{2}(-2)\%=-1\%$$

[$\because$ decrease in $l=2\%$]

Percentage error in $T=-1\%$.

Example 13 ***In an acute*** ΔABC, ***if sides*** a ***and*** b ***are constants and the base angles*** A ***and*** B ***vary, show that*** $\dfrac{dA}{\sqrt{a^2-b^2\sin^2A}}=\dfrac{dB}{\sqrt{b^2-a^2\sin^2B}}$.

Sol. $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}$ or $b\sin A=a\sin B$

$$\Rightarrow\quad b\cos A\,dA=a\cos B\,dB$$

$$\Rightarrow\quad \frac{dA}{a\cos B}=\frac{dB}{b\cos A}$$

$$\Rightarrow\quad \frac{dA}{a\sqrt{1-\sin^2B}}=\frac{dB}{b\sqrt{1-\sin^2A}}$$

$$\Rightarrow\quad \frac{dA}{a\sqrt{1-\dfrac{b^2\sin^2A}{a^2}}}=\frac{dB}{b\sqrt{1-\dfrac{a^2\sin^2B}{b^2}}}$$

$$\Rightarrow\quad \frac{dA}{\sqrt{a^2-b^2\sin^2A}}=\frac{dB}{\sqrt{b^2-a^2\sin^2B}}$$

Exercise for Session 2

1. Use differential to approximate $\sqrt{51}$.
2. Use differential to approximate $\log(9.01)$. (Given, $\log 3 = 1.0986$)
3. If the error committed in measuring the radius of a circle is 0.01%, find the corresponding error in calculating the area.
4. The pressure p and the volume V of a gas are connected by the relation $pV^{1/4} = a$ (constant). Find the percentage increase in pressure corresponding to a decrease of $\left(\frac{1}{2}\right)\%$ in volume.
5. If in a ΔABC, the side c and the angle C remain constant, while the remaining elements are changed slightly. Using differential, show that $\frac{da}{\cos A} + \frac{db}{\cos B} = 0$.
6. If a ΔABC, inscribed in a fixed circle, is slightly varied in such a way as to have its vertices always on the circle, show that $\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0$.

Session 3

Slope of Tangent and Normal, Equation of Tangent, Equation of Normal

Slope of Tangent and Normal

Slope of Tangent

Let $y = f(x)$ be a continuous curve and let $P(x_1, y_1)$ be a point on it.

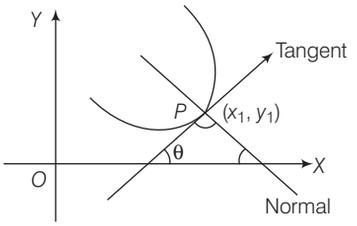


Figure 7.3

Then, $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$ **is the slope of tangent to the curve $y = f(x)$ at a point $P(x_1, y_1)$.**

$\Rightarrow \quad \left(\frac{dy}{dx}\right)_P = \tan\theta =$ Slope of tangent at P

where, θ is the angle which the tangent at $P(x_1, y_1)$ forms with the positive direction of X-axis as shown in the figure.

Remarks

(i) **Horizontal tangent** If tangent is parallel to X-axis, then

$$\theta = 0° \Rightarrow \tan\theta = 0$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$$

(ii) **Vertical tangent** If tangent is perpendicular to X-axis or parallel to Y-axis, then

$$\theta = 90° \Rightarrow \tan\theta = \infty \quad \text{or} \quad \cot\theta = 0$$

$$\therefore \quad \left(\frac{dx}{dy}\right)_{(x_1, y_1)} = 0$$

Slope of Normal

We know that the normal to the curve at $P(x_1, y_1)$ is a line perpendicular to tangent at $P(x_1, y_1)$ and passes through P.

$\therefore$ **Slope of the normal at**

$$P = -\frac{1}{\text{Slope of the tangent at } P}$$

$$\Rightarrow \text{ Slope of normal at } P(x_1, y_1) = -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}$$

$$\text{or} \quad \text{Slope of normal at } P(x_1, y_1) = -\left(\frac{dx}{dy}\right)_{(x_1, y_1)}$$

Remarks

(i) **Horizontal normal** If normal is parallel to X-axis, then

$$-\left(\frac{dx}{dy}\right)_{(x_1, y_1)} = 0 \text{ or } \left(\frac{dx}{dy}\right)_{(x_1, y_1)} = 0$$

(ii) **Vertical normal** If normal is perpendicular to X-axis or parallel to Y-axis, then $-\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$

Example 14 ***Find the slopes of the tangent and normal to the curve $x^3 + 3xy + y^3 = 2$ at (1, 1).***

Sol. Given equation of curve is $x^3 + 3xy + y^3 = 2$.

Differentiating it w.r.t. x, we get

$$3x^2 + 3x\frac{dy}{dx} + 3y + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{(3x^2 + 3y)}{(3x + 3y^2)} \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{(x^2 + y)}{(x + y^2)}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(1,1)} = -\left(\frac{2}{2}\right) = -1$$

$$\therefore \quad \text{Slope of tangent at } (1,1) = \left(\frac{dy}{dx}\right)_{(1,1)} = -1$$

$$\text{and slope of normal at } (1,1) = -\frac{1}{\left(\frac{dy}{dx}\right)_{(1,1)}} = \frac{-1}{-1} = 1$$

Example 15 ***Find the point on the curve $y = x^3 - 3x$ at which tangent is parallel to X-axis.***

Sol. Let the point at which tangent is parallel to X-axis be $P(x_1, y_1)$.

Then, it must lie on curve.

Therefore, we have $y_1 = x_1^3 - 3x_1$ ...(i)

Differentiating $y = x^3 - 3x$ w.r.t. x, we get

$$\frac{dy}{dx} = 3x^2 - 3 \quad \Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 3x_1^2 - 3$$

Since, the tangent is parallel to X-axis.

$$\therefore \quad \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} = 0 \;\Rightarrow\; 3x_1^2 - 3 = 0$$

$$\Rightarrow \quad x_1 = \pm 1 \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

When $x_1 = 1$, then $y_1 = 1 - 3 = -2$

When $x_1 = -1$, then $y_1 = -1 + 3 = 2$

$\therefore$ Points at which tangent is parallel to X-axis are $(1, -2)$ and $(-1, 2)$.

Example 16 Find the point on the curve $y = x^3 - 2x^2 - x$ at which the tangent line is parallel to the line $y = 3x - 2$.

Sol. Let $P(x_1, y_1)$ be the required point.

Then, we have $\quad y_1 = x_1^3 - 2x_1^2 - x_1 \qquad \text{...(i)}$

Differentiating the curve $y = x^3 - 2x^2 - x$ w.r.t. x, we get

$$\frac{dy}{dx} = 3x^2 - 4x - 1 \;\Rightarrow\; \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} = 3x_1^2 - 4x_1 - 1$$

Since, tangent at (x_1, y_1) is parallel to the line $y = 3x - 2$.

$\therefore$ Slope of the tangent at $P(x_1, y_1)$ = Slope of the line

$$y = 3x - 2 \;\Rightarrow\; \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} = 3$$

$$\Rightarrow \quad 3x_1^2 - 4x_1 - 1 = 3 \;\Rightarrow\; 3x_1^2 - 4x_1 - 4 = 0$$

$$\Rightarrow \quad (x_1 - 2)(3x_1 + 2) = 0 \;\Rightarrow\; x_1 = 2, -2/3 \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

When $x_1 = 2$, then

$$y_1 = 8 - 8 - 2 \;\Rightarrow\; y_1 = -2$$

When $\quad x_1 = -2/3$, then $y_1 = x_1^3 - 2x_1^2 - x_1$

$$\Rightarrow \quad y_1 = \frac{-8}{27} - \frac{8}{9} + \frac{2}{3} \;\Rightarrow\; y_1 = \frac{-14}{27}$$

Thus, the point at which tangent is parallel to $y = 3x - 2$ are $(2, -2)$ and $\left(-\frac{2}{3}, -\frac{14}{27}\right)$.

Example 17 *In which of the following cases, the function $f(x)$ has a vertical tangent at $x = 0$?*

(i) $f(x) = x^{1/3}$ (ii) $f(x) = \text{sgn } x$

(iii) $f(x) = x^{2/3}$ (iv) $f(x) = \sqrt{|x|}$

(v) $f(x) = \begin{cases} 0, \text{ if } x < 0 \\ 1, \text{ if } x \geq 0 \end{cases}$

Sol. **Vertical Tangent**

Concept $y = f(x)$ has a vertical tangent at the points $x = x_0$, if

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \infty \text{ or } -\infty \text{ but not both.}$$

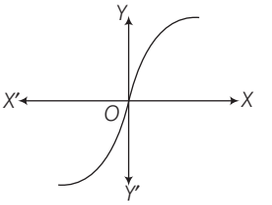
Here, the functions $f(x) = x^{1/3}$ and $f(x) = \text{sgn } x$ both have a vertical tangent at $x = 0$, but $f(x) = x^{2/3}, f(x) = \sqrt{|x|}$ and $f(x) = \begin{cases} 0, \text{ if } x < 0 \\ 1, \text{ if } x \geq 0 \end{cases}$ have no vertical tangent.

Explanation

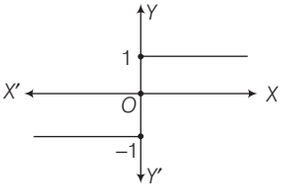
(i) $f(x) = x^{1/3}$

$$\left.\begin{aligned} f'(0^+) &= \lim_{h \to 0} \frac{h^{1/3}}{h} = \frac{1}{h^{2/3}} \to \infty \\ f'(0^-) &= \lim_{h \to 0} \frac{(-h)^{1/3}}{-h} = \frac{1}{(-h)^{2/3}} = \frac{1}{h^{2/3}} \to \infty \end{aligned}\right\}$$

$\Rightarrow$ $f(x)$ has a vertical tangent at $x = 0$.



(ii) $f(x) = \text{sgn } x = \begin{cases} 1, \text{ if } x > 0 \\ 0, \text{ if } x = 0 \\ -1, \text{ if } x < 0 \end{cases}$



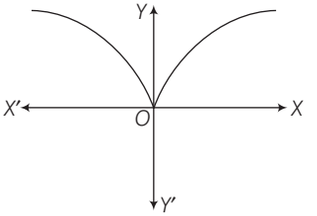
$$\left.\begin{aligned} f'(0^+) &= \lim_{h \to 0} \frac{1 - 0}{h} \to \infty \\ f'(0^-) &= \lim_{h \to 0} \frac{-1}{-h} \to \infty \end{aligned}\right\}$$

$\Rightarrow$ $f(x)$ has a vertical tangent at $x = 0$.

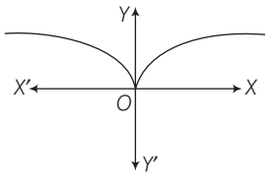
(iii) $f(x) = x^{2/3}$

$$\left.\begin{aligned} f'(0^+) &= \lim_{h \to 0} \frac{h^{2/3}}{h} \to \infty \\ f'(0^-) &= \lim_{h \to 0} \frac{(-h)^{2/3}}{-h} = -\frac{1}{h^{1/3}} \to -\infty \end{aligned}\right\}$$

$\Rightarrow$ No vertical tangent at $x = 0$.



(iv) $f(x) = \sqrt{|x|} = \begin{cases} \sqrt{x}, \text{ if } x \geq 0 \\ \sqrt{-x}, \text{ if } x < 0 \end{cases}$

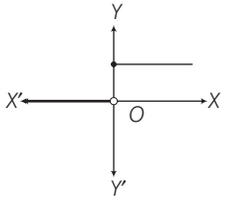


$$\left.\begin{aligned} f'(0^+) &= \lim_{h \to 0} \frac{\sqrt{h} - 0}{h} \to \infty \\ f'(0^-) &= \lim_{h \to 0} \frac{\sqrt{h}}{-h} \to -\infty \end{aligned}\right\}$$

$\Rightarrow$ No vertical tangent at $x = 0$.

(v) $f(x) = \begin{cases} 0, \text{ if } x < 0 \\ 1, \text{ if } x \geq 0 \end{cases}$

$$\left.\begin{aligned} f'(0^+) &= \lim_{h \to 0} \frac{1-1}{h} = 0 \\ f'(0^-) &= \lim_{h \to 0} \frac{0-1}{h} \to -\infty \end{aligned}\right\}$$



$\Rightarrow$ No vertical tangent at $x = 0$.

Equation of Tangent

Let $y = f(x)$ be the equation of curve and point (x_1, y_1) be any point on the curve. Let PT be the tangent at point (x_1, y_1).

Since, tangent is a line passing through the point $P(x_1, y_1)$ and having slope $m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$, therefore by coordinate geometry, the equation of tangent is

$$\boldsymbol{y - y_1 = m(x - x_1) \Rightarrow y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} \cdot (x - x_1).}$$

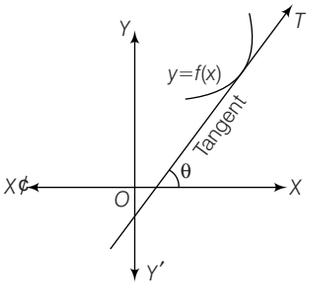


Figure 7.4

Tangent from External Point

If a point $P(a, b)$ does not lie on the curve $y = f(x)$, then the equation of all possible tangents to the curve $y = f(x)$, passing through (a, b), can be found by solving for the point of contact Q.

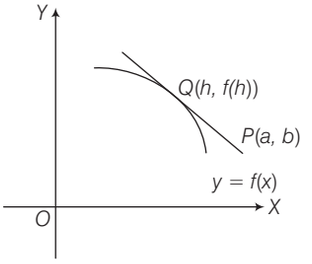


Figure 7.5

Then, $f'(h) = \dfrac{f(h) - b}{h - a}$

and equation of tangent is

$$\boldsymbol{y - b = \frac{f(h) - b}{h - a} \cdot (x - a)}$$

Example 18 Find value of c such that line joining points $(0, 3)$ and $(5, -2)$ becomes tangent to $y = \dfrac{c}{x+1}$.

Sol. Equation of line joining $A(0, 3)$ and $B(5, -2)$ is $x + y = 3$.

Solving the line and curve, we get

$$3 - x = \frac{c}{x+1}$$

$\Rightarrow$ $x^2 - 2x + (c - 3) = 0$ ...(i)

For tangency, roots of this equation must be equal.

Hence, discriminant of quadratic equation = 0.

$\Rightarrow$ $4 = 4(c - 3) \Rightarrow c = 4$

Hence, required value of c is 4.

Equation of Normal

We know that normal to curve at any point is a straight line passes through that point and is perpendicular to the tangent to the curve at that point.

Since, the slope of tangent at $P(x_1, y_1) = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$

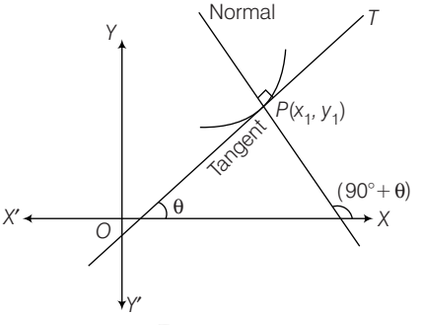


Figure 7.6

Now, as normal is perpendicular to the tangent.

$\therefore$ Slope of normal at $P(x_1, y_1) = \dfrac{-1}{\left(\dfrac{dy}{dx}\right)_{(x_1, y_1)}}$

Hence, from coordinate geometry equation of normal is

$$y - y_1 = -\frac{1}{\left(\dfrac{dy}{dx}\right)_{(x_1, y_1)}}(x - x_1)$$

or $$y - y_1 = -\left(\frac{dx}{dy}\right)_{(x_1, y_1)}(x - x_1)$$

Example 19 Find the equation of tangent and normal to the curve $2y = 3 - x^2$ at (1,1).

Sol. The equation of given curve is,

$$2y = 3 - x^2 \quad \text{...(i)}$$

Differentiating Eq. (i) w.r.t. x, we get

$$2\left(\frac{dy}{dx}\right) = -2x \Rightarrow \frac{dy}{dx} = -x$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,1)} = -1 \quad \text{...(ii)}$$

Now, the equation of tangent at (1, 1) is,

$$\Rightarrow \frac{y-1}{x-1} = \left(\frac{dy}{dx}\right)_{(1,1)} \Rightarrow \frac{y-1}{x-1} = -1$$

$$\Rightarrow y - 1 = -x + 1$$

$$\Rightarrow y + x = 2 \quad \text{[required equation of tangent]}$$

and the equation of the normal at (1, 1) is,

$$\frac{y-1}{x-1} = -\frac{1}{(dy/dx)_{(1,1)}} = 1$$

$$\therefore y - x = 0 \quad \text{[required equation of normal]}$$

Example 20 ***Find the equation of tangent and normal to the parabola*** $y^2 = 4ax$ ***at the point*** $(at^2, 2at)$.

Sol. The equation of given curve is,

$$y^2 = 4ax \quad \text{...(i)}$$

Differentiating Eq. (i) w.r.t. x, we get $2y\dfrac{dy}{dx} = 4a$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(at^2, 2at)} = \frac{4a}{4at} = \frac{1}{t} \quad \text{...(ii)}$$

Now, the equation of tangent at $(at^2, 2at)$ is

$$\frac{y - 2at}{x - at^2} = \left(\frac{dy}{dx}\right)_{(at^2, 2at)} = \frac{1}{t} \quad \text{[using Eq. (ii)]}$$

$$\Rightarrow (y - 2at)t = x - at^2$$

$$\Rightarrow yt - 2at^2 = x - at^2 \Rightarrow yt = x + at^2$$

[required equation of tangent]

and the equation of normal at $(at^2, 2at)$ is,

$$\frac{y - 2at}{x - at^2} = -\frac{1}{\left(\dfrac{dy}{dx}\right)_{(at^2, 2at)}} = -t \quad \text{[using Eq. (ii)]}$$

$$\Rightarrow y - 2at = -xt + at^3$$

$$\Rightarrow y + xt = 2at + at^3 \text{ [required equation of normal]}$$

Example 21 ***Find the point on the curve*** $y - e^{xy} + x = 0$ ***at which we have vertical tangent.***

Sol. The equation of given curve is,

$$y - e^{xy} + x = 0 \quad \text{...(i)}$$

Differentiating Eq. (i) w.r.t. x, we get

$$\frac{dy}{dx} - e^{xy}\left\{1 \cdot y + x \cdot \frac{dy}{dx}\right\} + 1 = 0$$

$$\Rightarrow \frac{dy}{dx}(1 - xe^{xy}) = -1 + y \cdot e^{xy}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1 + y \cdot e^{xy}}{1 - xe^{xy}} \quad \text{...(ii)}$$

Let at point (x_1, y_1) on the curve, we have a vertical tangent (i.e. a tangent parallel to Y-axis). Then,

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \infty \text{ or } \left(\frac{dx}{dy}\right)_{(x_1, y_1)} = 0$$

$$\Rightarrow \frac{1 - x_1 e^{x_1 y_1}}{-1 + y_1 e^{x_1 y_1}} = 0 \Rightarrow 1 - x_1 e^{x_1 y_1} = 0$$

$$\Rightarrow x_1 e^{x_1 y_1} = 1,$$

which is possible only if $x_1 = 1$ and $y_1 = 0$.

Thus, the required point is (1, 0).

Remarks

For standard curves students are advised to use direct method of finding equation of tangent.

For the curve of the form

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, replace

x^2 by xx_1; $2x$ by $x + x_1$;

y^2 by yy_1; $2y$ by $y + y_1$

and xy by $\dfrac{xy_1 + x_1 y}{2}$

Then, equation of tangent is,

$axx_1 + h(xy_1 + yx_1) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0$

e.g.

(i) Find equation of tangent to the curve $2y = x^2 + 3$ at (x_1, y_1).

Sol. On replacing $2y$ by $y + y_1$ and x^2 by xx_1, we get

$(y + y_1) = xx_1 + 3$, which is the required equation of tangent.

(ii) Find equation of tangent to the curve $y^2 = 4ax$ at $(at^2, 2at)$.

Sol. Clearly, the equation of curve $y^2 = 4ax$ is a standard equation of curve, therefore on replacing y^2 by yy_1 and $2x$ by $x + x_1$,

we can get the equation of tangent. Thus, the required equation is given by

$$yy_1 = 2a(x + x_1)$$

where $(x_1, y_1) = (at^2, 2at)$

Hence, the required equation of tangent is

$$y(2at) = 2a(x + at^2) \Rightarrow yt = x + at^2$$

(iii) Find the equation of tangent, at point $P(x_1, y_1)$, to the curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Sol. Clearly, the equation of curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is a standard equation of curve, therefore equation of the tangent can be obtained by replacing x^2 *by* xx_1 and y^2 *by* yy_1.

i.e. $$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

Some Important Points Regarding Tangent and Normal

1. If a curve passes through the origin, then the equation of the tangent at the origin can be directly written by equating the lowest degree terms appearing in the equation of the curve to zero.

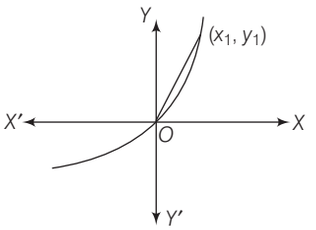


Figure 7.7

Proof Let the equation of the curve be

$$a_1x + b_1y + a_2x^2 + b_2xy + c_2y^2 = 0 \quad \ldots(i)$$

Tangent $$y - 0 = \lim_{\substack{x_1 \to 0 \\ y_1 \to 0}} \frac{y_1}{x_1}(x - 0)$$

Now, Eq. (i) becomes

$$a_1 + b_1\frac{y_1}{x_1} + a_2x_1 + b_2\frac{y_1}{x_1}\cdot x_1 + c_2\frac{y_1}{x_1}\cdot y_1 = 0 \quad \ldots(ii)$$

at $x_1 \to 0$ and $y_1 \to 0, \frac{y_1}{x_1} \to m$

From Eq. (ii), $a_1 + b_1m = 0 \quad \left[\because\ m = -\frac{a_1}{b_1}\right]$

Hence, tangent is $y = -\frac{a_1}{b_1}x$

$\Rightarrow \quad a_1x + b_1y = 0$

e.g.

(i) Equation of tangent at origin, to the curve $x^2 + y^2 + 2gx + 2fy = 0$ is $gx + fy = 0$.

(ii) Equation of tangent at origin to the curve $x^3 + y^3 - 3x^2y + 3xy^2 + x^2 - y^2 = 0$ is $x^2 - y^2 = 0$.

(iii) Equation of tangent at origin, to the curve $x^3 + y^3 - 3xy = 0$ is $xy = 0$.

2. If the curve is $x^4 + y^4 = x^2 + y^2$, then the equation of the tangent would be $x^2 + y^2 = 0$ which would indicate that the origin is an isolated point on the graph.

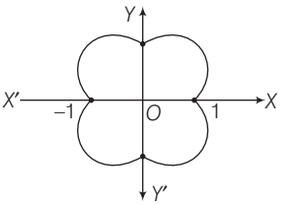


Figure 7.8

3. Same line could be the tangent as well as normal to a given curve at a given point.

e.g. In $x^3 + y^3 - 3xy = 0$ [folium of descartes]

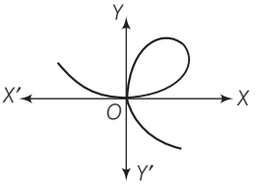


Figure 7.9

The line pair $xy = 0$ is both the tangent as well as normal at $x = 0$.

4. Some common parametric coordinates on a curve

(i) For $x^{2/3} + y^{2/3} = a^{2/3}$, take parametric coordinate $x = a\cos^3\theta$ and $y = a\sin^3\theta$.

(ii) For $\sqrt{x} + \sqrt{y} = \sqrt{a}$, take $x = a\cos^4\theta$ and $y = a\sin^4\theta$.

(iii) For $\frac{x^n}{a^n} + \frac{y^n}{b^n} = 1$, take $x = a(\cos\theta)^{2/n}$ and $y = b(\sin\theta)^{2/n}$.

(iv) For $c^2(x^2 + y^2) = x^2y^2$, take $x = c\sec\theta$ and $y = c\,\mathrm{cosec}\,\theta$.

(v) For $y^2 = x^3$, take $x = t^2$ and $y = t^3$.

Example 22 ***Find the sum of the intercepts on the axes of coordinates by any tangent to the curve*** $\sqrt{x} + \sqrt{y} = 2$.

Sol. Here, equation of curve is $\sqrt{x} + \sqrt{y} = 2$.

Whose parametric coordinates are given by,

$$\sqrt{x} = 2\cos^2\theta$$

and $\sqrt{y} = 2\sin^2\theta$

i.e. $x = 4\cos^4\theta$

and $y = 4\sin^4\theta$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{4 \times 4\sin^3\theta \cdot \cos\theta}{4 \times 4\cos^3\theta(-\sin\theta)} = -\tan^2\theta$

Now, equation of tangent is $\dfrac{y - 4\sin^4\theta}{x - 4\cos^4\theta} = -\tan^2\theta$

$\therefore$ x-intercept, $\left|4\cos^4\theta + \dfrac{4\sin^4\theta}{\tan^2\theta}\right| = 4\cos^2\theta$

and y-intercept, $\left|4\sin^4\theta + \dfrac{4\sin^4\theta}{\tan^2\theta}\right| = 4\sin^2\theta$

Hence, the sum of intercepts made on the axes of coordinates is,

$$4\cos^2\theta + 4\sin^2\theta = 4$$

Example 23 T***he*** tangent, represented by the graph of the function $y = f(x)$, at the point with abscissa $x = 1$ form an angle of $\pi/6$, at the point $x = 2$ form an angle of $\pi/3$ and at the point $x = 3$ form an angle of $\pi/4$. T***hen,*** find the value of,

$$\int_1^3 f'(x)\, f''(x)\, dx + \int_2^3 f''(x)\, dx.$$

Sol. Given, at $x = 1$, $\dfrac{dy}{dx} = \tan \pi/6 = 1/\sqrt{3}$

or at $x = 1$, $f'(1) = \tan \pi/6 = 1/\sqrt{3}$

Also, at $x = 2$, $f'(2) = \tan \pi/3 = \sqrt{3}$

and at $x = 3$, $f'(3) = \tan \pi/4 = 1$

Then, $\int_1^3 f'(x)\, f''(x)\, dx + \int_2^3 f''(x)\, dx$

$= \int_{f'(1)}^{f'(3)} t\, dt + (f'(x))_2^3$

[putting $f'(x) = t \Rightarrow f''(x)\, dx = dt$]

$= \dfrac{1}{2}(t^2)_{f'(1)}^{f'(3)} + \{f'(3) - f'(2)\}$

$= \dfrac{1}{2}\{(f'(3))^2 - (f'(1))^2\} + \{f'(3) - f'(2)\}$

$= \dfrac{1}{2}\left\{(1)^2 - \left(\dfrac{1}{\sqrt{3}}\right)^2\right\} + \{1 - \sqrt{3}\} = \dfrac{1}{2}\left(1 - \dfrac{1}{3}\right) + (1 - \sqrt{3})$

$= \dfrac{4}{3} - \sqrt{3} = \dfrac{4 - 3\sqrt{3}}{3}$

Example 24 Find the equation for family of curves for which the length of normal is equal to the radius vector.

Sol. Let $P(x, y)$ be the point on the curve.

OP = radius vector $= \sqrt{x^2 + y^2}$

PN = length of normal

Now, $\tan\phi = -\dfrac{1}{\left(\dfrac{dy}{dx}\right)}$ and $PN = \dfrac{y}{\sin\phi}$

It is given $OP = PN$

$\Rightarrow \quad \sqrt{x^2 + y^2} = y\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$

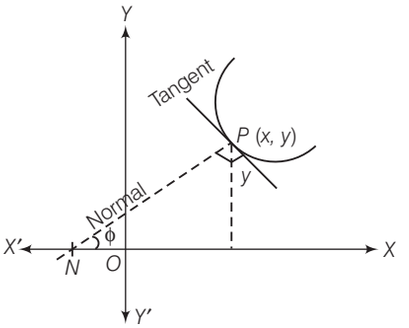


Figure 7.10

$\Rightarrow \quad x^2 + y^2 = y^2\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]$

$\Rightarrow \quad x^2 = y^2\left(\dfrac{dy}{dx}\right)^2 \Rightarrow \dfrac{dy}{dx} = \pm\dfrac{x}{y}$

or $y\, dy = \pm\, x\, dx$

Integrating both the sides, we get

$y^2 = \pm\, x^2 + C$, which is the required family of curves.

Example 25 ***Find the condition that the line*** $x\cos\alpha + y\sin\alpha = p$ ***may touch the curve*** $\left(\dfrac{x}{a}\right)^m + \left(\dfrac{y}{b}\right)^m = 1$.

Sol. Given equation of curve is $\left(\dfrac{x}{a}\right)^m + \left(\dfrac{y}{b}\right)^m = 1$

Differentiating the equation of curve w.r.t. x, we get

$$m\left(\frac{x}{a}\right)^{m-1} \cdot \frac{1}{a} + m\left(\frac{y}{b}\right)^{m-1} \cdot \frac{1}{b} \cdot \frac{dy}{dx} = 0$$

On simplifying, we get $\dfrac{dy}{dx} = \dfrac{-b^m x^{m-1}}{a^m y^{m-1}}$

Now, at any point $P(x_1, y_1)$ on the curve,

slope of tangent $= \left(\dfrac{dy}{dx}\right)_{(x_1, y_1)} = \dfrac{-b^m x_1^{m-1}}{a^m y_1^{m-1}}$

$\therefore$ Equation of tangent at P is, $y - y_1 = \dfrac{-b^m x_1^{m-1}}{a^m y_1^{m-1}}(x - x_1)$

$\Rightarrow \quad \dfrac{yy_1^{m-1}}{b^m} - \dfrac{y_1^m}{b^m} = -\dfrac{xx_1^{m-1}}{a^m} + \dfrac{x_1^m}{a^m}$

i.e. $\dfrac{x}{a}\left(\dfrac{x_1}{a}\right)^{m-1} + \dfrac{y}{b}\left(\dfrac{y_1}{b}\right)^{m-1} = \left(\dfrac{x_1}{a}\right)^m + \left(\dfrac{y_1}{b}\right)^m = 1$

Since, P lies on the curve, therefore the equation of tangent at $P(x_1, y_1)$ on the curve is,

$$\frac{x}{a}\left(\frac{x_1}{a}\right)^{m-1}+\frac{y}{b}\left(\frac{y_1}{b}\right)^{m-1}=1 \quad \text{...(i)}$$

Also, we have $x\cos\alpha + y\sin\alpha = p$ ...(ii)

If Eq. (ii) is the equation of tangent, then coefficients of Eqs. (i) and (ii) must be proportional for point (x_1, y_1).

i.e. $$\frac{\cos\alpha}{\frac{1}{a}\left(\frac{x_1}{a}\right)^{m-1}}=\frac{\sin\alpha}{\frac{1}{b}\left(\frac{y_1}{b}\right)^{m-1}}=\frac{p}{1}$$

This gives $$\frac{x_1}{a}=\left(\frac{a\cos\alpha}{p}\right)^{\frac{1}{m-1}}, \frac{y_1}{b}=\left(\frac{b\sin\alpha}{p}\right)^{\frac{1}{m-1}}$$

Since, point $P(x_1, y_1)$ lies on the curve.

Therefore, we have $\left(\frac{x_1}{a}\right)^m+\left(\frac{y_1}{b}\right)^m=1$

i.e. $$\left(\frac{a\cos\alpha}{p}\right)^{\frac{m}{m-1}}+\left(\frac{b\sin\alpha}{p}\right)^{\frac{m}{m-1}}=1$$

i.e. $$(a\cos\alpha)^{\frac{m}{m-1}}+(b\sin\alpha)^{\frac{m}{m-1}}=p^{\frac{m}{m-1}},$$

which is the required condition.

Example 26 If tangent and normal to the curve $y=2\sin x+\sin 2x$ are drawn at $P\left(x=\frac{\pi}{3}\right)$, then area of the quadrilateral formed by the tangent, the normal at P and the coordinate axes is

(a) $\frac{\pi}{3}$ (b) 3π

(c) $\frac{\pi\sqrt{3}}{2}$ (d) None of these

Sol. Here, $\frac{dy}{dx}=0$ at $\left(x=\frac{\pi}{3}, y=\frac{3\sqrt{3}}{2}\right)$

$\Rightarrow$ Tangent at $x=\frac{\pi}{3}$ is parallel to X-axis.

$\Rightarrow$ Equation of tangent is, $y=\frac{3\sqrt{3}}{2}$

Also, equation of normal is, $x=\frac{\pi}{3}$

Now, area of quadrilateral $=\frac{\pi}{3}\cdot\frac{3\sqrt{3}}{2}=\frac{\pi\sqrt{3}}{2}$ sq unit

Hence, (c) is the correct answer.

Example 27 *The* ***maximum value of the sum of the intercepts made by any tangent to the curve*** $(a\sin^2\theta, 2a\sin\theta)$ ***with the axes is***

(a) $2a$ (b) $a/4$ (c) $a/2$ (d) a

Sol. The curve is $(a\sin^2\theta, 2a\sin\theta) \Rightarrow x=a\sin^2\theta, y=2a\sin\theta$

Now, the equation of any tangent to the curve is,

$$\frac{y-2a\sin\theta}{x-a\sin^2\theta}=\frac{1}{\sin\theta} \Rightarrow y\sin\theta = x+a\sin^2\theta$$

$$\Rightarrow \quad \frac{x}{-a\sin^2\theta}+\frac{y}{a\sin\theta}=1$$

$\Rightarrow$ Sum of intercepts $= a(\sin^2\theta+\sin\theta)$

$$=a\left\{\left(\sin\theta+\frac{1}{2}\right)^2-\frac{1}{4}\right\}$$

Which is maximum, when $\sin\theta=1$

i.e. $(\text{Sum of intercept})_{\max}=2a$

Hence, (a) is the correct answer.

Example 28 ***If*** $g(x)$ ***is a curve which is obtained by the reflection of*** $f(x)=\frac{e^x-e^{-x}}{2}$ ***by the line*** $y=x$, ***then***

(a) $g(x)$ has more than one tangent parallel to X-axis

(b) $g(x)$ has more than one tangent parallel to Y-axis

(c) $y=-x$ is a tangent to $g(x)$ at $(0, 0)$

(d) $g(x)$ has no extremum

Sol. As $g(x)$ is a curve, obtained by the reflection of $f(x)=\frac{e^x-e^{-x}}{2}$ on $y=x$.

so $g(x)$ is inverse of $f(x)$

$\therefore \quad g(x)=\log(x+\sqrt{1+x^2})=f^{-1}(x)$

$$\Rightarrow \quad g'(x)=\frac{1}{x+\sqrt{1+x^2}}\cdot\left(1+\frac{2x}{2\sqrt{1+x^2}}\right)$$

$$=\frac{1}{\sqrt{1+x^2}}\neq 0, \forall\, x\in R$$

$\Rightarrow g(x)$ has no tangent parallel to X-axis. Also, $g'(x)$ is always defined, $\forall\, x\in R$.

$\Rightarrow g(x)$ has no tangent parallel to Y-axis.

Since $g'(x)>0\ \forall x\in R$,

therefore $g(x)$ doesn't have any extremum.

Hence, (d) is the correct answer.

Exercise for Session 3

1. If the line $ax + by + c = 0$ is normal to the $xy + 5 = 0$, then a and b have

(a) same sign (b) opposite sign

(c) cannot be discussed (d) None of these

2. The equation of tangent drawn to the curve $y^2 - 2x^3 - 4y + 8 = 0$ from the point (1, 2) is given by

(a) $y - 2(1 \pm \sqrt{2}) = \pm 2\sqrt{3}\,(x - 2)$ (b) $y - 2(1 \pm \sqrt{3}) = \pm 2\sqrt{2}\,(x - 2)$

(c) $y - 2(1 \pm \sqrt{3}) = \pm 2\sqrt{3}\,(x - 2)$ (d) None of these

3. The equation of the tangents to the curve $(1 + x^2)y = 1$ at the points of its intersection with the curve $(x + 1)y = 1$, is given by

(a) $x + y = 1, y = 1$ (b) $x + 2y = 2, y = 1$

(c) $x - y = 1, y = 1$ (d) None of these

4. The tangent lines for the curve $y = \int_0^x 2|t|\,dt$ which are parallel to the bisector of the first coordinate angle, is given by

(a) $y = x + \frac{3}{4}, y = x - \frac{1}{4}$ (b) $y = -x + \frac{1}{4}, y = -x + \frac{3}{4}$

(c) $x + y = 2, x - y = 1$ (d) None of these

5. The equation of normal to $x + y = x^y$, where it intersects X-axis, is given by

(a) $x + y = 1$ (b) $x - y - 1 = 0$

(c) $x - y + 1 = 0$ (d) None of these

6. The equation of normal at any point θ to the curve $x = a\cos\theta + a\theta\sin\theta, y = a\sin\theta - a\theta\cos\theta$ is always at a distance of

(a) $2a$ unit from origin (b) a unit from origin

(c) $\frac{1}{2}a$ unit from origin (d) None of these

7. If the tangent at (x_0, y_0) to the curve $x^3 + y^3 = a^3$ meets the curve again at (x_1, y_1), then $\frac{x_1}{x_0} + \frac{y_1}{y_0}$ is equal to

(a) a (b) $2a$

(c) 1 (d) None of these

8. The area bounded by the axes of reference and the normal to $y = \log_e x$ at $(1, 0)$, is

(a) 1 sq unit (b) 2 sq units

(c) $\frac{1}{2}$ sq unit (d) None of these

9. If $\frac{x}{a} + \frac{y}{b} = 2$ touches the curve $\frac{x^n}{a^n} + \frac{y^n}{b^n} = 2$ at the point (α, β), then

(a) $\alpha = a^2, \beta = b^2$ (b) $\alpha = a, \beta = b$

(c) $\alpha = -2a, \beta = 4b$ (d) $\alpha = 3a, \beta = -2b$

10. The equation of tangents to the curve $y = \cos(x + y)$, $-2\pi \le x \le 2\pi$ that are parallel to the line $x + 2y = 0$, is

(a) $x + 2y = \frac{\pi}{2}$ and $x + 2y = -\frac{3\pi}{2}$ (b) $x + 2y = \frac{\pi}{2}$ and $x + 2y = \frac{3\pi}{2}$

(c) $x + 2y = 0$ and $x + 2y = \pi$ (d) None of these

Session 4

Angle of Intersection of Two Curves, Length of Tangent, Subtangent, Normal and Subnormal

Angle of Intersection of Two Curves

The angle of intersection of two curves is defined as the angle between the tangents to the two curves at their point of intersection.

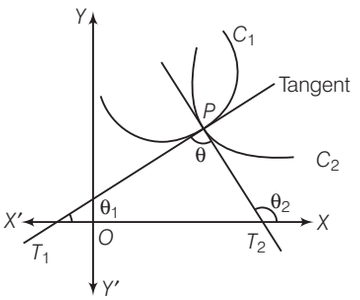


Figure 7.11

Let C_1 and C_2 be two curves having equations $y = f(x)$ and $y = g(x)$, respectively.

Let PT_1 and PT_2 be tangents to the curves C_1 and C_2 at their point of intersection.

Let θ be the angle between the two tangents PT_1 and PT_2 and θ_1 and θ_2 are the angles made by tangents with the positive direction of X-axis in anti-clockwise sense.

Then,
$$m_1 = \tan\theta_1 = \left(\frac{dy}{dx}\right)_{C_1}$$
$$m_2 = \tan\theta_2 = \left(\frac{dy}{dx}\right)_{C_2}$$

From the figure it follows, $\theta = \theta_2 - \theta_1$

$$\Rightarrow \quad \tan\theta = \tan(\theta_2 - \theta_1) = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_1}$$

$$\Rightarrow \quad \tan\theta = \left|\frac{\left(\frac{dy}{dx}\right)_{C_1} - \left(\frac{dy}{dx}\right)_{C_2}}{1 + \left(\frac{dy}{dx}\right)_{C_1}\left(\frac{dy}{dx}\right)_{C_2}}\right|$$

Angle of intersection of these curves is defined as acute angle between the tangents.

Orthogonal Curves If the angle of intersection of two curves is a right angle, then the two curves are said to be orthogonal and the curves are called orthogonal curves.

$\therefore$ If the curves are orthogonal, then $\theta = \pi/2$

$$\Rightarrow \quad 1 + \left(\frac{dy}{dx}\right)_{C_1}\left(\frac{dy}{dx}\right)_{C_2} = 0$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{C_1}\left(\frac{dy}{dx}\right)_{C_2} = -1$$

Condition for Two Curves to Touch

If two curves touch each other, then

$$\theta = 0$$

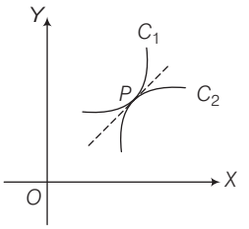


Figure 7.12

$$\Rightarrow \quad \tan\theta = 0$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{C_1} - \left(\frac{dy}{dx}\right)_{C_2} = 0$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{C_1} = \left(\frac{dy}{dx}\right)_{C_2}$$

Example 29 ***Find the angle of intersection of the curves*** $y = x^2$ ***and*** $y = 4 - x^2$.

Sol. For the intersection points of the given curves, consider

$$x^2 = 4 - x^2$$

$$\Rightarrow \quad x = \pm\sqrt{2}$$

Now, at $x = \sqrt{2}$, $\frac{dy}{dx}$ for first curve $= 2x|_{\text{at } x=\sqrt{2}} = 2\sqrt{2}$

While at $x = \sqrt{2}$,

$\frac{dy}{dx}$ for second curve $= -2x|_{at\ x=\sqrt{2}} = -2\sqrt{2}$

Hence, if θ_1 is the acute angle of intersection of the curves, then

$$\tan\theta_1 = \left|\frac{2\sqrt{2}-(-2\sqrt{2})}{1+2\sqrt{2}(-2\sqrt{2})}\right| = \left|\frac{4\sqrt{2}}{-7}\right|$$

$$\therefore \quad \theta_1 = \tan^{-1}\left(\frac{4\sqrt{2}}{7}\right) \qquad ...(i)$$

Again, at $x=-\sqrt{2}$,

$$\left(\frac{dy}{dx}\right) \text{ for first curve} = -2\sqrt{2}$$

and $$\left(\frac{dy}{dx}\right) \text{ for second curve} = 2\sqrt{2}$$

Hence, if θ_2 is the acute angle of intersection of the curves, then

$$\tan\theta_2 = \left|\frac{-2\sqrt{2}-2\sqrt{2}}{1+(-2\sqrt{2})(2\sqrt{2})}\right|$$

$$\therefore \quad \theta_2 = \tan^{-1}\left(\frac{4\sqrt{2}}{7}\right) \qquad ...(ii)$$

From Eqs. (i) and (ii) the two acute angles are equal.

Example 30 ***Find the acute angle between the curves*** $y=|x^2-1|$ ***and*** $y=|x^2-3|$ ***at their points of intersection when*** $x>0$.

Sol. For the intersection of the given curves,

$$|x^2-1| = |x^2-3| \Rightarrow (x^2-1)^2 = (x^2-3)^2$$

$$\Rightarrow \quad (x^2-1)^2-(x^2-3)^2 = 0$$

$$\Rightarrow \quad [(x^2-1)-(x^2-3)][(x^2-1)+(x^2-3)] = 0$$

$$\Rightarrow \quad 2[2x^2-4] = 0 \Rightarrow 2x^2 = 4 \Rightarrow x = \pm\sqrt{2}$$

Neglecting $x=-\sqrt{2}$, as $x>0$

Now, we have, point of intersection as $x=\sqrt{2}$.

Here, $y=|x^2-1|=(x^2-1)$ in the neighbourhood of $x=\sqrt{2}$ and $y=-(x^2-3)$ in the neighbourhood of $x=\sqrt{2}$.

Now, at $x=\sqrt{2}$, $\frac{dy}{dx}$ for first curve

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{C_1} = 2x = 2\sqrt{2}$$

and $\frac{dy}{dx}$ for second curve $\left(\frac{dy}{dx}\right)_{C_2} = -2x = -2\sqrt{2}$

Hence, if θ is angle between them, then

$$\tan\theta = \left|\frac{2\sqrt{2}-(-2\sqrt{2})}{1+2\sqrt{2}(-2\sqrt{2})}\right| = \left|\frac{4\sqrt{2}}{-7}\right| = \left(\frac{4\sqrt{2}}{7}\right)$$

$$\therefore \quad \theta = \tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$$

Example 31 Find the angle of intersection of curves, $y=[|\sin x|+|\cos x|]$ and $x^2+y^2=5$, where $[\cdot]$ denotes the greatest integral function.

Sol. We know that, $1 \le |\sin x| + |\cos x| \le \sqrt{2}$

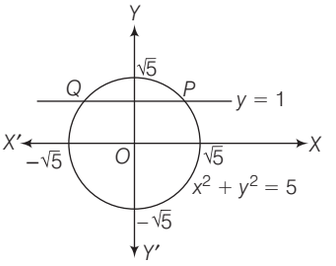
$$\therefore \quad y = [|\sin x| + |\cos x|] = 1$$

Let P and Q be the points of intersection of given curves.

Clearly, the given curves meet at points where $y=1$, so we get $x^2+1=5$

$$\Rightarrow \quad x = \pm 2$$

Now, we have $P(2, 1)$ and $Q(-2, 1)$



Differentiating $x^2+y^2=5$ w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(2,\,1)} = -2$$

and $$\left(\frac{dy}{dx}\right)_{(-2,\,1)} = 2$$

Clearly, the slope of line $y=1$ is zero and the slope of the tangents at P and Q are (-2) and (2), respectively.

Thus, the angle of intersection is $\tan^{-1}(2)$.

Length of Tangent, Subtangent, Normal and Subnormal

Length of Tangent The length of the segment PT, i.e. the portion of the tangent intercepted between the point of contact and X-axis is called the length of tangent.

Subtangent and its Length The projection of the segment PT along X-axis is called the subtangent. Here, ST is the length of subtangent.

Length of Normal The length of segment PN, i.e. the portion of the normal intercepted between the point on the curve and X-axis is called the length of normal.

Subnormal and its Length

The projection of the segment PN along X-axis is called the subnormal. Here, SN is the length of sub normal.

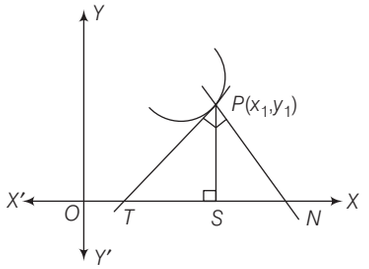


Figure 7.13

From the figure, if PT makes an angle θ with X-axis, then

$$\tan\theta = \left(\frac{dy}{dx}\right)_{(x_1,\,y_1)}$$

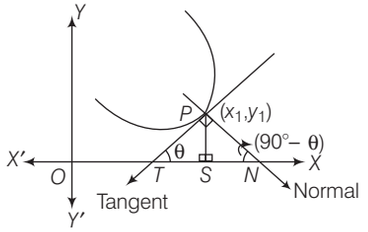


Figure 7.14

Also, we have $\dfrac{ST}{PS} = \cot\theta \Rightarrow ST = PS\cot\theta$

or **subtangent** $= ST = \left| y_1 \left(\dfrac{dx}{dy}\right)_{(x_1,\,y_1)} \right|$

Similarly, $\dfrac{SN}{PS} = \cot(90^\circ - \theta)$

$\Rightarrow$ Subnormal $= SN = PS\tan\theta$

$\Rightarrow$ **Subnormal** $= SN = \left| y_1 \left(\dfrac{dy}{dx}\right)_{(x_1,\,y_1)} \right|$

Now, **length of tangent**

$$= PT = \left| \sqrt{y_1^2 + y_1^2 \left(\frac{dx}{dy}\right)^2_{(x_1,\,y_1)}} \right|$$

$$\Rightarrow \quad PT = \left| y_1 \sqrt{1 + \left(\frac{dx}{dy}\right)^2_{(x_1,\,y_1)}} \right|$$

and **length of normal** $= PN = \left| \sqrt{y_1^2 + y_1^2 \left(\dfrac{dy}{dx}\right)^2_{(x_1,\,y_1)}} \right|$

$$\Rightarrow \quad PN = \left| y_1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2_{(x_1,\,y_1)}} \right|$$

Example 32 ***Show that for the curve $y = be^{x/a}$, the subtangent is of constant length and the subnormal varies as the square of ordinate.***

Sol. The equation of given curve is $y = be^{x/a}$

Let us consider a point (x_1, y_1) on the curve.

Then, we have $y_1 = be^{x_1/a}$ ...(i)

Differentiating the curve $y = be^{x/a}$ w.r.t. x, we get

$$\frac{dy}{dx} = be^{x/a}\cdot\frac{1}{a}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{(x_1,\,y_1)} = \frac{b}{a}e^{x_1/a} \qquad \text{...(ii)}$$

Thus, the length of subtangent $= \left| y_1 \left(\dfrac{dx}{dy}\right)_{(x_1,\,y_1)} \right|$

$$= \left| y_1 \cdot \frac{a}{be^{x_1/a}} \right| = \left| be^{x_1/a} \cdot \frac{a}{be^{x_1/a}} \right|$$

$= a$ (constant) [using Eqs. (i) and (ii)]

$\Rightarrow$ Subtangent is of constant length a.

Again, length of subnormal $= \left| y_1 \left(\dfrac{dy}{dx}\right)_{(x_1,\,y_1)} \right|$

$$= \left| be^{x_1/a} \cdot \frac{be^{x_1/a}}{a} \right| = \frac{1}{a}(be^{x_1/a})^2 = \frac{1}{a}y^2$$ [using Eq. (i)]

Therefore, subnormal varies as the square of ordinate.

Example 33 ***Find the length of tangent, subtangent, normal and subnormal to $y^2 = 4ax$ at $(at^2, 2at)$.***

Sol. The equation of given curve is

$$y^2 = 4ax \qquad \text{...(i)}$$

Differentiating Eq. (i) w.r.t. x, we get

$$2y\frac{dy}{dx} = 4a$$

$$\Rightarrow \quad \left[\frac{dy}{dx}\right]_{(at^2,\,2at)} = \frac{4a}{4at} = \frac{1}{t} \qquad \text{...(ii)}$$

Now, the length of tangent at $(at^2, 2at)$ is

$$= y_1\sqrt{1 + \left(\frac{dx}{dy}\right)^2_{(x_1,\,y_1)}} = 2at\sqrt{1 + t^2}$$ [using Eq. (ii)]

length of normal at $(at^2, 2at)$ is

$$= y_1\sqrt{1 + \left(\frac{dy}{dx}\right)^2_{(x_1,\,y_1)}} = 2at\sqrt{1 + 1/t^2} = 2a\sqrt{t^2 + 1}$$

length of subtangent is

$$\frac{y_1}{\left[\dfrac{dy}{dx}\right]_{(x_1,\,y_1)}} = \frac{2at}{1/t} = 2at^2$$

and length of subnormal is

$$y_1\left[\frac{dy}{dx}\right]_{(x_1,\ y_1)} = 2at \cdot \frac{1}{t} = 2a$$

Example 34 Find the equation of tangent and normal, the length of subtangent and subnormal of the circle $x^2 + y^2 = a^2$ at the point (x_1, y_1).

Sol. The equation of given curve is,

$$x^2 + y^2 = a^2 \qquad ...(i)$$

Differentiating Eq. (i) w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \qquad \left(\frac{dy}{dx}\right)_{(x_1,\ y_1)} = -\frac{x_1}{y_1} \qquad ...(ii)$$

Thus, the equation of tangent is,

$$y - y_1 = \left[\frac{dy}{dx}\right]_{(x_1,\ y_1)}(x - x_1)$$

$$\Rightarrow \qquad y - y_1 = -\frac{x_1}{y_1}(x - x_1) \qquad \text{[from Eq. (ii)]}$$

$$\Rightarrow \qquad yy_1 - y_1^2 = -xx_1 + x_1^2$$

$$\Rightarrow \qquad xx_1 + yy_1 = x_1^2 + y_1^2$$

$$\Rightarrow \qquad xx_1 + yy_1 = a^2 \qquad \text{[using Eq. (i) as } (x_1, y_1) \text{ lies on } x^2 + y^2 = a^2 \Rightarrow x_1^2 + y_1^2 = a^2\text{]}$$

While the equation of normal is,

$$y - y_1 = \frac{y_1}{x_1}(x - x_1)$$

$$\Rightarrow \qquad x_1y - x_1y_1 = xy_1 - x_1y_1 \Rightarrow xy_1 - x_1y = 0$$

The length of subtangent $= \left|y_1 \cdot \left(\frac{dx}{dy}\right)_{(x_1,\ y_1)}\right|$

$$= \left|y_1 \cdot \left(\frac{-y_1}{x_1}\right)\right| \qquad \text{[using Eq. (ii)]}$$

$\Rightarrow$ The length of subtangent $= \left|\frac{y_1^2}{x_1}\right|$

While the length of subnormal $= \left|y_1 \cdot \left(\frac{dy}{dx}\right)_{(x_1,\ y_1)}\right|$

$$= \left|y_1\left(-\frac{x_1}{y_1}\right)\right| = |x_1|$$

Example 35 If the relation between subnormal SN and subtangent ST on the curve, $by^2 = (x + a)^3$ is $p(SN) = q(ST)^2$, then find the value of $\frac{p}{q}$.

Sol. The equation of given curve is $by^2 = (x + a)^3$

Differentiating both the sides w.r.t. x, we get

$$2by\frac{dy}{dx} = 3(x + a)^2 \cdot 1$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{3}{2}\frac{(x + a)^2}{by}$$

$\therefore$ Length of subnormal

$$\Rightarrow \qquad SN = y\frac{dy}{dx} = \frac{3}{2} \cdot \frac{(x + a)^2}{b} \qquad ...(i)$$

and Length of subtangent

$$\Rightarrow \qquad ST = y\frac{dx}{dy} = \frac{2by^2}{3(x + a)^2} \qquad ...(ii)$$

$$\because \qquad \frac{p}{q} = \frac{(ST)^2}{(SN)} \qquad \text{[given]}$$

$$\therefore \qquad \frac{p}{q} = \frac{(2by^2)^2 \cdot 2b}{\{3(x + a)^2\}^2 \cdot 3(x + a)^2} \qquad \text{[using Eqs. (i) and (ii)]}$$

$$= \frac{8b}{27} \cdot \frac{\{(x + a)^3\}^2}{(x + a)^6} = \frac{8b}{27} \qquad \text{[using, } by^2 = (x + a)^3\text{]}$$

$$\therefore \qquad \frac{p}{q} = \frac{8b}{27}$$

Example 36 ***If the length of subnormal is equal to length of subtangent at any point*** $(3, 4)$ ***on the curve*** $y = f(x)$ ***and the tangent at (3, 4) to*** $y = f(x)$ ***meets the coordinate axes at A and B, then maximum area of the*** ΔOAB ***where O is origin, is***

(a) $\frac{45}{2}$ (b) $\frac{49}{2}$ (c) $\frac{25}{2}$ (d) $\frac{81}{2}$

Sol. Length of subnormal = Length of subtangent

$$\Rightarrow \qquad \frac{dy}{dx} = \pm 1$$

If $\frac{dy}{dx} = 1$, then equation of tangent is,

$$y - 4 = x - 3 \Rightarrow y - x = 1$$

$\therefore$ Area of $\Delta OAB = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$ ...(i)

If $\frac{dy}{dx} = -1$, then equation of tangent is,

$$y - 4 = -x + 3 \Rightarrow x + y = 7$$

$\therefore$ Area of $\Delta OAB = \frac{1}{2} \times 7 \times 7 = \frac{49}{2}$ ...(ii)

Clearly, maximum area $= \frac{49}{2}$

Hence, (b) is the correct answer.

Exercise for Session 4

1. The angle of intersection of $y = a^x$ and $y = b^x$, is given by

(a) $\tan\theta = \left|\frac{\log(ab)}{1-\log(ab)}\right|$ (b) $\tan\theta = \left|\frac{\log(a/b)}{1+\log a \log b}\right|$

(c) $\tan\theta = \left|\frac{\log(a/b)}{1-\log(a/b)}\right|$ (d) None of these

2. The angle between the curves $x^2 + 4y^2 = 32$ and $x^2 - y^2 = 12$, is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{2}$

3. If $ax^2 + by^2 = 1$ cuts $a'x^2 + b'y^2 = 1$ orthogonally, then

(a) $\frac{1}{a} - \frac{1}{a'} = \frac{1}{b} - \frac{1}{b'}$ (b) $\frac{1}{a} + \frac{1}{a'} = \frac{1}{b} + \frac{1}{b'}$

(c) $\frac{1}{a} + \frac{1}{b} = \frac{1}{a'} + \frac{1}{b'}$ (d) None of these

4. The length of subtangent to the curve, $y = e^{x/a}$, is

(a) $2a$ (b) a (c) $a/2$ (d) $a/4$

5. The length of normal to the curve $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$, at $\theta = \frac{\pi}{2}$ is

(a) $2a$ (b) a (c) $\sqrt{2}a$ (d) $2\sqrt{2}a$

6. The length of tangent to the curve $y = x^3 + 3x^2 + 4x - 1$ at a point $x = 0$, is

(a) $\frac{1}{4}$ (b) $\frac{\sqrt{17}}{4}$ (c) $\frac{3}{4}$ (d) $\frac{\sqrt{15}}{4}$

7. The possible values of p such that the equation $px^2 = \log x$ has exactly one solution, are

(a) R (b) R^+ (c) $(-\infty, 0] \cup \left\{\frac{1}{2e}\right\}$ (d) $[0, \infty)$

8. If curve $y = 1 - ax^2$ and $y = x^2$ intersect orthogonally then a is

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) 2 (d) 3

9. The length of tangent to the curve $x = a\left(\cos t + \log\tan\frac{t}{2}\right)$, $y = a(\sin t)$, is

(a) ax (b) ay (c) a (d) xy

10. The lines tangent to the curves $y^3 - `x^2y + 5y - 2x = 0$ and $x^4 - x^3y^2 + 5x + 2y = 0$ at the origin intersect at an angle θ equal to

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

Session 5

Rolle's Theorem, Lagrange's Mean Value Theorem

Rolle's Theorem

The theorem was named after the French Mathematician Michel Rolle (1652–1719). Who first gave it in his book *Methode pour resoudre lesegalites* (1691).

Statement

Let f be a real-valued function defined on the closed interval $[a, b]$ such that

(i) $f(x)$ is continuous in the closed interval $[a, b]$

(ii) $f(x)$ is differentiable in the open interval $]a, b[$ and

(iii) $f(a) = f(b)$

Then, there is atleast one value c of x in open interval $]a, b[$ for which $f'(c) = 0$.

Analytical Proof

Now, Rolle's theorem is valid for a function such that

(i) $f(x)$ is continuous in the closed interval $[a, b]$

(ii) $f(x)$ is differentiable in open interval $]a, b[$ and

(iii) $f(a) = f(b)$

So, generally two cases arise in such circumstances.

***Case* I** $f(x)$ is constant in the interval $[a, b]$, then $f'(x) = 0$ for all $x \in [a, b]$.

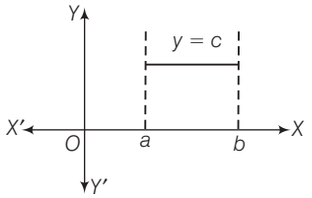


***Figure* 7.15**

Hence, Rolle's theorem follows and we can say, $f'(c) = 0$, where $a < c < b$

***Case* II** $f(x)$ is not constant in the interval $[a, b]$ and since $f(a) = f(b)$.

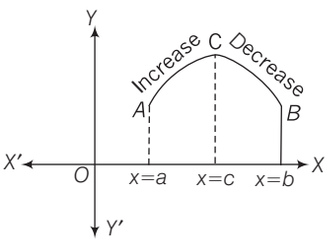


***Figure* 7.16**

The function should either increase or decrease when x assumes values slightly greater than a.

Now, let $f(x)$ increases for $x > a$.

Since, $f(a) = f(b)$, hence the function increase at some value $x = c$ and decrease upto $x = b$.

Clearly, at $x = c$ function has maximum value.

Now, let h be a small positive quantity, then from definition of maximum value of the function,

$$f(c + h) - f(c) < 0$$

and $$f(c - h) - f(c) < 0$$

$\therefore$ $$\frac{f(c + h) - f(c)}{h} < 0$$

and $$\frac{f(c - h) - f(c)}{-h} > 0$$

So, $$\lim_{h \to 0} \frac{f(c + h) - f(c)}{h} \leq 0$$

and $$\lim_{h \to 0} \frac{f(c - h) - f(c)}{-h} \geq 0 \quad \text{...(i)}$$

But, if $$\lim_{h \to 0} \frac{f(c + h) - f(c)}{h} \neq \lim_{h \to 0} \frac{f(c - h) - f(c)}{-h}$$

Then, Rolle's theorem cannot be applied because in such case,

$$\text{RHD at } x = c \neq \text{LHD at } x = c.$$

Hence, $f(x)$ is not differentiable at $x = c$, which contradicts the condition of Rolle's theorem.

$\therefore$ Only one possible solution arises, when

$$\lim_{h \to 0} \frac{f(c + h) - f(c)}{h} = \lim_{h \to 0} \frac{f(c - h) - f(c)}{-h} = 0$$

which implies that, $f'(c) = 0$ where $a < c < b$.

Hence, Rolle's theorem is proved.

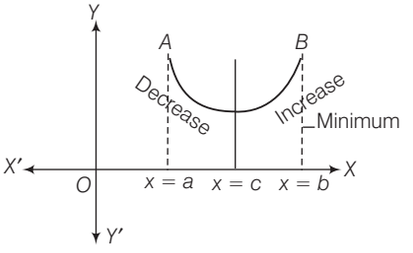


***Figure* 7.17**

Similarly, the case where $f(x)$ decreases in the interval $a < x < c$ and then increases in the interval $c < x < b$, $f'(c) = 0$, but when $x = c$, the minimum value of $f(x)$ exists in the interval $[a, b]$.

Geometrical Proof

Consider the portion AB of the curve $y = f(x)$, lying between $x = a$ and $x = b$, such that

(i) it goes continuously from A to B

(ii) it has tangent at every point between A and B

(iii) ordinate of A = ordinate of B

From the figure, it is clear that $f(x)$ **increases in the interval** AC_1, which implies that $f'(x) > 0$ in this region and decreases in the interval C_1B which implies $f'(x) < 0$ in this region. Now, **since there is unique tangent to be drawn on the curve lying in between A and B and since each of them has a unique slope, i.e. unique value of $f'(x)$.**

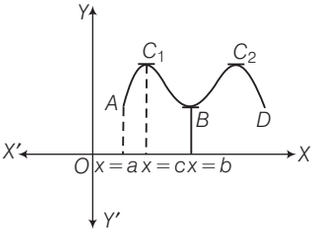


Figure 7.18

So, due to continuity and differentiability of the function $f(x)$ in the region A to B, there is a point $x = c$, where $f'(c)$ should be zero.

Hence, $f'(c) = 0$, where $a < c < b$

Thus, Rolle's theorem is proved.

Remarks

Generally, two types of problems are formulated on Rolle's theorem.

(i) To check the applicability of Rolle's theorem to a given function on a given interval

(ii) To verify Rolle's theorem for a given function in a given interval.

In both the types of problems, first we check whether $f(x)$ satisfies the conditions of Rolle's theorem or not.

Some Important Results Regarding Rolle's Theorem

(i) A polynomial function is everywhere continuous and differentiable.

(ii) The exponential function, sine and cosine functions are everywhere continuous and differentiable.

(iii) Logarithmic functions is continuous and differentiable in its domain.

(iv) $\tan x$ is not continuous and differentiable at $x = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots\ldots$

(v) $|x|$ is not differentiable at $x = 0$.

(vi) If $f'(x)$ tends to $\pm\infty$ as $x \to k$, then $f(x)$ is not differentiable at $x = k$.

e.g. If $f(x) = (2x - 1)^{1/2}$, then $f'(x) = \dfrac{1}{\sqrt{2x - 1}}$ is such that as

$$x \to \left(\frac{1}{2}\right)^{+} \quad \Rightarrow \quad f'(x) \to \infty$$

So, $f(x)$ is not differentiable at $x = 1/2$.

Example 37 Verify Rolle's theorem for the function $f(x) = x^3 - 3x^2 + 2x$ in the interval $[0, 2]$.

Sol. Here, we observe that

(a) $f(x)$ is polynomial and since polynomials are always continuous, so $f(x)$ is continuous in the interval $[0, 2]$.

(b) $f'(x) = 3x^2 - 6x + 2$, which exists for all $x \in (0, 2)$. So, $f(x)$ is differentiable for all $x \in (0, 2)$ and

(c) $f(0) = 0$, $f(2) = 2^3 - 3 \cdot (2)^2 + 2(2) = 0$

$\therefore \quad f(0) = f(2)$

Thus, all the conditions of Rolle's theorem are satisfied.

So, there must exists some $c \in (0, 2)$ such that $f'(c) = 0$

$\Rightarrow \quad f'(c) = 3c^2 - 6c + 2 = 0 \Rightarrow c = 1 \pm \dfrac{1}{\sqrt{3}}$.

Where, $c = 1 \pm \dfrac{1}{\sqrt{3}} \in (0, 2)$, thus Rolle's theorem is verified.

Example 38 If $ax^2 + bx + c = 0$, $a, b, c \in R$, then find the condition that this equation would have at least one root in (0, 1).

Sol. Let $f'(x) = ax^2 + bx + c$

Integrating both sides, we get

$$f(x) = \frac{ax^3}{3} + \frac{bx^2}{2} + cx + d$$

$$\Rightarrow \quad f(0) = d \text{ and } f(1) = \frac{a}{3} + \frac{b}{2} + c + d$$

Now, for Rolle's theorem to be applicable, we should have

$$f(0) = f(1)$$

$$\Rightarrow \quad d = \frac{a}{3} + \frac{b}{2} + c + d$$

$$\Rightarrow \quad 2a + 3b + 6c = 0$$

Here, the required condition is $2a + 3b + 6c = 0$.

Example 39 If $f(x)$ and $g(x)$ are continuous functions in $[a, b]$ and they are differentiable in (a, b), then prove that there exists $c \in (a, b)$ such that

$$\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = (b - a) \begin{vmatrix} f(a) & f'(c) \\ g(a) & g'(c) \end{vmatrix}$$

Sol. Consider a function, $\phi(x) = f(a)\,g(x) - f(x)\,g(a)$ for all $x \in [a, b]$.

As, $\phi(x)$ is continuous and differentiable on (a, b).

$$\therefore \quad \phi'(c) = \frac{\phi(b) - \phi(a)}{b - a} \text{ for some } c \in (a, b). \qquad \text{...(i)}$$

Now, as $\phi(x) = f(a)\,g(x) - f(x)\,g(a)$

$$\therefore \quad \phi'(x) = f(a)\,g'(x) - f'(x)\,g(a)$$

$$\Rightarrow \quad \phi'(c) = f(a)\,g'(c) - f'(c)\,g(a)$$

$$\Rightarrow \quad \phi'(c) = \begin{vmatrix} f(a) & f'(c) \\ g(a) & g'(c) \end{vmatrix}$$

Also, $$\phi(b) = f(a)\,g(b) - f(b)\,g(a) = \begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix}$$

and $$\phi(a) = f(a)\,g(a) - f(a)\,g(a) = 0$$

$\therefore$ Eq. (i) reduces to;

$$\begin{vmatrix} f(a) & f'(c) \\ g(a) & g'(c) \end{vmatrix} = \frac{1}{(b-a)} \begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix}$$

or $$\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = (b-a) \begin{vmatrix} f(a) & f'(c) \\ g(a) & g'(c) \end{vmatrix}$$

Example 40 Use Rolle's theorem to find the condition for the polynomial equation $f(x) = 0$ to have a repeated real roots. Hence, or otherwise prove that the equation;

$1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} = 0$, cannot have repeated roots.

Sol. By Rolle's theorem, we can say that between any two roots of a polynomial there is always a root of its derivative. Thus, if α is a repeated root of a polynomial $f(x)$, then there must be a root of $f'(x)$ in the interval.

$$\Rightarrow \quad f'(\alpha) = 0$$

i.e. $f(\alpha) = f'(\alpha) = 0$, for α to be a repeated root.

Let $\phi(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}$ has a repeated root α.

$$\Rightarrow \quad \phi(\alpha) = 0$$

and $$\phi'(\alpha) = 0$$

$$\Rightarrow \quad 1 + \frac{\alpha}{1!} + \frac{\alpha^2}{2!} + \ldots + \frac{\alpha^n}{n!} = 0$$

and $$1 + \alpha + \frac{\alpha^2}{2!} + \ldots + \frac{\alpha^{n-1}}{(n-1)!} = 0$$

Solving above equations, we get

$$\frac{\alpha^n}{n!} = 0$$

or $\alpha = 0$, thus 0 is the repeated root of $\phi(x) = 0$.

But, 0 doesn't satisfy $\phi(x)$.

$\therefore$ There is no repeated root of $\phi(x) = 0$.

Example 41 Let $f(x)$ and $g(x)$ be differentiable functions such that $f'(x)g(x) \neq f(x)g'(x)$ for any real x. Show that between any two real solutions of $f(x) = 0$, there is atleast one real solution of $g(x) = 0$.

Sol. Let a, b be the solutions of $f(x) = 0$.

Suppose $g(x)$ is not equal to zero for any x belonging to $[a, b]$.

Now, consider $h(x) = f(x) / g(x)$

Since, $g(x)$ is not equal to zero, therefore

$h(x)$ is differentiable and continuous in $[a, b]$.

Also, $h(a) = h(b) = 0$

[as $f(a) = 0$ and $f(b) = 0$ but $g(a)$ and $g(b) \neq 0$]

Applying Rolle's theorem for $h(x)$ in $[a, b]$, we get

$h'(c) = 0$ for some c belonging to (a, b)

$$\Rightarrow \quad f(c)g'(c) = f'(c)g(c), \text{ for some } c \in (a, b).$$

This gives the contradiction. **Hence proved**.

Example 42 Consider the function $f(x) = \begin{cases} x \sin\frac{\pi}{x}, & \text{for } x > 0 \\ 0, & \text{for } x = 0 \end{cases}$, then the number of points in (0, 1), where the derivative $f'(x)$ tends to zero is

(a) 0 (b) 1 (c) 2 (d) infinite ***[IIT 2010]***

Sol. $f(x)$ tends to zero at points, where $\sin\frac{\pi}{x} = 0$

i.e. $\frac{\pi}{x} = k\pi$, $k = 1, 2, 3, 4, \ldots$

Hence, $x = \frac{1}{k}$.

Also, $$f'(x) = \sin\frac{\pi}{x} - \frac{\pi}{x}\cos\frac{\pi}{x}, \quad \text{if } x \neq 0$$

Since, the function has a derivative at any interior point of the interval (0, 1), also continuous in [0, 1] and $f(0) = f(1)$, therefore Rolle's theorem is applicable to any one of the intervals $\left[\frac{1}{2}, 1\right], \left[\frac{1}{3}, \frac{1}{2}\right], \ldots, \left[\frac{1}{k+1}, \frac{1}{k}\right]$.

Hence, there exists some c in each of these intervals, where $f'(c) = 0 \Rightarrow$ Infinite points.

Hence, (d) is the correct answer.

Lagrange's Mean Value Theorem

First Form

If a function $f(x)$,

(i) is continuous in the closed interval $[a, b]$ and

(ii) is differentiable in the open interval $]a, b[$

Then, there is at least one value $c \in (a, b)$, such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof Consider the function,

$$\phi(x) = f(x) - \frac{f(b) - f(a)}{b - a}\, x$$

Since, $f(x)$ is continuous in $[a, b]$.

$\therefore$ $\phi(x)$ is also continuous in $[a, b]$.

Since, $f'(x)$ exists in (a, b), therefore $\phi'(x)$ also exists in (a, b) and

$$\phi'(x) = f'(x) - \frac{f(b) - f(a)}{b - a} \qquad ...(i)$$

Also, we have $\phi(a) = \phi(b)$.

Thus, $\phi(x)$ satisfies all the conditions of Rolle's theorem.

$\therefore$ There is atleast one value of c of x between a and b, such that $\phi'(c) = 0$

On substituting $x = c$ in Eq. (i), we get

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \text{ which proves the theorem.}$$

Second Form

If we write $b = a + h$, then $c = a + \theta h$, where $0 < \theta < 1$.

$[\because \ a < c < b]$

Thus, the mean value theorem can be stated as follows:

If (i) $f(x)$ is continuous in closed interval $[a, a + h]$.

(ii) $f'(x)$ exists in the open interval $]a, a + h[$, then there exists at least one number θ $(0 < \theta < 1)$, such that $f(a + h) = f(a) + hf'(a + \theta h)$

Geometrical Interpretation of Lagrange's Theorem

Let A, B be the points on the curve $y = f(x)$ corresponding to $x = a$ and $x = b$, so that $A = [a, f(a)]$ and $B = [b, f(b)]$

Now, slope of chord $AB = \dfrac{f(b) - f(a)}{b - a}$

The slope of the chord $AB = f'(c)$, the slope of the tangent to the curve at $x = c$.

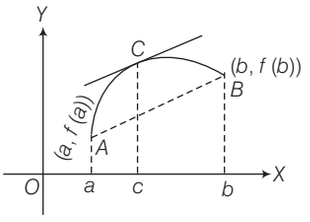


Figure 7.19

Hence, the Lagrange's mean value theorem asserts that if a curve has a tangent at each of its points, then there is a point c on this curve in between A and B, the tangent at which is parallel to the chord AB.

Example 43 ***Find*** c ***of the Lagrange's mean value theorem for which*** $f(x) = \sqrt{25 - x^2}$ ***in*** $[1, 5]$ **.**

Sol. It is clear that $f(x)$ has a definite and unique value for each $x \in [1, 5]$.

Thus, for every point in the interval $[1, 5]$ the value of $f(x)$ is equal to the limit of $f(x)$.

So, $f(x)$ is continuous in the interval $[1, 5]$.

Also, $f'(x) = \dfrac{-x}{\sqrt{25 - x^2}}$, which clearly exists for all x in open interval $(1, 5)$

Hence, $f(x)$ is differentiable in $(1, 5)$.

So, there must be a value $c \in (1, 5)$ such that,

$$f'(c) = \frac{f(5) - f(1)}{5 - 1} = \frac{0 - \sqrt{24}}{4} = \frac{-\sqrt{6}}{2}$$

But $\quad f'(c) = \dfrac{-c}{\sqrt{25 - c^2}}$

$$\therefore \quad \frac{-c}{\sqrt{25 - c^2}} = \frac{-\sqrt{6}}{2} \ \Rightarrow \ 4c^2 = 6(25 - c^2)$$

$$\Rightarrow \quad c = \pm\sqrt{15}$$

Clearly, $c = \sqrt{15} \in (1, 5)$. So, for $c = \sqrt{15}$ Lagrange's theorem is satisfied.

Example 44 ***Let*** $f(x)$ ***and*** $g(x)$ ***be differentiable for*** $0 \le x \le 2$ ***such that*** $f(0) = 2, g(0) = 1$ ***and*** $f(2) = 8$***. Let there exists a real number c in [0, 2] such that*** $f'(c) = 3g'(c)$***, then the value of*** $g(2)$ ***must be***

(a) 2 (b) 3 (c) 4 (d) 5

Sol. As $f(x)$ and $g(x)$ are continuous and differentiable in $[0, 2]$, therefore there exists atleast one value 'c' such that

$$\frac{f'(c)}{g'(c)} = \frac{f(2) - f(0)}{g(2) - g(0)} \Rightarrow \frac{8 - 2}{g(2) - 1} = 3$$

$$\Rightarrow \quad g(2) - 1 = 2 \ \Rightarrow \ g(2) = 3$$

Hence, (b) is the correct answer.

Example 45 ***If*** $f(x) = \log_e x$, $g(x) = x^2$ ***and*** $c \in (4, 5)$, ***then*** $c \log\left(\dfrac{4^{25}}{5^{16}}\right)$ ***is equal to***

(a) $c \log_e 5 - 8$ (b) $2(c^2 \log_e 4 - 8)$

(c) $2(c^2 \log_e 5 - 8)$ (d) $c \log_e 4 - 8$

Sol. Let $\phi(x) = x^2 (\log_e 4) - 16 \log_e x$, which is continuous in $[4, 5]$ and differentiable in $(4, 5)$.

Then, by Lagrange's theorem, we get

$$\frac{\phi(5) - \phi(4)}{5 - 4} = \phi'(c), \text{ for some } c \in (4, 5)$$

Now, $$\phi(5) - \phi(4) = \log_e\left(\frac{4^{25}}{5^{16}}\right)$$

Also, $$\phi'(c) = \frac{2}{c}(c^2 \log_e 4 - 8)$$

$$\therefore \quad \phi'(c) = \frac{\phi(5) - \phi(4)}{5 - 4}$$

$$\Rightarrow \quad \frac{2}{c}(c^2 \log_e 4 - 8) = \log_e\left(\frac{4^{25}}{5^{16}}\right)$$

or $$c \log\left(\frac{4^{25}}{5^{16}}\right) = 2(c^2 \log 4 - 8)$$

Hence, (b) is the correct answer.

Example 46 *If* $0 < a < b < \frac{\pi}{2}$ *and* $f(a,b) = \frac{\tan b - \tan a}{b - a}$, *then*

(a) $f(a, b) \geq 2$ (b) $f(a, b) > 1$

(c) $f(a, b) \leq 1$ (d) None of these

Sol. Consider the function $f(x) = \tan x$, defined on $[a, b]$ such that $a, b \in \left(0, \frac{\pi}{2}\right)$.

Applying Lagrange's mean value theorem, we have

$$f'(c) = \frac{f(b) - f(a)}{b - a} \text{ for some } c \in (a, b)$$

$$\Rightarrow \quad \sec^2 c = \frac{\tan b - \tan a}{b - a}$$

$$\Rightarrow \quad f(a, b) = \sec^2 c$$

$$\Rightarrow \quad f(a, b) > 1 \qquad [\because \sec^2 c > 1 \text{ as } c \in (0, \pi/2)]$$

Hence, (b) is the correct answer.

Example 47 In $[0, 1]$ Lagrange's mean value theorem is not applicable to *[IIT JEE 2003]*

(a) $f(x) = \begin{cases} \frac{1}{2} - x, & x < \frac{1}{2} \\ \left(\frac{1}{2} - x\right)^2, & x \geq \frac{1}{2} \end{cases}$

(b) $f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

(c) $f(x) = x|x|$

(d) $f(x) = |x|$

Sol. For the function $f(x)$ given in option (a), we have (LHD at $x = 1/2) = -1$ and (RHD at $x = 1/2) = 0$.

So, it is not differentiable at $x = 1/2 \in (0, 1)$.

$\therefore$ Lagrange's mean value theorem is not applicable.

Hence, (a) is the correct answer.

Example 48 Let $f(x)$ satisfy the requirements of Lagrange's mean value theorem in $[0, 2]$. If $f(0) = 0$ and $|f'(x)| \leq 1/2$ for all $x \in [0, 2]$, then

(a) $f(x) \leq 2$ (b) $|f(x)| \leq 2x$

(c) $|f(x)| \leq 1$ (d) $f(x) = 3$, for atleast one $x \in [0, 2]$

Sol. Let $x \in (0, 2)$. Since, $f(x)$ satisfies the requirements of Lagrange's mean value theorem in $[0, 2]$. So, it also satisfies in $[0, x]$. Consequently, there exists $c \in (0, x)$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0} \Rightarrow f'(c) = \frac{f(x)}{x}$$

$$\Rightarrow \quad \left|\frac{f(x)}{x}\right| = |f'(c)| \leq 1/2 \qquad [\because |f'(x)| \leq 1/2]$$

$$\Rightarrow \quad |f(x)| \leq \frac{x}{2}$$

$$\Rightarrow \quad |f(x)| \leq 1 \qquad [\because x \in (0, 2), \ \therefore x \leq 2]$$

Hence, (c) is the correct answer.

Example 49 Let $f : [2, 7] \to [0, \infty)$ be a continuous and differentiable function. T*hen,* the value of $(f(7) - f(2)) \frac{(f(7))^2 + (f(2))^2 + f(2) \cdot f(7)}{3}$ is (where $c \in (2, 7)$)

(a) $3f^2(c) f'(c)$ (b) $5f^2(c) \cdot f(c)$

(c) $5f^2(c) \cdot f'(c)$ (d) None of these

Sol. Let $g(x) = f^3(x) \Rightarrow g'(x) = 3f^2(x) \cdot f'(x)$

$\because \quad f : [2, 7] \to [0, \infty) \quad \therefore \quad g : [2, 7] \to [0, \infty)$

Using Lagrange's mean value theorem on $g(x)$, we get

$$g'(c) = \frac{g(7) - g(2)}{5}, \ c \in (2, 7)$$

$$\Rightarrow 3f^2(c) f'(c) = \frac{f^3(7) - f^3(2)}{5}$$

$$\Rightarrow 5f^2(c) \cdot f'(c) = \frac{(f(7) - f(2))(f^2(7) + f^2(2) + f(7) \cdot f(2))}{3}$$

Hence, (c) is the correct answer.

Example 50 T*he* ***equation*** $\sin x + x \cos x = 0$ ***has atleast one root in the interval .***

(a) $\left(-\frac{\pi}{2}, 0\right)$ (b) $(0, \pi)$ (c) $\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ (d) None of these

Sol. Consider the function given by,

$$f(x) = \int (\sin x + x\cos x)\, dx = x\sin x$$

we observe that $f(0) = f(\pi) = 0$

$\therefore$ 0 and π are two roots of $f(x) = 0$.

Consequently, $f'(x) = 0$, i.e. $\sin x + x\cos x = 0$ has at least one root in $(0, \pi)$.

Hence, (b) is the correct answer.

Example 51 Between any two real roots of the equation $e^x \sin x - 1 = 0$, the equation $e^x \cos x + 1 = 0$ has

(a) at least one root (b) at most one root

(c) exactly one root (d) no root

Sol. Let $f(x) = e^{-x} - \sin x$ and α and β be two roots of the equation $e^x \sin x - 1 = 0$ such that $\alpha < \beta$.

Then, $e^{\alpha}\sin\alpha = 1$ and $e^{\beta}\sin\beta = 1$

$\Rightarrow$ $e^{-\alpha} - \sin\alpha = 0$ and $e^{-\beta} - \sin\beta = 0$ ...(i)

Clearly, $f(x)$ is continuous on $[\alpha, \beta]$ and differentiable on (α, β). Also, $f(\alpha) = f(\beta) = 0$ [using Eq. (i)]

$\therefore$ By Rolle's theorem there exists $c \in (\alpha, \beta)$ such that

$$f'(c) = 0$$

$\Rightarrow$ $-e^{-c} - \cos c = 0$

$\Rightarrow$ $e^c \cos c + 1 = 0$

$\Rightarrow$ $x = c$ is root of $e^x \cos x + 1 = 0$; where $c \in (\alpha, \beta)$

Hence, (a) is the correct answer.

Exercise for Session 5

1. If $f(x) = x^{\alpha}\log x$ and $f(0) = 0$, then the value of 'α' for which Rolle's theorem can be applied in [0, 1], is

(a) -2 (b) -1

(c) 0 (d) $\frac{1}{2}$

2. If a, b, c are non-zero real numbers such that $\int_0^1 (1+\cos^8 x)(ax^2 + bx + c)\, dx$

$= \int_0^2 (1+\cos^8 x)(ax^2 + bx + c)\, dx = 0$, then the equation $ax^2 + bx + c = 0$ will have

(a) one root between 0 and 1 and another between 1 and 2

(b) both the roots between 0 and 1

(c) both the roots between 1 and 2

(d) None of the above

3. Let $n \in N$, if the value of c prescribed in Rolle's theorem for the function $f(x) = 2x\,(x-3)^n$ on [0, 3] is $\frac{3}{4}$, then n is equal to

(a) 1 (b) 3

(c) 5 (d) 7

4. If $f(x) > x\,; \forall\, x \in R$. Then, the equation $f(f(x)) - x = 0$, has ***[IIT]***

(a) atleast one real root

(b) more than one real root

(c) no real root if $f(x)$ is a polynomial and one real root if $f(x)$ is not a polynomial

(d) no real root

5. If $f(x)$ is twice differentiable function such that $f(a) = 0, f(b) = 2, f(c) = -1, f(d) = 2, f(e) = 0$, where $a < b < c < d < e$, then the minimum number of zeroes of $g(x) = (f'(x))^2 + f''(x)\cdot f(x)$ in the interval $[a, e]$, is ***[IIT]***

(a) 4 (b) 5

(c) 6 (d) 7

Session 6

Application of Cubic Functions

Application of Cubic Functions

The cubic function $y = ax^3 + bx^2 + cx + d$ will have

1. One real and two imaginary roots
(always monotonic), $\forall\ x \in R$

Condition $f'(x) \geq 0$ or $f'(x) \leq 0$ together with either $f'(x) = 0$ has no root (i.e. $D < 0$) or $f'(x) = 0$ has a root $x = \alpha$, then $f(\alpha) = 0$.

e.g. $y = x^3 - 2x^2 + 5x + 4$

$y' = 3x^2 - 4x + 5 \qquad [D < 0]$

$y = (x-2)^3 \ \Rightarrow\ y' = 3(x-2)^2 = 0$

$\Rightarrow \quad x = 2$, also $f(2) = 0$ gives $x = 2, y(2) = 0$

In this case, if $f'(x) = 0$ has a root $x = \alpha$ and $f(\alpha) = 0$ this would mean $f(x) = 0$ has repeated roots, which is dealt with separately.

2. Exactly one root and non-monotonic

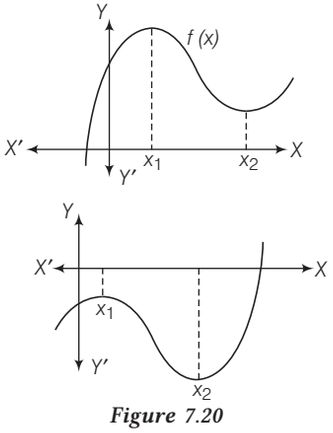


Figure 7.20

$f(x_1) \cdot f(x_2) > 0$

where x_1 and x_2 are the roots of $f'(x) = 0$

3. Two coincident and One different $f(x_1) \cdot f(x_2) = 0$

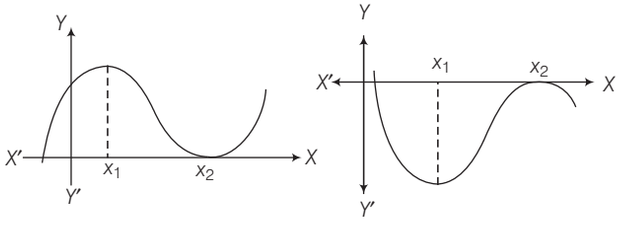


Figure 7.21

where x_1 and x_2 are the roots of $f'(x) = 0$

4. All three distinct real roots

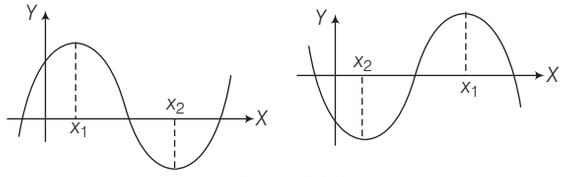


Figure 7.22

$f(x_1) \cdot f(x_2) < 0$

where x_1 and x_2 are the roots of $f'(x) = 0$

5. All the three roots are coincident

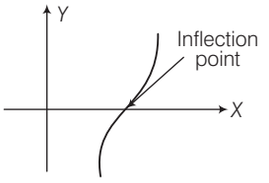


Figure 7.23

$f'(x) \geq 0 \quad$ or $\quad f'(x) \leq 0$

and $\quad f(\alpha) = 0$

where α is a root of $f'(x) = 0$

e.g. $y = (x-1)^3$

Remarks

(i) Graph of every cubic polynomial must have exactly one point of inflection.

(ii) In case (4), if $f(a)$, $f(b)$, $f(c)$ and $f(d)$ alternatively change sign.

Example 52 If the cubic function $f(x) = x^3 + px + q$ has 3 distinct real roots, then prove that $4p^3 + 27q^2 < 0$.

Sol. $f'(x) = 3x^2 + p = 3x^2 + 0 \cdot x + p$

Now, we have $x_1 + x_2 = 0$ and $x_1 x_2 = \frac{p}{3}$, where x_1, x_2 are roots of $f'(x) = 0$

According to given condition, we have

$(x_1^3 + px_1 + q)(x_2^3 + px_2 + q) < 0$

$\Rightarrow \quad x_1^3 \cdot x_2^3 + px_1^3 x_2 + qx_1^3 + p^2 x_1 x_2 + px_1 x_2^3 + qx_2^3 + pqx_1 + q^2 + pqx_2 < 0$

$\Rightarrow (x_1 x_2)^3 + px_1 x_2 (x_1^2 + x_2^2) + q(x_1^3 + x_2^3) + pq(x_1 + x_2) + p^2 x_1 x_2 + q^2 < 0$

$\Rightarrow (x_1 x_2)^3 + px_1 x_2 \{(x_1 + x_2)^2 - 2x_1 x_2\} + q\{(x_1 + x_2)^3 - 3(x_1 x_2)(x_1 + x_2)\} + pq(x_1 + x_2) + p^2 x_1 x_2 + q^2 < 0$

$$\Rightarrow \quad \frac{p^3}{27} + \frac{p^2}{3}\left\{-\frac{2p}{3}\right\} + p^2\left\{\frac{p}{3}\right\} + q^2 < 0$$

$$\Rightarrow \quad 4p^3 + 27q^2 < 0$$

Example 53 ***If*** $f(x)$ ***is a polynomial of degree 5 with real coefficients such that*** $f(|x|)=0$ ***has 8 real roots, then*** $f(x)=0$ ***has***

(a) 4 real roots

(b) 5 real roots

(c) 3 real roots

(d) nothing can be said

Sol. Given that $f(|x|)=0$ has 8 real roots.

$\Rightarrow$ $f(x)=0$ has 4 positive roots.

Since, $f(x)$ is a polynomial of degree 5, $f(x)$ cannot have even number of real roots.

$\Rightarrow f(x)$ has all the five roots real, in which four positive and one root is negative.

Hence, (b) is the correct answer.

Example 54 If the function $f(x)=x^3-9x^2+24x+c$ has three real and distinct roots α, β and γ, then the value of $[\alpha]+[\beta]+[\gamma]$ are

(a) 5, 6

(b) 6, 7

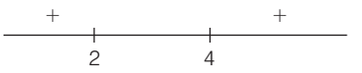
(c) 7, 8

(d) None of the above

Sol. Take $y = x^3 - 9x^2 + 24x$

$$\Rightarrow \quad \frac{dy}{dx} = 3x^2 - 18x + 24 = 3(x^2 - 6x + 8)$$

$$= 3(x-2)(x-4)$$



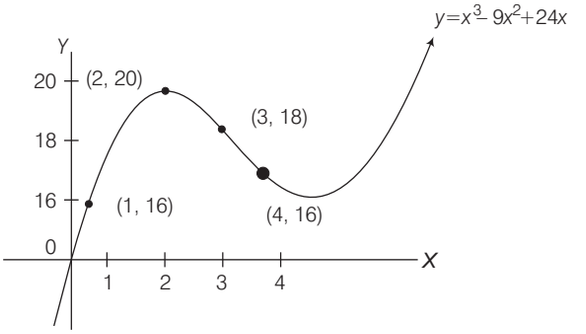
for three real roots of

$f(x) = x^3 - 9x^2 + 24x + c, f(2)f(4) < 0$

$\therefore \ (c+20)(c+16) < 0$

$c \in (-20, -16)$

Now, if $c \in (-20, -18)$, $\alpha \in (1,2), \beta \in (2,3)$, $\gamma \in (4,5)$



$\Rightarrow \quad [\alpha]+[\beta]+[\gamma]=7$

If $\quad c \in (-18, -16)$

$\Rightarrow \quad \alpha \in (1,2), \beta \in (3,4), \gamma \in (4,5)$

$\Rightarrow \quad [\alpha]+[\beta]+[\gamma]=8$

Hence, (c) is the correct answer.

Exercise for Session 6

1. Find all the possible values of the parameter a, so that $x^3 - 3x + a = 0$ has three real and distinct roots.

2. $f(x)$ is a polynomial of degree 4 with real coefficients such that $f(x)=0$ is satisfied by $x = 1, 2, 3$ only, then $f'(1)\cdot f'(2)\cdot f'(3)$ is equal to

   (a) 0 (b) 2

   (c) –1 (d) None of these

3. If the function $f(x) = |x^2 + a|x| + b|$ has exactly three points of non-differentiability, then which of the following can be true?

   (a) $b = 0, a < 0$ (b) $b < 0, a \in R$

   (c) $b > 0, a \in R$ (d) All of these

4. If the equation $e^{||x|-2|+b} = 2$ has four solutions, then b lies in

   (a) (log 2, – log2) (b) (log2 – 2 , log2)

   (c) (– 2, log2) (d) (0, log2)

JEE Type Solved Examples :
Single Option Correct Type Questions

● **Ex. 1** *Number of integral value(s) of k for which the equation $4x^2 - 16x + k = 0$ has one root lie between 1 and 2 and other root lies between 2 and 3, is*

(a) 1 (b) 2 (c) 3 (d) 4

Sol. (*c*) Here, $f(1)\cdot f(2) < 0$

$\Rightarrow \quad (k-12)(k-16) < 0$

$\Rightarrow \quad 12 < k < 16$ ...(i)

Also, $f(2)\cdot f(3) < 0$

$\Rightarrow \quad (k-16)(k-12) < 0 \Rightarrow 12 < k < 16$ ...(ii)

From Eqs. (i) and (ii), we get

$k = \{13, 14, 15\}$

$\therefore$ Number of integral values of $k = 3$

● **Ex. 2** *Let a, b, c∈ R such that two of them are equal and satisfy* $\begin{vmatrix} 2a & b & c \\ b & c & 2a \\ c & 2a & b \end{vmatrix} = 0$, *then equation* $24ax^2 + 4bx + c = 0$ *has*

(a) at least one root in $(0, 1/2)$
(b) at least one root in $(-1/2, 1/2)$
(c) at least one root in $(-1, 0)$
(d) at least two roots in $(0, 2)$

Sol. (*a*) Given determinant is

$$2a(bc - 4a^2) - b(b^2 - 2ac) + c(2ab - c^2) = 0$$

$\Rightarrow \quad 6abc - 8a^3 - b^3 - c^3 = 0$

or $\quad (2a + b + c)[(2a - b)^2 + (b - c)^2 + (c - 2a)^2] = 0$

$\Rightarrow \quad 2a + b + c = 0$ ...(i)
$[\because b \neq c]$

Let $f(x) = 8ax^3 + 2bx^2 + c\ x \Rightarrow f'(x) = 24ax^2 + 4bx + c$

Here, $f(0) = 0$ and $f(1/2) = a + b/2 + c/2 = \dfrac{2a + b + c}{2}$

$\therefore \quad f(1/2) = 0$ [using Eq. (i)]

So, $f(x)$ satisfy Rolle's theorem and hence $f'(x) = 0$ must have at least one root in $(0, 1/2)$.

● **Ex. 3** *The set of values of 'a' for which the equation* $\log_e(a\log_e x) = \log_e x$ *has more than one solution is*

(a) $(1, \infty)$ (b) (e, ∞) (c) $(0, e)$ (d) $(1, e)$

Sol. (*b*) Here, $\log_e(a\log_e x) = \log_e x$

$\Rightarrow \quad a\log_e x = e^{\log_e x} \Rightarrow a\log_e x = x$

If we put $a = e$, then $\log_e x = x/e$ will have only one solution at $x = e$.

$[\because$ the line $y = \dfrac{x}{e}$ is tangent to the curve $y = \log_e x$ at $x = e]$

$\Rightarrow \log_e x = \dfrac{x}{a}$ has more than one solution for $a \in (e, \infty)$.

● **Ex. 4** *The tangent to the hyperbola* $y = \dfrac{x+9}{x+5}$ *passing through the origin is*

(a) $x + 25y = 0$ (b) $5x + y = 0$
(c) $5x - y = 0$ (d) $x - 25y = 0$

Sol. (*a*) Here, $y = 1 + \dfrac{4}{x+5} \Rightarrow (dy/dx)$ at $(x_1, y_1) = -\dfrac{4}{(x_1+5)^2}$

Now, equation of tangent, is

$$y - \left(1 + \frac{4}{x_1 + 5}\right) = -\frac{4}{(x_1+5)^2}(x - x_1)$$

Since, it passes through $(0, 0)$, therefore

$$-1 - \frac{4}{x_1 + 5} = \frac{4x_1}{(x_1+5)^2}$$

$\Rightarrow \quad -(x_1+5)^2 - 4(x_1+5) = 4x_1$

$\Rightarrow \quad (x_1+5)^2 + 4(x_1+5) + 4x_1 = 0$

$\Rightarrow \quad x_1^2 + 18x_1 + 45 = 0$

$\Rightarrow \quad (x_1+15)(x_1+3) = 0 \Rightarrow x_1 = -15$ or -3

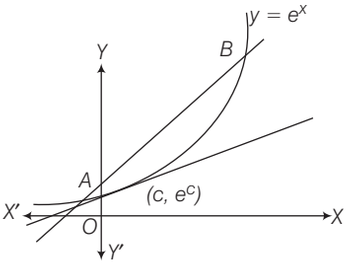
So, equation of tangents are

$$x + 25y = 0 \text{ or } x + y = 0$$

● **Ex. 5** *The tangent to the curve* $y = e^x$ *drawn at the point* (c, e^c) *intersects the line joining the points* $(c-1, e^{c-1})$ *and* $(c+1, e^{c+1})$ ***[IIT JEE 2007]***

(a) on the left of $x = c$ (b) on the right of $x = c$
(c) at no point (d) at all points

Sol. (*a*) Slope of the line joining the points $(c - 1, e^{c-1})$ and $(c + 1, e^{c+1})$ is equal to $\dfrac{e^{c+1} - e^{c-1}}{2} > e^c$



$\Rightarrow$ Tangent to the curve $y = e^x$ will intersect the given line to the left of the line $x = c$.

Aliter

The equation of the tangent to the curve $y = e^x$ at (c, e^c) is

$$y - e^c = e^c (x - c) \quad \ldots(i)$$

Equation of the line joining the given points is

$$y - e^{c-1} = \frac{e^c (e - e^{-1})}{2} [x - (c - 1)] \quad \ldots(ii)$$

Eliminating y from Eqs. (i) and (ii), we get

$$[x - (c - 1)] [2 - (e - e^{-1})] = 2e^{-1}$$

or $$x - c = \frac{e + e^{-1} - 2}{2 - (e - e^{-1})} < 0$$

$\Rightarrow$ $$x < c$$

JEE Type Solved Examples :
More than One Correct Option Type Questions

● ***Ex. 6*** *The coordinate of the point(s) on the graph of the function,* $f(x) = \frac{x^3}{3} - \frac{5x^2}{2} + 7x - 4$ *where the tangent drawn cuts-off intercepts from the coordinate axes which are equal in magnitude but opposite in sign, is*

(a) $\left(2, \frac{8}{3}\right)$ (b) $\left(3, \frac{7}{2}\right)$

(c) $\left(1, \frac{5}{6}\right)$ (d) None of these

Sol. (a, b) Since, intercepts are equal in magnitude but opposite in sign, therefore

$$\left[\frac{dy}{dx}\right]_P = 1$$

Now, $\frac{dy}{dx} = x^2 - 5x + 7 = 1$

$\Rightarrow$ $x^2 - 5x + 6 = 0$

$\therefore$ $x = 2$ or 3

● ***Ex. 7*** *Let* $f : [0, 1] \to R$ *be a differentiable function with non-increasing derivative such that* $f(0) = 0$, $f'(1) > 0$, *then*

(a) $f(1) \geq f'(1)$

(b) $f'(c) \neq 0$ for any $c \in (0, 1)$

(c) $f(1/2) > f(1)$

(d) None of the above

Sol. (a, b) By Lagrange's mean value theorem,

$$\frac{f(1) - f(0)}{1 - 0} = f'(c), c \in (0, 1)$$

$\because f'(x)$ is non-increasing.

$\therefore$ $f'(c) \geq f'(1)$

$\Rightarrow$ $f'(c) > 0 [\because f'(1) > 0]$

Also, $f(1) = f(1) - f(0) = f'(c) \geq f'(1)$

$\therefore$ $f(1) \geq f'(1)$.

● ***Ex. 8*** *If* $f(x)$ *is continuous and derivable,* $\forall\, x \in R$ *and* $f'(c) = 0$ *for exactly* 2 *real values of 'c', then the number of real and distinct values of 'd' for which* $f(d) = 0$ *can be*

(a) 1 (b) 2 (c) 3 (d) 4

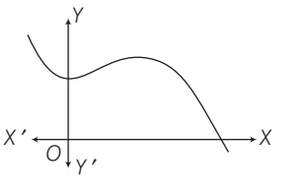
Sol. (a, b, c) If $f'(x) = 0$ has n real roots, then

$f(x) = 0$ has atmost $(n + 1)$ real roots.

Now, if $f'(c) = 0$ for exactly 2 real values of c.

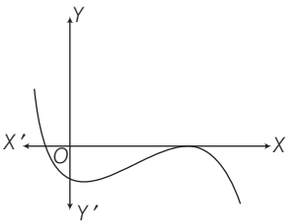
Then, following cases may arise

(a)



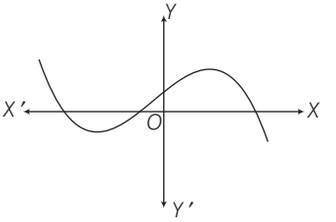
$\Rightarrow$ $f(x) = 0$ have 1 real root.

(b)



$\Rightarrow$ $f(x) = 0$ has 2 real roots.

(c)



$\Rightarrow$ $f(x) = 0$ has 3 real roots.

● ***Ex. 9*** *Which of the following statements are true, where* $\phi(x)$ *is a polynomial?*

(a) Between any two roots of $\phi(x)=0$ there exists at least one root of $\phi'(x)+\lambda\phi(x)=0$

(b) Between any two roots of $\phi(x)=0$ there exists at least one root of $x\phi'(x)+\lambda\phi(x)=0$

(c) Between any two roots of $\phi(x)=0$ there exists at least one root of $(x^2+1)\phi'(x)+\phi(x)=0$

(d) Between any two roots of $\phi(x)=0$ there exists at least one root of $\phi'(x)+x\phi(x)=0$

Sol. (c, d) If α and β are the consecutive roots of $\phi(x)=0$, then $\phi'(\alpha)\cdot\phi'(\beta)\le 0$

Let $\phi(x)=(x-1)(x-2)$

$\phi'(x)=2x-3$

$\therefore\quad \phi'(x)+\lambda\phi(x)=0$

$\Rightarrow\quad (2x-3)+\lambda(x-1)(x-2)=0$

Must have at least one root $\in(\alpha,\beta)$.

Similarly, $(x^2+1)\phi'(x)+\phi(x)=0$

and $\phi'(x)+x\phi(x)=0$ has at least one root $\in(\alpha,\beta)$.

● ***Ex. 10*** *Let* $f(x)$ *be twice differentiable function such that* $f''(x)<0$ *in* $[0,2]$. *Then,*

(a) $f(0)+f(2)=2f(c), 0<c<2$

(b) $f(0)+f(2)=2f(1)$

(c) $f(0)+f(2)>2f(1)$

(d) $f(0)+f(2)<2f(1)$

Sol. (a, d) By intermediate mean value theorem, we get

$$\frac{f(0)+f(2)}{2}=f(c),\quad 0<c<2 \qquad ...(i)$$

By Lagrange's mean value theorem, we get

$$f(1)-f(0)=f'(c_1),\quad 0<c_1<1 \qquad ...(ii)$$

$$f(2)-f(1)=f'(c_2),\quad 1<c_2<2 \qquad ...(iii)$$

On subtracting Eq. (ii) from Eq. (iii), we get

$$f(2)+f(0)-2f(1)=f'(c_2)-f'(c_1) \qquad ...(iv)$$

Again, by Lagrange's mean value theorem, we get

$$f''(c_3)=\frac{f'(c_2)-f'(c_1)}{c_2-c_1},\ \text{for some } c_3\in(c_1,c_2)$$

$\Rightarrow\quad f'(c_2)-f'(c_1)=(c_2-c_1)f''(c_3)<0\quad [f''(x)<0]$ ...(v)

From Eqs. (iv) and (v), we get

$$f(2)+f(0)-2f(1)<0\ \Rightarrow\ f(0)+f(2)<2f(1)$$

JEE Type Solved Examples :
Statements I and II Type Questions

■ **Directions** (Q. Nos. 11 to 13) *For the following questions, choose the correct answers from the option* (*a*), (*b*), (*c*) *and* (*d*) *defined as follows*

(a) Statement I is true, Statement II is also true; Statement II is the correct explanation of Statement I

(b) Statement I is true, Statement II is also true; Statement II is not the correct explanation of Statement I

(c) Statement I is true, Statement II is false

(d) Statement I is false, Statement II is true

● ***Ex. 11*** **Statement I** *The tangent at* $x=1$ *to the curve* $y=x^3-x^2-x+2$ *again meets the curve at* $x=-2$.

Statement II *When an equation of a tangent solved with the curve, repeated roots are obtained at the point of tangency.*

Sol. (*d*) When $x=1$, then $y=1$. Also, $y'=3x^2-2x-1$

$\Rightarrow\quad y'\big|_{x=1}=0\ \Rightarrow$ equation of tangent is $y=1$.

Solving $x^3-x^2-x+2=1$

or $\quad x^3-x^2-x+1=0$, we get $x=-1,1$

$\therefore$ The tangent meets the curve again at $x=-1$.

$\therefore$ Statement I is false and Statement II is true.

● ***Ex. 12*** **Statement I** *The ratio of length of tangent to length of normal is inversely proportional to the ordinate of the point of tengency at the curve* $y^2=4ax$.

Statement II *Length of normal and tangent to a curve*

$$y=f(x)\ is\ \left|y\sqrt{1+m^2}\right|\ and\ \left|\frac{y\sqrt{1+m^2}}{m}\right|,$$

where $m=\dfrac{dy}{dx}$.

Sol. (*a*) Let the slope of the tangent be denoted by $\tan\psi$ and the length of tangent $= y\ \text{cosec}\ \psi$.

Then, length of normal $= y\sec\psi$

$$\therefore\quad \frac{\text{Length of tangent}}{\text{Length of normal}}=\cot\psi\propto\frac{1}{y}$$

$\therefore$ Statement I is true.

$$\text{Length of normal}=\left|y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right|$$

$$= \left| y\sqrt{1+m^2} \right|$$

$$\text{Length of tangent} = \left| y\sqrt{1+\left(\frac{dx}{dy}\right)^2} \right| = \left| \frac{y\sqrt{1+m^2}}{m} \right|$$

$\therefore$ Statement II is true and explain Statement I.

● ***Ex. 13*** **Statement I** *Tangent drawn at the point* $(0, 1)$ *to the curve* $y = x^3 - 3x + 1$ *meets the curve thrice at one point only.*

Statement II *Tangent drawn at the point* $(1, -1)$ *to the curve* $y = x^3 - 3x + 1$ *meets the curve at one point only.*

Sol. (c) $\frac{dy}{dx} = 3x^2 - 3$

Statement I $\left(\frac{dy}{dx}\right)_{\text{at } (0,\,1)} = -3$

$\therefore$ Equation of the tangent is $y - 1 = -3(x - 0)$

i.e. $y = -3x + 1$

Now, $-3x + 1 = x^3 - 3x + 1 \Rightarrow x = 0$

$\therefore$ The tangent meets the curve at one point only.

$\therefore$ Statement I is true.

Statement II $\left(\frac{dy}{dx}\right)_{\text{at } (1,\,-1)} = 0$

$\therefore$ Equation of the tangent is $y + 1 = 0(x - 1)$

i.e. $y = -1$

Now, $-1 = x^3 - 3x + 1$

$\Rightarrow$ $x^3 - 3x + 2 = 0$

$\Rightarrow$ $(x - 1)(x^2 + x - 2) = 0$

$\Rightarrow$ The tangent meets the curve at two points.

$\therefore$ Statement II is false.

JEE Type Solved Examples : Passage Based Questions

Passage I (Ex. Nos. 14 to 18)

Let $f(x) = x^3 + ax^2 + bx + c$ *be the given cubic polynomial and* $f(x) = 0$ *be the corresponding cubic equation, where* $a, b, c \in R$. *Now,* $f'(x) = 3x^2 + 2ax + b$

Let $D = 4a^2 - 12b = 4(a^2 - 3b)$ *be the discriminant of the equation* $f'(x) = 0$.

● ***Ex. 14*** *If* $D = 4(a^2 - 3b) < 0$, *then*

(a) $f(x)$ has all real roots
(b) $f(x)$ has one real and two imaginary roots
(c) $f(x)$ has repeated roots
(d) None of the above

● ***Ex. 15*** *If* $D = 4(a^2 - 3b) > 0$ *and* $f(x_1) \cdot f(x_2) > 0$, *where* x_1, x_2 *are the roots of* $f(x)$, *then*

(a) $f(x)$ has all real and distinct roots
(b) $f(x)$ has three real roots but one of the roots would be repeated
(c) $f(x)$ would have just one real root
(d) None of the above

● ***Ex. 16*** *If* $D = 4(a^2 - 3b) > 0$ *and* $f(x_1) \cdot f(x_2) < 0$, *where* x_1, x_2 *are the roots of* $f(x)$, *then*

(a) $f(x)$ has all real and distinct roots
(b) $f(x)$ has three real roots but one of the roots would be repeated
(c) $f(x)$ would have just one real root
(d) None of the above

● ***Ex. 17*** *If* $D = 4(a^2 - 3b) > 0$ *and* $f(x_1) \cdot f(x_2) = 0$, *where* x_1, x_2 *are the roots of* $f(x)$, *then*

(a) $f(x)$ has all real and distinct roots
(b) $f(x)$ has three real roots but one of the roots would be repeated
(c) $f(x)$ would have just one real root
(d) $f(x)$ has three real roots but all are same

● ***Ex. 18*** *If* $D = 4(a^2 - 3b) = 0$, *then*

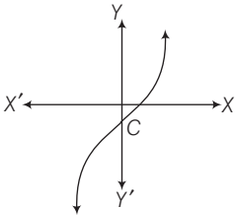
(a) $f(x)$ has all real and distinct roots
(b) $f(x)$ has three real roots but one of the roots would be repeated
(c) $f(x)$ would have just one real root
(d) None of the above

■ ***Sol.*** (Ex. Nos. 14 to 18)

Case I If $D < 0 \Rightarrow f'(x) > 0, \forall\ x \in R$

That means $f(x)$ would be an increasing function of x.

Also, $\lim_{x \to \infty} f(-x) = -\infty$ and $\lim_{x \to \infty} f(x) = \infty$



The graph of $f(x)$ would look like as shown in this figure.

It is clear that the graph of $f(x)$ would intersect the X-axis only once.

That means we would have just one real root (say x_0).

***Case* II** Clearly, $x_0 > 0$, if $c < 0$ and $x_0 < 0$, if $c > 0$.

If $D > 0$, $f'(x) = 0$ would have two real roots (say x_1 and x_2, let $x_1 < x_2$).

$\Rightarrow$ $f'(x) = 3(x - x_1)(x - x_2)$

Clearly, $f'(x) < 0,\ x \in (x_1, x_2)$

$f'(x) > 0,\ x \in (-\infty, x_1)\ (x_2, \infty)$

That means $f(x)$ would increase in $(-\infty, x_1)$ and (x_2, ∞) and would decrease in (x_1, x_2). Hence, $x = x_1$ would be a point of local maxima and $x = x_2$ would be a point of local minima. Thus, the graph of $y = f(x)$ could have these five possibilities.

Thus, the following results are obtained from the above graphs

(a) $f(x_1) \cdot f(x_2) > 0$, $f(x) = 0$ would have just one real root.

(b) $f(x_1) \cdot f(x_2) < 0$, $f(x) = 0$ would have three real and distinct roots.

(c) $f(x_1) \cdot f(x_2) = 0$, $f(x) = 0$ would have three real roots about one of the roots would be repeated.

***Case* III** If $D = 0$, $f'(x) = 3(x - x_1)^2$, where x_1 is root of $f'(x) = 0$

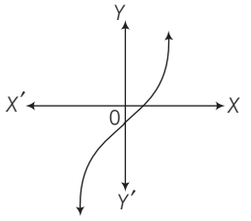
$\Rightarrow$ $f(x) = (x - x_1)^3 + k$

$\therefore$ $f(x) = 0$ has three real roots, if $k = 0$

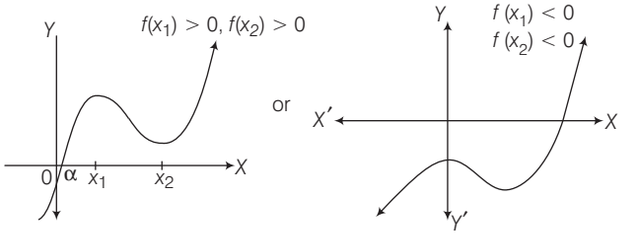
$f(x) = 0$ have one real root, if $k \neq 0$.

14. (*b*) Here, $f(x) = 0$ and $D < 0$

$\Rightarrow f'(x) > 0,\ \forall x \in R$



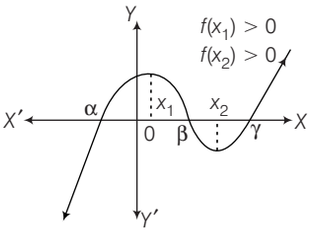
15. (*c*) Here, $f(x) = 0$ with one real root $x = \alpha$ and other two imaginary roots.



From both graph, $f(x_1)\ f(x_2) > 0$

$\therefore$ $f(x)$ would have one real roots.

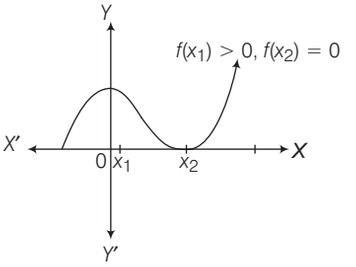
16. (*a*) Here, $f(x) = 0$ with three distinct root α, β, γ.



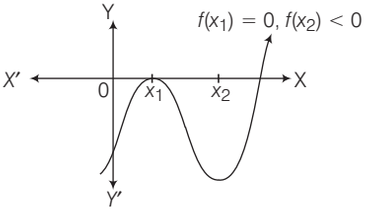
From graph, $f(x_1)\ f(x_2) < 0$

$\therefore$ $f(x)$ has real and distinct roots.

17. (*b*) Here, $f(x) = 0$ with three roots $x = \alpha, x_2$ (x_2 being repeated root).



18. (*b*) Here, $f(x) = 0$ with three real roots $x = x_1, \alpha$ (x_1 being repeated root).



Passage II (Ex. Nos. 19 to 21)

■ *If $y = f(x)$ is a curve and if there exists two points $A(x_1, f(x_1))$ and $B(x_2, f(x_2))$ on it such that $f'(x_1) = -\dfrac{1}{f'(x_2)} = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$, then the tangent at x_1 is normal at x_2 for that curve. Now, answer the following questions.*

● ***Ex. 19*** *Number of such lines on the curve $y = \sin x$ is*

(a) 1 (b) 0 (c) 2 (d) infinite

Sol. (*b*) Let $f(x) = y = \sin x$

Then, $f'(x) = \dfrac{dy}{dx} = \cos x$

$\therefore$ $\cos x_1 = -\dfrac{1}{\cos x_2} = \dfrac{\sin x_2 - \sin x_1}{x_2 - x_1}$

i.e. $\cos x_1 \cos x_2 = -1$

$\therefore$ $\sin x_1 = \sin x_2 = 0$

$\therefore$ There is no solution.

● **Ex. 20** *Number of such lines on the curve* $y = |\ln x|$ *is*

(a) 1 (b) 2 (c) 0 (d) infinite

Sol. (*c*) $f(x) = y = |\ln x|$

$$\therefore \quad f'(x_1) = -\frac{1}{f'(x_2)} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\Rightarrow \quad \frac{\ln x_1}{x_1 |\ln x_1|} = -\frac{x_2 |\ln x_2|}{\ln x_2} = \frac{|\ln x_2| - |\ln x_1|}{x_2 - x_1} \qquad ...(i)$$

$\Rightarrow \ln x_1 \cdot \ln x_2 < 0$

Let $0 < x_1 < 1$, then $1 < x_2$ and $x_1 \cdot x_2 = 1$

From Eq. (i), we get

$$-\frac{1}{x_1} = -x_2 = \frac{\ln x_2 + \ln x_1}{x_2 - x_1} = \frac{\ln x_1 x_2}{x_2 - x_1} = 0,$$

which is not possible.

∴ There is no solution.

● **Ex. 21** *Number of such lines on the curve* $y^2 = x^3$, *is*

(a) 1 (b) 2
(c) 3 (d) 0

Sol. (*b*) We know that, $y^2 = x^3$

$$\therefore \quad y = \sqrt{x^3} \text{ or } -\sqrt{x^3}$$

$$2y\frac{dy}{dx} = 3x^2$$

$$\frac{3x_1^2}{2y_1} = -\frac{2y_2}{3x_2^2} \Rightarrow \frac{9}{4}x_1^2 x_2^2 = -y_1 y_2$$

$$\therefore \quad y_1 y_2 < 0$$

Let $y_1 = \sqrt{x_1^3}$ and $y_2 = -\sqrt{x_2^3}$

Thus, $$\frac{3x_1^2}{2\sqrt{x_1^3}} = -\frac{2\sqrt{x_2^3}}{3x_2^2} = \frac{-\sqrt{x_2^3} - \sqrt{x_1^3}}{x_2 - x_1}$$

$$\Rightarrow \quad \frac{3\sqrt{x_1}}{2} = \frac{2}{3\sqrt{x_2}} = \frac{\sqrt{x_2^3} + \sqrt{x_1^3}}{x_1 - x_2}$$

$$\Rightarrow \quad \sqrt{x_1 x_2} = \frac{4}{9}$$

and $$3x_1\sqrt{x_1} - 3\sqrt{x_1}\,x_2 = 2\sqrt{x_2^3} + 2\sqrt{x_1^3}$$

$$\Rightarrow \quad 3(\sqrt{x_1})^3 - \frac{3 \times 16}{81\sqrt{x_1}} = 2 \cdot \frac{64}{729\sqrt{x_1^3}} + 2\sqrt{x_1^3}$$

$$\Rightarrow \quad 3x_1^3 - \frac{16}{27}x_1 = \frac{128}{729} + 2x_1^3$$

$$\Rightarrow \quad x_1^3 - \frac{16}{27}x_1 = \frac{128}{729}$$

$$\Rightarrow \quad 739x_1^3 - 432x_1 - 12 = 0$$

Consider, $h(t) = 729x^3 - 432t - 128$

$$h'(t) = 3 \times 729t^2 - 432 = 0$$

Gives $$t = \pm\frac{4}{9}$$

$$h\left(-\frac{4}{9}\right) = 0$$

∴ There are two distinct solutions of

$$729x_1^3 - 432x_1 - 128 = 0.$$

Passage III (Ex. Nos. 22 and 23)

■ *Let* $f(x) = \int_0^x (|t-1| - |t+2| + t - 2)\,dt$, *such that* $f''(a) \neq 1$. *If vectors* $a\hat{\mathbf{i}} - b^2\hat{\mathbf{j}}$ *and* $\hat{\mathbf{i}} + 3b\hat{\mathbf{j}}$ *are parallel for at least one* a, *then*

● **Ex. 22** *Number of integral values of 'b' can be*

(a) 5 (b) 10
(c) 11 (d) 13

● **Ex. 23** *Maximum value of* $(1 - 8b - b^2)$ *is*

(a) 4 (b) 8
(c) 12 (d) 16

Sol. (Ex. Nos. 22 and 23) Here, $f(x) = \int_0^x (|t-1| - |t+2| + t - 2)\,dt$

$$\Rightarrow \quad f'(x) = |x-1| - |x+2| + x - 2$$

or $$f'(x) = \begin{cases} x+1, & x < -2 \\ -x-3, & -2 \le x \le 1 \\ x-5, & x > 1 \end{cases}$$

Clearly, $f''(a) \neq 1$ only when $a \in [-2, 1]$ ...(i)

Since, vectors are parallel.

$$\therefore \quad \frac{a}{1} = \frac{-b^2}{3b} \Rightarrow b^2 + 3ab = 0 \qquad ...(ii)$$

Let $g(a) = b^2 + 3ab$ must have atleast one root in $[-2, 1]$.

$$\therefore \quad g(-2) \cdot g(1) \le 0$$

$$\Rightarrow \quad (b^2 - 6b)(b^2 + 3b) \le 0$$

$$\Rightarrow \quad b^2(b-6)(b+3) \le 0$$

$$\therefore \quad b \in [-3, 6] \cup \{0\}$$

$\Rightarrow$ Number of integral values of 'b' can be 11.

Now, let $h(b) = 1 - 8b - b^2$

$$h'(b) = -8 - 2b = 0$$

$$\Rightarrow \quad b = -4$$

∴ $h(b)$ is decreasing, when $b > -4$

$$\Rightarrow \quad (h(b)_{\max}) \text{ at } b = -3$$

$$\therefore \quad (h(-3))_{\max} = 16$$

22. (*c*)

23. (*d*)

JEE Type Solved Examples :
Matching Type Questions

● ***Ex. 24*** *Match the statements of Column* I *with values of Column* II.

	Column I		Column II
(A)	The sides of a triangle vary slightly in such a way that its circumradius remains constant, if $\frac{da}{\cos A}+\frac{db}{\cos B}+\frac{dc}{\cos C}+1=\lvert m\rvert$, then the value of m is	(p)	1
(B)	If the length of subtangent to the curve $x^2y^2=16$ at the point $(-2, 2)$ is $\lvert k\rvert$, then the value of k is	(q)	–1
(C)	If the curve $y=2e^{2x}$ intersects the Y-axis at an angle $\cot^{-1}\lvert(8n-4)/3\rvert$, then the value of n is	(r)	2
(D)	If the area of a triangle formed by normal at the point (1, 0) on the curve $x=e^{\sin y}$ with axes is $\lvert 2t+1\rvert/6$ sq units, then the value of t is	(s)	–2

Sol. (A) → (p, q); (B) → (r, s); (C) → (r, q); (D) → (p, s)

(A) We know that in any ΔABC,

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R$$

$$\Rightarrow \quad a=2R\sin A, b=2R\sin B, c=2R\sin C$$

$$\Rightarrow \quad \frac{da}{dA}=2R\cos A, \frac{db}{dB}=2R\cos B, \frac{dc}{dC}=2R\cos C$$

$$\Rightarrow \quad \frac{da}{\cos A}+\frac{db}{\cos B}+\frac{dc}{\cos C}=2R(dA+dB+dC)$$

Also, $A+B+C=\pi$

$\therefore \quad dA+dB+dC=0$

Hence, $\frac{da}{\cos A}+\frac{db}{\cos B}+\frac{dc}{\cos C}=0$

or $\frac{da}{\cos A}+\frac{db}{\cos B}+\frac{dc}{\cos C}+1=1$

$$\Rightarrow \quad \lvert m\rvert=1\Rightarrow m=\pm 1$$

(B) We have, $x^2y^2=16$

$$\Rightarrow \quad 2x^2y\frac{dy}{dx}+2xy^2=0 \quad \Rightarrow \quad \frac{dy}{dx}=-y/x$$

Length of subtangent $=\left\lvert y\frac{dx}{dy}\right\rvert=\left\lvert -y\frac{x}{y}\right\rvert=\lvert x\rvert$

$\Rightarrow$ Length of subtangent = 2

$\therefore \quad k=\pm 2$

(C) We have $y=2e^{2x}$, which intersect Y-axis at (0, 2)

Now, $\frac{dy}{dx}=4e^{2x}$

$\therefore \quad \left(\frac{dy}{dx}\right)_{(0,2)}=4$

$\therefore$ Angle of intersection with Y-axis

$$=\frac{\pi}{2}-\tan^{-1}4=\cot^{-1}4$$

Hence $\left\lvert\frac{8n-4}{3}\right\rvert=4, \ n=2$ or -1

(D) We have, $x=e^{\sin y}$

$$\Rightarrow \quad \frac{dx}{dy}=\cos y\, e^{\sin y}$$

Slope of normal at (1, 0) = -1

Equation of normal is $x+y=1$

Area of triangle $=\frac{1}{2}$

$$\therefore \quad \frac{\lvert 2t+1\rvert}{6}=\frac{1}{2}$$

$$\Rightarrow \quad t=1, -2$$

● ***Ex. 25*** *Match the statements of Column* I *with values of Column* II.

	Column I		Column II
(A)	Rolle's theorem in $[-1, 1]$ does not hold, if	(p)	$f(x)=\begin{cases}\left(\frac{1}{2}-x\right)^2, & x>\frac{1}{2}\\ \left(\frac{1}{2}-x\right), & x\le\frac{1}{2}\end{cases}$
(B)	LMVT in $[0, 1]$ does not hold, if	(q)	$f(x)=\lvert x\rvert$
(C)	Rolle's theorem in $[0, 1]$ does not hold, if	(r)	$f(x)=\begin{cases}1, & 0\le x<\frac{1}{2}\\ 2, & \frac{1}{2}\le x\le 1\end{cases}$
		(s)	$f(x)=\log\lvert x\rvert$
		(t)	$f(x)=x^2+x+1$

Sol. (A) → (p,q,r,s,t); (B) → (p, r, s); (C) → (p,r,s,t)

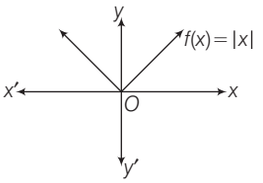
(p) $$f(x)=\begin{cases}\left(\frac{1}{2}-x\right)^2, & x>\frac{1}{2}\\ \left(\frac{1}{2}-x\right), & x\le\frac{1}{2}\end{cases}$$

$$\therefore\quad f'(x)=\begin{cases}-2\left(\frac{1}{2}-x\right), & x>\frac{1}{2}\\ -1, & x\le\frac{1}{2}\end{cases}$$

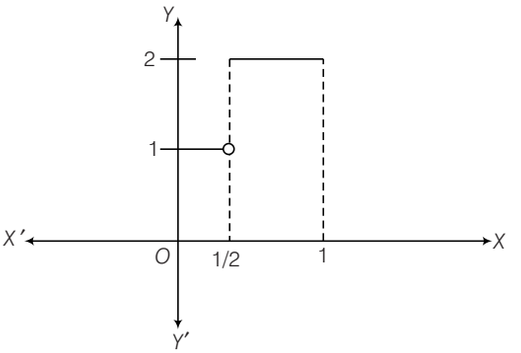
$$f'\left(\frac{1}{2}^{+}\right)\neq f'\left(\frac{1}{2}^{-}\right)$$

$\Rightarrow f(x)$ is not differentiable at $x=\frac{1}{2}$.

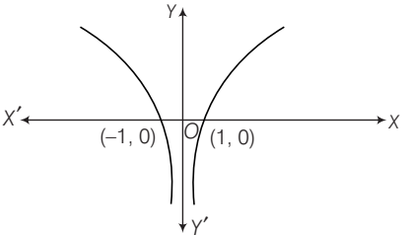
(q) $f(x)=|x|$ is not differential at $x=0$.



(r) $$f(x)=\begin{cases}1, & 0\le x<\frac{1}{2}\\ 2, & \frac{1}{2}\le x<1\end{cases}$$



$\Rightarrow f(x)$ is discontinuous at $x=\frac{1}{2}$.



$f(x)=\log|x|$ is discontinuous at $x=0$.

(t) $f(x)=x^2+x+1$

$\Rightarrow\quad f(-1)=1$

and $\quad f(1)=3$

$\Rightarrow\quad f(-1)\neq f(1)$

JEE Type Solved Examples :
Single Integer Answer Type Questions

● ***Ex. 26*** *Suppose that $f(0)=-3$ and $f'(x)\le 5$ for all values of x. Then, the largest value which $f(2)$ can assume is ..... .*

Sol. (7) Using LMVT in [0, 2]

$$\frac{f(2)-f(0)}{2-0}=f'(c)\qquad [\text{where } c\in(0,2)]$$

$$\Rightarrow\quad \frac{f(2)+3}{2}\le 5$$

$$\Rightarrow\quad f(2)\le 7$$

● ***Ex. 27*** *Let C be the curve $y=x^3$ (where x assumes all real values). The tangent at A meets the curve again at B. If the gradient at B is k times the gradient at A, then k is equal to ..... .*

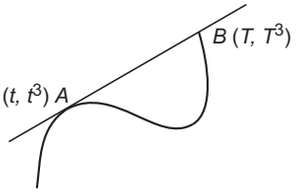
Sol. (4) $\quad \frac{dy}{dx}=3x^2=3t^2$ at 'A'

$$\therefore\quad 3t^2=\frac{T^3-t^3}{T-t}=T^2+Tt+t^2$$

$$\Rightarrow\quad T^2+Tt-2t^2=0$$

$$\Rightarrow\quad (T-t)(T+2t)=0$$

$$\Rightarrow\quad T=t \text{ or } T=-2t\qquad [T=t \text{ is not possible}]$$



Now, $\quad m_A=\frac{t^3}{t}=t^2;\ m_B=T^2$

$$\Rightarrow \quad \frac{m_B}{m_A} = \frac{T^2}{t^2} = \frac{4t^2}{t^2} \qquad [\text{using } T = -2t]$$

$$\therefore \quad k = 4$$

● ***Ex. 28*** *Consider the two graphs $y = 2x$ and $x^2 - xy + 2y^2 = 28$. The absolute value of the tangent of the angle between the two curves at the points where they meet, is ..... .*

Sol. (2) $y = 2x,\ x^2 - xy + 2y^2 = 28$

Solving the point of intersection are $(2, 4)$ and $(-2, -4)$.

For 1st curve, $\quad \frac{dy}{dx} = 2 = m_1 \qquad ...(i)$

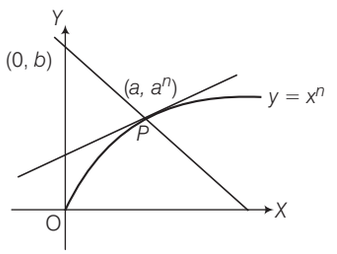
For 2nd curve, $\quad \frac{dy}{dx} = \frac{y - 2x}{4y - x} \Rightarrow m_2 = 0 \qquad ...(ii)$

$$\therefore \quad \tan\theta = 2$$

● ***Ex. 29*** *At the point $P(a, a^n)$ on the graph of $y = x^n$ $(n \in N)$ in the first quadrant a normal is drawn. The normal intersects the Y-axis at the point (0, b). If $\lim_{a \to 0} b = \frac{1}{2}$, then n equals ..... .*

Sol. (2) $y = x^n \Rightarrow \frac{dy}{dx} = n\,x^{n-1} = na^{n-1}$ at $p(a, a^n)$

Slope of normal $= -\frac{1}{na^{n-1}}$



Equation of normal is $y - a^n = -\frac{1}{na^{n-1}}(x - a)$

Put $x = 0$ to get y-intercept , then $y = a^n + \frac{1}{na^{n-2}}$

Hence, $\quad b = a^n + \frac{1}{na^{n-2}}$

Now, $\quad \lim_{a \to 0} b = \begin{cases} 0, & \text{if } n < 2 \\ \frac{1}{2}, & \text{if } n = 2 \\ \infty, & \text{if } n > 2 \end{cases} \Rightarrow \lim_{a \to 0} b = \frac{1}{2}, \text{ if } n = 2$

● ***Ex. 30*** *The maximum possible integral value of $\frac{\beta - \alpha}{\tan^{-1}\beta - \tan^{-1}\alpha}$, where $0 < \alpha < \beta < \sqrt{3}$ is*

Sol. (3) Let $f(x) = \tan^{-1}x,\ 0 < \alpha < \beta < \sqrt{3}$.

Then, by using LMVT, $\quad \frac{\tan^{-1}\beta - \tan^{-1}\alpha}{\beta - \alpha} = \frac{1}{1 + c^2}$,

where $0 < \alpha < c < \beta < \sqrt{3}$

So, $\quad \frac{1}{4} < \frac{1}{1 + c^2} < \frac{1}{1}$

$$\Rightarrow \quad \frac{1}{4} < \frac{\tan^{-1}\beta - \tan^{-1}\alpha}{\beta - \alpha} < 1$$

or $\quad 1 < \frac{\beta - \alpha}{\tan^{-1}\beta - \tan^{-1}\alpha} < 4$

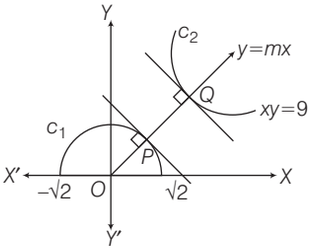
$\therefore$ Maximum possible integral value is 3.

● ***Ex. 31*** *Let P be a point on the curve $c_1 : y = \sqrt{2 - x^2}$ and Q be a point on the curve $c_2 : xy = 9$, both P and Q be in the first quadrant. If d denotes the minimum distance between P and Q, then d^2 is, ...*

Sol. (8) Here, c_1 is a semi-circle and c_2 is a rectangular hyperbola. PQ will be minimum if the normal at P on semi-circle is also a normal at Q on $xy = 9$.

Let the normal at P be $y = mx\ (m > 0)$. ...(i)

Solving with $xy = 9$



$$mx^2 = 9 \Rightarrow x = 3/\sqrt{m}$$

$$\Rightarrow \quad y = 9\sqrt{m}/3$$

$$\therefore \quad Q(3/\sqrt{m}, 3\sqrt{m})$$

Differentiating $xy = 9 \Rightarrow x\,dy/dx + y = 0$

$$\Rightarrow \quad dy/dx = -y/x$$

$$\therefore \quad |dy/dx|_Q = -\frac{3\sqrt{m}\sqrt{m}}{3} = -m$$

Now, tangent at P and Q must be parallel.

$$\therefore \quad -m = -1/m \Rightarrow m^2 = 1 \Rightarrow m = 1$$

$\therefore$ Normal at P and Q is $y = x$

Now, we get $P(1, 1)$ and $Q(3, 3)$.

$$\therefore \quad (PQ)^2 = d^2 = 4 + 4 = 8$$

Subjective Type Questions

● **Ex. 32** *If the line* $x\cos\alpha + y\sin\alpha = P$ *is the normal to the curve* $(x+a)y = c^2$*, then show that*

$$\alpha \in \left(2n\pi + \frac{\pi}{2}, (2n+1)\pi\right) \cup \left(2n\pi + \frac{3\pi}{2}, (2n+2)\pi\right).$$

Sol. Here, $y = \dfrac{c^2}{x+a} \Rightarrow \dfrac{dy}{dx} = -\dfrac{c^2}{(x+a)^2}$

Slope of normal is $\dfrac{(x+a)^2}{c^2} > 0$ for all x.

$\because$ $x\cos\alpha + y\sin\alpha = P$ is normal

$\therefore$ $-\dfrac{\cos\alpha}{\sin\alpha} > 0 \Rightarrow \cot\alpha < 0$

i.e. α lies in II or IV quadrant.

So, $\alpha \in \left(2n\pi + \dfrac{\pi}{2}, (2n+1)\pi\right) \cup \left(2n\pi + \dfrac{3\pi}{2}, (2n+2)\pi\right)$

● **Ex. 33** *Find the total number of parallel tangents of* $f_1(x) = x^2 - x + 1$ *and* $f_2(x) = x^3 - x^2 - 2x + 1$.

Sol. Here, $f_1(x) = x^2 - x + 1$

and $f_2(x) = x^3 - x^2 - 2x + 1$

$\Rightarrow$ $f_1'(x_1) = 2x_1 - 1$

and $f_2'(x_2) = 3x_2^2 - 2x_2 - 2$

Let the tangents drawn to the curves $y = f_1(x)$ and $y = f_2(x)$ at $(x_1, f_1(x_1))$ and $(x_2, f_2(x_2))$ respectively are parallel.

$\therefore$ $2x_1 - 1 = 3x_2^2 - 2x_2 - 2$ or $2x_1 = 3x_2^2 - 2x_2 - 1$

Which is possible for infinite numbers of ordered pairs.

$\therefore$ It will have infinite number of parallel tangents.

● **Ex. 34** *Find the point on the curve* $3x^2 - 4y^2 = 72$*, which is nearest to the line* $3x + 2y + 1 = 0$.

Sol. Slope of the given line $3x + 2y + 1 = 0$ is $(-3/2)$.

Let us locate the point (x_1, y_1) on the curve at which the tangent is parallel to the given line.

Differentiating the curve both sides with respect to x, we get

$$6x - 8y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{3x_1}{4y_1} = \frac{-3}{2} \qquad [\because \text{parallel to } 3x + 2y = 1]$$

Also, the point (x_1, y_1) lies on $3x^2 - 4y^2 = 72$

$\therefore$ $3x_1^2 - 4y_1^2 = 72$

$$\Rightarrow \quad 3\frac{x_1^2}{y_1^2} - 4 = \frac{72}{y_1^2} \Rightarrow 3(4) - 4 = \frac{72}{y_1^2} \quad \left[\text{as } \frac{x_1}{y_1} = -2\right]$$

$$\Rightarrow \quad y_1^2 = 9 \Rightarrow y_1 = \pm 3$$

Required points are $(-6, 3)$ and $(6, -3)$.

Distance of $(-6, 3)$ from the given line

$$= \frac{|-18 + 6 + 1|}{\sqrt{13}} = \frac{11}{\sqrt{13}}$$

and distance of $(6, -3)$ from the given line

$$= \frac{|18 - 6 + 1|}{\sqrt{13}} = \frac{13}{\sqrt{13}} = \sqrt{13}$$

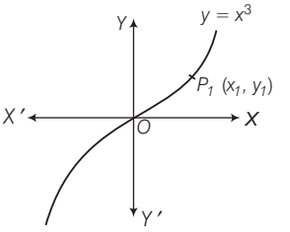
Thus, $(-6, 3)$ is the required point.

● **Ex. 35** *Tangent at a point* P_1 *(other than* $(0, 0)$*) on the curve* $y = x^3$ *meets the curve again at* P_2*. The tangent at* P_2 *meets the curve at* P_3 *and so on, show that the abscissa of* $P_1, P_2, P_3, \ldots, P_n$*, form a GP. Also, find the ratio of* $\dfrac{ar(\Delta P_1P_2P_3)}{ar(\Delta P_2P_3P_4)}$.

Sol. Let $P_1(x_1, y_1)$ be a point on the curve.

$$y = x^3 \qquad \ldots(i)$$

Then, we have $y_1 = x_1^3$ ...(ii)



Now, $\dfrac{dy}{dx} = 3x^2$

$\therefore$ Slope of the tangent at $P_1 = m_1 = 3x_1^2$

$\therefore$ Equation of the tangent at $P_1(x_1, y_1)$ is

$y - x_1^3 = 3x_1^2(x - x_1)$ or $y = 3x_1^2x - 2x_1^3$ ...(iii)

Solving Eqs. (i) and (iii), we get

$$x^3 - 3x_1^2x + 2x_1^3 = 0$$

i.e. $(x - x_1)\cdot(x^2 + xx_1 - 2x_1^2) = 0$

i.e. $(x - x_1)(x - x_1)(x + 2x_1) = 0$

$\therefore$ $x = x_1$ (neglecting) or $x = -2x_1$

$\therefore$ $x_2 = -2x_1, y_2 = x_2^3 = -8x_1^3$

$\therefore$ $P_2(x_2, y_2) = (-2x_1, -8x_1^3)$

Now, we find P_3, the point where the tangent, at P_2 meets the curve.

Slope of the tangent at $P_2 = \left(\dfrac{dy}{dx}\right)_{(x_2, y_2)}$

$$= 3x_2^2 = 3\cdot 4x_1^2 = 12x_1^2$$

$\therefore$ Equation of tangent at P_2 is,

$$y - x_2^3 = 3x_2^2(x - x_2) \qquad \text{...(iv)}$$

To get, $P_3 = (x_3, y_3)$, solve Eqs. (i) and (iv), we get

$P_3 = (x_3, y_3) = (-2x_2, -8x_2^3) = (4x_1, 64x_1^3)$ and so on.

$\therefore$ Abscissa of $P_1, P_2, P_3, \ldots$ are given by $x_1, -2x_1, 4x_1, -8x_1, \ldots$, which forms an GP with common ratio $= -2$.

Now, $$\text{a}\,(\Delta P_1P_2P_3) = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\Rightarrow \quad \text{ar}\,(\Delta P_1P_2P_3) = \frac{1}{2}\begin{vmatrix} x_1 & x_1^3 & 1 \\ -2x_1 & -8x_1^3 & 1 \\ 4x_1 & 64x_1^3 & 1 \end{vmatrix}$$

$$\Rightarrow \quad \text{ar}\,(\Delta P_1P_2P_3) = \frac{x_1^4}{2}\begin{vmatrix} 1 & 1 & 1 \\ -2 & -8 & 1 \\ 4 & 64 & 1 \end{vmatrix}$$

Similarly, $$\text{ar}\,(\Delta\, P_2P_3P_4) = \frac{1}{2}\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}$$

$$\Rightarrow \quad \text{ar}\,(\Delta\, P_2P_3P_4) = \frac{1}{2}\begin{vmatrix} -2x_1 & -8x_1^3 & 1 \\ 4x_1 & 64x_1^3 & 1 \\ -8x_1 & -512x_1^3 & 1 \end{vmatrix}$$

$$\Rightarrow \quad \text{ar}\,(\Delta\, P_2P_3P_4) = 8x_1^4\begin{vmatrix} 1 & 1 & 1 \\ -2 & -8 & 1 \\ 4 & 64 & 1 \end{vmatrix}$$

$$\therefore \quad \frac{\text{ar}\,(\Delta\, P_1P_2P_3)}{\text{ar}\,(\Delta\, P_2P_3P_4)} = \frac{1}{16}$$

● ***Ex. 36*** *Determine all polynomial $P(x)$ with rational coefficient so that for all x with $|x| \le 1$;*

$$P(x) = P\left(\frac{-x + \sqrt{3 - 3x^2}}{2}\right).$$

Sol. Here, $P(x) = P\left(\dfrac{-x + \sqrt{3 - 3x^2}}{2}\right)$ for $|x| \le 1$ ...(i)

Let $x = 0 \Rightarrow P(0) = P(\sqrt{3}/2)$

Which shows, $P(x) - P(0)$ is divisible by $x\left(x - \dfrac{\sqrt{3}}{2}\right)$.

Since, $P(x) - P(0)$ has rational coefficients and $\dfrac{\sqrt{3}}{2}$ is one of the roots.

$\therefore$ $-\dfrac{\sqrt{3}}{2}$ is also a root of $P(x) - P(0)$.

Thus, $x\left(x - \dfrac{\sqrt{3}}{2}\right)\left(x + \dfrac{\sqrt{3}}{2}\right) = x^3 - \dfrac{3}{4}x = \dfrac{4x^3 - 3x}{4}$ is factor of $P(x) - P(0)$.

$\therefore$ $P(x) = P(0) + (3x - 4x^3)\,P_1(x)$ for $|x| \le 1$ ...(ii)

As, $P(x) = P\left(\dfrac{-x + \sqrt{3 - 3x^2}}{2}\right)$

$$\Rightarrow \quad 3\left(\frac{-x + \sqrt{3 - 3x^2}}{2}\right) - 4\left(\frac{-x + \sqrt{3 - 3x^2}}{2}\right) = 3x - 4x^3$$

$$\therefore \quad P_1(x) = P_1\left(\frac{-x + \sqrt{3 - 3x^2}}{2}\right)$$

$\therefore$ $P_1(x) = P_1(0) + (3x - 4x^3)\,P_2(x)$ [using Eq. (ii)]

$\Rightarrow$ $P(x) = P(0) + (3x - 4x^3)\,(P_1(0)) + (3x - 4x^3)P_2(x))$

$= P(0) + (3x - 4x^3)\,P_1(0) + (3x - 4x^3)^2\,P_2(x)$

Thus, in general,

$$P(x) = a_0 + a_1(3x - 4x^3) + a_2\,(3x - 4x^3)^2 + \ldots + (3x - 4x^3)^k \cdot k\,(x)$$

Where $k\,(x)$ is a polynomial with rational coefficient.

● ***Ex. 37*** *Let $f(x) = (x - a)\,(x - b)\,(x - c), a < b < c$. Show that $f'(x) = 0$ has two roots one belonging to (a, b) and other belonging to (b, c).*

Sol. Here, $f(x)$ being a polynomial, is continuous and differentiable for all real values of x.

We also have, $f(a) = f(b) = f(c)$

If we apply Rolle's theorem to $f(x)$ in $[a, b]$ and $[b, c]$ we observe that $f'(x) = 0$ would have atleast one root in (a, b) and atleast one root in (b, c).

But $f'(x) = 0$ is a polynomial of degree two, hence $f'(x) = 0$ cannot have more than two roots.

It implies that exactly one root of $f'(x) = 0$ would lie in (a, b) and exactly one root of $f'(x) = 0$ would lie in (b, c).

● ***Ex. 38*** *If at each point of the curve $y = x^3 - ax^2 + x + 1$, the tangents is inclined at an acute angle with the positive direction of the X-axis, then find the interval of a.*

Sol. As, $y = x^3 - ax^2 + x + 1$ and the tangent is inclined at an acute angle with the positive direction of X-axis.

$$\therefore \quad \frac{dy}{dx} \ge 0 \;\Rightarrow\; 3x^2 - 2ax + 1 \ge 0, \text{ for all } x \in R$$

[we know, $ax^2 + bx + c \ge 0$ for all $x \in R$ $\Rightarrow a > 0$ and $D \le 0$]

$\Rightarrow$ $(2a)^2 - 4(3)(1) \le 0 \;\Rightarrow\; 4\,(a^2 - 3) \le 0$

$\Rightarrow$ $(a - \sqrt{3})(a + \sqrt{3}) \le 0 \quad \therefore \; -\sqrt{3} \le a \le \sqrt{3}$

● ***Ex. 39*** *Show that there is no cubic curve for which the tangent lines at two distinct points coincide.*

Sol. Suppose $y \equiv ax^3 + bx^2 + cx + d = 0\,(a \ne 0)$ be a cubic curve.

We assume that (x_1, y_1) and $(x_2, y_2), (x_1 < x_2)$ are two distinct points on the curve at which tangents coincide.

Then, by mean value theorem, there exists $x_3\,(x_1 < x_3 < x_2)$ such that

$$\frac{y_2 - y_1}{x_2 - x_1} = y'(x_3)$$

Since, tangent x_1, x_2, x_3 are solutions of equation

$$3ax^2 + 2bx + c = M$$

But, it is a quadratic and thus cannot have more than two roots. Therefore, no such cubic is possible.

● ***Ex. 40*** *The tangent at a point P to the rectangular hyperbola $xy = c^2$ meets the lines $x - y = 0, x + y = 0$ at Q and R, Δ_1 is the area of the Δ OQR, where O is the origin. The normal at P meets the X-axis at M and Y-axis at N. If Δ_2 is the area of the ΔOMN, show that Δ_2 varies inversely as the square of Δ_1.*

Sol. Tangent at $P(x_1, y_1)$ is $xy_1 + yx_1 = 2c^2$

The point of intersection of this line with $x - y = 0$ is given by,

$$x(x_1 + y_1) = 2c^2, \text{ i.e. } x = 2c^2/(x_1 + y_1)$$

$$\therefore \quad Q \text{ is} \left(\frac{2c^2}{x_1 + y_1}, \frac{2c^2}{x_1 + y_1}\right)$$

The point of intersection of the tangent with $x + y = 0$ is given by,

$$x(y_1 - x_1) = 2c^2, \quad x = \frac{2c^2}{y_1 - x_1}$$

$$\therefore \quad R \text{ is} \left(\frac{2c^2}{y_1 - x_1}, \frac{2c^2}{x_1 - y_1}\right)$$

Now, $\Delta_1 = \frac{1}{2}\{a_1b_2 - a_2b_1\} = \frac{1}{2}\{-a_1a_2 - a_2a_1\} = -a_1a_2$

$$\Delta_1 = \frac{4c^4}{x_1^2 - y_1^2}$$

The normal at (x_1, y_1) is $y - y_1 = \frac{x_1}{y_1}(x - x_1)$

Intersection with $y = 0$ is given by, $x - x_1 = -\frac{y_1^2}{x_1}$

$$\Rightarrow \quad x = \frac{x_1^2 - y_1^2}{x_1} \qquad \therefore \text{ M is} \left(\frac{x_1^2 - y_2^2}{x_1}, 0\right)$$

intersection with $x = 0$ is given by $\quad y - y_1 = -\frac{x_1^2}{y_1}$

$$\Rightarrow \quad y = \frac{y_1^2 - x_1^2}{y_1}. \qquad \therefore N \text{ is} \left(0, \frac{y_1^2 - x_1^2}{y_1}\right)$$

Now, $\Delta_2 = \frac{1}{2}\frac{(x_1^2 - y_1^2)}{x_1} \cdot \frac{(y_1^2 - x_1^2)}{y_1}$

$$= -\frac{1}{2}\frac{(x_1^2 - y_1^2)^2}{x_1y_1} = -\frac{1}{2}\frac{(x_1^2 - y_1^2)^2}{c^2}$$

$$\therefore \quad \Delta_1^2 \Delta_2 = \frac{16c^8}{(x_1^2 - y_1^2)^2}\left(-\frac{1}{2}\right)\frac{(x_1^2 - y_1^2)^2}{c^2} = -8c^6 = \text{constant}$$

$$\Rightarrow \quad \Delta_2 \propto \frac{1}{\Delta_1^2}$$

or Δ_2 varies inversely as the square of Δ_1.

● ***Ex. 41*** *If the function of $f : [0, 4] \to R$ is differentiable, then show that*

(i) $(f(4))^2 - (f(0))^2 = 8 f'(a) f(b)$ for some $a, b \in (0, 4)$.

(ii) $\int_0^4 f(t)\, dt = 2\{\alpha f(\alpha^2) + \beta f(\beta^2)\}$ for some $0 < \alpha, \beta < 2$.

[IIT JEE 2003]

Sol. (i) Here, f is differentiable $\Rightarrow f$ is also continuous.

Now, by Lagrange's mean value theorem, there exist $a \in (0, 4)$ such that

$$f'(a) = \frac{f(4) - f(0)}{4 - 0} = \frac{f(4) - f(0)}{4} \qquad \ldots(i)$$

Also, by intermediate mean value theorem, there exists $b \in (0, 4)$ such that

$$f(b) = \frac{f(4) + f(0)}{2} \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$f'(a)\, f(b) = \frac{f(4) - f(0)}{4} \times \frac{f(4) + f(0)}{2}$$

$$f'(a)\, f(b) = \frac{(f(4))^2 - (f(0))^2}{8}$$

$$\Rightarrow (f(4))^2 - (f(0))^2 = 8 f'(a) f(b) \text{ for some, } a, b \in (0, 4)$$

(ii) Putting $t = z^2$, we have

$$\int_0^4 f(t)\, dt = \int_0^2 2z\, f(z^2)\, dz$$

Consider the function $\phi(t)$ given by,

$$\phi(t) = \int_0^t 2z\, f(z^2)\, dz;\ t \in [0, 2]$$

Clearly, $\phi(t)$ being an integral function of a continuous function, is continuous and differentiable on $[0, 2]$.

$\therefore$ By Lagrange's mean value theorem, there exists $c \in (0, 2)$ such that $\frac{\phi(2) - \phi(0)}{2 - 0} = \phi'(c)$

$$\Rightarrow \frac{\int_0^2 2z\, f(z^2)\, dz - \int_0^0 2z\, f(z^2)\, dz}{2 - 0} = 2cf(c^2)$$

[using $\phi'(t) = 2t\, f(t^2)$]

$$\Rightarrow \quad \int_0^2 2z\, f(z^2)\, dz = 4cf(c^2) \qquad \ldots(i)$$

Also, by intermediate mean value theorem for $c \in (0, 2)$, there exist $\alpha, \beta \in (0, 2)$ such that

$$\frac{\phi'(\alpha) + \phi'(\beta)}{2} = \phi'(c), \text{ where } 0 < \alpha < c < \beta < 2$$

$$\Rightarrow \quad 2\alpha f(\alpha^2) + 2\beta\, f(\beta^2) = 2\{2c\, f(c^2)\} \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$\int_0^2 2z\, f(z^2)\, dz = 2\alpha\, f(\alpha^2) + 2\beta\, f(\beta^2),$$

for some $0 < \alpha, \beta < 2$

$$\Rightarrow \int_0^4 f(t)\, dt = 2(\alpha\, f(\alpha^2) + \beta\, f(\beta^2)),$$

for some $0 < \alpha, \beta < 2$

dy/dx as Rate Measurer and Tangents, Normals Exercise 1 : Single Option Correct Type Questions

1. Consider the cubic equation $f(x) = x^3 - nx + 1 = 0$ where $n \geq 3, n \in N$, then $f(x) = 0$ has
(a) at least one root in $(0, 1)$ (b) at least one root in $(1, 2)$
(c) at least one root in $(-1, 0)$ (d) data insufficient

2. If the normal to $y = f(x)$ at $(0, 0)$ is given by $y - x = 0$, then $\lim_{x \to 0} \dfrac{x^2}{f(x^2) - 20f(9x^2) + 2f(99x^2)}$ is
(a) $1/19$ (b) $-1/19$ (c) $1/2$ (d) does not exist

3. Tangent to a curve intersects the Y-axis at a point. A line perpendicular to this tangent through P passes through another point $(1, 0)$. The differential equation of the curve is
(a) $y\dfrac{dy}{dx} - x\left(\dfrac{dy}{dx}\right)^2 = 1$ (b) $x\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = 1$
(c) $y\dfrac{dy}{dx} + x = 1$ (d) None of these

4. The number of points in the rectangle $\{(x, y) : |x| \leq 9, |y| \leq 3\}$ which lie on the curve $y^2 = x + \sin x$ and at which the tangent to the curve is parallel to X-axis, is
(a) 3 (b) 2 (c) 4 (d) None of these

5. If $a, b, c, d \in R$ such that $\dfrac{a + 2c}{b + 3d} = -\dfrac{4}{3}$, then the equation $ax^3 + bx^2 + cx + d = 0$ has
(a) at least one root in $(-1, 0)$ (b) at least one root in $(0, 1)$
(c) no root in $(-1, 1)$ (d) no root in $(0, 2)$

6. If $3(a + 2c) = 4(b + 3d) \neq 0$, then the equation $ax^3 + bx^2 + cx + d = 0$ will have
(a) no real solution
(b) at least one real root in $(-1, 0)$
(c) at least one real root in $(0, 1)$
(d) None of the above

7. Let $f(x)$ be a differentiable function in the interval $(0, 2)$, then the value of $\int_0^2 f(x)\,dx$
(a) $f(c)$ where $c \in (0, 2)$ (b) $2f(c)$, where $c \in (0, 2)$
(c) $f'(c)$, where $c \in (0, 2)$ (d) None of these

8. Let $f(x)$ be a fourth differentiable function such that $f(2x^2 - 1) = 2xf(x), \forall\ x \in R$, then $f^{iv}(0)$ is equal to (where $f^{iv}(0)$ represents fourth derivative of $f(x)$ at $x = 0$)
(a) 0 (b) 1
(c) -1 (d) data insufficient

9. The curve $x + y - \ln(x + y) = 2x + 5$ has a vertical tangent at the point (α, β). Then, $\alpha + \beta$ is equal to
(a) -1 (b) 1 (c) 2 (d) -2

10. Let $y = f(x)$, $f : R \to R$ be an odd differentiable function such that $f'''(x) > 0$ and
$g(\alpha, \beta) = \sin^8 \alpha + \cos^8 \beta + 2 - 4\sin^2 \alpha \cos^2 \beta$.
If $f''(g(\alpha, \beta)) = 0$, then $\sin^2 \alpha + \sin^2 \beta$ is equal to
(a) 0 (b) 1 (c) 2 (d) 3

11. A polynomial of 6th degree $f(x)$ satisfies $f(x) = f(2 - x), \forall\ x \in R$, if $f(x) = 0$ has 4 distinct and 2 equal roots, then sum of the roots of $f(x) = 0$ is
(a) 4 (b) 5 (c) 6 (d) 7

12. Let a curve $y = f(x)$, $f(x) \geq 0, \forall\ x \in R$ has property that for every point P on the curve length of subnormal is equal to abscissa of P. If $f(1) = 3$, then $f(4)$ is equal to
(a) $-2\sqrt{6}$ (b) $2\sqrt{6}$ (c) $3\sqrt{5}$ (d) None of these

13. If a variable tangent to the curve $x^2y = c^3$ makes intercepts, a, b on X and Y-axes respectively, then the value of a^2b is
(a) $27c^3$ (b) $\dfrac{4}{27}c^3$
(c) $\dfrac{27}{4}c^3$ (d) $\dfrac{4}{9}c^3$

14. Let $f(x) = \begin{vmatrix} 1 & 1 & 1 \\ 3 - x & 5 - 3x^2 & 3x^3 - 1 \\ 2x^2 - 1 & 3x^5 - 1 & 7x^8 - 1 \end{vmatrix}$. Then, the equation $f(x) = 0$ has
(a) no real root
(b) atmost one real root
(c) at least two real roots
(d) exactly one real root in $(0, 1)$ and no other real root

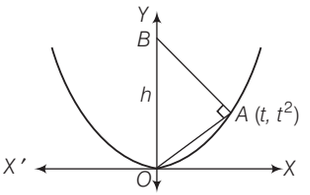
15. The graphs $y = 2x^3 - 4x + 2$ and $y = x^3 + 2x - 1$ intersect at exactly 3 distinct points. The slope of the line passing through two of these points is
(a) equal to 4 (b) equal to 6
(c) equal to 8 (d) not unique

16. In which of the following functions Rolle's theorem is applicable?
(a) $f(x) = \begin{cases} x, & 0 \leq x < 1 \\ 0, & x = 1 \end{cases}$ on $[0, 1]$
(b) $f(x) = \begin{cases} \dfrac{\sin x}{x}, & -\pi \leq x < 0 \\ 0, & x = 0 \end{cases}$ on $[-\pi, 0]$

(c) $f(x) = \dfrac{x^2 - x - 6}{x - 1}$ on $[-2, 3]$

(d) $f(x) = \begin{cases} \dfrac{x^3 - 2x^2 - 5x + 6}{x - 1}, & \text{if } x \neq 1 \\ -6, & \text{if } x = 1 \end{cases}$ on $[-2, 3]$

17. The figure shows a right triangle with its hypotenuse OB along the Y-axis and its vertex A on the parabola $y = x^2$.



Let h represents the length of the hypotenuse which depends on the x-coordinate of the point A. The value of $\lim\limits_{t \to 0} (h)$ is equal to

(a) 0 (b) 1/2 (c) 1 (d) 2

18. Number of positive integral value(s) of 'a' for which the curve $y = a^x$ intersects the line $y = x$ is

(a) 0 (b) 1 (c) 2 (d) more than 2

19. If $f(x) = 4 - \left(\dfrac{1}{2} - x\right)^{2/3}$, $g(x) = \begin{cases} \dfrac{\tan [x]}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

$h(x) = \{x\}$, $k(x) = 5^{\log_2 (x+3)}$, where $[x]$ and $\{x\}$ denote the greatest integer and fraction part function, then in $[0, 1]$, Lagrange's mean value theorem is not applicable to

(a) f, g, h (b) h, k (c) f, g (d) g, h, k

20. If the function $f(x) = x^4 + bx^2 + 8x + 1$ has a horizontal tangent and a point of inflection for the same value of x, then the value of b is equal to

(a) -1 (b) 1 (c) 6 (d) -6

21. Coffee is coming out from a conical filter, with height and diameter both are 15 cm into a cylindrical coffee pot with a diameter 15 cm. The rate at which coffee comes out from the filter into the pot is 100 cu cm/min.

The rate (in cm/min) at which the level in the pot is rising at the instance when the coffee in the pot is 10 cm, is

(a) $\dfrac{9}{16\pi}$ (b) $\dfrac{25}{9\pi}$ (c) $\dfrac{5}{3\pi}$ (d) $\dfrac{16}{9\pi}$

22. A horse runs along a circle with a speed of 20 km/h. A lantern is at the centre of the circle. A fence is there along the tangent to the circle at the point at which the horse starts. The speed with which the shadow of the horse moves along the fence at the moment when it covers 1/8 of the circle (in km/h) is

(a) 20 (b) 40 (c) 30 (d) 60

23. Water runs into an inverted conical tent at the rate of 20 cu ft/min and leaks out at the rate of 5 cu ft/min. The height of the water is three times the radius of the water's surface. The radius of the water surface is increasing when the radius is 5 ft, is

(a) $\dfrac{1}{5\pi}$ ft/min (b) $\dfrac{1}{10\pi}$ ft/min (c) $\dfrac{1}{15\pi}$ ft/min (d) None of these

24. Let $f(x) = x^3 - 3x^2 + 2x$. If the equation $f(x) = k$ has exactly one positive and one negative solution, then the value of k equals

(a) $-\dfrac{2\sqrt{3}}{9}$ (b) $-\dfrac{2}{9}$

(c) $\dfrac{2}{3\sqrt{3}}$ (d) $\dfrac{1}{3\sqrt{3}}$

25. The x-intercept of the tangent at any arbitrary point of the curve $\dfrac{a}{x^2} + \dfrac{b}{y^2} = 1$ is proportional to

(a) square of the abscissa of the point of tangency
(b) square root of the abscissa of the point of tangency
(c) cube of the abscissa of the point of tangency
(d) cube root of the abscissa of the point of tangency

26. If $f(x)$ is continuous and differentiable over $[-2, 5]$ and $-4 \leq f'(x) \leq 3$ for all x in $(-2, 5)$, then the greatest possible value of $f(5) - f(-2)$ is

(a) 7 (b) 9
(c) 15 (d) 21

27. A curve is represented parametrically by the equations $x = t + e^{at}$ and $y = -t + e^{at}$, where $t \in R$ and $a > 0$. If the curve touches the axis of x at the point A, then the coordinates of the point A are

(a) (1, 0) (b) $(1/e, 0)$
(c) $(e, 0)$ (d) $(2e, 0)$

28. At any two points of the curve represented parametrically by $x = a(2\cos t - \cos 2t)$, $y = a(2\sin t - \sin 2t)$, the tangents are parallel to the axis of x corresponding to the values of the parameter t differing from each other by

(a) $2\pi / 3$ (b) $3\pi / 4$ (c) $\pi / 2$ (d) $\pi / 3$

29. Let $F(x) = \int_{\sin x}^{\cos x} e^{(1 + \sin^{-1}(t))^2}\, dt$ on $\left[0, \dfrac{\pi}{2}\right]$, then

(a) $F''(c) = 0$ for all $c \in \left(0, \dfrac{\pi}{2}\right)$

(b) $F''(c) = 0$ for some $c \in \left(0, \dfrac{\pi}{2}\right)$

(c) $F'(c) = 0$ for some $c \in \left(0, \dfrac{\pi}{2}\right)$

(d) $F(c) \neq 0$ for all $c \in \left(0, \dfrac{\pi}{2}\right)$

30. If $f'(1) = 1$ and $\frac{d}{dx}(f(2x)) = f'(x), \forall\ x > 0$. If $f'(x)$ is differentiable, then there exists a number $c \in (2, 4)$ such that $f''(c)$ is equal to
(a) $-1/4$
(b) $-1/8$
(c) $1/4$
(d) $1/8$

31. Let $f(x)$ and $g(x)$ be two functions which are defined and differentiable for all $x \geq x_0$. If $f(x_0) = g(x_0)$ and $f'(x) > g'(x)$ for all $x > x_0$, then
(a) $f(x) < g(x)$ for some $x > x_0$
(b) $f(x) = g(x)$ for some $x > x_0$
(c) $f(x) > g(x)$ only for some $x > x_0$
(d) $f(x) > g(x)$ for all $x > x_0$

32. The range of values of m for which the line $y = mx$ and the curve $y = \frac{x}{x^2 + 1}$ enclose a region, is
(a) $(-1, 1)$ (b) $(0, 1)$
(c) $[0, 1]$ (d) $(1, \infty)$

33. Let S be a square with sides of length x. If we approximate the change in size of the area of S by $h \cdot \left.\frac{dA}{dx}\right|_{x = x_0}$, when the sides are changed from x_0 to $x_0 + h$, then the absolute value of the error in our approximation, is
(a) h^2 (b) $2hx_0$ (c) x_0^2 (d) h

34. Consider $f(x) = \int_1^x \left(t + \frac{1}{t}\right) dt$ and $g(x) = f'(x)$ for $x \in \left[\frac{1}{2}, 3\right]$. If P is a point on the curve $y = g(x)$ such that the tangent to this curve at P is parallel to a chord joining the points $\left(\frac{1}{2}, g\left(\frac{1}{2}\right)\right)$ and $(3, g(3))$ of the curve, then the coordinates of the point P are
(a) can't be found out (b) $\left(\frac{7}{4}, \frac{65}{28}\right)$
(c) $(1, 2)$ (d) $\left(\sqrt{\frac{3}{2}}, \frac{5}{\sqrt{6}}\right)$

dy/dx as Rate Measurer & Tangents, Normals Exercise 2 : More than One Option Correct Type Questions

35. For the curve represented parametrically by the equation, $x = 2\log(\cot t) + 1$ and $y = \tan t + \cot t$, then
(a) tangent at $t = \frac{\pi}{4}$ is parallel to X-axis
(b) normal at $t = \frac{\pi}{4}$ is parallel to Y-axis
(c) tangent at $t = \frac{\pi}{4}$ is parallel to $y = x$
(d) normal at $t = \frac{\pi}{4}$ is parallel to $y = x$

36. Consider the curve $f(x) = x^{1/3}$, then
(a) the equation of tangent at $(0, 0)$ is $x = 0$
(b) the equation of normal at $(0, 0)$ is $y = 0$
(c) normal to the curve does not exist at $(0, 0)$
(d) $f(x)$ and its inverse meet at exactly 3 points

37. The angle at which the curve $y = ke^{kx}$ intersects Y-axis is
(a) $\tan^{-1}(k^2)$ (b) $\cot^{-1}(k^2)$
(c) $\sin^{-1}\left(\frac{1}{\sqrt{1 + k^4}}\right)$ (d) $\sec^{-1}(\sqrt{1 + k^4})$

38. Let $f(x) = 8x^3 - 6x^2 - 2x + 1$, then
(a) $f(x) = 0$ has no root in $(0, 1)$
(b) $f(x) = 0$ has at least one root in $(0, 1)$
(c) $f'(c)$ vanishes for some $c \in (0, 1)$
(d) None of the above

39. If $f(0) = f(1) = f(2) = 0$ and function $f(x)$ is twice differentiable in $(0, 2)$ and continuous in $[0, 2]$, then which of the following is/are definitely true?
(a) $f''(c) = 0; \forall\ c \in (0, 2)$
(b) $f'(c) = 0$; for at least two $c \in (0, 2)$
(c) $f'(c) = 0$; for exactly one $c \in (0, 2)$
(d) $f''(c) = 0$; for at least one $c \in (0, 2)$

40. Equation $\frac{1}{(x + 1)^3} - 3x + \sin x = 0$ has
(a) no real root
(b) two real and distinct roots
(c) exactly one negative root
(d) exactly one root between -1 and ∞

41. If f is an odd continuous function in $[-1, 1]$ and differentiable in $(-1, 1)$, then
(a) $f'(A) = f(1)$ for some $A \in (-1, 0)$
(b) $f'(B) = f(1)$ for some $B \in (0, 1)$
(c) $n(f(A))^{n-1} f'(A) = (f(1))^n$ for some $A \in (-1, 0), n \in N$
(d) $n(f(B))^{n-1} f'(B) = (f(1))^n$ for some $B \in (0, 1), n \in N$

42. The parabola $y = x^2 + px + q$ intersects the straight line $y = 2x - 3$ at a point with abscissa 1. If the distance between the vertex of the parabola and the X-axis is least, then
(a) $p = 0$ and $q = -2$
(b) $p = -2$ and $q = 0$

(c) least distance between the vertex of the parabola and X-axis is 2
(d) least distance between the vertex of the parabola and X-axis is 1

43. The abscissa of the point on the curve $\sqrt{xy} = a + x$, the tangent at which cuts off equal intersects from the coordinate axes, is $(a > 0)$

(a) $\frac{a}{\sqrt{2}}$ (b) $-\frac{a}{\sqrt{2}}$ (c) $a\sqrt{2}$ (d) $-a\sqrt{2}$

44. If the side of a triangle vary slightly in such a way that its circumradius remains constant, then $\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C}$ is equal to

(a) $6R$ (b) $2R$
(c) 0 (d) $2R\,(dA + dB + dC)$

45. Let $f(x)$ satisfy the requirements of Lagrange's mean value theorem in $[0, 1]$, $f(0) = 0$ and $f'(x) \le 1 - x, \forall\ x \in (0, 1)$, then

(a) $f(x) \ge x$ (b) $|f(x)| \ge 1$
(c) $f(x) \le x(1 - x)$ (d) $f(x) \le 1/4$

46. For function $f(x) = \frac{\ln x}{x}$, which of the following statements are true?

(a) $f(x)$ has horizontal tangent at $x = e$
(b) $f(x)$ cuts the X-axis only at one point
(c) $f(x)$ is many-one function
(d) $f(x)$ has one vertical tangent

47. Equation of a line which is tangent to both the curves $y = x^2 + 1$ and $y = -x^2$ is

(a) $y = \sqrt{2}\,x + \frac{1}{2}$ (b) $y = \sqrt{2}\,x - \frac{1}{2}$
(c) $y = -\sqrt{2}\,x + \frac{1}{2}$ (d) $y = -\sqrt{2}\,x - \frac{1}{2}$

48. Let $F(x) = (f(x))^2 + (f'(x))^2$, $F(0) = 6$ where $f(x)$ is thrice differentiable function such that $|f(x)| \le 1$ for all $x \in [-1, 1]$, then choose the correct statement(s).

(a) There is at least one point in each of the intervals $(-1, 0)$ and $(0, 1)$ where $|f'(x)| \le 2$
(b) There is at least one point in each of the intervals $(-1, 0)$ and $(0, 1)$ where $F(x) \le 5$
(c) There is no point of local maxima of $F(x)$ in $(-1, 1)$
(d) For some $c \in (-1, 1)$, $F(c) \ge 6$, $F'(c) = 0$ and $F''(c) \le 0$

49. If the Rolle's theorem is applicable to the function f defined by

$$f(x) = \begin{cases} ax^2 + b, & |x| \le 1 \\ 1, & |x| = 1 \\ \frac{c}{|x|}, & |x| > 1 \end{cases}$$

in the interval $[-3, 3]$, then which of the following alternative(s) is/are correct?

(a) $a + b + c = 2$ (b) $|a| + |b| + |c| = 3$
(c) $2a + 4b + 3c = 8$ (d) $4a^2 + 4b^2 + 5c^2 = 15$

dy/dx as Rate Measurer & Tangents, Normals Exercise 3 : Statements I and II Type Questions

■ **Directions** (Q. Nos. 50 to 56) *For the following questions, choose the correct answer from the options (a), (b), (c) and (d) defined as follows*

(a) Statement I is true, Statement II is also true; Statement II is the correct explanation of Statement I
(b) Statement I is true, Statement II is also true; Statement II is not the correct explanation of Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

50. **Statement I** If $g(x)$ is a differentiable function $g(1) \ne 0$, $g(-1) \ne 0$ and Rolle's theorem is not applicable to $f(x) = \frac{x^2 - 1}{g(x)}$ in $[-1, 1]$, then $g(x)$ has atleast one root in $(-1, 1)$.

Statement II If $f(a) = f(b)$, then Rolle's theorem is applicable for $x \in (a, b)$.

51. **Statement I** Shortest distance between $|x| + |y| = 2$ and $x^2 + y^2 = 16$ is $4 - \sqrt{2}$.

Statement II Shortest distance between the two smooth curves lies along the common normal.

52. **Statement I** If $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are the n real roots of a polynomial equation of nth degree with real coefficients such that sum of the roots taken $r\,(1 \le r \le n)$ at a time is positive, then all the roots are positive.

Statement II The number of times sign of coefficients change while going left to right of a polynomial equation is the number of maximum positive roots.

53. **Statement I** Tangents at two distinct points of a cubic polynomial cannot coincide.

Statement II If $P(x)$ is a polynomial of degree $n\,(n \ge 2)$, then $P'(x) + k$ cannot hold for n or more distinct values of x.

54. **Statement I** For $f(x)=\begin{cases}\frac{1}{2}-x, & x<\frac{1}{2}\\ \left(\frac{1}{2}-x\right)^2, & x\geq\frac{1}{2}\end{cases}$, mean value theorem is applicable in the interval $[0, 1]$.

Statement II For application of mean value theorem, $f(x)$ must be continuous in $[0, 1]$ and differentiable in $(0, 1)$.

55. Let $f(x)=\ln(2+x)-\frac{2x+2}{x+3}$.

Statement I The function $f(x)=0$ has a unique solution in the domain of $f(x)$.

Statement II If $f(x)$ is continuous in $[a, b]$ and is strictly monotonic in (a, b), then f has a unique root in (a, b).

56. Consider the polynomial function

$$f(x)=\frac{x^7}{7}-\frac{x^6}{6}+\frac{x^5}{5}-\frac{x^4}{4}+\frac{x^3}{3}-\frac{x^2}{2}+x$$

Statement I The equation $f(x)=0$ cannot have two or more roots.

Statement II Rolle's theorem is not applicable for $y=f(x)$ on any interval $[a, b]$, where $a, b\in R$.

dy/dx as Rate Measurer & Tangents, Normals Exercise 4 : Passage Based Questions

Passage I (Q. Nos. 57 to 59)

We say an equation $f(x)=g(x)$ is consistent, if the curves $y=f(x)$ and $y=g(x)$ touch or intersect at at least one point. If the curves $y=f(x)$ and $y=g(x)$ do not intersect or touch, then the equation $f(x)=g(x)$ is said to be inconsistent, i.e. has no solution.

57. The equation $\cos x+\cos^{-1}x=\sin x+\sin^{-1}x$ is

(a) consistent and has infinite number of solutions
(b) consistent and has finite number of solutions
(c) inconsistent
(d) None of the above

58. The equation $\sin x=x^2+x+1$ is

(a) consistent and has infinite number of solutions
(b) consistent and has finite number of solutions
(c) inconsistent
(d) consistent and has unique solution

59. Among the following equations, which is consistent in $(0, \pi/2)$?

(a) $\sin x+x^2=0$ (b) $\cos x=x$
(c) $\tan x=x$ (d) All of these

Passage II (Q. Nos. 60 to 62)

To find the point of contact $P\equiv(x_1, y_1)$ of a tangent to the graph of $y=f(x)$ passing through origin O, we equate the slope of tangent to $y=f(x)$ at P to the slope of OP. Hence, we solve the equation $f'(x_1)=\frac{f(x_1)}{x_1}$ to get x_1 and y_1.

60. The equation $|\ln mx|=px$, where m is a positive constant, has a single root for

(a) $0<p<\frac{m}{e}$ (b) $p<\frac{e}{m}$
(c) $0<p<\frac{e}{m}$ (d) $p>\frac{m}{e}$

61. The equation $|\ln mx|=px$, where m is a positive constant, has exactly two roots for

(a) $p=\frac{m}{e}$ (b) $p=\frac{e}{m}$
(c) $0<p\leq\frac{e}{m}$ (d) $0<p\leq\frac{m}{e}$

62. The equation $|\ln mx|=px$, where m is a positive constant, has exactly three roots for

(a) $p<\frac{m}{e}$ (b) $0<p<\frac{m}{e}$
(c) $0<p<\frac{e}{m}$ (d) $p<\frac{e}{m}$

Passage III (Q. Nos. 63 to 64)

Consider the family of circles: $x^2+y^2-3x-4y-c_i=0$, $c_i\in N$ $(i=1, 2, 3, \ldots, n)$.
Also, let all circles intersects X-axis at integral points only and $c_1<c_2<c_3<c_4\ldots<c_n$. A point (x, y) is said to be integral point, if both coordinates x and y are integers.

63. If circle $x^2+y^2-3x-4y-(c_2-c_1)=0$ and circle $x^2+y^2=r^2$ have only one common tangent, then

(a) $r=1/2$
(b) tangent passes through $(10, 0)$
(c) $(3, 4)$ lies outside the circle $x^2+y^2=r^2$
(d) $c_2=2r+c_1$

64. The ellipse $4x^2+9y^2=36$ and hyperbola $a^2x^2-y^2=4$ intersect orthogonally, then the equation of circle through the points of intersection of two conics is

(a) $x^2+y^2=(c_5)^2$ (b) $x^2+y^2=\frac{c_4}{7}$
(c) $x^2+y^2=c_3-2c_1$ (d) $x^2+y^2=c_7/14$

dy/dx as Rate Measurer & Tangents, Normals Exercise 5 : Matching Type Questions

65. Match the statements of Column I with values of Column II.

	Column I		Column II
(A)	The equation $x \log x = 3 - x$ has at least one root in	(p)	(0, 1)
(B)	If $27a + 9b + 3c + d = 0$, then the equation $4ax^3 + 3bx^2 + 2cx + d = 0$ has at least one root in	(q)	(1, 3)
(C)	If $c = \sqrt{3}$ and $f(x) = x + \frac{1}{x}$, then interval of x in which LMVT is applicable for $f(x)$ is	(r)	(0, 3)
(D)	If $c = \frac{1}{2}$ and $f(x) = 2x - x^2$, then the interval of x in which LMVT is applicable for $f(x)$, is	(s)	(− 1, 1)

66. Match the statements of Column I with values of Column II.

	Column I		Column II
(A)	A circular plate is expanded by heat from radius 5 cm to 5.06 cm. Approximate increase in area is	(p)	4
(B)	If an edge of a cube increases by 1%, then percentage increase in volume is	(q)	$0.6\,\pi$
(C)	If the rate of decrease of $\frac{x^2}{2} - 2x + 5$ is twice the rate of decrease of x, then x is equal to (rate of decrease is non-zero)	(r)	3
(D)	Rate of increase in area of equilateral triangle of side 15 cm, when each side is increasing at the rate of 0.1 cm/s, is	(s)	$\frac{3\sqrt{3}}{4}$

dy/dx as Rate Measurer & Tangents, Normals Exercise 6 : Single Integer Answer Type Questions

67. A point is moving along the curve $y^3 = 27x$. The interval in which the abscissa changes at slower rate than ordinate, is (a, b). Then, $(a + b)$ is ..... .

68. The slope of the curve $2y^2 = ax^2 + b$ at $(1, -1)$ is -1. Then, $(a - b)$ is ..... .

69. Let $f(1) = -2$, $f'(x) \geq 4.2$ for $1 \leq x \leq 6$. The smallest possible value of $f(6) - 16$ is ..... .

70. Let $f(x) = \begin{cases} -x^2, & \text{for } x < 0 \\ x^2 + 8, & \text{for } x \geq 0 \end{cases}$. Then, the absolute value of x-intercept of the line that is tangent to the graph of $f(x)$, is ..... .

71. The tangent to the graph of the function $y = f(x)$ at that point with abscissa, $x = a$ forms with the X- axis at an angle of $\frac{\pi}{3}$ and the point with abscissa at $x = b$ at an angle $\frac{\pi}{4}$, then the value of $\left| \int_a^b f'(x) \cdot f''(x)\, dx \right|$ is ..... .

72. Two curves $C_1 : y = x^2 - 3$ and $C_2 : y = kx^2,\ k \in R$ intersect each other at two different points. The tangent drawn to C_2 at one of the points of intersection $A(a, y_1), (a > 0)$ meets C_1 again at $B(1, y_2)$.
The value of a is ..... .

73. Consider the function $f(x) = 8x^2 - 7x + 5$ on the interval $[-6, 6]$. The value of c that satisfies the conclusion of the mean value theorem, is ..... .

74. Suppose that f is differentiable for all x and that $f'(x) \leq 2$ for all x. If $f(1) = 2$ and $f(4) = 8$, then $f(2)$ has the value equal to ..... .

75. Suppose a, b and c are positive integers with $a < b < c$ such that $1/a + 1/b + 1/c = 1$. The value of $(a + b + c - 5)$ is ..... .

Subjective Type Questions

76. Show that a tangent to an ellipse whose segment intercepted by the axes is the shortest, is divided at the point of tangency into two parts respectively, is equal to the semi-axes of the ellipse.

77. Tangents are drawn from the origin to the curve $y = \sin x$. Prove that points of contact lie on $y^2 = \dfrac{x^2}{1+x^2}$.

78. If f is a continuous function with $\int_0^x f(t)\,dt \to \infty$ as $|x| \to \infty$, then show that every line $y = mx$ intersects the curve $y^2 + \int_0^x f(t)\,dt = 2$.

79. Find the equation of the straight line which is a tangent at one point and normal at another point to the curve $y = 8t^3 - 1,\ x = 4t^2 + 3$.

80. Let a curve $y = f(x)$ passes through (1, 1), at any point P on the curve tangent and normal are drawn to intersect the X-axis at Q and R, respectively. If $QR = 2$, find the equation of all such possible curves.

81. Show that the angle between the tangent at any point P and the line joining P to the origin 'O' is the same at all points of the curve $\log(x^2 + y^2) = c\,\tan^{-1}(y/x)$, where c is constant.

82. If the equation of two curves are $y^2 = 4ax$ and $x^2 = 4ay$

(i) Find the angle of intersection of two curves.
(ii) Find the equation of common tangents to these curves.

83. A straight line intersects the three concentric circles at A, B, C. If the distance of the line from the centre of the circles is 'P', prove that the area of the triangle formed by the tangents to the circle at A, B, C is $\left(\dfrac{1}{2P}\cdot AB\cdot BC\cdot CA\right)$.

84. Find the equation of all possible curves such that length of intercept made by any tangent on X-axis is equal to the square of x-coordinate of the point of tangency. Given that the curve passes through $(2, 1)$.

85. The tangent to the curve $y = x - x^3$ at a point P meets the curve again at Q. Prove that one point of trisection of PQ lies on the Y-axis. Find the locus of the other points of trisection.

86. Determine all the curves for which the ratio of the length of the segment intercepted by any tangent on the Y-axis to the length of the radius vector is constant.

87. If t be a real number satisfying $2t^3 - 9t^2 + 30 - a = 0$, then the values of the parameter a for which the equation $x + \dfrac{1}{x} = t$ gives six real and distinct values of x.

dy/dx as Rate Measurer & Tangents, Normals Exercise 8 : Questions Asked in Previous 10 Years' Exams

(i) JEE Advanced & IIT-JEE

88. The slope of the tangent to the curve $(y - x^5)^2 = x(1 + x^2)^2$ at the point (1, 3) is **[Integer Type Question 2014]**

89. Let $f(x) = 2 + \cos x$, for all real x.

Statement I For each real t, there exists a point c in $[t, t + \pi]$, such that $f'(c) = 0$. Because
Statement II $f(t) = f(t + 2\pi)$ for each real t. **[Assertion and Reason 2007]**

(a) Statement I is correct, Statement II is also correct; Statement II is the correct explanation of Statement I
(b) Statement I is correct, Statement II is also correct; Statement II is not the correct explanation of Statement I
(c) Statement I is correct; Statement II is incorrect
(d) Statement I is incorrect; Statement II is correct

90. If $|f(x_1) - f(x_2)| \le (x_1 - x_2)^2, \forall x_1, x_2 \in R$. Find the equation of tangent to the curve $y = f(x)$ at the point (1, 2). **[Analytical Descriptive 2005]**

91. The point(s) on the curve $y^3 + 3x^2 = 12y$, where the tangent is vertical, is (are) **[One Correct Option 2002]**

(a) $\left(\pm\dfrac{4}{\sqrt{3}}, -2\right)$ (b) $\left(\pm\sqrt{\dfrac{11}{3}}, 0\right)$ (c) (0, 0) (d) $\left(\pm\dfrac{4}{\sqrt{3}}, 2\right)$

92. If the normal to the curve $y = f(x)$ at the point (3, 4) makes an angle $\frac{3\pi}{4}$ with the positive X-axis, then $f'(3)$ is equal to **[One Correct Option 2000]**

(a) −1 (b) −3/4 (c) 4/3 (d) 1

(ii) JEE Main & AIEEE

93. The normal to the curve $y(x-2)(x-3) = x+6$ at the point where the curve intersects the Y-axis passes through the point **[2017 JEE Main]**

(a) $\left(\frac{1}{2}, \frac{1}{3}\right)$ (b) $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ (c) $\left(\frac{1}{2}, \frac{1}{2}\right)$ (d) $\left(\frac{1}{2}, -\frac{1}{3}\right)$

94. The normal to the curve $x^2 + 2xy - 3y^2 = 0$ at (1,1) **[2015 JEE Main]**

(a) does not meet the curve again
(b) meets in the curve again the second quadrant
(c) meets the curve again in the third quadrant.
(d) meets the curve again in the fourth quadrant

Answers

Exercise for Session 1

1. 6 cm^3/s **2.** (1, 5/3) and (−1, 1/3)

3. 8 cm/s **4.** 750 cm^3/s **5.** $\frac{200\,\pi r^3}{(r+5)^2}$ km^2/h

6. $\frac{1}{48\,\pi}$ cm/s **7.** $-\frac{1}{3\pi}$ cm/s **8.** $\frac{1}{10\pi}$ cm/s

9. 20 m/s **10.** $\frac{-3}{10}$ rad/s **11.** (a) $\frac{1}{40\pi}$ cm/s

(b) $\frac{1}{30}$ cm^2/s (c) $\frac{\sqrt{10}}{30}$ cm^2/s

12. $\frac{0.004}{\pi}$ m/s and $\frac{0.02}{\pi}$ m/s **13.** $\left(3, \frac{16}{3}\right), \left(-3, \frac{-16}{3}\right)$

14. $\left(\frac{2a}{b}\right)$ units/s **15.** $-\frac{3}{8}$ m/s

Exercise for Session 2

1. $\frac{50}{7}$ **2.** 2.1983 **3.** 0.02%

4. $\left(\frac{1}{8}\right)\%$

Exercise for Session 3

1. (a) **2.** (c) **3.** (b) **4.** (a) **5.** (b)
6. (b) **7.** (d) **8.** (c) **9.** (b) **10.** (a)

Exercise for Session 4

1. (b) **2.** (d) **3.** (a) **4.** (b) **5.** (c)
6. (b) **7.** (c) **8.** (b) **9.** (c) **10.** (d)

Exercise for Session 5

1. (d) **2.** (a) **3.** (b) **4.** (d) **5.** (c)

Exercise for Session 6

1. Real and distinct roots if $a \in (-2,2)$.
2. (a) **3.** (a) **4.** (b)

Chapter Exercises

1. (a) **2.** (b) **3.** (a) **4.** (b) **5.** (b)
6. (b) **7.** (b) **8.** (a) **9.** (b) **10.** (b)
11. (c) **12.** (b) **13.** (c) **14.** (c) **15.** (c)
16. (d) **17.** (c) **18.** (b) **19.** (a) **20.** (d)
21. (d) **22.** (b) **23.** (a) **24.** (a) **25.** (c)
26. (d) **27.** (d) **28.** (a) **29.** (b) **30.** (b)
31. (d) **32.** (b) **33.** (a) **34.** (d)
35. (a, b) **36.** (a, b, d) **37.** (b, c) **38.** (b, c)
39. (b, d) **40.** (b, c, d)
41. (a, b, d) **42.** (b, d) 43. (a, b) **44.** (c, d)
45. (c, d) **46.** (a, b, c) **47.** (a, c)
48. (a, b, d) **49.** (a, b, c, d) **50.** (c)
51. (d) **52.** (a) **53.** (d) **54.** (d) **55.** (c)
56. (a) **57.** (b) **58.** (c) **59.** (b) **60.** (d) **61.** (a)
62. (b) **63.** (b) **64.** (d)
65. (A) → (q); (B) → (r); (C) → (q); (D) → (p)
66. (A) → (q); (B) → (r); (C) → (p); (D) → (s)
67. (0) **68.** (2) **69.** (3) **70.** (1) **71.** (1)
72. (3) **73.** (0) **74.** (4) **75.** (6)
79. $\pm\sqrt{2}(27x - 105)$
80. $\log y - x = \pm\left(\log - \left|\frac{1-\sqrt{1-y^2}}{y}\right| + \sqrt{1-y^2}\right) - 1$
82. (i) $Q = \tan^{-1}\left(\frac{3}{4}\right)$ (ii) $x + y = a = 0$
84. Possible curve are $y = \frac{x}{2(x-1)}$ or $y = \frac{3x}{2(1+x)}$.
85. $y = x - 5x^3$ **86.** $(y + \sqrt{x^2 + y^2})\, x^{k-1} = c_1$
87. No real value **88.** 8 **89.** (b) **90.** $y - 2 = 0$
91. (d) **92.** (d) **93.** (c) **94.** (d)

Solutions

1. Here, $f(x) = x^3 - nx + 1$

$f(0) = 1$ and $f(1) = 1 - n + 1 = 2 - n$

$\because \quad n \geq 3$

$\therefore \quad f(1) < 0$ and also we have, $f(0) > 0$

$\therefore$ $f(x)$ must have at least one real root in $(0, 1)$.

2. Given, slope of normal to $y = f(x)$ is 1.

$$\Rightarrow \quad \left(-\frac{1}{f'(x)}\right)_{(0,\,0)} = 1$$

$$\Rightarrow \quad f'(0) = -1 \qquad \text{...(i)}$$

Now, $\lim\limits_{x\to 0} \dfrac{x^2}{f(x^2) - 20f(9x^2) + 2f(99x^2)}$

$$= \lim_{x\to 0} \frac{2x}{f'(x^2)\cdot 2x - 20f'(9x^2)\cdot 18x + 2f'(99x^2)\cdot(198x)}$$

[using L' Hospital's rule]

$$= \lim_{x\to 0} \frac{1}{f'(x^2) - 180f'(9x^2) + f'(99x^2)\cdot(198)}$$

$$= \frac{1}{f'(0) - 180\cdot f'(0) + 198\cdot f'(0)}$$

$$= \frac{1}{-1 + 180 - 198}$$

$$= -\frac{1}{19}$$

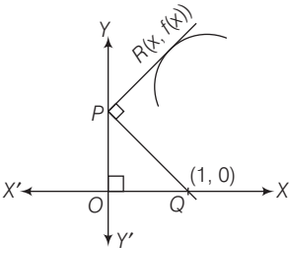
3. The equation of the tangent at the point $R(x, f(x))$ is

$$Y - f(x) = f'(x)(X - x) \qquad \text{...(i)}$$

The point of intersection on Y-axis, is $P(0, f(x) - x f'(x))$.

The slope of line perpendicular to tangent at R, is

$$m_{PQ} = \frac{(f(x) - xf'(x)) - 0}{0 - 1} \qquad \text{...(ii)}$$



$$\therefore \quad f'(x)\cdot m_{PQ} = -1$$

$$\Rightarrow \quad f'(x)\cdot\frac{(f(x) - xf'(x))}{-1} = -1$$

$$\Rightarrow \quad f(x)f'(x) - x(f'(x))^2 = 1$$

4. Here, $y^2 = x + \sin x$

$$\Rightarrow \quad 2y\,dy/dx = 1 + \cos x$$

For horizontal tangent, $dy/dx = 0$

$$\therefore \quad \cos x = -1 \Rightarrow x = (2n+1)\pi$$

Since, $y^2 = x + \sin x$ and $|y| \leq 3$

$$\Rightarrow \quad 0 \leq x \leq 9$$

$$\Rightarrow \quad 0 \leq (2n+1)\pi \leq 9$$

$$\therefore \quad n = 0 \Rightarrow x = \pi$$

$$\Rightarrow \quad y^2 = \pi \text{ or } y = \pm\sqrt{\pi}$$

Thus, points are $(\pi, \sqrt{\pi})$ and $(\pi, -\sqrt{\pi})$.

$\therefore$ Number of points is 2.

5. Here, $\dfrac{a + 2c}{b + 3d} = -\dfrac{4}{3}$

$$\Rightarrow \quad 3a + 6c = -4b - 12d$$

$$\Rightarrow \quad 3a + 4b + 6c + 12d = 0$$

or $$\frac{a}{4} + \frac{b}{3} + \frac{c}{2} + d = 0 \qquad \text{...(i)}$$

Consider, $f(x) = \dfrac{ax^4}{4} + \dfrac{bx^3}{3} + \dfrac{cx^2}{2} + dx$

Then, $f(0) = 0 = f(1)$ [using Eq. (i)] ...(ii)

$\therefore$ $f(x)$ satisfy the condition of Rolle's theorem in $[0, 1]$.

Hence, $f'(x) = 0$ has at least one solution in $(0, 1)$.

6. Let $f(x) = \dfrac{ax^4}{4} + \dfrac{bx^3}{3} + \dfrac{cx^2}{2} + dx$,

which is continuous and differentiable.

Also, $$f(0) = 0,\ f(-1) = \frac{a}{4} - \frac{b}{3} + \frac{c}{2} - d$$

$$= \frac{1}{4}(a + 2c) - \frac{1}{3}(b + 3d) = 0$$

So, according to Rolle's theorem, there exists at least one root of $f'(x) = 0$ in $(-1, 0)$.

7. Let us consider $g(t) = \int_0^t f(x)\,dx$

Applying LMVT in $(0, 2)$, we get

$$\frac{g(2) - g(0)}{2 - 0} = g'(c), \text{ where } c \in (0, 2)$$

$$\Rightarrow \quad \int_0^2 f(x)\,dx = 2f(c), \text{ where } c \in (0, 2)$$

8. Replace x by $-x$

$\Rightarrow$ $x[f(x) + f(-x)] = 0 \Rightarrow$ $f(x)$ is an odd function.

$\Rightarrow$ $f^{iv}(x)$ is also odd $\Rightarrow$ $f^{iv}(0) = 0$

9. Given, $x + y - \ln(x + y) = 2x + 5$

$$\Rightarrow \quad 1 + \frac{dy}{dx} - \frac{1}{x + y}\left(1 + \frac{dy}{dx}\right) = 2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x + y + 1}{x + y - 1}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(\alpha,\,\beta)} = \frac{\alpha + \beta + 1}{\alpha + \beta - 1} = \infty, \text{ when } \alpha + \beta = 1$$

10. Since, $f''(x)$ is odd function.

$\therefore \quad g(\alpha, \beta) = 0$

$\Rightarrow \quad (\sin^4 \alpha - 1)^2 + (\cos^4 \beta - 1)^2 + 2(\sin^2 \alpha - \cos^2 \beta)^2 = 0$

$\Rightarrow \quad \sin^2 \alpha + \sin^2 \beta = 1$

11. Let α be the root of $f(x) = 0$.

$\Rightarrow \quad f(\alpha) = f(2 - \alpha) = 0$

$\therefore \quad f(x)$ has 4 distinct and 2 equal roots.

$\therefore$ Sum of the roots $= 6$

12. Given, $y\dfrac{dy}{dx} = x$

$\Rightarrow \quad y\,dy = x\,dx \Rightarrow \; y^2 = x^2 + c$

$\because \quad f(1) = 3$

$\therefore \quad 9 = 1 + c \;\Rightarrow\; c = 8 \;\Rightarrow\; y^2 = x^2 + 8$

$\Rightarrow \quad f(x) = \sqrt{x^2 + 8}$

$\Rightarrow \quad f(4) = \sqrt{16 + 8} = 2\sqrt{6}$

13. Given, $x^2y = c^3$

$\Rightarrow \quad x^2\dfrac{dy}{dx} + 2xy = 0$

$\Rightarrow \quad \dfrac{dy}{dx} = -\dfrac{2y}{x}$

Equation of tangent at (x, y), is $Y - y = -\dfrac{2y}{x}(X - x)$

$Y = 0$, gives $X = \dfrac{3x}{2} = a$

and $X = 0$, gives $Y = 3y = b$

Now, $a^2b = \dfrac{9x^2}{4} \cdot 3y = \dfrac{27}{4}x^2y = \dfrac{27}{4}c^3$

14. Clearly, $f(0) = f(1) = 0$ and $f(x)$ is a polynomial of degree 10.

Therefore, by LMVT, we must have at least one root in $(0, 1)$.

Since, the degree of $f(x)$ is even.

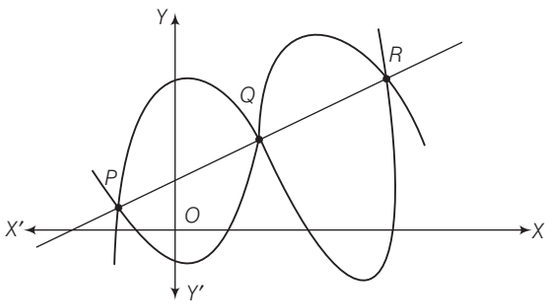
$\therefore$ It has at least two real roots.

15. Let (x_1, y_1) and (x_2, y_2) be two intersect points.

Given, $y = x^3 + 2x - 1$ and $y = 2x^3 - 4x + 2$

$\therefore \quad y_1 = 2x_1^3 - 4x_1 + 2$ ...(i)

and $2y_1 = 2x_1^3 + 4x_1 - 2$ ...(ii)



Subtracting Eq. (i) from Eq. (ii), we get

$y_1 = 8x_1 - 4$ ...(iii)

Similarly, $y_2 = 8x_2 - 4$ ...(iv)

Now, from Eqs. (iii) and (iv), we get $y_2 - y_1 = 8(x_2 - x_1)$

$\therefore \quad \dfrac{y_2 - y_1}{x_2 - x_1} = 8$

16. (a) Discontinuous at $x = 1$, therefore not applicable.

(b) Discontinuous at $x = 0$, therefore not applicable.

(c) Discontinuity at $x = 1 \;\Rightarrow$ not applicable.

(d) Note that $x^3 - 2x^2 - 5x + 6 = (x - 1)(x^2 - x - 6)$.

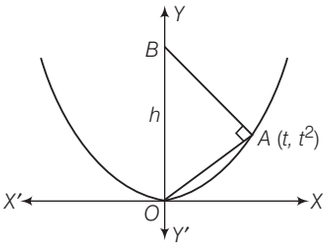
Hence, $f(x) = x^2 - x - 6$, if $x \neq 1$ and $f(1) = -6$

$\Rightarrow \; f$ is continuous at $x = 1$.

So, $f(x) = x^2 - x - 6$ is continuous throughout the interval $[-2, 3]$.

Also, note that $f(-2) = f(3) = 0$. Hence, Rolle's theorem is applicable.

17. Let $A = (t, t^2)$. Then, $m_{OA} = t, m_{AB} = -\dfrac{1}{t}$



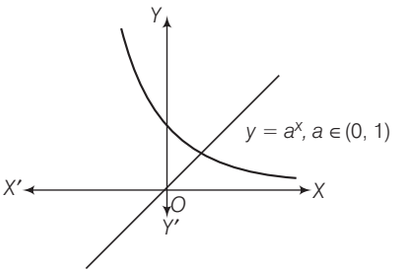
Equation of AB, $y - t^2 = -\dfrac{1}{t}(x - t)$

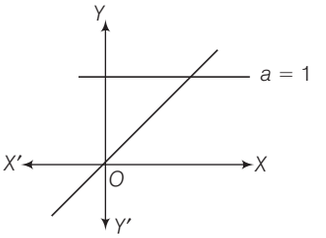
Putting $x = 0$

$\Rightarrow \quad h = t^2 + 1$

Now, $\lim\limits_{t \to 0}(h) = \lim\limits_{t \to 0}(1 + t^2) = 1$

18. For $0 < a \leq 1$, the line always intersects $y = a^x$



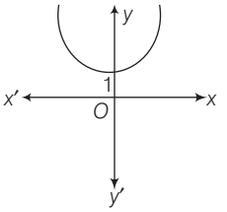


For $a > 1$, say $a = e$. Consider $f(x) = e^x - x$, $f'(x) = e^x - 1$

$f'(x) > 0$ for $x > 0$ and $f'(x) < 0$ for $x < 0$

$\therefore$ $f(x)$ is increasing $(\uparrow)$ for $x > 0$

and decreasing $(\downarrow)$ for $x < 0$



$y = e^x$ always lies above $y = x$, i.e. $e^x - x \geq 1$ for $a > 1$

Hence, the line $y = x$ intersect when $a \in (0, 1]$.

19. $\because$ f is not differentiable at $x = \frac{1}{2}$,

g is not continuous in $[0, 1]$ at $x = 0$ and 1,

and h is not continuous in $[0, 1]$ at $x = 1$

$$k(x) = (x+3)^{\ln_2 5} = (x+3)^p, \text{ where } 2 < p < 3$$

$\therefore$ None of these, f, g, h follows Lagrange's mean value theorem.

20. $f'(x) = 0$ and $f''(x) = 0$ for the same $x = x_1$ [say]

Now, $f'(x) = 4x^3 + 2bx + 8$

$$f'(x_1) = 2[2x_1^3 + bx_1 + 4] = 0 \quad \ldots(i)$$

$$f''(x_1) = 2[6x_1^2 + b] = 0 \quad \ldots(ii)$$

From Eq. (ii), $b = -6x_1^2$

Substituting this value of b in Eq. (i), we get

$$2x_1^3 + (-6x_1^3) + 4 = 0$$

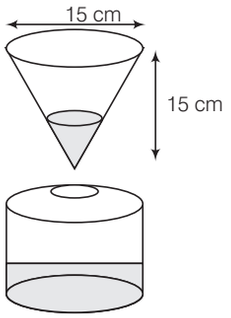
$$\Rightarrow \quad 4x_1^3 = 4$$

$$\Rightarrow \quad x_1 = 1$$

Hence, $b = -6$

21. For a cylindrical pot, $V = \pi r^2 h$

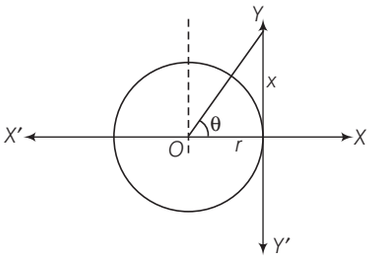
$$\Rightarrow \quad \frac{dV}{dt} = \pi\left[r^2\frac{dh}{dt} + h\cdot 2r\frac{dr}{dt}\right] \qquad \left[r = \text{constant}, \frac{dr}{dt} = 0\right]$$



$$\Rightarrow \quad 100 = \pi r^2\frac{dh}{dt} \Rightarrow 100 = \pi\cdot\frac{225}{4}\cdot\frac{dh}{dt} \qquad \left[\because\ r = \frac{15}{2}\text{ cm}\right]$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{400}{225\pi} = \frac{16}{9\pi}\text{ cm/min}$$

22. $\tan\theta = x/r \ \Rightarrow \ x = r\tan\theta$

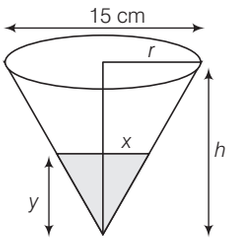


$$\Rightarrow \quad dx/dt = r\sec^2\theta\,(d\theta/dt) = r\,\omega\sec^2\theta = v\sec^2\theta$$

when, $\theta = 2\pi/8$, $dx/dt = v\sec^2(\pi/4) = 2$ $v = 40$ km/h.

23. $\frac{dV}{dt} = 15$, $h = 3r$, $V = \frac{1}{3}\pi x^2 y$, $\frac{dx}{dt} = ?$, when $x = 5$

$$\frac{x}{y} = \frac{r}{h} = \frac{1}{3} \ \Rightarrow V = \frac{2}{3}\pi r^2 3x = \pi x^3$$



$$\frac{dV}{dt} = 3\pi x^2\frac{dx}{dt}$$

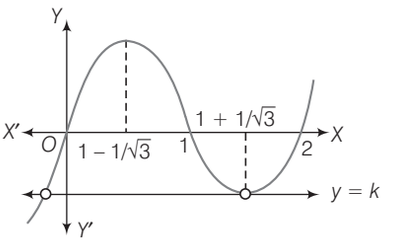
$$\Rightarrow \quad 15 = 3\pi\cdot 25\frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{1}{5\pi}$$

24. Given, $f(x) = x(x^2 - 3x + 2)$

$\Rightarrow f(x) = x(x-2)(x-1)$

Graph of $y = f(x)$ is shown as



Now, for exactly one positive and one negative solution of the equation $f(x) = K$

We should have, $k = f\left(1 + \frac{1}{\sqrt{3}}\right)$

$$\left[\because\ -\frac{1}{\sqrt{3}} \text{ and } 1 + \frac{1}{\sqrt{3}} \text{ are the roots of } f'(x) = 0\right]$$

$$\therefore \quad k = \underbrace{\left(1 + \frac{1}{\sqrt{3}}\right)}_{x}\underbrace{\left(\frac{1}{\sqrt{3}} - 1\right)}_{x-2}\underbrace{\left(\frac{1}{\sqrt{3}}\right)}_{x-1}$$

$$= \left(\frac{1}{3} - 1\right)\left(\frac{1}{\sqrt{3}}\right) = -\frac{2}{3\sqrt{3}}$$

$$= -\frac{2\sqrt{3}}{3}$$

25. We have, $\frac{a}{x^2}+\frac{b}{y^2}=1$

$$\Rightarrow \quad -\frac{2a}{x^3}-\frac{2b}{y^3}\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{ay^3}{bx^3} \quad \ldots(i)$$

Equation of tangent is $Y-y=-\frac{ay^3}{bx^3}(X-x)$

For x-intercept, put $Y=0$

$$\therefore \quad X=\frac{bx^3}{ay^2}+x$$

$$\Rightarrow \quad X=x\left[\frac{bx^2+ay^2}{ay^2}\right]=x\left[\frac{x^2y^2}{ay^2}\right]=\frac{x^3}{a}$$

[using Eq. (i)]

$\Rightarrow$ x-intercept is proportional to cube of abscissa.

26. Using LMVT in $[-2, 5]$, we get

$$-4\le\frac{f(5)-f(-2)}{7}\le 3$$

$$\Rightarrow \quad -28\le f(5)-f(-2)\le 21$$

27. Given, $x=t+e^{at}$, $y=-t+e^{at}$

$$\Rightarrow \quad \frac{dx}{dt}=1+ae^{at}, \ \frac{dy}{dt}=-1+ae^{at}, \ \frac{dy}{dx}=\frac{-1+ae^{at}}{1+ae^{at}}$$

At the point A, $y=0$ and $\frac{dy}{dx}=0$ for some $t=t_1$

$$\therefore \quad ae^{at_1}=1 \quad \ldots(i)$$

Also, $0=-t_1+e^{at_1}$

$$\therefore \quad e^{at_1}=t_1 \quad \ldots(ii)$$

Putting this value in Eq. (i), we get

$$at_1=1 \Rightarrow t_1=\frac{1}{a}$$

Now from Eq. (i), $ae=1$

$$\Rightarrow \quad a=\frac{1}{e}$$

Hence, $x_A=t_1+e^{at_1}=e+e=2e$

$$\Rightarrow \quad A\equiv(2e, 0)$$

28. Given, $x=a(2\cos t-\cos 2t)$, $y=a(2\sin t-\sin 2t)$

$$\therefore \quad \frac{dy}{dx}=\frac{\cos t-\cos 2t}{\sin 2t-\sin t}=0$$

$$\Rightarrow \quad \cos 2t=\cos t$$

$$\Rightarrow \quad \cos 2t=\cos(2\pi-t)$$

$$\Rightarrow \quad t=2\pi/3$$

29. $F'(x)=e^{(1+\sin^{-1}(\cos x))^2}\cdot(-\sin x)-e^{(1+x)^2}\cdot\cos x$

$$\therefore \quad F'(0)=0-e=-e$$

and $F'\left(\frac{\pi}{2}\right)=-e-0=-e$

Hence, Rolle's theorem is applicable for the function $F'(x)$.

So, there exists c in $\left(0, \frac{\pi}{2}\right)$ for which $F''(c)=0$ as

Rolle's theorem is applicable for $F'(x)$ in $\left[0, \frac{\pi}{2}\right]$.

Also, $F(0)=\int_0^1 f(t)\,dt$ and $F\left(\frac{\pi}{2}\right)=\int_1^0 f(t)\,dt$

Hence, $F(0)$ and $F\left(\frac{\pi}{2}\right)$ have opposite signs.

$$\Rightarrow \quad F(c)=0 \text{ for some } c\in\left(0, \frac{\pi}{2}\right)$$

30. Given, $f'(1)=1, 2\cdot f'(2x)=f'(x)$

Put $x=1$, $f'(2)=\frac{f'(1)}{2}=\frac{1}{2}$

and $f'(4)=\frac{1}{2}f'(2)=\frac{1}{4}$

Applying LMVT for $y=f'(x)$ in $[2, 4]$, we get

$$f''(c)=\frac{f'(4)-f'(2)}{2}=\frac{\frac{1}{4}-\frac{1}{2}}{2}=-\frac{1}{8}$$

31. Consider $\phi(x)=f(x)-g(x)\Rightarrow\phi'(x)=f'(x)-g'(x)>0$.

Clearly, $\phi(x)$ is also continuous and derivable in $[x_0, x]$.

Using LMVT for $\phi(x)$ in $[x_0, x]$, we get

$$\phi'(c)=\frac{\phi(x)-\phi(x_0)}{x-x_0}$$

Since, $\phi'(x)=f'(x)-g'(x)>0$ for all $x>x_0$

$$\therefore \quad \phi'(c)>0$$

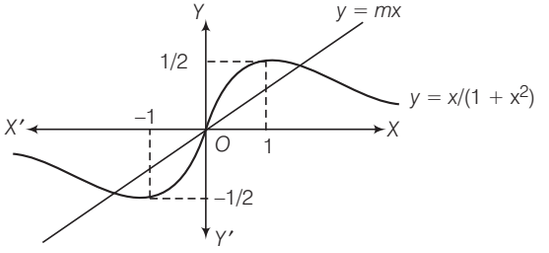
Hence, $\phi(x)-\phi(x_0)>0$

$$\Rightarrow \quad \phi(x)>\phi(x_0)$$

$$\Rightarrow \quad \phi(x)>0 \quad [\because \phi(x_0)=f(x_0)-g(x_0)=0]$$

$$\Rightarrow \quad f(x)-g(x)>0$$

32. Solving, $mx=\frac{x}{x^2+1} \Rightarrow x^2+1=\frac{1}{m}$ or $x=0$



$$\Rightarrow \quad x^2=\frac{1}{m}-1>0 \text{ for a region}$$

$$\Rightarrow \quad \frac{m-1}{m}<0 \Rightarrow m\in(0, 1)$$

Remark

For $m=0$ or 1, the line does not enclose a region.

33. $\because$ Side $=x$

$$\therefore \text{Area } A=x^2 \Rightarrow \frac{dA}{dx}=2x. \text{ So, } \left\{\left(\frac{dA}{dx}\right)_{x=x_0}\times h\right\}=2x_0h$$

The exact change in the area of S when x is changed from x_0 to x_0+h, is

$$(x_0+h)^2-x_0^2=x_0^2+2x_0h+h^2-x_0^2=2x_0h+h^2$$

The difference between the exact change and the approximate change is $2x_0h+h^2-2x_0h=h^2$

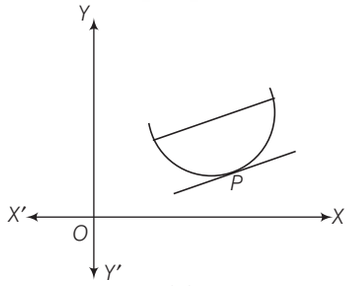
34. $f(x)=\int_1^x\left(t+\frac{1}{t}\right)dt \Rightarrow f'(x)=x+\frac{1}{x}$

$\therefore \quad g(x)=x+\frac{1}{x}$

For $x\in\left[\frac{1}{2},3\right]$,

$g\left(\frac{1}{2}\right)=2+\frac{1}{2}=\frac{5}{2},\ g(3)=3+\frac{1}{3}=\frac{10}{3}$

Let $P\equiv(c,\ g(c)),\ c\in\left[\frac{1}{2},3\right]$



By LMVT, $g'(c)=\frac{g(3)-g\left(\frac{1}{2}\right)}{3-\frac{1}{2}}$

$\therefore \quad 1-\frac{1}{c^2}=\frac{\frac{10}{3}-\frac{5}{2}}{3-\frac{1}{2}}$

$\Rightarrow \quad c^2=\frac{3}{2}$

$\Rightarrow \quad c=\sqrt{\frac{3}{2}}$

$\therefore \quad g(c)=\sqrt{\frac{3}{2}}+\frac{1}{\sqrt{\frac{3}{2}}}=\frac{5}{\sqrt{6}}$

Thus, $P\equiv\left(\sqrt{\frac{3}{2}},\frac{5}{\sqrt{6}}\right)$

35. Here, $\frac{dx}{dt}=\frac{2(-\operatorname{cosec}^2 t)}{\cot t}$

$\Rightarrow \quad \left.\frac{dx}{dt}\right|_{\text{at } t=\frac{\pi}{4}}=-4$

and $\frac{dy}{dt}=\sec^2 t-\operatorname{cosec}^2 t$

$\Rightarrow \quad \left.\frac{dy}{dt}\right|_{\text{at } t=\frac{\pi}{4}}=0$

$\therefore \quad \frac{dy}{dx}=0$

Hence, tangent is parallel to X-axis and its normal is parallel to Y-axis.

36. Here, $f'(x)=\frac{1}{3x^{2/3}}$

$\Rightarrow f'(0)\to\infty$ and tangent is vertical at $x=0$.

Equation of tangent at $(0,\ 0)$ is $x=0$.

Equation of normal is $y=0$.

Now, $f(x)=f^{-1}(x) \Rightarrow x^{1/3}=x^3$

$\Rightarrow \quad x^9=x \Rightarrow x=0,\ 1,\ -1$

37. Here, $\frac{dy}{dx}=k^2\,e^{kx}$

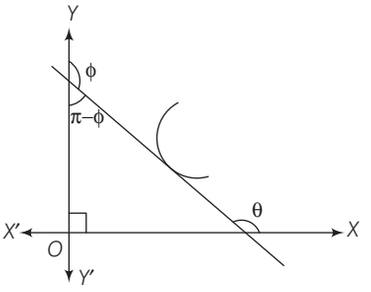
$\Rightarrow \left(\frac{dy}{dx}\right)_{x=0}=k^2=\tan\theta$, where θ is angle made by X- axis.

Let ϕ be the angle made by Y-axis.

$\therefore \quad \tan\theta=\tan\left(\frac{3\pi}{2}-\phi\right)=\cot\phi$

$\Rightarrow \quad \cot\phi=k^2 \Rightarrow \phi=\cot^{-1}k^2$

$\Rightarrow \quad \phi=\sin^{-1}\left(\frac{1}{\sqrt{1+k^4}}\right)$



38. As, $f(x)$ is continuous in $[0,\ 1]$ and differentiable in $(0,\ 1)$ and $f(0)=f(1)=1$.

$\therefore$ By Rolle's theorem, there must exist at least one $x=c\in(0,\ 1)$ such that $f'(c)=0$

$\therefore f'(c)$ vanishes for some $c\in(0,\ 1)$.

Now, $f(0)=1$, $f(1/2)=-\frac{1}{2}$ and $f(1)=1$

$\therefore$ By intermediate value theorem, $f(x)$ must have one root belongs to $\left(0,\frac{1}{2}\right)$ and other in the interval $\left(\frac{1}{2},1\right)$.

39. Here, $f(0)=f(1)$ and f is continuous in $[0,\ 1]$ and derivable in $(0,\ 1)$.

$\therefore f'(c_1)=0$ for at least one $c_1\in(0,\ 1)$

Similarly, as $f(1)=f(2)$

$\therefore\ f'(c_2)=0$ for at least one $c_2\in(1,\ 2). \Rightarrow f'(c_1)=f'(c_2)$

$\Rightarrow f''(c)=0$ for at least one $c\in(c_1,\ c_2)$.

40. Let $f(x)=\frac{1}{(x+1)^3}-3x+\sin x$

Domain of f is $(-\infty,-1)\cup(-1,\infty)$.

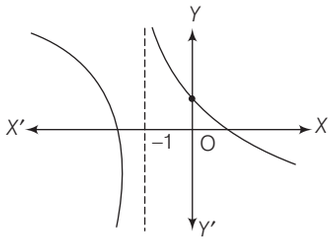
$$f'(x)=-3\left[\frac{1}{(x+1)^4}+1\right]+\cos x \Rightarrow f'(x)<0$$

$\Rightarrow f$ is decreasing.

Also, $\lim\limits_{x\to-1^+}f(x)\to\infty,\quad \lim\limits_{x\to-1^-}f(x)\to-\infty$

and $\lim\limits_{x\to\infty}f(x)\to-\infty,\quad \lim\limits_{x\to-\infty}f(x)\to\infty$

$\Rightarrow f(x) = 0$ has exactly two roots.



41. Clearly, $f(-1) = -f(1)$ and $f(0) = 0$. For (a) and (b) apply LMVT for the function $f(x)$ in $(-1, 0)$ and $(0, 1)$, respectively. For (d) apply LMVT for $(f(x))^n$ in $(0, 1)$.

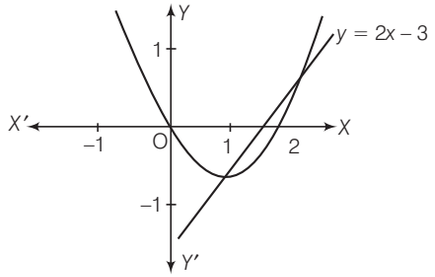
42. When $x = 1$, $y = -1$ [from the line]

Thus, it must lie on the parabola $y = x^2 + px + q$

$\Rightarrow -1 = 1 + p + q \Rightarrow p + q = -2$

$\therefore$ Now, distance of the vertex of the parabola from the X-axis is

$$d = f\left(-\frac{p}{2}\right) = \frac{p^2}{4} - \frac{p^2}{2} + q = q - \frac{p^2}{4}$$



Substituting $q = -2 - p$, we get

$$d = -2 - p - \frac{p^2}{4}$$

Now, take $g(p) = -2 - p - \frac{p^2}{4}$

So, $g'(p) = -1 - \frac{p}{2} = 0 \Rightarrow p = -2$

Hence, $q = 0$

Note that least distance of the vertex from X-axis is 1.

43. Given, $\sqrt{xy} = a + x \Rightarrow xy = a^2 + x^2 + 2ax$

$$\Rightarrow y = \frac{a^2}{x} + x + 2a$$

$$\Rightarrow \frac{dy}{dx} = -\frac{a^2}{x^2} + 1 = -1$$

$$\Rightarrow 2x^2 = a^2$$

$$\Rightarrow x = \pm\frac{a}{\sqrt{2}}$$

44. Given, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$ [say]

$\therefore$ $da = 2R\cos A\, dA$, $db = 2R\cos B\, dB$,

$dc = 2R\cos C\, dC$

$$\therefore \frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 2R(dA + dB + dC) \quad \ldots(i)$$

Also, $A + B + C = \pi$

So, $dA + dB + dC = 0$ ...(ii)

From Eqs. (i) and (ii), we get

$$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0$$

45. $\frac{f(x) - f(0)}{x - 0} = f'(c) \le 1 - x$ for some $c \in (0, 1)$.

$\Rightarrow f(x) \le x(1 - x) \le 1/4$

46. Here, $f(x) = \frac{\ln x}{x}$ ...(i)

$\therefore$ Domain is R^+.

$\because$ $$f'(x) = \frac{x \cdot \frac{1}{x} - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

(a) For horizontal tangent, $f'(x) = 0$

$\Rightarrow \ln x = 1 \Rightarrow x = e$ [true]

(b) If $f(x)$ intersects the X-axis.

$$\frac{\ln x}{x} = 0 \Rightarrow x = 1$$ [true]

(c) $f'(x)$ is positive, if $x \in (0, e)$ and $f'(x)$ is negative, if $x \in (e, \infty)$

$\therefore$ $f(x)$ is not monotonic.

$\therefore$ $f(x)$ is many-one.

(d) For vertical tangent, $f'(x) = \infty$ [true]

$$\Rightarrow \frac{1 - \ln x}{x^2} = \infty \Rightarrow \frac{x^2}{1 - \ln x} = 0$$

$\Rightarrow x = 0$ which is not in the domain of $f(x)$. [false]

47. Let the tangent line be $y = ax + b$

The equation for its intersection with the upper parabola is

$$x^2 + 1 = ax + b$$

$$\Rightarrow x^2 - ax + (1 - b) = 0$$

This has a double root when $a^2 - 4(1 - b) = 0$

or $$a^2 + 4b = 4$$

For the lower parabola, $ax + b = -x^2$

$$\Rightarrow x^2 + ax + b = 0$$

This has a double root when $a^2 - 4b = 0$

On subtracting these two equations, we get $8b = 4$ or $b = \frac{1}{2}$

On adding these equations, we get $2a^2 = 4$ or $a = \pm\sqrt{2}$

The tangent lines are $y = \sqrt{2}\,x + \frac{1}{2}$ and $y = -\sqrt{2}\,x + \frac{1}{2}$

48. For some $\alpha \in (0, 1)$, $|f'(\alpha)| = \left|\frac{f(1) - f(0)}{1 - 0}\right| \le |f(1)| + |f(0)|$

$$\Rightarrow |f'(\alpha)| \le 1 + 1 = 2$$

Similarly, for some $\beta \in (-1, 0)$, $|f'(\beta)| \le 2$

Also, $F(x) = (f(x))^2 + (f'(x))^2$

$$\Rightarrow F(\alpha) = (f(\alpha))^2 + (f'(\alpha))^2 \le 1 + 4 \le 5$$

Similarly, $F(\beta) \le 5$ for some $\beta \in (-1, 0)$

As, $F(0) = 6$, so there must be a point of local maxima for $F(x)$ in $(-1, 1)$ and at the point of maxima, say $x = c$,

$$F(c) \geq 6 \Rightarrow F'(c) = 0 \text{ and } F''(c) \leq 0$$

49. As, Rolle's theorem is applicable, the function should be continuous and differentiable in $[-3, 3]$.

So, at $x = 1$ it is continuous

$$\Rightarrow \quad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$$

$$\Rightarrow \quad a + b = c = 1 \qquad \text{...(i)}$$

Since, differentiable at $x = 1$, therefore

$$f'(1^+) = f'(1^-)$$

$$\Rightarrow \quad \lim_{h \to 0} \frac{\frac{c}{1+h} - 1}{h} = 2a$$

$$\Rightarrow \quad 2a = \lim_{h \to 0} \frac{c - 1 - h}{h(h+1)} \text{ exists only when } c = 1$$

$$\Rightarrow \quad 2a = \lim_{h \to 0} \frac{-1}{(1+h)} = -1$$

$$\therefore \quad a = -1/2 \text{ and } c = 1 \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$b = 3/2 \qquad \text{...(iii)}$$

$$\therefore \quad a + b + c = 2 \qquad \text{...(iv)}$$

$$|a| + |b| + |c| = 3 \qquad \text{...(v)}$$

$$2a + 4b + 3c = 8 \qquad \text{...(vi)}$$

$$4a^2 + 4b^2 + 5c^2 = 15 \qquad \text{...(vii)}$$

50. **Statement I** As, $f(-1) = f(1)$ and Rolle's theorem is not applicable, then it implies that $f(x)$ is either discontinuous or $f'(x)$ does not exist at at least one point in $(-1, 1)$.

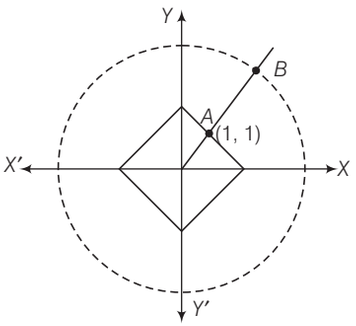
$\Rightarrow$ $g(x) = 0$ for at at least one value of x in $(-1, 1)$.

Statement II is false. Consider the example in Statement I.

51. Common normal is $y = x$

Solving, $y = x$ with $x + y = 2$, we get $A(1, 1)$

and with $x^2 + y^2 = 16$, we get $B(2\sqrt{2}, 2\sqrt{2})$



The shortest distance between the given curves is $AB = 4 - \sqrt{2}$.

But as the curves are not smooth, check at slope points. The coordinates in 1st quadrant are (2, 0) and (4, 0) and here distance = 2.

$\therefore$ $4 - \sqrt{2}$ is not the shortest.

52. If $P(x) = 0$ is a polynomial equation, then $P(-x) = 0$ has no positive root.

$\Rightarrow$ $P(x) = 0$ cannot have negative roots.

53. Let $A(a, P(a))$, $B(b, P(b))$, then slope of $AB = P'(a) = P'(b)$ from LMVT there exists $c \in (a, b)$, where $P'(c) =$ slope of AB.

54.
$$f(x) = \begin{cases} \frac{1}{2} - x, & x < \frac{1}{2} \\ \left(\frac{1}{2} - x\right)^2, & x \geq \frac{1}{2} \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} -1, & x < \frac{1}{2} \\ 2\left(\frac{1}{2} - x\right)(-1), & x > \frac{1}{2} \end{cases}$$

Left hand derivative at $x = 1/2$ is (-1) and right hand derivative at $x = 1/2$ is 0, so the function is not differentiable at $x = 1/2$.

55. $f(x) = \ln(2 + x) - \dfrac{2x + 2}{x + 3}$ is continuous in $[-2, \infty)$.

$$f'(x) = \frac{1}{x+2} - \frac{4}{(x+3)^2} = \frac{(x+3)^2 - 4(x+2)}{(x+2)(x+3)^2}$$

$$= \frac{x^2 + 2x + 1}{(x+2)(x+3)^2} = \frac{(x+1)^2}{(x+2)(x+3)^2} > 0$$

$$[f'(x) = 0 \text{ at } x = -1]$$

$\Rightarrow$ f is increasing in $(-2, \infty)$.

Also, $\lim_{x \to -2^+} f(x) \to -\infty$

and $\lim_{x \to \infty} f(x) \to \infty$ $\Rightarrow$ unique root.

56. Let $f(x) = 0$ has two roots say $x = r_1$ and $x = r_2$, where $r_1, r_2 \in [a, b]$.

$$\Rightarrow \quad f(r_1) = f(r_2)$$

Hence, there must exist some $c \in (r_1, r_2)$, where $f'(c) = 0$

But $f'(x) = x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$

for $x \geq 1$, $f'(x) = (x^6 - x^5) + (x^4 - x^3) + (x^2 - x) + 1 > 0$

for $x \leq 1$, $f'(x) = (1 - x) + (x^2 - x^3) + (x^4 - x^5) + x^6 > 0$

Hence, $f'(x) > 0$ for all x.

$\therefore$ Rolle's theorem fails.

$\Rightarrow$ $f(x) = 0$ cannot have two or more roots.

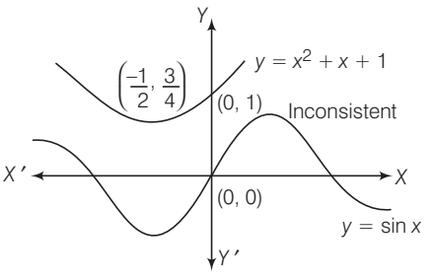
57. Let $f(x) = \cos x - \sin x + \cos^{-1} x - \sin^{-1} x$, $x \in [-1, 1]$

$\because$ $f(-1)\, f(1) < 0$

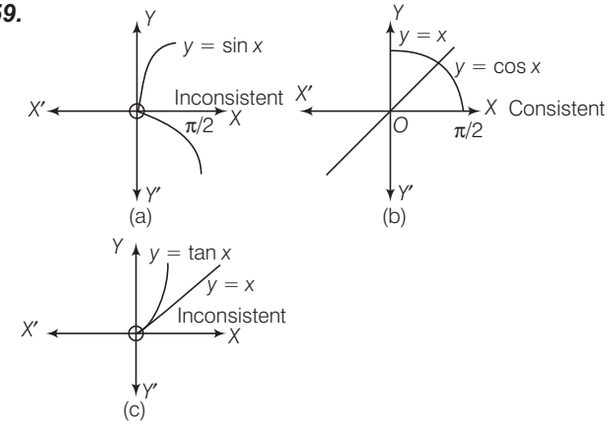
$\therefore$ There exists at least one $c \in (-1, 1)$ such that $f(c) = 0$.

Hence, the curves $y = \cos x + \cos^{-1} x$ and $y = \sin x + \sin^{-1} x$ intersect each other at at least one point.

58. Given, $\sin x = x^2 + x + 1 = \left(x + \dfrac{1}{2}\right)^2 + \dfrac{3}{4}$



59.

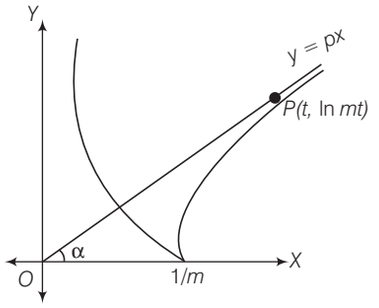


Sol. (Q. Nos. 60 to 62)

Slope of tangent at P = Slope of OP

$$\Rightarrow \quad \frac{1}{t} = \frac{\ln mt}{t}$$

$$\Rightarrow \quad t = \frac{e}{m}$$



$$\Rightarrow \quad P \equiv \left(\frac{e}{m}, 1\right)$$

$$\Rightarrow \quad \tan\alpha = p = \frac{m}{e}$$

60. $p > m/e$ **61.** $p = m/e$

62. $0 < p < m/e$

63. Putting $y = 0$, we get $x^2 - 3x - c_i = 0$

As roots are integers and $D = 9 + 4c_i$ must be perfect square, therefore $9 + 4c_i = (2\lambda + 1)^2$, $\lambda \in I$.

$$\Rightarrow \quad c_i = \lambda^2 + \lambda - 2$$

$$\Rightarrow \quad c_k = k(k+3), \quad k = 1, 2, 3, \ldots$$

$$\therefore \quad c_1 = 4, c_2 = 10, \ldots$$

$$\Rightarrow \quad x^2 + y^2 - 3x - 4y - (c_2 - c_1) = 0$$

or $\quad x^2 + y^2 - 3x - 4y - 6 = 0$

and $x^2 + y^2 = r^2$ will touch each other, if

$$\sqrt{9/4 + 4} = |r - \sqrt{9/4 + 10}|$$

$$\Rightarrow \quad |r - 7/2| = 5/2$$

$$\Rightarrow \quad r = 6$$

$\therefore$ Common tangent is $3x + 4y - 30 = 0$ and passes through $(10, 0)$.

64. An ellipse and hyperbola intersect orthogonally.

They must be confocal $\Rightarrow$ $a = 2$

Let point $P(\alpha, \beta)$ lies on both the curves, then

$$4\alpha^2 + 9\beta^2 = 36 \quad \ldots(i)$$

and $\quad 4\alpha^2 - \beta^2 = 4 \quad \ldots(ii)$

On adding Eqs. (i) and (ii), we get

$$8\alpha^2 + 8\beta^2 = 40$$

$$\Rightarrow \quad \alpha^2 + \beta^2 = 5$$

or $\quad x^2 + y^2 = \frac{c_7}{14}$, as $c_7 = 70$.

65. (A) $f'(x) = \log x - \frac{3}{x} + 1$

$$\Rightarrow \quad f(x) = (x-3)\log x + c$$

$$\Rightarrow \quad f(1) = f(3)$$

(B) $f'(x) = 4ax^3 + 3bx^2 + 2cx + d$

$$\Rightarrow \quad f(x) = ax^4 + bx^3 + cx^2 + dx + e$$

$$\Rightarrow \quad f(0) = f(3) \qquad [\because 27a + 9b + 3c + d = 0]$$

(C) $\frac{f(b) - f(a)}{b - a} = f'(\sqrt{3}) = \frac{2}{3}$

$$\Rightarrow \quad \frac{ab - 1}{ab} = \frac{2}{3}$$

(D) $\frac{f(b) - f(a)}{b - a} = f'\left(\frac{1}{2}\right) = 1$

$$\Rightarrow \quad a + b = 1$$

66. (A) Here, $r = 5$ cm, $\Delta r = 0.06$

$$\because \quad A = \pi r^2,$$

$$\therefore \quad dA = 2\pi r\, dr = 2\pi r \cdot \Delta r = 10\pi \times 0.06 = 0.6\,\pi$$

(B) $v = x^3$, $dv = 3x^2\, dx$

$$\frac{dv}{v} \times 100 = 3\frac{dx}{x} \times 100 = 3 \times 1 = 3$$

(C) $(x-2)\frac{dx}{dt} = 2\frac{dx}{dt} \Rightarrow x = 4$

(D) $A = \frac{\sqrt{3}}{4}x^2$

$$\frac{dA}{dt} = \frac{\sqrt{3}}{2} \cdot x \cdot \frac{dx}{dt}$$

$$= \frac{\sqrt{3}}{2} \cdot 15 \cdot \frac{1}{10} = \frac{3\sqrt{3}}{4}\ \text{cm}^2/\text{s}$$

67. $\because \left|\frac{dx}{dt}\right| < \left|\frac{dy}{dt}\right| \Rightarrow \left|\frac{dy}{dx}\right| > 1$

and $\quad 3y^2 \cdot \frac{dy}{dx} = 27$ or $\frac{dy}{dx} = \frac{9}{y^2}$

$$\therefore \quad \frac{9}{y^2} > 1 \Rightarrow y^2 < 9$$

$$\Rightarrow \quad -3 < y < 3 \Rightarrow -27 < y^3 < 27$$

$$\Rightarrow \quad -27 < 27x < 27 \Rightarrow -1 < x < 1$$

$$\therefore \quad x \in (-1, 1) \Rightarrow a + b = 0$$

68. Here, $4y\dfrac{dy}{dx} = 2ax$

$\therefore \quad \left.\dfrac{dy}{dx}\right|_{(1,\,-1)} = \dfrac{-a}{2} = -1$

$\Rightarrow \quad a = 2$

Also, $2y^2 = ax^2 + b$ at $(1, -1)$ is

$2 = a + b$

$\Rightarrow \quad b = 0$

$\therefore \quad a - b = 2$

69. Using LMVT, for some $c \in (1, 6)$, we get

$$f'(c) = \frac{f(6) - f(1)}{5}$$

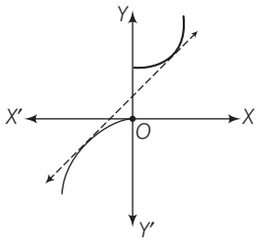
$$= \frac{f(6) + 2}{5} \geq 4.2$$

$\Rightarrow \quad f(6) \geq 19$

$\Rightarrow \quad f(6) - 16 \geq 3$

$\therefore$ Least value of $f(6) - 16 = 3$

70. Let $y = mx + c$ be a tangent to $f(x)$.



For, $x \geq 0$, intersection point is given by

$mx + c = x^2 + 8 \qquad [\because\ y = x^2 + 8, \text{ for } x \geq 0]$

$\Rightarrow \quad x^2 - mx + (8 - c) = 0$

For line to be tangent, $D = 0$

$\therefore \quad m^2 = 4(8 - c) \qquad \ldots(i)$

Again, for $x < 0$

$mx + c = -x^2 \qquad [\because\ y = -x^2, \text{ for } x < 0]$

$\Rightarrow \quad x^2 + mx + c = 0$

Now, $D = 0$

$\Rightarrow \quad m^2 = 4c \qquad \ldots(ii)$

From Eqs. (i) and (ii), we get

$c = 4, m = 4$

$\therefore$ Tangent is $y = 4x + 4$

Putting $y = 0$, we get

$x = -1$

$\therefore$ Absolute value of x-intercept is 1.

71. Here, $f'(a) = \sqrt{3}$ and $f'(b) = 1$

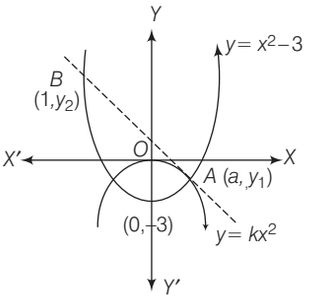
$$\therefore \quad \left|\int_a^b f'(x)\cdot f''(x)\,dx\right| = \left|\left(\frac{(f'(x))^2}{2}\right)_a^b\right| = \frac{1}{2}\left|(f'(b))^2 - (f'(a))^2\right|$$

$$= \frac{1}{2}|1 - 3| = 1$$

72. Point $A(a, y_1)$ lies on C_1 and C_2.

$\therefore \quad y_1 = a^2 - 3$ and $y_1 = ka^2$.

$\Rightarrow \quad a^2 - 3 = ka^2 \qquad \ldots(i)$



Now, $y = kx^2$

$\Rightarrow \quad \dfrac{dy}{dx} = 2kx$

$\therefore \quad \left(\dfrac{dy}{dx}\right)_{(a,\,y_1)} = 2ka = \dfrac{y_2 - y_1}{1 - a}$

But, $y_2 = 1 - 3 = -2$

$\therefore \quad 2ka = \dfrac{-2 - (a^2 - 3)}{1 - a}$

$\Rightarrow \quad 2ka = \dfrac{1 - a^2}{1 - a} = 1 + a$

$\Rightarrow \quad 2ak = 1 + a \qquad \ldots(ii)$

Substituting $k = \dfrac{a^2 - 3}{a^2}$ from Eq. (i) in Eq. (ii), we get

$$\frac{2a(a^2 - 3)}{a^2} = 1 + a$$

$\Rightarrow \quad 2a^2 - 6 = a + a^2$

$\Rightarrow \quad a^2 - a - 6 = 0$

$\Rightarrow \quad a = 3, -2$

$\therefore \quad a = 3 \qquad [\because\ -2 \text{ is rejected as } a > 0]$

73. $f'(c) = 16c - 7 = \dfrac{f(6) - f(-6)}{12}$

$$= \frac{(8\cdot 36 - 7\cdot 6 + 5) - (8\cdot 36 + 7\cdot 6 + 5)}{12}$$

$$= -\frac{2\cdot 7\cdot 6}{12} = -7$$

$\Rightarrow \ 16c = 0 \ \Rightarrow \ c = 0$

74. Using LMVT for f in $[1, 2]$, we get, for some $c \in (1, 2)$

$$\frac{f(2) - f(1)}{2 - 1} = f'(c) \leq 2$$

$$f(2) - f(1) \leq 2$$

$\Rightarrow \quad f(2) \leq 4 \qquad \ldots(i)$

Again, using LMVT in $[2, 4]$, we get, for some $d \in (2, 4)$

$\Rightarrow \quad \dfrac{f(4) - f(2)}{4 - 2} = f'(d) \leq 2$

$\therefore \quad f(4)-f(2)\le 4, \quad 8-f(2)\le 4, \quad 4\le f(2)$

$\Rightarrow \quad f(2)\ge 4 \qquad ...(ii)$

From Eqs. (i) and (ii), we get $f(2)=4$

75. As, a, b and c are positive integers.

We must have $\frac{1}{a}<1$, so $a>1$. $\left[\text{similarly, } \frac{1}{b}<1, \frac{1}{c}<1\right]$

Since, $\frac{1}{a}>\frac{1}{b}>\frac{1}{c}$

$\Rightarrow \quad \frac{1}{a}>\frac{1}{3} \Rightarrow a<3 \Rightarrow a=2$

$\therefore \quad \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1 \Rightarrow \frac{1}{b}+\frac{1}{c}=\frac{1}{2}$, where $2<b<c$.

Similarly, $\frac{1}{b}>\frac{1}{4}$, so $b<4 \Rightarrow b=3$

Now, $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$

$\Rightarrow \quad \frac{1}{2}+\frac{1}{3}+\frac{1}{c}=1$

$\Rightarrow \quad c=6$

$\therefore \quad (a+b+c-5)=2+3+6-5=6$

76. Equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Equation of tangent at $(a\cos\theta, b\sin\theta)$ is

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$$

Intercept on the X-axis $=(a\sec\theta)$

Intercept on the Y-axis $=(b\operatorname{cosec}\theta)$

Length of intercept of the tangent by the axes

$$=\sqrt{a^2\sec^2\theta+b^2\operatorname{cosec}^2\theta}$$

Let $l=a^2\sec^2\theta+b^2\operatorname{cosec}^2\theta$

$\Rightarrow \quad \frac{dl}{d\theta}=2a^2\sec^2\theta\tan\theta-2b^2\operatorname{cosec}^2\theta\cot\theta$

Now, $\frac{dl}{d\theta}=0$

$\Rightarrow \quad a^2\sin^4\theta=b^2\cos^4\theta \Rightarrow \frac{a}{b}=\cot^2\theta$

$\Rightarrow \quad \sin^2\theta=\frac{b}{a+b}, \ \cos^2\theta=\frac{a}{a+b}$

Distance between $(a\sec\theta, 0)$ and point of tangency $(a\cos\theta, b\sin\theta)$ is

$$=\sqrt{a^2(\sec\theta-\cos\theta)^2+b^2\sin^2\theta}$$

$$=\sqrt{a^2\cos^2\theta(\sec^2\theta-1)^2+b^2\sin^2\theta}$$

$$=\sqrt{a^2\cos^2\theta\left(\frac{a+b}{a}-1\right)^2+b^2\sin^2\theta}$$

$$=\sqrt{a^2\cos^2\theta\frac{b^2}{a^2}+b^2\sin^2\theta}=b$$

Similarly, distance between $(0, b\operatorname{cosec}\theta)=a$

77. Let (x_1, y_1) be a point of contact of tangents from the origin $(0, 0)$ to the curve $y=\sin x$.

Here, $y=\sin x$

$\therefore \quad \frac{dy}{dx}=\cos x$

$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)}=\cos x_1$

Now, equation of tangent at (x_1, y_1) is

$$y-y_1=\left(\frac{dy}{dx}\right)_{(x_1, y_1)}(x-x_1)$$

$\Rightarrow \quad y-y_1=(\cos x_1)(x-x_1)$

$\because$ It passes through $(0, 0)$.

$\therefore \quad -y_1=(\cos x_1)(-x_1) \qquad ...(i)$

Also, (x_1, y_1) lies on the curve.

So, $y_1=\sin x_1 \qquad ...(ii)$

Squaring and adding Eqs. (i) and (ii), we get

$$\left(\frac{y_1}{x_1}\right)^2+y_1^2=\cos^2 x_1+\sin^2 x_1=1$$

or $y_1^2+x_1^2y_1^2=x_1^2$

i.e. $(x_1^2+1)y_1^2=x_1^2$

$\therefore$ The point of contact (x_1, y_1) lies on the curve $y^2=\frac{x^2}{x^2+1}$.

78. We must show that for a given $m\in R$ these exists $x\in R$ such that,

$$m^2x^2+\int_0^x f(t)\,dt=2$$

Let $f(x)=\int_0^x[2m^2t+f(t)]\,dt-2, \ x\in R$

Since, $f(x)$ is continuous and $2m^2t^2$ is continuous, therefore

$$\int_0^x[2m^2t+f(t)]\,dt \text{ continuous on } R,$$

$\therefore \quad f$ is continuous on R, also

$$f(0)=\int_0^0[2m^2t+f(t)]\,dt-2=0-2=-2$$

and $f(x)=\int_0^x[2m^2t+f(t)]\,dt-2$

where, $f(x)=m^2x^2+\int_0^x f(t)\,dt-2\to\infty$

$$\left[\because \int_0^x f(t)\,dt\to\infty\right]$$

As, $|x|\to\infty$

Thus, these exists some $a\in R$ such that;

$$f(x)>1, \ \text{for}\,|x|>a$$

Note that f is a continuous on $[0, a+1]$ and $f(0)\,f(a+1)<0$. By the intermediate value theorem of continuous functions, we have that there exists some $b\in(0, a+1)$ such that $f(b)=0$, i.e. there exists a real β which satisfies the equation

$$m^2x^2+\int_0^x f(t)\,dt=2$$

79. Tangent at any point $P(t_1)$, i.e. $(4t_1^2+3, 8t_1^3-1)$ be normal to the curve at $Q(t_2)$, i.e. $(4t_2^2+3, 8t_2^3-1)$.

The equation of the tangent at t_1 is

$$y-(8t_1^3-1)=\left(\frac{dy}{dx}\right)_{t_1}\cdot\{x-(4t_1^2+3)\}$$

or $$y-(8t_1^3-1)=\left(\frac{dy/dt}{dx/dt}\right)_{t_1}\cdot\{x-(4t_1^2+3)\}$$

or $$y-(8t_1^3-1)=\frac{24t_1^2}{8t_1}\cdot\{x-(4t_1^2+3)\}$$

or $$y-(8t_1^3-1)=3t_1\{x-(4t_1^2+3)\} \quad ...(i)$$

Clearly, slope of tangent at t_1 = slope of tangent at t_2.

$$\therefore \quad \left(\frac{dy}{dx}\right)_{t_1}=\frac{-1}{\left(\frac{dy}{dx}\right)_{t_2}} \; i.e.\; 3t_1=\frac{-1}{3t_2} \quad ...(ii)$$

$\Rightarrow$ Equation of normal at t_2 is $y-(8t_2^3-1)=3t_1\{x-(4t_2^3+3)\}$ ...(iii)

On subtracting Eq. (iii) from Eq. (i), we get

$$(8t_2^3-1)-(8t_1^3-1)=3t_1\{(4t_2^2+3)-(4t_1^2+3)\}$$

$$\Rightarrow \quad 2t_2^2=t_1t_2+t_1^2$$

$$\Rightarrow \quad 2\cdot\left(\frac{-1}{9t_1}\right)^2=-\frac{1}{9}+t_1^2 \quad \text{[using Eq. (ii)]}$$

$$\Rightarrow \quad 2=-9t_1^2+81t_1^4$$

$$\therefore \quad 81t_1^4-9t_1^2-2=0$$

$$\Rightarrow \quad t_1=\pm\frac{\sqrt{2}}{3}$$

Putting in Eq. (i), the equation is

$$27(y+1)\mp16\sqrt{2}=\pm\sqrt{2}(27x-105)$$

80. Equation of tangent at (x, y) is

$$Y-y=\frac{dx}{dy}(X-x)$$

$$\therefore \quad Q=\left(x-y\frac{dx}{dy},\,0\right)$$

Equation of normal at (x, y) is

$$Y-y=-\frac{dx}{dy}(X-x)$$

$$\therefore \quad R=\left(x+y\frac{dy}{dx},\,0\right)$$

Given, $QR=2$

$$\Rightarrow \quad y\frac{dy}{dx}+y\frac{dx}{dy}=2$$

$$\Rightarrow \quad y\left(\frac{dy}{dx}\right)^2-2\left(\frac{dy}{dx}\right)+y=0$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{2\pm\sqrt{4-4y^2}}{2y}=\frac{1\pm\sqrt{1-y^2}}{y}$$

$$\Rightarrow \quad \frac{y\,dy}{1\pm\sqrt{1-y^2}}=dx \text{ or } \frac{1\mp\sqrt{1-y^2}}{y}dy=dx$$

On integrating both the sides, we get

$$\Rightarrow \quad \log y\mp\left(\log\left|\frac{1-\sqrt{1-y^2}}{y}\right|+\sqrt{1-y^2}\right)=x+c$$

The curve passes through (1,1), so $c=-1$

Hence, the possible curves

$$\log y-x=\pm\left(\log\left|\frac{1-\sqrt{1-y^2}}{y}\right|+\sqrt{1-y^2}\right)-1$$

81. Let the point $P(x, y)$ be on the curve,

$$\log(x^2+y^2)=c\tan^{-1}\left(\frac{y}{x}\right)$$

Differentiating both the sides w.r.t. 'x', we get

$$\frac{2x+2yy'}{(x^2+y^2)}=\frac{c(xy'-y)}{(x^2+y^2)}$$

$$\Rightarrow \quad y'=\frac{2x+cy}{cx-2y}=m_1 \text{ (say)}$$

Slope of $OP=\frac{y}{x}=m_2$ (say)

Let the angle between the tangent at P and OP be θ.

Then, $$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|=\left|\frac{\frac{2x+cy}{cx-2y}-\frac{y}{x}}{1+\frac{2xy+cy^2}{x(cx-2y)}}\right|=\frac{2}{c}$$

$$\therefore \quad \theta=\tan^{-1}\left(\frac{2}{c}\right)=\text{constant.}$$

Hence, the angle between the tangent at any point P and the line joining P to the origin O is the same.

82. (i) The given curves are

$$y^2=4ax \quad ...(i)$$

and $$x^2=4ay \quad ...(ii)$$

Point of intersection of Eqs. (i) and (ii) are $(0, 0)$ and $(4a, 4a)$.

From Eq. (i), $$\frac{dy}{dx}=\frac{2a}{y}=m_1 \text{ (say)}$$

From Eq. (ii), $$\frac{dy}{dx}=\frac{x}{2a}=m_2 \text{ (say)}$$

Let the angle of intersection of two curves is θ, then

$$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|=\left|\frac{\frac{2a}{y}-\frac{x}{2a}}{1+\frac{x}{y}}\right|=\left|\frac{4a^2-xy}{2a(x+y)}\right|$$

$$\therefore \quad (\tan\theta)_{(0,0)}=\infty \quad \text{or} \quad \theta=90^\circ$$

$$\text{and } (\tan\theta)_{(4a,\,4a)}=\left|\frac{\frac{1}{2}-2}{1+1}\right|=\left|-\frac{3}{4}\right|=\frac{3}{4}$$

Hence, $$\theta=\tan^{-1}\left(\frac{3}{4}\right)$$

(ii) The given curves are $y^2 = 4ax$ ...(i)

and $x^2 = 4ay$ ...(ii)

Tangents of Eq. (i) in terms of slope m is

$$y = mx + \frac{a}{m} \quad ...(iii)$$

Now, Eq. (iii) is also tangent of Eq. (ii).

Eliminating y from Eqs. (ii) and (iii), we have

$$x^2 = 4a\left(mx + \frac{a}{m}\right)$$

$$\Rightarrow \quad x^2 - 4amx - \frac{4a^2}{m} = 0 \quad ...(iv)$$

$$\Rightarrow \quad B^2 - 4AC = 0$$

$$\Rightarrow \quad 16a^2m^2 - 4\left(-\frac{4a^2}{m}\right) = 0$$

$$\Rightarrow \quad m^3 = -1 \text{ or } m = -1$$

From Eq. (iii) common tangent is

$$y = -x - a \text{ or } y + x + a = 0$$

Hence, the common tangent is $x + y + a = 0$.

83. Let the coordinate system be chosen such that the given straight line is $x = p$ and the equations of the circles are $x^2 + y^2 = a^2$, $x^2 + y^2 = b^2$, $x^2 + y^2 = c^2$.

The line $x = p$ cuts these circles at A, B and C, respectively.

The coordinates of these points are $A(p, \sqrt{a^2 - p^2})$, $B(p, \sqrt{b^2 - p^2})$ and $C(p, \sqrt{c^2 - p^2})$.

Equations of the tangents at these points are

$$px + \sqrt{a^2 - p^2}\,y = a^2,\ px + \sqrt{b^2 - p^2}\,y = b^2$$

and $px + \sqrt{c^2 - p^2}\,y = c^2$.

These tangents intersect at

$$\left[\frac{p^2 - \sqrt{a^2 - p^2}\sqrt{b^2 - p^2}}{p}, \sqrt{a^2 - p^2} + \sqrt{b^2 - p^2}\right],$$

$$\left[\frac{p^2 - \sqrt{b^2 - p^2}\sqrt{c^2 - p^2}}{p}, \sqrt{b^2 - p^2} + \sqrt{c^2 - p^2}\right],$$

$$\left[\frac{p^2 - \sqrt{c^2 - p^2}\sqrt{c^2 - p^2}}{p}, \sqrt{c^2 - p^2} + \sqrt{a^2 - p^2}\right].$$

Area of Δ formed by the tangents at A, B, C is

$$\Delta = \frac{1}{2}\begin{vmatrix} \frac{p^2 - \sqrt{a^2 - p^2}\sqrt{b^2 - p^2}}{p} & \sqrt{a^2 - p^2} + \sqrt{b^2 - p^2} & 1 \\ \frac{p^2 - \sqrt{b^2 - p^2}\sqrt{c^2 - p^2}}{p} & \sqrt{b^2 - p^2} + \sqrt{c^2 - p^2} & 1 \\ \frac{p^2 - \sqrt{c^2 - p^2}\sqrt{c^2 - p^2}}{p} & \sqrt{c^2 - p^2} + \sqrt{a^2 - p^2} & 1 \end{vmatrix}$$

$$= \frac{(\sqrt{a^2 - p^2} - \sqrt{c^2 - p^2})(\sqrt{c^2 - p^2} - \sqrt{b^2 - p^2})(\sqrt{b^2 - p^2} - \sqrt{a^2 - p^2})}{2p}$$

$$= \frac{CA \cdot BC \cdot AB}{2p}$$

84. Let the curve be $y = f(x)$ and tangent drawn at $P(x, y)$ meets the X-axis at T.

We have, $OT = x^2$

Equation of tangent at $P(x, y)$;

$$Y - y = f'(x)(X - x)$$

$$\Rightarrow \quad T \equiv \left(\left|x - \frac{y}{f'(x)}\right|, 0\right)$$

$$\Rightarrow \quad \left|x - \frac{y}{f'(x)}\right| = x^2$$

$$\Rightarrow \quad x - \frac{f(x)}{f'(x)} = \pm x^2$$

$$\Rightarrow \quad \frac{xf'(x) - f(x)}{x^2} = \pm f'(x)$$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{f(x)}{x}\right) = \pm f'(x)$$

On integrating both the sides, we get $\frac{f(x)}{x} = \pm f(x) + c$

Since, the curve passes through (2, 1).

$$\therefore \quad \frac{1}{2} = \pm 1 + c$$

$$\Rightarrow \quad c = -\frac{1}{2}, \frac{3}{2}$$

$$\Rightarrow \quad f(x) = \frac{x}{2(x - 1)} \text{ or } f(x) = \frac{3x}{2(1 + x)}$$

Hence, possible curves are $y = \frac{x}{2(x - 1)}$ and $y = \frac{3x}{2(1 + x)}$.

85. For $y = x - x^3$, $\frac{dy}{dx} = 1 - 3x^2$

Therefore, the equation of the tangent at the point $P(x_1, y_1)$ is

$$y - y_1 = (1 - 3x_1^2)(x - x_1)$$

It meets the curve again at $Q(x_2, y_2)$.

Hence, $x_2 - x_2^3 - (x_1 - x_1^3) = (1 - 3x_1^2)(x_2 - x_1)$

$$\Rightarrow \quad (x_2 - x_1)[1 - (x_2^2 + x_1x_2 + x_1^2)] = (x_2 - x_1)(1 - 3x_1^2)$$

$$\Rightarrow \quad 1 - x_2^2 - x_1x_2 - x_1^2 = 1 - 3x_1^2$$

$$\Rightarrow \quad x_2^2 + x_1x_2 - 2x_1^2 = 0$$

$$\Rightarrow \quad \left(x_2 + \frac{x_1}{2}\right)^2 = \frac{9x_1^2}{4} \Rightarrow x_2 + \frac{x_1}{2} = \pm\frac{3x_1}{2}$$

Since, $x_1 \neq x_2$, we have $x_2 = -2x_1$

$\Rightarrow$ Q is $(-2x_1, -2x_1 + 8x_1^3)$.

If $L_1(\alpha, \beta)$ is the point of trisection of PQ, then $\alpha = \frac{2x_1 - 2x_1}{3} = 0$. Hence, L_1 lies on the Y-axis. If $L_2(h, k)$ is the other point of trisection, then $h = \frac{x_1 - 4x_1}{3} = -x_1$ and

$$k = \frac{y_1 - 4x_1 + 16x_1^3}{3}$$

i.e. $$k = \frac{x_1 - x_1^3 - 4x_1 + 16x_1^3}{3} = -x_1 + 5x_1^3$$

$$\Rightarrow \quad k = h - 5h^3$$

$\therefore$ Locus of (h, k) is $y = x - 5x^3$.

86. Let the curve be $y = f(x)$

The equation of the tangent at any point (x, y) is

$$Y - y = \frac{dy}{dx}(X - x)$$

Its intercept on the y-axis is given by $(X = 0)$

$$Y = y - x\frac{dy}{dx} = k\sqrt{x^2 + y^2}$$

So, $x\frac{dy}{dx} - y + k\sqrt{x^2 + y^2} = 0$ is the differential equation governing the curve. This can be written as

$$\frac{dy}{dx} = \frac{y - k\sqrt{x^2 + y^2}}{x}$$

Let $y = vx$, so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

The differential equation becomes

$$v + x\frac{dv}{dx} = v - k\sqrt{1 + v^2}$$

$$\Rightarrow \quad \frac{dv}{\sqrt{1 + v^2}} + k\frac{dx}{x} = 0$$

$$\Rightarrow \quad \log|v + \sqrt{1 + v^2}| + k\log|x| = c \quad \text{(on integrating)}$$

$$\Rightarrow \quad \log\left|\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}}\right| + k\log|x| = c$$

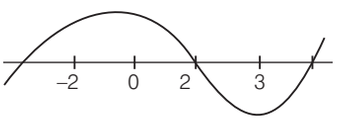
$$\Rightarrow \quad \log|y + \sqrt{x^2 + y^2}| + (k-1)\log|x| = c$$

$$\Rightarrow \quad (y + \sqrt{x^2 + y^2})\, x^{k-1} = c_1$$

87. We have, $2t^3 - 9t^2 + 30 - a = 0$

Any real root t_0 of this equation gives two real and distinct values of x if $|t_0| > 2$.

Thus, we need to find the condition for the equation in t to have three real and distinct roots none of which lies in $[-2,2]$.



Let $f(t) = 2t^3 - 9t^2 + 30 - a$

$$\Rightarrow \quad f'(t) = 6t^2 - 8t = 0 \quad \Rightarrow \quad t = 0, 3$$

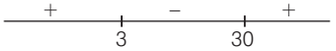
So, the equation $f(t) = 0$ has three real and distinct roots, if $f(0) \cdot f(3) < 0$

$$\Rightarrow \quad (30 - a)(54 - 81 + 30 - a) < 0$$

$$\Rightarrow \quad (30 - a)(3 - a) < 0$$

$$\Rightarrow \quad (a - 3)(a - 30) < 0, \text{ using number line rule i.e.,}$$

$$\Rightarrow \quad a \in (3, 30) \qquad \ldots(i)$$



Also, none of the roots lie in $[-2,2]$

if $f(-2) > 0$ and $f(2) > 0$

$$\Rightarrow \quad (-16 - 36 + 30 - a) > 0 \quad \text{and} \quad (16 - 36 + 30 - a) > 0$$

$$\Rightarrow \quad a + 22 < 0 \quad \text{and} \quad a - 10 < 0$$

$$\Rightarrow \quad a < -22 \text{ and } a < 10$$

$$\Rightarrow \quad a < -22 \qquad \ldots(ii)$$

From Eqs. (i) and (ii) no real value of a exists.

88. Slope of tangent at the point (x_1, y_1) is given by $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$.

Given curve, $(y - x^5)^2 = x(1 + x^2)^2$

$$\Rightarrow \quad 2(y - x^5)\left(\frac{dy}{dx} - 5x^4\right) = (1 + x^2)^2 + 2x(1 + x^2)\cdot 2x$$

Put $x = 1$ and $y = 3$, then

$$dy/dx = 8$$

89. Given, $f(x) = 2 + \cos x, \forall x \in R$

Statement I There exists a point $\in [t, t + \pi]$, where $f'(c) = 0$

Hence, Statement I is true.

Statement II $f(t) = f(t + 2\pi)$ is true. But Statement II is not correct explanation for Statement I.

90. As, $|f(x_1) - f(x_2)| \le (x_1 - x_2)^2, \forall x_1, x_2 \in R$

$$\Rightarrow \quad |f(x_1) - f(x_2)| \le |x_1 - x_2|^2 \qquad [\text{as } x^2 = |x|^2]$$

$$\therefore \quad \left|\frac{f(x_1) - f(x_2)}{x_1 - x_2}\right| \le |x_1 - x_2|$$

$$\Rightarrow \quad \lim_{x_1 \to x_2}\left|\frac{f(x_1) - f(x_2)}{x_1 - x_2}\right| \le \lim_{x_1 \to x_2}|x_1 - x_2|$$

$$\Rightarrow \quad |f'(x_1)| \le 0, \forall x_1 \in R$$

$\therefore \quad |f'(x)| \le 0$, which shows $|f'(x)| = 0$

[as modulus is non-negative or $|f'(x)| \ge 0$]

$\therefore \quad f'(x) = 0$ or $f(x)$ is constant function.

$\Rightarrow$ Equation of tangent at (1, 2) is

$$\frac{y - 2}{x - 1} = f'(x)$$

or $\qquad y - 2 = 0 \qquad [\because f'(x) = 0]$

$\therefore \quad y - 2 = 0$ is required equation of tangent.

91. Given, $\qquad y^3 + 3x^2 = 12y \qquad \ldots(i)$

On differentiating w.r.t. x, we get

$$3y^2\frac{dy}{dx} + 6x = 12\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{6x}{12 - 3y^2}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{12 - 3y^2}{6x}$$

For vertical tangent,

$$\frac{dx}{dy} = 0$$

$$\Rightarrow \quad 12 - 3y^2 = 0$$

$$\Rightarrow \quad y = \pm 2$$

On putting, $y = 2$ in Eq. (i), we get $x = \pm\frac{4}{\sqrt{3}}$ and again putting $y = -2$ in Eq. (i), we get $3x^2 = -16$, no real solution.

So, the required point is $\left(\pm\frac{4}{\sqrt{3}}, 2\right)$.

92. Slope of tangent to the curve, $y = f(x)$ is

$$\left.\frac{dy}{dx}\right|_{(3,\,4)} = f'(x)_{(3,\,4)}$$

Therefore, slope of normal $= -\dfrac{1}{f'(x)_{(3,4)}} = -\dfrac{1}{f'(3)}$

But $\quad -\dfrac{1}{f'(3)} = \tan\left(\dfrac{3\pi}{4}\right)$ [given]

$\Rightarrow \quad \dfrac{-1}{f'(3)} = \tan\left(\dfrac{\pi}{2} + \dfrac{\pi}{4}\right) = -1$

$f'(3) = 1$

93. We have, $y = \dfrac{x+6}{(x-2)(x-3)}$

Point of intersection with Y-axis $(0, 1)$

$$y' = \frac{(x^2 - 5x + 6)(1) - (x+6)(2x-5)}{(x^2 - 5x + 6)^2}$$

$\Rightarrow y' = 1$ at point $(0, 1)$.

$\therefore$ Slope of normal is -1.

Hence, equation of normal is $x + y = 1$.

$\therefore \left(\dfrac{1}{2}, \dfrac{1}{2}\right)$ satisfy it.

94. Given equation of curve is

$$x^2 + 2xy - 3y^2 = 0 \quad \ldots(i)$$

On differentiating w.r.t. x, we get

$$2x + 2xy' + 2y - 6yy' = 0 \Rightarrow y' = \frac{x+y}{3y-x}$$

At $\quad x = 1, y = 1, y' = 1$ i.e. $\left(\dfrac{dy}{dx}\right)_{(1,1)} = 1$

Equation of normal at $(1, 1)$ is

$$y - 1 = -\frac{1}{1}(x-1) \Rightarrow y - 1 = -(x-1)$$

$$\Rightarrow \quad x + y = 2 \quad \ldots(ii)$$

On solving Eqs. (i) and (ii) simultaneously, we get

$$\Rightarrow \quad x^2 + 2x(2-x) - 3(2-x)^2 = 0$$

$$\Rightarrow \quad x^2 + 4x - 2x^2 - 3(4 + x^2 - 4x) = 0$$

$$\Rightarrow \quad -x^2 + 4x - 12 - 3x^2 + 12x = 0$$

$$\Rightarrow \quad -4x^2 + 16x - 12 = 0$$

$$\Rightarrow \quad 4x^2 - 16x + 12 = 0$$

$$\Rightarrow \quad x^2 - 4x + 3 = 0$$

$$\Rightarrow \quad (x-1)(x-3) = 0$$

$$\therefore \quad x = 1, 3$$

Now, when $x = 1$, then $y = 1$

and when $x = 3$, then $y = -1$.

$\therefore \quad P = (1, 1)$ and $Q = (3, -1)$

Hence, normal meets the curve again at $(3, -1)$ in fourth quadrant.

Aliter

Given, $\quad x^2 + 2xy - 3y^2 = 0$

$$\Rightarrow \quad (x-y)(x+3y) = 0$$

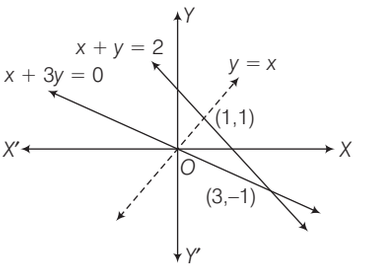
$$\Rightarrow \quad x - y = 0 \text{ or } x + 3y = 0$$

Equation of normal at $(1, 1)$ is

$$y - 1 = -1(x-1)$$

$$\Rightarrow \quad x + y - 2 = 0$$

It intersects $x + 3y = 0$ at $(3, -1)$ and hence normal meets the curve again in fourth quadrant.



CHAPTER

08

Monotonicity, Maxima and Minima

Learning Part

Session 1

- Monotonicity

Session 2

- Critical Points

Session 3

- Comparison of Functions Using Calculus

Session 4

- Introduction to Maxima and Minima
- Methods of Finding Extrema of Continuous Functions
- Convexity/Concavity and Point of Inflection
- Concept of Global Maximum/Minimum

Session 5

- Maxima and Minima of Discontinuous Functions

Practice Part

- JEE Type Examples
- Chapter Exercises

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Exercises with this symbol can be practised on your mobile. See title inside to activate for free.

Session 1

Monotonicity

Monotonicity

A function is said to be monotonic if it is either increasing or decreasing in its domain.

Strictly Increasing

Consider a function represented by the following graph.

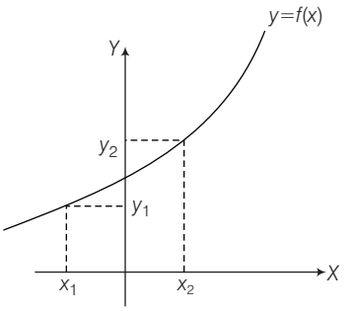


Figure 8.1

i.e. $x_1 < x_2$

implies $f(x_1) < f(x_2)$

Such a function is called a strictly increasing function or a monotonically increasing function.

For two different input arguments x_1 and x_2, where $x_1 < x_2$, $y_1 = f(x_1)$ will always be less than $y_2 = f(x_2)$.

The word 'monotonically' apparently has its origin in the word monotonous.

e.g. A monotonous routine is one in which one follows the same routine repeatedly or continuously.

Similarly, a monotonically increasing function is one that increases continuously.

e.g. consider $f(x) = [x]$.

For this function $x_1 < x_2$ does not always imply $f(x_1) < f(x_2)$.

However, $x_1 < x_2$ does imply $f(x_1) \le f(x_2)$.

In other words, $f(x) = [x]$ is not strictly (or monotonically) increases. It will nevertheless be termed **increasing**.

Strictly Decreasing

Now, we consider a function represented by the following graph.

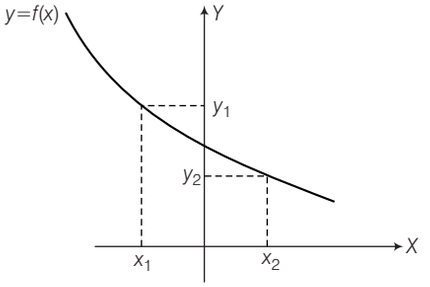


Figure 8.2

For two different input arguments x_1 and x_2, where $x_1 < x_2$, $y_1 = f(x_1)$ will always be greater than $y_2 = f(x_2)$.

i.e. $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

Such a function is called a strictly decreasing function or a monotonically decreasing function. e.g. consider $f(x) = -[x]$.

For this function $x_1 < x_2$ does not imply $f(x_1) > f(x_2)$.

However, $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$

In other words, $f(x) = -[x]$ is not strictly decreasing. It would only be termed **decreasing**. The following table list down a few Examples of functions and their behaviour in different intervals.

Function	Behaviour
$f(x) = x$	Strictly increasing on R
$f(x) = x^2$	Strictly decreasing on $(-\infty, 0]$ Strictly increasing on $[0, \infty)$
$f(x) = \sqrt{x}$	Strictly increasing on $[0, \infty)$
$f(x) = x^3$	Strictly increasing on R
$f(x) = \lvert x \rvert$	Strictly decreasing on $(-\infty, 0]$
	Strictly increasing on $[0, \infty)$
$f(x) = \frac{1}{x}$	Neither decreasing nor increasing on R. Strictly decreasing on $(-\infty, 0)$, Strictly decreasing on $(0, \infty)$
$f(x) = [x]$	Increasing on R
$f(x) = \{x\}$	Neither increasing nor decreasing on R. However, strictly increasing on $[n, n+1)$, where $n \in Z$
$f(x) = \sin x$	Neither increasing nor decreasing on R. Strictly increasing on $\left[\left(2n - \frac{1}{2}\right)\pi, \left(2n + \frac{1}{2}\right)\pi\right]; n \in Z$ Strictly decreasing on $\left[\left(2n + \frac{1}{2}\right)\pi, \left(2n + \frac{3}{2}\right)\pi\right]; n \in Z$

Function	Behaviour
$f(x) = \cos x$	Neither increasing nor decreasing on R. Strictly increasing on $[(2n-1)\pi, 2n\pi]; n \in Z$ Strictly decreasing on $[2n\pi, (2n+1)\pi]; n \in Z$
$f(x) = \tan x$	Neither increasing nor decreasing on R. Strictly increasing on $\left[\left(n-\frac{1}{2}\right)\pi, \left(n+\frac{1}{2}\right)\pi\right]; n \in Z$
$f(x) = e^x$	Strictly increasing on R
$f(x) = e^{-x}$	Strictly decreasing on R
$f(x) = \ln x$	Strictly increasing on $(0, \infty)$

Monotonicity with the Help of Derivative

Let us now deduce the condition(s) on the derivative of a function $f(x)$ which determines whether $f(x)$ is increasing/decreasing on a given interval. We are assuming that $f(x)$ is everywhere differentiable.

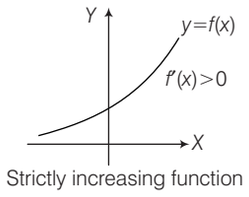


Figure 8.3

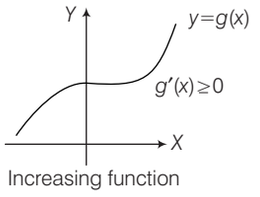


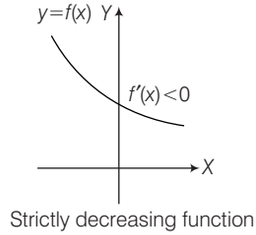
Figure 8.4

In figure 8.3, the function $y = f(x)$ is a strictly increasing function. Notice that the slope of the tangent drawn at any point on this curve is always positive.

Hence, a sufficient condition for $f(x)$ to be **strictly increasing** on a given domain D is $\boldsymbol{f'(x) > 0\ \forall\ x \in D}$ (Later on, we will see that this is not a necessary condition for a function to be strictly increasing).

In figure 8.4, the function $y = g(x)$ is not strictly increasing though it is increasing. Notice that $g'(x) > 0$ or $g'(x) = 0, \forall x$. $g'(x)$ is never negative.

Hence, a sufficient condition for $g(x)$ to be **increasing** on a given domain D is $\boldsymbol{g'(x) \geq 0,\ \forall\ x \in D}$. Now, consider $f(x)$ and $g(x)$ in given figure.



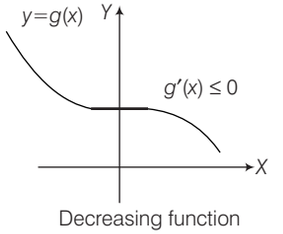


Figure 8.5

Extending as above, we get the conditions for a **strictly decreasing and decreasing** functions

Strictly decreasing : $f'(x) < 0 \quad \forall\ x \in D$

Decreasing : $g'(x) \leq 0 \quad \forall\ x \in D$

Note

(i) Above conditions on the derivatives to be applied, the function must be differentiable in the given domain. However, these conditions will hold good even if the function is non differentiable, at a finite number (or infinitely countable number) of points. e.g. $f(x) = [x] + \sqrt{\{x\}}$ is strictly increasing on R. However, $f(x)$ is non-differentiable at all integers (a countable set).

(ii) A function must be continuous for above conditions to be applied. Consider $y = \{x\}$. This is non-differentiable (due to discontinuities) at all integers. At all other points, $y' = 1 > 0$. However, we know that $y = \{x\}$ is not strictly increasing.

Similarly, $y = \frac{1}{x}$ is non-differentiable (and non-continuous) at $x = 0$. At all other points, $y' = \frac{-1}{x^2} < 0$, so y should be strictly decreasing on $R \setminus \{0\}$. However, it is not strictly decreasing on $R \setminus \{0\}$ although it is strictly decreasing in the separate intervals $(-\infty, 0)$ and $(0, \infty)$.

Therefore, we see that discontinuous functions cannot be subjected to the derivative condition. Even though they may be discontinuous only at finite (on infinitely countable) number of points.

Properties

(i) If $f(x)$ is strictly increasing, then $f^{-1}(x)$ is also strictly increasing. Similarly, if $f(x)$ is strictly decreasing, then $f^{-1}(x)$ is also strictly decreasing.

(ii) If $f(x)$ and $g(x)$ have the same monotonicity (both increasing or decreasing) on $[a, b]$, then $f(g(x))$ and $g(f(x))$ are monotonically increasing on $[a, b]$.

(iii) If $f(x)$ and $g(x)$ have opposite monotonicity on $[a, b]$, then $f(g(x))$ and $g(f(x))$ are strictly decreasing on $[a, b]$.

(iv) The inverse of a continuous function is continuous

(v) If $f'(x) > 0, \forall\ x \in (a, b)$ except for a finite (or an infinitely countable) number of points, where $f'(x) = 0$, $f(x)$ is still strictly increasing on (a, b). That is why, we said earlier that $f'(x) > 0, \forall\ x \in D$ is not necessary condition for strict increase. e.g. In a later Example, we will consider the graph of the function $f(x) = x + \cos x$. We will see that $f'(x) = 1 - \sin x$ is not always positive $\left(\text{at } x = 2n\pi + \frac{\pi}{2}, n \in Z, f'(x) = 0\right)$; even then, $f(x)$ increases strictly because the points at which $f'(x) = 0$ are countable.

(vi) Similarly if $f'(x) < 0, \forall\ x \in (a, b)$ except for a finite (or an infinitely countable) number of points, where $f'(x) = 0$, $f(x)$ is still strictly decreasing on (a, b).

Classification of Strictly Increasing Functions

Increasing functions can be classified as

(i) **Concave up** When $f'(x) > 0$ and $f''(x) > 0$, $\forall x \in$ domain.

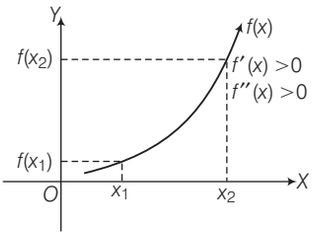


Figure 8.6

From the figure 8.6, it is clear that the graph of $f(x)$ is concave up and increasing as x increases.

(ii) **Concave down** When $f'(x) > 0$ and $f''(x) < 0$, $\forall x \in$ domain.

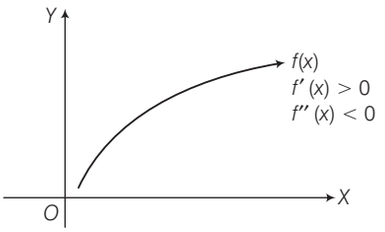


Figure 8.7

From the figure 8.7 it is clear that the graph of $f(x)$ is concave down and increasing as x increases.

(iii) When $f'(x) > 0$ and $f''(x) = 0$, $\forall x \in$ domain.

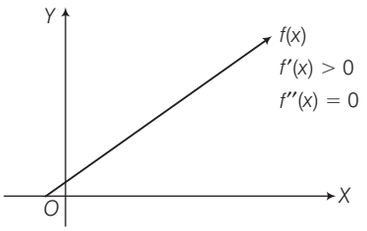


Figure 8.8

From the figure 8.8, it is clear that the graph of $f(x)$ is neither concave up nor concave down but still increasing as x increases.

Classification of Strictly Decreasing Functions

Decreasing functions can be classified as follows :

(i) **Concave up** When $f'(x) < 0$ and $f''(x) > 0$, $\forall x \in$ domain.

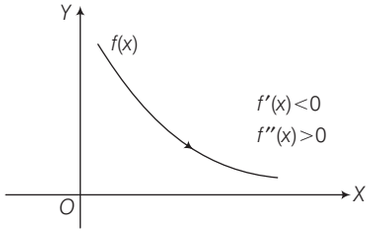


Figure 8.9

From the figure 8.9, it is clear that the graph of the function $f(x)$ is concave up and decreasing as x increases.

(ii) **Concave down** When $f'(x) < 0$ and $f''(x) < 0$, $\forall x \in$ domain.

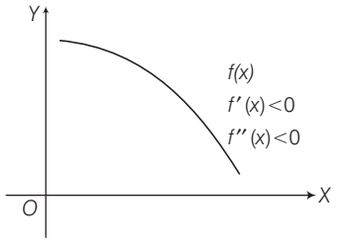


Figure 8.10

From the figure 8.10, it is clear that the graph of $f(x)$ is concave down and $f(x)$ is decreasing as x increases.

(iii) When $f'(x) < 0$ and $f''(x) = 0$, $\forall x \in$ domain.

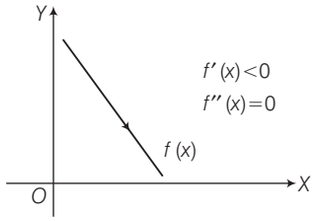


Figure 8.11

From the figure 8.11, it is clear that the graph of $f(x)$ is neither concave up nor concave down but $f(x)$ is decreasing as x increases.

Example 1 Find the interval in which $f(x) = 2x^3 + 3x^2 - 12x + 1$ is increasing.

Sol. Given, $f(x) = 2x^3 + 3x^2 - 12x + 1$

Differentiating both the sides, we have

$$f'(x) = 6x^2 + 6x - 12$$

$$\Rightarrow \quad f'(x) = 6(x^2 + x - 2)$$

$$\Rightarrow \quad f'(x) = 6(x+2)(x-1)$$

Using number line rule, we have

Hence, $f'(x) \geq 0$

when $x \in (-\infty, -2] \cup [1, \infty)$

$\Rightarrow f(x)$ is increasing when $x \in (-\infty, -2] \cup [1, \infty)$

Example 2 Find the interval in which $f(x) = x^3 - 3x^2 - 9x + 20$ is strictly increasing or strictly decreasing.

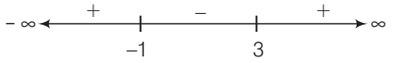
Sol. Given, $f(x) = x^3 - 3x^2 - 9x + 20$

$$f'(x) = 3x^2 - 6x - 9$$

$$\Rightarrow \quad f'(x) = 3(x^2 - 2x - 3)$$

$$\Rightarrow \quad f'(x) = 3(x-3)(x+1)$$

Using number line method as shown in figure,



$f'(x) > 0$

For $x \in (-\infty, -1) \cup (3, \infty)$ and $f'(x) < 0$,

For $x \in (-1, 3)$

Thus, $f(x)$ is strictly increasing for $x \in (-\infty, -1) \cup (3, \infty)$ and strictly decreasing for $x \in (-1, 3)$.

Example 3 Show that the function $f(x) = x^2$ is a strictly increasing function on $(0, \infty)$.

Sol. Given, $f(x) = x^2 \quad \Rightarrow \quad f'(x) = 2x$

$\Rightarrow \quad f'(x) > 0$ for $x \in (0, \infty) \qquad [\because \ x \in (0, \infty) \Rightarrow 2x > 0]$

Thus, $f(x)$ is strictly increasing for $x \in (0, \infty)$.

Example 4 ***Find the interval of increasing or decreasing of the*** $f(x) = \int_{-1}^{x} (t^2 + 2t)(t^2 - 1)\, dt$.

Sol. Given, $f(x) = \int_{-1}^{x} (t^2 + 2t)(t^2 - 1)\, dt$

On differentiating both the sides, we have

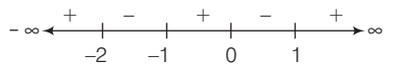
$$f'(x) = (x^2 + 2x)(x^2 - 1)\left\{\frac{d}{dx}(x)\right\} - (1-2)(1-1)\left\{\frac{d}{dx}(-1)\right\}$$

[using Leibnitz rule]

$$\Rightarrow \quad f'(x) = (x^2 + 2x)(x^2 - 1)$$

$$\Rightarrow \quad f'(x) = x(x+2)(x+1)(x-1)$$

Using number line rule as shown in figure,



Clearly, $f'(x) \geq 0$ when $x \in (-\infty, -2] \cup [-1, 0] \cup [1, \infty)$

and $f'(x) \leq 0$ when $x \in [-2, -1] \cup [0, 1]$.

Hence, $f(x)$ is increasing, when $x \in (-\infty, -2] \cup [-1, 0] \cup [1, \infty)$ and $f(x)$ is decreasing, when $x \in [-2, -1] \cup [0, 1]$.

Remark

In above example, Leibnitz rule is used which is stated as

$$\frac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)} f(t)\, dt\right] = f(\psi(x))\left\{\frac{d}{dx}\psi(x)\right\} - f(\phi(x))\left\{\frac{d}{dx}\phi(x)\right\}.$$

Example 5 ***The function*** $f(x) = \sin^4 x + \cos^4 x$ ***increasing, if***

(a) $0 < x < \pi/8$ (b) $\pi/4 < x < 3\pi/8$

(c) $3\pi/8 < x < 5\pi/8$ (d) $5\pi/8 < x < 3\pi/4$

[IIT JEE 1999]

Sol. Here, $f(x) = \sin^4 x + \cos^4 x$

$$\Rightarrow \quad f'(x) = 4\sin^3 x \cdot \cos x + 4\cos^3 x\,(-\sin x)$$

$$f'(x) = 4\sin x \cos x\,(\sin^2 x - \cos^2 x)$$

$$f'(x) = 2(\sin 2x)(-\cos 2x) \Rightarrow f'(x) = -\sin 4x$$

Now, $f'(x) \geq 0$, if $\sin 4x \leq 0 \quad \Rightarrow \pi \leq 4x \leq 2\pi$

$\Rightarrow \quad \pi/4 \leq x \leq \pi/2$

Here, $\left(\frac{\pi}{4}, \frac{3\pi}{8}\right)$ is only subset of $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$.

Hence, (b) is the correct answer.

Example 6 ***Let*** $f(x) = \int_0^x e^t (t-1)(t-2)\, dt$. ***Then, f decreases in the interval*** ***[IIT JEE 2000]***

(a) $(-\infty, -2)$ (b) $(-2, -1)$

(c) $[1, 2]$ (d) $(2, \infty)$

Sol. Here, $f(x) = \int_0^x e^t (t-1)(t-2)\, dt$

$f'(x) = e^x (x-1)(x-2)$, [using Leibnitz rule]

Using number line rule for $f'(x)$, we get

$f'(x) \leq 0$ when $1 \leq x \leq 2$, as e^x is always positive.

$\therefore$ f decreases when $1 \leq x \leq 2$

Hence, (c) is the correct answer.

Example 7 If $f(x) = x \cdot e^{x(1-x)}$, then $f(x)$ is ***[IIT JEE 2000]***

(a) increasing on $\left[-\frac{1}{2}, 1\right]$ (b) decreasing on R

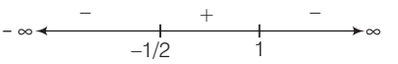
(c) increasing on R (d) decreasing on $\left[-\frac{1}{2}, 1\right]$

Sol. Here, $f'(x) = x \cdot e^{x(1-x)} \cdot (1-2x) + 1 \cdot e^{x(1-x)}$

$$f'(x) = e^{x(1-x)}[x - 2x^2 + 1]$$

$$f'(x) = -e^{x(1-x)}(x-1)(2x+1)$$

Using number line rule for $f'(x)$ we get, $f'(x) \geq 0$, when $x \in \left[-\frac{1}{2}, 1\right]$ as shown in figure.



Hence, (a) is the correct answer.

Example 8 Find the interval for which $f(x) = x - \sin x$ is increasing or decreasing.

Sol. Given, $f(x) = x - \sin x$,

Differentiating both the sides w.r.t. x, we have

$$f'(x) = 1 - \cos x$$

We know, $-1 \le \cos x \le 1$

or $\cos x \le 1$

$\Rightarrow$ $1 - \cos x \ge 0$

Therefore, $f'(x) \ge 0, \forall\, x \in R$

Which shows $f(x)$ is increasing for the entire number scale.

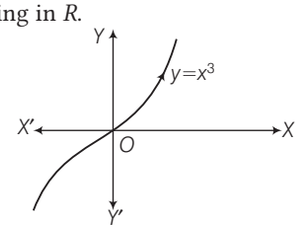
i.e. all real numbers.

Example 9 ***Discuss the nature of following functions graphically.***

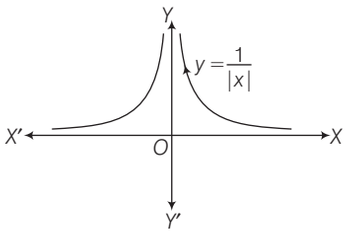
(i) $f(x) = x^3$ (ii) $f(x) = \dfrac{1}{|x|}$

(iii) $f(x) = e^x$ (iv) $f(x) = [x]$

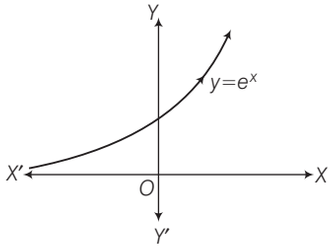
Sol. (i) $f(x) = x^3$ can be graphically plotted as shown in the following figure, which shows $f(x) = x^3$ is strictly increasing in R.



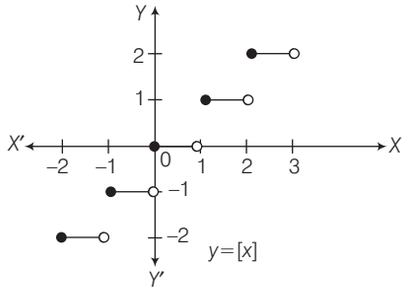
(ii) $f(x) = \dfrac{1}{|x|}$ can be graphically plotted as shown in the following figure, which shows $f(x) = \dfrac{1}{|x|}$ is strictly increasing in $]-\infty, 0[$ and strictly decreasing in $]0, \infty[$.



(iii) $f(x) = e^x$ can be graphically plotted as shown in the following figure, which shows $f(x) = e^x$ is strictly increasing in R.



(iv) $f(x) = [x]$ can be graphically plotted as shown in the following figure, which shows $f(x) = [x]$ is increasing but not strictly increasing i.e. non-decreasing in R.



Example 10 If $H(x_0) = 0$ for some $x = x_0$ and $\dfrac{d}{dx} H(x) > 2cx\, H(x)$ for all $x \ge x_0$, where $c > 0$, then prove that $H(x)$ cannot be zero for any $x > x_0$.

Sol. Given that, $\dfrac{d}{dx} H(x) > 2cx\, H(x)$

$\Rightarrow$ $\dfrac{d}{dx} H(x) - 2cx\, H(x) > 0$

$\Rightarrow$ $\dfrac{d}{dx}\{H(x)\}e^{-cx^2} - 2cxe^{-cx^2} \cdot H(x) > 0$

$\Rightarrow$ $\left\{\dfrac{d}{dx} H(x)\right\} e^{-cx^2} + H(x)\left\{\dfrac{d}{dx} e^{-cx^2}\right\} > 0$

$\Rightarrow$ $\left\{\dfrac{d}{dx} H(x) \cdot e^{-cx^2}\right\} > 0$

$\therefore$ $H(x)\, e^{-cx^2}$ is an strictly increasing function.

But $H(x_0) = 0$ and e^{-cx^2} is always positive.

$\Rightarrow$ $H(x_0) > 0$ for all $x > x_0$

$\Rightarrow$ $H(x)$ cannot be zero for any $x > x_0$.

Example 11 If $f(x)$ is a decreasing function and attain positive values, then the set of value of 'k', for which the major axis of the ellipse $\dfrac{x^2}{f(k^2 + 2k + 5)} + \dfrac{y^2}{f(k + 11)} = 1$, is the X-axis, is $(-a, b)$, then $(a + b)$ is

Sol. Here, $f(x)$ is decreasing function and major axis is X-axis

$\Rightarrow$ $f(k^2 + 2k + 5) > f(k + 11)$

[$\because$ for the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ major axis lies along X-axis, then $a^2 > b^2$]

As, $f(x)$ is decreasing, therefore

$$k^2 + 2k + 5 < k + 11$$

$\Rightarrow$ $k^2 + k - 6 < 0$

$\Rightarrow$ $(k + 3)(k - 2) < 0$

$\Rightarrow$ $k \in (-3, 2) = (-a, b)$

$\therefore$ $a + b = 5$

Example 12 Let $f(x)=3x+5$, then show that $f(x)$ is strictly increasing and $f^{-1}(x)$ exists and is strictly increasing for $x\in R$.

Sol. Here, $f(x)=3x+5$

$f'(x)=3>0$, which is strictly increasing for $x\in R$.

Now, finding $f^{-1}(x)$, let $f(x)=y$, $y=3x+5$

or $$x=\frac{y-5}{3}$$

or $$f^{-1}(y)=\frac{y-5}{3} \quad [\text{as } f(x)=y\Rightarrow x=f^{-1}(y)]$$

$$f^{-1}(x)=\frac{x-5}{3}$$

Which shows $f^{-1}(x)=\frac{x-5}{3}$ exists for all $x\in R$ and is strictly increasing as $\frac{d}{dx}(f^{-1}(x))=\frac{1}{3}>0$ for all $x\in R$.

Example 13 Let $\phi(x)=\sin(\cos x)$, then check whether it is increasing or decreasing in $[0,\pi/2]$.

Sol. I. Given, $\phi(x)=\sin(\cos x)$

$$\Rightarrow \quad \phi'(x)=\cos(\cos x)\cdot(-\sin x)$$

$$\Rightarrow \quad \phi'(x)=-\cos(\cos x)\cdot\sin x$$

$$\left[\because \cos(\cos x)>0 \text{ and } \sin x>0, \forall x\in\left(0,\frac{\pi}{2}\right)\right]$$

Therefore, it is clearly decreasing for $x\in[0,\pi/2]$ as $\phi'(x)\le 0$.

Aliter

Here, $f(x)=\sin x$ and $g(x)=\cos x$ are increasing and decreasing in $[0,\pi/2]$.

$\Rightarrow (fog)(x)=\phi(x)=\sin(\cos x)$ is decreasing.

Example 14 Let $\phi(x)=\cos(\cos x)$, then check whether it is increasing or decreasing in $[0,\pi/2]$.

Sol. Given, $\phi(x)=\cos(\cos x)$

$$\Rightarrow \quad \phi'(x)=-\sin(\cos x)\cdot(-\sin x)$$

$$\Rightarrow \quad \phi'(x)=\sin x\sin(\cos x)$$

$$\left[\because \sin(\cos x)>0 \text{ and } \sin x>0, \forall x\in\left(0,\frac{\pi}{2}\right)\right]$$

Therefore, it is clearly increasing for $x\in[0,\pi/2]$ as $\phi'(x)\ge 0$.

Aliter

Here, $f(x)=\cos x$ and $g(x)=\cos x$ are decreasing in $[0,\pi/2]$.

$\Rightarrow (fog)(x)=\cos(\cos x)=\phi(x)$ is increasing.

Example 15 ***Let*** $f(x)=\begin{cases} xe^{ax}; & x\le 0 \\ x+ax^2-x^3; & x>0 \end{cases}$ ***where a is positive constant. Find the interval in which*** $f'(x)$ ***is increasing.*** ***[IIT JEE 1996]***

Sol. Given, $f(x)=\begin{cases} xe^{ax}; & x\le 0 \\ x+ax^2-x^3; & x>0 \end{cases}$

Differentiating both the sides, we have

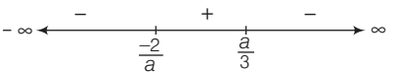
$$f'(x)=\begin{cases} axe^{ax}+e^{ax}; & x\le 0 \\ 1+2ax-3x^2; & x>0 \end{cases}$$

Again, differentiating both sides, we have

$$f''(x)=\begin{cases} 2ae^{ax}+a^2xe^{ax}; & x\le 0 \\ 2a-6x; & x>0 \end{cases}$$

Now, $f''(x)=0$, then in the interval $x\le 0$ the root is $x=-\frac{2}{a}$ and in interval $x>0$ root is $x=\frac{a}{3}$

Using sign scheme or number line rule as shown in figure, we get



$f'(x)$ decreases on $\left(-\infty,-\frac{2}{a}\right]\cup\left[\frac{a}{3},\infty\right)$ and increases on $\left[-\frac{2}{a},\frac{a}{3}\right]$.

Example 16 If $a<0$ and $f(x)=e^{ax}+e^{-ax}$ is monotonically decreasing. Find the interval to which x belongs.

Sol. Given, $a<0$ and $f(x)=e^{ax}+e^{-ax}$ is monotonically decreasing. ...(i)

$$\Rightarrow \quad f'(x)<0$$

$$\Rightarrow \quad ae^{ax}-ae^{-ax}<0$$

$$\Rightarrow \quad a\left(\frac{e^{2ax}-1}{e^{ax}}\right)<0 \qquad \text{...(ii)}$$

As from Eq. (i), $a<0$

$$\Rightarrow \quad (e^{2ax}-1)>0$$

$$\Rightarrow \quad e^{2ax}>1$$

$$\Rightarrow \quad 2ax>0$$

$$\Rightarrow \quad ax>0$$

$$\Rightarrow \quad x<0 \qquad [\text{as } a<0]$$

Thus, $f(x)$ is monotonically decreasing, if $x\in(-\infty,0)$.

Example 17 If $0<\alpha<\frac{\pi}{6}$, then the value of $(\alpha\ \text{cosec}\ \alpha)$ is

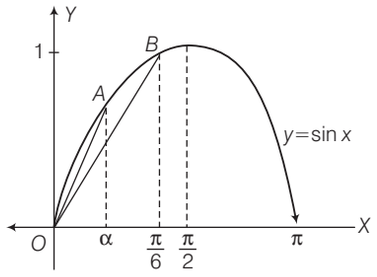
(a) less than $\frac{\pi}{3}$ (b) more than $\frac{\pi}{3}$

(c) less than $\frac{\pi}{6}$ (d) more than $\frac{\pi}{6}$

Sol. From the figure, we can say

Slope of $OA >$ Slope of OB

where $A = (\alpha, \sin\alpha)$ and $B\left(\frac{\pi}{6}, \sin\frac{\pi}{6}\right)$.



$$\Rightarrow \quad \frac{\sin\alpha - 0}{\alpha - 0} > \frac{\sin\frac{\pi}{6} - 0}{\frac{\pi}{6} - 0}$$

$$\Rightarrow \quad \frac{\sin\alpha}{\alpha} > \frac{3}{\pi} \quad \text{or} \quad \frac{\alpha}{\sin\alpha} < \frac{\pi}{3} \Rightarrow \alpha \operatorname{cosec}\alpha < \frac{\pi}{3}$$

Hence, (a) is the correct answer.

Example 18 If $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c and d are real numbers and $3b^2 < c^2$, is an increasing cubic function and $g(x) = af'(x) + bf''(x) + c^2$, then

(a) $\int_a^x g(t)\,dt$ is a decreasing function

(b) $\int_a^x g(t)\,dt$ is an increasing function

(c) $\int_a^x g(t)\,dt$ is a neither increasing nor a decreasing function

(d) None of the above

Sol. $f'(x) = 3ax^2 + 2bx + c > 0$ [since, $f(x)$ is increasing]

$$\Rightarrow \quad a > 0 \text{ and } b^2 - 3ac < 0 \Rightarrow a > 0 \text{ and } b^2 < 3ac$$

Also, $g(x) = af'(x) + bf''(x) + c^2$

$$g(x) = 3a^2x^2 + 2abx + ac + 6abx + 2b^2 + c^2$$

$$g(x) = 3a^2x^2 + 8abx + (2b^2 + c^2 + ac)$$

where $D = 64a^2b^2 - 4 \cdot 3a^2 \cdot (2b^2 + c^2 + ac)$

$$= 4a^2(16b^2 - 6b^2 - 3c^2 - 3ac)$$

$$= 4a^2(10b^2 - 3c^2 - 3ac) < 4a^2(10b^2 - 3c^2 - b^2)$$

$$[\text{as } 3ac > b^2 \Rightarrow -3ac < -b^2]$$

$$= 4a^2(9b^2 - 3c^2) = 12a^2(3b^2 - c^2) \,[\text{given } 3b^2 < c^2]$$

$\therefore \quad D < 0 \Rightarrow g(x) > 0, \forall\ x \in R$

$\therefore \int_a^x g(t)\,dt$ is an increasing function.

$$\left[\because \frac{d}{dx}\left(\int_a^x g(t)dt\right) = g(x) > 0\right]$$

Hence, (b) is the correct answer.

Example 19 If $f : R \to R$, $f(x)$ is a differentiable bijective function, then which of the following is true?

(a) $(f(x) - x)\, f''(x) < 0, \forall\ x \in R$

(b) $(f(x) - x)\, f''(x) > 0, \forall\ x \in R$

(c) If $(f(x) - x)\, f''(x) > 0$, then $f(x) = f^{-1}(x)$ has no solution

(d) If $(f(x) - x)\, f''(x) > 0$, then $f(x) = f^{-1}(x)$ has at least one real solution

Sol. As, $(f(x) - x)\, f''(x) < 0, \forall\ x \in R$

$$\Rightarrow \quad (f(x) - x) > 0 \text{ and } f''(x) < 0)$$

or $\quad (f(x) - x) < 0 \text{ and } f''(x) > 0)$

Can't be true as $f(x) - x > 0$ and $f'(x)$ are decreasing. Then, $f(x)$ has to intersect the line $y = x$.

Similarly, $f(x) - x < 0$ and $f'(x)$ is increasing, is not possible.

Also, $\quad f(x) - x \neq 0$

$$\Rightarrow \quad f(x) = f^{-1}(x) \text{ has no solution.}$$

Hence, (c) is the correct answer.

Example 20 ***If*** $f(x)$ ***and*** $g(x)$ ***are two positive and increasing functions, then***

(a) $(f(x))^{g(x)}$ is always increasing

(b) if $(f(x))^{g(x)}$ is decreasing, then $f(x) < 1$

(c) if $(f(x))^{g(x)}$ is increasing, then $f(x) > 1$

(d) if $f(x) > 1$, then $(f(x))^{g(x)}$ is increasing

Sol. Let $h(x) = (f(x))^{g(x)}$

$$\Rightarrow \quad \log(h(x)) = g(x)\{\log f(x)\}$$

$$\Rightarrow \quad \frac{1}{h(x)} \cdot h'(x) = \frac{g(x)}{f(x)} \cdot f'(x) + \{\log f(x)\}\, g'(x)$$

and $h(x)$ is increasing, if $\log(f(x)) > 0 \Rightarrow f(x) > 1$

Hence, (d) is the correct answer.

Example 21 ***If the function*** $y = \sin(f(x))$ ***is monotonic for all values of x [where*** $f(x)$ ***is continuous], then the maximum value of the difference between the maximum and the minimum value of*** $f(x)$***, is***

(a) π (b) 2π (c) $\frac{\pi}{2}$ (d) None of these

Sol. As, $y = \sin(f(x))$ is monotonic for

$$f(x) \in \left[2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right]$$

or $\quad \left[2n\pi + \frac{\pi}{2}, 2n\pi + \frac{3\pi}{2}\right]$

$\therefore$ The maximum value of difference is π.

Hence, (a) is the correct answer.

Example 22 ***If*** $f''(x) > 0$ ***and*** $f'(1) = 0$ ***such that*** $g(x) = f(\cot^2 x + 2\cot x + 2)$, ***where*** $0 < x < \pi$, ***then the interval in which*** $g(x)$ ***is decreasing is***

(a) $(0, \pi)$ (b) $\left(\frac{\pi}{2}, \pi\right)$

(c) $\left(\frac{3\pi}{4}, \pi\right)$ (d) $\left(0, \frac{3\pi}{4}\right)$

Sol. Here, $g(x) = f(\cot^2 x + 2\cot x + 2)$

$$\Rightarrow \quad g'(x) = f'(\cot^2 x + 2\cot x + 2) \cdot \{-2\cot x \ \text{cosec}^2 x - 2\ \text{cosec}^2\ x\}$$

for $g(x)$ to be decreasing, $g'(x) < 0$

$$\Rightarrow \quad f'\{(\cot x + 1)^2 + 1\} \cdot (-2\ \text{cosec}^2\ x)(\cot x + 1) < 0$$

$$\Rightarrow \quad f'\{(\cot x + 1)^2 + 1\} \cdot (\cot x + 1) > 0 \quad \ldots(\text{i})$$

As $f''(x) > 0 \Rightarrow f'(x)$ is increasing, then

$$f'\{(\cot x + 1)^2 + 1\} > f'(1) = 0, \ \forall \ \ x \in \left(0, \frac{3\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \pi\right)$$

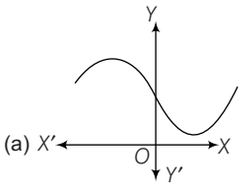
Thus, Eq. (i) holds, if $\cot x + 1 > 0$

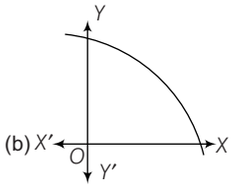
$$\Rightarrow \quad \cot x > -1, \forall \ x \in \left(0, \frac{3\pi}{4}\right)$$

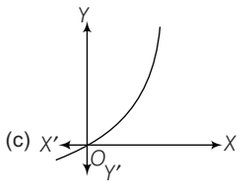
Hence, (d) is the correct answer.

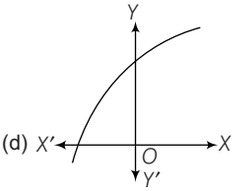
Exercise for Session 1

1. The curve $y = f(x)$ which satisfies the condition $f'(x) > 0$ and $f''(x) < 0$ for all real x, is

(a) 

(b) 

(c) 

(d) 

2. The interval in which $f(x) = \cot^{-1} x + x$ increases, is

(a) R (b) $(0, \infty)$ (c) $R - \{n\pi\}$ (d) None of these

3. The interval in which $f(x) = 3\cos^4 x + 10\cos^3 x + 6\cos^2 x - 3$ increase or decrease in $(0, \pi)$

(a) decreases on $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$ and increases on $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{2\pi}{3}, \pi\right)$ (b) decreases on $\left(\frac{\pi}{2}, \pi\right)$ and increases on $\left(0, \frac{\pi}{2}\right)$

(c) decreases on $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{2\pi}{3}, \pi\right)$ and increases on $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$ (d) decreases on $\left(0, \frac{\pi}{2}\right)$ and increases on $\left(\frac{\pi}{2}, \pi\right)$

4. The interval in which $f(x) = \int_0^x \{(t+1)(e^t - 1)(t-2)(t+4)\}\, dt$ increases and decreases

(a) increases on $(-\infty, -4) \cup (-1, 0) \cup (2, \infty)$ and decreases on $(-4, -1) \cup (0, 2)$
(b) increases on $(-\infty, -4) \cup (-1, 2)$ and decreases on $(-4, -1) \cup (2, \infty)$
(c) increases on $(-\infty, -4) \cup (2, \infty)$ and decreases on $(-4, 2)$
(d) increases on $(-4, -1) \cup (0, 2)$ and decreases on $(-\infty, -4) \cup (-1, 0) \cup (2, \infty)$

5. The interval of monotonicity of the function $f(x) = \frac{x}{\log_e x}$, is

(a) increases when $x \in (e, \infty)$ and decreases when $x \in (0, e)$
(b) increases when $x \in (e, \infty)$ and decreases when $x \in (0, e) - \{1\}$
(c) increases when $x \in (0, e)$ and decreases when $x \in (e, \infty)$
(d) increases when $x \in (0, e) - \{1\}$ and decreases when $x \in (e, \infty)$

6. Let $f(x) = x^3 + ax^2 + bx + 5\ \sin^2 x$ be an increasing function on the set R. Then,

(a) $a^2 - 3b + 15 > 0$ (b) $a^2 - 3b + 5 < 0$ (c) $a^2 - 3b + 15 < 0$ (d) $a^2 - 3b + 5 > 0$

7. Let $g(x) = f(x) + f(1-x)$ and $f''(x) > 0, \forall\ x \in (0, 1)$. Then, $g(x)$ is

(a) increasing on $\left(0, \frac{1}{2}\right)$ and decreasing on $\left(\frac{1}{2}, 1\right)$ (b) increasing on $\left(\frac{1}{2}, 1\right)$ and decreasing on $\left(0, \frac{1}{2}\right)$

(c) increasing on $(0, 1)$ (d) decreasing on $(0, 1)$

Session 2

Critical Points

Critical Points

It is a collection of points for which,

(i) $f(x)$ does not exist (ii) $f'(x)$ does not exist

(iii) $f'(x)=0$.

All the values of x obtained from above conditions are said to be the critical points.

It should be noted that critical points are the interior points of an interval.

Example 23 Find the critical points for $f(x)=(x-2)^{2/3}(2x+1)$.

Sol. Given, $f(x)=(x-2)^{2/3}(2x+1)$

$$\Rightarrow \quad f'(x)=\frac{2}{3}(x-2)^{-1/3}(2x+1)+(x-2)^{2/3}\cdot 2$$

$$\text{or} \quad f'(x)=2\left[\frac{(2x+1)}{3(x-2)^{1/3}}+\frac{(x-2)^{2/3}}{1}\right]$$

Clearly, $f'(x)$ is not defined at $x=2$, so $x=2$ is a critical point.

Another critical point is given by

$$f'(x)=0$$

$$\text{i.e.} \quad 2\left[\frac{(2x+1)+3(x-2)}{(x-2)^{1/3}}\right]=0$$

$$\Rightarrow \quad 5x-5=0 \Rightarrow \quad x=1$$

Hence, $x=1$ and $x=2$ are two critical points of $f(x)$.

Example 24 Find all the values of a for which the function $f(x)=(a^2-3a+2)\cos\left(\frac{x}{2}\right)+(a-1)x$, possess critical points.

Sol. Given, $f(x)=(a^2-3a+2)\cos\left(\frac{x}{2}\right)+(a-1)x$

$$\Rightarrow \quad f'(x)=-\frac{1}{2}(a-1)(a-2)\sin\left(\frac{x}{2}\right)+(a-1)$$

$$\Rightarrow \quad f'(x)=(a-1)\left[1-\frac{1}{2}(a-2)\sin\left(\frac{x}{2}\right)\right]$$

If $f(x)$ possess critical points, then

$$f'(x)=0$$

$$\Rightarrow \quad (a-1)\left[1-\left(\frac{a-2}{2}\right)\sin\frac{x}{2}\right]=0$$

$$\Rightarrow \quad a=1 \text{ and } 1-\left(\frac{a-2}{2}\right)\sin\frac{x}{2}=0,$$

but at $a=1 \Rightarrow f(x)=0$

Thus, $\sin\left(\frac{x}{2}\right)=\frac{2}{a-2}$

$$\Rightarrow \quad \left|\frac{2}{a-2}\right|\le 1$$

$$\Rightarrow \quad |a-2|\ge 2 \Rightarrow a-2\ge 2$$

or $a-2\le -2$

$a\ge 4$ or $a\le 0$

Therefore, $a\in(-\infty,0]\cup[4,\infty)$

Example 25 The set of all values of 'b' for which the function $f(x)=(b^2-3b+2)(\cos^2 x-\sin^2 x)+(b-1)x+\sin 2$ does not possess stationary points is

(a) $[1,\infty)$ (b) $(0,1)\cup(1,4)$

(c) $\left(\frac{3}{2},\frac{5}{2}\right)$ (d) None of these

Sol. Here, $f(x)=(b^2-3b+2)(\cos^2 x-\sin^2 x)+(b-1)x+\sin 2$

$$=(b^2-3b+2)\cos 2x+(b-1)x+\sin 2$$

$$\Rightarrow \quad f'(x)=(b^2-3b+2)\cdot(-2\sin 2x)+(b-1).$$

As, $f(x)$ does not possess stationary points.

$$\Rightarrow \quad f'(x)\ne 0$$

$$\Rightarrow \quad (b-1)(b-2)(-2\sin 2x)+(b-1)\ne 0,$$

for any $x\in R$

$$\Rightarrow \quad (b-1)\{1-2(b-2)\sin 2x\}\ne 0$$

$$\Rightarrow \quad \left|\frac{1}{2(b-2)}\right|>1 \text{ and } b\ne 1$$

$$\Rightarrow \quad -\frac{1}{2}<b-2<\frac{1}{2} \text{ and } b\ne 1$$

$$\Rightarrow \quad \frac{3}{2}<b<\frac{5}{2} \Rightarrow b\in\left(\frac{3}{2},\frac{5}{2}\right)$$

Hence, (c) is the correct answer.

Example 26 ***Find the set of critical points of the function***

$$f(x) = x - \log x + \int_2^x \left(\frac{1}{z} - 2 - 2\cos 4z\right) dz.$$

Sol. Here, $f(x) = x - \log x + \int_2^x \left(\frac{1}{z} - 2 - 2\cos 4z\right) dz$

$\Rightarrow \quad f'(x) = 1 - \frac{1}{x} + \left(\frac{1}{x} - 2 - 2\cos 4x\right)(1) - 0$

$= -1 - 2\cos 4x$ [using Leibnitz rule]

Put $f'(x) = 0$,

$\Rightarrow \quad \cos 4x = -\frac{1}{2}$ or $\cos 4x = \cos\left(\pi - \frac{\pi}{3}\right)$

or $4x = 2n\pi + \frac{2\pi}{3}, n \in \text{Integer}$

$\Rightarrow \quad x = \frac{n\pi}{2} \pm \frac{\pi}{6}, n \in I$

But $\log x$ is defined for $x > 0$

$\therefore$ For $n = 0$, $x = \pm\frac{\pi}{6}$

$\Rightarrow \quad x = \frac{\pi}{6}$ [neglecting $x = -\pi/6$]

$\therefore$ Set of critical points $= \left\{\frac{\pi}{6}, \frac{n\pi}{2} \pm \frac{\pi}{6}\right\}$, where $n \in N$.

Exercise for Session 2

1. Determine all the critical points for the function $f(x) = 6x^5 + 33x^4 - 30x^3 + 100$.
2. Find the critical points of $f(x) = x^{2/3}(2x - 1)$.
3. Determine all the critical points for the function $f(x) = xe^{x^2}$.
4. The number of critical points of $f(x) = \max\{\sin x, \cos x\}, \forall\ x \in (-2\pi, 2\pi)$ is

   (a) 5 (b) 6

   (c) 7 (d) 8

Session 3

Comparison of Functions Using Calculus

Comparison of Functions Using Calculus

If we want to compare $\phi(x)$ and $g(x)$ then we consider a function $f(x)=\phi(x)-g(x)$ or $f(x)=g(x)-\phi(x)$ and check whether $f(x)$ is increasing or decreasing in given domain of $\phi(x)$ and $g(x)$. The procedure is illustrated by following examples :

Example 27 Using calculus, find the order relation between x and $\tan^{-1}x$ when $x\in[0,\infty)$.

Sol. Let $f(x)=x-\tan^{-1}(x)$

$$\Rightarrow \quad f'(x)=1-\frac{1}{1+x^2}$$

$$\Rightarrow \quad f'(x)=\frac{x^2}{1+x^2}\geq 0,\ \forall x\in[0,\infty)$$

Thus, $f(x)$ is an increasing function.

As, we know $x_1\leq x_2\Rightarrow f(x_1)\leq f(x_2)$ for increasing function

$$\therefore \quad x\geq 0,\ \forall x\in[0,\infty)$$

$$\Rightarrow \quad f(x)\geq f(0),\ \forall x\in[0,\infty)$$

$$\Rightarrow \quad x-\tan^{-1}x\geq 0-\tan^{-1}(0),\ \forall x\in[0,\infty)$$

$$\Rightarrow \quad x\geq\tan^{-1}x,\forall x\in[0,\infty)$$

Thus, the above relation is the order relation between x and $\tan^{-1}x$.

Example 28 ***Using calculus, find the order relation between x and*** $\tan^{-1}x$ ***when*** $x\in(-\infty,0]$.

Sol. Let $f(x)=(x)-\tan^{-1}(x)$

$$\Rightarrow \quad f'(x)=1-\frac{1}{1+x^2}$$

$$\Rightarrow \quad f'(x)=\frac{x^2}{1+x^2}\geq 0,\ \forall x\in(-\infty,0]$$

Thus, $f(x)$ is increasing function.

As, we know $x_1\leq x_2\Rightarrow f(x_1)\leq f(x_2)$ for increasing function.

$$\therefore \quad x\leq 0,\ \forall x\in(-\infty,0]$$

$$\Rightarrow \quad f(x)\leq f(0),\ \forall x\in(-\infty,0]$$

$$\Rightarrow \quad x-\tan^{-1}x\leq 0,\ \forall x\in(-\infty,0]$$

$$\Rightarrow \quad x\leq\tan^{-1}x,\ \forall x\in(-\infty,0]$$

Thus, the above relation is the order relation between x and $\tan^{-1}x$.

Example 29 For all $x\in(0,1)$ ***[IIT JEE 2000]***

(a) $e^x<1+x$ (b) $\log_e(1+x)<x$

(c) $\sin x>x$ (d) $\log_e x>x$

Sol. (a) Let $f(x)=e^x-1-x$

$$\Rightarrow\ f'(x)=e^x-1>0,\ \forall x\in(0,1)$$

So, $f(x)$ is increasing, when $0<x<1$

$$\Rightarrow \quad f(x)>f(0) \text{ or } e^x-1-x>0$$

$$\Rightarrow \quad e^x>1+x$$

Hence, (a) is false.

(b) Let $g(x)=\log(1+x)-x$

$$\Rightarrow\ g'(x)=\frac{1}{1+x}-1=\frac{-x}{1+x}<0,\ \forall x\in(0,1)$$

So, $g(x)$ is decreasing, when $0<x<1$

$$\Rightarrow \quad g(0)>g(x)\ \Rightarrow\ \log(1+x)<x$$

Hence, (b) is true.

(c) Let $h(x)=\sin x-x\ \Rightarrow\ h'(x)=\cos x-1<0,\ \forall x\in(0,1)$

So, $h(x)$ is decreasing, when $0<x<1\ \Rightarrow\ h(x)<h(0)$

$$\Rightarrow \quad \sin x<x$$

Hence, (c) is false.

(d) Let $g(x)=\log x-x\ \Rightarrow\ g'(x)=\frac{1}{x}-1$

$$\therefore\ g'(x)>0,\forall x\in(0,1) \text{ or } g(x)<g(1)$$

$$\Rightarrow \quad \log x-x<-1$$

$$\Rightarrow\ x-1>\log x \text{ or } x>\log x$$

Hence, (d) is false.

Thus, (b) is the correct answer.

Example 30 Prove that $\left(\tan^{-1}\frac{1}{e}\right)^2+\frac{2e}{\sqrt{e^2+1}}<(\tan^{-1}e)^2+\frac{2}{\sqrt{e^2+1}}$.

Sol. Let us consider a function $f(x)$,

i.e. $f(x)=(\tan^{-1}x)^2+\frac{2}{\sqrt{x^2+1}}$ for all $x\in R$

$$\therefore \quad f'(x)=\frac{2\tan^{-1}x}{1+x^2}-\frac{2x}{(x^2+1)^{3/2}}$$

$$=\frac{2}{1+x^2}\left[\tan^{-1}x-\frac{x}{\sqrt{x^2+1}}\right]$$

$\Rightarrow \quad f'(x) = \dfrac{2}{1+x^2}\, g(x),$

where $g(x) = \tan^{-1} x - \dfrac{x}{\sqrt{x^2+1}}$ ...(i)

$\therefore \quad g'(x) = \dfrac{1}{1+x^2} - \dfrac{1}{\sqrt{x^2+1}} + \dfrac{x^2}{(x^2+1)^{3/2}}$

$= \dfrac{1}{1+x^2} - \dfrac{1}{(x^2+1)^{3/2}}$

$= \dfrac{1}{1+x^2}\left(1 - \dfrac{1}{\sqrt{(x^2+1)}}\right) > 0$ for all $x \in R$

$\Rightarrow$ $g(x)$ is increasing for $x \in R$

$\Rightarrow$ $g(x) > g(0)$ for all $x \in R$

$\Rightarrow \quad \tan^{-1} x - \dfrac{x}{\sqrt{x^2+1}} > 0$ ...(ii)

$\therefore$ From Eqs. (i) and (ii), $f'(x) > 0$ for all $x > 0$

$\therefore$ $f(x)$ is increasing for all $x > 0$

$\Rightarrow \quad f(1/e) < f(e)$

$\Rightarrow \quad \left(\tan^{-1}\dfrac{1}{e}\right)^2 + \dfrac{2e}{\sqrt{e^2+1}} < (\tan^{-1} e)^2 + \dfrac{2}{\sqrt{e^2+1}}$

Exercise for Session 3

1. Show that $\sin x < x < \tan x$ for $0 < x < \pi/2$.

2. Show that $\dfrac{x}{(1+x)} < \log(1+x) < x$ for $x > 0$.

3. Show that $x - \dfrac{x^3}{6} < \sin x$ for $0 < x < \dfrac{\pi}{2}$.

4. If $ax^2 + \dfrac{b}{x} \geq c$ for all positive x, where $a, b > 0$, then

(a) $27ab^2 \geq 4c^3$ (b) $27ab^2 < 4c^3$

(c) $4ab^2 \geq 27c^3$ (d) None of these

5. If $ax + \dfrac{b}{x} \geq c$ for all positive x, where $a, b > 0$, then

(a) $ab < \dfrac{c^2}{4}$ (b) $ab \geq \dfrac{c^2}{4}$

(c) $ab \geq \dfrac{c}{4}$ (d) None of these

Session 4

Introduction to Maxima and Minima, Methods of Finding Extrema of Continuous Functions, Convexity/Concavity & Point of Inflection, Concept of Global Maximum/Minimum

Introduction to Maxima and Minima

By the maximum/minimum value of function $f(x)$ we mean local or regional maximum/minimum value and not the greatest/least value attainable by the function. It is also possible in a function that local maximum at one point is smaller than local minimum at another point. Sometimes, we use the word extrema for maxima and minima.

Definition A function $f(x)$ is said to have a maximum at $x = a$, if $f(a)$ is greatest of all values in the suitably small neighbourhood of a, where $x = a$ is an interior point in the domain of $f(x)$.

Analytically, this means $\boldsymbol{f(a) \geq f(a+h)}$ and $\boldsymbol{f(a) \geq f(a-h)}$, where $h \geq 0$ (very small quantity).

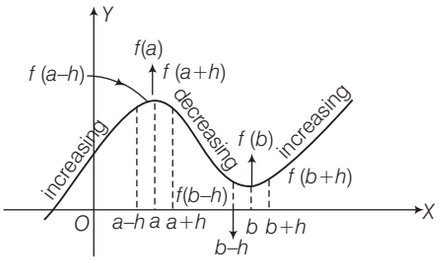


Figure 8.12

Similarly, a function $y = f(x)$ is said to have a minimum at $x = b$, if $f(b)$ is smallest of all values in the suitably small neighbourhood of b, where $x = b$ is an interior point in the domain of $f(x)$.

Analytically, $\boldsymbol{f(b) \leq f(b+h)}$ and $\boldsymbol{f(b) \leq f(b-h)}$, where $h \geq 0$ (very small quantity).

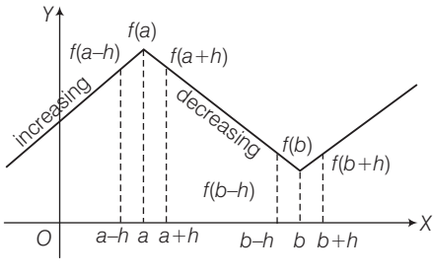


Figure 8.13

Methods of Finding Extrema of Continuous Functions

The following tests apply to a continuous function in order to get the extrema

First Derivative Test

As we know the function attains maximum, when it has assumed its maximum value and attains minimum, when it has assumed its minimum value which could be shown as

(i) At a critical point $x = a$

(a) **When $\boldsymbol{f(x)}$ attains maximum at $\boldsymbol{(x = a)}$**

Consider the following graph

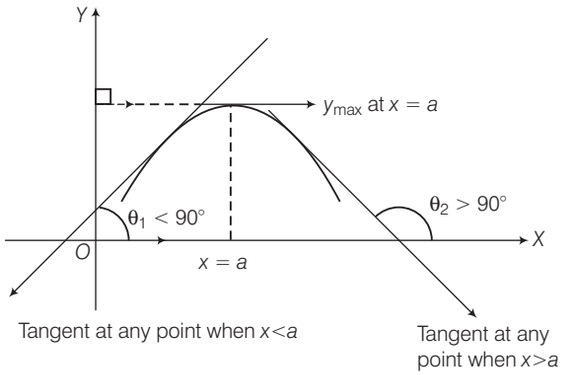


Figure 8.14

From above graph, we see that

$$\begin{cases} \text{for } x < a, \theta_1 < 90° \Rightarrow \tan\theta_1 > 0 \text{ or increasing for } x < a \\ \text{for } x = a, \tan\theta = 0 \Rightarrow \text{neither increasing nor decreasing for } x = a \\ \text{for } x > a, \theta_2 > 90° \Rightarrow \tan\theta_2 < 0 \text{ or decreasing for } x > a \end{cases}$$

Thus, we can say,

$f(x)$ is maximum at some point $(x = a)$.

$$\Rightarrow \quad \begin{cases} f(x) \text{ is increasing for } x < a \\ f(x) \text{ is decreasing for } x > a \end{cases}$$

(b) **When $f(x)$ attains minimum at $(x = a)$**
Consider the following graph

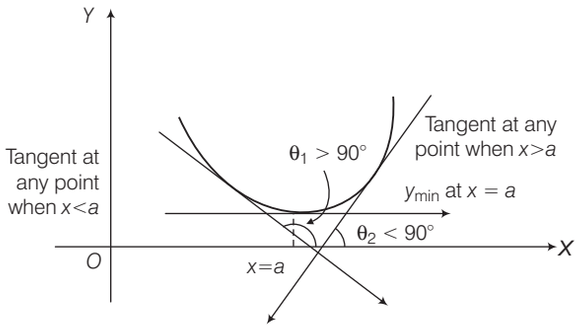


Figure 8.15

From the above graph, we see that

$$\begin{cases} \text{for } x < a, \theta_1 > 90^\circ \Rightarrow \tan\theta_1 < 0 \text{ or decreasing when } x < a \\ \text{for } x = a, \tan\theta = 0 \Rightarrow \text{neither increasing nor} \\ \text{decreasing for } x = a \\ \text{for } x > a, \theta_2 < 90^\circ \Rightarrow \tan\theta_2 > 0 \text{ or increasing when } x > a \end{cases}$$

Thus, we can say,

$f(x)$ is minimum at some points '$x = a$',

$$\Rightarrow \quad \begin{cases} f(x) \text{ is decreasing for } x < a \\ f(x) \text{ is increasing for } x > a \end{cases}$$

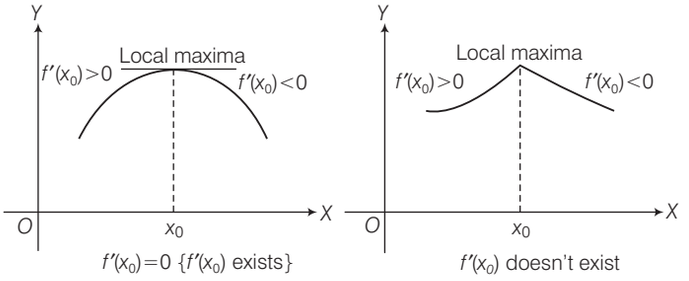
Here, some of the examples are given to make it more clear.

Remark *(Ex. Nos. 31-33)*

Statement of example Number 31, 32, 33 can be used directly as result.

Example 31 If $f'(x)$ changes from positive to negative at x_0 while moving from left to right,
i.e. $f'(x) > 0, \quad x < x_0$
$f'(x) < 0, \quad x > x_0$, *then $f(x)$ has local maximum value at $x = x_0$.*

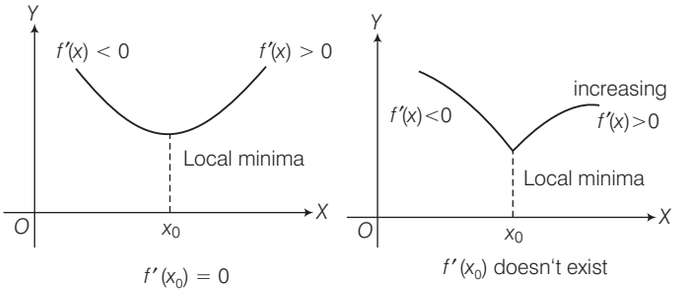
Sol. Consider the following graph



In both the graph we see that $f'(x)$ changes its sign from positive to negative when we move through x_0 and the function has local maxima at $x = x_0$.

Example 32 If $f'(x)$ changes from negative to positive at x_0 while moving from left to right,
i.e. $f'(x) < 0, \; x < x_0$
$f'(x) > 0, \; x > x_0$,
then $f(x)$ has local minimum value at $x = x_0$

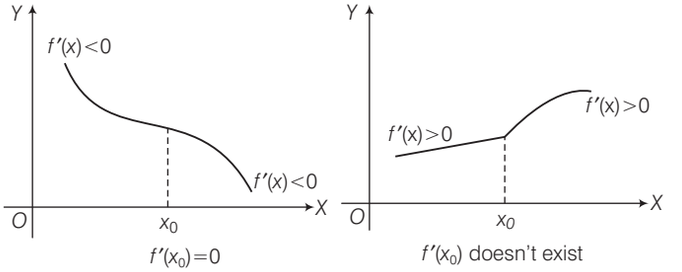
Sol. Consider the following graph



In both the graph we see that $f'(x)$ changes its sign from negative to positive when we move through x_0 and the curve has local minima at $x = x_0$.

Example 33 If sign of $f'(x)$ doesn't change at x_0, while moving from left to right, then $f(x)$ has neither a maximum nor a minimum at x_0.

Sol. Consider the following graph



In both the graph we see that $f'(x)$ has same sign when we move through x_0, therefore $f(x)$ has neither maxima nor minima at $x = x_0$.

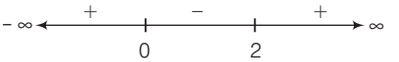
Example 34 Let $f(x) = x^3 - 3x^2 + 6$ find the point at which $f(x)$ assumes local maximum and local minimum.

Sol. Given, $\quad f(x) = x^3 - 3x^2 + 6$

$$\Rightarrow \quad f'(x) = 3x^2 - 6x = 3x(x - 2)$$

Using number line rule,

$f'(x)$ changes sign from +ve to –ve at $x = 0$ and $f'(x)$ changes sign from –ve to +ve at $x = 2$.



Therefore, at $x = 0$, we have local maxima and at $x = 2$, we have local minima.

Example 35 Let $f(x) = x^3$ find the point at which $f(x)$ assumes local maximum and local minimum.

Sol. Given, $f(x) = x^3$

$\Rightarrow$ $f'(x) = 3x^2$

Here, $f'(x) \geq 0$ for all x, it does not change sign from negative to positive or positive to negative.

Thus, $f(x)$, we has neither maximum nor minimum.

Example 36 ***Let*** $f(x) = x + \frac{1}{x}$, $x \neq 0$***. Discuss the maximum and minimum value of*** $f(x)$***.***

Sol. Here, $f(x) = x + \frac{1}{x}$

$\Rightarrow$ $f'(x) = 1 - \frac{1}{x^2}$

$$f'(x) = \frac{x^2 - 1}{x^2} = \frac{(x-1)(x+1)}{x^2}$$

Using number line rule, we have maximum at $x = -1$ and minimum at $x = 1$

$\therefore$ At $x = -1$, we have local maximum $\Rightarrow f(x)_{\max} = -2$

and at $x = 1$, we have local minimum $\Rightarrow f(x)_{\min} = 2$

Example 37 The function $f(x) = \int_{-1}^{x} t(e^t - 1)(t-1)(t-2)^3(t-3)^5 dt$ has a local maximum at x equals ***[IIT JEE 1999]***

(a) 0 (b) 1

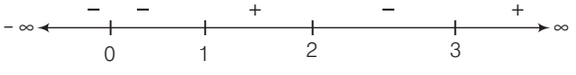
(c) 2 (d) 3

Sol. Given, $f(x) = \int_{-1}^{x} t(e^t - 1)(t-1)(t-2)^3(t-3)^5 dt$

$$f'(x) = x(e^x - 1)(x-1)(x-2)^3(x-3)^5$$

[using Leibnitz rule]

Using number line rule for $f'(x)$, we get the following figure which shows local maxima at $x = 2$ as $f'(x)$ changes from (+ve) to (−ve) and local minima at $x = 1$ and $x = 3$ as $f'(x)$ changes from (−ve) to (+ve).



$\therefore$ Local minima at $x = 1,3$ and local maxima at $x = 2$.

Hence, (c) is the correct answer.

(ii) At left end point a and right end point b in $[a, b]$

Let $f(x)$ be defined on $[a, b]$ and if $f'(x) < 0$ for $x > a$, then $f(x)$ has local maximum at $x = a$ and local minimum at $x = b$.

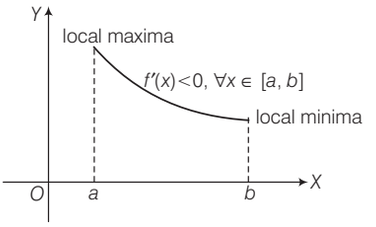


Figure 8.16

Again, if $f'(x) > 0$ for $x > a$, then $f(x)$ has local minimum at $x = a$ and local maximum at $x = b$.

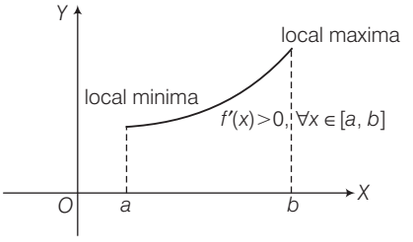


Figure 8.17

We can summarise above result as

(a) If a function is strictly increasing in $[a, b]$, then
$\begin{cases} f(a) \text{ is local minimum} \\ f(b) \text{ is local maximum} \end{cases}$

(b) If a function is strictly decreasing in $[a, b]$, then
$\begin{cases} f(a) \text{ is local maximum} \\ f(b) \text{ is local minimum} \end{cases}$

Example 38 Find the local maximum and local minimum of $f(x) = x^3 + 3x$ in $[-2, 4]$.

Sol. Given, $f(x) = x^3 + 3x$

$\Rightarrow$ $f'(x) = 3x^2 + 3$ which is strictly increasing for all $x \in R$ and thus, increasing for $[-2, 4]$.

Hence, local minimum is $f(-2) = (-2)^3 + 3(-2) = -14$

and local maximum is $f(4) = (4)^3 + 3(4) = 76$

Example 39 ***If*** $f(x) = \begin{cases} 3x^2 + 12x - 1, & -1 \leq x \leq 2 \\ 37 - x, & 2 < x \leq 3, \end{cases}$

then

(a) $f(x)$ is increasing on $[-1, 2]$ ***[IIT JEE 1993]***

(b) $f(x)$ is continuous on $[-1, 3]$

(c) $f'(x)$ does not exist at $x = 2$

(d) $f(x)$ has the maximum value at $x = 2$

Sol. Given, $f(x) = \begin{cases} 3x^2 + 12x - 1, & -1 \leq x \leq 2 \\ 37 - x, & 2 < x \leq 3 \end{cases}$

$\Rightarrow$ $f'(x) = \begin{cases} 6x + 12, & -1 \leq x < 2 \\ -1, & 2 < x \leq 3 \end{cases}$

(a) Which shows $f'(x) > 0$ for $x \in [-1, 2]$

So, $f(x)$ is increasing on $[-1, 2]$. Hence, (a) is correct.

(b) For continuity of $f(x)$. [check at $x = 2$]

$\because \quad f(x) = 3x^2 + 12x - 1, -1 \le x \le 2$

It is a polynomial, therefore continuous in $[-1, 2]$.

Also, for (2,3], $f(x) = 37 - x$ which is again a polynomial, therefore continuous in (2,3].

Now, we have to check continuity of $f(x)$ at $x = 2$

$$\text{LHL} = \lim_{h\to 0} f(2-h) = \lim_{h\to 0} 3(2-h)^2 + 12(2-h) - 1 = 35$$

$$\text{RHL} = \lim_{h\to 0} f(2+h) = \lim_{h\to 0} 37 - (2+h) = 35$$

RHL = 35 , LHL = 35 and $f(2) = 35$

So, (b) is correct.

(c) As discussed in previous chapter,

$$Rf'(2) = \lim_{h\to 0} \frac{37-(2+h)-35}{h} = -1$$

and $$Lf'(2) = \lim_{h\to 0} \frac{3(2-h)^2 + 12(2-h) - 1 - 35}{-h} = 24$$

So, not differentiable at $x = 2$

Hence, (c) is correct.

(d) We know $f(x)$ is increasing on $[-1, 2]$ and decreasing on (2, 3]. Thus, maximum at $x = 2$

Hence, (d) is correct.

$\therefore$ Hence, (a), (b), (c) and (d) all are correct answers.

Example 40 ***Let*** $f(x) = \sin x - x$ ***on*** $[0, \pi/2]$ ***, find local maximum and local minimum.***

Sol. Given, $f(x) = \sin x - x$

$$f'(x) = \cos x - 1, \ \forall x \in [0, \pi/2]$$

$$(\cos x - 1) \le 0, \forall x \in [0, \pi/2], \text{ as } \cos x \le 1$$

$\therefore \quad f'(x) \le 0, \forall x \in [0, \pi/2]$

$\therefore$ $f(x)$ is decreasing for $x \in [0, \pi/2]$.

Hence, maximum value of $f(x)$ is at $x = 0$

i.e. $f_{\max}(0) = \sin 0 - 0 = 0$

and minimum value of $f(x)$ is at $x = \pi/2$

i.e. $$f_{\min}\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} - \frac{\pi}{2} = 1 - \frac{\pi}{2}$$

"n^{th} Derivative Test"

First we find the root of $f'(x) = 0$. Suppose $x = a$ is one of the roots of $f'(x) = 0$.

Now, find $f''(x)$ at $x = a$

(i) If $f''(a) =$ negative, then $f(x)$ is maximum at $x = a$

(ii) If $f''(a) =$ positive, then $f(x)$ is minimum at $x = a$

(iii) If $f''(a) =$ zero

Then, we find $f'''(x)$ at $x = a$

If $f'''(a) \ne 0$, then $f(x)$ has neither maximum nor minimum (inflexion point), at $x = a$.

But, if $f'''(a) = 0$, then find $f^{iv}(a)$

If $f^{iv}(a) =$ positive, then $f(x)$ is minimum at $x = a$.

If $f^{iv}(a) =$ negative, then $f(x)$ is maximum at $x = a$.

and so on, process is repeated till point is discussed.

Convexity/Concavity and Point of Inflection

Observe the two graphs sketched in the figure below. What is the difference between them? Although they are both increasing, the first graph's rate of increase in itself increasing whereas the rate of increase is decreasing in case of the second graph.

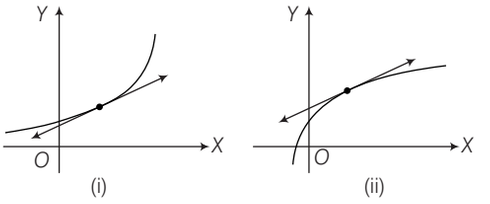


Figure 8.18

On graph (i), if you draw a tangent any where, the entire curve will lie above this tangent. Such a curve is called a **concave upwards curve.**

For graph (ii), the entire curve will lie below any tangent drawn to itself. Such a curve is called a **concave downwards curve.**

The concavity's nature can of course be restricted to particular intervals. e.g. A graph might be concave upwards in some interval while concave downwards in another shown in fig. 8.19.

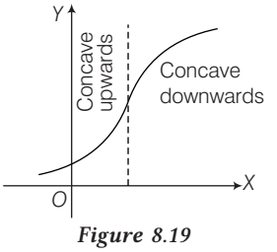


Figure 8.19

Relation of Concavity with the Derivative

Let us again consider a graph in figure. This is a concave upwards curve. We see that the rate of increase of the graph itself increases with increasing x, i.e. rate of increase of slope is positive.

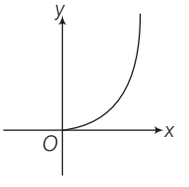


Figure 8.20

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) > 0 \Rightarrow \frac{d^2y}{dx^2} > 0$$

Similarly, for **a concave downwards curve,** $\frac{d^2y}{dx^2} < 0$

The nature of concavity is simply related to the sign of the second derivative.

We now finally come to what we mean by point of inflection.

Consider $f(x) = x^3$ again $f'(0) = f''(0) = 0$

$f''(x) < 0$ for $x < 0$

and > 0 for $x > 0$

$\Rightarrow f''(x)$ changes sign as x crosses 0.

$\Rightarrow f(x)$ changes the nature of its concavity as x crosses 0.

Such a point is called a **point of inflection**, a point at which the concavity of the graph changes.

Notice that $f''(a) = 0$ alone is not sufficient to guarantee a point of inflection at $x = a$. $f''(x)$ must also change sign as x crosses a.

e.g. In $f(x) = x^4$, $f''(0) = 0$, but $x = 0$ is not a point of inflection since $f''(x)$ does not change its sign as x crosses 0. From the higher order derivative test, we know that $x = 0$ is a local minimum for $f(x) = x^4$.

A summary of results for maxima, minima and point of Inflection

	First order derivative test	Second order derivative test	Higher order derivative test
Max.	$f'(a) = 0$ $f'(x)$ changes sign from +ve to –ve as x crosses a	$f'(a) = 0$ $f''(a) < 0$	$f'(a) = 0$ $f''(a) = 0$ $\vdots$ $f^{n-1}(a) = 0$ $f^n(a) < 0$ where n is even (If n is odd, $x = a$ is not an extremum point; it is a point of inflextion)
Min.	$f'(a) = 0$ $f'(x)$ changes sign from –ve to +ve as x crosses a	$f'(a) = 0$ $f''(a) > 0$	$f'(a) = 0$ $f''(a) = 0$ $\vdots$ $f^{n-1}(a) = 0$ $f^n(a) > 0$ where n is even (If n is odd, $x = a$ is not an extremum point; it is a point of inflextion)
Point of inflection			$f''(x)$ change sign at $x = a$

Example 41 Let $f(x) = x(x-1)^2$, find the point at which $f(x)$ assumes maximum and minimum.

Sol. Given, $f(x) = x(x-1)^2$

$$f'(x) = 2x(x-1) + (x-1)^2$$

$$f'(x) = (x-1)[2x + x - 1]$$

$$f'(x) = (x-1)(3x-1)$$

Using number line rule for $f'(x)$, we have the following figure which shows $f'(x)$ changes sign from + ve to – ve at $x = 1/3$. Hence, at $x = 1/3$, we have maximum and $f'(x)$ changes sign from – ve to + ve at $x = 1$.

Hence, $f(x)$ is minimum at $x = 1$.

Aliter

We have, $f'(x) = (x-1)(3x-1)$ and $f''(x) = 6x - 4$

Let $f'(x) = 0 \Rightarrow x = 1, 1/3$ [critical points]

$\therefore$ $f''(1) = 2 > 0$, i.e. minimum at $x = 1$

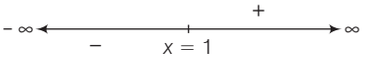
and $f''(1/3) = -2 < 0$, i.e. maximum at $x = 1/3$

Example 42 ***Let*** $f(x) = (x-1)^4$ ***discuss the point at which*** $f(x)$ ***assumes maximum or minimum value.***

Sol. Given, $f(x) = (x-1)^4$

$$f'(x) = 4(x-1)^3$$

Using number line rule.



Shows $f'(x)$ changes sign from –ve to + ve.

Therefore, $f(x)$ assumes minimum at $x = 1$.

Aliter

Given, $f(x) = (x-1)^4 \Rightarrow f'(x) = 4(x-1)^3$

Let $f'(x) = 0 \Rightarrow x = 1$

Now, $f''(x) = 12(x-1)^2$ which is zero at $x = 1$

i.e. $f''(1) = 0$

Thus, finding $f'''(x) = 24(x-1)$ which is again zero at $x = 1$

i.e. $f'''(1) = 0$

Again, finding $f^{iv}(x) = 24$ which is positive at $x = 1$,

i.e. $f^{iv}(1) > 0$

Therefore, minimum at $x = 1$.

Example 43 Discuss the function $f(x) = x^6 - 3x^4 + 3x^2 - 5$, and plot the graph.

Sol. The domain is obviously R.

$f(x)$ is an even function.

Since $f(x)$ is a polynomial function, it is continuous and differentiable on R.

$f'(x) = 6x^5 - 12x^3 + 6x = 6x(x^4 - 2x^2 + 1) = 6x(x^2 - 1)^2$

$f'(x) = 0 \Rightarrow 0, \pm 1$

$f''(x) = 6(x^2 - 1)^2 + 24x^2(x^2 - 1)$

$= (x^2 - 1)\{6(x^2 - 1) + 24x^2\}$

$= (x^2 - 1)(30x^2 - 6)$ ...(i)

$= 6(5x^4 - 6x^2 + 1)$

$f''(0) = 6,\ f''(\pm 1) = 0$

$\Rightarrow x = 0$ is a point of local minimum and $x = \pm 1$ are points of inflection

(verify that $f''(x)$ does not change sign as x crosses ± 1).

Now, $f'(x) > 0$ if $x > 0$ and $f'(x) < 0$ if $x < 0$.

Therefore, $f(x)$ decreases on $(-\infty, 0)$ and increases on $(0, -\infty)$.

There is one more important fact we must take into account. $f''(x)$ has roots ± 1 and additionally, $\pm \frac{1}{\sqrt{5}}$.

[from Eq. (i)]

Therefore, at these four points the convexity of the graph changes

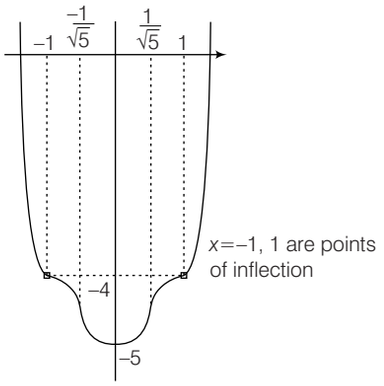
$\Rightarrow f''(x) > 0 \forall\ x \in (-\infty, -1) \cup \left(\frac{-1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) \cup (1, \infty)$ so that $f(x)$ is concave upwards in these intervals.

$\Rightarrow f''(x) < 0 \forall\ x \in \left(-1, \frac{-1}{\sqrt{5}}\right) \cup \left(\frac{1}{\sqrt{5}}, 1\right)$ so that $f(x)$ is concave downwards in these intervals.

$f(0) = -5,\ f(\pm 1) = -4,\ f(\pm 2) = 23$

Therefore one root each of $f(x)$ lies in $(-2, -1)$ and $(1, 2)$.

This information is sufficient to accurately draw the graph of the given function.



Example 44 Discuss the function $f(x) = \frac{1}{2}\sin 2x + \cos x$., and plot its graph.

Sol. The domain of $f(x)$ is R.

$f(x)$ is periodic with period 2π and therefore we need to analyse it only in $[0, 2\pi]$.

$f(x)$ is continuous and differentiable on R.

$f'(x) = \cos 2x - \sin x$

$= 1 - 2\sin^2 x - \sin x$

$= (1 + \sin x)(1 - 2\sin x)$

This is 0 in $[0, 2\pi]$ when $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

$f''(x) = -2\sin 2x - \cos x$

Now, $f''\left(\frac{\pi}{6}\right) < 0,\ f''\left(\frac{5\pi}{6}\right) > 0$ and $f''\left(\frac{3\pi}{2}\right) = 0$

$\Rightarrow x = \frac{\pi}{6}$ is a local maximum for $f(x)$; $f\left(\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{4}$

$x = \frac{5\pi}{6}$ is a local minimum for $f(x)$; $f\left(\frac{5\pi}{6}\right) = \frac{-3\sqrt{3}}{4}$

$x = \frac{3\pi}{2}$ is a point of inflection; $f\left(\frac{3\pi}{2}\right) = 0$

We now need to analyse the sign of $f''(x)$.

$f''(x) = -2\sin 2x - \cos x$

$= -4\sin x \cos x - \cos x$

$= -\cos x(1 + 4\sin x)$

This is 0 in $[0, 2\pi]$ when

$x = \frac{\pi}{2}, \pi + \sin^{-1}\frac{1}{4}, \frac{3\pi}{2}, 2\pi - \sin^{-1}\frac{1}{4}$

We see that $f(x)$ will change its convexity at four different points.

$\Rightarrow f''(x) > 0 \forall\ x \in \left(\frac{\pi}{2}, \pi + \sin^{-1}\frac{1}{4}\right) \cup \left(\frac{3\pi}{2}, 2\pi - \sin^{-1}\frac{1}{4}\right)$

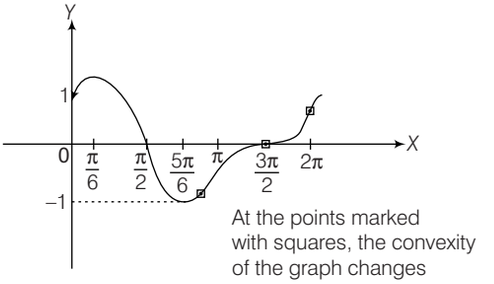
So that $f(x)$ is concave upwards in these intervals

$\Rightarrow \quad f''(x) < 0\ \forall\ x \in \left(0, \frac{\pi}{2}\right) \cup \left(\pi + \sin^{-1}\frac{1}{4}, \frac{3\pi}{2}\right) \cup \left(2\pi - \sin^{-1}\frac{1}{4}, 2\pi\right)$

So that, $f(x)$ is concave downwards in these intervals.

$f(0) = 1,\ f\left(\frac{\pi}{2}\right) = 0,\ f(2\pi) = 1$

The graph has been plotted below for $[0, 2\pi]$



Example 45 Discuss the function $y = x + \ln(x^2 - 1)$ and plot its graph.

Sol. The domain is given by $x^2 - 1 > 0$

$\Rightarrow$ $D = R \setminus [-1, 1]$

$f(x)$ is continuous and differentiable on D.

$\therefore$ $\lim\limits_{x \to 1^+} y = -\infty$; $\lim\limits_{x \to 1^-} y = -\infty$

$\Rightarrow x = \pm 1$ are vertical asymptotes to the curve.

Verify that the graph has no other asymptotes.

$$y' = 1 + \frac{2x}{x^2 - 1}$$

$\Rightarrow$ $y' = 0$ when $x^2 + 2x - 1 = 0$

$\Rightarrow$ $x = -1 \pm \sqrt{2}$

$x = -1 + \sqrt{2}$ does not belong to D.

$x = -1 - \sqrt{2}$ is an extremum point.

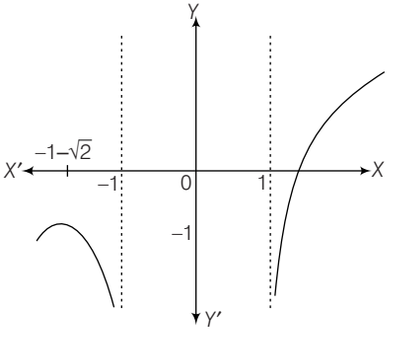
$\Rightarrow$ $$y'' = -\frac{2(x^2 + 1)}{(x^2 - 1)^2} < 0, \forall\ x$$

$\Rightarrow$ The curve is always concave downwards so that $x = -1 - \sqrt{2}$ is a point of local maximum.

$$\lim_{x \to +\infty} y = \infty$$

$$\lim_{x \to -\infty} y = -\infty$$

$\therefore$ Based on this data, the graph can be plotted as shown below.



Concept of Global Maximum/Minimum

Let $y = f(x)$ be a given function with domain D.

Let $[a, b] \subseteq D$, then global maximum/minimum of $f(x)$ in $[a, b]$ is basically the greatest/least value of $f(x)$ in $[a, b]$.

Global maxima/minima in $[a, b]$ would always occur at critical points of $f(x)$ within $[a, b]$ or at the end points of the interval.

Global Maximum/Minimum in $[a, b]$

In order to find the global maximum and minimum of $f(x)$ in $[a, b]$, find out all critical points of $f(x)$ in $[a, b]$ (i.e. all points at which $f'(x) = 0$).

Let $c_1, c_2, c_3, \ldots, c_n$ be the points at which $f'(x) = 0$

and let $f(c_1), f(c_2), \ldots, f(c_n)$ be the values of the function at these points,

$$\max\{f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b)\} = M_1 \quad \text{(say)}$$

$$\min\{f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b)\} = M_2 \quad \text{(say)}$$

Then, M_1 is the greatest value or global maxima in $[a, b]$ and M_2 is the least value or global minima in $[a, b]$.

Example 46 Let $f(x) = 2x^3 - 9x^2 + 12x + 6$. Discuss the global maxima and minima of $f(x)$ in $[0, 2]$.

Sol. Given, $f(x) = 2x^3 - 9x^2 + 12x + 6$

$\Rightarrow$ $f'(x) = 6x^2 - 18x + 12$

$\Rightarrow$ $f'(x) = 6(x^2 - 3x + 2)$

$\Rightarrow$ $f'(x) = 6(x - 1)(x - 2)$

Put $f'(x) = 0$

$\therefore$ $x = 1, 2$ [say c_1 and c_2]

Then, for global maximum or global minimum.

We have, $f(0) = 6,\ f(1) = 11,\ f(2) = 10,$

$\therefore$ Global maximum $\Rightarrow$ $M_1 = \max\{6, 10, 11\} = 11$

and global minimum $\Rightarrow$ $M_2 = \min\{6, 10, 11\} = 6$

$\therefore$ $f(1) = 11$ global maximum and $f(0) = 6$ global minimum.

Global Maximum/Minimum in (a, b)

Method for obtaining the greatest and least values of $f(x)$ in (a, b) is almost the same as the method used for obtaining the greatest and least values in $[a, b]$.

However a caution may be exercised

let $M_1 = \max\{f(c_1), f(c_2), \ldots, f(c_n)\}$

and $M_2 = \min\{f(c_1), f(c_2), \ldots, f(c_n)\}$

Now, M_1 and M_2 are global maximum and global minimum, respectively.

But, if $\lim\limits_{x \to a^+} f(x) > M_1$

or $\lim\limits_{x \to b^-} f(x) < M_2$

$\Rightarrow f(x)$ would not possess global maximum or global minimum in (a, b).

This means that if **limiting values at the end points are greater than M_1 or less than M_2, then global maximum or global minimum does not exist in (a, b).**

Example 47 ***Let*** $f(x)=2x^3-9x^2+12x+6$. ***Discuss the global maxima and global minima of*** $f(x)$ ***in (1, 3).***

Sol. Given, $f(x)=2x^3-9x^2+12x+6$

$\Rightarrow \quad f'(x)=6x^2-18x+12 \quad \Rightarrow \quad f'(x)=6(x-1)(x-2)$

Put $f'(x)=0 \Rightarrow x=1,2$

$\therefore \quad f(1)=11 \quad$ and $\quad f(2)=10 \quad$ ...(i)

Let us consider the open interval (1, 3).

Clearly, $x=2$ is the only point in (1, 3).

And $\quad f(2)=10 \quad$ [from Eq. (i)]

Now, $\lim\limits_{x\to 1^+} f(x)=\lim\limits_{h\to 0} f(1+h)=\lim\limits_{h\to 0} 2(1+h)^3-9(1+h)^2$
$+12(1+h)+6=11$

and $\lim\limits_{x\to 3^-} f(x)=\lim\limits_{h\to 0} f(3-h)=\lim\limits_{h\to 0} 2(3-h)^3-9(3-h)^2$
$+12(3-h)+6=15$

Thus, $x=2$ is the point of global minima in (1, 3) and global maxima does not exist in (1, 3).

Remark

Based on the above discussion, we can summarize things in a single graph as given below.

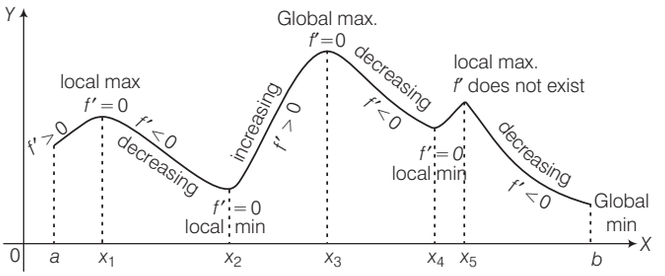


Figure 8.21

Exercise for Session 4

1. The minimum value of x^x is attained when x is equal to

(a) e (b) e^{-1} (c) 1 (d) e^2

2. The function f is defined by $f(x)=x^p(1-x)^q$ for all $x\in R$, where p,q are positive integers, has a maximum value, for x is equal to

(a) $\dfrac{pq}{p+q}$ (b) 1 (c) 0 (d) $\dfrac{p}{p+q}$

3. The least area of the circle circumscribing any right triangle of area S is

(a) πS (b) $2\pi S$ (c) $\sqrt{2}\pi S$ (d) $4\pi S$

4. The coordinates of the point on the curve $x^2=4y$, which is atleast distance from the line $y=x-4$ is

(a) (2, 1) (b) (– 2, 1)

(c) (– 2, –1) (d) none of these

5. The largest area of a rectangle which has one side on the x-axis and the two vertices on the curve $y=e^{-x^2}$ is

(a) $\sqrt{2}\,e^{-1/2}$ (b) $2\,e^{-1/2}$

(c) $e^{-1/2}$ (d) none of these

6. Let $f(x)=\ln(2x-x^2)+\sin\dfrac{\pi x}{2}$. Then,

(a) graph of f is symmetrical about the line $x=1$ (b) graph of f is symmetrical about the line $x=2$

(c) maximum value of f is 1 (d) minimum value of f does not exist

7. The sum of the legs of a right triangle is 9 cm. When the triangle rotates about one of the the legs, a cone results which has the maximum volume. Then,

(a) slant height of such a cone is $3\sqrt{5}$ (b) maximum volume of the cone is $32\,\pi$

(c) curved surface of the cone is $18\sqrt{5}\,\pi$ (d) semi vertical angle of cone is $\tan^{-1}\sqrt{2}$

8. Least value of the function $f(x)=2^{x^2}-1+\dfrac{2}{2^{x^2}+1}$ is

(a) 0 (b) $\dfrac{3}{2}$ (c) $\dfrac{2}{3}$ (d) 1

9. The greatest and the least value of the function, $f(x) = \sqrt{1-2x+x^2} - \sqrt{1+2x+x^2},\ x \in (-\infty, \infty)$ are

(a) 2, – 2 (b) 2, – 1

(c) 2, 0 (d) none

10. The minimum value of the polynomial $x(x+1)(x+2)(x+3)$ is

(a) 0 (b) $\frac{9}{16}$

(c) – 1 (d) $-\frac{3}{2}$

11. The difference between the greatest and least values of the functions, $f(x) = \cos x + \frac{1}{2}\cos 2x - \frac{1}{3}\cos 3x$ is

(a) $\frac{4}{3}$ (b) 1

(c) $\frac{9}{4}$ (d) $\frac{1}{6}$

12. The point at which the slope of the tangent of the function, $f(x) = e^x \cdot \cos x$ attains minima, when $x \in [0, 2\pi]$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$

(c) $\frac{3\pi}{4}$ (d) π

13. If λ, μ are real numbers such that, $x^3 - \lambda x^2 + \mu x - 6 = 0$ has its real roots and positive, then the minimum value of μ is

(a) $3(6)^{1/3}$ (b) $3(6)^{2/3}$

(c) $(6)^{1/3}$ (d) $(6)^{2/3}$

14. The points for which the function $f(x) = \int_1^x \{2(t-1)(t-2)^3 + 3(t-1)^2(t-2)^2\}\, dt$ attains maxima and minima, is

(a) maximum when $x = \frac{7}{5}$ and minimum when $x = 1$ (b) maximum when $x = 1$and minimum when $x = 0$

(c) maximum when $x = 1$and minimum when $x = 2$ (d) maximum when $x = 1$and minimum when $x = \frac{7}{5}$

15. The set of values of 'a' for which the function $f(x) = \frac{ax^3}{3} + (a+2)x^2 + (a-1)x + 2$ possess a negative point of inflection.

(a) $(-\infty, -2) \cup (0, \infty)$ (b) $\{-4/5\}$

(c) $(-2, 0)$ (d) empty set

Session 5

Maxima and Minima of Discontinuous Functions

Maxima and Minima of Discontinuous Functions

In discontinuous functions we don't apply any of the methods discussed earlier but we observe certain conditions and their graphs which would give you more clear picture.

Minimum of Discontinuous Functions

Let $f(x)$ be a function discontinuous or not differentiable at $x = a$, then the following four cases arise for minimum at $x = a$

Case (i) From figure 8.22

$$f(a) < f(a+h) \qquad f(a) < f(a-h)$$

Case (ii) From figure 8.23

$$f(a) < f(a+h) \qquad f(a) < f(a-h)$$

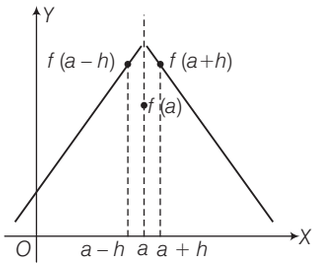


Figure 8.22

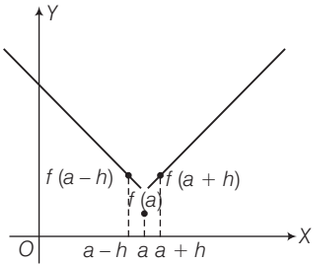


Figure 8.23

Case (iii) From figure 8.24

$$f(a) < f(a+h) \qquad f(a) \le f(a-h)$$

Case (iv) From figure 8.25

$$f(a) \le f(a+h) \qquad f(a) < f(a-h)$$

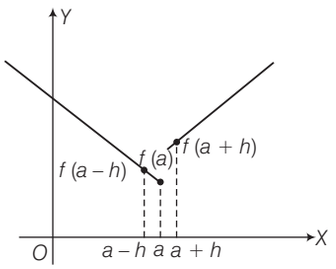


Figure 8.24

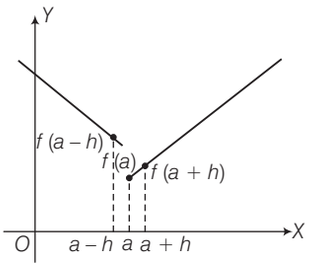


Figure 8.25

from all the four above mentioned cases for minimum of discontinuous functions, we have

$$f(a) \le f(a+h) \text{ and } f(a) \le f(a-h)$$

Example 48 Discuss the minima of $f(x) = \{x\}$, (where $\{.\}$ denotes the fractional part of x) for $x = 6$.

Sol. As we have discussed for discontinuous functions, minimum at $x = a$ is attained when,

$f(a) \le f(a+h)$ and $f(a) \le f(a-h) \Rightarrow f(x)$ is minimum at $x = a$

Here, $f(x) = \{x\}$ is discontinuous function at $x = 6$

where, $f(6) = 0$

$f(6+h) > 0$ and $f(6-h) > 0$

So, $f(6) < f(6+h)$ and $f(6) < f(6-h)$

$\Rightarrow$ $f(x)$ is minimum at $x = 6$.

$\therefore$ $f(x)$ attains local minima for $x = 6$.

Example 49 Let $f(x) = \begin{cases} |x-2| + a^2 - 9a - 9, & \text{if } x < 2 \\ 2x - 3, & \text{if } x \ge 2 \end{cases}$.

Then, find the value of 'a' for which $f(x)$ has local minimum at $x = 2$.

Sol. We have, $f(x) = \begin{cases} |x-2| + a^2 - 9a - 9, & \text{if } x < 2 \\ 2x - 3, & \text{if } x \ge 2 \end{cases}$

$f(x)$ has local minima at $x = 2$.

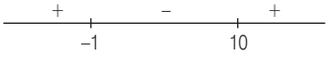
Since, $f(x) = 2x - 3$ for $x \ge 2$ [is strictly increasing]

$\therefore$ $\lim\limits_{x \to 2^-} f(x) \ge f(2)$ or $\lim\limits_{h \to 0} f(2-h) \ge f(2)$

$\Rightarrow \lim\limits_{h \to 0} \{|2-h-2| + a^2 - 9a - 9\} \ge 1 \quad [\because f(2) = 2 \times 2 - 3 = 1]$

$\Rightarrow$ $a^2 - 9a - 10 \ge 0$

$\Rightarrow$ $(a+1)(a-10) \ge 0$



$\Rightarrow$ $a \le -1$ or $a \ge 10$

Maximum of Discontinuous Functions

Let $f(x)$ be a function discontinuous or not differentiable at $x = a$, then the following four cases arise for maximum at $x = a$.

(i) From figure 8.26

$$f(a) > f(a+h) \qquad f(a) > f(a-h)$$

(ii) From figure 8.27

$$f(a) > f(a+h) \qquad f(a) > f(a-h)$$

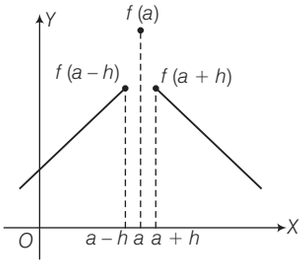


Figure 8.26

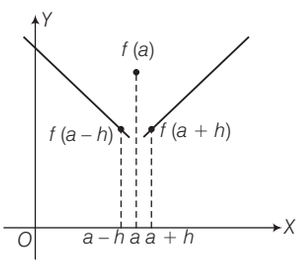


Figure 8.27

(iii) From figure 8.28

$f(a) > f(a+h)$

$f(a) \geq f(a-h)$

(iv) From figure 8.29

$f(a) \geq f(a+h)$

$f(a) > f(a-h)$

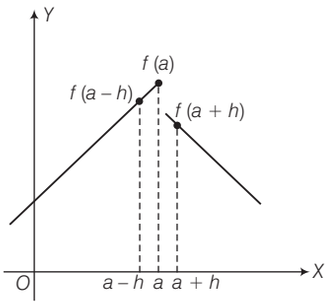


Figure 8.28

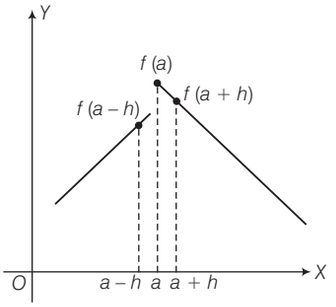


Figure 8.29

From all the above four mentioned cases for maximum of discontinuous functions, we have

$$f(a) \geq f(a+h),\ f(a) \geq f(a-h)$$

Neither Maximum Nor Minimum for Discontinuous Functions

Let $f(x)$ be a function discontinuous or not differentiable at $x = a$, then the following cases arise for neither maximum nor minimum at $x = a$

(i) $f(a) < f(a-h)$

$f(a) \geq f(a+h)$

(ii) $f(a) \leq f(a+h)$

$f(a) > f(a-h)$

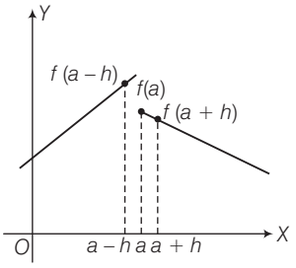


Figure 8.30

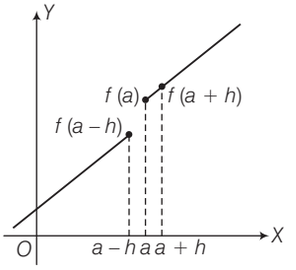


Figure 8.31

(iii) $f(a) < f(a+h)$

$f(a) \geq f(a-h)$

(iv) $f(a) \leq f(a-h)$

$f(a) > f(a+h)$

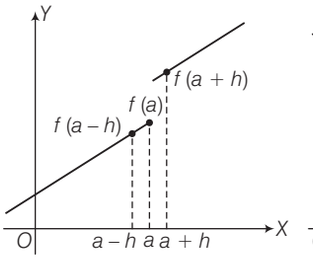


Figure 8.32

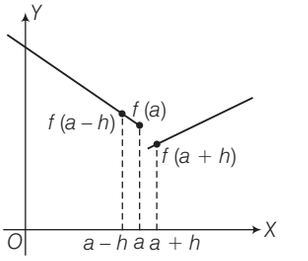


Figure 8.33

(v) $f(a) < f(a-h)$

$f(a) > f(a+h)$

(vi) $f(a) < f(a-h)$

$f(a) \geq f(a+h)$

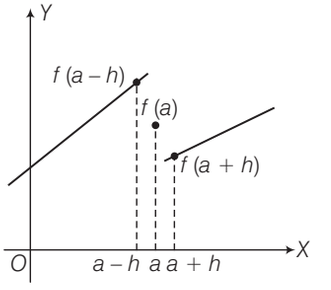


Figure 8.34

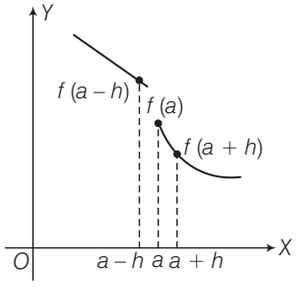
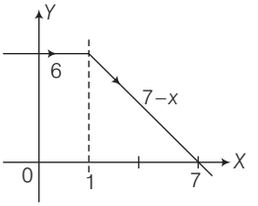


Figure 8.35

In all above cases no extremum exist.

Example 50 Let $f(x) = \begin{cases} 6, & x \leq 1 \\ 7-x, & x > 1 \end{cases}$, then for $f(x)$ at $x = 1$ discuss maxima and minima.

Sol. Here, $f(x) = \begin{cases} 6, & x \leq 1 \\ 7-x, & x > 1 \end{cases}$



Clearly, $f(x)$ is not differentiable at $x = 1$

$\Rightarrow \quad f(1) = 6 \ \Rightarrow \ f(1-h) = 6$

and $\quad f(1+h) = 7-(1+h) = 6-h < 6$

Thus, $x = 1$ is a point of maxima.

Example 51 Find the values of 'a' for which,

$$f(x) = \begin{cases} 4x - x^3 + \log(a^2 - 3a + 3), & 0 \leq x < 3 \\ x - 18, & x \geq 3 \end{cases}$$

has a local maxima at $x = 3$.

Sol. Given, $f(x) = \begin{cases} 4x - x^3 + \log(a^2 - 3a + 3), & 0 \leq x < 3 \\ x - 18, & x \geq 3 \end{cases}$

Since, function attains maxima at $x = 3$

$\Rightarrow \quad f(3) \geq f(3-0)$

$\Rightarrow \quad -15 \geq 12 - 27 + \log(a^2 - 3a + 3)$

$\Rightarrow \quad \log(a^2 - 3a + 3) \le 0$

Where for 'log' to exists,

$a^2 - 3a + 3 > 0$ and $\log(a^2 - 3a + 3) \le 0$

$\Rightarrow \quad 0 < a^2 - 3a + 3 \le 1$

i.e. $(a-2)(a-1) \le 0$

Using number line rule, we have

i.e. $1 \le a \le 2$

$\therefore$ $f(x)$ attains local maxima at $x = 3$, when $a \in [1, 2]$.

Nature of Roots of Cubic Polynomials

Let $f(x) = ax^3 + bx^2 + cx + d$ be the given cubic polynomial and $f(x) = 0$ be the corresponding cubic equations, where $a, b, c, d \in R$ and $a > 0$.

Now, $f'(x) = 3ax^2 + 2bx + c$

Let $D = 4b^2 - 12ac = 4(b^2 - 3ac)$ be the discriminant of the equations $f'(x) = 0$.

Now, we have the following three cases.

***Case* I** If $D < 0 \Rightarrow f'(x) > 0, \forall x \in R$, i.e. $f(x)$ would be an increasing function of x. Also, $\lim\limits_{x \to \infty} f(x) = \infty$ and $\lim\limits_{x \to -\infty} f(x) = -\infty$.

Since, $f(x)$ is an increasing function, so it will intersect the X-axis at only once.

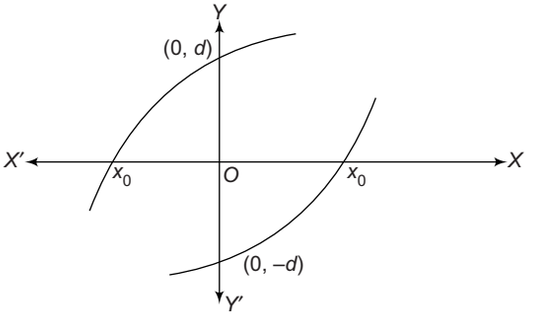


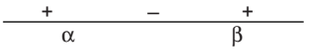
Figure 8.36

Clearly, $x_0 > 0$ if $d < 0$ and $x_0 < 0$, if $d > 0$

***Case* II** If $D > 0, f'(x) = 0$ would have two real roots (say) α and β (where $\alpha < \beta$)

$\Rightarrow \quad f'(x) = 3a(x - \alpha)(x - \beta)$

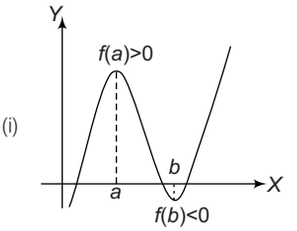
$$\Rightarrow \quad \begin{cases} f'(x) < 0, x \in (\alpha, \beta) \\ f'(x) > 0, x \in (-\infty, \alpha) \cup (\beta, \infty) \\ f'(x) = 0, x \in \{\alpha, \beta\} \end{cases}$$



Here, $x = \alpha$, will be point of local maxima and $x = \beta$ will be point of local minima.

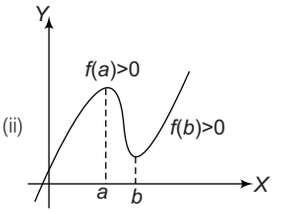
***Case* III** If $D = 0$, $f'(x) = 3a(x - \alpha)^2$, where α is a root of $f'(x) = 0$, then $f(x) = a(x - \alpha)^3 + K$. If $K = 0$, then $f(x) = 0$ has three equal real roots. if $K \neq 0$, then $f(x) = 0$ has one real root.

Graphical Representation of Roots of Cubic Polynomials



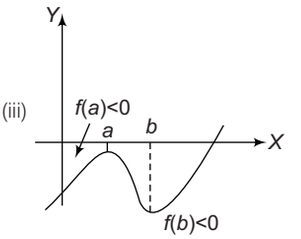
$f(x)$ has three distinct roots

Figure 8.37



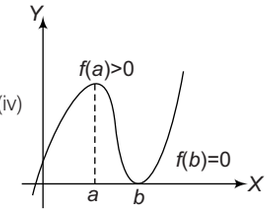
$f(x)$ has one real root

Figure 8.38



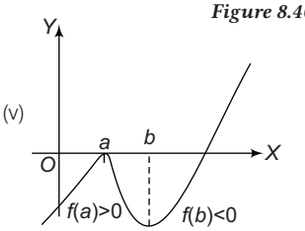
$f(x)$ has one real root

Figure 8.39



$f(x)$ has three real roots
(b being repeated)

Figure 8.40



$f(x)$ has three real roots
(a being repeated)

Figure 8.41

Conclusions

(i) If $f(a) \cdot f(b) < 0$, then $f(x)$ has three distinct roots (see figure 8.37).

(ii) If $f(a) \cdot f(b) > 0$, then $f(x)$ has one real and two imaginary roots (see figure 8.38 and 8.39).

(iii) If $f(a) \cdot f(b) = 0$, then $f(x)$ has three roots but two roots are identical (see figure 8.40 and 8.41).

Example 52 If $4x^3 - 3x - p = 0$, where $-1 \le p \le 1$ has unique root in $\left[\frac{1}{2}, 1\right]$, then the root is *[IIT JEE 2005]*

(a) $\frac{\cos^{-1} p}{3}$ (b) $\cos\left(\frac{1}{3}\cos^{-1} p\right)$

(c) $\cos(\cos^{-1} p)$ (d) None of these

Sol. Given that, $-1 \le p \le 1$

Let $f(x) = 4x^3 - 3x - p$

Now, $f\left(\frac{1}{2}\right) = \frac{1}{2} - \frac{3}{2} - p = -1 - p \le 0 \quad [\because p \ge -1]$

and $f(1) = 4 - 3 - p = 1 - p \ge 0 \quad [\because p \le 1]$

$\therefore$ $f(x)$ has atleast one real root between $\left[\frac{1}{2}, 1\right]$.

To find a root we observe $f(x)$ contains $4x^3 - 3x$, which is multiple angle formula for $\cos 3\theta$.

$\therefore$ We put $x = \cos\theta$, then

$$4\cos^3\theta - 3\cos\theta - p = 0$$

$$\Rightarrow \quad p = \cos 3\theta$$

$$\Rightarrow \quad \theta = \frac{1}{3}\cos^{-1}(p)$$

$\therefore$ Root is $\cos\left(\frac{1}{3}\cos^{-1} p\right)$

Hence, (b) is the correct answer.

Example 53 The number of distinct real roots of $x^4 - 4x^3 + 12x^2 + x - 1 = 0$ *[IIT JEE 2011]*

Sol. Given, $f(x) = x^4 - 4x^3 + 12x^2 + x - 1$

$$\Rightarrow \quad f'(x) = 4x^3 - 12x^2 + 24x + 1$$

$$\Rightarrow \quad f''(x) = 12x^2 - 24x + 24$$

$$\Rightarrow \quad f''(x) = 12(x^2 - 2x + 2)$$

$$\Rightarrow \quad f''(x) = 12((x-1)^2 + 1) \Rightarrow f''(x) > 0$$

$\Rightarrow$ $f'(x)$ is an increasing cubic function.

$\Rightarrow$ $f'(x)$ has only one real root and two imaginary root.

$\therefore$ $f(x)$ cannot have all distinct roots, atmost two real roots.

Now, $f(-1) = 15, f(0) = -1, f(1) = 9$

$\therefore$ $f(x)$ must have one root in $(-1, 0)$ and other in $(0,1)$.

Hence, there are 2 real roots.

Exercise for Session 5

1. Let $f(x) = \begin{cases} x^3 - x^2 + 10x - 5, & x \le 1 \\ -2x + \log_2(b^2 - 2), & x > 1 \end{cases}$ the set of values of b for which $f(x)$ have greatest value at $x = 1$ is given by

(a) $1 < b \le 2$ (b) $b = \{1,2\}$ (c) $b \in (-\infty, -1)$ (d) $[-\sqrt{130}, -\sqrt{2}) \cup (\sqrt{2}, \sqrt{130}]$

2. Number of solution(s) satisfying the equations, $3x^2 - 2x^3 = \log_2(x^2 + 1) - \log_2 x$ is

(a) 1 (b) 2 (c) 3 (d) None of these

3. Let $f(x) = \cos 2\pi x + x - [x]$([.] denotes the greatest integer function). Then, number of points in [0, 10] at which $f(x)$ assumes its local maximum value, is

(a) 0 (b) 10 (c) 9 (d) infinite

4. If $f(x) = |x| + |x - 1| + |x - 2|$, then

(a) $f(x)$ has minima at $x = 1$ (b) $f(x)$ has maxima at $x = 0$

(c) $f(x)$ has neither maxima nor minima at $x = 3$ (d) None of these

5. The function $f(x) = 1 + [\cos x]x$, in $0 < x \le \frac{\pi}{2}$

(a) has a minimum value 0 (b) has a maximum value 2

(c) is continuous in $\left[0, \frac{\pi}{2}\right]$ (d) is not differentiable at $x = \frac{\pi}{2}$

6. If $\lim_{x \to a} f(x) = \lim_{x \to a} [f(x)]$([.] denotes the greater integer function) and $f(x)$ is non-constant continuous function, then

(a) $\lim_{x \to a} f(x)$ is irrational (b) $\lim_{x \to a} f(x)$ is non-integer

(c) $f(x)$ has local maxima at $x = a$ (d) $f(x)$ has local minima at $x = a$

7. Find the value of a if $x^3 - 3x + a = 0$ has three distinct real roots.

8. Prove that there exist exactly three non- similar isosceles ΔABC such that $\tan A + \tan B + \tan C = 100$.

JEE Type Solved Examples :
Single Option Correct Type Questions

● ***Ex. 1*** *The values of 'K' for which the point of minimum of the function* $f(x)=1+K^2x-x^3$ *satisfy the inequality* $\frac{(x^2+x+2)}{(x^2+5x+6)}<0$, *belongs*

(a) $(-3\sqrt{3},\infty)$

(b) $(-3\sqrt{3},-2\sqrt{3})\cup(0,\infty)$

(c) $(-3\sqrt{3},-2\sqrt{3})\cup(2\sqrt{3},3\sqrt{3})$

(d) $(0,\infty)$

Sol. Here, $\frac{x^2+x+2}{x^2+5x+6}<0 \Rightarrow \frac{\left(x+\frac{1}{2}\right)^2+\frac{7}{4}}{(x+2)(x+3)}<0$

$+ \quad - \quad +$

$-3 \qquad -2$

where $\left(x+\frac{1}{2}\right)^2+\frac{7}{4}$ is always positive.

$\therefore$ Using number line rule for $(x+2)(x+3)$ as shown above, we get

$$\frac{x^2+x+2}{x^2+5x+6}<0 \quad \Rightarrow \quad -3<x<-2 \qquad ...(i)$$

Now, consider $f(x)=1+K^2x-x^3$

$$f'(x)=K^2-3x^2, \quad f''(x)=-6x$$

For maximum/minimum let $f'(x)=0$,

$$\Rightarrow \qquad x=\pm\frac{|K|}{\sqrt{3}}$$

Let $\qquad x_1=\frac{|K|}{\sqrt{3}}$ and $x_2=-\frac{|K|}{\sqrt{3}}$

$\therefore \qquad f''(x_1)<0$ and $f''(x_2)>0$

$\Rightarrow$ $f(x)$ is maximum at $x=x_1$ and $f(x)$ is minimum at $x=x_2$.

$\therefore \qquad -3<x_2<-2$

$$\Rightarrow \qquad -3<\frac{-|K|}{\sqrt{3}}<-2 \qquad \text{[from Eq. (i)]}$$

$$\Rightarrow \qquad 3\sqrt{3}>|K|>2\sqrt{3}$$

$$\Rightarrow \qquad K\in(-3\sqrt{3},-2\sqrt{3})\cup(2\sqrt{3},3\sqrt{3})$$

Hence, (c) is the correct answer.

● ***Ex. 2*** *The values of a and b for which all the extrema of the function,* $f(x)=a^2x^3-0.5ax^2-2x-b$, *is positive and the minimum is at the point* $x_0=\frac{1}{3}$, *are*

(a) when $a=-2\Rightarrow b<-\frac{11}{27}$ and when $a=3\Rightarrow b<-\frac{1}{2}$

(b) when $a=3\Rightarrow b<-\frac{11}{27}$ and when $a=2\Rightarrow b<-\frac{1}{2}$

(c) when $a=-2\Rightarrow b<-\frac{1}{2}$ and when $a=3\Rightarrow b<-\frac{11}{27}$

(d) None of the above

Sol. Given, $\qquad f(x)=a^2x^3-0.5ax^2-2x-b$

$$\Rightarrow \qquad f'(x)=3a^2x^2-ax-2$$

For extrema, $f'(x)=0$

$$\Rightarrow \quad 3a^2x^2-ax-2=0 \text{ at } x=\frac{1}{3}$$

$$\left[\text{as at } x=\frac{1}{3} \text{ function is minimum}\right]$$

$$\therefore \qquad 3a^2\left(\frac{1}{3}\right)^2-a\left(\frac{1}{3}\right)-2=0 \quad \Rightarrow \quad \frac{a^2}{3}-\frac{a}{3}-2=0$$

$$\Rightarrow \qquad a^2-a-6=0 \quad \text{or} \quad a=-2,3$$

So, their arise two cases as

***Case* I** At $a=3$, if function attains minimum and is positive.

Then, $\qquad 9\left(\frac{1}{3}\right)^3-(0.5)(3)\left(\frac{1}{3}\right)^2-2\left(\frac{1}{3}\right)-b>0$

$$\left[\because \text{minimum at } x=\frac{1}{3} \text{and} \, a=3\Rightarrow f\left(\frac{1}{3}\right)>0 \text{ when} \, a=3\right]$$

$$\Rightarrow \qquad b<\frac{1}{3}-\frac{1.5}{9}-\frac{2}{3} \quad \Rightarrow \quad b<-\frac{1}{2}$$

***Case* II** At $a=-2$, if function attains minimum and is positive.

Then, $\qquad (-2)^2\left(\frac{1}{3}\right)^3-(0.5)(-2)\left(\frac{1}{3}\right)^2-2\left(\frac{1}{3}\right)-b>0$

$$\left[\begin{array}{l}\text{since minimum at } x=\frac{1}{3}, \text{ when } a=-2\\ \Rightarrow f\left(\frac{1}{3}\right)>0, \text{ when } a=-2\end{array}\right]$$

$$\Rightarrow \qquad b<\frac{4}{27}+\frac{1}{9}-\frac{2}{3} \quad \text{or} \quad b<-\frac{11}{27}$$

$\therefore$ When $a=3\Rightarrow b<-\frac{1}{2}$ and when $a=-2\Rightarrow b<-\frac{11}{27}$

Hence, (a) is the correct answer.

● ***Ex. 3*** *If* $f''(x)+f'(x)+f^2(x)=x^2$ *is the differential equation of a curve and let P be the point of maxima, then number of tangents which can be drawn from P to* $x^2-y^2=a^2$ *is/are*

(a) 2 (b) 1

(c) 0 (d) either 1 or 2

Sol. At point of maxima $f'(x) = 0$ and $f''(x) < 0$

$\Rightarrow$ $f''(x) = x^2 - f^2(x) \le 0$

(Since, the curve is $x^2 - y^2 = a^2$ and $x^2 - f^2(x) \le 0$

$\therefore x_1^2 - y_1^2 < a^2 \Rightarrow$ point lies outside the hyperbola)

$\Rightarrow$ Point $P(x, f(x))$ lies outside $x^2 - y^2 = a^2$

$\therefore$ Two tangents can be drawn.

Hence, (a) is the correct answer.

● ***Ex. 4*** *Let* $f(x) = \int_0^x \cos\left(\frac{t^2 + 2t + 1}{5}\right) dt, 0 < x < 2$, *then* $f(x)$

(a) increases monotonically

(b) decreasing monotonically

(c) has one point of local maximum

(d) has one point of local minima

Sol. We have, $f'(x) = \cos\left(\frac{x^2 + 2x + 1}{5}\right) \times 1 = \cos\left(\frac{(x+1)^2}{5}\right)$

Since, $0 < x < 2 \Rightarrow 1/5 < \frac{(x+1)^2}{5} > 9/5$

Now, $\cos\left(\frac{(x+1)^2}{5}\right) = 0$

$\Rightarrow$ $\frac{(x+1)^2}{5} = \frac{\pi}{2}$

$f'(x) > 0$, if $0 < x < (\sqrt{5\pi/2} - 1)$

and $f'(x) < 0$, if $(\sqrt{5\pi/2} - 1) < x < 2$

$\therefore$ $x = (\sqrt{5\pi/2} - 1)$ is a point of local maxima

Hence, (c) is the correct answer.

● ***Ex. 5*** *As 'x' ranges over the interval* $(0, \infty)$, *the function* $f(x) = \sqrt{9x^2 + 173x + 900} - \sqrt{9x^2 + 77x + 900}$, *ranges over*

(a) (0, 4) (b) (0, 8)

(c) (0, 12) (d) (0, 16)

Sol. $\because \sqrt{9x^2 + 173x + 900} > \sqrt{9x^2 + 77x + 900}, \forall\ x \in (0, \infty)$

$\Rightarrow$ $f(x) > 0, \forall\ x \in (0, \infty)$

Now, $f(x) = \frac{96x}{\sqrt{9x^2 + 173x + 900} + \sqrt{9x^2 + 77x + 900}}$

[rationalise]

$\therefore$ $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{96}{\sqrt{9 + \frac{173}{x} + \frac{900}{x^2}} + \sqrt{9 + \frac{77}{x} + \frac{900}{x^2}}}$

$= \frac{96}{6} = 16$

$\therefore$ Range of $f(x) \in (0, 16)$

Hence, (d) is the correct answer.

● ***Ex. 6*** *Let* $g:[1, 6] \to [0, \infty)$ *be a real valued differentiable function satisfying* $g'(x) = \frac{2}{x + g(x)}$ *and* $g(1) = 0$, *then the maximum value of g cannot exceed*

(a) $\log 2$ (b) $\log 6$ (c) $6 \log 2$ (d) $2 \log 6$

Sol. Here, $g'(x) = \frac{2}{x + g(x)} > 0$, for all $x \in [1, 6]$

$\Rightarrow$ $g(x)$ is increasing function on $[1, 6]$

$\therefore$ $\int_1^6 g'(x)\, dx \le \int_1^6 \frac{2}{x} dx$

$\Rightarrow$ $(g(x))_1^6 \le 2 (\log_e x)_1^6$

$\Rightarrow$ $g(6) - g(1) \le 2 \log 6$

$\therefore$ $g(6) \le 2 \log 6$, as $g(1) = 0$

$\Rightarrow$ g cannot exceed $2 \log 6$.

Hence, (d) is the correct answer.

● ***Ex. 7*** *The minimum value of the function,* $f(x) = x^{3/2} + x^{-3/2} - 4\left(x + \frac{1}{x}\right)$. *for all permissible real values of x, is*

(a) -10 (b) -6 (c) -7 (d) -8

Sol. Here, $f(x) = x^{3/2} + x^{-3/2} - 4\left(x + \frac{1}{x}\right)$

$f(x) = \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^3 - 3\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) - 4\left(\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 - 2\right)$

Let $\sqrt{x} + \frac{1}{\sqrt{x}} = t$, $[x > 0]$

$\therefore$ $g(t) = t^3 - 3t - 4(t^2 - 2)$

Now, $g(t) = t^3 - 4t^2 - 3t + 8$, where $t \in [2, \infty)$

$g'(t) = 3t^2 - 8t - 3 = (t - 3)(3t + 1)$

$g'(t) = 0 \Rightarrow t = 3 (t \ne -1/3)$

$g''(t) = 6t - 8$

$\Rightarrow$ $g''(3) = 10 > 0$

$\therefore$ Minimum value is

$g(3) = 27 - 36 - 9 + 8 = -10$

Hence, (a) is the correct answer.

● ***Ex. 8*** *The tangent to the curve* $y = 1 - x^2$ *at* $x = \alpha$, *where* $0 < \alpha < 1$, *meets the axes at P and Q. If* α *varies and the minimum value of the area of the* ΔOPQ *is k times the area bounded by the axis and the part of the curve for* $0 < x < 1$, *then k is equal to*

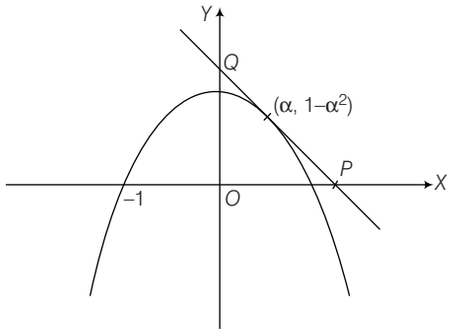
(a) $2/\sqrt{3}$ (b) $75/16$ (c) $25/18$ (d) $2/3$

Sol. Here, equation of tangent

$y + 2\alpha x = \alpha^2 + 1$ ...(i)

From Eq. (i) intersect the axes at

$$P\left(\frac{\alpha^2+1}{2\alpha}, 0\right) \text{ and } Q(0, \alpha^2+1)$$



$$\therefore \text{ Area of } \Delta OPQ = \frac{1}{2}\cdot\frac{(\alpha^2+1)^2}{2\alpha} = \frac{(\alpha^2+1)^2}{4\alpha}$$

$$f(\alpha) = \frac{1}{4}(\alpha^3 + 2\alpha + 1/\alpha)$$

$$f'(\alpha) = \frac{1}{4}(3\alpha^2 + 2 - 1/\alpha^2) = \frac{1}{4}\frac{(3\alpha^4 + 2\alpha^2 - 1)}{\alpha^2}$$

$$= \frac{1}{4}\cdot\frac{(\alpha^2+1)(3\alpha^2-1)}{\alpha^2}$$

$$= \frac{1}{4}\frac{(\alpha^2+1)(\sqrt{3}\alpha-1)(\sqrt{3}\alpha+1)}{\alpha^2}$$

Using number line rule, we have

$\therefore$ Minimum when $\alpha = 1/\sqrt{3}$

Thus, minimum area of triangle $= 4/3\sqrt{3}$

Given, $\frac{4}{3\sqrt{3}} = k\int_0^1 (1-x^2)\,dx = k\left(x - \frac{x^3}{3}\right)_0^1 = \frac{2k}{3}$

$$\Rightarrow \qquad k = 2/\sqrt{3}$$

● ***Ex. 9*** *Least natural number a for which* $x + ax^{-2} > 2, \forall\ x \in [0, \infty)$ *is*

(a) 1 (b) 2 (c) 5 (d) None of these

Sol. Here, $x + ax^{-2} > 2$

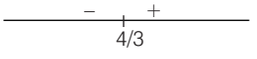
$$\Rightarrow \qquad x^3 - 2x^2 + a > 0$$

Let $f(x) = x^3 - 2x^2 + a$

Since, $f(x) > 0, \forall\ x \in [0, \infty), f(0) > 0$

and $\min f(x) > 0$

$\Rightarrow$ $a > 0$ and for minimum $f(x)$



$$f'(x) = 3x^2 - 4x = 0$$

$$\Rightarrow \qquad x = 0, \frac{4}{3}$$

$\therefore$ $f(x)$ is minimum at $x = \frac{4}{3}$

$$f\left(\frac{4}{3}\right) > 0 \Rightarrow a > \frac{32}{27}$$

Hence, (b) is the correct answer.

● ***Ex. 10*** *If* $k\sin^2 x + \frac{1}{k}\operatorname{cosec}^2 x = 2, x \in \left(0, \frac{\pi}{2}\right)$, *then* $\cos^2 x + 5\sin x\cos x + 6\sin^2 x$ *is equal to*

(a) $\frac{k^2+5k+6}{k^2}$ (b) $\frac{k^2-5k+6}{k^2}$

(c) 6 (d) None of these

Sol. Given, $k\sin^2 x + \frac{1}{k\sin^2 x} = 2$

$$\Rightarrow \qquad \left(\sqrt{k}\sin x - \frac{1}{\sqrt{k}\sin x}\right)^2 = 0$$

$$\Rightarrow \qquad \sin^2 x = \frac{1}{k}$$

So, $\cos^2 x + 5\sin x\cos x + 6\sin^2 x$

$$= \frac{k-1}{k} + \frac{5\sqrt{k-1}}{k} + \frac{6}{k}$$

$$= \frac{k+5+5\sqrt{k-1}}{k}$$

Hence, (d) is the correct answer.

● ***Ex. 11*** *The least value of the expression* $x^2 + 4y^2 + 3z^2 - 2x - 12y - 6z + 14$ *is*

(a) 0

(b) 1

(c) no least value

(d) None of the above

Sol. Let $f(x, y, z) = x^2 + 4y^2 + 3z^2 - 2x - 12y - 6z + 14$

$$= (x-1)^2 + (2y-3)^2 + 3(z-1)^2 + 1$$

For least value of $f(x, y, z)$

$$x - 1 = 0, 2y - 3 = 0 \text{ and } z - 1 = 0$$

$$\therefore \qquad x = 1, \ y = \frac{3}{2}, \ z = 1$$

Hence, least value of $f(x, y, z)$ is $f\left(1, \frac{3}{2}, 1\right) = 1$

Hence, (b) is the correct answer.

● **Ex. 12** *On the interval* $\left[\frac{5\pi}{4}, \frac{4\pi}{3}\right]$, *the least value of the function* $f(x) = \int_{5\pi/4}^{x} (3\sin t + 4\cos t)\, dt$ *is*

(a) $\frac{3}{2} + \frac{1}{\sqrt{2}} - 2\sqrt{3}$ (b) $\frac{3}{2} - \frac{1}{\sqrt{2}} + 2\sqrt{3}$

(c) $\frac{3}{2} - \frac{1}{\sqrt{2}} - 2\sqrt{3}$ (d) None of these

Sol. Given, $f(x) = \int_{5\pi/4}^{x} (3\sin t + 4\cos t)\, dt,\ x \in \left[\frac{5\pi}{4}, \frac{4\pi}{3}\right]$.

$$f'(x) = 3\sin x + 4\cos x$$

$f'(x) < 0$ as $\sin x, \cos x$ are negative for $x \in \left[\frac{5\pi}{4}, \frac{4\pi}{3}\right]$.

$$\Rightarrow \quad f(x)|_{\min} = f\left(\frac{4\pi}{3}\right) = \int_{5\pi/4}^{4\pi/3} (3\sin t + 4\cos t)\, dt$$

$$= \frac{3}{2} + \frac{1}{\sqrt{2}} - 2\sqrt{3}$$

Hence, (a) is the correct answer.

● **Ex. 13** *For any real* θ, *the maximum value of* $\cos^2(\cos\theta) + \sin^2(\sin\theta)$ *is*

(a) 1 (b) $1 + \sin^2 1$ (c) $1 + \cos^2 1$ (d) does not exist

Sol. The maximum value of $\cos^2(\cos\theta)$ is 1 and that of $\sin^2(\sin\theta)$ is $\sin^2 1$, both exists for $\theta = \frac{\pi}{2}$.

So, maximum value is $1 + \sin^2 1$.

Hence, (b) is the correct answer.

● **Ex. 14** *If* $\sin\theta + \cos\theta = 1$, *then the minimum value of* $(1 + \text{cosec}\,\theta)(1 + \sec\theta)$ *is*

(a) 3 (b) 4 (c) 6 (d) 9

Sol. We know that, AM $\geq$ GM

$$\frac{\sin\theta + \cos\theta}{2} \geq \sqrt{\sin\theta\cos\theta} \quad \Rightarrow \quad \sin\theta\cos\theta \leq \frac{1}{4}$$

Now, let $\sin\theta = x, \cos\theta = y$

and $(1 + \text{cosec}\,\theta)(1 + \sec\theta) \geq p$

$$\Rightarrow \quad \left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right) \geq p$$

$$\Rightarrow \quad \left(\frac{1+x}{x}\right)\left(\frac{1+y}{y}\right) \geq p$$

$$\Rightarrow \quad xy + x + y + 1 \geq pxy$$

$$\Rightarrow \quad x + y + 1 \geq (p-1)\,xy$$

$$\Rightarrow \quad 2 \geq (p-1)\,xy \qquad [\text{since, } x + y = 1]$$

$$\Rightarrow \quad xy \leq \frac{2}{p-1}$$

$$\Rightarrow \frac{2}{p-1} = \frac{1}{4} \Rightarrow p - 1 = 8 \ \Rightarrow \ p = 9$$

Hence, (d) is the correct answer.

● **Ex. 15** *The coordinates of the point on the curve* $x^3 = y(x-a)^2, a > 0$, *where the ordinate is minimum*

(a) $(2a, 8a)$ (b) $\left(-2a, \frac{-8a}{9}\right)$

(c) $\left(3a, \frac{27a}{4}\right)$ (d) $\left(-3a, \frac{-27a}{16}\right)$

Sol. The ordinates of any point on the curve is given by

$$y = \frac{x^3}{(x-a)^2}$$

$$\frac{dy}{dx} = \frac{x^2(x-3a)}{(x-a)^3}$$

Now, $\frac{dy}{dx} = 0$

$$\Rightarrow \quad x = 0 \text{ or } x = 3a$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=0} = 0 \quad \text{and} \quad \left[\frac{d^2y}{dx^2}\right]_{x=3a} = \frac{72a^5}{(2a)^6} > 0$$

So, y is minimum at $x = 3a$ and is equal to $\frac{27a}{4}$.

Hence, (c) is the correct answer.

● **Ex. 16** *If* $x^2 + y^2 + z^2 = 1$ *for* $x, y, z \in R$, *then the maximum value of* $x^3 + y^3 + z^3 - 3xyz$ *is*

(a) $\frac{1}{2}$ (b) 1 (c) 2 (d) 3

Sol. Let $t = xy + yz + zx$, so $-\frac{1}{2} \leq t \leq 1$

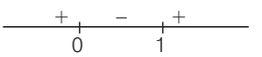
$$\therefore \quad x^3 + y^3 + z^3 - 3xyz$$

$$= (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$$

$$= \sqrt{(1+2t)}\,(1-t)$$

Let $f(t) = (1+2t)(1-t)^2$

$$f'(t) = 6t(t-1) = 0$$



Clearly, $t_{\max} = f(0) = 1$

Hence, (b) is the correct answer.

● **Ex. 17** *If* $a, b \in R$ *distinct numbers satisfying* $|a-1| + |b-1| = |a| + |b| = |a+1| + |b+1|$, *then the minimum value of* $|a-b|$ *is*

(a) 3 (b) 0

(c) 1 (d) 2

Sol. Let $a < b$ and $f(x) = |x - a| + |x - b|, \forall\ x \in R$

So, $f(x)$ is decreasing in $(-\infty, a]$ constant in $[a, b]$ and increasing in $[b, \infty)$, we have

$$f(0) = f(1) = f(-1)$$

$\Rightarrow$ $\{-1, 0, 1\} \in [a, b]$

$\therefore$ $|a - b|_{\min} = 2$

Hence, (d) is the correct answer.

● **Ex. 18** *Let $f(x) = x^4 + ax^3 + 3x^2 + bx + 1, a, b \in R$. If $f(x) \geq 0, \forall\ x \in R$, then the maximum value of $a^2 + b^2$ is equal to*

(a) 10 (b) 12

(c) 16 (d) None of these

Sol. Given, $f(x) = x^4 + ax^3 + 3x^2 + bx + 1$ and $f(x) \geq 0$

$$\Rightarrow \left(x^2 + \frac{a}{2}x\right)^2 + \left(3 - \frac{a^2 + b^2}{4}\right)x^2 + \left(\frac{b}{2}x + 1\right)^2 \geq 0$$

which holds only when $3 - \frac{a^2 + b^2}{4} \geq 0$

$$\Rightarrow a^2 + b^2 \leq 12$$

Hence, (b) is the correct answer.

● **Ex. 19** *The maximum value of $(\sqrt{-3 + 4x - x^2} + 4)^2 + (x - 5)^2$ (where $1 \leq x \leq 3$) is*

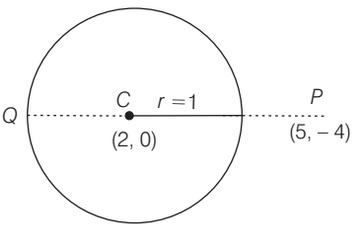
(a) 34 (b) 36

(c) 32 (d) 20

Sol. Here, $(\sqrt{-3 + 4x - x^2} + 4)^2 + (x - 5)^2$, represents the square of the distance between circle $y = \sqrt{-3 + 4x - x^2}$ and point $(5, -4)$.

Clearly, the centre and radius of the circle respectively are $(2, 0)$ and 1.

i.e. Maximum distance between $x^2 + y^2 - 4x + 3 = 0$ and $(5, -4)$,



$$\Rightarrow PQ^2 = (PC + \text{radius})^2$$

$$= (\sqrt{(5-2)^2 + (4-0)^2} + 1)^2$$

$$= 6^2 = 36$$

Hence, (b) is the correct answer.

● **Ex. 20** *If $a > b > 0$ and $f(\theta) = \frac{(a^2 - b^2)\cos\theta}{a - b\sin\theta}$, then the maximum value of $f(\theta)$, is*

(a) $2\sqrt{a^2 + b^2}$ (b) $\sqrt{a^2 + b^2}$

(c) $\sqrt{a^2 - b^2}$ (d) $\sqrt{b^2 - a^2}$

Sol. Here, $f(\theta) = \frac{(a^2 - b^2)\cos\theta}{a - b\sin\theta} = \frac{(a^2 - b^2)}{a\sec\theta - b\tan\theta}$

or $f(\theta) = \frac{a^2 - b^2}{h(\theta)}$, where $h(\theta) = a\sec\theta - b\tan\theta$

$\therefore$ $f(\theta)$ is maximum and minimum as $h(\theta)$ is minimum and maximum respectively.

$\Rightarrow$ $h(\theta) = a\sec\theta - b\tan\theta$

$\Rightarrow$ $h'(\theta) = \sec\theta\,(a\tan\theta - b\sec\theta)$

For maximum and minimum put $h'(\theta) = 0$

$\Rightarrow$ $\sin\theta = \frac{b}{a}$ $[\sec\theta \neq 0]$ ...(i)

Also, $h''(\theta) = a\sec^3\theta + a\sec\theta \cdot \tan^2\theta - 2b\sec^2\theta\tan\theta$

or $h''(\theta) = \frac{a + a\sin^2\theta - 2b\sin\theta}{\cos^3\theta}$

When $\sin\theta = \frac{b}{a}$

$$(h''(\theta)) = \frac{a + a\cdot\frac{b^2}{a^2} - 2b\cdot\frac{b}{a}}{\left(1 - \frac{b^2}{a^2}\right)^{3/2}} = \frac{a^2 - b^2}{a\left(\frac{a^2 - b^2}{a^2}\right)^{3/2}} > 0$$

[as $a > b$]

$\Rightarrow h(\theta)$ is minimum, when $\sin\theta = \frac{b}{a}$

$\therefore$ $f(\theta)$ is maximum, when

$$\sin\theta = \frac{b}{a} \Rightarrow f_{\max}(\theta) = \sqrt{a^2 - b^2}$$

Hence, (c) is the correct answer.

● **Ex. 21** *A solid cylinder of height H has a conical portion of same height and radius 1/3rd of height removed from it. Rain water is accumulating in it, at the rate equal to π times the instantaneous radius of the water surface inside the hole, the time after which hole will filled with water is*

(a) $\frac{H^2}{3}$ (b) H^2 (c) $\frac{H^2}{6}$ (d) $\frac{H^2}{4}$

Sol. Here radius, $r = \frac{H}{3}, \frac{x}{r} = \frac{y}{H} \Rightarrow 3x = y \Rightarrow \frac{dv}{dt} = \pi x$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{3}\pi x^2 y\right) = \pi x \Rightarrow 3\int_0^r x\,dx = \int_0^t dt$$

$$\Rightarrow 3\frac{r^2}{2} = t \Rightarrow \frac{3}{2}\cdot\frac{H^2}{9} = t \Rightarrow t = \frac{H^2}{6}$$

Hence, (c) is the correct answer.

● **Ex. 22** *If composite function* $f_1(f_2(f_3(....(f_n(x)))))$ *n times, is an increasing function and if r of* f_i*'s are decreasing function while rest are increasing, then maximum value of function is*

(a) $\frac{n^2-1}{4}$ when n is an even number

(b) $\frac{n^2}{4}$ when n is an odd number

(c) $\frac{n^2-1}{4}$ when n is an odd number

(d) None of the above

Sol. r must be an even integer because two decreasing are required to make it increasing function.

Let $y = r(n-r)$,

When n is odd

$$r = \frac{n-1}{2} \text{ or } \frac{n+1}{2} \text{ for maximum value of } y$$

When n is even

$$r = \frac{n}{2} \text{ for maximum value of } y$$

$\therefore$ Maximum $(y) = \frac{n^2-1}{4}$, when n is odd and $\frac{n^2}{4}$

When n is even.

Hence, (c) is the correct answer.

JEE Type Solved Examples :
More than One Correct Option Type Questions

● **Ex. 23** *Let* $f(x) = \sin x + ax + b$*. Then,* $f(x) = 0$ *has*

(a) only one real root which is positive, if $a > 1, b < 0$

(b) only one real root which is negative, if $a > 1, b > 0$

(c) only one real root which is negative, if $a < -1, b < 0$

(d) None of the above

Sol. Given, $f(x) = \sin x + ax + b$

$\Rightarrow f'(x) = -\cos x + a$, if $a > 1$, then $f(x)$ is entirely increasing. So, $f(x) = 0$ has only one real root, which is positive, if $f(x) < 0$ and negative, if $f(0) > 0$. Similarly, when $a < -1$ then, $f(x)$ entirely decreasing. Therefore, $f(x)$ has only one real root which is positive, if $f(0) < 0$ and negative, if $f(0) > 0$.

Hence, (a), (b) and (c) are the correct answers.

● **Ex. 24** *If* $a > 0, b > 0, c > 0$ *and* $a + b + c = abc$*, then atleast one of the numbers* a, b, c *exceeds*

(a) $\frac{3}{2}$ (b) $\frac{17}{10}$

(c) 2 (d) $\frac{13}{10}$

Sol. We may suppose $a \geq b \geq c$

$$abc = a + b + c \geq 3c$$

So, $$ab \geq 3, a \geq b \text{ and } a \geq \sqrt{3} > \frac{17}{10}$$

Aliter $(a+b+c)^3 \geq 27\ abc \geq 27\ (a+b+c)$

$\therefore \quad a + b + c \geq 3\sqrt{3}$

$\Rightarrow$ Atleast one of them $\geq \sqrt{3}$

Hence, (a), (b) and (d) are the correct answers.

● **Ex. 25** *Let* $f(x, y) = x^2 + 2xy + 3y^2 - 6x - 2y$*, where* $x, y \in R$*, then*

(a) $f(x, y) \geq -11$ (b) $f(x, y) \geq -10$

(c) $f(x, y) > -11$ (d) $f(x, y) > -12$

Sol. Let $z = x^2 + 2xy + 3y^2 - 6x - 2y$

$\Rightarrow \quad x^2 + 2xy + 3y^2 - 6x - 2y - z = 0$ as $x \in R$

$\therefore \quad D \geq 0$

$\Rightarrow \quad 4(y-3)^2 - 4(3y^2 - 2y - z) \geq 0$

$\Rightarrow \quad y^2 + 9 - 6y - 3y^2 + 2y + z \geq 0$

$\Rightarrow \quad -2y^2 - 4y + 9 + z \geq 0$

$\Rightarrow \quad z \geq 2(y^2 + 2y + 1) - 11 = 2(y+1)^2 - 11$

$\Rightarrow \quad z \geq -11$

Hence, (a), (c) and (d) are the correct answers.

● **Ex. 26** *Let* $g(x) = f(\tan x) + f(\cot x), \forall\ x \in \left(\frac{\pi}{2}, \pi\right)$*. If* $f''(x) < 0, \forall\ x \in \left(\frac{\pi}{2}, \pi\right)$*, then*

(a) $g(x)$ is increasing in $\left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$

(b) $g(x)$ has local minimum at $x = \frac{3\pi}{4}$

(c) $g(x)$ is decreasing in $\left(\frac{3\pi}{4}, \pi\right)$

(d) $g(x)$ has local maximum at $x = \frac{3\pi}{4}$

Sol. Given, $g(x) = f(\tan x) + f(\cot x)$

$$g'(x) = f'(\tan x)\sec^2 x - f'(\cot x)\operatorname{cosec}^2 x$$

For increasing $g'(x) > 0$

$$f'(\tan x) > f'(\cot x)$$

$$[\because f''(x) < 0 \text{ and } \tan x < \cot x, \forall\ x \left(\frac{\pi}{4}, \frac{\pi}{2}\right)]$$

Also, $\sec^2 x > \text{cosec}^2 x,\ x \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$

$$g'(x) > 0 \Rightarrow g(x) \text{ is increasing in } \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$$

Similarly, $g(x)$ is decreasing in $\left(\frac{3\pi}{4}, \pi\right)$

Also, $g(x)$ has local maximum at $x = \frac{3\pi}{4}$. Hence, (a), (c) and (d) are the correct answers.

● ***Ex. 27*** *The function* $f(x) = \int_0^x \sqrt{1-t^4}\ dt$ *is such that*

(a) it is defined on the interval $[-1, 1]$
(b) it is an increasing function
(c) it is an odd function
(d) the point $(0, 0)$ is the point of inflection

Sol. $f'(x) = \sqrt{1-x^4} > 0$ in $(-1, 1) \Rightarrow f$ is increasing

Now, $f(x) + f(-x) = \int_0^x \sqrt{1-t^4}\ dt + \int_0^{-x} \sqrt{1-t^4}\ dt$

$$\Rightarrow \quad \int_0^x \sqrt{1-t^4}\ dt + \left(-\int_0^x \sqrt{1-y^4}\ dy\right)(t = -y) = 0$$

$\Rightarrow f(x)$ is odd

Again, $f''(x) = \frac{-4x^3}{2\sqrt{1-x^4}}$ which vanished at $x = 0$

and changes sign $\Rightarrow (0, 0)$ is point of inflection since, f is well defined in $[-1, 1]$.

Hence, (a), (b), (c) and (d) are the correct answers.

● ***Ex. 28*** *The function* $\frac{\sin(x+a)}{\sin(x+b)}$ *has no maxima or minima, if*

(a) $b - a = n\pi, n \in I$ (b) $b - a = (2n+1)\pi, n \in I$
(c) $b - a = 2n\pi, n \in I$ (d) None of these

Sol. Given, $f(x) = \frac{\sin(x+a)}{\sin(x+b)}$

$$f'(x) = \frac{\sin(x+b) \times \cos(x+a) - \sin(x+a)\cos(x+b)}{\sin^2(x+b)}$$

$$= \frac{\sin(b-a)}{\sin^2(x+b)}$$

If $\sin(b-a) = 0$, then $f'(x) = 0$

$\Rightarrow f(x)$ will be constant.

i.e. $b - a = n\pi$ or $b - a = (2n+1)\pi$

or $b - a = 2n\pi$,

then $f(x)$ has no minima.

Hence, (a), (b) and (c) are the correct answers.

● ***Ex. 29*** *Let* $F(x) = 1 + f(x) + (f(x))^2 + (f(x))^3$, *where* $f(x)$ *is an increasing differentiable function and* $F(x) = 0$ *has a positive root, then*

(a) $F(x)$ is an increasing function (b) $F(0) \le 0$
(c) $f(0) \le -1$ (d) $F'(0) > 0$

Sol. Given, $F(x) = 1 + f(x) + (f(x))^2 + (f(x))^3$

$F'(x) = (1 + 2f(x) + 3(f(x))^2)\ f'(x) > 0$, so $F(x)$ is increasing.

So, $F(0) < 0 \Rightarrow (1 + f(0))(1 + f(0))^2) < 0 \Rightarrow f(0) < -1$

Hence, (a), (b), (c) and (d) are the correct answers.

● ***Ex. 30*** *The extremum values of the function* $f(x) = \frac{1}{\sin x + 4} - \frac{1}{\cos x - 4}$, *where* $x \in R$ *is*

(a) $\frac{4}{8-\sqrt{2}}$ (b) $\frac{2\sqrt{2}}{8-\sqrt{2}}$ (c) $\frac{2\sqrt{2}}{4\sqrt{2}+1}$ (d) $\frac{4\sqrt{2}}{8+\sqrt{2}}$

Sol. Given, $f(x) = \frac{1}{\sin x + 4} - \frac{1}{\cos x - 4}$

$$\Rightarrow \quad f'(x) = \frac{-\cos x}{(\sin x + 4)^2} + \frac{\sin x}{(\cos x - 4)^2}$$

$$\Rightarrow \quad f'(x) = 0 \ \Rightarrow\ (\sin x + \cos x)\ (\text{non-zero quantity})$$

$$= 0 \ \Rightarrow\ \tan x = -1$$

$$\Rightarrow \quad x = \frac{3\pi}{4} \quad \text{or} \quad \frac{7\pi}{4}$$

Global minimum $= x = 2n\pi + \left(\frac{3\pi}{4}\right)$

Global maximum $= x = 2n\pi + \left(\frac{7\pi}{4}\right)$

$$M = \frac{4}{8-\sqrt{2}},\ m = \frac{4}{8+\sqrt{2}} = \frac{2\sqrt{2}}{4\sqrt{2}+1}$$

Hence, (a) and (c) are the correct answers.

● ***Ex. 31*** *The function* $f(x) = x^{1/3}(x-1)$

(a) has 2 inflection points

(b) is strictly increasing for $x > \frac{1}{4}$ and strictly decreasing for $x < \frac{1}{4}$

(c) is concave down in $\left(-\frac{1}{2}, 0\right)$

(d) area enclosed by the curve lying in the fourth quadrant is $\frac{9}{28}$

Sol. $y = x^{1/3}(x-1)$

$$\frac{dy}{dx} = \frac{4}{3}x^{1/3} - \frac{1}{3} \cdot \frac{1}{x^{2/3}} = \frac{1}{3x^{2/3}}[4x - 1]$$

$x^{2/3}$ is always positive and $x = \frac{1}{4}$ has a local minima.

Hence, f is increasing for $x > \frac{1}{4}$

and f is decreasing for $x < \frac{1}{4}$

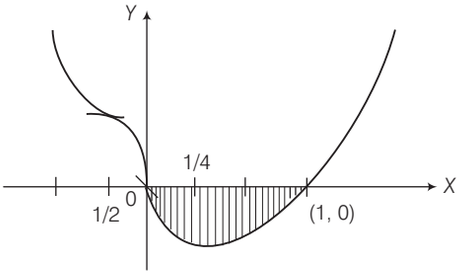
Now, $f'(x) = \frac{4}{3}x^{1/3} - \frac{1}{3}\cdot x^{-2/3}$

[non-existence at $x = 0$, vertical tangent]

$$f''(x) = \frac{4}{9}\cdot\frac{1}{x^{2/3}} + \frac{1}{3}\cdot\frac{2}{3}\cdot\frac{1}{x^{5/3}} = \frac{2}{9x^{2/3}}\left[2 + \frac{1}{x}\right]$$

$$= \frac{2}{9x^{2/3}}\left[\frac{2x+1}{x}\right]$$

$\therefore$ $f''(x) = 0$ at $x = -\frac{1}{2}$ [inflection point]



Graph of $f(x)$ is as $A = \int_0^1 (x^{4/3} - x^{1/3})\,dx$

$$= \left[\frac{3}{7}x^{3/7} - \frac{3}{4}x^{4/3}\right]_0^1 = \left|\frac{3}{7} - \frac{3}{4}\right| = 3\left|\frac{4-7}{28}\right| = \frac{9}{28}$$

Hence, (a), (b), (c) and (d) are the correct answers.

● ***Ex. 32*** *Assume that inverse of the function f is denoted by g, then which of the following statement hold good?*

(a) If f is increasing, then g is also increasing
(b) If f is decreasing, then g is increasing
(c) The function f is injective
(d) The function g is onto

Sol. If f and g are inverse, then $(fog)(x) = x$

$$f'[g(x)]g'(x) = 1$$

If f is increasing $\Rightarrow f' > 0 \Rightarrow$ Sign of g' is also positive.

Therefore, option (a) is correct.

If f is decreasing $\Rightarrow f' < 0 \Rightarrow$ Sign of g' is negative.

Therefore, option (b) is false.

Since, f has an inverse.

$\Rightarrow$ f is bijective $\Rightarrow$ f is injective

Therefore, option (c) is correct.

Inverse of a bijective mapping is bijective.

$\Rightarrow$ g is also bijective $\Rightarrow$ g is onto

Therefore, option (d) is correct.

Hence, (a), (c) and (d) are the correct answers

JEE Type Solved Examples :
Statements I and II Type Questions

■ **Directions** (Ex. Nos. 33 to 39) *For the following questions, choose the correct answers from the codes (a), (b), (c) and (d) defined as follows*

(a) Statement I is true, Statement II is also true, Statement II is the correct explanation of Statement I.
(b) Statement I is true, Statement II is also true, Statement II is not the correct explanation of Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

● ***Ex. 33*** **Statement I** *Among all the rectangles of the given perimeter, the square has the largest area. Also, among all the rectangles of given area, the square has the least perimeter.*

Statement II *For $x > 0, y > 0$, if $x + y$ = constant, then xy will be maximum for $y = x$ and if xy = constant, then $x + y$ will be minimum for $y = x$.*

Sol. Statement II $x + y = k$

then $xy = x(k - x) = f(x)$, $f'(x) = k - 2x = 0$

$\Rightarrow$ $x = \frac{k}{2}, y = \frac{k}{2}$

$\Rightarrow$ $y = x$

$$x + y = x + \frac{k}{x} = f(x),\ f'(x) = 1 - \frac{k}{x^2} = 0$$

$\Rightarrow$ $x = \sqrt{k}, y = \sqrt{k}$

So, Statement II is true and it explains Statement I.

Hence, (a) is the correct answer.

● ***Ex. 34*** **Statement I** *The function $f(x) = (x^3 + 3x - 4)(x^2 + 4x - 5)$ has local extremum at $x = 1$.*

Statement II *$f(x)$ is continuous and differentiable and $f'(1) = 0$.*

Sol. Statement I is correct because $f(1)^- > f(1) < f(1)^+$.

Statement II is correct as $f(x)$ has a repeated root at $x = 1$.

Statement II is not the correct explanation of Statement I as $f'(c) = 0$ doesn't imply that f has an extrema at $x = c$.

Hence, (b) is the correct answer.

● ***Ex. 35*** **Statement I** *If $f(x)$ is increasing function with upward concavity, then concavity of $f^{-1}(x)$ is also upwards.*

Statement II *If $f(x)$ is decreasing function with upwards concavity, then concavity of $f^{-1}(x)$ is also upwards.*

Sol. Let $g(x)$ be the inverse function of $f(x)$.

Then, $f(g(x)) = x$

$\therefore$ $f'(g(x)) \cdot g'(x) = 1$

i.e. $g'(x) = \dfrac{1}{f'(g(x))}$

$\therefore$ $g'' = -\dfrac{1}{(f'(g(x)))^2} \cdot f''(g(x)) \cdot g'(x)$

In Statement I $f''(g(x)) > 0$

and $g'(x) > 0$

$\Rightarrow$ $g''(x) < 0$

$\Rightarrow$ Concavity of $f^{-1}(x)$ is downwards.

$\therefore$ Statement I is false.

In Statement II $f''(g(x)) > 0$

and $g'(x) < 0$

$\Rightarrow$ $g''(x) > 0$

$\Rightarrow$ Concavity of $f^{-1}(x)$ is upwards.

$\therefore$ Statement II is true.

Hence, (d) is the correct answer.

● ***Ex. 36*** **Statement I** *The minimum distance of the fixed point $(0, y_0)$, where $0 \le y_0 \le \dfrac{1}{2}$, from the curve $y = x^2$ is y_0.*

Statement II *Maxima and minima of a function is always a root of the equation $f'(x) = 0$.*

Sol. Let the point on the parabola be (t, t^2).

Let d be the distance between (t, t^2) and $(0, y_0)$,

then $d^2 = t^2 + (t^2 - y_0)^2 = t^4 + (1 - 2y_0)\, t^2 + y_0^2$

$= z^2 + (1 - 2y_0)\, z + y_0^2, z \ge 0$

Its vertex is at $x = y_0 - \dfrac{1}{2} < 0$

$\therefore$ The minimum value of d^2 is at $z = 0$, i.e. $t^2 = 0$

$\therefore$ $d = y_0$

Statement I is true. Statement II is false because extremum can occur at a point where $f'(x)$ does not exist.

Hence, (c) is the correct answer.

● ***Ex. 37*** *Let $f : R \to R$ is differentiable and strictly increasing function throughout its domain.*

Statement I *If $|f(x)|$ is also strictly increasing function, then $f(x) = 0$ has no real roots.*

Statement II *At ∞ or $-\infty$, $f(x)$ may approach to 0, but cannot be equal to zero.*

Sol. Suppose $f(x) = 0$ has a real root say $x = a$, then $f(x) < 0$ for $x < a$. Thus, $|f(x)|$ becomes strictly decreasing on $(-\infty, a)$, which is a contradiction.

Hence, (a) is the correct answer.

● ***Ex. 38*** **Statement I** *$f(x) = x + \cos x$ is strictly increasing.*

Statement II *If $f(x)$ is strictly increasing, then $f'(x)$ may tend to zero at some finite number of points.*

Sol. Given, $f(x) = x + \cos x$

$\therefore$ $f'(x) = 1 - \sin x > 0, \ \forall \ x \in R,$

except at $x = 2n\pi + \dfrac{\pi}{2}$

and $f'(x) = 0$ at $x = 2n\pi + \dfrac{\pi}{2}$

$\therefore$ $f(x)$ is strictly increasing.

Statement II is true but does not explain Statement I.

Statement II gives $f'(x)$ may tend to zero at finite number of points but in Statement I $f'(x)$ tend to zero at infinite number of points.

Hence, (b) is the correct answer.

● ***Ex. 39*** **Statement I** *The largest term in the sequence*

$$a_n = \frac{n^2}{n^3 + 200}, n \in N \text{ is } \frac{(400)^{2/3}}{600}$$

Statement II $f(x) = \dfrac{x^2}{x^3 + 200}, x > 0$, *then at* $x = (400)^{1/3}$, *$f(x)$ is maximum.*

Sol. **Statement II** Given, $f(x) = \dfrac{x^2}{x^3 + 200}$

$$f'(x) = \frac{(x^3 + 200)\, 2x - 3x^2 x^2}{(x^3 + 200)^2} = \frac{-x^4 + 400x}{(x^3 + 200)^2}$$

$x \to 0^+ \ f(x) = 0^+ \ \Rightarrow \ x = 400^{1/3} \ f(x) = \dfrac{400^{2/3}}{600}$

$x \to \infty \ f(x) \to 0$

So, Statement II is true. But Statement I is false as $x \in N$.

Hence, (d) is the correct answer.

JEE Type Solved Examples : Passage Based Questions

Passage I

(Ex. Nos. 40 to 42)

Let x_1, x_2, x_3, x_4 be the roots (real or complex) of the equation $x^4 + ax^3 + bx^2 + cx + d = 0$. If $x_1 + x_2 = x_3 + x_4$ and $a, b, c, d \in R$, then

● **Ex. 40** *If $a = 2$, then the value of $b - c$ is*

(a) -1 (b) 1 (c) -2 (d) 2

● **Ex. 41** *If $b < 0$, then how many different values of a, we may have*

(a) 3 (b) 2 (c) 1 (d) 0

● **Ex. 42** *If $b + c = 1$ and $a \neq -2$, then for real values of $a, c \in$*

(a) $\left(-\infty, \frac{1}{4}\right)$ (b) $(-\infty, 3)$

(c) $(-\infty, 1)$ (d) $(-\infty, 4)$

■ **Sol.** (Ex. Nos. 40 to 42)

Let $x^4 + ax^3 + bx^2 + cx + d = (x - x_1)(x - x_2)(x - x_3)(x - x_4)$

Let $(x - x_1)(x - x_2) = x^2 + px + q$

and $(x - x_3)(x - x_4) = x^2 + px + r$

$\therefore \quad q = x_1x_2$ and $r = x_3x_4$

$x^4 + ax^3 + bx^2 + cx + d$

$= x^4 + 2px^3 + (p^2 + q + r)x^2 + p(q + r)x + qr$

$\therefore \quad a = 2p, b = p^2 + q + r, c = p(q + r), d = qr$

Clearly, $a^3 - 4ab + 8c = 0$ ...(i)

40. If $a = 2 \Rightarrow b - c = 1$

Hence, (b) is the correct answer.

41. Investigating the nature of the cubic equation of a.

Let $f(a) = a^3 - 4ab + 8c$

$f'(a) = 3a^2 - 4b$

If $b < 0 \Rightarrow f'(a) > 0$

$\therefore$ The equation $a^3 - 4ab + 8c = 0$, has only one real root.

Hence, (c) is the correct answer.

42. Substituting $c = 1 - b$ in Eq. (i), we have

$(a + 2)[(a - 1)^2 + 3 - 4b] = 0$

$\Rightarrow \quad 4b - 3 > 0$

$\Rightarrow \quad b > \frac{3}{4} \quad \Rightarrow \quad c < \frac{1}{4}.$

Hence, (a) is the correct answer.

Passage II

(Ex. Nos. 43 to 45)

Consider a ΔOAB formed by the point $O(0, 0), A(2, 0), B(1, \sqrt{3})$. $P(x, y)$ is an arbitrary interior point of triangle moving in such a way that $d(P, OA) + d(P, AB) + d(P, OB) = \sqrt{3}$, where $d(P, OA), d(P, AB), d(P, OB)$ represent the distance of P from the sides OA, AB and OB , respectively.

● **Ex. 43** *Area of region representing all possible position of point P is equal to*

(a) $2\sqrt{3}$ (b) $\sqrt{6}$ (c) $\sqrt{3}$ (d) None of these

● **Ex. 44** *If the point P moves in such a way that $d(P, OA) \leq \min(d(P, OB), d(P, AB))$, then area of region representing all possible position of point P is equal to*

(a) $\sqrt{3}$ (b) $\sqrt{6}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\frac{1}{\sqrt{6}}$

● **Ex. 45** *If the point P moves in such a way that $d(P, OA) \geq \min(d(P, OB), d(P, AB))$, then area of region representing all possible position of point P is equal to*

(a) $\sqrt{3}$ (b) $\sqrt{6}$ (c) $1/\sqrt{3}$ (d) $\frac{1}{\sqrt{6}}$

■ **Sol.** (Ex. Nos. 43 to 45)

43. ΔOAB is clearly equilateral

$\Delta\, OAB = \Delta\, OPA + \Delta\, APB + \Delta\, OPB = \frac{\sqrt{3}}{4} \times 4 = \frac{1}{2} \cdot 2$

$(d(P, OA) + (P, AB) + (P, OB))$

$d(P, OA) + d(P, AB) + d(P, OB) = \frac{4}{\sqrt{3}}$

Hence, (c) is the correct answer.

44. We must have, $d(P, OA) \leq d(P, OB)$ as well as $d(P, OA) \leq d(P, AB)$, then P lies either on or below the angle bisector of $\angle BOA$ and $\angle BAO$ area

$= \frac{1}{3}\Delta\, OAB = \frac{1}{\sqrt{3}}.$

Hence, (c) is the correct asnwer.

45. We must have $d(P, OA) \geq d(P, OB)$ as well as $d(P, OA) \geq d(P, AB)$, then P must be above bisector of $\angle BOA$ and $\angle BAO$.

Area of triangle $= \frac{1}{3}\Delta\, OAB = \frac{1}{\sqrt{3}}$

Hence, (c) is the correct answer.

Passage III

(Ex. Nos. 46 to 48)

Let $f(x) = ax^2 + bx + c, a, b, c \in R$. *It is given* $|f(x)| \le 1, |x| \le 1$

● ***Ex. 46*** *The possible value of* $|a + c|$, *if* $\frac{8}{3}a^2 + 2b^2$ *is maximum, is given by*

(a) 1 (b) 0 (c) 2 (d) 3

● ***Ex. 47*** *The possible value of* $|a + b|$, *if* $\frac{8}{3}a^2 + 2b^2$ *is maximum, is given by*

(a) 1 (b) 0 (c) 2 (d) 3

● ***Ex. 48*** *The possible maximum value of* $\frac{8}{3}a^2 + 2b^2$ *is given by*

(a) 32 (b) $\frac{32}{3}$

(c) $\frac{2}{3}$ (d) $\frac{16}{3}$

■ ***Sol.*** (Ex. Nos. 46 to 48)

We know that for $|u| \le 1; |v| \le 1$, then $|u - v| \le 2$

Now, $|f(1) - f(0)| \le 2 \Rightarrow |a + b| \le 2$

$\Rightarrow (a + b)^2 \le 4$ ...(i)

Also, $|f(-1) - f(0)| \le 2 \Rightarrow |a - b| \le 2$

$\Rightarrow (a - b)^2 \le 4$ ...(ii)

Now, $4a^2 + 3b^2 = 2(a + b)^2 + 2(a - b)^2 - b^2 \le 16$

$\Rightarrow (4a^2 + 3b^2)_{\max} = 16$ when $b = 0$

$\Rightarrow |a + b| = |a - b| = |a| = 2$

Also, $|f(1) - f(0)| = |(a + c) - c| = |a| = 2$

$\Rightarrow |a + c| = |c| = 1$

The possible ordered triplet (a, b, c) are $(2, 0, -1)$ or $(-2, 0, 1)$.

Also, $\frac{8}{3}a^2 + 2b^2 = \frac{2}{3}(4a^2 + 3b^2) \le \frac{2}{3} + 16$

Ans. 46. (a) **47.** (c) **48.** (b)

Passage IV

(Ex. Nos. 49 to 53)

The absolute maximum and minimum values of functions can be found by their monotonic and asymptotic behaviour provided they exist. We may agree that finite limiting values may be regarded as absolute maximum or minimum. For instance the absolute maximum value of $\frac{1}{1 + x^2}$ *is unity. It is attained at* $x = 0$ *while absolute minimum value of the same function is zero which is a limiting value of* $(x \to \infty$ *or* $x \to -\infty)$

● ***Ex. 49*** *The function* $x^4 - 4x + 1$ *will have*

(a) absolute maximum value
(b) absolute minimum value
(c) both absolute maximum and minimum values
(d) None of the above

● ***Ex. 50*** *The absolute minimum value of the function* $\frac{x - 2}{\sqrt{x^2 + 1}}$ *is*

(a) -1 (b) $\frac{1}{2}$ (c) $-\sqrt{5}$ (d) None of these

● ***Ex. 51*** *The absolute minimum and maximum values of the function* $\frac{x^2 - x + 1}{x^2 + x + 1}$ *is*

(a) 1 and 3 (b) $\frac{1}{2}$ and 3

(c) $\frac{1}{3}$ and 3 (d) None of these

● ***Ex. 52*** *The function* $f(x) = \frac{4}{x - 1} - \frac{9}{x + 1}$ *will*

(a) have absolute maximum value $-\frac{1}{2}$
(b) have absolute minimum value $-\frac{25}{2}$
(c) not lie between $-\frac{25}{2}$ and $-\frac{1}{2}$
(d) always be negative

● ***Ex. 53*** *Which of the following functions will have absolute minimum value?*

(a) $\cot(\sin x)$ (b) $\tan(\log x)$

(c) $x^{2005} - x^{1947} + 1$ (d) $x^{2006} + x^{1947} + 1$

■ ***Sol.*** (Ex. Nos. 49 to 53)

49. Since, $f'(x) = 4x^3 - 4 = 0 \Rightarrow x = 1$

$\Rightarrow$ There is only one extrema which is minima.

$\Rightarrow$ 1 is a point of absolute minima.

Hence, (b) is the correct answer.

50. Since, $f'(x) = \frac{1 + 2x}{(x^2 + 1)^{3/2}}$

$\Rightarrow f(x)$ is increasing in $\left(-\frac{1}{2}, \infty\right)$ and decreasing in $\left(-\infty, -\frac{1}{2}\right)$

$\Rightarrow$ Absolute minimum occurs at $x = -\frac{1}{2}$, we have $= -\sqrt{5}$

Hence, (c) is the correct answer.

51. $yx^2 + yx + y = x^2 - x + 1$

$x^2(1-y) - x(1+y) + (1-y) = 0 \ \because D \geq 0$

$\Rightarrow (1+y)^2 - 4(1-y)^2 \geq 0 \Rightarrow -(3y-1)(y-3) \geq 0$

$\Rightarrow \quad \frac{1}{3} \leq y \leq 3$

Hence, (c) is the correct answer.

52. Should be the correct choice (we can prove by using monotonically that f cannot lie between $-\frac{25}{2}$ and $-\frac{1}{2}$. This is an Ex. of a function whose maximum (local) value is smaller than minimum value).

Hence, (c) is the correct answer.

53. As even degree polynomial will have absolute minimum essentially.

Hence, (d) is the correct answer.

Passage V

(Ex. Nos. 54 to 56)

We are given the curves $y = \int_{-\infty}^{x} f(t)\,dt$ *through the point* $\left(0, \frac{1}{2}\right)$ *and* $y = f(x)$, *where* $f(x) > 0$ *and* $f(x)$ *is differentiable,* $\forall\ x \in R$ *through* $(0,1)$. *If tangents drawn to both the curves at the points with equal abscissae intersect on the same point on the X-axis, then*

● ***Ex. 54*** *Number of solutions* $f(x) = 2ex$ *is equal to*

(a) 0 (b) 1
(c) 2 (d) None of these

● ***Ex. 55*** $\lim_{x \to \infty} (f(x))^{f(-x)}$ *is*

(a) 3 (b) 6
(c) 1 (d) None of these

● ***Ex. 56*** *The function* $f(x)$ *is*

(a) increasing for all x (b) non-monotonic
(c) decreasing for all x (d) None of these

■ ***Sol.*** (Ex. Nos. 54 to 56)

We have, the equations of the tangents to the curves $y = \int_{-\infty}^{x} f(t)\,dt$ and $y = f(x)$ at arbitrary points on them are

$$Y - \int_{-\infty}^{x} f(t)\,dt = f(x)(X - x) \qquad \ldots(i)$$

and $$Y - f(x) = f'(x)(X - X) \qquad \ldots(ii)$$

As Eqs. (i) and (ii), intersect at the same point on X-axis.

On putting $Y = 0$ and equating x-coordinates, we have

$$x - \frac{f(x)}{f'(x)} = x - \frac{\int_{-\infty}^{x} f(t)\,dt}{f(x)} \Rightarrow \frac{f(x)}{\int_{-\infty}^{x} f(t)\,dt} = \frac{f'(x)}{f(x)}$$

$$\Rightarrow \quad \int_{-\infty}^{x} f(t) = cf(x) \qquad \ldots(iii)$$

As, $f(0) = 1 \Rightarrow \int_{-\infty}^{0} f(t)\,dt = \frac{1}{2} = c \times 1 \Rightarrow c = \frac{1}{2}$

$\Rightarrow \int_{-\infty}^{x} f(t)\,dt = \frac{1}{2} f(x)$, differentiating both the sides and on integrating and using boundary condition, we get $f(x) = e^{2x}$, $y = 2ex$ is tangent to $y = e^{2x}$

$\Rightarrow$ Number of solutions = 1

Clearly, $f(x)$ is increasing for all x.

$$\lim_{x \to \infty} (e^{2x})^{e^{-2x}} = 1 \qquad [\infty^0 \text{ form}]$$

Ans. 54. (b) **55.** (c) **56.** (a)

Passage VI

(Ex. Nos. 57 to 59)

Let $f(x) = \left(1 + \frac{1}{x}\right)^x \ (x > 0)$ *and*

$$g(x) = \begin{cases} x \ln(1 + (1/x)), & \text{if } 0 < x \leq 1 \\ 0, & \text{if } x = 0 \end{cases}$$

● ***Ex. 57*** $\lim_{x \to 0^+} g(x)$

(a) is equal to 0 (b) is equal to 1
(c) is equal to e (d) is non-existent

● ***Ex. 58*** *The function* f

(a) has a maxima but not minima
(b) has a minima but not maxima
(c) has both of maxima and minima
(d) is a monotonic

● ***Ex. 59*** $\lim_{n \to \infty} \left\{ f\left(\frac{1}{n}\right) \cdot f\left(\frac{2}{n}\right) \cdot f\left(\frac{3}{n}\right) \ldots f\left(\frac{n}{n}\right) \right\}^{1/n}$ *equals*

(a) $\sqrt{2}e$ (b) $\sqrt{2e}$
(c) $2\sqrt{e}$ (d) $\sqrt{e}$

■ ***Sol.*** (Ex. Nos. 57 to 59)

57. $\lim_{x \to 0^+} x \ln\left(1 + \frac{1}{x}\right) = \lim_{x \to 0^+} \frac{\ln\left(\frac{x+1}{x}\right)}{\frac{1}{x}} \qquad \left[\frac{\infty}{\infty} \text{form}\right]$

Using L'Hospital's rule,

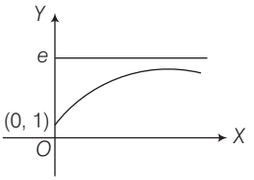
$$l = \lim_{x \to 0} -\left(\frac{1}{x+1} - \frac{1}{x}\right) x^2 = \lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{x+1}\right) \cdot x^2$$

$$= \lim_{x \to 0} \frac{1}{x(x+1)} \cdot x^2 = \lim_{x \to 0} \frac{x}{(x+1)} = 0$$

Hence, (a) is the correct answer.

58. $\lim\limits_{x \to 0} f(x) = 1$ (can be verified)

$\lim\limits_{x \to \infty} f(x) = e$



Also, f is increasing for all

$x > 0$

$\Rightarrow$ (d) (can be verified)

Hence, (d) is the correct answer.

59. Given, $\lim\limits_{n \to \infty}\left(f\left(\frac{1}{n}\right) f\left(\frac{2}{n}\right) \ldots f\left(\frac{n}{n}\right)\right)^{1/n}$

$$l = \left(\prod_{k=1}^{n} \left(1 + \frac{n}{k}\right)^{k/n} \right)^{1/n}$$

$$[\text{given } f(x) = (1 + 1/x)^x \text{ and } f(k/n) = \left(1 + \frac{n}{k}\right)^{k/n}]$$

Taking log,

$$\ln l = \lim_{n \to \infty} \frac{1}{n} \cdot \sum_{k=1}^{n} \ln \left(1 + \frac{n}{k}\right)^{k/n}$$

$$= \lim_{n \to \infty} \frac{1}{n} \cdot \sum_{k=1}^{n} \frac{k}{n} \ln \left(1 + \frac{1}{k/n}\right)$$

$$= \int_0^1 \underbrace{x}_{\text{II}} \underbrace{\ln \left(1 + \frac{1}{x}\right)}_{\text{I}} dx = \left[\ln \left(1 + \frac{1}{x}\right) \cdot \frac{x^2}{2} \right]_0^1 + \int_0^1 \left(\frac{1}{x} - \frac{1}{x+1} \right) \cdot \frac{x^2}{2} dx$$

$$= \left(\frac{1}{2} \ln 2 - 0 \right) + \frac{1}{2} \int_0^1 \frac{x + 1 - 1}{x + 1} dx$$

$$= \frac{1}{2} \ln 2 + \frac{1}{2} [x - \ln (x + 1)]_0^1$$

$$= \frac{1}{2} \ln 2 + \frac{1}{2} [(1 - \ln 2) - 0] = \frac{1}{2}; \quad l = \sqrt{e}$$

Hence, (d) is the correct answer.

Passage VII

(Ex. Nos. 60 to 62)

Consider the cubic $f(x) = 8x^3 + 4ax^2 + 2bx + a$, where $a, b \in R$.

● ***Ex. 60*** *For $a = 1$, if $y = f(x)$ is strictly increasing, $\forall\ x \in R$, then maximum range of the values of b is*

(a) $\left(-\infty, \frac{1}{3}\right]$ (b) $\left(\frac{1}{3}, \infty\right)$

(c) $\left[\frac{1}{3}, \infty\right)$ (d) $(-\infty, \infty)$

● ***Ex. 61*** *For $b = 1$, if $y = f(x)$ is non-monotonic, then the sum of the integral values of $a \in [1, 100]$, is*

(a) 4950 (b) 5049

(c) 5050 (d) 5047

● ***Ex. 62*** *If the sum of the base 2 logarithms of the roots of the cubic $f(x) = 0$ is 5, then the value of a is*

(a) -64 (b) -8 (c) -128 (d) -256

■ ***Sol.*** (Ex. Nos. 60 to 62)

60. $a = 1$

$$f(x) = 8x^3 + 4x^2 + 2bx + 1$$

$$f'(x) = 24x^2 + 8x + 2b = 2(12x^2 + 4x + b)$$

For increasing function $f'(x) \geq 0, \forall\ x \in R$

$\therefore \quad D \leq 0$

$\Rightarrow \quad 16 - 48b \leq 0 \quad \Rightarrow \quad b \geq \frac{1}{3}$

Hence, (c) is the correct answer.

61. If $b = 1$

$$f(x) = 8x^3 + 4ax^2 + 2x + a$$

$$f'(x) = 24x^2 + 8ax + 2 \quad \text{or} \quad 2(12x^2 + 4ax + 1)$$

For non-monotonic $f'(x) = 0$ must have distinct roots hence,

$$D > 0, \quad \text{i.e.} \quad 16a^2 - 48 > 0$$

$\Rightarrow \quad a^2 > 3$

$\therefore \quad a > \sqrt{3}$ or $a < -\sqrt{3}$

$\therefore \quad a \in 2, 3, 4, \ldots,$

Sum $= 5050 - 1 = 5049$

Hence, (b) is the correct answer.

62. If x_1, x_2 and x_3 are the roots, then

$$\log_2 x_1 + \log_2 x_2 + \log_2 x_3 = 5$$

$$\log_2 (x_1 x_2 x_3) = 5$$

$$x_1 x_2 x_3 = 32$$

$$-\frac{a}{8} = 32 \Rightarrow a = -256$$

Hence, (d) is the correct answer.

Passage VIII

(Ex. Nos. 63 to 65)

Let $f(x) = \begin{cases} \max & \{t^3 - t^2 + t + 1, 0 \leq t \leq x\}, 0 \leq x \leq 1 \\ \min & \{3 - t, 1 < t \leq x\}, 1 < x \leq 2 \end{cases}$ *and*

$g(x) = \begin{cases} \max & \{3/8\, t^4 + 1/2\, t^3 - 3/2\, t^2 + 1, 0 \leq t \leq x\}, 0 \leq x \leq 1 \\ \min & \{3/8\, t + 1/32 \sin^2 \pi t + 5/8, 1 \leq t \leq x\}, 1 \leq x \leq 2 \end{cases}$

Define $f(x)$ and $g(x)$ expilicitly and then answer the following questions.

● ***Ex. 63*** *The function $f(x), \forall\ x \in [0, 2]$ is*

(a) continuous and differentiable

(b) continuous but not differentiable

(c) discontinuous and not differentiable

(d) None of the above

● ***Ex. 64*** *Which of the following is true.*

(a) $\lim_{x\to 1^-}(fog)(x) > \lim_{x\to 1^+}(gof)(x)$

(b) $\lim_{x\to 1^-}(fog)(x) < \lim_{x\to 1^+}(gof)(x)$

(c) $\lim_{x\to 1^-}(fog)(x) = \lim_{x\to 1^+}(gof)(x)$

(d) None of the above

● ***Ex. 65*** *Let* $z(x)=\dfrac{d}{dx}f(x)^{g(x)}$ *and* $y(x)=\dfrac{d}{dx}g(x)^{f(x)}$, *then* $z(x)$ *and* $y(x)$ *vanish simultaneously at*

(a) $x=-1/3$ (b) $x=0$

(c) $x=1$ (d) No real value of x

■ ***Sol.*** (Ex. Nos. 63 to 65)

Consider $f(x)$ in $[0,1]$

$$f'(t)=3t^2-2t+1>0,\ \forall\ t\in(0,1)$$

$\Rightarrow$ $f(t)$ is increasing on $(0,1)$

Maximum occurs at $t=x$

and $$f(x)=\begin{cases}x^3-x^2+x+1, & 0\le x\le 1\\ 3-x, & 1<x\le 2\end{cases}$$

Again, consider $g(x)$ in $(0,1)$

$$g(t)=3/8t^4+1/2t^3-3/2t^2+1$$

$$g'(t)=3/2t^3+3/2t^2-3t=3/2t(t^2+t-2)$$

$$=3/2t(t-1)(t+2)$$

$g(t)$ decreases in $[0,1)\Rightarrow$ maximum occurs when $t=0$ and $g(0)=1$, again consider $g(x)$ function in $[1,2]$.

$$g(t)=3/8t+1/32\sin^2\pi t+5/8$$

$$g'(t)=3/8+\pi/32\sin(2\pi t)>0,\ \forall\ t\in R$$

$\therefore$ $g(t)$ is an increasing function in $[1,2]$.

$\Rightarrow$ Minimum occurs when $t=1$ and $g(1)=1$

Hence, $$g(x)=\begin{cases}1, & 0\le x\le 1\\ 1, & 1<x\le 2\end{cases}=1$$

$\Rightarrow$ $$g(x)=1,\ \forall\ x\in[0,2]$$

63. $f(x)$ is continuous but not differentiable at $x=1$

64. $\lim_{x\to 1^-} fog(x)=f(1)$ and $\lim_{x\to 1^+} gof(x)=1$, also $f(1)>1$

65. $z(x)=\dfrac{d}{dx}f(x)^{g(x)}=\dfrac{d}{dx}(f(x))^1,\ \forall\ x\in[0,1)\cup(1,2]$

$=f'(x),\ \forall\ x\in[0,1)\cup(1,2]$

and $$y(x)=\frac{d}{dx}(g(x))^{f(x)}$$

$$=\frac{d}{dx}(1)^{f(x)}=0,$$

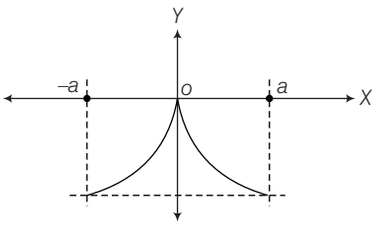
$$\forall\ x\in[0,1)\cup(1,2]$$

Hence, the functions $y(x)$ and $z(x)$ can vanish simultaneously at $f'(x)=0$, which is not possible for any real x.

Passage IX

(Ex. Nos. 66 to 69)

The graph of derivative of a function $f(x)$ *is given* (i.e. $y=f'(x)$). *Analyse the graph in the given domain and answer the following questions, if it is given that* $f(0)=0$



● ***Ex. 66*** *The function* $f'(x)$ *is*

(a) even function (b) odd function

(c) neither even nor odd (d) indefinite

● ***Ex. 67*** *The function* $f(x)$ *is*

(a) even function (b) odd function

(c) neither even nor odd (d) indefinite

● ***Ex. 68*** *The graph of* $y=f(x)$ *has*

(a) no inflexion point (b) one point of inflexion

(c) one extreme point (d) two extreme points

● ***Ex. 69*** *The function* $f(x)$ *for* $-a\le x\le a$, *is*

(a) always decreasing

(b) always increasing

(c) increasing for $(-a,0)$ and decreasing for $(0,a)$

(d) increasing for $(0,a)$ and decreasing for $(-a,0)$

■ ***Sol.*** (Ex. Nos 66 to 69)

66. The graph of $y=f'(x)$ is symmetrical about Y-axis, so $f(x)$ is an even function.

67. $f(0)=0$, so $f(x)$ is an odd function (i.e. derivative of an odd function is an even function)

68. Here, $f''(x)>0$, as $x<0$ and $f''(x)<0$, as $x>0$

$\therefore$ $x=0$, is the point of inflexion

69. $f'(x)\le 0,\ \forall\ x$

So, $f(x)$ is always decreasing.

Passage X

(Ex. Nos. 70 to 72)

If a function (continuous and twice differentiable) is always concave upward in an interval, then its graph lies always below the segment joining extremities of the graph in that interval and vice-versa.

● ***Ex. 70*** *If* $\sin x+x\ge|k|x^2,\ \forall\ x\in[0,\pi/2]$, *then the greatest value of k, is*

(a) $\dfrac{-2(2+\pi)}{\pi^2}$ (b) $\dfrac{2(2+\pi)}{\pi^2}$

(c) can't be determined finitely

(d) zero

● ***Ex. 71*** *Let $f(x)$, $f'(x)$ and $f''(x)$ are all positive, $\forall\, x \in [0,7]$. If $f^{-1}(x)$ exists, then $3f^{-1}(4) - f^{-1}(2) - 2f^{-1}(5)$, is*

(a) always positive (b) always negative
(c) non-negative (d) non-positive

● ***Ex. 72*** *Let $f: R^+ \to R^+$ is such that $f''(x) \geq 0$, $\forall\, x \in [a,b]$, then value of $\int_a^b f(x)dx$, cannot exceed*

(a) $\dfrac{(f(a)+f(b))(b-a)}{3}$ (b) $\dfrac{(f(b)-f(a))(b-a)}{2}$

(c) $\dfrac{(f(b)+f(a))(b-a)}{2}$ (d) None of these

■ ***Sol.*** (Ex. Nos. 70 to 72)

70. $f(x) = \sin x + x \Rightarrow f'(x) = \cos x + 1$

$\Rightarrow f''(x) = -\sin x$

f is concave downward for $x \in [0, \pi/2]$

Irrespective of k, $g(x) = |k|x^2$ is concave upward

So, if $g(\pi/2) \leq f(\pi/2)$, then $f(x) \geq g(x)$

$\Rightarrow \quad 1 + \pi/2 \geq |k|\dfrac{\pi^2}{4} \;\Rightarrow\; k \in \left[\dfrac{(-2\pi+4)}{\pi^2}, \dfrac{(2\pi+4)}{\pi^2}\right]$

71. $f''(x) > 0$

$\Rightarrow$ f is concave upwards $\Rightarrow f^{-1}$ is concave downwards

Consider point dividing the join of this segment in ratio 2:1 is given as $\left(4, \dfrac{2f^{-1}(5) + f^{-1}(2)}{3}\right)$ when upon a point $(4,\ f^{-1}(4))$ on graph of $f^{-1}(x)$ is always above it.

$\Rightarrow \qquad 3f^{-1}(4) - f^{-1}(2) - 2f^{-1}(5) > 0$

72. $\int_a^b f(x)dx$, is area bounded by curve in first quadrant.

$\dfrac{(f(a)+f(b))(b-a)}{2}$ is area of trapezium $ABCD$, which is always more than or equal to $\int_a^b f(x)dx$.

Passage XI

(Ex. Nos. 73 to 74)

Consider a twice differentiable function $f(x)$ of degree four symmetrical to line $x = 1$, defined as $f: R \to R$ and $f''(2) = 0$, then

● ***Ex. 73*** *The sum of roots of the cubic, $f'(x) = 0$, is*

(a) 0 (b) 1 (c) 2 (d) 9/5

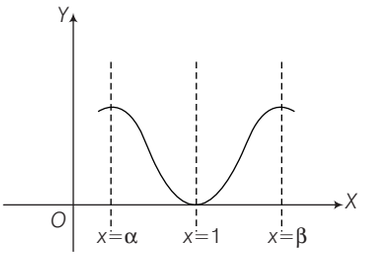
● ***Ex. 74*** *If $f(1) = 0$, $f(2) = 1$, then the value of $f(3)$, is*

(a) 6/7 (b) 7/5 (c) 8/5 (d) 9/5

■ ***Sol.*** (Ex. Nos. 73 to 74)

Since, $f(x)$ is symmetric about $x = 1$ and it is twice differentiable.

So, $f'(x)$ must have one root $x = 1$



$\therefore \qquad f'(x) = a(x-1)(x-\alpha)(x-\beta)$

$= a(x-1)[x^2 - (\alpha+\beta)x + \alpha\beta]$

$= a(x-1)[x^2 - 2x + \alpha\beta]$, as $\dfrac{\alpha+\beta}{2} = 1$

$\therefore \qquad f'(x) = a(x^3 - 2x^2 + \alpha\beta x - x^2 + 2x - \alpha\beta)$

$f''(x) = a(3x^2 - 6x + \alpha\beta + 2)$

$f''(2) = a(12 - 12 + \alpha\beta + 2) = a(\alpha\beta + 2) = 0$

$\therefore \qquad \alpha\beta = -2$ and $\alpha + \beta = 2$

$\Rightarrow \qquad \alpha - \dfrac{2}{\alpha} = 2 \;\Rightarrow\; \alpha^2 - 2\alpha - 2 = 0$

$\therefore \qquad \alpha = 1 - \sqrt{3}, \beta = 1 + \sqrt{3}$

So, sum $= 1 + \alpha + \beta = 3$

$f'(x) = a(x-1)(x^2 - 2x - 2) = a(x-1)[(x-1)^2 - 3]$

On integrating, we get

$f(x) = a\left[\dfrac{(x-1)^4}{4} - \dfrac{3(x-1)^2}{2}\right] + C$

$\{\because f(1) = 0$, so $C = 0$ and $f(2) = 1$, so $a = -4/5\}$

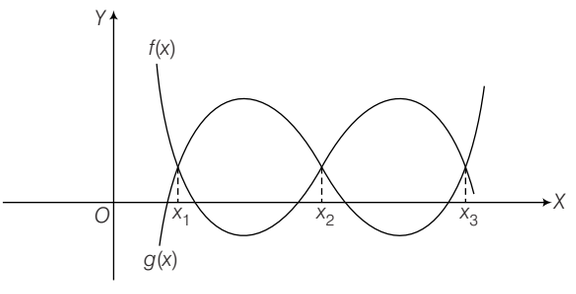
$\Rightarrow \qquad f(x) = -4/5\left[\dfrac{(x-1)^4}{4} - \dfrac{3(x-1)^2}{2}\right]$

$f(3) = -\dfrac{4}{5}\left[\dfrac{16}{4} - \dfrac{3}{2}(2)^2\right] = 8/5$

Passage XII

(Ex. Nos. 75 to 76)

Let $y = f(x)$ and $y = g(x)$ be the two functions, then the number of solutions of these two functions means the number of values of x for which these two function gives same of y as it is shown below.



● ***Ex. 75*** *Let* $\max\{|x+y|, |x-y|\} = 1$ *and* $y = x - [x]$ *be two equations which x and y satisfy, then the number of ordered pairs* (x, y) *is*

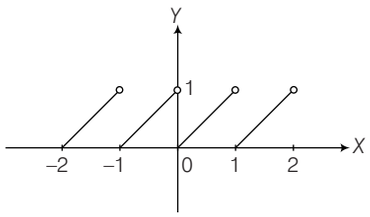
(a) 6 (b) 8 (c) 0 (d) ∞

● ***Ex. 76*** *If* $f(x) = \left\{\frac{1}{x}\right\}$ *and* $g(x) = \{x^2\}$, *then the number of positive roots satisfying the equations* $f(x) = g(x)$ *such that* $2 < x^2 < 3$

(a) 1 (b) 0 (c) 3 (d) 2

■ ***Sol.*** (Ex. Nos. 75 to 76)

75. Here, $y = x - [x] = \{x\}$ and we know graph of fractional part of x.



Also, we have $\max\{|x+y|, |x-y|\} = 1$

If $|x+y| \geq |x-y|$, then $|x+y| = 1$

On squaring above inequality, we get

$$x^2 + y^2 + 2xy \geq x^2 + y^2 - 2xy$$

$\Rightarrow 4xy \geq 0$, then $|x+y| = 1$, i.e. if function lies in I or III quadrant,

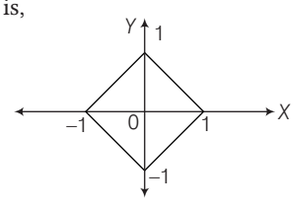
$$|x+y| = 1 \Rightarrow x + y = \pm 1 \quad \text{...(i)}$$

and of $|x+y| \leq |x-y|$, then $|x-y| = 1$

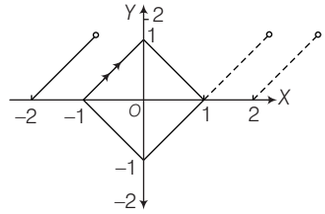
On squaring both sides $4xy \leq 0$ means x and y lies in II or IV quadrant. So, if curves lies in II or IV quadrant, then

$$|x-y| = 1 \Rightarrow x - y = \pm 1 \quad \text{...(ii)}$$

So, graph is,



Combining both graph,
when $-1 \leq x < 0$, then both graph coincide.



So, number of solutions is infinite.

76. If $2 < x^2 < 3 \Rightarrow \sqrt{2} < x < \sqrt{3}$ [as $x > 0$]

So, if $\sqrt{2} < x < \sqrt{3}$, then $\frac{1}{\sqrt{3}} < \frac{1}{x} < \frac{1}{\sqrt{2}}$

$$\Rightarrow \quad \left\{\frac{1}{x}\right\} = \frac{1}{x} \quad \text{...(i)}$$

and $$\{x^2\} = x^2 - 2 \quad \text{...(ii)}$$

So, from Eqs. (i) and (ii), we get

$$\left\{\frac{1}{x}\right\} = \{x^2\}$$

$$\therefore \quad \frac{1}{x} = x^2 - 2 \Rightarrow x^3 - 2x - 1 = 0$$

$$\Rightarrow (x+1)(x^2 - x - 1) = 0$$

So, $$x = -1, \frac{1 \pm \sqrt{5}}{2}, \text{ but as } x > 0$$

$$\therefore \quad x = \frac{1+\sqrt{5}}{2} \qquad \left(\frac{1-\sqrt{5}}{2} \text{ and } -1 \text{ are neglected}\right)$$

JEE Type Solved Examples :
Matching Type Questions

● ***Ex. 77*** *Match the statements of Column I with values of Column II.*

	Column I		Column II
(A)	Number of the values of x lying in $\left(0, \frac{\pi}{2}\right)$ at which $f(x) = \ln(\sin x)$ is not monotonic, is	(p)	0
(B)	If the greatest interval of decrease of the function $f(x) = x^3 - 3x + 2$ is $[a, b]$, then $a + b$ equals	(q)	2
(C)	Let $f(x) = \frac{x^2+2}{[x]}$, $1 \leq x \leq 3$, where [.] greatest integer function, then least value of $f(x)$ is	(r)	-3
(D)	Set of all possible values of a such that $f(x) = e^{2x} - (a+1)e^x + 2x$ is monotonically increasing for all $x \in R$ is $(-\infty, a]$, then a equals	(s)	3

Sol. (A) → (p), (B) → (p), (C) → (s), (D) → (s)

(A) $f(x) = \ln(\sin x) \Rightarrow f'(x) = \dfrac{\cos x}{\sin x} > 0$

$\therefore$ Required number of values of x is 0.

(B) $f'(x) = 3x^2 - 3 \le 0$, if $-1 \le x \le 1$

$\therefore \quad a = -1, b = 1 \quad \therefore \quad a + b = 0$

(C) $f(x) = \begin{cases} x^2 + 2, & 1 \le x < 2 \\ \dfrac{x^2+2}{2}, & 2 \le x < 3 \\ \dfrac{x^2+2}{3}, & x = 3 \end{cases} \Rightarrow f'(x) = \begin{cases} 2x, & 1 < x < 2 \\ x, & 2 < x < 3 \end{cases}$

$\therefore$ Least value of $f(x)$ in $[1, 2]$ is 3.

Least value of $f(x)$ in $[2, 3)$ is 3.

$$f(3) = \frac{11}{3}$$

$\therefore$ Least value of $f(x)$ is 3.

(D) $f(x) = e^{2x} - (a+1)e^x + 2x$

$f'(x) = 2e^{2x} - (a+1)e^x + 2$

Now, $2e^{2x} - (a+1)e^x + 2 \ge 0$, for all $x \in R$

$\Rightarrow \quad 2\left(e^x + \dfrac{1}{e^x}\right) - (a+1) \ge 0$, for all $x \in R$

$\Rightarrow \quad 4 - (a+1) \ge 0 \Rightarrow a \le 3 \quad \therefore \quad a = 3$

● ***Ex. 78*** *Match the statements of Column I with values of Column II.*

	Column I	Column II	
(A)	The dimensions of the rectangle of perimeter 36 cm, which sweeps out the largest volume when revolved about one of its sides, are	(p)	6
(B)	Let $A(-1, 2)$ and $B(2, 3)$ be two fixed points, A point P lying on $y = x$ such that perimeter of ΔPAB is minimum, then sum of the abscissa and ordinate of point P, is	(q)	12
(C)	If x_1 and x_2 are abscissae of two points on the curve $f(x) = x - x^2$ in the interval $[0, 1]$, then maximum value of expression $(x_1 + x_2) - (x_1^2 + x_2^2)$ is	(r)	4
(D)	The number of non-zero integral values of a for which the function $f(x) = x^4 + ax^3 + \dfrac{3x^2}{2} + 1$ is concave upward along the entire real line is	(s)	$\dfrac{1}{2}$

Sol. (A) → (p, q), (B) → (r), (C) → (s), (D) → (r)

(A) Perimeter of the rectangle is 36 cm.

If one side is x, then the other side $= 18 - x$

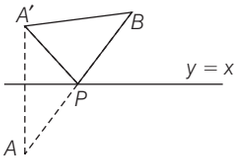
If the rectangle is revolved around the side x, then volume swept out $V = \pi x(18 - x)^2$

$$\frac{dV}{dx} = \pi[(18-x)^2 - 2x(18-x)]$$

$$= \pi(18-x)(18-x-2x)$$

$\therefore \quad x = 6$ and $y = 12$

(B) $A(-1, 2)$, $B(2, 3)$ and P in a point on $y = x$. Perimeter of ΔPAB is minimum when $PA + PB$ is minimum image of $A(-1, 2)$ in the line $y = x$ is $A'(2, -1)$. Equation of $A'B$ is $x = 2$, hence P is $(2, 2)$.



(C) Let (x_1, y_1) and (x_2, y_2) are two points.

$\therefore \quad y_1 + y_2 = (x_1 + x_2) - (x_1^2 + x_2^2)$

Now, $\quad y = x - x^2 = -\left(x - \dfrac{1}{2}\right)^2 + \dfrac{1}{4}$

$\therefore \quad (y_1 + y_2)_{\max} = 2 \times \dfrac{1}{4} = \dfrac{1}{2}$

(D) $f''(x) = 12x^2 + 6ax + 3 \ge 0, \forall\ x \in R \Rightarrow a \in [-2, 2]$

$\Rightarrow$ Number of non-zero integer values of a is 4.

JEE Type Solved Examples :
Single Integer Answer Type Questions

● ***Ex. 79*** *The function* $S(x) = \int_0^x \sin\left(\dfrac{\pi t^2}{2}\right) dt$ *has two critical points in the interval* $[1, 2.4]$*. One of the critical points is a local minimum and the other is a local maximum. The local minimum occurs at x equals ..... .*

Sol. Given, $S(x) = \int_0^x \sin\left(\dfrac{\pi t^2}{2}\right) dt$, $S'(x) = \sin\left(\dfrac{\pi x^2}{2}\right) = 0$

$\Rightarrow \quad \dfrac{\pi x^2}{2} = n\pi \quad \Rightarrow \quad x^2 = 2n \quad [1 \le x^2 \le 5.76$ as is given$]$

Hence, $\quad n = 1$ or 2

$$x = \sqrt{2} \text{ or } x = 2, S''(x) = \cos\left(\frac{\pi x^2}{2}\right) \cdot \pi x$$

$S''(\sqrt{2}) < 0$ and $S''(2) > 0$

$\Rightarrow \quad$ minimum at $x = 2$

● ***Ex. 80*** *The radius of a right circular cylinder increases at a constant rate. Its altitude is a linear function of the radius and increases three times as fast as radius. When the radius is* 1 *cm the altitude is* 6 *cm. When the radius is* 6 *cm, the volume is increasing at the rate of* 1 *cu cm/s. When the radius is* 36 *cm, the volume is increasing at a rate of n cu cm/s. The value of n/*11 *is equal to ..... .*

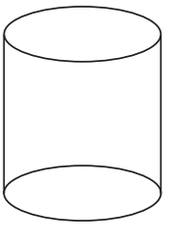
Sol. Here, $\frac{dr}{dt} = c$ and $h = ar + b$

Also, $\frac{dh}{dt} = 3\frac{dr}{dt}$ [given]

$\therefore \quad a\frac{dr}{dt} = 3\frac{dr}{dt} \Rightarrow a = 3$

Hence, $h = 3r + b$

when $r = 1, h = 6 \Rightarrow 6 = 3 + b \Rightarrow b = 3$



$\therefore \quad h = 3(r + 1)$

$V = \pi r^2 h = 3\pi r^2 (r + 1) = 3\pi (r^3 + r^2)$

$\frac{dV}{dt} = 3\pi (3r^2 + 2r)\frac{dr}{dt}$

where $r = 6, \frac{dV}{dt} = 1$ cc/s

$\therefore \quad 1 = 3\pi (108 + 12)\frac{dr}{dt} \Rightarrow 360\,\pi\frac{dr}{dt} = 1$

Again when $r = 23, \frac{dV}{dt} = n$

$n = 3\pi ((3.36)^2 + 2.36)\frac{dr}{dt}$

$n = 3\pi \cdot 36(110)\cdot\frac{1}{360\pi} \Rightarrow n = 33$

$\Rightarrow \quad \frac{n}{11} = \frac{33}{11} = 3$

● ***Ex. 81*** *The set of all points where* $f(x)$ *is increasing is* $(a, b) \cup (c, \infty)$*, then find* $[a + b + c]$ *{where* $[.]$ *denotes GIF}. Given that*

$$f(x) = 2f\left(\frac{x^2}{2}\right) + f(6 - x^2), \forall\, x \in R$$

and $f''(x) > 0, \forall\, x \in R.$

Sol. Here, $f(x) = 2f\left(\frac{x^2}{2}\right) + f(6 - x^2)$

$f'(x) = 2f'\left(\frac{x^2}{2}\right)\cdot x - 2x\,f'(6 - x^2)$

$f'(x) = 2x\left(f'\left(\frac{x^2}{2}\right) - f'(6 - x^2)\right)$

$f'\left(\frac{x^2}{2}\right) > f'(6 - x)^2$, if $\frac{x^2}{2} > 6 - x^2$

[$\because f'(x)$ is increasing]

$\frac{x^2}{2} > 6 - x^2 \Rightarrow x^2 > 4$

$\Rightarrow \quad f'\left(\frac{x^2}{2}\right) - f'(6 - x^2) > 0$

when $x < -2$ or $x > 2 \Rightarrow f'(x) > 0$

when $x \ (-2, 0) \cup (2, \infty)$

$\therefore \quad a + b + c = 0$

$\Rightarrow \quad [a + b + c] = 0$

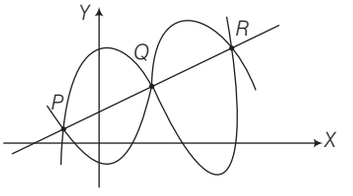
● ***Ex. 82*** *The graphs* $y = 2x^3 - 4x + 2$ *and* $y = x^3 + 2x - 1$ *intersect at exactly* 3 *distinct points. The slope of the line passing through two of these points is equal* to..... .

Sol. Let (x_1, y_1) and (x_2, y_2) be two of these points. Given, $y = x^3 + 2x - 1$ and $y = 2x^3 - 4x + 2$

$\therefore \quad y_1 = 2x_1^3 - 4x_1 + 2$ ...(i)

and $2y_1 = 2x_1^3 + 4x_1 - 2$ ...(ii)

Subtracting Eq. (i) from Eq. (ii), we get



$y_1 = 8x_1 - 4$ ...(iii)

Similarly, $y_2 = 8x_2 - 4$ ...(iv)

$y_2 - y_1 = 8(x_2 - x_1)$

$\frac{y_2 - y_1}{x_2 - x_1} = 8$

● ***Ex. 83*** *The length of the shortest path that begins at the point* (2, 5)*, touches the X-axis and then ends at a point on the circle* $x^2 + y^2 + 12x - 20y + 120 = 0$*, is..... .*

Sol. Circles with centre $(-6, 10)$ and radius

$= \sqrt{36 + 100 - 120} = 4$

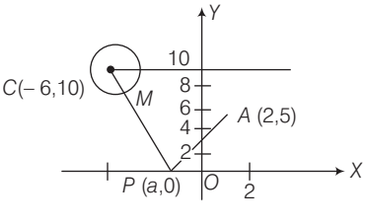
Now, let $(a, 0)$ be a point on the X-axis. If y is the distance from A to P and P to M

$y = \sqrt{(a - 2)^2 + 25} + \sqrt{(a + 6)^2 + 100} - 4$

$\frac{dy}{dx} = \frac{2(a - 2)}{2\sqrt{(a - 2)^2 + 25}} + \frac{2(a + 6)}{2\sqrt{(a + 6)^2 + 100}}$

$\frac{dy}{da}$ can be zero only if $a - 2 > 0$ and $a + 6 < 0$ not possible or $a - 2 < 0$ and $a + 6 > 0$, hence $a \in (-6, 2)$

Solving $\frac{dy}{da}=0$, gives $a=10$ (rejected) or $a=-\frac{2}{3}$



Hence, $y_{\min}=\sqrt{\frac{64}{9}+25}+\sqrt{\frac{256}{9}+100}-4$

$=\frac{17}{3}+\frac{\sqrt{1156}}{3}-4=\frac{17}{3}+\frac{34}{3}-4=17-4=13$

● ***Ex. 84*** *If $f:[1,\infty)\to R: f(x)$ is monotonic and differentiable function and $f(1)=1$, then number of solutions of the equation $f(f(x))=\frac{1}{x^2-2x+2}$ is/are ....*

Sol. Let $g(x)=f(f(x))$, given $f(x)$ is monotonic and differentiable

$\therefore \quad g'(x)=f'(f(x))\cdot f'(x)$, since $f(x)$ is monotonic

$\Rightarrow \quad f'(x)\cdot f'(f(x))>0$ for all $x\geq 1$

Here, $g(1)=f(f(1))$

$\therefore \quad g(1)=f(1)=1$ [given, $f(1)=1$] ...(i)

Since, $g(x)$ is increasing for $x\geq 1$.

$\therefore \quad g(x)\geq g(1)\Rightarrow \frac{1}{x^2-2x+2}\geq 1$

$\Rightarrow \quad \frac{1}{(x-1)^2+1}\geq 1$, i.e. only possible when $x=1$.

$\therefore$ Number of solutions of $f(f(x))$ is 1.

● ***Ex. 85*** *The set of values of 'a' for which the equation $x^4+4x^3+ax^2+4x+1=0$ has all its roots real is given by $(a_1,a_2]\cup\{a_3\}$. Then, $|a_3+a_2|$ is*

Sol. Here, $x^4+4x^3+ax^2+4x+1=0$

dividing by x^2 $\quad \left(x^2+\frac{1}{x^2}\right)+4\left(x+\frac{1}{x}\right)+a=0$

$\Rightarrow \quad \left(x+\frac{1}{x}\right)^2+4\left(x+\frac{1}{x}\right)+(a-2)=0$

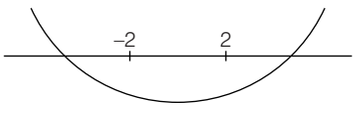
Put $x+\frac{1}{x}=y$, $(y\geq 2$ or $y\leq -2)$

$\therefore \quad y^2+4y+(a-2)=0$

For y to be real $D\geq 0$.

$16-4(a-2)\geq 0 \Rightarrow a\leq 6$ ...(i)

Also, if $f(y)=y^2+4y+(a-2) \Rightarrow f(-2)\cdot f(2)\geq 0$



$\Rightarrow \quad (4+8+a-2)(4-8+a-2)\geq 0$

$\Rightarrow \quad (a+10)(a-6)\geq 0$ ...(ii)

From Eqs. (i) and (ii), we get $a\in(-\infty,-10]\cup\{6\}$

when $a=6$, $y^2+4y+4=0 \Rightarrow y=-2$

$\therefore \quad a\in(-\infty,-10]\cup\{6\}$

$a_2=-10, a_3=6 \Rightarrow |a_2+a_3|=4$

● ***Ex. 86*** *Let $f(x)$ be a cubic polynomial defined by $f(x)=\frac{x^3}{3}+(a-3)x^2+x-13$. Then, the sum of all possible value (s) of 'a' for which $f(x)$ has negative point of local minimum in the interval $[1,5]$, is*

Sol. We have, $f(x)=\frac{x^3}{3}+(a-3)x^2+x-13$

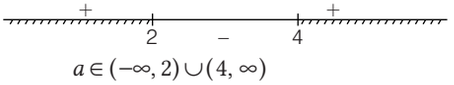
$\therefore$ For $f(x)$ have negative point of local minimum, the equation $f'(x)=0$ must have two distinct negative roots.

Now, $f'(x)=x^2+2(a-3)x+1$

$\therefore$ Following condition(s) must be satisfied simultaneously

(i) $D>0$ (ii) $\alpha+\beta<0$ (iii) $\alpha\beta>0$

Now, $D>0 \Rightarrow 4(a-3)^2-4>0 \Rightarrow (a-2)(a-4)>0$



$\therefore \quad a\in(-\infty,2)\cup(4,\infty)$ ...(i)

Also, $-2(a-3)<0 \Rightarrow a-3>0 \Rightarrow a>3$ ...(ii)

and product of root(s) $=1>0, \forall\, a\in R$

$\therefore$ (i) $\cap$ (ii) $\cap$ (iii) $\Rightarrow a\in(4,\infty)$ ...(iii)

$\therefore \quad a=5$, as $a\in[1,5]$

Thus, sum of all possible values is 5

● ***Ex. 87*** *Consider a polynomial $P(x)$ of the least degree that has a maximum equal to 6 at $x=1$ and a minimum equal to 2 at $x=3$. Then, the value of $P(2)+P'(0)-7$, is*

Sol. The polynomial is an everywhere differentiable function. Therefore, the points of extremum can only be roots of derivative. Further more, the derivative of a polynomial is a polynomial.

The polynomial of the least degree with roots $x_1=1$ and $x_2=3$ has the form $a(x-1)(x-3)$.

Hence, $P'(x)=a(x-1)(x-3)=a(x^2-4x+3)$

Since, at $x=1$, there must be $P(1)=6$, we have

$P(x)=\int_1^x P'(x)dx+6=a\int_1^x(x^2-4x+3)dx+6$

$=a\left(\frac{x^3}{3}-2x^2+3x-\frac{4}{3}\right)+6$

Also, $P(3)=2 \Rightarrow a=3$

Hence, $P(x)=x^3-6x^2+9x+2$

Now, $P(2)=8-24+18+2=4$, also

$P'(x)=3(x^2-4x+3) \Rightarrow P'(0)=9$

$\therefore \quad P(2)+P'(0)-7=6$

Subjective Type Questions

● **Ex. 88** *Let $g'(x) > 0$ and $f'(x) < 0, \forall\, x \in R$, then show*

(i) $g(f(x+1)) < g(f(x-1))$ (ii) $f(g(x+1)) < f(g(x-1))$

Sol. Here, $g'(x) > 0$ and $f'(x) < 0, \forall\, x \in R$

i.e. $g(x)$ is increasing [or if $x_1 > x_2 \Rightarrow g(x_1) > g(x_2)$]

and $f(x)$ is decreasing

[or if $x_1 > x_2 \Rightarrow f(x_1) < f(x_2), \forall\, x \in R$]

$\therefore$ $f(x+1) < f(x-1)$ ...(i)

and $g(x+1) > g(x-1)$

[as $(x+1) > (x-1)$] ...(ii)

***Case* I** As, $g(x)$ is increasing (so greater input gives greater output)

$\Rightarrow$ $g(f(x-1)) > g(f(x+1))$ [using Eq. (i)]

or $g(f(x+1)) < g(f(x-1))$

***Case* II** $f(x)$ is decreasing (so greater input gives smaller output)

$\Rightarrow$ $f(g(x+1)) < f(g(x-1))$ [using Eq. (ii)]

● **Ex. 89** *Let $f'(\sin x) < 0$ and $f''(\sin x) > 0, \forall\, x \in \left(0, \frac{\pi}{2}\right)$ and $g(x) = f(\sin x) + f(\cos x)$, then find the interval in which $g(x)$ is increasing and decreasing.*

Sol. Here, $f'(\sin x) < 0$ and $f''(\sin x) > 0, \forall\, x \in \left(0, \frac{\pi}{2}\right)$ ...(i)

and $g(x) = f(\sin x) + f(\cos x)$

$\Rightarrow$ $g'(x) = f'(\sin x) \cdot \cos x + f'(\cos x)(-\sin x)$

$\Rightarrow$ $g''(x) = \{-f'(\sin x) \cdot \sin x + f''(\sin x) \cdot \cos^2 x\}$

$- \{f'(\cos x) \cdot \cos x - f''(\cos x) \cdot \sin^2 x\}$ ...(ii)

As, $f'(\sin x) < 0$, $f''(\sin x) > 0$,

$\sin x > 0, \cos x > 0, \forall\, x \in \left(0, \frac{\pi}{2}\right)$ ...(iii)

$\therefore$ From Eqs. (ii) and (iii), we can say

$$g''(x) = \underbrace{\{-f'(\sin x) \cdot \cos x\}}_{+\text{ve}} + \underbrace{\{f''(\sin x) \cdot \cos^2 x\}}_{+\text{ve}} + \underbrace{\{f''(\cos x) \cdot \sin^2 x\}}_{+\text{ve}} + \underbrace{\{-f'(\cos x) \cdot \cos x\}}_{+\text{ve}}$$

$\Rightarrow$ $g''(x) > 0, \forall\, x \in \left(0, \frac{\pi}{2}\right)$ ...(iv)

$\Rightarrow$ $g'(x)$ is increasing in $\left(0, \frac{\pi}{2}\right)$ ...(v)

Now, putting $g'(x) = 0$,

$g'(x) = f'(\sin x) \cdot \cos x - f'(\cos x) \cdot \sin x = 0$

$\Rightarrow$ $x = \frac{\pi}{4}$

and $g'(x) > 0$, when $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

$g'(x) < 0$, when $x \in \left(0, \frac{\pi}{4}\right)$

$\therefore$ $g(x)$ is increasing, when $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

$g(x)$ is decreasing, when $x \in \left(0, \frac{\pi}{4}\right)$.

● **Ex. 90** *If $f(x) = \frac{x}{\sin x}$ and $g(x) = \frac{x}{\tan x}$, where $0 < x \leq 1$, then in this interval find that $f(x)$ and $g(x)$ are increasing or decreasing.*

Sol. Here, $f(x) = \frac{x}{\sin x}$

$\Rightarrow$ $f'(x) = \frac{\sin x \cdot 1 - x \cdot \cos x}{\sin^2 x}$ ...(i)

where $\sin^2 x$ is always positive when $0 < x \leq 1$.

But to check numerator we again let, $h(x) = \sin x - x \cos x$

$\Rightarrow$ $h'(x) = \cos x - 1 \cdot \cos x + x \sin x = x \sin x$, which is +ve for $0 < x \leq 1$

$\therefore$ $h'(x) > 0$

$\Rightarrow h(x)$ is increasing, when $0 < x \leq 1$

$\Rightarrow$ $h(0) < h(x)$

$\Rightarrow$ $0 < \sin x - x \cos x$

$\therefore$ In Eq. (i), $(\sin x - x \cos x) > 0$,

$\Rightarrow$ $f'(x) > 0$, when $0 < x \leq 1$

Hence, $f(x)$ is increasing, when $0 < x \leq 1$

Again, $g(x) = \frac{x}{\tan x}$ [given]

$\Rightarrow$ $g'(x) = \frac{\tan x \cdot 1 - x \cdot \sec^2 x}{\tan^2 x}$ ...(ii)

where $\tan^2 x > 0$

we let $\phi(x) = \tan x - x \sec^2 x$

$\Rightarrow$ $\phi'(x) = \sec^2 x - \sec^2 x - x\,(2 \sec x) \cdot (\sec x \tan x)$

$\phi'(x) = -2x \sec^2 x \tan x$

As $\phi'(x) < 0, \forall 0 < x \leq 1$

[$\because$ $x \in (0,1], \sec^2 x > 0, \tan x > 0$]

$\therefore$ $\phi(x)$ is decreasing, when $0 < x \leq 1$.

$\Rightarrow$ $\phi(0) > \phi(x)$ or $0 > \tan x - x \sec^2 x$

$\therefore$ In Eq. (ii), $(\tan x - x \sec^2 x) < 0$

$\Rightarrow$ $g'(x) < 0$, when $0 < x \leq 1$

$\therefore$ $g(x)$ is decreasing, when $0 < x \leq 1$.

● ***Ex. 91*** *Let $f:[0,\infty)\to[0,\infty)$ and $g:[0,\infty)\to[0,\infty)$ be non-increasing and non-decreasing function and $h(x)=g(f(x))$ and if f and g are differentiable for all points in their respective domains and $h(0)=0$. Then, show $h(x)$ is always identically zero.*

Sol. Here, $h(x)=g(f(x))$, since, $g(x)\in[0,\infty)$

$h(x)\geq 0, \forall\ x\in$ domain

Also, $h'(x)=g'(f(x))\cdot f'(x)\leq 0$, as $g'(x)\geq 0$

and $h(x)\leq 0, \forall\ x\in$ domain as $h(0)=0$

Hence, $h(x)=0, \forall\ x\in$ domain

● ***Ex. 92*** *A cubic function $f(x)$ tends to zero at $x=-2$ and has relative maximum/ minimum at $x=-1$ and $x=\frac{1}{3}$. If $\int_{-1}^{1} f(x)\,dx=\frac{14}{3}$. Find the cubic function $f(x)$.* ***[IIT JEE 1992]***

Sol. $f(x)$ is a cubic polynomial. Therefore, $f'(x)$ is a quadratic polynomial and $f(x)$ has relative maximum and minimum at $x=\frac{1}{3}$ and $x=-1$, respectively.

$\therefore$ -1 and $\frac{1}{3}$ are roots of $f'(x)=0$

$$\Rightarrow \quad f'(x)=a(x+1)\left(x-\frac{1}{3}\right)=a\left(x^2+\frac{2}{3}x-\frac{1}{3}\right)$$

Now, integrating w.r.t. x, we get

$$f(x)=a\left[\frac{x^3}{3}+\frac{x^2}{3}-\frac{x}{3}\right]+c,\ c \text{ is constant of integration.}$$

Again, $f(-2)=0$ [given]

$$\therefore \quad f(-2)=a\left(\frac{-8}{3}+\frac{4}{3}+\frac{2}{3}\right)+c=0$$

$$\Rightarrow \quad \frac{-2a}{3}+c=0 \quad \text{or} \quad c=\frac{2a}{3}$$

$$\Rightarrow \quad f(x)=a\left(\frac{x^3}{3}+\frac{x^2}{3}-\frac{x}{3}\right)+\frac{2a}{3}$$

$$\Rightarrow \quad f(x)=a\left(\frac{x^3}{3}+\frac{x^2}{3}-\frac{x}{3}+\frac{2}{3}\right)$$

Again, $\int_{-1}^{1} f(x)\,dx=\frac{14}{3}$ [given]

$$\Rightarrow \quad \frac{a}{3}\int_{-1}^{1}(x^3+x^2-x+2)\,dx=\frac{14}{3}$$

Because, $y=x^3$ and $y=-x$ are odd functions.

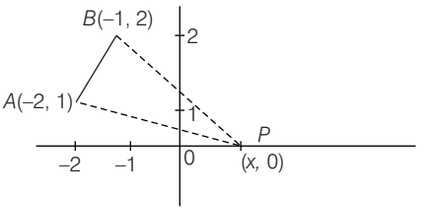
So, $\int_{-1}^{1} x^3dx=\int_{-1}^{1} x\,dx=0$

$$\therefore \quad \frac{a}{3}\left[\frac{2x^3}{3}+4x\right]_0^1=\frac{14}{3} \quad \Rightarrow \quad a=3$$

$$\therefore \quad f(x)=x^3+x^2-x+2$$

● ***Ex. 93*** *Given that, $S=|\sqrt{x^2+4x+5}-\sqrt{x^2+2x+5}|$ for all real x, then find the maximum value of S^4.*

Sol. Here, $S=\left|\sqrt{(x+2)^2+1}-\sqrt{(x+1)^2+4}\right|$



$$\therefore \quad S=\left|\sqrt{(x-(-2))^2+(1-0)^2}-\sqrt{(x-(-1))^2+(2-0)^2}\right|$$

$$=|PA-PB| \qquad \ldots(i)$$

Since, $|PA-PB|\leq AB$, using triangle law.

$$\therefore \quad |PA-PB|_{\max}=\sqrt{1+1}=\sqrt{2}$$

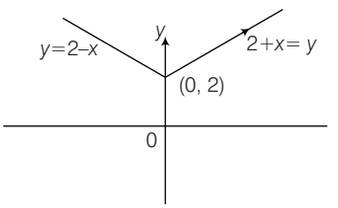
$$[\because AB=\sqrt{-1+2^2+(2-1)^2}=\sqrt{2}] \ldots(ii)$$

$$\therefore \quad S_{\max}=\sqrt{2}$$

$$\Rightarrow \quad S^4=4$$

● ***Ex. 94*** *If $f(x)=\max|2\sin y-x|$, (where $y\in R$), then find the minimum value of $f(x)$.*

Sol. Here, $f(x)=\max|2\sin y-x|=\begin{cases}2-x, & x\leq 0\\ 2+x, & x>0\end{cases}$



$\therefore$ Minimum value of $f(x)=2$

● ***Ex. 95*** *Find the maximum and minimum value of*

$$f(x)=\frac{40}{3x^4+8x^3-18x^2+60}.$$

Sol. Given, $f(x)=\frac{40}{3x^4+8x^3-18x^2+60}$ is maximum or minimum according by $3x^4+8x^3-18x^2+60$ is minimum or maximum.

Then, $f_{\max}$, if $3x^4+8x^3-18x^2+60$ is minimum.

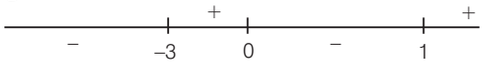
$f_{\min}$, if $3x^4+8x^3-18x^2+60$ is maximum.

Let $g(x)=3x^4+8x^3-18x^2+60$

$\Rightarrow \quad g'(x) = 12x(x^2 + 2x - 3)$

$\Rightarrow \quad g'(x) = 12x(x+3)(x-1)$

Using number line rule,



which indicates $g'(x)$ changes sign from −ve to +ve at $x = -3, 1$.

$\therefore$ Local minimum at $x = -3, 1$

and local maximum at $x = 0$ [as changes from +ve to −ve]

$\Rightarrow \quad g(x)$ is maximum at $x = 0$

i.e. $g_{\max}(0) = 60$

and for $g(x)$ to be minimum,

$g_{\min}(-3) = 243 - 216 - 162 + 60 = -75$

$g_{\min}(1) = 3 + 8 - 18 + 60 = 53$

Substituting these values in Eq. (i), we get

$f_{\max}(x)$ when $g_{\min}$, i.e. $f(x) = \dfrac{40}{-75}$ and $\dfrac{40}{53}$

Maximum value $= \dfrac{40}{53}, \dfrac{-8}{15}$

$f_{\min}(x)$ when $g_{\max}$ i.e. $f(x) = \dfrac{40}{60} = \dfrac{2}{3}$

Minimum value $= \dfrac{2}{3}$

● ***Ex. 96*** *Use the function $f(x) = x^{1/x}$, $x > 0$ and determine the bigger of the two numbers e^{π} and π^e.*

Sol. Given, $f(x) = x^{1/x}$, $x > 0$

Let $y = f(x) = x^{1/x}$

Taking log on both sides, we have $\log y = \dfrac{1}{x}\log x$

Differentiating both the sides, we have

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{x^2} + \log x\left(-\frac{1}{x^2}\right)$$

or $$\frac{dy}{dx} = y\left[\frac{1 - \log x}{x^2}\right]$$

$\therefore \quad f'(x) = \dfrac{x^{1/x}}{x^2}[1 - \log x]$

Let $f'(x) = 0$,

$\Rightarrow \quad \log x = 1 \quad$ or $\quad x = e$

Again, $f''(x) = \dfrac{x^{1/x}}{x^2}\left[0 - \dfrac{1}{x}\right] + (1 - \log x)\dfrac{d}{dx}\left(\dfrac{x^{1/x}}{x^2}\right)$

$\therefore \quad f''(e) = \dfrac{e^{1/e}}{e^2}\left(-\dfrac{1}{e}\right) + 0$

$\Rightarrow \quad f''(e) < 0$

$\therefore$ 'f' has a maximum at $x = e$

But $x = e$ is the only extreme value.

$\therefore \quad f$ has the greatest value at $x = e$

$\Rightarrow \quad f(e) > f(\pi), \quad$ for all $x > 0$

$\Rightarrow \quad (e)^{\frac{1}{e}} > (\pi)^{\frac{1}{\pi}}$

$\Rightarrow \quad e^{\pi} > \pi^e$

● ***Ex. 97*** *Using the relation $2(1 - \cos x) < x^2$, $x \neq 0$ or otherwise prove that $\sin(\tan x) \geq x$ for all $x \in \left[0, \dfrac{\pi}{4}\right]$.* ***[IIT JEE 2003]***

Sol. Let $f(x) = \sin(\tan x) - x$

Then, $f'(x) = \cos(\tan x)\cdot\sec^2 x - 1$

$\Rightarrow \quad f'(x) = \cos(\tan x)\{1 + \tan^2 x\} - 1$

$\Rightarrow \quad f'(x) = \tan^2 x \cos(\tan x) - \{1 - \cos(\tan x)\}$

[using $2(1 - \cos x) < x^2$]

$$f'(x) > \tan^2 x \cos(\tan x) - \frac{1}{2}\tan^2 x$$

$\Rightarrow \quad f'(x) > \dfrac{1}{2}\tan^2 x\{2\cos(\tan x) - 1\}$

[again, using $2(1 - \cos x) < x^2$]

$$f'(x) > \frac{1}{2}\tan^2 x(1 - \tan^2 x)$$

$\Rightarrow \quad f'(x) \geq 0, \forall \quad x \in \left[0, \dfrac{\pi}{4}\right]$

$\Rightarrow \quad f(x)$ is increasing function for all $x \in \left[0, \dfrac{\pi}{4}\right]$.

$\therefore \quad f(x) \geq f(0),$ for all $x \in \left[0, \dfrac{\pi}{4}\right]$

$\Rightarrow \quad \sin(\tan x) - x > \sin(\tan 0) - 0$

$\Rightarrow \quad \sin(\tan x) \geq x,$ for all $x \in \left[0, \dfrac{\pi}{4}\right]$

● ***Ex. 98*** *If $P(1) = 0$ and $\dfrac{d}{dx}\{P(x)\} > P(x)$ for all $x \geq 1$, then prove that $P(x) > 0$ for all $x > 1$.* ***[IIT JEE 2003]***

Sol. Here, $\dfrac{d}{dx}\{P(x)\} > P(x)$, for all $x \geq 1$

or $\dfrac{d}{dx}\{P(x)\}e^{-x} > P(x)e^{-x}$, for all $x \geq 1$

$\Rightarrow \quad \dfrac{d}{dx}\{P(x)\}e^{-x} - P(x)e^{-x} > 0,$ for all $x \geq 1$

$\Rightarrow \quad \dfrac{d}{dx}\{e^{-x}P(x)\} > 0,$ for all $x \geq 1$

$\Rightarrow \quad P(x)e^{-x}$ is an increasing function for all $x \geq 1$

$\Rightarrow \quad P(x)\,e^{-x} > P(1)\,e^{-1}$, for all $x > 1$

$\Rightarrow \quad P(x)\,e^{-x} > 0$, for all $x > 1$ [as $P(1) = 0$, given]

Thus, $P(x) > 0$, for all $x > 1$ [as $e^{-x} > 0$, for all x]

● ***Ex. 99*** *In the graph of the function* $y = \frac{3}{\sqrt{2}}\,x \log x$, *where* $x \in [e^{-1.5}, \infty[$; *find the point* $P(x, y)$ *such that the segment of the tangent to the graph of the function at the point, intercepted between the point P and Y-axis, is shortest.*

Sol. Given, $y = \frac{3}{\sqrt{2}}\,x \log x$

Differentiating w.r.t. x, the given function, we have

$$\frac{dy}{dx} = \frac{3}{\sqrt{2}}\left(x \cdot \frac{1}{x} + \log_e x\right) = \frac{3}{\sqrt{2}}(1 + \log_e x)$$

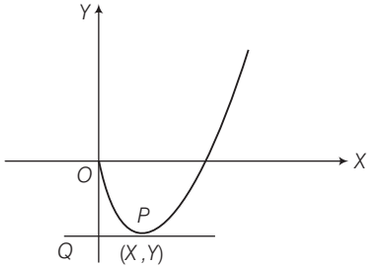
Let the point $P(x, y) \Rightarrow P\left(x, \frac{3}{\sqrt{2}}\,x \log_e x\right)$

Hence, the equation of tangent to the curve at the point $P\left(x, \frac{3}{\sqrt{2}}\,x \log_e x\right)$ is

$$Y - \frac{3}{\sqrt{2}}\,x \log_e x = \frac{3}{\sqrt{2}}(1 + \log_e x)(X - x)$$

When it cuts Y-axis, $X = 0$

Thus, $y = \frac{3}{\sqrt{2}}\,x \log_e x - \frac{3}{\sqrt{2}}(1 + \log_e x)\,x$



So, $y = -\frac{3}{\sqrt{2}}\,x$

Hence, the given tangent intersects axis at

$$Q\left(0, -\frac{3}{\sqrt{2}}\,x\right)$$

Now, $PQ^2 = x^2 + \left(y + \frac{3}{\sqrt{2}}\,x\right)^2$

$$= x^2 + \left(\frac{3}{\sqrt{2}}\,x\,(1 + \log_e x)\right)^2$$

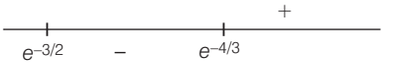
$$PQ^2 = x^2\left[1 + \frac{9}{2}(1 + \log_e x)^2\right]$$

Differentiating w.r.t. x, we have

$$\frac{d(PQ^2)}{dx} = x^2\left[\frac{9}{2} \cdot 2(1 + \log_e x) \cdot \frac{1}{x}\right] + \left[1 + \frac{9}{2}(1 + \log_e x)^2\right]2x$$

$$= x\,[9(1 + \log_e x) + 2 + 9(1 + \log_e x)^2]$$

For extremum $\frac{d(PQ^2)}{dx} = 0$



Since, $x \neq 0$, $9(1 + \log x)^2 + 9(1 + \log x) + 2 = 0$

$\Rightarrow \quad (1 + \log x) = -1/3, -2/3$

$\Rightarrow \quad \log x = -4/3, -5/3$

$\Rightarrow \quad x = e^{-4/3}, e^{-5/3}$

$x = e^{-5/3}$ lie outside the interval $(e^{-1.5}, \infty)$.

The sign scheme for $\frac{d(PQ^2)}{dx}$ is shown in figure, which shows that PQ^2 is minimum.

Therefore, PQ is minimum when $x = e^{-4/3}$.

● ***Ex. 100*** *John has x children from his first wife. Mary has* $(x + 1)$ *children from her first husband. They marry and have children of their own. The whole family has* 24 *children. Assuming that the two children of same parents do not fight, then find the maximum possible number of fights that can take place.*

Sol. Since, the whole family has 24 children, those of John and Mary are

$$24 - x - (x + 1)$$

i.e. $(23 - 2x)$

Now, F = Total number of fights.

= (number of fights when a John's child fights a Mary's child) + (number of fights when a John child fights a John-Mary's child) + (number of fights when a Mary's child fights a John-Mary's child)

$= x(x + 1) + x(23 - 2x) + (x + 1)(23 - 2x)$

$= 23 + 45x - 3x^2$

For maximum, $\frac{dF}{dx} = 0$

$\Rightarrow \quad 45 - 6x = 0$ or $x = 7.5$

$\Rightarrow \quad \frac{d^2F}{dx^2} = 45 > 0$

$\therefore f(x)$ is minimum when $x = 7.5$

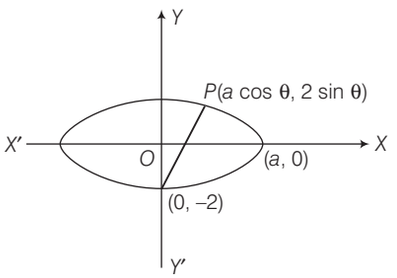
But in this case fractional value is not possible. The nearest integral values are $x = 7$ and $x = 8$.

In either case the total number of fights $= 23 + 45 \times 7 - 3(7)^2$
$= 191$

● **Ex. 101** *Find the point on the curve* $4x^2 + a^2y^2 = 4a^2$; $4 < a^2 < 8$ *that is farthest from the point* $(0, -2)$.

Sol. The equation of given curve can be expressed as shown in the figure

$\frac{x^2}{a^2} + \frac{y^2}{4} = 1$, where $4 < a^2 < 8$ which is equation of ellipse.



Hence, let us consider a point $P(a\cos\theta, 2\sin\theta)$ on the ellipse.

Let the distance of $P(a\cos\theta, 2\sin\theta)$ from $(0, -2)$ be L.

Then, $L^2 = (a\cos\theta - 0)^2 + (2\sin\theta + 2)^2$

Differentiating w.r.t. θ, we have

$$\frac{d(L^2)}{d\theta} = a^2 \cdot 2\cos\theta(-\sin\theta) + 4 \cdot 2(\sin\theta + 1)\cdot\cos\theta$$

$$= \cos\theta\,[-2a^2\sin\theta + 8\sin\theta + 8] = 0$$

$\Rightarrow$ Either $\cos\theta = 0$ or $(8 - 2a^2)\sin\theta + 8 = 0$

i.e. $\theta = \pi/2$ or $\sin\theta = \frac{4}{a^2 - 4}$

Since, $a^2 < 8$

$\Rightarrow$ $a^2 - 4 < 4$

$\Rightarrow$ $\frac{4}{a^2 - 4} > 1 \Rightarrow \sin\theta > 1$, which is not possible.

Further, $\frac{d^2(L^2)}{d\theta^2} = \cos\theta\,[-2a^2\cos\theta + 8\cos\theta]$

$+ (-\sin\theta)[-2a^2\sin\theta + 8\sin\theta + 8]$

At $\theta = \pi/2$, $\frac{d^2(L^2)}{d\theta^2} = 0 - [16 - 2a^2] = 2(a^2 - 8) < 0$ [as $a^2 < 8$]

Hence, L is maximum at $\theta = \pi/2$ and the farthest point is $(0, 2)$.

● **Ex. 102** *Let* $f(x) = \sin^{-1}\left(\frac{2\phi(x)}{1 + \phi^2(x)}\right)$, *then find the interval in which* $f(x)$ *is increasing or decreasing.*

Sol. Here, $f(x) = \sin^{-1}\left(\frac{2\phi(x)}{1 + \phi^2(x)}\right)$

***Case* I** $|\phi(x)| < 1$

Let $\phi(x) = \tan\theta$

$\therefore$ $f(x) = \sin^{-1}\left(\frac{2\tan\theta}{1 + \tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta$

$\Rightarrow$ $f(x) = 2\tan^{-1}\{\phi(x)\}$

$\Rightarrow$ $f'(x) = \frac{2}{1 + \{\phi(x)^2\}} \cdot \phi'(x)$

where $\phi'(x) > 0 \Rightarrow f'(x) > 0 \Rightarrow f(x)$ is increasing
[since $\phi(x) = \tan\theta$, which is increasing]

***Case* II** When $|\phi(x)| > 1$ or $\left|\frac{1}{\phi(x)}\right| < 1$

Now, put $\frac{1}{\phi(x)} = \tan\theta$

$\therefore$ $f(x) = \sin^{-1}\left(\frac{2\cdot\frac{1}{\phi(x)}}{1 + \left(\frac{1}{\phi(x)}\right)^2}\right)$

$\Rightarrow$ $f(x) = \sin^{-1}\left(\frac{2\tan\theta}{1 + \tan^2\theta}\right)$

$f(x) = \sin^{-1}(\sin 2\theta) = 2\theta = 2\tan^{-1}\left(\frac{1}{\phi(x)}\right)$

$= 2\cot^{-1}(\phi(x))$

$\Rightarrow$ $f'(x) = -2 \cdot \frac{1}{1 + (\phi(x))^2} \cdot \phi'(x)$

where $\phi'(x) > 0$

$\Rightarrow$ $f'(x) < 0$ for all $|\phi(x)| > 1$

Hence, $f(x)$ is increasing, when $|\phi(x)| < 1$ and $f(x)$ is decreasing, when $|\phi(x)| > 1$.

● **Ex. 103** *Find the minimum value of*

$$f(x) = \big||x + 2| - 2|x - 2|\big| + |x|.$$

Sol. Here, $f(x) = \big||x + 2| - 2|x - 2|\big| + |x|$, which gives rise to four cases as

***Case* I** $x < -2$

$f(x) = |-(x + 2) + 2(x - 2)| - x$

$= |-x - 2 + 2x - 4| - x$

$= |x - 6| - x = -(x - 6) - x$

$\Rightarrow$ $f(x) = -2x + 6$ ...(i)

***Case* II** $-2 \le x < 0$

$f(x) = |(x + 2) + 2(x - 2)| - x$

$= |3x - 2| - x = -(3x - 2) - x$

$\Rightarrow$ $f(x) = -4x + 2$ ...(ii)

***Case* III** $0 \le x < 2$

$$f(x) = |x + 2 + 2(x-2)| + x$$
$$= |3x - 2| + x$$
$$= \begin{cases} -(3x-2) + x, \text{ for } 0 \le x < \frac{2}{3} \\ (3x-2) + x, \quad \text{for } \frac{2}{3} \le x < 2 \end{cases}$$

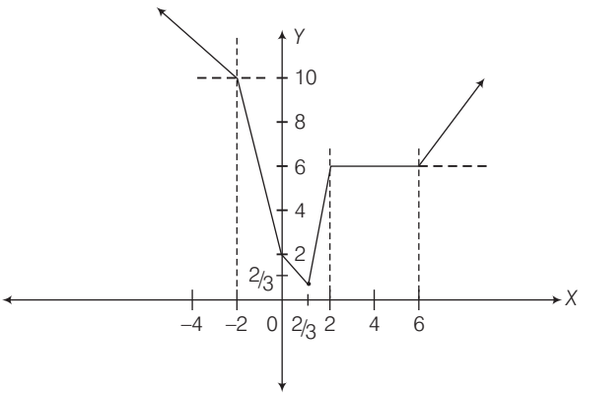
$$\Rightarrow \quad f(x) = \begin{cases} -2x + 2, 0 \le x < 2/3 \\ 4x - 2, 2/3 \le x < 2 \end{cases} \quad \text{...(iii)}$$

***Case* IV** $x \ge 2$

$$f(x) = |x + 2 - 2(x-2)| + x = |-x + 6| + x = |x - 6| + x$$
$$= \begin{cases} -(x-6) + x, 2 \le x < 6 \\ (x-6) + x, \ x \ge 6 \end{cases}$$

$$f(x) = \begin{cases} 6, & 2 \le x < 6 \\ 2x - 6, & x \ge 6 \end{cases} \quad \text{...(iv)}$$

From Eqs. (i), (ii), (iii) and (iv), we have the following figure.



From the graph, minimum value of $f(x) = \frac{2}{3}$.

● ***Ex. 104*** *Which normal to the curve $y = x^2$ forms the shortest chord?* ***[IIT JEE 1992]***

Sol. Let (t, t^2) be any point on the parabola $y = x^2$

Now, $\frac{dy}{dx} = 2x \Rightarrow \left(\frac{dy}{dx}\right)_{(t, t^2)} = 2t$, which is the slope of tangent.

So, the slope of the normal to $y = x^2$ at (t, t^2) is $\left(-\frac{1}{2t}\right)$.

$\therefore$ The equation of the normal to $y = x^2$ at (t, t^2) is

$$y - t^2 = \left(-\frac{1}{2t}\right)(x - t) \quad \text{...(i)}$$

Suppose Eq. (i) meets the curve again at $B(t_1, t_1^2)$, then

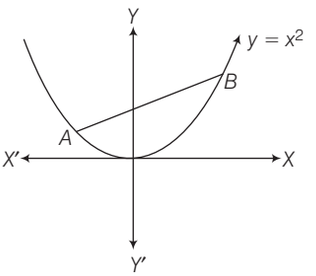
$$t_1^2 - t^2 = -\frac{1}{2t}(t_1 - t) \Rightarrow t_1 + t = -\frac{1}{2t}$$

$$\Rightarrow \quad t_1 = -t - \frac{1}{2t} \quad \text{...(ii)}$$

Let L be the length of the chord AB (as normal)

$$L = AB^2 = (t - t_1)^2 + (t^2 - t_1^2)^2$$
$$= (t - t_1)^2 [1 + (t + t_1)^2]$$
$$= \left(t + t + \frac{1}{2t}\right)^2 \left[1 + \left(t - t - \frac{1}{2t}\right)^2\right] \quad \text{[using Eq. (ii)]}$$
$$= \left(2t + \frac{1}{2t}\right)^2 \left(1 + \frac{1}{4t^2}\right)$$

$$L = 4t^2 \left(1 + \frac{1}{4t^2}\right)^3$$



$$\Rightarrow \quad \frac{dL}{dt} = 8t\left(1 + \frac{1}{4t^2}\right)^3 + 12t^2\left(1 + \frac{1}{4t^2}\right)^2 \cdot \left(-\frac{2}{4t^3}\right)$$

$$\Rightarrow \quad \frac{dL}{dt} = 2\left(1 + \frac{1}{4t^2}\right)^2 \left[4t\left(1 + \frac{1}{4t^2}\right) - \frac{3}{t}\right]$$

$$\Rightarrow \quad \frac{dL}{dt} = 2\left(1 + \frac{1}{4t^2}\right)^2 \left(4t - \frac{2}{t}\right) = 4\left(1 + \frac{1}{4t^2}\right)^2 \left(2t - \frac{1}{t}\right)$$

For extremum, let $\frac{dL}{dt} = 0$

$$\Rightarrow \quad t = \pm\frac{1}{\sqrt{2}}$$

Again,

$$\frac{d^2L}{dt^2} = 8\left(1 + \frac{1}{4t^2}\right)\left(-\frac{1}{2t^2}\right)\left(2t - \frac{1}{t}\right) + 4\left(1 + \frac{1}{4t^2}\right)^2 \left(2 + \frac{1}{t^2}\right)$$

$$\Rightarrow \quad \left(\frac{d^2L}{dt^2}\right)_{t = \pm\frac{1}{\sqrt{2}}} > 0$$

$\therefore$ Minimum when $t = \pm\frac{1}{\sqrt{2}}$

Thus, points are $A\left(\pm\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ and $B(\mp\sqrt{2}, 2)$.

$\Rightarrow$ Equation of normal AB is $\sqrt{2}x + 2y - 2 = 0$ and $\sqrt{2}x - 2y + 2 = 0$.

● ***Ex. 105*** *Let $f(x) = \sin^3 x + \lambda \sin^2 x$ where $-\pi/2 < x < \pi/2$. Find the intervals in which λ should lie in order that $f(x)$ has exactly one minimum.*

Sol. Given, $f(x) = \sin^3 x + \lambda \sin^2 x$

$\therefore \quad f'(x) = 3\sin^2 x (\cos x) + \lambda \cdot 2 \sin x (\cos x)$

$= \sin x \cos x (3 \sin x + 2\lambda)$

For extremum, let $f'(x) = 0$

$\therefore \quad \sin x = 0, \quad \cos x = 0,$

$\sin x = -2\lambda/3$

Since, $-\pi/2 < x < \pi/2$

$\therefore \quad \cos x \neq 0$

$\Rightarrow \quad \sin x = 0 \quad \Rightarrow \quad x = 0$

and $\sin x = \frac{-2\lambda}{3} \Rightarrow x = \sin^{-1}\left(\frac{-2\lambda}{3}\right)$ ...(i)

One of these from Eq. (i) will give maximum and one minimum, provided

$$-1 < \sin x = \frac{-2\lambda}{3} < 1$$

i.e. $$-1 < \frac{-2\lambda}{3} < 1$$

$\Rightarrow \quad -3 < -2\lambda < 3 \Rightarrow -3 < 2\lambda < 3$

i.e. $-3/2 < \lambda < 3/2$

But, if $\lambda = 0$, then $\sin x = 0$ has only one solution.

$\therefore \quad \lambda \in (-3/2, 3/2) - \{0\}$

$\Rightarrow \quad \lambda \in (-3/2, 0) \cup (0, 3/2)$

For this value of λ there are two distinct solutions.

Since, $f(x)$ is continuous, these solutions give one maximum and one minimum because for a continuous function, between two maxima there must lie one minima and *vice-versa*.

● ***Ex. 106*** *Determine the points of maxima and minima of the function, $f(x) = \frac{1}{8}\log x - bx + x^2, x > 0$ when $b \geq 0$ is a constant.* ***[IIT JEE 1996]***

Sol. Here, $f(x) = \frac{1}{8}\log x - bx + x^2$ is defined and continuous for all $x > 0$.

Then, $f'(x) = \frac{1}{8x} - b + 2x$

or $f'(x) = \frac{16x^2 - 8bx + 1}{8x}$

For extrema let $f'(x) = 0$

$\Rightarrow \quad 16x^2 - 8bx + 1 = 0$

So, $x = \frac{8b \pm \sqrt{64(b^2 - 1)}}{2 \times 16}$

or $x = \frac{b \pm \sqrt{b^2 - 1}}{4}$

Obviously the roots are real, if $b^2 - 1 \geq 0$

$\Rightarrow \quad b > 1 \quad [\text{as } b > 0]$

Hence, when $b > 1$, then using number line rule for $f'(x)$ as shown in given figure.

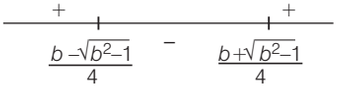
We know $f'(x)$ changes sign from +ve to −ve at

$$x = \frac{b - \sqrt{b^2 - 1}}{4}$$

$\therefore \quad f(x)_{\max}$ at $x = \frac{b - \sqrt{b^2 - 1}}{4}$

and $f'(x)$ changes sign from −ve to +ve at

$$x = \frac{b + \sqrt{b^2 - 1}}{4}$$



$\therefore \quad f(x)_{\min}$ at $x = \frac{b + \sqrt{b^2 - 1}}{4}$

Also, if $b = 1$ $f'(x) = \frac{16x^2 - 8x + 1}{8x} = \frac{(4x - 1)^2}{8x}$ no change in sign.

$\therefore$ Neither maximum nor minimum, if $b = 1$

Thus,

$$f(x) = \begin{cases} f(x)_{0_{\max}} \text{ when } x = \frac{b - \sqrt{b^2 - 1}}{4} \text{ and } b > 1 \\ f(x)_{\min} \text{ when } x = \frac{b + \sqrt{b^2 - 1}}{4} \text{ and } b > 1 \\ f(x) \text{ neither maximum nor minimum when } b = 1 \end{cases}$$

● ***Ex. 107*** *Find the points on the curve $ax^2 + 2bxy + ay^2 = c$, $0 < a < b < c$, whose distance from the origin is minimum.*

Sol. Let $P(x, y)$ be a point on the curve $ax^2 + 2bxy + ay^2 = c$, whose distance from the origin is r.

$\therefore \quad x = r \cos\theta \quad \text{and} \quad y = r \sin\theta$

As, $r\cos\theta$ and $r\sin\theta$ lies on $ax^2 + 2bxy + ay^2 = c$

$\Rightarrow \quad ar^2 \cos^2\theta + 2br^2 \sin\theta \cos\theta + ar^2 \sin^2\theta = c$

$\Rightarrow \quad (a + b\sin 2\theta) r^2 = c$

$\Rightarrow \quad r^2 = \frac{c}{a + b\sin 2\theta}$ ...(i)

From Eq. (i) r is minimum when $(a + b \sin 2\theta)$ is maximum, i.e. $\sin 2\theta$ is maximum.

i.e. $$2\theta = \frac{\pi}{2} \text{ or } \frac{5\pi}{2} \Rightarrow \theta = \frac{\pi}{4}, \frac{5\pi}{4}$$

For θ, maximum value of $(a + b \sin 2\theta) = a + b$

$$\therefore \quad r_{\min} = \sqrt{\frac{c}{a+b}}$$

Also, when $\theta = \frac{\pi}{4}$, $P\left(\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}\right) = \left(\sqrt{\frac{c}{2(a+b)}}, \sqrt{\frac{c}{2(a+b)}}\right)$

Again, when $\theta = \frac{5\pi}{4}$,

$$P\left(\frac{-r}{\sqrt{2}}, \frac{-r}{\sqrt{2}}\right) = \left(-\sqrt{\frac{c}{2(a+b)}}, -\sqrt{\frac{c}{2(a+b)}}\right)$$

Thus, the required points are $\pm\left(\sqrt{\frac{c}{2(a+b)}}, \sqrt{\frac{c}{2(a+b)}}\right)$.

● ***Ex. 108*** *Find the value of n, for which* $f(x) = (x^2 - 4)^n (x^2 - x + 1)$, $n \in N$ *assumes a local minima at* $x = 2$.

Sol. Here, $f(x) = (x^2 - 4)^n (x^2 - x + 1)$ assumes local minima at $x = 2$

$\Rightarrow$ $f(2) < f(2 - h)$ and $f(2) < f(2 + h)$, where $h > 0$

where $f(2) = 0$

$\Rightarrow$ $f(2 - h) > 0$ and $f(2 + h) > 0, \forall\ h > 0$

$\Rightarrow$ $(-h)^n (4 - h)^n \cdot \{h^2 - 3h + 1\} > 0$

and $h^n (4 + h)^n (h^2 + 5h + 1) > 0$

i.e. $(-h)^n > 0$

$[\because (4 - h) > 0, h^2 - 3h + 1 > 0, 4 + h > 0, h^2 + 5h + 1 > 0, \forall\ h > 0]$

$\Rightarrow n \in$ even number.

● ***Ex. 109*** *The interval to which b may belong so that the function,*

$$f(x) = \left(1 - \frac{\sqrt{21 - 4b - b^2}}{b+1}\right) x^3 + 5x + \sqrt{16},$$

increases for all x.

Sol. If $f(x) = \left(1 - \frac{\sqrt{21 - 4b - b^2}}{b+1}\right) x^3 + 5x + \sqrt{16}$, increases we must have $f'(x) > 0, \forall\ x \in$ real number.

Then, $f'(x) = \left(1 - \frac{\sqrt{21 - 4b - b^2}}{b+1}\right) 3x^2 + 5 > 0, \forall\ x \in R$

[as we know $ax^2 + bx + c > 0, \forall\ x \in R$, we must have $a > 0$ and $D < 0$]

$$\therefore \quad 1 - \frac{\sqrt{21 - 4b - b^2}}{b+1} > 0$$

and $(0)^2 - 4 \times 3\left(1 - \frac{\sqrt{21 - 4b - b^2}}{b+1}\right) 5 < 0$

$$\Rightarrow \quad 1 - \frac{\sqrt{21 - 4b - b^2}}{b+1} > 0$$

The above inequality holds, when, (i) $b + 1 < 0$ and (ii)

$$21 - 4b - b^2 > 0$$

$\therefore$ $b < -1$ and $b^2 + 4b - 21 < 0$

$\Rightarrow$ $b < -1$ and $(b + 7)(b - 3) < 0$

$\Rightarrow$ $b < -1$ and $-7 < b < 3$ [using number line rule]

$\therefore$ $b \in (-7, -1)$ ...(i)

Again, when $b + 1 > 0$, $f(x)$ will be increasing for all x, if

$$21 - 4b - b^2 > 0 \quad \text{and} \quad 1 > \frac{\sqrt{21 - 4b - b^2}}{b+1}$$

$\Rightarrow$ $b^2 + 4b - 21 < 0$

and $(b + 1)^2 > (21 - 4b - b^2)$ [as $b + 1 > 0$]

$\Rightarrow$ $(b + 7)(b - 3) < 0$ and $b^2 + 3b - 10 > 0$

$\Rightarrow$ $(-7 < b < 3)$ and $(b < -5 \text{ or } b > 2)$

$\Rightarrow$ $2 < b < 3$ ...(ii)

From Eqs. (i) and (ii), we have concluded that.

$$b \in (-7, -1) \cup (2, 3)$$

● ***Ex. 110*** *Find the set of all values of 'a' for which* $f(x) = \left(\frac{\sqrt{a+4}}{1-a} - 1\right) x^5 - 3x + \log 5$ *monotonically decreases for all x.*

Sol. Given, $f(x) = \left(\frac{\sqrt{a+4}}{1-a} - 1\right) x^5 - 3x + \log 5$ decreases for all x.

Then, $f'(x) = \left(\frac{\sqrt{a+4}}{1-a} - 1\right) 5x^4 - 3 < 0, \forall\ x \in R$

i.e. $5\left(\frac{\sqrt{a+4}}{1-a} - 1\right) x^4 - 3 < 0, \forall\ x \in R$

$\Rightarrow$ $\left(\frac{\sqrt{a+4}}{1-a} - 1\right) < 0, \forall\ x \in R$

[as $ax^2 + bx + c < 0 \Rightarrow a < 0$ and $D < 0$] ...(i)

Now, two cases arise

Case **I** If $1 - a < 0 \Rightarrow a > 1$,

then $\sqrt{a+4} > (1 - a)$ and $a + 4 > 0$

Which is always true as LHS > 0 and RHS < 0

$\therefore$ Above inequality is true for all $a > 1$ ...(ii)

***Case* II** If $1-a>0 \Rightarrow a<1$

$\Rightarrow \sqrt{a+4}<1-a$ and $a>-4$

$\Rightarrow a^2-3a-3>0$ and $a>-4$

$\Rightarrow a<\frac{3-\sqrt{21}}{2}$

or $a>\frac{3+\sqrt{21}}{2}$ and $a>-4$

$\Rightarrow -4<a<\frac{3-\sqrt{21}}{2}$ ...(iii)

From Eqs. (ii) and (iii), we conclude

$$a\in\left(-4,\frac{3-\sqrt{21}}{2}\right)\cup(1,\infty)$$

● ***Ex. 111*** *Let $a+b=4$, where $a<2$ and let $g(x)$ be a differentiable function. If $\frac{dg}{dx}>0, \forall\ x$ prove that $\int_0^b g(x)\,dx+\int_0^a g(x)\,dx$ increases as $(b-a)$ increases.*

[IIT JEE 1997]

Sol. Let $(b-a)=t$ and since $a+b=4$, we have $a=\frac{4-t}{2}$

and $b=\frac{t+4}{2}$ ...(i)

where $t>0$ (as $a<2$ and $b>2$)

Let $\int_0^a g(x)\,dx+\int_0^b g(x)\,dx=\phi(t)$

$\Rightarrow \phi(t)=\int_0^{\frac{4-t}{2}} g(x)\,dx+\int_0^{\frac{4+t}{2}} g(x)\,dx$

$\therefore \phi'(t)=g\left(\frac{4-t}{2}\right).\left(\frac{-1}{2}\right)+g\left(\frac{4+t}{2}\right)\left(\frac{1}{2}\right)$

[using Leibnitz rule]

$\Rightarrow \phi'(t)=\frac{1}{2}\left[g\left(\frac{4+t}{2}\right)-g\left(\frac{4-t}{2}\right)\right]$ ...(ii)

Since, $g(x)$ is increasing and we know

$x_1>x_2$

$\Rightarrow g(x_1)>g(x_2)\quad\left[\because \frac{dg}{dx}>0\right]$

Here, $\frac{4+t}{2}>\frac{4-t}{2}$ and $g(x)$ is increasing.

$\therefore g\left(\frac{4+t}{2}\right)>g\left(\frac{4-t}{2}\right)$ ...(iii)

Then, $\phi'(t)=\frac{1}{2}\left[g\left(\frac{4+t}{2}\right)-g\left(\frac{4-t}{2}\right)\right]>0$ [using Eq. (iii)]

$\Rightarrow \phi'(t)>0$

Hence, $\phi(t)$ increases as t increases.

or $\int_0^a g(x)\,dx+\int_0^b g(x)\,dx$ is increasing as $(b-a)$ increases.

● ***Ex. 112*** *Let $g(x)=2f\left(\frac{x}{2}\right)+f(2-x)$ and $f''(x)<0$, $\forall\ x\in(0,2)$. Find the intervals of increase and decrease of $g(x)$.*

Sol. We have, $g(x)=2f\left(\frac{x}{2}\right)+f(2-x)$

$\Rightarrow g'(x)=2f'\left(\frac{x}{2}\right)\cdot\left(\frac{1}{2}\right)+f'(2-x)\cdot(-1)$

$\Rightarrow g'(x)=f'\left(\frac{x}{2}\right)-f'(2-x)$ ...(i)

We are given that $f''(x)<0, \forall\ x\in(0,2)$

It means that $f'(x)$ would be decreasing on $(0, 2)$.

Now, two cases arise

***Case* I** $\frac{x}{2}>(2-x)$ and $f'(x)$ is decreasing.

$\Rightarrow f'\left(\frac{x}{2}\right)<f'(2-x), \forall\ x>\frac{4}{3}$

$\left[\text{As } \frac{x}{2}>2-x\Rightarrow x>\frac{4}{3}\right]$

or $g'(x)=f'\left(\frac{x}{2}\right)-f'(2-x)<0, \forall\ \frac{4}{3}<x<2$

$\therefore$ $g(x)$ is decreasing in $\left(\frac{4}{3},2\right)$. ...(ii)

***Case* II** $\frac{x}{2}<(2-x)$ and $f'(x)$ is decreasing.

$\Rightarrow f\left(\frac{x}{2}\right)>f(2-x), \forall\ x<\frac{4}{3}\quad\left[\text{As } \frac{x}{2}<2-x\Rightarrow x<\frac{4}{3}\right]$

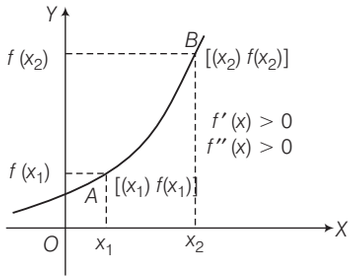
or $g'(x)=f\left(\frac{x}{2}\right)-f(2-x)>0, \forall\ 0<x<\frac{4}{3}$

$\therefore$ $g(x)$ is increasing in $\left(0,\frac{4}{3}\right)$. ...(iii)

From Eqs. (ii) and (iii), we conclude $g(x)$ is increasing in $\left(0,\frac{4}{3}\right)$ and decreasing in $\left(\frac{4}{3},2\right)$.

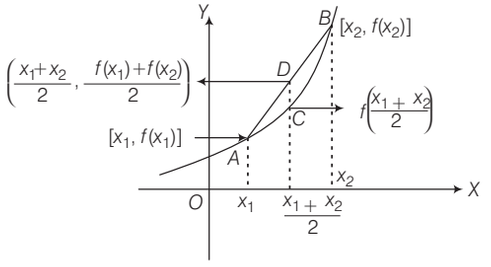
● ***Ex. 113*** *Let $f'(x) > 0$ and $f''(x) > 0$ where $x_1 < x_2$. Then, show $f\left(\frac{x_1 + x_2}{2}\right) < \frac{f(x_1) + f(x_2)}{2}$.*

Sol. As we have discussed in theory, if $f'(x) > 0$ and $f''(x) > 0$, then graphically it can be expressed as shown in the following figure



We know, $x_1 < \frac{x_1 + x_2}{2} < x_2$

and $\left(\frac{x_1 + x_2}{2}, \frac{f(x_1) + f(x_2)}{2}\right)$ is mid-point of the chord joining A and B.

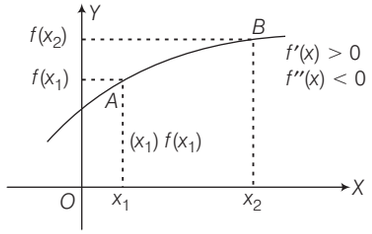


Thus, it can be expressed, from the figure as

$$f\left(\frac{x_1 + x_2}{2}\right) < \frac{f(x_1) + f(x_2)}{2}$$

● ***Ex. 114*** *Let $f'(x) > 0$ and $f''(x) < 0$ where $x_1 < x_2$. Then, show $f\left(\frac{x_1 + x_2}{2}\right) > \frac{f(x_1) + f(x_2)}{2}$.*

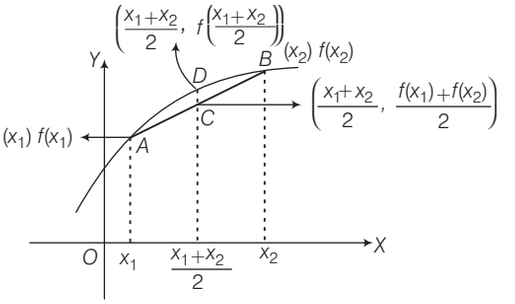
Sol. As we know, if $f'(x) > 0$ and $f''(x) < 0$. Then, it could be expressed graphically as shown in the following figure



We know, $x_1 < \frac{x_1 + x_2}{2} < x_2$

and $\left(\frac{x_1 + x_2}{2}, \frac{f(x_1) + f(x_2)}{2}\right)$ is the mid-point of chord joining A and B.

Thus, it can be expressed as shown in figure



From the adjacent figure, $f\left(\frac{x_1 + x_2}{2}\right) > \frac{f(x_1) + f(x_2)}{2}$.

● ***Ex. 115*** *If $f(x)$ is monotonically increasing function for all $x \in R$, such that $f''(x) > 0$ and $f^{-1}(x)$ exists, then prove that $\frac{f^{-1}(x_1) + f^{-1}(x_2) + f^{-1}(x_3)}{3} < f^{-1}\left(\frac{x_1 + x_2 + x_3}{3}\right)$*

Sol. Let $g(x) = f^{-1}(x)$

Since, g is the inverse of f.

$\Rightarrow \quad fog(x) = gof(x) = x \Rightarrow g'(x) = \frac{1}{f'(g(x))}$

$\Rightarrow \quad g'(x) > 0$ [as $f(x)$ is increasing] ...(i)

$\Rightarrow$ $g(x)$ is increasing for all $x \in R$.

$\Rightarrow$ $f^{-1}(x)$ is increasing for all $x \in R$.

Again, $g'(x) = \frac{1}{f'(g(x))}$

$\Rightarrow \quad g''(x) = -\frac{1}{(f'(g(x)))^2} f''(g(x))\, g'(x)$, for all $x \in R$

$\Rightarrow \quad g''(x) < 0$,

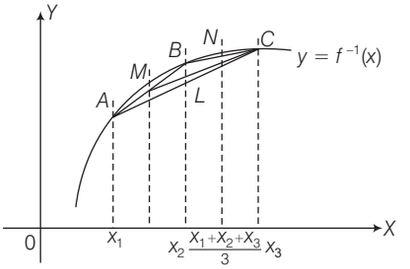
{as $g'(x) > 0$, from relation

(i) $f''(x) > 0$, given

$\Rightarrow$ $g'(x)$ is decreasing for all $x \in R$.

$\Rightarrow$ $f^{-1}(x)$ is increasing and $\frac{d}{dx}\{f^{-1}(x)\}$ is decreasing.

Thus, the graph for $f^{-1}(x)$ could be plotted as



In above figure, we have taken three points A, B, C as; $A\,(x_1, f^{-1}(x_1))$, $B\,(x_2, f^{-1}(x_2))$, $C\,(x_3, f^{-1}(x_3))$.

Also, M is the mid-point of AB as

$$\left(\frac{x_1+x_2}{2}, \frac{f^{-1}(x_1)+f^{-1}(x_2)}{2}\right)$$

and L as the centroid of ΔABC,

i.e. $$L=\left(\frac{x_1+x_2+x_3}{3}, \frac{f^{-1}(x_1)+f^{-1}(x_2)+f^{-1}(x_3)}{3}\right)$$

Correspondingly a point N lies on the curve;

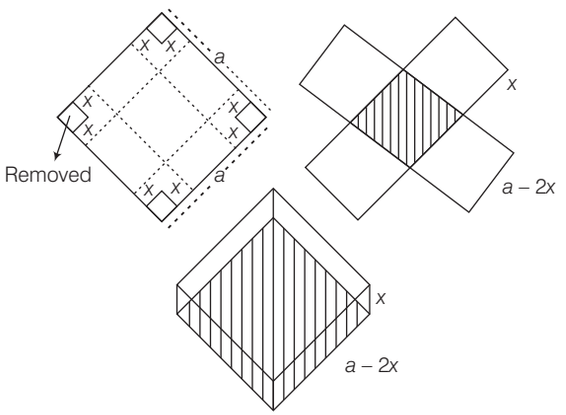
$$N=\left(\frac{x_1+x_2+x_3}{3}, f^{-1}\left(\frac{x_1+x_2+x_3}{3}\right)\right)$$

Also, from above figure it is clear that ordinate of $N >$ ordinate of L

$$\Rightarrow\quad f^{-1}\left(\frac{x_1+x_2+x_3}{3}\right) > \frac{f^{-1}(x_1)+f^{-1}(x_2)+f^{-1}(x_3)}{3}$$

● ***Ex. 116*** *A box of maximum volume with top open is to be made by cutting out four equal squares from four corners of a square tin sheet of side length a feet and then folding up the flaps. Find the side of the square cut-off.*

Sol. Volume of the box is, $V=(a-2x)^2\cdot x$ i.e. squares of side x are cut out, then we will get a box with a square base of side $(a-2x)$ and height x.



Volume of box $(V)=(2a-x)^2\cdot x$

$$\therefore\quad \frac{dV}{dx}=(a-2x)^2+x\cdot 2(a-2x)\cdot(-2)=(a-2x)(a-6x)$$

For V to be extremum, $\dfrac{dV}{dx}=0$

$$\Rightarrow\quad x=a/2, a/6$$

But when $x=a/2$; $V=0$ (minimum) and we know minimum and maximum occurs simultaneously in a continuous function.

Hence, V is maximum when $x=a/6$.

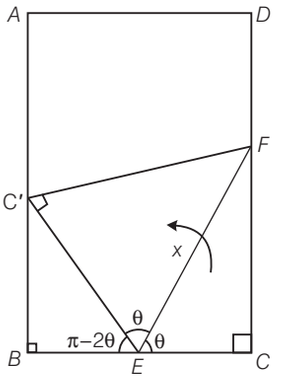
● ***Ex. 117*** *One corner of a long rectangular sheet of paper of width* 1 *unit is folded over so as to reach the opposite edge of the sheet. Find the minimum length of the crease.*

Sol. Let $ABCD$ be the rectangular sheet whose corner C is folded over along EF so as to reach the edge AB at C'.

Let $EF=x$

$$\angle FEC=\theta=\angle FEC'$$

$$\therefore\quad EC=x\cos\theta=EC'$$



From $\Delta BEC'$, we have $BE=C'E\cos(\pi-2\theta)$

$$\Rightarrow\quad BE=-x\cos\theta\cdot\cos 2\theta$$

$$\therefore\quad BC=BE+EC$$

$$1=-x\cos\theta\cos 2\theta+x\cos\theta$$

$$\Rightarrow\quad x=\frac{1}{\cos\theta(1-\cos 2\theta)} \qquad ...(i)$$

Let $Z=\cos\theta(1-\cos 2\theta)\,x$ to be minimum, Z has to be maximum.

$$Z=\cos\theta(1-\cos 2\theta) \qquad ...(ii)$$

Differentiating Eq. (ii) w.r.t. θ, we get

$$\frac{dZ}{d\theta}=\cos\theta(0+2\sin 2\theta)-\sin\theta(1-\cos 2\theta)$$

and $$\frac{d^2Z}{d\theta^2}=\cos\theta(4\cos 4\theta)-2\sin 2\theta\cdot\sin\theta$$

$$-\sin\theta(2\sin 2\theta)-\cos\theta(1-\cos 2\theta)$$

For maximum/minimum,

$$\frac{dZ}{d\theta}=0$$

$$\Rightarrow\quad 2\sin\theta(2-3\sin^2\theta)=0$$

$$\therefore\quad \sin\theta=+\sqrt{2/3} \qquad [\because \sin\theta\neq 0]$$

When $\sin\theta=\sqrt{\dfrac{2}{3}}$

$$\Rightarrow\quad \frac{d^2Z}{d\theta^2}=-\frac{5}{5\sqrt{3}}-\frac{16}{3\sqrt{3}}-\frac{1}{\sqrt{3}}=-\frac{8}{\sqrt{3}}<0$$

Hence, Z is maximum.

$\Rightarrow$ $x = \dfrac{1}{Z}$ is minimum [from Eq. (i)]

$\therefore$ x is minimum.

$\Rightarrow$ $x_{\min} \Rightarrow \dfrac{1}{Z} = \dfrac{1}{(1/\sqrt{3})\left(1+\dfrac{1}{3}\right)} = \dfrac{3\sqrt{3}}{4}$ unit

● ***Ex. 118*** *Find the volume of the greatest right circular cone that can be described by the revolution about a side of a right angled triangle of hypotenuse* 1 *ft.*

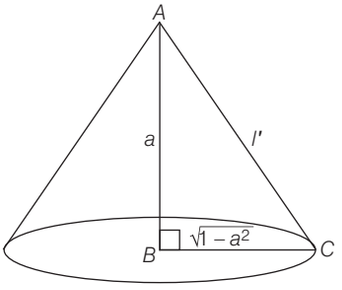
Sol. Let ABC be right angled triangle .

Let the cone be revolved about AB.

$AC = 1$ ft [given]

Let $AB = a$ = height of cone

$\therefore$ $BC = \sqrt{1-a^2}$ = radius of cone



Volume of cone $= \dfrac{1}{3}\pi(1-a^2)a$

$$V = \frac{1}{3}\pi(a - a^3)$$

$\Rightarrow$ $\dfrac{dV}{da} = \dfrac{1}{3}\pi(1-3a^2)$

and $\dfrac{d^2V}{da^2} = \dfrac{1}{3}\pi(-6a) < 0$

$\therefore$ Maximum volume when $\dfrac{dV}{da} = 0$

i.e. when $a = \dfrac{1}{\sqrt{3}}$

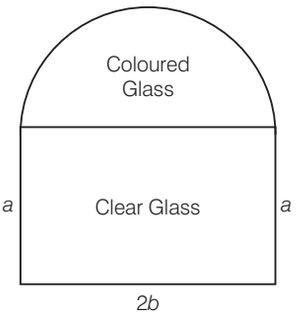
Putting $a = \dfrac{1}{\sqrt{3}}$, we get $V_{\max} = \dfrac{2\sqrt{3}}{27}\pi$ cu ft

● ***Ex. 119*** *A window of fixed perimeter (including the base of the arc) is in the form of a rectangle surmounted by a semi-circle. The semi-circular portion is fitted with coloured glass while the rectangular portion is fitted with clear glass. The clear glass transmits three times as much light per square metre as the coloured glass does. What is the ratio of the sides of the rectangle so that the window transmits the maximum light?* ***[IIT JEE 1991]***

Sol. Let $2b$ be the diameter of the circular portion and a be the lengths of the other side of the rectangle.

Total perimeter $= 2a + 4b + \pi b = K$ [say]...(i)

Now, let the light transmission rate (per square metre) of the coloured glass be L and Q be the total amount of transmitted light.



Then, $Q = 2ab\cdot(3L) + \dfrac{1}{2}\pi b^2\cdot(L)$

$$Q = \frac{L}{2}[\pi b^2 + 12ab]$$

$$Q = \frac{L}{2}[\pi b^2 + 6b(K - 4b - \pi b)]$$

$$Q = \frac{L}{2}[6Kb - 24b^2 - 5\pi b^2]$$

$\therefore$ $\dfrac{dQ}{db} = \dfrac{L}{2}[6K - 48b - 10\pi b] = 0$

$$b = \frac{6K}{48 + 10\pi} \quad \text{...(ii)}$$

and $\dfrac{d^2Q}{db^2} = \dfrac{L}{2}[-48 - 10\pi] < 0$

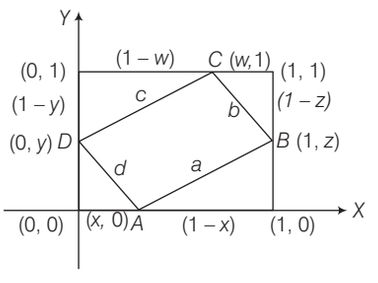
Thus, Q is maximum and from Eqs. (i) and (ii)

$$(48 + 10\pi)\, b = 6K = 6\{2a + 4b + \pi b\}$$

Thus, the ratio $= \dfrac{2b}{a} = \dfrac{6}{6+\pi}$

● ***Ex. 120*** *Let S be a square of unit area. Consider any quadrilateral which has one vertex on each side of S. If a, b, c and d denote the lengths of the sides of the quadrilateral, prove that* $2 \le a^2 + b^2 + c^2 + d^2 \le 4$ ***[IIT JEE 1997]***

Sol. Let S be the square of unit area and $ABCD$ be the quadrilateral of sides a, b, c and d.



Here, $a^2 = (1-x^2) + z^2,\ b^2 = w^2 + (1-z)^2$

$c^2 = (1-w^2) + (1-y)^2, d^2 = x^2 + y^2$

Adding all the above, $a^2 + b^2 + c^2 + d^2$

$= \{x^2 + (1-x)^2\} + \{y^2 + (1-y)^2\} + \{z^2 + (1-z)^2\}$

$+ \{w^2 + (1-w)^2\}$

where $0 \le x, y, z, w \le 1$

Let us consider a function,

$$f(x) = x^2 + (1-x)^2, 0 \le x \le 1$$

Then, $f'(x) = 2x - 2(1-x)$

Let $f'(x) = 0$ for maximum/minimum.

$\Rightarrow \quad 4x - 2 = 0 \Rightarrow x = 1/2$

Again, $f''(x) = 4 > 0$ when $x = 1/2$

$\therefore$ $f(x)$ is minimum at $x = 1/2$ and maximum at $x = 1$

$\Rightarrow \quad 2 \le a^2 + b^2 + c^2 + d^2 \le 4$

● ***Ex. 121*** *Show that a triangle of maximum area that can be inscribed in a circle of radius a is an equilateral triangle.*

Sol. Let BC be one of the sides of the triangle and the third vertex A should be in a position that the altitude AD is maximum (for area of the triangle to be maximum).

For that the ΔABC must be symmetric about AD.

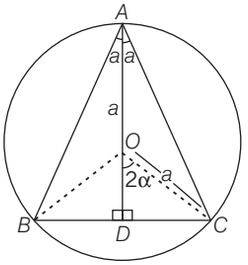
i.e. D should be the mid-point of BC.

Let $\angle A = 2\alpha$

$\therefore \quad \angle BOD = \angle COD = 2\alpha$

Thus, $AD = AO + OD = a + a\cos 2\alpha$

and $CD = a \sin 2\alpha$



Hence, area of $\Delta ABC = A = \frac{1}{2} \cdot AD \cdot BC$

Area, $A = \frac{1}{2} \cdot a(1 + \cos 2\alpha) \cdot a \sin 2\alpha$

$$A = \frac{a^2}{2}(\sin 2\alpha + \frac{1}{2}\sin 4\alpha)$$

Differentiating w.r.t. α, we get

$$\frac{dA}{d\alpha} = \frac{a^2}{2}[2\cos 2\alpha + 2\cos 4\alpha] = 0$$

$\therefore \quad 2a^2 \cos 3\alpha \cdot \cos \alpha = 0$

$\Rightarrow$ Either $\cos 3\alpha = 0$ or $\cos \alpha = 0$

$\Rightarrow \quad \alpha = \pi/6, \pi/2$

But $\alpha = \pi/2$ is not possible.

Now, $\frac{d^2A}{d\alpha^2} = 2a^2[-\sin 2\alpha - 4\sin 4\alpha] = -\text{ve at } \alpha = \pi/6$

$\therefore$ A is maximum when $\alpha = \pi/6$.

Also, $\angle A = \pi/3$ and triangle is isosceles.

Hence, ΔABC must be equilateral.

● ***Ex. 122*** *Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius a is $2a/\sqrt{3}$.*

Sol. Let h be the height and r be the radius of the cylinder. Let O be the centre of the sphere.

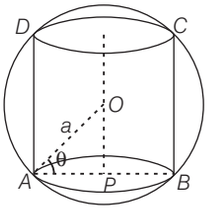
From the figure,

$$OA^2 = OP^2 + PA^2$$

i.e. $$a^2 = \frac{h^2}{4} + r^2 \qquad \text{...(i)}$$

Now, volume of the cylinder,

$$V = \pi r^2 h = \pi h\left(a^2 - \frac{h^2}{4}\right) \qquad \text{[using Eq. (i)]}$$



Differentiating w.r.t. h, we have

$$\frac{dV}{dh} = \pi\left[a^2 - \frac{3h^2}{4}\right] = 0 \text{ for extremum}$$

$\Rightarrow \quad h = \frac{2a}{\sqrt{3}}$

Also, $\frac{d^2V}{dh^2} = -\frac{3\pi}{2}h < 0$

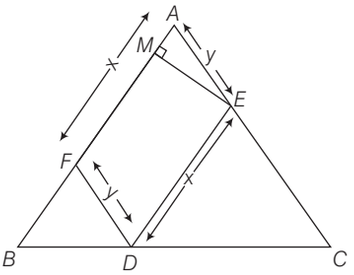
at $h = \frac{2a}{\sqrt{3}}$

Thus, volume is maximum when $h = \frac{2a}{\sqrt{3}}$.

● ***Ex. 123*** *Let $A(p^2, p), B(q^2, -q)$ and $C(r^2, -r)$ be the vertices of the ΔABC. A parallelogram AFDE is drawn with vertices D, E, F on the line segments BC, CA and AB, respectively. Show that maximum area of such a parallelogram is $\frac{1}{4}(p-q)(q-r)(r-p)$.*

Sol. Let $AF = x = DE$ and $AE = y = DF$

Since, $\Delta's$ CAB and CED are similar.



We have, $\frac{CE}{CA} = \frac{DE}{AB}$ [as shown in figure]

$\Rightarrow$ $\frac{b-y}{b} = \frac{x}{c}$

[here, $BC = a$, $AC = b$ and $AB = c$] ...(i)

Now, area of parallelogram,

$$S = AF \cdot EM = xy \sin A$$

$$S = x \cdot b\left(1 - \frac{x}{c}\right)\sin A \quad \text{[from Eq. (i)]} \ldots(ii)$$

Differentiating w.r.t. x, we have

$$\frac{dS}{dx} = \frac{b}{c}(c - 2x)\sin A \quad [\text{where } \sin A \text{ is constant}]$$

For extremum, $\frac{dS}{dx} = 0$

$\Rightarrow$ $x = \frac{c}{2}$

Also, $\frac{d^2S}{dx^2} = -\frac{2b}{c} < 0$

at $x = \frac{c}{2}$

Hence, S is maximum when $x = \frac{c}{2}$

Now, $S_{\max} = \frac{1}{4}bc \sin A$ [from Eq. (ii)]

$$S_{\max} = \frac{1}{2}\left(\frac{1}{2}bc \sin A\right)$$

$$S_{\max} = \frac{1}{2}(\text{area of } \Delta ABC)$$

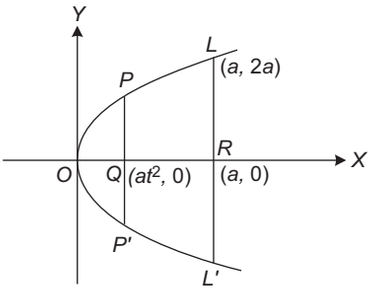
$$S_{\max} = \frac{1}{4}\begin{vmatrix} p^2 & -p & 1 \\ q^2 & -q & 1 \\ r^2 & -r & 1 \end{vmatrix}$$

$$S_{\max} = \frac{1}{4}(p-q)(q-r)(r-p)$$

● **Ex. 124** *LL' be latusrectum of the parabola $y^2 = 4ax$ and PP' is a double ordinate between the vertex and the latusrectum. Show that the area of trapezium $PP'L'L$ is maximum when the distance of PP' from the vertex is $\frac{a}{9}$.*

Sol. Let the double ordinate PP' be drawn at a distance $x = at^2$ from the origin. (vertex).

Thus, the coordinate of P is $(at^2, 2at)$ as shown in figure



Hence, the area of trapezium is,

$$A = \frac{1}{2}(PP' + LL') \cdot QR$$

$$= \frac{1}{2}(4at + 4a) \cdot (a - at^2)$$

$$A = 2a^2(t+1)(1-t^2)$$

$\therefore$ $\frac{dA}{dt} = 2a^2(t+1)(1-3t) = 0$ [for extremum]

$\Rightarrow$ $t = -1, 1/3$

Also, $\frac{d^2A}{dt^2} = 2a^2(-6t - 2)$

Thus, A is maximum when $t = 1/3$ as $\frac{d^2A}{dt^2} < 0$

Hence, $x = at^2 = \frac{a}{9}$ is the point at which area of trapezium is maximum.

● **Ex. 125** *The circle $x^2 + y^2 = 1$ intersects the X-axis at P and Q. Another circle with centre at Q and variable radius intersects the first circle at R, above the X-axis and the line segment PQ at S. Find the maximum area of the ΔQSR.*

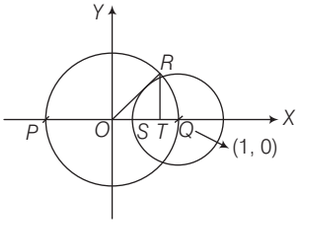
[IIT JEE 1994]

Sol. The centre of the circle

$$x^2 + y^2 = 1 \quad \ldots(i)$$

is (0, 0) and radius $OP = 1 = OQ$

So, coordinates of Q are (1,0).



Let the radius of the variable circle be r.

Hence, its equation is

$$(x-1)^2 + (y)^2 = r^2 \quad \ldots(ii)$$

Subtracting Eq. (ii) from Eq. (i), we get

$$2x - 1 = 1 - r^2$$

$$x = 1 - \frac{r^2}{2} = OT \quad \text{...(iii)}$$

Now, $RT = \sqrt{OR^2 - OT^2} = \sqrt{1 - \left(1 - \frac{r^2}{2}\right)^2} = \sqrt{r^2 - \frac{r^4}{4}}$ ...(iv)

Now, the area of ΔQSR is, $A = \frac{1}{2} \cdot QS \cdot RT$

$$\therefore \quad A^2 = \frac{1}{4}(QS^2) \cdot (RT^2)$$

$$\Rightarrow \quad A^2 = \frac{1}{4} r^2 \left(r^2 - \frac{r^4}{4}\right)$$

[using Eqs. (ii) and (iv)]

$$\Rightarrow \quad A^2 = \frac{1}{16}(4r^4 - r^6) = f(r) \quad \text{(say)}$$

Thus, $$\frac{df(r)}{dr} = \frac{1}{16}(16r^3 - 6r^5) = 0$$

[for extremum]

$$\Rightarrow \quad r = 2\sqrt{\frac{2}{3}}$$

Also, $$\frac{d^2 f(r)}{dr^2} = \frac{1}{16}(48r^2 - 30r^4) = -\frac{16}{3} < 0$$

where $$r = 2\sqrt{\frac{2}{3}}$$

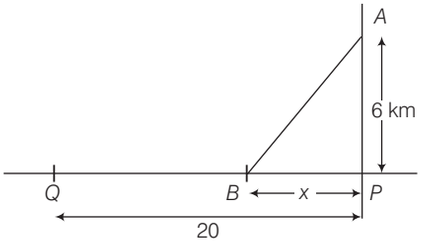
Hence, area is maximum at $r = 2\sqrt{\frac{2}{3}}$ and $A_{\max} = \frac{4}{3\sqrt{3}}$ sq unit.

● ***Ex. 126*** *A despatch rider is in open country at a distance of 6 km from the nearest point P of a straight road. He wishes to proceed as quickly as possible to a point Q on the road 20 km from P. If his maximum speed across country is 40 km/h, 50 km/h on road, then at what distance from P, he should touch the road.*

Sol. Let A be the initial position of rider.

Then, $PB = x \Rightarrow QB = 20 - x$

and $AB = \sqrt{AP^2 + BP^2} = \sqrt{6^2 + x^2}$ km



$\therefore$ Total time T,

$$T = \frac{\sqrt{x^2 + 36}}{40} + \frac{20 - x}{50}$$

$$\Rightarrow \quad \frac{dT}{dx} = \frac{1}{40} \cdot \frac{(2x)}{2\sqrt{x^2 + 36}} - \frac{1}{50}$$

For maximum and minimum value, we must have

$$\frac{dT}{dx} = 0$$

$$\Rightarrow \quad \frac{x}{40\sqrt{x^2 + 36}} = \frac{1}{50}$$

or $$\frac{5x}{4} = \sqrt{x^2 + 36}$$

or $$\frac{25x^2}{16} - x^2 = 36$$

$$\Rightarrow \quad x^2 = \frac{36 \times 16}{9}$$

$$\Rightarrow \quad x = \frac{6 \times 4}{3} = 8 \text{ km}$$

● ***Ex. 127*** *From point A located on a highway a boy has to get his bus to his school B located in the field at a distance l from the highway in the least possible time. At what distance from D should the bus leave the highway when the bus moves n times slower in the field than on the highway?*

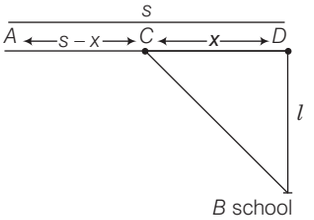
Sol. Let $AD = s$ and $CD = x$, where C is a point where the bus leaves the highway.

Also, let the speed of the bus is 'v' on the highway.

$\therefore$ Total time taken,

$$t = \frac{AC}{v} + \frac{BC}{(v/n)}$$

$$\Rightarrow \quad t = \frac{s - x}{v} + \frac{n\sqrt{l^2 + x^2}}{v}$$

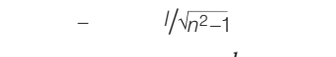


Differentiating w.r.t. x, we have

$$\frac{dt}{dx} = \frac{1}{v}\left[-1 + \frac{n}{2\sqrt{l^2 + x^2}}(+2x)\right] = 0 \quad \text{[for extremum]}$$

$$\Rightarrow \quad n^2 x^2 = l^2 + x^2$$

$$\Rightarrow \quad x = \frac{l}{\sqrt{n^2 - 1}}$$



Thus, 't' is minimum when $x = \frac{l}{\sqrt{n^2 - 1}}$, as shown in figure.

● ***Ex. 128*** *Two men are walking on a path $x^3 + y^3 = a^3$ when the first man arrives at a point (x_1, y_1), he finds the second man in the direction of his own instantaneous motion. If the coordinates of the second man are (x_2, y_2), then show that*

$$\frac{x_2}{x_1} + \frac{y_2}{y_1} + 1 = 0.$$

Sol. Since, (x_1, y_1) and (x_2, y_2) lies on the curve.

$\therefore$ $x_1^3 + y_1^3 = a^3$ ...(i)

and $x_2^3 + y_2^3 = a^3$ ...(ii)

Subtracting Eqs. (i) from (ii), we get

$$(x_2^3 - x_1^3) + (y_2^3 - y_1^3) = 0$$

or $x_2^3 - x_1^3 = -(y_2^3 - y_1^3)$ ...(iii)

Now, differentiating both sides of $x^3 + y^3 = a^3$ w.r.t. x, we get

$$3x^2 + 3y^2 \frac{dy}{dx} = 0$$

Slope of tangent at $(x_1, y_1) = -\frac{x_1^2}{y_1^2}$

$\therefore$ The equation of tangent at (x_1, y_1)

$$y - y_1 = -\frac{x_1^2}{y_1^2}(x - x_1)$$

It passes through (x_2, y_2).

$\therefore$ $y_2 - y_1 = \frac{-x_1^2}{y_1^2}(x_2 - x_1)$

or $x_1^2(x_2 - x_1) = -y_1^2(y_2 - y_1)$ ...(iv)

Dividing Eq. (iii) by Eq. (iv), we get

$$\frac{x_2^3 - x_1^3}{x_1^2(x_2 - x_1)} = \frac{-(y_2^3 - y_1^3)}{-y_1^2(y_2 - y_1)}$$

$$\Rightarrow \quad \frac{x_2^2 + x_1^2 + x_1x_2}{x_1^2} = \frac{y_2^2 + y_1^2 + y_1y_2}{y_1^2}$$

$$\Rightarrow \quad \left(\frac{x_2}{x_1}\right)^2 + \left(\frac{x_2}{x_1}\right) + 1 = \left(\frac{y_2}{y_1}\right)^2 + \left(\frac{y_2}{y_1}\right) + 1$$

$$\Rightarrow \quad \left(\frac{x_2}{x_1}\right)^2 - \left(\frac{y_2}{y_1}\right)^2 = \left(\frac{y_2}{y_1}\right) - \left(\frac{x_2}{x_1}\right)$$

$$\Rightarrow \quad \left(\frac{x_2}{x_1} - \frac{y_2}{y_1}\right) \cdot \left(\frac{x_2}{x_1} + \frac{y_2}{y_1}\right) + \left(\frac{x_2}{x_1} - \frac{y_2}{y_1}\right) = 0$$

$$\Rightarrow \quad \left(\frac{x_2}{x_1} - \frac{y_2}{y_1}\right) \cdot \left(\frac{x_2}{x_1} + \frac{y_2}{y_1} + 1\right) = 0$$

i.e. either $\frac{x_2}{x_1} = \frac{y_2}{y_1}$

or $\frac{x_2}{x_1} + \frac{y_2}{y_1} + 1 = 0$

But, $\frac{x_2}{x_1} \neq \frac{y_2}{y_1}$

$\therefore$ $\frac{x_2}{x_1} + \frac{y_2}{y_1} + 1 = 0$

● ***Ex. 129*** *In still water a boat moves with a velocity which is K times less than velocity the river has current. At what angle to the stream direction must the boat move to minimize drifting.*

Sol. Let the flow velocity of river be u and the velocity of boat in still water be v.

Thus, $v = u/K$

Also, let the boat moves at an angle θ with direction of stream.

Now, the velocity of boat in the river is vector resultant of the velocity of boat and velocity of following water or water current, which can be written as,

$$\bar{v}_B = (u - v\cos\alpha)\hat{\mathbf{i}} + (u\sin\alpha)\hat{\mathbf{j}} = (u + v\cos\theta)\hat{\mathbf{i}} + (u\sin\theta)\hat{\mathbf{j}}$$

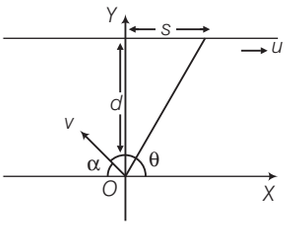
Hence, the time taken to cross the river

$$= \frac{d}{u\sin\theta} \qquad [\text{where, } d = \text{width of river}]$$

Thus, the drift $s = (u + v\cos\theta)\cdot d$

$$\Rightarrow \quad s = d\left(\operatorname{cosec}\theta + \frac{v}{u}\cot\theta\right)$$

$$\Rightarrow \quad \frac{ds}{d\theta} = d\left(-\operatorname{cosec}\theta\cot\theta - \frac{v}{u}\operatorname{cosec}^2\theta\right) = 0$$



$$\Rightarrow \quad -\frac{v}{u}\operatorname{cosec}^2\theta = \operatorname{cosec}\theta\cot\theta$$

$$\Rightarrow \quad \operatorname{cosec}\theta = -1/K$$

$$\Rightarrow \quad \theta = \cos^{-1}(-1/K)$$

● ***Ex. 130*** *Consider a square with vertices at* (1, 1) (−1, 1) (−1, −1) *and* (1, −1). *Let S be the region consisting of all points inside the square which are nearer to the origin than to any edge. Sketch the region S and find its area.* ***[IIT JEE 1995]***

Sol. Let the square $ABCD$ and the equations of the sides of the square are as follows

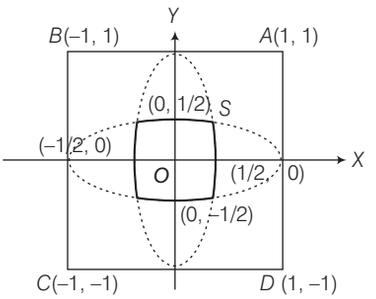
$AB : y = 1, \quad BC : x = -1, \quad CD : y = -1, \quad DA : x = 1$

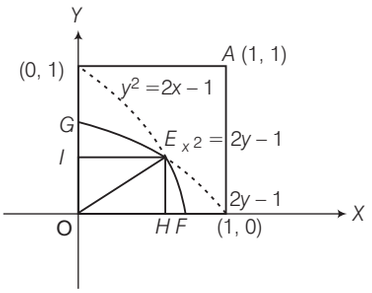
Let the region be S and (x, y) is any point inside it. Then, according to given conditions,

$$\sqrt{x^2+y^2} < |1-x|, |1+x|, |1-y|, |1+y|$$

$$\Rightarrow (x^2+y^2) < x^2-2x+1, x^2+2x+1, y^2-2y+1, y^2+2y+1$$

$$\Rightarrow y^2 < 1-2x, y^2 < 1+2x, x^2 < 1-2y \text{ and } x^2 < 2y+1$$





Now, in $y^2 = 1-2x$ and $y^2 = 2x+1$, the first equation represents a parabola with vertex $(1/2,0)$ and second equation represents a parabola with vertex $(-1/2, 0)$ and in $x^2 = 1-2y$ and $x^2 = 1+2y$, the first equation represents parabola with vertex at $(0, 1/2)$ and second equation represents a parabola with vertex at $(0,-1/2)$.

So, the region S is the region lying inside the four parabolas.

$$y^2 = 1-2x$$
$$y^2 = 1+2x$$
$$x^2 = 1-2y$$
$$x^2 = 1+2y$$

Now, S is symmetrical in all four quadrants.

$\therefore$ $S = 4 \times$ area lying in first quadrant.

Now, $y^2 = 1-2x$

and $x^2 = 1-2y$ intersect on $y = x$

The point of intersection is $E(\sqrt{2}-1, \sqrt{2}-1)$.

$\therefore$ Area of region $OEFO$ = Area of ΔOEH + Area of region $HEFH$

$$= \frac{1}{2}(\sqrt{2}-1)^2 + \int_{\sqrt{2}-1}^{1/2} \sqrt{1-2x}\, dx$$

$$= \frac{1}{2}(2+1-2\sqrt{2}) + \frac{2}{3}[(1-2x)^{3/2}]_{\sqrt{2}-1}^{1/2}$$

$$= \frac{1}{2}(3-2\sqrt{2}) + \frac{1}{3}(3-2\sqrt{2})^{3/2}$$

$$= \frac{1}{2}(3-2\sqrt{2}) + \frac{1}{3}(\sqrt{2}-1)^3$$

$$= \frac{1}{2}(3-2\sqrt{2}) + \frac{1}{3}(5\sqrt{2}-7) = \frac{1}{6}[4\sqrt{2}-5]$$

Similarly, area of region $OEGO = \frac{1}{6}(4\sqrt{2}-5)$

So, area of S lying in first quadrant $= \frac{2}{6}(4\sqrt{2}-5)$

Hence, $S = \frac{4}{3}(4\sqrt{2}-5)$

● ***Ex. 131*** *If x is increasing at the rate of 2 cm/s at the instant when $x = 3$ cm and $y = 1$ cm, at what rate y must be changing in order that the quantity $(2xy - 3x^2y)$ shall be neither increasing nor decreasing.*

If $S = \{(a, b) \in R \times R : x = a, y = b, 2xy - 3x^2y = \text{constant} \Rightarrow \frac{dy}{dx} > 0\}$

$S' = \{(x, y) \in A \times B : -1 \le A \le 1$ and $-1 \le B \le 1\}$, then find the area $S \cap S'$.

Sol. Here, $x = 3$ cm, $y = 1$ cm and $\frac{dx}{dt} = 2$ cm/s

Let $f = 2xy - 3x^2y$

$$\Rightarrow \frac{df}{dt} = 2y\frac{dx}{dt} + 2x\frac{dy}{dt} - 3x^2\frac{dy}{dt} - 6xy\frac{dx}{dt}$$

$$= (2x-3x^2)\frac{dy}{dx} + (2y-6xy)\frac{dx}{dt} \quad \ldots(i)$$

$\because f$ is neither increasing nor decreasing.

$$\therefore \left(\frac{df}{dt}\right)_{(3,1)} = 0$$

$$\Rightarrow (2\times 3 - 27)\frac{dy}{dt} + (2-18)2 = 0$$

$$\Rightarrow dy/dt = -32/21$$

Now, for S, f = constant

$$\Rightarrow 0 = (2x-3x^2)\frac{dy}{dt} + (2y-6xy)\frac{dx}{dt}$$

$$\Rightarrow \frac{dy}{dx} = \frac{6xy-2y}{2x-3x^2}$$

$$\Rightarrow \frac{dy}{dx} > 0 \Rightarrow \frac{6xy-2y}{2x-3x^2} > 0$$

$$\Rightarrow \frac{y(3x-1)}{x(3x-2)} < 0,$$

Thus, there arise two cases

***Case* I** $y(3x-1) > 0$ and $x(3x-2) < 0$

$\Rightarrow$ $(y > 0$ and $x > 1/3)$ or $(y < 0$ and $x < 1/3)$

and $\left(0 < x < \frac{2}{3}\right)$

$$\Rightarrow \left(y > 0 \text{ and } \frac{1}{3} < x < \frac{2}{3}\right) \text{ or } \left(y < 0 \text{ and } 0 < x < \frac{1}{3}\right)$$

***Case* II** $y(3x-1)<0$ and $x(3x-2)>0$

$\Rightarrow$ $(y>0$ and $x<1/3)$

or $(y<0$ and $x>1/3)$

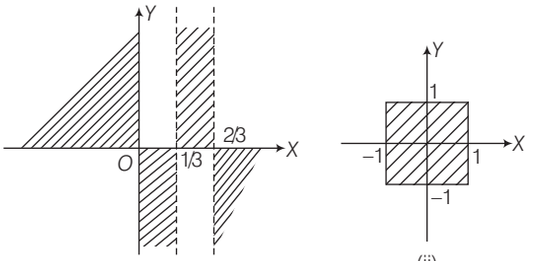
and $(x<0$ or $x>2/3)$

$\Rightarrow$ $(y>0$ and $x<0)$

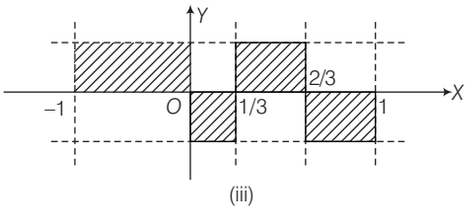
or $(y<0$ and $x>2/3)$

Thus, S is shaded portion shown in figure (i).

Also, S' represents the portion shown in figure (ii).



Thus, area for $S \cap S'$ is shown in figure (iii).



Thus, the area of $S \cap S' = 2$

● ***Ex. 132*** *In how many parts an integer $N \geq 5$ should be divide so that the product of the parts is maximized?*

Sol. Let $x_1 + x_2 + x_3 + ... + x_n = N$

$\therefore$ To maximize the value of $x_1x_2x_3 \ldots x_n$

Using AM > GM

$$\frac{x_1 + x_2 + \ldots + x_n}{n} > (x_1x_2 \ldots x_n)^{1/n}$$

$$\Rightarrow \quad x_1x_2 \ldots x_n \leq \left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right)^n$$

which shows maximum value of $x_1x_2 \ldots x_n$ is obtained when $x_1 = x_2 = x_3 = \ldots = x_n$

Now, function to be maximized is $\left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right)^n$ which is discrete function of n. In order to arrive at some possible neighbourhood first we make it continuous. Thus, by changing the variable n to x.

We write $f(x) = \left(\frac{N}{x}\right)^x$

For maxima $f'(x) = 0$

i.e. $\left(\frac{N}{x}\right)^x \left\{\log \frac{N}{x} - 1\right\} = 0 \Rightarrow \frac{N}{x} = e$

or $x = \frac{N}{e}$, now as $x \in$ integer

$\Rightarrow$ The nearest integer is $\left[\frac{N}{e}\right]$ or $\left[\frac{N}{e}\right] + 1$.

Monotonicity, Maxima and Minima Exercise 1 :
Single Option Correct Type Questions

1. If $f : [1, 10] \to [1, 10]$ is a non-decreasing function and $g : [1, 10] \to [1, 10]$ is a non-increasing function. Let $h(x) = f(g(x))$ with $h(1) = 1$. Then, $h(2)$
(a) lies in (1, 2) (b) is more than two
(c) is equal to one (d) is not defined

2. P is a variable point on the curve $y = f(x)$ and A is a fixed point in the plane not lying on the curve. If PA^2 is minimum, then the angle between PA and the tangent at P is
(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$
(c) $\frac{\pi}{2}$ (d) None of these

3. Let $f(x) = \begin{cases} 1+\sin x, & x < 0 \\ x^2 - x + 1, & x \geq 0 \end{cases}$. Then,
(a) f has a local maximum at $x = 0$
(b) f has a local minimum at $x = 0$
(c) f is increasing everywhere
(d) f is decreasing everywhere

4. If m and n are positive integers and $f(x) = \int_1^x (t-a)^{2n}(t-b)^{2m+1}dt,\ a \neq b$, then
(a) $x = b$ is a point of local minimum
(b) $x = b$ is a point of local maximum
(c) $x = a$ is a point of local minimum
(d) $x = a$ is a point of local maximum

5. Let $f(x) = x^{n+1} + a \cdot x^n$, where a is a positive real number. Then, $x = 0$ is a point of
(a) local minimum for any integer n
(b) local maximum for any integer n
(c) local minimum if n is an even integer
(d) local minimum if n is an odd integer

6. If $f(x) = \begin{cases} 2x^2 + \frac{2}{x^2}, & 0 < |x| \leq 2 \\ 3, & x > 2 \end{cases}$, then
(a) $x = 1, -1$ are the points of global minima
(b) $x = 1, -1$ are the points of local minima
(c) $x = 0$ is the points of local minima
(d) None of the above

7. Given a function $y = x^x,\ x > 0$ and $0 < x < 1$. The values of x for which the function attain values exceeding the values of its inverse are
(a) $0 < x < 1$ (b) $1 < x < \infty$
(c) $0 < x < 2$ (d) None of these

8. $\sin x + \cos x = y^2 - y + a$ has no value of x for any y, if a belongs
(a) $(0, \sqrt{3})$ (b) $(-\sqrt{3}, 0)$
(c) $(-\infty, -\sqrt{3})$ (d) $(\sqrt{3}, \infty)$

9. If $f : R \to R$ is the function defined by $f(x) = \frac{e^{x^2} - e^{-x^2}}{e^{x^2} + e^{-x^2}}$, then
(a) $f(x)$ is an increasing function
(b) $f(x)$ is a decreasing function
(c) $f(x)$ is a onto
(d) None of the above

10. Let $f(x)$ be a quadratic expression positive for all real x. If $g(x) = f(x) - f'(x) + f''(x)$, then for any real x
(a) $g(x) > 0$ (b) $g(x) \leq 0$
(c) $g(x) \geq 0$ (d) $g(x) < 0$

11. $f(x) = \min\{1, \cos x, 1 - \sin x\},\ -\pi \leq x \leq \pi$, then
(a) $f(x)$ is differentiable at 0
(b) $f(x)$ is differentiable at $\frac{\pi}{2}$
(c) $f(x)$ has local maxima at 0
(d) None of the above

12. $f(x) = \begin{cases} 2 - |x^2 + 5x + 6|, & x \neq -2 \\ a^2 + 1, & x = -2 \end{cases}$, then the range of a, so that $f(x)$ has maxima at $x = -2$ is
(a) $|a| \geq 1$ (b) $|a| < 1$ (c) $a > 1$ (d) $a < 1$

13. Maximum number of real solution for the equation $ax^n + x^2 + bx + c = 0$, where $a,\ b,\ c \in R$ and n is an even positive number, is
(a) 2 (b) 3 (c) 4 (d) infinite

14. Maximum area of rectangle whose two sides are $x = x_0,\ x = \pi - x_0$ and which is inscribed in a region bounded by $y = \sin x$ and X-axis is obtained when $x_0 \in$
(a) $\left(\frac{\pi}{4}, \frac{\pi}{3}\right)$ (b) $\left(\frac{\pi}{6}, \frac{\pi}{4}\right)$
(c) $\left(0, \frac{\pi}{6}\right)$ (d) None of these

15. $f(x) = -1 + kx + k$ neither touches nor intersect the curve $f(x) = \ln x$, then minimum value of $k \in$
(a) $\left(\frac{1}{e}, \frac{1}{\sqrt{e}}\right)$ (b) (e, e^2)
(c) $\left(\frac{1}{\sqrt{e}}, e\right)$ (d) None of these

16. Let $f(x)$ be a polynomial with real coefficients satisfies $f(x)=f'(x)\times f'''(x)$. If $f(x)=0$ satisfies $x=1, 2, 3$ only, then the value of $f'(1)\times f'(2)\times f'(3)$ is equal to
(a) positive (b) negative
(c) 0 (d) inadequate data

17. A curve whose concavity is directly proportional to the logarithm of its x-coordinates at any of the curve, is given by
(a) $c_1\cdot x^2(2\log x-3)+c_2x+c_3$
(b) $c_1x^2(2\log x+3)+c_2x+c_3$
(c) $c_1x^2(2\log x)+c_2$
(d) None of the above

18. $f(x)=4\tan x-\tan^2 x+\tan^3 x, \forall\, x\neq n\pi+\dfrac{\pi}{2}, \forall\ n\in I$, then
(a) $f(x)$ is increasing for all $x\in R$
(b) $f(x)$ is decreasing for all $x\in R$
(c) $f(x)$ is increasing in its domain
(d) None of the above

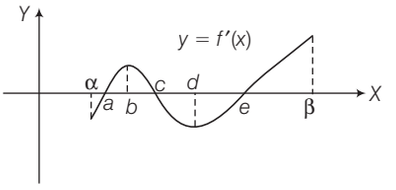
19. If $f(x)=\begin{cases}3+|x-k|, & \text{for } x\le k\\ a^2-2+\dfrac{\sin(x-k)}{x-k}, & \text{for } x>k\end{cases}$ has minimum at $x=k$, then
(a) $a\in R$ (b) $|a|<2$
(c) $|a|>2$ (d) $1<|a|<2$

20. Let f be a linear function with properties $f(1)\le f(2)$, $f(3)\ge f(4)$ and $f(5)=5$, then which of the following is true?
(a) $f(0)<0$ (b) $f(0)=0$
(c) $f(1)<f(0)<f(-1)$ (d) $f(0)=5$

21. If $P(x)$ is polynomial satisfying $P(x^2)=x^2\ P(x)$ and $P(0)=-2$, $P'(3/2)=0$ and $P(1)=0$.
The maximum value of $P(x)$ is
(a) $-\dfrac{1}{3}$ (b) $-\dfrac{1}{4}$
(c) $-\dfrac{1}{2}$ (d) None of these

22. If the curve $x^2=-4(y-a)$ does not intersect the curve $y=[x^2-x+1]$ (where [.] denotes the greatest integer function) in $\left[0, \dfrac{1+\sqrt{5}}{2}\right]$, then
(a) $\dfrac{1}{3}<a<1$ (b) $-1<a<1$
(c) $\dfrac{1}{4}<a<1$ (d) None of these

23. Analyze the following graph of $f'(x)$



which is incorrect about $f(x)$ for $\alpha<x<\beta$?
(a) only three extreme points
(b) two inflexion points
(c) $f'''(x)>0$ for $d<x<e$
(d) $x=e$ is the point of local maxima

24. Let $f(x)=x^2-2x$ and $g(x)=f(f(x)-1)+f(5-f(x))$, then
(a) $g(x)<0, \forall\ x\in R$
(b) $g(x)<0$, for some $x\in R$
(c) $g(x)\ge 0$, for some $x\in R$
(d) $g(x)\ge 0, \forall\ x\in R$

25. Let $f: N\to N$ be such that $f(n+1)>f(f(n))$ for all $n\in N$, then
(a) $f(x)=x^2-x+1$
(b) $f(x)=x-1$
(c) $f(x)=x^2+1$
(d) None of the above

26. The equation $|2ax-3|+|ax+1|+|5-ax|=\dfrac{1}{2}$ possesses
(a) infinite number of real solutions for some $a\in R$
(b) finitely many real solutions for some $a\in R$
(c) no real solutions for some $a\in R$
(d) no real solutions for all $a\in R$

27. If $\int_0^x 2x f^2(t)\,dt=\left(\int_0^x 2f(x-t)\,dt\right)^2$ for $f(1)=1$ and $f(x)$ is continuous function for $x>0$ and $\{a_n\}$ is a sequence such that $a_{n+1}=a_n+\sqrt{1+a_n^2}$ for $a=0$; if $f(x)$ is an increasing function, then $\lim\limits_{n\to\infty}\dfrac{a_k}{2^{n-1}}=$
(where $k=f(n^{\sqrt{2}-1})$) is
(a) $\pi/4$ (b) $4/\pi$ (c) π (d) $\pi/2$

28. A function f is defined by $f(x)=|x|^m|x-1|^n, \forall\ x\in R$. The maximum value of the function is $(m, n\in N)$
(a) 1 (b) $m^n\cdot n^m$
(c) $\dfrac{m^m\cdot n^n}{(m+n)^{m+n}}$ (d) $\dfrac{(mn)^{mn}}{(m+n)^{m+n}}$

Monotonicity, Maxima and Minima Exercise 2 : More than One Option Correct Type Questions

29. Which of the following is/are true?

(you may use $f(x)=\frac{\ln(\ln x)}{\ln x}$)

(a) $(\ln 2.1)^{\ln 2.2} > (\ln 2.2)^{\ln 2.1}$

(b) $(\ln 4)^{\ln 5} < (\ln 5)^{\ln 4}$

(c) $(\ln 30)^{\ln 31} > (\ln 31)^{\ln 30}$

(d) $(\ln 28)^{30} < (\ln 30)^{\ln 28}$

30. If $\lim_{x\to a} f(x) = \lim_{x\to a}[f(x)]$ (where $[\cdot]$ denotes the greatest integer function) and $f(x)$ is non-constant continuous function, then

(a) $\lim_{x\to a} f(x)$ is an integer

(b) $\lim_{x\to a} f(x)$ is non-integer

(c) $f(x)$ has local maximum at $x = a$

(d) $f(x)$ has local minimum at $x = a$

31. Let S be the set of real values of parameter λ for which the equation $f(x) = 2x^3 - 3(2+\lambda)x^2 + 12\lambda x$ has exactly one local maximum and exactly one local minimum. Then, S is a subset of

(a) $(-4, \infty)$ (b) $(-3, 3)$

(c) $(3, \infty)$ (d) $(-\infty, 0)$

32. $h(x) = 3f\left(\frac{x^2}{3}\right) + f(3-x^2),\ \forall\ x \in (-3,\ 4)$, where $f''(x) > 0,\ \forall\ x \in (-3,\ 4)$, then $h(x)$ is

(a) increasing in $\left(\frac{3}{2}, 4\right)$

(b) increasing in $\left(-\frac{3}{2}, 0\right)$

(c) decreasing in $\left(-3, -\frac{3}{2}\right)$

(d) decreasing in $\left(0, \frac{3}{2}\right)$

33. $f(x) = \tan^{-1}(\sin x + \cos x)$, then $f(x)$ is increasing in

(a) $\left(-\frac{\pi}{2}, \frac{\pi}{4}\right)$ (b) $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

(c) $\left(\frac{5\pi}{4}, \frac{3\pi}{2}\right)$ (d) $\left(-2\pi, -\frac{7\pi}{4}\right)$

34. If maximum and minimum values of the determinant

$$\begin{vmatrix} 1+\sin^2 x & \cos^2 x & \sin 2x \\ \sin^2 x & 1+\cos^2 x & \sin 2x \\ \sin^2 x & \cos^2 x & 1+\sin 2x \end{vmatrix}$$

are α and β, then

(a) $\alpha + \beta^{99} = 4$

(b) $\alpha^3 - \beta^{17} = 26$

(c) $(\alpha^{2n} - \beta^{2n})$ is always an even integer for $n \in N$

(d) a triangle can be drawn having it's sides as α, β and $\alpha - \beta$

35. Let $f(x) = \begin{cases} x^2 + 4x, & -3 \le x \le 0 \\ -\sin x, & 0 < x \le \frac{\pi}{2} \\ -\cos x - 1, & \frac{\pi}{2} < x \le \pi \end{cases}$. Then,

(a) $x = -2$ is the point of global minima

(b) $x = \pi$ is the point of global maxima

(c) $f(x)$ is non-differentiable at $x = \frac{\pi}{2}$

(d) $f(x)$ is discontinuous at $x = 0$

36. Let $f(x) = ab\sin x + b\sqrt{1-a^2}\cos x + c$, where $|a| < 1,\ b > 0$, then

(a) maximum value of $f(x)$ is b, if $c = 0$

(b) difference of maximum and minimum value of $f(x)$ is $2b$

(c) $f(x) = c$, if $x = -\cos^{-1} a$

(d) $f(x) = c$, if $x = \cos^{-1} a$

37. If $f(x) = \int_{x^m}^{x^n} \frac{dt}{\ln t},\ x > 0$ and $n > m$, then

(a) $f'(x) = \frac{x^{m-1}(x-1)}{\ln x}$

(b) $f(x)$ is decreasing for $x > 1$

(c) $f(x)$ is increasing in $(0, 1)$

(d) $f(x)$ is increasing for $x > 1$

38. $f(x) = \sqrt{x-1} + \sqrt{2-x}$ and $g(x) = x^2 + bx + c$ are two given functions such that $f(x)$ and $g(x)$ attain their maximum and minimum values respectively for same value of x, then

(a) $f(x)_{\max}$ at $x = \frac{1}{2}$ (b) $f(x)_{\max}$ at $x = \frac{3}{2}$

(c) $b = 3$ (d) $b = -3$

39. If $f(x) = a^{\{a^{|x|} \operatorname{sgn} x\}}$; $g(x) = a^{[a^{|x|} \operatorname{sgn} x]}$ for $a > 0, a \neq 1$ and $x \in R$, where {} and [·] denote the fractional part and integral part functions respectively, then which of the following statements can hold good for the function $h(x)$? where $(\ln a)\, h(x) = (\ln f(x) + \ln g(x))$

(a) h is even and increasing

(b) h is odd and decreasing

(c) h is even and decreasing

(d) h is odd and increasing

40. For the function $f(x) = \ln(1 - \ln x)$, which of the following do not hold good?

(a) increasing in (0, 1) and decreasing in $(1, e)$

(b) decreasing in (0, 1) and increasing in $(1, e)$

(c) $x = 1$ is the critical number for $f(x)$

(d) f has two asymptotes

41. The function $f(x) = \begin{cases} x + 2, & \text{if } x < -1 \\ x^2, & \text{if } -1 \le x < 1 \\ (x-2)^2, & \text{if } x \ge 1 \end{cases}$

(a) is continuous for all $x \in R$

(b) is continuous but not differentiable, $\forall\ x \in R$

(c) is such that $f'(x)$ change its sign exactly twice

(d) has two local maxima and two local minima

42. A function f is defined by $f(x) = \int_0^{\pi} \cos t \cos(x - t)\, dt$, $0 \le x \le 2\pi$, then which of the following hold(s) good?

(a) $f(x)$ is continuous but not differentiable in $(0, 2\pi)$

(b) Maximum value of f is π

(c) There exists atleast one $c \in (0, 2\pi)$ if $f'(c) = 0$

(d) Minimum value of f is $-\dfrac{\pi}{2}$

43. Let $f(x) = \dfrac{x-1}{x^2}$, then which of the following is correct?

(a) $f(x)$ has minima but no maxima

(b) $f(x)$ increases in the interval $(0, 2)$ and decreases in the interval $(-\infty, 0) \cup (2, \infty)$

(c) $f(x)$ can come down in $(-\infty, 0) \cup (2, 3)$

(d) $x = 3$ is the point of inflection

44. Let $f(x)$ be differentiable function on the interval $(-\infty, \infty)$ such that $f(1) = 5$ and $\lim\limits_{a \to x} \dfrac{af(x) - xf(a)}{a - x} = 2$, for all $x \in R$. Then, which of the following alternative(s) is/are correct?

(a) $f(x)$ has an inflection point

(b) $f'(x) = 3, \forall\ x \in R$

(c) $\int_0^2 f(x)\, dx = 10$

(d) Area bounded by $f(x)$ with coordinate axes is $(2/3)$

45. Let $f : (0, \infty) \to (0, \infty)$ be a derivable function and $F(x)$ is the primitive of $f(x)$ such that $2(F(x) - f(x)) = f^2(x)$ for any real positive x

(a) f is strictly increasing (b) $\lim\limits_{x \to \infty} \dfrac{f(x)}{x} = 1$

(c) f is strictly decreasing (d) f is non-monotonic

Monotonicity, Maxima and Minima Exercise 3 : Statements I and II Type Questions

Directions (Q. Nos. 46 to 54) *For the following questions, choose the correct answers from the codes (a), (b), (c) and (d) defined as follows.*

(a) Statement I is true, Statement II is also true, Statement II is the correct explanation of Statement I

(b) Statement I is true, Statement II is also true, Statement II is not the correct explanation of Statement I

(c) Statement I is true, Statement II is false

(d) Statement I is false, Statement II is true

46. **Statement I** The equation $3x^2 + 4ax + b = 0$ has atleast one root in (0, 1), if $3 + 4a = 0$.

Statement II $f(x) = 3x^2 + 4x + b$ is continuous and differentiable in (0, 1).

47. **Statement I** For the function $f(x) = \begin{cases} 15 - x, & x < 2 \\ 2x - 3, & x \ge 2 \end{cases}$ $x = 2$ has neither a maximum nor a minimum point.

Statement II $f'(x)$ does not exist at $x = 2$.

48. **Statement I** $\phi(x) = \int_0^x (3\sin t + 4\cos t)\, dt, x \in \left[\dfrac{\pi}{6}, \dfrac{\pi}{3}\right]$ $\phi(x)$ attains its maximum value at $x = \dfrac{\pi}{3}$.

Statement II $\phi(x) = \int_0^x (3\sin t + 4\cos t)\, dt, \phi(x)$ is increasing function in $\left[\dfrac{\pi}{6}, \dfrac{\pi}{3}\right]$.

49. Let $f(x)$ be a twice differentiable function in $[a, b]$, given that $f(x)$ and $f''(x)$ has same sign in $[a, b]$.

Statement I $f'(x) = 0$ has at the most one real root in $[a, b]$.

Statement II An increasing function can intersect the X-axis at the most once.

50. Let $u = \sqrt{c+1} - \sqrt{c}$ and $v = \sqrt{c} - \sqrt{c-1}$, $c > 1$ and let $f(x) = \ln(1+x)$, $\forall\ x \in (-1, \infty)$.

Statement I $f(u) > f(v)$, $\forall\ c > 1$ because

Statement II $f(x)$ is increasing function, hence for $u > v$, $f(u) > f(v)$.

51. Let $f(0) = 0$, $f\left(\frac{\pi}{2}\right) = 1$, $f\left(\frac{3\pi}{2}\right) = -1$ be a continuous and twice differentiable function.

Statement I $|f''(x)| \le 1$ for atleast one $x \in \left(0, \frac{3\pi}{2}\right)$ because

Statement II According to Rolle's theorem, if $y = g(x)$ is continuous and differentiable, $\forall x \in [a, b]$ and $g(a) = g(b)$, then there exists atleast one c such that $g'(c) = 0$.

52. **Statement I** For any ΔABC

$$\sin\left(\frac{A+B+C}{3}\right) \ge \frac{\sin A + \sin B + \sin C}{3}$$

Statement II $y = \sin x$ is concave downward for $x \in (0,\ \pi]$.

53. **Statement I** If $f(x) = [x](\sin x + \cos x - 1)$ (where $[\cdot]$ denotes the greatest integer function), then $f'(x) = [x](\cos x - \sin x)$ for any $x \in$ integer.

Statement II $f'(x)$ does not exist for any $x \in$ integer.

54. $f(x)$ is a polynomial of degree 3 passing through origin having local extrema at $x = \pm a$.

Statement I Ratio of areas in which $f(x)$ cuts the circle $x^2 + y^2 = 36$ is 1 : 1.

Statement II Both $y = f(x)$ and the circle are symmetric about origin.

Monotonicity, Maxima and Minima Exercise 4 : Passage Based Questions

Passage I

(Q. Nos. 55 to 57)

Let $f(x) = \frac{1}{1+x^2}$, let m be the slope, a be the x-intercept and b be the y-intercept of a tangent to $y = f(x)$.

55. Abscissa of the point of contact of the tangent for which m is greatest, is

(a) $\frac{1}{\sqrt{3}}$ (b) 1

(c) -1 (d) $-\frac{1}{\sqrt{3}}$

56. Value of b for the tangent drawn to the curve $y = f(x)$ whose slope is greatest, is

(a) $\frac{9}{8}$ (b) $\frac{3}{8}$

(c) $\frac{1}{8}$ (d) $\frac{5}{8}$

57. Value of a for the tangent drawn to the curve $y = f(x)$ whose slope is greatest, is

(a) $-\sqrt{3}$

(b) 1

(c) -1

(d) $\sqrt{3}$

Passage II

(Q. Nos. 58 to 60)

Consider the function $f(x) = \max\{x^2, (1-x)^2, 2x(1-x)\}$, $x \in [0, 1]$

58. The interval in which $f(x)$ is increasing, is

(a) $\left(\frac{1}{3}, \frac{2}{3}\right)$ (b) $\left(\frac{1}{3}, \frac{1}{2}\right)$

(c) $\left(\frac{1}{3}, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \frac{2}{3}\right)$ (d) $\left(\frac{1}{3}, \frac{1}{2}\right) \cup \left(\frac{2}{3}, 1\right)$

59. Let RMVT is applicable for $f(x)$ on $(a,\ b)$, then $a+b+c$ is (where c is the point such that $f'(c) = 0$)

(a) $\frac{2}{3}$ (b) $\frac{1}{3}$

(c) $\frac{1}{2}$ (d) $\frac{3}{2}$

60. The interval in which $f(x)$ is decreasing, is

(a) $\left(\frac{1}{3}, \frac{2}{3}\right)$ (b) $\left(\frac{1}{3}, \frac{1}{2}\right)$

(c) $\left(0, \frac{1}{3}\right) \cup \left(\frac{1}{2}, \frac{2}{3}\right)$ (d) $\left(\frac{1}{3}, \frac{1}{2}\right) \cup \left(\frac{2}{3}, 1\right)$

Passage III

(Q. Nos. 61 to 63)

$f(x)$, $g(x)$, $h(x)$ all are continuous and differentiable functions in $[a, b]$ also $a < c < b$ and $f(a) = g(a) = h(a)$. Point of intersection of the tangent at $x = c$ with chord joining $x = a$ and $x = b$ is on the left of c in $y = f(x)$ and on the right in $y = h(x)$. And tangent at $x = c$ is parallel to the chord in case of $y = g(x)$.
Now, answer the following questions.

61. If $f'(x) > g'(x) > h'(x)$, then

(a) $f(b) < g(b) < h(b)$ (b) $f(b) > g(b) > h(b)$
(c) $f(b) \le g(b) \le h(b)$ (d) $f(b) \ge g(b) \ge h(b)$

62. If $f(b) = g(b) = h(b)$, then

(a) $f'(c) = g'(c) = h'(c)$ (b) $f'(c) > g'(c) > h'(c)$
(c) $f'(c) < g'(c) < h'(c)$ (d) None of these

63. If $c = \dfrac{a+b}{2}$ for each b, then

(a) $g(x) = Ax^2 + Bx + C$
(b) $g(x) = \log x$
(c) $g(x) = \sin x$
(d) $g(x) = e^x$

Passage IV

(Q. Nos. 64 to 66)

In the non-decreasing sequence of odd integers $(a_1, a_2, a_3, \ldots) = \{1, 3, 3, 3, 5, 5, 5, 5, 5, \ldots\}$ each positive odd integer k appears k times. It is a fact that there are integers b, c and d such that for all positive integer n, $a_n = b[\sqrt{n + c}] + d$ (where $[.]$ denotes greatest integer function).

64. The possible value of $b + c + d$ is

(a) 0 (b) 1 (c) 2 (d) 4

65. The possible value of $\dfrac{b-2d}{8}$ is

(a) 0 (b) 1
(c) 2 (d) 4

66. The possible value of $\dfrac{c+d}{2b}$ is

(a) 0 (b) 1
(c) 2 (d) 4

Passage V

(Q. Nos. 67 to 69)

Let $g(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ and $f(x) = \sqrt{g(x)}$, $f(x)$ have its non-zero local minimum and maximum values at -3 and 3, respectively. If $a_3 \in$ the domain of the function

$$h(x) = \sin^{-1}\left(\frac{1 + x^2}{2x}\right)$$

67. The value of $a_1 + a_2$ is equal to

(a) 30 (b) -30
(c) 27 (d) -27

68. The value of a_0 is

(a) equal to 50
(b) greater than 54
(c) less than 54
(d) less than 50

69. $f(-10)$ is defined for

(a) $a_0 > 830$
(b) $a_0 < 830$
(c) $a_0 = 830$
(d) None of the above

Passage VI

(Q. Nos. 70 to 74)

If $f : D \to R$, $f(x) = \dfrac{x^2 + bx + c}{x^2 + b_1x + c_1}$, where α, β are the roots of the equation $x^2 + bx + c = 0$ and α_1, β_1 are the roots of $x^2 + b_1x + c_1 = 0$. Now, answer the following questions for $f(x)$. A combination of graphical and analytical approach may be helpful in solving these problems. (If α_1 and β_1 are real, then $f(x)$ has vertical asymptote at $x = \alpha_1, \beta_1$). Then,

70. If $\alpha_1 < \alpha < \beta_1 < \beta$, then

(a) $f(x)$ is increasing in (α_1, β_1)
(b) $f(x)$ is decreasing in (α, β)
(c) $f(x)$ is decreasing in (β_1, β)
(d) $f(x)$ is decreasing in $(-\infty, \alpha)$

71. If $\alpha_1 < \beta_1 < \alpha < \beta$, then

(a) $f(x)$ has a maxima in $[\alpha_1, \beta_1]$ and a minima is $[\alpha, \beta]$
(b) $f(x)$ has a minima in (α_1, β_1) and a maxima in (α, β)
(c) $f'(x) > 0$ where ever defined
(d) $f'(x) < 0$ where ever defined

72. If the equations $x^2 + bx + c = 0$ and $x^2 + b_1x + c_1 = 0$ do not have real roots, then

(a) $f'(x) = 0$ has real and distinct roots
(b) $f'(x) = 0$ has real and equal roots
(c) $f'(x) = 0$ has imaginary roots
(d) nothing can be said

73. In the above problem, $\lim\limits_{x \to \infty} [f(x)] \cdot \lim\limits_{x \to \infty} [f(x)]$ (where $[.]$ denotes the greatest integer function) is equal to

(a) 1
(b) 0
(c) -1
(d) does not exist

74. In the last problem, if $b > b_1$, then

(a) x-coordinate of point of minima is greater than the x-coordinate of point of maxima
(b) x-coordinate of point of minima is less than x-coordinate of point of maxima
(c) it also depends upon c and c_1
(d) nothing can be said

Passage VII

(Q. Nos. 75 to 77)

Consider the function $f(x) = \dfrac{x^2}{x^2 - 1}$

75. The interval in which f is increasing is

(a) $(-1, 1)$
(b) $(-\infty, -1) \cup (-1, 0)$
(c) $(-\infty, \infty) - \{-1, 1\}$
(d) $(0, 1) \cup (1, \infty)$

76. If f is defined from $R - \{-1, 1\} \to R$, then f is

(a) injective but not surjective
(b) surjective but not inective
(c) injective as well as surjective
(d) neither injective nor surjective

77. f has

(a) local maxima but not local minima
(b) local minima but not local maxima
(c) both local maxima and local minima
(d) neither local maxima nor local minima

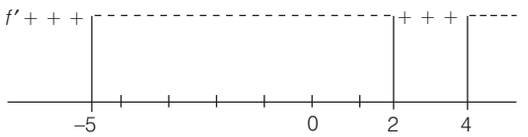
Passage VIII

(Q. Nos. 78 to 80)

Suppose you do not know the function $f(x)$*, however some information about* $f(x)$ *is listed below.*

Read the following carefully before attempting the questions

(i) $f(x)$ *is continuous and defined for all real numbers*
(ii) $f'(-5) = 0$, $f'(2)$ *is not defined and* $f'(4) = 0$
(iii) $(-5, 12)$ is a point which lies on the graph of $f(x)$
(iv) $f''(2)$ *is undefined, but* $f''(x)$ *is negative everywhere else*
(v) *The signs of* $f'(x)$ *is given below*



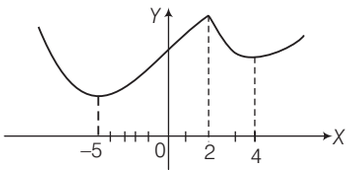
78. On the possible graph of $y = f(x)$, we have

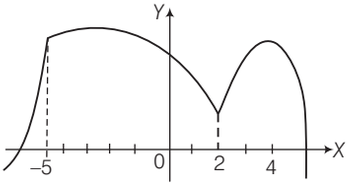
(a) $x = -5$ is a point of relative minima
(b) $x = 2$ is a point of relative maxima
(c) $x = 4$ is a point of relative minima
(d) graph of $y = f(x)$ must have a geometrically sharp corner

79. From the possible graph of $y = f(x)$, we can say that

(a) there is exactly one point of inflection on the curve
(b) $f(x)$ increases on $-5 < x < 2$ and $x > 4$ and decreases on $-\infty < x < -5$ and $2 < x < 4$
(c) the curve is always concave down
(d) curve always concave up

80. Possible graph of $y = f(x)$ is

(a)


(b)


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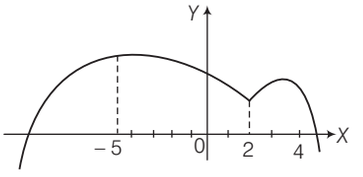
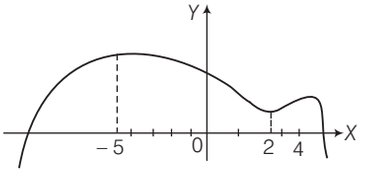


(d)


Passage IX

(Q. Nos. 81 to 83)

Let $f(x) = e^{(p+1)x} - e^x$ *for real number* $p > 0$*, then*

81. The value of $x = s_p$ for which $f(x)$ is minimum, is

(a) $\dfrac{-\log_e(p+1)}{p}$ (b) $-\log_e(p+1)$

(c) $-\log_e p$ (d) $\log_e\left(\dfrac{p+1}{p}\right)$

82. Let $g(t) = \int_t^{t+1} f(x)\, e^{t-x}\, dx$. The value of $t = t_p$, for which $g(t)$ is minimum, is

(a) $-\log_e \dfrac{(e^p - 1)}{p}$ (b) $-\dfrac{1}{p}\log_e\left(\dfrac{e^p - 1}{p}\right)$

(c) $-\dfrac{1}{p}\log_e\left(\dfrac{(p+1)(e^p - 1)}{p}\right)$ (d) $-\log_e((p+1)(e^p - 1))$

83. Use the fact that $1 + \dfrac{p}{2} \le \dfrac{e^p - 1}{p} \le 1 + \dfrac{p}{2} + p^2$ $(0 < p \le 1)$, the value of $\lim\limits_{p \to 0^+} (s_p - t_p)$ is

(a) 0 (b) $\dfrac{1}{2}$ (c) 1 (d) non-existent

Passage X

(Q. Nos. 84 to 86)

Consider f, g and h be three real valued function defined on R.

Let $f(x)=\sin 3x+\cos x$, $g(x)=\cos 3x+\sin x$ *and* $h(x)=f^2(x)+g^2(x)$.

84. The length of a longest interval in which the function $g=h(x)$ is increasing, is

(a) $\pi/8$ (b) $\pi/4$ (c) $\pi/6$ (d) $\pi/2$

85. The general solution of the equation $h(x)=4$, is

(a) $(4n+1)\pi/8$ (b) $(8n+1)\pi/8$
(c) $(2n+1)\pi/4$ (d) $(7n+1)\pi/4$

86. Number of point(s) where the graphs of the two function, $y=f(x)$ and $y=g(x)$ intersects in $[0,\pi]$, is

(a) 2 (b) 3
(c) 4 (d) 5

Passage XI

(Q. Nos. 87 to 89)

Consider f, g and h be three real valued functions defined on R.

Let $f(x)=\begin{cases}-1, & x<0\\ 0, & x=0\\ 1, & x>0\end{cases}$, $g(x)=x(1-x^2)$ *and* $h(x)$ *be such that* $h''(x)=6x-4$.

Also, $h(x)$ *has local minimum value* 5 *at* $x=1$.

87. The equation of tangent at $m(2,7)$ to the curve $y=h(x)$, is

(a) $5x+y=17$ (b) $x+5y=37$
(c) $x-5y+33=0$ (d) $5x-y=3$

88. The area bounded by $y=h(x)$, $y=g(f(x))$ between $x=0$ and $x=2$ equals

(a) 23/2 (b) 20/3 (c) 32/3 (d) 40/3

89. Range of function $\sin^{-1}\sqrt{(fog(x))}$ is

(a) $(0,\pi/2)$ (b) $\{0,\pi/2\}$
(c) $\{-\pi/2,0,\pi/2\}$ (d) $\{\pi/2\}$

Passage XII

(Q. Nos. 90 to 92)

Consider f, g and h be three real valued differentiable functions defined on R.

Let $g(x)=x^3+g''(1)x^2+(3g'(1)-g''(1)-1)x+3g'(1)$

$f(x)=x\,g(x)-12x+1$

and $f(x)=(h(x))^2$, *where* $h(0)=1$.

90. The function $y=f(x)$ has

(a) Exactly one local minima and no local maxima
(b) Exactly one local maxima and no local minima
(c) Exactly one local maxima and two local minima
(d) Exactly two local maxima and one local minima

91. Which of the following is/are true for the function $y=g(x)$?

(a) $g(x)$ monotonically decreases inEREDEDDD $\left(-\infty,2-\frac{1}{\sqrt{3}}\right)\cup\left(2+\frac{1}{\sqrt{3}},\infty\right)$

(b) $g(x)$ monotonically increases in $\left[2-\frac{1}{\sqrt{3}},2+\frac{1}{\sqrt{3}}\right)$

(c) There exists exactly one tangent to $y=g(x)$ which is parallel to the chord joining the points $(1,\ g(1))$ and $(3,\ g(3))$

(d) There exists exactly two distinct lagrange's mean value in $(0,\ 4)$ for the function $y=g(x)$

92. Which one of the following does not hold good for $y=h(x)$?

(a) Exactly one critical point
(b) No point of inflexion
(c) Exactly one real zero in $(0,3)$
(d) Exactly one tangent parallel to X-axis

Monotonicity, Maxima and Minima Exercise 5 : Matching Type Questions

93. Match the following :

	Column I	Column II
(A)	The maximum value attained by $y=10-\lvert x-10\rvert$, $-1\le x\le 3$ is	(p) 3
(B)	If $P(t^2,2t)$, $t\in[0,2]$ is an arbitrary point on parabola $y^2=4x$, Q is foot of perpendicular from focus S on the tangent at P, then maximum area of ΔPQS is	(q) $\frac{1}{3}$
(C)	If $a+b=1$, $a,b>0$, then maximum value of $\sqrt{\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)}$ is	(r) 5
(D)	For real values of x, the greatest and least value of expression $\frac{x+2}{2x^2+3x+6}$ is	(s) $-\frac{1}{13}$

94. Match the entries of the following two columns.

	Column I	Column II
(A)	The least value of the function $f(x) = 2\cdot 3^{3x} - 3^{2x}\cdot 4 + 2\cdot 3^{x}$ in $[-1, 1]$ is	(p) 5
(B)	The minimum value of the polynomial $f(x) = (x-1)x(x+1)$ is	(q) -1
(C)	The value of the polynWmial $\int_{-1}^{3}(\|x-2\|-[x])dx$ (where $[\cdot]$ denotes the greatest integer function) is	(r) 3
(D)	If period of the function $f(x) = \sin 36x \tan 42x$ is p, then $\frac{18p}{\pi}$ equals	(s) 0

95. Match the following :

	Column I	Column II
(A)	$f(x) = \int_{0}^{2x^2}(t-2)(t+1)^3(t-3)^2\,dt$ is/has in $(-1, 1)$	(p) local maxima
(B)	$f(x)=\begin{cases}\sin\left(\frac{\pi x}{4}\right), x\le 2\\ 9-4x, \quad x>2\end{cases}$ is/has in $(0,2)$	(q) local minima
(C)	$f(x) = \{2x\}$ denotes fractional part of x is/has in $(0, 1)$	(r) continuous
(D)	$f(x)=\begin{cases}\|\|x-2\|-2\|, & x<2\\ [x], & x\ge 2\end{cases}$ (where $[\cdot]$ denotes the greatest integer function), then $f(x)$ is/has in $(-1, 4)$	(s) non-differentiable

96. Match the statements of Column I with the values of Column II.

	Column I	Column II
(A)	$f(x) = \cos\pi x + 10x + 3x^2 + x^3$, $-2 \le x \le 3$. The absolute minimum value of $f(x)$ is	(p) 3/4
(B)	If $x \in [-1, 1]$, then the minimum value of $f(x) = x^2 + x + 1$ is	(q) 2
(C)	Let $f(x) = \frac{4}{3}x^3 - 4x$, $0 \le x \le 2$. Then, the global minimum value of the function is	(r) -15
(D)	Let $f(x) = 6 - 12x + 9x^2 - 2x^3$, $1 \le x \le 4$. Then, the absolute maximum value of $f(x)$ in the interval is	(s) $-8/3$

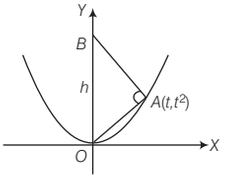
Monotonicity, Maxima and Minima Exercise 6 : Single Integer Answer Type Questions

97. A particular substance is being cooled by a streem of cold air (temperature of the air is constant and is 5°C) where rate of cooling is directly proportional to square of difference of temperature of the substance and the air. If the substance is cooled from 40°C to 30°C in 15 min and temperature after 1 hour is T°C, then find the value of $[T]/2$, where $[.]$ represents the greatest integer function.

98. The minimum value of $\dfrac{\tan\left(x+\frac{\pi}{6}\right)}{\tan x}$ is ..... .

99. The figure shows a right triangle with its hypotenuse OB along the Y-axis and its vertex A on the parabola $y = x^2$. Let h represents the length of the hypotenuse which depends on the x-coordinate of the point A.

The value of $\lim\limits_{x\to 0}(h)$ equals ..... .

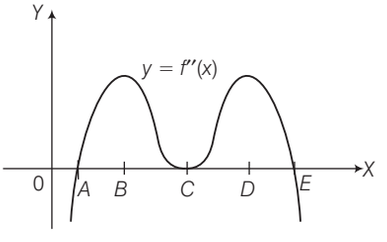


100. Number of positive integral values of a for which the curve $y = a^x$ intersects the line $y = x$ is ..... .

101. The least value of a for which the equation, $\dfrac{4}{\sin x} + \dfrac{1}{1-\sin x} = a$ has atleast one solution in the interval $(0, \pi/2)$ is ..... .

102. Let $f(x)=\begin{cases}x^{3/5} & \text{if } x\le 1\\ -(x-2)^3 & \text{if } x>1\end{cases}$, then the number of critical points on the graph of the function are ..... .

103. The graph of $y = f''(x)$ for a function f is shown. Number of points of inflection for $y = f(x)$ is ..... .



104. Number of critical points of the function, $f(x)=\frac{2}{3}\sqrt{x^3}-\frac{x}{2}+\int_1^x\left(\frac{1}{2}+\frac{1}{2}\cos 2t-\sqrt{t}\right)dt$ which lie in the interval $[-2\pi, 2\pi]$ is ..... .

105. Let $f(x)$ and $g(x)$ be two continuous functions defined from $R\to R$, such that $f(x_1)>f(x_2)$ and $g(x_1)<g(x_2), \forall\ x_1>x_2$, then the least integral value of α for which $f(g(\alpha^2-2\alpha))>f(g(3\alpha-4))$ is ..... .

106. If the function $f(x)=\frac{t+3x-x^2}{x-4}$, where t is a parameter, has a minimum and a maximum, then the greatest value of t is ..... .

107. If $f(x)=(x-a)(x-b)$ for $a, b\in R$, then minimum number of roots of equation $\pi(f'(x))^2\cos(\pi f(x))+\sin(\pi f(x))\cdot f''(x)=0$ in (α, β), where $f(\alpha)=3=f(\beta)$ is ..... . (here, $\alpha<a<b<\beta$)

108. If absolute maximum value of $f(x)=\frac{1}{|x-4|+1}+\frac{1}{|x+8|+1}$ is $\frac{p}{q}$, (p, q are coprime) the $(p-q)$ is ..... .

Monotonicity, Maxima and Minima Exercise 7 : Questions Asked in Previous 10 Years' Exams

(i) JEE Advanced & IIT-JEE

109. The least value of $\alpha\in R$ for which $4\alpha x^2+\frac{1}{x}\geq 1$, for all $x>0$, is **[One Correct Option 2016 Adv.]**

(a) $\frac{1}{64}$ (b) $\frac{1}{32}$

(c) $\frac{1}{27}$ (d) $\frac{1}{25}$

110. The number of points in $(-\infty, \infty)$ for which $x^2-x\ \sin x-\cos x=0$, is **[One Correct Option, 2013 Adv.]**

(a) 6 (b) 4

(c) 2 (d) 0

111. Let $f: R\to(0, \infty)$ and $g: R\to R$ be twice differentiable functions such that f'' and g'' are continuous funcitons of R. Suppose $f'(2)=g(2)=0, f''(2)\neq 0$ and $g'(2)\neq 0$. If $\lim_{x\to 2}\frac{f(x)g(x)}{f'(x)g'(x)}=1$, then **[More than One Correct option, 2016 Adv.]**

(a) f has a local minimum at $x=2$

(b) f has a local maximum at $x=2$

(c) $f''(2)>f(2)$

(d) $f(x)-f''(x)=0$ for atleast one $x\in R$

112. If $f:(0, \infty)\to R$ be given by $f(x)=\int_{1/x}^{x}e^{-\left(t+\frac{1}{t}\right)}\frac{dt}{t}$.

Then, **[More than One Correct Option, 2014 Adv.]**

(a) $f(x)$ is monotonically increasing on $[1, \infty)$

(b) $f(x)$ is monotonically decreasing on $[0,1)$

(c) $f(x)+f\left(\frac{1}{x}\right)=0, \forall x\in(0, \infty)$

(d) $f(2^x)$ is an odd function of x on R

113. The function $f(x)=2|x|+|x+2|-||x+2|-2|x||$ has a local minimum or a local maximum at x is equal to **[More than One Correct Option, 2013 Adv.]**

(a) -2 (b) $\frac{-2}{3}$

(c) 2 (d) 2/3

114. A rectangular sheet of fixed perimeter with sides having their lengths in the ratio 8 : 15 is converted into an open rectangular box by folding after removing squares of equal area from all four corners. If the total area of removed squares is 100, the resulting box has maximum volume. The lengths of the sides of the rectangular sheet are **[More than One Correct Option, 2013 Adv.]**

(a) 24 (b) 32 (c) 45 (d) 60

115. A vertical line passing through the point $(h, 0)$ intersects the ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$ at the points P and Q. If the tangents to the ellipse at P and Q meet at the point R. If $\Delta(h)$ = area of the ΔPQR, $\Delta_1=\max_{1/2\leq h\leq 1}\Delta(h)$ and $\Delta_2=\min_{1/2\leq h\leq 1}\Delta(h)$, then $\frac{8}{\sqrt{5}}\Delta_1-8\Delta_2$ is equal to **[Integer Type Question, 2013 Adv.]**

116. Let f, g and h be real-valued functions defined on the interval [0, 1] by $f(x)=e^{x^2}+e^{-x^2}$, $g(x)=xe^{x^2}+e^{-x^2}$ and $h(x)=x^2e^{x^2}+e^{-x^2}$. If a, b and c denote respectively, the absolute maximum of f, g and h on [0, 1], then **[One Correct Option, 2010]**

(a) $a=b$ and $c\neq b$ (b) $a=c$ and $a\neq b$

(c) $a\neq b$ and $c\neq b$ (d) $a=b=c$

117. The total number of local maxima and local minima of the function $f(x)=\begin{cases}(2+x)^3, & -3<x\le -1\\ x^{\frac{2}{3}}, & -1<x<2\end{cases}$ is **[One Correct Option, 2008]**

(a) 0 (b) 1 (c) 2 (d) 3

118. If the function $g:(-\infty,\infty)\to\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ is given by $g(u)=2\tan^{-1}(e^u)-\frac{\pi}{2}$. Then, g is **[One Correct Option, 2008)**

(a) even and is strictly increasing in $(0,\infty)$
(b) odd and is strictly decreasing in $(-\infty,\infty)$
(c) odd and is strictly increasing in $(-\infty,\infty)$
(d) neither even nor odd but is strictly increasing in $(-\infty,\infty)$

119. The second degree polynomial $f(x)$, satisfying $f(0)=0$, $f(1)=1$, $f'(x)>0\ \forall x\in(0,1)$ **[One Correct Option, 2005]**

(a) $f(x)=\phi$
(b) $f(x)=ax+(1-a)x^2, \forall\ a\in(0,\infty)$
(c) $f(x)=ax+(1-a)x^2, a\in(0,2)$
(d) No such polynomial

120. If $f(x)=x^3+bx^2+cx+d$ and $0<b^2<c$, then in $(-\infty,\infty)$ **[One Correct Option, 2004]**

(a) $f(x)$ is strictly increasing function
(b) $f(x)$ has a local maxima
(c) $f(x)$ is strictly decreasing function
(d) $f(x)$ is bounded

121. If $f(x)=x^2+2bx+2c^2$ and $g(x)=-x^2-2cx+b^2$, such that $\min f(x)>\max g(x)$, then the relation between b and c, is **[One Correct Option, 2003]**

(a) No real value of b and c (b) $0<c<b\sqrt{2}$
(c) $|c|<|b|\sqrt{2}$ (d) $|c|>|b|\sqrt{2}$

122. The length of a longest interval in which the function $3\sin x-4\sin^3 x$ is increasing, is **[One Correct Option, 2002]**

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{2}$ (d) π

123. The maximum value of $(\cos\alpha_1)\cdot(\cos\alpha_2)\cdot\ldots\cdot(\cos\alpha_n)$, under the restrictions $0\le\alpha_1,\alpha_2\ldots,\alpha_n\le\frac{\pi}{2}$ and $(\cot\alpha_1)\cdot(\cot\alpha_2)\cdot\ldots\cdot(\cot\alpha_n)=1$ is **[One Correct Option, 2001]**

(a) $\frac{1}{2^{n/2}}$ (b) $\frac{1}{2^n}$ (c) $\frac{1}{2n}$ (d) 1

124. If $f(x)=\begin{cases}e^x, & 0\le x\le 1\\ 2-e^{x-1}, & 1<x\le 2\\ x-e, & 2<x\le 3\end{cases}$ and $g(x)=\int_0^x f(t)\,dt$, $x\in[1,3]$, then **[More than One Correct Option, 2006]**

(a) $g(x)$ has local maxima at $x=1+\log_e 2$ and local minima at $x=e$
(b) $f(x)$ has local maxima at $x=1$ and local minima at $x=2$
(c) $g(x)$ has no local minima
(d) $f(x)$ has no local maxima

125. If $f(x)$ is a cubic polynomial which has local maximum at $x=-1$. If $f(2)=18$, $f(1)=-1$ and $f'(x)$ has local minimum at $x=0$, then **[More than One Correct Option, 2006]**

(a) the distance between $(-1, 2)$ and $(a, f(a))$, where $x=a$ is the point of local minima, is $2\sqrt{5}$
(b) $f(x)$ is increasing for $x\in[1,2\sqrt{5}]$
(c) $f(x)$ has local minima at $x=1$
(d) the value of $f(0)=5$

126. Consider the function $f:(-\infty,\infty)\to(-\infty,\infty)$ defined by $f(x)=\frac{x^2-ax+1}{x^2+ax+1}; 0<a<2$. Which of the following is true ? **[Passage Based Question, 2008]**

(a) $(2+a)^2 f''(1)+(2-a)^2 f''(-1)=0$
(b) $(2-a)^2 f''(1)-(2+a)^2 f''(-1)=0$
(c) $f'(1)\,f'(-1)=(2-a)^2$
(d) $f'(1)\,f'(-1)=-(2+a)^2$

127. Which of the following is true? **[Passage Based Question, 2008]**

(a) $f(x)$ is decreasing on $(-1, 1)$ and has a local minimum at $x=1$
(b) $f(x)$ is increasing on $(-1, 1)$ and has a local maximum at $x=1$
(c) $f(x)$ is increasing on $(-1, 1)$ but has neither a local maximum nor a local minimum at $x=1$
(d) $f(x)$ is decreasing on $(-1, 1)$ but has neither a local maximum nor a local minimum at $x=1$

128. Let $g(x)=\int_0^{e^x}\frac{f'(t)}{1+t^2}dt$. Which of the following is true? **[Passage Based Question, 2008]**

(a) $g'(x)$ is positive on $(-\infty, 0)$ and negative on $(0,\infty)$
(b) $g'(x)$ is negative on $(-\infty, 0)$ and positive on $(0,\infty)$
(c) $g'(x)$ changes sign on both $(-\infty, 0)$ and $(0,\infty)$
(d) $g'(x)$ does not change sign $(-\infty,\infty)$

129. For the circle $x^2+y^2=r^2$, find the value of r for which the area enclosed by the tangents drawn from the point $P(6, 8)$ to the circle and the chord of contact is maximum. **[Subjective Type Question, 2003]**

130. Find a point on the curve $x^2+2y^2=6$ whose distance from the line $x+y=7$, is minimum. **[Subjective Type Question, 2003]**

131. Let $f:R\to R$ be defined as $f(x)=|x|+|x^2-1|$. Total number of points at which f attains either a local maximum or a local minimum is **[Integer Type Question, 2012]**

132. Let $p(x)$ be a real polynomial of least degree which has a local maximum at $x=1$ and a local minimum at $x=3$. If $p(1)=6$ and $p(3)=2$, then $p'(0)$ is equal to **[Integer Type Question, 2012]**

133. Let f be a function defined on R (the set of all real numbers) such that $f'(x) = 2010(x-2009)(x-2010)^2 (x-2011)^3 (x-2012)^4, \forall x \in R$. If g is a function defined on R with values in the interval $(0, \infty)$ such that $f(x) = \ln(g(x)), \forall x \in R$, then the number of points in R at which g has a local maximum is ..... .

[Integer Type Question, 2010]

134. The maximum value of the expression $\frac{1}{\sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta}$ is ..... .

[Integer Type Question, 2010]

135. The maximum value of the function $f(x) = 2x^3 - 15x^2 + 36x - 48$ on the set $A = \{x \mid x^2 + 20 \le 9x\}$ is ... .

[Integer Type Question, 2009]

(ii) JEE Main & AIEEE

136. A wire of length 2 units is cut into two parts which are bent respectively to form a square of side = x units and a circle of radius = r units. If the sum of the areas of the square and the circle so formed is minimum, then

[2016 JEE Main]

(a) $2x = (\pi + 4)r$ (b) $(4 - \pi)x = \pi r$

(c) $x = 2r$ (d) $2x = r$

137. If $x = -1$ and $x = 2$ are extreme points of $f(x) = \alpha \log|x| + \beta x^2 + x$, then **[2014 JEE Main]**

(a) $\alpha = -6, \beta = \frac{1}{2}$ (b) $\alpha = -6, \beta = -\frac{1}{2}$

(c) $\alpha = 2, \beta = -\frac{1}{2}$ (d) $\alpha = 2, \beta = \frac{1}{2}$

138. Let $a, b \in R$ be such that the function f given by $f(x) = \log|x| + bx^2 + ax, x \neq 0$ has extreme values at $x = -1$ and $x = 2$.

Statement I f has local maximum at $x = -1$ and at $x = 2$.

Statement II $a = \frac{1}{2}$ and $b = \frac{-1}{4}$ **[2012 AIEEE]**

(a) Statement I is false, Statement II is true

(b) Statement I is true, Statement II is true; Statement II is a correct explanation of Statement I

(c) Statement I is true, Statement II is true; Statement II is not a correct explanation of Statement I

(d) Statement I is true, Statement II is false

Answers

Exercise for Session 1

1. (d) 2. (a) 3. (c) 4. (a) 5. (b) 6. (c) 7. (b)

Exercise for Session 2

1. $-5, 0, \frac{3}{5}$ 2. $0, \frac{1}{5}$ 3. No critical points 4. (c)

Exercise for Session 3

4. (a) 5. (b)

Exercise for Session 4

1. (b) 2. (d) 3. (a) 4. (a) 5. (a)
6. (c, d) 7. (a,c) 8. (d) 9. (a) 10. (c)
11. (c) 12. (d) 13. (b) 14. (d) 15. (a)

Exercise for Session 5

1. (d) 2. (a) 3. (b) 4. (a, c) 5. (c) 6. (d)
7. $-2 < a < 2$

Chapter Exercises

1. (c) 2. (c) 3. (a) 4. (a) 5. (c)
6. (b) 7. (a) 8. (d) 9. (d)
10. (a) 11. (c) 12. (a) 13. (d) 14. (b)
15. (a) 16. (c) 17. (a) 18. (c) 19. (c)
20. (d) 21. (b) 22. (c) 23. (c) 24. (d)
25. (d) 26. (d) 27. (b) 28. (c) 29. (b,c)
30. (a,d) 31. (c,d) 32. (a,b,c,d)
33. (a,b,c,d) 34. (a,b,c) 35. (a,b,c)
36. (a,b,c) 37. (c,d) 38. (b,d)
39. (b,d) 40. (a,b,c) 41. (a,b,d)
42. (a,b) 43. (b,c,d) 44. (b,c,d)
45. (a,b) 46.(d) 47. (d) 48. (a) 49. (a)
50. (d) 51. (a) 52. (b) 53. (b) 54. (a)
55. (d) 56. (a) 57. (a) 58. (d) 59. (d)
60. (c) 61. (b) 62. (c) 63. (a) 64. (c)
65. (a) 66. (a) 67. (c) 68. (b) 69. (a)
70. (a) 71. (a) 72. (a) 73. (b) 74. (b)
75. (b) 76. (d) 77. (a) 78. (d) 79. (c)
80. (c) 81. (a) 82. (c) 83. (b) 84. (b)
85. (a) 86. (c) 87. (d) 88. (c) 89. (b)
90. (c) 91. (d) 92. (c)
93. (A) $\to$ (p), (B) $\to$ (r), (C) $\to$ (p), (D) $\to$ (q, s)
94. (A) $\to$ (s), (B) $\to$ (q), (C) $\to$ (r), (D) $\to$ (r)
95. (A) $\to$ (p, r), (B) $\to$ (p, r, s), (C) $\to$ (p, s), (D) $\to$ (q, s)
96. (A) $\to$ (r), (B) $\to$ (p), (C) $\to$ (s), (D) $\to$ (q)
97. (9) 98. (3) 99. (1) 100. (1) 101. (9)
102. (3) 103. (2) 104. (4) 105. (2) 106. (3)
107. (4) 108. (1) 109. (c) 110. (c) 111. (a,d) 112. (a, c, d)
113. (a,b) 114. (a,c) 115. (9) 116. (d) 117. (a) 118. (c)
119. (c) 120. (a) 121. (d) 122. (a) 123. (a) 124. (a,b)
125. (b,c) 126. (a) 127. (d) 128. (b) 129. (5) 130. (2,1)
131. (5) 132. (9) 133. (1) 134. (2) 135. (7) 136. (c)
137. (c) 138. (c)

Solutions

1. Since, f is non-decreasing and g is non-increasing, so h is a non-increasing function. Now, $h(1) = 1$

$\Rightarrow$ $h(x)$ is a constant function

$\Rightarrow$ $h(2) = 1$

2. Since, maximum or minimum distance between two curve always measure along common normal. AP is perpendicular on the tangent drawn to the curve.

3. f is continuous at 0 and $f'(0-) > 0$ and $f'(0+) < 0$. Thus, f has a local maximum at 0.

4. Here, $f(x) = \int_1^x (t-a)^{2n}(t-b)^{2m+1}dt,\ a \neq b$

$f'(x) = (x-a)^{2n}(x-b)^{2m+1}$.

Obviously, $f'(a-),\ f'(a+) > 0$

while $f'(b-) < 0$ and $f'(b+) > 0$

Hence, $x = b$ is a point of local minima.

5. $f'(x) = x^{n-1}[(n+1)x + an]$.

If n is even then, $f'(0-) < 0$ and $f'(0+) > 0$

Hence, 0 is a point of minimum when n is even.

6. $f'(x) = 4x - \frac{4}{x^3} = \frac{4}{x^3}(x^4 - 1) = \frac{4}{x^3}(x-1)(x+1)(x^2+1)$

$\Rightarrow$ $f'(x) = 0$ at $x = 1$ and -1

$f''(x) = 4 + \frac{12}{x^4} > 0$ for $x = \pm 1$

$\Rightarrow$ $x = 1$ and -1 are points of local minima.

7. Since, $y_{\min} = \frac{1}{e^{1/e}} > \frac{1}{e}$. So, graph always lie above the line $y = x$.

Hence, $0 < x < 1$ is correct answer.

8. $y^2 - y + a = \left(y - \frac{1}{2}\right)^2 + a - \frac{1}{4}$

Since, $-\sqrt{2} \le \sin x + \cos x \le \sqrt{2}$, given equation will have no real value of x for any y, if $a - \frac{1}{4} > \sqrt{2}$

i.e. $a \in \left(\sqrt{2} + \frac{1}{4}, \infty\right)$

$\Rightarrow$ $a \in (\sqrt{3}, \infty)$ $\left[\text{as } \sqrt{2} + \frac{1}{4} < \sqrt{3}\right]$

9. $f(x) = 1 - \frac{2e^{-x^2}}{e^{x^2} + e^{-x^2}} = 1 - \frac{2}{e^{2x^2} + 1}$

As $x \to +\infty,\ f(x) \to 1$

As $x \to -\infty,\ f(x) \to 1$

$\Rightarrow f(x)$ is increasing as well as decreasing in some intervals. Since, the range of $f(x)$ is $[0, 1)$ which does not coincide with the co-domain R and hence f is not an onto function.

10. Let $f(x) = ax^2 + bx + c > 0,\ \forall\ x \in R$

$\Rightarrow$ $b^2 - 4ac < 0$ and $a > 0$ ... (i)

Now, $g(x) = f(x) - f'(x) + f''(x)$

$= (ax^2 + bx + c) - (2ax + b) + 2a$

$= ax^2 + (b - 2a)x + (2a - b + c)$

$\Rightarrow$ $\Delta = (b-2a)^2 - 4a(2a - b + c)$

$= (b^2 - 4ac) - 4a^2 < 0$ [using Eq. (i)]

$\Rightarrow$ $g(x) > 0,\ \forall\ x \in R$

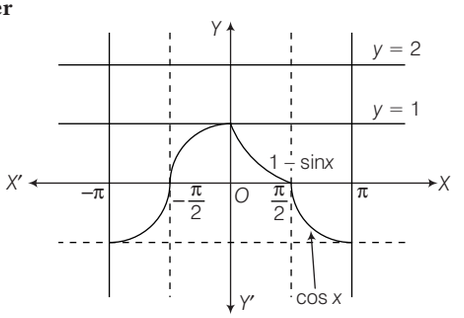
11. We have, $f(x) = \min\{1, \cos x, 1 - \sin x\}$

$$f(x) = \begin{cases} \cos x, & -\pi \le x \le 0 \\ 1 - \sin x, & 0 < x \le \frac{\pi}{2} \\ \cos x, & \frac{\pi}{2} < x \le \pi \end{cases} \Rightarrow f'(x) = \begin{cases} -\sin x, & -\pi \le x \le 0 \\ -\cos x, & 0 < x \le \frac{\pi}{2} \\ -\sin x, & \frac{\pi}{2} < x \le \pi \end{cases}$$

Therefore, $f'(0) = 0$

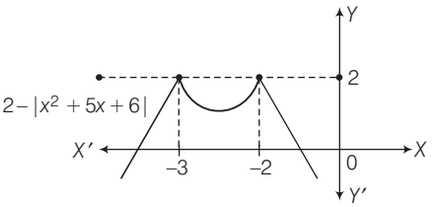
Hence, $f(x)$ has local maximum at 0 and $f(x)$ is not differentiable at $x = 0$ and $\frac{\pi}{2}$

Aliter



From the graph it is clear that $f(x)$ is not differentiable at $x = 0, \frac{\pi}{2}$ and $f(x)$ has occurs local maximum at $x = 0$.

12. $f(x)$ will have maxima only, if $a^2 + 1 \ge 2 \Rightarrow |a| \ge 1$

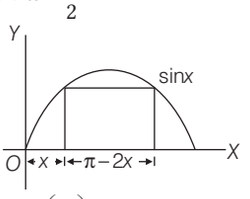


13. $ax^n = -x^2 - bx - c$. For $a = -1$ and $b = c = 0$ and $n = 2$, it will have infinite solutions.

14. $A = \text{Area} = \sin x(\pi - 2x)$ (as $l = \pi - 2x,\ b = \sin x$)

$\frac{dA}{dx} = (\pi - 2x)\cos x - 2\sin x = 0 \Rightarrow \tan x = \frac{\pi}{2} - x$

Let $f(x) = \tan x + x - \frac{\pi}{2}$

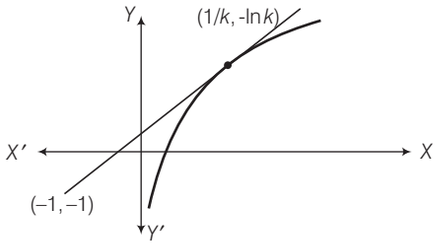


$f\left(\frac{\pi}{6}\right)$ is negative, $f\left(\frac{\pi}{4}\right)$ is positive.

So, one root lies between $\left(\frac{\pi}{6}, \frac{\pi}{4}\right)$.

15. $f(x)+1=k(x+1)$ always passes through $(-1,-1)$. Clearly, its maximum slope can go upto ∞. For minimum slope this line should touch $y=\ln x$.

$\frac{dy}{dx}=\frac{1}{x}=k$. So, $\left(\frac{1}{k},-\ln k\right)$ is point of tangency.



Now, $\frac{-\ln k+1}{\frac{1}{k}+1}=k \quad \Rightarrow \quad -\ln k+1=k+1$

$\Rightarrow \quad -\ln k=k$

Let $f(x)=k+\ln k$

$f\left(\frac{1}{e}\right)=\frac{1}{e}-1$ [negative]

$f\left(\frac{1}{\sqrt{e}}\right)=\frac{1}{\sqrt{e}}-\frac{1}{2}$ [positive]

So, one root must lie between $\frac{1}{e}$ and $\frac{1}{\sqrt{e}}$.

16. $f(x)=f'(x)\times f'''(x)$ is satisfied only by the polynomial of degree 4. Since, $f(x)=0$ satisfies $x=1,2,3$ only. It is clear that one of the roots is repeated twice.

$\Rightarrow \quad f'(1)f'(2)f'(3)=0$

17. $\frac{d^2y}{dx^2}=k\log x \Rightarrow \frac{dy}{dx}=k(x\log x-x)+A$

$=kx(\log x-1)+A$

$=k\left[(\log x-1)\frac{x^2}{2}-\frac{x^2}{4}\right]+Ax+B$

$=k\frac{x^2}{4}(2\log x-3)+Ax+B$

$\Rightarrow y=c_1(2\log x-3)x^2+c_2x+c_3$

18. $f'(x)=3\sec^2\left[\left(\tan x-\frac{1}{3}\right)^2+\frac{11}{9}\right]>0$ for all x in its domain.

19. $\lim\limits_{x\to k^-} f(x)=\lim\limits_{h\to 0}3+h=3 \Rightarrow f(k)=3$

and $\lim\limits_{x\to k^+} f(x)=\lim\limits_{h\to 0} a^2-2+\frac{\sin h}{h}=a^2-2+1$

Since, $f(x)$ has minimum at $x=k$

$f(k^-)>f(k)$

and $f(k^+)>f(k) \Rightarrow \alpha^2-2+1>3$

$|a|>2$

20. Let $f(x)=mx+b$

$\because \quad f(1)\le f(2) \Rightarrow m\ge 0$, similarly

$f(3)\ge f(4) \Rightarrow m\le 0 \Rightarrow m=0$

$\therefore \quad f(0)=f(5)=5$

21. Let $P(x)$ is of degree n, then

$\Rightarrow \quad 2n=n+2$ [as $P(x^2)=x^2P(x)$]

$\Rightarrow \quad n=2$

$P(x)=ax^2+bx+c$ form

$P(0)=-2$

$\Rightarrow \quad c=-2$ ...(i)

$P(1)=0$

$\Rightarrow \quad a+b+c=0$ ...(ii)

$P'\left(\frac{3}{2}\right)=0 \quad \Rightarrow \quad 3a+b=0$

$\Rightarrow \quad \frac{a}{b}=\frac{-1}{3} \quad \Rightarrow P(x)=-x^2+3x-2$

Hence, maximum $P(x)=-\frac{1}{4}$

22. In $[0,1]$, $[x^2-x+1]=0$

$\ln\left[1,\frac{1+\sqrt{5}}{2}\right)$, $[x^2-x+1]=1 \Rightarrow x^2=-4(y-a)$

Put $y=0$, $x^2=4(a-1) \Rightarrow 4a>1$

Put $y=1$, $x^2=4(a-1) \Rightarrow a<1$

$\Rightarrow \quad \frac{1}{4}<a<1.$

23. Clearly, $f'(a)=0$, $f'(c)=0$, $f'(e)=0$, $x=b$ and $x=d$ two points of inflexion $f'''(x)>0$, $d<x<e$.

$x=e$ is point of local minima.

24. $g(x)=f(x^2-2x-1)+f(5-x^2+2x)$

$=2x^4-8x^3-4x^2+24x+18$

$g'(x)=8x^3-24x^2-8x+24$

$g'(x)=0 \quad \Rightarrow \quad x=-1,1,3$

We observe that $g(x)\ge \min\{g(-1),g(1),g(3)\}=0$

$\therefore \quad g(x)\ge 0, \forall\ x\in R$

25. Using the fact that every set of natural numbers have the smallest element.

$\therefore$ $f(f(1))$ is the smallest element in $\{f(f(1)),f(2),f(f(2)),\ldots\}$

Same argument implies that $f(1)=1$

Repeating the argument for $f\{n\ge 2\}\to\{n\ge 2\}$, we get $f(2)=2$

$\therefore$ Clearly, $f(x)=x$

26. $\frac{1}{2}=|2ax-3|+|ax+1|+|5-ax|\ge|2ax-3-(ax+1)+5-ax|=1$

which is impossible.

27. Given, $\int_0^x 2xf^2(t)dt=\left(\int_0^x 2f(x-t)dt\right)^2$

$\Rightarrow \quad x\cdot\int_0^x f^2(t)dt=2\left(\int_0^x f(t)dt\right)^2$, using $\int_0^a f(x)dx$

$=\int_0^a f(a-x)dx$

Differentiating, we get

$x\cdot f^2(x)+\int_0^x f^2(t)dt=4\int_0^x f(t)dt\cdot f(x)$

[using Leibnitz rule]

$\Rightarrow \quad x^2f^2(x)+x\cdot\int_0^x f^2(t)dt=4x\cdot f(x)\cdot\int_0^x f(t)dt$

$$\Rightarrow \quad x^2 f^2(x) + 2\left(\int_0^x f(t)\,dt\right)^2 = 4x\,f(x)\cdot\int_0^x f(t)\,dt$$

$$\Rightarrow \quad 2\left(\int_0^x f(t)\,dt\right)^2 - 4x\cdot f(x)\cdot\int_0^x f(t)\,dt + x^2 f^2(x) = 0$$

$$\int_0^x f(t)\,dt = \frac{4x\,f(x) \pm \sqrt{16x^2 f^2(x) - 8x^2 f^2(x)}}{4} \qquad \left(\begin{matrix}\because ax^2 + bx + c = 0\\ \Rightarrow x = \dfrac{-b \pm \sqrt{d}}{2a}\end{matrix}\right)$$

$$\Rightarrow \qquad \sqrt{2}\int_0^x f(t)\,dt = (\sqrt{2} \pm 1)\,x\,f(x)$$

Again, differentiating, we get

$$\sqrt{2}\,f(x) = (\sqrt{2}\pm 1)(f(x) + x\,f'(x))$$

$$\Rightarrow \quad \sqrt{2}\,f(x) = (\sqrt{2}\pm 1)\,f(x) + (\sqrt{2}\pm 1)\,x\cdot f'(x)$$

$$\Rightarrow \quad \pm\,f(x) = (\sqrt{2}\pm 1)\cdot x\,f'(x)$$

$$\Rightarrow \quad \int \pm\frac{1}{\sqrt{2}\pm 1}\cdot\frac{1}{x}\,dx = \int\frac{f'(x)}{f(x)}\,dx$$

$$\Rightarrow \quad \pm(\sqrt{2}\pm 1)\cdot\log x = \log|f(x)| + C, \text{ as } f(1) = 1 \Rightarrow C = 0$$

$$\Rightarrow \quad f(x) = x^{\sqrt{2}\pm 1}$$

$$\therefore \quad f(x) = x^{\sqrt{2}+1} \qquad [\text{as } f'(x) > 0]$$

$$\therefore \quad f(n^{\sqrt{2}-1}) = n \qquad \ldots(\text{i})$$

Now, let $a_n = \cot\theta_n$

$$\Rightarrow \quad a_{n+1} = \cot\left(\frac{\theta_n}{2}\right)$$

$$\Rightarrow \quad \theta_{n+1} = \frac{1}{2}\theta_n$$

$$\because \quad \theta_0 = \pi/2$$

$$\Rightarrow \quad \theta_1 = \frac{1}{2}\left(\frac{\pi}{2}\right),\ \theta_2 = \frac{1}{2^2}\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \quad \theta_n = \frac{1}{2^n}\times\frac{\pi}{2} \qquad \therefore\ a_n = \cot\left(\frac{\pi}{2^{n+1}}\right)$$

$$\Rightarrow \quad \lim_{n\to\infty}\frac{a_n}{2^{n-1}} = \frac{1}{2^{-1}\cdot\pi/2} = 4/\pi$$

28. We have, $f(x) = |x|^m|x-1|^n$

$$f(x) = \begin{cases}(-1)^{m+n}x^m\cdot(x-1)^n, & \text{if} \quad x < 0\\ (-1)^n x^m(x-1)^n, & \text{if} \quad 0 \le x < 1\\ x^m(x-1)^n, & \text{if} \quad x \ge 1\end{cases}$$

Let, $g(x) = x^m(x-1)^n$, then

$$g'(x) = mx^{m-1}\cdot(x-1)^n + n\,x^m\cdot(x-1)^{n-1}$$

$$= x^{m-1}\cdot(x-1)^{n-1}\cdot(m(x-1) + n\cdot x) = 0$$

Now, $f'(x) = 0$

$\Rightarrow \quad g'(x) = 0$

$$\Rightarrow \quad x = 0 \text{ or } \frac{m}{m+n} \text{ or } 1.$$

$$f(0) = 0,\ f(1) = 0$$

$$f\left(\frac{m}{m+n}\right) = (-1)^n\cdot\frac{m^m\cdot n^n\cdot(-1)^n}{(m+n)^{m+n}} \qquad \left[\text{as, } 0 < \frac{m}{m+1} < 1\right]$$

$$\therefore \text{ Maximum value} = \frac{m^m\cdot n^n}{(m+n)^{m+n}}$$

29. Let $f(x) = \dfrac{\ln(\ln x)}{\ln x} \Rightarrow f'(x) = \dfrac{1-\ln(\ln x)}{x(\ln x)^2} < 0,\ \forall\ x > e^e$

So, $f(x)$ is increasing in $(1, e^e)$ and decreasing in (e^e, ∞).

Therefore, $x > y$

$$\Rightarrow \quad (\ln x)^{\ln y} < (\ln y)^{\ln x},\ \forall\ x, y \in (e^e, \infty) \quad \text{and} \quad x < y$$

$$\Rightarrow \quad (\ln x)^{\ln y} < (\ln y)^{\ln x},\ \forall\ x, y \in (1, e^e)$$

30. $\lim\limits_{x\to a}[f(x)]$ can exist only when $f(x)$ either increases or decreases at both sides of the point $x = a$.

Since, $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a}[f(x)]$

So, this can occur only when $\lim\limits_{x\to a} f(x)$ is an integer.

31. $f(x) = 2x^3 - 3(2+\lambda)x^2 + 12\lambda x$

$$\Rightarrow \quad f'(x) = 6x^2 - 6(2+\lambda)\,x + 12\lambda$$

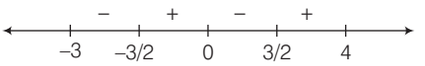
$$\Rightarrow \quad f'(x) = 0 \ \Rightarrow\ x = 2, \lambda$$

If $f(x)$ has exactly one local maximum and exactly one local minimum, then $\lambda \neq 2.1$

32. $h(x) = 3f\left(\dfrac{x^2}{3}\right) + f(3-x^2) \ \Rightarrow\ h'(x) = 2x\left(f'\left(\dfrac{x^2}{3}\right) - f'(3-x^2)\right)$

$$\Rightarrow \quad f'\left(\frac{x^2}{3}\right) > f'(3-x^2),\ \forall\ x \text{ such that } \frac{x^2}{3} > 3 - x^2 \Rightarrow x^2 > \frac{9}{4}$$

$$\Rightarrow \quad f'\left(\frac{x^2}{3}\right) < f'(3-x^2),\ \forall\ x \text{ such that } x^2 < \frac{9}{4}$$



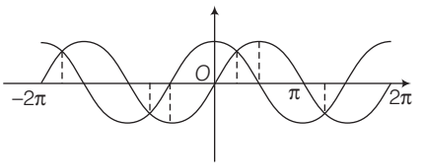
using number line rule

$$\Rightarrow\ h(x) \text{ increases in } \left(-\frac{3}{2}, 0\right)\cup\left(\frac{3}{2}, 4\right)$$

and $h(x)$ decreases in $\left(-3, -\dfrac{3}{2}\right)\cup\left(0, \dfrac{3}{2}\right)$

33. $f'(x) = \dfrac{\cos x - \sin x}{1 + (\sin x + \cos x)^2} \ge 0 \Rightarrow \cos x \ge \sin x$

$$\Rightarrow \quad x \in \left[-2\pi, -\frac{7\pi}{4}\right]\cup\left[-\frac{3\pi}{4}, \frac{\pi}{4}\right]\cup\left[\frac{5\pi}{4}, 2\pi\right]$$



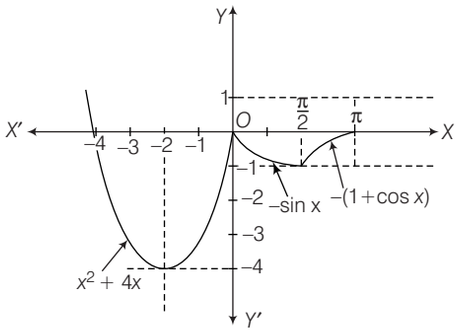
34. Applying $C_1 \to C_1 + C_2$, we get $\begin{vmatrix}2 & \cos^2 x & \sin 2x\\ 2 & 1+\cos^2 x & \sin 2x\\ 1 & \cos^2 x & 1+\sin 2x\end{vmatrix}$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\begin{vmatrix}2 & \cos^2 x & \sin 2x\\ 0 & 1 & 0\\ -1 & 0 & 1\end{vmatrix} = 2 + \sin 2x$$

Since, the maximum value of $\sin 2x$ is 1 and minimum value of $\sin 2x$ is (-1). Therefore, $\alpha = 3,\ \beta = 1$.

35.



From above figure clearly options (a), (b) and (c) are correct.

36. $f(x) = \sqrt{a^2b^2 + b^2 - b^2a^2}\sin(x + \alpha) + c$

$= b\sin(x + \alpha) + c$, where $\tan\alpha = \dfrac{\sqrt{1-a^2}}{a}$

$f_{\max} = b + c$

$f_{\min} = -b + c$

$(f(x))_{\max} - (f(x))_{\min} = 2b$

Also, at $x = -\cos^{-1}a$, $f(x) = c$.

37. $f'(x) = \dfrac{n \cdot x^{n-1}}{(\ln x^n)} - \dfrac{1 \cdot mx^{m-1}}{(\ln x^m)}$,

Clearly, (c) and (d) are the answers.

38. $f(x)$ occurs maximum at $x = \dfrac{3}{2}$

$g(x)$ occurs maximum at $x = \dfrac{-b}{2}$

$\therefore \qquad \dfrac{3}{2} = \dfrac{-b}{2} \Rightarrow b = -3$

39. $h(x) = \dfrac{\ln(f(x)\cdot g(x))}{\ln a} = \dfrac{\ln a^{\{a^{|x|}\ \text{sgn}\ x\} + [a^{|x|}\cdot \text{sgn}\ x]}}{\ln a}$

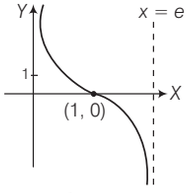
$= \{a^{|x|}\cdot \text{sgn}\ x\} + [a^{|x|}\cdot \text{sgn}\ x] = a^{|x|}\ \text{sgn}\ x$

$$= \begin{cases} a^x & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -a^{-x} & \text{for } x < 0 \end{cases} \qquad [\because \{y\} + [y] = y]$$

$\Rightarrow h(x)$ is an odd function.

40. $f(x) = \ln(1 - \ln x)$, Domain $(0, e)$

$$f'(x) = -\frac{1}{(1-\ln x)}\cdot\frac{1}{x} < 0$$



$\Rightarrow$ Decreasing, $\forall\ x$ in its domain

$\Rightarrow$ (a) and (b) are incorrect.

$f'(1) = -1 \Rightarrow$ (c) is also incorrect.

Also, $f(1) = 0$, $\lim\limits_{x\to e} \to -\infty$, $\lim\limits_{x\to 0^+} f(x) \to \infty$

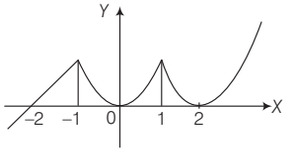
$$f''(x) = \frac{-\ln x}{x^2(1-\ln x)^2}$$

$f''(1) = 0$ which is a point of inflection as shown in graph

Y-axis and $x = e$ are two asymptotes.

41. f is obvious continuous, $\forall\ x \in R$ and not derivable at -1 and 1.

$f'(x)$ changes sign 4 times at $-1, 0, 1, 2$



Local maxima at 1 and -1.

Local minima at $x = 0$ and 2.

42. $f(x) = \int_0^{\pi}\cos t\cos(x - t)\,dt$ ...(i)

$= \int_0^{\pi} -\cos t\cdot\cos(x - \pi + t)\,dt$

$f(x) = \int_0^{\pi} -\cos t\cdot\cos(x + t)\,dt$ ...(ii)

On adding Eqs. (i) and (ii), we get

$2f(x) = \int_0^{\pi}\cos t\,(2\cos x\cdot\cos t)\,dt$

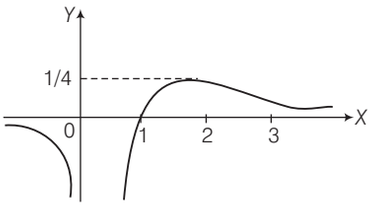
$2f(x) = 2\cos x\int_0^{\pi/2}\cos^2 t\,dt$

$f(x) = \dfrac{\pi\cos x}{2}$ Now, verify.

Only (a) and (b) are correct.

Aliter Convert the integer and into sum of two cosine functions.

43. $f'(x) = \dfrac{2-x}{x^3}$ and $f''(x) = \dfrac{x-3}{x^4}$ Now, interpret



44. We have, $\lim\limits_{a\to x}\dfrac{af(x) - xf(a)}{a - x}$ $\left[\dfrac{0}{0}\text{form}\right]$

$\Rightarrow \lim\limits_{a\to x}\dfrac{f(x) - xf'(a)}{1} = 2$

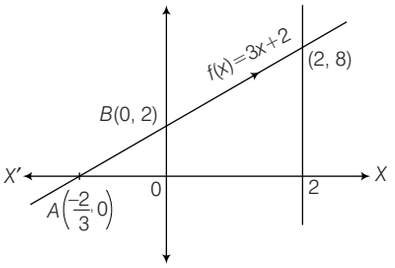
$\Rightarrow f(x) - xf'(x) = 2$

$\Rightarrow \dfrac{f'(x)}{f(x)-2} = \dfrac{1}{x}$

On integrating both sides w.r.t. x, we get

$$\int\frac{f'(x)}{f(x)-2}dx = \int\frac{1}{x}dx$$

$\Rightarrow \quad \log(f(x)-2) = \log x + \log c$



$\Rightarrow \quad f(x) = cx + 2$

As, $f(1) = 5$, so $5 = c + 2 \Rightarrow c = 3$

Hence, $f(x) = 3x + 2$

Clearly, area $(\Delta OAB) = \frac{1}{2}\left(\frac{2}{3}\right)(2) = \frac{2}{3}$

Also, $\int_0^2 (3x+2)\,dx = \left(\frac{3x^2}{2} + 2x\right)_0^2$

$= 6 + 4 = 10$

45. Given, $2(F(x) - f(x)) = f^2(x)$ and $\frac{d\,F(x)}{dx} = f(x)$ ...(i)

$\therefore \quad F(x) = \frac{f^2(x)}{2} + f(x)$

$\Rightarrow \quad F'(x) = f(x)\cdot f'(x) + f'(x)$

$\therefore \quad f(x) = f(x)\cdot f'(x) + f'(x)$, using $F'(x) = f(x)$

$\Rightarrow \quad f'(x) = \frac{f(x)}{1+f(x)}$, as $f(0,\infty) \to (0,\infty)$

$\therefore \quad f'(x) > 0$

Hence, f is strictly increasing and $\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{f'(x)}{1}$

[using L'Hospital's rule]

$= \lim_{x\to\infty}\left(1 + \frac{1}{1+f(x)}\right)$, as $x \to \infty; f(x) \to \infty$

$\therefore \quad \lim_{x\to 0} \frac{f(x)}{x} = 1$

46. If $b < 0$, then $f(0) = b < 0, f(1) = b < 0$

$\therefore \quad 0,\ 1$ lies between the roots, Statement I is false.

47. $x = 2$ is a point of local minima.

48. $\phi'(x) = 3\sin x + 4\cos x > 0$

$\phi(x)$ is increasing in $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$.

$\therefore\ f(x)$ attain maximum value at $x = \frac{\pi}{3}$.

49. Let $g(x) = f(x)\cdot f'(x)$

$\Rightarrow\ g'(x) > 0$ in $[a, b]$.

50. Let $g(x) = \sqrt{x} - \sqrt{x-1},\ x > 1$

$\Rightarrow g'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x-1}} = \frac{1}{2}\left(\frac{\sqrt{x-1} - \sqrt{x}}{\sqrt{x(x-1)}}\right)$

$= \frac{-1}{2\sqrt{x}\sqrt{x-1}(\sqrt{x} + \sqrt{x-1})} < 0, \forall\ x > 1$

Hence, $g(x)$ is a decreasing function.

$\Rightarrow \quad c + 1 > c$

$g(c+1) < g(c)$

$\Rightarrow \quad f(u) < f(v)$

51. Let $h(x) = f(x) - \sin x$ or $x \in \left[0, \frac{\pi}{2}\right]$

$\Rightarrow \quad h(0) = h\left(\frac{\pi}{2}\right) = 0$

According to Rolle's theorem for atleast one $c \in \left(0, \frac{\pi}{2}\right)$.

$h'(c) = f'(c) - \sin c = 0$

for $x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right], h\left(\frac{\pi}{2}\right) = h\left(\frac{5\pi}{2}\right) = 0$

According to Rolle's theorem for atleast one $d \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

$h'(d) = f'(d) - \cos d = 0$

for $x \in [c, d],\ h'(c) = h'(d) = 0$

According to Rolle's theorem for atleast one $x \in (c, d)$.

$h'(x) = f''(x) + \sin x = 0$

$\Rightarrow\ |f''(x)| \le 1$ for atleast one $x \in \left(0, \frac{3\pi}{2}\right)$.

52. Statements I and II both are true but Statement II does not explain Statement I.

53. Statement II is true as $f'(a)^+ = f'(a)^-, \forall\ a \in I$.

Statement I is true and is obtained from differentiable rule.

54. Statement II is correct as $y = f(x)$ is odd and hence Statement I is correct.

Sol. **(Q. Nos. 55 to 57)**

Here, we have $f'(x) = \frac{-2x}{(1+x^2)^2}$ and $f''(x) = \frac{6x^2 - 2}{(1+x^2)^3}$

$\therefore\ f'(x)$ is maximum at $x = -\frac{1}{\sqrt{3}}$

If m is greatest, then $m = \frac{3\sqrt{3}}{8}$

y-coordinate of the point of contact is $\frac{3}{4}$.

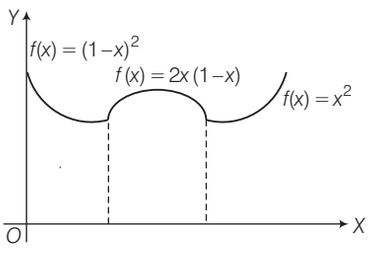
$\therefore$ Equation of the tangent is $y - \frac{3}{4} = \frac{3\sqrt{3}}{8}\left(x + \frac{1}{\sqrt{3}}\right)$

$\therefore \quad a = -\sqrt{3}$ and $b = \frac{9}{8}$

55. (d) **56.** (a) **57.** (a)

Sol. **(Q. Nos. 58 to 60)**

We draw the graphs of $f_1(x) = x^2$, $f_2(x) = (1-x)^2$ and $f_3(x) = 2x(1-x)$. Here, $f(x)$ is redefined as



$$f(x)=\begin{cases}(1-x)^2, & 0\le x<\dfrac{1}{3}\\ 2x(1-x), & \dfrac{1}{3}\le x\le\dfrac{2}{3}\\ x^2, & \dfrac{2}{3}<x\le 1\end{cases}$$

Interval of increase of $f(x)$ is $\left(\frac{1}{3},\frac{1}{2}\right)\cup\left(\frac{2}{3},1\right)$.

Interval of decrease of $f(x)$ is $\left(0,\frac{1}{3}\right)\cup\left(\frac{1}{2},\frac{2}{3}\right)$.

Clearly, Rolle's theorem is applicable on $\left[\frac{1}{3},\frac{2}{3}\right]$, where $f(x)=2x(1-x)$.

$\Rightarrow$ $f'(c)=2-4c=0 \Rightarrow c=\frac{1}{2}$

$\Rightarrow$ $a+b+c=\frac{1}{3}+\frac{2}{3}+\frac{1}{2}=\frac{3}{2}$

58. (d) ***59.*** (d) ***60.*** (c)

Sol. **(Q. Nos. 61 to 63)**

According to paragraph

$\frac{f(b)-f(a)}{b-a}>f'(c)$, $\frac{g(b)-g(a)}{b-a}=g'(c)$ and $\frac{h(b)-h(a)}{b-a}<h'(c)$

As, $f'(x)>g'(x)>h'(x)$

$\Rightarrow$ $\frac{f(b)-f(a)}{b-a}>\frac{g(b)-g(a)}{b-a}>\frac{h(b)-h(a)}{b-a}$

If $g(x)=Ax^2+Bx+C$

$\Rightarrow$ $\frac{g(b)-g(a)}{b-a}=\frac{A(b^2-a^2)+B(b-a)}{b-a}$

$\Rightarrow$ $2A\frac{(b+a)}{2}+B=g'\left(\frac{b+a}{2}\right)$

61. (b) ***62.*** (c) ***63.*** (a)

Sol. **(Q. Nos. 64 to 66)**

$a_n>a_{n-1}$ iff $n+c$ is a perfect square, since $a_2>a_1$ and $a_5>a_4$.

$\Rightarrow$ $2+c$ and $5+c$ are perfect squares

$\Rightarrow$ $c=-1$ ...(i)

Now, $a_2=3=b\,[\sqrt{2+c}]+d$

$\therefore$ $b+d=2$ ...(ii)

Also, $a_{10}>a_9$

$\Rightarrow$ $a_{10}=7=3b+d$ ...(iii)

On solving Eqs. (ii) and (iii), we get $b=2,\ d=1$

64. (c) ***65.*** (a) ***66.*** (a)

Sol. **(Q. Nos 67 to 69)**

$D_h=\{-1,1\}$, as minimum occurs before maxima

$\therefore$ $a_3=-1$

Now, $g(x)=a_0+a_1x+a_2x^2-x^3$

$g'(x)=a_1+2a_2x-3x^2=-3(x-3)(x+3)=-3x^2=27$

$\therefore$ $a_1=27, a_2=0$

$\therefore$ $a_1+a_2=27$

Also, $g(-3)>0$ and $g(3)>0 \Rightarrow a_0>54$ and $a_0<-54$

$\therefore$ $a_0>54$

Now, $g(x)=a_0+27x-x^3 \Rightarrow f(x)=\sqrt{a_0+27x-x^3}$

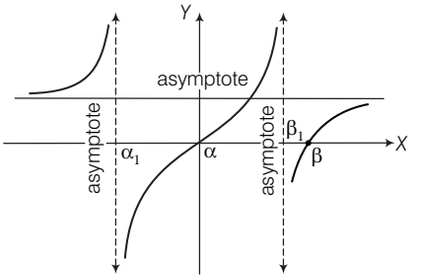
$f(-10)=\sqrt{a_0+270-1000}$

Clearly, $f(-10)$ is defined for $a_0>830$

67. (c) ***68.*** (b) ***69.*** (a)

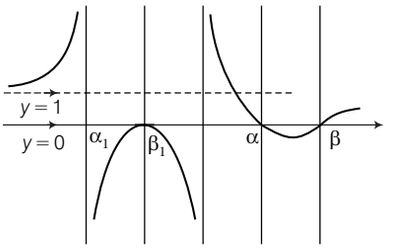
Sol. (Q. Nos. 70 to 74)

70. Graph of $f(x)$ is shown.



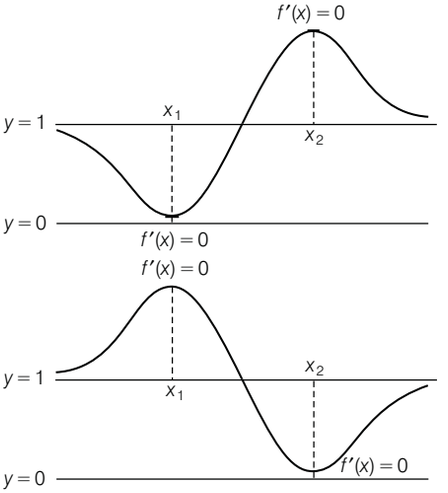
Clearly, $f(x)$ is increasing in (α_1,β_1).

71. Clearly, $f(x)$ has a maximum in $[\alpha_1,\beta_1]$ and a minima in $[\alpha,\beta]$, shown as.



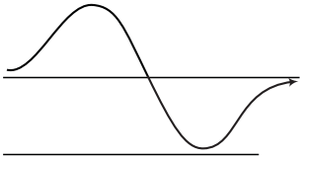
72. $f(x)$ has one of the two graphs.

$\Rightarrow$ $f'(x)=0$ has real and distinct roots.



73. Clearly, $\lim\limits_{x\to\infty}[f(x)]=0$ and $\lim\limits_{x\to\infty}[f(x)]=1$

$\therefore$ $\lim\limits_{x\to\infty}[f(x)]\lim\limits_{x\to\infty}[f(x)]=0$



74. $f(x)=\dfrac{x^2+bx+c}{x^2+b_1x+c_1}=1+\dfrac{(b-b_1)x+(c-c_1)}{x^2+b_1x+c_1}$

$$=1+\frac{\dfrac{(b-b_1)}{x}+\dfrac{(c-c_1)}{x^2}}{1+\dfrac{b_1}{x}+\dfrac{c_1}{x^2}}$$

For $b>b_1$, $\lim\limits_{x\to\infty} f(x)\to 1^+$

$\Rightarrow$ Point of maxima is greater than point of minima.

Sol. **(Q. Nos. 75 to 77)**

$$y=\frac{x^2}{x^2-1},\ \text{not defined at } x=\pm 1$$

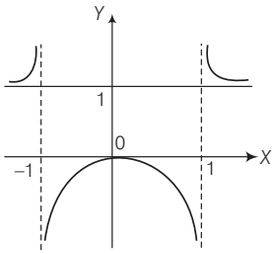
$$=1+\frac{1}{x^2-1}\Rightarrow y'=-\frac{2x}{(x^2-1)^2}$$

$$\frac{dy}{dx}=0$$

$\Rightarrow$ $x=0$ [point of maxima]

as $x\to 1^+, y\to\infty,\ x\to 1^-, y\to-\infty$

Similarly, $x\to -1^+, y\to-\infty,\ x\to -1^-, y\to\infty$



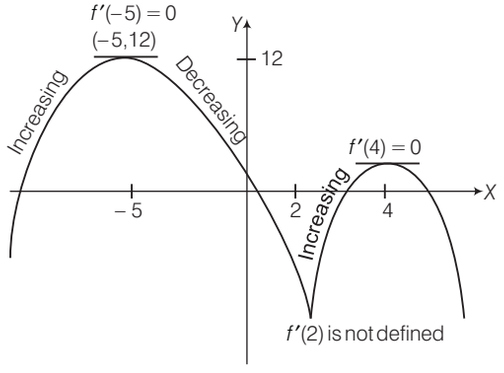
The graph of $y=\dfrac{x^2}{x^2-1}$ is as shown

verify all alternatives from the graph.

75. (b) ***76.*** (d) ***77.*** (a)

Sol. **(Q. Nos. 78 to 80)**

From given statements (i) to (v), one of the graph of $f(x)$ can be plotted as

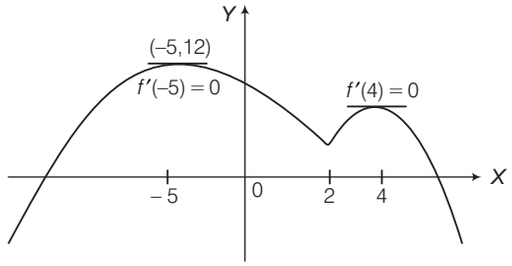


78. Since, $f'(2)$ is not defined and continuous for $x\in R$.

$\Rightarrow y=f(x)$ must have a geometrically sharp corner at $x=2$.

79. At $x=-5$, $f'(x)$ changes from positive to negative and at $x=4$, $f'(x)$ change sign for positive to negative, hence maxima at $x=-5$ and 4. f is continuous and $f'(2)$ is not defined, hence $x=2$ must be geometrical sharp corner.

80. Clearly, one of the graph of $f(x)$ is



Sol. **(Q. Nos. 81 to 83)**

$$f(x)=e^{(p+1)x}-e^x=e^x(e^{px}-1)$$

$$f'(x)=e^x(e^{px}-1)+e^x\cdot pe^{px}$$

$$=e^x[(p+1)e^{px}-1]=0 \qquad [e^x\neq 0]$$

$\Rightarrow$ $e^{px}=\dfrac{1}{p+1}\Rightarrow px=-\log_e(p+1)$

$$x=-\frac{\log_e(p+1)}{p} \qquad \text{...(i)}$$

Now, $f''(x)=(p+1)^2e^{(p+1)x}-e^x$

$$=e^x[(p+1)^2e^{px}-1]$$

$\therefore$ $f''\left(-\dfrac{\log(p+1)}{p}\right)=e^x[(p+1)-1]>0$

Hence, $x=s_p=-\dfrac{\log_e(p+1)}{p}$

$$g(t)=\int_t^{t+1}(e^{px}\cdot e^x-e^x)e^t\cdot e^{-x}\,dx=e^t\int_t^{t+1}(e^{px}-1)\,dx$$

On integrating, we get

$$g(t)=e^t\cdot\left(\frac{e^{px}}{p}-x\right)_t^{t+1}=e^t\left[\frac{e^{pt}(e^p-1)}{p}-1\right]$$

$\therefore$ $g(t)=\dfrac{(e^p-1)e^{(p+1)t}}{p}-e^t$

$$g'(t)=(p+1)\frac{(e^p-1)}{p}e^{(p+1)t}-e^t=0$$

$\Rightarrow$ $\dfrac{(p+1)(e^p-1)}{p}e^{pt}=1\Rightarrow e^{pt}=\dfrac{p}{(p+1)(e^p-1)}$

$$pt=\log_e\left(\frac{p}{(p+1)(e^p-1)}\right)$$

$\therefore$ $t=-\dfrac{1}{p}\log\left(\dfrac{(p+1)(e^p-1)}{p}\right)=t_p$

Also, $s_p-t_p=\dfrac{1}{p}\log\left(\dfrac{(p+1)(e^p-1)}{p}\right)-\dfrac{\log(p+1)}{p}$

$$=\frac{1}{p}\log\left(\frac{e^p-1}{p}\right)$$

Hence, $\lim\limits_{p\to 0^+}(s_p-t_p)=\lim\limits_{p\to 0^+}\dfrac{1}{p}\log\left(\dfrac{e^p-1}{p}\right)$

$$=\lim_{p\to 0}\frac{\log\left(1+\dfrac{e^p-p-1}{p}\right)}{\left(\dfrac{e^p-p-1}{p}\right)\cdot p}\cdot\left(\frac{e^p-p-1}{p}\right)=\lim_{p\to 0^+}\left(\frac{e^p-p-1}{p^2}\right)=\frac{1}{2}$$

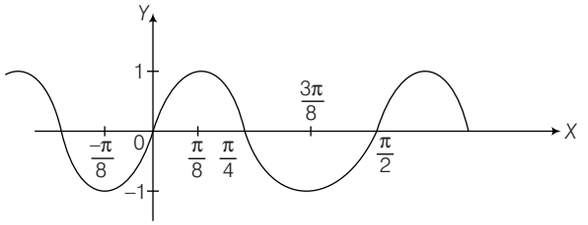
81. (c) ***82.*** (c) ***83.*** (b)

Sol. (Q. Nos. 84 to 86)

84. Here, $h(x) = f^2(x) + g^2(x)$

$= (\sin 3x + \cos x)^2 + (\cos 3x + \sin x)^2 = 2 + 2\sin 4x$

Graph of $y = \sin 4x$



Clearly, $h(x)$ is periodic function with period $\pi/2$ and from above graph, the length of a longest interval in which the function $y = h(x)$ is increasing $= \pi/8 - (-\pi/8) = \pi/4$

85. We have, $h(x) = 4$

$\Rightarrow \quad 2 + 2\sin 4x = 4 \Rightarrow \sin 4x = 1$

$\therefore \quad 4x = 2n\pi + \pi/2 = (4n+1)\pi/2$

$\Rightarrow \quad x = (4n+1)\pi/8, n \in I$

86. We have, $f(x) = g(x)$

$\Rightarrow \quad \sin 3x + \cos x = \cos 3x + \sin x$

$\Rightarrow \quad \sin 3x - \sin x = \cos 3x - \cos x$

$\Rightarrow \quad 2\sin x(\cos 2x + \sin 2x) = 0$

$\Rightarrow \quad$ either $\sin x = 0$ or $\tan 2x = -1$

$\Rightarrow \quad x = 0, \pi, 3\pi/8, 7\pi/8$

$\therefore$ Number of solution = 4

Sol. (Q. Nos. 87 to 89)

87. We have, $h''(x) = 6x - 4$

$\Rightarrow \quad h'(x) = 3x^2 - 4x + c$

As, $h'(1) = 0 \Rightarrow 0 = -1 + c$ or $c = 1$

So, $h'(x) = 3x^2 - 4x + 1$

$\Rightarrow \quad h(x) = x^3 - 2x^2 + x + k$

Also, $h(1) = 5 \Rightarrow k = 5$

$\therefore \quad h(x) = x^3 - 2x^2 + x + 5$

Now, $h'(2) = 5$

$\therefore$ The equation of tangent at $m(2, 7)$ to $y = h(x)$, is

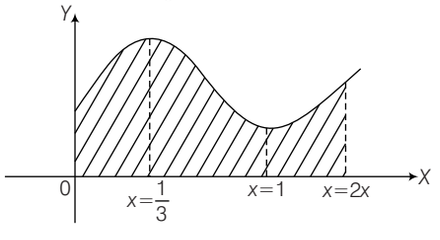
$(g - 7) = 5(x - 2)$

$\Rightarrow \quad 5x - y = 3$

88. Also, $g(f(x)) = 0, \forall\, x \in R$

$\therefore$ Required area $= \int_0^2 h(x)\,dx = \left(\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2} + 5x\right)_0^2$

$= 4 - \frac{16}{3} + 2 + 10 = 32/3$



89. Also, range of $f(g(x)) = \{-1, 0, 1\}$

$\therefore \quad \sqrt{f(g(x))} \in \{0, 1\}$

Hence, range of $\sin^{-1}\sqrt{(fog)(x)} \in \{0, \pi/2\}$

Sol. (Q. Nos. 90 to 92)

We have, $g(x) = x^3 + g''(1)x^2 + \{3g'(1) - g''(1) - 1\}x + 3g'(1)$

Let $\quad g'(1) = a,\ g''(1) = b$, then

$g(x) = x^3 + bx^2 + (3a - b - 1)x + 3a$

Differentiating both sides w.r.t. x, we get

$g'(x) = 3x^2 + 2bx + (3a - b - 1)$

Put $x = 1$,

$\Rightarrow \quad g'(1) = 3 + 2b + 3a - b - 1 \qquad [\because g'(1) = a]$

$\Rightarrow \quad a = b + 3a + 2$ or $2a + b = -2 \qquad$ ...(i)

Again differentiating, we get

$g''(x) = 6x + 2b$

Put $x = 1$,

$\Rightarrow \quad g''(1) = 6 + 2b \qquad [\therefore g''(1) = b]$

$\Rightarrow \quad b = 6 + 2b$

$\Rightarrow \quad b = -6 \qquad$ ...(ii)

From Eq. (i), $a = 2$

$\therefore \quad g(x) = x^3 - 6x^2 + 11x + 6$

Given, $\quad f(x) = x\,g(x) - 12x + 1$

$= x^4 - 6x^3 + 11x^2 + 6x - 12x + 1$

$= x^4 - 6x^3 + 11x^2 - 6x + 1$

$= (x^2+1)^2 - 2x^2 + 11x^2 - 6x^3 - 6x$

$= (x^2+1)^2 - 6x(x^2+1) + (3x)^2$

$= (x^2 + 1 - 3x)^2 = \{h(x)\}^2$, given

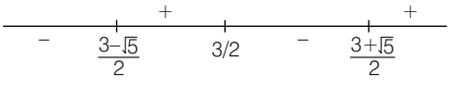
$\because \quad h(0) = 1$

$\therefore \quad h(x) = x^2 - 3x + 1$

90. $f(x) = (x^2 - 3x + 1)^2$

$\because \quad f'(x) = 2(x^2 - 3x + 1)(2x - 3) = 0$

$\Rightarrow \quad x = \frac{3}{2}, \frac{3 \pm \sqrt{5}}{2}$



Clearly, $f(x)$ has local maxima at $x = 3/2$ and local minima at $x = \frac{3 \pm \sqrt{5}}{2}$

$\therefore \quad f(x)$ has exactly one local maxima and two local minima

91. We have,

$g(x) = x^3 - 6x^2 + 11x + 6$

$g'(x) = 3x^2 - 12x + 11$

$= 3(x-2)^2 - 1 = 3[(x-2)^2 - 1/3]$

$\therefore \quad g'(x) > 0 \Rightarrow x \in \left(-\infty, 2 - \frac{1}{\sqrt{3}}\right) \cup \left(2 + \frac{1}{\sqrt{3}}, \infty\right)$

and $\quad g'(x) < 0 \Rightarrow x \in \left(2 - \frac{1}{\sqrt{3}}, 2 + \frac{1}{\sqrt{3}}\right)$

$\therefore$ $g(x)$ monotonically increases for
$x \in \left(-\infty, 2-\frac{1}{\sqrt{3}}\right) \cup \left(2+\frac{1}{\sqrt{3}}, \infty\right)$ and monotonically decreases for $x \in \left(2-\frac{1}{\sqrt{3}}, 2+\frac{1}{\sqrt{3}}\right)$

For $x \in [1, 3]$

$$g(x) = (x-1)(x-2)(x-3)+12$$

$\Rightarrow$ $g(1) = 12$ and $g(3) = 12$

$\therefore$ By Rolle's theorem in $[1, 3]$, we have $g'(c) = 0$

$\Rightarrow$ $c = 2 \pm \frac{1}{\sqrt{3}}$ [both $\in (1, 3)$]

$\therefore$ There exists two distinct tangents to the curve $y = g(x)$ which are parallel to the chord joining $(1, g(1))$ and $(3, g(3))$.

For $x \in [0, 4]$

$$g(0) = 6 \text{ and } g(4) = 18$$

$\therefore$ By LMVT

$$g'(c) = \frac{g(4)-g(0)}{4-0} = \frac{18-6}{4} = 3$$

$\therefore$ $3c^2 - 12c + 11 = 3$

$\Rightarrow$ $3c^2 - 12c + 8 = 0$

$\Rightarrow$ $c = 2 \pm 2/\sqrt{3}$ [both $\in (0, 4)$]

$\therefore$ There exists exactly two distinct lagrange's mean value in $(0, 4)$ for $y = g(x)$.

92. We have, $h(x) = x^2 - 3x + 1 = \left(x - \frac{3}{2}\right)^2 - \frac{5}{4}$

The curve $y = h(x)$ is an upward parabola, intersecting X-axis at two distinct points.

$\therefore$ $h(x)$ has exactly one critical point (i.e. the vertex) and no point of inflexion.

Also, $h(x) = 0$

$\Rightarrow$ $x = \frac{3 \pm \sqrt{5}}{2}$ [both $\in (0, 3)$]

$\therefore$ $h(x) = 0$ has exactly two distinct zeroes in $(0, 3)$.

93. (A) $y = 10 - (10 - x)$, $-1 \le x \le 3$
$= x$

$\therefore$ Maximum value $= 3$

(B) Equation of tangent at $(t^2, 2t)$

$$ty = x + t^2$$

this tangent means Y-axis at $Q(0, t)$

Here, $QS \times PQ = t \times \left(\frac{-1}{t}\right) = -1$

Area of triangle $PQS, \Delta = \frac{1}{2}\begin{vmatrix} t^2 & 2t & 1 \\ 0 & t & 1 \\ 1 & 0 & 1 \end{vmatrix}$

$\Rightarrow$ $\Delta = \frac{1}{2}[3t^2 + t]$

for $\in [0, 2]$, $\frac{d\Delta}{dt} \ge 0$

Δ is maximum at $t = 2$

$\therefore$ max $\Delta = \frac{1}{2}[2^3 + 2] = 5$

(C) Let $y = \left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right) = \left(1+\frac{1}{1-b}\right)\left(1+\frac{1}{b}\right)$

$\Rightarrow b^2(y-1) + b(1-y) + 2 = 0$

Then, $(1-y)^2 - 8(y-1) \le 0$

$\Rightarrow$ $1 \le \sqrt{y} \le 3$

(D) $y = \frac{x+2}{2x^2+3x+6}$

$\Rightarrow$ $2yx^2 + (3y-1)x + 6y - 2 = 0$

$(3y-1)^2 - 4 \times 2y(6y-2) \ge 0$

$\Rightarrow$ $(3y-1)(13y+1) \le 0$

$\Rightarrow$ $\frac{-1}{13} \le y \le \frac{1}{3}$

94. (A) $3^x = t \Rightarrow Q(t) = 2t^3 - 4t^2 + 2t$ in $t \in \left[\frac{1}{3}, 3\right]$

$\Rightarrow Q'(t) < 0$ in $\left(\frac{1}{3}, 1\right)$ and $Q'(t) > 0$ in $(1, 3) \Rightarrow Q(t)_{\min} = Q(1) = 0$

(B) Take $x^2 + x = t \Rightarrow Q(t) = t(t-2) \Rightarrow Q(t)_{\min} = -1$

(C) $\int_{-1}^{3} |x-2|\,dx = 5$ and $\int_{-1}^{3} [x]dx = 2 \Rightarrow I = 3$

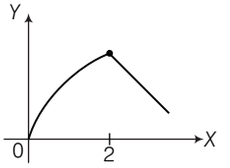
(D) Period of $\sin 36x = \frac{\pi}{18}$

Period of $\tan 42x = \frac{\pi}{42} \Rightarrow p = \frac{\pi}{6}$

95. (A) $f'(x) = 8x(x-1)(x+1)$ [positive factor]

$\Rightarrow$ Local maxima at $x = 0$

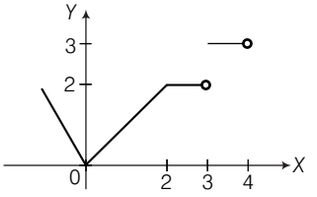
(B) Clearly, we see (p), (r), (s)



(C) $f\left(\frac{1}{2}\right)^- > f\left(\frac{1}{2}\right) < f\left(\frac{1}{2}\right)^+$

Also, $f(x)$ is non-differentiable at $x = \frac{1}{2}$

$\Rightarrow$ Clearly, we see (p), (s).



(D) Clearly, we see (q), (s).

96. (A) $f'(x) = -\pi \sin \pi x + 10 + 6x + 3x^2$

$= 3(x+1)^2 + 7 - \pi \sin \pi x > 0$ for all x

$\therefore$ $f(x)$ is minimum in $-2 \le x \le 3$. So, the absolute minimum

$= f(-2) = -15$

(B) $f'(x) = 2x + 1$. Therefore, for $-1 \le x < -\frac{1}{2}$, we get $f'(x) < 0$ and for $-\frac{1}{2} < x \le 1$, we get $f'(x) > 0$

$\therefore$ $f(x)$ is minimum decrease in $f\left[-1, -\frac{1}{2}\right]$.

and minimum decrease in $\left(-\frac{1}{2}, 1\right]$.

$\therefore$ $\min f(x) = f\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right) + 1 = \frac{3}{4}$

(C) $f'(x) = 4(x^2 - 1)$. So, for $0 \le x < 1$, we get $f'(x) < 0$, i.e. $f(x)$ is monotonically decreasing and for $1 < x \le 2$, we get $f'(x) > 0$, $f(x)$ is monotonically increasing.

$\therefore$ $\min f(x) = f(1) = \frac{4}{3} - 4 = -\frac{8}{3}$.

(D) $f'(x) = -12 + 18x - 6x^2$

$= -6(x^2 - 3x + 2) = -6(x-1)(x-2)$

$\therefore$ $f'(x) > 0$, if $1 < x < 2$

and $f'(x) < 0$, if $2 < x \le 4$

$\therefore$ $f(x)$ is monotonically increasing in $1 < x < 2$ and monotonically decreasing in $2 < x \le 4$.

$\therefore$ Absolute maximum = The greatest among $\{f(1), f(2)\}$

= The greatest among $\{1, 2\} = 2$

97. $\frac{dT}{dt} = k(T-5)^2$

$$\frac{dT}{(T-5)^2} = k\,dt$$

$\Rightarrow$ $$\frac{(T-5)^{-1}}{-1} = kt + c$$

$\therefore$ $-(40-5)^{-1} = c$ i.e. $c = -\frac{1}{35}$

$\therefore$ $$-\frac{1}{T-5} = kt - \frac{1}{35}$$

After 15 min

$$-\frac{1}{30-5} = 15k - \frac{1}{35}$$

$\therefore$ $k = \frac{1}{15}\left(\frac{1}{35} - \frac{1}{25}\right) = \frac{-10}{15 \times 35 \times 25} = \frac{-2}{75 \times 35}$

Temperature after 60 min is given by

$$-\frac{1}{T-5} = 60\left(\frac{-2}{75 \times 35}\right) - \frac{1}{35}$$

$\Rightarrow$ $$\frac{1}{T-5} = \frac{120 + 75}{75 \times 35} + 5 = \frac{195}{75 \times 35}$$

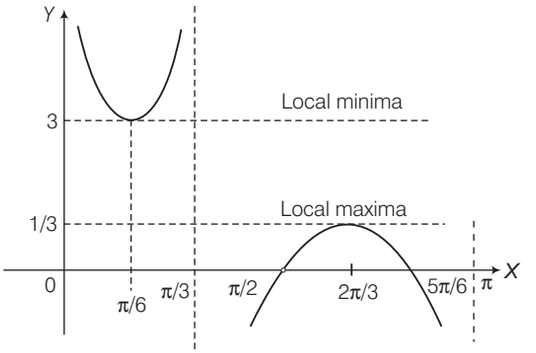
$\Rightarrow$ $T = \frac{75 \times 35}{195} + 5 = \frac{5 \times 35}{13} + 5 = \frac{175}{13} + 5$

$\Rightarrow$ $[T] = 18$

$\therefore$ $[T]/2 = 18/2 = 9$

98. $y = \frac{2\sin\left(x + \frac{\pi}{6}\right)\cos x}{2\sin x \cos\left(x + \frac{\pi}{6}\right)} = \frac{\sin\left(2x + \frac{\pi}{6}\right) + \sin\frac{\pi}{6}}{\sin\left(2x + \frac{\pi}{6}\right) - \sin\frac{\pi}{6}}$

$= 1 + \frac{1}{\sin\left(2x + \frac{\pi}{6}\right) - \sin\frac{\pi}{6}}$



y is minimum, if $2x + \frac{\pi}{6} = \frac{\pi}{2} \Rightarrow x = \frac{\pi}{6}$

$\Rightarrow$ $y_{\min} = 1 + 2 = 3$

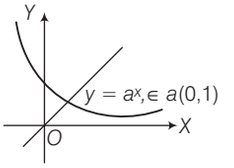
99. Let $A = (t, t^2)$, $m_{OA} = t$, $m_{AB} = -\frac{1}{t}$

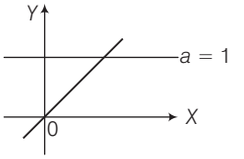
Equation of AB, $y - t^2 = -\frac{1}{t}(x - t)$

Put $x = 0, h = t^2 + 1$

Now, $\lim_{x \to 0}(h) = \lim_{t \to 0}(1 + t^2) = 1$ [as $x \to 0$, then $t \to 0$]

100. For $0 < a \le 1$ the line always intersects $y = a^x$



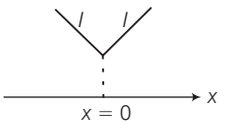


For $a > 1$ say $a = e$ consider $f(x) = e^x - x$

$f'(x) = e^x - 1$

$f'(x) > 0$ for $x > 0$ and $f'(x) < 0$ for $x < 0$

$\therefore f(x)$ is increasing ($\uparrow$) for $x > 0$ and decreasing ($\downarrow$) for $x < 0$



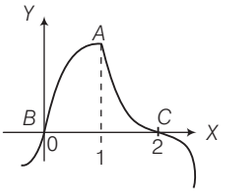
$y = e^x$ always lies above $y = x$, i.e. $e^x - x \ge 1$ for $a > 1$, hence never intersects $\Rightarrow a \in (0, 1]$

101. $\frac{dy}{dx} = -\frac{4\cos x}{\sin^2 x} + \frac{\cos x}{(1-\sin x)^2} = 0$ gives $\sin x = \frac{2}{3}$

note that $f(x) \to \infty$ as $x \to 0^+$ or $x \to \frac{\pi^-}{2}$ and between two maxima, we have a minima.

So, $a = \frac{4}{2/3} + \frac{1}{1-2/3} = 9$

102.



A, B, C are the 3 critical points of $y = f(x)$.

103. Only at A and E, $f(x) = 0$ but does not change sign.

104. Note that f is defined for $x > 0$

$$f'(x) = \frac{1}{2}\cos 2x = 0 \quad \Rightarrow \quad x = n\pi \pm \frac{\pi}{4}$$

105. Obviously, f is increasing and g is decreasing in (x_1, x_2), hence $f(g(\alpha^2 - 2\alpha)) > f(g(3\alpha - 4))$ as f is increasing

$\Rightarrow \quad g(\alpha^2 - 2\alpha) > g(3\alpha - 4)$

$\therefore \quad \alpha^2 - 2\alpha < 3\alpha - 4$ as g is decreasing.

$$\alpha^2 - 5\alpha + 4 < 0$$

$\Rightarrow \quad (\alpha - 1)(\alpha - 4) < 0 \quad \Rightarrow \quad \alpha \in (1, 4)$

106. $f(x) = \frac{t + 3x - x^2}{x - 4}$, $f'(x) = \frac{(x-4)(3-2x) - (t + 3x - x^2)}{(x-4)^2}$

For maximum or minimum, $f'(x) = 0$

$$-2x^2 + 11x - 12 - t - 3x + x^2 = 0$$

$$-x^2 + 8x - (12 + t) = 0$$

For one M and m, $D > 0$

$$64 - 4(12 + t) > 0$$

$$16 - 12 - t > 0 \quad \Rightarrow \quad 4 > t \text{ or } t < 4$$

Hence, the greatest value of t is 3.

107. Let $g(x) = f'(x)\sin(\pi f(x))$ ...(i)

$\therefore \; g'(x) = \pi(f'(x))^2 \cos[\pi f(x)] + \sin(\pi f(x)) \cdot f''(x)$ ...(ii)

Here, $g(a) = 0 = g(b)$

and $g(\alpha) = g(\beta) = 0$

$$g\left(\frac{\alpha + \beta}{2}\right) = 0$$

$\therefore$ According to Rolle's theorem,

$g'(x) = 0$ has atleast one root in $(\alpha, a), \left(a, \frac{a+b}{2}\right), \left(\frac{a+b}{2}, b\right)$ and (b, β), i.e. minimum of 4 roots.

108. Let $g(x) = \frac{1}{|x-4|+1}$

$\Rightarrow \quad g(x+12) = \frac{1}{|x+8|+1}$

When $x < -8$ both $g(x)$ and $g(x+12)$ are increasing, hence maximum value can't occur in this interval.

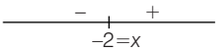
Similarly, for $x \in (4, \infty)$ both $g(x)$ and $g(x+12)$ are decreasing, hence maximum value can't occur in this interval.

So, now for all values of $x \in (-8, 4)$

$$f(x) = \frac{1}{x+9} + \frac{1}{5-x}$$

$\Rightarrow \quad f'(x) = -\frac{1}{(x+9)^2} + \frac{1}{(5-x)^2} = \frac{(x+9)^2 - (5-x)^2}{(x+9)^2(5-x)^2}$

$\therefore \quad f'(x) = \frac{14(2x+4)}{(x+9)^2(5-x)^2}$



$\therefore$ Minimum at $x = -2$, and maximum occurs at $x = +4$ or $x = -8$.

Here, $f(4) = f(-8) = \frac{14}{13} = \frac{p}{q}$

$\therefore \quad (p - q) = 1$

109. Here, to find the least value of $\alpha \in R$, for which $4\alpha x^2 + \frac{1}{x} \geq 1$, for all $x > 0$.

i.e. to find the minimum value of α when $y = 4\alpha x^2 + \frac{1}{x}$; $x > 0$ attains minimum value of α.

$\therefore \quad \frac{dy}{dx} = 8\alpha x - \frac{1}{x^2}$ ...(i)

Now, $\frac{d^2y}{dx^2} = 8\alpha + \frac{2}{x^3}$ ...(ii)

When $\frac{dy}{dx} = 0$, then $8x^3\alpha = 1$

At $x = \left(\frac{1}{8\alpha}\right)^{1/3}$, $\frac{d^2y}{dx^2} = 8\alpha + 16\alpha = 24\alpha$, Thus, y attains minimum when

$$x = \left(\frac{1}{8\alpha}\right)^{1/3}; \; \alpha > 0.$$

$\therefore \; y$ attains minimum when $x = \left(\frac{1}{8\alpha}\right)^{1/3}$.

i.e. $4\alpha\left(\frac{1}{8\alpha}\right)^{2/3} + (8\alpha)^{1/3} \geq 1$

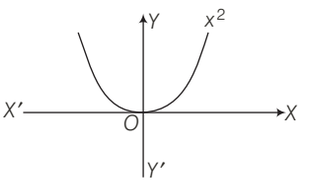
$\Rightarrow \quad \alpha^{1/3} + 2\alpha^{1/3} \geq 1$

$\Rightarrow \quad 3\alpha^{1/3} \geq 1 \Rightarrow \alpha \geq \frac{1}{27}$

Hence, the least value of α is $\frac{1}{27}$.

110. **Plan** The given equation contains algebraic and trigonometric functions called transcendental equation. To solve transcendental equations we should always plot the graph for LHS and RHS.

Here, $x^2 = x\sin x + \cos x$



Let $f(x) = x^2$ and $g(x) = x\sin x + \cos x$

We know that, the graph for $f(x) = x^2$

To plot, $g(x) = x\sin x + \cos x$

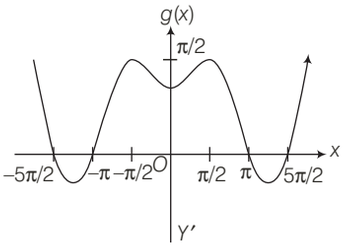
$g'(x) = x\cos x + \sin x - \sin x$

$g'(x) = x\cos x$ ...(i)

$g''(x) = -x\sin x + \cos x$ ...(ii)

Put $g'(x) = 0 \Rightarrow x\cos x = 0$

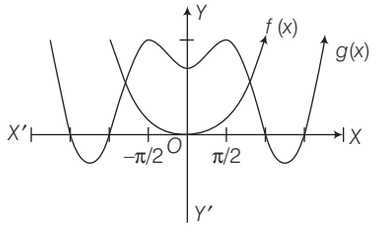
$\therefore$ $x = 0, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$



At $x = 0, \frac{3\pi}{2}, \frac{7\pi}{2}, ..., f''(x) > 0$, so minimum

At $x = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, ..., f'(x) < 0$, so maximum

So, graph of $f(x)$ and $g(x)$ are shown as



So, number of solutions are 2.

111. Here, $\lim_{x\to 2} \frac{f(x)\cdot g(x)}{f'(x)\cdot g'(x)} = 1$

$\Rightarrow \lim_{x\to 2} \frac{f(x)g'(x) + f'(x)g(x)}{f''(x)g'(x) + f'(x)g''(x)} = 1$ [using L'Hospital's rule]

$\Rightarrow \frac{f(2)g'(2) + f'(2)g(2)}{f''(2)g'(2) + f'(2)g''(2)} = 1$

$\Rightarrow \frac{f(2)g'(2)}{f''(2)g'(2)} = 1$ [$\because f'(2) = g(2) = 0$]

$\Rightarrow f(2) = f''(2)$ ...(i)

$\therefore$ $f(x) - f''(x) = 0$ for atleast one $x \in R$.

$\Rightarrow$ Option (d) is correct.

Also, $f: R \to (0, \infty)$

$\Rightarrow f(2) > 0$

$\therefore f''(2) = f(2) > 0$ [from Eq. (i)]

Since, $f'(2) = 0$ and $f''(2) > 0$

$\therefore f(x)$ attains local minimum at $x = 2$.

$\Rightarrow$ Option (a) is correct.

112. Given, $f(x) = \int_{\frac{1}{x}}^{x} \frac{e^{-\left(t+\frac{1}{t}\right)}}{t} dt$

$$f'(x) = 1\cdot \frac{e^{-\left(x+\frac{1}{x}\right)}}{x} - \left(\frac{-1}{x^2}\right)\frac{e^{-\left(\frac{1}{x}+x\right)}}{1/x}$$

$$= \frac{e^{-\left(x+\frac{1}{x}\right)}}{x} + \frac{e^{-\left(x+\frac{1}{x}\right)}}{x} = \frac{2e^{-\left(x+\frac{1}{x}\right)}}{x}$$

As $f'(x) > 0, \forall x \in (0, \infty)$

$\therefore$ $f(x)$ is monotonically increasing on $(0, \infty)$.

$\Rightarrow$ Option (a) is correct and option (b) is wrong.

Now, $f(x) + f\left(\frac{1}{x}\right) = \int_{1/x}^{x} \frac{e^{-\left(t+\frac{1}{t}\right)}}{t} dt + \int_{x}^{1/x} \frac{e^{-\left(t+\frac{1}{t}\right)}}{t} dt$

$= 0, \forall x \in (0, \infty)$

Now, let $g(x) = f(2^x) = \int_{2^{-x}}^{2^x} \frac{e^{-\left(t+\frac{1}{t}\right)}}{t} dt$

$$g(-x) = f(2^{-x}) = \int_{2^x}^{2^{-x}} \frac{e^{-\left(t+\frac{1}{t}\right)}}{t} dt = -g(x)$$

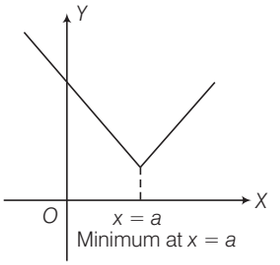
$\therefore$ $f(2^x)$ is an odd function of x on R.

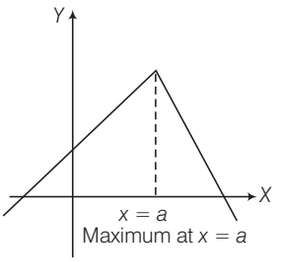
113. **Plan**

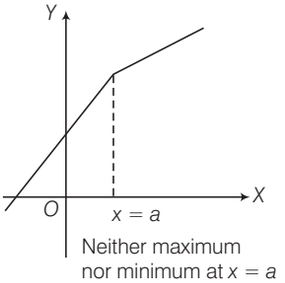
We know that, $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

$\Rightarrow$ $|x - a| = \begin{cases} x - a, & \text{if } x \geq a \\ -(x - a), & \text{if } x < a \end{cases}$

and for non-differentiable continuous function, the maximum or minimum can be checked with graph as





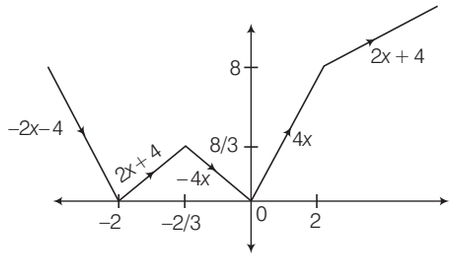


Here, $f(x) = 2|x| + |x + 2| - ||x + 2| - 2|x||$

$$= \begin{cases} -2x - (x + 2) + (x - 2), & \text{if when } x \leq -2 \\ -2x + x + 2 + 3x + 2, & \text{if when } -2 < x \leq -2/3 \\ -4x, & \text{if when } -\frac{2}{3} < x \leq 0 \\ 4x, & \text{if when } 0 < x \leq 2 \\ 2x + 4, & \text{if when } x > 2 \end{cases}$$

$$= \begin{cases} -2x-4, & \text{if} \quad x \le -2 \\ 2x+4, & \text{if} \quad -2 < x \le -2/3 \\ -4x, & \text{if} \quad -\frac{2}{3} < x \le 0 \\ 4x, & \text{if} \quad 0 < x \le 2 \\ 2x+4, & \text{if} \quad x > 2 \end{cases}$$

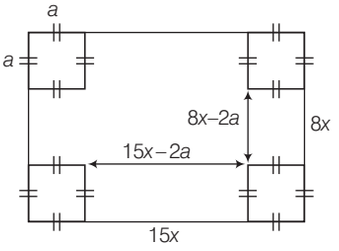
Graph for $y = f(x)$ is shown as



114. **Plan** The problem is based on the concept to maximise volume of cuboid, i.e. to form a function of volume, say $f(x)$ find $f'(x)$ and $f''(x)$.

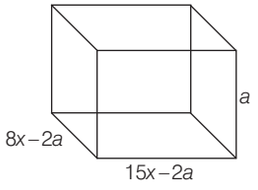
Put $f'(x) = 0$ and check $f''(x)$ to be + ve or − ve for minimum and maximum, respectively.

Here, $l = 15x - 2a$, $b = 8x - 2a$ and $h = a$



$\therefore$ Volume $= (8x - 2a)(15x - 2a)a$

$$V = 2a \cdot (4x - a)(15x - 2a) \qquad \ldots(i)$$



On differentiating Eq. (i) w.r.t. a, we get

$$\frac{dv}{da} = 6a^2 - 46ax + 60x^2$$

Again, differentiating, we get $\dfrac{d^2v}{da^2} = 12a - 46x$

Here, $\left(\dfrac{dv}{da}\right) = 0$

$\Rightarrow$ $6x^2 - 23x + 15 = 0$

At $a = 5$, $x = 3, \dfrac{5}{6}$

$\Rightarrow$ $\left(\dfrac{d^2v}{da^2}\right) = 2(30 - 23x)$

At $x = 3, \left(\dfrac{d^2v}{da^2}\right) = 2(30 - 69) < 0$

$\therefore$ Maximum when $x = 3$, also at $x = \dfrac{5}{6}$

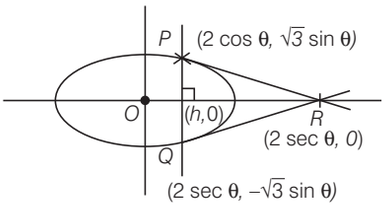
$\Rightarrow$ $\left(\dfrac{d^2v}{da^2}\right) > 0$

$\therefore$ At $x = 5/6$, volume is minimum.

Thus, sides are $8x = 24$ and $15x = 45$

115. **Plan** As to maximise or minimise area of triangle, we should find area in terms of parametric coordinates and use second derivative test.

Here, tangent at $P(2\cos\theta, \sqrt{3}\sin\theta)$ is



$$\frac{x}{2}\cos\theta + \frac{y}{\sqrt{3}}\sin\theta = 1$$

$\therefore$ $R(2\sec\theta, 0)$

$\Rightarrow$ $\Delta =$ Area of ΔPQR

$$= \frac{1}{2}(2\sqrt{3}\sin\theta)(2\sec\theta - 2\cos\theta)$$

$$= 2\sqrt{3} \cdot \sin^3\theta / \cos\theta \qquad \ldots(i)$$

Since, $\dfrac{1}{2} \le h \le 1$

$\therefore$ $\dfrac{1}{2} \le 2\cos\theta \le 1$

$\Rightarrow$ $\dfrac{1}{4} \le \cos\theta \le \dfrac{1}{2}$ $\qquad \ldots(ii)$

$$\therefore \frac{d\Delta}{d\theta} = \frac{2\sqrt{3}\{\cos\theta \cdot 3\sin^2\theta\cos\theta - \sin^3\theta(-\sin\theta)\}}{\cos^2\theta}$$

$$= \frac{2\sqrt{3}\cdot\sin^2\theta}{\cos^2\theta}[3\cos^2\theta + \sin^2\theta] = \frac{2\sqrt{3}\sin^2\theta}{\cos^2\theta}\cdot[2\cos^2\theta + 1]$$

$$= 2\sqrt{3}\tan^2\theta(2\cos^2\theta + 1) > 0$$

When $\dfrac{1}{4} \le \cos\theta \le \dfrac{1}{2}$,

$\therefore$ $\Delta_1 = \Delta_{\max}$ occurs at $\cos\theta = \dfrac{1}{4} = \left(\dfrac{2\sqrt{3}\cdot\sin^3\theta}{\cos\theta}\right)$

When $\cos\theta = \dfrac{1}{4} = \dfrac{45\sqrt{5}}{8}$

$\Delta_2 = \Delta_{\min}$ occurs at $\cos\theta = \dfrac{1}{2} = \left(\dfrac{2\sqrt{3}\sin^3\theta}{\cos\theta}\right)$

When $\cos\theta = \dfrac{1}{2} = \dfrac{9}{2}$

$\therefore$ $\dfrac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 = 45 - 36 = 9$

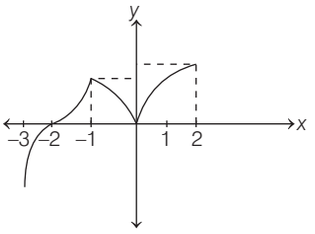
116. Given function, $f(x) = e^{x^2} + e^{-x^2}$, $g(x) = xe^{x^2} + e^{-x^2}$ and $h(x) = x^2e^{x^2} + e^{-x^2}$ are strictly increasing on $[0, 1]$.

Hence, at $x = 1$, the given function attains absolute maximum all equal to $e + 1/e$.

$\Rightarrow$ $a = b = c$

117. Given, $f(x) = \begin{cases} (2+x)^3, & \text{if } -3 < x \le -1 \\ x^{2/3}, & \text{if } -1 < x < 2 \end{cases}$

$\Rightarrow$ $f'(x) = \begin{cases} 3(x+2)^2, & \text{if } -3 < x < -1 \\ \frac{2}{3}x^{-\frac{1}{3}}, & \text{if } -1 < x < 2 \end{cases}$



Clearly, $f'(x)$ changes its sign at $x = -1$ from +ve to –ve and so $f(x)$ has local maxima at $x = -1$.

Also, $f'(0)$ does not exist but $f'(0^-) < 0$ and $f'(0^+) > 0$. It can only be inferred that $f(x)$ has a possibility of a minima at $x = 0$. Hence, the given function has one local maxima at $x = -1$ and one local minima at $x = 0$.

118. Given, $g(u) = 2\tan^{-1}(e^u) - \frac{\pi}{2}$ for $u \in (-\infty, \infty)$

$$g(-u) = 2\tan^{-1}(e^{-u}) - \frac{\pi}{2} = 2(\cot^{-1}(e^u)) - \frac{\pi}{2}$$

$$= 2\left(\frac{\pi}{2} - \tan^{-1}(e^u)\right) - \frac{\pi}{2}$$

$$= \pi/2 - 2\tan^{-1}(e^u) = -g(u)$$

$\therefore$ $g(-u) = -g(u)$

$\Rightarrow$ $g(u)$ is an odd function.

We have, $g(u) = 2\tan^{-1}(e^u) - \pi/2$

$$g'(u) = \frac{2e^u}{1 + e^{2u}}$$

$g'(u) > 0, \forall\ x \in R$ $\quad [\because e^u > 0]$

So, $g'(u)$ is increasing for all $x \in R$.

119. Let $f(x) = bx^2 + ax + c$

$\because$ $f(0) = 0 \Rightarrow c = 0$

and $f(1) = 1 \Rightarrow a + b = 1$

$\therefore$ $f(x) = ax + (1 - a)x^2$

Also, $f'(x) > 0$, for $x \in (0, 1)$

$\Rightarrow$ $a + 2(1 - a)x > 0$

$\Rightarrow$ $a(1 - 2x) + 2x > 0$

$\Rightarrow$ $a > \frac{2x}{2x - 1} \Rightarrow 0 < a < 2$, since $x \in (0, 1)$

$\therefore$ $f(x) = ax + (1 - a)x^2$, $0 < a < 2$

120. Given, $f(x) = x^3 + bx^2 + cx + d$

$\Rightarrow$ $f'(x) = 3x^2 + 2bx + c$

As we know that, if $ax^2 + bx + c > 0, \forall x$, then $a > 0$ and $D < 0$.

Now, $D = 4b^2 - 12c = 4(b^2 - c) - 8c$

[where, $b^2 - c < 0$ and $c > 0$]

$\therefore$ $D = (-\text{ve})$ or $D < 0$

$\Rightarrow f'(x) = 3x^2 + 2bx + c > 0\ \forall x \in (-\infty, \infty)$ [as $D < 0$ and $a > 0$]

Hence, $f(x)$ is strictly increasing function.

121. Given $f(x) = x^2 + 2bx + 2c^2$ and $g(x) = -x^2 - 2cx + b^2$

Then, $f(x)$ is minimum and $g(x)$ is maximum at $\left(x = \frac{-b}{4a} \text{ and } f(x) = \frac{-D}{4a}\right)$, respectively.

$\therefore$ $\min f(x) = \frac{-(4b^2 - 8c^2)}{4} = (2c^2 - b^2)$

and $\max g(x) = -\frac{(4c^2 + 4b^2)}{4(-1)} = (b^2 + c^2)$

Now, $\min f(x) > \max g(x)$

$\Rightarrow$ $2c^2 - b^2 > b^2 + c^2 \Rightarrow c^2 > 2b^2 \Rightarrow |c| > \sqrt{2}|b|$

122. Let $f(x) = 3\sin x - 4\sin^3 x = \sin 3x$

The longest interval in which $\sin x$ is increasing is of length π. So, the length of largest interval in which $f(x) = \sin 3x$ is increasing is $\frac{\pi}{3}$.

123. Given, $\cot\alpha_1 \cdot \cot\alpha_2 \cdot \ldots \cdot \cot\alpha_n = 1$

$\Rightarrow$ $\frac{\cos\alpha_1}{\sin\alpha_1} \cdot \frac{\cos\alpha_2}{\sin\alpha_2} \cdot \frac{\cos\alpha_3}{\sin\alpha_3} \cdot \ldots \cdot \frac{\cos\alpha_n}{\sin\alpha_n} = 1$

Let $\cos\alpha_1 \cdot \cos\alpha_2 \cdot \cos\alpha_3 \cdot \ldots \cdot \cos\alpha_n = k$ ...(i)

and $\sin\alpha_1 \cdot \sin\alpha_2 \cdot \sin\alpha_3 \cdot \ldots \cdot \sin\alpha_n = k$ ... (ii)

Again, on multiplying Eqs. (i) and (ii), we get

$(\cos\alpha_1 \cdot \cos\alpha_2 \cdot \cos\alpha_3 \cdot \ldots \cdot \cos\alpha_n)$

$\times (\sin\alpha_1 \cdot \sin\alpha_2 \cdot \sin\alpha_3 \cdot \ldots \cdot \sin\alpha_n) = k^2$

$$k^2 = \frac{1}{2 \times 2 \times \ldots n \text{ times}} (2\sin\alpha_1\cos\alpha_1)$$

$$(2\sin\alpha_2\cos\alpha_2)\ldots(2\sin\alpha_n\cos\alpha_n)$$

$\Rightarrow$ $k^2 = \frac{1}{2^n}(\sin 2\alpha_1)(\sin 2\alpha_2) \ldots (\sin 2\alpha_n)$

$$\le \frac{1}{2^n}\sin 2\alpha_i \le 1, \forall 1 \le i < n \Rightarrow k \le \frac{1}{2^{n/2}}$$

124. Given, $f(x) = \begin{cases} e^x, & \text{if } 0 \le x \le 1 \\ 2 - e^{x-1}, & \text{if } 1 < x \le 2 \\ x - e, & \text{if } 2 < x \le 3 \end{cases}$

and $g(x) = \int_0^x f(t)\,dt$

$\Rightarrow$ $g'(x) = f(x)$

Put $g'(x) = 0 \Rightarrow x = 1 + \log_e 2$ and $x = e$.

Also, $g''(x) = \begin{cases} e^x, & \text{if } 0 \le x \le 1 \\ -e^{x-1}, & \text{if } 1 < x \le 2 \\ 1, & \text{if } 2 < x \le 3 \end{cases}$

At $x = 1 + \log_e 2$, $g''(1 + \log_e 2) = -e^{\log_e 2} < 0$, $g(x)$ has a local maximum.

Also, at $x = e$,

$g''(e) = 1 > 0$, $g(x)$ has a local minima.

$\because$ $f(x)$ is discontinuous at $x = 1$, then we get local maxima at $x = 1$ and local minima at $x = 2$.

Hence, (a) and (b) are correct answers.

125. Since, $f(x)$ has local maxima at $x = -1$ and $f'(x)$ has local minima at $x = 0$.

$\therefore$ $f''(x) = \lambda x$

On integrating, we get

$$f'(x) = \lambda \frac{x^2}{2} + c \qquad [\because f'(-1) = 0]$$

$$\Rightarrow \quad \frac{\lambda}{2} + c = 0 \Rightarrow \lambda = -2c \qquad \ldots(i)$$

Again, integrating on both sides, we get

$$f(x) = \lambda \frac{x^3}{6} + cx + d$$

$$\Rightarrow \quad f(2) = \lambda\left(\frac{8}{6}\right) + 2c + d = 18 \qquad \ldots(ii)$$

and $$f(1) = \frac{\lambda}{6} + c + d = -1 \qquad \ldots(iii)$$

From Eqs. (i), (ii) and (iii), we get

$$f(x) = \frac{1}{4}(19x^3 - 57x + 34)$$

$$\therefore \quad f'(x) = \frac{1}{4}(57x^2 - 57) = \frac{57}{4}(x-1)(x+1)$$

For maxima or minima, put $f'(x) = 0$

$$\Rightarrow \quad x = 1, -1$$

Now $$f''(x) = \frac{1}{4}(114x)$$

At $x = 1$, $f''(x) > 0$, minima

At $x = -1$, $f''(x) < 0$, maxima

$\therefore$ $f(x)$ is increasing for $[1, 2\sqrt{5}]$.

$\therefore$ $f(x)$ has local maxima at $x = -1$ and $f(x)$ has local minima at $x = 1$.

Also, $$f(0) = 34/4$$

Hence, (b) and (c) are the correct answers.

126. $$f(x) = \frac{(x^2 + ax + 1) - 2ax}{x^2 + ax + 1} = 1 - \frac{2ax}{x^2 + ax + 1}$$

$$f'(x) = -\left[\frac{(x^2 + ax + 1)\cdot 2a - 2ax(2x + a)}{(x^2 + ax + 1)^2}\right]$$

$$= -\left[\frac{-2ax^2 + 2a}{(x^2 + ax + a)^2}\right] = 2a\left[\frac{(x^2 - 1)}{(x^2 + ax + 1)^2}\right] \qquad \ldots(i)$$

$$f''(x) = 2a\left[\frac{(x^2 + ax + 1)^2(2x) - 2(x^2 - 1)(x^2 + ax + 1)(2x + a)}{(x^2 + ax + 1)^4}\right]$$

$$= 2a\left[\frac{2x(x^2 + ax + 1) - 2(x^2 - 1)(2x + a)}{(x^2 + ax + 1)^3}\right] \qquad \ldots(ii)$$

Now, $$f''(1) = \frac{4a(a+2)}{(a+2)^3} = \frac{4a}{(a+2)^2}$$

and $$f''(-1) = \frac{4a(a-2)}{(2-a)^3} = \frac{-4a}{(a-2)^2}$$

$\therefore$ $(2 + a)^2 f''(1) + (2 - a)^2 f''(-1) = 4a - 4a = 0$

127. When $x \in (-1, 1)$,

$$x^2 < 1$$

$\Rightarrow x^2 - 1 < 0$

$\therefore$ $f'(x) < 0$, $f(x)$ is decreasing.

Also, at $x = 1$, $f''(1) = \frac{4a}{(a+2)^2} > 0$ $\qquad [\because 0 < a < 2]$

So, $f(x)$ has local minimum at $x = 1$.

128. $$g'(x) = \frac{f'(e^x)}{1 + (e^x)^2}\cdot e^x = 2a\left[\frac{e^{2x} - 1}{(e^{2x} + ae^x + 1)^2}\right]\left(\frac{e^x}{1 + e^{2x}}\right)$$

$g'(x) = 0$, if $e^{2x} - 1 = 0$,

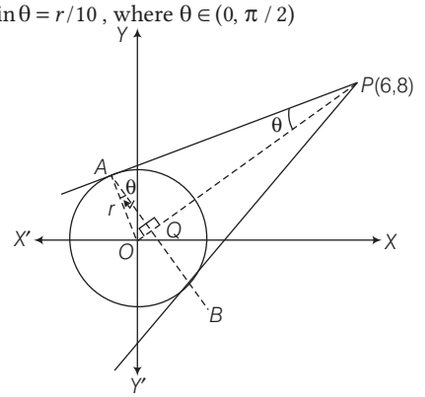
i.e. $$x = 0$$

If $$x < 0, e^{2x} < 1$$

$$\Rightarrow \quad g'(x) < 0$$

129. To maximize area of ΔAPB, we know that, $OP = 10$

and $\sin\theta = r/10$, where $\theta \in (0, \pi/2)$ $\qquad \ldots$ (i)



$\therefore$ $$\text{Area} = \frac{1}{2}(2AQ)(PQ)$$

$$= AQ \cdot PQ = (r\cos\theta)(10 - OQ)$$

$$= (r\cos\theta)(10 - r\sin\theta)$$

$$= 10\sin\theta\cos\theta(10 - 10\sin^2\theta) \qquad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad A = 100\cos^3\theta\sin\theta$$

$$\Rightarrow \quad \frac{dA}{d\theta} = 100\cos^4\theta - 300\cos^2\theta\cdot\sin^2\theta$$

Put $$\frac{dA}{d\theta} = 0$$

$$\Rightarrow \quad \cos^2\theta = 3\sin^2\theta$$

$$\Rightarrow \quad \tan\theta = 1/\sqrt{3}$$

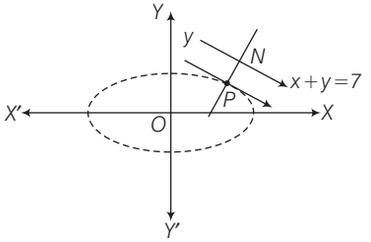
$$\Rightarrow \quad \theta = \pi/6$$

At which $\frac{dA}{d\theta} < 0$, thus when $\theta = \pi/6$, area is maximum

From Eq. (i), $r = 10\sin\frac{\pi}{6} = 5$ units

130. Let us take a point $P(\sqrt{6}\cos\theta, \sqrt{3}\sin\theta)$ on $\frac{x^2}{6}+\frac{y^2}{3}=1$.

Now, to minimise the distance from P to given straight line $x+y=7$, shortest distance exists along the common normal.



Slope of normal at $P=\frac{a^2/x_1}{b^2/y_1}=\frac{\sqrt{6}\sec\theta}{\sqrt{6}\,\mathrm{cosec}\,\theta}=\sqrt{2}\tan\theta=1$

So, $\cos\theta=\sqrt{\frac{2}{3}}$ and $\sin\theta=\frac{1}{\sqrt{3}}$

Hence, required point is $P(2,1)$.

131. **Plan**

(i) Local maximum and local minimum are those points at which $f'(x)=0$, when defined for all real numbers.

(ii) Local maximum and local minimum for piecewise functions are also been checked at sharp edges.

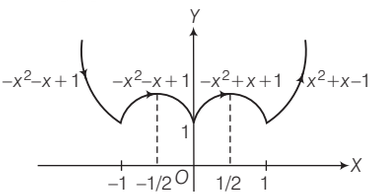
Description of Situation $y=|x|=\begin{cases}x, & \text{if } x\geq 0\\ -x, & \text{if } x<0\end{cases}$

Also, $y=|x^2-1|=\begin{cases}(x^2-1), & \text{if } x\leq -1 \text{ or } x\geq 1\\ (1-x^2), & \text{if } -1\leq x\leq 1\end{cases}$

$$y=|x|+|x^2-1|=\begin{cases}-x+1-x^2, & \text{if } x\leq -1\\ -x+1-x^2, & \text{if } -1\leq x\leq 0\\ x+1-x^2, & \text{if } 0\leq x\leq 1\\ x+x^2-1, & \text{if } x\geq 1\end{cases}$$

$$=\begin{cases}-x^2-x+1, & \text{if } x\leq -1\\ -x^2-x+1, & \text{if } -1\leq x\leq 0\\ -x^2+x+1, & \text{if } 0\leq x\leq 1\\ x^2+x-1, & \text{if } x\geq 1\end{cases}$$

which could be graphically shown as



Thus, $f(x)$ attains maximum at $x=\frac{1}{2},\frac{-1}{2}$ and $f(x)$ attains minimum at $x=-1,0,1$.

$\Rightarrow$ Total number of points = 5

132. **Plan** If $f(x)$ is least degree polynomial having local maximum and local minimum at α and β.

Then, $f'(x)=\lambda(x-\alpha)(x-\beta)$

Here, $p'(x)=\lambda(x-1)(x-3)=\lambda(x^2-4x+3)$

On integrating both sides between 1 to 3, we get

$$\int_1^3 p'(x)\,dx=\int_1^3\lambda(x^2-4x+3)\,dx$$

$$\Rightarrow\quad (p(x))_1^3=\lambda\left(\frac{x^3}{3}-2x^2+3x\right)_1^3$$

$$\Rightarrow\quad p(3)-p(1)=\lambda\left((9-18+9)-\left(\frac{1}{3}-2+3\right)\right)$$

$$\Rightarrow\quad 2-6=\lambda\left\{\frac{-4}{3}\right\}\quad\Rightarrow\quad\lambda=3$$

$$\Rightarrow\quad p'(x)=3(x-1)(x-3)$$

$$\therefore\quad p'(0)=9$$

133. Let $g(x)=e^{f(x)},\ \forall\ x\in R\ \Rightarrow\ g'(x)=e^{f(x)}\cdot f'(x)$

$\Rightarrow f'(x)$ changes its sign from positive to negative in the neighbourhood of $x=2009$

$\Rightarrow f(x)$ has local maxima at $x=2009$.

So, the number of local maximum is one.

134. Let $f(\theta)=\frac{1}{\sin^2\theta+3\sin\theta\cos\theta+5\cos^2\theta}$

Again let, $g(\theta)=\sin^2\theta+3\sin\theta\cos\theta+5\cos^2\theta$

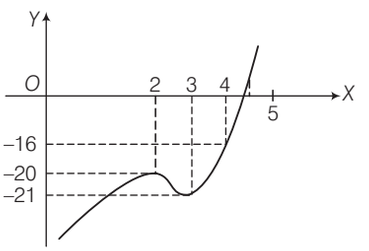
$$=\frac{1-\cos2\theta}{2}+5\left(\frac{1+\cos2\theta}{2}\right)+\frac{3}{2}\sin2\theta$$

$$=3+2\cos2\theta+\frac{3}{2}\sin2\theta$$

$$\therefore\quad g(\theta)_{\min}=3-\sqrt{4+\frac{9}{4}}=3-\frac{5}{2}=\frac{1}{2}$$

$\therefore$ Maximum value of $f(\theta)=\frac{1}{1/2}=2$

135. Given, $A=\{x|x^2+20\leq 9x\}=\{x|x\in[4,5]\}$



Now, $f'(x)=6(x^2-5x+6)$

Put $f'(x)=0\ \Rightarrow\ x=2,3$

$f(2)=-20,\ f(3)=-21,\ f(4)=-16,\ f(5)=7$

From graph, maximum value of $f(x)$ on set A is $f(5)=7$.

136. According to given information, we have

Perimeter of square + Perimeter of circle = 2 units

$$\Rightarrow\quad 4x+2\pi r=2$$

$$\Rightarrow\quad r=\frac{1-2x}{\pi}\qquad\ldots(i)$$

Now, let A be the sum of the areas of the square and the circle.

Then, $A=x^2+\pi r^2=x^2+\pi\frac{(1-2x)^2}{\pi^2}$

$$\Rightarrow\quad A(x)=x^2+\frac{(1-2x)^2}{\pi}$$

Now, for minimum value of $A(x)$, $\frac{dA}{dx} = 0$

$$\Rightarrow \quad 2x + \frac{2(1-2x)}{\pi} \cdot (-2) = 0 \Rightarrow \quad x = \frac{2-4x}{\pi}$$

$$\Rightarrow \qquad \pi x + 4x = 2$$

$$\Rightarrow \qquad x = \frac{2}{\pi + 4} \qquad \text{...(ii)}$$

Now, from Eq. (i), we get

$$r = \frac{1 - 2 \cdot \frac{2}{\pi + 4}}{\pi} = \frac{\pi + 4 - 4}{\pi(\pi + 4)} = \frac{1}{\pi + 4} \qquad \text{...(iii)}$$

From Eqs. (ii) and (iii), we get $x = 2r$

137. Here, $x = -1$ and $x = 2$ are extreme points of $f(x) = \alpha \log|x| + \beta x^2 + x$, then

$$f'(x) = \frac{\alpha}{x} + 2\beta x + 1$$

$$\therefore \quad f'(-1) = -\alpha - 2\beta + 1 = 0 \qquad \text{...(i)}$$

[at extreme point, $f'(x) = 0$]

and $\quad f'(-2) = \frac{\alpha}{2} + 4\beta + 1 = 0 \qquad$ ...(ii)

On solving Eqs. (i) and (ii), we get

$$\alpha = 2, \beta = -\frac{1}{2}$$

138. Given

(i) A function f, such that

$$f(x) = \log|x| + bx^2 + ax, x \neq 0$$

(ii) The function f has extrema at $x = -1$ and $x = 2$, *i.e.,* $f'(-1) = f'(2) = 0$ and $f''(-1) \neq 0 \neq f''(2)$.

Now, given function f is given by

$$f(x) = \log|x| + bx^2 + ax$$

$$\Rightarrow \qquad f'(x) = \frac{1}{x} + 2bx + a \Rightarrow \quad f''(x) = \frac{-1}{x^2} + 2b$$

Since, f has extrema at $x = -1$ and $x = 2$.

Hence, $\qquad f'(-1) = 0 = f'(2)$

$$f'(-1) = 0$$

$$\Rightarrow \qquad a - 2b = 1 \qquad \text{...(i)}$$

and $\qquad f'(2) = 0$

$$\Rightarrow \qquad a + 4b = \frac{-1}{2} \qquad \text{...(ii)}$$

On solving Eqs. (i) and (ii), we get

$$a = \frac{1}{2} \quad \text{and} \quad b = \frac{-1}{4}$$

$$\Rightarrow \qquad f''(x) = \frac{-1}{x^2} + \frac{-1}{2} = -\left(\frac{x^2 + 2}{2x^2}\right)$$

$$\Rightarrow \quad f''(-1) < 0 \quad \text{and } f''(2) < 0$$

Hence, f has local maxima at both $x = -1$ and $x = 2$.

Hence, Statement I is correct.

Also, while solving for Statement I, we found the values of a and b, which justify that Statement II is also correct.

However, Statement II does not explain Statement I in any way.

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PREFACE

"YOU CAN DO ANYTHING IF YOU SET YOUR MIND TO IT, I TEACH CALCULUS TO JEE ASPIRANTS BUT BELIEVE THE MOST IMPORTANT FORMULA IS COURAGE + DREAMS = SUCCESS"

It is a matter of great pride and honour for me to have received such an overwhelming response to the previous editions of this book from the readers. In a way, this has inspired me to revise this book thoroughly as per the changed pattern of JEE Main & Advanced. I have tried to make the contents more relevant as per the needs of students, many topics have been re-written, a lot of new problems of new types have been added in etcetc. All possible efforts are made to remove all the printing errors that had crept in previous editions. The book is now in such a shape that the students would feel at ease while going through the problems, which will in turn clear their concepts too.

A Summary of changes that have been made in Revised & Enlarged Edition

- Theory has been completely updated so as to accommodate all the changes made in JEE Syllabus & Pattern in recent years.
- The most important point about this new edition is, now the whole text matter of each chapter has been divided into small sessions with exercise in each session. In this way the reader will be able to go through the whole chapter in a systematic way.
- Just after completion of theory, Solved Examples of all JEE types have been given, providing the students a complete understanding of all the formats of JEE questions & the level of difficulty of questions generally asked in JEE.
- Along with exercises given with each session, a complete cumulative exercises have been given at the end of each chapter so as to give the students complete practice for JEE along with the assessment of knowledge that they have gained with the study of the chapter.
- Last 10 Years questions asked in JEE Main & Adv, IIT-JEE & AIEEE have been covered in all the chapters.

However I have made the best efforts and put my all calculus teaching experience in revising this book. Still I am looking forward to get the valuable suggestions and criticism from my own fraternity i.e. the fraternity of JEE teachers.

I would also like to motivate the students to send their suggestions or the changes that they want to be incorporated in this book.

All the suggestions given by you all will be kept in prime focus at the time of next revision of the book.

Amit M. Agarwal

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SYLLABUS

JEE MAIN

Integral Calculus

Integral as an anti - derivative. Fundamental integrals involving algebraic, trigonometric, exponential and logarithmic functions. Integration by substitution, by parts and by partial fractions. Integration using trigonometric identities.

Evaluation of simple integrals of the type

$$\int \frac{dx}{x^2 \pm a^2}, \quad \int \frac{dx}{\sqrt{x^2 \pm a^2}}, \quad \int \frac{dx}{a^2 - x^2}, \quad \int \frac{dx}{\sqrt{a^2 - x^2}},$$

$$\int \frac{dx}{ax^2 + bx + c}, \quad \int \frac{dx}{\sqrt{ax^2 + bx + c}}, \quad \int \frac{(px + q)\,dx}{ax^2 + bx + c},$$

$$\int \frac{(px + q)\,dx}{\sqrt{ax^2 + bx + c}}, \quad \int \sqrt{a^2 \pm x^2}\,dx \quad \text{and} \quad \int \sqrt{x^2 - a^2}\,dx$$

Integral as limit of a sum. Fundamental Theorem of Calculus. Properties of definite integrals. Evaluation of definite integrals, determining areas of the regions bounded by simple curves in
standard form.

Differential Equations

Ordinary differential equations, their order and degree. Formation of differential equations. Solution of differential equations by the method of separation of variables, solution of homogeneous and linear differential equations of the type $\frac{dy}{dx} + p(x)y = q(x)$

JEE ADVANCED

Integral Calculus

Integration as the inverse process of differentiation, indefinite integrals of standard functions, definite integrals and their properties, application of the Fundamental Theorem of Integral Calculus.

Integration by parts, integration by the methods of substitution and partial fractions, application of definite integrals to the determination of areas involving simple curves.

Formation of ordinary differential equations, solution of homogeneous differential equations, variables separable method, linear first order differential equations.

CHAPTER

01

Indefinite Integral

Learning Part

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Session 2
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Session 3
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Practice Part
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It was in this aspect that the process of integration was treated by Leibnitz, the symbol of $\int$ being regarded as the initial letter of the word sum, in the same way as the symbol of differentiation d is the initial letter in the word difference.

Definition

If f and g are functions of x such that $g'(x) = f(x)$, then the function g is called a anti-derivative (or primitive function or simply integral) of f w.r.t. x. It is written symbolically, $\int f(x)\,dx = g(x)$, where, $\frac{d}{dx} g(x) = f(x)$

Remarks

1. In other words, $\int f(x)\,dx = g(x)$ iff $g'(x) = f(x)$
2. $\int f(x)\,dx = g(x) + c$, where c is constant, $[\because (g(x) + C)' = g'(x) = f(x)]$ and C is called constant of integration.

Example 1 If $\frac{d}{dx}[x^{n+1} + c] = (n+1)\,x^n$, then find $\int x^n\,dx$.

Sol. As, $\frac{d}{dx}[x^{n+1} + C] = (n+1)\,x^n$

$\Rightarrow$ $(x^{n+1} + C)$ is anti-derivative or integral of $(n+1)\,x^n$.

$\therefore$ $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

Example 2 If $\frac{d}{dx}(\sin x + c) = \cos x$, then find $\int \cos x\,dx$.

Sol. As, $\frac{d}{dx}(\sin x + C) = \cos x$

$\Rightarrow \sin x + C$ is anti-derivative or integral of $\cos x$.

$\therefore$ $\int \cos x\,dx = (\sin x) + C$

Session 1

Fundamental of Indefinite Integral

Fundamental of Indefinite Integral

Since, $\frac{d}{dx}\{g(x)\} = f(x)$

$\Leftrightarrow$ $\int f(x)\,dx = g(x) + C$

Therefore, based upon this definition and various standard differentiation formulas, we obtain the following integration formulae

(i) $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n, n \neq -1 \Rightarrow \int x^n\,dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$

(ii) $\frac{d}{dx}(\log|x|) = \frac{1}{x} \Rightarrow \int \frac{1}{x}\,dx = \log|x| + C$, when $x \neq 0$

(iii) $\frac{d}{dx}(e^x) = e^x \Rightarrow \int e^x\,dx = e^x + C$

(iv) $\frac{d}{dx}\left(\frac{a^x}{\log_e a}\right) = a^x, a > 0, a \neq 1$

$\Rightarrow \int a^x\,dx = \frac{a^x}{\log_e a} + C$

(v) $\frac{d}{dx}(-\cos x) = \sin x \Rightarrow \int \sin x\,dx = -\cos x + C$

(vi) $\frac{d}{dx}(\sin x) = \cos x \Rightarrow \int \cos x\,dx = \sin x + C$

(vii) $\frac{d}{dx}(\tan x) = \sec^2 x \Rightarrow \int \sec^2 x\,dx = \tan x + C$

(viii) $\frac{d}{dx}(-\cot x) = \operatorname{cosec}^2 x \Rightarrow \int \operatorname{cosec}^2 x\,dx = -\cot x + C$

(ix) $\frac{d}{dx}(\sec x) = \sec x \tan x$

$\Rightarrow \int \sec x \tan x\,dx = \sec x + C$

(x) $\frac{d}{dx}(-\operatorname{cosec} x) = \operatorname{cosec} x \cot x$

$\Rightarrow \int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + C$

(xi) $\frac{d}{dx}(\log|\sin x|) = \cot x$

$\Rightarrow \int \cot x\,dx = \log|\sin x| + C$

(xii) $\frac{d}{dx}(-\log|\cos x|)=\tan x$

$\Rightarrow \int \tan x\,dx=-\log|\cos x|+C$

(xiii) $\frac{d}{dx}(\log|\sec x+\tan x|)=\sec x$

$\Rightarrow \int \sec x\,dx=\log|\sec x+\tan x|+C$

(xiv) $\frac{d}{dx}(\log|\operatorname{cosec} x-\cot x|)=\operatorname{cosec} x$

$\Rightarrow \int \operatorname{cosec} x\,dx=\log|\operatorname{cosec} x-\cot x|+C$

(xv) $\frac{d}{dx}\left(\sin^{-1}\frac{x}{a}\right)=\frac{1}{\sqrt{a^2-x^2}}$

$\Rightarrow \int \frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\left(\frac{x}{a}\right)+C$

(xvi) $\frac{d}{dx}\left(\cos^{-1}\frac{x}{a}\right)=\frac{-1}{\sqrt{a^2-x^2}}$

$\Rightarrow \int \frac{-1}{\sqrt{a^2-x^2}}dx=\cos^{-1}\left(\frac{x}{a}\right)+C$

(xvii) $\frac{d}{dx}\left(\frac{1}{a}\tan^{-1}\frac{x}{a}\right)=\frac{1}{a^2+x^2}$

$\Rightarrow \int \frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$

(xviii) $\frac{d}{dx}\left(\frac{1}{a}\cot^{-1}\frac{x}{a}\right)=\frac{-1}{a^2+x^2}$

$\Rightarrow \int \frac{-1}{a^2+x^2}dx=\frac{1}{a}\cot^{-1}\left(\frac{x}{a}\right)+C$

(xix) $\frac{d}{dx}\left(\frac{1}{a}\sec^{-1}\frac{x}{a}\right)=\frac{1}{x\sqrt{x^2-a^2}}$

$\Rightarrow \int \frac{dx}{x\sqrt{x^2-a^2}}=\frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)+C$

(xx) $\frac{d}{dx}\left(\frac{1}{a}\operatorname{cosec}^{-1}\frac{x}{a}\right)=\frac{-1}{x\sqrt{x^2-a^2}}$

$\Rightarrow \int \frac{-dx}{x\sqrt{x^2-a^2}}=\frac{1}{a}\operatorname{cosec}^{-1}\left(\frac{x}{a}\right)+C$

Example 3 Evaluate

(i) $\int \frac{x^2+5x-1}{\sqrt{x}}dx$ (ii) $\int (x^2+5)^3\,dx$

Sol. (i) $I=\int \frac{x^2+5x-1}{\sqrt{x}}dx=\int\left(\frac{x^2}{x^{1/2}}+\frac{5x^1}{x^{1/2}}-\frac{1}{x^{1/2}}\right)dx$

$=\int (x^{3/2}+5x^{1/2}-x^{-1/2})\,dx$

$\left[\text{using}\int x^n\,dx=\frac{x^{n+1}}{n+1}+C\right]$

$=\frac{x^{3/2+1}}{3/2+1}+\frac{5x^{1/2+1}}{1/2+1}-\frac{x^{-1/2+1}}{-1/2+1}+C$

$\Rightarrow \quad I=\frac{2}{5}x^{5/2}+\frac{2}{3}\cdot 5x^{3/2}-2x^{1/2}+C$

(ii) $I=\int (x^2+5)^3\,dx$ [using $(a+b)^3=a^3+3a^2b+3ab^2+b^3$]

$I=\int (x^6+15x^4+75x^2+125)\,dx$

$I=\frac{x^7}{7}+\frac{15x^5}{5}+\frac{75x^3}{3}+125x+C$

$I=\frac{x^7}{7}+3x^5+25x^3+125x+C$

Example 4 Evaluate

(i) $\int \tan^2 x\,dx$ (ii) $\int \frac{dx}{\sin^2 x\cos^2 x}$

(iii) $\int \frac{\sin^6 x+\cos^6 x}{\sin^2 x\cos^2 x}dx$ (iv) $\int \frac{\cos x-\cos 2x}{1-\cos x}dx$

Sol. (i) $I=\int \tan^2 x\,dx \Rightarrow I=\int(\sec^2 x-1)\,dx$

$I=\int \sec^2 x\,dx-\int 1\,dx$ [using $\int \sec^2 x dx=\tan x+C$]

$\Rightarrow I=\tan x-x+C$

(ii) $I=\int \frac{1}{\sin^2 x\cos^2 x}dx$

$I=\int \frac{\sin^2 x+\cos^2 x}{\sin^2 x\cos^2 x}dx$ [Using $\sin^2 x+\cos^2 x=1$]

$I=\int \frac{\sin^2 x}{\sin^2 x\cos^2 x}dx+\int \frac{\cos^2 x}{\sin^2 x\cos^2 x}dx$

$I=\int \sec^2 x\,dx+\int \operatorname{cosec}^2 x\,dx$

$I=\tan x-\cot x+C$

(iii) $I=\int \frac{\sin^6 x+\cos^6 x}{\sin^2 x\cos^2 x}dx$

$I=\int \frac{(\sin^2 x)^3+(\cos^2 x)^3}{\sin^2 x\cos^2 x}dx$

[using $(a+b)^3=a^3+b^3+3ab(a+b)$]

$I=\int \frac{(\sin^2 x+\cos^2 x)^3-3\sin^2 x\cos^2 x(\sin^2 x+\cos^2 x)}{\sin^2 x\cos^2 x}dx$

$I=\int \frac{1-3\sin^2 x\cos^2 x}{\sin^2 x\cos^2 x}dx$

$$I = \int \frac{1}{\sin^2 x \cos^2 x}\,dx - \int 3\,dx$$

$$I = \int \frac{(\sin^2 x + \cos^2 x)}{\sin^2 x \cos^2 x}\,dx - 3x + C$$

$$I = \int \sec^2 x\,dx + \int \operatorname{cosec}^2 x\,dx - 3x + C$$

$$I = \tan x - \cot x - 3x + C$$

(iv) $I = \int \frac{\cos x - \cos 2x}{1 - \cos x}\,dx$ [using $\cos 2x = 2\cos^2 x - 1$.]

$$I = \int \frac{\cos x - (2\cos^2 x - 1)}{1 - \cos x}\,dx$$

$$= \int \frac{-2\cos^2 x + \cos x + 1}{1 - \cos x}\,dx$$

$$\Rightarrow I = \int \frac{-(2\cos x + 1)(\cos x - 1)}{-(\cos x - 1)}\,dx$$

[as $-2\cos^2 x + \cos x - 1 = -(2\cos x + 1)\cdot(\cos x - 1)$]

$$\Rightarrow I = \int (2\cos x + 1)\,dx$$

$$\therefore \quad I = 2\sin x + x + C$$

Remark

In rational algebraic functions if the degree of numerator is greater than or equal to degree of denominator, then always divide the numerator by denominator and use the result of integration.

Example 5 Evaluate

(i) $\int \frac{x^3}{x+2}\,dx$ (ii) $\int \frac{x^2}{x^2+5}\,dx$

Sol. (i) $I = \int \frac{x^3}{x+2}\,dx = \int \frac{x^3 + 8 - 8}{x+2}\,dx$

$$I = \int \left(\frac{(x^3 + 2^3)}{x+2} - \frac{8}{x+2} \right) dx$$

$$\Rightarrow I = \int \left(\frac{(x+2)(x^2 - 2x + 4)}{x+2} - \frac{8}{x+2} \right) dx$$

$$I = \int \left(x^2 - 2x + 4 - \frac{8}{x+2} \right) dx$$

$$\therefore \quad I = \frac{x^3}{3} - x^2 + 4x - 8\log|x+2| + C$$

(ii) $I = \int \frac{x^2}{x^2+5}dx = \int \frac{x^2+5-5}{x^2+5}dx = \int \left(\frac{x^2+5}{x^2+5} - \frac{5}{x^2+5} \right) dx$

$$I = \int \left(1 - \frac{5}{x^2+5} \right) dx = x - 5\int \frac{dx}{x^2 + (\sqrt{5})^2}$$

$$I = x - \frac{5}{\sqrt{5}} \tan^{-1}\left(\frac{x}{\sqrt{5}} \right) + C$$

$$I = x - \sqrt{5}\tan^{-1}\left(\frac{x}{\sqrt{5}} \right) + C$$

Example 6 *Evaluate*

(i) $\int 5^{\log_e x}\,dx$ (ii) $\int 2^{\log_4 x}\,dx$

Sol. (i) $I = \int 5^{\log_e x}\,dx = \int x^{\log_e 5}\,dx$ [Using $a^{\log_e b} = b^{\log_c a}$]

$$= \frac{x^{\log_e 5 + 1}}{(\log_e 5 + 1)} + C$$

$$\therefore \quad \int 5^{\log_e x}\,dx = \frac{x^{\log_e 5 + 1}}{\log_e 5 + 1} + C$$

(ii) $I = \int 2^{\log_4 x}\,dx = \int 2^{\log_{2^2} x}\,dx = \int 2^{1/2 \log_2 x}\,dx$

$$\left[\text{using } \log_{b^n} x = \frac{1}{n}\log_b x \right]$$

$$= \int 2^{\log_2 \sqrt{x}}\,dx = \int \sqrt{x}\,dx \qquad [\text{using } a^{\log_a b} = b]$$

$$= \frac{x^{3/2}}{3/2} + C$$

$$\therefore \quad \int 2^{\log_4 x}\,dx = \frac{2}{3}x^{3/2} + C$$

Example 7 Evaluate $\int \frac{(\sqrt{x}+1)(x^2 - \sqrt{x})}{x\sqrt{x} + x + \sqrt{x}}\,dx$.

Sol. Here, $I = \int \frac{(\sqrt{x}+1)\cdot\sqrt{x}(x^{3/2} - 1)}{\sqrt{x}(x + \sqrt{x} + 1)}\,dx$

$$\therefore \quad I = \int \frac{(\sqrt{x}+1)[(\sqrt{x})^3 - 1^3]}{(x + \sqrt{x} + 1)}\,dx$$

$$= \int \frac{(\sqrt{x}+1)(\sqrt{x}-1)(x + \sqrt{x} + 1)}{(x + \sqrt{x} + 1)}\,dx$$

[Using, $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$]

$$= \int (x - 1)\,dx = \frac{x^2}{2} - x + C$$

Example 8 Evaluate

(i) $\int \frac{1 + 2x^2}{x^2(1 + x^2)}\,dx$ (ii) $\int \frac{x^6 - 1}{(x^2 + 1)}\,dx$

Sol. (i) Here, $I = \int \frac{1 + 2x^2}{x^2(1 + x^2)}\,dx = \int \frac{1 + x^2 + x^2}{x^2(1 + x^2)}\,dx$

$$= \int \frac{1 + x^2}{x^2(1 + x^2)}\,dx + \int \frac{x^2}{x^2(1 + x^2)}\,dx$$

$$= \int \frac{1}{x^2}\,dx + \int \frac{1}{1 + x^2}\,dx = -\frac{1}{x} + \tan^{-1} x + C$$

(ii) Here, $I = \int \frac{x^6 - 1}{x^2 + 1}\,dx = \int \frac{x^6 + 1 - 2}{x^2 + 1}\,dx$

$$= \int \frac{(x^2)^3 + 1^3}{x^2 + 1}\,dx - \int \frac{2}{x^2 + 1}\,dx$$

$$I=\int\frac{(x^2+1)(x^4-x^2+1)}{(x^2+1)}\,dx-2\int\frac{dx}{x^2+1}$$

[Using, $a^3+b^3=(a+b)(a^2-ab+b^2)$]

$$=\int(x^4-x^2+1)\,dx-2\int\frac{1}{x^2+1}\,dx$$

$$=\frac{x^5}{5}-\frac{x^3}{3}+x-2\tan^{-1}x+C$$

Example 9 Evaluate

(i) $\int\left(\frac{1-x^{-2}}{x^{1/2}-x^{-1/2}}-\frac{2}{x^{3/2}}+\frac{x^{-2}-x}{x^{1/2}-x^{-1/2}}\right)dx$

(ii) $\int\left(\frac{x^{-6}-64}{4+2x^{-1}+x^{-2}}\cdot\frac{x^2}{4-4x^{-1}+x^{-2}}-\frac{4x^2(2x+1)}{1-2x}\right)dx$

Sol. (i) Here, $I=\int\left(\frac{1-x^{-2}}{x^{1/2}-x^{-1/2}}-\frac{2}{x^{3/2}}+\frac{x^{-2}-x}{x^{1/2}-x^{-1/2}}\right)dx$

$$=\int\left(\frac{(1-x^{-2})+(x^{-2}-x)}{x^{1/2}-x^{-1/2}}-\frac{2}{x^{3/2}}\right)dx$$

$$=\int\left(\frac{1-x}{\sqrt{x}-\frac{1}{\sqrt{x}}}-\frac{2}{x^{3/2}}\right)dx=\int\left(\frac{1-x}{\frac{x-1}{\sqrt{x}}}-\frac{2}{x^{3/2}}\right)dx$$

$$=\int(-\sqrt{x}-2x^{-3/2})\,dx$$

$$=\left(-\frac{x^{3/2}}{3/2}-2\cdot\frac{x^{-1/2}}{-1/2}\right)+C=-\frac{2}{3}x^{3/2}+\frac{4}{\sqrt{x}}+C$$

(ii) Here

$$I=\int\left(\frac{x^{-6}-64}{4+2x^{-1}+x^{-2}}\cdot\frac{x^2}{4-4x^{-1}+x^{-2}}-\frac{4x^2(2x+1)}{1-2x}\right)dx$$

$$=\int\left(\frac{\frac{1-64x^6}{x^6}}{\frac{4x^2+2x+1}{x^2}}\cdot\frac{x^2}{\frac{4x^2-4x+1}{x^2}}-\frac{4x^2(2x+1)}{(1-2x)}\right)dx$$

$$=\int\left(\frac{1-(4x^2)^3}{x^6\cdot(4x^2+2x+1)}\cdot\frac{x^6}{(4x^2-4x+1)}-\frac{4x^2(2x+1)}{(1-2x)}\right)dx$$

$$=\int\left(\frac{(1-4x^2)(1+4x^2+16x^4)}{(4x^2+2x+1)(4x^2-4x+1)}-\frac{4x^2(2x+1)}{(1-2x)}\right)dx$$

$$=\int\left(\frac{(1-4x^2)\cdot(4x^2+2x+1)(4x^2-2x+1)}{(4x^2+2x+1)(2x-1)^2}-\frac{4x^2(2x+1)}{(1-2x)}\right)dx$$

[using, $16x^4+4x^2+1=16x^4+8x^2+1-4x^2$

$=(4x^2+1)^2-(2x)^2=(4x^2+1+2x)$]

$$=\int\left(\frac{(1-2x)(1+2x)(4x^2-2x+1)}{(1-2x)^2}-\frac{4x^2(2x+1)}{(1-2x)}\right)dx$$

$$=\int\left(\frac{(1+2x)\cdot(4x^2-2x+1)}{(1-2x)}-\frac{4x^2(2x+1)}{(1-2x)}\right)dx$$

$$=\int\frac{(2x+1)\{4x^2-2x+1-4x^2\}\,dx}{1-2x}$$

$$=\int\frac{(2x+1)(1-2x)}{(1-2x)}\,dx$$

$$=\int(2x+1)\,dx=x^2+x+C$$

Example 10 Evaluate

(i) $\int\frac{1}{\sin(x-a)\cos(x-b)}\,dx$

(ii) $\int\frac{1}{\cos(x-a)\cos(x-b)}\,dx$

Sol. (i) $I=\int\frac{1}{\sin(x-a)\cos(x-b)}\,dx$

$$I=\frac{\cos(a-b)}{\cos(a-b)}\cdot\int\frac{dx}{\sin(x-a)\cos(x-b)}$$

$$=\frac{1}{\cos(a-b)}\cdot\int\frac{\cos\{(x-b)-(x-a)\}}{\sin(x-a)\cos(x-b)}\,dx$$

$$=\frac{1}{\cos(a-b)}\cdot\int\left\{\frac{\cos(x-b)\cdot\cos(x-a)}{\sin(x-a)\cos(x-b)}+\frac{\sin(x-b)\cdot\sin(x-a)}{\sin(x-a)\cos(x-b)}\right\}dx$$

$$=\frac{1}{\cos(a-b)}\int\{\cot(x-a)+\tan(x-b)\}\,dx$$

$$=\frac{1}{\cos(a-b)}\{\log|\sin(x-a)|-\log|\cos(x-b)|\}+C$$

$$=\frac{1}{\cos(a-b)}\log_e\left|\frac{\sin(x-a)}{\cos(x-b)}\right|+C$$

(ii) $I=\int\frac{1}{\cos(x-a)\cos(x-b)}\,dx$

$$=\frac{1}{\sin(a-b)}\int\frac{\sin(a-b)}{\cos(x-a)\cos(x-b)}dx$$

$$=\frac{1}{\sin(a-b)}\int\frac{\sin\{(x-b)-(x-a)\}}{\cos(x-a)\cos(x-b)}\,dx$$

$$=\frac{1}{\sin(a-b)}\int\left\{\frac{\sin(x-b)\cos(x-a)}{\cos(x-a)\cos(x-b)}-\frac{\cos(x-b)\sin(x-a)}{\cos(x-a)\cos(x-b)}\right\}dx$$

$$=\frac{1}{\sin(a-b)}\int\{\tan(x-b)-\tan(x-a)\}\,dx$$

$$=\frac{1}{\sin(a-b)}[-\log|\cos(x-b)|+\log|\cos(x-a)|]+C$$

$$=\frac{1}{\sin(a-b)}\left[\log\left|\frac{\cos(x-a)}{\cos(x-b)}\right|\right]+C$$

Example 11 Evaluate $\int \frac{\sin(x+a)}{\sin(x+b)}\,dx$.

Sol. Let $I = \int \frac{\sin(x+a)}{\sin(x+b)}\,dx$. Put $x + b = t \Rightarrow dx = dt$

$$\therefore \quad I = \int \frac{\sin(t-b+a)}{\sin t}\,dt$$

$$= \int \left\{ \frac{\sin t \cos(a-b)}{\sin t} + \frac{\cos t \sin(a-b)}{\sin t} \right\} dt$$

$$= \cos(a-b)\int 1\,dt + \sin(a-b)\int \cot(t)\,dt$$

$$= t\cos(a-b) + \sin(a-b)\log|\sin t| + C$$

$$= (x+b)\cos(a-b) + \sin(a-b)\log|\sin(x+b)| + C$$

Example 12

(i) If $f'(x) = \frac{x}{2} + \frac{2}{x}$ and $f(1) = \frac{5}{4}$, then find $f(x)$.

(ii) The gradient of the curve is given by $\frac{dy}{dx} = 2x - \frac{3}{x^2}$.

The curve passes through (1, 2) find its equation.

Sol. (i) Given, $f'(x) = \frac{x}{2} + \frac{2}{x}$,

On integrating both sides w.r.t. x,

we get $\int f'(x)\,dx = \int \left(\frac{x}{2} + \frac{2}{x}\right) dx$

$$\Rightarrow \quad f(x) = \frac{1}{2}\cdot\frac{x^2}{2} + 2\log|x| + c \qquad \text{...(i)}$$

Now, as $f(1) = \frac{5}{4}$ (called as initial value problem

i.e. when $x = 1, y = \frac{5}{4}$ or $f(1) = \frac{5}{4}$)

Putting, $x = 1$ in Eq. (i),

$$f(1) = \frac{1}{4} + 2\log|1| + C, \text{ but } f(1) = \frac{5}{4}$$

$$\therefore \quad \frac{5}{4} = \frac{1}{4} + C \Rightarrow C = 1$$

$$\Rightarrow \quad f(x) = \frac{x^2}{4} + 2\log|x| + 1$$

(ii) Given, $\frac{dy}{dx} = 2x - \frac{3}{x^2}$ or $dy = \left(2x - \frac{3}{x^2}\right) dx$,

On integrating both sides w.r.t. x, we get

$$\int dy = \int \left(2x - \frac{3}{x^2}\right) dx$$

$$\Rightarrow \quad y = \frac{2x^2}{2} + \frac{3}{x} + C$$

Since, curve passes through (1, 1).

$$\Rightarrow \quad 1 = 1 + 3 + C \Rightarrow C = -3$$

$$\therefore \quad f(x) = x^2 + \frac{3}{x} - 3$$

Important Points Related to Integration

1. $\int k\,f(x)\,dx = k\int f(x)\,dx$, where k is constant. i.e. the integral of the product of a constant and a function = the constant × integral of the function

2. $\int \{f_1(x) \pm f_2(x) \pm \ldots \pm f_n(x)\}\,dx$

$$= \int f_1(x)\,dx \pm \int f_2(x)\,dx \pm \ldots \pm \int f_n(x)\,dx.$$

i.e. the integral of the sum or difference of a finite number of functions is equal to the sum or difference of the integrals of the various functions.

3. **Geometrical interpretation of constant of integration** By adding C means the graph of function would shift in upward or downward direction along y-axis as C is +ve or − ve respectively.

e.g. $$y = \int x\,dx = \frac{x^2}{2} + C$$

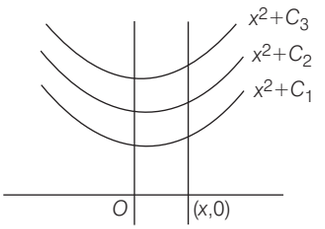

Figure. 1.1

$$\therefore \quad y = \int f(x)\,dx = F(x) + C$$

$$\Rightarrow \quad F'(x) = f(x);\ F'(x_1) = f(x_1)$$

Hence, $y = \int f(x)\,dx$ denotes a family of curves such that the slope of the tangent at $x = x_1$ on every member is same i.e. $F'(x_1) = f(x)$ [when x_1 lies in the domain of $f(x)$]

Hence, anti-derivative of a function is not unique. If $g_1(x)$ and $g_2(x)$ are two anti-derivatives of a function $f(x)$ on $[a, b]$, then they differ only by a constant.

i.e. $$g_1(x) - g_2(x) = C$$

Anti-derivative of a continuous function is differentiable.

4. If $f(x)$ is continuous, then

$$\int f(x)\,dx = F(x) + C$$

$$\Rightarrow \quad F'(x) = f(x)$$

$\Rightarrow$ always exists and is continuous.

$$\Rightarrow \quad F'(x)$$

5. If integral is discontinuous at $x = x_1$, then its anti-derivative at $x = x_1$ need not be discontinuous.
   e.g. $\int x^{-1/3} dx$. Here, $x^{-1/3}$ is discontinuous at $x = 0$. But $\int x^{-1/3} dx = \frac{3}{2} x^{2/3} + C$ is continuous at $x = 0$.
6. Anti-derivative of a periodic function need not be a periodic function. e.g. $f(x) = \cos x + 1$ is periodic but $\int (\cos x + 1)\, dx = \sin x + x + C$ is a periodic.

Daily Life Applications

The Derivative			The Integral		
Function	**Its derivative function**	**In symbols**	**Function**	**It's Anti-derivative Function**	**In symbols**
Distance (s)	Velocity (v)	$v = \frac{ds}{dt}$	Velocity	Distance	$s = \int v(t)\, dt$
Velocity (v)	Acceleration (a)	$a = \frac{dv}{dt}$	Acceleration	Velocity	$v = \int a(t)\, dt$
Mass (μ)	Liner Density (ρ)	$\rho = \frac{d\mu}{dx}$	Linear Density	Mass	$\mu = \int \rho(x)\, dx$
Population (P)	Instantaneous growth	$\frac{dP}{dt}$	Instantaneous Growth	Population	$\rho = \int \left(\frac{dP}{dt}\right) dt$
Cost (C)	Marginal cost (MC)	$\text{MC} = \frac{dC}{dq}$	Marginal Cost	Cost	$C(q) = \int \left(\frac{dC}{dq}\right) dq$
Revenue (R)	Marginal Revenue (MR)	$\text{MR} = \frac{dR}{dq}$	Marginal Revenue	Revenue	$R(q) = \int \left(\frac{dR}{dq}\right) dq$

Here, q is quantity of products.

Exercise for Session 1

- Evaluate the following integration

1. $\int \frac{dx}{\sqrt{x+1} - \sqrt{x}}$
2. $\int \frac{x^2 + 3}{x^6 (x^2 + 1)} dx$
3. $\int \frac{(1+x)^2}{x(1+x^2)} dx$
4. $\int \frac{x^4}{1+x^2} dx$
5. $\int \frac{x^4 + x^2 + 1}{2(1+x^2)} dx$
6. $\int \frac{(x^2 + \sin^2 x)\sec^2 x}{(1+x^2)} dx$
7. $\int \frac{x^2}{(a+bx)^2} dx$
8. $\int 2^x \cdot e^x \cdot dx$
9. $\int \frac{e^{3x} + e^{5x}}{e^x + e^{-x}} dx$
10. $\int (e^{a \log x} + e^{x \log a})\, dx$
11. $\int \frac{1 + \cos 4x}{\cot x - \tan x} dx$
12. $\int \tan x \tan 2x \tan 3x\, dx$
13. $\int \frac{\sin 4x}{\sin x} dx$
14. $\int \cos^3 x\, dx$
15. $\int \sin^3 x \cos^3 x\, dx$

Session 2

Methods of Integration

Methods of Integration

If the integral is not a derivative of a simple function, then the corresponding integrals cannot be found directly. In order to find the integral of complex problems.

e.g. $\int \frac{\sin x}{x}\,dx, \int \frac{\cos x}{x}\,dx, \int \frac{1}{\log x}\,dx$

Some Integrals which Cannot be Found

Any function continuous on an interval (a, b) has an anti-derivative in that interval. In other words, there exists a function $F(x)$ such that $F'(x) = f(x)$.

However, not every anti-derivative $F(x)$, even when it exists, is expressible in closed form in terms of elementary functions such as polynomials, trigonometric, logarithmic, exponential functions etc. Then, we say that such anti-derivatives or integrals "cannot be found''.

Some typical examples are

(i) $\int \frac{\sin x}{x}\,dx$ (ii) $\int \frac{\cos x}{x}\,dx$

(iii) $\int \frac{1}{\log x}\,dx$ (iv) $\int \sqrt{1-k^2\sin^2 x}\,dx$

(v) $\int \sqrt{\sin x}\,dx$ (vi) $\int \sin(x^2)\,dx$

(vii) $\int \cos(x^2)\,dx$ (viii) $\int x\tan x\,dx$

(ix) $\int e^{-x^2}\,dx$ (x) $\int e^{x^2}\,dx$

(xi) $\int \frac{x^2}{1+x^5}\,dx$ (xii) $\int \sqrt[3]{1+x^2}\,dx$

(xiii) $\int \sqrt{1+x^3}\,dx$ etc.

Integration by Substitution

(or by change of the independent variable)

If $g(x)$ is a continuously differentiable function, then to evaluate integrals of the form,

$$I = \int f(g(x)).g'(x)\,dx,$$

we substitute $g(x) = t$ and $g'(x)\,dx = dt$

The substitution reduces the integral to $\int f(t)\,dt$. After evaluating this integral we substitute back the value of t.

Example 13 Prove that

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{(n+1)a} + C, n \neq 1.$$

Sol. Putting, $ax + b = t$, we get

$$adx = dt \text{ or } dx = \frac{1}{a}dt$$

$$\therefore \quad I = \int (ax+b)^n\,dx = \int t^n \cdot \frac{dt}{a} = \frac{1}{a}\cdot\frac{t^{n+1}}{n+1} + C$$

$$= \frac{1}{a(n+1)}(ax+b)^{n+1}dx + C.$$

Remarks

1. If $\int f(x)\,dx = g(x) + C$, then $\int f(ax+b)\,dx = \frac{1}{a}g(ax+b) + C$

2. If $\int \frac{1}{x}\,dx = \log|x| + C$, then $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log|ax+b| + C$

Thus, in any fundamental integral formulae given in article fundamental integration formulae if in place of x we have $(ax + b)$, then same formula is applicable but we must divide by coefficient of x or derivative of $(ax + b)$ i.e. a.

Here is the list of some of frequently used formulae

(i) $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C, n \neq -1$

(ii) $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log|ax+b| + C$

(iii) $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$

(iv) $\int a^{bx+c}\,dx = \frac{1}{b}\cdot\frac{a^{bx+c}}{\log a} + C$

(v) $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$

(vi) $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$

(vii) $\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C$

(viii) $\int \text{cosec}^2\,(ax+b)\,dx = -\frac{1}{a}\cot(ax+b)+C$

(ix) $\int \sec(ax+b)\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b)+C$

(x) $\int \text{cosec}\,(ax+b)\cot(ax+b)\,dx = -\frac{1}{a}\text{cosec}\,(ax+b)+C$

(xi) $\int \tan(ax+b)\,dx = -\frac{1}{a}\log|\cos(ax+b)|+C$

(xii) $\int \cot(ax+b)\,dx = \frac{1}{a}\log|\sin(ax+b)|+C$

(xiii) $\int \sec(ax+b)dx = \frac{1}{a}\log|\sec(ax+b)+\tan(ax+b)|+C$

(xiv) $\int \text{cosec}(ax+b)\,dx = \frac{1}{a}\log|\text{cosec}(ax+b)-\cot(ax+b)|+C$

Example 14 Evaluate

(i) $\int \frac{1}{\sqrt{3x+4}-\sqrt{3x+1}}\,dx$ (ii) $\int \frac{8x+13}{\sqrt{4x+7}}\,dx.$

(iii) $\int (7x-2)\sqrt{3x+2}\,dx.$ (iv) $\int \frac{2+3x^2}{x^2(1+x^2)}\,dx.$

Sol. (i) Here, $I = \int \frac{dx}{\sqrt{3x+4}-\sqrt{3x+1}}$

$$= \int \frac{(\sqrt{3x+4}+\sqrt{3x+1})}{(\sqrt{3x+4}-\sqrt{3x+1})(\sqrt{3x+4}+\sqrt{3x+1})}\,dx$$

$$= \int \frac{(\sqrt{3x+4}+\sqrt{3x+1})}{(3x+4)-(3x+1)}\,dx$$

$$= \frac{1}{3}\int \sqrt{3x+4}\,dx + \frac{1}{3}\int \sqrt{3x+1}\,dx$$

$$= \frac{1}{3}\left\{\frac{(3x+4)^{3/2}}{3/2\times 3}\right\} + \frac{1}{3}\left\{\frac{(3x+1)^{3/2}}{3/2\times 3}\right\} + C$$

Using, $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{(n+1)\,a}+C$

$$= \frac{2}{27}[(3x+4)^{3/2}+(3x+1)^{3/2}]+C$$

(ii) Here, $I = \int \frac{8x+13}{\sqrt{4x+7}}\,dx = \int \frac{8x+14-1}{\sqrt{4x+7}}\,dx$

$$= \int \frac{2(4x+7)-1}{\sqrt{4x+7}}dx$$

$$= 2\int \frac{4x+7}{\sqrt{4x+7}}\,dx - \int \frac{dx}{\sqrt{4x+7}}$$

$$= 2\int (\sqrt{4x+7})dx - \int (4x+7)^{-1/2}dx$$

$$= 2\left(\frac{(4x+7)^{3/2}}{3/2\times 4}\right) - \left(\frac{(4x+7)^{1/2}}{1/2\times 4}\right)+C$$

$$= \frac{1}{3}(4x+7)^{3/2} - \frac{1}{2}(4x+7)^{1/2}+C$$

(iii) Here, $I = \int (7x-2)\sqrt{3x+2}\,dx = 7\int \left(x-\frac{2}{7}\right)\sqrt{3x+2}\,dx$

$$= \frac{7}{3}\int \left(3x-\frac{6}{7}\right)\sqrt{3x+2}\,dx$$

$$= \frac{7}{3}\int \left(3x+2-2-\frac{6}{7}\right)\sqrt{3x+2}\,dx$$

$$= \frac{7}{3}\int (3x+2)\sqrt{3x+2}\,dx - \frac{20}{3}\int \sqrt{3x+2}\,dx$$

$$= \frac{7}{3}\int (3x+2)^{3/2}\,dx - \frac{20}{3}\int \sqrt{3x+2}\,dx$$

$$= \frac{7}{3}\left(\frac{(3x+2)^{5/2}}{\frac{5}{2}\times 3}\right) - \frac{20}{3}\left(\frac{(3x+2)^{3/2}}{\frac{3}{2}\times 3}\right)+C$$

$$= \frac{14}{45}(3x+2)^{5/2} - \frac{40}{27}(3x+2)^{3/2}+C$$

(iv) Here, $I = \int \frac{(2+3x^2)\,dx}{x^2(1+x^2)} = \int \frac{2+2x^2+x^2}{x^2(1+x^2)}\,dx$

$$= \int \left(\frac{2(1+x^2)}{x^2(1+x^2)} + \frac{x^2}{x^2(1+x^2)}\right)dx$$

$$= \int \left(\frac{2}{x^2}+\frac{1}{1+x^2}\right)dx = 2\int x^{-2}dx + \int \frac{1}{1+x^2}\,dx$$

$$= 2\cdot\frac{x^{-1}}{-1} + \tan^{-1}x + C = -\frac{2}{x} + \tan^{-1}x + C$$

Example 15 Evaluate

(i) $\int \frac{\sin(\log x)}{x}\,dx$ (ii) $\int \left(\frac{3\sin x+4\cos x}{4\sin x-3\cos x}\right)dx$

(iii) $\int \frac{e^{m\tan^{-1}x}}{1+x^2}\,dx$ (iv) $\int x\sin(4x^2+7)\,dx$

Sol. (i) $I = \int \frac{\sin(\log x)}{x}\,dx$

We know that, $\frac{d}{dx}(\log x) = \frac{1}{x}$

Thus, let $\log x = t$

$\Rightarrow \quad \frac{1}{x}\,dx = dt$ ...(i)

$\therefore \quad I = \int \sin(t)\,dt = -\cos(t)+C$

$= -\cos(\log x)+C$ [using Eq. (i)]

(ii) $I = \int \frac{3\sin x+4\cos x}{4\sin x-3\cos x}\,dx$

We know, $\frac{d}{dx}(4\sin x-3\cos x) = (4\cos x+3\sin x)$

Thus, let $4\sin x-3\cos x = t$ ...(i)

$\Rightarrow \quad (4\cos x+3\sin x)\,dx = dt$

$$\therefore \quad I = \int \frac{dt}{t} = \log|t| + C$$

$$= \log|4\sin x - 3\cos x| + C \quad \text{[using Eq. (i)]}$$

(iii) $I = \int \frac{e^{m\tan^{-1}x}}{1+x^2}\,dx$ Let $m\tan^{-1}x = t$

$$\Rightarrow \quad \frac{m}{1+x^2}\,dx = dt \quad \Rightarrow \quad \frac{1}{1+x^2}\,dx = \frac{1}{m}\,dt$$

$$\therefore \quad I = \int e^t \cdot \frac{dt}{m} \quad \Rightarrow \quad I = \frac{1}{m}\int e^t\,dt$$

$$I = \frac{1}{m}e^t + C = \frac{1}{m}e^{m\tan^{-1}x} + C$$

(iv) $I = \int x\sin(4x^2+7)\,dx$

Let $4x^2 + 7 = t \Rightarrow 8x\,dx = dt \Rightarrow x\,dx = \frac{1}{8}dt$

$$\therefore \quad I = \int \sin(t)\frac{dt}{8} = -\frac{1}{8}\cos(t) + C$$

$$= -\frac{1}{8}\cos(4x^2+7) + C$$

Remarks

While solving product of trignometric, it is expedient to use the following trigonometric identities

1. $\sin mx\cos nx = \frac{1}{2}\{\sin(m-n)x + \sin(m+n)x\}$
2. $\cos mx\sin nx = \frac{1}{2}\{\sin(m+n)x - \sin(m-n)x\}$
3. $\sin mx\sin nx = \frac{1}{2}\{\cos(m-n)x - \cos(m+n)x\}$
4. $\cos mx\cos nx = \frac{1}{2}\{\cos(m-n)x + \cos(m+n)x\}$

Example 16 Evaluate

(i) $\int \cos 4x\cos 7x\,dx$ (ii) $\int \cos x\cos 2x\cos 5x\,dx$

Sol. When calculating such integrals it is advisable to use the trigonometric product formulae.

(i) $\int \cos 4x\cos 7x\,dx$

Here, $\cos 4x\cos 7x = \frac{1}{2}(\cos 3x + \cos 11x)$,

$[\text{using } \cos mx\cdot\cos nx = \frac{1}{2}\{\cos(m-n)x + \cos(m+n)x\}]$

$$\therefore \quad I = \int \cos 4x\cos 7x\,dx = \frac{1}{2}\int(\cos 3x + \cos 11x)\,dx$$

$$= \frac{1}{2}\int \cos 3x\,dx + \frac{1}{2}\int \cos 11x\,dx$$

$$= \frac{\sin 3x}{6} + \frac{\sin 11x}{22} + C$$

(ii) $\int \cos x\cos 2x\cos 5x\,dx$,

We have $(\cos x\cos 2x)\cos 5x = \frac{1}{2}(\cos x + \cos 3x)\cos 5x$,

$[\text{using } \cos mx.\cos nx = \frac{1}{2}\{\cos(m-n)x + \cos(m+n)x\}]$

$$= \frac{1}{4}\{2\cos x\cos 5x + 2\cos 3x\cos 5x\}$$

$$= \frac{1}{4}\{(\cos 4x + \cos 6x) + (\cos 2x + \cos 8x)\}$$

$$\therefore \quad \cos x\cos 2x\cos 5x = \frac{1}{4}\{\cos 2x + \cos 4x + \cos 6x + \cos 8x\}$$

$$\therefore \quad I = \int(\cos x\cos 2x\cos 5x)\,dx$$

$$= \frac{1}{4}\int(\cos 2x + \cos 4x + \cos 6x + \cos 8x)\,dx$$

$$= \frac{\sin 2x}{8} + \frac{\sin 4x}{16} + \frac{\sin 6x}{24} + \frac{\sin 8x}{32} + C$$

Example 17 Evaluate

(i) $\int \sin x\cos x\cdot\cos 2x\cdot\cos 4x\,dx$

(ii) $\int \frac{1-\tan^2 x}{1+\tan^2 x}\,dx$ (iii) $\int \frac{1+\cos^2 x}{1+\cos 2x}\,dx$

(iv) $\int \frac{\cos 2x}{\cos^2 x\sin^2 x}\,dx$ (v) $\int \frac{\sec 2x - 1}{\sec 2x + 1}\,dx$

Sol. (i) Here, $I = \int \sin x\cos x\cdot\cos 2x\cos 4x\,dx$

$$= \frac{1}{2}\int 2\sin x\cdot\cos x\cdot\cos 2x\cdot\cos 4x\,dx$$

$[\text{using, } \sin 2x = 2\sin x\cdot\cos x]$

$$= \frac{1}{2\times 2}\int 2\sin 2x\cdot\cos 2x\cdot\cos 4x\cdot dx$$

$$= \frac{1}{4}\int \sin 4x\cdot\cos 4x\,dx = \frac{1}{2\times 4}\int 2\sin 4x\cos 4x\,dx$$

$$= \frac{1}{8}\int \sin 8x\,dx = \frac{-\cos 8x}{64} + C$$

(ii) Here, $I = \int \frac{1-\tan^2 x}{1+\tan^2 x}\,dx$

$$\therefore \quad I = \int \cos 2x\,dx \qquad \text{Using, } \cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}$$

$$= \frac{\sin 2x}{2} + c$$

(iii) Here, $I = \int \frac{1+\cos^2 x}{1+\cos 2x}\,dx$,

$$= \int \frac{1+\cos^2 x}{1+2\cos^2 x - 1}\,dx = \int \frac{1+\cos^2 x}{2\cos^2 x}\,dx$$

$[\text{Using, } \cos 2x = 2\cos^2 x - 1]$

$$= \frac{1}{2}\int(\sec^2 x + 1)\,dx = \frac{1}{2}(\tan x + x) + C$$

(iv) Here, $I = \int \frac{\cos 2x}{\cos^2 x\cdot\sin^2 x}\,dx$

$$I = \int \frac{(\cos^2 x - \sin^2 x)\,dx}{\cos^2 x\cdot\sin^2 x}$$

$[\text{Using, } \cos 2x = \cos^2 x - \sin^2 x]$

$$= \int \left(\frac{1}{\sin^2 x} - \frac{1}{\cos^2 x} \right) dx$$

$$= \int (\operatorname{cosec}^2 x - \sec^2 x)\, dx = -\cot x - \tan x + C$$

(v) Here, $I = \int \frac{\sec 2x - 1}{\sec 2x + 1} \cdot dx = \int \frac{1 - \cos 2x}{1 + \cos 2x}\, dx,$

We get, $I = \int \frac{2\sin^2 x}{2\cos^2 x}\, dx$

[using, $1 - \cos 2x = 2\sin^2 x$ and $1 + \cos 2x = 2\cos^2 x$]

$$= \int \tan^2 x \cdot dx$$

As, $\tan^2 x = \sec^2 x - 1$

$\therefore \quad I = \int (\sec^2 x - 1)\, dx = \tan x - x + C$

Example 18 ***Evaluate***

$$I = \int \left(\frac{\cot^2 2x - 1}{2\cot 2x} - \cos 8x \cdot \cot 4x \right) dx.$$

Sol. Here, $I = \int \left(\frac{\cot^2 2x - 1}{2\cot 2x} - \cos 8x \cdot \cot 4x \right) dx$

$$I = \int (\cot 4x - \cos 8x \cdot \cot 4x)\, dx$$

$$\left[\text{using } \cot 2A = \frac{\cot^2 A - 1}{2\cot A} \right]$$

$$= \int \cot 4x (1 - \cos 8x)\, dx$$

[using $1 - \cos 2A = 2\sin^2 A$]

$$= \int \cot 4x \cdot 2\sin^2 (4x)\, dx$$

$$= \int \frac{\cos 4x}{\sin 4x} \cdot 2\sin^2 4x\, dx$$

$$= \int 2\sin 4x \cdot \cos 4x\, dx,$$

using $\sin 2A = 2\sin A \cos A$

$$= \int \sin 8x\, dx = \frac{-\cos 8x}{8} + C$$

Exercise for Session 2

- Solve the following integration

1. $\int \frac{dx}{1 + \sin x}$

2. $\int \frac{\cos x - \sin x}{\cos x + \sin x} \cdot (2 + 2\sin 2x)\, dx$

3. $\int (3\sin x \cos^2 x - \sin^3 x)\, dx$

4. $\int \cos x^\circ\, dx$

5. $\int \frac{\sin x + \cos x}{\sqrt{1 + \sin 2x}} \cdot dx$, here $(\sin x + \cos x) > 0$

6. $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} \cdot dx$

7. $\int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cdot \cos^2 x}\, dx$

8. $\int \sec^2 x \cdot \operatorname{cosec}^2 x\, dx$

9. $\int \sqrt{1 - \sin 2x}\, dx$

10. $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cdot \cos^2 x}\, dx$

11. $\int \left(\sin^2 \left(\frac{9\pi}{8} + \frac{x}{4} \right) - \sin^2 \left(\frac{7\pi}{8} + \frac{x}{4} \right) \right) dx$

12. $\int \frac{\cos 4x + 1}{\cot x - \tan x}\, dx$

13. $\int \left(\sin \alpha \cdot \sin (x - \alpha) + \sin^2 \left(\frac{x}{2} - \alpha \right) \right) dx$

14. $\int \frac{\sin 2x + \sin 5x - \sin 3x}{\cos x + 1 - 2\sin^2 2x}\, dx$

15. $\int \frac{\cos^4 x + \sin^4 x}{\sqrt{1 + \cos 4x}}\, dx$, here $\cos 2x > 0$

Session 3

Some Special Integrals

Some Special Integrals

(i) $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$

(ii) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+C$

(iii) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+C$

(iv) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)+C$

(v) $\int \frac{dx}{\sqrt{a^2+x^2}} = \log|x+\sqrt{x^2+a^2}|+C$

(vi) $\int \frac{dx}{\sqrt{x^2-a^2}} = \log|x+\sqrt{x^2-a^2}|+C$

(vii) $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2}+\frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right)+C$

(viii) $\int \sqrt{a^2+x^2}\,dx$

$= \frac{1}{2}x\sqrt{a^2+x^2}+\frac{1}{2}a^2\log|x+\sqrt{a^2+x^2}|+C$

(ix) $\int \sqrt{x^2-a^2}\,dx$

$= \frac{1}{2}x\sqrt{x^2-a^2}-\frac{1}{2}a^2\log|x+\sqrt{x^2-a^2}|+C$

Some Important Substitutions

Expression	Substitution
a^2+x^2	$x = a\tan\theta$ or $a\cot\theta$
a^2-x^2	$x = a\sin\theta$ or $a\cos\theta$
x^2-a^2	$x = a\sec\theta$ or $a\operatorname{cosec}\theta$
$\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	$x = a\cos 2\theta$
$\sqrt{\frac{x-\alpha}{\beta-x}}$ or $\sqrt{(x-\alpha)(x-\beta)}$	$x = \alpha\cos^2\theta+\beta\sin^2\theta$

Application of these Formulae

The above standard integrals are very important. Given below are integrals which are applications of these.

Type I

(i) $\int \frac{dx}{ax^2+bx+c}$ (ii) $\int \frac{dx}{\sqrt{ax^2+bx+c}}$

(iii) $\int \sqrt{ax^2+bx+c}\,dx$

If ax^2+bx+c can be factorised, then the integration is easily done by the method of partial fractions (explained later). If the denominator cannot be factorised, then express it as the sum or difference of two squares by the method of completing the square

$$ax^2+bx+c = a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right) = a\left\{\left(x+\frac{b}{2a}\right)^2+\left(\frac{c}{a}\right)-\frac{b^2}{4a^2}\right\}$$

make the substitution $x+\frac{b}{2a}=t$.

Example 19 Evaluate

(i) $\int \frac{1}{x^2-x+1}dx$ (ii) $\int \frac{1}{2x^2+x-1}dx$

(iii) $\int \frac{1}{\sqrt{x^2-2x+3}}dx$ (iv) $\int \sqrt{2x^2-3x+1}\,dx$

Sol. (i) $I = \int \frac{dx}{x^2-x+1}$,

completing x^2-x+1 into perfect square.

$$= \int \frac{dx}{x^2-x+1/4-1/4+1} = \int \frac{dx}{(x-1/2)^2+3/4}$$

$$\therefore\ I = \int \frac{dx}{(x-1/2)^2+(\sqrt{3}/2)^2}$$

$$= \frac{1}{\sqrt{3}/2}\tan^{-1}\left(\frac{x-1/2}{\sqrt{3}/2}\right)+C$$

$$\therefore I = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+C$$

$$\left[\text{using } \int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C\right]$$

(ii) $I = \int \frac{1}{2x^2+x-1}dx = \frac{1}{2}\int \frac{1}{x^2+x/2-1/2}dx$

$$= \frac{1}{2}\int \frac{1}{x^2+x/2+1/16-1/16-1/2}dx$$

$$= \frac{1}{2}\int \frac{dx}{(x+1/4)^2 - 9/16} = \frac{1}{2}\int \frac{dx}{(x+1/4)^2 - (3/4)^2}$$

$$= \frac{1}{2}\cdot\frac{1}{2(3/4)}\log\left|\frac{x+1/4-3/4}{x+1/4+3/4}\right| + C$$

$$\left[\text{using}\int\frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C\right]$$

$$= \frac{1}{3}\log\left|\frac{x-1/2}{x+1}\right| + C$$

$$\therefore \quad I = \frac{1}{3}\log\left|\frac{2x-1}{2(x+1)}\right| + C$$

(iii) $I = \int \frac{dx}{\sqrt{x^2-2x+3}} = \int \frac{dx}{\sqrt{x^2-2x+1-1+3}}$

$$= \int \frac{dx}{\sqrt{(x-1)^2+(\sqrt{2})^2}}$$

$$= \log|(x-1) + \sqrt{(x-1)^2+(\sqrt{2})^2}|$$

$$\left[\text{using}\int\frac{dx}{\sqrt{x^2+a^2}} = \log|x+\sqrt{x^2+a^2}| + C\right]$$

$$\therefore \quad I = \log|(x-1)+\sqrt{x^2-2x+3}| + C$$

(iv) $I = \int(\sqrt{2x^2-3x+1})\,dx = \sqrt{2}\int(\sqrt{x^2-3x/2+1/2})\,dx$

$$= \sqrt{2}\int(\sqrt{x^2-3x/2+9/16-9/16+1/2})\,dx$$

$$= \sqrt{2}\int(\sqrt{(x-3/4)^2-1/16})\,dx$$

$$= \sqrt{2}\left\{\begin{matrix}\frac{1}{2}(x-3/4)\sqrt{(x-3/4)^2-1/16} \\ -\frac{1}{16\times 2}\log|(x-3/4)+\sqrt{(x-3/4)^2-1/16}\end{matrix}\right\} + C$$

$$= \sqrt{2}\left\{\begin{matrix}\frac{1}{8}(4x-3)\sqrt{x^2-3x/2+1/2} \\ -\frac{1}{32}\log|(x-3/4)+\sqrt{x^2-3x/2+1/2}\end{matrix}\right\} + C$$

Example 20 Evaluate

(i) $\int \frac{1}{\sqrt{1-e^{2x}}}\,dx$ (ii) $\int \frac{2x}{\sqrt{1-x^2-x^4}}\,dx$

(iii) $\int \frac{a^x}{\sqrt{1-a^{2x}}}\,dx$ (iv) $\int \sqrt{\frac{x}{a^3-x^3}}\,dx$

Sol. (i) Here, $I = \int \frac{dx}{\sqrt{1-e^{2x}}}$

Let, $1 - e^{2x} = t^2$

Then, $-2e^{2x}dx = 2t\,dt$

$$\Rightarrow \quad dx = -\frac{-t\,dt}{e^{2x}},$$

$$dx = \frac{t\,dt}{t^2-1} = \int \frac{1}{\sqrt{t^2}}\cdot\frac{t}{t^2-1}\,dt$$

$$I = \int\frac{dt}{t^2-1} = \frac{1}{2\cdot 1}\log\left|\frac{t-1}{t+1}\right| + C$$

$$= \frac{1}{2}\log\left|\frac{\sqrt{1-e^{2x}}-1}{\sqrt{1-e^{2x}}+1}\right| + C$$

(ii) Here, $I = \int \frac{2x}{\sqrt{1-x^2-x^4}}\,dx$

Let, $x^2 = t, 2x\,dx = dt$

$$\therefore \quad I = \int\frac{dt}{\sqrt{1-t-t^2}} = \int\frac{dt}{\sqrt{1+\frac{1}{4}-\frac{1}{4}-t-t^2}}$$

$$I = \int\frac{dt}{\sqrt{5/4-(t+1/2)^2}} = \int\frac{dt}{\sqrt{(\sqrt{5}/2)^2-(t+1/2)^2}}$$

$$= \sin^{-1}\left(\frac{t+1/2}{\sqrt{5}/2}\right) + C = \sin^{-1}\left(\frac{2x^2+1}{\sqrt{5}}\right) + C$$

(iii) Here, $I = \int \frac{a^x}{\sqrt{1-a^{2x}}}\,dx$. Let, $a^x = t$

$$\therefore \quad a^x \log a\,dx = dt, \; a^x\,dx = \frac{dt}{\log a}$$

$$\therefore \quad I = \int\frac{1}{\sqrt{1-t^2}}\cdot\frac{dt}{\log a} = \frac{1}{\log a}\cdot\sin^{-1}(t) + C$$

$$I = \frac{1}{\log a}\sin^{-1}(a^x) + C$$

(iv) Here, $I = \int\sqrt{\frac{x}{a^3-x^3}}\,dx = \int\frac{x^{1/2}\,dx}{\sqrt{(a^{3/2})^2-(x^{3/2})^2}}$

Let, $x^{3/2} = t, \frac{3}{2}x^{1/2}\,dx = dt, x^{1/2}\,dx = \frac{2}{3}dt$

$$= \int\frac{\frac{2}{3}dt}{\sqrt{(a^{3/2})^2-(t)^2}}$$

$$= \frac{2}{3}\cdot\sin^{-1}\left(\frac{t}{a^{3/2}}\right) + C = \frac{2}{3}\sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right) + C$$

Example 21 Evaluate

(i) $\int \frac{\cos x}{\sqrt{\sin^2 x - 2\sin x - 3}}\,dx$. (ii) $\int\sqrt{\frac{\sin(x-\alpha)}{\sin(x+\alpha)}}\,dx$.

(iii) $\int \frac{2\sin 2x - \cos x}{6-\cos^2 x - 4\sin x}\,dx$.

Sol. (i) Let $I = \int \frac{\cos x\, dx}{\sqrt{\sin^2 x - 2\sin x - 3}}$

Put $\sin x = t \quad \therefore \cos x\, dx = dt$

$$\Rightarrow I = \int \frac{dt}{\sqrt{t^2 - 2t - 3}} = \int \frac{dt}{\sqrt{t^2 - 2t + 1 - 1 - 3}}$$

$$I = \int \frac{dt}{\sqrt{(t-1)^2 - (2)^2}}$$

$$= \log|(t-1) + \sqrt{(t-1)^2 - (2)^2}| + C$$

$$\therefore \quad I = \log|(\sin x - 1) + \sqrt{\sin^2 x - 2\sin x - 3}| + C$$

(ii) Let $I = \int \sqrt{\frac{\sin(x-\alpha)}{\sin(x+\alpha)}}\, dx$

$$I = \int \sqrt{\frac{\sin(x-\alpha)}{\sin(x+\alpha)} \times \frac{\sin(x-\alpha)}{\sin(x-\alpha)}}\, dx$$

$$= \int \frac{\sin(x-\alpha)}{\sqrt{\sin^2 x - \sin^2 \alpha}}\, dx$$

$$I = \int \frac{\sin x \cos\alpha - \cos x \sin\alpha}{\sqrt{\sin^2 x - \sin^2 \alpha}}\, dx$$

$$= \cos\alpha \int \frac{\sin x\, dx}{\sqrt{\sin^2 x - \sin^2 \alpha}} - \sin\alpha \int \frac{\cos x\, dx}{\sqrt{\sin^2 x - \sin^2 \alpha}}$$

$$= \cos\alpha \int \frac{\sin x\, dx}{\sqrt{1 - \cos^2 x - \sin^2 \alpha}} - \sin\alpha \int \frac{\cos x\, dx}{\sqrt{\sin^2 x - \sin^2 \alpha}}$$

$$= \cos\alpha \int \frac{\sin x\, dx}{\sqrt{\cos^2\alpha - \cos^2 x}} - \sin\alpha \int \frac{\cos x\, dx}{\sqrt{\sin^2 x - \sin^2 \alpha}}$$

In the first part, put $\cos x = t$, so that $-\sin x\, dx = dt$ and in second part, put $\sin x = u$, so that $\cos x\, dx = du$

$$\therefore \quad I = -\cos\alpha \int \frac{dt}{\sqrt{\cos^2\alpha - t^2}} - \sin\alpha \int \frac{du}{\sqrt{u^2 - \sin^2\alpha}}$$

$$= -\cos\alpha \cdot \sin^{-1}\left(\frac{t}{\cos\alpha}\right) - \sin\alpha \cdot \log |u - \sqrt{(u^2 - \sin^2\alpha)}| + C$$

$$= -\cos\alpha \cdot \sin^{-1}\left(\frac{\cos x}{\cos\alpha}\right) - \sin\alpha \cdot \log |\sin x - \sqrt{\sin^2 x - \sin^2\alpha}| + C$$

(iii) $I = \int \frac{2\sin 2x - \cos x}{6 - \cos^2 x - 4\sin x}\, dx$

$$= \int \frac{(4\sin x - 1)\cos x}{6 - (1 - \sin^2 x) - 4\sin x}\, dx$$

$$= \int \frac{(4\sin x - 1)\cos x}{\sin^2 x - 4\sin x + 5}\, dx$$

Put $\sin x = t$, so that $\cos x\, dx = dt$

$$\therefore \quad I = \int \frac{(4t-1)\, dt}{(t^2 - 4t + 5)} \qquad \ldots(i)$$

Now, let $\quad (4t - 1) = \lambda(2t - 4) + \mu$

Comparing coefficients of like powers of t, we get

$$2\lambda = 4, -4\lambda + \mu = -1$$

$$\Rightarrow \quad \lambda = 2, \mu = 7 \qquad \ldots(ii)$$

$$\therefore I = \int \frac{2(2t-4) + 7}{t^2 - 4t + 5}\, dt \qquad \text{[using Eqs. (i) and (ii)]}$$

$$= 2\int \frac{2t - 4}{t^2 - 4t + 5}\, dt + 7\int \frac{dt}{t^2 - 4t + 5}$$

$$= 2\log|t^2 - 4t + 5| + 7\int \frac{dt}{t^2 - 4t + 4 - 4 + 5}$$

$$= 2\log|t^2 - 4t + 5| + 7\int \frac{dt}{(t-2)^2 + (1)^2}$$

$$= 2\log|t^2 - 4t + 5| + 7\tan^{-1}(t-2) + C$$

$$= 2\log|\sin^2 x - 4\sin x + 5| + 7\tan^{-1}(\sin x - 2) + C$$

Type II

(i) $\int \frac{(px+q)\, dx}{ax^2 + bx + c}$ (ii) $\int \frac{(px+q)}{\sqrt{ax^2 + bx + c}}\, dx$

(iii) $\int (px+q)\sqrt{ax^2 + bx + c}\, dx$

The linear factor $(px + q)$ is expressed in terms of the derivative of the quadratic factor $ax^2 + bx + c$ together with a constant as $px + q = \frac{\lambda d}{dx}(ax^2 + bx + c) + \mu$

$$\Rightarrow \quad px + q = \lambda(2ax + b) + \mu$$

Here, we have to find λ and μ and replace $(px + q)$ by $\{\lambda(2ax + b) + \mu\}$ in (i), (ii) and (iii).

Example 22 Evaluate

(i) $\int \sqrt{\frac{a-x}{a+x}}\, dx$ (ii) $\int x\sqrt{\frac{a^2 - x^2}{a^2 + x^2}}\, dx$

Sol. (i) Let $I = \int \sqrt{\frac{a-x}{a+x}}\, dx$

$$I = \int \sqrt{\frac{a-x}{a+x} \times \frac{a-x}{a-x}}\, dx = \int \frac{a-x}{\sqrt{a^2 - x^2}}\, dx$$

$$I = \int \frac{a}{\sqrt{a^2 - x^2}}\, dx - \int \frac{x}{\sqrt{a^2 - x^2}}\, dx$$

$$= a \cdot \sin^{-1}\left(\frac{x}{a}\right) + \int \frac{t\, dt}{t}$$

Put $\quad a^2 - x^2 = t^2 \Rightarrow -2x\, dx = 2t\, dt \Rightarrow x\, dx = -t\, dt$

$$= a \cdot \sin^{-1}\left(\frac{x}{a}\right) + t + C$$

$$= a \cdot \sin^{-1}\left(\frac{x}{a}\right) + \sqrt{a^2 - x^2} + C$$

(ii) Let $I = \int x \sqrt{\frac{a^2 - x^2}{a^2 + x^2}}\, dx$

Put $x^2 = t \Rightarrow 2x\, dx = dt$

$$\therefore \quad I = \int \sqrt{\frac{a^2 - t}{a^2 + t}} \cdot \frac{dt}{2} = \frac{1}{2}\int \sqrt{\frac{a^2 - t}{a^2 + t} \cdot \frac{a^2 - t}{a^2 - t}}\, dt$$

$$= \frac{1}{2}\int \frac{a^2 - t}{\sqrt{a^4 - t^2}}\, dt$$

$$= \frac{1}{2} a^2 \int \frac{dt}{\sqrt{(a^2)^2 - (t)^2}} - \frac{1}{2}\int \frac{t\, dt}{\sqrt{a^4 - t^2}}$$

$$= \frac{1}{2} a^2 \cdot \sin^{-1}\left(\frac{t}{a^2}\right) + \frac{1}{4}\int \frac{du}{\sqrt{u}},$$

where $a^4 - t^2 = u \implies -2t\, dt = du$

$$= \frac{1}{2} a^2 \cdot \sin^{-1}\left(\frac{t}{a^2}\right) + \frac{1}{4} \cdot \left(\frac{u^{1/2}}{1/2}\right) + C$$

[where $t = x^2$ and $u = a^4 - x^4$]

$$= \frac{1}{2} a^2 \cdot \sin^{-1}\left(\frac{x^2}{a^2}\right) + \frac{1}{2} \cdot \sqrt{a^4 - x^4} + C$$

Integrals of the Form $\int \frac{k(x)}{ax^2 + bx + c}\, dx$, where $k(x)$ is a Polynomial of Degree Greater than 2

To evaluate this type of integrals we divide the numerator by denominator and express the integral as

$Q(x) + \frac{R(x)}{ax^2 + bx + c}$, where $R(x)$ is a linear function of x.

Example 23 Evaluate

(i) $\int x \sqrt{1 + x - x^2}\, dx$ (ii) $\int (x + 1) \sqrt{1 - x - x^2}\, dx$

Sol. (i) Let $I = \int x \sqrt{1 + x - x^2}\, dx$

Put $x = \lambda \left\{\frac{d}{dx}(1 + x - x^2)\right\} + \mu$

Then, comparing the coefficients of like powers of x, we get

$1 = -2\lambda$ and $\lambda + \mu = 0 \Rightarrow \lambda = -1/2, \mu = 1/2$

$\therefore \quad I = \int x \sqrt{1 + x - x^2}\, dx$

$$= \int \left\{-\frac{1}{2}(1 - 2x) + \frac{1}{2}\right\} \sqrt{1 + x - x^2}\, dx$$

$$= -\frac{1}{2}\int (1 - 2x) \sqrt{1 + x - x^2}\, dx + \frac{1}{2}\int \sqrt{1 + x - x^2}\, dx$$

$$= -\frac{1}{2}\int \sqrt{t}\, dt + \frac{1}{2}\int \sqrt{-(x^2 - x + \frac{1}{4} - \frac{1}{4} - 1)}\, dx$$

[where $t = 1 + x - x^2$]

$$= -\frac{1}{2}\left(\frac{t^{3/2}}{3/2}\right) + \frac{1}{2}\int \sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2}\, dx$$

$$= -\frac{1}{3} t^{3/2} + \frac{1}{2}$$

$$\left[\frac{(x - 1/2)}{2}\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2} + \frac{1}{2}\left(\frac{\sqrt{5}}{2}\right)^2 \sin^{-1}\left(\frac{x - 1/2}{\sqrt{5}/2}\right)\right] + C$$

$$= -\frac{1}{3}(1 + x - x^2)^{3/2}$$

$$+ \frac{1}{2}\left[\frac{(x - 1/2)}{2}\sqrt{1 + x - x^2} + \frac{5}{8}\sin^{-1}\left(\frac{2x - 1}{\sqrt{5}}\right)\right] + C$$

(ii) Let $I = \int (x + 1) \sqrt{1 - x - x^2}\, dx$

Put, $(x + 1) = \lambda \cdot \left(\frac{d}{dx}(1 - x - x^2)\right) + \mu$

Then, $(x + 1) = \lambda(-1 - 2x) + \mu$ comparing the coefficients of like powers of x, we get $-2\lambda = 1$ and $\mu - \lambda = 1 \Rightarrow \lambda = -1/2$ and $\mu = 1/2$.

$\therefore \quad (x + 1) = -\frac{1}{2}(-1 - 2x) + \frac{1}{2}$

So, $\int (x + 1) \sqrt{1 - x - x^2}\, dx = \int \left\{-\frac{1}{2}(-1 - 2x) + \frac{1}{2}\right\} \sqrt{1 - x - x^2}\, dx$

$$= -\frac{1}{2}\int (-1 - 2x) \sqrt{1 - x - x^2}\, dx + \frac{1}{2}\int \sqrt{1 - x - x^2}\, dx$$

$$= -\frac{1}{2}\int (-1 - 2x) \sqrt{1 - x - x^2}\, dx + \frac{1}{2}\int \sqrt{\left\{1 - \left(x^2 + x + \frac{1}{4} - \frac{1}{4}\right)\right\}}\, dx$$

$$= -\frac{1}{2}\sqrt{t}\, dt + \frac{1}{2}\int \sqrt{(\sqrt{5}/2)^2 - (x + 1/2)^2}\, dx$$

[where $t = 1 + x - x^2$]

$$= -\frac{1}{2}\left(\frac{t^{3/2}}{3/2}\right) + \frac{1}{2}\left\{\frac{1}{2}\left(x + \frac{1}{2}\right)\sqrt{1 - x - x^2} + \frac{1}{2} \cdot \frac{5}{4}\sin^{-1}\left(\frac{x + 1/2}{\sqrt{5}/2}\right)\right\} + C$$

$$= -\frac{1}{3}(1 - x - x^2)^{3/2} + \frac{1}{8}(2x + 1)\sqrt{1 - x - x^2} + \frac{5}{16}\sin^{-1}\left(\frac{2x + 1}{\sqrt{5}}\right) + C$$

Example 24 Evaluate $\int \frac{x^2+x+3}{x^2-x-2}\,dx$.

Sol. Let $I = \int \frac{x^2+x+3}{x^2-x-2}\,dx$

$\therefore \quad I = \int \left(1 + \frac{2x+5}{x^2-x-2}\right) dx$

$\Rightarrow \quad I = \int 1\,dx + \int \frac{2x+5}{x^2-x-2}\,dx$

$I = x + \int \frac{2x+5}{x^2-x-2}\,dx$

Put, $2x+5 = \lambda\left\{\frac{d}{dx}(x^2-x-2)\right\} + \mu$. Then, obtaining $2x+5 = \lambda(2x-1) + \mu$, comparing the coefficients of like terms. We get, $2 = 2\lambda$ and $5 = \mu - \lambda$

$\therefore \quad \lambda = 1$ and $\mu = 6$

$\therefore \quad I = x + \int \frac{\lambda(2x-1)+\mu}{x^2-x-2}\,dx$

$= x + \int \frac{2x-1}{x^2-x-2}\,dx + 6\int \frac{1}{x^2-x-2}\,dx$

$= x + \int \frac{1}{t}\,dt + 6\int \frac{1}{x^2-x+\frac{1}{4}-\frac{1}{4}-2}$

[where $t = x^2 - x - 2$]

$= x + \log|t| + 6\int \frac{dx}{(x-1/2)^2-(3/2)^2}\,dx$

$= x + \log|x^2-x-2| + 6\cdot\frac{1}{2(3/2)}\log\left|\frac{x-\frac{1}{2}-\frac{3}{2}}{x-\frac{1}{2}+\frac{3}{2}}\right| + C$

$= x + \log|x^2-x-2| + 2\cdot\log\left|\frac{x-2}{x+1}\right| + C$

Integrals of the Type

1. $\int \frac{ax^2+bx+c}{(px^2+qx+r)}\,dx$
2. $\int \frac{(ax^2+bx+c)}{\sqrt{px^2+qx+r}}\,dx$
3. $\int (ax^2+bx+c)\sqrt{px^2+qx+r}\,dx$

In above cases; substitute

$$ax^2+bx+c = \lambda(px^2+qx+r) + \mu\left\{\frac{d}{dx}(px^2+qx+r)\right\} + \gamma.$$

Find λ, μ and γ. These integrations reduces to integration of three independent functions.

Example 25 ***Evaluate*** $\int \frac{2x^2+5x+4}{\sqrt{x^2+x+1}}\,dx$.

Sol. Let $I = \int \frac{2x^2+5x+4}{\sqrt{x^2+x+1}}\,dx$

Put $2x^2+5x+4 = \lambda(x^2+x+1) + \mu\left\{\frac{d}{dx}(x^2+x+1)\right\} + \gamma$

or $2x^2+5x+4 = \lambda(x^2+x+1) + \mu(2x+1) + \gamma$

Comparing the coefficients of like terms, we get

$2 = \lambda, \quad 5 = \lambda + 2\mu, \quad 4 = \lambda + \mu + \gamma$

$\therefore \quad \lambda = 2, \quad \mu = 3/2, \quad \gamma = 1/2$

Hence, the above integral reduces to

$I = \int \frac{2x^2+5x+4}{\sqrt{x^2+x+1}}\,dx$

$= \int \left(2\cdot\frac{(x^2+x+1)}{\sqrt{x^2+x+1}} + \frac{3}{2}\cdot\frac{(2x+1)}{\sqrt{x^2+x+1}} + \frac{1}{2}\cdot\frac{1}{\sqrt{x^2+x+1}}\right) dx$

$= 2\int \sqrt{x^2+x+1}\,dx + \frac{3}{2}\int \frac{dt}{\sqrt{t}} + \frac{1}{2}\int \frac{dx}{\sqrt{x^2+x+1}}$

[where $t = x^2 + x + 1$]

$= 2\int \sqrt{(x+1/2)^2+(\sqrt{3}/2)^2}\,dx + \frac{3}{2}\cdot\frac{t^{1/2}}{1/2} + \frac{1}{2}\int \frac{dx}{\sqrt{(x+1/2)^2+(\sqrt{3}/2)^2}}$

$= 2\left[\frac{1}{2}\left(x+\frac{1}{2}\right)\sqrt{x^2+x+1} + \frac{1}{2}\cdot\frac{3}{4}\cdot\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right|\right]$

$+ 3\sqrt{x^2+x+1} + \frac{1}{2}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right| + C$

$\therefore \quad I = \left(x+\frac{1}{2}\right)\sqrt{x^2+x+1} + \frac{3}{4}\log\left\{\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right\}$

$+ 3\sqrt{x^2+x+1} + \frac{1}{2}\log\left\{\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right\} + C$

$\Rightarrow I = \left(x+\frac{7}{2}\right)\sqrt{x^2+x+1} + \frac{5}{4}\log\left\{\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right\} + C$

Trigonometric Integrals

(a) Integrals of the Form

$$\int \frac{1}{a\cos^2 x + b\sin^2 x}\,dx, \int \frac{1}{a+b\sin^2 x}\,dx, \int \frac{1}{a+b\cos^2 x}\,dx,$$

$$\int \frac{1}{(a\sin x + b\cos x)^2}\,dx, \int \frac{1}{a+b\sin^2 x + c\cos^2 x}\,dx$$

To evaluate this type of integrals, divide numerator and denominator both by $\cos^2 x$, replace $\sec^2 x$, if any, in denominator by $(1+\tan^2 x)$ and put $\tan x = t$. So that $\sec^2 x\, dx = dt$.

Example 26 Evaluate

(i) $\int \frac{1}{4\sin^2 x + 9\cos^2 x}\, dx$ (ii) $\int \frac{\sin x}{\sin 3x}\, dx$

Sol. (i) $I = \int \frac{dx}{4\sin^2 x + 9\cos^2 x}$

Here, dividing numerator and denominator by $\cos^2 x$. We get

$$I = \int \frac{\sec^2 x}{4\tan^2 x + 9} dx$$

Put $\tan x = t$

$\Rightarrow \sec^2 x\, dx = dt$

$$\therefore \quad I = \int \frac{dt}{4t^2+9} = \frac{1}{4}\int \frac{dt}{t^2 + (3/2)^2}$$

$$= \frac{1}{4}\cdot\frac{1}{3/2}\tan^{-1}\left(\frac{t}{3/2}\right) + C$$

$$I = \frac{1}{6}\tan^{-1}\left(\frac{2\tan x}{3}\right) + C$$

(ii) Let $I = \int \frac{\sin x}{\sin 3x}\, dx = \int \frac{\sin x}{3\sin x - 4\sin^3 x}\, dx$

$$I = \int \frac{dx}{3 - 4\sin^2 x}$$

Dividing numerator and denominator by $\cos^2 x$, we get

$$I = \int \frac{\sec^2 x\, dx}{3\sec^2 x - 4\tan^2 x} = \int \frac{\sec^2 x\, dx}{3(1+\tan^2 x) - 4\tan^2 x}$$

$$I = \int \frac{\sec^2 x\, dx}{3 - \tan^2 x}$$

Let $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$\therefore \quad I = \int \frac{dt}{(\sqrt{3})^2 - (t)^2} = \frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+t}{\sqrt{3}-t}\right| + C$$

$$I = \frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right| + C$$

(b) Integrals of the Form

$$\int \frac{1}{a\sin x + b\cos x}\, dx, \int \frac{1}{a + b\sin x}\, dx,$$

$$\int \frac{1}{a + b\cos x}\, dx, \int \frac{1}{a\sin x + b\cos x + c}\, dx$$

To evaluate this type of integrals we put

$\sin x = \frac{2\tan x/2}{1+\tan^2 x/2}$ and $\cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}$ and replace $\tan x/2 = t$, by performing these steps the integral reduces to the form $\int \frac{1}{at^2+bt+c}\, dt$ which can be evaluated by methods discussed earlier.

Example 27 Evaluate

(i) $\int \frac{dx}{2 + \sin x + \cos x}$ (ii) $\int \frac{dx}{\sqrt{3}\sin x + \cos x}$

Sol. For this type we use, $\sin x = \frac{2\tan x/2}{1+\tan^2 x/2}$,

$$\cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}.$$

(i) Let $I = \int \frac{dx}{2 + \sin x + \cos x}$

$$= \int \frac{dx}{2 + \frac{2\tan x/2}{1+\tan^2 x/2} + \frac{1-\tan^2 x/2}{1+\tan^2 x/2}}$$

$$= \int \frac{\sec^2 \frac{x}{2}\, dx}{2 + 2\tan^2 \frac{x}{2} + 2\tan\frac{x}{2} + 1 - \tan^2\frac{x}{2}}$$

$$I = \int \frac{\sec^2 \frac{x}{2}\, dx}{\tan^2 \frac{x}{2} + 2\tan\frac{x}{2} + 3}$$

Put $\tan\frac{x}{2} = t$

$$\Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\, dx = dt = \int \frac{2\, dt}{t^2+2t+3} = 2\int \frac{dt}{t^2+2t+1+2}$$

$$= 2\int \frac{dt}{(t+1)^2 + (\sqrt{2})^2}$$

$$= 2\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t+1}{\sqrt{2}}\right) + C$$

$$I = \sqrt{2}\tan^{-1}\left(\frac{\tan x/2 + 1}{\sqrt{2}}\right) + C$$

(ii) Let $I = \int \frac{dx}{\sqrt{3}\sin x + \cos x}$

$$= \int \frac{dx}{\frac{\sqrt{3}\cdot 2\tan x/2}{1+\tan^2 x/2} + \frac{1-\tan^2 x/2}{1+\tan^2 x/2}}$$

$$= \int \frac{\sec^2 \frac{x}{2}\, dx}{2\sqrt{3}\tan\frac{x}{2} + 1 - \tan^2\frac{x}{2}}$$

Put $\tan\frac{x}{2}=t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\,dx=dt$

$$\therefore \quad I=\int\frac{2dt}{-t^2+2\sqrt{3}t+1}=2\int\frac{dt}{-t^2+2\sqrt{3}t-3+3+1}$$

$$=2\int\frac{dt}{4-(t-\sqrt{3})^2}=2\int\frac{dt}{(2)^2-(t-\sqrt{3})^2}$$

$$=2\cdot\frac{1}{2(2)}\log\left|\frac{2+t-\sqrt{3}}{2-t+\sqrt{3}}\right|+C$$

$$\therefore \quad I=\frac{1}{2}\log\left|\frac{2-\sqrt{3}+\tan x/2}{2+\sqrt{3}-\tan x/2}\right|+C$$

(c) Alternative Method to Evaluate the Integrals of the Form

$$\int\frac{1}{a\sin x+b\cos x}\,dx$$

To evaluate this type of integrals we substitute $a=r\cos\theta$, $b=r\sin\theta$ and so

$$r=\sqrt{a^2+b^2},\quad \theta=\tan^{-1}\left(\frac{b}{a}\right)$$

$$\therefore \quad a\sin x+b\cos x=r\sin(x+\theta)$$

So, $$\int\frac{1}{a\sin x+b\cos x}\,dx=\frac{1}{r}\int\frac{1}{\sin(x+\theta)}\,dx$$

$$=\frac{1}{r}\int\operatorname{cosec}(x+\theta)\,dx=\frac{1}{r}\log\left|\tan\left(\frac{x}{2}+\frac{\theta}{2}\right)\right|+C$$

$$\therefore \quad \int\frac{1}{a\sin x+b\cos x}\,dx=\frac{1}{\sqrt{a^2+b^2}}\log\left|\tan\left(\frac{x}{2}+\frac{1}{2}\tan^{-1}\frac{b}{a}\right)\right|+C$$

Example 28 Evaluate $\int\frac{1}{\sqrt{3}\sin x+\cos x}\,dx$.

Sol. Let $\sqrt{3}=r\sin\theta$ and $1=r\cos\theta$.

Then, $r=\sqrt{(\sqrt{3})^2+(1)^2}=2$ and $\tan\theta=\frac{\sqrt{3}}{1}\Rightarrow\theta=\frac{\pi}{3}$

$$\therefore \quad I=\int\frac{1}{\sqrt{3}\sin x+\cos x}\,dx$$

$$=\int\frac{1}{r\sin\theta\sin x+r\cos\theta\cos x}\,dx$$

$$=\frac{1}{r}\int\frac{dx}{\cos(x-\theta)}=\frac{1}{r}\int\sec(x-\theta)\,dx$$

$$=\frac{1}{r}\log\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}-\frac{\theta}{2}\right)\right|+C$$

$$=\frac{1}{2}\log\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}-\frac{\pi}{6}\right)\right|+C$$

$$=\frac{1}{2}\log\left|\tan\left(\frac{x}{2}+\frac{\pi}{12}\right)\right|+C$$

(d) Integrals of the Form

$$\int\frac{p\cos x+q\sin x+r}{a\cos x+b\sin x+c}\,dx,\ \int\frac{p\cos x+q\sin x}{a\cos x+b\sin x}\,dx$$

Rule for (i) In this integral express numerator as,

λ (denominator) + μ (diffn. of denominator) + γ.

Find λ,μ and γ by comparing coefficients of $\sin x, \cos x$ and constant term and split the integral into sum of three integrals.

$$\lambda\int dx+\mu\int\frac{\text{d.c. of (denominator)}}{\text{denominator}}\,dx+\gamma\int\frac{dx}{a\sin x+b\cos x+c}$$

Rule for (ii) Express numerator as λ (denominator) + μ (differentiation of denominator) and find λ and μ as above.

Example 29 Evaluate $\int\frac{(2+3\cos x)}{\sin x+2\cos x+3}\,dx$.

Sol. Write the numerator = λ (denominator) + μ (d.c. of denominator) + γ

$$\Rightarrow \quad 2+3\cos x=\lambda(\sin x+2\cos x+3)+\mu(\cos x-2\sin x)+\gamma$$

Comparing the coefficients of $\sin x$, $\cos x$ and constant terms, we get

$$0=\lambda-2\mu,\ 3=2\lambda+\mu,\ 2=3\lambda+\gamma$$

$$\Rightarrow \quad \lambda=6/5,\ \mu=3/5,\ \gamma=-8/5$$

Hence, $$I=\frac{6}{5}\int 1\,dx+\frac{3}{5}\int\frac{\cos x-2\sin x}{\sin x+2\cos x+3}\,dx-\frac{8}{5}\int\frac{dx}{\sin x+2\cos x+3}$$

$$=\frac{6}{5}\cdot x+\frac{3}{5}\log|\sin x+2\cos x+3|-\frac{8}{5}I_3 \qquad \ldots(i)$$

Where, $I_3=\int\frac{dx}{\sin x+2\cos x+3}$

$$=\int\frac{dx}{\frac{2\tan x/2}{1+\tan^2 x/2}+\frac{2(1-\tan^2 x/2)}{1+\tan^2 x/2}+3}$$

$$=\int\frac{\sec^2\frac{x}{2}\,dx}{2\tan\frac{x}{2}+2-2\tan^2\frac{x}{2}+3+3\tan^2\frac{x}{2}}$$

$$=\int\frac{\sec^2\frac{x}{2}\,dx}{\tan^2\frac{x}{2}+2\tan\frac{x}{2}+5},\quad \text{let } \tan\frac{x}{2}=t$$

$$\Rightarrow \quad \frac{1}{2}\sec^2\frac{x}{2}\,dx = dt = \int \frac{2dt}{t^2+2t+5} = 2\int \frac{dt}{(t+1)^2+2^2}$$

$$I_3 = 2\cdot\frac{1}{2}\cdot\tan^{-1}\left(\frac{t+1}{2}\right) = \tan^{-1}\left(\frac{\tan\frac{x}{2}+1}{2}\right) \quad ...(ii)$$

From Eqs. (i) and (ii),

$$I = \frac{6}{5}x + \frac{3}{5}\log|\sin x + 2\cos x + 3| - \frac{8}{5}\tan^{-1}\left|\left(\frac{\tan\frac{x}{2}+1}{2}\right)\right| + C$$

Example 30 The value of $\int\{1+\tan x\cdot\tan(x+A)\}\,dx$ is equal to

(a) $\cot A\cdot\log\left|\dfrac{\sec x}{\sec(x+A)}\right| + C$

(b) $\tan A\cdot\log|\sec(x+A)| + C$

(c) $\cot A\cdot\log\left|\dfrac{\sec(x+A)}{\sec(x)}\right| + C$

(d) None of the above

Sol. Let $I = \int\{1+\tan x\cdot\tan(x+A)\}\,dx$

$$= \int\left\{1+\frac{\sin x\cdot\sin(x+A)}{\cos x\cdot\cos(x+A)}\right\}dx$$

$$= \int\frac{\cos x\cdot\cos(x+A)+\sin x\cdot\sin(x+A)}{\cos x\cdot\cos(x+A)}\,dx$$

$$= \int\frac{\cos(x+A-x)\,dx}{\cos x\cdot\cos(x+A)}$$

$$= \cos A\cdot\int\frac{dx}{\cos x\cdot\cos(x+A)}$$

Multiplying and dividing by $\sin A$, we get

$$= \cot A\cdot\int\frac{\sin A\,dx}{\cos x\cdot\cos(x+A)}$$

$$= \cot A\cdot\int\frac{\sin(x+A-x)\,dx}{\cos x\cdot\cos(x+A)}$$

$$= \cot A\cdot\int\left\{\frac{\sin(x+A)\cdot\cos x}{\cos x\cos(x+A)} - \frac{\cos(x+A)\cdot\sin x}{\cos x\cdot\cos(x+A)}\right\}dx$$

$$= \cot A\cdot\{\int\tan(x+A)dx - \int\tan x\,dx\}$$

$$= \cot A\cdot\{\log|\sec(x+A)| - \log|\sec x|\} + C$$

Hence, (c) is the correct answer.

Example 31 The value of $\int\frac{\sqrt{\cos 2x}}{\sin x}\,dx$, is equal to

(a) $\log\left|\cot x+\sqrt{\cot^2 x-1}\right| + \sqrt{2}\log\left|\cos x + \sqrt{\cos^2 x - 1/2}\right| + C$

(b) $-\log\left|\cot x+\sqrt{\cot^2 x-1}\right| + \sqrt{2}\log\left|\cos x + \sqrt{\cos^2 x - 1/2}\right| + C$

(c) $\log\left|\cot x+\sqrt{\cot^2 x-1}\right| + 2\log\left|\cos x + \sqrt{\cos^2 x - 1/2}\right| + C$

(d) $-\log\left|\cot x+\sqrt{\cot^2 x-1}\right| + 2\log\left|\cos x + \sqrt{\cos^2 x - 1/2}\right| + C$

Sol. Let $I = \int\frac{\sqrt{\cos 2x}}{\sin x}\,dx = \int\frac{\cos 2x}{\sin x\sqrt{\cos 2x}}\,dx$

$$= \int\frac{1-2\sin^2 x}{\sin x\sqrt{\cos 2x}}\,dx$$

$$= \int\frac{1}{\sin x\sqrt{\cos^2 x - \sin^2 x}}\,dx - 2\int\frac{\sin x}{\sqrt{2\cos^2 x - 1}}\,dx$$

$$= \int\frac{\operatorname{cosec}^2 x}{\sqrt{\cot^2 x - 1}}\,dx - \frac{2}{\sqrt{2}}\int\frac{\sin x}{\sqrt{\cos^2 x - 1/2}}\,dx$$

$$= -\int\frac{dt}{\sqrt{t^2-1}} - \sqrt{2}\int\frac{-ds}{\sqrt{s^2-(1/\sqrt{2})^2}}$$

[where $t = \cot x$ and $s = \cos x$]

$$= -\log|t+\sqrt{t^2-1}| + \sqrt{2}\log|s+\sqrt{s^2-1/2}| + C$$

$$= -\log|\cot x+\sqrt{\cot^2 x-1}| + \sqrt{2}\log|\cos x + \sqrt{\cos^2 x - 1/2}| + C$$

Hence, (b) is the correct answer.

Exercise for Session 3

■ Evaluate the following integrals

1. $\int \frac{x\,dx}{9-16x^4}$

2. $\int \frac{x^2}{9+16x^6}\,dx$

3. $\int \frac{x^3\,dx}{16x^8-25}$

4. $\int \sqrt{\frac{x}{a^3-x^3}}\,dx$

5. $\int \sqrt{\frac{x^4}{a^6+x^6}}\,dx$

6. $\int \frac{1}{4e^{-x}-9e^{x}}\,dx$

7. $\int \frac{2^x}{\sqrt{4^x-25}}\,dx$

8. $\int \frac{8x-11}{\sqrt{5+2x-x^2}}\,dx$

9. $\int \frac{x+2}{x^2+2x+2}\,dx$

10. $\int \frac{x-3}{3-2x-x^2}\,dx$

11. $\int \frac{3x-1}{4x^2-4x+17}\,dx$

12. $\int \frac{\sqrt{x}\,dx}{\sqrt{2x+3}}$

13. $\int \sqrt{\frac{a-x}{x-b}}\,dx$

14. $\int \sqrt{\frac{1-x}{1+x}}\,dx$

15. $\int \frac{x^2+2x+3}{\sqrt{x^2+x+1}}\,dx$

16. $\int \frac{dx}{1+\sin x+\cos x}$

17. $\int \frac{dx}{\sin x+\sqrt{3}\cos x}$

18. $\int \frac{\cos^2 x \sin x}{\sin x-\cos x}\,dx$

19. $\int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}}\,dx$

20. $\int \sqrt{\frac{\cos x-\cos^3 x}{(1-\cos^3 x)}}\,dx$

21. Evaluate $\int \frac{3\sin x+2\cos x}{3\cos x+2\sin x}\,dx$

22. Evalute $\int (2x-4)\sqrt{4+3x-x^2}dx$.

23. $\int \frac{(2x^2+5x+9)dx}{(x+1)\sqrt{x^2+x+1}}$

24. The value of $\int \frac{dx}{\sec x+\operatorname{cosec} x}$, is equal to

(a) $\left\{(\sin x+\cos x)+\frac{1}{\sqrt{2}}\log\left|\frac{\tan x/2-1-\sqrt{2}}{\tan x/2-1+\sqrt{2}}\right|\right\}+C$

(b) $2\left\{(\sin x+\cos x)+\frac{1}{\sqrt{2}}\log\left|\frac{\tan x/2-1-\sqrt{2}}{\tan x/2-1+\sqrt{2}}\right|\right\}+C$

(c) $\frac{1}{2}\left\{(\sin x-\cos x)+\frac{1}{\sqrt{2}}\log\left|\frac{\tan x/2-1-\sqrt{2}}{\tan x-1+\sqrt{2}}\right|\right\}+C$

(d) None of these

Session 4

Integration by Parts

Integration by Parts

Theorem If u and v are two functions of x, then

$$\int uv\,dx = u\int v\,dx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx$$

i.e. The integral of product of two functions = (first function) × (integral of second function) − integral of (differential of first function × integral of second function).

Proof For any two functions $f(x)$ and $g(x)$, we have

$$\frac{d}{dx}\{f(x)\cdot g(x)\} = f(x)\cdot\frac{d}{dx}\{g(x)\} + g(x)\cdot\frac{d}{dx}\{f(x)\}$$

$$\therefore \int\left(f(x)\cdot\frac{d}{dx}\{g(x)\}+g(x)\cdot\frac{d}{dx}\{f(x)\}\right)dx = \int f(x)\cdot g(x)\,dx$$

$$\Rightarrow \int\left(f(x)\cdot\frac{d}{dx}\{g(x)\}\right)dx + \int\left(g(x)\cdot\frac{d}{dx}\{f(x)\}\right)dx = \int f(x)\cdot g(x)dx$$

$$\Rightarrow \int\left(f(x)\cdot\frac{d}{dx}\{g(x)\}\right)dx = \int f(x)\cdot g(x)\,dx - \int\left(g(x)\cdot\frac{d}{dx}\{f(x)\}\right)dx$$

Let $f(x) = u$ and $\frac{d}{dx}\{g(x)\} = v$

So that, $g(x) = \int v\,dx$

$$\therefore \int uv\,dx = u\cdot\int v\,dx - \int\left\{\frac{du}{dx}\cdot\int v\,dx\right\}\cdot dx$$

Remarks

While applying the above rule, care has to be taken in the selection of first function (u) and selection of second function (v). Normally we use the following methods :

1. If in the product of the two functions, one of the functions is not directly integrable (e.g. $\log|x|$, $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, ..., etc.) Then, we take it as the first function and the remaining function is taken as the second function. i.e. In the integration of $\int x\tan^{-1}x\,dx$, $\tan^{-1}x$ is taken as the first function and x as the second function.
2. If there is no other function, then unity is taken as the second function. e.g. In the integration of $\int\tan^{-1}x\,dx$, $\tan^{-1}x$ is taken as first function and 1 as the second function.
3. If both of the function are directly integrable, then the first function is chosen in such a way that the derivative of the function thus obtained under integral sign is easily integrable.

Usually we use the following preference order for selecting the first function. (Inverse, Logarithmic, Algebraic, Trigonometric, Exponent).

In above stated order, the function on the left is always chosen as the first function. This rule is called as **ILATE**.

Example 32 Evaluate

(i) $\int\sin^{-1}x\,dx$ (ii) $\int\log_e|x|\,dx$

Sol. (i) $I = \int\sin^{-1}x\,dx = \int\underset{\text{I}}{\sin^{-1}x}\cdot\underset{\text{II}}{1}\,dx$

Here, we know by definition of integration by parts that order of preference is taken according to ILATE. So, '$\sin^{-1}x$' should be taken as first and '1' as the second function to apply by parts.

Applying integration by parts, we get

$$I = \sin^{-1}x\cdot(x) - \int\frac{1}{\sqrt{1-x^2}}\cdot x\,dx$$

$$= x\cdot\sin^{-1}x + \frac{1}{2}\int\frac{dt}{t^{1/2}}$$

Let $1-x^2 = t$

$$-2x\,dx = dt \Rightarrow x\,dx = -\frac{1}{2}dt$$

$$= x\sin^{-1}x + \frac{1}{2}\cdot\frac{t^{1/2}}{1/2} + C$$

$$I = x\sin^{-1}x + \sqrt{1-x^2} + C$$

$$\therefore \int\sin^{-1}x\,dx = x\sin^{-1}x + \sqrt{1-x^2} + C$$

(ii) $I = \int\log_e|x|\,dx = \int\underset{\text{I}}{\log_e|x|}\cdot\underset{\text{II}}{1}\,dx$

Applying integration by parts, we get

$$= \log|x|\cdot x - \int\frac{1}{x}\cdot x\,dx$$

$$= x\log|x| - \int 1\,dx$$

$$I = x\log|x| - x + C$$

Example 33 Evaluate

(i) $\int x\cos x\,dx$ (ii) $\int x^2\cos x\,dx$

Sol. (i) $\int x\cos x\,dx$, $I = \int\underset{\text{I}}{x}\,\underset{\text{II}}{\cos x}\,dx$

Applying integration by parts,

$$I = x\left(\int \cos x\, dx\right) - \int \left\{\frac{d}{dx}(x)\right\}\left\{\int (\cos x)\, dx\right\} dx$$

$$I = x \sin x - \int 1 \cdot \sin x\, dx = x \sin x + \cos x + C$$

(ii) $I = \underset{\text{I}}{\int x^2} \underset{\text{II}}{\cos x}\, dx$

Applying integration by parts,

$$I = x^2\left(\int \cos x\, dx\right) - \int \left\{\frac{d}{dx}(x^2)\right\} \cdot \{\int \cos x\} dx$$

$$= x^2 \sin x - \int 2x \cdot (\sin x)\, dx$$

$$= x^2 \sin x - 2\int x\,(\sin x)\, dx$$

We again have to integrate $\int x \sin x\, dx$ using integration by parts,

$$= x^2 \cdot \sin x - 2 \int \underset{\text{I}}{x} \cdot \underset{\text{II}}{\sin x}\, dx$$

$$= x^2 \sin x - 2\left\{x\left(\int \sin x\, dx\right) - \int \left(\frac{dx}{dx}\right)\left(\int \sin x\, dx\right) dx\right\}$$

$$= x^2 \sin x - 2\{-x \cos x - \int 1 \cdot (-\cos x)\, dx\}$$

$$I = x^2 \sin x + 2x \cos x - 2 \sin x + C$$

Example 34 Evaluate $\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}\, dx$.

Sol. Let $I = \int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}\, dx$

$$= \int \frac{\sin^{-1}\sqrt{x} - (\pi/2 - \sin^{-1}\sqrt{x})}{\pi/2}\, dx$$

$[\because \sin^{-1}\theta + \cos^{-1}\theta = \pi/2]$

$$\Rightarrow \quad I = \frac{2}{\pi}\int (2\sin^{-1}\sqrt{x} - \pi/2)\, dx$$

$$= \frac{4}{\pi}\int \sin^{-1}\sqrt{x}\, dx - \int 1\, dx$$

$$I = \frac{4}{\pi}\int \sin^{-1}\sqrt{x}\, dx - x + C \qquad \ldots(i)$$

Let $x = \sin^2\theta$, then $dx = 2\sin\theta\cos\theta\, d\theta = \sin 2\theta\, d\theta$

$$\therefore \quad \int \sin^{-1}\sqrt{x}\, dx = \int \underset{\text{I}}{\theta} \cdot \underset{\text{II}}{\sin 2\theta}\, d\theta$$

Applying integration by parts

$$\int \sin^{-1}\sqrt{x}\, dx = -\theta \cdot \frac{\cos 2\theta}{2} + \int \frac{1}{2}\cos 2\theta\, d\theta$$

$$= \frac{-\theta}{2} \cdot \cos 2\theta + \frac{1}{4}\sin 2\theta$$

$$= \frac{-1 \cdot \theta}{2} \cdot (1 - 2\sin^2\theta) + \frac{1}{2} \cdot \sin\theta \cdot \sqrt{1 - \sin^2\theta}$$

$$= \frac{-1}{2}\sin^{-1}\sqrt{x}\,(1 - 2x) + \frac{1}{2} \cdot \sqrt{x}\,\sqrt{1 - x} \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$I = \frac{4}{\pi}\left\{\frac{-1}{2}(\sin^{-1}\sqrt{x})(1 - 2x) + \frac{1}{2}\sqrt{x}\sqrt{1 - x}\right\} - x + C$$

$$= \frac{2}{\pi}\{\sqrt{x - x^2} - (1 - 2x)\sin^{-1}\sqrt{x}\} - x + C$$

Integral of Form $\int e^x \{f(x) + f'(x)\} dx$

Theorem Prove that

$$\int e^x \{f(x) + f'(x)\}\, dx = e^x f(x) + C$$

Proof We have, $\int e^x \{f(x) + f'(x)\}\, dx$

$$= \int \underset{\text{II}}{e^x} \cdot \underset{\text{I}}{f(x)}\, dx + \int e^x \cdot f'(x)\, dx$$

$$= f(x) \cdot e^x - \int f'(x) \cdot e^x dx + \int e^x \cdot f'(x)\, dx + C$$

$$= f(x) \cdot e^x + C$$

Thus, to evaluate the integrals of the type

$$\int e^x \{f(x) + f'(x)\}\, dx,$$

we first express the integral as the sum of two integrals $\int e^x f(x)\, dx$ and $\int e^x f'(x)\, dx$ and then integrate the integral involving $e^x f(x)$ as integral by parts taking e^x as second function.

Remark

The above theorem is also true, if we have e^{kx} in place of e^x

i.e. $\int e^{kx}\{f(kx) + f'(kx)\}\, dx = e^{kx} f(kx) + C$

General Concept

$$\int e^{g(x)} \{f(x)g'(x) + f'(x)\}\, dx$$

Proof $I = \int \underset{\text{II}}{e^{g(x)}}\, \underset{\text{I}}{f(x)}\, \underset{\text{II}}{g'(x)}\, dx + \underbrace{\int e^{g(x)} f'(x)\, dx}_{\text{as it is}}$

Using, $\int e^{g(x)} \cdot g'(x)\, dx = e^{g(x)}$, we get

$$= f(x) \cdot e^{g(x)} - \int f'(x) \cdot e^{g(x)} dx + \int e^{g(x)} \cdot f'(x)\, dx$$

$$= f(x) \cdot e^{g(x)} + C$$

e.g. $$= \int e^{(x\sin x + \cos x)}\left(\frac{x^2\cos^2 x - (x\sin x + \cos x)}{x^2}\right) dx$$

$$\Rightarrow \quad \int e^{(x\sin x + \cos x)}\left(\cos^2 x - \frac{x\sin x + \cos x}{x^2}\right) dx$$

$$\Rightarrow \int e^{(x\sin x+\cos x)}\left(x\cos x\left(\frac{\cos x}{x}\right)+\left(\frac{\cos x}{x}\right)'\right)dx$$

$$\Rightarrow e^{(x\sin x+\cos x)}\cdot\frac{\cos x}{x}+C$$

e.g. $$=\int e^{\tan x}(\sin x-\sec x)\,dx$$

$$=\int e^{\tan x}\sin x\,dx-\int e^{\tan x}\sec x\,dx$$

$$\Rightarrow -e^{\tan x}\cdot\cos x+\int e^{\tan x}\sec^2 x\cos x dx-\int e^{\tan x}\sec x dx$$

$$\Rightarrow -e^{\tan x}\cdot\cos x$$

Example 35 Evaluate

(i) $\int e^x\left(\frac{1+\sin x\cos x}{\cos^2 x}\right)dx$ (ii) $\int e^{2x}\left(\frac{1+\sin 2x}{1+\cos 2x}\right)dx$

Sol. (i) $I=\int e^x\left(\frac{1+\sin x\cos x}{\cos^2 x}\right)dx$

$$I=\int e^x\left\{\frac{1}{\cos^2 x}+\frac{\sin x\cos x}{\cos^2 x}\right\}dx$$

$$I=\int e^x\{\tan x+\sec^2 x\}\,dx$$

$$I=\int \underset{\text{II}}{e^x}\cdot\underset{\text{I}}{\tan x}\,dx+\int e^x(\sec^2 x)\,dx$$

$$I=\tan x\cdot e^x-\int\sec^2 x\cdot e^x\,dx+\int e^x\cdot\sec^2 x\,dx+C$$

$$I=e^x\tan x+C$$

(ii) $I=\int e^{2x}\left\{\frac{1+\sin 2x}{1+\cos 2x}\right\}dx$

$$=\int e^{2x}\left\{\frac{1+2\sin x\cos x}{2\cos^2 x}\right\}dx$$

$$=\int e^{2x}\left\{\frac{1}{2\cos^2 x}+\frac{2\sin x\cos x}{2\cos^2 x}\right\}dx$$

$$=\int e^{2x}\left\{\frac{1}{2}\sec^2 x+\tan x\right\}dx$$

$$=\int \underset{\text{II}}{e^{2x}}\cdot\underset{\text{I}}{\tan x}\,dx+\frac{1}{2}\int e^{2x}\cdot\sec^2 x\,dx$$

$$=\tan x\cdot\frac{e^{2x}}{2}-\int\sec^2 x\cdot\frac{e^{2x}}{2}dx+\frac{1}{2}\int e^{2x}\cdot\sec^2 x\,dx$$

$$I=\frac{1}{2}e^{2x}\cdot\tan x+C$$

Example 36 ***Evaluate*** $\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx$.

Sol. $I=\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx=\int e^x\frac{(1-2x+x^2)}{(1+x^2)^2}dx$

$$=\int e^x\left\{\frac{1+x^2}{(1+x^2)^2}-\frac{2x}{(1+x^2)^2}\right\}dx$$

$$=\int e^x\left\{\frac{1}{1+x^2}-\frac{2x}{(1+x^2)^2}\right\}dx\left[\text{as}\,\frac{d}{dx}\left(\frac{1}{1+x^2}\right)=-\frac{2x}{(1+x^2)^2}\right]$$

$$=\frac{e^x}{1+x^2}+C$$

$$\therefore \quad I=\frac{e^x}{1+x^2}+C$$

Integrals of the Form $\int e^{ax}\sin bx\,dx, \int e^{ax}\cos bx\,dx$

Let $$I=\int e^{ax}(\sin bx)\,dx$$

Then, $$I=\int \underset{\text{I}}{\sin bx}\cdot\underset{\text{II}}{e^{ax}}\,dx$$

$$=\sin bx\cdot\left(\frac{e^{ax}}{a}\right)-\int b\cos bx\cdot\frac{e^{ax}}{a}dx$$

$$=\frac{1}{a}\sin bx\cdot e^{ax}-\frac{b}{a}\left\{\cos bx\cdot\frac{e^{ax}}{a}-\int(-b\sin bx)\cdot\frac{e^{ax}}{a}dx\right\}$$

$$=\frac{1}{a}\sin bx\cdot e^{ax}-\frac{b}{a^2}\cos bx\cdot e^{ax}-\frac{b^2}{a^2}\int\sin bx\cdot e^{ax}\,dx$$

$$I=\frac{1}{a}\sin bx\cdot e^{ax}-\frac{b}{a^2}\cos bx\cdot e^{ax}-\frac{b^2}{a^2}I$$

$$\therefore \quad I+\frac{b^2}{a^2}I=\frac{1\cdot e^{ax}}{a^2}\cdot(a\sin bx-b\cos bx)$$

$$\Rightarrow \quad I\left(\frac{a^2+b^2}{a^2}\right)=\frac{e^{ax}}{a^2}(a\sin bx-b\cos bx)$$

or $$I=\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)+C$$

Thus, $\int e^{ax}\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)+C$

Similarly, $\int e^{ax}\cos bx\,dx=\frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)+C$

Aliter Use Euler's equation

Let $P=\int e^{ax}\cos bx\,dx$ and $Q=\int e^{ax}\sin bx\,dx$

Hence, $P+iQ=\int e^{ax}\cdot e^{ibx}\,dx=\int e^{(a+ib)x}\,dx$

$$P+iQ=\frac{1}{a+ib}e^{(a+ib)x}=\frac{a-ib}{a^2+b^2}e^{ax}(\cos bx+i\sin bx)$$

$$= \frac{(ae^{ax}\cos bx + be^{ax}\sin bx) - i(ae^{ax}\sin bx - be^{ax}\cos bx)}{a^2 + b^2}$$

$$\therefore \quad P = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2 + b^2}$$

$$Q = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2 + b^2}$$

Example 37 *Evaluate*

(i) $\int e^x \cos^2 x\, dx$

(ii) $\int \sin(\log x)\, dx$

Sol. (i) $I = \int e^x \cdot \cos^2 x\, dx = \int e^x \cdot \left\{\frac{1 + \cos 2x}{2}\right\} dx$

$$I = \frac{1}{2}\int e^x\, dx + \frac{1}{2}\int \cos 2x \cdot e^x\, dx$$

$$I = \frac{1}{2}e^x + \frac{1}{2}I_1 \qquad \ldots(i)$$

where, $I_1 = \int \cos 2x \cdot e^x\, dx$

$$I_1 = \int \underset{\text{I}}{\cos 2x} \cdot \underset{\text{II}}{e^x}\, dx = \cos 2x \cdot e^x - \int -2\sin 2x \cdot e^x\, dx$$

$$= e^x \cdot \cos 2x + 2\int \underset{\text{I}}{\sin 2x} \cdot \underset{\text{II}}{e^x}\, dx$$

$$= e^x \cdot \cos 2x + 2\{\sin 2x \cdot e^x - \int 2\cos 2x \cdot e^x\, dx\}$$

$$= e^x \cdot \cos 2x + 2\sin 2x \cdot e^x - 4I_1$$

$$\therefore \quad I_1 = \frac{1}{5}\{e^x \cos 2x + 2\sin 2x\, e^x\} \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$I = \frac{1}{2}e^x + \frac{1}{2}\cdot\frac{1}{5}\{e^x \cos 2x + 2\sin 2x \cdot e^x\}$$

$$I = \frac{1}{2}e^x + \frac{1}{10}e^x\{\cos 2x + 2\sin 2x\} + C$$

(ii) $I = \int \sin(\log x)\, dx$

Let $\log x = t$

$\Rightarrow \quad x = e^t$ or $dx = e^t\, dt$

$$\therefore \quad I = \int \underset{\text{I}}{(\sin t)} \cdot \underset{\text{II}}{e^t}\, dt = \sin t \cdot e^t - \int \underset{\text{I}}{\cos t} \cdot \underset{\text{II}}{e^t}\, dt$$

$$I = \sin t \cdot e^t - \{\cos t \cdot e^t - \int -\sin t \cdot e^t\, dt\}$$

$$I = e^t \cdot \sin t - e^t \cdot \cos t - I$$

$$\therefore \quad I = \frac{1}{2}e^t(\sin t - \cos t) + C$$

$$I = \frac{x}{2}\{\sin(\log x) - \cos(\log x)\} + C$$

Example 38 *Evaluate* $\int \frac{x^2 dx}{(x\sin x + \cos x)^2}$.

Sol. Let $I = \int \frac{x^2}{(x\sin x + \cos x)^2}\, dx$

Multiplying and dividing it by $(x\cos x)$, we get

$$I = \int \underset{\text{I}}{(x\sec x)} \cdot \underset{\text{II}}{\frac{(x\cos x)}{(x\sin x + \cos x)^2}}\, dx$$

$$I = x\sec x \cdot \int \frac{x\cos x}{(x\sin x + \cos x)^2}\, dx$$

$$- \int \left\{\frac{d}{dx}(x\sec x)\right\}\left\{\int \frac{x\cos x}{(x\sin x + \cos x)^2}\, dx\right\} dx$$

$$= x\sec x \cdot \frac{-1}{(x\sin x + \cos x)}$$

$$- \int (x\sec x\tan x + \sec x) \cdot \frac{-1}{(x\sin x + \cos x)}\, dx$$

$$= \frac{-x\sec x}{(x\sin x + \cos x)} + \int \frac{(x\sin x + \cos x)}{\cos^2 x \cdot (x\sin x + \cos x)}\, dx$$

$$= \frac{-x\sec x}{(x\sin x + \cos x)} + \int \sec^2 x\, dx$$

$$I = \frac{-x\sec x}{(x\sin x + \cos x)} + \tan x + C$$

Example 39 The value of $\int \sqrt{\frac{3-x}{3+x}} \cdot \sin^{-1}\left(\frac{1}{\sqrt{6}}\sqrt{3-x}\right) dx$, is equal to

(a) $\frac{1}{4}\left\{-3\left(\cos^{-1}\left(\frac{x}{3}\right)\right)^2 + 2\sqrt{9-x^2}\cdot\cos^{-1}\left(\frac{x}{3}\right) + 2x\right\} + C$

(b) $\frac{1}{4}\left\{-3\left(\cos^{-1}\left(\frac{x}{3}\right)\right)^2 + 2\sqrt{9-x^2}\sin^{-1}\left(\frac{x}{3}\right) + 2x\right\} + C$

(c) $\frac{1}{4}\left\{-3\left(\sin^{-1}\left(\frac{x}{3}\right)\right)^2 + 2\sqrt{9-x^2}\sin^{-1}\left(\frac{x}{3}\right) + 2x\right\} + C$

(d) None of the above

Sol. Here, $I = \int \sqrt{\frac{3-x}{3+x}} \cdot \sin^{-1}\left(\frac{1}{\sqrt{6}}\sqrt{3-x}\right) dx$

Put $\quad x = 3\cos 2\theta$

$\Rightarrow \quad dx = -6\sin 2\theta\, d\theta$

$$\therefore \quad I = \int \sqrt{\frac{3-3\cos 2\theta}{3+3\cos 2\theta}} \cdot \sin^{-1}\left(\frac{1}{\sqrt{6}}\sqrt{3-3\cos 2\theta}\right)(-6\sin 2\theta)\, d\theta$$

$$= \int \frac{\sin\theta}{\cos\theta} \cdot \sin^{-1}(\sin\theta) \cdot (-6\sin 2\theta)\, d\theta$$

$$= -6\int \theta \cdot (2\sin^2\theta)\, d\theta = -6\int \theta(1 - \cos 2\theta)\, d\theta$$

$$= -6\left\{\frac{\theta^2}{2} - \int \theta \cos 2\theta \, d\theta\right\}$$

$$= -6\left\{\frac{\theta^2}{2} - \left(\theta \frac{\sin 2\theta}{2} - \int 1 \cdot \left(\frac{\sin 2\theta}{2}\right) d\theta\right)\right\}$$

$$= -3\theta^2 + 6\left\{\frac{\theta \sin 2\theta}{2} + \frac{\cos 2\theta}{4}\right\} + C$$

$$= \frac{1}{4}\left\{-3\left(\cos^{-1}\left(\frac{x}{3}\right)\right)^2 + 2\sqrt{9-x^2} \cdot \cos^{-1}\left(\frac{x}{3}\right) + 2x\right\} + C$$

Hence, (a) is the correct answer.

Example 40 The value of $\int \frac{\sec x\,(2+\sec x)}{(1+2\sec x)^2}\,dx$, is equal to

(a) $\frac{\sin x}{2+\cos x} + C$ (b) $\frac{\cos x}{2+\cos x} + C$

(c) $\frac{-\sin x}{2+\sin x} + C$ (d) $\frac{\cos x}{2+\sin x} + C$

Sol. Let $I = \int \frac{\sec x\,(2+\sec x)}{(1+2\sec x)^2}\,dx = \int \frac{2\cos x + 1}{(\cos x + 2)^2}\,dx$

$$= \int \frac{\cos x\,(\cos x + 2) + \sin^2 x}{(2+\cos x)^2}\,dx$$

$$= \int \frac{\cos x}{2+\cos x}\,dx + \int \frac{\sin^2 x}{(2+\cos x)^2}\,dx$$

$$= \int \underset{\text{II}}{\cos x} \cdot \underset{\text{I}}{\frac{1}{(2+\cos x)}}\,dx + \int \frac{\sin^2 x}{(2+\cos x)^2}\,dx$$

Applying integration by parts to first by taking $\cos x$ as second function, keeping $\int \frac{\sin^2 x}{(2+\cos x)^2}\,dx$ as it is.

$$\therefore \quad I = \frac{1}{2+\cos x} \cdot (\sin x) - \int \sin x \cdot \frac{\sin x}{(2+\cos x)^2}\,dx + \int \frac{\sin^2 x}{(2+\cos x)^2}\,dx$$

$$\therefore \quad I = \frac{\sin x}{2+\cos x} + C$$

Hence, (a) is the correct answer.

Example 41 The value of $\int \log\left(\sqrt{1-x} + \sqrt{1+x}\right) dx$, is equal to

(a) $x \log\left(\sqrt{1-x} + \sqrt{1+x}\right) + \frac{1}{2}x - \frac{1}{2}\sin^{-1}(x) + C$

(b) $x \log\left(\sqrt{1-x} + \sqrt{1+x}\right) + \frac{1}{2}x + \frac{1}{2}\sin^{-1}(x) + C$

(c) $x \log\left(\sqrt{1-x} + \sqrt{1+x}\right) - \frac{1}{2}x + \frac{1}{2}\sin^{-1}(x) + C$

(d) None of the above

Sol. Here, we have only one function. This can be solved easily by applying integration by parts taking unity as second function.

If we take $u = \log\left(\sqrt{1-x} + \sqrt{1+x}\right)$ as the first function and $v = 1$ as the second function.

Then,

$$I = \int 1 \cdot \log\left(\sqrt{1-x} + \sqrt{1+x}\right) dx$$

$$= \{\log\left(\sqrt{1-x} + \sqrt{1+x}\right)\} \cdot x - \int \frac{1}{\sqrt{1-x} + \sqrt{1+x}} \left(-\frac{1}{2\sqrt{1-x}} + \frac{1}{2\sqrt{1+x}}\right) \cdot x\,dx$$

$$= x \log\left(\sqrt{1-x} + \sqrt{1+x}\right) - \frac{1}{2}\int \frac{\sqrt{1-x} - \sqrt{1+x}}{\sqrt{1-x} + \sqrt{1+x}} \cdot \frac{1}{\sqrt{1-x^2}} \cdot x\,dx$$

$$= x \log\left(\sqrt{1-x} + \sqrt{1+x}\right) - \frac{1}{2}\int \frac{(1-x) + (1+x) - 2\sqrt{1-x^2}}{(1-x) - (1+x)} \cdot \frac{1}{\sqrt{1-x^2}} \cdot 2x\,dx$$

$$= x \log\left(\sqrt{1-x} + \sqrt{1+x}\right) - \frac{1}{2}\int \frac{\sqrt{1-x^2} - 1}{\sqrt{1-x^2}}\,dx$$

$$= x \log\left(\sqrt{1-x} + \sqrt{1+x}\right) - \frac{1}{2}\int 1\,dx + \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= x \log\left(\sqrt{1-x} + \sqrt{1+x}\right) - \frac{1}{2}x + \frac{1}{2}\sin^{-1} x + C$$

Hence, (c) is the correct answer.

Example 42 The value of $\int e^x \left(\frac{x^4+2}{(1+x^2)^{5/2}}\right) dx$, is equal to

(a) $\frac{e^x\,(x+1)}{(1+x^2)^{3/2}}$ (b) $\frac{e^x\,(1-x+x^2)}{(1+x^2)^{3/2}}$

(c) $\frac{e^x\,(1+x)}{(1+x^2)^{3/2}}$ (d) None of these

Sol. Let $I = \int e^x \left(\frac{x^4+2}{(1+x^2)^{5/2}}\right) dx$

$$= \int e^x \left(\frac{1}{(1+x^2)^{1/2}} + \frac{1-2x^2}{(1+x^2)^{5/2}}\right) dx$$

$$= \int e^x \left(\frac{1}{(1+x^2)^{1/2}} - \frac{x}{(1+x^2)^{3/2}} + \frac{x}{(1+x^2)^{3/2}} + \frac{1-2x^2}{(1+x^2)^{5/2}}\right) dx$$

$$= \frac{e^x}{(1+x^2)^{1/2}} + \frac{xe^x}{(1+x^2)^{3/2}} + C = \frac{e^x\,\{1+x^2+x\}}{(1+x^2)^{3/2}} + C$$

Hence, (d) is the correct answer.

Example 43 If $\int (\sin 3\theta + \sin\theta)\, e^{\sin\theta} \cos\theta\, d\theta = (A\sin^3\theta + B\cos^2\theta + C\sin\theta + D\cos\theta + E)\, e^{\sin\theta} + F$, then

(a) $A = -4, B = 12$ (b) $A = -4, B = -12$

(c) $A = 4, B = 12$ (d) $A = 4, B = -12$

Sol. Let $I = \int (\sin 3\theta + \sin\theta)\, e^{\sin\theta} \cdot \cos\theta\, d\theta$

$= \int (3\sin\theta - 4\sin^3\theta) \cdot e^{\sin\theta} \cdot \cos\theta\, d\theta;$

Put $\sin\theta = t \Rightarrow \cos\theta\, d\theta = dt$

$= \int (3t - 4t^3)\, e^t\, dt$ ...(i)

As, $I = (A\sin^3\theta + B\cos^2\theta + C\sin\theta + D\cos\theta + E)e^{\sin\theta} + F$

$= (A\sin^3\theta - B\sin^2\theta + C\sin\theta + D\cos\theta + B + E)e^{\sin\theta} + F$

When, $\sin\theta = t$

$I = (At^3 - Bt^2 + Ct + B + E)\, e^t + F$

as by Eq. (i) $D = 0$ ...(ii)

From Eqs. (i) and (ii),

$\int (3t - 4t^3)\, e^t\, dt = \underbrace{(At^3 - Bt^2 + Ct + B + E)}_{f(t)}\, e^t + F,$

differentiating both sides

$(3t - 4t^3)\, e^t = (At^3 - Bt^2 + Ct + B + E)e^t + (3At^2 - 2Bt + C)e^t$

$\Rightarrow \quad A = -4$ and $3A = B \Rightarrow B = -12$

Hence, (b) is the correct answer.

Example 44 The value of $\int e^{(x\sin x + \cos x)} \cdot \left(\frac{x^4\cos^3 x - x\sin x + \cos x}{x^2\cos^2 x}\right) dx$, is equal to

(a) $e^{(x\sin x + \cos x)} \cdot \left(x + \frac{1}{x\cos x}\right) + C$

(b) $e^{(x\sin x + \cos x)} \cdot \left(x\cos x + \frac{1}{x}\right) + C$

(c) $e^{(x\sin x + \cos x)} \cdot \left(x - \frac{1}{x\cos x}\right) + C$

(d) None of the above

Sol. Let $I = \int e^{(x\sin x + \cos x)} \cdot \left(\frac{x^4\cos^3 x - x\sin x + \cos x}{x^2\cos^2 x}\right) dx$

$= \int (x \cdot e^{(x\sin x + \cos x)} \cdot x\cos x)\, dx - \int e^{(x\sin x + \cos x)} \cdot \left(\frac{x\sin x - \cos x}{(x\cos x)^2}\right) dx$

Applying integration by parts

$= \{x \cdot e^{(x\sin x + \cos x)} - \int e^{(x\sin x + \cos x)}\, dx\} - \left\{e^{(x\sin x + \cos x)} \cdot \frac{1}{x\cos x} - \int e^{(x\sin x + \cos x)}\, dx\right\}$

$= e^{(x\sin x + \cos x)} \left(x - \frac{1}{x\cos x}\right) + C$

Hence, (c) is the correct answer.

Example 45 ***Evaluate*** $\int \sin^{-1}\left\{\frac{2x+2}{\sqrt{4x^2+8x+13}}\right\} dx.$

[IIT JEE 2001]

Sol. Here, $I = \int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right) dx$

$= \int \sin^{-1}\left(\frac{2x+2}{\sqrt{(2x+2)^2+3^2}}\right) dx$

Put $2x + 2 = 3\tan\theta \Rightarrow 2\, dx = 3\sec^2\theta\, d\theta$

$= \int \sin^{-1}\left(\frac{3\tan\theta}{3\sec\theta}\right)\frac{3}{2}\sec^2\theta\, d\theta = \frac{3}{2}\int \theta\sec^2\theta\, d\theta$

$= \frac{3}{2}\{\theta\tan\theta - \int \tan\theta\, d\theta\}$

$= \frac{3}{2}\{\theta\tan\theta - \log|\sec\theta|\} + C$

$\Rightarrow I = \frac{3}{2}\left\{\frac{2x+2}{3}\tan^{-1}\left(\frac{2x+2}{3}\right) - \log\left(\sqrt{1 + \left(\frac{2x+2}{3}\right)^2}\right)\right\} + C$

$= \frac{3}{2}\left\{\frac{2}{3}(x+1)\tan^{-1}\left(\frac{2}{3}(x+1)\right) - \log\sqrt{4x^2+8x+13}\right\} + C$

$\Rightarrow I = (x+1)\tan^{-1}\left(\frac{2}{3}(x+1)\right) - \frac{3}{4}\log(4x^2+8x+13) + C$

Example 46 ***Evaluate*** $\int \frac{x^2(x\sec^2 x + \tan x)\, dx}{(x\tan x + 1)^2}.$

Sol. Here, $I = \int x^2 \cdot \frac{x\sec^2 x + \tan x}{(x\tan x + 1)^2}\, dx$

$= x^2\left(-\frac{1}{(x\tan x + 1)}\right) - \int 2x \cdot \left(-\frac{1}{(x\tan x + 1)}\right) dx$ ...(i)

$\left[\text{using } \int \frac{x\sec^2 x + \tan x}{(x\tan x + 1)^2}\, dx = \int \frac{dt}{t^2} = -\frac{1}{t} = -\frac{1}{(x\tan x + 1)}\right]$

$\Rightarrow I = -\left(\frac{x^2}{x\tan x + 1}\right) + \int \frac{2x(\cos x)}{x\sin x + \cos x}\, dx$

[put, $x\sin x + \cos x = u$

$\Rightarrow (x\cos x + \sin x - \cos x)\, dx = du$]

$\Rightarrow \quad I = -\frac{x^2}{(x\tan x + 1)} + 2\int \frac{du}{u}$

$= -\frac{x^2}{x\tan x + 1} + 2\log|u| + C$

$= -\frac{x^2}{x\tan x + 1} + 2\log|x\sin x + \cos x| + C$

Exercise for Session 4

1. $\int x^2 e^x dx$

2. $\int x^2 \sin x \, dx$

3. $\int \log x \cdot dx$

4. $\int (\log x)^2 dx$

5. $\int (\tan^{-1} x)\, dx$

6. $\int (\sec^{-1} x)\, dx$

7. $\int x \tan^{-1} x \, dx$

8. $\int \frac{\log x}{x^2} dx$

9. $\int \frac{x - \sin x}{1 - \cos x} dx$

10. $\int \log (1 + x^2)\, dx$

11. $\int e^x (\tan x + \log \sec x)\, dx$

12. $\int e^x \left\{ \frac{1 + \sin x \cos x}{\cos^2 x} \right\} dx$

13. $\int \left(\log (\log x) + \frac{1}{(\log x)^2} \right) dx$

14. $\int e^{2x} \cdot \left\{ \frac{1 + \sin 2x}{1 + \cos 2x} \right\} dx$

15. $\int e^x \frac{(1 - x)^2}{(1 + x^2)^2} dx$

16. $\int \frac{e^x \cdot (2 - x^2)}{(1 - x)\sqrt{1 - x^2}} dx$

17. $\int e^{ax} \cdot \cos (bx + c)\, dx$

18. $\int \sec^3 x \, dx$

19. $\int \sin \sqrt{x} \, dx$

20. $\int (\sin^{-1} x)^2 dx$

21. $\int \cot^{-1} (1 - x + x^2)\, dx$

22. $\int \sin^{-1} \sqrt{\frac{x}{a + x}} dx$

23. $\int \frac{\sqrt{x^2 + 1}\{\log (x^2 + 1) - 2 \log x\}}{x^4} dx$

24. $\int \frac{\cos^2 x + \sin 2x}{(2 \cos x - \sin x)^2} dx$

25. $\int e^{\sin x} \left(\frac{x \cos^2 x - \sin x}{\cos^2 x} \right) dx$

Session 5

Integration Using Partial Fraction

This section deals with the integration of general algebraic rational functions, of the form $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are both polynomials. We already have seen some examples of this form. For example, we know how to integrate functions of the form $\frac{1}{Q(x)}$ or $\frac{L(x)}{Q(x)}$ or $\frac{P(x)}{Q(x)}$ where $L(x)$ is a linear factor, $Q(x)$ is a quadratic factor and $P(x)$ is a polynomial of degree $n \geq 2$. We intend to generalise that previous discussion in this section.

We are assuming the scanario where $g(x)$ (the denominator) is decomposible into linear or quadratic factors. These are the only cases relevant to us right now. Any linear or quadratic factor in $g(x)$ might also occur repeatedly.

Thus, $g(x)$ could be of the following general forms.

- $g(x) = L_1(x)L_2(x)\ldots L_n(x)$ (n linear factors)
- $g(x) = L_1(x)\ldots L_r^k(x)\ldots L_n(x)$ (n linear factors; the rth factor is repeated k times)
- $g(x) = L_1^{k_1}(x)L_2^{k_2}(x)\ldots L_n^{k_n}(x)$ (n linear factors, the ith factor is repeated k_i times)
- $g(x) = L_1(x)L_2(x)\ldots L_n(x)Q(x)Q_2(x)\ldots Q_m(x)$ (n linear factors and m quadratic factors)
- $g(x) = \ldots Q_r^k(x)\ldots$ (a particular quadratic factor repeats more than once)
- A combination of any of the above

Suppose that the degree of $g(x)$ is n and that of $f(x)$ is m. If $m \geq n$,we can always divide $f(x)$ by $g(x)$ to obtain a quotient $q(x)$ and a remainder $r(x)$ whose degree would be less than n.

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)} \qquad \ldots(\text{i})$$

If $m < n$, $\frac{f(x)}{g(x)}$ is termed a proper rational function.

The partial fraction expansion technique says that a proper rational function can be expressed as a sum of simpler rational functions each possessing one of the factors of $g(x)$. The simpler rational functions are called partial fractions.

From now on, we consider only proper rational functions. If $\frac{f(x)}{g(x)}$ is not proper, we make it proper $\left(\frac{r(x)}{g(x)}\right)$ by the procedure described in (1) above. Let us consider a few examples.

Let $g(x)$ be a product of non-repeated, linear factors :

$$g(x) = L_1(x)L_2(x)\ldots L_n(x)$$

Then, we can expand $\frac{f(x)}{g(x)}$ in terms of partial fractions as

$$\frac{f(x)}{g(x)} = \frac{A_1}{L_1(x)} + \frac{A_2}{L_2(x)} + \ldots + \frac{A_n}{L_n(x)}$$

where the $A_i's$ are all constants that need to be determined.

Suppose $f(x) = x + 1$ and $g(x) = (x-1)(x-2)(x-3)$. Let us write down the partial fraction expansion of $\frac{f(x)}{g(x)}$:

$$\frac{f(x)}{g(x)} = \frac{x+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

We need to determine A, B and C. Cross multiplying in the expression above, we obtain :

$$(x+1) = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

A, B, C can now be determined by comparing coefficients on both sides. More simply since this relation that we have obtained should held true for all x,we substitute those values of x that would straight way give us the required values of A, B and C. These values are obviously the roots of $g(x)$.

$$\begin{aligned}
x = 1 \quad &\Rightarrow \quad 2 = A(-1)(-2) + B(0) + C(0) \\
&\Rightarrow \quad A = 1 \\
x = 2 \quad &\Rightarrow \quad 3 = A(0) + B(1)(-1) + C(0) \\
&\Rightarrow \quad B = -3 \\
x = 3 \quad &\Rightarrow \quad 4 = A(0) + B(0) + C(2)(1) \\
&\Rightarrow \quad C = 2
\end{aligned}$$

Thus, $A = 1$, $B = -3$ and $C = 2$.

We can therefore write $\frac{f(x)}{g(x)}$ as a sum of partial fractions.

$$\frac{f(x)}{g(x)} = \frac{1}{x-1} - \frac{3}{x-2} + \frac{2}{x-3}$$

Integrating $\frac{f(x)}{g(x)}$ is now a simple matter of integrating the partial fractions. This was our sole motive in writing such an expansion, so that integration could be carried out easily. In the example above :

$$\int \frac{f(x)}{g(x)}\,dx = \ln(x-1) - 3\ln(x-2) + 2\ln(x-3) + C$$

Now, suppose that $g(x)$ contains all linear factors, but a particular factor, say $L_1(x)$, is repeated k times.

Thus, $g(x) = L_1^k(x)L_2(x)\ldots L_n(x)$

$\frac{f(x)}{g(x)}$ can now be expanded into partial fractions as follows

$$\frac{f(x)}{g(x)} = \underbrace{\frac{A_1}{L_1(x)} + \frac{A_2}{L_1^2(x)} + \frac{A_3}{L_1^3(x)} + \ldots \frac{A_k}{L_1^k(x)}}_{k \text{ partial fractions corresponding to } L_1(x)} + \frac{B_2}{L_2(x)} + \ldots + \frac{B_n}{L_n(x)}$$

This means that we will have k terms corresponding to $L_1(x)$. The rest of the linear factors will have single corresponding terms in the expansion. Here are some examples.

$\Rightarrow$ $\dfrac{1}{(x-1)^2(x-2)}$

can be expanded as $\dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x-2}$

$\Rightarrow$ $\dfrac{1}{(x-1)^3(x-2)(x-3)}$

can be expanded as

$\Rightarrow$ $\dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{(x-1)^3} + \dfrac{D}{(x-2)} + \dfrac{C}{(x-3)}$

$\Rightarrow$ $\dfrac{1}{(x-1)^2(x+5)^3}$

can be expanded as

$$\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x+5)} + \frac{D}{(x+5)^2} + \frac{E}{(x+5)^3}$$

Example 47 $\frac{2x+1}{(x+1)(x-2)}$ into partial fractions

Sol. Here, $\frac{2x+1}{(x+1)(x-2)}$ has $Q(x) = (x+1)(x-2)$ i.e linear and non-repeated roots.

$\therefore$ $\dfrac{2x+1}{(x+1)(x-2)} = \dfrac{A}{x+1} + \dfrac{B}{x-2}$

$\Rightarrow$ $(2x+1) = A(x-2) + B(x+1)$

On putting, $x = 2$ we get

$$5 = A(0) + B(3) \;\Rightarrow\; B = \frac{5}{3}$$

Again, let $x = -1$

$\Rightarrow$ $2(-1) + 1 = A(-1-2) + B(0)$

$\therefore$ $A = \dfrac{1}{3}$.

$\therefore$ $\dfrac{2x+1}{(x+1)(x-2)} = \dfrac{1/3}{x+1} + \dfrac{5/3}{x-2}$

Example 48 Resolve $\frac{1}{(x-1)(x+2)(2x+3)}$ into partial fractions.

Sol. Let $\dfrac{1}{(x-1)(x+2)(2x+3)} = \dfrac{A}{x-1} + \dfrac{B}{x+2} + \dfrac{C}{2x+3}$,

where A, B, C are constants.

$1 = A(x+2)(2x+3) + B(x-1)(2x+3) + C(x-1)(x+2)\ldots$(i)

For finding A, let $x - 1 = 0$ or $x = 1$ in Eq. (i), we get

$1 = A(1+2)(2+3) + B(0) + C(0)$

$\therefore$ $A = \dfrac{1}{15}$

Similarly, for getting B, let $x + 2 = 0$ or $x = -2$ in Eq. (i), we get

$1 = A(0) + B(-2-1)(-4+3) + C(0)$

$\Rightarrow$ $B = \dfrac{1}{3}$

For getting C, let $2x + 3 = 0$ or $x = -3/2$ in Eq. (i), we get

$$1 = A(0) + B(0) + C\left(-\frac{3}{2} - 1\right)\left(-\frac{3}{2} + 2\right)$$

$\Rightarrow$ $C = -\dfrac{4}{5}$

Hence, $\dfrac{1}{(x-1)(x+2)(2x+3)} = \dfrac{1}{15(x-1)} + \dfrac{1}{3(x+2)} - \dfrac{4}{5(2x+3)}$

Example 49 Resolve $\frac{3x^3 + 2x^2 + x + 1}{(x+1)(x+2)}$ into partial fractions.

Sol. This is not a proper fraction. Hence, by division process it is to be expressed as the sum of an integral polynomial and a fraction.

Now, $3x^3 + 2x^2 + x + 1 = 3x(x^2 + 3x + 2)$

$\qquad - 7(x^2 + 3x + 2) + (16x + 15)$

So, the given polynomial

$$\frac{3x^3 + 2x^2 + x + 1}{(x+1)(x+2)} = (3x - 7) + \frac{(16x+15)}{(x+1)(x+2)} \quad \ldots(i)$$

Now, the second term is proper fraction hence it can be expressed as a sum of partial fractions.

$$\frac{16x+15}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

To find A put $x+1=0$, *ie*, $x=-1$ in the fraction except in the factor $(x+1)$.

$$\therefore \quad \frac{16(-1)+15}{(-1+2)}=A \quad \Rightarrow \quad A=-1 \qquad \text{...(ii)}$$

To find B put $x+2=0$, *ie*, $x=-2$ in the fraction except in the factor $(x+2)$.

$$\therefore \quad \frac{16(-2)+15}{(-2+1)}=B \quad \Rightarrow \quad B=17 \qquad \text{...(iii)}$$

$$\Rightarrow \text{The given expression} = (3x-7)-\frac{1}{x+1}+\frac{17}{x+2}$$

[using Eqs. (i), (ii) and (iii)]

Case II When the denominator $g(x)$ is expressible as the product of the linear factors such that some of them are repeating. (Linear and Repeated)

Let $Q(x)=(x-a)^k(x-a_1)(x-a_2)\ldots(x-a_r)$. Then, we assume that

$$\frac{P(x)}{Q(x)}=\frac{A_1}{(x-a)}+\frac{A_2}{(x-a)^2}+\ldots+\frac{A_k}{(x-a)^k}+\frac{B_1}{(x-a_1)}+\frac{B_2}{(x-a_2)}+\ldots+\frac{B_r}{(x-a_r)}$$

Example 50 Expression $\frac{x+5}{(x-2)^2}$ has repeated (twice) linear factors in denominator, so find partial fractions.

Sol. Let $\quad \frac{x+5}{(x-2)^2}=\frac{A}{(x-2)}+\frac{B}{(x-2)^2}$

$$\therefore \quad (x+5)=A(x-2)+B$$

Comparing the like terms, $A=1, -2A+B=5$ or $B=7$

$$\therefore \quad \frac{x+5}{(x-2)^2}=\frac{1}{(x-2)}+\frac{7}{(x-2)^2}$$

Example 51 Resolve $\frac{3x-2}{(x-1)^2(x+1)(x+2)}$ into partial fractions.

Sol. Let $\frac{3x-2}{(x-1)^2(x+1)(x+2)}$

$$=\frac{A}{(x-1)}+\frac{B}{(x-1)^2}+\frac{C}{(x+1)}+\frac{D}{(x+2)}$$

$$\therefore 3x-2=A(x-1)(x+1)(x+2)+B(x+1)(x+2)+C(x-1)^2(x+2)+D(x-1)^2(x+1)$$

Putting $(x-1)=0$, we get $B=1/6$,

Putting $(x+1)=0$, we get $C=-5/4$

Putting $(x+2)=0$, we get $D=8/9$

Now, equating the coefficient of x^3 on both the sides, we get

$$0=A+C+D \quad \Rightarrow \quad A=\frac{5}{4}-\frac{8}{9}=\frac{13}{36}$$

$$\therefore \frac{3x-2}{(x-1)^2(x+1)(x+2)}=\frac{13}{36(x-1)}+\frac{1}{6(x-1)^2}-\frac{5}{4(x+1)}+\frac{8}{9(x+2)}$$

Case III When some of the factors in denominator are quadratic but non-repeating. Corresponding to each quadratic factor ax^2+bx+c, we assume the partial fraction of the type $\frac{Ax+B}{ax^2+bx+c}$, where A and B are constants to be determined by comparing coefficients of similar powers of x in numerator of both the sides.

Example 52 ***Resolve*** $\frac{2x+7}{(x+1)(x^2+4)}$ ***into partial fractions.***

Sol. Let $\quad \frac{2x+7}{(x+1)(x^2+4)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+4}$

$$\therefore \quad 2x+7=A(x^2+4)+(Bx+C)(x+1)$$

Put $\quad x=-1$

$$\therefore \quad 5=5A \text{ or } A=1$$

Comparing the terms, $\quad 0=A+B \quad \Rightarrow \quad B=-1$

$$7=4A+C \quad \Rightarrow \quad C=3$$

$$\therefore \quad \frac{2x+7}{(x+1)(x^2+4)}=\frac{1}{x+1}+\frac{(-x+3)}{x^2+4}$$

Aliter To obtain values of A, B and C from

$$2x+7=A(x^2+4)+(Bx+C)(x+1)$$

i.e., $\quad 2x+7=(A+B)x^2+(B+C)x+4A+C$

Equating the coefficients of identical powers of x, we get

$$A+B=0, B+C=2 \text{ and } 4A+C=7.$$

Solving, we get $A=1, B=-1, C=3$

Example 53 ***Find the partial fraction*** $\frac{2x+1}{(3x+2)(4x^2+5x+6)}$.

Sol. Let $\frac{2x+1}{(3x+2)(4x^2+5x+6)}=\frac{A}{(3x+2)}+\frac{Bx+C}{(4x^2+5x+6)}$,

then $2x+1=A(4x^2+5x+6)+(Bx+C)(3x+2)$

where A, B, C are constants.

For A, let $\quad 3x+2=0$,

i.e., $\quad x=-2/3$

$$2\left(-\frac{2}{3}\right)+1=A\left\{4\cdot\frac{4}{9}-\frac{10}{3}+6\right\}+\left\{B\left(-\frac{2}{3}\right)+C\right\}(0)$$

$$-\frac{1}{3}=A\left(\frac{40}{9}\right) \Rightarrow \quad A=-\frac{3}{40}$$

Comparing coefficients of x^2 and constant term on both the sides for B and C, we get

$$4A + 3B = 0,$$

$$\therefore \quad B = -\frac{4}{3}A \quad \Rightarrow \quad B = \frac{1}{10} \quad \text{and} \quad 6A + 2C = 1,$$

$$\therefore \quad C = \frac{1-6A}{2} \quad \Rightarrow C = \frac{29}{40}$$

$$\therefore \frac{2x+1}{(3x+2)(4x^2+5x+6)} = \frac{-3}{40(3x+2)} + \frac{\left(x+\frac{29}{4}\right)}{10(4x^2+5x+6)}$$

Case IV When some of the factors of the denominator are quadratic and repeating. For every quadratic repeating factor of the type $(ax^2+bx+c)^k$, we assume :

$$\frac{A_1x+A_2}{ax^2+bx+c} + \frac{A_3\,x+A_4}{(ax^2+bx+c)^2} + \ldots + \frac{A_{2k-1}\,x+A_{2k}}{(ax^2+bx+c)^k}$$

Example 54 Resolve $\dfrac{2x^4+2x^2+x+1}{x(x^2+1)^2}$ into partial fractions.

Sol. Let $\dfrac{2x^4+2x^2+x+1}{x(x^2+1)^2} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1} + \dfrac{Dx+E}{(x^2+1)^2}$

or $2x^4+2x^2+x+1 = A(x^2+1)^2 + (Bx+C)\,x(x^2+1) + (Dx+E)\,x$

Comparing coefficients of x^4, x^3, x^2, x and constant term

$$\therefore \quad A+B=2, C=0, 2A+D+B=2, E=1, A=1$$

$\therefore$ we get $A=1, B=1, C=0, D=-1, E=1$

Hence, the partial fraction,

$$\frac{2x^4+2x^2+x+1}{x(x^2+1)^2} = \frac{1}{x} + \frac{x}{1+x^2} + \frac{1-x}{(1+x^2)^2}$$

Example 55 Evaluate the following integrals:

(i) $\displaystyle\int \frac{(1-x^2)\,dx}{x(1-2x)}$ (ii) $\displaystyle\int \frac{3x-1}{(x-2)^2}\,dx$

(iii) $\displaystyle\int \frac{x^2+x+1}{x^2(x+2)}\,dx$ (iv) $\displaystyle\int \frac{8dx}{(x+2)(x^2+4)}$

Sol. (i) Let, $\dfrac{1-x^2}{x(1-2x)} = \dfrac{1}{2} + \dfrac{A}{x} + \dfrac{B}{(1-2x)}$

$$\Rightarrow \quad (1-x^2) = \frac{1}{2}x(1-2x) + A(1-2x) + B(x)$$

On putting $x=0$ and $x=\dfrac{1}{2}$, we get

$$1 = A \text{ and } 1 - \frac{1}{4} = B \cdot \frac{1}{2} \Rightarrow A = 1, B = \frac{3}{2}$$

$$\therefore \quad \int \frac{1-x^2}{x(1-2x)}\,dx = \int \left(\frac{1}{2} + \frac{1}{x} + \frac{3}{2(1-2x)}\right) dx$$

$$= \frac{1}{2}x + \log|x| + \frac{3}{2}\frac{\log|1-2x|}{-2} + C$$

$$= \frac{1}{2}x + \log|x| - \frac{3}{4}\log|1-2x| + C.$$

(ii) Let, $\dfrac{3x-1}{(x-2)^2} = \dfrac{A}{(x-2)} + \dfrac{B}{(x-2)^2}$

$$\Rightarrow \quad 3x-1 = A(x-2) + B, \qquad \ldots(i)$$

On putting $x=2$ in Eq. (i), we get $B=5$.

On equating coefficients of x on both sides of (i), we get $A=3$,

$$\therefore \quad \int \frac{3x-1}{(x-2)^2}\,dx = \int \left(\frac{3}{(x-2)} + \frac{5}{(x-2)^2}\right) dx$$

$$= 3\log|x-2| - \frac{5}{x-2} + C$$

(iii) Let, $\dfrac{x^2+x+1}{x^2(x+2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+2}$

$$\Rightarrow \quad x^2+x+1 = Ax(x+2) + B(x+2) + C(x^2) \quad \ldots(i)$$

On putting, $x=-2$ and $x=0$ in Eq. (i), we get

$$C = 3/4 \text{ and } B = 1/2$$

On equation Coefficient of x^2 on both sides of (i), we get $1 = A + C \Rightarrow A = 1/4$.

$$\therefore \quad \int \frac{x^2+x+1}{x^2(x+2)}\,dx = \int \left(\frac{1/4}{x} + \frac{1/2}{x^2} + \frac{3/4}{x+2}\right) dx$$

$$= \frac{1}{4}\log|x| - \frac{1}{2x} + \frac{3}{4}\log|x+2| + C$$

(iv) Let, $\dfrac{8}{(x+2)(x^2+4)} = \dfrac{A}{x+2} + \dfrac{Bx+C}{x^2+4}$

$$\Rightarrow \quad 8 = A(x^2+4) + (Bx+C)(x+2) \qquad \ldots(i)$$

On putting $x=-2$ in Eq. (i), we get $A=1$.

On equating coefficient of x^2 on both sides we get,

$$0 = A + B \Rightarrow B = -1$$

On equating constant term on both sides, we get,

$$8 = 4A + 2C \Rightarrow C = 2$$

$$\therefore \quad \int \frac{8}{(x+2)(x^2+4)}\,dx = \int \left(\frac{1}{x+2} + \frac{(-x+2)}{x^2+4}\right) dx$$

$$= \int \frac{1}{x+2}\,dx - \frac{1}{2}\int \frac{2x\,dx}{x^2+4} + 2\int \frac{dx}{x^2+4}$$

$$= \log|x+2| - \frac{1}{2}\log|x^2+4| + 2\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C.$$

Example 56 Evaluate $\displaystyle\int \frac{1}{\sin x - \sin 2x}\,dx$.

Sol. Let $I = \displaystyle\int \frac{1}{\sin x - \sin 2x}\,dx = \int \frac{1}{\sin x - 2\sin x\cos x}\,dx$

$$= \int \frac{1}{\sin x(1-2\cos x)}\,dx = \int \frac{\sin x}{\sin^2 x(1-2\cos x)}\,dx$$

$$= \int \frac{\sin x}{(1-\cos^2 x)(1-2\cos x)}\,dx$$

(put $\cos x = t \Rightarrow -\sin x\,dx = dt$)

$$\therefore \quad I = \int \frac{-dt}{(1-t^2)(1-2t)} = \int \frac{-1}{(1-t)(1+t)(1-2t)}\,dt \quad ...(i)$$

Here, in Eq. (i) we have linear and non-repeated factors thus we use partial fractions for;

$$\frac{-1}{(1-t)(1+t)(1-2t)} = \frac{A}{(1-t)} + \frac{B}{(1+t)} + \frac{C}{(1-2t)}$$

or $-1 = A(1+t)(1-2t) + B(1-t)(1-2t) + C(1-t)(1+t)$ ...(ii)

Putting $(t+1) = 0$ or $t = -1$, we get

$$-1 = B(2)(1+2) \Rightarrow B = -\frac{1}{6}$$

Putting $(1-t) = 0$ or $t = 1$, we get

$$-1 = A(2)(-1) \quad \Rightarrow \quad A = \frac{1}{2}$$

Putting $(1-2t) = 0$ or $t = 1/2$, we get

$$-1 = C\left(1-\frac{1}{2}\right)\left(1+\frac{1}{2}\right) \Rightarrow C = -\frac{4}{3}$$

$$\therefore \quad \frac{-1}{(1-t)(1+t)(1-2t)} = \frac{1}{2(1-t)} - \frac{1}{6(1+t)} - \frac{4}{3(1-2t)}$$

So, Eq. (i) reduces to

$$I = \frac{1}{2}\int \frac{1}{1-t}\,dt - \frac{1}{6}\int \frac{1}{1+t}\,dt - \frac{4}{3}\int \frac{1}{1-2t}\,dt$$

$$= -\frac{1}{2}\log|1-t| - \frac{1}{6}\log|1+t| - \frac{4}{3}\times -\frac{1}{2}\log|1-2t| + C$$

$$= -\frac{1}{2}\log|1-\cos x| - \frac{1}{6}\log|1+\cos x| + \frac{2}{3}\log|1-2\cos x| + C$$

Example 57 Evaluate $\int \frac{(1-x\sin x)\,dx}{x(1-x^3 e^{3\cos x})}$.

Sol. Here, $I = \int \frac{(1-x\sin x)\,dx}{x(1-(xe^{\cos x})^3)}$

Put $xe^{\cos x} = t$

$\Rightarrow \quad (xe^{\cos x}\cdot(-\sin x) + e^{\cos x})\,dx = dt$

$$\therefore \quad I = \int \frac{dt}{t(1-t^3)} = \int \frac{dt}{t(1-t)(1+t+t^2)}$$

$$= \int\left(\frac{A}{t} + \frac{B}{1-t} + \frac{Ct+D}{1+t+t^2}\right)dt$$

Comparing coefficients, we get

$$A = 1, B = \frac{1}{3}, C = -\frac{2}{3}, D = -\frac{1}{3}$$

$$\therefore \quad I = \int \frac{dt}{t} + \frac{1}{3}\int \frac{dt}{1-t} + \int \frac{\left(-\frac{2}{3}t - \frac{1}{3}\right)}{1+t+t^2}\,dt$$

$$= \log|t| - \frac{1}{3}\log|1-t| - \frac{1}{3}\log|1+t+t^2|$$

[where, $t = xe^{\cos x}$]

Example 58 Evaluate $\int \sin 4x \cdot e^{\tan^2 x}\,dx$.

Sol. The given integral could be written as,

$$I = \int 4\sin x\cdot\cos x\cdot\cos 2x\cdot e^{\tan^2 x}dx$$

$$= 4\int \tan x\cdot\cos^2 x(\cos^2 x - \sin^2 x)\cdot e^{\tan^2 x}dx$$

$$= 4\int \tan x\cdot\cos^4 x(1-\tan^2 x)\cdot e^{\tan^2 x}\,dx$$

$$= 4\int \frac{\tan x}{(\sec^2 x)^2}\cdot(1-\tan^2 x)\cdot e^{\tan^2 x}\,dx$$

Put $\tan^2 x = t$

$\Rightarrow \quad 2\tan x\cdot\sec^2 x\,dx = dt$

$$\Rightarrow \quad I = 4\int \frac{(1-t)e^t}{(1+t)^3}\cdot\frac{dt}{2} = 2\int \frac{(1-t)e^t}{(1+t)^3}\,dt$$

$$= 2\left\{\int \frac{2e^t - (1+t)e^t}{(1+t)^3}dt\right\}$$

$$= 2\left\{\int e^t\frac{2}{(1+t)^3} - \frac{1}{(1+t)^2}\right\}dt$$

$$= -2\frac{1}{(1+t)^2}e^t + C$$

[using $\int e^x(f(x)) + f'(x))dx = e^x.f(x) + c$]

$$= -\frac{2e^{\tan^2 x}}{(1+\tan^2 x)^2} + C$$

$$I = -2\cos^4 x\cdot e^{\tan^2 x} + C$$

Example 59 Solve $\int \frac{1+x\cos x}{x(1-x^2e^{2\sin x})}\,dx$.

Sol. Let $I = \int \frac{1+x\cos x}{x(1-x^2e^2\sin x)}\,dx$

Put $(x\,e^{\sin x}) = t$

Differentiating both the sides, we get

$(x\,e^{\sin x}\cdot\cos x + e^{\sin x})\,dx = dt$

$\Rightarrow \quad e^{\sin x}(x\cos x + 1)\,dx = dt$

$$\Rightarrow \quad I = \int \frac{dt}{t(1-t^2)}$$

$$= \int \frac{dt}{t(1-t)(1+t)} \qquad \text{[using partial fraction]}$$

$$= \int\left\{\frac{1}{t} + \frac{1}{2(1-t)} - \frac{1}{2(1+t)}\right\}dt$$

$$= \log|t| - \frac{1}{2}\log|1-t| - \frac{1}{2}\log|1+t| + C$$

$$= \log|x\,e^{\sin x}| - \frac{1}{2}\log|1-x^2e^{2\sin x}| + C$$

Example 60 Evaluate;

$$\int \frac{1}{x}\{\log e^{ex} \cdot \log e^{e^2x} \cdot \log e^{e^3x}\}\, dx$$

Sol. We have,

$$I = \int \frac{1}{x}\{\log e^{ex} \cdot \log e^{e^2x} \cdot \log_e e^3 x\}\, dx$$

$$= \int \frac{1}{x}\left\{\frac{1}{\log e^{ex}} \cdot \frac{1}{\log e^{e^2x}} \cdot \frac{1}{\log e^{e^3x}}\right\} dx$$

$$= \int \frac{dx}{x\{\log e^{e} + \log e^{x}\}\{\log e^{e^2} + \log e^{x^2}\}\{\log e^{e^3} + \log x^2\}}$$

$$= \int \frac{dx}{x\{1 + \log e^{x}\}\{2 + \log e^{x}\}\{3 + \log e^{x}\}}$$

Put $\log e^x = t \Rightarrow \frac{1}{x}dx = dt$

$$I = \int \frac{dt}{(1+t)(2+t)(3+t)} = \int\left(\frac{1}{2}\cdot\frac{1}{(1+t)} - \frac{1}{(2+t)} + \frac{1}{(3+t)}\right)dt$$

[using partial fraction]

$$= \frac{1}{2}\log|1+t| - \log|2+t| + \log|3+t| + C.$$

$$= \frac{1}{2}\log|1+\log e^x| - \log|2+\log e^x| + \log|3+\log e^x| + C.$$

Exercise for Session 5

- Evaluate the following Integrals :

1. $\int \frac{x^2}{(x-1)(x-2)(x-3)}\, dx$

2. $\int \frac{dx}{1+x^3}$

3. $\int \frac{dx}{x(x^n+1)}$

4. $\int \frac{2x}{(x^2+1)(x^2+3)}$

5. $\int \frac{\cos x}{(1+\sin x)(2+\sin x)}\, dx$

6. $\int \frac{dx}{\sin x\,(3+2\cos x)}\, dx$

7. $\int \frac{\sec x}{1+\operatorname{cosec} x}\, dx$

8. $\int \frac{\tan x + \tan^3 x}{1+\tan^3 x}\, dx$

9. $\int \frac{dx}{x\,|6(\log x)^2 + 7\log x + 2|}$

10. $\int \frac{\tan^{-1}}{x^2}.\, dx$

Session 6

Indirect and Derived Substitutions

Indirect and Derived Substitutions

(i) Indirect Substitution

If the integral is of the form $f(x)\cdot g(x)$, where $g(x)$ is a function of the integral of $f(x)$, then put integral of $f(x)=t$.

Example 61 The value of $\int \frac{d(x^2+1)}{\sqrt{x^2+2}}$, is

(a) $2\sqrt{x^2+2}+C$ (b) $\sqrt{x^2+2}+C$

(c) $x\sqrt{x^2+2}+C$ (d) None of these

Sol. Here, $I=\int \frac{d(x^2+1)}{\sqrt{x^2+2}}$

We know, $d(x^2+1)=2x\,dx$

$\therefore \quad I=\int \frac{2x\,dx}{\sqrt{x^2+2}}$

Put, $x^2+2=t^2$

$\therefore \quad 2x\,dx=2t\,dt \Rightarrow I=\int \frac{2t\,dt}{t}=2t+C$

$\Rightarrow \quad I=2\sqrt{x^2+2}+C$

Hence, (a) is the correct answer.

Example 62 If $\int \frac{(\sqrt{x})^5}{(\sqrt{x})^7+x^6}dx=a\log\left(\frac{x^k}{1+x^k}\right)+c$, then a and k are

(a) $2/5, 5/2$ (b) $1/5, 2/5$

(c) $5/2, 1/2$ (d) $2/5, 1/2$

Sol. Here, $I=\int \frac{(\sqrt{x})^5}{(\sqrt{x})^7+x^6}dx = \int \frac{dx}{(\sqrt{x})^2+(\sqrt{x})^7}$

$$=\int \frac{dx}{x^{7/2}\left\{1+\frac{1}{x^{5/2}}\right\}},$$

Put $\frac{1}{x^{5/2}}=y \Rightarrow -\frac{5}{2x^{7/2}}dx=dy$

$$I=-\frac{2}{5}\int \frac{dy}{1+y}$$

$$=-\frac{2}{5}\log|1+y|+C=\frac{2}{5}\log\left|\frac{1}{1+y}\right|+C$$

$$=\frac{2}{5}\log\left|\frac{x^{5/2}}{x^{5/2}+1}\right|+C \quad \text{...(i)}$$

where, $I=a\log\left(\frac{x^k}{1+x^k}\right)+C$ (given) ...(ii)

$\therefore$ From Eqs. (i) and (ii), we get

$$a\log\left(\frac{x^k}{1+x^k}\right)+C=\frac{2}{5}\log\left(\frac{x^{5/2}}{1+x^{5/2}}\right)+C$$

$\Rightarrow \quad a=2/5$ and $k=5/2$

Hence, (a) is the correct answer.

Example 63 Evaluate $\int \frac{5x^4+4x^5}{(x^5+x+1)^2}dx$.

Sol. Here, $I=\int \frac{5x^4+4x^5}{(x^5+x+1)^2}dx=\int \frac{x^4(5+4x)}{x^{10}\left(1+\frac{1}{x^4}+\frac{1}{x^5}\right)^2}dx$

$$=\int \frac{5/x^6+4/x^5}{\left(1+\frac{1}{x^4}+\frac{1}{x^5}\right)^2}dx$$

Put $1+\frac{1}{x^4}+\frac{1}{x^5}=t$

$\Rightarrow \quad \left(-\frac{4}{x^5}-\frac{5}{x^6}\right)dx=dt$

$$I=\int -\frac{dt}{t^2}=\frac{1}{t}+C=\frac{1}{1+\frac{1}{x^4}+\frac{1}{x^5}}+C$$

$$=\frac{x^5}{x^5+x+1}+C$$

Example 64 For any natural number m, evaluate $\int (x^{3m}+x^{2m}+x^m)(2x^{2m}+3x^m+6)^{1/m}\,dx,\ x>0$

[IIT JEE 2002]

Sol. Here, $I=\int (x^{3m}+x^{2m}+x^m)(2x^{2m}+3x^m+6)^{1/m}\,dx$

$$=\int (x^{3m}+x^{2m}+x^m)\frac{(2x^{3m}+3x^{2m}+6x^m)^{1/m}}{x}dx$$

$$=\int (x^{3m-1}+x^{2m-1}+x^{m-1})(2x^{3m}+3x^{2m}+6x^m)^{1/m}dx \quad \text{...(i)}$$

Put $2x^{3m} + 3x^{2m} + 6x^m = t$

$\Rightarrow \quad 6m\,(x^{3m-1} + x^{2m-1} + x^{m-1})\,dx = dt$

$\therefore$ Eq. (i) becomes,

$$I = \int t^{1/m}\,\frac{dt}{6m} = \frac{1}{6m}\cdot\frac{t^{(1/m)+1}}{(1/m)+1} + C$$

$$I = \frac{1}{6\,(m+1)}\{2x^{3m} + 3x^{2m} + 6x^m\}^{\frac{m+1}{m}} + C$$

Example 65 $\int \frac{x\,dx}{\sqrt{1+x^2+\sqrt{(1+x^2)^3}}}$ is equal to

(a) $\frac{1}{2}\ln(1+\sqrt{1+x^2}) + C$ (b) $2\sqrt{1+\sqrt{1+x^2}} + C$

(c) $2(1+\sqrt{1+x^2}) + C$ (d) None of these

Sol. $\int \frac{x\,dx}{\sqrt{(1+x^2)}\sqrt{1+\sqrt{1+x^2}}}$

Put $1+\sqrt{1+x^2} = t^2 \;\Rightarrow\; \frac{2x}{2\sqrt{1+x^2}}\,dx = 2t\,dt$

$\therefore \quad \frac{x\,dx}{\sqrt{1+x^2}} = 2t\,dt$

$\therefore \quad I = \int \frac{2t\,dt}{t} = 2t + C = 2\sqrt{1+\sqrt{1+x^2}} + C$

Hence, (b) is the correct answer.

Example 66 $\int \frac{(2x+1)}{(x^2+4x+1)^{3/2}}\,dx$

(a) $\frac{x^3}{(x^2+4x+1)^{1/2}} + C$ (b) $\frac{x}{(x^2+4x+1)^{1/2}} + C$

(c) $\frac{x^2}{(x^2+4x+1)^{1/2}} + C$ (d) $\frac{1}{(x^2+4x+1)^{1/2}} + C$

Sol. $\int \frac{2x+1}{(x^2+4x+1)^{3/2}}\,dx = \int \frac{2x+1}{x^3\left(1+\frac{4}{x}+\frac{1}{x^2}\right)^{3/2}}\,dx$

$$= \int \frac{2x^{-2}+x^{-3}}{\left(1+\frac{4}{x}+\frac{1}{x^2}\right)^{3/2}}\,dx$$

Now, put $\frac{1}{x^2}+\frac{4}{x}+1 = t^2 \;\Rightarrow\; \left(-\frac{2}{x^3}-\frac{4}{x^2}\right)dx = 2t\,dt$

$\therefore \quad I = \int \frac{-t\,dt}{t^3} = \frac{1}{t} + C$

$$= \frac{x}{\sqrt{x^2+4x+1}} + C$$

Hence, (b) is the correct answer.

Example 67 *The evaluation of* $\int \frac{p\,x^{p+2q-1} - qx^{q-1}}{x^{2p+2q} + 2x^{p+q} + 1}\,dx$ *is*

(a) $-\frac{x^p}{x^{p+q}+1} + C$ (b) $\frac{x^q}{x^{p+q}+1} + C$

(c) $-\frac{x^q}{x^{p+q}+1} + C$ (d) $\frac{x^p}{x^{p+q}+1} + C$

Sol. Here, $I = \int \frac{px^{p+2q-1} - qx^{q-1}}{x^{2p+2q} + 2x^{p+q} + 1}$

$$\int \frac{px^{p+2q-1} - qx^{q-1}}{(x^{p+q}+1)^2}\,dx = \int \frac{px^{p-1} - qx^{-q-1}}{(x^p + x^{-q})^2}\,dx$$

Taking x^q as x^{2q} common from denominator and take it in numerator.

Put $x^p + x^{-q} = t \;\Rightarrow\; (px^{p-1} - qx^{-q-1})\,dx = dt$

$\therefore \quad I = \int \frac{dt}{t^2} = -\frac{1}{t} + C = -\left(\frac{x^q}{x^{p+q}+1}\right) + C$

Hence, (c) is the correct answer.

Example 68 $\int \frac{x^2(1-\ln x)}{\ln^4 x - x^4}\,dx$ is equal to

(a) $\frac{1}{2}\ln\left(\frac{x}{\ln x}\right) - \frac{1}{4}\ln(\ln^2 x - x^2) + C$

(b) $\frac{1}{4}\ln\left(\frac{\ln x - x}{\ln x + x}\right) - \frac{1}{2}\tan^{-1}\left(\frac{\ln x}{x}\right) + C$

(c) $\frac{1}{4}\ln\left(\frac{\ln x + x}{\ln x - x}\right) + \frac{1}{2}\tan^{-1}\left(\frac{\ln x}{x}\right) + C$

(d) $\frac{1}{4}\left(\ln\left(\frac{\ln x - x}{\ln x + x}\right) + \tan^{-1}\left(\frac{\ln x}{x}\right)\right) + C$

Sol. Here, $I = \int \frac{x^2(1-\ln x)}{\ln x^4 x - x^4}$

$$I = \int \frac{x^2(1-\ln x)}{x^4\left(\left(\frac{\ln x}{x}\right)^4 - 1\right)}\,dx = \int \frac{1-\ln x}{x^2\left(\left(\frac{\ln x}{x}\right)^4 - 1\right)}\,dx$$

Put $\frac{\ln x}{x} = t \;\Rightarrow\; \frac{1-\ln x}{x^2} = dt$

$$I = \int \frac{dt}{(t^4-1)} = \int \frac{dt}{(t^2+1)(t^2-1)}$$

$$= \frac{1}{2}\int \frac{(t^2+1)-(t^2-1)}{(t^2+1)(t^2-1)}\,dt$$

$$I = \frac{1}{2}\left(\int \frac{dt}{t^2-1} - \int \frac{dt}{t^2+1}\right) = \frac{1}{2}\left(\frac{1}{2}\ln\frac{t-1}{t+1} - \tan^{-1}t\right)$$

$$= \frac{1}{4}\ln\left(\frac{\ln x - x}{\ln x + x}\right) - \frac{1}{2}\tan^{-1}\left(\frac{\ln x}{x}\right) + C$$

Hence, (b) is the correct answer.

Example 69 $\int \frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}}\,dx$ is equal to

[IIT JEE 2006]

(a) $\frac{\sqrt{2x^4-2x^2+1}}{x^2}+C$ (b) $\frac{\sqrt{2x^4-2x^2+1}}{x^3}+C$

(c) $\frac{\sqrt{2x^4-2x^2+1}}{x}+C$ (d) $\frac{\sqrt{2x^4-2x^2+1}}{2x^2}+C$

Sol. Let $I = \int \frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}}\,dx$

$$= \int \frac{x^2-1}{x^5}\cdot\frac{dx}{\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}}}$$

$$= \frac{1}{4}\int \frac{\frac{4}{x^3}-\frac{4}{x^5}}{\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}}}\,dx = \frac{1}{4}\cdot 2\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}}$$

$$\left(\because \int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)}+C\right)$$

$$= \frac{\sqrt{2x^4-2x^2+1}}{2x^2}+C$$

Hence, (d) is the correct answer.

Example 70 ***Let*** $f(x)=\frac{x}{(1+x^n)^{1/n}}$ ***for*** $n\geq 2$ ***and*** $g(x)=\underbrace{fofo\ldots of}_{n\text{ times}}(x)$, ***then*** $\int x^{n-2}\,g(x)\,dx$ ***equals to***

[IIT JEE 2007]

(a) $\frac{1}{n(n-1)}(1+nx^n)^{1-\frac{1}{n}}+C$ (b) $\frac{1}{n-1}(1+nx^n)^{1-\frac{1}{n}}+C$

(c) $\frac{1}{n(n+1)}(1+nx^n)^{1+\frac{1}{n}}+C$ (d) $\frac{1}{n+1}(1+nx^n)^{1+\frac{1}{n}}+C$

Sol. $\because$ $f(x)=\frac{x}{(1+x^n)^{1/n}}$

$\therefore$ $f(f(x))=\frac{y}{(1+x^n)^{1/n}}$,

Where, $y=\frac{x}{(1+x^n)^{1/n}}=\frac{x}{(1+2x^n)^{1/n}}$

Similarly, $f(f(f(x)))=\frac{x}{(1+3x^n)^{1/n}}$

and $\underbrace{(fofofo\ldots of)}_{n\text{ times}}(x)=g(x)=\frac{x}{(1+nx^n)^{1/n}}$

$$\therefore \int x^{n-2}g(x)\,dx = \int \frac{x^{n-1}}{(1+nx^n)^{1/n}}\,dx$$

$$= \frac{1}{n^2}\int n^2\cdot x^{n-1}\cdot(1+nx^n)^{-1/n}\,dx$$

$$= \frac{1}{n^2}\cdot\frac{(1+nx^n)^{1-\frac{1}{n}}}{1-\frac{1}{n}}+C$$

$$= \frac{(1+nx^n)^{1-\frac{1}{n}}}{n(n-1)}+C$$

Hence, (a) is the correct answer.

Derived Substitutions

Some times it is useful to write the integral as a sum of two related integrals which can be evaluated by making suitable substitutions.

Examples of such integrals are

Type I

(a) Algebraic Twins

$$\int \frac{2x^2}{x^4+1}\,dx = \int \frac{x^2+1}{x^4+1}\,dx + \int \frac{x^2-1}{x^4+1}\,dx$$

$$\int \frac{2}{x^4+1}\,dx = \int \frac{x^2+1}{x^4+1}\,dx - \int \frac{x^2-1}{x^4+1}\,dx$$

$$\int \frac{2x^2}{x^4+1+kx^2}\,dx, \int \frac{2}{(x^4+1+kx^2)}\,dx$$

(b) Trigonometric Twins

$$\int \sqrt{\tan x}\,dx, \int \sqrt{\cot x}\,dx,$$

$$\int \frac{1}{(\sin^4 x+\cos^4 x)}\,dx, \int \frac{1}{\sin^6 x+\cos^6 x}\,dx,$$

$$\int \frac{\pm\sin x\pm\cos x}{a+b\sin x\cos x}\,dx$$

Method of evaluating these integral are illustrated by mean of the following examples :

Integral of the Form

1. $\int f\left(x+\frac{1}{x}\right)\left(1-\frac{1}{x^2}\right)dx$

Put $x+\frac{1}{x}=t \Rightarrow \left(1-\frac{1}{x^2}\right)dx = dt$

2. $\int f\left(x-\frac{1}{x}\right)\left(1+\frac{1}{x^2}\right)dx.$

   Put $x-\frac{1}{x}=t \Rightarrow \left(1+\frac{1}{x^2}\right)dx=dt$

3. $\int \frac{x^2+1}{x^4+kx^2+1}\,dx.$

   Divide numerator and denominator by x^2.

4. $\int \frac{x^2-1}{x^4+kx^2+1}\,dx$

   Divide numerator and denominator by x^2.

Example 71 Evaluate $\int \frac{5}{1+x^4}\,dx.$

Sol. Let $I=\int \frac{5}{1+x^4}\,dx=\frac{5}{2}\int \frac{2}{1+x^4}\,dx$

$$=\frac{5}{2}\left[\int \frac{1+x^2}{1+x^4}\,dx+\int \frac{1-x^2}{1+x^4}\,dx\right]$$

$$=\frac{5}{2}\left[\int \frac{x^2+1}{x^4+1}\,dx-\int \frac{x^2-1}{x^4+1}\,dx\right]$$

Remark

Here, dividing Numerator and Denominator by x^2 and converting Denominator into perfect square so as to get differential in Numerator

i.e. $$I=\frac{5}{2}\left[\int \frac{1+1/x^2}{x^2+1/x^2}\,dx-\int \frac{1-1/x^2}{x^2+1/x^2}\,dx\right]$$

$$=\frac{5}{2}\left[\int \frac{1+1/x^2}{(x-1/x)^2+2}\,dx-\int \frac{1-1/x^2}{(x+1/x)^2-2}\,dx\right]$$

$$=\frac{5}{2}\left[\int \frac{dt}{t^2+(\sqrt{2})^2}-\int \frac{du}{u^2-(\sqrt{2})^2}\right],$$

$$[\text{where } t=x-\frac{1}{x} \text{ and } u=x+\frac{1}{x}]$$

$$=\frac{5}{2}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)-\frac{1}{2\sqrt{2}}\log\left|\frac{u-\sqrt{2}}{u+\sqrt{2}}\right|\right]+C$$

$$\therefore \quad I=\frac{5}{2}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-1/x}{\sqrt{2}}\right)-\frac{1}{2\sqrt{2}}\log\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right|\right]+C$$

Example 72 Evaluate $\int \frac{1}{x^4+5x^2+1}\,dx.$

Sol. Let $I=\frac{1}{2}\int \frac{2}{x^4+5x^2+1}\,dx$

$$\Rightarrow I=\frac{1}{2}\int \frac{1+x^2}{x^4+5x^2+1}\,dx+\frac{1}{2}\int \frac{1-x^2}{x^4+5x^2+1}\,dx$$

$$=\frac{1}{2}\int \frac{1+1/x^2}{x^2+5+1/x^2}\,dx-\frac{1}{2}\int \frac{1-1/x^2}{x^2+5+1/x^2}\,dx$$

[(dividing Numerator and Denominator by x^2)]

$$=\frac{1}{2}\int \frac{(1+1/x^2)}{(x-1/x)^2+7}\,dx-\frac{1}{2}\int \frac{(1-1/x^2)}{(x+1/x)^2+3}\,dx$$

$$=\frac{1}{2}\int \frac{dt}{t^2+(\sqrt{7})^2}-\frac{1}{2}\int \frac{du}{u^2+(\sqrt{3})^2}$$

where $t=x-\frac{1}{x}$ and $u=x+\frac{1}{x}$

$$\therefore \quad I=\frac{1}{2}\cdot\frac{1}{\sqrt{7}}\left(\tan^{-1}\frac{t}{\sqrt{7}}\right)-\frac{1}{2}\cdot\frac{1}{\sqrt{3}}\left(\tan^{-1}\frac{u}{\sqrt{3}}\right)+C$$

$$=\frac{1}{2}\left[\frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{x-1/x}{\sqrt{7}}\right)-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x+1/x}{\sqrt{3}}\right)\right]+C$$

Example 73 Evaluate $\int \sqrt{\tan x}\,dx.$

Sol. Here $I=\int \sqrt{\tan x}\,dx$

Put $\tan x=t^2$

$$\Rightarrow \sec^2 x\,dx=2t\,dt \quad \Rightarrow dx=\frac{2t\,dt}{1+t^4}$$

$$\therefore \quad I=\int t\cdot\frac{2t}{1+t^4}\,dt=\int \frac{2t^2}{t^4+1}\,dt$$

$$=\int \frac{t^2+1}{t^4+1}\,dt+\int \frac{t^2-1}{t^4+1}\,dt$$

$$=\int \frac{1+1/t^2}{t^2+1/t^2}\,dt+\int \frac{1-1/t^2}{t^2+1/t^2}\,dt$$

$$=\int \frac{1+1/t^2}{(t-1/t)^2+2}\,dt+\int \frac{1-1/t^2}{(t+1/t)^2-2}\,dt$$

$$I=\int \frac{ds}{s^2+(\sqrt{2})^2}+\int \frac{dr}{r^2-(\sqrt{2})^2}, \left[s=t-\frac{1}{t} \text{ and } r=t+\frac{1}{t}\right]$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{s}{\sqrt{2}}\right)+\frac{1}{2\sqrt{2}}\log\left|\frac{r-\sqrt{2}}{r+\sqrt{2}}\right|+C$$

$$=\frac{1}{\sqrt{2}}\left[\tan^{-1}\left(\frac{t-1/t}{\sqrt{2}}\right)+\frac{1}{2}\log\left(\frac{t+\frac{1}{t}-\sqrt{2}}{t+\frac{1}{t}+\sqrt{2}}\right)\right]+C$$

[(where $t=\sqrt{\tan x}$)]

Example 74 Evaluate $\int \frac{4}{\sin^4 x+\cos^4 x}\,dx.$

Sol. Let $I=4\int \frac{dx}{\sin^4 x+\cos^4 x}$

Dividing numerator and denominator by $\cos^4 x$, we get

$$I = 4\int \frac{\sec^4 x}{\tan^4 x + 1}\,dx$$

$$I = 4\int \frac{\sec^2 x\,(1+\tan^2 x)}{1+\tan^4 x}\,dx$$

Put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$$\therefore \quad I = 4\int \frac{t^2+1}{t^4+1}\,dt$$

$$= 4\int \frac{1+1/t^2}{t^2+1/t^2}\,dt = 4\int \frac{1+1/t^2}{(1-1/t)^2+2}\,dt$$

Again, put $z = t - \frac{1}{t}$

$$\therefore \quad I = 4\int \frac{dz}{z^2+2} = \frac{4}{\sqrt{2}}\tan^{-1}\left(\frac{z}{\sqrt{2}}\right)+C$$

$$I = 2\sqrt{2}\tan^{-1}\left(\frac{\tan x - 1/\tan x}{\sqrt{2}}\right)+C$$

Example 75 The value of $\int \frac{(ax^2-b)\,dx}{x\sqrt{c^2x^2-(ax^2+b)^2}}$, is equal to

(a) $\frac{1}{c}\sin^{-1}\left(ax\frac{b}{x}\right)+k$ (b) $c\sin^{-1}\left(ax+\frac{b}{x}\right)+k$

(c) $\sin^{-1}\left[\frac{ax+b/x}{c}\right]+k$ (d) None of these

Sol. Here, $I = \int \frac{(ax^2-b)\,dx}{x\sqrt{c^2x^2-(ax^2+b)^2}} = \int \frac{\left(a-\frac{b}{x^2}\right)dx}{\sqrt{c^2-\left(ax+\frac{b}{x}\right)^2}}$

Put $ax + \frac{b}{x} = t$

$$\therefore \quad \left(a-\frac{b}{x^2}\right)dx = dt$$

$$I = \int \frac{dt}{\sqrt{c^2-t^2}} = \sin^{-1}\left(\frac{t}{c}\right)+k$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{ax+b/x}{c}\right)+k$$

Hence (c) is the correct answer.

Example 76 $\int \frac{x^x\,(x^{2x}+1)(\ln x+1)}{x^{4x}+1}\,dx$

Sol. $I = \int \frac{x^x\,(x^{2x}+1)(\ln x+1)}{x^{4x}+1}\,dx$

Put $x^x = y \Rightarrow x^x(\ln x+1)\,dx = dy$

$$I = \int \frac{y^2+1}{y^4+1}\,dy = \int \frac{1+\frac{1}{y^2}}{y^2+\frac{1}{y^2}}\,dy = \int \frac{1+\frac{1}{y^2}}{\left(y-\frac{1}{y}\right)^2+2}\,dy$$

Put, $y - \frac{1}{y} = t \Rightarrow \left(1+\frac{1}{y^2}\right)dy = dt$

$$I = \int \frac{dt}{t^2+2} = \frac{1}{\sqrt{2}}\left(\tan^{-1}\frac{t}{\sqrt{2}}\right)+C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{y-\frac{1}{y}}{\sqrt{2}}\right)+C = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^x-\frac{1}{x^x}}{\sqrt{2}}\right)+C$$

Example 77 Evaluate

$$\int \frac{(x^2-1)\,dx}{(x^4+3x^2+1)\tan^{-1}\left(x+\frac{1}{x}\right)}.$$

Sol. Here $I = \int \frac{(x^2-1)}{(x^4+x^3+1)\tan^{-1}\left(x+\frac{1}{x}\right)}$

The given integral can be written as

$$I = \int \frac{(1-1/x^2)\,dx}{(x^2+3+1/x^2)\tan^{-1}\left(x+\frac{1}{x}\right)}$$

(dividing numerator and denominator by x^2)

$$I = \int \frac{(1-1/x^2)\,dx}{\{(x+1/x)^2+1\}\tan^{-1}\left(x+\frac{1}{x}\right)}$$

Put, $x + \frac{1}{x} = t \Rightarrow \left(1-\frac{1}{x^2}\right)dx = dt$

$$\therefore \quad I = \int \frac{dt}{(t^2+1)\cdot\tan^{-1}(t)} \qquad \ldots(i)$$

Now, make one more substitution

$\tan^{-1} t = u$. Then, $\frac{dt}{t^2+1} = du$

$\therefore$ Eq. (i) becomes, $I = \int \frac{du}{u} = \log|u| + C$

$\Rightarrow I = \log|\tan^{-1} t| + C = \log|\tan^{-1}(x+1/x)| + C$

Example 78 $\int \frac{(x^{-7/6}-x^{5/6})\,dx}{x^{1/3}(x^2+x+1)^{1/2}-x^{1/2}(x^2+x+1)^{1/3}}$

Sol. $I = \int \frac{x^{7/6}(x^{-7/6}-x^{5/6})\,dx}{x^{7/6}\cdot x^{1/3}(x^2+x+1)^{1/2}-x^{1/2}\cdot x^{7/6}(x^2+x+1)^{1/3}}$

$$= \int \frac{(1-x^2)\,dx}{x^{3/2}(x^2+x+1)^{1/2}-x^{5/3}(x^2+x+1)^{1/3}}$$

$$= \int \frac{-\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}+1\right)^{1/2}-\left(x+\frac{1}{x}+1\right)^{1/3}} \left[\begin{array}{l}\text{Putting } x+\frac{1}{x}=t \\ \Rightarrow \left(1-\frac{1}{x^2}\right)dx = dt\end{array}\right]$$

$$= -\int \frac{dt}{(t+1)^{1/2}-(t+1)^{1/3}}$$

Substitute, $(t+1) = u^6$

$$= -\int \frac{6u^5\,du}{u^3-u^2} = -6\int \frac{u^3}{u-1}\,du,$$

Put $u-1=z$

$$dz = -6\int \frac{(z+1)^3}{z}\,dz$$

$$= -6\int \frac{z^3+3z^2+3z+1}{z}\,dz$$

$$= -6\int \left(z^2+3z+3+\frac{1}{z}\right)dz$$

$$= -6\left\{\frac{z^3}{3}+\frac{3z^2}{2}+3z+\log|z|\right\}+C$$

where, $z = \left(x+\frac{1}{x}+1\right)^{1/6}-1$

Example 79 The value of $\int \{\{[x]\}\}\,dx$, where {.} and [.] denotes fractional part of x and greatest integer function, is equal to

(a) 0 (b) 1 (c) 2 (d) –1

Sol. Let $I = \int \{\{[x]\}\}\,dx$

where, $[x] =$ Integer and we know $\{n\} = 0; n \in$ Integer.

$\therefore \quad I = \int 0\,dx = 0$

Hence, (a) is the correct answer.

Type II

Integration of Some Special Irrational Algebraic Functions

In this case we shall discuss four integrals of the form $\int \frac{\phi(x)}{P\sqrt{Q}}\,dx$, where P and Q are polynomial functions of x and $\phi(x)$ is polynomial in x.

(a) Integrals of the Form $\int \frac{\phi(x)}{P\sqrt{Q}}\,dx$, where P and Q are both linear of x

To evaluate this type of integrals we put $Q = t^2$, i.e. to evaluate integrals of the form $\int \frac{1}{(ax+b)\sqrt{cx+d}}\,dx$, put $cx+d = t^2$.

The following examples illustrate the procedure :

Example 80 Evaluate $\int \frac{1}{(x+1)\sqrt{x-2}}\,dx$.

Sol. Let $I = \int \frac{1}{(x+1)\sqrt{x-2}}\,dx$

Here, P and Q both are linear, so we put $Q = t^2$

i.e. $x-2 = t^2$

So that $dx = 2t\,dt$

$$\therefore \quad I = \int \frac{1}{(t^2+2+1)\sqrt{t^2}}\cdot 2t\,dt = 2\int \frac{dt}{t^2+3}$$

$$= 2\times\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)+C$$

$$\therefore \quad I = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{x-2}}{\sqrt{3}}\right)+C$$

(b) Integrals of the Form $\int \frac{\phi(x)}{P\sqrt{Q}}\,dx$, where P is a quadratic expression and Q is a linear expression

To evaluate this type of integrals we put $Q = t^2$ i.e. to evaluate the integrals of the form

$$\int \frac{1}{(ax^2+bx+c)\sqrt{px+q}}\,dx$$

put $px+q = t^2$.

Example 81 ***Evaluate*** $\int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}}\,dx$.

Sol. Let $I = \int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}}\,dx$

Put $x+1 = t^2 \Rightarrow dx = 2t\,dt$

$$\therefore \quad I = \int \frac{(t^2-1)+2}{\{(t^2-1)^2+3(t^2-1)+3\}\sqrt{t^2}}\cdot(2t)\,dt$$

$$= 2\int \frac{t^2+1}{t^4+t^2+1}\,dt = 2\int \frac{1+1/t^2}{t^2+1+1/t^2}\,dt$$

$$= 2\int \frac{1+1/t^2}{(t-1/t)^2+(\sqrt{3})^2}\,dt = 2\int \frac{du}{u^2+(\sqrt{3})^2} \qquad \left[\text{where } u = t-\frac{1}{t}\right]$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right)+C$$

$$\therefore \quad I = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t^2-1}{\sqrt{3}t}\right)+C$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3(x+1)}}\right)+C$$

(c) Integral of the Form $\int \frac{1}{(ax+b)} \cdot \frac{dx}{\sqrt{px^2+qx+r}}$, where in $\int \frac{dx}{P\sqrt{Q}}$ P is linear and Q is a quadratic put, $ax+b=\frac{1}{t}$.

Example 82 Evaluate $\int \frac{dx}{(x-1)\sqrt{x^2+x+1}}$.

Sol. Let $I=\int \frac{dx}{(x-1)\sqrt{x^2+x+1}}$

Put $x-1=\frac{1}{t} \Rightarrow dx=-\frac{1}{t^2}dt$

$$\therefore \quad I=\int \frac{-1/t^2\,dt}{1/t\sqrt{\left(\frac{1}{t}+1\right)^2+\left(\frac{1}{t}+1\right)+1}}$$

$$=-\int \frac{dt}{\sqrt{3t^2+3t+1}}=-\frac{1}{\sqrt{3}}\int \frac{dt}{\sqrt{\left(t+\frac{1}{2}\right)^2+\frac{1}{12}}}$$

$$=-\frac{1}{\sqrt{3}}\log\left|(t+1/2)+\sqrt{(t+1/2)^2+1/12}\right|+C$$

$$=-\frac{1}{\sqrt{3}}\log\left|\left(\frac{1}{x-1}+\frac{1}{2}\right)+\sqrt{\frac{12\left(\frac{1}{x-1}+\frac{1}{2}\right)^2+1}{12}}\right|+C$$

(d) Integrals of the Form $\int \frac{dx}{P\sqrt{Q}}$, where P and Q both are pure quadratic expression in x, i.e. $P=ax^2+b$ and $Q=cx^2+d$, i.e. $\int \frac{dx}{(ax^2+b)\sqrt{cx^2+d}}$.

To evaluate this type of integrals of the form we put $x=\frac{1}{t}$

Example 83 Evaluate $\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$.

Sol. Let $I=\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

Put $x=\frac{1}{t}$, so that $dx=-\frac{1}{t^2}\,dt$

$$\therefore \quad I=\int \frac{-1/t^2\,dt}{(1+1/t^2)\sqrt{1-1/t^2}}=-\int \frac{t\,dt}{(t^2+1)\sqrt{t^2-1}}$$

Again, $t^2=u \Rightarrow 2t\,dt=du$.

$=-\frac{1}{2}\int \frac{du}{(u+1)\sqrt{u-1}}$ which reduces to the form $\int \frac{dx}{P\sqrt{Q}}$ where both P and Q are linear so that we put $u-1=z^2$ so that $du=2z\,dz$

$$\therefore \quad I=-\frac{1}{2}\int \frac{2z\,dz}{(z^2+1+1)\sqrt{z^2}}=-\int \frac{dz}{(z^2+2)}$$

$$I=-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{z}{\sqrt{2}}\right)+C$$

$$I=-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{u-1}}{\sqrt{2}}\right)+C$$

$$=-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{t^2-1}}{\sqrt{2}}\right)+C$$

$$=-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{1-x^2}}{\sqrt{2}x}\right)+C$$

Aliter Put $x=\cos\theta, dx=-\sin\theta\,d\theta$

$$\therefore \quad I=-\int \frac{\sin\theta\,d\theta}{(1+\cos^2\theta)\sin\theta}=-\int \frac{d\theta}{1+\cos^2\theta}$$

$$=-\int \frac{\sec^2\theta\,d\theta}{\sec^2\theta+1}=-\int \frac{\sec^2\theta\,d\theta}{\tan^2\theta+2}$$

Put $\tan\theta=t. \Rightarrow \sec^2\theta\,d\theta=dt$

$$\therefore \quad I=-\int \frac{dt}{t^2+2}=-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)+C$$

$$=-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan\theta}{\sqrt{2}}\right)+C$$

$$=-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{1-x^2}}{\sqrt{2}\,x}\right)+C \quad \left[\begin{array}{l}\text{where, } \cos\theta=x \\ \qquad \sin\theta=\sqrt{1-x^2} \\ \therefore \quad \tan\theta=\frac{\sqrt{1-x^2}}{x}\end{array}\right]$$

Example 84 Evaluate

$$I=\int \frac{(x-1)\sqrt{x^4+2x^3-x^2+2x+1}}{x^2(x+1)}\,dx.$$

Sol. Here, $I=\int \frac{(x^2-1)\sqrt{x^4+2x^3-x^2+2x+1}}{x^2(x+1)^2}\,dx$

$$=\int \frac{\left(1-\frac{1}{x^2}\right)\sqrt{x^2\left(x^2+2x-1+\frac{2}{x}+\frac{1}{x^2}\right)}}{\frac{x^2(x^2+2x+1)}{x^2}}\,dx$$

$$=\int \frac{\left(1-\frac{1}{x^2}\right)\sqrt{\left(x^2+\frac{1}{x^2}\right)+2\left(x+\frac{1}{x}\right)-1}}{\left(x+\frac{1}{x}+2\right)}\,dx$$

Put $x+\frac{1}{x}=t$, i.e. $\left(1-\frac{1}{x^2}\right)dx=dt$

$$=\int \frac{\sqrt{(t^2-2)}+2t-1}{(t+2)}\,dt=\int \frac{\sqrt{t^2+2t-3}}{(t+2)}\,dt$$

$$= \int \frac{t^2 + 2t - 3}{(t+2)\sqrt{t^2+2t-3}}\, dt$$

$$= \int \frac{t(t+2)}{(t+2)\sqrt{t^2+2t-3}}\, dt - 3\int \frac{dt}{(t+2)\sqrt{t^2+2t-3}}$$

$$I = I_1 - 3I_2 \quad \text{...(i)}$$

Where, $I_1 = \int \frac{t\, dt}{\sqrt{t^2+2t-3}}$ and $I_2 = \int \frac{dt}{(t+2)\sqrt{t^2+2t-3}}$

$$\therefore \quad I_1 = \int \frac{t\, dt}{\sqrt{(t+1)^2 - 4}}$$

Put, $t + 1 = z = \int \frac{(z-1)\, dz}{\sqrt{z^2 - 2^2}}$

$$= \int \frac{z dz}{\sqrt{z^2-2^2}} - \int \frac{dz}{\sqrt{z^2-2^2}}$$

$$= \sqrt{z^2-2^2} - \log|z + \sqrt{z^2-4}|$$

$$= \sqrt{t^2+2t-3} - \log|(t+1) + \sqrt{t^2+2t+3}| \quad \text{...(ii)}$$

Also, $I_2 = \int \frac{dy}{y^2 \cdot \frac{1}{y}\sqrt{\left(\frac{1}{y}-2\right)^2 + 2\left(\frac{1}{y}-2\right) - 3}}$

Put $t + 2 = \frac{1}{y} = \int \frac{dy}{\sqrt{1-2y-3y^2}} = \frac{1}{\sqrt{3}} \int \frac{dy}{\sqrt{\left(\frac{2}{3}\right)^2 - \left(y + \frac{1}{3}\right)^2}}$

$$= \frac{1}{\sqrt{3}} \sin^{-1}\left(\frac{y + \frac{1}{3}}{\frac{2}{3}}\right) = \frac{1}{\sqrt{3}} \sin^{-1}\left(\frac{5+t}{2+t}\right) \quad \text{...(iii)}$$

$$\therefore \quad I = \sqrt{t^2+2t-3} - \log(t + 1 + \sqrt{t^2+2t-3}) - \sqrt{3} \sin^{-1}\left(\frac{t+5}{t+2}\right)$$

where, $t = x + \frac{1}{x}$

(e) Integrals of the Form $\int \frac{dx}{(x-k)^r \sqrt{ax^2 + bx + c}}$, where $r \geq 2$ and $r \in I$

Here, we substitute, $x - k = \frac{1}{t}$

Example 85 Evaluate $\int \frac{dx}{(x-3)^3 \sqrt{x^2 - 6x + 10}}$.

Sol. Substitute $(x-3) = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}\, dt$

We get, $\int \frac{dx}{(x-3)^3 \sqrt{x^2 - 6x + 10}}$

$$= \int \frac{-1/t^2\, dt}{1/t^3 \sqrt{(1/t+3)^2 - 6(1/t+3) + 10}}$$

$$= -\int \frac{t^2\, dt}{\sqrt{1+t^2}} = \int \frac{dt}{\sqrt{1+t^2}} - \int \sqrt{1+t^2}\, dt$$

$$= \log|t + \sqrt{1+t^2}| - \frac{t}{2}\sqrt{1+t^2} - \frac{1}{2}\log|t + \sqrt{1+t^2}| + C$$

$$= \frac{1}{2}\log|t + \sqrt{1+t^2}| - \frac{t}{2}\sqrt{1+t^2} + C$$

$$= \frac{1}{2}\left[\log\left|\frac{1 + \sqrt{x^2 - 6x + 10}}{|x-3|}\right| - \frac{\sqrt{x^2-6x+10}}{|x-3|^2}\right] + C$$

(f) Integrals of the Form $\int \frac{ax^2 + bx + c}{(dx+e)\sqrt{fx^2 + gx + h}}\, dx$

Here, we write

$$ax^2 + bx + c = A_1(dx+e)(2fx+g) + B_1(dx+e) + C_1$$

Where A_1, B_1 and C_1 are constants which can be obtained by comparing the coefficients of like terms on both the sides.

Example 86 ***Evaluate*** $\int \frac{2x^2 + 5x + 9}{(x+1)\sqrt{x^2+x+1}}\, dx$.

Sol. Let $2x^2 + 5x + 9 = A(x+1)(2x+1) + B(x+1) + C$

or $2x^2 + 5x + 9 = x^2(2A) + x(3A + B) + (A + B + C)$

$\Rightarrow \quad A = 1, B = 2, C = 6$

Thus, $\int \frac{2x^2 + 5x + 9}{(x+1)\sqrt{x^2+x+1}}\, dx$

$$= \int \frac{(x+1)(2x+1)}{(x+1)\sqrt{x^2+x+1}}\, dx + 2\int \frac{x+1}{(x+1)\sqrt{x^2+x+1}}\, dx + 6\int \frac{dx}{(x+1)\sqrt{x^2+x+1}}$$

$$= \int \frac{2x+1}{\sqrt{x^2+x+1}}\, dx + 2\int \frac{dx}{\sqrt{x^2+x+1}} + 6\int \frac{dx}{(x+1)\sqrt{x^2+x+1}}$$

$$= \int \frac{du}{\sqrt{u}} + 2\int \frac{dx}{\sqrt{(x+1/2)^2 + (3/4)}} + 6\int \frac{-dt}{\sqrt{t^2 - t + 1}}$$

[where $u = x^2 + x + 1$ and $\frac{1}{t} = x + 1$]

$$= 2\sqrt{x^2+x+1} + 2 \cdot 1 \log|(x + 1/2) + \sqrt{x^2+x+1}| - 6\int \frac{dt}{\sqrt{(t-1/2)^2 + 3/4}}$$

$$= 2\sqrt{x^2+x+1} + 2\log\left|\left(x + \frac{1}{2}\right) + \sqrt{x^2+x+1}\right| - 6\log\left|\left(t - \frac{1}{2}\right) + \sqrt{t^2-t+1}\right| + C$$

$$= 2\sqrt{x^2+x+1} + 2\log\left|\left(x + \frac{1}{2}\right) + \sqrt{x^2+x+1}\right| - 6\log\left|\frac{1 - x + \sqrt{x^2+x+1}}{2(x+1)}\right| + C$$

Type III

Integration of Type $\int (\sin^m x \cdot \cos^n x)\, dx$

(i) Where m, n belongs to natural number.
(ii) If one of them is odd, then substitute for term of even power.
(iii) If both are odd, substitute either of them.
(iv) If both are even, use trigonometric identities only.
(v) If m and n are rational numbers and $\left(\frac{m+n-2}{2}\right)$ is a negative integer, then substitute $\cot x = p$ or $\tan x = p$ which so ever is found suitable.

Example 87 Evaluate $\int \sin^3 x \cdot \cos^5 x\, dx$.

Sol. $I = \int \sin^3 x \cdot \cos^5 x\, dx$

Let $\cos x = t \quad \Rightarrow \quad -\sin x\, dx = dt$

$$I = -\int (1 - t^2) \cdot t^5 dt$$

$$I = \int t^7\, dt - \int t^5\, dt = \frac{t^8}{8} - \frac{t^6}{6} + C$$

$$I = \frac{\cos^8 x}{8} - \frac{\cos^6 x}{6} + C$$

Aliter $I = \int R^3 (1 - R^2)^2\, dR,$

if $\sin x = R,\ \cos x\, dx = dR$

$$I = \int R^3\, dR - \int 2R^5\, dR + \int R^7 dR$$

$$I = \frac{\sin^4 x}{4} - \frac{2\sin^6 x}{6} + \frac{\sin^8 x}{8} + C$$

Remark

This problem can also be handled by successive reduction or by trigonometrical identities. Answers will be in different form but identical with modified constant of integration.

Example 88 Evaluate $\int \sin^{-11/3} x \cdot \cos^{-1/3} x\, dx$.

Sol. Here, $\int \sin^{-11/3} x \cdot \cos^{-1/3} x\, dx$ i.e. $\left(\frac{-\frac{11}{3} - \frac{1}{3} - 2}{2} = -3\right)$

$$\therefore \quad I = \int \frac{\cos^{-1/3} x}{\sin^{-1/3} x \cdot \sin^4 x}\, dx = \int (\cot^{-1/3} x)(\operatorname{cosec}^2 x)^2 dx$$

$$I = \int (\cot^{-1/3} x)(1 + \cot^2 x) \operatorname{cosec}^2 x\, dx$$

$$= -\int t^{-1/3} (1 + t^2)\, dt \ = -\int (t^{-1/3} + t^{5/3})\, dt$$

[Put $\cot x = t, \Rightarrow -\operatorname{cosec}^2 x\, dx = dt$]

$$= -\left\{\frac{3}{2} t^{2/3} + \frac{3}{8} t^{8/3}\right\} + C$$

$$= -\left\{\frac{3}{2} (\cot^{2/3} x) + \frac{3}{8} (\cot^{8/3} x)\right\} + C$$

Example 89 Evaluate $\int \frac{dx}{2\sin x + \sec x}$.

Sol. Let $I = \int \frac{dx}{2\sin x + \sec x} = \int \frac{\cos x\, dx}{\sin 2x + 1} = \frac{1}{2}\int \frac{2\cos x\, dx}{1 + \sin 2x}$

$$= \frac{1}{2}\int \frac{(\cos x + \sin x) + (\cos x - \sin x)}{(\sin^2 + \cos^2 x + 2\sin x \cos x)}\, dx$$

$$= \frac{1}{2}\int \frac{\cos x + \sin x}{(\sin x + \cos x)^2}\, dx + \frac{1}{2}\int \frac{(\cos x - \sin x)}{(\sin x + \cos x)^2}\, dx$$

$$= \frac{1}{2}\int \frac{dx}{\sin x + \cos x} + \frac{1}{2}\int \frac{dv}{v^2}, \text{ where, } v = \sin x + \cos x$$

$$= \frac{1}{2\sqrt{2}}\int \frac{dx}{\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x} - \frac{1}{2v} + C$$

$$= \frac{1}{2\sqrt{2}}\int \frac{dx}{\sin\left(x + \frac{\pi}{4}\right)} - \frac{1}{2(\sin x + \cos x)} + C$$

$$= \frac{1}{2\sqrt{2}} \log\left|\operatorname{cosec}\left(x + \frac{\pi}{4}\right) - \cot\left(x + \frac{\pi}{4}\right)\right| - \frac{1}{2(\sin x + \cos x)} + C$$

Type IV

Integrals of the Form $\int x^m (a + bx^n)^P dx$

Case I If $P \in N$. We expand using binomial and integrate.

Case II If $P \in I^-$ (*ie*, negative integer), write $x = t^k$, where k is the LCM of m and n.

Case III If $\frac{m+1}{n}$ is an integer and $P \leftrightarrow$ fraction, put $(a + b\, x^n) = t^k$, where k is denominator of the fraction P.

Case IV If $\left(\frac{m+1}{n} + P\right)$ is an integer and $P \in$ fraction.

We put $(a + b\, x^n) = t^k x^n$, where k is denominator of the fraction P.

Example 90 Evaluate $\int x^{1/3} (2 + x^{1/2})^2\, dx$.

Sol. $I = \int x^{1/3} (2 + x^{1/2})^2\, dx$

Since, P is natural number.

$$\therefore \quad I = \int x^{1/3} (4 + x + 4x^{1/2})\, dx$$

$$= \int (4x^{1/3} + x^{4/3} + 4x^{5/6})\, dx$$

$$= \frac{4x^{4/3}}{4/3} + \frac{x^{7/3}}{7/3} + \frac{4x^{11/6}}{11/6} + C$$

$$= 3x^{4/3} + \frac{3}{7} x^{7/3} + \frac{24}{11} x^{11/6} + C$$

Example 91 Evaluate $\int x^{-2/3}\,(1+x^{2/3})^{-1}\,dx$.

Sol. If we substitute $x = t^3$ (as we know $P \in$ negative integer)

$\therefore$ Let $x = t^k$, where k is the LCM of m and n.

$\therefore \quad x = t^3 \Rightarrow dx = 3t^2\,dt$

or $\quad I = \int \frac{3t^2}{t^2(1+t^2)}\,dt = 3\int \frac{dt}{t^2+1} = 3\tan^{-1}(t) + C$

$\Rightarrow \quad I = 3\tan^{-1}(x^{1/3}) + C$

Example 92 Evaluate $\int x^{-2/3}\,(1+x^{1/3})^{1/2}\,dx$.

Sol. If we substitute $1 + x^{1/3} = t^2$, then $\frac{1}{3x^{2/3}}\,dx = 2t\,dt$

$\therefore \quad I = \int \frac{t \cdot 6t\,dt}{1} = 6\int t^2\,dt = 2t^3 + C$

or $\quad I = 2(1 + x^{1/3})^{3/2} + C$

Example 93 Evaluate $\int \sqrt{x}\,(1+x^{1/3})^4\,dx$.

Sol. Here, $m = \frac{1}{2}$ and $n = \frac{1}{3}$

Put $\quad x = t^6 \Rightarrow dx = 6t^5\,dt$

$\Rightarrow \quad I = \int t^3\,(1+t^2)^4\,6t^5\,dt$

$\Rightarrow \quad I = 6\int t^8\,(1 + 4t^2 + 6t^4 + 4t^6 + t^8)\,dt$

$$= 6\int (t^8 + 4t^{10} + 6t^{12} + 4t^{14} + t^{16})\,dt$$

$$= 6\left\{\frac{t^9}{9} + \frac{4t^{11}}{11} + \frac{6t^{13}}{13} + \frac{4t^{15}}{15} + \frac{t^{17}}{17}\right\} + C$$

$$I = 6\left\{x^{2/3} + \frac{4}{11}x^{11/6} + \frac{6}{13}x^{13/6} + \frac{4}{15}x^{5/2} + \frac{1}{17}x^{17/6}\right\} + C$$

Example 94 Evaluate $\int x^5\,(1+x^3)^{2/3}\,dx$.

Sol. Here, $\int x^5\,(1+x^3)^{2/3}\,dx$ have $m = 5, n = 3$ and $p = \frac{2}{3}$

$\therefore \quad \frac{m+1}{n} = \frac{6}{3} = 2$ [an integer]

So, we substitute $1 + x^3 = t^2$ and $3x^2\,dx = 2t\,dt$

$$\therefore \int x^5(1+x^3)^{2/3}dx = \int x^3\,(1+x^3)^{2/3}x^2dx$$

$$= \int (t^2-1)\,(t^2)^{2/3}\,\frac{2}{3}\,t\,dt$$

$$= \frac{2}{3}\int (t^2-1)\,t^{7/3}dt = \frac{2}{3}\int (t^{13/3} - t^{7/3})\,dt$$

$$= \frac{2}{3}\left\{\frac{3}{16}t^{16/3} - \frac{3}{10}t^{10/3}\right\} + C$$

$$= \frac{1}{8}(1+x^3)^{8/3} - \frac{1}{5}(1+x^3)^{5/3} + C$$

Example 95 Evaluate $\int x^{-11}\,(1+x^4)^{-1/2}\,dx$.

Sol. Here, $\left(\frac{m+1}{n} + p\right) = \left[\frac{-11+1}{4} - \frac{1}{2}\right] = -3$

If we substitute $(1 + x^4) = t^2\,x^4$,

then $1 + \frac{1}{x^4} = t^2$ and $\frac{-4}{x^5}\,dx = 2t\,dt$

$$\therefore \quad I = \int \frac{dx}{x^{11}\,(1+x^4)^{1/2}} = \int \frac{dx}{x^{11}\cdot x^2\,(1+1/x^4)^{1/2}}$$

$$= \int \frac{dx}{x^{13}\,(1+1/x^4)^{1/2}} = -\frac{1}{4}\int \frac{2t\,dt}{x^8\,t}$$

$$= -\frac{1}{2}\int (t^2-1)^2\,dt = -\frac{1}{2}\int (t^4 - 2t^2 + 1)\,dt$$

$$= -\frac{1}{2}\left[\frac{t^5}{5} - \frac{2t^3}{3} + t\right] + C$$

Where $t = \sqrt{1 + \frac{1}{x^4}}$

Example 96 Evaluate $\int \frac{1}{\sqrt[3]{x} + \sqrt[4]{x}}\,dx$.

Sol. Let $\quad I = \int \frac{1}{\sqrt[3]{x} + \sqrt[4]{x}}\,dx$

Put $\quad x^{1/12} = t, \quad \Rightarrow \quad x = t^{12}$ and $dx = 12t^{11}\,dt$

$$\therefore \quad I = \int \frac{1}{t^4 + t^3}\cdot 12\,t^{11}\,dt = 12\int \frac{t^8}{t+1}\,dt$$

Again put $(t + 1) = y$

$$\therefore\ dt = dy = 12\int \frac{(y-1)^8}{y}\,dy$$

$$= 12\int \frac{(y^8 - 8y^7 + 28y^6 - 56y^5 + 70y^4 - 56y^3 + 28y^2 - 8y + 1)}{y}dy$$

[using binomial]

$$= 12\int (y^7 - 8y^6 + 28y^5 - 56y^4 + 70y^3 - 56y^2 + 28y - 8 + 1/y)dy$$

$$= 12\left(\frac{y^8}{8} - \frac{8y^7}{7} + \frac{28y^6}{6} - \frac{56y^5}{5} + \frac{70y^4}{4} - \frac{56y^3}{3} + \frac{28y^2}{2} - 8y + \log|y|\right) + C_1$$

Where $\quad y = x^{1/12} + 1$

Exercise for Session 6

■ Evaluate the following integrals

1. $\int \frac{x^4 - 1}{x^{2(}x^4 + x^2 + 1)^{1/2}}\, dx$

2. $\int \frac{(x+2)\, dx}{(x^2 + 3x + 3)\sqrt{x+1}}$

3. $\int \frac{dx}{(x+1)^{1/3} + (x+1)^{1/2}}$

4. $\int \frac{dx}{(x+a)^{8/7}(x-b)^{6/7}}$

5. $\int \frac{\sec x\, .dx}{\sqrt{\sin(x+2A) + \sin A}}$

6. The value of $\int [\{x\}]\, dx$; (where $[.]$ and $\{.\}$ denotes greatest integer and fractional part of x) is equal to

(a) 0 (b) 1
(c) 2 (d) – 1

7. If $\int f(x)\cos x\, dx = \frac{1}{2} f^2(x) + C$, then $f(x)$ can be

(a) x (b) 1
(c) $\cos x$ (d) $\sin x$

8. The value of, $\int \frac{\sin x + \cos x}{9 + 16\sin 2x}\, dx$ is

(a) $\frac{1}{40}\log\left|\frac{5+4(\sin x - \cos x)}{5-4(\sin x - \cos x)}\right| + C$ (b) $\log\left|\frac{5+4(\sin x - \cos x)}{5-4(\sin x - \cos x)}\right| + C$

(c) $\frac{1}{10}\log\left|\frac{5+4(\sin x + \cos x)}{5-4(\sin x + \cos x)}\right| + C$ (d) None of these

9. The value of $\int \frac{\cos 7x - \cos 8x}{1 + 2\cos 5x}\, dx$, is

(a) $\frac{\sin 2x}{2} + \frac{\cos 3x}{3} + C$ (b) $\sin x - \cos x + C$

(c) $\frac{\sin 2x}{2} - \frac{\cos 3x}{3} + C$ (d) None of these

10. The value of $\int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x}\, dx$, is

(a) $\sin x + \sin 2x + C$ (b) $\sin x - \frac{\sin 2x}{2} + C$

(c) $-\sin x - \frac{\sin 2x}{2} + C$ (d) None of these

Session 7

Euler's Substitution, Reduction Formula and Integration Using Diffrentiation

Euler's Substitution, Reduction and Integration Using Diffrentiation

Integration Using Euler's Substitutions

Integrals of the form $\int f(x), \sqrt{ax^2+bx+c}\,dx$ are calculated with the aid of one of the three Euler's substitutions

(i) $\sqrt{ax^2+bx+c} = t \pm x\sqrt{a}$, if $a>0$.

(ii) $\sqrt{ax^2+bx+c} = tx+\sqrt{c}$, if $c>0$.

(iii) $\sqrt{ax^2+bx+c} = (x-\alpha)\,t$, if $ax^2+bx+c = a(x-\alpha)(x-\beta)$, i.e. If α is real root of (ax^2+bx+c).

Remark

The Euler's substitutions often lead to rather cumbersome calculations, therefore they should be applied only when it is difficult to find another method for calculating a given integral.

Example 97 *Evaluate* $I = \int \frac{x\,dx}{(\sqrt{7x-10-x^2})^3}$.

Sol. In this case $a<0$ and $c<0$. Therefore, neither (I) nor (II) Euler's Substitution is applicable. But the quadratic $7x-10-x^2$ has real roots $\alpha=2, \beta=5$.

$\therefore$ We use the substitution (III)

i.e. $\sqrt{7x-10-x^2} = \sqrt{(x-2)(5-x)} = (x-2)\,t$

Where $(5-x) = (x-2)\,t^2$

or $5+2t^2 = x(1+t^2)$

$\therefore$ $x = \frac{5+2t^2}{1+t^2}$

$(x-2)\,t = \left(\frac{5+2t^2}{1+t^2}-2\right)t = \frac{3t}{1+t^2}$

$\therefore$ $dx = \frac{-6t}{(1+t^2)^2}\,dt$

Hence, $I = \int \frac{x\,dx}{(\sqrt{7x-10-x^2})^3} = \int \frac{\left(\frac{5+2t^2}{1+t^2}\right)\cdot\frac{-6t}{(1+t^2)^2}\,dt}{\left(\frac{3t}{1+t^2}\right)^3}$

$= \frac{-6}{27}\int \frac{5+2t^2}{t^2}\,dt$

$= \frac{-2}{9}\int\left(\frac{5}{t^2}+2\right)dt = \frac{-2}{9}\left[\frac{-5}{t}+2t\right]+C$

$\therefore$ $\int \frac{x\,dx}{(\sqrt{7x-10-x^2})^3} = \frac{-2}{9}\left(\frac{-5}{t}+2t\right)+C$,

where, $t = \frac{\sqrt{7x-10-x^2}}{x-2}$

Example 98 Evaluate $\int \frac{dx}{x+\sqrt{x^2-x+1}}$.

Sol. Since, here $c=1$, we can apply the second Euler's Substitution.

$\sqrt{x^2-x+1} = tx-1$

Therefore, $(2t-1)\,x = (t^2-1)\,x^2 \Rightarrow x = \frac{2t-1}{t^2-1}$

$\therefore$ $dx = -\frac{2(t^2-t+1)\,dt}{(t^2-1)^2}$ and $x+\sqrt{x^2-x+1} = \frac{t}{t-1}$

$\therefore$ $I = \int \frac{dx}{x+\sqrt{x^2-x+1}} = \int \frac{-2t^2+2t-2}{t(t-1)(t+1)^2}\,dt$

Using partial fractions, we have

$\frac{-2t^2+2t-2}{t(t-1)(t+1)^2} = \frac{A}{t}+\frac{B}{t-1}+\frac{C}{(t+1)}+\frac{D}{(t+1)^2}$

or $(-2t^2+2t-2) = A(t-1)(t+1)^2 + Bt(t+1)^2 + C(t-1)(t+1)t + Dt$

we get $A=2, \quad B=-1/2, \quad C=-3/2, \quad D=-3$

Hence, $I = 2\int\frac{dt}{t} - \frac{1}{2}\int\frac{dt}{t-1} - \frac{3}{2}\int\frac{dt}{(t+1)} - 3\int\frac{dt}{(t+1)^2}$

$= 2\log_e|t| - \frac{1}{2}\log_e|t-1| - \frac{3}{2}\log_e|t+1| + \frac{3}{(t+1)} + C$

$\left[\text{where } t = \frac{\sqrt{x^2-x+1}+1}{x}\right]$

Introduction of Reduction Formulae (a recursive relation) Over Indefinite Integrals

Reduction formulae makes it possible to reduce an integral depending on the index $n > 0$, called the order of the integral, to an integral of the same type with smaller index. (i.e. To reduce the integrals into similar integrals of order less than or greater than given integral).

Application of reduction formula is given with the help of some examples.

Reduction Formula for $\int \sin^n x\,dx$

Let $I_n = \int \sin^n x\,dx = \int \underset{\text{I}}{\sin^{n-1} x} \underset{\text{II}}{\sin x}\,dx$

$= -\sin^{n-1} x \cos x + \int (n-1)\sin^{n-2} x \cos^2 x\,dx$

$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)\,dx$

$= -\sin^{n-1} x \cos x + (n-1)\int (\sin^{n-2} x - \sin^n x)\,dx$

$= -\sin^{n-1} x \cos x + (n-1) I_{n-2} - (n-1) I_n$

$\therefore n I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$

$\Rightarrow I_n = -\dfrac{\sin^{n-1} x \cos x}{n} + \dfrac{n-1}{n} I_{n-2}$

Thus, $\int \sin^n x\,dx = \dfrac{-\sin^{n-1} x \cos x}{n} + \dfrac{n-1}{n}\int \sin^{n-2} x\,dx$

Reduction Formula for $\int \cos^n x\,dx$

Let $I_n = \int \cos^n x\,dx = \int \underset{\text{I}}{\cos^{n-1} x} \underset{\text{II}}{\cos x}\,dx$

$= \cos^{n-1} x \sin x + \int (n-1)\cos^{n-2} x \sin^2 x\,dx$

$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x (1-\cos^2 x)\,dx$

$= \cos^{n-1} x \sin x + (n-1) I_{n-2} - (n-1) I_n$

$\therefore \quad nI_n = \cos^{n-1} x \sin x + (n-1) I_{n-2}$

or $\int \cos^n x\,dx = \dfrac{\cos^{n-1} x \sin x}{n} + \dfrac{n-1}{n}\int \cos^{n-2} x\,dx$

Reduction Formula for $\int \tan^n x\,dx$

Let $I_n = \int \tan^n x\,dx$

$\Rightarrow I_n = \int \tan^{n-2} x \tan^2 x\,dx = \int \tan^{n-2} x (\sec^2 x - 1)\,dx$

$= \int \tan^{n-2} x \sec^2 x - I_{n-2} = \int t^{n-2}\,dt - I_{n-2}$

where, $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$I_n = \dfrac{t^{n-1}}{n-1} - I_{n-2}$

$\therefore \quad I_n = \dfrac{\tan^{n-1} x}{n-1} - I_{n-2}$

$\Rightarrow \quad \int \tan^n x\,dx = \dfrac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx$

Reduction Formula for $\int \text{cosec}^n x\,dx$

Let $I_n = \int \text{cosec}^n x\,dx = \int \underset{\text{I}}{\text{cosec}^{n-2} x} \underset{\text{II}}{\text{cosec}^2 x}\,dx$

$= \text{cosec}^{n-2} x\,(-\cot x) - \int (n-2)\,\text{cosec}^{n-2} x\,(\text{cosec}^2 x - 1)\,dx$

$= -\text{cosec}^{n-2} x \cot x - (n-2)\int (\text{cosec}^n x - \text{cosec}^{n-2} x)\,dx$

$= -\text{cosec}^{n-2} x \cot x - (n-2) I_n + (n-2) I_{n-2}$

$\therefore \quad (n-1) I_n = -\text{cosec}^{n-2} x \cot x + (n-2) I_{n-2}$

or $I_n = -\dfrac{\text{cosec}^{n-2} x \cot x}{n-1} + \dfrac{n-2}{n-1} I_{n-2}$

$\therefore \int \text{cosec}^n x\,dx = -\dfrac{\text{cosec}^{n-2} x \cot x}{n-1} + \dfrac{n-2}{n-1}\int \text{cosec}^{n-2} x\,dx$

Reduction Formula for $\int \sec^n x\,dx$

Let $I_n = \int \sec^n x\,dx = \int \underset{\text{I}}{\sec^{n-2} x} \underset{\text{II}}{\sec^2 x}\,dx$

$= \sec^{n-2} x \tan x - \int (n-2)\sec^{n-3} x \sec x \tan x \cdot \tan x\,dx$

$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x (\sec^2 x - 1)\,dx$

$= \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$

$\Rightarrow (n-1) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$

or $I_n = \dfrac{\sec^{n-2} x \tan x}{(n-1)} + \dfrac{(n-2)}{(n-1)} I_{n-2}$

$\therefore \int \sec^n x\,dx = \dfrac{\sec^{n-2} x \tan x}{(n-1)} + \dfrac{(n-2)}{(n-1)}\int \sec^{n-2} x\,dx$

Reduction Formula for $\int \cot^n x\,dx$

Let $I_n = \int \cot^n x\,dx = \int \cot^{n-2} x \cot^2 x\,dx$

$= \int \cot^{n-2} x\,(\text{cosec}^2 x - 1)\,dx$

$= \int \cot^{n-2} x\,(\text{cosec}^2 x - 1)\,dx = \int \cot^{n-2} x\,dx$

$= \int t^{n-2}\,dt - I_{n-2}$, where $t = \cot x$

$$I_n = -\frac{\cot^{n-1} x}{n-1} - I_{n-2}$$

$$\therefore \quad \int \cot^n x\, dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\, dx$$

Reduction Formula for $\int \sin^m x \cos^n x\, dx$

Let $A = \sin^{m-1} x \cos^{n+1} x$

$$\therefore \quad \frac{dA}{dx} = (m-1)\sin^{m-2} x \cos^{n+2} x - (n+1)\sin^m x \cos^n x$$

$$= (m-1)\sin^{m-2} x \cos^n x\,(1-\sin^2 x) - (n+1)\sin^m x \cos^n x$$

$$= (m-1)\sin^{m-2} x \cos^n x - (m-1+n+1)\sin^m x \cos^n x$$

$$\Rightarrow \quad \frac{dA}{dx} = (m-1)\sin^{m-2} x \cos^n x - (m+n)\sin^m x \cos^n x$$

Integrating with respect to x on both the sides, we get

$$A = (m-1)\int \sin^{m-2} x \cos^n x\, dx - (m+n)\int \sin^m x \cos^n x\, dx$$

$$\Rightarrow \quad (m+n)\int \sin^m x \cos^n x\, dx = (m-1)\int \sin^{m-2} x \cos^n x\, dx - P$$

$$\Rightarrow \quad \int \sin^m x \cos^n x\, dx = \frac{(m-1)}{(m+n)}\int \sin^{m-2} x \cos^n x\, dx - \frac{\sin^{m-1} x \cos^{n+1} x}{m+n}$$

or $$I_{m,n} = \frac{(m-1)}{(m+n)} I_{m-2,n} - \frac{\sin^{m-1} x \cos^{n+1} x}{(m+n)}$$

Remarks

Similarly, we can show

1. $\int \sin^m x \cos^n x\, dx = \frac{\sin^{m+1} x \cos^{n+1} x}{m+n} + \frac{n-1}{m+n}\int \sin^m x \cos^{n-2} x\, dx$

2. $\int \sin^m x \cos^n x\, dx = \frac{\sin^{m+1} x \cos^{n+1} x}{m+1} + \frac{m+n+2}{m+1}\int \sin^{m+2} x \cos^n x\, dx$

3. $\int \sin^m x \cos^n x\, dx = \frac{\sin^{m+1} x \cos^{n+1} x}{n+1} + \frac{m+n+2}{n+1}\int \sin^m x \cos^{n+2} x\, dx$

4. $\int \sin^m x \cos^n x\, dx = -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2} x \cos^{n+2} x\, dx$

5. $\int \sin^m x \cos^n x\, dx = \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1}\int \sin^{m+2} x \cos^{n-2} x\, dx$

Reduction Formula for $\int \cos^m x \sin nx\, dx$

Let $I_{m,n} = \int \underset{\text{I}}{\cos^m x} \underset{\text{II}}{\sin nx}\, dx$

$$= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \cos^{m-1} x \sin x \cos nx\, dx$$

$$= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \cos^{m-1} x \{\sin nx \cos x - \sin(n-1)x\}\, dx$$

[using $\sin(n-1)x = \sin nx \cos x - \cos nx \sin x$

$\Rightarrow \sin x \cos nx = \sin nx \cos x - \sin(n-1)x$]

$$= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \cos^m x \sin nx\, dx + \frac{m}{n}\int \cos^{m-1} x \sin(n-1)x\, dx$$

$$I_{m,n} = -\frac{\cos^m x \cos nx}{n} - \frac{m}{n} I_{m,n} + \frac{m}{n} I_{m-1,n-1}$$

$$\Rightarrow \quad \frac{m+n}{n} I_{m,n} = -\frac{\cos^m x \cos nx}{n} + \frac{m}{n} I_{m-1,n-1}$$

or $$I_{m,n} = -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$$

Remarks

Similarly, we can show

1. $\int \cos^m x \cos nx\, dx = \frac{\cos^m x \sin nx}{m+n} + \frac{m}{m+n}\int \cos^{m-1} x \cos(n-1)x\, dx$

2. $\int \sin^m x \sin nx\, dx = \frac{n \sin^m x \cos nx}{m^2-n^2} - \frac{m \sin^{m-1} x \cos x \cos nx}{m^2-n^2} + \frac{m(m-1)}{m^2-n^2}\int \sin^{m-2} x \sin nx\, dx$

3. $\int \sin^m x \cos nx\, dx = \frac{n \sin^m x \sin nx}{m^2-n^2} - \frac{m \sin^{m-1} x \cos x \cos nx}{m^2-n^2} + \frac{m(m-1)}{m^2-n^2}\int \sin^{m-2} x \cos nx\, dx$

Example 99 Evaluate $I_n = \int \frac{dx}{(x^2+a^2)^n}$.

Sol. Here, $I_n = \int \frac{dx}{(x^2+a^2)^n} = \int \frac{1}{(x^2+a^2)^n} \cdot 1\, dx$

Applying Integration by parts, we get

$$= \frac{1}{(x^2+a^2)^n} \cdot x - \int \frac{(2x)}{(x^2+a^2)^{n+1}} \cdot (-n) \cdot (x)\, dx$$

$$= \frac{x}{(x^2+a^2)^n} + 2n\int \frac{x^2}{(x^2+a^2)^{n+1}}\, dx$$

$$= \frac{x}{(x^2+a^2)^n} + 2n\int \frac{x^2+a^2-a^2}{(x^2+a^2)^{n+1}}\, dx$$

$$\therefore \quad I_n = \frac{x}{(x^2+a^2)^n} + 2n\int \frac{1}{(x^2+a^2)^n}\,dx - 2a^2 n\int \frac{dx}{(x^2+a^2)^{n+1}}$$

$$I_n = \frac{x}{(x^2+a^2)^n} + 2n\,I_n - 2n\,a^2\,I_{n+1}$$

$$\therefore \quad 2n\,a^2\,I_{n+1} = \frac{x}{(x^2+a^2)^n} + (2n-1)\,I_n$$

$$\text{or} \qquad I_{n+1} = \frac{1}{2n\,a^2}\cdot\frac{x}{(x^2+a^2)^n} + \frac{(2n-1)}{2n}\cdot\frac{1}{a^2}\,I_n$$

Remark

Above obtained formula reduces the calculations of the integral I_{n+1} to the calculations of the integral I_n and consequently, allows us to calculate completely an integral with natural index, as

$$I_1 = \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$\therefore$ From above formula

Let $n = 1$

$$I_2 = \int \frac{dx}{(x^2+a^2)^2} = \frac{1}{2a^2}\cdot\frac{x}{x^2+a^2} + \frac{1}{2a^2}\cdot I_1$$

$$= \frac{1}{2a^2}\cdot\frac{x}{x^2+a^2} + \frac{1}{2a^2}\cdot\frac{1}{a}\cdot\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$= \frac{1}{2a^2}\cdot\frac{x}{x^2+a^2} + \frac{1}{2a^3}\cdot\tan^{-1}\left(\frac{x}{a}\right) + C$$

Let $n = 2$

$$I_3 = \int \frac{dx}{(x^2+a^2)^3} = \frac{1}{4a^2}\cdot\frac{x}{(x^2+a^2)^2} + \frac{3}{4a^2}\,I_2$$

$$= \frac{1}{4a^2}\cdot\frac{x}{(x^2+a^2)^2} + \frac{3}{8a^4}\cdot\frac{x}{x^2+a^2} + \frac{3}{8a^5}\tan^{-1}\left(\frac{x}{a}\right) + C$$

...and so on.

Example 100 Derive reduction formula for $I_{(n,m)} = \int \frac{\sin^n x}{\cos^m x}\,dx$.

Sol. Using Integration by parts for $I_{(n,m)}$, we get

$$I_{(n,m)} = \int \underset{\text{I}}{\sin^{n-1} x}\;\underset{\text{II}}{\frac{\sin x}{\cos^m x}}\,dx$$

$$= \sin^{n-1} x\cdot\frac{(\cos x)^{-m+1}}{(m-1)} - \int (n-1)\sin^{n-2} x\cdot\cos x\cdot\frac{(\cos x)^{-m+1}}{(m-1)}\,dx$$

$$= \frac{1}{m-1}\cdot\frac{\sin^{n-1} x}{\cos^{m-1} x} - \frac{(n-1)}{(m-1)}\cdot\int\frac{\sin^{n-2} x}{\cos^{m-2} x}\,dx$$

$$I_{n,m} = \frac{1}{(m-1)}\cdot\frac{\sin^{n-1} x}{\cos^{m-1} x} - \frac{(n-1)}{(m-1)}\cdot I_{(n-2,\,m-2)}$$

is required reduction formula.

Integration Using Differentiation

In $\int\frac{dx}{(a+b\cos x)^2}$, $\int\frac{dx}{(a+b\sin x)^2}$, $\int\frac{dx}{(\sin x + a\sec x)^2}$, $\int\frac{a+b\sin x}{(b+a\sin x)^2}\,dx$,... we follow the following method.

1. Let $A = \frac{\sin x}{a+b\cos x}$ or $A = \frac{\cos x}{a+b\sin x}$ according to the integral to evaluated is of the form
$$\int\frac{dx}{(a+b\cos x)^2} \quad \text{or} \quad \int\frac{dx}{(a+b\sin x)^2}$$
2. Find $\frac{dA}{dx}$ and express it in terms of $\frac{1}{a+b\cos x}$ or $\frac{1}{a+b\sin x}$ as the case may be.
3. Integrate both the sides of the expression obtained in step 2 to obtain the value of the required integral.

Example 101 ***Evaluate*** $\int\frac{dx}{(5+4\cos x)^2}$.

Sol. Here, $A = \frac{\sin x}{5+4\cos x}$, then

$$\frac{dA}{dx} = \frac{(5+4\cos x)(\cos x) - \sin x(-4\sin x)}{(5+4\cos x)^2}$$

$$\Rightarrow \quad \frac{dA}{dx} = \frac{5\cos x + 4}{(5+4\cos x)^2} = \frac{\frac{5}{4}(4\cos x + 5) + 4 - \frac{25}{4}}{(5+4\cos x)^2}$$

$$\Rightarrow \quad \frac{dA}{dx} = \frac{5}{4}\cdot\frac{1}{(5+4\cos x)} - \frac{9}{4}\cdot\frac{1}{(5+4\cos x)^2}$$

Integrating both the sides w.r.t. 'x', we get

$$A = \frac{5}{4}\int\frac{dx}{5+4\cos x} - \frac{9}{4}\int\frac{dx}{(5+4\cos x)^2}$$

$$\Rightarrow \quad \frac{9}{4}\int\frac{dx}{(5+4\cos x)^2} = \frac{5}{4}\int\frac{dx}{5+4\cos x} - A$$

$$= \frac{5}{4}\int\frac{dx}{5+4\frac{(1-\tan^2 x/2)}{(1+\tan^2 x/2)}} - \frac{\sin x}{(5+4\cos x)}$$

$$\Rightarrow \int\frac{dx}{(5+4\cos x)^2} = \frac{5}{9}\int\frac{1+\tan^2 x/2}{9+\tan^2 x/2}\,dx - \frac{4}{9}\cdot\frac{\sin x}{5+4\cos x}$$

$$\Rightarrow \int\frac{dx}{(5+4\cos x)^2} = \frac{5}{9}\int\frac{2\,dt}{9+t^2} - \frac{4}{9}\cdot\frac{\sin x}{5+4\cos x}$$

(where $\tan x/2 = t$)

$$\Rightarrow \int \frac{dx}{(5+4\cos x)^2} = \frac{10}{9}\cdot\frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right) - \frac{4}{9}\cdot\frac{\sin x}{5+4\cos x}$$

$$\Rightarrow \int \frac{dx}{(5+4\cos x)^2} = \frac{10}{27}\tan^{-1}\left(\frac{\tan x/2}{3}\right) - \frac{4}{9}\left(\frac{\sin x}{5+4\cos x}\right) + C$$

Example 102 Evaluate $\int \frac{dx}{(16+9\sin x)^2}$.

Sol. Let $A = \frac{\cos x}{16+9\sin x}$ ...(i)

$$\Rightarrow \frac{dA}{dx} = \frac{(16+9\sin x)(-\sin x) - \cos x(9\cos x)}{(16+9\sin x)^2}$$

$$\Rightarrow \frac{dA}{dx} = \frac{-16\sin x - 9}{(16+9\sin x)^2}$$

$$\Rightarrow \frac{dA}{dx} = \frac{-\frac{16}{9}(9\sin x + 16) + \frac{256}{9} - 9}{(16+9\sin x)^2}$$

$$\Rightarrow \frac{dA}{dx} = -\frac{16}{9}\cdot\frac{1}{(16+9\sin x)} + \frac{175}{9(16+9\sin x)^2} \quad ...(ii)$$

Integrating both the sides of Eq. (ii) w.r.t. 'x', we get

$$A = -\frac{16}{9}\int \frac{dx}{16+9\sin x} + \frac{175}{9}\int \frac{dx}{(16+9\sin x)^2}$$

$$\Rightarrow \frac{175}{9}\int \frac{dx}{(16+9\sin x)^2} = A + \frac{16}{9}\int \frac{(1+\tan^2 x/2)\,dx}{16+16\tan^2 x/2 + 18\tan x/2}$$

$$\Rightarrow \frac{175}{9}\int \frac{dx}{(16+9\sin x)^2} = A + \frac{16}{9}\int \frac{2\,dt}{16t^2+18t+16}$$

[where $\tan x/2 = t$]

$$\Rightarrow \frac{175}{9}\int \frac{dx}{(16+9\sin x)^2} = A + \frac{2}{9}\int \frac{dt}{t^2 + \frac{9}{8}t + 1}$$

$$= A + \frac{2}{9}\int \frac{dt}{\left(t+\frac{9}{16}\right)^2 + \left(\frac{\sqrt{175}}{16}\right)^2}$$

$$= A + \frac{2}{9}\times\frac{16}{\sqrt{175}}\tan^{-1}\left(\frac{16t+9}{\sqrt{175}}\right)$$

$$\Rightarrow \int \frac{dx}{(16+9\sin x)^2} = \frac{9}{175}\cdot\frac{\cos x}{(16+9\sin x)} + \frac{2}{(175)^{3/2}}\tan^{-1}\left(\frac{16\tan x/2+9}{\sqrt{175}}\right) + C$$

Example 103 Evaluate $\int \frac{dx}{(\sin x + a\sec x)^2}$ when $|a| > 1/2$.

Sol. Here, $I = \int \frac{dx}{(\sin x + a\sec x)^2}$ or $I = \int \frac{\cos^2 x\,dx}{(\sin x\cos x + a)^2}$

$$= \int \frac{\cos^2 x\,dx}{a^2 + 2a\sin x\cos x + \sin^2 x\cos^2 x}$$

$$= \int \frac{\cos^2 x\,dx}{a^2 + a\sin 2x + \frac{1}{4}\sin^2 2x}$$

$$= \int \frac{4\cos^2 x\,dx}{(4a^2 + 4a\sin 2x + \sin^2 2x)} = 2\int \frac{(1+\cos 2x)\,dx}{(2a+\sin 2x)^2}$$

$$= 2\int \frac{dx}{(2a+\sin 2x)^2} + 2\int \frac{\cos 2x\,dx}{(2a+\sin 2x)^2}$$

$$\Rightarrow I = 2I_1 + \int \frac{dt}{t^2} \quad [\text{where } (2a+\sin 2x) = t, (2\cos 2x)\,dx = dt]$$

$$\Rightarrow I = 2I_1 - \frac{1}{(2a+\sin 2x)} + C$$

where $I_1 = \int \frac{dx}{(2a+\sin 2x)^2}$ ...(i)

Put $A = \frac{\cos 2x}{2a+\sin 2x}$

$$\Rightarrow \frac{dA}{dx} = \frac{(2a+\sin 2x)(-2\sin 2x) - \cos 2x(2\cos 2x)}{(2a+\sin 2x)^2}$$

$$\Rightarrow \frac{dA}{dx} = \frac{-4a\sin 2x - 2}{(2a+\sin 2x)^2}$$

$$\Rightarrow \frac{dA}{dx} = \frac{-4a(\sin 2x + 2a) - 2 + 8a^2}{(2a+\sin 2x)^2}$$

$$\Rightarrow \frac{dA}{dx} = -\frac{4a}{(2a+\sin 2x)} + \frac{(8a^2-2)}{(2a+\sin 2x)^2}$$

Integrating both the sides w.r.t. 'x', we get

$$\Rightarrow A = -4a\int \frac{dx}{(2a+\sin 2x)} + (8a^2-2)\,I_1$$

$$\Rightarrow (8a^2-2)\,I_1 = A + 4a\int \frac{\sec^2 x\,dx}{2a + 2\tan x + 2a\tan^2 x}$$

$$= A + \frac{4a}{2a}\int \frac{dt}{t^2 + \frac{t}{a} + 1}$$

$$= A + 2\int \frac{dt}{\left(t+\frac{1}{2a}\right)^2 + \left(1 - \frac{1}{4a^2}\right)}$$

$$= A + 2\frac{(2a)}{\sqrt{4a^2-1}}\tan^{-1}\left(\frac{(2at+1)}{\sqrt{4a^2-1}}\right)$$

$$\Rightarrow (8a^2-2)\,I_1 = \frac{\cos 2x}{2a+\sin 2x} + \frac{4a}{\sqrt{4a^2-1}}\tan^{-1}\left(\frac{(2a\tan x+1)}{\sqrt{4a^2-1}}\right) \quad ...(ii)$$

From Eqs. (i) and (ii)

$$I = \frac{1}{(4a^2-1)}\cdot\frac{\cos 2x}{(2a+\sin 2x)} + \frac{4a}{(4a^2-1)^{3/2}}\tan^{-1}\left(\frac{2a\tan x+1}{\sqrt{4a^2-1}}\right) - \frac{1}{(2a+\sin 2x)} + C$$

JEE Type Solved Examples :
Single Option Correct Type Questions

● **Ex. 1** *The value of* $\int \frac{dx}{\cos^6 x + \sin^6 x}$, *is equal to*

(a) $\tan^{-1}(2\cot 2x) + C$ (b) $\tan^{-1}(\cot 2x) + C$

(c) $\tan^{-1}\left(\frac{1}{2}\cot 2x\right) + C$ (d) $\tan^{-1}(-2\cot 2x) + C$

Sol. Let $I = \int \frac{dx}{\cos^6 x + \sin^6 x} = \int \frac{\sec^6 x\, dx}{1 + \tan^6 x}$

$$= \int \frac{(1 + \tan^2 x)^2 \cdot \sec^2 x}{1 + \tan^6 x}\, dx$$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt = \int \frac{(1 + t^2)^2}{1 + t^6}\, dt$

$$= \int \frac{(1 + t^2)^2}{(1 + t^2)(1 - t^2 + t^4)}\, dt$$

$$= \int \frac{1 + t^2}{1 - t^2 + t^4}\, dt = \int \frac{(1 + 1/t^2)\, dt}{(1/t^2 - 1 + t^2)} = \int \frac{\left(1 + \frac{1}{t^2}\right) dt}{(t - 1/t)^2 + 1},$$

Put $t - \frac{1}{t} = z$

$\therefore \left(1 + \frac{1}{t^2}\right) dt = dz = \int \frac{dz}{z^2 + 1} = \tan^{-1}(z) + C$

$$= \tan^{-1}\left(\frac{t^2 - 1}{t}\right) + C = \tan^{-1}\left(\frac{\tan^2 x - 1}{\tan x}\right) + C$$

$$= \tan^{-1}(-2\cot 2x) + C$$

Hence, (d) is the correct answer.

● **Ex. 2** $\int \frac{e^{\tan^{-1} x}}{(1 + x^2)}\left[(\sec^{-1}\sqrt{1 + x^2})^2 + \cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right)\right] dx,$ *$(x > 0)$ is equal to*

(a) $e^{\tan^{-1} x} \cdot \tan^{-1} x + C$

(b) $\frac{e^{\tan^{-1} x} \cdot (\tan^{-1} x)^2}{2} + C$

(c) $e^{\tan^{-1} x} \cdot (\sec^{-1}(\sqrt{1 + x^2}))^2 + C$

(d) $e^{\tan^{-1} x} \cdot (\operatorname{cosec}^{-1}(\sqrt{1 + x^2}))^2 + C$

Sol. Note that $\sec^{-1}\sqrt{1 + x^2} = \tan^{-1} x$; $\cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right) = 2\tan^{-1} x$,

For $x > 0$

$$\Rightarrow \quad I = \int \frac{e^{\tan^{-1} x}}{1 + x^2}\{(\tan^{-1} x)^2 + 2\tan^{-1} x\}\, dx,$$

Put $\tan^{-1} x = t$

$$= \int e^t (t^2 + 2t)\, dt = e^t \cdot t^2 = e^{\tan^{-1} x}(\tan^{-1} x)^2 + C$$

Hence, (c) is the correct answer.

● **Ex. 3** *Let* $I = \int \frac{e^x}{e^{4x} + e^{2x} + 1}\, dx$, $J = \int \frac{e^{-x}}{e^{-4x} + e^{-2x} + 1}\, dx.$ *Then, for an arbitrary constant c, the value of* $J - I$ *equals to* **[IIT JEE 2008]**

(a) $\frac{1}{2}\log\left(\frac{e^{4x} - e^{2x} + 1}{e^{4x} + e^{2x} + 1}\right) + C$ (b) $\frac{1}{2}\log\left(\frac{e^{2x} + e^x + 1}{e^{2x} - e^x + 1}\right) + C$

(c) $\frac{1}{2}\log\left(\frac{e^{2x} - e^x + 1}{e^{2x} + e^x + 1}\right) + C$ (d) $\frac{1}{2}\log\left(\frac{e^{4x} + e^{2x} + 1}{e^{4x} - e^{2x} + 1}\right) + C$

Sol. $J = \int \frac{e^{3x}}{1 + e^{2x} + e^{4x}}\, dx$

$$J - I = \int \frac{(e^{3x} - e^x)}{1 + e^{2x} + e^{4x}}\, dx = \int \frac{(u^2 - 1)}{1 + u^2 + u^4}\, du \qquad (u = e^x)$$

$$= \int \frac{\left(1 - \frac{1}{u^2}\right) du}{1 + \frac{1}{u^2} + u^2} = \int \frac{\left(1 - \frac{1}{u^2}\right) du}{\left(u + \frac{1}{u}\right)^2 - 1} = \int \frac{dt}{t^2 - 1} \qquad \left(t = u + \frac{1}{u}\right)$$

$$= \frac{1}{2}\log\left|\frac{t - 1}{t + 1}\right| + C = \frac{1}{2}\log\left|\frac{u^2 - u + 1}{u^2 + u + 1}\right| + C$$

$$= \frac{1}{2}\log\left|\frac{e^{2x} - e^x + 1}{e^{2x} + e^x + 1}\right| + C$$

Hence, (c) is the correct answer.

● **Ex. 4** *Integral of* $\sqrt{1 + 2\cot x(\cot x + \operatorname{cosec} x)}$ *w.r.t. x, is*

(a) $2\ln\cos\frac{x}{2} + C$ (b) $2\ln\sin\frac{x}{2} + C$

(c) $\frac{1}{2}\ln\cos\frac{x}{2} + C$ (d) $\ln\sin x - \ln(\operatorname{cosec} x - \cot x) + C$

Sol. $I = \int \sqrt{1 + 2\operatorname{cosec} x \cot x + 2\cot^2 x}\, dx$

$$= \int \sqrt{\operatorname{cosec}^2 x + 2\operatorname{cosec} x \cot x + \cot^2 x}\, dx$$

$$= \int (\operatorname{cosec} x + \cot x)\, dx$$

$$= \int \frac{1 + \cos x}{\sin x}\, dx = \int \cot\left(\frac{x}{2}\right) dx = 2\log\left|\sin\frac{x}{2}\right| + C$$

Hence, (b) is the correct answer.

● **Ex. 5** *If* $I_n = \int \cot^n x\, dx$, *then* $I_0 + I_1 + 2(I_2 + I_3 + \ldots + I_8) + I_9 + I_{10}$ *equals to (where* $u = \cot x$*)*

(a) $u + \frac{u^2}{2} + \ldots + \frac{u^9}{9}$ (b) $-\left(u + \frac{u^2}{2} + \ldots + \frac{u^9}{9}\right)$

(c) $-\left(u + \frac{u^2}{2!} + \ldots + \frac{u^9}{9!}\right)$ (d) $\frac{u}{2} + \frac{2u^2}{3} + \ldots + \frac{9u^9}{10}$

Sol. $I_n = \int \cot^n x\, dx = \int \cot^{n-2} x \cdot (\operatorname{cosec}^2 x - 1)\, dx$

$$I_n = -\frac{u^{n-1}}{n-1} - I_{n-2} \quad \text{or} \quad I_n + I_{n-2} = -\frac{u^{n-1}}{n-1} \quad [\text{put } n = 2, 3, 4, \ldots, 10]$$

$$I_2 + I_0 = -\frac{u}{1}$$

$$I_3 + I_1 = -\frac{u^2}{2}$$

$$I_4 + I_2 = -\frac{u^3}{3}$$

.....

.....

$$I_{10} + I_9 = -\frac{u^9}{9}$$

Adding, $I_0 + I_1 + 2(I_2 + I_3 + \ldots + I_8) + I_9 + I_{10}$

$$= -\left(u + \frac{u^2}{2} + \ldots + \frac{u^9}{9}\right)$$

Hence, (b) is the correct answer.

● ***Ex. 6*** *Let $f(x) = x + \sin x$. Suppose g denotes the inverse function of f. The value of $g'\left(\frac{\pi}{4} + \frac{1}{\sqrt{2}}\right)$ has the value equal to*

(a) $\sqrt{2} - 1$ (b) $\dfrac{\sqrt{2}+1}{\sqrt{2}}$

(c) $2 - \sqrt{2}$ (d) $\sqrt{2} + 1$

Sol. $f(x) = y = x + \sin x$

$$\Rightarrow \quad \frac{dy}{dx} = 1 + \cos x$$

$$g'(y) = \frac{dx}{dy} = \frac{1}{1 + \cos x}$$

where $y = \dfrac{\pi}{4} + \dfrac{1}{\sqrt{2}} = x + \sin x \;\Rightarrow\; x = \dfrac{\pi}{4}$

$$\therefore \quad g'\left(\frac{\pi}{4} + \frac{1}{\sqrt{2}}\right) = \frac{1}{1 + (1/\sqrt{2})}$$

$$= \frac{\sqrt{2}}{\sqrt{2}+1} = \sqrt{2}(\sqrt{2} - 1) = 2 - \sqrt{2}$$

Hence, (c) is the correct answer.

● ***Ex. 7*** *The value of $\int \dfrac{dx}{\sqrt{(x-a)(b-x)}}$, is*

(a) $2\sin^{-1}\sqrt{\dfrac{x-a}{b-a}} + C$ (b) $2\sin^{-1}\sqrt{\dfrac{x-b}{b-a}} + C$

(c) $\sin^{-1}\sqrt{\dfrac{x-a}{b-a}} + C$ (d) None of these

Sol. Let $x = a\cos^2\theta + b\sin^2\theta$ in the given integral.

So that, $dx = a(2\cos\theta)(-\sin\theta) + b(2\sin\theta)(\cos\theta)\, d\theta$

$$dx = 2(b-a)\sin\theta\cos\theta\, d\theta$$

$$\therefore \quad I = \int \frac{2(b-a)\sin\theta\cos\theta\, d\theta}{\sqrt{(a\cos^2\theta + b\sin^2\theta - a)(b - a\cos^2\theta - b\sin^2\theta)}}$$

$$= 2(b-a)\int \frac{\sin\theta\cos\theta\, d\theta}{\sqrt{(b\sin^2\theta - a\sin^2\theta)(b\cos^2\theta - a\cos^2\theta)}}$$

$$= 2(b-a)\int \frac{\sin\theta\cos\theta\, d\theta}{(b-a)\sin\theta\cos\theta} = 2\int 1\, d\theta$$

$$= 2\theta + C = 2\sin^{-1}\sqrt{\frac{x-a}{b-a}} + C$$

Hence, (a) is the correct answer.

● ***Ex. 8*** *The value of $\int \dfrac{(x-1)}{(x+1)\sqrt{x^3+x^2+x}}\, dx$, is*

(a) $2\tan^{-1}\sqrt{\dfrac{x+1}{x}} + C$ (b) $\tan^{-1}\sqrt{\dfrac{x^2+x+1}{x}} + C$

(c) $2\tan^{-1}\sqrt{\dfrac{x^2+x+1}{x}} + C$ (d) None of these

Sol. Let $I = \int \dfrac{(x-1)}{(x+1)\sqrt{x^3+x^2+x}}\, dx$

$$= \int \frac{(x^2-1)}{(x+1)^2\sqrt{x^3+x^2+x}}\, dx$$

$$= \int \frac{x^2(1 - 1/x^2)}{(x^2+2x+1)\sqrt{x^3+x^2+x}}\, dx$$

$$= \int \frac{x^2(1 - 1/x^2)}{x(x+2+1/x)\cdot x\sqrt{x+1+1/x}}\, dx$$

Put $x + \dfrac{1}{x} = t$,

$\Rightarrow (1 - 1/x^2)\, dx = dt$

$$= \int \frac{dt}{(t+2)\sqrt{t+1}}, \text{ which reduces to } \int \frac{dx}{P\sqrt{Q}}.$$

Let $t + 1 = z^2$

$$\therefore \quad dt = 2z\, dz = \int \frac{2z\, dz}{(z^2+1)\sqrt{z^2}}$$

$$= 2\int \frac{dz}{z^2+1} = 2\tan^{-1}(z) + C$$

$$= 2\tan^{-1}(\sqrt{t+1}) + C = 2\tan^{-1}\sqrt{\frac{x^2+x+1}{x}} + C$$

Hence, (c) is the correct answer.

● ***Ex. 9*** *The value of $\int \dfrac{(1+x^2)\, dx}{(1-x^2)\sqrt{1+x^2+x^4}}$, is*

(a) $-\dfrac{1}{2\sqrt{3}}\log\left|\dfrac{\sqrt{x^4+x^2+1} - \sqrt{3}x}{\sqrt{x^4+x^2+1} + \sqrt{3}x}\right| + C$

(b) $\dfrac{1}{2\sqrt{3}}\log\left|\dfrac{\sqrt{x^4+x^2+1} + \sqrt{2}x}{\sqrt{x^4+x^2+1} - \sqrt{2}x}\right| + C$

(c) $\dfrac{1}{2\sqrt{3}}\log\left|\dfrac{\sqrt{x^4-x^2+1} - \sqrt{3}x}{\sqrt{x^4+x^2+1} + \sqrt{3}x}\right| + C$

(d) None of the above

Sol. Let $I = \int \frac{(1+x^2)\,dx}{(1-x^2)\sqrt{1+x^2+x^4}}$

$$= \int \frac{x^2\left(1+\frac{1}{x^2}\right)dx}{x\left(\frac{1}{x}-x\right)x\sqrt{\frac{1}{x^2}+1+x^2}}$$

$$= -\int \frac{(1+1/x^2)\,dx}{(1-1/x)\sqrt{(x-1/x)^2+3}}$$

Put $x - \frac{1}{x} = t = \left(1+\frac{1}{x^2}\right)dx = dt \Rightarrow -\int \frac{dt}{t\sqrt{t^2+3}}$

Again, put $t^2 + 3 = s^2$

$\Rightarrow \quad 2t\,dt = 2s\,ds = -\int \frac{s\,ds}{s(s^2-3)} = -\int \frac{ds}{s^2-(\sqrt{3})^2}$

$$= -\frac{1}{2\sqrt{3}}\log\left|\frac{s-\sqrt{3}}{s+\sqrt{3}}\right| + C$$

$$= -\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{t^2+3}-\sqrt{3}}{\sqrt{t^2+3}+\sqrt{3}}\right| + C$$

$$= -\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{(x-1/x)^2+3}-\sqrt{3}}{\sqrt{(x-1/x)^2+3}+\sqrt{3}}\right| + C$$

$$= -\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{x^2+\frac{1}{x^2}+1}-\sqrt{3}}{\sqrt{x^2+\frac{1}{x^2}+1}+\sqrt{3}}\right| + C$$

$$= -\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{x^4+x^2+1}-\sqrt{3}x}{\sqrt{x^4+x^2+1}+\sqrt{3}x}\right| + C$$

Hence (a) is the correct answer.

● **Ex. 10** *The value of* $I = \int \frac{dx}{(a+bx^2)\sqrt{b-ax^2}}$, *is*

(a) $\frac{1}{\sqrt{a(a^2+b^2)}}\tan^{-1}\left(\frac{x\sqrt{a^2+b^2}}{a\sqrt{b-ax^2}}\right)+C$

(b) $\frac{1}{\sqrt{(a^2+b^2)}}\tan^{-1}\left(\frac{x\sqrt{a^2+b^2}}{a\sqrt{b-ax^2}}\right)+C$

(c) $\frac{1}{\sqrt{a(a^2+b^2)}}\tan^{-1}\left(\frac{x\sqrt{a^2+b^2}}{a}\right)+C$

(d) None of the above

Sol. Substituting $ax^2 = b\sin^2\theta$

$\Rightarrow \quad dx = \sqrt{\frac{b}{a}}\cos\theta\,d\theta$

$$\therefore \quad I = \int \frac{\sqrt{\frac{b}{a}}\cos\theta\,d\theta}{\left(a+\frac{b^2}{a}\sin^2\theta\right)\sqrt{b-b\sin^2\theta}}$$

$$= \sqrt{a}\int \frac{\cos\theta\,d\theta}{(a^2+b^2\sin^2\theta)\cdot\cos\theta}$$

$$= \sqrt{a}\int \frac{d\theta}{a^2+b^2\sin^2\theta},$$

dividing numerator and denominator by $\cos^2\theta$, we get

$$= \sqrt{a}\int \frac{\sec^2\theta\,d\theta}{a^2\sec^2\theta + b^2\tan^2\theta}, \text{ put } \tan\theta = t$$

$$= \sqrt{a}\int \frac{dt}{a^2(1+t^2)+b^2t^2} = \sqrt{a}\int \frac{dt}{(a^2+b^2)t^2+a^2}$$

$$= \frac{\sqrt{a}}{a^2+b^2}\int \frac{dt}{t^2+\frac{a^2}{a^2+b^2}}$$

$$= \frac{\sqrt{a}}{a^2+b^2}\cdot\left(\frac{\sqrt{a^2+b^2}}{a}\right)\tan^{-1}\left(\frac{t\sqrt{a^2+b^2}}{a}\right)+C$$

$$= \frac{1}{\sqrt{a(a^2+b^2)}}\cdot\tan^{-1}\left(\frac{x\sqrt{a^2+b^2}}{a\sqrt{b-ax^2}}\right)+C \qquad \left[\because\ t=\tan\theta=\frac{x}{\sqrt{b-ax^2}}\right]$$

Hence (a) is the correct answer.

● **Ex. 11** *The value of* $I = \int \frac{dx}{2x\sqrt{1-x}\sqrt{(2-x)+\sqrt{1-x}}}$

$$= -\frac{1}{2}\left\{\log\left(z+\frac{3}{2}+\sqrt{z^2+3z+3}\right)\right\}+\frac{1}{2}\left|\log s-\frac{1}{2}+\sqrt{s^2-s+1}\right|+C$$

and $s - z = \frac{k}{x}$, *then value of k, is*

(a) 1 (b) 2 (c) 3 (d) 4

Sol. Here, $I = \int \frac{dx}{2x\sqrt{1-x}\sqrt{(2-x)+\sqrt{1-x}}}$,

put $(1-x) = t^2 - dx = 2t\,dt$

$$= -\int \frac{2t\,dt}{2(1-t^2)\cdot t\sqrt{1+t^2+t}}$$

$$= -\int \frac{dt}{(1-t^2)\sqrt{t^2+t+1}}$$

$$= \int \frac{dt}{(t-1)(t+1)\sqrt{t^2+t+1}} = \frac{1}{2}\int\left(\frac{1}{t-1}-\frac{1}{t+1}\right)\cdot\frac{dt}{\sqrt{t^2+t+1}}$$

$\because \ \frac{1}{(t-1)(t+1)} = \frac{1}{2}\left(\frac{1}{t-1}-\frac{1}{t+1}\right)$

$$= \frac{1}{2}\int \frac{1}{(t-1)\sqrt{t^2+t+1}}\,dt - \frac{1}{2}\int \frac{1}{(t+1)\sqrt{t^2+t+1}}\,dt$$

Let $I = \frac{1}{2}I_1 - \frac{1}{2}I_2$ ...(i)

where, $I_1 = \int \frac{dt}{(t-1)\sqrt{t^2+t+1}}$

and $I_2 = \int \frac{dt}{(t+1)\sqrt{t^2+t+1}}$

For I_1, put $(t-1) = \frac{1}{z} \Rightarrow dt = -\frac{1}{z^2}\,dz$

$$I_1 = \int \frac{-1/z^2\,dz}{\frac{1}{z}\sqrt{\left(1+\frac{1}{z}\right)^2+\left(1+\frac{1}{z}\right)+1}} = -\int \frac{dz}{\sqrt{\left(z+\frac{3}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}$$

$$= -\log\left|\left(z+\frac{3}{2}\right)+\sqrt{z^2+3z+3}\right| \quad ...(ii)$$

For I_2, put $(t+1)=\frac{1}{S}$

$\Rightarrow \quad dt = -\frac{1}{s^2}ds$

$$I_2 = -\int \frac{ds}{\sqrt{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}}$$

$$= -\log\left|\left(s-\frac{1}{2}\right)+\sqrt{s^2-s+1}\right| \quad ...(iii)$$

$$\therefore \quad I = -\frac{1}{2}\left\{\log\left(z+\frac{3}{2}+\sqrt{z^2+3z+3}\right)\right\} + \frac{1}{2}\log\left|\left(s-\frac{1}{2}\right)+\sqrt{s^2-s+1}\right| + C$$

where, $z=\frac{1}{\sqrt{1-x}-1}$ and $s=\frac{1}{\sqrt{1-x}+1}$

$$\therefore \quad s-z=\frac{1}{\sqrt{1-x}+1}-\frac{1}{\sqrt{1-x}-1}=\frac{2}{x}\Rightarrow k=2$$

Hence (b) is the correct answer.

● ***Ex. 12*** *If* $\int \frac{dx}{(x^2+a^2)^2} = \frac{1}{ka^2}\left\{\frac{x}{x^2+a^2}+\frac{1}{a}\tan^{-1}\frac{x}{a}\right\}+C$. *Then the value of k, is*

(a) 1 (b) 2 (c) 3 (d) 4

Sol. Here, we know

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}+C \quad ...(i)$$

Also, $$\int \frac{1}{x^2+a^2}\cdot 1\,dx = \frac{1}{x^2+a^2}x - \int \frac{-2x}{(x^2+a^2)^2}x\,dx$$

$$= \frac{x}{x^2+a^2}+2\int \frac{x^2+a^2-a^2}{(x^2+a^2)^2}dx$$

I II

$$\int \frac{dx}{x^2+a^2} = \frac{x}{x^2+a^2}+2\int \frac{dx}{x^2+a^2}-2a^2\int \frac{dx}{(x^2+a^2)^2} \quad ...(ii)$$

From Eqs. (i) and (ii), we get

$$\frac{1}{a}\tan^{-1}\frac{x}{a} = \frac{x}{x^2+a^2}+2\frac{1}{a}\tan^{-1}\frac{x}{a}-2a^2\int \frac{dx}{(x^2+a^2)^2}$$

$$\Rightarrow \quad 2a^2\int \frac{dx}{(x^2+a^2)^2} = \frac{x}{x^2+a^2}+\frac{1}{a}\tan^{-1}\frac{x}{a}$$

or $$\int \frac{dx}{(x^2+a^2)^2} = \frac{1}{2a^2}\left\{\frac{x}{x^2+a^2}+\frac{1}{a}\tan^{-1}\frac{x}{a}\right\}+C$$

$$= \frac{1}{ka^2}\left\{\frac{x}{x^2+a^2}+\frac{1}{a}\tan^{-1}\frac{x}{a}\right\}+C$$

$$\therefore \quad k=2$$

Hence, (b) is the correct answer.

● ***Ex. 13*** *If* $\int \frac{dx}{(x^2+a^2)^3} = \frac{x}{4a^2(x^2+a^2)}+\frac{m}{na^2}\left\{\frac{x}{2a^2(x^2+a^2)}+\frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right)\right\}+C$. *Then* $|m-n|$ *is equal to*

(a) 4 (b) 3 (c) 2 (d) 1

Sol. Let $$I = \int \frac{dx}{(x^2+a^2)^3} \quad ...(i)$$

and $$I_1 = \int \frac{1}{(x^2+a^2)^2}dx \quad ...(ii)$$

$$= \int \frac{1}{(x^2+a^2)^2}\cdot 1\,dx = \frac{1}{(x^2+a^2)^2}\cdot x - \int \frac{-2(2x)}{(x^2+a^2)^3}x\,dx$$

I II

$$= \frac{x}{(x^2+a^2)^2}+4\int \frac{x^2+a^2-a^2}{(x^2+a^2)^3}dx$$

$$= \frac{x}{(x^2+a^2)^2}+4\int \frac{1}{(x^2+a^2)^2}dx - 4a^2\int \frac{dx}{(x^2+a^2)^3}$$

$\Rightarrow \quad I_1 = \frac{x}{(x^2+a^2)^2}+4I_1-4a^2\cdot I$ [using Eqs. (i) and (ii)]

$\Rightarrow 4a^2 I = \frac{x}{(x^2+a^2)^2}+3I_1$

$\Rightarrow \quad I = \frac{x}{4a^2(x^2+a^2)^2}+\frac{3}{4a^2}I_1 \quad ...(iii)$

[using previous example,

$$I_1 = \int \frac{dx}{(x^2+a^2)^2} = \frac{x}{2a^2(x^2+a^2)}+\frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right)+C\Bigg]$$

$$\Rightarrow \quad I = \frac{x}{4a^2(x^2+a^2)^2}+\frac{3}{4a^2}\left\{\frac{x}{2a^2(x^2+a^2)}+\frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right)\right\}+C \quad ...(iv)$$

$m=3$ and $n=4$

$|m-n| = |3-4| = |-1| = 1$

Hence, (d) is the correct answer.

● ***Ex. 14*** *If* $y(x-y)^2 = x$, *then* $\int \frac{dx}{(x-3y)} = \frac{m}{n}\ln[(x-y)^2-1]$. *Then* $(m+2n)$ *is equal to*

(a) 1 (b) 3 (c) 5 (d) 7

Sol. Let $$P = \int \frac{dx}{(x-3y)} = \frac{1}{2}\ln[(x-y)^2-1]$$

$$\therefore \quad \frac{dP}{dx} = \frac{1}{x-3y} = \frac{(x-y)\left\{1-\frac{dy}{dx}\right\}}{\{(x-y)^2-1\}} \quad ...(i)$$

Given, $y(x-y)^2 = x$, on differentiating both the sides, we get

$$\frac{dy}{dx} = \frac{1-2y(x-y)}{(x-y)(x-3y)} \quad ...(ii)$$

$$\therefore \quad \frac{dP}{dx} = \frac{(x-y)\left\{1-\frac{1-2y(x-y)}{(x-y)(x-3y)}\right\}}{\{(x-y)^2-1\}}$$

$$= \frac{(x-y)(x-3y)-1+2y(x-y)}{(x-3y)\{(x-y)^2-1\}} = \frac{(x-y)^2-1}{(x-3y)\{(x-y)^2-1\}}$$

$$\therefore \frac{dP}{dx} = \frac{1}{(x-3y)} \quad \ldots(iii)$$

which is true as given

$$\therefore \int \frac{dx}{(x-3y)} = \frac{1}{2}\log\{(x-y)^2-1\},$$

$$\therefore \quad m = 1, n = 2$$

$$\Rightarrow \quad m + 2n = 5$$

Hence, (c) is the correct answer.

● **Ex. 15** *If* $\int (x+\sqrt{1+x^2})^n\, dx.$

$$= \frac{1}{a(n+1)}\{x+\sqrt{1+x^2}\}^{n+1} + \frac{1}{-b(n-1)}\{x+\sqrt{1+x^2}\}^{n-1} + C$$

Then $(a+b)$ *is equal to*

(a) 2 (b) 3 (c) 4 (d) 5

Sol. Let $I = \int (x+\sqrt{1+x^2})^n\, dx$

Put $$x + \sqrt{1+x^2} = t \quad \ldots(i)$$

$$\Rightarrow \quad \left(1 + \frac{1}{2\sqrt{1+x^2}} \cdot 2x\right) dx = dt$$

$$\Rightarrow \quad \left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right) dx = dt \quad \ldots(ii)$$

We know, $t = x + \sqrt{1+x^2} = x + \sqrt{1+x^2} \times \dfrac{x-\sqrt{1+x^2}}{x-\sqrt{1+x^2}}$

$$t = \frac{-1}{x-\sqrt{1+x^2}} \Rightarrow t = x + \sqrt{1+x^2}$$

$$-\frac{1}{t} = x - \sqrt{1+x^2}$$

Subtracting, we get

$$2\sqrt{1+x^2} = t + \frac{1}{t} \quad \text{or} \quad \frac{1}{\sqrt{1+x^2}} = \frac{2t}{t^2+1} \quad \ldots(iii)$$

From Eqs. (i), (ii) and (iii), we get

$$dx = \frac{t^2+1}{2t^2}\, dt$$

$$\therefore \quad I = \int t^n \cdot \frac{t^2+1}{(2t^2)}\, dt = \frac{1}{2}\int (t^n + t^{n-2})\, dt$$

$$= \frac{1}{2}\left[\frac{t^{n+1}}{n+1} + \frac{t^{n-1}}{n-1}\right] + C$$

$$\Rightarrow \quad I = \frac{1}{2(n+1)}[x+\sqrt{(1+x^2)}]^{n+1} + \frac{1}{2(n-1)}(x+\sqrt{(1+x^2)})^{n-1} + C \quad \ldots(iv)$$

Then comparing the values of a and b by Eq. (iv) $a = 2, b = 2$

$$\therefore \quad (a+b) = (2+2) = 4$$

Hence, (c) is the correct answer.

● **Ex. 16** *If* $\int \frac{f(x)}{x^3-1}\, dx$, *where* $f(x)$ *is a polynomial of degree 2 in x such that* $f(0) = f(1) = 3f(2) = -3$ *and* $\int \frac{f(x)}{x^3-1}\, dx = -\log|x-1| + \log|x^2+x+1|$

$$+ \frac{m}{\sqrt{n}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C.$$ *Then* $(2m+n)$ *is*

(a) 3 (b) 5 (c) 7 (d) 9

Sol. Let $f(x) = ax^2 + bx + c$

Given, $f(0) = f(1) = 3f(2) = -3$

$$\therefore \quad f(0) = c = -3$$

$$f(1) = a + b + c = -3$$

$$3f(2) = 3(4a + 2b + c) = -3$$

On solving, we get $a = 1, b = -1, c = -3$

$$\therefore \quad f(x) = x^2 - x - 3$$

$$\Rightarrow \quad I = \int \frac{f(x)}{x^3-1}\, dx = \int \frac{x^2-x-3}{(x-1)(x^2+x+1)}\, dx$$

Using partial fractions, we get

$$\frac{(x^2-x-3)}{(x-1)(x^2+x+1)} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+x+1)}$$

we get, $A = -1, \quad B = 2, \quad C = 2$

$$\therefore I = \int -\frac{1}{x-1}\, dx + \int \frac{(2x+2)}{(x^2+x+1)}\, dx$$

$$= -\log|x-1| + \int \frac{(2x+1)\, dx}{x^2+x+1} + \int \frac{1\, dx}{x^2+x+1}$$

$$= -\log|x-1| + \log|x^2+x+1| + \int \frac{dx}{(x+1/2)^2 + (\sqrt{3}/2)^2}$$

$$= -\log|x-1| + \log|x^2+x+1| + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

$\therefore$ On comparing $m = 2, n = 3 \Rightarrow 2m + n = 7.$

Hence, (c) is the correct answer.

● **Ex. 17** *The value of* $\int \frac{(1+x)}{x(1+xe^x)^2}\, dx$, *is equal to*

(a) $\log\left|\dfrac{x}{1+xe^x}\right| + \dfrac{1}{(1+xe^x)} + C$

(b) $\log\left|\dfrac{xe^x}{1+xe^x}\right| + \dfrac{1}{1+xe^x} + C$

(c) $\log\left|\dfrac{xe^x}{1+e^x}\right| + \dfrac{1}{1+xe^x} + C$

(d) None of the above

Sol. Let $I = \int \frac{(1+x)}{x(1+xe^x)^2}\, dx = \int \frac{(1+x)\, e^x}{(xe^x)(1+xe^x)^2}\, dx,$

put $1 + xe^x = t$

$$\therefore \quad (1+x)\, e^x\, dx = dt = \int \frac{dt}{(t-1)\cdot t^2}, \text{ applying partial fraction,}$$

we get $\dfrac{1}{(t-1)\, t^2} = \dfrac{A}{t-1} + \dfrac{B}{t} + \dfrac{C}{t^2}$

$\Rightarrow \quad 1 = A(t^2) + Bt(t-1) + C(t-1)$

For $t = 1 \Rightarrow A = 1$

For $t = 0 \Rightarrow C = -1$ and $B = -1$

$$\therefore \quad I = \int \left\{ \frac{1}{t-1} - \frac{1}{t} - \frac{1}{t^2} \right\} dt = \log|t-1| - \log|t| + \frac{1}{t} + C$$

$$= \log|xe^x| - \log|1 + xe^x| + \frac{1}{1+xe^x} + C$$

$$= \log\left|\frac{xe^x}{1+xe^x}\right| + \frac{1}{1+xe^x} + C$$

Hence, (b) is the correct answer.

● ***Ex. 18*** *The value of* $\int \frac{dx}{x + \sqrt{a^2 - x^2}}$, *is equal to*

(a) $\frac{1}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}\log|x + \sqrt{a^2 - x^2}| + C_1$

(b) $\frac{1}{2}\sin^{-1}\left(\frac{x}{a}\right) - \frac{1}{2}\log|x + \sqrt{a^2 - x^2}| + C_1$

(c) $\frac{1}{2}\sin^{-1}\left(\frac{x}{a}\right) - \log|x + \sqrt{a^2 - x^2}| + C_1$

(d) $\frac{1}{2}\cos^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}\log|x + \sqrt{a^2 - x^2}| + C_1$

Sol. Let $I = \int \frac{dx}{x + \sqrt{a^2 - x^2}}$, Put $x = a\sin\theta$

$$\therefore \quad dx = a\cos\theta\, d\theta = \int \frac{a\cos\theta\, d\theta}{a\sin\theta + \sqrt{a^2 - a^2\sin^2\theta}}$$

$$= \int \frac{\cos\theta\, d\theta}{\sin\theta + \cos\theta} = \frac{1}{2}\int \frac{\cos\theta + \sin\theta + \cos\theta - \sin\theta}{\sin\theta + \cos\theta} d\theta$$

$$= \frac{1}{2}\int 1\, d\theta + \frac{1}{2}\int \frac{\cos\theta - \sin\theta}{\sin\theta + \cos\theta} d\theta$$

$$= \frac{1}{2}\cdot\theta + \frac{1}{2}\log(\sin\theta + \cos\theta) + C$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}\log\left|\frac{x}{a} + \sqrt{1 - \frac{x^2}{a^2}}\right| + C$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}\{\log|x + \sqrt{a^2 - x^2}|\} - \frac{1}{2}\log a + C$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}\log|x + \sqrt{a^2 - x^2}| + C_1$$

$$\left[\text{where } C_1 = C - \frac{1}{2}\log a\right]$$

Hence, (a) is the correct answer.

● ***Ex. 19*** *The value of* $\int \frac{x^2 - 1}{(x^2+1)\sqrt{x^4+1}} dx$, *is equal to*

(a) $\frac{1}{\sqrt{2}}\sec^{-1}\left(\frac{x^2+1}{\sqrt{2}x}\right) + C$ (b) $\sqrt{2}\sec^{-1}\left(\frac{x^2+1}{\sqrt{2}x}\right) + C$

(c) $\frac{1}{\sqrt{2}}\operatorname{cosec}^{-1}\left(\frac{x^2+1}{\sqrt{2}x}\right) + C$ (d) $\sqrt{2}\operatorname{cosec}^{-1}\left(\frac{x^2+1}{\sqrt{2}x}\right) + C$

Sol. Let $I = \int \frac{(x^2-1)}{(x^2+1)\sqrt{x^4+1}} dx = \int \frac{x^2(1 - 1/x^2)\, dx}{x^2\left(x + \frac{1}{x}\right)\sqrt{x^2 + \frac{1}{x^2}}}$

$$= \int \frac{(1 - 1/x^2)\, dx}{\left(x + \frac{1}{x}\right)\sqrt{\left(x + \frac{1}{x}\right)^2 - 2}}$$

Put $x + \frac{1}{x} = t \quad \Rightarrow \quad \left(1 - \frac{1}{x^2}\right) dx = dt$

$$= \int \frac{dt}{t\sqrt{t^2 - 2}} = \frac{1}{\sqrt{2}}\sec^{-1}\left(\frac{t}{\sqrt{2}}\right) + C$$

$$= \frac{1}{\sqrt{2}}\sec^{-1}\left(\frac{x^2+1}{\sqrt{2}x}\right) + C$$

Hence, (a) is the correct answer.

JEE Type Solved Examples :
More than One Correct Option Type Questions

● ***Ex. 20*** $\int \frac{\sqrt{4 + x^2}}{x^6} dx = \frac{A(4 + x^2)^{3/2}(Bx^2 - 6)}{x^5} + C$, *then*

(a) $A = \frac{1}{120}$ (b) $B = 1$

(c) $A = -\frac{1}{120}$ (d) $B = -1$

Sol. Here, $I = \int \frac{\sqrt{4 - x^2}}{x^6} dx = \int \frac{\sqrt{1 + \frac{4}{x^2}}}{x^5} dx = \int \frac{\sqrt{1 + \frac{4}{x^2}}}{x^2 \cdot x^3} dx$

Put $\quad t = \sqrt{1 + \frac{4}{x^2}} \Rightarrow t^2 = 1 + \frac{4}{x^2}$

$$\therefore \quad 2t\, dt = -\frac{8}{x^3} dx$$

$$\Rightarrow \quad I = \frac{1}{16}\int (t^2 - t^4)\, dx = \frac{1}{16}\left\{\frac{t^3}{3} - \frac{t^5}{5}\right\} + C$$

$$= \frac{1}{120}\cdot\frac{(4 + x^2)^{3/2}}{x^5}(x^2 - 6) + C$$

$$A = \frac{1}{120},\ B = 1$$

Hence, (a) and (b) are the correct answers.

● **Ex. 21** *The value of the integral* $\int e^{\sin^2 x}(\cos x + \cos^3 x)\sin x\,dx$ *is*

(a) $\frac{1}{2}e^{\sin^2 x}(3 - \sin^2 x) + C$

(b) $e^{\sin^2 x}\left(1 + \frac{1}{2}\cos^2 x\right) + C$

(c) $e^{\sin^2 x}(3\cos^2 x + 2\sin^2 x) + C$

(d) $e^{\sin^2 x}(2\cos^2 x + 3\sin^2 x) + C$

Sol. Put $t = \sin^2 x$

The integral reduces to $I = \frac{1}{2}\int e^t(2 - t)\,dt = \frac{3}{2}e^t - \frac{te^t}{2} + C$

$= \frac{1}{2}e^{\sin^2 x}(3 - \sin^2 x) + C$ [option (a)]

$= e^{\sin^2 x}\left(1 + \frac{1}{2}\cos^2 x\right) + C$ [option (b)]

Hence, (a) and (b) are the correct answers.

● **Ex. 22** *If* $I = \int(\sqrt{\tan x} + \sqrt{\cot x})\,dx = f(x) + c$, *then* $f(x)$ *is equal to*

(a) $\sqrt{2}\sin^{-1}(\sin x - \cos x)$ (b) $\frac{\pi}{\sqrt{2}} - \sqrt{2}\cos^{-1}(\sin x - \cos x)$

(c) $\sqrt{2}\tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2}\sqrt{\tan x}}\right)$ (d) None of these

Sol. $I = \int(\sqrt{\tan x} + \sqrt{\cot x})\,dx = \int\sqrt{2}\cdot\frac{\sin x + \cos x}{\sqrt{2\sin x\cos x}}\,dx$

If $\sin x - \cos x = p$, then $(\cos x + \sin x)\,dx = dp$

$I = \sqrt{2}\int\frac{dp}{\sqrt{1 - p^2}} = \sqrt{2}\sin^{-1}p + c = \sqrt{2}\sin^{-1}(\sin x - \cos x) + c$

$= \frac{\pi}{\sqrt{2}} - \sqrt{2}\cos^{-1}(\sin x - \cos x) = \sqrt{2}\tan^{-1}\frac{\sin x - \cos x}{\sqrt{1 - (\sin x - \cos x)^2}}$

$= \sqrt{2}\tan^{-1}\frac{\sin x - \cos x}{\sqrt{2\sin x\cos x}} = \sqrt{2}\tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2\tan x}}\right)$

Hence, (a), (b) and (c) are the correct answers.

JEE Type Solved Examples : Passage Based Questions

Passage

(Q. Nos. 23 to 25)

For integral $\int f\left(x - \frac{a}{x}\right)\cdot\left(1 + \frac{a}{x^2}\right)dx$, *put* $x - \frac{a}{x} = t$

For integral $\int f\left(x + \frac{a}{x}\right)\cdot\left(1 - \frac{a}{x^2}\right)dx$, *put* $x + \frac{a}{x} = t$

For integral $\int f\left(x^2 - \frac{a}{x^2}\right)\cdot\left(x + \frac{a}{x^3}\right)dx$, *put* $x^2 - \frac{a}{x^2} = t$

For integral $\int f\left(x^2 + \frac{a}{x^2}\right)\cdot\left(x - \frac{a}{x^3}\right)dx$, *put* $x^2 + \frac{a}{x^2} = t$

many integrands can be brought into above forms by suitable reductions or transformations.

● **Ex. 23** $\int\frac{x^4 - 2}{x^2\sqrt{x^4 + x^2 + 2}}\,dx$

(a) $\sqrt{x^2 + 1 + \frac{1}{x^2}} + C$ (b) $\sqrt{x^2 + 1 + \frac{2}{x^2}} + C$

(c) $\sqrt{x^2 + \frac{1}{x^2}} + C$ (d) $\sqrt{x^2 + \frac{2}{x^2}} + C$

Sol. Here, $I = \int\frac{x - \frac{2}{x^3}}{\sqrt{x^2 + 1 + \frac{2}{x^2}}}\,dx$

Put $x^2 + \frac{2}{x^2} + 1 = t \Rightarrow 2\left(x - \frac{2}{x^3}\right)dx = dt$

$= \frac{1}{2}\int\frac{dt}{\sqrt{t}} = \frac{1}{2}\cdot\frac{t^{1/2}}{1/2} + C = \sqrt{t} + C$

$= \sqrt{x^2 + \frac{2}{x^2} + 1} + C$

Hence, (b) is the correct answer.

● **Ex. 24** $\int\frac{(x - 1)}{(x + 1)\sqrt{x^3 + x^2 + x}}\,dx$

(a) $\tan^{-1}\left(x + \frac{1}{x} + 1\right) + C$ (b) $\tan^{-1}\sqrt{x + \frac{1}{x} + 1} + C$

(c) $2\tan^{-1}\sqrt{x + \frac{1}{x} + 1} + C$ (d) None of these

Sol. $\int\frac{x^2 - 1}{(x + 1)^2\sqrt{x^3 + x^2 + x}}\,dx = \int\frac{\left(1 - \frac{1}{x^2}\right)}{\left(x + \frac{1}{x} + 2\right)\sqrt{x + \frac{1}{x} + 1}}\,dx$

Put $x + \frac{1}{x} + 1 = t^2 \Rightarrow \left(1 - \frac{1}{x^2}\right)dx = 2t\,dt$

$= \int\frac{2t\,dt}{(t^2 + 1)t} = 2\int\frac{1}{(t^2 + 1)}\,dt$

$= 2\cdot\tan^{-1}(t) + C = 2\tan^{-1}\left(\sqrt{x + \frac{1}{x} + 1}\right) + C$

Hence, (c) is the correct answer.

● ***Ex. 25*** $\int \frac{5x^4 + 4x^5}{(x^5 + x + 1)^2}\, dx$

(a) $x^5 + x + 1 + C$ (b) $\frac{x^5}{x^5 + x + 1} + C$

(c) $x^{-4} + x^{-5} + C$ (d) $\frac{x^5}{x^5 + x + 1} + C$

Sol. Here, $I = \sqrt{\frac{5x^4 + 4x^5}{(x^5 + x + 1)^2}}$

Divide numerator and denominator by x^{10}, we get

$$I = \int \frac{5x^{-6} + 4x^{-5}}{(1 + x^{-4} + x^{-5})^2}\, dx$$

Put $1 + x^{-4} + x^{-5} = t \Rightarrow (-4x^{-5} - 5x^{-6})\, dx = dt$

$$\therefore \quad I = -\int \frac{dt}{t^2} = \frac{1}{t} + C = \frac{1}{1 + x^{-4} + x^{-5}} + C$$

$$= \frac{x^5}{x^5 + x + 1} + C$$

Hence, (d) is the correct answer.

JEE Type Solved Examples :
Matching Type Questions

● ***Ex. 26*** *If $x \in (0, 1)$ then match the entries of Column I with Column II considering 'c' as an arbitrary constant of integration.*

Column I	Column II
(A) $\int \tan\left(2\tan^{-1}\sqrt{\frac{\sqrt{1+\sqrt{x}}-1}{\sqrt{1+\sqrt{x}}+1}}\right) dx$	(p) $\frac{4}{3}x^{3/4} + C$
(B) $\int \cot\left(2\tan^{-1}\sqrt{\frac{\sqrt{1+\sqrt{x}}-\sqrt[4]{x}}{\sqrt{1+\sqrt{x}}+\sqrt[4]{x}}}\right) dx$	(q) $\frac{4}{5}x^{5/4} + C$
(C) $\int \left(\frac{1-\tan\left(\frac{1}{2}\sin^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)\right)}{1+\tan\left(\frac{1}{2}\sin^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)\right)}\right) dx$	(r) $\frac{2}{3}x^{3/4} + C$
(D) $\int \sqrt{x}\tan\left(2\tan^{-1}\left(\frac{\sqrt{\sqrt{1+\sqrt{x}}+1}-\sqrt{\sqrt{1+\sqrt{x}}-1}}{\sqrt{\sqrt{1+\sqrt{x}}+1}+\sqrt{\sqrt{1+\sqrt{x}}-1}}\right)\right) dx$	(s) $\frac{2}{5}x^{5/4} + C$

Sol. Let $\sqrt{x} = \tan^2\theta$

$$x = \tan^4\theta \quad \Rightarrow \quad \tan\theta = x^{1/4}$$

$$\therefore \quad dx = 4\tan^3\theta\sec^2\theta\, d\theta \qquad \begin{bmatrix} \because\ x \in (0, 1) \\ \therefore\ \theta \in (0, \pi/4) \end{bmatrix}$$

$$\therefore \quad \sqrt{1+\sqrt{x}} = \sec\theta$$

(A) $I = \int \tan\left(2\tan^{-1}\sqrt{\frac{\sqrt{1+\sqrt{x}}-1}{\sqrt{1+\sqrt{x}}+1}}\right) dx$

$$\Rightarrow \tan\left(2\tan^{-1}\sqrt{\frac{\sqrt{1+\sqrt{x}}-1}{\sqrt{1+\sqrt{x}}+1}}\right) = \tan\left(2\tan^{-1}\sqrt{\frac{\sec\theta - 1}{\sec\theta + 1}}\right)$$

$$= \tan\left(2\tan^{-1}\left(\tan\frac{\theta}{2}\right)\right) = \tan\theta$$

$$\therefore \quad I = \int \tan\theta \cdot 4\tan^3\theta\sec^2\theta\, d\theta$$

$$= \frac{4}{5}\tan^5\theta + C = \frac{4}{5}(x^{5/4}) + C$$

(B) $I = \int \cot\left(2\tan^{-1}\sqrt{\frac{\sqrt{1+\sqrt{x}}-\sqrt[4]{x}}{\sqrt{1+\sqrt{x}}+\sqrt[4]{x}}}\right) dx$

$$\therefore \quad \cot\left(2\tan^{-1}\sqrt{\frac{\sqrt{1+\sqrt{x}}-\sqrt[4]{x}}{\sqrt{1+\sqrt{x}}+\sqrt[4]{x}}}\right) = \cot\left(2\tan^{-1}\sqrt{\frac{\sec\theta - \tan\theta}{\sec\theta + \tan\theta}}\right)$$

$$= \cot\left(2\tan^{-1}\sqrt{(\sec\theta - \tan\theta)^2}\right)$$

$$= \cot\left(2\tan^{-1}\frac{1-\sin\theta}{\cos\theta}\right)$$

If $\theta \in (0, \pi/4)$, then $\sec\theta - \tan\theta > 0$

$$= \cot\left(2\tan^{-1}\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right) = \cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$$

$$\therefore \quad I = \int \tan\theta \cdot 4\tan^3\theta\sec^2\theta\, d\theta$$

$$= \frac{4}{5}\tan^5\theta + C = \frac{4}{5}(x^{5/4}) + C$$

(C) $\frac{1}{2}\sin^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right) = \frac{1}{2}\sin^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$$= \frac{1}{2}\sin^{-1}(\cos 2\theta) = \frac{1}{2}\sin^{-1}\sin\left(\frac{\pi}{2} - 2\theta\right) = \frac{\pi}{4} - \theta$$

$$\therefore \quad \int \left(\frac{1-\tan\left(\frac{1}{2}\sin^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)\right)}{1+\tan\left(\frac{1}{2}\sin^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)\right)}\right) dx$$

$$= \int \frac{1-\tan\left(\frac{\pi}{4} - \theta\right)}{1+\tan\left(\frac{\pi}{4} - \theta\right)} 4\tan^3\theta\sec^2\theta\, d\theta$$

$$= \int \tan\left(\frac{\pi}{4} - \left(\frac{\pi}{4} - \theta\right)\right) 4\tan^3\theta\sec^2\theta\, d\theta$$

$$= \int 4\tan^4\theta\sec^2\theta\, d\theta$$

$$= \frac{4}{5}\tan^5\theta + C = \frac{4}{5}(x^{5/4}) + C$$

(D) Let $\sqrt{x} = \tan^2\theta \Rightarrow x = \tan^4\theta$

$\therefore \quad dx = 4\tan^3\theta\sec^2\theta\, d\theta \qquad \left[\because\ x \in (0,1)\ \therefore\ \theta \in (0, \pi/4)\right]$

$\therefore \sqrt{1+\sqrt{x}} = \sec\theta$

$$I = \int \sqrt{x}\tan\left(2\tan^{-1}\left(\frac{\sqrt{\sqrt{1+\sqrt{x}}+1} - \sqrt{\sqrt{1+\sqrt{x}}-1}}{\sqrt{\sqrt{1+\sqrt{x}}+1} + \sqrt{\sqrt{1+\sqrt{x}}-1}}\right)\right)dx$$

$$\therefore \sqrt{x}\tan\left(2\tan^{-1}\left(\frac{\sqrt{\sqrt{1+\sqrt{x}}+1} - \sqrt{\sqrt{1+\sqrt{x}}-1}}{\sqrt{\sqrt{1+\sqrt{x}}+1} + \sqrt{\sqrt{1+\sqrt{x}}-1}}\right)\right)$$

$$= \tan^2\theta\cdot\tan\left(2\tan^{-1}\left(\frac{\sqrt{\sec\theta+1} - \sqrt{\sec\theta-1}}{\sqrt{\sec\theta+1} + \sqrt{\sec\theta-1}}\right)\right)$$

$$= \tan^2\theta\cdot\tan\left(2\tan^{-1}\left(\frac{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}\right)\right)$$

$$= \tan^2\theta\cdot\tan\left(2\tan^{-1}\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right)$$

$$= \tan^2\theta\cdot\tan\left(\frac{\pi}{2} - \theta\right) = \tan^2\theta\cdot\cot\theta = \tan\theta$$

$\therefore \quad I = \int \tan\theta\cdot 4\tan^3\theta\sec^2\theta\, d\theta = \frac{4\tan^5\theta}{5} + C = \frac{4}{5}(x^{5/4}) + C$

(A) → (q); (B) → (q); (C) → (q); (D) → (q)

JEE Type Solved Examples :
Single Integer Answer Type Questions

● ***Ex. 27*** *If the primitive of the function* $f(x) = \dfrac{x^{2009}}{(1+x^2)^{1006}}$ *w.r.t. x is equal to* $\dfrac{1}{n}\left(\dfrac{x^2}{1+x^2}\right)^m + C$, *then* $\dfrac{n}{m}$ *is equal to .....*

Sol. $f(x) = \int \dfrac{x^{2009}}{(1+x^2)^{1006}}\,dx$

Put $1 + x^2 = t \ \Rightarrow\ 2x\,dx = dt$

$$\therefore \quad I = \frac{1}{2}\int \frac{(t-1)^{1004}\,dt}{t^{1006}} = \frac{1}{2}\int\left(1-\frac{1}{t}\right)^{1004}\cdot\frac{1}{t^2}\,dt$$

Put $1 - \dfrac{1}{t} = y \ \Rightarrow\ \dfrac{1}{t^2}\,dt = dy$

$$\therefore \quad I = \frac{1}{2}\int y^{1004}dy = \frac{1}{2}\cdot\frac{y^{1005}}{1005} + C$$

$$= \frac{1}{2010}\cdot\left(\frac{t-1}{t}\right)^{1005} + C = \frac{1}{2010}\cdot\left(\frac{x^2}{1+x^2}\right)^{1005} + C$$

$$\Rightarrow \quad m = 1005,\ n = 2010 \ \Rightarrow \frac{n}{m} = \frac{2010}{1005} = 2$$

● ***Ex. 28*** *Suppose* $\begin{vmatrix} f'(x) & f(x) \\ f''(x) & f'(x) \end{vmatrix} = 0$ *where* $f(x)$ *is continuous differentiable function with* $f'(x) \neq 0$ *and satisfies* $f(0) = 1$ *and* $f'(0) = 2$, *then* $f(x) = e^{\lambda x} + k$, *then* $\lambda + k$ *is equal to .....*

Sol. $f'(x)\cdot f'(x) - f(x)\cdot f''(x) = 0$ or $\dfrac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2} = 0$

$$\frac{d}{dx}\left[\frac{f(x)}{f'(x)}\right] = 0$$

Integrating, $\dfrac{f(x)}{f'(x)} + C$ ...(i)

Put $x = 0,\ \dfrac{f(0)}{f'(0)} = C \ \Rightarrow\ C = \dfrac{1}{2}$.

Hence, $\dfrac{f(x)}{f'(x)} = \dfrac{1}{2}$ ...(ii)

From Eq. (i), $2f(x) = f'(x) \ \therefore\ \dfrac{f'(x)}{f(x)} = 2$

Again, integrating, $\ln[f(x)] = 2x + k$

Put $x = 0$ to get, $k = 0$

$$f(x) = e^{2x} \ \Rightarrow\ \lambda + k = 2 + 0 = 2$$

● ***Ex. 29*** $\int\{\sin(101x)\cdot\sin^{99}x\}\,dx = \dfrac{\sin(100x)(\sin x)^{\lambda}}{\mu}$, *then* $\dfrac{\lambda}{\mu}$ *is equal to .....*

Sol. (1) $I = \int\{\sin(100\,x + x)\cdot(\sin x)^{99}\}\,dx$

$$= \int\{\sin(100\,x)\cos x + \cos 100\,x\sin x\}(\sin x)^{99}\,dx$$

$$= \int \underbrace{\sin(100x)}_{\text{I}}\ \underbrace{\cos x\cdot(\sin x)^{99}}_{\text{II}}\,dx + \int\cos(100x)\cdot(\sin x)^{100}\,dx$$

$$= \frac{\sin(100x)(\sin x)^{100}}{100}$$

$$- \frac{100}{100}\int\cos(100x)(\sin x)^{100}\,dx + \int\cos(100x)(\sin x)^{100}\,dx$$

$$= \frac{\sin(100x)(\sin x)^{100}}{100} + C$$

$$\Rightarrow \lambda = 100,\ \mu = 100 \ \Rightarrow\ \frac{\lambda}{\mu} = \frac{100}{100} = 1$$

Subjective Type Questions

● **Ex. 30** *If I_n denotes $\int z^n\, e^{1/z}\, dz$, then show that* $(n+1)!\, I_n = I_0 + e^{1/z}\,(1!\, z^2 + 2!\, z^3 + \ldots + n!\, z^{n+1})$.

Sol. $I_n = \int z^n\, e^{1/z}\, dz$, applying integration by parts taking $e^{1/z}$ as first function and z^n as second function. We get,

$$I_n = \frac{e^{1/z}\cdot z^{n+1}}{(n+1)} - \int e^{1/z}\left(-\frac{1}{z^2}\right)\cdot\frac{z^{n+1}}{n+1}\,dz$$

$$= \frac{e^{1/z}\cdot z^{n+1}}{(n+1)} + \frac{1}{(n+1)}\int e^{1/z}\cdot z^{n-1}\,dz$$

$$I_n = \frac{e^{1/z}\cdot z^{n+1}}{(n+1)} + \frac{I_{n-1}}{(n+1)}$$

$$= \frac{e^{1/z}\cdot z^{n+1}}{(n+1)} + \frac{1}{(n+1)}\left[\frac{e^{1/z}\cdot z^n}{n} + \frac{1}{n}\,I_{n-2}\right]$$

$$= \frac{e^{1/z}\,(z)^{n+1}}{(n+1)} + \frac{e^{1/z}\cdot(z)^n}{(n+1)\,n} + \frac{1}{(n+1)\,n}\,I_{n-2}$$

$$= \frac{e^{1/z}\cdot(z)^{n+1}}{(n+1)} + \frac{e^{1/z}\cdot(z)^n}{(n+1)n} + \frac{e^{1/z}\cdot(z)^{n-1}}{(n+1)n\cdot(n-1)} + \frac{1}{(n+1)n(n-1)}\,I_{n-3}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \frac{e^{1/z}\cdot(z)^{n+1}}{n+1} + \frac{e^{1/z}\cdot(z)^n}{(n+1)\,n} + \ldots + \frac{e^{1/z}\cdot(z)^1}{(n+1)\,n\ldots 3\cdot 2} + \frac{1}{(n+1)\,n\,(n-1)\ldots 3\cdot 2}\,I_0$$

Multiplying both the sides by $(n+1)!$. We get,

$$(n+1)!\, I_n = (e^{1/z}\cdot z^{n+1}\cdot n! + e^{1/z}\cdot z^n\,(n-1)! + \ldots + \ldots + e^{1/z}\cdot z^3\cdot(2)! + e^{1/z}\cdot z^2\cdot 1!) + I_0$$

$$\Rightarrow \quad I_n\,(n+1)! = I_0 + e^{1/z}\,(1!\, z^2 + 2!\, z^3 + \ldots + n!\, z^{n+1})$$

Hence Proved.

● **Ex. 31** *If $I_n = \int x^n\,\sqrt{a^2 - x^2}\,dx$, prove that* $I_n = -\dfrac{x^{n-1}\,(a^2-x^2)^{3/2}}{(n+2)} + \dfrac{(n-1)}{(n+2)}\,a^2\,I_{n-2}$.

Sol. $I_n = \int x^n\,\sqrt{a^2-x^2}\,dx = \int \underset{\text{I}}{x^{n-1}}\cdot\underset{\text{II}}{\{x\,\sqrt{a^2-x^2}\}}\,dx$

Applying integration by parts, we get

$$= x^{n-1}\cdot\left\{-\frac{(a^2-x^2)^{3/2}}{3}\right\} + \int (n-1)\,x^{n-2}\cdot\left\{-\frac{(a^2-x^2)^{3/2}}{3}\right\}dx$$

$$= -\frac{x^{n-1}\,(a^2-x^2)^{3/2}}{3} + \frac{(n-1)}{3}\int x^{n-2}\cdot(a^2-x^2)\,\sqrt{a^2-x^2}\,dx$$

$$\Rightarrow \quad I_n = -\frac{x^{n-1}\,(a^2-x^2)^{3/2}}{3} + \frac{(n-1)\,a^2}{3}\,I_{n-2} - \frac{(n-1)}{3}\,I_n$$

$$\therefore \quad I_n + \frac{(n-1)}{3}\,I_n = -\frac{x^{n-1}\,(a^2-x^2)^{3/2}}{3} + \frac{(n-1)\,a^2}{3}\,I_{n-2}$$

$$\left(\frac{n+2}{3}\right)I_n = -\frac{x^{n-1}\,(a^2-x^2)^{3/2}}{3} + \frac{(n-1)\,a^2}{3}\,I_{n-2}$$

$$I_n = -\frac{x^{n-1}\,(a^2-x^2)^{3/2}}{(n+2)} + \frac{(n-1)\,a^2}{(n+2)}\,I_{n-2}$$

Hence Proved.

● **Ex. 32** *If $I_m = \int (\sin x + \cos x)^m\,dx$, then show that* $m\,I_m = (\sin x + \cos x)^{m-1}\cdot(\sin x - \cos x) + 2\,(m-1)\,I_{m-2}$

Sol. $\because\ I_m = \int (\sin x + \cos x)^m\,dx$

$$= \int (\sin x + \cos x)^{m-1}\cdot(\sin x + \cos x)\,dx,$$

Applying integration by parts

$$= (\sin x + \cos x)^{m-1}(\cos x + \sin x) - \int (m-1)\,(\sin x + \cos x)^{m-2}\cdot(\cos x - \sin x)\cdot(\sin x - \cos x)\,dx$$

$$= (\sin x + \cos x)^{m-1}(\sin x - \cos x) + (m-1)\int (\sin x + \cos x)^{m-2}\,(\sin x - \cos x)^2\,dx$$

As we know, $(\sin x + \cos x)^2 + (\sin x - \cos x)^2 = 2$,

$$\therefore I_m = (\sin x + \cos x)^{m-1}\,(\sin x - \cos x) + (m-1)\int (\sin x + \cos x)^{m-2}\cdot\{2 - (\sin x + \cos x)^2\}\,dx$$

$$= (\sin x + \cos x)^{m-1}\,(\sin x - \cos x) + (m-1)\int 2\,(\sin x + \cos x)^{m-2}\,dx - (m-1)\int (\sin x + \cos x)^m\,dx$$

$$I_m = (\sin x + \cos x)^{m-1}\,(\sin x - \cos x) + 2\,(m-1)\,I_{m-2} - (m-1)\,I_m$$

or $(m-1)I_m + I_m = (\sin x + \cos x)^{m-1}(\sin x - \cos x) + 2(m-1)I_{m-2}$

or $\quad m\,I_m = (\sin x + \cos x)^{m-1}\,(\sin x - \cos x) + 2\,(m-1)\,I_{m-2}$

Hence Proved.

● **Ex. 33** *If $I_{m,n} = \int \cos^m x\cdot\cos nx\,dx$, show that* $(m+n)\,I_{m,n} = \cos^m x\cdot\sin nx + m\,I_{(m-1,\,n-1)}$

Sol. We have,

$$I_{m,n} = \int \underset{\text{I}}{\cos^m x}\cdot\underset{\text{II}}{\cos n\,x}\,dx$$

$$= (\cos^m x)\left[\frac{\sin nx}{n}\right] - \int m\cos^{m-1}x\,(-\sin x)\cdot\frac{\sin nx}{n}\,dx$$

$$= \frac{1}{n}\cos^m x\cdot\sin nx + \frac{m}{n}\int \cos^{m-1}x\,\{\sin x\cdot\sin nx\}\,dx$$

As we have, $\cos(n-1)\,x = \cos nx\cos x + \sin nx\sin x$

$$\therefore\ I_{m,n} = \frac{1}{n}\cos^m x\cdot\sin x + \frac{m}{n}\int \cos^{m-1}x\,\{\cos(n-1)\,x - \cos nx\cdot\cos x\}\,dx$$

$$= \frac{1}{n}\cos^m x\cdot\sin x + \frac{m}{n}\int \cos^{m-1}x\cdot\cos(n-1)\,x\,dx - \frac{m}{n}\int \cos^m x\cdot\cos n\,x\,dx$$

$$= \frac{1}{n}\cos^m x\cdot\sin nx + \frac{m}{n}\,I_{m-1,\,n-1} - \frac{m}{n}\,I_{m,n}$$

$$I_{m,n} + \frac{m}{n} I_{m,n} = \frac{1}{n} [\cos^m x \cdot \sin nx + m I_{m-1,n-1}]$$

$$\left(\frac{m+n}{n}\right) I_{m,n} = \frac{1}{n} [\cos^m x \cdot \sin nx + m I_{m-1,n-1}]$$

$$(m+n) I_{m,n} = \cos^m x \cdot \sin nx + m I_{m-1,n-1}$$

● **Ex. 34** *Evaluate* $\int \frac{\tan\left(\frac{\pi}{4} - x\right)}{\cos^2 x \sqrt{\tan^3 x + \tan^2 x + \tan x}} dx.$

Sol. $I = \int \frac{\tan\left(\frac{\pi}{4} - x\right) dx}{\cos^2 x \sqrt{\tan^3 x + \tan^2 x + \tan x}}$

$$= \int \frac{(1 - \tan^2 x)\, dx}{(1 + \tan x)^2 \cos^2 x \sqrt{\tan^3 x + \tan^2 x + \tan x}}$$

$$I = \int \frac{-\left(1 - \frac{1}{\tan^2 x}\right) \sec^2 x\, dx}{\left(\tan x + 2 + \frac{1}{\tan x}\right) \sqrt{\tan x + 1 + \frac{1}{\tan x}}}$$

let $\quad y = \sqrt{\tan x + 1 + \frac{1}{\tan x}}$

$$\Rightarrow \quad 2y\, dy = \left(\sec^2 x - \frac{1}{\tan^2 x} \cdot \sec^2 x\right) dx$$

$$\therefore \quad I = \int \frac{-2y\, dy}{(y^2+1) \cdot y} = -2 \int \frac{dy}{1+y^2} = -2 \tan^{-1} y + C$$

$$= -2 \tan^{-1}\left(\sqrt{\tan x + 1 + \frac{1}{\tan x}}\right) + C$$

● **Ex. 35** *Evaluate* $\int \frac{x^2 + n(n-1)}{(x \sin x + n \cos x)^2} dx.$

Sol. Here, $\quad I = \int \frac{x^2 + n(n-1)}{(x \sin x + n \cos x)^2} dx$

Multiplying and dividing by x^{2n-2}, we get

$$I = \int \frac{\{x^2 + n(n-1)\}\, x^{2n-2}}{(x^n \sin x + n x^{n-1} \cos x)^2} dx$$

We know $x^n \sin x + n x^{n-1} \cos x = t$

$$\Rightarrow \{(n x^{n-1} \sin x) + (x^n \cos x) + n(n-1) x^{n-2} \cos x - (n x^{n-1} \sin x)\, dx = dt\}$$

$$\Rightarrow x^{n-2} \cos x \cdot \{x^2 + n(n-1)\}\, dx = dt$$

Keeping this in mind, we put

$$I = \int \frac{\{x^2 + n(n-1)\} \cdot x^{n-2} \cdot \cos x}{(x^n \sin x + n x^{n-1} \cos x)^2} \cdot x^n \cdot \sec x\, dx$$

II I

Applying integration by parts, we get

$$= x^n \sec x \cdot \left(-\frac{1}{(x^n \sin x + n x^{n-1} \cos x)}\right) + \int \frac{(x^n \sec x \tan x + n x^{n-1} \cdot \sec x)}{(x^n \sin x + n x^{n-1} \cos x)} dx$$

$$= -\frac{(x \sec x)}{(x \sin x + n \cos x)} + \int \sec^2 x\, dx$$

$$= -\frac{(x \sec x)}{(x \sin x + n \cos x)} + \tan x + C \qquad \textbf{Hence Proved.}$$

● **Ex. 36** *If* $\cos\theta > \sin\theta > 0$, *then evaluate*

$$\int \left\{ \log\left(\frac{1+\sin 2\theta}{1-\sin 2\theta}\right)^{\cos^2\theta} + \log\left(\frac{\cos 2\theta}{1+\sin 2\theta}\right)\right\} d\theta$$

Sol. Here, $\quad I = \int \left\{ \log\left(\frac{1+\sin 2\theta}{1-\sin 2\theta}\right)^{\cos^2\theta} + \log\left(\frac{\cos 2\theta}{1+\sin 2\theta}\right)\right\} d\theta$

$$= \int \left\{ 2\cos^2\theta \log\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right) - \log\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right)\right\} d\theta$$

$$= \int (2\cos^2\theta - 1) \log\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right) d\theta$$

$$= \int \underset{\text{II}}{\cos 2\theta} \cdot \log\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right) \underset{\text{I}}{d\theta}, \text{ applying integration by parts}$$

$$= \log\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right) \cdot \frac{\sin 2\theta}{2} - \int \frac{2}{\cos 2\theta} \cdot \frac{\sin 2\theta}{2} d\theta$$

$$= \frac{\sin 2\theta}{2} \log\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right) + \frac{1}{2} \log|\cos 2\theta| + C$$

● **Ex. 37** *Evaluate* $\int \frac{\tan^{-1} x}{x^4} dx.$

Sol. $I = \int \frac{\tan^{-1} x}{x^4} dx = \int \underset{\text{I}}{\tan^{-1} x} \cdot \underset{\text{II}}{\frac{1}{x^4}} dx$

$$= (\tan^{-1} x)\left(-\frac{1}{3x^3}\right) - \int \frac{1}{1+x^2} \cdot \frac{1}{(-3x^3)} dx$$

$$= -\frac{\tan^{-1} x}{3x^3} + \frac{1}{3} \int \frac{dx}{x^3 (1+x^2)},$$

Put $1 + x^2 = t$,

$$2x\, dx = dt = -\frac{\tan^{-1} x}{3x^3} + \frac{1}{6} \int \frac{dt}{(t-1)^2 \cdot t}$$

$$I = -\frac{\tan^{-1} x}{3x^3} + \frac{1}{6} I_1 \qquad \text{...(i)}$$

Where, $I_1 = \int \frac{1}{(t-1)^2 \cdot t} dt = \int \left\{ \frac{A}{t-1} + \frac{B}{(t-1)^2} + \frac{C}{t} \right\} dt$

Comparing coefficients, we get

$$A = -1,\ B = 1,\ C = 1$$

$$\therefore \quad I_1 = \int \left\{ -\frac{1}{(t-1)} + \frac{1}{(t-1)^2} + \frac{1}{t} \right\} dt$$

$$= -\log|t-1| - \frac{1}{(t-1)} + \log|t| \qquad \text{...(ii)}$$

$\therefore$ From Eqs. (i) and (ii), we get

$$I = -\frac{\tan^{-1} x}{3x^3} + \frac{1}{6}\left\{-\log|x^2| - \frac{1}{x^2} + \log|1+x^2|\right\} + C$$

$$= -\frac{\tan^{-1} x}{3x^3} - \frac{1}{6}\log\left|\frac{x^2+1}{x^2}\right| - \frac{1}{6x^2} + C$$

● ***Ex. 38*** *Evaluate* $\int x^2 \log(1-x^2)\,dx$, *and hence prove that* $\frac{1}{1\cdot 5} + \frac{1}{2\cdot 7} + \frac{1}{3\cdot 9} + \ldots = \frac{2}{3}\log 2 - \frac{8}{9}$.

Sol. We know, $\log(1-x) = -\left\{x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots \infty\right\}$

Put x^2 instead of x in the above identity,

$$\Rightarrow \quad \log(1-x^2) = -\left\{x^2 + \frac{x^4}{2} + \frac{x^6}{3} + \frac{x^8}{4} + \ldots \infty\right\}$$

$$\Rightarrow \quad x^2\log(1-x^2) = -\left\{x^4 + \frac{x^6}{2} + \frac{x^8}{3} + \frac{x^{10}}{4} + \ldots \infty\right\}$$

Integrating both the sides, we get

$$\int x^2\log(1-x^2)\,dx = -\left\{\frac{x^5}{1\cdot 5} + \frac{x^7}{2\cdot 7} + \frac{x^9}{3\cdot 9} + \ldots \infty\right\} + C$$

Now, to find constant of integration, put $x = 0$

$$\Rightarrow \quad 0 = 0 + C$$

$$\Rightarrow \quad C = 0$$

$$\therefore \quad \int \underset{\text{II}}{x^2}\ \underset{\text{I}}{\log(1-x^2)}\,dx = -\left\{\frac{x^5}{1\cdot 5} + \frac{x^7}{2\cdot 7} + \frac{x^9}{3\cdot 9} + \ldots \infty\right\}$$

Applying integration by parts, and taking limits 0 to 1 for LHS

$$\Rightarrow \quad \left(\frac{x^3}{3}\log(1-x^2)\right)_0^1 - \int_0^1 \frac{x^3}{3}\cdot\frac{1(-2x)}{1-x^2}\,dx$$

$$\Rightarrow \quad \left(\frac{x^3}{3}\log(1-x^2)\right)_0^1 + \frac{2}{3}\left(-\frac{x^3}{3} - x + \frac{1}{2}\log\left|\frac{1+x}{1-x}\right|\right)_0^1$$

Taking $\log(1-x^2) = \log(1+x) + \log(1-x)$

and $\quad \log\left(\frac{1+x}{1-x}\right) = \log(1+x) - \log(1-x)$

$$\Rightarrow \quad \frac{1}{3}\log 2 + \frac{1}{3}\log 2 - \frac{2}{3} - \frac{2}{9} + \lim_{x\to 1}\left(\frac{x^3}{3} - \frac{1}{3}\right)\log(1-x)$$

$$\left[\because \lim_{x\to 1}(x^3-1)\log(1-x) = 0\right]$$

$$\Rightarrow \quad \frac{2}{3}\log 2 - \frac{8}{9} = \text{RHS}$$

$$\therefore \quad \frac{1}{1\cdot 5} + \frac{1}{2\cdot 7} + \frac{1}{3\cdot 9} + \ldots = \frac{2}{3}\log(2) - \frac{8}{9}$$

● ***Ex. 39*** *Evaluate* $\int \frac{a + b\sin x}{(b + a\sin x)^2}\,dx$

Sol. Here, $I = \int \frac{a + b\sin x}{(b + a\sin x)^2}\,dx = \frac{b}{a}\int \frac{\frac{a^2}{b} - b + (b + a\sin x)}{(b + a\sin x)^2}\,dx$

$$I = \frac{a^2 - b^2}{a}\int \frac{dx}{(b + a\sin x)^2} + \frac{b}{a}\int \frac{dx}{(b + a\sin x)} \qquad \ldots(\text{i})$$

Now, let $A = \frac{\cos x}{b + a\sin x} \Rightarrow \frac{dA}{dx} = \frac{-b\sin x - a}{(b + a\sin x)^2}$

$$\Rightarrow \quad \frac{dA}{dx} = -\frac{b}{a}\left\{\frac{a\sin x + b + \frac{a^2}{b} - b}{(b + a\sin x)^2}\right\}$$

$$\Rightarrow \quad \frac{dA}{dx} = -\frac{b}{a}\left\{\frac{1}{b + a\sin x} + \frac{a^2 - b^2}{b(b + a\sin x)^2}\right\}$$

Integrating both the sides w.r.t. 'x', we get

$$A = -\frac{b}{a}\int \frac{dx}{b + a\sin x} - \frac{(a^2 - b^2)}{a}\int \frac{dx}{(b + a\sin x)^2}$$

$$\Rightarrow \quad \frac{a^2 - b^2}{a}\int \frac{dx}{(b + a\sin x)^2} = -\frac{b}{a}\int \frac{dx}{b + a\sin x} - A \qquad \ldots(\text{ii})$$

From Eqs. (i) and (ii), we get

$$I = -\frac{b}{a}\int \frac{dx}{(b + a\sin x)} - A + \frac{b}{a}\int \frac{dx}{(b + a\sin x)}$$

$$\Rightarrow \quad I = -A + C \quad \Rightarrow \quad I = -\left(\frac{\cos x}{b + a\sin x}\right) + C$$

● ***Ex. 40*** *Evaluate* $\int \frac{dx}{(x-1)^{3/4}(x+2)^{5/4}}$.

Sol. Let $I = \int \frac{dx}{(x-1)^{3/4}(x+2)^{5/4}} = \int \frac{dx}{(x+2)^2\left(\frac{(x-1)}{(x+2)}\right)^{3/4}}$

Let $\quad \frac{x-1}{x+2} = t$

So that, $\quad \frac{3}{(x+2)^2}\,dx = dt$

$$\therefore \quad I = \int \frac{dt}{3\,t^{3/4}} = \frac{1}{3}\int t^{-3/4}\,dt$$

$$= \frac{1}{3}\cdot\frac{t^{1/4}}{1/4} + C = \frac{4}{3}\left(\frac{x-1}{x+2}\right)^{1/4} + C$$

Indefinite Integral Exercise 1 :
Single Option Correct Type Questions

1. Let $f(x) = \int \frac{x^2}{(1+x^2)(1+\sqrt{1+x^2})}\,dx$ and $f(0) = 0$. Then $f(1)$ is equal to

(a) $\log_e(1+\sqrt{2})$ (b) $\log_e(1+\sqrt{2}) - \frac{\pi}{4}$

(c) $\log_e(1+\sqrt{2}) + \frac{\pi}{4}$ (d) None of these

2. If $\int f(x)\,dx = f(x)$, then $\int \{f(x)\}^2\,dx$ is equal to

(a) $\frac{1}{2}\{f(x)\}^2$ (b) $\{f(x)\}^3$

(c) $\frac{\{f(x)\}^3}{3}$ (d) $\{f(x)\}^2$

3. If $\int f(x)\,dx = F(x)$, then $\int x^3 f(x^2)\,dx$ is equal to

(a) $\frac{1}{2}[x^2\{F(x)\}^2 - \int \{F(x)\}^2\,dx]$

(b) $\frac{1}{2}[x^2F(x^2) - \int F(x^2)\,d(x^2)]$

(c) $\frac{1}{2}[x^2F(x) - \frac{1}{2}\int \{F(x)\}^2\,dx]$

(d) None of the above

4. If n is an odd positive integer, then $\int |x^n|\,dx$ is equal to

(a) $\left|\frac{x^{n+1}}{n+1}\right| + C$ (b) $\frac{x^{n+1}}{n+1} + C$

(c) $\frac{|x|^n x}{n+1} + C$ (d) None of these

5. Let $F(x)$ be the primitive of $\frac{3x+2}{\sqrt{x-9}}$ w.r.t. x. If $F(10) = 60$, then the value of $F(13)$ is

(a) 66 (b) 132

(c) 248 (d) 264

6. $\int (x^x)^x\,(2x\log_e x + x)\,dx$ is equal to

(a) $x^{(x^x)} + C$ (b) $(x^x)^x + C$

(c) $x^2 \cdot \log_e x + C$ (d) None of these

7. The value of $\int x\log x(\log x - 1)\,dx$ is equal to

(a) $2(x\log x - x)^2 + C$

(b) $\frac{1}{2}(x\log x - x)^2 + C$

(c) $(x\log x)^2 + C$

(d) $\frac{1}{2}(x\log x)^3 + C$

8. $\int \frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}}\,dx$ is equal to

(a) $\frac{\sqrt{2x^4-2x^5+1}}{x^2} + C$ (b) $\frac{\sqrt{2x^4-2x^2+1}}{x^3} + C$

(c) $\frac{\sqrt{2x^4+2x^2+1}}{x} + C$ (d) $\frac{\sqrt{2x^4-2x^2+1}}{2x^2} + C$

9. Let $f(x)$ be a polynomial satisfying $f(0) = 2$, $f'(0) = 3$ and $f''(x) = f(x)$. Then $f(4)$ is equal to

(a) $\frac{5(e^8+1)}{2e^4}$ (b) $\frac{5(e^8-1)}{2e^4}$

(c) $\frac{2e^4}{5(e^8-1)}$ (d) $\frac{2e^4}{5(e^8+1)}$

10. $\int \frac{e^{(x^2+4\ln x)} - x^3 e^{x^2}}{x-1}\,dx$ is equal to

(a) $\left(\frac{e^{3\ln x} - e^{\ln x}}{2x}\right)e^{x^2} + C$ (b) $\frac{(x-1)xe^{x^2}}{2} + C$

(c) $\frac{(x^2-1)}{2x} - e^{x^2} + C$ (d) None of these

11. $\int \tan^4 x\,dx = A\tan^3 x + B\tan x + f(x)$, then

(a) $A = \frac{1}{3}$, $B = -1$, $f(x) = x + C$

(b) $A = \frac{2}{3}$, $B = -1$, $f(x) = x + C$

(c) $A = \frac{1}{3}$, $B = 1$, $f(x) = x + C$

(d) $A = \frac{2}{3}$, $B = 1$, $f(x) = -x + C$

12. If the anti-derivative of $\int \frac{\sin^4 x}{x}\,dx$ is $f(x)$, then $\int \frac{\sin^4\{(p+q)x\}}{x}\,dx$ in terms of $f(x)$ is

(a) $f\{(p+q)x\}$ (b) $\frac{f\{(p+q)x\}}{p+q}$

(c) $f\{(p+q)x\}(p+q)$ (d) None of these

13. $\int\left(\frac{\sin\theta}{\cos 3\theta} + \frac{\sin 3\theta}{\cos 9\theta} + \frac{\sin 9\theta}{\cos 27\theta}\right)d\theta$ is equal to

(a) $\frac{1}{2}\log\left|\frac{\sec 27\theta}{\sec\theta}\right| + C$ (b) $\frac{1}{2}\log\left|\frac{\sec\theta}{\sec 27\theta}\right| + C$

(c) $\frac{1}{2}\log\left|\frac{\sqrt[27]{\sec 27\theta}}{\sec\theta}\right| + C$ (d) None of these

14. Let $x^2 \neq n\pi - 1, n \in N$. Then, the value of $\int x\sqrt{\frac{2\sin(x^2+1)-\sin 2(x^2+1)}{2\sin(x^2+1)+\sin 2(x^2+1)}}\,dx$ is equal to

(a) $\log\left|\frac{1}{2}\sec(x^2+1)\right|+C$ (b) $\log\left|\sec\left(\frac{x^2+1}{2}\right)\right|+C$

(c) $\frac{1}{2}\log|\sec(x^2+1)|+C$ (d) None of these

15. $\int \frac{dx}{\cos(2x)\cos(4x)}$ is equal to

(a) $\frac{1}{2\sqrt{2}}\log\left|\frac{1+\sqrt{2}\sin 2x}{1-\sqrt{2}\sin 2x}\right|-\frac{1}{2}(\log|\sec 2x-\tan 2x|)+C$

(b) $\frac{1}{2\sqrt{2}}\log\left|\frac{1+\sqrt{2}\sin 2x}{1+\sqrt{2}\sin x}\right|-\frac{1}{2}(\log|\sec 2x-\tan 2x|)+C$

(c) $\frac{1}{\sqrt{2}}\log\left|\frac{1+\sqrt{2}\sin 2x}{1-\sqrt{2}\sin 2x}\right|-\frac{1}{2}(\log|\sec 2x-\tan 2x|)+C$

(d) None of the above

16. $\int \frac{1-7\cos^2 x}{\sin^7 x\cos^2 x}\,dx=\frac{f(x)}{(\sin x)^7}+C$, then $f(x)$ is equal to

(a) $\sin x$ (b) $\cos x$ (c) $\tan x$ (d) $\cot x$

17. $\int \frac{\sin^3 x}{(\cos^4 x+3\cos^2 x+1)\tan^{-1}(\sec x+\cos x)}\,dx$ is equal to

(a) $\tan^{-1}(\sec x+\cos x)+C$ (b) $\log_e|\tan^{-1}(\sec x+\cos x)|+C$

(c) $\frac{1}{(\sec x+\cos x)^2}+C$ (d) None of these

18. The primitive of the function $f(x)=x|\cos x|$, when $\frac{\pi}{2}<x<\pi$ is given by

(a) $\cos x+x\sin x+C$

(b) $-\cos x-x\sin x+C$

(c) $x\sin x-\cos x+C$

(d) None of the above

19. The primitive of the function $f(x)=(2x+1)|\sin x|$, when $\pi<x<2\pi$ is

(a) $-(2x+1)\cos x+2\sin x+C$

(b) $(2x+1)\cos x-2\sin x+C$

(c) $(x^2+x)\cos x+C$

(d) None of the above

20. Given, $f(x)=\begin{vmatrix} 0 & x^2-\sin x & \cos x-2 \\ \sin x-x^2 & 0 & 1-2x \\ 2-\cos x & 2x-1 & 0 \end{vmatrix}$, then $\int f(x)\,dx$ is equal to

(a) $\frac{x^3}{3}-x^2\sin x+\sin 2x+C$

(b) $\frac{x^3}{3}-x^2\sin x-\cos 2x+C$

(c) $\frac{x^3}{3}-x^2\cos x-\cos 2x+C$

(d) None of the above

Indefinite Integral Exercise 2 :
More than One Option Correct Type Questions

21. $\int \frac{dx}{(x+1)(x-2)}=A\log(x+1)+B\log(x-2)+C$, where

(a) $A+B=0$ (b) $AB=0$

(c) $A/B=-1$ (d) None of these

22. If $\int \frac{dx}{(x^2+1)(x^2+4)}=k\tan^{-1}x+l\tan^{-1}\frac{x}{2}+C$, then

(a) $k=\frac{1}{3}$ (b) $l=\frac{2}{3}$ (c) $k=-\frac{1}{3}$ (d) $l=-\frac{1}{6}$

23. If $\int x\log(1+x^2)\,dx=\phi(x)\log(1+x^2)+x(\psi)+C$, then

(a) $\phi(x)=\frac{1+x^2}{2}$ (b) $\psi(x)=\frac{1+x^2}{2}$

(c) $\psi(x)=-\frac{1+x^2}{2}$ (d) $\phi(x)=-\frac{1+x^2}{2}$

24. If $\int \frac{4e^x+6e^{-x}}{9e^x-4e^{-x}}\,dx=Ax+B\log_e(9e^{2x}-4)+C$, then

(a) $A=\frac{3}{2}$ (b) $B=\frac{35}{36}$

(c) C is indefinite (d) $A+B=-\frac{19}{36}$

25. If $\int \tan^5 x\,dx=A\tan^4 x+B\tan^2 x+g(x)+C$, then

(a) $A=\frac{1}{4}, B=-\frac{1}{2}$

(b) $g(x)=\ln|\sec x|$

(c) $g(x)=\ln|\cos x|$

(d) $A=-\frac{1}{4}, B=\frac{1}{3}$

Indefinite Integral Exercise 3 :
Statement I and II Type Questions

■ **Directions** (Q. Nos. 26 to 30) For the following questions, choose the correct answers from the codes (a), (b), (c) and (d) defined as follows :

(a) Statement I is true, Statement II is also true; Statement II is the correct explanation of Statement I.

(b) Statement I is true, Statement II is also true; Statement II is not the correct explanation of Statement I.

(c) Statement I is true, Statement II is false.

(d) Statement I is false, Statement II is true.

26. **Statement I** If y is a function of x such that $y(x-y)^2 = x$, then $\int \frac{dx}{x-3y} = \frac{1}{2}[\log(x-y)^2 - 1]$

Statement II $\int \frac{dx}{x-3y} = \log(x-3y) + C$

27. **Statement I** Integral of an even function is not always an odd function.

Statement II Integral of an odd function is an even function.

28. **Statement I** If $a>0$ and $b^2 - 4ac < 0$, then the value of the integral $\int \frac{dx}{ax^2+bx+c}$ will be of the type $\mu \tan^{-1}\frac{x+A}{B} + C$, where A, B, C, μ are constants.

Statement II If $a>0, b^2 - 4ac < 0$, then $ax^2 + bx + C$ can be written as sum of two squares.

29. **Statement I** $\int \left(\frac{1}{1+x^4}\right) dx = \tan^{-1}(x^2) + C$

Statement II $\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$

30. **Statement I** $\int 2^{\tan^{-1} x} d(\cot^{-1} x) = \frac{2^{\tan^{-1} x}}{\ln 2} + C$

Statement II $\frac{d}{dx}(a^x + C) = a^x \ln a$

Indefinite Integral Exercise 4 :
Passage Based Questions

Passage I

(Q. Nos. 31 to 33)

Let us consider the integral of the following forms

$$f(x_1, \sqrt{mx^2 + nx + p})^{1/2}$$

Case I *If* $m>0$, *then put* $\sqrt{mx^2 + nx + C} = u \pm x\sqrt{m}$

Case II *If* $p>0$, *then put* $\sqrt{mx^2 + nx + C} = ux \pm \sqrt{p}$

Case III *If quadratic equation* $mx^2 + nx + p = 0$ *has real roots* α *and* β *there put* $\sqrt{mx^2 + nx + p} = (x-\alpha)u$ or $(x-\beta)u$

31. If $I = \int \frac{dx}{x - \sqrt{9x^2 + 4x + 6}}$ to evaluate I, one of the most proper substitution could be

(a) $\sqrt{9x^2 + 4x + 6} = u \pm 3x$

(b) $\sqrt{9x^2 + 4x + 6} = 3u \pm x$

(c) $x = \frac{1}{t}$

(d) $9x^2 + 4x + 6 = \frac{1}{t}$

32. $\int \frac{(x + \sqrt{1+x^2})^{15}}{\sqrt{1+x^2}} dx$ is equal to

(a) $\frac{(x + \sqrt{1+x^2})^{16}}{10} + C$

(b) $\frac{1}{15(\sqrt{1+x^2} + x)} + C$

(c) $\frac{15}{(\sqrt{1+x^2} - x)} + C$

(d) $\frac{(x + \sqrt{1+x^2})^{15}}{15} + C$

33. To evaluate $\int \frac{dx}{(x-1)\sqrt{-x^2 + 3x - 2}}$ one of the most suitable substitution could be

(a) $\sqrt{-x^2 + 3x - 2} = u$

(b) $\sqrt{-x^2 + 3x - 2} = (ux\sqrt{2})$

(c) $\sqrt{-x^2 + 3x - 2} = u(1-x)$

(d) $\sqrt{-x^2 + 3x - 2} = u(x+2)$

Passage II

(Q. Nos. 34 to 36)

Let $I_{n,m} = \int \sin^n x \cos^m x\, dx$. *Then, we can relate* $I_{n,m}$ *with each of the following* :

(i) $I_{n-2,m}$ (ii) $I_{n+2,m}$

(iii) $I_{n,m-2}$ (iv) $I_{n,m+2}$

(v) $I_{n-2,m+2}$ (vi) $I_{n+2,m-2}$

Suppose we want to establish a relation between $I_{n,m}$ *and* $I_{n,m-2}$ *, then we get*

$$P(x) = \sin^{n+1} x \cos^{m-1} x \qquad \ldots(i)$$

In $I_{n,m}$ *and* $I_{n,m-2}$ *the exponent of* $\cos x$ *is* m *and* $m-2$ *respectively, the minimum of the two is* $m-2$, *adding* 1 *to the minimum we get* $m-2+1=m-1$. *Now, choose the exponent* $m-1$ *of* $\cos x$ *in* $P(x)$. *Similarly, choose the exponent of* $\sin x$ *for* $P(x) = (nH)\sin^n x \cos^m x - (m-1)\sin^{n+2} x \cos^{m-2} x$.

Now, differentiating both the sides of Eq. (i), *we get*

$$= (n+1)\sin^n x \cos^m x - (m-1)\sin^n x (1-\cos^2 x)\cos^{m-2} x$$

$$= (n+1)\sin^n x \cos^m x - (m-1)\sin^n x \cos^{m-2} x + (m-1)\sin^n x \cos^n x$$

$$= (n+m)\sin^n x \cos^m x - (m-1)\sin^n x \cos^{m-2} x$$

Now, integrating both the sides, we get

$$\sin^{n+1} x \cos^{m-1} x = (n+m) I_{n,m} - (m-1) I_{n,m-2}$$

Similarly, we can establish the other relations.

34. The relation between $I_{4,2}$ and $I_{2,2}$ is

(a) $I_{4,2} = \frac{1}{6}(-\sin^3 x \cos^3 x + 3 I_{2,2})$

(b) $I_{4,2} = \frac{1}{6}(\sin^3 x \cos^3 x + 3 I_{2,2})$

(c) $I_{4,2} = \frac{1}{6}(\sin^3 x \cos^3 x - 3 I_{2,2})$

(d) $I_{4,2} = \frac{1}{4}(-\sin^3 x \cos^3 x + 2 I_{2,2})$

35. The relation between $I_{4,2}$ and $I_{6,2}$ is

(a) $I_{4,2} = \frac{1}{5}(\sin^5 x \cos^3 x + 8 I_{6,2})$

(b) $I_{4,2} = \frac{1}{5}(-\sin^5 x \cos^3 x + 8 I_{6,2})$

(c) $I_{4,2} = \frac{1}{5}(\sin^5 x \cos^3 x - 8 I_{6,2})$

(d) $I_{4,2} = \frac{1}{6}(\sin^5 x \cos^3 x + 8 I_{6,2})$

36. The relation between $I_{4,2}$ and $I_{4,4}$ is

(a) $I_{4,2} = \frac{1}{3}(\sin^5 x \cos^3 x + 8 I_{4,4})$

(b) $I_{4,2} = \frac{1}{3}(-\sin^5 x \cos^3 x + 8 I_{4,4})$

(c) $I_{4,2} = \frac{1}{3}(\sin^5 x \cos^3 x - 8 I_{4,4})$

(d) $I_{4,2} = \frac{1}{3}(\sin^5 x \cos^3 x + 6 I_{4,4})$

Passage III

(Q. Nos. 37 to 38)

If $f : R \to (0, \infty)$ be a differentiable function $f(x)$ satisfying $f(x+y) - f(x-y) = f(x)\cdot\{f(y) - f(y) - y\}$, $\forall\, x, y \in R$, $(f(y) \neq f(-y)$ for all $y \in R)$ and $f'(0) = 2010$.

Now, answer the following questions.

37. Which of the following is true for $f(x)$

(a) $f(x)$ is one-one and into

(b) $\{f(x)\}$ is non-periodic, where $\{\cdot\}$ denotes fractional part of x.

(c) $f(x) = 4$ has only two solutions.

(d) $f(x) = f^{-1}(x)$ has only one solution.

38. let $g(x) = \log_e(\sin x)$, and $\int f(g(x)) \cos x\, dx = h(x) + c$, (where c is constant of integration), then $h\left(\frac{\pi}{2}\right)$ is equal to

(a) 0 (b) $\frac{1}{2010}$

(c) 1 (d) $\frac{1}{2011}$

Passage IV

(Q. Nos. 39 to 41)

Let $f : R \to R$ *be a function as* $f(x) = (x-1)(x+2)(x-3)(x-6) - 100$. *If* $g(x)$ *is a polynomial of degree* ≤ 3 *such that* $\int \frac{g(x)}{f(x)}\, dx$ *does not contain any logarithm function and* $g(-2) = 10$. *Then*

39. The equation $f(x) = 0$ has

(a) all four distinct roots

(b) three distinct real roots

(c) two real and two imaginary

(d) all four imaginary roots

40. The minimum value of $f(x)$ is

(a) -136 (b) -100

(c) -84 (d) -68

41. $\int \frac{g(x)}{f(x)}\, dx$, equals

(a) $\tan^{-1}\left(\frac{x-2}{2}\right) + c$ (b) $\tan^{-1}\left(\frac{x-1}{1}\right) + c$

(c) $\tan^{-1}(x) + c$ (d) None of these

Indefinite Integral Exercise 5 :
Matching Type Questions

42. Match the following :

	Column I		Column II
(A)	If $I = \int \frac{\sin x - \cos x}{\lvert \sin x - \cos x \rvert}\, dx$, where $\frac{\pi}{4} < x < \frac{3\pi}{8}$, then I is equal to	(p)	$\sin x$
(B)	If $\int \frac{x^2}{(x^3+1)(x^3+2)}\, dx = \frac{1}{3} f\left(\frac{x^3+1}{x^3+2}\right) + C$, then $f(x)$ is equal to	(q)	$x + C$
(C)	If $\int \sin^{-1} x \cdot \cos^{-1} x\, dx = f^{-1}(x)\left[\frac{\pi}{2} x - x f^{-1}(x) - 2\sqrt{1-x^2}\right] + 2x + C$, then $f(x)$ is equal to	(r)	$\ln \lvert x \rvert$
(D)	If $\int \frac{dx}{x f(x)} = f(f(x)) + C$, then $f(x)$ is equal to	(s)	$\sin^{-1} x$

43. Match the following :

	Column I		Column II
(A)	If $\int \left(\frac{x^2 + \cos^2 x}{1+x^2}\right) \operatorname{cosec}^2 x\, dx = A \cot^{-1} x + B \frac{\operatorname{cosec} x}{\sec x}$, then	(p)	$A = 1$
(B)	If $\int \sqrt{x + \sqrt{x^2+2}}\, dx = \frac{A}{3}(x + \sqrt{x^2+2})^{3/2} - \frac{B}{(x + \sqrt{x^2+2})}$, then	(q)	$B = -1$
(C)	If $\int \frac{\sqrt{2-x-x^2}}{x^2}\, dx = A\sqrt{\frac{2-x-x^2}{x}} + \frac{B}{4\sqrt{2}} \log\left(\frac{4 - x + 4\sqrt{2-x-x^2}}{x}\right) - \sin^{-1}\left(\frac{2x+1}{3}\right)$, then	(r)	$B = 2$
(D)	If $\int \frac{\sin 2x}{\sin^4 x + \cos^4 x}\, dx = B \cot^{-1}(\tan^2 x)$, then	(s)	$A = -1$

Indefinite Integral Exercise 6 :
Single Integer Answer Type Questions

44. If $\int \frac{(2x+3)\, dx}{x(x+1)(x+2)(x+3)+1} = C - \frac{1}{f(x)}$, where $f(x)$ is of the form of $ax^2 + bx + c$, then $(a+b+c)$ equals to ..... .

45. Let $F(x)$ be the primitive of $\frac{3x+2}{\sqrt{x-9}}$ w.r.t. x. If $F(10) = 60$, then the sum of digits of the value of $F(13)$, is ..... .

46. Let $u(x)$ and $v(x)$ are differentiable function such that $\frac{u(x)}{v(x)} = 7$. If $\frac{u'(x)}{v'(x)} = p$ and $\left(\frac{u(x)}{v(x)}\right)' = q$, then $\frac{p+q}{p-q}$ has the value equal to ..... .

47. If $\int \frac{1}{(x^2-1)} \ln\left(\frac{x-1}{x+1}\right) dx = 6A\left[\ln\left(\frac{x-1}{x+1}\right)\right]^2 + C$, then find $24A$.

48. If $\int \frac{e^x (2-x^2)}{(1-x)\sqrt{1-x^2}}\, dx = \mu e^x \left(\frac{1+x}{1-x}\right)^{\lambda} + C$, then $2(\lambda + \mu)$ is equal to ..... .

49. If $\int \frac{\cos x - \sin x + 1 - x}{e^x + \sin x + x}\,dx = \ln\{f(x)\} + g(x) + C$, where C is the constant of integrating and $f(x)$ is positive, then $\frac{f(x)+g(x)}{e^x + \sin x}$ is equal to ..... .

50. Suppose $A = \int \frac{dx}{x^2 + 6x + 25}$ and $B = \int \frac{dx}{x^2 - 6x - 27}$.

If $12(A+B) = \lambda \cdot \tan^{-1}\left(\frac{x+3}{4}\right) + \mu \cdot \ln\left|\frac{x-9}{x+3}\right| + C$, then the value of $(\lambda + \mu)$ is ... .

51. If $\int \frac{\cos 6x + \cos 9x}{1 - 2\cos 5x}\,dx = -\frac{\sin 4x}{k} - \sin x + C$, then the value of k is ..... .

52. The value of $\int \frac{\tan x}{1 + \tan x + \tan^2 x}\,dx = x - \frac{1}{\sqrt{A}}\tan^{-1}\left(\frac{2\tan x + 1}{\sqrt{A}}\right) + C$, then the value of A is..... .

53. $\int \sin^{5/2} x \cos^3 x\,dx = 2\sin^{A/2} x\left[\frac{1}{B} - \frac{1}{C}\sin^2 x\right] + C$, then the value of $(A+B) - C$ is equal to ..... .

54. If $\int (x^{2010} + x^{804} + x^{402})(2x^{1608} + 5x^{402} + 10)^{1/402}\,dx = \frac{1}{10a}(2x^{2010} + 5x^{804} + 10x^{402})^{a/402}$. Then $(a - 400)$ is equal to .....

55. If $\int e^{x^3 + x^2 - 1}(3x^4 + 2x^3 + 2x)\,dx = h(x) + c$ Then the value of $h(1)\cdot h(-1)$, is .....

Indefinite Integral Exercise 7 :
Subjective Type Questions

56. Evaluate $e^{(x\sin x + \cos x)}dx\left[\frac{x^4\cos^3 x - x\sin x + \cos x}{x^2\cos^2 x}\right]$

57. Evaluate $\int \sqrt{x + \sqrt{x^2 + 2}}\,dx$.

58. Evaluate $\int \frac{dx}{(\sqrt{(x-\alpha)^2 - \beta^2})(ax+b)}$.

59. Evaluate $\int \frac{\sqrt{1 + \sqrt[3]{x}}}{\sqrt[3]{x^2}}\,dx$.

60. Evaluate $\int \frac{\sin^3(\theta/2)\,d\theta}{\cos\theta/2\sqrt{\cos^3\theta + \cos^2\theta + \cos\theta}}$.

61. Evaluate $\int \frac{(2\sin\theta + \sin 2\theta)\,d\theta}{(\cos\theta - 1)\sqrt{\cos\theta + \cos^2\theta + \cos^3\theta}}$.

62. Connect $\int x^{m-1}(a + bx^n)^p\,dx$ with $\int x^{m-n-1}(a+bx^n)^p\,dx$ and evaluate $\int \frac{x^8\,dx}{(1-x^3)^{1/3}}$.

63. Evaluate $\int \operatorname{cosec}^2 x \ln(\cos x + \sqrt{\cos 2x})\,dx$.

64. Evaluate $\int \frac{dx}{(\sin x + a\sec x)^2}$, $a \in N$.

65. Evaluate $\int \frac{dx}{x - \sqrt{x^2 + 2x + 4}}$.

66. Evaluate $\int \frac{dx}{1 + \sqrt{x^2 + 2x + 2}}$.

67. Evaluate $\int \frac{x^4 + 1}{x^6 + 1}\,dx$.

68. Evaluate $\int \frac{dx}{(1-x^3)^{1/3}}\,dx$.

69. Evaluate $\int \frac{(x + \sqrt{1+x^2})^{15}}{\sqrt{1+x^2}}\,dx$.

70. If $y^2 = ax^2 + 2bx + c$, and $u_n = \int \frac{x^n}{y}\,dx$, prove that $(n+1)\,a\,u_{n+1} + (2n+1)\,bu_n + {}^nc\,y_{n-1} = x^n y$ and deduce that $au_1 = y - b\,u_0$; $2a^2u_2 = y(ax - 3b) - (ac - 3b^2)\,u_0$.

Indefinite Integrals Exercise 8 : Questions Asked in Previous 10 Years' Exams

(i) JEE Advanced & IIT-JEE

71. $\int \frac{\sec^2 x}{(\sec x + \tan x)^{9/2}}\,dx$ equals to (for some arbitrary constant K) **[Only One Correct Option 2012]**

(a) $\frac{-1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} - \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

(b) $\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} - \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

(c) $\frac{-1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} + \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

(d) $\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} + \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

72. If $I = \int \frac{e^x}{e^{4x} + e^{2x} + 1}\,dx$, $J = \int \frac{e^{-x}}{e^{-4x} + e^{-2x} + 1}\,dx$. Then, for an arbitrary constant c, the value of $J - I$ equals **[Only One Correct Option 2008]**

(a) $\frac{1}{2}\log\left|\frac{e^{4x} - e^{2x} + 1}{e^{4x} + e^{2x} + 1}\right| + C$ (b) $\frac{1}{2}\log\left|\frac{e^{2x} + e^x + 1}{e^{2x} - e^x + 1}\right| + C$

(c) $\frac{1}{2}\log\left|\frac{e^{2x} - e^x + 1}{e^{2x} + e^x + 1}\right| + C$ (d) $\frac{1}{2}\log\left|\frac{e^{4x} + e^{2x} + 1}{e^{4x} - e^{2x} + 1}\right| + C$

(ii) JEE Main & AIEEE

73. The integral $\int \frac{2x^{12} + 5x^9}{(x^5 + x^3 + 1)^3}\,dx$ is equal to **[2016 JEE Main]**

(a) $\frac{-x^5}{(x^5 + x^3 + 1)^2} + C$ (b) $\frac{x^{10}}{2(x^5 + x^3 + 1)^2} + C$

(c) $\frac{x^5}{2(x^5 + x^3 + 1)^2} + C$ (d) $\frac{-x^{10}}{2(x^5 + x^3 + 1)^2} + C$

where, C is an arbitrary constant.

74. The integral $\int \frac{dx}{x^2(x^4 + 1)^{\frac{3}{4}}}$ equals **[2015 JEE Main]**

(a) $\left(\frac{x^4 + 1}{x^4}\right)^{\frac{1}{4}} + C$ (b) $(x^4 + 1)^{\frac{1}{4}} + C$

(c) $-(x^4 + 1)^{\frac{1}{4}} + C$ (d) $-\left(\frac{x^4 + 1}{x^4}\right)^{\frac{1}{4}} + C$

75. The integral $\int \left(1 + x - \frac{1}{x}\right) e^{x + \frac{1}{x}}\,dx$ is equal to **[2014 JEE Main]**

(a) $(x - 1)\,e^{x + \frac{1}{x}} + C$ (b) $x e^{x + \frac{1}{x}} + C$

(c) $(x + 1)\,e^{x + \frac{1}{x}} + C$ (d) $-x e^{x + \frac{1}{x}} + C$

76. If $\int f(x)\,dx = \psi(x)$, then $\int x^5 f(x^3)\,dx$ is equal to **[2013 JEE Main]**

(a) $\frac{1}{3}[x^3\psi(x^3) - \int x^2\psi(x^3)dx] + C$

(b) $\frac{1}{3}x^3\psi(x^3) - 3\int x^3\psi(x^3)\,dx + C$

(c) $\frac{1}{3}x^3\psi(x^3) - \int x^2\psi(x^3)\,dx + C$

(d) $\frac{1}{3}[x^3\psi(x^3) - \int x^3\psi(x^3)\,dx] + C$

77. If the integral $\int \frac{5\tan x}{\tan x - 2}\,dx = x + a\log|\sin x - 2\cos x| + k$, then a is equal to **[2012 AIEEE]**

(a) -1 (b) -2

(c) 1 (d) 2

78. The value of $\sqrt{2}\int \frac{\sin x\,dx}{\sin\left(x - \frac{\pi}{4}\right)}$ is **[2012 AIEEE]**

(a) $x + \log\left|\cos\left(x - \frac{\pi}{4}\right)\right| + C$ (b) $x + \log\left|\sin\left(x - \frac{\pi}{4}\right)\right| + C$

(c) $x - \log\left|\sin\left(x - \frac{\pi}{4}\right)\right| + C$ (d) $x - \log\left|\cos\left(x - \frac{\pi}{4}\right)\right| + C$

Answers

Exercise for Session 1

1. $\frac{2}{3}(x+1)^{3/2}+\frac{2}{3}x^{3/2}+c$
2. $-\frac{2}{x}+\frac{2}{3x^3}-\frac{3}{5x^5}-2\tan^{-1}x+c$
3. $\log x+2\tan^{-1}x+c$
4. $\frac{x^3}{3}-x+\tan^{-1}x+c$
5. $\frac{1}{2}\left(\frac{x^3}{3}+\tan^{-1}x\right)+c$
6. $\tan x-\tan^{-1}x+c$
7. $\frac{1}{b^3}\left\{bx-2a\log|bx+a|-\frac{a^2}{a+bx}\right\}+c$
8. $\frac{2^x e^x}{1+\log_e 2}+c$
9. $\frac{e^{4x}}{4}+c$
10. $\frac{x^{a+1}}{a+1}+\frac{a^x}{\log a}+c$
11. $-\frac{1}{8}\cos 4x+C$
12. $\frac{1}{3}\ln|\sec 3x|-\frac{1}{2}\ln|\sec 2x|-\ln|\sec x|+e$
13. $\frac{2}{3}\sin 3x+2\sin x+C$
14. $\frac{3}{4}\sin x+\frac{1}{12}\sin 3x+C$
15. $-\frac{3}{64}\cos 2x+\frac{1}{192}\cos 6x+C$

Exercise for Session 2

1. $\tan x-\sec x+C$
2. $\sin 2x+C$
3. $-\frac{\cos 3x}{3}+C$
4. $\frac{180}{\pi}\sin x^\circ+C$
5. $x+C$
6. $2(\sin x+x\cos\alpha)+C$
7. $\sec x-\operatorname{cosec} x+C$
8. $\tan x-\cot x+C$
9. $(\sin x+\cos x)\sin(\cos x-\sin x)+C$
10. $\tan x-\cot x-3x+C$
11. $-\sqrt{2}\cos\left(\frac{x}{2}\right)+C$
12. $\frac{-\cos 4x}{8}+C$
13. $\frac{1}{2}(x-\sin x)+C$
14. $-2\cos x+C$
15. $\frac{x}{\sqrt{2}}+C$

Exercise for Session 3

1. $\frac{1}{48}\tan^{-1}\left(\frac{3x^4}{4}\right)+C$
2. $\frac{1}{36}\tan^{-1}\left(\frac{4x^3}{3}\right)+C$
3. $\frac{1}{160}\log\left|\frac{4x^4-5}{4x^4+5}\right|+C$
4. $\frac{2}{3}\sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right)+C$
5. $\frac{1}{3}\log|x^3+\sqrt{a^6+x^6}|+C$
6. $\frac{1}{12}\log\left|\frac{2+3e^x}{2-3e^x}\right|+C$
7. $\frac{1}{\log_e 2}\log|2^x+\sqrt{4^x-25}|+C$
8. $-8\sqrt{5+2x-x^2}-3\sin^{-1}\left(\frac{x-1}{\sqrt{6}}\right)+C$
9. $\frac{1}{2}\log|x^2+2x+2|+\tan^{-1}(x+1)+C$
10. $-\sqrt{3-2x-x^2}-4\sin^{-1}\left(\frac{x-1}{\sqrt{6}}\right)+C$
11. $\frac{3}{8}\left\{\log|4x^2-4x+17|+\frac{1}{6}\tan^{-1}\left(\frac{2x-1}{4}\right)+C\right\}$
12. $\frac{1}{2}\sqrt{2x^2+3x}-\frac{3}{4\sqrt{2}}\log\left|x+\frac{3}{4}\sqrt{x^2+\frac{3}{2}x}\right|+C$
13. $\sqrt{a-x(x-b)}-(a-b)\tan^{-1}\sqrt{\frac{a-x}{x-b}}+C$
14. $\sin^{-1}x+\sqrt{1-x^2}+C$
15. $\frac{(2x+5)}{4}\sqrt{x^2+x+1}+\frac{15}{8}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right|+C$
16. $\log\left|\tan\frac{x}{2}+1\right|+C$
17. $\frac{1}{2}\log\left|\tan\left(\frac{x}{2}+\frac{\pi}{6}\right)\right|+C$
18. $\frac{1}{4}\log|\sin x-\cos x|+\frac{1}{8}\cos 2x+\frac{1}{8}\sin 2x+C$
19. $\sin^{-1}\left(\frac{e^x+2}{3}\right)+C$
20. $-\frac{2}{3}\sin^{-1}(\cos^{3/2}x)+C$
21. $\frac{12}{13}(3\sin x-2\cos x)-\frac{5}{13}\log(3\cos x+2\sin x)+C$
22. $-\frac{2}{3}(4+3x-x^2)^{3/2}-\frac{1}{2}\left[\left(x-\frac{3}{2}\right)\sqrt{4+3x-x^2}+\frac{25}{4}\sin^{-1}\left(\frac{2x-3}{5}\right)\right]+C$
23. $2\sqrt{x^2+x+1}+2\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right|-6\log\left|\frac{1-x+\sqrt{x^2+x+1}}{2(x+1)}\right|+C$
24. (c)

Exercise for Session 4

1. $x^2e^x-2(xe^x-e^x)+C$
2. $-x^2\cos x+2(x\sin x+\cos x)+C$
3. $x(\log x)-x+C$
4. $x(\log x)^2-2(x\log x-x)+C$
5. $x\tan^{-1}x-\frac{1}{2}\log|1+x^2|+C$
6. $x(\sec^{-1}x)-\log|x+\sqrt{x^2-1}|+C$
7. $\frac{x^2}{2}\tan^{-1}-\frac{1}{2}(x-\tan^{-1}x)+C$
8. $-\frac{1}{x}(1+\log x)+C$
9. $-x\cot\frac{x}{2}+C$
10. $x\log(x^2+1)-2x+2\tan^{-1}x+C$
11. $e^x\cdot\log(\sec x)+C$
12. $e^x\tan x+C$
13. $x\log(\log x)-\frac{x}{\log x}+C$
14. $\frac{1}{2}e^{2x}\tan x+C$
15. $\frac{e^x}{x^2+1}+C$
16. $e^x\cdot\sqrt{\frac{1+x}{1-x}}+C$
17. $\frac{e^{ax}}{a^2+b^2}\{a\cos(bx+c)+b\sin(bx+c)\}+C$
18. $\frac{1}{2}\sec x\tan x+\frac{1}{2}\log|\sec x+\tan x|+C$
19. $-2(-\sqrt{x}\cos\sqrt{x}+\sin\sqrt{x})+C$
20. $x(\sin^{-1}x)^2-2(-\sin^{-1}x\cdot\sqrt{1-x^2}+x)+C$
21. $x\tan^{-1}x-\frac{1}{2}\log(1+x^2)-(1-x)\tan^{-1}(1-x)+\frac{1}{2}\log\{1+(1+x^2)\}+C$
22. $a\left\{\frac{x}{a}\tan^{-1}\sqrt{\frac{x}{a}}-\sqrt{\frac{x}{a}}+\tan^{-1}\sqrt{\frac{x}{a}}\right\}+C$
23. $-\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{3/2}\cdot\log\left(1+\frac{1}{x^2}\right)+\frac{2}{9}\left(1+\frac{1}{x^2}\right)^{3/2}+C$
24. $\frac{\cos x}{2\cos x-\sin x}-\frac{x}{5}-\frac{2}{5}\log|2\cos x-\sin x|+C$
25. $e^{\sin x}(x-\sec x)+C$

Exercise for Session 5

1. $\frac{1}{2}\log|x-1| - 4\log|x-2| + \frac{9}{2}\log|x-3| + C$

2. $\frac{1}{3}\log|1+x| - \frac{1}{6}\log|1-x+x^2| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C.$

3. $\frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right| + C$

4. $\frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right| + C$

5. $\log\left|\frac{1+\sin x}{2+\sin x}\right| + C$

6. $\frac{1}{10}\log|1-\cos x| - \frac{1}{2}\log|1+\cos x| + \frac{2}{5}\log|3+2\cos x| + C$

7. $\frac{1}{4}\log\left|\frac{1+\sin x}{1-\sin x}\right| + \frac{1}{2(1+\sin x)} + C$

8. $-\frac{1}{3}\log|1+\tan x| + \frac{1}{6}\log|\tan^2 x - \tan x + 1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan x - 1}{\sqrt{3}}\right) + C$

9. $\log\left|\frac{2\log x + 1}{3\log x + 2}\right| + C$

10. $-\frac{\tan^{-1} x}{x} + \log|x| - \frac{1}{2}\log|1+x^2| + C$

Exercise for Session 6

1. $\sqrt{\frac{x^4+x^2+1}{x^2}} + C,$

2. $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}\sqrt{x+1}}\right) + C.$

3. $2t^3 - 3t^2 + 6t - 6\log|1+t| + C$, where, $t = (x+1)^{1/6}$.

4. $\frac{7}{(a+b)}\left(\frac{x-b}{x+a}\right)^{1/7} + C.$

5. $\sqrt{2}\sec A\left(\sqrt{\tan x \sin A + \cos A}\right) + C$ **6.** (a) **7.** (d)

8. (c) **9.** (d) **10.** (c)

Chapter Exercises

1. (b)	**2.** (a)	**3.** (b)	**4.** (c)	**5.** (b)	**6.** (b)
7. (b)	**8.** (d)	**9.** (b)	**10.** (d)	**11.** (a)	**12.** (a)
13. (c)	**14.** (b)	**15.** (a)	**16.** (c)	**17.** (b)	**18.** (b)
19. (b)	**20.** (d)	**21.** (a,c)	**22.** (a, d)	**23.** (a, c)	**24.** (b,c,d)
25. (a, b)	**26.** (c)	**27.** (c)	**28.** (a)	**29.** (d)	**30.** (d)

31. (a)	**32.** (d)	**33.** (c)	**34.** (a)	**35.** (a)	**36.** (b)
37. (b)	**38.** (d)	**39.** (c)	**40.** (c)	**41.** (a)	

42. A → q; B → r; C → p; D → r

43. A → p, q; B → p, r; C → r; D → q **44.** (5) **45.** (6)

46. (1)	**47.** (1)	**48.** (3)	**49.** (1)	**50.** (4)	**51.** (4)
52. (3)	**53.** (3)	**54.** (3)	**55.** (1)		

56. $e^{(x\sin x + \cos x)}\cdot\left[\left(x - \frac{1}{x\cos x}\right)\right] + C$

57. $\frac{1}{3}(x+\sqrt{x^2+2})^{3/2} - 2\frac{1}{\sqrt{x+\sqrt{x^2+2}}} + C$ **58.** $\tan\frac{\theta}{2} = t$

59. $2(1+x^{1/3})^{3/2} + C$ **60.** $\tan^{-1}(\cos\theta + \sec\theta + 1)^{1/2} + C$

61. $-\frac{2}{3}\log\left|\frac{\sqrt{\cos\theta+\sec\theta+1}-\sqrt{3}}{\sqrt{\cos\theta+\sec\theta+1}+\sqrt{3}}\right| + C$

62. $-\frac{1}{8}x^6(1-x^3)^{2/3} - \frac{3}{20}x^3(1-x^3)^{2/3} - \frac{9}{40}(1-x^3)^{2/3} + C$

63. $-\cot x\log(\cos x + \sqrt{\cos 2x}) - \cot x - x + \sqrt{\cot^2 x - 1} + C$

64. $\frac{1}{(4a^2-1)^{3/2}}\left\{2a\sin^{-1}\left(\frac{2a\sin 2x + 1}{2a+\sin 2x}\right) + \sqrt{1-\left(\frac{2a\sin 2x+1}{2a+\sin 2x}\right)}\right\} - \frac{1}{(2a+\sin 2x)} + C$

65. $2\ln|\sqrt{x^2+2x+4} - x| - \frac{3\log}{2(\sqrt{x^2+2x+4}-(x+1))} - \frac{3}{2}\ln\sqrt{x^2+2x+4} - x - 1 + C$

66. $\log(x+1+\sqrt{x^2+2x+2}) + \frac{2}{(x+2)+\sqrt{(x^2+2x+2)}} + C$

67. $\tan^{-1}\left(x - \frac{1}{x}\right) - \frac{2}{3}\tan^{-1}(x^3) + C$

68. $\frac{1}{3}\log\left|\frac{(1-x^3)^{1/3}+x}{x}\right| - \frac{1}{6}\log\left|\frac{(1-x^3)^{2/3} - x(1-x^3)^{1/3} + x^2}{x^2}\right| - \frac{1}{\sqrt{3}}\tan^{-1}\left\{\frac{2(1-x^3)}{\sqrt{3x}} - x\right\}^{1/3} + C$

69. $\frac{(x+\sqrt{1+x^2})^{15}}{15} + C$ **71.** (c) **72.** (c) **73.** (b)

74. (d) **75.** (b) **76.** (c) **77.** (d) **78.** (b)

Solutions

1. Let $I = \sqrt{\dfrac{x^2}{(1+x^2)1+(1+x^2)}}\,dx$

Put $x = \tan\theta \Rightarrow dx = \sec^2 d\theta$

Putting, $x = \tan\theta$, $I = \int \dfrac{\tan^2\theta}{1+\sec\theta}\,d\theta$

$= \int (\sec\theta - 1)\,d\theta$

$= \log(\sec\theta + \tan\theta) - \theta + C$

$f(x) = \log_e(x + \sqrt{x^2+1}) - \tan^{-1} x + C$

$f(0) = 0 \Rightarrow C = 0$

$\Rightarrow \quad f(1) = \log_e(1+\sqrt{2}) - \tan^{-1} 1$

$= \log_e(1+\sqrt{2}) - \dfrac{\pi}{4}$

2. We have, $\int f(x)\,dx = f(x)$

$\Rightarrow \quad \dfrac{d}{dx}\{f(x)\} = f(x) \quad \Rightarrow \quad \dfrac{1}{f(x)}\,d\{f(x)\} = dx$

$\Rightarrow \quad \log\{f(x)\} = x + \log C \quad \Rightarrow \quad f(x) = Ce^x$

$\Rightarrow \quad \{f(x)\}^2 = C^2e^{2x}$

$\Rightarrow \quad \int \{f(x)\}^2\,dx = \int C^2e^{2x}\,dx = \dfrac{C^2e^{2x}}{2} = \dfrac{1}{2}\{f(x)\}^2$

3. We have, $\int f(x)\,dx = F(x)$

$\therefore \quad \int x^3 f(x^2)\,dx = \dfrac{1}{2}\int \underbrace{x^2}_{\text{I}}\ \underbrace{f(x^2)\,d(x^2)}_{\text{II}}$

$= \dfrac{1}{2}[x^2F(x^2) - \int F(x^2)\,d(x^2)]$

4. We have the following cases :

***Case* I** When $x \geq 0$

In this case, we have

$\int |x^n|\,dx = \int |x|^n\,dx = \int x^n\,dx \qquad [\because |x| = x]$

$= \dfrac{x^{n+1}}{n+1} + C = \dfrac{|x|^n x}{n+1} + C \qquad [\because x \geq 0 \Rightarrow |x| = x]$

***Case* II** When $x \leq 0$

In this case, we have $|x| = -x$

$\therefore \quad \int |x^n|\,dx = \int |x|^n\,dx$

$= \int (-x)^n\,dx = -\int x^n\,dx \qquad [\because n \text{ in odd}]$

$= -\dfrac{x^{n+1}}{n+1} + C = \dfrac{(-x)^n x}{n+1} + C = \dfrac{|x|^n x}{n+1} + C$

Hence, $\int |x^n|\,dx = \dfrac{|x|^n x}{n+1} + C$

5. $F(x) = \int \dfrac{3x+2}{\sqrt{x-9}}\,dx$. Let $x - 9 = t^2$

$\Rightarrow \quad dx = 2t\,dt$

$\therefore \quad F(x) = \int \left(\dfrac{3(t^2+9)+2}{t}\cdot 2t\right) dt$

$= 2\int (29 + 3t^2)\,dt = 2[29t + t^3]$

$F(x) = 2[29\sqrt{x-9} + (x-9)^{3/2}] + C$

Given, $F(10) = 60 = 2[29+1] + C \Rightarrow C = 0$

$\therefore \quad F(x) = 2[29\sqrt{x-9} + (x-9)^{3/2}]$

$F(13) = 2[29\times 2 + 4\times 2] = 132$

6. We have, $\int (x^x)^x\,(2x\log_e x + x)\,dx$

$= \int x^{x^2}(2x\log_e x + x)\,dx$

$= \int 1\cdot d(x^{x^2}) = x^{x^2} + C = (x^x)^x + C$

7. We have, $\int x\log x(\log x - 1)\,dx$

$= \int \log x\,(x\log x - x)\,dx$

$= \int (x\log x - x)\,d(x\log x - x)$

$= \dfrac{(x\log x - x)^2}{2} + C$

8. $\int \dfrac{\left(\dfrac{1}{x^3} - \dfrac{1}{x^5}\right)dx}{\sqrt{2 - \dfrac{2}{x^2} + \dfrac{1}{x^4}}}$

Let $2 - \dfrac{2}{x^2} + \dfrac{1}{x^4} = z \Rightarrow \dfrac{1}{4}\int \dfrac{dz}{\sqrt{z}}$

$\Rightarrow \quad \dfrac{1}{2}\sqrt{z} + C \Rightarrow \dfrac{1}{2}\sqrt{2 - \dfrac{2}{x^2} + \dfrac{1}{x^4}} + C$

or $\sqrt{\dfrac{2x^4 - 2x^2 + 1}{2x^2}} + C$

9. We have, $f''(x) = f(x)$

$\Rightarrow \quad 2f'(x)\,f''(x) = 2f(x)\,f'(x)$

$\Rightarrow \quad \dfrac{d}{dx}\{f'(x)\}^2 = \dfrac{d}{dx}\{f(x)\}^2$

$\Rightarrow \quad \{f'(x)\}^2 = \{f(x)\}^2 + C \qquad \ldots(i)$

Now, $f(0) = 2$ and $f'(0) = 3$. Therefore, from Eq. (i), we get

$\{f'(0)\}^2 = \{f(0)\}^2 + C$

$\Rightarrow \quad 9 = 4 + C \Rightarrow C = 5$

$\therefore \quad \{f'(x)\}^2 = \{f(x)\}^2 + 5$

$\Rightarrow \quad f'(x) = \sqrt{5 + \{f(x)\}^2}$

$\Rightarrow \quad \int \dfrac{1}{\sqrt{(\sqrt{5})^2 + \{f(x)\}^2}}\,d\{f(x)\} = \int dx$

$\Rightarrow \quad \log\left|f(x) + \sqrt{5 + \{f(x)\}^2}\right| = x + C_1$

$\therefore \quad f(0)=2 \Rightarrow \log|2+3|=C_1$

$\Rightarrow \quad C_1=\log 5$

$\therefore \quad \log|f(x)+\sqrt{5+\{f(x)\}^2}|=x+\log 5$

$\Rightarrow \log\left\{\frac{f(x)+\sqrt{5+\{f(x)\}^2}}{5}\right\}=x$

$\Rightarrow f(x)+\sqrt{5+\{f(x)\}^2}=5e^x$

$\Rightarrow \sqrt{5+\{f(x)\}^2}+f(x)=5e^x$ and $\sqrt{5+\{f(x)\}^2}-f(x)=5e^{-x}$

$\Rightarrow \quad 2f(x)=5(e^x-e^{-x})$

$\Rightarrow \quad f(x)=\frac{5}{2}(e^x-e^{-x})$

$\Rightarrow \quad f(4)=\frac{5}{2}(e^4-e^{-4}) \Rightarrow f(4)=\frac{5(e^8-1)}{2e^4}$

10. We have, $\int \frac{e^{x^2+4\ln x}-x^3e^{x^2}}{x-1}dx=\int \frac{e^{x^2}\cdot x^4-x^3e^{x^2}}{x-1}dx$

$=\int x^3e^{x^2}dx=\frac{1}{2}\int te^t dt$, [where $t=x^2$]

$=\frac{1}{2}(t-1)e^t+C=\frac{1}{2}(x^2-1)e^{x^2}+C$

11. $\int \tan^4 x\,dx=\int(\tan^2 x\sec^2 x-\sec^2 x+1)\,dx$

$=\frac{\tan^3 x}{3}-\tan x+x+C$

$\Rightarrow \quad A=\frac{1}{3}, B=-1$ and $f(x)=x+C$

12. $\int \frac{\sin^4 x}{x}dx=f(x) \Rightarrow \int \frac{\sin^4(p+q)x}{(p+q)x}dx=\frac{f\{(p+q)x\}}{p+q}$

$\int \frac{\sin^4(p+q)x}{x}dx=f\{(p+q)x\}$

13. On solving, $\frac{\sin\theta}{\cos 3\theta}=\frac{1}{2}[\tan 3\theta-\tan\theta]$

$\frac{\sin 3\theta}{\cos 9\theta}=\frac{1}{2}[\tan 9\theta-\tan 3\theta]$

$\frac{\sin 9\theta}{\cos 27\theta}=\frac{1}{2}[\tan 27\theta-\tan 9\theta]$

$\therefore \int\left(\frac{\sin\theta}{\cos 3\theta}+\frac{\sin 3\theta}{\cos 9\theta}+\frac{\sin 9\theta}{\cos 27\theta}\right)d\theta=\frac{1}{2}\int(\tan 27\theta-\tan\theta)\,d\theta$

$=\frac{1}{2}\left\{\frac{1}{27}\log(\sec 127\theta)-\log(\sec\theta)\right\}+C$

$=\frac{1}{2}\log\frac{\sqrt[27]{\sec 27\theta}}{\sec\theta}+C$

14. We have, $\int x\sqrt{\frac{2\sin(x^2+1)-\sin 2(x^2+1)}{2\sin(x^2+1)+\sin 2(x^2+1)}}dx$

$=\int x\sqrt{\frac{2\sin(x^2+1)-2\sin(x^2+1)\cos(x^2+1)}{2\sin(x^2+1)+2\sin(x^2+1)\cos(x^2+1)}}dx$

$=\int x\sqrt{\frac{1-\cos(x^2+1)}{1+\cos(x^2+1)}}dx=\int x\tan\left(\frac{x^2+1}{2}\right)dx$

$=\int \tan\left(\frac{x^2+1}{2}\right)d\left(\frac{x^2+1}{2}\right)=\log\left|\sec\left(\frac{x^2+1}{2}\right)\right|+C$

15. $\int \frac{\sin(4x-2x)\,dx}{\sin(2x)\cos(2x)\cos(4x)}=\int \frac{\sin(4x)\,dx}{\sin(2x)\cos(4x)}-\int \sec 2x\,dx$

$=2\int \frac{\cos 2x\,dx}{\cos 4x}-\frac{1}{2}(\log|\sec 2x-\tan 2x|)$

$=2\int \frac{\cos 2x}{(1-2\sin^2 2x)}dx-\frac{1}{2}(\log|\sec 2x-\tan 2x|)$

$=\frac{2}{2\sqrt{2}}\left[\frac{1}{2\times 1}\log\left|\frac{1+\sqrt{2}\sin 2x}{1-\sqrt{2}\sin 2x}\right|\right]-\frac{1}{2}\log|\sec 2x-\tan 2x|+C$

$=\frac{1}{2\sqrt{2}}\left[\log\left|\frac{1+\sqrt{2}\sin 2x}{1-\sqrt{2}\sin 2x}\right|\right]-\frac{1}{2}\log|\sec 2x-\tan 2x|+C$

16. $\int \frac{1-7\cos^2 x}{\sin^7 x\cos^2 x}dx=\int\left(\frac{\sec^2 x}{\sin^7 x}-\frac{7}{\sin^7 x}\right)dx$

$=\int \frac{\sec^2 x}{\sin^7 x}dx-\int \frac{7}{\sin^7 x}dx=I_1+I_2$

Now, $I_1=\int \frac{\sec^2 x}{\sin^7 x}dx=\frac{\tan x}{\sin^7 x}+7\int \frac{\tan x\cdot\cos x}{\sin^8 x}dx$

$=\frac{\tan x}{\sin^7 x}+I_2$

$\therefore \quad I_1+I_2=\frac{\tan x}{\sin^2 x}+C \Rightarrow f(x)=\tan x$

17. We have, $\int \frac{\sin^3 x}{(\cos^4 x+3\cos^2 x+1)\tan^{-1}(\sec x+\cos x)}dx$

$=\int \frac{\frac{\sin^3 x}{\cos^2 x}}{(\cos^2 x+3+\sec^2 x)\tan^{-1}(\sec x+\tan x)}dx$

$=\int \frac{1}{1+(\sec x+\cos x)^2}\times\frac{\sin x(1-\cos^2 x)}{\cos^2 x}$

$\times\frac{1}{\tan^{-1}(\sec x+\tan x)}dx$

$=\int \frac{1}{\tan^{-1}(\sec x+\cos x)}\times\frac{1}{1+(\sec x+\cos x)^2}$

$(\tan x\sec x-\sin x)\,dx$

$=\int \frac{1}{\tan^{-1}(\sec x+\cos x)}d|\tan^{-1}(\sec x+\cos x)|$

$=\log_e|\tan^{-1}(\sec x+\cos x)|+C$

18. We have, $f(x)=x|\cos x|, \frac{\pi}{2}<x<\pi$

$\Rightarrow \quad f(x)=-x\cos x$ [$\because \cos x<0$ for $x\in(\pi/2,\pi)$]

Hence, required primitive is given by

$\int f(x)\,dx=-\int x\cos x\,dx+C=-x\sin x-\cos x+C$

19. We have, $f(x) = (2x+1)\,|\sin x|,\ \pi < x < 2\pi$

$\Rightarrow \quad f(x) = -(2x+1)\sin x$

Hence, required primitive is given by

$$-\int \underset{\text{I}}{(2x+1)}\ \underset{\text{II}}{\sin x}\,dx = -[-(2x+1)\cos x + 2\sin x] + C$$

$$= (2x+1)\cos x - 2\sin x + C$$

20. We have, $f(x) = \begin{vmatrix} 0 & x^2-\sin x & \cos x - 2 \\ \sin x - x^2 & 0 & 1-2x \\ 2-\cos x & 2x-1 & 0 \end{vmatrix}$

$$\Rightarrow \quad f(x) = \begin{vmatrix} 0 & \sin x - x^2 & 2-\cos x \\ x^2-\sin x & 0 & 2x-1 \\ \cos x - 2 & 1-2x & 0 \end{vmatrix}$$

[Interchanging rows and columns]

$$\Rightarrow \quad f(x) = (-1)^3 \begin{vmatrix} 0 & x^2-\sin x & \cos x - 2 \\ \sin x - x^2 & 0 & 1-2x \\ 2-\cos x & 2x-1 & 0 \end{vmatrix}$$

[Taking (-1) common from each column]

$\Rightarrow \quad f(x) = -f(x)$

$\Rightarrow \quad f(x) = 0$

$\Rightarrow \quad \int f(x)\,dx = 0$

21. $$\int \frac{dx}{(x+1)(x-2)} = \int \left(-\frac{1}{3(x+1)} + \frac{1}{3(x-2)}\right) dx$$

$$= -\frac{1}{3}\log(x+1) + \frac{1}{3}\log(x-2) + C$$

$\therefore \quad A = -\frac{1}{3},\ B = \frac{1}{3}$

$\therefore \quad A + B = 0 \ \Rightarrow \ \frac{A}{B} = \frac{-\frac{1}{3}}{\frac{1}{3}} = -1$

22. $$\frac{1}{3}\int \left(\frac{1}{x^2+1} - \frac{1}{x^2+4}\right) dx$$

$$\Rightarrow \ \frac{1}{3}\tan^{-1}x - \frac{1}{6}\tan^{-1}\frac{x}{2} = k\tan^{-1}x + l\tan^{-1}\frac{x}{2}$$

$\therefore \quad k = \frac{1}{3} \quad \text{and} \quad l = -\frac{1}{6}$

23. $$I = \int \underset{\text{I}}{x}\ \underbrace{\log(1+x^2)}_{\text{II}}\,dx = \log(1+x^2)\cdot\frac{x^2}{2} - \int \frac{1}{1+x^2}\cdot 2x\cdot\frac{x^2}{2}\,dx$$

$$= \frac{x^2}{2}\log(1+x^2) - \int \frac{x^3}{x^2+1}\,dx$$

$$I = \left(\frac{x^2+1}{2}\right)\log(1+x^2) - \left(\frac{x^2+1}{2}\right) + C$$

$\therefore \quad \phi(x) = \frac{x^2+1}{2},\ \psi(x) = -\left(\frac{1+x^2}{2}\right)$

24. $$I = \int \frac{4e^2 + 6^{-x}}{9e^x - 4e^{-x}}\,dx$$

$$I = \int \frac{4e^{2x} + 6}{9e^{2x} - 4}\,dx, \text{ put } 9e^{2x} - 4 = z \ \Rightarrow \ 18e^{2x}dx = dz$$

Then, $$I = \int \frac{1}{z}\left\{4\cdot\frac{z+4}{9} + 6\right\}\frac{1}{18}\,\frac{dz}{\frac{z+4}{9}}$$

$$= \int \frac{1}{z(z+4)}\left\{\frac{2z+8}{9} + 3\right\} dz$$

$$= \frac{1}{9}\int \frac{2z+35}{z(z+4)}\,dz = \frac{1}{9}\int \frac{2(z+4)+27}{z(z+4)}\,dz$$

$$= \frac{2}{9}\int \frac{dz}{z} + \frac{3}{4}\int\left(\frac{1}{z} - \frac{1}{z+4}\right) dz$$

$$= \left(\frac{2}{9} + \frac{3}{4}\right)\log z - \frac{3}{4}\log(z+4) + C$$

$$= \frac{35}{36}\log(9e^{2x} - 4) = \frac{3}{4}\log(e^{2x}) + C$$

$$= -\frac{3}{2}x + \frac{35}{36}\log(9e^{2x} - 4) - \frac{3}{2}\log 3 + C$$

25. $$\int \tan^3 x\,(\sec^2 x - 1)\,dx = \int \tan^3 x\sec^2 x\,dx - \int \tan^3 x\,dx$$

$$I_1 = \int \tan^3 x\sec^2 x\,dx = \frac{\tan^4 x}{4} + C_1$$

$$I_2 = \int \tan x\,(\sec^2 x - 1)\,dx$$

$$= \frac{\tan^2 x}{2} - \ln|\sec x| + C_2$$

26. The Statement II is false since while writing

$$\int \frac{dx}{x-3y} = \log(x-3y) + C,$$

we are assuming that y is a constant. We will know prove the Statement I. From the given relation $(x-y)^2 = \frac{x}{y}$ and

$$2\log(x-y) = \log x - \log y.$$

Also, $\frac{dy}{dx} = \left(-\frac{y}{x}\right)\cdot\frac{x+y}{x-3y}$. To prove the integral relation it is sufficient to show that $\frac{d}{dx}\text{RHS} = \frac{1}{x-3y}$.

Now, $\text{RHS} = \frac{1}{2}\log\left[\frac{x}{y} - 1\right] \qquad \left[\because (x-y)^2 = \frac{x}{y}\right]$

$$= \frac{1}{2}[\log(x-y) - \log y]$$

$$= \frac{1}{2}\left[\frac{\log x - \log y}{2} - \log y\right] = \frac{1}{4}[\log x - 3\log y]$$

$$\Rightarrow \ \frac{d}{dx}\text{RHS} = \frac{1}{4}\left[\frac{1}{x} - \frac{3}{y}\frac{dy}{dx}\right]$$

$$= \frac{1}{4}\left[\frac{1}{x} - \frac{3}{y}\left(-\frac{y}{x}\right)\frac{x+y}{x-3y}\right] = \frac{1}{x-3y}$$

Thus, Statement I is true. Hence, choice (c) is correct.

27. Let $g(x) = f(x) + f(-x)$

Assuming, $\int f(x)\,dx = F(x) + C$

$$\int g(x)\,dx = \int \{f(x) + f(-x)\}\,dx$$
$$= \int f(x)\,dx + \int f(-x)\,dx$$
$$= F(x) + C + \{-F(-x) + C'\}$$
$$= F(x) - F(-x) + C + C'$$

which may be an odd function, if $C + C' = 0$.

Similarly, integral of an odd function is not always an even function.

Hence, Statement I is true and Statement II is false.

28. If $a > 0$ and $b^2 - 4ac < 0$, then

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

$$\Rightarrow \quad \int \frac{dx}{ax^2 + bx + c} = \int \frac{dx}{a\left(x + \frac{b}{2a}\right)^2 + k^2}$$

where $k^2 = \dfrac{4ac - b^2}{4a} > 0$. which will have an answer of the type

$$\frac{1}{a} \cdot \frac{1}{k/\sqrt{a}} \tan^{-1} \frac{x + \frac{b}{2a}}{k/\sqrt{a}} + C \quad \text{or} \quad \mu \tan^{-1} \frac{x + A}{B} + C$$

Thus, choice (a) is correct.

29. $I = \int \dfrac{dx}{x^4 + 1} \Rightarrow I = \dfrac{1}{2} \int \dfrac{\frac{2}{x^2}}{x^2 + \frac{1}{x^2}}\,dx$

$$= \frac{1}{2} \int \frac{\left(1 + \frac{1}{x^2}\right)}{\left(x - \frac{1}{x}\right)^2 + 2}\,dx - \frac{1}{2} \int \frac{\left(1 - \frac{1}{x^2}\right)}{\left(x + \frac{1}{x}\right)^2 - 2}\,dx$$

$$I = \frac{1}{2} \cdot \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x - \frac{1}{x}}{2}\right) + \frac{1}{2} \cdot \frac{1}{2} \log\left(\frac{x + \frac{1}{x} - 1}{x + \frac{1}{x} + 1}\right) + C$$

$\therefore$ Statement I is false.

30. Since, $\cot^{-1} x = \dfrac{\pi}{2} - \tan^{-1} x,$

$\therefore \quad d(\cot^{-1} x) = -\,d\,(\tan^{-1} x)$

Thus, $\int 2^{\tan^{-1} x}\,d(\cot^{-1} x) = -\int 2^{\tan^{-1} x}\,d(\tan^{-1} x)$

$$= -\frac{2^{\tan^{-1} x}}{\ln 2} + C$$

$\therefore$ Statement I is false. Statement II is true.

31. As $m = 9 > 0$, hence, we can substitute

$$\sqrt{9x^2 + 4x + 6} = u \pm 3x$$

32. Here, as per notations given, we can substitute

$$\sqrt{1 + x^2} = (u - x)$$

As $m = 1 > 0$ and $p = 1 > 0$

$$\Rightarrow \quad I = \int \frac{u^{15}}{u}\,du = \int u^{14} du = \frac{1}{15} u^{15} + C$$
$$= \frac{1}{15}(x + \sqrt{1 + x^2})^{15} + C$$

33. Here, $m = -1 < 0$

$p = -2 < 0$

Also, $-x^2 + 3x - 2 = -(x - 1)(x - 2)$

We can use case III

$\Rightarrow$ Putting, $\sqrt{-x^2 + 3x - 2} = u(x - 2)$

or $\quad (x - 1)\,u \quad$ or $\quad u(1 - x)$

34. Let $P = \sin^3 x \cos^3 x$

$$\frac{dP}{dx} = 3\sin^2 x \cos^4 x - 3\sin^4 x \cos^2 x$$
$$= 3\sin^2 x (1 - \sin^2 x)\cos^2 x - 3\sin^4 x \cos^2 x$$
$$= 3\sin^2 x \cos^2 x - 6\sin^4 x \cos^2 x$$

$\therefore \quad P = 3I_{2,2} - 6I_{4,2}$

$\therefore \quad I_{4,2} = \dfrac{1}{6}(-P + 3I_{2,2})$

35. Let $P = \sin^5 x \cos^3 x$

$\therefore \quad \dfrac{dP}{dx} = 5\sin^4 x \cos^4 x - 3\sin^6 x \cos^2 x$

$$= 5\sin^4 x (1 - \sin^2 x)\cos^2 x - 3\sin^6 x \cos^2 x$$
$$= 5\sin^4 x \cos^2 x - 8\sin^6 x \cos^2 x$$

$\therefore \quad P = 5I_{4,2} - 8I_{6,2}$

$\therefore \quad I_{4,2} = \dfrac{1}{5}(P + 8I_{6,2})$

36. Let $P = \sin^5 x \cos^3 x$

$\therefore \quad \dfrac{dP}{dx} = 5\sin^4 x \cos^4 x - 3\sin^6 x \cos^2 x$

$$= 5\sin^4 x \cos^4 x - 3\sin^4 x (1 - \cos^2 x)\cos^2 x$$
$$= 8\sin^4 x \cos^4 x - 3\sin^4 x \cos^2 x$$

$\therefore \quad P = 8I_{4,4} - 3I_{4,2}$

$\therefore \quad I_{4,2} = \dfrac{1}{3}(-P + 8I_{4,4})$

37. Here, $2f'(x) = \lim\limits_{h \to 0}\left(\dfrac{f(x + h) - f(x)}{h} + \dfrac{f(x\,x - h) - f(x)}{-h}\right)$

$$= \lim_{h \to 0}\left(\frac{f(x + h) - f(x - h)}{h}\right) \qquad \ldots(i)$$

$\therefore \quad 2f'(0) \lim\limits_{h \to 0}\left(\dfrac{f(h) - f(0)}{h} + \dfrac{f(-h) - f(0)}{-h}\right)$

$$= \lim_{h \to 0} \frac{f(h) - f(-h)}{h} \ldots(ii)$$

Now by given relation, we have

$f(h) - f(-h) = \dfrac{f(x + h) - f(x - h)}{-h}$ and $f(0) = 1$

From Eqs. (i) and (ii), we have $\frac{f'(x)}{f(x)} = 2010$

$\Rightarrow \quad f(x) = e^{2010e}, f(0) = 1$

$\therefore$ $\{f(x)\}$ is non-periodic.

38. Here, $\int f(g(x)) \cos x\, dx = \int f(\log(\sin x)) . \cos x\, dx$

$$= \int e^{2010 \log (\sin x)} . \cos x\, dx$$

$$= \int (\sin x)^{2010} . \cos x\, dx$$

$$= \int (\sin x)^{2010} . \cos x\, dx$$

$$= \frac{(\sin x)^{2011}}{2011} + C$$

$\therefore \quad h(x) = \frac{(\sin x)^{2011}}{2011}$

$\Rightarrow \quad h\left(\frac{\pi}{2}\right) = \frac{1}{2011}$

Sol. (Q.Nos. 39 to 41)

Here, $f(x) = (x-1)(x+2)(x-3)(x-6) - 100$

$$= (x^2 - 4x + 3)(x^2 - 4x - 12) - 100$$

$$= (x^2 - 4x)^2 - 9(x^2 - 4x) - 136$$

$$= (x^2 - 4x + 8)(x^2 - 4x - 17)$$

39. $\therefore f(x) = 0 \Rightarrow \underbrace{(x^2 - 4x + 8)}_{D > 0} \underbrace{(x^2 - 4x + 17}_{D < 0} = 0$

$\therefore$ Equation has two distinct and two imaginary roots.

40. $f(x) = (x^2 - 4x - 17)(x^2 - 4x + 8)$

$$= \{((x-2)^2 - 21\}\{(x-2)^2 + 4\}$$

$\because \quad (f(x))_{\min} = (-21)(4) = -84$

which occurs at $x = 2$

41. $\because \quad \int \frac{g(x)}{f(x)} = \frac{g(x)}{(x^2 - 4x - 17)(x^2 - 4x + 8)}$

$$= \frac{Ax + B}{x^2 - 4x - 17} + \frac{Cx + D}{x^2 - 4x + 8}$$

Clearly, A, B and C must be zero.

$\therefore \quad \frac{g(x)}{(x^2 - 4x - 17)(x^2 - 4x + 8)} = \frac{D}{x^2 - 4x + 8}$

$\therefore \quad g(x) = D(x^2 - 4x - 17)$

$g(-2) = D(4 + 8 - 17) = -10$ [given]

$\Rightarrow \quad \frac{g(x)}{f(x)} = \frac{2(x^2 - 4x - 17)}{(x^2 - 4x - 17)(x^2 - 4x + 8)} = \frac{2}{x^2 - 4x + 8}$

$\therefore \quad \int \frac{g(x)}{f(x)}\, dx = \int \frac{2}{x^2 - 4x + 8}\, dx = 2\int \frac{dx}{(x-2)^2 + (2)^2}$

$$= 2 \cdot \frac{1}{2} \tan^{-1}\left(\frac{x-2}{2}\right) + C = \tan^{-1}\left(\frac{x-2}{2}\right) + C$$

42. (A) If $\frac{\pi}{4} < x < \frac{3\pi}{8}$, then $\sin x > \cos x$

$\therefore \quad \int \frac{\sin x - \cos x}{|\sin x - \cos x|}\, dx = \int 1\, dx = x + C$

(B) $\int \frac{x^2 dx}{(x^3 + 1)(x^3 + 2)} = \frac{1}{3}\int 3x^2 \left(\frac{1}{x^3 + 1} - \frac{1}{x^3 + 2}\right) dx$

$$= \frac{1}{3} \ln \left|\frac{x^3 + 1}{x^3 + 2}\right| + C$$

(C) $\int \sin^{-1} x \cos^{-1} x\, dx = \int \left[\frac{\pi}{2} \sin^{-1} x - (\sin^{-1} x)^2\right] dx$

$\Rightarrow \frac{\pi}{2}(x \sin^{-1} x + \sqrt{1 - x^2}) - \{x(\sin^{-1} x)^2$

$$+ \sin^{-1} x \sqrt{1 - x^2} - x\} + C \text{ (by parts)}$$

$\Rightarrow \sin^{-1} x \left[\frac{\pi}{2} x - x \sin^{-1} x - 2\sqrt{1 - x^2}\right] + \frac{\pi}{2}\sqrt{1 - x^2} + 2x + C$

$\therefore \quad f^{-1}(x) = \sin^{-1} x,\ f(x) = \sin x$

(D) $\int \frac{dx}{x \ln |x|} = \ln |\ln |x|| + C$

$\therefore \quad f(x) = \ln |x|$

43. (A) $I = \int \left(\frac{x^2 + \cos^2 x}{1 + x^2}\right) \cdot \operatorname{cosec}^2 x\, dx$

$$= \int \left(\frac{x^2 + 1 - \sin^2 x}{1 + x^2}\right) \cdot \operatorname{cosec}^2 x\, dx$$

$$= \int \left(\operatorname{cosec}^2 x - \frac{1}{1 + x^2}\right) dx = -\cot x + \cot^{-1} x + k$$

$\Rightarrow \quad A = 1,\ B = -1$

(B) $I = \int \sqrt{x + \sqrt{x^2 + 2}}\, dx$

Put $\sqrt{x^2 + 2} + x = t \quad \Rightarrow \quad \sqrt{x^2 + 2} - x = \frac{2}{t}$

$\therefore \quad 2x = t - \frac{2}{t} \quad \Rightarrow \quad 2\, dx = \left(1 + \frac{2}{t^2}\right) dt$

$$= \frac{1}{2}\int \left(\sqrt{t} + \frac{2}{t^{3/2}}\right) dt$$

$\Rightarrow \quad I = \frac{1}{2} \cdot \frac{t^{3/2}}{\frac{3}{2}} + \frac{1}{-\frac{1}{2} t^{1/2}} + k$

$$= \frac{1}{3}(x + \sqrt{x^2 + 2})^{3/2} - \frac{2}{\sqrt{x + \sqrt{x^2 + 2}}} + k$$

$\Rightarrow \quad A = 1$ and $B = 2$

(C) $I = \int \frac{2 - x - x^2}{x^2 \sqrt{2 - x - x^2}}\, dx$

$$= -\int \frac{dx}{\sqrt{2 - x - x^2}} + 2\int \frac{dx}{x^2 \sqrt{2 - x - x^2}} - \int \frac{dx}{x\sqrt{2 - x - x^2}}$$

$$= -\int \frac{dx}{\sqrt{\frac{9}{4} - \left(x + \frac{1}{2}\right)^2}} + 2\int -\frac{dt}{\sqrt{2 - \frac{1}{t} - \frac{1}{t^2}}} + \int \frac{\frac{1}{t}dt}{\sqrt{2 - \frac{1}{t} - \frac{1}{t^2}}} \quad \left(\text{put } x = \frac{1}{t}\right)$$

$$= -\sin^{-1}\left(\frac{2x+1}{3}\right) - \frac{1}{2}\int \frac{(4t-1)dt}{\sqrt{2t^2 - t - 1}} + \frac{1}{2}\int \frac{dt}{\sqrt{2t^2 - t - 1}}$$

$$= -\sin^{-1}\left(\frac{2x+1}{3}\right) - \frac{1}{2}\frac{\sqrt{2t^2 - t - 1}}{\frac{1}{2}} + \frac{1}{2\sqrt{2}}\int \frac{dt}{\sqrt{t^2 - \frac{t}{2} - \frac{1}{2} + \frac{1}{16} - \frac{1}{16}}}$$

$$= -\sin^{-1}\left(\frac{2x+1}{3}\right) - \frac{\sqrt{2 - x - x^2}}{x} + \frac{1}{2\sqrt{2}}\log\left|\left(t - \frac{1}{4}\right) + \sqrt{2t^2 - t - 1}\right| + K$$

$$= -\frac{\sqrt{2 - x - x^2}}{x} + \frac{1}{2\sqrt{2}}\log\left|\frac{(4-x) + \sqrt{2 - x - x^2}}{4x}\right| - \sin^{-1}\left(\frac{2x+1}{3}\right) + K$$

(D) $\int \frac{\sin 2x}{\sin^4 x + \cos^4 x}\,dx$, dividing N^r and D^r by $\cos^4 x$

$$I = \int \frac{2\tan x \cdot \sec^2 x}{\tan^4 x + 1}dx, \text{ put } \tan^2 x = t$$

$\Rightarrow 2\tan x \cdot \sec^2 x\,dx = dt = \int \frac{dt}{t^2+1} = -\cot^{-1}(t) + C$

$\Rightarrow -\cot^{-1}(\tan^2 x) + C \quad \therefore B = -1$

44. $\int \frac{2x+3}{(x^2+3x)(x^2+3x+2)+1}\,dx$

Put $x^2 + 3x = t \Rightarrow (2x+3)\,dx = dt$

$$\int \frac{dt}{t(t+2)+1} = \int \frac{dt}{(t+1)^2} = C - \frac{1}{t+1} = C - \frac{1}{x^2+3x+1}$$

$\Rightarrow \quad a = 1, b = 3, c = 1 \Rightarrow a + b + c = 5$

45. $F(x) = \int \frac{3x+2}{\sqrt{x-9}}\,dx.$

Let $x - 9 = t^2 \Rightarrow dx = 2t\,dt$

$\therefore \quad F(x) = \int \left(\frac{3(t^2+9)+2}{t} \cdot 2t\right) dt$

$$= 2\int (29 + 3t^2)\,dt = 2[29t + t^3]$$

$$F(x) = 2[29\sqrt{x-9} + (x-9)^{3/2}] + C$$

Given, $F(10) = 60 = 2[29 + 1] + C$

$\Rightarrow \quad C = 0$

$\therefore \quad F(x) = 2[29\sqrt{x-9} + (x-9)^{3/2}]$

$$F(13) = 2[29 \times 2 + 4 \times 2]$$

$$= 4 \times 33 = 132$$

Hence, sum of digits $= 1 + 3 + 2 = 6$

46. $u(x) = 7v(x) \Rightarrow u'(x) = 7v'(x)$

$\Rightarrow \quad p = 7$ (given)

Again, $\frac{u(x)}{v(x)} = 7 \Rightarrow \left(\frac{u(x)}{v(x)}\right)' = 0$

$\Rightarrow \quad q = 0$

Now, $\frac{p+q}{p-q} = \frac{7+0}{7-0} = 1$

47. Let $t = \ln\left(\frac{x-1}{x+1}\right) \Rightarrow \frac{dt}{dx} = \frac{2}{x^2-1}$

$$I = \int \frac{1}{2}t\,dt = \frac{1}{4}t^2 + C$$

$$I = \frac{1}{4}\left[\ln\left(\frac{x-1}{x+1}\right)\right]^2 + C$$

$\Rightarrow \quad 6A = \frac{1}{4} \Rightarrow 24A = 1$

48. $I = \int e^x \left(\frac{1}{(1-x)\sqrt{1-x^2}} + \sqrt{\frac{1+x}{1-x}}\right) dx$

As $\frac{d}{dx}\left(\sqrt{\frac{1+x}{1-x}}\right) = \frac{1}{(1-x)\sqrt{1-x^2}}$

$\therefore \quad I = e^x\sqrt{\frac{1+x}{1-x}} + C \Rightarrow I = e^x\left(\frac{1+x}{1-x}\right)^{1/2} + C$

$\Rightarrow \quad \mu = 1, \lambda = \frac{1}{2}$

$\Rightarrow \quad 2(\mu + \lambda) = 2\left(1 + \frac{1}{2}\right) = 2 \times \frac{3}{2} = 3$

49. $I = \int \frac{(e^x + \cos x + 1) - (e^x + \sin x + x)}{e^x + \sin x + x}\,dx$

$$= \ln(e^x + \sin x + x) - x + C$$

$\therefore \quad f(x) = e^x + \sin x + x$ and $g(x) = -x$

$\Rightarrow \quad f(x) + g(x) = e^x + \sin x$

$\Rightarrow \quad \frac{f(x) + g(x)}{e^x + \sin x} = 1$

50. $12\left[\frac{1}{4}\tan^{-1}\frac{x+3}{4} + \frac{1}{2\cdot 6}\ln\left|\frac{x-9}{x+3}\right|\right] = 3\tan^{-1}\left(\frac{x+3}{4}\right) + \ln\left|\frac{x-9}{x+3}\right|$

$\Rightarrow \quad \lambda = 3, \mu = 1$

$\Rightarrow \quad \lambda + \mu = 4$

51. $\frac{2\cos\frac{15}{2}x\cos\frac{3}{2}x}{1 - 2\left(2\cos^2\frac{5x}{2} - 1\right)} = \frac{2\left(4\cos^3\frac{5x}{2} - 3\cos\frac{5x}{2}\right)\cos\frac{3x}{2}}{3 - 4\cos^2\frac{5x}{2}}$

$$= -2\cos\frac{5x}{2}\cos\frac{3x}{2}$$

$$= -(\cos 4x + \cos x)$$

$$I = -\frac{\sin 4x}{4} - \sin x + C$$

52. $I=\int \frac{\tan x}{1+\tan x+\tan^2 x}\,dx=\int \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos^2 x}+\frac{\sin x}{\cos x}}\,dx$

$=\int \frac{\sin 2x}{2+\sin 2x}\,dx=\int dx-2\int \frac{dx}{2+\sin 2x}$

$=x-2\int \frac{\sec^2 x}{2\sec^2 x+2\tan x}\,dx$

Let $t=\tan x, dt=\sec^2 x\,dx$

$=x-\frac{2}{2}\int \frac{dx}{t^2+t+1}=x-\int \frac{dt}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$

$I=x-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan x+1}{\sqrt{3}}\right)+C$

53. $\int \sin^{5/2}x\cos^3 x\,dx=\int \sin^2 x\sin^{1/2}x\cos^2 x\cos x\,dx$

$=\int \sin^2 x\sin^{1/2}x(1-\sin^2 x)\cos x\,dx$

$=\int t^2 t^{1/2}(1-t^2)\,dt=\int [t^{5/2}-t^{9/2}]dt$

$=\int t^{5/2}dt-\int t^{9/2}dt=\frac{t^{7/2}}{7/2}-\frac{t^{11/2}}{11/2}+C$

$=\frac{2}{7}t^{7/2}-\frac{2}{11}t^{11/2}+C=\frac{2}{7}\sin^{7/2}x-\frac{2}{11}\sin^{11/2}x+C$

$=2\sin^{7/2}x\left[\frac{1}{7}-\frac{1}{11}\sin^2 x\right]+C$

$\Rightarrow \quad A=7, B=7, C=11$

$\Rightarrow \quad (A+B)-C=(7+7)-11=3$

54. Let $I=\int (x^{2010}+x^{804}+x^{402})(2x^{1608}+5x^{402}+10)^{1/402}dx$

$=\int x(x^{2009}+x^{508}+x^{401})\cdot(2x^{1608}+5x^{402}+10)^{1/402}dx$

$=\int (x^{2009}+x^{803}+x^{401})\cdot(2x^{2010}+5x^{804}+10^{402})^{1/402}dx$

Put $2x^{2010}+5x^{804}+10^{x^{402}}=t$

$\Rightarrow \quad 4020(x^{2009}+x^{803}+x^{401})\,dx=dt$

$\therefore \quad I=\int \frac{1}{4020}\cdot(t)^{1/402}dt=\frac{1}{4020}\cdot\frac{t^{1/420+1}}{1/402+1}$

$=\frac{1}{4020}\cdot\frac{t^{403/402}}{403/402}$

$=\frac{1}{4030}(2x^{2010}+5x^{804}+10^{402})^{403/402}$

$\therefore \; a-400=3$

55. Let $e^{x^3+x^2-1}(3x^4+2x^3+2x)\,dx$

$=\int x^2\cdot e^{x^3+x^2-1}\cdot(3x^2+2x)\,dx+\int e^{x^3+x^2-1}\cdot(2x)\,dx$

I II

Applying by parts in first internal, we get

$I=x\cdot e^{x^3+x^2-1}-\int 2x\cdot e^{x^3+x^2-1}dx+\int e^{x^3-x^2-1}(2x)\,dx$

$=x^2\cdot e^{x^3+x^2-1}+C=h(x)+C$

$\therefore \quad h(x)=x^2\cdot e^{x^3+x^2-1}$

$\Rightarrow \quad h(1)\cdot h(-1)=e^1\cdot e^{-1}=1$

56. $I=\int e^{(x\sin x+\cos x)}\left\{\frac{x^4\cos^3 x-x\sin x+\cos x}{x^2\cos^2 x}\right\}dx$

$=\int e^{(x\sin x+\cos x)}\cdot(x^2\cos x)\,dx$

$-\int e^{(x\sin x+\cos x)}\cdot\frac{d}{dx}\left(\frac{1}{x\cos x}\right)dx$

$=e^{(x\sin x+\cos x)}\cdot\left(x-\frac{1}{x\cos x}\right)+C$

57. $I=\int \sqrt{x+\sqrt{x^2+2}}\,dx$

Let $x+\sqrt{x^2+2}=p$ or $x^2+2=p^2+x^2-2px$

$\Rightarrow \quad x=\frac{p^2-2}{2p}$ or $dx=\frac{(p^2+2)\,dp}{2p^2}$

$I=\int \frac{p^{1/2}(p^2+2)}{2p^2}\,dp=\frac{1}{2}\int p^{1/2}\,dp+\int p^{-3/2}\,dp$

$=\frac{1}{3}(x+\sqrt{x^2+2})^{3/2}-2\frac{1}{\sqrt{x+\sqrt{x^2+2}}}+C$

58. $I=\int \frac{dx}{(ax+b)\sqrt{(x-\alpha)^2-\beta^2}}$

Put $(x-\alpha)=\beta\sec\theta \Rightarrow dx=\beta\sec\theta\tan\theta\,d\theta$

$\therefore \quad I=\int \frac{d\theta}{a(\alpha\cos\theta+\beta)+b\cos\theta}$

$=\int \frac{d\theta}{(a\alpha+b)\cos\theta+a\beta}$

$=\frac{1}{(a\alpha+b)}\int \frac{d\theta}{\cos\theta+\left(\frac{a\beta}{a\alpha+b}\right)}$

if $\left|\frac{a\beta}{a\alpha+b}\right|<1$ ...(i)

Then, $I=\frac{1}{(a\alpha+b)}\cdot\operatorname{cosec}\alpha^1\log\left|\frac{\cot\frac{\alpha^1}{2}+1}{\cot\frac{\alpha^1}{2}-1}\right|+C$

where $\frac{a\beta}{a\alpha+b}=\cos\alpha^1,\ \tan\frac{\theta}{2}=t$

Again, if $\left|\frac{a\beta}{a\alpha+b}\right|>1$

$I=\frac{1}{a\alpha+b}\cot\alpha^1\tan^{-1}\left(t\tan\frac{\alpha^1}{2}\right)$

where $\sec\alpha^1=\frac{a\beta}{a\alpha+b} \Rightarrow \tan\frac{\theta}{2}=t$

59. $I=\int \frac{\sqrt{1+\sqrt[3]{x}}}{\sqrt[3]{x^2}}\,dx=\int x^{-2/3}(1+x^{1/3})^{1/2}\,dx,$

$m=-\frac{2}{3}, n=\frac{1}{3}, p=\frac{1}{2}$

$\therefore \quad \dfrac{m+1}{n} = 1$ ie, integer

$\therefore$ Let us make the substitution,

$$1 + x^{1/3} = t^2 \quad \therefore \quad \frac{1}{3} x^{-2/3}\, dx = 2t\, dt$$

Hence, $$I = 6 \int t^2\, dt = 2t^3 + C = 2(1 + x^{1/3})^{3/2} + C$$

60. $$I = \int \frac{\sin^3(\theta/2)}{\cos\theta/2 \sqrt{\cos^3\theta + \cos^2\theta + \cos\theta}}\, d\theta$$

$$= \frac{1}{2} \int \frac{2 \sin\theta/2 \cdot \cos\theta/2 \cdot 2\sin^2\theta/2}{2\cos^2\theta/2 \sqrt{\cos^3\theta + \cos^2\theta + \cos\theta}}\, d\theta$$

$$= \frac{1}{2} \int \frac{\sin\theta(1 - \cos\theta)}{(1 + \cos\theta)\sqrt{\cos^3\theta + \cos^2\theta + \cos\theta}}\, d\theta$$

Put $\cos\theta = t \Rightarrow -\sin\theta\, d\theta = dt$

$\therefore$ $$I = \frac{1}{2} \int \frac{(t-1)}{(t+1)\sqrt{t^3 + t^2 + t}}\, dt$$

$$= \frac{1}{2} \int \frac{t^2 - 1}{(t+1)^2 \sqrt{t^3 + t^2 + t}}\, dt$$

$$= \frac{1}{2} \int \frac{(1 - 1/t^2)}{(t + \frac{1}{t} + 2)\sqrt{t + 1 + \frac{1}{t}}}\, dt$$

Put $t + 1 + \dfrac{1}{t} = u^2 \Rightarrow \left(1 - \dfrac{1}{t^2}\right) dt = 2u\, du$

$$I = \frac{1}{2} \int \frac{2u\, du}{(u^2 + 1) \cdot u} = \int \frac{du}{u^2 + 1} = \tan^{-1}(u) + C$$

$$= \tan^{-1}\left(t + 1 + \frac{1}{t}\right)^{1/2} + C$$

$$= \tan^{-1}(\cos\theta + \sec\theta + 1)^{1/2} + C$$

61. $$I = \int \frac{(2\sin\theta + \sin 2\theta)\, d\theta}{(\cos\theta - 1)\sqrt{\cos\theta + \cos^2\theta + \cos^3\theta}}$$

Put $\cos\theta = x^2$

$\Rightarrow \quad -\sin\theta\, d\theta = 2x\, dx$

$$= 2 \int \frac{(1 + x^2)}{(1 - x^2)} \cdot \frac{2x\, dx}{\sqrt{x^2 + x^4 + x^6}}$$

$$= 4 \int \frac{(1 + 1/x^2)\, dx}{(1/x - x)\sqrt{(1/x - x)^2 + 3}},$$

Put $\dfrac{1}{x} - x = t$

$\Rightarrow \left(-\dfrac{1}{x^2} - 1\right) dx = dt$

$$= -4 \int \frac{dt}{t\sqrt{t^2 + 3}}$$

Again, put $t^2 + 3 = u^2$

$\Rightarrow \quad 2t\, dt = 2u\, du$

$\therefore$ $$I = 4 \int \frac{-u\, du}{u(u^2 - 3)} = -4 \int \frac{du}{u^2 - 3}$$

$$= -\frac{2}{\sqrt{3}} \log\left|\frac{u - \sqrt{3}}{u + \sqrt{3}}\right| + C$$

$$= -\frac{2}{\sqrt{3}} \log\left|\frac{\sqrt{t^2 + 3} - \sqrt{3}}{\sqrt{t^2 + 3} + \sqrt{3}}\right| + C$$

$$= -\frac{2}{\sqrt{3}} \log\left|\frac{\sqrt{x^2 + 1/x^2 + 1} - \sqrt{3}}{\sqrt{x^2 + 1/x^2 + 1} + \sqrt{3}}\right| + C$$

$$= -\frac{2}{3} \log\left|\frac{\sqrt{\cos\theta + \sec\theta + 1} - \sqrt{3}}{\sqrt{\cos\theta + \sec\theta + 1} + \sqrt{3}}\right| + C$$

62. Let $$I_{m-1} = \int x^{m-1}(a + bx^n)^p\, dx$$

and $$I_{m-n-1} = \int x^{m-n-1}(a + bx^n)^p\, dx$$

Let $$p = x^{\lambda+1}(a + bx^n)^{\mu+1}$$

where λ and μ are the smaller indices of x and $(a + bx^n)$.

Here, $\lambda = m - n - 1, \mu = p$

$\therefore$ $$p = x^{m-n}(a + bx^n)^{p+1}$$

Differentiating w.r.t. x, we have

$$\frac{dp}{dx} = x^{m-n}(p+1)(a + bx^n)^p (n\, bx^{n-1}) + (a + bx^n)^{p+1}(m - n)\, x^{m-n-1}$$

$$= nb(p+1)\, x^{m-1}(a + bx^n)^p + (m - n)\, x^{m-n-1}(a + bx^n)^p \{a + bx^n\}$$

$$= nb(p+1)\, x^{m-1}(a + bx^n)^p + a(m - n)\, x^{m-n-1}(a + bx^n)^p + b(m - n)\, x^{m-1}(a + bx^n)^p$$

$= b(np + m)\, x^{m-1}(a + bx^n)^p + a(m - n)\, x^{m-n-1}(a + bx^n)^p$Integrating both the sides w.r.t. x, we get

$$p = b(np + m)\, I_{m-1} + a(m - n)\, I_{m-n-1}$$

$\therefore\ x^{m-n}(a + bx^n)^{p+1} = b(np + m)\, I_{m-1} + a(m - n)\, I_{m-n-1}$

or $$I_{m-1} = \frac{x^{m-n}(a + bx^n)^{p+1}}{b(np + m)} - \frac{a(m - n)}{b(np + m)} \cdot I_{m-n-1}$$

Hence Proved.

Again, $$\int \frac{x^8}{(1 - x^3)^{1/3}}\, dx = \int x^{9-1}(1 - x^3)^{-1/3}\, dx$$

Here, $m = 9, \quad b = -1, \quad n = 3, \quad p = -1/3, \quad a = 1$

$\therefore$ $$I_8 = \frac{(x^6(1 - x^3)^{2/3})}{-8} + \frac{6}{8} I_5 \qquad \text{...(i)}$$

$\Rightarrow$ $$I_5 = \frac{x^3(1 - x^3)^{2/3}}{-5} + \frac{3}{5} I_2 \qquad (\text{here } m = 6)$$

$\Rightarrow$ $$I_2 = \frac{(1 - x^3)^{2/3}}{2}$$

Hence, $$I_8 = -\frac{1}{8}x^6(1 - x^3)^{2/3} - \frac{3}{20}x^3(1 - x^3)^{2/3} - \frac{9}{40}(1 - x^3)^{2/3} + C$$

63. Let $$I = \int \operatorname{cosec}^2 x \ln(\cos x + \sqrt{\cos 2x})\, dx$$

$$= \int \operatorname{cosec}^2 x \cdot \ln\{\sin x(\cot x + \sqrt{\cot^2 x - 1})\}\, dx$$

$$= \int \text{cosec}^2 x \ln(\sin x)\, dx + \int \text{cosec}^2 x \cdot \ln(\cot x + \sqrt{\cot^2 x - 1})\, dx$$

In second integral put $\cot x = 1$

$\therefore \quad \text{cosec}^2 x\, dx = dt$

$\therefore \quad I = \int \text{cosec}^2 x \cdot \ln(\sin x)\, dx - \int \ln(t + \sqrt{t^2 - 1})\, dt$

In first integral (integrating by parts taking $\text{cosec}^2 x$ as second integral) and in second integral (integration by parts taking unity as second function).

We have, $(\ln \sin x)(-\cot x) - \int \cot x (-\cot x)\, dx$

$$- \ln(t + \sqrt{t^2 - 1})\, t + \int \frac{t\, dt}{\sqrt{t^2 - 1}}$$

$$= -\cot x (\ln \sin x) - \cot x - x - t \ln(t + \sqrt{t^2 - 1}) + \sqrt{t^2 - 1} + C$$

$$= -\cot x (\ln \sin x) - \cot x - x - \cot x \{\ln(\cot x + \sqrt{\cot^2 x - 1})\} + \sqrt{\cot^2 x - 1} + C$$

$$= -\cot x \cdot \ln(\cos x + \sqrt{\cos 2x}) - \cot x - x + \sqrt{\cot^2 x - 1} + C$$

64. $I = \int \frac{dx}{(\sin x + a \sec x)^2}\ a \in N = \int \frac{\cos^2 x\, dx}{(a + \sin x \cos x)^2}$

$$= \int \frac{\cos^2 x\, dx}{(a^2 + \sin^2 x \cdot \cos^2 x + 2a \sin x \cdot \cos x)}$$

$$= \int \frac{4 \cos^2 x\, dx}{4a^2 + \sin^2 2x + 4a \sin 2x} = 2 \int \frac{1 + \cos 2x}{(2a + \sin 2x)^2}\, dx$$

$$= 2 \int \frac{1}{(2a + \sin 2x)^2}\, dx + 2 \int \frac{\cos 2x}{(2a + \sin 2x)^2}\, dx$$

$$= 2I_1 - \frac{1}{(2a + \sin 2x)} \qquad \ldots(\text{i})$$

$$I_1 = \int \frac{dx}{(2a + \sin 2x)^2},$$

we know

$$\therefore \quad I_1 = \int \frac{du}{(4a^2 - 1)\frac{\sqrt{4a^2 - 1}\sqrt{1 - u^2}}{(2a - u)}}$$

$$u = 2a \frac{\sin 2x + 1}{2a + \sin 2x} = \frac{1}{(4a^2 - 1)^{3/2}} \int \frac{(2a - u)}{\sqrt{1 - u^2}}\, du$$

$$\Rightarrow \frac{du}{dx} = \frac{(4a^2 - 1)\cos 2x}{(2a + \sin 2x)^2}$$

$$\Rightarrow \frac{1}{(4a^2 - 1)^{3/2}} [2a \sin^{-1} u + \sqrt{1 - u^2}] = I_1 \text{ and } \sin 2x = \frac{2au - 1}{2a - u}$$

$$\therefore \quad I = \frac{1}{(4a^2 - 1)^{3/2}} \left[2a \sin^{-1} \left(\frac{2a \sin 2x + 1}{2a + \sin 2x} \right) + \sqrt{1 - \left(\frac{2a \sin 2x + 1}{2a + \sin 2x} \right)} \right] - \frac{1}{(2a + \sin 2x)} + C$$

65. $I = \int \frac{dx}{x - \sqrt{x^2 + 2x + 4}}$

Put $\sqrt{x^2 + 2x + 4} = t + x$

$$\Rightarrow \quad x^2 + 2x + 4 = t^2 + x^2 + 2tx$$

$$\Rightarrow \quad 2x - 2tx = t^2 - 4$$

$$\Rightarrow \quad x = \frac{t^2 - 4}{2 - 2t} = \frac{1}{2} \frac{(t^2 - 4)}{(1 - t)}$$

$$\Rightarrow \quad dx = -\frac{1}{2} \left[\frac{t^2 - 2t + 4}{(1 - t)^2} \right]$$

$$\therefore \quad I = -\frac{1}{2} \int \frac{t^2 - 2t + 4}{-t(1 - t)^2}\, dt$$

$$= \frac{1}{2} \int \left[\frac{4}{t} + \frac{3}{(1 - t)} + \frac{3}{(1 - t)^2} \right] dt$$

$$= \frac{1}{2} \left[4 \log |t| - 3 \log |1 - t| + \frac{3}{(1 - t)} \right]$$

$$= 2 \log |\sqrt{x^2 + 2x + 4} - x| - \frac{3}{2} \log |1 - \sqrt{x^2 + 2x + 4} + x| + \frac{3}{2(1 - \sqrt{x^2 + 2x + 4} + x)}$$

$$= 2 \log |\sqrt{x^2 + 2x + 4} - x| - \frac{3}{2} \log |\sqrt{x^2 + 2x + 4} - 1 - x| - \frac{3}{2(\sqrt{x^2 + 2x + 4} - x - 1)}$$

66. $I = \int \frac{dx}{1 + \sqrt{x^2 + 2x + 2}}$

$\sqrt{x^2 + 2x + 2} = t - x$, squaring both the sides, we get

$$x^2 + 2x + 2 = t^2 + x^2 - 2tx$$

$$2x + 2tx = t^2 - 2$$

$$x = \frac{t^2 - 2}{2(1 + t)}$$

$$\Rightarrow \quad dx = \frac{t^2 + 2t + 2}{2(1 + t)^2}\, dt$$

$$\therefore \quad 1 + \sqrt{x^2 + 2x + 2} = 1 + t - \frac{t^2 - 2}{2(1 + t)} = \frac{t^2 + 4t + 4}{2(1 + t)}$$

$$\Rightarrow \quad I = \int \frac{2(1 + t)(t^2 + 2t + 2)}{(t^2 + 4t + 4) \cdot 2 \cdot (1 + t)^2}\, dt = \int \frac{(t^2 + 2t + 2)}{(1 + t)(t + 2)^2}\, dt$$

Using partial fractions, we get

$$I = \int \frac{dt}{t + 1} - 2 \int \frac{dt}{(t + 2)^2}$$

$$= \ln |t + 1| + \frac{2}{(t + 2)} + C$$

$$I = \ln(x + 1 + \sqrt{x^2 + 2x + 2}) + \frac{2}{(x + 2) + \sqrt{x^2 + 2x + 2}} + C$$

67. $$I=\int \frac{x^4+1}{x^6+1}\,dx=\int \frac{(x^2+1)^2-2x^2}{(x^2+1)(x^4-x^2+1)}\,dx$$

$$=\int \frac{(1+1/x^2)\,dx}{(x^2+1/x^2-1)}-2\int \frac{x^2\,dx}{(x^3)^2+1}$$

$$=\int \frac{(1+1/x^2)\,dx}{(x-1/x)^2+1}-2\int \frac{x^2\,dx}{(x^3)^2+1}$$

In first integral put $x-1/x=t$ and in second integral put $x^3=u$

$$=\int \frac{dt}{t^2+1}-\frac{2}{3}\int \frac{du}{u^2+1}$$

$$=\tan^{-1}(t)-\frac{2}{3}\tan^{-1}(u)+C$$

$$=\tan^{-1}(x-1/x)-\frac{2}{3}\tan^{-1}(x^3)+C$$

68. $I=\int \frac{dx}{(1-x^3)^{1/3}}$

Put $x=1/t,\ dx=-1/t^2\,dt$

$$\therefore \quad I=-\int \frac{dt}{t^2(1-1/t^3)^{1/3}}=-\int \frac{dt}{t(t^3-1)^{1/3}}$$

Again, put $t^3-1=u^3 \Rightarrow 3t^2\,dt=3u^2\,du$

$$=-\int \frac{u^2\,du}{(1+u^3)\cdot u}=-\int \frac{u\,du}{(1+u)(1-u+u^2)}$$

$$=\frac{1}{3}\int \frac{du}{1+u}-\frac{1}{3}\int \frac{u+1}{u^2-u+1}\,du$$

(using partial fractions)

$$=\frac{1}{3}\int \frac{du}{u+1}-\frac{1}{3}\int \frac{1/2(2u-1)+3/2}{(u^2-u+1)}\,du$$

$$=\frac{1}{3}\int \frac{du}{u+1}-\frac{1}{6}\int \frac{2u-1}{u^2-u+1}\,du-\frac{1}{2}\int \frac{du}{(u-1/2)^2+3/4}$$

$$=\frac{1}{3}\log|u+1|-\frac{1}{6}\log|u^2-u+1|-\frac{1}{2}\cdot\frac{1}{\sqrt{3}/2}\tan^{-1}\left\{\frac{2u-1}{\sqrt{3}}\right\}+C$$

$$=\frac{1}{3}\log|(t^3-1)^{1/3}+1|-\frac{1}{6}\log|(t^3-1)^{2/3}-(t^3-1)^{1/3}+1|-\frac{1}{\sqrt{3}}\tan^{-1}\left\{\frac{2(t^3-1)^{1/3}-1}{\sqrt{3}}\right\}+C$$

$$=\frac{1}{3}\log\left|\frac{(1-x^3)^{1/3}+x}{x}\right|-\frac{1}{6}\log\left|\frac{(1-x^3)^{2/3}-(1-x^3)^{1/3}+x^2}{x^2}\right|-\frac{1}{\sqrt{3}}\tan^{-1}\left\{\frac{2(1-x^3)^{1/3}-x}{\sqrt{3}x}\right\}+C$$

69. $I=\int \frac{(x+\sqrt{1+x^2})^{15}}{\sqrt{1+x^2}}\,dx$

Put $(x+\sqrt{1+x^2})=t$

$$\therefore \left(1+\frac{x}{\sqrt{1+x^2}}\right)dx=dt \quad \text{or} \quad \frac{t\,dx}{\sqrt{1+x^2}}=dt$$

$$\therefore \ I=\int t^{15}\cdot\frac{dt}{t}=\int t^{14}\,dt=\frac{t^{15}}{15}+C=\frac{(x+\sqrt{1+x^2})^{15}}{15}+C$$

70. $\because u_{n+1}=\int \frac{x^{n+1}}{y}\,dx=\int \frac{x^{n+1}}{\sqrt{ax^2+2bx+c}}\,dx$

$$=\frac{1}{2a}\int \frac{x^n(2ax+2b)-2bx^n}{\sqrt{ax^2+2bx+c}}\,dx$$

$$=\frac{1}{2a}\int x^n\frac{(2ax+2b)\,dx}{\sqrt{ax^2+2bx+c}}-\frac{b}{a}\int \frac{x^n}{\sqrt{ax^2+2bx+c}}\,dx$$

$$u_{n+1}=\frac{1}{2a}\int x^n\frac{(2ax+2b)}{\sqrt{ax^2+2bx+c}}-\frac{b}{a}u_n$$

I II

$$u_{n+1}=\frac{1}{2a}\left[x^n\cdot 2\sqrt{ax^2+2bx+c}-\int nx^{n-1}\cdot 2\sqrt{(ax^2+bx+c)}\,dx\right]-\frac{b}{a}u_n$$

$$=\frac{1}{a}x^n y-\frac{n}{a}\int x^{n-1}\cdot\sqrt{ax^2+2bx+c}\,dx-\frac{b}{a}u_n$$

$$au_{n+1}=x^n y-n\int \frac{x^{n-1}(ax^2+2bx+c)}{\sqrt{ax^2+2bx+c}}\,dx-\frac{b}{a}u_n$$

$$au_{n+1}=x^n y-n[au_{n+1}+2bu_n+cu_{n-1}]-bu_n$$

$$\Rightarrow (n+1)a\,u_{n+1}+(2n+1)bu_n+nc\cdot u_{n-1}=x^n y \qquad \ldots(i)$$

Now, putting, $n=0$ in both the sides, we get

$$au_1+bu_0=x^0 y$$

$$au_1=y-bu_0 \qquad \ldots(ii)$$

Putting $n=1$ in Eq. (i), we get

$$2au_2+3bu_1+cu_0=xy$$

$$2au_2+3b\left(\frac{y-bu_0}{a}\right)+cu_0=xy \qquad \text{[from Eq. (ii)]}$$

$$\Rightarrow 2a^2u_2+3by-3b^2u_0+acu_0=axy$$

$$\Rightarrow 2a^2u_2=y(ax-3b)+(3b^2-ac)u_0$$

71. **Plan** Integration by Substitution

i.e. $$I=\int f\{g(x)\}\cdot g'(x)\,dx$$

Put $$g(x)=t \ \Rightarrow g'(x)\,dx=dt$$

$$\therefore \quad I=\int f(t)\,dt$$

Description of Situation Generally, students gets confused after substitution, i.e. $\sec x+\tan x=t$.

Now, for $\sec x$, we should use

$$\sec^2 x-\tan^2 x=1$$

$$\Rightarrow \quad (\sec x-\tan x)(\sec x+\tan x)=1$$

$$\Rightarrow \quad \sec x-\tan x=\frac{1}{t}$$

Here, $I=\int \frac{\sec^2 dx}{(\sec x+\tan x)^{9/2}}$

Put $\sec x+\tan x=t$

$\Rightarrow \quad (\sec x\tan x+\sec^2 x)dx=dt$

$\Rightarrow \quad \sec x\cdot t\,dx=dt$

$\Rightarrow \quad \sec x\,dx=\frac{dt}{t}$

$\therefore \quad \sec x-\tan x=\frac{1}{t} \Rightarrow \sec x=\frac{1}{2}\left(t+\frac{1}{t}\right)$

$\therefore \quad I=\int \frac{\sec x\cdot\sec x\,dx}{(\sec x+\tan x)^{9/2}}$

$\Rightarrow \quad I=\int \frac{\frac{1}{2}\left(t+\frac{1}{t}\right)\cdot\frac{dt}{t}}{t^{9/2}}=\frac{1}{2}\int\left(\frac{1}{t^{9/2}}+\frac{1}{t^{13/2}}\right)dt$

$$=-\frac{1}{2}\left\{\frac{2}{7t^{7/2}}+\frac{2}{11t^{11/2}}\right\}+K$$

$$=-\left[\frac{1}{7(\sec x+\tan x)^{7/2}}+\frac{1}{11(\sec x+\tan x)^{11/2}}\right]+K$$

$$=\frac{-1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}+\frac{1}{7}(\sec x+\tan x)^2\right\}+K$$

72. Since, $I=\int \frac{e^x}{e^{4x}+e^{2x}+1}dx$ and $J=\int \frac{e^{3x}}{1+e^{2x}+e^{4x}}dx$

$\therefore \quad J-I=\int \frac{(e^{3x}-e^x)}{1+e^{2x}+e^{4x}}dx$

Put $e^x=u \Rightarrow e^x dx=du$

$\therefore \quad J-I=\int \frac{(u^2-1)}{1+u^2+u^4}du=\int \frac{\left(1-\frac{1}{u^2}\right)}{1+\frac{1}{u^2}+u^2}du$

$$=\int \frac{\left(1-\frac{1}{u^2}\right)}{\left(u+\frac{1}{u}\right)^2-1}\,du$$

Put $u+\frac{1}{u}=t \Rightarrow \left(1-\frac{1}{u^2}\right)du=dt$

$$=\int \frac{dt}{t^2-1}=\frac{1}{2}\log\left|\frac{t-1}{t+1}\right|+C$$

$$=\frac{1}{2}\log\left|\frac{u^2-u+1}{u^2+u+1}\right|+C=\frac{1}{2}\log\left|\frac{e^{2x}-e^x+1}{e^{2x}+e^x+1}\right|+C$$

73. Let $I=\int \frac{2x^{12}+5x^9}{(x^5+x^3+1)^3}dx=\int \frac{2x^{12}+5x^9}{x^{15}(1+x^{-2}+x^{-5})^3}dx$

$$=\int \frac{2x^{-3}+5x^{-6}}{(1+x^{-2}+x^{-5})^3}dx$$

Now, put $1+x^{-2}+x^{-5}=t$

$\Rightarrow \quad (-2x^{-3}-5x^{-6})\,dx=dt$

$\Rightarrow \quad (2x^{-3}+5x^{-6})\,dx=-dt$

$\therefore \quad I=-\int \frac{dt}{t^3}=-\int t^{-3}\,dt$

$$=-\frac{t^{-3+1}}{-3+1}+C=\frac{1}{2t^2}+C=\frac{x^{10}}{2(x^5+x^3+1)^2}+C$$

74. $\int \frac{dx}{x^2(x^4+1)^{\frac{3}{4}}}=\int \frac{dx}{x^5\left(1+\frac{1}{x^4}\right)^{\frac{3}{4}}}$

Put $1+\frac{1}{x^4}=t^4 \Rightarrow -\frac{4}{x^5}dx=4t^3dt$

$\Rightarrow \quad \frac{dx}{x^5}=-t^3dt$

$\therefore \quad I=\int \frac{-t^3dt}{t^3}=-\int dt=-t+C=-\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}}+C$

75. $\int\left(1+x-\frac{1}{x}\right)e^{x+\frac{1}{x}}dx=\int e^{x+\frac{1}{x}}dx+\int x\left(1-\frac{1}{x^2}\right)e^{x+\frac{1}{x}}dx$

$$=\int e^{x+\frac{1}{x}}dx+xe^{x+\frac{1}{x}}-\int \frac{d}{dx}(x)e^{x+\frac{1}{x}}dx$$

$$=\int e^{x+\frac{1}{x}}dx+xe^{x+\frac{1}{x}}-\int e^{x+\frac{1}{x}}dx$$

$$\left[\because \int\left(1-\frac{1}{x^2}\right)e^{x+\frac{1}{x}}dx=e^{x+\frac{1}{x}}\right]$$

$$=\int e^{x+\frac{1}{x}}dx+xe^{x+\frac{1}{x}}-\int ex^{x+\frac{1}{x}}dx$$

$$=xe^{x+\frac{1}{x}}+C$$

76. Given, $\int f(x)\,dx=\psi(x)$

Let $I=\int x^5 f(x^3)\,dx$

Put $x^3=t$

$\Rightarrow \quad x^2dx=\frac{dt}{3}$ ...(i)

$\therefore \quad I=\frac{1}{3}\int t\,f(t)\,dt$

$$=\frac{1}{3}\left[t\int f(t)\,dt-\int\left\{\frac{d}{dtt}(t)\,dt\right\}dt\right]$$

[Integration by parts]

$$=\frac{1}{3}[t\,\psi(t)-\int \psi(t)\,dt]$$

$$=\frac{1}{3}[x^3\psi(x^3)-3\int x^2\psi(x^3)\,dx]+C \quad \text{[from Eq. (i)]}$$

$$=\frac{1}{3}x^3\psi(x^3)-\int x^2\psi(x^3)\,dx+C$$

77. **Given** Integral is $\int \frac{5 \tan x}{\tan x - 2}\, dx$

To find The value of a, if

$\int \frac{5 \tan x}{\tan x - 2}\, dx = x + a \log |\sin x - 2 \cos x| + k$...(i)

Now, let us assume that $I = \int \frac{5 \tan x}{\tan x - 2}\, dx$

Multiplying by $\cos x$ in numerator and denominator, we get

$$I = \int \frac{5 \sin x}{\sin x - 2 \cos x}\, dx$$

This special integration requires special substitution of type

$$N^r = A(D^r) + B\left(\frac{dD^r}{dx}\right)$$

$\Rightarrow$ Let $5 \sin x = A(\sin x - 2 \cos x) + B(\cos x + 2 \sin x)$

$\Rightarrow 0 \cos x + 5 \sin x = (A + 2B) \sin x + (B - 2A) \cos x$

Comparing the coefficients of $\sin x$ and $\cos x$, we get

$$A + 2B = 5 \quad \text{and} \quad B - 2A = 0$$

Solving the above two equations in A and B, we get

$$A = 1 \quad \text{and} \quad B = 2$$

$\Rightarrow 5 \sin x = (\sin x - 2 \cos x) + 2(\cos x + 2 \sin x)$

$\Rightarrow I = \int \frac{5 \sin x}{\sin x - 2 \cos x}\, dx$

$$= \int \frac{(\sin x - 2 \cos x) + 2(\cos x + 2 \sin x)}{(\sin x - 2 \cos x)}\, dx$$

$\Rightarrow I = \int \frac{\sin x - 2 \cos x}{\sin x - 2 \cos x}\, dx + 2 \int \frac{(\cos x + 2 \sin x)}{(\sin x - 2 \cos x)}\, dx$

$\Rightarrow I = \int 1\, dx + 2 \int \frac{d(\sin x - 2 \cos x)}{(\sin x - 2 \cos x)}$

$\Rightarrow I = x + 2 \log |(\sin x - 2 \cos x)| + k$ ...(ii)

where, k is the constant of integration.

Now, by comparing the value of I in Eqs. (i) and (ii), we get $a = 2$.

78. Let $I = \sqrt{2} \int \frac{\sin x}{\sin\left(x - \frac{\pi}{4}\right)}\, dx$

Put $x - \frac{\pi}{4} = t \quad \Rightarrow \quad dx = dt$

$\therefore$ $$I = \sqrt{2} \int \frac{\sin\left(\frac{\pi}{4} + t\right) dt}{\sin t}$$

$$= \sqrt{2} \int \left[\frac{1}{\sqrt{2}} \cot t + \frac{1}{\sqrt{2}}\right] dt$$

$$= 1 + \log |\sin t| + C$$

$$= x + \log \left|\sin\left(x - \frac{\pi}{4}\right)\right| + C$$

CHAPTER

02

Definite Integral

Learning Part

Session 1

- Integration Basics
- Geometrical Interpretation of Definite Integral
- Evaluation of Definite Integrals by Substitution

Session 2

- Properties of Definite Integral

Session 3

- Applications of Piecewise Function Property

Session 4

- Applications of Even-Odd Property and Half the Integral Limit Property

Session 5

- Applications of Periodic Functions and Newton-Leibnitz's Formula

Session 6

- Integration as Limit of a Sum
- Applications of Inequality
- Gamma Function
- Beta Function
- Walli's Formula

Practice Part

- JEE Type Examples
- Chapter Exercise

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Session 1

Integration Basics, Geometrical Interpretation of Definite Integral, Evaluation of Definite Integrals by Substitution

Integration Basics

What is Definite Integral ?

Let f be a function of x defined in the closed interval $[a, b]$ and ϕ be another function, such that $\phi'(x) = f(x)$ for all x in the domain of f, then

$$\int_a^b f(x)\,dx = [\phi(x) + c]_a^b = \phi(b) - \phi(a)$$

is called the definite integral of the function $f(x)$ over the interval $[a, b]$, a and b are called the limits of integration, a being the lower limit and b be the upper limit.

Remark

In definite integrals constant of integration is never present.

Working Rules

To evaluate definite integral $\int_a^b f(x)\,dx$.

1. First evaluate the indefinite integral $\int f(x)\,dx$ and suppose the result is $g(x)$.
2. Next find $g(b)$ and $g(a)$.
3. Finally, the value of the definite integral is obtained by subtracting $g(a)$ from $g(b)$.

   Thus, $\int_a^b f(x)\,dx = [g(x)]_a^b = g(b) - g(a)$

Example 1 Evaluate

(i) $\int_0^1 \frac{1}{3+4x}\,dx$ (ii) $\int_0^{\pi/2} \sin^4 x\,dx$

Sol. (i) Here, $I = \int_0^1 \frac{1}{3+4x}\,dx = \left[\frac{\ln(3+4x)}{4}\right]_0^1$

$$= \frac{1}{4}[\ln 7 - \ln 3] = \frac{1}{4}\ln\left(\frac{7}{3}\right)$$

(ii) Let $I = \int_0^{\pi/2} \sin^4 x\,dx$

$$= \frac{1}{4}\int_0^{\pi/2} (2\sin^2 x)^2\,dx = \frac{1}{4}\int_0^{\pi/2} (1-\cos 2x)^2\,dx$$

$$= \frac{1}{4}\int_0^{\pi/2} (1 - 2\cos 2x + \cos^2 2x)\,dx$$

$$= \frac{1}{4}\int_0^{\pi/2} \left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right)dx$$

$$= \frac{1}{4}\int_0^{\pi/2} \left(\frac{3 - 4\cos 2x + \cos 4x}{2}\right)dx$$

$$= \frac{1}{8}\left[3x - \frac{4}{2}\sin 2x + \frac{\sin 4x}{4}\right]_0^{\pi/2}$$

$$= \frac{1}{8}\left[\left(\frac{3\pi}{2} - 2\sin\pi + \frac{1}{4}\sin 2\pi\right) - 0\right]$$

$$= \frac{1}{8}\left(\frac{3\pi}{2} - 0 + 0\right) = \frac{3\pi}{16}$$

Example 2 The value of $\int_{-1}^1 \left[\frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right)\right]dx$ is

(a) $\pi/2$ (b) $\pi/4$ (c) $-\pi/2$ (d) None of these

Sol. Let $I = \int_{-1}^1 \left[\frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right)\right]dx$

Here, $\frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right) = \frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$

$\therefore \quad I = \int_{-1}^1 -\frac{1}{1+x^2}\,dx = -\int_{-1}^1 \frac{1}{1+x^2}\,dx$

$$= -(\tan^{-1}x)_{-1}^1 = -[\tan^{-1}(1) - \tan^{-1}(-1)]$$

$$= -\left(\frac{\pi}{4} + \frac{\pi}{4}\right) = -\frac{\pi}{2}$$

Hence, (c) is the correct answer.

Remark

Note that $\int_{-1}^1 \left(\frac{d}{dx}\tan^{-1}\frac{1}{x}\right)dx = \left(\tan^{-1}\frac{1}{x}\right)_{-1}^1 = \tan^{-1}(1) - \tan^{-1}(-1)$

$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

is incorrect, because $\tan^{-1}\left(\frac{1}{x}\right)$ is not a anti-derivative (primitive) of $\frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right)$ on the interval $[-1, 1]$.

Example 3 *If* $I_n = \int_1^e (\log x)^n\, dx$, *then* $I_n + nI_{n-1}$ *is equal to*

(a) $\frac{1}{e}$ (b) e (c) $e-1$ (d) None of these

Sol. We have, $I_n = \int_1^e (\log x)^n\, dx = \int_1^e \underbrace{(\log x)^n}_{\text{I}} \cdot \underbrace{1}_{\text{II}}\, dx$

$$\therefore \quad I_n = [x\cdot(\log x)^n]_1^e - \int_1^e n\cdot(\log x)^{n-1}\cdot\frac{1}{x}\cdot x\,dx$$

$$= (e-0) - n\int_1^e (\log_e x)^{n-1}dx = e - n\cdot I_{n-1}$$

$\therefore\ I_n + n\cdot I_{n-1} = e$

Hence, (b) is the correct answer.

Example 4 All the values of 'a' for which $\int_1^2 \{a^2 + (4-4a)x + 4x^3\}dx \le 12$ are given by

(a) $a = 3$ (b) $a \le 4$

(c) $0 \le a < 3$ (d) None of these

Sol. We have, $\int_1^2 \{a^2 + (4-4a)x + 4x^3\}dx \le 12$

$$\Rightarrow \quad [a^2x + (2-2a)x^2 + x^4]_1^2 \le 12$$

$$\Rightarrow a^2(2-1) + (2-2a)(4-1) + (2^4 - 1^4) \le 12$$

$$\Rightarrow \quad a^2 + 3(2-2a) + 15 \le 12$$

$$\Rightarrow \quad a^2 - 6a + 9 \le 0$$

$$\Rightarrow \quad (a-3)^2 \le 0$$

$$\therefore \quad a = 3$$

Hence, (a) is the correct answer.

Geometrical Interpretation of Definite Integral

If $f(x) > 0$ for all $x \in [a, b]$, then $\int_a^b f(x)\, dx$ is numerically equal to the area bounded by the curve $y = f(x)$, the X-axis and the straight lines $x = a$ and $x = b$ i.e. $\int_a^b f(x)\, dx$

In general, $\int_a^b f(x)\, dx$ represents the algebraic sum of the areas of the figures bounded by the curve $y = f(x)$, the X-axis and the straight lines $x = a$ and $x = b$. The areas above X-axis are taken with plus sign and the areas below X-axis are taken with minus sign,

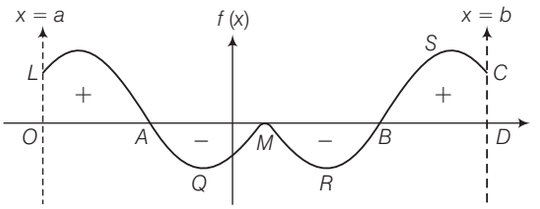


Figure 2.1

i.e. $\int_a^b f(x)\, dx$ = Area (OLA) − Area (AQM) − Area (MRB) + Area $(BSCD)$

Remark

$\int_a^b f(x)\, dx$, represents algebraic sum of areas means that area of function $y = f(x)$ is asked between a to b.

$\Rightarrow$ Area bounded $= \int_a^b |f(x)|\, dx$ and not been represented by $\int_a^b f(x)\, dx$. e.g. If someone asks for the area of $y = x^3$ between -1 to 1, then $y = x^3$ could be plotted as

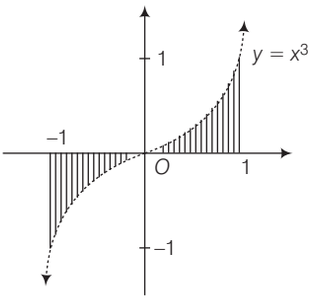


Figure 2.2

$$\therefore \quad \text{Area} = \int_{-1}^0 -x^3\, dx + \int_0^1 x^3\, dx = \frac{1}{2}$$

or using above definition, area $= \int_{-1}^1 |x^3|\, dx = 2\int_0^1 x^3\, dx$

$$= 2\left[\frac{x^4}{4}\right]_0^1 = \frac{1}{2}$$

But, if we integrate x^3 between -1 to 1.

$\Rightarrow \quad \int_{-1}^1 x^3\, dx = 0$ which does not represent the area.

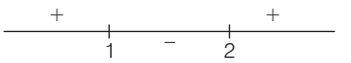
Thus, students are adviced to make difference between area and definite integral.

Example 5 Evaluate $\int_0^3 |(x-1)(x-2)|\, dx$.

Sol. Let $I = \int_0^3 |(x-1)(x-2)|\, dx$

We know,

$$|(x-1)(x-2)| = \begin{cases} (x-1)(x-2), & x < 1 \text{ or } x > 2 \\ -(x-1)(x-2), & 1 < x < 2 \end{cases}$$



Using number line rule,

$$I = \int_0^3 |(x-1)(x-2)|\, dx$$

$$= \int_0^1 (x-1)(x-2)\,dx - \int_1^2 (x-1)(x-2)\,dx + \int_2^3 (x-1)(x-2)\,dx$$

$$= \int_0^1 (x^2-3x+2)\,dx - \int_1^2 (x^2-3x+2)\,dx + \int_2^3 (x^2-3x+2)\,dx$$

$$= \left[\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right]_0^1 - \left[\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right]_1^2 + \left[\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right]_2^3$$

$$= \left(\frac{1}{3} - \frac{3}{2} + 2\right) - \left(\frac{8}{3} - \frac{12}{2} + 4 - \frac{1}{3} + \frac{3}{2} - 2\right) + \left(\frac{27}{3} - \frac{27}{2} + 6 - \frac{8}{3} + \frac{12}{2} - 4\right) = \frac{11}{6}$$

Evaluation of Definite Integrals by Substitution

Sometimes, the indefinite integral may need substitution, say $x = \phi(t)$. Then, in that case don't forget to change the limits of integration a and b corresponding to the new variable t. The substitution $x = \phi(t)$ is not valid, if it is not continuous in the interval $[a, b]$.

Example 6 Show that

$$\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2 \sin^2 x} = \frac{\pi}{2ab};\ a, b > 0.$$

Sol. Let $$I = \int_{x=0}^{x=\pi/2} \frac{dx}{a^2\cos^2 x + b^2 \sin^2 x}$$

$$= \int_{x=0}^{x=\pi/2} \frac{\sec^2 x\,dx}{a^2 + b^2\tan^2 x}$$

(divide numerator and denominator by $\cos^2 x$)

Put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$$\therefore \quad I = \int_{t=0}^{t=\infty} \frac{dt}{a^2 + b^2 t^2}$$

We find the new limits of integration $t = \tan x \Rightarrow t = 0$ when $x = 0$ and $t = \infty$ when $x = \pi/2$.

$$\Rightarrow \quad I = \frac{1}{b^2}\int_0^\infty \frac{dt}{\left(\frac{a}{b}\right)^2 + t^2} = \frac{1}{b^2}\cdot\frac{1}{a/b}\left[\tan^{-1}\frac{bt}{a}\right]_0^\infty$$

$$= \frac{1}{ab}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{2ab}$$

Example 7 Evaluate $\int_{-2}^{2} \frac{dx}{4+x^2}$ directly as well as by the substitution $x = 1/t$. Examine as to why the answer don't valid?

Sol. Let $$I = \int_{-2}^{2} \frac{dx}{4+x^2}$$

$$= \left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{2} = \frac{1}{2}\left[\tan^{-1}(1) - \tan^{-1}(-1)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi}{4}$$

$$\Rightarrow \quad I = \frac{\pi}{4}$$

On the other hand; if $x = 1/t$, then

$$I = \int_{-2}^{2} \frac{dx}{4+x^2} = -\int_{-1/2}^{1/2} \frac{dt}{t^2(4 + 1/t^2)} = -\int_{-1/2}^{1/2} \frac{dt}{4t^2+1}$$

$$= -\left[\frac{1}{2}\tan^{-1}(2t)\right]_{-1/2}^{1/2}$$

$$= -\frac{1}{2}\tan^{-1}(1) - \left(-\frac{1}{2}\tan^{-1}(-1)\right)$$

$$= -\frac{\pi}{8} - \frac{\pi}{8} = -\frac{\pi}{4}$$

$\therefore\ I = -\frac{\pi}{4}$, when $x = \frac{1}{t}$

In above two results, $I = -\pi/4$ is wrong. Since, the integrand $\frac{1}{4+x^2} > 0$ and therefore the definite integral of this function cannot be negative.

Since, $x = 1/t$ is discontinuous at $t = 0$, then substitution is not valid. ($\because I = \pi/4$)

Remark

It is important that the substitution must be continuous in the interval of integration.

Example 8 Evaluate $\int_0^{1/2} x \cdot \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx$.

Sol. Let $I = \int_0^{1/2} x \cdot \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx$

Put $\sin^{-1} x = \theta$, then $x = \sin\theta \Rightarrow dx = \cos\theta\,d\theta$

Also when $x = 0$, then $\theta = 0$ and when $x = \frac{1}{2}$,

then $\theta = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$$\therefore \quad I = \int_0^{\pi/6} \sin\theta \cdot \frac{\theta}{\sqrt{1-\sin^2\theta}} \cdot \cos\theta\,d\theta = \int_0^{\pi/6} \theta \cdot \sin\theta\,d\theta$$

$$= (-\theta\cdot\cos\theta)_0^{\pi/6} + \int_0^{\pi/6} \cos\theta\,d\theta,$$

using integeration by parts.

$$= (-\theta\cos\theta)_0^{\pi/6} + (\sin\theta)_0^{\pi/6}$$

$$= -\frac{\pi}{6}\cos\frac{\pi}{6} + 0 + \sin\frac{\pi}{6} - 0 = \frac{-\sqrt{3}\pi}{12} + \frac{1}{2}$$

Example 9 For any $n > 1$, evaluate the integral $\int_0^{\infty} \frac{1}{(x+\sqrt{x^2+1})^n}\,dx$.

Sol. Let $I = \int_0^{\infty} \frac{1}{(x+\sqrt{1+x^2})^n}\,dx$

Put $x+\sqrt{1+x^2} = t \quad \Rightarrow \sqrt{1+x^2} = t - x$

$\Rightarrow 1+x^2 = (t-x)^2 \quad \Rightarrow x = \frac{t^2-1}{2t}$ or $x = \frac{1}{2}\left(t - \frac{1}{t}\right)$

$\therefore \quad dx = \frac{1}{2}\left(1+\frac{1}{t^2}\right)dt$

$\therefore \quad I = \int_{t=1}^{\infty} \frac{1}{t^n}\cdot\frac{1}{2}\left(1+\frac{1}{t^2}\right)dt = \frac{1}{2}\int_1^{\infty}(t^{-n}+t^{-n-2})dt$

$$= \frac{1}{2}\left[\frac{t^{1-n}}{1-n} + \frac{t^{-n-1}}{-(n+1)}\right]_1^{\infty} = \frac{1}{2}\left[0-\left(\frac{1}{1-n}-\frac{1}{n+1}\right)\right]$$

$$= \frac{1}{2}\left[\frac{-2n}{1-n^2}\right] = \frac{n}{n^2-1}$$

Example 10 The value of $\int_0^{e-1} \frac{e^{\frac{x^2+2x-1}{2}}}{(x+1)}\,dx + \int_1^{e} x\log x \cdot e^{\frac{x^2-2}{2}}\,dx$ is equal to

(a) $(\sqrt{e})^{(e^2+1)}$ (b) $(\sqrt{e})^{e^2-1}$ (c) 0 (d) $(\sqrt{e})^{e^2-2}$

Sol. Let $I = \int_0^{e-1} \frac{e^{\frac{x^2+2x-1}{2}}}{(x+1)}\,dx + \int_1^{e} x\log x \cdot e^{\frac{x^2-2}{2}}\,dx$

Put $x+1 = t$ in first integral

$$\therefore \quad I = \int_1^{e} \frac{e^{\frac{t^2-2}{2}}}{t}\,dt + \int_1^{e} x\log x \cdot e^{\frac{x^2-2}{2}}\,dx$$

$$= \int_1^{e} e^{\frac{t^2-2}{2}}\left\{\frac{1}{t} + t\cdot\log t\right\}dt = \left(\log t \cdot e^{\frac{t^2-2}{2}}\right)_1^{e}$$

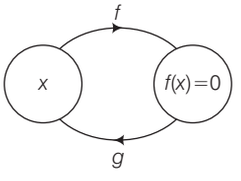
$$= (\sqrt{e})^{e^2-2}$$

Hence, (d) is the correct answer.

Example 11 Let $f(x) = \int_2^x \frac{dt}{\sqrt{1+t^4}}$ and g be the inverse of f. Then, the value of $g'(0)$ is

(a) 1 (b) 17 (c) $\sqrt{17}$ (d) None of these

Sol. Here, $f'(x) = \frac{1}{\sqrt{1+x^4}} = \frac{dy}{dx}$



Now, $g'(x) = \frac{dx}{dy} = \sqrt{1+x^4}$

When $y = 0$ i.e. $\int_2^x \frac{dt}{\sqrt{1+t^4}} = 0$, then $x = 2$

Therefore, $g'(0) = \sqrt{1+16} = \sqrt{17}$

Hence, (c) is the correct answer.

Example 12 Let $a_n = \int_0^{\pi/2} (1-\sin t)^n \sin 2t$, then $\lim_{n\to\infty} \sum_{n=1}^{n} \frac{a_n}{n}$ is equal to

(a) 1/2 (b) 1
(c) 4/3 (d) 3/2

Sol. We have, $a_n = \int_0^{\pi/2} (1-\sin t)^n \sin 2t\,dt$

Let $1-\sin t = u \Rightarrow -\cos t\,dt = du$

$$\therefore \quad a_n = 2\int_0^1 u^n(1-u)du = 2\left(\int_0^1 u^n du - \int_0^1 u^{n+1}du\right)$$

$$= 2\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

Therefore, $\frac{a_n}{n} = 2\left(\frac{1}{n(n+1)} - \frac{1}{n(n+2)}\right)$

$$\therefore \lim_{n\to\infty} \sum_{n=1}^{n} \frac{a_n}{n} = 2\left[\sum\left(\frac{1}{n}-\frac{1}{n+1}\right) - \frac{1}{2}\sum\left(\frac{1}{n}-\frac{1}{n+2}\right)\right]$$

$$= 2\sum_{n=1}^{n}\left(\frac{1}{n}-\frac{1}{n+1}\right) - \sum_{n=1}^{n}\left(\frac{1}{n}-\frac{1}{n+2}\right)$$

$$= 2(1) - \left[\left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\ldots\right] = 2-\frac{3}{2} = \frac{1}{2}$$

Hence, (a) is the correct answer.

Example 13 The value of $x > 1$ satisfying the equation $\int_1^x t\ln t\,dt = \frac{1}{4}$ is

(a) $\sqrt{e}$ (b) e (c) e^2 (d) $e-1$

Sol. Let $I = \int_1^x t\ln t\,dt = \left[\ln t \cdot \frac{t^2}{2}\right]_1^x$

$$-\frac{1}{2}\int_1^x \frac{1}{t}\cdot t^2\,dt = \frac{x^2}{2}\ln x - \frac{1}{2}\left[\frac{t^2}{2}\right]_1^x$$

$$= \frac{x^2\ln x}{2} - \frac{1}{4}[x^2-1] = \frac{1}{4}$$

$$\therefore \frac{x^2\ln x}{2} - \frac{1}{4}x^2 = 0 \Rightarrow [2\ln x - 1] = 0 \qquad (\text{as } x > 1)$$

$$\Rightarrow \quad \ln x = \frac{1}{2} \Rightarrow x = \sqrt{e}$$

Hence, (a) is the correct answer.

Example 14 If $\lim_{a\to\infty}\frac{1}{a}\int_0^{\infty}\frac{x^2+ax+1}{1+x^4}\cdot\tan^{-1}\left(\frac{1}{x}\right)dx$ is equal to $\frac{\pi^2}{k}$, where $k\in N$, then k equals to

(a) 4 (b) 8 (c) 16 (d) 32

Sol. Let $I=\int_0^{\infty}\frac{x^2+ax+1}{1+x^4}\cdot\tan^{-1}\left(\frac{1}{x}\right)dx$

Put $x=1/t$ and adding, we get

[using $\tan^{-1}(1/x)+\cot^{-1}x=\pi/2$]

$$I=\frac{\pi}{4}\int_0^{\infty}\frac{(x^2+1)+ax}{1+x^4}dx$$

$$=\frac{\pi}{4}\left[\int_0^{\infty}\frac{(x^2+1)}{1+x^4}dx+a\int_0^{\infty}\frac{x\,dx}{1+x^4}\right]$$

$$=\frac{\pi}{4}\left[\frac{\pi}{2\sqrt{2}}+\frac{a\pi}{4}\right]=\left[\frac{\pi^2}{8\sqrt{2}}+\frac{\pi^2 a}{16}\right]$$

$$\therefore\quad l=\lim_{a\to\infty}\frac{1}{a}\left[\frac{\pi^2}{8\sqrt{2}}+\frac{\pi^2 a}{16}\right]=\lim_{a\to\infty}\left[\frac{\pi^2}{(8\sqrt{2})a}+\frac{\pi^2}{16}\right]=\frac{\pi^2}{16}$$

$$\Rightarrow\quad k=16$$

Hence, (c) is the correct answer.

Example 15 If the value of definite integral $\int_1^a x\cdot a^{-[\log_a x]}\,dx$, where $a>1$ and $[x]$ denotes the greatest integer, is $\frac{e-1}{2}$, then the value of 'a' equals to

(a) $\sqrt{e}$ (b) e (c) $\sqrt{e+1}$ (d) $e-1$

Sol. Let $I=\int_1^a x\cdot a^{-[\log_a x]}\,dx$

Put $\log_a x=t\quad\Rightarrow\quad a^t=x$

$$\therefore\quad I=\ln a\cdot\int_0^1(a^t\cdot a^{-[t]}\cdot a^t)\,dt=\ln a\cdot\int_0^1(a^{t-[t]}\cdot a^t)dt$$

$$=\ln a\cdot\int_0^1(a^{\{t\}}\cdot a^t)dt=\ln a\cdot\int_0^1 a^{2t}\,dt$$

$$=\left[\frac{\ln a}{2}\cdot\frac{a^{2t}}{\ln a}\right]_0^1=\frac{1}{2}(a^2-1)\qquad[\text{as }\{t\}=t,\text{ if }t\in(0,1)]$$

$$\therefore\quad\frac{1}{2}(a^2-1)=\frac{e-1}{2}\ \Rightarrow\ a=\sqrt{e}$$

Aliter $\quad x\in(1,a)$

$$\Rightarrow\quad\log_a x\in(0,1)\ \Rightarrow\ [\log_a x]=0$$

$$\therefore\quad I=\int_1^a x\,dx=\frac{1}{2}(a^2-1)=\frac{e-1}{2}\ \Rightarrow\ a=\sqrt{e}$$

Hence, (a) is the correct answer.

Exercise for Session 1

1. $\int_0^{\pi/4}\cos^2 x\,dx$
2. $\int_0^{\pi/2}\frac{dx}{1+\cos x}$
3. $\int_0^{\pi/2}\sqrt{1+\cos x}\,dx$
4. $\int_0^{\pi/6}\sin 2x\cdot\cos x\,dx$
5. $\int_1^2\frac{dx}{\sqrt{x}-\sqrt{x-1}}$
6. $\int_0^1\log x\,dx$
7. $\int_0^{\pi/4}\frac{(\sin x+\cos x)}{9+16\sin 2x}dx$
8. $\int_a^b\frac{1}{\sqrt{(x-a)(b-x)}}dx, b>a$
9. $\int_a^b\sqrt{\frac{x-a}{b-x}}dx$
10. $\int_0^{\pi/4}\sqrt{\tan x}\,dx$
11. $\int_0^{\pi}\cos 2x.\log(\sin x)dx$
12. $\int_0^{\pi/4}e^{\sin x}\left(\frac{x\cos^3 x-\sin x}{\cos^2 x}\right)dx$
13. If $f(x)$ is a function satisfying $f\left(\frac{1}{x}\right)+x^2f(x)=0$ for all non-zero x, then $\int_{\sin\theta}^{\operatorname{cosec}\theta}f(x)dx$ is equal to
14. The value of $\int_0^1\left(\prod_{r=1}^{n}(n+r)\right)\left(\sum_{k=1}^{n}\frac{1}{x+k}\right)dx$ equals to

    (a) n (b) $n!$ (c) $(n+1)!$ (d) $n\cdot n!$
15. The true set of values of 'a' for which the inequality $\int_x^0(3^{-2x}-2\cdot 3^{-x})\,dx\ge 0$ is true, is

    (a) $[0, 1]$ (b) $[-\infty, -1]$ (c) $[0, \infty]$ (d) $[-\infty, -1]\cup[1, \infty]$

Session 2

Properties of Definite Integral

Properties of Definite Integrals

Property I. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

i.e. The integration is independent of the change of variable.

Proof Let $\phi(x)$ be a primitive of $f(x)$, then

$$\frac{d}{dx}[\phi(x)] = f(x) \Rightarrow \frac{d}{dt}[\phi(t)] = f(t)$$

Therefore, $\int_a^b f(x)\,dx = [\phi(x)]_a^b = \phi(b) - \phi(a)$ ...(i)

and $\int_a^b f(t)\,dt = [\phi(t)]_a^b = \phi(b) - \phi(a)$ ...(ii)

From Eqs. (i) and (ii), we have

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

Property II. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

i.e. if the limits of definite integral are interchanged, then its value changes by minus sign only.

Proof Let $\phi(x)$ be a primitive of $f(x)$, then

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a)$$

and $-\int_b^a f(x)\,dx = -[\phi(a) - \phi(b)] = \phi(b) - \phi(a)$

$\therefore$ $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Property III. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ (King's property)

Proof On RHS put $(a - x) = t$, so that $dx = -dt$

Also, when $x = 0$, then $t = a$ and when $x = a$, then $t = 0$

$\therefore$ $\int_0^a f(a-x)\,dx = -\int_a^0 f(t)\,dt = \int_0^a f(t)\,dt = \int_0^a f(x)\,dx$

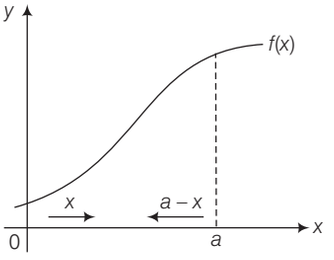
$\therefore$ $\int_0^a f(a-x)\,dx = \int_0^a f(x)\,dx$

Remark

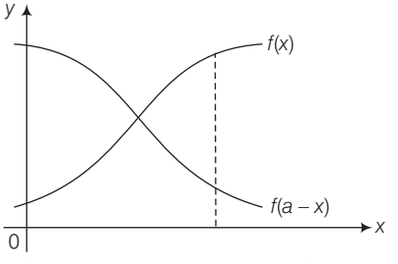
This property is useful to evaluate a definite integral without first finding the corresponding indefinite integrals which may be difficult or sometimes impossible to find.

Geometrically $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

This property says that when integrating from 0 to a, we will get the same result whether we use the function $f(x)$ or $f(a - x)$. The justification for this property will become clear from the figures below :



As x progresses from 0 to a, the variable $a - x$ progresses from a to 0. Thus, whether we use x or $a - x$, the entire interval $[0, a]$ is still covered.



The function $f(a - x)$ can be obtained from the function $f(x)$ by first flipping $f(x)$ along the y-axis and then shifting it right by a units. Notice that in the interval $[0, a]$, $f(x)$ and $f(a - x)$ describe precisely the same area.

There are two ways to look at the justification of this property, as described in the figures on the left and right respectively.

Example 16 Show that

(i) $\int_0^{\pi/2} f(\sin x)\,dx = \int_0^{\pi/2} f(\cos x)\,dx$

(ii) $\int_0^{\pi/2} f(\tan x)\,dx = \int_0^{\pi/2} f(\cot x)\,dx$

(iii) $\int_0^{\pi/2} f(\sin 2x)\sin x\,dx = \int_0^{\pi/2} \sqrt{2}\, f(\cos 2x)\cdot\cos x\,dx$

$= \int_0^{\pi/2} f(\sin 2x)\cdot\cos x\,dx$

(iv) $\int_0^{\pi} x f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$

[IIT JEE 1982]

Sol. (i) We know,

$$\int_0^{\pi/2} f(\sin x)\,dx = \int_0^{\pi/2} f\left[\sin\left(\frac{\pi}{2}-x\right)\right]dx$$

$$\left[\text{using } \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi/2} f(\cos x)\,dx$$

$$\therefore \int_0^{\pi/2} f(\sin x)\,dx = \int_0^{\pi/2} f(\cos x)\,dx$$

(ii) $\int_0^{\pi/2} f(\tan x)\,dx = \int_0^{\pi/2} f\left(\tan\left(\frac{\pi}{2}-x\right)\right)dx$

$$\left[\text{using } \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi/2} f(\cot x)\,dx$$

$$\therefore \int_0^{\pi/2} f(\tan x)\,dx = \int_0^{\pi/2} f(\cot x)\,dx$$

(iii) We know, $I = \int_0^{\pi/2} f(\sin 2x)\sin x\,dx$ ...(i)

$$= \int_0^{\pi/2} f\left[\sin 2\left(\frac{\pi}{2}-x\right)\right]\cdot\sin\left(\frac{\pi}{2}-x\right)dx$$

$$\left[\text{using } \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi/2} f(\sin(\pi-2x))\cos x\,dx$$

$$I = \int_0^{\pi/2} f(\sin 2x)\cos x\,dx \quad \text{...(ii)}$$

Adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi/2} f(\sin 2x)(\sin x + \cos x)\,dx$$

$$= \sqrt{2}\int_0^{\pi/2} f(\sin 2x)\sin\left(x+\frac{\pi}{4}\right)dx$$

Put $x + \frac{\pi}{4} = \left(\frac{\pi}{2} - \theta\right)$ $\left(\text{i.e. } x = \frac{\pi}{4} - \theta\right)$

$$\Rightarrow \quad 2I = -\sqrt{2}\int_{\pi/4}^{-\pi/4} f(\cos 2\theta)\cos\theta\,d\theta$$

$$= \sqrt{2}\int_{-\pi/4}^{\pi/4} f(\cos 2\theta)\cos\theta\,d\theta$$

$$= 2\sqrt{2}\int_0^{\pi/4} f(\cos 2\theta)\cos\theta\,d\theta \text{ (since it is even)}$$

$$\therefore \quad I = \sqrt{2}\int_0^{\pi/4} f(\cos 2\theta)\cos\theta\,d\theta$$

(iv) Let $I = \int_0^{\pi} x\,f(\sin x)\,dx$ ...(i)

Replacing x by $(\pi - x)$, we get

$$I = \int_0^{\pi} (\pi - x)\,f(\sin(\pi - x))\,dx$$

$$\Rightarrow \quad I = \int_0^{\pi} (\pi - x)\,f(\sin x)\,dx \quad \text{...(ii)}$$

Adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi} \pi\,f(\sin x)\,dx \Rightarrow I = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$$

$$\therefore \int_0^{\pi} x\,f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$$

Example 17 If f and g are continuous functions satisfying $f(x) = f(a-x)$ and $g(x) + g(a-x) = 2$, then show that $\int_0^a f(x)\,g(x)\,dx = \int_0^a f(x)\,dx$.

Sol. Let $I = \int_0^a f(x)\,g(x)\,dx = \int_0^a f(a-x)\,g(a-x)\,dx$

$$= \int_0^a f(x)\cdot[2 - g(x)]\,dx$$

$$\left[\text{using } \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$\because \quad f(x) = f(a-x)$ and $g(a-x) + g(x) = 2$ (given)

$$\therefore \quad \int_0^a f(x)\cdot g(x)\,dx = 2\int_0^a f(x)\,dx - \int_0^a f(x)\cdot g(x)\,dx$$

or $2\int_0^a f(x)\cdot g(x)\,dx = 2\int_0^a f(x)\,dx$

$$\Rightarrow \quad \int_0^a f(x)\,g(x)\,dx = \int_0^a f(x)\,dx$$

Example 18 Evaluate

(i) $\int_0^{\pi/2} \frac{dx}{1+\sqrt{\tan x}}$ (ii) $\int_0^{\pi/2} \log(\tan x)\,dx$

(iii) $\int_0^{\pi/4} \log(1+\tan x)\,dx$ (iv) $\int_0^{\pi/2} \frac{\sin x - \cos x}{1+\sin x\cos x}\,dx$

Sol. (i) Let $I = \int_0^{\pi/2} \frac{dx}{1+\sqrt{\tan x}} = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$ ...(i)

Then, $I = \int_0^{\pi/2} \frac{\sqrt{\cos(\pi/2 - x)}}{\sqrt{\cos(\pi/2-x)}+\sqrt{\sin(\pi/2-x)}}\,dx$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx \quad \text{...(ii)}$$

Adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = \int_0^{\pi/2} 1\,dx$$

$$= [x]_0^{\pi/2} = \frac{\pi}{2} - 0 \Rightarrow 2I = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

(ii) Let $I = \int_0^{\pi/2} \log(\tan x)\,dx$ ...(i)

Then, $I = \int_0^{\pi/2} \log\left\{\tan\left(\frac{\pi}{2}-x\right)\right\}dx$

$$\Rightarrow \quad I = \int_0^{\pi/2} \log(\cot x)\,dx \quad \text{...(ii)}$$

Adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi/2} \log(\tan x)\,dx + \int_0^{\pi/2} \log(\cot x)\,dx$$

$$= \int_0^{\pi/2} (\log\tan x + \log\cot x)\,dx$$

$$= \int_0^{\pi/2} \log(\tan x\cdot\cot x)\,dx = \int_0^{\pi/2} \log(1)\,dx$$

$$\Rightarrow \quad 2I = 0 \Rightarrow I = 0$$

(iii) Let $I = \int_0^{\pi/4} \log(1+\tan x)\,dx$ ...(i)

$$= \int_0^{\pi/4} \log[1+\tan(\pi/4 - x)]\,dx$$

$$= \int_0^{\pi/4} \log\left(1+\frac{\tan\pi/4 - \tan x}{1+\tan(\pi/4)\cdot\tan x}\right)dx$$

$$= \int_0^{\pi/4} \log\left(\frac{1+\tan x+1-\tan x}{1+\tan x}\right)dx$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan x}\right)dx$$

$$= \int_0^{\pi/4} \log(2)\,dx - \int_0^{\pi/4} \log(1+\tan x)\,dx$$

$\Rightarrow I = (\log 2)(x)_0^{\pi/4} - I$ [using Eq. (i)]

$\Rightarrow 2I = \frac{\pi}{4}\log 2 \Rightarrow I = \frac{\pi}{8}\log 2$

(iv) Let $I = \int_0^{\pi/2} \frac{\sin x - \cos x}{1+\sin x\cos x}\,dx$ ...(i)

Then, $I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-x\right) - \cos\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}\,dx$

$\Rightarrow I = \int_0^{\pi/2} \frac{\cos x - \sin x}{1+\cos x\cdot\sin x}\,dx$ ...(ii)

Adding Eqs. (i) and (ii), we get

$2I = \int_0^{\pi/2} \frac{\sin x - \cos x}{1+\sin x\cos x}\,dx + \int_0^{\pi/2} \frac{\cos x - \sin x}{1+\sin x\cos x}\,dx \Rightarrow$

$2I = 0 \Rightarrow I = 0$

Example 19 The value of $\int_0^a \log(\cot a + \tan x)\,dx$, where $a \in (0, \pi/2)$ is equal to

(a) $a\log(\sin a)$ (b) $-a\log(\sin a)$

(c) $-a\log(\cos a)$ (d) None of these

Sol. Let $I = \int_0^a \log(\cot a + \tan x)\,dx$

$$= \int_0^a \log\left(\frac{\cos a}{\sin a} + \frac{\sin x}{\cos x}\right)dx$$

$$= \int_0^a \log\left(\frac{\cos(a-x)}{\sin a\cos x}\right)dx$$

$$= \int_0^a \log[\cos(a-x)]\,dx - \int_0^a \log(\sin a)\,dx - \int_0^a \log(\cos x)\,dx$$

$$= \int_0^a \log\cos(x)\,dx - \int_0^a \log(\sin a)\,dx - \int_0^a \log(\cos x)\,dx$$

$\left[\text{using } \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx \text{ to first integral}\right]$

$$= -\log(\sin a)\int_0^a dx = -a\log(\sin a)$$

Hence, (b) is the correct answer.

Property IV. $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$ (King's property)

Proof Put $x = a+b-t \Rightarrow dx = -dt$

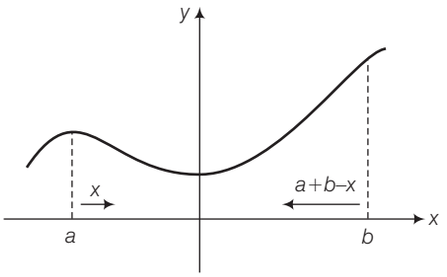
Also, when $x = a$, then $t = b$, and when $x = b$

$$\therefore \int_a^b f(x)\,dx = \int_b^a f(a+b-t)(-dt) = -\int_b^a f(a+b-t)\,dt$$

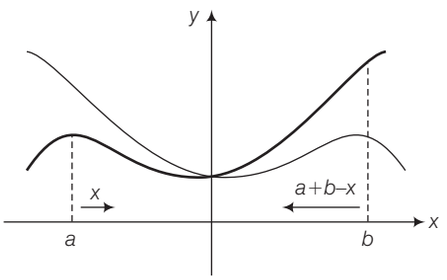
$$= \int_a^b f(a+b-t)\,dt = \int_a^b f(a+b-x)\,dx$$

$$\therefore \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

Geometrically $\int_0^a f(x)dx = \int^{a^b} f(a+b-x)\,dx$



As the variable x varies from a to b, the variable $a+b-x$ varies from b to a. Thus, whether we use x or $a+b-x$, the entire interval $[a, b]$ is covered in both the cases and the areas will be the same.



The graph of $f(a+b-x)$ can be obtained from the graph of $f(x)$ by first flipping the graph of $f(x)$ along the y-axis and then shifting it $(a+b)$ units towards the right; the areas described by $f(x)$ and $f(a+b-x)$ in the interval $[a, b]$ are precisely the same.

Example 20 ***Evaluate*** $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$.

Sol. Let $I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}} = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}\cdot dx}{\sqrt{\cos x}+\sqrt{\sin x}}$ ... (i)

then, $I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos(\pi/2-x)}}{\sqrt{\cos(\pi/2-x)}+\sqrt{\sin(\pi/2-x)}}\,dx$

$(\because a+b = \pi/2)$

$\Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$ ...(ii)

Adding Eqs. (i) and (ii), we get

$$2I = \int_{\pi/6}^{\pi/3} 1\,dx = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6} \Rightarrow I = \frac{\pi}{12}$$

Example 21 Prove that

$$\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx = \frac{b-a}{2}.$$

Sol. Let $$I = \int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx \quad \text{...(i)}$$

then, $$I = \int_a^b \frac{f(a+b-x)}{f(a+b-x)+f(a+b-(a+b-x))}\,dx$$

$$\Rightarrow \quad I = \int_a^b \frac{f(a+b-x)}{f(a+b-x)+f(x)}\,dx \quad \text{...(ii)}$$

Adding Eqs. (i) and (ii), we get

$$2I = \int_a^b \frac{f(a+b-x)+f(x)}{f(a+b-x)+f(x)}\,dx$$

$$\Rightarrow \quad 2I = \int_a^b 1\,dx = (b-a) \Rightarrow I = \frac{b-a}{2}$$

Example 22 Solve

$$I = \int_{\cos^4 t}^{-\sin^4 t} \frac{\sqrt{f(z)}\,dz}{\sqrt{f(\cos 2t - z)}+\sqrt{f(z)}}.$$

Sol. We have, $$I = \int_{\cos^4 t}^{-\sin^4 t} \frac{\sqrt{f(z)}\,dz}{\sqrt{f(\cos 2t - z)}+\sqrt{f(z)}} \quad \text{...(i)}$$

$$\therefore \quad I = \int_{\cos^4 t}^{-\sin^4 t} \frac{\sqrt{f(\cos 2t - z)}\,dz}{\sqrt{f(\cos 2t - z)}+\sqrt{f(z)}} \quad \text{..(ii)}$$

$$\left[\text{using } \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx\right]$$

Adding Eqs. (i) and (ii), we get

$$2I = \int_{\cos^4 t}^{-\sin^4 t} dz \Rightarrow 2I = (z)_{\cos^4 t}^{-\sin^4 t}$$

$$\therefore \quad I = -\frac{1}{2}(\sin^4 t + \cos^4 t) = -\frac{1}{2}(1 - 2\sin^2 t \cos^2 t)$$

$$= -\frac{1}{2}\left(1 - \frac{1}{2}\sin^2 2t\right) = -\frac{1}{2} + \frac{1}{4}\sin^2 2t$$

Directions (Ex 23-25) *Let the function f satisfies*

$$f(x)\cdot f'(-x) = f(-x)\cdot f'(x) \text{ for all } x \text{ and } f(0) = 3.$$

Example 23 The value of $f(x)\cdot f(-x)$ for all x is

(a) 4 (b) 9
(c) 12 (d) 16

Sol. Given, $$f(x)\cdot f'(-x) = f(-x)\cdot f'(x)$$

$$\Rightarrow \quad \frac{f'(x)}{f(x)} = \frac{f'(-x)}{f(-x)}$$

Integrating both sides, we get

$$\ln f(x) = -f(-x) + C$$

$$\ln[f(x)\cdot f(-x)] = C$$

$$f(x)\cdot f(-x) = C$$

But $$f(0) = 3$$

$$\Rightarrow \quad f^2(0) = C \therefore C = 9$$

$$\therefore \quad f(x)\cdot f(-x) = 9$$

Aliter $$f(x)\cdot f'(x) - f(-x)\cdot f'(x) = 0$$

$$\Rightarrow \quad \frac{d}{dx}[f(-x)\cdot f(x)] = 0$$

Integrating both sides, we get $f(x)\cdot f(-x) = \text{Constant}$

Hence, (b) is the correct answer.

Example 24 $\int_{-51}^{51} \frac{dx}{3+f(x)}$ has the value equal to

(a) 17 (b) 34
(c) 102 (d) 0

Sol. Let $$I = \int_{-51}^{51} \frac{dx}{3+f(x)} = \int_{-51}^{51} \frac{dx}{3+f(-x)}$$

$$\left[\text{using } \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right]$$

$$2I = \int_{-51}^{51} \frac{6+f(x)+f(-x)}{[3+f(x)][3+f(-x)]}dx$$

$$= \int_{-51}^{51} \frac{6+f(x)+f(-x)}{9+3[f(x)+f(-x)]+f(x)\cdot f(-x)}\,dx$$

$$= \int_{-51}^{51} \frac{6+f(x)+f(-x)}{18+3[f(x)+f(-x)]}\,dx$$

$$= \frac{1}{3}\int_{-51}^{51} dx = \frac{2\cdot 51}{3}$$

$$\Rightarrow \quad I = \frac{51}{3} = 17$$

Hence, (a) is the correct answer.

Example 25 Number of roots of $f(x) = 0$ in $[-2, 2]$ is

(a) 0 (b) 1
(c) 2 (d) 4

Sol. Let $x = \alpha$ be the root of $f(x) = 0$.

$$\therefore \quad f(\alpha) = 0$$

$$f(x)\cdot f(-x) = 9$$

Put $x = \alpha$, then $0 = 9$ (impossible)

Therefore, $f(x)$ has no root but $f(0) = 3$.

$\therefore f(x) > 0, \forall\ x \in R$ as f is continuous possible function $f(x) = 3e^{-x}$.

Hence, (a) is the correct answer.

Exercise for Session 2

1. The value of $\int_0^{\pi/4} \log(1+\tan\theta)\,d\theta$ is equal to

(a) $\frac{\pi}{2}\log 2$ (b) $-\frac{\pi}{4}\log 2$ (c) $\frac{\pi}{8}\log 2$ (d) None of these

2. For any integer n, the value of $\int_0^{\pi} e^{\cos^2 x}\cdot\cos^3(2n+1)x\cdot dx$ is equal to

(a) 0 (b) 1 (c) –1 (d) None of these

3. The value of $\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}\,dx$ is equal to

(a) 1/2 (b) 1/3 (c) 1/4 (d) None of these

4. The value of $\int_0^2 \frac{dx}{(17+8x-4x^2)(e^{6(1-x)}+1)}$ is equal to

(a) $-\frac{1}{8\sqrt{21}}\log\left|\frac{2-\sqrt{21}}{2+\sqrt{21}}\right|$ (b) $-\frac{1}{8\sqrt{21}}\log\left|\frac{2+\sqrt{21}}{\sqrt{21}-2}\right|$

(c) $-\frac{1}{8\sqrt{21}}\left\{\log\left|\frac{2-\sqrt{21}}{2+\sqrt{21}}\right|-\log\left|\frac{2+\sqrt{21}}{\sqrt{21}-2}\right|\right\}$ (d) None of these

5. If f is an odd function, then the value of $\int_{-a}^{a} \frac{f(\sin x)}{f(\cos x)+f(\sin^2 x)}\,dx$ is equal to

(a) 0 (b) $f(\cos x)+f(\sin x)$ (c) 1 (d) None of these

6. If $[x]$ stands for the greatest integar function, then $\int_4^{10} \frac{[x^2]\,dx}{[x^2-28x+196]+[x^2]}$ is

(a) 1 (b) 2 (c) 3 (d) 4

7. The value of $\int_0^{\pi} \frac{x\,dx}{1+\cos\alpha\cdot\sin\alpha}(0<\alpha<\pi)$ is

(a) $\frac{\pi}{\sin\alpha}$ (b) $\frac{\pi\alpha}{\sin\alpha}$ (c) $\frac{\alpha}{\sin\alpha}$ (d) $\frac{\sin\alpha}{\alpha}$

8. If f, g, h be continuous functions on $[0, a]$ such that $f(a-x)=f(x)$, $g(a-x)=-g(x)$ and $3h(x)=5+4h(a-x)$, then the value of $\int_0^a f(x)\cdot g(x)\cdot h(x)dx$ is

(a) 0 (b) 1 (c) a (d) $2a$

9. If $2f(x)+f(-x)=\frac{1}{x}\sin\left(x-\frac{1}{x}\right)$, then the value of $\int_{1/e}^{e} f(x)dx$ is

(a) 0 (b) e (c) $1/e$ (d) $e+1/e$

10. Prove that $\int_0^{\pi} xf(\sin x)dx=\frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$.

11. Evaluate $\int_0^{\pi} \frac{x^2\sin 2x\cdot\sin\left(\frac{\pi}{2}\cos x\right)dx}{(2x-\pi)}$.

12. Number of positive continuous function $f(x)$ defined on $[0, 1]$ for which $\int_0^1 f(x)dx=1$, $\int_0^1 xf(x)dx=2$ and $\int_0^1 x^2 f(x)dx=4$.

13. Let $I_1=\int_0^1 \frac{e^x}{1+x}dx$ and $I_2=\int_0^1 \frac{x^2}{e^{x^3}(2-x^3)}dx$. Then $\frac{I_1}{I_2}$ is equal to

(a) $\frac{3}{e}$ (b) $\frac{3}{e}$ (c) $3e$ (d) $\frac{1}{3e}$

14. If $f(x)=\frac{e^x}{1+e^x}$, $I_1=\int_{f(-a)}^{f(x)} x\cdot g\{x(1-x)\}dx$ and $I_2=\int_{f(-a)}^{f(a)} g\{x(1-x)\}dx$, then the value of $\frac{I_2}{I_1}$ is

(a) 1 (b) –3 (c) –1 (d) 2

Session 3

Applications of Piecewise Function Property

Applications of Piecewise Function Property

Property V (a). $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$

where $c \leftrightarrow R$

Proof Let $\phi(x)$ be primitive of $f(x)$, then

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a) \qquad \ldots(i)$$

and $\int_a^c f(x)dx + \int_c^b f(x)dx = [\phi(c) - \phi(a)] + [\phi(b) - \phi(c)]$

$$= \phi(b) - \phi(a) \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Generalisation Property V(a) can be generalised into the following form

$$\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)\,dx + \ldots + \int_{c_n}^b f(x)dx$$

where, $a < c_1 < c_2 < \ldots < c_{n-1} < c_n < b$

Property V (b).

$$\int_0^a f(x)\,dx = \int_0^{a/2} f(x)\,dx + \int_0^{a/2} f(a-x)\,dx$$

Proof As we know,

$$\int_0^a f(x)\,dx = \int_0^{a/2} f(x)\,dx + \int_{a/2}^a f(x)\,dx$$

Put $x = a - t \Rightarrow dx = -dt$ in the second integral also, when $x = a/2$, then $t = a/2$ and when $x = a$, then $t = 0$.

$$\therefore \quad \int_0^a f(x)\,dx = \int_0^{a/2} f(x)\,dx + \int_{a/2}^0 f(a-t)\,(-dt)$$

$$= \int_0^{a/2} f(x)\,dx + \int_0^{a/2} f(a-t)\,dt$$

$$\int_0^a f(x)\,dx = \int_0^{a/2} f(x)\,dx + \int_0^{a/2} f(a-x)\,dx$$

Property V (c).

$$\int_a^b f(x)\,dx = \begin{cases} 0, & \text{if } f(a+x) = -f(b-x) \\ 2\int_a^{\frac{a+b}{2}} f(x)\,dx, & \text{if } f(a+x) = f(b-x) \end{cases}$$

Proof Let us consider the function $f(x)$ on $[a, b]$ when $f(a+x) = f(b-x)$, then f is even symmetric about the mid-point $x = \dfrac{a+b}{2}$ on the interval $[a, b]$ when $f(a+x) = -f(b-x)$, then f is odd symmetric about the mid-point $x = \dfrac{a+b}{2}$ of the interval $[a, b]$.

$$\therefore \text{Using } \int_{-a}^a f(x)\,dx = \begin{cases} 0, & \text{if } f \text{ is an odd function} \\ 2f(x)dx, & \text{if } f \text{ is an even function} \end{cases}$$

$$\Rightarrow \int_a^b f(x)dx = \begin{cases} 0, & \text{if } f(a+x) = -f(b-x) \\ 2\int_a^{\frac{a+b}{2}} f(x)\,dx, & \text{if } f(a+x) = f(b-x) \end{cases}$$

Example 26 ***Given function,*** $f(x) = \begin{cases} x^2, \text{ for } 0 \le x < 1 \\ \sqrt{x}, \text{ for } 1 \le x \le 2 \end{cases}$.

Evaluate $\int_0^2 f(x)\,dx$.

Sol. Here, $\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$

$$\therefore \quad \int_0^2 f(x)\,dx = \int_0^1 x^2\,dx + \int_1^2 \sqrt{x}\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{x^{3/2}}{3/2}\right]_1^2$$

$$= \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{2}{3}x\sqrt{x}\right]_1^2$$

$$= \left[\frac{1}{3} - 0\right] + \frac{2}{3}[2\sqrt{2} - 1]$$

$$= \frac{1}{3} + \frac{4\sqrt{2}}{3} - \frac{2}{3} = \frac{4\sqrt{2}}{3} - \frac{1}{3} = \frac{1}{3}(4\sqrt{2} - 1)$$

Example 27 Evaluate the integral $I = \int_0^2 |1 - x|\,dx$.

Sol. By definition, $|a-b| = \begin{cases} b-a, & \text{if } a < b \\ a-b, & \text{if } a > b \end{cases}$

Then, $|1-x| = \begin{cases} (1-x), & 0 \le x \le 1 \\ (x-1), & 1 \le x \le 2 \end{cases}$

$$\therefore \int_0^2 |1-x|\,dx = \int_0^1 (1-x)\,dx + \int_1^2 (x-1)\,dx$$

$$= \left[x - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^2$$

$$= \left\{\left(1-\frac{1}{2}\right) - (0-0)\right\} + \left\{\left(\frac{4}{2}-2\right) - \left(\frac{1}{2}-1\right)\right\}$$

$$= \frac{1}{2} + \left\{0 + \frac{1}{2}\right\} = 1$$

Example 28 Evaluate

(i) $\int_0^{\pi} |\cos x|\,dx$ (ii) $\int_0^2 |x^2+2x-3|\,dx$

Sol. (i) $\int_0^{\pi} |\cos x|\,dx = \int_0^{\pi/2} |\cos x|\,dx + \int_{\pi/2}^{\pi} |\cos x|\,dx$

$$= \int_0^{\pi/2} (\cos x)\,dx - \int_{\pi/2}^{\pi} (\cos x)\,dx$$

$$= [\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{\pi}$$

$$= (1-0) - (0-1) = 2$$

(ii) $\int_0^2 |x^2+2x-3|\,dx$

$$= \int_0^1 |x^2+2x-3|\,dx + \int_1^2 |x^2+2x-3|\,dx \quad \ldots(i)$$

We have, $x^2+2x-3 = (x+3)(x-1)$

$\therefore \quad x^2+2x-3 > 0$ for $x < -3$ or $x > 1$

and $x^2+2x-3 < 0$ for $-3 < x < 1$

So, $|x^2+2x-3|$

$$= \begin{cases} (x^2+2x-3), & \text{for } x < -3 \text{ or } x > 1 \\ -(x^2+2x-3), & \text{for } -3 < x < 1 \end{cases}$$

$\therefore$ Eq. (i) becomes

$$I = \int_0^1 -(x^2+2x-3)\,dx + \int_1^2 (x^2+2x-3)\,dx$$

$$= -\left[\frac{x^3}{3} + x^2 - 3x\right]_0^1 + \left[\frac{x^3}{3} + x^2 - 3x\right]_1^2$$

$$= 4$$

Example 29 Evaluate $\int_{-1}^1 (x - [x])\,dx$, where [.] denotes the greatest integral part of x.

Sol. Let $I = \int_{-1}^1 (x-[x])\,dx = \int_{-1}^1 x \cdot dx - \int_{-1}^1 [x]\,dx$

$$= \left(\frac{x^2}{2}\right)_{-1}^1 - \left(\int_{-1}^0 [x]\,dx + \int_0^1 [x]\,dx\right)$$

$$= \frac{1}{2}(1-1) - \left(\int_{-1}^0 -1\,dx + \int_0^1 0\,dx\right)$$

$$= 0 + (x)_{-1}^1 - 0 = 1$$

Example 30 Evaluate $\int_0^2 \{x\}\,dx$, where $\{x\}$ denotes the fractional part of x.

Sol. $\int_0^2 \{x\}\,dx = \int_0^2 (x-[x])\,dx = \int_0^2 x\,dx - \int_0^2 [x]\,dx$

$$= \left(\frac{x^2}{2}\right)_0^2 - \left(\int_0^1 [x]\,dx + \int_1^2 [x]\,dx\right)$$

$$= \frac{1}{2}(4-0) - \left(\int_0^1 0\,dx + \int_1^2 1\,dx\right)$$

$$= 2 - (x)_1^2 = 2 - (1) = 1$$

Remark

In above example, for greatest integer less than or equal to x, it is compulsory to break it at integral limits.

Example 31 Evaluate $\int_0^9 \{\sqrt{x}\}\,dx$, where $\{x\}$ denotes the fractional part of x.

Sol. $\int_0^9 \{\sqrt{x}\}\,dx = \int_0^9 \left(\sqrt{x} - [\sqrt{x}]\right)dx$

$$= \int_0^9 (x^{1/2})\,dx - \int_0^9 [\sqrt{x}]\,dx$$

$$= \frac{2}{3}(x^{3/2})_0^9 - \int_0^9 [\sqrt{x}]\,dx$$

$$= \frac{2}{3}[27] - \int_0^9 [\sqrt{x}]\,dx$$

As $\int_0^9 [\sqrt{x}]\,dx \Rightarrow 0 \le x \le 9$ and $0 \le \sqrt{x} \le 3$

Thus, it should be divided into three parts

$$0 \le \sqrt{x} \le 1, 1 \le \sqrt{x} \le 2, 2 \le \sqrt{x} \le 3$$

i.e. $I = 2(9) - \int_0^9 [\sqrt{x}]\,dx$

$$= 18 - \left[\int_0^1 [\sqrt{x}]\,dx + \int_1^4 [\sqrt{x}]\,dx + \int_4^9 [\sqrt{x}]\,dx\right]$$

$$= 18 - \left(\int_0^1 0\,dx + \int_1^4 1\,dx + \int_4^9 2\,dx\right)$$

$$= 18 - \left(0 + (x)_1^4 + (2x)_4^9\right) = 18 - (3+10) = 5$$

Example 32 If for a real number y, $[y]$ is the greatest integer less than or equal to y, then find the value of the integral $\int_{\pi/2}^{3\pi/2} [2\sin x]\,dx$.

Sol. We know, $-1 \le \sin x \le 1$ as $x \in [\pi/2, 3\pi/2]$

$\Rightarrow \quad -2 \le 2\sin x \le 2$

$\therefore$ $2\sin x$ must be divided (or broken) at $x = 5\pi/6, \pi, 7\pi/6$.

As $2\sin x = +1, 0, -1$ at these points.

$$\therefore \int_{\pi/2}^{3\pi/2} [2\sin x]\,dx = \int_{\pi/2}^{5\pi/6} [2\sin x]\,dx + \int_{5\pi/6}^{\pi} [2\sin x]\,dx$$

$$+ \int_{\pi}^{7\pi/6} [2\sin x]\,dx + \int_{7\pi/6}^{3\pi/2} [2\sin x]\,dx$$

$$= \int_{\pi/2}^{5\pi/6} 1\,dx + \int_{5\pi/6}^{\pi} 0\,dx + \int_{\pi}^{7\pi/6} (-1)\,dx + \int_{7\pi/6}^{3\pi/2} (-2)\,dx$$

$$= \left(\frac{5\pi}{6} - \frac{\pi}{2}\right) + 0 - \left(\frac{7\pi}{6} - \pi\right) - 2\left(\frac{3\pi}{2} - \frac{7\pi}{6}\right) = -\frac{\pi}{2}$$

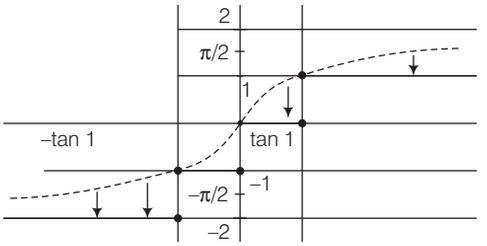
Example 33 The value of $\int_0^{100} [\tan^{-1} x]\,dx$ is equal to (where [.] denotes the greatest integer function)

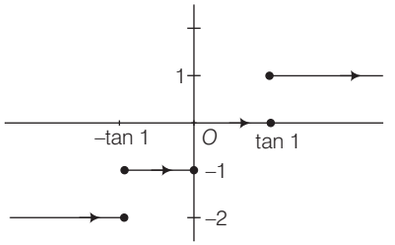
(a) tan 1 – 100 (b) $\pi/2 - \tan 1$

(c) 100 – tan 1 (d) None of these

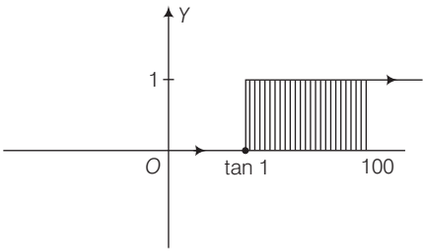
Sol. Let $I = \int_0^{100} [\tan^{-1} x]\,dx$

where $[\tan^{-1} x]$ is shown as





$\therefore \int_0^{100} [\tan^{-1} x]\,dx$ is shown as



$$= 1 \times (100 - \tan 1)$$

Hence, (c) is the correct answer.

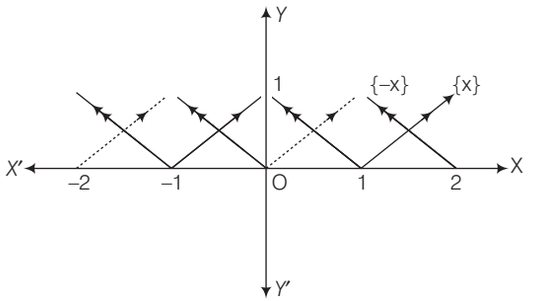
Example 34 The value of $\int_{-2}^{2} \min\{x - [x], -x - [-x]\}\,dx$ is equal to (where [.] denotes the greatest integer function)

(a) 1/2 (b) 1

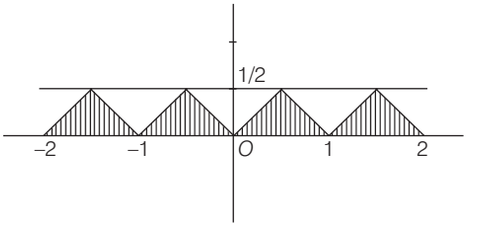
(c) 3/2 (d) 2

Sol. Let $f(x) = \min(x - [x], -x - [-x]) = \min(\{x\}, \{-x\})$

Graphically, $\{x\}$ and $\{-x\}$ could be plotted as;



From the above graph, we need $\int_{-2}^{2} \min(\{x\}, \{-x\})$ shown as



$$\therefore \quad \int_{-2}^{2} f(x)\,dx = 4\int_0^1 f(x)\,dx = 4 \times \frac{1}{2} \times 1 \times \frac{1}{2} = 1$$

Hence, (b) is the correct answer.

Example 35 ***The value of*** $\int_1^2 (x^{[x^2]} + [x^2]^x)\,dx$ ***is equal to where [.] denotes the greatest integer function***

(a) $\frac{5}{4} + \sqrt{3} + (2^{\sqrt{3}} - 2^{\sqrt{2}}) + \frac{1}{\log 3}(9 - 3^{\sqrt{3}})$

(b) $\frac{5}{4} + \sqrt{3} + \frac{\sqrt{2}}{3} + \frac{1}{\log 2}(2^{\sqrt{3}} - 2^{\sqrt{2}}) + \frac{1}{\log 3}(9 - 3^{\sqrt{3}})$

(c) $\frac{5}{4} + \frac{\sqrt{2}}{3} + \frac{1}{\log 2}(2^{\sqrt{3}} - 2^{\sqrt{2}}) + \frac{1}{\log 3}(9 - 3^{\sqrt{3}})$

(d) None of the above

Sol. Let $I = \int_1^2 (x^{[x^2]} + [x^2]^x)\,dx$

$$= \int_1^{\sqrt{2}} (x+1)\,dx + \int_{\sqrt{2}}^{\sqrt{3}} (x^2 + 2^x)\,dx + \int_{\sqrt{3}}^{2} (x^3 + 3^x)\,dx$$

$$= \left[\frac{x^2}{2} + x\right]_1^{\sqrt{2}} + \left[\frac{x^3}{3} + \frac{2^x}{\log 2}\right]_{\sqrt{2}}^{\sqrt{3}} + \left[\frac{x^4}{4} + \frac{3^x}{\log 3}\right]_{\sqrt{3}}^{2}$$

$$= \frac{5}{4} + \sqrt{3} + \frac{\sqrt{2}}{3} + \frac{1}{\log 2}(2^{\sqrt{3}} - 2^{\sqrt{2}}) + \frac{1}{\log 3}(3^2 - 3^{\sqrt{3}})$$

Hence, (b) is the correct answer.

Example 36 The value of $\int_0^{2\pi} [|\sin x| + |\cos x|]\,dx$ is equal to

(a) $\frac{\pi}{2}$ (b) π (c) $\frac{3\pi}{2}$ (d) 2π

Sol. Let $f(x) = [|\sin x| + |\cos x|]$

As, $|\sin x| \geq \sin^2 x$ and $|\cos x| \geq \cos^2 x$

$\therefore$ $|\sin x| + |\cos x| \geq 1$

and $|\sin x| + |\cos x| \leq \sqrt{1^2 + 1^2}$

$\Rightarrow$ $1 \leq |\sin x| + |\cos x| \leq \sqrt{2}$

Thus, $[|\sin x| + |\cos x|] = 1$

$\therefore$ $\int_0^{2\pi} [|\sin x| + |\cos x|]\, dx = \int_0^{2\pi} 1\, dx = 2\pi$

Hence, (d) is the correct answer.

Example 37 The value of the definite integral $\int_0^{\pi/2} \sin|2x - \alpha|\, dx$, where $\alpha \in [0, \pi]$, is

(a) 1 (b) $\cos\alpha$ (c) $\frac{1+\cos\alpha}{2}$ (d) $\frac{1-\cos\alpha}{2}$

Sol. Let $I = \int_{-\alpha}^{(\pi-\alpha)} \sin|t|\, dt$, where $2x - \alpha = t \Rightarrow dx = \frac{dt}{2}$

$$= \frac{1}{2}\int_{-\alpha}^{0} -\sin t\, dt + \frac{1}{2}\int_0^{\pi-\alpha} \sin t\, dt$$

$$= \left[\frac{1}{2}\cos t\right]_{-\alpha}^{0} - \left[\frac{1}{2}\cos t\right]_0^{\pi-\alpha}$$

$$= \frac{1}{2}[1 - \cos\alpha] - \frac{1}{2}[-\cos\alpha - 1]$$

$$= \frac{1}{2}(1 - \cos\alpha) + \frac{1}{2}(1 + \cos\alpha) = 1$$

$$= \frac{1}{2}(1 - \cos\alpha) + \frac{1}{2}(1 + \cos\alpha) = 1$$

Hence, (a) is the correct answer.

Example 38 Let f be a continuous functions satisfying

$$f'(\ln x) = \begin{cases} 1 & \text{for} \quad 0 < x \leq 1 \\ e^x - 1 & \text{for} \quad x > 1 \end{cases}$$

and $f(0) = 0$, then $f(x)$ can be defined as

(a) $f(x) = \begin{cases} 1, & \text{if} \quad x \leq 0 \\ 1 - e^x, & \text{if} \quad x > 0 \end{cases}$ (b) $f(x) = \begin{cases} 1, & \text{if} \quad x \leq 0 \\ e^x - 1, & \text{if} \quad x > 0 \end{cases}$

(c) $f(x) = \begin{cases} x, & \text{if} \quad x < 0 \\ e^x, & \text{if} \quad x > 0 \end{cases}$ (d) $f(x) = \begin{cases} x, & \text{if} \quad x \leq 0 \\ e^x - 1, & \text{if} \quad x > 0 \end{cases}$

Sol. $f'(\ln x) = \begin{cases} 1, & \text{for} \quad 0 < x \leq 1 \\ x, & \text{for} \quad x > 1 \end{cases}$

Put $\ln x = t \Rightarrow x = e^t$

For $x > 1$; $f'(t) = e^t$ for $t > 0$

Integrating $f(t) = e^t + C$; $f(0) = e^0 + C \Rightarrow C = -1$ [given, $f(0) = 0$]

$\therefore$ $f(t) = e^t - 1$, for $t > 0$ (corresponding to $x > 1$)

Therefore, $f(x) = e^x - 1$, for $x > 0$ ...(i)

Again, for $0 < x \leq 1$,

$$f'(\ln x) = 1 \qquad (\because x = e^t)$$

$f'(t) = 1$, for $t \leq 0$

$f(t) = t + C$

$f(0) = 0 + C \Rightarrow C = 0 \Rightarrow f(t) = t$, for $t \leq 0$

$\Rightarrow$ $f(x) = x$, for $x \leq 0$

Hence, (d) is the correct answer.

Example 39 The integral $\int_{\pi/4}^{5\pi/4} (|\cos t| \sin t + |\sin t| \cos t)\, dt$ has the value equal to

(a) 0 (b) 1/2

(c) $1/\sqrt{2}$ (d) 1

Sol. Let

$$I = \int_{\pi/4}^{\pi/2} 2\sin t \cos t\, dt + \underbrace{\int_{\pi/2}^{\pi} [(-\sin t \cos t) + (\sin t \cos t)]\, dt}_{\text{zero}} + \int_{\pi}^{5\pi/4} (-2\sin t \cos t)\, dt$$

$$= \int_{\pi/4}^{\pi/2} \sin 2t\, dt - \int_{\pi}^{5\pi/4} \sin 2t\, dt$$

These two integrals cancels

$\Rightarrow$ Zero.

Hence, (a) is the correct answer.

Example 40 The value of $\int_0^2 f(x)\, dx$, where

$f(x) = \begin{cases} 0, \text{when } x = \frac{n}{n+1}, & n = 1, 2, 3 \ldots \\ 1, \text{elsewhere} \end{cases}$ 3 is equal to

(a) 1 (b) 2

(c) 3 (d) None of these

Sol. Here, $\int_0^2 f(x)\, dx = \int_0^{1/2} 1\, dx + \int_{1/2}^{2/3} 1\, dx + \int_{2/3}^{3/4} 1\, dx + \ldots + \int_{\frac{n}{n+1}}^{\frac{n}{n-1}} 1\, dx + \ldots + \int_1^2 1\, dx$

$$= \left(\frac{1}{2}\right) + \left(\frac{2}{3} - \frac{1}{2}\right) + \left(\frac{3}{4} - \frac{2}{3}\right) + \ldots + \left(\frac{n}{n+1} - \frac{n-1}{n}\right) + \ldots + 1$$

$$= \frac{n}{n+1} + \ldots + 1, \text{ as } n \to \infty$$

We take, limit $n \to \infty$

We have, $\int_0^2 f(x)\, dx = 1 + 1 = 2$

Hence, (b) is the correct answer.

Exercise for Session 3

1. The value of $\int_{-1}^{3} \{|x-2|+[x]\}\,dx$, where [.] denotes the greatest integer function, is equal to

(a) 5 (b) 6 (c) 7 (d) None of these

2. The value of $\int_{-1}^{3} (|x|+|x-1|)\,dx$ is equal to

(a) 9 (b) 6 (c) 3 (d) None of these

3. Let $f(x)=x-[x]$, for every real number x (where, $[x]$ is integral part of x). Then, the value of $\int_{-1}^{1} f(x)\,dx$ is equal to

(a) 0 (b) 1 (c) 2 (d) None of these

4. The value of $\int_{0}^{2} [x+[x+[x]]]\,dx$ (where, [.] denotes the greatest integer function) is equal to

(a) 2 (b) 3 (c) –3 (d) None of these

5. The value of $\int_{0}^{[x]} \frac{2^x}{2^{[x]}}\,dx$ is equal to (where, [.] denotes the greatest integer function)

(a) $\frac{[x]}{\log 2}$ (b) $\frac{[x]}{2\log 2}$ (c) $\frac{[x]}{4\log 2}$ (d) None of these

6. The value of $\int_{0}^{4} \{x\}\,dx$ (where, {.} denotes fractional part of x) is equal to

(a) $\frac{4}{3}$ (b) $\frac{5}{3}$ (c) $\frac{7}{3}$ (d) None of these

7. The value of $\int_{1}^{4} \{x\}^{[x]}\,dx$ (where, [.] and {.} denote the greatest integer and fractional part of x) is equal to

(a) $\frac{11}{12}$ (b) $\frac{13}{12}$

(c) $\frac{7}{12}$ (d) $\frac{19}{12}$

8. The value of $\int_{0}^{x} [t+1]^3\,dt$ (where, [.] denotes the greatest integer function of x) is equal to

(a) $\left(\frac{[x]([x]+1)}{2}\right)^2 + ([x]+1)^3\{x\}$ (b) $\left(\frac{[x]([x]+1)}{2}\right)^3 + ([x]+1)^3\{x\}$

(c) $\left(\frac{[x]([x]+1)}{2}\right)^3 + ([x]+1)^2\{x\}$ (d) None of these

9. The value of $\int_{0}^{10\pi} [\tan^{-1}x]\,dx$ (where, $[\cdot]$ denotes the greatest integer function of x) is equal to

(a) $\tan 1$ (b) 10π

(c) $10\pi - \tan 1$ (d) None of these

10. If $f(x)=\min\{|x-1|,|x|,|x+1|$, then the value of $\int_{-1}^{1} f(x)\,dx$ is equal to

(a) 1 (b) $\frac{1}{2}$

(c) $\frac{1}{4}$ (d) $\frac{1}{8}$

11. The value of $\int_{0}^{\infty} [2e^{-x}]\,dx$ (where, $[\cdot]$ denotes the greatest integer function of x) is equal to

(a) 1 (b) $\log_e 2$

(c) 0 (d) $\frac{1}{e}$

12. The value of $\int_1^{10\pi}([\sec^{-1}x]+[\cot^{-1}x])dx$ (where, $[\cdot]$ denotes the greatest integer function) is equal to

(a) $(\sec 1)-10\pi$ (b) $10\pi-\sec 1$
(c) $\pi-\sec 1$ (d) None of these

13. The value of $\int_{-\pi/2}^{\pi/2}[\cot^{-1}x]dx$ (where, $[\cdot]$ denotes greatest integer function) is equal to

(a) $\pi+\cot 1$ (b) $\pi+\cot 2$
(c) $\pi+\cot 1+\cot 2$ (d) $\cot 1+\cot 2$

14. The value of $\int_0^{\pi/4}(\tan^n(x-[x])+\tan^{n-2}(x-[x]))dx$ (where, $[\cdot]$ denotes greatest integer function) is equal to

(a) $\frac{1}{n}$ (b) $\frac{1}{n-1}$ (c) $\frac{1}{n(n-1)}$ (d) $\frac{1}{n(n+1)}$

15. The value of $\int_0^2[x^2-x+1]dx$ (where, $[\cdot]$ denotes the greatest integer function) is equal to

(a) $\frac{5+\sqrt{5}}{2}$ b) $\frac{1+\sqrt{5}}{2}$ (c) $\frac{1-\sqrt{5}}{2}$ (d) $\frac{5-\sqrt{5}}{2}$

16. Evaluate $\int_0^a[x^n]dx$, (where, $[\cdot]$ denotes the greatest integer function).

17. Prove that $\int_0^x[x]dx=x[x]-\frac{1}{2}[x]([x]+1)$, where $[\cdot]$ denotes the greatest integer function.

18. If $f(n)=\frac{\int_0^n[x]dx}{\int_0^n\{x\}dx}$ (where, $[\cdot]$ and $\{\}$ denotes greatest integer and fractional part of x and $n\in N$). Then, the value of $f(4)$ is ...

19. If $f(n)=\int_0^x[\cos t]dt$, where $x\in\left(2n\pi,2n\pi+\frac{\pi}{2}\right);n\in N$ and $[\cdot]$ denotes the greatest integer function. Then, the value of $\left|f\left(\frac{1}{\pi}\right)\right|$ is ...

20. If $\int_0^x[x]dx=\int_0^{[x]}xdx, x\notin$ integer (where, $[\cdot]$ and $\{\}$ denotes the greatest integer and fractional parts respectively, then the value of $4\{x\}$ is equal to ...

Session 4

Applications of Even-Odd Property and Half the Integral Limit Property

Applications of Even-Odd Property and Half the Integral Limit Property

Property VI.

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is an even function} \\ 0, & \text{if } f(x) \text{ is an odd function} \end{cases}$$

Proof We know,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \text{ if } a < c < b$$

$$\therefore \quad \int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx \qquad \ldots(i)$$

Now, $\int_{-a}^{0} f(x)\,dx = \int_a^0 f(-t)(-dt)$, where $t = -x$

$$= -\int_a^0 f(-t)\,dt = \int_0^a f(-t)\,dt$$

$$= \int_0^a f(-x)\,dx \text{ (using properties I and II)}$$

$$= \begin{cases} \int_0^a f(x)\,dx, & \text{if } f(x) \text{ is even} \\ -\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is odd} \end{cases} \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & f(x) \text{ is even} \\ 0, & \text{if } f(x) \text{ is odd} \end{cases}$$

Example 41 Evaluate $\int_{-1}^{1} (x^3 + 5x + \sin x)dx$.

Sol. Let, $f(x) = x^3 + 5x + \sin x$

$\therefore \quad f(-x) = -x^3 - 5x - \sin x = -f(x)$

So, $f(x)$ is an odd function).

Hence, $\int_{-1}^{1} (x^3 + 5x + \sin x)dx = 0$

$$\left[\text{using } \int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)dx, f(x) \text{ is even} \\ 0, f(x) \text{ is odd} \end{cases}\right]$$

Example 42 Evaluate $\int_{-\pi/4}^{\pi/4} x^3 \sin^4 x\,dx$.

Sol. Let $f(x) = x^3 \sin^4 x$, then

$$f(-x) = (-x)^3 \sin^4(-x) = -x^3 [\sin(-x)]^4$$

$$= -x^3 (-\sin x)^4 = -x^3 \sin^4 x = -f(x)$$

So, $f(x)$ is an odd function.

Hence, $\int_{-\pi/4}^{\pi/4} f(x)\,dx = 0$

i.e. $\int_{-\pi/4}^{\pi/4} x^3 \sin^4 x\,dx = 0$

Example 43 Evaluate $\int_{-\pi/2}^{\pi/2} \sin^2 x\,dx$.

Sol. Let $f(x) = \sin^2 x$, then

$$f(-x) = \sin^2(-x) = [\sin(-x)]^2 = (-\sin x)^2$$

$$= \sin^2 x = f(x)$$

So, $f(x)$ is an even function, hence

$$\int_{-\pi/2}^{\pi/2} (\sin^2 x)\,dx = 2\int_0^{\pi/2} (\sin^2 x)\,dx$$

$$= 2\int_0^{\pi/2} \frac{1 - \cos 2x}{2}\,dx$$

$$= \left(x - \frac{\sin 2x}{2}\right)_0^{\pi/2} = \frac{\pi}{2}$$

$$\therefore \int_{-\pi/2}^{\pi/2} \sin^2 x\,dx = \frac{\pi}{2}$$

Example 44 ***The value of*** $\int_{-1}^{1} \log\left(\frac{2-x}{2+x}\right)dx$ ***is equal to***

(a) $\frac{1}{2}$ (b) 1 (c) –1 (d) 0

Sol. Let $f(x) = \log\left(\frac{2-x}{2+x}\right)$

Now, $f(-x) = \log\left(\frac{2+x}{2-x}\right) = \log\left(\frac{2-x}{2+x}\right)^{-1}$

$$= -\log\left(\frac{2-x}{2+x}\right)$$

$$\therefore \quad f(-x) = -\log\left(\frac{2-x}{2+x}\right) = -f(x)$$

i.e. $f(x)$ is an odd function.

So, $\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} \log\left(\frac{2-x}{2+x}\right)dx = 0$

$$\left[\because \int_{-a}^{a} f(x)\,dx = \begin{cases} 0, & \text{if } f(x) \text{ is odd} \\ 2\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is even} \end{cases}\right]$$

Hence, (d) is the correct answer.

Example 45 The value of

$\int_0^{\pi} \frac{x\sin(2x)\cdot\sin\left(\frac{\pi}{2}\cos x\right)}{(2x-\pi)}dx$ is equal to

(a) $\frac{8}{\pi}$ (b) $\frac{\pi}{8}$ (c) $\frac{8}{\pi^2}$ (d) $\frac{\pi^2}{8}$

Sol. Let $I = \int_0^{\pi} \frac{x\sin(2x)\cdot\sin\left(\frac{\pi}{2}\cos x\right)}{(2x-\pi)}dx$ ...(i)

$$I = \int_0^{\pi} \frac{(\pi-x)\cdot\sin 2(\pi-x)\cdot\sin\left(\frac{\pi}{2}\cos(\pi-x)\right)}{2(\pi-x)-\pi}dx$$

$$I = \int_0^{\pi} \frac{(\pi-x)\cdot\sin(2x)\cdot\sin\left(\frac{\pi}{2}\cos x\right)}{-(2x-\pi)}dx \quad \text{...(ii)}$$

Adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi} \frac{(2x-\pi)\sin 2x\cdot\sin\left(\frac{\pi}{2}\cos x\right)}{(2x-\pi)}dx$$

$$\therefore\ I = \frac{1}{2}\int_0^{\pi} 2\sin x\cos x\cdot\sin\left(\frac{\pi}{2}\cos x\right)dx$$

(put $\cos x = t$, then $-\sin x\,dx = dt$)

$$= -\int_1^{-1} t\sin\left(\frac{\pi}{2}t\right)dt$$

$$= 2\int_0^1 t\cdot\sin\left(\frac{\pi}{2}t\right)dt \quad \text{(using by parts)}$$

$$= 2\left\{\left(t\cdot\frac{\cos\left(\frac{\pi}{2}t\right)}{-\frac{\pi}{2}}\right)_0^1 - \int_0^1 1\cdot\frac{\cos\left(\frac{\pi}{2}t\right)}{-\frac{\pi}{2}}dt\right\}$$

$$= 2\left\{0+\frac{2}{\pi}\cdot\left(\frac{\sin\left(\frac{\pi}{2}t\right)}{\frac{\pi}{2}}\right)_0^1\right\} = \frac{8}{\pi^2}$$

Hence, (c) is the correct answer.

Example 46 ***If*** $f(x) = \begin{vmatrix} \cos x & e^{x^2} & 2x\cos^2 x/2 \\ x^2 & \sec x & \sin x + x^3 \\ 1 & 2 & x+\tan x \end{vmatrix}$, ***then value of*** $\int_{-\pi/2}^{\pi/2} (x^2+1)[f(x)+f''(x)]\,dx$ ***is equal to***

(a) 1 (b) –1 (c) 2 (d) None of these

Sol. As, $f(x) = \begin{vmatrix} \cos x & e^{x^2} & 2x\cos^2 x/2 \\ x^2 & \sec x & \sin x + x^3 \\ 1 & 2 & x+\tan x \end{vmatrix}$

$\Rightarrow f(-x) = -f(x)$

$\Rightarrow f(x)$ is odd. $\Rightarrow f'(x)$ is even.

$\Rightarrow f''(x)$ is odd.

Thus, $f(x)+f''(x)$ is odd function, let

$$\phi(x) = (x^2+1)\cdot\{f(x)+f''(x)\}$$

$$\Rightarrow \quad \phi(-x) = -\phi(x)$$

i.e. $\phi(x)$ is odd.

$$\therefore \quad \int_{-\pi/2}^{\pi/2} \phi(x)\,dx = 0$$

Hence, (d) is the correct answer.

Example 47 The value of

$\int_{-1}^{1} \frac{x}{\sqrt{1-x^2}}\cdot\sin^{-1}(2x\sqrt{1-x^2})\,dx$ is equal to

(a) $4\sqrt{2}$ (b) $4(\sqrt{2}-1)$ (c) $4(\sqrt{2}+1)$ (d) None of these

Sol. Let $I = \int_{-1}^{1} \frac{x}{\sqrt{1-x^2}}\cdot\sin^{-1}(2x\sqrt{1-x^2})\,dx$

$$= 2\int_0^1 \frac{x}{\sqrt{1-x^2}}\cdot\sin^{-1}(2x\sqrt{1-x^2})\,dx$$

$$\left[\text{using } \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx, \text{if } f(-x) = f(x)\right]$$

Put $x = \sin\theta \Rightarrow dx = \cos\theta\,d\theta$

$$\therefore \quad I = 2\int_0^{\pi/2} \frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\cdot\sin^{-1}(2\sin\theta\cos\theta)\cdot\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/4} \sin\theta\cdot 2\theta\,d\theta + 2\int_{\pi/4}^{\pi/2} (\pi-2\theta)\sin\theta\,d\theta$$

$$\left[\text{using } \sin^{-1}(\sin 2\theta) = \begin{cases} 2\theta, & 0<\theta<\pi/4 \\ \pi-2\theta, & \pi/4<\theta<\pi/2 \end{cases}\right]$$

$$= 4\int_0^{\pi/4} \theta\cdot\sin\theta\,d\theta + 2\pi\int_{\pi/4}^{\pi/2} \sin\theta\,d\theta - 4\int_{\pi/4}^{\pi/2} \theta\cdot\sin\theta\,d\theta$$

$$= 4\{\theta(-\cos\theta)\}_0^{\pi/4} - 4\int_0^{\pi/4} 1\cdot(-\cos\theta)d\theta + 2\pi(-\cos\theta)_{\pi/4}^{\pi/2}$$

$$- 4\{\theta(-\cos\theta)\}_{\pi/4}^{\pi/2} + 4\int_{\pi/4}^{\pi/2} (-\cos\theta)\,d\theta$$

$$= -\frac{\pi}{\sqrt{2}} + 2\sqrt{2} + \sqrt{2}\pi - \frac{\pi}{\sqrt{2}} - 4 + 2\sqrt{2} = 4\sqrt{2} - 4$$

$$= 4(\sqrt{2}-1)$$

Hence, (b) is the correct answer.

Example 48 Suppose the function $g_n(x) = x^{2n+1} + a_n x + b_n (n \in N)$ satisfies the equation $\int_{-1}^{1} (px+q) g_n(x)\,dx = 0$ for all linear functions $(px+q)$, then

(a) $a_n = b_n = 0$ (b) $b_n = 0\,; a_n = -\frac{3}{2n+3}$

(c) $a_n = 0\,; b_n = -\frac{3}{2n+3}$ (d) $a_n = \frac{3}{2n+3}; b_n = -\frac{3}{2n+3}$

Sol. We have, $\int_{-1}^{1} (px+q)(x^{2n+1} + a_n x + b_n)\,dx = 0$

Equating the odd component to be zero and integrating, we get

$$\frac{2p}{2n+3} + \frac{2a_n p}{3} + 2b_n q = 0 \text{ for all } p, q$$

Therefore, $b_n = 0$ and $a_n = -\frac{3}{2n+3}$

Hence, (b) is the correct answer.

Property VII (a).

$$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$$

Proof We know,

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx \qquad \ldots(\text{i})$$

Consider the integral $\int_0^{2a} f(x)\,dx$; putting $x = 2a - t$, so that $dx = -dt$

Also, when $x = a$, then $t = a$ and when $x = 2a$, then $t = 0$.

$$\therefore \int_a^{2a} f(x)\,dx = \int_a^0 f(2a-t)(-dt) = -\int_a^0 f(2a-t)\,dt$$

$$= \int_0^a f(2a-t)\,dt = \int_0^a f(2a-x)\,dx$$

$$= \begin{cases} \int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \\ -\int_0^a f(x)\,dx, & \text{if } f(2a-x) = -f(x) \end{cases} \ldots(\text{ii})$$

From Eqs. (i) and (ii), we have

$$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$$

Example 49 Evaluate $\int_0^{\pi} \frac{x}{1+\cos^2 x}\,dx$.

Sol. Let $I = \int_0^{\pi} \frac{x}{1+\cos^2 x}\,dx$

$$= \int_0^{\pi} \frac{(\pi - x)\,dx}{1+\cos^2(\pi - x)} = \int_0^{\pi} \frac{\pi\,dx}{1+\cos^2 x} - \int_0^{\pi} \frac{x\,dx}{1+\cos^2 x}$$

$$\therefore \quad I = \pi \int_0^{\pi} \frac{dx}{1+\cos^2 x} - I$$

$$\Rightarrow \quad 2I = \pi \int_0^{\pi} \frac{dx}{1+\cos^2 x} = 2\pi \int_0^{\pi/2} \frac{dx}{1+\cos^2 x}$$

$$\left[\text{using} \int_0^{2a} f(x)\,dx = \begin{cases} 0, & f(2a-x) = -f(x) \\ 2\int_0^a f(x)\,dx, & f(2a-x) = f(x) \end{cases}\right]$$

$$\Rightarrow \quad I = \pi \int_0^{\pi/2} \frac{\sec^2 x}{\sec^2 x + 1}\,dx$$

(dividing numerator and denominator by $\cos^2 x$)

$$I = \pi \int_0^{\pi/2} \frac{\sec^2 x}{2 + \tan^2 x}\,dx$$

Put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

Also, when $x = 0$, then $t = 0$ and when $x = \pi/2$, then $t = \infty$

Hence, $$I = \pi \int_0^{\infty} \frac{dt}{2+t^2} = \frac{\pi}{\sqrt{2}} \left(\tan^{-1} \frac{t}{\sqrt{2}} \right)_0^{\infty}$$

$$= \frac{\pi}{\sqrt{2}} \left(\frac{\pi}{2} - 0 \right) = \frac{\pi^2}{2\sqrt{2}}$$

Example 50 Prove that $\int_0^{\pi/2} \log(\sin x)\,dx = \int_0^{\pi/2} \log(\cos x)\,dx = -\frac{\pi}{2}\log 2$.

Sol. Let $I = \int_0^{\pi/2} \log(\sin x)\,dx$ ...(i)

Then, $I = \int_0^{\pi/2} \log\sin(\pi/2 - x)\,dx$

$$= \int_0^{\pi/2} \log(\cos x)\,dx \quad \ldots(\text{ii})$$

Adding eqs. (i) and (ii), we get

$$2I = \int_0^{\pi/2} \log\sin x\,dx + \int_0^{\pi/2} \log\cos x\,dx$$

$$= \int_0^{\pi/2} (\log\sin x + \log\cos x)\,dx$$

$$= \int_0^{\pi/2} \log(\sin x \cos x)\,dx = \int_0^{\pi/2} \log\left(\frac{2\sin x \cos x}{2}\right) dx$$

$$= \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right) dx$$

$$= \int_0^{\pi/2} \log(\sin 2x)\,dx - \int_0^{\pi/2} (\log 2)\,dx$$

$$= \int_0^{\pi/2} \log\sin 2x\,dx - (\log 2)(x)_0^{\pi/2}$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \log(\sin 2x)\,dx - \frac{\pi}{2}\log 2 \qquad \ldots(\text{iii})$$

Let $I_1 = \int_0^{\pi/2} \log(\sin 2x)\,dx$

$$I_1 = \int_0^{\pi} \log \sin t \, \frac{dt}{2} = \frac{1}{2}\int_0^{\pi} \log \sin t \, dt \text{ (putting } 2x = t)$$

$$= \frac{1}{2} \cdot 2 \int_0^{\pi/2} \log (\sin t)\, dt$$

$$\left[\text{using } \int_0^{2a} f(x)\, dx = \begin{cases} 0, & f(2a - x) = -f(x) \\ 2\int_0^a f(x)\, dx, & f(2a - x) = f(x), \end{cases}\right.$$

$$= \int_0^{\pi/2} \log (\sin x)\, dx$$

$\therefore$ Eq. (iii) becomes $2I = I - \frac{\pi}{2}\log 2$

Hence, $\int_0^{\pi/2} \log \sin x \, dx = -\frac{\pi}{2}\log 2.$

Remark

Students are advised to learn

$\int_0^{\pi/2} \log(\sin x)\, dx = \int_0^{\pi/2} \log(\cos x)\, dx = -\frac{\pi}{2}\log 2.$

Example 51 If $f(x) = -\int_0^x \log(\cos t)\, dt$, then the value of $f(x) - 2f\left(\frac{\pi}{4} + \frac{x}{2}\right) + 2f\left(\frac{\pi}{4} - \frac{x}{2}\right)$ is equal to

(a) $-x\log 2$ (b) $\frac{x}{2}\log 2$

(c) $\frac{x}{3}\log 2$ (d) None of these

Sol. Here, $f\left(\frac{\pi}{4} + \frac{x}{2}\right) = -\int_0^{\pi/4 + x/2} \log(\cos t)\, dt$

$$= -\int_0^{\pi/4} \log(\cos t)\, dt - \int_{\pi/4}^{\pi/4 + x/2} \log(\cos t)\, dt \quad \text{...(i)}$$

$$f\left(\frac{\pi}{4} - \frac{x}{2}\right) = -\int_0^{\pi/4 - x/2} \log(\cos t)\, dt$$

$$= -\int_0^{\pi/4} \log(\cos t)\, dt - \int_{\pi/4}^{\pi/4 - x/2} \log(\cos t)\, dt \quad \text{...(ii)}$$

$\therefore$ $2f\left(\frac{\pi}{4} + \frac{x}{2}\right) - 2f\left(\frac{\pi}{4} - \frac{x}{2}\right)$

$$= 2\int_{\pi/4}^{\pi/4 - x/2} \log(\cos t)\, dt - 2\int_{\pi/4}^{\pi/4 + x/2} \log(\cos t)\, dt$$

Put $t = \frac{\pi}{4} - z$ in first integral and $t = \frac{\pi}{4} + z$ in second integral, we get

$$= -2\int_0^{x/2} \log\cos\left(\frac{\pi}{4} - z\right) dz - 2\int_0^{x/2} \log\cos\left(z + \frac{\pi}{4}\right) dz$$

$$= -2\int_0^{x/2} \log\left(\frac{1}{2}(\cos^2 z - \sin^2 z)\right) dz$$

$$= 2\int_0^{x/2} (\log 2)\, dx - 2\int_0^{x/2} \log(\cos 2z)\, dz$$

$$= x\log 2 - 2\int_0^{x/2} \log(\cos 2z)\, dz$$

$\therefore$ $2f\left(\frac{\pi}{4} + \frac{x}{2}\right) - 2f\left(\frac{\pi}{4} - \frac{x}{2}\right) = x(\log 2) - 2\int_0^{x/2} \log(\cos 2z)\, dz$

$\Rightarrow$ $2f\left(\frac{\pi}{4} + \frac{x}{2}\right) - 2f\left(\frac{\pi}{4} - \frac{x}{2}\right) = x(\log 2) + f(x)$

or $f(x) - 2f\left(\frac{\pi}{4} + \frac{x}{2}\right) + 2f\left(\frac{\pi}{4} - \frac{x}{2}\right) = -x\log 2$

Hence, (a) is the correct answer.

Example 52 If $\int_0^{\pi} \left(\frac{x}{1 + \sin x}\right)^2 dx = A$, then the value for $\int_0^{\pi} \frac{2x^2 \cdot \cos^2 x/2}{(1 + \sin x)^2} dx$ is equal to

(a) $A + 2\pi - \pi^2$ (b) $A - 2\pi + \pi^2$

(c) $2\pi - A - \pi^2$ (d) None of these

Sol. Let $B = \int_0^{\pi} \frac{2x^2 \cos^2 x/2}{(1 + \sin x)^2} dx$

$$\therefore \quad B - A = \int_0^{\pi} \frac{x^2 (2\cos^2 x/2 - 1)}{(1 + \sin x)^2} dx$$

$$= \int_0^{\pi} \frac{x^2 \cdot \cos x}{(1 + \sin x)^2} dx$$

Using by parts,

$$B - A = \left\{ -\frac{x^2}{1 + \sin x} \right\}_0^{\pi} + 2\int_0^{\pi} \frac{x}{(1 + \sin x)} dx$$

$$B - A = -\pi^2 + 2K \quad \text{...(i)}$$

where, $K = \int_0^{\pi} \frac{x\, dx}{1 + \sin x} = \int_0^{\pi} \frac{(\pi - x)\, dx}{1 + \sin x}$

$$\therefore \quad K = \pi\int_0^{\pi} \frac{dx}{1 + \sin x} - K,$$

$$\left[\text{using } \int_0^{2a} f(x)\, dx = \begin{cases} 0, & f(2a - x) = -f(x) \\ 2\int_0^a f(x)\, dx, & f(2a - x) = f(x), \end{cases}\right.$$

$$\Rightarrow 2K = 2\pi\int_0^{\pi/2} \frac{dx}{1 + \sin x} = 2\pi\int_0^{\pi/2} \frac{dx}{1 + \cos x}$$

$$= 2\pi\int_0^{\pi/2} \frac{1}{2}\sec^2\frac{x}{2}\, dx = 2\pi\left\{\tan\frac{x}{2}\right\}_0^{\pi/2} = 2\pi$$

$\therefore$ $K = \pi$

Thus, $B - A = -\pi^2 + 2K$

$\Rightarrow$ $B = A - \pi^2 + 2\pi$

Hence, (a) is the correct answer.

Example 53 ***The value of*** $\int_{-a}^{a} (\cos^{-1} x - \sin^{-1}\sqrt{1-x^2})\,dx\ (a>0)$ ***(where,*** $\int_0^a \cos^{-1} x\,dx = A$***) is***

(a) $\pi a - A$ (b) $\pi a + 2A$

(c) $\pi a - 2A$ (d) $\pi a + A$

Sol. Let $I = \int_{-a}^{a} (\cos^{-1} x - \sin^{-1}\sqrt{1-x^2})\,dx$ (as $a>0$)

$$= \int_{-a}^{0} (\cos^{-1} x - \sin^{-1}\sqrt{1-x^2})\,dx + \int_0^a (\cos^{-1} x - \sin^{-1}\sqrt{1-x^2})\,dx$$

$$= \int_{-a}^{0} \cos^{-1} x\,dx + A - 2\int_0^a \sin^{-1}\sqrt{1-x^2}\,dx$$

$$= -\int_a^0 (\pi - \cos^{-1} x)\,dx + A - 2A$$

$$= \pi a - \int_0^a \cos^{-1} x\,dx - A$$

$$= \pi a - A - A = \pi a - 2A$$

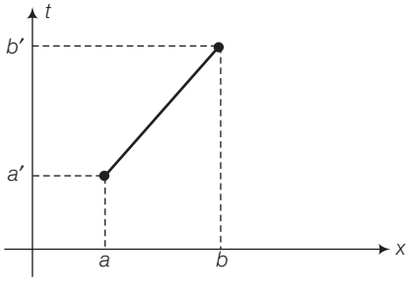
Hence, (c) is the correct answer.

Property VII (b)

$$\int_a^b f(x)\,dx = (b-a)\int_0^1 f[(b-a)\,x + a]\,dx$$

Sometimes, it is convenient to change the limits of integration into some other limits. For example, suppose we have to add two definite integrals I_1 and I_2; the limits of integration of these integrals are different. if we could somehow change the limits of I_2 into those of I_1 or vice-versa, or infact change the limits of both I_1 and I_2 into a third (common) set of limits, the addition could be accomplished easily.

Suppose that $I = \int_a^b f(x)\,dx$. We need to change the limits (a to b) to (a' to b'). As x varies from a to b, we need a new variable t (in terms of x) which varies from a' to b'.



As x varies from a to b, t varies from a to b, t varies from a' to b'.

Thus, $$\frac{t-a'}{x-a} = \frac{b'-a'}{b-a}$$

As described in the figure above, the new variable t is given by,

$$t = a' + \left(\frac{b'-a'}{b-a}\right)(x-a).$$

Thus, $$dt = \frac{b'-a'}{b-a}dx$$

$$\Rightarrow \quad I = \int_0^b f(x)\,dx = \int_{a'}^{b'} f\left(a + \left(\frac{b-a}{b'-a'}\right)(t-a')\right)\left(\frac{b-a}{b'-a'}\right)dt$$

The modified integral has the limits (a' to b'). A particular case of this property is modifying the arbitrary integration limits (a to b) to (0 to 1) i.e. $a' = 0$ and $b' = 1$. For this case,

$$I = \int_a^b f(x)\,dx = (b-a)\int_0^1 f(a + (b-a)t)\,dt$$

Example 54 Evaluate

$$\int_{-4}^{-5} e^{(x+5)^2}\,dx + 3\int_{1/3}^{2/3} e^{9\left(x-\frac{2}{3}\right)^2}\,dx.$$

Sol. Here, we know $\int e^{x^2}\,dx$ cannot be evaluated by indefinite integral.

Let $I_1 = \int_{-4}^{-5} e^{(x+5)^2}\,dx = (-5+4)\int_0^1 e^{[(-5+4)x - 4 + 5]^2}\,dx$

$$\therefore \quad I_1 = -\int_0^1 e^{(x-1)^2}\,dx \qquad \ldots(i)$$

Again, let $I_2 = \int_{1/3}^{2/3} e^{9(x-2/3)^2}dx$

$$= \left(\frac{2}{3} - \frac{1}{3}\right)\int_0^1 e^{9\left[\left(\frac{2}{3}-\frac{1}{2}\right)x + \frac{1}{3} - \frac{2}{3}\right]^2}dx$$

$$= \frac{1}{3}\int_0^1 e^{(x-1)^2}dx = \frac{1}{3}(-I_1) \qquad \ldots(ii)$$

where, $I = I_1 + 3I_2 = I_1 + 3\left(-\frac{I_1}{3}\right) = I_1 - I_1 = 0$

$$\therefore \quad \int_{-4}^{-5} e^{(x+5)^2}\,dx + 3\int_{1/3}^{2/3} e^{9(x-2/3)^2}\,dx = 0$$

Property VII (c) If $f(t)$ is an odd function, then $\phi(x) = \int_a^x f(t)\,dt$ is an even function.

Proof We have, $\phi(-x) = \int_a^{-x} f(t)\,dt$

$$\Rightarrow \quad \phi(-x) = \int_a^{-a} f(t)\,dt + \int_{-a}^{-x} f(t)\,dt$$

$$\Rightarrow \quad \phi(-x) = 0 + \int_{-a}^{-x} f(t)\,dt$$

$$\left[\because f(t) \text{ is an odd function, then } \int_{-a}^{a} f(t)\,dt = 0\right]$$

$\Rightarrow \quad \phi(-x) = -\int_a^x f(-y)\,dy$, where $t = -y$

$\Rightarrow \quad \phi(-x) = \int_a^x f(y)\,dy$

[$\because f$ is an odd function, then $f(-y) = -f(y)$]

$\Rightarrow \quad \phi(-x) = \int_a^x f(t)\,dt$

$\Rightarrow \quad \phi(-x) = \phi(x)$

Hence, $\phi(x) = \int_a^x f(t)\,dt$ is an even function, if $f(t)$ is odd.

Example 55 ***If*** $f(x) = \int_0^x \log\left(\frac{1-t}{1+t}\right)dt$, ***then discuss whether even or odd?***

Sol. Let $\phi(t) = \log\left(\frac{1-t}{1+t}\right)$

$\therefore \quad \phi(-t) = \log\left(\frac{1+t}{1-t}\right) = -\log\left(\frac{1-t}{1+t}\right) = -\phi(t)$

$\Rightarrow \quad \phi(-t) = -\phi(t)$, i.e. $\phi(t)$ is odd function

$\therefore \quad \phi(x) = \int_0^x \log\left(\frac{1-t}{1+t}\right)dt$ is an even function.

Property VII (d) If $f(t)$ is an even function, then $\phi(x) = \int_0^x f(t)\,dt$ is an odd function.

Proof We have, $\phi(-x) = \int_0^{-x} f(t)\,dt = -\int_0^x f(-y)\,dy$, where $t = -y$

$\Rightarrow \quad \phi(-x) = -\int_0^x f(y)\,dy$

[$\because f$ is even function $\Rightarrow f(-y) = f(y)$]

$\Rightarrow \quad \phi(-x) = -\int_0^x f(t)\,dt$

$\Rightarrow \quad \phi(-x) = -\phi(x)$

Hence, $\phi(x)$ is an odd function.

Remark

If $f(t)$ is an even function, then for non-zero 'a', $\int_a^x f(t)\,dt$ is not necessarily an odd function. It will be an odd function, if $\int_0^a f(t)\,dt = 0$, because, if $\phi(x) = \int_a^x f(t)\,dt$, it is an odd function.

$\Rightarrow \quad \phi(-x) = -\phi(x)$

$\Rightarrow \quad \int_a^{-x} f(t)\,dt = -\int_a^x f(t)\,dt$

$\Rightarrow \quad \int_a^0 f(t)\,dt + \int_0^{-x} f(t)\,dt = -\int_a^0 f(t)\,dt - \int_0^x f(t)\,dt$

$\Rightarrow \quad \int_a^0 f(t)\,dt - \int_0^x f(-y)\,dy = -\int_0^a f(t)\,dt - \int_0^x f(t)\,dt$

[where, $y = -t$ in second integral of LHS $\Rightarrow f(-y) = f(y)$]

$\Rightarrow \quad 2\int_a^0 f(t)\,dt = \int_0^x f(y)\,dy - \int_0^x f(t)\,dt$

$\Rightarrow \quad 2\int_a^0 f(t)\,dt = 0$

$\Rightarrow \quad -\int_0^a f(t)\,dt = 0 \Rightarrow \int_0^a f(t)\,dt = 0$

or $\phi(x) = \int_a^x f(t)\,dt$ is an odd function, [when $f(t)$ is even.]

Only, if $\int_0^a f(t)\,dt = 0$

Exercise for Session 4

1. Let $f : R \to R$ and $g : R \to R$ be continous functions, then the value of $\int_{-\pi/2}^{\pi/2} [f(x) + f(-x)][g(x) - g(-x)]\,dx$ is equal to

(a) -1 (b) 0 (c) 1 (d) None of these

2. The value of $\int_{-1}^{1} (x|x|)\,dx$ is equal to

(a) 1 (b) $\frac{1}{2}$ (c) 0 (d) None of these

3. The value of $\int_{-1}^{1} \left(\frac{x^2 + \sin x}{1 + x^2}\right) dx$ is equal to

(a) $2 - \pi$ (b) $\pi - 2$ (c) $2 - \frac{\pi}{2}$ (d) None of these

4. If $f(x)$ is an odd function, then the value of $\int_{-a}^{a} \frac{f(\sin x)}{f(\cos x) + f(\sin^2 x)} dx$ is equal to

(a) 0 (b) $f(\cos x) + f(\sin x)$ (c) 1 (d) None of these

5. The value of $\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{\cos^{-1}\left(\frac{2x}{1+x^2}\right) + \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx}{1 + e^x}$ is equal to

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{\sqrt{3}}$ (c) $\frac{\pi}{2\sqrt{3}}$ (d) $\frac{\pi}{3\sqrt{3}}$

6. The value of $\int_{-\pi}^{\pi} \frac{\cos^2 x}{1 + a^x} dx$, where $a > 0$, is *[IITJEE 2001]*

(a) π (b) $a\pi$ (c) 2π (d) $\frac{\pi}{2}$

7. The integral $\int_{-1/2}^{1/2} \left\{[x] + \log_e\left(\frac{1+x}{1-x}\right)\right\} dx$ is equal to (where, $[\cdot]$ denotes greatest integer function)

(a) $-\frac{1}{2}$ (b) 0 (c) 1 (d) $2 \log\left(\frac{1}{2}\right)$

8. The value of $\int_{-\pi/2}^{\pi/2} \frac{1}{e^{\sin x} + 1} dx$ is equal to

(a) 0 (b) 1 (c) $\frac{\pi}{2}$ (d) $-\frac{\pi}{2}$

9. If $[\cdot]$ denotes greatest integer function, then the value of $\int_{-\pi/2}^{\pi/2} \left(\left[\frac{\pi}{2}\right] + 0.5\right) dx$ is

(a) π (b) $\frac{\pi}{2}$ (c) 0 (d) $-\frac{\pi}{2}$

10. The equation $\int_{-\pi/4}^{\pi/4} \left\{a|\sin x| + \frac{b \sin x}{1 + \cos^2 x} + c\right\} dx = 0$, where a, b, c are constants gives a relation between

(a) a, b and c (b) a and c (c) a and b (d) b and c

11. The value of $\int_{-2}^{2} \frac{\sin^2 x}{\left[\frac{x}{\pi}\right] + \frac{1}{2}} dx$, where $[\cdot]$ denotes greatest integer function, is

(a) 1 (b) 0 (c) $4 - \sin 4$ (d) None of these

12. Let $f(x)$ be a continuous function such that $\int_{m}^{n+1} f(x)dx = n^3, n \in Z$. Then, the value of $\int_{-3}^{3} f(x)dx$ is

(a) 9 (b) -27 (c) -9 (d) 27

13. Let $f(x)=\dfrac{e^x+1}{e^x-1}$ and $\int_0^1 x^3\cdot\dfrac{e^x+1}{e^x-1}dx=\alpha$. Then, $\int_{-1}^{1} t^3 f(t)\,dt$ is equal to

(a) 0 (b) α (c) 2α (d) None of these

14. Let $f:R\to R$ be a continuous function given by $f(x+y)=f(x)+f(y)$ for all $x, y\in R$. if $\int_0^2 f(x)\,dx=\alpha$, then $\int_{-2}^{2} f(x)\,dx$ is equal to

(a) 2α (b) α (c) 0 (d) None of these

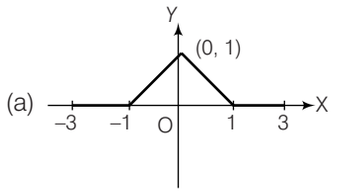
15. The value of $\int_{-2}^{2} |[x]|\,dx$ is equal to

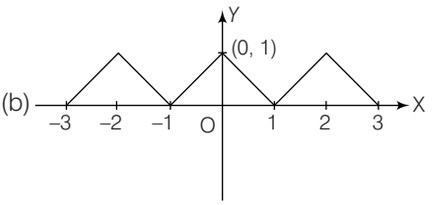
(a) 1 (b) 2 (c) 3 (d) 4

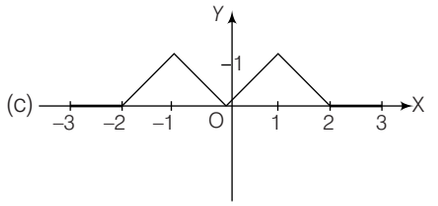
■ **Directions** (Q. Nos. 16 to 17)

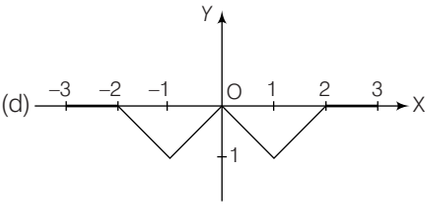
Let $f(x)=\begin{cases}1-|x|, & |x|\le 1\\ 0, & |x|>1\end{cases}$ and $g(x)=f(x+1)+f(x-1)$ for all $x\in R$.

16. The graph for $g(x)$ is given by

(a)


(b)


(c)


(d)


17. The value of $\int_{-3}^{3} g(x)dx$ is

(a) 2 (b) 3 (c) 4 (d) 5

■ **Direction** (Q. Nos. 18 to 20)

Let $I_n=\int_{-\pi}^{\pi}\dfrac{\sin x}{(1+\pi^x)\sin x}\,dx;\ n=0, 1, 2, \ldots$

18. The value of $I_{n+2}-I_n$ is equal to

(a) $n\pi$ (b) π (c) $-\pi$ (d) 0

19. The value of $\sum_{m=1}^{10} I_{2m+1}$ is equal to

(a) 0 (b) 5π
(c) 10π (d) None of these

20. The value of $\sum_{m=1}^{10} I_{2m}$ is equal to

(a) 0 (b) 5π
(c) 10π (d) None of these

Session 5

Applications of Periodic Functions and Newton- Leibnitz's Formula

Applications of Periodic Functions and Newton-Leibnitz's Formula

If $f(x)$ is a periodic function with period T, then the area under $f(x)$ for n periods would be n times the area under $f(x)$ for one period, i.e.

$$\int_0^{nT} f(x)\,dx = n\int_0^{T} f(x)\,dx$$

Now, consider the periodic function $f(x) = \sin x$ as an example. The period of $\sin x$ is 2π.

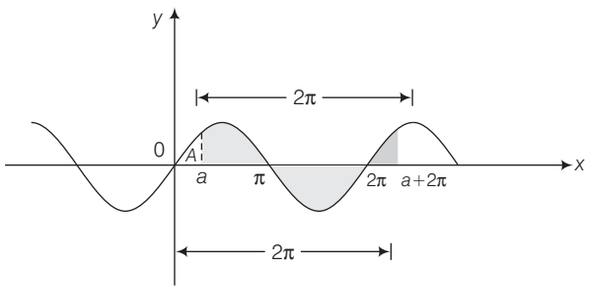


Figure 2.6

Suppose we intend to calculate $\int_a^{a+2\pi} \sin x\,dx$ as depicted above. Notice that the darkly shaded area in the interval $[2\pi, a+2\pi]$ can precisely cover the area marked as

Thus, $$\int_a^{a+2\pi} \sin x\,dx = \int_0^{2\pi} \sin x\,dx$$

This will hold true for every periodic function, i.e.

$$\int_a^{a+T} f(x)\,dx = \int_0^{T} f(x)\,dx$$

(where T is the period of $f(x)$)

This also implies that

$$\int_a^{a+nT} f(x)\,dx = \int_0^{nT} f(x)dx = n\int_0^{T} f(x)\,dx$$

and $$\int_{a+nT}^{b+nT} f(x)\,dx = \int_a^b f(x)\,dx$$

and $$\int_a^{b+nT} f(x)\,dx = \int_a^b f(x)dx + n\int_0^{T} f(x)\,dx$$

Example 56 Evaluate $\int_0^{4\pi} |\cos x|\,dx$.

Sol. Note that $|\cos x|$ is a periodic with period π.

Let $$I = 4\int_0^{\pi} |\cos x|\,dx$$

$$\left[\text{using property, } \int_0^{nT} f(x)\,dx = n\int_0^{T} f(x)\,dx\right]$$

$$= 4\left\{\int_0^{\pi/2} \cos x\,dx - \int_{\pi/2}^{\pi} \cos x\,dx\right\}$$

$$= 4\left\{\left(\sin x\right)_0^{\pi/2} - \left(\sin x\right)_{\pi/2}^{\pi}\right\} = 4\{1+1\} = 8$$

Example 57 Prove that $\int_0^{25} e^{x-[x]}\,dx = 25(e-1)$.

Sol. Since, $x - [x]$ is a periodic function with period one.

Therefore, $e^{x-[x]}$ has period one

$$\therefore \quad I = \int_0^{25\times 1} e^{x-[x]}\,dx = 25\int_0^{1} e^{x-[x]}\,dx$$

$$\left[\text{using property, } \int_0^{nT} f(x)\,dx = n\int_0^{T} f(x)\,dx\right]$$

$$= 25\int_0^{1} e^{x-0}\,dx = 25\left(e^x\right)_0^1 = 25\left(e^1 - e^0\right)$$

$$= 25(e-1)$$

Example 58 ***The value of*** $\int_0^{2n\pi} [\sin x + \cos x]\,dx$ ***is equal to***

(a) $-n\pi$ (b) $n\pi$ (c) $-2n\pi$ (d) None of these

(where [.] denotes the greatest integer function)

Sol. Let $I = \int_0^{2n\pi} [\sin x + \cos x]\,dx$

We know, $\sin x + \cos x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$ ...(i)

$$\therefore \quad \sin x + \cos x = \begin{cases} 1, & 0 \le x \le \pi/2 \\ 0, & \pi/2 < x \le 3\pi/4 \\ -1, & 3\pi/4 < x \le \pi \\ -2, & \pi < x < 3\pi/2 \\ -1, & 3\pi/2 \le x < 7\pi/4 \\ 0, & 7\pi/4 \le x \le 2\pi \end{cases}$$

$$\therefore \int_0^{2\pi} [\sin x + \cos x]\,dx = \int_0^{\pi/2} (1)\,dx + \int_{\pi/2}^{3\pi/4} (0)\,dx$$

$$+\int_{3\pi/4}^{\pi}(-1)dx+\int_{\pi}^{3\pi/2}(-2)\,dx+\int_{3\pi/2}^{7\pi/4}(-1)\,dx+\int_{7\pi/4}^{2\pi}(0)\,dx$$

$$=\left(\frac{\pi}{2}\right)+(0)-\left(\pi-\frac{3\pi}{4}\right)-2\left(\frac{3\pi}{2}-\pi\right)-1\left(\frac{7\pi}{4}-\frac{3\pi}{2}\right)+0$$

$$=-\pi$$

Since, $\sin x+\cos x$ has the period 2π.

So, $I=\int_0^{2n\pi}[\sin x+\cos x]\,dx=n\int_0^{2\pi}[\sin x+\cos x]\,dx$

$=-n\pi$

Hence, (a) is the correct answer.

Example 59 The value of $\int_{-5}^{5} f(x)\,dx$; where $f(x)=$ *minimum* $(\{x+1\},\{x-1\}),\forall\ x\in R$ and $\{.\}$ denotes fractional part of x, is equal to

(a) 3 (b) 4 (c) 5 (d) 6

Sol. We know, $\{x+1\}=\{x-1\}=\{x\}$

Thus, $f(x)=\text{minimum}\,(\{x+1\},\{x-1\})=\{x\}$

$$\Rightarrow\quad \int_{-5}^{5} f(x)\,dx=\int_{-5}^{5}\{x\}\,dx=(5-(-5))\int_0^1\{x\}\,dx$$

[as $\{x\}$ is periodic with period '1']

$$=10\int_0^1(x-[x])\,dx=10\int_0^1 x\,dx$$

$$=10\left(\frac{x^2}{2}\right)_0^1=5$$

Hence, (c) is the correct answer.

Example 60 Show $\int_0^{n\pi+V}|\sin x|\,dx=(2n+1)-\cos V$, where n is a positive integer and $0\le V<\pi$.

[IIT JEE 1994, 2004]

Sol. $\int_0^{n\pi+V}|\sin x|\,dx=\int_0^{V}|\sin x|\,dx+\int_V^{n\pi+V}|\sin x|\,dx$

$$=\int_0^V\sin x\,dx+n\int_0^{\pi}|\sin x|\,dx$$

$$\left[\text{using property, }\int_a^{a+nT}f(x)dx=n\int_0^T f(x)dx\ \text{ i.e. }\int_V^{n\pi V}|\sin x|dx=n\int_V^{\pi}|\sin x|dx\right]$$

$$=\left(-\cos x\right)_0^V+n\int_0^{\pi}\sin x\,dx$$

$$=(-\cos V+1)+n\left(-\cos x\right)_0^{\pi}$$

$$=-(\cos V)+1+n\,(1+1)=(2n+1)-\cos V$$

$\therefore\ \int_0^{n\pi+V}|\sin x|\,dx=(2n+1)-\cos V$, where n is a positive integer and $0\le V<\pi$.

Example 61 The value of $\int_{-2\pi}^{5\pi}\cot^{-1}(\tan x)\,dx$ is equal to

(a) $\frac{7\pi}{2}$ (b) $\frac{7\pi^2}{2}$

(c) $\frac{3\pi}{2}$ (d) π^2

Sol. Let $I=\int_{-2\pi}^{5\pi}\cot^{-1}(\tan x)\,dx$

$$\Rightarrow\quad I=7\int_0^{\pi}\cot^{-1}\left(\cot\left(\frac{\pi}{2}-x\right)\right)dx \qquad ...(i)$$

$$\left[\because\ \int_{nT}^{mT}f(x)\,dx=(m-n)\int_0^T f(x)\,dx\right]$$

As we know, $\cot^{-1}(\cot x)=\begin{cases}x, & 0<x<\pi/2\\ \pi+x, & \pi/2<x<\pi\end{cases}$

$$\therefore\quad I=7\left\{\int_0^{\pi/2}\left(\frac{\pi}{2}-x\right)dx+\int_{\pi/2}^{\pi}\left(\pi+\frac{\pi}{2}-x\right)dx\right\}$$

$$=7\left\{\left(\frac{\pi}{2}x-\frac{x^2}{2}\right)_0^{\pi/2}+\left(\frac{3\pi}{2}x-\frac{x^2}{2}\right)_{\pi/2}^{\pi}\right\}$$

$$=7\left\{\left(\frac{\pi^2}{4}-\frac{\pi^2}{8}\right)+\left(\frac{3\pi^2}{2}-\frac{\pi^2}{2}-\frac{3\pi^2}{4}+\frac{\pi^2}{8}\right)\right\}=\frac{7\pi^2}{2}$$

Hence, (b) is the correct answer.

Example 62 Let $g(x)$ be a continuous and differentiable function such that $\int_0^2\left\{\int_{\sqrt{2}}^{\sqrt{5/2}}[2x^2-3]\,dx\right\}\cdot g(x)\,dx=0$, then $g(x)=0$ when $x\in(0,2)$ has (where, [.] denotes the greatest integer function)

(a) exactly one real root (b) atleast one real root

(c) no real root (d) None of these

Sol. As, $1<2x^2-3<2,\ \forall\ x\in(\sqrt{2},\sqrt{5/2})$

$$\Rightarrow\quad \int_{\sqrt{2}}^{\sqrt{5/2}}[2x^2-3]\,dx>0,\forall\ x\in(0,2)$$

$\Rightarrow\ g(x)=0$ should have atleast one root in (0, 2).

$[\because\ g'(x)\ne 0]$

Hence, (b) is the correct answer.

Example 63 The value of x satisfying $\int_0^{2[x+14]}\left\{\frac{x}{2}\right\}dx=\int_0^{\{x\}}[x+14]\,dx$ is equal to (where, [.] and {.} denotes the greatest integer and fractional part of x)

(a) $[-14,-13)$ (b) (0, 1)

(c) $(-15,-14]$ (d) None of these

Sol. Given, $\int_0^{2[x+14]} \left\{\frac{x}{2}\right\} dx = \int_0^{\{x\}} [x+14]\, dx$

$\Rightarrow \quad \int_0^{28+2[x]} \left\{\frac{x}{2}\right\} dx = \int_0^{\{x\}} (14+[x])\, dx$

$\Rightarrow \quad \int_0^{28} \left\{\frac{x}{2}\right\} dx + \int_{28}^{28+2[x]} \left\{\frac{x}{2}\right\} dx = (14+[x])\{x\}$

$\Rightarrow \quad 14\int_0^{2} \left\{\frac{x}{2}\right\} dx + \int_0^{2[x]} \left\{\frac{x}{2}\right\} dx = (14+[x])\{x\}$

$\left[\text{using } \int_0^{nT} f(x)\,dx = n\int_0^{T} f(x)\,dx \quad \text{and} \right.$
$\left. \int_a^{a+nT} f(x)\,dx = \int_0^{nT} f(x)\,dx; \text{ where } T \text{ is period of } f(x)\right]$

$\Rightarrow \quad 14 + [x] = (14+[x])\{x\}$

$\Rightarrow \quad (14+[x])(1-\{x\}) = 0$

$\Rightarrow \quad [x] = -14$

$\Rightarrow \quad x \in [-14, -13)$

Hence, (a) is the correct answer.

Property IX. Leibnitz's Rule for the Differentiation under the Integral Sign

(i) If the functions $\phi(x)$ and $\psi(x)$ are defined on $[a, b]$ and are differentiable at a point $x \in (a, b)$ and $f(x, t)$ is continuous, then

$$\frac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)} f(x,t)\,dt\right] = \int_{\phi(x)}^{\psi(x)} \frac{\partial}{\partial x} f(x,t)\,dt + \left\{\frac{d^{\psi(x)}}{dx}\right\} f^{(x,\,\psi(x))} - \left\{\frac{d^{\phi(x)}}{dx}\right\} f^{(x, f(x))}$$

(ii) If the functions $\phi(x)$ and $\psi(x)$ are defined on $[a, b]$ and differentiable at a point $x \in (a, b)$ and $f(t)$ is continuous on $[\phi(a), \phi(b)]$, then

$$\frac{d}{dx}\left(\int_{\phi(x)}^{\psi(x)} f(t)\,dt\right) = \frac{d}{dx}\{\psi(x)\}\, f(\psi(x)) - \frac{d}{dx}\{\phi(x)\}\, f(\phi(x))$$

Example 64 Find the derivative of the following with respect to x.

(i) $\int_0^x \cos t\, dt$ (ii) $\int_0^{x^2} \cos t^2 dt$

Sol. (i) Let $f(x) = \int_0^x \cos t\, dt$

$\therefore \frac{d}{dx}(f(x)) = \cos(x)\left\{\frac{d}{dx}(x)\right\} - \cos 0 \cdot \left\{\frac{d}{dx}(0)\right\} = \cos x$

$\left[\text{using Leibnitiz's rule, } \frac{d}{dx}\int_{\phi(x)}^{\psi(x)} f(t)dt = \frac{d}{dx}\{\psi(x)\}f(\psi(x)) - \frac{d}{dx}\{\phi(x)\} + f(\phi(x))\right]$

$\Rightarrow \quad \frac{d}{dx}\left\{\int_0^x \cos t\, dt\right\} = \cos x$

(ii) Let $f(x) = \int_0^{x^2} \cos t^2\, dt$

$\therefore \frac{d}{dx}(f(x)) = \cos(x^2)^2 \cdot \left\{\frac{d}{dx}(x^2)\right\} - \cos(0)^2\left\{\frac{d}{dx}(0)\right\}$

$= 2x \cdot \cos x^4$

$\Rightarrow \quad \frac{d}{dx}\left(\int_0^{x^2} (\cos t^2)\, dt\right) = 2x \cos x^4$

Example 65 Evaluate $\frac{d}{dx}\left(\int_{1/x}^{\sqrt{x}} \cos t^2\, dt\right)$.

Sol. Let $f(x) = \int_{1/x}^{\sqrt{x}} \cos t^2\, dt$

$\therefore \frac{d}{dx}(f(x)) = \cos(\sqrt{x})^2 \cdot \left\{\frac{d}{dx}(\sqrt{x})\right\} - \cos\left(\frac{1}{x}\right)^2\left\{\frac{d}{dx}\left(\frac{1}{x}\right)\right\}$

$= \frac{1}{2\sqrt{x}} \cos x + \frac{1}{x^2} \cdot \cos\left(\frac{1}{x^2}\right)$

$\Rightarrow \quad \frac{d}{dx}\left(\int_{1/x}^{\sqrt{x}} \cos t^2\, dt\right) = \frac{1}{2\sqrt{x}} \cos x + \frac{1}{x^2} \cos\left(\frac{1}{x^2}\right)$

Example 66 *If* $\frac{d}{dx}\left(\int_0^y e^{-t^2}\, dt + \int_0^{x^2} \sin^2 t\, dt\right) = 0$, *find* $\frac{dy}{dx}$.

Sol. We know, $\frac{d}{dx}\left(\int_0^y e^{-t^2}\, dt + \int_0^{x^2} \sin^2 t\, dt\right) = 0$

$\Rightarrow \quad e^{-y^2} \cdot \left\{\frac{d}{dx}(y)\right\} - e^{-0}\left\{\frac{d}{dx}(0)\right\} + \sin^2(x^2)\left\{\frac{d}{dx}(x^2)\right\} - \sin^2 0\left\{\frac{d}{dx}(0)\right\} = 0$

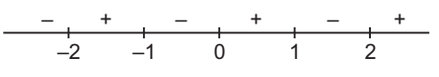
$\Rightarrow \quad e^{-y^2}\frac{dy}{dx} + 2x \sin^2 x^2 = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -2x\, e^{y^2} \sin^2 x^2$

Example 67 Find the points of maxima/minima of $\int_0^{x^2} \frac{t^2 - 5t + 4}{2 + e^t}\, dt$.

Sol. Let $f(x) = \int_0^{x^2} \frac{t^2 - 5t + 4}{2 + e^t}\, dt$

$\therefore f'(x) = \frac{x^4 - 5x^2 + 4}{2 + e^{x^2}} \cdot 2x - 0$

$= \frac{(x-1)(x+1)(x-2)(x+2)\cdot 2x}{2 + e^{x^2}}$



From the wavy curve, it is clear that $f'(x)$ changes its sign at $x = \pm 2, \pm 1, 0$ and hence the points of maxima are $-1, 1$ (as sign changes from + ve to − ve) and of the minima are $-2, 0, 2$ (as sign changes from − ve to + ve).

Example 68 *Find* $\frac{d}{dx}\left(\int_{x^2}^{x^3} \frac{1}{\log t}\,dt\right)$.

Sol. $\frac{d}{dx}\left(\int_{x^2}^{x^3} \frac{1}{\log t}\,dt\right) = \frac{1}{\log x^3}\cdot\frac{d}{dx}(x^3) - \frac{1}{\log x^2}\cdot\frac{d}{dx}(x^2)$

$$= \frac{3x^2}{3\log x} - \frac{2x}{2\log x}$$

$$\therefore \quad \frac{d}{dx}\left(\int_{x^2}^{x^3} \frac{1}{\log t}\,dt\right) = \frac{1}{\log x}(x^2 - x)$$

Example 69 If $y = \int_0^x f(t)\sin\{K(x-t)\}\,dt$, then prove that $\frac{d^2y}{dx^2} + K^2y = Kf(x)$.

Sol. We have, $y = \int_0^x f(t)\sin\{K(x-t)\}\,dt$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \int_0^x \frac{\partial}{\partial x}\{f(t)\sin K(x-t)\}\,dt + \frac{d}{dx}(x)$$

$$\cdot\{f(x)\sin K(x-x)\} - \frac{d}{dx}(0)\{f(0)\sin K(x-0)\}$$

$$= K\int_0^x f(t)\cos K(x-t)\,dt + 0 - 0$$

$$\Rightarrow \frac{dy}{dx} = K\int_0^x f(t)\cos K(x-t)\,dt$$

Again, differentiating both the sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = K\left\{\int_0^x \frac{\partial}{\partial x}\{f(t)\cos K(x-t)\}\,dt\right.$$

$$\left.+\left\{f(x)\cdot\cos K(x-x)\cdot\frac{d}{dx}(x)\right\} - \{f(0)\cos K(x-0)\cdot\frac{d}{dx}(0)\}\right\}$$

$$= K\left[-K\int_0^x f(t)\sin K(x-t)\,dt + f(x) - 0\right]$$

$$= -K^2\int_0^x f(t)\sin K(x-t)\,dt + Kf(x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = -K^2y + Kf(x) \;\Rightarrow\; \frac{d^2y}{dx^2} + K^2y = Kf(x)$$

Example 70 *If* $\int_{\pi/3}^x \sqrt{3-\sin^2 t}\,dt + \int_0^y \cos t\,dt = 0$, *then evaluate* $\frac{dy}{dx}$.

Sol. Differentiating w.r.t. x, we have

$$\frac{d}{dx}\left(\int_{\pi/3}^x \sqrt{3-\sin^2 t}\,dt\right) + \frac{d}{dx}\int_0^y (\cos t)\,dt = 0$$

$$\Rightarrow \sqrt{3-\sin^2 x}\cdot\frac{d}{dx}(x) - \sqrt{3-\sin^2\frac{\pi}{3}}\cdot\frac{d}{dx}\left(\frac{\pi}{3}\right) + \cos y\cdot\frac{dy}{dx}$$

$$-\cos(0)\frac{d}{dx}(0) = 0$$

$$\Rightarrow \quad \sqrt{3-\sin^2 x} + \cos y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{\sqrt{3-\sin^2 x}}{\cos y}$$

Example 71 Let $\frac{d}{dx}(F(x)) = \frac{e^{\sin x}}{x}$, $x > 0$. If $\int_1^4 \frac{2e^{\sin x^2}}{x}\,dx = F(K) - F(1)$, then the possible value of K is

(a) 10 (b) 14 (c) 16 (d) 18

Sol. We have, $\frac{d}{dx}(F(x)) = \frac{e^{\sin x}}{x}$

$$\Rightarrow \quad \int \frac{e^{\sin x}}{x}\,dx = F(x) \qquad \ldots(i)$$

Now, $\int_1^4 \frac{2e^{\sin x^2}}{x}\,dx = \int_1^4 \frac{e^{\sin(x^2)}}{x^2}\,d(x^2) = \int_1^{16} \frac{e^{\sin t}}{t}\,dt$

$(\because x^2 = t)$

$$= [F(t)]_1^{16} = F(16) - F(1)$$

$$\Rightarrow \quad K = 16$$

Hence, (c) is the correct answer.

Example 72 The function $f(x) = \int_0^x \log_{|\sin t|}\left(\sin t + \frac{1}{2}\right)dt$, where $x \in (0, 2\pi)$, then $f(x)$ strictly increases in the interval

(a) $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$ (b) $\left(\frac{5\pi}{6}, 2\pi\right)$

(c) $\left(\frac{\pi}{6}, \frac{7\pi}{6}\right)$ (d) $\left(\frac{5\pi}{6}, \frac{7\pi}{6}\right)$

Sol. Here, $f'(x) = \log_{|\sin x|}\left(\sin x + \frac{1}{2}\right) > 0$

$$\Rightarrow \quad \sin x + \frac{1}{2} < 1 \quad \text{and} \quad \left(\sin x + \frac{1}{2}\right) > 0$$

$$\Rightarrow \quad 0 < \sin x + \frac{1}{2} < 1$$

$$\Rightarrow \quad -1/2 < \sin x < 1/2$$

$$\Rightarrow \quad x \in \left(\frac{5\pi}{6}, \frac{7\pi}{6}\right) \text{ as } x \in (0, 2\pi)$$

Hence, (d) is the correct answer.

Example 73 Let $f:(0,\infty)\to R$ and $F(x)=\int_0^x t\,f(t)\,dt$. If $F(x^2)=x^4+x^5$, then $\sum_{r=1}^{12} f(r^2)$ is equal to

(a) 216 (b) 219 (c) 221 (d) 223

Sol. Here, $F(x^2)=x^4+x^5=\int_0^{x^2} t\,f(t)\,dt$

Differentiating w.r.t. x, we get

$$x^2\,f(x^2)\cdot 2x=4x^3+5x^4$$

$$\Rightarrow \quad f(x^2)=2+\frac{5}{2}x \;\Rightarrow\; f(r^2)=2+\frac{5}{2}r$$

$$\therefore \quad \sum_{r=1}^{n} f(r^2)=2n+\frac{5}{2}n(n+1)=\frac{n(5n+13)}{4}$$

$$\Rightarrow \quad \sum_{r=1}^{12} f(r^2)=\frac{12(60+13)}{4}=219$$

Hence, (b) is the correct answer.

Example 74 A function $f(x)$ satisfies $f(x)=\sin x+\int_0^x f'(t)(2\sin t-\sin^2 t)\,dt$, then $f(x)$ is

(a) $\frac{x}{1-\sin x}$ (b) $\frac{\sin x}{1-\sin x}$ (c) $\frac{1-\cos x}{\cos x}$ (d) $\frac{\tan x}{1-\sin x}$

Sol. Differentiating both the sides w.r.t. x, we get

$$f'(x)=\cos x+f'(x)(2\sin x-\sin^2 x)$$

$$\Rightarrow (1+\sin^2 x-2\sin x)f'(x)=\cos x$$

$$\Rightarrow \quad f'(x)=\frac{\cos x}{1+\sin^2 x-2\sin x}=\frac{\cos x}{(1-\sin x)^2}$$

Integrating, we get $f(x)=\int\frac{\cos x}{(1-\sin x)^2}\,dx$

Put $1-\sin x=t$, $f(x)=-\int\frac{dt}{t^2}=\frac{1}{t}=\frac{1}{1-\sin x}+C$

Also, $f(0)=0$, hence $C=-1$

$$f(x)=\frac{1}{1-\sin x}-1=\frac{1-1+\sin x}{1-\sin x}=\frac{\sin x}{1-\sin x}$$

Hence, (b) is the correct answer.

Example 75 If $F(x)=\int_1^x f(t)\,dt$, where $f(t)=\int_1^{t^2}\frac{\sqrt{1+u^4}}{u}\,du$, then the value of $F''(2)$ equals to

(a) $\frac{7}{4\sqrt{17}}$ (b) $\frac{15}{\sqrt{17}}$ (c) $\sqrt{257}$ (d) $\frac{15\sqrt{17}}{68}$

Sol. Here, $f'(t)=\frac{\sqrt{1+t^8}\cdot 2t}{t^2}=\frac{2\sqrt{1+t^8}}{t}$ ...(i)

Now, $F(x)=\int_1^x f(t)dt \;\Rightarrow\; F'(x)=f(x)$

$$F''(x)=f'(x) \;\Rightarrow\; F''(2)=f'(2)$$

From Eq. (i), $f'(2)=\sqrt{256+1}=\sqrt{257}$

Hence, (c) is the correct answer.

Property X. Let a function $f(x,\alpha)$ be continuous for $a\le x\le b$ and $c\le\alpha\le d$, then for any $\alpha\in[c,d]$, if $I(\alpha)=\int_a^b f(x,\alpha)\,dx$, then $\frac{dI(\alpha)}{d\alpha}=\int_a^b\frac{\partial f(x,\alpha)}{\partial\alpha}\,dx$

Example 76 Evaluate $I(b)=\int_0^1\frac{x^b-1}{\ln x}\,dx; b\ge 0$.

Sol. We have, $I(b)=\int_0^1\frac{x^b-1}{\ln x}\,dx$

$$\Rightarrow \quad \frac{d}{db}(I(b))=\int_0^1\frac{\partial}{\partial b}\left(\frac{x^b-1}{\ln x}\right)dx+0-0$$

$$=\int_0^1\frac{x^b\ln x}{\ln x}\,dx=\int_0^1(x^b)\,dx=\left(\frac{x^{b+1}}{b+1}\right)_0^1$$

$$=\frac{1}{b+1}[1-0]=\frac{1}{b+1}$$

$$\therefore \quad \frac{d}{dx}(I(b))=\frac{1}{b+1}$$

Integrating both the sides w.r.t. b, we get

$$I(b)=\log(b+1)+C \qquad ...(i)$$

Given, $I(b)=\int_0^1\frac{x^b-1}{\ln x}\,dx$

$$\therefore \quad I(0)=0 \qquad (\text{when } b=0) \quad ...(ii)$$

Also, from Eq. (i), $I(0)=\log(1)+C$

$$\therefore \quad I(0)=C \qquad ...(iii)$$

From Eqs. (ii) and (iii), $C=0$

$$\Rightarrow \quad I(b)=\log(b+1)$$

Example 77 Prove that $\int_0^{\pi/2}\frac{dx}{(a^2\sin^2 x+b^2\cos^2 x)^2}=\frac{\pi(a^2+b^2)}{4a^3b^3}$.

Sol. Let $I=\int_0^{\pi/2}\frac{1\,dx}{a^2\sin^2 x+b^2\cos^2 x}$

Then, $I=\int_0^{\pi/2}\frac{\sec^2 x}{a^2\tan^2 x+b^2}\,dx=\int_0^{\infty}\frac{dt}{a^2t^2+b^2}$

(where, $t=\tan x$)

$$\Rightarrow \quad I=\frac{1}{ab}\left[\tan^{-1}\left(\frac{at}{b}\right)\right]_0^{\infty}=\frac{1}{ab}\left(\frac{\pi}{2}-0\right)=\frac{\pi}{2ab}$$

Thus, $\int_0^{\pi/2}\frac{1}{a^2\sin^2 x+b^2\cos^2 x}\,dx=\frac{\pi}{2ab}=f(a,b)$ ...(i)

Differentiating both the sides w.r.t. 'a', we get

$$\int_0^{\pi/2} \frac{-2a\sin^2 x}{(a^2\sin^2 x + b^2\cos^2 x)^2}\,dx = -\frac{\pi}{2a^2b}$$

$$\Rightarrow \quad \int_0^{\pi/2} \frac{\sin^2 x}{(a^2\sin^2 x + b^2\cos^2 x)^2}\,dx = \frac{\pi}{4a^3b} \qquad \ldots(ii)$$

Similarly, by differentiating Eq. (i) w.r.t. 'b', we get

$$\int_0^{\pi/2} \frac{\cos^2 x}{(a^2\sin^2 x + b^2\cos^2 x)^2}\,dx = \frac{\pi}{4ab^3} \qquad \ldots(iii)$$

Adding Eqs. (ii) and (iii), we get

$$\int_0^{\pi/2} \frac{(\sin^2 x + \cos^2 x)\,dx}{(a^2\sin^2 x + b^2\cos^2 x)^2} = \frac{\pi}{4a^3b} + \frac{\pi}{4ab^3}$$

$$\Rightarrow \quad \int_0^{\pi/2} \frac{dx}{(a^2\sin^2 x + b^2\cos^2 x)^2} = \frac{\pi}{4a^3b^3}(a^2 + b^2)$$

Example 78 The value of $\int_0^{\pi/2} \frac{\log(1+x\sin^2\theta)}{\sin^2\theta}\,d\theta\,;\, x \geq 0$ is equal to

(a) $\pi(\sqrt{1+x}-1)$ (b) $\pi(\sqrt{1+x}-2)$

(c) $\sqrt{\pi}(\sqrt{1+x}-1)$ (d) None of these

Sol. Given, $f(x) = \int_0^{\pi/2} \frac{\log(1+x\sin^2\theta)}{\sin^2\theta}\,d\theta;\ x \geq 0$

As above integral is function of x. Thus, differentiating both the sides w.r.t. 'x', we get

$$f'(x) = \int_0^{\pi/2} \frac{1}{1+x\sin^2\theta}\cdot\frac{\sin^2\theta}{\sin^2\theta}\,d\theta + 0 - 0$$

(using Newton-Leibnitz's formula)

$$= \int_0^{\pi/2} \frac{d\theta}{1+x\sin^2\theta} = \int_0^{\pi/2} \frac{\operatorname{cosec}^2\theta\,d\theta}{\cot^2\theta + (1+x)}$$

$$= \left(-\frac{1}{\sqrt{1+x}}\cdot\tan^{-1}\frac{\cot\theta}{\sqrt{1+x}}\right)_0^{\pi/2} = \frac{\pi}{2\sqrt{1+x}}$$

$$\therefore \quad f'(x) = \frac{\pi}{2\sqrt{1+x}}$$

$$\Rightarrow \quad f(x) = \pi\sqrt{1+x} + C$$

where, $f(0) = 0 \Rightarrow C = -\pi$

$$\Rightarrow \quad f(x) = \pi\sqrt{1+x} - \pi = \pi(\sqrt{1+x} - 1)$$

Hence, (a) is the correct answer.

Example 79 Let $f(x)$ be a continuous functions for all x, such that $(f(x))^2 = \int_0^x f(t)\cdot\frac{2\sec^2 t}{4+\tan t}\,dt$ and $f(0) = 0$, then

(a) $f\left(\frac{\pi}{4}\right) = \log\frac{5}{4}$ (b) $f\left(\frac{\pi}{4}\right) = \frac{3}{4}$

(c) $f\left(\frac{\pi}{2}\right) = 2$ (d) None of these

Sol. Here, $(f(x))^2 = \int_0^x f(t)\cdot\frac{2\sec^2 t}{4+\tan t}\,dt,$

On differentiating both the sides w.r.t. x, we get

$$2f(x)\cdot f'(x) = f(x)\cdot\frac{2\sec^2 x}{4+\tan x}$$

$$\Rightarrow \quad f'(x) = \frac{\sec^2 x}{4+\tan x}$$

Integrating both the sides, we get

$$f(x) = \int \frac{\sec^2 x}{4+\tan x}\,dx = \log(4+\tan x) + C$$

Since, $f(0) = 0$

$$\Rightarrow \quad 0 = \log(4) + C$$

$$\Rightarrow \quad C = -\log 4$$

$$\therefore \quad f(x) = \log(4+\tan x) - \log(4)$$

$$\Rightarrow \quad f\left(\frac{\pi}{4}\right) = \log(4+1) - \log(4) = \log\frac{5}{4}$$

Hence, (a) is the correct answer.

Exercise for Session 5

1. The value of $\int_{-1}^{10} \text{sgn}(x-[x])dx$ is equal to (where, $[\cdot]$ denotes the greatest integer function)

(a) 9 (b) 10 (c) 11 (d) 12

2. The value of $\int_{0}^{[x]} (x-[x])dx$ (where, $[\cdot]$ denotes the greatest integer function)

(a) $[x]$ (b) $\frac{[x]}{2}$ (c) $x[x]$ (d) None of these

3. The value of $\int_{-\pi/4}^{n\pi-\pi/4} |\sin x + \cos x| dx$ is equal to

(a) $2\sqrt{2}n$ (b) $\sqrt{2}n$ (c) $\frac{1}{2\sqrt{2}}n$ (d) None of these

4. Let $f(x) = x-[x]$ for every real number x (where, $[\cdot]$ denotes greatest integer function), then $\int_{-1}^{1} f(x)dx$ is equal to

(a) 1 (b) 2 (c) 0 (d) $\frac{1}{2}$

5. If $\int_{0}^{x} f(t)dt = x + f\int_{x}^{1} tf(t)\,dt$, then the value of $f(1)$ is

(a) $\frac{1}{2}$ (b) 0 (c) 1 (d) $-\frac{1}{2}$

6. The least value of the function $\phi(x) = \int_{5\pi/4}^{x} (3\sin t + 4\cos t)dt$ on the interval $\left[\frac{5\pi}{4}, \frac{4\pi}{3}\right]$ is

(a) $\sqrt{3} + \frac{3}{2}$ (b) $-2\sqrt{3} + \frac{3}{2} + \frac{1}{\sqrt{2}}$

(c) $\frac{3}{2} + \frac{1}{\sqrt{2}}$ (d) None of these

7. The points of extremum of $\phi(x) = \int_{1}^{x} e^{-t^2/2}(1-t^2)dt$ are

(a) $x = 1, -1$ (b) $x = -1, 2$ (c) $x = 2, 1$ (d) $x = -2, 1$

8. If $f(x)$ is periodic function, with period T, then

(a) $\int_{a}^{b} f(x)dx = \int_{a}^{b+T} f(x)dx$ (b) $\int_{a}^{b} f(x)dx = \int_{a+T}^{b+T} f(x)dx$

(c) $\int_{a}^{b} f(x)dx = \int_{a+T}^{b} f(x)dx$ (d) $\int_{a}^{b} f(x)dx = \int_{a+T}^{b+2T} f(x)dx$

9. Let $\frac{d}{dx}(F(x)) = \frac{e^{\sin x}}{x}$, $x > 0$. If $\int_{1}^{4} \frac{2e^{\sin x^2}}{x} dx = F(k) - F(1)$, then one of the possible value of k is

(a) 4 (b) 8 (c) 16 (d) 32

10. Let $f:(0,\infty) \to R$ and $F(x) = \int_{0}^{x} f(t)dt$. If $F(x^2) = x^2(1+x)$, then $f(4)$ is equal to

(a) $\frac{5}{4}$ (b) 7 (c) 4 (d) 2

11. Let $T > 0$ be a fixed real number. Suppose f is continuous function such that $f(x+T) = f(x)$ for all $x \in R$. If $I = \int_{0}^{T} f(x)\,dx$, then the value of $\int_{3}^{3+3T} f(2x)\,dx$ is

(a) $\frac{3}{2}I$ (b) $2I$ (c) $3I$ (d) $6I$

12. Let $f(x) = \int_{1}^{x} \sqrt{2-t^2}dt$, then the real roots of the equation $x^2 - f'(x) = 0$ are

(a) ± 1 (b) $\pm\frac{1}{\sqrt{2}}$ (c) $\pm\frac{1}{2}$ (d) $\pm\sqrt{2}$

13. Let $f(x)$ be an odd continuous function which is periodic with period 2. If $g(x) = \int_0^x f(t)\,dt$, then

(a) $g(x)$ is an odd function
(b) $g(n) = 0$ for all $n \in N$
(c) $g(2n) = 0$ for all $n \in N$
(d) $g(x)$ is non-periodic

14. Let $f(x)$ be a function defined by $f(x) = \int_1^x t(t^2 - 3t + 2)dt,\ 1 \le x \le 3$. Then, the range of $f(x)$ is

(a) [0, 2] (b) $\left[-\frac{1}{4}, 4\right]$ (c) $\left[-\frac{1}{4}, 2\right]$ (d) None of these

15. The value of $\lim_{x \to 0} \frac{2\int_0^{\cos x} \cos^{-1}(t)}{2x - \sin 2x} dt$ is

(a) 0 (b) $\frac{1}{2}$ (c) $-\frac{1}{2}$ (d) $\frac{2}{3}$

16. If $\int_0^x \frac{bt\cos 4t - a\sin 4t}{t^2} dt = \frac{a\sin 4x}{x}$ for all $x \neq 0$, then a and b are given by

(a) $a = \frac{1}{4}, b = 1$
(b) $a = 2, b = 2$
(c) $a = -1, b = 4$
(d) $a = 2, b = 4$

17. If $f(x) = \int_0^x \{f(t)\}^{-1} dt$ and $\int_0^1 \{f(t)\}^{-1} = \sqrt{2}$, then

(a) $f(x) = \sqrt{2x}$
(b) $f(x) = \sqrt{2\log_e x}$
(c) $f(x) = \sqrt{3x - 1}$
(d) None of these

18. Let f be a real valued function defined on the interval $(-1, 1)$ such that $e^{-x} f(x) = 2 + \int_0^x \sqrt{t^4 + 1}\,dt$, for all $x \in (-1, 1)$ and let f^{-1} be the inverse of f. Then, $(f^{-1})'(2)$ is equal to *[IIT JEE 2010]*

(a) 1
(b) 1/3
(c) 1/2
(d) $1/e$

■ **Directions** (Q. Nos. 19 to 20) *Consider the function defined on* $[0, 1] \to R$

$$f(x) = \frac{\sin x - x\cos x}{x^2},\ \textit{if } x \neq 0 \textit{ and } f(0) = 0.$$

[IIT JEE 2012]

19. $\int_0^1 f(x)\,dx$ is equal to

(a) $1 - \sin(1)$ (b) $\sin(1) - 1$ (c) $\sin(1)$ (d) $-\sin(1)$

20. $\lim_{t \to 0} \frac{1}{t^2} \int_0^t f(x)\,dx$ is equal to

(a) 1/3 (b) 1/6 (c) 1/12 (d) 1/24

Session 6

Integration as Limit of a Sum, Applications of Inequality Gamma Function, Beta Function, Walli's Formula

Integration as Limit of a Sum

Applications of Inequality and Gamma Integrals

An alternative way of describing $\int_a^b f(x)\,dx$ is that the definite integral $\int_a^b f(x)\,dx$ is a limiting case of summation of an infinite series, provided $f(x)$ is continuous on $[a, b]$, i.e. $\int_a^b f(x)\,dx = \lim_{n\to\infty} h \sum_{r=1}^{n-1}$ $f(a + rh)$, where $h = \dfrac{b-a}{n}\cdot$ The converse is also true, *ie*, if we have an infinite series of the above form, it can be expressed as definite integral.

Method to Express the Infinite Series as Definite Integral

(i) Express the given series in the form $\sum \frac{1}{n} f\left(\frac{r}{n}\right)\cdot$

(ii) Then, the limit is its sum when $n \to \infty$, i.e. $\lim_{n\to\infty} \sum \frac{1}{n} f\left(\frac{r}{n}\right)\cdot$

(iii) Replace $\frac{r}{n}$ by x and $\frac{1}{n}$ by (dx) and $\lim_{n\to\infty} \sum$ by the sign of $\int$.

(iv) The lower and the upper limit of integration are limiting values of $\frac{r}{n}$ for the first and the last term of r respectively.

Some particular cases of the above are

(a) $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n} f\left(\frac{r}{n}\right)$

or $\lim_{n\to\infty} \sum_{r=1}^{n-1} \frac{1}{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx$

(b) $\lim_{n\to\infty} \sum_{r=1}^{Pn} \frac{1}{n} f\left(\frac{r}{n}\right) = \int_\alpha^\beta f(x)\,dx$

where, $\alpha = \lim_{n\to\infty} \frac{r}{n} = 0\,(\text{as } r = 1)$

and $\beta = \lim_{n\to\infty} \frac{r}{n} = p\,(\text{as } r = pn)$

Some Important Results to Remember

(i) $\sum_{r=1}^{n} r = \dfrac{n(n+1)}{2}$

(ii) $\sum_{r=1}^{n} r^2 = \dfrac{n(n+1)(2n+1)}{6}$

(iii) $\sum_{r=1}^{n} r^3 = \dfrac{n^2(n+1)^2}{4}$

(iv) In GP, sum of n terms, $S_n = \begin{cases} \dfrac{a(r^n-1)}{(r-1)}, & |r|>1 \\ an, & r=1 \\ \dfrac{a(1-r^n)}{(1-r)}, & |r|<1 \end{cases}$

(v) $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin[\alpha + (\overline{n-1})\beta]$
$= \dfrac{\sin n\beta/2}{\sin\beta/2}\cdot\sin[\alpha + (n-1)\beta/2]$

(vi) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos[\alpha + (\overline{n-1})\beta]$
$= \dfrac{\sin n\beta/2}{\sin\beta/2}\cdot\cos[\alpha + (n-1)\beta/2]$

(vii) $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \ldots\infty = \frac{\pi^2}{12}$

(viii) $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \ldots\infty = \frac{\pi^2}{6}$

(ix) $1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\infty = \frac{\pi^2}{8}$

(x) $\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \ldots\infty = \frac{\pi^2}{24}$

(xi) $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$, and $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

(xii) $\cos h\theta = \dfrac{e^{\theta} + e^{-\theta}}{2}$ and $\sin h\theta = \dfrac{e^{\theta} - e^{-\theta}}{2}$

Remark

The method of evaluate the integral, as limit of the sum of an infinite series is known as integration by first principles.

Example 80 Evaluate the following :

(i) $\lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\frac{3}{n^2}+\ldots+\frac{n-1}{n^2}\right)$

(ii) $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}\right)$

(iii) $\lim_{n\to\infty}\left(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\ldots+\frac{n}{2n^2}\right)$

(iv) $\lim_{n\to\infty}\frac{(1^P+2^P+\ldots+n^P)}{n^{P+1}}, P>0$

Sol. Let

(i) $S_n=\frac{1}{n^2}+\frac{2}{n^2}+\frac{3}{n^2}+\ldots+\frac{n-1}{n^2}$

$=\sum_{r=2}^{n}\frac{r-1}{n^2}=\sum_{r=2}^{n}\frac{1}{n}\left(\frac{r-1}{n}\right)$

$\Rightarrow\quad S=\lim_{n\to\infty}\sum_{r=2}^{n}\frac{1}{n}\left(\frac{r}{n}\right)\qquad\left(\text{as } n\to\infty\Rightarrow\frac{1}{n}\to 0\right)$

$=\int_0^1 x\,dx=\frac{1}{2}$

(ii) Let $S_n=\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}=\sum_{r=1}^{n}\frac{1}{n+r}$

$=\sum_{r=1}^{n}\frac{1}{n\left(1+\frac{r}{n}\right)}$

Hence, $S=\lim_{n\to\infty}S_n=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+\frac{r}{n}}$

$=\int_0^1\frac{dx}{1+x}=\left[\log(1+x)\right]_0^1=\log 2$

(iii) Let $S_n=\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\ldots+\frac{n}{2n^2}$

$=\sum_{r=1}^{n}\frac{n}{n^2+r^2}=\sum_{r=1}^{n}\frac{1}{n(1+r^2/n^2)}$

Hence, $S=\lim_{n\to\infty}S_n=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+\frac{r^2}{n^2}}$

$=\int_0^1\frac{dx}{1+x^2}=\left[\tan^{-1}x\right]_0^1=\frac{\pi}{4}$

(iv) Let $S_n=\frac{1^P+2^P+\ldots+n^P}{n^{P+1}}=\sum_{r=1}^{n}\frac{r^P}{n^{P+1}}=\sum_{r=1}^{n}\frac{1}{n}\left(\frac{r}{n}\right)^P$

$\therefore\quad S=\lim_{n\to\infty}S_n=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^P$

$=\int_0^1 x^P\,dx=\left[\frac{x^{P+1}}{P+1}\right]_0^1=\frac{1}{P+1}$

Example 81 ***Evaluate*** $S=\sum_{r=0}^{n-1}\frac{1}{\sqrt{4n^2-r^2}}$ ***as*** $n\to\infty$.

Sol. Let $S_n=\sum_{r=0}^{n-1}\frac{1}{\sqrt{4n^2-r^2}}=\sum_{r=0}^{n-1}\frac{1}{n\sqrt{4-(r/n)^2}}$

$=\frac{1}{n}\sum_{r=0}^{n-1}\frac{1}{\sqrt{4-(r/n)^2}}$

Hence, $S=\lim_{n\to\infty}S_n=\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\frac{1}{\sqrt{4-(r/n)^2}}$

$=\int_0^1\frac{dx}{\sqrt{4-x^2}}=\left[\sin^{-1}\frac{x}{2}\right]_0^1$

$\Rightarrow\qquad S=\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}\qquad\therefore\ S=\frac{\pi}{6}$

Example 82 Evaluate

$$\lim_{n\to\infty}\left(\frac{1}{2n+1}+\frac{1}{2n+2}+\ldots+\frac{1}{6n}\right).$$

Sol. Let $S_n=\left(\frac{1}{2n+1}+\frac{1}{2n+2}+\ldots+\frac{1}{6n}\right)$

$=\sum_{r=1}^{4n}\frac{1}{2n+r}=\sum_{r=1}^{4n}\frac{1}{n}\cdot\frac{1}{2+(r/n)}$

$\Rightarrow\quad S=\lim_{n\to\infty}S_n=\int_0^4\frac{dx}{2+x}=\left[\ln|2+x|\right]_0^4$

$=\log 6-\log 2=\log 3$

$\therefore\quad S=\ln 3$

Example 83 Evaluate $\int_1^4(ax^2+bx+c)\,dx$ from the first principle.

Sol. Let $x=1+rh$, where $h=\frac{4-1}{n}=\frac{3}{n}$

As $x\to 1, r\to 0$ and $x\to 4,\ r\to n$

$\therefore\quad\int_1^4(ax^2+bx+c)\,dx=\lim_{n\to\infty}\sum_{r=0}^{n}h\,f(1+rh)$

$=\lim_{n\to\infty}\sum_{r=0}^{n}h\left[a(1+rh)^2+b(1+rh)+c\right]$

$=\lim_{n\to\infty}\sum_{r=0}^{n}\frac{3}{n}\left[a\left(1+\frac{3r}{n}\right)^2+b\left(1+\frac{3r}{n}\right)+c\right]$

$=3\lim_{n\to\infty}\sum_{r=0}^{n}\frac{1}{n}\left[a\left(1+\frac{9r^2}{n^2}+\frac{6r}{n}\right)+b\left(1+\frac{3r}{n}\right)+c\right]$

$=3\lim_{n\to\infty}\frac{1}{n}\left[a\left(n+\frac{9n(n+1)(2n+1)}{6n^2}+\frac{6n(n+1)}{2n}\right)+b\left(n+\frac{3n(n+1)}{2n}\right)+cn\right]$

$=3\left[a(1+3+3)+b(1+3/2)+c\right]=21a+\frac{15}{2}b+3c$

Example 84 ***The value of*** $\lim_{n\to\infty}\left(\sin\frac{\pi}{2n}\cdot\sin\frac{2\pi}{2n}\cdot\sin\frac{3\pi}{2n}\ldots\sin\frac{(n-1)\pi}{n}\right)^{1/n}$ is equal to

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) None of these

Sol. Let $A=\left\{\lim_{n\to\infty}\sin\frac{\pi}{2n}\cdot\sin\frac{2\pi}{2n}\cdot\sin\frac{3\pi}{2n}\ldots\sin\frac{2(n-1)\pi}{2n}\right\}^{1/n}$

$$\therefore \log A=\lim_{n\to\infty}\frac{1}{n}\log\left(\sin\frac{\pi}{2n}\cdot\sin\frac{2\pi}{2n}\ldots\sin\frac{2(n-1)\pi}{2n}\right)$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{2(n-1)}\log\sin\frac{r\pi}{2n}$$

$$=\int_0^2\log\sin\left(\frac{\pi x}{2}\right)dx$$

$$\left[\text{using }\lim_{n\to\infty}\Sigma\frac{1}{n}f\left(\frac{r}{n}\right)=\int_a^b f(x)\,dx\right]$$

$$=\int_0^{\pi}\log(\sin t)\cdot\frac{2}{\pi}\,dt \qquad \left[\text{putting }\frac{\pi x}{2}=t\right]$$

$$=\frac{2\cdot2}{\pi}\int_0^{\pi/2}\log(\sin t)\,dt$$

$$\left[\text{using }\int_0^{2a}f(x)\,dx=2\int_0^a f(x)\,dx,\text{ if } f(2a-x)=f(x)\right]$$

$$=\frac{4}{\pi}\left\{-\frac{\pi}{2}\log 2\right\}\left[\text{using}\int_0^{\pi/2}\log(\sin x)dx=-\frac{\pi}{2}\log 2\right]$$

$$=-2\log 2$$

$$\therefore \log A=\log(1/4)\ \Rightarrow\ A=1/4$$

Hence, (c) is the correct answer.

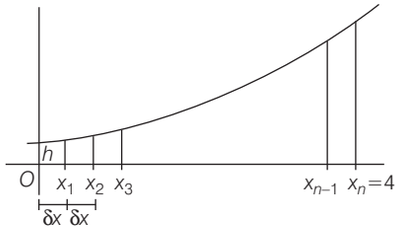
Example 85 ***The interval [0, 4] is divided into n equal sub-intervals by the points*** $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ ***where*** $0=x_0<x_1<x_2<x_3<\ldots<x_n=4$ **.**

If $\delta x=x_i-x_{i-1}$ for $i=1, 2, 3, \ldots, n$, then $\lim_{\delta x\to0}\sum_{i=1}^{n}x_i\delta x$ is equal to

(a) 4 (b) 8 (c) $\frac{32}{3}$ (d) 16

Sol. $\lim_{\delta x\to0}\delta x(x_1+x_2+x_3+\ldots+x_n)$

$$=\lim_{n\to\infty}\frac{4}{n}\left[\frac{4}{n}+\frac{8}{n}+\frac{12}{n}+\ldots\ldots+4\cdot\frac{n}{n}\right]\qquad\left(\because\delta x=\frac{4}{n}\right)$$



$$=\lim_{n\to\infty}\frac{16}{n^2}(1+2+3+\ldots+n)=\lim_{n\to\infty}\frac{16}{n^2}\cdot\frac{n(n+1)}{2}=8$$

Hence, (b) is the correct answer.

Example 86 The value of $\lim_{n\to\infty}\frac{1}{n}\cdot\sum_{r=1}^{2n}\frac{r}{\sqrt{n^2+r^2}}$ is equal to

(a) $1+\sqrt{5}$ (b) $-1+\sqrt{5}$ (c) $-1+\sqrt{2}$ (d) $1+\sqrt{2}$

Sol. We have, $\lim_{n\to\infty}\frac{1}{n}\cdot\sum_{r=1}^{2n}\frac{r}{\sqrt{n^2+r^2}}$

$$=\lim_{n\to\infty}\frac{1}{n}\cdot\sum_{r=1}^{2n}\frac{\frac{r}{n}}{\sqrt{1+\frac{r^2}{n^2}}}=\int_0^2\frac{x}{\sqrt{1+x^2}}dx=(\sqrt{x^2+1})_0^2=\sqrt{5}-1$$

Hence, (b) is the correct answer.

Applications of Inequality

Sometimes you are asked to prove inequalities involving definite integrals or to estimate the upper and lower boundary values of definite integral, where the exact value of the definite integral is difficult to find. Under these circumstances, we use the following types

Type I. If $f(x)$ is defined on $[a, b]$, then

$$\left|\int_a^b f(x)\,dx\right|\le\int_a^b|f(x)|\,dx.$$

Equality sign holds, where $f(x)$ is entirely of the same sign on $[a, b]$.

Example 87 ***Estimate the absolute value of the integral*** $\int_{10}^{19}\frac{\sin x}{1+x^8}dx$**.**

Sol. To find, $I=\left|\int_{10}^{19}\frac{\sin x}{1+x^8}dx\right|\le\int_{10}^{19}\left|\frac{\sin x}{1+x^8}\right|dx$ ...(i)

(using type I)

Since, $|\sin x|\le1$ for $x\ge10$

The inequality $\left|\frac{\sin x}{1+x^8}\right|\le\frac{1}{|1+x^8|}$ ...(ii)

Also, $10\le x\le19\ \Rightarrow\ 1+x^8>10^8$

$$\Rightarrow\quad\frac{1}{1+x^8}<\frac{1}{10^8}\ \text{ or }\ \frac{1}{|1+x^8|}<10^{-8}\qquad\text{...(iii)}$$

From Eqs. (ii) and (iii), $\left|\frac{\sin x}{1+x^8}\right|<10^{-8}$ is fulfilled.

$$\left|\int_{10}^{19}\frac{\sin x}{1+x^8}dx\right|<\int_{10}^{19}10^{-8}\,dx$$

$$\therefore\quad\left|\int_{10}^{19}\frac{\sin x}{1+x^8}dx\right|<(19-10).10^{-8}<10^{-7}$$

($\because$ the true value of integral $\approx10^{-8}$)

Example 88 The minimum odd value of 'a' ($a > 1$) for which $\int_{10}^{19} \frac{\sin x}{1+x^a}\,dx < \frac{1}{9}$, is equal to

(a) 1 (b) 3 (c) 5 (d) 9

Sol. Let $I = \int_{10}^{19} \frac{\sin x\,dx}{1+x^a} < \int_{10}^{19} \frac{dx}{1+x^a}$

$$\left(\because \sin x < 1 \Rightarrow \frac{\sin x}{1+x^a} < \frac{1}{1+x^a}\right)$$

$$\therefore \quad I < \int_{10}^{19} \frac{dx}{1+x^a} < \int_{10}^{19} \frac{dx}{1+10^a}$$

$$(\because\ 10 < x < 19 \Rightarrow 10^a + 1 < 1 + x^a < 19^a + 1)$$

$$\Rightarrow \quad I < \frac{9}{1+10^a}$$

$$\therefore \frac{9}{1+10^a} < \frac{1}{9} \Rightarrow 1 + 10^a > 81 \quad \text{or } 10^a > 80 \text{ i.e. } a = 2, 3, 4, 5, \ldots$$

$\therefore$ Minimum odd value of 'a' is 3.

Hence, (b) is the correct answer.

Type II.

$$\left|\int_a^b f(x)\,g(x)\,dx\right| \le \sqrt{\left(\int_a^b f^2(x)\,dx\right)\left(\int_a^b g^2(x)\,dx\right)},$$

where $f^2(x)$ and $g^2(x)$ are integrable on $[a, b]$.

Example 89 Prove that $\int_0^1 \sqrt{(1+x)(1+x^3)}\,dx$ cannot exceed $\sqrt{15/8}$.

Sol. $\int_0^1 \sqrt{(1+x)(1+x^3)}dx \le \sqrt{\left(\int_0^1 (1+x)dx\right)\left(\int_0^1 (1+x^3)\,dx\right)}$

$$\le \sqrt{\left(x + \frac{x^2}{2}\right)_0^1 \left(x + \frac{x^4}{4}\right)_0^1}$$

$$\le \sqrt{\frac{3}{2}\cdot\frac{5}{4}} \le \sqrt{\frac{15}{8}}$$

Type III. If $f(x) \mid g(x)$ on $[a, b]$, then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$. In particular, if $f(x) \ge 0$, then $\int_a^b f(x)\,dx \mid 0$.

Example 90 If $f(x)$ is a continuous function such that $f(x) \mid 0, \forall\ x \in [2, 10]$ and $\int_4^8 f(x)\,dx = 0$, then find $f(6)$.

Sol. $f(x)$ is above the X-axis or on the X-axis for all $x \in [2, 10]$.

If $f(x)$ is greater than zero at any sub-interval of [4, 8], then $\int_4^8 f(x)\,dx$ must be greater than zero. But $\int_4^8 f(x)\,dx = 0$, which shows $f(x)$ can't have any value greater than zero in the sub-interval [4, 8].

$\Rightarrow f(x)$ is constant in the sub-interval [4, 8] and has to be zero at all points, $x \in [4, 8]$.

$$\Rightarrow \quad f(x) = 0, \quad \forall\ x \in [4, 8]$$

$$\Rightarrow \quad f(6) = 0$$

Type IV. For a given function $f(x)$ continuous on $[a, b]$, if you are able to find two continuous functions $f_1(x)$ and $f_2(x)$ on $[a, b]$ such that $f_1(x) \le f(x) \le f_2(x), \forall\ x \in [a, b]$, then

$$\int_a^b f_1(x)\,dx \le \int_a^b f(x)\,dx \le \int_a^b f_2(x)\,dx.$$

Example 91 Prove that $\frac{\pi}{6} \le \int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} \le \frac{\pi}{4\sqrt{2}}$.

Sol. Since, $4 - x^2 \mid 4 - x^2 - x^3 \mid 4 - 2x^2 > 1, \forall\ x \in [0, 1]$

$$\sqrt{4 - x^2} \mid \sqrt{4 - x^2 - x^3} \mid \sqrt{4 - 2x^2} > 1, \forall\ x \in [0, 1]$$

$$\Rightarrow \frac{1}{\sqrt{4 - x^2}} \le \frac{1}{\sqrt{4 - x^2 - x^3}} \le \frac{1}{\sqrt{4 - 2x^2}}, \forall\ x \in [0, 1]$$

$$\Rightarrow \quad \int_0^1 \frac{dx}{\sqrt{4 - x^2}} \le \int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} \le \int_0^1 \frac{1}{\sqrt{4 - 2x^2}}\,dx, \forall\ x \in [0, 1]$$

$$\Rightarrow \quad \left(\sin^{-1}\frac{x}{2}\right)_0^1 \le \int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} \le \frac{1}{\sqrt{2}}\left(\sin^{-1}\frac{x}{\sqrt{2}}\right)_0^1$$

$$\Rightarrow \quad \frac{\pi}{6} \le \int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} \le \frac{\pi}{4\sqrt{2}}$$

Type V. If m and M be global minimum and global maximum of $f(x)$ respectively in $[a, b]$, then

$$m(b - a) \le \int_a^b f(x)\,dx \le M(b - a).$$

Proof We have, $m \le f(x) \le M$ for all $x \in [a, b]$

$$\Rightarrow \quad \int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx$$

$$\Rightarrow \quad m(b - a) \le \int_a^b f(x)\,dx \le M(b - a)$$

Example 92 Prove that $4 \le \int_1^3 \sqrt{3 + x^3}\,dx \le 2\sqrt{30}$.

Sol. Since, the function $f(x) = \sqrt{3 + x^3}$ increases monotonically on the interval [1, 3].

$$\therefore \quad M = \text{Maximum value of } \sqrt{3 + x^3} = \sqrt{3 + 3^3} = \sqrt{30}$$

and $\quad m = \text{Minimum value of } \sqrt{3 + x^3} = \sqrt{3 + 1^3} = 2$

$\therefore \quad m = 2, M = \sqrt{30}, b - a = 2$

Hence, $\quad 2 \cdot 2 \le \int_1^3 \sqrt{3 + x^3}\, dx \le 2\sqrt{30}$

$\Rightarrow \quad 4 \le \int_1^3 \sqrt{3 + x^3}\, dx \le 2\sqrt{30}$

Example 93 Prove that $1 \le \int_0^1 e^{x^2} dx \le e$.

Sol. For $0 \le x \le 1$, we have $e^0 \le e^{x^2} \le e^1$

$\therefore \quad e^0 (1 - 0) \le \int_0^1 e^{x^2}\, dx \le e^1 (1 - 0)$

$\Rightarrow \quad 1 \le \int_0^1 e^{x^2} dx \le e$

Gamma Function

If n is a positive number, then the improper integral $\int_0^\infty e^{-x}\, x^{n-1}\, dx$ is defined as Gamma function and is denoted by Γn.

i.e. $\quad \Gamma n = \int_0^\infty e^{-x}\, x^{n-1}\, dx$, where $x \in Q^+$.

Example 94 Evaluate

(i) $\Gamma 1$ (ii) $\Gamma 2$

Sol. (i) $\Gamma 1 = \int_0^\infty e^{-x} \cdot x^{1-1}\, dx = \lim_{b \to \infty} \int_0^b e^{-x}\, dx$

$= \lim_{b \to \infty} \left[-e^{-x} \right]_0^b = \lim_{b \to \infty} (-e^{-b} + e^0)$

$= 0 + 1 = 1$

(ii) $\Gamma 2 = \int_0^\infty e^{-\infty}\, x^{2-1}\, dx = \lim_{b \to \infty} \int_0^b e^{-x} \cdot x\, dx$

$= \lim_{b \to \infty} \left[-x\, e^{-x} - e^{-x} \right]_0^b$

$= \lim_{b \to \infty} [(-b\, e^{-b} - e^{-b}) - (0 - 1)]$

$= \lim_{b \to \infty} \left[\frac{-b}{e^b} - \frac{1}{e^b} + 1 \right] = \lim_{b \to \infty} \frac{-(b+1)}{e^b} + 1$

$= \left(\lim_{b \to \infty} -\frac{1}{e^b} \right) + 1$ (using L'Hospital's rule)

$= 1$

Properties of Gamma Function

Gamma function has following properties :

(i) $\Gamma 1 = 1$, $\Gamma 0 = \infty$ and $\Gamma(n+1) = n\, \Gamma n$

e.g. $\Gamma 5 = 4\, \Gamma 4 = 4 \times 3\, \Gamma 3 = 4 \times 3 \times 2\, \Gamma 2$

$= 4 \times 3 \times 2 \times 1\, \Gamma 1 = 4 \times 3 \times 2 \times 1$

(ii) If $n \in N$, then $\Gamma(n+1) = (n)!$

(iii) $\Gamma(1/2) = \sqrt{\pi}$

(iv) $\int_0^{\pi/2} \sin^m x \cdot \cos^n x\, dx = \frac{\Gamma\left(\frac{m+1}{2}\right) \Gamma\left(\frac{n+1}{2}\right)}{2\, \Gamma\left(\frac{m+n+2}{2}\right)}$

(v) $\Gamma n\, \Gamma(1-n) = \frac{\pi}{\sin n\pi}, 0 < n < 1$

(vi) $\Gamma\, m\, \Gamma\left(m + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\, \Gamma\, 2m$

(vii) $\Gamma \frac{1}{n} \Gamma\left(\frac{2}{n}\right) \ldots \Gamma\left(\frac{n-1}{n}\right) = \frac{(2\pi)^{\frac{n-1}{2}}}{n^{1/2}}$

Example 95 ***Evaluate*** $\int_0^\infty e^{-x}\, x^3\, dx$.

Sol. By definition of Gamma function,

$\int_0^\infty e^{-x}\, x^3\, dx = \int_0^\infty e^{-x}\, x^{4-1}\, dx = \Gamma 4$

$\therefore \quad \int_0^\infty e^{-x}\, x^3\, dx = 6$

Example 96 Evaluate $\int_0^1 \left(\log \frac{1}{x} \right)^{n-1} dx$.

Sol. Let $I = \int_0^1 \left(\log \frac{1}{x} \right)^{n-1} dx$

Put $\log \frac{1}{x} = t \Rightarrow dx = -e^{-t}\, dt$

$\therefore \quad I = \int_\infty^0 t^{n-1} (-e^{-t})\, dt = \int_0^\infty e^{-t}\, t^{n-1}\, dt$

$\therefore \quad \int_0^1 \left(\log \frac{1}{x} \right)^{n-1} dx = \Gamma n$

Beta Function

The beta function is

$$B(m, n) = \int_0^1 x^{m-1} (1 - x)^{n-1}\, dx,$$

where $m, n > 0$

Properties of Beta Function

(i) $B(m, n) = B(n, m)$, where $m, n > 0$

(ii) $B(m, n) = \frac{\Gamma\, m\, \Gamma\, n}{\Gamma\, (m+n)}$, where $m, n > 0$

(iii) $B(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$

Example 97 Evaluate $\int_0^1 x^6 \sqrt{(1-x^2)}\, dx$.

Sol. Let, $I = \int_0^1 x^6 \sqrt{(1-x^2)}\, dx$

Put let $x^2 = t$

$\Rightarrow 2x\, dx = dt$

$\therefore \quad I = \frac{1}{2}\int_0^1 t^{5/2}\sqrt{(1-t)}\, dt$

$$I = \frac{1}{2} B\left(\frac{7}{2}, \frac{3}{2}\right) = \frac{1}{2}\cdot\frac{\Gamma\, 7/2\, \Gamma\, 3/2}{\Gamma\left(\frac{3}{2}+\frac{7}{2}\right)}$$

$$= \frac{1}{2}\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{4\cdot 3\cdot 2\cdot 1} = \frac{5\pi}{256}$$

Walli's Formula

An easy way to evaluate $\int_0^{\pi/2} \sin^m x\cdot\cos^n x\, dx$, where $m, n \in I_+$.

We have, $\int_0^{\pi/2} \sin^m x\cdot\cos^n x\, dx = \int_0^{\pi/2} \sin^n x\cdot\cos^m x\, dx$

$$= \frac{(m-1)(m-3)\ldots(1 \text{ or } 2)\,.\,(n-1)(n-3)\ldots(1 \text{ or } 2)}{(m+n)(m+n-2)\ldots(1 \text{ or } 2)}\cdot\frac{\pi}{2},$$

when both m and $n \in$ even integer.

$$= \frac{(m-1)(m-3)\ldots(1 \text{ or } 2)\cdot(n-1)(n-3)\ldots(1 \text{ or } 2)}{(m+n)(m+n-2)\ldots(1 \text{ or } 2)},$$

when either of m or $n \in$ odd integer.

Remark

If n be a positive integer, then

$$\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2},\ n \text{ is even.}$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{4}{5}\cdot\frac{2}{3}\cdot 1,\ n \text{ is odd.}$$

Example 98 Evaluate $\int_0^{\pi/2} \sin^4 x\cdot\cos^6 x\, dx$.

Sol. Using Gamma function, we have

$$\int_0^{\pi/4} \sin^4 x\cdot\cos^6 x\, dx = \frac{\Gamma(5/2)\,\Gamma(7/2)}{2\,\Gamma\left(\frac{4+6+2}{2}\right)} = \frac{\Gamma(5/2)\,\Gamma(7/2)}{2\,\Gamma 6}$$

$$= \frac{\left(\frac{3}{2}\times\frac{1}{2}\times\Gamma(1/2)\right)\left(\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\Gamma(1/2)\right)}{2\times 5!} = \frac{3\pi}{512}$$

Example 99 The value of $\int_0^\infty e^{-a^2x^2}\, dx$ is equal to

(a) $\frac{\sqrt{\pi}}{2a}$ (b) $\frac{\pi}{2a}$

(c) $\frac{\pi}{\sqrt{2a}}$ (d) None of these

Sol. Let $I = \int_0^\infty e^{-a^2x^2}\, dx$

$$\therefore \quad I = \int_0^\infty e^{-t}\cdot\frac{1}{2a\sqrt{t}}\, dt = \frac{1}{2a}\int_0^\infty e^{-t}\cdot t^{-1/2}\, dt$$

(put $a^2x^2 = t \Rightarrow 2a^2x\, dx = dt$)

$$= \frac{1}{2a}\int_0^\infty t^{1/2-1}\cdot e^{-t}\cdot dt \qquad (\text{using } \int_0^\infty e^{-x}\, x^{n-1} dx = \Gamma n)$$

$$= \frac{1}{2a}\Gamma\frac{1}{2} = \frac{1}{2a}\sqrt{\pi} \qquad (\text{using } \Gamma\frac{1}{2} = \sqrt{\pi})$$

Hence, (a) is the correct answer.

Exercise for Session 6

1. The value of $f(x)=\int_0^{\pi/2}\frac{\log(1+x\sin^2\theta)}{\sin^2\theta}d\theta, x\geq 0$ is equal to

(a) $\frac{1}{\pi}(\sqrt{1+x}-1)$ (b) $\sqrt{\pi}(\sqrt{1+x}-1)$ (c) $\pi(\sqrt{1+x}-1)$ (d) None of these

2. The value of $\lim_{n\to\infty}\frac{1}{n}\sum_{r-1}^{n}\left(\frac{r}{n+r}\right)$ is equal to

(a) 1– log 2 (b) log 4 – 1 (c) log 2 (d) None of these

3. The value of $\lim_{n\to\infty}\frac{1}{n}\{(n+1)(n+2)(n+3)\ldots(n+n)\}^{1/n}$ is equal to

(a) 4e (b) $\frac{e}{4}$ (c) $\frac{4}{e}$ (d) None of these

4. If $m,n\in N$, then the value of $\int_a^b(x-a)^m(b-x)^n\,dx$ is equal to

(a) $\frac{(b-a)^{m+n}\cdot m!n!}{(m+n)!}$ (b) $\frac{(b-a)^{m+n+1}\cdot m!n!}{(m+n+1)!}$

(c) $\frac{(b-a)^m\cdot m!}{m!}$ (d) None of these

5. The value of $\lim_{n\to\infty}\left(\frac{n!}{n^n}\right)^{\frac{2n^4+1}{5n^5+1}}$ is equal to

(a) e (b) $\frac{2}{e}$ (c) $\left(\frac{1}{e}\right)^{\frac{2}{5}}$ (d) None of these

6. The value of $\lim_{n\to\infty}n\left\{\frac{1}{3n^2+8n+4}+\frac{1}{3n^2+16n+16}+\ldots+\text{to }n\text{ terms}\right\}$ is equal to

(a) $\frac{1}{2}\log\left(\frac{9}{5}\right)$ (b) $\frac{1}{3}\log\left(\frac{9}{5}\right)$

(c) $\frac{1}{4}\log\left(\frac{9}{5}\right)$ (d) None of these

7. The value of $\lim_{n\to\infty}\frac{1}{n^2}\left\{\sin^3\frac{\pi}{4n}+2\sin^3\frac{2\pi}{4n}+\ldots+n\sin^3\frac{n\pi}{4n}\right\}$ is equal to

(a) $\frac{\sqrt{2}}{9\pi^2}(52-15n)$ (b) $\frac{2}{9\pi^2}(52-15n)$

(c) $\frac{1}{9\pi^2}(15n-15)$ (d) None of these

8. The value of $f(k)=\int_0^{\pi/2}\log(\sin^2\theta+k^2\cos^2\theta)\,d\theta$ is equal to

(a) $\pi\log(1+K)-\pi\log 2$ (b) $\pi\log 2-\log(1+K)$ (c) $\log(1+K)-\pi\log 2$ (d) None of these

9. If $m,n\in N$, then $I_{m,n}=\int_0^1 x^m(1-x)^n\,dx$ is equal to

(a) $\frac{m!n!}{(m+n+2)!}$ (b) $\frac{2m!n!}{(m+n+1)!}$ (c) $\frac{m!n!}{(m+n+1)!}$ (d) None of these

10. The value of $I(n)=\int_0^{\pi}\frac{\sin^2 n\theta}{\sin^2\theta}d\theta$ is $(\forall\, n\in N)$

(a) $n\pi$ (b) $\frac{n\pi}{2}$ (c) $\frac{n\pi}{4}$ (d) None of these

JEE Type Solved Examples :
Single Option Correct Type Questions

● **Ex. 1** *If* $\int_0^\infty f(x)dx = \frac{\pi}{2}$ *and* $f(x)$ *is an even function, then* $\int_0^\infty f\left(x-\frac{1}{x}\right)dx$ *is equal to*

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) π (d) None of these

Sol. Here, $\int_0^\infty f(x)dx = \frac{\pi}{2}$

Put $x = t - \frac{1}{t}$

$\Rightarrow \quad dx = \left(1+\frac{1}{t^2}\right)dt$

$\therefore \quad \int_0^\infty f(x)dx = \int_1^\infty f\left(t-\frac{1}{t}\right)\cdot\left(1+\frac{1}{t^2}\right)dt$

$= \int_1^\infty f\left(t-\frac{1}{t}\right)dt + \int_1^\infty f\left(t-\frac{1}{t}\right)\cdot\frac{1}{t^2}dt$

$= \int_1^\infty f\left(t-\frac{1}{t}\right)dt + \int_1^0 f\left(\frac{1}{y}-y\right)\cdot(-dy) \quad \left(\text{put } t=\frac{1}{y}\right)$

$= \int_1^\infty f\left(t-\frac{1}{t}\right)dt - \int_1^0 f\left(y-\frac{1}{y}\right)dy \quad$ as $f(x)$ is even

$= \int_1^\infty f\left(t-\frac{1}{t}\right)dt + \int_0^1 f\left(t-\frac{1}{t}\right)dt = \int_0^\infty f\left(t-\frac{1}{t}\right)dt$

$\therefore \quad \int_0^\infty f\left(x-\frac{1}{x}\right)dx = \int_0^\infty f(x)dx = \frac{\pi}{2}$

● **Ex. 2** *The value of* $\int_0^1 \left(\prod_{r=1}^{n}(x+r)\right)\left(\sum_{k=1}^{n}\frac{1}{x+k}\right)dx$ *equals to*

(a) n (b) $n!$ (c) $(n+1)!$ (d) $n\cdot n!$

Sol. The given integrand is perfect coefficient of $\prod_{r=1}^{n}(x+r)$

$\therefore \quad I = \left[\prod_{r=1}^{n}(x+r)\right]_0^1 = (n+1)! - n! = n\cdot n!$

Aliter $(x+1)(x+2)(x+3)\ldots(x+n) = e^t$

So that, when $x=0$, then $t = \ln n!$ when $x=1$, then $t = \ln(n+1)!$

$[\ln(x+1) + \ln(x+2) + \ldots\ldots + \ln(x+n)] = t$

$\therefore \quad \left(\frac{1}{(x+1)}+\frac{1}{(x+2)}+\ldots+\frac{1}{(x+n)}\right)dx = dt$

Therefore, $I = \int_{\ln n!}^{\ln(n+1)!} e^t dt = [e^t]_{\ln n!}^{\ln(n+1)!}$

$= e^{\ln(n+1)!} - e^{\ln n!} = (n+1)! - n! = n\cdot n!$

Hence, (d) is the correct answer.

● **Ex. 3** *The true set of values of 'a' for which the inequality* $\int_a^0 (3^{-2x} - 2\cdot 3^{-x})\,dx \geq 0$ *is true, is*

(a) $[0, 1]$ (b) $(-\infty, -1]$ (c) $[0, \infty)$ (d) $(-\infty, -1]\cup[0, \infty)$

Sol. We have, $\int_a^0 3^{-x}(3^{-x}-2)\,dx \geq 0$

Put $3^{-x} = t \Rightarrow 3^{-x}\ln 3\,dx = -dt$

$\Rightarrow \quad \ln 3\int_1^{3^{-a}}(t-2)dt \geq 0 \;= \left[\frac{t^2}{2}-2t\right]_1^{3^{-a}} \geq 0$

$\Rightarrow \quad \left(\frac{3^{-2a}}{2}-2\cdot 3^{-a}\right)-\left(\frac{1}{2}-2\right)\geq 0$

$\Rightarrow \quad 3^{-2a} - 4\times 3^{-a} + 3 > 0 \;\Rightarrow\; (3^{-a}-3)(3^{-a}-1) > 0$

$\Rightarrow \quad 3^{-a} > 3^1 \Rightarrow a < 1 \;$ or $\; 3^{-a} < 3^0 \Rightarrow a > 0$

Thus, $a \in (-\infty, -1]\cup[0, \infty)$

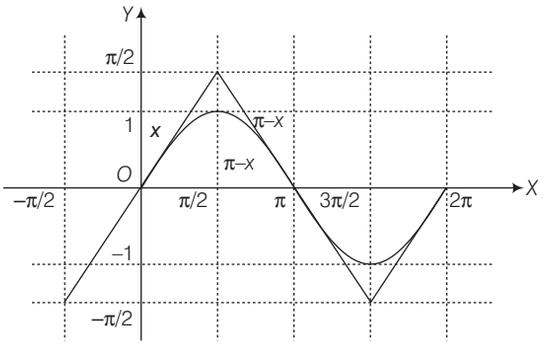
Hence, (d) is the correct answer.

● **Ex. 4** *The value of the definite integral* $\int_0^{2n\pi} \max(\sin x, \sin^{-1}(\sin x))dx$ *equals to (where,* $n \in I$*)*

(a) $\frac{n(\pi^2-4)}{2}$ (b) $\frac{n(\pi^2-4)}{4}$

(c) $\frac{n(\pi^2-8)}{4}$ (d) $\frac{n(\pi^2-2)}{4}$

Sol. Let $I = \int_0^{2n\pi} \max(\sin x, \sin^{-1}(\sin x))dx$



$= n\left[\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi}(\pi - x)dx + \int_{\pi}^{2\pi}(\sin x)\,dx\right]$

$= n\left[\frac{\pi^2}{8}+\frac{\pi^2}{2}-\frac{1}{2}\left(\pi^2-\frac{\pi^2}{4}\right)-2\right]$

$= n\left[\frac{\pi^2}{8}+\frac{\pi^2}{2}-\frac{3\pi^2}{8}-2\right] = \frac{n(\pi^2-8)}{4}$

Hence, (c) is the correct answer.

● **Ex. 5** $\lim_{n\to\infty} n^{-\frac{1}{2}\left(1+\frac{1}{n}\right)}\cdot(1^1\cdot 2^2\cdot 3^3.....n^n)^{\frac{1}{n^2}}$ *is equal to*

(a) $\sqrt{e}$ (b) $\frac{1}{\sqrt{e}}$ (c) $\frac{1}{\sqrt[4]{e}}$ (d) $\sqrt[4]{e}$

Sol. Let $L=\lim_{n\to\infty} n^{-\frac{1}{2}\left(1+\frac{1}{n}\right)}\cdot(1^1\cdot 2^2\cdot 3^3...n^n)^{\frac{1}{n^2}}$

$$\therefore \quad \ln L=\lim_{n\to\infty}-\frac{1}{2}\left(\frac{n+1}{n}\right)\ln n+\frac{1}{n^2}\sum_{k=1}^{n} k\ln k$$

$$=\lim_{n\to\infty}-\frac{1}{2}\left(\frac{n+1}{n}\right)\ln n+\frac{1}{n^2}\sum_{k=1}^{n}(k\ln k-k\ln n+k\ln n)$$

$$=\lim_{n\to\infty}-\frac{1}{2}\left(\frac{n+1}{n}\right)\ln n+\frac{1}{n^2}\sum_{k=1}^{n} k\ln\frac{k}{n}+\frac{\ln n}{n^2}\sum_{k=1}^{n} k$$

$$=\lim_{n\to\infty}-\frac{1}{2}\left(\frac{n+1}{n}\right)\ln n+\frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\ln\frac{k}{n}+\frac{\ln n}{n^2}\cdot\frac{n(n+1)}{2}$$

$$=-\frac{1}{2}\left(\frac{n+1}{n}\right)\ln n+\int_0^1 x\ln x\,dx+\frac{1}{2}\left(\frac{n+1}{n}\right)\ln n$$

$$=\int_0^1 \underbrace{x}_{\text{II}}\underbrace{\ln x}_{\text{I}}dx=-\frac{1}{4} \quad\Rightarrow\quad L=e^{-\frac{1}{4}}$$

Hence, (c) is the correct answer.

● **Ex. 6** $\lim_{x\to\frac{\pi}{4}}\frac{\int_2^{\sec^2 x} f(t)\,dt}{x^2-\frac{\pi^2}{16}}$ *is equal to* ***[IIT JEE 2007]***

(a) $\frac{8}{\pi}f(2)$ (b) $\frac{2}{\pi}f(2)$ (c) $\frac{2}{\pi}f\left(\frac{1}{2}\right)$ (d) $4f(2)$

Sol. $\lim_{x\to\frac{\pi}{4}}\frac{\int_2^{\sec^2 x} f(t)\,dt}{x^2-\frac{\pi^2}{16}}=\lim_{x\to\frac{\pi}{4}}\frac{f(\sec^2 x)\cdot 2\sec^2 x\tan x-0}{2x}$ (applying L'Hospital rule)

$$=\frac{f(2)\cdot 4}{\pi/2}=\frac{8f(2)}{\pi}$$

Hence, (a) is the correct answer.

● **Ex. 7** *Let f be a non-negative function defined on the interval* $[0,1]$. *If* $\int_0^x\sqrt{1-(f'(t))^2}\,dx=\int_0^x f(t)\,dt, 0\le x\le 1$ *and* $f(0)=0$, *then* ***[IIT JEE 2009]***

(a) $f\left(\frac{1}{2}\right)<\frac{1}{2}$ and $f\left(\frac{1}{3}\right)>\frac{1}{3}$ (b) $f\left(\frac{1}{2}\right)>\frac{1}{2}$ and $f\left(\frac{1}{3}\right)>\frac{1}{3}$

(c) $f\left(\frac{1}{2}\right)<\frac{1}{2}$ and $f\left(\frac{1}{3}\right)<\frac{1}{3}$ (d) $f\left(\frac{1}{2}\right)>\frac{1}{2}$ and $f\left(\frac{1}{3}\right)<\frac{1}{3}$

Sol. Given, $\int_0^x\sqrt{1-(f'(t))^2}dt=\int_0^x f(t)\,dt, 0\le x\le 1$

Applying Leibnitz theorem, we get

$$\sqrt{1-(f'(x))^2}=f(x)$$

$$\Rightarrow \quad 1-(f'(x))^2=f^2(x)$$

$$\Rightarrow \quad (f'(x))^2=1-f^2(x) \Rightarrow f'(x)=\pm\sqrt{1-f^2(x)}$$

$$\Rightarrow \quad \frac{dy}{dx}=\pm\sqrt{1-y^2}, \text{ where } y=f(x)$$

$$\Rightarrow \quad \frac{dy}{\sqrt{1-y^2}}=\pm dx$$

Integrating both the sides, we get $\sin^{-1}(y)=\pm x+C$

$$\therefore \quad f(0)=0 \Rightarrow C=0$$

$$\therefore \quad y=\pm\sin x$$

$$\Rightarrow \quad y=\sin x=f(x), \text{ given } f(x)\ge 0 \text{ for } x\in[0,1]$$

Its known that, $\sin x<x, \forall x\in R^+$

$$\sin\left(\frac{1}{2}\right)<\frac{1}{2} \quad\Rightarrow\quad f\left(\frac{1}{2}\right)<\frac{1}{2} \text{ and } \sin\left(\frac{1}{3}\right)<\frac{1}{3}$$

$$\Rightarrow \quad f\left(\frac{1}{3}\right)<\frac{1}{3}$$

Hence, (c) is the correct answer.

● **Ex. 8** *The value of* $\lim_{x\to 0}\frac{1}{x^3}\int_0^x\frac{t\ln(1+t)}{t^4+4}dt$ ***[IIT JEE 2010]***

(a) 0 (b) $\frac{1}{12}$ (c) $\frac{1}{24}$ (d) $\frac{1}{64}$

Sol. $\lim_{x\to 0}\frac{1}{x^3}\int_0^x\frac{t\log(1+t)}{4+t^4}dt$

Using L'Hospital's rule,

$$\lim_{x\to 0}\frac{\frac{x\log(1+x)}{4+x^4}}{3x^2}=\lim_{x\to 0}\frac{\log(1+x)}{3x}\cdot\frac{1}{4+x^4}=\frac{1}{12}$$

$$\left[\text{using } \lim_{x\to 0}\frac{\log(1+x)}{x}=1\right]$$

Hence, (b) is the correct answer.

● **Ex. 9** *The value(s) of* $\int_0^1\frac{x^4(1-x)^4}{1+x^2}dx$ *is (are)* ***[IIT JEE 2010]***

(a) $\frac{22}{7}-\pi$ (b) $\frac{2}{105}$ (c) 0 (d) $\frac{71}{15}-\frac{3\pi}{2}$

Sol. Let $I=\int_0^1\frac{x^4(1-x)^4}{1+x^2}dx$

$$=\int_0^1\frac{(x^4-1)(1-x)^4+(1-x)^4}{(1+x^2)}dx$$

$$=\int_0^1(x^2-1)(1-x)^4dx+\int_0^1\frac{(1+x^2-2x)^2}{(1+x^2)}dx$$

$$=\int_0^1\left\{(x^2-1)(1-x)^4+(1+x^2)-4x+\frac{4x^2}{(1+x^2)}\right\}dx$$

$$=\int_0^1\left((x^2-1)(1-x)^4+(1+x^2)-4x+4-\frac{4}{1-x^2}\right)dx$$

$$= \int_0^1 \left(x^6 - 4x^5 + 5x^4 + 4 - \frac{4}{1+x^2} \right) dx$$

$$= \frac{1}{7} - \frac{4}{6} + \frac{5}{5} - \frac{4}{3} + 4 - 4\left(\frac{\pi}{4} - 0 \right) = \frac{22}{7} - \pi$$

Hence, (a) is the correct answer.

● ***Ex. 10*** *If* $\int_{\sin\theta}^{\cos\theta} f(x\tan\theta)dx \left(\theta \neq \frac{n\pi}{2}, n \in I \right)$ *is equal to*

(a) $-\cos\theta \int_1^{\tan\theta} f(x\sin\theta)dx$

(b) $-\tan\theta \int_{\cos\theta}^{\sin\theta} f(x)dx$

(c) $\sin\theta \int_1^{\tan\theta} f(x\cos\theta)dx$

(d) $\frac{1}{\tan\theta} \int_{\sin\theta}^{\sin\theta\tan\theta} f(x)dx$

Sol. Let, $I = \int_{\sin\theta}^{\cos\theta} f(x\tan\theta)dx$. Put $x\tan\theta = z\sin\theta$

$\therefore \quad I = \int_{\tan\theta}^{1} f(z\sin\theta)\cos\theta\, dz$

$\Rightarrow \quad dx = \cos\theta\, dz$

$$= -\cos\theta \int_1^{\tan\theta} f(z\sin\theta)\, dz = -\cos\theta \int_1^{\tan\theta} f(x\sin\theta)\, dx$$

Hence, (a) is correct option.

JEE Type Solved Examples :
More than One Correct Option Type Questions

● ***Ex. 11*** *Let* $f(x)$ *be an even function which is mapped from* $(-\pi, \pi)$*. Then, the value of* $\int_{-\pi}^{\pi} \left(\int_0^x f(t)\, dt + [f(x)] \right) dx$ *can be (where,* $[\cdot]$ *denotes greatest integer function)*

(a) 0 (b) π

(c) 2π (d) 4π

Sol. As, $f(x)$ is an even function, then $\int_0^x f(x)\, dx$ is an odd function.

Also, $[f(x)] = 1, 2 \qquad [\text{as } 1 < f(x) < 3]$

$\therefore \quad g(x) = \int_{-\pi}^{\pi} \left(\int_0^x f(t)\, dt \right) dx + \int_{-\pi}^{\pi} [f(x)]\, dx$

$$= 0 + \int_{-\pi}^{\pi} [f(x)]\, dx, \qquad [\because \text{since } [f(x)] = 1 \text{ or } 2]$$

$$= \int_{-\pi}^{\pi} 1\, dx$$

or $\int_{-\pi}^{\pi} 2\, dx = 2\pi$ or 4π

Hence, options (c) and (d) are correct.

● ***Ex. 12*** *Let,* $A_1 = \int_n^{n+1} (\min\{|x-n|, |x-(n+1)|\})\, dx$,

$$A_2 = \int_{n+1}^{n+2} (|x-n| - |x-(n+1)|)\, dx,$$

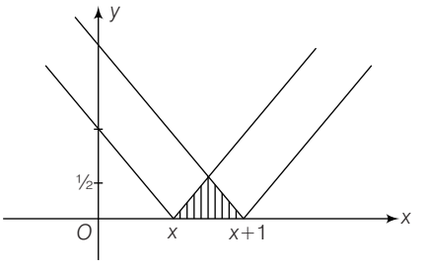
$$A_3 = \int_{n+2}^{n+3} (|x-(n+4)| - |x-(n+3)|)\, dx$$

and $g(x) = A_1 + A_2 + A_3$, then

(a) $A_1 + A_2 + A_3 = 9$ (b) $A_1 + A_2 + A_3 = \frac{9}{4}$

(c) $\sum_{n=1}^{100} g(x) = \frac{900}{4}$ (d) $\sum_{n=1}^{100} g(x) = 300$

Sol. Here, $\min\{|x-n|, |x-(n+1)|\}$ can be shown as



$\therefore \quad A_1 = \int_n^{n+1} (\min\{|x-n|, |x-(n+1)|\})\, dx$

$$= \frac{1}{2} \times 1 \times \frac{1}{2} = \frac{1}{4}$$

Now, $A_2 = \int_{n+1}^{n+2} (|x-n| - |x-(n+1)|)\, dx$

$$= \int_1^2 (|t| - |t-1|)dt \qquad [\text{put } x = n+t \Rightarrow dx = dt]$$

$$= \int_1^2 (t - (t-1))dt = \int_1^2 1dt = (t)_1^2 = 1$$

and $A_3 = \int_{n+2}^{n+3} (|x-(n+4)| - |x-(n+3)|)dx,$

$$= \int_2^3 (|t-4| - |t-3|)dt \qquad [\text{put } x = n+t \Rightarrow dx = dt]$$

$$= \int_2^3 ((4-t) - (3-t))dt = \int_2^3 1\, dt = 1$$

Also, $g(x) = A_1 + A_2 + A_3 = \frac{1}{4} + 1 + 1 = \frac{9}{4}$

$\therefore \quad \sum_{n=1}^{100} g(x) = g(1) + g(2) + g(3) + \ldots + g(100)$

$$= \frac{9}{4} + \frac{9}{4} + \ldots + \frac{9}{4} = \frac{900}{4}$$

JEE Type Solved Examples :
Statement I and II Type Questions

■ **Directions** (Q. Nos. 13 to 15) *For the following questions, choose the correct answers from the codes (a), (b), (c) and (d) defined as follows.*

(a) Statement I is true, Statement II is also true; Statement II is the correct explanation of Statement I.

(b) Statement I is true, Statement II is also true; Statement II is not the correct explanation of Statement I.

(c) Statement I is true, Statement II is false.

(d) Statement I is false, Statement II is true.

● ***Ex. 13*** **Statement I** *If* $f(x)=\int_0^1 (x f(t)+1)dt$, *then* $\int_0^3 f(x)dx=12$

Statement II $f(x)=3x+1$

Sol. Let $\int_0^1 f(t)dt=k$, so $f(x)=xk+1$

Now, $\int_0^1 (kt+1)dt=k \Rightarrow \frac{k}{2}+1=k$, so $k=2$

$\therefore \quad f(x)=2x+1$

Also $\int_0^3 f(x)dx=12$

Hence, (c) is the correct answer.

● ***Ex. 14*** **Statement I** *The function* $f(x)=\int_0^x \sqrt{1+t^2}\,dt$ *is an odd function and* $g(x)=f'(x)$ *is an even function.*

Statement II *For a differentiable function* $f(x)$ *if* $f'(x)$ *is an even function, then* $f(x)$ *is an odd function.*

Sol. If $f(x)$ is of odd, then $f'(x)$ is even but converse is not true.

e.g. If $f'(x)=x\sin x$, then $f(x)=\sin x - x\cos x + C$

$$f(-x)=-\sin x + x\cos x + C$$

On adding, $f(x)+f(-x)=$ constant which need not to be zero.

For Statement I $f(x)=\int_0^x \sqrt{1+t^2}dt;\ g(x)=\sqrt{1+x^2}$

$$f(-x)=\int_0^{-x}\sqrt{1+t^2}dt;\ t=-y$$

$$f(-x)=-\int_0^x \sqrt{1+y^2}dy$$

$\therefore \quad f(x)+f(-x)=0$

$\Rightarrow$ f is odd and g is obviously even.

Hence, (c) is the correct answer.

● ***Ex. 15*** *Given,* $f(x)=\sin^3 x$ *and* $P(x)$ *is a quadratic polynomial with leading coefficient unity.*

Statement I $\int_0^{2\pi} P(x)\cdot f'''(x)dx$ *vanishes.*

Statement II $\int_0^{2\pi} f(x)dx$ *vanishes.*

Sol. $P(x)=ax^2+bx+c;\ f(x)=\sin^3 x$

$$I=\int_0^{2\pi} \underbrace{P(x)}_{\text{I}}\cdot \underbrace{f'''(x)}_{\text{II}}dx$$

Using I.B.P. $\underbrace{P(x)\cdot f''(x)\big|_0^{2\pi}}_{\text{zero}} - \int_0^{2\pi} \underbrace{P'(x)}_{\text{I}}\cdot \underbrace{f''(x)}_{\text{II}}dx$

$$=-\left[[P'(x)\cdot f(x)]_0^{2\pi}-\int_0^{2\pi}P''(x)\cdot f(x)dx\right]$$

$$=\int_0^{2\pi}P''(x)\cdot f(x)dx=2\int_0^{2\pi}\sin^3 x\,dx=0$$

Hence, (a) is the correct answer.

JEE Type Solved Examples :
Passage Based Questions

Passage I

(Q. Nos. 16 to 18)

Suppose we define the definite integral using the following formula $\int_a^b f(x)dx=\frac{b-a}{2}(f(a)+f(b))$, *for more accurate result for* $c\in(a,b)$, $F(c)=\frac{c-a}{2}(f(a)+f(c))+\frac{b-c}{2}(f(b)+f(c))$.

When $c=\frac{a+b}{2}$, then $\int_a^b f(x)dx=\frac{b-a}{4}(f(a)+f(b)+2f(c))$.

[IIT JEE 2007]

● ***Ex. 16*** *$\int_0^{\pi/2} \sin x\,dx$ is equal to*

(a) $\frac{\pi}{8}(1+\sqrt{2})$ (b) $\frac{\pi}{4}(1+\sqrt{2})$

(c) $\frac{\pi}{8\sqrt{2}}$ (d) $\frac{\pi}{4\sqrt{2}}$

Sol. $\int_0^{\pi/2}\sin x\,dx = \frac{\frac{\pi}{2}-0}{4}\left(\sin(0)+\sin\left(\frac{\pi}{2}\right)+2\sin\left(\frac{0+\frac{\pi}{2}}{2}\right)\right)=\frac{\pi}{8}(1+\sqrt{2})$

Hence, (a) is the correct answer.

● ***Ex. 17*** *If $f(x)$ is a polynomial and if $\lim_{t\to a}\frac{\int_a^t f(x)dx-\frac{(t-a)}{2}(f(t)+f(a))}{(t-a)^3}=0$ for all a, then the degree of $f(x)$ can atmost be*

(a) 1 (b) 2

(c) 3 (d) 4

Sol. Applying L' Hospital's rule,

$$\lim_{t\to a}\frac{f(t)-\frac{1}{2}(f(t)+f(a))-\frac{(t-a)}{2}f'(t)}{3(t-a)^2}=0$$

$$\Rightarrow \lim_{t\to a}\frac{f(t)-\frac{1}{2}(f(t)+f(a))-\frac{(t-a)}{2}f'(t)}{3(t-a)^2}=0$$

$$\Rightarrow \lim_{t\to a}\frac{f'(t)-\frac{1}{2}(f(t)+f(a))-\frac{(t-a)}{2}f'(t)}{12(t-a)}=0$$

$$\Rightarrow \lim_{t\to a}\frac{f''(t)}{12}=0$$

$$\Rightarrow \lim_{t\to a}\frac{f'(t)-\frac{1}{2}(f(t)+f(a))-\frac{(t-a)}{2}f'(t)}{12(t-a)}=0$$

$\Rightarrow \lim_{t\to a}\frac{f''(t)}{12}=0$ $f''(a)=0$ for any a.

$\Rightarrow$ $f(a)$ is atmost of degree 1.

Hence, (a) is the correct answer.

● ***Ex. 18*** *If $f''(x)<0, \forall\, x\in(a,b)$ and c is a point such that $a<c<b$ and $(c,f(c))$ is the point on the curve for which $F(c)$ is maximum, then $f'(c)$ is equal to*

(a) $\frac{f(b)-f(a)}{b-a}$ (b) $\frac{2(f(b)-f(a))}{b-a}$

(c) $\frac{2f(b)-f(a)}{2b-a}$ (d) 0

Sol. $F'(c)=(b-a)'(c)+f(a)-f(b)$

$F''(c)=f''(c)(b-a)<0$

$\Rightarrow$ $F'(c)=0$

$\Rightarrow$ $f'(c)=\frac{f(b)-f(a)}{b-a}$

Hence, (a) is the correct answer.

Passage II

(Q. Nos. 19 to 20)

Let $f(\alpha,\beta)=\begin{vmatrix}\cos(\alpha+\beta) & -\sin(\alpha+\beta) & \cos 2\beta\\ \sin\alpha & \cos\alpha & \sin\beta\\ -\cos\alpha & \sin\alpha & \cos\beta\end{vmatrix}$.

● ***Ex. 19*** *The value of $I=\int_0^{\pi/2} e^{\beta}\left(f(0,0)+f\left(\frac{\pi}{2},\beta\right)+f\left(\frac{3\pi}{2},\frac{\pi}{2}-\beta\right)\right)d\beta$ is*

(a) $e^{\pi/2}$ (b) 0

(c) $2(2e^{\pi/2}-1)$ (d) None of these

Sol. Here, $f(\alpha,\beta)=2\cos\beta$ i.e. independent of α

$$\therefore\quad I=\int_0^{\pi/2} e^{\beta}\left(f(0,0)+f\left(\frac{\pi}{2},\pi\right)+f\left(\frac{3\pi}{2},\frac{\pi}{2}-\beta\right)\right)d\beta$$

$$=\int_0^{\pi/2} e^{\beta}(2+2\cos\beta+2\sin\beta)d\beta$$

Hence, $=[e^{\beta}(2+2\sin\beta)]_0^{\pi/2}$

$=4e^{\pi/2}-2=2(2e^{\pi/2}-1)$

Hence, (c) is the correct answer.

● ***Ex. 20*** *If $I=\int_{-\pi/2}^{\pi/2}\cos^2\beta\left(f(0,\beta)+f\left(0,\frac{\pi}{2}-\beta\right)\right)d\beta$, then $[I]$ is*

(a) $e^{\pi/2}$ (b) 2

(c) $2(2e^{\pi/2}-1)$ (d) None of these

Sol. Again, $I=\int_{-\pi/2}^{\pi/2}\cos^2\beta\left(f(0,\beta)+f\left(0,\frac{\pi}{2}-\beta\right)\right)d\beta$

$$=\int_{-\pi/2}^{\pi/2}\cos^2\beta(\cos\beta+\sin\beta)\,d\beta$$

$$=\int_{-\pi/2}^{\pi/2}\cos^3\beta\,d\beta+\int_{-\pi/2}^{\pi/2}\cos^2\beta\sin\beta\,d\beta$$

$$=2\int_0^{\pi/2}\cos^3\beta\,d\beta+0$$

$\Rightarrow$ $I=\frac{\theta}{3}\Rightarrow[I]=2$

Hence, (b) is the correct answer.

JEE Type Solved Examples :
Matching Type Questions

● **Ex. 21** *Match the following.*

	Column I	Column II
(A)	The function $f(x)=\dfrac{e^{x\cos x}-1-x}{\sin x^2}$ is not defined at $x=0$. The value of $f(0)$, so that f is continuous at $x=0$, is	(p) -1
(B)	The value of the definite integral $\int_0^1 \dfrac{dx}{\sqrt{x}+\sqrt[3]{x}}$ is equal to $a+b\ln 2$, where a and b are integers, then $(a+b)$ is equal to	(q) 0
(C)	Given, $e^n\int_0^n \dfrac{\sec^2\theta-\tan\theta}{e^\theta}\,d\theta=1$, then the value of tan n is equal to	(r) $1/2$
(D)	Let $a_n=\int_{\frac{n}{n+1}}^{1}\tan^{-1}(nx)\,dx$ and $b_n=\int_{\frac{n}{n+1}}^{1}\sin^{-1}(nx)\,dx$, then $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n}$ has the value equal to	(s) 1

Sol. (A) → (r), (B) → (p), (C) → (s), (D) → (r)

(A) $\lim\limits_{x\to 0}\dfrac{e^x\cos x-1-x}{x^2\cdot\dfrac{\sin x^2}{x^2}}=\lim\limits_{x\to 0}\dfrac{e^{x\cos x}-1-x}{x^2}$

(applying L'Hospital's twice)

$=\lim\limits_{x\to 0}\dfrac{e^{x\cos x}\cdot(-x\sin x+\cos x)-1}{2x}$

$=\lim\limits_{x\to 0}\dfrac{e^{x\cos x}(-x\cos x-\sin x-\sin x)+e^{x\cos x}(-x\sin x+\cos x)}{2}$

$=\dfrac{1}{2}$

(B) Put $x=u^6\Rightarrow dx=6u^5\,du$

$\therefore\quad I=\int_0^1\dfrac{6u^5\,du}{u^3+u^2}=6\int_0^1\dfrac{u^3+1-1}{u+1}\,du=5-6\ln 2$

$\Rightarrow a+b=5-6=-1$

(C) $e^n\int_0^n e^{-\theta}(\sec^2\theta-\tan\theta)\,d\theta=1$

Put $-\theta=t\Rightarrow d\theta=-dt$

$-e^n\int_0^{-n}e^t(\sec^2 t+\tan t)\,dt=1$

[use $\int e^x(f(x)+f'(x))=e^x f(x)$]

$\Rightarrow\ -e^n[e^t\tan t]_0^{-n}=1\Rightarrow -e^n[-e^{-n}\tan n]=1\Rightarrow\tan n=+1$

(D) $\lim\limits_{n\to\infty}\dfrac{\int_{1/n+1}^{1/n}\tan^{-1}(nx)\,dx}{\int_{1/n+1}^{1/n}\sin^{-1}(nx)\,dx}$

Put $nx=t\ \Rightarrow\ dx=\dfrac{1}{n}dt$

$\therefore\quad \lim\limits_{n\to\infty}\dfrac{\frac{1}{n}\int_{n/n+1}^{1}\tan^{-1}(t)\,dt}{\frac{1}{n}\int_{n/n+1}^{1}\sin^{-1}(t)\,dt}$ $\left(\dfrac{0}{0}\text{ form}\right)$

Use L'Hospital's rule, $\lim\limits_{n\to\infty}\dfrac{\tan^{-1}\left(\frac{n}{n+1}\right)}{\sin^{-1}\left(\frac{n}{n+1}\right)}=\dfrac{\frac{\pi}{4}}{\frac{\pi}{2}}=\dfrac{1}{2}$

● **Ex. 22** *Match the following* ***[IIT JEE 2006]***

	Column I	Column II
(A)	$\int_{-1}^{1}\dfrac{dx}{1+x^2}$	(p) $\dfrac{1}{2}\log\left(\dfrac{2}{3}\right)$
(B)	$\int_0^1\dfrac{dx}{\sqrt{1-x^2}}$	(q) $\dfrac{1}{2}\log\left(\dfrac{3}{2}\right)$
(C)	$\int_2^3\dfrac{dx}{1-x^2}$	(r) $\dfrac{\pi}{3}$
(D)	$\int_1^2\dfrac{dx}{x\sqrt{x^2-1}}$	(s) $\dfrac{\pi}{2}$

Sol. (A) → (s), (B) → (s), (C) → (q), (D) → (r)

(A) $\int_{-1}^{1}\dfrac{dx}{1+x^2}$

$\because\quad f(x)=\dfrac{1}{1+x^2}$ is an reven function.

$\therefore\quad I=2\int_0^1\dfrac{dx}{1+x^2}=[2(\tan^{-1}x)]_0^1=\dfrac{\pi}{2}$

(B) $\int_0^1\dfrac{dx}{\sqrt{1-x^2}}=[\sin^{-1}x]_0^1=\dfrac{\pi}{2}$

(C) $\int_2^3\dfrac{dx}{1-x^2}=-\int_2^3\dfrac{dx}{x^2-1}=-\left[\dfrac{1}{2}\ln\left|\dfrac{x-1}{x+1}\right|\right]_2^3$

$=-\dfrac{1}{2}\ln\dfrac{2}{3}=\dfrac{1}{2}\ln\left(\dfrac{3}{2}\right)$

(D) Let $I=\int_1^2\dfrac{dx}{x\sqrt{x^2-1}}$ [put $x=\sec\theta\ \Rightarrow\ dx=\sec\theta\tan\theta\,d\theta$]

$=\int_0^{\pi/3}\dfrac{\sec\theta\tan\theta\,d\theta}{\sec\theta\tan\theta}$

$=\int_0^{\pi/3}1\,d\theta=\dfrac{\pi}{3}$

JEE Type Solved Examples :
Single Integer Answer Type Questions

● **Ex. 23** *Let $f:R \to R$ be a continuous function which satisfies $f(x)=\int_0^x f(t)\,dt$. Then, the value of $f(\ln 5)$ is____.*

[IIT JEE 2009]

Sol. **(0)** From given integral equation, $f(0)=0$

Also, differentiating the given integral equation w.r.t. x, we get

$$f'(x)=f(x)$$

If $f(x)\neq 0$, then $\dfrac{f'(x)}{f(x)}=1 \Rightarrow f(x)=e^c e^x$

$\because$ $f(0)=0 \Rightarrow e^c=0$, a contradiction

$\therefore$ $f(x)=0, \forall\, x\in R \Rightarrow f(\ln 5)=0$

● **Ex. 24** *For any real number x, let $[x]$ denotes the largest integer less than or equal to x. Let f be a real valued function defined on the interval $[-10,10]$ by*

$$f(x)=\begin{cases} x-[x], & \text{if } [x] \text{ is odd} \\ 1+[x]-x, & \text{if } [x] \text{ is even} \end{cases}.$$

Then, the value of $\dfrac{\pi^2}{10}\int_{-10}^{10} f(x)\cos \pi x\,dx$ is.

[IIT JEE 2010]

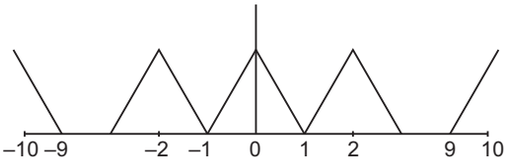
Sol. **(4)** We have, $f(x)=\begin{cases} x-[x], & \text{if } [x] \text{ is odd} \\ 1+[x]-x, & \text{if } [x] \text{ is even} \end{cases}$

$f(x)$ and $\cos \pi x$ are both periodic with period 2 and are both even.

$$\therefore \int_{-10}^{10} f(x)\cos \pi x\,dx = 2\int_0^{10} f(x)\cos \pi x\,dx$$

$$= 10\int_0^2 f(x)\cos \pi x\,dx$$

$$\int_0^1 f(x)\cos \pi x\,dx = \int_0^1 (1-x)\cos \pi x\,dx = -\int_0^1 u\cos \pi u\,du$$



$$\int_1^2 f(x)\cos \pi x\,dx = \int_1^2 (x-1)\cos \pi x\,dx = -\int_0^1 u\cos \pi u\,du$$

$$\therefore \int_{-10}^{10} f(x)\cos \pi x\,dx = -20\int_0^1 u\cos \pi u\,du = \frac{40}{\pi^2}$$

$$\Rightarrow \frac{\pi^2}{10}\int_{-10}^{10} f(x)\cos \pi x\,dx = 4$$

● **Ex. 25** *Let $f(x)$ be a differentiable function satisfying $f(x)+f\left(x+\dfrac{1}{2}\right)=1, \forall x\in R$ and $g(x)=\int_0^x f(t)dt$. If $g(1)=1$, then the value of $\sum_{n=2}^{\infty}\left(\dfrac{8}{\sum_{k=1}^{n}(g(x+k^2)-g(x+k))}\right)$ is.*

Sol. **(6)** Here, $f(x)+f\left(x+\dfrac{1}{2}\right)=1$

Replace x by $\left(x+\dfrac{1}{2}\right)$, we get $f\left(x+\dfrac{1}{2}\right)+f(x+1)=1$

On subtracting, $f(x)=f(x+1)$ ...(i)

Also, $g(x+1^2)=\int_0^{x+1^2} f(t)\,dt=\int_0^x f(t)\,dt+\int_n^{x+1^2} f(t)\,dt$

Since, $f(x+1)=f(x)$

$$\therefore \quad g(x+1^2)=\int_0^x f(t)\,dt+1^2\cdot\int_0^1 f(t)dt$$

$$g(x+2^2)=\int_0^x f(t)\,dt+2^2\cdot\int_0^1 f(t)dt$$

$$g(x+k^2)=\int_0^x f(t)\,dt+k^2\cdot\int_0^1 f(t)dt \quad \text{...(ii)}$$

and $g(x+k)=\int_0^x f(t)\,dt+k\cdot\int_0^1 f(t)dt$ ...(iii)

Thus, $\sum_{k=1}^{n}(g(x+k^2)-g(x+k))=\sum_{k=1}^{n}(k^2-k)\cdot\int_0^1 f(t)\,dt$

$$=\sum_{k=1}^{n}(k^2-k)\cdot g(1), \text{ given } g(x)=\int_0^x f(t)\,dt$$

$$=\left(\frac{n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}\right)\times 1$$

$$=\frac{n(n-1)(n+1)}{3}$$

$$\therefore \sum_{n=2}^{\infty}\frac{8}{\sum_{k=2}^{n}(g(x+k^2)-g(x+k))}=\sum_{n=2}^{\infty}\frac{8\times 3}{(n-1)n(n+1)}$$

$$=12\sum_{n=2}^{\infty}\left(\frac{(n+1)-(n-1)}{(n-1)\cdot n(n+1)}=\frac{1}{(n-1)n}-\frac{1}{n(n+1)}\right)$$

$$=12\left[\left(\frac{1}{1\times 2}-\frac{1}{2\times 3}\right)+\left(\frac{1}{2\times 3}-\frac{1}{3\times 4}\right)+\right.$$

$$\left.\ldots+\left(\frac{1}{(n-1)n}-\frac{1}{n(n+1)}\right)\right]$$

$$=12\left(\frac{1}{2}-\frac{1}{n(n+1)}\right)_{n\to\infty}=6$$

Subjective Type Questions

● **Ex. 26** *Find the error in steps to evaluate the following integral.*

$$\int_0^{\pi} \frac{dx}{1+2\sin^2 x} = \int_0^{\pi} \frac{\sec^2 x\,dx}{\sec^2 x + 2\tan^2 x} = \int_0^{\pi} \frac{\sec^2 x\,dx}{1+3\tan^2 x}$$

$$= \frac{1}{\sqrt{3}}\left[\tan^{-1}(\sqrt{3}\tan x)\right]_0^{\pi} = 0$$

Sol. Here, the Newton-Leibnitz's formula for evaluating the definite integrals is not applicable because the anti-derivative,

$$f(x) = \frac{1}{\sqrt{3}}\left[\tan^{-1}(\sqrt{3}\tan x)\right]$$

has a discontinuity at the point $x = \frac{\pi}{2}$ which lies in the interval $[0, \pi]$.

$$\text{LHL} = \lim_{h\to 0} \frac{1}{\sqrt{3}} \tan^{-1}\sqrt{3}\left[\tan\left(\frac{\pi}{2} - h\right)\right] \text{ at } x = \frac{\pi}{2}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{3}} \tan^{-1}(\sqrt{3}\cot h)$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{3}} \tan^{-1}(\infty) = \frac{\pi}{2\sqrt{3}} \quad \text{...(i)}$$

Also, $$\text{RHL} = \lim_{h\to \mathbf{0}} \frac{1}{\sqrt{3}} \tan^{-1}\sqrt{3}\left[\tan\left(\frac{\pi}{2} + h\right)\right] \text{at } x = \frac{\pi}{2}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{3}} \tan^{-1}(-\sqrt{3}\cot h)$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{3}} \tan^{-1}(-\infty) = -\frac{\pi}{2\sqrt{3}} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), LHL ≠ RHL at $x = \pi/2$

$\Rightarrow$ Anti-derivative $f(x)$ is discontinuous at $x = \pi/2$.

So, the correct solution for above integral;

$$I = \int_0^{\pi} \frac{dx}{1+2\sin^2 x} = \int_0^{\pi} \frac{\sec^2 x\,dx}{1+3\tan^2 x} \quad \text{...(iii)}$$

Using, $\int_0^{2a} f(x)\,dx = \begin{cases} 0, & f(2a-x) = -f(x) \\ 2\int_0^a f(x)\,dx, & f(2a-x) = f(x) \end{cases}$

We know, if $f(x) = \frac{\sec^2 x}{1+3\tan^2 x}$ $f(\pi - x) = f(x)$

$\therefore$ Eq. (iii) reduces to $I = 2\int_0^{\pi/2} \frac{\sec^2 x\,dx}{1+3\tan^2 x}$

$$= 2\cdot\frac{1}{\sqrt{3}}\int_0^{\infty} \frac{dt}{1+t^2} \quad (\text{put } \sqrt{3}\tan x = t \Rightarrow \sqrt{3}\sec^2 x\,dx = dt)$$

$$= \frac{2}{\sqrt{3}}\left[\tan^{-1}(t)\right]_0^{\infty}$$

$$= \frac{2}{\sqrt{3}}(\tan^{-1}\infty - \tan^{-1}0) = \frac{2}{\sqrt{3}}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{\sqrt{3}}$$

$\therefore$ $$\int_0^{\pi} \frac{dx}{1+2\sin^2 x} = \frac{\pi}{\sqrt{3}}$$

Remark

Students are advised to check continuity of anti-derivatives before substitution of integral limits.

● **Ex. 27** *If* $\int_a^b |\sin x|\,dx = 8$ *and* $\int_0^{a+b} |\cos x|\,dx = 9$, *then find the value of* $\int_a^b x\sin x\,dx$.

Sol. We know, $\int_a^b |\sin x|\,dx$ represents the area under the curve from $x = a$ to $x = b$.

We also know, area from $x = a$ to $x = a + \pi$ is 2.

$\therefore$ $$\int_a^b |\sin x|\,dx = 8$$

$\Rightarrow$ $$b - a = \frac{8\pi}{2} \quad \text{...(i)}$$

Similarly, $$\int_0^{a+b} |\cos x|\,dx = 9$$

$\Rightarrow$ $$a + b - 0 = \frac{9\pi}{2} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), $a = \pi/4, b = 17\pi/4$

Hence, $$\int_a^b x\sin x\,dx = \int_{\pi/4}^{17\pi/4} x\sin x\,dx$$

$$= [-x\cos x]_{\pi/4}^{17\pi/4} + \int_{\pi/4}^{17\pi/4} \cos x\,dx$$

$$= -\frac{17\pi}{4}\cos\frac{17\pi}{4} + \frac{\pi}{4}\cos\frac{\pi}{4} + [\sin x]_{\pi/4}^{17\pi/4}$$

$$= -\frac{4\pi}{\sqrt{2}}$$

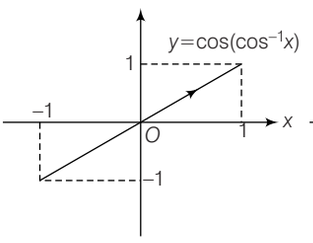
$\therefore$ $$\int_a^b x\sin x\,dx = -2\sqrt{2}\pi$$

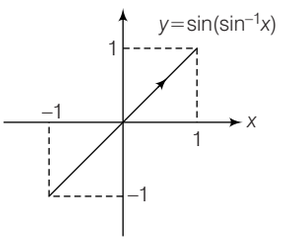
● **Ex. 28** *Evaluate* $\int_{\cos(\cos^{-1}\alpha)}^{\sin(\sin^{-1}\beta)} \left|\frac{\cos(\cos^{-1}x)}{\sin(\sin^{-1}x)}\right| dx$.

Sol. We know, $\cos(\cos^{-1}x)$ and $\sin(\sin^{-1}x)$ could be plotted as

$\therefore$ $\sin(\sin^{-1}x)$ and $\cos(\cos^{-1}x)$ are identical functions.

$\therefore$ $$\int_{\cos(\cos^{-1}\alpha)}^{\sin(\sin^{-1}\beta)} \frac{\cos(\cos^{-1}x)}{\sin(\sin^{-1}x)}\,dx = \int_{\cos(\cos^{-1}\alpha)}^{\sin(\sin^{-1}\beta)} 1\,dx = (\beta - \alpha)$$





● ***Ex. 29*** *Evaluate* $\int_\alpha^\beta \sqrt{\frac{x-\alpha}{\beta-x}}\,dx.$

Sol. Let $I=\int_\alpha^\beta \sqrt{\frac{x-\alpha}{\beta-x}}\,dx$ ...(i)

Put $x=\alpha\cos^2 t+\beta\sin^2 t$

$\therefore\quad x-\alpha=\alpha\cos^2 t+\beta\sin^2 t-\alpha=\beta\,[\sin^2(t)+\alpha(\cos^2 t-1)]$

$=\beta\sin^2 t-\alpha\sin^2 t \;\Rightarrow\; (x-\alpha)=(\beta-\alpha)\sin^2 t$

Similarly, $\beta-x=(\beta-\alpha)\cos^2 t$

$\therefore\quad dx=2(\beta-\alpha)\sin t\cos t\,dt$

where, $x=\alpha\Rightarrow\sin^2 t=0 \;\Rightarrow\; t=0$ $(\because\ \alpha<\beta)$

and $x=\beta\Rightarrow\cos^2 t=0 \;\Rightarrow\; t=\pi/2$

$\therefore$ Eq. (i) reduces to

$$I=\int_0^{\pi/2}\sqrt{\frac{(\beta-\alpha)\sin^2 t}{(\beta-\alpha)\cos^2 t}}\cdot 2(\beta-\alpha)\sin t\cos t\,dt$$

$$=\int_0^{\pi/2}\frac{\sin t}{\cos t}\cdot 2(\beta-\alpha)\sin t\cos t\,dt$$

$$\left[\because\ \sqrt{\frac{\sin^2 t}{\cos^2 t}}=+\frac{\sin t}{\cos t}\ \text{as}\ t\in[0,\pi/2]\right]$$

$$=(\beta-\alpha)\int_0^{\pi/2}2\sin^2 t\,dt=(\beta-\alpha)\int_0^{\pi/2}(1-\cos 2t)\,dt$$

$$=(\beta-\alpha)\left(t-\frac{\sin 2t}{2}\right)_0^{\pi/2}$$

$$=(\beta-\alpha)\left[\left(\frac{\pi}{2}-\frac{1}{2}\sin\pi\right)-\left(0-\frac{1}{2}\sin 0\right)\right]=(\beta-\alpha)\left[\frac{\pi}{2}\right]$$

$\therefore\quad I=\frac{\pi}{2}(\beta-\alpha)$

Aliter Let $I=\int_\alpha^\beta \sqrt{\frac{x-\alpha}{\beta-x}}\,dx$ $\left[\begin{array}{l}\text{put}\ \sqrt{\beta-x}=t\\ \text{or}\ \ \beta-x=t^2\\ \Rightarrow\ -dx=2t\,dt\end{array}\right]$

When $x=\alpha$, then $\beta-\alpha=t^2 \;\Rightarrow\; \sqrt{\beta-\alpha}=t$

When $x=\beta$, then $0=t^2 \;\Rightarrow\; 0=t$

$\therefore\quad I=\int_{\sqrt{\beta-\alpha}}^0 \frac{\sqrt{(\beta-t^2)-\alpha}}{t}(-2t)\,dt$

$$=-2\int_{\sqrt{\beta-\alpha}}^0\sqrt{(\beta-\alpha)-t^2}\,dt$$

$$=2\int_0^{\sqrt{\beta-\alpha}}\sqrt{(\sqrt{\beta-\alpha})^2-t^2}\,dt$$

$$=2\left\{\frac{t}{2}\sqrt{(\beta-\alpha)-t^2}+\frac{(\beta-\alpha)}{2}\sin^{-1}\left(\frac{t}{\sqrt{\beta-\alpha}}\right)\right\}_0^{\sqrt{\beta-\alpha}}$$

$$=2\left[\left\{0+\frac{(\beta-\alpha)}{2}\sin^{-1}(1)\right\}-\left\{0+\frac{(\beta-\alpha)}{2}\sin^{-1}(0)\right\}\right]$$

$$=\frac{2(\beta-\alpha)}{2}\cdot\frac{\pi}{2}=\frac{\pi}{2}(\beta-\alpha)$$

● ***Ex. 30*** *Evaluate* $\int_{\sqrt{(3a^2+b^2)/4}}^{\sqrt{(a^2+b^2)/2}}\frac{x}{\sqrt{(x^2-a^2)(b^2-x^2)}}\,dx.$

Sol. Let $I=\int_{\sqrt{(3a^2+b^2)/4}}^{\sqrt{(a^2+b^2)/2}}\frac{x}{\sqrt{(x^2-a^2)(b^2-x^2)}}\,dx$ ...(i)

Put $x^2=a^2\cos^2 t+b^2\sin^2 t$

$\Rightarrow\quad 2x\,dx=[2a^2\cos t(-\sin t)+2b^2\sin t(\cos t)]\,dt$

$\Rightarrow\quad x\,dx=\frac{1}{2}(b^2-a^2)\sin 2t\,dt$

For $x^2=\frac{a^2+b^2}{2}=a^2\cos^2 t+b^2\sin^2 t,$

$a^2+b^2=2(1-\sin^2 t)a^2+2b^2\sin^2 t$

or $(a^2+b^2)=2a^2+2(b^2-a^2)\sin^2 t$

$\Rightarrow\quad \sin^2 t=\frac{1}{2} \;\Rightarrow\; \cos 2t=0 \;\Rightarrow\; t=\frac{\pi}{4}$

For $x^2=\frac{3a^2+b^2}{4}=a^2\cos^2 t+b^2\sin^2 t,$

$3a^2+b^2=4a^2+4(b^2-a^2)\sin^2 t$

$\Rightarrow\quad \sin^2 t=\frac{1}{4} \;\Rightarrow\; \cos 2t=\frac{1}{2} \;\Rightarrow\; t=\frac{\pi}{6}$

$\therefore$ Eq. (i) reduces to

$$I=\int_{\pi/6}^{\pi/4}\frac{1}{2}\cdot\frac{(b^2-a^2)\sin 2t}{\sqrt{(b^2-a^2)\sin^2 t\,(b^2-a^2)\cos^2 t}}\,dt$$

$$=\int_{\pi/6}^{\pi/4}dt=\left(t\right)_{\pi/6}^{\pi/4}=\left(\frac{\pi}{4}-\frac{\pi}{6}\right)=\frac{\pi}{12}$$

● ***Ex. 31*** *Evaluate* $\int_0^{\pi/4}\frac{e^{\sec x}\left[\sin\left(x+\frac{\pi}{4}\right)\right]}{\cos x\,(1-\sin x)}\,dx.$

Sol. Let $I=\int_0^{\pi/4}\frac{e^{\sec x}\left[\sin\left(x+\frac{\pi}{4}\right)\right]}{\cos x(1-\sin x)}\,dx$

$$=\frac{1}{\sqrt{2}}\int_0^{\pi/4}\frac{e^{\sec x}(\sin x+\cos x)(1+\sin x)}{\cos x(1-\sin^2 x)}\,dx$$

$$=\frac{1}{\sqrt{2}}\int_0^{\pi/4}\frac{e^{\sec x}(\sin x+\cos x)(1+\sin x)}{\cos^2 x\cdot\cos x}\,dx$$

$$=\frac{1}{\sqrt{2}}\int_0^{\pi/4}e^{\sec x}\cdot(\sec x\tan x+\sec x)(\sec x+\tan x)\,dx$$

$$=\frac{1}{\sqrt{2}}\int_0^{\pi/4}e^{\sec x}\cdot[\sec x\tan x(\sec x+\tan x)+\sec^2 x+\sec x\tan x]\,dx$$

$$=\frac{1}{\sqrt{2}}\int_0^{\pi/4}e^{\sec x}\cdot[\sec x\tan x(\sec x+\tan x+1)+\sec^2 x]\,dx$$

$$=\frac{1}{\sqrt{2}}\int_0^{\pi/4}e^{\sec x}\cdot\sec x\tan x(\sec x+\tan x+1)+\frac{1}{\sqrt{2}}\int_0^{\pi/4}e^{\sec x}\cdot\sec^2 x\,dx$$

Applying integration by parts,

$$I=\frac{1}{\sqrt{2}}\{(\sec x+\tan x+1)\,e^{\sec x}\}_0^{\pi/4}$$
$$-\frac{1}{\sqrt{2}}\int_0^{\pi/4}(\sec x\tan x+\sec^2 x)\cdot e^{\sec x}dx+\frac{1}{\sqrt{2}}\int_0^{\pi/4}e^{\sec x}\cdot\sec^2 x\,dx$$
$$=\frac{1}{\sqrt{2}}\{(\sqrt{2}+1+1)\,e^{\sqrt{2}}-(1+1)\,e\}-\frac{1}{\sqrt{2}}[e^{\sec x}]_0^{\pi/4}$$
$$=\frac{1}{\sqrt{2}}(\sqrt{2}+2)\,e^{\sqrt{2}}-\sqrt{2}e-\frac{1}{\sqrt{2}}(e^{\sqrt{2}}-e)=\frac{1}{\sqrt{2}}(1+\sqrt{2})\,e^{\sqrt{2}}-\frac{e}{\sqrt{2}}$$

● ***Ex. 32*** *Evaluate* $\int_0^{\pi/4}\frac{x^2(\sin 2x-\cos 2x)\,dx}{(1+\sin 2x)\cos^2 x}$.

Sol. Let $I=\int_0^{\pi/4}\frac{x^2(\sin 2x-\cos 2x)\,dx}{(1+\sin 2x)\cos^2 x}$,

$$=\frac{1}{8}\int_0^{\pi/2}\frac{t^2(\sin t-\cos t)}{(1+\sin t)(\cos^2 t/2)}dt \qquad (\text{put } 2x=t)$$
$$I=\frac{1}{4}\int_0^{\pi/2}\frac{t^2(\sin t-\cos t)}{(1+\sin t)(1+\cos t)}dt \qquad \ldots(\text{i})$$
$$\left[\text{using, }\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx\right]$$
$$I=\frac{1}{4}\int_0^{\pi/2}\frac{\left(\frac{\pi}{2}-t\right)^2\cdot(\cos t-\sin t)}{(1+\cos t)(1+\sin t)}dt \qquad \ldots(\text{ii})$$

Adding Eqs. (i) and (ii), we get

$$2I=\frac{1}{4}\int_0^{\pi/2}\frac{\left(\pi t-\frac{\pi^2}{4}\right)(\sin t-\cos t)}{(1+\cos t)(1+\sin t)}dt \qquad \ldots(\text{iii})$$

where, $\int_0^{\pi/2}\frac{\sin t-\cos t}{(1+\cos t)(1+\sin t)}dt=0$

$$\left[\text{as }\int_0^{2a}f(x)\,dx=0,\text{ if } f(2a-x)=-f(x)\right]$$

∴ Eq. (iii) becomes

$$I=\frac{1}{8}\int_0^{\pi/2}\frac{\pi t(\sin t-\cos t)\,dt}{(1+\cos t)(1+\sin t)}+0$$
$$=\frac{\pi}{8}\int_0^{\pi/2}\frac{t\{(1+\sin t)-(1+\cos t)\}\,dt}{(1+\cos t)(1+\sin t)}$$
$$=\frac{\pi}{8}\int_0^{\pi/2}\frac{t\,dt}{1+\cos t}-\frac{\pi}{8}\int_0^{\pi/2}\frac{t\,dt}{1+\sin t}$$
$$=\frac{\pi}{8}\int_0^{\pi/2}\frac{t\,dt}{1+\cos t}-\frac{\pi}{8}\int_0^{\pi/2}\frac{(\pi/2-t)\,dt}{1+\cos t}$$
$$=\frac{2\pi}{8}\int_0^{\pi/2}\frac{t\,dt}{1+\cos t}-\frac{\pi^2}{16}\int_0^{\pi/2}\frac{dt}{1+\cos t}$$
$$=\frac{\pi}{8}\int_0^{\pi/2}t(\sec^2 t/2)\,dt-\frac{\pi^2}{16}\int_0^{\pi/2}(\sec^2 t/2)\,dt$$
$$=\frac{\pi}{8}\left\{\left(2t\cdot\tan\frac{t}{2}\right)_0^{\pi/2}-\int_0^{\pi/2}1\cdot 2\tan\frac{t}{2}dt\right\}-\frac{\pi^2}{16}\left(2\tan\frac{t}{2}\right)_0^{\pi/2}$$
$$=\frac{\pi}{8}\left\{(\pi-0)+4\left(\log\cos\frac{t}{2}\right)_0^{\pi/2}\right\}-\frac{\pi^2}{8}\{1-0\}$$
$$=\frac{\pi}{8}\left\{\pi+4\log\frac{1}{\sqrt{2}}\right\}-\frac{\pi^2}{8}=\frac{\pi^2}{8}+\pi\log\frac{1}{\sqrt{2}}-\frac{\pi^2}{8}$$
$$=\frac{\pi^2}{8}-\frac{1}{2}\pi\log 2=\frac{\pi^2}{16}-\frac{\pi}{4}\log 2$$

● ***Ex. 33*** *Evaluate* $\int_0^{\pi}\frac{x^2\cos x}{(1+\sin x)^2}dx$.

Sol. Let $I=\int_0^{\pi}\frac{x^2\cos x}{(1+\sin x)^2}dx=\int_0^{\pi}x^2\{(1+\sin x)^{-2}\cos x\}\,dx$

Applying integration by parts,

$$I=\left(x^2\frac{(1+\sin x)^{-1}}{-1}\right)_0^{\pi}-\int_0^{\pi}2x\cdot\frac{(1+\sin x)^{-1}}{-1}dx$$
$$=(-\pi^2+0)+2\int_0^{\pi}\frac{x}{1+\sin x}dx \qquad \ldots(\text{i})$$
$$\left[\text{using, }\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx\right]$$
$$I=-\pi^2+2\int_0^{\pi}\frac{(\pi-x)}{1+\sin x}dx \qquad \ldots(\text{ii})$$

Adding Eqs. (i) and (ii), we get

$$2I=-2\pi^2+2\int_0^{\pi}\frac{\pi}{1+\sin x}dx$$
$$=-2\pi^2+2\pi\int_0^{\pi}\frac{1-\sin x}{\cos^2 x}dx$$
$$=-2\pi^2+2\pi[\tan x-\sec x]_0^{\pi}$$
$$=-2\pi^2+2\pi[0-(-1-1)]=-2\pi^2+4\pi$$

∴ $I=-\pi^2+2\pi$

● ***Ex. 34*** *Compute the following integrals.*

(i) $\int_0^{\infty}f(x^n+x^{-n})\ln x\frac{dx}{x}=0$

(ii) $\int_0^{\infty}f(x^n+x^{-n})\ln x\frac{dx}{1+x^2}=0$

Sol. (i) Let $t=\ln x \Rightarrow x=e^t$

$$dx=e^t\,dt \quad \text{or} \quad \frac{dx}{x}=dt$$

Also, $x=0 \Rightarrow t=-\infty$

and $x=\infty \Rightarrow t=\infty$

∴ $\int_0^{\infty}f(x^n+x^{-n})\ln x\frac{dx}{x}=\int_{-\infty}^{\infty}f(e^{nt}+e^{-nt})\cdot t\,dt=0$

(∵ integrand is an odd function of t)

(ii) $\int_0^{\infty}f(x^n+x^{-n})\ln x\cdot\frac{dx}{1+x^2}=\int_{-\infty}^{\infty}f(e^{nt}+e^{-nt})\cdot t\cdot\frac{e^t}{1+e^{2t}}\cdot dt$

Now, if $h(t)=f(e^{nt}+e^{-nt})\cdot\dfrac{te^t}{1+e^{2t}}$

Then, $h(-t)=f(e^{-nt}+e^{nt})\cdot(-t)\cdot\dfrac{1}{e^t+e^{-t}}\quad\left(\because\dfrac{e^t}{1+e^{2t}}=\dfrac{1}{e^{-t}+e^t}\right)$

$h(-t)=-h(t)$

Thus, integrand is an odd function and hence

$$\int_0^{\infty} f(x^n+x^{-n})\ln x\cdot\frac{dx}{1+x^2}=0$$

● ***Ex. 35*** *Show that*

(a) $\int_0^{\infty}\sin x\,dx=1$ (b) $\int_0^{\infty}\cos x\,dx=0$

Sol. Let us first evaluate;

$$I=\int e^{-sx}\sin x\,dx \quad\text{and}\quad \mathcal{J}=\int e^{-sx}\cos x\,dx$$

Using integer by parts, we get

$$I=-e^{-sx}\cos x-s\mathcal{J} \qquad ...(i)$$

and $$\mathcal{J}=e^{-sx}\sin x+sI \qquad ...(ii)$$

Subtracting Eqs. (i) and (ii), we get

$$I=-e^{-sx}\left|\frac{\cos x+s\sin x}{1+s^2}\right|$$

$$\Rightarrow\quad \mathcal{J}=e^{-sx}\left|\sin x-\frac{s^2}{s^2+1}\sin x-\frac{s}{1+s^2}\cos x\right|$$

$$=e^{-sx}\left|\frac{\sin x-s\cos x}{1+s^2}\right|$$

Thus, $\int_0^{\infty}e^{-sx}\sin x\,dx=\dfrac{1}{s^2+1}$

and $\int_0^{\infty}e^{-sx}\cos x\,dx=\dfrac{s}{s^2+1}$

Now, $\int_0^{\infty}\sin x\,dx=\lim\limits_{s\to 0}\int_0^{\infty}e^{-sx}\sin x\,dx=\lim\limits_{s\to 0}\dfrac{1}{s^2+1}=1$

and $\int_0^{\infty}\cos x\,dx=\lim\limits_{s\to 0}\int_0^{\infty}e^{-sx}\cos x\,dx=\lim\limits_{s\to 0}\dfrac{s}{s^2+1}=0$

● ***Ex. 36*** *Find a function $g:R\to R$, continuous in $[0,\infty)$ and positive in $(0,\infty)$ satisfying $g(0)=1$ and* $\dfrac{1}{2}\int_0^x g^2(t)\,dt=\dfrac{1}{x}\left(\int_0^x g(t)\,dt\right)^2$.

Sol. Let $$f(x)=\int_0^x g(t)\,dt$$

$$\Rightarrow\quad f'(x)=g(x)$$

$$\Rightarrow\quad \frac{1}{2}\int_0^x (f'(t))^2\,dt=\frac{1}{2}\int_0^x g^2(t)\,dt \qquad ...(i)$$

$$\Rightarrow\quad \frac{1}{2}\left[\int_0^x g(t)\right]^2=\frac{1}{2}(f(x))^2$$

$$\Rightarrow\quad \frac{1}{2}\frac{\left[\int_0^x g(t)\right]^2}{x}=\frac{1}{2}\frac{(f(x))^2}{x}$$

$$\Rightarrow\quad \frac{1}{2}\cdot\frac{1}{2}\left[\int_0^x g^2(t)\right]dt=\frac{1}{2}\frac{(f(x))^2}{x}$$

$$\Rightarrow\quad \frac{1}{2}\int_0^x (f'(t))^2\,dt=\frac{1}{2}\frac{(f(x))^2}{x} \qquad \text{[using Eq. (i)]}$$

Differentiating both the sides w.r.t. x, we get

$$\frac{1}{2}[f'(x)]^2=\frac{x\,2f(x)f'(x)-(f(x))^2}{x^2}$$

$$\Rightarrow\quad \frac{1}{2}[g(x)]^2x^2=2xf(x)(g(x))-(f(x))^2$$

$$\Rightarrow\quad \left(\frac{xg(x)}{f(x)}\right)^2=4\left(\frac{xg(x)}{f(x)}\right)-2$$

$$\Rightarrow\quad t^2-4t+2=0,\ \text{where } t=\frac{xg(x)}{f(x)}$$

$$\Rightarrow\quad t=\frac{4\pm\sqrt{8}}{2}=2\pm\sqrt{2}\quad\text{or}\quad\frac{xf'(x)}{f(x)}=2\pm\sqrt{2}$$

$$\Rightarrow\quad \ln f(x)=2\pm\sqrt{2}\ln x+\text{constant}$$

$$\Rightarrow\quad f(x)=cx^{2+\sqrt{2}}\quad\text{or}\quad cx^{2-\sqrt{2}}$$

$$\Rightarrow\quad g(x)=f'(x)=c_1x^{1+\sqrt{2}}\quad\text{or}\quad c_2x^{1-\sqrt{2}}$$

where, c_1 and c_2 are constants of integration.

But g is continuous on $[0,\infty)$, then $c_2x^{1-\sqrt{2}}$ is ruled out.

Hence, $g(x)=c_1x^{1+\sqrt{2}}$

Also, $g(0)=c^1=1\ \Rightarrow\ g(x)=x^{1+\sqrt{2}}$

● ***Ex. 37*** *Let $I_n=\int_0^{\pi/4}\tan^n x\,dx\ (n>1$ and is an integer). Show that*

(i) $I_n+I_{n-2}=\dfrac{1}{n-1}$ (ii) $\dfrac{1}{2(n+1)}<I_n<\dfrac{1}{2(n-1)}$

Sol. (i) Given, $I_n=\int_0^{\pi/4}\tan^n x\,dx=\int_0^{\pi/4}\tan^{n-2}x\cdot\tan^2x\,dx$

$$=\int_0^{\pi/4}\sec^2x\cdot\tan^{n-2}x\,dx-\int_0^{\pi/4}\tan^{n-2}x\,dx$$

$$=\int_0^1 t^{n-2}\,dt-I_{n-2},\ \text{where } t=\tan x$$

$$I_n+I_{n-2}=\left(\frac{t^{n-1}}{n-1}\right)_0^1\qquad\therefore\ I_n+I_{n-2}=\frac{1}{n-1}$$

(ii) For $0<x<\pi/4$, we have $0<\tan^n x<\tan^{n-2}x$

So that; $$0<I_n<I_{n-2}$$

$$\therefore\quad I_n+I_{n+2}<2I_n<I_n+I_{n-2}$$

$$\Rightarrow\quad \frac{1}{n+1}<2I_n<\frac{1}{n-1}$$

$$\Rightarrow\quad \frac{1}{2(n+1)}<I_n<\frac{1}{2(n-1)}$$

Ex. 38 *If* $U_n = \int_0^{\pi} \frac{1-\cos nx}{1-\cos x}\,dx$, *where n is positive integer or zero, then show that* $U_{n+2} + U_n = 2U_{n+1}$. *Hence, deduce that* $\int_0^{\pi/2} \frac{\sin^2 n\theta}{\sin^2\theta} = \frac{1}{2}n\pi$.

Sol. $U_{n+2} - U_{n+1} = \int_0^{\pi} \frac{[1-\cos(n+2)x] - [1-\cos(n+1)x]}{1-\cos x}\,dx$

$$= \int_0^{\pi} \frac{\cos(n+1)x - \cos(n+2)x}{1-\cos x}\,dx$$

$$= \int_0^{\pi} \frac{2\sin\left(n+\frac{3}{2}\right)x \cdot \sin\frac{x}{2}}{2\sin^2\frac{x}{2}}\,dx$$

$$U_{n+2} - U_{n+1} = \int_0^{\pi} \frac{\sin\left(n+\frac{3}{2}\right)x}{\sin\frac{x}{2}}\,dx \quad \ldots(i)$$

$$\Rightarrow U_{n+1} - U_n = \int_0^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\frac{x}{2}}\,dx \quad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$(U_{n+2} - U_{n+1}) - (U_{n+1} - U_n) = \int_0^{\pi} \frac{\sin\left(n+\frac{3}{2}\right)x - \sin\left(n+\frac{1}{2}\right)x}{\sin\frac{x}{2}}\,dx$$

$$U_{n+2} + U_n - 2U_{n+1} = \int_0^{\pi} \frac{2\cos(n+1)x \cdot \sin x/2}{\sin x/2}\,dx$$

$$= 2\int_0^{\pi} \cos(n+1)x\,dx = 2\left(\frac{\sin(n+1)x}{n+1}\right)_0^{\pi} = 0$$

$\Rightarrow U_{n+2} + U_n = 2U_{n+1}$

$\Rightarrow U_n, U_{n+1}, U_{n+2}$ are in AP.

Now, $U_0 = \int_0^{\pi} \frac{1-1}{1-\cos x}\,dx = 0$

$$U_1 = \int_0^{\pi} \frac{1-\cos x}{1-\cos x}\,dx = \pi$$

$U_1 - U_0 = \pi$ (common difference)

$U_n = U_0 + n\pi = n\pi$

$U_n = n\pi$

Now, $I_n = \int_0^{\pi/2} \frac{\sin^2 n\theta}{\sin^2\theta}\,d\theta = \int_0^{\pi/2} \frac{1-\cos 2n\theta}{1-\cos 2\theta}\,d\theta$

$$= \frac{1}{2}\int_0^{\pi} \frac{1-\cos nx}{1-\cos x}\,dx = \frac{1}{2}n\pi$$

Ex. 39 *Prove that for any positive integer K,*

$$\frac{\sin 2Kx}{\sin x} = 2[\cos x + \cos 3x + \ldots + \cos(2K-1)x]$$

Hence, prove that $\int_0^{\pi/2} \sin 2Kx \cdot \cot x\,dx = \frac{\pi}{2}$.

Sol. To prove; $\sin 2Kx = 2\sin x[\cos x + \cos 3x + \cos 5x + \ldots + \cos(2K-1)x]$

Taking RHS, $[2\sin x\cos x + 2\sin x\cos 3x + 2\sin x\cos 5x + \ldots + 2\sin x\cos(2K-1)x]$

$= (\sin 2x) + (\sin 4x - \sin 2x) + [\sin 6x - \sin 4x) + \ldots + (\sin 2Kx - \sin(2K-2)x]$

$= \sin 2Kx$

Now, $\int_0^{\pi/2} \sin 2K\,x \cdot \cot x\,dx = \int_0^{\pi/2} \left(\frac{\sin 2Kx}{\sin x}\right) \cdot \cos x\,dx$

$$= \int_0^{\pi/2} 2\cos x[\cos x + \cos 3x + \ldots + \cos(2K-1)x]\,dx$$

$$= \int_0^{\pi/2} (1+\cos 2x)\,dx + \int_0^{\pi/2} (\cos 4x + \cos 2x)\,dx + \ldots + \int_0^{\pi/2} [\cos 2K\,x + \cos(2K-2)x]\,dx$$

But we know that,

$$\int_0^{\pi/2} (\cos 2nx)\,dx = 0, \forall\, n \in I \quad \text{and} \quad n \neq 0$$

$$\Rightarrow \int_0^{\pi/2} \sin 2Kx \cdot \cot x\,dx = \int_0^{\pi/2} 1\,dx + 0 = \frac{\pi}{2}$$

Ex. 40 *Evaluate* $\int_0^{\sqrt{3}} \frac{1}{1+x^2} \cdot \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$.

Sol. Let $I = \int_0^{\sqrt{3}} \frac{1}{1+x^2}\sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Now, $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x, & \text{if } -1 \le x \le 1 \\ \pi - 2\tan^{-1}x, & \text{if } x > 1 \end{cases}$

$$\therefore I = \int_0^1 \frac{1}{1+x^2}\sin^{-1}\left(\frac{2x}{1+x^2}\right)dx + \int_1^{\sqrt{3}} \frac{1}{1+x^2} \cdot \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$$

$$= \int_0^1 \frac{2\tan^{-1}x}{1+x^2}\,dx + \int_1^{\sqrt{3}} \frac{\pi - 2\tan^{-1}x}{1+x^2}\,dx$$

$$= 2\int_0^1 \frac{\tan^{-1}x}{1+x^2}\,dx + \pi\int_1^{\sqrt{3}} \frac{1}{1+x^2}\,dx - 2\int_1^{\sqrt{3}} \frac{\tan^{-1}x}{1+x^2}\,dx$$

$$= 2\int_0^{\pi/4} t\,dt + \pi(\tan^{-1}x)_1^{\sqrt{3}} - 2\int_{\pi/4}^{\pi/3} t\,dt \qquad (\text{put } \tan^{-1}x = t)$$

$$= 2\left(\frac{t^2}{2}\right)_0^{\pi/4} + \pi\{\tan^{-1}\sqrt{3} - \tan^{-1}1\} - 2\left(\frac{t^2}{2}\right)_{\pi/4}^{\pi/3}$$

$$= \frac{\pi^2}{16} + \pi\left\{\frac{\pi}{3} - \frac{\pi}{4}\right\} - \left\{\frac{\pi^2}{9} - \frac{\pi^2}{16}\right\} = \frac{7}{72}\pi^2$$

Ex. 41 *Prove that* $\int_0^x e^{xt} \cdot e^{-t^2}\,dt = e^{x^2/4}\int_0^x e^{-t^2/4}\,dt$.

Sol. Let $I = \int_0^x e^{xt} \cdot e^{-t^2}\,dt$. Put $t = \frac{x+z}{2} \Rightarrow dt = \frac{1}{2}dz$

$$\therefore I = \int_0^x e^{tx} \cdot e^{-t^2}\,dt = \frac{1}{2}\int_{-x}^x e^{\left(\frac{x+z}{2}\right)x} \cdot e^{-\left(\frac{x+z}{2}\right)^2}dz$$

$$= \frac{1}{2}\int_{-x}^x e^{\left(\frac{x+z}{2}\right)\left(x - \frac{x+z}{2}\right)}dz = \frac{1}{2}\int_{-x}^x e^{\left(\frac{x+z}{2}\right)\left(\frac{x-z}{2}\right)}dz$$

$$= \frac{1}{2}\int_{-x}^{x} e^{\frac{x^2}{4}-\frac{z^2}{4}}\,dz = \frac{1}{2}e^{x^2/4}\cdot\int_{-x}^{x} e^{-z^2/4}\,dz$$

$$= \frac{1}{2}e^{x^2/4}\int_{-x}^{x} e^{-t^2/4}\,dt = \frac{1}{2}e^{x^2/4}\cdot 2\int_{0}^{x} e^{-t^2/4}\,dt$$

$$\left[\text{using, } \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx, \text{ when } f(-x)=f(x)\right]$$

$$= e^{x^2/4}\int_{0}^{x} e^{-t^2/4}\,dt$$

$$\Rightarrow \quad \int_{0}^{x} e^{xt}\cdot x^{-t^2}\,dt = e^{x^2/4}\int_{0}^{x} e^{-t^2/4}\,dt$$

● ***Ex. 42*** *If* $f(x) = e^x + \int_0^1 (e^x + te^{-x})\, f(t)\,dt$, *find* $f(x)$.

Sol. We can write $f(x) = Ae^x + Be^{-x}$, where

$$A = 1 + \int_0^1 f(t)\,dt \quad \text{and} \quad B = \int_0^1 tf(t)\,dt$$

$$\therefore \quad A = 1 + \int_0^1 (Ae^t + Be^{-t})\,dt = 1 + (Ae^t - Be^{-t})_0^t$$

$$A = 1 + A(e^1 - 1) - B(e^{-1} - 1)$$

$$\Rightarrow \quad (2-e)A + (e^{-1}-1)B = 1 \qquad \ldots(\text{i})$$

$$B = \int_0^1 t(Ae^t + Be^{-t})\,dt$$

$$= A(te^t - e^t)_0^t + B(-te^{-t} - e^{-t})_0^1$$

$$B = A + B(1 - 2e^{-1})$$

$$\Rightarrow \quad A - 2e^{-1}B = 0 \qquad \ldots(\text{ii})$$

From Eqs. (i) and (ii), we get

$$A = \frac{2(e-1)}{4e-2e^2},\ B = \frac{e-1}{4-2e}$$

Hence, $$f(x) = \frac{2(e-1)}{4e-2e^2}\cdot e^x + \frac{e-1}{4-2e}\cdot e^{-x}$$

● ***Ex. 43*** *If* $|a| < 1$, *show that*

$$\int_0^{\pi} \frac{\log(1 + a\cos x)}{\cos x}\,dx = \pi \sin^{-1} a.$$

Sol. Given, $|a| < 1$

Let $$f(a) = \int_0^{\pi} \frac{\log(1 + a\cos x)}{\cos x}\,dx$$

$$\therefore \quad f'(a) = \int_0^{\pi} \frac{\cos x}{\cos x\cdot(1 + a\cos x)}\,dx = \int_0^{\pi} \frac{dx}{1 + a\cos x}$$

(differentiating w.r.t. 'a' using Leibnitz rule)

$$\left(\begin{array}{l}\text{put } \tan\dfrac{x}{2} = t \Rightarrow \dfrac{dx}{dt} = \dfrac{2}{1+t^2}, \text{ when } x = 0, t = 0\\ \text{and when } x = \pi, t \to \infty\end{array}\right)$$

$$\Rightarrow \quad f'(a) = \int_0^{\infty} \frac{\frac{2dt}{1+t^2}}{1 + a\left(\frac{1-t^2}{1+t^2}\right)}$$

$$= \int_0^{\infty} \frac{2dt}{(1+t^2) + a(1-t^2)} = \int_0^{\infty} \frac{2dt}{(1-a)t^2 + (1+a)}$$

$$= \frac{2}{1-a}\int_0^{\infty} \frac{dt}{t^2 + \left(\sqrt{\frac{1+a}{1-a}}\right)^2}$$

$$= \frac{2}{1-a}\cdot\frac{1}{\sqrt{(1+a)/(1-a)}}\cdot\left(\tan^{-1} t\sqrt{\frac{1-a}{1+a}}\right)_0^{\infty} = \frac{2}{\sqrt{1-a^2}}\cdot\frac{\pi}{2} = \frac{\pi}{\sqrt{1-a^2}}$$

(integrating both the sides w.r.t. 'a')

$$\Rightarrow \quad f(a) = \pi\sin^{-1} a + c$$

But $$f(0) = \int_0^{\pi} \frac{\log(1 + 0\cos x)\,dx}{\cos x} = 0$$

$$\Rightarrow \quad c = 0$$

$$\therefore \quad f(a) = \pi\sin^{-1} a$$

● ***Ex. 44*** *Evaluate* $\int_0^{\pi/2} \operatorname{cosec}\theta\,\tan^{-1}(c\sin\theta)\,d\theta$.

Sol. Let $$f(c) = \int_0^{\pi/2} \operatorname{cosec}\theta\cdot\tan^{-1}(c\sin\theta)\,d\theta$$

$$\therefore \quad f'(c) = \int_0^{\pi/2} \operatorname{cosec}\theta\cdot\frac{(\sin\theta)}{1 + c^2\sin^2\theta}\,d\theta + 0 - 0$$

$$= \int_0^{\pi/2} \frac{d\theta}{1 + c^2\sin^2\theta}$$

$$= \int_0^{\pi/2} \frac{\operatorname{cosec}^2\theta\,d\theta}{c^2 + \operatorname{cosec}^2\theta} = \int_0^{\pi/2} \frac{\operatorname{cosec}^2\theta\,d\theta}{(c^2+1) + \cot^2\theta}$$

$$= -\frac{1}{\sqrt{c^2+1}}\left(\tan^{-1}\frac{\cot\theta}{\sqrt{c^2+1}}\right)_0^{\pi/2}$$

$$f'(c) = \frac{\pi}{2\sqrt{c^2+1}}$$

Integrating both the sides, we get

$$f(c) = \int \frac{\pi\,dc}{2\sqrt{c^2+1}} = \frac{\pi}{2}\log(c + \sqrt{c^2+1}) + A$$

where, $f(0) = A$

But $$f(0) = \int_0^{\pi/2} \operatorname{cosec}\theta\tan^{-1}(0)\,d\theta = 0$$

$$\Rightarrow \quad f(c) = \frac{\pi}{2}\log(c + \sqrt{c^2+1})$$

● ***Ex. 45*** *Evaluate* $\int_0^{\pi/2} \sec\theta\cdot\tan^{-1}(a\cos\theta)\,d\theta$.

Sol. Given definite integral is a function of 'a'. Let its value be $I(a)$.

Then, $$I(a) = \int_0^{\pi/2} \sec\theta\cdot\tan^{-1}(a\cos\theta)\,d\theta$$

$$\Rightarrow \quad I'(a) = \int_0^{\pi/2} \sec\theta\cdot\frac{1}{1 + a^2\cos^2\theta}\cdot\cos\theta\,d\theta + 0 - 0$$

(using Leibnitz rule)

$$= \int_0^{\pi/2} \frac{1}{1 + a^2\cos^2\theta}\,d\theta = \int_0^{\pi/2} \frac{\sec^2\theta}{\sec^2\theta + a^2}\,d\theta$$

$$= \int_0^{\pi/2} \frac{\sec^2\theta}{1+\tan^2\theta + a^2}\, d\theta \qquad (\text{put } \tan\theta = t)$$

$$= \int_0^{\infty} \frac{dt}{t^2 + (a^2+1)} = \frac{1}{\sqrt{a^2+1}} \left(\tan^{-1} \frac{t}{\sqrt{a^2+1}} \right)_0^{\infty}$$

$$= \frac{1}{\sqrt{a^2+1}} \left(\frac{\pi}{2} - 0 \right) \Rightarrow I'(a) = \frac{\pi}{2\sqrt{a^2+1}}$$

Integrating both the sides w.r.t. 'a', we get

$$I(a) = \frac{\pi}{2} \log | a + \sqrt{a^2+1} | + C$$

Since, $I(0) = 0 + C$

$\Rightarrow$ $I(a) = \frac{\pi}{2} \log | a + \sqrt{1+a^2} |$

● **Ex. 46** *Let f be a continuous function on $[a, b]$. Prove that there exists a number $x \in [a, b]$ such that* $\int_a^x f(t)\, dt = \int_x^b f(t)\, dt$.

Sol. Let $g(x) = \int_a^x f(t)\, dt - \int_x^b f(t)\, dt,\ x \in [a, b]$

We have, $g(a) = -\int_a^b f(t)\, dt$ and $g(b) = \int_a^b f(t)\, dt$

$\Rightarrow$ $g(a) \cdot g(b) = -\left(\int_a^b f(t)\, dt \right)^2 \le 0$

Clearly, $g(x)$ is continuous in $[a, b]$ and $g(a) \cdot g(b) \le 0$.

It implies that $g(x)$ will become zero at least once in $[a, b]$.

Hence, $\int_a^x f(t)\, dt = \int_x^b f(t)\, dt$ for atleast one value of $x \in [a, b]$.

● **Ex. 47** *If $f(x) = x + \int_0^1 (xy^2 + x^2 y)\, (f(y))\, dy$, find $f(x)$.*

Sol. Given, $f(x) = x + x \int_0^1 y^2 f(y)\, dy + x^2 \int_0^1 y\, f(y)\, dy$

$$= x \left(1 + \int_0^1 y^2 f(y)\, dy \right) + x^2 \left(\int_0^1 y\, f(y)\, dy \right)$$

$\Rightarrow$ $f(x)$ is a quadratic expression.

$\Rightarrow$ $f(x) = ax + bx^2$ or $f(y) = ay + by^2$ ...(i)

where, $a = 1 + \int_0^1 y^2 f(y)\, dy = 1 + \int_0^1 y^2 (ay + by^2)\, dy$

$$= 1 + \left(\frac{ay^4}{4} + \frac{by^5}{5} \right)_0^1 \quad a = 1 + \left(\frac{a}{4} + \frac{b}{5} \right)$$

$\Rightarrow$ $20a = 20 + 5a + 4b$ or $15a - 4b = 20$ ...(ii)

and $b = \int_0^1 y\, f(y)\, dy = \int_0^1 y(ay + by^2)\, dy$

$$= \left(\frac{ay^3}{3} + \frac{by^4}{4} \right)_0^1 \Rightarrow b = \frac{a}{3} + \frac{b}{4}$$

$\Rightarrow$ $12b = 4a + 3b$ or $9b - 4a = 0$ ...(iii)

From Eqs. (ii) and (iii),

$$a = \frac{180}{119} \quad \text{and} \quad b = \frac{80}{119}$$

$\therefore$ Eq. (i) reduces to

$$f(x) = \frac{80x^2 + 180x}{119}$$

● **Ex. 48** *Prove that*

$$\int_0^x \{ \int_0^u f(t)\, dt \}\, du = \int_0^x (x - u)\, f(u)\, du.$$

Sol. Here, applying integration by parts to

$$\int_0^x \underbrace{1}_{\text{II}} \cdot \underbrace{\{ \int_0^u f(t)\, dt \}\, du}_{\text{I}}$$

i.e. taking '1' as second function and $\int_0^u f(t)\, dt$ as first function, we have

$$\int_0^x 1 \cdot \left\{ \int_0^u f(t)\, dt \right\} du = \left\{ \int_0^u f(t)\, dt \right\}_0^x \cdot (u)_0^x - \int_0^x f(u) \cdot u\, du$$

$$= \left(u \int_0^u f(t)\, dt \right)_0^x - \int_0^x uf(u)\, du$$

$$= x \int_0^x f(t)\, dt - \int_0^x uf(u)\, du$$

$$= \int_0^x (x - u)\, f(u)\, du$$

● **Ex. 49** *Evaluate $\int_0^{3\pi/2} (\log | \sin x |)\, (\cos(2nx))\, dx, n \in N$.*

Sol. Let $I(n) = \int_0^{3\pi/2} (\log | \sin x |)(\cos 2nx)\, dx$

$$I(n) = \left(\underset{\text{I}}{\log | \sin x |} \cdot \underset{\text{II}}{\frac{\sin 2nx}{2n}} \right)_0^{3\pi/2} - \int_0^{3\pi/2} \cot x \cdot \frac{\sin 2nx}{2n}\, dx$$

(using integration by parts)

$$I(n) = 0 - \frac{1}{2n} \int_0^{3\pi/2} \frac{\sin 2nx \cdot \cos x}{\sin x}\, dx = 0 - \frac{1}{2} I_1(n) \qquad ...(i)$$

Let $I_1(n) = \int_0^{3\pi/2} \frac{\sin(2nx) \cos x}{\sin x}\, dx$

$$I_1(n+1) = \int_0^{3\pi/2} \frac{\sin(2n+2)\, x \cdot \cos x}{\sin x}\, dx$$

$\therefore$ $I_1(n+1) - I_1(n) = \int_0^{3\pi/2} \frac{(\sin(2n+2)\, x - \sin 2nx) \cdot \cos x}{\sin x}\, dx$

$$= \int_0^{3\pi/2} \frac{2\cos(2n+1)\, x \cdot \sin x \cdot \cos x}{\sin x}\, dx$$

$$= \int_0^{3\pi/2} (\cos(2n+2)\, x + \cos 2nx)\, dx$$

$$= \left(\frac{\sin(2n+2)\, x}{(2n+2)} + \frac{\sin(2n)\, x}{2n} \right)_0^{3\pi/2} = 0 + 0$$

$\Rightarrow$ $I_1(n+1) = I_1(n) = \ldots = I_1(1)$

$\therefore$ $I_1(1) = \int_0^{3\pi/2} \frac{\sin 2x \cdot \cos x}{\sin x}\, dx = \int_0^{3\pi/2} \frac{2 \sin x \cos x \cdot \cos x}{\sin x}\, dx$

$$= \int_0^{3\pi/2} 2\cos^2 x\, dx = \int_0^{3\pi/2} (1+\cos 2x)\, dx$$

$$= \left(x + \frac{\sin 2x}{2}\right)_0^{3\pi/2}$$

$$\Rightarrow \quad I_1(1) = \frac{3\pi}{2} = I_1(n) \quad \Rightarrow \quad I(n) = -\frac{1}{2n} I_1(n) \qquad \text{[using Eq. (i)]}$$

$$\therefore \quad I(n) = -\frac{3\pi}{4n}$$

● ***Ex. 50*** *Evaluate* $\int_0^{\infty} e^{-x} \sin^n x\, dx$, *if n is an even integer.*

Sol. Here, $I_n = \int_0^{\infty} e^{-x} \sin^n x\, dx$

$$= (-\sin^n x\, e^{-x})_0^{\infty} + n\int_0^{\infty} \sin^{n-1} x \cos x\, e^{-x}\, dx$$

$$= n\int_0^{\infty} (\sin^{n-1} x \cdot \cos x)(e^{-x})\, dx \ [\text{where } (-\sin^n x\, e^{-x})_0^{\infty} = 0]$$

$$\Rightarrow \quad I_n = n[(-\sin^{n-1} \underset{\text{I}}{x} \cdot \cos x\, \underset{\text{II}}{e^{-x}})_0^{\infty}$$

$$+ (n-1)\int_0^{\infty} \sin^{n-2} x \cos^2 x\, e^{-x} dx - \int_0^{\infty} \sin^n x\, e^{-x}\, dx]$$

$$I_n = n(n-1)\int_0^{\infty} e^{-x} \sin^{n-2} x\, dx - n^2 \int_0^{\infty} e^{-x} \sin^n x\, dx$$

$$= n(n-1) I_{n-2} - n^2 I_n$$

$$\Rightarrow \quad (n^2+1) I_n = n(n-1) I_{n-2}$$

$$\therefore \quad I_n = \frac{n(n-1)}{n^2+1} I_{n-2}$$

Replacing n by $(n-2), (n-4), \ldots, 2$, we get

$$I_{n-2} = \frac{(n-2)(n-3)}{(n-2)^2+1} I_{n-4} = \frac{(n-1)(n-3)}{(n-2)^2+1} \cdot \frac{(n-4)(n-5)}{(n-4)^2+1} I_{n-6}$$

... and so on.

$$I_2 = \frac{2(2-1)}{2^2+1} I_0$$

$$\therefore \quad I_n = \frac{n(n-1)}{n^2+1} \cdot \frac{(n-2)(n-3)}{(n-2)^2+1} \cdot \frac{(n-4)(n-5)}{(n-3)^2+1} \cdots \frac{2(2-1)}{2^2+1} I_0$$

$$= \frac{n!}{\prod_{r=1}^{n/2} \{1+(2r)^2\}} \cdot I_0 = \frac{n!}{\prod_{r=1}^{n/2} 1+4r^2} \left(\because I_0 = \int_0^{\infty} e^{-x} dx = 1\right)$$

● ***Ex. 51*** *Evaluate* $\int_0^1 (tx+1-x)^n\, dx, n \in N$ *and t is independent of x. Hence, show*

$$\int_0^1 x^k (1-x)^{n-k}\, dx = \frac{1}{{}^nC_k (n+1)}.$$

Sol. Here, $I = \int_0^1 (tx+1-x)^n\, dx = \int_0^1 ((t-1)x+1)^n\, dx$

$$= \left\{\frac{((t-1)x+1)^{n+1}}{(n+1)(t-1)}\right\}_0^1$$

$$= \frac{1}{n+1}\{t^n + t^{n-1} + t^{n-2} + \ldots + t + 1\} \qquad \ldots(\text{i})$$

Again, $I = \int_0^1 (tx+1-x)^n\, dx = \int_0^1 \{(1-x)+tx\}^n\, dx$

$$= \int_0^1 \{{}^nC_0 (1-x)^n + {}^nC_1 (1-x)^{n-1}(tx) + {}^nC_2 (1-x)^{n-2}$$

$$(tx)^2 + \ldots + {}^nC_r (1-x)^{n-r} (tx)^r + \ldots + {}^nC_n (tx)^n\}\, dx$$

$$= \int_0^1 \left\{\sum_{r=0}^{n} {}^nC_r (1-x)^{n-r} (tx)^r\right\} dx$$

$$= \sum_{r=0}^{n} {}^nC_r\, t^r \int_0^1 \{(1-x)^{n-r} x^r\}\, dx \qquad \ldots(\text{ii})$$

From Eqs. (i) and (ii),

$$\sum_{r=0}^{n} {}^nC_r\, t^r \int_0^1 (1-x)^{n-r} x^r\, dx = \frac{1}{n+1}\{t^n + t^{n-1} + t^{n-2} + \ldots + t + 1\}$$

Equating coefficient of t^k on both the sides, we get

$${}^nC_k \int_0^1 (1-x)^{n-k} x^k\, dx = \frac{1}{n+1}$$

$$\Rightarrow \qquad \int_0^1 (1-x)^{n-k} x^k\, dx = \frac{1}{{}^nC_k (n+1)}$$

● ***Ex. 52*** *Given a real valued function* $f(x)$ *which is monotonic and differentiable, prove that for any real numbers a and b;*

$$\int_a^b \{f^2(x) - f^2(a)\}\, dx = \int_{f(a)}^{f(b)} 2x\{b - f^{-1}(x)\}\, dx$$

Sol. As, $f(x)$ is differentiable and monotonic.

$\therefore$ $f^{-1}(x)$ exists.

Let $\quad f^{-1}(x) = t \Rightarrow x = f(t)$ or $dx = f'(t)\, dt$

As, $\quad x = f(a) \Rightarrow f^{-1}\{f(a)\} = t \Rightarrow t = a$

and $\quad x = f(b) \Rightarrow f^{-1}\{f(b)\} = t \Rightarrow t = b$

$$\therefore \quad \int_{f(a)}^{f(b)} 2x\{b - f^{-1}(x)\}\, dx = \int_a^b 2f(t)(b-t) f'(t)\, dt$$

$$= \int_a^b (b-t)\{2f(t) f'(t)\}\, dt$$

$$= ((b-t)\{f(t)\}^2)_a^b + \int_a^b \{f(t)\}^2\, dt$$

$$= -(b-a)\{f(a)\}^2 + \int_a^b \{f(t)\}^2\, dt$$

$$= \int_a^b (\{f(t)\}^2 - \{f(a)\}^2)\, dt$$

$$= \int_a^b \{f^2(x) - f^2(a)\}\, dx$$

● ***Ex. 53*** *Evaluate*

$$\int_0^1 \frac{\sin\theta\,(\cos^2\theta - \cos^2 \pi/5)(\cos^2\theta - \cos^2 2\pi/5)}{\sin 5\theta}\, d\theta.$$

Sol. We know,

$$z^{10} - 1 = (z^2-1)\left[z^2 - 2\cos\left(\frac{\pi}{5}\right)z + 1\right]\left[z^2 - 2\left(\cos\frac{2\pi}{5}\right)z + 1\right]$$

$$\times\left(z^2 - 2\cos\frac{4\pi}{5}z + 1\right)\left(z^2 - 2\cos\frac{6\pi}{5}z + 1\right)$$

or $$z^5-\frac{1}{z^5}=\left(z-\frac{1}{z}\right)\left(z-2\cos\frac{\pi}{5}+\frac{1}{z}\right)\left(z-2\cos\frac{2\pi}{5}+\frac{1}{z}\right)$$
$$\times\left(z-2\cos\frac{4\pi}{5}+\frac{1}{z}\right)\left(z-2\cos\frac{6\pi}{5}+\frac{1}{z}\right)$$

Put $z=\cos\theta+i\sin\theta$

$\Rightarrow 2i\sin 5\theta=(2i\sin\theta)(2\cos\theta-2\cos\pi/5)(2\cos\theta-2\cos 2\pi/5)$
$\times(2\cos\theta-2\cos 4\pi/5)(2\cos\theta-2\cos 6\pi/5)$

$\Rightarrow \sin 5\theta=16\sin\theta(\cos^2\theta-\cos^2\pi/5)(\cos^2\theta-\cos^2 2\pi/5)$

$\Rightarrow \dfrac{\sin\theta(\cos^2\theta-\cos^2\pi/5)(\cos^2\theta-\cos^2 2\pi/5)}{\sin 5\theta}=\dfrac{1}{16}$

$\therefore \int_0^1 \dfrac{\sin\theta(\cos^2\theta-\cos^2\pi/5)(\cos^2\theta-\cos^2 2\pi/5)}{\sin 5\theta}d\theta$
$=\int_0^1 \dfrac{1}{16}d\theta=\dfrac{1}{16}$

● **Ex. 54** *Show that* $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\dfrac{{}^nC_K}{n^K(K+3)}=e-2$.

Sol. $\lim\limits_{n\to\infty}\sum\limits_{K=0}^{n}\dfrac{{}^nC_K}{n^K(K+3)}=\lim\limits_{n\to\infty}\sum\limits_{K=0}^{n}\dfrac{1}{K+3}{}^nC_K\cdot\dfrac{1}{n^K}$

$=\lim\limits_{n\to\infty}\sum\limits_{K=0}^{n}{}^nC_K\cdot\dfrac{1}{n^K}\int_0^1 x^{K+2}dx\left(\because\dfrac{1}{K+3}=\int_0^1 x^{K+2}dx\right)$

$=\int_0^1\left(x^2\lim\limits_{n\to\infty}\sum\limits_{K=0}^{n}{}^nC_K\cdot\left(\dfrac{x}{n}\right)^K\right)dx$

$=\int_0^1 x^2\left\{\lim\limits_{n\to\infty}\left(1+\dfrac{x}{n}\right)^n\right\}dx$

$=\int_0^1 x^2\cdot e^x dx\qquad\left[\because\lim\limits_{n\to\infty}\left(1+\dfrac{x}{n}\right)^n=e^x\right]$

$=(x^2\cdot e^x)_0^1-\int_0^1 2x\cdot e^x dx=e-2\left\{(xe^x)_0^1-\int_0^1 e^x dx\right\}$

$=e-2\{e-e+1\}=e-2$

● **Ex. 55** *Let* $I=\int_0^{\pi/2}\dfrac{\cos x}{a\cos x+b\sin x}dx$

and $J=\int_0^{\pi/2}\dfrac{\sin x}{a\cos x+b\sin x}dx$, *where* $a>0$ and $b>0$.

Compute the values of I and J.

Sol. $\because$ $$aI+bJ=\frac{\pi}{2}\qquad\text{...(i)}$$

and $$bI-aJ=\int_0^{\pi/2}\frac{b\cos x-a\sin x}{a\cos x+b\sin x}dx$$

$\therefore$ $$bI-aJ=\ln[a\cos x+b\sin x]_0^{\pi/2}$$

$\Rightarrow$ $$bI-aJ=\ln\left(\frac{b}{a}\right)\qquad\text{...(ii)}$$

From Eqs. (i) and (ii),

$$a^2I+abJ=\frac{a\pi}{2}$$
$$\underline{b^2I-abJ=b\ln(b/a)}$$
$$I=\frac{1}{a^2+b^2}\left[\frac{a\pi}{2}+b\ln\left(\frac{b}{a}\right)\right]$$

Again, $$abI+b^2I=\frac{b\pi}{2}$$

and $$\underline{abI-a^2J=a\ln(b/a)}$$

On subtracting, we get $J=\dfrac{1}{a^2+b^2}\left[\dfrac{b\pi}{2}-a\ln\left(\dfrac{b}{a}\right)\right]$

Aliter Convert $a\cos x+b\sin x$ into a single cosine say $\cos(x+\phi)$ and put $(x+\phi)=t$.

● **Ex. 56** *Evaluate* $\int_0^\infty\dfrac{\ln x\,dx}{x^2+2x+4}$.

Sol. $I=\int_0^\infty\dfrac{\ln x\,dx}{x^2+2x+4}$ (put $x=2t\Rightarrow dx=2dt$ to make coefficient of x^2 and constant term same)

$$=2\int_0^\infty\frac{\ln 2+\ln t}{4(t^2+t+1)}dt=\frac{\ln 2}{2}\underbrace{\int_0^\infty\frac{dt}{t^2+t+1}}_{I_1}+\frac{1}{2}\underbrace{\int_0^\infty\frac{\ln t\,dt}{t^2+t+1}}_{I_2=\text{zero}}$$

$$I_2=\int_0^\infty\frac{\ln t\,dt}{t^2+t+1}\left(\text{put } t=\frac{1}{y}\Rightarrow dt=-\frac{1}{y^2}dy\right)$$

$$I_2=\int_\infty^0\frac{\ln y\cdot(+1)}{\left(\dfrac{1}{y^2}+\dfrac{1}{y}+1\right)y^2}dy$$

$$I_2=\int_\infty^0\frac{\ln y\,dy}{y^2+y+1}=-\int_\infty^0\frac{\ln y\,dy}{y^2+y+1}=-I_2$$

Now, $I_1=\int_0^\infty\dfrac{dt}{\left(t+\dfrac{1}{2}\right)^2+\left(\dfrac{\sqrt{3}}{2}\right)^2}=\left[\dfrac{2}{\sqrt{3}}\tan^{-1}\dfrac{[t+(1/2)]^2}{\sqrt{3}}\right]_0^\infty$

$=\dfrac{2}{\sqrt{3}}\left[\dfrac{\pi}{2}-\dfrac{\pi}{6}\right]=\dfrac{2\pi}{3\sqrt{3}}\qquad\therefore I=\dfrac{\ln 2}{2}\cdot\dfrac{2\pi}{3\sqrt{3}}=\dfrac{\pi\ln 2}{3\sqrt{3}}$

Note For $a>0$, $I=\int_0^\infty\dfrac{\ln x\,dx}{ax^2+bx+a}=0$

[**Hint** By putting $x=1/t$, we get $I=-I$, so $I=0$]

● **Ex. 57** *Find a function* f, *continuous for all x (and not zero everywhere) such that* $f^2(x)=\int_0^x\dfrac{f(t)\sin t}{2+\cos t}dt$.

Sol. We have, $f^2(x)=\int_0^x\dfrac{f(t)\sin t}{2+\cos t}dt$

(Note that $f^2(x)$ being an integral function of a continuous function is continuous and differentiable)

$$2f(x)f'(x)=\frac{f(x)\cdot\sin x}{2+\cos x}$$

Integrating, we get $2f(x)=C-\ln(2+\cos x)$

$\because$ $x=0\Rightarrow f(0)=0\Rightarrow C=\ln 3$

$\therefore$ $$f(x)=\frac{1}{2}\ln\frac{3}{2+\cos x}$$

● ***Ex. 58*** *Evaluate* $\int_0^\infty \frac{\tan^{-1} ax - \tan^{-1} x}{x} dx$, *where a is a parameter.*

Sol. Let $I = \int_0^\infty \frac{\tan^{-1} ax - \tan^{-1} x}{x} dx$

$$\therefore \quad \frac{dI}{da} = \int_0^\infty \frac{1 \cdot x}{(1 + a^2x^2)x} dx = \int_0^\infty \frac{dx}{1 + a^2x^2}$$

$$= \frac{1}{a^2} \int_0^\infty \frac{dx}{x^2 + \frac{1}{a^2}} = \frac{1}{a^2} [a \tan^{-1} x]_0^\infty = \frac{\pi}{2a}$$

$$\int dI = \frac{\pi}{2} \int \frac{da}{a} \Rightarrow I = \frac{\pi}{2} \ln a + C$$

When $a = 1$ and $I = 0$ then $C = 0$

Hence, $I = \frac{\pi}{2} \ln a$

● ***Ex. 59*** *Evalute* $\int_0^1 \frac{\ln(1 - a^2x^2)}{x^2\sqrt{(1 - x^2)}} dx \, (a^2 < 1).$

Sol. Let $I = \int_0^1 \frac{\ln(1 - a^2x^2)}{x^2\sqrt{(1 - x^2)}} dx, |a| < 1$

$$\therefore \quad \frac{dI}{da} = \int_0^1 \frac{-2ax^2}{(1 - a^2x^2)} \cdot \frac{dx}{x^2\sqrt{1 - x^2}} \qquad (\text{put } x = \sin\theta)$$

$$= \int_0^{\pi/2} \frac{2a \, d\theta}{a^2 \sin^2\theta - 1} = \int_0^{\pi/2} \frac{2a \sec^2\theta \, d\theta}{a^2 \tan^2\theta - (1 + \tan^2\theta)}$$

$$= -\int_0^{\pi/2} \frac{2a \sec^2\theta \, d\theta}{(1 - a^2)\tan^2\theta + 1} \qquad (\text{put } \tan\theta = 1)$$

$$= \int_0^\infty \frac{2a \, dt}{(1 - a^2)t^2 + 1} \text{ or } -\frac{2a}{(1 - a^2)} \int_0^\infty \frac{dt}{t^2 + \left(\frac{1}{\sqrt{1 - a^2}}\right)^2}$$

$$= -\frac{2a}{\sqrt{1 - a^2}} [\tan^{-1} t\sqrt{1 - a^2}]_0^\infty = -\frac{\pi \cdot 2a}{2\sqrt{1 - a^2}} = -\frac{\pi a}{\sqrt{1 - a^2}}$$

or $I = \pi\sqrt{1 - a^2} + C$

If $a = 0$ and $I = 0$, then $C = -\pi$

$$\therefore \quad I = \pi\left(\sqrt{1 - a^2} - 1\right)$$

● ***Ex. 60*** *Evaluate* $\int_0^{\pi/2} \ln\left(\frac{1 + a\sin x}{1 - a\sin x}\right) \frac{dx}{\sin x} (|a| < 1).$

Sol. Let $I = \int_0^{\pi/2} \ln\left(\frac{1 + a\sin x}{1 - a\sin x}\right) \frac{dx}{\sin x} (|a| < 1)$

$$\therefore \quad \frac{dI}{da} = \int_0^{\pi/2} \frac{2\sin x}{(1 - a^2\sin^2 x)} \cdot \frac{dx}{\sin x}$$

$$= \int_0^{\pi/2} \frac{2\sec^2 x}{1 + \tan^2 x - a^2 \tan^2 x} dx$$

$$= \int_0^{\pi/2} \frac{2\sec^2 x}{(1 - a^2)\tan^2 x + 1} dx \qquad (\text{put } \tan x = t)$$

$$= \int_0^\infty \frac{2 \, dt}{(1 - a^2)t^2 + 1} = \frac{2}{(1 - a^2)} \int_0^\infty \frac{dt}{t^2 + \left(\frac{1}{\sqrt{1 - a^2}}\right)^2}$$

$$\Rightarrow \quad = \frac{2}{\sqrt{1 - a^2}} [\tan^{-1}(t\sqrt{1 - a^2})]_0^\infty = \frac{\pi}{\sqrt{1 - a^2}}$$

$$\Rightarrow \quad dI = \frac{\pi \, da}{\sqrt{1 - a^2}}$$

When $a = 0$ and $I = 0$, then $C = 0$

$$\therefore \quad I = \pi \sin^{-1}(a)$$

Hence, $I = \pi \sin^{-1} a + C$

Definite Integral Exercise 1 :
Single Option Correct Type Questions

1. The value of $\int_0^4 \frac{(y^2-4y+5)\sin(y-2)dy}{(2y^2-8y+1)}$ is

(a) 0 (b) 2
(c) -2 (d) None of these

2. Let $f(x)=x^2+ax+b$ and the only solution of the equation $f(x)=$ minimum $f(x)$ is $x=0$ and $f(x)=0$ has root α and β, then $\int_\alpha^\beta x^3\,dx$ is equal to

(a) $\frac{1}{4}(\beta^4+\alpha^4)$ (b) $\frac{1}{4}(a^2-b^2)$
(c) 0 (d) None of these

3. $\int_{\pi/2}^{x}\sqrt{(3-2\sin^2 t)}\,dt+\int_0^y \cos t\,dt=0$, then $\left(\frac{dy}{dx}\right)$ at $x=\pi$ and $y=\pi$ is

(a) $\sqrt{3}$ (b) $-\sqrt{2}$
(c) $-\sqrt{3}$ (d) None of these

4. $\int_{-4}^{4}\frac{\sin^{-1}(\sin x)+\cos^{-1}(\cos x)}{(1+x^2)\left(1+\left[\frac{x^2}{17}\right]\right)}dx=\log\left(\frac{(1+x^2)}{a}\right)+b\pi\tan^{-1}\left(\frac{c-\pi}{1+c\pi}\right)$ (where, [·] denotes greatest integer function), then number of ways in which $a-(2b+c)$ distinct object can distributed among $\frac{a-5}{c}$ persons equally, is

(a) $\frac{9!}{(3!)^3}$ (b) $\frac{12!}{(4!)^3}$ (c) $\frac{15!}{(5!)^3}$ (d) $\frac{10!}{(6!)^3}\times 3!$

5. The value of the definite integral $\int_0^\infty \frac{dx}{(1+x^a)(1+x^2)}\,(a>0)$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$
(c) π (d) Some function of a

6. The value of the definite integral $\int_0^{3\pi/4}[(1+x)\sin x+(1-x)\cos x]\,dx$ is

(a) $2\tan\frac{3\pi}{8}$ (b) $2\tan\frac{\pi}{4}$
(c) $2\tan\frac{\pi}{8}$ (d) 0

7. Let $C_n=\int_{1/n+1}^{1/n}\frac{\tan^{-1}(nx)}{\sin^{-1}(nx)}dx$, then $\lim_{n\to\infty} n^2\cdot C_n$ is equal to

(a) 1 (b) 0
(c) -1 (d) $\frac{1}{2}$

8. If x satisfies the equation $\left(\int_0^1 \frac{dt}{t^2+2t\cos\alpha+1}\right)x^2-\left(\int_{-3}^{3}\frac{t^2\sin 2t}{t^2+1}dt\right)x-2=0\,(0<\alpha<\pi)$, then the value of x is

(a) $\pm\sqrt{\frac{\alpha}{2\sin\alpha}}$ (b) $\pm\sqrt{\frac{2\sin\alpha}{\alpha}}$
(c) $\pm\sqrt{\frac{\alpha}{\sin\alpha}}$ (d) $\pm 2\sqrt{\frac{\sin\alpha}{\alpha}}$

9. If $f(x)=e^{g(x)}$ and $g(x)=\int_2^x\frac{t\,dt}{1+t^4}$, then $f'(2)$ is equal to

(a) 2/17 (b) 0
(c) 1 (d) Cannot be determined

10. If a, b and c are real numbers, then the value of $\lim_{t\to 0}\ln\left(\frac{1}{t}\int_0^t(1+a\sin bx)^{c/x}\,dx\right)$ equals

(a) abc (b) $\frac{ab}{c}$
(c) $\frac{bc}{a}$ (d) $\frac{ca}{b}$

11. The value of $\lim_{n\to\infty}\sum_{r=1}^{r=4n}\frac{\sqrt{n}}{\sqrt{r}\,(3\sqrt{r}+4\sqrt{n})^2}$ is equal to

(a) $\frac{1}{35}$ (b) $\frac{1}{14}$
(c) $\frac{1}{10}$ (d) $\frac{1}{5}$

12. Let $f(x)=\int_{-1}^{x}e^{t^2}\,dt$ and $h(x)=f(1+g(x))$ where $g(x)$ is defined for all x, $g'(x)$ exists for all x, and $g(x)<0$ for $x>0$. If $h'(1)=e$ and $g'(1)=1$, then the possible values which $g(1)$ can take

(a) 0 (b) -1
(c) -2 (d) -4

13. Let $f(x)$ be a function satisfying $f'(x)=f(x)$ with $f(0)=1$ and g be the function satisfying $f(x)+g(x)=x^2$.

The value of the integral $\int_0^1 f(x)\,g(x)\,dx$ is

(a) $e-\frac{1}{2}e^2-\frac{5}{2}$ (b) $e-e^2-3$
(c) $\frac{1}{2}(e-3)$ (d) $e-\frac{1}{2}e^2-\frac{3}{2}$

14. Let $f(x)=\int_0^{g(x)} \frac{dt}{\sqrt{1+t^2}}$, where $g(x)=\int_0^{\cos x}(1+\sin t^2)\,dt$. Also, $h(x)=e^{-|x|}$ and $f(x)=x^2\sin\frac{1}{x}$, if $x\neq 0$ and $f(0)=0$, then $f'\left(\frac{\pi}{2}\right)$ is equal to

(a) $l'(0)$ (b) $h'(0^-)$

(c) $h'(0^+)$ (d) $\lim_{x\to 0}\frac{1-\cos x}{x\sin x}$

15. For $f(x)=x^4+|x|$, let $I_1=\int_0^{\pi} f(\cos x)\,dx$ and $I_2=\int_0^{\pi/2} f(\sin x)\,dx$, then $\frac{I_1}{I_2}$ has the value equal to

(a) 1 (b) 1/2

(c) 2 (d) 4

16. Let f be a positive function. Let $I_1=\int_{1-k}^{k}(x)\,f(x(1-x))\,dx; I_2=\int_{1-k}^{k} f(x(1-x))\,dx$, where $2k-1>0$. Then, $\frac{I_2}{I_1}$ is

(a) k (b) 1/2

(c) 1 (d) 2

17. Suppose that the quadratic function $f(x)=ax^2+bx+c$ is non-negative on the interval $[-1,1]$. Then, the area under the graph of f over the interval $[-1,1]$ and the X-axis is given by the formula

(a) $A=f(-1)+f(1)$

(b) $A=f\left(-\frac{1}{2}\right)+f\left(\frac{1}{2}\right)$

(c) $A=\frac{1}{2}[f(-1)+2f(0)+f(1)]$

(d) $A=\frac{1}{3}[f(-1)+4f(0)+f(1)]$

18. Let $I(a)=\int_0^{\pi}\left(\frac{x}{a}+a\sin x\right)^2 dx$, where '$a$' is positive real. The value of 'a' for which $I(a)$ attains its minimum value, is

(a) $\sqrt{\pi\sqrt{\frac{2}{3}}}$ (b) $\sqrt{\pi\sqrt{\frac{3}{2}}}$ (c) $\sqrt{\frac{\pi}{16}}$ (d) $\sqrt{\frac{\pi}{13}}$

19. The set of values of 'a' which satisfy the equation $\int_0^2(t-\log_2 a)\,dt=\log_2\left(\frac{4}{a^2}\right)$, is

(a) $a\in R$ (b) $a\in R^+$

(c) $a<2$ (d) $a>2$

20. $\lim_{x\to\infty}\left(x^3\int_{-1/x}^{1/x}\frac{\ln(1+t^2)}{1+e^t}\,dt\right)$ is equal to

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$

(c) 1 (d) 0

21. The value of $\sqrt{\pi\left(\int_0^{2008} x\,|\sin\pi x|\,dx\right)}$ is equal to

(a) $\sqrt{2008}$ (b) $\pi\sqrt{2008}$ (c) 1004 (d) 2008

22. $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{n^2+k^2x^2}$, $x>0$ is equal to

(a) $x\tan^{-1}(x)$ (b) $\tan^{-1}(x)$

(c) $\frac{\tan^{-1}(x)}{x}$ (d) $\frac{\tan^{-1}(x)}{x^2}$

23. Let $a>0$ and $f(x)$ is monotonic increasing such that $f(0)=0$ and $f(a)=b$, then $\int_0^a f(x)\,dx+\int_0^b f^{-1}(x)\,dx$ is equal to

(a) $a+b$ (b) $ab+b$ (c) $ab+a$ (d) ab

24. If $\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{x^4}{1-x^4}\cos^{-1}\frac{2x}{1+x^2}\,dx=k\int_0^{1/\sqrt{3}}\frac{x^4}{1-x^4}$, then '$k$' is equal to

(a) π (b) 2π (c) 2 (d) 1

25. $\int_0^{\infty} f\left(x+\frac{1}{x}\right)\cdot\frac{\ln x}{x}\,dx$ is equal to

(a) 0 (b) 1

(c) $\frac{1}{2}$ (d) Cannot be evaluated

26. $\lim_{\lambda\to 0}\left(\int_0^1(1+x)^{\lambda}\,dx\right)^{1/\lambda}$ is equal to

(a) $2\ln 2$ (b) $\frac{4}{e}$ (c) $\ln\frac{4}{e}$ (d) 4

27. If $g(x)$ is the inverse of $f(x)$ and $f(x)$ has domain $x\in[1,5]$, where $f(1)=2$ and $f(5)=10$, then the values of $\int_1^5 f(x)\,dx+\int_2^{10} g(y)\,dy$ is equal to

(a) 48 (b) 64 (c) 71 (d) 52

28. The value of the definite integral $\int_0^{\pi/2}\sin x\sin 2x\sin 3x\,dx$ is equal to

(a) $\frac{1}{3}$ (b) $-\frac{2}{3}$ (c) $-\frac{1}{3}$ (d) $\frac{1}{6}$

29. If $f(x)=\int_0^x(f(t))^2\,dt$, $f: R\to R$ be differentiable function and $f(g(x))$ is differentiable function at $x=a$, then

(a) $g(x)$ must be differentiable at $x=a$

(b) $g(x)$ may be non-differentiable at $x=a$

(c) $g(x)$ may be discontinuous at $x=a$

(d) None of the above

30. The number of integral solutions of the equation $4\int_0^{\infty}\frac{\ln t\,dt}{x^2+t^2}-\pi\ln 2=0; x>0$, is

(a) 0 (b) 1 (c) 2 (d) 3

31. $\int_0^{16n^2/\pi}\cos\frac{\pi}{2}\left[\frac{x\pi}{n}\right]dx$ is equal to

(a) 0 (b) 1 (c) 2 (d) 3

32. If $\int_{-2}^{-1}(ax^2-5)\,dx=0$ and $5+\int_{1}^{2}(bx+c)\,dx=0$, then
(a) $ax^2-bx+c=0$ has atleast one root in (1, 2)
(b) $ax^2-bx+c=0$ has atleast one root in (−2, −1)
(c) $ax^2+bx+c=0$ has atleast one root in (−2, −1)
(d) None of the above

33. The value of $\int_0^6 \left(\sqrt{x+\sqrt{12x-36}}+\sqrt{x-\sqrt{12x-36}}\right)dx$ is equal to
(a) $6\sqrt{3}$ (b) $4\sqrt{3}$
(c) $12\sqrt{3}$ (d) None of these

34. If $I_n=\int_{-n}^{n}(\{x+1\}\{x^2+2\}+\{x^2+3\}\{x^3+4\})\,dx$, (where, {} denotes the fractional part), then I_1 is equal to
(a) $-\frac{1}{3}$ (b) $-\frac{2}{3}$ (c) $\frac{1}{3}$ (d) None of these

35. Let $I_n=\int_0^{\pi/2}\frac{\sin(2n-1)x}{\sin x}\,dx$, $\mathcal{J}_n=\int_0^{\pi/2}\frac{\sin^2 nx}{\sin^2 x}$, $n\in N$, then
(a) $\mathcal{J}_{n+1}-\mathcal{J}_n=I_n$ (b) $\mathcal{J}_{n+1}-\mathcal{J}_n=I_{n+1}$
(c) $\mathcal{J}_{n+1}+\mathcal{J}_n=\mathcal{J}_n$ (d) $\mathcal{J}_{n+1}-\mathcal{J}_{n+1}=\mathcal{J}_n$

Definite Integral Exercise 2 : More than One Option Correct Type Questions

36. If $f(x)=[\sin^{-1}(\sin 2x)]$ (where, [] denotes the greatest integer function), then
(a) $\int_0^{\pi/2}f(x)\,dx=\frac{\pi}{2}-\sin^{-1}(\sin 1)$
(b) $f(x)$ is periodic with period π
(c) $\lim_{x\to\frac{\pi}{2}}f(x)=-1$
(d) None of the above

37. Which of the following definite integral(s) vanishes?
(a) $\int_0^{\pi/2}\ln(\cot x)\,dx$ (b) $\int_0^{2\pi}\sin^3 x\,dx$
(c) $\int_{1/e}^{e}\frac{dx}{x(\ln x)^{1/3}}$ (d) $\int_0^{\pi}\sqrt{\frac{1+\cos 2x}{2}}\,dx$

38. The equation $10x^4-3x^2-1=0$ has
(a) atleast one root in (−1, 0)
(b) atleast one root in (0, 1)
(c) atleast two roots in (−1, 1)
(d) no root in (−1, 1)

39. Suppose $I_1=\int_0^{\pi/2}\cos(\pi\sin^2 x)\,dx$;
$I_2=\int_0^{\pi/2}\cos(2\pi\sin^2 x)\,dx$ and $I_3=\int_0^{\pi/2}\cos(\pi\sin x)\,dx$,
then
(a) $I_1=0$ (b) $I_2+I_3=0$
(c) $I_1+I_2+I_3=0$ (d) $I_2=I_3$

40. Let $f(x)=\int_{-1}^{1}(1-|t|)\cos(xt)\,dt$, then which of the following holds true?
(a) $f(0)$ is not defined
(b) $\lim_{x\to 0}f(x)$ exists and is equal to 2
(c) $\lim_{x\to 0}f(x)$ exists and is equal to 1
(d) $f(x)$ is continuous at $x=0$

41. The function f is continuous and has the property $f(f(x))=1-x$ for all $x\in[0,1]$ and $\mathcal{J}=\int_0^1 f(x)\,dx$, then
(a) $f\left(\frac{1}{4}\right)+f\left(\frac{3}{4}\right)=1$
(b) the value of $\mathcal{J}$ equals to 1/2
(c) $f\left(\frac{1}{3}\right)\cdot f\left(\frac{2}{3}\right)=1$
(d) $\int_0^{\pi/2}\frac{\sin x\,dx}{(\sin x+\cos x)^3}$ has the same value as $\mathcal{J}$

42. Let $f(x)$ is a real valued function defined by
$f(x)=x^2+x^2\int_{-1}^{1}tf(t)\,dt+x^3\int_{-1}^{1}f(t)\,dt$,
then which of the following hold(s) good?
(a) $\int_{-1}^{1}t\,f(t)\,dt=\frac{10}{11}$ (b) $f(1)+f(-1)=\frac{30}{11}$
(c) $\int_{-1}^{1}t\,f(t)\,dt>\int_{-1}^{1}f(t)\,dt$ (d) $f(1)-f(-1)=\frac{20}{11}$

43. Let $f(x)$ and $g(x)$ be differentiable functions such that $f(x)+\int_0^x g(t)\,dt=\sin x(\cos x-\sin x)$ and $(f'(x))^2+(g(x))^2=1$, then $f(x)$ and $g(x)$ respectively, can be
(a) $\frac{1}{2}\sin 2x, \sin 2x$ (b) $\frac{\cos 2x}{2}, \cos 2x$
(c) $\frac{1}{2}\sin 2x, -\sin 2x$ (d) $-\sin^2 x, \cos 2x$

44. Let $f(x)=\int_{-x}^{x}(t\sin at+bt+c)dt$, where a, b, c are non-zero real numbers, then $\lim_{x\to 0}\frac{f(x)}{x}$ is
(a) independent of a
(b) independent of a and b, and has the value equals to c
(c) independent a, b and c
(d) dependent only on c

45. Let $L=\lim_{n\to\infty}\int_a^{\infty}\frac{n\,dx}{1+n^2x^2}$, where $a\in R$, then L can be
(a) π (b) $\pi/2$ (c) 0 (d) 1

Definite Integral Exercise 3 : Passage Based Questions

Passage I

(Q. Nos. 46 to 48)

Suppose $\lim_{x\to 0}\dfrac{\int_0^x \dfrac{t^2\,dt}{(a+t^r)^{1/p}}}{bx-\sin x}=l$, *where* $p\in N,\ p\geq 2, a>0, r>0$ *and* $b\neq 0$.

46. If l exists and is non-zero, then
(a) $b>1$ (b) $0<b<1$
(c) $b<0$ (d) $b=1$

47. If $p=3$ and $l=1$, then the value of 'a' is equal to
(a) 8 (b) 3
(c) 6 (d) 3/2

48. If $p=2$ and $a=9$ and l exists, then the value of l is equal to
(a) $\frac{3}{2}$ (b) $\frac{2}{3}$ (c) $\frac{1}{3}$ (d) $\frac{7}{9}$

Passage II

(Q. Nos. 49 to 51)

Suppose $f(x)$ and $g(x)$ *are two continuous functions defined for* $0\leq x\leq 1$. *Given,* $f(x)=\int_0^1 e^{x+1}\cdot f(t)\,dt$ *and* $g(x)=\int_0^1 e^{x+t}\cdot g(t)\,dt+x$.

49. The value of $f(1)$ equals
(a) 0 (b) 1
(c) e^{-1} (d) e

50. The value of $g(0)-f(0)$ equals
(a) $\frac{2}{3-e^2}$ (b) $\frac{3}{e^2-2}$ (c) $\frac{1}{e^2-1}$ (d) 0

51. The value of $\frac{g(0)}{g(2)}$ equals
(a) 0 (b) $\frac{1}{3}$ (c) $\frac{1}{e^2}$ (d) $\frac{2}{e^2}$

Passage III

(Q. Nos. 52 to 54)

We are given the curves $y=\int_{-\infty}^x f(t)dt$ *through the point* $\left(0,\frac{1}{2}\right)$ *any* $y=f(x)$, *where* $f(x)>0$ *and* $f(x)$ *is differentiable,* $\forall\, x\in R$ *through* $(0,1)$. *Tangents drawn to both the curves at the points with equal abscissae intersect on the same point on the X-axis.*

52. The number of solutions $f(x)=2ex$ is equal to
(a) 0 (b) 1
(c) 2 (d) None of these

53. $\lim_{x\to\infty}(f(x))^{f(-x)}$ is
(a) 3 (b) 6
(c) 1 (d) None of these

54. The function $f(x)$ is
(a) increasing for all x (b) non-monotonic
(c) decreasing for all x (d) None of these

Passage IV

(Q. Nos. 55 to 57)

$f(x)=\int_0^x (4t^4-at^3)\,dt$ *and* $g(x)$ *is quadratic satisfying* $g(0)+6=g'(0)-c=g''(c)+2b=0$. $y=h(x)$ and $y=g(x)$ *intersect in* 4 *distinct points with abscissae* x_i; $i=1,2,3,4$ *such that* $\sum\frac{i}{x_i}=8, a, b, c\in R^+$ *and* $h(x)=f'(x)$.

55. Abscissae of point of intersection are in
(a) AP (b) GP
(c) HP (d) None of these

56. 'a' is equal to
(a) 6 (b) 8 (c) 20 (d) 12

57. 'c' is equal to
(a) 25 (b) 25/2 (c) 25/4 (d) 25/8

Passage V

(Q. Nos. 58 to 60)

Let $y=\int_{u(x)}^{v(x)} f(t)\,dt$, *let us define* $\frac{dy}{dx}$ *as* $\frac{dy}{dx}$ $=v'(x)\,f^2(v(x))-u'(x)\,f^2(u(x))$ *and the equation of the tangent at* (a,b) *and* $y-b=\left(\frac{dy}{dx}\right)_{(a,b)}(x-a)$.

58. If $y=\int_x^{x^2} t^2\,dt$, then the equation of tangent at $x=1$ is
(a) $x+y=1$ (b) $y=x-1$
(c) $y=x$ (d) $y=x+1$

59. If $y=\int_{x^2}^{x^4}(\ln t)dt$, then $\lim_{x\to 0^+}\frac{dy}{dx}$ is equal to
(a) 0 (b) 1 (c) 2 (d) −1

60. If $f(x)=\int_1^x e^{t^2/2}(1-t^2)\,dt$, then $\frac{d}{dx}f(x)$ at $x=1$ is
(a) 0 (b) 1
(c) 2 (d) −1

Passage VI (Q. Nos. 61 to 62)

Consider $f:(0,\infty)\to\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, *defined as* $f(x)=\tan^{-1}\left(\frac{\log_e x}{(\log_e x)^2+1}\right)$.

61. The about function can be classified as
(a) Injective but nor surjective
(b) Surjective but not bijective
(c) Neither injective nor surjective
(d) Both injective and surjective

62. The value of $\int_0^{\infty}[\tan]\,dx$ is equal to (where, [·] denotes the greatest integer function)
(a) $-\frac{\pi}{2}$ (b) $\frac{\pi}{2}$
(c) -1 (d) 1

Definite Integral Exercise 4 : Matching Type Questions

63. Let $\lim_{T\to\infty}\frac{1}{T}\int_0^T(\sin x+\sin ax)^2\,dx=L$, then

	Column I		Column II
(A)	For $a=0$, the value of L is	(p)	0
(B)	For $a=1$, the value of L is	(q)	1/2
(C)	For $a=-1$, the value of L is	(r)	1
(D)	For $a\in R-\{-1,0,1\}$, the value of L is	(s)	2

64. Let $f(\theta=\int_0^1(x+\sin\theta)^2\,dx$ and $g(\theta)=\int_0^1(x+\cos\theta)^2\,dx$ where $\theta\in[0,2\pi]$. The quantity $f(\theta)-g(\theta),\forall\,\theta$ in the interval given in column I, is

	Column I		Column II
(A)	$\left(\frac{\pi}{4},\frac{3\pi}{4}\right)$	(p)	negative
(B)	$\left(\frac{3\pi}{4},\pi\right]$	(q)	positive
(C)	$\left[\frac{3\pi}{2},\frac{7\pi}{4}\right)$	(r)	non-negative
(D)	$\left(0,\frac{\pi}{4}\right)\cup\left(\frac{7\pi}{4},2\pi\right)$	(s)	non-positive

65. Match the following.

	Column I		Column II
(A)	$\int_0^1(1+2008\,x^{2008})\,e^{x^{2008}}\,dx$ equals	(p)	e^{-1}
(B)	The value of the definite integral $\int_0^1 e^{-x^2}\,dx+\int_1^{1/e}\sqrt{-\ln x}\,dx$ is equal to	(q)	$e^{-1/4}$
(C)	$\lim_{n\to\infty}\left(\frac{1^1\cdot2^2\cdot3^3\ldots(n-1)^{n-1}\cdot n^n}{n^{1+2+3+\ldots+n}}\right)^{\frac{1}{n^2}}$ equals	(r)	$e^{1/2}$
		(s)	e

66. Match the following.

	Column I		Column II
(A)	If $f(x)=\int_0^{g(x)}\frac{dt}{\sqrt{1+t^3}}$, where $g(x)=\int_0^{\cos x}(1+\sin t^2)\,dt$, then value of $f'\left(\frac{\pi}{2}\right)$ is	(p)	-2
(B)	If $f(x)$ is a non-zero differentiable function such that $\int_0^x f(t)\,dt=\{f(x)\}^2$, $\forall x\in R$, then $f(2)$ is equal to	(q)	2
(C)	If $\int_a^b(2+x-x^2)\,dx$ is maximum, then $a+b$ is equal to	(r)	1
(D)	If $\lim_{x\to0}\left(\frac{\sin 2x}{x^3}+a+\frac{b}{x^2}\right)=0$, then $3a+b$ has the value	(s)	-1

Definite Integral Exercise 5 :
Single Integer Answer Type Questions

67. If $f(x)=\sum_{n=1}\frac{\sin nx}{4^n}$ and $\int_0^{\pi} f(x)\,dx=\log\left(\frac{m}{n}\right)$, then the value of $(m+n)$ is ...

68. The value of $I=\int_{-\pi/2}^{\pi/2}\frac{\cos x\,dx}{1+2[\sin^{-1}(\sin x)]}$ (where, [·] denotes greatest integer function) is ...

69. If $f(x)=\frac{1}{\pi}\int_0^{\pi/2}\frac{\sin^2 n\theta}{\sin^2\theta}d\theta\,(n\in N)$, then the value of $\frac{f(15)+f(13)}{f(15)-f(a)}$...

70. Let $f(x)=\int_{-2}^{x}e^{(1+t)^2}dt$ and $g(x)=(h/x)$, where $h(x)$ is defined for all $x\in h$. If $g^{1/2}=e^4$ and $h'(2)=1$. Then, absolute value of sun for all possible value of $h(2)$, is ...

71. If $=\int_0^{\pi/2}\sin x\cdot\log(\sin x)dx=\log\left(\frac{K}{e}\right)$. Then, the value of K is ...

Definite Integrals Exercise 6 :
Questions Asked in Previous 10 Years' Exams

(i) JEE Advanced & IIT-JEE

72. The value of $\int_{-\pi/2}^{\pi/2}\frac{x^2\cos x}{1+e^x}dx$ is equal to **[Only One Correct Option 2016]**

(a) $\frac{\pi^2}{4}-2$ (b) $\frac{\pi^2}{4}+2$

(c) $\pi^2-e^{-\pi/2}$ (d) $\pi^2+e^{\pi/2}$

73. The total number of distincts $x\in[0,1]$ for which $\int_0^x\frac{t^2}{1+t^4}dt=2x-1$ is **[Integer Type 2016]**

74. Let $f(x)=7\tan^8 x+7\tan^6 x-3\tan^4 x-3\tan^2 x$ for all $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. Then, the correct expression(s) is/are **[More than One Correct Option 2015]**

(a) $\int_0^{\pi/4}x f(x)\,dx=\frac{1}{12}$ (b) $\int_0^{\pi/4}f(x)\,dx=0$

(c) $\int_0^{\pi/4}x f(x)\,dx=\frac{1}{6}$ (d) $\int_0^{\pi/4}f(x)\,dx=1$

75. Let $f'(x)=\frac{192x^3}{2+\sin^4\pi x}$ for all $x\in R$ with $f\left(\frac{1}{2}\right)=0$. If $m\le\int_{t/2}^{1}f(x)\,dx\le M$, then the possible values of m and M are **[More than One Correct Option 2015]**

(a) $m=13,\ M=24$ (b) $m=\frac{1}{4},\ M=\frac{1}{2}$

(c) $m=-11,\ M=0$ (d) $m=1,\ M=12$

76. The option(s) with the values of a and L that satisfy the equation $\frac{\int_0^{4\pi}e^t(\sin^6 at+\cos^4 at)dt}{\int_0^{\pi}e^t(\sin^6 at+\cos^4 at)dt}=L$, is/are **[More than One Correct Option 2015]**

(a) $a=2,\ L=\frac{e^{4\pi}-1}{e^{\pi}-1}$ (b) $a=2,\ L=\frac{e^{4\pi}+1}{e^{\pi}+1}$

(c) $a=4,\ L=\frac{e^{4\pi}-1}{e^{\pi}-1}$ (d) $a=4,\ L=\frac{e^{4\pi}+1}{e^{\pi}+1}$

■ **Directions** (Q. Nos. 77 to 78) Let $F:R\to R$ be a thrice differentiable function. Suppose that $F(1)=0,\ F(3)=-4$ and $F'(x)<0$ for all $x\in(1,3)$. Let $f(x)=xF(x)$ for all $x\in R$.

77. The correct statement(s) is/are **[Passage Based Questions 2015]**

(a) $f'(1)<0$

(b) $f(2)<0$

(c) $f'(x)\neq 0$ for any $x\in(1,3)$

(d) $f'(x)=0$ for some $x\in(1,3)$

78. If $\int_1^3 x^2F'(x)\,dx=-12$ and $\int_1^3 x^3F''(x)\,dx=40$, then the correct expression(s) is/are

(a) $9f'(3)+f'(1)-32=0$ (b) $\int_1^3 f(x)\,dx=12$

(c) $9f'(3)-f'(1)+32=0$ (d) $\int_1^3 f(x)\,dx=-12$

79. Let $f: R \to R$ be a function defined by $f(x) = \begin{cases} [x], & x \le 2 \\ 0, & x > 2 \end{cases}$, where $[x]$ denotes the greatest integer less than or equal to x. If $I = \int_{-1}^{2} \frac{xf(x^2)}{2 + f(x+1)}\, dx$, then the value of $(4I - 1)$ is **[Integer Answer Type 2015]**

80. If $\alpha = \int_0^1 (e^{9x + 3\tan^{-1} x}) \left(\frac{12 + 9x^2}{1 + x^2}\right) dx$, where $\tan^{-1} x$ takes only principal values, then value of $\left(\log_e |1 + \alpha| - \frac{3\pi}{4}\right)$ is **[Integer Answer Type 2015]**

81. The integral $\int_{\pi/4}^{\pi/2} (2 \text{ cosec } x)^{17} dx$ is equal to **[Only One Correct Option 2014]**

(a) $\int_0^{\log(1+\sqrt{2})} 2(e^u + e^{-u})^{16} du$ (b) $\int_0^{\log(1+\sqrt{2})} (e^u + e^{-u})^{17} du$

(c) $\int_0^{\log(1+\sqrt{2})} (e^u - e^{-u})^{17} du$ (d) $\int_0^{\log(1+\sqrt{2})} 2(e^u - e^{-u})^{16} du$

82. Let $f: [0,2] \to R$ be a function which is continuous on $[0, 2]$ and is differentiable on $(0,2)$ with $f(0) = 1$. Let $F(x) = \int_0^{x^2} f(\sqrt{t})\, dt$, for $x \in [0,2]$. If $F'(x) = f'(x), \forall$ $x \in (0,2)$, then $F(2)$ equals **[Only One Correct Option 2014]**

(a) $e^2 - 1$ (b) $e^4 - 1$

(c) $e - 1$ (d) e^4

83. Match the conditions/expressions in Column I with statement in Column II. **[Matching Type 2014]**

	Column I		Column II
A.	$\int_{-1}^{1} \frac{dx}{1+x^2}$	p.	$\frac{1}{2}\log\left(\frac{2}{3}\right)$
B.	$\int_0^1 \frac{dx}{\sqrt{1-x^2}}$	q.	$2\log\left(\frac{2}{3}\right)$
C.	$\int_2^3 \frac{dx}{1-x^2}$	r.	$\frac{\pi}{3}$
D.	$\int_1^2 \frac{dx}{x\sqrt{x^2-1}}$	s.	$\frac{\pi}{2}$

84. Match List I with List II and select the correct answer using codes given below the lists. **[Matching Type 2014]**

	List I		List II
A.	The number of polynomials $f(x)$ with non-negative integer coefficients of degree ≤ 2, satisfying $f(0) = 0$ and $\int_0^1 f(x)\, dx = 1$, is	p.	8
B.	The number of points in the interval $[-\sqrt{13}, \sqrt{13}]$ at which $f(x) = \sin(x^2) + \cos(x^2)$ attains its maximum value, is	q.	2
C.	$\int_{-2}^{2} \frac{3x^2}{1 + e^x} dx$ equals	r.	4
D.	$\frac{\left(\int_{-1/2}^{1/2} \cos 2x \log\left(\frac{1+x}{1-x}\right) dx\right)}{\left(\int_0^{1/2} \cos\ 2x \log\left(\frac{1+x}{1-x}\right) dx\right)}$ equals	s.	0

Codes

	A	B	C	D		A	B	C	D
(a)	r	q	s	p	(b)	q	r	s	p
(c)	r	q	p	s	(d)	q	r	p	s

85. The value of $\int_0^1 4x^3 \left\{\frac{d^2}{dx^2}(1 - x^2)^5\right\} dx$ is **[Integer Answer Type 2014]**

86. The value of the integral $\int_{-\pi/2}^{\pi/2} \left(x^2 + \log\frac{\pi - x}{\pi + x}\right) \cos x\, dx$ is **[Only One Correct Option 2012]**

(a) 0 (b) $\frac{\pi^2}{2} - 4$

(c) $\frac{\pi^2}{2} + 4$ (d) $\frac{\pi^2}{2}$

87. The value of $\int_{\sqrt{\log 2}}^{\sqrt{\log 3}} \frac{x \sin x^2}{\sin x^2 + \sin(\log 6 - x^2)} dx$ is **[Integer Answer Type 2011]**

(a) $\frac{1}{4}\log\frac{3}{2}$ (b) $\frac{1}{2}\log\frac{3}{2}$

(c) $\log\frac{3}{2}$ (d) $\frac{1}{6}\log\frac{3}{2}$

88. Let $f: [1, \infty] \to [2, \infty]$ be differentiable function such that $f(1) = 2$. If $6\int_1^x f(x) = dt = 3x\ f(x) - x^3, \forall\ x \ge 1$ then the value of $f(2)$ is .... **[Integer Answer Type 2011]**

89. The value(s) of $\int_0^1 \frac{x^4 (1-x)^4}{1 + x^2} dx$ is (are) **[Only One Correct Option 2010]**

(a) $\frac{22}{7} - \pi$ (b) $\frac{2}{105}$

(c) 0 (d) $\frac{71}{15} - \frac{3\pi}{2}$

90. For $a \in R$ (the set of all real numbers), $a \ne -1$,

$$\lim_{n \to \infty} \frac{(1^a + 2^a + \ldots + n^a)}{(n+1)^{a-1}[(na+1) + (na+2) + \ldots + (na+n)]} = \frac{1}{60}.$$

Then, a is equal to **[More than One Correct Option 2010]**

(a) 5 (b) 7

(c) $\frac{-15}{2}$ (d) $\frac{-17}{2}$

■ **Directions** (Q Nos. 91 to 92) Let $f(x)=(1-x)^2\sin^2 x+x^2$, $\forall\, x\in R$ and $g(x)=\int_1^x\left(\frac{2(t-1)}{t+1}-\ln t\right)f(t)\,dt\ \forall\, x\in(1,\infty)$.

[Passage Based Questions 2010]

91. Consider the statements

P: There exists some $x\in R$ such that, $f(x)+2x=2(1+x^2)$.

Q: There exists some $x\in R$ such that $2f(x)+1=2x(1+x)$.

Then,

(a) both P and Q are true (b) P is true and Q is false

(c) P is false and Q is true (d) both P and Q are false

92. Which of the following is true?

(a) g is increasing on $(1,\infty)$

(b) g is decreasing on $(1,\infty)$

(c) g is increasing on $(1, 2)$ and decreasing on $(2,\infty)$

(d) g is decreasing on $(1, 2)$ and increasing on $(2,\infty)$

93. For any real number x, let $[x]$ denotes the largest integer less than or equal to x. Let f be a real valued function defined on the interval $[-10, 10]$ by

$$f(x)=\begin{cases}x-[x[, & \text{if } f(x) \text{ is odd}\\ 1+[x[-x, & \text{if } f(x) \text{ is even}\end{cases}$$

Then, the value of $\frac{\pi^2}{10}\int_{-10}^{10} f(x)\cos\pi x\,dx$ is.....

[Integer Answer Type 2010]

94. Let f be a non-negative function defined on the interval $[0, 1]$. If $\int_0^x\sqrt{1-(f'(t))^2}\,dt=\int_0^x f(t)\,dt$, $0\le x\le 1$ and $f(0)=0$, then **[Only One Option Correct 2009]**

(a) $f\left(\frac{1}{2}\right)<\frac{1}{2}$ and $f\left(\frac{1}{3}\right)>\frac{1}{3}$ (b) $f\left(\frac{1}{2}\right)>\frac{1}{2}$ and $f\left(\frac{1}{3}\right)>\frac{1}{3}$

(c) $f\left(\frac{1}{2}\right)<\frac{1}{2}$ and $f\left(\frac{1}{3}\right)<\frac{1}{3}$ (d) $f\left(\frac{1}{2}\right)>\frac{1}{2}$ and $f\left(\frac{1}{3}\right)<\frac{1}{3}$

95. If $I_n=\int_{-\pi}^{\pi}\frac{\sin nx}{(1+\pi^x)\sin x}\,dx$, $n=0,1,2,\ldots$, then

[More than One Correct 2009]

(a) $I_n=I_{n+2}$ (b) $\sum_{m=1}^{10} I_{2m+1}=10\pi$

(c) $\sum_{m=1}^{10} I_{2m}=0$ (d) $I_n=I_{n+1}$

96. Let $S_n=\sum_{k=0}^{n}\frac{n}{n^2+kn+k^2}$ and $T_n=\sum_{k=0}^{n-1}\frac{n}{n^2+kn+k^2}$, for $n=1,2,3,\ldots$, then **[More than One Correct Option 2008]**

(a) $S_n<\frac{\pi}{3\sqrt{3}}$ (b) $S_n>\frac{\pi}{3\sqrt{3}}$

(c) $T_n<\frac{\pi}{3\sqrt{3}}$ (d) $T_n>\frac{\pi}{3\sqrt{3}}$

(ii) JEE Main & AIEEE

97. The integral $\int_{\pi/4}^{3\pi/4}\frac{dx}{1+\cos x}$ is equal to **[2017 JEE Main]**

(a) -1 (b) -2

(c) 2 (d) 4

98. Let $I_n=\int\tan^n x\,dx,(n>1)$. $I_4+I_6=a\tan^5 x+bx^5+C$, where C is a constant of integration, then the ordered pair (a, b) is equal to **[2017 JEE Main]**

(a) $\left(-\frac{1}{5}, 0\right)$ (b) $\left(-\frac{1}{5}, 1\right)$ (c) $\left(\frac{1}{5}, 0\right)$ (d) $\left(\frac{1}{5}, -1\right)$

99. $\lim_{n\to\infty}\left[\frac{(n+1)(n+2)\ldots 3n}{n^{2n}}\right]^{1/n}$ is equal to **[2016 JEE Main]**

(a) $\frac{18}{e^4}$ (b) $\frac{27}{e^2}$ (c) $\frac{9}{e^2}$ (d) $3\log 3-2$

100. The integral $\int_2^4\frac{\log x^2}{\log x^2+\log(36-12x+x^2)}dx$ is equal to **[2015 JEE Main]**

(a) 2 (b) 4 (c) 1 (d) 6

101. The integral $\int_0^{\pi}\sqrt{1+4\sin^2\frac{x}{2}-4\sin\frac{x}{2}}\,dx$ is equal to **[2014 JEE Main]**

(a) $\pi-4$ (b) $\frac{2\pi}{3}-4-4\sqrt{3}$

(c) $4\sqrt{3}-4$ (d) $4\sqrt{3}-4-\pi/3$

102. **Statement I** The value of the integral $\int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\tan x}}$ is equal to $\pi/6$. **[2013 JEE Main]**

Statement II $\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx$

(a) Statement I is true; Statement II is true; Statement II is a true explanation for Statement I

(b) Statement I is true; Statement II is true; Statement II is not a true explanation for Statement I

(c) Statement I is true; Statement II is false

(d) Statement I is false; Statement II is true

103. The intercepts on X-axis made by tangents to the curve, $y=\int_0^x|t|\,dt$, $x\in R$, which are parallel to the line $y=2x$, are equal to **[2013 JEE Main]**

(a) ± 1 (b) ± 2

(c) ± 3 (d) ± 4

104. If $g(x)=\int_0^x\cos 4t\,dt$, then $g(x+\pi)$ equals **[2012 AIEEE]**

(a) $\frac{g(x)}{g(\pi)}$ (b) $g(x)+g(\pi)$ (c) $g(x)-g(\pi)$ (d) $g(x)\cdot g(\pi)$

105. The value of $\int_0^1\frac{8\log(1+x)}{1+x^2}\,dx$ is **[2011 AIEEE]**

(a) $\frac{\pi}{8}\log 2$ (b) $\frac{\pi}{2}\log 2$ (c) $\log 2$ (d) $\pi\log 2$

106. For $x \in \left(0, \frac{5\pi}{2}\right)$, define $f(x) = \int_0^x \sqrt{t} \sin t \, dt$. Then, f has **[2011 AIEEE]**

(a) local minimum at π and 2π
(b) local minimum at π and local maximum at 2π
(c) local maximum at π and local minimum at 2π
(d) local maximum at π and 2π

107. Let $p(x)$ be a function defined on R such that $\lim_{x \to \infty} \frac{f(3x)}{f(x)} = 1, p'(x) = p'(1-x)$, for all $x \in [0, 1], p(0) = 1$ and $p(1) = 41$. Then, $\int_0^1 p(x)\,dx$ equals **[2010 AIEEE]**

(a) $\sqrt{41}$ (b) 21
(c) 41 (d) 42

108. $\int_0^{\pi} [\cot x]\,dx$, [ ] denotes the greatest integer function, is equal to **[2009 AIEEE]**

(a) $\frac{\pi}{2}$ (b) 1
(c) -1 (d) $-\frac{\pi}{2}$

109. Let $I = \int_0^1 \frac{\sin x}{\sqrt{x}}\,dx$ and $J = \int_0^1 \frac{\cos x}{\sqrt{x}}\,dx$.

Then, which one of the following is true? **[2008 AIEEE]**

(a) $I > \frac{2}{3}$ and $J > 2$ (b) $I < \frac{2}{3}$ and $J < 2$
(c) $I < \frac{2}{3}$ and $J > 2$ (d) $I > \frac{2}{3}$ and $J < 2$

Answers

Exercise for Session 1

1. $\frac{\pi + 2}{8}$ 2. 1 3. 2 4. $\frac{2}{3} - \frac{\sqrt{3}}{4}$ 5. $\frac{4\sqrt{2}}{3}$
6. -1 7. $\frac{1}{20}\log_e 3$ 8. π 9. $\frac{\pi}{2}(b-a)$
10. $\frac{\pi}{2\sqrt{2}} + \frac{1}{\sqrt{2}}\log(\sqrt{2}-1)$ 11. $\frac{-\pi}{2}$ 12. $1 + \left(\frac{\pi}{4} - \sqrt{2}\right)e^{1/\sqrt{2}}$
13. 10 ! 14. (d) 15. (d)

Exercise for Session 2

1. (c) 2. (a) 3. (a) 4. (c) 5. (a) 6. (c)
7. (b) 8. (a) 9. (a) 11. $\frac{16}{\pi}$ 12. 0 13. (c)
14. (d)

Exercise for Session 3

1. (c) 2. (a) 3. (b) 4. (b) 5. (a) 6. (d)
7. (b) 8. (d) 9. (c) 10. (b) 11. (b)
12. (b) 13. (c) 14. (b) 15. (d)
16. $\sum_{r=1}^{k} (r-1)\cdot(r^{1/n} - (r-1)^{1/n}) + k\,(9 - k^{1/n})$
18. 3 19. 1 20. 2

Exercise for Session 4

1. (b) 2. (c) 3. (c) 4. (a) 5. (c) 6. (a)
7. (a) 8. (c) 9. (c) 10. (b) 11. (b) 12. (b)
13. (c) 14. (c) 15. (d) 16. (c) 17. (a) 18. (d)
19. (c) 20. (a)

Exercise for Session 5

1. (c) 2. (b) 3. (a) 4. (a) 5. (a) 6. (b)
7. (a) 8. (a) 9. (c) 10. (c) 11. (c) 12. (a)
13. (c) 14. (c) 15. (c) 16. (a) 17. (a) 18. (b)
19. (a) 20. (b)

Exercise for Session 6

1. (c) **2.** (a) **3.** (c) **4.** (b) **5.** (c) **6.** (c)
7. (a) **8.** (a) **9.** (c) **10.** (a)

Chapter Exercises

1. (a) **2.** (c) **3.** (a) **4.** (a) **5.** (a) **6.** (a)
7. (d) **8.** (d) **9.** (a) **10.** (a) **11.** (c) **12.** (c)
13. (d) **14.** (c) **15.** (c) **16.** (d) **17.** (d) **18.** (a)
19. (b) **20.** (a) **21.** (d) **22.** (c) **23.** (d) **24.** (a)
25. (a) **26.** (b) **27.** (a) **28.** (d) **29.** (a) **30.** (c)
31. (a) **32.** (b) **33.** (a) **34.** (b) **35.** (b) **36.** (a,b,c)
37. (c,d) **38.** (a) **39.** (a,b,c) **40.** (c,d) **41.** (a,b,d)
42. (b,d) **43.** (c) **44.** (d) **45.** (a,b,c) **46.** (d) **47.** (a)
48. (b) **49.** (a) **50.** (a) **51.** (b) **52.** (b) **53.** (c)
54. (a) **55.** (a) **56.** (c) **57.** (a) **58.** (b) **59.** (a)
60. (a) **61.** (c) **62.** (c)
63. (A) → (q), (B) → (s), (C) → (p), (D) → (r)
64. (A) → (q), (B) → (r), (C) → (s), (D) → (p)
65. (A) → (s), (B) → (p), (C) → (q)
66. (A) → (s), (B) → (r), (C) → (r), (D) → (q)
67. (8) **68.** (0) **69.** (3) **70.** (2) **71.** (2) **72.** (a)
73. (1) **74.** (a,b) **75.** (b) **76.** (a,c) **77.** (a,b,c) **78.** (c,d)
79. (0) **80.** (9) **81.** (a) **82.** (b)
83. A→s, B→s, C→p, D→r **84.** (d) **85.** (2) **86.** (b)
87. (a) **88.** (8/3) **89.** (a) **90.** (b,d) **91.** (c) **92.** (b)
93. (4) **94.** (c) **95.** (a,b,c) **96.** (d) **97.** (c) **98.** (c)
99. (b) **100.** (c) **101.** (d) **102.** (d) **103.** (a) **104.** (c)
105. (d) **106.** (c) **107.** (b) **108.** (d) **109.** (b)

Solutions

1. Let $I = \int_0^4 \frac{(y^2 - 4y + 5)\sin(y-2)dy}{(2y^2 - 8y + 1)}$

Put $y - 2 = z$

$\Rightarrow \quad dy = dz$

$\therefore \quad I = \int_{z=-2}^{2} \frac{(z^2+1)\sin z}{(2z^2 - 7)} dz$

$\Rightarrow \quad I = 0$, as $\int_{-a}^{a} f(x)\,dx = 0$, when $f(-x) = -f(x)$

and $f(z) = \frac{(z^2+1)\sin z}{(2z^2-7)} dz$ is an odd function.

2. Hence, $f(x) = x^2 + ax - b$

The solution of $f(x) = \text{minium } f(x)$ is $x = \frac{-a}{2}$.

Given, $f(x)_{\text{minimum}}$ is at $x = 0$.

$\Rightarrow \quad \frac{-a}{2} = 0 \quad \text{or} \quad a = 0$

$\therefore \quad f(x) = x^2 - b = 0$ has roots α and β.

$\Rightarrow \quad \alpha = \sqrt{b}$ and $\beta = -\sqrt{b}$

$\Rightarrow \quad \int_\alpha^\beta x^3 dx = \int_{\sqrt{b}}^{-\sqrt{b}} x^3 dx = 0$

$\left[\int_{-a}^{a} f(x)dx = 0, \text{ when } f(-x) = f(x)\right]$

3. Here, $\int_{\pi/2}^{x} \sqrt{(3 - 2\sin^2 t)}\,dt + \int_0^y \cos t\,dt = 0$

Differentiating both the sides, we get

$$(\sqrt{3 - 2\sin^2 x}) \cdot 1 + (\cos y)\left(\frac{dy}{dx}\right) = 0$$

$\Rightarrow \quad \frac{dy}{dx} = \frac{-\sqrt{3 - 2\sin^2 x}}{\cos y}$

$\therefore \quad \left(\frac{dy}{dx}\right)_{(\pi,\pi)} = \frac{-\sqrt{3}}{-1} = \sqrt{3}$

4. Here, $-4 \le x \le 4 \Rightarrow \ 0 \le \frac{x^2}{17} < 1$

$\therefore \ \left[\frac{x^2}{17}\right] = 0$ and $\sin^{-1}(\sin x)$ is an odd function.

Let, $I = \int_{-4}^{4} \frac{\sin^{-1}(\sin x)}{(1+x^2)\times 1} dx + \int_{-4}^{4} \frac{\cos^{-1}(\cos x)}{(1+x^2)\times 1} dx$

$= 0 + 2\int_0^\pi \frac{\cos^{-1}(\cos x)}{(1+x^2)} dx$

$I = 2\left[\int_0^\pi \frac{x}{1+x^2} dx + \int_\pi^4 \frac{2\pi}{(1+x^2)} dx\right]$

$= \log(1+\pi^2) + 2\pi(\tan^{-1} x)_\pi^4 - [\log(1+16) - \log(1+\pi^2)]$

$= \log(1+\pi^2) + 2\pi(\tan^{-1} 4 - \tan^{-1}\pi) - \log\frac{17}{1+\pi^2}$

$= \log\frac{(1+\pi^2)\cdot(1+\pi^2)}{17} + 2\pi\tan^{-1}\left[\tan^{-1}\left(\frac{4-\pi}{1+4\pi}\right)\right]$

$= \log\left(\frac{(1+\pi^2)^2}{17}\right) + 2\pi\tan^{-1}\left(\frac{4-\pi}{1+4\pi}\right)$

On comparing with $\log\left(\frac{(1+\pi^2)^2}{a}\right) + b\pi\tan^{-1}\left(\frac{c-\pi}{1+c\pi}\right)$, we get

$a = 17$, $b = 2$ and $c = 4$

$\therefore \quad a - (2b + c) = 17 - 8 = 9$ and $\frac{a-5}{c} = 3$

Thus, the number of ways to distribute 9 distinct bijective into 3 persons equally is $\frac{9!}{(3!)^3}$.

5. Put $x = \tan\theta$

$\therefore \quad I = \int_0^{\pi/2} \frac{d\theta}{1 + (\tan\theta)^a} = \int_0^{\pi/2} \frac{(\cos\theta)^a}{(\sin\theta)^a + (\cos\theta)^a} d\theta \Rightarrow I = \frac{\pi}{4}$

6. $I = \int_0^{3\pi/4} (\sin x + \cos x)\,dx + \int_0^{3\pi/4} \underbrace{x}_{\text{I}} \underbrace{(\sin x - \cos x)}_{\text{II}}\,dx$

$= \int_0^{3\pi/4} (\sin x + \cos x)\,dx$

$+ [x(-\cos x - \sin x)]_0^{3\pi/4} + \int_0^{3\pi/4} (\sin x + \cos x)\,dx$

$= 2\int_0^{3\pi/4} (\sin x + \cos x)\,dx + 0 = 2[-\cos x + \sin x]_0^{3\pi/4}$

$= 2\left[+\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 1\right] = 2(\sqrt{2}+1) = 2\tan\frac{3\pi}{8}$

7. We have, $C_n = \int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{\tan^{-1}(nx)}{\sin^{-1}(nx)} dx$

$C_n = \frac{1}{n}\int_{\frac{n}{n+1}}^{1} \frac{\tan^{-1}(t)}{\sin^{-1}(t)} dt \qquad (\text{put } nx = t)$

Now, $L = \lim_{n\to\infty} n^2 \cdot C_n = \lim_{n\to\infty} n\int_{\frac{n}{n+1}}^{1} \frac{\tan^{-1} t}{\sin^{-1} t} dt\,(\infty \times 0)$

$L = \frac{\int_{\frac{n}{n+1}}^{1} \frac{\tan^{-1} t}{\sin^{-1} t} dt}{\frac{1}{n}} \qquad \left(\frac{0}{0}\text{ form}\right)$

Applying Leibnitz rule,

$$L = \lim_{n\to\infty} \frac{0 - \frac{\tan^{-1}\frac{n}{n+1}}{\sin^{-1}\frac{n}{n+1}}\left(\frac{1}{(n+1)^2}\right)}{-\frac{1}{n^2}} = \frac{\pi}{4}\cdot\frac{2}{\pi} = \frac{1}{2}$$

8. $\int_{-3}^{3} \frac{t^2\sin 2t}{t^2+1} dt = 0$ as the integrand is an odd function.

Also, $\int_0^1 \frac{dt}{t^2 + 2t\cos\alpha + 1}$

$= \frac{1}{\sin\alpha}\left|\tan^{-1}\frac{t+\cos\alpha}{\sin\alpha}\right|_0^1 = \frac{\alpha}{2\sin\alpha}$

Thus, the given equation reduces to

$$x^2 \cdot \frac{\alpha}{2\sin\alpha} - 2 = 0 \quad \Rightarrow \quad x = \pm 2\sqrt{\frac{\sin\alpha}{\alpha}}$$

9. $f'(x) = e^{g(x)} \cdot g'(x)$ and $g'(x) = \dfrac{x}{1+x^4}$

$$\therefore \qquad f'(x) = e^{g(x)} \cdot \frac{x}{1+x^4}\ e^{g(2)} = e^0 = 1$$

Hence, $\qquad f'(2) = e^{g(2)} \cdot g'(2) = e^0 \cdot \dfrac{2}{17} = \dfrac{2}{17}$

10. $\lim\limits_{t\to 0} \ln\left(\dfrac{1}{t}\int_0^t (1 + a\sin bx)^{c/x}\,dx\right)$

$$= \ln \lim_{t\to 0} \frac{\int_0^t (1 + a\sin b\,x)^{c/x}\,dx}{t}$$

$$= \ln \lim_{t\to 0} \frac{(1 + a\sin bt)^{c/t}}{1} = \ln e^{\lim\limits_{t\to 0}\frac{a\sin bt}{t/c}} = \ln e^{abc} = abc$$

11. $T_r = \dfrac{1}{\sqrt{\dfrac{r}{n}} \cdot n\left(3\sqrt{\dfrac{r}{n}} + 4\right)^2}$

$$S = \frac{1}{n}\sum_1^{4n} \frac{1}{\left(3\sqrt{\frac{r}{n}} + 4\right)^2 \cdot \sqrt{\frac{r}{n}}} = \int_0^4 \frac{dx}{\sqrt{x}\,(3\sqrt{x} + 4)^2}$$

Put $3\sqrt{x} + 4 = t \ \Rightarrow\ \dfrac{3}{2} \cdot \dfrac{1}{\sqrt{x}}\,dx = dt$

$$= \frac{2}{3}\int_4^{10} \frac{dt}{t^2} = \frac{2}{3}\left[\frac{1}{t}\right]_{10}^4 = \frac{2}{3}\left[\frac{1}{4} - \frac{1}{10}\right] = \frac{2}{3} \cdot \frac{6}{40} = \frac{1}{10}$$

12. Given, $f(x) = \int_{-1}^{x} e^{t^2}\,dt$; $h(x) = f(1 + g(x))$; $g(x) < 0$ for $x > 0$

Now, $\quad h(x) = \int_{-1}^{1+g(x)} e^{t^2}\,dt = f(1 + g(x))$ (given)

Differentiating, we get $\quad h'(x) = e^{(1+g(x))^2} \cdot g'(x)$

Now, $\qquad h'(1) = e$ (given)

$\therefore \qquad e^{(1+g(1))^2} \cdot g'(1) = e$

$\Rightarrow \qquad (1 + g(1))^2 = 1$

$\Rightarrow \qquad 1 + g(1) = \pm 1$

$\Rightarrow \qquad g(1) = 0$ (not possible)

or $\qquad g(1) = -2$

13. Given, $f'(x) = f(x) \Rightarrow f(x) = Ce^x$

Since, $\quad f(0) = 1 \quad \therefore \quad 1 = f(0) = C$

$\therefore\ f(x) = e^x$ and hence $g(x) = x^2 - e^x$

Thus, $\int_0^1 f(x)\,g(x)\,dx = \int_0^1 (x^2e^x - e^{2x})\,dx$

$$= [x^2e^x]_0^1 - 2\int_0^1 xe^x\,dx - \left[\frac{e^{2x}}{2}\right]_0^1$$

$$= (e - 0) - 2\,[(xe^x)_0^1 - (e^x)_0^1\,] - \frac{1}{2}(e^2 - 1)$$

$$= (e - 0) - 2\,[(e - 0) - (e - 1)] - \frac{1}{2}(e^2 - 1)$$

$$= e - \frac{1}{2}e^2 - \frac{3}{2}$$

14. Here, $\quad f'(x) = \dfrac{1}{\sqrt{1 + g^2(x)}}\,g'(x)$;

Now, $\ f'\left(\dfrac{\pi}{2}\right) = \dfrac{g'(\pi/2)}{\sqrt{1 + g^2(\pi/2)}}$; $\quad g\left(\dfrac{\pi}{2}\right) = 0 = g'\left(\dfrac{\pi}{2}\right)$

But $\qquad g'(x) = [1 + \sin(\cos^2 x)](-\sin x)$

$$g'\left(\frac{\pi}{2}\right) = 1(-1) = -1$$

Hence, $\qquad f'\left(\dfrac{\pi}{2}\right) = -1$ as $h'(0^+) = -1$

15. Clearly, f is an even function, hence

$$I_1 = \int_0^{\pi} f[\cos(\pi - x)]\,dx = \int_0^{\pi} f(-\cos x)\,dx$$

$$= \int_0^{\pi} f(\cos x)\,dx$$

$$\therefore \quad I_1 = 2\int_0^{\pi/2} f(\cos x)\,dx \cdot 2\int_0^{\pi/2} f(\sin x)\,dx = 2I_2$$

$$\Rightarrow \frac{I_1}{I_2} = 2$$

Aliter Let $u = \cos x \Rightarrow du = -\sin x\,dx$

$$\therefore \qquad I_1 = \int_{-1}^{1} \frac{f(u)}{\sqrt{1 - u^2}}\,du$$

$$\Rightarrow \qquad I_1 = 2\int_0^1 \frac{f(u)}{\sqrt{1 - u^2}}\,du \qquad \text{...(i)}$$

Similarly, with $\sin t = t$,

$$I_2 = \int_0^1 \frac{f(t)}{\sqrt{1 - t^2}}\,dt \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), $\dfrac{I_1}{I_2} = 2$

16. Given, $I_1 = \int_{1-k}^{k} x\,f(x(1 - x))\,dx$ and

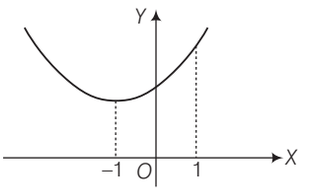
$$I_2 = \int_{1-k}^{k} f(x(1 - x))\,dx$$

Using King's property $\quad I_1 = \int_{1-k}^{k} (1 - x)\,f(x(1 - x))\,dx$

$$2I_1 = \int_{1-k}^{k} f(x(1 - x))\,dx = I_2$$

$$\therefore \qquad \frac{I_2}{I_1} = 2$$

17. $A = \int_{-1}^{1} (ax^2 + bx + c)\,dx = 2\int_0^1 (ax^2 + c)\,dx$



$$= 2\left[\frac{a}{3} + c\right] = \frac{1}{3}[2a + 6c]$$

$$\therefore \qquad A = \frac{1}{3}[f(-1) + 4f(0) + f(1)]$$

18. $I(a)=\int_0^{\pi}\left(\frac{x^2}{a^2}+a^2\sin^2 x+2x\sin x\right)dx \quad \left(\because \int_0^{\pi} x\sin x\,dx=\pi\right)$

$$\therefore \quad I(a)=\frac{\pi^3}{3a^2}+\frac{\pi a^2}{2}+2\pi=\pi\left[\frac{\pi^2}{3a^2}+\frac{a^2}{2}\right]+2\pi$$

$$=\pi\left[\left(\frac{\pi}{\sqrt{3}a}-\frac{a}{\sqrt{2}}\right)^2+\frac{2\pi}{\sqrt{6}}\right]+2\pi$$

$I(a)$ is minimum when $\frac{\pi}{\sqrt{3}a}=\frac{a}{\sqrt{2}} \Rightarrow a^2=\pi\sqrt{\frac{2}{3}}$

$$\Rightarrow \quad a=\sqrt{\pi\sqrt{\frac{2}{3}}}$$

Also, $[I(a)]_{\min}=2\pi+\pi^2\sqrt{\frac{2}{3}}$

19. $\left[\frac{t^2}{2}-\log_2 a\cdot t\right]_0^2=2-\log_2(a^2) \quad (\because a>0)$

$$\Rightarrow \quad (2-2\log_2 a)=2-2\log_2 a$$

$$\Rightarrow \quad 2\log_2 a=2\log_2 a \Rightarrow a\in R^+$$

20. Consider $I=\int_{-1/x}^{1/x}\frac{\ln(1+t^2)}{1+e^t}dt$ ...(i)

$$\Rightarrow \quad I=\int_{-1/x}^{1/x}\frac{\ln(1+t^2)}{1+e^{-t}}dt \quad \text{(using King' property)}$$

$$I=\int_{-1/x}^{1/x}\frac{\ln(1+t^2)\,e^t}{1+e^t}dt \quad ...(ii)$$

On adding Eqs. (i) and (ii), we get

$$2I=\int_{-1/x}^{1/x}\ln(1+t^2)\,dt=2\int_0^{1/x}\ln(1+t^2)\,dt$$

$$\Rightarrow \quad I=\int_0^{1/x}\ln(1+t^2)\,dt$$

Hence, $l=\lim_{x\to\infty}x^3\int_0^{1/x}\ln(1+t^2)\,dt$

$$=\lim_{x\to\infty}\frac{\int_0^{1/x}\ln(1+t^2)\,dt}{x^{-3}} \quad \left(\frac{0}{0}\text{ form}\right)$$

Using L' Hospital's rule,

$$l=\lim_{x\to 0}\frac{x^4\ln\left(1+\frac{1}{x^2}\right)\cdot\left(-\frac{1}{x^2}\right)}{-3}=\frac{1}{3}\lim_{x\to\infty}x^2\ln\left(1+\frac{1}{x^2}\right)$$

$$=\frac{1}{3}\lim_{x\to\infty}\ln\left(1+\frac{1}{x^2}\right)^{x^2} \quad (1^{\infty}\text{ form})$$

$$=\lim_{x\to\infty}\frac{1}{3}x^2\left(1+\frac{1}{x^2}-1\right)=\frac{1}{3}$$

21. Put $\pi x=t \Rightarrow dx=\frac{dt}{\pi}$

$$\therefore \quad I=\frac{1}{\pi}\cdot\frac{\pi}{\pi}\int_0^{2008\pi}t\,|\sin t|\,dt=\frac{1}{\pi}\int_0^{2008\pi}t\,|\sin t|\,dt \quad ...(i)$$

$$\Rightarrow \quad I=\frac{1}{\pi}\int_0^{2008\pi}(2008\pi-t)\,|\sin t|\,dt \quad ...(ii)$$

On adding Eqs. (i) and (ii), we get

$$2I=\frac{2008\pi}{\pi}\int_0^{2008\pi}|\sin t|\,dt=(2008)^2\cdot\int_0^{\pi}|\sin t|\,dt$$

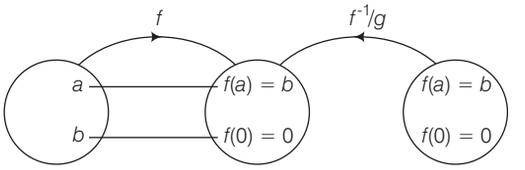
$$\Rightarrow \quad I=(2008)^2 \quad \Rightarrow \sqrt{I}=2008$$

22. $T_n=\frac{n}{n^2+k^2x^2}=\frac{1}{n[1+(k/n)^2x^2]}$;

$$S=\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\underbrace{(k/n)^2}_{t}x^2}=\int_0^1\frac{dt}{1+t^2x^2}$$

$$=\frac{1}{x^2}\int_0^1\frac{dt}{t^2+(1/x^2)}=\left[\frac{1}{x}\tan^{-1}(tx)\right]_0^1=\frac{\tan^{-1}(x)}{x}$$

23. Let $y=f(x) \Rightarrow x=g(y)$ and $dy=f'(x)\,dx$



$$I=\int_0^a f(x)\,dx+\int_0^b g(y)\,dy; y=f(x)$$

$$\Rightarrow \quad x=f^{-1}(y)=g(y)=\int_0^a f(x)dx+\int_0^a xf'(x)\,dx$$

$$=\int_0^a(f(x)+xf'(x))\,dx=[xf(x)]_0^a=a\,f(a)=ab$$

24. Let $I=\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{x^4}{1-x^4}\cos^{-1}\frac{2x}{1+x^2}dx$ ...(i)

$$\Rightarrow I=\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{x^4}{1-x^4}\cos^{-1}\left(\frac{-2x}{1-x^4}\right)dx \quad \text{(using King's property)}$$

$$\Rightarrow \quad I=\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{x^4}{1-x^4}\left(\pi-\cos^{-1}\frac{2x}{1-x^4}\right)dx \quad ...(ii)$$

On adding Eqs. (i) and (ii), we get

$$\therefore \quad 2I=\pi\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{x^4}{1-x^4}dx \Rightarrow 2I=2\pi\int_0^{1/\sqrt{3}}\frac{x^4}{1-x^4}dx$$

$$\therefore \quad k=\pi$$

25. Put $\ln x=t$ or $x=e^t \Rightarrow dx=e^t\,dt$

$$\therefore \quad I=\int_{-\infty}^{\infty}f(e^t+e^{-t})\frac{t}{e^t}e^t\,dt=\int_{-\infty}^{\infty}f(e^t+e^{-t})\,t\,dt=0$$

(as the function is odd)

Aliter I Put $x=\tan\theta$

$$\int_0^{\pi/2}f\left(\tan\theta+\frac{1}{\tan\theta}\right)\frac{\ln\tan\theta}{\tan\theta}\cdot\sec^2\theta\,d\theta$$

$$=\int_0^{\pi/2}f\left(\tan\theta+\frac{1}{\tan\theta}\right)\frac{\ln\tan\theta}{\sin\theta\cos\theta}d\theta$$

Aliter II Put $x=1/t \Rightarrow I=-1 \Rightarrow 2I=0 \Rightarrow I=0$

26. $\lim_{\lambda\to 0}\left(\int_0^1(1+x)^{\lambda}\,dx\right)^{1/\lambda}=\lim_{\lambda\to 0}\left(\left|\frac{(1+x)^{\lambda+1}}{\lambda+1}\right|_0^1\right)^{1/\lambda}$

$$=\lim_{\lambda\to 0}\left(\frac{2^{\lambda+1}-1}{\lambda+1}\right)^{1/\lambda} \quad (1^{\infty}\text{ form})$$

$$=e^{\lim_{\lambda\to 0}\frac{1}{\lambda}\left(\frac{2^{\lambda+1}-1-\lambda-1}{\lambda+1}\right)}=e^{\lim_{\lambda\to 0}\left(\frac{2^{\lambda+1}-2-\lambda}{\lambda(\lambda+1)}\right)}$$

$$=e^{\lim_{\lambda\to 0}\left(\frac{2(2^{\lambda}-1)}{\lambda}-1\right)}=e^{2\ln 2-1}=e^{\ln\left(\frac{4}{e}\right)}=\frac{4}{e}$$

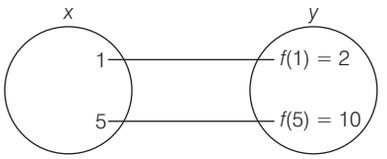
27. Let $y = f(x) \Rightarrow x = f^{-1}(y) = g(y)$

$$dy = f'(x)\,dx$$

$$\therefore \quad I = \int_1^5 f(x)\,dx + \int_1^5 xf'(x)\,dx$$

When $y = 2$, then $x = 1$

and when $y = 1$, then $x = 5$



$$\therefore \quad I = \int_1^5 (f(x) + x\,f'(x))dx = |x\,f(x)|_1^5$$

$$= 5f(5) - f(1) = 5 \cdot 10 - 2 = 48$$

28. $I = \int_0^{\pi/2} \sin x \sin 2x \sin 3x\,dx$ ...(i)

$$= \int_0^{\pi/2} \sin\left(\frac{\pi}{2} - x\right) \sin 2\left(\frac{\pi}{2} - x\right) \sin 3\left(\frac{\pi}{2} - x\right) dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} -\cos x \sin 2x \cos 3x\,dx \qquad \text{...(ii)}$$

$$\therefore \quad 2I = \int_0^{\pi/2} -\sin 2x\,(\cos x \cos 3x - \sin 3x \sin x)\,dx$$

$$= \int_0^{\pi/2} -\sin 2x \cos(4x)\,dx$$

$$= -\int_0^{\pi/2} \sin 2x\,(2\cos^2 2x - 1)\,dx$$

Put $\cos 2x = t \Rightarrow -\sin 2x \times 2\,dx = dt$

$$\therefore \quad 2I = \int_1^{-1} \frac{1}{2}(2t^2 - 1)\,dt = \frac{1}{2}\left[\frac{2t^3}{3} - t\right]_1^{-1}$$

$$= \frac{1}{2}\left[\frac{2}{3}(-1)^3 - (-1) - \left(\frac{2}{3}(1)^3 - 1\right)\right]$$

$$= \frac{1}{2}\left[-\frac{2}{3} + 1 - \frac{2}{3} + 1\right] = \frac{1}{2}\left[\frac{2}{3}\right] \quad \Rightarrow \quad I = \frac{1}{6}$$

29. Here, $f'(x) = (f(x))^2 > 0$; $\dfrac{d}{dx} f(g(x))|_{x=a}$

$$= f'(g(x)) \lim_{x \to a} \frac{g(x) - g(a)}{x - a}$$

It implies that $g(x)$ must be differentiable at $x = a$.

30. On putting $t = \dfrac{x^2}{z}$ and then solving,

$$\int_0^\infty \frac{\ln x}{x^2 + t^2}\,dt = \frac{2\pi \ln x}{x}$$

$$\Rightarrow \quad \frac{\ln x}{x} = \frac{\ln 2}{x}$$

$$\Rightarrow \quad x = 2 \text{ and } 4 \text{ i.e. two solutions.}$$

31. Put $\dfrac{x\pi}{n} = t$

$$\therefore \int_0^{\frac{16n^2}{\pi}} \cos\frac{\pi}{2}\left[\frac{\pi x}{n}\right] dx = \frac{n}{\pi}\int_0^{16n} \cos\frac{\pi}{2}[t]\,dt$$

$$= \frac{4n^2}{\pi}\int_0^4 \cos\frac{\pi}{2}[t]\,dt = 0$$

32. $\int_{-2}^{-1} (ax^2 - 5)\,dx + \int_1^2 (bx + c)\,dx + 5$

$$= \int_{-2}^{-1} (ax^2 - 5 - bx + c + 5)\,dx = 0$$

Hence, $ax^2 - bx + c = 0$ has atleast one root in $(-2, -1)$.

33. $I = \int_3^6 ((\sqrt{x-3} + \sqrt{3}) + (\sqrt{3} - \sqrt{x-3}))\,dx = 6\sqrt{3}$

34. $I_1 = \int_{-1}^1 (\{x\} + \{x^3\})\{x^2\}\,dx$

$$= -2\int_0^1 \{x^2\}\,dx = \left[-2 \times \frac{x^3}{3}\right]_0^1 = -\frac{2}{3}$$

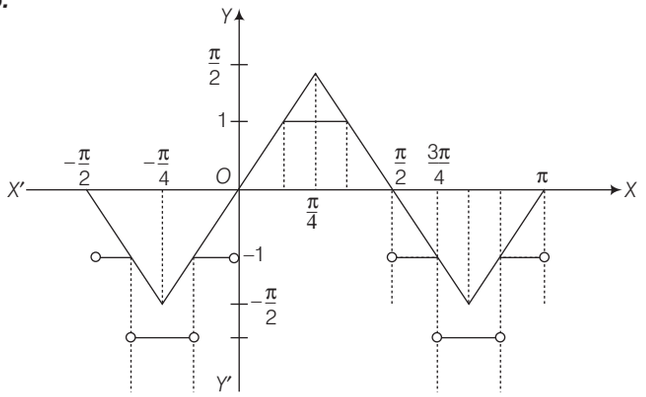
35. $\mathcal{J}_n - \mathcal{J}_{n-1} = \int_0^{\pi/2} \dfrac{\sin^2 nx - \sin^2(n-1)x}{\sin^2 x}\,dx$

$$= \int_0^{\pi/2} \frac{\sin(2n-1)x \cdot \sin x}{\sin^2 x}\,dx = I_n$$

i.e. $\quad \mathcal{J}_n - \mathcal{J}_{n-1} = I_n$

$\Rightarrow \quad \mathcal{J}_{n+1} - \mathcal{J}_n = I_{n+1}$

36.



37. (a) $I = \int_0^{\pi/2} \ln(\cot x)\,dx \Rightarrow I = \int_0^{\pi/2} \ln(\tan x)\,dx$

$$I = -\int_0^{\pi/2} \ln(\cot x)\,dx \Rightarrow I = -I \Rightarrow I = 0$$

(b) $I = \int_0^{2\pi} \sin^3 x\,dx = -\int_0^{2\pi} \sin^3 x\,dx \Rightarrow I = 0$

(c) At $x = \dfrac{1}{t}$, $I = \int_e^{1/e} \dfrac{-(1/t^2)\,dt}{-1/t\,(\ln t)^{1/3}} = -\int_{1/e}^e \dfrac{dt}{t(\ln t)^{1/3}}$

$$I = -1 \quad \text{or} \quad I = 0$$

(d) $\sqrt{\dfrac{1 + \cos 2x}{2}} > 0 \quad \Rightarrow \quad \int_0^\pi \sqrt{\dfrac{1 + \cos 2x}{2}}\,dx > 0$

38. $I = \int_0^1 (10x^4 - 3x^2 - 1)\,dx = [2x^5 - x^3 - x]_0^1 = 0$

Since, $f(x)$ is even, hence must have a root in $(-1, 0)$.

39. We have, $I_1 = \int_0^{\pi/2} \cos(\pi \sin^2 x)\,dx$

$$I_1 = \int_0^{\pi/2} \cos(\pi \cos^2 x)\,dx$$

On adding, $2I_1 = \int_0^{\pi/2} \cos(\pi \sin^2 x) + \cos(\pi \cos^2 x)\,dx$

$$= \int_0^{\pi/2} 2\cos\left(\frac{\pi}{2}\right) \cdot \cos\left(\frac{\pi}{2}\cos 2x\right) dx = 0$$

$$\Rightarrow \quad I_1 = 0 \qquad \text{...(i)}$$

Now, $I_2 = \int_0^{\pi/2} \cos\{\pi(1-\cos 2x)\}\,dx$

$= -\int_0^{\pi/2} \cos(\pi \cos 2x)\,dx$

$= -\frac{1}{2}\int_0^{\pi} \cos(\pi \cos t)\,dt$ (put $2x = t$)

$= -\frac{2}{2}\int_0^{\pi/2} \cos(\pi \cos t)\,dt = I_3$

$\Rightarrow \quad I_2 + I_3 = 0$

$I_3 = -\int_0^{\pi/2} \cos(\pi \sin t)\,dt$

$\therefore \quad I_2 + I_3 = 0$ ...(ii)

Hence, $I_1 + I_2 + I_3 = 0$

40. $f(x) = 2\int_0^1 \underbrace{(1-t)}_{\text{I}} \underbrace{\cos(xt)}_{\text{II}}\,dt$

$= 2\left[(1-t)\frac{\sin xt}{x}\Big|_0^t + \frac{1}{x}\int_0^1 \sin xt\,dt\right] = 2\left[\left|0 - \frac{1}{x^2}\cos xt\right|_0^1\right]$

$f(x) = 2\left[\frac{1-\cos x}{x^2}\right] \qquad (x \neq 0)$

If $x = 0$, then $f(x) = \int_{-1}^{1}(1-|t|)\,dt = 2\int_0^1 (1-t)\,dt = 1$

$\therefore$ option (c) is correct.

Hence, $f(x) = \begin{cases} 2\left(\frac{1-\cos x}{x^2}\right), & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$

$\therefore$ f is continuous at $x = 0$.

$\therefore$ option (d) is correct.

41. Given, $f(f(x)) = -x + 1$

Replacing x by $f(x)$, we get

$f(f(f(x))) = -f(x) + 1$

$f(1-x) = -f(x) + 1$

$f(x) + f(1-x) = 1$ ...(i)

Now, $\mathcal{I} = \int_0^1 f(x)\,dx = \int_0^1 f(1-x)\,dx$ (using King's property)

$\Rightarrow \quad 2\mathcal{I} = \int_0^1 (f(x) + f(1-x))\,dx$

$\Rightarrow \quad 2\mathcal{I} = \int_0^1 dx = 1 \Rightarrow \mathcal{I} = \frac{1}{2}$

Put $x = \frac{1}{4}$ in Eq. (i),

$F\left(\frac{1}{4}\right) + F\left(1 - \frac{1}{4}\right) = 1 \Rightarrow F\left(\frac{1}{4}\right) + F\left(\frac{3}{4}\right) = 1$

Now $I = \int_0^{\pi/2} \frac{\sin x}{(\sin x + \cos x)^3}\,dx$

$I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\left(\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)\right)^3}\,dx$

$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\cos x}{(\cos x + \sin x)^3}\,dx$

$\therefore \quad 2I = \int_0^{\pi/2} \frac{1}{(\sin x + \cos x)^2}\,dx = \frac{1}{2}\int_0^{\pi/2} \frac{dx}{\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right)^2}$

$= \frac{1}{2}\int_0^{\pi/2} \frac{dx}{\sin^2\left(\frac{\pi}{4} + x\right)}$

$= \frac{1}{2}\int_0^{\pi/2} \operatorname{cosec}^2\left(\frac{\pi}{4} + x\right)dx = -\frac{1}{2}\left[\cot\left(\frac{\pi}{4} + x\right)\right]_0^{\pi/2}$

$= -\frac{1}{2}[-1-1] = 1 \qquad \Rightarrow I = \frac{1}{2}$

42. We have, $f(x) = x^2 + ax^2 + bx^3$

where, $a = \int_{-1}^{1} t\,f(t)\,dt$ and $b = \int_{-1}^{1} f(t)\,dt$

Now, $a = \int_{-1}^{1} t\,[(a+1)t^2 + bt^3]\,dt$

$\Rightarrow \quad a = 2b\int_0^1 t^4 dt = \frac{2b}{5}$ . ..(i)

Again, $b = \int_{-1}^{1} f(t)\,dt = \int_{-1}^{1} ((a+1)t^2 + bt^3)\,dt$

$\Rightarrow \quad b = 2\int_0^1 (a+1)t^2\,dt$

$\Rightarrow \quad b = \frac{2(a+1)}{3}$ ...(ii)

From Eqs. (i) and (ii), $\frac{5a}{2} = \frac{2(a+1)}{3}$

$\Rightarrow \quad \left(\frac{5}{2} - \frac{2}{3}\right)a = \frac{2}{3} \quad \Rightarrow \quad \frac{11}{6}a = \frac{2}{3}$

$\Rightarrow \quad a = \frac{4}{11}$ and $b = \frac{10}{11}$

Hence, $\int_{-1}^{1} t\,f(t)\,dt = \frac{4}{11}$ and $\int_{-1}^{1} f(t)\,dt = \frac{10}{11}$

$\therefore \quad f(x) = (a+1)x^2 + bx^3$

$f(1) = (a+1) + b$

$f(-1) = (a+1) - b$

$\Rightarrow \quad f(1) + f(-1) = 2(a+1) = \frac{30}{11}$

and $f(1) - f(-1) = 2b = \frac{20}{11}$

43. Given, $(f'(x))^2 + (g(x))^2 = 1$

$f(x) + \int_0^x g(t)\,dt = \sin x(\cos x - \sin x)$

Differentiating both the sides, we get

$f'(x) + g(x) = \cos 2x - \sin 2x$ ...(i)

Squaring both the sides of Eq. (i), we get

$(f'(x))^2 + (g(x))^2 = 2f'(x)\cdot g(x) = 1 - \sin 4x$

$\Rightarrow \quad 1 + 2f'(x)\cdot g(x) = 1 - \sin 4x$

$\therefore \quad 2f'(x)\,g(x) = -\sin 4x$

Now, substituting $g(x) = -\frac{\sin 4x}{2f'(x)}$ in Eq. (i), we get

$$f'(x) - \frac{\sin 4x}{2f'(x)} = \cos 2x - \sin 2x$$

Put $f'(x) = t$

$\Rightarrow 2t^2 - 2(\cos 2x - \sin 2x)\, t - \sin 4x = 0$

$\Rightarrow t = \dfrac{2(\cos 2x - \sin 2x) \pm \sqrt{4(1 - \sin 4x) + 8 \sin 4x}}{4}$

$\therefore \quad 4t = 2(\cos 2x - \sin 2x) \pm \sqrt{4(1 - \sin 4x) + 8 \sin 4x}$

$\Rightarrow 2t = (\cos 2x - \sin 2x) \pm \sqrt{1 + \sin 4x}$

Taking + ve sign, $2t = \cos 2x - \sin 2x + \cos 2x + \sin 2x$

$\Rightarrow \quad t = \cos 2x$

Taking – ve sign, $t = -\sin 2x$

Since, $f'(x) = \cos 2x$ or $f'(x) = -\sin 2x$

$$f(x) = \frac{1}{2}\sin 2x + C_1 \quad \text{or} \quad f(x) = \frac{\cos 2x}{2} + C_2$$

$f(0) = 0$

$\Rightarrow \quad C_1 = 0$ and $C_2 = -1/2$

$\therefore \quad f(x) = \dfrac{1}{2}\sin 2x$ or $f(x) = \dfrac{\cos 2x - 1}{2}$

If $f'(x) = \cos 2x$, then $g(x) = -\sin 2x$

If $f'(x) = -\sin 2x$, then $g(x) = \cos 2x$

i.e. $f(x) = \dfrac{1}{2}\sin 2x$ and $g(x) = -\sin 2x$

$\Rightarrow f(x) = \dfrac{\cos 2x - 1}{2}$ and $g(x) = \cos 2x$

44. Consider $f(x) = \int_{-x}^{x} (\underbrace{t \sin at}_{\text{even}} + \underbrace{bt}_{\text{odd}} + \underbrace{c}_{\text{even}})\, dt$

$$= 2\int_0^x (t \sin at + c)\, dt$$

$$= 2\left[\left|-t\,\frac{\cos at}{a}\right|_0^x + \int_0^x \frac{\cos at}{a}\, dt + |ct|_0^x\right] \text{(using I.B.P.)}$$

$$= 2\left[\frac{-x\cos ax}{a} + \frac{1}{a^2}\sin ax + cx\right]$$

$$\therefore \quad \lim_{x\to 0}\frac{f(x)}{x} = \lim_{x\to 0} 2\left[-\frac{\cos ax}{a} + \frac{\sin ax}{a \cdot ax} + c\right]$$

$$= 2\left[-\frac{1}{a} + \frac{1}{a} + c\right] = 2c$$

45. Consider $I = \int_a^{\infty} \dfrac{n\, dx}{n^2\left(x^2 + \dfrac{1}{n^2}\right)} = \dfrac{1}{n} \cdot n\,(\tan^{-1} nx)_0^{\infty} = \left(\dfrac{\pi}{2} - \tan^{-1} an\right)$

$$\therefore \quad L = \lim_{n\to\infty}\left(\frac{\pi}{2} - \tan^{-1} an\right) = \begin{cases} \pi, & \text{if } a < 0 \\ \pi/2, & \text{if } a = 0 \\ 0, & \text{if } a > 0 \end{cases}$$

46. $\lim\limits_{x\to 0} \dfrac{\int_0^x \dfrac{t^2 dt}{(a + t^r)^{1/p}}}{bx - \sin x} = \lim\limits_{x\to 0} \dfrac{\dfrac{x^2}{(a + x^r)^{1/p}}}{b - \cos x}$ Using L'Hospital's rule

For existence of limit, $\lim\limits_{x\to 0}$ denominator = 0

$\therefore \quad b - 1 = 0 \Rightarrow \quad b = 1$

47. $l = \lim\limits_{x\to 0} \dfrac{x^2}{(a + x^r)^{1/p}} \cdot \dfrac{x^2}{(1 - \cos x)} \cdot \dfrac{1}{x^2}$

$$= 2\lim_{x\to 0} \frac{1}{(a + x^r)^{1/p}} = \frac{2}{a^{1/p}}$$

If $p = 3$ and $l = 1$, then $1 = \dfrac{2}{a^{1/3}} \Rightarrow a = 8$

48. If, $p = 2$ and $a = 9$, then $l = \dfrac{2}{9^{1/2}} = \dfrac{2}{3}$

49. Here, $f(x) = e^x \underbrace{\int_0^1 e^t \cdot f(t)\, dt}_{A\,(\text{say})}$

$f(x) = Ae^x$ ...(i)

$\Rightarrow \quad f(t) = Ae^t$

where, $A = \int_0^1 e^t \cdot f(t)\, dt$

$\Rightarrow \quad A = \int_0^1 e^t \cdot Ae^t\, dt; \quad A = A\int_0^1 e^{2t}\, dt$

Now, $A\left[\int_0^1 e^{2t} dt - 1\right] = 0 \Rightarrow A = 0$, as $\int_0^1 e^{2t}\, dt \neq 0$

Hence, $f(x) = 0 \Rightarrow f(1) = 0$

50. Again, $g(x) = e^x \int_0^1 e^t\, g(t)\, dt + x$

$\Rightarrow \quad g(x) = Be^x + x$ ...(ii)

$\Rightarrow \quad g(t) = Be^t + t$

where, $B = \int_0^1 e^t\, g(t)\, dt \Rightarrow B = \int_0^t e^t\,(Be^t + t)\, dt$

$\Rightarrow \quad B = B\int_0^1 e^{2t}\, dt + \int_0^1 e^t \cdot t\, dt$

But $\int_0^1 e^{2t}\, dt = \dfrac{1}{2}(e^2 - 1)$ and $\int_0^1 te^t\, dt = 1$

$\therefore \quad B = \dfrac{B}{2}(e^2 - 1) + 1$

$\Rightarrow \quad 2B = B(e^2 - 1) + 2$

$\Rightarrow \quad 3B = Be^2 + 2 \quad \Rightarrow \quad B = \dfrac{2}{3 - e^2}$

From Eq. (ii), $g(x) = \left(\dfrac{2}{3 - e^2}\right)e^x + x \Rightarrow g(0) = \dfrac{2}{3 - e^2}$

Also, $f(0) = 0$

$\therefore \quad g(0) - f(0) = \dfrac{2}{3 - e^2} - 0 = \dfrac{2}{3 - e^2}$

51. $g(2) = \dfrac{2e^2}{3 - e^2} + 2 = \dfrac{6}{3 - e^2}$

$\therefore \quad \dfrac{g(0)}{g(2)} = \dfrac{2}{3 - e^2} \cdot \dfrac{3 - e^2}{6} = \dfrac{1}{3}$

Solutions (Q. Nos. 52 to 54)

We have the equations of the tangents to the curve $y = \int_{-\infty}^{x} f(t)dt$ and $y = f(x)$ at arbitrary points on them are

$$Y - \int_{-x}^{x} f(t)\, dt = f(x)\,(X - x) \quad \text{...(i)}$$

and $Y - f(x) = f'(x)\,(X - x)$ ...(ii)

As Eqs. (i) and (ii) intersect at the same point on the X-axis

Putting $Y = 0$ and equating x-coordinates, we have

$$x - \frac{f(x)}{f'(x} = x - \frac{\int_{-\infty}^{x} f(t)\,dt}{f(x)}$$

$$\Rightarrow \quad \frac{f(x)}{\int_{-\infty}^{x} f(t)\,dt} = \frac{f'(x)}{f(x)}$$

$$\Rightarrow \quad \int_{-\infty}^{x} f(t)\,dt = cf(x) \qquad \text{...(iii)}$$

As, $f(0) = 1 \Rightarrow \int_{-\infty}^{0} f(t)\,dt = c \times 1 \Rightarrow c = \frac{1}{2}$

$\Rightarrow \int_{-\infty}^{x} f(t)\,dt = \frac{1}{2} f(x)$; differentiating both the sides and integrating and using boundary conditions, we get $f(x) = e^{2x}$; $y = 2ex$ is tangent to $y = e^{2x}$.

$\therefore$ Number of solutions = 1.

Cearly, $f(x)$ is increasing for all x.

$\therefore \quad \lim_{x \to \infty} (e^{2x})^{e^{-2x}} = 1 \qquad (\infty^0 \text{ form})$

Solutions (Q. Nos. 55 to 57)

We have, $g(x) = g(0) + xg'(0) + \frac{x^2}{2} g''(0) = -bx^2 + cx - 6$

$h(x) = g(x) = 4x^4 - ax^3 + bx^2 - cx + 6 = 0$ has 4 distinct real roots. Using Descarte's rule of signs.

Given biquadratic equation has 4 distinct positive roots.

Let the roots be x_1, x_2, x_3 and x_4.

Now, $\frac{\frac{1}{x_1} + \frac{2}{x_2} + \frac{3}{x_3} + \frac{4}{x_4}}{4} \geq \sqrt[4]{\frac{24}{x_1\ x_2\ x_3\ x_4}}$

$\Rightarrow \quad 2 \geq 2 \Rightarrow \frac{1}{x_1} = \frac{2}{x_2} = \frac{3}{x_3} = \frac{4}{x_4} = k$

$\Rightarrow \quad \frac{1}{x_1} \cdot \frac{2}{x_2} \cdot \frac{3}{x_3} \cdot \frac{4}{x_4} = k^4$

$\Rightarrow \quad \frac{24}{3/2} = k^4 \ \Rightarrow \ k = 2$

$\therefore$ Roots are $\frac{1}{2}$, 1, $\frac{3}{2}$ and 2. Also, $a = 20$ and $c = 25$

58. At $x = 1$, $y = 0$, $\frac{dy}{dx} = 2x \cdot (x^4)^2 - (x^2)^2 = 1$

$\therefore$ Equation of the tangent is $y = x - 1$.

59. $\frac{dy}{dx} = 4x^3(\ln x^4)^2 - 3x^2 (\ln x^3)^2$

$= 64x^3 (\ln x)^2 - 27x^2 (\ln x)^2$

$\therefore \lim_{x \to 0^+} \frac{dy}{dx} = 64 \lim_{x \to 0^+} x^3 (\ln x)^2 - 27 \lim_{x \to 0^+} x^2 (\ln x)^2 = 0$

60. We have, $f(x) = \int_1^x e^{t^2/2} (1 - t^2)\,dt$

$\therefore \quad f'(x) = [e^{x^2/2} (1 - x^2)]^2$

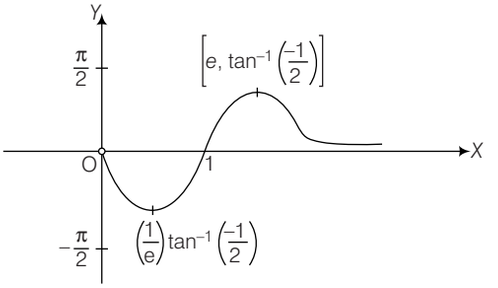
Now, $f'(1) = e^{1/2} \cdot 0 = 0$

61. Here, $f(x) = \tan^{-1}\left(\frac{\log_e x}{\log_e x + 1}\right), 0 < x < \infty$

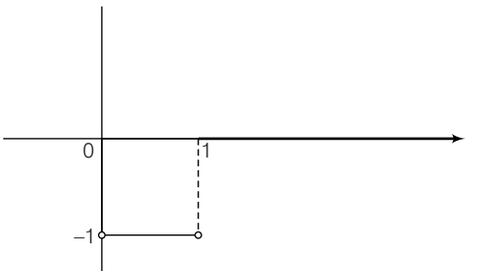
attains minimum at $x = \frac{1}{e}$ and maximum at $x = e$.

Also, $f(x)$ has $y = 0$ as asymptotes.

$\therefore \quad f(x)$ can be shown as



Clearly, $f(x)$ is neither injective nor subjective, also graph for $[f(x)]$ can be shown, as



62. $\int_0^{\infty} [f(x)]dx = \int_0^1 -1 \cdot dx + \int_1^{\infty} 0 \cdot dx = -(x)\frac{1}{0} = -1$

63. (A) For $a = 0$, $I = \int_0^T \sin^2 x\,dx$

$= \int_0^T \frac{1 - \cos 2x}{2}\,dx$

$= \left[\frac{x}{2} - \frac{1}{4} \sin 2x\right]_0^T = \frac{T}{2} - \frac{1}{4} \sin 2T$

$\therefore \quad L = \frac{1}{2} - \lim_{T \to \infty} \frac{1}{4} \frac{\sin 2T}{T} = \frac{1}{2}$

(B) For $a = 1$, $\int_0^T 4 \sin^2 x\,dx \Rightarrow L = 2$

(C) For $a = -1$, $\int_0^T 0\,dx = 0 \Rightarrow L = 0$

(D) For $a \neq 0, -1, 1$,

$I = \int_0^T (\sin^2 x + \sin^2 ax + 2 \sin x \cdot \sin ax)\,dx$

$= \int_0^T \left(\frac{1 - \cos 2x}{2} + \frac{1 - \cos 2ax}{2} + \cos(a-1)x - \cos(a+1)x\right)dx$

$= \left[x - \frac{1}{4} \sin 2x - \frac{1}{4a} \sin 2ax \frac{\sin(a-1)\,x}{a-1} - \frac{\sin(a+1)\,x}{a+1}\right]_0^T$

$\therefore L = \lim_{T \to \infty} \frac{T}{T} - \lim_{T \to \infty} \frac{1}{T}$

$\left[\frac{1}{4} \sin 2x - \frac{1}{4a} \sin 2ax + \frac{\sin(a-1)\,x}{a-1} - \frac{\sin(a+1)\,x}{a+1}\right]_0^T$

$\Rightarrow \quad L = 1$

64. $\because \quad f(\theta)=\left[\frac{(x+\sin\theta)^3}{3}\right]_0^1=\frac{(1+\sin\theta)^3-\sin^3\theta}{3}$

and $g(\theta)=\left[\frac{(x+\cos\theta)^3}{3}\right]_0^1=\frac{(1+\cos\theta)^3-\cos^3\theta}{3}$

$\therefore f(\theta)=\frac{1+3\sin\theta+3\sin^2\theta}{3}$ and $g(\theta)=\frac{1+3\cos\theta+3\cos^2\theta}{3}$

Now, $f(\theta)-g(\theta)=(\sin\theta-\cos\theta)+(\sin^2\theta-\cos^2\theta)$

and $f(\theta)-g(\theta)=(\sin\theta-\cos\theta)(1+\sin\theta+\cos\theta)$

Now verify all matchings.

65. (A) Let $a=2008$, then

$I=\int_0^1(1+ax^a)\,e^{x^a}\,dx$

$I=\int_0^1(e^{x^a}+ax^a\,e^{x^a})\,dx \qquad (\text{note}: ax^a=ax\cdot x^{a-1})$

$\therefore \quad I=\int_0^1(e^{x^a}+e^{x^a}\cdot x\cdot ax^{a-1})\,dx$

$=\int_0^1(f(x)+x\,f'(x))\,dx$, where $f(x)=e^{x^a}$

Hence, $I=[xe^{x^a}]_0^1=e$

(B) $\quad I=I_1+I_2$

Consider $I_2=\int_1^{1/e}\sqrt{-\ln x}\,dx$

Put $\sqrt{-\ln x}=t\Rightarrow -\ln x=t^2\Rightarrow x=e^{-t^2}$

$\Rightarrow \qquad dx=-2t\,e^{-t^2}\,dt$

$\therefore \quad I_2=\int_0^1\underbrace{t}_{\text{I}}\cdot\underbrace{(-2te^{-t^2})}_{\text{II}}\,dt=[te^{-t^2}]_0^1-\int_0^1e^{-t^2}dt=\frac{1}{e}-\int_0^1e^{-t^2}dt$

Hence, $\quad I_2=\int_0^1e^{-x}\,dx+\frac{1}{e}-\int_0^1e^{-t^2}\,dt=\frac{1}{e}=e^{-1}$

Note that, if $f(x)=e^{-x^2}$, then $f^{-1}(x)=\sqrt{-\ln x}$

(C) $L=\lim_{n\to\infty}\left[\left(\frac{1}{n}\right)^1\cdot\left(\frac{2}{n}\right)^2\cdot\left(\frac{3}{n}\right)^3\cdots\left(\frac{n}{n}\right)^n\right]^{\frac{1}{n^2}}$

$\Rightarrow \quad \ln L=\lim_{n\to\infty}\frac{1}{n^2}$

$\left[1\cdot\ln\left(\frac{1}{n}\right)+2\cdot\ln\left(\frac{2}{n}\right)+3\cdot\ln\left(\frac{3}{n}\right)+\ldots+n\cdot\ln\left(\frac{n}{n}\right)\right]$

General term of $\ln L=\frac{r}{n^2}\ln\frac{r}{n}$

$$\text{Sum}=\frac{1}{n}\cdot\sum_{r=1}^{n}\frac{r}{n}\ln\left(\frac{r}{n}\right)$$

$\ln L=\int_0^1 x\ln x\,dx=\left[\frac{x^2\cdot\ln x}{2}\right]_0^1-\frac{1}{2}\int_0^1x^2\,\frac{1}{x}\,dx$

$=\left[0-\frac{1}{2}\,\frac{x^2}{2}\right]_0^1=-\frac{1}{4}$

$\therefore \quad L=e^{-1/4}$

66. (A) We have, $f'(x)=\frac{g'(x)}{\sqrt{1+g^3(x)}}$

and $g'(x)=(1+\sin(\cos^2 x))(-\sin x)$

Hence, $\quad f'(x)=\frac{(1+\sin(\cos^2 x))(-\sin x)}{\sqrt{1+g^3(x)}}$

$$f'\left(\frac{\pi}{2}\right)=\frac{1+0}{\sqrt{1+g^3\left(\frac{\pi}{2}\right)}}=-1,\ g\left(\frac{\pi}{2}\right)=0$$

$\therefore \quad f'\left(\frac{\pi}{2}\right)=-1$

(B) We have, $\int_0^x f(x)\,dx=\{f(x)\}^2$

Differentiating both the sides, we get

$$f(x)=2f(x)\cdot f'(x)\ \Rightarrow\ f'(x)=\frac{1}{2}$$

Integrating both the sides, $f(x)=\frac{1}{2}x+C$

where, $\quad f(0)=0\ \Rightarrow\ C=0$

$\Rightarrow \qquad f(x)=\frac{x}{2}\ \Rightarrow\ f(2)=1$

(C) Maximum when $a=-1, b=2\Rightarrow a+b=1$

(D) If $\quad \lim_{x\to0}\frac{\sin 2x}{x^3}+a+\frac{b}{x^2}=0$, then

$$\lim_{x\to0}\frac{\sin 2x+ax^3+bx}{x^3}=0$$

For limit to exist $2+b=0\Rightarrow b=-2$

$\therefore \quad \lim_{x\to0}\frac{\sin 2x+ax^3-2x}{x^3}=0$

Using left hand rule and solving, we get $a=\frac{4}{3}$

$\therefore \qquad 3a+b=2$

67. Let $B=\sum_{n=1}\frac{\sin nx}{4^n}=\frac{\sin x}{4}+\frac{\sin 2x}{4^2}+\ldots$

and $\quad A=1+\cos\frac{x}{4}+\cos\frac{2x}{4^2}+\ldots$

$\therefore A+iB=1+\frac{e^{ix}}{4}+\frac{e^{i2x}}{4^2}+\ldots=\frac{1}{1-\frac{e^{ix}}{4}}$

Thus, B imaginary part of $\frac{1}{1-\frac{e^{ix}}{4}}=\frac{4}{4-\cos x-i\sin x}$

$\therefore \qquad B=\frac{4\sin x}{17-\theta\cos x}\Rightarrow f(x)=\frac{4\sin x}{17-\theta\cos x}$

and $\int_0^\pi f(x)dx=\int_0^\pi\frac{4\sin x}{17-\theta\cos x}=\frac{1}{2}\log\left(\frac{25}{9}\right)$

$=\log\left(\frac{5}{3}\right)=\log\left(\frac{m}{n}\right)$

$\therefore \qquad m+n=8$

68. Here, $I=\int_{-\pi/2}^{\pi/2}\frac{\cos x dx}{1+2[\sin^{-1}(\sin x)]}$

$=\int_{-\pi/2}^{-1}\frac{\cos x}{-3}dx+\int_{-1}^{0}\frac{\cos x\,dx}{-1}+\int_0^1\cos x\,dx+\int_1^{\pi/2}\frac{\cos x}{3}dx$

$=\frac{1}{3}\int_{-1}^{-\pi/2}\cos x dx+\int_0^{-1}\cos x\,dx+\int_0^1\cos x\,dx+\frac{1}{3}\int_1^{\pi/2}\cos x\,dx$

$=\frac{1}{3}\int_0^{\pi/2}\cos t(-dt)+\int_0^1\cos(t)\cdot(-dt)$

$+\int_0^1\cos x\,dx+\frac{1}{3}\int_1^{\pi/2}\cos x\,dx$

$=0$

69. Here, $f(x+1)-f(x)=\frac{1}{\pi}\int_0^{\pi/2}\frac{\sin^2(n+1)\theta-\sin^2 n\theta}{\sin^2\theta}d\theta$

$=\frac{1}{\pi}\int_0^{\pi/2}\frac{\sin(2n+1)\theta\cdot\sin\theta}{\sin^2\theta}d\theta$

$=\frac{1}{\pi}\int_0^{\pi/2}\frac{\sin(2n+1)\theta}{\sin^2\theta}d\theta$

$=\frac{1}{\pi}\left(\int_0^{\pi/2}\frac{\sin 2n\theta\cdot\cos\theta}{\sin\theta}d\theta+\int_0^{\pi/2}\cos 2n\theta\, d\theta\right)$

Using, $\cos\theta+\cos3\theta+\cos5\theta+....+\cos(2n-1)\theta$

$=\frac{\sin\frac{n\cdot(2\theta)}{2}}{\sin\left(\frac{2\theta}{2}\right)}\cdot\cos(\theta+(n-1)\theta)$

i.e. $\cos\theta+\cos3\theta+\cos5\theta+...+\cos(2n-1)\theta=\frac{1}{2}\left(\frac{\sin2n\theta}{\sin\theta}\right)$

$\therefore f(n+1)-f(n)=\frac{1}{\pi}$

$\left[2\int_0^{\pi/2}\cos\theta+\cos3\theta+...+\cos(2n-1)\theta\cdot\cos\theta\,d\theta+0\right]$

$=\frac{1}{\pi}\int_0^{\pi/2}\{(2\cos\theta\cos\theta)+(2\cos3\theta\cos\theta)$

$+\ldots(2\cos(2n-1)\cdot\cos\theta)\}d\theta$

$=\frac{1}{\pi}\int_0^{\pi/2}1\cdot d\theta,\ \text{as}\int_0^{\pi/2}\cos2n\theta\,d\theta=0$

$\therefore\quad f(n+1)-f(n)=\frac{1}{2}$

If $n=1$, $f(2)-f(1)=\frac{1}{2}\left(\text{as}\,f(1)=\frac{1}{2}\right)\Rightarrow f(2)=\frac{2}{2}$

If $n=2$, $f(3)-f(2)=\frac{1}{2}$

$\Rightarrow f(3)=\frac{3}{2}$ and so on $\quad\therefore f(n)=\frac{n}{2}$

Hence, $\frac{f(15)+f(3)}{f(15)-f(9)}=\frac{\frac{15}{2}+\frac{3}{2}}{\frac{15}{2}-\frac{9}{2}}=3$

70. Here, $g(x)=\int_{-2}^{h(x)}e^{(1+t)^2}dt$

$\Rightarrow\quad g'(x)=h'(x)\cdot e^{(1+h(x))}$

$\Rightarrow\quad g'(x)=h'(x)\cdot e$

$\Rightarrow\quad g'(2)=h'(2)\cdot e^{(1+h(x))^2}$

$\Rightarrow e^4=1\cdot e^{(1+h(2))^2}$, given $g'(2)=e^4$ and $h'(2)=1$

$\therefore\quad (1+h(2))^2=4$

$\Rightarrow\quad 1+h(2)=2,-2$

$\therefore\quad h(2)=-3,1$

$\therefore$ Absolute sum for all possible values of $h(2)=|-3+1|=2$

71. Let $I=\int_0^{\pi/2}\sin x\cdot\log(\sin x)dx=\frac{1}{2}\int_0^{\pi/2}\sin x\cdot\log(\sin^2 x)dx$

$=\frac{1}{2}\int_0^{\pi/2}\sin x\cdot\log(1-\cos^2 x)dx$

Put $\cos x=t\Rightarrow-\sin x\,dx=dt$

$\therefore\quad I=\frac{1}{2}\int_0^1\log(1-t^2)dt$

$=\frac{1}{2}\int_0^1\left(-t^2-\frac{(-t^2)^2}{2}+\frac{(-t^2)^3}{2}...\right)dt$

$=\frac{1}{2}\left[\frac{-t^3}{3}-\frac{t^5}{10}+\frac{t^7}{21}...\right]_0^1=\frac{1}{2}\left[\frac{-1}{3}-\frac{1}{10}+\frac{1}{21}...\right]$

$=-\frac{1}{2}\left[\frac{1}{3}+\frac{1}{10}+\frac{1}{21}+...\right]=-\left[\frac{1}{2\times3}+\frac{1}{20}+\frac{1}{42}+...\right]$

$=-\left[\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{6}-\frac{1}{7}\right)+...\right]$

$=\log_e 2-1=\log_e\left(\frac{2}{e}\right)$

$\therefore\quad K=2$

72. Let $I=\int_{-\pi/2}^{\pi/2}\frac{x^2\cos x}{1+e^x}dx$ ...(i)

$\left[\because\int_a^b f(x)dx=\int_a^b f(a+b-x)\,dx\right]$

$\Rightarrow\quad I=\int_{-\pi/2}^{\pi/2}\frac{x^2\cos(-x)}{1+e^{-x}}dx$ ...(ii)

On adding Eqs. (i) and (ii), we get

$2I=\int_{-\pi/2}^{\pi/2}x^2\cos x\left[\frac{1}{1+e^x}+\frac{1}{1+e^{-x}}\right]dx$

$=\int_{-\pi/2}^{\pi/2}x^2\cos x\cdot(1)\,dx$

$\left[\because\int_{-a}^{a}f(x)dx=2\int_0^a f(x)dx,\ \text{when}\ f(-x)=f(x)\right]$

$\Rightarrow\quad 2I=2\int_0^{\pi/2}x^2\cos x\,dx$

Using integration by parts, we get

$2I=2\left[x^2(\sin x)-(2x)(-\cos x)+(2)(-\sin x)\right]_0^{\pi/2}$

$\Rightarrow\quad 2I=2\left[\frac{\pi^2}{4}-2\right]$

$\therefore\quad I=\frac{\pi^2}{4}-2$

73. Let $f(x)=\int_0^x\frac{t^2}{1+t^4}dt$

$\Rightarrow f'(x)=\frac{x^2}{1+x^4}>0$, for all $x\in[0,1]$

$\therefore f(x)$ is increasing.

At $x=0$, $f(0)=0$ and at $x=1$,

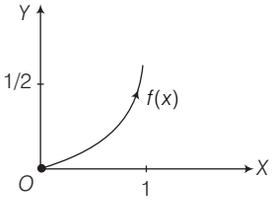
$f(1)=\int_0^1\frac{t^2}{1+t^4}dt$

Because, $\quad 0<\frac{t^2}{1+t^4}<\frac{1}{2}$

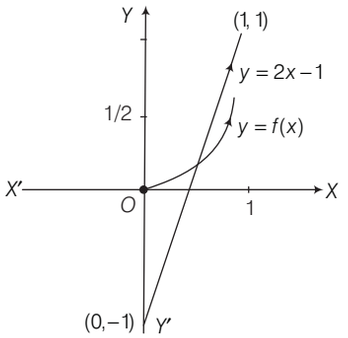
$\Rightarrow\quad\int_0^1 0\cdot dt<\int_0^1\frac{t^2}{1+t^4}dt<\int_0^1\frac{1}{2}\cdot dt$

$\Rightarrow\quad 0<f(1)<\frac{1}{2}$

Thus, $f(x)$ can be plotted as



$\therefore$ $y = f(x)$ and $y = 2x - 1$ can be shown as



From the graph, the total number of distinct solutions for $x \in (0, 1] = 1$. [as they intersect only at one point]

74. Here, $f(x) = 7\tan^8 x + 7\tan^6 x - 3\tan^4 x - 3\tan^2 x$

for all $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$$\therefore \quad f(x) = 7\tan^6 x \sec^2 x - 3\tan^2 x \sec^2 x$$
$$= (7\tan^6 x - 3\tan^2 x)\sec^2 x$$

Now, $\int_0^{\pi/4} x f(x)dx = \int_0^{\pi/4} \underset{\text{I}}{x}\,(7\tan^6 x - \underset{\text{II}}{3\tan^2 x})\sec^2 x\,dx$

$$= [x(\tan^7 x - \tan^3 x)]_0^{\pi/4} - \int_0^{\pi/4} 1\,(\tan^7 x - \tan^3 x)dx$$
$$= 0 - \int_0^{\pi/4} \tan^3 x(\tan^4 x - 1)dx$$
$$= -\int_0^{\pi/4} \tan^3 x\,(\tan^2 x - 1)\sec^2 x\,dx$$

Put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$$\therefore \quad \int_0^{\pi/4} x f(x)dx = -\int_0^1 t^3(t^2 - 1)dt$$
$$= \int_0^1 (t^3 - t^5)dt = \left[\frac{t^4}{4} - \frac{t^5}{5}\right]_0^1 = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$$

Also, $\int_0^{\pi/4} f(x)dx = \int_0^{\pi/4} (7\tan^6 x - 3\tan^2 x)\ \sec^2 x\,dx$

$$= \int_0^1 (7t^6 - 3t^2)dt = [t^7 - t^3]_0^1 = 0$$

75. Here, $f'(x) = \dfrac{192\,x^3}{2 + \sin^4 \pi x}$

$$\therefore \quad \frac{192x^3}{3} \le f'(x) \le \frac{192x^3}{2}$$

On integrating between the limits $\frac{1}{2}$ to x, we get

$$\int_{1/2}^x \frac{192x^3}{3}dx \le \int_{1/2}^x f'(x)dx \le \int_{1/2}^x \frac{192x^3}{2}dx$$

$$\Rightarrow \quad \frac{192}{12}\left(x^4 - \frac{1}{16}\right) \le f(x) - f(0) \le 24x^4 - \frac{3}{2}$$

$$\Rightarrow \quad 16x^4 - 1 \le f(x) \le 24x^4 - \frac{3}{2}$$

Again, integrating between the limits $\frac{1}{2}$ to 1, we get

$$\int_{1/2}^1 (16x^4 - 1)dx \le \int_{1/2}^1 f(x)dx \le \int_{1/2}^1 \left(24x^4 - \frac{3}{2}\right)dx$$

$$\Rightarrow \quad \left[\frac{16x^5}{5} - x\right]_{1/2}^1 \le \int_{1/2}^1 f(x)dx \le \left[\frac{24x^5}{5} - \frac{3}{2}x\right]_{1/2}^1$$

$$\Rightarrow \quad \left(\frac{11}{5} + \frac{2}{5}\right) \le \int_{1/2}^1 f(x)dx \le \left(\frac{33}{10} + \frac{6}{10}\right)$$

$$\Rightarrow \quad 2.6 \le \int_{1/2}^1 f(x)dx \le 3.9$$

76. Let $I_1 = \int_0^{4\pi} e^t(\sin^6 at + \cos^6 at)dt$

$$= \int_0^{\pi} e^t(\sin^6 at + \cos^6 at)dt + \int_{\pi}^{2\pi} e^t(\sin^6 at + \cos^6 at)dt$$
$$+ \int_{2\pi}^{3\pi} e^t(\sin^6 at + \cos^6 at)dt + \int_{3\pi}^{4\pi} e^t(\sin^6 at + \cos^6 at)dt$$

$$\therefore \quad I_1 = I_2 + I_3 + I_4 + I_5 \qquad \ldots(\text{i})$$

Now, $I_3 = \int_{\pi}^{2\pi} e^t(\sin^6 at + \cos^6 at)dt$

Put $t = \pi + t \Rightarrow dt = dt$

$$\therefore \quad I_3 = \int_0^{\pi} e^{\pi + t} \cdot (\sin^6 at + \cos^6 at)dt$$
$$= e^t \cdot I_2 \qquad \ldots(\text{ii})$$

Now, $I_4 = \int_{2\pi}^{3\pi} e^t(\sin^6 at + \cos^6 at)dt$

Put $t = 2\pi + t \Rightarrow dt = dt$

$$\therefore \quad I_4 = \int_0^{\pi} e^{t + 2\pi}(\sin^6 at + \cos^6 at)dt$$
$$= e^{2\pi} \cdot I_2 \qquad \ldots(\text{iii})$$

and $I_5 = \int_{3\pi}^{4\pi} e^t(\sin^6 at + \cos^6 at)dt$

Put $t = 3\pi + t$

$$\therefore \quad I_5 = \int_0^{\pi} e^{3\pi + t}(\sin^6 at + \cos^6 at)dt$$
$$= e^{3\pi} \cdot I_2 \qquad \ldots(\text{iv})$$

From Eqs. (i), (ii), (iii) and (iv), we get

$$I_1 = I_2 + e^{\pi} \cdot I_2 + e^{2\pi} \cdot I_2 + e^{3\pi} \cdot I_2 = (1 + e^{\pi} + e^{2\pi} + e^{3\pi})\, I_2$$

$$\therefore \quad L = \frac{\int_0^{4\pi} e^t(\sin^6 at + \cos^6 at)dt}{\int_0^{\pi} e^t(\sin^6 at + \cos^6 at)dt}$$
$$= (1 + e^{\pi} + e^{2\pi} + e^{3\pi}) = \frac{1 \cdot (e^{4\pi} - 1)}{e^{\pi} - 1} \text{ for } a \in R$$

77. According to the given data, $F(x) < 0,\ \forall x \in (1, 3)$

We have, $f(x) = xF(x)$

$$\Rightarrow \quad f'(x) = F(x) + xF'(x) \qquad \ldots(\text{i})$$

$\Rightarrow$ $f'(1) = F(1) + F'(1) < 0$ [given $F(1) = 0$ and $F'(x) < 0$]

Also, $f(2) = 2F(2) < 0$ [using $F(x) < 0, \forall x \in (1, 3)$]

Now, $f'(x) = F(x) + xF'(x) < 0$ [using $F(x) < 0,\ \forall x \in (1,3)$]

$\Rightarrow$ $f'(x) < 0$

78. Given, $\int_1^3 x^2F'(x)dx=-12$

$\Rightarrow \quad [x^2F(x)]_1^3-\int_1^3 2x\cdot F(x)dx=-12$

$\Rightarrow \quad 9F(3)-F(1)-2\int_1^3 f(x)dx=-12\ [\because xF(x)=f(x), \text{given}]$

$\Rightarrow \quad -36-0-2\int_1^3 f(x)dx=-12$

$\therefore \quad \int_1^3 f(x)dx=-12$

and $\int_1^3 x^3F''(x)dx=40$

$\Rightarrow \quad [x^3F'(x)]_1^3-\int_1^3 3x^2F'(x)dx=40$

$\Rightarrow \quad [x^2(xF'(x)]_1^3-3\times(-12)=40$

$\Rightarrow \quad \{x^2\cdot[f'(x)-F(x)]\}_1^3=4$

$\Rightarrow 9[f'(3)-F(3)]-[f'(1)-F(1)]=4$

$\Rightarrow \quad 9[f'(3)+4]-[f'(1)-0]=4$

$\Rightarrow \quad 9f'(3)-f'(1)=-32$

79. Here, $f(x)=\begin{cases}[x], & x\le 2\\ 0, & x>2\end{cases}$

$\therefore \quad I=\int_{-1}^2 \frac{xf(x^2)}{2+f(x+1)}dx$

$=\int_{-1}^0 \frac{xf(x^2)}{2+f(x+1)}dx+\int_0^1 \frac{xf(x^2)}{2+f(x+1)}dx$

$+\int_1^{\sqrt2} \frac{xf(x^2)}{2+f(x+1)}dx+\int_{\sqrt2}^{\sqrt3} \frac{xf(x^2)}{2+f(x+1)}dx$

$+\int_{\sqrt3}^2 \frac{xf(x^2)}{2+f(x+1)}dx$

$=\int_{-1}^0 0\,dx+\int_0^1 0\,dx+\int_1^{\sqrt2} \frac{x\cdot1}{2+0}dx+\int_{\sqrt2}^{\sqrt3} 0\,dx+\int_{\sqrt3}^2 0\,dx$

$\because -1<x<0\Rightarrow 0<x^2<1\Rightarrow[x^2]=0,$

$0<x<1\Rightarrow 0<x^2<1\Rightarrow[x^2]=0,$

$1<x<\sqrt2\Rightarrow\begin{cases}1<x^2<2 & \Rightarrow[x^2]=1\\ 2<x+1<1+\sqrt2 & \Rightarrow f(x+1)=0,\end{cases}$

$\sqrt2<x<\sqrt3\ \Rightarrow 2<x^2<3\Rightarrow f(x^2)=0,$

and $\sqrt3<x<2\Rightarrow 3<x^2<4\Rightarrow f(x^2)=0$

$\Rightarrow \quad I=\int_1^{\sqrt2}\frac{x}{2}dx=\left[\frac{x^2}{4}\right]_1^{\sqrt2}=\frac14(2-1)=\frac14$

$\therefore \quad 4I=1\ \Rightarrow\ 4I-1=0$

80. Here, $\alpha=\int_0^1 e^{(9x+3\tan^{-1}x)}\left(\frac{12+9x^2}{1+x^2}\right)dx$

Put $9x+3\tan^{-1}x=t$

$\Rightarrow \quad \left(9+\frac{3}{1+x^2}\right)dx=dt$

$\therefore \quad \alpha=\int_0^{9+3\pi/4} e^t\,dt=[e^t]_0^{9+3\pi/4}=e^{9+3\pi/4}-1$

$\Rightarrow \quad \log_e|1+\alpha|=9+\frac{3\pi}{4}$

$\Rightarrow \quad \log_e|\alpha+1|-\frac{3\pi}{4}=9$

81. **Plan** This type of question can be done using appropriate substitution.

Given, $I=\int_{\pi/4}^{\pi/2}(2\ \text{cosec}\ x)^{17}\,dx$

$=\int_{\pi/4}^{\pi/2}\frac{2^{17}(\text{cosec}\ x)^{16}\ \text{cosec}\ x(\text{cosec}\ x+\cot x)}{(\text{cosec}\ x+\cot x)}dx$

Let $\text{cosec}\ x+\cot x=t$

$\Rightarrow (-\text{cosec}\ x\cdot\cot x-\text{cosec}^2x)\,dx=dt$

and $\text{cosec}\ x-\cot x=1/t$

$\Rightarrow \quad 2\ \text{cosec}\ x=t+\frac1t$

$\therefore \quad I=-\int_{\sqrt2+1}^1 2^{17}\left(\frac{t+\frac1t}{2}\right)^{16}\frac{dt}{t}$

Let $t=e^u\ \Rightarrow\ dt=e^u du.$ When $t=1, e^u=1\Rightarrow u=0$

and when $t=\sqrt2+1, e^u=\sqrt2+1$

$\Rightarrow \quad u=\ln(\sqrt2+1)$

$\Rightarrow \quad I=-\int_{\ln(\sqrt2+1)}^0 2(e^u+e^{-u})^{16}\frac{e^u du}{e^u}$

$=2\int_0^{\ln(\sqrt2+1)}(e^u+e^{-u})^{16}\,du$

82. **Plan** Newton-Leibnitz's formula

$\frac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)} f(t)\,dt\right]=f\{\psi(x)\}\left\{\frac{d}{dx}\psi(x)\right\}-f\{\phi(x)\}\left\{\frac{d}{dx}\phi(x)\right\}$

Given, $F(x)=\int_0^{x^2} f(\sqrt t)\,dt$

$\therefore \quad F'(x)=2x\,f(x)$

Also, $F'(x)=f'(x)$

$\Rightarrow \quad 2x\,f(x)=f'(x)$

$\Rightarrow \quad \frac{f'(x)}{f(x)}=2x$

$\Rightarrow \quad \int\frac{f'(x)}{f(x)}dx=\int 2x\,dx$

$\Rightarrow \quad \text{In}\ f(x)=x^2+c\ \Rightarrow\ f(x)=e^{x^2+c}$

$\Rightarrow \quad f(x)=K\,e^{x^2}\qquad [K=e^c]$

Now, $f(0)=1$

$\therefore \quad 1=K$

Hence, $f(x)=e^{x^2}$

$F(2)=\int_0^4 e^t dt=[e^t]_0^4=e^4-1$

83. (A) Let $I=\int_{-1}^1\frac{dx}{1+x^2}$

Put $x=\tan\theta$

$\Rightarrow dx=\sec^2\theta\,d\theta$

$\therefore \quad I=2\int_0^{\pi/4}d\theta=\frac{\pi}{2}$

(B) Let $I = \int_0^1 \frac{dx}{\sqrt{1-x^2}}$

Put $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

$\therefore \quad I = \int_0^{\pi/2} 1\, d\theta = \frac{\pi}{2}$

(C) $\int_2^3 \frac{dx}{1-x^2} = \frac{1}{2}\left[\log\left(\frac{1+x}{1-x}\right)\right]_2^3$

$= \frac{1}{2}\left[\log\left(\frac{4}{-2}\right) - \log\left(\frac{3}{-1}\right)\right] = \frac{1}{2}\left[\log\left(\frac{2}{3}\right)\right]$

(D) $\int_1^2 \frac{dx}{x\sqrt{x^2-1}} = [\sec^{-1} x]_1^2 = \frac{\pi}{3} - 0 = \frac{\pi}{3}$

84. (A) **Plan**

(p) A polynomial satisfying the given conditions is taken.

(q) The other conditions are also applied and the number of polynomial is taken out.

Let $f(x) = ax^2 + bx + c$

$f(0) = 0 \Rightarrow c = 0$

Now, $\int_0^1 f(x)\, dx = 1 \Rightarrow \left(\frac{ax^3}{3} + \frac{bx^2}{2}\right)_0^1 = 1$

$\Rightarrow \quad \frac{\alpha}{3} + \frac{\beta}{2} = 1 \Rightarrow \quad 2a + 3b = 6$

As a, b are non-negative integers.

So, $a = 0, b = 2$ or $a = 3, b = 0$

$\therefore \quad f(x) = 2x$ or $f(x) = 3x^2$

(B) **Plan** Such type of questions are converted into only sine or cosine expression and then the number of points of maxima in given interval are obtained.

$f(x) = \sin(x^2) + \cos(x^2)$

$= \sqrt{2}\left[\frac{1}{\sqrt{2}}\cos(x^2) + \frac{1}{\sqrt{2}}\sin(x^2)\right]$

$= \sqrt{2}\left[\cos x^2 \cos\frac{\pi}{4} + \sin\frac{\pi}{4}\sin(x^2)\right] = \sqrt{2}\cos\left(x^2 - \frac{\pi}{4}\right)$

For maximum value, $x^2 - \frac{\pi}{4} = 2n\pi \Rightarrow x^2 = 2n\pi + \frac{\pi}{4}$

$\Rightarrow \quad x = \pm\sqrt{\frac{\pi}{4}}$, for $n = 0$

$x = \pm\sqrt{\frac{9\pi}{4}}$, for $n = 1$

So, $f(x)$ attains maximum at 4 points in $[-\sqrt{13}, \sqrt{13}]$.

(C) **Plan**

(p) $\int_{-a}^a f(x)\, dx = \int_{-a}^a f(-x)\, dx$

(q) $\int_{-a}^a f(x)\, dx = 2\int_0^a f(x)\, dx$, if $f(-x) = f(x)$, i.e. f is an even function.

$I = \int_{-2}^2 \frac{3x^2}{1+e^x}\, dx$ and $I = \int_{-2}^2 \frac{3x^2}{1+e^{-x}}\, dx$

$\Rightarrow \quad 2I = \int_{-2}^2 \left(\frac{3x^2}{1+e^x} + \frac{3x^2(e^x)}{e^x+1}\right) dx$

$2I = \int_{-2}^2 3x^2\, dx \quad \Rightarrow \quad 2I = 2\int_0^2 3x^2\, dx$

$I = [x^3]_0^2 = 8$

(D) **Plan** $\int_{-a}^a f(x)\, dx = 0$

If $f(-x) = -f(x)$, i.e. $f(x)$ is an odd function.

Let $f(x) = \cos 2x \log\left(\frac{1+x}{1-x}\right)$

$f(-x) = \cos 2x \log\left(\frac{1-x}{1+x}\right) = -f(x)$

Hence, $f(x)$ is an odd function.

So, $\int_{-1/2}^{1/2} f(x)\, dx = 0$

(A) $\to$ (q); (B) $\to$ (r); (C) $\to$ (p); (D) $\to$ (s)

85. **Plan** Integration by parts

$$\int f(x)\, g(x)\, dx = f(x)\int g(x)\, dx - \int\left(\frac{d}{dx}[f(x)]\int g(x)\, dx\right) dx$$

Given, $I = \int_0^1 \underset{\text{I}}{4x^3}\, \frac{d^2}{dx^2}\underset{\text{II}}{(1-x^2)^5}\, dx$

$= \left[4x^3\frac{d}{dx}(1-x^2)^5\right]_0^1 - \int_0^1 12x^2\frac{d}{dx}(1-x^2)^5\, dx$

$= \left[4x^3 \times 5(1-x^2)^4(-2x)\right]_0^1$

$- 12\left[[x^2(1-x^2)^5]_0^1 - \int_0^1 2x(1-x^2)^5\, dx\right]$

$= 0 - 0 - 12(0-0) + 12\int_0^1 2x(1-x^2)^5\, dx$

$= 12 \times \left[-\frac{(1-x^2)^6}{6}\right]_0^1 = 12\left[0 + \frac{1}{6}\right] = 2$

86. $I = \int_{-\pi/2}^{\pi/2}\left[x^2 + \log\left(\frac{\pi - x}{\pi + x}\right)\right]\cos x\, dx$

As, $\int_{-a}^a f(x)\, dx = 0$, when $f(-x) = -f(x)$

$\therefore \quad I = \int_{-\pi/2}^{\pi/2} x^2 \cos x\, dx + 0 = 2\int_0^{\pi/2}(x^2\cos x)\, dx$

$= 2\{(x^2\sin x)_0^{\pi/2} - \int_0^{\pi/2} 2x\cdot\sin x\, dx\}$

$= 2\left[\frac{\pi^2}{4} - 2\{(-x\cdot\cos x)_0^{\pi/2} - \int_0^{\pi/2} 1\cdot(-\cos x)\, dx\}\right]$

$= 2\left[\frac{\pi^2}{4} - 2(\sin x)_0^{\pi/2}\right] = 2\left[\frac{\pi^2}{4} - 2\right] = \left(\frac{\pi^2}{2} - 4\right)$

87. Put $x^2 = t \Rightarrow x\, dx = dt/2$

$\therefore \quad I = \int_{\log 2}^{\log 3} \frac{\sin t \cdot \frac{dt}{2}}{\sin t + \sin(\log 6 - t)}$ ...(i)

Using, $\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$

$= \frac{1}{2}\int_{\log 2}^{\log 3} \frac{\sin(\log 2 + \log 3 - t)}{\sin(\log 2 + \log 3 - t) + \sin(\log 6 - (\log 2 + \log 3 - t))}\, dt$

$= \frac{1}{2}\int_{\log 2}^{\log 3} \frac{\sin(\log 6 - t)}{\sin(\log 6 - t) + \sin(t)}\, dt$

$\therefore \quad I = \int_{\log 2}^{\log 3} \frac{\sin(\log 6 - t)}{\sin(\log 6 - t) + \sin t}\, dt$ ...(ii)

On adding Eqs. (i) and (ii), we get

$$2I = \frac{1}{2}\int_{\log 2}^{\log 3} \frac{\sin t + \sin(\log 6 - t)}{\sin(\log 6 - t) + \sin t}\, dt$$

$$\Rightarrow \quad 2I = \frac{1}{2}(t)_{\log 2}^{\log 3} = \frac{1}{2}(\log 3 - \log 2)$$

$$\therefore \quad I = \frac{1}{4}\log\left(\frac{3}{2}\right)$$

88. Given, $f(1) = \frac{1}{3}$ and $6\int_1^x f(t)dt = 3x\, f(x) - x^3,\ \forall\ x \geq 1$

Using Newton-Leibnitz formula.

Differentiating both sides

$$\Rightarrow \quad 6f(x)\cdot 1 - 0 = 3f(x) + 3xf'(x) - 3x^2$$

$$\Rightarrow\ 3xf'(x) - 3f(x) = 3x^2 \quad \Rightarrow \quad f'(x) - \frac{1}{x}f(x) = x$$

$$\Rightarrow \quad \frac{xf'(x) - f'(x)}{x^2} = 1 \qquad \Rightarrow \qquad \frac{d}{dx}\left\{\frac{x}{x}\right\} = 1$$

On integrating both sides, we get

$$\Rightarrow \quad \frac{f(x)}{x} = x + c \qquad \left[\because\ f(1) = \frac{1}{3}\right]$$

$$\frac{1}{3} = 1 + c \ \Rightarrow\ c = \frac{2}{3} \text{ and } f(x) = x^2 - \frac{2}{3}x$$

$$\therefore \quad f(2) = 4 - \frac{4}{3} = \frac{8}{3}$$

Note Here, $f(1) = 2$, does not satisfy given function.

$$\therefore \quad f(1) = \frac{1}{3}$$

For that $f(x) = x^2 - \frac{2}{3}x$ and $f(2) = 4 - \frac{4}{3} = \frac{8}{3}$

89. Let $I = \int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx$

$$= \int_0^1 \frac{(x^4-1)(1-x)^4 + (1-x)^4}{(1+x^2)}dx$$

$$= \int_0^1 (x^2-1)(1-x)^4\, dx + \int_0^1 \frac{(1+x^2-2x)^2}{(1+x^2)}dx$$

$$= \int_0^1 \left\{(x^2-1)(1-x)^4 + (1+x^2) - 4x + \frac{4x^2}{(1+x^2)}\right\}dx$$

$$= \int_0^1 \left((x^2-1)(1-x)^4 + (1+x^2) - 4x + 4 - \frac{4}{1+x^2}\right)dx$$

$$= \int_0^1 \left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2}\right)dx$$

$$= \left[\frac{x^7}{7} - \frac{4x^6}{6} + \frac{5x^5}{5} - \frac{4x^3}{3} + 4x - 4\tan^{-1}x\right]_0^1$$

$$= \frac{1}{7} - \frac{4}{6} + \frac{5}{5} - \frac{4}{3} + 4 - 4\left(\frac{\pi}{4} - 0\right) = \frac{22}{7} - \pi$$

90. Converting infinite series into definite integral

i.e. $\lim_{n\to\infty} \frac{h(n)}{n}$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{r=g(n)}^{h(n)} f\left(\frac{r}{n}\right) = \int f(x)dx$$

$\lim_{n\to\infty} \frac{g(n)}{n}$, where, $\frac{r}{n}$ is replaced with x.

Σ is replaced with integral.

Here, $\lim_{n\to\infty} \frac{1^a + 2^a + \ldots + n^a}{(n+1)^{a-1}\{(na+1) + (na+2) + \ldots + (na+n)\}} = \frac{1}{60}$

$$\Rightarrow \lim_{n\to\infty} \frac{\sum_{r=1}^{n} r^a}{(n+1)^{a-1}\cdot\left[n^2a + \frac{n(n+1)}{2}\right]} = \frac{1}{60}$$

$$\Rightarrow \lim_{n\to\infty} \frac{2\sum_{r=1}^{n}\left(\frac{r}{n}\right)^a}{\left(1+\frac{1}{n}\right)^{a-1}\cdot(2na+n+1)} = \frac{1}{60}$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n}\left(2\sum_{r=1}^{n}\left(\frac{r}{n}\right)^a\right)\cdot \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{a-1}\cdot\left(2a+1+\frac{1}{n}\right)} = \frac{1}{60}$$

$$\Rightarrow \quad 2\int_0^1 (x^a)dx \cdot \frac{1}{1\cdot(2a+1)} = \frac{1}{60}$$

$$\Rightarrow \quad \frac{2\cdot[x^{a+1}]_0^1}{(2a+1)\cdot(a+1)} = \frac{1}{60}$$

$$\therefore \quad \frac{2}{(2a+1)(a+1)} = \frac{1}{60}$$

$$\Rightarrow \quad (2a+1)(a+1) = 120$$

$$\Rightarrow \quad 2a^2 + 3a + 1 - 120 = 0$$

$$\Rightarrow \quad 2a^2 + 3a - 119 = 0$$

$$\Rightarrow \quad (2a+17)(a-7) = 0$$

$$\Rightarrow \quad a = 7, \frac{-17}{2}$$

91. Here, $f(x) + 2x = (1-x)^2\cdot\sin^2 x + x^2 + 2x$ ...(i)

where, $P: f(x) + 2x = 2(1+x)^2$ ...(ii)

$$\therefore \quad 2(1+x^2) = (1-x)^2\sin^2 x + x^2 + 2x$$

$$\Rightarrow \quad (1-x)^2\sin^2 x = x^2 - 2x + 2$$

$$\Rightarrow \quad (1-x)^2\sin^2 x = (1-x)^2 + 1 \ \Rightarrow (1-x)^2\cos^2 x = -1$$

which is never possible.

$\therefore$ P is false.

Again, let $Q: h(x) = 2f(x) + 1 - 2x(1+x)$

where, $h(0) = 2f(0) + 1 - 0 = 1$

$h(1) = 2f(1) + 1 - 4 = -3$, as $h(0)\,h(1) < 0$

$\Rightarrow$ $h(x)$ must have a solution.

$\therefore$ Q is true.

92. Here, $f(x) = (1-x)^2\cdot\sin^2 x + x^2 \geq 0,\ \forall\ x.$

and $g(x) = \int_1^x \left(\frac{2(t-1)}{t+1} - \log t\right) f(t)\, dt$

$$\Rightarrow \quad g'(x) = \left\{\frac{2(x-1)}{(x+1)} - \log x\right\}\cdot \underbrace{f(x)}_{+\text{ ve}} \qquad \ldots(i)$$

For $g'(x)$ to be increasing or decreasing,

let $\phi(x) = \frac{2(x-1)}{(x+1)} - \log x$

$$\phi'(x)=\frac{4}{(x+1)^2}-\frac{1}{x}=\frac{-(x-1)^2}{x(x+1)^2}$$

$\phi'(x)<0$, for $x>1 \Rightarrow \phi(x)<\phi(1) \Rightarrow \phi(x)<0$ ...(ii)

From Eqs. (i) and (ii), we get

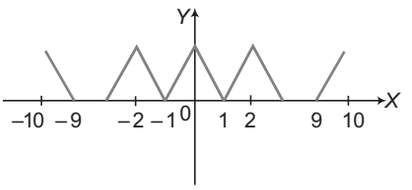
$$g'(x)<0 \text{ for } x\in(1,\infty)$$

$\therefore$ $g(x)$ is decreasing for $x\in(1,\infty)$.

93. Given, $f(x)=\begin{cases} x-[x], & \text{if } [x] \text{ is odd.}\\ 1+[x]-x, & \text{if } [x] \text{ is even.}\end{cases}$

$f(x)$ and $\cos\pi x$ both are periodic with period 2 and both are even.

$$\therefore \quad \int_{-10}^{10} f(x)\cos\pi x\,dx=2\int_0^{10} f(x)\cos\pi x\,dx$$



$$=10\int_0^2 f(x)\cos\pi x\,dx$$

Now, $\int_0^1 f(x)\cos\pi x\,dx=\int_0^1(1-x)\cos\pi x\,dx=-\int_0^1 u\cos\pi u\,du$

and $\int_1^2 f(x)\cos\pi x\,dx=\int_1^2(x-1)\cos\pi x\,dx=-\int_0^1 u\cos\pi u\,du$

$$\therefore \quad \int_{-10}^{10} f(x)\cos\pi x\,dx=-20\int_0^1 u\cos\pi u\,du=\frac{40}{\pi^2}$$

$$\Rightarrow \quad \frac{\pi^2}{10}\int_{-10}^{10} f(x)\cos\pi x\,dx=4$$

94. Given $\int_0^x\sqrt{1-\{f'(t)\}^2}\,dt=\int_0^x f(t)\,dt,\ 0\le x\le 1$

Differentiating both sides w.r.t. x by using Leibnitz's rule, we get

$$\sqrt{1-\{f'(x)\}^2}=f(x) \Rightarrow f'(x)=\pm\sqrt{1-\{f(x)\}^2}$$

$$\Rightarrow \int\frac{f'(x)}{\sqrt{1-\{f(x)\}^2}}\,dx=\pm\int dx$$

$$\Rightarrow \quad \sin^{-1}\{f(x)\}=\pm x+c$$

Put $x=0 \Rightarrow \sin^{-1}\{f(0)\}=c$

$\Rightarrow$ $c=\sin^{-1}(0)=0$ $[\because f(0)=0]$

$\therefore$ $f(x)=\pm\sin x$

but $f(x)\ge 0,\ \forall\ x\in[0,1]$

$\therefore$ $f(x)=\sin x$

As we know that,

$$\sin x<x,\ \forall\ x>0$$

$$\therefore \quad \sin\left(\frac{1}{2}\right)<\frac{1}{2} \text{ and } \sin\left(\frac{1}{3}\right)<\frac{1}{3}$$

$$\Rightarrow \quad f\left(\frac{1}{2}\right)<\frac{1}{2} \text{ and } f\left(\frac{1}{3}\right)<\frac{1}{3}$$

95. Given $I_n=\int_{-\pi}^{\pi}\frac{\sin nx}{(1+\pi^x)\sin x}\,dx$ ...(i)

Using $\int_a^b f(x)\,dx=\int_a^b f(b+a-x)\,dx$, we get

$$I_n=\int_{-\pi}^{\pi}\frac{\pi^x\sin nx}{(1+\pi^x)\sin x}\,dx \quad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we have

$$2I_n=\int_{-\pi}^{\pi}\frac{\sin nx}{\sin x}\,dx=2\int_0^{\pi}\frac{\sin nx}{\sin x}\,dx$$

$$[\because f(x)=\frac{\sin nx}{\sin x} \text{ is an even function}]$$

$$\Rightarrow \quad I_n=\int_0^{\pi}\frac{\sin nx}{\sin x}\,dx$$

Now, $I_{n+2}-I_n=\int_0^{\pi}\frac{\sin(n+2)x-\sin nx}{\sin x}\,dx$

$$=\int_0^{\pi}\frac{2\cos(n+1)\,x\cdot\sin x}{\sin x}\,dx$$

$$=2\int_0^{\pi}\cos(n+1)x\,dx=2\left[\frac{\sin(n+1)\,x}{(n+1)}\right]_0^{\pi}=0$$

$\therefore$ $I_{n+2}=I_n$ ...(iii)

Since, $I_n=\int_0^{\pi}\frac{\sin nx}{\sin x}\,dx \Rightarrow I_1=\pi$ and $I_2=0$

From Eq. (iii) $I_1=I_3=I_5=....=\pi$ and $I_2=I_4=I_6=...=0$

$$\Rightarrow \quad \sum_{m=1}^{10} I_{2m+1}=10\pi \text{ and } \sum_{m=1}^{10} I_{2m}=0$$

$\therefore$ Correct options are (a), (b), (c).

96. Given, $S_n=\sum_{k=0}^{n}\frac{n}{n^2+kn+k^2}$

$$=\sum_{k=0}^{n}\frac{1}{n}\cdot\left(\frac{1}{1+\frac{k}{n}+\frac{k^2}{n^2}}\right)<\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{n}\left(\frac{1}{1+\frac{k}{n}+\left(\frac{k}{n}\right)^2}\right)$$

$$=\int_0^1\frac{1}{1+x+x^2}\,dx=\left[\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)\right]_0^1$$

$$=\frac{2}{\sqrt{3}}\cdot\left(\frac{\pi}{3}-\frac{\pi}{6}\right)=\frac{\pi}{3\sqrt{3}} \quad \text{i.e. } S_n<\frac{\pi}{3\sqrt{3}}$$

Similarly, $T_n>\frac{\pi}{3\sqrt{3}}$

97. $I=\int_{\pi/4}^{3\pi/4}\frac{dx}{1+\cos x}$ ...(i)

$$I=\int_{\pi/4}^{3\pi/4}\frac{dx}{1-\cos x} \quad \text{...(ii)}$$

Adding Eqs. (i) and (ii)

$$2I=\int_{\pi/4}^{3\pi/4}\frac{2}{\sin^2 x}\,dx \Rightarrow I=\int_{\pi/4}^{3\pi/4}\text{cosec}^2\,x\,dx$$

$$I=-(\cot\,x)_{\pi/4}^{3\pi/4}=2$$

98. $I_4+I_6=\int(\tan^4 x+\tan^6 x)\,dx=\int\tan^4 x\sec^2 x\,dx$

$$=\frac{1}{5}\tan^5 x+c \quad \Rightarrow \quad a=\frac{1}{5},\ b=0$$

99. (b) Let $l=\lim_{n\to\infty}\left[\frac{(n+1)\cdot(n+2)\ldots(3n)}{n^{2n}}\right]^{\frac{1}{n}}$

$$=\lim_{n\to\infty}\left[\frac{(n+1)\cdot(n+2)\ldots(n+2n)}{n^{2n}}\right]^{\frac{1}{n}}$$

$$=\lim_{n\to\infty}\left[\left(\frac{n+1}{n}\right)\left(\frac{n+2}{n}\right)\cdots\left(\frac{n+2n}{n}\right)\right]^{\frac{1}{n}}$$

Taking log on both sides, we get

$$\log l=\lim_{n\to\infty}\frac{1}{n}\left[\log\left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\left(1+\frac{2n}{n}\right)\right\}\right]$$

$$\Rightarrow\ \log l=\lim_{n\to\infty}\frac{1}{n}\left[\log\left(1+\frac{1}{n}\right)+\log\left(1+\frac{2}{n}\right)+\ldots+\log\left(1+\frac{2n}{n}\right)\right]$$

$$\Rightarrow\ \log l=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{2n}\log\left(1+\frac{r}{n}\right)$$

$$\Rightarrow\quad \log l=\int_0^2\log(1+x)\,dx$$

$$\Rightarrow\quad \log l=\left[\log(1+x)\cdot x-\int\frac{1}{1+x}\cdot x\,dx\right]_0^2$$

$$\Rightarrow\quad \log l=[\log(1+x)\cdot x]_0^2-\int_0^2\frac{x+1-1}{1+x}\,dx$$

$$\Rightarrow\quad \log l=2\cdot\log 3-\int_0^2\left(1-\frac{1}{1+x}\right)dx$$

$$\Rightarrow\quad \log l=2\cdot\log 3-\left[x-\log|1+x|\right]_0^2$$

$$\Rightarrow\quad \log l=2\cdot\log 3-[2-\log 3]$$

$$\Rightarrow\quad \log l=3\cdot\log 3-2\ \Rightarrow\ \log l=\log 27-2$$

$$\therefore\quad l=e^{\log 27-2}=27\cdot e^{-2}=\frac{27}{e^2}$$

100. **Central Idea** Apply the property $\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$ and then add. Let

$$I=\int_2^4\frac{\log x^2}{\log x^2+\log(36-12x+x^2)}dx$$

$$=\int_2^4\frac{2\log x}{2\log x+\log(6-x)^2}dx$$

$$=\int_2^4\frac{2\log x\,dx}{2[\log x+\log(6-x)]}$$

$$\Rightarrow\quad I=\int_2^4\frac{\log x\,dx}{[\log x+\log(6-x)]}\qquad\ldots(i)$$

$$\Rightarrow\quad I=\int_2^4\frac{\log(6-x)}{\log(6-x)+\log x}dx\qquad\ldots(ii)$$

$$\left[\because\int_a^b f(x)dx=\int_a^b f(a+b-x)dx\right]$$

On adding Eqs. (i) and (ii), we get

$$2I=\int_2^4\frac{\log x+\log(6-x)}{\log x+\log(6-x)}dx$$

$$\Rightarrow\quad 2I=\int_2^4 dx=[x]_2^4$$

$$\Rightarrow\quad 2I=2\ \Rightarrow\ I=1$$

101. **Plan** Use the formula, $|x-a|=\begin{cases}x-a, & x\geq a\\ -(x-a), & x<a\end{cases}$ to break given integral in two parts and then integrate separately.

$$\int_0^\pi\sqrt{\left(1-2\sin\frac{x}{2}\right)^2}dx=\int_0^\pi\left|1-2\sin\frac{x}{2}\right|dx$$

$$=\int_0^{\frac{\pi}{3}}\left(1-2\sin\frac{x}{2}\right)dx-\int_{\frac{\pi}{3}}^\pi\left(1-2\sin\frac{x}{2}\right)dx$$

$$=\left(x+4\cos\frac{x}{2}\right)_0^{\frac{\pi}{3}}-\left(x+4\cos\frac{x}{2}\right)_{\frac{\pi}{3}}^\pi=4\sqrt{3}-4-\frac{\pi}{3}$$

102. Let $\quad I=\int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\tan x}}\qquad\ldots(i)$

$$\therefore\quad I=\int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\tan\left(\frac{\pi}{2}-x\right)}}=\int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\cot x}}$$

$$\Rightarrow\quad I=\int_{\pi/6}^{\pi/3}\frac{\sqrt{\tan x}\,dx}{1+\sqrt{\tan x}}\qquad\ldots(ii)$$

On adding Eqs. (i) and (ii), we get

$$2I=\int_{\pi/6}^{\pi/3}dx\ \Rightarrow\ 2I=[x]_{\pi/6}^{\pi/3}dx$$

$$\Rightarrow\quad I=\frac{1}{2}\left[\frac{\pi}{3}-\frac{\pi}{6}\right]=\frac{\pi}{12}$$

Statement I is false.

But $\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$ is a true statement by property of definite integrals.

103. Given, $y=\int_0^x|t|\,dt$

$$\therefore\quad \frac{dy}{dx}=|x|\cdot 1-0=|x|\qquad\text{[by Leibnitz's rule]}$$

$\because$ Tangent to the curve $y=\int_0^x|t|\,dt$, $x\in R$ are parallel to the line $y=2x$

$\therefore$ Slope of both are equal $\Rightarrow\ x=\pm 2$

Points, $\quad y=\int_0^{\pm 2}|t|\,dt=\pm 2$

Equation of tangent is

$$y-2=2(x-2)\ \text{ and }\ y+2=2(x+2)$$

For x intercept put $y=0$, we get

$$0-2=2(x-2)\ \text{ and }\ 0+2=2(x+2)$$

$$\Rightarrow\quad x=\pm 1$$

104. Given integral $g(x)=\int_0^x\cos 4t\,dt$

To find $g(x+\pi)$ in terms of $g(x)$ and $g(\pi)$.

$$g(x)=\int_0^x\cos 4t\,dt$$

$$\Rightarrow\quad g(x+\pi)=\int_{t=0}^{t=x+\pi}\cos 4t\,dt$$

$$=\int_0^x\cos 4t\,dt+\int_x^{x+\pi}\cos 4t\,dt$$

$$=g(x)+I_1\qquad\text{(say)}$$

$$I_1=\int_x^{x+\pi}\cos 4t\,dt=\int_0^\pi\cos 4t\,dt$$

(definite integral property)

$$=g(\pi)$$

$\Rightarrow \quad g(x+\pi) = g(x) + g(\pi)$

But the value of I_1 is zero.

$\Rightarrow \quad I_1 = \left[\frac{\sin 4t}{4}\right]_0^{\pi} = \left(\frac{\sin 4\pi}{4} - \frac{\sin 0}{4}\right) = 0$

$\Rightarrow \quad g(x+\pi) = g(x) - g(\pi)$

In my opinion, the examiner has made this question keeping $g(x) + g(\pi)$ as the only answer in his/her mind. However, he/she did not realise that the value of the integral I_1 is actually zero. Hence, it does not matter whether you add to or subtract from $g(x)$.

105. $I = \int_0^1 \frac{8\log(1+x)}{(1+x^2)}\,dx$

Put $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

When $x = 0 \Rightarrow \tan\theta = 0$

$\therefore \quad \theta = 0$

When $x = 1 = \tan\theta$

$\Rightarrow \quad \theta = \frac{\pi}{4}$

$\therefore \quad I = \int_0^{\pi/4} \frac{8\log[1+\tan\theta]}{1+\tan^2\theta} \cdot \sec^2\theta\, d\theta$

$I = 8\int_0^{\pi/4} \log(1+\tan\theta)\, d\theta \qquad \ldots(i)$

Using $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, we get

$I = 8\int_0^{\pi/4} \log\left\{1+\tan\left(\frac{\pi}{4}-\theta\right)\right\} d\theta$

$= 8\int_0^{\pi/4} \log\left\{1+\frac{1-\tan\theta}{1+\tan\theta}\right\} d\theta$

$= 8\int_0^{\pi/4} \log\left\{\frac{2}{1+\tan\theta}\right\} d\theta \qquad \ldots(ii)$

Adding Eqs. (i) and (ii), we get

$2I = 8\int_0^{\pi/4} \left[\log\{1+\tan\theta\} + \log\left\{\frac{2}{1+\tan\theta}\right\}\right] d\theta$

$\Rightarrow \quad I = 4\int_0^{\pi/4} \log 2\, d\theta = 4\cdot\log 2(\theta)_0^{\pi/4}$

$= 4\log 2\cdot\left(\frac{\pi}{4}-0\right) = \pi\log 2$

106. If $\phi(x)$ and $\psi(x)$ are defined on $[a, b]$ and differentiable for every x and $f(t)$ is continuous, then

$$\frac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)} f(t)dt\right] = f[\psi(x)]\cdot\frac{d}{dx}\psi(x) - f[\phi(x)]\frac{d}{dx}\phi(x)$$

Here, $f(x) = \int_0^x \sqrt{t}\sin t\, dt$, where $x \in \left(0, \frac{5\pi}{2}\right)$

$f'(x) = \{\sqrt{x}\sin x - 0\} \qquad \ldots(i)$

(using Newton-Leibnitz formula)

$\therefore \quad f'(x) = \sqrt{x}\sin x = 0$

$\Rightarrow \quad \sin x = 0$

$\therefore \quad x = \pi, 2\pi$

$f''(x) = \sqrt{x}\cos x + \frac{1}{2\sqrt{x}}\sin x$

At $x = \pi$, $f''(\pi) = -\sqrt{\pi} < 0$

$\therefore$ Local maximum at $x = \pi$. At $x = 2\pi$, $f''(2\pi) = \sqrt{2\pi} > 0$

$\therefore$ Local minimum at $x = 2\pi$.

107. We have, $p'(x) = p'(1-x),\ \forall\ x \in [0,1],\ p(0) = 1,\ p(1) = 41$

$\Rightarrow \quad p(x) = -p(1-x) + C$

$\Rightarrow \quad 1 = -41 + C$

$\Rightarrow \quad C = 42$

$\therefore \quad p(x) + p(1-x) = 42$

Now, $I = \int_0^1 p(x)dx = \int_0^1 p(1-x)dx$

$\Rightarrow \quad 2I = \int_0^1 (p(x)+p(1-x))dx = \int_0^1 42\,dx = 42$

$\Rightarrow \quad I = 21$

108. Let $I = \int_0^{\pi} [\cot x]\,dx \qquad \ldots(i)$

$\Rightarrow I = \int_0^{\pi} [\cot(\pi - x)]\,dx = \int_0^{\pi} [-\cot x]\,dx \qquad \ldots(ii)$

On adding Eqs. (i) and (ii),

$2I = \int_0^{\pi} [\cot x]\,dx + \int_0^{\pi} [-\cot x]\,dx = \int_0^{\pi} (-1)\,dx$

$[\because [x] + [-x] = -1$, if $x \notin z$ and 0, if $x \in z]$

$= [-x]_0^{\pi} = -\pi$

$\therefore \quad I = -\frac{\pi}{2}$

109. Since, $I = \int_0^1 \frac{\sin x}{\sqrt{x}}dx < \int_0^1 \frac{x}{\sqrt{x}}dx$,

because in $x \in (0,1), x > \sin x$.

$I < \int_0^1 \sqrt{x}\,dx = \frac{2}{3}[x^{3/2}]_0^1 \Rightarrow I < \frac{2}{3}$

and $\quad J = \int_0^1 \frac{\cos x}{\sqrt{x}}dx < \int_0^1 x^{-\frac{1}{2}}dx = 2$

$J < 2$

CHAPTER

03

Area of Bounded Regions

Learning Part

Session 1

- Sketching of Some Common Curves
- Some More Curves which Occur Frequently in Mathematics in Standard Forms
- Asymptotes
- Areas of Curves Given by Cartesian Equations

Session 2

- Area Bounded by Two or More Curves

Practice Part

- JEE Type Examples
- Chapter Exercises

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Session 1

Sketching of Some Common Curves, Some More Curves which Occur Frequently in Mathematics in Standard Forms, Asymptotes, Areas of Curves Given by Cartesian Equations

Sketching of Some Common Curves

For finding the area of a given region, we require the knowledge of some standard curves.

(i) Straight Line

Every first degree equation in x, y represents a straight line. So, the general equation of a line is $ax + by + c = 0$. To draw a straight line find the points, where it meets with the coordinate axes by putting $y = 0$ and $x = 0$ respectively in its equation.

By joining these two points we get the sketch of the line. Sometimes the equation of a line is given in the form $y = mx$. This equation represents a line passing through the origin and inclined at an angle $\tan^{-1} m$ with the positive direction of X-axis. The equation of the form $x = a$ and $y = b$ represents straight lines parallel to Y-axis and X-axis, respectively.

Region Represented by a Linear Inequality

To find the region represented by linear inequality $ax + by \le c$ and $ax + by \ge c$, we proceed as follows

(i) Convert the inequality into equality to obtain a linear equation in x, y.

(ii) Draw the straight line represented by it.

(iii) The straight line obtained in (ii) divides the XY-plane in two parts.

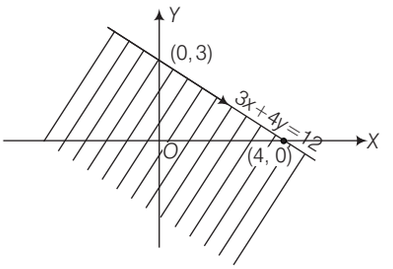
To determine the region represented by the inequality choose some convenient points; e.g. origin or some points on the coordinate axes.

If the coordinates of a point satisfy the inequality, then region containing the points is the required region, otherwise the region not containing the point is required region.

Example 1 Mark the region represented by $3x + 4y \le 12$.

Sol. Converting the inequality into equation, we get $3x + 4y = 12$.

This line meets the coordinate axes at (4, 0) and (0, 3), respectively. Join these points to obtain straight line represented by $3x + 4y = 12$.



This straight line divides the plane in two parts. One part contains the origin and the other does not contains the origin. Clearly, (0, 0) satisfy the inequality $3x + 4y < 12$. So, the region represented by $3x + 4y < 12$ is region containing the origin as shown in the figure.

(ii) Circle

The general equation of a circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$\therefore$ The second degree equation in x, y such that coeff. of x^2 = coeff. of y^2 and there is no term containing xy; it always represents a circle. To draw a sketch of a circle, we write the equation instandard form $(x - h)^2 + (y - k)^2 = r^2$, whose centre is (h, k) and radius is r.

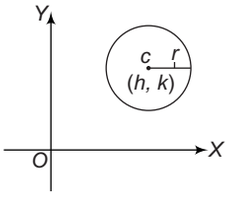


Figure 3.1

Remarks

1. The inequality $(x - a)^2 + (y - b)^2 < r^2$ represents the interior of a circle.
2. The inequality $(x - a)^2 + (y - b)^2 > r^2$ represents the exterior of a circle (i.e. region lying outside the circle).

(iii) Parabola

It is the locus of points such that its distance from a fixed point is equal to its distance from a fixed straight line.

Taking the fixed straight line $x = -a, a > 0$ and fixed point $(a, 0)$, we get the equation of parabola $y^2 = 4ax$.

Steps to Sketch the Curve

(i) It passes through (0, 0).

(ii) It is symmetrical about axis of X.

(iii) No part of the curve lies on the negative side of axis of X.

(iv) Curve turns at (0, 0) which is called the vertex of the curve.

(v) The curve extends to infinity. It is not a closed curve.

(1) $y^2 = 4ax$ (Standard equation of parabola)

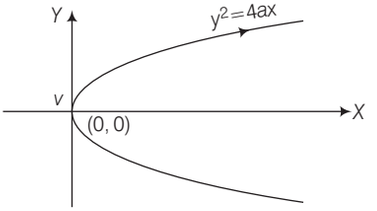


Figure 3.2

(2) $y^2 = 4a(x - h)$; where a and h are positive.

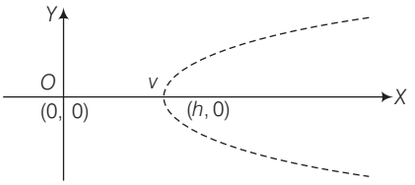


Figure 3.3

(3) $y^2 = 4a(x + h)$; where a and h are positive.

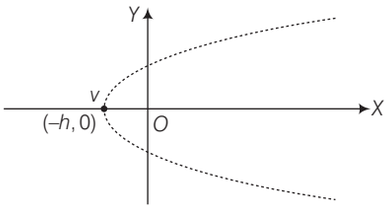


Figure 3.4

(4) $y^2 = -4ax$; $a > 0$

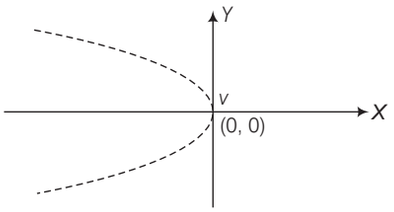


Figure 3.5

(5) $y^2 = -4a(x - h)$; $a, h > 0$

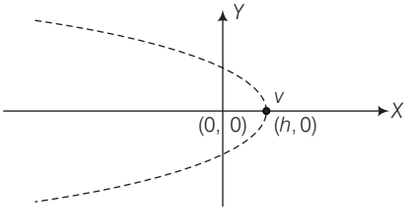


Figure 3.6

(6) $y^2 = -4a(x + h)$; $a, h > 0$

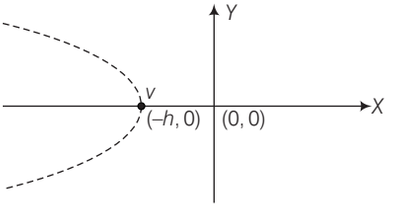


Figure 3.7

(7) $x^2 = 4ay$; $a > 0$

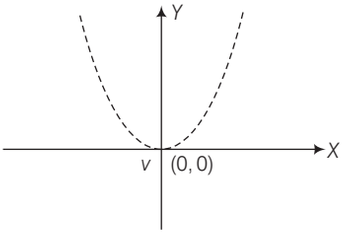


Figure 3.8

(8) $x^2 = 4a(y + k)$; $a > 0, k > 0$

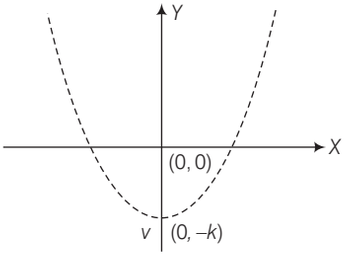


Figure 3.9

(9) $x^2 = 4a(y - k)$; $a, k > 0$

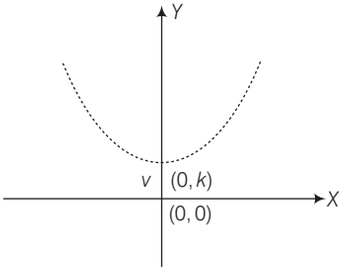


Figure 3.10

(10) $x^2 = -4ay;\ a > 0$

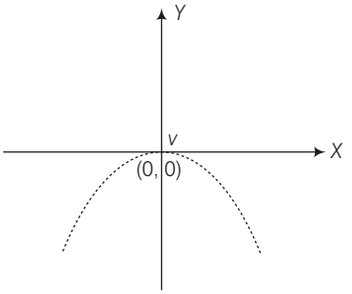


Figure 3.11

(11) $x^2 = -4a(y+k);\ a, k > 0$

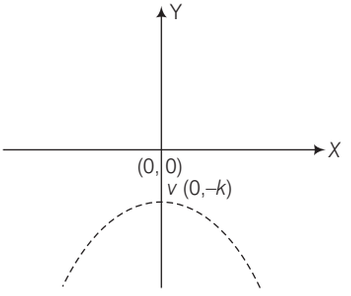


Figure 3.12

(12) $x^2 = -4a(y-k);\ a, k > 0$

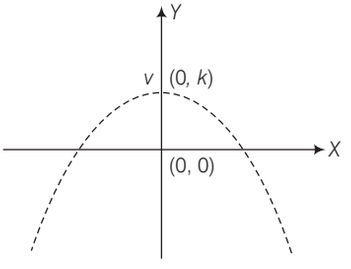


Figure 3.13

(iv) Ellipse

Basics of Ellipse

Definition 1. An ellipse is the locus of a moving point such that the ratio of its distance from a fixed point to its distance from a fixed line is a constant less than unity. This constant is termed the eccentricity of the ellipse. The fixed point is the focus while the fixed line is the directrix.

The symmetrical nature of the ellipse ensures that there will be two foci and two directrices.

Definition 2. An ellipse is the locus of a moving point such that the sum of the its distances from two fixed points is constant. The two fixed points are the two foci of the ellipse. To plot the ellipse, we can use the peg-and-thread method described earlier.

Stantard Equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

	If $a > b$	**If $a < b$**
Vertices	$(a, 0)$ and $(-a, 0)$	$(0, b)$ and $(0, -b)$
Foci	$(ae, 0)$ and $(-ae, 0)$	$(0, be)$ and $(0, -be)$
Major axis	$2a$ (along x-axis)	$2b$ (along y-axis)
Minor axis	$2b$ (along y-axis)	$2a$ (along x-axis)
Directrices	$x = \frac{a}{e}$ and $x = -\frac{a}{e}$	$y = \frac{b}{e}$ and $y = -\frac{b}{e}$
Eccentricity e	$\sqrt{1 - \frac{b^2}{a^2}}$	$\sqrt{1 - \frac{a^2}{b^2}}$
Latus-rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Focal distances of (x, y)	$a \pm ex$	$b \pm ey$

And lastly, if the equation of the ellipse is

$$\frac{(x-\alpha)^2}{a^2} + \frac{(y-\beta)^2}{b^2} = 1$$

instead of the usual standard form, we can use the transformation $X \to x - \alpha$ and $Y \to y - \beta$ (basically a translation of the axes so the axes so that the origin of the new system coincides with (α, β). The equation then becomes

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$$

We can now work on this form, use all the standard formulae that we'd like to and obtain whatever it is that we wish to obtain. The final result (in the x-y system) is obtained using the reverse transformation $x \to X + \alpha$ and $y \to Y + \beta$.

(1) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;\ a > 0, b > 0$ (Standard equation of the ellipse)

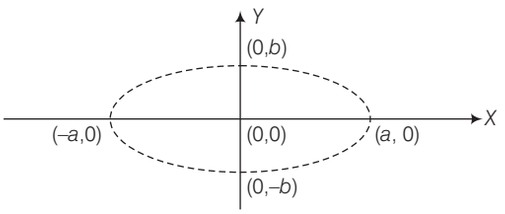


Figure 3.14

(2) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1;\ b>a>0$ (Conjugate ellipse)

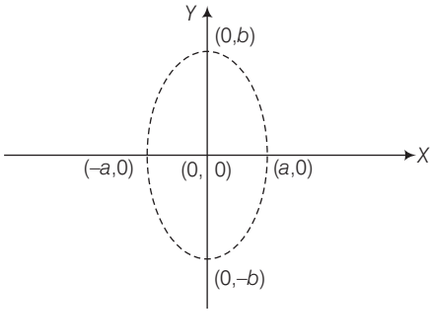


Figure 3.15

(3) $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1;\ a>0, b>0$ and $a>b$

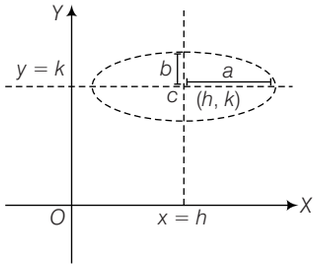


Figure 3.16

(4) $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, where $a<b$

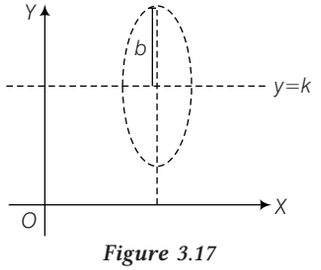


Figure 3.17

(v) Hyperbola

A hyperbola is the locus of a moving point such that the **difference** of its distances from two fixed points is always constant. The two fixed point are called the foci of the hyperbola. Constrast this with the definition of the ellipse where we had the sum of focal distances (instead of difference) as constant. As in the case of the ellipse, we have

Focal distance of $P(x, y)$

$$d_1 = e(PF) = e\left(x-\frac{a}{e}\right) = ex - a$$

$$d_2 = e(PF') = e\left(x+\frac{a}{e}\right) = ex + a$$

Latus-Rectum

The chord(s) of the hyperbola passing through the focus F (or F') and perpendicular to the transverse axis. The length of the latus-rectum can be evaluated by substituting $x = \pm ae$ in the equation for the hyperbola :

$$\frac{x^2}{a^2}-\frac{y^2}{a^2(e^2-1)}=1$$

$$\Rightarrow \quad e^2-\frac{y^2}{a^2(e^2-1)}=1$$

$$\Rightarrow \quad y^2 = a^2(e^2-1)^2 = \frac{a^2\cdot b^4}{a^4}=\frac{b^4}{a^2}$$

$$\Rightarrow \quad y=\pm\frac{b^2}{a}.$$

Thus, the length of the latus-rectum is $\frac{2b^2}{a}$.

We discussed in the unit of Ellipose that an ellipse with centre at (α, β) instead of the origin and the major and minor axis parallel to the coordinate axes will have the equation

$$\frac{(x-\alpha)^2}{a^2}+\frac{(y-\beta)}{b^2}=1 \text{ or } \frac{X^2}{a^2}+\frac{Y^2}{b^2}=1$$

where $X \to x-\alpha$ and $Y \to y-\beta$.

The same holds true for a hyperbola. Any hyperbola with centre at (α, β) and the transverse and conjugate axis parallel to the coordinate axes will have the form

$$\frac{(x-\alpha)^2}{a^2}-\frac{(y-\beta)^2}{b^2}=1$$

or

$$\frac{X^2}{a^2}-\frac{Y^2}{b^2}=1$$

where $X \to x-\alpha$ and $Y \to y-\beta$.

We can, using the definition of a hyperbola, write the equation of any hyperbola with an arbitrary focus and directrix, but we will rarely have the occassion to use it.

(1) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ (Standard equation of hyperbola)

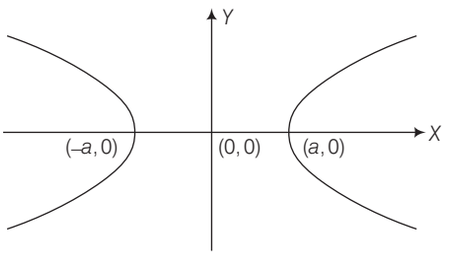


Figure 3.18

(2) $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ (Conjugate hyperbola)

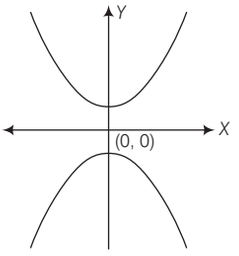


Figure 3.19

(3) $xy = c^2$ (Rectangular hyperbola)

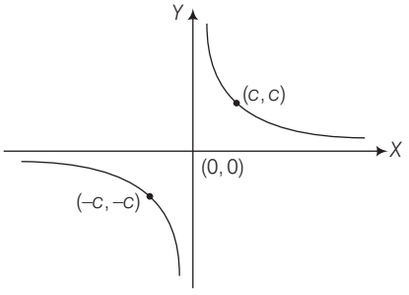


Figure 3.20

(4) $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1; a > b > 0$

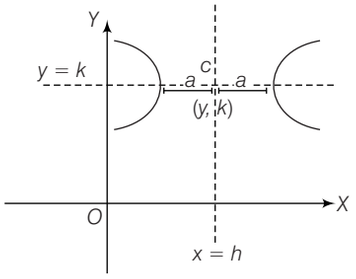


Figure 3.21

(5) $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = -1$

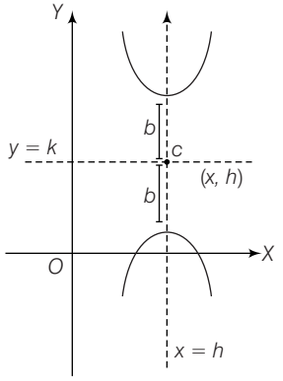


Figure 3.22

Some More Curves which Occur Frequently in Mathematics in Standard Forms

1.

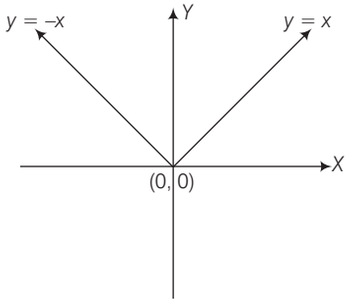


Figure 3.23

Modulus function, $y = |x|$

2.

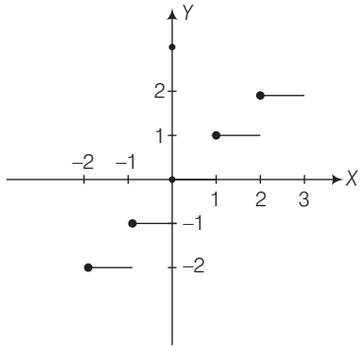


Figure 3.24

Greatest integer function $y = [x]$

3.

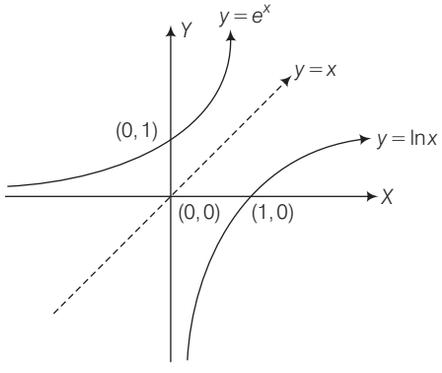


Figure 3.25

4.

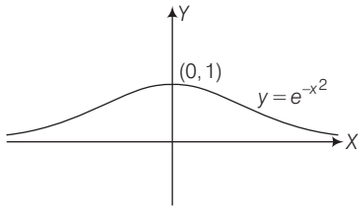


Figure 3.26

5.

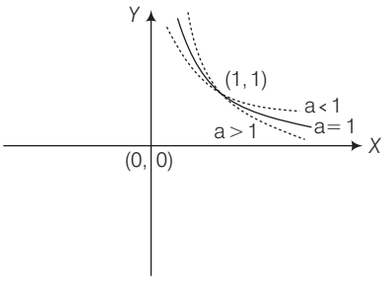


Figure 3.27

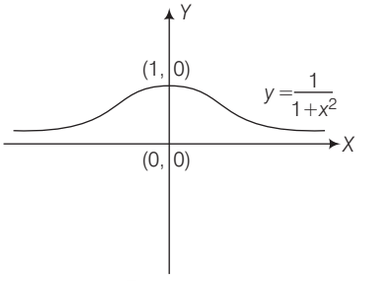


Figure 3.28

$$y = \frac{1}{x^{\alpha}}, \alpha > 0$$

7. The Astroid

Its cartesian equation is $x^{2/3} + y^{2/3} = a^{2/3}$

Its parametric equation is $x = a\cos^3 t, y = a\sin^3 t$

and it could be plotted as

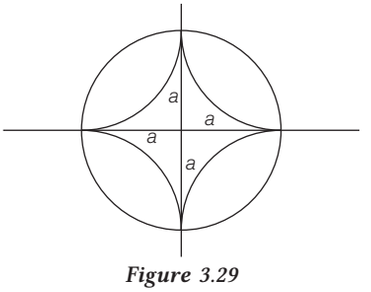


Figure 3.29

Curve Sketching

For the evaluation of area of bounded regions it is very essential to know the rough sketch of the curves. The following points are very useful to draw a rough sketch of a curve.

(i) Symmetry

(a) **Symmetry about *X*-axis** If all powers of y in the equation of the given curve are even, then it is symmetric about X-axis, i.e. the shape of the curve above X-axis is exactly identical to its shape below X-axis. **e.g.** $y^2 = 4ax$ is symmetric about X-axis.

(b) **Symmetry about *Y*-axis** If all powers of x in the equation of the given curve are even, then it is symmetric about Y-axis. **e.g.** $x^2 = 4ay$ is symmetric about Y-axis.

(c) **Symmetry in opposite quadrants** If by putting $-x$ for x and $-y$ for y, the equation of curve remains same, then it is symmetric in opposite quadrants.

e.g. $xy = c^2$, $x^2 + y^2 = a^2$ are symmetric in opposite quadrants.

(d) **Symmetric about the line $y = x$** If the equation of a given curve remains unaltered by interchanging x and y, then it is symmetric about the line $y = x$ which passes through the origin and makes an angle of 45° with positive direction of X-axis.

(ii) Origin and Tangents at the Origin

See whether the curve passes through origin or not. If the point (0, 0) satisfies the equation of the curve, then it passes through the origin and in such a case to find the equations of the tangents at the origin, equate the lowest degree term to zero. **e.g.** $y^2 = 4ax$ passes through the origin. The lowest degree term in this equation is $4ax$. Equating $4ax$ to zero, we get $x = 0$.

So, $x = 0$ i.e. Y-axis is tangent at the origin to $y^2 = 4ax$.

(iii) Points of Intersection of Curve with the Coordinate Axes

By putting $y = 0$ in the equation of the given curve, find points where the curve crosses the X-axis. Similarly, by putting $x = 0$ in the equation of the given curve we can find points where the curve crosses the Y-axis.

e.g. To find the points where the curve $xy^2 = 4a^2(2a - x)$ meets X-axis, we put $y = 0$ in the equation which gives $4a^2(2a - x) = 0$ or $x = 2a$. So the curve $xy^2 = 4a^2(2a - x)$, meets X-axis at $(2a, 0)$. This curve does not intersect Y-axis, because by putting $x = 0$ in the equation of the given curve get an absurd result.

(iv) Regions where the Curve Does Not Exist

Determine the regions in which the curve does not exist. For this, find the value of y in terms of x from the equation of the curve and find the value of x for which y is imaginary. Similarly, find the value of x in terms of y and determine the values of y for which x is imaginary. The curve does not exist for these values of x and y.

e.g. The values of y obtained from $y^2 = 4ax$ are imaginary for negative values of x. So, the curve does not exist on the left side of Y-axis. Similarly, the curve $a^2y^2 = x^2(a - x)$ does not exist for $x > a$ as the values of y are imaginary for $x > a$.

(v) Special Points

Find the points at which $\frac{dy}{dx} = 0$. At these points the tangent to the curve is parallel to X-axis.

Find the points at which $\frac{dx}{dy} = 0$. At these points the tangents to the curve is parallel to Y-axis.

(vi) Sign of dy/dx and Points of Maxima and Minima

Find the interval in which $\frac{dy}{dx} > 0$. In this interval, the function is monotonically increasing, find the interval in which $\frac{dy}{dx} < 0$. In this interval, the function is monotonically decreasing.

Put $\frac{dy}{dx} = 0$ and check the sign of $\frac{d^2y}{dx^2}$ at the points so obtained to find the points of maxima and minima.

Keeping the above facts in mind and plotting some points on the curve one can easily have a rough sketch of the curve. Following examples will clear the procedure.

Example 2 *Sketch the curve* $y = x^3$.

Sol. We observe the following points about the given curve

(i) The equation of the curve remains unchanged, if x is replaced by $-x$ and y by $-y$. So, it is symmetric in opposite quadrants. Consequently, the shape of the curve is similar in the first and the third quadrants.

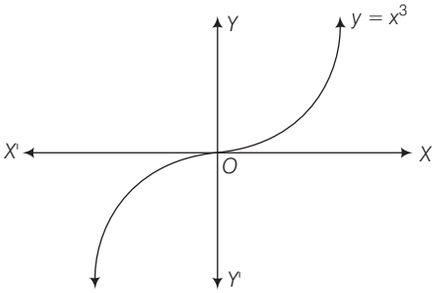
(ii) The curve passes through origin. Equating lowest degree term y to zero, we get $y = 0$ i.e. X-axis is the tangent at the origin.

(iii) Putting $y = 0$ in the equation of the curve, we get $x = 0$. Similarly, when $x = 0$, we get $y = 0$. So, the curve meets the coordinate axes at (0, 0) only.

(iv) $y = x^3 \Rightarrow \frac{dy}{dx} = 3x^2, \frac{d^2y}{dx^2} = 6x$ and $\frac{d^3y}{dx^3} = 6$

Clearly, $\frac{dy}{dx} = 0 = \frac{d^2y}{dx^2}$ at the origin but $\frac{d^3y}{dx^3} \neq 0$.

So, the origin is a point of inflexion.



(v) As x increases from 0 to ∞, y also increases from 0 to ∞. Keeping all the above points in mind, we obtain a sketch of the curve as shown in figure.

Example 3 Sketch the curve $y = x^3 - 4x$.

Sol. We note the following points about the curve

(i) The equation of the curve remains same, if x is replaced by $(-x)$ and y by $(-y)$, so it is symmetric in opposite quadrants.

Consequently, the curve in the first quadrant is identical to the curve in third quadrant and the curve in second quadrant is similar to the curve in fourth quadrant.

(ii) The curve passes through the origin. Equating the lowest degree term $y + 4x$ to zero, we get $y + 4x = 0$ or $y = -4x$. So, $y = -4x$ is tangent to the curve at the origin.

(iii) Putting $y = 0$ in the equation of the curve, we obtain $x^3 - 4x = 0 \Rightarrow x = 0, \pm 2$. So, the curve meets X-axis at (0, 0), (2, 0), (−2, 0).

Putting $x = 0$ in the equation of the curve, we get $y = 0$. So, the curve meets Y-axis at (0, 0) only.

(iv) $y = x^3 - 4x \Rightarrow \frac{dy}{dx} = 3x^2 - 4$

Now, $\frac{dy}{dx} > 0 \Rightarrow 3x^2 - 4 > 0$

$\Rightarrow \left(x - \frac{2}{\sqrt{3}}\right)\left(x + \frac{2}{\sqrt{3}}\right) > 0$

$\Rightarrow x < -\frac{2}{\sqrt{3}}$ or $x > \frac{2}{\sqrt{3}}$ (using number line rule)

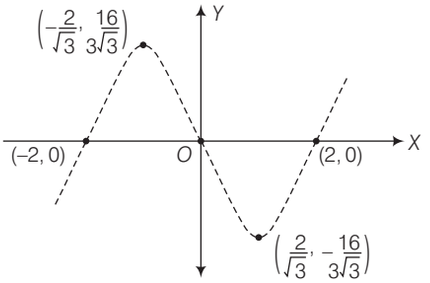
and $\frac{dy}{dx} < 0 \Rightarrow -\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}}$

So, the curve is decreasing in the interval $(-2/\sqrt{3}, 2/\sqrt{3})$ and increasing for $x > \frac{2}{\sqrt{3}}$ or $x < -\frac{2}{\sqrt{3}}$.

$x = -\frac{2}{\sqrt{3}}$ is a point of local maximum and $x = \frac{2}{\sqrt{3}}$ is point of local minimum.

When $x = \frac{2}{\sqrt{3}}$, then $y = -\frac{16}{3\sqrt{3}}$

When $x = -\frac{2}{\sqrt{3}}$, then $y = \frac{16}{3\sqrt{3}}$



Keeping above points in mind, we sketch the curve as shown in figure.

Example 4 Sketch the curve $y = (x-1)(x-2)(x-3)$.

Sol. We note the following points about the given curve

(i) The curve does not have any type of symmetry about the coordinate axes and also in opposite quadrants.

(ii) The curve does not pass through the origin.

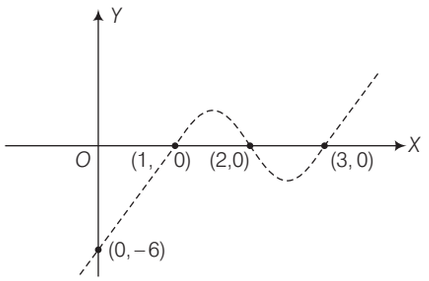
(iii) Putting $y = 0$ in the equation of the curve, we get $(x-1)(x-2)(x-3) = 0 \Rightarrow x = 1, 2, 3$. So, the curve meets X-axis at (1, 0), (2, 0) and (3, 0).

Putting $x = 0$ in the equation of the curve, we get $y = -6$. So, the curve crosses Y-axis at $(0, -6)$.

We observe that

$$x < 1 \quad \Rightarrow \quad y < 0$$
$$1 < x < 2 \quad \Rightarrow \quad y > 0$$
$$2 < x < 3 \quad \Rightarrow \quad y < 0$$

and $$x > 3 \quad \Rightarrow \quad y > 0$$



Clearly, y decreases as x decreases for all $x < 1$ and y increases as x increases for $x > 3$.

Keeping all the above points in mind, we sketch the curve as shown in figure.

Example 5 Sketch the graph for $y = x^2 - x$.

Sol. We note the following points about the curve

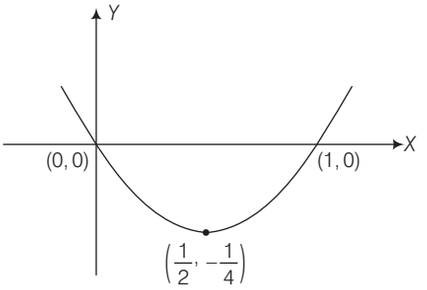
(i) The curve does not have any kind of symmetry.

(ii) The curve passes through the origin and the tangent at the origin is obtained by equating the lowest degree term to zero.

The lowest degree term is $x + y$. Equating it to zero, we get $x + y = 0$ as the equation of tangent at the origin.

(iii) Putting $y = 0$ in the equation of curve, we get $x^2 - x = 0 \Rightarrow x = 0, 1$. So, the curve crosses X-axis at (0, 0) and (1, 0).

Putting $x = 0$ in the equation of the curve, we obtain $y = 0$. So, the curve meets Y-axis at (0, 0) only.



(iv) $y = x^2 - x \Rightarrow \frac{dy}{dx} = 2x - 1$ and $\frac{d^2y}{dx^2} = 2$

Now, $$\frac{dy}{dx} = 0 \Rightarrow x = \frac{1}{2}$$

At $$x = \frac{1}{2}, \ \frac{d^2y}{dx^2} > 0$$

So, $x = \frac{1}{2}$ is point of local minima.

(v) $\frac{dy}{dx} > 0 \Rightarrow 2x - 1 > 0 \Rightarrow x > \frac{1}{2}$

So, the curve increases for all $x > \frac{1}{2}$ and decreases for all $x < \frac{1}{2}$. Keeping above points in mind, we sketch the curve as shown in figure.

Example 6 Sketch the curve $y = \sin 2x$.

Sol. We note the following points about the curve

(i) The equation of the curve remains unchanged, if x is replaced by $(-x)$ and y by $(-y)$, so it is symmetric in opposite quadrants. Consequently, the shape of the curve is similar in opposite quadrants.

(ii) The curve passes through origin.

(iii) Putting $x = 0$ in the equation of the curve, we get $y = 0$. So, the curve crosses the Y-axis at (0, 0) only.

Putting $y = 0$ in the equation of the curve, we get

$$\sin 2x = 0 \Rightarrow 2x = n\pi, \ n \in Z$$

$$\Rightarrow \quad x = \frac{n\pi}{2}, n \in Z$$

So, the curve cuts the X-axis at the points

$$\ldots, (-\pi, 0), (-\pi/2, 0), (0, 0), (\pi/2, 0), (\pi, 0), \ldots$$

(iv) $y = \sin 2x \Rightarrow \dfrac{dy}{dx} = 2\cos 2x$ and $\dfrac{d^2y}{dx^2} = -4\sin 2x$

Now, $\dfrac{dy}{dx} = 0 \Rightarrow \cos 2x = 0$

$\Rightarrow 2x = \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}, \ldots$

$\Rightarrow x = \pm\dfrac{\pi}{4}, \pm\dfrac{3\pi}{4}, \ldots$

Clearly, $\dfrac{d^2y}{dx^2} < 0$ at $x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}, \ldots$

and at $x = -\dfrac{3\pi}{4}, -\dfrac{7\pi}{4}, \ldots$

and $\dfrac{d^2y}{dx^2} > 0$ at $x = \dfrac{3\pi}{4}, \dfrac{7\pi}{4}, \ldots$

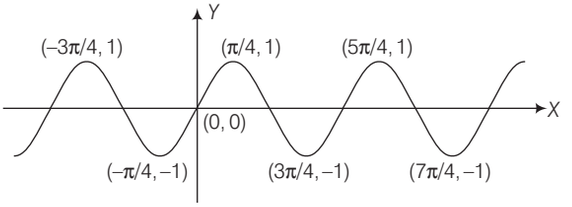
and at $x = -\dfrac{\pi}{4}, -\dfrac{5\pi}{4}, \ldots$

So, the points $x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}, \ldots$

and $x = -\dfrac{3\pi}{4}, -\dfrac{7\pi}{4}, \ldots$ are points of local maximum and local maximum values at these points are 1.

Similarly, $x = -\dfrac{\pi}{4}, -\dfrac{5\pi}{4}, \ldots$ and $x = \dfrac{3\pi}{4}, \dfrac{7\pi}{4}, \ldots$ are points of local minimum and local minimum value at these points is (-1).

(v) $\sin 2(x + \pi) = \sin 2x$ for all x. So, the periodicity of the function is π. This means that the pattern of the curve repeats at intervals of length π.



Thus, keeping in mind, we sketch the curve as shown in figure.

Example 7 Sketch the curve $y = \sin^2 x$.

Sol. We note the following points about the curve

(i) The equation of the curve remains same, if x is replaced by $(-x)$. So, the curve is symmetric about Y-axis, i.e. the curve on the left side of Y-axis is identical to the curve on its right side.

(ii) The curve meets the coordinate axes at the same points where $y = \sin x$ meets them.

(iii) $y = \sin^2 x \Rightarrow \dfrac{dy}{dx} = \sin 2x$ and $\dfrac{d^2y}{dx^2} = 2\cos 2x$

Now, $\dfrac{dy}{dx} = 0 \Rightarrow \sin 2x = 0 \Rightarrow 2x = n\pi, n \in Z$

$\Rightarrow x = \dfrac{n\pi}{2}, \ n \in Z \Rightarrow x = \pm\pi/2, \pm\pi, \pm 3\pi/2, \pm 2\pi, \ldots$

Clearly, $\dfrac{d^2y}{dx^2} < 0$ at $x = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$

and $\dfrac{d^2y}{dx^2} > 0$ at $x = \pm\pi, \pm 2\pi, \pm 3\pi \ldots$

So, $x = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$ are the points of local maximum and local maximum value at these points is 1. Points $x = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$ are points of local minimum and the local minimum value at these points is 0.

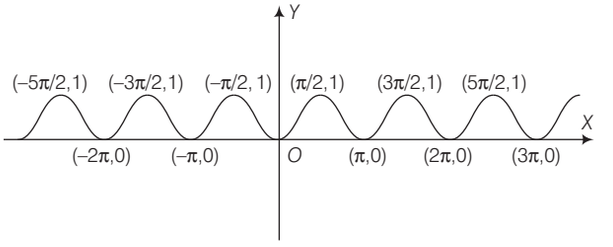
(iv) $y = \sin^2 x \Rightarrow \dfrac{dy}{dx} = \sin 2x$

Clearly, $\dfrac{dy}{dx} > 0$ when $0 < x < \dfrac{\pi}{2}$

and $\dfrac{dy}{dx} < 0$ when $\dfrac{\pi}{2} < x < \pi$.

So, the given curve is increasing in the interval $[0, \pi/2]$ and decreasing in $[\pi/2, \pi]$.

(v) $\sin^2(\pi + x) = \sin^2 x$ for all x. So, the periodicity of the function is π. This means that the shape of the curve repeats at the interval of length π.



Keeping the above facts in mind, we sketch the curve as shown in figure.

Asymptotes

The straight line AB is called the asymptote of curve $y = f(x)$, if the distance MK from M a point on the curve $y = f(x)$ to the straight line AB tends to zero as M recedes infinity.

In other words, the straight line AB meets the curve $y = f(x)$ at infinity (K is a point on AB). Thus,

1. If $f(x) \to \pm\infty$ for $x \to a$, then the straight line $x = a$ is the asymptote of the curve $y = f(x)$.

2. If in the right hand member of the equation of the curve $y = f(x)$ it is possible to single out a linear part so that the remaining part tends to zero as $x \to \pm\infty$, i.e. if $y = f(x) = Kx + b + g(x)$ and $g(x) \to 0$ for $x \to \pm\infty$, then the straight line $y = Kx + b$ is the asymptote of the curve.
3. If there exist finite limits $\lim_{x \to \pm\infty} \frac{f(x)}{x} = K$ and $\lim_{x \to \pm\infty} [f(x) - Kx] = b$, then the straight line $y = Kx + b$ is the asymptote of the curve.

Methods to Sketch Curves

While constructing the graphs of functions, it is expedient to follow the procedure given below

(1) Find the domain of definition of the function.
(2) Determine the odd-even nature of the function.
(3) Find the period of the function if its periodic.
(4) Find the asymptotes of the function.
(5) Check the behaviour of the function for $x \to 0 \pm$
(6) Find the values of x, if possible for which $f(x) \to 0$.
(7) The interval of increase and decrease of the function in its range. Hence, determine the greatest and the least values of the function if any.

Remark

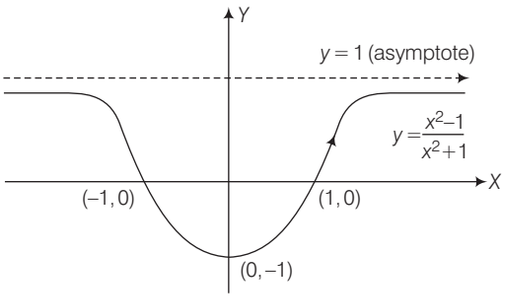
(5), (6) and (7) gives the points where the function cuts the coordinate axes.

Example 8 *Construct the graph for* $f(x) = \frac{x^2 - 1}{x^2 + 1}$.

Sol. Here, $f(x) = \frac{x^2 - 1}{x^2 + 1} = 1 - \frac{2}{x^2 + 1}$

(1) The function $f(x)$ is well defined for all real x.
$\Rightarrow$ Domain of $f(x) \in R$.
(2) $f(-x) = f(x)$, so it is an even function.
(3) Since, algebraic $\to$ non-periodic function.
$f(x) \to 1$ for $x \to \pm\infty$
and $f(x) \to -1$ for $x \to 0 \pm$
It may be observed that $f(x) < 1$ for any $x \in R$ and consequently its graph lies below the line $y = 1$ which is asymptote to the graph of the given function.
Again, $\frac{2}{x^2 + 1}$ decreases for $(0, \infty)$ and increases for $(-\infty, 0)$, thus $f(x)$ increases for $(0, \infty)$ and decreases for $(-\infty, 0)$ in its range.
(4) The greatest value $\to 1$ for $x \to \pm\infty$ and the least value is -1 for $x = 0$.

Thus, its graph is as shown in figure



Example 9 Construct the graph for $f(x) = x + \frac{1}{x}$.

Sol. The function is defined for all x except for $x = 0$.

It is an odd function for $x \neq 0$.

It is not a periodic function.

For $x \to 0+$, $f(x) \to +\infty$; for $x \to 0-$, $f(x) \to -\infty$

For $x \to -\infty$, $f(x) \to -\infty$; for $x \to \infty$, $f(x) \to \infty$

$\therefore \lim_{x \to \pm\infty} (f(x) - x) = 0$

$\therefore$ The straight lines $x = 0$ and $y = x$ are the asymptotes of the graph of the given function.

Now, consider $f(x_2) - f(x_1)$ (for $x_2 > x_1$)

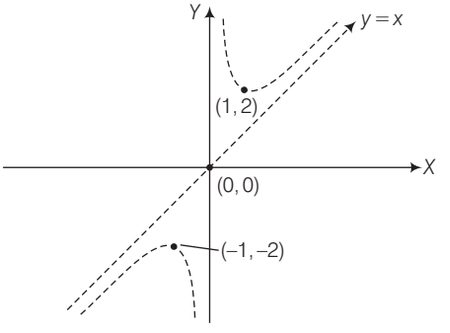
$$= (x_2 - x_1) + \frac{1}{x_2} - \frac{1}{x_1}$$

$$= (x_2 - x_1)\left[1 - \frac{1}{x_1 x_2}\right] < 0 \text{ for } x_1 x_2 \in (0, 1]$$

and it is > 0 for $x_1 x_2 \in [1, \infty)$.

Thus, $f(x)$ increases for $x \in [1, \infty)$ and decreases for $x \in (0, 1]$.

Thus, the least value of the function is at $x = 1$ which is $f(1) = 2$. Thus, its graph can be drawn as



Example 10 Construct the graph for $f(x) = \frac{1}{1 + e^{1/x}}$.

Sol. The function is defined for all x except for $x = 0$. It is neither even nor an odd function. It is not a periodic function.

For $x \to 0+$ $f(x) \to 0$; for $x \to 0-$, $f(x) \to 1$

For $x \to \infty \quad f(x) \to \frac{1}{2}$; for $x \to -\infty, \quad f(x) \to \frac{1}{2}$

$\therefore \quad \lim_{x \to \pm\infty} f(x) = \frac{1}{2}$

$\therefore$ The straight line $y = \frac{1}{2}$ is asymptote of the graph of the given function.

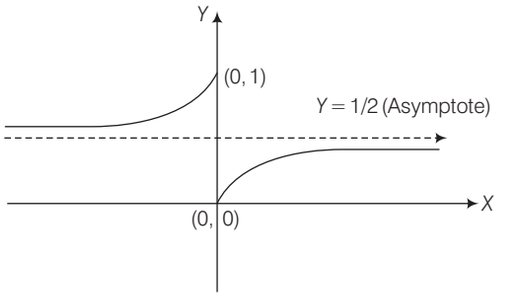
As x increases from $(0, \infty)$, $\frac{1}{x}$ decreases from $(0, \infty)$ and $e^{1/x}$ decreases from $(0, \infty)$. Thus, $(1 + e^{1/x})$ decreases from $(2, \infty)$.

$\therefore$ $f(x)$ increases from $\left(0, \frac{1}{2}\right)$ for $x \in (0, \infty)$.

Similarly, $f(x)$ increases from $(1/2, 1)$ for $x \to (-\infty, 0)$.

i.e. $f(x)$ is an increasing function except for $x = 0$.

Thus, its graph can be drawn as shown in figure



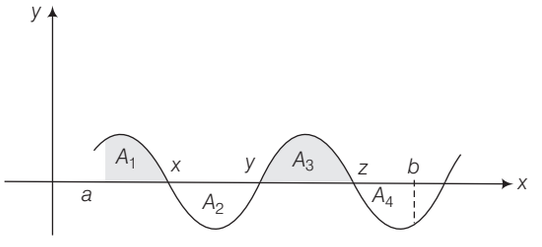
Areas of Curves

(1) Suppose that $f(x) < 0$ on some interval $[a, b]$. Then, the area under the curve $y = f(x)$ from $x = a$ to $x = b$ will be negative in sign, i.e.

$$\int_a^b f(x)\, dx \leq 0$$

This is obvious once you consider how the definite integral was arrived at in the first place; as a limit of the sum of the n rectangles $(n \to \infty)$. Thus, if $f(x) < 0$ in some interval then the area of the rectangles in that interval will also be negative.

This property means that for example, if $f(x)$ has the following form



then $\int_a^b f(x)\, dx$ will equal $A_1 - A_2 + A_3 - A_4$ and not $A_1 + A_2 + A_3 + A_4$.

If we need to evaluate $A_1 + A_2 + A_3 + A_4$ (the magnitude of the bounded area), we will have to calculate

$$\int_a^x f(x)\, dx + \left| \int_x^y f(x)\, dx \right| \int_y^z f(x)\, dx + \left| \int_z^b f(x)\, dx \right|$$

From this, it should also be obvious that

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx$$

(2) The area under the curve $y = f(x)$ from $x = a$ to $x = b$ is equal in magnitude but opposite in sign to the area under the same curve from $x = b$ to $x = a$, i.e.

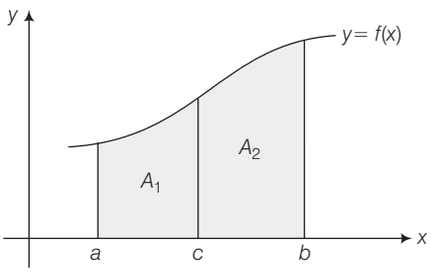
$$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$$

This property is obvious if you consider the Newton-Leibnitz formula. If $g(x)$ is the anti-derivative of $x(f)$, then $\int_a^b f(x)\, dx$ is $g(b) - g(a)$ while $\int_a^b f(x)\, dx$ is $g(a) - g(b)$

(3) The area under the curve $y = f(x)$ from $x = a$ to $x = b$ can be written as the sum of the area under the curve from $x = a$ to $x = c$ and from $x = c$ to $x = b$, that is

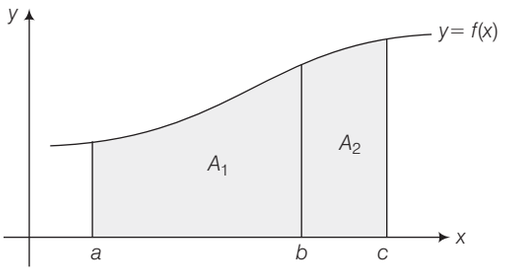
$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$

Let us consider an example of this. Let $c \in (a, b)$



It is clear that the area under the curve from $x = a$ to $x = b$, A is $A_1 + A_2$.

Note that c need not lie between a and b for this relation to hold true. Suppose that $c > b$.



Observe that $A = \int_a^b f(x)\,dx = (A + A_1) - A_1$

$$= \int_a^c f(x)\,dx - \int_b^c f(x)\,dx$$

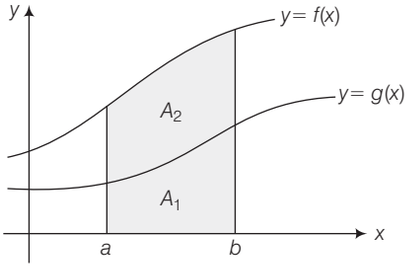
$$= \int_a^c f(x)\,dx + \int_a^b f(x)\,dx$$

Analytically, this relation can be proved easily using the Newton Leibnitz's formula.

(4) Let $f(x) > g(x)$ on the interval $[a, b]$. Then,

$$\int_a^b f(x)\,dx > \int_a^b g(x)\,dx$$

This is because the curve of $f(x)$ lies above the curve of $g(x)$, or equivalently, the curve of $f(x) - g(x)$ lies above the x-axis for $[a, b]$



This is an example where $f(x) > g(x) > 0$.

$$\int_a^b f(x)\,dx = A_1 + A_2$$

while $$\int_a^b g(x)\,dx = A_1$$

Similarly, if $f(x) < g(x)$ on the interval $[a, b]$, then

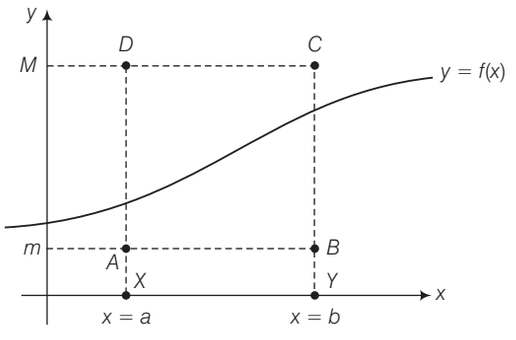
$$\int_a^b f(x)\,dx < \int_a^b g(x)\,dx$$

(5) For the interval $[a, b]$, suppose $m < f(x) < M$.

That is, m is a lower-bound for $f(x)$ while M is an upper bound.

Then $$m(b - a) < \int_a^b f(x)\,dx < M(b - a)$$

This is obvious once we consider the figure below :



Observe that area (rect $AXYB$) $< \int_a^b f(x)\,dx <$ area (rect $DXYC$)

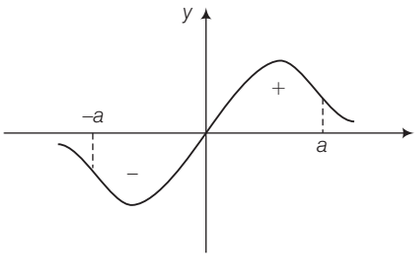
(6) Let us consider the integral of $f_1(x) + f_2(x)$ from $x = a$ to $x = b$. To evaluate the area under $f_1(x) + f_2(x)$, we can separately evaluate the area under $f_1(x)$ and the area under $f_2(x)$ and add the two area (algebraically).

Thus, $\int_a^b (f_1(x) + f_2(x))\,dx = \int_a^b f_1(x)\,dx + \int_a^b f_2(x)\,dx$

Now consider the integral of $kf(x)$ from $x = a$ to $x = b$. To evaluate the area under $kf(x)$, we can first evaluate the area under $f(x)$ and then multiply it by k, that is :

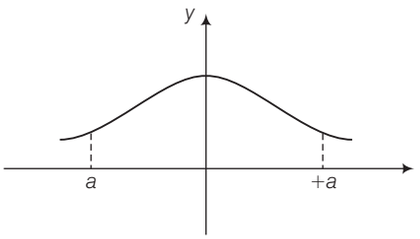
$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$

(7) Consider the odd function $f(x)$, i.e. $f(x) = -f(-x)$. This meas that the graph of $f(x)$ is symmetric about the origin.

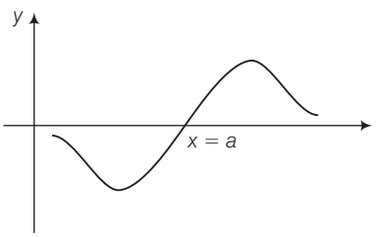


From the figure, it should be obvious that $\int_{-a}^a f(x)\,dx = 0$, because the area on the left side and that on the right algebraically add to 0.

Similarly, if $f(x)$ was even, i.e. $f(x) = f(-x)$



$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$ because the graph is symmetrical about the y-axis. If you recall the discussion in the unit on functions, a function can also be even or odd about any arbitrary point $x = a$. Let us suppose that $f(x)$ is odd about $x = a$, i.e. $f(x) = -f(2a - x)$

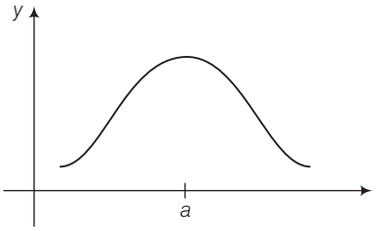


The points x and $2a - x$ lie equidistant from $x = a$ at either sides of it.

Suppose for example, that we need to calculate $\int_a^{2a} f(x)\,dx$.

It is obvious that this will be 0, since we are considering equal variation on either side of $x = a$, the area from $x = 0$ to $x = a$ and the area from $x = a$ to $x = 2a$ will add algebraically to 0.

Similarly, if $f(x)$ is even about $x = a$, i.e. $f(x) = f(2a - x)$

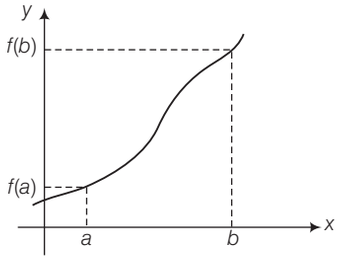


then we have, for example

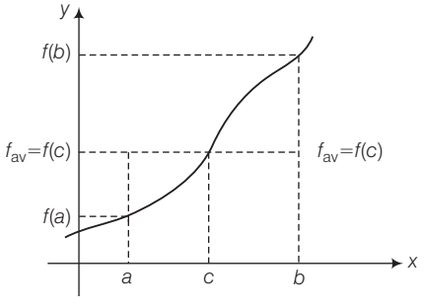
$$\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

From the discussion, you will get a general idea as to how to approach such issues regarding even/odd functions.

(8) Let us consider a function $f(x)$ on $[a, b]$



We want to somehow define the "average" value that $f(x)$ takes on the interval $[a, b]$. What would be an appropriate way to define such an average? Let f_{av} be the average value that we are seeking. Let it be such that it is obtained at some $x = c \in [a, b]$



We can measure f_{av} by saying that the area under $f(x)$ from $x = a$ to $x = b$ should equal the area under the average value from $x = a$ to $x = b$. This seems to be the only logical way to define the average (and this is how it is actually defined!).

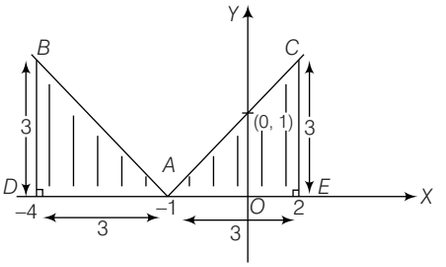
Thus $f_{av}(b - a) = \int_a^b f(x)\,dx \Rightarrow f_{av} = \frac{1}{b-a}\int_a^b f(x)\,dx$

This value is attained for at least one $c \in (a, b)$ (under the constraint that f is continuous, of course).

Example 11 ***Sketch the graph*** $y = |x + 1|$. ***Evaluate*** $\int_{-4}^{2} |x+1|\,dx$. ***What does the value of this integral represents on the graph.***

Sol. Here, $y = |x + 1| = \begin{cases} (x+1), & \text{if } x \geq -1 \\ -(x+1), & \text{if } x \leq -1 \end{cases}$

which can be shown as



$$\therefore \int_{-4}^{2} |x+1|dx = \int_{-4}^{-1} |x+1|dx + \int_{-1}^{2} |x+1|dx$$

$$= \int_{-4}^{-1} -(x+1)dx + \int_{-1}^{2} |x+1|dx + \int_{-1}^{2} (x+1)dx$$

$$= -\left[\frac{x^2}{2} + x\right]_{-4}^{-1} + \left[\frac{x^2}{2} + x\right]_{-1}^{2} = 9$$

Representation of the value 9 of integral on graph.

$\therefore \int_{-4}^{2} |x+1|dx = 9$ represents the area bounded by the curve $y = |x + 1|$. X-axis and the lines $x = -4$ and $x = 2$, i.e. if is equal to the sum of the ares of ΔABD and ACE,

i.e. $$\frac{1}{2}(3)(3) + \frac{1}{3}(3)(3) = \frac{9}{2} + \frac{9}{2} = 9$$

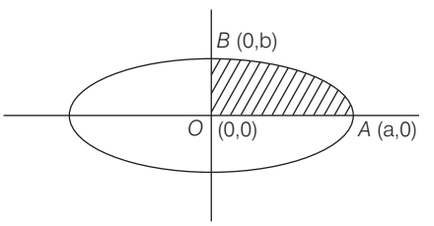
$$(\because \text{ area of triangle} = \frac{1}{2} \times \text{base} \times \text{height})$$

Example 12 Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Sol. Using the symmetry of the figure; required area is given by

$$A = 4\,(\text{area } OABO)$$

$$= 4\int_0^a y\,dx, \text{ where } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



$$\therefore \quad \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\Rightarrow \quad y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

$$\Rightarrow \quad y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$$

In the first quadrant,

$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$

$$A = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx = 4\frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\, dx$$

$$= 4\frac{b}{a}\left\{\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right\}_0^a$$

$$= 4\frac{b}{a}\left[\left\{0 + \frac{a^2}{2}\sin^{-1}\frac{a}{a}\right\} - \{0+0\}\right] = \frac{4b}{a}\cdot\frac{a^2}{2}\sin^{-1}(1)$$

$$= 2ab\left(\frac{\pi}{2}\right)$$

$$A = \pi ab \text{ sq units}$$

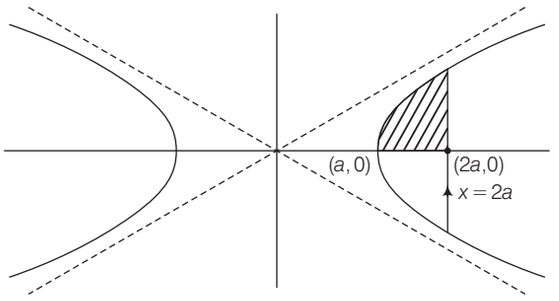
Example 13 Find the area bounded by the hyperbola $x^2 - y^2 = a^2$ between the straight lines $x = a$ and $x = 2a$.

Sol. We use the symmetry of figure.

Required area, $A = 2\int_a^{2a} y\, dx$, where $x^2 - y^2 = a^2$

i.e. $x^2 - a^2 = y^2$

$$\therefore \quad y = \pm\sqrt{x^2 - a^2}$$



In the first quadrant; $y = +\sqrt{x^2 - a^2}$

$$\therefore\ A = 2\int_a^{2a}\sqrt{x^2 - a^2}\, dx$$

$$= 2\left[\frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left(x + \sqrt{x^2 - a^2}\right)\right]_a^{2a}$$

$$= 2\left[\left\{a\sqrt{4a^2 - a^2} - \frac{a^2}{2}\log\left(2a + \sqrt{4a^2 - a^2}\right)\right\} - \left\{0 - \frac{a^2}{2}\log a\right\}\right]$$

$$= 2\left[a\sqrt{3a^2} - \frac{a^2}{2}\log(2a + a\sqrt{3}) + \frac{a^2}{2}\log a\right]$$

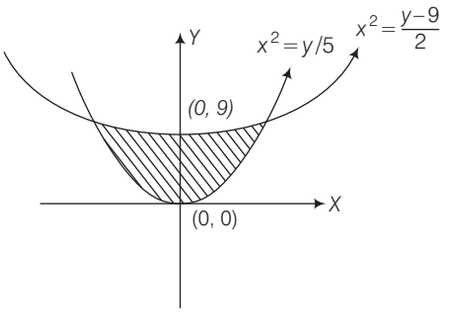
$$= 2\left[a^2\sqrt{3} - \frac{a^2}{2}\log\left(\frac{2a + a\sqrt{3}}{a}\right)\right]$$

$$= 2a^2\sqrt{3} - a^2\log(2 + \sqrt{3}) \text{ sq units}$$

Example 14 Find the area common to the parabola $5x^2 - y = 0$ and $2x^2 - y + 9 = 0$.

Sol. Given curves are $y = 5x^2$ ...(i)

and $y = 2x^2 + 9$ ...(ii)



Remark

In such examples, figure is the most essential thing. Without figure it just becomes difficult to judge whether y_1 to be subtracted from y_2 or otherwise.

Let us solve Eqs. (i) and (ii) simultaneously,

$$\therefore \quad 5x^2 = 2x^2 + 9$$

$$\Rightarrow \quad 3x^2 = 9 \Rightarrow x^2 = 3$$

$$\therefore \quad x = -\sqrt{3}$$

or $x = \sqrt{3}$

In the usual notations, the required area is given by

$$A = \int_{-\sqrt{3}}^{\sqrt{3}} (y_1 - y_2)\, dx$$

We have to find which curve is above and which is below w.r.t. X-axis in order to decide y_1 and y_2.

Take any point between $x = -\sqrt{3}$ and $x = \sqrt{3}$

Let us take $x = 0$, which lies between

$$x = -\sqrt{3} \text{ and } x = \sqrt{3}$$

When $x = 0$ from Eq. (i) $y = 0$

When $x = 0$ from Eq. (ii) $y = 9$

Now, $9 > 0$

$\therefore$ Parabola Eq. (ii) is above parabola Eq. (i) between

$$x = -\sqrt{3}$$

and $x = \sqrt{3}$.

$\therefore$ y of the curve (ii) is to be taken as y_1 and y of the curve (i) is to be taken as y_2.

$$\therefore \quad \text{Area}\,(A) = \int_{-\sqrt{3}}^{\sqrt{3}} \{(2x^2 + 9) - 5x^2\}\, dx$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} (9 - 3x^2)\, dx$$

$$= 2\int_0^{\sqrt{3}} (9 - 3x^2)\, dx$$

$$= 2\,[9x - x^3]_0^{\sqrt{3}}$$

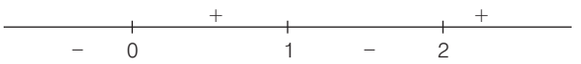
$$= 2\,[9\sqrt{3} - 3\sqrt{3}]$$

$$\text{Area} = 12\sqrt{3} \text{ sq units}$$

Example 15 Find the area enclosed by $y = x(x-1)(x-2)$ and X-axis.

Sol. The given curve is $y = x(x-1)(x-2)$. It passes through (0, 0), (1, 0) and (2, 0).

The sign scheme for $y = x(x-1)(x-2)$ is as shown in figure.



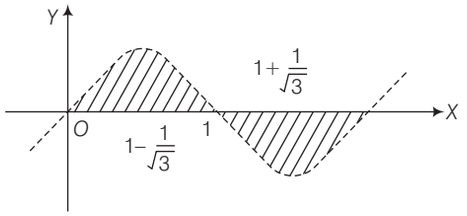
From the sign scheme it is clear that the curve is + ve when $0 < x < 1$ or $x > 2$, hence in these regions the curve lies above X-axis while in the rest regions the curve lies below X-axis.

Remark

Sometimes the discussion of monotonicity of function helps us in sketching polynomials. In the present case, $\frac{dy}{dx} = 3x^2 - 6x + 2$

When $\frac{dy}{dx} = 0$, then ; $x = 1 \pm \frac{1}{\sqrt{3}}$. Sign scheme for $\frac{dy}{dx}$ is

Thus, it is clear that the curve increases in $\left(-\infty, 1-\frac{1}{\sqrt{3}}\right)$, decreases in $\left(1-\frac{1}{\sqrt{3}}, 1+\frac{1}{\sqrt{3}}\right)$ and again increases in $\left(1+\frac{1}{\sqrt{3}}, \infty\right)$. Therefore, the graph of the curve is as below



Hence, required area

$$= \int_0^1 x(x-1)(x-2)\, dx + \left| \int_1^2 x(x-1)(x-2)\, dx \right|$$

$$= \int_0^1 (x^3 - 3x^2 + 2x)\, dx + \left| \int_1^2 (x^3 - 3x^2 + 2x)\, dx \right|$$

$$= \left(\frac{x^4}{4} - x^3 + x^2 \right)_0^1 + \left| \left(\frac{x^4}{4} - x^3 + x^2 \right)_1^2 \right|$$

$$= \left(\frac{1}{4} - 1 + 1 \right) + \left| (4 - 8 + 4) - \left(\frac{1}{4} - 1 + 1 \right) \right|$$

$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \text{ sq unit}$$

Example 16 ***Find the area between the curves $y = 2x^4 - x^2$, the X-axis and the ordinates of two minima of the curve.***

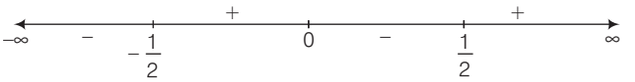
Sol. The given curve is $y = 2x^4 - x^2$.

When $y = 0$, then $x = 0, 0, \pm\frac{1}{\sqrt{2}}$

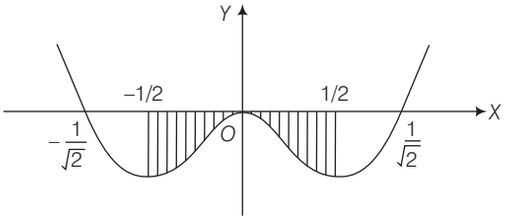
The sign scheme is as shown below

Therefore, it is clear that the curve cuts the X-axis at $x = -\frac{1}{\sqrt{2}}, 0$ and $\frac{1}{\sqrt{2}}$.

The curve is –ve in $\left(-\frac{1}{\sqrt{2}}, 0\right)$ and $\left(0, \frac{1}{\sqrt{2}}\right)$ while positive in the rest. Now, $\frac{dy}{dx} = 8x^3 - 2x$. The sign scheme for $\frac{dy}{dx}$ is as below



i.e. The curve decreases in $(-\infty, -1/2)$ and $(0, 1/2)$ and increases in the rest of portions. Also, the function possess minimum at $x = -1/2$ and $1/2$ while maximum at $x = 0$. Therefore, the graph of the curve is as shown below



$$\therefore \text{ Required area} = 2\int_0^{1/2} |\, 2x^4 - x^2 \,|\, dx$$

$$= 2\int_0^{1/2} -(2x^4 - x^2)dx = -2\left[2\frac{x^5}{5} - \frac{x^3}{3} \right]_0^{1/2}$$

$$= \frac{7}{120} \text{ sq unit}$$

Exercise for Session 1

1. Draw a rough sketch of $y = \sin 2x$ and determine the area enclosed by the curve, X-axis and the lines $x = \pi/4$ and $x = 3\pi/4$.
2. Find the area under the curve $y = (x^2 + 2)^2 + 2x$ between the ordinates $x = 0$ and $x = 2$.
3. Find the area of the region bounded by the curve $y = 2x - x^2$ and the X-axis.
4. Find the area bounded by the curve $y^2 = 2y - x$ and the Y-axis.
5. Find the area bounded by the curve $y = 4 - x^2$ and the line $y = 0$ and $y = 3$.
6. Find the area bounded by $x = at^2$ and $y = 2at$ between the ordinates corresponding to $t = 1$ and $t = 2$.
7. Find the area of the parabola $y^2 = 4ax$ and the latusrectum.
8. Find the area bounded by $y = 1 + 2\sin^2 x$, X-axis, $x = 0$ and $x = \pi$.
9. Sketch the graph of $y = \sqrt{x} + 1$ in [0, 4] and determine the area of the region enclosed by the curve, the axis of X and the lines $x = 0, x = 4$.
10. Find the area of the region bounded by the curve $xy - 3x - 2y - 10 = 0$, X-axis and the lines $x = 3, x = 4$.

Session 2

Area Bounded by Two or More Curves

Area Bounded by Two or More Curves

Area bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$.

Let the curves $y = f(x)$ and $y = g(x)$ be represented by AB and CD, respectively. We assume that the two curves do not intersect each other in the interval $[a, b]$.

Thus, shaded area = Area of curvilinear trapezoid $APQB$ − Area of curvilinear trapezoid $CPQD$

$$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b \{f(x) - g(x)\}\,dx$$

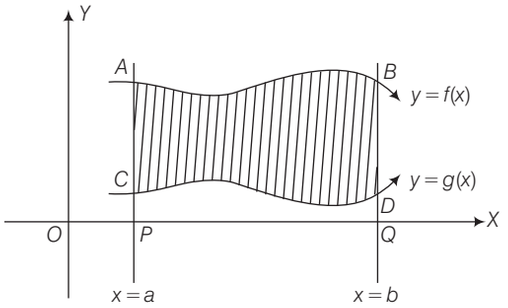


Figure 3.32

Now, consider the case when $f(x)$ and $g(x)$ intersect each other in the interval $[a, b]$.

First of all we should find the intersection point of $y = f(x)$ and $y = g(x)$. For that we solve $f(x) = g(x)$. Let the root is $x = c$. (We consider only one intersection point to illustrate the phenomenon).

Thus, required (shaded) area

$$= \int_a^c \{f(x) - g(x)\}dx + \int_c^b \{g(x) - f(x)\}\,dx$$

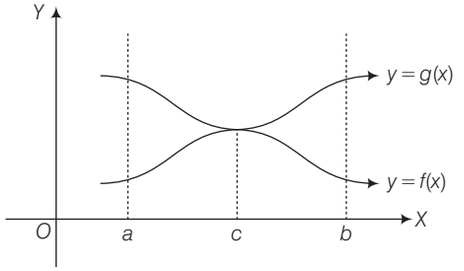


Figure 3.33

If confusion arises in such case evaluate $\int_a^b |f(x) - g(x)|\,dx$ which gives the required area.

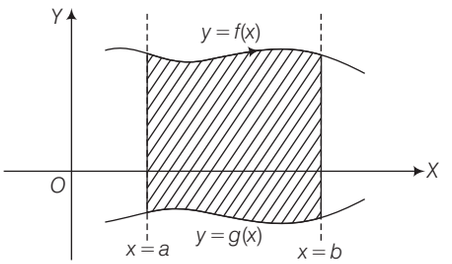


Figure 3.34

Area between two curves $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$ is always given by $\int_a^b \{f(x) - g(x)\}\,dx$ provided $f(x) > g(x)$ in $[a, b]$; the position of the graph is immaterial. As shown in Fig. 3.34, Fig. 3.35, Fig. 3.36.

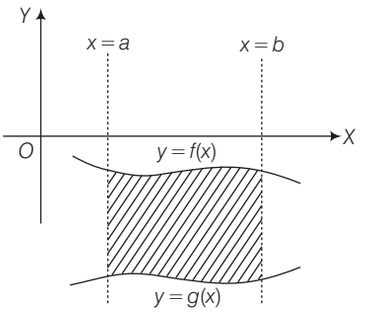


Figure 3.35

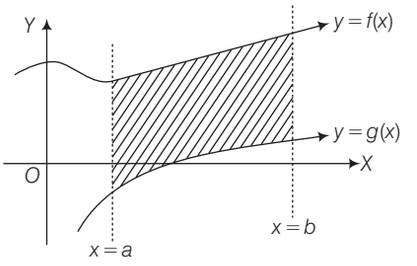
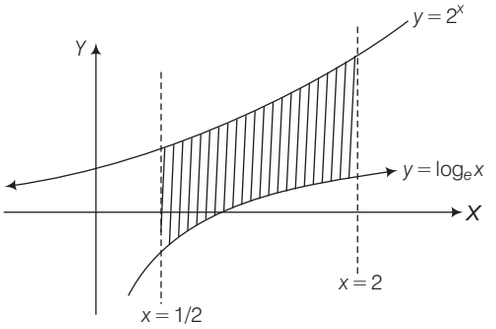


Figure 3.36

Example 17 Sketch the curves and identify the region bounded by $x = 1/2$, $x = 2$, $y = \log_e x$ and $y = 2^x$. Find the area of this region. ***[IIT JEE 1991]***

Sol. The required area is the shaded portion in the following figure.



In the region $\frac{1}{2} \le x \le 2$; the curve $y = 2^x$ lies above as compared to $y = \log_e x$.

Hence, required area $= \int_{1/2}^{2} (2^x - \log x)\, dx$

$$= \left[\frac{2^x}{\log 2} - (x \log x - x) \right]_{1/2}^{2}$$

$$= \left(\frac{4 - \sqrt{2}}{\log 2} - \frac{5}{2} \log 2 + \frac{3}{2} \right) \text{sq units}$$

Example 18 Find the area given by $x + y \le 6, x^2 + y^2 \le 6y$ and $y^2 \le 8x$.

Sol. Let us consider the curves

$$P \equiv y^2 - 8x = 0 \quad \text{...(i)}$$

$$C \equiv x^2 + y^2 = 6y$$

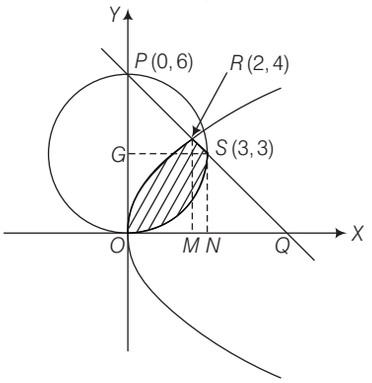
i.e. $$x^2 + (y - 3)^2 - 9 = 0 \quad \text{...(ii)}$$

and $$S \equiv x + y - 6 = 0 \quad \text{...(iii)}$$

The intersection points of the curves (ii) and (iii) are given by

$$(6 - y)^2 + y^2 - 6y = 0$$

i.e. $$y = 3, 6$$



Therefore, the points are (0, 6) and (3, 3). The intersection points of the curves (i) and (iii) are given by

$$y^2 = 8(6 - y), \quad \text{i.e. } y = 4, -12$$

Therefore, the point of intersection in 1st quadrant is (2, 4).

Now, we know that

$C \le 0$ denotes the region, inside the circle $C = 0$.

$P \le 0$ denotes the region, inside the parabola $P = 0$.

$S \le 0$ denotes the region, which is negative side of the line $S = 0$.

$\therefore$ Required area = Area of curvilinear $\Delta OMRO$ + Area of trapezium $MNSR$ − Area of curvilinear $\Delta ONSO$

$$= \int_0^2 \sqrt{8x}\, dx + \frac{1}{2}(MR + NS) \cdot MN - (\text{Area of square } ONSG - \text{Area of sector } OSGO)$$

$$= \int_0^2 \sqrt{8x}\, dx + \frac{1}{2}(4 + 3) \cdot 1 - \left(3^2 - \frac{\pi \cdot 3^2}{4} \right)$$

$$= \left(\frac{9\pi}{4} - \frac{1}{6} \right) \text{sq units}$$

Example 19 Find the area of the region $\{(x, y) : 0 \le y \le x^2 + 1, 0 \le y \le x + 1, 0 \le x \le 2\}$.

Sol. Let $R = \{(x, y) : 0 \le y \le x^2 + 1, 0 \le y \le x + 1, 0 \le x \le 2\}$

$$= \{(x, y) : 0 \le y \le x^2 + 1\} \cap \{(x, y) : 0 \le y \le x + 1\} \cap \{(x, y) : 0 \le x \le 2\}$$

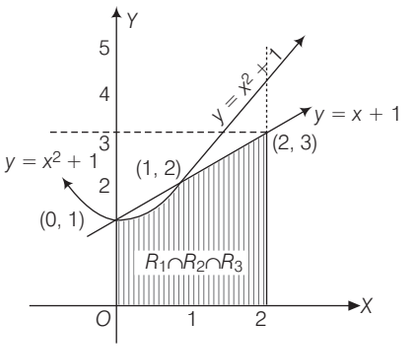
$$= R_1 \cap R_2 \cap R_3$$

where, $R_1 = \{(x, y) : 0 \le y \le x^2 + 1\}$

$R_2 = \{(x, y) : 0 \le y \le x + 1\}$

and $R_3 = \{(x, y) : 0 \le x \le 2\}$

Thus, the sketch of R_1, R_2 and R_3 are



From the above figure,

$$\text{Required area} = \int_0^1 (x^2 + 1)\, dx + \int_1^2 (x + 1)\, dx$$

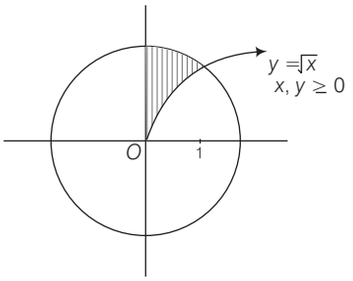
$$= \left(\frac{x^3}{3} + x \right)_0^1 + \left(\frac{x^2}{2} + x \right)_1^2 = \frac{23}{6} \text{ sq units}$$

Example 20 The area common to the region determined by $y \ge \sqrt{x}$ and $x^2 + y^2 < 2$ has the value

(a) π sq units (b) $(2\pi - 1)$ sq units

(c) $\left(\frac{\pi}{4} - \frac{1}{6} \right)$ sq units (d) None of these

Sol. The region formed by $y \geq \sqrt{x}$ is the outer region of the parabola $y^2 = x$, when $y \geq 0$ and $x \geq 0$ and $x^2 + y^2 < 2$ is the region inner to circle $x^2 + y^2 = 2$ shown as in figure.



Now, to find the point of intersection put $y^2 = x$ in $x^2 + y^2 = 2$.

$$\Rightarrow \quad x^2 + x - 2 = 0$$

$$\Rightarrow \quad (x+2)(x-1) = 0$$

$$\Rightarrow \quad x = 1, \text{ as } x \geq 0$$

$$\therefore \text{ Required area} = \int_0^1 (\sqrt{2 - x^2} - \sqrt{x})\, dx$$

$$= \left[\frac{x\sqrt{2-x^2}}{2} + \sin^{-1}\frac{x}{\sqrt{2}} - \frac{2}{\sqrt{3}} x^{3/2} \right]_0^1$$

$$= \frac{1}{2} + \frac{\pi}{4} - \frac{2}{3} = \left(\frac{\pi}{4} - \frac{1}{6} \right) \text{ sq units}$$

Hence, (c) is the correct answer.

Example 21 Find the area of the region enclosed by the curve $5x^2 + 6xy + 2y^2 + 7x + 6y + 6 = 0$.

Sol. Comparing $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get $a = 5, b = 2, h = 3, g = 7/2, f = 3$ and $c = 6$

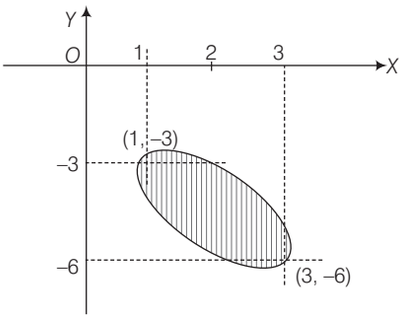
$$\Rightarrow \quad h^2 - ab = -1 < 0$$

So, the above equation represents an ellipse.

$$\therefore \quad 2y^2 + 6(1+x)y + (5x^2 + 7x + 6) = 0$$

$$\Rightarrow \quad y = \frac{-3(1+x) \pm \sqrt{(3-x)(x-1)}}{2}$$

Clearly, the values of y are real for all $x \in [1, 3]$. Thus, the graph is as shown below



Thus, required area

$$= \left| \int_1^3 \left(\frac{-3(1+x) - \sqrt{(3-x)(x-1)}}{2} \right) - \left(\frac{-3(1+x) + \sqrt{(3-x)(x-1)}}{2} \right) dx \right|$$

$$= \left| -\int_1^3 \sqrt{(3-x)(x-1)}\, dx \right| = \left| \int_1^3 \sqrt{1^2 - (x-2)^2}\, dx \right|$$

$$= \left| -\left\{ \frac{1}{2}(x-2)\sqrt{-x^2 + 4x - 3} + \frac{1}{2}\sin^{-1}\left(\frac{x-2}{1} \right) \right\}_1^3 \right|$$

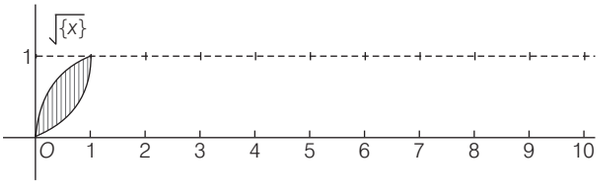
$$= \frac{\pi}{2} \text{ sq units}$$

Example 22 ***If*** $f(x) = \begin{cases} \sqrt{\{x\}}, & x \notin Z \\ 1, & x \in Z \end{cases}$ ***and*** $g(x) = \{x\}^2$ ***(where, {.} denotes fractional part of*** x***), then the area bounded by*** $f(x)$ ***and*** $g(x)$ ***for*** $x \in [0, 10]$ ***is***

(a) $\frac{5}{3}$ sq units (b) 5 sq units

(c) $\frac{10}{3}$ sq units (d) None of these

Sol. As, $f(x) = \begin{cases} \sqrt{\{x\}}, & x \notin z \\ 1, & x \in z \end{cases}$ and $g(x) = \{x\}^2$, where both $f(x)$ and $g(x)$ are periodic with period '1' shown as



Thus, required area $= 10\int_0^1 [\sqrt{\{x\}} - \{x\}^2]\, dx$

$$= 10\int_0^1 [(x)^{1/2} - x^2]\, dx$$

$$= 10\left[\frac{x^{3/2}}{3/2} - \frac{x^3}{3} \right]_0^1$$

$$= 10\left(\frac{2}{3} - \frac{1}{3} \right) = \frac{10}{3} \text{ sq units}$$

Hence, (c) is the correct answer.

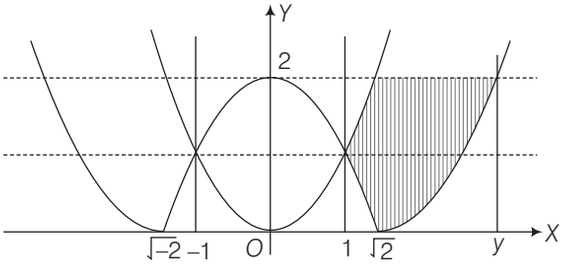
Example 23 Find the area of the region bounded by the curves $y = x^2$, $y = |2 - x^2|$ and $y = 2$, which lies to the right of the line $x = 1$. ***[IIT JEE 2002]***

Sol. The region bounded by given curves on the right side of $x = 1$ is shown as

Required area $= \int_1^{\sqrt{2}} \{x^2 - (2 - x^2)\}\, dx + \int_{\sqrt{2}}^2 \{4 - x^2\}\, dx$

$$= \int_1^{\sqrt{2}} (2x^2 - 2)\, dx + \int_{\sqrt{2}}^{2} (4 - x^2)\, dx$$

$$= \left(2 \cdot \frac{x^3}{3} - 2x\right)_1^{\sqrt{2}} + \left(4x - \frac{x^3}{3}\right)_{\sqrt{2}}^{2}$$



$$= \left(\frac{4\sqrt{2}}{3} - 2\sqrt{2}\right) - \left(\frac{2}{3} - 2\right) + \left(8 - \frac{8}{3}\right) - \left(4\sqrt{2} - \frac{2\sqrt{2}}{3}\right)$$

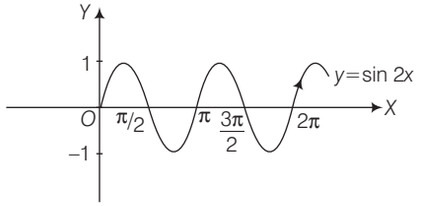
$$= \left(-4\sqrt{2} + \frac{20}{3}\right) = \left(\frac{20 - 12\sqrt{2}}{3}\right) \text{ sq units}$$

Example 24 The area enclosed by the curve $|y| = \sin 2x$, when $x \in [0, 2\pi]$ is

(a) 1 sq unit (b) 2 sq units

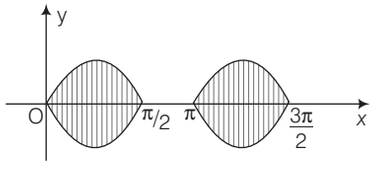
(c) 3 sq units (d) 4 sq units

Sol. As, we know $y = \sin 2x$ could be plotted as



Thus, $|y| = \sin 2x$ is whenever positive, y can have both positive and negative values, i.e. the curve is symmetric about the axes.

$\sin 2x$ is positive only in $0 \le x \le \frac{\pi}{2}$ and $\pi \le x \le \frac{3\pi}{2}$. Thus, the curve consists of two loops one in $\left[0, \frac{\pi}{2}\right]$ and another in $\left[\pi, \frac{3\pi}{2}\right]$.



Thus, required area $= 4\int_0^{\pi/2} (\sin 2x)\, dx$

$$= 4\left(-\frac{\cos 2x}{2}\right)_0^{\pi/2} = -2(\cos \pi - \cos 0)$$

$$= -2(-1 - 1) = 4 \text{ sq units}$$

Hence, (d) is the correct answer.

Example 25 Let $f(x) = x^2$, $g(x) = \cos x$ and $\alpha, \beta\ (\alpha < \beta)$ be the roots of the equation $18x^2 - 9\pi x + \pi^2 = 0$. Then, the area bounded by the curves $y = fog(x)$, the ordinates $x = \alpha$, $x = \beta$ and the X-axis is

(a) $\frac{1}{2}(\pi - 3)$ sq units (b) $\frac{\pi}{3}$ sq units

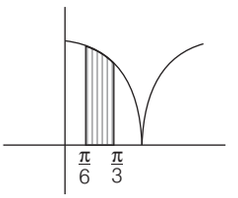
(c) $\frac{\pi}{4}$ sq units (d) $\frac{\pi}{12}$ sq units

Sol. Here, $y = fog(x) = f\{g(x)\} = (\cos x)^2 = \cos^2 x$

Also, $18x^2 - 9\pi x + \pi^2 = 0$

$\Rightarrow$ $(3x - \pi)(6x - \pi) = 0$

$\Rightarrow$ $x = \frac{\pi}{6}, \frac{\pi}{3}$ (as α, β)



$\therefore$ Required area of curve

$$= \int_{\pi/6}^{\pi/3} \cos^2 x\, dx = \frac{1}{2}\int_{\pi/6}^{\pi/3} (1 + \cos 2x)\, dx$$

$$= \frac{1}{2}\left\{x + \frac{\sin 2x}{2}\right\}_{\pi/6}^{\pi/3} = \frac{1}{2}\left\{\left(\frac{\pi}{3} - \frac{\pi}{6}\right) + \frac{1}{2}\left(\sin\frac{2\pi}{3} - \sin\frac{2\pi}{6}\right)\right\}$$

$$= \frac{1}{2}\left\{\frac{\pi}{6} + \frac{1}{2}\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right)\right\} = \frac{\pi}{12}$$

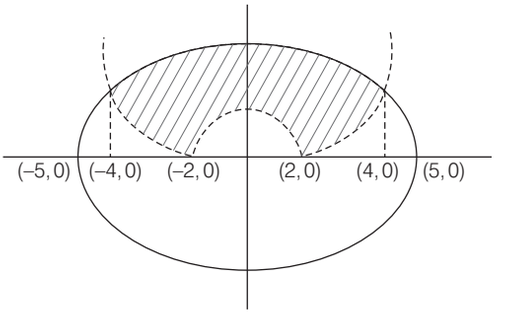
Hence, (d) is the correct answer.

Example 26 ***Find the area bounded by the curves*** $x^2 + y^2 = 25, 4y = |4 - x^2|$ ***and*** $x = 0$ ***above the*** X***-axis.***

Sol. The 1st curve is a circle of radius 5 with centre at (0, 0).

The 2nd curve is $y = \left|\frac{4 - x^2}{4}\right| = \left|1 - \frac{x^2}{4}\right|$

which can be traced easily by graph transformation.



When the two curves intersect each other, then

$$x^2 + \left(1 - \frac{x^2}{4}\right)^2 = 25 \Rightarrow x = \pm 4$$

Hence, required area $= 2\int_0^4 \left(\sqrt{25 - x^2} - \left|1 - \frac{x^2}{4}\right|\right) dx$

$$= 2\left[\int_0^4 \sqrt{25 - x^2}\,dx - \int_0^2 \left(1 - \frac{x^2}{4}\right)dx + \int_2^4 \left(1 - \frac{x^2}{4}\right)dx\right]$$

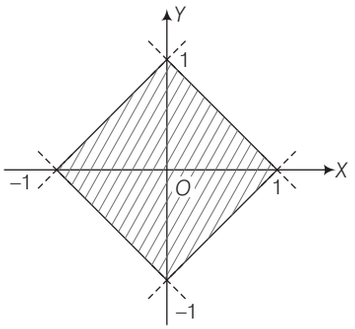
$$= 2\left[6 + \frac{25}{2}\sin^{-1}\left(\frac{4}{5}\right) - \frac{4}{3} - \frac{8}{3}\right] = \left\{25\sin^{-1}\left(\frac{4}{5}\right) + 4\right\}$$

Example 27 ***Find the area enclosed by*** $|x| + |y| = 1$.

Sol. From the given equation, we have

$$|y| = 1 - |x| \geq 0 \qquad [\because |y| \geq 0]$$

$$\Rightarrow \qquad -1 \leq x \leq 1$$



Therefore, the curve exists for $x \in [-1, 1]$ only

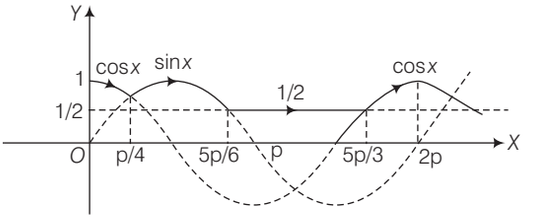
and for $-1 \leq x \leq 1$; $\quad y = \pm(1 - |x|)$ i.e. $y = \begin{cases} |x| - 1 \\ -(|x| - 1) \end{cases}$

Thus, the required graph is as given in figure.

$\therefore$ Required area $= (\sqrt{2})^2 = 2$ sq units

Example 28 Let $f(x) = \max\left\{\sin x, \cos x, \frac{1}{2}\right\}$, then determine the area of region bounded by the curves $y = f(x)$, X-axis, Y-axis and $x = 2\pi$.

Sol. We have, $f(x) = \max\left\{\sin x, \cos x, \frac{1}{2}\right\}$. Graphically, $f(x)$ could be drawn as



Here, the graph is plotted between 0 to 2π and between the points of intersection the maximum portion is included, thus the shaded part is required area

	Interval	Value of $f(x)$
i.e. for	$0 \leq x < \pi/4$	$\cos x$
for	$\pi/4 \leq x < 5\pi/6$	$\sin x$
for	$5\pi/6 \leq x < 5\pi/3$	$1/2$
for	$5\pi/3 \leq x < 2\pi$	$\cos x$

Hence, required area

$$I = \int_0^{\pi/4} \cos x\,dx + \int_{\pi/4}^{5\pi/6} \sin x\,dx + \int_{5\pi/6}^{5\pi/3} \frac{1}{2}\,dx + \int_{5\pi/3}^{2\pi} \cos x\,dx$$

$$= (\sin x)_0^{\pi/4} - (\cos x)_{\pi/4}^{5\pi/6} + \frac{1}{2}(x)_{5\pi/6}^{5\pi/3} + (\sin x)_{5\pi/3}^{2\pi}$$

$$= \left(\frac{1}{\sqrt{2}} - 0\right) - \left(-\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\right) + \frac{1}{2}\left(\frac{5\pi}{3} - \frac{5\pi}{6}\right) + \left(0 + \frac{\sqrt{3}}{2}\right)$$

$$= \left(\frac{5\pi}{12} + \sqrt{2} + \sqrt{3}\right) \text{ sq units}$$

Exercise for Session 2

1. The area of the region bounded by $y^2 = 2x + 1$ and $x - y - 1 = 0$ is

(a) 2/3 (b) 4/3 (c) 8/3 (d) 16/3

2. The area bounded by the curve $y = 2x - x^2$ and the straight line $y = x$ is given by

(a) 9/2 (b) 43/6 (c) 35/6 (d) None of these

3. The area bounded by the curve $y = x|x|$, X-axis and the ordinates $x = -1$, $x = 1$ is given by

(a) 0 (b) 1/3 (c) 2/3 (d) None of these

4. Area of the region bounded by the curves $y = 2^x$, $y = 2x - x^2$, $x = 0$ and $x = 2$ is given by

(a) $\frac{3}{\log 2} - \frac{4}{3}$ (b) $\frac{3}{\log 2} + \frac{4}{3}$ (c) $3\log 2 - \frac{4}{3}$ (d) None of these

5. The area of the figure bounded by the cuves $y = e^x$, $y = e^{-x}$ and the straight line $x = 1$ is

(a) $e + \frac{1}{e}$ (b) $e - \frac{1}{e}$ (c) $e + \frac{1}{e} - 2$ (d) None of these

6. Area of the region bounded by the curve $y^2 = 4x$, Y-axis and the line $y = 3$ is

(a) 2 (b) 9/4 (c) $6\sqrt{3}$ (d) None of these

7. The area of the figure bounded by $y = \sin x$, $y = \cos x$ is the first quadrant is

(a) $2(\sqrt{2-1})$ (b) $\sqrt{3}+1$ (c) $2(\sqrt{3}-1)$ (d) None of these

8. The area bounded by the curves $y = xe^x$, $y = xe^{-x}$ and the line $x = 1$ is

(a) $\frac{2}{e}$ (b) $1-\frac{2}{e}$ (c) $\frac{1}{e}$ (d) $1-\frac{1}{e}$

9. The areas of the figure into which the curve $y^2 = 6x$ divides the circle $x^2 + y^2 = 16$ are in the ratio

(a) $\frac{2}{3}$ (b) $\frac{4\pi-\sqrt{3}}{8\pi+\sqrt{3}}$ (c) $\frac{4\pi+\sqrt{3}}{8\pi-\sqrt{3}}$ (d) None of these

10. The area bounded by the Y-axis, $y = \cos x$ and $y = \sin x$, $0 \geq x \leq \pi/2$ is

(a) $2(\sqrt{2}-1)$ (b) $\sqrt{2}-1$ (c) $(\sqrt{2}+1)$ (d) $\sqrt{2}$

11. The area bounded by the curve $y = \frac{3}{|x|}$ and $y + |2-x| = 2$ is

(a) $\frac{4-\log 27}{3}$ (b) $2-\log 3$ (c) $2+\log 3$ (d) None of these

12. The area bounded by the curves $y = x^2 + 2$ and $y = 2|x| - \cos + x$ is

(a) 2/3 (b) 8/3 (c) 4/3 (d) 1/3

13. The are bounded by the curve $y^2 = 4x$ and the circle $x^2 + y^2 - 2x - 3 = 0$ is

(a) $2\pi + \frac{8}{3}$ (b) $4\pi + \frac{8}{3}$ (c) $\pi + \frac{8}{3}$ (d) $\pi - \frac{8}{3}$

14. A point P moves inside a triangle formed by $A(0,0), B\left(1, \frac{1}{\sqrt{3}}\right), C(2, 0)$ such that min $\{PA, PB, PC\} = 1$, then the area bounded by the curve traced by P, is

(a) $3\sqrt{3} - \frac{3\pi}{2}$ (b) $\sqrt{3} + \frac{\pi}{2}$ (c) $\sqrt{3} - \frac{\pi}{2}$ (d) $3\sqrt{3} + \frac{3\pi}{2}$

15. The graph of $y^2 + 2xy + 40|x| = 400$ divides the plane into regions. The area of the bounded region is

(a) 400 (b) 800 (c) 600 (c) None of these

16. The area of the region defined by $||x| - |y|| \leq 1$ and $x^2 + y^2 \leq 1$ in the xy- plane is

(a) π (b) 2π (c) 3π (d) 1

17. The area of the region defined by $1 \leq |x-2| + |y+1| \leq 2$ is

(a) 2 (b) 4 (c) 6 (d) None of these

18. The area of the region enclosed by the curve $|y| = -(1-|x|)^2 + 5$, is

(a) $\frac{8}{3}(7+5\sqrt{5})$ sq units (b) $\frac{2}{3}(7+5\sqrt{5})$ sq units (c) $\frac{2}{3}(5\sqrt{5}-7)$ sq units (d) None of these

19. The area bounded by the curve $f(x) = ||\tan x + \cot x| - |\tan x - \cot x||$ between the lines $x = 0, x = \frac{\pi}{2}$ and the X-axis is

(a) log 4 (b) $\log\sqrt{2}$ (c) 2log2 (d) $\sqrt{2}\log 2$

20. If $f(x) = \max\left\{\sin x, \cos x, \frac{1}{2}\right\}$, then the area of the region bounded by the curves $y = f(x)$, X-axis, Y-axis and $x = \frac{5\pi}{3}$ is

(a) $\left(\sqrt{2}-\sqrt{3}+\frac{5\pi}{12}\right)$ sq units (b) $\left(\sqrt{2}+\sqrt{3}+\frac{5\pi}{2}\right)$ sq units

(c) $\left(\sqrt{2}+\sqrt{3}+\frac{5\pi}{2}\right)$ sq units (d) None of these

JEE Type Solved Examples :
Single Option Correct Type Questions

● **Ex. 1** *If A denotes the area bounded by* $f(x)=\left|\frac{\sin x+\cos x}{x}\right|$, *X-axis,* $x=\pi$ *and* $x=3\pi$, *then*

(a) $1<A<2$ (b) $0<A<2$

(c) $2<A<3$ (d) None of these

Sol. $\frac{2\sqrt{2}}{2\pi}<\int_{\pi}^{2\pi}\frac{|\sin x+\cos x|}{x}dx<\frac{2\sqrt{2}}{2\pi}$

$\left[\because \pi<x<2\pi \Rightarrow \frac{1}{2\pi}<\frac{1}{x}<\frac{1}{\pi}\right]$ ...(i)

$\frac{2\sqrt{2}}{3\pi}<\int_{2\pi}^{3\pi}\frac{|\sin x+\cos x|}{x}dx<\frac{2\sqrt{2}}{2\pi}$ ...(ii)

On adding Eqs. (i) and (ii), we get

$$\frac{5\sqrt{2}}{3\pi}<A<\frac{3\sqrt{2}}{\pi}$$

$\Rightarrow$ $0.75<A<1.3$

Hence, (b) is the correct answer.

● **Ex. 2** *If* $f(x)\geq 0, \forall x\in(0,2)$ *and* $y=f(x)$ *makes positive intercepts of* 2 *and* 1 *unit on X and Y-axes respectively and encloses an area of* 3/4 *unit with axes, then* $\int_0^2 xf'(x)\,dx$ *is*

(a) $\frac{3}{4}$ (b) 1 (c) $\frac{5}{4}$ (d) $-\frac{3}{4}$

Sol. $I=xf(x)|_0^2-\int_0^2 f(x)\,dx=0-\frac{3}{4}=-\frac{3}{4}$

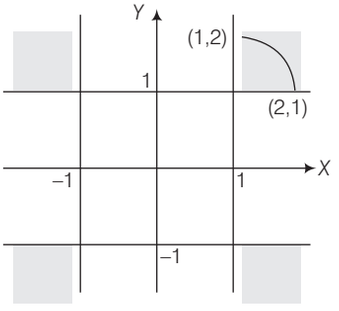
Hence, (d) is the correct answer.

● **Ex. 3** *The area of the region included between the regions satisfying* $\min(|x|,|y|)\geq 1$ *and* $x^2+y^2\leq 5$ *is*

(a) $\frac{5}{2}\left(\sin^{-1}\frac{2}{\sqrt{5}}-\sin^{-1}\frac{1}{\sqrt{5}}\right)-4$ (b) $10\left(\sin^{-1}\frac{2}{\sqrt{5}}-\sin^{-1}\frac{1}{\sqrt{5}}\right)-4$

(c) $\frac{2}{5}\left(\sin^{-1}\frac{2}{\sqrt{5}}-\sin^{-1}\frac{1}{\sqrt{5}}\right)-4$ (d) $15\left(\sin^{-1}\frac{2}{\sqrt{5}}-\sin^{-1}\frac{1}{\sqrt{5}}\right)-4$

Sol. Shaded region depicts $\min(|x|,|y|)\geq 1$



Required area $=4\int_1^2(\sqrt{5-x^2}-1)\,dx$

$$=10\left(\sin^{-1}\frac{2}{\sqrt{5}}-\sin^{-1}\frac{1}{\sqrt{5}}\right)-4$$

Hence, (b) is the correct answer.

● **Ex. 4** *The area of the region between the curves* $y=\sqrt{\frac{1+\sin x}{\cos x}}$ *and* $y=\sqrt{\frac{1-\sin x}{\cos x}}$ *and bounded by the lines* $x=0$ *and* $x=\frac{\pi}{4}$ *is* ***[IIT JEE 2008]***

(a) $\int_0^{\sqrt{2}-1}\frac{t}{(1+t^2)\sqrt{1-t^2}}dt$

(b) $\int_0^{\sqrt{2}-1}\frac{4t}{(1+t^2)\sqrt{1-t^2}}dt$

(c) $\int_0^{\sqrt{2}+1}\frac{4t}{(1+t^2)\sqrt{1-t^2}}dt$

(d) $\int_0^{\sqrt{2}+1}\frac{t}{(1+t^2)\sqrt{1-t^2}}dt$

Sol. Required area $=\int_0^{\pi/4}\left(\sqrt{\frac{1+\sin x}{\cos x}}-\sqrt{\frac{1-\sin x}{\cos x}}\right)dx$

$\left(\because \frac{1+\sin x}{\cos x}>\frac{1-\sin x}{\cos x}>0\right)$

$$=\int_0^{\pi/4}\left(\sqrt{\frac{1+\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}{\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}}-\sqrt{\frac{1-\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}{\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}}\right)dx$$

$$=\int_0^{\pi/4}\left(\sqrt{\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}}-\sqrt{\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}}\right)dx$$

$$=\int_0^{\pi/4}\frac{1+\tan\frac{x}{2}-1+\tan\frac{x}{2}}{\sqrt{1-\tan^2\frac{x}{2}}}dx=\int_0^{\pi/4}\frac{2\tan\frac{x}{2}}{\sqrt{1-\tan^2\frac{x}{2}}}dx$$

Put $\tan\frac{x}{2}=t$

$\therefore$ Area $=\int_0^{\sqrt{2}-1}\frac{4t}{(1+t^2)\sqrt{1-t^2}}dt$

Hence, (b) is the correct answer.

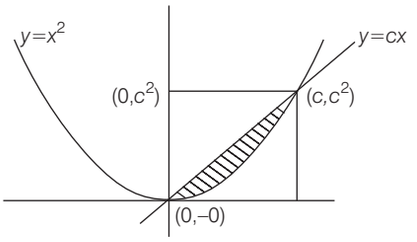
JEE Type Solved Examples :
More than One Correct Option Type Questions

● **Ex. 5** *Let T be the triangle with vertices* $(0,0),(0,c^2)$ *and* (c,c^2) *and let R be the region between* $y=cx$ *and* $y=x^2$, *where* $c>0$, *then*

(a) Area $(R)=\dfrac{c^3}{6}$ (b) Area of $R=\dfrac{c^3}{3}$

(c) $\lim\limits_{c\to 0^+}\dfrac{\text{Area}\,(T)}{\text{Area}\,(R)}=3$ (d) $\lim\limits_{c\to 0^+}\dfrac{\text{Area}\,(T)}{\text{Area}\,(R)}=\dfrac{3}{2}$

Sol. Area $(T)=\dfrac{c\cdot c^2}{2}=\dfrac{c^3}{2}$



$$\text{Area}\,(R)=\frac{c^3}{2}-\int_0^c x^2\,dx=\frac{c^3}{2}-\frac{c^3}{3}=\frac{c^3}{6}$$

$$\therefore\quad \lim_{c\to 0^+}\frac{\text{Area}(T)}{\text{Area}\,(R)}=\lim_{c\to 0^+}\frac{c^3}{2}\cdot\frac{6}{c^3}=3$$

Hence, (a) and (c) are the correct answers.

● **Ex. 6** *Suppose f is defined from* $R\to[-1,1]$ *as* $f(x)=\dfrac{x^2-1}{x^2+1}$ *where R is the set of real number. Then, the statement which does not hold is*

(a) f is many-one onto

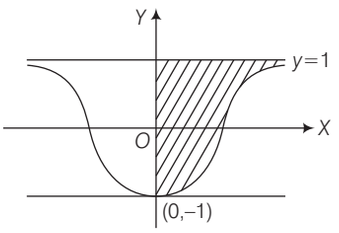
(b) f increases for $x>0$ and decreases for $x<0$

(c) minimum value is not attained even though f is bounded

(d) the area included by the curve $y=f(x)$ and the line $y=1$ is π sq units

Sol. $y=f(x)=\dfrac{x^2-1}{x^2+1}=1-\dfrac{2}{x^2+1}$

$f'(x)=\dfrac{4x}{(x^2+1)^2}$ $x>0$, f is increasing and $x<0$ f is decreasing.



$\Rightarrow$ (b) is true; range is $[-1,1)$ $\Rightarrow$ into $\Rightarrow$ (a) is false; minimum value occurs at $x=0$ and $f(0)=-1$ $\Rightarrow$ (c) is false.

$$A=2\int_0^\infty\left(1-\frac{x^2-1}{x^2+1}\right)dx=4\int_0^\infty\frac{dx}{x^2+1}$$

$$=[4\cdot\tan^{-1}x]_0^\infty=4\cdot\frac{\pi}{2}=2\pi\ \Rightarrow\ \text{(d) is false.}$$

Hence, (a), (c) and (d) are the correct answers.

● **Ex. 7** *Consider* $f(x)=\begin{cases}\cos\,x, & 0\le x<\dfrac{\pi}{2}\\ \left(\dfrac{\pi}{2}-x\right)^2, & \dfrac{\pi}{2}\le x<\pi\end{cases}$ *such that f is periodic with period* π, *then*

(a) the range of f is $\left[0,\dfrac{\pi^2}{4}\right)$

(b) f is continuous for all real x, but not differentiable for some real x

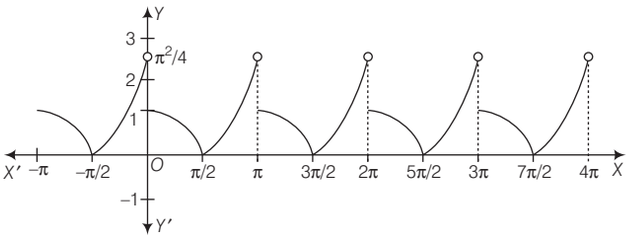
(c) f is continuous for all real x

(d) the area bounded by $y=f(x)$ and the X-axis from $x=-n\pi$ to $x=n\pi$ is $2n\left(1+\dfrac{\pi^2}{24}\right)$ for a given $n\in N$

Sol. Given, $f(x)=\begin{cases}\cos x, & 0\le x<\dfrac{\pi}{2}\\ \left(\dfrac{\pi}{2}-x\right)^2, & \dfrac{\pi}{2}\le x<\pi\end{cases}$ and f is periodic with period π. Let us draw the graph of $y=f(x)$

From the graph, the range of the function is $\left[0,\dfrac{\pi^2}{4}\right)$.

It is discontinuous at $x=n\pi$, $n\in I$. It is not differentiable at $x=\dfrac{n\pi}{2}$, $n\in I$.



Area bounded by $y=f(x)$ and the X-axis from $-n\pi$ to $n\pi$ for $n\in N$

$$=2n\int_0^\pi f(x)\,dx=2n\left[\int_0^{\pi/2}\cos x\,dx+\int_{\pi/2}^{\pi}\left(\frac{\pi}{2}-x\right)^2dx\right]$$

$$=2n\left(1+\frac{\pi^3}{24}\right)$$

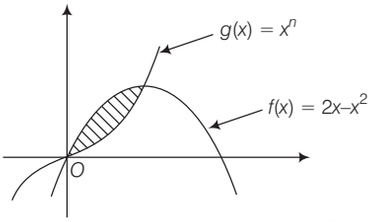
Hence, (a) and (d) are the correct answers.

● **Ex. 8** *Consider the functions $f(x)$ and $g(x)$, both defined from $R \to R$ and are defined as $f(x) = 2x - x^2$ and $g(x) = x^n$ where $n \in N$. If the area between $f(x)$ and $g(x)$ is 1/2, then n is a divisor of*

(a) 12 (b) 15 (c) 20 (d) 30

Sol. Solving, $f(x) = 2x - x^2$ and $g(x) = x^n$ we have

$2x - x^2 = x^n \Rightarrow x = 0$ and $x = 1$



$$A = \int_0^1 (2x - x^2 - x^n)\, dx = \left[x^2 - \frac{x^3}{3} - \frac{x^{n+1}}{n+1}\right]_0^1$$

$$= 1 - \frac{1}{3} - \frac{1}{n+1} = \frac{2}{3} - \frac{1}{n+1}$$

Since, $\frac{2}{3} - \frac{1}{n+1} = \frac{1}{2} \Rightarrow \frac{2}{3} - \frac{1}{2} = \frac{1}{n+1}$

$\Rightarrow \frac{4-3}{6} = \frac{1}{n+1} \Rightarrow n + 1 = 6$

$\Rightarrow n = 5$

Thus, n is a divisor of 15, 20, 30.

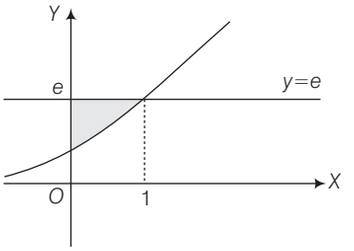
Hence, (b), (c) and (d) are the correct answers.

● **Ex. 9** *Area of the region bounded by the curve $y = e^x$ and lines $x = 0$ and $y = e$ is* **[IIT JEE 2009]**

(a) $e - 1$ (b) $\int_1^e \ln(e + 1 - y)\, dy$

(c) $e - \int_0^1 e^x\, dx$ (d) $\int_1^e \ln y\, dy$

Sol. Shaded area $= e - \left(\int_0^1 e^x\, dx\right) = 1$



Also, $\int_1^e \ln(e + 1 - y)\, dy$

Put $e + 1 - y = t$

$\Rightarrow -dy = dt$

$$= \int_0^1 \ln t(-dt) = \int_0^e \ln t\, dt$$

$$= \int_1^e \ln y\, dy = 1$$

Hence, (b), (c) and (d) are the correct answers.

JEE Type Solved Examples : Passage Based Questions

Passage I

(Q. Nos. 10 to 12)

Consider the function $f(x) = x^3 - 8x^2 + 20x - 13$.

● **Ex. 10** *Number of positive integers x for which $f(x)$ is a prime number, is*

(a) 1 (b) 2
(c) 3 (d) 4

Sol. $f(x) = (x-1)(x^2 - 7x + 13)$ for $f(x)$ to be prime atleast one of the factors must be prime.

Therefore, $x - 1 = 1$

$\Rightarrow x = 2$

or $x^2 - 7x + 13 = 1$

$\Rightarrow x^2 - 7x + 12 = 0$

$\Rightarrow x = 3$ or 4

$\Rightarrow x = 2, 3, 4$

Hence, (c) is the correct answer.

● **Ex. 11** *The function $f(x)$ defined for $R \to R$*

(a) is one-one onto
(b) is many-one onto
(c) has 3 real roots
(d) is such that $f(x_1) \cdot f(x_2) < 0$ where x_1 and x_2 are the roots of $f'(x) = 0$

Sol. $f(x)$ is many-one as it increases and decreases, also range of $f(x) \in R \Rightarrow$ many-one onto.

Hence, (b) is the correct answer.

● **Ex. 12** *Area enclosed by $y = f(x)$ and the coordinate axes is*

(a) 65/12 (b) 13/12
(c) 71/12 (d) None of these

Sol. $A = \left|\int_0^1 f(x)\, dx\right| = -\int_0^1 (x^3 - 8x^2 + 20x - 13)\, dx = \frac{65}{12}$

Hence, (a) is the correct answer.

Passage II

(Q. Nos. 13 to 15)

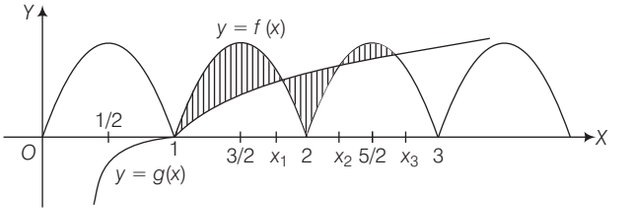
Let $h(x) = f(x) - g(x)$, where $f(x) = \sin^4 \pi x$ and $g(x) = \ln x$. Let $x_0, x_1, x_2, \ldots, x_{n+1}$ be the roots *of $f(x) = g(x)$ in increasing order.*

● ***Ex. 13*** *The absolute area enclosed by $y = f(x)$ and $y = g(x)$ is given by*

(a) $\sum_{r=0}^{n} \int_{x_r}^{x_{r+1}} (-1)^r h(x)\,dx$ (b) $\sum_{r=0}^{n} \int_{x_r}^{x_{r+1}} (-1)^{r+1} h(x)\,dx$

(c) $2\sum_{r=0}^{n} \int_{x_r}^{x_{r+1}} (-1)^r h(x)\,dx$ (d) $\frac{1}{2}\sum_{r=0}^{n} \int_{x_r}^{x_{r+1}} (-1)^{r+1} h(x)\,dx$

Sol.



Hence, (a) is the correct answer.

● ***Ex. 14*** *In the above question, the value of n is*

(a) 1 (b) 2 (c) 3 (d) 4

Sol. $x_{n+1} = x_3 \Rightarrow n = 2$.

Hence, (b) is the correct answer.

● ***Ex. 15*** *The whole area bounded by $y = f(x), y = g(x)$ and $x = 0$ is*

(a) $\frac{11}{8}$ (b) $\frac{8}{3}$ (c) 2 (d) $\frac{13}{3}$

Sol. Required area $= \int_0^1 \sin^4 \pi x\,dx - \int_0^1 \ln\ x\,dx = \frac{11}{8}$

Hence, (a) is the correct answer.

Passage III

(Q. Nos. 16 to 18)

Consider the function defined implicitly by the equation $y^3 - 3y + x = 0$ on various intervals in the real line. If $x \in (-\infty, -2) \cup (2, \infty)$, the equation implicitly defines a unique real-valued differentiable function $y = f(x)$. If $x \in (-2, 2)$, the equation implicitly defines a unique real-valued differentiable function $y = g(x)$ satisfying $g(0) = 0$. ***[IIT JEE 2008]***

● ***Ex. 16*** *If $f(-10\sqrt{2}) = 2\sqrt{2}$, then $f''(-10\sqrt{2})$ is equal to*

(a) $\frac{4\sqrt{2}}{7^3 3^2}$ (b) $-\frac{4\sqrt{2}}{7^3 3^2}$ (c) $\frac{4\sqrt{2}}{7^3 3}$ (d) $-\frac{4\sqrt{2}}{7^3 3}$

Sol. $\therefore$ $y^3 - 3y + x = 0$

On differentiating, we get $3y^2 y' - 3y' - 1 = 0$

$$\Rightarrow \quad y' = \frac{1}{3(1 - y^2)}$$

$$\Rightarrow \quad y'(-10\sqrt{2}) = \frac{1}{3\{1 - (2\sqrt{2})^2\}} \qquad \ldots(i)$$

$$\Rightarrow \quad y'(-10\sqrt{2}) = \frac{1}{3\{1 - (2\sqrt{2})^2\}} = \frac{1}{3(1-8)} = -\frac{1}{21}$$

Again differentiating Eq. (i), we get

$$y'' = \frac{6yy'^2}{3(1 - y^2)}$$

$$y''(-10\sqrt{2}) = \frac{6 \cdot 2\sqrt{2} \cdot \left(\frac{1}{21}\right)^2}{3(1-8)} = -\frac{4\sqrt{2}}{7^3 \cdot 3^2}$$

Hence, (b) is the correct answer.

● ***Ex. 17*** *The area of the region bounded by the curve $y = f(x)$, the X-axis and the line $x = a$ and $x = b$, where $-\infty < a < b < -2$ is*

(a) $\int_a^b \frac{x}{3[\{f(x)\}^2 - 1]}dx + bf(b) - af(a)$

(b) $-\int_a^b \frac{x}{3[\{f(x)\}^2 - 1]}dx - bf(b) + af(a)$

(c) $\int_a^b \frac{x}{3[\{f(x)\}^2 - 1]}dx - bf(b) + af(a)$

(d) $-\int_a^b \frac{x}{3[\{f(x)\}^2 - 1]}dx - bf(b) + af(a)$

Sol. Required area $= \int_a^b f(x)\,dx = [xf(x)]_a^b - \int_a^b xf'(x)\,dx$

$$= bf(b) - af(a) + \int_a^b \frac{x}{3[\{f(x)\}^2 - 1]}\,dx$$

Hence, (a) is the correct answer.

● ***Ex. 18*** $\int_{-1}^{1} g'(x)\,dx$ *is equal to*

(a) $2g(-1)$ (b) 0 (c) $-2g(1)$ (d) $2g(1)$

Sol. $I = \int_{-1}^{1} g'(x)\,dx = [g(x)]_{-1}^{1} = g(1) - g(-1)$

Since, $y^3 - 3y + x = 0$ ...(i)

and $y = g(x)$

Since, $\{g(x)\}^3 - 3g(x) + x = 0$ [by Eq. (i)]

At $x = 1$, $\{g(1)\}^3 - 3g(1) + 1 = 0$ ...(ii)

At $x = -1$, $\{g(-1)\}^3 - 3g(-1) - 1 = 0$ ...(iii)

On adding Eqs. (i) and (ii), we get

$$\{g(1)\}^3 + \{g(-1)\}^3 - 3\{g(1) + g(-1)\} = 0$$

$$\{g(1) + g(-1)\}\{g(1)^2 + g(-1)^2 - g(1)g(-1) - 3\} = 0$$

$\Rightarrow$ $g(1) + g(-1) = 0,\ g(1) = -g(-1)$

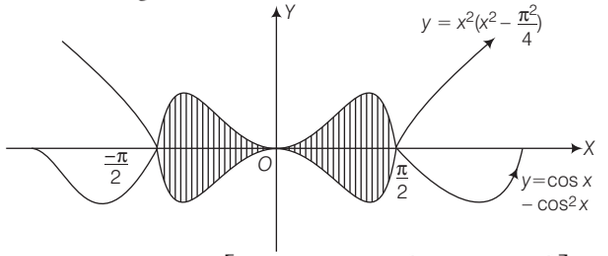
$\Rightarrow$ $I = g(1) - g(-1) = g(1) - (-g(1)) = 2g(1)$

Hence, (d) is the correct answer.

Subjective Type Questions

● **Ex. 19** *Find the total area bounded by the curves* $y = \cos x - \cos^2 x$ *and* $y = x^2\left(x^2 - \frac{\pi^2}{4}\right)$.

Sol. Here, $y = \cos x - \cos^2 x$ and $y = x^2\left(x^2 - \frac{\pi^2}{4}\right)$ could be drawn as in figure.



Thus, the area $= 2\int_0^{\pi/2}\left[(\cos x - \cos^2 x) - \left\{x^2\left(x^2 - \frac{\pi^2}{4}\right)\right\}\right]dx$

$= 2\int_0^{\pi/2}\left(\cos x - \cos^2 x - x^4 + \frac{\pi^2}{4}x^2\right)dx$

$= 2\left[\sin x - \frac{x}{2} - \frac{\sin 2x}{4} - \frac{x^5}{5} + \frac{\pi^2 x^3}{12}\right]_0^{\pi/2}$

$= \left(2 - \frac{\pi}{2} + \frac{\pi^5}{120}\right)$

● **Ex. 20** *A curve* $y = f(x)$ *passes through the point* $P(1,1)$, *the normal to the curve at* P *is* $a(y-1)+(x-1)=0$. *If the slope of the tangent at any point on the curve is proportional to the ordinate of that point, determine the equation of the curve. Also obtain the area bounded by the Y-axis, the curve and the normal to the curve at P.*

Sol. Here, slope of the normal at $P(x, y)$.

$\Rightarrow$ Slope of the line $a(y-1)+(x-1)=0$ is $-\frac{1}{a}$

$\therefore$ Slope of the tangent at $P = a$,

$$\left(\frac{dy}{dx}\right)_P = a \qquad \text{...(i)}$$

It is given that the slope of the tangent at any point on the curve $y = f(x)$ is proportional to the ordinate of the point.

$\therefore \quad \frac{dy}{dx} \propto y \Rightarrow \frac{dy}{dx} = \lambda y$

$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(1,1)} = \lambda \Rightarrow a = \lambda$

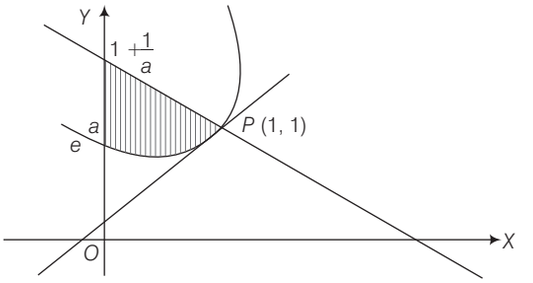
$\therefore \quad \frac{dy}{dx} = ay \Rightarrow \frac{dy}{y} = a\,dx$

$\Rightarrow \quad \log y = ax + \log c$

$\Rightarrow \quad y = ce^{ax}$, which passes through $P(1, 1)$

$\therefore \quad c = e^{-a}$

$\Rightarrow$ Curve is $y = e^{-a}e^{ax} \Rightarrow y = e^{a(x-1)}$



$\therefore$ Required area $= \frac{1}{a}\int_{e^{-a}}^{1}(\log y + a)\,dy + \int_1^{1+1/a}\{(1+a) - ay\}dy$

$= \frac{1}{a}\{y(\log y - 1) + ay\}_{e^{-a}}^{1} + \left\{(1+a)y - \frac{a}{2}y^2\right\}_a^{1+\frac{1}{a}}$

$= \frac{1}{a}\{(-1+a) - e^{-a}(-a-1) - ae^{-a}\}$

$+\left[(1+a)\left(1+\frac{1}{a}-1\right) - \frac{a}{2}\left\{\left(1+\frac{1}{a}\right)^2 - 1^2\right\}\right]$

$= \frac{1}{a}[-1+a+ae^{-a}+e^{-a}-ae^{-a}] + \left[\frac{1+a}{a} - \frac{a}{2}\left(2+\frac{1}{a}\right)\frac{1}{a}\right]$

$= \frac{1}{a}(-1+a+e^{-a}) + \left(\frac{1}{a}+1-1-\frac{1}{2a}\right) = \left(1 + \frac{e^{-a}}{a} - \frac{1}{2a}\right)$ sq units

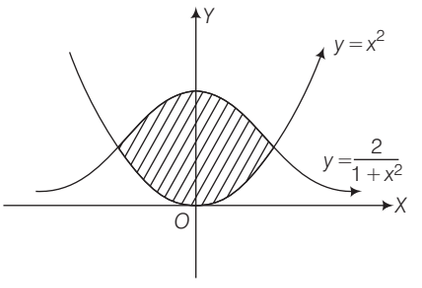
● **Ex. 21** *Sketch the region bounded by the curves* $y = x^2$ *and* $y = \frac{2}{1+x^2}$. *Find the area.*

Sol. For intersection point, $x^2 = \frac{2}{1+x^2}$

i.e. $x^4 + x^2 - 2 = 0$

i.e. $(x^2+2)(x^2-1) = 0$

i.e. $x = \pm 1$



Hence, required area $= 2\int_0^1 \left(\frac{2}{1+x^2} - x^2\right) dx$

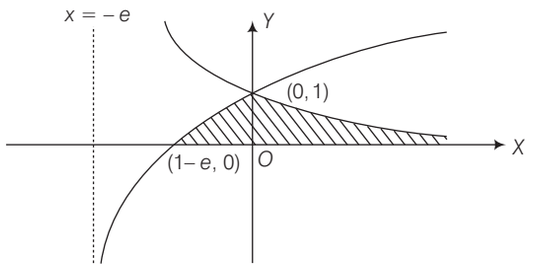
$$= 2\left[2\tan^{-1} x - \frac{x^3}{3}\right]_0^1 = 2\left(\frac{\pi}{2} - \frac{1}{3}\right)$$

● ***Ex. 22*** *Find the area enclosed between the curves* $y = \log(x+e);\ x = \log_e\left(\frac{1}{y}\right)$ *and X-axis.*

Sol. The given curves are $y = \log(x+e)$ and

$$x = \log_e\left(\frac{1}{y}\right) \Rightarrow \frac{1}{y} = e^x \Rightarrow y = e^{-x}$$

Using graph transformation we can sketch the curves.



Hence, required area $= \int_{1-e}^{0} \log(x+e)\,dx + \int_0^{\infty} e^{-x}\,dx$

$$= \int_1^e \log(t)\,dt + \int_0^{\infty} e^{-x}\,dx \qquad \text{(putting } x+e=t\text{)}$$

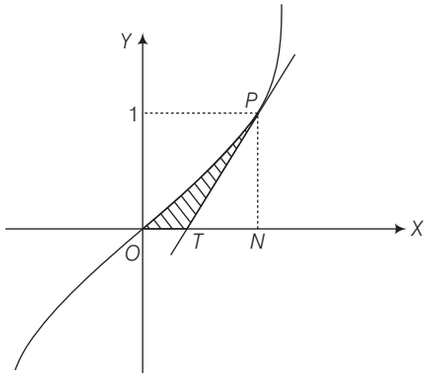
$$= [t\log t - t]_1^e - [e^{-x}]_0^{\infty} = 1 + 1 = 2$$

● ***Ex. 23*** *Find the area of the region bounded by the curve* $c: y = \tan x$, *tangent drawn to c at* $x = \pi/4$ *and the X-axis.*

Sol. The given curve is $y = \tan x$

$$\therefore \quad \frac{dy}{dx} = \sec^2 x$$

$$\left(\frac{dy}{dx}\right)_{x=\pi/4} = \sec^2\frac{\pi}{4} = 2$$



Also, at $x = \frac{\pi}{4};\ y = 1$

$\therefore$ The equation of the tangent to the curve at the point $\left(\frac{\pi}{4}, 1\right)$ is $\quad y - 1 = 2\left(x - \frac{\pi}{4}\right)$

when $\qquad y = 0;\ x = \frac{\pi}{4} - \frac{1}{2} = OT$

Now, the required area = Area of curvilinear ΔOPN − Area of ΔPTN

$$= \int_0^{\pi/4} (\tan x)\,dx - \frac{1}{2}\cdot NT\cdot PN$$

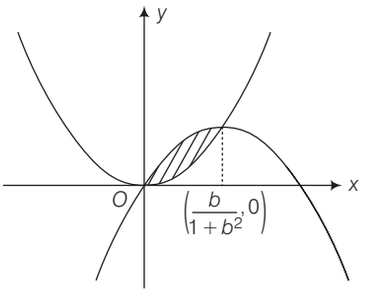
$$= [\log(\sec x)]_0^{\pi/4} - \frac{1}{2}\left(\frac{\pi}{4} - \frac{\pi}{4} + \frac{1}{2}\right)\cdot 1 = \frac{1}{2}\left(\log 2 - \frac{1}{2}\right)$$

● ***Ex. 24*** *Find all the possible values of* $b > 0$, *so that the area of the bounded region enclosed between the parabolas* $y = x - bx^2$ *and* $y = \frac{x^2}{b}$ *is maximum.*

Sol. Eliminating y from $y = \frac{x^2}{b}$ and $y = x - bx^2$, we get

$$x^2 = bx - b^2x^2$$

$$\Rightarrow \qquad x = 0,\ \frac{b}{1+b^2}$$



Thus, the area enclosed between the parabolas,

$$A = \int_0^{b/(1+b^2)} \left(x - bx^2 - \frac{x^2}{b}\right) dx$$

$$= \int_0^{b/1+b^2} \left\{x - x^2\left(\frac{1+b^2}{b}\right)\right\} dx$$

$$= \left(\frac{x^2}{2} - \frac{x^3}{3}\cdot\frac{1+b^2}{b}\right)_0^{b/1+b^2} = \frac{1}{6}\cdot\frac{b^2}{(1+b^2)^2}$$

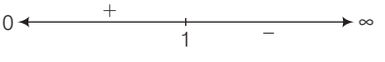
For maximum value of A, $\frac{dA}{db} = 0$

But $\frac{dA}{db} = \frac{1}{6}\cdot\frac{(1+b^2)^2\cdot 2b - b^2\cdot 2(1+b^2)\cdot 2b}{(1+b^2)^4} = \frac{1}{3}\cdot\frac{b(1-b^2)}{(1+b^2)^3}$

Hence, $\frac{dA}{db} = 0$ gives $b = -1,\ 0,\ 1$ since $b > 0$

Therefore, we consider only $b = 1$

Sign scheme for $\frac{dA}{db}$ around $b = 1$ is as below



From sign scheme it is clear that A is maximum.

● **Ex. 25** *Let C_1 and C_2 be the graphs of the function $y = x^2$ and $y = 2x, 0 \le x \le 1$, respectively. Let C_3 be the graph of a function $y = f(x); 0 \le x \le 1, f(0) = 0$. For a point P on C_1, let the lines through P parallel to the axes, meets C_2 and C_3 at Q and R respectively. If for every position of P on (C_1), the areas of the shaded region OPQ and ORP are equal, determine the function $f(x)$.* ***[IIT JEE 1998]***

Sol. On the curve C_1, i.e. $y = x^2$.

Let P be (α, α^2). So, ordinate of point Q on C_2 is also α^2.

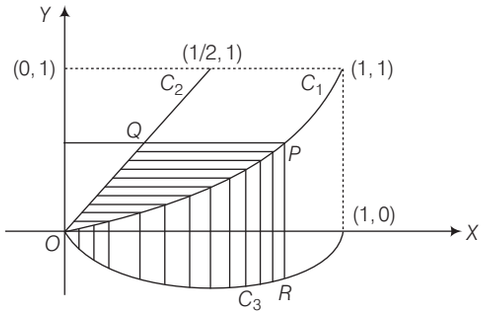
Now, C_2 ($y = 2x$) the abscissae of Q is given by $x = \dfrac{y}{2} = \dfrac{\alpha^2}{2}$.

$\therefore$ Q is $\left(\dfrac{\alpha^2}{2}, \alpha\right)$ and R on C_3 is $\{\alpha, f(x)\}$.

Now, area of $\Delta OPQ = \int_0^{\alpha^2} (x_1 - x_2)\, dy = \int_0^{\alpha^2} \left(\sqrt{y} - \dfrac{y}{2}\right) dy$

$$= \frac{2}{3}\alpha^3 - \frac{\alpha^4}{4} \qquad \ldots(i)$$

Again, area of $\Delta ORP = \int_0^{\alpha} (y_1 - y_2) dx = \int_0^{\alpha^2} \{x^2 - f(x)\}\, dx$ ...(ii)



Thus, from Eqs. (i) and (ii), we get

$$\frac{2\alpha^3}{3} - \frac{\alpha^4}{4} = \int_0^{\alpha} \{x^2 - f(x)\}\, dx$$

Differentiating both the sides w.r.t. α, we get

$$2\alpha^2 - \alpha^3 = \alpha^2 - f(\alpha)$$

$$\Rightarrow \qquad f(\alpha) = \alpha^3 - \alpha^2 \;\Rightarrow\; f(x) = x^3 - x^2$$

● **Ex. 26** *Find the area of the region bounded by the curves $y = ex \log x$ and $y = \dfrac{\log x}{ex}$.* ***[IIT JEE 1990]***

Sol. Both the curves are defined for $x > 0$. Both are positive when $x > 1$ and negative when $0 < x < 1$.

We know that, $\lim\limits_{x \to 0^+} \log x \to -\infty$

Therefore, $\lim\limits_{x \to 0^+} \dfrac{\log x}{ex} \to -\infty$

Thus, Y-axis is asymptote of second curve.

and $\lim\limits_{x \to 0^+} ex \log x$ [(0) $(-\infty)$ form]

$$= \lim_{x \to 0^+} \frac{e \log x}{1/x} \qquad \left(-\frac{\infty}{\infty} \text{ form}\right)$$

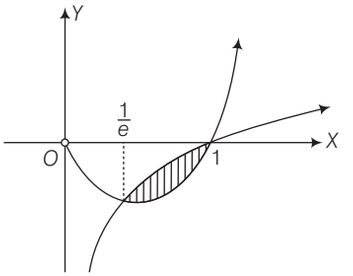
$$= \lim_{x \to 0^+} \frac{e(1/x)}{(-1/x^2)} = 0 \quad \text{(using L'Hospital's rule)}$$

Thus, the first curve starts from (0, 0) but does not include (0, 0). Now, the given curves intersect therefore

$$ex \log x = \frac{\log x}{ex}$$

i.e. $\qquad (e^2x^2 - 1) \log x = 0$

i.e. $\qquad x = 1, \dfrac{1}{e} \qquad (\because\ x > 0)$



Therefore, using the above results figure could be drawn as

$\therefore$ Required area $= \int_{1/e}^{1} \left(\dfrac{\log x}{ex} - ex \log x\right) dx$

$$= \frac{1}{e}\left[\frac{(\log x)^2}{2}\right]_{1/e}^{1} - e\left[\frac{x^2}{4}(2\log x - 1)\right]_{1/e}^{1} = \frac{e^2 - 5}{4e}$$

● **Ex. 27** *Let A_n be the area bounded by the curve $y = (\tan x)^n$ and the lines $x = 0, y = 0$ and $x = \dfrac{\pi}{4}$. Prove that for $n > 2$; $A_n + A_{n-2} = \dfrac{1}{n-1}$ and deduce that $\dfrac{1}{2n+2} < A_n < \dfrac{1}{2n-2}$.*

Sol. **First part** We have, $A_n = \int_0^{\pi/4} (\tan x)^n\, dx$

Hence, $\qquad A_{n-2} = \int_0^{\pi/4} (\tan x)^{n-2}\, dx$

$\therefore$ $\qquad A_n + A_{n-2} = \int_0^{\pi/4} (\tan x)^{n-2} (\tan^2 x + 1)\, dx$

$$= \int_0^{\pi/4} (\tan x)^{n-2} \cdot \sec^2 x\, dx$$

Let $\tan x = t$, so that $\sec^2 x\, dx = dt$

$$\therefore \qquad A_n + A_{n-2} = \int_0^1 t^{n-2}\, dt = \left(\frac{t^{n-1}}{n-1}\right)_0^1 = \frac{1}{n-1} \qquad \ldots(i)$$

Second part

Since, $\qquad 0 \le x \le \pi/4 \;\;\therefore\;\; 0 \le \tan x \le 1$

$$\Rightarrow \qquad \tan^{n+2} x < \tan^n x < \tan^{n-2} x$$

$$\Rightarrow \quad \int_0^{\pi/4} \tan^{n+2} x\, dx < \int_0^{\pi/4} \tan^n x\, dx < \int_0^{\pi/4} \tan^{n-2} x\, dx$$

$$\Rightarrow \qquad A_{n+2} < A_n < A_{n-2} \Rightarrow A_n + A_{n+2} < 2A_n < A_n + A_{n-2}$$

$$\therefore \quad \frac{1}{n+1} < 2A_n < \frac{1}{n-1}$$

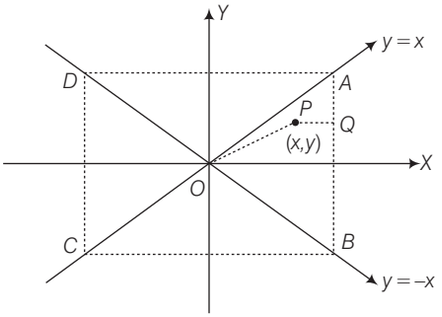
$$\Rightarrow \quad \frac{1}{2(n+1)} < A_n < \frac{1}{2(n-1)} \qquad \text{[using Eq. (i)]}$$

● ***Ex. 28*** *Consider a square with vertices at* (1, 1) (−1, 1), (−1, −1) *and* (1, −1). *Let S be the region consisting of all points inside the square which are nearer to the origin than to any edge. Sketch the region S and find its area.*

[IIT JEE 1995]

Sol. For the points lying in the ΔOAB the edge AB, i.e. $x = 1$ is the closest edge. Therefore, if the distance of a point $P(x, y)$ (lying in the ΔOAB) from origin is less than that of its distance from the edge $x = 1$ it will fall in the region S.

$$\therefore \quad OP \le PQ$$



$$\Rightarrow \quad \sqrt{x^2 + y^2} \le 1 - x$$

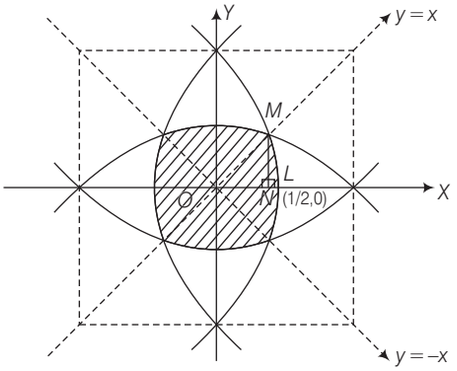
$$\Rightarrow \quad x^2 + y^2 \le x^2 - 2x + 1$$

$$\Rightarrow \quad y^2 \le 1 - 2x$$

Similarly, for points lying in the ΔOAD the side $y = 1$ is the closest side and therefore the region S is determined by

$$x^2 \le 1 - 2y$$

Since, the edges are symmetric about the origin. Hence, by the above inequality and by symmetry, the required area will be the shaded portion in the figure given below



Now, when the curves $y^2 = 1 - 2x$ and $y = x$ intersect each other, then

$$x^2 = 1 - 2x \;\Rightarrow\; x^2 + 2x - 1 = 0$$

$$\Rightarrow \quad x = \sqrt{2} - 1, -\sqrt{2} - 1$$

Hence, the intersection points in the first quadrant is $(\sqrt{2} - 1, \sqrt{2} - 1)$.

$\therefore$ Required area = 8 [Area of curvilinear ΔOLM]

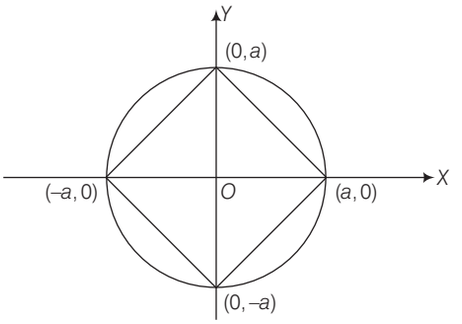
$$= 8\left[\frac{1}{2}(\sqrt{2}-1)(\sqrt{2}-1) + \int_{\sqrt{2}-1}^{1/2} \sqrt{1-2x}\, dx\right]$$

$$= 8\left[\frac{1}{2}(3 - 2\sqrt{2}) + \left(\frac{2(1-2x)^{3/2}}{3(-2)}\right)_{\sqrt{2}-1}^{1/2}\right]$$

$$= \frac{4}{3}(4\sqrt{2} - 5)$$

● ***Ex. 29*** *Sketch the region included between the curves* $x^2 + y^2 = a^2$ *and* $\sqrt{|x|} + \sqrt{|y|} = \sqrt{a}\ (a > 0)$ *and find its area.*

Sol. The graphs $|x| + |y| = a$ and $|x|^2 + |y|^2 = a^2$ are as shown in figure.



From the figure it can be concluded that when powers of $|x|$ and $|y|$ both is reduced to half the straight lines get stretched inside taking the shape as above.

Thus, required area = 4 [shaded area in the first quadrant]

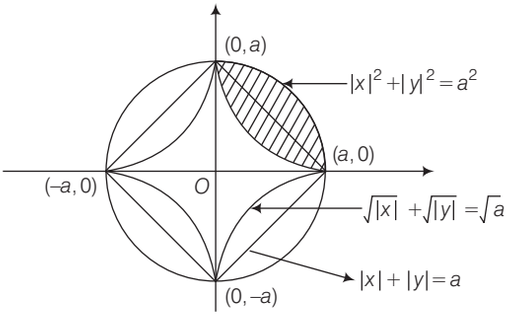
$$= 4\left[\frac{\pi a^2}{4} - \int_0^a (\sqrt{a} - \sqrt{x})^2\, dx\right]$$

(since in 1st quadrant $x, y > 0$), hence

$$\sqrt{|x|} + \sqrt{|y|} = \sqrt{a}$$

$$\Rightarrow \quad \sqrt{x} + \sqrt{y} = \sqrt{a}$$

$$\Rightarrow \quad y = (\sqrt{a} - \sqrt{x})^2$$



Hence, required the area $= \left(\pi - \frac{2}{3}\right) a^2$

● ***Ex. 30*** *Show that the area included between the parabolas* $y^2 = 4a(x+a)$ *and* $y^2 = 4b(b-x)$ *is* $\frac{8}{3}(a+b)\sqrt{ab}$.

Sol. Given parabolas are

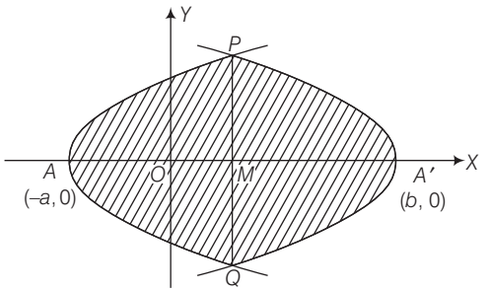
$$y^2 = 4a(x+a) \quad \ldots(i)$$

and $$y^2 = 4b(b-x) \quad \ldots(ii)$$

Solving Eqs. (i) and (ii), we get

$$x = (b-a) \quad \text{and} \quad y = \pm 2\sqrt{ab}$$

$\therefore$ P and Q are $(b-a, 2\sqrt{ab})$ and $(b-a, -2\sqrt{ab})$ respectively, and the points A and A' are $(-a, 0)$ and $(b, 0)$, respectively.



Now, required area = Area $APA'QA$ = 2 Area $APA'A$

$$= 2[\text{Area } APMA + \text{Area } MPA'M]$$

$$= 2\left[\int_{-a}^{b-a} 2\sqrt{a(a+x)}\,dx + \int_{b-a}^{b} 2\sqrt{b(b-x)}\,dx\right]$$

$$= 4\sqrt{a}\int_{-a}^{b-a}\sqrt{a+x}\,dx + 4\sqrt{b}\int_{b-a}^{b}\sqrt{b-x}\,dx$$

$$= 4\sqrt{a}\left[\frac{2}{3}(a+x)^{3/2}\right]_{-a}^{b-a} + 4\sqrt{b}\left[-\frac{2}{3}(b-x)^{3/2}\right]_{b-a}^{b}$$

$$= \frac{8}{3}\sqrt{a}\,(b)^{3/2} + \frac{8\sqrt{b}}{3}(a)^{3/2} = \frac{8}{3}\sqrt{ab}(a+b) \text{ sq units}$$

● ***Ex. 31*** *Determine the area of the figure bounded by two branches of the curve* $(y-x)^2 = x^3$ *and the straight line* $x = 1$.

Sol. Given curves are $(y-x)^2 = x^3$

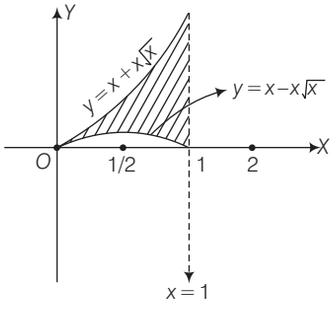
$$y - x = \pm x\sqrt{x}$$

$$y = x + x\sqrt{x} \quad \ldots(i)$$

$$y = x - x\sqrt{x} \quad \ldots(ii)$$

and $$x = 1 \quad \ldots(iii)$$

Which could be drawn as; shown in figure.



Hence, required area

$$= \int_0^1 \{(x + x\sqrt{x}) - (x - x\sqrt{x})\}\,dx$$

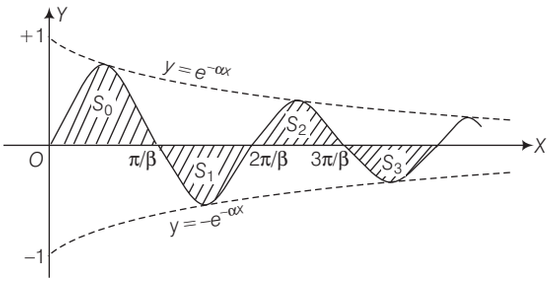
$$= \int_0^1 (2x\sqrt{x})\,dx$$

$$= 2\int_0^1 x^{3/2}\,dx = \frac{4}{5} \text{ sq unit}$$

● ***Ex. 32*** *Prove that the areas* $S_0, S_1, S_2, \ldots$ *bounded by the X-axis and half-waves of the curve* $y = e^{-\alpha x}\sin\beta x$, $x \mid 0$. *form a geometric progression with the common ratio* $g = e^{-\pi\alpha/\beta}$.

Sol. The curve $y = e^{-\alpha x}\sin\beta x$ intersects the positive X-axis at the points where $y = 0$.

$\therefore$ $$e^{-\alpha x}\sin\beta x = 0$$



$$\Rightarrow \quad \sin\beta x = 0 \;\Rightarrow\; x_n = \frac{n\pi}{\beta},\ n = 0, 1, 2, \ldots$$

The function $y = e^{-\alpha x}\sin\beta x$ is positive in the interval (x_{2K}, x_{2K+1}) and negative in (x_{2K+1}, x_{2K+2}), i.e. the sign of the function in the interval (x_n, x_{n+1}), therefore

$$S_n = \left|\int_{n\pi/\beta}^{(n+1)\pi/\beta} e^{-\alpha x}\sin\beta x\,dx\right|$$

$$= \left|\left\{\frac{(-1)^{n+1}e^{-\alpha x}}{\alpha^2+\beta^2}(\alpha\sin\beta x + \beta\cos\beta x)\right\}_{n\pi/\beta}^{(n+1)\pi/\beta}\right|$$

$$= \frac{\beta e^{-n\pi\alpha/\beta}}{(\alpha^2+\beta^2)}\{1 + e^{-\pi\alpha/\beta}\}$$

Hence, $$g = \frac{S_{n+1}}{S_n} = \frac{\dfrac{\beta e^{-(n+1)\pi\alpha/\beta}\{1+e^{-\pi\alpha/\beta}\}}{(\alpha^2+\beta^2)}}{\dfrac{\beta e^{-n\pi\alpha/\beta}\{1+e^{-\pi\alpha/\beta}\}}{\alpha^2+\beta^2}} = e^{-\pi\alpha/\beta}$$

which completes the proof.

● ***Ex. 33*** *Let* $b \neq 0$ *and for* $j = 0, 1, 2, \ldots, n$. *Let* S_j *be the area of the region bounded by Y-axis and the curve* $x \cdot e^{ay} = \sin by$, $\frac{j\pi}{b} \le y \le \frac{(j+1)\pi}{b}$. *Show that* $S_0, S_1, S_2, \ldots, S_n$ *are in geometric progression. Also, find their sum for* $a = -1$ *and* $b = \pi$.

Sol. Here, $$S_j = \left|\int_{j\pi/b}^{(j+1)\pi/b} x\,dy\right| = \left|\int_{j\pi/b}^{(j+1)\pi/b} e^{-ay}\sin by\,dy\right|$$

$$= \left| \left\{ \frac{e^{-ay}}{a^2+b^2}(-a\sin by - b\cos by) \right\}_{j\pi/b}^{(j+1)\pi/b} \right|$$

$$= \left| \frac{e^{-a(j+1)\pi/b}}{a^2+b^2} \times (-b)(-1)^{j+1} - \frac{e^{-aj\pi/b}}{a^2+b^2}(-b)(-1)^{j} \right|$$

$$= |b| \left\{ \frac{e^{-a(j+1)\pi/b}}{a^2+b^2} + \frac{e^{-aj\pi/b}}{a^2+b^2} \right\}$$

$$= |b| \frac{e^{-aj\pi/b}}{a^2+b^2}\{e^{-a\pi/b}+1\};\ \{j = 0, 1, 2, ..., n\}$$

Now, $\frac{S_{j+1}}{S_j} = |b| \frac{e^{-a(j+1)\pi/b}}{a^2+b^2}\{e^{-a\pi/b}+1\} \mid b \mid \frac{e^{-aj\pi/b}}{a^2+b^2}\{e^{-a\pi/b}+1\}$

$= e^{-a\pi/b}$ for all $j = 0, 1, 2, ..., n$

Hence, $S_0, S_1, S_2, ..., S_n$ are in GP with common ratio $e^{-a\pi/b}$.

For $a = -1$ and $b = \pi$, we have

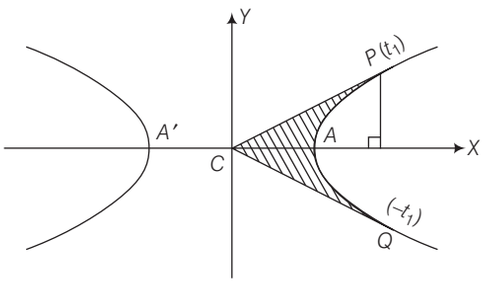
$$S_j = \frac{e^j \pi (e+1)}{\pi^2+1};\ j = 0, 1, 2, ..., n$$

$$\therefore \quad \sum_{j=0}^{n} S_j = \sum_{j=0}^{n} \frac{e^{j\pi}(e+1)}{\pi^2+1} = \frac{(e+1)\pi}{\pi+1}\left\{\frac{e^{n+1}-1}{e-1}\right\}$$

● ***Ex. 34*** *For any real* $t, x = 2 + \frac{e^t + e^{-t}}{2}, y = 2 + \frac{e^t - e^{-t}}{2}$ *is a point on the hyperbola* $x^2 - y^2 - 4x + 4y - 1 = 0$. *Find the area bounded by the hyperbola and the lines joining the centre to the points corresponding to* t_1 *and* $-t_1$.

Sol. The points $x = 2 + \frac{e^t + e^{-t}}{2}, y = 2 + \frac{e^t - e^{-t}}{2}$ is on the curve

$(x-2)^2 - (y-2)^2 = 1$ or $x^2 - y^2 - 4x + 4y - 1 = 0$



Put $(x-2) = x, (y-2) = y$

$$\therefore \quad x^2 - y^2 = 1 \quad \text{and} \quad x = \frac{e^t + e^{-t}}{2}, y = \frac{e^t - e^{-t}}{2}$$

We have to find the area of the region bounded by the curve $x^2 - y^2 = 1$ and the lines joining the centre $x = 0, y = 0$ to the point (t_1) and $(-t_1)$.

$$\therefore \text{Required area} = 2\left[\text{Area of } \Delta PCN - \int_1^{\frac{e^{t_1}+e^{-t_1}}{2}} y\,dx\right]$$

$$= 2\left[\frac{1}{2}\left(\frac{e^{t_1}+e^{-t_1}}{2}\right)\left(\frac{e^{t_1}-e^{-t_1}}{2}\right) - \int_1^{t_1} y\frac{dx}{dt}\,dt\right]$$

$$= 2\left[\left(\frac{e^{2t_1}-e^{-2t_1}}{8}\right) - \int_0^{t_1}\left(\frac{e^t - e^{-t}}{2}\right)^2 dt\right]$$

$$= \frac{e^{2t_1}-e^{-2t_1}}{4} - \frac{1}{2}\int_0^{t_1}(e^{2t} + e^{-2t} - 2)\,dt$$

$$= \frac{e^{2t_1}-e^{-2t_1}}{4} - \frac{1}{2}\left[\frac{e^{2t_1}}{2} - \frac{e^{-2t_1}}{2} - 2t_1\right]_0^{t_1}$$

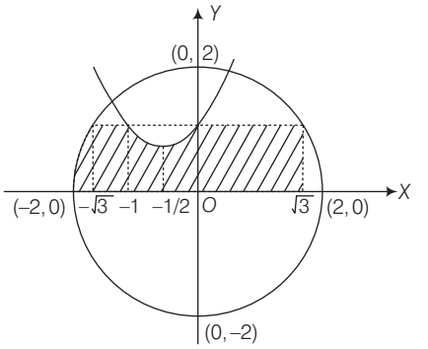
$$= \frac{e^{2t_1}-e^{-2t_1}}{4} - \frac{1}{2}\left[\frac{e^{2t_1}}{2} - \frac{e^{-2t_1}}{2} - 2t_1\right] = t_1$$

● ***Ex. 35*** *Find the area enclosed by circle* $x^2 + y^2 = 4$, *parabola* $y = x^2 + x + 1$, *the curve* $y = \left[\sin^2\frac{x}{4} + \cos\frac{x}{4}\right]$ *and X-axis (where, [.] is the greatest integer function).*

Sol. $\because \quad y = \left[\sin^2\frac{x}{4} + \cos\frac{x}{4}\right]$

$\because \quad 1 < \sin^2\frac{x}{4} + \cos\frac{x}{4} < 2 \quad$ for $x \in (-2, 2]$

$\therefore \quad y = \left[\sin^2\frac{x}{4} + \cos\frac{x}{4}\right] = 1$



Now, we have to find out the area enclosed by the circle $x^2 + y^2 = 4$, parabola $\left(y - \frac{3}{4}\right) = \left(x + \frac{1}{2}\right)^2$, line $y = 1$ and X-axis. Required area is shaded area in the figure.

Hence, required area

$$= \sqrt{3}\times 1 + (\sqrt{3}-1)\times 1 + \int_{-1}^{0}(x^2 + x + 1)\,dx + 2\int_{\sqrt{3}}^{2}(\sqrt{4-x^2})\,dx$$

$$= (2\sqrt{3}-1) + \left[\frac{x^3}{3} + \frac{x^2}{2} + x\right]_{-1}^{0} + 2\left[\frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\left(\frac{x}{2}\right)\right]_{\sqrt{3}}^{2}$$

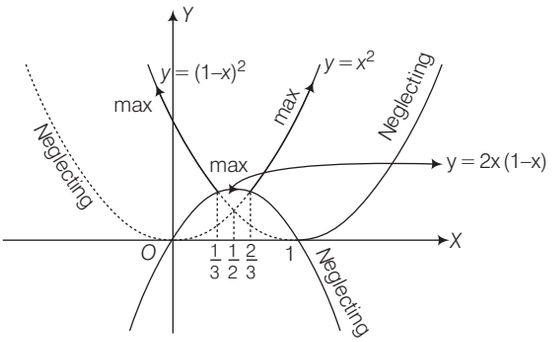
$$= (2\sqrt{3}-1) + \left[0 - \left(-\frac{1}{3} + \frac{1}{2} - 1\right)\right] + 2\left[(0+\pi) - \left(\frac{\sqrt{3}}{2} + \frac{2\pi}{3}\right)\right]$$

$$= (2\sqrt{3}-1) + \frac{5}{6} + \frac{2\pi}{3} - \sqrt{3} = \left(\frac{2\pi}{3} + \sqrt{3} - \frac{1}{6}\right) \text{ sq units}$$

● ***Ex. 36*** *Let* $f(x) = \max\{x^2, (1-x)^2, 2x(1-x)\}$, *where* $0 \le x \le 1$. *Determine the area of the region bounded by the curves* $y = f(x)$, *X-axis,* $x = 0$ *and* $x = 1$. ***[IIT JEE 1997]***

Sol. We have, $f(x) = \max\{x^2, (1-x^2), 2x(1-x)\}$

Graphically it could be shown as;



For figure it is clear that maximum graph (i.e. above max graph is considered and others are neglected).

$\therefore$ For $x \in \left[0, \frac{1}{3}\right]$, $x^2 \le 2x(1-x) \le (1-x)^2$

For $x \in \left[\frac{1}{3}, \frac{1}{2}\right]$, $x^2 \le (1-x)^2 \le 2x(1-x)$

For $x \in \left[\frac{1}{2}, \frac{2}{3}\right]$, $(1-x)^2 \le x^2 \le 2x(1-x)$

For $x \in \left[\frac{2}{3}, 1\right]$, $(1-x)^2 \le 2x(1-x) \le x^2$

Hence, $f(x)$ can be written as

$$f(x) = \begin{cases} (1-x)^2, & \text{for } 0 \le x \le 1/3 \\ 2x(1-x), & \text{for } 1/3 \le x \le 2/3 \\ x^2, & \text{for } 2/3 \le x \le 1 \end{cases}$$

Hence, the area bounded by the curve $y = f(x)$; X-axis and the lines $x = 0$ and $x = 1$ is given by

$$= \int_0^{1/3} (1-x)^2\,dx + \int_{1/3}^{2/3} 2x(1-x)\,dx + \int_{2/3}^{1} (x^2)\,dx$$

$$= \frac{17}{27} \text{ sq unit}$$

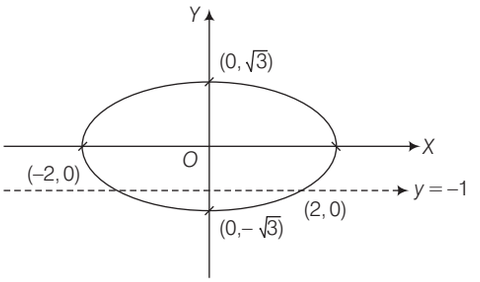
● ***Ex. 37*** *Find the ratio in which the curve,* $y = [-0.01x^4 - 0.02x^2]$ *[where, [·] denotes the greatest integer function) divides the ellipse* $3x^2 + 4y^2 = 12$.

Sol. Here, $y = [-0.01x^4 - 0.02x^2]$

i.e. $y = -1$, when $-2 < x < 2$

$y = -1$ cut the ellipse $3x^2 + 4y^2 = 12$

At $x^2 = \frac{8}{3}$ or $x = \pm\frac{2\sqrt{2}}{\sqrt{3}}$



$$\text{Required area} = \int_{-2\sqrt{6}/3}^{2\sqrt{6}/3} \left\{\sqrt{\frac{12-3x^2}{4}} - 1\right\} dx$$

$$= \frac{\sqrt{3}}{2}\int_{-2\sqrt{6}/3}^{2\sqrt{6}/3} \sqrt{4-x^2}\,dx - \int_{-2\sqrt{6}/3}^{2\sqrt{6}/3} 1\,dx$$

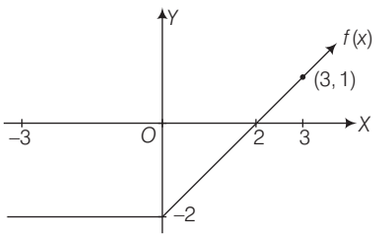
$$= 2\sin^{-1}\sqrt{\frac{2}{3}} + \frac{2}{3} - \frac{4}{3}\sqrt{6}$$

● ***Ex. 38*** *Let* $f(x) = \begin{cases} -2, -3 \le x \le 0 \\ x - 2, 0 < x \le 3 \end{cases}$, *where* $g(x) = \min\{f(|x|) + |f(x)|, f(|x|) - |f(x)|\}$. *Find the area bounded by the curve* $g(x)$ *and the X-axis between the ordinates at* $x = 3$ and $x = -3$.

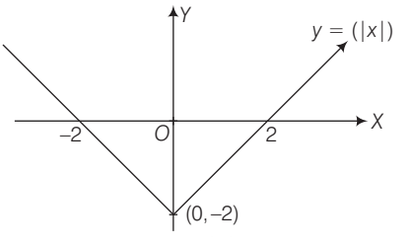
Sol. Here, $f(|x|) = \begin{cases} -x - 2, -3 \le x \le 0 \\ x - 2, 0 < x \le 3 \end{cases}$

$$|f(x)| = \begin{cases} 2, & -3 \le x \le 0 \\ -x + 2, & 0 < x \le 2 \\ x - 2, & 2 < x \le 3 \end{cases}$$

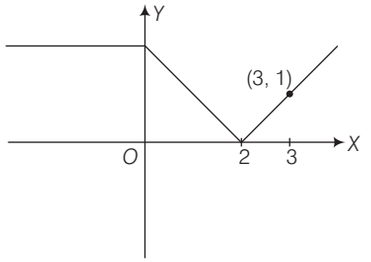
$$\therefore\ f(|x|) - |f(x)| = \begin{cases} -x - 4, -3 \le x \le 0 \\ 2x - 4, & 0 < x \le 2 \\ 0, & 2 < x \le 3 \end{cases}$$



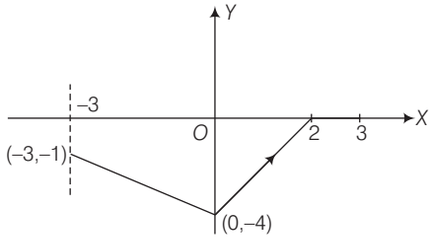
Graph of $f(x)$



Graph of $f(|x|)$



Graph of $|f(x)|$



Graph of $g(x)$

Since, $|f(x)|$ is always positive.

$$g(x) = f(|x|) - |f(x)|$$

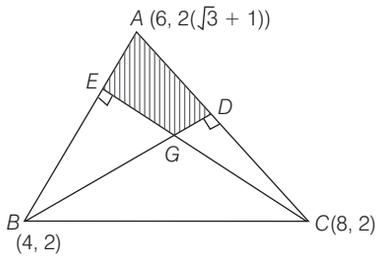
where the graphs could be drawn as shown in above figures.

From the graph, required area

$$= \frac{1}{2}(1+4)\times 3 + \left(\frac{1}{2}\times 2\times 4\right) + 0 = \frac{23}{2} \text{ sq units}$$

● ***Ex. 39*** *Let ABC be a triangle with vertices* $A \equiv (6, 2(\sqrt{3}+1)), B \equiv (4, 2)$ *and* $C \equiv (8, 2)$. *Let R be the region consisting of all those points P inside* ΔABC *which satisfy* $d(P, BC) \geq \max\{d(P, AB), d(P, AC)\}$, where $d(P, L)$ denotes the distance of the point P from the line L. Sketch the region R and find its area.

Sol. It is easy to see that ABC is an equilateral triangle with side of length 4. BD and CE are angle bisectors of angle B and C, respectively. Any point inside the ΔAEC is nearer to AC than BC and any point inside the ΔBDA is nearer to AB than BC. So any point inside the quadrilateral $AEGC$ will satisfy the given condition. Hence, shaded region is the required region, whose area is to be found, shown as in figure



Thus, required area $= 2$ Area of $\Delta EAG = 2\times\frac{1}{2}\,AE\times EG$

$$= \frac{1}{2}\,AB\times\frac{1}{3}\,CE = \frac{1}{6}\times 4\times\sqrt{4^2-2^2}$$

$$= \frac{4\sqrt{3}}{3} \text{ sq units}$$

● ***Ex. 40*** *Let* $O(0,0)$, $A(2,0)$ *and* $B\left(1, \frac{1}{\sqrt{3}}\right)$ *be the vertices of a triangle. Let R be the region consisting of all those points P inside* ΔOAB *which satisfy* $d(P, OA) \leq min\{d(P, OB), d(P, AB)\}$, *when 'd' denotes the distance from the point to the corresponding line. Sketch the region R and find its area.* ***[IIT JEE 1997]***

Sol. Let the coordinate of P be (x, y).

Equation of line $OA \equiv y = 0$

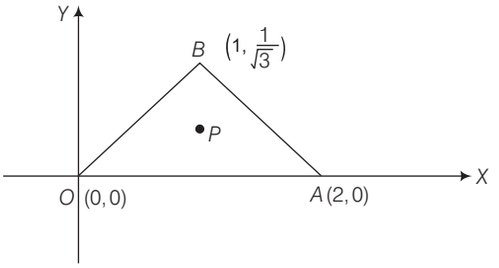
Equation of line $OB \equiv \sqrt{3}y = x$

Equation of line $AB = \sqrt{3}y = 2 - x$

$d(P, OA) =$ Distance of P from line $OA = y$

$d(P, OB) =$ Distance of P from line $OB = \dfrac{|\sqrt{3}y - x|}{2}$

$d(P, AB) =$ Distance of P from line $AB = \dfrac{|\sqrt{3}y + x - 2|}{2}$



Given, $d(P, OA) \leq \min\{d(P, OB), d(P, AB)\}$

$$y \leq \min\left\{\frac{|\sqrt{3}y - x|}{2}, \frac{|\sqrt{3}y + x - 2|}{2}\right\}$$

$$\Rightarrow \quad y \leq \frac{|\sqrt{3}y - x|}{2} \qquad \text{...(i)}$$

and $$y \leq \frac{|\sqrt{3}y + x - 2|}{2} \qquad \text{...(ii)}$$

Case I If $y \leq \dfrac{|\sqrt{3}y - x|}{2}$

$$\therefore \quad y \leq \frac{x - \sqrt{3}y}{2}, \text{ i.e. } x > \sqrt{3}y \qquad (\because \sqrt{3}y - x < 0)$$

$$\Rightarrow \quad (2+\sqrt{3})\,y \leq x$$

$$\Rightarrow \quad y \leq (2-\sqrt{3})\,x$$

$$\Rightarrow \quad y \leq x\tan 15^\circ \qquad \text{...(iii)}$$

$(\because y = x\tan 15^\circ$ is an acute angle bisector of $\angle AOB)$

Case II If $y \leq \dfrac{|\sqrt{3}y + x - 2|}{2}$

$$\Rightarrow \quad 2y \leq 2 - x - \sqrt{3}y \qquad (\text{i.e. } \sqrt{3}y + x - 2 < 0)$$

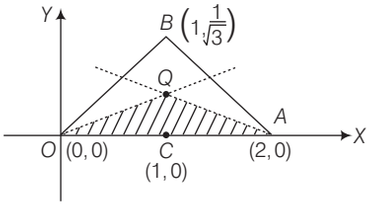
$$\Rightarrow \quad (2+\sqrt{3})\,y \leq 2 - x$$

$$\Rightarrow \quad y \leq -(2-\sqrt{3})\,(x-2)$$

$$\Rightarrow \quad y \leq -(\tan 15^\circ)\,(x-2) \qquad \text{...(iv)}$$

$[\because y = (x-2)\tan 15^\circ$ is an acute angle bisector of $CA]$

From Eqs. (iii) and (iv), P moves inside the triangle as shown in figure.



As $\angle QOB = \angle OBQ = 15°$, ΔOQB is an isosceles triangle

$\Rightarrow$ $OC = AC = 1$ unit

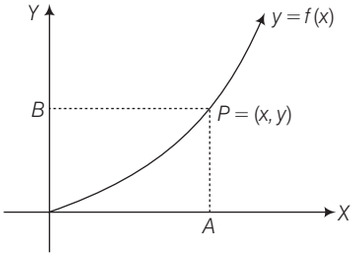
Area of shaded region = area of $\Delta OQA = \frac{1}{2}$ (base) $\times$ (height)

$$= \frac{1}{2}(2)(1 \tan 15°) = \tan 15°$$

$$= (2 - \sqrt{3}) \text{ sq units}$$

● ***Ex. 41*** *A curve $y = f(x)$ passes through the origin and lies entirely in the first quadrant. Through any point $P(x, y)$ on the curve, lines are drawn parallel to the coordinate axes. If the curve divides the area formed by these lines and coordinate axes in $m : n$, find $f(x)$.*

Sol. Area of $(OAPB) = xy$, Area of $(OAPO) = \int_0^x f(t)\,dt$



Therefore, area of $(OBPO) = xy - \int_0^x f(t)\,dt$

According to the given condition,

$$\frac{xy - \int_0^x f(t)\,dt}{\int_0^x f(t)\,dt} = \frac{m}{n}$$

$\Rightarrow$ $n\,xy = (m + n)\int_0^x f(t)\,dt$

Differentiating w.r.t. x, we get

$$n\left(x\frac{dy}{dx} + y\right) = (m + n)\,f(x) = (m + n)\,y, \text{ as } y = f(x)$$

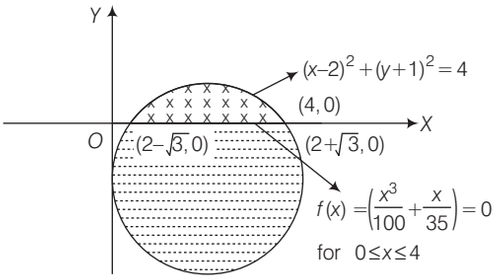
$$\frac{m}{n} \cdot \frac{dx}{x} = \frac{dy}{y} \Rightarrow \frac{m}{n} \cdot (\log x) = \log y - \log c, \text{ where } c \text{ is a constant.}$$

$\Rightarrow$ $y = cx^{m/n}$

● ***Ex. 42*** *Find the ratio of the areas in which the curve $y = \left[\frac{x^3}{100} + \frac{x}{35}\right]$ divides the circle $x^2 + y^2 - 4x + 2y + 1 = 0$ (where, [.] denotes the greatest integer function).*

Sol. We have, $x^2 + y^2 - 4x + 2y + 1 = 0$

or $(x - 2)^2 + (y + 1)^2 = 4$ ...(i)



Now, for $0 \le x \le 4$,

$$0 \le \frac{x^3}{100} + \frac{x}{35} < 1 \Rightarrow \left[\frac{x^3}{100} + \frac{x}{35}\right] = 0$$

So, we have to find out the ratio in which X-axis divides the circle (i).

Now, at X-axis, $y = 0$

So, $(x - 2)^2 = 3$

So, it cuts the X-axis at $(2 - \sqrt{3}, 0)$ and $(2 + \sqrt{3}, 0)$.

Therefore, required area, $A = \int_{2-\sqrt{3}}^{2+\sqrt{3}} (\sqrt{4 - (x - 2)^2} - 1)\,dx$

$$= \frac{4\pi - 3\sqrt{3}}{3}$$

$\therefore$ Required ratio $= \frac{A}{4\pi - A} = \frac{4\pi - 3\sqrt{3}}{8\pi + 3\sqrt{3}}$

● ***Ex. 43*** *Area bounded by the line $y = x$, curve $y = f(x), (f(x) > x, \forall x > 1)$ and the lines $x = 1, x = t$ is $(t + \sqrt{1 + t^2}) - (1 + \sqrt{2})$ for all $t > 1$. Find $f(x)$.*

Sol. The area bounded by $y = f(x)$ and $y = x$ between the lines $x = 1$ and $x = t$ is $\int_{+1}^{t} (f(x) - x)\,dx$. But it is equal to $(t + \sqrt{1 + t^2}) - (1 + \sqrt{2})$.

So, $\int_1^t (f(x) - x)\,dx = (t + \sqrt{1 + t^2}) - (1 + \sqrt{2})$

Differentiating both the sides w.r.t. t, we get

$$f(t) - t = 1 + \frac{t}{\sqrt{1 + t^2}} \Rightarrow f(t) = 1 + t + \frac{t}{\sqrt{1 + t^2}}$$

or $f(x) = 1 + x + \frac{x}{\sqrt{1 + x^2}}$

● ***Ex. 44*** *The area bounded by the curve $y = f(x)$, X-axis and ordinates $x = 1$ and $x = b$ is $(b - 1)\sin(3b + 4)$, find $f(x)$.*

Sol. We know that the area bounded by the curve $y = f(x)$, X-axis and the ordinates $x = 1$ and $x = b$ is $\int_1^b f(x)\,dx$.

From the question; $\int_1^b f(x)\,dx = (b - 1)\sin(3b + 4)$

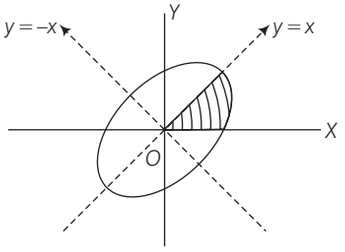
Differentiating w.r.t. b, we get

$$f(b) \cdot 1 = 3(b - 1)\cos(3b + 4) + \sin(3b + 4)$$

$\Rightarrow$ $f(x) = 3(x - 1)\cos(3x + 4) + \sin(3x + 4)$

● ***Ex. 45*** *Find the area of region enclosed by the curve* $\frac{(x-y)^2}{a^2}+\frac{(x+y)^2}{b^2}=2\,(a>b)$, the line $y=x$ and the positive X-axis.

Sol. The given curve $\frac{(x-y)^2}{a^2}+\frac{(x+y)^2}{b^2}=2$ is an ellipse major and minor axes are $x-y=0$ and $x+y=0$, respectively.
The required area is shown with shaded region.



Instead of directly solving the problem we can solve equivalent problem with equivalent ellipse whose axes are $x=0$ and $y=0$.
The equivalent region is shown as $(OA'B'O)$ where the equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

$\therefore$ Required area = Area $(\Delta OA'A''+A'A''B')$

where, $A'=\left(\frac{ab}{\sqrt{a^2+b^2}},\frac{ab}{\sqrt{a^2+b^2}}\right)$

Area $\Delta OA'A''=\frac{1}{2}\times\frac{ab}{\sqrt{a^2+b^2}}\times\frac{ab}{\sqrt{a^2+b^2}}=\frac{1}{2}\left(\frac{a^2b^2}{a^2+b^2}\right)$ ...(i)

Area of $A'B'A''=\int_{\frac{ab}{\sqrt{a^2+b^2}}}^{a} b\sqrt{\left(1-\frac{x^2}{a^2}\right)}\,dx$

$=\frac{b}{a}\cdot\int_{\frac{ab}{\sqrt{a^2+b^2}}}^{a}\sqrt{a^2-x^2}\,dx$

$=\frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{ab/\sqrt{a^2+b^2}}^{a}$

$=\frac{b}{a}\left[0+\frac{a^2}{2}\cdot\frac{\pi}{2}-\frac{ab}{2\sqrt{a^2+b^2}}\cdot\sqrt{\left(a^2-\frac{a^2b^2}{a^2+b^2}\right)}\right.$
$\left.-\frac{a^2}{2}\sin^{-1}\frac{b}{\sqrt{a^2+b^2}}\right]$

$=\frac{b}{a}\left[\frac{\pi a^2}{4}-\frac{a^2}{2}\sin^{-1}\left(\frac{b}{\sqrt{a^2+b^2}}\right)-\frac{a^3b}{2(a^2+b^2)}\right]$

$=\frac{\pi ab}{4}-\frac{ab}{2}\sin^{-1}\left(\frac{b}{\sqrt{a^2+b^2}}\right)-\frac{a^2b^2}{2(a^2+b^2)}$ ...(ii)

Hence, required area = Sum of Eqs. (i) and (ii)

$=\frac{\pi ab}{4}-\frac{ab}{2}\sin^{-1}\left(\frac{b}{\sqrt{a^2+b^2}}\right)$

● ***Ex. 46*** *Let* $f(x)$ *be a function which satisfy the equation* $f(xy)=f(x)+f(y)$ *for all* $x>0, y>0$ *such that* $f'(1)=2$. *Find the area of the region bounded by the curves* $y=f(x), y=|x^3-6x^2+11x-6|$ *and* $x=0$.

Sol. Take $x=y=1\Rightarrow f(1)=0$

Now, $y=\frac{1}{x}$

$\Rightarrow\quad 0=f\left(x\cdot\frac{1}{x}\right)=f(x)+f\left(\frac{1}{x}\right)\Rightarrow f\left(\frac{1}{x}\right)=-f(x)$

$\therefore f\left(\frac{x}{y}\right)=f(x)+f\left(\frac{1}{y}\right)=f(x)-f(y)$ ...(i)

Now, $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$=\lim_{h\to 0}\frac{f\left(\frac{x+h}{x}\right)}{h}$ [using Eq. (i)]

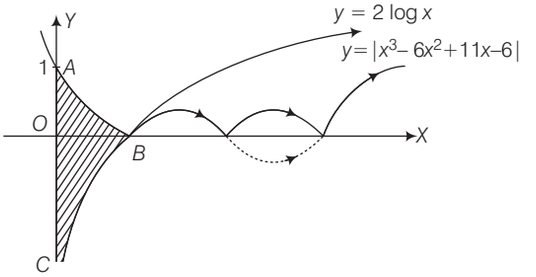
$=\lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)-f(1)}{\frac{h}{x}\cdot x}=\frac{f'(1)}{x}$ [$\because f(1)=0$]

$f'(x)=\frac{2}{x}$

$\Rightarrow\quad f(x)=2\log x+c$ [since, $f(1)=0\Rightarrow c=0$]

$\Rightarrow\quad f(x)=2\log x$

Thus, $f(x)=2\log x$ and $y=|x^3-6x^2+11x-6|$ could be plotted as



Hence, required area

$=\int_0^1(x^3-6x^2+11x-6)\,dx+\int_{-\infty}^{0}e^{y/2}\,dy$

$=\left(\frac{x^4}{4}-\frac{6x^3}{3}+\frac{11x^2}{2}-6x\right)_0^1+2\,(e^{y/2})_{-\infty}^{0}$

$=\left(\frac{1}{4}-2+\frac{11}{2}-6\right)-(0)+2\,(e^0-e^{-\infty})$

$=\frac{1}{4}+\frac{11}{2}-8+2=-\frac{1}{4}$ sq unit

● ***Ex. 47*** *Find the area of the region which contains all the points satisfying condition* $|x-2y|+|x+2y|\le 8$ *and* $xy\ge 2$.

Sol. The line $y=\pm\frac{x}{2}$ divide the xy plane in four parts

Region I $2y-x\le 0$ and $2y+x\ge 0$

So that, $|x-2y|+|x+2y|\le 8$

$\Rightarrow\quad (x-2y)+(x+2y)\le 8\ \Rightarrow\ 0\le x\le 4$

Region II $2y - x \mid 0$ and $2y + x \geq 0$

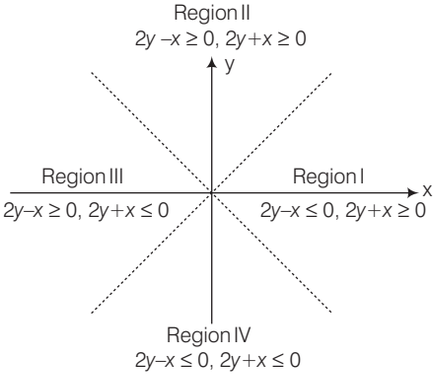
So that, $\quad | x - 2y | + | x + 2y | \leq 8$

$\Rightarrow \quad -(x - 2y) + (x + 2y) \leq 8 \Rightarrow 0 \leq y \leq 2$

Region III $2y + x \leq 0, 2y - x \geq 0$

So that, $\quad | x - 2y | + | x + 2y | \leq 8$

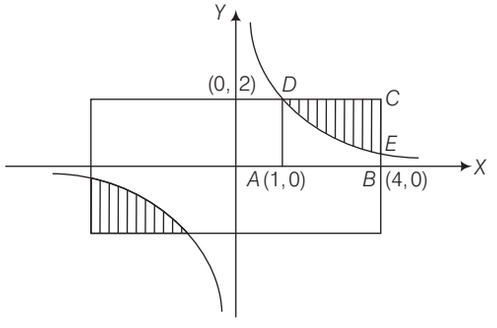
$\Rightarrow \quad -(x - 2y) - (x + 2y) \leq 8 \Rightarrow -4 \leq x \leq 0$



Region IV $2y + x \leq 0, 2y - x \leq 0$

So that, $\quad | x - 2y | + | x + 2y | \leq 8 \Rightarrow -2 \leq y \leq 0$

Here, all the points lie in the rectangle.



Also, the hyperbola $xy = 2$ meets the sides of the rectangle at the points (1, 2) and (4, 1 / 2) in the 1st quadrant graphically.

Hence, required area

$= 2$ (Area of rectangle $ABCD$ − Area of $ABEDA$)

$$= 2\left(3 \times 2 - \int_1^4 \frac{2}{x}\, dx \right) = 2 (6 - 2 \log 4) \text{ sq units}$$

● ***Ex. 48*** *Consider the function*

$$f(x) = \begin{cases} x - [x] - \dfrac{1}{2}, & \text{if } x \notin I \\ 0, & \text{if } x \in I \end{cases}$$

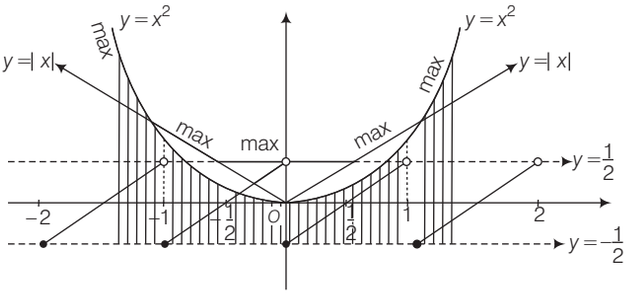
, where [.] denotes the greatest integer function and I is the set of integers. If $g(x) = \max \{x^2, f(x), | x |\}, -2 \leq x \leq 2$, *then find the area bounded by* $g(x)$ *when* $-2 \leq x \leq 2$.

Sol. Here, $\quad f(x) = \begin{cases} x - [x] - \dfrac{1}{2}, & \text{if } x \notin I \\ 0, & \text{if } x \in I \end{cases}$

$$= \begin{cases} \{x\} - \dfrac{1}{2}, & \text{if } x \notin I \\ 0, & \text{if } x \in I \end{cases}$$

Thus, $\quad g(x) = \max \{x^2, f(x), | x |\}$

which could be graphically expressed as



Clearly, $\quad g(x) = \begin{cases} x^2, & -2 \leq x \leq -1 \\ -x, & -1 \leq x \leq -1/4 \\ x + \dfrac{1}{2}, & -1/4 \leq x \leq 0 \\ x, & 0 \leq x \leq 1 \\ x^2, & 1 \leq x \leq 2 \end{cases}$

Hence, required area $= \int_{-2}^{2} g(x)\, dx$

$$= \int_{-2}^{-1} (x^2)\, dx + \int_{-1}^{-1/4} (-x)\, dx + \int_{-1/4}^{0} \left(x + \frac{1}{2} \right) dx + \int_0^1 x\, dx + \int_1^2 x^2\, dx$$

$$= \left(\frac{x^3}{3} \right)_{-2}^{-1} - \left(\frac{x^2}{2} \right)_{-1}^{-1/4} + \left(\frac{x^2}{2} + \frac{1}{2} x \right)_{-1/4}^{0} + \left(\frac{x^2}{2} \right)_0^1 + \left(\frac{x^3}{3} \right)_1^2 = \frac{275}{48} \text{ sq units}$$

● ***Ex. 49*** *Find the area of the region bounded by* $y = f(x), y = | g(x) |$ *and the lines* $x = 0, x = 2$, *where* f, g *are continuous functions satisfying*

$$f(x + y) = f(x) + f(y) - 8xy, \ \forall\, x, y \in R$$

and $\quad g(x + y) = g(x) + g(y) + 3xy\,(x + y), \ \forall\, x, y \in R$

Also, $f'(0) = 8$ and $g'(0) = -4$.

Sol. Here, $f(x + y) = f(x) + f(y) - 8xy$

Replacing $x, y \to 0$, we get $f(0) = 0$

Now, $\quad f'(x) = \lim_{h \to 0} \dfrac{f(x + h) - f(x)}{h} = \lim_{y \to 0} \dfrac{f(x + y) - f(x)}{y}$

$$= \lim_{y \to 0} \frac{f(x) + f(y) - 8xy - f(x)}{y}$$

$$= \lim_{y \to 0} \left\{ \frac{f(y)}{y} - \frac{8xy}{y} \right\}$$

$$= \lim_{y \to 0} \left(\frac{f'(y)}{1} \right) - 8x \text{ (using L'Hospital's rule)}$$

$$= f'(0) - 8x = 8 - 8x \qquad [\text{given, } f'(0) = 8]$$

$\Rightarrow \quad f'(x) = 8 - 8x$

Integrating both the sides, we get

$$f(x) = 8x - 4x^2 + c$$

As $\quad f(0) = 0 \Rightarrow c = 0$

$\Rightarrow \quad f(x) = 8x - 4x^2 \quad ...(i)$

Also, $\quad g(x + y) = g(x) + g(y) + 3xy\,(x + y)$

Replacing $x, y \Rightarrow 0$, we get $g(0) = 0$

Now, $\quad g'(x) = \lim\limits_{y \to 0} \dfrac{g(x+y) - g(x)}{y}$

$$= \lim_{y \to 0} \frac{g(x) + g(y) + 3x^2 y + 3xy^2 - g(x)}{y}$$

$$= \lim_{y \to 0} \left[\frac{g(y)}{y} + \frac{y(3x^2 + 3xy)}{y} \right]$$

$$= \frac{g'(0)}{1} + 3x^2 = -4 + 3x^2$$

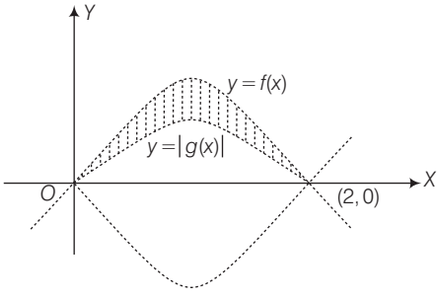
$\therefore \quad g'(x) = -4 + 3x^2$

$\Rightarrow \quad g(x) = x^3 - 4x \quad [\text{as } g(0) = 0] ...(ii)$

Points where $f(x)$ and $g(x)$ meets, we have

$$f(x) = g(x) \quad \text{or} \quad 8x - 4x^2 = x^3 - 4x$$

$\Rightarrow \quad x = 0, 2, -6$



Now, $\quad |g(x)| = \begin{cases} x^3 - 4x, & x \in [-2, 0] \cup (2, \infty) \\ 4x - x^3, & x \in [-\infty, -2] \cup (0, 2) \end{cases}$

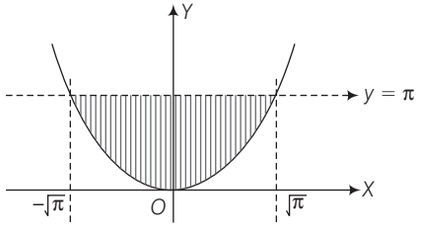
$\therefore$ Area bounded by $y = f(x)$ and $y = |g(x)|$ between $x = 0$ to $x = 2$

$$= \int_0^2 \{(8x - 4x^2) - (4x - x^3)\}\, dx$$

$$= \int_0^2 (x^3 - 4x^2 + 4x)\, dx = \frac{4}{3} \text{ sq units}$$

● ***Ex. 50*** *Find the area of the region bounded by the curve* $y = x^2$ *and* $y = \sec^{-1}[-\sin^2 x]$, *where [.] denotes the greatest integer function.*

Sol. As we know, $[-\sin^2 x] = 0$ or -1. But $\sec^{-1}(0)$ is not defined.



$\Rightarrow \quad \sec^{-1}[-\sin^2 x] = \sec^{-1}(-1) = \pi$

Thus, to find the area bounded between

$$y = x^2 \quad \text{and} \quad y = \pi$$

i.e. when $x^2 = \pi$ or $(x = -\sqrt{\pi}$ to $x = \sqrt{\pi})$

$\therefore$ Required area $= \int_{-\sqrt{\pi}}^{\sqrt{\pi}} (\pi - x^2)\, dx = \left(\pi x - \dfrac{x^3}{3} \right)_{-\sqrt{\pi}}^{\sqrt{\pi}}$

$$= \pi(\sqrt{\pi} + \sqrt{\pi}) - \frac{1}{3}(\pi\sqrt{\pi} + \pi\sqrt{\pi}) = \frac{4\pi}{3}\sqrt{\pi}$$

● ***Ex. 51*** *Sketch the graph of* $\cos^{-1}(4x^3 - 3x)$ *and find the area enclosed between* $y = 0$, $y = f(x)$ *and* $x \geq -1/2$.

Sol. Here, $\quad f(x) = \cos^{-1}(4x^3 - 3x)$

Let $\quad x = \cos\theta \quad$ and $\quad 0 \leq \theta \leq \pi$

$\Rightarrow \quad f(x) = \cos^{-1}(4\cos^3\theta - 3\cos\theta)$

$$= \cos^{-1}(\cos 3\theta);\ 0 \leq 3\theta \leq 3\pi$$

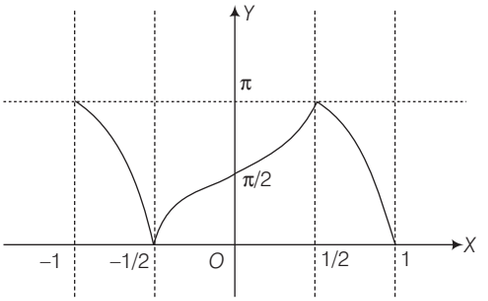
$\Rightarrow \quad f(x) = \begin{cases} 3\theta, & 0 \leq 3\theta \leq \pi \\ 2\pi - 3\theta, & \pi < 3\theta \leq 2\pi \\ 3\theta - 2\pi, & 2\pi < 3\theta \leq 3\pi \end{cases}$

$$= \begin{cases} 3\cos^{-1} x, & 1/2 \leq x \leq 1 \\ 2\pi - 3\cos^{-1} x, & -1/2 \leq x < 1 \\ 3\cos^{-1} x - 2\pi, & -1 \leq x < -1/2 \end{cases}$$

$\therefore \quad f'(x) = \begin{cases} -3/\sqrt{1 - x^2}, & 1/2 < x < 1 \\ 3/\sqrt{1 - x^2}, & -1/2 < x < 1/2 \\ -3/\sqrt{1 - x^2}, & -1 < x < -1/2 \end{cases}$

and $\quad f''(x) = \begin{cases} \dfrac{-3x}{(1 - x^2)^{3/2}}, & 1/2 < x < 1 \\ \dfrac{3x}{(1 - x^2)^{3/2}}, & -1/2 < x < 1/2 \\ \dfrac{-3x}{(1 - x^2)^{3/2}}, & -1 < x < -1/2 \end{cases}$

Thus, the graph for $f(x) = \cos^{-1}(4x^3 - 3x)$ is



Thus, required area $= \int_{-1/2}^{1} f(x)\, dx$

$$= \int_{-1/2}^{1/2} (2\pi - 3\cos^{-1} x)\, dx + \int_{1/2}^{1} (3\cos^{-1} x)\, dx$$

$$= 2\pi - 3\int_{-1/2}^{1/2}\left(\frac{\pi}{2} - \sin^{-1} x\right)dx + 3\int_{1/2}^{1}(\cos^{-1} x)\,dx$$

$$= \frac{\pi}{2} + 3\int_{1/2}^{1}\cos^{-1} x\,dx \qquad \left[\text{as}\int_{-1/2}^{1/2}\sin^{-1} x\,dx = 0\right]$$

$$= \frac{\pi}{2} + 3\left\{(x\cos^{-1} x)_{1/2}^{1} + \int_{1/2}^{1}\frac{x}{\sqrt{1-x^2}}\,dx\right\}$$

On solving, we get $= \frac{3\sqrt{3}}{2}$ sq units

● ***Ex. 52*** *Consider two curves $y^2 = 4a(x-\lambda)$ and $x^2 = 4a(y-\lambda)$, where $a > 0$ and λ is a parameter. Show that*

(i) there is a single positive value of λ for which the two curves have exactly one point of intersection in the 1st quadrant find it.

(ii) there are infinitely many negative values of λ for which the two curves have exactly one point of intersection in the 1st quadrant.

(iii) if $\lambda = -a$, then find the area of the bounded by the two curves and the axes in the 1st quadrant.

Sol. The two curves are inverse of each other. Hence, the two curves always meet along the line $y = x$.

Consider, $y^2 = 4a(x-\lambda)$ and put $x = y$

$\Rightarrow \quad y^2 - 4ay + 4a\lambda = 0$

$$\Rightarrow \quad y = \frac{4a \pm 4\sqrt{a^2 - a\lambda}}{2} = 2a \pm \sqrt{a^2 - a\lambda}$$

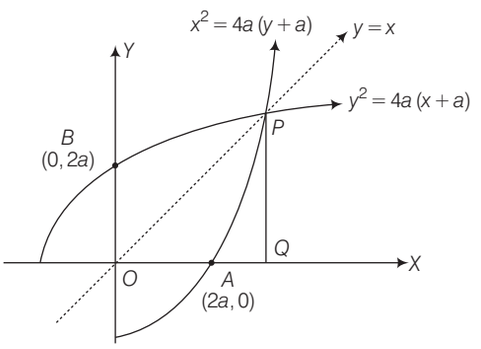
Since, y is real $\Rightarrow a^2 - a\lambda \mid 0$ or $\lambda \le a$

(i) If $0 < \lambda < a$, then there are two distinct values of y and both $2(a + \sqrt{a^2 - a\lambda})$ and $2(a - \sqrt{a^2 - a\lambda})$ are positive, i.e. both points lie in the first quadrant.

If $\lambda = a$, then $y = 2a$ only, i.e. only one point of intersection $(2a, 2a)$.

Hence, there is exactly one point of intersection in 1st quadrant for $\lambda = a$. It is infact the points of tangency of the two curves.

(ii) If $\lambda < 0$, then $y = 2(a + \sqrt{a^2 - a\lambda}) > 0$ and $y = 2(a - \sqrt{a^2 - a\lambda}) < 0$. i.e. the only point of intersection is in the first quadrant, the other in the 3rd quadrant. Hence, there are infinitely many such values.



(iii) For $\lambda = -a$, we have $y^2 = 4a(x+a)$

$$x^2 = 4a(y+a)$$

The point of intersection in the 1st quadrant $P = ((2+2\sqrt{2})\,a, (2+2\sqrt{2})\,a)$

Required area = 2 (area of ΔOPQ − area $\overline{APQA}$)

of $\Delta OPQ = \frac{1}{2}\cdot(2+2\sqrt{2})^2\,a^2 = 2(1+\sqrt{2})^2\,a^2$

Area of $APQA = \int_{2a}^{(2+2\sqrt{2})a} y\,dx = \int_{2a}^{(2+2\sqrt{2})a}\left(\frac{x^2}{4a} - a\right)dx$

$$= \frac{1}{4a}\left[\frac{x^3}{3}\right]_{2a}^{(2+2\sqrt{2})a} - a\,(x)_{2a}^{(2+2\sqrt{2})a}$$

$$= \frac{1}{12a}[(2+2\sqrt{2})^3\cdot a^3 - 8a^3] - 2\sqrt{2}\,a^2$$

$$= \left(\frac{4\sqrt{2}+12}{3}\right)a^2$$

$\therefore$ Required area $= 2\left(2(1+\sqrt{2})^2 + \frac{4\sqrt{2}+12}{3}\right)a^2$

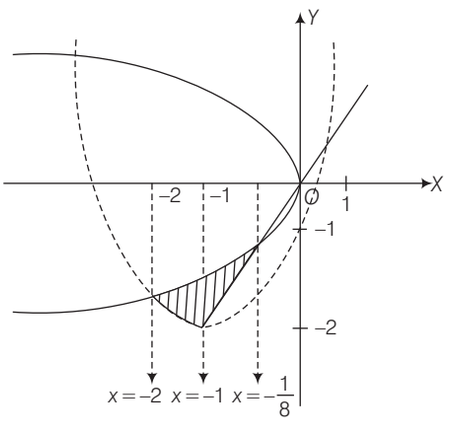
$$= \frac{4}{3}(15 + 8\sqrt{2})\,a^2$$

● ***Ex. 53*** *Let $f(x)$ be continuous function given by*

$$f(x) = \begin{cases} 2x, & |x| \le 1 \\ x^2 + ax + b, & |x| > 1 \end{cases}$$

Find the area of the region in the third quadrant bounded by the curves $x = -2y^2$ and $y = f(x)$ lying on the left of the line $8x + 1 = 0$. ***[IIT JEE 1999]***

Sol. Given, a continuous function $f(x)$, given by



$$f(x) = \begin{cases} 2x, & |x| \le 1 \\ x^2 + ax + b, & |x| > 1 \end{cases}$$

i.e. $$f(x) = \begin{cases} x^2 + ax + b, & -\infty < x < -1 \\ 2x, & -1 \le x \le 1 \\ x^2 + ax + b, & 1 < x < \infty \end{cases}$$

$\because$ f is continuous.

$\therefore$ It is continuous at $x = -1$ and $x = 1$.

$\therefore \quad (-1)^2 + a(-1) + b = 2(-1)$

and $\quad (1)^2 + a(1) + b = 2(1)$

$\Rightarrow \quad 1 - a + b = -2$

and $\quad 1 + a + b = 2$

$\Rightarrow \quad -a + b = -3$

and $\quad a + b = 1$

$\Rightarrow \quad b = -1$ and $a = 2$

$\therefore \quad f(x) = \begin{cases} x^2 + 2x - 1, & \text{when } -\infty < x < -1 \\ 2x, & \text{when } -1 \le x \le 1 \end{cases}$

Now, $\quad y = x^2 + 2x - 1 = (x+1)^2 - 2$

or $\quad (y+2) = (x+1)^2$

We need the area of the region in third quadrant bounded by the curves $a = -2y^2$, $y = f(x)$ lying on the left of the line $8x + 1 = 0$.

$\Rightarrow \quad (y+2) = (x+1)^2$

Cuts the Y-axis at $(0, -1)$ and the X-axis at $(-1-\sqrt{2}, 0)$ and $(-1+\sqrt{2}, 0)$.

When $x = 1$ and $y = 2$

Solving $x = -2y^2$ and $(y+2) = (x+1)^2$, we get

$$(y+2) = (-2y^2 + 1)^2$$

$\Rightarrow \quad 4y^4 - 4y^2 + 1 - y - 2 = 0$

$\Rightarrow \quad 4y^4 - 4y^2 - y - 1 = 0$

For $\quad y = -1,\ 4y^4 - 4y^2 - y - 1 = 0$

At $\quad y = -1,\ x = -2$

$\therefore \quad$ Required area $= \int_{-2}^{-1} \left\{ -\sqrt{-\frac{x}{2}} - (x+1)^2 + 2 \right\} dx + \int_{-1}^{-1/8} \left\{ -\sqrt{-\frac{x}{2}} - 2x \right\} dx$

$$= \left\{ \frac{(-x/2)^{3/2}}{\frac{3}{2}(1/2)} - \frac{(x+1)^3}{3} + 2x \right\}_{-2}^{-1} + \left\{ \frac{(-x/2)^{3/2}}{\frac{3}{2}(1/2)} - \frac{2x^2}{2} \right\}_{-1}^{-1/8}$$

$$= \left\{ \frac{4}{3}\left(-\frac{x}{2}\right)^{3/2} - \frac{(x+1)^3}{3} + 2x \right\}_{-2}^{-1} + \left\{ \frac{4}{3}\left(-\frac{x}{2}\right)^{3/2} - x^2 \right\}_{-1}^{-1/8}$$

$$= \left\{ \frac{4}{3}\left(\frac{1}{2}\right)^{3/2} - 0 - 2 \right\} - \left\{ \frac{4}{3}(1)^{3/2} - \frac{(-1)^3}{3} - 4 \right\} - \left\{ \frac{4}{3}\left(\frac{1}{2}\right)^{3/2} - (-1)^2 \right\} + \left\{ \frac{4}{3}\left(\frac{1}{16}\right)^{3/2} - \frac{1}{64} \right\}$$

$$= \left\{ \frac{4}{3\cdot 2\sqrt{2}} - 2 - \frac{4}{3} - \frac{1}{3} + 4 - \frac{4}{3\cdot 2\sqrt{2}} + 1 + \frac{4}{3(64)} - \frac{1}{64} \right\}$$

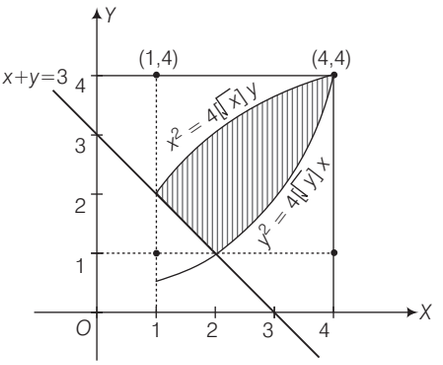
$$= \frac{4}{3} + \frac{1}{3(64)} = \frac{257}{192} \text{ sq units}$$

● ***Ex. 54*** *Let $[x]$ denotes the greatest integer function. Draw a rough sketch of the portions of the curves $x^2 = 4[\sqrt{x}]\,y$ and $y^2 = 4[\sqrt{y}]\,x$ that lie within the square $\{(x, y) \mid 1 \le x \le 4, 1 \le y \le 4\}$. Find the area of the part of the square that is enclosed by the two curves and the line $x + y = 3$.*

Sol. We have, $\quad 1 \le x \le 4$ and $1 \le y \le 4$

$\Rightarrow \quad 1 \le \sqrt{x} \le 2$ and $1 \le \sqrt{y} \le 2$

$\Rightarrow [\sqrt{x}] = 1$ and $[\sqrt{y}] = 1$ for all (x, y) lying with in the square.



Thus, $\quad x^2 = 4[\sqrt{x}]\,y$ and $y^2 = 4[\sqrt{y}]\,x$

$\Rightarrow \quad x^2 = 4y$ and $y^2 = 4x$ when $1 \le x, y \le 4$ which could be plotted as;

Thus, required area

$$= \int_1^2 (2\sqrt{x} - 3 + x)\,dx + \int_2^4 \left(2\sqrt{x} - \frac{x^2}{4} \right) dx$$

$$= \left(\frac{4}{3}x^{3/2} - 3x + \frac{x^2}{2} \right)_1^2 + \left(\frac{4}{3}x^{3/2} - \frac{x^3}{12} \right)_2^4$$

$$= \left(\frac{8\sqrt{2}}{3} - 6 + 2 \right) - \left(\frac{4}{3} - 3 + \frac{1}{2} \right) + \left(\frac{32}{3} - \frac{64}{12} \right) - \left(\frac{8\sqrt{2}}{3} - \frac{8}{12} \right)$$

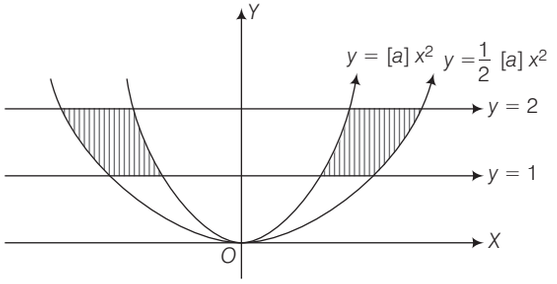
$$= \frac{19}{6} \text{ sq units}$$

● ***Ex. 55*** *Find all the values of the parameter a $(a \ge 1)$ for which the area of the figure bounded by pair of straight lines $y^2 - 3y + 2 = 0$ and the curves $y = [a]\,x^2, y = \frac{1}{2}[a]\,x^2$ is greatest, where [.] denotes the greatest integer function.*

Sol. The curves $y = [a]\,x^2$ and $y = \frac{1}{2}[a]\,x^2$ represent parabolas which are symmetric about Y-axis.

The equation $y^2 - 3y + 2 = 0$ gives a pair of straight lines $y = 1$, $y = 2$ which are parallel to X-axis.

Thus, the area bounded is shown as



From the above figure,

$$\text{Required area} = 2\int_{y=1}^{y=2} (x_2 - x_1)\, dy$$

$$= 2\int_1^2 \left(\sqrt{\frac{2y}{[a]}} - \sqrt{\frac{y}{[a]}} \right) dy$$

$$= \frac{2(\sqrt{2}-1)}{[a]} \int_1^2 \sqrt{y}\, dy = \frac{2(\sqrt{2}-1)}{[a]} \frac{2}{3} (y^{3/2})_1^2$$

$$= \frac{4}{3} \frac{(\sqrt{2}-1)}{[a]} \cdot (2^{3/2} - 1)$$

$$= \frac{4}{3} \frac{(5 - \sqrt{2} - 2\sqrt{2})}{[a]} = \frac{4}{3} \frac{(5 - 3\sqrt{2})}{[a]}$$

$\therefore$ Area is the greatest when $[a]$ is least, i.e. 1.

$\therefore$ Area is the greatest when $[a] = 1$

Hence, $a \in [1, 2)$

● ***Ex. 56*** *Find the area in the first quadrant bounded by* $[x] + [y] = n$, *where* $n \in N$ *and* $y = i$ *(where,* $i \in N\ \forall\ i \le n+1$*),* $[\cdot]$ *denotes the greatest integer less than or equal to* x.

Sol. As we know, $[x] + [y] = n \Rightarrow [x] = n - [y]$

When $y = 0$, $[x] = n \Rightarrow n \le x < n + 1$

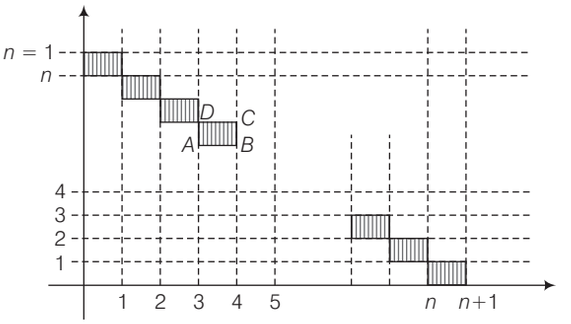
When $y = 1$, $[x] = n - 1 \Rightarrow n - 1 \le x < n - 2$

When $y = 2$, $[x] = n - 2$

$\Rightarrow$ $n - 2 \le x < n - 3$. ... and so on.

When $y = n$, $[x] = 0 \Rightarrow 0 \le x < 1$

which could be shown as



From the above figure,

$= (n + 1)$ area of square $ABCD = (n + 1) \cdot 1$

Required area $= (n + 1)$ sq units

Area of Bounded Regions Exercise 1 :
Single Option Correct Type Questions

1. A Point $P(x, y)$ moves such that $[x + y + 1] = [x]$. (where $[\cdot]$ denotes greatest integer function) and $x \in (0, 2)$, then the area represented by all the possible positions of P, is
(a) $\sqrt{2}$ (b) $2\sqrt{2}$
(c) $4\sqrt{2}$ (d) 2

2. If $f: [-1, 1] \to \left[-\frac{1}{2}, \frac{1}{2}\right]$, $f(x) = \frac{x}{1 + x^2}$. The area bounded by $y = f^{-1}(x)$, X-axis, $x = \frac{1}{2}$, $x = -\frac{1}{2}$ is
(a) $\frac{1}{2}\log e$ (b) $\log\left(\frac{e}{2}\right)$
(c) $\frac{1}{2}\log\frac{e}{3}$ (d) $\frac{1}{2}\log\left(\frac{e}{2}\right)$

3. If the length of latusrectum of ellipse
$E_1 : 4(x + y - 1)^2 + 2(x - y + 3)^2 = 8$
and $E_2 : \frac{x^2}{p} + \frac{y^2}{p^2} = 1, (0 < p < 1)$ are equal, then area of ellipse E_2, is
(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{\sqrt{2}}$
(c) $\frac{\pi}{2\sqrt{2}}$ (d) $\frac{\pi}{4}$

4. The area of bounded by the curve
$4\,|\,x - 2017^{2017}\,| + 5\,|\,y - 2017^{2017}\,| \le 20$, is
(a) 60 (b) 50
(c) 40 (d) 30

5. If the area bounded by the corve $y = x^2 + 1$, $y = x$ and the pair of lines $x^2 + y^2 + 2xy - 4x - 4y + 3 = 0$ is K units, then the area of the region bounded by the curve $y = x^2 + 1, y = \sqrt{x - 1}$ and the pair of lines $(x + y - 1)(x + y - 3) = 0$, is
(a) K (b) $2K$
(c) $\frac{K}{2}$ (d) None of these

6. Suppose $y = f(x)$ and $y = g(x)$ are two functions whose graphs intersect at the three points (0, 4), (2, 2) and (4, 0) with $f(x) > g(x)$ for $0 < x < 2$ and $f(x) < g(x)$ for $2 < x < 4$. If $\int_0^4 [f(x) - g(x)]\,dx = 10$
and $\int_2^4 [g(x) - f(x)]\,dx = 5$, then the area between two curves for $0 < x < 2$, is
(a) 5 (b) 10
(c) 15 (d) 20

7. Let 'a' be a positive constant number. Consider two curves $C_1 : y = e^x$, $C_2 : y = e^{a-x}$. Let S be the area of the part surrounding by C_1, C_2 and the Y-axis, then $\lim_{a \to 0} \frac{S}{a^2}$ equals
(a) 4 (b) 1/2
(c) 0 (d) 1/4

8. 3 points $O(0,0), P(a, a^2)$, $Q(-b, b^2)(a > 0, b > 0)$ are on the parabola $y = x^2$. Let S_1 be the area bounded by the line PQ and the parabola and let S_2 be the area of the ΔOPQ, the minimum value of S_1/S_2 is
(a) 4/3 (b) 5/3
(c) 2 (d) 7/3

9. Area enclosed by the graph of the function $y = \ln^2 x - 1$ lying in the 4th quadrant is
(a) $\frac{2}{e}$ (b) $\frac{4}{e}$
(c) $2\left(e + \frac{1}{e}\right)$ (d) $4\left(e - \frac{1}{e}\right)$

10. The area bounded by $y = 2 - |2 - x|$ and $y = \frac{3}{|x|}$ is
(a) $\frac{4 + 3\ln 3}{2}$ (b) $\frac{19}{8} - 3\ln 2$
(c) $\frac{3}{2} + \ln 3$ (d) $\frac{1}{2} + \ln 3$

11. Suppose $g(x) = 2x + 1$ and $h(x) = 4x^2 + 4x + 5$ and $h(x) = (fog)(x)$. The area enclosed by the graph of the function $y = f(x)$ and the pair of tangents drawn to it from the origin, is
(a) 8/3 (b) 16/3
(c) 32/3 (d) None of these

12. The area bounded by the curves $y = -\sqrt{-x}$ and $x = -\sqrt{-y}$ where $x, y \le 0$
(a) cannot be determined
(b) is 1/3
(c) is 2/3
(d) is same as that of the figure bounded by the curves $y = \sqrt{-x}$; $x \le 0$ and $x = \sqrt{-y}$; $y \le 0$

13. $y = f(x)$ is a function which satisfies
(i) $f(0) = 0$ (ii) $f''(x) = f'(x)$ and (iii) $f'(0) = 1$
Then, the area bounded by the graph of $y = f(x)$, the lines $x = 0, x - 1 = 0$ and $y + 1 = 0$, is
(a) e (b) $e - 2$
(c) $e - 1$ (d) $e + 1$

14. Area of the region enclosed between the curves $x = y^2 - 1$ and $x = |y|\sqrt{1-y^2}$ is

(a) 1 (b) 4/3
(c) 2/3 (d) 2

15. The area bounded by the curve $y = x e^{-x}$; $xy = 0$ and $x = c$ where c is the x-coordinate of the curve's inflection point, is

(a) $1-3e^{-2}$ (b) $1-2e^{-2}$
(c) $1-e^{-2}$ (d) 1

16. If $(a,0); a>0$ is the point where the curve $y = \sin 2x - \sqrt{3}\sin x$ cuts the X-axis first, A is the area bounded by this part of the curve, the origin and the positive X-axis, then

(a) $4A + 8\cos a = 7$ (b) $4A + 8\sin a = 7$
(c) $4A - 8\sin a = 7$ (d) $4A - 8\cos a = 7$

17. The curve $y = ax^2 + bx + c$ passes through the point $(1,2)$ and its tangent at origin is the line $y = x$. The area bounded by the curve, the ordinate of the curve at minima and the tangent line is

(a) $\frac{1}{24}$ (b) $\frac{1}{12}$ (c) $\frac{1}{8}$ (d) $\frac{1}{6}$

18. A function $y = f(x)$ satisfies the differential equation $\frac{dy}{dx} - y = \cos x - \sin x$, with initial condition that y is bounded when $x \to \infty$. The area enclosed by $y = f(x), y = \cos x$ and the Y-axis in the 1st quadrant

(a) $\sqrt{2}-1$ (b) $\sqrt{2}$
(c) 1 (d) $1/\sqrt{2}$

19. If the area bounded between X-axis and the graph of $y = 6x - 3x^2$ between the ordinates $x = 1$ and $x = a$ is 19 sq units, then 'a' can take the value

(a) 4 or -2
(b) two values are in $(2,3)$ and one in $(-1,0)$
(c) two values one in $(3,4)$ and one in $(-2,-1)$
(d) None of the above

20. Area bounded by $y = f^{-1}(x)$ and tangent and normal drawn to it at the points with abscissae π and 2π, where $f(x) = \sin x - x$ is

(a) $\frac{\pi^2}{2} - 1$ (b) $\frac{\pi^2}{2} - 2$
(c) $\frac{\pi^2}{2} - 4$ (d) $\frac{\pi^2}{2}$

21. If $f(x) = x - 1$ and $g(x) = |f(|x|) - 2|$, then the area bounded by $y = g(x)$ and the curve $x^2 - 4y + 8 = 0$ is equal to

(a) $\frac{4}{3}(4\sqrt{2} - 5)$ (b) $\frac{4}{3}(4\sqrt{2} - 3)$
(c) $\frac{8}{3}(4\sqrt{2} - 3)$ (d) $\frac{8}{3}(4\sqrt{2} - 5)$

22. Let $S = \left\{(x,y): \frac{y(3x-1)}{x(3x-2)} < 0\right\}$,
$S' = \{(x,y) \in A \times B: -1 \le A \le 1, -1 \le B \le 1\}$,
then the area of the region enclosed by all points in $S \cap S'$ is

(a) 1 (b) 2
(c) 3 (d) 4

23. The area of the region bounded between the curves $y = e^{||x|\ln|x||}$, $x^2 + y^2 - 2(|x| + |y|) + 1 \ge 0$ and X-axis where $|x| \le 1$, if α is the x-coordinate of the point of intersection of curves in 1st quadrant, is

(a) $4\left[\int_0^{\alpha} ex\ln x\,dx + \int_{\alpha}^{1}(1 - \sqrt{1-(x-1)^2})\,dx\right]$
(b) $4\left[\int_0^{\alpha} ex\ln x\,dx + \int_{1}^{\alpha}(1 - \sqrt{1-(x-1)^2})\,dx\right]$
(c) $4\left[-\int_0^{\alpha} ex\ln x\,dx + \int_{\alpha}^{1}(1 - \sqrt{1-(x-1)^2})\,dx\right]$
(d) $2\left[\int_0^{\alpha} ex\ln x\,dx + \int_{1}^{\alpha}(1 - \sqrt{1-(x-1)^2})\,dx\right]$

24. A point P lying inside the curve $y = \sqrt{2ax - x^2}$ is moving such that its shortest distance from the curve at any position is greater than its distance from X-axis. The point P enclose a region whose area is equal to

(a) $\frac{\pi a^2}{2}$ (b) $\frac{a^2}{3}$
(c) $\frac{2a^2}{3}$ (d) $\left(\frac{3\pi - 4}{6}\right)a^2$

Area of Bounded Regions Exercise 2 :
More than One Option Correct Type Questions

25. The triangle formed by the normal to the curve $f(x) = x^2 - ax + 2a$ at the point (2, 4) and the coordinate axes lies in second quadrant, if its area is 2 sq units, then a can be
(a) 2
(b) 17/4
(c) 5
(d) 19/4

26. Let f and g be continuous function on $a \le x \le b$ and set $p(x) = \max\{f(x),\ g(x)\}$ and $q(x) = \min\{f(x), g(x)\}$, then the area bounded by the curves $y = p(x)$, $y = q(x)$ and the ordinates $x = a$ and $x = b$ is given by

(a) $\int_a^b |f(x) - g(x)|\, dx$ (b) $\int_a^b |p(x) - q(x)|\, dx$

(c) $\int_a^b \{f(x) - g(x)\}\, dx$ (d) $\int_a^b \{p(x) - q(x)\}\, dx$

27. The area bounded by the parabola $y = x^2 - 7x + 10$ and X-axis equals
(a) area bounded by $y = -x^2 + 7x - 10$ and X-axis
(b) 1/6 sq units
(c) 5/6 sq units
(d) 9/2 sq units

28. Area bounded by the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ is equal to
(a) 6π sq units
(b) 3π sq units
(c) 12π sq units
(d) area bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$

29. There is a curve in which the length of the perpendicular from the origin to tangent at any point is equal to abscissa of that point. Then,
(a) $x^2 + y^2 = 2$ is one such curve
(b) $y^2 = 4x$ is one such curve
(c) $x^2 + y^2 = 2cx$ (c parameters) are such curves
(d) there are no such curves

Area of Bounded Regions Exercise 3 :
Statement I and II Type Questions

■ **Direction** (Q. No. 30-34) *For the following questions, choose the correct answers from the codes* (*a*), (*b*), (*c*) *and* (*d*) *defined as follows* :

(a) Statement I is true, Statement II is also true; Statement II is the correct explanation of Statement I
(b) Statement I is true, Statement II is also true; Statement II is not the correct explanation of Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

30. **Statement I** The area of the curve $y = \sin^2 x$ from 0 to π will be more than that of the curve $y = \sin x$ from 0 to π.

Statement II $x^2 > x$, if $x > 1$.

31. **Statement I** The area bounded by the curves $y = x^2 - 3$ and $y = kx + 2$ is least if $k = 0$.

Statement II The area bounded by the curves $y = x^2 - 3$ and $y = kx + 2$ is $\sqrt{k^2 + 20}$.

32. **Statement I** The area of region bounded parabola $y^2 = 4x$ and $x^2 = 4y$ is $\frac{32}{3}$ sq units.

Statement II The area of region bounded by parabola $y^2 = 4ax$ and $x^2 = 4by$ is $\frac{16}{3}ab$.

33. **Statement I** The area by region $|x + y| + |x - y| \le 2$ is 8 sq units.

Statement II Area enclosed by region $|x + y| + |x - y| \le 2$ is symmetric about X-axis.

34. **Statement I** Area bounded by $y = x(x - 1)$ and $y = x(1 - x)$ is $\frac{1}{3}$.

Statement II Area bounded by $y = f(x)$ and $y = g(x)$ is $\left| \int_a^b (f(x) - g(x))\, dx \right|$ is true when $f(x)$ and $g(x)$ lies above X-axis. (Where a and b are intersection of $y = f(x)$ and $y = g(x)$).

Area of Bounded Regions Exercise 4 : Passage Based Questions

Passage I

(Q. Nos. 35 to 37)

Let $f(x)=\dfrac{ax^2+bx+c}{x^2+1}$ *such that* $y=-2$ *is an asymptote of the curve* $y=f(x)$. *The curve* $y=f(x)$ *is symmetric about Y-axis and its maximum values is 4. Let* $h(x)=f(x)-g(x)$, *where* $f(x)=\sin^4 \pi x$ *and* $g(x)=\log_e x$. *Let* $x_0, x_1, x_2, \ldots . x_{n+1}$ *be the roots of* $f(x)=g(x)$ *in increasing order.*

35. Then, the absolute area enclosed by $y=f(x)$ and $y=g(x)$ is given by

(a) $\sum_{r=0}^{n} \int_{x_r}^{x_{r+1}} (-1)^r \cdot h(x)dx$

(b) $\sum_{r=0}^{n} \int_{x_r}^{x_{r+1}} (-1)^{r+1} \cdot h(x)dx$

(c) $2\sum_{r=0}^{n} \int_{x_r}^{x_{r-r+1}} (-1)^r \cdot h(x)dx$

(d) $\frac{1}{2}\cdot\sum_{r=0}^{n} \int_{x_r}^{x_{r+1}} (-1)^{r+1} h(x)dx$

36. In above inquestion the value of *n*, is

(a) 1 (b) 2
(c) 3 (d) 4

37. The whole area bounded by $y=f(x), y=g(x)\ x=0$ is

(a) 11/8 (b) 8/3
(c) 2 (d) 13/3

Passage II

(Q. Nos. 38 to 40)

Consider the function $f:(-\infty,\infty)\to(-\infty,\infty)$ *defined by* $f(x)=\dfrac{x^2-ax+1}{x^2+ax+1}, 0<a<2.$

38. Which of the following is true?

(a) $(2+a)^2 f''(1)+(2-a)^2 f''(-1)=0$

(b) $(2-a)^2 f''(1)-(2+a)^2 f''(-1)=0$

(c) $f'(1)f'(-1)=(2-a)^2$

(d) $f'(1)f'(-1)=-(2+a)^2$

39. Which of the following is true?

(a) $f(x)$ is decreasing on $(-1,1)$ and has a local minimum at $x=1$

(b) $f(x)$ is increasing on $(-1,1)$ and has a local maximum at $x=1$

(c) $f(x)$ is increasing on $(-1,1)$ but has neither a local maximum nor a local minimum at $x=1$

(d) $f(x)$ is decreasing on $(-1,1)$ but has neither a local maximum nor a local minimum at $x=1$

40. Let $g(x)=\int_0^{e^x} \dfrac{f'(t)}{1+t^2}\,dt$. Which of the following is true?

(a) $g'(x)$ is positive on $(-\infty,0)$ and negative on $(0,\infty)$

(b) $g'(x)$ is negative on $(-\infty,0)$ and positive on $(0,\infty)$

(c) $g'(x)$ change sign on both $(-\infty,0)$ and $(0,\infty)$

(d) $g'(x)$ does not change sign on $(-\infty,\infty)$

Passage III

(Q. Nos. 41 to 43)

Computing areas with parametrically represented boundaries : *If the boundary of a figure is represented by parametric equations* i.e. $x=x(t)$, $y=y(t)$, *then the area of the figure is evaluated by one of the three formulas*

$$S=-\int_\alpha^\beta y(t)\cdot x'(t)\,dt$$

$$S=\int_\alpha^\beta x(t)\cdot y'(t)\,dt$$

$$S=\frac{1}{2}\int_\alpha^\beta (xy'-yx')\,dt$$

where α *and* β *are the values of the parameter '*t*' corresponding, respectively to the beginning and the end of the traversal of the curve corresponding to increasing '*t*'.*

41. The area enclosed by the asteroid $\left(\dfrac{x}{a}\right)^{2/3}+\left(\dfrac{y}{a}\right)^{2/3}=1$ is

(a) $\frac{3}{4}a^2\pi$ (b) $\frac{3}{18}\pi a^2$
(c) $\frac{3}{8}\pi a^2$ (d) $\frac{3}{4}a\pi$

42. The area of the region bounded by an arc of the cycloid $x=a(t-\sin t), y=a(1-\cos t)$ and the *X*-axis is

(a) $6\pi a^2$ (b) $3\pi a^2$
(c) $4\pi a^2$ (d) None of these

43. Area of the loop described as $x=\dfrac{t}{3}(6-t), y=\dfrac{t^2}{8}(6-t)$ is

(a) $\frac{27}{5}$ (b) $\frac{24}{5}$
(c) $\frac{27}{6}$ (d) $\frac{21}{5}$

Area of Bounded Regions Exercise 5 : Matching Type Questions

44. Match the statements of Column I with values of Column II.

	Column I		Column II
(A)	The area bounded by the curve $y = x + \sin x$ and its inverse function between the ordinates $x = 0$ to $x = 2\pi$ is 4s. Then, the value of s is	(p)	2
(B)	The area bounded by $y = x e^{\lvert x\rvert}$ and lies $\lvert x\rvert = 1$, $y = 0$ is	(q)	1
(C)	The area bounded by the curves $y^2 = x^3$ and $\lvert y\rvert = 2x$ is	(r)	$\frac{16}{5}$
(D)	The smaller are included between the curves $\sqrt{\lvert x\rvert} + \sqrt{\lvert y\rvert} = 1$ and $\lvert x\rvert + \lvert y\rvert = 1$ is	(s)	$\frac{1}{3}$

45. Match the following :

	Column I		Column II
(A)	Area enclosed by $y = \lvert x\rvert$, $\lvert x\rvert = 1$ and $y = 0$ is	(p)	2
(B)	Area enclosed by the curve $y = \sin x$, $x = 0$, $x = \pi$ and $y = 0$ is	(q)	4
(C)	If the area of the region bounded by $x^2 \le y$ and $y \le x + 2$ is $\frac{k}{4}$, then k is equal to	(r)	27
(D)	Area of the quadrilateral formed by tangents at the ends of latusrectum of ellipse of ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$ is	(s)	18

Area of Bounded Regions Exercise 6 : Single Integer Answer Type Questions

46. Consider $f(x) = x^2 - 3x + 2$. The area bounded by $\lvert y\rvert = \lvert f(\lvert x\rvert)\rvert$, $x \ge 1$ is A, then find the value of $3A + 2$.

47. The value of $c + 2$ for which the area of the figure bounded by the curve $y = 8x^2 - x^5$; the straight lines $x = 1$ and $x = c$ and X-axis is equal to $\frac{16}{3}$, is .....

48. The area bounded by $y = 2 - \lvert 2 - x\rvert$; $y = \frac{3}{\lvert x\rvert}$ is $\frac{k - 3\ln 3}{2}$, then k is equal to .....

49. The area of the ΔABC, coordinates of whose vertices are $A(2, 0)$, $B(4, 5)$ and $C(6, 3)$ is .....

50. A point P moves in XY-plane in such a way that $[\lvert x\rvert] + [\lvert y\rvert] = 1$, where $[\cdot]$ denotes the greatest integer function. Area of the region representing all possible of the point P is equal to ..... .

51. Let $f : [0, 1] \to \left[0, \frac{1}{2}\right]$ be a function such that $f(x)$ is a polynomial of 2nd degree, satisfy the following condition :
(a) $f(0) = 0$
(b) has a maximum value of $\frac{1}{2}$ at $x = 1$.
If A is the area bounded by $y = f(x)$; $y = f^{-1}(x)$ and the line $2x + 2y - 3 = 0$ in 1st quadrant, then the value of $24A$ is equal to ..... .

52. Let $f(x) = \min\left\{\sin^{-1} x, \cos^{-1} x, \frac{\pi}{6}\right\}$, $x \in [0, 1]$. If area bounded by $y = f(x)$ and X-axis, between the lines $x = 0$ and $x = 1$ is $\frac{a - X}{b(\sqrt{3} + 1)}$. Then, $(a - b)$ is ..... .

53. Let f be a real valued function satisfying $f\left(\frac{x}{y}\right) = f(x) - f(y)$ and $\lim\limits_{x \to 0} \frac{f(1 + x)}{x} = 3$. Find the area bounded by the curve $y = f(x)$, the Y-axis and the line $y = 3$, where $x, y \in R^+$.

Area of Bounded Regions Exercise 7 :
Subjective Type Questions

54. Find a continuous function 'f', $(x^4 - 4x^2) \le f(x) \le (2x^2 - x^3)$ such that the area bounded by $y = f(x)$, $y = x^4 - 4x^2$, the Y-axis and the line $x = t, (0 \le t \le 2)$ is k times the area bounded by $y = f(x)$, $y = 2x^2 - x^3$, Y-axis and line $x = t, (0 \le t \le 2)$.

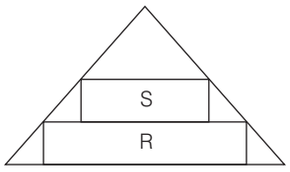
55. Let $f(t) = |t-1| - |t| + |t+1|, \forall\, t \in R$ and $g(x) = \max\{f(t) : x+1 \le t \le x+2\}; \forall\, x \in R$. Find $g(x)$ and the area bounded by the curve $y = g(x)$, the X-axis and the lines $x = -3/2$ and $x = 5$.

56. Let $f(x) = \text{minimum}\{e^x, 3/2, 1 + e^{-x}\}, 0 \le x \le 1$. Find the area bounded by $y = f(x)$, X-axis, Y-axis and the line $x = 1$.

57. Find the area bounded by $y = f(x)$ and the curve $y = \dfrac{2}{1+x^2}$, where f is a continuous function satisfying the conditions $f(x) \cdot f(y) = f(xy), \forall\, x, y \in R$ and $f'(1) = 2$, $f(1) = 1$.

58. Find out the area bounded by the curve $y = \int_{1/8}^{\sin^2 x} (\sin^{-1}\sqrt{t})\, dt + \int_{1/8}^{\cos^2 x} (\cos^{-1}\sqrt{t})\, dt \; (0 \le x \le \pi/2)$ and the curve satisfying the differential equation $y(x + y^3)\, dx = x(y^3 - x)\, dy$ passing through $(4, -2)$.

59. Let T be an acute triangle. Inscribe a pair R, S of rectangles in T as shown :

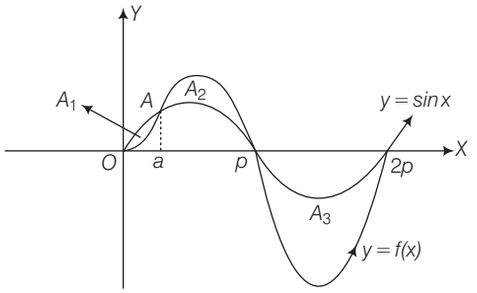


Let $A(x)$ denote the area of polygon X find the maximum value (or show that no maximum exists), of $\dfrac{A(R) + A(S)}{A(T)}$, where T ranges over all triangles and R, S over all rectangles as above.

60. Find the maximum area of the ellipse that can be inscribed in an isosceles triangles of area A and having one axis lying along the perpendicular from the vertex of the triangles to its base.

61. In the adjacent figure the graphs of two function $y = f(x)$ and $y = \sin x$ are given. $y = \sin x$ intersects, $y = f(x)$ at $A(a, f(a))$; $B(\pi, 0)$ and $C(2\pi, 0)$.

A_i $(i = 1, 2, 3)$ is the area bounded by the curves $y = f(x)$ and $y = \sin x$. between $x = 0$ and $x = a$; $i = 1$



between $x = a$ and $x = \pi$; $i = 2$ between $x = \pi$ and $x = 2\pi$; $i = 3$.

If $A_1 = 1 - \sin a + (a-1)\cos a$, determine the function $f(x)$. Hence, determine a and A_1. Also, calculate A_2 and A_3.

62. Find the area of the region bounded by curve $y = 25^x + 16$ and the curve $y = b \cdot 5^x + 4$, whose tangent at the point $x = 1$ make an angle $\tan^{-1}(40 \ln 5)$ with the X-axis.

63. If the circles of the maximum area inscribed in the region bounded by the curves $y = x^2 - x - 3$ and $y = 3 + 2x - x^2$, then the area of region $y - x^2 + 2x + 3 \le 0, y + x^2 - 2x - 3 \le 0$ and $s \le 0$.

64. Find limit of the ratio of the area of the triangle formed by the origin and intersection points of the parabola $y = 4x^2$ and the line $y = a^2$, to the area between the parabola and the line as a approaches to zero.

65. Find the area of curve enclosed by : $|x + y| + |x - y| \le 4, |x| \le 1, y \ge \sqrt{x^2 - 2x + 1}$.

66. Calculate the area enclosed by the curve $4 \le x^2 + y^2 \le 2(|x| + |y|)$.

67. Find the area enclosed by the curve $[x] + [y] = 4$ in 1st quadrant (where [.] denotes greatest integer function).

68. Sketch the region and find the area bounded by the curves $|y + x| \le 1, |y - x| \le 1$ and $2x^2 + 2y^2 = 1$.

69. Find the area of the region bounded by the curve, $2^{|x|}|y| + 2^{|x|-1} \le 1$, with in the square formed by the lines $|x| \le 1/2, |y| \le 1/2$.

70. Find all the values of the parameter $a\,(a \le 1)$ for which the area of the figure bounded by the pair of straight lines $y^2 - 3y + 2 = 0$ and the curves $y = [a]x^2$, $y = \dfrac{1}{2}[a]x^2$ is the greatest, where [.] denotes greatest integer function.

71. If $f(x)$ is positive for all positive values of X and $f'(x) < 0, f''(x) > 0, \forall\, x \in R^+$, prove that $\dfrac{1}{2}f(1) + \int_1^n f(x)\, dx < \sum_{r=1}^{n} f(r) < \int_1^n f(x)\, dx + f(1)$.

Area of Bounded Regions Exercise 8 : Questions Asked in Previous 10 Years' Exams

(i) JEE Advanced & IIT-JEE

72. Area of the region $\{(x, y)\} \in R^2 : y \geq \sqrt{|x+3|}$, $5y \leq (x+9) \leq 15\}$ is equal to **[Single Correct Option 2016]**

(a) $\frac{1}{6}$ (b) $\frac{4}{3}$ (c) $\frac{3}{2}$ (d) $\frac{5}{3}$

73. Let $F(x) = \int_x^{x^2 + \frac{\pi}{6}} 2\cos^2 t \, dt$ for all $x \in R$ and $f : \left[0, \frac{1}{2}\right] \to [0, \infty)$ be a continuous function. For $a \in \left[0, \frac{1}{2}\right]$, if $F'(a) + 2$ is the area of the region bounded by $x = 0$, $y = 0$, $y = f(x)$ and $x = a$, then $f(0)$ is **[Integer Answer Type 2015]**

74. The common tangents to the circle $x^2 + y^2 = 2$ and the parabola $y^2 = 8x$ touch the circle at the points P, Q and the parabola at the points R, S. Then, the area (in sq units) of the quadrilateral $PQRS$ is **[Single Correct Option 2014]**

(a) 3 (b) 6 (c) 9 (d) 15

75. The area enclosed by the curves $y = \sin x + \cos x$ and $y = |\cos x - \sin x|$ over the interval $\left[0, \frac{\pi}{2}\right]$ is **[Single Correct Option 2014]**

(a) $4(\sqrt{2} - 1)$ (b) $2\sqrt{2}(\sqrt{2} - 1)$

(c) $2(\sqrt{2} + 1)$ (d) $2\sqrt{2}(\sqrt{2} + 1)$

76. If S be the area of the region enclosed by $y = e^{-x^2}$, $y = 0$, $x = 0$ and $x = 1$. Then, **[More than One Option Correct 2012]**

(a) $S \geq \frac{1}{e}$ (b) $S \geq 1 - \frac{1}{e}$

(c) $S \leq \frac{1}{4}\left(1 + \frac{1}{\sqrt{e}}\right)$ (d) $S \leq \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}}\left(1 - \frac{1}{\sqrt{2}}\right)$

77. Let $f : [-1, 2] \to [0, \infty)$ be a continuous function such that $f(x) = f(1-x), \forall x \in [-1, 2]$. If $R_1 = \int_{-1}^{2} x f(x)\, dx$ and R_2 are the area of the region bounded by $y = f(x)$, $x = -1$, $x = 2$ and the X-axis. Then, **[Single Correct Option 2011]**

(a) $R_1 = 2R_2$ (b) $R_1 = 3R_2$ (c) $2R_1 = R_2$ (d) $3R_1 = R_2$

78. If the straight line $x = b$ divide the area enclosed by $y = (1-x)^2$, $y = 0$ and $x = 0$ into two parts $R_1 (0 \leq x \leq b)$ and $R_2 (b \leq x \leq 1)$ such that $R_1 - R_2 = \frac{1}{4}$. Then, b equals to **[Single Correct Option 2011]**

(a) $\frac{3}{4}$ (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) $\frac{1}{4}$

79. Area of the region bounded by the curve $y = e^x$ and lines $x = 0$ and $y = e$ is **[More than One Option Correct 2009]**

(a) $e - 1$ (b) $\int_1^e \ln(e + 1 - y)\, dy$

(c) $e - \int_0^1 e^x\, dx$ (d) $\int_1^e \ln y\, dy$

80. The area of the region between the curves $y = \sqrt{\frac{1+\sin x}{\cos x}}$ and $y = \sqrt{\frac{1-\sin x}{\cos x}}$ and bounded by the lines $x = 0$ and $x = \frac{\pi}{4}$ is **[Single Correct Option 2008]**

(a) $\int_0^{\sqrt{2}-1} \frac{t}{(1+t^2)\sqrt{1-t^2}}\, dt$ (b) $\int_0^{\sqrt{2}-1} \frac{4t}{(1+t^2)\sqrt{1-t^2}}\, dt$

(c) $\int_0^{\sqrt{2}+1} \frac{4t}{(1+t^2)\sqrt{1-t^2}}\, dt$ (d) $\int_0^{\sqrt{2}+1} \frac{t}{(1+t^2)\sqrt{1-t^2}}\, dt$

■ **Directions** (Q. Nos. 81 to 83) Consider the functions defined implicity by the equation $y^3 - 3y + x = 0$ on various intervals in the real line. If $x \in (-\infty, -2) \cup (2, \infty)$, the equation implicitly defines a unique real-valued differentiable function $y = f(x)$. If $x \in (-2, 2)$, the equation implicitly defines a unique real-valued differentiable function $y = g(x)$, satisfying $g(0) = 0$. **[Passage Based Questions 2008]**

81. If $f(-10\sqrt{2}) = 2\sqrt{2}$, then $f''(-10\sqrt{2})$ is equal to

(a) $\frac{4\sqrt{2}}{7^3 3^2}$ (b) $-\frac{4\sqrt{2}}{7^3 3^2}$

(c) $\frac{4\sqrt{2}}{7^3 3}$ (d) $-\frac{4\sqrt{2}}{7^3 3}$

82. The area of the region bounded by the curve $y = f(x)$, the X-axis and the lines $x = a$ and $x = b$, where $-\infty < a < b < -2$, is

(a) $\int_a^b \frac{x}{3[\{f(x)\}^2 - 1]}\, dx + bf(b) - af(a)$

(b) $-\int_a^b \frac{x}{3[\{f(x)\}^2 - 1]}\, dx + bf(b) - af(a)$

(c) $\int_a^b \frac{x}{3[\{f(x)\}^2 - 1]}\, dx - bf(b) + af(a)$

(d) $-\int_a^b \frac{x}{3[\{f(x)\}^2 - 1]}\, dx - bf(b) + af(a)$

83. $\int_{-1}^{1} g'(x)\, dx$ is equal to

(a) $2g(-1)$ (b) 0 (c) $-2g(1)$ (d) $2g(1)$

(ii) JEE Main & AIEEE

84. The area (in sq. units) of the region **[2017 JEE Main]**
$\{(x, y): x \geq 0, x + y \leq 3, x^2 \leq 4y\}$ and $y \leq 1 + \sqrt{x}\}$ is
(a) $\frac{5}{2}$ (b) $\frac{59}{12}$ (c) $\frac{3}{2}$ (d) $\frac{7}{3}$

85. The area (in sq units) of the region $\{(x, y): y^2 \geq 2x$ and $x^2 + y^2 \leq 4x, x \geq 0, y \geq 0\}$ is **[2016 JEE Main]**
(a) $\pi - \frac{4}{3}$ (b) $\pi - \frac{8}{3}$
(c) $\pi - \frac{4\sqrt{2}}{3}$ (d) $\frac{\pi}{2} - \frac{2\sqrt{2}}{3}$

86. The area (in sq units) of the region described by $\{(x, y): y^2 \leq 2x$ and $y \geq 4x - 1\}$ is **[2015 JEE Main]**
(a) $\frac{7}{32}$ (b) $\frac{5}{64}$
(c) $\frac{15}{64}$ (d) $\frac{9}{32}$

87. The area (in sq units) of the quadrilateral formed by the tangents at the end points of the latusrectum to the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$ is **[2015 JEE Main]**
(a) $\frac{27}{4}$ (b) 18
(c) $\frac{27}{2}$ (d) 27

88. The area of the region described by $A = \{(x, y): x^2 + y^2 \leq 1$ and $y^2 \leq 1 - x\}$ is **[2014 JEE Main]**
(a) $\frac{\pi}{2} + \frac{4}{3}$ (b) $\frac{\pi}{2} - \frac{4}{3}$
(c) $\frac{\pi}{2} - \frac{2}{3}$ (d) $\frac{\pi}{2} + \frac{2}{3}$

89. The area (in sq units) bounded by the curves $y = \sqrt{x}$, $2y - x + 3 = 0$, X-axis and lying in the first quadrant is **[2013 JEE Main]**
(a) 9 (b) 36
(c) 18 (d) $\frac{27}{4}$

90. The area bounded between the parabolas $x^2 = \frac{y}{4}$ and $x^2 = 9y$ and the straight line $y = 2$ is **[2012 AIEEE]**
(a) $20\sqrt{2}$ (b) $\frac{10\sqrt{2}}{3}$
(c) $\frac{20\sqrt{2}}{3}$ (d) $10\sqrt{2}$

91. The area of the region enclosed by the curves $y = x$, $x = e$, $y = \frac{1}{x}$ and the positive X-axis is **[2011 AIEEE]**
(a) 1 sq unit (b) $\frac{3}{2}$ sq units
(c) $\frac{5}{2}$ sq units (d) $\frac{1}{2}$ sq unit

92. The area bounded by the curves $y = \cos x$ and $y = \sin x$ between the ordinates $x = 0$ and $x = \frac{3\pi}{2}$ is **[2010 AIEEE]**
(a) $(4\sqrt{2} - 2)$ sq units (b) $(4\sqrt{2} + 2)$ sq units
(c) $(4\sqrt{2} - 1)$ sq units (d) $(4\sqrt{2} + 1)$ sq units

93. The area of the region bounded by the parabola $(y - 2)^2 = x - 1$, the tangent to the parabola at the point $(2, 3)$ and the X-axis is **[2009 AIEEE]**
(a) 6 sq units (b) 9 sq units
(c) 12 sq units (d) 3 sq units

94. The area of the plane region bounded by the curves $x + 2y^2 = 0$ and $x + 3y^2 = 1$ is equal to **[2008 AIEEE]**
(a) $\frac{5}{3}$ sq units (b) $\frac{1}{3}$ sq unit
(c) $\frac{2}{3}$ sq unit (d) $\frac{4}{3}$ sq units

Answers

Exercise for Session 1

1. 1 sq unit **2.** $\frac{436}{15}$ sq units **3.** $\frac{4}{3}$ sq units

4. $\frac{4}{3}$ sq units **5.** $\frac{28}{3}$ sq units **6.** $\frac{56}{3}a^2$ sq units

7. $\frac{8}{3}a^2$ sq units **8.** 2z sq units **9.** $\frac{28}{3}$ sq units

10. 3+16 log2 sq units

Exercise for Session 2

1. (d) **2.** (a) **3.** (c) **4.** (d) **5.** (a)
6. (b) **7.** (a) **8.** (a) **9.** (c) **10.** (b)
11. (d) **12.** (b) **13.** (a) **14.** (c) **15.** (b)
16. (a) **17.** (c) **18.** (a) **19.** (a) **20.** (b)

Chapter Exercises

1. (d) **2.** (b) **3.** (b) **4.** (c) **5.** (b)
6. (c) **7.** (d) **8.** (a) **9.** (b) **10.** (b)
11. (b) **12.** (b) **13.** (c) **14.** (d) **15.** (a)
16. (a) **17.** (a) **18.** (a) **19.** (c) **20.** (b)
21. (a) **22.** (b) **23.** (d) **24.** (c) **25.** (b, c)
26. (a, b, d) **27.** (a, d) **28.** (a, d) **29.** (a, c)
30. (d) **31.** (c) **32.** (d) **33.** (b) **34.** (c)
35. (a) **36.** (b) **37.** (a) **38.** (a) **39.** (a)
40. (b) **41.** (c) **42.** (b) **43.** (a)

44. (A) $\rightarrow$ (p); (B) $\rightarrow$ (p); (C) $\rightarrow$ (r); (D) $\rightarrow$ (s)

45. (A) $\rightarrow$ (p); (B) $\rightarrow$ (p); (C) $\rightarrow$ (r); (D) $\rightarrow$ (r)

46. (7) **47.** (1) **48.** (4) **49.** (7)
50. (8) **51.** (5) **52.** (3)

53. 3 e sq units **54.** $f(x) = \frac{1}{k+1}[x^4 - kx^3 + (2k-4)x^2]$

55. $g(x) = \begin{cases} -x-1, & x \le -5/2 \\ 4+x, & -5/2 < x \le -2 \\ 2, & -2 < x \le -1 \\ 1-x, & -1 < x \le -1/2 \\ 1+x, & x > -1/2 \end{cases}$

and area = $\frac{101}{4}$ sq units

56. $\left[2 + \log\left(\frac{4}{3\sqrt{3}}\right) - \frac{1}{e}\right]$ sq units

57. $\left(\pi - \frac{2}{3}\right)$ sq units **58.** $\frac{1}{8}\left(\frac{3\pi}{16}\right)^4$ sq units

59. Required maximum ratio = $\frac{2}{3}$

60. $(A)_{\max} = \frac{\sqrt{3}\,\pi A}{9}$ sq units

61. $A_1 = 1 - \sin 1$, $A_2 = \pi - 1 - \sin 1$, $A_3 = 3\pi - 2$

62. $4\log_5\left(\frac{e^4}{27}\right)$ sq units

63. $4\left(\frac{16}{3} - 4\pi\right)$ sq units

64. $\frac{3}{2}$ **65.** 2 sq units

66. 8 sq units **67.** 5 sq units

68. $\left(2 - \frac{\pi}{2}\right)$ sq units

69. $\left[\frac{4}{\log 2}(1 - 2^{-1/2}) - 1\right]$ sq units

70. $a \in [1, 2)$ **72.** (c) **73.** (3) **74.** (d) **75.** (b)
76. (b,d) **77.** (c) **78.** (b) **79.** (b,c,d) **80.** (b) **81.** (b)
82. (a) **83.** (d) **84.** (a) **85.** (b) **86.** (d) **87.** (d)
88. (a) **89.** (a) **90.** (c) **91.** (b) **92.** (a) **93.** (b)
94. (d)

Solutions

1. Here, $[x+y]=[x]-1$

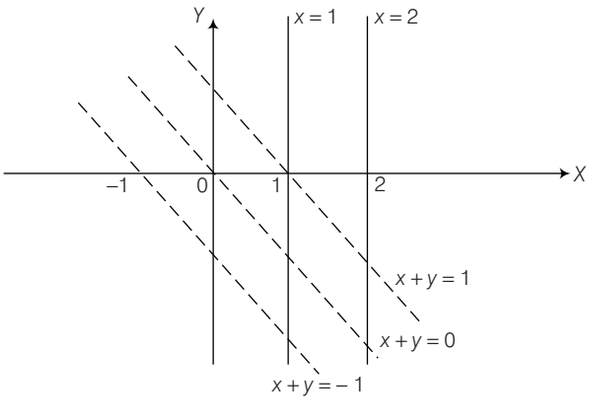
when $x \in [0,1) \Rightarrow [x+y]=-1$

$$-1 \le x+y<0 \quad \text{...(i)}$$

when $x \in [1,2) \Rightarrow [x+y]=0$

$$\therefore \quad 0 \le x+y<1 \quad \text{...(ii)}$$

which can be shown, as



$\therefore$ Required area $= 2$

2. Required area $= 2\int_0^{1/2} f^{-1}(x)\,dx$

Let, $f^{-1}(x)=t \Rightarrow x=f(t)$

$$dx=f'(t)dt$$

$$\therefore \quad A=2\int_0^1 (t\cdot f'(t)dt$$

$$=2\left[\left(t\cdot f\ (t)\right)_0^1-\int_0^1 1\cdot f(t)dt\right]$$

$$=2\left[f(1)-\int_0^1 \frac{t}{1+t^2}\,dt\right]$$

$$=2\left[f(1)-\frac{1}{2}\left(\log(1+t^2)\right)_0^1\right]$$

$$=2\left[f(1)-\frac{1}{2}\log 2\right]$$

$$=2\left[\frac{1}{2}-\frac{1}{2}\log 2\right]=1-\log 2=\log\left(\frac{e}{2}\right)$$

3. Here, $E_1:\dfrac{\left(\frac{x+y-1}{\sqrt{2}}\right)^2}{12}+\dfrac{\left(\frac{x-y+3}{\sqrt{2}}\right)^2}{(\sqrt{2})^2}=1$

Length of latusrectum $= 2\dfrac{a^2}{b}=\dfrac{2}{\sqrt{2}}=\sqrt{2}$

Now, $\dfrac{2p^2}{\sqrt{p}}=\sqrt{2} \Rightarrow p^{3/2}=2^{-1//2}$

$\Rightarrow \quad P=2^{-1/3}$

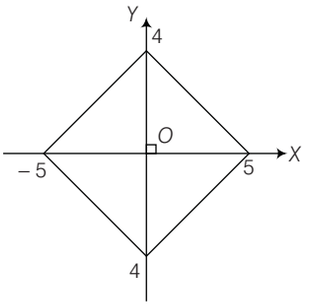
$\therefore \quad E_2:\dfrac{x^2}{2^{-1/3}}+\dfrac{y^2}{4^{-1/3}}=1$

Area of ellipse E_2, is

$$\pi\cdot\sqrt{p}\cdot p=\pi\ p^{3/2}=\frac{\pi}{\sqrt{2}}$$

4. Area of bounded region by $4\,|x-2017^{2017}|+5\,|y-2017^{2017}|\le 20$, is same as area of the region bounded by $4|x|+5|y|\le 20$.

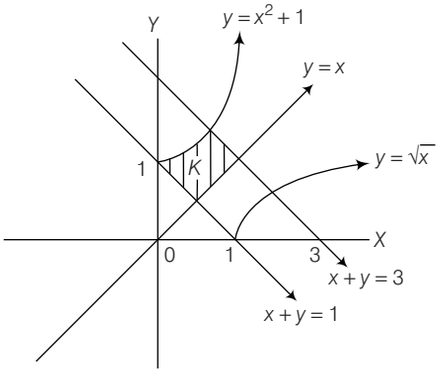
$\Rightarrow \quad 4\times\dfrac{1}{2}\times 4\times 5=40$



5. Here, $y=x^2+1$ and $y=\sqrt{x-1}$ are inverse of each other.

The shaded area is given K units

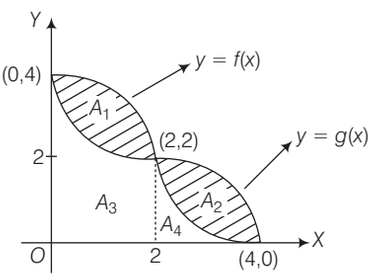
$\Rightarrow$ Area of the region bounded by $y=x^2+1, y=\sqrt{x-1}$ and $(x+y-1)(x+y-3)=0$, is $2K$ units.



6. Given, $\int_0^4 f(x)\,dx-\int_0^4 g(x)\,dx=10$

$$(A_1+A_3+A_4)-(A_2+A_3+A_4)=10$$

$$A_1-A_2=10 \quad \text{...(i)}$$



Again, $\int_2^4 g(x)\,dx-\int_2^4 f(x)\,dx=5$

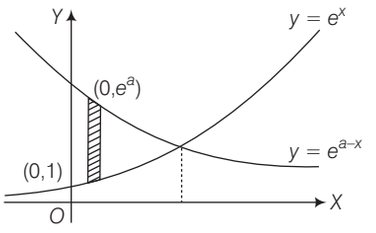
$$(A_2+A_4)-A_4=5$$

$$A_2=5 \quad \text{...(ii)}$$

Adding Eqs. (i) and (ii),

$$A_1=15$$

7. Solving, $e^x = e^{a-x}$, we get $e^{2x} = e^a \Rightarrow x = \frac{a}{2}$



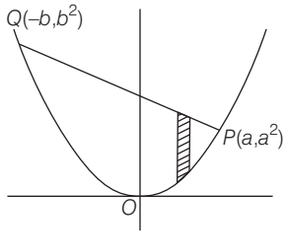
$$S = \int_0^{a/2} (e^a \cdot e^{-x} - e^x)\, dx = [-(e^a \cdot e^{-x} + e^x)]_0^{a/2}$$

$$= (e^a + 1) - (e^{a/2} + e^{a/2}) = e^a - 2e^{a/2} + 1 = (e^{a/2} - 1)^2$$

$$\therefore \quad \frac{S}{a^2} = \left(\frac{e^{a/2}-1}{a}\right)^2 = \frac{1}{4}\left(\frac{e^{a/2}-1}{a/2}\right)^2$$

$$\therefore \quad \lim_{a\to 0} \frac{S}{a^2} = \frac{1}{4}$$

8. $m_{PQ} = \frac{a^2 - b^2}{a+b} = a - b$ equation of PQ



$$y - a^2 = \frac{a^2 - b^2}{a+b}(x - a) \quad \text{or} \quad y - a^2 = (a-b)(x-a)$$

$$y = a^2 + x(a-b) - a^2 + ab$$

$$y = (a-b)x + ab$$

$$\therefore \quad S_1 = \int_{-b}^{a} ((a-b)x + ab - x^2)\, dx$$

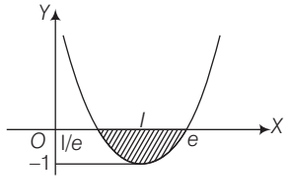
which simplifies to $\frac{(a+b)^3}{6}$. ...(i)

Also, $S_2 = \frac{1}{2}\begin{vmatrix} a & a^2 & 1 \\ -b & b^2 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{2}[ab^2 + a^2 b] = \frac{1}{2}ab\,(a+b)$ ...(ii)

$$\therefore \quad \frac{S_1}{S_2} = \frac{(a+b)^3}{6} \cdot \frac{2}{ab\,(a+b)} = \frac{(a+b)^2}{3ab} = \frac{1}{3}\left[\frac{a}{b} + \frac{b}{a} + 2\right]$$

$$\therefore \quad \left.\frac{S_1}{S_2}\right|_{\min} = \frac{4}{3}$$

9. $y = \ln^2 x - 1 \Rightarrow y' = \frac{2\ln x}{x} = 0 \quad \Rightarrow \quad x = 1$



$x > 1$, y increasing and $0 < x < 1$, y is decreasing

$$A = \left| \int_{1/e}^{e} (\ln^2 x - 1)\, dx \right|$$

$$= \left| \int_{1/e}^{e} \ln^2 x\, dx - \int_{1/e}^{e} dx \right|$$

$$= \left| [x \ln^2 x]_{1/e}^{e} - 2\int_{1/e}^{e} \left(\frac{\ln x}{x}\right) \cdot x\, dx - \left(e - \frac{1}{e}\right) \right|$$

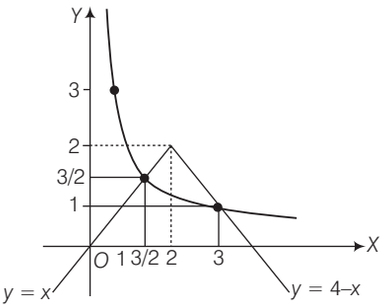
$$= \left| \left(e - \frac{1}{e}\right) - 2\int_{1/e}^{e} \left(\frac{\ln x}{x}\right) \cdot x\, dx - \left(e - \frac{1}{e}\right) \right|$$

$$= \left| -2\left[[x \ln x]_{1/e}^{e} - \int_{1/e}^{e} dx \right] \right|$$

$$= \left| -2\left[\left(e + \frac{1}{e}\right) - \left(e - \frac{1}{e}\right) \right] \right| = \left|\frac{4}{e}\right| = \frac{4}{e}$$

10. $y = \begin{cases} 2 - (2 - x), & \text{if } x \le 2 = x \\ 2 - (x - 2), & \text{if } x \ge 2 = 4 - x \end{cases} = \begin{cases} x, & \text{if } x \le 2 \\ 4 - x, & \text{if } x \ge 2 \end{cases}$

Also, $y = \begin{cases} \frac{3}{x}, & \text{if } x > 0 \\ -\frac{3}{x}, & \text{if } x < 0 \end{cases}$



$$A = \int_{3/2}^{2} \left(x - \frac{3}{x}\right) dx + \int_{2}^{3} \left((4 - x) - \frac{3}{x}\right) dx$$

$$= \left[\frac{x^2}{2} - 3\ln x\right]_{3/2}^{2} + \left[4x - \frac{x^2}{2} - 3\ln x\right]_{2}^{3}$$

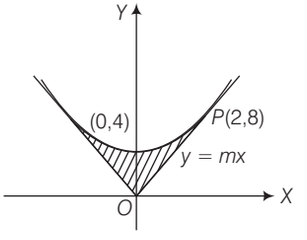
$$= \left[2 - 3\ln 2 - \left(\frac{9}{8} - 3\ln\frac{3}{2}\right)\right] + \left[12 - \frac{9}{2} - 3\ln 3 - (8 - 2 - \ln 2)\right]$$

$$= \frac{7}{8} - 3\ln 2 + 3\ln 3 - 3\ln 2 + \frac{3}{2} - 3\ln 3 + 3\ln 2 = \frac{19}{8} - 3\ln 2$$

11. Given, $g(x) = 2x + 1$; $h(x) = (2x+1)^2 + 4$

Now, $h(x) = f[g(x)]$

$$(2x+1)^2 + 4 = f(2x+1)$$



Let $2x+1=t \Rightarrow f(t)=t^2+4$

$\therefore$ $f(x)=x^2+4$ ...(i)

Solving, $y=mx$ and $y=x^2+4$

$x^2-mx+4=0$

Put $D=0$; $m^2=16 \Rightarrow m=\pm 4$

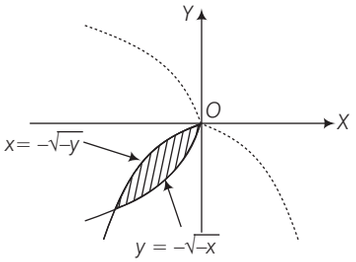
Tangents are $y=4x$ and $y=-4x$

$$A=2\int_0^2[(x^2+4)-4x]\,dx=2\int_0^2(x-2)^2dx$$

$$=\left[\frac{2}{3}(x-2)^3\right]_0^2=\frac{16}{3}\text{ sq units}$$

12. $y=-\sqrt{-x} \Rightarrow y^2=-x$, where x and y both negative

$x=-\sqrt{-y} \Rightarrow x^2=-y$, where x and y both negative

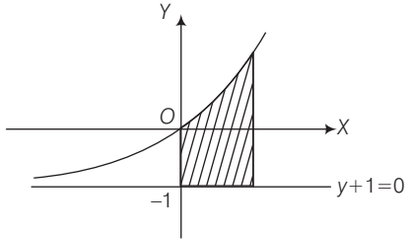


Since, $A=\frac{16ab}{3}$ where, $a=b=\frac{1}{4}$

$\therefore$ $A=\frac{1}{3}$

13. $\frac{f''(x)}{f'(x)}=1.$

Integrating, $\ln f'(x)=x+C$, $f'(0)=1 \Rightarrow C=0$

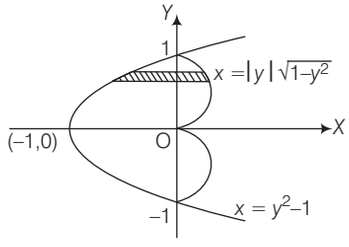


$f'(x)=e^x f(x)=e^x+k$, $f(0)=0 \Rightarrow k=-1$

$f(x)=e^x-1$

Area $=\int_0^1(e^x-1+1)\,dx=[e^x]_0^1=e-1$

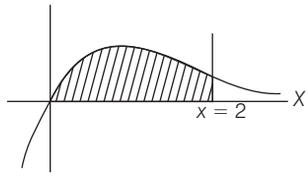
14. $A=2\int_0^1[y\sqrt{1-y^2}-(y^2-1)]\,dy=2$



15. $y=xe^{-x}$

$y'=e^{-x}-xe^{-x}=(1-x)e^{-x}$ increasing for $x<1$

$y''=-e^{-x}-[e^{-x}-xe^{-x}]=e^{-x}[-1-1+x]=(x-2)e^{-x}$



For point of inflection $y''=0 \Rightarrow x=2$

$$A=\int_0^2 xe^{-x}dx=[-xe^{-x}]_0^2+\int_0^2 e^{-x}=(-2e^{-2})-(e^{-x})_0^2$$

$$=-2e^{-2}-(e^{-2}-1)=1-e^{-2}-2e^{-2}=1-3e^{-2}$$

16. $(a, 0)$ lies on the given curve

$\therefore$ $0=\sin 2a-\sqrt{3}\sin a \Rightarrow \sin a=0$ or $\cos a=\sqrt{3}/2$

$\Rightarrow$ $a=\frac{\pi}{6}$ (as $a>0$ and the first point of intersection with positive X-axis)

and $A=\int_0^{\pi/6}(\sin 2x-\sqrt{3}\sin x)\,dx=\left(-\frac{\cos 2x}{2}+\sqrt{3}\cos x\right)_0^{\pi/6}$

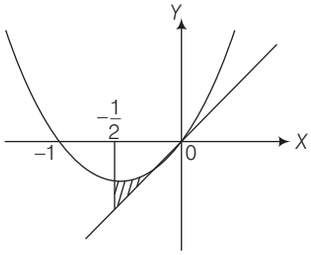
$$=\left(-\frac{1}{4}+\frac{3}{2}\right)-\left(-\frac{1}{2}+\sqrt{3}\right)=\frac{7}{4}-\sqrt{3}=\frac{7}{4}-2\cos a$$

$\Rightarrow$ $4A+8\cos a=7$

17. $x=1; y=2$

$2=a+b+c$ ...(i)

$x=0, y=0 \Rightarrow c=0 \Rightarrow a+b=2$



Now, $\left.\frac{dy}{dx}\right|_{(0,0)}=2a\ x+b=1$

$\therefore$ $b=1, a=1$

Hence, the curve is $y=x^2+x$

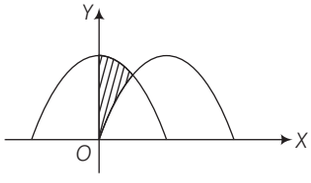
$$A=\int_{-\frac{1}{2}}^0(x^2+x-x)\,dx=\int_{-\frac{1}{2}}^0(x^2)\,dx=\frac{1}{24}\text{ sq units}$$

18. IF $=e^{-x}$

$\therefore$ $ye^{-x}=\int e^{-x}(\cos x-\sin x)\,dx$. Put $-x=t$

$$=-\int e^t(\cos t+\sin t)\,dt=-e^t\sin t+C$$

$ye^{-x}=e^{-x}\sin x+C$



Since, y is bounded when $x\to\infty \Rightarrow C=0$

$\therefore \quad y = \sin x$

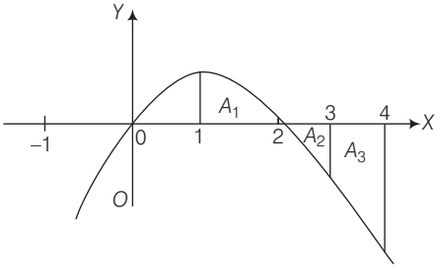
$$\text{Area} = \int_0^{\pi/4} (\cos x - \sin x)\, dx = \sqrt{2} - 1$$

19. $I = \int (6x - 3x^2)\, dx = \dfrac{6x^2}{2} - \dfrac{3x^3}{3} = 3x^2 - x^3 = x^2(3 - x)$

$A_1 = I(2) - I(1) = 4 - 2 = 2$ units

$A_2 = I(2) - I(3) = 4 - 0 = 4$ units

$A_3 = I(3) - I(4) = 0 - (-16) = 16$ units



$\Rightarrow$ One value of a will lie in (3, 4).

Using symmetry, other will lie in (−2, −1).

20. Required area, $A = \int_{\pi}^{2\pi} ((\sin x - x) + 2\pi)\, dx$

$$= \frac{\pi^2}{2} - 2 \text{ sq units}$$

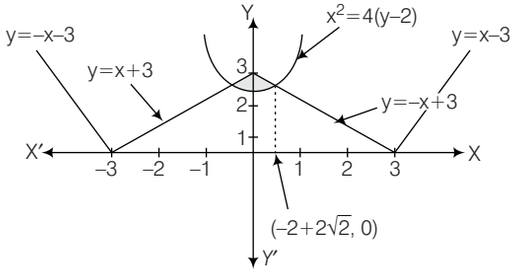
21. $g(x) = |\, f(|\, x\,|) - 2\,| = |\,|\, x\,| - 1 - 2\,| = |\,|\, x\,| - 3\,|$

$$= \begin{cases} (|\,x\,| - 3), & x < -3 \\ -(|\,x\,| - 3), & -3 \le x < 3 \\ |\,x\,| - 3 & x \ge 3 \end{cases} = \begin{cases} -x - 3, & x < -3 \\ -(-x - 3), & -3 \le x < 0 \\ -(x - 3), & 0 \le x < 3 \\ x - 3, & x \ge 3 \end{cases}$$

$$= \begin{cases} -x - 3, & x < -3 \\ x + 3, & -3 \le x < 0 \\ -x + 3, & 0 \le x < 3 \\ x - 3, & x \ge 3 \end{cases}$$

Now, $\quad x^2 - 4y + 8 = 0$

$\Rightarrow \quad x^2 = 4(y - 2)$



For point of intersection,

$$x^2 - 4(-x + 3) + 8 = 0$$

$\Rightarrow \quad x^2 + 4x - 4 = 0$

$\Rightarrow \quad x = \dfrac{-4 \pm \sqrt{16 + 16}}{2} = -2 \pm 2\sqrt{2}$

$\therefore$ Point of intersection is at $x = -2 + 2\sqrt{2}$

$\therefore$ Required area $= 2\int_0^{-2+2\sqrt{2}} \left[(-x + 3) - \left(\dfrac{x^2 + 8}{4}\right)\right] dx$

$$= \frac{2}{4}\int_0^{-2+2\sqrt{2}} (4 - 4x - x^2)\, dx = \frac{1}{2}\left[4x - 2x^2 - \frac{x^3}{3}\right]_0^{-2+2\sqrt{2}}$$

$$= \frac{1}{6}\,[12x - 6\,x^2 - x^3]_0^{-2+2\sqrt{2}}$$

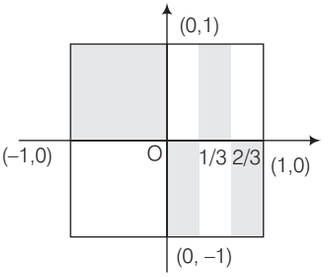
$$= \frac{1}{6}[12(-2 + 2\sqrt{2}) - 6(-2 + 2\sqrt{2})^2 - (-2 + 2\sqrt{2})^3]$$

$$= \frac{1}{6}\,[-24 + 24\sqrt{2} - 6(4 + 8 - 8\sqrt{2}) - (-8 + 16\sqrt{2}) + 24\sqrt{2} - 48]$$

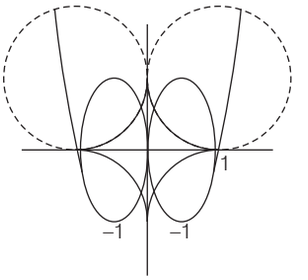
$$= \frac{1}{6}\,[-24 + 24\sqrt{2} - 72 + 48\sqrt{2} + 56 - 40\sqrt{2}]$$

$$= \frac{1}{6}\,[32\sqrt{2} - 40] = \frac{8}{6}(4\sqrt{2} - 5) = \frac{4}{3}(4\sqrt{2} - 5)$$

22. Shaded region represents $S \cap S'$ clearly area enclosed is 2 sq units.



23. Required area is $2\left[\int_0^{\alpha} ex \ln x\, dx + \int_1^{\alpha} (1 - \sqrt{1 - (x-1)^2})\, dx\right]$



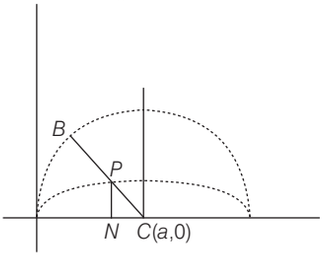
24. $y = \sqrt{2ax - x^2} \Rightarrow (x - a)^2 + y^2 = a^2$

Let $P(h,k)$ be a point, then $BP > PN$

For the bounded condition $BP = PN = k$

Now, $\quad AP = a - k = \sqrt{(h - a)^2 + k^2}$

$\Rightarrow \quad k = h - \dfrac{h^2}{2a}$



$\therefore$ Boundary of the region is $y = x - \frac{x^2}{2a}$

Required area $= 2\int_0^a \left(x - \frac{x^2}{2a}\right) dx = \frac{2a^2}{3}$

25. $f'(x) = 2x - a$. At (2, 4), $f'(x) = 4 - a$

Equation of normal at (2, 4) is $(y-4) = -\frac{1}{(4-a)}(x-2)$

Let point of intersection with X and Y-axes be A and B respectively, then

$$A \equiv (-4a + 18, 0)\ B \equiv \left(0, \frac{4a-18}{a-4}\right)$$

Since, $a > \frac{9}{2}$ as

$\therefore$ Area of triangle $= \frac{1}{2}(4a-18)\frac{(4a-18)}{(a-4)} = 2$

$\Rightarrow$ $(4a-17)(a-5) = 0$

$\Rightarrow$ $a = 5$ or $\frac{17}{4}$

26. Max $\{f(x), g(x)\} = \frac{1}{2}[|f(x)+g(x)|+|f(x)-g(x)|]$

Min $\{f(x), g(x)\} = \frac{1}{2}[|f(x)+g(x)|-|f(x)-g(x)|]$

$\therefore$ Area $= \int_a^b [\max\{f(x), g(x)\} - \min\{f(x), g(x)\}]\, dx$

27. Area bounded by parabola $y = x^2 - 7x + 10$ and X-axis is given by

$$\int_2^5 |x^2 - 7x + 10|\, dx = \int_2^5 |-x^2 + 7x - 10|\, dx = \frac{9}{2} \text{ sq units}$$

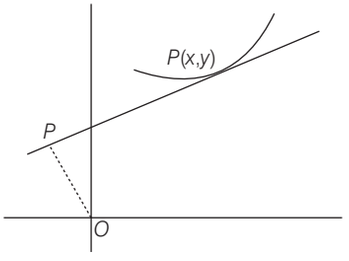
28. Area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ will be the same as the area bounded by the ellipse $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ and πab

$\therefore$ Required area $= \pi(2)(3) = 6\pi$ sq units

29. $OP = x$

If slope is $\frac{dy}{dx}$, then equation of tangent is

$$Y - t = \frac{dy}{dx}(X - x)$$



Length of perpendicular from origin to this tangent is

$$x = \frac{Y - x\frac{dy}{dx}}{\sqrt{1 + \left(\frac{dy}{x}\right)^2}}$$

$$x^2\left\{1 + \left(\frac{dy}{dx}\right)^2\right\} = y^2 + x^2\left(\frac{dy}{dx}\right)^2 = 2xy\frac{dy}{dx}$$

$\Rightarrow$ $\frac{dy}{dx} = -\frac{x^2 - y^2}{2xy}$ [homogeneous form]

$$\frac{dy}{dx} = \frac{1-x}{y} = \frac{x - x^2}{xy} = \frac{\left[\frac{x^2+y^2}{2}\right] - x^2}{xy} = \frac{y^2 - x^2}{2xy}$$

Since, option (a) is true (equation of circle).

If option (a) is true (b) can't be (It is parabola).

If option (a) is true, option (c) is also true where $c = 1$.

30. $\therefore$ $\sin^2 x \le \sin x$, $\forall\ x \in (0, \pi)$

Therefore, area of $y = \sin^2 x$ will be lesser from area of $y = \sin x$.

Statement II is obviously true.

Hence, (d) is the correct answer.

31. Let the line $y = kx + 2$ cuts $y = x^2 - 3$ at $x = \alpha$ and $\alpha = \beta$, area bounded by the curves $= \int_\alpha^\beta (y_1 - y_2) = \int_\alpha^\beta \{(kx+2) - (x^2 - 3)\}\, dx$

$\Rightarrow$ $f(x) = \frac{(k^2+20)^{3/2}}{6}$

which clearly, shows the Statement II is false but $f(k)$ is least when $k = 0$.

Hence, (c) is the correct answer.

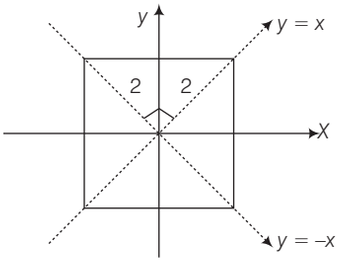
32. As of region bounded by parabola $y^2 = 4x$ and $x^2 = 4y$ is

$$\int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx = \left[\frac{4}{3}x^{3/2} - \frac{x^3}{12}\right]_0^4 = \frac{32}{3} - \frac{16}{3} = \frac{16}{3} \text{ sq units}$$

Hence, Statement I is false.

33. As the area enclosed by $|x| + |y| \le a$ is the area of square (i.e., $2a^2$).

$\therefore$Area enclosed by $|x+y| + |x-y| \le 2$ is area of square shown as



$\therefore$ Area $= 4\left(\frac{1}{2} \times 2 \times 2\right) = 8$ sq units

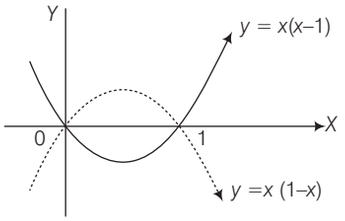
Also, the area enclosed by $|x+y| + |x-y| \le 2$ is symmetric about X-axis, Y-axis, $y = x$ and $y = -x$.

$\therefore$Both the Statements are true but Statement II is not the correct explanation of Statement I.

34. $\left|\int_a^b \{f(x) - g(x)\}\, dx\right|$ is true for all quadrants.

$\therefore$ Statement II is false.

The area bounded by $y = x(x-1)$ and $y = x(1-x)$.



$\therefore$ Area enclosed $= 2\int_0^1 x(1-x)\,dx = 2\left\{\frac{x^2}{2}-\frac{x^3}{3}\right\}_0^1$

$= 2\left(\frac{1}{2}-\frac{1}{3}\right) = 2\left(\frac{3-2}{6}\right) = \frac{1}{3}$

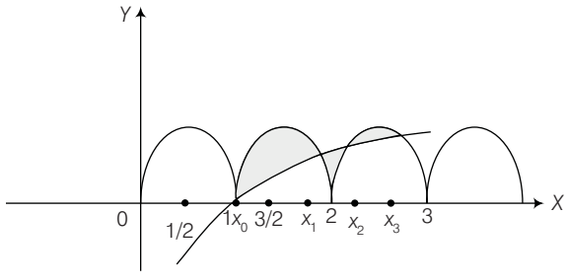
35. Since, absolute area

$= \int_{x_0}^{x_1} h(x)\,dx + \int_{x_1}^{x_2} -h(x)\,dx + \int_{x_2}^{x_3} h(x)\,dx$

$= \sum_{r=0}^{n} \int_{x_r}^{x_{r+1}} (-1)^r \cdot h(x)\,dx$

36. Also, $x_{n+1} = x_3 \Rightarrow n = 2$

37. Required area $= \int_0^1 \sin^4 \pi x\,dx - \int_0^1 \log_e x\,dx = \frac{11}{8}$



38. $f(x) = \frac{x^2-ax+1}{x^2+ax+1}$

For differentiation, better write $f(x)$ as

$$f(x) = 1 - \frac{2ax}{x^2+ax+1}$$

Now, on differentiation $f(x) = \frac{2a(x^2-1)}{(x^2+ax+1)^2}$ ...(i)

$$f'(1) = 0 = f'(-1)$$

Then, the options (b) and (d) are eliminated.

Again, for the differentiating Eq. (i) gives

$$(x^2+ax+1)^2 \cdot 2x - (x^2-1)$$

$$f''(x) = 2a \cdot \frac{2(x^2+ax+1)(2x+a)}{(x^2+ax+1)^4}$$

$$f''(-1) = -\frac{4a}{(2-a)^2},\ f''(1) = -\frac{4a}{(2+a)^2}$$

Combining both $f''(1)(2+a)^2 + f''(-1)(2-a)^2 = 0$

39. $f'(x) = 2a\frac{(x^2-1)}{(x^2+ax+1)^2} = 2a\frac{(x-1)(x+1)}{(x^2+ax+1)^2}$

If is easily seen that $f(x)$ decreases on $(-1,1)$ and has a local minimum at $x = 1$, because the derivatives changes its sign from negative to positive.

40. $\because$ $g'(x) = \frac{f'(e^x)e^x}{1+e^{2x}}$

Now, $f'(e^x) = \frac{2a\cdot(e^{2x}-1)}{(e^{2x}+ae^x+1)^2}$

It is seen from the above that $f'(e^x)$ and so $g'(x)$ is positive on $(0,\infty)$ and negative on $(-\infty,0)$.

41. Clearly, $x = a\sin^3 t, y = a\cos^3 t, (0 \le t \le 2\pi)$

$$S = \int_0^{2\pi} a\sin^3 t \cdot a \cdot 3\sin^2 t(-\sin t)\,dt$$

$$= -3a^2 \times \int_0^{\pi/2} \sin^4 t\cos^2 t\,dt$$

$$= -3a^2 \times \frac{\sqrt{\frac{5}{2}}\sqrt{\frac{3}{2}}}{2\sqrt{\frac{4+2+2}{2}}} = -3a^2\frac{\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}\times\frac{1}{2}\sqrt{\pi}}{2\times3\times2\times1}$$

$$= -\frac{3}{8}\pi a^2 = \frac{3}{8}\pi a^2 \qquad \text{[absolute value]}$$

42. $S = -\int_0^{2\pi} a(1-\cos t)a(1-\cos t)\,dt$

$$= -a^2\int_0^{2\pi}(1-2\cos t+\cos^2 t)\,dt$$

$$= -a^2\int_0^{2\pi}\left(1-2\cos t+\left(\frac{1+\cos 2t}{2}\right)\right)dt$$

$$= -\frac{a^2}{2}\int_0^{2\pi}[3-4\cos t+\cos 2t]_0^{2\pi}$$

$$= -3\pi a^2 = 3\pi a^2 \qquad \text{[absolute value]}$$

43. $x = \frac{t}{3}(6-t) = \frac{1}{3}(6t-t^2),\ \ y = \frac{t^2}{8}(6-t) = \frac{1}{8}(6t^2-t^3)$

$\therefore$ Area $= \int_0^6 \frac{t}{3}(6-t)\cdot\frac{1}{8}(12t-13t^2)dt$

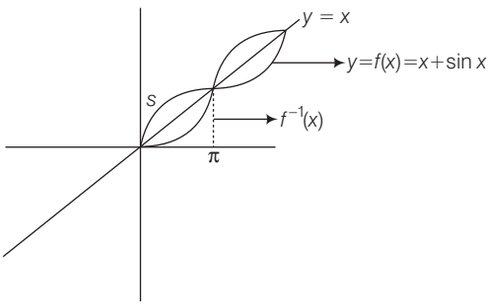
$$= \frac{1}{8}\int_0^6 (t^4+24t^2-10t^3)\,dt = \frac{1}{8}\left[\frac{t^5}{5}+8t^3-\frac{5}{2}t^4\right]_0^6$$

$$= \frac{1}{8}\left[\frac{7776}{5}+1728-3240\right] = \frac{216}{5\times8} = \frac{27}{5} \text{ sq units}$$

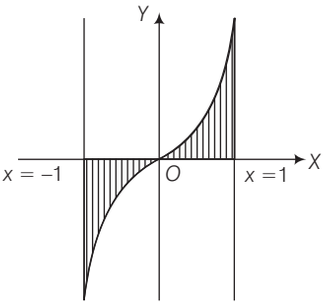
44. (A) Required area = 4s

$$s = \int_0^{\pi}(x+\sin x)\,dx - \int_0^{\pi} x\,dx$$

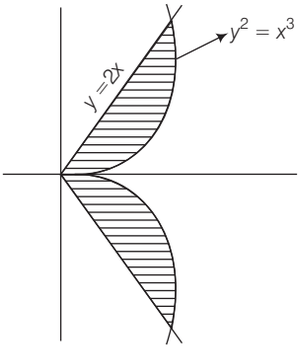
$$= \frac{\pi^2}{2} - \cos\pi + \cos 0 - \frac{\pi^2}{2} = 2 \text{ sq units}$$



(B) Required area $= 2\int_0^1 xe^x\,dx = 2[xe^x - e^x]_0^1 = 2$



(C) $y^2 = x^3$ and $|y| = 2x$ both the curve are symmetric about Y-axis

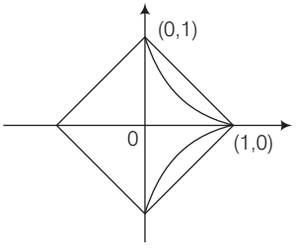


$4x^2 = x^3 \Rightarrow x = 0, 4$

Required area $= 2\int_0^4 (2x - x^{3/2})\,dx = \frac{16}{5}$

(D) $\sqrt{x} + \sqrt{|y|} = 1$

Above curve is symmetric about X-axis



$\sqrt{|y|} = 1 - \sqrt{x}$ and $\sqrt{x} = 1 - \sqrt{|y|}$

$\Rightarrow$ for $x > 0, y > 0, \sqrt{y} = 1 - \sqrt{x}$

$$\frac{1}{2\sqrt{y}}\frac{dy}{dx} = -\frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = -\sqrt{\frac{x}{y}}$$

$$\frac{dy}{dx} < 0$$

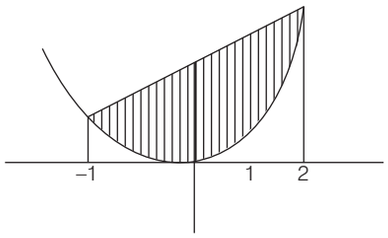
Function is decreasing required area $= \int_0^1 (2\sqrt{x} - 2x) = \frac{1}{3}$.

45. (A) The area = 2 unit

(B) Area enclosed $= \int_0^{\pi} \sin x\,dx = 2$

(C) The line $y = x + 2$ intersects $y = x^2$ at $x = -1$ and $x = 2$

The given region is shaded region area

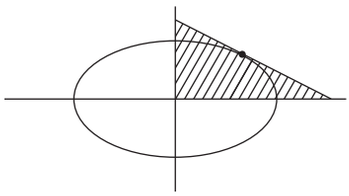


$$= \frac{15}{2} - \int_{-1}^{2} x^2\,dx = \frac{9}{2}$$

(D) Here, $a^2 = 9, b^2 = 5, b^2 = a^2(1 - e^2)$

$\Rightarrow \quad e^2 = \frac{4}{9} \Rightarrow = \frac{2}{3}$

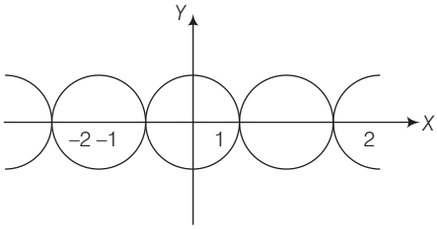
Equation of tangent at $\left(2, \frac{5}{3}\right)$ is $\frac{2x}{9} + \frac{y}{3} = 1$



x-intercept $= \frac{9}{2}$, y-intercept = 3

Area $= 4 \times \frac{9}{2} \times 3 \times \frac{1}{2} = 27$ sq units

46.



Area is given by $A = 2\int_0^1 (x^2 - 3x + 2)\,dx = \frac{5}{3}$

$\Rightarrow \quad 3A + 2 = 3 \cdot \frac{5}{3} + 2 = 7$

47. For $c < 1; \int_c^1 (8x^2 - x^5)\,dx = \frac{1}{3}$

$\Rightarrow \quad \frac{8}{3} - \frac{1}{6} - \frac{8c^3}{3} - \frac{c^6}{6} = \frac{16}{6}$

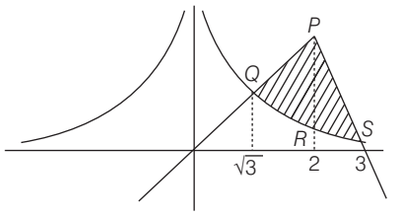
$\Rightarrow \quad c^3\left[-\frac{8}{3} + \frac{c^3}{6}\right] = \frac{16}{3} - \frac{8}{3} + \frac{1}{6} = \frac{17}{6}$

$\Rightarrow \quad c = -1$

Again, for $c \geq 1$ none of the values of c satisfy the required condition that

$\int_1^c (8x^2 - x^5)\,dx = \frac{16}{3} \Rightarrow c + 2 = 1$

48. $y=\begin{cases} x; & x<2 \\ 4-x; & x\geq 2 \end{cases}$ and $y=\begin{cases} 3/x; & x>0 \\ -3/x & x<2 \end{cases}$



Required area = $PQRSP$ = Area $PQRP$ + Area $PRSP$

$$=\left|\int_{\sqrt{3}}^{2}\left(x-\frac{3}{x}\right)dx\right|+\left|\int_{2}^{3}\left((4-x)-\frac{3}{x}\right)dx\right|$$

$$=\frac{4-3\ln 3}{2}\text{ sq units}$$

49. Equation of AB $y=\frac{5}{2}(x-2)$

Equation of BC $y-5=\frac{3-5}{6-4}(x-4)\Rightarrow y=-x+9$

Equation of CA $y-3=\frac{0-3}{2-6}(x-6)\Rightarrow y=\frac{3}{4}(x-2)$

Required area

$$=\frac{5}{2}\int_{2}^{4}(x-2)\,dx+\int_{4}^{6}-(x-9)\,dx-\frac{3}{4}\int_{2}^{6}(x-2)\,dx$$

$$=\frac{5}{2}\left[\frac{(x-2)^2}{2}\right]_2^4-\left[\frac{(x-9)^2}{2}\right]_4^6-\frac{3}{4}\left[\frac{(x-2)^2}{2}\right]_2^6$$

$$=\frac{5}{2}[2^2-0]-\frac{1}{2}[(-3)^2-(-5)^2]-\frac{3}{8}[4^2-0]$$

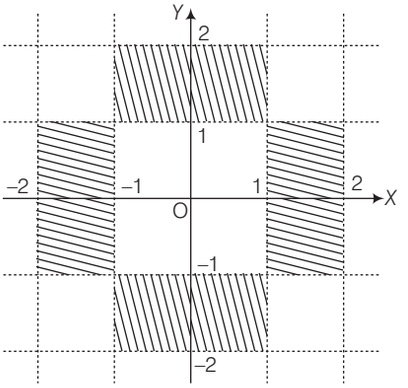
$$=\frac{5}{4}\times 4-\frac{1}{2}[9-25]-\frac{3}{8}[16-0]$$

$$=5-\frac{1}{2}[-16]-\frac{3}{8}\times 16=5+8-6=7\text{ sq units}$$

50. If $[|x|]=1$ and $[|y|]=0$, then $1\leq|x|<2, 0\leq|y|<1$

$\Rightarrow$ $x\in(-2,-1]\cup[1,2),\ y\in(-1,1)$

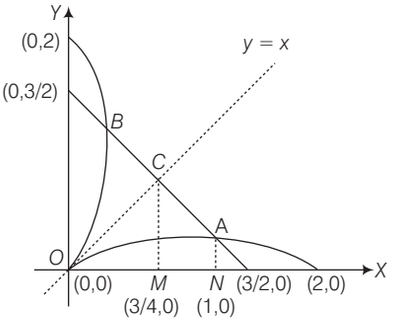
If $[|x|]=0, [|y|]=1$



Then, $x\in(-1,1),\ y\in(-2,-1]\cup[1,2]$

Area of required region $=4(2-1)(1-(-1))=8$ sq units

51. Clearly, $f(x)=\frac{2x-x^2}{2}$



Since, $2x+2y=3$; passes through $A\left(1,\frac{1}{2}\right)$ and $B\left(\frac{1}{2},1\right)$

so bounded area A

= Area OAB = 2 [Area OCM + Area $CMNA$ − Area ONA]

$$=2\left[\frac{1}{2}\times\frac{3}{4}\times\frac{3}{4}+\frac{1}{2}\left(\frac{3}{4}+\frac{1}{2}\right)\times\frac{1}{4}-\frac{1}{2}\int_0^1(2x-x^2)\,dx\right]=\frac{5}{24}$$

$\Rightarrow$ $24A=5$

52. $f(x)=\min\left\{\sin^{-1}x,\cos^{-1},x\frac{\pi}{6}\right\},\ x\in[0,1]$

$$\therefore\ \text{Area}=\int_0^{1/2}\sin^{-1}x,dx+\frac{1}{2}\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)+\int_{\sqrt{3}/2}^{1}\cos^{-1}x\,dx$$

$$=(x\sin^{-1}x+\sqrt{1-x^2})_0^{1/2}+\frac{\sqrt{3}-1}{4}$$
$$+(x\cos^{-1}x-\sqrt{1-x^2})_{\sqrt{3}/2}$$

$$=\frac{\pi}{2}(1-\sqrt{3})+\frac{\sqrt{3}-1}{2}+\frac{\sqrt{3}-1}{2}$$

$$=(\sqrt{3}-1)\left(\frac{3}{4}-\frac{\pi}{12}\right)=\frac{9-\pi}{6(\sqrt{3}+1)}=\frac{a-\pi}{b(\sqrt{3}+1)}$$

$\therefore$ $a=9, b=6\ \Rightarrow\ a-b=3$

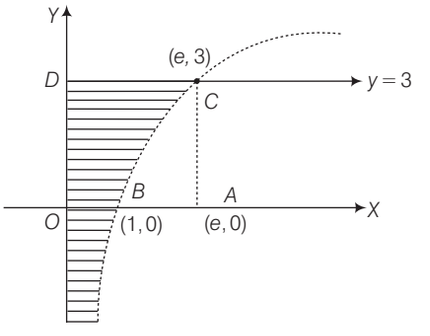
53. Given, $f\left(\frac{x}{y}\right)=f(x)-f(y)$ ...(i)

Putting, $x=y=1$, $f(1)=0$

Now, $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)}{h}$ [from Eq. (i)]

$$=\lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)}{\frac{h}{x}\cdot x}$$

$\Rightarrow$ $f'(x)=\frac{3}{x}$ $\left[\text{since, }\lim_{x\to 0}\frac{f(1+x)}{x}=3\right]$



$\Rightarrow \quad f(x) = 3 \log x + c$

Putting $\quad x = 1 \Rightarrow c = 0$

$\Rightarrow \quad f(x) = 3 \log x = y$ [say]

$\therefore$ Required area $= \int_{-\infty}^{3} x\, dy = \int_{-\infty}^{3} e^{y/3}\, dy = 3\,[e^{y/3}]_{-\infty}^{3}$

$= 3(e - 0) = 3e$ sq units

54. According to given conditions,

$$\int_0^t [f(x) - (x^4 - 4x^2)]\, dx = k \int_0^t [(2x^2 - x^3) - f(x)]\, dx$$

Differentiable both the sides w.r.t. t, we get

$$f(t) - (t^4 - 4t^2) = k((2t^2 - t^3) - f(t))$$

or $\quad (1 + k) f(t) = 2kt^2 - kt^3 + t^4 - 4t^2.$

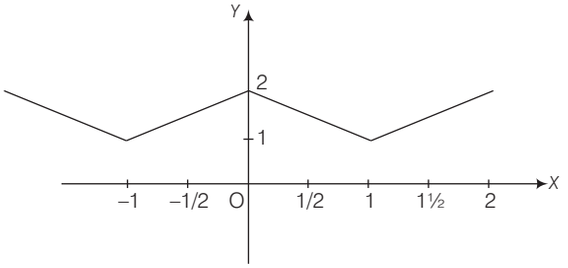
$$\Rightarrow \quad f(t) = \frac{1}{k+1}\{t^4 - kt^3 + (2k - 4)t^2\}$$

Hence, required f is given by;

$$f(x) = \frac{1}{k+1}[x^4 - kx^3 + (2k - 4)x^2]$$

55. $f(t) = |t - 1| = |t| + |t + 1|$

$$\Rightarrow \quad f(t) = \begin{cases} -t, & 1 \le -1 \\ 2 + t, & -1 < t \le 0 \\ 2 - t, & 0 < t \le 1 \\ t, & t > 1 \end{cases}$$



***Case* I** $\quad x + 2 \le -\frac{1}{2} \Rightarrow x \le -\frac{5}{2}$

$g(x) = \max\{f(t) : x + 1 \le t \le x + 2\}$

$= f(x + 1) = -x - 1,\ x \le -5/2$

***Case* II** $\quad -\frac{1}{2} < x + 2 \le 0 \Rightarrow -\frac{5}{2} < x \le -2$

$g(x) = f(x + 2) = 4 + x, -\frac{5}{2} < x \le -2$

***Case* III** $\quad 0 < x + 2 \le 1 \Rightarrow -2 < x \le -1$

$g(x) = 2$

***Case* IV** $\quad 1 < x + 2 \le 3/2 \Rightarrow -1 < x \le -1/2$

$g(x) = f(x + 1) = 1 - x$

***Case* V** $\quad x + 2 > 3/2 \Rightarrow x > -1/2$

$g(x) = f(x + 2) = 2 + x$

Hence, $\quad g(x) = \begin{cases} -x - 1, & x \le -5/2 \\ 4 + x, & -5/2 < x \le -2 \\ 2, & -2 < x \le -1 \\ 1 - x, & -1 < x \le -1/2 \\ 2 + x, & x > -1/2 \end{cases}$

Now, required area $= \int_{-3/2}^{5} g(x)\, dx$

$= \int_{-3/2}^{-1} (2)\, dx + \int_{-1}^{-1/2} (1 - x)\, dx + \int_{-1/2}^{5} (2 + x)\, dx$

$= 2\left(-1 + \frac{3}{2}\right) + \left(-\frac{1}{2} - \frac{1}{8}\right) - \left(-1 - \frac{1}{2}\right) + 2\left(5 + \frac{1}{2}\right) + \frac{1}{2}\left(25 - \frac{1}{4}\right)$

$= \frac{101}{4}$ sq units

56. It is easy to see that,

$$f(x) = \begin{cases} e^x, & 0 \le x < \log(3/2) \\ 3/2, & \log(3/2) \le x < \log(2) \\ 1 + e^x, & \log(2) \le x \le 1 \end{cases}$$

Let A be the required area. Then,

$$A = \int_0^{(\log 3/2)} e^x\, dx + \int_{\log 3/2}^{\log 2} \frac{3}{2}\, dx + \int_{\log 2}^{1} (1 + e^{-x})\, dx$$

$$= (e^x)_0^{\log 3/2} + \frac{3}{2}(x)_{\log 3/2}^{\log 2} + (x - e^{-x})_{\log 2}^{1}$$

$$= \left(\frac{3}{2} - 1\right) + \frac{3}{2}\left(\log 2 - \log\frac{3}{2}\right) + \left(1 - \frac{1}{e} - \log 2 + \frac{1}{2}\right)$$

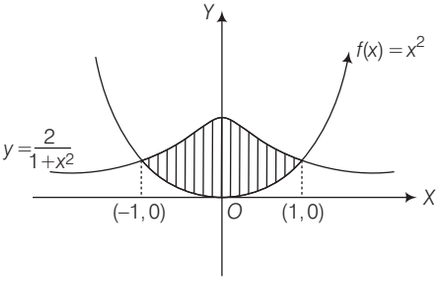
$$= \left[2 + \log\left(\frac{4}{3\sqrt{3}}\right) - \frac{1}{e}\right] \text{sq units}$$

57. $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{f(x(1 + h/x)) - f(x)}{h}$$

$$= \frac{f(x)}{x} \lim_{h \to 0} \frac{f(1 + h/x) - f(1)}{h/x} = \frac{f(x)}{x} \cdot f'(1)$$

$\therefore f'(x) = \frac{2f(x)}{x}$ or $\frac{f'(x)}{f(x)} = \frac{2}{x}$



Integrating both the sides, we get $f(x) = Cx^2$, since $f(1) = 1 \Rightarrow C = 1$.

So, $\quad f(x) = x^2$

Now, $\quad \frac{2}{1 + x^2} = x^2 \Rightarrow x^4 + x^2 - 2 = 0$

$\Rightarrow \quad x^2 = 1 \Rightarrow x = \pm 1$

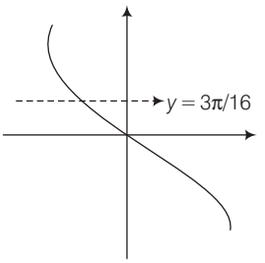
Required area $= 2\left[\int_0^1 \left(\frac{2}{1 + x^2} - x^2\right) dx\right]$

$$= 2\left[2\tan^{-1}x - \frac{x^3}{3}\right]_0^1 = 2\left[\frac{\pi}{2} - \frac{1}{3}\right]$$

$$= \left(\pi - \frac{2}{3}\right) \text{sq units}$$

58. $y(x+y^3)\,dx = x(y^3-x)\,dy$

$\Rightarrow\ xy\,dx + y^2\,dx = xy^3\,dy - x^2\,dy$



$$\Rightarrow \qquad xd(xy) = x^2y^3\left(\frac{1}{x}\,dy - \frac{y}{x^2}\,dx\right)$$

$$\Rightarrow \qquad \frac{d(xy)}{(xy)^2} = \frac{y}{x}\cdot d\left(\frac{y}{x}\right) \Rightarrow -\frac{1}{xy} = \frac{1}{2}\left(\frac{y}{x}\right)^2 + C$$

At, $x = 4,\ y = -2$

So, $$\frac{1}{8} = \frac{1}{2}\left(-\frac{1}{2}\right)^2 + C \Rightarrow C = 0$$

Hence, $$y^3 + 2x = 0$$

So, $$f(x) = (-2x)^{1/3}$$

The second equation given is

$$y = \int_{1/8}^{\sin^2 x} \sin^{-1}\sqrt{t}\,dt + \int_{1/8}^{\cos^2 x} \cos^{-1}\sqrt{t}\,dt$$

$\Rightarrow\ y' = x\cdot 2\sin x\cos x + x\cdot 2\cos x(-\sin x) = 0$

So, y is constant.

Put $\sin x = \cos x = \dfrac{1}{\sqrt{2}}$

Hence, $y = \displaystyle\int_{1/8}^{1/2} (\sin^{-1}\sqrt{t} + \cos^{-1}\sqrt{t})\,dt$

$$= \int_{1/8}^{1/2}\left(\frac{\pi}{2}\right)dt = \frac{\pi}{2}\cdot\frac{3}{8} = \frac{3\pi}{16},\ \text{and}\ g(x) = \frac{3\pi}{16}$$

So, we must find the area between $y = f(x), y = \dfrac{3\pi}{16}$

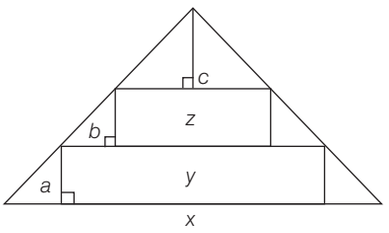
At $y = \dfrac{3\pi}{16}$; $x = -\dfrac{1}{2}\left(\dfrac{3\pi}{16}\right)^3 = P$ (say)

Hence, $$\text{area} = \int_P^0 \left(\frac{3\pi}{16} + (2x)^{1/3}\right)dx$$

$$= \left(\frac{3\pi}{16}x + 2^{1/3}\cdot\frac{x^{4/3}}{4/3}\right)_P^0 = \frac{1}{8}\left(\frac{3\pi}{16}\right)^4 \text{ sq units}$$

59. As in the figure $\dfrac{A(R)+A(S)}{A(T)} = \dfrac{ay+bz}{h\,x/2}$

where $h = a + b + c$, the altitude of T.



By similar triangles $\dfrac{x}{h} = \dfrac{y}{b+c} = \dfrac{z}{c}$,

So, $$\frac{A(R)+A(S)}{A(T)} = \frac{\frac{a(b+c)\,x}{h} + b\cdot\frac{cx}{h}}{hx/2}$$

$$= \frac{2}{h^2}(ab + ac + bc)$$

We need to maximize $(ab + bc + ca)$ subject to $a + b + c = h$. One way to do this is first to fix a, so $b + c = h - a$.

Then, $$(ab + bc + ac) = a(h - a) + bc$$

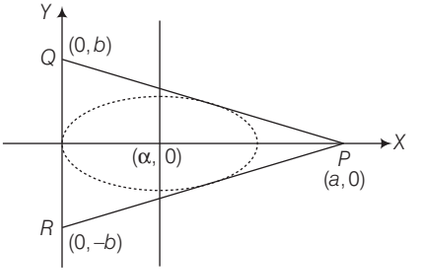
and bc is maximized when $b = c$. We now wish to maximize $2ab + b^2$ subject to $a + 2b = h$. This is a straight forward calculus problem giving $a = b = c = 1/3$. Hence, the maximum ratio is 2/3 (independent of T).

60. Consider a coordinate system with vertex P of the isosceles ΔPQR at $(a, 0)$ and Q and R at $(0, b)$ and $(0, -b)$ respectively.

$$A = \frac{1}{2}\,a\cdot 2b = ab \qquad \ldots(i)$$

Let the centre of ellipse be $(\alpha, 0)$ and the axes be of lengths, 2α and 2β.

So, the equation of ellipse is $\dfrac{(x-\alpha)^2}{\alpha^2} + \dfrac{(y)^2}{\beta^2} = 1$



Now, the line PQ is tangent to the ellipse. To apply condition of tangency, let us take a new system $x'y'$ whose origin is at $(\alpha, 0)$.

Then, $x = x' + \alpha$ and $y = y'$.

So, the ellipse becomes $\dfrac{x'^2}{\alpha^2} + \dfrac{y'^2}{\beta^2} = 1$

and the line PQ becomes $\dfrac{x'+\alpha}{a} + \dfrac{y'}{b} = 1$

which can be written as $y' = -\dfrac{b}{a}x' + b\left(1 - \dfrac{\alpha}{a}\right)$

So, $$b^2\left(1-\frac{\alpha}{a}\right)^2 = \alpha^2\left(-\frac{b}{a}\right)^2 + \beta^2$$

$$\Rightarrow \qquad \beta^2 = b^2\left(1 - \frac{2\alpha}{a}\right) \qquad \ldots(ii)$$

Now, area of ellipse $= \pi\alpha\beta \Rightarrow A^2 = \pi^2\alpha^2\beta^2$

Using Eq. (ii), $A^2 = \pi^2\alpha^2 b^2\left(1 - \dfrac{2\alpha}{a}\right) = f(x)$ (say)

$$f'(x) = \pi^2 b^2\left(2\alpha - \frac{6\alpha^2}{a}\right)$$

$$f'(\alpha) = 0 \Rightarrow 2\alpha = \frac{6\alpha^2}{a} \Rightarrow \alpha = \frac{a}{3}\,(\alpha \neq 0)$$

Since, $f''(\alpha) = \pi^2 b^2\left(2 - \frac{12\alpha}{a}\right) = -2\pi^2 b^2 = (-\text{ve})$

$(\text{at}\ \alpha = a/3)$

So, $f\left(\frac{a}{3}\right) = \pi^2 \cdot \frac{a^2}{9} \cdot b^2\left(1 - \frac{2}{3}\right) = \frac{\pi^2 A^2}{27}$ [using Eq. (i)]

Hence, $(A)_{\max} = \frac{\pi A}{3\sqrt{3}} = \frac{\sqrt{3}\pi A}{9}$ sq units

61. $A_1 = \int_0^a \{\sin x - f(x)\}\, dx = -[\cos x]_0^a - \int_0^a f(x)\, dx$

$= -(\cos a - 1) - \int_0^a f(x)\, dx = 1 - \sin a + (a-1)\cos a$ (given)

$\Rightarrow -\cos a - \int_0^a f(x)\, dx = -\sin a + (a-1)\cos a$

$\Rightarrow -\int_0^a f(x)\, dx = -\sin a + a\cos a$

$\Rightarrow \int_0^a f(x)\, dx = \sin a - a\cos a$

Differentiating w.r.t. a.

$f(a) = \cos a - (\cos a - a\sin a) = a\sin a$

$\therefore\ f(x) = x\sin x$

Now, $y = \sin x$ and $y = f(x)$ intersects at

$\Rightarrow\ a\sin a = \sin a \Rightarrow (a-1)\sin a = 0$

$\Rightarrow\ a = 1$ [as $\sin a = 0$]

Hence, $A_1 = 1 - \sin 1$

$A_2 = \int_1^{\pi} (x\sin x - \sin x)\, dx = \pi - 1 - \sin 1$

$A_3 = \int_{\pi}^{2\pi} |x\sin x - \sin x|\, dx = 3\pi - 2$

62. For $x = 1$, $y = b\cdot 5^x + 4 = 5b + 4$

and $\frac{dy}{dx} = b\cdot 5^x \log 5 \Rightarrow 5b\log 5 = 40\log 5 \Rightarrow b = 8$

The two curves intersects at points where

$8\cdot 5^x + 4 = 25^x + 16$

$\Rightarrow\ 5^{2x} - 8\cdot 5 + 12 = 0 \Rightarrow x = \log_5 2,\ x = \log_5 6$

Hence, the area of the given region;

$= \int_{\log_5 2}^{\log_5 6} \{8\cdot 5^x + 4 - (25^x + 16)\}\, dx$

$= \int_{\log_5 2}^{\log_5 6} (8\cdot 5^x - 25^x - 12)\, dx$

$= \left[\frac{8\cdot 5^x}{\log_e 5} - 12x - \frac{25^x}{\log_e 25}\right]_{\log_5 2}^{\log_5 6}$

$= \frac{-5^{\log_5 36}}{\log_e 25} + \frac{8\cdot 5^{\log_5 6}}{\log_e 5} - 12(\log_5 6 - \log_5 2) + \frac{5^{\log_5 4}}{\log_e 25} - \frac{8\cdot 5^{\log_5 2}}{\log_e 5}$

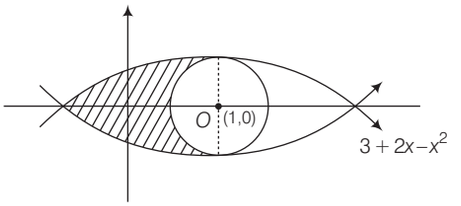
$= \frac{-36}{2\log_e 5} + \frac{48}{\log_e 5} - 12[\log_5 3] + \frac{4}{2(\log_e 5)} - \frac{16}{(\log_e 5)}$

$= \frac{16}{\log_e 5} - 12\log_5 3 = 4\log_5 e^4 - 4\log_5 25$

$= 4\log_5\left[\frac{e^4}{27}\right]$ sq units

63. By the symmetry of the figure circle(s) of maximum area will have the end point of diameter at the vertex of the two parabola.

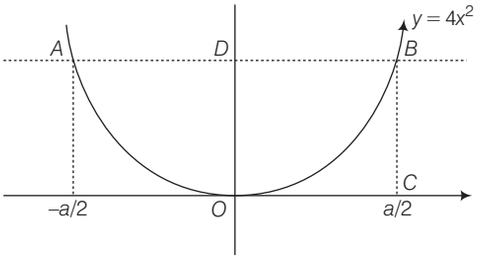
$\Rightarrow$ Radius of circle $= \frac{1}{2}\times AB = \frac{1}{2}\times 8 = 4$ units



So, the area of shaded region $= 4\times\left[\int_1^3 (3 + 2x - x^2)\, dx - \frac{1}{4}\right.$

area of circle $= 4\left(\frac{16}{3} - 4\pi\right)$ sq units

64. Area of $\Delta AOB = 2\times\left(\frac{1}{2}\times\frac{a}{2}\times a^2\right) = \frac{a^3}{2}$

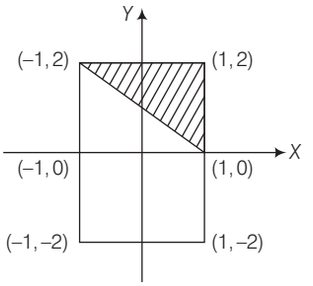


Area = Area $OCBD - \int_0^{a/2} 4x^2\, dx = \frac{a^3}{2} - \frac{a^3}{6} = \frac{a^3}{3}$

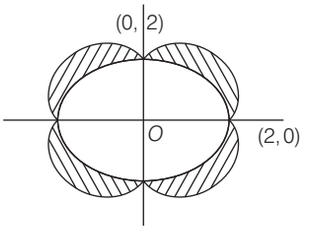
$= \lim_{x\to 0} \frac{\text{Area of triangle}}{\text{Area between the line and parabola}}$

$= \lim_{x\to 0} \frac{a^3/2}{a^3/3} = \frac{3}{2}$

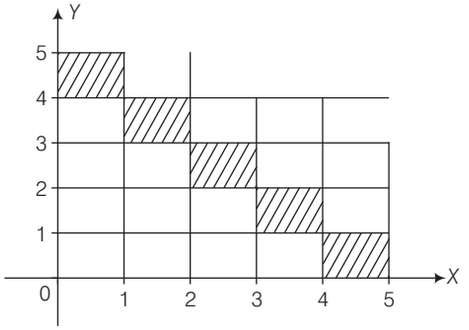
65. Required area $= \frac{1}{2}\times 2\times 2 = 2$ sq units



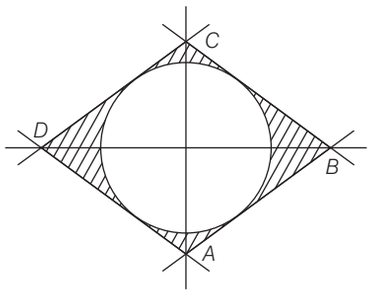
66. Required area $= 4\times\left[\frac{\pi(\sqrt{2})^2}{2} - (\pi - 2)\right] = 8$ sq units



67. 5 sq units



68. Area of the square $ABCD = 2$ sq units



Area of the circle $= \pi \times \frac{1}{2} = \frac{\pi}{2}$ sq units.

Required area $= \left(2 - \frac{\pi}{2}\right)$ sq units

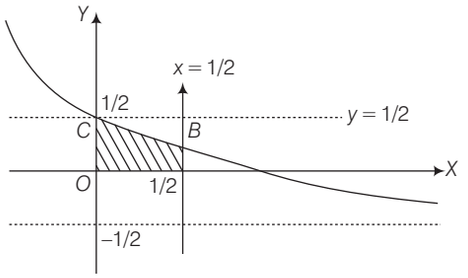
69. $2^{|x|} \cdot |y| + 2^{|x|-1} \le 1$ ...(i)

Clearly, this region is symmetrical about X and Y-axes.

Let $x < 0$, Eq. (i) gives,

$$2^{x} - y + 2^{x-1} \le 1$$

$$\Rightarrow \quad y \le \frac{1 - 2^{x-1}}{2^{x}} = 2^{-x} - \frac{1}{2}$$



Clearly, bounded region in the first quadrant is $OABC$. The required area is 4 times the area of the region $OABC$.

$$\text{Required area} = 4\int_0^{1/2}\left(2^{-x} - \frac{1}{2}\right)dx = 4\left(-\frac{2^{-x}}{\ln 2} - \frac{x}{2}\right)_0^{1/2}$$

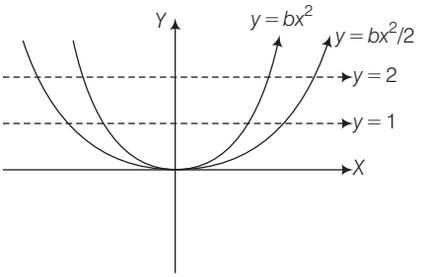
$$= \left[\frac{4}{\ln 2}(1 - 2^{-1/2}) - 1\right] \text{sq units}$$

70. Pair of lines $y^2 - 3y + 2 = 0$

$\Rightarrow$ Lines are $y = 2$, $y = 1$.

Let $[a] = b$

Now, curves are $y = bx^2$ and $y = \frac{b}{2}x^2$



$$\text{Bounded area} = 2\left[\int_1^2 (x_2 - x_1)\,dy\right]$$

$$= 2\left[\int_1^2 \left(\sqrt{\frac{2y}{b}} - \sqrt{\frac{y}{b}}\right)dy\right]$$

$$= 2\left[\sqrt{\frac{2}{b}} \cdot \frac{y^{3/2}}{3/2} - \frac{1}{\sqrt{b}} \cdot \frac{y^{3/2}}{3/2}\right]_1^2$$

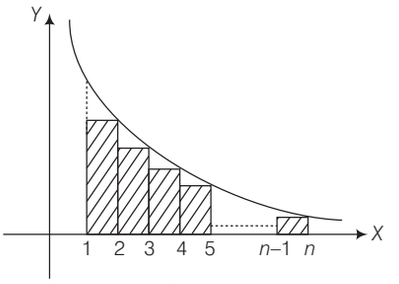
$$= \frac{4}{3\sqrt{b}}\left((\sqrt{2} - 1)\,y^{3/2}\right)_1^2 = \frac{4}{3\sqrt{b}}(\sqrt{2} - 1)(2\sqrt{2} - 1)$$

Area will be maximum when $b = [a]$ is least.

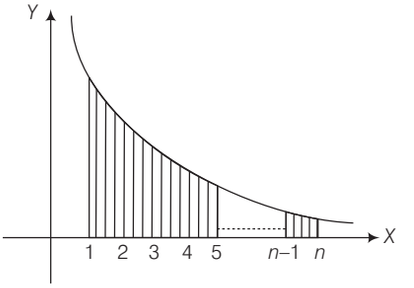
As $a \ge 2 \Rightarrow [a]_{\text{least}} = 1 \Rightarrow 1 \le a < 2$

71. Since, $f'(x) < 0 \Rightarrow f(x)$ is a decreasing function and also $f''(x) > 0 \Rightarrow f(x)$ is concave upwards.

Hence, the graph of the function $y = f(x)$ is as follows :



(i)



(ii)

Let S_L denotes the shaded area in figure (i).

$$\Rightarrow \quad S_L = f(2) + f(3) + \ldots + f(n) = \sum_{r=1}^{n} f(r) - f(1)$$

From the figure (i) it is clear that, $S_L < \int_1^n f(x)\,dx$

$$\Rightarrow \quad \sum_{r=1}^{n} f(r) - f(1) < \int_1^n f(x)\,dx$$

$$\Rightarrow \quad \sum_{r=1}^{n} f(r) < \int_1^n f(x)\,dx + f(1) \qquad \ldots(i)$$

Let S_u denotes the area of the shaded region in figure (ii).

$$\Rightarrow \quad S_u = \left(\frac{1}{2}f(1) + f(2)\right) + \frac{1}{2}(f(2) + f(3)) + \ldots + \frac{1}{2}(f(n-1) + f(n))$$

$$= [(f(1) + f(2) + f(3)) + \ldots + f(n-1) + f(n)] - \frac{1}{2}(f(1) + f(n))$$

$$= \sum_{r=1}^{n} f(r) - \frac{1}{2}f(1) - \frac{1}{2}f(n)$$

From figure (ii) it is clear that;

$$S_u > \int_1^n f(x)\,dx$$

$$\Rightarrow \quad \sum_{r=1}^{n} f(r) - \frac{1}{2}(f(1) + f(n)) > \int_1^n f(x)\,dx$$

$$\Rightarrow \quad \sum_{r=1}^{n} f(r) > \int_1^n f(x)\,dx + \frac{1}{2}(f(1) + f(n))$$

$$> \int_1^n f(x)\,dx + 4\cdot\frac{1}{2}f(1) \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$\frac{1}{2}f(1) + \int_1^n f(x)\,dx < \sum_{r=1}^{n} f(r) < \int_1^n f(x)\,dx + f(1).$$

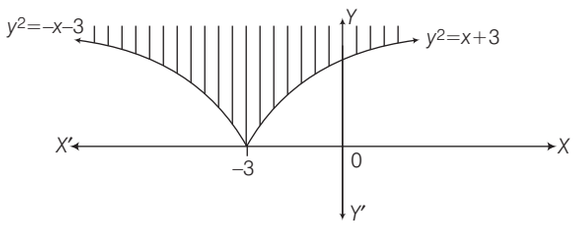
72. Here, $\{(x, y) \in R^2 : y \geq \sqrt{|x+3|}, 5y \leq (x+9) \leq 15\}$

$$\therefore \quad y \geq \sqrt{|x+3|}$$

$$\Rightarrow \quad y \geq \begin{cases} \sqrt{x+3}, \text{ when } x \geq -3 \\ \sqrt{-x-3}, \text{ when } x \leq -3 \end{cases}$$

or $$y^2 \geq \begin{cases} x+3, \text{ when } x \geq -3 \\ -3-x, \text{ when } x \leq -3 \end{cases}$$

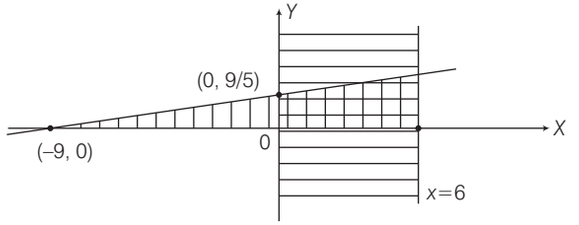
Shown as



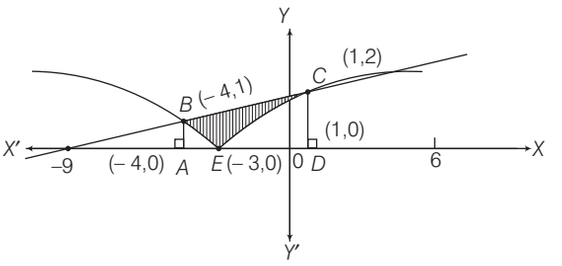
Also, $5y \leq (x+9) \leq 15$

$\Rightarrow \quad (x+9) \geq 5y$ and $x \leq 6$

Shown as



$\therefore \quad \{(x, y) \in R^2 : y \geq \sqrt{|x+3|}, 5y \leq (x+9) \leq 15\}$



$\therefore$ Required area = Area of trapezium $ABCD$ − Area of ABE under parabola − Area of CDE under parabola

$$= \frac{1}{2}(1+2)(5) - \int_{-4}^{-3}\sqrt{-(x+3)}\,dx - \int_{-3}^{1}\sqrt{(x+3)}\,dx$$

$$= \frac{15}{2} - \left[\frac{(-3-x)^{3/2}}{-\frac{3}{2}}\right]_{-4}^{-3} - \left[\frac{(x+3)^{3/2}}{\frac{3}{2}}\right]_{-3}^{1}$$

$$= \frac{15}{2} + \frac{2}{3}[0-1] - \frac{2}{3}[8-0] = \frac{15}{2} - \frac{2}{3} - \frac{16}{3} = \frac{15}{2} - \frac{18}{3} = \frac{3}{2}$$

73. Since, $F'(a) + 2$ is the area bounded by $x = 0$, $y = 0$, $y = f(x)$ and $x = a$.

$$\therefore \quad \int_0^a f(x)dx = F'(a) + 2$$

Using Newton-Leibnitz formula,

$$f(a) = F''(a) \quad \text{and} \quad f(0) = F''(0) \qquad \ldots(i)$$

Given, $$F(x) = \int_x^{x^2 + \pi/6} 2\cos^2 t\,dt$$

On differentiating,

$$F'(x) = 2\cos^2\left(x^2 + \frac{\pi}{6}\right)\cdot 2x - 2\cos^2 x \cdot 1$$

Again differentiating,

$$F''(x) = 4\left\{\cos^2\left(x^2+\frac{\pi}{6}\right) - 2x\cos\left(x^2+\frac{\pi}{6}\right)\sin\left(x^2+\frac{\pi}{6}\right)2x\right\} + \{4\cos x \cdot \sin x\}$$

$$= 4\left\{\cos^2\left(x^2 + \frac{\pi}{6}\right) - 4x^2\cos\left(x^2 + \frac{\pi}{6}\right)\sin\left(x^2 + \frac{\pi}{6}\right)\right\} + 2\sin 2x$$

$$\therefore \quad F''(0) = 4\left\{\cos^2\left(\frac{\pi}{6}\right)\right\} = 3$$

$$\therefore \quad f(0) = 3$$

74. Let equation of tangent to parabola be $y = mx + \frac{2}{m}$

It also touches the circle $x^2 + y^2 = 2$.

$$\therefore \quad \left|\frac{2}{m\sqrt{1+m^2}}\right| = \sqrt{2}$$

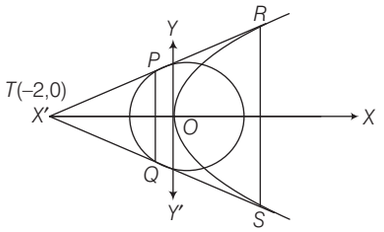
$$\Rightarrow \quad m^4 + m^2 = 2 \quad \Rightarrow \quad m^4 + m^2 - 2 = 0$$

$$\Rightarrow \quad (m^2 - 1)(m^2 + 2) = 0$$

$$\Rightarrow \quad m = \pm 1, m^2 = -2 \qquad [\text{rejected } m^2 = -2]$$

So, tangents are $y = x + 2$, $y = -x - 2$.

They intersect at $(-2, 0)$.



Equation of chord PQ is $-2x = 2 \Rightarrow x = -1$

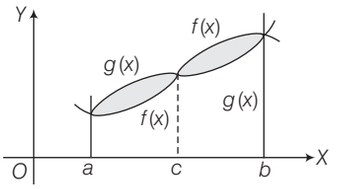
Equation of chord RS is $0 = 4(x-2) \Rightarrow x = 2$

$\therefore$ Coordinates of P, Q, R, S are

$$P(-1, 1),\ Q(-1, -1),\ R(2, 4),\ S(2, -4)$$

$\therefore$ Area of quadrilateral $= \dfrac{(2+8)\times 3}{2} = 15$ sq units

75. To find the bounded area between $y = f(x)$ and $y = g(x)$ between $x = a$ to $x = b$.



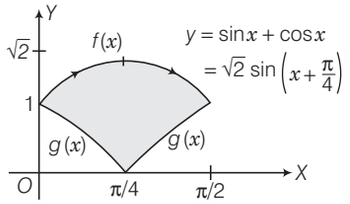
$\therefore$ Area bounded $= \int_a^c [g(x) - f(x)]dx + \int_c^b [f(x) - g(x)]dx$

$$= \int_a^b |f(x) - g(x)|dx$$

Here, $f(x) = y = \sin x + \cos x$, when $0 \le x \le \dfrac{\pi}{2}$

and $g(x) = y = |\cos x - \sin x| = \begin{cases} \cos x - \sin x, & 0 \le x \le \dfrac{\pi}{4} \\ \sin x - \cos x, & \dfrac{\pi}{4} \le x \le \dfrac{\pi}{2} \end{cases}$

could be shown as



$\therefore$ Area bounded $= \int_0^{\pi/4} \{(\sin x + \cos x) - (\cos x - \sin x)\}dx$

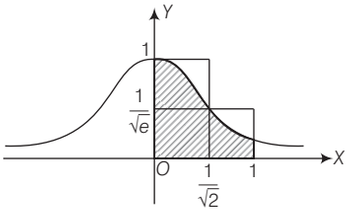
$$+ \int_{\pi/4}^{\pi/2} \{(\sin x + \cos x) - (\sin x - \cos x)\}dx$$

$$= \int_0^{\pi/4} 2\sin x\,dx + \int_{\pi/4}^{\pi/2} 2\cos x\,dx$$

$$= -2[\cos x]_0^{\pi/4} + 2[\sin x]_{\pi/4}^{\pi/2}$$

$$= 4 - 2\sqrt{2} = 2\sqrt{2}(\sqrt{2} - 1) \text{ sq units}$$

76. Graph for $y = e^{-x^2}$



Since, $x^2 \le x$ when $x \in [0, 1]$

$$\Rightarrow \quad -x^2 \ge -x \quad \text{or} \quad e^{-x^2} \ge e^{-x}$$

$$\therefore \quad \int_0^1 e^{-x^2}dx \ge \int_0^1 e^{-x}dx$$

$$\Rightarrow \quad S \ge -(e^{-x})_0^1 = 1 - \frac{1}{e} \qquad \ldots(i)$$

Also, $\int_0^1 e^{-x^2}dx \le$ Area of two rectangles

$$\le \left(1\times\frac{1}{\sqrt{2}}\right) + \left(1 - \frac{1}{\sqrt{2}}\right)\times\frac{1}{\sqrt{e}}$$

$$\le \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}}\left(1 - \frac{1}{\sqrt{2}}\right) \qquad \ldots(ii)$$

$$\therefore \quad \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}}\left(1 - \frac{1}{\sqrt{2}}\right) \ge S \ge 1 - \frac{1}{e} \qquad \text{[from Eqs. (i) and (ii)]}$$

77. $R_1 = \int_{-1}^{2} x\,f(x)\,dx \qquad \ldots(i)$

Using $\int_a^b f(x)\,dx = \int_a^b f(a + b - x)\,dx$

$$R_1 = \int_{-1}^{2} (1 - x)\,f(1 - x)\,dx$$

$$\therefore \quad R_1 = \int_{-1}^{2} (1 - x)\,f(x)\,dx \qquad \ldots(ii)$$

$[f(x) = f(1 - x)$, given$]$

Given, R_2 is area bounded by $f(x)$, $x = -1$ and $x = 2$.

$$\therefore \quad R_2 = \int_{-1}^{2} f(x)\,dx \qquad \ldots(iii)$$

On adding Eqs. (i) and (ii), we get

$$2R_1 = \int_{-1}^{2} f(x)\,dx \qquad \ldots(iv)$$

From Eqs. (iii) and (iv), we get

$$2R_1 = R_2$$

78. Here, area between 0 to b is R_1 and b to 1 is R_2.

$$\therefore \quad \int_0^b (1 - x)^2\,dx - \int_b^1 (1 - x)^2\,dx = \frac{1}{4}$$

$$\Rightarrow \quad \left[\frac{(1-x)^3}{-3}\right]_0^b - \left[\frac{(1-x)^3}{-3}\right]_b^1 = \frac{1}{4}$$

$$\Rightarrow -\frac{1}{3}[(1-b)^3 - 1] + \frac{1}{3}[0 - (1-b)^3] = \frac{1}{4}$$

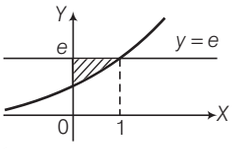
$$\Rightarrow \quad -\frac{2}{3}(1-b)^3 = -\frac{1}{3} + \frac{1}{4} = -\frac{1}{12}$$

$$\Rightarrow \quad (1-b)^3 = \frac{1}{8}$$

$$\Rightarrow \quad (1-b) = \frac{1}{2} \Rightarrow b = \frac{1}{2}$$

79. Shaded area $= e - \left(\int_0^1 e^x\,dx\right) = 1$

Also, $\int_1^e \ln(e + 1 - y)\,dy$ [put $e + 1 - y = t \Rightarrow -dy = dt$]



$$= \int_e^1 \ln\ t(-dt) = \int_1^e \ln t\,dt = \int_1^e \ln y\,dy = 1$$

80. Required area $= \int_0^{\pi/4}\left(\sqrt{\frac{1+\sin x}{\cos x}}-\sqrt{\frac{1-\sin x}{\cos x}}\right)dx$

$\left[\because \frac{1+\sin x}{\cos x} > \frac{1-\sin x}{\cos x} > 0\right]$

$$= \int_0^{\pi/4}\left(\sqrt{\frac{1+\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}{\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}}-\sqrt{\frac{1-\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}{\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}}\right)dx$$

$$= \int_0^{\pi/4}\left(\sqrt{\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}}-\sqrt{\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}}\right)dx$$

$$= \int_0^{\pi/4}\frac{1+\tan\frac{x}{2}-1+\tan\frac{x}{2}}{\sqrt{1-\tan^2\frac{x}{2}}}dx = \int_0^{\pi/4}\frac{2\tan\frac{x}{2}}{\sqrt{1-\tan^2\frac{x}{2}}}dx$$

Put $\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$

$$= \int_0^{\tan\frac{\pi}{8}}\frac{4t\,dt}{(1+t^2)\sqrt{1-t^2}}$$

As $\int_0^{\sqrt{2}-1}\frac{4t\,dt}{(1+t^2)\sqrt{1-t^2}}$ $\left[\because \tan\frac{\pi}{8} = \sqrt{2}-1\right]$

81. Given, $y^3 - 3y + x = 0$

$$\Rightarrow 3y^2\frac{dy}{dx} - 3\frac{dy}{dx} + 1 = 0 \quad \text{...(i)}$$

$$\Rightarrow 3y^2\left(\frac{d^2y}{dx^2}\right) + 6y\left(\frac{dy}{dx}\right)^2 - 3\frac{d^2y}{dx^2} = 0 \quad \text{...(ii)}$$

At $x = -10\sqrt{2},\ y = 2\sqrt{2}$

On substituting in Eq. (i) we get

$$3(2\sqrt{2})^2\cdot\frac{dy}{dx} - 3\cdot\frac{dy}{dx} + 1 = 0 \Rightarrow \frac{dy}{dx} = -\frac{1}{21}$$

Again, substituting in Eq. (ii), we get

$$3(2\sqrt{2})^2\frac{d^2y}{dx^2} + 6(2\sqrt{2})\cdot\left(-\frac{1}{21}\right)^2 - 3\cdot\frac{d^2y}{dx^2} = 0$$

$$\Rightarrow 21\cdot\frac{d^2y}{dx^2} = -\frac{12\sqrt{2}}{(21)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-12\sqrt{2}}{(21)^3} = \frac{-4\sqrt{2}}{7^3\cdot 3^2}$$

82. Required area $= \int_a^b y\,dx = \int_a^b f(x)\,dx = [f(x)\cdot x]_a^b - \int_a^b f'(x)x\,dx$

$$= bf(b) - af(a) - \int_a^b f'(x)x\,dx$$

$$= bf(b) - af(a) + \int_a^b \frac{x\,dx}{3[\{f(x)\}^2 - 1]}$$

$$\left[\because f'(x) = \frac{dy}{dx} = \frac{-1}{3(y^2-1)} = \frac{-1}{3[\{f(x)\}^2-1]}\right]$$

83. Let $I = \int_{-1}^{1} g'(x)\,dx = [g(x)]_{-1}^{1} = g(1) - g(-1)$

Since, $y^3 - 3y + x = 0$ ...(i)

and $y = g(x)$

$\therefore$ $\{g(x)\}^3 - 3g(x) + x = 0$ [from Eq. (i)]

At $x = 1$, $\{g(1)\}^3 - 3g(1) + 1 = 0$ ...(ii)

At $x = -1$, $\{g(-1)\}^3 - 3g(-1) - 1 = 0$ ...(iii)

On adding Eqs. (i) and (ii), we get

$\{g(1)\}^3 + \{g(-1)\}^3 - 3\{g(1) + g(-1)\} = 0$

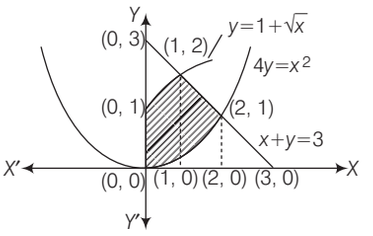
$\Rightarrow [g(1) + g(-1)][\{g(1)\}^2 + \{g(-1)\}^2 - g(1)g(-1) - 3] = 0$

$\Rightarrow g(1) + g(-1) = 0$

$\Rightarrow g(1) = -g(-1)$

$\therefore I = g(1) - g(-1) = g(1) - \{-g(1)\} = 2g(1)$

84. Required area $= \int_0^1 (1+\sqrt{x})dx + \int_1^2 (3-x)dx - \int_0^2 \frac{x^2}{4}dx$



$$= \left[x + \frac{x^{3/2}}{3/2}\right]_0^1 + \left[3x - \frac{x^2}{2}\right]_1^2 - \left[\frac{x^3}{12}\right]_0^2$$

$$= \left(1 + \frac{2}{3}\right) + \left(6 - 2 - 3 + \frac{1}{2}\right) - \left(\frac{8}{12}\right)$$

$$= \frac{5}{3} + \frac{3}{2} - \frac{2}{3} = 1 + \frac{3}{2} = \frac{5}{2} \text{ sq units}$$

85. Given equations of curves are $y^2 = 2x$, ...(i)

which is a parabola with vertex (0, 0) and axis parallel to X-axis.

And $x^2 + y^2 = 4x$ ...(ii)

which is a circle with centre (2, 0) and radius = 2

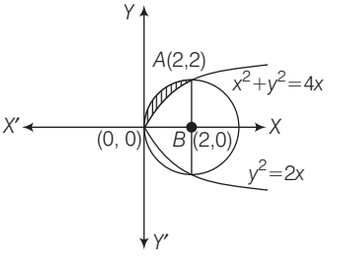
On substituting $y^2 = 2x$ in Eq. (ii), we get

$$x^2 + 2x = 4x \Rightarrow x^2 = 2x$$

$\Rightarrow x = 0$ or $x = 2$

$\Rightarrow y = 0$ or $y = \pm 2$ [using Eq. (i)]

Now, the required area is the area of shaded region, i.e.



Required area $= \frac{\text{Area of a circle}}{4} - \int_0^2 \sqrt{2x}\,dx$

$$= \frac{\pi(2)^2}{4} - \sqrt{2}\int_0^2 x^{1/2}dx = \pi - \sqrt{2}\left[\frac{x^{3/2}}{3/2}\right]_0^2$$

$$= \pi - \frac{2\sqrt{2}}{3}[2\sqrt{2} - 0] = \left(\pi - \frac{8}{3}\right) \text{ sq units}$$

86. Given region is $\{(x, y) : y^2 \le 2x \text{ and } y \ge 4x - 1\}$

$y^2 \le 2x$ represents a region inside the parabola

$$y^2 = 2x \quad \ldots(i)$$

and $y \ge 4x - 1$ represents a region to the left of the line

$$y = 4x - 1 \quad \ldots(ii)$$

The point of intersection of the curve (i) and (ii) is

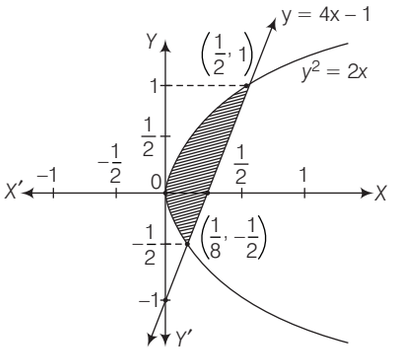
$$(4x - 1)^2 = 2x$$

$$\Rightarrow \quad 16x^2 + 1 - 8x = 2x$$

$$\Rightarrow \quad 16x^2 - 10x + 1 = 0$$

$$\Rightarrow \quad x = \frac{1}{2}, \frac{1}{8}$$

$\therefore$ The points where these curves intersect, are $\left(\frac{1}{2}, 1\right)$ and $\left(\frac{1}{8}, -\frac{1}{2}\right)$.



Hence, required area $= \int_{-1/2}^{1}\left(\frac{y+1}{4} - \frac{y^2}{2}\right)dy$

$$= \frac{1}{4}\left(\frac{y^2}{2} + y\right)_{-1/2}^{1} - \frac{1}{6}(y^3)_{-1/2}^{1}$$

$$= \frac{1}{4}\left\{\left(\frac{1}{2} + 1\right) - \left(\frac{1}{8} - \frac{1}{2}\right)\right\} - \frac{1}{6}\left\{1 + \frac{1}{8}\right\}$$

$$= \frac{1}{4}\left\{\frac{3}{2} + \frac{3}{8}\right\} - \frac{1}{6}\left\{\frac{9}{8}\right\} = \frac{1}{4} \times \frac{15}{8} - \frac{3}{16} = \frac{9}{32}$$

87. Given equation of ellipse is

$$\frac{x^2}{9} + \frac{y^2}{5} = 1 \quad \ldots(i)$$

$\therefore \quad a^2 = 9, b^2 = 5 \quad \Rightarrow \quad a = 3, b = \sqrt{5}$

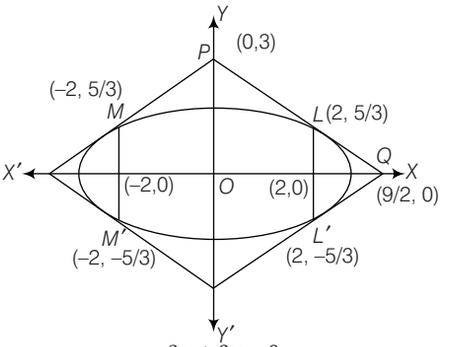
Now, $\quad e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{5}{9}} = \frac{2}{3}$

Foci $= (\pm ae, 0) = (\pm 2, 0)$ and $\frac{b^2}{a} = \frac{5}{3}$

$\therefore$ Extremities of one of latusrectum are $\left(2, \frac{5}{3}\right)$ and $\left(2, \frac{-5}{3}\right)$.

$\therefore$ Equation of tangent at $\left(2, \frac{5}{3}\right)$ is,

$$\frac{x(2)}{9} + \frac{y(5/3)}{5} = 1$$



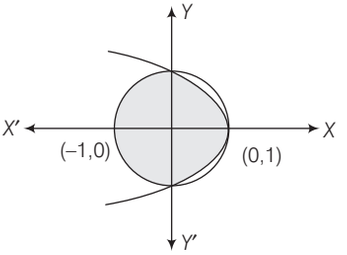
or $\quad 2x + 3y = 9 \quad \ldots(ii)$

Eq.(ii) intersects X and Y-axes at $\left(\frac{9}{2}, 0\right)$ and $(0, 3)$, respectively.

$\therefore$ Area of quadrilateral $= 4 \times$ Area of ΔPOQ

$$= 4 \times \left(\frac{1}{2} \times \frac{9}{2} \times 3\right) = 27 \text{ sq units}$$

88. Given, $A = \{(x, y) : x^2 + y^2 \le 1 \text{ and } y^2 \le 1 - x\}$

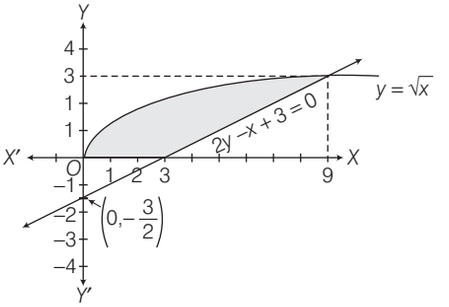


Required area $= \frac{1}{2}\pi r^2 + 2\int_0^1 (1 - y^2)dy = \frac{1}{2}\pi(1)^2 + 2\left(y - \frac{y^3}{3}\right)_0^1$

$$= \frac{\pi}{2} + \frac{4}{3}$$

89. Given curves are $y = \sqrt{x} \quad \ldots(i)$

and $\quad 2y - x + 3 = 0 \quad \ldots(ii)$



On solving Eqs. (i) and (ii), we get $2\sqrt{x} - (\sqrt{x})^2 + 3 = 0$

$$\Rightarrow \quad (\sqrt{x})^2 - 2\sqrt{x} - 3 = 0$$

$$\Rightarrow \quad (\sqrt{x} - 3)(\sqrt{x} + 1) = 0$$

$$\Rightarrow \quad \sqrt{x} = 3 \quad [\because \sqrt{x} = -1 \text{ is not possible}]$$

$$\Rightarrow \quad y = 3$$

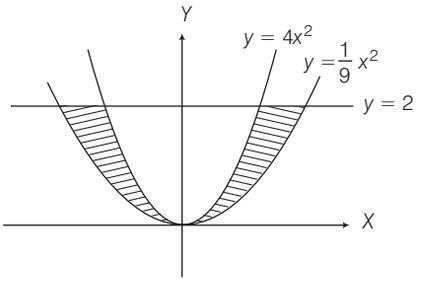
$\therefore$ Required area $= \int_0^3 (\text{line} - \text{curve})\, dy = \int_0^3 \{(2y + 3) - y^2\}\, dy$

$$= \left[y^2 + 3y - \frac{y^3}{3}\right]_0^3 = 9 + 9 - 9 = 9$$

90. **Given** Two parabolas $x^2 = \frac{y}{4}$ and $x^2 = 9y$

To find The area bounded between the parabolas and the straight line $y = 2$.

The required area is equal to the shaded region in the drawn figure.



The area of the shaded region (which can be very easily found by using integration) is twice the area shaded in first quadrant.

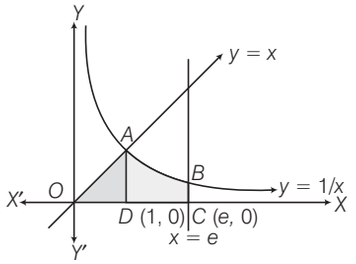
$$\text{Required area} = 2\int_0^2 \left(3\sqrt{y} - \frac{\sqrt{y}}{2}\right) dy = 2\int_0^2 \left(\frac{5}{2}\sqrt{y}\right) dy$$

$$= 5\left[\frac{y^{3/2}}{3/2}\right]_{y=0}^{y=2} = \frac{10}{3}(2^{3/2} - 0) = \frac{20\sqrt{2}}{3}$$

91. Given, $y = x$, $x = e$ and $y = \frac{1}{x}$, $x \geq 0$

Since, $y = x$ and $x \geq 0 \quad \Rightarrow \quad y \geq 0$

$\therefore$ Area to be calculated in I quadrant shown as



$\therefore$ Area = Area of ΔODA + Area of $DABCD$

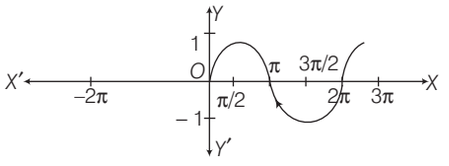
$$= \frac{1}{2}(1 \times 1) + \int_1^e \frac{1}{x}\, dx$$

$$= \frac{1}{2} + (\log|x|)_1^e$$

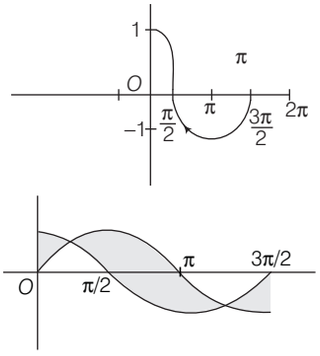
$$= \frac{1}{2} + \{\log|e| - \log 1\} \qquad [\because \log|e| = 1]$$

$$= \frac{1}{2} + 1 = \frac{3}{2} \text{ sq units}$$

92. Graph of $y = \sin x$ is



and graph of $y = \cos x$ is

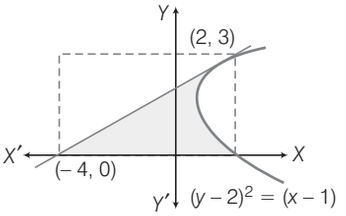


$$\text{Required area} = \int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\, dx + \int_{5\pi/4}^{3\pi/2} (\cos x - \sin x)\, dx$$

$$= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{5\pi/4} + [\sin x + \cos x]_{5\pi/4}^{3\pi/2}$$

$$= (4\sqrt{2} - 2) \text{ sq units}$$

93. The equation of tangent at (2, 3) to the given parabola is $x = 2y - 4$



$\therefore$ Required area $= \int_0^3 \{(y-2)^2 + 1 - 2y + 4\} dy$

$$= \left[\frac{(y-2)^3}{3} - y^2 + 5y\right]_0^3$$

$$= \frac{1}{3} - 9 + 15 + \frac{8}{3} = 9 \text{ sq units}$$

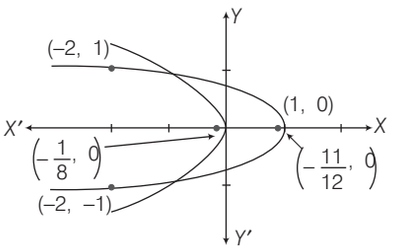
94. Given, equations of curves are $x + 3y^2 = 1$ ... (i)

and $x + 2y^2 = 0$ ... (ii)

On solving Eqs. (i) and (ii), we get

$$y = \pm 1 \text{ and } x = -2$$

$\therefore$ Required area $= \left|\int_{-1}^{1} (x_1 - x_2) dy\right|$



$$= \left|\int_{-1}^{1} (1 - 3y^2 + 2y^2) dy\right| = \left|\int_{-1}^{1} (1 - y^2) dy\right|$$

$$= \left|2\int_0^1 (1 - y^2) dy\right| = \left|2\left[y - \frac{y^3}{3}\right]_0^1\right| = \left|2\left(1 - \frac{1}{3}\right)\right| = \frac{4}{3} \text{ sq units}$$

CHAPTER

04

Differential Equations

Learning Part

Session 1

- Solution of a Differential Equation

Session 2

- Solving of Varibale Seperable Form
- Homogeneous Differential Equation

Session 3

- Solving of Linear Differential Equations
- Bernoulli's Equation
- Orthogonal Trajectory

Session 4

- Exact Differential Equations

Session 5

- Solving of First Order and Higher Degrees
- Application of Differential Equations
- Application of First Order Differential Equations

Practice Part

- JEE Type Examples
- Chapter Exercises

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A differential equation can simply be said to be an equation involving derivatives of an unknown function. For example, consider the equation

$$\frac{dy}{dx} + xy = x^2$$

This is a differential equation since it involves the derivative of the funtion $y(x)$ which we may wish to determine. We must first understand why and how differnetial equations arise and why we need them at all. In general, we can say that a differential equation describes the behaviour of some continuously varying quantity.

Scenario 1 : A Freely Falling Body

A body is release at rest from a heigh h. How do we described the motion of this body?

The height x of the body is a function of time. Since the acceleration of the body is g, we have $\frac{d^2x}{dt^2} = g$

This is the differential equation describing the motion of the body. Along with the initial condition $x(0) = h$, it completely describes the motion of the body at all instants after the body starts falling.

Scenario 2 : Radioactive disintegration

Experimental evidence shows that the rate of decay of any ratioactive substance is proportional to the amount of the substance present,

i.e. $$\frac{dm}{dt} = -\lambda m$$

where m is the mass of the radioactive substance and a function of t. If we know $m(0)$, the initial mass, we can use this differential equation to determine the mass of the substance remaining at any later time instant.

Scenario 3 : Population Growth

The growth of population (of say, a biological culture) in a closed environment is dependent on the birth and death rates. The birth rate will contribute to increaseing the population while the death rate will contribute to its decrease. It has been found that for low populations, the birth rate is the dominant influence in population growth and the growth rate is linearly dependent on the current population. For high populations, there is a competition among the population for the limited resources available, and thus death rate becomes dominant. Also, the death rate shows a quadratic dependence on the current population.

Thus, if $N(t)$ represents the population at time t, the different equation describing the population variation is of the form

$$\frac{dN}{dt} = \lambda_1 N - \lambda_2 N^2$$

where λ_1 and λ_2 are constants.

Along with the initial population $N(0)$, this equation can tell us the population at any later time instant.

Session 1

Solution of a Differential Equation

These three examples should be sufficient for you to realise why and how differential equations arise and why they are important.

In all the three equations mentioned above, there is only independent variable (the time t in all the three cases). Such equations are termed **ordinary differential equations.** We might have equations involving more than one independent variable :

$$\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = x^2$$

where the notation $\frac{\partial}{\partial x}$ stands for the partial derivative, i.e. the term $\frac{\partial f}{\partial x}$ would imply that we differentiate the function f with respect to the independent variable x as the variable (while treating the other independent variable y as a constant). A similar interpretation can be attached to $\frac{\partial}{\partial y}$.

Such equations are termed **partial differential equations** but we shall not be concerned with them in this chapter.

Consider the ordinary differnetial equation

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + x^2 = c$$

The order of the highest derivative present in this equation is two; thus we shall call it a second order differential equation (*DE*, for convenience).

The **order** of a *DE* is the order of the highest derivative that occurs in the equation

Again, consider the *DE*

$$\frac{d^2y}{dx^3}+\frac{dy}{dx}=x^2y^2$$

The degree of the highest order derivative in this DE is two, so this is a DE of degree two (and order three).

The degree of a DE is the degree of the highest order derivative that occurs in the equation, **when all the derivatives in the equation are made of free of fractional powers.**

$$\sqrt{\left(\frac{dy}{dx}\right)^2-1}+x\left(\frac{d^2y}{dx^2}\right)=k$$

is not of degree two. When we make this equation free of fractional powers, by the following rearrangement,

$$\left(\frac{dy}{dx}\right)^2-1+\left\{k-x\left(\frac{d^2y}{dx^2}\right)^2\right\}^2$$

we see that the degree of the highest order derivative will become four. Thus, this is a DE of degree four (and order two).

Finally, an n^{th} **linear DE** (degree one) is an equation of the form

$$a_0\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}}{dx^{n-1}}+\ldots+a_{n-1}\frac{dy}{dx}+a_ny=b$$

where the $a_i's$ and b are functions of x.

Solving an n^{th} order DE to evaluate the unknown function will essentially consists of doing n integrations on the DE. Each integration step will introduce an arbitrary constant. Thus, you can expect in general that the **solution of an n^{th} order DE will contain n independnet arbitrary constants.**

By n independent constants, we mean to say that the most general solution of the DE cannot be expressed in fewer that n constants. As an example, the second order DE

$$\frac{d^y}{dx^2}+y=0$$

has its most general solution of the form

$$y=A\cos x+B\sin x. \qquad \ldots(1)$$

(verify that this is a solution by explicit substitution).

Thus, two arbitrary and independent constants must be included in the general solution. We cannot reduce (1) to a relation containing only one arbitrary constant. On the other hand, it can be verified that the function

$$y=ae^{x+b}$$

is a solution to the second-order DE

$$\frac{d^2y}{dx^2}=y$$

but even through it (seems to) contain two arbitrary constants, it is not the general solution to this DE. This is because it can be reduced to a relation involving only one arbitrary constant :

$$y=ae^{x+b}=ae^x.e^b=ce^x \quad (\text{where } c=a\cdot e^b)$$

Let us summarise what we have seen till now : the most general solution of an n^{th} order DE will consist of n orbitrary constants; conversely, from a functional relation involving n arbitrary constants, an n^{th} order DE can be generated (we shall soon see how to do this). We are generally interested in solutions of the DE satisfying some particular constraints (say, some initial values). Since the most general solution of the DE involves n arbitrary constant, we see that the maximum member of independent conditions which can be imposed on a solution of the DE is n. As a first example, consider the functional relation

$$y=x^2+c_1e^{2x}+c_2e^{3x} \qquad \ldots(i)$$

This curve's equation contains two arbitrary constants; as we vary c_1 and c_2, we obtain different curves; those curves constitute a family of curves. All members of this family will satisfy the DE that we can generate from this general relation; this DE will be second order since the relation contains two arbitrary constants.

We now see how to generate the DE. Differentiate the given relation twice to obtain

$$y'=2x+2c_1e^{2x}+3c_2e^{3x} \qquad \ldots(ii)$$

$$y''=2+4c_1e^{2x}+9c_2e^{3x} \qquad \ldots(iii)$$

From Eqs. (i), (ii) and (iii), c_1 and c_2 can be eliminated to obtain

$$\begin{vmatrix} e^{2x} & e^{3x} & x^2-y \\ 2e^{2x} & 3e^{3x} & 2x-y' \\ 4e^{2x} & 9e^{3x} & 2-y'' \end{vmatrix}$$

$$\Rightarrow \quad \begin{vmatrix} 1 & 1 & x^2-y \\ 2 & 3 & 2x-y' \\ 4 & 9 & 2-y'' \end{vmatrix}$$

$$\Rightarrow 6-3''-18x+9y'+8x-4y'-4+2y''+7x^2-6y=0$$

$$\Rightarrow \quad y''-5y'+6y=6x^2-10x+2 \qquad \ldots(iv)$$

This is the required DE; it corresponds to the family of curves given by Eq. (i). Differently put, the most general solution of this DE is given by Eq. (i).

As an exercise for the reader, show that the DE corrersponding to the general equation

$$y = Ae^{2x} + Be^{x} + C$$

where A, B, C are arbitrary constants, is

$$y''' - 3y'' + 2y' = 0$$

By expected, the three arbitrary constants cause the DE to the third order.

Example 1 Find the order and degree (if defined) of the following differential equations :

(i) $y = 1 + \left(\frac{dy}{dx}\right) + \frac{1}{2!}\left(\frac{dy}{dx}\right)^2 + \frac{1}{3!}\left(\frac{dy}{dx}\right)^3 + \ldots$

(ii) $\left(\frac{d^3y}{dx^3}\right)^{2/3} = \frac{dy}{dx} + 2$ (iii) $\frac{d^2y}{dx^2} = x \ln\left(\frac{dy}{dx}\right)$

Sol. (i) The given differential equation can be rewritten as

$$y = e^{dy/dx}$$

$$\Rightarrow \qquad \frac{dy}{dx} = \ln y.$$

Hence, its order is 1 and degree 1.

(ii) The given differential equation can be rewritten as

$$\left(\frac{d^3y}{dx^3}\right)^2 = \left(\frac{dy}{dx} + 2\right)^3.$$

Hence, its order is 3 and degree 2.

(iii) Its order is obviously 2.

Since, the given differential equation cannot be written as a polynomial in all the differential coefficients, the degree of the equation is not defined.

Example 2 Find the order and degree (if defined) of the following differential equations :

(i) $\sqrt{\frac{d^2y}{dx^2}} = \sqrt[3]{\frac{dy}{dx} + 3}$ (ii) $\frac{d^2y}{dx^2} = \sin\left(\frac{dy}{dx}\right)$

(iii) $\frac{dy}{dx} = \sqrt{3x + 5}$

Sol. (i) The given differential equation can be rewritten as

$$\left(\frac{d^2y}{dx^2}\right)^3 = \left(\frac{dy}{dx} + 3\right)^2$$

Hence, order is 2 and degree is 3.

(ii) The given differential equation has the order 2. Since, the given differential equation cannot be written as a polynomial in the differential coefficients, the degree of the equation is not defined.

(iii) Its order is obviously 1 and degree 1.

Linear and Non-linear Differential Equation

A differential equation is a linear differential equation if it is expressible in the form

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_{n-1} \frac{dy}{dx} + any = Q$$

where $a_0, a_1, a_2, \ldots, a_n$ and Q are either constants or functions of independent variable x.

Thus, if a differential equation when expressed in the form of a polynomial involves the derivatives and dependent variable in the first power and there ae no product of these, and also the coefficient of the various terms are either constants or functions of the independent variable, then it is said to be linear differential equation. otherwise, it i a non-linear differential equation.

The differentiable equation $\left(\frac{d^3y}{dx^3}\right)^2 - 6\left(\frac{d^2y}{dx^2}\right) - 4y = 0$, is a non-linear differential equation, because its degree is 2, more than one.

e.g. The differential equation, $\left(\frac{d^3y}{dx^3}\right) + 2\left(\frac{dy}{dx}\right)^2 + 9y = x$, is non-linear differential equation, because differential coefficient $\frac{dy}{dx}$ has exponent 2.

e.g. The differential equation $(x^2 + y^2)\,dx - 2xydy = 0$ is a non-linear differential equation, because the exponent of dependent variable y is 2 and it involves the product of y and $\frac{dy}{dx}$. e.g. Consider the differential equation

$$\left(\frac{d^2y}{dx^2}\right) - 5\left(\frac{dy}{dx}\right) + 6y = \sin x$$

This is a linear differential equation of order 2 and degree.

Formation of Differential Equations

If an equation in independent and dependent variables involving some arbitrary constants is given, then a differential equation is obtained as follows :

(i) Differentiate the given equation w.r.t. the independent variable (say x) as many times as the number of arbitrary constants in it.

(ii) Eliminate the arbitrary constants.

(iii) The eliminant is the required differential equation. i.e. If we have an equation $f(x, y, c_1, c_2, \ldots, c_n) = 0$

Containing n arbitrary constants $c_1, c_2, c_3, \ldots, c_n$, then by differentiating this n times, we shall get n-equations.

Now, among these n-equations and the given equation, in all $(n+1)$ equations, if the n arbitrary constants $c_1, c_2, c_3, \ldots, c_n$ are eliminated, we shall evidently get a differential equation of the nth order. For there being n differentiation, the resulting equation must contain a derivative of the nth order.

Algorithm for Formation of Differential Equations

Step I Write the given equation involving independent variable x (say), dependent variable y (say) and the arbitrary constant.

Step II Obtain the number of arbitrary constants in Step I. Let there be n arbitrary constants.

Step III Differentiate the relation in step I, n times with respect to x.

Step IV Eliminate arbitrary consstants with the help of n equations involving differential coefficients obtained in step III and an equation in step I.

The equation so obtained is the desired differential equation. The following examples will illustrate the above procedure.

Example 3 Form the differential equation, if $y^2 = 4a(x+b)$, where a, b are arbitrary constants.

Sol. Differentiating $y^2 = 4a(x+b)$ w.r.t. x,

$$2y \cdot \frac{dy}{dx} = 4a \quad \text{i.e.} \quad y\frac{dy}{dx} = 2a$$

Again, differentiating w.r.t. x, we get

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

which is the required differential equation. Thus, the elimination of arbitrary leads to the formation of a differential equation.

Example 4 Find the differential equation whose solution represents the family $xy = ae^x + be^{-x}$.

Sol.
$$xy = ae^x + be^{-x} \quad \ldots(i)$$

Differentiating Eq. (i) w.r.t. x, we get

$$x\frac{dy}{dx} + y = ae^x - be^{-x} \quad \ldots(ii)$$

Differentiating Eq. (ii) w.r.t. x, we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot 1 + \frac{dy}{dx} = ae^x + be^{-x} \quad \ldots(iii)$$

Using Eqs. (i) and (iii), we get

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = xy$$

Which is the required differential equation.

Example 5 Find the differential equation whose solution represents the family $c(y+c)^2 = x^3$.

Sol. Differentiating $c(y+c)^2 = x^3$ ...(i)

We get $c[2(y+c)]\frac{dy}{dx} = 3x^2$ but from Eq. (i), we have

$$\frac{2x^3}{(y+c)^2}(y+c)\frac{dy}{dx} = 3x^2$$

$$\Rightarrow \quad \frac{2x^3}{y+c}\cdot\frac{dy}{dx} = 3x^2 \quad \text{i.e.} \quad \frac{2x}{y+c}\cdot\frac{dy}{dx} = 3$$

$$\Rightarrow \quad \frac{2x}{3}\left[\frac{dy}{dx}\right] = y + c \quad \therefore \quad c = \frac{2x}{3}\left[\frac{dy}{dx}\right] - y$$

Substituting c in Eq. (i), we get

$$\left[\frac{2x}{3}\left(\frac{dy}{dx}\right) - y\right]\left[\frac{2x}{3}\frac{dy}{dx}\right]^2 = x^3$$

Which is the required differential equation.

Example 6 ***Find the differential equation whose solution represents the family*** $y = ae^{3x} + be^x$.

Sol.
$$y = ae^{3x} + be^x \quad \ldots(i)$$

Differentiating the given equation twice, we get

$$\frac{dy}{dx} = 3ae^{3x} + be^x \quad \text{and} \quad \frac{d^2y}{dx^2} = 9ae^{3x} + be^x$$

From the three equations by eliminating a and b, we obtain

$$\frac{d^2y}{dx^2} - \frac{4dy}{dx} + 3y = 0$$

Remark

The order of the differential equation will be equal to number of independent parameters and is not equal to the number of all the parameters in the family of curves.

Example 7 ***Find the order of the family of curves*** $y = (c_1 + c_2)e^x + c_3 e^{x+c_4}$.

Sol. Here, the number of arbitrary parameters is 4 but the order of the corresponding differential equation will not be 4 as it can be rewritten as, $y = (c_1 + c_2 + c_3 e^{c_4})e^x$, which is of the form $y = Ae^x$. Hence, the corresponding differential equation will be of order 1.

Example 8 The differential equation of all non-horizontal lines in a plane is given by

(a) $\frac{d^2y}{dx^2} = 0$ (b) $\frac{d^2x}{dy^2} = 0$

(c) $\frac{d^2y}{dx^2} = 0$ and $\frac{d^2x}{dy^2} = 0$ (d) All of these

Sol. The equation of the family of all non-horizontal lines in a plane is given by,

$$ax + by = 1 \qquad \text{(where } a \neq 0\text{)} \quad \ldots(i)$$

Differentiating w.r.t. y, we get

$$a\frac{dx}{dy} + b = 0 \qquad (\text{as } a \neq 0 \text{ and } b \in R)$$

Again, differentiating w.r.t. y, we get

$$a\frac{d^2x}{dy^2} = 0 \qquad (\text{as } a \neq 0 \text{ and } b \in R)$$

$$\Rightarrow \qquad \frac{d^2x}{dy^2} = 0 \qquad (\text{as } a \neq 0)$$

$\therefore$ Differential equation of all non-horizontal lines in a plane is $\frac{d^2x}{dy^2} = 0$. Hence, (b) is the correct answer.

Example 9 The differential equation of all non-vertical lines in a plane is given by

(a) $\frac{d^2y}{dx^2} = 0$ (b) $\frac{d^2x}{dy^2} = 0$

(c) $\frac{d^2x}{dy^2} = 0$ and $\frac{d^2y}{dx^2} = 0$ (d) All of these

Sol. The equation of the family of all non-vertical lines in a plane is given by $ax + by = 1$, where $b \neq 0$ and $a \in R$.

Differentiating both the sides w.r.t. x, we get

$$a + b\frac{dy}{dx} = 0 \qquad (\text{as } b \neq 0 \text{ and } a \in R)$$

Again, differentiating both the sides w.r.t. x, we get

$$b\frac{d^2y}{dx^2} = 0 \qquad (\text{as } b \neq 0 \text{ and } a \in R)$$

$$\Rightarrow \qquad \frac{d^2y}{dx^2} = 0 \qquad (\text{as } b \neq 0)$$

$\therefore$ Differential equation of all non-vertical lines in a plane.

$$\Rightarrow \qquad \frac{d^2y}{dx^2} = 0$$

Hence, (a) is the correct answer.

Example 10 ***The differential equation of all straight lines which are at a constant distance*** p ***from the origin, is***

(a) $(y + xy_1)^2 = p^2(1 + y_1^2)$ (b) $(y - xy_1^2) = p^2(1 + y_1)^2$

(c) $(y - xy_1)^2 = p^2(1 + y_1^2)$ (d) None of these

Sol. As, we know $x\cos\alpha + y\sin\alpha = p$ ...(i)

Represents the family of straight lines which are at a constant distance p from origin. Differentiating Eq. (i) w.r.t. x, we get

$$\cos\alpha + \sin\alpha \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \qquad \tan\alpha = -\frac{1}{y_1} \quad \text{or} \quad \sin\alpha = \frac{1}{\sqrt{1 + y_1^2}}$$

and $$\cos\alpha = -\frac{y_1}{\sqrt{1 + y_1^2}} \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$\frac{-xy_1}{\sqrt{1 + y_1^2}} + \frac{y}{\sqrt{1 + y_1^2}} = p$$

$\Rightarrow (y - xy_1)^2 = p^2(1 + y_1^2)$ is required differential equations.

Hence, (c) is the correct answer.

Example 11 The differential equation of all circles of radius r, is given by

(a) $\{1 + (y_1)^2\}^2 = r^2 y_2^3$ (b) $\{1 + (y_1)^2\}^3 = r^2 y_2^3$

(c) $\{1 + (y_1)^2\}^3 = r^2 y_2^2$ (d) None of these

Sol. Equation of circle of radius r,

$$(x - a)^2 + (y - b)^2 = r^2 \qquad \ldots(i)$$

(Here, a, b are two arbitrary constants)

Differentiating Eq. (i), we get

$$2(x - a) + 2(y - b)y_1 = 0 \qquad \ldots(ii)$$

Again, differentiating Eq. (ii), we get

$$1 + (y - b)y_2 + y_1^2 = 0$$

$$\Rightarrow \qquad (y - b) = -\left(\frac{1 + y_1^2}{y_2}\right) \qquad \ldots(iii)$$

Putting $(y - b)$ in Eq. (ii), we get

$$(x - a) = \frac{(1 + y_1^2)y_1}{y_2} \qquad \ldots(iv)$$

From Eqs. (i), (iii) and (iv), we get

$$\frac{(1 + y_1^2)^2 \cdot y_1^2}{y_2^2} + \frac{(1 + y_1^2)^2}{y_2^2} = r^2$$

$$\Rightarrow \qquad (1 + (y_1)^2)^3 = r^2 y_2^2$$

Hence, (c) is the correct answer.

Example 12 The differential equations of all circles touching the x-axis at origin is

(a) $(y^2 - x^2) = 2xy\left(\frac{dy}{dx}\right)$

(b) $(x^2 - y^2)\frac{dy}{dx} = 2xy$

(c) $(x^2 - y^2) = 2xy\left(\frac{dy}{dx}\right)$

(d) None of the above

Sol. The equation of circle touches x-axis at origin.

$$\Rightarrow \qquad (x - 0)^2 + (y - a)^2 = a^2$$

$$\text{or} \qquad x^2 + y^2 - 2ay = 0 \qquad \ldots(i)$$

Differentiating w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} - 2a\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx}(a - y) = x$$

$$\Rightarrow \quad a = \frac{x + y\left(\frac{dy}{dx}\right)}{\left(\frac{dy}{dx}\right)} \quad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$x^2 + y^2 - 2y\left[\frac{x + y\left(\frac{dy}{dx}\right)}{\left(\frac{dy}{dx}\right)}\right] = 0$$

or $$(x^2 - y^2)\frac{dy}{dx} = 2xy$$

Hence, (b) is the correct answer.

Example 13 The differential equation of all circles in the first quadrant which touch the coordinate axes is

(a) $(x - y)^2\,(1 + (y')^2) = (x + yy')^2$

(b) $(x + y)^2\,(1 + (y')^2) = (x + y')^2$

(c) $(x - y)^2\,(1 + y') = (x + yy')^2$

(d) None of these

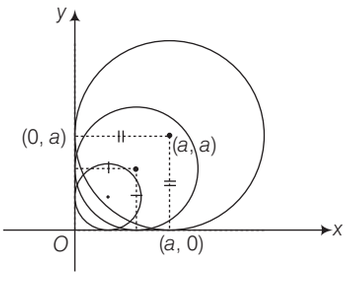
Sol. Equation of circles touching coordinate axes is

$$(x - a)^2 + (y - a)^2 = a^2 \quad \ldots(i)$$

Differentiating, we get

$$2(x - a) + 2(y - a)y' = 0$$

$$\Rightarrow \quad a = \frac{x + yy'}{1 + y'}; \quad \text{where } y' = \frac{dy}{dx} \quad \ldots(ii)$$



From Eqs. (i) and (ii),

$$\left(x - \frac{x + yy'}{1 + y'}\right)^2 + \left(y - \frac{x + yy'}{1 + y'}\right)^2 = \left(\frac{x + yy'}{1 + y'}\right)^2$$

$$\Rightarrow \quad \left(\frac{xy' - yy'}{1 + y'}\right)^2 + \left(\frac{y - x}{1 + y'}\right)^2 = \left(\frac{x + yy'}{1 + y'}\right)^2$$

$$\Rightarrow \quad (x - y)^2\,(y')^2 + (y - x)^2 = (x + yy')^2$$

$$\Rightarrow \quad (x - y)^2\,(1 + (y')^2) = (x + yy')^2$$

Hence, (a) is the correct answer.

Example 14 The differential equation satisfying the curve $\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1$, where λ being arbitrary unknown, is

(a) $(x + yy_1)(xy_1 - y) = (a^2 - b^2)\,y_1$

(b) $(x + yy_1)(x - yy_1) = y_1$

(c) $(x - yy_1)(xy_1 + y) = (a^2 - b^2)\,y_1$

(d) None of these

Sol. Here, $$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1 \quad \ldots(i)$$

Differentiating both the sides, we get

$$\frac{2x}{a^2 + \lambda} + \frac{2y}{b^2 + \lambda}\cdot\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x(b^2 + \lambda) + y(a^2 + \lambda)\,y_1 = 0$$

$$\Rightarrow \quad \lambda = -\left(\frac{xb^2 + a^2yy_1}{x + yy_1}\right)$$

$$\therefore \quad a^2 + \lambda = a^2 - \frac{xb^2 + a^2yy_1}{x + yy_1}$$

$$\Rightarrow \quad a^2 + \lambda = \frac{(a^2 - b^2)\,x}{x + yy_1} \quad \ldots(ii)$$

Also, $$b^2 + \lambda = \frac{(a^2 - b^2)\,yy_1}{x + yy_1} \quad \ldots(iii)$$

From Eqs. (i), (ii) and (iii), we get

$$\frac{x^2(x + yy_1)}{(a^2 - b^2)\,x} + \frac{y^2(x + yy_1)}{(a^2 - b^2)\,yy_1} = 1$$

$$\Rightarrow \quad (x + yy_1)(xy_1 - y) = (a^2 - b^2)\,y_1$$

Hence, (a) is the correct answer.

Example 15 The differential equation of all conics whose centre lies at origin, is given by

(a) $(3xy_2 + x^2y_3)(y - xy_1) = 3xy_2\,(y - xy_1 - x^2y_2)$

(b) $(3xy_1 + x^2y_2)(y_1 - xy_3) = 3xy_1\,(y - xy_2 - x^2y_3)$

(c) $(3xy_2 + x^2y_3)(y_1 - xy) = 3xy_1\,(y - xy_1 - x^2y_2)$

(d) None of the above

Sol. Equation of all conics whose centre lies at origin, is

$$ax^2 + 2hxy + by^2 = 1 \quad \ldots(i)$$

Differentiating Eq. (i) w.r.t. x, we get

$$2ax + 2hxy_1 + 2hy + 2byy_1 = 0$$

$$\Rightarrow \quad ax + h(y + xy_1) + byy_1 = 0$$

Multiplying by x equation becomes,

$$ax^2 + h(xy + x^2y_1) + bxyy_1 = 0 \quad \ldots(ii)$$

Subtracting Eqs. (i) and (ii), we get

$$h(xy - x^2y_1) + b(y^2 - xyy_1) = 1$$

$$\Rightarrow \quad (hx + by)\, y - xy_1\,(hx + by) = 1$$

$$\Rightarrow \quad (hx + by)\,(y - xy_1) = 1$$

$$\Rightarrow \quad hx + by = \frac{1}{y - xy_1} \qquad \ldots(\text{iii})$$

Again, differentiating w.r.t. x, we get

$$h + by_1 = -\frac{(y_1 - xy_2 - y_1)}{(y - xy_1)^2}$$

or
$$h + by_1 = \frac{xy_2}{(y - xy_1)^2} \qquad \ldots(\text{iv})$$

From Eqs. (iii) and (iv), we get

$$b\,(y - xy_1) = \frac{y - xy_1 - x^2y_2}{(y - xy_1)^2}$$

$$\Rightarrow \quad b = \frac{y - xy_1 - x^2y_2}{(y - xy_1)^3} \qquad \ldots(\text{v})$$

Again, differentiating both the sides w.r.t. x, we get

$$0 = \frac{y_1 - y_1 - 3xy_2 - x^2y_3}{(y - xy_1)^3} + \frac{3\,(y - xy_1 - x^2y_2)\,xy_2}{(y - xy_1)^4}$$

$$\Rightarrow \quad (3xy_2 + x^2y_3)\,(y - xy_1) = 3xy_2\,(y - xy_1 - x^2y_2)$$

Hence, (a) is the correct answer.

Exercise for Session 1

1. The differential equation of all parabolas whose axis of symmetry is along X-axis is of order.
(a) 2 (b) 3 (c) 1 (d) None of these

2. The order and degree of the differential equation of all tangent lines to the parabola $x^2 = 4y$ is
(a) 1, 2 (b) 2, 2 (c) 3, 1 (d) 4, 1

3. The degree of the differential equation $\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^2 = x^2 \log\left(\frac{d^2y}{dx^2}\right)$ is
(a) 1 (b) 2 (c) 3 (d) Not defined

4. The degree of the defferential equation satisfying the relation $\sqrt{1+x^2} + \sqrt{1+y^2} = \lambda(x\sqrt{1+y^2} - y\sqrt{1+x^2})$ is
(a) 1 (b) 2 (c) 3 (d) 4

5. The degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = x \sin\left(\frac{d^2y}{dx^2}\right)$ is
(a) 1 (b) 2 (c) 3 (d) Not defined

6. The differential equation of all circles touching the y-axis at origin, is
(a) $y^2 - x^2 = 2xy\frac{dy}{dx}$ (b) $y^2 - x^2 = 2xy\frac{dx}{dy}$ (c) $x^2 - y^2 = 2xy\frac{dy}{dx}$ (d) $x^2 - y^2 = 2xy\frac{dx}{dy}$

7. The differential equation of all parabolas having their axes of symmetry coincident with the axes of x, is
(a) $yy_2 + y_1^2 = y + y_1$ (b) $yy_2 + y_1^2 = 0$ (c) $yy_2 + y_1^2 = y_1$ (d) None of these

8. The differential equation of all conics whose axes coincide with the coordinate axes, is
(a) $xyy_2 + xy_1^2 - yy_1 = 0$ (b) $yy_2 + y_1^2 - yy_1 = 0$
(c) $xyy_2 + (x - y)\,y_1 = 0$ (d) None of these

9. The differential equation having $y = (\sin^{-1} x)^2 + A(\cos^{-1} x) + B$, where A and B are arbitrary constant, is
(a) $(1 - x^2)\,y_2 - xy_1 = 2$ (b) $(1 - x^2)\,y_2 + yy_1 = 0$
(c) $(1 - x)\,y_2 + xy_1 = 0$ (d) None of these

10. The differential equation of circles passing through the points of intersection of unit circle with centre at the origin and the line bisecting the first quadrant, is
(a) $y_1(x^2 + y^2 - 1) + (x + yy_1) = 0$ (b) $(y_1 - 1)(x^2 + y^2 - 1) + (x + yy_1)\,2(x - y) = 0$
(c) $(x^2 + y^2 - 1) + yy_2 = 0$ (d) None of these

Session 2

Solving of Variable Seperable Form, Homogeneous Differential Equation

Solving of Variable Seperable Form

Solution of a Differential Equation

The solution of the differential equation is a relation between the variables of the equation not containing the derivatives, but satisfying the given differential equation (i.e from which the given differential equation can be derived).

Thus, the solution of $\frac{dy}{dx} = e^x$ could be obtained by simply integrating both the sides, i.e. $y = e^x + C$ and that of, $\frac{dy}{dx} = px + q$ is $y = p\frac{x^2}{2} + qx + C$, where C is arbitrary constant.

(i) **A general solution or an integral** of a differential equation is a relation between the variables (not involving the derivatives) which contains the same number of the arbitrary constants as the order of the differential equation. For example, a general solution of the differential equation $\frac{d^2x}{dt^2} = -4x$ is $x = A\cos 2t + B\sin 2t$, where A and B are the arbitrary constants.

(ii) **Particular solution or particular integral** is that solution of the differential equation obtained from the general solution by assigning particular values to the arbitrary constant in the general solution.

For example, $x = 10\cot 2t + 5\sin 2t$ is a particular solution of differential equation $\frac{d^2x}{dt^2} = -4x$.

Differential Equations of the First Order and First Degree

In this section we shall discuss the differential equations which are of first order and first degree only.

A differential equation of first order and first degree is of the form $\frac{dy}{dx} = f(x, y)$.

Remark

All the differential equations, even of first order and first degree, cannot be solved. However, if they belong to any of the standard forms which we are going to discuss, in the subsequent articles they can be solved.

Equations in Which the Variables are Separable

The equation $\frac{dy}{dx} = f(x, y)$ is said to be in variables separable form, if we can express it in the form $f(x)\,dx = g(y)\,dy$. By integrating this, solution of the equation is obtained which is, $\int f(x)\,dx = \int g(y)\,dy + C$

Example 16 ***Solve***

$$\sec^2 x \tan y\,dx + \sec^2 y \tan x\,dy = 0.$$

Sol. Dividing the given equation by $\tan x \tan y$, we get

$$\frac{\sec^2 x}{\tan x}dx + \frac{\sec^2 y}{\tan y}dy = 0$$

This is variable-separable type

Integrating, $\int \frac{\sec^2 x}{\tan x}dx + \int \frac{\sec^2 y}{\tan y}dy = C'$

$$\ln|\tan x| + \ln|\tan y| = \ln C;\ \text{where } C' = \ln C$$

or $$\ln|\tan x \cdot \tan y| = \ln C;\ (C > 0)$$

$\Rightarrow$ $$|\tan x \cdot \tan y| = C$$

This is the general solution.

Example 17 Solve $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$.

Sol. Here, $\frac{dy}{dx} = \frac{e^x}{e^y} + \frac{x^2}{e^y} \Rightarrow e^y\,dy = (x^2 + e^x)\,dx$

This is variable-separable form,

$\therefore$ Integrating both the sides,

$$\int e^y\, dy = \int (x^2 + e^x)\, dx \Rightarrow e^y = \frac{x^3}{3} + e^x + C$$

Which is the general solution of the given differential equation, where C is an arbitrary constant.

Example 18 Solve $\sqrt{1+x^2+y^2+x^2y^2} + xy\dfrac{dy}{dx} = 0$.

Sol. The given differential equation can be written as

$$\sqrt{(1+y^2)(1+x^2)} = -xy\frac{dy}{dx}$$

$$\Rightarrow \quad -\frac{\sqrt{1+x^2}\,dx}{x} = \frac{y\,dy}{\sqrt{1+y^2}}$$

This is the variable-separable form.

$\therefore$ Integrating both the sides, we get

$$-\int \frac{\sqrt{1+x^2}}{x}\,dx = \int \frac{y}{\sqrt{1+y^2}}\,dy$$

$$\Rightarrow -\left[\sqrt{1+x^2} + \frac{1}{2}\log\left(\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right)\right] = \sqrt{1+y^2} + C$$

This is the general solution to the given differential equation.

Example 19 Solve $y - x\dfrac{dy}{dx} = a\left(y^2 + \dfrac{dy}{dx}\right)$.

Sol. Rewriting the given equation as

$$y - ay^2 = (x+a)\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{y(1-ay)} = \frac{dx}{(x+a)}$$

This is the variable-separable form.

Integrating both the sides, we get

$$\int \frac{dy}{y(1-ay)} = \int \frac{dx}{(x+a)}$$

$$\Rightarrow \quad \int \left(\frac{1}{y} + \frac{a}{1-ay}\right)dy = \int \frac{dx}{x+a}$$

$$\Rightarrow \quad \ln y - \frac{a}{a}\ln(1-ay) + \ln C = \ln(a+x)$$

$$\Rightarrow \quad \ln\left(\frac{(a+x)(1-ay)}{y}\right) = \ln(C)$$

or $Cy = (a+x)(1-ay)$ is the general solution.

Example 20 Solve $e^{dy/dx} = x+1$, given that when $x = 0, y = 3$.

Sol. This is an Example of particular solution.

$$e^{(dy/dx)} = x+1$$

$\therefore \quad \dfrac{dy}{dx} = \ln(x+1)$

$\therefore \quad \int dy = \int \ln(x+1)\,dx$ (integration by parts)

i.e. $\quad y = x\ln(x+1) - \int \dfrac{x}{x+1}\,dx$

i.e. $\quad y = x\ln(x+1) - x + \ln(x+1) + C$

This is the general solution.

To find the particular solution, put $x = 0, y = 3$ in the general equation.

$\therefore \quad 3 = 0 - 0 + 0 + C$

$\therefore \quad C = 3$

$\therefore$ The required particular solution is,

$$y = (x+1)\ln(x+1) - x + 3$$

Differential Equations Reducible to the Separable Variable Type

Sometimes differential equation of the first order cannot be solved directly by variable separation but by some substitution we can reduce it to a differential equation with separable variable. ''A differential equation of the form $\dfrac{dy}{dx} = f(ax+by+c)$ is solved by writing $ax+by+c = t$.''

Example 21 Solve $\dfrac{dy}{dx} = \sin^2(x+3y) + 5$.

Sol. Let $x+3y = t$, so that $1 + \dfrac{3dy}{dx} = \dfrac{dt}{dx}$

The given differential equation becomes,

$$\frac{1}{3}\left(\frac{dt}{dx} - 1\right) = \sin^2(t) + 5 \Rightarrow \frac{dt}{dx} = 3\sin^2 t + 16$$

$$\Rightarrow \quad \int \frac{dt}{3\sin^2 t + 16} = \int dx$$

$$\Rightarrow \quad \int \frac{\sec^2 t\, dt}{3\tan^2 t + 16\sec^2 t} = x + C$$

$$\Rightarrow \quad \int \frac{\sec^2 t\, dt}{19\tan^2 t + 16} = x + C$$

$$\Rightarrow \quad \int \frac{du}{19u^2 + 16} = x + C;\ \text{where } \tan t = u$$

$$\Rightarrow \quad \sec^2 t\, dt = du$$

$$\Rightarrow \quad \frac{1}{19}\int \frac{du}{u^2 + \frac{16}{19}} \Rightarrow \frac{1}{19}\frac{\sqrt{19}}{4}\tan^{-1}\left(\frac{\sqrt{19}u}{4}\right) + C$$

$$\Rightarrow \quad \frac{1}{4\sqrt{19}}\left\{\tan^{-1}\left(\frac{\sqrt{19}}{4}\tan(3y+x)\right)\right\} + C$$

Example 22 Solve $(x+y)^2 \dfrac{dy}{dx} = a^2$.

Sol. $$(x+y)^2 \frac{dy}{dx} = a^2 \quad \ldots(i)$$

Put $x + y = t \Rightarrow 1 + \dfrac{dy}{dx} = \dfrac{dt}{dx}$ or $\dfrac{dy}{dx} = \left(\dfrac{dt}{dx} - 1\right)$

$\therefore$ Eq. (i) reduces to $t^2 \left\{\dfrac{dt}{dx} - 1\right\} = a^2$

i.e. $t^2 \dfrac{dt}{dx} = a^2 + t^2$, separating the variable and integrating.

$$\int dx = \int \frac{t^2}{a^2 + t^2}\, dt = \int \left(1 - \frac{a^2}{a^2 + t^2}\right) dt$$

$$\therefore \quad x = t - a \tan^{-1}\left(\frac{t}{a}\right) + C$$

i.e. $$x = x + y - a \tan^{-1}\left(\frac{x+y}{a}\right) + C$$

i.e. $y = a \tan^{-1}\left(\dfrac{x+y}{a}\right) - C$ is the required general solution.

Example 23 ***Solve***

$(2x + 3y - 1)\,dx + (4x + 6y - 5)\,dy = 0.$

Sol. $(2x + 3y - 1)\,dx + (4x + 6y - 5)\,dy = 0 \quad \ldots(i)$

Substitute $u = 2x + 3y - 1$

$$\therefore \quad \frac{du}{dx} = 2 + \frac{3dy}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{3}\left(\frac{du}{dx} - 2\right)$$

$\therefore$ Eq. (i) reduces to

$$u + (2u - 3)\frac{1}{3}\left(\frac{du}{dx} - 2\right) = 0$$

i.e. $$u - \frac{2}{3}(2u - 3) + \frac{1}{3}(2u - 3)\frac{du}{dx} = 0$$

i.e. $$\frac{-u + 6}{3} + \frac{1}{3}(2u - 3)\frac{du}{dx} = 0$$

Writing this in the variable-separable form

$$\left(\frac{2u - 3}{u - 6}\right) du = dx$$

$$\therefore \quad \int dx = \int \frac{2u - 3}{u - 6}\, du$$

$$\therefore \quad x + C = \int \frac{2(u-6) + 9}{(u-6)}\, du$$

$$\therefore \quad x + C = 2u + 9 \ln|u - 6|$$

$$\therefore \quad x + C = 2(2x + 3y - 1) + 9 \ln|2x + 3y - 7|$$

$3x + 6y - 2 + 9 \ln|2x + 3y - 7| = C$ is the general solution.

Remarks

Sometimes transformation to the polar coordinates facilitates separation of varibales. It is convenient to remember the following differentials.

1. $x\,dx + y\,dy = r\,dr$ **2.** $x\,dy - y\,dx = r^2\,d\theta$

3. $dx^2 + dy^2 = dr^2 + r^2 d\,\theta^2$

Example 24 Solve $\dfrac{x\,dx + y\,dy}{x\,dy - y\,dx} = \sqrt{\dfrac{a^2 - x^2 - y^2}{x^2 + y^2}}$.

Sol. Let $x = r\cos\theta, y = r\sin\theta$

So that $$x^2 + y^2 = r^2 \quad \ldots(i)$$

and $$\tan\theta = \frac{y}{x} \quad \ldots(ii)$$

From Eq. (i), we have $d(x^2 + y^2) = d(r^2)$

i.e. $$x\,dx + y\,dy = r\,dr \quad \ldots(iii)$$

From Eq. (ii), we have $d\left(\dfrac{y}{x}\right) = d(\tan\theta)$

i.e. $$\frac{x\,dy - y\,dx}{x^2} = \sec^2\theta\, d\theta$$

i.e. $x\,dy - y\,dx = x^2 \sec^2\theta\, d\theta = r^2 \cos^2\theta \sec^2\theta\, d\theta \ldots(iv)$

Using Eqs. (iii) and (iv) in the given equation, we get

$$\frac{r\,dr}{r^2\,d\theta} = \sqrt{\frac{a^2 - r^2}{r^2}} \text{ i.e. } \frac{dr}{\sqrt{a^2 - r^2}} = d\theta$$

i.e. $$\sin^{-1}\left(\frac{r}{a}\right) = \theta + C \quad \text{or} \quad r = a\sin(\theta + C)$$

or $$\sqrt{x^2 + y^2} = a\sin\{C + \tan^{-1}(y/x)\}$$

It is advised to remember the results (iii) and (iv).

Homogeneous Differential Equation

By definition, a homogeneouos function $f(x, y)$ of degree n satisfies the property

$$f(\lambda x, \lambda y) = \lambda^n f(x, y)$$

For example, the functions

$$f_1(x, y) = x^3 + y^3$$

$$f_2(x, y) = x^2 + xy + y^2$$

$$f_3(x, y) = x^3 e^{x/y} + xy^2$$

are all homogeneous functions, of degrees three, two and three respectively (verify this assertion).

Observe that any homogeneous function $f(x, y)$ of degree n can be equivalnetly written as follows :

$$f(x, y) = x^n f\left(\frac{y}{x}\right) = y^n f\left(\frac{x}{y}\right)$$

For example, $f_{(x,\ y)} = x^3 + y^3$

$$= x^3\left(1 + \left(\frac{y}{x}\right)^3\right) = y^3\left(1 + \left(\frac{x}{y}\right)^3\right)$$

Having seen homogeneous functions we define homogeneous DEs as follows :

Any DE of the form $M(x, y)\,dx + N(x, y)dy = 0$

or $\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}$ is called homogeneous if $M(x, y)$ and $N(x, y)$ are homogeneous functions of the same degree.

What is so special about homogeneous DEs? Well, it turns out that they areextremely simple to solve. To see how, we express both $M(x, y)$ and $N(x, y)$ as, say $x^n M\left(\frac{y}{x}\right)$ and $x^n N\left(\frac{y}{x}\right)$. This can be done sicne $M(x, y)$ and $N(x, y)$ are both homogeneous function of degree n. Doing this reduces our DE to

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)} = \frac{x^n M\left(\frac{y}{x}\right)}{x^n N\left(\frac{y}{x}\right)} = -\frac{M\left(\frac{y}{x}\right)}{N\left(\frac{y}{x}\right)} = P\left(\frac{y}{x}\right)$$

(The function $P(t)$ stands for $\frac{-M(t)}{N(t)}$)

Now, the simple substitution $y = vx$ reduces this DE to a VS form

$$y = vx$$

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dy}{dx}$$

Thus, $\frac{dy}{dx} = p\left(\frac{y}{x}\right)$ transforms to

$$v + x\frac{dv}{dx} = P(v)$$

$$\Rightarrow \quad \frac{dv}{P(v) - v} = \frac{dx}{x}$$

This can now be integraed directly since it is in VS form. Let us see some examples of solving homogeneous DEs.

Alogorithm for Solving Homogeneous Differential Equation

***Step* I** Put the differential equation in the form

$$\frac{dy}{dx} = \frac{\phi(x,y)}{\psi(x,y)}$$

***Step* II** Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in the equation in step I and can out x from the right hand side. The equation reduces to the form $v + x\frac{dv}{dx} = f(v)$.

***Step* III** Shift v on R.H.S and seperate the variables in v and x.

***Step* IV** Integrate both sides to obtain the solution in terms of v and x.

***Step* V** Replace v by $\frac{y}{x}$ in the solution obtained in step IV to obtain the solution in terms of x and y.

Following examples illustrate the procedure.

Example 25 Solve $y\,dx + (2\sqrt{xy} - x)\,dy = 0$.

Sol. $$y\,dx + (2\sqrt{xy} - x)\,dy = 0 \qquad \ldots(i)$$

This is homogeneous type. Substitute $y = ux$

$$\therefore \quad \frac{dy}{dx} = u + \frac{x\,du}{dx}$$

$\therefore$ Equation $ux\,dx + (2\sqrt{x^2 u} - x)(u\,dx + x\,du) = 0$

i.e. $x \cdot \{udx + (2\sqrt{u} - 1)\,u\,dx + x\,du\,(2\sqrt{u} - 1)\} = 0$

i.e. $$dx\,(2u^{3/2} - u + u) + x\,du\,(2\sqrt{u} - 1) = 0$$

Separating the variables, $\frac{dx}{x} + \left(\frac{2\sqrt{u} - 1}{2u^{3/2}}\right)du = 0$

Integrating both the sides $\ln|x| + \ln|u| + \frac{1}{\sqrt{u}} = C$

or $\ln|xu| + \frac{1}{\sqrt{u}} = C$ or $\ln|y| + \sqrt{\frac{x}{y}} = C \quad \left(\because u = \frac{y}{x}\right)$

Which is the general solution.

Example 26 Solve $(x^2 + y^2)\,dx - 2xy\,dy = 0$.

Sol. Here, $\frac{dy}{dx} = \frac{x^2 + y^2}{2xy} = \frac{1}{2}\left[\frac{x}{y} + \frac{y}{x}\right]$

With $y = ux$, $\frac{dy}{dx} = u + x\frac{du}{dx}$, so that the differential equation becomes

$$u + x\frac{du}{dx} = \frac{1}{2}\left(\frac{1}{u} + u\right)$$

$$\Rightarrow \quad \frac{x\,du}{dx} = \frac{1 + u^2}{2u} - u$$

$$\Rightarrow \quad \frac{x\,du}{dx} = \frac{1 - u^2}{2u}$$

$$\Rightarrow \quad \int \frac{2u}{1 - u^2}\,du = \int \frac{dx}{x}$$

$$\Rightarrow \quad -\log|1 - u^2| = \log|x| - \log|C|$$

$$\Rightarrow \quad x(1 - u^2) = C$$

$$\Rightarrow \quad \frac{x^2 - y^2}{x} = C \qquad \left(\because u = \frac{y}{x}\right)$$

Hence, $x^2 - y^2 = xC$, is the required solution.

Example 27 Solve $\dfrac{2\,dy}{dx} = \dfrac{y}{x} + \dfrac{y^2}{x^2}$.

Sol. The above equation is homogeneous so that we put $y = ux$.

$$\Rightarrow \quad 2\left[u + x\frac{du}{dx}\right] = u + u^2 \;\Rightarrow\; 2u + 2x\frac{du}{dx} = u + u^2$$

$$\Rightarrow \quad 2x\frac{du}{dx} = u^2 - u \;\Rightarrow\; \frac{du}{u^2 - u} = \frac{dx}{2x}$$

$$\Rightarrow \quad \int \frac{du}{u(u-1)} = \frac{1}{2}\int \frac{dx}{x}$$

$$\Rightarrow \quad \int \frac{1}{u-1}\,du - \int \frac{1}{u}\,du = \frac{1}{2}\int \frac{dx}{x}$$

$$\Rightarrow \quad \log|u-1| - \log|u| = \frac{1}{2}\log|x| + \log|C|$$

$$\Rightarrow \quad \log\left|\frac{u-1}{u}\right| = \log|C\sqrt{x}|$$

$$\Rightarrow \quad \frac{u-1}{u} = C\sqrt{x} \;\Rightarrow\; \frac{y-x}{y} = C\sqrt{x}$$

$$\Rightarrow \quad y - x = C\sqrt{x}\cdot y$$

Which is the required solution.

Example 28 ***Solve***

$(1+2\,e^{x/y})\,dx + 2\,e^{x/y}\,(1 - x/y)\,dy = 0.$

Sol. The appearance of x/y in the equation suggests the substitution $x = vy$ or $dx = v\,dy + y\,dv$.

$\therefore$ The given equation is

$$(1+2e^v)(v\,dy + y\,dv) + 2e^v(1-v)\,dy = 0$$

i.e. $$y(1+2e^v)\,dv + (v + 2e^v)\,dy = 0$$

i.e. $$\frac{1+2e^v}{v+2e^v}\,dv + \frac{dy}{y} = 0$$

Integrating, $$\int \frac{1+2e^v}{v+2e^v}\,dv + \int \frac{dy}{y} = 0$$

$$\log|v + 2e^v| + \log|y| = \log|C|$$

$$\Rightarrow \quad (v + 2e^v)\,y = C \qquad \left(v = \frac{x}{y}\right)$$

$$\Rightarrow \quad \left(\frac{x}{y} + 2e^{x/y}\right)y = C$$

$$\Rightarrow \quad (x + 2ye^{x/y}) = C, \text{ is required solution.}$$

Example 29 ***Show that any equation of the form***

$$y\,f(xy)\,dx + x\,g(xy)\,dy = 0$$

can be converted to variable separable form by substituting $xy = v$.

Sol. Since, $xy = v$, $y = \dfrac{v}{x}$ and $d(x\,y) = d\,v$

i.e. $$x\,dy + y\,dx = dv$$

and $$dy = d\left(\frac{v}{x}\right) = \frac{x\,dv - v\,dx}{x^2}$$

i.e. $$x\,dy = dv - \frac{v}{x}\,dx$$

$\therefore$ $$\frac{v}{x}f(v)\,dx + g(v)\left\{dv - \frac{v}{x}\,dx\right\} = 0$$

$\therefore$ $$\frac{v\{f(v) - g(v)\}}{x}\,dx + g(v)\,dv = 0$$

i.e. $$\frac{dx}{x} + \frac{g(v)\,dv}{v\{f(v) - g(v)\}} = 0$$

Which is in variables separable form.

Reducible to Homogeneous Form

Type I

Many a times, the DE specified may not be homogeneous but some suibtale manipulation might reduce it to a homogeneous form. Generally, such equations involve a function of a rational expression whose numerator and denominator are linear functions of the variable, i.e., of the form

$$\frac{dy}{dx} = f\left(\frac{ax + by + c}{dx + cy + f}\right) \qquad \ldots(i)$$

Note that the presence of the constant c and f causes this DE to be non-homogeneous.

To make it homogeneous, we use the substitutions

$$x \to X + h$$
$$y \to Y + k$$

and select h and k so that

$$\left.\begin{aligned} ah + bk + c &= 0 \\ dh + ek + f &= 0 \end{aligned}\right\} \qquad \ldots(ii)$$

This can always be done $\left(\text{if } \dfrac{a}{b} \neq \dfrac{d}{e}\right)$. The RHS of the DE in (i) now reduces to $= f\left(\dfrac{a(X+h) + b(Y+k) + c}{d(X+h) + e(Y+k+f)}\right)$

$$= f\left(\frac{aX + bY}{dX + eY}\right) \qquad \text{(Using Eq. (ii))}$$

This expression is clearly homogeneous! The LHS of Eq. (i) is $\dfrac{dy}{dx}$ which equals $\dfrac{dy}{dY}\cdot\dfrac{dY}{dX}\cdot\dfrac{dX}{dx}$. Since $\dfrac{dy}{dY}\cdot\dfrac{dx}{dX} = 1$, the LHS $\dfrac{dy}{dx}$ equals $\dfrac{dY}{dX}$. Thus, our equation becomes

$$\frac{dY}{dX} = f\left(\frac{aX + bY}{dX + eY}\right) \quad \text{...(iii)}$$

We have thus succeeded in transforming the non-homogeneous DE in Eq. (i) to the homogeneous DE in Eq. (iii). This can now be solved as described earlier.

Example 30 Solve the DE $\frac{dy}{dx} = \frac{2y - x - 4}{y - 3x + 3}$.

Sol. We substitute $x \to X + h$ and $y \to Y + k$ where h, k need to be determined

$$\frac{dy}{dx} = \frac{dY}{dX} = \frac{(2Y - X) + (2k - h - 4)}{(Y - 3X) + (k - 3h + 3)}$$

h and k must be chosen so that

$$2k - h - 4 = 0$$
$$k - 3h + 3 = 0$$

This gives $h = 2$ and $k = 3$. Thus,

$$x = X + 2$$
$$y = Y + 3$$

Our DE now reduces to

$$\frac{dY}{dX} = \frac{2Y - X}{Y - 3X}$$

Using the substitution $Y = vX$, and simplifying, we have (verify),

$$\frac{v - 3}{v^2 \;\; 5v + 1} dv = \frac{-dX}{X}$$

We now integrate this DE which is VS; the left-hand side can be integrated by the techniques described in the unit of Indefinite Integration.

Finally, we substitute $v = \frac{Y}{X}$ and

$$X = x - 2$$
$$Y = y - 3$$

to obtain the general solution.

Type II

Suppose our DE is of the form

$$\frac{dy}{dx} = f\left(\frac{ax + by + c}{dx + ey + f}\right)$$

We try to find h, k so that

$$ah + bk + c = 0$$
$$dh + ek + f = 0$$

What if this system does not yield a solution? Recall that this will happen if $\frac{a}{b} = \frac{d}{e}$. How do we reduce the DE to a homogeneous one in such a case?

Let $\frac{a}{d} = \frac{b}{e} = \lambda$ (say).

Thus,

$$\frac{ax + by + c}{dx + ey + f} = \frac{\lambda(dx + ey) + c}{dx + ey + f}$$

This suggests the substitution $dx + ey = v$, which will give

$$d + e\frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{e}\left(\frac{dv}{dx} - d\right)$$

Thus, our DE reduces to

$$\frac{1}{e}\left(\frac{dv}{dx} - d\right) = \frac{\lambda v + c}{v + f}$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{\lambda ev + ec}{v + f} + d$$

$$= \frac{(\lambda e + d)v + (ec + d)}{v + f}$$

$$\Rightarrow \quad \frac{(v + f)}{(\lambda e + d)\, v + ec + df} dv = dx$$

which is in VS form and hence can be solved.

Example 31 Solve the DE $\frac{dy}{dx} = \frac{x + 2y - 1}{x + 2y + 1}$.

Sol. Note that h, k do not exist in this case which can reduce this DE to homogeneous form. Thus, we use the substitution

$$x + 2y = v$$

$$\Rightarrow \quad 1 + 2\frac{dy}{dx} = \frac{dv}{dx}$$

Thus, our DE becomes

$$\frac{1}{2}\left(\frac{dv}{dx} - 1\right) = \frac{v - 1}{v + 1}$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{2v - 2}{v + 1} + 1 = \frac{3v - 1}{v + 1}$$

$$\Rightarrow \quad \frac{v + 1}{3v - 1} dv = dx$$

$$\Rightarrow \quad \frac{1}{3}\left(1 + \frac{4}{3v - 1}\right) dv = dx$$

Integrating, we have

$$\frac{1}{3}\left(v + \frac{4}{3}\ln(3v - 1)\right) = x + C_1$$

Substituting $v = x + 2y$, we have

$$x + 2y + \frac{4}{3}\ln(3x + 6y - 1) = 3x + C_2$$

$$\Rightarrow \quad y - x + \frac{2}{3}\ln(3x + 6y - 1) = C$$

Example 32 ***The solution of the differential equation*** $\frac{dy}{dx}=\frac{\sin y+x}{\sin 2y-x\cos y}$ ***is***

(a) $\sin^2 y = x\sin y+\frac{x^2}{2}+C$

(b) $\sin^2 y = x\sin y-\frac{x^2}{2}+C$

(c) $\sin^2 y = x+\sin y+\frac{x^2}{2}+C$

(d) $\sin^2 y = x-\sin y+\frac{x^2}{2}+C$

Sol. Here, $\frac{dy}{dx}=\frac{\sin y+x}{\sin 2y-x\cos y}$

$\Rightarrow \cos y\frac{dy}{dx}=\frac{\sin y+x}{2\sin y-x}$, put $\sin y=t$

$\Rightarrow \frac{dt}{dx}=\frac{t+x}{2t-x}$, put $t=vx$

$$\frac{x\,dv}{dx}+v=\frac{vx+x}{2vx-x}=\frac{v+1}{2v-1}$$

$\therefore \quad x\frac{dv}{dx}=\frac{v+1}{2v-1}-v=\frac{v+1-2v^2+v}{2v-1}$

or $\frac{2v^2-v}{-2v^2+2v+1}dv=\frac{dx}{x}$

On solving, we get

$$\sin^2 y = x\sin y+\frac{x^2}{2}+C$$

Hence, (a) is the correct answer.

Example 33 The equation of curve passing through (1, 0) and satisfying $\left(y\frac{dy}{dx}+2x\right)^2=(y^2+2x^2)\left(1+\left(\frac{dy}{dx}\right)^2\right)$, is given by

(a) $\sqrt{2}\,x^{\pm\frac{1}{\sqrt{2}}}=\frac{y+\sqrt{y^2+2x^2}}{x}$

(b) $\sqrt{2}x^{\pm\sqrt{2}}=\frac{y+\sqrt{y^2+\sqrt{2}\,x^2}}{x}$

(c) $\sqrt{2}\,y^{\pm\frac{1}{\sqrt{2}}}=\frac{y+\sqrt{x^2+2y^2}}{x}$

(d) None of the above

Sol. The given differential equation can be written as

$$y^2\left(\frac{dy}{dx}\right)^2+4x^2+4xy\cdot\frac{dy}{dx}=(y^2+2x^2)\left(1+\left(\frac{dy}{dx}\right)^2\right)$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{y}{x}\pm\sqrt{\frac{1}{2}\left(\frac{y}{x}\right)^2+1} \qquad \ldots(i)$$

Let $y=vx$

$$\Rightarrow \quad v+x\frac{dv}{dx}=\frac{dy}{dx}$$

$\therefore$ Eq. (i) becomes

$$v+x\frac{dv}{dx}=v\pm\sqrt{\frac{1}{2}v^2+1}$$

or $\int\frac{dv}{\sqrt{\frac{1}{2}v^2+1}}=\int\frac{dx}{x}$

$$\Rightarrow \quad \sqrt{2}\log|v+\sqrt{v^2+2}|=\log|xC|$$

$$\Rightarrow \sqrt{2}\log\left|\frac{y+\sqrt{y^2+2x^2}}{x}\right|=\log|xC|,$$

putting $x=1$ and $y=0$

$$\Rightarrow \quad C=(\sqrt{2})^{\sqrt{2}}$$

$\therefore$ Curves are given by $\frac{y+\sqrt{y^2+2x^2}}{x}=\sqrt{2}\,x^{\pm\frac{1}{\sqrt{2}}}$

Hence, (a) is the correct answer.

Exercise for Session 2

1. The solution of $\frac{dy}{dx} = \frac{(x+y)^2}{(x+2)(y-2)}$, is given by

(a) $(x+2)^4\left(1+\frac{2y}{x}\right) = ke^{2y/x}$ (b) $(x+2)^4\left(1+\frac{2(y-2)}{(x+2)}\right) = k\,e^{\frac{2(y-2)}{(x+2)}}$

(c) $(x+2)^3\left(1+2\frac{(y-2)}{x+2}\right) = ke^{\frac{2(y-2)}{x+2}}$ (d) None of these

2. If $(y^3 - 2x^2y)\,dx + (2xy^2 - x^3)\,dy = 0$, then the value of $xy\sqrt{y^2-x^2}$, is

(a) $y^2 + x$ (b) xy^2

(c) any constant (d) None of these

3. The solution of $dy/dx = \cos(x+y) + \sin(x+y)$, is given by

(a) $\log\left|1+\tan\left(\frac{x+y}{2}\right)\right| = x + C$ (b) $\log|1+\tan(x+y)| = x + C$

(c) $\log|1-\tan(x+y)| = x + C$ (d) None of these

4. The solution of $\frac{dy}{dx} = (x+y-1) + \frac{x+y}{\log(x+y)}$, is given by

(a) $\{1+\log(x+y)\} - \log\{1+\log(x+y)\} = x + C$ (b) $\{1-\log(x+y)\} - \log\{1-\log(x+y)\} = x + C$

(c) $\{1+\log(x+y)\}^2 - \log\{1+\log(x+y)\} = x + C$ (d) None of these

5. The solution of $(2x^2+3y^2-7)\,xdx - (3x^2+2y^2-8)\,ydy = 0$, is given by

(a) $(x^2+y^2-1) = (x^2+y^2-3)^5\,C$ (b) $(x^2+y^2-1)^2 = (x^2+y^2-3)^5\,C$

(c) $(x^2+y^2-3) = (x^2+y^2-1)^5\,C$ (d) None of these

6. The solution of $\frac{dy}{dx} = \frac{(x-1)^2 + (y-2)^2\tan^{-1}\left(\frac{y-2}{x-1}\right)}{(xy-2x-y+2)\tan^{-1}\left(\frac{y-2}{x-1}\right)}$, is equal to

(a) $\{(x-1)^2+(y-1)^2\}\tan^{-1}\left(\frac{y-2}{x-1}\right) - 2(x-1)(y-2) = 2(x-1)^2\log C(x-1)$

(b) $\{(x-1)^2+(y-1)^2\} - 2(x-1)(y-2)\tan^{-1}\left(\frac{y-2}{x-1}\right) = 2(x-1)^2\log C$

(c) $\{(x-1)^2+(y-1)^2\}\tan^{-1}\left(\frac{y-2}{x-1}\right) - 2(x-1)(y-2) = \log C(x-1)$

(d) None of the above

7. The solution of $\frac{dy}{dx} = \left(\frac{x+2y-3}{2x+y+3}\right)^2$, is

(a) $(x+3)^3 - (y-3)^3 = C(x-y+6)^4$ (b) $(x+3)^3 - (y-3)^3 = C$

(c) $(x+3)^4 + (y-3)^4 = C$ (d) None of these

8. The solution of $\frac{dy}{dx} = \frac{-\cos x(3\cos y - 7\sin x - 3)}{\sin y(3\sin x - 7\cos y + 7)}$, is

(a) $(\cos y - \sin x - 1)^2(\sin x + \cos y - 1)^5 = C$ (b) $(\cos y - \sin x - 1)^2(\sin x + \cos y - 1)^3 = C$

(c) $(\cos y - \sin x - 1)^2(\sin x + \cos y - 1)^7 = C$ (d) None of these

9. A curve C has the property that if the tangent drawn at any point P on C meets. The coordinate axes at A and B, then P is the mid point of AB. The curve passes through the point $(1, 1)$. Then the equation of curve is

(a) $xy = 1$ (b) $\frac{x}{y} = 1$

(c) $2x = xy - 1$ (d) None of these

10. The family of curves whose tangent form an angle $\frac{\pi}{4}$ with the hyperbola $xy = 1$, is

(a) $y = x - 2\tan^{-1}(x) + K$ (b) $y = x + 2\tan^{-1}(x) + K$

(c) $y = 2x - \tan^{-1}(x) + K$ (d) $y = 2x + \tan^{-1}(x) + K$

11. A and B are two separate reservoires of mater capacity of reservoir are filled completely with water their inlets are closed and then the water is released simultaneously from both the reservoirs. The rate of flow of water out of each reservoir at any instant of time is proportional to the quantity of water in the reservoir at that time. One hour after the water is released, the quantity of water in the reservoir A is $1\frac{1}{2}$ times the quantity of the water in reservoir B. The time after which do both the reservoirs have the same quantity of water, is

(a) $\log_{3/4}(2)$ (b) $\log_{3/4}\left(\frac{1}{2}\right)$

(c) $\log_{1/2}\left(\frac{1}{2}\right)$ (d) None of these

12. A curve passes through $(2, 1)$ and is such that the square of the ordinate is twice the rectangle contained by the abscissa and the intercept of the normal. Then the equation of curve is

(a) $x^2 + y^2 = 9x$ (b) $4x^2 + y^2 = 9x$

(c) $4x^2 + 2y^2 = 9x$ (d) None of these

13. A normal at $P(x, y)$ on a curve neets the X-axis at Q and N is the foot of the ordinate at P. If $NQ = \frac{x(1+y^2)}{(1+x^2)}$.

Then the equation of curve passing through (3, 1) is

(a) $5(1+y^2) = (1+x^2)$ (b) $(1+y^2) = 5(1+x^2)$

(c) $(1+x^2) = (1+y^2).x$ (d) None of these

14. The curve for which the ratio of the length of the segment intercepted by any tangent on the Y-axis to the length of the radius vector is constant (k), is

(a) $(y + \sqrt{x^2 - y^2})x^{k-1} = c$ (b) $(y + \sqrt{x^2 + y^2})x^{k-1} = c$

(c) $(y - \sqrt{x^2 - y^2})x^{k-1} = c$ (d) $(y - \sqrt{x^2 + y^2})x^{k-1} = c$

15. A point $P(x, y)$ nores on the curve $x^{2/3} + y^{2/3} = a^{2/3}, a > 0$ for each position (x, y) of p, perpendiculars are drawn from origin upon the tangent and normal at P, the length (absolute valve) of them being $P_1(x)$ and $P_2(x)$ brespectively, then

(a) $\frac{dp_1}{dx}.\frac{dp_2}{dx} < 0$ (b) $\frac{dp_1}{dx}.\frac{dp_2}{dx} \le 0$

(c) $\frac{dp_1}{dx}.\frac{dp_2}{dx} > 0$ (d) $\frac{dp_1}{dx}.\frac{dp_2}{dx} \ge 0$

Session 3

Solving of Linear Differential Equations, Bernoulli's Equation, Orthogonal Trajectory

Solving of Linear Differential Equations

First Order Linear Differential Equations

A differential equation is said to be linear if an unknown variable and its derivative occur only in the first degree.

An equation of the form

$$\frac{dy}{dx} + P(x) \cdot y = Q(x).$$

Where $P(x)$ and $Q(x)$ are functions of x only or constant is called a linear equation of the first order.

To get the general solution of the above equation we proceeds as follows. By multiplying both the sides of the above equation by $e^{\int Pdx}$, we get

$$e^{\int Pdx} \cdot \frac{dy}{dx} + yP \cdot e^{\int Pdx} = Q\, e^{\int Pdx}$$

i.e. $$e^{\int Pdx} \cdot \frac{dy}{dx} + y \frac{d}{dx}(e^{\int Pdx}) = Q\, e^{\int Pdx}$$

i.e. $$\frac{d}{dx}(y\, e^{\int Pdx}) = Q \cdot e^{\int Pdx}$$

$\therefore$ Integrating, we get $y\, e^{\int Pdx} = \int Q\, e^{\int Pdx} dx + C$

Here, the term $e^{\int Pdx}$ which converts the left hand expression of the equation into a perfect differential is called an Integrating factor. In short it is written as IF. Thus, we remember the solution of the above equation as

$$y\,(\text{IF}) = \int Q\,(\text{IF})\, dx + C.$$

Algoritm for Solving A Linear Differential Equation

Step I Write the differential equation in the form $dy / dx + Py = Q$ and obtain P and Q.

Step II Find integrating factor (I.F.) given by I.F. $= e^{\int Pdx}$.

Step III Multiply both sides of equation in Step I by I.F.

Step IV Integrate both sides of the equation obtained in step III. w.r.t x to obtain y (I.F.) $= \int Q.(\text{I.F.})\, dx + C$

This gives the required solution following examples illustrate the procedure.

Example 34 Solve $\frac{dy}{dx} + 2y = \cos x$.

Sol. It is a linear equation of the form

$$\frac{dy}{dx} + Py = Q(x)$$

where $P = 2$ and $Q = \cos x$

Then $$\text{IF} = e^{\int Pdx} = e^{\int 2dx} = e^{2x}$$

Hence, the general solution is $y\,(\text{IF}) = \int Q\,(\text{IF})\, dx$

i.e. $$y \cdot e^{2x} = \int e^{2x} \cos x\, dx + C$$

$$y \cdot e^{2x} = \frac{e^{2x}}{5}[2 \cos x + \sin x] + C$$

Example 35 Solve $\frac{dy}{dx} + \frac{y}{x} = \log x$.

Sol. It is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q(x)$$

Here, $$P = \frac{1}{x}, Q = \log x$$

Then $$\text{IF} = e^{\int Pdx} = e^{\int 1/x\, dx} = e^{\log x} = x$$

Hence, the general solution is

$$y\,(\text{IF}) = \int Q\,(\text{IF})\, dx + C$$

i.e. $$yx = \int (\log x)\, x\, dx + C$$

i.e. $$yx = (\log x) \cdot \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} + C$$

i.e. $$yx = \frac{x^2}{2}(\log x) - \frac{x^2}{4} + C$$

Example 36 *Solve* $\dfrac{dy}{dx} = \dfrac{y}{2y \ln y + y - x}$.

Sol. The equation can be written as

$$\frac{dx}{dy} = \frac{2y \ln y + y - x}{y} = (2 \ln y + 1) - \frac{x}{y}$$

i.e. $$\frac{dx}{dy} + \frac{1}{y} \cdot x = (2 \ln y + 1)$$

In this equation it is clear that $P = \dfrac{1}{y}$ and $Q = (2 \ln y + 1)$.

Which are function of y only because equation contains derivatives of x with respect to y.

$$\text{IF} = e^{\int P dy} = e^{\int 1/y \, dy} = e^{\ln y} = y$$

$\therefore$ The solution is; $x \, (\text{IF}) = \int (2 \ln y + 1) \, (\text{IF}) \, dy$

i.e. $$xy = \int (2 \ln y + 1) \cdot y \, dy = y^2 \ln y + C$$

i.e. $$x = y \ln y + \frac{C}{y}$$

Note In some cases a linear differential equation may be of the form $\dfrac{dx}{dy} + P_1 x = Q_1$, where P_1 and Q_1 are function of y alone. In such a case the integrating factor is $e^{\int P_1 \, dy}$.

Example 37 Solve $\cos^2 x \dfrac{dy}{dx} - y \tan 2x = \cos^4 x$, where $|x| < \dfrac{\pi}{4}$ and $y\left(\dfrac{\pi}{6}\right) = \dfrac{3\sqrt{3}}{8}$.

Sol. The given equation can be written as

$$\frac{dy}{dx} - \sec^2 x \cdot \tan 2x \cdot y = \cos^2 x$$

$$\text{IF} = e^{\int -\tan 2x \cdot \sec^2 x \, dx} = e^{\int + \frac{2 \tan x}{\tan^2 x - 1} \sec^2 x \, dx}$$

$$= e^{\int \frac{dt}{t}}, \text{ where } t = \tan^2 x - 1$$

$$= e^{\ln |t|} = |t| = |\tan^2 x - 1|$$

It is given that $|x| < \dfrac{\pi}{4}$ and for this region $\tan^2 x < 1$.

$\therefore$ $$\text{IF} = (1 - \tan^2 x)$$

$\therefore$ The solution is

$$y (1 - \tan^2 x) = \int \cos^2 x \, (1 - \tan^2 x) \, dx$$

$$= \int (\cos^2 x - \sin^2 x) \, dx$$

$$= \int (\cos 2x) \, dx = \frac{\sin 2x}{2} + C$$

Now, when $x = \dfrac{\pi}{6}, y = \dfrac{3\sqrt{3}}{8}$

$\therefore$ $$\frac{3\sqrt{3}}{8}\left(1 - \frac{1}{3}\right) = \frac{1}{2} \cdot \frac{\sqrt{3}}{2} + C \implies C = 0$$

$\therefore$ $$y = \frac{\sin 2x}{2(1 - \tan^2 x)}$$

Example 38 Solve $\dfrac{dy}{dx} + y \, \phi'(x) = \phi(x) \cdot \phi'(x)$, where $\phi(x)$ is a given function.

Sol. Here, $P = \phi'(x)$ and $Q = \phi(x) \, \phi'(x)$

$$\text{IF} = e^{\int \phi'(x) \, dx} = e^{\phi(x)}$$

$\therefore$ The solution is

$$y \, e^{\phi(x)} = \int \phi(x) \cdot \phi'(x) \cdot e^{\phi(x)} \, dx = \int t \cdot e^t \, dt,$$

where $\phi(x) = t$

$$y \, e^{\phi(x)} = e^t (t - 1) + C$$

i.e. $$y e^{\phi(x)} = \{\phi(x) - 1\} e^{\phi(x)} + C$$

Bernoulli's Equation

Sometimes a differential equation is not linear but it can be converted into a linear differential equation by a suitable substitution. An equation of the form

$$\frac{dy}{dx} + Py = Q \, y^n. \qquad (n \neq 0, 1)$$

Where P and Q are functions of x only, is known as Bernoulli's equation (for $n = 0$ the equation is linear.)

It is easy to reduce the above equation into linear form as below :

Dividing both the sides by y^n, we get

$$y^{-n} \frac{dy}{dx} + P \, y^{1-n} = Q$$

Putting $y^{1-n} = z$ and hence, $(1-n) \, y^{-n} \dfrac{dy}{dx} = \dfrac{dz}{dx}$ the equation becomes $\dfrac{dz}{dx} + (1-n) \, Pz = (1-n) \, Q$ which is linear in z.

Here, $$\text{IF} = e^{\int (1-n) \, P \, dx}$$

$\therefore$ The solution is,

$$z \, e^{\int (1-n) \, P dx} = \int \{(1-n) \cdot Q \cdot e^{\int (1-n) \, P dx}\} \, dx$$

Example 39 Solve $(y \log x - 1)\, y\, dx = x\, dy$.

Sol. The given differential equation can be written as

$$x\frac{dy}{dx} + y = y^2 \log x \quad \text{...(i)}$$

Dividing by xy^2, hence

$$\frac{1}{y^2}\cdot\frac{dy}{dx} + \frac{1}{xy} = \frac{1}{x}\log x$$

Let $$\frac{1}{y} = v \Rightarrow -\frac{1}{y^2}\cdot\frac{dy}{dx} = \frac{dv}{dx}$$

So that $$\frac{dv}{dx} - \frac{1}{x}v = -\frac{1}{x}\log x$$

Which is the standard linear differential equations, with

$$P = -\frac{1}{x},\ Q = -\frac{1}{x}\log x$$

$$\text{IF} = e^{\int -1/x\, dx} = e^{-\ln x} = e^{\ln x^{-1}} = \frac{1}{x}$$

The solution is given by

$$v\cdot\frac{1}{x} = \int\frac{1}{x}\left(-\frac{1}{x}\log x\right)dx = -\int\frac{\log x}{x^2}\,dx$$

$$= \frac{\log x}{x} - \int\frac{1}{x}\cdot\frac{1}{x}\,dx = \frac{\log x}{x} + \frac{1}{x} + C$$

$$\Rightarrow \quad v = 1 + \log x + Cx = \log(ex) + Cx$$

or $$\frac{1}{y} = \log(ex) + Cx \quad \text{or} \quad y\{\log(ex) + Cx\} = 1$$

Example 40 Solve $\dfrac{dy}{dx} + xy = xy^2$.

Sol. Dividing by y^2, we get

$$y^{-2}\frac{dy}{dx} + \frac{1}{y}\cdot x = x$$

Let $$\frac{1}{y} = z$$

So that $$-\frac{1}{y^2}\cdot\frac{dy}{dx} = \frac{dz}{dx}$$

$\therefore$ The given equation reduces to

$$\frac{dz}{dx} - xz = -x$$

$$\text{IF} = e^{\int -x\, dx} = e^{-x^2/2}$$

$\therefore$ The solution is

$$z\, e^{-x^2/2} = \int -x\cdot e^{-x^2/2}\,dx = e^{-x^2/2} + C$$

i.e. $$\frac{1}{y} = 1 + Ce^{x^2/2}$$

Example 41 Solve $\dfrac{dy}{dx} = \dfrac{y\,\phi'(x) - y^2}{\phi(x)}$, where $\phi(x)$ is a given function.

Sol. The equation can be written as

$$\frac{dy}{dx} - \frac{\phi'(x)}{\phi(x)}y = -\frac{y^2}{\phi(x)}$$

i.e. $$-y^{-2}\frac{dy}{dx} + \frac{\phi'(x)}{\phi(x)}\cdot\frac{1}{y} = \frac{1}{\phi(x)}$$

Let $\dfrac{1}{y} = z$. So that, $-\dfrac{1}{y^2}\dfrac{dy}{dx} = \dfrac{dz}{dx}$

$\therefore$ $$\frac{dz}{dx} + \frac{\phi'(x)}{\phi(x)}\cdot z = \frac{1}{\phi(x)}$$

$$\text{IF} = e^{\int\frac{\phi'(x)}{\phi(x)}dx} = e^{\ln\phi(x)} = \phi(x)$$

$\therefore$ The solution is $z\cdot\phi(x) = \int\dfrac{1}{\phi(x)}\cdot\phi(x)\,dx = x + C$

i.e. $$\frac{\phi(x)}{y} = x + C \quad ie, \quad \frac{\phi(x)}{x + C} = y$$

Remark

Another type of equation which is reducible to the linear form is

$$f'(y)\frac{dy}{dx} + P(x)\cdot f(y) = Q(x)$$

An equation of this type can be easily reduced the linear form by taking $z = f(y)$.

Example 42 Solve $\sec^2 y\dfrac{dy}{dx} + 2x\tan y = x^3$.

Sol. Let $\tan y = z$ so that $\sec^2 y\cdot\dfrac{dy}{dx} = \dfrac{dz}{dx}$

Thus, the given equation reduces to

$$\frac{dz}{dx} + 2x\cdot z = x^3$$

$$\text{IF} = e^{\int 2x\,dx} = e^{x^2}$$

$\therefore$ The solution is, $z\cdot e^{x^2} = \int x^3\cdot e^{x^2}dx$

i.e. $$\tan y\cdot e^{x^2} = \frac{1}{2}\int x^2\, e^{x^2}\cdot(2x)\,dx = \frac{1}{2}\int t\cdot e^t\,dt,$$

where $t = x^2$

$$\tan y\cdot e^{x^2} = \frac{1}{2}(t\cdot e^t - e^t) + C$$

i.e. $$\tan y = Ce^{-x^2} + \frac{e^{-x^2}}{2}\cdot e^{x^2}(x^2 - 1)$$

$$\tan y = Ce^{-x^2} + \frac{1}{2}(x^2 - 1)$$

Example 43 Solve $\frac{dy}{dx} + x(x+y) = x^3(x+y)^3 - 1$.

Sol. The given equation can be written as

$$\left(\frac{dy}{dx} + 1\right) + x(x+y) = x^3(x+y)^3$$

i.e. $$\frac{d(x+y)}{dx} + x(x+y) = x^3(x+y)^3$$

i.e. $(x+y)^{-3} \cdot \frac{d(x+y)}{dx} + x(x+y)^{-2} = x^3$

Let $(x+y)^{-2} = z$ so that $-2(x+y)^{-3}\frac{d(x+y)}{dx} = \frac{dz}{dx}$

The given equation reduces to

$$-\frac{1}{2}\frac{dz}{dx} + xz = x^3$$

i.e. $$\frac{dz}{dx} - 2xz = -2x^3$$

$$\text{IF} = e^{\int -2x\,dx} = e^{-x^2}$$

$\therefore$ The solution is

$$z \cdot e^{-x^2} = \int -2x^3 \cdot e^{-x^2}\,dx = (x^2+1)e^{-x^2} + C$$

$$\frac{1}{(x+y)^2} = Ce^{x^2} + x^2 + 1$$

Example 44 Solve $\sin y \cdot \frac{dy}{dx} = \cos y(1 - x\cos y)$.

Sol. The given differential equation is

$$\sin y \frac{dy}{dx} = \cos y(1 - x\cos y)$$

or $$\sin y \frac{dy}{dx} - \cos y = -x\cos^2 y$$

Dividing by $\cos^2 y$, we get

$$\tan y \cdot \sec y \cdot \frac{dy}{dx} - \sec y = -x$$

Let $\sec y = v \Rightarrow \sec y \tan y \cdot \frac{dy}{dx} = \frac{dv}{dx}$

So that $$\frac{dv}{dx} - v = -x$$

Which is linear differential equation with $P = -1, Q = -x$

$$\text{IF} = e^{\int P dx} = e^{\int -1\,dx} = e^{-x}$$

The solution is given by

$$v \cdot e^{-x} = \int -x \cdot e^{-x}\,dx = xe^{-x} + e^{-x} + C$$

$$= e^{-x}(x+1) + C$$

or $$v = (1+x) + Ce^x$$

or $$\sec y = (1+x) + Ce^x$$

Orthogonal Trajectory

Any curve, which cuts every member of a given family of curves at right angles, is called an orthogonal trajectory of the family. For example, each straight line passing through the origin, *ie*, $y = kx$ is an orthogonal trajectory of the family of the circles $x^2 + y^2 = a^2$.

Procedure for Finding the Orthogonal Trajectory

(i) Let $f(x, y, c) = 0$ be the equation of the given family of curves, where c is an arbitrary parameter.

(ii) Differentiate $f = 0$; w.r.t. 'x' and eliminate 'c', i.e. form a differential equation.

(iii) Substitute $-\frac{dx}{dy}$ for $\frac{dy}{dx}$ in the above differential equation. This will give the differential equation of the orthogonal trajectories.

(iv) By solving this differential equation, we get the required orthogonal trajectories.

Example 45 Find the orthogonal trajectories of the hyperbola $xy = C$.

Sol. The equation of the given family of curves is $xy = c$ ...(i)

Differentiating Eq. (i) w.r.t. x, we get

$$\frac{x\,dy}{dx} + y = 0 \quad \text{...(ii)}$$

Substitute $-\frac{dx}{dy}$ for $\frac{dy}{dx}$ in Eq. (ii), we get

$$-\frac{x\,dx}{dy} + y = 0 \quad \text{...(iii)}$$

This is the differential equation for the orthogonal trajectory of given family of hyperbola. Eq. (iii) can be rewritten as $x\,dx = y\,dy$, which on integration gives

$$x^2 - y^2 = C.$$

This is the family of required orthogonal trajectories.

Example 46 Find the orthogonal trajectories of the curves $y = cx^2$.

Sol. Here, $$y = cx^2 \quad \text{...(i)}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 2cx \quad \text{...(ii)}$$

Eliminating c from Eqs. (i) and (ii),

$$y = \left(\frac{1}{2x}\frac{dy}{dx}\right)x^2$$

$$\Rightarrow \quad 2y = x\frac{dy}{dx} \qquad \text{...(iii)}$$

This is the differential equation of the family of curves given in Eq. (i).

Now, to obtain orthogonal trajectory replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in Eq. (iii).

$$\Rightarrow \quad 2y = -x\frac{dx}{dy}$$

or $\quad 2y\,dy = -x\,dx$

Integrating both the sides, we get

$$y^2 = -\frac{x^2}{2} + C_1$$

$\Rightarrow x^2 + 2y^2 = C_1$, is the required family of orthogonal trajectory.

Example 47 Find the equation of all possible curves that will cut each member of the family of circles $x^2 + y^2 - 2cx = 0$ at right angle.

Sol. Here, $\quad x^2 + y^2 - 2cx = 0 \qquad \text{...(i)}$

Differentiating w.r.t. x, we get

$$2x + 2yy_1 - 2c = 0$$

$$\Rightarrow \quad c = x + yy_1 \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we eliminate c

$$\Rightarrow \quad x^2 + y^2 - 2(x + yy_1)\,x = 0$$

or $\quad -x^2 + y^2 - 2xy\frac{dy}{dx} = 0$

This is the differential equation representing the given family of circles. To find differential equation of the orthogonal trajectories, we replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$.

$$\Rightarrow \quad y^2 - x^2 = -2xy\frac{dx}{dy}$$

$$\Rightarrow \quad y^2\,dy = x^2\,dy - 2xy\,dx$$

$$\Rightarrow \quad -dy = \frac{yd(x^2) - x^2\,dy}{y^2}$$

$$\Rightarrow \quad -dy = d\left(\frac{x^2}{y}\right)$$

Integrating both the sides, we get

$$-y = \frac{x^2}{y} + C \;\Rightarrow\; x^2 + y^2 + Cy = 0$$

Represents family of orthogonal trajectory.

Example 48 ***Find the orthogonal trajectory of the circles***

$$x^2 + y^2 - ay = 0.$$

Sol. Here, $\quad x^2 + y^2 - ay = 0 \qquad \text{...(i)}$

Differentiating, we get

$$2x + 2yy_1 - ay_1 = 0$$

$$\Rightarrow \quad a = \frac{2(x + yy_1)}{y_1} \qquad \text{...(ii)}$$

Substituting 'a' in Eq. (i), we get

$$x^2 + y^2 - \frac{2(x + yy_1)}{y_1}y = 0$$

$$\Rightarrow \quad (x^2 - y^2)\,y_1 - 2xy = 0$$

This is the differential equation of the family of circles given in Eq. (i).

$\therefore$ The differential representing the orthogonal trajectory is obtained by replacing $\frac{dy}{dx}$ by $-\frac{dy}{dx}$.

i.e. $\quad -(x^2 - y^2)\frac{dx}{dy} - 2xy = 0$

$$\Rightarrow \quad 2xy\,dy - y^2\,dx = -x^2\,dx$$

$$\Rightarrow \quad \frac{x\,d(y^2) - y^2\,dx}{x^2} = -dx$$

$$\Rightarrow \quad d\left(\frac{y^2}{x}\right) = -dx$$

Integrating both the sides, we get

$y^2 + x^2 = Cx$, is required family of orthogonal trajectories.

Exercise for Session 3

1. The solution of $(1+x^2)\frac{dy}{dx}+y=e^{\tan^{-1}x}$, is given by

(a) $2ye^{\tan^{-1}x}=e^{2\tan^{-1}x}+C$ (b) $ye^{\tan^{-1}y}=e^{2\tan^{-1}x}+C$

(c) $2ye^{\tan^{-1}y}=e^{2\tan^{-1}y}+C$ (d) None of these

2. The solution of $\frac{dy}{dx}+\frac{x}{1-x^2}y=x\sqrt{y}$, is given by

(a) $3\sqrt{y}+(1-x^2)=C(1-x^2)^{1/4}$ (b) $\frac{3}{2}\sqrt{y}+(1-x^2)=C(1-x^2)^{3/2}$

(c) $3\sqrt{y}-(1-x^2)=C(1-x^2)^{3/2}$ (d) None of these

3. The solution of $\frac{dy}{dx}+x\sin 2y=x^3\cos^2 y$, is

(a) $e^{x^2}=(x^2-1)e^{x^2}\tan y+C$ (b) $e^{x^2}\tan y=\frac{1}{2}(x^2-1)e^{x^2}+C$

(c) $e^{x^2}\tan y=(x^2-1)\tan y+C$ (d) None of these

4. The solution of $3x(1-x^2)y^2\,dy/dx+(2x^2-1)y^3=ax^3$ is

(a) $y^3=ax+C\sqrt{1-x^2}$ (b) $y^3=ax+Cx\sqrt{1-x^2}$

(c) $y^2=ax+C\sqrt{1-x^2}$ (d) None of these

5. The solution of $\frac{dy}{dx}+\frac{y}{x}\log y=\frac{y}{x^2}(\log y)^2$, is

(a) $x=\frac{1}{2x}\log y+C$ (b) $x^2+\log y=C$ (c) $\frac{1}{x\log y}=\frac{1}{2x^2}+C$ (d) None of these

6. The solution of $\frac{dy}{dx}+yf'(x)-f(x)\cdot f'(x)=0, y\neq f(x)$ is

(a) $y=f(x)+1+ce^{-f(x)}$ (b) $y-ce^{-f(x)}$

(c) $y=f(x)-1+ce^{-f(x)}$ (d) None of these

7. The solution of $(x^{2-1})\,dy/dx\cdot\sin y-2x\cdot\cos y=2x-2x^3$, is

(a) $(x^2-1)\cos y=\frac{x^4}{2}-x^2+C$ (b) $(x^2-1)\sin y=\frac{x^4}{2}-x^2+C$

(c) $(x^2-1)\cos y=\frac{x^4}{4}-\frac{x^2}{2}+C$ (d) $(x^2-1)\sin y=\frac{x^4}{4}-\frac{x^2}{2}+C$

8. The Curve possessing the property text the intercept made by the tangent at any point of the curve on the y-axis is equal to square of the abscissa of the point of tangency, is given by

(a) $y^2=x+C$ (b) $y=2x^2+cx$

(c) $y=-x^2+cx$ (d) None of these

9. The tangent at a point P of a curve meets the y-axis at A, and the line parallel to y-axis at A, and the line parallel to y-axis through P meets the x-axis at B. If area of ΔOAB is constant (O being the origin), Then the curve is

(a) $cx^2-xy+k=0$ (b) $x^2+y^2=cx$

(c) $3x^2+4y^2=k$ (d) $xy-x^2y^2+kx=0$

10. The value of k such that the family of parabolas $y=cx^2+k$ is the orthogonal trajectory of the family of ellipse $x^2+2y^2-y=C$, is

(a) y_2 (b) y_3 (c) y_4 (d) y_5

Session 4

Exact Differential Equations

Exact Differential Equations

A differential equation of the form $M(x, y)\,dx + N(x, y)\,dy = 0$ is said to be exact (or total) if its left hand expression is the exact differential of some function $u(x, y)$.

i.e. $$du = M \cdot dx + N \cdot dy$$

Hence, its solution is $u(x, y) = c$ (where c is an arbitrary constant). But then there is a question that how do we confirm whether the above mentioned equation is exact. The answer to this question is the following theorem.

Theorem The necessary and sufficient condition for the differential equation $M\,dx + N\,dy = 0$ to be exact is $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

The solution of $M\,dx + N\,dy = 0$ is ,

$$\int_{y-\text{constant}} M\,dx + \int(\text{terms of } N \text{ not containing } x)\,dy = C$$

provided $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

Example 49 Solve $(x^2 - ay)\,dx + (y^2 - ax)\,dy = 0$.

Sol. Here, we have $M = x^2 - ay$ and $N = y^2 - ax$

$$\therefore \quad \frac{\partial M}{\partial y} = -a = \frac{\partial N}{\partial x}$$

Thus, the equation is exact.

$\therefore$ The solution is, $\int_{y-\text{constant}} (x^2 - ay)\,dx + \int y^2\,dy = C$

$$\Rightarrow \quad \frac{x^3}{3} - axy + \frac{y^3}{3} = C$$

Example 50 Solve $(2x \log y)\,dx + \left(\dfrac{x^2}{y} + 3y^2\right)dy = 0$.

Sol. Here, we have $M = 2x \log y$ and $N = \dfrac{x^2}{y} + 3y^2$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{2x}{y}$$

and $\dfrac{\partial N}{\partial x} = \dfrac{2x}{y}$ and hence the equation is exact.

$\therefore$ The solution is,

$$\int_{y-\text{constant}} (2x \log y)\,dx + \int 3y^2\,dy = C$$

$$\Rightarrow \quad x^2 \log y + y^3 = C$$

Equations Reducible to the Exact Form

Sometimes a differential equation of the form $M\,dx + N\,dy = 0$ which is not exact can be reduced to an exact form by multiplying by a suitable function $f(x, y)$ which is not identically zero. This function $f(x, y)$ which then multiplied to a non-exact differential equation makes it exact is known as integrating factor.

One can find integrating factors by inspection but for that some experience and practice is required.

For finding the integrating factors by inspection, the following identities must be remembered.

1. $x\,dy + y\,dx = d(xy)$
2. $x\,dx + y\,dy = \dfrac{1}{2}\,d(x^2 + y^2)$
3. $\dfrac{x\,dy - y\,dx}{x^2} = d\left(\dfrac{y}{x}\right)$
4. $\dfrac{y\,dx - x\,dy}{y^2} = d\left(\dfrac{x}{y}\right)$
5. $\dfrac{x\,dy - y\,dx}{xy} = \dfrac{dy}{y} - \dfrac{dx}{x} = d\left[\log\left(\dfrac{y}{x}\right)\right]$
6. $\dfrac{y\,dx - x\,dy}{xy} = d\left[\log\left(\dfrac{x}{y}\right)\right]$
7. $\dfrac{x\,dy - y\,dx}{x^2 + y^2} = \dfrac{\dfrac{x\,dy - y\,dx}{x^2}}{1 + \dfrac{y^2}{x^2}} = \dfrac{d\left(\dfrac{y}{x}\right)}{1 + \left(\dfrac{y}{x}\right)^2} = d\left[\tan^{-1}\left(\dfrac{y}{x}\right)\right]$
8. $\dfrac{dx + dy}{x + y} = d[\ln(x + y)]$
9. $d(\ln(xy)) = \dfrac{x\,dy + y\,dx}{xy}$
10. $d\left(\dfrac{1}{2}\ln(x^2 + y^2)\right) = \dfrac{x\,dx + y\,dy}{x^2 + y^2}$

11. $d\left(-\frac{1}{xy}\right)=\frac{x\,dy+y\,dx}{x^2\,y^2}$

12. $d\left(\frac{e^y}{x}\right)=\frac{x\,e^y\,dy-e^y\,dx}{x^2}$

13. $d\left(\frac{e^x}{y}\right)=\frac{ye^x dx-e^x dy}{y^2}$

14. $d(x^m\,y^n)=x^{m-1}\,y^{n-1}\,(my\,dx+nx\,dy)$

15. $\frac{d[f(x,y)]^{1-n}}{1-n}=\frac{f'(x,y)}{(f(x,y))^n}$

Example 51 Solve $(x^2-ay)\,dx+(y^2-ax)\,dy=0$.

Sol. The given differential equation is

$$x^2dx+y^2dy-a(y\,dx+x\,dy)=0$$

$$\Rightarrow\quad d\left(\frac{x^3}{3}\right)+d\left(\frac{y^3}{3}\right)-ad(xy)=0$$

Integrating, we get $\frac{x^3}{3}+\frac{y^3}{3}-a\,xy=k$

$$\Rightarrow\quad x^3+y^3-3axy=3k=C$$

Example 52 Solve $(2x\log y)\,dx+\left(\frac{x^2}{y}+3y^2\right)dy=0$.

Sol. The given differential equation is

$$(\log y)\,2x\,dx+\frac{x^2}{y}\,dy+3y^2\,dy=0$$

$$\Rightarrow\quad (\log y)\,d(x^2)+x^2\,d(\log y)+d(y^3)=0$$

$$\Rightarrow\quad d(x^2\log y)+d(y^3)=0$$

$$\Rightarrow\quad x^2\log y+y^3=C$$

(integrating both the sides)

Example 53 Solve $x\,dx+y\,dy=x\,dy-y\,dx$.

Sol. The given equation can be written as

$$\frac{1}{2}d(x^2+y^2)=x^2\,d\left(\frac{y}{x}\right)$$

$$\Rightarrow\quad \frac{d(x^2+y^2)}{x^2+y^2}=\frac{2x^2\,d\left(\frac{y}{x}\right)}{x^2+y^2}$$

$$\Rightarrow\quad \frac{d(x^2+y^2)}{x^2+y^2}=\frac{2d(y/x)}{1+(y/x)^2}$$

$$\Rightarrow\quad \log(x^2+y^2)=2\tan^{-1}\left(\frac{y}{x}\right)+C$$

Example 54 ***Solve***

$$\frac{y+\sin x\cos^2(xy)}{\cos^2(xy)}\,dx+\left(\frac{x}{\cos^2(xy)}+\sin y\right)dy=0.$$

Sol. The given differential equation can be written as

$$\frac{y\,dx+x\,dy}{\cos^2(xy)}+\sin x\,dx+\sin y\,dy=0$$

$$\Rightarrow\quad \sec^2(xy)\,d(xy)+\sin x\,dx+\sin y\,dy=0$$

$$d(\tan(xy))+d(-\cos x)+d(-\cos y)=0$$

$$\Rightarrow\quad \tan(xy)-\cos x-\cos y=C$$

Example 55 Solve $\frac{x+y\frac{dy}{dx}}{y-x\frac{dy}{dx}}=x^2+2y^2+\frac{y^4}{x^2}$.

Sol. The given equation can be written as

$$\frac{x\,dx+y\,dy}{(x^2+y^2)^2}=\frac{y\,dx-x\,dy}{y^2}\cdot\frac{y^2}{x^2}$$

$$\Rightarrow\quad \int\frac{d(x^2+y^2)}{(x^2+y^2)^2}=2\int\frac{1}{x^2/y^2}\,d\left(\frac{x}{y}\right)$$

Integrating both the sides, we get

$$-\frac{1}{(x^2+y^2)}=-\frac{1}{(x/y)}+C$$

$$\Rightarrow\quad \frac{y}{x}-\frac{1}{x^2+y^2}=C$$

Example 56 The solution of

$$e^{x\frac{(y^2-1)}{y}}\{xy^2\,dy+y^3dx\}+\{ydx-xdy\}=0,\text{ is}$$

(a) $e^{xy}+e^{x/y}+C=0$ (b) $e^{xy}-e^{x/y}+C=0$

(c) $e^{xy}+e^{y/x}+C=0$ (d) $e^{xy}-e^{y/x}+C=0$

Sol. Here, $e^{x\frac{(y^2-1)}{y}}\cdot y^2\{xdy+ydx\}+\{ydx-xdy\}=0$

$$\Rightarrow\quad e^{xy}\cdot y^2\cdot\{xdy+ydx\}+e^{x/y}\{ydx-xdy\}=0$$

or $$e^{xy}\cdot\{xdy+ydx\}+e^{x/y}\frac{\{ydx-xdy\}}{y^2}=0$$

or $$e^{xy}\cdot d(xy)+e^{x/y}\cdot d\left(\frac{x}{y}\right)=0$$

or $$d(e^{xy})+d(e^{x/y})=0$$

Integrating both the sides, we get

$$e^{xy}+e^{x/y}+C=0$$

Hence, (a) is the correct answer.

Example 57 *The solution of* $x^2dy - y^2dx + xy^2(x-y)\,dy = 0$, *is*

(a) $\log\left|\frac{x-y}{xy}\right| = \frac{y^2}{2} + C$ (b) $\log\left|\frac{xy}{x-y}\right| = \frac{x^2}{2} + C$

(c) $\log\left|\frac{x-y}{xy}\right| = \frac{x^2}{2} + C$ (d) $\log\left|\frac{x-y}{xy}\right| = x + C$

Sol. Here, $x^2y^2\left(\frac{dy}{y^2} - \frac{dx}{x^2}\right) + x^2y^3\left(\frac{1}{y} - \frac{1}{x}\right)dy = 0$

$$\Rightarrow \quad -d\left(\frac{1}{y} - \frac{1}{x}\right) + y\left(\frac{1}{y} - \frac{1}{x}\right)dy = 0$$

$$\Rightarrow \quad -\frac{d\left(\frac{1}{y} - \frac{1}{x}\right)}{\left(\frac{1}{y} - \frac{1}{x}\right)} + ydy = 0 \text{ or } \frac{d\left(\frac{1}{y} - \frac{1}{x}\right)}{\left(\frac{1}{y} - \frac{1}{x}\right)} = d\left(\frac{y^2}{2}\right)$$

Integrating both the sides, we get

$$\log\left|\frac{1}{x} - \frac{1}{y}\right| = \frac{y^2}{2} + C$$

Hence, (a) is the correct answer.

Example 58 *The solution of the differential equation* $ydx - xdy + xy^2\,dx = 0$, is

(a) $\frac{x}{y} + x^2 = \lambda$ (b) $\frac{x}{y} + \frac{x^2}{2} = \lambda$

(c) $\frac{x}{2y^2} + \frac{x^2}{4} = \lambda$ (d) None of these

Sol. Given equation is, $ydx - xdy + xy^2\,dx = 0$

Which could be converted into exact form

i.e. $$\frac{ydx - xdy}{y^2} + xdx = 0$$

$$\Rightarrow \quad d\left(\frac{x}{y}\right) + d\left(\frac{x^2}{2}\right) = 0$$

Integrating both the sides, we get

$$\frac{x}{y} + \frac{x^2}{2} = \text{constant}$$

or $$\frac{x}{y} + \frac{x^2}{2} = \lambda$$

Hence, (b) is the correct answer.

Example 59 The solution of differential equation $xdy\,(y^2e^{xy} + e^{x/y}) = ydx\,(e^{x/y} - y^2e^{xy})$, is

(a) $xy = \log(e^x + \lambda)$ (b) $x^2/y = \log(e^{x/y} + \lambda)$

(c) $xy = \log(e^{x/y} + \lambda)$ (d) $xy^2 = \log(e^{x/y} + \lambda)$

Sol. The given equation is

$$(xy^2e^{xy})\,dy + (xe^{x/y})\,dy = (ye^{x/y})\,dx - (y^3e^{xy})\,dx$$

$$\Rightarrow \quad y^2e^{xy}(xdy + ydx) = e^{x/y}(ydx - xdy)$$

$$\Rightarrow \quad e^{xy}(d(xy)) = e^{x/y}\cdot\left(\frac{ydx - xdy}{y^2}\right)$$

$$\Rightarrow \quad e^{xy}(d(xy)) = e^{x/y}\cdot d\left(\frac{x}{y}\right)$$

$$\Rightarrow \quad d(e^{xy}) = d(e^{x/y})$$

Integrating both the sides, we get

$$e^{xy} = e^{x/y} + \lambda$$

$$\Rightarrow \quad xy = \log(e^{x/y} + \lambda)$$

Hence, (c) is the correct answer.

Example 60 *The solution of the differential equation*

$(y + x\sqrt{xy}\,(x+y))\,dx + (y\sqrt{xy}\,(x+y) - x)\,dy = 0$, is

(a) $\frac{x^2+y^2}{2} + 2\tan^{-1}\sqrt{\frac{x}{2y}} = C$

(b) $\frac{x^2+y^2}{2} + 2\tan^{-1}\sqrt{\frac{x}{y}} = C$

(c) $\frac{x^2+y^2}{\sqrt{2}} + 2\tan^{-1}\sqrt{\frac{x}{y}} = C$

(d) None of these

Sol. The given equation can be written as

$$(ydx - xdy) + x\sqrt{xy}\,(x+y)\,dx + y\sqrt{xy}\,(x+y)\,dy = 0$$

$$\Rightarrow \quad (ydx - xdy) + (x+y)\sqrt{xy}\,(xdx + ydy) = 0$$

$$\Rightarrow \quad \frac{ydx - xdy}{y^2} + \left(\frac{x}{y} + 1\right)\cdot\sqrt{\frac{x}{y}}\,d\left(\frac{x^2+y^2}{2}\right) = 0$$

$$\Rightarrow \quad d\left(\frac{x}{y}\right) + d\left(\frac{x^2+y^2}{2}\right)\left(\frac{x}{y} + 1\right)\sqrt{\frac{x}{y}} = 0$$

$$\Rightarrow \quad d\left(\frac{x^2+y^2}{2}\right) + \frac{d\left(\frac{x}{y}\right)}{\left(\frac{x}{y} + 1\right)\cdot\sqrt{\frac{x}{y}}} = 0$$

or $$d\left(\frac{x^2+y^2}{2}\right) + 2d\left(\tan^{-1}\left(\sqrt{\frac{x}{y}}\right)\right) = 0$$

Integrating both the sides, we get

$$\Rightarrow \quad \frac{x^2+y^2}{2} + 2\tan^{-1}\sqrt{\frac{x}{y}} = C$$

Hence, (b) is the correct answer.

Exercise for Session 4

1. The solution of $xdy + ydx + 2x^3dx = 0$, is

(a) $xy + x^4 = C$ (b) $xy + \frac{1}{2}x^4 = C$ (c) $\frac{x^2}{y} + \frac{x^4}{4} = C$ (d) None of these

2. The solution of, $ydx - xdy + (1 + x^2)\,dx + x^2 \sin y\, dy = 0$, is given by

(a) $x + 1 - y^2 + \cos y + C = 0$ (b) $y + 1 - x^2 + x \cos y + C = 0$

(c) $\frac{x}{y} + \frac{1}{y} - y + \cos y + C = 0$ (d) $\frac{y}{x} + \frac{1}{x} - x + \cos y + C = 0$

3. The solution of $(1 + x\sqrt{x^2 + y^2})\,dx + (-1 + \sqrt{x^2 + y^2})\,ydy = 0$, is

(a) $2x - y^2 + \frac{2}{3}(x^2 + y^2)^{3/2} = C$ (b) $2x - y + \frac{2}{3}(x^2 + y^2)^{3/2} = C$

(c) $2y - x^2 + \frac{2}{3}(x^2 + y^2)^{3/2} = C$ (d) None of these

4. The solution of, $\frac{xdy}{x^2 + y^2} = \left(\frac{y}{x^2 + y^2} - 1\right)dx$, is given by

(a) $\tan^{-1}\left(\frac{x}{y}\right) + x = C$ (b) $\tan^{-1}\left(\frac{y}{x}\right) + x = C$ (c) $\tan^{-1}\left(\frac{y}{x}\right) + xy = C$ (d) $\tan^{-1}\left(\frac{y}{x}\right) + x^2 = C$

5. The solution of $ye^{x/y}\,dx = (xe^{x/y} + y^2 \sin y)\,dy$, is given by

(a) $e^{x/y} = -\cos y + C$ (b) $e^{x/y} + 2\cos y = C$ (c) $e^{x/y} = x \cos y + C$ (d) $e^{x/y} = 2\cos y\, e^{x/y} + C$

6. The solution of $x \sin\left(\frac{y}{x}\right)dy = \left\{ y \sin\left(\frac{y}{x}\right) - x \right\}dx$, is given by

(a) $\log x - \cos\left(\frac{y}{x}\right) = \log C$ (b) $\log x - \sin\left(\frac{y}{x}\right) = C$ (c) $\log\left(\frac{x}{y}\right) - \cos\left(\frac{y}{x}\right) = \log C$ (d) None of these

7. The solution of $\frac{xdx + ydy}{xdy - ydx} = \sqrt{\frac{a^2 - x^2 - y^2}{x^2 + y^2}}$, is given by

(a) $\sin^{-1}(\sqrt{x^2 + y^2}) = a\,\tan^{-1}\left(\frac{y}{x}\right) + C$ (b) $\sin^{-1}(\sqrt{x^2 + y^2}) = \frac{1}{a}\tan^{-1}\left(\frac{y}{x}\right) + C$

(c) $\sin^{-1}\left(\frac{\sqrt{x^2 + y^2}}{a}\right) = \tan^{-1}\left(\frac{y}{x}\right) + C$ (d) None of the above

8. The solution of $(1 + e^{x/y})\,dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0$, is given by

(a) $x - ye^{x/y} = C$ (b) $x + ye^{x/y} = C$ (c) $y - \frac{x}{y}e^{x/y} = C$ (d) None of these

9. The solution of $\frac{x + y\,dy/dx}{y - x\,dy/dx} = \frac{x \sin^2(x^2 + y^2)}{y^3}$, is given by

(a) $-\cot(x^2 + y^2) = \left(\frac{x}{y}\right)^2 + C$ (b) $\tan(x^2 + y^2) = x^2y^2 + C$

(c) $\cot(x^2 + y^2) = \frac{x}{y} + C$ (d) None of these

10. The solution of $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{(1 + \log x + \log y)^2}$, is given by

(a) $xy\,(1 + \log(xy)) = C$ (b) $xy^2\,(1 + \log(xy)) = C$ (c) $xy\,(1 + \log(xy))^2 = C$ (d) $xy\,(1 + (\log xy)^2) = C$

Session 5

Solving of First Order and Higher Degrees, Application of Differential Equations, Application of First Order Differential Equations

Solving of First Order and Higher Degrees

Differential Equation of First Order and Higher Degrees

A differential equation of first order is of the form $f(x, y, P)$ where $P = dy / dx$. If in the equation degree of P is greater than one, then the equation is of first order and higher degree.

The differential equation of first order and higher degree can be written in the form

$$P^n + F_1(x, y)\, P^{n-1} + \ldots + F_{n-1}(x, y)\, P + F_n(x, y) = 0$$

The differential equations of these category can be solved by one or more of the following methods :

(i) Equations solvable for P.

(ii) Equations solvable for y.

(iii) Equations solvable for x.

(iv) Clairaut's equations.

Now, we shall discuss these cases.

(i) Equations Solvable For P

If the equation

$$P^n + F_1(x, y)\, P^{n-1} + \ldots + F_{n-1}(x, y)\, P + F_n(x, y) = 0,$$

is solvable for P, then LHS expression can be resolved into n linear factors and hence can be put in the form

$$(P - f_1(x, y))\,(P - f_2(x, y)) \ldots (P - f_n(x, y)) = 0.$$

Equating each of these factors to zero, we get n differential equations of the first order and first degree.

$$\frac{dy}{dx} = f_1(x, y), \frac{dy}{dx} = f_2(x, y), \ldots, \frac{dy}{dx} = f_n(x, y)$$

Let the solutions of these obtained equations are

$$\phi_1(x, y, c_1) = 0, \phi_2(x, y, c_2) = 0, \ldots, \phi_n(x, y, c_n) = 0$$

respectively.

Hence, the general solution is given by

$$\phi_1(x, y, c), \phi_2(x, y, c), \ldots, \phi_n(x, y, c) = 0$$

Here, the arbitrary constant $c_1, c_2, \ldots, c_n$ are replaced by a single arbitrary constant c because every first order equation has only one arbitrary constant in its solution.

Example 61 Solve $(p - x)(p - e^x)(p - 1/y) = 0$; where $p = \dfrac{dy}{dx}$.

Sol. The component linear equations are $p = x$, $p = e^x$, $p = \dfrac{1}{y}$

If $\dfrac{dy}{dx} = x$, then $dy = x\, dx \Rightarrow y = \dfrac{x^2}{2} + C_1$

If $\dfrac{dy}{dx} = e^x$, then $dy = e^x\, dx \Rightarrow y = e^x + C_2$

If $\dfrac{dy}{dx} = \dfrac{1}{y}$, then $y\, dy = dx \Rightarrow \dfrac{y^2}{2} = x + C_3$

$\therefore$ The required solution is

$$\left(y - \frac{x^2}{2} + C\right)(y - e^x + C)\left(\frac{y^2}{2} - x + C\right) = 0$$

Example 62 Solve $x^2p^2 + xyp - 6y^2 = 0$.

Sol. The given equation is

$$x^2p^2 + xyp - 6y^2 = 0$$

Solving as a quadratic in p, we get

$$p = \frac{(-xy \pm \sqrt{x^2y^2 + 24x^2y^2})}{2x^2} = \frac{2y}{x}, -\frac{3y}{x}$$

If $p = \dfrac{2y}{x}$, then $\dfrac{dy}{dx} = \dfrac{2y}{x} \Rightarrow \dfrac{dy}{y} = \dfrac{2dx}{x}$

$$\Rightarrow \quad \log\left|\frac{y}{x^2}\right| = k \Rightarrow y = C_1 x^2$$

If $p = -\dfrac{3y}{x}$, then $\dfrac{dy}{dx} = -\dfrac{3y}{x} \Rightarrow \dfrac{dy}{y} = -\dfrac{3dx}{x}$

$$\Rightarrow \quad x^3y = C_2$$

$\therefore$ The required solution is $(y - Cx^2)(x^3y - C) = 0$.

Example 63 Solve $xy^2(p^2+2)=2py^3+x^3$.

Sol. The given equation can be written as

$$(xy^2p^2-x^3)+2(xy^2-py^3)=0$$

$$\Rightarrow \quad x(y^2p^2-x^2)+2y^2(x-py)=0$$

$$\Rightarrow \quad (py-x)\{x(py+x)-2y^2\}=0$$

If $py-x=0$, then $y\,dy-x\,dx=0 \Rightarrow y^2-x^2=C_1$

If $xyp+x^2-2y^2=0$, then $2y\dfrac{dy}{dx}-\dfrac{4y^2}{x}=-2x$

$$\Rightarrow \quad \frac{dt}{dx}-\frac{4}{x}t=-2x,$$

where $t=y^2$ IF $=e^{-\int \frac{4}{x}dx}=e^{-4\ln x}=\dfrac{1}{x^4}$

Its solution is $t\left(\dfrac{1}{x^4}\right)=\int -2x\cdot\dfrac{1}{x^4}\,dx$

i.e. $\dfrac{t}{x^4}=\dfrac{1}{x^2}+C_2$ i.e. $y^2=x^2+C_2x^4$

Hence, the required solution is

$$(y^2-x^2-C)(y^2-x^2-Cx^4)=0$$

(ii) Equations Solvable For y

Equation that comes under this category, can be expressed in the form

$$y=g(x,p)$$

(i.e. an explicit function y in term of x and p) ...(i)

Differentiating Eq. (i) w.r.t. x, we get

$$\frac{dy}{dx}=p=F\left(x,p,\frac{dp}{dx}\right)$$

Which is a differential equation of the first order containing x and p. Let us suppose that its solution is

$$\phi(x,p,c)=0 \qquad \text{...(ii)}$$

Then, the solution is obtained by eliminating p between $y=g(x,p)$ and $\phi(x,p,c)=0$. However, if eliminating of p is difficult express x and y as a function of the parameter p.

Example 64 Solve $xp^2-2yp+ax=0$.

Sol. The given equation can be written as,

$$y=\frac{xp}{2}+\frac{ax}{2p} \qquad \text{...(i)}$$

$$\frac{dy}{dx}=\frac{p}{2}+\frac{x}{2}\cdot\frac{dp}{dx}+\frac{a}{2p}-\frac{ax}{2p^2}\cdot\frac{dp}{dx}$$

$$\Rightarrow \quad p(p^2-a)=x(p^2-a)\cdot\frac{dp}{dx}$$

$$\Rightarrow \quad \frac{dp}{dx}=\frac{p}{x}\Rightarrow p=Cx$$

(The equation $p^2-a=0$ gives us singular solution in which we are not interested).

The substitute p in Eq. (i), we get the required solution

$$2y=Cx^2+\frac{a}{C}$$

Example 65 Solve $y=2px-p^2$.

Sol. Differentiating the given equation w.r.t. x, we get

$$\frac{dy}{dx}=2p+2x\frac{dp}{dx}-2p\frac{dp}{dx}$$

or $\quad 2(x-p)\dfrac{dp}{dx}+p=0$

or $\quad \dfrac{dx}{dp}+\dfrac{2}{p}x=2$

It is a linear equation in x and p.

$$\text{IF}=e^{\int \frac{2}{p}dx}=e^{2\log p}=p^2$$

$\therefore$ The solution is $xp^2=\int p^2\cdot 2\,dp=\dfrac{2}{3}p^3+C$

Thus, the solution of the given equation is

$$x=\frac{2}{3}p+Cp^{-2}\text{, where } p \text{ is parameter.}$$

(iii) Equations Solvable For x

This type of equation can be put in the form

$$x=g(y,p) \qquad \text{...(i)}$$

Differentiating w.r.t. y, we get

$$\frac{1}{p}=\frac{dx}{dy}=G\left(x,p,\frac{dp}{dx}\right)$$

which is a differential equation of 1st order containing y and p and its solution is

$$\phi(y,p,c)=0$$

Then, the solution is obtained by eliminating p between $x=g(y,p)$ and $\phi(y,p,c)=0$. However, if eliminating of p is difficult express x and y as a function of the parameter p.

Example 66 Solve $y=2px+y^2p^3$.

Sol. Solving for x, we get

$$x=\frac{y}{2p}-\frac{y^2p^2}{2} \qquad \text{...(i)}$$

Differentiating Eq. (i) w.r.t. y, we get

$$\frac{dx}{dy}=\frac{1}{2p}-\frac{y}{2p^2}\cdot\frac{dp}{dy}-yp^2-y^2p\cdot\frac{dp}{dy}$$

or $$\frac{1}{p}-\frac{1}{2p}+yp^2=-y\left(\frac{1}{2p^2}+yp\right)\cdot\frac{dp}{dy}$$

or $$(1+2yp^3)\,p=-y\,(1+2p^3y)\cdot\frac{dp}{dy}$$

or $$\frac{dp}{p}+\frac{dy}{y}=0\Rightarrow py=C\Rightarrow p=\frac{C}{y}$$

Substituting this in the Eq. (i), we get

$$x=\frac{y^2}{2C}-\frac{C^2}{2}\;\Rightarrow\;y^2=2Cx+C^2$$

(iv) Clairaut's Equation

The differential equation $y=px+f(p)$ is known as Clairaut's equation. The solution of equation of this type is given by $y=cx+f(c)$.

Which is obtained by replacing p by c in the given equation.

Remark

Some equations can be reduced to Clairaut's form by suitable substitution.

Example 67 Solve $y=px+\dfrac{p}{\sqrt{1+p^2}}$.

Sol. Its solution is, $y=cx+\dfrac{c}{\sqrt{1+c^2}}$

Example 68 Solve $\sqrt{1+p^2}=\tan(px-y)$.

Sol. The given equation is

$$\sqrt{1+p^2}=\tan(px-y)$$

or $$px-y=\tan^{-1}(\sqrt{1+p^2})$$

or $$y=px-\tan^{-1}(\sqrt{1+p^2})$$

Its solution is, $$y=cx-\tan^{-1}(\sqrt{1+c^2})$$

Example 69 Solve $y^2\log y=pxy+p^2$.

Sol. Let $\log y=t$. Then $\dfrac{1}{y}\dfrac{dy}{dx}=\dfrac{dt}{dx}$

So, if $\dfrac{dt}{dx}=p$, then $\dfrac{p}{y}=p$

Substituting these in the given equation, we have

$$y^2t=y\cdot p\;xy+p^2y^2\quad\text{or}\quad t=px+p^2$$

Which is in Clairaut's form.

Thus, the required solution is

$$t=cx+c^2\quad\text{or}\quad\log y=cx+c^2$$

(c being an arbitrary constant.)

Application of Differential Equations

Differential Equation of First Order But not of First Degree

1. The most general form of a first order and higher degree differential equation is $p^n+P_1\,p^{n-1}+P_2\,p^{n-2}+\ldots\ldots+P_n=0$ where $P_1, P_2,$ ....., P_n are function of x, y and $p=dy/dx$. If a 1st order any degree equation can be resolved into differential equation (involving p) of first degree and 1st order, in such case we say that the equation is solvable for p.

   Let their solution be $g_1(x,y,c_1)\times g_2(x,y,c_2)\times\ldots\times g_n(x,y,c_n)=0$, (where $c_1, c_2,$ ....., c_n, are arbitrary constant) we take $c_1=c_2=\ldots\ldots=c_n=c$ because the differential equation of 1st order 1st degree contain only one arbitrary constant. So solution is

   $$g_1(x,y,c)\times g_2(x,y,c)\times\ldots\times g_n(x,y,c)=0$$

2. The most general form of a first order and higher degree differential equation is $p^n+P_1\,p^{n-1}+P_2\,p^{n-2}+\ldots+P_n=0$, where $P_1, P_2,$ ....., P_n are function of x, y and $p=dy/dx$. If differential equation is expressible in the form $y=f(x,p)$, then

   ***Step* 1** Differentiate w.r.t. x, we get $p\dfrac{dy}{dx}=f\left(x,p,\dfrac{dp}{dx}\right)$.

   ***Step* 2** Solving this we obtain $\phi(x,p,c)=0$.

   ***Step* 3** The solution of differential equation is obtained by eliminating p.

Application of First Order Differential Equations

Growth and Decay Problems

Let $N(t)$ denotes the amount of substance (or population) that is either growing or decaying. If we assume that dN/dt, the time rate of change of this amount of substance, is proportional to the amount of substance present, then

$$\frac{dN}{dt}=kN\quad\text{or}\quad\frac{dN}{dt}-kN=0\qquad\ldots(i)$$

Where k is the constant of proportionality. We are assuming that $N(t)$ is a differentiable, hence continuous, function of time.

Example 70 The population of a certain country is known to increase at a rate proportional to the number of people presently living in the country. If after two years the population has doubled and after three years the population is 20000, estimate the number of people initially living in the country.

Sol. Let N denotes the number of people living in the country at any time t, and let N_0 denote the number of people initially living in the country. Then, from Eq. (i)

$$\frac{dN}{dt} - kN = 0$$

Which has the solution $N = Ce^{kt}$ ...(i)

At $t = 0$, $N = N_0$; hence, it follows from Eq. (i) that $N_0 = Ce^{k(0)}$ or that $C = N_0$.

Thus, $N = N_0 e^{kt}$ ...(ii)

At $t = 2$, $N = 2N_0$. Substituting these values into Eq. (ii), we have

$$2N_0 = N_0 e^{2k} \text{ from which } k = \frac{1}{2}\ln 2 = 0.347$$

Substituting this value into Eq. (i) gives

$$N = N_0 e^{(0.347)\,t} \quad \text{...(iii)}$$

At $t = 3$, $N = 20000$. Substituting these values into Eq. (iii), we obtain

$$20000 = N_0 e^{(0.347)(3)}$$

Example 71 ***A certain radioactive material is known to decay at a rate proportional to the amount present. If initially there is 50 mg of the material present and after two hours it is observed that the material has lost 10% of its original mass, find (a) and expression for the mass of the material remaining at any time t, (b) the mass of the material after four hours, and (c) the time at which the material has decayed to one half of its initial mass.***

Sol. (a) Let N denotes the amount of material present at time t. Then, from Eq. (i)

$$\frac{dN}{dt} - kN = 0$$

This differential equation is separable and linear, its solution is

$$N = Ce^{kt} \quad \text{...(i)}$$

At $t = 0$, we are given that $N = 50$. Therefore, from Eq. (i), $50 = Ce^{k(0)}$ or $C = 50$.

Thus, $N = 50\, e^{kt}$ ...(ii)

At $t = 2$, 10% of the original mass of 50 mg or 5 mg has decayed. Hence, at $t = 2$, $N = 50 - 5 = 45$. Substituting these values into Eq. (ii) and solving for k, we have

$$45 = 50e^{2k} \quad \text{or} \quad k = \frac{1}{2}\ln\frac{45}{50} = -0.053$$

Substituting this value into Eq. (ii), we obtain the amount of mass present at any time t as

$$N = 50e^{-0.053\,t} \quad \text{...(iii)}$$

Where t is measured in hours.

(b) We require N at $t = 4$. Substituting $t = 4$ into Eq. (iii) and then solving for N, we find $N = 50e^{(-0.053)\,(4)}$

(c) We require t when $N = 50/2 = 25$. Substituting $N = 25$ into Eq. (iii) and solving for t, we find $25 = 50e^{-0.053t}$

or $-0.053\ t = \ln\frac{1}{2}$ or $t = 13$ h

Example 72 Five mice in a stable population of 500 are intentionally infected with a contagious disease to test a theory of epidemic spread that postulates the rate of change in the infected population is proportional to the product of the number of mice who have the disease with the number that are disease free. Assuming the theory is correct, how long will it take half the population to contract the disease?

Sol. Let $N(t)$ denotes the number of mice with the disease at time t. We are given that $N(0) = 5$, and it follows that $500 - N(t)$ is the number of mice without the disease at time t. The theory predicts that

$$\frac{dN}{dt} = kN\,(500 - N) \quad \text{...(i)}$$

Where k is a constant of proportionality. This equation is different from Eq. (i) because the rate of change is no longer proportional to just the number of mice who have the disease. Eq. (i) has the differential form

$$\frac{dN}{N(500 - N)} - kdt = 0 \quad \text{...(ii)}$$

Which is separable. Using partial fraction decomposition, we have

$$\frac{1}{N(500 - N)} = \frac{1/500}{N} + \frac{1/500}{500 - N}$$

Hence, Eq. (ii) may be rewritten as

$$\frac{1}{500}\left(\frac{1}{N} + \frac{1}{500 - N}\right)dN - kdt = 0$$

It solution is $\frac{1}{500}\int\left(\frac{1}{N} + \frac{1}{500 - N}\right)dN - \int kdt = C$

or $\frac{1}{500}(\ln|N| - \ln|500 - N|) - kt = C$

Which may be rewritten as

$$\ln\left|\frac{N}{500 - N}\right| = 500\,(C + kt)$$

$$\frac{N}{500-N} = e^{500(C+kt)} \qquad \ldots(iii)$$

But $e^{500(C+kt)} = e^{500}\ e^{kt}$. Setting $C_1 = e^{500C}$, we can write Eq. (iii) as

$$\frac{N}{500-N} = C_1 e^{500kt} \qquad \ldots(iv)$$

At $t = 0$, $N = 5$. Substituting these values into Eq. (iv), we find

$$\frac{4}{495} = C_1\, e^{500k(0)} = C_1$$

So, $C_1 = 1/99$ and Eq. (iv) becomes

$$\frac{N}{500-N} = \frac{1}{99}\, e^{500kt} \qquad \ldots(v)$$

We could solve Eq. (v) for N, but this is not necessary. We seek a value of t when $N = 250$, one half the population. Substituting $N = 250$ into Eq. (v) and solving for t, we obtain

$$1 = \frac{1}{99}\, e^{500kt}; \qquad \ln 99 = 500kt$$

or $t = 0.0091/k$ time units. Without additional information, we cannot obtain a numerical value for the constant of proportionality k or be more definitive about t.

Geometrical Applications

Let $P(x_1, y_1)$ be any point on the curve $y = f(x)$, then slope of the tangent at $P\ (= \tan\psi) = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$ and hence we find the following facts.

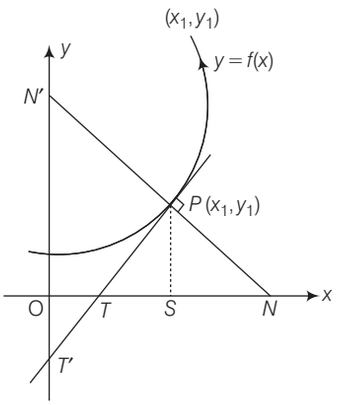


Fig. 4.2

(i) The equation of the tangent at P is,

$y - y_1 = \frac{dy}{dx}(x - x_1)$ when it cuts x-axis, $y = 0$.

$\therefore$ x-intercept of the tangent $= x_1 - y_1\left(\frac{dx}{dy}\right)$

y-intercept of the tangent $= y_1 - x_1\frac{dy}{dx}$

(ii) The equation of normal at P is,

$y - y_1 = -\frac{1}{(dy/dx)}(x - x_1)$ x and y-intercepts of normal are; $x_1 + y_1\frac{dy}{dx}$ and $y_1 + x_1\frac{dx}{dy}$

(iii) Length of tangent $= PT = |y_1|\sqrt{1 + (dx/dy)^2_{(x_1, y_1)}}$

(iv) Length of normal $= PN = |y_1|\sqrt{1 + (dy/dx)^2_{(x_1, y_1)}}$

(v) Length of subtangent $= ST = \left|y_1\left(\frac{dx}{dy}\right)_{(x_1, y_1)}\right|$

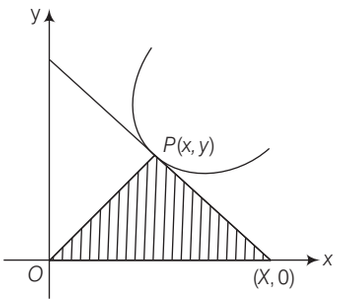
(vi) Length of subnormal $= SN = \left|y_1\left(\frac{dy}{dx}\right)_{(x_1, y_1)}\right|$

(vii) Length of radius vector $= \sqrt{x_1^2 + y_1^2}$

Example 73 Find the curve for which the area of the triangle formed by the x-axis tangent drawn at any point on the curve and radius vector of the point of tangency is constant equal to a^2.

Sol. Tangent drawn at any point (x, y) is

$$Y - y = \frac{dy}{dx}(X - x)$$



When $Y = 0$, $X = x - y\frac{dx}{dy}$

Area of $\Delta = 2a^2$ (given)

i.e. $\left|\frac{1}{2}\cdot X \cdot Y\right| = 2a^2$

i.e. $\left|xy - y^2\frac{dx}{dy}\right| = 2a^2$

i.e. $xy - y^2\frac{dx}{dy} = \pm 2a^2$

i.e. $\frac{dx}{dy} - \frac{x}{y} = \pm\frac{2a^2}{y^2}$

$$\text{IF} = e^{\int -\frac{1}{y}\,dy} = \frac{1}{y}$$

$\therefore$ The solution is $x \cdot \frac{1}{y} = \int \pm \frac{2a^2}{y^2} \cdot \frac{1}{y}\,dy$

$$\frac{x}{y} = \pm \frac{2a^2 \cdot y^{-2}}{-2} + C, \text{ i.e. } x = Cy + \frac{a^2}{y}$$

Example 74 Find the curve for which the intercept cut off by any tangent on y-axis is proportional to the square of the ordinate of the point of tangency.

Sol. The equation of tangent at any point (x, y) is

$$Y - y = \frac{dy}{dx}(X - x)$$

When $X = 0; Y = y - x\frac{dy}{dx} = y$-intercept

It is given $Y \propto y^2$. i.e. $Y = ky^2$

(k being constant of proportionality)

i.e. $$y - x\frac{dy}{dx} = ky^2$$

i.e. $$\frac{dy}{dx} - \frac{1}{x}y = -\frac{ky^2}{x}$$

i.e. $$-y^{-2}\frac{dy}{dx} + \frac{1}{x}\cdot\frac{1}{y} = \frac{k}{x}$$

Let $\frac{1}{y} = z$ so that, $-\frac{1}{y^2}\cdot\frac{dy}{dx} = \frac{dz}{dx}$ $\therefore$ $\frac{dz}{dx} + \frac{z}{x} = \frac{k}{x}$

$$\text{IF} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$$

$\therefore$ The solution is $zx = \int \frac{kx}{x}\,dx = kx + C$

$\Rightarrow$ $$\frac{x}{y} = kx + C$$

$\Rightarrow$ $$x = kxy + Cy$$

$\Rightarrow$ $x - Cy = kxy \Rightarrow \frac{1}{ky} + \frac{-C}{k}\cdot\frac{1}{x} = 1$

$\Rightarrow$ $\frac{C_1}{x} + \frac{C_2}{y} = 1$ $\left(\text{where } \frac{1}{k} = C_2 \text{ and } -\frac{C}{k} = C_1\right)$

Example 75 For any differential function $y = f(x)$, the value of $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 \cdot \frac{d^2x}{dy^2}$ is equal to

(a) $2y\frac{dy}{dx}$ (b) $y^2\frac{dy}{dx}$

(c) $y\frac{dy}{dx} + \left(\frac{d^2x}{dy^2}\right)^2$ (d) None of these

Sol. We know, $\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1}$, differentiating both the sides

or $$\frac{d^2y}{dx^2} = -\left(\frac{dx}{dy}\right)^{-2}\cdot\frac{d}{dy}\left(\frac{dx}{dy}\right)\cdot\frac{dy}{dx}$$

$$= -\left(\frac{dx}{dy}\right)^{-2}\cdot\frac{d^2x}{dy^2}\cdot\frac{dy}{dx}$$

$\Rightarrow$ $$\frac{d^2y}{dx^2} = -\frac{d^2x}{dy^2}\cdot\left(\frac{dy}{dx}\right)^3$$

or $$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3\cdot\frac{d^2x}{dy^2} = 0$$

Hence, (d) is the correct answer.

Example 76 ***The solution of*** $y = x\frac{dy}{dx} + \frac{dy}{dx} - \left(\frac{dy}{dx}\right)^2$, ***is***

(a) $y = (x-1)^2$ (b) $4y = (x+1)^2$

(c) $(y-1)^2 = 4x$ (d) None of these

Sol. The given equation can be written as

$$y = xp + p - p^2; \quad \text{where } p = \frac{dy}{dx} \quad \ldots(i)$$

Differentiating both the sides w.r.t. x, we get

$$p = p + \frac{xdp}{dx} + \frac{dp}{dx} - 2p\frac{dp}{dx}$$

$\therefore$ $$\frac{dp}{dx}(x + 1 - 2p) = 0$$

$\therefore$ Either $\frac{dp}{dx} = 0$ i.e. $p = C$ ...(ii)

or $x + 1 - 2p = 0$ *ie*, $p = \frac{1}{2}(x+1)$ ...(iii)

Eliminating p between Eqs. (i) and (ii), we get

$$y = Cx + C - C^2$$

As the complete solution and eliminating p between Eqs. (i) and (iii)

$$y = \frac{1}{2}(x+1)\,x + \frac{1}{2}(x+1) - \frac{1}{4}(x+1)^2$$

ie, $4y = (x+1)^2$ as the singular solution.

Hence, (b) is the correct answer.

Example 77 The solution of $\left(\frac{dy}{dx}\right)^2 + (2x + y)\frac{dy}{dx} + 2xy = 0$, is

(a) $(y + x^2 - C_1)(x + \log y + y^2 + C_2) = 0$

(b) $(y + x^2 - C_1)(x - \log y - C_2) = 0$

(c) $(y + x^2 - C_1)(x + \log y - C_2) = 0$

(d) None of the above

Sol. The given equation can be written as

$$p^2 + (2x + y)\, p + 2xy = 0, \text{ where } p = \frac{dy}{dx}$$

i.e. $(p + 2x)(p + y) = 0$

$\therefore$ $p + 2x = 0$, otherwise $p + y = 0$

$\Rightarrow$ $\frac{dy}{dx} + 2x = 0$ or $\frac{dy}{dx} + y = 0$

$\Rightarrow$ $\int dy + 2\int x\, dx = C_1$ or $\int \frac{dy}{y} + \int dx = C_2$

$\Rightarrow$ $y + x^2 = C_1$ or $\log y + x = C_2$

$\therefore$ $(y + x^2 - C_1) = 0$

or $(x + \log y - C_2) = 0$

$\Rightarrow$ $(y + x^2 - C_1)(x + \log y - C_2) = 0$

is the required solution.

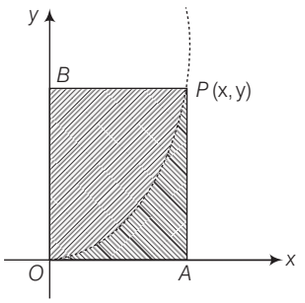
Hence, (c) is the correct answer.

Example 78 A curve $y = f(x)$ passes through the origin. Through any point (x, y) on the curve, lines are drawn parallel to the coordinate axes. If the curve divides the area formed by these lines and coordinate axes in the ratio $m : n$. Then the equation of curve is

(a) $y = Cx^{m/n}$ (b) $my^2 = Cx^{m/n}$

(c) $y^3 = Cx^{m/n}$ (d) None of these

Sol. $\frac{\text{Area of } OBPO}{\text{Area of } OPAO} = \frac{m}{n}$



$\Rightarrow$ $\frac{xy - \int_0^x y\, dx}{\int_0^x y\, dx} = \frac{m}{n} \Rightarrow nxy = (m + n)\int_0^x y\, dx$

Differentiating w.r.t. x, we get

$$n\left(x\frac{dy}{dx} + y\right) = (m + n)\, y$$

$\Rightarrow$ $nx\frac{dy}{dx} = my$

$\Rightarrow$ $\frac{m}{n}\cdot\frac{dx}{x} = \frac{dy}{y}$, integrating both the sides

$$y = Cx^{m/n}$$

Hence, (a) is the correct answer.

Example 79 The equation of the curve passing through the points $(3a, a)\,(a > 0)$ in the form $x = f(y)$ which satisfy the differential equation;

$\frac{a^2}{xy}\cdot\frac{dx}{dy} = \frac{x}{y} + \frac{y}{x} - 2$, is

(a) $x = y + a\left(\frac{1 + e^{y-k}}{1 - 2e^{y-k}}\right)$ (b) $x = y + a\left(\frac{1 + e^{y-k}}{1 - e^{y-k}}\right)$

(c) $y = x + a\left(\frac{1 + e^{y-k}}{1 - e^{y-k}}\right)$ (d) None of these

Sol. Here, $\frac{a^2}{xy}\cdot\frac{dx}{dy} = \frac{x}{y} + \frac{y}{x} - 2$

$\Rightarrow$ $a^2 = \frac{dy}{dx}(x^2 + y^2 - 2xy)$

$\Rightarrow$ $(x - y)^2\cdot\frac{dy}{dx} = a^2$, put $x - y = v$ $\therefore 1 - \frac{dy}{dx} = \frac{dv}{dx}$

$\Rightarrow$ $v^2\left(1 - \frac{dv}{dx}\right) = a^2$

$\Rightarrow$ $v^2 - a^2 = v^2\frac{dv}{dx} \Rightarrow \frac{v^2 dv}{v^2 - a^2} = dx$

$\Rightarrow$ $\left(1 + \frac{a^2}{v^2 - a^2}\right)dv = dx$

Integrating both the sides, we get

$$v + \frac{a}{2}\log\left|\frac{v - a}{v + a}\right| = x + C$$

$\Rightarrow$ $(x - y) + \frac{a}{2}\log\left(\frac{x - y - a}{x - y + a}\right) = x + C$

$\Rightarrow$ $y + C = \frac{a}{2}\log\left(\frac{x - y - a}{x - y + a}\right)$ ...(i)

It passes through $(3a, a)$

$\Rightarrow$ $a + C = \frac{a}{2}\log\left(\frac{1}{3}\right)$

$\Rightarrow$ $C = -a + \frac{a}{2}\log\left(\frac{1}{3}\right)$

$\Rightarrow$ $C = -a\left(\frac{2 + \log 3}{2}\right)$

$\therefore$ $y = \frac{a}{2}(2 + \log 3) + \log\left(\frac{x - y - a}{x - y + a}\right)$

$\Rightarrow$ $\frac{x - y - a}{x - y + a} = e^{y-k}$, where $k = \frac{a}{2}(2 + \log 3)$

$\therefore$ $\frac{x - y}{a} = \frac{1 + e^{y-k}}{1 - e^{y-k}}$

$\Rightarrow$ $x = y + a\left(\frac{1 + e^{y-k}}{1 - e^{y-k}}\right)$, where $k = \frac{a}{2}(2 + \log 3)$

Which is required equation of curve.

Hence, (b) is the correct answer.

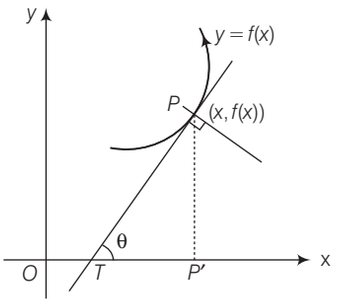
Example 80 ***The family of curves, the subtangent at any point of which is the arithmetic mean of the coordinates of the point of tangency, is given by***

(a) $(x-y)^2 = Cy$ (b) $(y-x)^2 = Cx$

(c) $(x-y)^2 = Cxy$ (d) None of these

Sol. Let the family of curves be $y = f(x)$

$$\tan\theta = \frac{l(PP')}{l(TP')}$$



$$\tan\theta = \frac{l(PP')}{l(TP')}$$

$$\therefore \quad l(\text{subtangent}) = \frac{f(x)}{f'(x)}$$

$$\therefore \quad \frac{y}{y'} = \frac{x+y}{2} \text{ (given)}$$

$$\therefore \quad y' = \frac{2y}{x+y}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2y}{x+y} \quad \therefore \quad \frac{dx}{dy} = \frac{x+y}{2xy} \qquad \ldots(i)$$

It is a homogeneous differential equation.

$\therefore$ Put $x = vy$

Differentiating w.r.t. y, we get

$$\frac{dx}{dy} = v + y\frac{dv}{dy} \qquad \ldots(ii)$$

In Eq. (i) replacing $\frac{dx}{dy}$ by Eq. (ii), we get

$$v + y\frac{dv}{dy} = \frac{vy+y}{2y} = \frac{1+v}{2}$$

$$\Rightarrow \quad y\frac{dv}{dy} = \frac{1+v}{2} - v = \frac{1+v-2v}{2} = \frac{1-v}{2}$$

$$\Rightarrow \quad \frac{2}{1-v}dv = \frac{dy}{y}$$

Integrating, $\frac{2\log|1-v|}{-1} = \log|y| + \log C_1 (C_1 > 0)$

$$\therefore \quad -2\log|y-x| + 2\log|y| = \log|y| + \log C_1$$

$$\Rightarrow \quad \log|y-x|^2 = \log|y| - \log C_1$$

$$\Rightarrow \quad \log|y-x|^2 = \log|y| + \log C,$$

where $\log C = -\log C_1$

$$\Rightarrow \quad \log|y-x|^2 = \log|yC|$$

$\Rightarrow (x-y)^2 = Cy$, is the required equation of family of curves.

Hence, (a) is the correct answer.

Exercise for Session 5

1. The equation of curve for which the normal at every point passes through a fixed point, is

(a) a circle
(b) an ellipse
(c) a hyperbola
(d) None of these

2. If the tangent at any point P of a curve meets the axis of x in T. Then the curve for which $OP = PT$, O being the origin is

(a) $x = Cy^2$
(b) $x = Cy^2$ or $x = C / y^2$
(c) $x = Cy$ or $x = C / y$
(d) None of these

3. According to Newton's law, the rate of cooling is proportional to the difference between the temperature of the body and the temperature of the air. If the temperature of the air is 20°C and body cools for 20 min from 100°C to 60°C, then the time it will take for it temperature to drop to 30°C, is

(a) 30 min
(b) 40 min
(c) 60 min
(d) 80 min

4. Let $f(x, y)$ be a curve in the x-y plane having the property that distance from the origin of any tangent to the curve is equal to distance of point of contact from the y-axis. If $f(1, 2) = 0$, then all such possible curves are

(a) $x^2 + y^2 = 5x$
(b) $x^2 - y^2 = 5x$
(c) $x^2y^2 = 5x$
(d) All of these

5. Given the curves $y = f(x)$ passing through the point $(0, 1)$ and $y = \int_{-\infty}^{x} f(t)$ passing through the point $\left(0, \frac{1}{2}\right)$. The tangents drawn to both the curves at the points with equal abscissae intersect on the x-axis. Then the curve $y = f(x)$, is

(a) $f(x) = x^2 + x + 1$
(b) $f(x) = \frac{x^2}{e^x}$
(c) $f(x) = e^{2x}$
(d) $f(x) = x - e^x$

6. A curve passing through (1, 0) is such that the ratio of the square of the intercept cut by any tangent on the y-axis to the Sub-normal is equal to the ratio of the product of the Coordinates of the point of tangency to the product of square of the slope of the tangent and the subtangent at the same point, is given by

(a) $x = e^{\pm 2\sqrt{y/x}}$
(b) $x = e^{\pm \sqrt{y/x}}$
(c) $y = e^{\pm \sqrt{y/x}} - 1$
(d) $xy + e^{y/x} - 1 = 0$

7. Consider a curve $y = f(x)$ in xy-plane. The curve passes through (0, 0) and has the property that a segment of tangent drawn at any point $P(x, f(x))$ and the line $y = 3$ gets bisected by the line $x + y = 1$. then the equation of curve, is

(a) $y^2 = 9(x - y)$
(b) $(y - 3)^2 = 9(1 - x - y)$
(c) $(y + 3)^2 = 9(1 - x - y)$
(d) $(y - 3)^2 - 9(1 + x + y)$

8. Consider the curved mirror $y = f(x)$ passing through (0, 6) having the property that all light rays emerging from origin, after getting reflected from the mirror becomes parallel to x-axis, then the equation of curve, is

(a) $y^2 = 4(x - y)$ or $y^2 = 36(9 + x)$
(b) $y^2 = 4(1 - x)$ or $y^2 = 36(9 - x)$
(c) $y^2 = 4(1 + x)$ or $y^2 = 36(9 - x)$
(d) None of these

JEE Type Solved Examples :
Single Option Correct Type Questions

● **Ex. 1** *The order of the differential equation of family of curves* $y = C_1 \sin^{-1}x + C_2 \cos^{-1}x + C_3 \tan^{-1}x + C_4 \cot^{-1}x$ *(where* C_1, C_2, C_3 *and* C_4 *are arbitrary constants) is*

(a) 2 (b) 3

(c) 4 (d) None of these

Sol. Here, $y = C_1\sin^{-1}x + C_2\cos^{-1}x + C_3\tan^{-1}x + C_4\cot^{-1}x$

$$\Rightarrow y = C_1\sin^{-1}x + C_2\left(\frac{\pi}{2} - \sin^{-1}x\right) + C_3\tan^{-1}x + C_4\left(\frac{\pi}{2} - \tan^{-1}x\right)$$

$$= (C_1 - C_2)\sin^{-1}x + (C_3 - C_4)\tan^{-1}x + (C_3 - C_4)\frac{\pi}{2}$$

There are only two independent arbitrary constant order of the differential equation is 2.

Hence, (c) is the correct answer.

● **Ex. 2** *The solution of the differential equation* $\frac{dy}{dx} = \frac{1}{xy(x^2\sin y^2 + 1)}$ *is*

(a) $x^2(\cos y^2 - \sin y^2 - 2ce^{-y^2}) = 2$

(b) $y^2(\sin x^2 - \cos y^2 - 2ce^{-y^2}) = 2$

(c) $x^2(\cos y^2 - \sin y^2 - e^{-y^2}) = 4c$

(d) None of the above

Sol. Here, $\frac{dx}{dy} = xy(x^2\sin y^2 + 1)$

$$\Rightarrow \quad \frac{1}{x^3}\frac{dx}{dy} - \frac{1}{x^2}y = y\sin y^2$$

Let, $\quad -\frac{1}{x^2} = t \;\Rightarrow\; \frac{2}{x^3}\frac{dx}{dy} = \frac{dt}{dy}$

$$\Rightarrow \quad \frac{dt}{dy} + 2t\cdot y = y\cdot\sin y^2, I.F. = e^{\int +2y\,dy = e^{+y^2}}$$

So, required solution is

$$t\cdot e^{y^2} = \int 2y\sin y^2 \times e^{y^2}dy = \frac{1}{2}e^{y^2}(\sin y^2 - \cos y^2) + C$$

$$\Rightarrow \quad 2t = (\sin y^2 - \cos y^2) + 2C\,e^{-y^2}$$

$$\Rightarrow \quad 2 = -x^2(\sin y^2 - \cos y^2 + 2ce^{-y^2})$$

$$\Rightarrow \quad x^2(\cos y^2 - \sin y^2 - 2ce^{-y^2}) = 2$$

● **Ex. 3** *The curve satisfying the differential equation* $\frac{dy}{dx} = \frac{y(x+y^3)}{x(y^3 - x)}$ *and passing through* $(4, -2)$ *is*

(a) $y^2 = -2x$ (b) $y = -2x$

(c) $y^3 = -2x$ (d) None of these

Sol. Here, $(xy^3 - x^2)dy = (xy + y^4)dx$

$$\Rightarrow \quad y^3(xdy - ydx) - x(xdy + ydx = 0)$$

$$\Rightarrow \quad x^2y^3d\left(\frac{y}{x}\right) - x\,d(xy) = 0$$

dividing by x^3y^2, we get

$$\Rightarrow \quad \frac{y}{x}d\left(\frac{y}{x}\right) - \frac{1}{x^2y^2}\cdot d\left(\frac{x}{y}\right) = 0$$

$$\Rightarrow \quad \frac{1}{2}d\left(\frac{y}{x}\right)^2 + d\left(\frac{1}{xy}\right) = 0$$

Now integrating, we get

$$\frac{1}{2}\left(\frac{y}{x}\right)^2 + \frac{1}{xy} = c$$

It passes through $(4, -2)$

$$\Rightarrow \quad \frac{1}{8} - \frac{1}{8} = c \;\Rightarrow\; C = 0$$

$\therefore\ y^3 = -2x$ is required curve.

● **Ex. 4** *Spherical rain drop evaporates at a rate proportional to its surface area. The differential equation corresponding to the rate of change of the radius of the rain drop, if the constant of proportionality is* $K > 0$*, is*

(a) $\frac{dr}{dt} + K = 0$ (b) $\frac{dr}{dt} - K = 0$

(c) $\frac{dr}{dt} = Kr$ (d) None of these

Sol. $$\frac{dV}{dt} = -k4\pi r^2 \qquad ...(i)$$

But $$V = \frac{4}{3}\pi r^3$$

$$\Rightarrow \quad \frac{dV}{dt} = 4\pi r^2\frac{dr}{dt} \qquad ...(ii)$$

Therefore, $$\frac{dr}{dt} = -K$$

Hence, (a) is the correct answer.

● **Ex. 5** *A function* $y = f(x)$ *satisfies the differential equation* $f(x)\cdot\sin 2x - \cos x + (1 + \sin^2 x)\,f'(x) = 0$ *with initial condition* $y(0) = 0$*. The value of* $f(\pi/6)$ *is equal to*

(a) 1/5 (b) 3/5

(c) 4/5 (d) 2/5

Sol. $y\sin 2x - \cos x + (1 + \sin^2 x)\frac{dy}{dx} = 0$ where $y = f(x)$

$$\frac{dy}{dx} + \left(\frac{\sin 2x}{1 + \sin^2 x}\right)y = \frac{\cos x}{1 + \sin^2 x}$$

$$\text{IF} = e^{\int \frac{\sin 2x}{1+\sin^2 x}\,dx} = e^{\int \frac{dt}{t}} = e^{\ln(1+\sin^2 x)}$$

$$= 1 + \sin^2 x \quad (\text{by putting } 1 + \sin^2 x = t)$$

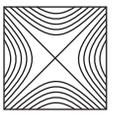
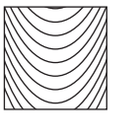
$$y(1 + \sin^2 x) = \int \cos x\, dx$$

$$y(1 + \sin^2 x) = \sin x + C;\ y(0) = 0 \Rightarrow C = 0$$

Therefore, $$y = \frac{\sin x}{1 + \sin^2 x};\ y\left(\frac{\pi}{6}\right) = \frac{2}{5}$$

Hence, (d) is the correct answer.

● **Ex. 6** *The general solution of the differential equation* $\frac{dy}{dx} = \frac{1-x}{y}$ *is a family of curves which looks most like which of the following?*

(a) (b) (c)  (d) 

Sol. $$\int y\,dy = \int (1 - x)\,dx$$

$$\frac{y^2}{2} = x - \frac{x^2}{2} + C$$

$$x^2 + y^2 - 2x = C$$

$$(x-1)^2 + y^2 = C + 1 = C$$

Hence, (b) is the correct answer.

Remark

Family of concentric circles with (1, 0) as the centre and variable radius.

● **Ex. 7** *Water is drained from a vertical cylindrical tank by opening a valve at the base of the tank. It is known that the rate at which the water level drops is proportional to the square root of water depth y, where the constant of proportionality* $k > 0$ *depends on the acceleration due to gravity and the geometry of the hole. If t is measured in minutes and* $k = \frac{1}{15}$*, then the time to drain the tank, if the water is 4 m deep to start with is*

(a) 30 min (b) 45 min (c) 60 min (d) 80 min

Sol. $\frac{dy}{dt} = -k\sqrt{y}$; when $t = 0$; $y = 4$

$$\int_4^0 \frac{dy}{\sqrt{y}} = -k\int_0^t dt$$

$$[2\sqrt{y}]_4^0 = -kt = -\frac{t}{15}$$

$$0 - 4 = -\frac{t}{15}$$

$$\Rightarrow \quad t = 60 \text{ min}$$

Hence, (c) is the correct answer.

● **Ex. 8** *Number of straight lines which satisfy the differential equation* $\frac{dy}{dx} + x\left(\frac{dy}{dx}\right)^2 - y = 0$ *is*

(a) 1 (b) 2 (c) 3 (d) 4

Sol. $y = kx + b;\ \frac{dy}{dx} = k$

$$\Rightarrow \quad kx + b \equiv k + xk^2 \quad \Rightarrow\ k = k^2 \text{ and } b = k$$

$$k = 0 \text{ or } k = 1$$

Hence, (b) is the correct answer.

● **Ex. 9** *Consider the two statements :*

Statement I $y = \sin kt$ *satisfy the differential equation* $y'' + 9y = 0$.

Statement II $y = e^{kt}$ *satisfy the differential equation* $y'' + y' - 6y = 0$.

The value of k for which both the statements are correct is

(a) −3 (b) 0 (c) 2 (d) 3

Sol. **Statement I** $y = \sin kt,\ y' = k\cos kt;\ y'' = -k^2 \sin kt$

$$\therefore \quad -k^2 \sin kt + 9\sin kt = 0$$

$$\sin kt\,[9 - k^2] = 0 \ \Rightarrow\ k = 0, k = 3, k = -3$$

Statement II $y = e^{kt},\ y' = ke^{kt};\ y'' = k^2 e^{kt}$

$$\therefore \quad k^2 e^{kt} + ke^{kt} - 6e^{kt} = 0$$

$$e^{kt}[k^2 + k - 6] = 0$$

$$(k + 3)(k - 2) = 0$$

$$k = -3 \text{ or } 2$$

Common value is $k = -3$.

Hence, (a) is the correct answer.

● **Ex. 10** *If* $y = \frac{x}{\ln|cx|}$ *(where c is an arbitrary constant) is the general solution of the differential equation* $\frac{dy}{dx} = \frac{y}{x} + \phi\left(\frac{x}{y}\right)$*, then the function* $\phi\left(\frac{x}{y}\right)$ *is*

(a) $\frac{x^2}{y^2}$ (b) $-\frac{x^2}{y^2}$ (c) $\frac{y^2}{x^2}$ (d) $-\frac{y^2}{x^2}$

Sol. $\ln c + \ln|x| = \frac{x}{y}$

Differentiating w.r.t. x, $\frac{1}{x} = \frac{y - xy_1}{y^2}$

$$\frac{y^2}{x} = y - x\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{y}{x} - \frac{y^2}{x^2} \ \Rightarrow\ \phi\left(\frac{x}{y}\right) = -\frac{y^2}{x^2}$$

Hence, (d) is the correct answer.

● **Ex. 11** *If* $\int_a^x ty(t)\,dt = x^2 + y(x)$, *then* y *as a function of* x *is*

(a) $y = 2 - (2 + a^2)\,e^{\frac{x^2 - a^2}{2}}$ (b) $y = 1 - (2 + a^2)\,e^{\frac{x^2 - a^2}{2}}$

(c) $y = 2 - (1 + a^2)\,e^{\frac{x^2 - a^2}{2}}$ (d) None of these

Sol. Differentiating both the sides, we get

$$xy\,(x) = 2x - y'(x)$$

Hence, $\dfrac{dy}{dx} - xy = -2x \qquad \left[y'(x) = \dfrac{dy}{dx};\ y(x) = y\right]$

$$\text{IF} = e^{\int -x\,dx} = e^{\frac{-x^2}{2}}$$

$$ye^{\frac{-x^2}{2}} = \int -2xe^{\frac{-x^2}{2}}\,dx$$

Let $e^{-\frac{x^2}{2}} = t \implies -xe^{-\frac{x^2}{2}}\,dx = dt$

$$I = \int 2dt$$

$$ye^{-\frac{x^2}{2}} = 2e^{-\frac{x^2}{2}} + C$$

$$y = 2 + Ce^{\frac{x^2}{2}}$$

If $x = a \implies a^2 + y = 0 \implies y = -a^2$ (from the given equation)

Hence, $-a^2 = 2 + Ce^{\frac{a^2}{2}};\ Ce^{\frac{a^2}{2}} = -(2 + a^2)$

$$C = -(2 + a^2)\,e^{-\frac{a^2}{2}};\ y = 2 - (2 + a^2)\,e^{\frac{x^2 - a^2}{2}}$$

Hence, (a) is the correct answer.

● **Ex. 12** *The differential equation* $\dfrac{dy}{dx} = \dfrac{\sqrt{1 - y^2}}{y}$ *determines a family of circles with* ***[IIT JEE 2007]***

(a) variable radii and a fixed centre at (0, 1)

(b) variable radii and fixed centre at (0, −1)

(c) fixed radius 1 and variable centres along the x-axis

(d) fixed radius 1 and variable centres along the y-axis

Sol. $\because \quad \dfrac{dy}{dx} = \dfrac{\sqrt{1 - y^2}}{y}$

$$\Rightarrow \quad \int \frac{y}{\sqrt{1 - y^2}}\,dy = \int dx \implies -\frac{1}{2} \cdot 2\sqrt{1 - y^2} = x + C$$

$$\Rightarrow \quad -\sqrt{1 - y^2} = x + C \implies 1 - y^2 = (x + C)^2$$

$$\Rightarrow \quad (x + C)^2 + y^2 = 1$$

Therefore, the differential equation represents a circle of fixed radius 1 and variable centres along the x-axis. Hence, (c) is the correct answer.

JEE Type Solved Examples :
More than One Correct Option Type Questions

● **Ex. 13** *A curve* $y = f(x)$ *has the property that the perpendicular distance of the origin from the normal at any point P of the curve is equal to the distance of the point P from the x-axis. Then the differential equation of the curve*

(a) is homogeneous

(b) can be converted into linear differential equation with some suitable substitution

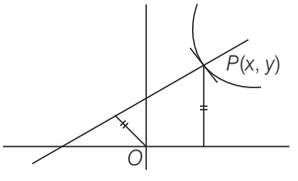
(c) is the family of circles touching the x-axis at the origin

(d) the family of circles touching the y-axis at the origin

Sol. Equation of normal

$$Y - y = -\frac{1}{m}(X - x) \implies -my + my = X - x$$

$$X + my - (x + my) = 0$$



$$\text{Perpendicular from } (0, 0) = \left|\frac{x + my}{\sqrt{1 + m^2}}\right| = y \implies x^2 + 2xym = y^2$$

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy} \implies \text{homogeneous}$$

Also, $x \cdot 2y \cdot \dfrac{dy}{dx} - x^2 = y^2$

Put $y^2 = t$; $\quad 2y\dfrac{dy}{dx} = \dfrac{dt}{dx};\ x \cdot \dfrac{dt}{dx} + x^2 = t$

$\dfrac{dt}{dx} - \dfrac{1}{x}t = -x$ which is linear differential equation.

Hence, (a), (b) and (d) are the correct answers.

● **Ex. 14** *A differentiable function satisfies* $f(x) = \int_0^x \{f(t)\cos t - \cos(t - x)\}\,dt$. *Which is of the following hold good?*

(a) $f(x)$ has a minimum value $1 - e$

(b) $f(x)$ has a maximum value $1 - e^{-1}$

(c) $f''\left(\dfrac{\pi}{2}\right) = e$ (d) $f'(0) = 1$

Sol. $f(x) = \int_0^x \{f(t)\cos t - \cos(t - x)\}\,dx$

$$= \int_0^x f(t)\cos t\,dt - \int_0^x \cos(-t)\,dt \left[\int_0^a f(x)dx = \int_0^a f(a - x)\,dx\right]$$

$$f(x) = \int_0^x f(t)\cos t\,dt - \sin x$$

Differentiating both the sides, we get

$$f'(x) = f(x)\cos x - \cos x$$

Let $f(x) = y;\ f'(x) = \dfrac{dy}{dx}$

$$\frac{dy}{dx} - y\cos x = -\cos x \qquad \text{(L.D.E.)}$$

$$\text{IF} = e^{-\int \cos x\,dx} = e^{-\sin x}$$

Therefore, $y \cdot e^{-\sin x} = -\int e^{-\sin x}\cos x\,dx;$

$$y \cdot e^{-\sin x} = C + e^{-\sin x};\ y = Ce^{\sin x} + 1$$

If $x = 0;$ $y = 0$ (from the given relation)

$\Rightarrow$ $C = -1$

Therefore, $f(x) = 1 - e^{\sin x}$

Now, minimum value $= 1 - e$ (when $x = \pi/2$)

Maximum value $= 1 - e^{-1}$ (when $x = -\pi/2$)

$$f'(x) = -e^{\sin x}\cos x$$

Therefore, $f'(0) = -1$

$$f''(x) = -[\cos^2 x \cdot e^{\sin x} - e^{\sin x}\cdot\sin x]$$

$$f''\left(\frac{\pi}{2}\right) = e$$

Hence, (a), (b) and (c) are the correct answers.

● ***Ex. 15*** *Let* $\dfrac{dy}{dx} + y = f(x)$ *where* y *is a continuous function of* x *with* $y(0) = 1$ *and* $f(x) = \begin{cases} e^{-x}, & \text{if } 0 \le x \le 2 \\ e^{-2}, & \text{if } x > 2 \end{cases}$. *Which is of the following hold(s) good?*

(a) $y(1) = 2e^{-1}$ (b) $y'(1) = -e^{-1}$

(c) $y(3) = -2e^{-3}$ (d) $y'(3) = -2e^{-3}$

Sol. $\dfrac{dy}{dx} + y = f(x) \Rightarrow \text{IF} = e^x$

$$ye^x = \int e^x f(x)\,dx + C$$

Now, if $0 \le x \le 2$, then $ye^x = \int e^x e^{-x}\,dx + C$

$\Rightarrow$ $ye^x = x + C$

$x = 0,\ y(0) = 1,\ C = 1$

$\therefore$ $ye^x = x + 1$ ...(i)

$$y = \frac{x+1}{e^x};\ y(1) = \frac{2}{e} \Rightarrow y' = \frac{e^x - (x+1)e^x}{e^{2x}}$$

$$y'(1) = \frac{e - 2e}{e^2} = \frac{-e}{e^2} = -\frac{1}{e}$$

If $x > 2$, $ye^x = \int e^{x-2}\,dx$

$$ye^x = e^{x-2} + C$$

$$y = e^{-2} + Ce^{-x}$$

As y is continuous.

$\therefore$ $\lim\limits_{x\to 2} \dfrac{x+1}{e^x} = \lim\limits_{x\to 2}(e^{-2} + Ce^{-x})$

$$3e^{-2} = e^{-2} + Ce^{-2} \Rightarrow C = 2$$

$\therefore$ for $x > 2$

$$y = e^{-2} + 2e^{-x}$$

Hence, $y(3) = 2e^{-3} + e^{-2} = e^{-2}(2e^{-1} + 1)$

$$y' = -2e^{-x}$$

$$y'(3) = -2e^{-3}$$

Hence, (a), (b) and (d) are the correct answers.

● ***Ex. 16*** *A curve* $y = f(x)$ *passes through (1, 1) and tangent at* $P(x, y)$ *cuts the x-axis and y-axis at A and B respectively such that* $BP : AP = 3 : 1$, *then* ***[IIT JEE 2006]***

(a) equation of curve is $xy' - 3y = 0$

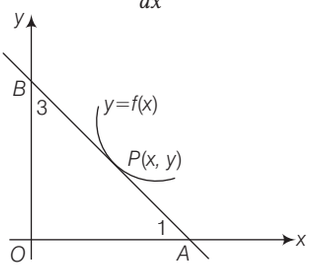
(b) normal at (1, 1) is $x + 3y = 4$

(c) curve passes through $\left(2, \dfrac{1}{8}\right)$

(d) equation of curve is $xy' + 3y = 0$

Sol. Equation of the tangent to the curve $y = f(x)$ at (x, y) is

$$Y - y = \frac{dy}{dx}(X - x)$$



$\because$ Thus, cuts the x-axis at A and y-axis at B.

$$\therefore \quad A\left(\frac{x\frac{dy}{dx} - y}{\frac{dy}{dx}}, 0\right) \text{ and } B\left(0, -x\frac{dy}{dx} + y\right)$$

$\because$ $BP : PA = 3 : 1$

$$\Rightarrow \quad x = \frac{3\left(x\frac{dy}{dx} - y\right)}{(dy/dx)} + 1 \times 0}{4}$$

$$\Rightarrow \quad x\frac{dy}{dx} + 3y = 0$$

$$\Rightarrow \quad \int \frac{dy}{y} = \int -3\frac{dx}{x}$$

$$\Rightarrow \quad \log y = -3\log x + \log C$$

$$\Rightarrow \quad y = \frac{C}{x^3}$$

$\because$ Curve passes through (1, 1) $\therefore$ $C = 1$

$\therefore$ Curve is $x^3y = 1$ which also passes through $\left(2, \dfrac{1}{8}\right)$.

Hence, (c) and (d) are the correct answers.

JEE Type Solved Examples :
Statement I and II Type Questions

● **Ex. 17** *Let a solution $y = y(x)$ of the differential equation $x\sqrt{x^2-1}\,dy - y\sqrt{y^2-1}\,dx = 0$ satisfy $y(2) = \frac{2}{\sqrt{3}}$.* ***[IIT JEE 2008]***

Statement I $y(x) = \sec\left(\sec^{-1} x - \frac{\pi}{6}\right)$.

Statement II $y(x)$ *is given by* $\frac{1}{y} = \frac{2\sqrt{3}}{x} - \sqrt{1-\frac{1}{x^2}}$.

(a) Statement I is true, Statement II is also true; Statement II is the correct explanation of Statement I.
(b) Statement I is true, Statement II is also true; Statement II is not the correct explanation of Statement I.
(c) Statement I is true, Statement II is false.
(d) Statement I is false, Statement II is true.

Sol. $\because \quad x\sqrt{x^2-1}\,dy - y\sqrt{y^2-1}\,dx = 0$

Which can be rewritten as $\frac{dx}{x\sqrt{x^2-1}} = \frac{dy}{y\sqrt{y^2-1}}$

Integration yields, $\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{dy}{y\sqrt{y^2-1}}$

$\Rightarrow \quad \sec^{-1} x = \sec^{-1} y + C$

$\Rightarrow \quad \sec^{-1}(2) = \sec^{-1}(2/\sqrt{3}) + C$

$\Rightarrow \quad \frac{\pi}{3} = \frac{\pi}{6} + C \;\Rightarrow\; C = \frac{\pi}{6}$

Thus, $\quad \sec^{-1} x = \sec^{-1} y + \frac{\pi}{6}$

$\Rightarrow \quad y = \sec\left(\sec^{-1} x - \frac{\pi}{6}\right)$

$$= \frac{1}{\cos\left(\cos^{-1}\frac{1}{x} - \frac{\pi}{6}\right)} = \frac{1}{\frac{1}{x}\cdot\frac{\sqrt{3}}{2} + \sqrt{1-\frac{1}{x^2}}\cdot\frac{1}{2}}$$

$\Rightarrow \quad \frac{1}{y} = \frac{\sqrt{3}}{2x} + \frac{\sqrt{1-\frac{1}{x^2}}}{2}$

Hence, (c) is the correct answer.

JEE Type Solved Examples :
Passage Based Questions

Passage
(Q. Nos. 18 to 20)

A curve $y = f(x)$ satisfies the differential equation $(1+x^2)\frac{dy}{dx} + 2yx = 4x^2$ and passes through the origin.

18 The function $y = f(x)$
(a) is strictly increasing, $\forall\, x \in R$
(b) is such that it has a minima but no maxima
(c) is such that it has a maxima but no minima
(d) has no inflection point

19 The area enclosed by $y = f^{-1}(x)$, the x-axis and the ordinate at $x = 2/3$ is
(a) $2\ln 2$ (b) $\frac{4}{3}\ln 2$ (c) $\frac{2}{3}\ln 2$ (d) $\frac{1}{3}\ln 2$

20 For the function $y = f(x)$ which one of the following does not hold good?
(a) $f(x)$ is a rational function
(b) $f(x)$ has the same domain and same rage
(c) $f(x)$ is a transcendental function
(d) $y = f(x)$ is a bijective mapping

Sol. (Q. Nos. 18 to 20)

$$\frac{dy}{dx} + \left(\frac{2x}{1+x^2}\right)y = \frac{4x^2}{1+x^2}$$

$$\text{IF} = e^{\int \frac{2x}{1+x^2}dx}$$

$$= e^{\ln(1+x^2)} = (1+x^2)$$

$\therefore \quad y(1+x^2) = \int 4x^2\,dx = \frac{4x^3}{3} + C$

Passing through $(0, 0) \Rightarrow C = 0$

$\therefore \quad y = \frac{4x^3}{3(1+x^2)}$

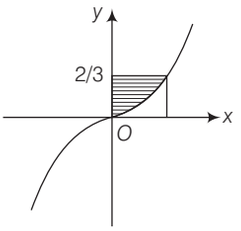
$$\frac{dy}{dx} = \frac{4}{3}\left[\frac{(1+x^2)3x^2 - x^3\cdot 2x}{(1+x^2)^2}\right]$$

$$= \frac{4}{3}\left[\frac{3x^2+x^4}{(1+x^2)^2}\right] = \frac{4x^2(3+x^2)}{3(1+x^2)^2}$$

Hence, $\frac{dy}{dx} > 0, \forall\, x \neq 0; \frac{dy}{dx} = 0$ at $x = 0$

and it does not change sign $\Rightarrow x = 0$ is the point of inflection
$y = f(x)$ is increasing for all $x \in R$.

$x \to \infty;\ y \to \infty;\ x \to -\infty;\ y \to -\infty$



Area enclosed by $y = f^{-1}(x)$, x-axis and ordinate at $x = \frac{2}{3}$

$$A = \frac{2}{3} - \frac{4}{3}\int_0^1 \frac{x^3}{1+x^2}\,dx$$

Put $1 + x^2 = t \implies 2x\,dx = dt$

$$A = \frac{2}{3} - \frac{2}{3}\int_1^2 \frac{(t-1)}{t}\,dt$$

$$= \frac{2}{3} - \frac{2}{3}\int_1^2 \left(1 - \frac{1}{t}\right)dt$$

$$= \frac{2}{3} - \frac{2}{3}[t - \ln t]_1^2 = \frac{2}{3} - \frac{2}{3}[(2 - \ln 2) - 1]$$

$$= \frac{2}{3} - \frac{2}{3}[1 - \ln 2] = \frac{2}{3}\ln 2$$

JEE Type Solved Examples :
Single Integer Answer Type Questions

● ***Ex. 21*** *Let $y = f(x)$ be a curve passing through (4, 3) such that slope of normal at any point lying in the first quadrant is negative and the normal and tangent at any point P cuts the Y-axis at A and B respectively such that the mid-point of AB is origin, then the number of solutions of $y = f(x)$ and $y = |5 - |x||$, is*

Sol. Equation of tangent at any point (x_1, y_1) of curve $y = f(x)$ is $(y - y_1) = f'(x_1)(x - x_1)$, so $B\,(0, y_1 - x_1 f'(x_1))$

Equation of normal at (x_1, y_1) is $(y, y_1) = -\frac{1}{f'(x_1)}(x, x_1)$ so,

$A = \left(0, y_1 + \frac{x_1}{f'(x_1)}\right)$ mid point of AB is origin, so

$$2y_1 - x_1\left(f'(x_1) - \frac{1}{f'(x_1)}\right) = 0$$

Thus differential equation of curve $y = f(x)$, is

$$x\left(\frac{dy}{dx}\right)^2 - 2y\frac{dy}{dx} - x = 0$$

Thus, $$\frac{dy}{dx} = \frac{y \pm \sqrt{x^2 + y^2}}{x}$$

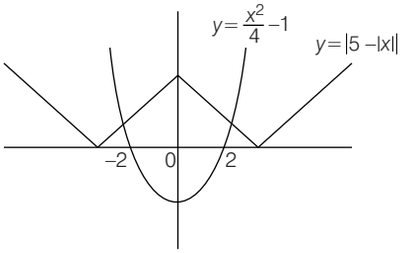
In first quadrant, $x > 0, y > 0, \frac{dy}{dx} > 0$,

So $$\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x}, \text{ put } y = vx$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{vx + x\sqrt{1 + v^2}}{x}$$

$$\Rightarrow \quad \int \frac{dv}{\sqrt{1 + v^2}} = \int \frac{dx}{x}$$

on solving we get $y = \frac{x^2}{4} - 1$



$\therefore$ number of solutions for $y = f(x)$ and $y = |5 - |x||$

$\therefore$ number of solutions are 2.

● ***Ex. 22*** *A real valued function, $f(x)$, $f : \left(0, \frac{\pi}{2}\right) \to R^+$ satisfies the differential equation $xf'(x) = 1 + f(x)\{x^2 f(x)^{-1}\}$ and $f\left(\frac{\pi}{4}\right) = \frac{4}{\pi}$, then $\lim_{x\to 0} f(x)$, is*

Sol. Here, $xf'(x) = 1 + x^2 f^2(x) - f(x)$

$$\Rightarrow \quad \frac{x\,f'(x) + f(x)}{1 + x^2 f^2(x)} = 1$$

Integrating both sides

$$\int \frac{(x\,f'(x) + f(x))dx}{1 + (x\,f(x))^2} = x + C$$

$$\Rightarrow \quad \tan^{-1}(xf(x) = x + C, \text{ as } f\left(\frac{\pi}{4}\right) = \frac{4}{\pi}.$$

$$\Rightarrow \quad \tan^{-1}1 = \frac{\pi}{4} + C$$

$$\Rightarrow \quad C = 0$$

$$\therefore \quad x\,f(x) = \tan x$$

$$\Rightarrow \quad f(x) = \frac{\tan x}{x}$$

and $$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{\tan x}{x} = 1$$

● ***Ex. 23*** *If the area bounded by* $y = f(x), x = \frac{1}{2}, x = \frac{\sqrt{3}}{2}$ *and the X-axis is A sq units where* $f(x) = x + \frac{2}{3}x^3 + \frac{2}{3}\cdot\frac{4}{5}x^5 + \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}x^7 + \ldots\infty, |x|<1$, *Then the value of* $[4A]$ *is (where* $[\cdot]$ *is G.I.F)*

Sol. Here, $f'(x) = 1 + 2x^2 + \frac{2}{3}\cdot 4x^4 + \frac{2}{3}\cdot\frac{4}{5}\cdot 6x^6 + \ldots\infty$

$= 1 + x\left(\frac{d}{dx}(xf(x))\right)$

$\Rightarrow \quad f'(x) = 1 + x\,|xf'(x) + f(x)|$

$\Rightarrow (1 - x^2)f'(x) = 1 + xf(x)$

$\Rightarrow \quad \frac{dy}{dx} - \frac{x}{1-x^2}\cdot y = \frac{1}{x^2}$, I.F. $= e^{\int \frac{-x}{1-x^2}dx} = e^{\frac{1}{2}\log|1-x^2|} = \sqrt{1-x^2}$

$\therefore \quad y\cdot\sqrt{1-x^2} = \int \frac{1}{1-x^2}\cdot\sqrt{1-x^2} + C$

$\Rightarrow \quad y\sqrt{1-x^2} = \sin^{-1}x + C$, as $f(0) = 0 \Rightarrow C = 0$

$\Rightarrow \quad y = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$

$\Rightarrow \quad A = \int_{1/2}^{\sqrt{3}/2} \frac{\sin^{-1}x}{\sqrt{1-x^2}}dx = \int_{\pi/6}^{\pi/3} t\,dt = \left(\frac{t^2}{2}\right)_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \frac{1}{2}\left[\frac{\pi^2}{4} - \frac{\pi^2}{36}\right]$

$\therefore \quad [4A] = 1$

Subjective Type Questions

● ***Ex. 24*** *For a certain curve* $y = f(x)$ *satisfying* $\frac{d^2y}{dx^2} = 6x - 4$; $f(x)$ *has a local minimum value 5 when* $x = 1$. *Find the equation of the curve and also the global maximum and global minimum values of* $f(x)$ *given that* $0 \le x \le 2$.

Sol. Integrating, $\frac{d^2y}{dx^2} = 6x - 4$, we get $\frac{dy}{dx} = 3x^2 - 4x + C$

when $x = 1, \frac{dy}{dx} = 0$. So that $C = 1$

Hence, $\frac{dy}{dx} = 3x^2 - 4x + 1$ ...(i)

Integrating, we get

$y = x^3 - 2x^2 + x + C_1$, when $x = 1, y = 5$,

so that $C_1 = 5$

Thus, we have $y = x^3 - 2x^2 + x + 5$

Form Eq. (i), we get the critical points $x = \frac{1}{3}, x = 1$

At the critical point $x = \frac{1}{3}, \frac{d^2y}{dx^2}$ is (–ve).

Therefore, at $x = \frac{1}{3}$, y has a local maximum.

At $x = 1, \frac{d^2y}{dx^2}$ is (+ ve).

Therefore, at $x = 1$, y has a local minimum.

Also, $f(1) = 5$

$\Rightarrow \quad f\left(\frac{1}{3}\right) = \frac{139}{27}$

$f(0) = 5, f(2) = 7$

Hence, the global maximum value = 7, the global minimum value = 5.

● ***Ex. 25*** *If* $\phi(x)$ *is a differentiable real-valued function satisfying* $\phi'(x) + 2\,\phi(x) \le 1$, *prove that* $\phi(x) - \frac{1}{2}$ *is a non-increasing function of x.*

Sol. $\phi'(x) + 2\,\phi(x) \le 1$

$\Rightarrow \quad e^{2x}\,\phi'(x) + 2\,\phi(x)\,e^{2x} \le e^{2x}$

$\Rightarrow \quad \frac{d}{dx}\left(e^{2x}\,\phi(x) - \frac{1}{2}e^{2x}\right) \le 0$

$\Rightarrow \quad e^{2x}\left(\phi(x) - \frac{1}{2}\right)$ is a non-increasing function of x.

$\Rightarrow \quad \phi(x) - \frac{1}{2}$ is a non-increasing function of x.

● ***Ex. 26*** *Determine all curve for which the ratios of the length of the segment intercepted by any tangent on the y-axis to the length of the radius vector is a constant.*

Sol. Let $y = f(x)$ be the equation of the required curve.

Given that $\frac{\left|y - x\frac{dy}{dx}\right|}{\sqrt{x^2+y^2}} = k$ (a constant)

$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} \pm k\sqrt{1 + \left(\frac{y}{x}\right)^2}$

Let $y = vx$, then $v + x\frac{dv}{dx} = v \pm k\sqrt{1+v^2}$

$\Rightarrow \quad \frac{dv}{\sqrt{1+v^2}} = \pm k\frac{dx}{x}$, integrating we get

$\Rightarrow \quad \log\left|v + \sqrt{1+v^2}\right| = \pm k\ln x + C$

$\Rightarrow \quad \log\left|\frac{xy + \sqrt{x^2+y^2}}{x}\right| = \pm k\ln x + C$

Which are the equations of the required curves.

● **Ex. 27** *Let $u(x)$ and $v(x)$ satisfy the differential equations $\frac{du}{dx} + p(x)\,u = f(x)$ and $\frac{dv}{dx} + p(x)\,v = g(x)$, where $p(x)$, $f(x)$ and $g(x)$ are continuous functions. If $u(x_1) > v(x_1)$ for some x_1 and $f(x) > g(x)$, for all $x > x_1$, prove that any point (x, y), where $x > x_1$, does not satisfy the equation $y = u(x)$ and $y = v(x)$.* **[IIT JEE 1997]**

Sol. Given that

$$\frac{du}{dx} + p(x)\cdot u = f(x) \quad \text{and} \quad \frac{dv}{dx} + p(x)\cdot v = g(x)$$

Subtracting, we get

$$\frac{d(u-v)}{dx} + p(x)\cdot(u-v) = f(x) - g(x)$$

Multiplying by $e^{\int p(x)\,dx}$, we get

$$e^{\int p(x)\,dx}\cdot\frac{d(u-v)}{dx} + (u-v)\cdot p(x)\cdot e^{\int p(x)\,dx} = \{f(x)-g(x)\}\cdot e^{\int p(x)\,dx}$$

i.e. $$\frac{d}{dx}\{(u-v)\,e^{\int p\,dx}\} = \{f(x)-g(x)\}\,e^{\int p(x)\,dx}$$

Since, exponential function takes only positive values and

$f(x) > g(x)$ for all $x > x_1$, RHS is + ve; $x > x_1$

$$\therefore \quad \frac{d}{dx}\{(u-v)\cdot e^{\int p(x)\,dx}\} = 0$$

ie, $(u-v)\cdot e^{\int p(x)\,dx}$ is increasing function.

Hence, if $e^{\int p(x)\,dx} = \phi(x)$, then for $x > x_1$

We have, $\{u(x)-v(x)\}\,\phi(x) > \{u(x_1)-v(x_1)\}\,\phi(x_1)$

i.e. $$u(x)-v(x) > \frac{\{u(x_1)-v(x_1)\}\cdot\phi(x_1)}{\phi(x)} > 0 \quad [\because u(x_1) > v(x_1)]$$

Thus, $u(x) > v(x),\ \forall\ x > x_1$

i.e. $u(x) \neq v(x),\ \forall\ x > x_1$

Hence, no point (x, y) such that $x > x_1$ can satisfy the equations $y = u(x)$ and $y = v(x)$.

● **Ex. 28** *A normal is drawn at a point $P(x, y)$ of a curve. It meets the x-axis at Q. If PQ is of constant length k, then show that the differential equation describing such curves is, $y\frac{dy}{dx} = \pm\sqrt{k^2 - y^2}$ and the equation of such a curve passing through $(0, k)$.* **[IIT JEE 1994]**

Sol. Let $y = f(x)$ be the curve such that the normal at $P(x, y)$ to this curve meets x-axis at Q. Then,

PQ = length of the normal at P

$$= y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

But $PQ = k$

$$\therefore \quad y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = k$$

$$\Rightarrow \quad y^2 + y^2\left(\frac{dy}{dx}\right)^2 = k^2 \quad \text{or} \quad y\frac{dy}{dx} = \pm\sqrt{k^2 - y^2}$$

$$\Rightarrow \quad \frac{y\ dy}{\sqrt{k^2 - y^2}} = \pm\,dx$$

Integrating both the sides, we get

$-\sqrt{k^2 - y^2} = \pm\ x + C$, since it passes through $(0, k) \to C = 0$.

$$\therefore \quad -\sqrt{k^2 - y^2} = \pm\ x$$

or $$k^2 - y^2 = x^2$$

$\Rightarrow$ $x^2 + y^2 = k^2$, is required equation of the curve.

● **Ex. 29** *A curve passing through the point (1, 1) has the property that the perpendicular distance of the normal at any point P on the curve from the origin is equal to the distance of P from x-axis. Determine the equation of the curve.* **[IIT JEE 1999]**

Sol. Let $P(x, y)$ be any point on the curve $y = f(x)$. Then, the equation of the normal at P is,

$$Y - y = -\frac{1}{(dy/dx)}(X - x)$$

or $$X + Y\frac{dy}{dx} - \left(y\frac{dy}{dx} + x\right) = 0 \qquad \text{...(i)}$$

It is given that distance of Eq. (i) from origin = Distance from x-axis (i.e. y)

i.e. $$\left|\frac{0 - \left(y\frac{dy}{dx} + x\right)}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}\right| = y$$

$$\Rightarrow \quad \left(y\frac{dy}{dx} + x\right)^2 = y^2\left[1 + \left(\frac{dy}{dx}\right)^2\right]$$

$$\Rightarrow \quad x^2 + 2xy\frac{dy}{dx} = y^2$$

or $$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

which is homogeneous differential equation and we can solve by homogeneous or by total differential.

Here, using total differential,

$$2xy\,dy - y^2\,dx = -x^2\,dx$$

$$\Rightarrow \quad \frac{x\,d(y^2) - y^2\,dx}{x^2} = -\,dx$$

$$\Rightarrow \quad d\left(\frac{y^2}{x}\right) = -\,dx$$

Integrating both the sides, we get

$$\Rightarrow \quad \frac{y^2}{x} = -\,x + C \qquad \text{...(ii)}$$

It passes through $(1, 1) \Rightarrow C = 2$

$$\therefore \quad \frac{y^2}{x} = -\,x + 2 \ \text{ or } \ y^2 = -\,x^2 + 2x$$

$\Rightarrow$ $x^2 + y^2 - 2x = 0$, is required equation of curve.

● ***Ex. 30*** *A country has a food deficit of 10%. Its population grows continuously at a rate of 3% per year. Its annual food production every year is 4% more than that of the last year. Assuming that the average food requirement per person remains constant, prove that the country will become self-sufficient in food after n years, where n is the smallest integer bigger than or equal to* $\frac{\log_e 10 - \log_e 9}{(\log_e 1.04) - 0.03}$ ***[IIT JEE 2000]***

Sol. Let P_0 be the initial population, Q_0 be its initial food production.

Let P be the population of the country in year t and Q be its food production in year t.

$$\Rightarrow \quad \frac{dP}{dt} = \frac{3P}{100} \quad \text{or} \quad \frac{dP}{P} = \frac{3}{100}\,dt$$

Integrating, we get

$$\log P = \frac{3}{100}\,t + C$$

At $t = 0$, we have $P = P_0$

$$\Rightarrow \quad C = \log P_0$$

$$\Rightarrow \quad P = P_0\, e^{0.03\,t} \qquad \text{...(i)}$$

It is given that the annual food production every year is 4% more than that of last year.

$$\Rightarrow \quad Q = Q_0\left(1 + \frac{4}{100}\right)^t$$

Let the average consumption per person be k units.

$$\Rightarrow \quad Q_0 = kP_0\left(\frac{90}{100}\right) = 0.9\,kP_0$$

$$\therefore \quad Q = 0.9\,kP_0\,(1.04)^t \qquad \text{...(ii)}$$

This gives quantity of food available in year t. The population in year t is,

$$P = P_0\, e^{0.03\,t} \qquad \text{[from Eq. (i)]}$$

$\therefore$ Consumption in year, $t = kP_0 e^{0.03\,t}$ ...(iii)

The country will be self sufficient, if

$$Q \geq P$$

$$\Rightarrow \quad 0.9k\,P_0\,(1.04)^t \geq kP_0 e^{0.03\,t}$$

$$\Rightarrow \quad \frac{9}{10}\,(1.04)^t \geq e^{0.03\,t}$$

$$\Rightarrow \quad (1.04)^t\, e^{-0.03\,t} \geq \frac{10}{9}$$

$$\Rightarrow \quad t\log(1.04) - 0.03\,t \geq \log\left(\frac{10}{9}\right)$$

$$\Rightarrow \quad t\{\log(1.04) - 0.03\} \geq \log\left(\frac{10}{9}\right)$$

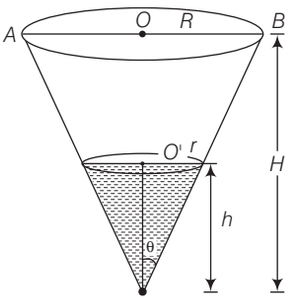
$$\Rightarrow \quad t \geq \frac{\log 10 - \log 9}{\log(1.04) - 0.03}$$

Thus, the least number of year in which country becomes self sufficient.

$$\Rightarrow \quad t = \frac{\log 10 - \log 9}{\log(1.04) - 0.03}$$

● ***Ex. 31*** *A right circular cone with radius R and height H contains a liquid which evaporates at a rate proportional to its surface area in contact with air (proportionality constant = k > 0), find the time after which the cone is empty.* ***[IIT JEE 2003]***

Sol. Let the semi-vertical angle of the cone be θ and let the height of the liquid at time 't' be 'h' from the vertex V and radius of the liquid cone be r. Let V be the volume at time t. Then,



$$V = \frac{1}{3}\,\pi r^2 h$$

$$\Rightarrow \quad V = \frac{1}{3}\,\pi r^3 \cot\theta \qquad \left(\because \tan\theta = \frac{r}{h}\right)$$

Let S be the surface area of the liquid in contact with air at time t.

Then, $\quad S = \pi r^2$

$$\Rightarrow \quad \frac{dV}{dt} \propto S$$

$$\Rightarrow \quad \frac{dV}{dt} = -kS,\ k \text{ is constant of proportionality.}$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{1}{3}\,\pi r^3 \cot\theta\right) = -k\pi r^2$$

$$\Rightarrow \quad \pi r^2 \frac{dr}{dt}\cot\theta = -k\pi r^2 \Rightarrow \cot\theta\,dr = -k\,dt$$

On integrating, we get $\cot\theta \int_R^0 dr = -k\int_0^T dt$

$\Rightarrow$ $R\cot\theta = +kT$, where T is required time.

$\Rightarrow$ $T = H/k$ (as $\tan\theta = R/H$)

● ***Ex. 32*** *Solve the equation*

$$x\int_0^x y(t)\,dt = (x+1)\int_0^x t\,y(t)\,dt,\ x > 0.$$

Sol. Differentiating the equation w.r.t. x, we get

$$xy(x) + 1\cdot\int_0^x y(t)\,dt = (x+1)\,xy(x) + 1\cdot\int_0^x t\,y(t)\,dt$$

i.e. $$\int_0^x y(t)\,dt = x^2\,y(x) + \int_0^x t\,y(t)\,dt$$

Again, differentiating w.r.t. x, we get

$$y(x) = x^2\,y'(x) + 2xy(x) + xy(x)$$

i.e. $$(1-3x)\,y(x) = \frac{x^2\,dy(x)}{dx}$$

i.e. $$\frac{(1-3x)\,dx}{x^2} = \frac{dy(x)}{y(x)}$$

Integrating, we get $y = \frac{C}{x^3}\,e^{-1/x}$

● **Ex. 33** *If (y_1, y_2) are two solutions of the differential equation* $\dfrac{dy}{dx} + P(x) \cdot y = Q(x)$

Then prove that $y = y_1 + C(y_1 - y_2)$ is the general solution of the equation where C is any constant. For what relation between the constant α, β will the linear combination $\alpha y_1 + \beta y_2$ also be a solution.

Sol. As y_1, y_2 are the solutions of the differential equation;

$$\frac{dy}{dx} + P(x) \cdot y = Q(x) \quad \ldots(i)$$

$$\therefore \quad \frac{dy_1}{dx} + P(x) \cdot y_1 = Q(x) \quad \ldots(ii)$$

and $$\frac{dy_2}{dx} + P(x) \cdot y_2 = Q(x) \quad \ldots(iii)$$

From Eqs. (i) and (ii), $\left(\dfrac{dy}{dx} - \dfrac{dy_1}{dx}\right) + P(x)(y - y_1) = 0$

$$\therefore \quad \frac{d}{dx}(y - y_1) + P(x) \cdot (y - y_1) = 0 \quad \ldots(iv)$$

From Eqs. (ii) and (iii), $\dfrac{d}{dx}(y_1 - y_2) + P(x)(y_1 - y_2) = 0$ ...(v)

From Eqs. (iv) and (v), $\dfrac{\frac{d}{dx}(y - y_1)}{\frac{d}{dx}(y_1 - y_2)} = \dfrac{y - y_1}{y_1 - y_2}$

$$\Rightarrow \quad \frac{\frac{d}{dx}(y - y_1)}{y - y_1} = \frac{\frac{d}{dx}(y_1 - y_2)}{y_1 - y_2}$$

Integrating both the sides, we get

$$\log(y - y_1) = \log(y_1 - y_2)$$

$$\therefore \quad y = y_1 + C(y_1 - y_2)$$

Now, $y = \alpha y_1 + \beta y_2$ will be a solutions, if

$$\frac{d}{dx}(\alpha y_1 + \beta y_2) + P(x)(\alpha y_1 + \beta y_2) = Q(x)$$

or $$\alpha\left(\frac{dy_1}{dx} + P(x)\, y_1\right) + \beta\left(\frac{dy_2}{dx} + P(x)\, y_2\right) = Q(x)$$

or $\alpha Q(x) + \beta Q(x) = Q(x)$ [using Eqs. (ii) and (iii)]

$\therefore \quad (\alpha + \beta) Q(x) = Q(x)$

Hence, $\alpha + \beta = 1$

● **Ex. 34** *Find a pair of curves such that*

(a) the tangents drawn at points with equal abscissae intersect on the y-axis.

(b) the normal drawn at points with equal abscissae intersect on x-axis.

(c) one curve passes through (1, 1) and other passes through (2, 3).

Sol. Let the curve be $y = f_1(x)$ and $y = f_2(x)$ equation of tangents with equal abscissa, x are

$$(y - f_1(x)) = f'_1(x)(X - x)$$

and $$Y - f_2(x) = f'_2(x)(X - x)$$

These tangent intersect at y-axis,

$$\Rightarrow \quad -x f'_1(x) + f_1(x) = -x f'_2(x) + f_2(x)$$

$$\Rightarrow \quad f_1(x) - f_2(x) = x(f'_1(x) - f'_2(x))$$

Integrating both the sides, we get

$$\Rightarrow \quad \ln|f_1(x) - f_2(x)| = \ln|x| + C$$

$$\Rightarrow \quad f_1(x) - f_2(x) = \pm C_1 x \quad \ldots(i)$$

Now, equations of normal with equal abscissa x, are

$$y - f_1(x) = -\frac{1}{f'_1(x)}(X - x)$$

and $$(y - f_2(x)) = -\frac{1}{f'_2(x)}(X - x)$$

As these normal intersect on the x-axis,

$$x + f_1(x) \cdot f'_1(x) = x + f_2(x) \cdot f'_2(x)$$

$$\Rightarrow \quad f_1(x) \cdot f_1'(x) = f_2(x) \cdot f_2'(x) \text{ Integrating}$$

$$\Rightarrow \quad f'^2_1(x) - f_2^2(x) = C_2$$

$$\Rightarrow \quad f_1(x) + f_2(x) = \frac{C_2}{f_1(x) - f_2(x)} = \pm\frac{C_2}{C_1 x} = \pm\frac{\lambda_2}{x}$$

[using Eq. (i)] ...(ii)

From Eqs. (i) and (ii), we get

$$2f_1(x) = \pm\left(\frac{\lambda_2}{x} + C_1 x\right), 2f_2(x) = \pm\left(\frac{\lambda_2}{x} - C_1 x\right)$$

We have, $f_1(1) = 1$ and $f_2(2) = 3$

$$\Rightarrow \quad f_1(x) = \frac{2}{x} - x \quad \text{and} \quad f_2(x) = \frac{2}{x} + x$$

● **Ex. 35** *Given two curves $y = f(x)$ passing through (0, 1) and $y = \int_{-\infty}^{x} f(t)\, dt$ passing through $(0, 1/n)$. The tangents drawn to both the curves at the points with equal abscissae intersect on the x-axis find the curve $y = f(x)$.*

Sol. Equation of the tangent to the curve; $y = f(x)$ is

$$(Y - y) = f'(x)(X - x)$$

Equation of tangent to the curve $g(x) = y_1 = \int_{-\infty}^{x} f(t)\, dt$ is

$$(Y - y_1) = g'(x)(X - x) = f(x)(X - x)$$

Given that tangent with equal abscissas intersect on the x-axis.

$$\Rightarrow \quad x - \frac{y}{f'(x)} = x - \frac{y_1}{f(x)}$$

$$\Rightarrow \quad \frac{f(x)}{f'(x)} = \frac{y_1}{f(x)} \qquad [\because y = f(x)]$$

$$\Rightarrow \quad \frac{f(x)}{y_1} = \frac{f'(x)}{f(x)} \Rightarrow \frac{g'(x)}{g(x)} = \frac{f'(x)}{f(x)}$$

$$\Rightarrow \quad \frac{g'(x)}{g(x)} = k \quad \Rightarrow \quad g(x) = Ce^{kx}$$

$$\Rightarrow \quad g'(x) = k\, Ce^{kx} \Rightarrow f(x) = k\, Ce^{kx}$$

$y = f(x)$ passes through $(0, 1) \Rightarrow kC = 1$

$y_1 = g(x)$ passes through

$(0, 1/n) \Rightarrow C = \dfrac{1}{n} \Rightarrow k = n$

$$\Rightarrow \quad f(x) = e^{nx}$$

● ***Ex. 36*** *A normal is drawn at a point $P(x, y)$ of a curve. It meets the x-axis and the y-axis in point A and B, respectively, such that $\frac{1}{OA}+\frac{1}{OB}=1$, where O is the origin, find the equation of such a curve passing through (5, 4).*

Sol. The equation of the normal at (x, y) is

$$(X-x)+(Y-y)\frac{dy}{dx}=0$$

$$\Rightarrow \quad \frac{X}{x+y\frac{dy}{dx}}+\frac{Y}{\frac{(x+y\,dy/dx)}{dy/dx}}=1$$

$$\Rightarrow \quad OA=x+y\frac{dy}{dx},\ OB=\frac{\left(x+y\frac{dy}{dx}\right)}{\frac{dy}{dx}}$$

Also, $\frac{1}{OA}+\frac{1}{OB}=1 \Rightarrow 1+\frac{dy}{dx}=x+y\frac{dy}{dx}$

$$\Rightarrow \quad (y-1)\frac{dy}{dx}+(x-1)=0$$

Integrating, we get

$$(y-1)^2+(x-1)^2=C$$

Since, the curve passes through (5, 4), $C = 25$.

Hence, the curve is $(x-1)^2+(y-1)^2=25$.

● ***Ex. 37*** *A line is drawn from a point $P(x, y)$ on curve $y = f(x)$, making an angle with the x-axis which is supplementary to the one made by the tangent to the curve at $P(x, y)$. The line meets the x-axis at A. Another line perpendicular to the first, is drawn from $P(x, y)$ meeting the y-axis at B. If OA = OB, where O is origin, find all curve which passes through (1, 1).*

Sol. The equation of the line through $P(x, y)$ making an angle with the x-axis which is supplementary to the angle made by the tangent at $P(x, y)$ is

$$Y-y=-\frac{dy}{dx}(X-x) \qquad ...(i)$$

where it meets the x-axis.

$$Y=0, \quad X=x+\frac{y}{dy/dx} \Rightarrow OA=x+y\frac{dx}{dy} \qquad ...(ii)$$

The line through $P(x, y)$ and perpendicular to Eq. (i) is

$$Y-y=\frac{dx}{dy}(X-x)$$

where it meets the y-axis.

$$X=0,\ Y=y-x\frac{dx}{dy} \Rightarrow OB=y-x\frac{dx}{dy} \qquad ...(iii)$$

Since, $OA = OB$

$$\Rightarrow \quad x+y\frac{dx}{dy}=y-x\frac{dx}{dy}$$

or $$(y-x)=(y+x)\frac{dx}{dy}$$

or $$\frac{dy}{dx}=\frac{y+x}{y-x},\ \text{put } y=vx$$

$$\Rightarrow \quad x\frac{dv}{dx}=\frac{1+2v-v^2}{v-1}$$

$$\Rightarrow \quad \frac{(1-v)\,dv}{1+2v-v^2}+\frac{dx}{x}=0$$

$$\Rightarrow \quad \log(1+2v-v^2)+\log x=C_1$$

$$\Rightarrow \quad x^2+2xy-y^2=C$$

where $C_1=\log\sqrt{C}$

Since, curve passes through $(1, 1) \rightarrow C = 2$

$\therefore$ Required curve, $x^2-y^2+2xy=2$

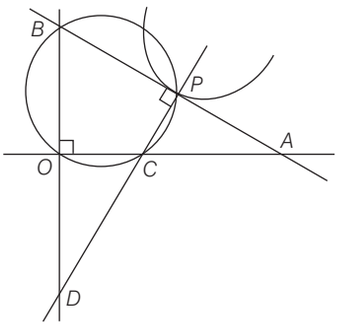
● ***Ex. 38*** *The tangent and a normal to a curve at any point P meet the x and y axes at A, B, C and D respectively. Find the equation of the curve passing through (1, 0) if the centre of circle through O, C, P and B lies on the line y = x (where O is origin).*

Sol. Let $P(x, y)$ be a point on the curve.

$$\Rightarrow \quad C\equiv\left(x+y\frac{dy}{dx},\ 0\right)$$

$$B\equiv\left(0,\ y-x\frac{dy}{dx}\right)$$

Circle passing through O, C, P and B has its centre at mid-point of BC.



Let the centre of the circle be (α, β).

$$\Rightarrow \quad 2\alpha=x+y\frac{dy}{dx}$$

and $$2\beta=y-x\frac{dy}{dx}$$

and since $\beta=\alpha,\ y-x\frac{dy}{dx}=x+y\frac{dy}{dx}$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{y-x}{y+x}$$

Let $y = vx \Rightarrow x\frac{dv}{dx}=-\frac{(1+v^2)}{1+v}$

$$\Rightarrow \quad \frac{1+v}{v^2+1}\,dv=-\frac{dx}{x}$$

Integrating both the sides, we get

$$\int \frac{1+v}{v^2+1}\,dv = -\int \frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{2}\int \frac{2v}{v^2+1}\,dv + \int \frac{dv}{v^2+1} = -\int \frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{2}\log|v^2+1| + \tan^{-1}|v| = -\log x + C$$

$$\Rightarrow \quad \log\{(\sqrt{v^2+1})\,x\} + \tan^{-1} v = C$$

$$\Rightarrow \quad \log\sqrt{x^2+y^2} + \tan^{-1}\frac{y}{x} = C$$

As $x = 1$ and $y = 0$,

$$\log 1 + \tan^{-1} 0 = C \Rightarrow C = 0$$

$\therefore$ Required curve, $(\log\sqrt{x^2+y^2}) + \tan^{-1}\left(\frac{y}{x}\right) = 0$

● ***Ex. 39*** *If $f(x)$ be a positive, continuous and differentiable on the interval (a, b). If $\lim_{x\to a^+} f(x) = 1$ and $\lim_{x\to b^-} f(x) = 3^{1/4}$. Also, $f'(x) \geq f^3(x) + \frac{1}{f(x)}$, then*

(a) $b - a \geq \pi/4$ (b) $b - a \leq \pi/4$

(c) $b - a \leq \pi/24$ (d) None of these

Sol. Since,

$$f'(x) \geq f^3(x) + \frac{1}{f(x)}$$

$$\Rightarrow \quad f'(x)\,f(x) \geq 1 + f^4(x)$$

$$\Rightarrow \quad \frac{f(x)\,f'(x)}{1+f^4(x)} \geq 1$$

On integrating w.r.t. 'x' from $x = a$ to $x = b$.

$$\frac{1}{2}(\tan^{-1}(f^2(x)))_a^b \geq (b-a)$$

or $$(b-a) \leq \frac{1}{2}\left\{\lim_{x\to b^-}(\tan^{-1}(f^2(x))) - \lim_{x\to a^+}(\tan^{-1}(f^2(x)))\right\}$$

or $$(b-a) \leq \pi/24$$

Hence, (c) is the correct answer.

Differential Equations Exercise 1 :
Single Option Correct Type Questions

1. If the differential equation of the family of curve given by $y = Ax + Be^{2x}$, where A and B are arbitrary constants, is of the form $(1-2x)\frac{d}{dx}\left(\frac{dy}{dx}+ly\right)+k\left(\frac{dy}{dx}+ly\right)=0$, then the ordered pair (k, l) is

(a) $(2, -2)$ (b) $(-2, 2)$
(c) $(2, 2)$ (d) $(-2, -2)$

2. A curve passes through the point $\left(1, \frac{\pi}{4}\right)$ and its slope at any point is given by $\frac{y}{x} - \cos^2\left(\frac{y}{x}\right)$. Then, the curve has the equation

(a) $y = x\tan^{-1}\left(\ln\frac{e}{x}\right)$ (b) $y = x\tan^{-1}(\ln 2)$
(c) $y = \frac{1}{x}\tan^{-1}\left(\ln\frac{e}{x}\right)$ (d) None of these

3. The x-intercept of the tangent to a curve is equal to the ordinate of the point of contact. The equation of the curve through the point (1, 1) is

(a) $ye^{\frac{x}{y}} = e$ (b) $xe^{\frac{x}{y}} = e$
(c) $xe^{\frac{y}{x}} = e$ (d) $ye^{\frac{y}{x}} = e$

4. A function $y = f(x)$ satisfies the condition $f'(x)\sin x + f(x)\cos x = 1$, $f(x)$ being bounded when $x \to 0$. If $I = \int_0^{\pi/2} f(x)\,dx$, then

(a) $\frac{\pi}{2} < I < \frac{\pi^2}{4}$ (b) $\frac{\pi}{4} < I < \frac{\pi^2}{2}$
(c) $1 < I < \frac{\pi}{2}$ (d) $0 < I < 1$

5. A curve is such that the area of the region bounded by the coordinate axes, the curve and the ordinate of any point on it is equal to the cube of that ordinate. The curve represents

(a) a pair of straight lines (b) a circle
(c) a parabola (d) an ellipse

6. The value of the constant 'm' and 'c' for which $y = mx + c$ is a solution of the differential equation $D^2y - 3Dy - 4y = -4x$.

(a) is $m = -1; c = 3/4$ (b) is $m = 1; c = -3/4$
(c) no such real m, c (d) is $m = 1; c = 3/4$

7. The real value of m for which the substitution, $y = u^m$ will transform the differential equation, $2x^4y\frac{dy}{dx} + y^4 = 4x^6$ into a homogeneous equation is

(a) $m = 0$ (b) $m = 1$
(c) $m = 3/2$ (d) No value of m

8. The solution of the differential equation,

$$x^2\frac{dy}{dx}\cdot\cos\frac{1}{x} - y\sin\frac{1}{x} = -1,$$

where $y \to -1$ as $x \to \infty$ is

(a) $y = \sin\frac{1}{x} - \cos\frac{1}{x}$ (b) $y = \frac{x+1}{x\sin\frac{1}{x}}$
(c) $y = \sin\frac{1}{x} + \cos\frac{1}{x}$ (d) $y = \frac{x+1}{x\cos\frac{1}{x}}$

9. A wet porous substance in the open air loses its moisture at a rate proportional to the moisture content. If a sheet hung in the wind loses half its moisture during the first hour, then the time when it would have lost 99.9% of its moisture is (weather conditions remaining same)

(a) more than 100 h
(b) more than 10 h
(c) approximately 10 h
(d) approximately 9 h

10. A curve C passes through origin and has the property that at each point (x, y) on it, the normal line at that point passes through (1, 0). The equation of a common tangent to the curve C and the parabola $y^2 = 4x$ is

(a) $x = 0$ (b) $y = 0$
(c) $y = x + 1$ (d) $x + y + 1 = 0$

11. A function $y = f(x)$ satisfies

$$(x+1)\cdot f'(x) - 2(x^2+x)\,f(x) = \frac{e^{x^2}}{(x+1)},\ \forall\ x > -1.$$

If $f(0) = 5$, then $f(x)$ is

(a) $\left(\frac{3x+5}{x+1}\right)\cdot e^{x^2}$ (b) $\left(\frac{6x+5}{x+1}\right)\cdot e^{x^2}$
(c) $\left(\frac{6x+5}{(x+1)^2}\right)\cdot e^{x^2}$ (d) $\left(\frac{5-6x}{x+1}\right)\cdot e^{x^2}$

12. The curve, with the property that the projection of the ordinate on the normal is constant and has a length equal to 'a', is

(a) $x - a\ln\left(\sqrt{y^2 - a^2} + y\right) = C$

(b) $x + \sqrt{a^2 - y^2} = C$

(c) $(y - a)^2 = Cx$

(d) $ay = \tan^{-1}(x + C)$

13. The differential equation corresponding to the family of curves $y = e^x(ax + b)$ is

(a) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - y = 0$ (b) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$

(c) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$ (d) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - y = 0$

14. The equation to the orthogonal trajectories of the system of parabolas $y = ax^2$ is

(a) $\frac{x^2}{2} + y^2 = C$ (b) $x^2 + \frac{y^2}{2} = C$

(c) $\frac{x^2}{2} - y^2 = C$ (d) $x^2 - \frac{y^2}{2} = C$

15. A function $f(x)$ satisfying $\int_0^1 f(tx)\,dt = n\,f(x)$, where $x > 0$, is

(a) $f(x) = C \cdot x^{\frac{1-n}{n}}$ (b) $f(x) = C \cdot x^{\frac{n}{n-1}}$

(c) $f(x) = C \cdot x^{\frac{1}{n}}$ (d) $f(x) = C \cdot x^{(1-n)}$

16. The substitution $y = z^\alpha$ transforms the differential equation $(x^2y^2 - 1)\,dy + 2xy^3\,dx = 0$ into a homogeneous differential equation for

(a) $\alpha = -1$ (b) 0

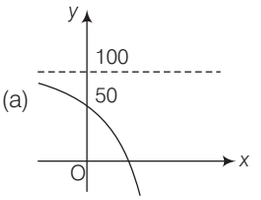
(c) $\alpha = 1$ (d) No value of α

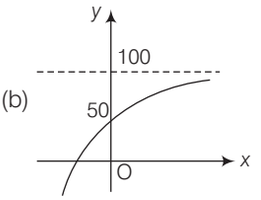
17. A curve passing through (2, 3) and satisfying the differential equation $\int_0^x ty(t)\,dt = x^2y(x), (x > 0)$ is

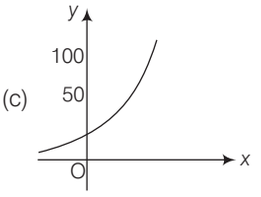
(a) $x^2 + y^2 = 13$ (b) $y^2 = \frac{9}{2}x$

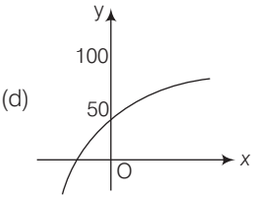
(c) $\frac{x^2}{8} + \frac{y^2}{18} = 1$ (d) $xy = 6$

18. Which one of the following curves represents the solution of the initial value problem $Dy = 100 - y$, where $y(0) = 50$?

(a)


(b)


(c)


(d)


Differential Equations Exercise 2 : More than One Option Correct Type Questions

19. The differential equation $x\frac{dy}{dx} + \frac{3}{\frac{dy}{dx}} = y^2$

(a) is of order 1 (b) is of degree 2

(c) is linear (d) is non-linear

20. The function $f(x)$ satisfying the equation $f^2(x) + 4f'(x)\cdot f(x) + [f'(x)]^2 = 0$.

(a) $f(x) = C \cdot e^{(2-\sqrt{3})x}$

(b) $f(x) = C \cdot e^{(2+\sqrt{3})x}$

(c) $f(x) = C \cdot e^{(\sqrt{3}-2)x}$

(d) $f(x) = C \cdot e^{-(2+\sqrt{3})x}$

where C is an arbitrary constant.

21. Which of the following pair(s) is/are orthogonal?

(a) $16x^2 + y^2 = C$ and $y^{16} = kx$

(b) $y = x + Ce^{-x}$ and $x + 2 = y + ke^{-y}$

(c) $y = Cx^2$ and $x^2 + 2y^2 = k$

(d) $x^2 - y^2 = C$ and $xy = k$

22. Family of curves whose tangent at a point with its intersection with the curve $xy = c^2$ form an angle of $\frac{\pi}{4}$ is

(a) $y^2 - 2xy - x^2 = k$

(b) $y^2 + 2xy - x^2 = k$

(c) $y = x - 2c\tan^{-1}\left(\frac{x}{c}\right) + k$

(d) $y = c\ln\left|\frac{c + x}{c - x}\right| - x + k$

23. The general solution of the differential equation, $x\left(\frac{dy}{dx}\right) = y \cdot \log\left(\frac{y}{x}\right)$ is

(a) $y = xe^{1-Cx}$

(b) $y = xe^{1+Cx}$

(c) $y = ex \cdot e^{Cx}$

(d) $y = xe^{Cx}$

where C is an arbitrary constant.

24. Which of the following equation(s) is/are linear?

(a) $\frac{dy}{dx} + \frac{y}{x} = \ln x$

(b) $y\left(\frac{dy}{dx}\right) + 4x = 0$

(c) $dx + dy = 0$

(d) $\frac{d^2y}{dx^2} = \cos x$

25. The equation of the curve passing through (3, 4) and satisfying the differential equation,

$$y\left(\frac{dy}{dx}\right)^2 + (x - y)\frac{dy}{dx} - x = 0$$

can be

(a) $x - y + 1 = 0$

(b) $x^2 + y^2 = 25$

(c) $x^2 + y^2 - 5x - 10 = 0$

(d) $x + y - 7 = 0$

26. Identify the statement(s) which is/are true?

(a) $f(x, y) = e^{y/x} + \tan\frac{y}{x}$ is homogeneous of degree zero.

(b) $x \cdot \log\frac{y}{x}\,dx + \frac{y^2}{x}\sin^{-1}\frac{y}{x}\,dy = 0$ is homogeneous differential equation.

(c) $f(x, y) = x^2 + \sin x \cdot \cos y$ is not homogeneous.

(d) $(x^2 + y^2)\,dx - (xy^2 - y^3)\,dy = 0$ is a homogeneous differential equation.

27. The graph of the function $y = f(x)$ passing through the point $(0, -1)$ and satisfying the differential equation $\frac{dy}{dx} + y\cos x = \cos x$ is such that

(a) it is a constant function

(b) it is periodic

(c) it is neither an even nor an odd function

(d) it is continuous and differentiable for all x

28. A function $y = f(x)$ satisfying the differential equation

$$\frac{dy}{dx} \cdot \sin x - y\cos x + \frac{\sin^2 x}{x^2} = 0$$

is such that, $y \to 0$ as $x \to \infty$, then the statement which is correct?

(a) $\lim\limits_{x \to 0} f(x) = 1$ (b) $\int_0^{\pi/2} f(x)\,dx$ is less than $\frac{\pi}{2}$

(c) $\int_0^{\pi/2} f(x)\,dx$ is greater than unity

(d) $f(x)$ is an odd function

29. Identify the statement(s) which is/are true?

(a) The order of differential equation $\sqrt{1 + \frac{d^2y}{dx^2}} = x$ is 1.

(b) Solution of the differential equation $xdy - ydx = \sqrt{x^2 + y^2}\,dx$ is $y + \sqrt{x^2 + y^2} = Cx^2$.

(c) $\frac{d^2y}{dx^2} = 2\left(\frac{dy}{dx} - y\right)$ is differential equation of family of curves $y = e^x(A\cos x + B\sin x)$.

(d) The solution of differential equation $(1 + y^2) + (x - 2e^{\tan^{-1} y})\frac{dy}{dx} = 0$ is $xe^{\tan^{-1} y} = e^{2\tan^{-1} y} + k$.

30. Let $y = (A + Bx)e^{3x}$ is a solution of the differential equation $\frac{d^2y}{dx^2} + m\frac{dy}{dx} + ny = 0$, $m, n \in I$, then

(a) $m = -6$ (b) $n = -6$

(c) $m = 9$ (d) $n = 9$

Differential Equations Exercise 3 :
Statement I and II Type Questions

Directions
(Q. Nos. 31 to 40)

For the following questions, choose the correct answers from the codes (*a*), (*b*), (*c*) *and* (*d*) *defined as follows* :

(a) Statement I is true, Statement II is true and Statement II is the correct explanation for Statement I.
(b) Statement I is true, Statement II is true and Statement II is not the correct explanation for Statement I.
(c) Statement I is true, Statement II is false.
(d) Statement I is false, Statement II is true.

31. A curve C has the property that its initial ordinate of any tangent drawn is less than the abscissa of the point of tangency by unity.

Statement I Differential equation satisfying the curve is linear.

Statement II Degree of differential equation is one.

32. **Statement I** Differential equation corresponding to all lines, $ax + by + c = 0$ has the order 3.

Statement II General solution of a differential equation of nth order contains n independent arbitrary constants.

33. **Statement I** Integral curves denoted by the first order linear differential equation $\frac{dy}{dx} - \frac{1}{x}y = -x$ are family of parabolas passing through the origin.

Statement II Every differential equation geometrically represents a family of curve having some common property.

34. **Statement I** The solution of $(y\,dx - x\,dy)\cot\left(\frac{x}{y}\right) = ny^2\,dx$ is $\sin\left(\frac{x}{y}\right) = Ce^{nx}$

Statement II Such type of differential equations can only be solved by the substitution $x = vy$.

35. **Statement I** The order of the differential equation whose general solution is $y = c_1\cos 2x + c_2\sin^2 x + c_4e^{2x} + c_5e^{2x + c_6}$ is 3.

Statement II Total number of arbitrary parameters in the given general solution in the Statement I is 6.

36. Consider differential equation $(x^2 + 1)\cdot\frac{d^2y}{dx^2} = 2x\cdot\frac{dy}{dx}$

Statement I For any member of this family $y \to \infty$ as $x \to \infty$.

Statement II Any solution of this differential equation is a polynomial of odd degree with positive coefficient of maximum power.

37. **Statement I** Order of differential equation of family of parallel whose axis is parallel to Y-axis and latusrectum is fixed is 2.

Statement II Order of first equation is same as actual number of arbitrary constant present in differential equation.

38. **Statement I** The differential equation of all non-vertical lines in a plane is $\frac{d^2x}{dy^2} = 0$.

Statement II The general equation of all non-vertical lines in a plane is $ax + by = 1$, where $b \neq 0$.

39. **Statement I** The order of differential equation of all conics whose centre lies at origin is, 2.

Statement II The order of differential equation is same as number of arbitrary unknowns in the given curve.

40. **Statement I** $y = a\sin x + b\cos x$ is general solution of $y'' + y = 0$.

Statement II $y = a\sin x + b\cos x$ is a trigonometric function.

Differential Equations Exercise 4 : Passage Based Questions

Passage I

(Q. Nos. 41 to 43)

Let $y = f(x)$ *satisfies the equation*

$f(x) = (e^{-x} + e^{x})\cos x - 2x - \int_0^x (x - t)\, f'(t)\, dt.$

41. y satisfies the differential equation

(a) $\frac{dy}{dx} + y = e^{x}(\cos x - \sin x) - e^{-x}(\cos x + \sin x)$

(b) $\frac{dy}{dx} - y = e^{x}(\cos x - \sin x) + e^{-x}(\cos x + \sin x)$

(c) $\frac{dy}{dx} + y = e^{x}(\cos x + \sin x) - e^{-x}(\cos x - \sin x)$

(d) $\frac{dy}{dx} - y = e^{x}(\cos x - \sin x) + e^{-x}(\cos x - \sin x)$

42. The value of $f'(0) + f''(0)$ equals to

(a) -1 (b) 2
(c) 1 (d) 0

43. $f(x)$ as a function of x equals to

(a) $e^{-x}(\cos x - \sin x) + \frac{e^{x}}{5}(3\cos x + \sin x) + \frac{2}{5}e^{-x}$

(b) $e^{-x}(\cos x + \sin x) + \frac{e^{x}}{5}(3\cos x - \sin x) - \frac{2}{5}e^{-x}$

(c) $e^{-x}(\cos x - \sin x) + \frac{e^{x}}{5}(3\cos x - \sin x) + \frac{2}{5}e^{-x}$

(d) $e^{-x}(\cos x + \sin x) + \frac{e^{x}}{5}(3\cos x - \sin x) - \frac{2}{5}e^{-x}$

Passage II

(Q. Nos. 44 to 46)

For certain curves $y = f(x)$ *satisfying* $\frac{d^2y}{dx^2} = 6x - 4$, $f(x)$ *has local minimum value 5 when* $x = 1$.

44. Number of critical point for $y = f(x)$ for $x \in [0, 2]$

(a) 0 (b) 1
(c) 2 (d) 3

45. Global minimum value of $y = f(x)$ for $x \in [0, 2]$ is

(a) 5 (b) 7
(c) 8 (d) 9

46. Global maximum value of $y = f(x)$ for $x \in [0, 2]$ is

(a) 5 (b) 7
(c) 8 (d) 9

Passage III

(Q. Nos. 47 to 49)

If any differential equation in the form

$f(f_1(x, y))\, d(f_1(x, y)) + \phi(f_2(x, y))\, d(f_2(x, y)) + \ldots = 0$

then each term can be integrated separately.

For example,

$$\int \sin xy\, d(xy) + \int \left(\frac{x}{y}\right) d\left(\frac{x}{y}\right) = -\cos xy + \frac{1}{2}\left(\frac{x}{y}\right)^2 + C$$

47. The solution of the differential equation

$x dy - y\, dx = \sqrt{x^2 - y^2}\, dx$ is

(a) $Cx = e^{\sin^{-1}\frac{y}{x}}$ (b) $xe^{\sin^{-1}\frac{y}{x}} = C$

(c) $x + e^{\sin^{-1}\frac{y}{x}} = C$ (d) None of these

48. The solution of the differential equation

$(xy^4 + y)\, dx - x dy = 0$ is

(a) $\frac{x^3}{4} + \frac{1}{2}\left(\frac{x}{y}\right)^2 = C$ (b) $\frac{x^4}{4} + \frac{1}{3}\left(\frac{x}{y}\right)^3 = C$

(c) $\frac{x^4}{4} - \frac{1}{2}\left(\frac{x}{y}\right)^3 = C$ (d) $\frac{x^3}{4} - \frac{1}{2}\left(\frac{x}{y}\right)^2 = C$

49. Solution of differential equation

$(2x\cos y + y^2\cos x)\, dx + (2y\sin x - x^2\sin y)\, dy = 0$ is

(a) $x^2\cos y + y^2\sin x = C$

(b) $x\cos y - y\sin x = C$

(c) $x^2\cos^2 y + y^2\sin^2 x = C$

(d) None of the above

Passage IV

(Q. Nos. 50 to 52)

Differential equation $\frac{dy}{dx} = f(x)\, g(x)$ *can be solved by separating variable* $\frac{dy}{g(y)} = f(x)\, dx$.

50. The equation of the curve to the point (1, 0) which satisfies the differential equation $(1 + y^2)\, dx - xy\, dy = 0$ is

(a) $x^2 + y^2 = 1$ (b) $x^2 - y^2 = 1$
(c) $x^2 + y^2 = 2$ (d) $x^2 - y^2 = 2$

51. Solution of the differential equation $\frac{dy}{dx}+\frac{1+y^2}{\sqrt{1-x^2}}=0$ is

(a) $\tan^{-1} y+\sin^{-1} x=C$

(b) $\tan^{-1} x+\sin^{-1} y=C$

(c) $\tan^{-1} y\cdot\sin^{-1} x=C$

(d) $\tan^{-1} y-\sin^{-1} x=C$

52. If $\frac{dy}{dx}=1+x+y+xy$ and $y(-1)=0$, then y is equal to

(a) $e^{\frac{(1-x)^2}{2}}$ (b) $e^{\frac{(1+x)^2}{2}}-1$

(c) $\ln(1+x)-1$ (d) $1+x$

Passage V

(Q. Nos. 53 to 55)

Let C be the set of curves having the property that the point of intersection of tangent with y-axis is equidistant from the point of tangency and origin (0, 0)

53. If $C_1, C_2 \in C$

C_1: Curve is passing through (1, 0)

C_2: Curve is passing through $(-1,0)$

The number of common tangents for C_1 and C_2 is

(a) 1 (b) 2

(c) 3 (d) None of these

54. If $C_3 \in C$

C_3: is passing through (2, 4). If $\frac{x}{a}+\frac{y}{b}=1$. is tangent to C_3, then

(a) $25a+10b^2-ab^2=0$ (b) $25a+10b-13ab=0$

(c) $13a+25b-16ab=0$ (d) $29a+b+13ab=0$

55. If common tangents of C_1 and C_2 form an equilateral triangle, where $C_1, C_2 \in C$ and C_1: Curve passes through (2, 0), then C_2 may passes through

(a) $(-1/3, 1/3)$ (b) $(-1/3, 1)$

(c) $(-2/3, 4)$ (d) $(-2/3, 2)$

Differential Equations Exercise 5 : Matching Type Questions

56. Match the following :

	Column I		Column II
(A)	$\frac{xdx+ydy}{xdy-ydx}=\sqrt{\frac{a^2-x^2-y^2}{x^2+y^2}}$	(p)	$y=\frac{C}{x^3}e^{-1/x}$
(B)	Solution of $\cos^2 x\frac{dy}{dx}-\tan 2x\cdot y=\cos^4 x$, where $\lvert x\rvert<\frac{\pi}{4}$ and $y\left(\frac{\pi}{6}\right)=\frac{3\sqrt{3}}{8}$	(q)	$\sqrt{x^2+y^2}=a\sin\left\{C+\tan^{-1}\left(\frac{y}{x}\right)\right\}$
(C)	The equation of all possible curves that will cut each member of the family of circles $x^2+y^2-2cx=0$ at right angle	(r)	$x^2+y^2+Cy=0$
(D)	Solution of the equation $x\int_0^x y(t)\,dt=(x+1)\int_0^x ty(t)\,dt, x>0$ is	(s)	$y=\frac{\sin 2x}{2(1-\tan^2 x)}$

57. Match the following :

	Column I		Column II
(A)	Circular plate is expanded by heat from radius 5 cm to 5.06 cm. Approximate increase in area is	(p)	4
(B)	Side of cube increasing by 1%, then percentage increase in volume is	(q)	0.6π
(C)	If the rate of decrease of $\frac{x^2}{2}-2x+5$ is twice the rate of decrease of x, then x is equal to	(r)	3
(D)	Rate of increase in area of equilateral triangle of side 15 cm, when each side is increasing at the rate of 0.1 cm/s; is	(s)	$\frac{3\sqrt{3}}{4}$

58. Match the following :

	Column I		Column II
(A)	The differential equation of the family of curves $y = e^x (A\cos x + B\sin x)$, where A, B are arbitrary constants, has the degree n and order m. Then, the values of n and m are, respectively	(p)	2, 1
(B)	The degree and order of the differential equation of the family of all parabolas whose axis is the x-axis, are respectively	(q)	1, 1
(C)	The order and degree of the differential equations of the family of circles touching the x-axis at the origin, are respectively	(r)	2, 2
(D)	The degree and order of the differential equation of the family of ellipse having the same foci, are respectively.	(s)	1, 2

Differential Equations Exercise 6 :
Single Integer Answer Type Questions

59. Find the constant of integration by the general solution of the differential equation $(2x^2y - 2y^4)\,dx + (2x^3 + 3xy^3)\,dy = 0$ if curve passes through (1, 1).

60. A tank initially contains 50 gallons of fresh water. Brine contains 2 pounds per gallon of salt, flows into the tank at the rate of 2 gallons per minutes and the mixture kept uniform by stirring, runs out at the same rate. If it will take for the quantity of salt in the tank to increase from 40 to 80 pounds (in seconds) is 206λ, then find λ.

61. If $f : R - \{-1\} \to$ and f is differentiable function which satisfies :
$f\{x + f(y) + xf(y)) = y + f(x) + yf(x) \;\forall\; x, y \in R - \{-1\}$,
then find the value of $2010\,[1 + f(2009)]$.

62. If $\phi(x)$ is a differential real-valued function satisfying $\phi'(x) + 2\phi(x) \le 1$, then the value of $2\phi(x)$ is always less than or equal to ..... .

63. The degree of the differential equation satisfied by the curves $\sqrt{1+x} - a\sqrt{1+y} = 1$, is ..... .

64. Let $f(x)$ be a twice differentiable bounded function satisfayi $2f^5(x)\cdot f'(x) + 2(f'(x))^3 \cdot f^5(x) = f''(x)$. If $f(x)$ is bounded in between $y = k$, and $y = k_2$, Then the number of integers between k_1 and k_2 is/are (where $f(0) = f'(0) = 0$).

65. Let $y(x)$ be a function satisfying $d^2y/dx^2 - dy/dx + e^{2x} = 0, y(0) = 2$ and $y'(0) = 1$. If maximum value of $y(x)$ is $y(\alpha)$, Then Integral part of (2α) is ..... .

Differential Equations Exercise 7 :
Subjective Type Questions

66. Find the time required for a cylindrical tank of radius r and height H to empty through a round hole of area 'a' at the bottom. The flow through a hole is according to the law $U(t) = u\sqrt{2gh(t)}$ where $v(t)$ and $h(t)$ are respectively the velocity of flow through the hole and the height of the water level above the hole at time t and g is the acceleration due to gravity.

67. The hemispherical tank of radius 2 m is initially full of water and has an outlet of 12 cm^2 cross-sectional area at the bottom. The outlet is opened at some instant. The flow through the outlet is according to the law $v(t) = 0\cdot 6\sqrt{2gh(t)}$, where $v(t)$ and $h(t)$ are respectively velocity of the flow through the outlet and the height of water level above the outlet at time t and g is the acceleration due to gravity. Find the time it takes to empty the tank.

68. Let $f : R^+ \to R$ satisfies the functional equation $f(xy) = e^{xy-x-y}(e^y f(x) + e^x f(y)) \forall x, y \in R^+$. If $f'(1) = e$, determine $f(x)$.

69. Let $y = f(x)$ be curve passing through $(1, \sqrt{3})$ such that tangent at any point P on the curve lying in the first quadrant has positive slope and the tangent and the normal at the point P cut the x-axis at A and B respectively so that the mid-point of AB is origin. Find the differential equation of the curve and hence determine $f(x)$.

70. If y_1 and y_2 are the solution of differential equation $dy/dx + Py = Q$, where P and Q are function of x alone and $y_2 = y_1 z$, then prove that $z = 1 + c \cdot e^{-\int \frac{Q}{y_1} dx}$, where c is an arbitrary constant.

71. Let $y = f(x)$ be a differentiable function $\forall x \in R$ and satisfies :
$f(x) = x + \int_0^1 x^2 z f(z)\, dz + \int_0^1 x z^2 f(z)\, dz.$
Determine the function.

72. If $f : R - \{-1\} \to R$ and f is differentiable function satisfies :
$f((x) + f(y) + x f(y)) = y + f(x) + yf(x) \forall x$, $y \in R - \{-1\}$ Find $f(x)$.

Differential Equations Exercise 8 : Questions Asked in Previous 10 Years' Exams

(i) JEE Advanced & IIT-JEE

73. A solution curve of the differential equation $(x^2 + xy + 4x + 2y + 4)\frac{dy}{dx} - y^2 = 0, x > 0$, passes through the point $(1, 3)$. Then, the solution curve **[More than One Correct 2016]**

(a) intersects $y = x + 2$ exactly at one point
(b) intersects $y = x + 2$ exactly at two points
(c) intersects $y = (x + 2)^2$
(d) does not intersect $y = (x + 3)^2$

74. Let $f : (0, \infty) \to R$ be a differentiable function such that $f'(x) = 2 - \frac{f(x)}{x}$ for all $x \in (0, \infty)$ and $f(1) \neq 1$. Then **[More than One Correct 2016]**

(a) $\lim_{x \to 0+} f'\left(\frac{1}{x}\right) = 1$
(b) $\lim_{x \to 0+} x f\left(\frac{1}{x}\right) = 2$
(c) $\lim_{x \to 0+} x^2 f'(x) = 0$
(d) $|f(x)| \leq 2$ for all $x \in (0, 2)$

75. Let $y(x)$ be a solution of the differential equation $(1 + e^x)y' + ye^x = 1$. If $y(0) = 2$, then which of the following statement(s) is/are true? **[More than One Correct 2015]**

(a) $y(-4) = 0$
(b) $y(-2) = 0$
(c) $y(x)$ has a critical point in the interval $(-1, 0)$
(d) $y(x)$ has no critical point in the interval $(-1, 0)$

76. Consider the family of all circles whose centres lie on the straight line $y = x$. If this family of circles is represented by the differential equation $Py'' + Qy' + 1 = 0$, where P, Q are the functions of x, y and y' (here, $y' = \frac{dy}{dx}$, $y'' = \frac{d^2y}{dx^2}$), then which of the following statement(s) is/are true? **[More than One correct 2015]**

(a) $P = y + x$
(b) $P = y - x$
(c) $P + Q = 1 - x + y + y' + (y')^2$
(d) $P - Q = x + y - y' - (y')^2$

77. The function $y = f(x)$ is the solution of the differential equation $\frac{dy}{dx} + \frac{xy}{x^2 - 1} = \frac{x^4 + 2x}{\sqrt{1 - x^2}}$ in $(-1, 1)$ satisfying $f(0) = 0$. Then, $\int_{-\frac{\sqrt{3}}{2}}^{\sqrt{3}/2} f(x)\, dx$ is **[Only One correct 2014]**

(a) $\frac{\pi}{3} - \frac{\sqrt{3}}{2}$ (b) $\frac{\pi}{3} - \frac{\sqrt{3}}{4}$ (c) $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$ (d) $\frac{\pi}{6} - \frac{\sqrt{3}}{2}$

78. Let $f : [1/2, 1] \to R$ (the set of all real numbers) be a positive, non-constant and differentiable function such that $f'(x) < 2f(x)$ and $f(1/2) = 1$. Then, the value of $\int_{1/2}^{1} f(x)\, dx$ lies in the interval **[Only One Correct 2013]**

(a) $(2e - 1, 2e)$ (b) $(e - 1, 2e - 1)$
(c) $\left(\frac{e-1}{2}, e - 1\right)$ (d) $\left(0, \frac{e-1}{2}\right)$

79. A curve passes through the point $\left(1, \frac{\pi}{6}\right)$. Let the slope of the curve at each point (x, y) be $\frac{y}{x} + \sec\left(\frac{y}{x}\right)$, $x > 0$. Then, the equation of the curve is **[Only One Correct 2013]**

(a) $\sin\left(\frac{y}{x}\right) = \log x + \frac{1}{2}$ (b) $\operatorname{cosec}\left(\frac{y}{x}\right) = \log x + 2$

(c) $\sec\left(\frac{2y}{x}\right) = \log x + 2$ (d) $\cos\left(\frac{2y}{x}\right) = \log x + \frac{1}{2}$

■ **Directions** (Q. Nos. 80 to 83) Let $f : [0, 1] \to R$ (the set of all real numbers) be a function. Suppose the function f is twice differentiable, $f(0) = f(1) = 0$ and satisfies

$$f''(x) - 2f'(x) + f(x) \geq e^x, \; x \in [0, 1]$$

[Passage Based Questions 2013]

80. If the function $e^{-x} f(x)$ assumes its minimum in interval $[0, 1]$ at $x = 1/4$, then which of the following is true?

(a) $f'(x) < f(x), \frac{1}{4} < x < \frac{3}{4}$ (b) $f'(x) > f(x), 0 < x < \frac{1}{4}$

(c) $f'(x) < f(x), 0 < x < \frac{1}{4}$ (d) $f'(x) < f(x), \frac{3}{4} < x < 1$

81. Which of the following is true?

(a) $0 < f(x) < \infty$ (b) $-\frac{1}{2} < f(x) < \frac{1}{2}$

(c) $-\frac{1}{4} < f(x) < 1$ (d) $-\infty < f(x) < 0$

82. Which of the following is true?

(a) g is increasing on $(1, \infty)$

(b) g is decreasing on $(1, \infty)$

(c) g is increasing on $(1, 2)$ and decreasing on $(2, \infty)$

(d) g is decreasing on $(1, 2)$ and increasing on $(2, \infty)$

83. Consider the statements.

I. There exists some $x \in R$ such that, $f(x) + 2x = 2(1 + x^2)$

II. There exists some $x \in R$ such that, $2f(x) + 1 = 2x(1 + x)$

(a) Both I and II are true (b) I is true and II is false

(c) I is false and II is true (d) Both I and II are false

84. If $y(x)$ satisfies the differential equation $y' - y \tan x = 2x \sec x$ and $y(0)$, then **[More than One Correct 2016]**

(a) $y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8\sqrt{2}}$ (b) $y'\left(\frac{\pi}{4}\right) = \frac{\pi^2}{18}$

(c) $y\left(\frac{\pi}{3}\right) = \frac{\pi^2}{9}$ (d) $y'\left(\frac{\pi}{3}\right) = \frac{4\pi}{3} + \frac{2\pi^2}{3\sqrt{3}}$

85. Let $y'(x) + y(x)g'(x) = g(x)g'(x)$, $y(0) = 0$, $x \in R$, where $f'(x)$ denotes $\frac{d f(x)}{dx}$ and $g(x)$ is a given non-constant differentiable function on R with $g(0) = g(2) = 0$. Then, the value of $y(2)$ is ..... **[Integer Type 2011]**

86. Let $f : R \to R$ be a continuous function, which satisfies $f(x) = \int_0^x f(t)\, dt$. Then, the value of $f(\ln 5)$ is ... . **[Integer Type 2009]**

■ **Direction** For the following question, choose the correct answer from the codes (a), (b), (c) and (d) defined as follows

(a) Statement I is true, Statement II is also true; Statement II is the correct explanation of Statement I.

(b) Statement I is true, Statement II is also true; Statement II is not the correct explanation of Statement I.

(c) Statement I is true; Statement II is false.

(d) Statement I is false; Statement II is true.

87. Let a solution $y = y(x)$ of the differential equation $x\sqrt{x^2 - 1}\, dy - y\sqrt{y^2 - 1}\, dx = 0$ satisfy $y(2) = \frac{2}{\sqrt{3}}$

Statement I $y(x) = \sec\left(\sec^{-1} x - \frac{\pi}{6}\right)$ and

Statement II $y(x)$ is given by $\frac{1}{y} = \frac{2\sqrt{3}}{x} - \sqrt{1 - \frac{1}{x^2}}$

[Statement Based Questions 2008]

(ii) JEE Main & AIEEE

88. If a curve $y = f(x)$ passes through the point $(1, -1)$ and satisfies the differential equation, $y(1 + xy)dx = x\, dy$, then $f\left(-\frac{1}{2}\right)$ is equal to **[2016 JEE Main]**

(a) $-\frac{2}{5}$ (b) $-\frac{4}{5}$ (c) $\frac{2}{5}$ (d) $\frac{4}{5}$

89. Let $y(x)$ be the solution of the differential equation $(x \log x)\frac{dy}{dx} + y = 2x \log x, (x \geq 1)$. Then, $y(e)$ is equal to **[2015 JEE Main]**

(a) e (b) 0 (c) 2 (d) $2e$

90. Let the population of rabbits surviving at a time t be governed by the differential equation $\frac{dp(t)}{dt} = \frac{1}{2}p(t) - 200$. If $p(0) = 100$, then $p(t)$ is equal to **[2014 JEE Main]**

(a) $400 - 300e^{\frac{t}{2}}$

(b) $300 - 200e^{-\frac{t}{2}}$

(c) $600 - 500e^{\frac{t}{2}}$

(d) $400 - 300e^{-\frac{t}{2}}$

91. At present, a firm is manufacturing 2000 items. It is estimated that the rate of change of production P with respect to additional number of workers x is given by $\frac{dP}{dx} = 100 - 12\sqrt{x}$. If the firm employees 25 more workers, then the new level of production of items is **[2013 JEE Main]**

(a) 2500 (b) 3000
(c) 3500 (d) 4500

92. The population $p(t)$ at time t of a certain mouse species satisfies the differential equation $\frac{dp(t)}{dt} = 0.5(t) - 450$. If $p(0) = 850$, then the time at which the population becomes zero is **[2012 AIEEE]**

(a) 2 log 18 (b) log 9
(c) $\frac{1}{2}$ log 18 (d) log 18

93. If $\frac{dy}{dx} = y + 3 > 0$ and $y(0) = 2$, then $y(\log 2)$ is equal to **[2011 JEE Main]**

(a) 5 (b) 13
(c) −2 (d) 7

94. Let I be the purchase value of an equipment and $V(t)$ be the value after it has been used for t years. The value $V(t)$ depreciates at a rate given by differential equation $\frac{dV(t)}{dt} = -k(T - t)$, where $k > 0$ is a constant and T is the total life in years of the equipment. Then, the scrap value $V(T)$ of the equipment is **[2010 AIEEE]**

(a) $I - \frac{kT^2}{2}$ (b) $I - \frac{k(T-t)^2}{2}$
(c) e^{-kT} (d) $T^2 - \frac{1}{k}$

95. Solution of the differential equation $\cos x\, dy = y(\sin x - y)\, dx,\ 0 < x < \frac{\pi}{2}$, is **[2010 AIEEE]**

(a) $\sec x = (\tan x + C)y$
(b) $y \sec x = \tan x + C$
(c) $y \tan x = \sec x + C$
(d) $\tan x = (\sec x + C)y$

96. The differential equation which represents the family of curves $y = c_1 e^{c_2 x}$, where c_1 and c_2 are arbitrary constants, is **[2009 AIEEE]**

(a) $y' = y^2$ (b) $y'' = y'y$
(c) $yy'' = y'$ (d) $yy'' = (y')^2$

97. The differential equation of the family of circles with fixed radius 5 units and centre on the line $y = 2$ is **[AIEEE 2008]**

(a) $(x-2)y'^2 = 25 - (y-2)^2$
(b) $(y-2)y'^2 = 25 - (y-2)^2$
(c) $(y-2)^2 y'^2 = 25 - (y-2)^2$
(d) $(x-2)^2 y'^2 = 25 - (y-2)^2$

Answers

Exercise for Session 1

1. (a) **2.** (a) **3.** (d) **4.** (a) **5.** (d)
6. (a) **7.** (b) **8.** (a) **9.** (a) **10.** (b)

Exercise for Session 2

1. (b) **2.** (c) **3.** (a) **4.** (a) **5.** (c)
6. (a) **7.** (a) **8.** (a) **9.** (a) **10.** (a)
11. (b) **12.** (c) **13.** (a) **14.** (b) **15.** (b)

Exercise for Session 3

1. (a) **2.** (a) **3.** (b) **4.** (b) **5.** (c)
6. (c) **7.** (a) **8.** (c) **9.** (a) **10.** (c)

Exercise for Session 4

1. (b) **2.** (d) **3.** (a) **4.** (b) **5.** (a)
6. (a) **7.** (c) **8.** (b) **9.** (a) **10.** (d)

Exercise for Session 5

1. (a) **2.** (c) **3.** (c) **4.** (a) **5.** (c)
6. (a) **7.** (b) **8.** (c)

Chapter Exercises

1. (a) **2.** (a) **3.** (a) **4**. (a) **5.** (c)
6. (b) **7.** (c) **8.** (a) **9**. (c) **10.** (a)
11. (b) **12.** (a) **13.** (b) **14.** (a) **15.** (a)
16. (a) **17**. (d) **18.** (b) **19.** (a,b,d)
20. (c,d) **21.** (a,b,c,d) **22.** (b,c,d)
23. (a,b,c) **24.** (a,c,d) **25.** (a,b)
26. (a,b,c) **27.** (a,b,d) **28.** (a,b,c)
29. (b,c,d) **30.** (a,d) **31.** (b)
32. (d) **33.** (d) **34.** (c) **35.** (c) **36.** (a)
37. (b) **38.** (d) **39.** (d) **40**. (b) **41.** (a)
42. (d) **43.** (c) **44.** (c) **45.** (a) **46.** (b)
47. (a) **48.** (b) **49.** (a) **50.** (b) **51.** (a)
52. (b) **53.** (c) **54.** (a) **55.** (a)
56. (A) $\rightarrow$ (q), (B) $\rightarrow$ (s), (C) $\rightarrow$ (r), (D) $\rightarrow$ (p)
57. (A) $\rightarrow$ (q), (B) $\rightarrow$ (r), (C) $\rightarrow$ (p), (D) $\rightarrow$ (s)
58. (A) $\rightarrow$ (s), (B) $\rightarrow$ (s), (C) $\rightarrow$ (q), (D) $\rightarrow$ (p)
59. (1) **60.** (8) **61.** (1) **62.** (1) **63.** (1)
64. (3) **65.** (1)

66. $t = \dfrac{\pi r^2}{\mu a}\sqrt{\dfrac{2H}{g}}$

67. $t = \dfrac{7\pi \times 10^5}{135\sqrt{g}}$

68. $f(x) = e^x \log x$

69. $x + f(x)f'(x) = \sqrt{x^2 + f^2(x)}$ and $f^2(x) = 1 + 2x$

71. $f(x) = \dfrac{20x}{119}(4 + 9x)$

72. $f(x) = \dfrac{-x}{1+x}$

73. (a, d) **74.** (a,d) **75.** (a, c) **76.** (b, c) **77.** (b)
78. (d) **79.** (a) **80.** (c) **81**. (d) **82.** (b)
83. (c) **84.** (d) **85.** (0) **86.** (0) **87.** (c)
88. (d) **89.** (c) **90.** (a) **91.** (c) **92.** (a)
93. (d) **94.** (a) **95.** (a) **96.** (d) **97.** (c)

Solutions

1. $y \cdot e^{-2x} = Ax\, e^{-2x} + B$

$e^{-2x} \cdot y_1 - 2ye^{-2x} = A(e^{-2x} - 2x\, e^{-2x})$

Cancelling e^{-2x} throughout

$$y_1 - 2y = A(1 - 2x) \qquad \ldots(i)$$

Differentiating again $y_2 - 2y_1 = -2A$

$$\Rightarrow \qquad A = \frac{2y_1 - y_2}{2}$$

On substituting A in Eq. (i)

$$2(y_2 - 2y) = (2y_1 - y_2)(1 - 2x)$$

$$2y_1 - 4y = 2y_1(1 - 2x) - (1 - 2x)y_2$$

$$(1 - 2x)\frac{d}{dx}\left(\frac{dy}{dx} - 2y\right) + 2\left(\frac{dy}{dx} - 2y\right) = 0$$

Hence, $k = 2$ and $l = -2$

$\Rightarrow$ Ordered pair $(k, l) \equiv (2, -2)$

2. $\frac{dy}{dx} = \frac{y}{x} - \cos^2\frac{y}{x}$ $y = vx$

$$v + x\frac{dv}{dx} = v - \cos^2 v$$

$$\int \frac{dv}{\cos^2 v} + \int \frac{dx}{x} = C \;\Rightarrow\; \tan v + \ln x = C$$

$$\tan\frac{y}{x} + \ln x = C$$

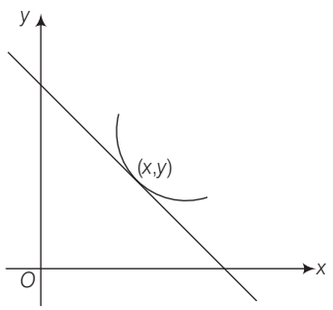
If $x = 1, y = \frac{\pi}{4} \Rightarrow C = 1 \Rightarrow \tan\frac{y}{x} = 1 - \ln x = \ln\frac{e}{x}$

$$y = x\tan^{-1}\left(\ln\frac{e}{x}\right)$$

3. $Y - y = m(X - x)$. For X-intercept $Y = 0$

$$X = x - \frac{y}{m}$$

Therefore, $x - \frac{y}{m} = y$



or $$\frac{dy}{dx} = \frac{y}{x - y}$$

Put $y = vx$ $\quad v + x\frac{dv}{dx} = \frac{v}{1 - v}$

$$x\frac{dv}{dx} = \frac{v}{1 - v} - v = \frac{v - v + v^2}{1 - v}$$

$$\int \frac{1 - v}{v^2}\, dv = \int \frac{dx}{x}$$

$$-\frac{1}{v} - \ln v = \ln x + C$$

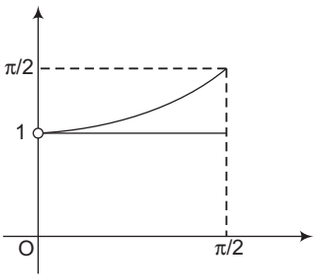
$$-\frac{x}{y} - \ln\frac{y}{x} = \ln x + C \;\Rightarrow\; -\frac{x}{y} = \ln y + C$$

$x = 1, y = 1 \Rightarrow C = -1$

$$1 - \frac{x}{y} = \ln y \;\Rightarrow\; y = e \cdot e^{-x/y}$$

$$e^{-x/y} = \frac{e}{y} \;\Rightarrow\; ye^{x/y} = e$$

4. $\sin x\frac{dy}{dx} + y\cos x = 1$



$$\frac{dy}{dx} + y\cot x = \text{cosec}\ x$$

$$\text{IF} = e^{\int \cot x\, dx} = e^{\ln(\sin x)} = \sin x$$

$$y\sin x = \int \text{cosec}\ x \cdot \sin x\, dx$$

$$y\sin x = x + C$$

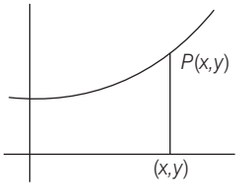
If $x = 0$, y is finite $\;\therefore\;\; C = 0$

$$y = x\,(\text{cosec}\ x) = \frac{x}{\sin x}$$

Now, $\qquad I < \frac{\pi^2}{4}$ and $I > \frac{\pi}{2}$

Hence, $\qquad \frac{\pi}{2} < I < \frac{\pi^2}{4}$

5. $\int_0^x f(x)\, dx = y^3$. Differentiating $f(x) = 3y^2 \cdot \frac{dy}{dx}$



$$y = 3y^2\frac{dy}{dx} \;\Rightarrow\; y = 0 \qquad \text{(rejected)}$$

or $\qquad 3y\, dy = dx$

$$\frac{3y^2}{2} = x + C \;\Rightarrow\; \text{Parabola}$$

6. $y = mx + c; \dfrac{dy}{dx} = m; \dfrac{d^2y}{dx^2} = 0$

Substituting in $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} - 4y = -4x$

$$0 - 3m - 4(mx + c) = -4x$$

$$-3m - 4c - 4mx = -4x$$

$$-(3m + 4c) = 4x(m - 1) \qquad \ldots(i)$$

Eq. (i) is true for all real x, if $m = +1$ and $c = -3/4$.

7. $y = u^m \Rightarrow \dfrac{dy}{dx} = mu^{m-1}\dfrac{dy}{dx}$

Since, $2x^4 \cdot u^m \cdot mu^{m-1} \cdot \dfrac{du}{dx} + u^{4m} = 4x^6$

$$\frac{du}{dx} = \frac{4x^6 - u^{4m}}{2m\, x^4\, u^{2m-1}}$$

$$\Rightarrow \qquad 4m = 6 \;\Rightarrow\; m = \frac{3}{2}$$

and $\qquad 2m - 1 = 2 \;\Rightarrow\; m = \dfrac{3}{2}$

8. $\dfrac{dy}{dx} - \dfrac{y}{x^2}\tan\dfrac{1}{x} = -\sec\dfrac{1}{x}\cdot\dfrac{1}{x^2}$

$$\text{IF} = e^{-\int \frac{1}{x^2}\tan\frac{1}{x}dx} = \sec\frac{1}{x} \;\Rightarrow\; y\cdot\sec\frac{1}{x}$$

$$= -\int \sec^2\left(\frac{1}{x}\right)\frac{1}{x^2}\,dx = \tan\frac{1}{x} + C$$

If $y \to -1$, then $x \to \infty$

$$\Rightarrow \qquad C = -1 \;\Rightarrow\; y = \sin\frac{1}{x} - \cos\frac{1}{x}$$

9. $\dfrac{dM}{dt} = -KM \;\Rightarrow\; M = Ce^{-kt}$ when $t = 0$; $M = M_0$

$$\Rightarrow \qquad C = M_0$$

$$\Rightarrow \qquad M = M_0e^{-kt}$$

when $t = 1$, $M = \dfrac{M_0}{2}$

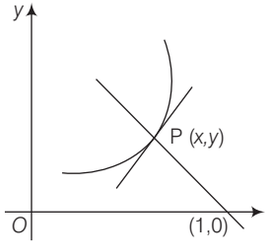
$$\Rightarrow \qquad k = \ln 2$$

Therefore, $M = M_0e^{-t\ln 2}$

when $\qquad M = \dfrac{M_0}{1000}$, then $t = \log_2 1000 = 9.98$

$= 10$ h approximately

10. Slope of the normal $= \dfrac{y}{x-1}$



$$\therefore \qquad \frac{dy}{dx} = \frac{1-x}{y}$$

$$\frac{y^2}{2} = x - \frac{x^2}{2} + C \qquad \ldots(ii)$$

Eq. (ii) passes through (0, 0).

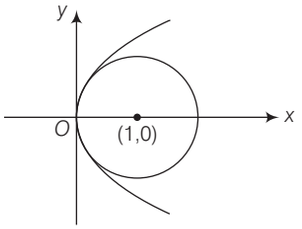
Thus, $C = 0$

$$x^2 + y^2 - 2x = 0$$

Now, tangent to $y^2 = 4x$

$$y = mx + \frac{1}{m} \qquad \ldots(iii)$$

If it touches the circle



$$x^2 + y^2 - 2x = 0$$

Then, $\left|\dfrac{m + (1/m)}{\sqrt{1+m^2}}\right| = 1$

$$\Rightarrow \qquad 1 + m^2 = m^2$$

$$\Rightarrow \qquad m \to \infty$$

Hence, tangent is y-axis *ie*, $x = 0$.

11. $f'(x) - \dfrac{2x(x+1)}{x+1}f(x) = \dfrac{e^{x^2}}{(x+1)^2}$

$$\text{IF} = e^{\int -2x\,dx} = e^{-x^2}$$

$$\therefore \qquad f(x)\cdot e^{-x^2} = \int \frac{dx}{(x+1)^2}$$

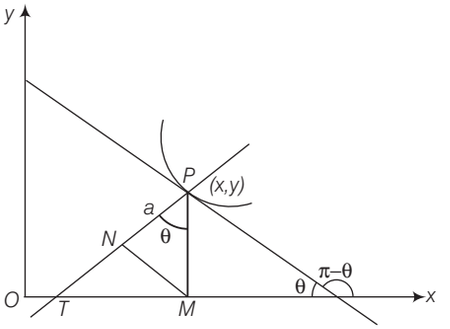
$$\Rightarrow \qquad f(x)\cdot e^{-x^2} = -\frac{1}{x+1} + C$$

At $x = 0$, $f(0) = 5 \;\Rightarrow\; C = 6$

$$\therefore \qquad f(x) = \left(\frac{6x+5}{x+1}\right)\cdot e^{x^2}$$

12. Ordinate = PM. Let $P \equiv (x, y)$

Projection of ordinate on normal = PN



$\therefore \quad PN = PN\cos\theta = a$ (given)

$\therefore \quad \dfrac{y}{\sqrt{1+\tan^2\theta}} = a$

$\Rightarrow \quad y = a\sqrt{1+(y_1)^2}$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{\sqrt{y^2-a^2}}{a}$

$\Rightarrow \quad \int \dfrac{a\,dy}{\sqrt{y^2-a^2}} = \int dx$

$\Rightarrow \quad a\ln|y+\sqrt{y^2-a^2}| = x + C$

13. $y = e^x(ax+b)$ ...(i)

Differentiating w.r.t. x, we get

$\dfrac{dy}{dx} = e^x(ax+b) + e^x\cdot a$ or $\dfrac{dy}{dx} = y + ae^x$ ...(ii)

Again, differentiating both the sides

$\dfrac{d^2y}{dx^2} = \dfrac{dy}{dx} + ae^x$ ...(iii)

From Eq. (iii) – Eq. (ii)

$\dfrac{d^2y}{dx^2} - \dfrac{2dy}{dx} + y = 0$

is required differential equation.

14. $\dfrac{dy}{dx} = 2ax = 2x\cdot\dfrac{y}{x^2};\ \dfrac{dy}{dx} = \dfrac{2y}{x};$

Now, $m\dfrac{dy}{dx} = -1 \ \Rightarrow\ m = -\dfrac{x}{2y} \ \Rightarrow\ \dfrac{dy}{dx} = -\dfrac{x}{2y}$

$y^2 = -\dfrac{x^2}{2} + C$

15. $\int_0^1 f(tx)\,dt = n\cdot f(x)$

Put, $x = y \ \Rightarrow\ dt = \dfrac{1}{x}\,dy$

$\therefore \quad \dfrac{1}{x}\int_0^x f(y)\,dy = nf(x)$

$\therefore \quad \int_0^x f(y)\,dy = x\cdot n\cdot f(x)$

Differentiating, $f(x) = n\,[f(x) + xf'(x)]$

$f(x)(1-n) = nx\,f'(x)$

$\therefore \quad \dfrac{f'(x)}{f(x)} = \dfrac{1-n}{nx}$

Integrating, $\ln f(x) = \left(\dfrac{1-n}{n}\right)\ln Cx = \ln(Cx)^{\frac{1-n}{n}};$

$\therefore \quad f(x) = Cx^{\frac{1-n}{n}}$

16. $(x^2z^{2\alpha}-1)\,\alpha z^{\alpha-1}\,dz + 2x\,z^{3\alpha}\,dx = 0$

or $\alpha(x^2\,z^{3\alpha-1} - z^{\alpha-1})\,dz + 2x\,z^{3\alpha}\,dx = 0$

for homogeneous every term must be of the same degree,

$3\alpha + 1 = \alpha - 1 \ \Rightarrow\ \alpha = -1$

17. Differentiate, $xy\,(x) = x^2y'(x) + 2\,xy(x)$

or $xy(x) + x^2\,y'(x) = 0$

$x\dfrac{dy}{dx} + y = 0$

$\ln y + \ln x = \ln C$

$xy = C$

At point (2, 3),

$2\times 3 = C \Rightarrow C = 6$

$\therefore \quad xy = 6$

18. $\int \dfrac{dy}{100-y} = \int dx$

$-\ln(100-y) = x + C$

$\ln(100-y) = -x + C$

Also $x = 0,\ y = 50$

$\therefore \quad C = \ln 50$

$x = \ln 50 - \ln(100-y)$

$\Rightarrow \quad \ln\dfrac{50}{100-y} = x$

$\Rightarrow \quad \dfrac{50}{100-y} = e^x$

$\Rightarrow \quad 100 - y = 50e^{-x}$

$\Rightarrow \quad y = 100 - 50e^{-x}$

19. Here, $x\left(\dfrac{dy}{dx}\right)^2 - 3y^2\left(\dfrac{dy}{dx}\right) + 3 = 0$

has order 1, degree 2 and non-linear.

20. Here, $(f'(x))^2 + 4f'(x)\cdot f(x) + (f(x))^2 = 0$

$\Rightarrow \quad \left(\dfrac{f'(x)}{f(x)}\right)^2 + 4\left(\dfrac{f'(x)}{f(x)}\right) + 1 = 0$

$\Rightarrow \quad \dfrac{f'(x)}{f(x)} = \dfrac{-4\pm\sqrt{16-4}}{2}$

$\Rightarrow \quad \dfrac{f'(x)}{f(x)} = -2\pm\sqrt{3}$

Integrating both the sides

$\log|f(x)| = (-2\pm\sqrt{3})\,x + C_1$

$\Rightarrow \quad f(x) = e^{(-2\pm\sqrt{3})x}\cdot C$

21. (a) $32x + 2y\dfrac{dy}{dx} = 0$

$\Rightarrow \quad \dfrac{dy}{dx} = m_1 = -\dfrac{16x}{y}$

and $16y^{15}\dfrac{dy}{dx} = k$

$\Rightarrow \quad \dfrac{dy}{dx} = m_2 = \dfrac{k}{16y^{15}}$

$m_1m_2 = -\dfrac{16x}{y}\cdot\dfrac{k}{16y^{15}} = -\dfrac{x}{y^{16}}\cdot k$

$= -\dfrac{x}{y^{16}}\cdot\dfrac{y^{16}}{x} = -1$

(b) $\dfrac{dy}{dx} = 1 - ce^{-x} = 1 - (y-x) = -(y-x-1)$ [using $ce^{-x} = y - x$]

and $\dfrac{dy}{dx} - k\cdot\dfrac{dy}{dx}e^{-y} = 1$

$\dfrac{dy}{dx}[1 - ke^{-y}] = 1$

or $[1-(x+2-y)]\dfrac{dy}{dx}=1 \qquad [\text{using } ke^{-y}=x-y+2]$

$$\frac{dy}{dx}=m_2=\frac{1}{y-x-1}$$

$\Rightarrow \quad m_1m_2=-1$

(c) $\dfrac{dy}{dx}=2cx=2x\cdot\dfrac{y}{x^2}=\dfrac{2y}{x}=m_1$

Also, $2x+4y\dfrac{dy}{dx}=0 \Rightarrow \dfrac{dy}{dx}=-\dfrac{x}{2y}=m_2$

Hence, $m_1m_2=-1$

(d) $x^2-y^2=c$

$$2x-2y\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{x}{y}=m_1$$

$$xy=k$$

$$x\frac{dy}{dx}+y=0 \Rightarrow \frac{dy}{dx}=-\frac{y}{x}=m_2$$

$\therefore \quad m_1m_2=-1$

Hence, (a), (b), (c) and (d) are all correct.

22. Let $m=\dfrac{dy}{dx}$ be the slope of tangent (x, y) to the required curve.

m_1 = slope of the tangent at $xy=c^2$

$$=-\frac{c^2}{x^2}=-\frac{y}{x}$$

Hence, $\dfrac{m+\dfrac{c^2}{x^2}}{1-\dfrac{c^2}{x^2}m}=\pm 1 \quad \text{or} \quad \dfrac{m+\dfrac{y}{x}}{1-\dfrac{y}{x}m}=\pm 1 \qquad \ldots(i)$

Consider $y^2-2xy-x^2=k$

$\Rightarrow \; 2y\dfrac{dy}{dx}-2\left(y+x\dfrac{dy}{dx}\right)-2x=0$

$$\Rightarrow \quad \frac{dy}{dx}(y-x)=x+y$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{x+y}{y-x}=m \qquad \text{(say)}$$

From Eq. (i)

$$\text{LHS}=\frac{\dfrac{x+y}{y-x}+\dfrac{y}{x}}{1-\dfrac{y}{x}\left(\dfrac{x+y}{y-x}\right)}$$

$$=\frac{x^2+y^2}{x^2+y^2}=1 \text{ RHS}$$

Similarly option, (b), (c) and (d) safisfy

23. $x\dfrac{dy}{dx}=y\log\left(\dfrac{y}{x}\right)\dfrac{dy}{dx}=\dfrac{y}{x}\log\left(\dfrac{y}{x}\right)$, put $y=vx$

$$\Rightarrow \quad \frac{dy}{dx}=v+x\frac{dv}{dx}$$

$$\therefore \quad v+x\frac{dv}{dx}=v\log v \Rightarrow x\frac{dv}{dx}=v(\log v-1)$$

$$\Rightarrow \quad \int\frac{dv}{v(\log v-1)}=\int\frac{dx}{x} \Rightarrow \int\frac{dt}{t-1}=\int\frac{dx}{x},$$

let $\log v=t$

$\Rightarrow \quad \log(t-1)=\log(x)+\log C$

$\Rightarrow \quad \log(t-1)=\log(xC)$

$\Rightarrow \quad t=1+xC$

$\Rightarrow \quad \log\dfrac{y}{x}=1+xC$

$\Rightarrow \quad y=x\cdot e^{1+Cx}$

or $\quad y=x\cdot e^{1-Cx}$

and $\quad \log\dfrac{y}{x}=\log e+Cx$

$\Rightarrow \quad y=ex\cdot e^{Cx}$

24. Clearly, (a) and (c) are of the form $\dfrac{dy}{dx}+Py=Q$, which is linear in y.

Also, (d) is $\dfrac{d^2y}{dx^2}=\cos x$, on integrating $\dfrac{dy}{dx}=\sin x+C$

which is also linear in y.

25. $\dfrac{dy}{dx}=\dfrac{(y-x)\pm\sqrt{(x-y)^2+4xy}}{2y}$

$$\Rightarrow \quad \frac{dy}{dx}=1 \quad \text{or} \quad \frac{dy}{dx}=-\frac{x}{y}$$

$\Rightarrow \quad x-y+1=0 \quad \text{and} \quad x^2+y^2=25$

26. (a) $f(x, tx)=e^t+\tan^{-1}(t)$, independent of x.

$\Rightarrow$ Homogeneous differential equation.

(b) $\log\left(\dfrac{y}{x}\right)dx+\dfrac{y^2}{x^2}\sin^{-1}\dfrac{y}{x}\cdot dy=0$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{\log\left(\dfrac{y}{x}\right)}{\dfrac{y^2}{x^2}\sin^{-1}\left(\dfrac{y}{x}\right)}$$

$$f(x, y)=\frac{\log\left(\dfrac{y}{x}\right)}{\dfrac{y^2}{x^2}\sin^{-1}\left(\dfrac{y}{x}\right)}$$

$$\therefore \quad f(x, tx)=\frac{\log(t)}{t^2\sin^{-1}(t)}, \text{ independent of } x$$

$\Rightarrow$ Homogeneous differential equation.

(c) $f(x, y)=x^2+\sin x\cdot\cos y$

$f(x, tx)=x^2+\sin x\cdot\cos(tx)$, not independent of x.

$\Rightarrow$ Not homogeneous differential equation.

(d) $f(x, y)=\dfrac{x^2+y^2}{xy^2-y^3}$

$\Rightarrow f(x, tx)$ is not independent of x.

$\Rightarrow$ Not homogeneous differential equation.

27. Integrating Factor $= e^{\int \cos x\, dx} = e^{\sin x}$

or $\quad y \cdot e^{\sin x} = \int e^{+\sin x} \cos x = -e^{+\sin x} + C$

At point $(0, -1)$,

$$-1\, e^0 = -e^0 + C \Rightarrow 0$$

$\therefore \quad ye^{\sin x} = -e^{\sin x}$

$$y = -1$$

28. $\dfrac{dy}{dx} - y \cot x = -\dfrac{\sin x}{x^2}$

Integrating Factor $= e^{\int -\cot x\, dx} = e^{-\log \sin x} = \dfrac{1}{\sin x}$

$\therefore$ Required solution is $y \cdot \dfrac{1}{\sin x} = \int -\dfrac{\sin x}{x^2} \cdot \dfrac{1}{\sin x}\, dx + C$

$$y \cdot \frac{1}{\sin x} = \frac{1}{x} + C$$

As $x \to \infty, y \to 0 \Rightarrow C = 0$

$\therefore \quad y = \dfrac{\sin x}{x} \Rightarrow \lim\limits_{x \to 0} f(x) = 1 \qquad \ldots(i)$

$\therefore \quad I = \int_0^{\pi/2} \dfrac{\sin x}{x}\, dx$

Since, $\dfrac{\sin x}{x}$ is decreasing, when $x > 0$

$\Rightarrow \quad f(x) < f(0)$

$\Rightarrow \quad \int_0^{\pi/2} f(x) < \dfrac{\pi}{2}$ and $x < \dfrac{\pi}{2}$

$\Rightarrow \quad f(x) > \int\left(\dfrac{\pi}{2}\right) \Rightarrow \int_0^{\pi/2} f(x)\, dx > 1$

29. (a) Order of the differential equation is 2.

(b) $\dfrac{xdy - ydx}{\sqrt{x^2 + y^2}} = dx \Rightarrow \dfrac{\dfrac{xdy - ydx}{x^2}}{\sqrt{\left(1 + \dfrac{y^2}{x^2}\right)}} = \dfrac{dx}{x}$

$\therefore \quad \ln\left|\dfrac{y}{x} + \sqrt{1 + \dfrac{y^2}{x^2}}\right| = \ln|Cx|$

$\therefore \quad \dfrac{y}{x} + \dfrac{\sqrt{x^2 + y^2}}{x} = Cx$

i.e. $\quad y + \sqrt{x^2 + y^2} = Cx^2$

(c) $y = e^x (A \cos x + B \sin x)$

$$\frac{dy}{dx} = e^x (A \cos x + B \sin x) + e^x(-A \sin x + B \cos x)$$

$$= y + e^x (-A \sin x + B \cos x)$$

$\therefore \quad \dfrac{d^2y}{dx^2} = \dfrac{dy}{dx} + e^x (-A \sin x + B \cos x)$

$$+ e^x (-A \cos x - B \sin x)$$

$\therefore \quad \dfrac{d^2y}{dx^2} = \dfrac{dy}{dx} + e^x (-A \sin x + B \cos x) - y$

$$\frac{dy}{dx} + \frac{dy}{dx} - y - y = 2\left(\frac{dy}{dx} - y\right)$$

(d) $(1 + y^2)\dfrac{dy}{dx} + x = 2e^{\tan^{-1} y}$

$$\Rightarrow \frac{dx}{dy} + \frac{1}{1 + y^2} x = \frac{e^{\tan^{-1} y}}{1 + y^2}$$

$$\text{IF} = e^{\int \frac{1}{1 + y^2}} = e^{\tan^{-1} y}$$

$$\Rightarrow \quad xe^{\tan^{-1} y} = 2 \int e^{\tan^{-1} y} \cdot \frac{e^{\tan^1 y}}{1 + y^2}\, dy$$

$$\Rightarrow \quad xe^{\tan^{-1} y} = e^{2 \tan^{-1} y} + k$$

30. $\dfrac{dy}{dx} = 3(A + Bx)\, e^{3x} = Be^{3x}$

$$\Rightarrow \frac{dy}{dx} + my = (3 + m)(A + Bx)\, e^{3x} + Be^{3x}$$

$$\Rightarrow \frac{d^2y}{dx^2} + m\frac{dy}{dx} + ny = (9 + 3m + n)(A + Bx)\, e^{3x}$$

$$+ B(6 + m)\, e^{3x} = 0$$

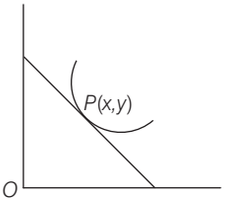
$\Rightarrow \quad 3m + n + 9 = 0 \quad$ and $\quad m + 6 = 0$

$\Rightarrow \quad m = -6 \quad$ and $\quad n = 9$

31. Equation of tangent

$$Y - y = m(X - x)$$

Put $X = 0, \qquad Y = y - mx$



Since, initial ordinate is

$$y - mx = x - 1$$

$\Rightarrow \quad mx - y = 1 - x$

$\dfrac{dy}{dx} - \dfrac{1}{x}y = \dfrac{1 - x}{x}$ which is a linear differential equation.

Hence, Statement I is correct and its degree is 1.

$\Rightarrow$ Statement II is also correct. Since, every 1st degree differential equation need not be linear, hence Statement II is not the correct explanation of Statement I.

32. **Statement I** The order of differential equation is 2.

$\therefore$ Statement I is false.

33. Integral curves are $y = cx - x^2$

The differential equation does not represent all the parabolas passing through origin but it represents all parabolas through origin with axis of symmetry parallel to y-axis and coefficient of x^2 as -1, hence Statement I is false.

Statement II is universally true.

34. $\dfrac{ydx - xdy}{y^2} \cdot \cot \dfrac{x}{y} = x\, dx$

or $\quad \int \cot \dfrac{x}{y} \cdot d\left(\dfrac{x}{y}\right) = \int x\, dx$

or $$\int \cot t\, dx = nx + C$$

$$\ln(\sin t) = nx + C; \sin\frac{x}{y} = Ce^{nx}$$

35. $y = c_1\cos 2x + c_2\sin^2 x + c_3\cos^2 x + c_4e^{2x} + c_5$

$$e^{2x+c_6} = c_1\cos 2x + c_2\left[\frac{1-\cos 2x}{2}\right] + c_3\left[\frac{\cos 2x+1}{2}\right] + c_4e^{2x} + c_5e^{2x}\cdot e^{c_6}$$

$$= \left(c_1 - \frac{c_2}{2} + \frac{c_3}{2}\right)\cos 2x + \left(\frac{c_2}{2} + \frac{c_3}{2}\right) + (c_4 + c'_5)\, e^{2x}$$

$$= \lambda_1\cos 2x + \lambda_2e^{2x} + \lambda_3$$

$\Rightarrow$ Total number of independent parameters in the given general solution is 3.

36. The given differential equation is

$$\frac{d\left(\frac{dy}{dx}\right)}{\frac{dy}{dx}} = \frac{2x}{x^2+1}\,dx$$

$$\Rightarrow \quad \ln\left(\frac{dy}{dx}\right) = \ln(x^2+1) + \ln C,\ C > 0$$

$$\Rightarrow \quad \frac{dy}{dx} = C(x^2+1)$$

$$\Rightarrow \quad y = C\left(\frac{x^3}{3} + x\right) + C',\ C' \in R$$

Obviously $y \to \infty$, as $x \to \infty$; as $C > 0$

37. $(x-h)^2 = 4b(y-k)$ here b is constant and h, k are parameters.

38. The general equation of all non-vertical lines in a plane is $ax + by = 1$, where $b \neq 0$.

Now, $$ax + by = 1$$

$$\Rightarrow \quad a + b\frac{dy}{dx} = 0 \qquad \text{(differentiating w.r.t. } x\text{)}$$

$$\Rightarrow \quad b\frac{d^2y}{dx^2} = 0 \qquad \text{(differentiating w.r.t. } x\text{)}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 0 \qquad (\text{as } b \neq 0)$$

Hence, the differential equation is $\frac{d^2y}{dx^2} = 0$.

39. The equation $ax^2 + 2hxy + by^2 = 1$ represents the family of all conics whose centre lies at the origin for different values of a, h, b.

$\therefore$ Order is 3.

Thus, Statement I is false and Statement II is true.

Hence, option (d) is the correct answer.

40. $y = a\sin x + b\cos x$

$$\frac{dy}{dx} = a\cos x - b\sin x$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -a\cos x - b\sin x = -y$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} + y = 0$$

But Statement II is not the correct explanation of the Statement I.

41. $f(0) = 2$

$$f(x) = (e^x + e^{-x})\cos x - 2x - \left[x\int_0^x f'(t)\,dt - \int_0^x \underbrace{t}_{\text{I}}\ \underbrace{f'(t)\,dt}_{\text{II}}\right]$$

$$f(x) = (e^x + e^{-x})\cos x - 2x - [x(f(x) - f(0)) - \{t\cdot f(t)|_0^x - \int_0^x f(t)dt\}]$$

$$f(x) = (e^x + e^{-x})\cos x - 2x - xf(x) + 2x + \left[x\ f(x) - \int_0^x f(t)\,dt\right]$$

$$f(x) = (e^x + e^{-x})\cos x - \int_0^x f(t)\,dt \qquad \ldots(\text{i})$$

On differentiating Eq. (i)

$$f'(x) + f(x) = \cos x\,(e^x - e^{-x}) - (e^x + e^{-x})\sin x \qquad \ldots(\text{ii})$$

Hence, $$\frac{dy}{dx} + y = e^x(\cos x - \sin x) - e^x(\cos x + \sin x)$$

42. $f'(0) + f(0) = 0 - 2\cdot 0 = 0$

43. IF of Eq. (i) is e^x

$$y\cdot e^x = \int e^{2x}(\cos x - \sin x)\,dx - \int(\cos x + \sin x)\,dx$$

$$y\cdot e^x = \int e^{2x}(\cos x - \sin x)\,dx - (\sin x - \cos x) + C$$

Let $$I = \int e^{2x}(\cos x - \sin x)\,dx = e^{2x}(A\cos x + B\sin x)$$

Solving, $A = 3/5$ and $B = -1/5$ and $C = 2/5$

$\therefore$ $$y = e^x\left(\frac{3}{5}\cos x - \frac{1}{5}\sin x\right) - (\sin x - \cos x)\,e^{-x} + \frac{2}{5}e^{-x}$$

Solutions (Q. Nos. 44 to 46)

Integrating, $\frac{d^2y}{dx^2} = 6x - 4$,

we get $\frac{dy}{dx} = 3x^2 - 4x + A$

When $x = 1$, $\frac{dy}{dx} = 0$ so that $A = 1$.

Hence, $$\frac{dy}{dx} = 3x^2 - 4x + 1 \qquad \ldots(\text{i})$$

Integrating, we get $y = x^3 - 2x^2 + x + B$.

When $x = 1$, $y = 5$, so that $B = 5$

Thus, we have $y = x^3 - 2x^2 + x + 5$

From Eq. (i), we get the critical points $x = \frac{1}{3}$, $x = 1$

At the critical point $x = \frac{1}{3}$, $\frac{d^2y}{dx^2}$ is negative.

Therefore, at $x = \frac{1}{3}$, y has a local maximum.

At $x = 1$, $\frac{d^2y}{dx^2}$ is positive. Therefore, at $x = 1$, y has a local minimum.

Also, $f(1) = 5$, $f\left(\frac{1}{3}\right) = \frac{157}{27}$, $f(0) = 5$, $f(2) = 7$

Hence, the global maximum value = 7.
and the global minimum value = 5.

47. $xdy - ydx = \sqrt{x^2 - y^2}\, dx$

$\Rightarrow \quad d\left(\frac{y}{x}\right) = \frac{\sqrt{1-(y/x)^2}}{x}\, dx \;\Rightarrow\; \int \frac{d(y/x)}{\sqrt{1-(y/x)^2}} = \int \frac{dx}{x}$

$\Rightarrow \sin^{-1}\frac{y}{x} = \ln x = \ln C$ or $Cx = e^{\sin^{-1} y/x}$

48. $(xy^4 + y)\, dx - x\, dy = 0$

$\Rightarrow \quad xy^4 dx + y\, dx - x\, dy = 0$

$\Rightarrow \quad \int x^3\, dx - \int \left(\frac{x}{y}\right)^2 d\left(\frac{x}{y}\right) = 0$

$\Rightarrow \quad \frac{x^4}{4} + \frac{1}{3}\left(\frac{x}{y}\right)^3 = C$

49. $2x\cos y\, dx - x^2 \sin y\, dy + y^2 \cos x\, dx + 2y \sin x\, dy = 0$

$\Rightarrow \quad \int d(x^2 \cos y) + \int d(y^2 \sin x) = 0$

$\Rightarrow \quad x^2 \cos y + y^2 \sin x = C$

50. $\frac{dx}{x} = \frac{ydy}{1+y^2}$

$\Rightarrow \quad \ln x = \frac{1}{2}\cdot \ln(1+y^2) + C$

From the given condition, $C = 0$

$\therefore \quad x^2 - y^2 = 1$

51. $\frac{dy}{dx} + \frac{1+y^2}{\sqrt{1-x^2}} = 0$

$\Rightarrow \quad \frac{dy}{1+y^2} + \frac{dx}{\sqrt{1-x^2}} = 0$

$\Rightarrow \quad \tan^{-1} y + \sin^{-1} x = C$

52. $\frac{dy}{dx} = (1+x)\cdot(1+y)$ gives $y = e^{\frac{(1+x)^2}{2}} - 1$

53. Tangent at point $P(x,y)$, is $y - y = f'(x)(x - x)$

$Q:(0, y - x f'(x))$

Then, $PQ = OQ$

$\Rightarrow \; x^2 + x^2(f'(x))^2 = y^2 + x^2(f'(x))^2 - 2xy(f'(x))$

$\Rightarrow \; x^2 = y^2 - 2xy \cdot f'(x)$

or $dy/dx = \frac{y^2 - x^2}{2xy}$, put $y = tx$

$\Rightarrow \quad t + x\frac{dt}{dx} = \frac{dy}{dx}$

$\therefore \quad t + x\frac{dt}{dx} = \frac{t^2 - 1}{2t}$

$\Rightarrow \quad x\frac{dt}{dx} = \frac{-(1+t^2)}{2t} \Rightarrow \left|\frac{2t}{1+t^2}dt\right| = -\left|\frac{dx}{x}\right|$

$\Rightarrow \quad \frac{c}{x} = 1 + \frac{y^2}{x^2} \Rightarrow x^2 + y^2 - cx = 0$

$\therefore \quad C_1 : x^2 + y^2 - x = 0$

$C_2 : x^2 + y^2 + x = 0$

C_1 and C_2 touch externally $\Rightarrow$ number of common tangents = 3.

54. $C_3 : (x-5)^2 + y^2 = 5^2$

Tangent, $bx + ay - ab = 0$, length of perpendicular to tangent from centre = radius.

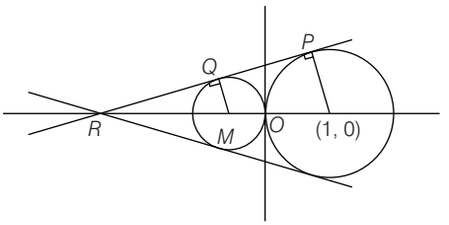
$\Rightarrow \quad |5b - ab| = 5\sqrt{a^2 + b^2}$

$\Rightarrow \quad a^2b^2 - 10ab^2 = 25a^2$

$\Rightarrow \quad ab^2 - 10b^2 = 25a$

55. $C_1 : x^2 + y^2 - 2x = 0$

$C_2 : x^2 + y^2 - Cx = 0$



$\angle QRM = 30°$

$\sin 30° = \frac{1}{1+OR} \Rightarrow OR = 1$

$\sin 30° = \frac{QM}{1-QM} \Rightarrow QM = 1/3$

radius of $C_2 = 1/3 \Rightarrow C = -2/3, C_2 : x^2 + y^2 + \frac{2}{3}x = 0$

Point $(-1/3, 1/3)$ will satisfy C_2.

56. (A) Let $x = r\cos\theta$, $y = r\sin\theta$

$\therefore \quad x^2 + y^2 = r^2(\sin^2\theta + \cos^2\theta) = r^2$ ...(i)

and $\tan\theta = \frac{y}{x}$ ...(ii)

$\therefore \quad d(x^2 + y^2) = d(r^2)$

From Eq. (i), $x\, dx + y\, dy = rdr$ ...(iii)

From Eq. (ii),

$d\left(\frac{y}{x}\right) = d(\tan\theta)$

$\Rightarrow \quad \frac{xdy - ydx}{x^2} = \sec^2\theta\, d\theta$

$\therefore \quad x\, dy - y\, dx = x^2 \sec^2\theta\, d\theta$

$= r^2\cos^2\theta \sec^2\theta\, d\theta = r^2 d\theta$ ...(iv)

From Eqs. (iii) and (iv)

$\frac{rdr}{r^2 d\theta} = \sqrt{\frac{a^2 - r^2}{r^2}}$

or $\frac{dr}{\sqrt{a^2 - r^2}} = d\theta$

$\sin^{-1}\left(\frac{r}{a}\right) = \theta + C$

$\Rightarrow \quad r = a\sin(\theta + C)$

$\Rightarrow \quad \sqrt{x^2 + y^2} = a\sin\left\{C + \tan^{-1}\left(\frac{y}{x}\right)\right\}$

(B) $\dfrac{dy}{dx} - \sec^2 x \tan 2xy = \cos^2 x$

$$\text{IF} = e^{\int \tan 2x \sec^2 x\, dx} = e^{\int \frac{2\tan x}{\tan^2 x - 1} \times \sec^2 x\, dx}$$

$$= e^{\int \frac{dt}{t}}, \text{ where } t = \tan^2 x - 1$$

$$= e^{\ln|t|} = |t| = |\tan^2 x - 1|$$

Given that, $|x| < \dfrac{\pi}{4}$

$\therefore \quad \tan^2 x < 1$

$\therefore \quad \text{IF} = 1 - \tan^2 x$

Solution is $y(1 - \tan^2 x) = \int \cos^2 x (1 - \tan^2 x)\, dx$

$$= \int (\cos^2 x - \sin^2 x)\, dx$$

$$= \int \cos 2x\, dx = \frac{\sin 2x}{2} + C$$

when $x = \dfrac{\pi}{6}, y = \dfrac{3\sqrt{3}}{8}$

$\therefore \quad \dfrac{3\sqrt{3}}{8}\left(1 - \dfrac{1}{3}\right) = \dfrac{1}{2} \times \dfrac{\sqrt{3}}{2} + C$

$\therefore \quad C = 0 \Rightarrow y = \dfrac{\sin 2x}{2(1 - \tan^2 x)}$

(C) $x^2 + y^2 - 2cx = 0$ ...(i)

$\therefore \quad 2x + 2y\dfrac{dy}{dx} - 2c = 0$

or $C = x + \dfrac{ydy}{dx}$ ...(ii)

From Eqs. (i) and (ii), $x^2 + y^2 - 2\left(x + \dfrac{ydy}{dx}\right)x = 0$

or $-x^2 + y^2 - 2xy\dfrac{dy}{dx} = 0$ is the differential equation representing the given family of circles. To find the orthogonal trajectories.

$$y^2 - x^2 = -2xy\frac{dx}{dy}$$

or $y^2\, dy = x^2\, dy - 2xy\, dx$

or $-dy = \dfrac{yd(x^2) - x^2\, dy}{y^2}$

or $-dy = d\left(\dfrac{x^2}{y}\right)$

or $-y = \dfrac{x^2}{y} + C$

$\Rightarrow \quad x^2 + y^2 + Cy = 0$

$\Rightarrow$ Orthogonal trajectory.

(D) Differentiating the equation w.r.t. x, we get

$$xy(x) + 1\int_0^x y(t)\, dt = (x+1)\, xy(x) + 1\int_0^x ty(t)\, dt$$

Again, differentiating, w.r.t. x, we get

$$y(x) = x^2 y'(x) + 2xy(x) + xy(x)$$

or $(1 - 3x)\, y(x) = \dfrac{x^2 dy(x)}{dx}$

or $\dfrac{(1-3x)\, dx}{x^2} = \dfrac{dy(x)}{y(x)}$ or $y = \dfrac{C}{x^3}e^{-1/x}$

57. (A) $r = 5$ cm, $\delta r = 0.06$, $A = \pi r^2$

$\delta A = 2\pi r\, \delta r = 10\pi \times 0.06 = 0.6\pi$

(B) $v = x^3$

$\delta v = 3x^2\, \delta x$

$\dfrac{\delta v}{v} \times 100 = 3\dfrac{\delta x}{x} \times 100 = 3 \times 1 = 3$

(C) $(x-2)\dfrac{dx}{dt} = 2\dfrac{dx}{dt} \Rightarrow x = 4$

(D) $A = \dfrac{\sqrt{3}}{4}x^2$

$$\frac{dA}{dt} = \frac{\sqrt{3}}{2}x\frac{dx}{dt} = \frac{\sqrt{3}}{2} \cdot 15 \cdot \frac{1}{10} = \frac{3\sqrt{3}}{4}$$

58. (A) $\dfrac{dy}{dx} = e^x(A\cos x + B\sin x) + e^x(-A\sin x + B\cos x)$

$$= y + e^x(-A\sin x + B\cos x)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{dy}{dx} + e^x(-A\sin x + B\cos x) + e^x(-A\cos x - B\sin x)$$

$$= \frac{dy}{dx} + \left(\frac{dy}{dx} - y\right) - y = 2\left(\frac{dy}{dx} - y\right)$$

So, degree = 1 and order = 2

(B) The equation of the family is $y^2 = 4a(x - b)$ where a, b are arbitrary constants.

$\therefore \quad 2y\dfrac{dy}{dx} = 4a$ or $\left(\dfrac{dy}{dx}\right)^2 + y\dfrac{d^2y}{dx^2} = 0$

So, degree = 1 and order = 2

(C) The equation of the family is

$$(x-c)^2 + y^2 = c^2$$

or $x^2 + y^2 - 2cx = 0$ or $\dfrac{x^2 + y^2}{x} = 2c$

$$\Rightarrow \quad \frac{\left(2x + 2y\dfrac{dy}{dx}\right)x - (x^2 + y^2)\cdot 1}{x^2} = 0$$

So, degree = 1 and order = 1

(D) The equation of the family is $\dfrac{x^2}{a^2 + \lambda} + \dfrac{y^2}{b^2 + \lambda} = 1$ because they have the same foci $(\pm\sqrt{a^2 - b^2}, 0)$.

On differentiating, $\dfrac{2x}{d^2 + \lambda} + \dfrac{2y}{b^2 + \lambda} \cdot \dfrac{dy}{dx} = 0$

or $\dfrac{x}{a^2 + \lambda} + \dfrac{yp}{b^2 + \lambda} = 0 \qquad \left(\text{Let } p = \dfrac{dy}{dx}\right)$

or $x(b^2 + \lambda) + yp(a^2 + \lambda) = 0$

$$\Rightarrow \qquad \lambda = \frac{-b^2x - a^2yp}{x + yp}$$

$\therefore$ The differential equation is

$$\frac{x^2}{a^2 + \dfrac{-b^2x - a^2yp}{x + yp}} + \frac{y^2}{b^2 + \dfrac{-b^2x - a^2yp}{x + yp}} = 1$$

or $$\frac{x(x + yp)}{a^2 - b^2} + \frac{y(x + yp)}{(b^2 - a^2)p} = 1$$

So, order = 1 and degree = 2

59. $2x^2y\,dx - 2y^4\,dx + 2x^3dy + 3xy^3\,dy = 0$

Dividing throughout by x^3y, we get

$$2\frac{dx}{x} - \frac{2y^3}{x^3}dx + 2\cdot\frac{dy}{y} + \frac{3y^2}{x^2}dy = 0$$

$$\Rightarrow \qquad 2\frac{dx}{x} + 2\frac{dy}{y} + \frac{3y^2x^4\,dy - 2y^3x^3dx}{x^6} = 0$$

$$\Rightarrow \qquad 2\frac{dx}{x} + 2\frac{dy}{y} + \frac{3y^2x^2\,dy - 2y^3x\,dx}{x^4} = 0$$

$$\Rightarrow \qquad 2d(\ln x) + 2d(\ln y) + d\left(\frac{y^3}{x^2}\right) = 0$$

$$\Rightarrow \qquad 2\ln|x| + 2\ln|y| + \frac{y^3}{x^2} = C$$

when $x = 1, y = 1$. So, $C = 1$

60. Let the salt content at time 't' be u lb, so that its rate of change is du/dt.

$$= 2 \text{ gal} \times 2 \text{ lb} = 4 \text{ lb / min}$$

If c be the concentration of the brine at time t, the rate at which the salt content decreases due to the out flow

$$= 2 \text{ gal} \times \text{c lb / min}$$
$$= 2\text{c lb / min}$$

$$\therefore \qquad \frac{du}{dt} = 4 - 2c \qquad \ldots(i)$$

Also, since there is no increase in the volume of the liquid, the concentrations $\quad c = \dfrac{u}{50}$

$\therefore$ Eq. (i) becomes $\dfrac{du}{dt} = 4 - \dfrac{2u}{50}$

Separating the variables and integrating, we have

$$\int dt = 25\int \frac{du}{100 - u}$$

or $$t = -25\ln(100 - u) + K \qquad \ldots(ii)$$

Initially when $t = 0, u = 0$

$$0 = -25\ln 100 + K \qquad \ldots(iii)$$

Eliminating 'K' from Eqs. (ii) and (iii), we get

$$t = 25\ln\left(\frac{100}{100 - u}\right)$$

Taking $t = t_1$ when $u = 40$ and $t = t_2$ when $u = 80$, we have

$$t_1 = 25\ln\left(\frac{100}{60}\right) \quad \text{and} \quad t_2 = 25\ln\left(\frac{100}{20}\right)$$

$\therefore$ The required time $(t_2 - t_1) = 25\left(\ln 5 - \ln\dfrac{5}{3}\right) = 25\ln 3$

$$= 25 \times 1.0986 = 26 \text{ min } 28 \text{ s}$$
$$= 1648 \text{ s} = 206 \times 8 = 206 \times \lambda$$

$\therefore \qquad \lambda = 8$

61. $f(x + f(y) + xf(y)) = y + f(x) + yf(x)$ ...(i)

Differentiating w.r.t. x and y is constant

$$f'(x + f(y) + xf(y))(1 + f(y)) = f'(x) + yf'(x) \qquad \ldots(ii)$$

From Eq. (i) again differentiating w.r.t. y as x is constant

$$f'(x + f(y) + xf(y))(1 + x)\,f'(y) = 1 + f(x) \qquad \ldots(iii)$$

From Eqs. (ii) and (iii)

$$\frac{1 + f(y)}{(1 + x)\,f'(y)} = \frac{(1 + y)\,f'(x)}{1 + f(x)}$$

$$\frac{(1 + y)\,f'(y)}{1 + f(y)} = \frac{1 + f(x)}{(1 + x)\,f'(x)} = \lambda$$

$\therefore \quad f'(y) = \dfrac{1 + f(y)}{1 + y}\lambda$ and $f'(x) = \dfrac{1 + f(x)}{\lambda(1 + x)}$

$\therefore \qquad \lambda = \dfrac{1}{\lambda} \Rightarrow \lambda = \pm 1 \;\therefore\; \dfrac{f'(x)}{1 + f(x)} = \pm\dfrac{1}{1 + x}$

Integrating both the sides $f(x) = C(1 + x)^{\pm 1} - 1$

From Eq. (i) put $x = y = 0$

$$f(f(0)) = f(0)$$

From Eq. (ii), $f(0) = C - 1$

So, $$f(C - 1) = C - 1$$

$\therefore \qquad C = 0, 1$ (taking +ve sign)

So, $f(x) = -1$ and $f(x) = (1 + x) - 1 = x$ and $C = 1$

$\therefore \qquad f(x) = (1 + x)^{-1} - 1 = \dfrac{-x}{1 + x}$

$$1 + f(x) = \frac{1}{1 + x}$$

$\therefore \qquad 1 + f(2009) = \dfrac{1}{2010} \;\therefore\; (2010)(1 + f(2009)) = 1$

62. $\phi'(x) + 2\phi(x) \le 1$

$$e^{2x}\,\phi'(x) + 2\phi(x)\,e^{2x} \le e^{2x}$$

$$\frac{d}{dx}\left(e^{2x}\,\phi(x) - \frac{1}{2}e^{2x}\right) \le 0$$

$\therefore \left(e^{2x}\,\phi(x) - \dfrac{1}{2}\right)$ is a non-increasing function of x.

$\Rightarrow \phi(x) - \dfrac{1}{2}$ is a non-increasing function of x.

$\therefore 2\phi(x)$ is always less than or equal to 1.

63. $\dfrac{1}{2}(1 + x)^{-1/2} - \dfrac{a}{2}(1 + y)^{-1/2}\dfrac{dy}{dx} = 0$

$$\Rightarrow \quad \frac{1}{\sqrt{1 + x}} = \frac{a}{\sqrt{1 + y}}\cdot\frac{dy}{dx} \;\Rightarrow\; a = \frac{\sqrt{1 + y}}{\sqrt{1 + x}}\cdot\frac{1}{dy/dx}$$

Putting this value in the given equation,

$$\frac{dy}{dx}\sqrt{1 + x} - \frac{1 + y}{\sqrt{1 + x}} = \frac{dy}{dx}$$

$$\Rightarrow \quad (1+x)\frac{dy}{dx} = 1 + y + \sqrt{1+x}\,\frac{dy}{dx}$$

$$\Rightarrow \quad (1 + x - \sqrt{1+x})\frac{dy}{dx} = 1 + y$$

$\therefore$ Degree of the given equation is 1.

64. Here, $2f^5(x)\cdot f'(x)\cdot\left|1+(f'(x))^2\right| = f''(x)$

$$\Rightarrow \quad f^4(x)d(f^2(x)) = \frac{f''(x)}{1+(f'(x))^2}$$

$$\Rightarrow \quad \left|f^4(x)d(f^2(x))\right| = \int d(\tan^{-1}(f'(x)))$$

$$\Rightarrow \quad \frac{f^6(x)}{3} + C = \tan^{-1}(f'(x)), \text{ as } f(0) = f'(0) = 0 \Rightarrow c = 0$$

$$\Rightarrow \quad \frac{f^6(x)}{3} = \tan^{-1}(f'(x))$$

$$\Rightarrow \quad -\frac{\pi}{2} < \frac{f^6(x)}{3} < \frac{\pi}{2}$$

$$\therefore \quad -\left(\frac{3\pi}{2}\right)^{1/6} < f(x) < \left(\frac{3\pi}{2}\right)^{1/6}$$

$\Rightarrow$ Number of integers between k_1 and k_2 are 3.

65. Put, $dy/dx = t$

$\therefore$ $dt/dx - t + e^{2x} = 0$, I.F. $= e^{\int -dx = e^{-x}}$

$\therefore$ Solution is, $t\cdot e^{-x} = \int -e^{2x}\cdot e^{-x}\,dx + C$

$\Rightarrow$ $te^{-x} = -e^x + C$ $y'(0) = 1 \Rightarrow C = 2$

$$\therefore \quad dy/dx = \frac{2-e^x}{e-x}\int dy = \int (2e^x - e^{2x})dx$$

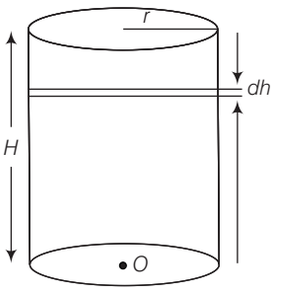
$$\Rightarrow \quad y = 2^{ex} - \frac{e^{2x}}{2} + C^1, y(0) = 2 \Rightarrow C^1 = 1/2$$

$$\therefore \quad y(x) = 2e^x - \frac{e^{2x}}{2} + \frac{1}{2} \quad \Rightarrow \quad y(x) \le \frac{5}{2} \text{ for } x = \log 2$$

$\therefore$ $[2x] = [2\log 2] = [\log 4] = 1$

66. Let in time dt the decrease in water level in the tank is dh, then amount of water flown out in time $dt = \pi r^2\cdot dh$

Now, through the hole the amount of water flown = (Volume of cylinder of cross sectional area 'a' and length vdt).



$$= a\cdot v\cdot dt = a\,\mu\sqrt{2gh}\,dt$$

Hence, $\mu a\sqrt{2gh}\,dt = -\pi r^2 dh$

$$\Rightarrow \quad dt = \frac{-\pi r^2}{\mu a\sqrt{2g}}\cdot h^{-1/2}\,dh$$

Now, when $t = 0$, $h = H$ and when $t = t$, $h = 0$

Thus, $$\int_0^t dt = \frac{-\pi r^2}{\mu a\sqrt{2g}}\int_H^0 h^{-1/2}\,dh$$

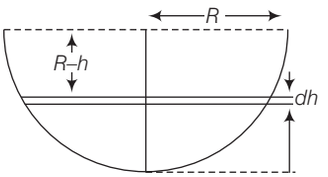
$$\Rightarrow \quad t = \frac{-\pi r^2}{\mu a\sqrt{2g}}\cdot 2(\sqrt{h})_H^0$$

$$\Rightarrow \quad t = \frac{\pi r^2}{\mu a}\cdot\sqrt{\frac{2H}{g}}$$

67. Let at time t the depth of water is h; the radius of water surface is r.

Then, $r^2 = R^2 - (R-h)^2$

$\Rightarrow$ $r^2 = 2Rh - h^2$



Now, if in time dt the decrease in water level is dh, then

$$-\pi r^2\,dh = 0.6\sqrt{2gh}\cdot a\,dt$$

(a is cross-sectional area of the outlet)

$$\Rightarrow \quad \frac{-\pi}{(0.6)\,a\sqrt{2g}}(2Rh - h^2)\frac{dh}{\sqrt{h}} = dt$$

$$\Rightarrow \quad \frac{\pi}{(0.6)\,a\sqrt{2g}}\int_R^0 (h^{3/2} - 2R\,h^{1/2})\,dh = \int_0^t dt$$

$$\Rightarrow \quad \frac{\pi}{(0.6)\,a\sqrt{2g}}\left[\frac{2}{5}h^{5/2} - \frac{4}{3}Rh^{3/2}\right]_R^0 = t$$

$$\Rightarrow \quad t = \frac{\pi}{(0.6)\,a\sqrt{2g}}\left[0 - R^{5/2}\left(\frac{2}{5} - \frac{4}{3}\right)\right] = \frac{7\pi\times 10^5}{135\sqrt{g}}$$

68. We have been given,

$f(xy) = e^{xy-x-y}\{e^y f(x) + e^x f(y)\}, \forall\ x, y \in R^+$ ...(i)

Replacing $x = 1$, $y = 1$ in Eq. (i), we get

$f(1) = e^{1-1-1}\{e f(1) + e f(1)\} \Rightarrow f(1) = 0$

Now, $$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f\left(x\left(1+\frac{h}{x}\right)\right) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{e^{x+h-x-1-\frac{h}{x}}(e^{1+h/x}\cdot f(x) + e^x\cdot f(1+h/x)) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{e^h f(x) + e^{h-1-\frac{h}{x}+x}\cdot f\left(1+\frac{h}{x}\right) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x)(e^h - 1) + e^{h-1-\frac{h}{x}+x}\cdot\left(f\left(1+\frac{h}{x}\right) - f(1)\right)}{h}$$

$$= \lim_{h \to 0} \frac{f(x)(e^h - 1)}{h} + \lim_{h \to 0} \frac{e^{h-1-\frac{h}{x}+x}\left(f\left(1+\frac{h}{x}\right) - f(1)\right)}{\frac{h}{x} \cdot x}$$

$$= f(x) + \frac{e^{x-1} \cdot f'(1)}{x} = f(x) + \frac{e^x}{x}$$

$$\Rightarrow \quad f'(x) - f(x) = \frac{e^x}{x}$$

$$\Rightarrow \quad \frac{e^x f'(x) - f(x) e^x}{e^{2x}} = \frac{1}{x}$$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{f(x)}{e^x}\right) = \frac{1}{x}$$

Integrating both the sides, we get

$$\frac{f(x)}{e^x} = \ln|x| + C$$

Since, $f(1) = 0 \Rightarrow C = 0$

Thus, $f(x) = e^x \cdot \ln(x)$

69. Let $P(x, f(x))$ be a point lying on the curve in first quadrant.

Equation of normal and tangent at P are

$$(Y - f(x)) = -\frac{1}{f'(x)}(X - x) \quad \text{and}$$

$(Y - f(x)) = f'(x)(X - x)$ respectively.

$$\Rightarrow \quad A \equiv \left(x - \frac{f(x)}{f'(x)}, 0\right), B \equiv (x + f(x) f'(x), 0)$$

Since, the mid-point of segment AB is origin.

$$\Rightarrow \quad 2x - \frac{f(x)}{f'(x)} + f(x) \cdot f'(x) = 0$$

$$\Rightarrow \quad f(x)(f'(x))^2 + 2x f'(x) - f(x) = 0$$

$$\Rightarrow \quad f'(x) = \frac{-2x \pm \sqrt{4x^2 + 4f^2(x)}}{2f(x)} = \frac{-x + \sqrt{x^2 + f^2(x)}}{f(x)}$$

Negative sign been neglected as $f'(x) > 0$

Thus, we have, $x + f(x) \cdot f'(x) = \sqrt{x^2 + f^2(x)}$

$$\Rightarrow \quad \frac{d}{2dx}(x^2 + f^2(x)) = \sqrt{x^2 + f^2(x)}$$

$$\Rightarrow \quad \frac{d(x^2 + f^2(x))}{2\sqrt{x^2 + f^2(x)}} = dx$$

Integrating both the sides, we get $\sqrt{x^2 + f^2(x)} = x + \lambda$

It passes through $(1, \sqrt{3})$.

Hence, $\lambda + 1 = 2 \Rightarrow \lambda = 1$

$\therefore$ Curve is $x^2 + f^2(x) = (x + 1)^2$ or $y^2 = 1 + 2x$

70. We have been given,

$$\frac{dy_1}{dx} + Py_1 = Q, \frac{dy_2}{dx} + Py_2 = Q$$

Now, $y_2 = y_1 z \Rightarrow \frac{dy_2}{dx} = y_1 \frac{dz}{dx} + z \frac{dy_1}{dx}$

$$\Rightarrow \quad y_1 \frac{dz}{dx} + z \frac{dy_1}{dx} + Py_1 z = Q$$

$$\Rightarrow \quad y_1 \cdot \frac{dz}{dx} + z\left(\frac{dy_1}{dx} + py_1\right) = Q$$

$$\Rightarrow \quad y_1 \cdot \frac{dz}{dx} + z\,Q = Q$$

$$\Rightarrow \quad y_1 \frac{dz}{dx} = Q(1 - z)$$

$$\Rightarrow \quad \frac{dz}{1 - z} = \frac{Q}{y_1} dx$$

$$\Rightarrow \quad \ln|z - 1| = -\int \frac{Q}{y_1} dx + \lambda, \text{ '}\lambda\text{' being constant of integration}$$

$$\Rightarrow \quad z = 1 + c\, e^{\int -\frac{Q}{y_1} dx}$$

71. We have, $f(x) = x + x^2 \int_0^1 z f(z)\, dz + x \int_0^1 z^2 f(z)\, dz$

$$f(x) = x + x^2 \lambda_1 + x \lambda_2 \quad \text{(say)}$$

Now, $\lambda_1 = \int_0^1 z f(z)\, dz = \int_0^1 ((1 + \lambda_2) z + z^2 \lambda_1) z\, dz$

$$= \frac{1 + \lambda_2}{3} + \frac{\lambda_1}{4}$$

$$\Rightarrow \quad 9\lambda_1 - 4\lambda_2 = 4 \quad \ldots(i)$$

Also, $\lambda_2 = \int_0^1 z^2 f(z)\, dz$

$$= \int_0^1 ((1 + \lambda_2) z^3 + z^4 \lambda_1)\, dz$$

$$= \frac{(1 + \lambda_2)}{4} + \frac{\lambda_1}{5}$$

$$\Rightarrow \quad 15\lambda_2 - 4\lambda_1 = 5 \quad \ldots(ii)$$

From Eqs. (i) and (ii),

$$\lambda_1 = \frac{80}{119} \quad \text{and} \quad \lambda_2 = \frac{61}{119}$$

$$\Rightarrow \quad f(x) = x + \frac{80}{119} x^2 + \frac{61}{119} x = \frac{20x}{119}(4 + 9x)$$

72. $f(x + f(y) + x f(y)) = y + f(x) + y f(x)$

Keep y constant and differentiating the expression w.r.t. 'x', we get

$$f'(x + f(y) + x(y))(1 + f(y)) = f'(x)(1 + y) \quad \ldots(i)$$

Similarly, differentiate the expression w.r.t. 'y' and keep x constant, we get

$$f'(x + f(y) + x f(y))(f'(y)(1 + x)) = (1 + f(x)) \quad \ldots(ii)$$

Dividing Eq. (i) by Eq. (ii), we get

$$\frac{1 + f(y)}{f'(y)(1 + x)} = \frac{f'(x)(1 + y)}{(1 + f(x))}$$

$$\Rightarrow \quad \frac{1 + f(y)}{f'(y)(1 + y)} = \frac{f'(x)(1 + x)}{(1 + f(x))} = C$$

$$\Rightarrow \quad f'(y) = \frac{1}{C}\left[\frac{1 + f(y)}{1 + y}\right] \text{ and } f'(x) = C\frac{1 + f(x)}{(1 + x)}$$

$$\Rightarrow \quad C = \frac{1}{C} \Rightarrow C = \pm 1$$

$$\therefore \quad \frac{f'(x)}{1 + f(x)} = \pm \frac{1}{1 + x}$$

$$\Rightarrow \quad f(x) = \lambda_1 (1 + x)^{\pm 1} - 1$$

Replacing $x, y \to 0$ in the Eq. (i), we get

$$f(f(0)) = f(0)$$

Now, $f(x) = \lambda_1 (1+x)^{\pm 1} - 1 \Rightarrow f(0) = \lambda_1 - 1$

and $f(f(0)) = f(\lambda_1 - 1) = \lambda_1 \lambda_1^{\pm 1} - 1$

Since, $f(f(0)) = f(0),\ \lambda_1 \lambda_1^{\pm 1} - 1 = \lambda_1 - 1$

$\Rightarrow \lambda_1^{1 \pm 1} = \lambda_1$

By taking +ve sign, we get $\lambda_1 = 0, 1$

$\Rightarrow f(x) = -1$ or $f(x) = x$

By taking − ve sign, we get $\lambda_1 = 1$

$\Rightarrow f(x) = \frac{1}{1+x} - 1 = -\frac{x}{1+x}$

73. Given, $(x^2 + xy + 4x + 2y + 4)\frac{dy}{dx} - y^2 = 0$

$\Rightarrow [(x^2 + 4x + 4) + y(x+2)]\frac{dy}{dx} - y^2 = 0$

$\Rightarrow [(x+2)^2 + y(x+2)]\frac{dy}{dx} - y^2 = 0$

Put $x + 2 = X$ and $y = Y$, then

$$(X^2 + XY)\frac{dY}{dX} - Y^2 = 0$$

$\Rightarrow X^2 dY + XY dY - Y^2 dX = 0$

$\Rightarrow X^2 dY + Y(X dY - Y dX) = 0$

$\Rightarrow -\frac{dY}{Y} = \frac{X dY - Y dX}{X^2}$

$\Rightarrow -d(\log |Y|) = d\left(\frac{Y}{X}\right)$

On integrating both sides, we get

$$-\log |Y| = \frac{Y}{X} + C, \text{ where } x + 2 = X \text{ and } y = Y$$

$\Rightarrow -\log |y| = \frac{y}{x+2} + C$ ...(i)

Since, it passes through the point (1, 3).

$\therefore -\log 3 = 1 + C$

$\Rightarrow C = -1 - \log 3 = -(\log e + \log 3)$

$= -\log 3e$

$\therefore$ Eq. (i) becomes

$$\log |y| + \frac{y}{x+2} - \log(3e) = 0$$

$\Rightarrow \log\left(\frac{|y|}{3e}\right) + \frac{y}{x+2} = 0$ ...(ii)

Now, to check option (a), $y = x + 2$ intersects the curve.

$\Rightarrow \log\left(\frac{|x+2|}{3e}\right) + \frac{x+2}{x+2} = 0$

$\Rightarrow \log\left(\frac{|x+2|}{3e}\right) = -1$

$\Rightarrow \frac{|x+2|}{3e} = e^{-1} = \frac{1}{e}$

$\Rightarrow |x+2| = 3$ or $x + 2 = \pm 3$

$\therefore x = 1, -5$ (rejected), as $x > 0$ [given]

$\therefore x = 1$ only one solution.

Thus, (a) is the correct answer.

To check option (c), we have $y = (x+2)^2$

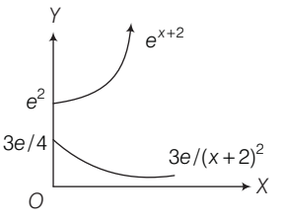
and $\log\left(\frac{|y|}{3e}\right) + \frac{y}{x+2} = 0$

$\Rightarrow \log\left[\frac{|x+2|^2}{3e}\right] + \frac{(x+2)^2}{x+2} = 0$

$\Rightarrow \log\left[\frac{|x+2|^2}{3e}\right] = -(x+2)$

$\Rightarrow \frac{(x+2)^2}{3e} = e^{-(x+2)}$ or $(x+2)^2 \cdot e^{x+2} = 3e$

$\Rightarrow e^{x+2} = \frac{3e}{(x+2)^2}$



Clearly, they have no solution.

To check option (d), $y = (x+3)^2$

i.e. $\log\left[\frac{|x+3|^2}{3e}\right] + \frac{(x+3)^2}{(x+2)} = 0$

To check the number of solutions.

Let $g(x) = 2\log(x+3) + \frac{(x+3)^2}{(x+2)} - \log(3e)$

$\therefore g'(x) = \frac{2}{x+3} + \left(\frac{(x+2)\cdot 2(x+3) - (x+3)^2 \cdot 1}{(x+2)^2}\right) - 0$

$= \frac{2}{x+3} + \frac{(x+3)(x+1)}{(x+2)^2}$

Clearly, when $x > 0$, then, $g'(x) > 0$

$\therefore$ $g(x)$ is increasing, when $x > 0$.

Thus, when $x > 0$, then $g(x) > g(0)$

$$g(x) > \log\left(\frac{3}{e}\right) + \frac{9}{4} > 0$$

Hence, there is no solution.

Thus, option (d) is true.

74. Here, $f'(x) = 2 - \frac{f(x)}{x}$

or $\frac{dy}{dx} + \frac{y}{x} = 2$ [i.e. linear differential equation in y]

Integrating Factor, IF $= e^{\int \frac{1}{x} dx} = e^{\log x} = x$

$\therefore$ Required solution is $y \cdot (\text{IF}) = \int Q(\text{IF}) dx + C$

$\Rightarrow y(x) = \int 2(x)\, dx + C$

$\Rightarrow yx = x^2 + C$

$\therefore y = x + \frac{C}{x}$ [$\because C \neq 0$, as $f(1) \neq 1$]

(a) $\lim_{x \to 0^+} f'\left(\frac{1}{x}\right) = \lim_{x \to 0^+} (1 - Cx^2) = 1$

$\therefore$ Option (a) is correct.

(b) $\lim_{x \to 0^+} x\, f\left(\frac{1}{x}\right) = \lim_{x \to 0^+} (1 + Cx^2) = 1$

$\therefore$ Option (b) is incorrect.

(c) $\lim_{x \to 0^+} x^2 f'(x) = \lim_{x \to 0^+} (x^2 - C) = -C \neq 0$

$\therefore$ Option (c) is incorrect.

(d) $f(x) = x + \frac{C}{x}, C \neq 0$

For $C > 0$, $\lim_{x \to 0^+} f(x) = \infty$

$\therefore$ Function is not bounded in $(0, 2)$.

$\therefore$ Option (d) is incorrect.

75. Here, $(1 + e^x)y' + y e^x = 1$

$\Rightarrow \quad \frac{dy}{dx} + e^x \cdot \frac{dy}{dx} + ye^x = 1$

$\Rightarrow \quad dy + e^x dy + ye^x\, dx = dx$

$\Rightarrow \quad dy + d(e^x y) = dx$

On integrating both sides, we get

$$y + e^x y = x + C$$

Given, $y(0) = 2$

$\Rightarrow \quad 2 + e^0 \cdot 2 = 0 + C$

$\Rightarrow \quad C = 4$

$\therefore \quad y(1 + e^x) = x + 4$

$\Rightarrow \quad y = \frac{x + 4}{1 + e^x}$

Now at $x = -4, y = \frac{-4 + 4}{1 + e^{-4}} = 0$

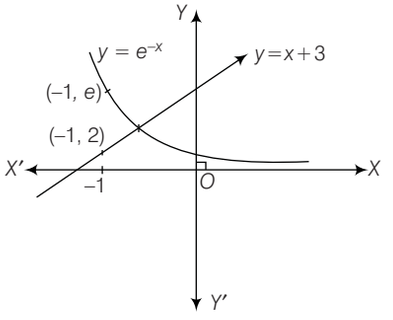
$\therefore \quad y(-4) = 0 \qquad \ldots(i)$

For critical points, $\frac{dy}{dx} = 0$

i.e. $\frac{dy}{dx} = \frac{(1 + e^x) \cdot 1 - (x + 4)e^x}{(1 + e^x)^2} = 0$

$\Rightarrow \quad e^x (x + 3) - 1 = 0$

or $e^{-x} = (x + 3)$



Clearly, the intersection point lies between $(-1, 0)$.

$\therefore$ $y(x)$ has a critical point in the interval $(-1, 0)$.

76. Since, centre lies on $y = x$.

$\therefore$ Equation of circle is $x^2 + y^2 - 2ax - 2ay + c = 0$

On differentiating, we get

$$2x + 2yy' - 2a - 2ay' = 0$$

$\Rightarrow \quad x + yy' - a - ay' = 0$

$\Rightarrow \quad a = \frac{x + yy'}{1 + y'}$

Again differentiating, we get

$$0 = \frac{(1 + y')[1 + yy'' + (y')^2] - (x + yy') \cdot (y'')}{(1 + y')^2}$$

$\Rightarrow (1 + y')\,[1 + (y')^2 + yy'']\, - (x + yy')\,(y'') = 0$

$\Rightarrow \quad 1 + y'\,[(y')^2 + y' + 1] + y''(y - x) = 0$

On comparing with $Py'' + Qy' + 1 = 0$, we get

$$P = y - x$$

and $Q = (y')^2 + y' + 1$

77. (i) Solution of the differential equation $\frac{dy}{dx} + Py = Q$ is

$$y \cdot (\text{IF}) = \int Q \cdot (\text{IF})\, dx + c$$

where, $\text{IF} = e^{\int P\, dx}$

(ii) $\int_{-a}^{a} f(x)\, dx = 2 \int_0^a f(x)\, dx$, if $f(-x) = f(x)$

Given differential equation

$$\frac{dy}{dx} + \frac{x}{x^2 - 1} y = \frac{x^4 + 2x}{\sqrt{1 - x^2}}$$

This is a linear differential equation.

$$\text{IF} = e^{\int \frac{x}{x^2 - 1} dx} = e^{\frac{1}{2} \ln |x^2 - 1|} = \sqrt{1 - x^2}$$

$\Rightarrow$ Solution is

$$y\sqrt{1 - x^2} = \int \frac{x(x^3 + 2)}{\sqrt{1 - x^2}} \cdot \sqrt{1 - x^2}\, dx$$

or $y\sqrt{1 - x^2} = \int (x^4 + 2x)\, dx = \frac{x^5}{5} + x^2 + c$

$$f(0) = 0 \Rightarrow c = 0$$

$\Rightarrow f(x)\sqrt{1 - x^2} = \frac{x^5}{5} + x^2$

Now, $\int_{-\sqrt{3}/2}^{\sqrt{3}/2} f(x)dx = \int_{-\sqrt{3}/2}^{\sqrt{3}/2} \frac{x^2}{\sqrt{1 - x^2}}\, dx$ [using property]

$$= 2 \int_0^{\sqrt{3}/2} \frac{x^2}{\sqrt{1 - x^2}}\, dx$$

$$= 2 \int_0^{\pi/3} \frac{\sin^2 \theta}{\cos \theta} \cos \theta\, d\theta \text{ [taking } x = \sin \theta]$$

$$= 2 \int_0^{\pi/3} \sin^2 \theta\, d\theta = \int_0^{\pi/3} (1 - \cos 2\theta)\, d\theta$$

$$= \left(\theta - \frac{\sin 2\theta}{2}\right)_0^{\pi/3} = \frac{\pi}{3} - \frac{\sin 2\pi/3}{2} = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$$

78. Whenever we have linear differential equation containing inequality, we should always check for increasing or decreasing,

i.e. for $\frac{dy}{dx} + Py < 0 \quad \Rightarrow \quad \frac{dy}{dx} + Py > 0$

Multiply by integrating factor, i.e. $e^{\int P dx}$ and convert into total differential equation.

Here, $f'(x) < 2f(x)$, multiplying by $e^{-\int 2dx}$

$f'(x)\cdot e^{-2x} - 2e^{-2x} f(x) < 0 \;\Rightarrow\; \frac{d}{dx}(f(x)\cdot e^{-2x}) < 0$

$\therefore \quad \phi(x) = f(x)e^{-2x}$ is decreasing for $x \in \left[\frac{1}{2}, 1\right]$

Thus, when $x > \frac{1}{2}$

$$\phi(x) < \phi\left(\frac{1}{2}\right) \Rightarrow e^{-2x} f(x) < e^{-1}\cdot f\left(\frac{1}{2}\right)$$

$$\Rightarrow \quad f(x) < e^{2x-1}\cdot 1, \quad \text{given } f\left(\frac{1}{2}\right) = 1$$

$$\Rightarrow \quad 0 < \int_{1/2}^{1} f(x)dx < \int_{1/2}^{1} e^{2x-1}dx$$

$$\Rightarrow \quad 0 < \int_{1/2}^{1} f(x)dx < \left(\frac{e^{2x-1}}{2}\right)_{1/2}^{1}$$

$$\Rightarrow \quad 0 < \int_{1/2}^{1} f(x)dx < \frac{e-1}{2}$$

79. To solve homogeneous differential equation,

i.e. substitute $\frac{y}{x} = v$

$$\therefore \quad y = vx \;\Rightarrow\; \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Here, slope of the curve at (x, y) is

$$\frac{dy}{dx} = \frac{y}{x} + \sec\left(\frac{y}{x}\right)$$

Put $\frac{y}{x} = v$

$$\therefore \quad v + x\frac{dv}{dx} = v + \sec(v)$$

$$\Rightarrow \quad x\frac{dv}{dx} = \sec(v) \Rightarrow \int \frac{dv}{\sec v} = \int \frac{dx}{x}$$

$$\Rightarrow \quad \int \cos v\, dv = \int \frac{dx}{x} \;\Rightarrow\; \sin v = \log x + \log c$$

$$\Rightarrow \quad \sin\left(\frac{y}{x}\right) = \log(cx)$$

As it passes through $\left(1, \frac{\pi}{6}\right) \;\Rightarrow\; \sin\left(\frac{\pi}{6}\right) = \log c$

$$\Rightarrow \quad \log c = \frac{1}{2}$$

$$\therefore \quad \sin\left(\frac{y}{x}\right) = \log x + \frac{1}{2}$$

80. Let $\phi(x) = e^{-x} f(x)$

Here, $\phi'(x) < 0,\ x \in \left(0, \frac{1}{4}\right)$

and $\phi'(x) > 0,\ x \in \left(\frac{1}{4}, 1\right)$

$$\Rightarrow e^{-x} f'(x) - e^{-x} f(x) < 0,\ x \in \left(0, \frac{1}{4}\right)$$

$$\Rightarrow f'(x) < f(x), 0 < x < \frac{1}{4}$$

81. Here, $f''(x) - 2f'(x) + f(x) \geq e^x$

$$\Rightarrow \quad f''(x)e^{-x} - f'(x)e^{-x} - f'(x)e^{-x} + f(x)e^{-x} \geq 0$$

$$\Rightarrow \quad \frac{d}{dx}\{f'(x)e^{-x}\} - \frac{d}{dx}\{f(x)e^{-x}\} \geq 1$$

$$\Rightarrow \quad \frac{d}{dx}\{f'(x)e^{-x} - f(x)e^{-x}\} \geq 1$$

$$\Rightarrow \quad \frac{d^2}{dx^2}\{e^{-x} f(x)\} \geq 1, \forall x \in [0, 1]$$

$\therefore \phi(x) = e^{-x} f(x)$ is concave function.

$$f(0) = f(1) = 0$$

$$\Rightarrow \quad \phi(0) = 0 = f(1)$$

$$\Rightarrow \quad \phi(x) < 0$$

$$\Rightarrow \quad e^{-x} f(x) < 0$$

$$\therefore \quad f(x) < 0$$

82. Here, $f(x) = (1-x)^2 \cdot \sin^2 x + x^2 \geq 0, \forall\ x$

and $g(x) = \int_1^x \left(\frac{2(t-1)}{t+1} - \log t\right) f(t)dt$

$$\Rightarrow \quad g'(x) = \left\{\frac{2(x-1)}{(x+1)} - \log x\right\} \cdot \underbrace{f(x)}_{+\text{ ve}} \qquad \ldots(i)$$

For $g'(x)$ to be increasing or decreasing.

Let $\phi(x) = \frac{2(x-1)}{x+1} - \log x$

$$\phi'(x) = \frac{4}{(x+1)^2} - \frac{1}{x} = \frac{-(x-1)}{x(x+1)^2}$$

$$\phi'(x) < 0, \forall\ x > 1$$

$$\Rightarrow \quad \phi(x) < \phi(1) \;\Rightarrow\; \phi(x) < 0 \qquad \ldots(ii)$$

From Eqs. (i) and (ii), $g'(x) < 0,\ x \in (1, \infty)$

$\therefore g(x)$ is decreasing on $x \in (1, \infty)$.

83. Here, $f(x) + 2x = (1-x)^2 \cdot \sin^2 x + x^2 + 2x \qquad \ldots(i)$

where, I : $f(x) + 2x = 2(1+x)^2 \qquad \ldots(ii)$

$$\therefore \quad 2(1+x^2) = (1-x)^2 \sin^2 x + x^2 + 2x$$

$$\Rightarrow \quad (1-x)^2 \sin^2 x = x^2 - 2x + 2$$

$$\Rightarrow \quad (1-x)^2 \sin^2 x = (1-x)^2 + 1$$

$$\Rightarrow \quad (1-x)^2 \cos^2 x = -1$$

which is never possible.

$\therefore$ I is false.

Again, let $h(x) = 2f(x) + 1 - 2x(1+x)$

where, $h(0) = 2f(0) + 1 - 0 = 1$

$h(1) = 2(1) + 1 - 4 = -3$ as $[h(0)h(1) < 0]$

$\Rightarrow$ $h(x)$ must have a solution.

$\therefore$ II is true.

84. Linear differential equation under one variable.

$$\frac{dy}{dx}+Py=Q;\quad \text{IF}=e^{\int P\,dx}$$

$\therefore$ Solution is, $y\,(\text{IF})=\int Q\cdot(\text{IF})dx+C$

$$y'-y\tan x=2x\sec x \text{ and } y(0)=0$$

$\Rightarrow$ $\dfrac{dy}{dx}-y\tan x=2x\sec x$

$\therefore$ $\text{IF}=\int e^{-\tan x}\,dx=e^{\log|\cos x|}=\cos x$

Solution is $y\cdot\cos x=\int 2x\sec x\cdot\cos x\,dx+C$

$\Rightarrow$ $y\cdot\cos x=x^2+C$

As $y(0)=0$

$\Rightarrow$ $C=0$

$\therefore$ $y=x^2\sec x$

Now, $y\left(\dfrac{\pi}{4}\right)=\dfrac{\pi^2}{8\sqrt{2}}$

$\Rightarrow$ $y'\left(\dfrac{\pi}{4}\right)=\dfrac{\pi}{\sqrt{2}}+\dfrac{\pi^2}{8\sqrt{2}}\Rightarrow y\left(\dfrac{\pi}{3}\right)=\dfrac{2\pi^2}{9}$

$\Rightarrow$ $y'\left(\dfrac{\pi}{3}\right)=\dfrac{4\pi}{3}+\dfrac{2\pi^2}{3\sqrt{3}}$

85. $\dfrac{dy}{dx}+y\cdot g'(x)=g(x)\,g'(x)$

$$\text{IF}=e^{\int g'(x)\,dx}=e^{g(x)}$$

$\therefore$ Solution is $y\,(e^{g(x)})=\int g(x)\cdot g'(x)\cdot e^{g(x)}\,dx+C$

Put $g(x)=t,\ g'(x)\,dx=dt$

$$y(e^{g(x)})=\int t\cdot e^t\,dt+C=t\cdot e^t-\int 1\cdot e^t\,dt+C$$

$$=t\cdot e^t-e^t+C$$

$$y\,e^{g(x)}=(g(x)-1)\,e^{g(x)}+C \qquad \ldots(\text{i})$$

Given, $y(0)=0,\ g(0)=g(2)=0$

$\therefore$ Eq. (i) becomes,

$$y(0)\cdot e^{g(0)}=(g(0)-1)\cdot e^{g(0)}+C$$

$\Rightarrow$ $0=(-1)\cdot 1+C\Rightarrow C=1$

$\therefore$ $y(x)\cdot e^{g(x)}=(g(x)-1)\,e^{g(x)}+1$

$\Rightarrow$ $y(2)\cdot e^{g(2)}=(g(2)-1)\,e^{g(2)}+1,$ where $g(2)=0$

$\Rightarrow$ $y(2)\cdot 1=(-1)\cdot 1+1$

$$y(2)=0$$

86. From given integral equation, $f(0)=0$.

Also, differentiating the given integral equation w.r.t. x

$$f'(x)=f(x)$$

If $f(x)\neq 0$

$\Rightarrow$ $\dfrac{f'(x)}{f(x)}=1\Rightarrow \log f(x)=x+c$

$\Rightarrow$ $f(x)=e^c e^x$

$\because$ $f(0)=0\Rightarrow e^c=0,$ a contradiction

$\therefore$ $f(x)=0,\ \forall\ x\in R\Rightarrow f(\ln 5)=0$

Alter

Given, $f(x)=\int_0^x f(t)\,dt$

$\Rightarrow$ $f(0)=0$ and $f'(x)=f(x)$

If $f(x)\neq 0$

$\Rightarrow$ $\dfrac{f'(x)}{f(x)}=1\Rightarrow \ln f(x)=x+c$

$\Rightarrow$ $f(x)=e^c\cdot e^x$

$\because$ $f(0)=0$

$\Rightarrow e^c=0,$ a contradiction

$\therefore$ $f(x)=0,\ \forall\ x\in R\Rightarrow f(\ln 5)=0$

87. Given, $\dfrac{dy}{dx}=\dfrac{y\sqrt{y^2-1}}{x\sqrt{x^2-1}}$

$$\int\frac{dy}{y\sqrt{y^2-1}}=\int\frac{dx}{x\sqrt{x^2-1}}$$

$\Rightarrow$ $\sec^{-1}y=\sec^{-1}x+c$

At $x=2,\ y=\dfrac{2}{\sqrt{3}};\ \dfrac{\pi}{6}=\dfrac{\pi}{3}+c\Rightarrow c=-\dfrac{\pi}{6}$

Now, $y=\sec\left(\sec^{-1}x-\dfrac{\pi}{6}\right)=\cos\left[\cos^{-1}\dfrac{1}{x}-\cos^{-1}\dfrac{\sqrt{3}}{2}\right]$

$$=\cos\left[\cos^{-1}\left(\frac{\sqrt{3}}{2x}+\sqrt{1-\frac{1}{x^2}}\sqrt{1-\frac{3}{4}}\right)\right]$$

$$y=\frac{\sqrt{3}}{2x}+\frac{1}{2}\sqrt{1-\frac{1}{x^2}}$$

88. Given differential equation is

$$y(1+xy)\,dx=x\,dy$$

$\Rightarrow$ $y\,dx+xy^2\,dx=x\,dy$

$\Rightarrow$ $\dfrac{x\,dy-y\,dx}{y^2}=x\,dx$

$\Rightarrow$ $-\dfrac{(y\,dx-x\,dy)}{y^2}=x\,dx\Rightarrow -d\left(\dfrac{x}{y}\right)=x\,dx$

On integrating both sides, we get

$$-\frac{x}{y}=\frac{x^2}{2}+C \qquad \ldots(\text{i})$$

$\because$ It passes through $(1,-1)$.

$\therefore$ $1=\dfrac{1}{2}+C\Rightarrow C=\dfrac{1}{2}$

Now, from Eq. (i) $-\dfrac{x}{y}=\dfrac{x^2}{2}+\dfrac{1}{2}$

$\Rightarrow$ $x^2+1=-\dfrac{2x}{y}\Rightarrow y=-\dfrac{2x}{x^2+1}$

$\therefore$ $f\left(-\dfrac{1}{2}\right)=\dfrac{4}{5}$

89. Given differential equation is

$$(x\log x)\frac{dy}{dx}+y=2x\log x, \qquad (x\geq 1)$$

$\Rightarrow$ $\dfrac{dy}{dx}+\dfrac{y}{x\log x}=2$

This is a linear differential equation.

$$\therefore \quad \text{IF} = e^{\int \frac{1}{x \log x} dx} = e^{\log(\log x)} = \log x$$

Now, the solution of given differential equation is given by

$$y \cdot \log x = \int \log x \cdot 2dx$$

$$\Rightarrow \quad y \cdot \log x = 2\int \log x \, dx$$

$$\Rightarrow \quad y \cdot \log x = 2[x \log x - x] + c$$

At $x = 1, c = 2$

$$\Rightarrow \quad y \cdot \log x = 2[x \log x - x] + 2$$

At $x = e$,

$y = 2(e - e) + 2$

$\Rightarrow \quad y = 2$

90. Given differential equation $\frac{dp}{dt} - \frac{1}{2}p(t) = -200$ is a linear differential equation.

Here, $p(t) = \frac{-1}{2}, Q(t) = -200$

$$\text{IF} = e^{\int -\left(\frac{1}{2}\right)dt} = e^{-\frac{t}{2}}$$

Hence, solution is

$$p(t).\text{IF} = \int Q(t)\,\text{IF}\,dt$$

$$p(t) \cdot e^{-\frac{t}{2}} = \int -200 \cdot e^{-\frac{t}{2}} dt$$

$$p(t) \cdot e^{-\frac{t}{2}} = 400 e^{-\frac{t}{2}} + K$$

$$\Rightarrow \quad p(t) = 400 + ke^{-1/2}$$

If $p(0) = 100$, then $k = -300$

$$\Rightarrow \quad p(t) = 400 - 300e^{\frac{t}{2}}$$

91. Given, $\frac{dP}{dx} = (100 - 12\sqrt{x})$

$$\Rightarrow \quad dP = (100 - 12\sqrt{x})\,dx$$

On integrating both sides, we get

$$\int dP = \int (100 - 12\sqrt{x})\,dx$$

$$P = 100x - 8x^{3/2} + C$$

When $x = 0$, then $P = 2000$

$$\Rightarrow \quad C = 2000$$

Now, when $x = 25$, then

$$P = 100 \times 25 - 8 \times (25)^{3/2} + 2000$$

$$= 2500 - 8 \times 125 + 2000$$

$$= 4500 - 1000 = 3500$$

92. **Given**

(i) The population of mouse at time 't' satisfies the differential equation $p'(t) = \frac{dp(t)}{dt} = 0.5p(t) - 450$

(ii) Population of mouse at time $t = 0$ is

$$p(0) = 850$$

To find The time at which the population of the mouse will become zero, i.e. to find the value of 't' at which $p(t) = 0$.

Let's solve the differential equation first

$$p'(t) = \frac{dp(t)}{dt} = 0.5\,p(t) - 450$$

$$\Rightarrow \quad \frac{2dp(t)}{p(t) - 900} = dt$$

$$\Rightarrow \quad \int \frac{2dp(t)}{p(t) - 900} = \int dt$$

$\Rightarrow 2\log|\,p(t) - 900|\, = t + C$, where C is the constant of integration.

To find the value of 'C', let's substitute $t = 0$.

$\Rightarrow \; 2\log|\,p(0) - 900\,| = 0 + C$

$$\Rightarrow \quad C = 2\log|\,850 - 900\,|$$

$$\Rightarrow \quad C = 2\log 50$$

Now, substituting the value of C back in the solution, we get

$$2\log|\,p(t) - 900\,| = t + 2\log 50$$

Here, since we want to find the value of t at which $p(t) = 0$, hence substituting $p(t) = 0$, we get

$$2\log|\,0 - 900\,| = t + 2\log 50$$

$$\Rightarrow \quad t = 2\log\left|\frac{900}{50}\right|$$

$$\Rightarrow \quad t = 2\log 18$$

93. Here, $\frac{dy}{dx} = y + 3 > 0$ and $y(0) = 2$

$$\Rightarrow \quad \int \frac{dy}{y + 3} = \int dx$$

$$\Rightarrow \quad \log|y + 3| = x + C$$

Since, $y(0) = 2$

$$\Rightarrow \quad \log_e|2 + 3| = 0 + C$$

$$\therefore \quad C = \log_e 5$$

$$\Rightarrow \quad \log_e|y + 3| = x + \log_e 5$$

When $x = \log_e 2$

$$\Rightarrow \quad \log_e|y + 3| = \log_e 2 + \log_e 5 = \log_e 10$$

$$\Rightarrow \quad y + 3 = 10$$

$$\Rightarrow \quad y = 7$$

94. Given, $\frac{d\{V(t)\}}{dt} = -k(T - t)$

$$\therefore \quad d\{V(t)\} = -k(T - t)\,dt$$

On integrating both sides, we get

$$V(t) = -k\frac{(T - t)^2}{2(-1)} + C$$

$$\Rightarrow \quad V(t) = \frac{k}{2}(T - t)^2 + C$$

$\because$ At $t = 0, V(t) = I$

$$\therefore \quad I = \frac{k}{2}(T - 0)^2 + C$$

$$\Rightarrow \quad C = I - \frac{k}{2}T^2$$

$$\therefore \quad V(t) = \frac{k}{2}(T-t)^2 + I - \frac{k}{2}T^2$$

Now, $V(T) = I - \frac{k}{2}T^2$

95. Since, $\cos x\, dy = y \sin x\, dx - y^2\, dx$

$$\Rightarrow \quad \frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y}\tan x = -\sec x$$

Put $-\frac{1}{y} = z$

$$\Rightarrow \quad \frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow \quad \frac{dz}{dx} + (\tan x)z = -\sec x$$

This is a linear differential equation.

Therefore,

$$\text{IF} = e^{\int \tan x\, dx} = e^{\log \sec x} = \sec x$$

Hence, the solution is

$$z \cdot (\sec x) = \int -\sec x \cdot \sec x\, dx + C_1$$

$$\Rightarrow \quad -\frac{1}{y}\sec x = -\tan x + C_1$$

$$\Rightarrow \quad \sec x = y(\tan x + C)$$

96. Given, $y = c_1 e^{c_2 x}$

$$\Rightarrow \quad y' = c_2 c_1 e^{c_2 x}$$

$$\Rightarrow \quad y' = c_2 y \qquad \ldots(\text{i})$$

$$\Rightarrow \quad y'' = c_2 y' \qquad \ldots(\text{ii})$$

$$\Rightarrow \quad y'' = \frac{(y')^2}{y} \qquad \left[\text{from Eq. (i), } c_2 = \frac{y'}{y}\right]$$

$$\Rightarrow \quad yy'' = (y')^2$$

97. Equation of circle having centre (h, k) and radius a is $(x-h)^2 + (y-k)^2 = a^2$.

The equation of family of circles with centre on $y = 2$ and of radius 5 is

$$(x-\alpha)^2 + (y-2)^2 = 5^2 \qquad \ldots(\text{i})$$

$$\Rightarrow \quad x^2 + \alpha^2 - 2\alpha x + y^2 + 4 - 4y = 25$$

On differentiating w.r.t. x, we get

$$2x - 2\alpha + 2y\frac{dy}{dx} - 4\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \alpha = x + \frac{dy}{dx}(y-2)$$

On putting the value of α in Eq. (i), we get

$$\left[x - x - \frac{dy}{dx}(y-2)\right]^2 + (y-2)^2 = 5^2$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 (y-2)^2 = 25 - (y-2)^2$$

$$\Rightarrow \quad y'^2(y-2)^2 = 25 - (y-2)^2$$

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PREFACE

It is a matter of great pleasure and pride for me to introduce to you this book "Play with Graphs". As a teacher, guiding the Engineering aspirants for over a decade now, I have always been in the lookout for right approach to understand various mathematical problems. I had always felt the need of a book that can develop and sharpen the ideas of the students within a very short span of time.

The book in your hands, aims to help you solve various mathematical problems by the use of graphs. Ways to draw different types of graphs are very easy and can be understood by even an average student. I feel glad to mention that the use of graphs in solving various mathematical problems has been well illustrated in this book.

I would like to take this opportunity to thank
M/s Arihant Prakashan for assigning this work to me.
It is their inspiration that has encouraged me to bring this book in this present form.

I would also like to thank Arihant DTP Unit for the nice laser typesetting.

Valuable suggestions from the students and teachers are always welcome, and these will find due places in the ensuing editions.

Amit M. Agarwal

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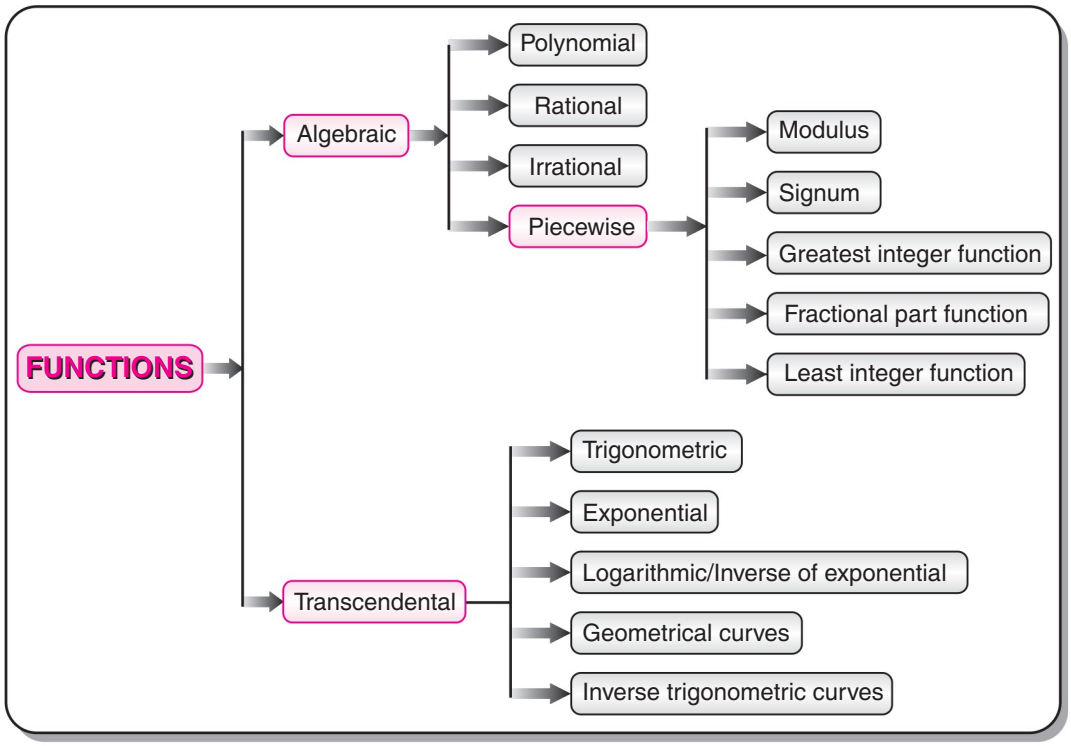
Chapter 1

INTRODUCTION OF GRAPHS

➥ **In this section, we shall revise some basic curves which are given as.**



1.1 ALGEBRAIC FUNCTIONS

1. Polynomial Function

A function of the form:

$$f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n ;$$

where $n \in N$ and $a_0, a_1, a_2, \ldots, a_n \in R$.

Then, *f* is called a polynomial function. "f(x) is also called polynomial in *x*".

Some of basic polynomial functions are

(i) Identity function/Graph of $f(x) = x$

A function f defined by $f(x) = x$ for all $x \in R$, is called the identity function.

Here, $y = x$ clearly represents a straight line passing through the origin and inclined at an angle of 45° with x-axis shown as:

The domain and range of identity functions are both equal to R.

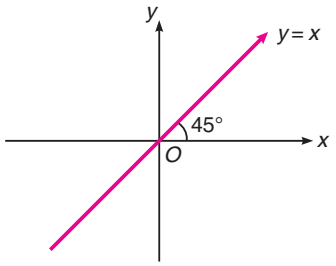


Fig. 1.1

(ii) Graph of $f(x) = x^2$

A function given by $f(x) = x^2$ is called the square function.

The domain of square function is R and its range is $R^+ \cup \{0\}$ or $[0, \infty)$

Clearly $y = x^2$, is a parabola. Since $y = x^2$ is an even function, so its graph is **symmetrical about y-axis**, shown as:

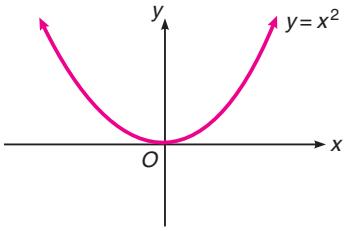


Fig. 1.2

(iii) Graph of $f(x) = x^3$

A function given by $f(x) = x^3$ is called the cube function.

The domain and range of cube are both equal to R.

Since, $y = x^3$ is an odd function, so its graph is **symmetrical about opposite quadrant**, *i.e.,* **"origin",** shown as:

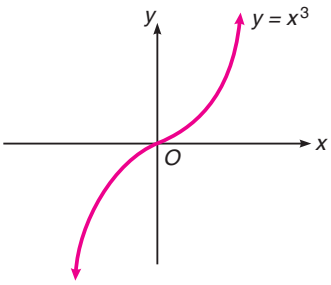


Fig. 1.3

(iv) Graph of $f(x) = x^{2n}$; $n \in N$

If $n \in N$, then function f given by $f(x) = x^{2n}$ is an even function. So, its graph is always **symmetrical about y-axis.**

Also, $x^2 > x^4 > x^6 > x^8 > \ldots$ for all $x \in (-1, 1)$

and $x^2 < x^4 < x^6 < x^8 < \ldots$ for all $x \in (-\infty, -1) \cup (1, \infty)$

Graphs of $y = x^2$, $y = x^4$, $y = x^6$,..., etc. are shown as:

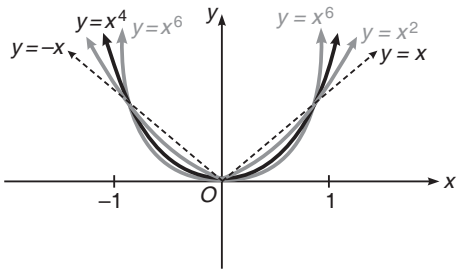


Fig. 1.4

(v) Graph of $f(x) = x^{2n-1}$; $n \in N$

If $n \in N$, then the function f given by $f(x) = x^{2n-1}$ is an odd function. So, its graph is **symmetrical about origin or opposite quadrants.**

Here, comparison of values of x, x^3, x^5,... for

$x \in (1, \infty)$ $\quad x < x^3 < x^5 < \ldots$

$x \in (0, 1)$ $\quad x > x^3 > x^5 > \ldots$

$x \in (-1, 0)$ $\quad x < x^3 < x^5 < \ldots$

$x \in (-\infty, -1)$ $\quad x > x^3 > x^5 > \ldots$

Graphs of $f(x) = x$, $f(x) = x^3$, $f(x) = x^5$,... are shown as in Fig. 1.5

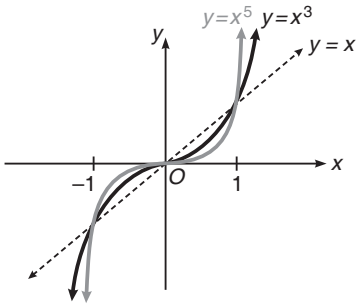


Fig. 1.5

2. Rational Expression

A function obtained by dividing a polynomial by another polynomial is called a rational function.

$$\Rightarrow \quad f(x) = \frac{P(x)}{Q(x)}$$

$$\text{Domain} \in R - \{x \mid Q(x) = 0\}$$

i.e., domain $\in$ R except those points for which denominator = 0.

Graphs of some Simple Rational Functions

(i) Graph of $f(x) = \frac{1}{x}$

A function defined by $f(x) = \frac{1}{x}$ is called the reciprocal function or rectangular hyperbola, with coordinate axis as asymptotes. The domain and range of $f(x) = \frac{1}{x}$ is $R - \{0\}$.

Since, f(x) is odd function, so its graph is **symmetrical about opposite quadrants.** Also, we observe

$$\lim_{x \to 0^+} f(x) = +\infty \quad \text{and} \quad \lim_{x \to 0^-} f(x) = -\infty \quad .$$

and $\quad \text{as } x \to \pm\infty \Rightarrow f(x) \to 0$

Thus, $f(x) = \frac{1}{x}$ could be shown as in Fig. 1.6.

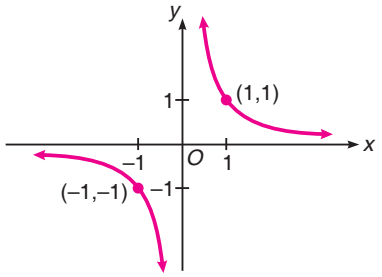


Fig. 1.6

(ii) Graph of $f(x) = \frac{1}{x^2}$

Here, $f(x) = \frac{1}{x^2}$ is an even function, so its graph is **symmetrical about *y*-axis**.

Domain of f(x) is $R - \{0\}$ and range is $(0, \infty)$.

Also, as $y \to \infty$ as $\lim_{x \to 0^+} f(x)$ or $\lim_{x \to 0^-} f(x)$.

and $\quad y \to 0$ as $\lim_{x \to \pm\infty} f(x)$.

Thus, $f(x) = \frac{1}{x^2}$ could be shown as in Fig. 1.7.

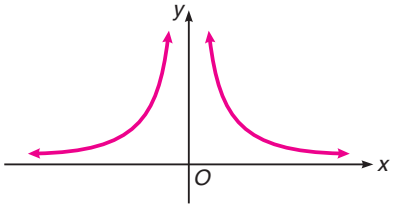


Fig. 1.7

(iii) Graph of $f(x) = \frac{1}{x^{2n-1}}$; $n \in N$

Here, $f(x) = \frac{1}{x^{2n-1}}$ is an odd function, so its graph is **symmetrical in opposite quadrants.**

Also, $y \to \infty$ when $\lim_{x \to 0^+} f(x)$ and

$y \to -\infty$ when $\lim_{x \to 0^-} f(x)$.

Thus, the graph for $f(x) = \frac{1}{x^3}$; $f(x) = \frac{1}{x^5}, \ldots$, etc. will be similar to the graph of $f(x) = \frac{1}{x}$ **which has asymptotes as coordinate axes,** shown as in Fig. 1.8

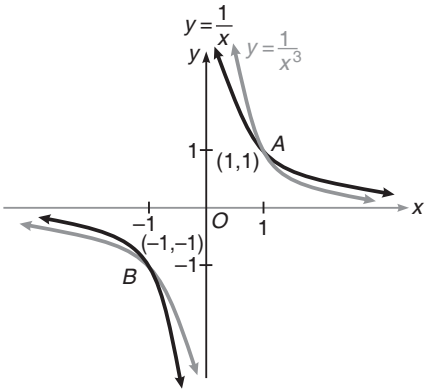


Fig. 1.8

(iv) Graph of $f(x) = \frac{1}{x^{2n}}$; $n \in N$

We observe that the function $f(x) = \frac{1}{x^{2n}}$ is an even function, so its graph is **symmetrical about *y*-axis.**

Also, $y \to \infty$ as $\lim_{x \to 0^+} f(x)$ or $\lim_{x \to 0^-} f(x)$

and $y \to 0$ as $\lim_{x \to -\infty} f(x)$ or $\lim_{x \to +\infty} f(x)$.

The values of y decrease as the values of x increase. Thus, the graph of $f(x) = \frac{1}{x^2}$; $f(x) = \frac{1}{x^4}, \ldots$, etc. will be similar as the graph of $f(x) = \frac{1}{x^2}$, which has **asymptotes as coordinate axis.** Shown as in Fig. 1.9.

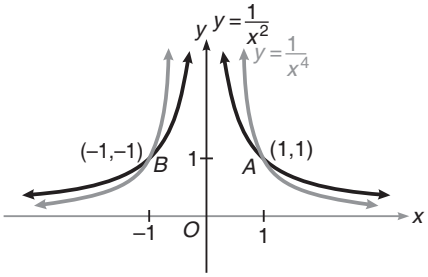


Fig. 1.9

3. Irrational Function

The algebraic function containing terms having non-integral rational powers of x are called irrational functions.

Graphs of Some Simple Irrational Functions

(i) Graph of $f(x) = x^{1/2}$

Here; $f(x) = \sqrt{x}$ is the portion of the parabola $y^2 = x$, which lies above x-axis.

Domain of $f(x) \in R^+ \cup \{0\}$ or $[0, \infty)$

and range of $f(x) \in R^+ \cup \{0\}$ or $[0, \infty)$.

Thus, the graph of $f(x) = x^{1/2}$ is shown as;

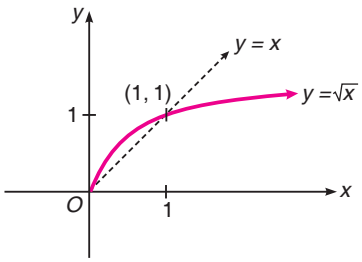


Fig. 1.10

Note If $f(x) = x^n$ and $g(x) = x^{1/n}$, then $f(x)$ and $g(x)$ are inverse of each other.
$\therefore$ $f(x) = x^n$ and $g(x) = x^{1/n}$ is the mirror image about $y = x$.

(ii) Graph of $f(x) = x^{1/3}$

As discussed above, if $g(x) = x^3$. Then $f(x) = x^{1/3}$ is image of $g(x)$ about $y = x$.

where domain $f(x) \in R$.

and range of $f(x) \in R$.

Thus, the graph of $f(x) = x^{1/3}$ is shown in Fig. 1.11;

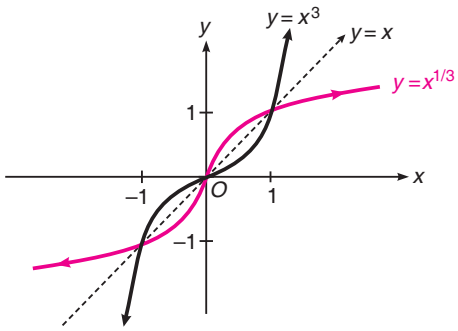


Fig. 1.11

(iii) Graph of $f(x) = x^{1/2n}$; $n \in N$

Here, $f(x) = x^{1/2n}$ is defined for all $x \in [0, \infty)$ and the values taken by $f(x)$ are positive.

So, domain and range of $f(x)$ are $[0, \infty)$.

Here, the graph of $f(x) = x^{1/2n}$ is the mirror image of the graph of $f(x) = x^{2n}$ about the line $y = x$, when $x \in [0, \infty)$.

Thus, $f(x) = x^{1/2}$, $f(x) = x^{1/4}$, … are shown as;

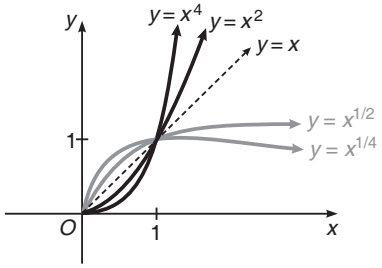


Fig. 1.12

(iv) Graph of $f(x) = x^{1/2n-1}$, when $n \in N$

Here, $f(x) = x^{1/2n-1}$ is defined for all $x \in R$. So, domain of $f(x) \in R$, and range of $f(x) \in R$. Also the graph of $f(x) = x^{1/2n-1}$ is the mirror image of the graph of $f(x) = x^{2n-1}$ about the line $y = x$ when $x \in R$.

Thus, $f(x) = x^{1/3}$, $f(x) = x^{1/5}$, …, are shown as;

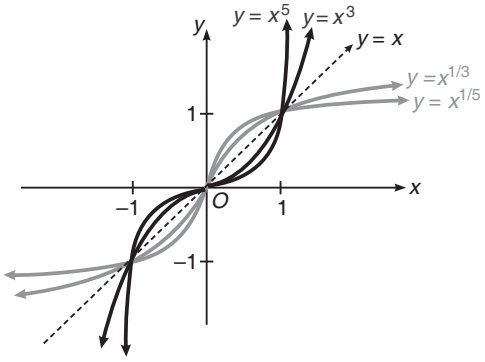


Fig. 1.13

Note We have discussed some of the simple curves for Polynomial, Rational and Irrational functions. Graphs of the some more difficult rational functions will be discussed in chapter 3. Such as;

$$y = \frac{x}{x+1} \quad y = \frac{1}{x^2-1}, \quad y = \frac{x^2+x+1}{x^2-x+1}, \ldots .$$

4. Piecewise Functions

As discussed piecewise functions are:

(a) Absolute value function (or modulus function),
(b) Signum function.
(c) Greatest integer function.
(d) Fractional part function.
(e) Least integer function.

(a) Absolute value function (or modulus function)

$$y = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

"It is the numerical value of x".

"It is symmetric about y-axis" where domain $\in R$ and range $\in [0, \infty)$.

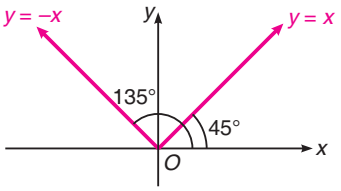


Fig. 1.14

Properties of modulus functions

(i) $|x| \le a \Rightarrow -a \le x \le a$; $(a \ge 0)$

(ii) $|x| \ge a \Rightarrow x \le -a$ or $x \ge a$; $(a \ge 0)$

(iii) $|x \pm y| \le |x| + |y|$

(iv) $|x \pm y| \ge ||x| - |y||$.

(b) Signum function; $y = Sgn(x)$

It is defined by;

$$y = \text{Sgn}(x) = \begin{cases} \dfrac{|x|}{x} \text{ or } \dfrac{x}{|x|}; & x \neq 0 \\ 0; & x = 0. \end{cases} = \begin{cases} +1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \end{cases}$$

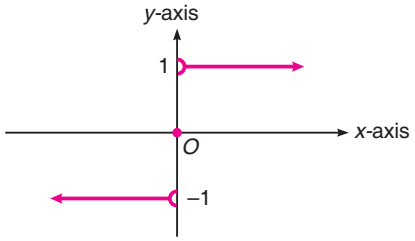


Fig. 1.15

Here, Domain of $f(x) \in R$.

and Range of $f(x) \in \{-1, 0, 1\}$.

(c) Greatest integer function

$[x]$ indicates the integral part of x which is nearest and smaller integer to x. It is also known as **floor of x**.

Thus, $[2.3] = 2$, $[0.23] = 0$, $[2] = 2$, $[-8.0725] = -9, \ldots$.

In general;

$n \le x < n + 1$ $(n \in \text{Integer}) \Rightarrow [x] = n$.

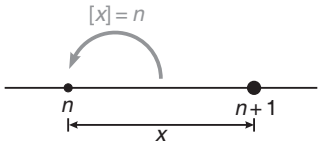


Fig. 1.16

Here, $f(x) = [x]$ could be expressed graphically as;

x	[x]
$0 \le x < 1$	0
$1 \le x < 2$	1
$2 \le x < 3$	2

Thus, $f(x) = [x]$ could be shown as;

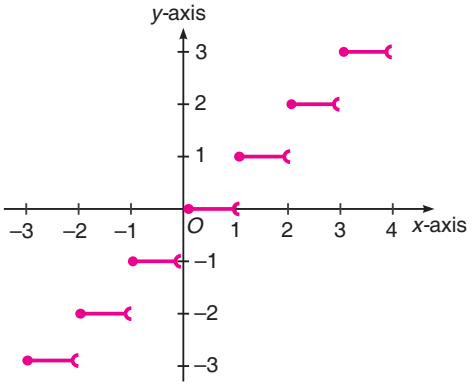


Fig. 1.17

Properties of greatest integer function

(i) $[x] = x$ holds, if x is integer.

(ii) $[x + I] = [x] + I$, if I is integer.

(iii) $[x + y] \ge [x] + [y]$.

(iv) If $[\phi(x)] \ge I$, then $\phi(x) \ge I$.

(v) If $[\phi(x)] \le I$, then $\phi(x) < I + 1$.

(vi) $[-x] = -[x]$, if $x \in$ integer.

(vii) $[-x] = -[x] - 1$, if $x \notin$ integer.

"It is also known as **stepwise function/floor of x.**"

(d) Fractional part of function

Here, {.} denotes the fractional part of x. Thus, in $y = \{x\}$.

$$x = [x] + \{x\} = I + f \; ; \text{ where } I = [x] \text{ and } f = \{x\}$$

$\therefore$ $y = \{x\} = x - [x]$, where $0 \le \{x\} < 1$; shown as:

x	$\{x\}$
$0 \le x < 1$	x
$1 \le x < 2$	$x - 1$
$2 \le x < 3$	$x - 2$
$-1 \le x < 0$	$x + 1$
$-2 \le x < -1$	$x + 2$

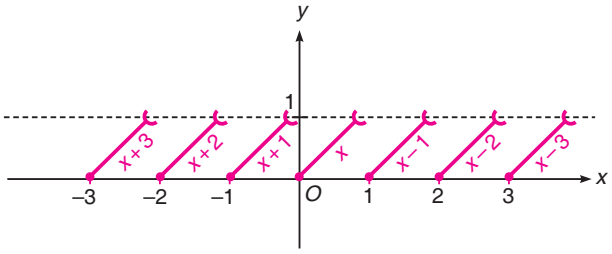


Fig. 1.18

Properties of fractional part of x

(i) $\{x\} = x$; if $0 \le x < 1$
(ii) $\{x\} = 0$; if $x \in$ integer.
(iii) $\{-x\} = 1 - \{x\}$; if $x \in$ integer.

(e) Least integer function

$$y = (x) = \lceil x \rceil,$$

(x) or $\lceil x \rceil$ indicates the integral part of x which is nearest and greatest integer to x.

It is known as ceiling of x.

Thus, $\lceil 2.3023 \rceil = 3$, $(0.23) = 1$, $(-8.0725) = -8$, $(-0.6) = 0$
In general, $n < x \le n + 1$ $(n \in \text{integer}))$
i. e., $\lceil x \rceil$ or $(x) = n + 1$
shown as;
Here, $f(x) = (x) = \lceil x \rceil$, can be expressed graphically as:

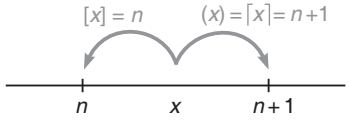


Fig. 1.19

x	$\lceil x \rceil = (x)$
$-1 < x \le 0$	0
$0 < x \le 1$	1
$1 < x \le 2$	2
$-2 < x \le -1$	-1
$-3 < x \le -2$	-2

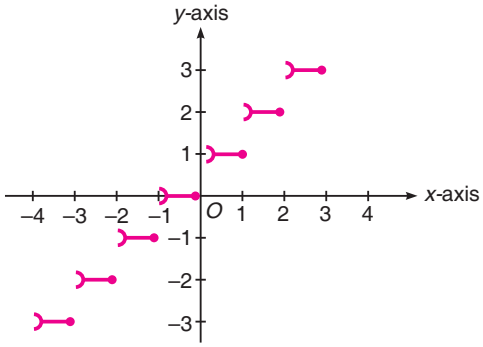


Fig. 1.20

Properties of least integer function

(i) $(x) = \lceil x \rceil = x$, if x is integer.
(ii) $(x + I) = \lceil x + I \rceil = (x) + I$; if $I \in$ integer.
(iii) Greatest integer converts $x = I + f$ to $[x] = I$ while $\lceil x \rceil$ converts to $(I + 1)$.

Note We shall discuss the curves:
$y = \{\sin x\}$, $y = \{x^3\}$, $y = \{\sin^{-1}(\sin x)\}$ $y = [\sin x]$, etc. in **chapter 2. (Curvature and Transformations)**.

1.2 TRANSCENDENTAL FUNCTIONS

1. Trigonometric Function

(a) Sine function

Here, f(x) = sin x can be discussed in two ways *i.e.,* Graph diagram and Circle diagram where Domain of sine function is "*R*" and range is [–1, 1].

Graph diagram

(On *x*-axis and *y*-axis)

f(x) = sin x, increases strictly from –1 to 1 as *x* increases from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, decreases strictly from 1 to –1 as x increases from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$ and so on. We have graph as;

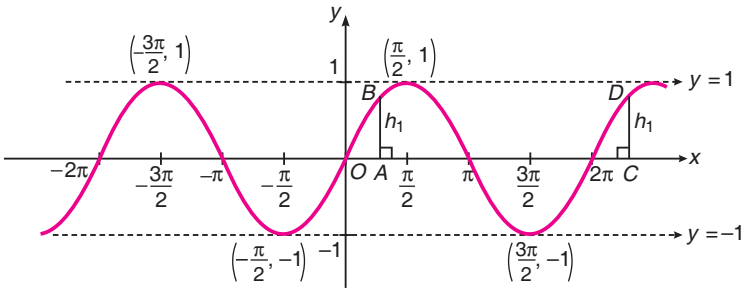


Fig. 1.21

Here, the height is same after every interval of 2π. (*i.e.,* In above figure, AB = CD after every interval of 2π).

∴ sin *x* is called periodic function with period 2π.

Circle diagram

(On trigonometric plane or using quadrants). Let a circle of radius '1', *i.e.,* unit circle.

Then, $\sin\alpha = \frac{a}{1}$,

$\sin\beta = \frac{b}{1}$,

$\sin\gamma = -\frac{c}{1}$,

$\sin\delta = -\frac{d}{1}, \ldots,$ shown as.

∴ sin x generates a circle of radius '1'.

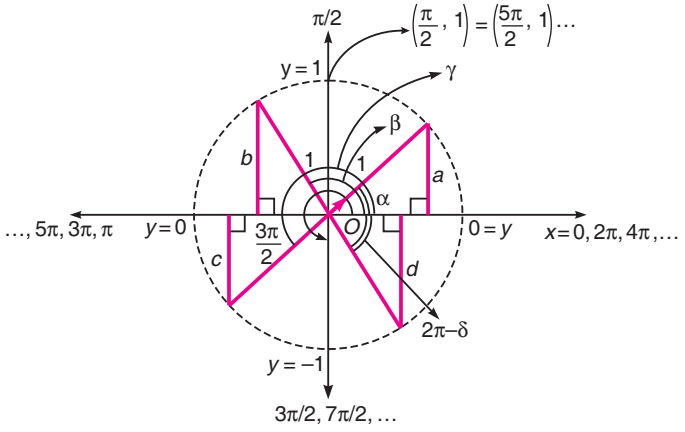


Fig. 1.22

(b) Cosine function

Here, f(x) = cos x

The domain of cosine function is *R* and the range is [–1, 1].

Graph diagram (on *x*-axis and *y*-axis)

As discussed, cos x decreases strictly from 1 to –1 as *x* increases from 0 to π, increases strictly from –1 to 1 as *x* increases from π to 2π and so on. Also, cos x is periodic with period 2π.

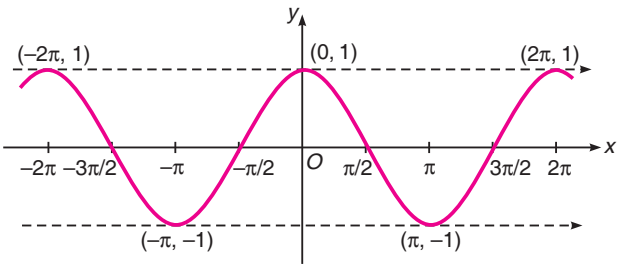


Fig. 1.23

Circle diagram

Let a circle of radius '1', *i.e.,* a unit circle.

Then, $\cos\alpha = \frac{a}{1}$, $\cos\beta = -\frac{b}{1}$,

$\cos\gamma = \frac{c}{1}$, $\cos\delta = -\frac{d}{1}$

$\therefore$ cos x generates a circle of **radius '1'**.

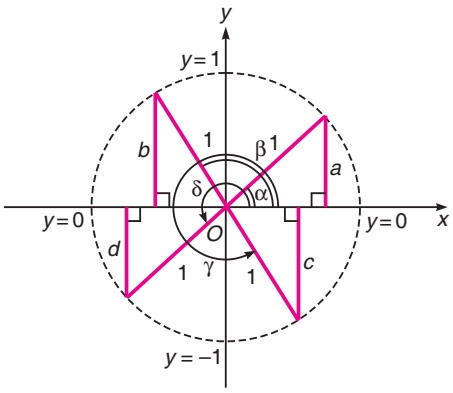


Fig. 1.24

(c) Tangent function

$f(x) = \tan x$

The domain of the function $y = \tan x$ is;

$R - \left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots\right\}$

i.e., $R - \left\{(2n+1)\frac{\pi}{2}\right\}$

and Range $\in$ R or $(-\infty, \infty)$.

The function $y = \tan x$ increases strictly from $-\infty$ to $+\infty$ as *x* increases from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, $\frac{\pi}{2}$ to $\frac{3\pi}{2}$, $\frac{3\pi}{2}$ to $\frac{5\pi}{2}$, ... and so on.

The graph is shown as :

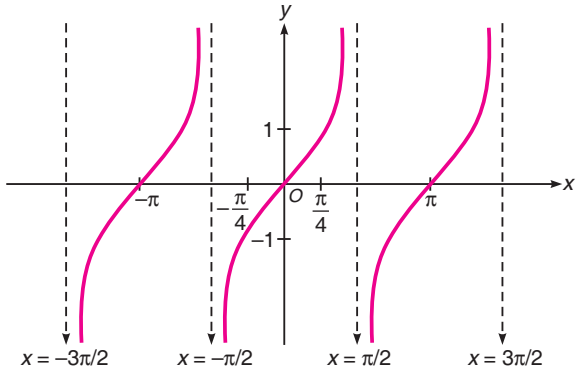


Fig. 1.25

Note Here, the curve tends to meet at $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2} \pm \frac{5\pi}{2}, \ldots$ but never meets or tends to infinity.

$\therefore$ $x = \pm\frac{\pi}{2}$, $\pm\frac{3\pi}{2}$, $\pm\frac{5\pi}{2}$ ... are asymptotes to $y = \tan x$.

(d) Cosecant function

$f(x) = \text{cosec } x$

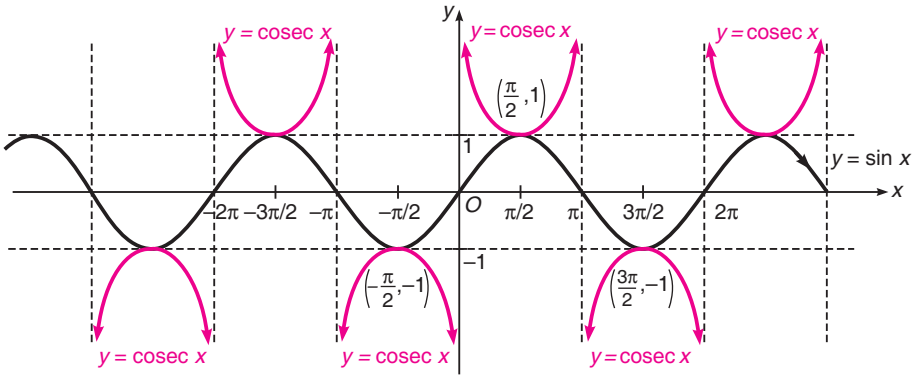


Fig. 1.26

Here, domain of y = cosec x is,

$$R - \{0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots\}$$

i. e., $R - \{n\pi \mid n \in z\}$ and range $\in R - (-1, 1)$.

as shown in Fig. 1.26.

The function y = cosec x is periodic with period 2π.

(e) Secant function

$$f(x) = \sec x$$

Here, $$\text{domain} \in R - \left\{(2n+1)\frac{\pi}{2} \;\middle|\; n \in z\right\}$$

$$\text{Range} \in R - (-1, 1)$$

Shown as:

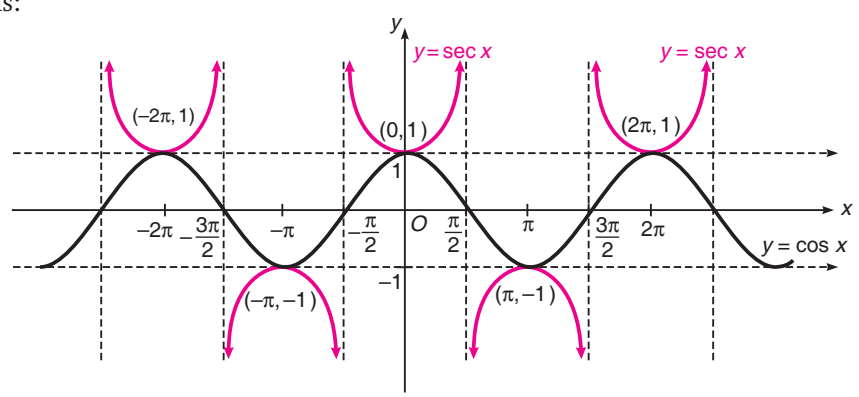


Fig. 1.27

The function y = sec x is periodic with period 2π.

Note (i) The curve $y = \text{cosec } x$ tends to meet at $x = 0, \pm\pi, \pm 2\pi, \ldots$ at infinity.

$\therefore$ $x = 0, \pm\pi, \pm 2\pi, \ldots$

or $x = n\pi$, $n \in$ integer are asymptote to $y = \text{cosec } x$.

(ii) The curve $y = \sec x$ tends to meet at $x = \pm\frac{\pi}{2} \pm \frac{3\pi}{2}, \ldots$ at infinity.

$\therefore x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$ or $x = (2n+1)\frac{\pi}{2}$, $n \in$ integer are asymptote to $y = \text{cosec } x$.

Here, we have used the notation of asymptotes of a curve in the context of special curves, but we would have a detailed discussion in chapter 3.

(f) Cotangent function

$$f(x) = \cot x$$

Here, domain $\in R - \{n\pi \mid n \in z\}$ Range $\in R$.

which is periodic with period π, and has $x = n\pi$, $n \in z$ as asymptotes. As shown in Fig. 1.28;

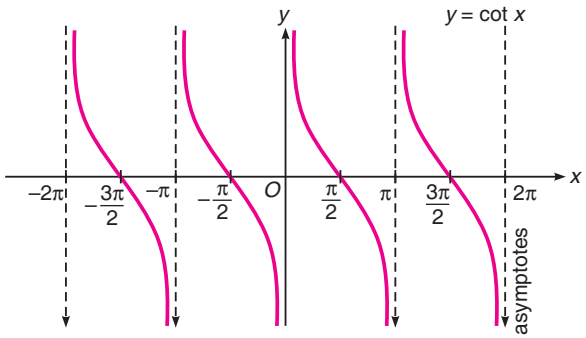


Fig. 1.28

2. Exponential Function

Here, $f(x) = a^x$, $a > 0$, $a \neq 1$, and $x \in R$, where domain $\in R$,

Range $\in (0, \infty)$.

Case I. $a > 1$

Here, $f(x) = y = a^x$ increase with the increase in x, *i.e.*, $f(x)$ is increasing function on R.

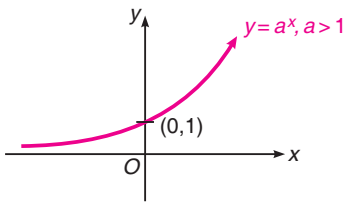


Fig. 1.29

For example;

$y = 2^x$, $y = 3^x$, $y = 4^x$,... have;

$2^x < 3^x < 4^x < \ldots$ for $x > 1$

and $2^x > 3^x > 4^x > \ldots$ for $0 < x < 1$.

and they can be shown as;

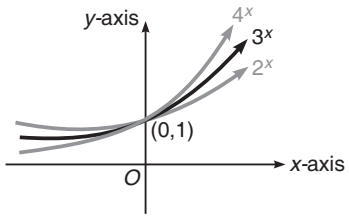


Fig. 1.30

Case II. $0 < a < 1$

Here, $f(x) = a^x$ decrease with the increase in x, *i.e.*, $f(x)$ is decreasing function on R.

"In general, exponential function increases or decreases as $(a > 1)$ or $(0 < a < 1)$ **respectively".**

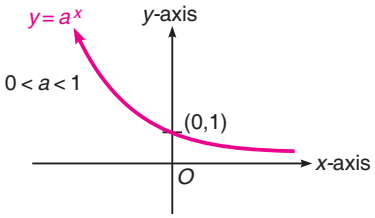


Fig. 1.31

3. Logarithmic Function

(Inverse of Exponential)

The function $f(x) = \log_a x$; $(x, a > 0)$ and $a \neq 1$ is a logarithmic function.

Thus, the domain of logarithmic function is all real positive numbers and their range is the set R of all real numbers.

We have seen that $y = a^x$ is strictly increasing **when** $a > 1$ and strictly decreasing **when** $0 < a < 1$.

Thus, the function is invertible. The inverse of this function is denoted by $\log_a x$, we write

$y = a^x \Rightarrow x = \log_a y$;

where $x \in R$ and $y \in (0, \infty)$

writing $y = \log_a x$ in place of $x = \log_a y$, we have the graph of $y = \log_a x$.

Thus, logarithmic function is also known as inverse of exponential function.

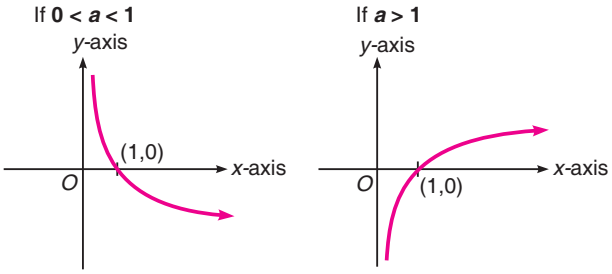


Fig. 1.32

Properties of logarithmic function

1. $\log_e (ab) = \log_e a + \log_e b$ $\{a, b > 0\}$

2. $\log_e \left(\frac{a}{b}\right) = \log_e a - \log_e b$ $\{a, b > 0\}$

3. $\log_e a^m = m \log_e a$ $\{a > 0 \text{ and } m \in R\}$

4. $\log_a a = 1$ $\{a > 0 \text{ and } a \neq 1\}$

5. $\log_{b^m} a = \frac{1}{m} \log_b a$ $\{a, b > 0, b \neq 1 \text{ and } m \in R\}$

6. $\log_b a = \frac{1}{\log_a b}$ $\{a, b > 0 \text{ and } a, b \neq 1\}$

7. $\log_b a = \frac{\log_m a}{\log_m b}$ $\{a, b > 0 \neq \{1\} \text{ and } m > 0\}$

8. $a^{\log_a m} = m$ $\{a, m > 0 \text{ and } a \neq 1\}$

9. $a^{\log_c b} = b^{\log_c a}$ $\{a, b, c > 0 \text{ and } c \neq 1\}$

10. If $\log_m x > \log_m y \Rightarrow \begin{cases} x > y, & \text{if } m > 1 \\ x < y, & \text{if } 0 < m < 1 \end{cases}$ $\{m, x, y, > 0 \text{ and } m \neq 1\}$

which could be graphically shown as;

If $m > 1$ (Graph of $\log_m a$)

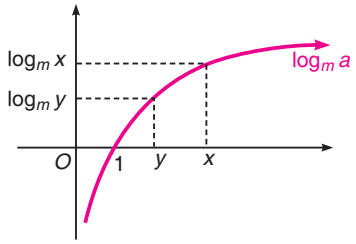


Fig. 1.33

$\Rightarrow \log_m x > \log_m y$ when $x > y$ and $m > 1$.

Again **if** $0 < m < 1$. **(Graph of** $\log_m a$**)**

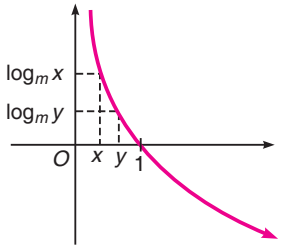


Fig. 1.34

$\Rightarrow \log_m x > \log_m y$; when $x < y$ and $0 < m < 1$.

11. $\log_m a = b \Rightarrow a = m^b$ $\{a, m > 0 ; m \neq 1 ; b \in R\}$

12. $\log_m a > b \Rightarrow \begin{cases} a > m^b ; & \text{if } m > 1 \\ a < m^b ; & \text{if } 0 < m < 1 \end{cases}$

13. $\log_m a < b \Rightarrow \begin{cases} a < m^b ; & \text{if } m > 1 \\ a > m^b ; & \text{if } 0 < m < 1 \end{cases}$.

4. Geometrical Curves

(a) Straight line

ax + by + c = 0 (represents general equation of straight line). We know,

$$y = -\frac{c}{b} \quad \text{when} \quad x = 0$$

and $$x = -\frac{c}{a} \quad \text{when} \quad y = 0$$

joining above points we get required straight line.

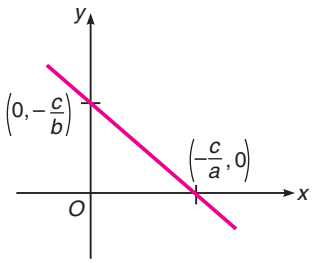


Fig. 1.35

(b) Circle

We know,

(i) $x^2 + y^2 = a^2$ is circle with centre (0, 0) and radius r.

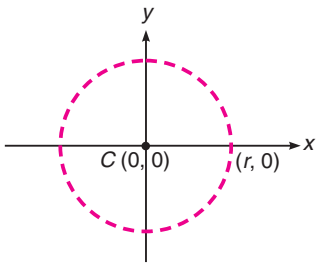


Fig. 1.36

(ii) $(x - a)^2 + (y - b)^2 = r^2$, circle with centre (a, b) and radius r.

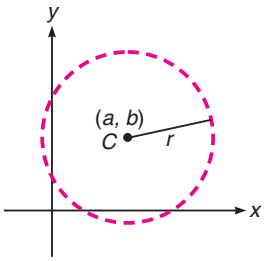


Fig. 1.37

(iii) $x^2 + y^2 + 2gx + 2fy + c = 0$;

centre $(-g, -f)$; radius $\sqrt{g^2 + f^2 - c}$.

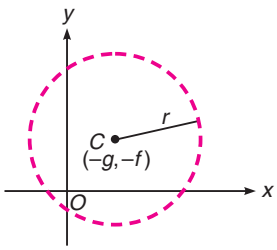


Fig. 1.38

(iv) $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$;

End points of diameter are (x_1, y_1) and (x_2, y_2).

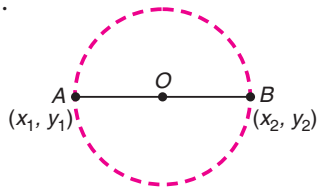


Fig. 1.39

(c) Parabola

(i) $y^2 = 4ax$

Vertex	:	(0, 0)
Focus	:	$(a, 0)$
Axis	:	x-axis or y = 0
Directrix	:	x = – a

(ii) $y^2 = -4ax$

Vertex	:	(0, 0)
Focus	:	$(-a, 0)$
Axis	:	x-axis or y = 0
Directrix	:	x = a

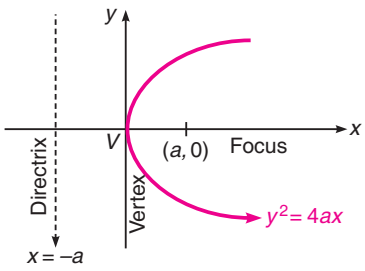


Fig. 1.40

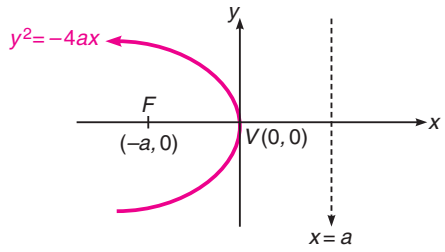


Fig. 1.41

(iii) $x^2 = 4ay$

Vextex : (0, 0)
Focus : (0, a)
Axis : y-axis or x = 0
Directrix : y = – a

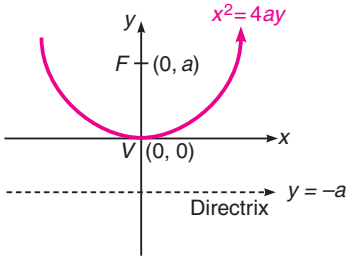


Fig. 1.42

(iv) $x^2 = -4ay$

Vertex : (0, 0)
Focus : (0, – a)
Axis : y-axis or x = 0
Directrix : y = a

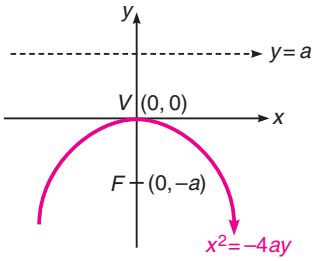


Fig. 1.43

(v) $(y - k)^2 = 4a(x - h)$

Vertex : (h, k)
Focus : (h + a, k)
Axis : x = h
Directrix : x = h – a

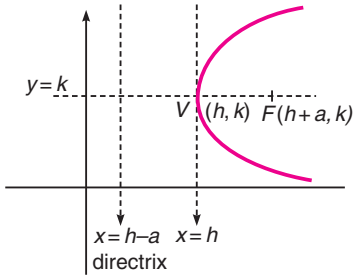


Fig. 1.44

(d) Ellipse

(i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a^2 > b^2)$

Centre : (0, 0)
Focus : (±ae, 0)
Vertex : (±a, 0)
Eccentricity : $e = \sqrt{1 - \frac{b^2}{a^2}}$
Directrix : $x = \pm \frac{a}{e}$

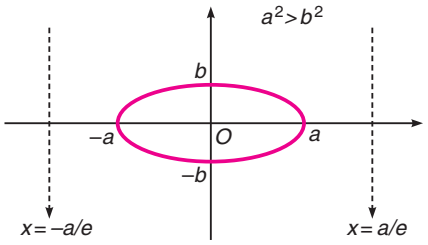


Fig. 1.45

(ii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a^2 < b^2)$

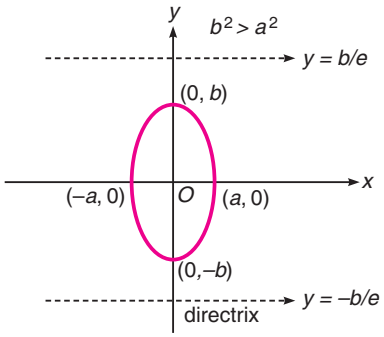


Fig. 1.46

(iii) $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \quad (a^2 > b^2)$

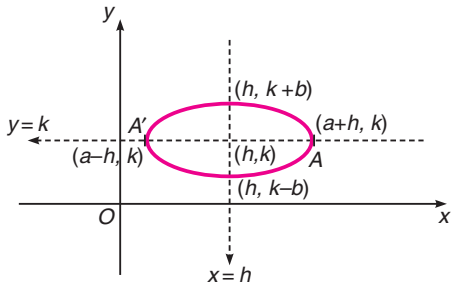


Fig. 1.47

(e) Hyperbola

(i) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Centre : (0, 0)
Focus : (±ae, 0)
Vertices : (±a, 0)
Eccentricity : $e = \sqrt{1 + \frac{b^2}{a^2}}$

Directrix : $x = \pm \frac{a}{e}$

In above figure asymptotes are $y = \pm \frac{b}{a} x$.

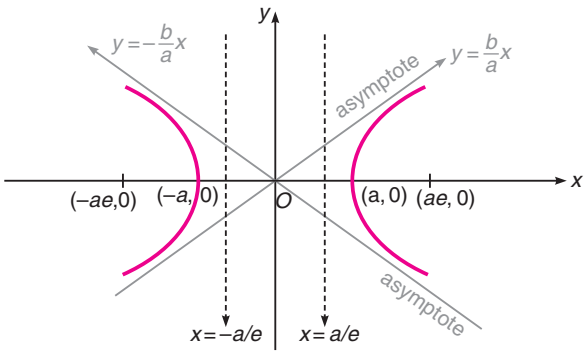


Fig. 1.48

(ii) $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

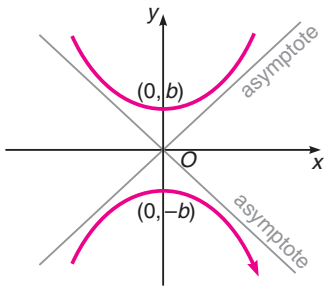


Fig. 1.49

(iii) $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

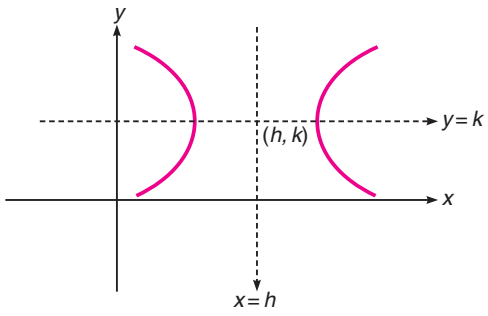


Fig. 1.50

(iv) $x^2 - y^2 = a^2$ (Rectangular hyperbola)

As asymptotes are perpendicular. Therefore, called rectangular hyperbola.

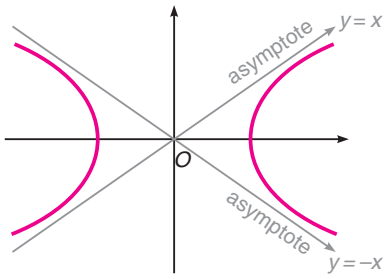


Fig. 1.51

(v) $xy = c^2$

Here, the asymptotes are x-axis and y-axis.

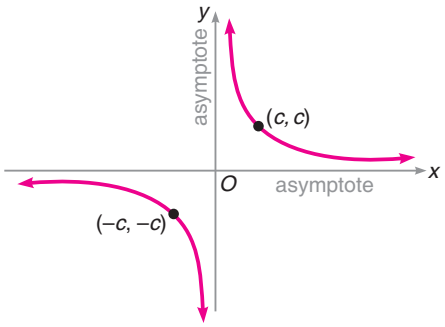


Fig. 1.52

Note In above curves we have used the name asymptotes for its complete definition see **chapter 3.**

Inverse Trigonometric Curves

As we know **trigonometric functions** are **many one** in their domain, **hence, they are not invertible.**

But their inverse can be obtained by restricting the domain so as to make invertible.

Note Every inverse trigonometric is been converted to a function by shortening the domain.

For example: Let $f(x) = \sin x$

We know, $\sin x$ is not invertible for $x \in R$.

In order to get the inverse we have to define domain as:

$$x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$\therefore$ If $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ defined by $f(x) = \sin x$ is invertible and inverse can be represented by:

$$y = \sin^{-1} x. \quad \left(-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}\right)$$

Similarly,

$y = \cos x$ becomes invertible when $f : [0, \pi] \to [-1, 1]$

$y = \tan x$; becomes invertible when $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to (-\infty, \infty)$

$y = \cot x$; becomes invertible when $f : (0, \pi) \to (-\infty, \infty)$

$y = \sec x$; becomes invertible when $f : [0, \pi] - \left\{\frac{\pi}{2}\right\} \to R - (-1, 1)$

$y = \operatorname{cosec} x$; becomes invertible when $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\} \to R - (-1, 1)$

(i) Graph of $y = \sin^{-1} x$;

where,

$$x \in [-1, 1]$$

and $$y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

As the graph of $f^{-1}(x)$ is mirror image of $f(x)$ about $y = x$.

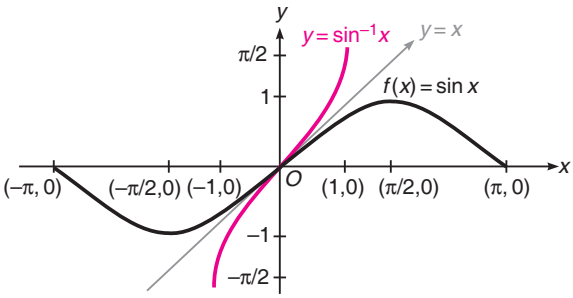


Fig. 1.53

(ii) Graph of $y = \cos^{-1} x$;

Here,

domain $\in [-1, 1]$

Range $\in [0, \pi]$

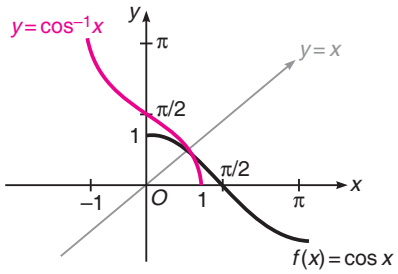


Fig. 1.54

(iii) Graph of $y = \tan^{-1} x$;

Here, domain $\in$ R, Range $\in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

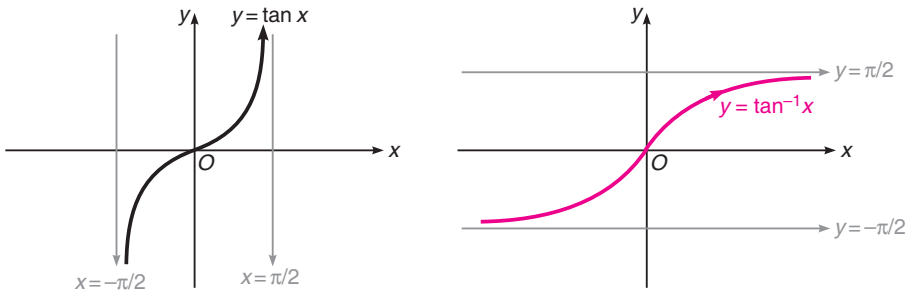


Fig. 1.55

As we have discussed earlier, "graph of inverse function is image of f(x) about y = x" or "by interchanging the coordinate axes".

(iv) Graph of $y = \cot^{-1} x$;

We know that the function $f : (0, \pi) \to R$, given by $f(\theta) = \cot \theta$ is invertible.

$\therefore$ Thus, domain of $\cot^{-1}$ $x \in R$ and Range $\in (0, \pi)$.

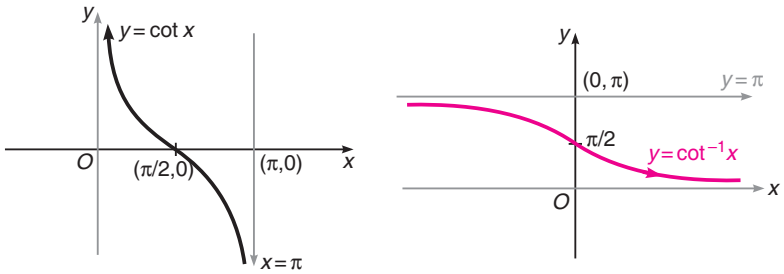


Fig. 1.56

(v) Graph for $y = \sec^{-1} x$;

The function $f : [0, \pi] - \left\{\frac{\pi}{2}\right\} \rightarrow (-\infty, -1] \cup [1, \infty)$ given by $f(\theta) = \sec \theta$ is invertible.

$\therefore$ $y = \sec^{-1} x$, has domain $\in R - (-1, 1)$ and range $\in [0, \pi] - \left\{\frac{\pi}{2}\right\}$: shown as

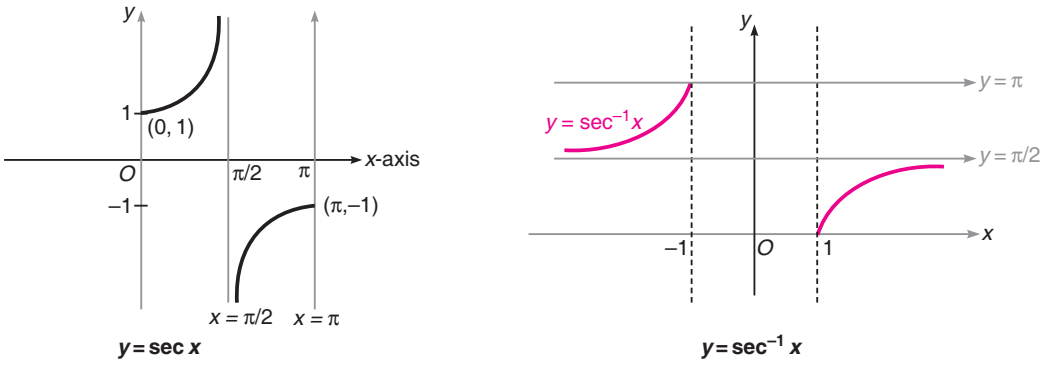


Fig. 1.57

(vi) Graph for $y = \text{cosec}^{-1} x$;

As we know, $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\} \rightarrow R - (-1, 1)$ is invertible given by $f(\theta) = \cos \theta$.

$\therefore$ $y = \text{cosec}^{-1} x$; domain $\in R - (-1, 1)$

Range $\in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

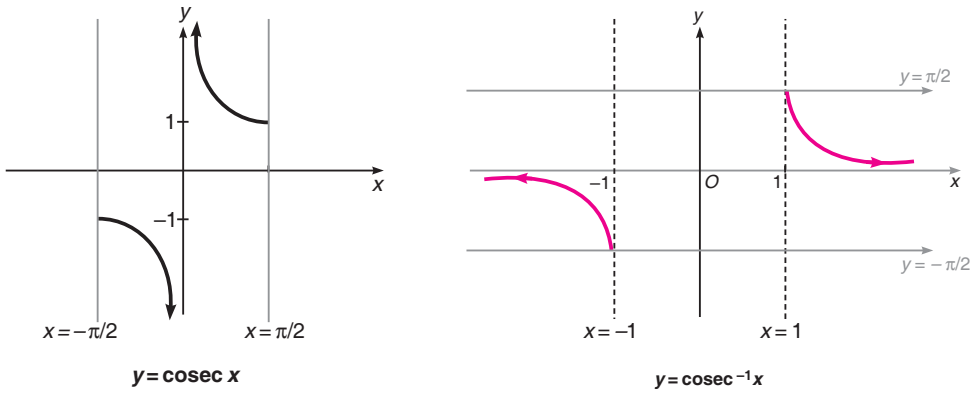


Fig. 1.58

Note If no branch of an inverse trigonometric function is mentioned, then it means the principal value branch of that function.

In case no branch of an inverse trigonometric function is mentioned, it will mean the principal value branch of that function. (*i.e.,*)

	Function	Domain	Range	Principal value branch
1.	$\sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, where $y = \sin^{-1} x$
2.	$\cos^{-1} x$	$[-1, -1]$	$[0, \pi]$	$0 \le y \le \pi$, where $y = \cos^{-1} x$
3.	$\tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$, where $y = \tan^{-1} x$
4.	$\text{cosec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$; $y \ne 0$, where $y = \text{cosec}^{-1} x$
5.	$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$	$0 \le y \le \pi$; $y \ne \left\{\frac{\pi}{2}\right\}$, where $y = \sec^{-1} x$
6.	$\cot^{-1} x$	R	$(0, \pi)$	$0 < y < \pi$; where $y = \cot^{-1} x$.

1.3 TRIGONOMETRIC INEQUALITIES

To solve trigonometric inequalities including trigonometric functions, it is good to practice periodicity and monotonicity of functions.

Thus, first solve the inequality for the periodicity and then get the set of all solutions by adding numbers of the form $2n\pi$; $n \in z$, to each of the solutions obtained on that interval.

EXAMPLE 1 *Solve the inequality;* $\sin x > -\frac{1}{2}$.

SOLUTION As the function $\sin x$ has least positive period 2π. {That is why it is sufficient to solve inequality of the form $\sin x > a$, $\sin x \ge a$, $\sin x < a$, $\sin x \le a$ first on the interval of length 2π, and then get the solution set by adding numbers of the form $2\pi n$, $n \in z$, to each of the solutions obtained on that interval}. Thus, let us solve this inequality on the interval $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$, where graph of $y = \sin x$ and $y = -\frac{1}{2}$ are taken two curves on x-y plane.

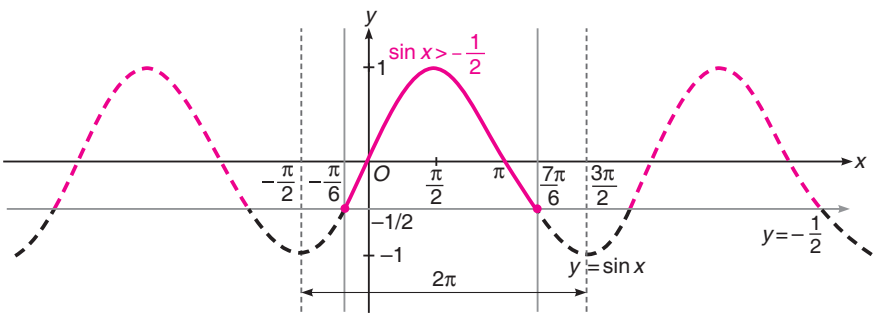


Fig. 1.59

$y = \sin x$ and $y = -\frac{1}{2}$

From above figure, $\sin x > -\frac{1}{2}$ when $-\frac{\pi}{6} < x < \frac{7\pi}{6}$.

Thus, on generalising above solution;

$$2n\pi - \frac{\pi}{6} < x < 2n\pi + \frac{7\pi}{6}; \quad n \in z.$$

which implies that those and only those values of x each of which satisfies these two inequalities for a certain $n \in z$ can serve as solutions to the original inequality.

EXAMPLE 2 *Solve the inequality:* $\cos x \le -\frac{1}{2}$.

SOLUTION As discussed in previous example, cos x is periodic with period 2π. So, to check the solution in $[0, 2\pi]$.

It is clear from figure, $\cos x \le -\frac{1}{2}$ when;

$\frac{2\pi}{3} \le x \le \frac{4\pi}{3}$.

On generalising above solution;

$$2n\pi + \frac{2\pi}{3} \le x \le 2n\pi + \frac{4\pi}{3}; \quad n \in z$$

$\therefore$ Solution of $\cos x \le -\frac{1}{2}$

$\Rightarrow$ $$x \in \left[2n\pi + \frac{2\pi}{3}, 2n\pi + \frac{4\pi}{3}\right]; \quad n \in z.$$

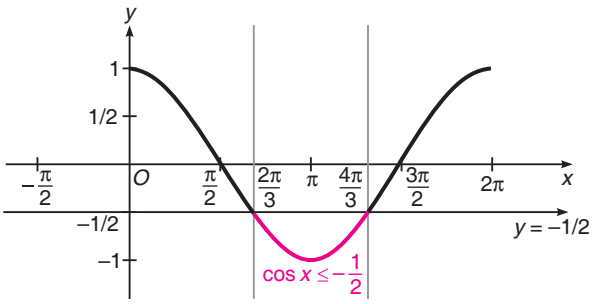


Fig. 1.60

EXAMPLE 3 *Solve the inequality:* $\tan x < 2$.

SOLUTION We know tan x is periodic with period π. So, to check the solution on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

It is clear from figure, $\tan x < 2$ when;

$-\frac{\pi}{2} < x < \tan^{-1} 2$ or $-\frac{\pi}{2} < x < \text{arc tan } 2$

$\Rightarrow$ General solution

$$2n\pi - \frac{\pi}{2} < x < 2n\pi + \tan^{-1} 2$$

$\Rightarrow$ $$n \in \left(2n\pi - \frac{\pi}{2}, 2n\pi + \text{arc tan } 2\right)$$

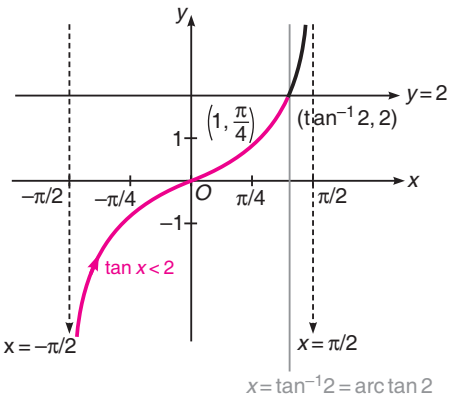


Fig. 1.61

EXAMPLE 4 *Solve the inequality:* $\sin\left(\frac{3x}{2} + \frac{\pi}{12}\right) < \frac{1}{\sqrt{2}}$.

SOLUTION Here, $\sin\left(\frac{3x}{2} + \frac{\pi}{12}\right) < \frac{1}{\sqrt{2}}$; put $\frac{3x}{2} + \frac{\pi}{12} = t$

$\therefore \sin t < \frac{1}{\sqrt{2}}$, now sin t is periodic with period 2π, thus to check on

$\left[\frac{\pi}{2}, \frac{5\pi}{2}\right]$ or $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$

From figure,

$\sin t < \frac{1}{\sqrt{2}}$, when $\frac{3\pi}{4} < t < \frac{9\pi}{4}$.

$\therefore$ generalsolution

$2n\pi + \frac{3\pi}{4} < t < 2n\pi + \frac{9\pi}{4}$; $n \in z$

Substituting $t = \frac{3x}{2} + \frac{\pi}{12}$

$$2n\pi + \frac{3\pi}{4} < \frac{3x}{2} + \frac{\pi}{12} < 2n\pi + \frac{9\pi}{4}$$

$$\Rightarrow \quad \frac{4}{9}\pi + \frac{4}{3}\pi n < x < \frac{13}{9}\pi + \frac{4}{3}n\pi \; ; \; n \in z.$$

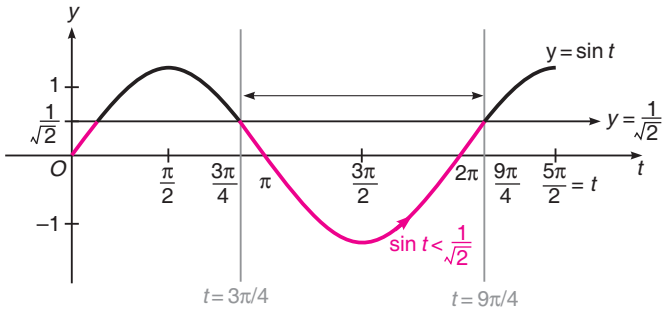


Fig. 1.62

EXAMPLE 5 *Solve the inequality :* $\cos 2x - \sin 2x \geq 0$.

SOLUTION Here, $\cos 2x - \sin 2x$ can be reduced to,

$$\sqrt{2}\left\{\frac{1}{\sqrt{2}}\cos 2x - \frac{1}{\sqrt{2}}\sin 2x\right\} \quad \Rightarrow \quad \sqrt{2}\left\{\cos\frac{\pi}{4}\cos 2x - \sin\frac{\pi}{4}\sin 2x\right\}$$

$$\Rightarrow \quad \sqrt{2}\cos\left(\frac{\pi}{4} + 2x\right)$$

$\therefore$ $\cos 2x - \sin 2x \geq 0$ or $\cos\left(\frac{\pi}{4} + 2x\right) \geq 0$; put $2x + \frac{\pi}{4} = t$

$\therefore \cos(t) \geq 0$, solving graphically,

Clearly; $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$

or $2n\pi - \frac{\pi}{2} < t < 2n\pi + \frac{\pi}{2}$

where $t = 2n + \frac{\pi}{4}$

$\therefore \; 2n\pi - \frac{\pi}{2} \leq 2n + \frac{\pi}{4} \leq 2n\pi + \frac{\pi}{2}$

$n\pi - \frac{3\pi}{8} \leq n \leq n\pi + \frac{\pi}{8}$; $n \in z$.

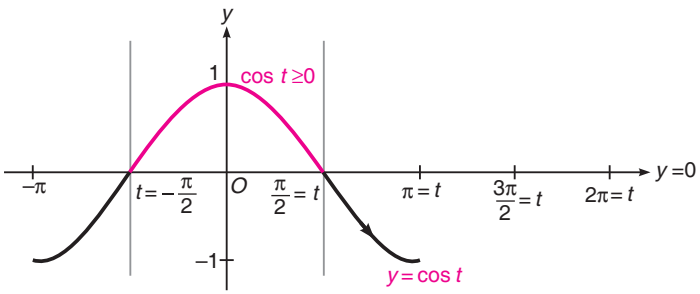


Fig. 1.63

EXAMPLE 6 *If* $A + B + C = \pi$, *then prove that;* $\cos A + \cos B + \cos C < \frac{3}{2}$; *where* A, B, C *are distinct.*

SOLUTION Here, we have the three trigonometric functions as cos A, cos B and cos C.

$\therefore$ let $f(x) = \cos x$; which can be plotted as;

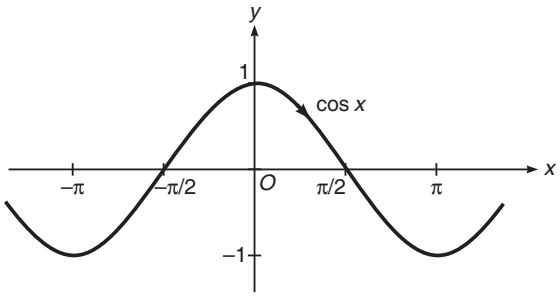


Fig. 1.64

Now, let us suppose any three points x = A, x = B, x = C on f(x) = cos x. So that A + B + C = π or on the interval of length π.

where G, be centroid of Δ given by $\left(\frac{A+B+C}{3}, \frac{\cos A+\cos B+\cos C}{3}\right)$

Thus, from figure points Q, G, P are collinear,

where; ordinate of GQ < ordinate of PQ.

$$\frac{\cos A+\cos B+\cos C}{3} < \cos\left(\frac{A+B+C}{3}\right)$$

$$\Rightarrow \quad \cos A+\cos B+\cos C < 3\cos\left(\frac{\pi}{3}\right)$$

$$\Rightarrow \quad \cos A+\cos B+\cos C < \frac{3}{2}$$

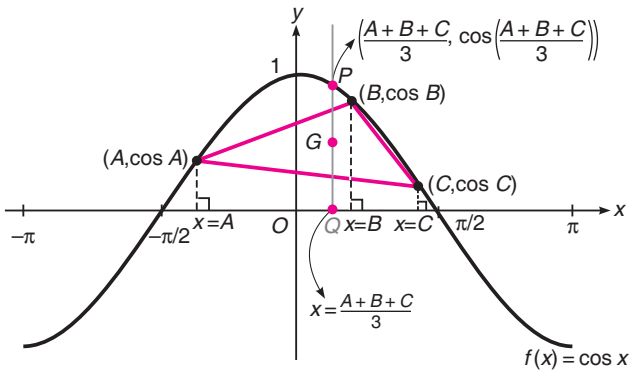


Fig. 1.65

Note Here, a particular case arises when $A = B = C$ (*i.e.*, when A, B, C are non-distinct)

$$\cos A = \cos B = \cos C \quad \text{and} \quad A + B + C = \pi$$

$$\Rightarrow \quad A + A + A = \pi \quad \text{or} \quad A = \pi/3.$$

$$\therefore \quad \cos A = \cos B = \cos C = \frac{1}{2}$$

$$\therefore \quad \cos A + \cos B + \cos C = \frac{3}{2} \qquad \text{(only when } A = B = C\text{)}$$

EXAMPLE 7 *Solve the inequality:* $\sin x \cos x + \frac{1}{2}\tan x \geq 1$.

SOLUTION Here; left hand side is defined for all x, except $x = n\pi + \frac{\pi}{2}$, where $n \in z$.

$$\therefore \quad 2\sin x \cos x + \tan x \geq 2$$

$$\Rightarrow \quad \frac{2\tan x}{1+\tan^2 x} + \tan x \geq 2 \qquad [\text{Let, } \tan x = y]$$

$$\Rightarrow \quad \frac{2y}{1+y^2} + y \geq 2$$

$$\Rightarrow \quad \frac{2y + y(1+y^2) - 2(1+y^2)}{(1+y^2)} \geq 0 \qquad \{\because\ 1+y^2 \geq 0\}$$

$$\therefore \quad 2y + y(1+y^2) - 2(1+y^2) \geq 0$$

$$\Rightarrow \quad y^3 - 2y^2 + 3y - 2 \geq 0$$

$$\Rightarrow \quad y^2(y-1) - y(y-1) + 2(y-1) \geq 0$$

$$\text{or} \quad (y-1)(y^2 - y + 2) \geq 0$$

$$\Rightarrow \quad y - 1 \geq 0$$

$$\{\because\ y^2 - y + 2 = \left(y - \frac{1}{2}\right)^2 + \frac{7}{4} > 0, \text{ for all } y\}$$

$\therefore \tan x \geq 1$, shown as:

from given figure;

$$\frac{\pi}{4} \leq x < \frac{\pi}{2}$$

$$\text{or} \quad n\pi + \frac{\pi}{4} \leq x < n\pi + \frac{\pi}{2} \quad ; \quad n \in z$$

$$\therefore \quad x \in \left[n\pi + \frac{\pi}{4}, n\pi + \frac{\pi}{2}\right]; \ n \in z$$

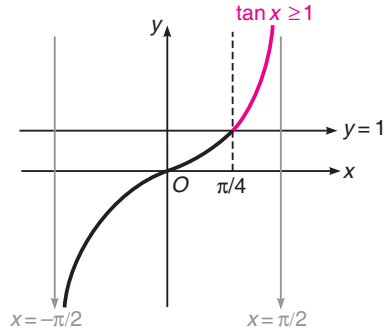


Fig. 1.66

EXAMPLE 8 *If* $A + B + C = \pi$, *then prove that*

$$\tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2} > 1.$$

SOLUTION Here, $\tan^2\frac{A}{2}$, $\tan^2\frac{B}{2}$ and $\tan^2\frac{C}{2}$ are three same function. So consider $f(x) = \tan^2\frac{x}{2}$, whose period is 2π.

$\therefore$ plotting $\tan^2\frac{x}{2}$ for $x \in (-\pi, \pi)$.

In given curve let us consider any three points A, B, C such that

$$A + B + C = \pi.$$

Now, centroid of ΔRST;

$$G\left(\frac{A+B+C}{3}, \frac{\tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2}}{3}\right)$$

also, $$M\left(\frac{A+B+C}{3}, \tan^2\left(\frac{A+B+C}{2(3)}\right)\right)$$

where; $GN > MN$.

$$\Rightarrow \quad \frac{\tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2}}{3} > \tan^2\left(\frac{A+B+C}{6}\right) \quad \Rightarrow \quad \tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2} > 1.$$

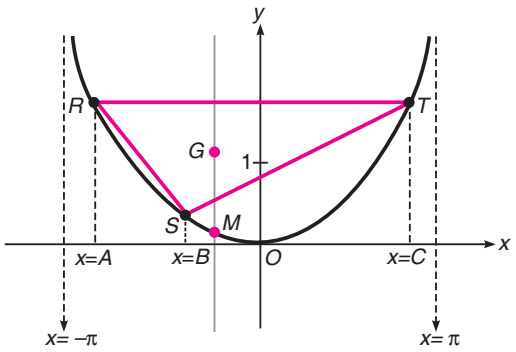


Fig. 1.67

1.4 SOLVING EQUATIONS GRAPHICALLY

Here, we sketch both left hand and right hand side of equality and the numbers of intersections are required solutions.

EXAMPLE 9 *Find the number of solutions of;* $\sin x = \frac{x}{10}$.

SOLUTION Here, let $f(x) = \sin x$ and $g(x) = \frac{x}{10}$

also we know; $-1 \le \sin x \le 1$

$\therefore \quad -1 \le \frac{x}{10} \le 1 \quad \Rightarrow \quad -10 \le x \le 10$

Thus, to sketch both curves when $x \in [-10, 10]$

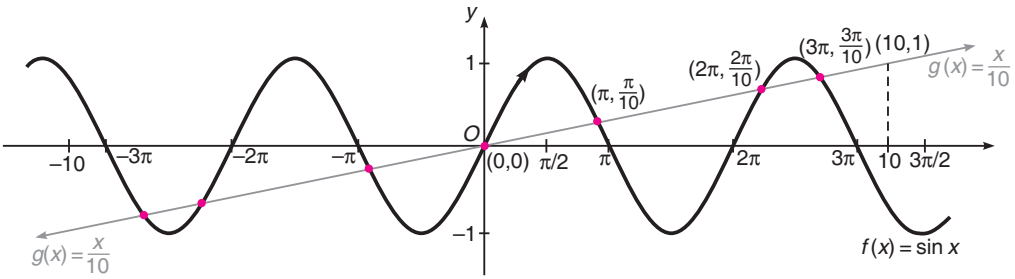


Fig. 1.68

From above figure $f(x) = \sin x$ and $g(x) = \frac{x}{10}$ intersect at 7 points. So, **numbers of solutions are 7.**

EXAMPLE 10 *Find the least positive value of x, satisfying* $\tan x = x + 1$ *lies in the interval.*

SOLUTION Let; $f(x) = \tan x$ and $g(x) = x + 1$; which could be shown as:

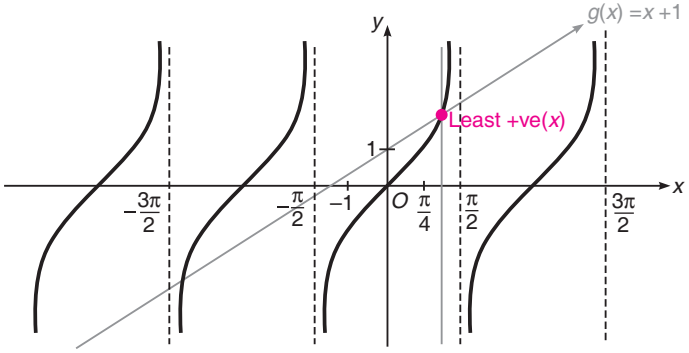


Fig. 1.69

From the above figure $\tan x = x + 1$ has infinitely many solutions but the least positive value of $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

EXAMPLE 11 *Find the number of solutions of the equation,*

$$\sin x = x^2 + x + 1.$$

SOLUTION Let; $f(x) = \sin x$ and $g(x) = x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$

which could be shown as;

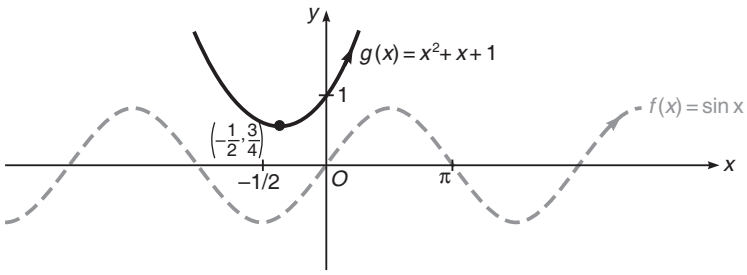


Fig. 1.70

which do not intersect at any point, therefore **no solution**.

EXAMPLE 12 *Find the number of solutions of:* $e^x = x^4$.

SOLUTION Let; $f(x) = e^x$ and $g(x) = x^4$, which could be shown as;

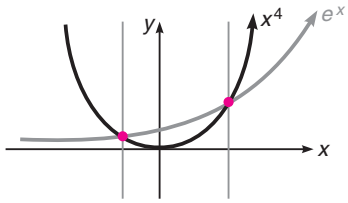


Fig. 1.71

From the figure, it is clear they intersect at two points, therefore **two solutions**.

EXAMPLE 13 *Find the number of solutions of;* $\log_{10} x = x$.

SOLUTION Let; $f(x) = \log_{10} x$ and $g(x) = x$; which could be shown as;

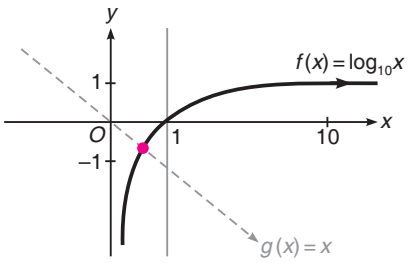


Fig. 1.72

From above figure, it is clear they intersect at one points, therefore **1 solution**.

SOME MORE SOLVED EXAMPLES

EXAMPLE 1 *Sketch the graph for* $y = \sin^{-1}(\sin x)$.

SOLUTION As, $y = \sin^{-1}(\sin x)$ is periodic with period 2π.

$\therefore$ to draw this graph we should draw the graph for one interval of length 2π and repeat for entire values of x.

As we know;
$$\sin^{-1}(\sin x) = \begin{cases} x\,; & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ (\pi - x)\,; & -\frac{\pi}{2} \le \pi - x < \frac{\pi}{2} \quad \left(\text{i.e.}, \frac{\pi}{2} \le x \le \frac{3\pi}{2}\right) \end{cases}$$

or
$$\sin^{-1}(\sin x) = \begin{cases} x\,, & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ \pi - x\,, & \frac{\pi}{2} \le x \le \frac{3\pi}{2}, \end{cases}$$

which is defined for the interval of length 2π, plotted as;

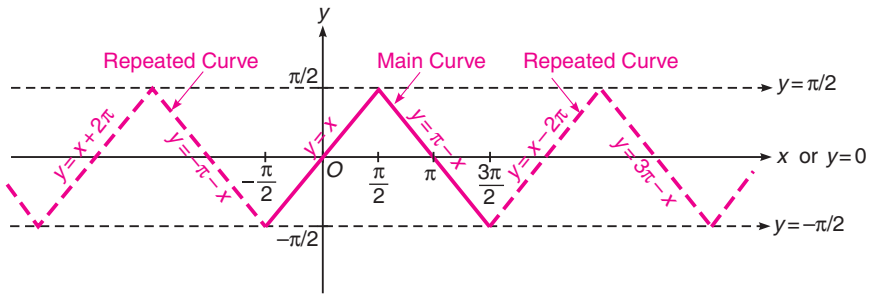


Fig. 1.73

Thus, the graph for $y = \sin^{-1}(\sin x)$, is a straight line up and a straight line down with slopes 1 and –1 respectively lying between $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Note : Students are adviced to learn the definition of $\sin^{-1}(\sin x)$ as,

$$y = \sin^{-1}(\sin x) = \begin{cases} x + 2\pi & ; \quad -\frac{5\pi}{2} \le x \le -\frac{3\pi}{2} \\ -\pi - x & ; \quad -\frac{3\pi}{2} \le x \le -\frac{\pi}{2} \\ x & ; \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ \pi - x & ; \quad \frac{\pi}{2} \le x \le \frac{3\pi}{2} \\ x - 2\pi & ; \quad \frac{3\pi}{2} \le x \le \frac{5\pi}{2} \end{cases} \ldots \text{and so on}$$

EXAMPLE 2 *Sketch the graph for* $y = \cos^{-1}(\cos x)$.

SOLUTION As, $y = \cos^{-1}(\cos x)$ is periodic with period 2π.

$\therefore$ to draw this graph we should draw the graph for one interval of length 2π and repeat for entire values of x of length 2π.

As we know;

$$\cos^{-1}(\cos x) = \begin{cases} x; & 0 \le x \le \pi \\ 2\pi - x; & 0 \le 2\pi - x \le \pi, \end{cases}$$

or

$$\cos^{-1}(\cos x) = \begin{cases} x\,; & 0 \le x \le \pi \\ 2\pi - x\,; & \pi \le x \le 2\pi. \end{cases}$$

Thus, it has been defined for $0 < x < 2\pi$ that has length 2π. So, its graph could be plotted as;

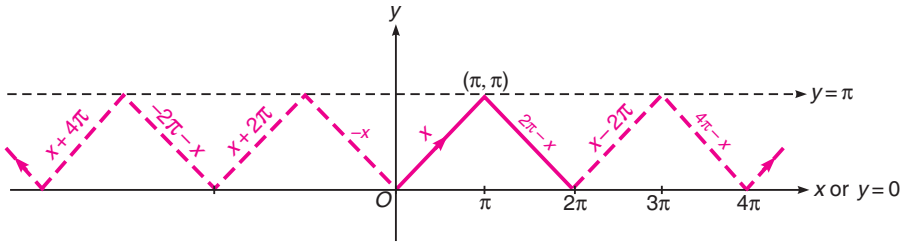


Fig. 1.74

Thus, the curve $y = \cos^{-1}(\cos x)$.

EXAMPLE 3 *Sketch the graph for* $y = \tan^{-1}(\tan x)$.

SOLUTION As $y = \tan^{-1}(\tan x)$ is periodic with period π.

$\therefore$ to draw this graph we should draw the graph for one interval of length π and repeat for entire values of x.

As we know;

$$\tan^{-1}(\tan x) = \left\{ x\,; -\frac{\pi}{2} < x < \frac{\pi}{2} \right\}$$

Thus, it has been defined for $-\frac{\pi}{2} < x < \frac{\pi}{2}$ that has length π. So, its graph could be plotted as;

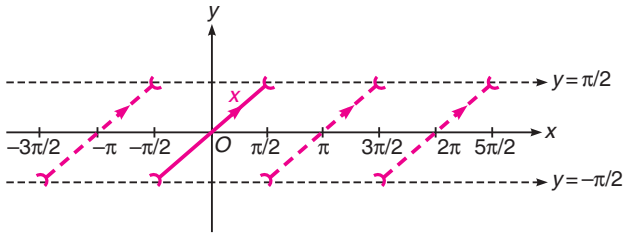


Fig. 1.75

Thus, the curve for $y = \tan^{-1}(\tan x)$, where y is not defined for $x \in (2n + 1)\frac{\pi}{2}$.

EXAMPLE 4 *Sketch the graph for* $y = \text{cosec}^{-1}(\text{cosec}\, x)$.

SOLUTION As $y = \text{cosec}^{-1}(\text{cosec } x)$ is periodic with period 2π.

$\therefore$ to draw this graph we should draw the graph for one interval of length 2π and repeat for entire values of x.

As we know;

$$\text{cosec}^{-1}(\text{cosec } x) = \begin{cases} x\ ; & -\frac{\pi}{2} \le x < 0 \quad \text{or} \quad 0 < x \le \frac{\pi}{2} \\ \pi - x; & -\frac{\pi}{2} \le \pi - x < 0 \quad \text{or} \quad 0 < \pi - x \le \frac{\pi}{2} \end{cases}$$

or

$$\text{cosec}^{-1}(\text{cosec } x) = \begin{cases} x; & x \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right] \\ \pi - x; & x \in \left[\frac{\pi}{2}, \pi\right) \cup \left(\pi, \frac{3\pi}{2}\right] \end{cases}$$

Thus, it has been defined for $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] - \{0, \pi\}$ that has length 2π. So, its graph could be plotted as

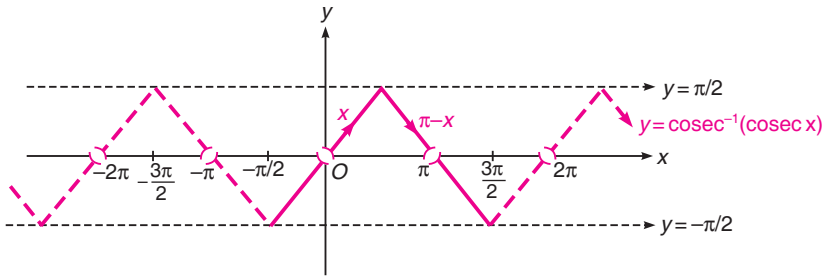


Fig. 1.76

EXAMPLE 5 *Sketch the graph for* $y = \sec^{-1}(\sec x)$.

SOLUTION As $y = \sec^{-1}(\sec x)$ is periodic with period 2π.

$\therefore$ to draw this graph we should draw the graph for one interval of length 2π and repeat for entire values of x.

As we know;

$$\sec^{-1}(\sec x) = \begin{cases} x\ ; & x \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right] \\ 2\pi - x\ ; & 2\pi - x \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right] \end{cases}$$

or

$$\sec^{-1}(\sec x) = \begin{cases} x\ ; & 0 \le x < \frac{\pi}{2} \quad \text{or} \quad \frac{\pi}{2} < x \le \pi \\ 2\pi - x\ ; & \pi \le x < \frac{3\pi}{2} \quad \text{or} \quad \frac{3\pi}{2} < x \le 2\pi \end{cases}.$$

Thus, it has been defined for $[0, 2\pi] - \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$ that has length 2π. So, its graph could be plotted as;

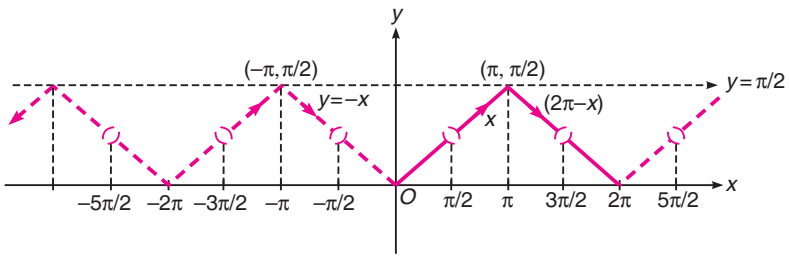


Fig. 1.77

Thus, the curve for $y = \sec^{-1}(\sec x)$.

EXAMPLE 6 *Sketch the graph for* $y = \cot^{-1}(\cot x)$.

SOLUTION As, $y = \cot^{-1}(\cot x)$ is periodic with period π.

$\therefore$ to draw this graph we should draw the graph for one interval of length π and repeat for entire values of x.

As we know

$$\cot^{-1}(\cot x) = \{x ;\ 0 < x < \pi\}$$

which is defined for length π, *i.e.,* $x \in (0, \pi)$ and $x \notin \{n\pi, n \in z\}$.

So, its graph could be plotted as;

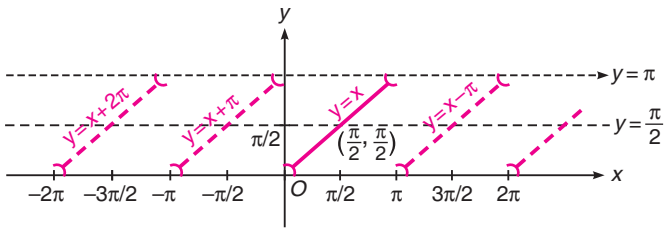


Fig. 1.78

Thus, the curve for $y = \cot^{-1}(\cot x)$.

EXAMPLE 7 *Sketch the graph for:*

(i) $\sin(\sin^{-1} x)$ *(ii)* $\cos(\cos^{-1} x)$ *(iii)* $\tan(\tan^{-1} x)$

(iv) $\text{cosec}(\text{cosec}^{-1} x)$ *(v)* $\sec(\sec^{-1} x)$ *(vi)* $\cot(\cot^{-1} x)$

SOLUTION As we know, all the above mentioned six curves are **non-periodic, but have restricted domain and range.**

So, we shall first define each curve for its domain and range and then sketch these curves.

(i) Sketch for $y = \sin(\sin^{-1} x)$

We know; domain, $x \in [-1, 1]$ (*i.e.,* $-1 \le x \le 1$)

and range $y = x \Rightarrow y \in [-1, 1]$

Hence, we should sketch $y = \sin(\sin^{-1} x)$ only when $x \in [-1, 1]$ and $y = x$. So, its graph could be plotted as shown in figure.

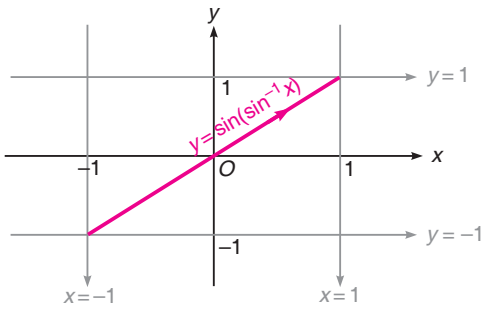


Fig. 1.79

Thus, the graph for $y = \sin(\sin^{-1} x)$.

(ii) Sketch for the curve $y = \cos(\cos^{-1} x)$.

We know, domain, $x \in [-1, 1]$ (*i.e.*, $-1 \le x \le 1$)

and range $y = x \Rightarrow y \in [-1, 1]$

Hence, we should sketch $y = \cos(\cos^{-1} x) = x$ only when $x \in [-1, 1]$. So, its graph could be plotted as shown in Fig. 1.80.

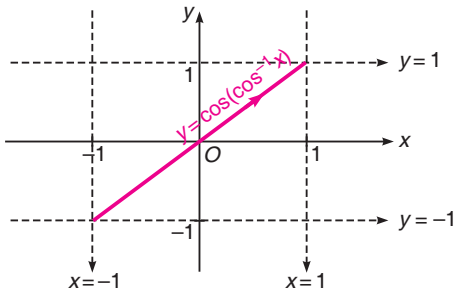


Fig. 1.80

Thus, the graph for $y = \cos(\cos^{-1} x)$.

(iii) Sketch for the curve $y = \tan(\tan^{-1} x)$

We know,

domain, $x \in R$ (*i.e.*, $-\infty < x < \infty$) and

Range $y = x \Rightarrow y \in R$.

Hence, we should sketch

$y = \tan(\tan^{-1} x) = x, \forall\ x \in R$.

So, its graph could be plotted as shown;

Thus, the graph for $y = \tan(\tan^{-1} x)$.

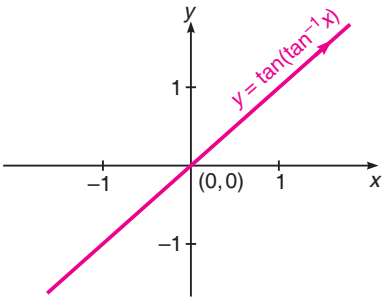


Fig. 1.81

(iv) Sketch for y = cosec (cosec^{-1}x)

We know;

domain $\in$ R – (–1, 1)

(*i.e.*, $-\infty < x \le -1$ or $1 \le x < \infty$)

and range $y = x \Rightarrow y \in R - (-1, 1)$.

Hence, we should sketch

y = cosec (cosec^{-1}x) = x only when $x \in (-\infty, -1] \cup [1, \infty)$.

So, its graph could be plotted as shown in Fig. 1.82;

Thus, the graph for

y = cosec (cosec^{-1}x).

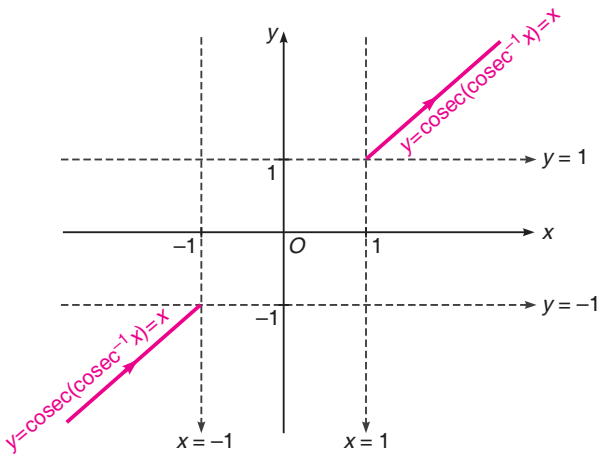


Fig 1.82

(v) Sketch for y = sec (sec^{-1}x)

We know, domain $\in$ R – (–1, 1)

(*i.e.*, $-\infty < x \le -1$ or $1 \le x < \infty$)

and range $y = x \Rightarrow y \in R - (-1, 1)$.

Hence, we should sketch

y = sec (sec^{-1}x) = x,

only when $x \in (-\infty, -1] \cup [1, \infty)$

So, its graph could be plotted as shown in Fig. 1.83.

Thus, the graph for

y = sec (sec^{-1}x) = x

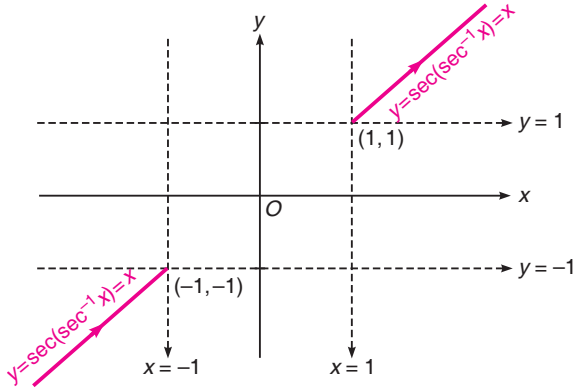


Fig. 1.83

(vi) Sketch for y = cot (cot^{-1}x)

We know; Domain $\in$ R (i.e., $-\infty < x < \infty$)

and Range $y = x \Rightarrow y = R$.

Hence, we should sketch

y = cot (cot^{-1} x) = x, $\forall$ $x \in R$.

Shown as in Fig. 1.84.

Thus, the graph for y = cot (cot^{-1} x).

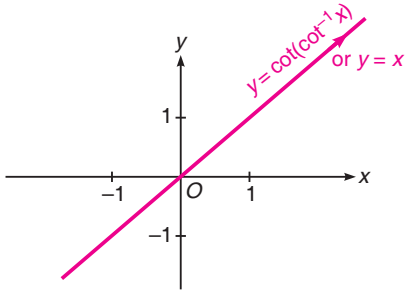


Fig. 1.84

Note From previous discussions, we learn that if:

(i) The function is periodic then find period and trace the curve.

(ii) If non-periodic, then define for their domain and find range to trace the curve.

Now, before going ahead you must revise previous curves of inverse trigonometry as;

$y = \sin^{-1} x$, $y = \cos^{-1} x$, $y = \tan^{-1} x$, $y = \cot^{-1} x$, $y = \text{cosec}^{-1} x$, $y = \sec^{-1} x$

with their domain and range.

EXAMPLE 8 *Sketch the graph for:*

(i) $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ *(ii)* $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ *(iii)* $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$

(iv) $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$ *(v)* $\sin^{-1}(3x-4x^3)$ *(vi)* $\cos^{-1}(4x^3-3x)$.

SOLUTION As we know, all the above mentioned six curves are **non-periodic,** but have **Restricted domain and Range.**

So, we shall first define each curve for its domain and range and then sketch these curves.

(i) Sketch for $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

Here, **for domain** $\left|\frac{2x}{1+x^2}\right| \le 1$

$\Rightarrow$ $2|x| \le 1 + x^2$ $\{\because 1 + x^2 > 0 \text{ for all } x\}$

$\Rightarrow$ $|x|^2 - 2|x| + 1 \ge 0$ $\{\because x^2 = |x|^2\}$

$\Rightarrow$ $(|x|^2 - 1)^2 \ge 0$

$\Rightarrow$ $x \in R$.

For range: $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$\Rightarrow$ $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $\left\{\text{as}; \; y = \sin^{-1}\theta \Rightarrow y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right\}$

Defining the curve: Let, $x = \tan\theta$

$\Rightarrow$ $y = \sin^{-1}(\sin 2\theta) = \begin{cases} \pi - 2\theta & ; \; 2\theta > \frac{\pi}{2} \\ 2\theta & ; \; -\frac{\pi}{2} \le 2\theta \le \frac{\pi}{2} \\ -\pi - 2\theta & ; \; 2\theta < -\frac{\pi}{2} \end{cases}$ {See Ex. 1}

or $y = \begin{cases} \pi - 2\tan^{-1}x & ; \; \tan^{-1}x > \frac{\pi}{4} \\ 2\tan^{-1}x & ; \; -\frac{\pi}{4} \le \tan^{-1}x \le \frac{\pi}{4} \\ -\pi - 2\tan^{-1}x & ; \; \tan^{-1}x < -\frac{\pi}{4} \end{cases}$ $\{\because \tan\theta = x \Rightarrow \theta = \tan^{-1}x\}$

or $y = \begin{cases} \pi - 2\tan^{-1}x & ; \; x > 1 \\ 2\tan^{-1}x & ; \; -1 \le x \le 1 \\ -\pi - 2\tan^{-1}x & ; \; x < -1 \end{cases}$ ...(i)

Thus, $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ is defined for $x \in R$, where $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so the graph for Eq. (i) could be shown as in Fig. 1.85.

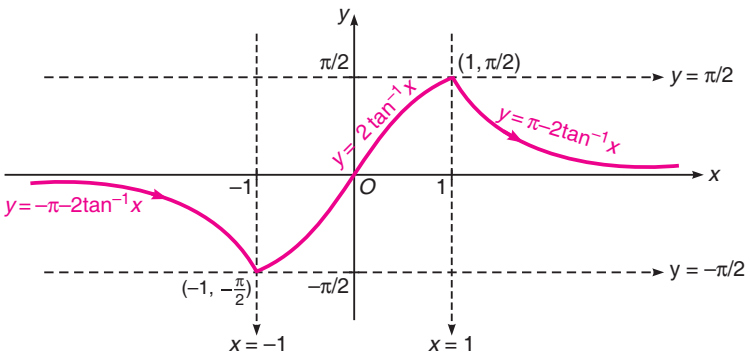


Fig. 1.85

Thus, the graph for $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

Note As in later section (*i.e.,* chapter 2) we shall discuss that functions having sharp edges and gaps are not differentiable at that point.

So, in previous curve $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, we know it has sharp edge at $x = -1$ and $x = 1$.

So, not differentiable.

(ii) Sketch for $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Here, **for domain** $\left|\frac{1-x^2}{1+x^2}\right| \le 1$

$\Rightarrow$ $|1-x^2| \le 1+x^2$ $\{\because\ 1+x^2 > 0,\ \forall\ x \in R\}$

which is true for all x; as $1+x^2 > 1-x^2$

$\therefore$ $x \in R$

For range: $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) \Rightarrow y \in (0, \pi)$

Define the curve : Let, $x = \tan\theta$

$\therefore$ $y = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \cos^{-1}(\cos 2\theta)$

$= \begin{cases} 2\theta; & 2\theta \ge 0 \\ -2\theta; & 2\theta < 0 \end{cases}$. {See Example 2}

$\Rightarrow$ $y = \begin{cases} 2\tan^{-1}x; & \tan^{-1}x \ge 0 \\ -2\tan^{-1}x; & \tan^{-1}x < 0 \end{cases}$ $\{\because\ \tan\theta = x \Rightarrow \theta = \tan^{-1}x\}$

So, the graph of $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \begin{cases} 2\tan^{-1} x; & x \geq 0 \\ -2\tan^{-1} x; & x < 0, \end{cases}$ is shown as:

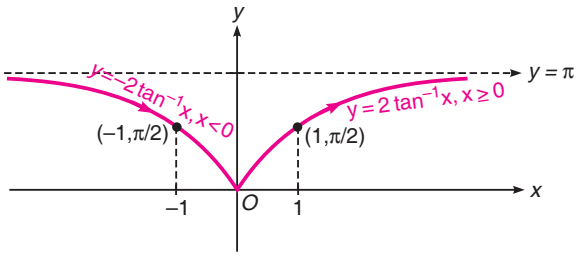


Fig. 1.86

Thus, the graph for $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \begin{cases} 2\tan^{-1} x, & x \geq 0 \\ -2\tan^{-1} x, & x < 0 \end{cases}$.

From above figure it is clear $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ is not differentiable at x = 0.

(iii) Sketch for $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Here, **for domain** $\frac{2x}{1-x^2} \in R$ except; $1 - x^2 = 0$

i.e., $x \neq \pm 1$ or $x \in R - \{1, -1\}$

For range $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

$\Rightarrow$ $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ $\left\{\text{as } y = \tan^{-1}\theta \Rightarrow y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right\}$

Defining the curve

Let $x = \tan\theta$

$\Rightarrow$ $y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}(\tan 2\theta) = \begin{cases} \pi + 2\theta; & 2\theta < -\frac{\pi}{2} \\ 2\theta; & -\frac{\pi}{2} < 2\theta < \frac{\pi}{2} \\ -\pi + 2\theta; & 2\theta > \frac{\pi}{2} \end{cases}$ {See Example 3}

$= \begin{cases} \pi + 2\tan^{-1} x; & \tan^{-1} x < -\frac{\pi}{4} \\ 2\tan^{-1} x; & -\frac{\pi}{4} < \tan^{-1} x < \frac{\pi}{4} \\ -\pi + 2\tan^{-1} x; & \tan^{-1} x > \frac{\pi}{4} \end{cases}$ {as $\tan\theta = x \Rightarrow \theta = \tan^{-1} x$}

$$= \begin{cases} \pi + 2\tan^{-1} x; & x < -1 \\ 2\tan^{-1} x; & -1 < x < 1 \\ -\pi + 2\tan^{-1} x; & x > 1 \end{cases}$$

So, the graph of;

$$y = \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \begin{cases} \pi + 2\tan^{-1} x; & x < -1 \\ 2\tan^{-1} x; & -1 < x < 1 \\ -\pi + 2\tan^{-1} x; & x > 1 \end{cases} \text{ is shown as;}$$

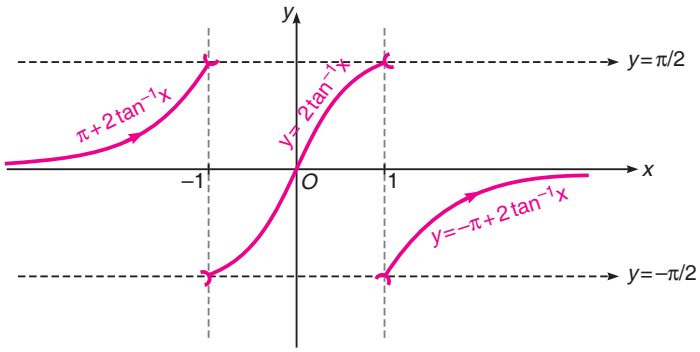


Fig. 1.87

Thus, the graph for $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \begin{cases} \pi + 2\tan^{-1} x, & x < -1 \\ 2\tan^{-1} x; & -1 < x < 1 \\ -\pi + 2\tan^{-1} x, & x > 1 \end{cases}$

which is neither continuous nor differentiable at $x = \{-1, 1\}$.

(v) Sketch for the curve $y = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$.

Here, **for domain** $y = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$

$\Rightarrow$ $x \in R$ except $1 - 3x^2 = 0 \Rightarrow x \neq \pm\frac{1}{\sqrt{3}}$

$\therefore$ $x \in R - \left\{\pm\frac{1}{\sqrt{3}}\right\}$

For range $y = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$

$\Rightarrow$ $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ $\left\{\text{as } y = \tan^{-1}\theta \Rightarrow y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right\}$

Defining the curve: Let; $x = \tan\theta$

$$\Rightarrow \quad y = \tan^{-1}(\tan 3\theta) = \begin{cases} \pi + 3\theta; & 3\theta < -\frac{\pi}{2} \\ 3\theta; & -\frac{\pi}{2} < 3\theta < \frac{\pi}{2} \\ -\pi + 3\theta; & 3\theta > \frac{\pi}{2} \end{cases} = \begin{cases} \pi + 3\tan^{-1} x; & \tan^{-1} x < -\frac{\pi}{6} \\ 3\tan^{-1} x; & -\frac{\pi}{6} < \tan^{-1} x < \frac{\pi}{6} \\ -\pi + 3\tan^{-1} x; & \tan^{-1} x > \frac{\pi}{6} \end{cases}$$

$$= \begin{cases} \pi + 3\tan^{-1} x; & x < -\frac{1}{\sqrt{3}} \\ 3\tan^{-1} x; & -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}} \\ -\pi + 3\tan^{-1} x; & x > \frac{1}{\sqrt{3}} \end{cases}$$

So, the graph of; $y = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right) = \begin{cases} \pi + 3\tan^{-1} x; & x < -\frac{1}{\sqrt{3}} \\ 3\tan^{-1} x; & -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}} \\ -\pi + 3\tan^{-1} x; & x > \frac{1}{\sqrt{3}} \end{cases}$

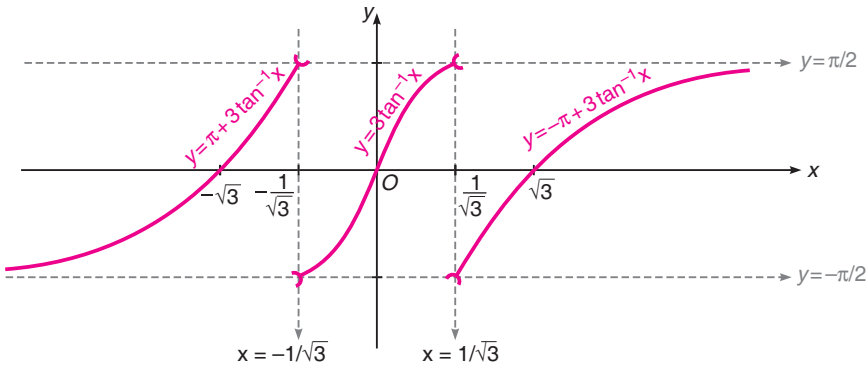


Fig. 1.88

Thus, the curve for $y = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right) = \begin{cases} \pi + 3\tan^{-1} x, & x < -\frac{1}{\sqrt{3}} \\ 3\tan^{-1} x, & -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}} \\ -\pi + 3\tan^{-1} x; & x > \frac{1}{\sqrt{3}} \end{cases}$

which is neither continuous nor differentiable for $x = \left\{\pm \frac{1}{\sqrt{3}}\right\}$.

(v) Sketch the curve $y = \sin^{-1}(3x - 4x^3)$

Defining the curve: Let $x = \sin\theta$,

$$\Rightarrow \quad y = \sin^{-1}(\sin 3\theta) = \begin{cases} \pi - 3\theta; & \frac{\pi}{2} \le 3\theta \le \frac{3\pi}{2} \\ 3\theta; & -\frac{\pi}{2} \le 3\theta \le \frac{\pi}{2} \\ -\pi - 3\theta; & -\frac{3\pi}{2} \le 3\theta \le -\frac{\pi}{2} \end{cases} = \begin{cases} \pi - 3\sin^{-1} x; & \frac{\pi}{6} \le \sin^{-1} x \le \frac{\pi}{2} \\ 3\sin^{-1} x; & -\frac{\pi}{6} \le \sin^{-1} x \le \frac{\pi}{6} \\ -\pi - 3\sin^{-1} x; & -\frac{\pi}{2} \le \sin^{-1} x \le -\frac{\pi}{6} \end{cases}$$

$$\therefore \qquad y = \sin^{-1}(3x - 4x^3) = \begin{cases} \pi - 3\sin^{-1} x; & \frac{1}{2} \le x \le 1 \\ 3\sin^{-1} x; & -\frac{1}{2} \le x \le \frac{1}{2} \\ -\pi - 3\sin^{-1} x; & -1 \le x \le -\frac{1}{2} \end{cases}$$

For domain $y = \sin^{-1}(3x - 4x^3) \quad \Rightarrow \quad x \in [-1, 1]$

For range $y = \sin^{-1}(3x - 4x^3) \quad \Rightarrow \quad y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

So, the graph of; $y = \sin^{-1}(3x - 4x^3) = \begin{cases} \pi - 3\sin^{-1} x; & \frac{1}{2} \le x \le 1 \\ 3\sin^{-1} x; & -\frac{1}{2} \le x \le \frac{1}{2} \\ -\pi - 3\sin^{-1} x; & -1 \le x \le -\frac{1}{2} \end{cases}$

is shown as:

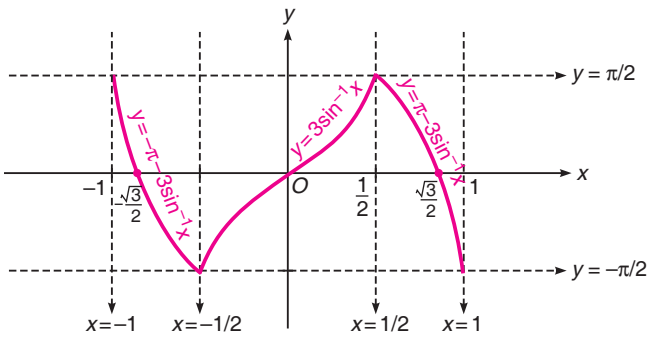


Fig. 1.89

Thus, the curve for $y = \sin^{-1}(3x - 4x^3)$

which is not differentiable at $x = \left\{\pm \frac{1}{2}\right\}$.

(vi) Sketch the curve $y = \cos^{-1}(4x^3 - 3x)$

Here, **domain** $\in [-1, 1]$ **range** $\in [0, \pi]$

Now, **defining the curve**

Let $x = \cos\theta$

$$\Rightarrow \quad y = \cos^{-1}(\cos 3\theta) = \begin{cases} 2\pi - 3\theta; & \pi \le 3\theta \le 2\pi \\ 3\theta; & 0 \le 3\theta \le \pi \\ -2\pi + 3\theta; & -\pi \le 3\theta \le 0 \end{cases} = \begin{cases} 2\pi - 3\cos^{-1} x; & \frac{\pi}{3} \le \cos^{-1} x \le \frac{2\pi}{3} \\ 3\cos^{-1} x; & 0 \le \cos^{-1} x \le \frac{\pi}{3} \\ -2\pi + 3\cos^{-1} x; & -\frac{\pi}{3} \le \cos^{-1} x \le 0 \end{cases}$$

$$= \begin{cases} 2\pi - 3\cos^{-1} x; & -\frac{1}{2} \le x \le \frac{1}{2} \\ 3\cos^{-1} x; & \frac{1}{2} \le x \le 1 \\ -2\pi + 3\cos^{-1} x; & -1 \le x \le -\frac{1}{2} \end{cases}$$

{$\because$ If $0 \le \theta \le \frac{\pi}{3} \Rightarrow \cos\frac{\pi}{3} \le \cos\theta \le \cos 0$ or $\frac{1}{2} \le \cos\theta \le 1$. Here, the interval changed since, cos x is decreasing in $[0, \pi]$}

So, the graph of; $y = \cos^{-1}(4x^3 - 3x) = \begin{cases} 2\pi - 3\cos^{-1} x; & -\frac{1}{2} \le x \le \frac{1}{2} \\ 3\cos^{-1} x; & \frac{1}{2} \le x \le 1 \\ -2\pi + 3\cos^{-1} x; & -1 \le x \le -\frac{1}{2}, \end{cases}$

is shown as;

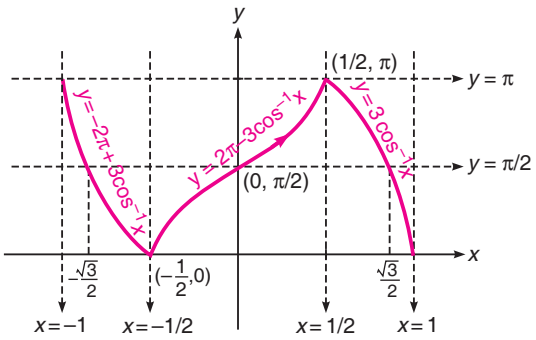


Fig. 1.90

Thus, the curve for $y = \cos^{-1}(4x^3 - 3x)$,
which is not differentiable at $x = \left\{\pm \frac{1}{2}\right\}$.

EXAMPLE 9 *Sketch the graph for:*

(i) sin x. cosec x *(ii)* cos x · sec x *(iii)* tan x · cot x

SOLUTION As we know for the above curves each is equal to 1, but for different domain as;

(i) $y = \sin x \cdot \operatorname{cosec} x = 1; \ \forall \ x \in R - \{n\pi; \ n \in z\}$

(ii) $y = \cos x \cdot \sec x = 1;$ $\quad x \in R - \left\{(2n+1)\frac{\pi}{2};\ n \in z\right\}$

(iii) $y = \tan x \cdot \cot x = 1;$ $\quad x \in R - \left\{n\pi,\ n\pi + \frac{\pi}{2};\ n \in z\right\}.$

Thus, they could be plotted as:

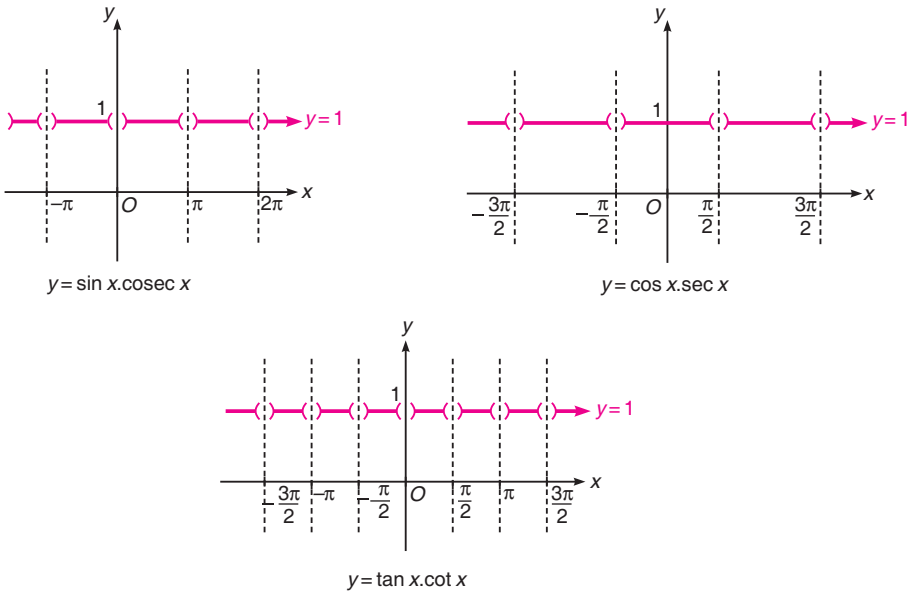


Fig. 1.91

Note From above example it becomes clear that $y = \sin x \cdot \operatorname{cosec} x = 1$, $y = \cos x \cdot \sec x = 1$, $y = \tan x \cdot \cot x = 1$ but they are not equal, as their domains are different.

$\therefore$ **Equal functions :** Those functions which have same domain and range are equal functions.

EXAMPLE 10 *Sketch the graph for:*

(i) $\dfrac{|\sin x|}{\sin x}$ $\qquad$ *(ii)* $\dfrac{|\cos x|}{\cos x}$ $\qquad$ *(iii)* $\sin^{-1}\left(\dfrac{1+x^2}{2x}\right)$

(iv) $\log_{1/4}\left(x - \dfrac{1}{4}\right) + \dfrac{1}{2}\log_4(16x^2 - 8x + 1)$

(v) $1 + 3\,(\log|\sin x| + \log|\operatorname{cosec} x|)$ $\qquad$ *(vi)* $1 + 3\,(\log \sin x + \log \operatorname{cosec} x)$

SOLUTION As we know, to plot above curves we must check periodicity domain and range;

(i) $y = \dfrac{|\sin x|}{\sin x}$

Here, $\quad y = \begin{cases} 1; & \sin x > 0 \\ -1; & \sin x < 0 \end{cases}$

$$y = \begin{cases} 1; & 2n\pi < x < (2n+1)\pi; \quad n \in z \\ -1; & (2n+1)\pi < x < (2n+2)\pi; \quad n \in z \end{cases}$$

So, from above,

domain $\in R - \{n\pi;\ n \in z\}$

Range $\in \begin{cases} 1; & 2n\pi < x < (2n+1)\pi \\ -1; & (2n+1)\pi < x < (2n+2)\pi \end{cases}$

$\therefore$ it could be plotted as shown in Fig. 1.92.

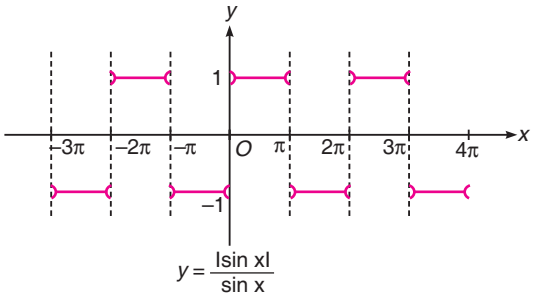


Fig. 1.92

(ii) Sketch for $y = \frac{|\cos x|}{\cos x}$

Here, $y = \begin{cases} 1; & \cos x > 0 \\ -1; & \cos x < 0 \end{cases} \Rightarrow y = \begin{cases} 1; & 2n\pi - \frac{\pi}{2} < x < 2n\pi + \frac{\pi}{2} \\ -1; & 2n\pi + \frac{\pi}{2} < x < 2n\pi + \frac{3\pi}{2} \end{cases}$

So, it could be plotted as:

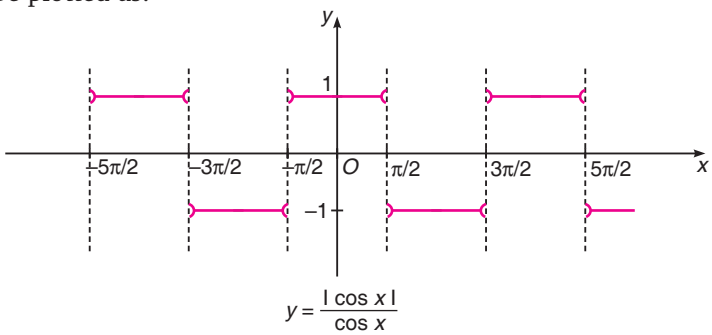


Fig. 1.93

(iii) Sketch for $y = \sin^{-1}\left(\frac{1+x^2}{2x}\right)$

Here $y = \sin^{-1}\left(\frac{1+x^2}{2x}\right)$ is defined;

when; $\left|\frac{1+x^2}{2x}\right| \le 1$ {as; $\sin^{-1} x$ is defined when $|x| \le 1$}

$\Rightarrow$ $1 + x^2 \le 2|x|$ {as; $1 + x^2 > 0$}

$\Rightarrow$ $x^2 - 2|x| + 1 \le 0$ {as; $x^2 = |x|^2$}

$\Rightarrow$ $|x|^2 - 2|x| + 1 \le 0$

$\Rightarrow$ $(|x| - 1)^2 \le 0$

$\Rightarrow$ $(|x| - 1)^2 = 0$ {as; $(|x| - 1)^2 < 0$ is not possible}

$\Rightarrow \quad x = \pm 1$

$\therefore \quad$ **Domain** $\in \{\pm 1\}$

$\therefore \quad$ For range $y = \sin^{-1}\left(\frac{1+x^2}{2x}\right)$, where $x = +1, -1$

$\therefore \quad y = \sin^{-1}(1)$ and $y = \sin^{-1}(-1)$

$\Rightarrow \quad y = \pm \frac{\pi}{2}$

$\therefore \quad$ **Range** $\in \left\{\pm \frac{\pi}{2}\right\}$

Hence, the graph for $y = \sin^{-1}\left(\frac{1+x^2}{2x}\right)$ is **only two points.** Shown as:

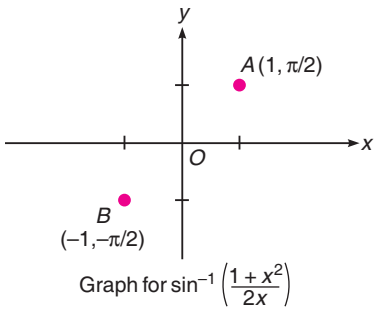


Fig. 1.94

Thus, the sketch for $y = \sin^{-1}\left(\frac{1+x^2}{2x}\right)$ is only two points ***A*** and ***B***.

(iv) Sketch for $y = \log_{1/4}\left(x - \frac{1}{4}\right) + \frac{1}{2}\log_4(16x^2 - 8x + 1)$

Here, $y = \log_{1/4}\left(x - \frac{1}{4}\right) + \frac{1}{2}\log_4 (4x-1)^2$

$\Rightarrow \quad y = \log_{1/4}\left(x - \frac{1}{4}\right) + \frac{1}{2}\log_4 16 + \frac{1}{2}\log_4\left(x - \frac{1}{4}\right)^2$

or $\quad y = -\log_4\left(x - \frac{1}{4}\right) + \frac{1}{2}\log_4 4^2 + \frac{2}{2}\log_4\left(x - \frac{1}{4}\right)$ $\quad \left\{\text{as; } \log_{b^n} a^m = \frac{m}{n}\log_b a\right\}$

$\Rightarrow \quad y = -\log_4\left(x - \frac{1}{4}\right) + \log_4\left(x - \frac{1}{4}\right) + \frac{2}{2}\log_4 4$

$\Rightarrow y = 1$, whenever; $\left(x - \frac{1}{4}\right) > 0$ {as; $\log_a x$ exists only when a, $x > 0$ and $a \neq 1$}

Thus, $y = \log_{1/4}\left(x - \frac{1}{4}\right) + \frac{1}{2}\log_4 (4x-1)^2$

$\Rightarrow$ **Domain** $\in \left(\frac{1}{4}, \infty\right)$

Range $\in \{1\}$

Thus, the graph is shown as:

Thus, the graph for

$y = \log_{1/4}\left(x - \frac{1}{4}\right) + \frac{1}{2}\log_4 (4x-1)^2$.

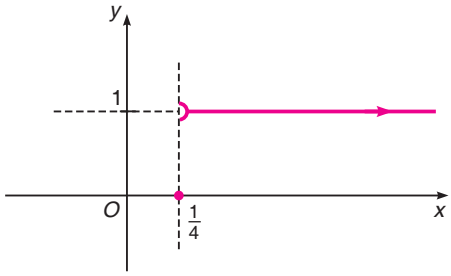


Fig. 1.95

(v) Sketch for $y = 1 + 3(\log|\sin x| + \log|\text{cosec } x|)$

Here $\quad y = 1 + 3[\log(|\sin x| \cdot |\text{cosec } x|)]$

whenever $|\sin x| \neq 0$ and $|\text{cosec } x| \neq 0$

i.e., $y = 1 + 3(\log 1)$; whenever $x \notin n\pi;\ n \in z.$

$\Rightarrow$ $y = 1$ {as; $\log 1 = 0$}

$\therefore$ **Domain** $\in R - \{n\pi;\ n \in z\}$

Range $\in \{1\}$

$\therefore$ it could be plotted as:

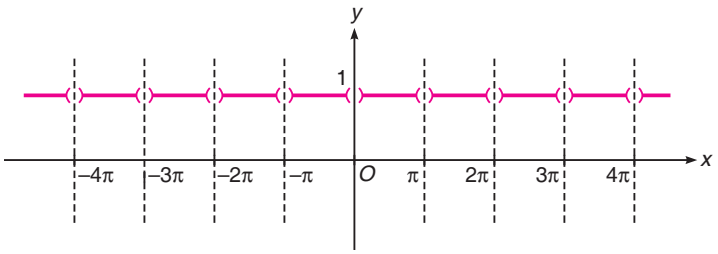


Fig. 1.96

Thus, the curve for $y = 1 + 3(\log|\sin x| + \log|\text{cosec } x|)$.

(vi) Sketch for $y = 1 + 3$ (log sin x + log cosec x)

Here $y = 1 + 3(\log \sin x \cdot \text{cosec } x)$ whenever $\sin x > 0$ and $\text{cosec } x > 0$

$\Rightarrow$ $y = 1 + 3 \log 1$; $x \in (2n\pi, (2n + 1)\pi)$

or $y = 1$; whenever $x \in (2n\pi, (2n + 1)\pi)$

$\therefore$ $y = 1 + 3(\log \sin x + \log \text{cosec } x) = \{1;\ 2n\pi < x < (2n + 1)\pi$ is shown as;

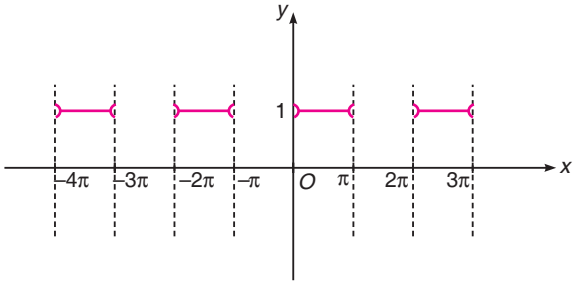


Fig. 1.97

Thus, the curve for $y = 1 + 3(\log \sin x + \log \text{cosec } x)$

EXAMPLE 11 *Sketch the curve for* $\cos y = \cos x$.

SOLUTION Here, $\cos y = \cos x \Rightarrow y = 2n\pi \pm x;\ n \in z$

$\therefore$ $\cos y = \cos x$, represents two straight lines;

$$y = \begin{cases} x + 2n\pi; & n \in z \\ -x + 2n\pi; & n \in z \end{cases}$$

i.e., two infinite set of perpendicular straight lines which could be shown as:

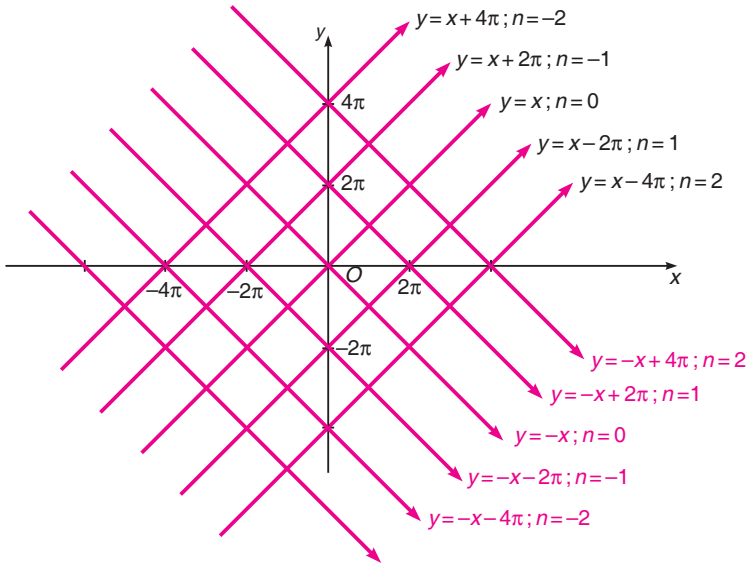


Fig. 1.98

Thus, graph for $\cos y = \cos x$; represents two infinite set of perpendicular straight lines which have infinite number of points of intersections; (So, if asked number of solutions then they are infinite).

EXAMPLE 12 *Sketch the curve for* $\sin y = \sin x$.

SOLUTION Here $\sin y = \sin x \Rightarrow y = n\pi + (-1)^n x;\ n \in z$

$\therefore$ $\sin y = \sin x$; represent two straight lines;

$$y = \begin{cases} n\pi + x; & n \text{ even integer} \\ n\pi - x; & n \text{ odd integer} \end{cases}$$

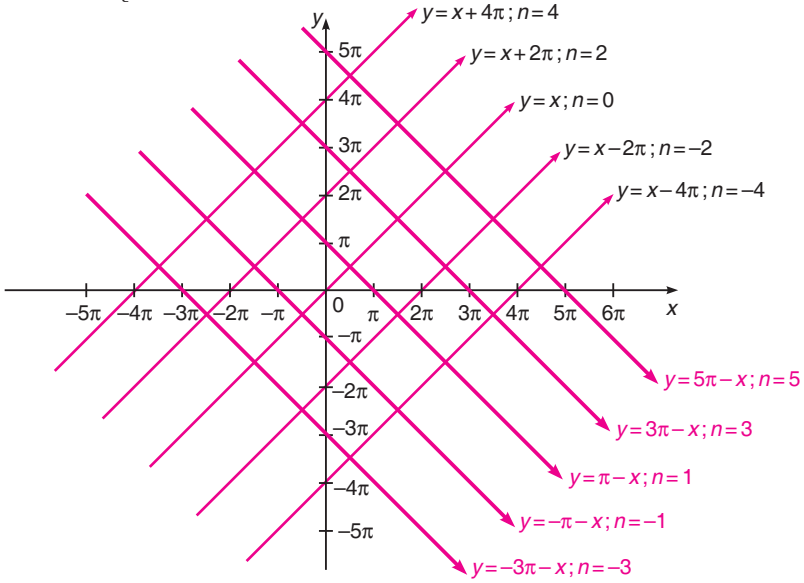


Fig. 1.99

i.e., two infinite set of perpendicular straight lines as shown in Fig. 1.99:
Thus, the graph for $\sin y = \sin x$.

EXAMPLE 13 *Find the number of solutions for;* $\sin\frac{\pi x}{2} = \frac{99x}{500}$.

SOLUTION Let $f(x) = \sin\frac{\pi x}{2}$ and $g(x) = \frac{99x}{500}$,
to find number of solutions; we shall plot both the curves as;

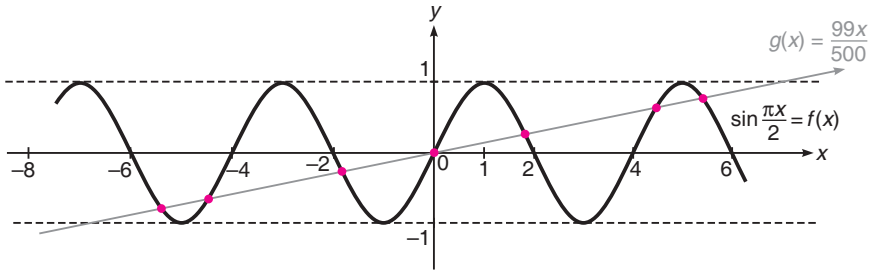


Fig. 1.100

Clearly, from the above figure, **the number of solutions are 7.**

EXAMPLE 14 *Find the number of solutions for;* $\cos x = x$

SOLUTION As, $\cos x = x$
$\therefore$ to plot the curve for $y = \cos x$; $y = x$ and find the point of intersection as to obtain number of solutions.
Here, the two curves intersect at a point *A*.
So, $\cos x = x$ has **only one solution.**

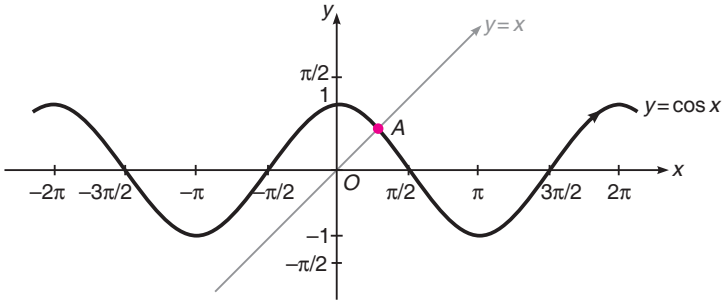


Fig. 1.101

EXAMPLE 15 *Find the number of solutions for;* $[x] = \{x\}$. *where* $[\bullet]$, $\{\bullet\}$ *represents greatest integer and fractional part of* x.

SOLUTION As, $[x] = \{x\}$

$\therefore$ to plot $y = [x]$; $y = \{x\}$

and find point of intersection.

Here, the only point of intersection is $x = 0$,

$\therefore$ only one **solutions**.

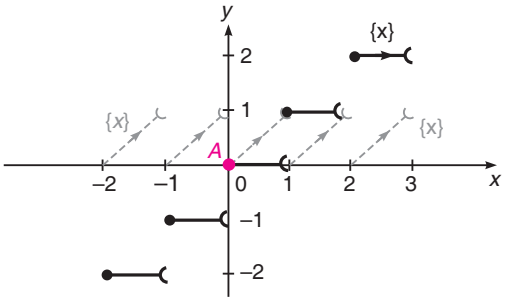


Fig. 1.102

EXAMPLE 16 *Find the number of solutions of*

$$4\{x\} = x + [x]$$

where $\{\cdot\}$, $[\cdot]$ *represents fractional part and greatest integer function.*

SOLUTION As we know, to find number of solutions of two curves we should find the point of intersection of two curves.

$\therefore \quad 4\{x\} = x + [x]$

$\Rightarrow \quad 4(x - [x]) = x + [x]$

$\{\because\ x = [x] + \{x\}\}$

$\Rightarrow \quad 4x - x = 4[x] + [x]$

$\Rightarrow \quad 3x = 5[x]$

$\Rightarrow \quad [x] = \frac{3}{5}x \qquad \ldots(i)$

$\therefore$ To plot the graph of both

$$y = [x] \quad \text{and} \quad y = \frac{3}{5}x.$$

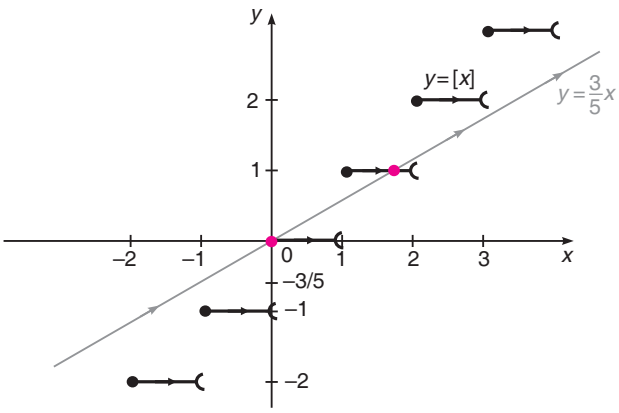


Fig. 1.103

Clearly, the two graphs intersects when

$$[x] = 0 \quad \textbf{and} \quad [x] = 1 \qquad \ldots(ii)$$

$\therefore \quad x = \frac{5}{3}[x]$ [from Eqs. (i) and (ii)]

$$x = \frac{5}{3}\cdot 0 \quad \text{and} \quad x = \frac{5}{3}(1)$$

$\therefore\ x = 0$ and $x = \frac{5}{3}$ are the only two **solutions**.

EXAMPLE 17 *Find the value of x graphically satisfying;* $[x] - 1 + x^2 \geq 0$*; where* $[\cdot]$ *denotes the greatest integer function.*

SOLUTION As, $[x] - 1 + x^2 \geq 0 \quad \Rightarrow \quad x^2 - 1 \geq -[x]$

Thus, to find the points for which $f(x) = x^2 - 1$ is greater than or equals to

$$g(x) = -[x].$$

where two functions $f(x)$ and $g(x)$ could be plotted as shown in Fig. 1.104;

From the adjoining figure; the solution set lies when

$$x \leq A \quad \text{or} \quad x \geq B.$$

Thus, to find A and B.

It is clear that $f(x)$ and $g(x)$ intersects when; $-[x] = 2$.

$\therefore \quad x^2 - 1 = 2$

$x = \pm\sqrt{3} \quad \Rightarrow \quad x = -\sqrt{3}$

(neglecting $x = +\sqrt{3}$ as A lies for $x < 0$)

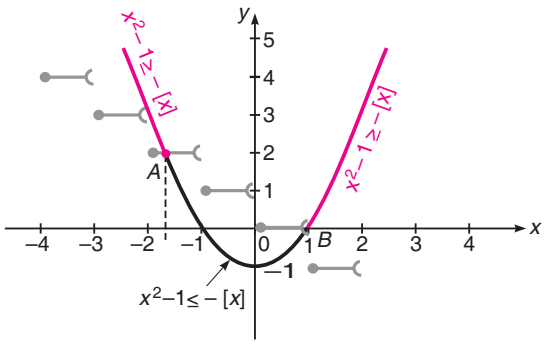


Fig. 1.104

Thus, **for *A* :** $x = -\sqrt{3}$ and **for *B* :** $x = 1$.

$\therefore$ Solution set for which $x^2 - 1 \geq -[x]$ holds.

$\Rightarrow$ $x \in (-\infty, -\sqrt{3}] \cup [1, \infty)$.

Note The method discussed in previous example is very important as it reduces your calculations, so students should practice these forms.

EXAMPLE 18 *Find the values of* x *graphically which satisfy;* $-1 \leq [x] - x^2 + 4 \leq 2$*; where* $[\cdot]$ *denotes the greatest integer function.*

SOLUTION As, $-1 \leq [x] - x^2 + 4 \leq 2 \quad \Rightarrow \quad x^2 - 5 \leq [x] \leq x^2 - 2$

Thus, to find the points for which $f(x) = x^2 - 5$ is less than or equal to $g(x) = [x]$ and $g(x) = [x]$ is less than or equal to $h(x) = x^2 - 2$, where the three functions $f(x)$, $g(x)$ and $h(x)$ could be plotted as;

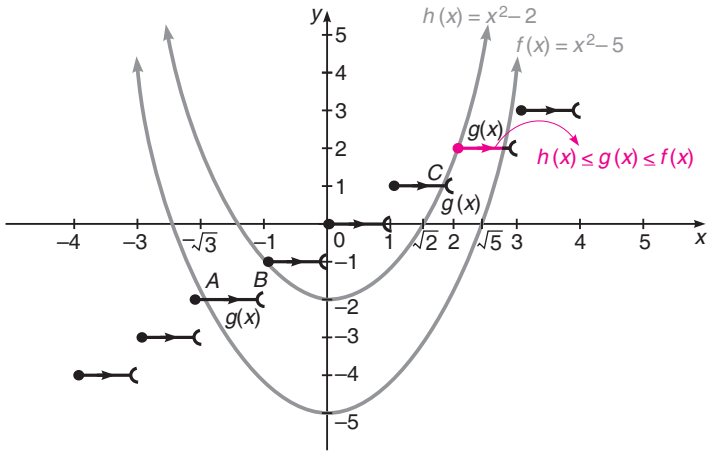


Fig. 1.105

Thus, from the above graph;

$x^2 - 5 \leq [x] \leq x^2 - 2$ **when** $x \in [A, B] \cup [C, D]$

where *A* and *D* is the point of intersection if;

$x^2 - 5 = \pm 2 \quad \Rightarrow \quad x = -\sqrt{3}, \sqrt{7}$

and *C* is point of intersection of

$x^2 - 2 = 1 \quad \Rightarrow \quad x = \sqrt{3}$.

$\therefore \quad A = -\sqrt{3}, \quad B = -1, \quad C = \sqrt{3}$ and $D = \sqrt{7}$.

$\therefore \quad -1 \leq [x] - x^2 + 4 \leq 2$ is satisfied;

when $x \in [-\sqrt{3}, -1] \cup [\sqrt{3}, \sqrt{7}]$.

EXAMPLE 19 *If* $0 \leq a \leq 3$, $0 \leq b \leq 3$ *and the equation;* $x^2 + 4 + 3\cos(ax + b) = 2x$ *has atleast one solution then find the value of* $(a + b)$.

SOLUTION Here, $x^2 + 4 + 3\cos(ax + b) = 2x$ or $x^2 - 2x + 4 = -3\cos(ax + b)$

$\Rightarrow \quad (x - 1)^2 + 3 = -3\cos(ax + b)$

for above equation to have atleast one solution; plot $f(x) = (x-1)^2 + 3$ and $g(x) = -3\cos(ax+b)$ in such a way that they touch each other.

From figure the two curves could atmost touch at one point only when $-3\cos(a+b) = 3$

$\Rightarrow \quad \cos(a+b) = -1$

$\Rightarrow \quad a+b = \pi, \ 3\pi, \ 5\pi, \ldots$

But $\quad 3\pi > 6$

$\Rightarrow \quad a+b = \pi$ as $0 \le a, b \le 3$.

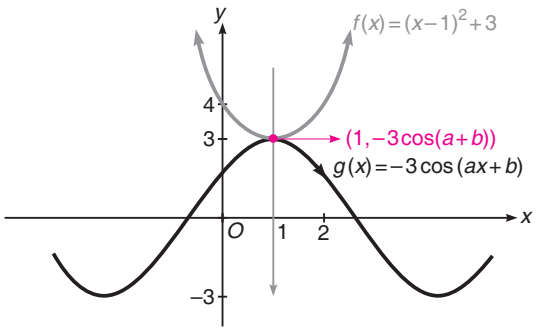


Fig. 1.106

EXAMPLE 20 *If* $A + B + C = \pi$ *and* *A, B, C are angles of* Δ*; then show*

$$\sin A + \sin B + \sin C < \frac{3\sqrt{3}}{2}.$$

SOLUTION Here; we have three trigonometric ratios $\sin A$, $\sin B$, $\sin C$.
$\therefore$ Let $y = \sin x$, on which there are three points $x = A$, $x = B$ and $x = C$ shown as;
As from the figure; **In** ΔPQR,
Centroid of Δ formed by $P(A, \sin A)$ $Q(B, \sin B)$ $R(C, \sin C)$ is;

$$G = \left(\frac{A+B+C}{3}, \frac{\sin A + \sin B + \sin C}{3}\right)$$

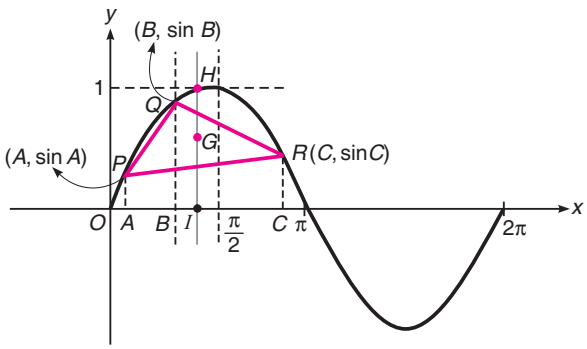


Fig. 1.107

where; G, H and I are collinear

$$I\left(\frac{A+B+C}{3}, 0\right); \ G\left(\frac{A+B+C}{3}, \frac{\sin A + \sin B + \sin C}{3}\right)$$

and $$H\left(\frac{A+B+C}{3}, \sin\left(\frac{A+B+C}{3}\right)\right)$$

From figure; $\quad HI > GI$

i.e., $\quad$ ordinate of H > ordinate of *G*

$$\Rightarrow \quad \sin\left(\frac{A+B+C}{3}\right) > \frac{\sin A + \sin B + \sin C}{3}$$

$$\Rightarrow \quad \frac{3\sqrt{3}}{2} > \sin A + \sin B + \sin C.$$

EXAMPLE 21 *If* $0 < A < \frac{\pi}{6}$, *then show* $A(\text{cosec } A) < \frac{\pi}{3}$.

SOLUTION Here, graph for $y = \sin x$ is shown as;

where $P(A, \sin A)$ and $Q\left(\frac{\pi}{6}, \sin\frac{\pi}{6}\right)$

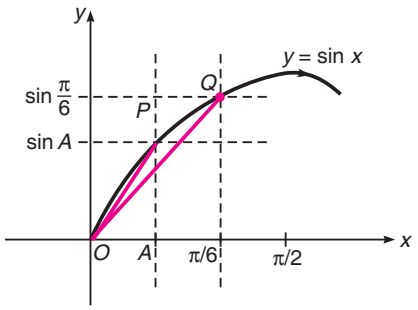


Fig. 1.108

From adjoining figure;

slope of OP > slope of OQ

$$\Rightarrow \quad \frac{\sin A - 0}{A - 0} > \frac{\sin\frac{\pi}{6} - 0}{\frac{\pi}{6} - 0}$$

$$\Rightarrow \quad \frac{\sin A}{A} > \frac{3}{\pi} \quad \text{or} \quad \frac{A}{\sin A} < \frac{\pi}{3}$$

or $$A(\text{cosec } A) < \frac{\pi}{3}.$$

EXAMPLE 22 *If* $0 < A, B, C < \frac{\pi}{2}$, *then show that:* $A \text{ cosec } A + B \text{ cosec } B + C \text{ cosec } C < \frac{3\pi}{2}$.

SOLUTION Here, graph for $y = \sin x$ is shown as;

where $P(B, \sin B)$; $Q(C, \sin C)$; $R(A, \sin A)$; $S\left(\frac{\pi}{2}, \sin\frac{\pi}{2}\right)$.

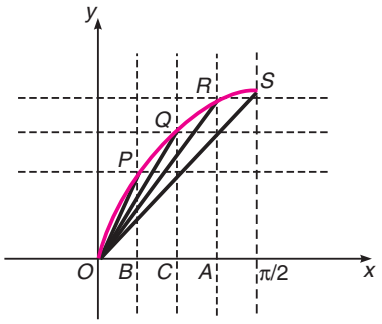


Fig. 1.109

From figure;

slope of OP > **slope of *OS*:**

$$\Rightarrow \quad \frac{\sin B - 0}{B - 0} > \frac{\sin\frac{\pi}{2} - 0}{\frac{\pi}{2}}$$

$$\Rightarrow \quad \frac{\sin B}{B} > \frac{2}{\pi} \quad \text{or} \quad \frac{B}{\sin B} < \frac{\pi}{2}$$

or $$B \text{ cosec } B < \frac{\pi}{2} \qquad \ldots(i)$$

Similarly, slope of OQ < slope of *OS* and slope of OR < slope of *OS*.

$$\Rightarrow \quad C \text{ cosec } C < \pi/2 \qquad \ldots(ii)$$

$$A \text{ cosec } A < \pi/2 \qquad \ldots(iii)$$

Adding Eqs. (i), (ii) and (iii), we get

$$A \text{ cosec } A + B \text{ cosec } B + C \text{ cosec } C < \frac{3\pi}{2}.$$

Note Students must practice above method in different questions of trigonometric inequality as it saves time.

EXERCISE

1. Construct the graph for;

(i) $f(x)=\begin{cases}x-1; & x<0\\ \frac{1}{4}; & x=0\\ x^2; & x>0\end{cases}$

(ii) $f(x)=\begin{cases}2x+3; & -3\le x<-2\\ x+1; & -2\le x<0\\ x+2; & 0\le x\le 1\end{cases}$

2. Construct the graph of the function:

(i) $f(x)=|x-1|+|x+1|$

(ii) $f(x)=\begin{cases}3^x; & -1\le x\le 1\\ 4-x; & 1\le x<4\end{cases}$

(iii) $f(x)=[x]+|x-1|; \quad -1\le x\le 3$

(where $[\cdot]$ denotes greatest integer function)

(iv) $f(x)=\begin{cases}x^4; & x^2<1\\ x; & x^2\ge 1\end{cases}$

3. Is $f(x)=x^2+x+1$ invertible? If not in which region it is invertible.

4. If f is defined by $y=f(x)$; where $x=2t-|t|$, $y=t^2+t|t|$, $t\in R$. Then construct the graph for $f(x)$.

5. Construct the graph for $f(x)=[[x]-x]$; where $[\cdot]$ denotes greatest integer function.

6. If $0<\alpha<1$; then show; $\dfrac{\tan^{-1}\alpha}{\alpha}>\dfrac{\pi}{4}$.

7. Find the number of solutions for; $\cos^{-1}(\cos x)=[x]$ where $[\cdot]$ denotes the greatest integer function.

8. Find the number of solutions for; $[[x]-x]=\sin x$; where $[\cdot]$ denotes the greatest integer function.

9. Find the values of x graphically which satisfy $\left|\dfrac{x^2}{x-1}\right|\le 1$.

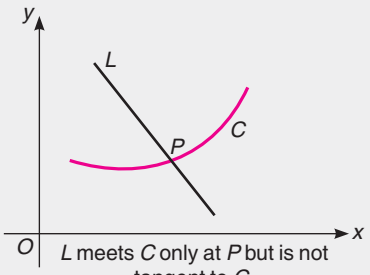
10. Find the value of x for which $x^3-[x]=3$, where $[\cdot]$ denotes the greatest integer function.

ANSWERS

3. $x\ge -\frac{1}{2}$ **7.** 5 solutions. **8.** infinite **9.** $x\in\left[\frac{-1-\sqrt{5}}{2},\frac{-1+\sqrt{5}}{2}\right]$ **10.** $x=2^{2/3}$.

Myths About Jangent Lins

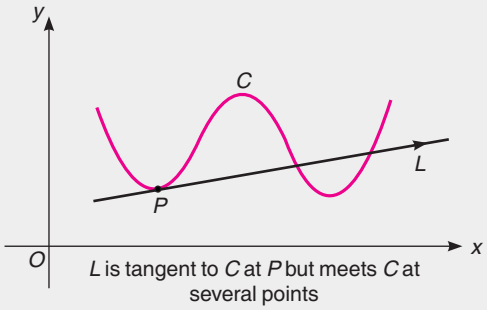
Remark 1



L meets *C* only at *P* but is not tangent to *C*.

Fig. 1

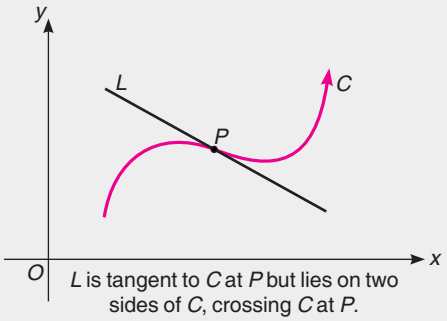
Remark 2



L is tangent to *C* at *P* but meets *C* at several points

Fig. 2

Remark 3



L is tangent to *C* at *P* but lies on two sides of *C*, crossing *C* at *P*.

Fig. 3

Chapter 2

CURVATURE AND TRANSFORMATIONS

In this chapter we shall study:

- The bending of curves at different points.
- Transformations of curves.

2.1 CURVATURE

"**The study of bending of curves at different points is known as curvature.**" or "**Rate at which the curve curves**".

Consider a curve and a point P on it and let Q be a point near P. Let A be a point on the curve.

arc AP = s, arc AQ = s + δs

$\therefore$ arc PQ = δs

Let ψ and ψ + δψ be angles which the tangents at P and Q makes with x-axis.

$\therefore$ δψ is called **the total curvature of the arc *PQ*.**

$\frac{\delta\psi}{\delta s}$ is called **the average curvature of the arc *PQ*.**

$\lim_{\delta s \to 0} \frac{\delta\psi}{\delta s} = \frac{d\psi}{ds}$ = **curvature of the curve at *P*.**

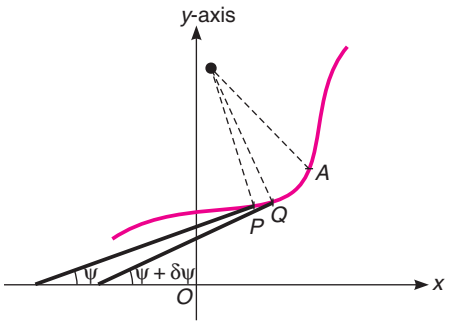


Fig. 2.1

Note To study curvature we shall define $\frac{d\psi}{ds}$ or $\frac{d^2y}{dx^2}$.

2.2 CONCAVITY, CONVEXITY AND POINTS OF INFLEXION

(a) Concave upwards

If in the neighbourhood of a point P on a curve is above the tangent at P, it is said to be concave upwards.

Mathematically

$\frac{dy}{dx}$ increases as x increases. $\Rightarrow$ $\frac{d^2y}{dx^2} > 0$

Geometrically

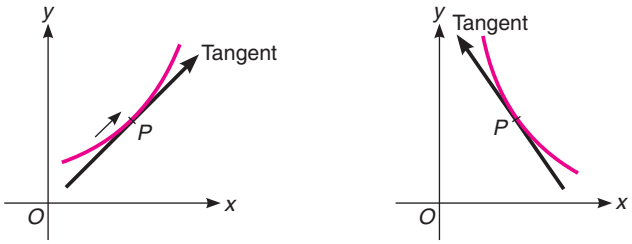


Fig. 2.2

(b) Convex upwards or Concave downwards

If the curve is below the tangent at P, it is said to be convex upward or concave downward.

Mathematically

$\frac{dy}{dx}$ decreases as x decreases. $\Rightarrow$ $\frac{d^2y}{dx^2} < 0$

Geometrically

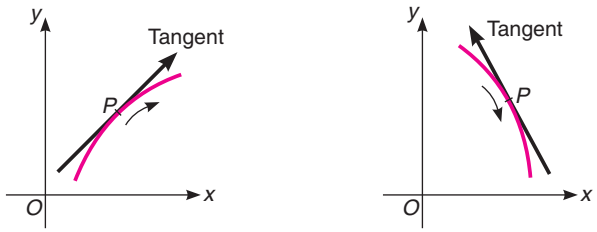


Fig. 2.3

(c) Point of Inflexion

If at a point P, a curve changes its concavity from upwards to downwards or vice versa. Then P is called point of inflexion.

Geometrically

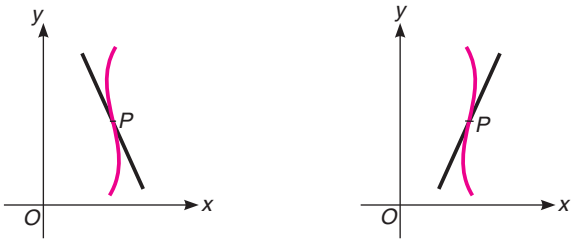


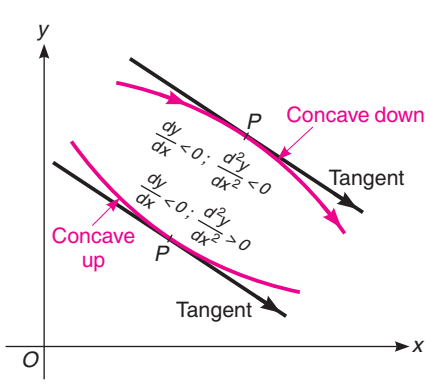
Fig. 2.4

Mathematically

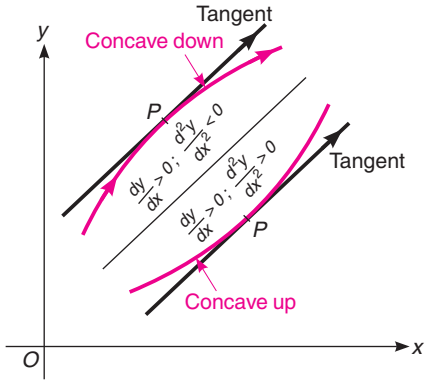
(i) $\frac{d^2y}{dx^2} = 0$ at the point.

(ii) $\frac{d^2y}{dx^2}$ changes its sign as x increases through the value at which $\frac{d^2y}{dx^2} = 0$; *i.e.*, $\frac{d^3y}{dx^3} \neq 0$.

Note In general we can represent concavity as;.



Here; $\frac{dy}{dx} < 0$ for both curves.



Here; $\frac{dy}{dx} > 0$ for both curves.

Fig. 2.5

2.3 PLOTTING OF ALGEBRAIC CURVES USING CONCAVITY

Here; if $y = f(x) = (x - \alpha)(x - \beta)$. $0 < \alpha < \beta$. Then we know it has roots α and β and would be plotted as shown in Fig. 2.6.

From above discussion it becomes clear that to plot curves we require;

(i) Point of intersection on x-axis. (*i.e.*, $y = 0$)

(ii) Point of maximum and minimum value.

(iii) Interval for which function increases or decreases.

(iv) Point at which concave up, concave down and point of inflexion.

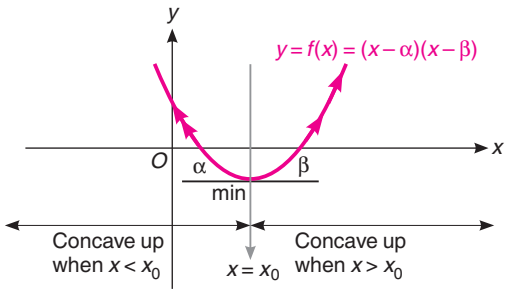


Fig. 2.6

EXAMPLE 1 *Sketch* $y = (x - 1)(x - 2)$.

SOLUTION Here; $y = (x - 1)(x - 2)$

(i) Put $y = 0 \Rightarrow x = 1, 2$. (for point of intersection on x-axis.)

(ii) $y = x^2 - 3x + 2$

$\Rightarrow \quad \frac{dy}{dx} = 2x - 3$ and $\frac{d^2y}{dx^2} = 2$

$\therefore$ minimum at $x = \frac{3}{2}$ $\left(\text{as } \frac{d^2y}{dx^2} > 0\right)$

(iii) Increases when $x > \frac{3}{2}$ and decreases when $x < \frac{3}{2}$.

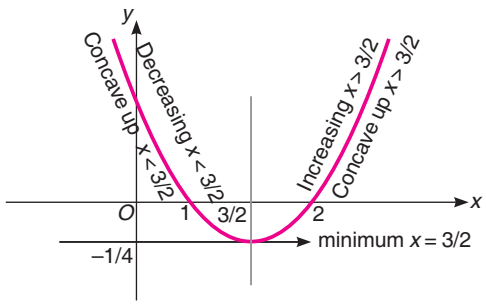


Fig. 2.7

(iv) Concave upwards for $x > \frac{3}{2}$ or $x < \frac{3}{2}$.

$\therefore$ Graph is sketched as shown in Fig. 2.7.

EXAMPLE 2 *Sketch the curve* $y = (x-1)(x-2)(x-3)$.

SOLUTION Here; $y = (x-1)(x-2)(x-3)$

(i) Put $y = 0 \Rightarrow x = 1, 2, 3$.

(ii) $y = x^3 - 6x^2 + 11x - 6$

$$\Rightarrow \frac{dy}{dx} = 3x^2 - 12x + 11 \quad \text{and} \quad \frac{d^2y}{dx^2} = 6x - 12$$

when $\frac{dy}{dx} = 0 \Rightarrow x = \frac{6 \pm \sqrt{3}}{3}$

$\therefore$ maximum when; $x = \frac{6-\sqrt{3}}{3}$ as $\frac{d^2y}{dx^2} = -2\sqrt{3}$

minimum when; $x = \frac{6+\sqrt{3}}{3}$ as $\frac{d^2y}{dx^2} = 2\sqrt{3}$

(iii) Here; $\frac{dy}{dx} = 3x^2 - 12x + 11$

$$= 3\left(x - \frac{6-\sqrt{3}}{3}\right)\left(x - \frac{6+\sqrt{3}}{3}\right)$$

$\Rightarrow$ Increases when; $x < \frac{6-\sqrt{3}}{3}$

or $x > \frac{6+\sqrt{3}}{3}$

decreases when; $\frac{6-\sqrt{3}}{3} < x < \frac{6+\sqrt{3}}{3}$

(iv) Concave upwards when $x > 2$ and concave down when $x < 2$.

$\therefore$ Graph is sketched as shown in Fig. 2.8.

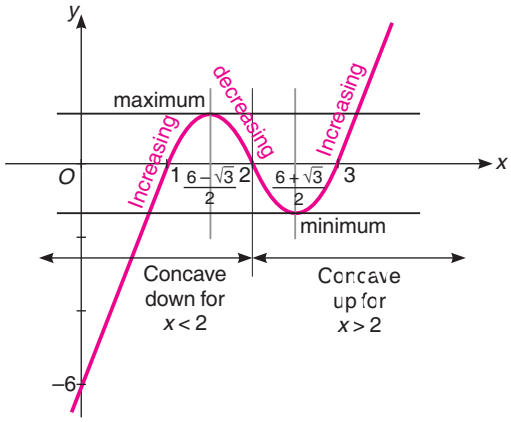


Fig. 2.8

EXAMPLE 3 *Sketch the curve* $y = (x-1)^2(x-2)$.

SOLUTION Here; $y = (x-1)^2(x-2)$

(i) Put $y = 0 \Rightarrow x = 1, 1, 2$

(ii) $y = x^3 - 4x^2 + 5x - 2$

$$\Rightarrow \frac{dy}{dx} = 3x^2 - 8x + 5 \quad \text{and} \quad \frac{d^2y}{dx^2} = 6x - 8$$

when $\frac{dy}{dx} = 0 \Rightarrow x = 1, \frac{5}{3}$

$\therefore$ maximum when; $x = 1$ as $\frac{d^2y}{dx^2} = -2$.

minimum when; $x = \frac{5}{3}$ as $\frac{d^2y}{dx^2} = 2$.

(iii) Here; $\frac{dy}{dx} = 3x^2 - 8x + 5 = 3(x-1)\left(x - \frac{5}{3}\right)$

$\Rightarrow$ Increases when; $x < 1$ or $x > \frac{5}{3}$.

Decreases when; $1 < x < \frac{5}{3}$.

(iv) Concave up when $x > \frac{4}{3}$ and

concave down when $x < \frac{4}{3}$.

$\therefore$ Graph is sketched as shown in Fig. 2.9.

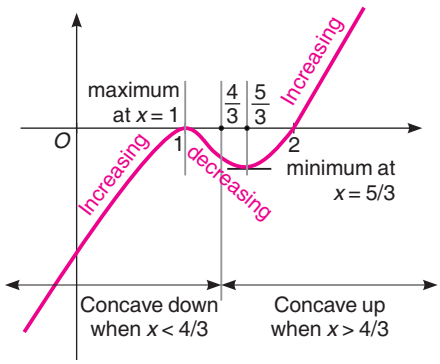


Fig. 2.9

EXAMPLE 4 *Sketch the curve* $y = 3x^2 - 2x^3$.

SOLUTION Here; $y = 3x^2 - 2x^3$

$\Rightarrow$ $x = 0, \frac{3}{2}$ when $y = 0$ ...(i)

also $\frac{dy}{dx} = 6x - 6x^2 = 6x(1-x)$

and $\frac{d^2y}{dx^2} = 6 - 12x = 6(1-2x)$

$\Rightarrow$ maximum when; $x = 1$ as $\frac{d^2y}{dx^2} = -6$

and minimum when; $x = 0$ as $\frac{d^2y}{dx^2} = 6$ } ...(ii)

y increases when; $0 < x < 1$.

y decreases when; $x < 0$ and $x > 1$ } ...(iii)

Concave up when; $x < \frac{1}{2}$

Concave down when; $x > \frac{1}{2}$ } ...(iv)

$\therefore$ Graph is sketched as shown in Fig. 2.10.

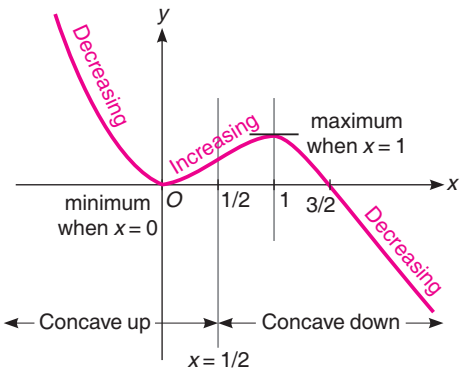


Fig. 2.10

EXAMPLE 5 *Sketch the graph for the function:* $f(x) = |x+3|(x+1)$.

SOLUTION Here; $y = |x+3|(x+1) = \begin{cases} (x+3)(x+1); & x \geq -3 \\ -(x+3)(x+1); & x < -3 \end{cases}$

$\Rightarrow$ $x = -1, -3$ when $y = 0$ ...(i)

also; $\frac{dy}{dx} = \begin{cases} +2x+4; & x \geq -3 \\ -2x-4; & x < -3 \end{cases}$ and $\frac{d^2y}{dx^2} = \begin{cases} 2; & x \geq -3 \\ -2; & x < -3 \end{cases}$

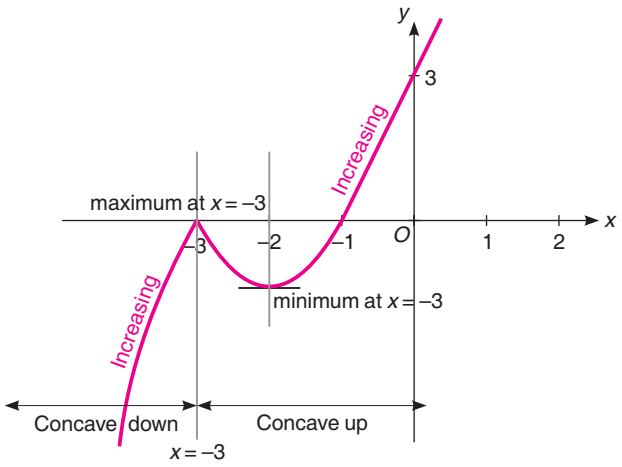


Fig. 2.11

$\Rightarrow$ Increasing when $x < -3$ or $x > -2$; decreasing when $-3 < x < -2$ …(ii)

maximum at $x = -3$; minimum at $x = -2$ …(iii)

concave up when $x < -3$; concave down when $x > -3$ …(iv)

Note Above example could also be solved by using transformations discussed in later part of chapter.

EXAMPLE 6 *Sketch the graph for:* $f(x) = \dfrac{x+1}{x^2+3}$

SOLUTION Here; $y = \dfrac{x+1}{x^2+3} \Rightarrow$ $x = -1$ when $y = 0$; $y = \dfrac{1}{3}$ when $x = 0$ …(i)

$\dfrac{dy}{dx} = \dfrac{-x^2-2x+3}{(x^2+3)^2} = \dfrac{-(x+3)(x-1)}{(x^2+3)^2} \Rightarrow$ Increasing when; $-3 < x < 1$; Decreasing when; $x < -3$ or $x > 1$ …(ii)

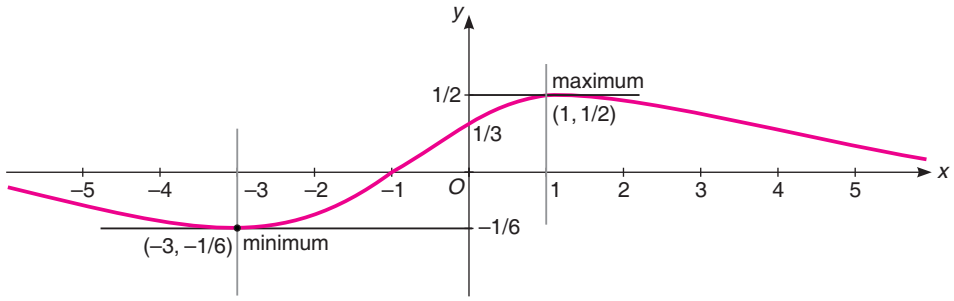
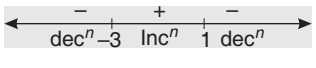


Fig. 2.12

{Using number line rule for $\frac{dy}{dx}$, 

}

also; $\frac{d^2y}{dx^2} = \frac{2(x^3 + 3x^2 - 9x - 3)}{(x^2 + 3)^3}$

minimum at $x = -3$ as; $\frac{d^2y}{dx^2} = +\frac{1}{36} > 0$

maximum at $x = 1$ as; $\frac{d^2y}{dx^2} = -\frac{1}{4} < 0$ …(iii)

Note In above curve *x*-axis works as asymptote, *i.e.*, the curve would never meet *x*-axis. For detail refer chapter 3.

2.4 GRAPHICAL TRANSFORMATIONS

Here, we shall discuss the transformations as;

(i) $f(x)$ transforms to $f(x) \pm a$
(ii) $f(x)$ transforms to $f(x \pm a)$.
(iii) $f(x)$ transforms to $(a\ f(x))$
(iv) $f(x)$ transforms to $f(ax)$.
(v) $f(x)$ transforms to $f(-x)$.
(vi) $f(x)$ transforms to $-f(x)$.
(vii) $f(x)$ transforms to $-f(-x)$

(viii) $f(x)$ transforms to $|f(x)|$.
(ix) $f(x)$ transforms to $f(|x|)$.
(x) $f(x)$ transforms to $|f(|x|)|$.
(xi) $y = f(x)$ transforms to $|y| = f(x)$.
(xii) $y = f(x)$ transforms to $|y| = |f(x)|$.
(xiii) $y = f(x)$ transforms to $|y| = |f(|x|)|$.

} Where $|\bullet|$ means modulus or absolute value function.

(xiv) $y = f(x)$ transforms to $y = [f(x)]$.
(xv) $y = f(x)$ transforms to $y = f([x])$.
(xvi) $y = f(x)$ transforms to $y = [f([x])]$.
(xvii) $y = f(x)$ transforms to $[y] = f(x)$.
(xviii) $y = f(x)$ transforms to $[y] = [f(x)]$.

} Where $[\bullet]$ denotes greatest integer less than or equal to *x*.

(xix) $y = f(x)$ transforms to $y = f(\{x\})$.
(xx) $y = f(x)$ transforms to $y = \{f(x)\}$.
(xxi) $y = f(x)$ transforms to $y = \{f(\{x\})\}$.
(xxii) $y = f(x)$ transforms to $\{y\} = f(x)$.
(xxiii) $y = f(x)$ transforms to $\{y\} = \{f(x)\}$.

} Where $\{\bullet\}$ denotes fractional part of x.

(xxiv) $y = f(x)$ transforms to $y = f^{-1}(x)$, $f^{-1}(x)$ represents inverse of $f(x)$.

Now, we shall study the following cases as;

(i) When $f(x)$, transforms to $f(x) \pm a$. (where a is + ve)

i.e., $f(x) \longrightarrow f(x) + a$

shift the given graph of f(x) **upward** through 'a' units

again, $f(x) \longrightarrow f(x) - a$

shift the given graph of f(x) **downward** through 'a' units.

Graphically it could be stated as:

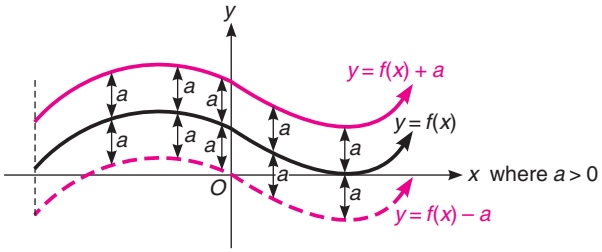


Fig. 2.13

EXAMPLE 1 *Plot* $y = e^x + 1$; $y = e^x - 1$, *with the help of* $y = e^x$.

SOLUTION We know; $y = e^x$ (exponential function) could be plotted as;

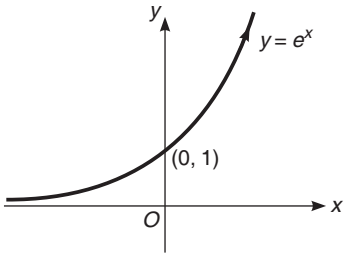


Fig. 2.14

$\Rightarrow$ $y = e^x + 1$, is shifted **upwards** by 1 unit, shown as

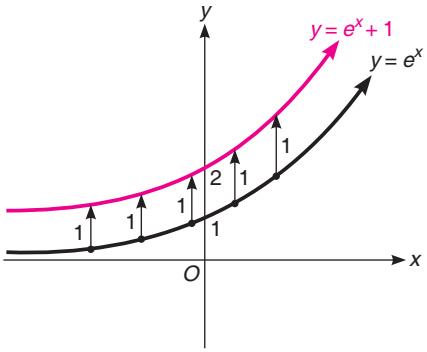


Fig. 2.15

Also $y = e^x - 1$, is shifted downwards by 1 unit, shown as

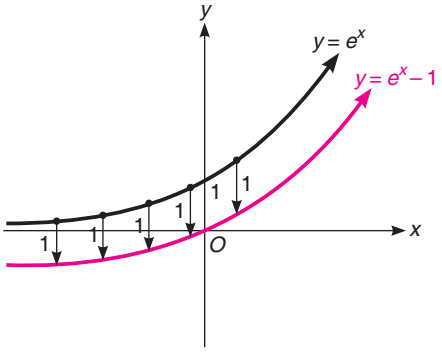


Fig. 2.16

EXAMPLE 2 *Plot* $y = |x| + 2$ *and* $y = |x| - 2$, *with the help of* $y = |x|$.

SOLUTION We know; $y = |x|$ (modulus function) could be plotted as;

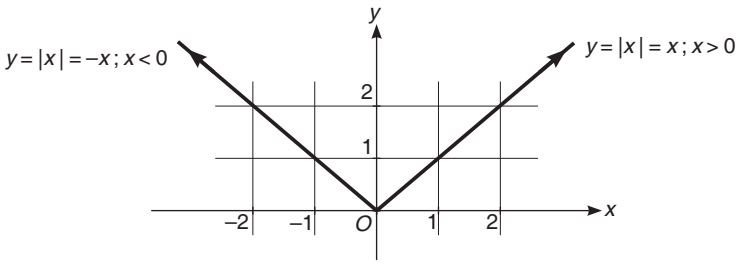


Fig. 2.17

$\Rightarrow$ $y = |x| + 2$ is shifted **upwards** by 2 units.

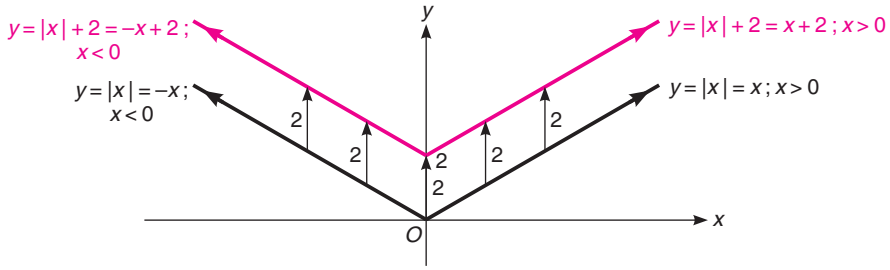


Fig. 2.18

also $y = |x| - 2$ is shifted **downwards** by 2 units.

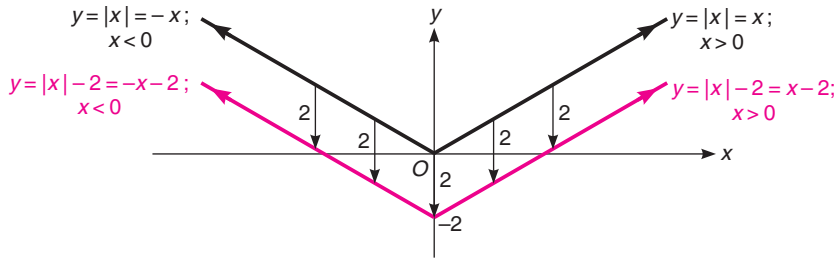


Fig. 2.19

EXAMPLE 3 *Plot* $y = \sin^{-1} x$; $y = (\sin^{-1} x) + 1$ *and* $y = (\sin^{-1} x) - 1$.

SOLUTION We know, $y = \sin^{-1} x$ (Inverse trigonometric) could be plotted as;

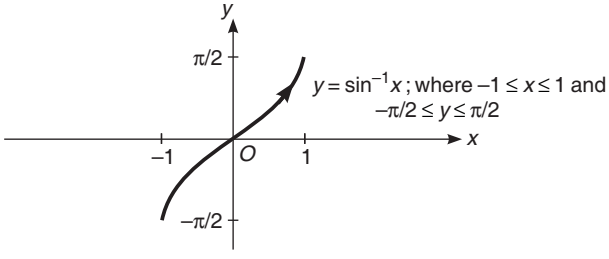


Fig. 2.20

$\Rightarrow$ $y = \sin^{-1} x + 1$, is shifted **upwards** by 1 unit.

and $y = \sin^{-1} x - 1$, is shifted **downwards** by 1 unit.

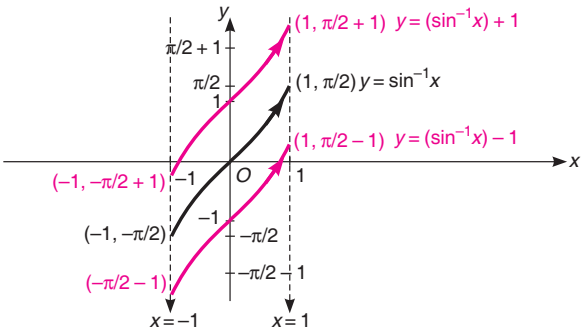


Fig. 2.21

(ii) $f(x)$ transforms to $f(x - a)$

i.e., $f(x) \longrightarrow f(x - a)$; a is positive. Shift the graph of $f(x)$ through 'a' unit towards right $f(x)$ **transforms to** $f(x + a)$**.**

i.e., $f(x) \longrightarrow f(x + a)$; a is positive. Shift the graph of $f(x)$ through 'a' units towards left.

Graphically it could be stated as

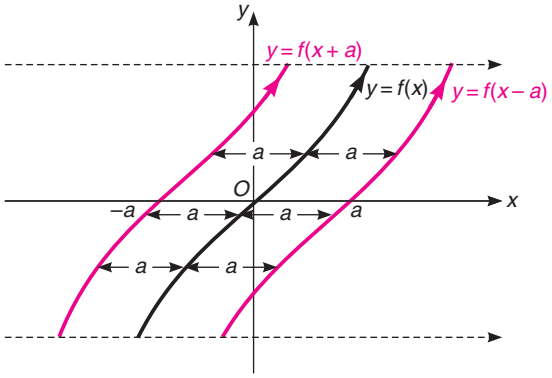


Fig. 2.22

EXAMPLE 1 *Plot* $y = |x|$, $y = |x - 2|$ *and* $y = |x + 2|$.

SOLUTION As discussed $f(x) \longrightarrow f(x - a)$; shift towards right.

$\Rightarrow$ $y = |x - 2|$ is shifted '2' units **towards right**.

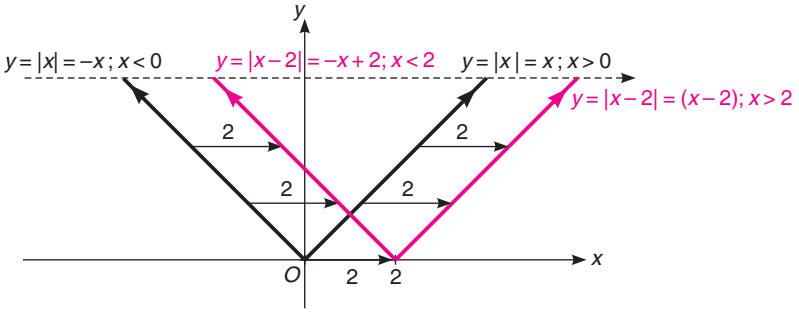


Fig. 2.23

also $y = |x + 2|$ is shifted '2' units **towards left**.

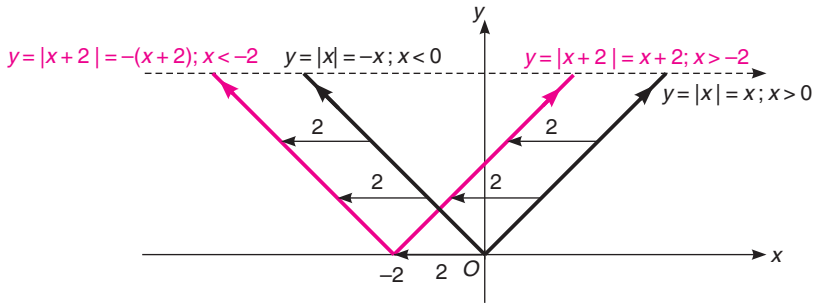


Fig. 2.24

EXAMPLE 2 *Plot* $y = \sin\left(x - \frac{\pi}{2}\right)$ *and* $y = \sin\left(x + \frac{\pi}{2}\right)$.

SOLUTION As we know; $y = \sin x$ could be plotted as;

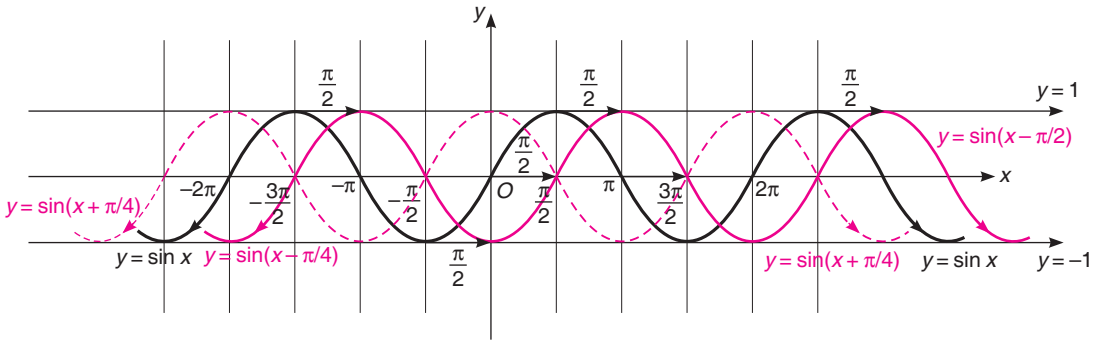


Fig. 2.25

EXAMPLE 3 *Plot* $y = \sin^{-1} x$; $y = \sin^{-1}(x - 1)$ *and* $y = \sin^{-1}(x + 1)$.

SOLUTION We know;

$y = \sin^{-1} x$ could be plotted as shown in Fig. 2.26.

$\Rightarrow$ $y = \sin^{-1}(x - 1)$ is shifted '1' unit **towards right**.

and $y = \sin^{-1}(x + 1)$ is shifted '1' unit **towards left**.

Shown as in Fig. 2.27.

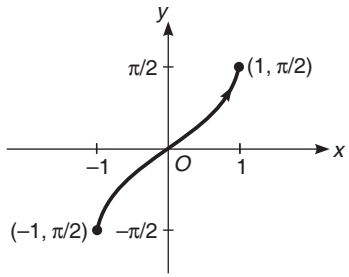


Fig. 2.26

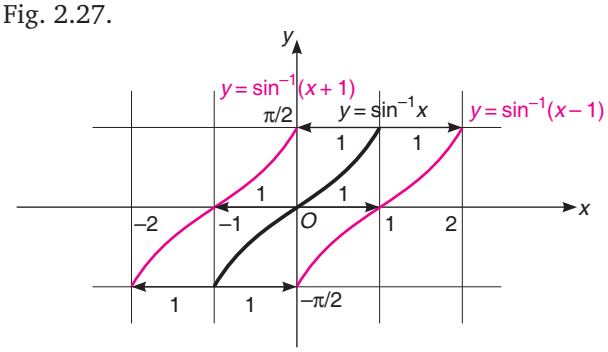


Fig. 2.27

(iii) $f(x)$ transforms to $a\,f(x)$

i.e., $f(x) \longrightarrow af(x);\ a > 1$

Stretch the graph of $f(x)$ 'a' times along y-axis.

$$f(x) \longrightarrow \frac{1}{a} f(x);\ a > 1.$$

Shrink the graph of $f(x)$ 'a' times along y-axis.

Graphically it could be stated as shown in Fig. 2.28.

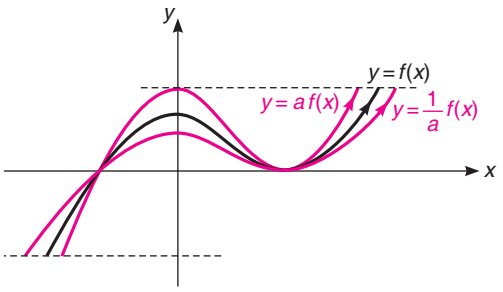


Fig. 2.28

EXAMPLE 1 *Plot* $y = x;\ y = 2x$ *and* $y = \frac{1}{2}x$.

SOLUTION As we know graph for $y = 2x$.

$\therefore$ $y = 2x$; is **stretch** of $f(x)$ '2' times along y-axis and $y = \frac{1}{2}x$; is **shrink** of $f(x)$ '2' times along y-axis.

Shown as in Fig. 2.29.

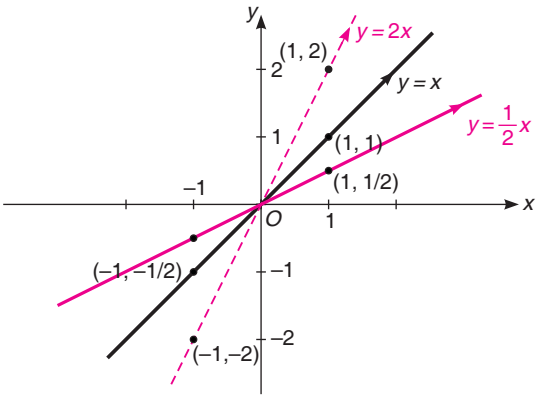


Fig. 2.29

EXAMPLE 2 *Plot* $y = \sin x$ *and* $y = 2\sin x$.

SOLUTION We know; $y = \sin x$ and $f(x) \to a\,f(x)$

$\Rightarrow$ **Stretch** the graph of $f(x)$ 'a' times along y-axis.

$\therefore$ $y = 2 \sin x$ $\Rightarrow$ **stretch** the graph of $\sin x$ '2' times along y-axis.

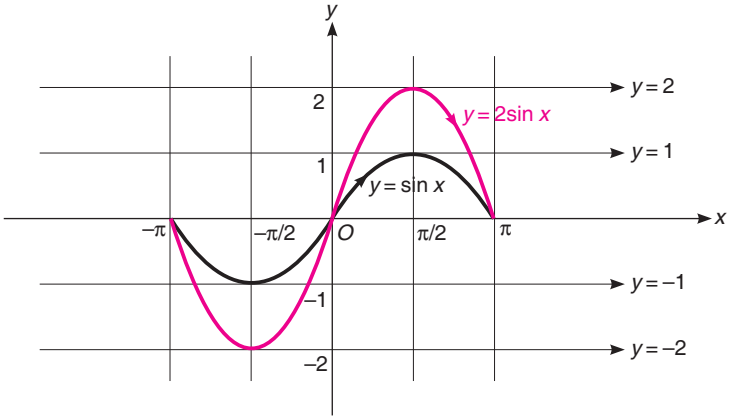


Fig. 2.30

Above curve is plotted for the interval $[-\pi, \pi]$ as periodic with period 2π.

EXAMPLE 3 *Plot* $y = \sin x$ *and* $y = \frac{1}{2}\sin x$.

SOLUTION As we know;

$$y = \frac{1}{a} f(x)$$

$\Rightarrow$ **shrink** the graph of $f(x)$ 'a' times along y-axis.

$\therefore \quad y = \frac{1}{2}\sin x$

$\Rightarrow$ **shrink** the graph of $f(x)$ '2' times along y-axis.

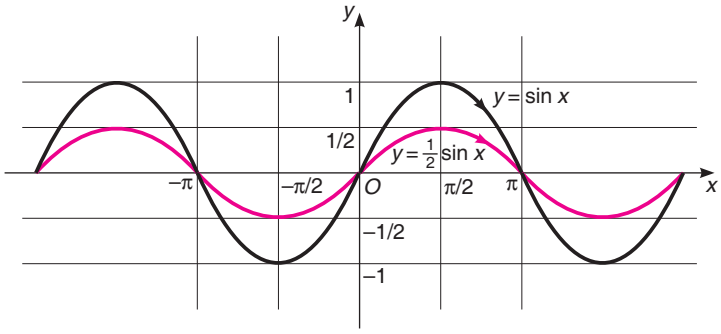


Fig. 2.31

(iv) $f(x)$ transforms to $f(ax)$

i.e., $f(x) \longrightarrow f(ax); \quad a > 1$

Shrink (or contract) the graph of $f(x)$ 'a' times along x-axis.

again $f(x) \longrightarrow f\left(\frac{1}{a}x\right); \quad a > 1$

Stretch (or expand) the graph of $f(x)$ 'a' times along x-axis.

Graphically it could be stated as shown in Fig. 2.32.

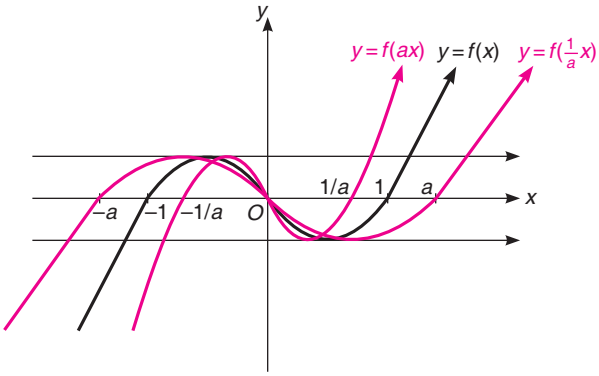


Fig. 2.32

EXAMPLE 1 *Plot* $y = \sin x$ *and* $y = \sin 2x$.

SOLUTION Here; $y = \sin 2x$, is to **shrink (or contract)** the graph of $\sin x$ by '2' units along x-axis. Shown as in Fig. 2.33.

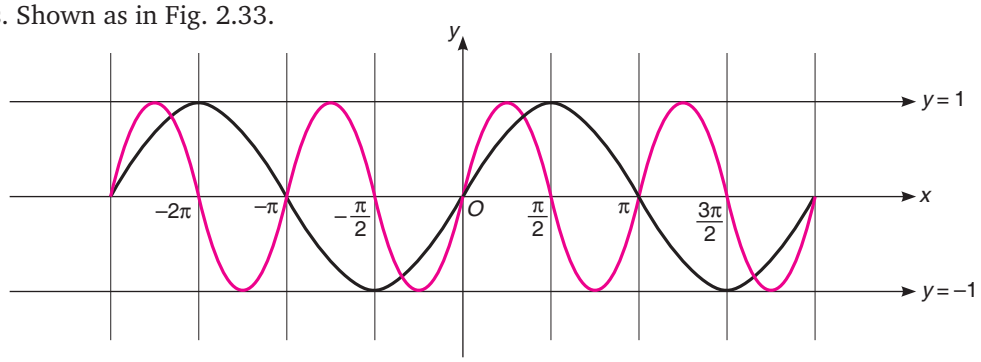


Fig. 2.33

From above figure $\sin x$ is periodic with period 2π and $\sin 2x$ with period π.

EXAMPLE 2 *Plot* $y = \sin x$ *and* $y = \sin\frac{x}{2}$.

SOLUTION Here; $y = \sin\left(\frac{x}{2}\right)$, is to **stretch (or expand)** the graph of $\sin x$ '2' times along x-axis. Shown as in Fig. 2.34.

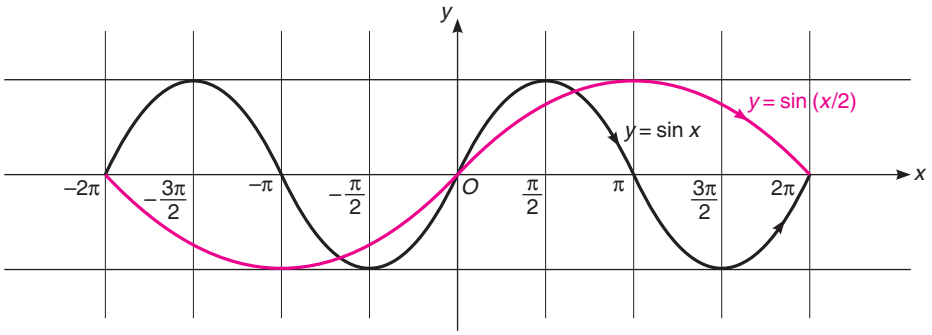


Fig. 2.34

From above figure $\sin x$ is periodic with period 2π and $\sin\left(\frac{x}{2}\right)$ is periodic with period 4π.

EXAMPLE 3 *Plot* $y = \sin^{-1} x$ *and* $y = \sin^{-1}(2x)$.

SOLUTION Here; $y = \sin^{-1}(2x)$, is to **shrink (or contract)** the graph of $\sin^{-1} x$ '2' times **along *x*-axis**.

Shown as in Fig. 2.35.

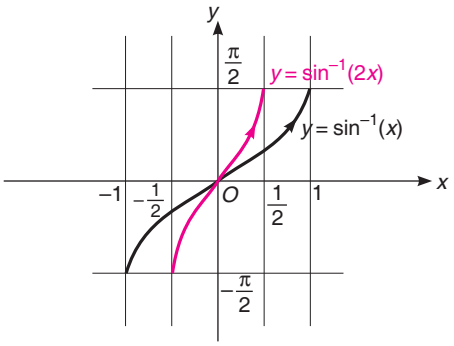


Fig. 2.35

EXAMPLE 4 *Plot* $y = \sin^{-1}\left(\frac{x}{3} - 1\right)$.

SOLUTION To plot $y = \sin^{-1}\left(\frac{x}{3} - 1\right)$. We should follow as;

(i) Plot $y = \sin^{-1} x$

(ii) Plot $y = \sin^{-1}\left(\frac{x}{3}\right)$, *i.e.*, stretch graph '3' units along x-axis.

(iii) Plot $y = \sin^{-1}\left(\frac{x}{3} - 1\right)$, *i.e.*, shift the graph (ii) by '3' unit towards right.

(i) Plotting $y = \sin^{-1} x$ **:**

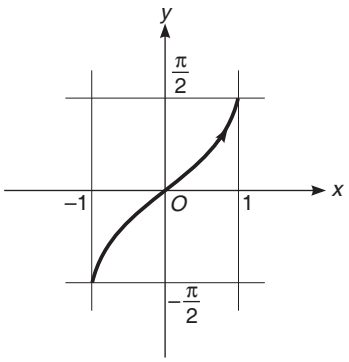


Fig. 2.36

(ii) Plot $y = \sin^{-1}\left(\frac{x}{3}\right)$ **:**

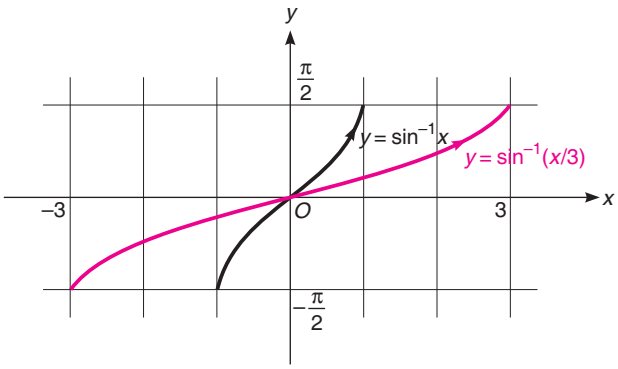


Fig. 2.37

(iii) Plot $y = \sin^{-1}\left(\frac{x}{3} - 1\right)$ **:**

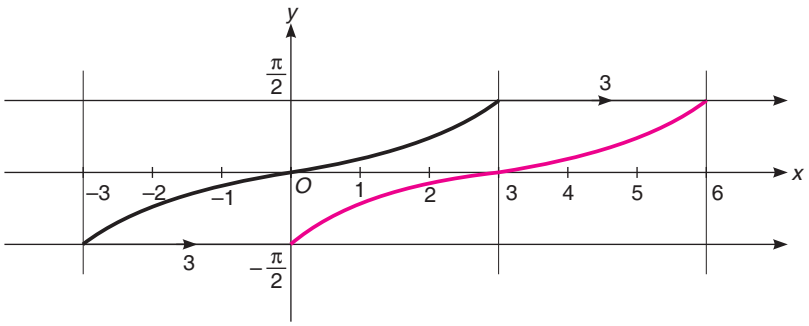


Fig. 2.38

(iv) $f(x)$ **transforms to** $f(-x)$

i.e., $f(x) \longrightarrow f(-x)$

To draw $y = f(-x)$, take the image of the curve $y = f(x)$ in y-axis as plane mirror.

OR

"Turn the graph of $f(x)$ **by 180° about** y**-axis."**

Graphically it is stated as;

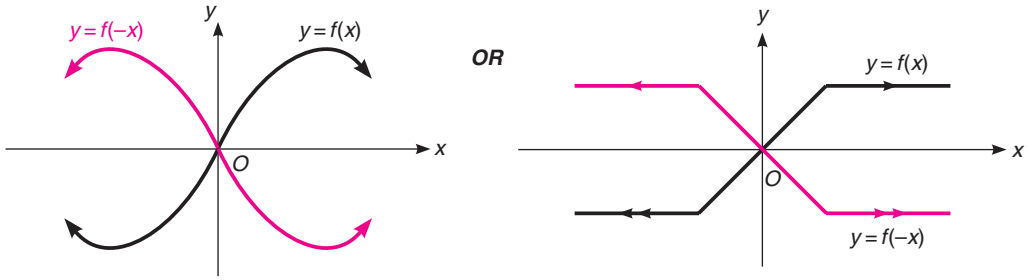


Fig. 2.39

EXAMPLE 1 *Plot* $y = e^{-x}$.

SOLUTION As $y = e^{x}$ is known; then $y = e^{-x}$ is the image in *y*-axis as plane mirror for $y = e^{x}$; shown as;

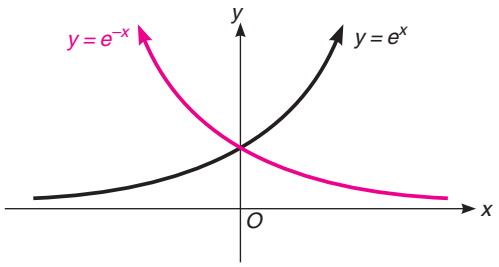


Fig. 2.40

EXAMPLE 2 *Plot the curve* $y = \log_e(-x)$.

SOLUTION Here; $y = \log_e(-x)$; is to take **mirror image** of $y = \log_e x$ about *y*-axis. Shown as;

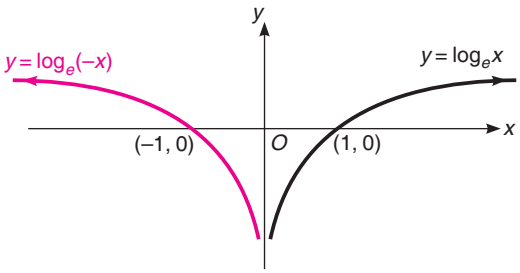


Fig. 2.41

EXAMPLE 3 *Plot the curve* $y = \sin^{-1}(-x)$.

SOLUTION Here; $y = \sin^{-1}(-x)$; is the **mirror image** of $y = \sin^{-1}(x)$ about *y*-axis. Shown as in Fig. 2.42.

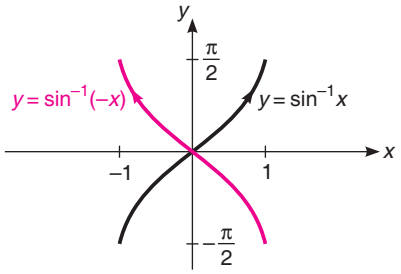


Fig. 2.42

(vi) $f(x)$ transforms to $-f(x)$

i.e., $f(x) \longrightarrow -f(x)$;

To draw $y = -f(x)$ take **image** of $y = f(x)$ in the *x*-axis as plane mirror.

OR

"Turn the graph of $f(x)$ by 180° about *x*-axis."

EXAMPLE 1 *Plot the curve* $y = -e^x$.

SOLUTION As $y = e^x$ is known;

$\therefore$ $y = -e^x$ take image of $y = e^x$ in the ***x*-axis as plane mirror**.

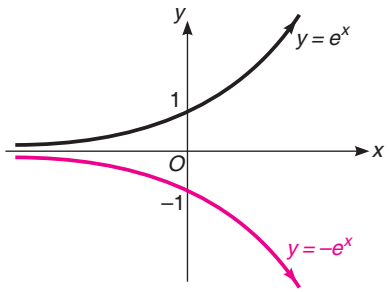


Fig. 2.43

EXAMPLE 2 *Plot the curve* $y = -(\log x)$.

SOLUTION As $y = \log x$ is given then $y = -\log x$ is the image of $y = \log x$ in the ***x*-axis as plane mirror.**

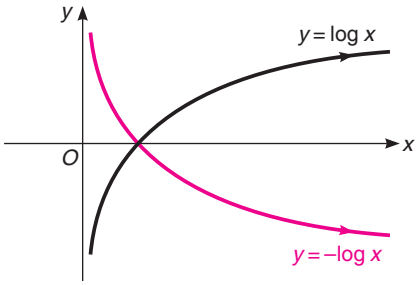


Fig. 2.44

EXAMPLE 3 *Plot the curve* $y = -\{x\}$*; where* $\{\cdot\}$ *denotes the fractional part of x.*

SOLUTION As $y = \{x\}$ is known;

$\therefore$ $y = -\{x\}$ is the image of $y = \{x\}$ about ***x*-axis as plane mirror**.

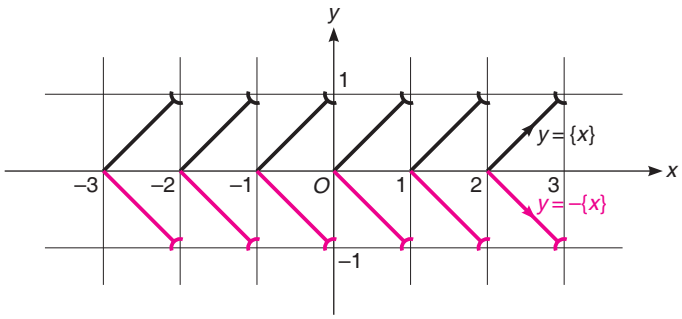


Fig. 2.45

(vii) $f(x)$ transforms to $-f(-x)$

i.e., $f(x) \longrightarrow -f(-x)$;

to draw $y = -f(-x)$ take image of $f(x)$ about y-axis to obtain $f(-x)$ and then take image of $f(-x)$ about x-axis to obtain $-f(-x)$.

$\therefore$ $f(x) \longrightarrow -f(-x)$

$\Rightarrow$ (i) **Image about *y*-axis.** (ii) **Image about *x*-axis.**

Graphically it could be stated as;

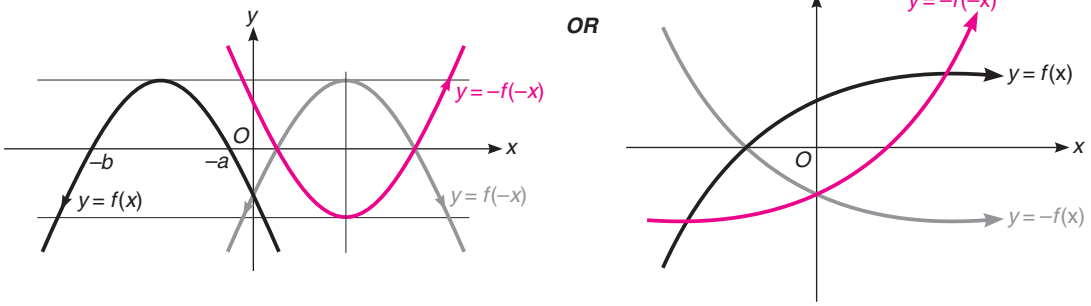


Fig. 2.46

EXAMPLE 1 *Plot the curve* $y = -e^{-x}$.

SOLUTION As $y = e^x$ is known;

(i) Take image about *y*-axis; for $y = e^{-x}$.

(ii) Take image of $y = e^{-x}$ about *x*-axis; for $y = -e^{-x}$. Shown as in Fig. 2.47.

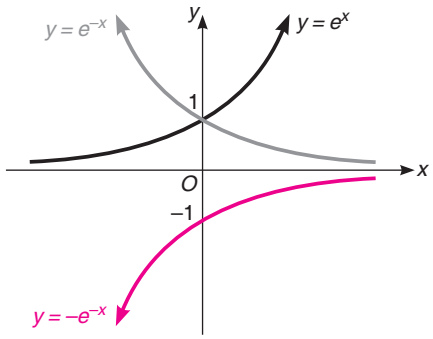


Fig. 2.47

EXAMPLE 2 *Plot the curve* $y = -\log(-x)$.

SOLUTION As $y = \log x$ is known;

(i) Take image about *y*-axis, for $y = \log(-x)$.

(ii) Take image of $y = \log(-x)$ about *x*-axis, for $y = -\log(-x)$.

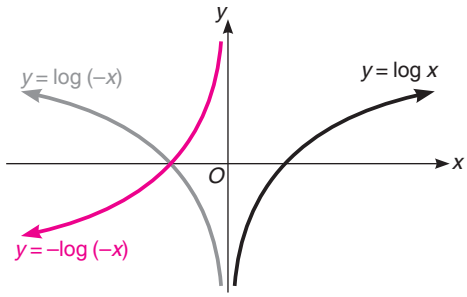


Fig. 2.48

EXAMPLE 3 *Plot the curve* $y = -\{-x\}$. *(where* $\{\cdot\}$ *denote fractional part).*

SOLUTION As we know the curve for $y = \{x\}$.

$\therefore$ **To plot** $y = -\{-x\}$

(i) Take image about x-axis.

(ii) Take image about y-axis.

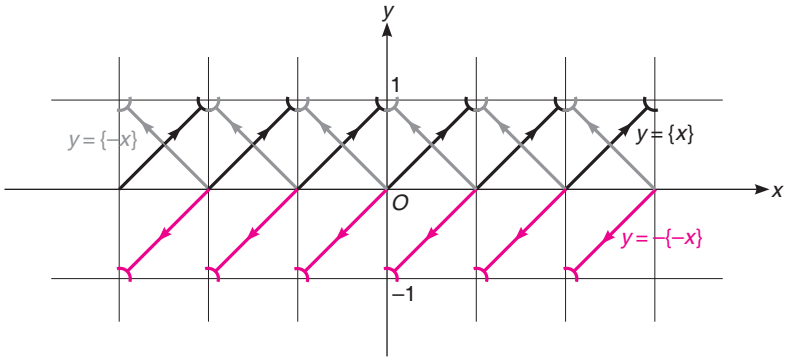


Fig. 2.49

EXAMPLE 4 *Plot the curve for* $y = -[-x]$. *(where* $[\cdot]$ *denotes the greatest integer function.)*

SOLUTION As we know the curve for $y = [x]$.

$\therefore$ to plot $y = -[-x]$

(i) Take image about x-axis.

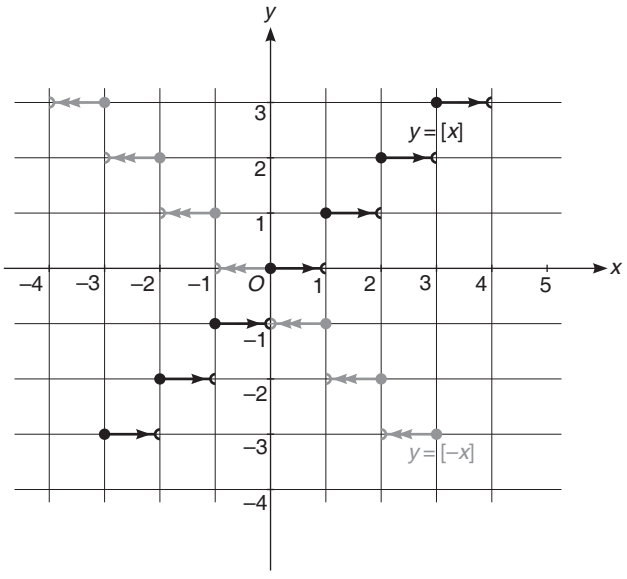


Fig. 2.50

(ii) Take image about y-axis.

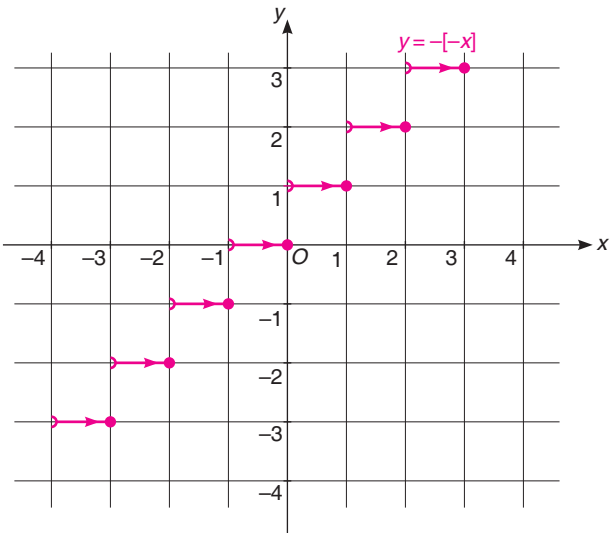


Fig. 2.51

(viii) $f(x)$ transforms to $|f(x)|$; (where $|\ \bullet\ |$ represents modulus function)

i.e., $f(x) \longrightarrow |f(x)|$

Here; $y = |f(x)|$ is drawn in two steps.

(a) In the I step, leave the positive part of $f(x)$, {*i.e.*, the part of $f(x)$ above x-axis) as it is.

(b) In the II step, take the mirror image of negative part of $f(x)$. {*i.e.*, the part of $f(x)$ below x-axis} in the x-axis as plane mirror.

OR

Take the mirror image (in x-axis) of the portion of the graph of $f(x)$ which lies below x-axis.

OR

Turn the portion of the graph of $f(x)$ lying below x-axis by 180° about x-axis.

Graphically it could be stated as

Graph of $f(x)$:

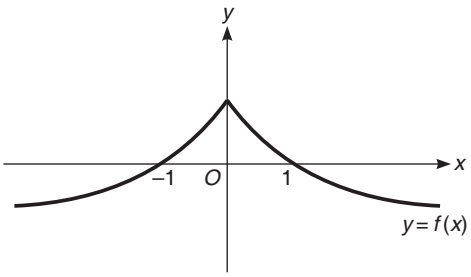


Fig. 2.52

Graph for $|f(x)|$:

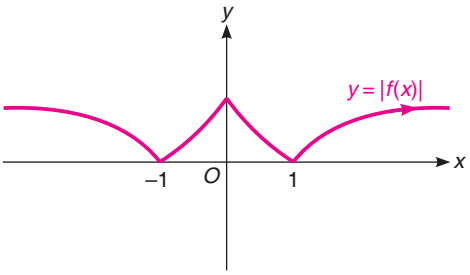


Fig. 2.53

Note Above transformation of graph is very important as to discuss differentiability of $f(x)$.
As from above example we could say $y = f(x)$ is differentiable for all $x \in R - \{0\}$.
But; $y = |f(x)|$ is differentiable for all $x \in R - \{-1, 0, 1\}$ as, "at sharp edges function is not differentiable."

EXAMPLE 1 *Draw the graph for* $y = |\log x|$.

SOLUTION To draw graph for $y = |\log x|$ we have to follow two steps:

(i) Leave the (+ ve) part of $y = \log x$, as it is

(ii) Take images of (–ve) part of $y = \log x$, *i.e.*, the part below x-axis in the x-axis as plane mirror.

Shown as:

Graph for $y = \log x$ **:**

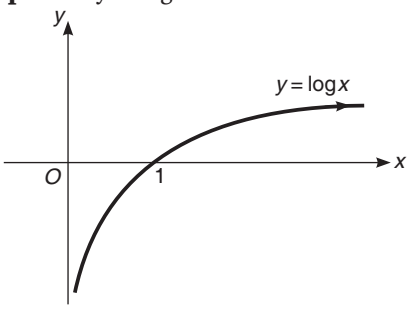


Fig. 2.54

which is differentiable for all $x \in (0, \infty)$

Graph for $y = |\log x|$ **:**

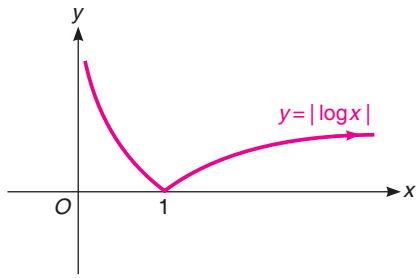


Fig. 2.55

which is clearly differentiable for all $x \in (0, \infty) - \{1\}$. **"as at** $x = 1$ **their is a sharp edge"**.

EXAMPLE 2 *Draw the graph for* $y = |x^2 - 2x - 3|$.

SOLUTION As we know the graph for $y = x^2 - 2x - 3 = (x - 3)(x + 1)$ is a parabola; so to sketch $y = |x^2 - 2x - 3|$ we have to follow two steps.

(i) Leave the positive part of $y = x^2 - 2x - 3$, as it is.

(ii) Take the **image of negative part** of $y = x^2 - 2x - 3$, *i.e.*, the part below x-axis in the ***x*-axis as plane mirror** shown as in Fig. 2.56.

Graph for $y = x^2 - 2x - 3 = (x - 3)(x + 1)$ **:**

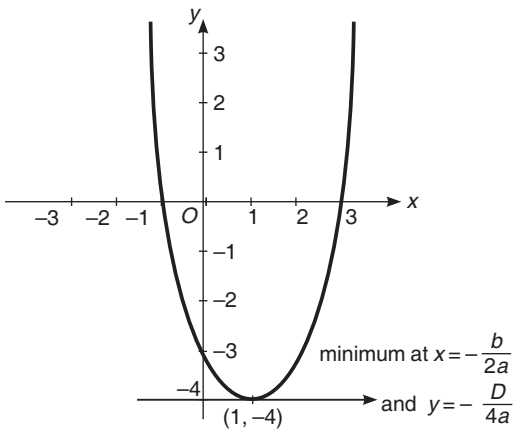


Fig. 2.56

Graph for $y = |x^2 - 2x - 3|$ **:**

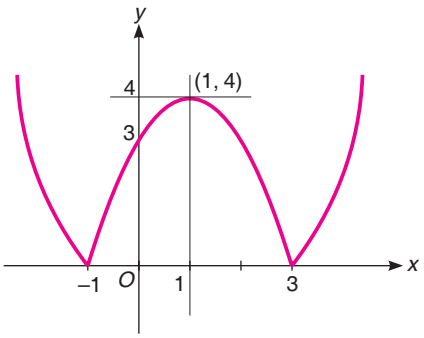


Fig. 2.57

Clearly above curve is differentiable for all $x \in R - \{-1, 3\}$.

EXAMPLE 3 *Sketch the graph for* $y = |\sin x|$.

SOLUTION Here; $y = \sin x$ is known.

$\therefore$ To draw $y = |\sin x|$, we take the mirror image (in x-axis) of the portion of the graph of $\sin x$ which lies below x-axis.

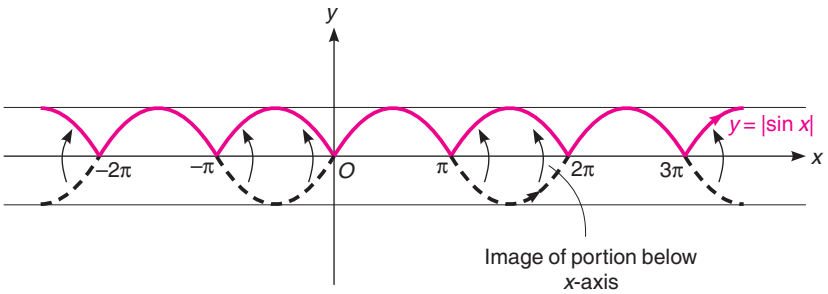


Fig. 2.58

From above figure it is clear;

$y = |\sin x|$ is differentiable for all $x \in R - \{n\pi;\ n \in \text{integer}\}$.

(ix) $f(x)$ transforms to $f(|x|)$

i.e., $f(x) \longrightarrow f(|x|)$.

If we know $y = f(x)$, then to plot $y = f(|x|)$, we should follow two steps:

(i) Leave the graph lying right side of the y-axis as it is.

(ii) Take the image of $f(x)$ in the right of y-axis as the plane mirror and the graph of $f(x)$ lying leftward of the y-axis (if it exists) is omitted.

OR

Neglect the curve for $x < 0$ **and take the images of curve for** $x \geq 0$ **about** y**-axis.**

Graphically shown as;

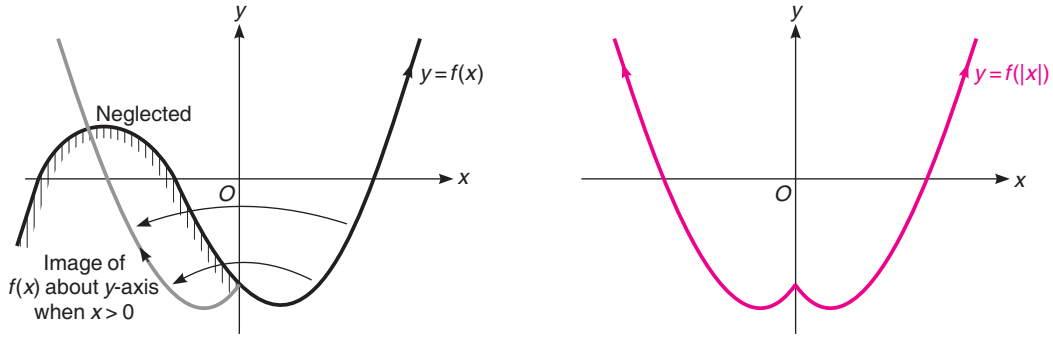


Fig. 2.59

EXAMPLE 1 *Sketch the curve* $y = \log|x|$.

SOLUTION As we know, the curve $y = \log x$.

$\therefore$ $y = \log|x|$ could be drawn in two steps:

(i) Leave the graph lying right side of y-axis as it is.

(ii) Take the image of $f(x)$ in the y-axis as plane mirror.

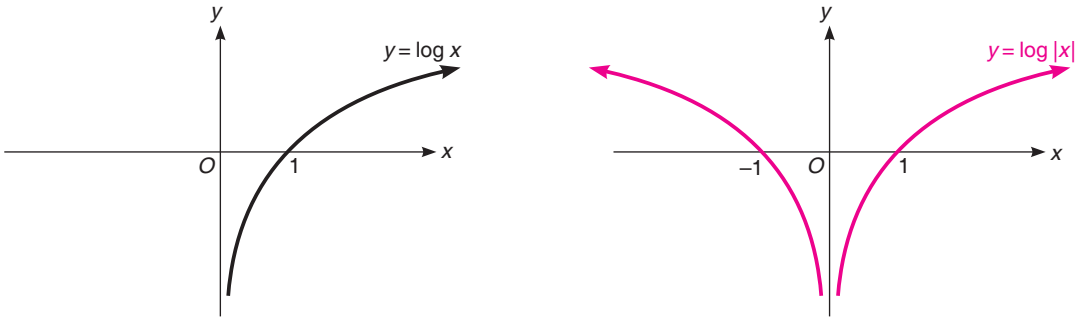


Fig. 2.60

EXAMPLE 2 *Plot the curve* $y = e^{|x|}$.

SOLUTION As we know the curve for $y = e^x$.

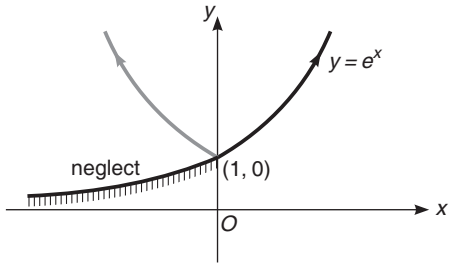


Fig. 2.61

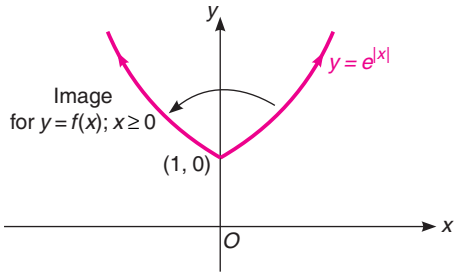


Fig 2.62

$\therefore$ To plot $y = e^{|x|}$, neglect the curve for $x < 0$ and take image about y-axis for $x \geq 0$. Shown as in Fig. 2.62.

EXAMPLE 3 *Plot the curve* $y = \sin|x|$.

SOLUTION

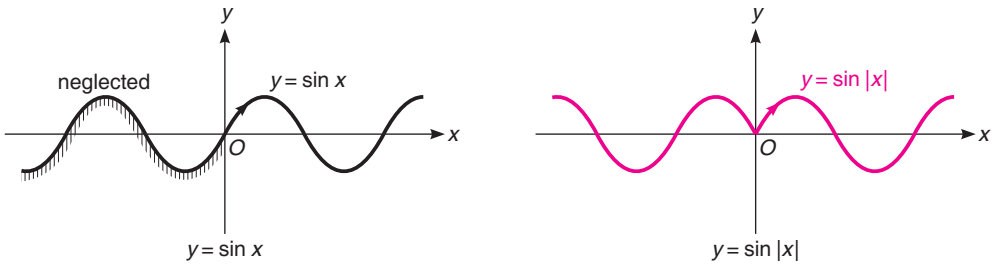


Fig. 2.63

EXAMPLE 4 *Plot the curve* $y = |x|^2 - 2|x| - 3$.

SOLUTION As we know, the curve for $y = x^2 - 2x - 3$ is plotted as shown in Fig. 2.64.

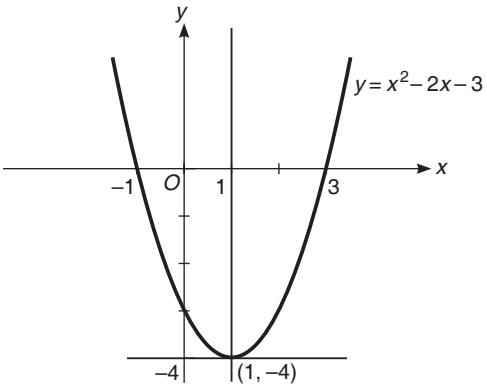


Fig. 2.64

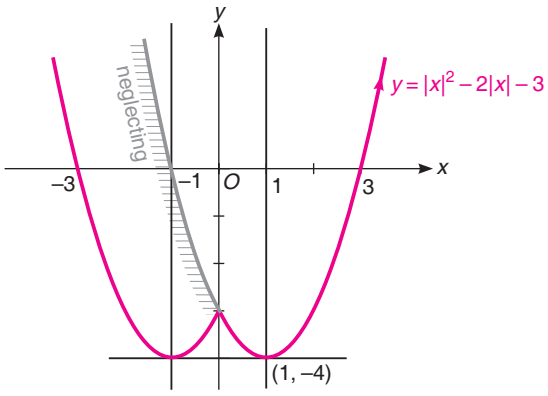


Fig. 2.65

$\therefore$ $y = f(|x|)$, *i.e.*, $y = |x|^2 - 2|x| - 3$ is to be plotted as shown in Fig. 2.65.

which shows $y = |x|^2 - 2|x| - 3$ is differentiable for all $x \in R - \{0\}$.

(x) $f(x)$ transforms to $|f(|x|)|$

i.e., $f(x) \longrightarrow |f(|x|)|$

Here, plot the curve in two steps;

(i) $f(x) \longrightarrow |f(x)|$ (ii) $|f(x)| \longrightarrow |f(|x|)|$

OR

(i) $f(x) \longrightarrow f(|x|)$

(ii) $f(|x|) \longrightarrow |f(|x|)|$., *i.e.*, (viii) and (ix) transformations.

Graphically it could be stated as shown in Fig. 2.66.

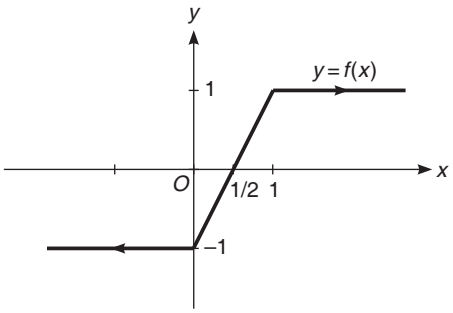


Fig. 2.66

(i) $y = |f(x)|$:

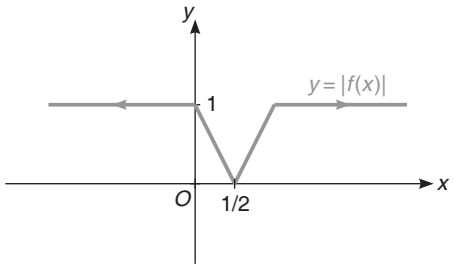


Fig. 2.67

(ii) $y = |f(|x|)|$:

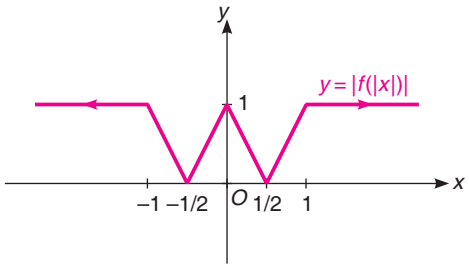


Fig. 2.68

EXAMPLE 1 *Sketch the curve for* $y = ||x|^2 - 2|x| - 3|$.

SOLUTION As we know the graph for $y = x^2 - 2x - 3$, shown as;

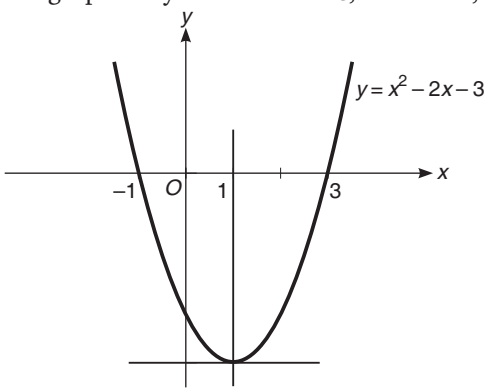


Fig. 2.69

(i) $y = x^2 - 2x - 3 \rightarrow y = |x|^2 - 2|x| - 3$.

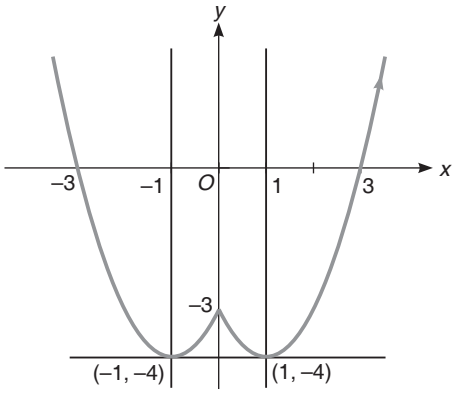


Fig. 2.70

(ii) $y = |x|^2 - 2|x| - 3 \rightarrow y = ||x|^2 - 2|x| - 3|$

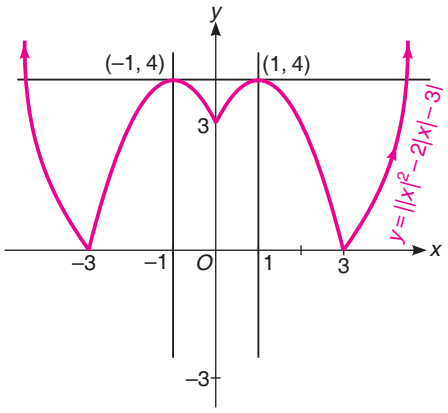


Fig. 2.71

Clearly, above figure is differentiable for all $x \in R - \{-3, 0, 3\}$.

EXAMPLE 2 *Sketch the graph for* $y = \left| e^{-|x|} - \frac{1}{2} \right|$.

SOLUTION As we know the graph for $y = e^{-x}$.

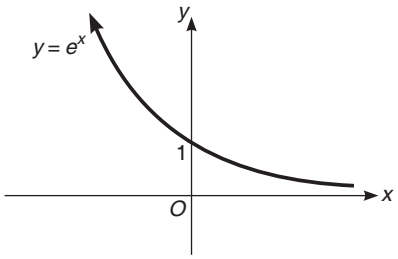


Fig. 2.72

(i) $y = e^{-x} \longrightarrow y = e^{-x} - \frac{1}{2}$.

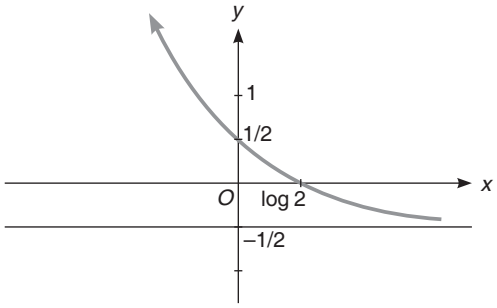


Fig. 2.73

(ii) $y = e^{-x} - \frac{1}{2} \longrightarrow y = e^{-|x|} - \frac{1}{2}$.

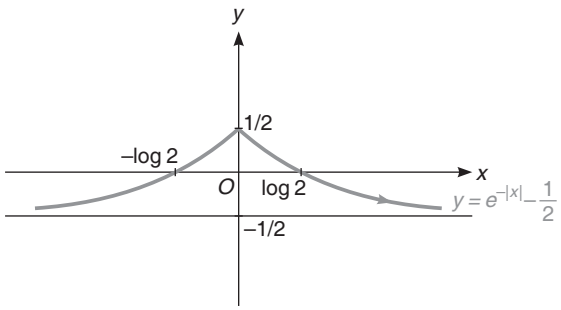


Fig. 2.74

(iii) $y = \left| e^{-|x|} - \frac{1}{2} \right|$

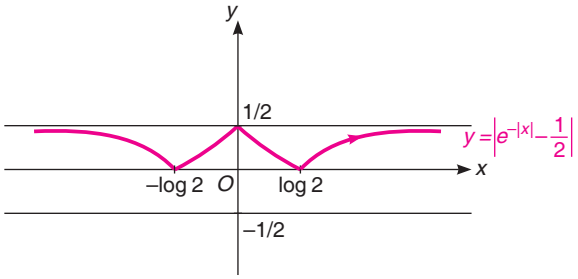


Fig. 2.75

(xi) $y = f(x)$ transforms to $|y| = f(x)$

Clearly $|y| \geq 0 \Rightarrow$ if $f(x) < 0$; graph of $|y| = f(x)$ would not exist.

if $f(x) \geq 0$; $|y| = f(x)$ would be given as $y = \pm f(x)$.

Hence, the graph of $|y| = f(x)$ exists only in the regions where $f(x)$ is non-negative and will be reflected about x-axis only when $f(x) \geq 0$. "Region where $f(x) < 0$ is neglected".

OR

(i) Remove (or neglect) the portion of the graph which lies below *x*-axis.
(ii) Plot the remaining portion of the graph, and also its mirror image in the *x*-axis.
Graphically it could be stated as shown in Fig. 2.76.

Graph for $y = f(x)$:

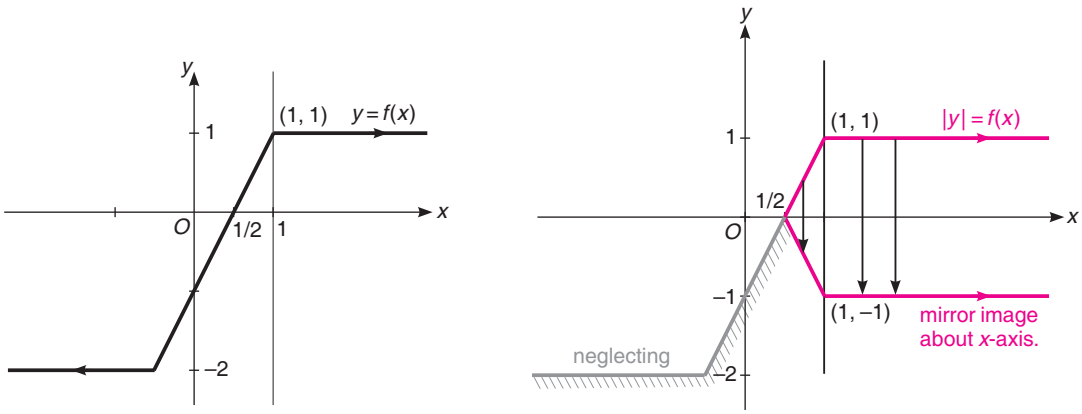


Fig. 2.76

EXAMPLE 1 *Sketch the curve* $|y| = (x - 1)(x - 2)$.

SOLUTION

As we know the graph for $y = (x - 1)(x - 2)$, is shown in Fig. 2.77.

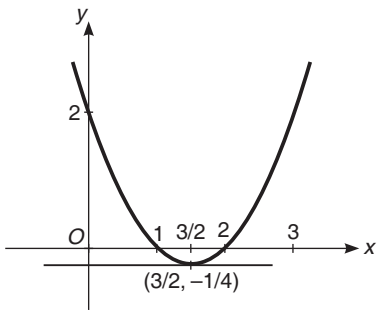


Fig. 2.77

$\Rightarrow y = (x - 1)(x - 2) \longrightarrow |y| = (x - 1)(x - 2)$, as shown in Fig. 2.78.

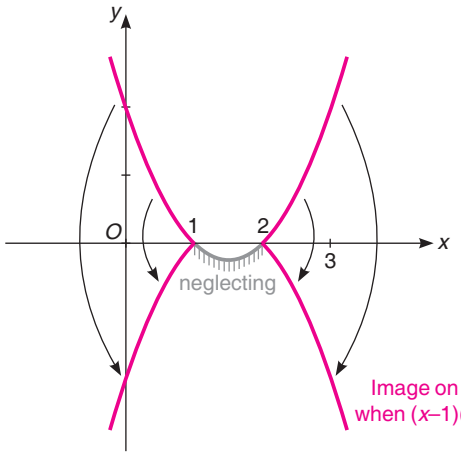


Fig. 2.78

EXAMPLE 2 *Plot the curve* $|y| = \sin x$.

SOLUTION Here, we know the curve for $y = \sin x$.

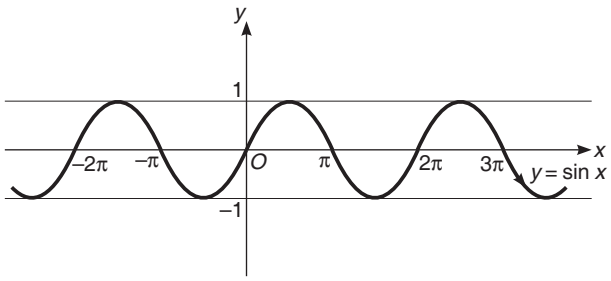


Fig. 2.79

$\therefore$ $y = \sin x \longrightarrow |y| = \sin x$

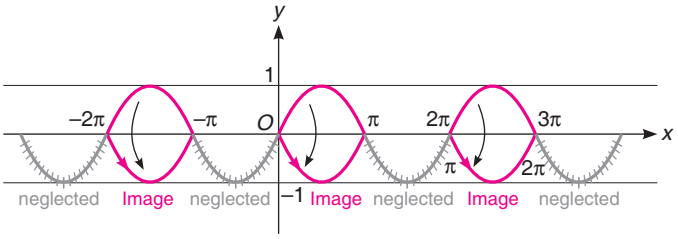


Fig. 2.80

EXAMPLE 3 *Sketch the curve* $|x| + |y| = 1$.

SOLUTION As the graph for $y = 1 - x$ is;

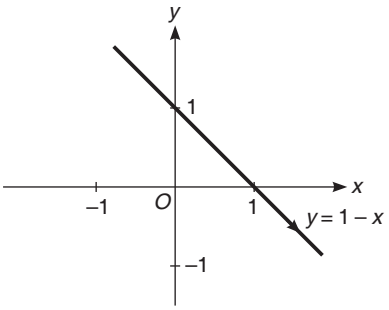


Fig. 2.81

(i) $y = 1 - x \longrightarrow y = 1 - |x|$.

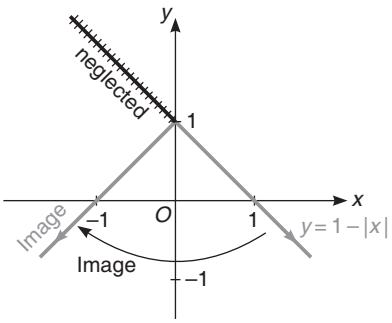


Fig. 2.82

(ii) $y = 1 - |x| \longrightarrow |y| = 1 - |x|$.

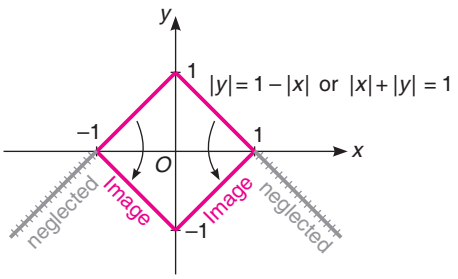
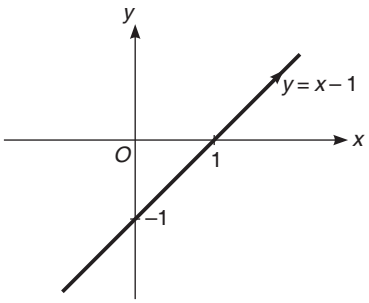


Fig. 2.83

Clearly above figure represents a square.

EXAMPLE 4 *Sketch the curve* $|x| - |y| = 1$.

SOLUTION As the graph for $y = x - 1$ is known;



Fit. 2.84

(i) $y = x - 1 \longrightarrow y = |x| - 1$

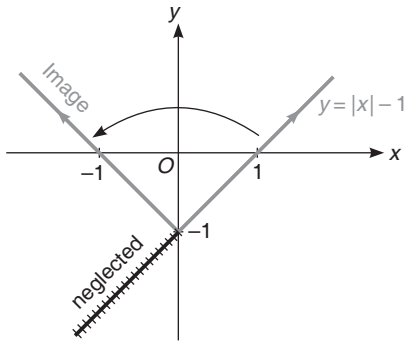


Fig. 2.85

(ii) $y = |x| - 1 \longrightarrow |y| = |x| - 1$

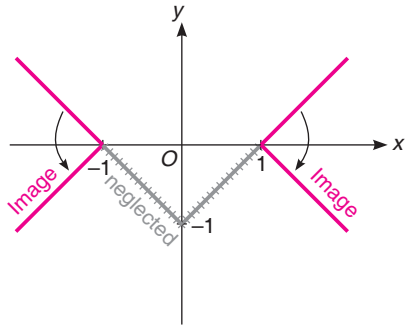


Fig. 2.86

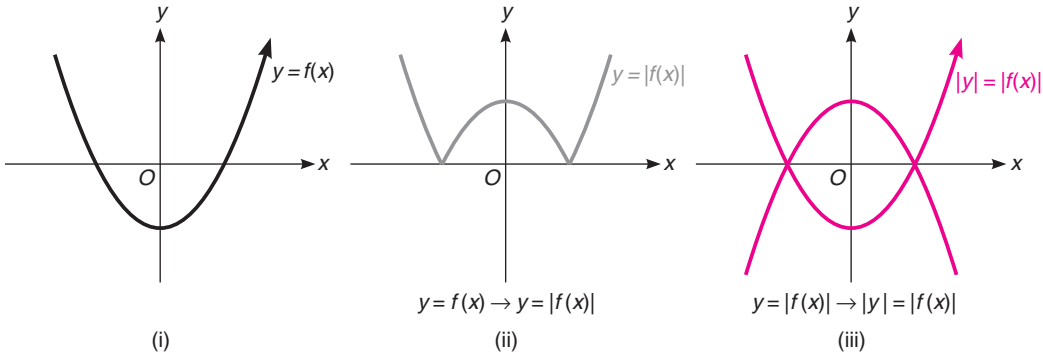
(xii) $y = f(x)$ transforms to $|y| = |f(x)|$

i.e., $y = f(x) \longrightarrow |y| = |f(x)|$; is plotted in two steps.

(i) $y = f(x) \longrightarrow y = |f(x)|$

(ii) $y = |f(x)| \longrightarrow |y| = |f(x)|$

Graphically it could be stated as;



(i) (ii) (iii)

Fig. 2.87

EXAMPLE 1 *Plot the curve for* $|y| = |e^{-x}|$.

SOLUTION Here; curve for $y = e^{-x}$ is shown as;

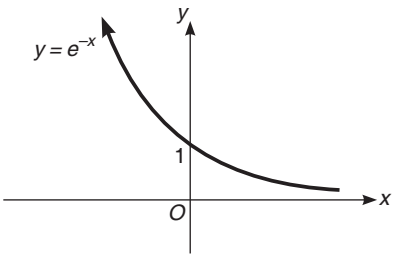


Fig. 2.88

(i) $y = e^{-x} \longrightarrow y = |e^{-x}|$

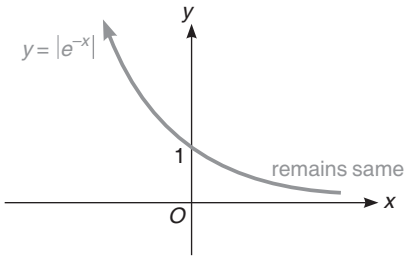


Fig. 2.89

(ii) $y = |e^{-x}| \longrightarrow |y| = |e^{-x}|$.

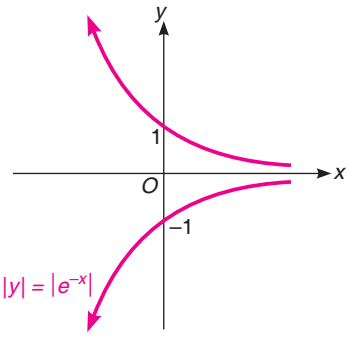


Fig. 2.90

EXAMPLE 2 *Plot the curve* $|y| = |e^x - 1|$.

SOLUTION As we know the curve for $y = e^x$ is shown as;

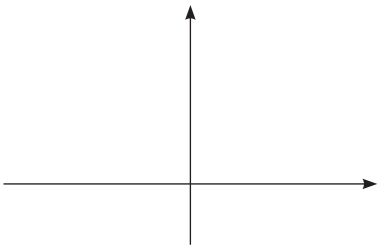


Fig. 2.91

(i) $y = e^x \longrightarrow y = e^x - 1$.

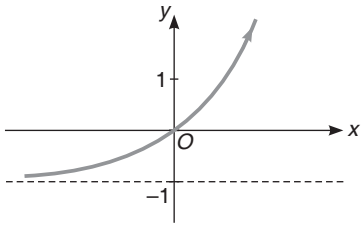


Fig. 2.92

(ii) $y = e^x - 1 \longrightarrow y = |e^x - 1|$.

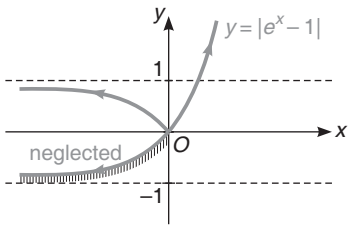


Fig. 2.93

(iii) $y = |e^x - 1| \longrightarrow |y| = |e^x - 1|$.

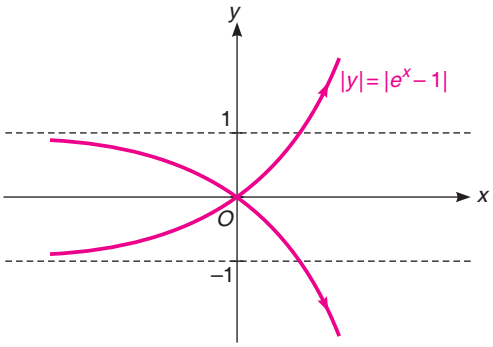


Fig. 2.94

EXAMPLE 3 *Plot the curve* $|y| = \left| \sin x + \frac{1}{2} \right|$.

SOLUTION Here; we know the graph for $y = \sin x$, is shown as

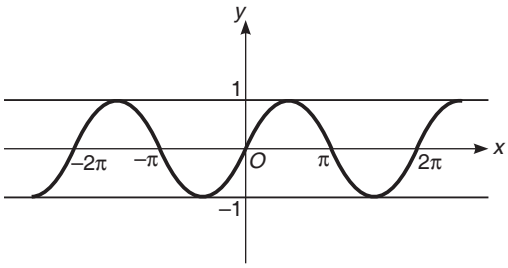


Fig. 2.95

(i) $y = \sin x \longrightarrow y = \sin x + \frac{1}{2}$.

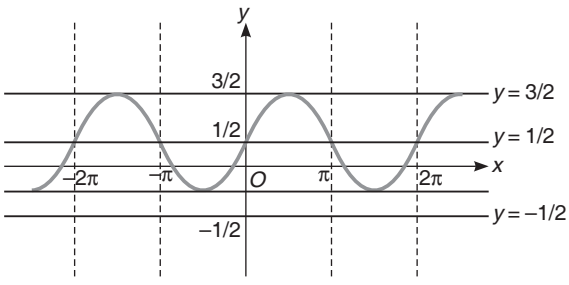


Fig. 2.96

(ii) $y = \sin x + \frac{1}{2} \longrightarrow y = \left| \sin x + \frac{1}{2} \right|$

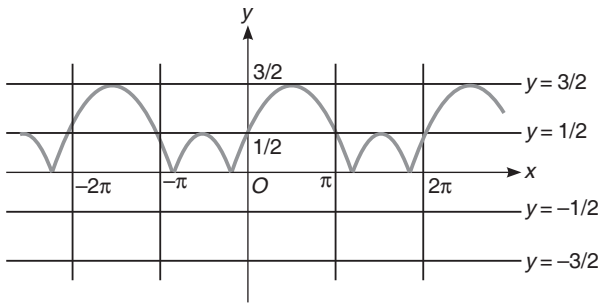


Fig. 2.97

(iii) $y = \left| \sin x + \frac{1}{2} \right| \longrightarrow |y| = \left| \sin x + \frac{1}{2} \right|$

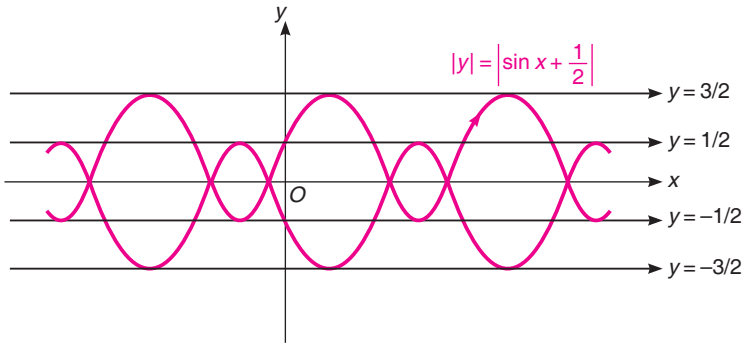


Fig. 2.98

(xiii) $y = f(x)$ transforms to $|y| = |f(|x|)|$

i.e., $y = f(x) \longrightarrow |y| = |f(x)|$.

The steps followed are:

(i) $y = f(x) \longrightarrow y = |f(x)|$.

(ii) $y = |f(x)| \longrightarrow y = |f(|x|)|$

(iii) $y = |f(|x|)| \longrightarrow |y| = |f(|x|)|$.

Graphically it could be stated as:

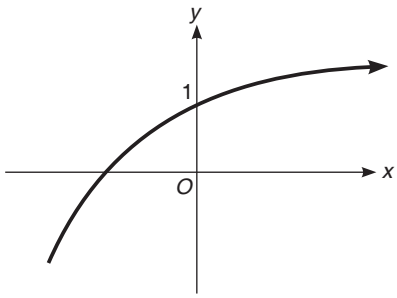


Fig. 2.99

(i) $y = f(x) \longrightarrow y = |f(x)|$

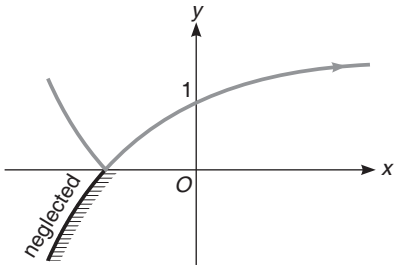


Fig. 2.100

(ii) $y = |f(x)| \longrightarrow y = |f(|x|)|$

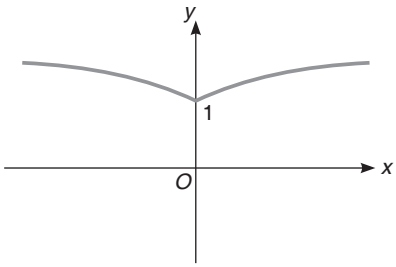


Fig. 2.101

(iii) $y = |f(|x|)| \longrightarrow |y| = |f(|x|)|$.

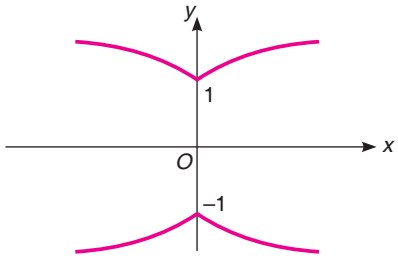


Fig. 2.102

EXAMPLE 1 *Plot the* $|y| = \left| e^{-|x|} - \frac{1}{2} \right|$.

SOLUTION Here; we know the graph for $y = e^{-x}$.

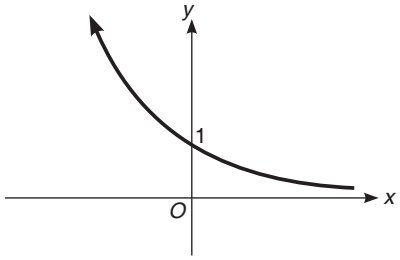


Fig. 2.103

(i) $y = e^{-x} \longrightarrow y = e^{-x} - \frac{1}{2}$

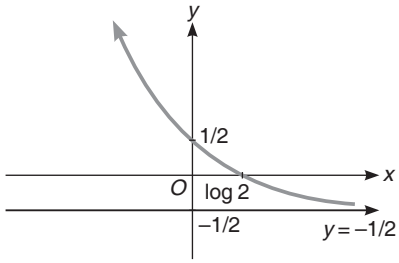


Fig. 2.104

(ii) $y = e^{-x} - \frac{1}{2} \longrightarrow y = e^{-|x|} - \frac{1}{2}$.

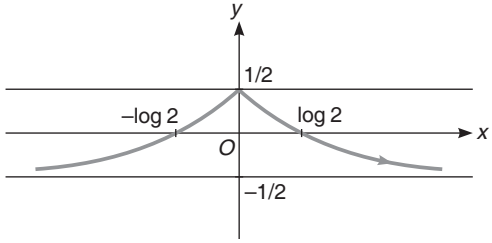


Fig. 2.105

(iii) $y = e^{-|x|} - \frac{1}{2} \longrightarrow y = \left| e^{-|x|} - \frac{1}{2} \right|$.

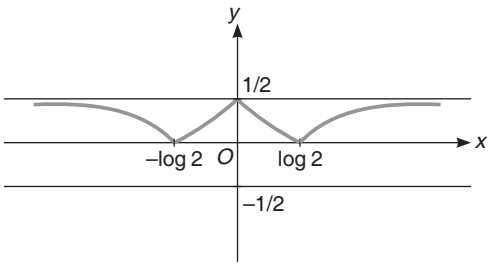


Fig. 2.106

(iv) $y = \left| e^{-|x|} - \frac{1}{2} \right| \longrightarrow |y| = \left| e^{-|x|} - \frac{1}{2} \right|$.

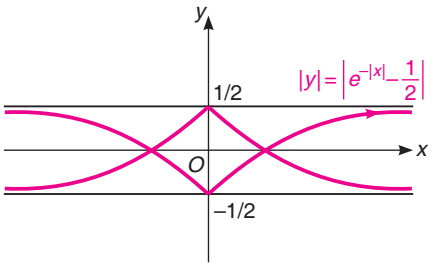


Fig. 2.107

EXAMPLE 2 *Plot* $|y| = |\log|x||$.

SOLUTION Here; $y = \log x$ is plotted as;

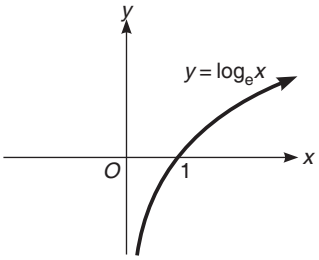


Fig. 2.108

(i) $y = \log x \longrightarrow y = \log|x|$

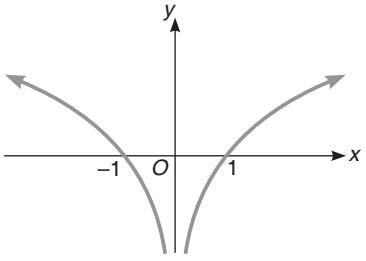


Fig. 2.109

(ii) $y = \log|x| \longrightarrow y = |\log|x||$

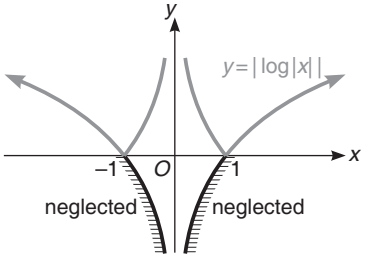


Fig. 2.110

(iii) $y = |\log|x|| \longrightarrow |y| = |\log|x||$

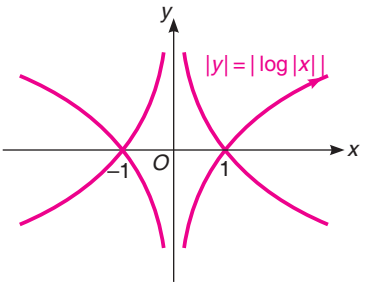


Fig. 2.111

EXAMPLE 3 *Sketch the curve* $|y| = ||x|^2 - 3|x| - 2|$.

SOLUTION As we know the graph for $y = x^2 - 3x - 2$.

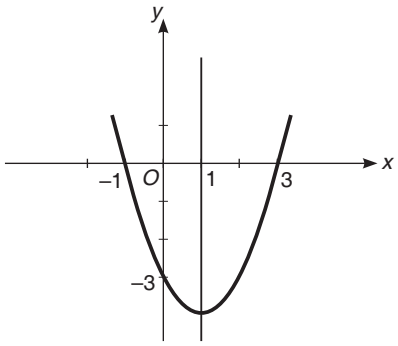


Fig. 2.112

(i) $y = x^2 - 3x - 2 \rightarrow y = |x|^2 - 3|x| - 2$ (ii)

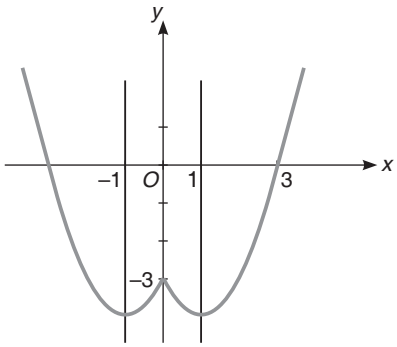


Fig. 2.113

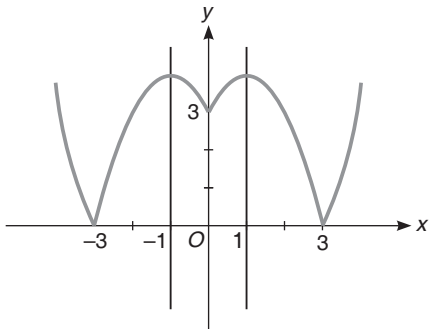


Fig. 2.114

$y = |x|^2 - 3|x| - 2 \rightarrow y = ||x|^2 - 3|x| - 2|$.

(iii) $|y| = ||x|^2 - 3|x| - 2|$

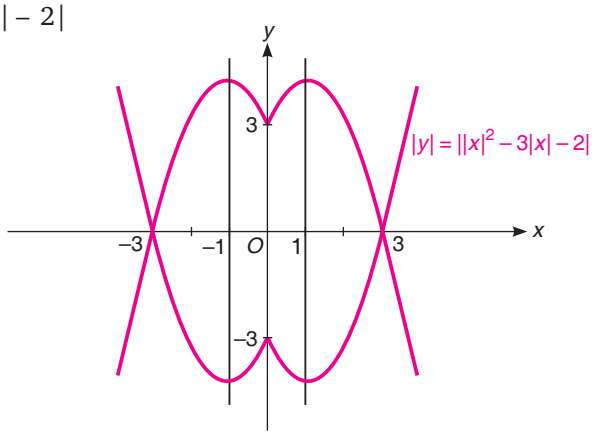


Fig. 2.115

(xiv) $y = f(x)$ transforms to $y = [f(x)]$; (where [•] denotes the greatest integer function)

i.e., $f(x) \longrightarrow [f(x)]$

Here; in order to draw $y = [f(x)]$ mark the integer on y-axis. Draw the horizontal lines through integers till they intersect the graph. Draw vertical dotted lines from these intersection points; finally draw horizontal lines parallel to x-axis from any intersection point to the nearest vertical dotted line with blank dot at right end in case $f(x)$ increase.

OR

Step 1. Plot $f(x)$.
Step 2. Mark the intervals of unit length with integers as end points on y-axis.
Step 3. Mark the corresponding intervals {with the help of graph of $f(x)$} on x-axis.
Step 4. Plot the value of $[f(x)]$ for each of the marked intervals.

Graphically it could be shown as:

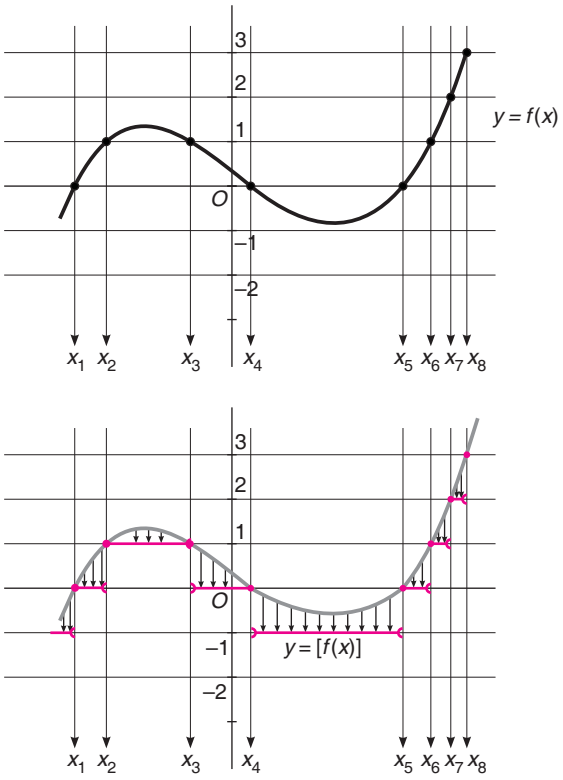


Fig. 2.116

EXAMPLE 1 *Sketch the curve* $y = [\sin x]$.

SOLUTION Here, sketch for $y = \sin x$ is shown as in Fig. 2.117.

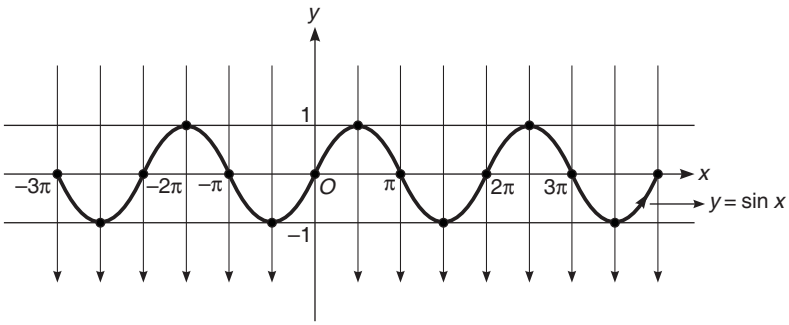


Fig. 2.117

$y = \sin x \rightarrow y = [\sin x]$

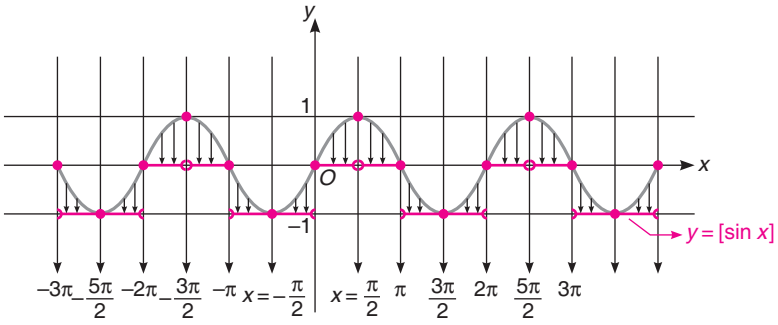


Fig. 2.118

EXAMPLE 2 *Sketch the curve* $y = [x^2 - 1]$. *(where [·] denotes greatest integer function). When* $-2 \le x \le 2$.

SOLUTION Here $y = x^2 - 1$ could be plotted as shown in Fig. 2.119.

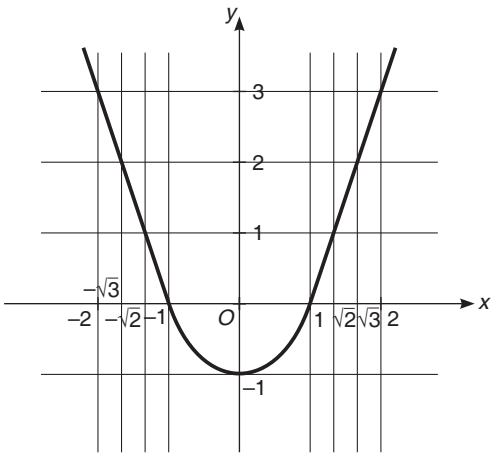


Fig. 2.119

(i) $y = x^2 - 1 \longrightarrow y = [x^2 - 1]$.

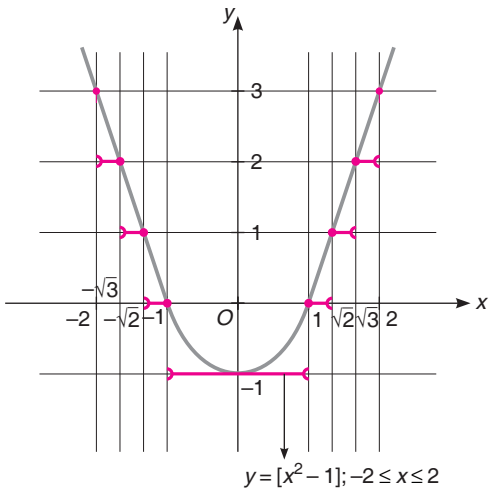
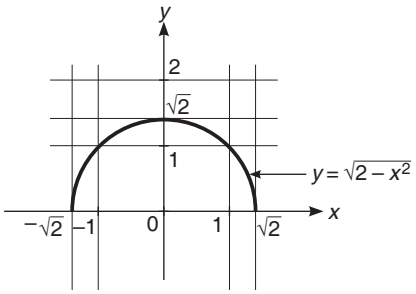


Fig. 2.120

EXAMPLE 3 *Sketch the curve;* $y = [\sqrt{2 - x^2}]$*; where* $[\cdot]$ *denotes the greatest integer function.*

SOLUTION We know, $y = \sqrt{2 - x^2}$ represents a circle for $y \geq 0$.

Shown as in Fig. 2.121.



Thus, the graph for $y = [\sqrt{2 - x^2}]$

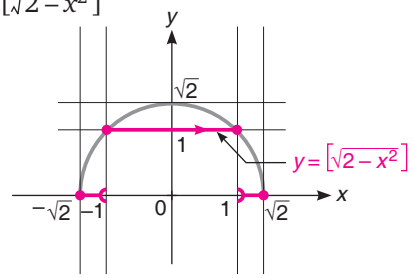


Fig. 2.121

(xv) $y = f(x)$ transforms to $y = f([x])$

Here, mark the integers on the x-axis. Draw vertical lines till they intersect the graph of f(x). From these intersection points draw horizontal lines (parallel to x-axis) to meet the nearest right vertical line, with a black dot on each nearest right vertical line which can be shown as in Fig. 2.122.

y = f(x)

y = f([x])

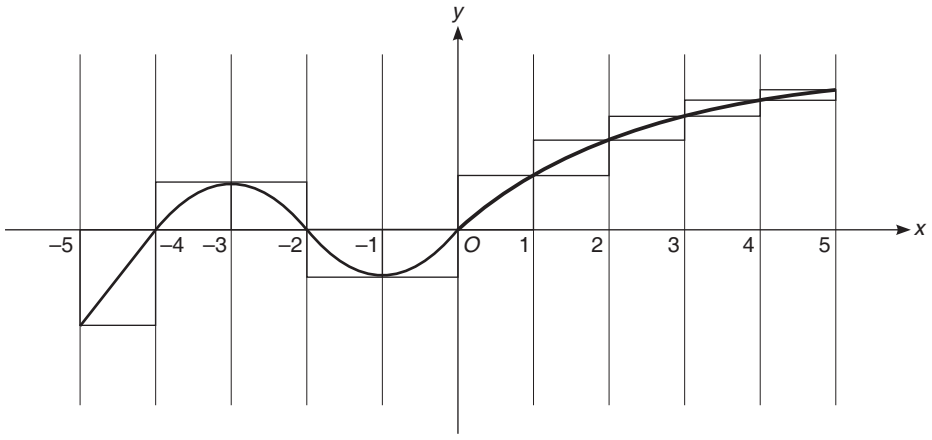


Fig. 2.122

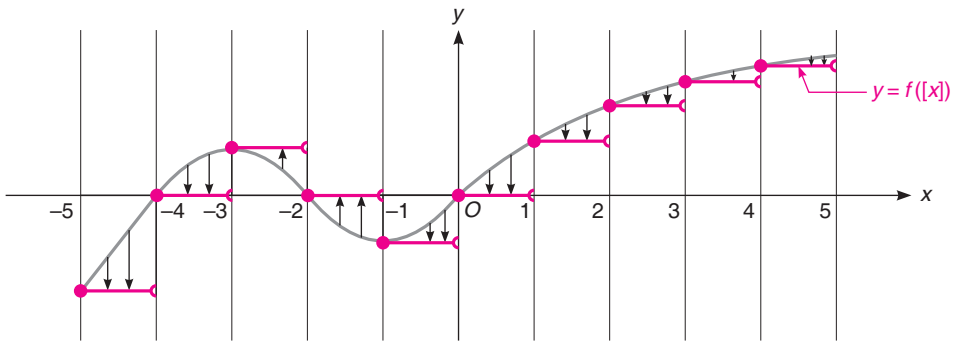


Fig. 2.123

OR

$$y = f(x) \longrightarrow y = f([x])$$

Step 1. Plot the straight lines parallel to y-axis for integral values of x (say $-3, -2, -1, 0, 1, 2, 3, \ldots$)

Step 2. Now mark the points at which $x = -3$, $x = -2$, $x = -1$, $x = 0$, $x = 1, \ldots$ on the curve.

Step 3. Take the lower marked point for x say if $n < x < n + 1$ then take the point at $x = n$ and draw a horizontal line to the nearest vertical line formed by $x = n + 1$, proceeding in this way we get required curve.

EXAMPLE 1 *Plot the curve* $y = e^{[x]}$.

SOLUTION Here the graph for $y = e^{[x]}$ is shown as;

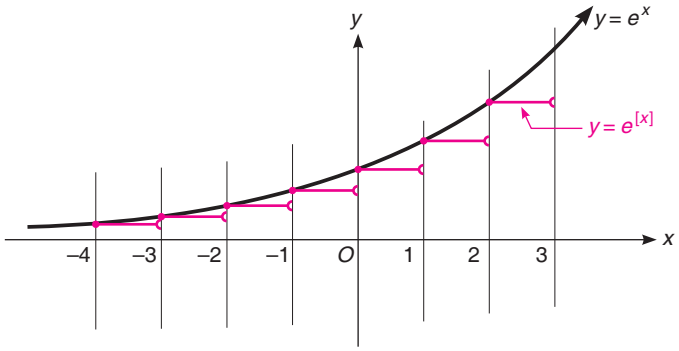


Fig. 2.124

EXAMPLE 2 *Sketch the curve* $y = \sin[x]$ *when* $-2\pi \le x \le 2\pi$.

SOLUTION The curve for

$y = \sin[x]$; could be plotted as shown in Fig. 2.125.

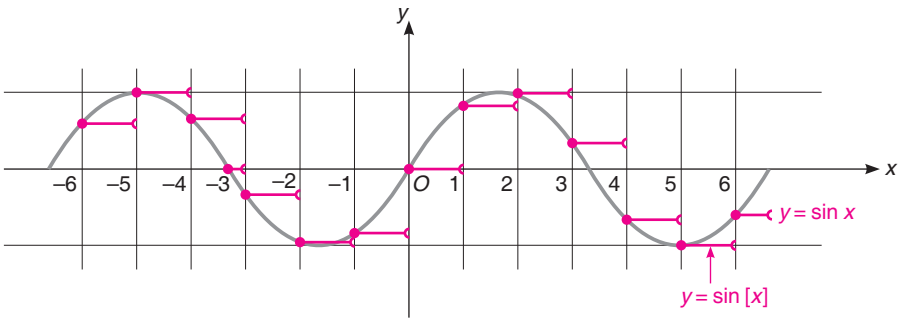


Fig. 2.125

EXAMPLE 3 *Sketch the curve for* $y = \cos[x]$; $-\pi \le x \le \pi$.

SOLUTION The curve for $y = \cos[x]$ could be plotted as;

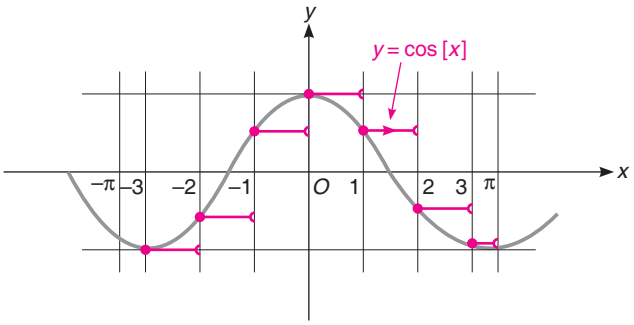


Fig. 2.126

EXAMPLE 4 *Plot the curve* $y = [x]^2$; $-2 \le x \le 2$.

SOLUTION The curve for $y = [x]^2$; $-2 \le x \le 2$ could be plotted as;

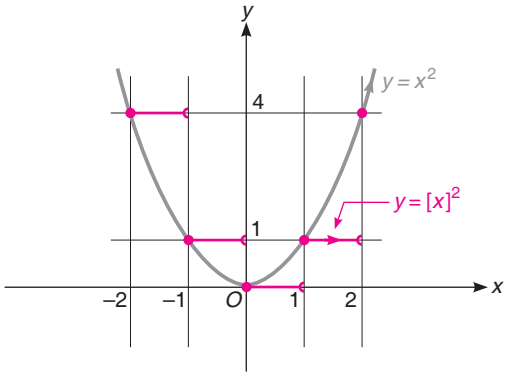


Fig. 2.127

(xvi) $y = f(x)$ transforms to $y = [f([x])]$

Here, we should follow two steps;

(i) $y = f(x) \longrightarrow y = f([x])$

(ii) $y = f([x]) \longrightarrow y = [f([x])]$

EXAMPLE 1 *Sketch the curve* $y = [\sin[x]]$*; where* $[\cdot]$ *denotes the greatest integral function when* $0 \le x \le \pi$.

SOLUTION Here; first we shall plot the curve for $y = \sin[x]$, when $x \in [0, \pi]$.

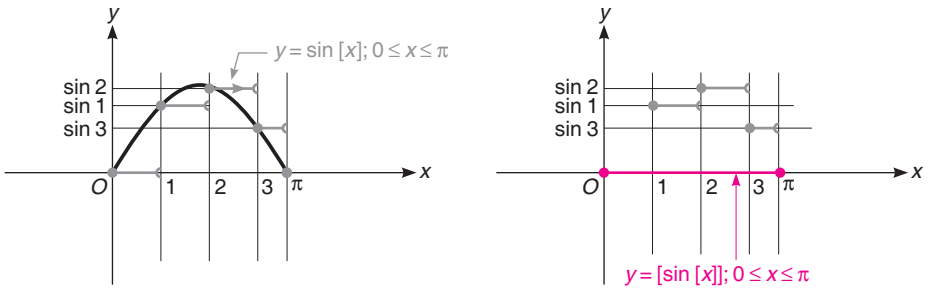


Fig. 2.128

From above figure we conclude that;

when $0 \le x \le \pi \Rightarrow y = \sin[x] \in [0, 1)$.

$\Rightarrow$ $y = [\sin[x]] \longrightarrow 0$ for all $0 \le x \le \pi$.

EXAMPLE 2 *Plot the curve* $y = [e^{[x]}]$*; when* $-4 < x < 2$.

SOLUTION Here to sketch $y = [e^{[x]}]$, we should follow the steps as;

(i) $y = e^x$

(ii) $y = e^{[x]}$

(iii) $y = [e^{[x]}]$

(i) $y = e^x$

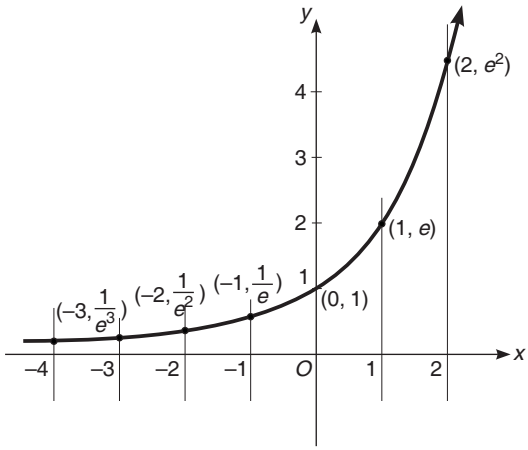


Fig. 2.129

(ii) $y = e^{[x]}$

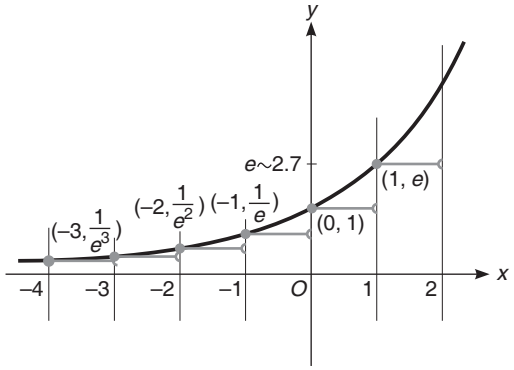


Fig. 2.130

(iii) $y = [e^{[x]}]$; from Fig. 2.130.

$$e^{[x]} = \begin{cases} 0 < e^{[x]} < 1; & x < 0 \\ e^{[x]} = 1 & ; \ 0 \le x < 1 \\ e^{[x]} = 2.7 & ; \ 1 \le x < 2 \end{cases}$$

Thus; $y = [e^{[x]}]$

$$\Rightarrow y = \begin{cases} 0; & x < 0 \\ 1; & 0 \le x < 1 \\ 2; & 1 \le x < 2 \end{cases}$$ shown as in Fig. 2.131.

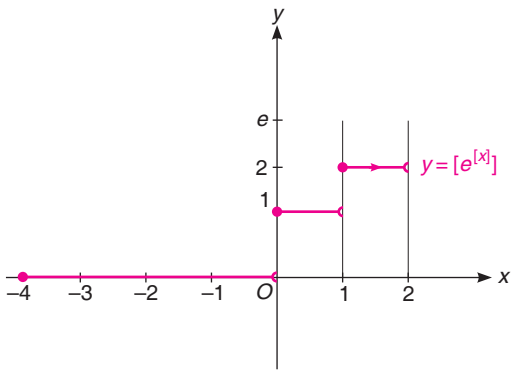


Fig. 2.131

Graph for; $y = [e^{[x]}]$; when $x < 2$.

(xvii) $y = f(x)$ transforms to $[y] = f(x)$ / Graph of Candles

Here, to plot $[y] = f(x)$; we check only those points for which $f(x) \in$ integers, as $[y] \in$ integers for all x.

Thus; $[y] = f(x)$ represents only integral values of y. Here, domain of $f(x)$ are set of values of x for which $f(x) \in$ integers.

EXAMPLE 1 *Sketch the curve;*

$$[y] = \sin x.$$

SOLUTION As we know; $-1 \le \sin x \le 1$ but since; $[y] = \sin x$.

$\Rightarrow$ $\sin x = -1, 0, 1$ are only solutions;

or $$x = \frac{n\pi}{2} \in \text{Domain of } [y] = \sin x.$$

Thus, $[y] = \sin x$ is shown as in Fig. 2.132.

Graph for $y = \sin x$ **:**

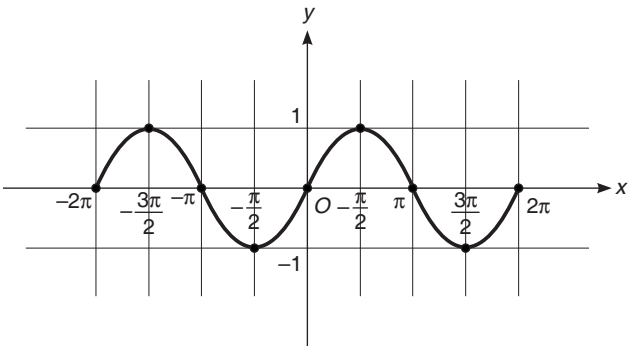


Fig. 2.132

Graph for [y] = sin x **:**

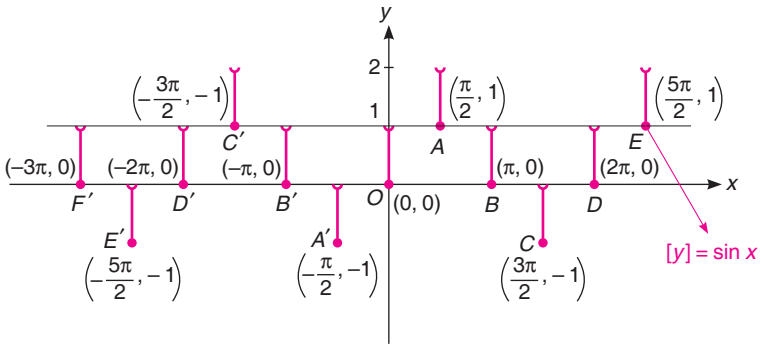


Fig. 2.133

From above figure, (the points marked O, *A*, *B*, *C*, *D*, *E*, A′, B′, C′, D′, E′,... is the graph for candles), or graph for [y] = sin x.

EXAMPLE 2 *Sketch the curve* $[y] = \sin^{-1} x$.

SOLUTION As we know the graph for $y = \sin^{-1} x$; shown as in Fig. 2.134 and 2.135.

Graph for $y = \sin^{-1} x$**;**

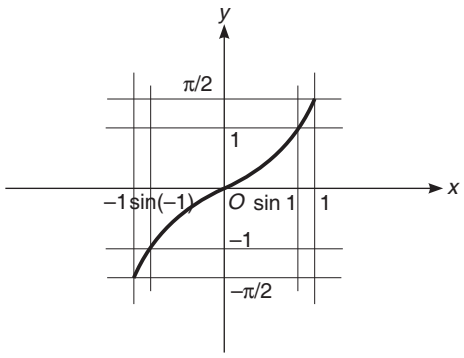


Fig. 2.134

Graph for $[y] = \sin^{-1} x$**;**

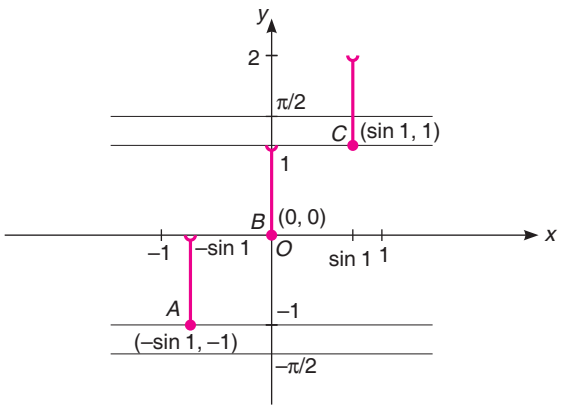


Fig. 2.135

From above figure, [y] = sin x.

(xviii) $y = f(x)$ transforms to $[y] = [f(x)]$

As we have earlier discussed y = [f(x)], *i.e.*, **transformation (xiv)**, and we know [y] implies only those values of *x* for which f(x) ∈ integer.

EXAMPLE 1 *Sketch the curve;* [y] = [sin x].

SOLUTION To sketch the curve [y] = [sin x] we first plot y = [sin x].

(i) Graph for $y = [\sin x]$

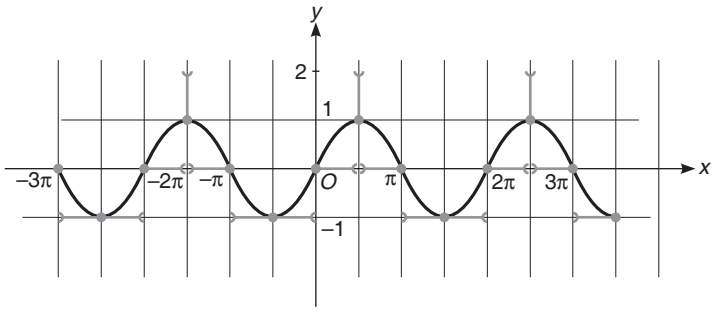


Fig. 2.136

(ii) Graph for $[y] = [\sin x]$

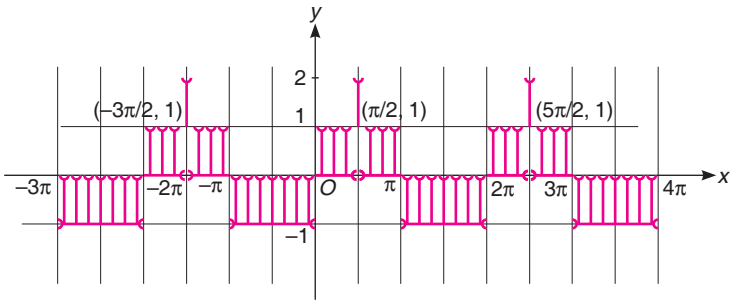


Fig. 2.137

From above figure it is clear $[y] = [\sin x]$ is periodic with period 2π.

(xix) $y = f(x)$ transforms to $y = f(\{x\})$; (where $\{\cdot\}$ denotes fractional part of x, *i.e.*, $\{x\} = x - [x]$)

$$f(x) \longrightarrow f(\{x\})$$

Graph of $f(x - [x])$ or $f(\{x\})$ can be obtained from the graph of $f(x)$ by following rule.

"Retain the graph of $f(x)$ for values of x lying between interval $[0, 1)$. Now it can be repeated for rest of the points. (taking periodicity 1).

New obtained function is graph for $y = f(\{x\})$".

Graphically it could be stated as;

Graph for $y = f(x)$

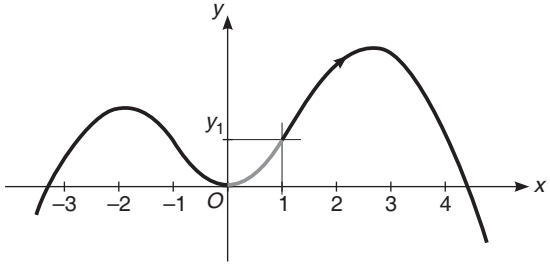


Fig. 2.138

Graph for $y = f(\{x\})$

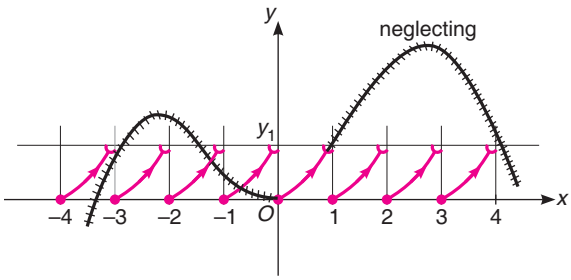


Fig. 2.139

EXAMPLE 1 *Sketch the curve* $y = (\{x\} - 1)^2$.

SOLUTION Here, we know the curve for $y = (x - 1)^2$ shown as;

Graph for $y = (x - 1)^2$**:**

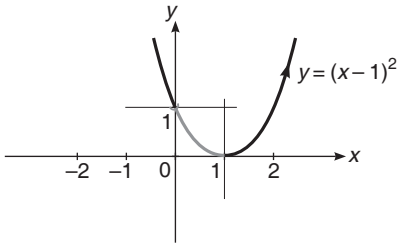


Fig. 2.140

Now; to plot $y = (\{x\} - 1)^2$ retain the graph for the interval $x \in [0, 1)$ and repeat for length 'one'.

Graph for $y = (\{x\} - 1)^2$ **:**

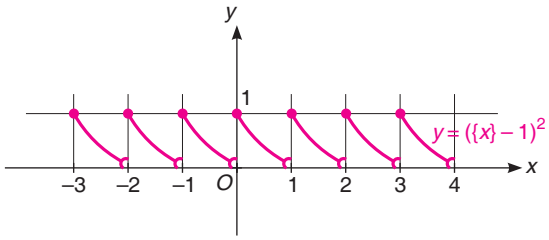


Fig. 2.141

EXAMPLE 2 *Sketch the curve* $|y| = (\{x\} - 1)^2$.

SOLUTION As discussed in above example $y = (\{x\} - 1)^2$.

Thus, $|y| = (\{x\} - 1)^2$ is image of $(\{x\} - 1)^2$ on x-axis whenever $(\{x\} - 1)^2$ is positive.

Graph of $|y| = (\{x\} - 1)^2$ **:**

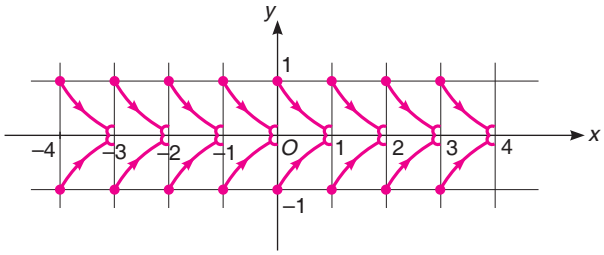


Fig. 2.142

EXAMPLE 3 *Sketch the graph of* $y = \dfrac{2^x}{2^{[x]}}$.

SOLUTION As we know 2^x is exponential function and we want to transform it to $2^{x-[x]}$, it retain the graph for $x \in [0, 1)$ and repeat for rest points.

Graph for $y = 2^x$

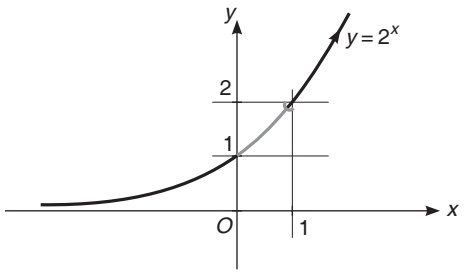


Fig. 2.143

Graph for $y = 2^{\{x\}}$

To retain graph between $x \in [0, 1)$.

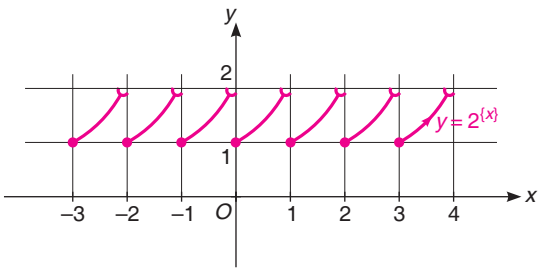


Fig. 2.144

(xx) $y = f(x)$ transforms to $y = \{f(x)\}$

Here, plot the horizontal lines for all integral values of y and for the point of intersection on $y = f(x)$ plot draw vertical lines and translate the graph for boundary $y = 0$ and $y = 1$.

EXAMPLE 1 *Sketch the curve* $y = \{\sin x\}$. *(where $\{\cdot\}$ denotes the fractional part of x).*

SOLUTION As we know the graph for $y = \sin x$. Shown as.

Graph of $\sin x$ **:**

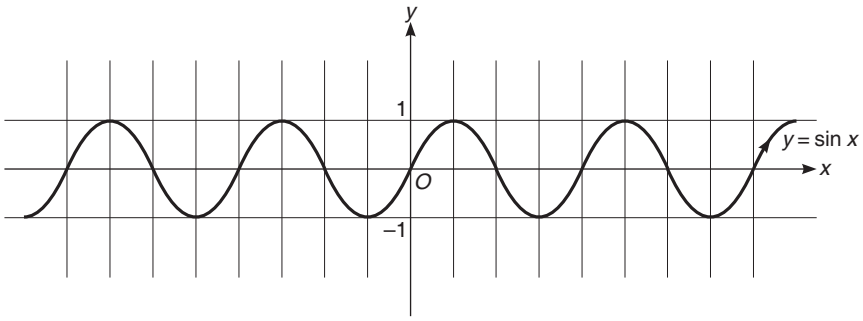


Fig. 2.145

As to retain the curve when $0 \le y < 1$; and shift other sections of graph between $y = 0$ to $y = 1$.

Graph for $y = \{\sin x\}$ **:**

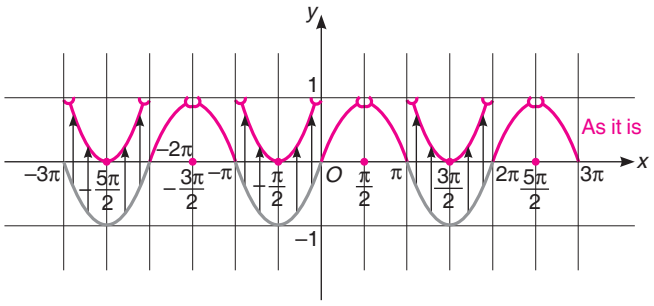


Fig. 2.146

EXAMPLE 2 *Sketch the curve* $y = \{x^2\}$.

SOLUTION As we know the curve $y = x^2$, is shown as:

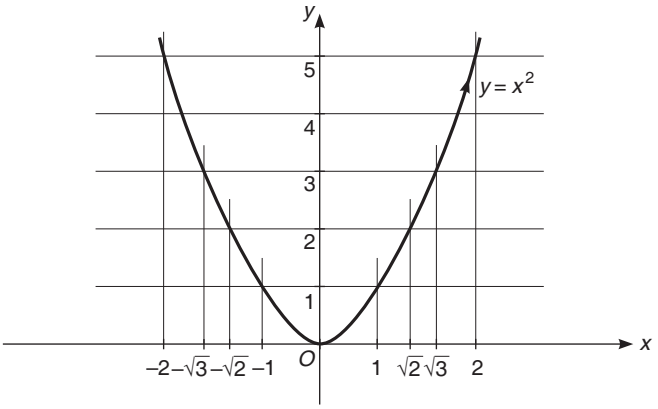


Fig. 2.147

Now to sketch $y = \{x^2\}$; retain the graph for $0 \le y < 1$ and for other intervals transform the graph between $0 \le y < 1$.

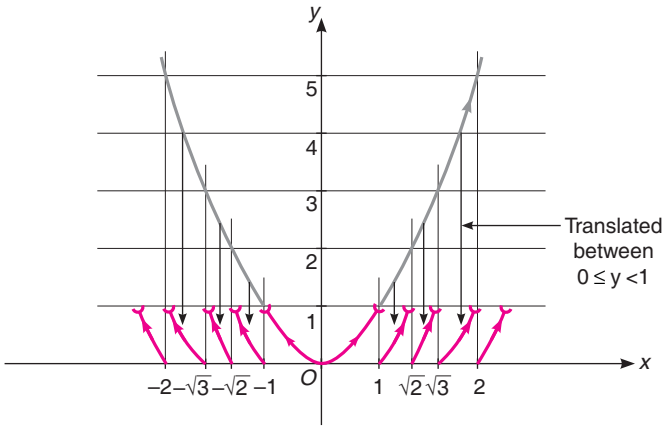


Fig. 2.148

EXAMPLE 3 *Sketch the curve* $y = \{e^x\}$.

SOLUTION As we know the curve $y = e^x$;

shown as:

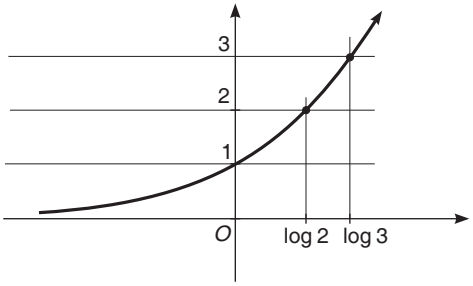


Fig. 2.149

Graph for $y = \{e^x\}$ **:**
To retain the graph for $0 \le y < 1$ and transform the others to $0 \le y < 1$.

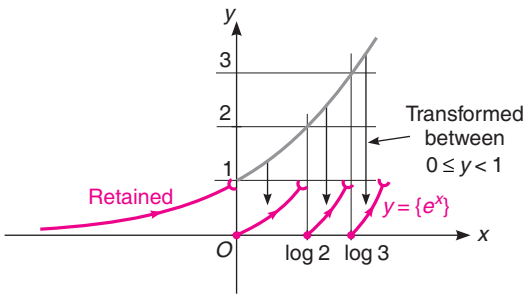


Fig. 2.150

(xxi) $y = f(x)$ transforms to $y = \{f(\{x\})\}$

Here, we have to follow two steps :

(i) Draw the graph for $y = f(\{x\})$.

(ii) $f(\{x\}) \longrightarrow \{f(\{x\})\}$

EXAMPLE 1 *Sketch the curve* $y = \{e^{\{x\}}\}$.

SOLUTION As we know the curve for $y = e^x$, is plotted as shown in Fig. 2.151.

Graph for $y = e^x$**:**

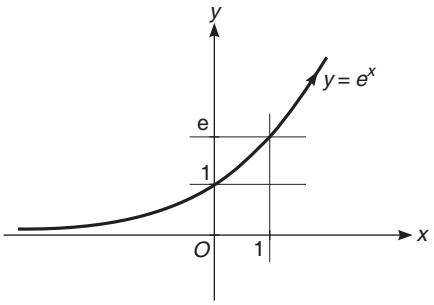


Fig. 2.151

Now to sketch $y = e^{\{x\}}$ retain the graph for $y = e^x$ between $0 \le x < 1$ and repeat for entire real x.

Graph for $y = e^{\{x\}}$**:**

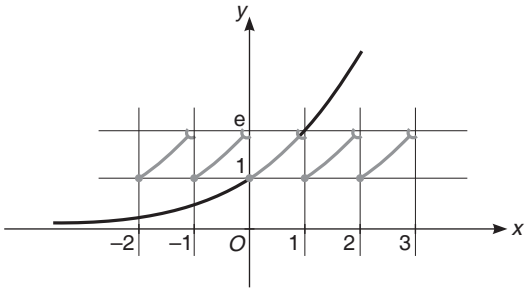


Fig. 2.152

Graph for $y = \{e^{\{x\}}\}$**:**

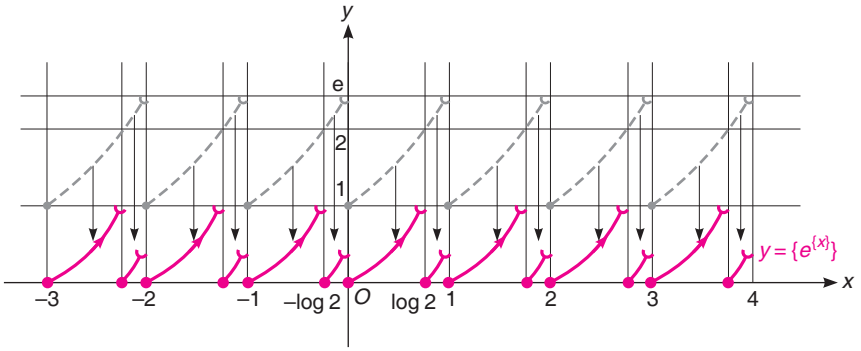


Fig. 2.153

Here, we know the graph for $y = e^{\{x\}}$, now to plot straight lines || to x-axis for integral values of y and retain the graph for $0 \le y < 1$ and transform the others between $0 \le y < 1$.

EXAMPLE 2 *Sketch the curve* $y = \{\sin\{x\}\}$.

SOLUTION As we know the curve $y = \sin x$; is plotted as;

Graph for $y = \sin x$ **:**

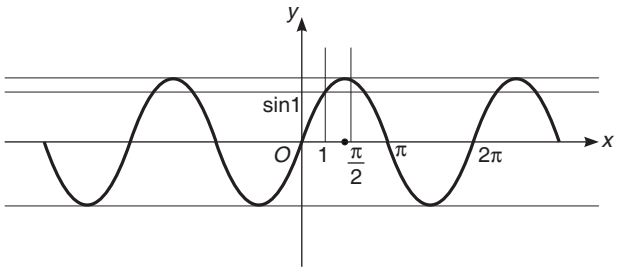


Fig. 2.154

Graph for y = sin{x} **:**

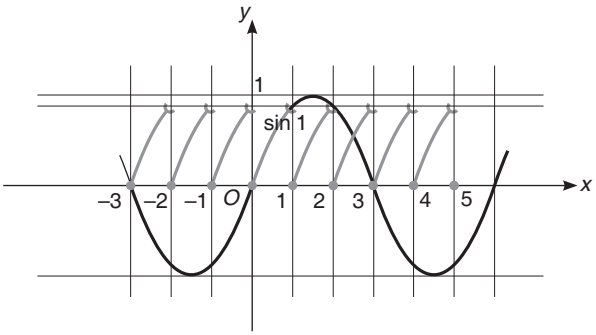


Fig. 2.155

Graph for y = {sin{x}} **:**

From above figure

$$y = \sin\{x\} \quad \Rightarrow \quad 0 \le y < 1.$$

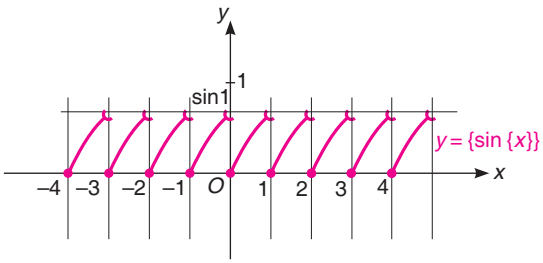


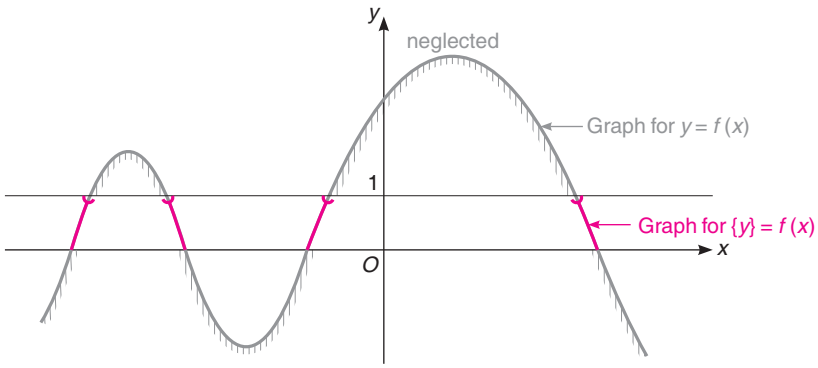
Fig. 2.156

So, the graph of y = sin{x} and y = {sin{x}} are same.

(xxii) $y = f(x)$ transforms to $\{y\} = f(x)$

Here; retain the graph of y = f(x) only when y = f(x) lies between $y \in [0, 1)$ and neglect the graph for other values.

Graphically it could be stated as;



Fgi. 2.157

EXAMPLE 1 *Sketch the curve* $\{y\} = x^2$.

SOLUTION As we know the curve for $y = x^2$, is plotted as:

Graph for $y = x^2$**:**

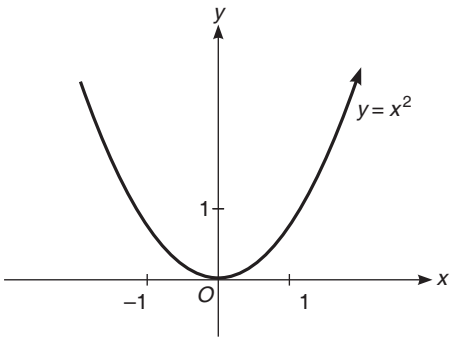


Fig. 2.158

Graph for $\{y\} = x^2$**:**

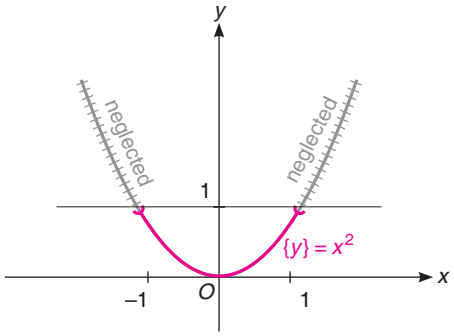


Fig. 2.159

EXAMPLE 2 *Sketch the curve* $\{y\} = \sin x$.

SOLUTION As we know the curve $y = \sin x$, is plotted as shown in Fig. 2.160 and 2.161.

Graph for $y = \sin x$ **:**

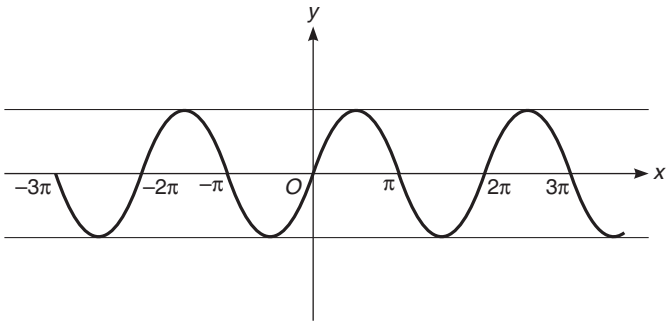


Fig. 2.160

Now to sketch $\{y\} = \sin x$: We retain the graph for $0 \le y < 1$ and neglect the graph for other values.

Graph for $\{y\} = \sin x$ **:**

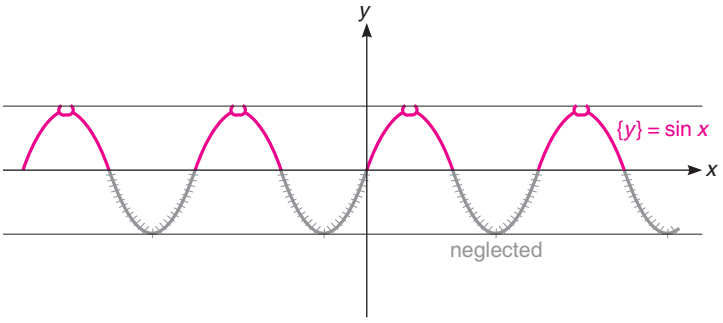


Fig. 2.161

(xxiii) $y = f(x)$ transforms to $\{y\} = \{f(x)\}$

As we have earlier discussed $y = \{f(x)\}$ (*i.e.*, transformation (xx), which shows $y = \{f(x)\}$ belongs to $[0, 1) \Rightarrow \{y\} = \{f(x)\}$. Thus, the graph of $y = \{f(x)\}$ and $\{y\} = \{f(x)\}$ are same.

EXAMPLE 1 *Sketch the curve* $\{y\} = \{x\}$.

SOLUTION As we know the curve for $y = x$.

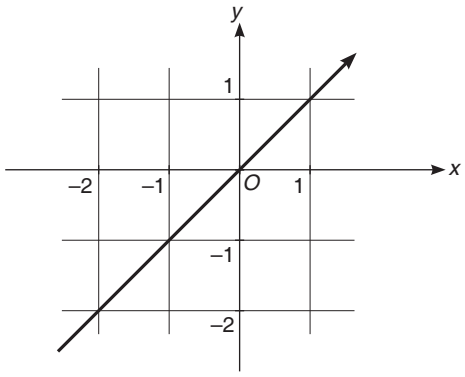


Fig. 2.162

Graph for $y = \{x\}$ **:**

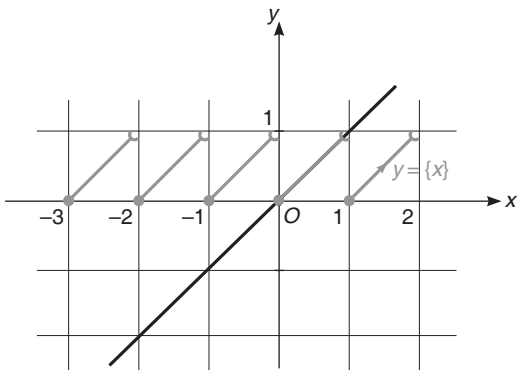


Fig. 2.163

Graph for $\{y\} = \{x\}$ **:** From above figure we can see $y = \{x\}$ attains all values between $[0, 1)$. Thus, graph remains same.

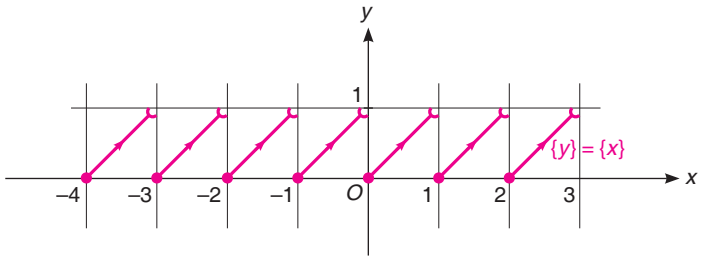


Fig. 2.164

EXAMPLE 2 *Sketch the curve* $\{y\} = \{\cos x\}$.

SOLUTION As we know the curve $y = \{\cos x\}$ is plotted as shown in Fig. 2.165.

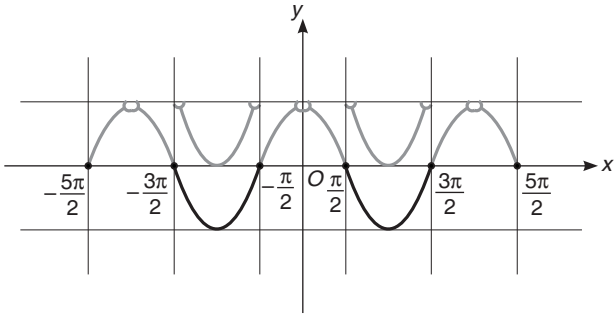


Fig. 2.165

From above figure; $y = \{\cos x\}$ lies between [0, 1), which shows $y = \{\cos x\}$ and $\{y\} = \{\cos x\}$ are same.

Thus, **graph for** $\{y\} = \{\cos x\}$**:**

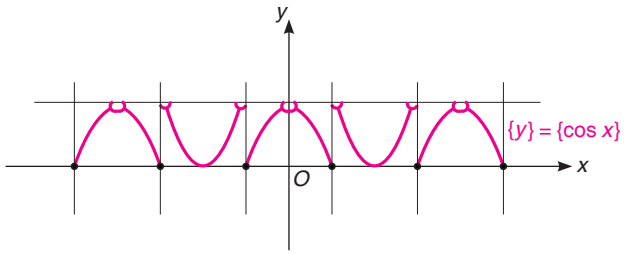


Fig. 2.166

(xxiv) $y = f(x)$ transforms to $y = f^{-1}(x)$

As discussed in chapter 1. $y = f^{-1}(x)$ is the mirror image of $y = f(x)$ about $y = x$.

OR

"Interchange x and y-axis when function is bijective."

Graphically it could be stated as :

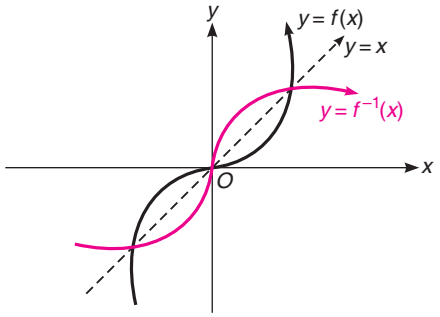


Fig. 2.167

2.4 SKETCHING $h(x)$ = MAXIMUM $\{f(x), g(x)\}$ AND $h(x)$ = MINIMUM $\{f(x), g(x)\}$

(i) $h(x) = \text{maximum}\{f(x), g(x)\}$

$$\Rightarrow \quad h(x) = \begin{cases} f(x); \text{ when } f(x) > g(x) \\ g(x); \text{ when } g(x) > f(x) \end{cases}$$

$\therefore$ Sketch $f(x)$ when its graph is above the graph of $g(x)$ and sketch $g(x)$ when its graph is above the graph of $f(x)$.

(ii) $h(x) = \text{minimum}\{f(x), g(x)\}$

$$\Rightarrow \quad h(x) = \begin{cases} f(x), \text{ when } f(x) < g(x) \\ g(x), \text{ when } g(x) < f(x) \end{cases}$$

$\therefore$ Sketch $f(x)$ when its graph is lower and otherwise sketch $g(x)$.

Note One must remember the formula we can write;

$$\max\{f(x), g(x)\} = \frac{f(x) + g(x)}{2} + \left|\frac{f(x) - g(x)}{2}\right|$$

$$\min\{f(x), g(x)\} = \frac{f(x) + g(x)}{2} - \left|\frac{f(x) - g(x)}{2}\right|$$

OR

"To draw the graph of functions of the form $y = \max\{f(x), g(x)\}$ **or** $y = \min\{f(x), g(x)\}$**."**

We first draw the graphs of both the functions $f(x)$ and $g(x)$ and their points of intersections.

Then we find any two consecutive points of intersection. In between these points either $f(x) > g(x)$ or $f(x) < g(x)$, then, in order to $\max\{f(x), g(x)\}$ **we take those segments of** $f(x)$ **for which** $f(x) > g(x)$**,** between any two consecutive points of intersection of $f(x)$ and $g(x)$.

Similarly, in order to $\min\{f(x), g(x)\}$**, we take those segments of** $f(x)$ **for which** $f(x) < g(x)$**,** between any two consecutive points of intersection of $f(x)$ and $g(x)$.

EXAMPLE 1 *Sketch the graph of* $y = \max\{\sin x, \cos x\}$, $\forall \quad x \in \left(-\pi, \frac{3\pi}{2}\right)$.

SOLUTION First plot both $y = \sin x$ and $y = \cos x$ by a dotted curve as can be seen from the graph in the interval $\left(-\pi, \frac{3\pi}{2}\right)$ and then darken those dotted lines for which $f(x) > g(x)$ or $g(x) > f(x)$.

From adjacent figure the point of intersections are A, B, C.

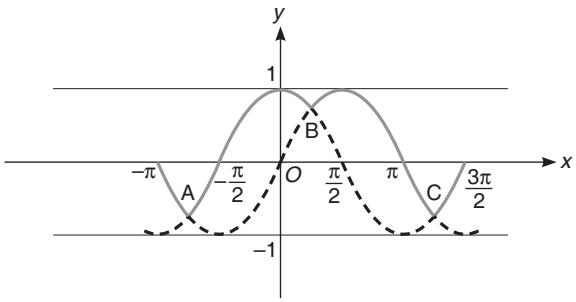


Fig. 2.168

∴ **Graph of max** $\{\sin x, \cos x\}$

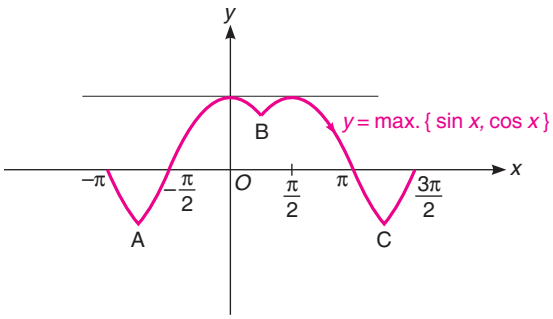


Fig. 2.169

EXAMPLE 2 *Sketch the graph for* $y = \min\{\tan x, \cot x\}$.

SOLUTION First plot both $f(x) = \tan x$ and $g(x) = \cot x$ by a dotted curves as can be seen from the graph and then darken those dotted lines for which $f(x) < g(x)$ and $g(x) < f(x)$.

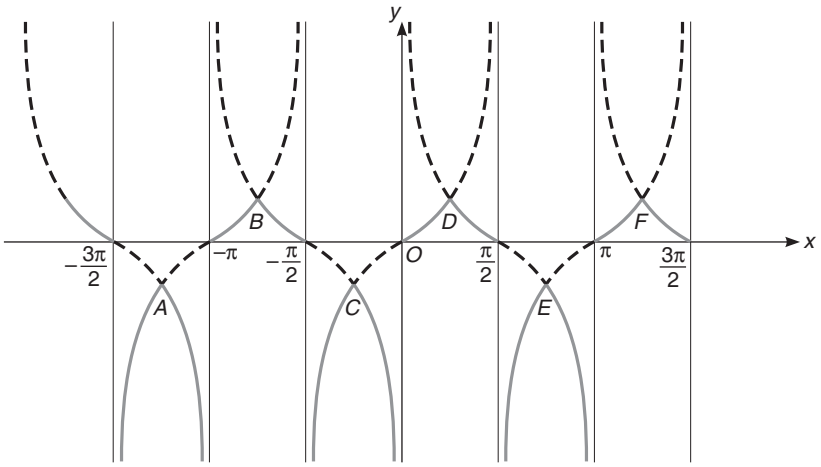


Fig. 2.170

From above figure we obtain the graph of min{tan x, cot x}.

Graph of min {tan x, cot x} :

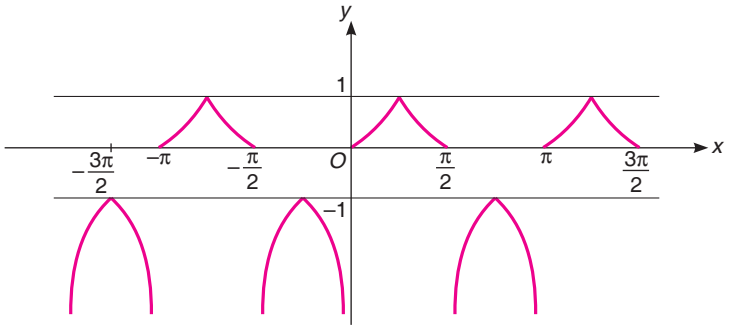


Fig. 2.171

EXAMPLE 3 *Sketch the curve* $y = \min\{|x|, |x-1|, |x+1|\}$.

SOLUTION First plot the graph for:

$y = |x|$, $y = |x-1|$, $y = |x+1|$

by a dotted curve as can be seen from the graph and then darken those dotted lines for which

$|x| < \{|x-1|, |x+1|\}$;

$|x-1| < \{|x|, |x+1|\}$

and $|x+1| < \{|x|, |x-1|\}$.

Graph for

$y = |x|$, $y = |x-1|$, $y = |x+1|$.

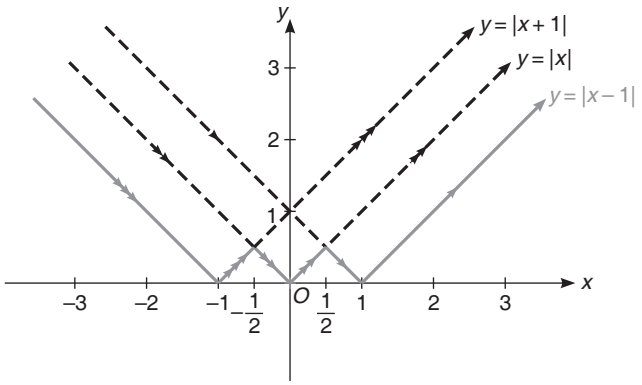


Fig. 2.172

As from the above curve graph for $y = \min\{|x-1|, |x|, |x+1|\}$ is plotted as;

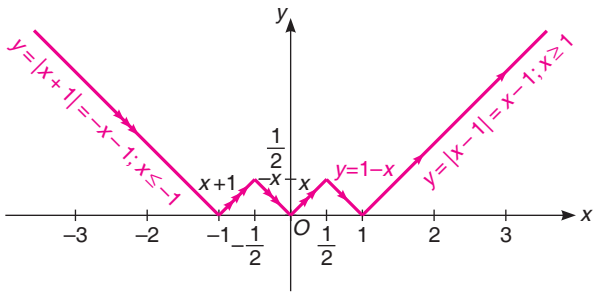


Fig. 2.173

From above figure;

$$\min\{|x-1|, |x|, |x+1|\} = \begin{cases} -(x+1); & x \le -1 \\ (x+1); & -1 \le x \le -\frac{1}{2} \\ -(x); & -\frac{1}{2} \le x \le 0 \\ (x); & 0 \le x \le \frac{1}{2} \\ (1-x); & \frac{1}{2} \le x \le 1 \\ (x-1); & x \ge 1 \end{cases}$$

2.5 WHEN $f(x), g(x) \longrightarrow f(x) + g(x) = h(x)$

There is no direct approach; but we can use following steps if minimum or maximum value of any one is known.

Step 1. Find maximum and minimum value of $g(x)$ say; $a \le g(x) \le b$.

Step 2. Plot the curve $h(x) = f(x) + g(x)$ between $f(x) + a$ to $f(x) + b$.

i.e., $f(x) + a \le h(x) \le f(x) + b$

Step 3. Check $g(x) = 0 \Rightarrow h(x) = f(x)$.

Step 4. When $g(x) > 0 \Rightarrow h(x) > f(x)$.

Step 5. When $g(x) < 0 \Rightarrow h(x) < f(x)$.

EXAMPLE 1 *Plot the curve* $y = x + \sin x$.

SOLUTION Here, $y = x + \sin x = f(x) + g(x)$

as we know; $g(x) = \sin x \in [-1, 1]$

$\therefore$ $x - 1 \le y \le x + 1$ …(i)

$\Rightarrow$ To sketch the curve between two parallel lines $y = x + 1$ and $y = x - 1$ (called bounded limits)

also; $g(x) = 0 \Rightarrow y = x$ …(ii)

$g(x) > 0 \Rightarrow y = x + \sin x > x$ …(iii)

$$g(x) < 0 \quad \Rightarrow \quad y = x + \sin x < x \qquad \ldots(iv)$$

Using Eqs. (i), (ii), (iii) and (iv), we get

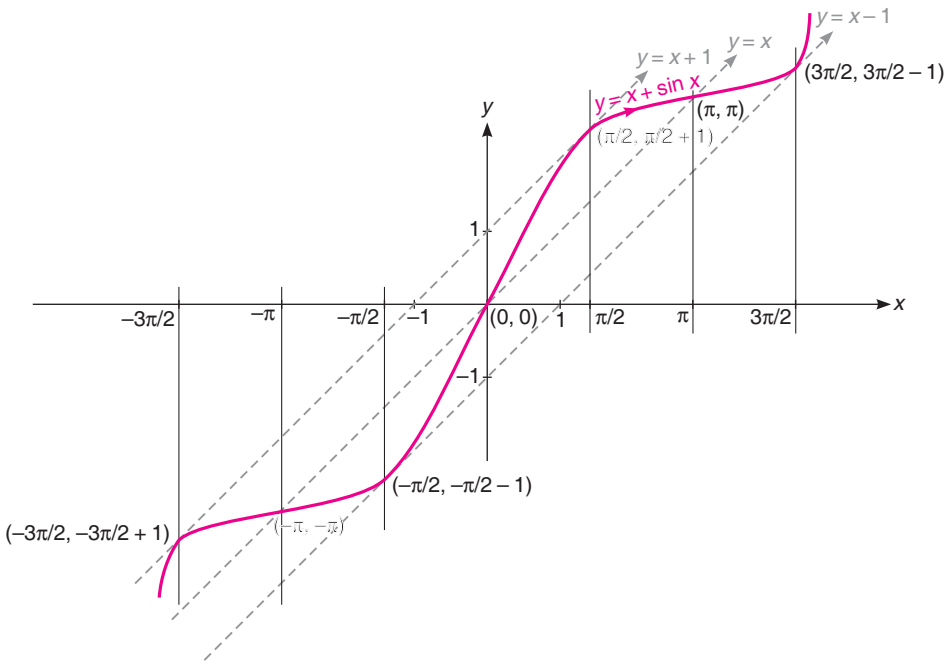


Fig. 2.174

2.6 WHEN $f(x), g(x) \longrightarrow f(x) \cdot g(x) = h(x)$

There is no direct approach but we can use the following steps if minimum and maximum of any one is known.

Step 1. Find the minimum and maximum of any one of them say $a \le g(x) \le b$.

Step 2. From step 1; $a \cdot f(x) \le h(x) \le b \cdot f(x)$

Step 3. Check $g(x) = 0 \quad \Rightarrow \quad h(x) = 0$.

EXAMPLE 1 *Sketch the curve;* $y = x \sin x$.

SOLUTION Here; $y = x \sin x$;

where, $-1 \le \sin x \le 1$

$\Rightarrow \quad -x \le y \le x \qquad \ldots(i)$

also; $\sin x = 0$

$\Rightarrow \quad x = -2\pi, -\pi, 0, \pi, 2\pi, \ldots$

$\therefore$ $y = 0$ when $x = -2\pi, -\pi, 0, \pi, \ldots$

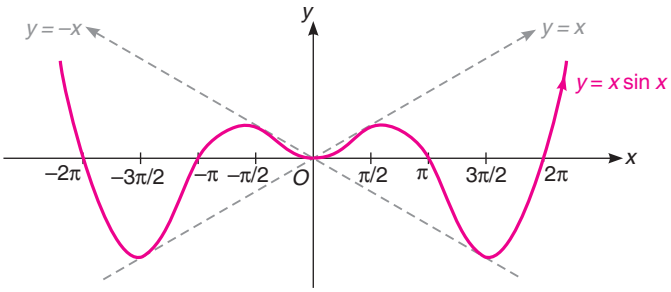


Fig. 2.175

and $\quad y = x \quad$ when $\quad x = \frac{\pi}{2}, \frac{5\pi}{2}, \ldots$

$\quad y = -x \quad$ when $\quad x = -\frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

EXAMPLE 2 *Draw the graph of the function* $y = \frac{\sin x}{x}$.

SOLUTION As we know; $-1 \le \sin x \le 1 \Rightarrow y = \frac{\sin x}{x}$ lies between $-\frac{1}{x}$ to $\frac{1}{x}$.

or to sketch the curve where $-\frac{1}{x} \le y \le \frac{1}{x}$ …(i)

Here at $x = 0$; y is not defined but as;

$$x \longrightarrow 0 \Rightarrow y = \frac{\sin x}{x} \longrightarrow 1 \quad \ldots(ii)$$

also; $y = 0$ at $x = n\pi$; $n \in z - \{0\}$ …(iii)

Using Eqs. (i), (ii) and (iii), the curve could be plotted as;

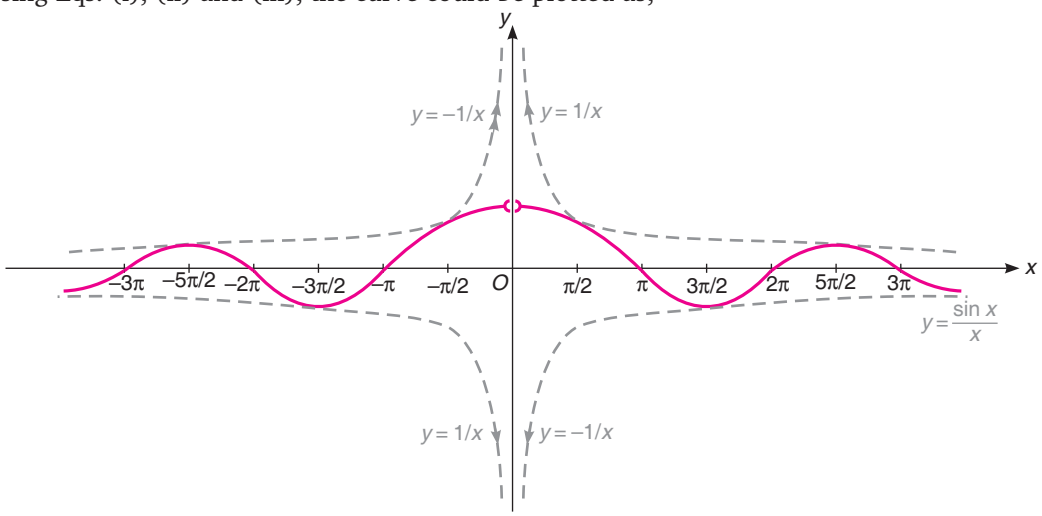


Fig. 2.176

EXAMPLE 3 *Draw the curve:* $y = e^{-x} \sin x$.

SOLUTION As we know;

$-1 \le \sin x \le 1$

$\therefore \quad y = e^{x} \sin x$

$\Rightarrow \quad -e^{-x} \le y \le e^{-x} \quad \ldots(i)$

Thus, y is bounded between

$y = -e^{-x}$ and $y = e^{-x}$.

Shown as in Fig. 2.177;

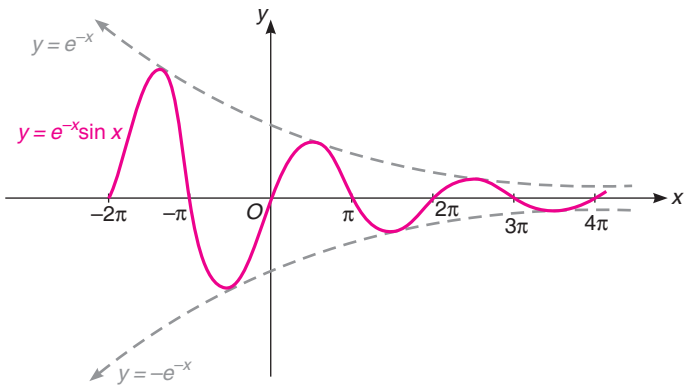


Fig. 2.177

SOME MORE SOLVED EXAMPLES

EXAMPLE 1 *Sketch the curves of the following :*

(i) $y = \sqrt{x - [x]}$ *(ii)* $y = [x] + \sqrt{x - [x]}$

(iii) $y = |[x] + \sqrt{x - [x]}|$; *where [•] represent greatest integer function*

SOLUTION **(i)** As we know that;

$$0 \le x - [x] < 1 \quad \text{for all} \quad x \in R.$$

Also, for any $x \in [0, 1)$, we have

$$x^2 \le x \le \sqrt{x}$$

$$\therefore \quad x - [x] \le \sqrt{x - [x]}$$

Now, $$y = \sqrt{x - [x]} = \begin{cases} \sqrt{x+1}, & \text{when } -1 \le x < 0 \\ \sqrt{x}, & \text{when } 0 \le x < 1 \\ \sqrt{x-1}, & \text{when } 1 \le x < 2 \\ \sqrt{x-2}, & \text{when } 2 \le x < 3 \quad \text{... and so on.} \end{cases}$$

Thus, from above we have to plot;

$y = \sqrt{x}$, when $0 \le x < 1$

$y = \sqrt{x-1}$, when $1 \le x < 2$

(*i.e.*, same as shifting $\sqrt{x}$ towards right by 1 unit.)

$y = \sqrt{x-2}$, when $2 \le x < 3$

(*i.e.*, same as shifting $\sqrt{x}$ towards right by 2 units)

$\therefore$ the curve for $y = \sqrt{x - [x]}$ is periodic with period '1'. Shown as in Fig. 2.178.

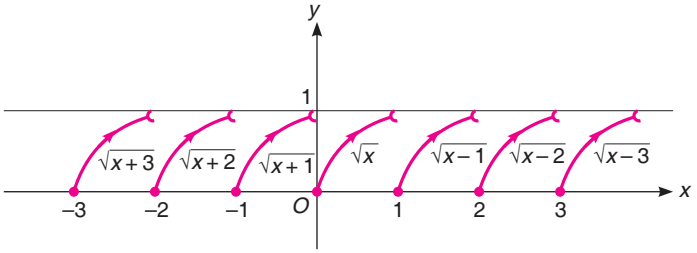


Fig. 2.178

(ii) As we know; $$y = [x] + \sqrt{x - [x]}$$

$$\Rightarrow \quad y = \begin{cases} -1 + \sqrt{x+1}, & \text{when } -1 \le x < 0 \\ \sqrt{x}, & \text{when } 0 \le x < 1 \\ 1 + \sqrt{x-1}, & \text{when } 1 \le x < 2 \\ 2 + \sqrt{x-2}, & \text{when } 2 \le x < 3 \quad \text{... and so on.} \end{cases}$$

Thus, the graph for $y = [x] + \sqrt{x - [x]}$ is obtained by the graph of $y = \sqrt{x - [x]}$ by translating it by $[x]$ units in upward or downward direction according as $[x] > 0$ or $[x] < 0$.

Thus, the curve for $y = [x] + \sqrt{x - [x]}$

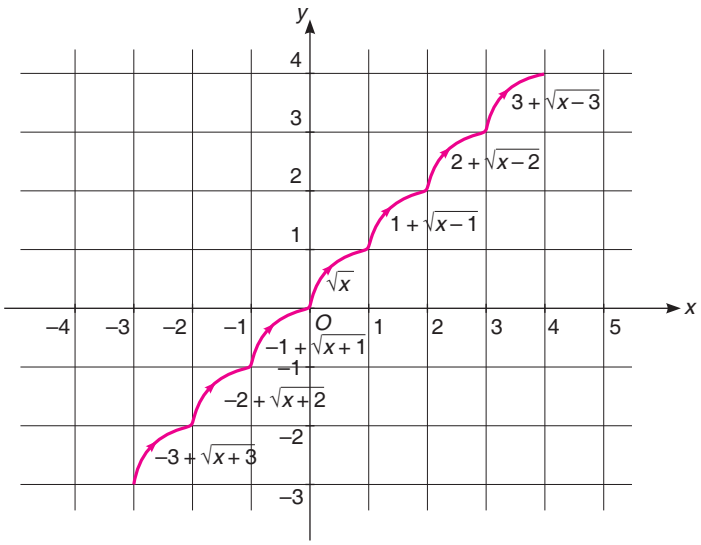


Fig. 2.179

From above curve we could discuss that;

$y = [x] + \sqrt{x - [x]}$ is continuous and differentiable for all x.

(iii) The graph for $y = \left|[x] + \sqrt{x - [x]}\right|$ is obtained by reflecting the portion lying below x-axis of the graph of $y = [x] + \sqrt{x - [x]}$ about x-axis and keeping the portion lying above x-axis (as it is).

Thus, the graph for $y = \left|[x] + \sqrt{x - [x]}\right|$.

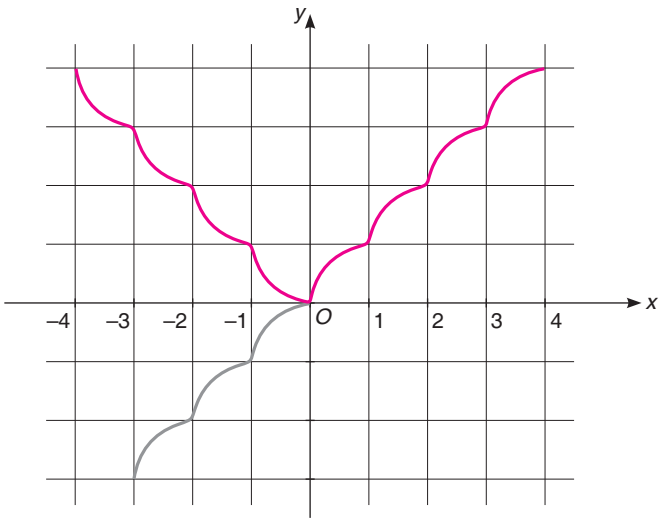


Fig. 2.180

EXAMPLE 2 *Sketch the curve* $y = (1 - x^{2/3})^{3/2}$.

SOLUTION Here;

(a) $f(-x) = f(x) \quad \therefore$ even function or symmetric about y-axis …(i)

(b) When $\left.\begin{array}{l} x = 0 \Rightarrow y = 1 \\ y = 0 \Rightarrow x = \pm 1 \end{array}\right\}$ …(ii)

(c) $\left.\begin{array}{l} \text{Domain} \in [-1, 1] \\ \text{Range} \in [0, 1] \end{array}\right\}$ …(iii)

(d) $\left.\begin{array}{l} \dfrac{dy}{dx} > 0 \text{ when } x \in [-1, 0] \\ \dfrac{dy}{dx} < 0 \text{ when } x \in [0, 1] \end{array}\right\}$ …(iv)

(e) $\left.\dfrac{d^2y}{dx^2} > 0 \text{ when } x \in [-1, 1]\right\}$ …(v)

From above;

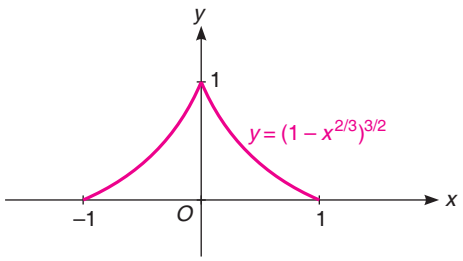


Fig. 2.181

EXAMPLE 3 *Sketch the graph for* $y = (x - 1)x^{2/3}$.

SOLUTION In $y = (x - 1)x^{2/3}$

(a) Not symmetric about any axis or origin. …(i)

(b) $\left.\begin{array}{l} \text{as } x = 0 \Rightarrow y = 0 \\ \text{as } y = 0 \Rightarrow x = 0, 1 \end{array}\right\}$ …(ii)

(c) $\left.\begin{array}{l} \text{Domain} \in R \\ \text{Range} \in R \end{array}\right\}$ …(iii)

(d) $\dfrac{dy}{dx} = x^{2/3} + \dfrac{2(x - 1)}{3x^{1/3}} = \dfrac{5x - 2}{3x^{1/3}}$

$\Rightarrow$ $\left.\begin{array}{l} \dfrac{dy}{dx} > 0, \text{ when } x < 0 \text{ or } x > \dfrac{2}{5} \\ \dfrac{dy}{dx} < 0, \text{ when } 0 < x < \dfrac{2}{5} \end{array}\right\}$ …(iv)

(e) $$\frac{d^2y}{dx^2} = \frac{1}{3}\left\{\frac{x^{1/3}\cdot 5 - (5x-2)\frac{1}{3}(x)^{-2/3}}{x^{2/3}}\right\}$$

$$= \frac{10x+2}{9x^{4/3}}$$

$$\Rightarrow \left.\begin{array}{l} \dfrac{d^2y}{dx^2} > 0, \text{ when } x < -\dfrac{2}{5} \text{ or } x > 0 \\ \dfrac{d^2y}{dx^2} < 0, \text{ when } -\dfrac{2}{5} < x < 0 \end{array}\right\} \quad \ldots(v$$

)

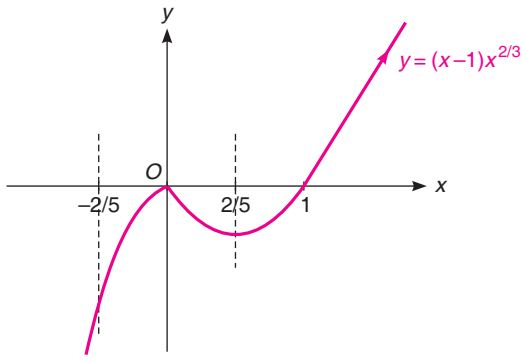


Fig. 2.182

Thus, curve for $y = (x-1)\,x^{2/3}$ is shown in Fig. 2.182.

EXAMPLE 4 *Draw the graph for:*

(i) $y = |1 - |x-1||$ *(ii)* $|y| = |1 - |x-1||$

SOLUTION As we know the graph for $y = x - 1$

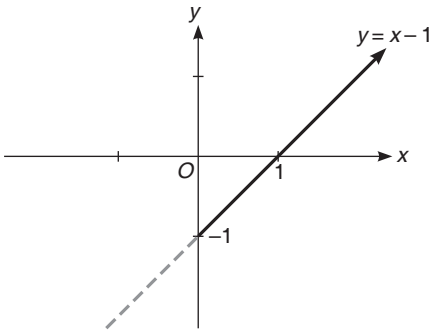


Fig. 2.183

(a) $y = (x-1) \longrightarrow y = |x-1|$

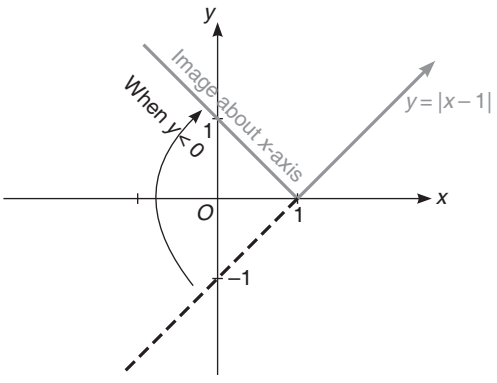


Fig. 2.184

(b) $y = |x-1| \longrightarrow y = -|x-1|$

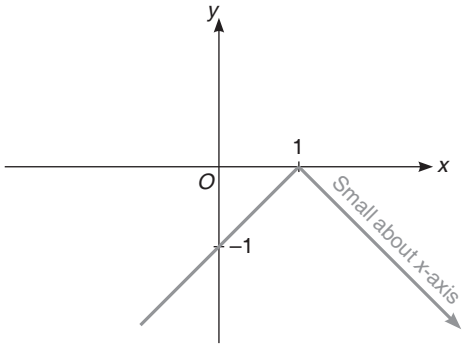


Fig. 2.185

(c) $y = -|x-1| \longrightarrow y = 1-|x-1|$

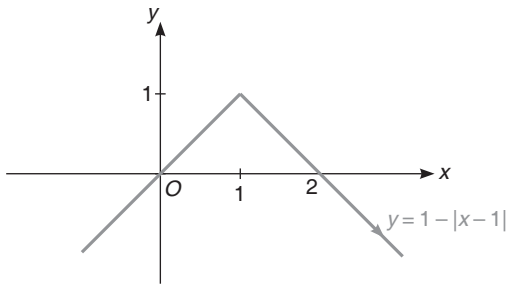


Fig. 2.186

(d) $y = 1-|x-1| \longrightarrow y = |1-|x-1||$

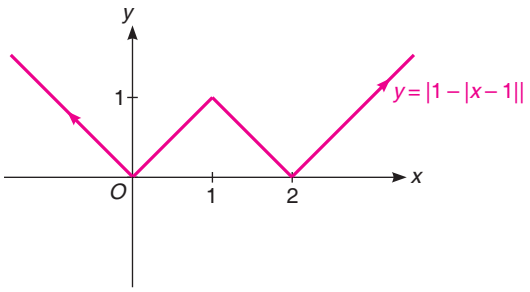


Fig. 2.187

(e) $y = |1-|x-1|| \longrightarrow |y| = |1-|x-1||$

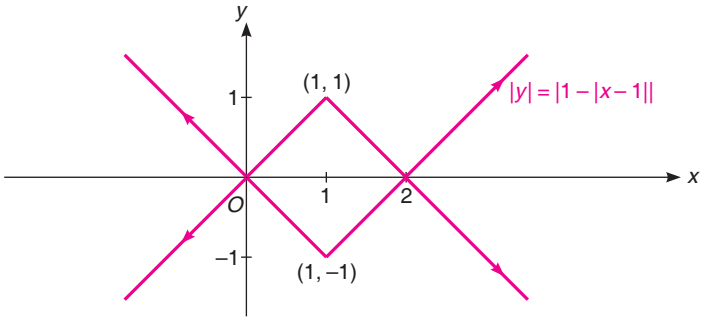


Fig. 2.188

EXAMPLE 5 *Draw the graph for:*

(i) $y = 2 - \dfrac{1}{|x-1|}$ *(ii)* $y = \left|2 - \dfrac{1}{|x-1|}\right|$ *(iii)* $|y| = \left|2 - \dfrac{1}{|x-1|}\right|$

SOLUTION We know the graph for $y = \dfrac{1}{x}$ is shown as;

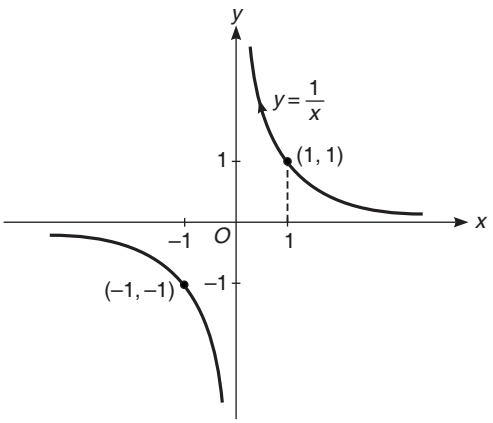


Fig. 2.189

(a) $y = \frac{1}{x} \longrightarrow y = \frac{1}{x-1}$

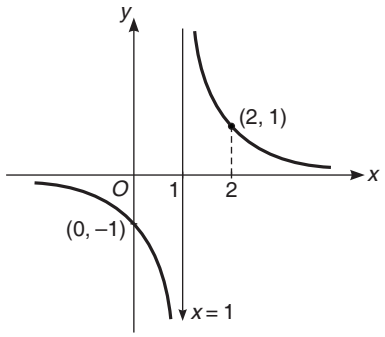


Fig. 2.190

(b) $y = \frac{1}{x-1} \longrightarrow y = \frac{1}{|x-1|}$

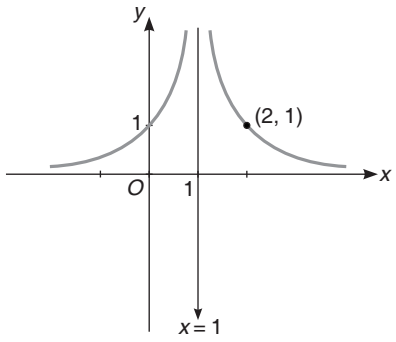


Fig. 2.191

(c) $y = \frac{1}{|x-1|} \longrightarrow y = -\frac{1}{|x-1|}$

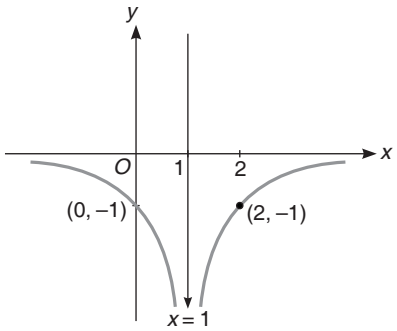


Fig. 2.192

(d) $y = -\frac{1}{|x-1|} \longrightarrow y = 2 - \frac{1}{|x-1|}$

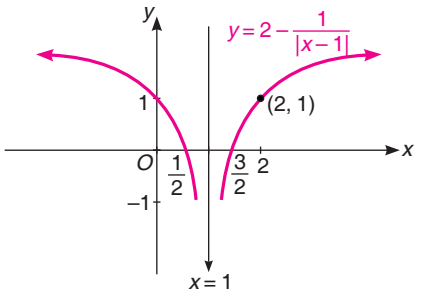


Fig. 2.193

(e) $y = 2 - \frac{1}{|x-1|} \longrightarrow y = \left| 2 - \frac{1}{|x-1|} \right|$

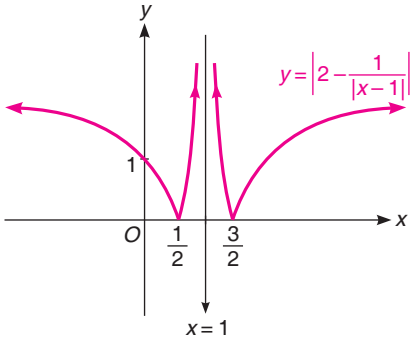


Fig. 2.194

(f) $y = \left|2 - \frac{1}{|x-1|}\right| \longrightarrow |y| = \left|2 - \frac{1}{|x-1|}\right|$

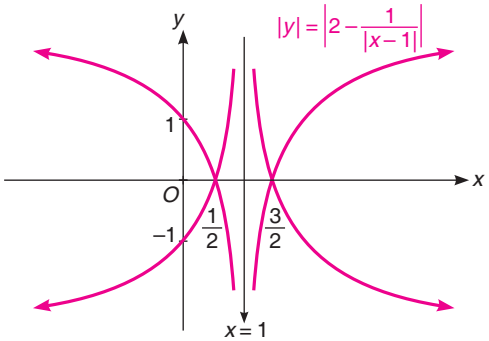


Fig. 2.195

EXAMPLE 6 *Sketch the graph for:*

(i) $y = e^{|x|} - e^{-x}$ *(ii)* $y = |1 + e^{|x|} - e^{-x}|$

(iii) $|y| = |1 + e^{|x|} - e^{-x}|$ *(iv)* $|y| \le |1 + e^{|x|} - e^{-x}|$

SOLUTION

(i) Here $y = e^{|x|} - e^{-x}$

$$\Rightarrow \quad y = \begin{cases} e^{x} - e^{-x} ; & x \ge 0 \\ e^{-x} - e^{-x} ; & x < 0 \end{cases}$$

$$y = \begin{cases} e^{x} - e^{-x}; & x \ge 0 \\ 0 ; & x < 0 \end{cases}$$

To discuss; $y = e^{x} - e^{-x}$

(i) when $x = 0 \Rightarrow y = 0$ **(it passes through origin)**

(ii) when $y = 0 \Rightarrow e^{2x} - 1 = 0 \Rightarrow x = 0$

(iii) $f(-x) = -f(x)$

as $y = f(x) = e^{x} - e^{-x}$

$\Rightarrow f(-x) = e^{-x} - e^{x} = -f(x);$

it shows $y = f(x) = e^{x} - e^{-x}$ is **odd function, *i.e.*, symmetric about origin.**

(iv) $y = e^{x} - e^{-x}$

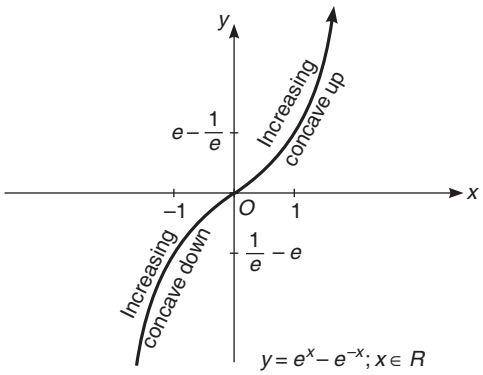
$$\Rightarrow \quad \frac{dy}{dx} = e^{x} + e^{-x} = \frac{e^{2x} + 1}{e^{x}} > 0 \quad \text{for all} \quad x \in R.$$

$\therefore$ y **is increasing for all *x*.**

(v) $$\frac{d^2y}{dx^2} = e^{x} - e^{-x} = \frac{e^{2x} - 1}{e^{x}}$$

$\Rightarrow \frac{d^2y}{dx^2} > 0$ **when** $x > 0$**; concave up** and increasing .

also $\frac{d^2y}{dx^2} < 0$ **when** $x < 0$**; concave down** and increasing from above discussion $y = e^{x} - e^{-x}$**;** is plotted as shown in Fig. 2.196.



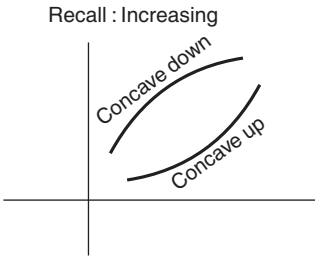


Fig. 2.196

Now;

$$y = e^{|x|} - e^{-x} = \begin{cases} e^{x} - e^{-x}; & x \geq 0 \\ 0 & ; \quad x < 0. \end{cases}$$

Thus from Fig. 2.197.

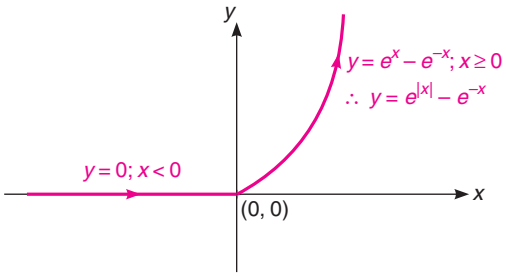


Fig. 2.197

(ii) Plotting of $y = |1 + e^{|x|} - e^{-x}|$

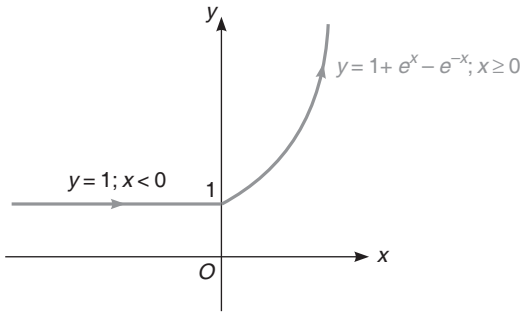


Fig. 2.198

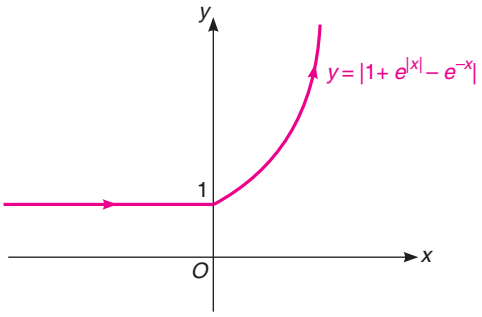


Fig. 2.199

From above Fig. 2.198.

$$y = e^{|x|} - e^{-x} \longrightarrow y = 1 + e^{|x|} - e^{-x}$$

$$\Rightarrow \quad y = \begin{cases} 1 + e^{x} - e^{-x}; & x \geq 0 \\ 1 & ; \quad x < 0 \end{cases}$$

Thus, the graph for; $y = |1 + e^{|x|} - e^{-x}|$

is same as; $y = 1 + e^{|x|} - e^{-x}$ {as $y \geq 1$ for all $x \in R$}

$\therefore$ Graph for $y = |1 + e^{|x|} - e^{-x}|$ is shown in Fig. 2.199.

(iii) Plotting of $|y| = |1 + e^{|x|} - e^{-x}|$

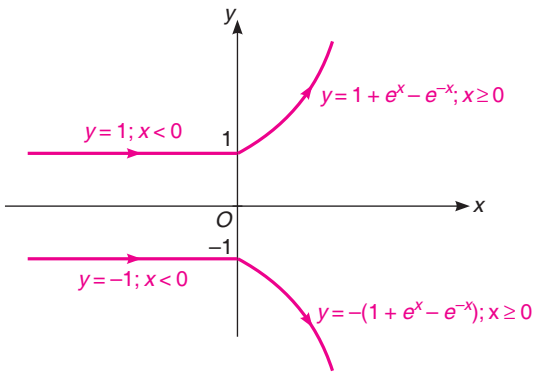


Fig. 2.200

(iv) Plotting of $|y| \le |1 + e^{|x|} - e^{-x}|$

From above figure check any point say (0, 0)

$\Rightarrow 0 \le |1|$ (True, thus to shade area towards (0, 0).

From given figure shade part represents the area bounded between two curves.

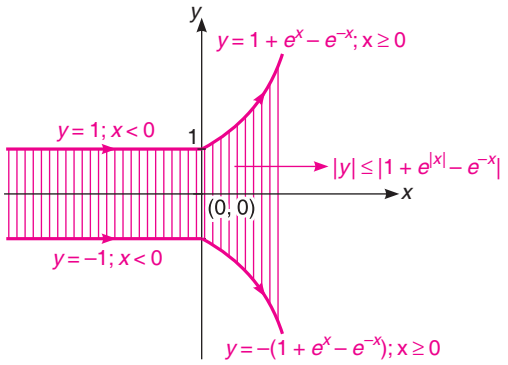


Fig. 2.201

EXAMPLE 7 *Sketch the curve of the following:*

(i) $|x + y| \ge 1$ *(ii)* $|x - y| \le 1$

(iii) $|x| + |y| \le 2$ *(iv)* $|x| - |y| \ge 1$

SOLUTION

(i) Here to sketch, $|x + y| \ge 1$

we know; $|x + y| \ge 1$

$$\Rightarrow \begin{cases} x + y \ge 1 \\ x + y \le -1 \end{cases}$$

or $x + y \ge 1$

and $x + y \le -1$

First we shall draw the graph for $x + y = 1$ and $x + y = -1$, now we shall consider any fixed point say (0, 0) and check $x + y \ge 1$ and $x + y \le -1$ holds or not.

As; $x + y \ge 1 \Rightarrow 0 \ge 1$ (false).

$\therefore$ shaded part is away from origin.

again as; $x + y \le -1$

$\Rightarrow 0 \le -1$ (false)

$\therefore$ Shaded part is away from origin shown as in Fig. 2.202.

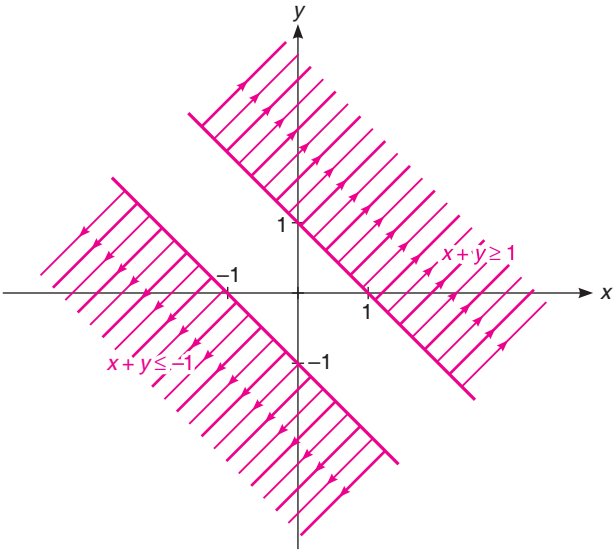


Fig. 2.202

(ii) To sketch $|x - y| \le 1$

we know $|x - y| \le 1$

$\Rightarrow -1 \le x - y \le 1$

$\Rightarrow -1 - x \le -y \le 1 - x$

or $(x - 1) \le y \le (x + 1)$

Thus, to plot y between $(x - 1)$ to $(x + 1)$.

In figure shaded parts is towards origin as; putting any fixed point say $(0, 0)$; in $|x - y| \le 1$.

$\Rightarrow \quad |0 - 0| \le 1$

$\Rightarrow \quad 0 \le 1 \qquad$ (true)

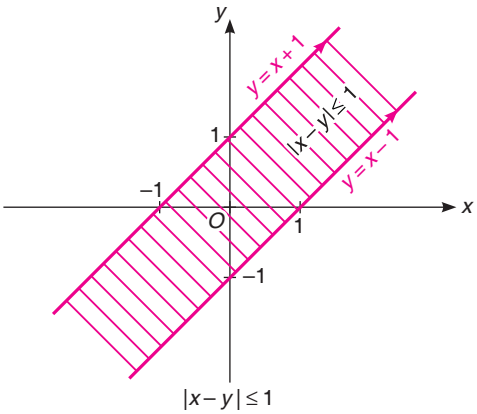


Fig. 2.203

(iii) To sketch $|x| + |y| \le 2$

Here; $|y| = 2 - |x|$ is plotted as;

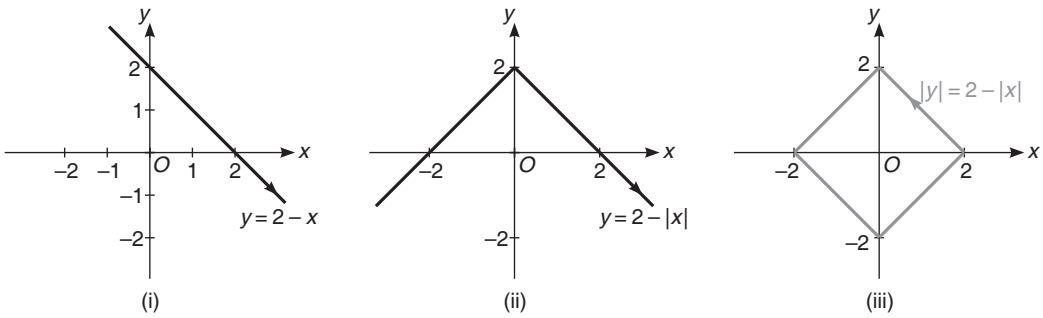


Fig. 2.204

From above graph of $|y| = 2 - |x|$, we can check shading of

$$|y| \le 2 - |x| \quad \text{or} \quad |x| + |y| \le 2.$$

as at $(0, 0) \Rightarrow 0 \le 2$ (true)

$\therefore$ shading towards origin; shown as in Fig. 2.205.

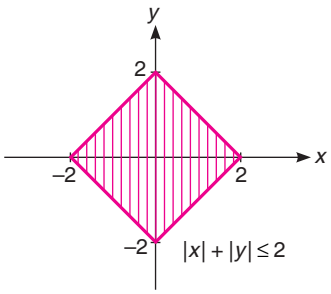
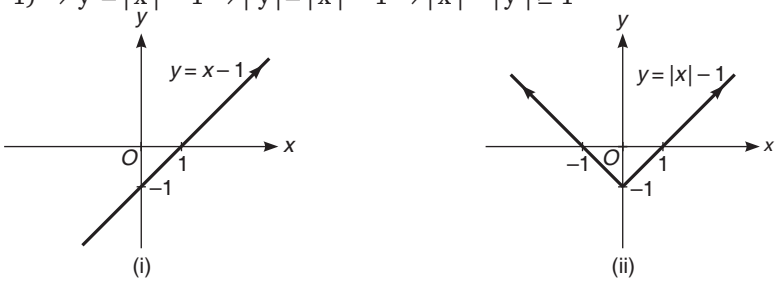


Fig. 2.205

(iv) Sketching of $|x| - |y| \ge 1$

To sketch $|x| - |y| \ge 1$; we proceed as:

$y = (x - 1) \rightarrow y = |x| - 1 \rightarrow |y| = |x| - 1 \rightarrow |x| - |y| \ge 1$



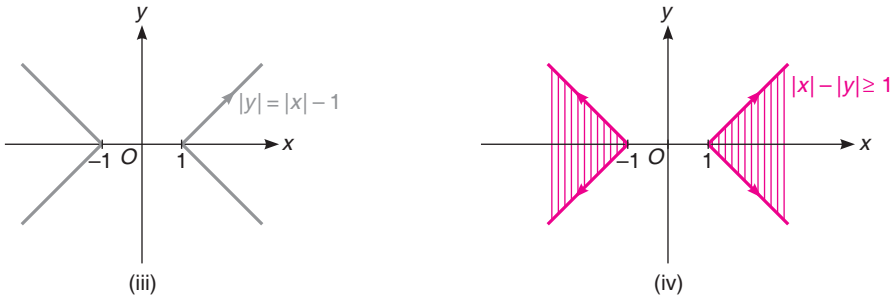


Fig. 2.206

EXAMPLE 8 *Sketch the curve;*

(i) $|x + y| + |x - y| \le 4$ *(ii)* $|x + y| + |x - y| \ge 4$

SOLUTION (i) As we know;

$$|x - y| = \begin{cases} |x| - |y|; & |x| \ge |y| \\ -(|x| - |y|); & |x| \le |y| \end{cases}$$

Thus; $|x + y| + |x - y| \le 4$

$$\Rightarrow \quad \begin{cases} x + y + x - y \le 4; & |x| \ge |y| \\ x + y - x + y \le 4; & |x| < |y| \end{cases}$$

$$\Rightarrow \quad \begin{cases} 2x \le 4; & |x| \ge |y| \\ 2y \le 4; & |x| < |y| \end{cases}$$

or $|x + y| + |x - y| \le 4$

$\Rightarrow \quad |x| \le 2$ and $|y| \le 2$

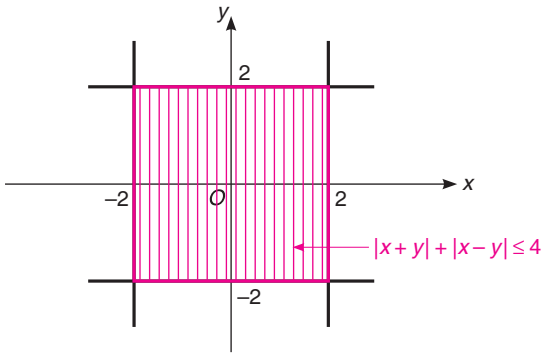


Fig. 2.207

Thus, to shade the portion when $-2 \le x \le 2$ and $-2 \le y \le 2$. Shown as in Fig. 2.207.

(ii) Again; $|x + y| + |x - y| \ge 4 \quad \Rightarrow \quad |x| \ge 2$ and $|y| \ge 2$

Thus, to shade the portion when $x \le -2$ or $x \ge 2$ and $y \le -2$ or $y \ge 2$. Shown as in Fig. 2.208.

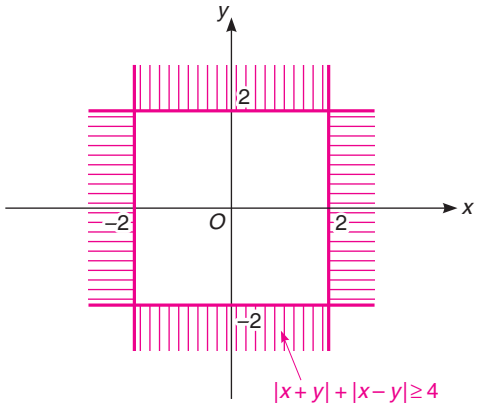


Fig. 2.208

EXAMPLE 9 *Sketch the curve for;*

(i) $2^{|x|}|y| + 2^{|x|-1} \le 1$ *(ii)* $2^{|x|}|y| + 2^{|x|-1} \le 1$; $|x| \le \frac{1}{2}$ and $|y| \le \frac{1}{2}$.

SOLUTION **(i)** Here $2^{|x|}|y| + 2^{|x|-1} \le 1 \Rightarrow 2^{|x|}\left\{|y| + \frac{1}{2}\right\} \le 1$

$$\Rightarrow \quad |y| + \frac{1}{2} \le 2^{-|x|}$$

Thus, to plot $|y| + \frac{1}{2} \le 2^{-|x|}$, we proceed as;

$$y = \underbrace{\left(2^{-x} - \frac{1}{2}\right)}_{\text{(i)}} \longrightarrow \underbrace{y = 2^{-|x|} - \frac{1}{2}}_{\text{(ii)}} \longrightarrow \underbrace{|y| = 2^{-|x|} - \frac{1}{2}}_{\text{(iii)}} \longrightarrow \underbrace{|y| \le 2^{-|x|} - \frac{1}{2}}_{\text{(iv)}}$$

Shown as;

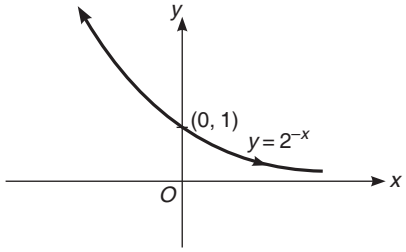
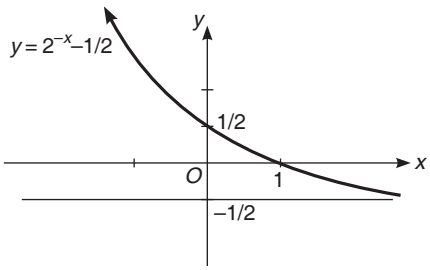
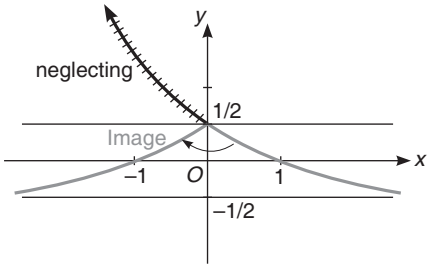


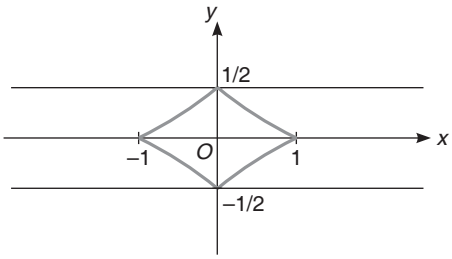
Fig. 2.209



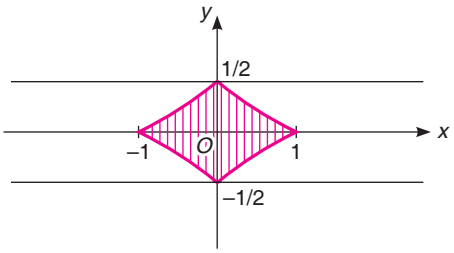
(i) $y = 2^{-x} - \frac{1}{2}$



(ii) $y = 2^{-|x|} - \frac{1}{2}$



(iii) $|y| = 2^{-|x|} - \frac{1}{2}$



(iv) $|y| \le 2^{-|x|} - \frac{1}{2}$

Fig. 2.210

Here, figure (iv) is shaded towards origin as putting $x = 0, y = 0$ in $|y| \le 2^{-|x|} - \frac{1}{2}$.

$\Rightarrow \quad 0 \le 1 - \frac{1}{2}$ or $0 \le \frac{1}{2}$, which is true, therefore, shaded towards origin.

(ii) Plotting of $2^{|x|}|y| + 2^{|x|-1} \le 1$;

$$|x| \le \frac{1}{2} \quad \text{and} \quad |y| \le \frac{1}{2}.$$

from above figure;

$$2^{|x|}|y| + 2^{|x|-1} \le 1$$

and $|x| \le \frac{1}{2}$, $|y| \le \frac{1}{2}$ shown as in Fig. 2.211.

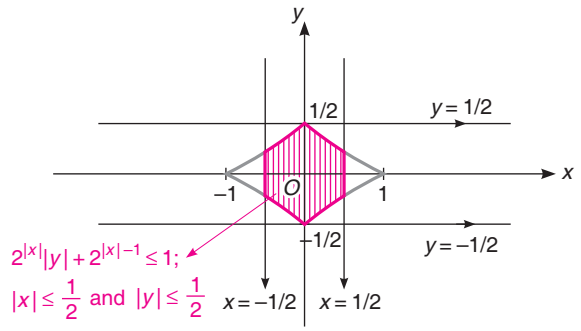


Fig. 2.211

EXAMPLE 10 *Sketch the graph of the function:*

$$f(x) = \log_2(1 - x^2)$$

SOLUTION Here, $\log_2(1 - x^2)$ exists when, $-1 < x < 1$

i.e., doman $\in (-1, 1)$

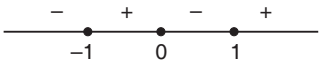
and range $\in (-\infty, 0]$ …(i)

as $x \longrightarrow \pm 1 \Rightarrow y \longrightarrow -\infty$ …(ii)

differentiating $y = \log_2(1 - x^2)$,

we get $$\frac{dy}{dx} = \frac{(-2x)}{1 - x^2} \cdot (\log_2 e) = 2\log_2 e \frac{x}{(x^2 - 1)}$$

Using number line rule;



$$\left.\begin{aligned} \frac{dy}{dx} &> 0, \quad \text{when } -1 < x < 0 \\ \frac{dy}{dx} &< 0, \quad \text{when } 0 < x < 1 \end{aligned}\right\} \quad \ldots(iii)$$

also, $\frac{d^2y}{dx^2} = \frac{-2\log_2 e(1 + x^2)}{(1 - x^2)^2} < 0$ for all $x \in (-1, 1)$

$\therefore$ $y = \log(1 - x^2)$ is concave down for $x \in (-1, 1)\}$ …(iv)

from above results we can draw

$y = \log_2(1 - x^2)$ as shown in Fig. 2.212.

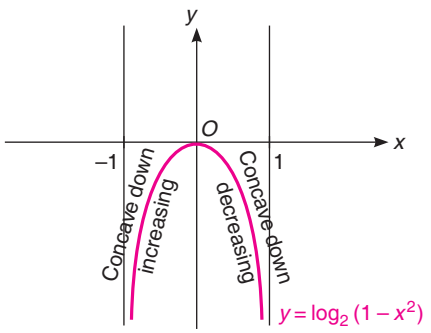


Fig. 2.212

EXAMPLE 11 *Sketch the graph of* $y = \log(\sin x)$.

SOLUTION Here;

(i) $y = \log(\sin x)$ is defined, when $\sin x > 0 \quad \Rightarrow \quad x \in (2n\pi, (2n+1)\pi)$.

(ii) Since, $\sin x$ is periodic with period 2π.

$\therefore$ to discuss $y = \log(\sin x)$, when $x \in (0, \pi)$ as exists when $x \in (0, \pi)$ and then plotted for entire number line.

(iii) as $x \longrightarrow 0 \quad \Rightarrow \quad y = \log(0) = -\infty$.

as $y \longrightarrow 0 \quad \Rightarrow \quad \sin x \longrightarrow 1 \quad \Rightarrow \quad x = \dfrac{\pi}{2}$.

(iv) $\dfrac{dy}{dx} = \dfrac{1}{\sin x} \cdot \cos x = \cot x$

$\Rightarrow \quad \dfrac{dy}{dx} > 0 \quad$ when $\quad x \in \left(0, \dfrac{\pi}{2}\right)$

$\dfrac{dy}{dx} < 0, \quad$ when $\quad x \in \left(\dfrac{\pi}{2}, \pi\right)$

(v) $\dfrac{d^2y}{dx^2} = -\operatorname{cosec}^2 x < 0 \quad$ for all $\quad x \in (0, \pi)$

Thus, **increasing and concave down** $x \in (0, \pi/2)$

decreasing and concave down $x \in (\pi/2, \pi)$.

and to plot the curve only when $x \in (2n\pi, (2n+1)\pi)$. Shown as in Fig. 2.213.

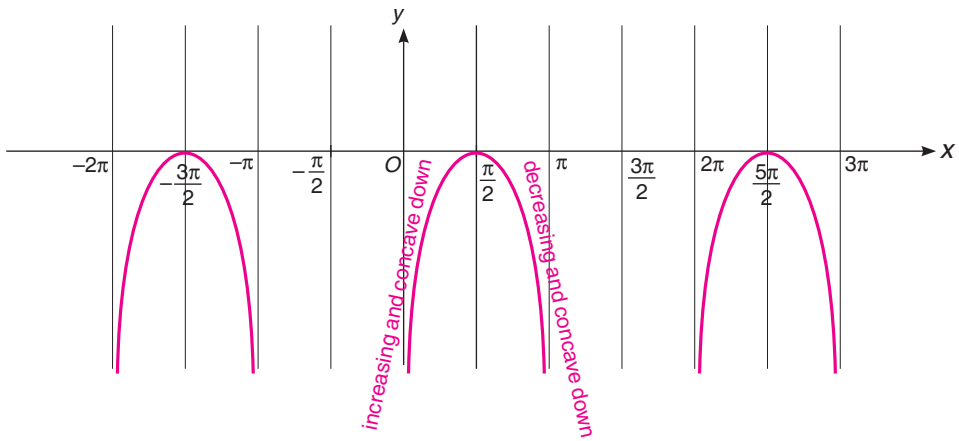


Fig. 2.213

EXAMPLE 12 *Sketch the graph for*

$$y = -\log_{\sin x} e.$$

SOLUTION (i) Here, $y = -\log_{\sin x} e = \dfrac{-1}{\log_e \sin x}$ is defined when,

$$\sin x > 0 \quad \text{and} \quad \sin x \neq 1.$$

i.e., $\quad x \in (2n\pi, (2n+1)\pi) - \{2n\pi + \pi/2\}; \; n \in N \qquad \ldots(i)$

(ii) $y = \dfrac{-1}{\log_e \sin x}$ is periodic with period 2π,

$\therefore$ to discuss the graph for $x \in (0, \pi) - \{\pi/2\}$ and sketch for the entire number line.

(iii) $\therefore \quad \dfrac{dy}{dx} = +\dfrac{1}{(\log \sin x)^2} \cdot \dfrac{1}{\sin x} \cdot \cos x \quad \Rightarrow \quad \dfrac{dy}{dx} = +\dfrac{\cot x}{(\log \sin x)^2}$

or $\dfrac{dy}{dx} > 0 \quad \text{when} \quad x \in \left(0, \dfrac{\pi}{2}\right)$

$\dfrac{dy}{dx} < 0 \quad \text{when} \quad x \in \left(\dfrac{\pi}{2}, \pi\right)$

(iv) $$\frac{d^2y}{dx^2} = +\left\{\frac{(\log \sin x)^2 \cdot (-\text{cosec}^2\, x) - \cot x \cdot 2\,(\log \sin x) \cdot \cot x}{(\log \sin x)^4}\right\}$$

$$= -\frac{\{(\log \sin x)^2 \text{cosec}^2 x + 2\,(\log \sin x)\cot^2 x\}}{(\log \sin x)^2}$$

$\Rightarrow \quad \dfrac{d^2y}{dx^2} > 0 \quad \text{for all} \quad x \in (0, \pi) - \left\{\dfrac{\pi}{2}\right\}$

Thus, $y = \dfrac{1}{\log \sin x} \Rightarrow$ $\begin{cases} \text{increasing and concave up } x \in \left(\dfrac{\pi}{2}, \pi\right) \\ \text{decreasing and concave up } x \in \left(0, \dfrac{\pi}{2}\right) \end{cases}$

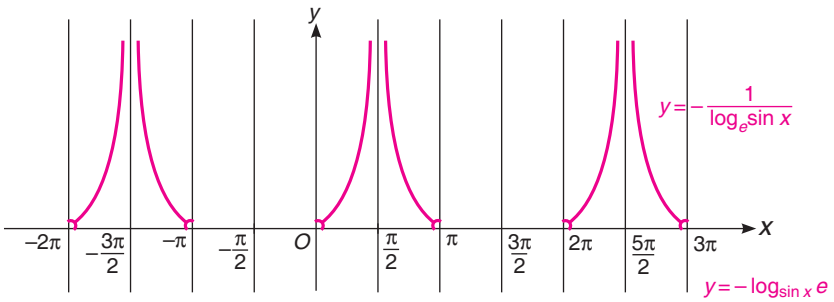


Fig. 2.214

EXAMPLE 13 *Sketch the curve:*

(i) $[y] = \cos x; \quad x \in [-2\pi, 2\pi]$

(ii) $[y] = [\cos x]; \quad x \in [-2\pi, 2\pi]$ *; where $[\cdot]$ denotes greatest integer function.*

SOLUTION As we know; $y = \cos x$ could be plotted as;

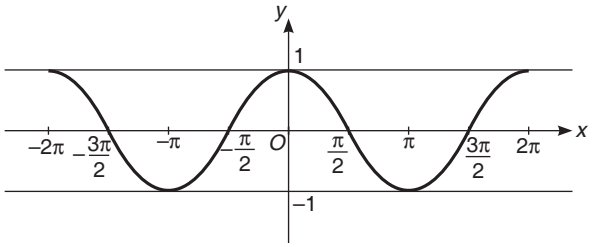


Fig. 2.215

(i) Sketching of $[y] = \cos x$

From above curve $\cos x = -1, 0, 1$ when $x = -2\pi, -\frac{3\pi}{2}, -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

$\Rightarrow$ $[y] = \cos x$ is possible only

when $x \in \left\{\pm 2\pi, \pm\frac{3\pi}{2}, \pm\pi, \pm\frac{\pi}{2}, 0\right\}$

Thus,

$\cos x = -1 \Rightarrow -1 \le y < 0$

$\cos x = 0 \Rightarrow 0 \le y < 0$

$\cos x = 1 \Rightarrow 1 \le y < 2$

Thus, $[y] = \cos x$ **is plotted as shown in Fig. 2.216.**

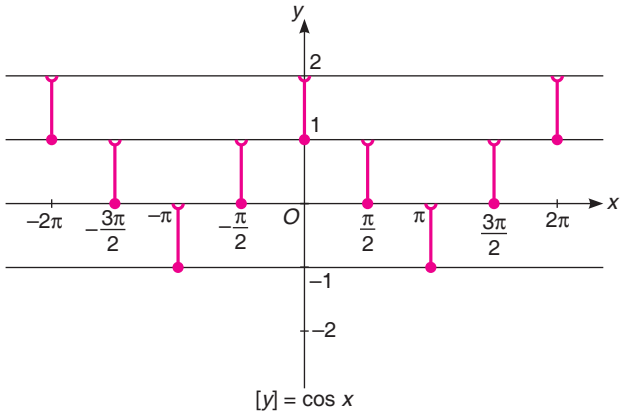


Fig. 2.216

(ii) Sketching of $[y] = [\cos x]$

First to plot $y = [\cos x]$. Shown as;

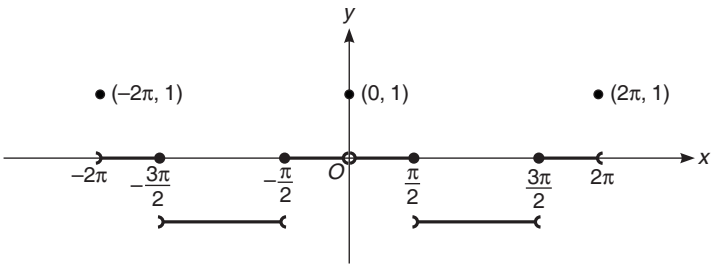


Fig. 2.217

From above figure $y = [\cos x]$

$$\Rightarrow \quad y = \begin{cases} -1 \ ; & \frac{\pi}{2} < |x| < \frac{3\pi}{2} \\ 0 \ ; & 0 < |x| \le \frac{\pi}{2} \text{ and } \frac{3\pi}{2} \le |x| < 2\pi \\ 1 \ ; & |x| = 0, 2\pi \end{cases}$$

Thus, when $y = 0 \Rightarrow [y] \in [0, 1)$

$y = 1 \Rightarrow [y] \in [1, 2)$

$y = -1 \Rightarrow [y] \in [-1, 0)$

Thus, **graph for** $[y] = [\cos x]$

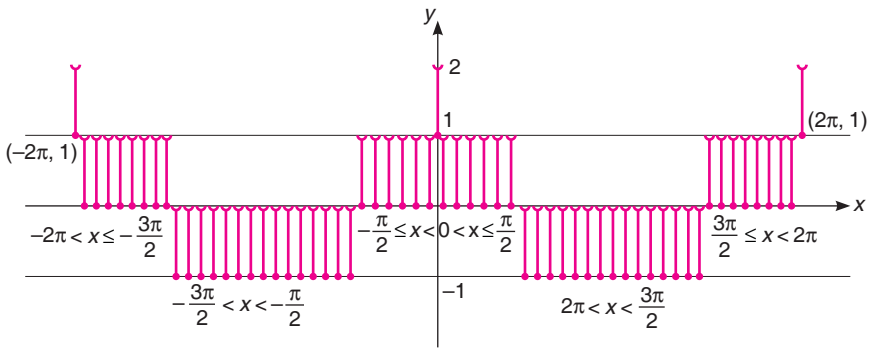


Fig. 2.218

EXAMPLE 14 *Sketch the curves*

(i) $y = 4 - [x]$

(ii) $[y] = 4 - [x]$

(iii) $[|y|] = 4 - [|x|]$

SOLUTION As we know;

(i) $y = 4 - [x]$ could be plotted with in two steps;

$$y = 4 - x \longrightarrow y = 4 - [x]$$

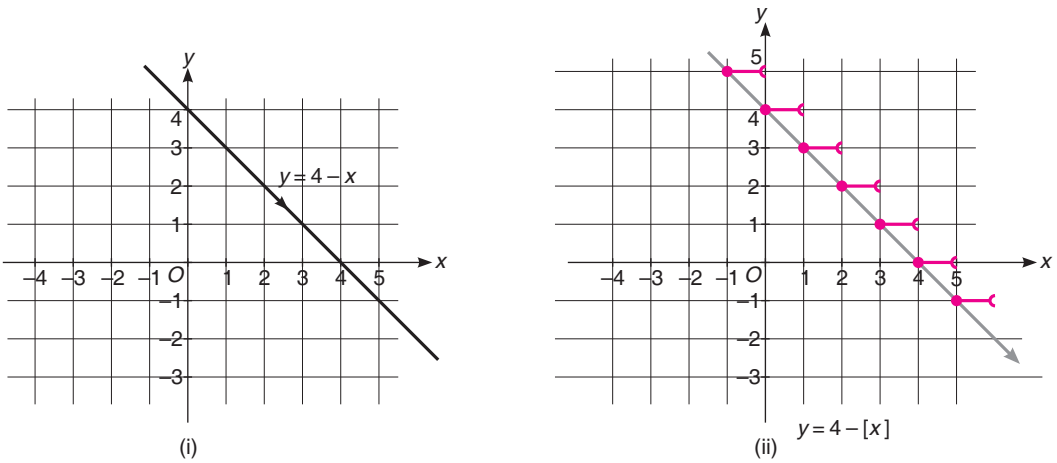


Fig. 2.219

(ii) Since, we know from above curve $y = 4 - [x]$

$\Rightarrow$ $y \in$ integer, thus, on taking integral part, say

$$[y] = I \quad \Rightarrow \quad I \le y < (I + 1)$$

∴ **Graph for** $[y] = 4 - [x]$

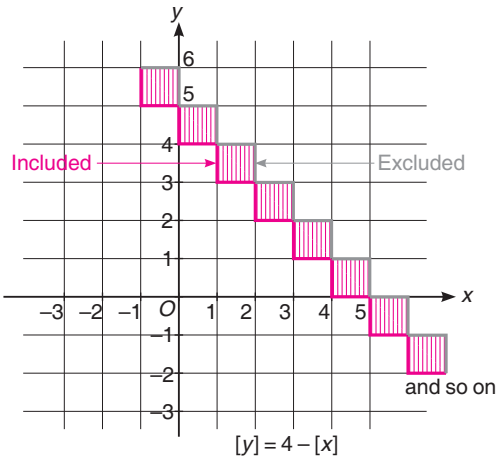


Fig. 2.220

In above curve lower boundary are included and upper are excluded.

(iii) To sketch $[|y|] = 4 - [|x|]$

From above figure we can say;

$$[y] = 4 - [x] \longrightarrow [y] = 4 - [|x|] \longrightarrow [|y|] = 4 - [|x|]$$

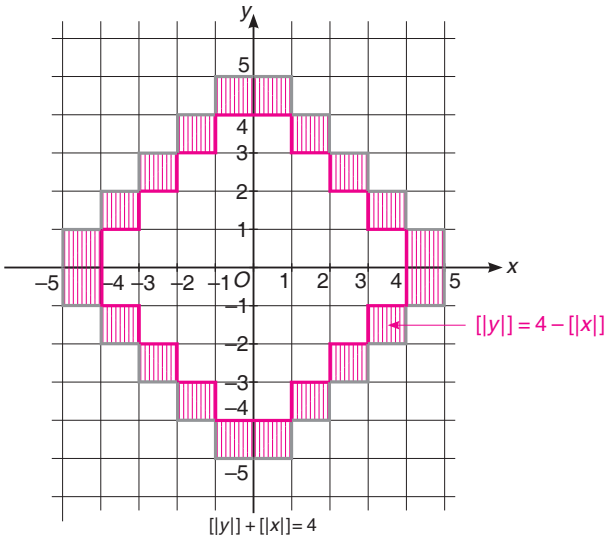


Fig. 2.221

EXAMPLE 15 *Shade the region whose co-ordinates x and y satisfy the equation.*

$$\cos x - \cos y > 0$$

SOLUTION Here, $\cos x - \cos y > 0$ can be written as, $2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{y-x}{2}\right) > 0$

$\Rightarrow \quad \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{y-x}{2}\right) > 0$

This inequality holds true for all points,

when $\sin\left(\frac{x+y}{2}\right)$ and $\sin\left(\frac{y-x}{2}\right)$ have same sign.

i.e., $\sin\left(\frac{x+y}{2}\right) > 0$ and $\sin\left(\frac{y-x}{2}\right) > 0$

or $\sin\left(\frac{x+y}{2}\right) < 0$ and $\sin\left(\frac{y-x}{2}\right) < 0$

$\Rightarrow \quad x + y > 2n\pi$ and $y - x > 2n\pi$

or $\quad x + y < 2n\pi$ and $y - x < 2n\pi$

$\Rightarrow \quad \begin{cases} y > 2n\pi - x \\ y > 2n\pi + x \end{cases}$ or $\begin{cases} y < 2n\pi - x \\ y < 2n\pi + x \end{cases}$ where $n \in z$

Here, the equation $x + y = 2k\pi$ represents a system of parallel straight lines corresponding to different values of k.

Say $k = 0, 1 \Rightarrow y = -x$ and $y = -x + 2\pi$.

i.e., the set of points whose coordinates satisfy the inequality

$$0 < x + y < 2\pi$$

Similarly, for general points: $2k\pi < x + y < (2k + 1)\pi$ ...(i)

Now, the set of points which satisfy $\sin\left(\frac{x+y}{2}\right) > 0$.

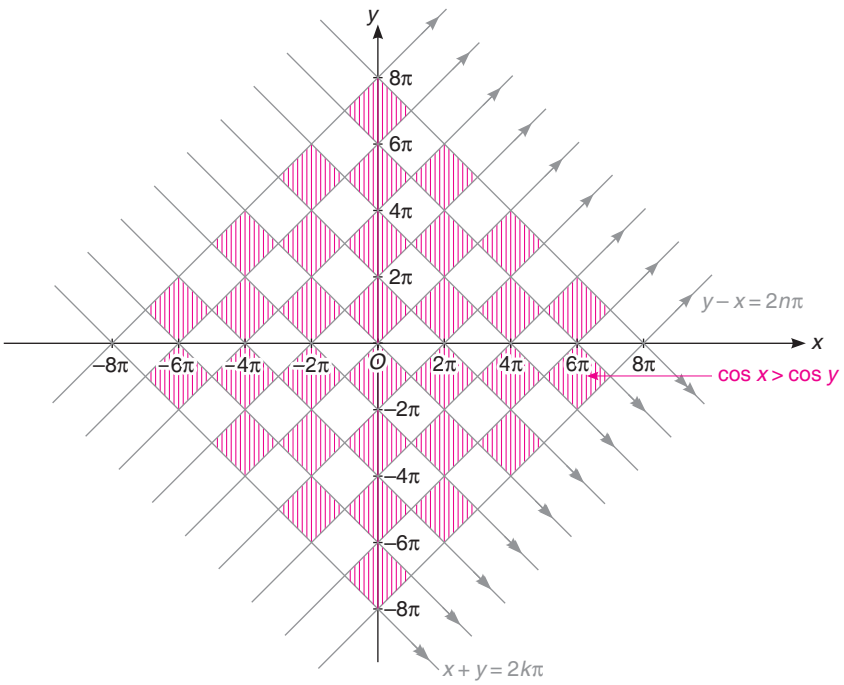


Fig. 2.222

$\Rightarrow \quad 2n\pi < \dfrac{x+y}{2} < (2n+1)\pi$

$\Rightarrow \quad 2(2n\pi) < x + y < 2(2n+1)\pi \quad \ldots(ii)$

$\Rightarrow$ only those points of x, y which satisfy Eq. (ii).

Similarly, $2(2p-1)\pi < y - x < 2(2p)\pi \quad \ldots(iii)$

From above results, graph for $\cos x > \cos y$ is shown in Fig. 2.222.

EXAMPLE 16 *Sketch the curve* $x^{2/3} + y^{2/3} = a^{2/3}$.

SOLUTION As to sketch, $x^{2/3} + y^{2/3} = a^{2/3}$.

(i) Curve is symmetric about y-axis (as when x is replaced by $(-x)$ curve remains same)

(ii) Curve is symmetric about x-axis (as when y is replaced by $(-y)$ curve remains same).

(iii) Curve is symmetric about origin (as when x is replaced by y and y by x curve remains same).

(iv) When, $x = 0 \Rightarrow y = \pm a$

when, $y = 0 \Rightarrow x = \pm a$.

(v) $y^{2/3} = (a^{2/3} - x^{2/3})$,

or $y^2 = (a^{2/3} - x^{2/3})^3$;

differentiating both sides, $2y\dfrac{dy}{dx} = 3(a^{2/3} - x^{2/3})^2 \cdot \left(-\dfrac{2}{3}x^{-1/3}\right)$

$\Rightarrow \dfrac{dy}{dx} < 0$ when $x > 0$, $y > 0$ [to discuss $0 < x \le a$ and to take symmetry for rest of graph using Eqs. (i), (ii) and (iii)].

(vi) From above $y\dfrac{dy}{dx} = -x^{-1/3}(a^{2/3} - x^{2/3})^2$

Differentiating again, we get;

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -x^{-1/3} \cdot 2(a^{2/3} - x^{2/3}) \cdot \left(-\frac{2}{3}x^{-1/3}\right) + \frac{1}{3}x^{-4/3} \cdot (a^{2/3} - x^{2/3})^2$$

$$= \frac{4}{3}x^{-2/3}(a^{2/3} - x^{2/3}) + \frac{1}{3}x^{-4/3}(a^{2/3} - x^{2/3})^2$$

$$= \frac{1}{3}x^{-2/3}(a^{2/3} - x^{2/3})\{4 + x^{-2/3}(a^{2/3} - x^{2/3})\}$$

$\dfrac{d^2y}{dx^2} = +\text{ve}$ whenever; $0 < x \le a$ and $y > 0$.

Thus; when $0 < x \le a$ and $y > 0$

$\Rightarrow \quad \dfrac{dy}{dx} < 0$ and $\dfrac{d^2y}{dx^2} > 0$,

i.e., **decreasing and concave up.**

$\therefore$ **Graph for** $x^{2/3} + y^{2/3} = a^{2/3}$

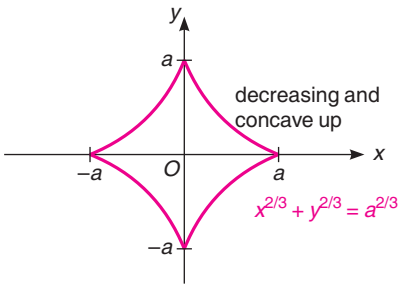


Fig. 2.223

Note Above curve is known as **Astroid** represented by the following parametric equations:

$$\left.\begin{aligned} x &= a\cos^3 t \\ y &= a\sin^3 t \end{aligned}\right\} \quad 0 \le t \le 2\pi, \quad a > 0$$

Eliminating cos t and sin t from above equations, we get $x^{2/3} + y^{2/3} = a^{2/3}$.

where the quantities x and y are defined for all values of t. But since, the functions $\cos^3 t$ and $\sin^3 t$ are periodic with period 2π, it is sufficient to discuss the curve for $t \in [0, 2\pi]$.

Thus, $\left.\begin{matrix}\text{Domain of function} \in [0, 2\pi] \\ \text{Range of function} \in [-a, a]\end{matrix}\right\}$ ...(i)

Now, $\left.\begin{matrix}\frac{dx}{dt} = -3a\cos^2 t \sin t \\ \frac{dy}{dt} = +3a\sin^2 t \cos t\end{matrix}\right\}$ ...(ii)

Here, $\frac{dx}{dt} = 0, \ \frac{dy}{dx} = 0 \ \Rightarrow \ t = 0, \frac{\pi}{2}, \ 8\pi, \frac{3\pi}{2}, 2\pi.$

$\therefore$ $\left.\frac{dy}{dx} = -\tan t\right\}$ ...(iii)

$\left.\frac{d^2y}{dx^2} = \frac{1}{3a\cos^4 t \sin t}\right\}$ ...(iv)

From Eqs. (i), (ii), (iii) and (iv), we construct a table;

Range of t	Domain (x)	Range (y)	Sign of $\frac{dy}{dx}$	Sign of $\frac{d^2y}{dx^2}$
$0 < t < \frac{\pi}{2}$	$0 < x < a$	$0 < y < a$	–	+
$\frac{\pi}{2} < t < \pi$	$-a < x < 0$	$0 < y < a$	+	+
$\pi < t < \frac{3\pi}{2}$	$-a < x < 0$	$-a < y < 0$	–	–
$\frac{3\pi}{2} < t < 2\pi$	$0 < x < a$	$-a < y < 0$	+	–

On the basis of above information we can sketch

$x = a\cos^3 t$

$y = b\sin^3 t$

Students are adviced to convert the cartesian into parametric if possible.

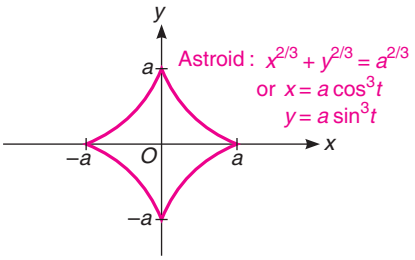


Fig. 2.224

EXAMPLE 17 *Sketch the curves:*

(i) $y = \frac{e^x + e^{-x}}{2}$ *(ii)* $y = \frac{e^x - e^{-x}}{2}$ *(iii)* $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ *(iv)* $y = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

SOLUTION (i) Sketching of $f(x) = y = \frac{e^x + e^{-x}}{2}$

(a) $f(-x) = f(x)$, $\therefore$ even or symmetric about y-axis ...(i)

(b) When $x = 0 \Rightarrow y = 1$ …(ii)

Differentiating, we get; $\frac{dy}{dx} = \frac{e^x - e^{-x}}{2} = \frac{e^{2x} - 1}{2e^x}$

$\Rightarrow$ $\frac{d^2y}{dx^2} = \frac{e^x + e^{-x}}{2} = \frac{e^{2x} + 1}{2e^x}$

From above; $\left.\begin{matrix}\frac{dy}{dx} > 0 \text{ whenever } x > 0 \\ \frac{dy}{dx} < 0 \text{ whenever } x < 0\end{matrix}\right\}$ …(iii)

$\left.\frac{d^2y}{dx^2} > 0 \text{ for all } x\right\}$ …(iv)

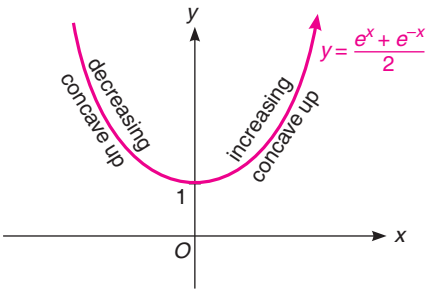


Fig. 2.225

Thus, from Eqs. (i), (ii), (iii) and (iv) graph of $y = \frac{e^x + e^{-x}}{2}$ is;

(ii) Sketching $y = \frac{e^x - e^{-x}}{2}$

(a) $f(-x) = -f(x)$,

$\therefore$ odd function or symmetric about origin …(i)

(b) $x = 0 \Rightarrow y = 0$ …(ii)

(c) $\frac{dy}{dx} = \frac{e^x + e^{-x}}{2} > 0$ for all x …(iii)

(d) $\frac{d^2y}{dx^2} = \frac{e^x - e^{-x}}{2}$

$\Rightarrow$ $\left.\begin{matrix}\frac{d^2y}{dx^2} > 0 \text{ when } x > 0 \\ \frac{d^2y}{dx^2} < 0 \text{ when } x < 0\end{matrix}\right\}$ …(iv)

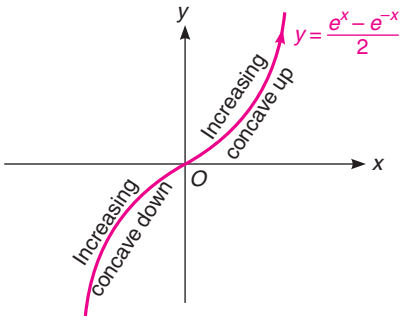


Fig. 2.226

From Eqs. (i), (ii), (iii) and (iv) graph of $y = \frac{e^x - e^{-x}}{2}$.

(iii) Sketching of $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

(a) $f(-x) = -f(x)$, $\therefore$ odd function or symmetric about origin …(i)

(b) as $x = 0 \Rightarrow y = 0$ …(ii)

(c) $\left.\begin{matrix}\text{Domain} \in R \\ \text{Range} \in (-1, 1)\end{matrix}\right\}$ …(iii)

(d) $\frac{dy}{dx} > 0$ for all $x \in R$ …(iv)

(e) $\left.\begin{aligned}\frac{d^2y}{dx^2} &< 0 \quad \text{for } x > 0\\ \frac{d^2y}{dx^2} &> 0 \quad \text{for } x < 0\end{aligned}\right\}$ ...(v)

From Eqs. (i), (ii), (iii), (iv) and (v) graph of $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ is shown as in Fig. 2.227.

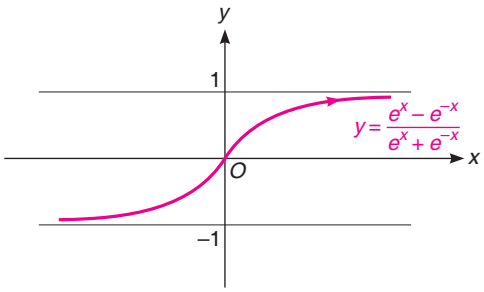


Fig. 2.227

(iv) Sketching $y = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

(a) $f(-x) = -f(x)$, $\therefore$ odd function or symmetric about origin ...(i)

(b) $\left.\begin{aligned}&\text{as } x \longrightarrow 0;\ y \longrightarrow \infty\\ &\text{as } y \longrightarrow 1;\ x \longrightarrow \infty\\ &y \longrightarrow -1;\ x \longrightarrow -\infty\end{aligned}\right\}$ ...(ii)

(c) $\left.\begin{aligned}&\text{Domain} \in R - \{0\}\\ &\text{Range} \in R - [-1, 1]\end{aligned}\right\}$ ...(iii)

(d) $\frac{dy}{dx} < 0$ for all $x \in R - \{0\}$ ...(iv)

(e) $\left.\begin{aligned}\frac{d^2y}{dx^2} &> 0 \quad \text{for } x > 0\\ \frac{d^2y}{dx^2} &< 0 \quad \text{for } x < 0\end{aligned}\right\}$ ...(v)

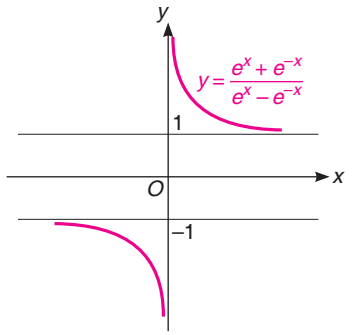


Fig. 2.228

From above information, graph for $y = \frac{e^x + e^{-x}}{e^x - e^{-x}}$ as shown in Fig. 2.228.

Note In many applications we come across exponential functions of the form $\frac{1}{2}(e^x - e^{-x})$ and $\frac{1}{2}(e^x + e^{-x})$ known as "hyperbolic functions" represented by;

$$\left.\begin{aligned}\sin hx &= \frac{e^x - e^{-x}}{2}\\ \cos hx &= \frac{e^x + e^{-x}}{2}\end{aligned}\right\} \quad ...(i)$$

called hyperbolic sine and hyperbolic cosine.

also,
$$\left.\begin{aligned}\tan hx &= \frac{e^x - e^{-x}}{e^x + e^{-x}}\\ \cot hx &= \frac{e^x + e^{-x}}{e^x - e^{-x}}\end{aligned}\right\} \quad ...(ii)$$

called hyperbolic tangent and hyperbolic cotangent. Where if; $x = \cos ht$, $y = \sin ht$

$\Rightarrow$ $\mathbf{x^2 - y^2 = \cos^2 ht - \sin^2 ht = 1}$ {Using Eq. (i)}

which is equation of hyperbola.

Thus, $\sin hx = \frac{e^x - e^{-x}}{2}$ and $\cos hx = \frac{e^x + e^{-x}}{2}$ are hyperbolic functions.

EXAMPLE 18 *Sketch the curve* $y = 2\sin x + \cos 2x$.

SOLUTION Since, the function is periodic with period 2π, it is sufficient to investigate the function in the interval $[0, 2\pi]$.

(a) $$\frac{dy}{dx} = 2\cos x - 2\sin 2x = 2(\cos x - 2\sin x \cos x)$$

$\therefore$ $$\frac{dy}{dx} = 2\cos x(1 - 2\sin x)$$

Here, $\frac{dy}{dx} = 0 \Rightarrow x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$

(b) $$\frac{d^2y}{dx^2} = -2\sin x - 4\cos 2x$$

$$\left(\frac{d^2y}{dx^2}\right)_{\text{at } x=\frac{\pi}{6}} = -3 < 0 \qquad \text{where} \quad y = \frac{3}{2}$$

$\therefore$ at $x = \frac{\pi}{6}$ is maximum at $x = \frac{\pi}{6}$. $\left(\frac{d^2y}{dx^2}\right)_{\text{at } x=\frac{\pi}{2}} = 2 > 0$;

$\therefore$ at $x = \frac{\pi}{2}$ is minimum at $x = \frac{\pi}{2}$, where $y = 1$

at $x = \frac{5\pi}{6}$ we have, $\frac{d^2y}{dx^2} = -3 < 0$

and $y = \frac{3}{2}$ (maximum)

at $x = \frac{3\pi}{2}$, we have, $\frac{d^2y}{dx^2} = 6 > 0$

and $y = -3$ (minimum)

Thus, curve for $y = 2\sin x + \cos 2x$;

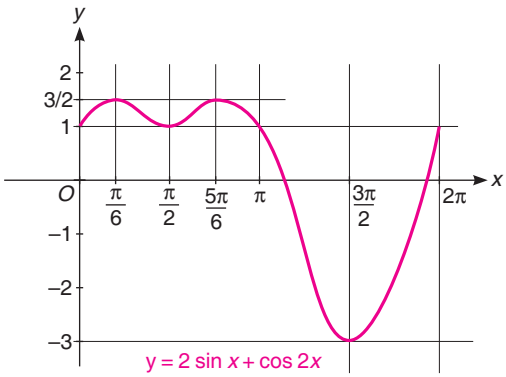


Fig. 2.229

EXAMPLE 19 *Sketch the curve* $y = e^{-x^2}$ *(Gaussian curve)*

SOLUTION As the curve $y = f(x) = e^{-x^2}$

(a) Symmetric about *y*-axis $\{\text{as } f(-x) = f(x)\}$ …(i)

(b) $$\left.\begin{array}{l} \text{as } x \longrightarrow 0 \Rightarrow y \longrightarrow 1 \\ \text{as } y \longrightarrow 0 \Rightarrow x \longrightarrow \pm\infty \end{array}\right\} \quad \ldots(ii)$$

(c) $$\frac{dy}{dx} = e^{-x^2}(-2x) \Rightarrow \left.\begin{array}{l} \frac{dy}{dx} < 0 \Rightarrow x > 0 \\ \frac{dy}{dx} > 0 \Rightarrow x < 0 \end{array}\right\} \quad \ldots(iii)$$

(d) $\dfrac{d^2y}{dx^2} = -2e^{-x^2} - 2x(-2xe^{-x^2}) = 2e^{x^2}(2x^2 - 1)$

$\Rightarrow$ $\left.\begin{aligned} &\dfrac{d^2y}{dx^2} > 0 \Rightarrow x < -\dfrac{1}{\sqrt{2}} \text{ or } x > \dfrac{1}{\sqrt{2}} \\ &\dfrac{d^2y}{dx^2} < 0 \Rightarrow -\dfrac{1}{\sqrt{2}} < x < \dfrac{1}{\sqrt{2}} \end{aligned}\right\}$ ...(iv)

(e) $y = e^{-x^2}$ assumes maximum at

$x = 0 \Rightarrow y = 1$

and Domain $\in R$
Range $\in (0, 1]$ ...(v)

From above discussion;

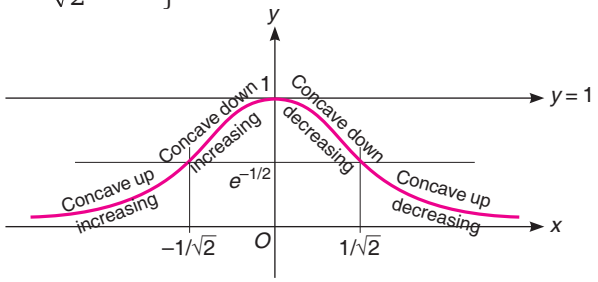


Fig. 2.230

EXAMPLE 20 *Sketch the curve* $y = \dfrac{x}{1 + x^2}$.

SOLUTION Here; (a) $\left.\begin{aligned} &\text{Domain} \in R \\ &\text{Range} \in [-0.5, \ 0.5] \end{aligned}\right\}$ ...(i)

(b) $f(-x) = -f(x)$, hence, $f(x)$ is odd, symmetric about origin ...(ii)

(c) When $x = 0 \Rightarrow y = 0$ ...(iii)

(d) $\dfrac{dy}{dx} = \dfrac{1 - x^2}{(1 + x^2)^2}$

Fig. 2.231

or $\left.\begin{aligned} &\dfrac{dy}{dx} > 0 \text{ when } -1 < x < 1 \\ &\dfrac{dy}{dx} < 0 \text{ when } x < -1 \text{ or } x > 1 \end{aligned}\right\}$ ...(iv)

From above maximum at $x = 1$ and minimum at $x = -1$} ...(v)

(e) $\dfrac{d^2y}{dx^2} = \dfrac{2^x(x^2 - 3)}{(1 + x^2)^3}$

Fig. 2.232

$\Rightarrow$ $\left.\begin{aligned} &\dfrac{d^2y}{dx^2} > 0, \text{ when } x \in (-\sqrt{3}, 0) \text{ and } (\sqrt{3}, \infty) \\ &\dfrac{d^2y}{dx^2} < 0, \text{ when } x \in (-\infty, -\sqrt{3}) \text{ and } (0, \sqrt{3}) \end{aligned}\right\}$ ...(vi)

From above conditions graph for $y = \dfrac{x}{1 + x^2}$

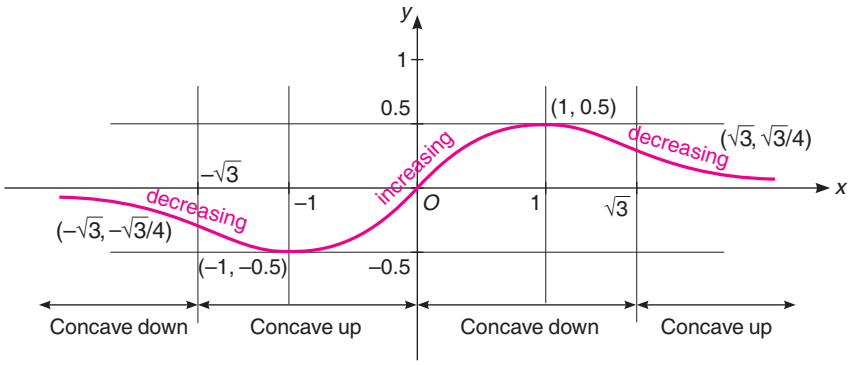


Fig. 2.233

In above figure :

x	dy/dx	d^2y/dx^2
$-\infty < x < -\sqrt{3}$	−	−
$-\sqrt{3} < x < -1$	−	+
$-1 < x < 0$	+	+
$0 < x < 1$	+	−
$1 < x < \sqrt{3}$	−	−
$\sqrt{3} < x < \infty$	−	+

EXAMPLE 21 *Sketch the curve* $y^2 = x^3$. *(Semicubical parabola).*

SOLUTION To plot the curve we discuss;

(i) Symmetric about x-axis (as $y \longrightarrow -y$ curve remains same).

(ii) Domain $\in [0, \infty)$

(iii) Range $\in$ R.

Here, to plot $y^2 = x^3 \Rightarrow y = \sqrt{x^3}$ and $y = -\sqrt{x^3}$, we draw $y = \sqrt{x^3}$ and take image about x-axis for $y = -\sqrt{x^3}$. (*i.e.*, to discuss curve when $x, y \geq 0$).

(iv) $\dfrac{dy}{dx} = \dfrac{3}{2}x^{1/2} = \dfrac{3}{2}\sqrt{x} > 0$ for all x, $y > 0$.

(v) $\dfrac{d^2y}{dx^2} = \dfrac{3}{4\sqrt{x}} > 0$ for all x, $y > 0$.

$\Rightarrow$ Increasing and concave up when $x > 0$.

From Eqs. (i), (ii), (iii), (iv) and (v).

x	y	dy/dx	d^2y/dx^2
+	+	+	+
+	−	−	−

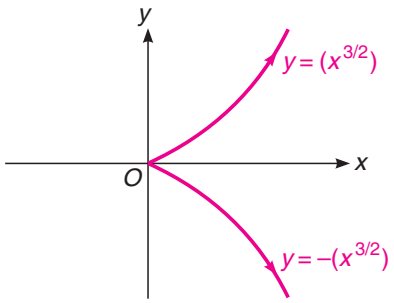


Fig. 2.234

Note In above curve $y^2 = x^3$ (semicubical parabola). For $x = 0$ we have $y = 0$ and $y' = 0$, thus, the branch of the curve has a tangent at $y = 0$ at origin. The second branch of $y = -\sqrt{x^3}$ also passes through origin and has the same tangent $y = 0$. Thus, two different branches of the curve meet at origin, have the same tangent, and situated at different sides of origin. This is known **as cusp of first kind.**

EXAMPLE 22 *Sketch the curve:* $y^2 = x^4 - x^6$.

SOLUTION Here; $y = \pm x^2\sqrt{1 - x^2}$

Thus, to plot the curve for $y = x^2\sqrt{1 - x^2}$ and take image about x-axis.

(i) Symmetric about x and y-axis.

(ii) Domain $\in [-1, 1]$

(iii) Range $\in \left[-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right]$.

(iv) When $x = 0 \Rightarrow y = 0$

$y = 0 \Rightarrow x = 0, \pm 1.$

(v) $2y\frac{dy}{dx} = 4x^3 - 6x^5 = 2x^3(2 - 3x^2) = 2x^3(\sqrt{2} - \sqrt{3}x)(\sqrt{2} + \sqrt{3}x)$

$\Rightarrow$ $\frac{dy}{dx} > 0$ when; $0 < x < \sqrt{\frac{2}{3}}$ and $y > 0$.

$\frac{dy}{dx} < 0$ when $\sqrt{\frac{2}{3}} < x < 1$ and $y > 0$.

Here, we are sketching the curve only when $x, y \geq 0$ and then take image about x and y-axis.

(vi) $\frac{d^2y}{dx^2} > 0$ when $0 < x < \sqrt{\frac{2}{5}}$ and $y > 0$.

$\frac{d^2y}{dx^2} < 0$ when $\sqrt{\frac{2}{5}} < x < 1$ and $y > 0$.

From above discussion.

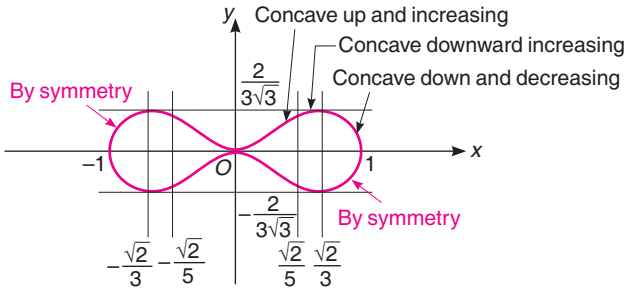


Fig. 2.235

Note At the origin (as the singular point) the two branches of the curve corresponding to plus and minus in front of the radical sign are mutually tangent. **Known as point of osculation or tacnode or double cusp**.

EXERCISE

1. Sketch the curves; (where $[\cdot]$ denotes the greatest integer function).

(i) $y = |2 - |x - 1||$

(ii) $y = 2 - \dfrac{4}{|x-1|}$

(iii) $|y| = 2 - \dfrac{4}{|x-1|}$

(iv) $y = \left| 2 - \dfrac{4}{|x-1|} \right|$

(v) $|y| = \left| 2 - \dfrac{4}{|x-1|} \right|$

(vi) $y = |e^{|x|} - 2|$

(vii) $|y| = |e^{|x|} - 2|$

(viii) $y = x - [x]$

(ix) $y = \sqrt{x - [x]}$

(x) $y = (x - [x])^2$

(xi) $|y| = \sqrt{x - [x]}$

(xii) $|y| = (x - [x])^2$

(xiii) $y = |x - 1| + |x + 1|$

(xiv) $|y| = |x - 1| + |x + 1|$

(xv) $y = [|x - 1|]$

(xvi) $|y| = [|x - 1|]$

(xvii) $y = x + [x]$

(xviii) $y = |x| + [|x|]$

(xix) $|y| = x + [x]$

(xx) $|y| = |x| + [|x|]$

2. Sketch the curves;

(i) $y = \sqrt{\sin x}$

(ii) $|y| = \sqrt{\sin x}$

(iii) $y = |\sin x| + |\cos x|$

(iv) $|y| = \cos x + |\cos x|$

(v) $y = \sin^2 x - 2\sin x$

(vi) $y = 2^{\sin x}$

(vii) $y = \log_2(|\sin x|)$

(viii) $|y| = \log_2|\sin x|$

(ix) $y = \log_{\sin x}\left(\dfrac{1}{2}\right)$

(x) $|y| = \log_{\sin x}\left(\dfrac{1}{2}\right)$

3. Sketch the curve $y = \sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

4. Sketch the curve $y = \dfrac{2x}{x^2+1}$

5. Sketch the curves;

(i) $y = x^2 - 2|x|$ (ii) $y = e^{-|x|}$

(iii) $y = e^{|x|}$ (iv) $|y| = x$

(v) $y = x^3 - x$ (vi) $y^2 = x - 1$

6. Construct the graph of the function

$$y = f(x-1) + f(x+1)$$

where $f(x) = \begin{cases} 1 - |x|, & \text{when } |x| \le 1 \\ 0, & \text{when } |x| > 1 \end{cases}$

7. Sketch the curves;

(i) $y = \dfrac{1}{x-2}$ (ii) $y = \dfrac{1}{|x|-2}$

(iii) $y = \left|\dfrac{1}{|x|-2}\right|$ (iv) $|y| = \left|\dfrac{1}{|x|-2}\right|$

8. Find number of solutions of $2\cos x = |\sin x|$. When $x \in [0, 4\pi]$.

9. Find the number of solutions of;

$$\sin \pi x = |\log|x||.$$

10. Sketch the curve

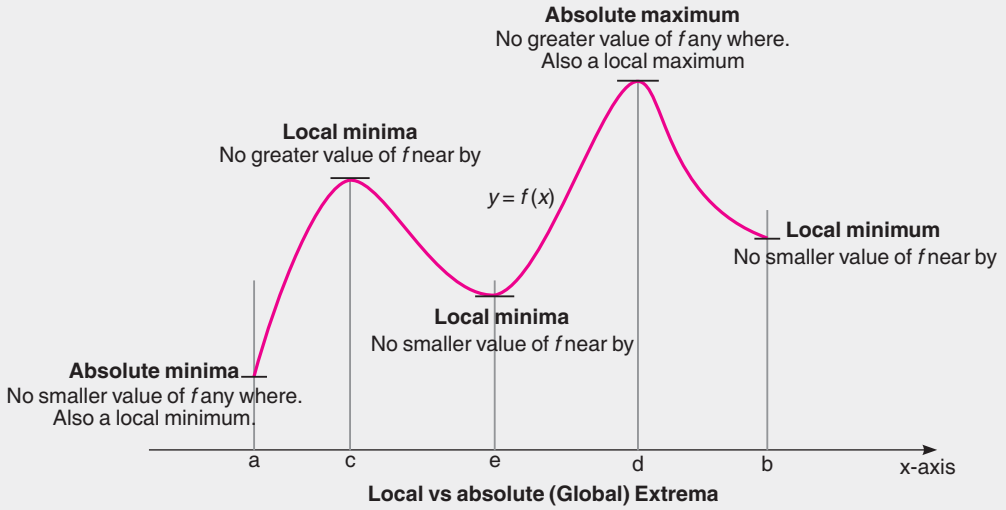
$$y = \frac{\sin 2x}{2} + \cos x.$$

ANSWERS

8. 2 solutions, **9.** 6 solutions.

Graph Jells Us

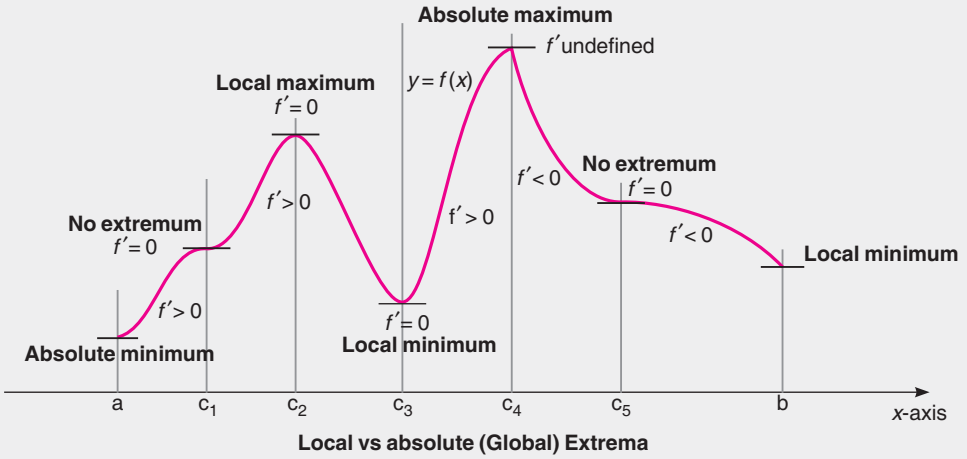
Remark 1



Local vs absolute (Global) Extrema

Fig. 1

Remark 2



Local vs absolute (Global) Extrema

Fig. 2

Remark 3

The First derivative Test for Local Extreme Value

(i) If f′ changes from positive to negative at C (f′ > 0 for x < c and f′ < 0 for x > c), then f has **a local maximum value at c.**

Fig. 3

(ii) If f′ changes from negative to positive at c (f′ < 0 for x < c and f′ > 0 for x > c), then f has **a local minimum at** x = c.

Fig. 4

(iii) If f′ does not change sign at c (f′ has the same sign on both sides of c), then f has **no local extreme value at c.**

Fig. 5

(iv) At a left end point 'a': If f′ < 0 (f′ > 0) for x > a, then f has **local maximum (minimum) value at** x = a.

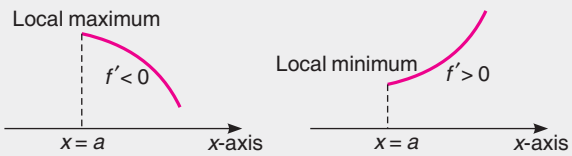


Fig. 6

(v) At a right end point 'b': If f′ < 0 (f′ > 0) for x < b, then f has **local minimum (maximum) at** x = b.

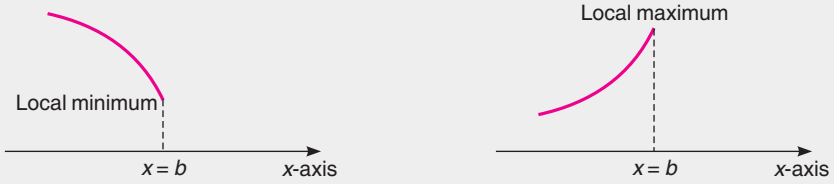


Fig. 7

Chapter 3

ASYMPTOTES, SINGULAR POINTS AND CURVE TRACING

In this chapter we shall study:

- Plotting of Rational and Irrational functions.
- Intersection of Curve and Straight line at Infinity.

3.1 ASYMPTOTES

A straight line, at a finite distance from origin, is said to be an asymptote of the curve $y = f(x)$, if the perpendicular distance of the point P on the curve from the line tends to zero when x or y both tends to infinity.

OR

A straight line A is called an asymptote to a curve, if the distance δ from the variable point M of the curve to this straight line approaches zero as the point M tends to infinity. Shown as:

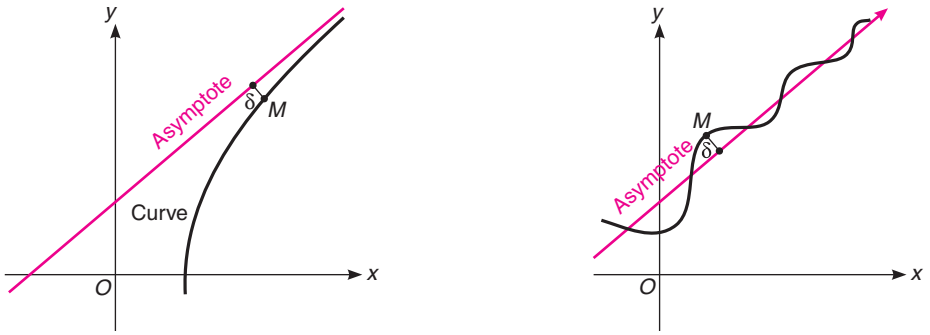


Fig. 3.1

Mathematically

Let $y = f(x)$ be a curve and let (x, y) be a point on it.

Tanget at (x, y) is given by;

$$Y - y = \frac{dy}{dx}(X - x)$$

$$Y = \frac{dy}{dx} \cdot X + \left(Y - x\frac{dy}{dx}\right) \qquad \ldots(i)$$

Now, if asymptote exists, then $x \to \infty$

$\Rightarrow$ $\frac{dy}{dx}$ and $\left(y - x\frac{dy}{dx}\right) \longrightarrow$ finite limit say m and c

Say $\frac{dy}{dx} \to m$ and $y - \frac{x\,dy}{dx} \to c$

$\therefore$ Eq. (i) reduces to, $Y = mX + c$ is asymptote of equation.

Now we shall discuss the following cases

(i) Asymptote parallel to x-axis.
(ii) Asymptote parallel to y-axis.
(iii) Asymptote of algebraic curves or oblique asymptotes.
(iv) Asymptote by inspection.
(v) Intersection of curve and its Asymptotes.
(vi) Asymptote by Expansion.
(vii) The position of the curve with respect to asymptote.

3.1 (i) Asymptote parallel to x-axis

Let the equation of curve be,

$$(a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \ldots + a_ny^n) + (b_1x^{n-1} + b_2x^{n-2}y + \ldots + b_ny^{n-1}) + (c_2x^{n-2} + c_3x^{n-2}y + \ldots + c_ny^{n-2}) + \ldots + \ldots = 0 \quad \ldots(i)$$

then it can be arranged in descending powers of x as follows:

$$a_0x^n + (a_1y + b_1)x^{n-1} + (a_2y^2 + b_2y + c_2)x^{n-2} + \ldots = 0 \quad \ldots(ii)$$

Now, if $a_0 = 0$, *i.e.*, the term consisting x^n is absent, then $a_1y + b_1 = 0$, *i.e.*, coefficient of $x^{n-1} = 0$ will make two roots of Eq. (i) infinite as coefficients of both x^n and x^{n-1} are zero.

Hence, $a_1y + b_1 = 0$ **is an asymptote parallel to *x*-axis.**

Again if; both x^n and x^{n-1} are absent, then $a_2y^2 + b_2y + c_2 = 0$, *i.e.*, coefficient of x^{n-2} being zero will make three roots of Eq. (ii) infinite hence, $a_2y^2 + b_2y + c_2 = 0$ **will give two asymptote parallel to *x*-axis.**

Method to find asymptote parallel to x-axis

To find the asymptote parallel to x-axis equate the coefficient of highest power of x to zero.

If the coefficient is constant, then there is no asymptote parallel to x-axis (horizontal).

3.1 (ii) Asymptote parallel to y-axis

From above article, if we need an asymptote parallel to y-axis, equate the coefficient of highest power of y to zero.

If this coefficient is constant, then there is no asymptote parallel to y-axis (vertical).

EXAMPLE 1 *Sketch the curve* $y = \frac{1}{x - 5}$

SOLUTION Here; $y(x - 5) = 1$

$\therefore$ **Asymptote parallel to *x*-axis.**

$\Rightarrow$ $y = 0$ (equating highest power of $x = 0$)

Asymptote parallel to *y*-axis.

$\Rightarrow$ $x = 5$ (equating highest power of $y = 0$)

Thus, $x = 5$ and y-axis are asymptotes shown as in figure.

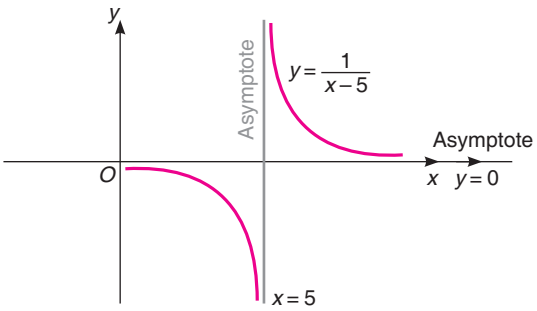


Fig. 3.2

EXAMPLE 2 *Show the curve* $y = \tan x$ *has an infinite number of vertical asymptote.*

SOLUTION $y = \tan x$

here $y \to \pm\infty$ as $x \to \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

or $\tan x \to \infty$ as $x \to \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

i.e., equating highest power of $y = 0$.

(as $y = \tan x \Rightarrow y \cot x = 1$, where $\cot x \to 0$).

Shown as:

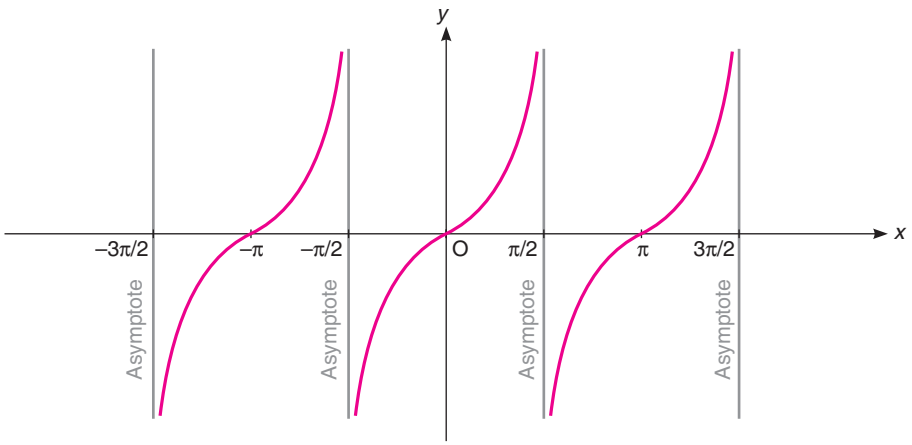


Fig. 3.3

EXAMPLE 3 *Show the curve* $y = e^{1/x}$ *has a vertical and horizontal asymptote.*

SOLUTION Here $y = e^{1/x}$

$\Rightarrow$ $y \cdot e^{-1/x} = 0$

or $e^{-1/x} \to 0$ as $x \to 0$ (Since, $\lim\limits_{x\to 0} e^{-1/x} \to 0$)

From adjoining figure

$$y = e^{1/x}$$

$$\Rightarrow \quad \frac{1}{x} = \log y$$

$$\Rightarrow \quad x = \frac{1}{\log y}$$

which shows $x(\log y) = 1$ has an asymptote parallel to x-axis as

$$\log y = 0 \quad \Rightarrow \quad y = 1.$$

Thus, $y = e^{1/x}$ has two asymptote

$$x = 0 \quad \text{and} \quad y = 1.$$

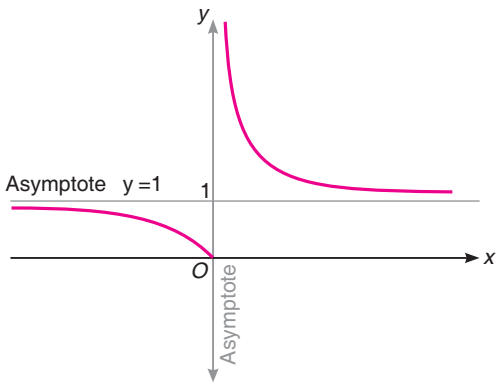


Fig. 3.4

3.1 (iii) Asymptote of algebraic curves or oblique asymptote

An asymptote which is not parallel to y-axis is called an oblique asymptote. Let $y = mx + c$ be an asymptote of $y = f(x)$, then

$$m = \lim_{\substack{x\to\infty \\ \text{or } x\to-\infty}} \frac{y}{x} \quad \text{and} \quad c = \lim_{\substack{x\to\infty \\ \text{or } x\to-\infty}} (y - mx)$$

Method to find oblique asymptote

Suppose $y = mx + c$ is an asymptote of the curve. Put $y = mx + c$ in the equation of the curve and arrange it in descending powers of x. Equate to zero the coefficients of two highest degree terms. Solve these two equations, find m and c. Put them in $y = mx + c$ to get asymptotes.

Note 1. Here, we will find non-parallel or non repeated asymptote only.
2. Neglect all imaginary values of m.

EXAMPLE 1 *Find the asymptotes to the curve* $y = x + \frac{1}{x}$ *and then sketch.*

SOLUTION Here, the given curve $y = x + \frac{1}{x}$

$$\Rightarrow \quad xy = x^2 + 1$$

or $$x^2 - xy + 1 = 0$$

(i) Asymptote parallel to x-axis

Equating highest power coefficient of x to zero in $x^2 - xy + 1 = 0$

$$\Rightarrow \quad 1 = 0 \qquad \text{(which is not true)}$$

$\therefore$ no asymptote parallel to x-axis.

(ii) Asymptote parallel to y-axis

Equating highest power coefficient of y to zero in

$$x^2 - xy + 1 = 0$$

$\Rightarrow$ $$-x = 0$$

or $$x = 0$$ (*i.e.*, *y*-axis) is asymptote for $y = x + \frac{1}{x}$

(iii) Oblique asymptote

Let $y = mx + c$ in $x^2 - xy + 1 = 0$

i.e., $$x^2 - mx^2 - xc + 1 = 0$$

$\Rightarrow$ $$x^2(1 - m) - (c)x + 1 = 0$$

Equating highest and second highest power of *x* to zero

i.e., $$1 - m = 0 \quad \text{and} \quad c = 0$$

$\therefore$ $$m = 1 \quad \text{and} \quad c = 0$$

or $$y = x$$

is oblique asymptote to $y = x + \frac{1}{x}$.

Now to trace the curve;

(iv) Symmetric about origin (as odd function)

(v) Domain $\in R - \{0\}$.

(vi) Range $\in (-\infty, -2] \cup [2, \infty)$

(vii) $\frac{dy}{dx} = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}$ {using number line rule,

}

$$\frac{dy}{dx} > 0, \quad \text{when} \quad x < -1 \quad \text{or} \quad x > 1$$

$$\frac{dy}{dx} < 0, \quad \text{when} \quad -1 < x < 1 - \{0\}$$

which shows y_{max} at $x = -1$

y_{min} at $x = 1$

(viii) Also, $$\frac{d^2y}{dx^2} = \frac{2}{x^3}$$

$\Rightarrow$ $$\frac{d^2y}{dx^2} > 0, \quad \text{when} \quad x > 0 \qquad \text{(concave up)}$$

$$\frac{d^2y}{dx^2} < 0, \quad \text{when} \quad x < 0 \qquad \text{(concave down)}$$

Using above information we can trace $y = x + \frac{1}{x}$ as:

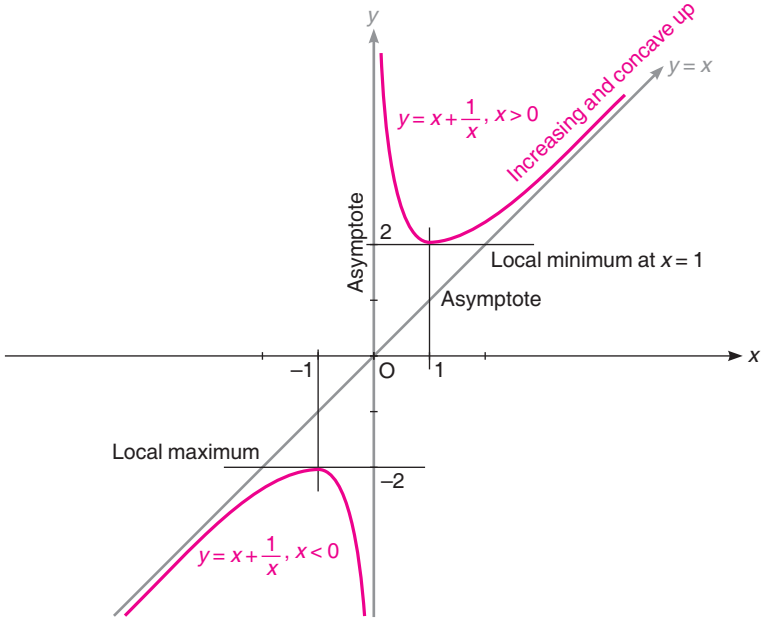


Fig. 3.5

EXAMPLE 2 *Find the asymptotes of the curve* $y = \frac{x^2 + 2x - 1}{x}$ *and hence, sketch.*

SOLUTION Here, the curve $y = \frac{x^2 + 2x - 1}{x}$ could be written as;

$$x^2 + 2x - yx - 1 = 0 \qquad \ldots(i)$$

(i) No asymptote parallel to x-axis.

(ii) Asymptote parallel to y-axis. $\Rightarrow$ $x = 0$.

(iii) Oblique asymptote

Let $y = mx + c$ be oblique asymptote

$\therefore$ $x^2 + 2x - x(mx + c) - 1 = 0$

$x^2 - mx^2 + 2x - cx - 1 = 0$

$\Rightarrow$ $x^2(1 - m) + x(2 - c) - 1 = 0$

For oblique asymptote equate highest power and second highest power of x to zero.

i.e., Coefficient of $x^2 = 0 \Rightarrow m = 1$

Coefficient of $x = 0 \Rightarrow c = 2$

$\therefore$ $y = x + 2$ is oblique asymptote to $y = x - \frac{1}{x} + 2$.

(iv) Neither symmetric about axis nor about origin.

(v) Domain $\in R - \{0\}$.

(vi) Range $\in R$.

(vii) $$\frac{dy}{dx} = 1 + \frac{1}{x^2}$$

$$\Rightarrow \quad \frac{dy}{dx} > 0, \quad \text{for all} \quad x \in R - \{0\}.$$

(viii) $$\frac{d^2y}{dx^2} = -\frac{2}{x^3}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} > 0, \quad \text{when} \quad x < 0 \qquad \textbf{(concave down)}$$

$$\frac{d^2y}{dx^2} < 0, \quad \text{when} \quad x > 0 \qquad \textbf{(concave up)}$$

Using above information, we can plot the curve $y = x - \frac{1}{x} + 2$ as;

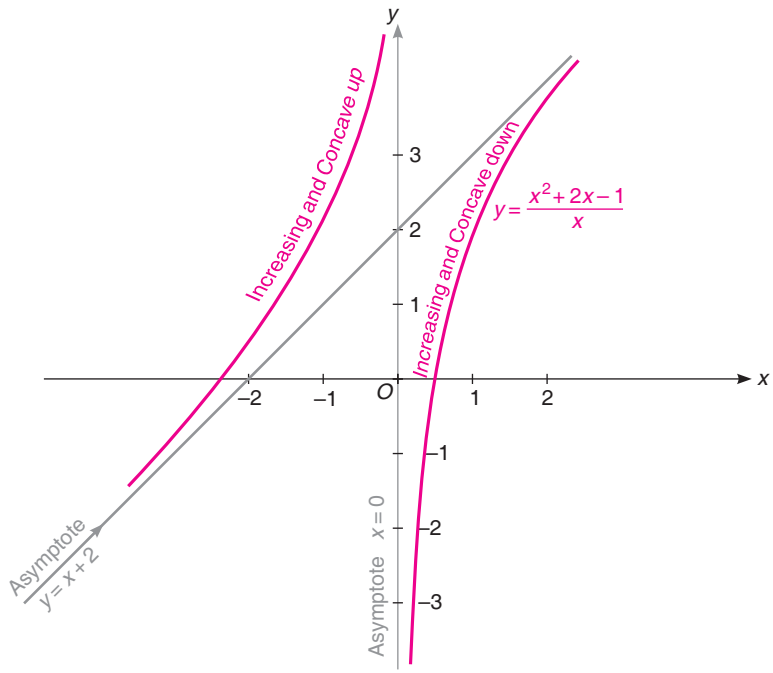


Fig. 3.6

3.1 (iv) Asymptote by inspection

If the equation of the curve be of the form $F_n + F_{n-2} = 0$, where F_n and F_{n-2} are expressions in x and y such that degree of $F_n = n$ and degree of $F_{n-2} \leq n - 2$, then every linear factor equated to zero will give an asymptote if no two straight lines represented by any other factor of F_n is parallel or coincident with it.

EXAMPLE 1 *Find the asymptote of the curve* $x^2y + xy^2 = a^3$.

SOLUTION Here, the given curve is.

$$x^2y + xy^2 = a^3 \quad \text{or} \quad x^2y + xy^2 - a^3 = 0$$

This equation is of the form $F_n + F_{n-2} = 0$

Here, $F_3 = x^2y + xy^2$ and $F_0 = -a^3$

$\therefore$ By inspection the asymptotes are given by

$$x^2y + xy^2 = 0 \quad \text{or} \quad xy(x + y) = 0$$

$\therefore$ The asymptotes are $x = 0$, $y = 0$, $x + y = 0$.

EXAMPLE 2 *Find the asymptote of the curve* $y = x + \frac{1}{x}$ *(by Inspection).*

SOLUTION Here, the given curve is $x^2 - xy + 1 = 0$

This equation is of the form $F_n + F_{n-2} = 0$

Here $F_2 = x^2 - xy$

$F_0 = 1$

$\therefore$ By inspection the asymptotes are given by

$$x^2 - xy = 0 \quad \text{or} \quad x(x - y) = 0$$

$\therefore$ The asymptotes are $x = 0$ and $x - y = 0$.

3.1 (v) Intersection of curve and its asymptote

An asymptote of curve of *n*th degree cut the curve in $(n - 2)$ points provided the asymptote is not parallel to any asymptote.

Hence, if there be *N* asymptotes of the curve, then they cut the curve in $N(n - 2)$ points.

Note The number of asymptotes of an algebraic curve of *n*th degree can not be more than *n*.

EXAMPLE 1 *Show the asymptote of the curve* $xy(x^2 - y^2) + x^2 + y^2 - 1 = 0$ *cut at 8 points.*

SOLUTION The equation of the curve is,

$$xy(x^2 - y^2) + x^2 + y^2 - 1 = 0 \quad \ldots(i)$$

Here $n = 4$

This equation is of the type $F_n + F_{n-2} = 0$

Hence, $F_n = xy(x^2 - y^2) = xy(x - y)(x + y)$

and $F_{n-2} = x^2 + y^2 - 1$

$\therefore$ $F_n = 0$

$\Rightarrow$ $x = 0$, $y = 0$, $x - y = 0$ and $x + y = 0$ are the equations of asymptotes.

The combined equation of the asymptotes is,

$$xy(x - y)(x + y) = 0 \quad \ldots(ii)$$

Subtracting Eq. (ii) from (i), we get

$$x^2 + y^2 - 1 = 0$$

Thus, intersection of curve and asymptotes lie on this curve since, there are 4 asymptotes, *i.e.*, N = 4.

∴ **Point of intersection of curve and asymptotes** = 4(4 – 2) = 8.

3.1 (vi) Asymptote by expansion

If the equation of the curve is of the form

$$y = mx + c + \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \ldots$$

Then $y = mx + c$ will be an asymptote of the given curve.

EXAMPLE 1 *Find the asymptote of the curve* $y^3 = x^2(x - a)$.

SOLUTION The curve is, $y^3 = x^2(x - a) = x^3\left(1 - \frac{a}{x}\right)$

$$\Rightarrow \quad y = x\left(1 - \frac{a}{x}\right)^{1/3} \quad \text{or} \quad y = x\left(1 - \frac{1}{3}\frac{a}{x} - \frac{1}{9}\frac{a^2}{x^2}\ldots\right)$$

or $$y = x - \frac{a}{3} - \frac{1}{9}\frac{a^2}{x}\ldots$$ which is of the form

$$y = mx + c + \frac{A}{x} + \frac{B}{x^2} + \ldots$$

$$\Rightarrow \quad y = mx + c$$ is asymptote

Hence, $y = x - \frac{a}{3}$ **is asymptote of the given curve.**

EXAMPLE 2 *Find the asymptote for* $y = x + \frac{1}{x}$.

SOLUTION Here; $y = x + \frac{1}{x}$ is of the form,

$$y = mx + c + \frac{A}{x} + \frac{B}{x^2} + \ldots$$

$\Rightarrow$ $y = x$ is asymptote of the curve

$$y = x + \frac{1}{x}.$$

Note Above method is useful to find oblique asymptote. Thus, students are adviced to find vertical and horizontal asymptote (*i.e.*, asymptote parallel to *x*-axis and *y*-axis).

3.1 (vii) The position of the curve with respect to an asymptote

Let the equation of the curve is of the form;

$$y = mx + c + \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \ldots, \text{ then}$$

(a) The curve lies above the asymptote if

(i) $A \neq 0$ and, A and x have same signs

or (ii) $A = 0,\ B > 0$

or (iii) $A = 0,\ B = 0,\ C \neq 0$ and C and x have same signs and

(b) The curve lies below the asymptote if

(i) $A \neq 0$ and, A and x have opposite signs.

or (ii) $A = 0,\ B < 0$

or (iii) $A = 0,\ B = 0,\ C \neq 0$ and C and x have opposite signs.

EXAMPLE 1 *For the curve* $y^5 = x^5 + 2x^4$*; show;*

(i) The curve lies above the asymptote $y = x + \frac{2}{5}$, *if* $x < 0$.

(ii) The curve lies below the asymptote $y = x + \frac{2}{5}$, *if* $x > 0$.

SOLUTION The given curve is, $y^5 = x^5 + 2x^4$

or $$y^5 = x^5\left(1 + \frac{2}{x}\right)$$

$$y = x\left(1 + \frac{2}{x}\right)^{1/5}$$

$$y = x\left(1 + \frac{2}{5}\cdot\frac{1}{x} - \frac{8}{25}\cdot\frac{1}{x^2}\ldots\right) = x + \frac{2}{5} - \frac{8}{25x} + \ldots$$

$\therefore$ The asymptote is $y = x + \frac{2}{5}$;

(i) Now if $A = -\frac{8}{25}$ and x have same sign $\Rightarrow$ $x < 0$. **Then the curve lie above the asymptote.**

(ii) Now if $A = -\frac{8}{25}$ and x have opposite sign $\Rightarrow$ $x > 0$. **Then the curve lie below the asymptote.**

EXAMPLE 2 *For the curve* $y = x + \frac{1}{x}$ *show,*

(i) The curve lies above the asymptote $y = x$, *if* $x > 0$

(ii) The curve lies below the asymptote $y = x$, *if* $x < 0$

SOLUTION The given curve is, $y = x + \frac{1}{x}$, is of the form

$$y = mx + c + \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3}\ldots$$

Thus, $y = x$ is the asymptote to $y = x + \frac{1}{x}$.

(i) Now if $A = 1$ and x have same sign $\Rightarrow x > 0$, **then the curve lies above the asymptote.**

(ii) Now if $A = 1$ and x have opposite sign $\Rightarrow x < 0$, **then the curve lies below the asymptote.**

3.2 SINGULAR POINTS

Here, we shall discuss the following

(i) Multiple points
(ii) Double points
Types of double points :
(a) Node (b) Cusp (c) Isolated point
(iii) Tangent at the origin.
(iv) Necessary conditions for existence of double points.
(v) Types of cusps.

3.2 (i) Multiple points

A point on a curve is said to be a multiple point of order r, if r branches of the curve pass through this point.

If P is the multiple point of order r, then there will be r tangents at P, one of each of the r branches. These r tangents may be real, imaginary, distinct, coincident.

3.2 (ii) Double points

A point on a curve is said to be a double point of the curve, if two branches of the curve pass through this point.

Double points have two tangents, they may be real, imaginary, distinct or coincident.

Types of Double points

(a) Node

If the two branches of a curve pass through the double point and the tangents to them at the point are real and distinct, then the double point is called **a node** as shown in Fig. 3.7.

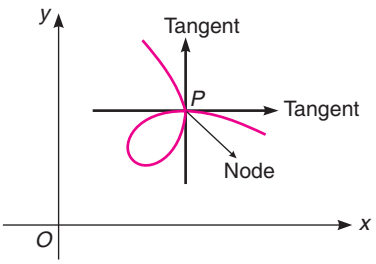


Fig. 3.7

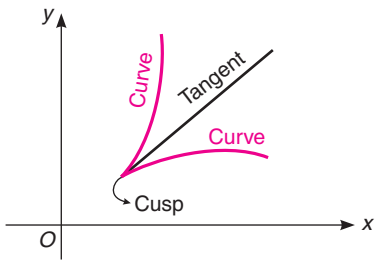


Fig. 3.8

(b) Cusp

If the two branches of the curve pass through the double point and the tangent to them are the point is real and coincident, then the double point is called **cusp** as shown in Fig. 3.8.

Cusps :

The graph of a continuous function $y = f(x)$ has a cusp at a point $x = c$ if the **concavity is same** on the both side of c and either.

1. $\lim\limits_{x \to c^-} f'(x) = \infty$ and $\lim\limits_{x \to c^+} f'(x) = -\infty$

OR

2. $\lim\limits_{x \to c^-} f'(x) = -\infty$ and $\lim\limits_{x \to c^+} f'(x) = \infty$ shown as:

1. $\lim\limits_{x \to c^-} f'(x) = \infty$ and $\lim\limits_{x \to c^+} f'(x) = -\infty$

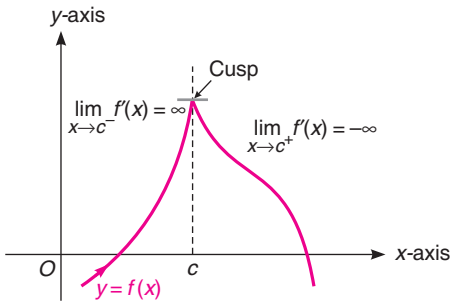


Fig. 3.9

2. $\lim\limits_{x \to c^-} f'(x) = -\infty$ and $\lim\limits_{x \to c^+} f'(x) = \infty$

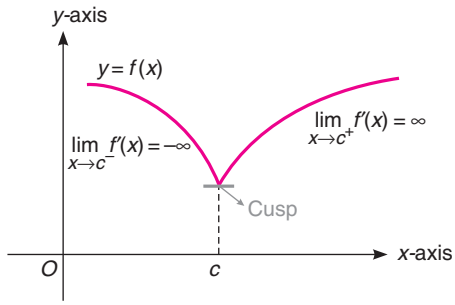


Fig. 3.10

Note A cusp can either be a local maximum (1) or a local minima as in (2).

(c) Isolated point

If there are no real point on the curve in the neighbourhood of a point P is called an **isolated or a conjugate point**.

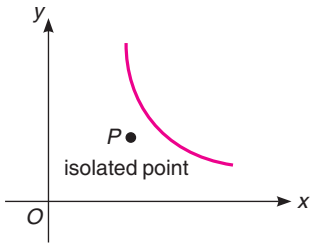


Fig. 3.11

3.2 (iii) Tangent at the origin

If an algebraic curve passes through the origin, the equation of tangent or tangents at the origin is obtained by equating to zero the lowest degree terms in the equation of the curve.

EXAMPLE 1 *Show that the curve* $y^2 = 4x^2 + 9x^4$ *has a node at origin and hence, sketch.*

SOLUTION The equation of the curve is,

$$y^2 = 4x^2 + 9x^4 \qquad \ldots(i)$$

It passes through the origin.
Now, equating to zero the lowest degree terms of the given curve, *i.e.*,

$$y^2 - 4x^2 = 0$$

$$\Rightarrow \quad y = 2x \quad \text{and} \quad y = -2x$$

There are two real and distinct tangents $y = 2x$ and $y = -2x$. Thus, two branches of curve passes through origin (0, 0).

$\therefore$ origin is node. …(ii)

Now to sketch;

(iii) Symmetric about x-axis, y-axis and origin.

(iv) As $x \to 0 \Rightarrow y \to 0$

(v) Domain $\in$ R.

(vi) Range $\in$ R.

Here, we shall discuss the behaviour of $y = x\sqrt{4 + 9x^2}$, $x \geq 0$ and use symmetry to construct $y^2 = x^2(4 + 9x^2)$.

(vii) $2y\dfrac{dy}{dx} = 8x + 36x^3 = 4x(2 + 9x^2)$

$$\Rightarrow \quad \frac{dy}{dx} > 0 \quad \text{for all} \quad x, y > 0$$

(viii) Also,

$$\left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} = 4 + 54x^2$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} > 0 \quad \text{for all } x$$

Thus, the graph for $y^2 = x^2(4 + 9x^2)$

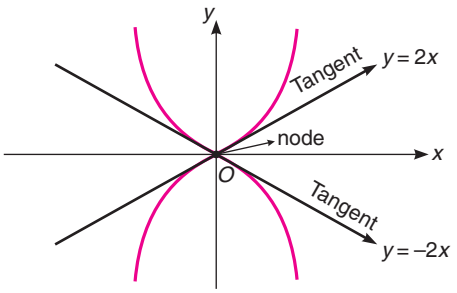


Fig. 3.12

EXAMPLE 2 *Show origin is a conjugate point for*

$$x^4 + y^3 + 2x^2 + 3y^2 = 0$$

SOLUTION The given curve is, $x^4 + y^3 + 2x^2 + 3y^2 = 0$

It passes through origin.

$\therefore$ To find equation of tangent at origin equating the lowest degree term to zero.

i.e., $2x^2 + 3y^2 = 0$

$$\Rightarrow \quad y = \pm i\sqrt{\frac{2}{3}}x$$

which are imaginary tangents.

Hence, **origin is a conjugate point of the curve.**

3.2 (iv) Necessary conditions for the existence of double points

Let (x, y) be a point on the given curve $f(x, y) = 0$.

The necessary and sufficient conditions for (x, y) to be a double points are:

$$f = 0, \quad \frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0 \quad \textbf{at} \quad (x, y)$$

Now, if $\frac{\partial^2 f}{\partial x\,\partial y}$, $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$ are not all zero, then,

(i) Double point will be a node if

$$\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2 - \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) > 0$$

or $$f_{xy}^2 - f_{xx}f_{yy} > 0$$

(ii) The double point will be an isolated point, if

$$f_{xy}^2 - f_{xx}f_{yy} < 0$$

(iii) The double point will be a cusp if

$$f_{xy}^2 - f_{xx}f_{yy} = 0.$$

Here, if $f_{xx} = f_{xy} = f_{yy} = 0$ at (x, y), then it will be a multiple point of order greater than 2.

EXAMPLE 1 *For the curve* $x^3 + x^2 + y^2 - x - 4y + 3 = 0$, *find the double point and hence, whether the point is node or isolated point.*

SOLUTION Let $f(x, y) = x^3 + x^2 + y^2 - x - 4y + 3 = 0$

$\therefore$ $f_x = 3x^2 + 2x - 1$

$f_y = 2y - 4$ for a double point $f_x = 0$, $f_y = 0$

$\therefore$ $f_x = 0 \Rightarrow 3x^2 + 2x - 1 = 0$

or $x = \frac{1}{3}, -1$

$f_y = 0 \Rightarrow 2y - 4 = 0 \Rightarrow y = 2.$

$\therefore$ Possible double points are $\left(\frac{1}{3}, 2\right)$, $(-1, 2)$

$\because$ $f\left(\frac{1}{3}, 2\right) \neq 0$ and $f(-1, 2) = 0$

$\therefore$ $f(-1, 2)$ **is a double point.**

$f_{xx} = 6x + 2 \Rightarrow f_{xx}$ at $(-1, 2) = -4$

$f_{xy} = 0 \Rightarrow f_{xy}$ at $(-1, 2) = 0$

$f_{yy} = 2 \Rightarrow f_{yy}$ at $(-1, 2) = 2$

$\therefore$ $f_{xy} - f_{xx}f_{yy} = 0 - (-4)(2) = 8 > 0$

$\therefore$ $(-1, 2)$ **may be node**.

For shifting origin to $(-1, 2)$, substitute $x = X - 1$, $y = Y + 2$ in the given equation, we get, $X^3 - 2X^2 + Y^2 = 0$

or $Y = \pm X\sqrt{2 - X}$

For numerically small values of X, Y is real.

$\therefore$ $(-1, 2)$ **is a node on the given curve**.

EXAMPLE 2 *For the curve* $x^3 + 2x^2 + 2xy - y^2 + 5x - 2y = 0$, *find the double point and hence, check whether node, cusp or isolated point.*

SOLUTION Let $f(x, y) = x^3 + 2x^2 + 2xy - y^2 + 5x - 2y = 0$ …(i)

$$f_x = \frac{\partial f}{\partial x} = 3x^2 + 4x + 2y + 5$$

$$f_y = \frac{\partial f}{\partial y} = 2x - 2y - 2$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = 6x + 4$$

$$f_{xy} = \frac{\partial^2 f}{\partial x\, \partial y} = 2$$

$$f_{yy} = \frac{\partial^2 f}{\partial y\, \partial x} = -2$$

For double points $f_x = f_y = f = 0$

$f_x = 0 \Rightarrow 3x^2 + 4x + 2y + 5 = 0$ …(ii)

$f_y = 0 \Rightarrow 2x - 2y - 2 = 0$

or $2y = 2x - 2$ …(iii)

Solving Eqs. (ii) and (iii), we get $3x^2 + 4x + 2x - 2 + 5 = 0$

$\Rightarrow$ $x = -1$

also $x = -1, \quad y = -2$ satisfies the given equation.

$\therefore$ $(-1, -2)$ **is a double point.**

At $(-1, -2)$, $f_{xx} = 6(-1) + 4 = -2$

$f_{xy} = 2, \quad f_{yy} = -2$

$\therefore$ $f_{xy}^2 - f_{xx} f_{yy}$ at $(-1, -2) = 0$

$\therefore$ $(-1, -2)$ may be a cusp.

For shifting the origin to $(-1, -2)$ substitute $x = X - 1, y = Y - 2$ in the given equation.

$$(X - 1)^3 + 2(X - 1)^2 + 2(X - 1)(Y - 2) - (Y - 2)^2 + 5(X - 1) - 2(Y - 2) = 0$$

or $X^3 - X^2 + 2XY - Y^2 = 0$ …(iv)

$\therefore$ $Y = X \pm X\sqrt{X}$

Y is real for all positive value of X.

$\therefore$ Two branches of (iv) pass through origin.

$\therefore$ Two branches of (i) pass through $(-1, -2)$.

$\Rightarrow$ $(-1, -2)$ **is a cusp.**

EXAMPLE 3 *Find for the curve* $y^2 = x \sin x$ *origin is node, cusp or isolated point.*

SOLUTION Let $f(x, y) = y^2 - x \sin x$

$f_x = -\sin x - x \cos x$

$f_y = 2y$

$f_{xx} = -\cos x + x \sin x - \cos x$

$$f_{xy} = 0$$
$$f_{yy} = 2$$

$\therefore$ **at** $x = 0$ **:** $f_{xx} = -2,\ f_{xy} = 0,\ f_{yy} = 2$

$\therefore$ $f_{xy}^2 - f_{xx} f_{yy}$ at $(0, 0)$

$\Rightarrow$ $0 + 2(2) = 4 > 0$

$f_{xy}^2 - f_{xx} f_{yy} > 0$ $\therefore$ **origin is node.**

3.2 (v) Types of cusps

When two branches of a curve pass through a cusp and the tangents at cusp are coincident. Therefore, normal to the branches at a cusp would also be coincident.

Cusp can be of five kinds

(a) Single cusp

If the branches of the curve lie on the same side of the common normal, then the cusp is called a single cusp.

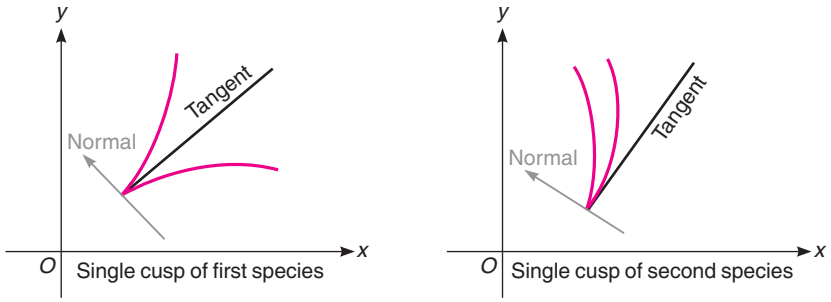


Fig. 3.13

(b) Double cusp

If the branches of the curve lie on the both sides of the common normal, then the cusp is called double cusp.

Here, both the branches of the curve lie on the both sides of common tangent, then the cusp is of first kind.

Also if, the branches of the curve lie on the same side of the common tangent, then the cusp is called **cusp of second species or Ramphoid cusp.**

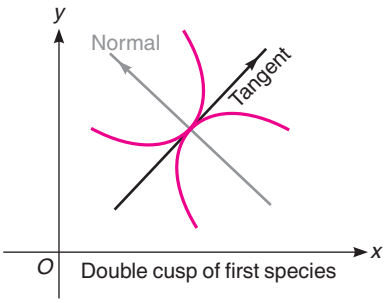


Fig. 3.14

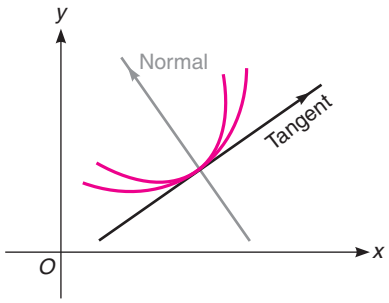


Fig. 3.15

(c) Point of oscu-inflexion

A double cusp of both the species is called a point of oscu-inflexion.

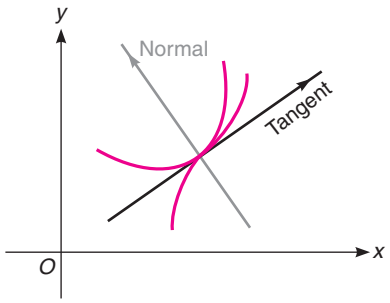


Fig. 3.16

EXAMPLE 1 *Sketch the curve* $y^2(a + x) = x^2(a - x)$.

SOLUTION Here, the curve is

$$y^2(a + x) = x^2(a - x) \qquad \ldots(i)$$

1. The curve is **symmetrical about *x*-axis**.

2. Curve passes through origin and cuts the x-axis at a point $(a, 0)$.

3. Equating to zero the lowest degree terms of Eq. (i), we get the tangents at origin.

$\therefore \quad y^2 = x^2 \quad$ or $\quad y = \pm x$

$\therefore$ **origin is node.**

4. Equating to zero the highest degree term, we get the asymptote

$$x + a = 0, \quad i.e., \quad x = -a.$$

5. From Eq. (i) $\qquad y = \pm x\sqrt{\dfrac{a - x}{a + x}}$

$\therefore$ y exists when $\quad -a < x \leq a \quad \Rightarrow \quad$ Domain $\in (-a, a]$.

6. As x increases from 0 to $a \Rightarrow y$ increases upto a point then decreases to zero.

$\Rightarrow$ y increases when $\quad x \in [0, a]$

y decreases when $\quad x \in (-a, 0]$

Thus, $\quad y^2(a + x) = x^2(a - x)$ could be plotted as;

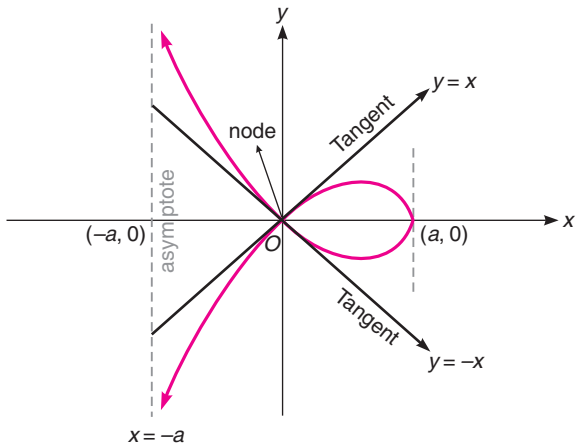


Fig. 3.17

3.3 REMEMBER FOR TRACING CARTESIAN EQUATION

1. Check symmetry

(a) A curve is symmetrical about x-axis, *i.e.*, y is replaced by $-y$ and curve remains same.

(b) A curve is symmetrical about y-axis, *i.e.*, $f(-x) = f(x)$.

(c) A curve is symmetrical about $y = x$, *i.e.*, on interchanging x and y curve remains same.

(d) A curve is symmetrical about $y = -x$. *i.e.*, on interchanging x by $-y$ and y by $-x$ curve remains same.

(e) The curve is symmetrical in opposite quadrants, *i.e.*, $f(-x) = -f(x)$.

2. Check for origin

Find whether origin lies on the curve or not.
If yes, check for multiple points (See Art. 3.2).

3. Point of intersection with x-axis and y-axis

Put $x = 0$ and find y, put $y = 0$ and find x. Also obtain the tangents at such points.

4. Asymptotes

Find the point at which asymptote meets the curve and equation of asymptote (see Art. 3.1)

5. Domain and range

To check in which part the curve lies.

6. Monotonicity and maxima minima

Find $\frac{dy}{dx}$ and check the interval in which y increases or decreases and the point at which it attains maximum or minimum.

7. Concavity and convexity

The interval in which,

$$\frac{d^2y}{dx^2} > 0$$

and

$$\frac{d^2y}{dx^2} < 0$$

Using all the above results we can sketch the curve

$$y = f(x).$$

SOME MORE SOLVED EXAMPLES

EXAMPLE 1 *Sketch the curve* $y^2(a^2 + x^2) = x^2(a^2 - x^2)$

SOLUTION Here, the curve is $y^2 = \dfrac{x^2(a^2 - x^2)}{(a^2 + x^2)}$

1. The curve is symmetric about x-axis and y-axis {as on replacing y by $-y$ curve remains same and on replacing x by $-x$ curve remains same thus, symmetric about x and y-axis respectively).
2. It passes through origin and $y = \pm x$ are two tangents at origin. **Thus, the origin is node**.
3. It meets x-axis at $(a, 0)$, $(0, 0)$ and $(-a, 0)$ and meets y-axis at $(0, 0)$ only.
   The tangents at $(a, 0)$ and $(-a, 0)$ are $x = a$ and $x = -a$ respectively.
4. The curve has no asymptote.
5. Here, $$y = \pm x\sqrt{\frac{a^2 - x^2}{a^2 + x^2}}$$

   $\therefore$ Domain $\in [-a, a]$
6. $$\frac{dy}{dx} = \frac{a^4 - 2a^2x^2 - x^4}{(a^2 + x^2)^{3/2}(a^2 - x^2)^{1/2}}$$

   $$\frac{dy}{dx} \to \infty \quad \text{as} \quad x \to \pm a$$

   Also $\dfrac{dy}{dx} = 0$ when $a^4 - 2a^2x^2 - x^4 = 0$

   i.e., $$\frac{dy}{dx} = \frac{a^4 - 2a^2x^2 - x^4}{(a^2 + x^2)^{3/2}(a^2 - x^2)^{1/2}}$$

   $$= \frac{-\{x^4 + 2a^2x^2 + a^4 - 2a^4\}}{(a^2 + x^2)^{3/2}(a^2 - x^2)^{1/2}}$$

   $$= \frac{-\{(x^2 + a^2)^2 - (\sqrt{2}a^2)^2\}}{(a^2 + x^2)^{3/2}(a^2 - x^2)^{1/2}}$$

   $$= \frac{-\{x - \sqrt{(-1 + \sqrt{2})}a\}\{x + \sqrt{(-1 + \sqrt{2})}a\}\{x^2 + (1 + \sqrt{2})a^2\}}{(a^2 + x^2)^{3/2}(a^2 - x^2)^{1/2}}$$

   $\Rightarrow$ $$\frac{dy}{dx} = \begin{cases} 0; & x = \pm\sqrt{(-1 + \sqrt{2})}\,a \\ +\text{ ve}; & x \in (-\sqrt{(-1 + \sqrt{2})}\,a, \sqrt{(-1 + \sqrt{2})}\,a) \\ -\text{ ve}; & x \in (-a, -\sqrt{(-1 + \sqrt{2})}\,a) \text{ or } (\sqrt{(-1 + \sqrt{2})}\,a, a) \end{cases}$$

   i.e., y increasing when $x \in (-\sqrt{(-1 + \sqrt{2})}\,a, \sqrt{(-1 + \sqrt{2})}\,a)$

   and y decreases when $x \in (-a, -\sqrt{(-1 + \sqrt{2})}\,a)$ or $(\sqrt{(-1 + \sqrt{2})}\,a, a)$

OR

$$\frac{dy}{dx} > 0, \quad \text{when} \quad x \in (-(0.6)\,a, (0.6)\,a)$$

$$\frac{dy}{dx} < 0, \quad \text{when} \quad x \in (-a, -(0.6)\,a) \quad \text{or} \quad ((0.6)\,a, a)$$

where $\sqrt{-1+\sqrt{2}} = (0.6)_{\text{approx}}$

Thus, the curve for $y^2(a^2+x^2) = x^2(a^2-x^2)$

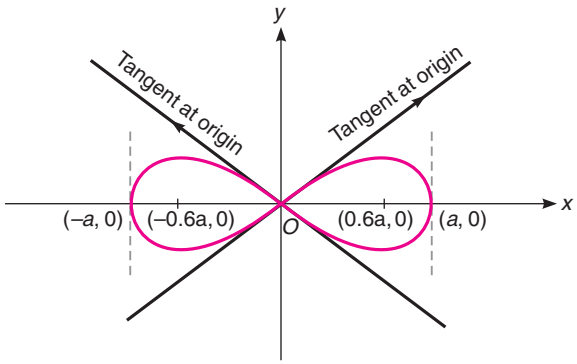


Fig. 3.18

EXAMPLE 2 *Sketch the curve* $y^2(x-a) = x^2(a+x)$.

SOLUTION Here, the curve is given by $y^2 = \dfrac{x^2(a+x)}{(x-a)}$

1. Symmetrical about x-axis only.
2. It passes through origin and $y^2 + x^2 = 0$, *i.e.*, $y = \pm ix$ are two imaginary tangents at origin. Thus, origin is **isolated point**.
3. It meets x-axis at $(-a, 0)$, $(0, 0)$ and y-axis at $(0, 0)$. The tangent at $(-a, 0)$ is $x = -a$.
4. $y = \pm(x-a)$ and $x = a$ are three asymptote.
5. $$y^2 = \frac{x^2(x+a)}{(x-a)} \quad \Rightarrow \quad y = \pm x\sqrt{\frac{x+a}{x-a}}$$

   Thus, for domain; $\dfrac{x+a}{x-a} \geq 0$ and $x \neq a$

   i.e., $x \leq -a$ and $x > a$

   or Domain $\in (-\infty, -a] \cup (a, \infty) \cup \{0\}$

6. $$\frac{dy}{dx} = \pm\frac{x^2 - ax - a^2}{(x-a)^{3/2}(x+a)^{1/2}} = \pm\frac{\left\{x - \frac{1}{2}(1+\sqrt{5})a\right\}\left\{x - \frac{1}{2}(1-\sqrt{5})a\right\}}{(x-a)^{3/2}(x+a)^{1/2}}$$

   $$\Rightarrow \quad \frac{dy}{dx} > 0, \quad \text{when} \quad x \in (-\infty, -a] \cup \left[\frac{1}{2}(1+\sqrt{5})a, \infty\right)$$

$$\frac{dy}{dx} < 0, \quad \text{when} \quad x \in \left(a, \frac{1}{2}(1+\sqrt{5})\, a\right]$$

Thus, the curve;

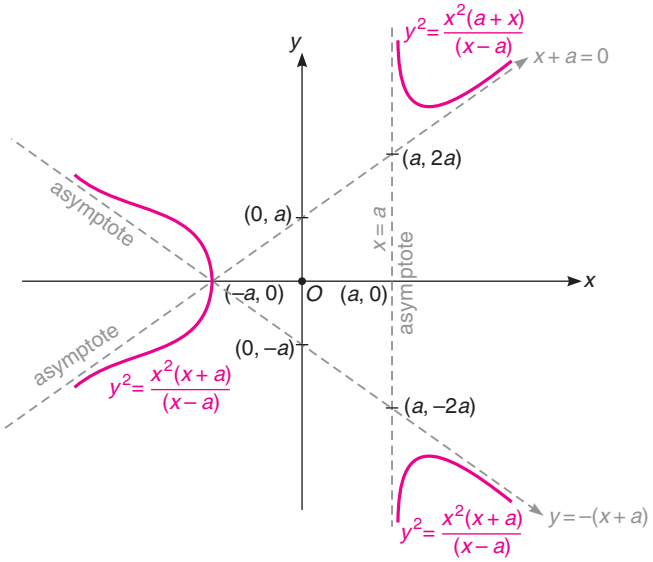


Fig. 3.19

EXAMPLE 3 *Sketch the curve* $y^2 = (x-1)(x-2)(x-3)$.

SOLUTION Here, $y^2 = (x-1)(x-2)(x-3)$

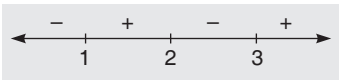
1. Symmetrical about *x*-axis.

2. It does not pass through origin.

3. It meets *x*-axis at (1, 0) (2, 0) and (3, 0) but it does not meet *y*-axis.

4. No asymptote.

5. For domain: $(x-1)(x-2)(x-3) \geq 0$



$\Rightarrow$ **Domain** $\in [1, 2] \cup [3, \infty)$

6. $y = \pm\sqrt{(x-1)(x-2)(x-3)}$

$$\therefore \quad \frac{dy}{dx} = \pm\frac{(3x^2 - 12x + 11)}{2\sqrt{(x-1)(x-2)(x-3)}} = \pm\frac{3\left(x - \frac{6-\sqrt{3}}{3}\right)\left(x - \frac{6+\sqrt{3}}{3}\right)}{2\sqrt{(x-1)(x-2)(x-3)}}$$

$$= \pm\frac{3(x-1.42)(x-2.5)}{2\sqrt{(x-1)(x-2)(x-3)}} \quad \left\{\text{as } \frac{6-\sqrt{3}}{3} = 1.42 \text{ and } \frac{6+\sqrt{3}}{3} = 2.5/\text{approx}\right\}$$

$\Rightarrow$ $\dfrac{dy}{dx} > 0$, when $x \in (1, 1.42) \cup (3, \infty)$

$\dfrac{dy}{dx} < 0$, when $x \in (1.42, 2)$

Thus, the curve $y^2 = (x-1)(x-2)(x-3)$

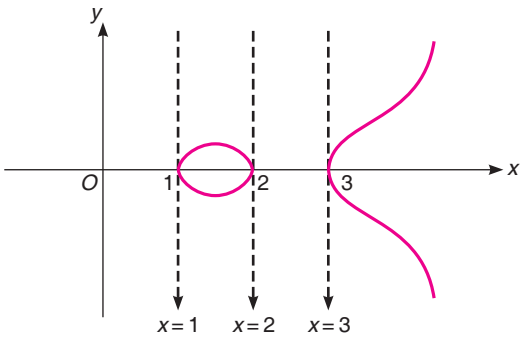


Fig. 3.20

EXAMPLE 4 *Sketch the curve* $y^2x^2 = x^2 - a^2$.

SOLUTION Here, $y^2 = \dfrac{x^2 - a^2}{x^2}$

1. Symmetrical about both the axis.
2. It does not pass through origin.
3. x-intercepts are $(a, 0)$ and $(-a, 0)$
   The tangent at $(a, 0)$ is $x = a$ and the tangent at $(-a, 0)$ is $x = -a$.
4. $y = \pm 1$ are the two asymptotes.
5. $$y = \pm \frac{\sqrt{x^2 - a^2}}{x} \quad \Rightarrow \quad \text{Domain} \in (-\infty, -a] \cup [a, \infty)$$
   $$\frac{dy}{dx} = \pm \frac{a^2}{x^2\sqrt{x^2 - a^2}} \quad \Rightarrow \quad \frac{dy}{dx} > 0, \quad \text{when} \quad x \in (-\infty, -a) \cup (a, \infty)$$

Thus, the curve for $y^2 = \dfrac{x^2 - a^2}{x^2}$ is,

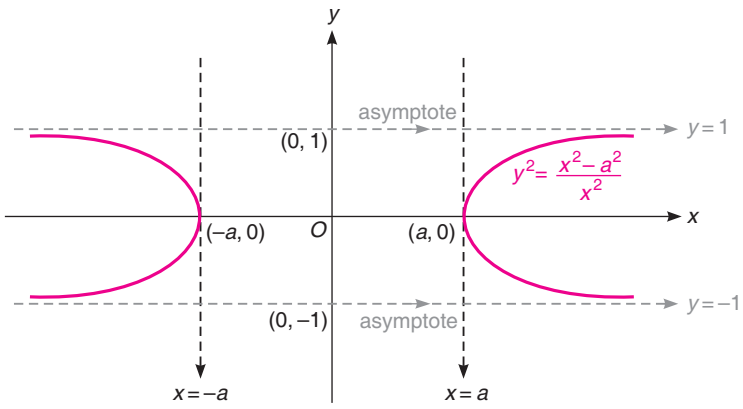


Fig. 3.21

EXAMPLE 5 *Sketch the curve* $y^2(x^2-1) = 2x-1$.

SOLUTION Here, $y^2 = \dfrac{2x-1}{x^2-1}$

1. Symmetrical about x-axis.

2. It does not pass through origin.

3. It meets x-axis in $\left(\frac{1}{2}, 0\right)$ and y-axis in $(0, 1)$ and $(0, -1)$ respectively.

The tangent at $\left(\frac{1}{2}, 0\right)$ is $x = \frac{1}{2}$.

4. $x = 1$, $x = -1$ and $y = 0$ are three asymptotes.

5. $y^2 = \dfrac{2x-1}{x^2-1} \Rightarrow$ Domain $\in \left(-1, \frac{1}{2}\right] \cup (1, \infty)$

6. $y = \pm\sqrt{\dfrac{2x-1}{x^2-1}} \Rightarrow \dfrac{dy}{dx} = \pm\left(\dfrac{-x^2+x+1}{(2x-1)^{1/2}(x^2-1)^{3/2}}\right)$

$\Rightarrow \dfrac{dy}{dx} < 0$ when $x \in \left(-1, \frac{1}{2}\right] \cup (1, \infty)$

$\therefore$ ***y* is decreasing in its domain.**

Thus, the graph for $y^2 = \dfrac{2x-1}{x^2-1}$ is,

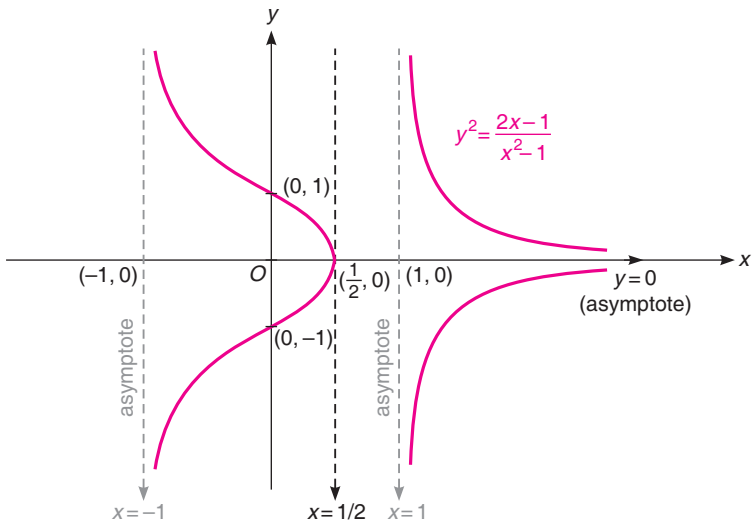


Fig. 3.22

EXAMPLE 6 *Sketch the curve :*

$$(x^2+y^2)x - a(x^2-y^2) = 0; \quad (a > 0)$$

SOLUTION Here, $y^2 = x^2\left(\dfrac{a-x}{a+x}\right)$

1. Symmetric about x-axis.
2. Origin lies on the curve and $y = \pm x$ are two tangents at origin. So, origin is node.
3. x-intercept are $(0, 0)$ and $(a, 0)$. The tangent at $(a, 0)$ is $x = a$.
4. $x = -a$ is the only asymptote.

5. $$y = \pm x\sqrt{\frac{a - x}{a + x}}$$

$\therefore$ Domain $\in (-a, a]$

6. $$\frac{dy}{dx} = \pm \frac{a^2 - ax - x^2}{(a + x)\sqrt{a^2 - x^2}}$$

$$\Rightarrow \frac{dy}{dx} > 0, \quad \text{when} \quad x \in \left(-a, \frac{-1 + \sqrt{5}}{2} a\right)$$

$$\Rightarrow \frac{dy}{dx} < 0, \quad \text{when} \quad x \in \left(\frac{-1 + \sqrt{5}}{2} a, a\right).$$

Thus, the graph for

$y^2 = x^2\left(\frac{a - x}{a + x}\right)$ as shown in Fig. 3.23.

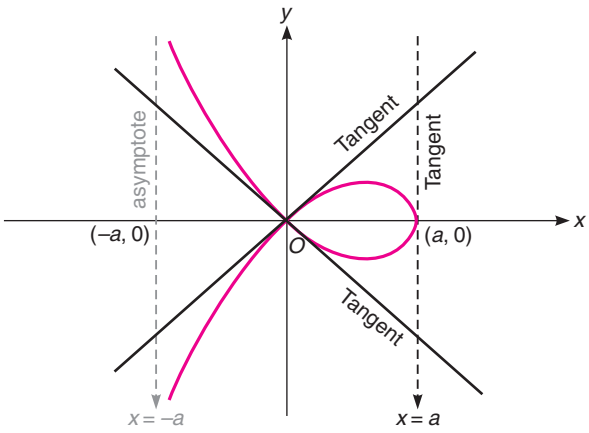


Fig. 3.23

EXAMPLE 7 *Sketch the curve* $x^3 + y^3 = 3ax^2 \quad (a > 0)$.

SOLUTION Here, $x^3 + y^3 = 3ax^2$

1. No line of symmetry.
2. Origin is cusp and $x = 0$ is tangent.
3. x-intercept, $(0, 0)$ $(3a, 0)$
The tangent at $(3a, 0)$ is $x = 3a$.

4. $y = a - x$ is asymptote and the curve meets asymptote at $\left(\frac{a}{3}, \frac{2a}{3}\right)$.

5. Here; $$x^3 + y^3 = 3ax^2$$

$\Rightarrow$ $$3ax^2 > 0 \qquad \text{(as, } a > 0\text{)}$$

$\therefore$ $$x^3 + y^3 > 0$$

i.e., x and y both cannot be negative (thus, curve would not lie in third quadrant).

6. $$y^2 \frac{dy}{dx} = x(2a - x)$$

$\Rightarrow$ $$\frac{dy}{dx} > 0, \quad \text{when} \quad x \in (0, 2a)$$

$$\frac{dy}{dx} < 0, \quad \text{when} \quad x \in (-\infty, 0) \cup (2a, \infty)$$

Thus, the curve $y^3 + x^3 = 3ax^2$ is,

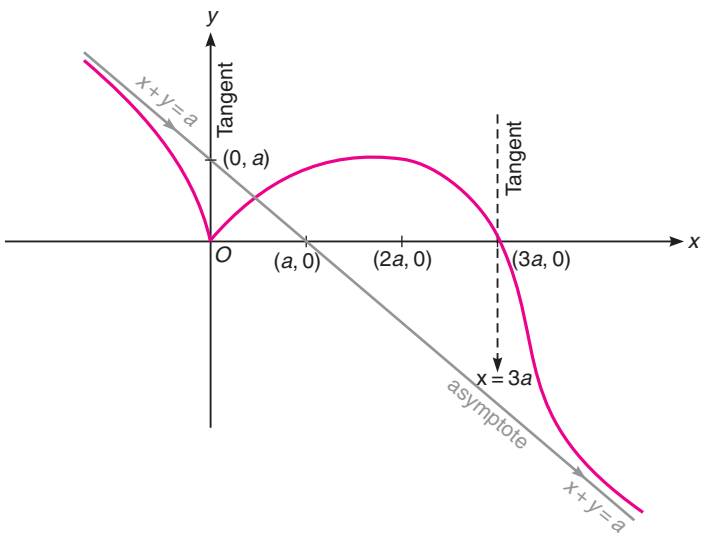


Fig. 3.24

EXAMPLE 8 *Sketch the curve with parametric equation* θ.

$$x = a(\theta + \sin\theta), \quad y = a(1 + \cos\theta); \quad x \in (-\pi, \pi).$$

SOLUTION Here, $x = a(\theta + \sin\theta)$ and $y = a(1 + \cos\theta)$ gives the following table for x and y with θ.

θ	$-\pi$	0	π
x	$-a\pi$	0	$a\pi$
y	0	2 a	0

So, that we have,

$$-\pi \le \theta \le 0$$

$\Rightarrow$ (x, y) starting from $(-a\pi, 0)$ moves to the right and upwards to $(0, 2a)$.

$$0 \le \theta \le \pi$$

$\Rightarrow$ the point (x, y) starting from $(0, 2a)$ moves to the right and downward to $(a\pi, 0)$.

Also $$\frac{dx}{d\theta} = a(1 + \cos\theta)$$

and $$\frac{dy}{d\theta} = -a\sin\theta$$

Now, $\frac{dx}{d\theta} = 0$ if $\theta = \pi, -\pi$

$$\frac{dy}{dx} = -\frac{\tan\theta}{2},$$

except for the values $\mp \pi$ of θ for which $\frac{dx}{d\theta} = 0$.

Also, tangent at $\theta = \pi$ and $\theta = -\pi$ are $x = \pi$ and $x = -\pi$.

Thus, the curve for $x = a(\theta + \sin\theta)$ and $y = a(1 + \cos\theta)$.

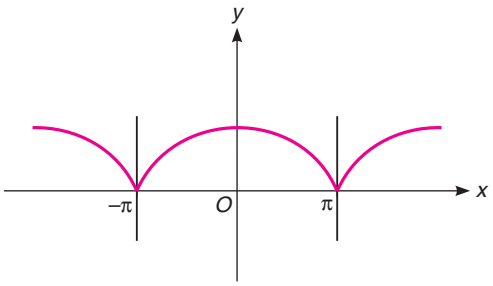


Fig. 3.25

EXAMPLE 9 *Sketch the curve :* $x^5 + y^5 = 5a^2xy^2$.

SOLUTION Here;

1. The curve is symmetrical in opposite quadrants.

2. The curve passes through origin and $x = 0$, $y = 0$ are tangents. Thus, origin is node.

3. It meets coordinate axis at origin.

4. $x + y = 0$ is an asymptote.

5. On transfering to polar coordinates, we get.

$$r^2 = \frac{5a^2 \cos\theta \sin\theta}{\cos^5\theta + \sin^5\theta}$$

when, $\theta = 0$, $r = 0$ when, $\theta = \frac{\pi}{2}$, $r = 0$

As θ increases from $\frac{\pi}{2}$ to $\frac{3\pi}{4}$, r^2 is negative and hence, r is imaginary.

$\therefore$ no portion of the curve lies in this region.

At $\theta = \frac{3\pi}{4}$, $r = \infty$ as θ increases from $\frac{3\pi}{4}$ to π $\Rightarrow$ r decreases from ∞ to 0.

$\therefore$ Curve $x^5 + y^5 = 5a^2xy^2$

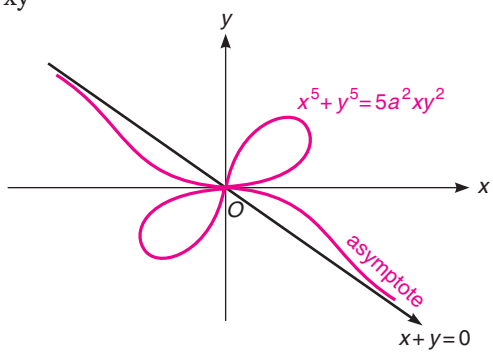


Fig. 3.26

EXAMPLE 10 *Sketch the curve* $y^4 - x^4 + xy = 0$.

SOLUTION Here, $y^4 - x^4 + xy = 0$

1. No line of symmetry.

2. It passes through origin two tangents at (0, 0) as $x = 0$ and $y = 0$,

$\therefore$ **origin is node**.

3. It cuts the coordinate axes at the origin only.

4. $y = x$, $y = -x$ are its asymptotes.

5. Converting into polar coordinates,

$$r^2 = \frac{1}{2}\tan 2\theta$$

6. When, $0 < \theta < \frac{\pi}{4}$ or $0 < 2\theta < \frac{\pi}{2}$ $\Rightarrow$ r^2 increases from 0 to ∞.

When, $\frac{\pi}{4} < \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < 2\theta < \pi$ $\Rightarrow$ r^2 is negative,

$\therefore$ no curve when $\frac{\pi}{4} < \theta < \frac{\pi}{2}$.

Hence, the curve

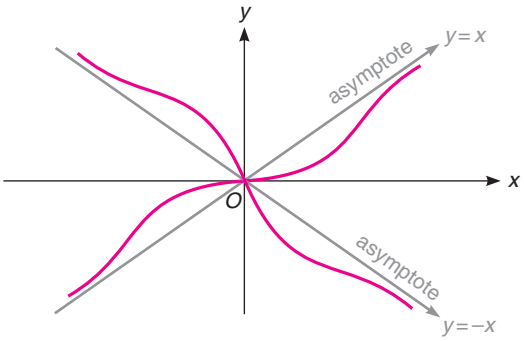


Fig. 3.27

EXERCISE

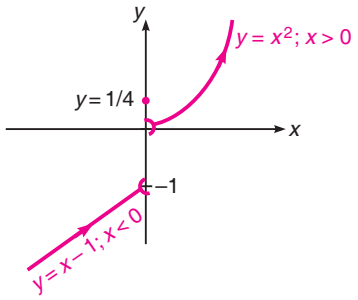
Plot the Curves :

1. $y = 1 + x^2 - \frac{1}{2}x^4$.
2. $y = (x+1)(x-2)^2$
3. $y = \frac{2}{5}x - \frac{1}{2}x^3 + \frac{1}{10}x^5$.
4. $y = (1-x^2)^{-1}$
5. $y = \frac{x^4}{(1+x)^3}$
6. $y = \frac{(1+x)^4}{(1-x)^4}$
7. $y = \frac{x^2(x-1)}{(x+1)^2}$
8. $y = \frac{x}{(1-x^2)^2}$
9. $y = 2x - 1 + \frac{1}{(x+1)}$
10. $y = \frac{x^2+1}{x^2-1}$
11. $y = \frac{a^2x}{a^2+x^2}$
12. $y^2 = x^2\left(\frac{a+x}{b-x}\right)$
13. $y = \frac{8a^3}{x^2+4a^2}$
14. $y = \frac{\cos x}{\cos 2x}$
15. $y = \text{arc cos}\left(\frac{1-x^2}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$
16. $y = \text{arc sin}(\sin x) = \sin^{-1}(\sin x)$
17. $y = \sin(\text{arc sin}\, x)$
18. $y = \text{arc tan}(\tan x)$
19. $y = \text{arc tan}\left(\frac{1}{x}\right)$
20. $y = (x+2)e^{1/x}$
21. $y = \frac{1}{2}(\sqrt{x^2+x+1} - \sqrt{x^2-x+1})$
22. $y = \sqrt{x^2+1} - \sqrt{x^2-1}$
23. $y = (x+2)^{2/3} - (x-2)^{2/3}$
24. $y = (x+1)^{2/3} + (x-1)^{2/3}$
25. $y^2 = 8x^2 - x^4$
26. $y^2 = (x-1)(x-2)(x-3)$
27. $y^2 = \frac{x-1}{x+1}$
28. $y^2 = \frac{x^2(1-x)}{(1+x)^2}$
29. $y^2 = x^4(x+1)$
30. $x^2(y-2)^2 + 2xy - y^2 = 0$
31. $x = \frac{1}{4}(t+1)^2,\ y = \frac{1}{4}(t-1)^2$
32. $x = \frac{t^2}{1-t^2},\quad y = \frac{1}{1+t^2}$
33. $x = \frac{t^2}{t-1},\ y = \frac{t}{t^2-1}$
34. $x = -5t^2 + 2t^5,\ y = -3t^2 + 2t^3$
35. $x = \frac{t^2+1}{4(1-t)},\ y = \frac{t}{t+1}$
36. $x = \frac{(t+2)^2}{(t+1)},\ y = \frac{(t-2)^2}{t-1}$
37. $x = \frac{t-t^2}{1+t^2},\ y = \frac{t^2-t^3}{1+t^2}$
38. $x^3 + y^3 = 3axy$, where $a > 0$.
39. $(x-a)^2(x^2+y^2) = b^2x^2$, where $a, b > 0$.
40. $x^{2/3} + y^{2/3} = a^{2/3}$, where $a > 0$
41. $x^6 + 2x^3y = y^3$
42. $4y^2 = 4x^2y + x^5$
43. $x^4 + 2y^3 = 4x^2y$
44. $x^3 - 2x^2y - y^2 = 0$
45. $x^2y^2 + y = 1$
46. $x^3 + y^3 = 3x^2$
47. $y^5 + x^4 = xy^2$
48. $x^4 - y^4 + xy = 0$
49. $x^5 + y^5 = xy^2$
50. $x = a\sin 2\theta(1+\cos 2\theta)$,
    $y = a\cos 2\theta(1-\cos 2\theta)$

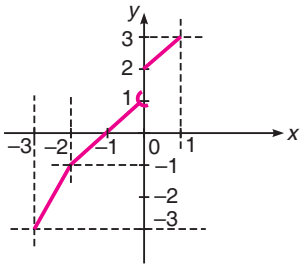
Hints and Solutions

INTRODUCTION OF GRAPHS

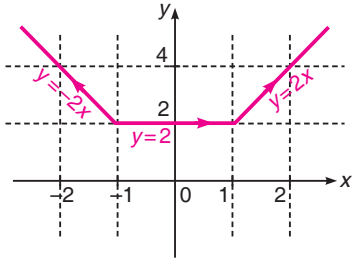
1. (i)



(ii)

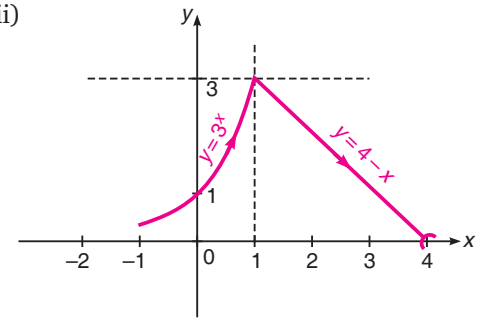


2. (i)



$$\text{Here; } f(x) = \begin{cases} -2x; & x < -1 \\ 2; & -1 \le x \le 1 \\ 2x; & x > 1 \end{cases}$$

(ii)



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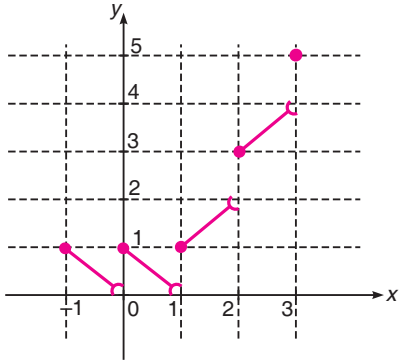
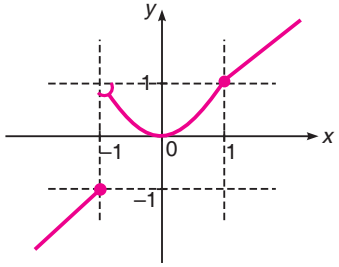


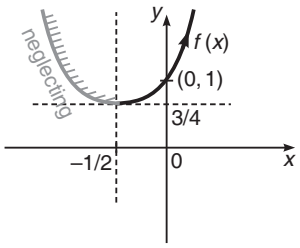
$$\text{Here, } f(x) = \begin{cases} -x; & -1 \le x < 0 \\ -x+1; & 0 \le x < 1 \\ x; & 1 \le x < 2 \\ x+1; & 2 \le x < 3 \\ 5; & x = 3 \end{cases}$$

(iv)

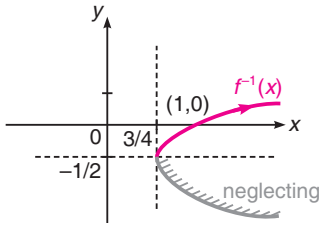


$$\text{Here, } f(x) = \begin{cases} x^4; & -1 < x < 1 \\ x; & x \le -1 \text{ or } x \ge 1 \end{cases}$$

3. We know, $y = x^2 + x + 1$ is shown as;



Graph for $f(x)$

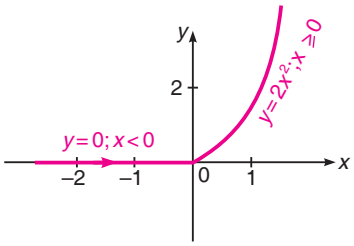


Graph for $f^{-1}(x)$

From above figures it is clear that the graph would exists only when; $x \geq -\frac{1}{2}$.

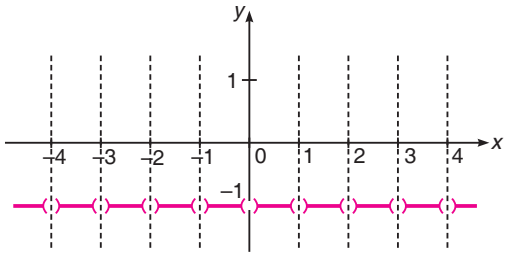
i.e., inverse for $f(x) = x^2 + x + 1$ would exists only when $x \geq -\frac{1}{2}$.

4.



Here, $f(x) = \begin{cases} 2x^2; & x \geq 0 \\ 0; & x \leq 0 \end{cases}$

5.

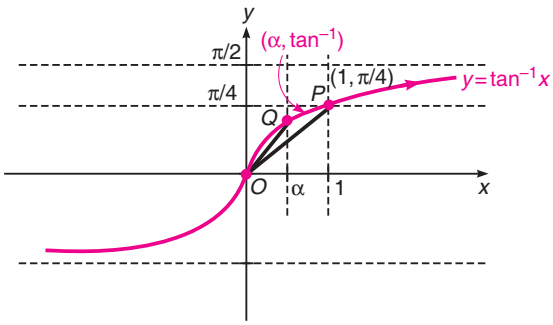


Here, $f(x) = [[x] - x] = [x] + [-x]$

$$= \begin{cases} x - x; & x \in \text{integer} \\ [x] + (-1 - [x]); & x \notin \text{integer} \end{cases}$$

$$= \begin{cases} 0; & x \in \text{integer} \\ -1; & x \notin \text{integer} \end{cases}$$

6.

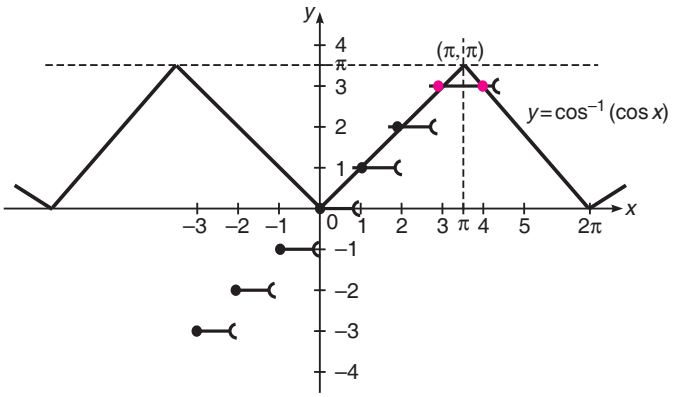


From figure;

Slope of OP < Slope of OQ.

$$\Rightarrow \quad \frac{\frac{\pi}{4} - 0}{1 - 0} < \frac{\tan^{-1}\alpha - 0}{\alpha - 0}$$

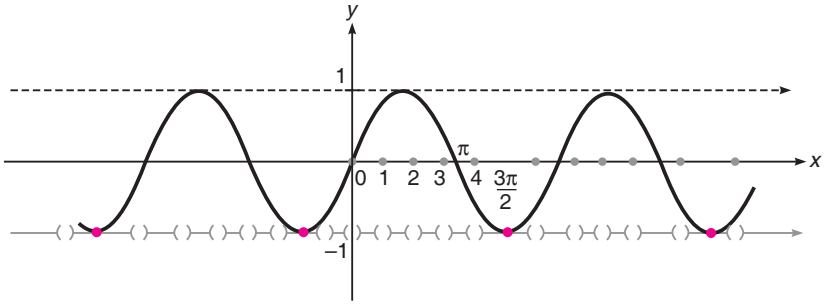
$$\Rightarrow \quad \frac{\tan^{-1}\alpha}{\alpha} > \frac{\pi}{4}.$$



Clearly, the above curves intersect at 5 points;

$\therefore$ number of solutions = 5.

8.

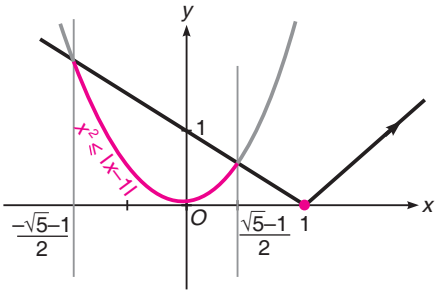


From above figure, number of solutions is infinite.

9. Clearly; $\left|\dfrac{x^2}{x-1}\right| \le 1$

or $x^2 \le |x-1|$

$\Rightarrow$ $x \in \left[\dfrac{-1-\sqrt{5}}{2}, \dfrac{-1+\sqrt{5}}{2}\right]$.



10. Here, $x^3 = 3 + [x]$

$\therefore$ to sketch $f(x) = x^3$

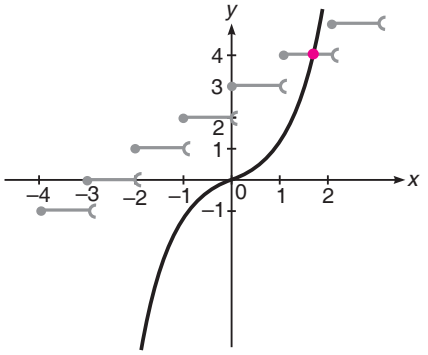
and $g(x) = 3 + [x]$

Clearly; from figure the two curves $f(x)$ and $g(x)$ intersects when $g(x) = 4$.

$\therefore$ $f(x) = 4$

$\Rightarrow$ $x^3 = 4$

or $x = 2^{2/3}$.



CURVATURE AND TRANSFORMATIONS

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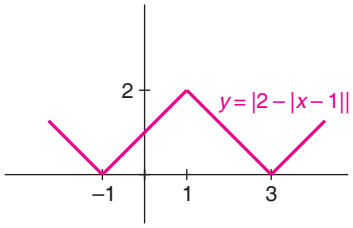

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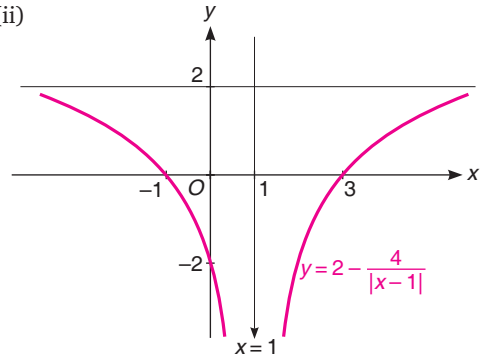

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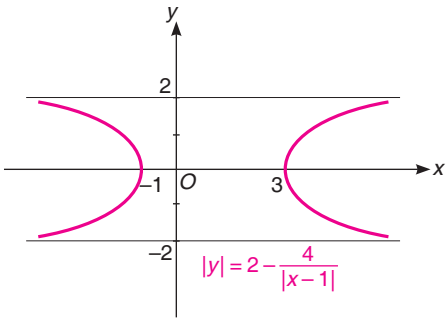

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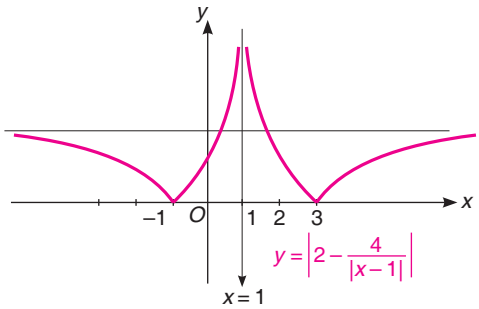

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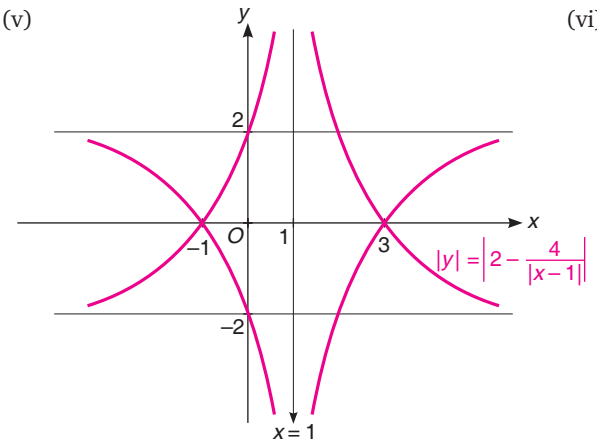

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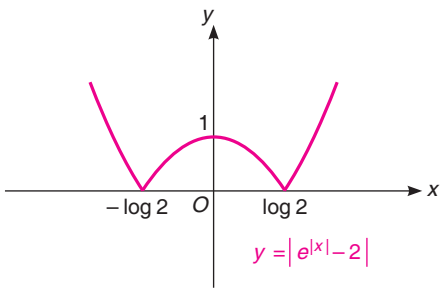

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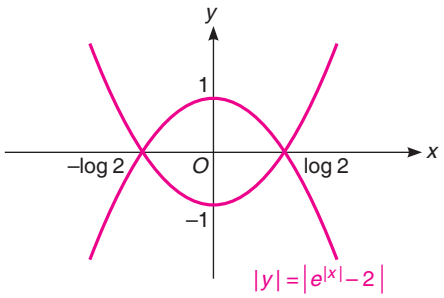

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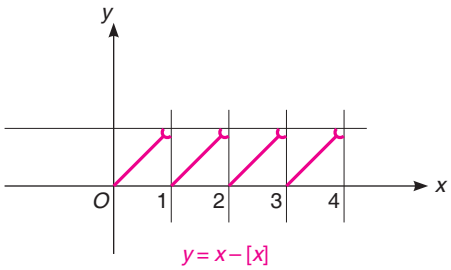
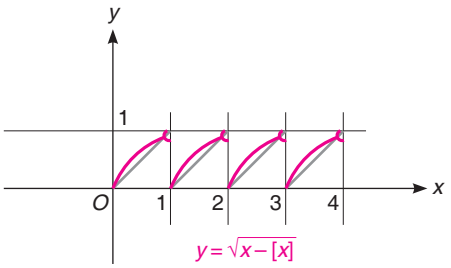
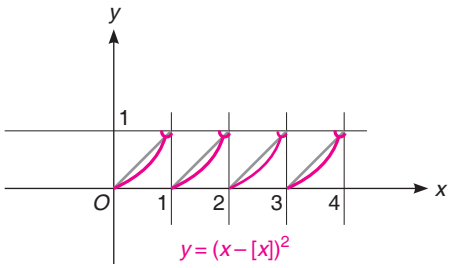


(ix)



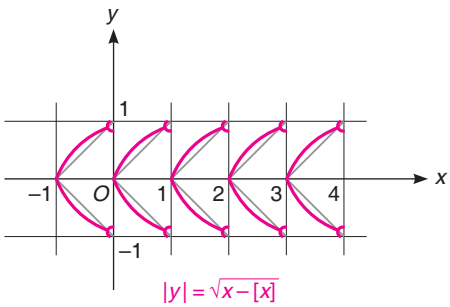
$y = \sqrt{x - [x]}$

(x)



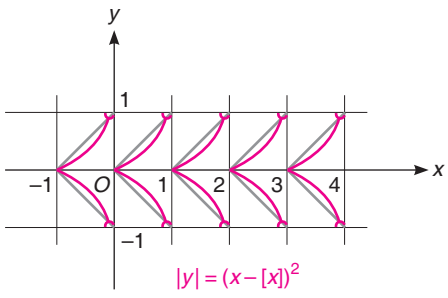
$y = (x - [x])^2$

(xi)



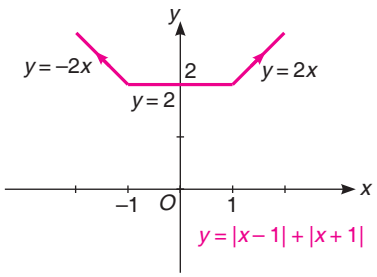
$|y| = \sqrt{x - [x]}$

(xii)



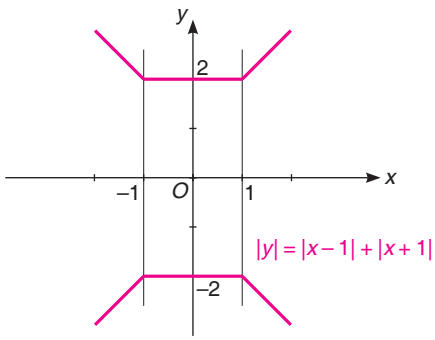
$|y| = (x - [x])^2$

(xiii)



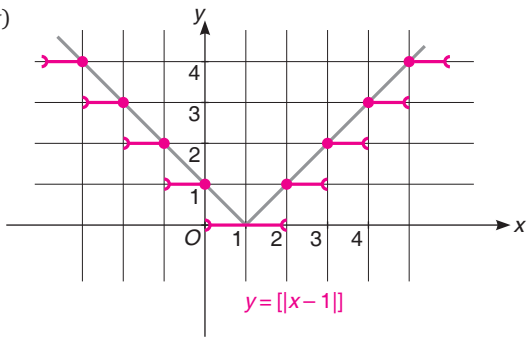
$y = |x - 1| + |x + 1|$

(xiv)



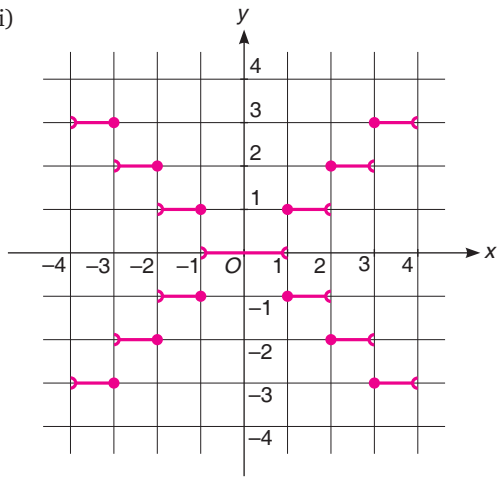
$|y| = |x - 1| + |x + 1|$

(xv)



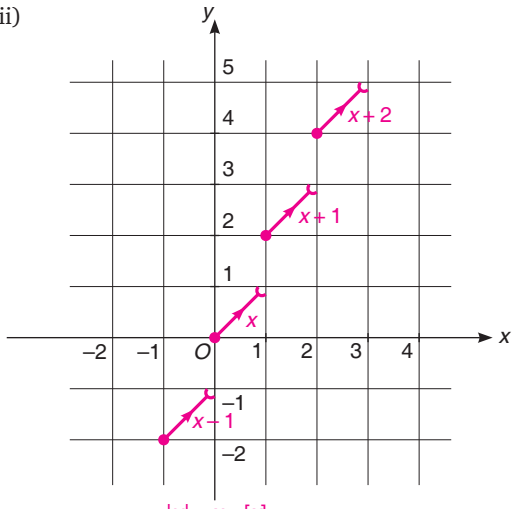
$y = [|x - 1|]$

(xvi)



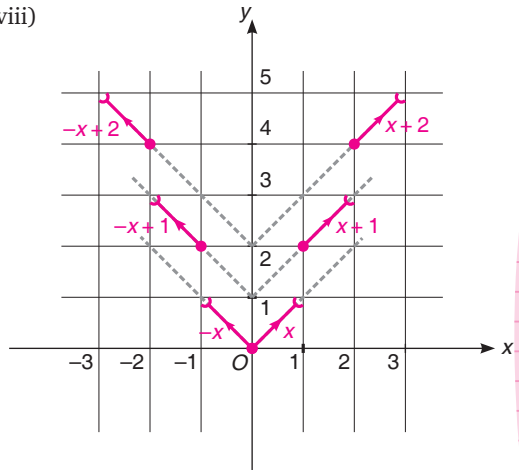
$|y| = [|x - 1|]$

(xvii)



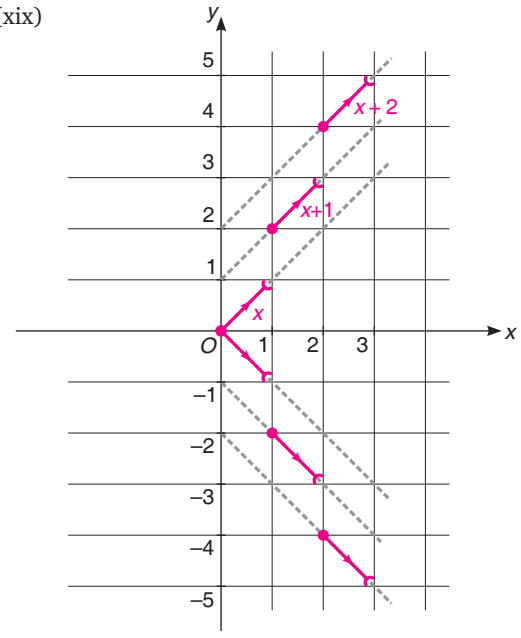
$|y| = x + [x]$

(xviii)



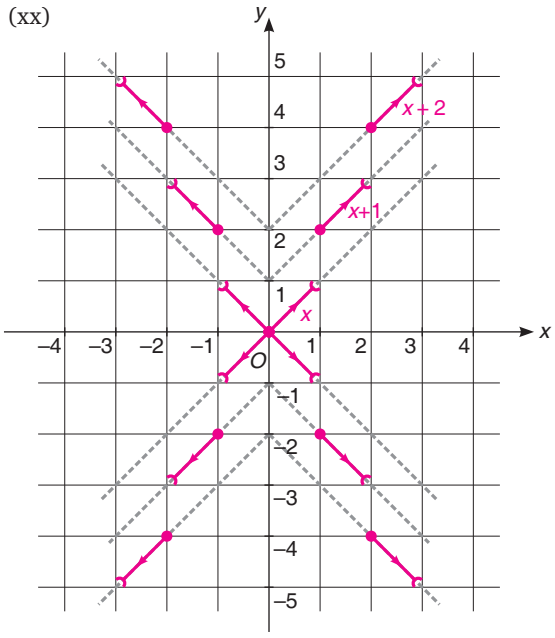
$y = |x| + [|x|]$

(xix)



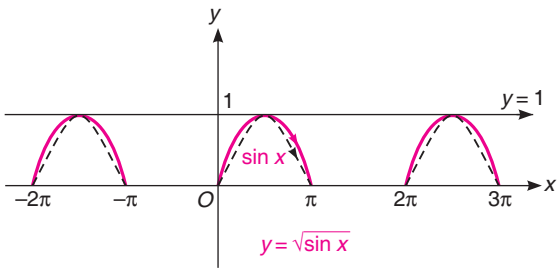
$|y| = x + [x]$

(xx)



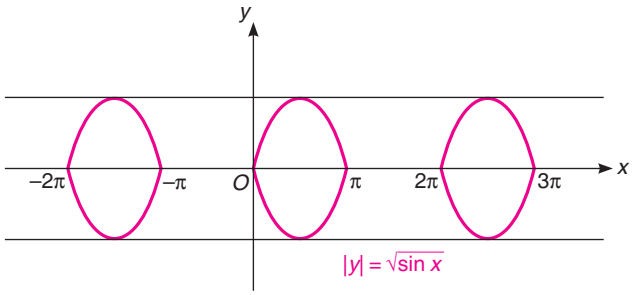
$|y| = |x| + [|x|]$

2. (i)

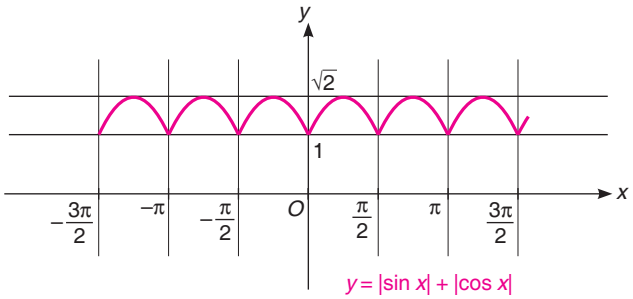


$y = \sqrt{\sin x}$

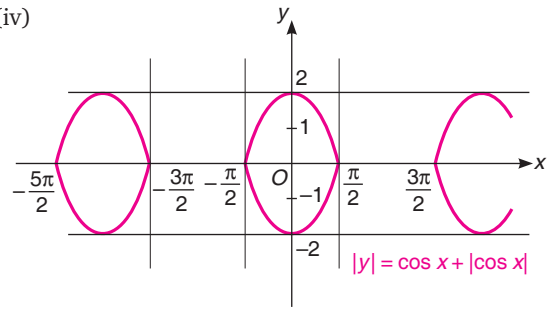
(ii)



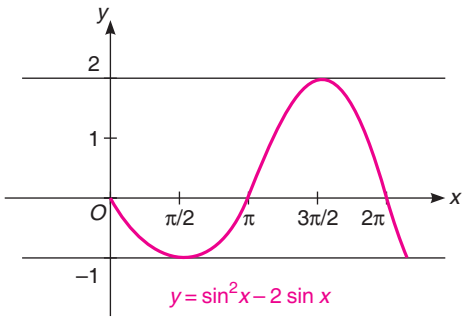
(iii)



(iv)

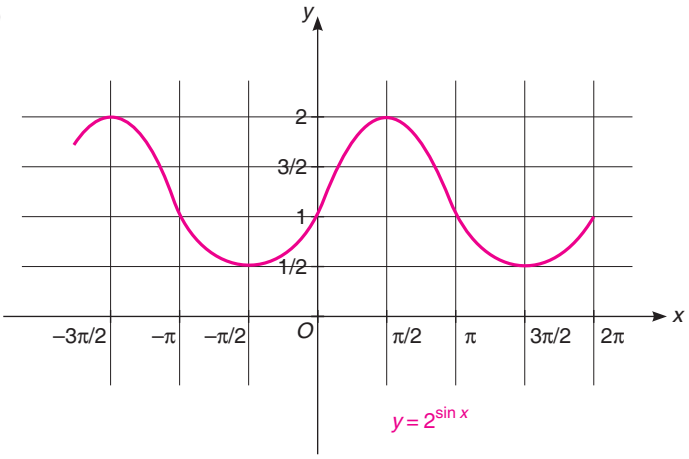


(v)

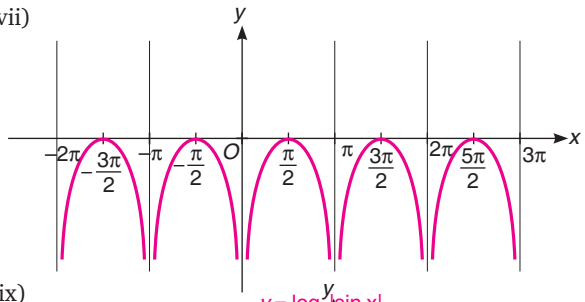


10.

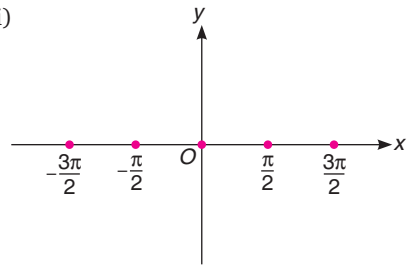
(vi)



(vii)

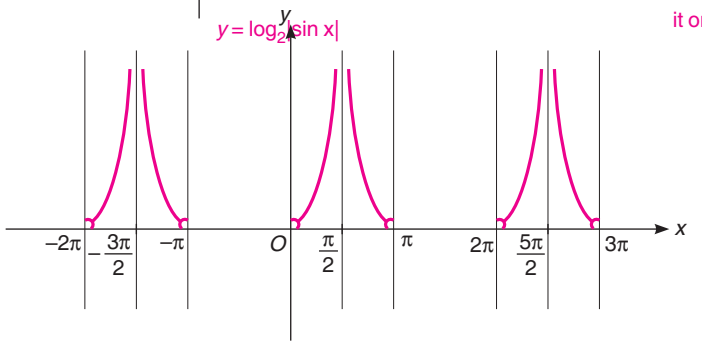


(viii)



it only represents points

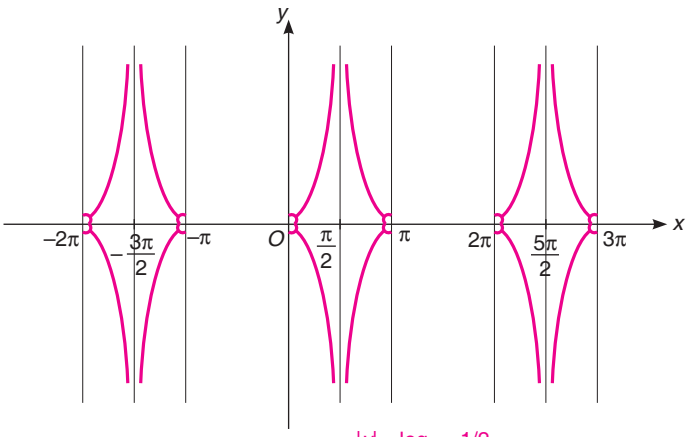
(ix)



$y = \log_{|\sin x|} 1/2$

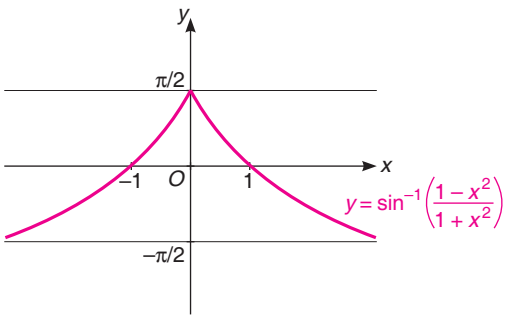
$y = \log_{\sin x} 1/2$

(x)



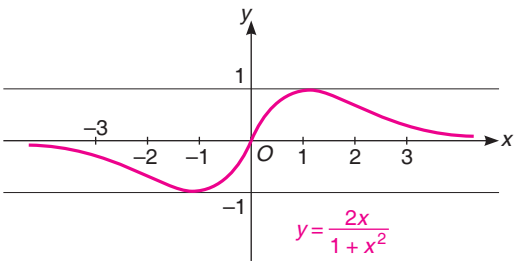
$|y| = \log_{\sin x} 1/2$

3.



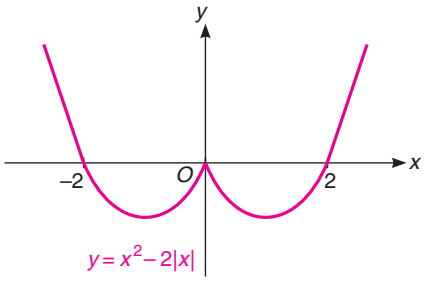
$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

4.



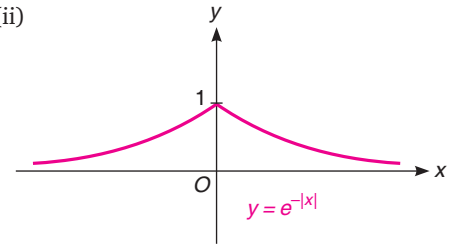
$y = \frac{2x}{1+x^2}$

5. (i)



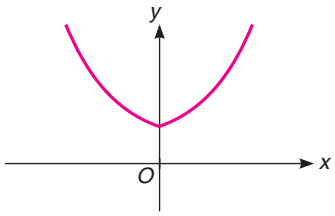
$y = x^2 - 2|x|$

(ii)



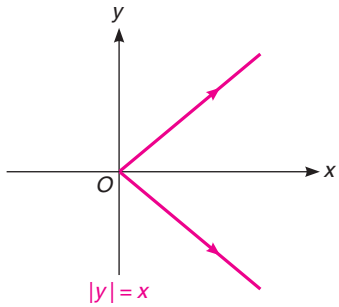
$y = e^{-|x|}$

(iii)



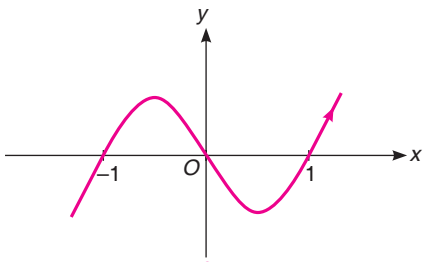
$y = e^{|x|}$

(iv)



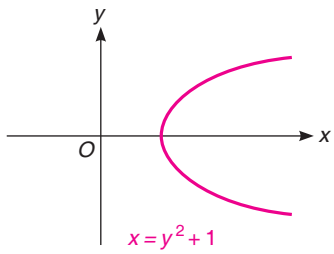
$|y| = x$

(v)



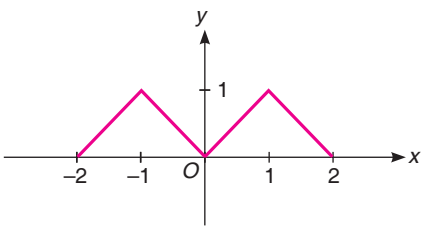
$y = x^3 - x$

(vi)



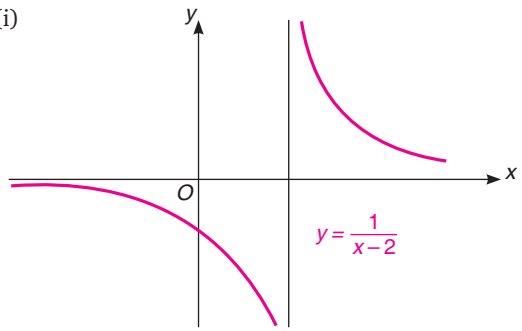
$x = y^2 + 1$

6.



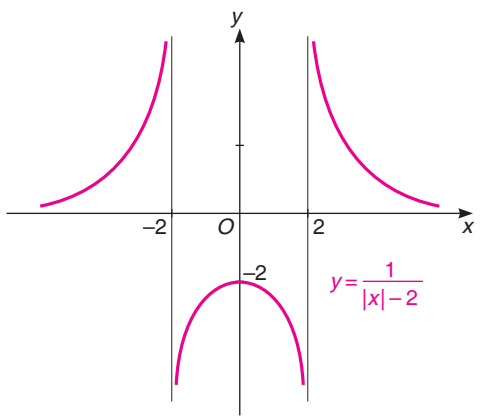
$y = f(x-1) + f(x+1)$

7. (i)



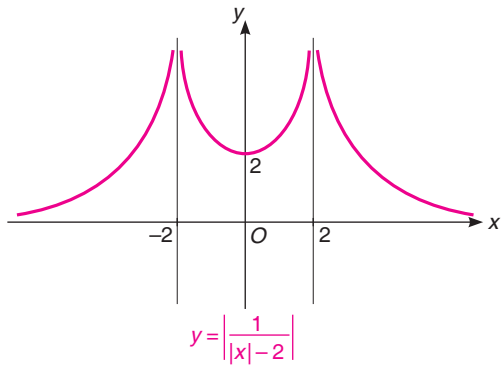
$y = \frac{1}{x-2}$

(ii)

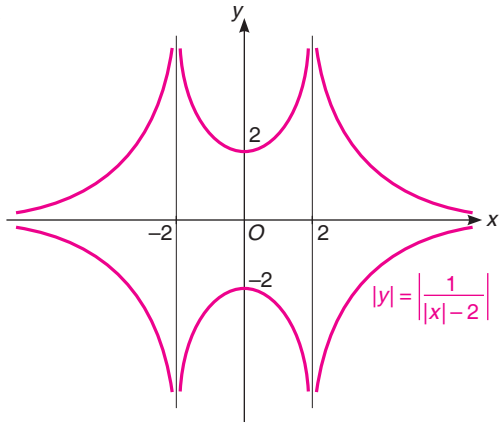


$y = \frac{1}{|x|-2}$

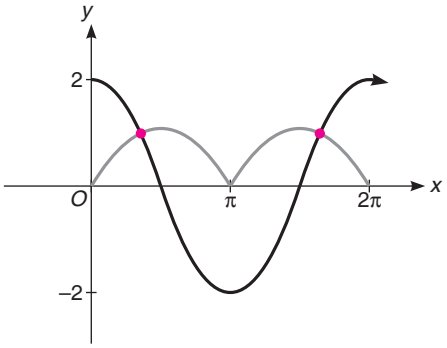
(iii)



(iv)



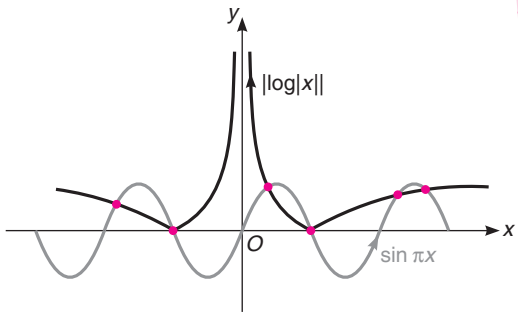
8.



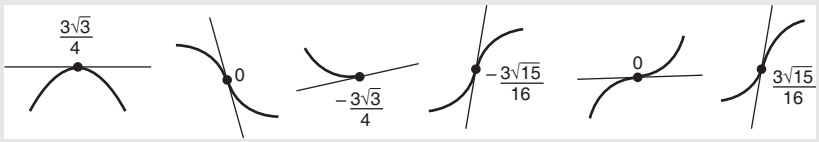
Since, $2\cos x$ and $|\sin x|$, intersects at two points for $x \in [0, 2\pi]$.

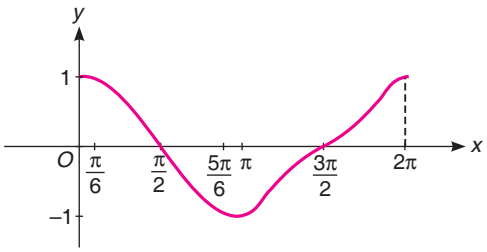
$\therefore$ number of solutions are four when $x \in [0, 4\pi]$.

9.



From above figure we have six solutions.

x	0	$\pi/6$	$\pi/2$	$5\pi/6$	x_2	$\left(x_2, \frac{3\pi}{2}\right)$	$\frac{3\pi}{2}$	$\left(\frac{3\pi}{2}, x_4\right)$	x_4		2π
y'		0	-2	0	9/8		0		9/8		
y''		$\frac{-3\sqrt{3}}{2}$	0	$\frac{3\sqrt{3}}{2}$	0	–	0	+	0	–	
y	1										1

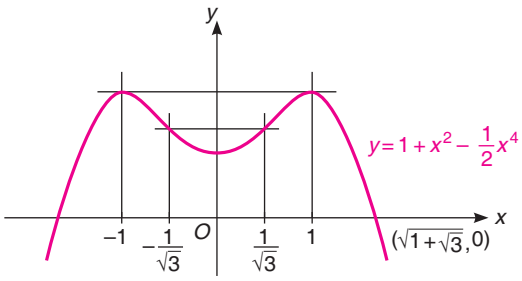


$y = \frac{1}{2}\sin 2x + \cos x$,

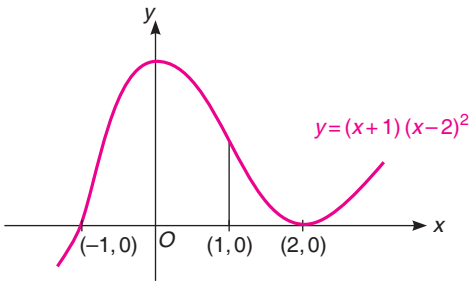
periodic with **period** 2π.

ASYMPTOTES, SINGULAR POINTS AND CURVE TRACING

1.



2.



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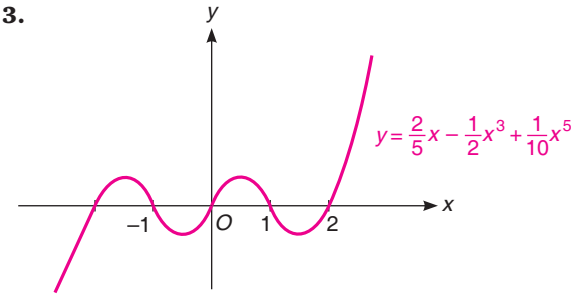
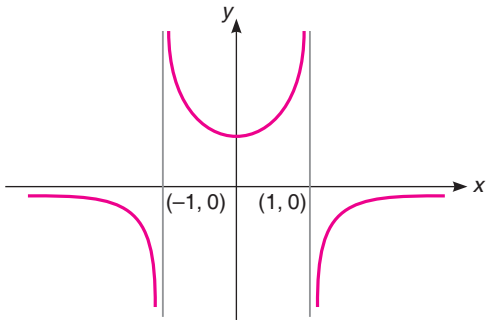
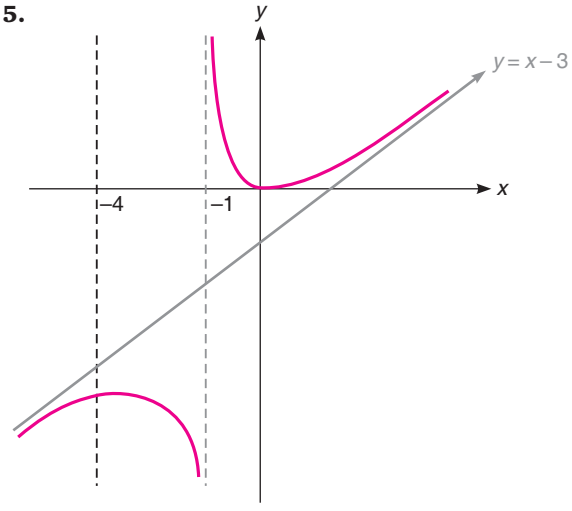


4.



5.



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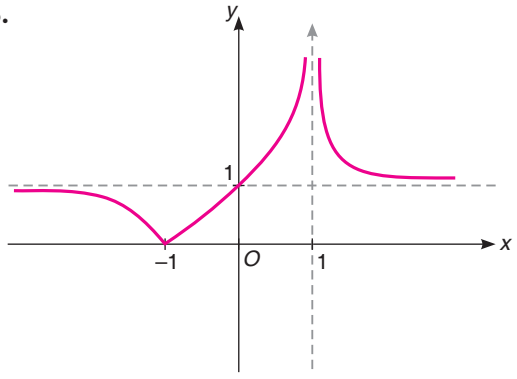

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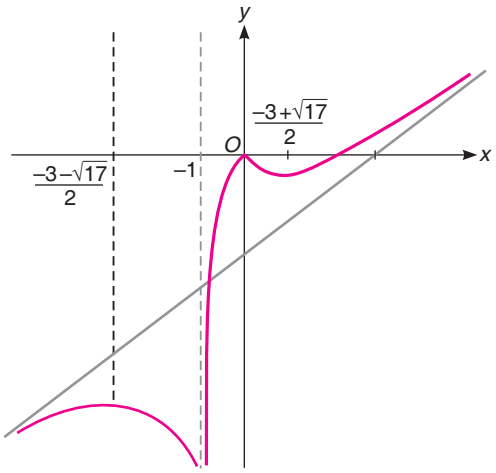
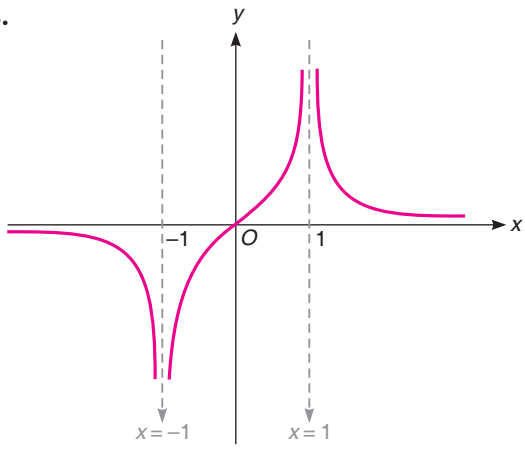
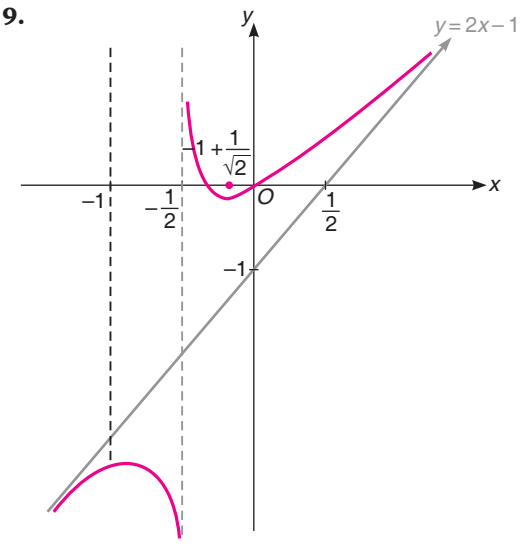

y
x
O
$\frac{-3+\sqrt{17}}{2}$
$\frac{-3-\sqrt{17}}{2}$
−1

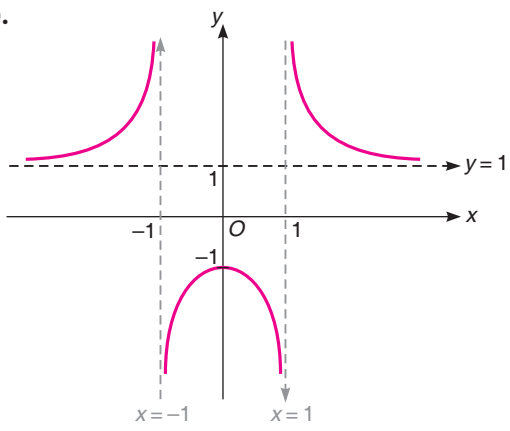
8.


y
x
O
−1
1
x = −1
x = 1

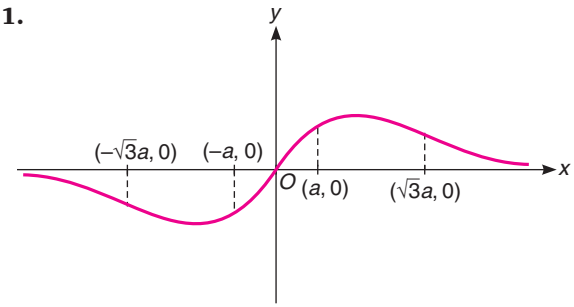
9.


y
x
y = 2x − 1
$-1+\frac{1}{\sqrt{2}}$
−1
$-\frac{1}{2}$
O
$\frac{1}{2}$
−1

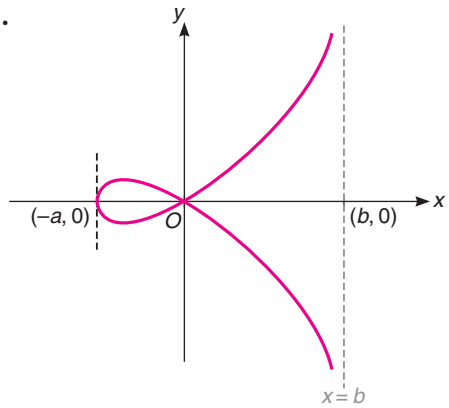
10.


y
x
y = 1
1
−1
O
1
−1
x = −1
x = 1

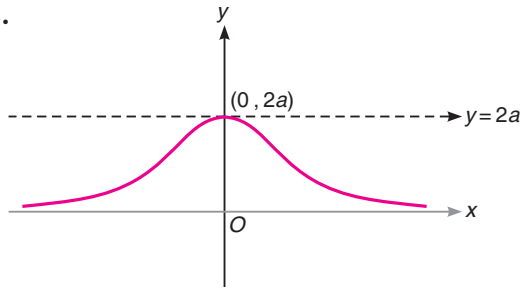
11.


y
x
$(-\sqrt{3}a, 0)$
$(-a, 0)$
O
$(a, 0)$
$(\sqrt{3}a, 0)$

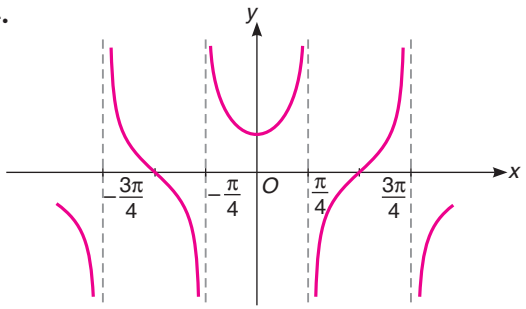
12.


y
x
$(-a, 0)$
O
$(b, 0)$
x = b

13.



14.



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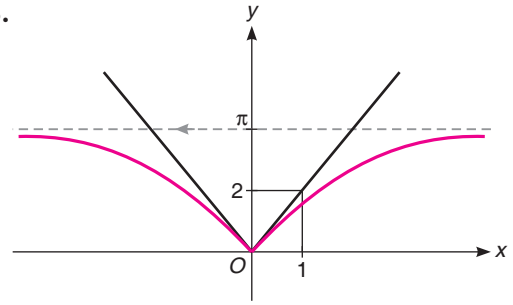
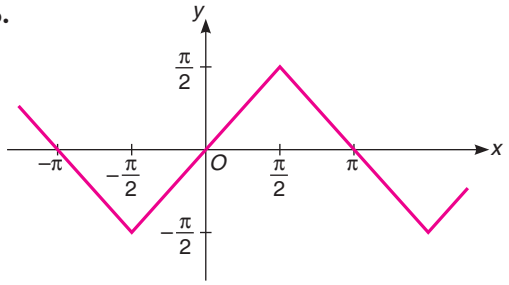


16.



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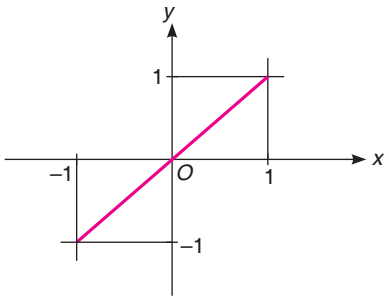

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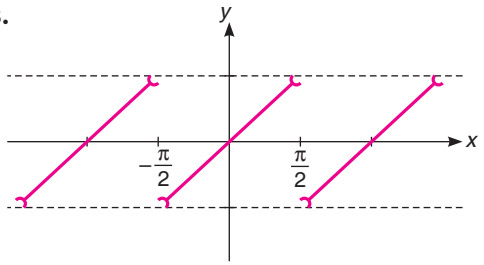

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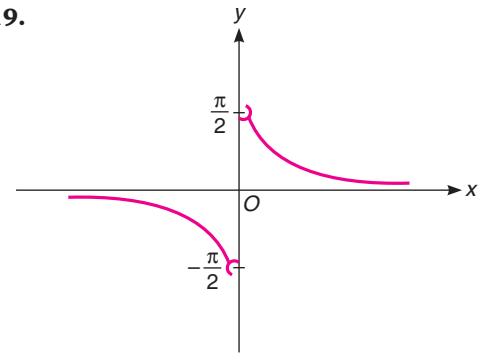

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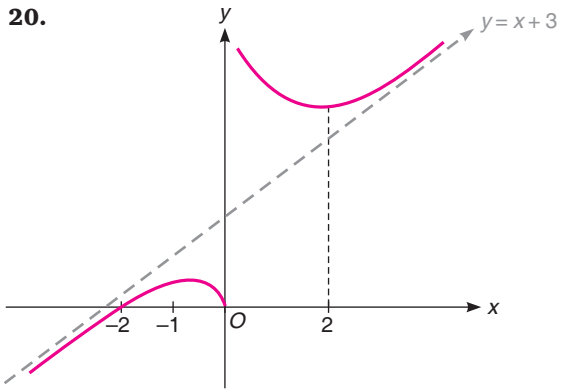

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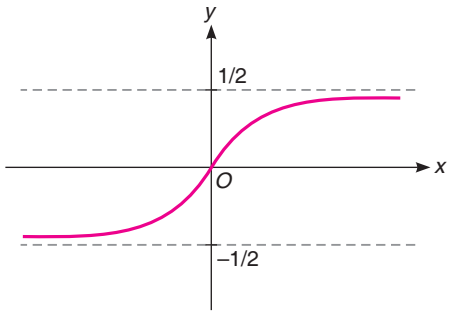

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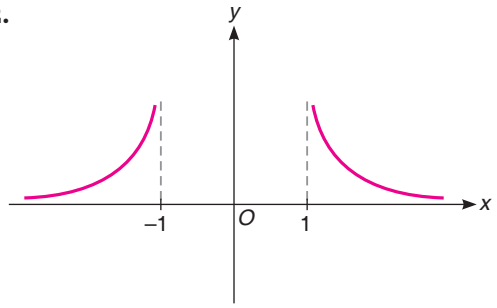

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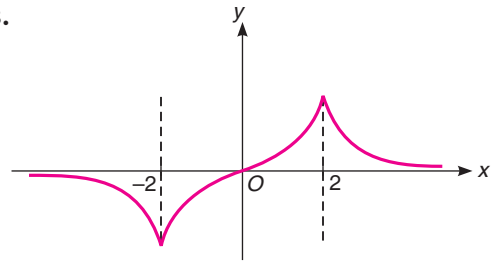

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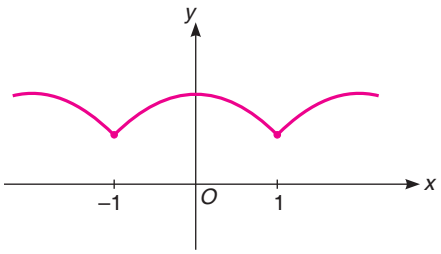

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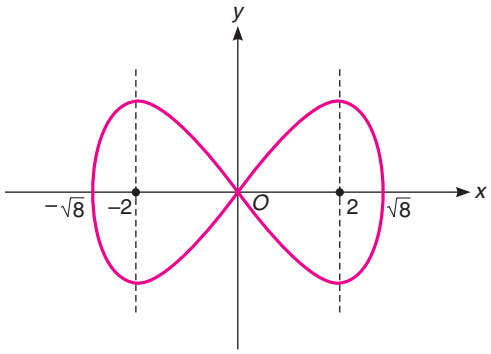

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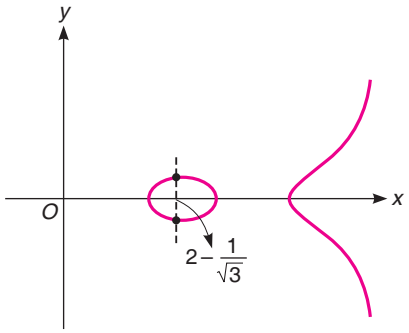

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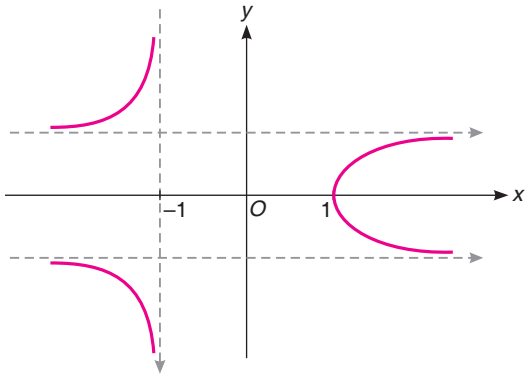

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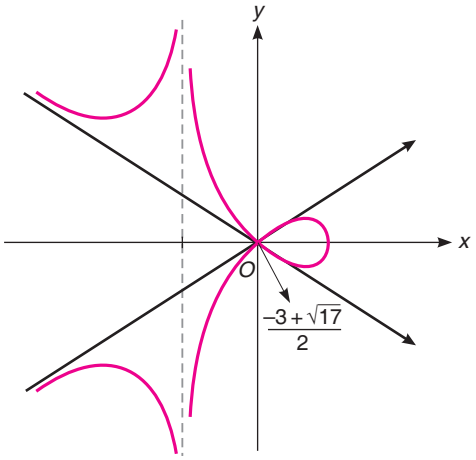

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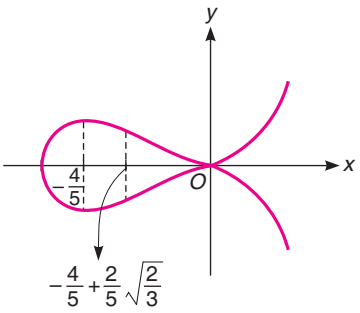

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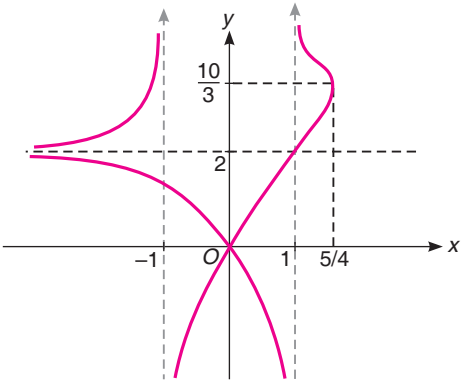

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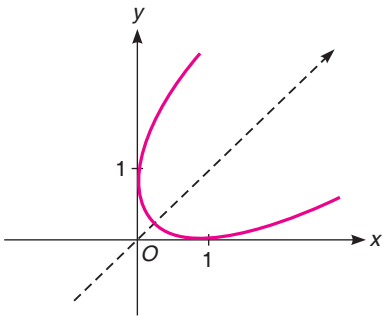

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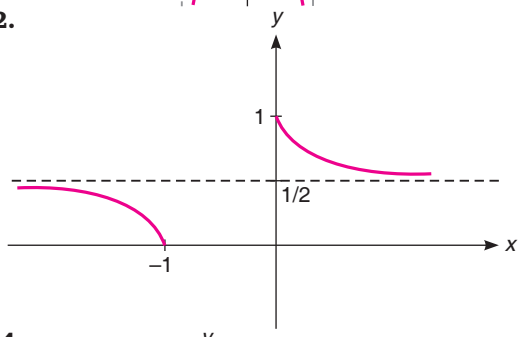

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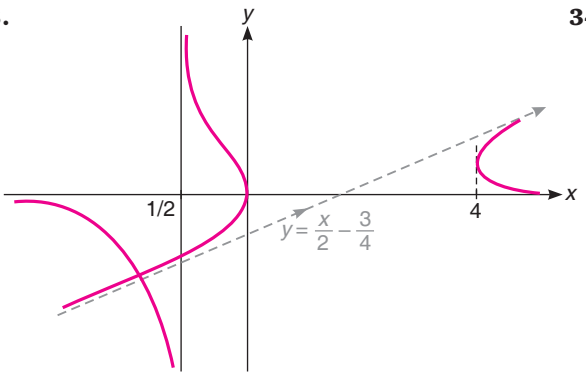

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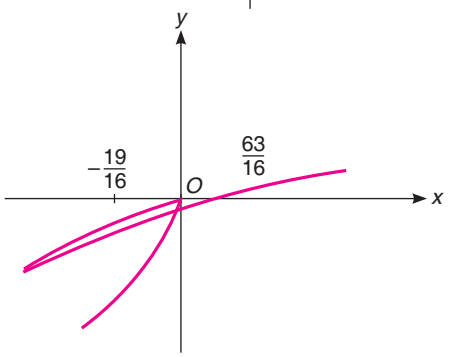

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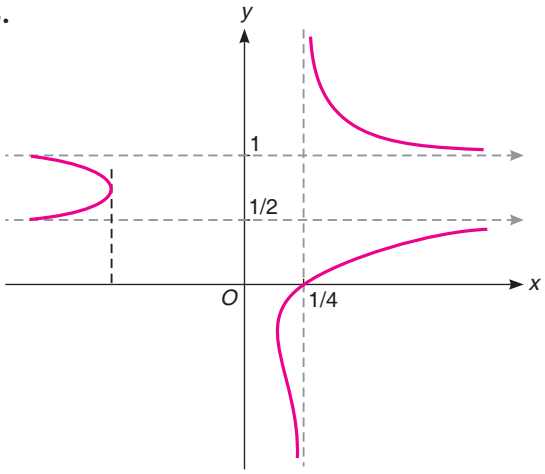

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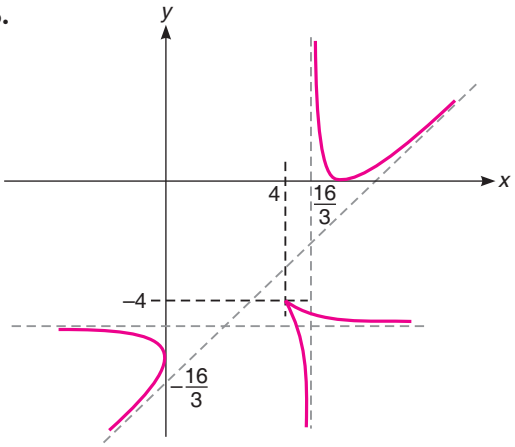

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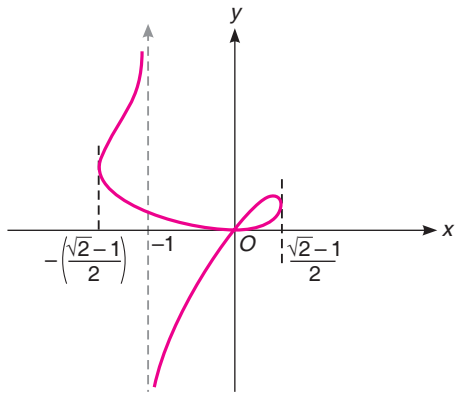

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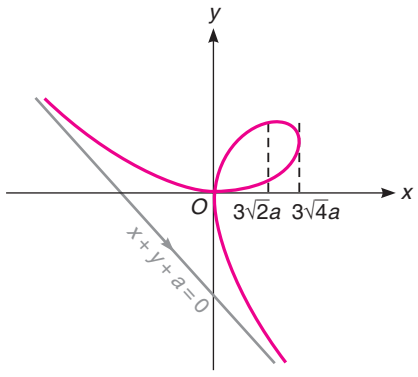

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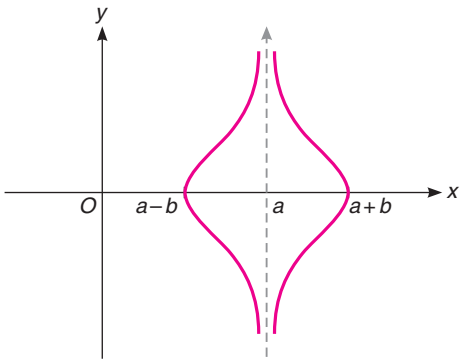

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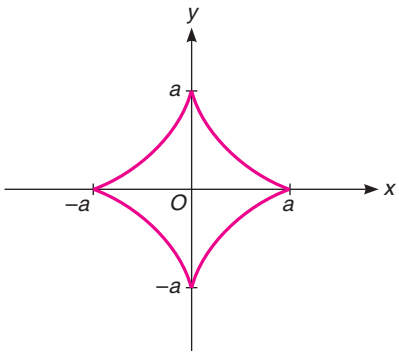

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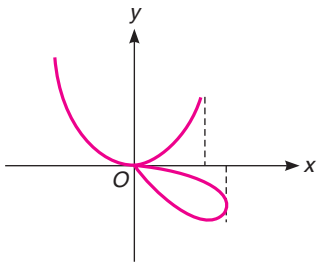

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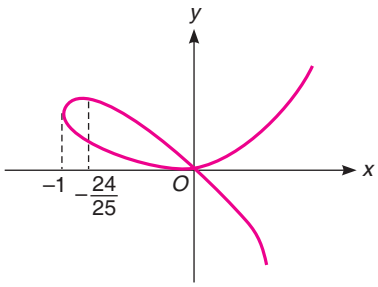

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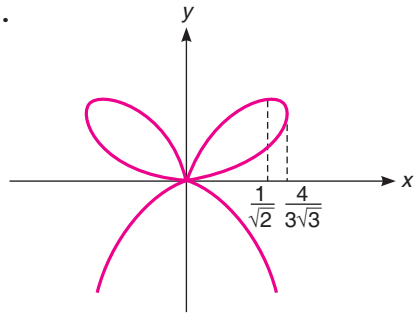

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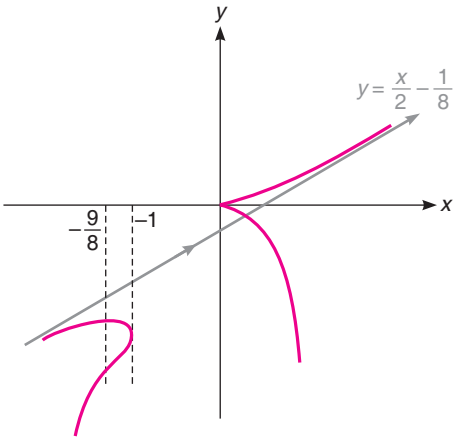

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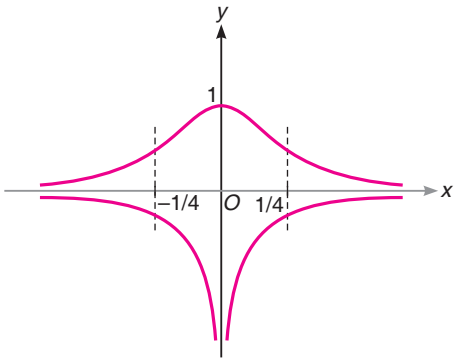

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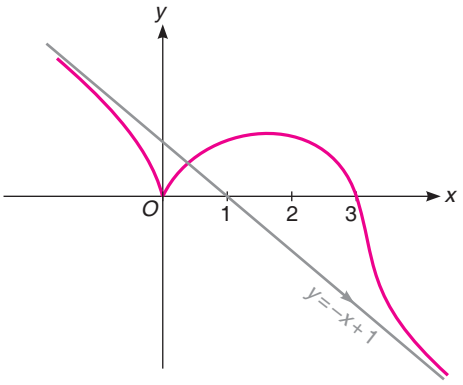

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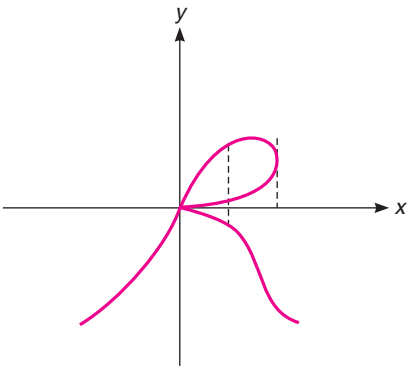

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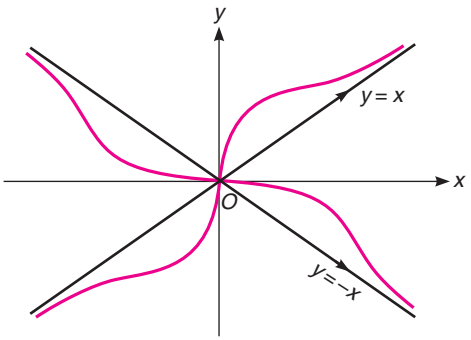

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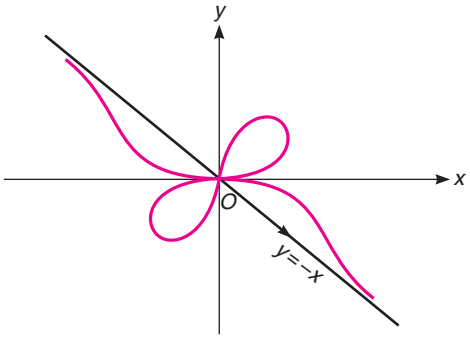

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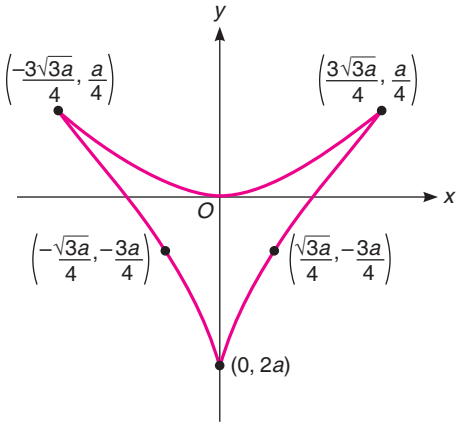

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Skills in Mathematics for

JEE Main & Advanced

Trigonometry

With Sessionwise Theory & Exercises

Amit M. Agarwal

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PREFACE

"YOU CAN DO ANYTHING IF YOU SET YOUR MIND TO IT, I TEACH TRIGONOMETRY TO JEE ASPIRANTS BUT BELIEVE THE MOST IMPORTANT FORMULA IS
COURAGE + DREAMS = SUCCESS"

It is a matter of great pride and honour for me to have received such an overwhelming response to the previous editions of this book from the readers. In a way, this has inspired me to revise this book thoroughly as per the changed pattern of JEE Main & Advanced. I have tried to make the contents more relevant as per the needs of students, many topics have been re-written, a lot of new problems of new types have been added in etc. All possible efforts are made to remove all the printing errors that had crept in previous editions. The book is now in such a shape that the students would feel at ease while going through the problems, which will in turn clear their concepts too.

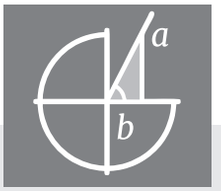
A Summary of changes that have been made in Revised & Enlarged Edition

- Theory has been completely updated so as to accommodate all the changes made in JEE Syllabus & Pattern in recent years.
- The most important point about this new edition is, now the whole text matter of each chapter has been divided into small sessions with exercise in each session. In this way the reader will be able to go through the whole chapter in a systematic way.
- Just after completion of theory, Solved Examples of all JEE types have been given, providing the students a complete understanding of all the formats of JEE questions & the level of difficulty of questions generally asked in JEE.
- Along with exercises given with each session, a complete cumulative exercises have been given at the end of each chapter so as to give the students complete practice for JEE along with the assessment of knowledge that they have gained with the study of the chapter.
- Previous Years questions asked in JEE Main & Adv, IIT-JEE & AIEEE have been covered in all the chapters.

However I have made the best efforts and put my all teaching experience in revising this book. Still I am looking forward to get the valuable suggestions and criticism from my own fraternity i.e. the fraternity of JEE teachers.

I would also like to motivate the students to send their suggestions or the changes that they want to be incorporated in this book. All the suggestions given by you all will be kept in prime focus at the time of next revision of the book.

Amit M. Agarwal



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SYLLABUS

JEE MAIN

Trigonometry

Trigonometrical identities and equations. Trigonometrical functions. Inverse trigonometrical functions and their properties. Heights and Distances.

JEE Advanced

Trigonometry

Trigonometric functions, their periodicity and graphs, addition and subtraction formulae, formulae involving multiple and sub-multiple angles, general solution of trigonometric equations

Relations between sides and angles of a triangle, sine rule, cosine rule half-angle formula and the area of a triangle, inverse trigonometric functions (principal value only).

CHAPTER

01

Trigonometric Functions and Identities

Learning Part

Session 1
- Measurement of Angles

Session 2
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- Application of Basic Trigonometry on Eliminating Variables or Parameters and Geometry

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Practice Part

- JEE Type Examples
- Chapter Exercises

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Exercises with the symbol can be practised on your mobile. See inside cover page to activate for free.

Session 1

Measurement of Angles

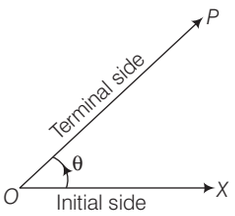
The word 'Trigonometry' is derived from two Greek words.

(i) trigonon (ii) metron

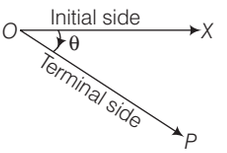
The word trigonon means a triangle and the word metron mean a measure. Hence, trigonometry means measuring the sides and angle of triangle. The subject was originally develop to solve geometric problems involving triangle.

Angle

In trigonometry, as in case of geometry. Angle is measure of rotation from the direction of one ray about its initial point. The original ray called the initial side and the final position of the ray after rotation is called the terminal side of the angle. The point of rotation is called the vertex. If the direction of rotation is anti-clockwise, the angle is said to be positive and if the direction of rotation is clockwise, then the angle is negative



(i) Positive angle



(ii) Negative angle

Measurement of Angles

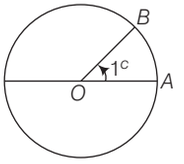
There are three systems used for the measurement of angles.

1. Sexagesimal system or English system (degree)
2. Circular measurement (radian)
3. Centesimal system or French system (grade)

We shall describe the units of measurement of angle which are most commonly used, i.e sexagesimal system (degree measure) and circular measurement (radian measure)

1. **Sexagesimal or Degree measure** If a rotation from the initial side to the terminal side is (1/360)th of a revolution, the angle is said to have a measure of one degree, written as 1°. A degree is divided into 60 minutes, and a minute is divided into 60 seconds. One sixtieth of a degree is called a minute, written as 1′; one sixtieth of minute is called a second, written as 1″. Thus, $1° = 60'$ and $1' = 60''$.

2. **Circular measurement or Radian measure** The angle subtended at the centre of a circle by an arc whose length is equal to the radius of the circle is called a radian and denoted by 1^c.



3. **Centesimal or French system** In this system of measurement a right angle is divided into 100 equal parts called **Grades**. Each grade is then divided into 100 equal parts called **minutes** and each minute is further divided into 100 equal parts called **Seconds**.

Thus, right angle $= 100^g$

$$1° = 100'$$
$$1' = 100''$$

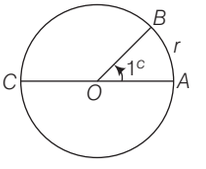
Note

Angle of 90° is called a right angle 1′ of centesimal system ≠ 1′ of sexagesimal system 1″ of centesimal system ≠ 1″ of sexagesimal system.

This system of measurement of angles is not commonly used and so here we will not study this system of measurement of angles.

Radian is a Constant Angle

Let ABC be a circle whose centre is O and radius is r. Let the length of arc AB of the circle by equal to r. Then by the definition of radian.



$\angle AOB = 1$ radian

Produce AO and let it cut the circle at C. Then AC is a diameter of the circle and arc ABC is equal to half the circumference of the circle.

Also $\angle AOC = 2$ right angle $= 180°$

By geometry, we know that angles subtended at the centre of a circle are proportional to the lengths of the arcs which subtend them

$$\therefore \quad \frac{\angle AOB}{\angle AOC} = \frac{\text{arc } AB}{\text{arc } ABC} \text{ or } \frac{1^c}{180^\circ} = \frac{r}{\frac{2\pi r}{2}}$$

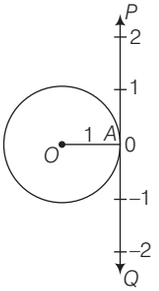
[$\because$ circumference of the circle $= 2\pi r$]

$$\therefore \quad 1 \text{ radian} = \frac{180^\circ}{\pi} = \frac{2 \text{ right angle}}{\pi} = \text{constant}$$

[since a right angle and π are constants]

Relation between Radians and Real Numbers

Consider a unit circle with center O. Let A be any point on the circle. Consider OA as the initial side of an angle. Then the length of an arc of the circle gives the radian measure of the angle which the arc subtends at the center of the circle. Consider line PAQ which is tangent to the circle at A. Let point A represents the real number zero, AP represents a positive real number, and AQ represents a negative real number. If we rope line AP in the counter-clockwise direction along the circle, and AQ in the clockwise direction, then every real number corresponds to a radian measure and conversely. Thus, radian measures and real numbers can be considered as one and the same.



Relation between Degree and Radian

It follows that the magnitude in radian of one complete revolution (360 degree) is the length of the entire circumference divided by the radius, or $\frac{2\pi r}{r}$ or 2π.

Therefore, 2π radian $= 360^\circ$

or π radian $= 180^\circ$

or 1 radian $= \frac{180^\circ}{\pi} = 57^\circ 16'$ (approximately)

Again, $180^\circ = \pi$ radian

$\therefore 1^\circ = \frac{\pi}{180}$ radian $= 0.01746$ radian (approximately)

Thus radian measure of an angle $= \frac{\pi}{180} \times$ degree measure of the angle and degree measure of an angle $= \frac{180}{\pi} \times$ radian measure of the angle.

Thus if the measure of an angle in degrees, and radians be D and C respectively, then

$$\frac{D}{180} = \frac{C}{\pi}$$

The Relation between Degree Measures and Radian Measures of Some Common Angles

Degree	30°	45°	60°	90°	180°	270°	360°
Radians	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

Note

(i) Radian is the unit to measure angle and **it does not means that π stands for 180°**, π is a real number. Where as π^c **stands for 180°**.

Remember the relation **π radians = 180 degrees = 200 grade**.

(ii) The number of radians in an angle subtended by an arc of a circle at the centre is equal to $\frac{\text{arc}}{\text{radius}}$.

$$\Rightarrow \quad \theta = \frac{s}{r}$$

Example 1. Convert 40°21′ into radian measure.

Sol. We know that $180^\circ = \pi$ radian.

Hence $40^\circ 21' = 40\frac{1}{3}$ degree

$$= \frac{\pi}{180} \times \frac{121}{3} \text{ radian} = \frac{121\pi}{540} \text{ radian.}$$

Therefore $40^\circ 21' = \frac{121\pi}{540}$ radian.

Example 2. Express the following angle in degrees.

(i) $\left(\frac{5\pi}{12}\right)^c$ (ii) $-\left(\frac{7\pi}{12}\right)^c$

(iii) $\frac{1^c}{3}$ (iv) $-\frac{2\pi^c}{9}$

Sol. (i) $\left(\frac{5\pi}{12}\right)^c = \left(\frac{5\pi}{12} \times \frac{180}{\pi}\right)^\circ = (5 \times 15)^\circ = 75^\circ$

(ii) $-\left(\frac{7\pi}{12}\right)^c = -\left(\frac{7\pi}{12} \times \frac{180}{\pi}\right)^\circ$

$= -(7 \times 15)^\circ = -105^\circ$

(iii) $\left(\frac{1}{3}\right)^c = -\left(\frac{1}{3} \times \frac{180}{\pi}\right)^\circ = -\left(\frac{60}{\pi}\right)^\circ = 19^\circ\ 5'\ 27''$

(iv) $-\frac{2\pi^c}{9} = -\left(\frac{2\pi}{9} \times \frac{180}{\pi}\right)^\circ = -(2 \times 20)^\circ = -40^\circ$

Example 3. Express the following angle in degrees, minutes and seconds form

$(321.9)^\circ$

Sol. $(321.9)^\circ = 321^\circ + 0.9^\circ$

$= 321^\circ + (0.9^\circ \times 60)'$

$= 321^\circ + 54' = 321^\circ 54'$

Example 4. In ΔABC, $m\angle A = \frac{2\pi^c}{3}$ and $m\angle B = 45°$. Find $m\angle C$ in both the systems.

Sol. $m\angle A = \frac{2\pi^c}{3} = \left(\frac{2\pi}{3} \times \frac{180}{\pi}\right)^\circ = 120°$

$m\angle B = 45°$

$= \left(45 \times \frac{\pi}{180}\right)^c = \frac{\pi^c}{4}$

In ΔABC, $m\angle A + m\angle B + m\angle C = 180°$

$\because$ The sum of angles of a triangle is 180°

$\Rightarrow \quad 120° + 45° + m\angle C = 180°$

$\Rightarrow \quad 165° + m\angle C = 180°$

$\Rightarrow \quad m\angle C = 180° - 165°$

$\Rightarrow \quad m\angle C = 15°$

$\Rightarrow \quad m\angle C = \left(15 \times \frac{\pi}{180}\right)^\circ$

$\therefore \quad m\angle C = \frac{\pi^c}{12}$

Example 5. The sum of two angles is $5\pi^c$ and their difference is 60°. Find the angles in degrees.

Sol. Let the angles be x and y in degrees.

Then, $x + y = 5\pi^c \Rightarrow x + y = \left(5\pi \times \frac{180}{\pi}\right)^\circ$

$\therefore \quad x + y = 900°$ ...(i)

$x - y = 60°$ ...(ii)

On adding Eqs. (i) and (ii), we get

$2x = 960°$

$\therefore \quad x = 480°$

On putting $x = 480°$ in Eq. (i), we get

$480° + y = 900°$

$\therefore \quad y = 420°$

$\therefore$ Hence, the angles are 480° and 420°.

Example 6. One angle of a quadrilateral has measure $\frac{2\pi^c}{5}$ and the measures of other three angles are in the ratio 2 : 3 : 4. Find their measures in radians and in degrees.

Sol. One angle $= \frac{2\pi^c}{5} = \left(\frac{2\pi}{5} \times \frac{180}{\pi}\right)^\circ = 72°$

Since, measures of other three angles are in the ratio 2 : 3 : 4. Let the angle be $2k$, $3k$ and $4k$ measured in degree.

$\therefore$ Sum of all angles of quadrilateral = 360°

$\Rightarrow \quad 72° + 2k + 3k + 4k = 360°$

$\Rightarrow \quad 9k = 288° \Rightarrow k = 32°$

$\therefore$ The other three angles are

$2k = 2 \times 32 = 64°$

$3k = 3 \times 32 = 96°$

$4k = 4 \times 32 = 128°$

$\therefore$ The other three angles measured in degree are 64°, 96° and 128°.

The angles in radians are

$64° = \left(64 \times \frac{\pi}{180}\right)^c = \frac{16\pi^c}{45}$

$96° = \left(96 \times \frac{\pi}{180°}\right)^c = \frac{8\pi^c}{15}$

$128° = \left(128 \times \frac{\pi}{180}\right)^c = \frac{32\pi^c}{45}$

$\therefore$ The other three angles measured in radian are

$\frac{16\pi^c}{45}, \frac{8\pi^c}{15}$ and $\frac{32\pi^c}{45}$.

Example 7. Express the following angles in radians.

(i) 120° (ii) $-600°$

(iii) $-144°$

Sol. (i) $120° = \left(120 \times \frac{\pi}{180}\right)^c = \frac{2\pi^c}{3}$

(ii) $-600° = -\left(600 \times \frac{\pi}{180}\right)^c = -\frac{10\pi}{3}$

(iii) $-144° = \left(-144 \times \frac{\pi}{180}\right) = -\frac{4\pi^c}{5}$

Example 8. If the three angles of a quadrilateral are 60°, 60^g and $\frac{5\pi}{6}$. Then, find the fourth angle.

Sol. First angle = 60°

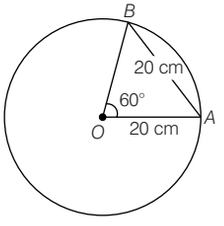
Second angle $= 60^g = 60 \times \frac{90}{100}$ degrees $= 54°$

Third angle $= \frac{5\pi}{6}$ radian $= \frac{5 \times 180}{6} = 150°$

$\therefore \quad$ Fourth angle $= 360° - (60° + 54° + 150°) = 96°$

Example 9. In a circle of diameter 40 cm, the length of a chord is 20 cm. Find the length of minor arc corresponding to the chord.

Sol. Let arc $AB = S$. It is given that $OA = 20$ cm and chord $AB = 20$ cm. Therefore, ΔOAB is an equilateral triangle.



Hence, $\angle AOB = 60°$

$= \left(60 \times \frac{\pi}{180}\right)^c = \left(\frac{\pi}{3}\right)^c$

Now, $\theta = \frac{\text{arc}}{\text{radius}}$

$\Rightarrow \quad \frac{\pi}{3} = \frac{S}{20} \Rightarrow S = \frac{20\pi}{3}$ cm

Example 10. In the circle of 5 cm. radius, what is the length of the arc which subtends and angle of 33°15′ at the centre.

Sol. Here, $r = 5$ cm; $15' = \frac{15}{60} = \left(\frac{1}{4}\right)^{\circ}$

$$\therefore \quad \theta = 33°15' = 33 + \frac{1}{4} = \frac{133}{4} \text{ degrees}$$

$$= \frac{133}{4} \times \frac{\pi}{180} = \frac{133}{4} \times \frac{22}{7 \times 180} = \frac{1463}{2520} \text{ radians}$$

Now, $\theta = \frac{l}{r}$

$$\therefore \quad l = \theta r = \frac{1463}{2520} \times 5 = 2\frac{65}{72} \text{ cm (approx.)}$$

Example 11. The minute hand of a watch is 35 cm long. How far does its tip move in 18 minutes? $\left(\text{use } \pi = \frac{22}{7}\right)$

Sol. The minute hand of a watch completes one revolution in 60 minutes. Therefore the angle traced by a minute hand in 60 minutes $= 360° = 2\pi$ radians.

$\therefore$ Angle traced by the minute hand in 18 minutes

$$= 2\pi \times \frac{18}{60} \text{ radians} = \frac{3\pi}{5} \text{ radians}$$

Let the distance moved by the tip in 18 minutes be l, then $l = r\theta$

$$= 35 \times \frac{3\pi}{5} = 21\pi = 21 \times \frac{22}{7} = 66 \text{ cm}$$

Example 12. The wheel of a railway carriage is 40 cm. in diameter and makes 6 revolutions in a second; how fast is the train going?

Sol. Diameter of the wheel = 40 cm

$\therefore$ radius of the wheel = 20 cm

Circumference of the wheel $= 2\pi r = 2\pi \times 20 = 40\pi$ cm

Number of revolutions made in 1 second = 6

$\therefore$ Distance covered in 1 second $= 40\pi \times 6 = 240\pi$ cm

$\therefore$ Speed of the train $= 240\pi$ cm/sec.

Example 13. Assuming that a person of normal sight can read print at such a distance that the letters subtend an angle of 5′ at his eye, find the height of the letters that he can read at a distance of 12 metres.

Sol. Let the height of the letters be h metres.

Now, h many be considered as the arc of a circle of radius 12 m, which subtends an angle of 5′ at its centre.

$$\therefore \quad \theta = 5' = \left(\frac{5}{60} \times \frac{\pi}{180}\right) \text{radians} = \left(\frac{\pi}{12 \times 180}\right) \text{ radian}$$

and $r = 12$ m

$$\therefore \quad h = r\theta = 12 \times \frac{\pi}{12 \times 180} = \left(\frac{\pi}{180}\right) \text{metres} \approx 1.7 \text{ cm}$$

Exercise for Session 1

1. The difference between two acute angles of a right angle triangle is $\frac{3\pi}{10}$ rad. Find the angles in degree.
2. Find the length of an arc of a circle of radius 6 cm subtending an angle of 15° at the centre.
3. A horse is tied to post by a rope. If the horse moves along circular path always keeping the tight and describes 88 m, when it has traced out 72° at centre, find the length of rope.
4. Find the angle between the minute hand and hour hand of a clock, when the time is 7 : 30 pm.
5. If OQ makes 4 revolutions in 1s, find the angular velocity in radians per second.
6. If a train is moving on the circular path of 1500 m radius at the rate of 66 km/h, find the angle in radian, if it has in 10 second.
7. Find the distance from the eye at which a coin of 2.2 cm diameter should be held so as to conceal the full moon with angular diameter 30′.
8. The wheel of a railway carriage is 40 cm in diameter and makes 7 revolutions in a second, find the speed of train.
9. Assuming that a person of normal sight can read print at such a distance that the letters subtend an angle of 5′ at his eye, find the height of letters that he can read a distance of 12 m.
10. For each natural number k, let C_K denotes the circle with radius k cm and centre at origin. On the circle C_K, a particle moves k cm in the counter-clockwise direction. After completing its motion on C_K, the particle moves on C_{K+1} in the radial direction. The motion of the particle continues in this manner. The particle starts at (1, 0). If the particle crosses the positive direction of the x-axis for the first time on the circle C_n, then n is equal to

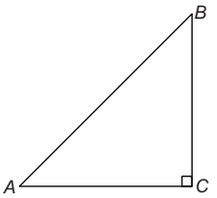
Session 2

Definition of Trigonometric Functions

Definition of Trigonometric Functions

An angle whose measure is greater than 0° but less than 90° is called an acute angle.

In a right angled triangle ABC, $\angle CAB = A$ and $\angle BCA = 90^\circ = \pi/2$. AC is the base, BC the altitude and AB is the hypotenuse. We refer to the base as the adjacent side and to the altitude as the opposite side. There are six trigonometric ratios, also called **trigonometric functions or circular functions** with reference to $\angle A$, the six ratio are



$\frac{BC}{AB} = \frac{\text{opposite side}}{\text{hypotenuse}}$, is called sine of A, and written as $\sin A$.

$\frac{AC}{AB} = \frac{\text{adjacent side}}{\text{hypotenuse}}$, is called the cosine of A, and written as $\cos A$.

$\frac{BC}{AC} = \frac{\text{opposite side}}{\text{adjacent side}}$, is called the tangent of A, and written as $\tan A$.

$\frac{AB}{BC} = \frac{\text{hypotenuse}}{\text{opposite side}}$, is called cosecant of A, and written as $\text{cosec } A$.

$\frac{AB}{AC} = \frac{\text{hypotenuse}}{\text{opposite side}}$, is called secant of A, and written as $\sec A$.

$\frac{AC}{BC} = \frac{\text{adjacent side}}{\text{opposite side}}$, is called cotangent of A, and written as $\cot A$.

Since, the hypotenuse is the greatest side in a right angle triangle, $\sin A$ and $\cos A$ can never be greater than unity and $\text{cosec } A$ and $\sec A$ can never be less than unity.

Hence, $|\sin A| \le 1$, $\cos A| \le 1|$, $|\text{cosec } A| \ge 1$, $|\sec A| \ge 1$, while $\tan A$ and $\cot A$ may have any numerical value lying between $-\infty$ to $+\infty$.

Note

Student must remember the following results

(i) $-1 \le \sin A \le 1$ (ii) $-1 \le \cos A \le 1$

(iii) $\text{cosec } A \ge 1$ or $\text{cosec } A \le -1$ (iv) $\sec A \ge 1$ or $\sec A \le -1$

(v) $\tan A \in R$ (vi) $\cot A \in R$

Some values of Trigonometrical Ratios

Students are already familiar with the values of sin, cos, tan, cot, sec and cosec of angles 0°, 30°, 45°, 60° and 90° which have been given in the following table

	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined
cot	undefined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	undefined
cosec	undefined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

Trigonometric Identities

Trigonometric identities are equalities that involve trigonometric functions that are true for every single value of the occurring variables. In other words, they are equations that hold true regardless of the value of the angles being chosen.

Trigonometric identities are as follows

1. $\sin^2 A + \cos^2 A = 1 \Rightarrow \cos^2 A = 1 - \sin^2 A$

   or $\sin^2 A = 1 - \cos^2 A$

2. $1 + \tan^2 A = \sec^2 A \Rightarrow \sec^2 A - \tan^2 A = 1$

3. $\cot^2 A + 1 = \text{cosec}^2 A$

   $\Rightarrow \text{cosec}^2 A - \cot^2 A = 1$

4. $\tan A = \dfrac{\sin A}{\cos A}$ and $\cot A = \dfrac{\cos A}{\sin A}$

5. Fundamental inequalities: For $0 < A < \pi/2$;

$$0 < \cos A < \frac{\sin A}{A} < \frac{1}{\cos A}$$

6. It is possible to express trigonometrical ratios in terms of any one of them as,

$$\sin\theta = \frac{1}{\sqrt{1+\cot^2\theta}},$$

$$\cos\theta = \frac{\cot\theta}{\sqrt{1+\cot^2\theta}}, \tan\theta = \frac{1}{\cot\theta},$$

$$\operatorname{cosec}\theta = \sqrt{1+\cot^2\theta}, \sec\theta = \frac{\sqrt{1+\cot^2\theta}}{\cot\theta}$$

i.e. all trigonometrical functions have been expressed in terms of $\cot\theta$.

Similarly, we can express all trigonometric function in other trigonometric ratios.

Example 14. Show that $2(\sin^6 x + \cos^6 x) - 3(\sin^4 x + \cos^4 x) + 1 = 0$.

Sol. $2(\sin^6 x + \cos^6 x) - 3(\sin^4 x + \cos^4 x) + 1$

$= 2[(\sin^2 x)^3 + (\cos^2 x)^3] - 3(\sin^4 x + \cos^4 x) + 1$

$= 2[(\sin^2 x + \cos^2 x)^3 - 3\sin^2 x \cos^2 x$

$(\sin^2 x + \cos^2 x) - 3[\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x] + 1$

$= 2[1 + 3\sin^2 x \cos^2 x] - 3[1 - 2\sin^2 x \cos^2 x] + 1 = 0$

Example 15. Show that

(i) $\sin^8 A - \cos^8 A = (\sin^2 A - \cos^2 A)(1 - 2\sin^2 A \cdot \cos^2 A)$

(ii) $\dfrac{1}{\sec A - \tan A} - \dfrac{1}{\cos A} = \dfrac{1}{\cos A} - \dfrac{1}{\sec A + \tan A}$

Sol. (i) L.H.S. $= \sin^8 A - \cos^8 A = (\sin^4 A)^2 - (\cos^4 A)^2$

$= (\sin^4 A - \cos^4 A)(\sin^4 A + \cos^4 A)$

$= (\sin^2 A - \cos^2 A)(\sin^2 A + \cos^2 A)$

$[(\sin^2 A + \cos^2 A)^2 - 2\sin^2 A \cos^2 A]$

$= (\sin^2 A - \cos^2 A)(1 - 2\sin^2 A \cos^2 A)$

$[\because \sin^2 A + \cos^2 A = 1]$

(ii) Given, $\dfrac{1}{\sec A - \tan A} - \dfrac{1}{\cos A} = \dfrac{1}{\cos A} - \dfrac{1}{\sec A + \tan A}$

or $\dfrac{1}{\sec A - \tan A} + \dfrac{1}{\sec A + \tan A} = \dfrac{1}{\cos A} + \dfrac{1}{\cos A}$

Here, R.H.S. $= \dfrac{2}{\cos A}$

Now L.H.S. $= \dfrac{1}{\sec A - \tan A} \dfrac{1}{\sec A + \tan A}$

$= \dfrac{\sec A + \tan A + \sec A - \tan A}{(\sec A - \tan A)(\sec A + \tan A)} = \dfrac{2}{\cos A}$

Thus, L.H.S. = R.H.S.

Example 16. If $\tan\theta + \sec\theta = 1.5$, find $\sin\theta$, $\tan\theta$ and $\sec\theta$.

Sol. Given, $\sec\theta + \tan\theta = \dfrac{3}{2}$ ...(i)

Now, $\sec\theta - \tan\theta = \dfrac{1}{\sec\theta + \tan\theta} = \dfrac{2}{3}$ ...(ii)

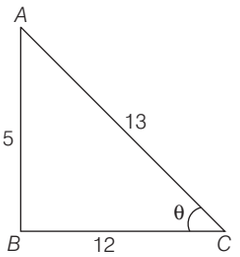
Adding Eqs. (i) and (ii), we get

$$2\sec\theta = \frac{3}{2} + \frac{2}{3} = \frac{13}{6}$$

$\therefore \quad \sec\theta = \dfrac{13}{12}$

$\therefore \quad \tan\theta = \dfrac{5}{12}$

and $\sin\theta = \dfrac{5}{13}$



Example 17. If $\dfrac{\cos^4 A}{\cos^2 B} + \dfrac{\sin^4 A}{\sin^2 B} = 1$, then prove that

(i) $\sin^4 A + \sin^4 B = 2\sin^2 A \sin^2 B$

(ii) $\dfrac{\cos^4 B}{\cos^2 A} + \dfrac{\sin^4 B}{\sin^2 A} = 1$

Sol. Given, $\dfrac{\cos^4 A}{\cos^2 B} + \dfrac{\sin^4 A}{\sin^2 B} = 1\,(\cos^2 A + \sin^2 A)$

or $\dfrac{\cos^4 A}{\cos^2 B} - \cos^2 A = \sin^2 A - \dfrac{\sin^4 A}{\sin^2 B}$

or $\dfrac{\cos^2 A(\cos^2 A - \cos^2 B)}{\cos^2 B} = \sin^2 A \dfrac{(\sin^2 B - \sin^2 A)}{\sin^2 B}$

or $\dfrac{\cos^2 A}{\cos^2 B}(\cos^2 A - \cos^2 B) = \dfrac{\sin^2 A}{\sin^2 B}$

$[(1 - \cos^2 B) - (1 - \cos^2 A)]$

or $\dfrac{\cos^2 A}{\cos^2 B}(\cos^2 A - \cos^2 B) = \dfrac{\sin^2 A}{\sin^2 B}(\cos^2 A - \cos^2 B)$

or $(\cos^2 A - \cos^2 B)\left(\dfrac{\cos^2 A}{\cos^2 B} - \dfrac{\sin^2 A}{\sin^2 B}\right) = 0$

When $\cos^2 A - \cos^2 B = 0$, we have

$$\cos^2 A = \cos^2 B \qquad \text{...(i)}$$

When $\dfrac{\cos^2 A}{\cos^2 B} - \dfrac{\sin^2 A}{\sin^2 B} = 0$, we have

$$\cos^2 A \sin^2 B = \sin^2 A \cos^2 B$$

or $\cos^2 A(1-\cos^2 B) = (1-\cos^2 A)\cos^2 B$

or $\cos^2 A - \cos^2 A\cos^2 B = \cos^2 B - \cos^2 A\cos^2 B$

or $\cos^2 A = \cos^2 B$ ...(ii)

Thus, in both the cases, $\cos^2 A = \cos^2 B$. Therefore,

$\therefore \quad 1-\sin^2 A = 1-\sin^2 B$ or $\sin^2 A = \sin^2 B$ ...(iii)

(i) L.H.S. $= \sin^4 A + \sin^4 B$

$= (\sin^2 A - \sin^2 B)^2 + 2\sin^2 A \sin^2 B$

$= 2\sin^2 A \sin^2 B$ = R.H.S. $[\because \sin^2 A = \sin^2 B]$

(ii) L.H.S. $= \dfrac{\cos^4 B}{\cos^2 A} + \dfrac{\sin^4 B}{\sin^2 A} = \dfrac{\cos^4 B}{\cos^2 B} + \dfrac{\sin^4 B}{\sin^2 B}$

$= \cos^2 B + \sin^2 B = 1$ = R.H.S.

Example 18. If $\tan^2\theta = 1-e^2$, prove that $\sec\theta + \tan^3\theta \operatorname{cosec}\theta = (2-e^2)^{\frac{3}{2}}$

Sol. Given, $\tan^2\theta = 1-e^2$

Now, L.H.S. $= \sec\theta + \tan^3\theta \operatorname{cosec}\theta$

$= \sec\theta\left(1 + \tan^3\theta \dfrac{\operatorname{cosec}\theta}{\sec\theta}\right)$

$= \sec\theta(1 + \tan^3\theta \cdot \cot\theta) = \sec\theta(1+\tan^2\theta) = \sec\theta\sec^2\theta$

$= \sec^3\theta = (\sec^2\theta)^{\frac{3}{2}} = (1+\tan^2\theta)^{\frac{3}{2}} = (1+1-e^2)^{\frac{3}{2}} = (2-e^2)^{\frac{3}{2}}$

Example 19. For what real values of x and y is the equation $\sec^2\theta = \dfrac{4xy}{(x+y)^2}$ possible?

Sol. Here, $\sec^2\theta = \dfrac{4xy}{(x+y)^2}$

We know $\sec^2\theta \geq 1$ and $\dfrac{4xy}{(x+y)^2} \leq 1$ [as AM $\geq$ GM]

$\Rightarrow \quad \sec^2\theta = \dfrac{4xy}{(x+y)^2}$ is only possible if $\sec^2\theta = 1$

i.e. $\dfrac{4xy}{(x+y)^2} = 1, \forall\ x, y \in R^+$

or $4xy = (x+y)^2 \quad \forall\ x, y \in R^+$

$\Rightarrow \quad x^2 + y^2 + 2xy - 4xy = 0, \quad \forall\ x, y \in R^+$

$\Rightarrow \quad (x-y)^2 = 0, \quad \forall\ x, y \in R^+$

or $x = y; \quad \forall\ x, y \in R^+$

Example 20. Show that the equation $\sin\theta = x + \dfrac{1}{x}$ is impossible if x is real.

Sol. Given, $\sin\theta = x + \dfrac{1}{x}$

$\therefore \quad \sin^2\theta = x^2 + \dfrac{1}{x^2} + 2x \cdot \dfrac{1}{x}$

$= x^2 + \dfrac{1}{x^2} + 2 \geq 2$

which is not possible since $\sin^2\theta \leq 1$

Exercise for Session 2

1. Prove that $(\operatorname{cosec}\theta - \sin\theta)(\sec\theta - \cos\theta)(\tan\theta + \cot\theta) = 1$.
2. If $\cos^2\alpha - \sin^2\alpha = \tan^2\beta$, then show that $\tan^2\alpha = \cos^2\beta - \sin^2\beta$.
3. If $\sin^6\theta + \cos^6\theta - 1 = \lambda\sin^2\theta\cos^2\theta$, find the value of λ.
4. If $a\cos\theta - b\sin\theta = c$, then find the value of $a\sin\theta + b\cos\theta$.
5. Find the value of $3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x)$.
6. If $\sin\theta + \operatorname{cosec}\theta = 2$, then find the value of $\sin^{20}\theta + \operatorname{cosec}^{20}\theta$.
7. Let $F_k(x) = \dfrac{1}{k}(\sin^k x + \cos^k x)$, where $x \in R$ and $k \geq 1$, then find the value of $F_4(x) - F_6(x)$.
8. If $\dfrac{\sin^4 x}{2} + \dfrac{\cos^4 x}{3} = \dfrac{1}{5}$, then show that $\dfrac{\sin^8 x}{8} + \dfrac{\cos^8 x}{27} = \dfrac{1}{125}$.
9. If $\cot\theta + \tan\theta = x$ and $\sec\theta - \cos\theta = y$, then show that $\sin\theta\cdot\cos\theta = \dfrac{1}{x}$ or $\sin\theta\cdot\tan\theta = y$ or $(x^2y)^{2/3} - (xy^2)^{2/3} = 1$.
10. If $\sin A + \sin^2 A + \sin^3 A = 1$, then find the value of $\cos^6 A - 4\cos^4 A + 8\cos^2 A$.

Session 3

Application of Basic Trigonometry on Eliminating Variables or Parameters and Geometry

Application of Basic Trigonometry on Eliminating Variables or Parameters

As we know, parameter are those values which could vary, e.g. θ if parameter could take any value as;

$$\theta = 0°, 30°, 45°, 60°, 90°, 120°, \ldots$$

Thus, to eliminate these parameter, we have to use basic trigonometric formulae, it could be more clear by some examples :

Example 21. If $\operatorname{cosec}\theta - \sin\theta = m$ and $\sec\theta - \cos\theta = n$, eliminate θ.

Sol. Given, $\operatorname{cosec}\theta - \sin\theta = m$ or, $\dfrac{1}{\sin\theta} - \sin\theta = m$

or, $\dfrac{1-\sin^2\theta}{\sin\theta} = m$ or $\dfrac{\cos^2\theta}{\sin\theta} = m$ ...(i)

Again $\sec\theta - \cos\theta = n$

or $\dfrac{1}{\cos\theta} - \cos\theta = n$

or $\dfrac{1-\cos^2\theta}{\cos\theta} = n$ or $\dfrac{\sin^2\theta}{\cos\theta} = n$ ...(ii)

From Eq. (i) $\sin\theta = \dfrac{\cos^2\theta}{m}$ ...(iii)

Putting in (ii), we get $\dfrac{\cos^4\theta}{m^2\cos\theta} = n$ or, $\cos^3\theta = m^2 n$

$\therefore \cos\theta = (m^2n)^{\frac{1}{3}}$ or, $\cos^2\theta = (m^2n)^{\frac{2}{3}}$ ...(iv)

From Eq. (iii), $\sin\theta = \dfrac{\cos^2\theta}{m} = \dfrac{(m^2n)^{\frac{2}{3}}}{m}$

$$= \frac{m^{\frac{4}{3}}n^{\frac{2}{3}}}{m} = m^{\frac{1}{3}}n^{\frac{2}{3}} = (mn^2)^{\frac{1}{3}}$$

$\therefore \sin^2\theta = (mn^2)^{\frac{2}{3}}$ ...(v)

Adding Eqs. (iv) and (v), we get

$$(m^2n)^{\frac{2}{3}} + (mn^2)^{\frac{2}{3}} = \cos^2\theta + \sin^2\theta$$

or, $(m^2n)^{\frac{2}{3}} + (mn^2)^{\frac{2}{3}} = 1$

Example 22. ***If*** $3\sin\theta + 4\cos\theta = 5$***, then find the value of*** $4\sin\theta - 3\cos\theta$.

Sol. Let $4\sin\theta - 3\cos\theta = a$ ...(i)

Thus, we want to eliminate θ from both $3\sin\theta + 4\cos\theta = 5$ and $4\sin\theta - 3\cos\theta = a$, i.e. squaring and adding these equations, we get

$$(3\sin\theta + 4\cos\theta)^2 + (4\sin\theta - 3\cos\theta)^2 = 25 + a^2$$

$$9\sin^2\theta + 16\cos^2\theta + 24\sin\theta\cos\theta + 16\sin^2\theta + 9\cos^2\theta - 24\cos\theta\sin\theta = 25 + a^2$$

$$9 + 16 = 25 + a^2 \quad \text{or} \quad a^2 = 0$$

$$a = 0$$

$\therefore$ $4\sin\theta - 3\cos\theta = 0$

Example 23. If $a\sec\alpha - c\tan\alpha = d$ and $b\sec\alpha + d\tan\alpha = c$, then eliminate α from above equations.

Sol. Here, $a\sec\alpha - c\tan\alpha = d$ and $b\sec\alpha + d\tan\alpha = c$ could be written as

$a = d\cos\alpha + c\sin\alpha$ ...(i)

and $b = c\cos\alpha - d\sin\alpha$ ...(ii)

On squaring and adding Eqs. (i) and (ii), we get

$$a^2 + b^2 = (d\cos\alpha + c\sin\alpha)^2 + (c\cos\alpha - d\sin\alpha)^2$$

$$\Rightarrow \quad a^2 + b^2 = d^2\cos^2\alpha + c^2\sin^2\alpha + 2dc\cos\alpha\sin\alpha + c^2\cos^2\alpha + d^2\sin^2\alpha - 2cd\cos\alpha\sin\alpha.$$

$$= d^2(\cos^2\alpha + \sin^2\alpha) + c^2(\sin^2\alpha + \cos^2\alpha)$$

$\therefore$ $a^2 + b^2 = c^2 + d^2$

Example 24. Eliminate θ between the equations $a\sec\theta + b\tan\theta + c = 0$ and $p\sec\theta + q\tan\theta + r = 0$.

Sol. Given $a\sec\theta + b\tan\theta + c = 0$ ...(i)

and $p\sec\theta + q\tan\theta + r = 0$ ...(ii)

Solving Eqs. (i) and (ii) by cross multiplication method, we have

$$\frac{\sec\theta}{br - qc} = \frac{\tan\theta}{pc - ar} = \frac{1}{aq - pb}$$

(i) (ii) (iii)

From Eqs. (i) and (iii), we get

$$\sec\theta = \frac{br - qc}{aq - pb} \quad \text{...(iii)}$$

From Eqs. (ii) and (iii), we get

$$\tan\theta = \frac{pc - ar}{pc - pb} \quad \text{...(iv)}$$

$\because \quad \sec^2\theta - \tan^2\theta = 1$

$$\therefore \quad \left(\frac{br - qc}{aq - pb}\right)^2 - \left(\frac{pc - ar}{aq - pb}\right)^2 = 1$$

or $\quad (br - qc)^2 - (pc - ar)^2 = (aq - pb)^2$

Example 25. If $x = \sec\theta - \tan\theta$ and $y = \text{cosec}\,\theta + \cot\theta$, then prove that $xy + 1 = y - x$.

Sol. $xy + 1 = \left(\frac{1 - \sin\theta}{\cos\theta}\right)\left(\frac{1 + \cos\theta}{\sin\theta}\right) + 1 = \frac{1 - \sin\theta + \cos\theta}{\sin\theta\cos\theta}$

$$= \frac{(\sin^2\theta + \cos^2\theta)}{\sin\theta\cos\theta} - \frac{(\sin\theta - \cos\theta)}{\sin\theta\cos\theta}$$

$$= (\tan\theta + \cot\theta) - (\sec\theta - \text{cosec}\,\theta)$$

$$= (\text{cosec}\,\theta + \cot\theta) - (\sec\theta - \tan\theta) = y - x$$

Example 26. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$. Find the value of $x^2 + y^2 + z^2$.

Sol. Here,

$$x^2 + y^2 + z^2 = r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta$$

$$= r^2\sin^2\theta(\cos^2\phi + \sin^2\phi) + r^2\cos^2\theta$$

$$= r^2\sin^2\theta + r^2\cos^2\theta$$

$$= r^2$$

$$\therefore \quad x^2 + y^2 + z^2 = r^2$$

Example 27. If $0 < \theta < \frac{\pi}{2}$, $x = \sum_{n=0}^{\infty}\cos^{2n}\theta$, $y = \sum_{n=0}^{\infty}\sin^{2n}\theta$ and $z = \sum_{n=0}^{\infty}\cos^{2n}\theta \cdot \sin^{2n}\theta$, then show $xyz = xy + z$.

Sol. Here, $x = \sum_{n=0}^{\infty}\cos^{2n}\theta = 1 + \cos^2\theta + \cos^4\theta + \cos^6\theta + \ldots\infty$

$$= \frac{1}{1 - \cos^2\theta} = \frac{1}{\sin^2\theta}$$

[using, $S_\infty = \frac{a}{1 - r}$ sum of infinite GP]

Similarly, $\quad y = \frac{1}{1 - \sin^2\theta} = \frac{1}{\cos^2\theta}$

and $\quad z = \frac{1}{1 - \sin^2\theta \cdot \cos^2\theta}$

$$\therefore \quad xyz = \frac{1}{\sin^2\theta\cos^2\theta(1 - \sin^2\theta\cos^2\theta)}$$

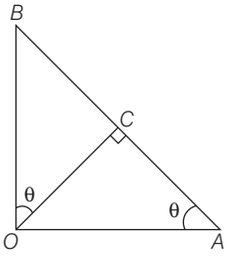
$$= \frac{(1 - \sin^2\theta\cos^2\theta) + (\sin^2\theta\cos^2\theta)}{\sin^2\theta\cos^2\theta(1 - \sin^2\theta\cos^2\theta)}$$

$$= \frac{1}{\sin^2\theta\cos^2\theta} + \frac{1}{1 - \sin^2\theta\cos^2\theta}$$

$$= xy + z$$

Application of Basic Trigonometry in Geometry

Example 28. If in given fig, $\tan(\angle BAO) = 3$, then find the ratio $BC : CA$.



Sol. From Fig., we have

$$\tan\theta = 3$$

In ΔOCA and ΔOCB respectively, we get

$$\therefore \quad \frac{OC}{AC} = \tan\theta, \frac{OC}{BC} = \cot\theta$$

On dividing, we get

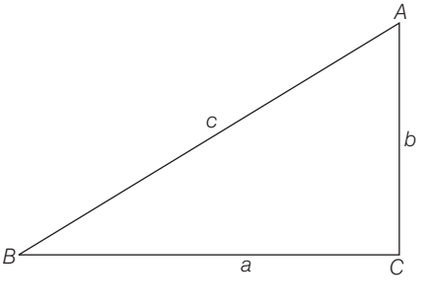
or $\quad \frac{BC}{AC} = \frac{\tan\theta}{\cot\theta} = \tan^2\theta = 9$

$$\Rightarrow \quad BC : AC = 9 : 1$$

Example 29. If angle C of triangle ABC is 90°, then prove that $\tan A + \tan B = \frac{c^2}{ab}$ (where, a, b, c are sides opposite to angles A, B, C, respectively).

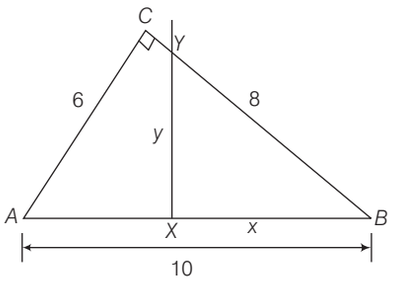
Sol. Draw ΔABC with $\angle C = 90^\circ$. We have

$$\tan A + \tan B = \frac{a}{b} + \frac{b}{a} = \frac{a^2 + b^2}{ab} = \frac{c^2}{ab}$$



Example 30. In triangle ABC, $BC = 8$, $CA = 6$, and $AB = 10$. A line dividing the triangle ABC into two regions of equal area is perpendicular to AB at point X. Find the value of $\frac{BX}{\sqrt{2}}$.

Sol.



We have area of $\Delta XYB = \frac{1}{2}$ area of ΔABC

$$\therefore \quad \frac{1}{2}(XY)\cdot(XB) = \frac{1}{2}\times\frac{1}{2}\times AC \times BC$$

$$2\left(\frac{x\times y}{2}\right) = \frac{8\times 6}{2} = 24$$

or $\quad x \times x \tan B = 24 \qquad [\because y = x\tan B]$

or $\quad x^2 \times \frac{3}{4} = 24 \qquad [\because \tan B = \frac{AC}{BC}]$

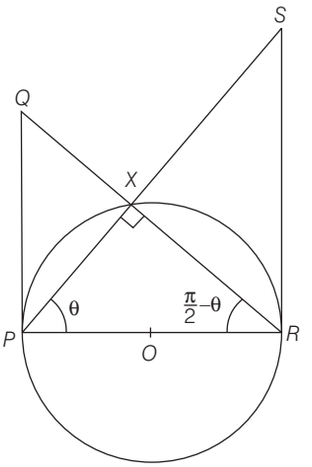
or $\quad x^2 = 32$ or $x = 4\sqrt{2}$

Example 31. Let PQ and RS be tangents at the extremities of the diameter PR or a circle of radius r. If PS and RQ intersect at a point X on the circumference of the circle, then prove that $2r = \sqrt{PQ \times RS}$.

Sol. From Fig. we have

$$\frac{PQ}{PR} = \tan\left(\frac{\pi}{2} - \theta\right)$$

$$= \cot\theta \text{ and } \frac{RS}{PR} = \tan\theta$$



$$\therefore \quad \frac{PQ}{PR}\times\frac{RS}{PR} = 1$$

or $\quad (PR)^2 = PQ \times PS$

or $\quad (2r)^2 = PQ \times PS$

or $\quad 2r = \sqrt{PQ \times PS}$

Exercise for Session 3

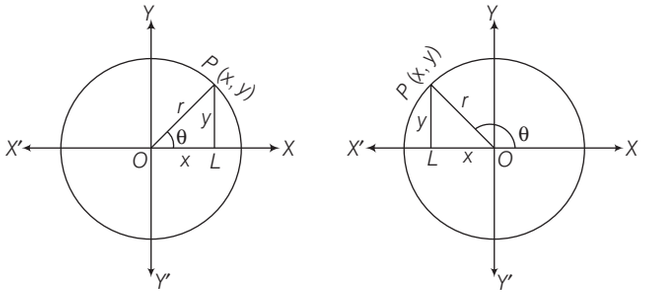
1. If $\sec\theta + \tan\theta = k$, find the value of $\cos\theta$.
2. If $x\sin^3\theta + y\cos^3\theta = \sin\theta\cos\theta$ and $x\sin\theta = y\cos\theta$, Find the value of $x^2 + y^2$.
3. If $\sin A + \cos A = m$ and $\sin^3 A + \cos^3 A = n$, prove that $m^3 - 3m + 2n = 0$.
4. If $\sin^2\theta = \frac{x^2 + y^2 + 1}{2x}$. Find the value of x and y.
5. If $\sin\theta - \sqrt{6}\cos\theta = \sqrt{7}\cos\theta$. Prove that $\cos\theta + \sqrt{6}\sin\theta - \sqrt{7}\sin\theta = 0$.
6. If $\sin x + \sin y + \sin z = 3$. Find the value of $\cos x + \cos y + \cos z$.
7. If $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$, $\frac{x}{a}\sin\theta - \frac{y}{b}\cos\theta = 1$, then eliminate θ.
8. If $a\sin^2 x + b\cos^2 x = c$, $b\sin^2 y + a\cos^2 y = d$ and $a\tan x = b\tan y$, then prove that $\frac{a^2}{b^2} = \frac{(d-a)(c-a)}{(b-c)(b-d)}$.
9. If $a + b\tan\theta = \sec\theta$ and $b - a\tan\theta = 3\sec\theta$, then find the value of $a^2 + b^2$.
10. Two circles of radii 4 cm and 1 cm touch each other externally and 8 is the angle contained by their direct common tangents. Find the value of $\sin\frac{\theta}{2} + \cos\frac{\theta}{2}$.

Session 4

Signs and Graph of Trigonometric Functions

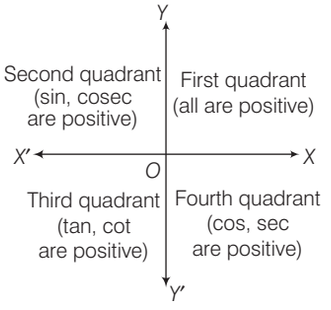
Signs of Trigonometric Functions

The signs of the trigonometric ratios of an angle depend on the quadrant in which the terminal side of the angle lies. We always take $OP = r$ to be positive (see figure). Thus the signs of all the trigonometric ratios depend on the signs of x and/or y.



An angle is said to be in that quadrant in which its terminal ray lies

For positive acute angles this definition gives the same result as in case of a right angled triangle since x and y are both positive for any point in the first quadrant and consequently they are the length of base and perpendicular of the angle θ.



1. Clearly in first quadrant $\sin\theta$, $\cos\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$ and $\operatorname{cosec}\theta$ are all positive as x, y are positive.
2. In second quadrant, x is negative and y is positive, therefore, only $\sin\theta$ and $\operatorname{cosec}\theta$ are positive.
3. In third quadrant, x and y are both negative, therefore, only $\tan\theta$ and $\cot\theta$ are positive.
4. In fourth quadrant, x is positive and y is negative, therefore, only $\cos\theta$ and $\sec\theta$ are positive.

Quadrant →	I	II	III	IV
$\sin\theta$	+	+	–	–
$\cos\theta$	+	–	–	+
$\tan\theta$	+	–	+	–
$\operatorname{cosec}\theta$	+	+	–	–
$\sec\theta$	+	–	–	+
$\cot\theta$	+	–	+	–

Variation in the Values of Trigonometric Functions in Different Quadrants

We observe that in the first quadrant, as x increases from 0 to $\frac{\pi}{2}$, $\sin x$ increases from 0 to 1 and in the second quadrant as x increases from $\frac{\pi}{2}$ to π, $\sin x$ decreases from 1 to 0.

In the third quadrant, as x increases from π to $\frac{3\pi}{2}$, $\sin x$ decreases from 0 to -1 and finally, in the fourth quadrant, $\sin x$ increases from -1 to 0 as x increase from $\frac{3\pi}{2}$ to 2π.

Function	1st quadrant	2nd quadrant	3rd quadrant	4th quadrant
$\sin\theta$	↑ from 0 to 1	↓ from 1 to 0	↓ from 0 to -1	↑ from -1 to 0
$\cos\theta$	↓ from 1 to 0	↓ from 0 to -1	↑ from -1 to 0	↑ from 0 to 1
$\tan\theta$	↑ from 0 to ∞	↑ from $-\infty$ to 0	↑ from 0 to ∞	↑ from $-\infty$ to 0
$\cot\theta$	↓ from ∞ to 0	↓ from 0 to $-\infty$	↓ from ∞ to 0	↓ from 0 to $-\infty$
$\sec\theta$	↑ from 1 to ∞	↑ from $-\infty$ to -1	↓ from -1 to $-\infty$	↓ from ∞ to 1
$\operatorname{cosec}\theta$	↓ from ∞ to 1	↑ from 1 to ∞	↑ from $-\infty$ to -1	↓ from -1 to $-\infty$

Note

$+\infty$ and $-\infty$ are two symbols. These are not real numbers. When we say that $\tan\theta$ increases from 0 to ∞ as θ varies from 0 to $\frac{\pi}{2}$, it means that $\tan\theta$ increases in the interval $\left(0, \frac{\pi}{2}\right)$ and it attains arbitrarily large positive values as θ tends to $\frac{\pi}{2}$. This rule applies to other trigonometric functions also.

Graphs of Trigonometric Functions

As in case of algebraic function, we can have some idea about the nature of a trigonometric function by its graph. Graph has many important applications in mathematical problems. We shall discuss the graphs of trigonometrical functions. We know that $\sin x$, $\cos x$, $\sec x$ and $\operatorname{cosec} x$ are periodic functions with period 2π and $\tan x$ and $\cot x$ are trigonometric functions of period π. Also if the period of function $f(x)$ is T, then period of $f(ax+b)$ is $\frac{T}{|a|}$.

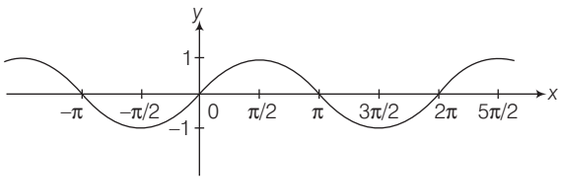
Graph and Other Useful Data of Trigonometric Functions

1. $y = f(x) = \sin x$

   Domain $\to R$,

   Range $\to [-1, 1]$

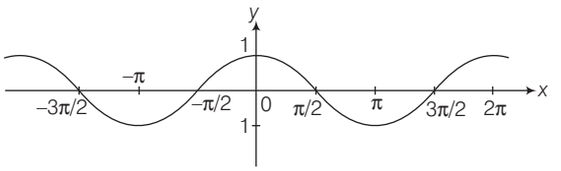
   Period $\to 2\pi$



2. $y = f(x) = \cos x$

   Domain $\to R$, Range $\to [-1, 1]$

   Period $\to 2\pi$

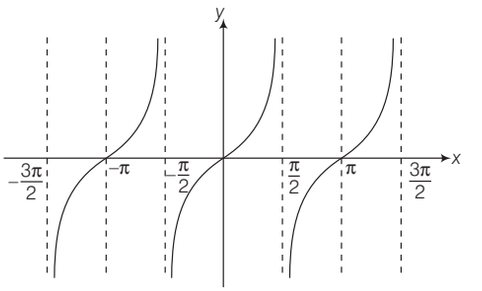


3. $y = f(x) = \tan x$

   Domain $\to R \sim (2n+1)\frac{\pi}{2}, n \in I$

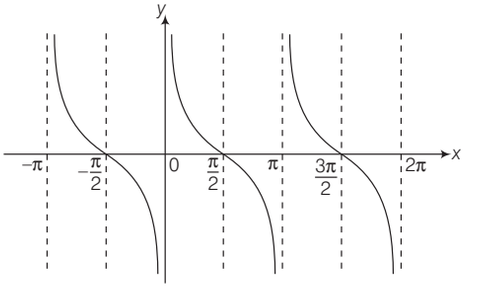
   Range $\to (-\infty, \infty)$

   Period $\to \pi$



4. $y = f(x) = \cot x$

   Domain $\to R \sim n\pi, n \in I$; Range $\to (-\infty, \infty)$; Period $\to \pi$,

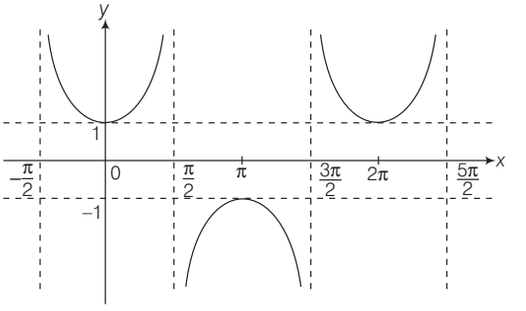


5. $y = f(x) = \sec x$

   Domain $\to R \sim (2n+1)\frac{\pi}{2}, n \in I$

   Range $\to (-\infty, -1] \cup [1, \infty)$

   Period $\to 2\pi$, $\sec^2 x$, $|\sec x| \in [1, \infty)$

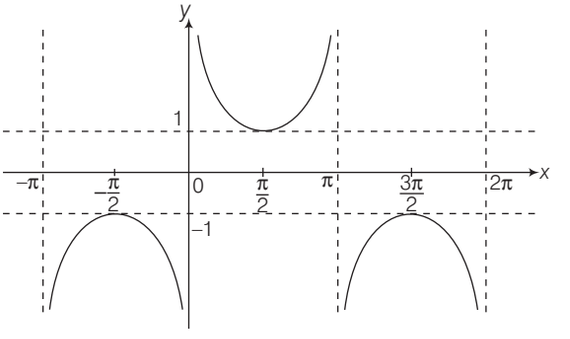


6. $y = f(x) = \operatorname{cosec} x$

   Domain $\to R \sim n\pi, n \in I$;

   Range $\to (-\infty, -1] \cup [1, \infty)$

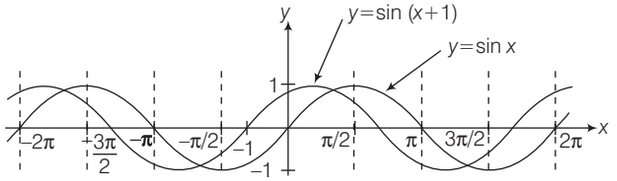
   Period $\to 2\pi$, $\operatorname{cosec}^2 x$, $|\operatorname{cosec} x| \in [1, \infty)$



Transformation of the Graphs of Trigonometric Functions

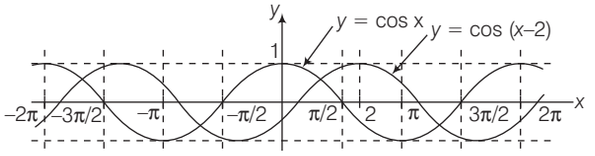
1. To draw the graph of $y = f(x + a); (a > 0)$ from the graph of $y = f(x)$, shift the graph of $y = f(x)$, a units left along the x-axis.

   Consider the following illustration.



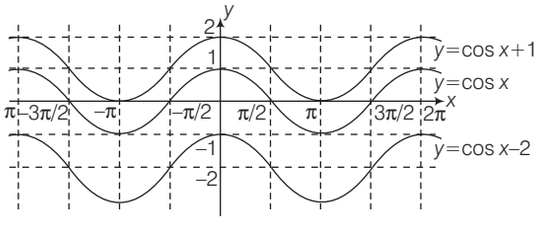
   To draw the graph of $y = f(x - a); (a > 0)$ from the graph of $y = f(x)$, shift the graph of $y = f(x)$, a units right along the x-axis.

   Consider the following illustration.

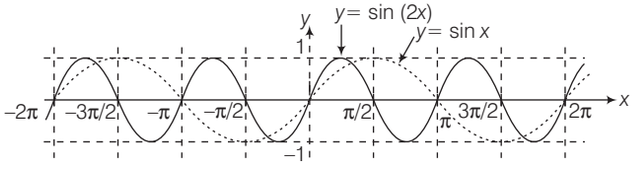


2. To draw the graph of $y = f(x) + a; (a > 0)$ from the graph of $y = f(x)$, shift the graph of $y = f(x)$, a units upwards along the y-axis.

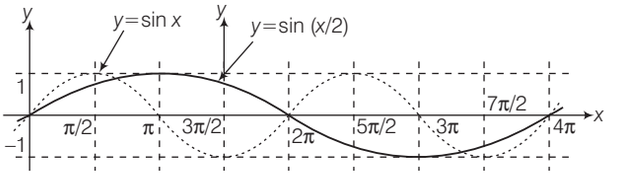
   To draw the graph of $y = f(x) - a; (a > 0)$ from the graph of $y = f(x)$, shift the graph of $y = f(x)$, a units downward along the y-axis.



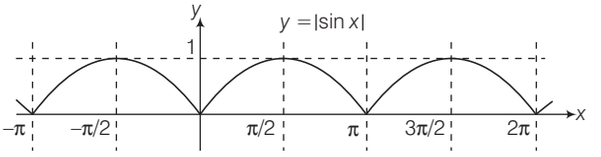
3. If $y = f(x)$ has period T, then period of $y = f(ax)$ is $\frac{T}{|a|}$.



   Period of $y = \sin(2x)$ is $\frac{2\pi}{2} = \pi$



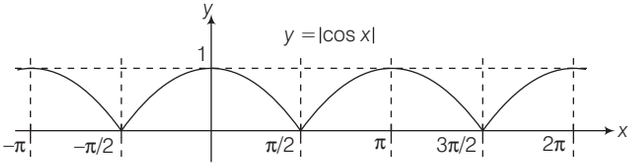
   Period of $y = \sin\left(\frac{x}{2}\right)$ is $\frac{2\pi}{\left(\frac{1}{2}\right)} = 4\pi$



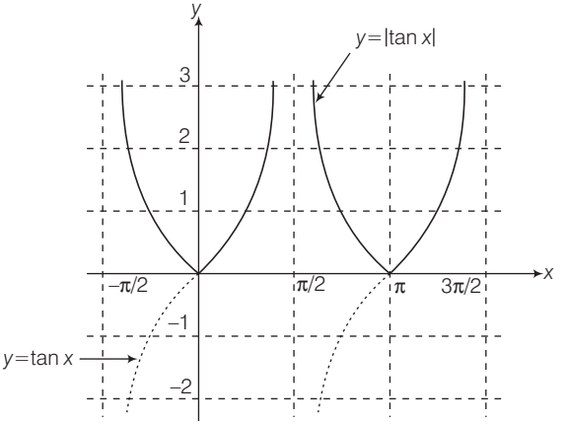
4. Since $y = |f(x)| \geq 0$, to draw the graph of $y = |f(x)|$, take the mirror of the graph of $y = f(x)$ in the x-axis for $f(x) < 0$, retaining the graph for $f(x) > 0$.

   Consider the following illustrations.

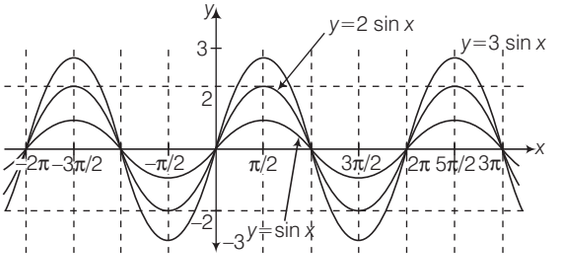
   Here, period of $f(x) = |\sin x|$ is π.

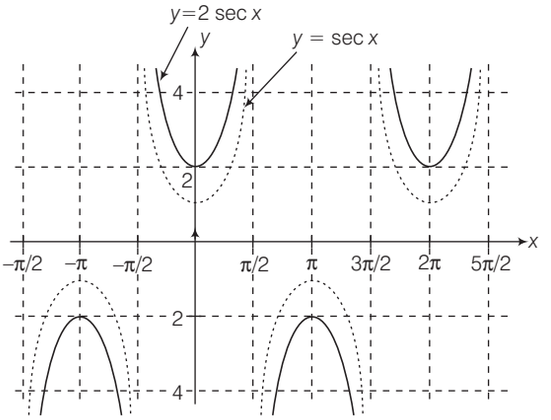


   Here, period of $f(x) = |\cos x|$ is π.



5. Graph of $y = af(x)$ from the graph of $y\ f(x)$

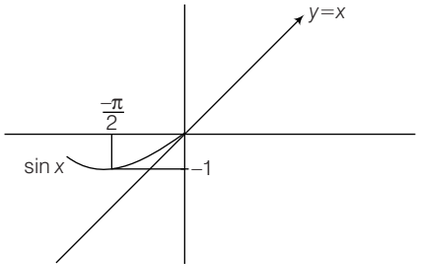




Some Important Graphical Deductions

To find relation between $\sin x$, x and $\tan x$

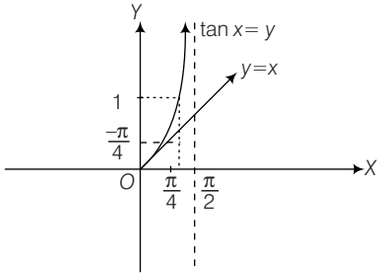
(i)



Thus, when $-\infty < x < 0$

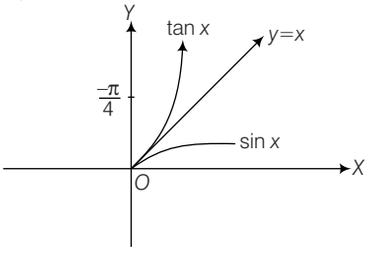
$\Rightarrow \sin x > x$

(ii)



$\therefore \quad \tan x > x$, when $0 > x > \dfrac{\pi}{2}$

(iii) In general,



Thus, $\tan x > x > \sin x, \forall\ x \in \left(0, \dfrac{\pi}{2}\right)$

and $\sin x > x > \tan x, \forall\ x \in \left(-\dfrac{\pi}{2}, 0\right)$.

Example 32. Find the values of the other five trigonometric functions in each of the following questions

(i) $\tan\theta = \dfrac{5}{12}$, where θ is in third quadrant.

(ii) $\sin\theta = \dfrac{3}{5}$, where θ is in second quadrant.

Sol. (i) Since θ is in third quadrant,

$\therefore$ Only $\tan\theta$ and $\cot\theta$ are positive

Now, $\tan\theta = \dfrac{5}{12}$

Therefore, $\cot\theta = \dfrac{12}{5}$,

$\sin\theta = -\dfrac{5}{13}$,

$\text{cosec} = -\dfrac{13}{5}$

$\cos\theta = -\dfrac{12}{13}$ and $\sec\theta = -\dfrac{13}{12}$.

(ii) Since θ is in the second quadrant,

$\therefore$ Only $\sin\theta$ and $\text{cosec}\,\theta$ will be positive.

Now, $\sin\theta = \dfrac{3}{5}$.

Therefore,

$\text{cosec}\,\theta = \dfrac{5}{3}$, $\cos\theta = -\dfrac{4}{5}$,

$\sec\theta = -\dfrac{5}{4}$, $\tan\theta = -\dfrac{3}{4}$

and $\cot\theta = -\dfrac{4}{3}$.

Example 33. If $\sin\theta = \dfrac{12}{13}$ and θ lies in the second quadrant, find the value of $\sec\theta + \tan\theta$.

Sol. We have $\sin^2\theta + \cos^2\theta = 1$

$\Rightarrow \quad \cos\theta = \pm\sqrt{1 - \sin^2\theta}$

In the second quadrant, $\cos\theta$ is negative

$\therefore \quad \cos\theta = -\sqrt{1 - \sin^2\theta}$

Now, $\sec\theta + \tan\theta = \dfrac{1}{\cos\theta} + \dfrac{\sin\theta}{\cos\theta} = \dfrac{1 + \sin\theta}{\cos\theta}$

$$= \frac{1 + \sin\theta}{-\sqrt{1 - \sin^2\theta}} = \frac{1 + \frac{12}{13}}{-\sqrt{1 - \left(\frac{12}{13}\right)^2}}$$

$$= \frac{\frac{25}{13}}{-\sqrt{\frac{25}{169}}} = \frac{\frac{25}{13}}{-\frac{5}{13}} = -5$$

Example 34. Draw the graph of $y = 3\sin 2x$.

Sol. $\sin x$ is a periodic function with period 2π, therefore, $\sin 2x$ will be a periodic function of period $\frac{2\pi}{|2|} = \pi$

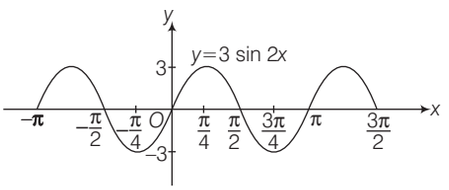
Also $\quad -1 \le \sin 2x \le 1$

$\therefore \quad -3 \le 3\sin 2x \le 3$

In order to draw the graph of $y = 3\sin 2x$, draw the graph of $y = \sin x$ and on X-axis change k to $\frac{k}{2}$, i.e. write $\frac{k}{2}$ wherever it is k. For example, write 15° in place of 30°, 45° in place of 90° etc.

On Y-axis change k to $3k$, i.e. write $3k$ wherever it is k for example, write 3 in place of 1, -3 in place of -1, 1.5 in place of 0.5 etc.

The graph of $y = 3\sin 2x$ will be as given in the figure.



Example 35. Draw the graph of $y = \cos\left(x - \frac{\pi}{4}\right)$

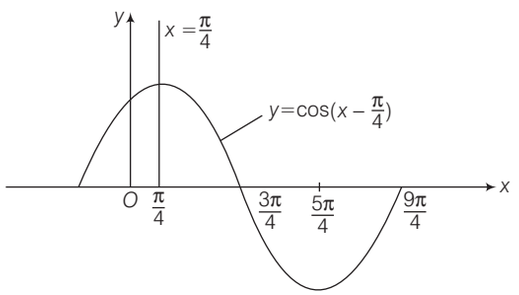
Sol. Given function is $y = \cos\left(x - \frac{\pi}{4}\right)$ ...(i)

Given function is $Y = \cos X$, where

$$X = x - \frac{\pi}{4} \text{ and } Y = y$$

or $\quad Y = 0 \Rightarrow y = 0$ and $X = 0$

$\Rightarrow \quad x - \frac{\pi}{4} = 0 \Rightarrow x = \frac{\pi}{4}$



In order to draw the graph of $y = \cos\left(x - \frac{\pi}{4}\right)$, we draw the graph of $y = \cos x$ and shift it on the right side through a distance of $\frac{\pi}{4}$ unit.

Example 36. Which of the following is the least?

(a) $\sin 3$ (b) $\sin 2$

(c) $\sin 1$ (d) $\sin 7$

Sol. (a) $\sin 3 = \sin[\pi - (\pi - 3)] = \sin(\pi - 3) = \sin(0.14)$

$\sin 2 = \sin[\pi - (\pi - 2)]$

$= \sin(\pi - 2) = \sin(1.14)$

$\sin 7 = \sin[2\pi + (7 - 2\pi)]$

$= \sin(7 - 2\pi) = \sin(0.72)$

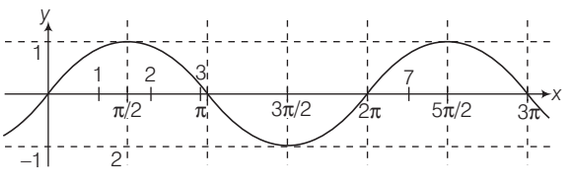
Now, $\quad 1.14 > 1 > 0.72 > 0.14$

$\Rightarrow \quad \sin(1.14) > \sin 1 > \sin(0.72) > \sin(0.14)$

[as 1.14, 0.72, 0.14 lie in the first quadrant and sine functions increase in the first quadrant]

Hence, among the given values, $\sin 3$ is the least.

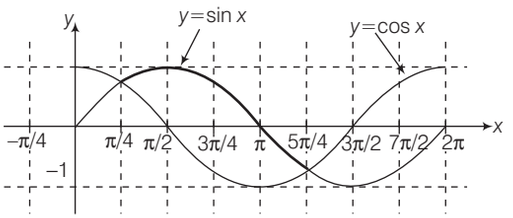
Alternate solution



From the graph, obviously $\sin 3$ is the least.

Example 37. ***Find the value of*** x ***for which*** $f(x) = \sqrt{\sin x - \cos x}$ ***is defined,*** $x \in [0, 2\pi]$.

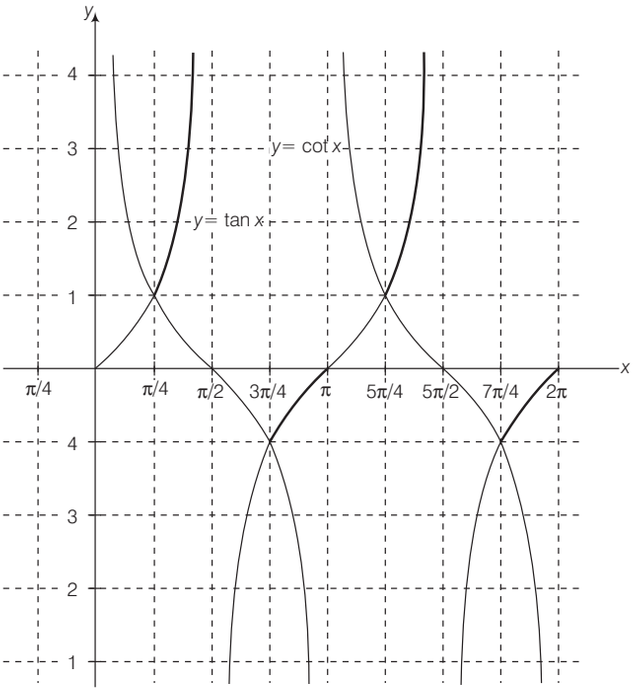
Sol. $f(x) = \sqrt{\sin x - \cos x}$ is defined if $\sin x \ge \cos x$.



From the graph, $\sin x \ge \cos x$, for $x \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$.

Example 38. ***Solve*** $\tan x > \cot x$, ***where*** $x \in [0, 2\pi]$.

Sol.



We find that $\tan x \geq \cot x$. Therefore, the values of $\tan x$ are more than the value of $\cot x$.

That is, the value of x for which graph of $y = \tan x$ is above the graph of $y = \cot x$.

From the graph, it is clear that

$$x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{4}, \pi\right) \cup \left(\frac{5\pi}{4}, \frac{3\pi}{2}\right) \cup \left(\frac{7\pi}{4}, 2\pi\right).$$

Exercise for Session 4

1. If $\tan x = -\frac{4}{3}, \frac{3\pi}{2} < x < 2\pi$, find the value of $9\sec^2 x - 4\cot x$.
2. Show that $\sin^2 x = p + \frac{1}{p}$ is impossible if x is real.
3. If $\cos x = \frac{3}{5}$ and x lies in the fourth quadrant find the values of $\operatorname{cosec} x + \cot x$.
4. Draw the graph of $y = \sin x$ and $y = \sin \frac{x}{2}$.
5. Draw the graph of $y = \sec^2 x - \tan^2 x$. Is $f(x)$ periodic? If yes, what is its fundamental period?
6. Prove that $\sin\theta < \theta < \tan\theta$ for $\theta \in \left(0, \frac{\pi}{2}\right)$.
7. Find the value of x for which $f(x) = \sqrt{\sin x - \cos x}$ is defined, $x \in [0, 2\pi]$.
8. Draw the graph of $y = \sin x$ and $y = \cos x, 0 \leq x \leq 2\pi$.
9. Draw the graph of $y = \tan(3x)$.
10. If $\cos x = -\frac{\sqrt{15}}{4}$ and $\frac{\pi}{2} < x < \pi$, find the value of $\sin x$.

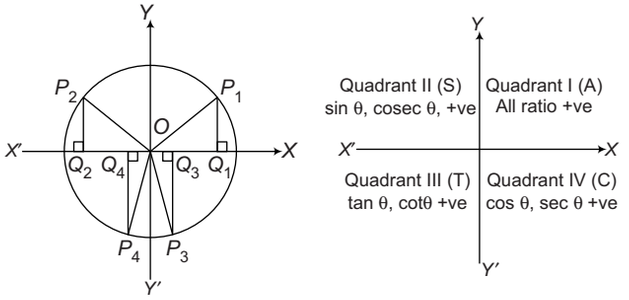
Session 5

Trigonometric Ratios of any Angle

Trigonometric Ratios of any Angle

Consider the system of rectangular coordinates axis dividing plane into four quadrants. A line OP makes angle θ with the positive x-axis. The angle θ is said to be positive if measured in counter clockwise direction from the positive x-axis and is negative if measured in clockwise direction.

The positive values of the trigonometric ratios in the various quadrants are shown, the signs of the other ratios may be derived.



Note that $\angle XOY = \frac{\pi}{2}$, $\angle XOY' = \pi$, $\angle XOY' = \frac{3\pi}{2}$

P_iQ_i is positive if above the x-axis, negative if below the x-axis, OP_i is always taken positive. OQ_i is positive if along x-axis, negative if in oppositive direction.

$$\sin \angle Q_iOP_i = \frac{P_iQ_i}{OP_i}.$$

$$\cos \angle Q_iOP_i = \frac{O_iQ_i}{OP_i}.$$

$$\tan \angle Q_iOP_i = \frac{P_iQ_i}{OQ_i} \qquad [i = 1, 2, 3]$$

Thus, depending on signs of OQ_i and P_iQ_i, the various trigonometrical ratios will have different signs given

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$$

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\tan(-\theta) = -\tan\theta \qquad \cot(-\theta) = -\cot\theta$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta \qquad \cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$$

$$\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta \qquad \sec(-\theta) = \sec\theta$$

$$\sec\left(\frac{\pi}{2} - \theta\right) = \operatorname{cosec}\theta \qquad \operatorname{cosec}\left(\frac{\pi}{2} - \theta\right) = \sec\theta$$

$$\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$$

$$\sin(\pi - \theta) = \sin\theta \qquad \cos(\pi - \theta) = -\cos\theta$$

$$\tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta \qquad \cot\left(\frac{\pi}{2} + \theta\right) = -\tan\theta$$

$$\tan(\pi - \theta) = -\tan\theta \qquad \cot(\pi - \theta) = -\cot\theta$$

$$\sec\left(\frac{\pi}{2} + \theta\right) = -\operatorname{cosec}\theta \qquad \operatorname{cosec}\left(\frac{\pi}{2} + \theta\right) = \sec\theta$$

$$\sec(\pi - \theta) = -\sec\theta \qquad \operatorname{cosec}(\pi - \theta) = \operatorname{cosec}\theta$$

$$\sin(\pi + \theta) = -\sin\theta \qquad \cos(\pi + \theta) = -\cos\theta$$

$$\sin\left(\frac{3\pi}{2} - \theta\right) = -\cos\theta \qquad \cos\left(\frac{3\pi}{2} - \theta\right) = -\sin\theta$$

$$\tan(\pi + \theta) = \tan\theta \qquad \cot(\pi + \theta) = \cot\theta$$

$$\tan\left(\frac{3\pi}{2} - \theta\right) = \cot\theta \qquad \cot\left(\frac{3\pi}{2} - \theta\right) = \tan\theta$$

$$\sec(\pi + \theta) = -\sec\theta \qquad \operatorname{cosec}(\pi + \theta) = -\operatorname{cosec}\theta$$

$$\sec\left(\frac{3\pi}{2} - \theta\right) = -\operatorname{cosec}\theta \qquad \operatorname{cosec}\left(\frac{3\pi}{2} - \theta\right) = -\sec\theta$$

$$\sin\left(\frac{3\pi}{2} + \theta\right) = -\cos\theta \qquad \cos\left(\frac{3\pi}{2} + \theta\right) = \sin\theta$$

$$\sin(2\pi - \theta) = -\sin\theta \qquad \cos(2\pi - \theta) = \cos\theta$$

$$\tan\left(\frac{3\pi}{2} + \theta\right) = -\cot\theta \qquad \cot\left(\frac{3\pi}{2} + \theta\right) = -\tan\theta$$

$$\tan(2\pi - \theta) = -\tan\theta \qquad \cot(2\pi - \theta) = -\cot\theta$$

$$\sec\left(\frac{3\pi}{2} + \theta\right) = \operatorname{cosec}\theta \qquad \operatorname{cosec}\left(\frac{3\pi}{2} + \theta\right) = -\sec\theta$$

$$\sec(2\pi - \theta) = \sec\theta \qquad \operatorname{cosec}(2\pi - \theta) = -\operatorname{cosec}\theta$$

$$\sin(2\pi + \theta) = \sin\theta \qquad \cos(2\pi + \theta) = \cos\theta$$

$$\tan(2\pi + \theta) = \tan\theta \qquad \cot(2\pi + \theta) = \cot\theta$$

$$\sec(2\pi + \theta) = \sec\theta \qquad \operatorname{cosec}(2\pi + \theta) = \operatorname{cosec}\theta$$

Allied angles (or numbers)

Two angles (or numbers) are called allied iff their sum or difference is a multiple of $\frac{\pi}{2}$. For example, $\frac{\pi}{3}$ and $\frac{\pi}{6}$ are allied, $\frac{5\pi}{6}$ and $-\frac{\pi}{6}$ are allied.

AID TO MEMORY

You must have been overwhelmed by large number of formulae for allied angles (or numbers). Instead of memorising all of them, use the following rules

1. Any trigonometric function of a real number $n\pi \pm x (n \in I)$, treating x as $0 < x < \frac{\pi}{2}$, is numerically equal to the same function of x, with sign depending upon the quadrant in which the arc length (on the unit circle) terminates. The proper sign can be ascertained by 'All – Sin – Tan – Cos' rule. For example, $\sin(\pi + x) = -\sin x$; $-ve$ sign was chosen because $\pi + x$ lies in the third quadrant and sin is – ve in the third quadrant.
2. Any trigonometric function of a real number $(2n + 1)\frac{\pi}{2} \pm x$m treating x as $0 < x < \frac{\pi}{2}$, is numerically equal to cofunction of x, with sign depending upon the quadrant in which the arc length (on the unit circle) terminates. Note that sin and cos are cofunctions of each other; tan and cot are cofunctions of each other; sec and cosec are cofunctions of each other. For example, $\sec\left(\frac{\pi}{2} + x\right) = -\operatorname{cosec} x$, – ve sign was chosen because $\frac{\pi}{2} + x$ lies in the second quadrant and sec is – ve in the second quadrant.

I. Method

To prove $\cos\left(\frac{\pi}{2} \pm \theta\right) = \mp\sin\theta$ and $\sin\left(\frac{\pi}{2} \pm \theta\right) = \cos\theta$

Proof

$$e^{i\left(\frac{\pi}{2} \pm \theta\right)} = \cos\left(\frac{\pi}{2} \pm \theta\right) + i\sin\left(\frac{\pi}{2} \pm \theta\right)$$

$$\Rightarrow \quad e^{i\frac{\pi}{2}} \cdot e^{\pm i\theta} = \cos\left(\frac{\pi}{2} \pm \theta\right) + i\sin\left(\frac{\pi}{2} \pm \theta\right)$$

$$\Rightarrow \quad i \cdot e^{i(\pm\theta)} = \cos\left(\frac{\pi}{2} \pm \theta\right) + i\sin\left(\frac{\pi}{2} \pm \theta\right)$$

$$\Rightarrow \quad i \cdot (\cos\theta \pm i\sin\theta) = \cos\left(\frac{\pi}{2} \pm \theta\right) + i\sin\left(\frac{\pi}{2} \pm \theta\right)$$

$$\Rightarrow \quad i \cdot \cos\theta \mp \sin\theta = \cos\left(\frac{\pi}{2} \pm \theta\right) + i\sin\left(\frac{\pi}{2} \pm \theta\right)$$

On comparing real and imaginary part of LHS and RHS, we get

$$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta \qquad \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$

II. Method

To prove $\cos(\pi \pm \theta) = -\cos\theta$ and $\sin(\pi \pm \theta) = \mp\sin\theta$

Since, $e^{i(\pi \pm \theta)} = \cos(\pi \pm \theta) + i\sin(\pi \pm \theta)$

$$\Rightarrow \quad e^{i\pi} \cdot e^{i(\pm\theta)} = \cos(\pi \pm \theta) + i\sin(\pi \pm \theta)$$

$$\Rightarrow \quad -(\cos(\pm\theta) + i\sin(\pm\theta)) = \cos(\pi \pm \theta) + i\sin(\pi \pm \theta)$$

On comparing real and imaginary part, we get

$$\cos(\pi + \theta) = -\cos\theta$$
$$\cos(\pi - \theta) = -\cos\theta$$
$$\sin(\pi + \theta) = -\sin\theta$$
$$\sin(\pi - \theta) = \sin\theta$$

Example 39. Prove that

(i) $\sin^2\frac{\pi}{6} + \cos^2\frac{\pi}{3} - \tan^2\frac{\pi}{4} = -\frac{1}{2}$

(ii) $2\sin^2\frac{\pi}{6} + \operatorname{cosec}^2\frac{7\pi}{6}\cos^2\frac{\pi}{3} = \frac{3}{2}$

(iii) $\cot^2\frac{\pi}{6} + \operatorname{cosec}\frac{5\pi}{6} + 3\tan^2\frac{\pi}{6} = 6$

(iv) $2\sin^2\frac{3\pi}{4} + 2\cos^2\frac{\pi}{4} + 2\sec^2\frac{\pi}{3} = 10$

Sol. (i) We have,

$$\sin^2\frac{\pi}{6} + \cos^2\frac{\pi}{3} - \tan^2\frac{\pi}{4} = -\frac{1}{2}$$

$$= \left(\sin\frac{\pi}{6}\right)^2 + \left(\cos\frac{\pi}{3}\right)^2 - \left(\tan\frac{\pi}{4}\right)^2$$

$$= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 - (1)^2$$

$$= \frac{1}{4} + \frac{1}{4} - 1 = \frac{1}{2} - 1 = -\frac{1}{2}$$

(ii) We have,

$$2\sin^2\frac{\pi}{6}+\operatorname{cosec}^2\frac{7\pi}{6}\cdot\cos^2\frac{\pi}{3}$$

$$=2\left(\sin\frac{\pi}{6}\right)^2+\left(\operatorname{cosec}\frac{7\pi}{6}\right)^2\cdot\left(\cos\frac{\pi}{3}\right)^2$$

$$=2\left(\sin\frac{\pi}{6}\right)^2+\left\{\operatorname{cosec}\left(\pi+\frac{\pi}{6}\right)\right\}^2\left(\cos\frac{\pi}{3}\right)^2$$

$$=2\left(\sin\frac{\pi}{6}\right)^2+\left\{-\operatorname{cosec}\frac{\pi}{6}\right\}^2\left(\cos\frac{\pi}{3}\right)^2$$

$$[\because \operatorname{cosec}(\pi+\theta)=-\operatorname{cosec}\theta]$$

$$=2\left(\frac{1}{2}\right)^2+(-2)^2\times\left(\frac{1}{2}\right)^2=\frac{1}{2}+1=\frac{3}{2}$$

(iii) We have,

$$\cot^2\frac{\pi}{6}+\operatorname{cosec}\frac{5\pi}{6}+3\tan^2\frac{\pi}{6}$$

$$=\left(\cot\frac{\pi}{6}\right)^2+\operatorname{cosec}\left(\pi-\frac{\pi}{6}\right)+3\left(\tan\frac{\pi}{6}\right)^2$$

$$=(\sqrt{3})^2+2+3\left(\frac{1}{\sqrt{3}}\right)^2$$

$$=3+2+1=6$$

(iv) We have,

$$2\sin^2\frac{3\pi}{4}+2\cos^2\frac{\pi}{4}+2\sec^2\frac{\pi}{3}$$

$$=2\left(\sin\frac{3\pi}{4}\right)^2+2\left(\cos\frac{\pi}{4}\right)^2+2\left(\sec\frac{\pi}{3}\right)^2$$

$$=2\left(\sin\frac{\pi}{4}\right)^2+2\left(\cos\frac{\pi}{4}\right)^2-2\left(\sec\frac{\pi}{3}\right)^2$$

$$\left[\because \sin\frac{3\pi}{4}=\sin\left(\pi-\frac{\pi}{4}\right)=\sin\frac{\pi}{4}\right]$$

$$=2\left(\frac{1}{\sqrt{2}}\right)^2+2\left(\frac{1}{\sqrt{2}}\right)^2+2(2)^2$$

$$=1+1+8=10$$

Example 40. Prove that

$$\frac{\cos(90^\circ+\theta)\sec(-\theta)\tan(180^\circ-\theta)}{\sec(360^\circ-\theta)\sin(180^\circ+\theta)\cot(90^\circ-\theta)}=-1.$$

Sol. L.H.S. $\dfrac{\cos(90^\circ+\theta)\sec(-\theta)\tan(180^\circ-\theta)}{\sec(360^\circ-\theta)\sin(180^\circ+\theta)\cot(90^\circ-\theta)}$

$$=\frac{(-\sin\theta)(\sec\theta)(-\tan\theta)}{(\sec\theta)(-\sin\theta)(\tan\theta)}$$

$$=-1$$

$$=\text{R.H.S.}$$

Example 41. Show that $\tan 1^\circ \tan 2^\circ \ldots \tan 89^\circ = 1$

Sol. L.H.S. $=(\tan 1^\circ \tan 89^\circ)(\tan 2^\circ \cdot \tan 88^\circ)\ldots$

$$=[\tan 1^\circ \tan(90^\circ-1^\circ)]\cdot[\tan 2^\circ \tan(90^\circ-2^\circ)]$$

$$\ldots[\tan 44^\circ \tan(90^\circ-44^\circ)]\tan 45^\circ$$

$$=(\tan 1^\circ\cdot\cot 1^\circ)(\tan 2^\circ\cdot\cot 2^\circ)$$

$$\ldots(\tan 44^\circ\cdot\cot 44^\circ)\tan 45^\circ$$

$$=1 \qquad [\because \tan\theta\cot\theta=1 \text{ and } \tan 45^\circ=1]$$

Example 42. Show that

$$\sin^2 5^\circ+\sin^2 10^\circ+\sin^2 15^\circ+\ldots+\sin^2 90^\circ=9\frac{1}{2}$$

Sol. L.H.S. $=(\sin^2 5^\circ+\sin^2 85^\circ)+(\sin^2 10^\circ+\sin^2 80^\circ)+\ldots+$
$(\sin^2 40^\circ+\sin^2 50^\circ)+\sin^2 45^\circ+\sin^2 90^\circ$

$$=(\sin^2 5^\circ+\cos^2 5^\circ)+(\sin^2 10+\cos^2 10^\circ)$$

$$+\ldots+(\sin^2 40^\circ+\cos^2 40^\circ)+\sin^2 45^\circ+\sin^2 90^\circ$$

$$=(1+1+1+1+1+1+1+1)+\left(\frac{1}{\sqrt{2}}\right)^2+1$$

$$=9\frac{1}{2}$$

Example 43. Find the value of

$$\cos^2\frac{\pi}{16}+\cos^2\frac{3\pi}{16}+\cos^2\frac{5\pi}{16}+\cos^2\frac{7\pi}{16}$$

Sol. L.H.S. $=\cos^2\dfrac{\pi}{16}+\cos^2\dfrac{3\pi}{16}+\cos^2\left(\dfrac{\pi}{2}-\dfrac{3\pi}{16}\right)+\cos^2\left(\dfrac{\pi}{2}-\dfrac{\pi}{16}\right)$

$$=\cos^2\frac{\pi}{16}+\cos^2\frac{3\pi}{16}+\sin^2\frac{3\pi}{16}+\sin^2\frac{\pi}{16}$$

$$=\left(\cos^2\frac{\pi}{16}+\sin^2\frac{\pi}{16}\right)+\left(\cos^2\frac{3\pi}{16}+\sin^2\frac{3\pi}{16}\right)$$

$$=1+1=2$$

Exercise for Session 5

1. Find the value of $\tan\frac{19\pi}{3}$.
2. Find the sign of $\sec 2000°$.
3. The value of $\cos 1° + \cos 2° + \cos 3° + \ldots + \cos 180°$.
4. Find the value of $\cos(270° + \theta)\cos(90° - \theta) - \sin(270° - \theta)\cos\theta$.
5. If $S_n = \cos^n\theta + \sin^n\theta$, find the value of $3S_4 - 2S_6$.
6. $\sin^2\theta = \frac{x^2 + y^2 + 1}{2x}$, then x must be.
7. If $\sin x + \operatorname{cosec} x = 2$, then find the value of $\sin^{10} x + \operatorname{cosec}^{10} x$.
8. $e^{\sin x} - e^{-\sin x} = 4$ then find the number of real solutions.
9. If $\pi < \alpha < \frac{3\pi}{2}$, then find the value of expression $\sqrt{4\sin^4\alpha + \sin^2 2\alpha} + 4\cos^2\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)$.
10. If $\sum_{i=1}^{n}\cos\theta_i = n$, then the value of $\sum_{i=1}^{n}\sin\theta_i$.

Session 6

Trigonometric Ratios of Compound Angles

Trigonometric Ratios of Compound Angles

Algebraic sum of two or more angles is called a compound angle. If A, B, C are any angles then $A + B$, $A - B$, $A + B + C$, $A - B + C$, $A - B - C$, $A + B - C$, etc., are all compound angles.

Till now, we have learnt the values of trigonometric ratios between 0° to 360°. Now, we are going to learn the values of trigonometric ratios of compound angles.

Note

Trigonometric ratios if i.e. sine, cosine, tan, cot, sec and cosec are not distributed over addition and substraction of 2 angles.

i.e. $\sin(A + B) \neq \sin A + \sin B$

Proof : $A = 60°, B = 30°$

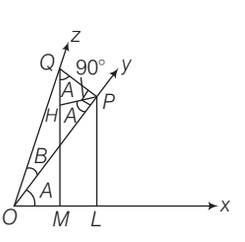
$\sin(90°) \neq \sin 60° + \sin 30°$

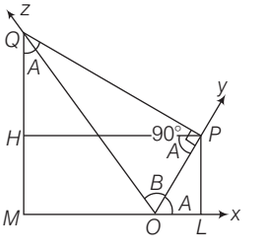
The Addition Formula

(i) $\sin(A + B) = \sin A\cos B + \cos A\sin B$

(ii) $\cos(A + B) = \cos A\cos B - \sin A\sin B$

(iii) $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$





Let the revolving line starting from the position OX describe first $\angle XOY = A$ and then proceed further so as to describe $\angle YOZ = B$ in its position OZ.

Then, $\angle XOZ = A + B$

In figure 6.1 $A + B < 90°$ and in figure 6.2 $A + B > 90°$

Let Q be a point on OZ. From Q draw $QM \perp OX$ and $QP \perp OY$. From P draw $PH \perp QM$.

Now, $\angle HPO = \angle POX = A$

$\because \quad \angle QPO = 90°$

$\therefore \quad \angle QPH = 90° - A$

$\therefore \quad \angle HQP = A$

In ΔQOM,

$$\sin(A+B) = \frac{QM}{OQ} = \frac{QH + HM}{OQ} = \frac{QH + PL}{OQ}$$

$$= \frac{QH}{OQ} + \frac{PL}{OQ} = \frac{QH}{QP} \cdot \frac{QP}{OQ} + \frac{PL}{OP} \cdot \frac{OP}{OQ}$$

$$= \frac{PL}{OP} \cdot \frac{OP}{OQ} + \frac{QH}{QP} \cdot \frac{QP}{OQ}$$

$$= \sin POL \cdot \cos POQ + \cos HQP \cdot \sin POQ$$

$$= \sin A \cos B + \cos A \sin B$$

From figure 6.1, $\cos(A+B) = \frac{OM}{Q} = \frac{OL - ML}{OQ} = \frac{OL - PH}{OQ}$

From figure 6.2, $\cos(A+B) = -\frac{OM}{OQ} = -\frac{ML - OL}{OQ}$

$$= \frac{OL - ML}{OQ} = \frac{OL - PH}{OQ}$$

$\therefore$ In both cases $\cos(A+B)$

$$= \frac{OL}{OQ} - \frac{PH}{OQ} = \frac{OL}{OP} \cdot \frac{OP}{OQ} - \frac{PH}{QP} \cdot \frac{QP}{OQ}$$

$$= \cos POL \cdot \cos POQ - \sin PQH \cdot \sin POQ$$

$$= \cos A \cos B - \sin A \sin B$$

In both cases

$$\tan(A+B) = \frac{QM}{OM} = \frac{QH + HM}{OL - ML} = \frac{QH + PL}{OL - PH}$$

$$= \frac{\frac{QH}{OL} + \frac{PL}{OL}}{1 - \frac{PH}{OL}} = \frac{\frac{QH}{OL} + \frac{PL}{OL}}{1 - \frac{PH}{PL} \cdot \frac{PL}{OL}} \quad \text{...(i)}$$

From similar ΔQPH and ΔOPL

$$\frac{QH}{OL} = \frac{PH}{PL} = \frac{PQ}{OP} \quad \text{...(ii)}$$

On putting the value from Eq. (ii) in Eq. (i), we get

$$\tan(A+B) = \frac{\frac{PQ}{OP} + \frac{PL}{OL}}{1 - \frac{PQ}{OP} \cdot \frac{PL}{OL}}$$

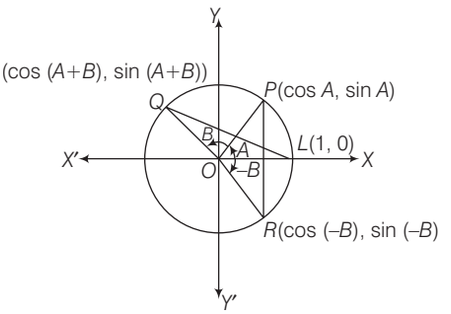
$$= \frac{\tan B + \tan A}{1 - \tan B \tan A} = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\left[\because \text{from } \Delta POQ, \frac{PQ}{OP} = \tan B, \text{ from } \Delta POL, \frac{PL}{OL} = \tan A\right]$$

Second Proof of Formulae

1. **$\cos(A+B) = \cos A \cos B - \sin A \sin B$**

Proof Let O be the centre of a unit circle.



Let $\angle LOP = A$ radian, $\angle POQ = B$ radian, $\angle LOR = -B$ radian

(This angle has been measured in clockwise direction)

Now $\angle LOQ = A + B$ and $\angle ROP = A - B$

Since radius of circle is unity

$\therefore \quad$ arc $LP = A$, arc $PQ = B$, arc $LR = |-B| = B$

[in formulae $\theta = \frac{l}{r}$, θ is always taken a positive]

Also as radius of the circle is 1.

$\therefore P \equiv (\cos A, \sin A)$,

$Q \equiv \cos(A+B), \sin(A+B)$,

$R \equiv (\cos(-B), \sin(-B)$ or $R \equiv (\cos B, -\sin B)$

$\therefore \quad \Delta LOQ \cong \Delta POR$

$\therefore \quad LQ = PR$

$\Rightarrow \quad LQ^2 = PR^2$

$\Rightarrow [1 - \cos(A+B)]^2 + [0 - \sin(A+B)]^2$

$$= [\cos A - \cos(-B)]^2 + [(\sin A - \sin(-B)]^2$$

$\Rightarrow 1 + \cos^2(A+B) - 2\cos(A+B) + \sin^2(A+B)$

$$= (\cos A - \cos B)^2 + (\sin A - \sin B)^2$$

$\Rightarrow 1 + \cos^2(A+B) + \sin^2(A+B) - 2\cos(A+B)$

$\Rightarrow \quad = \cos^2 A + \cos^2 B - 2\cos A \cos B + \sin^2 A + \sin^2 B + 2 \sin A \sin B$

$\Rightarrow \quad 2 - 2\cos(A+B) = (\cos^2 A + \sin^2 A) + (\cos^2 B + \sin^2 B) - 2(\cos A \cos B - \sin A \sin B)$

$\Rightarrow \quad 2 - 2\cos(A+B) = 2 - 2(\cos A \cos B - \sin A \sin B)$

$\Rightarrow \quad \cos(A+B) = \cos A \cos B - \sin A \sin B \quad \text{...(i)}$

2. **Putting $-B$ in place of B in (1), we get**

$$\cos(A-B) = \cos A \cos(-B) - \sin A \sin(-B)$$

$$= \cos A \cos B + \sin A \sin B \quad \text{...(ii)}$$

3. $\sin(A+B)=\cos\left[\frac{\pi}{2}-(A+B)\right]$

$=\cos\left[\left(\frac{\pi}{2}-A\right)-B\right]$

$=\cos\left(\frac{\pi}{2}-A\right)\cos B+\sin\left(\frac{\pi}{2}-A\right)\sin B$

$=\sin A\cos B+\cos A\sin B$ ...(iii)

4. $\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}$

$=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}$

$=\frac{\tan A+\tan B}{1-\tan A\tan B}$ ...(iv)

[dividing numerator and denominator by $\cos A\cos B$]

5. Putting $-B$ in place of B in (3), we get

$\sin(A-B)=\sin A\cos B-\cos A\sin B$ ...(v)

6. Putting $-B$ in place of B in (4), we get

$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$ ...(vi)

7. $\cot(A+B)=\frac{\cos(A+B)}{\sin(A+B)}$

$=\frac{\cos A\cos B-\sin A\sin B}{\sin A\cos B+\cos A\sin B}$

$=\frac{\cot A\cot B-1}{\cot B+\cot A}$ ...(vii)

[dividing numerator and denominator by $\sin A\sin B$]

8. Putting $-B$ in place of B in (7), we get

$\cot(A-B)=\frac{-\cot A\cot B-1}{-\cot B+\cot A}$

$=\frac{\cot A\cot B+1}{\cot B-\cot A}$ ...(viii)

Third Proof by Complex Number Method

The result of the sine, cosine and tangent of compound angle can also be derived using the concept of complex numbers as discussed.

$\cos(A\pm B)+i\sin(A\pm B)=e^{i(A\pm B)}$

$=e^{iA}\cdot e^{i(\pm B)}=(\cos A+i\sin A)(\cos(\pm B)+i\sin(\pm B))$

$=(\cos A\cos B\pm i\cos A\sin B+i\sin A\cos B\ \mp\sin A\sin B)$

$=(\cos A\cos B\mp\sin A\sin B)+i(\sin A\cos A\pm\cos A\sin B)$

Comparing real and imaginary parts of the left and right hand side, we get,

$\cos(A\pm B)=(\cos A\cos B\mp\sin A\sin B)$

$\sin(A\pm B)=(\sin A\cos B\pm\cos A\sin B)$

TWO VERY IMPORTANT IDENTITIES

(a) $\sin(A+B)\cdot\sin(A-B)$

$=\sin^2A-\sin^2B=\cos^2B-\cos^2A$

(b) $\cos(A+B)\cdot\cos(A-B)=\cos^2A-\sin^2B$

Proof : (a) $\sin(A+B)\cdot\sin(A-B)$

$=(\sin A\cos B+\cos A\sin B)(\sin A\cos B-\cos A\sin B)$

$=\sin^2A\cos^2B-\cos^2A\sin^2B$

$=\sin^2A(1-\sin^2B)-\sin^2B(1-\sin^2A)$

$=\sin^2A-\sin^2B$

(b) $\cos(A+B)\cdot\cos(A-B)$

$=(\cos A\cos B-\sin A\sin B)(\cos A\cos B+\sin A\sin B)$

$=\cos^2A\cos^2B-\sin^2A\sin^2B$

$=\cos^2A(1-\sin^2B)-(1-\cos^2A)\sin^2B$

$=\cos^2A-\sin^2B$

Example 44. ***Find the value of*** $\tan105°$.

Sol. $\tan105°=\tan(60°+45°)=\frac{\tan60°+\tan45°}{1-\tan60°\tan45°}$

$=\frac{\sqrt{3}+1}{1-\sqrt{3}\cdot1}=\frac{(\sqrt{3}+1)^2}{1-3}=-(2+\sqrt{3})$

$\therefore\quad\tan105°=-(2+\sqrt{3})$

Example 45. Prove that $\tan70°=\tan20°+2\tan50°$.

Sol. $\tan70°=\tan(20°+50°)=\frac{\tan20°+\tan50°}{1-\tan20°\tan50°}$

or $\tan70°-\tan20°\cdot\tan50°\cdot\tan70°=\tan20°+\tan50°$

or $\tan70°=\tan70°\tan50°\tan50°+\tan20°+\tan50°$

$=\cot20°\tan50°\tan20°+\tan20°+\tan50°$

$[\because\tan70°=\tan(90°-20°)=\cot20°]$

$=2\tan50°+\tan20°$

Example 46. If $A+B=45°$, then show that $(1+\tan A)(1+\tan B)=2$.

Sol. $\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$; $1=\frac{\tan A+\tan B}{1-\tan A\tan B}$

[as $A+B=45°$, $\tan(A+B)=1$]

$\therefore\quad\tan A+\tan B+\tan A\tan B=1$

or $1+\tan A+\tan B+\tan A\tan B=1+1$

[$\because$ adding '1' on both sides]

$\Rightarrow\quad(1+\tan A)+\tan B(1+\tan A)=2$

$\Rightarrow\quad(1+\tan A)(1+\tan B)=2$

Example 47. Find the value of $\dfrac{\tan 495^\circ}{\cot 855^\circ}$

Sol. $\tan 495^\circ = \tan(2.180^\circ + 135^\circ) = \tan 135^\circ = -1$

$\cot 855^\circ = \cot(4.180^\circ + 135^\circ)$

$= \cot 135^\circ = -1 \quad [\because \cot(4.180^\circ + \theta) = \cot\theta]$

$\therefore \quad \dfrac{\tan 495^\circ}{\cot 855^\circ} = \dfrac{-1}{-1} = 1$

Example 48. Evaluate $\sin\left\{n\pi + (-1)^n \dfrac{\pi}{4}\right\}$; where n is an integer.

Sol. $\because \quad \sin(\pi + \theta) = -\sin\theta$

$\therefore \quad \sin(n\pi + \theta) = (-1)^n \sin\theta \Rightarrow \sin\left\{n\pi + (-1)^n \dfrac{\pi}{4}\right\}$

$= (-1)^n \sin\left\{(-1)^n \dfrac{\pi}{4}\right\}$

$= (-1)^n (-1)^n \sin\dfrac{\pi}{4} \quad [\because \sin(-\theta) = -\sin\theta]$

$\therefore \sin\{(-1)^n\theta\} = (-1)^n \sin\theta$

$= (-1)^{2n} \sin\dfrac{\pi}{4} = \sin\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$

Example 49. Prove that $\cos 18^\circ - \sin 18^\circ = \sqrt{2}\sin 27^\circ$

Sol. R.H.S. $= \sqrt{2}\sin 27^\circ = \sqrt{2}\sin(45^\circ - 18^\circ)$

$= \sqrt{2}(\sin 45^\circ \cos 18^\circ - \cos 45^\circ \sin 18^\circ)$

$= \sqrt{2}\left(\dfrac{1}{\sqrt{2}}\cos 18^\circ - \dfrac{1}{\sqrt{2}}\sin 18^\circ\right)$

$= \cos 18^\circ - \sin 18^\circ$

$=$ L.H.S.

Example 50. Show that $\cot\left(\dfrac{\pi}{4} + x\right) \cdot \cot\left(\dfrac{\pi}{4} - x\right) = 1$

Sol. L.H.S. $= \dfrac{\cos\left(\dfrac{\pi}{4} + x\right)\cos\left(\dfrac{\pi}{4} - x\right)}{\sin\left(\dfrac{\pi}{4} + x\right)\sin\left(\dfrac{\pi}{4} - x\right)}$

$= \dfrac{\cos^2\dfrac{\pi}{4} - \sin^2 x}{\sin^2\dfrac{\pi}{4} - \sin^2 x} = \dfrac{\dfrac{1}{2} - \sin^2 x}{\dfrac{1}{2} - \sin^2 x} = 1$

Example 51. If $\sin\alpha\sin\beta - \cos\alpha\cos\beta + 1 = 0$, Prove that $1 + \cot\alpha\tan\beta = 0$

Sol. Given, $\sin\alpha\sin\beta - \cos\alpha\cos\beta + 1 = 0$

$\Rightarrow \quad \cos\alpha\cos\beta - \sin\alpha\sin\beta = 1$

$\Rightarrow \quad \cos(\alpha + \beta) = 1 \quad ...(i)$

Now, $1 + \cos\alpha\tan\beta = 1 + \dfrac{\cos\alpha}{\sin\alpha} \cdot \dfrac{\sin\beta}{\cos\beta}$

$= \dfrac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\sin\alpha\cos\beta} = \dfrac{\sin(\alpha + \beta)}{\sin\alpha\cos\beta}$

$= \dfrac{0}{\sin\alpha\cos\beta} = 0$

$[\because \sin^2(\alpha + \beta) = 1 - \cos^2(\alpha + \beta) = 1 - 1 = 0]$

Example 52. Prove that $\dfrac{\sin(B - C)}{\cos B\cos C} + \dfrac{\sin(C - A)}{\cos C\cos A} + \dfrac{\sin(A - B)}{\cos A\cos B} = 0$

Sol. First term of L.H.S.

$= \dfrac{\sin(B - C)}{\cos B\cos C} = \dfrac{\sin B\cos C - \cos B\sin C}{\cos B\cos C}$

$= \dfrac{\sin B\cos C}{\cos B\cos C} - \dfrac{\cos B\sin C}{\cos B\cos C}$

$= \tan B - \tan C$

Similarly, second term of L.H.S. $= \tan C - \tan A$ and 3rd term of L.H.S. $= \tan A - \tan B$

Now L.H.S. $= (\tan B - \tan C) + (\tan C - \tan A) + (\tan A - \tan B) = 0$

Example 53. Show that $\tan 75^\circ + \cot 75^\circ = 4$.

Sol. $\tan 75^\circ = \tan(45^\circ + 30^\circ) = \dfrac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ}$

$= \dfrac{1 + \dfrac{1}{\sqrt{3}}}{1 - \dfrac{1}{\sqrt{3}}} = \dfrac{\sqrt{3} + 1}{\sqrt{3} - 1} \quad ...(i)$

and $\cot 75^\circ = \dfrac{1}{\tan 75^\circ} = \dfrac{\sqrt{3} - 1}{\sqrt{3} + 1} \quad ...(ii)$

Now, L.H.S. $= \tan 75^\circ + \cot 75^\circ$

$= \dfrac{\sqrt{3} + 1}{\sqrt{3} - 1} + \dfrac{\sqrt{3} - 1}{\sqrt{3} + 1}$ [from Eqs. (i) and (ii)]

$= \dfrac{(\sqrt{3} + 1)^2 + (\sqrt{3} - 1)^2}{(\sqrt{3} - 1)(\sqrt{3} + 1)}$

$= \dfrac{(4 + 2\sqrt{3}) + (4 - 2\sqrt{3})}{3 - 1} = \dfrac{8}{2} = 4 =$ R.H.S.

Example 54. If $\tan\beta = \dfrac{n\sin\alpha\cos\alpha}{1 - n\sin^2\alpha}$. Prove that $\tan(\alpha - \beta) = (1 - n)\tan\alpha$.

Sol. $\tan\beta = \dfrac{n\sin\alpha\cos\beta}{1 - n\sin^2\alpha}$

$= \dfrac{\dfrac{n\sin\alpha\cos\alpha}{\cos^2\alpha}}{\dfrac{1}{\cos^2\alpha} - \dfrac{n\sin^2\alpha}{\cos^2\alpha}}$

[dividing numerator and denominator by $\cos\alpha$]

$$= \frac{n \tan\alpha}{\sec^2\alpha - n\tan^2\alpha}$$

$$= \frac{n \tan\alpha}{1 + \tan^2\alpha - n\tan^2\alpha}$$

$$= \frac{n \tan\alpha}{1 + (1-n)\tan^2\alpha} \quad ...(i)$$

Now, L.H.S. $= \tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha \tan\beta}$

$$= \frac{\tan\alpha - \dfrac{n\tan\alpha}{1+(1-n)\tan^2\alpha}}{1 + \tan\alpha \dfrac{n\tan\alpha}{1+(1-n)\tan^2\alpha}} \quad \text{[from Eqs. (i)]}$$

$$= \frac{\tan\alpha + (1-n)\tan^3\alpha - n\tan\alpha}{1 + (1-n)\tan^2\alpha + n\tan^2\alpha}$$

$$= \frac{(1-n)\tan\alpha + (1-n)\tan^3\alpha}{1 + \tan^2\alpha}$$

$$= \frac{(1-n)\tan\alpha\,(1+\tan^2\alpha)}{1+\tan^2\alpha}$$

$$= (1-n)\tan\alpha$$

Example 55. Show that $\cos^2\theta + \cos^2(\alpha+\theta) - 2\cos\alpha\cos\theta\cos(\alpha+\theta)$ in independent of θ.

Sol. $\cos^2\theta + \cos^2(\alpha+\theta) - 2\cos\alpha\cos\theta\cos(\alpha+\theta)$

$$= \cos^2\theta + \cos(\alpha+\theta)[\cos(\alpha+\theta) - 2\cos\alpha\cos\theta]$$

$$= \cos^2\theta + \cos(\alpha+\theta)$$

$$[\cos\alpha\cos\theta - \sin\alpha\sin\theta - 2\cos\alpha\cos\theta]$$

$$= \cos^2\theta - \cos(\alpha+\theta)[\cos\alpha\cos\theta + \sin\alpha\sin\theta]$$

$$= \cos^2\theta - \cos(\alpha+\theta)\cos(\alpha-\theta)$$

$$= \cos^2\theta - [\cos^2\alpha - \sin^2\theta]$$

$$= \cos^2\theta + \sin^2\theta - \cos^2\alpha$$

$$= 1 - \cos^2\alpha, \text{ which is independent of } \theta.$$

Example 56. If $3\tan\theta\tan\phi = 1$, then prove that $2\cos(\theta+\phi) = \cos(\theta-\phi)$.

Sol. Given, $3\tan\theta\tan\phi = 1$ or $\cot\theta\cot\phi = 3$

or $$\frac{\cos\theta\cos\phi}{\sin\theta\sin\phi} = \frac{3}{1}$$

By componendo and dividendo, we get

$$\frac{\cos\theta\cos\phi + \sin\theta\sin\phi}{\cos\theta\cos\phi - \sin\theta\sin\phi} = \frac{3+1}{3-1}$$

or $$\frac{\cos(\theta-\phi)}{\cos(\theta+\phi)} = 2$$

or $$2\cos(\theta+\phi) = \cos(\theta-\phi)$$

Example 57. Let A, B, C be the three angles such that $A+B+C = \pi$. If $\tan A \cdot \tan B = 2$, then find the value of $\dfrac{\cos A \cos B}{\cos C}$.

Sol. Given, $\tan A \cdot \tan B = 2$

Let $$y = \frac{\cos A\cos B}{\cos C} = -\frac{\cos A \cdot \cos B}{\cos(A+B)}$$

$$[\because \cos C = \cos(\pi - (A+B)) = -\cos(A+B)]$$

$$= \frac{\cos A \cdot \cos B}{\sin A \sin B - \cos A \cos B}$$

$$= \frac{1}{\tan A \tan B - 1} = \frac{1}{2-1} = 1$$

Example 58. Prove that $\dfrac{\cos 10^\circ + \sin 10^\circ}{\cos 10^\circ - \sin 10^\circ} = \tan 55^\circ$.

Sol. $\dfrac{\cos 10^\circ + \sin 10^\circ}{\cos 10^\circ - \sin 10^\circ} = \dfrac{1+\tan 10^\circ}{1-\tan 10^\circ} = \dfrac{\tan 45^\circ + \tan 10^\circ}{1 - \tan 45^\circ \tan 10^\circ}$

$$= \tan(45^\circ + 10^\circ) = \tan 55^\circ \text{ (dividing by } \cos 10^\circ)$$

Example 59. If $\sin(A-B) = \dfrac{1}{\sqrt{10}}$, $\cos(A+B) = \dfrac{2}{\sqrt{29}}$, find the value of $\tan 2A$ where A and B lie between 0 and $\dfrac{\pi}{4}$.

Sol. $\tan 2A = \tan[(A+B) + (A-B)]$

$$= \frac{\tan(A+B) + \tan(A-B)}{1 - \tan(A+B)\tan(A-B)} \quad ...(i)$$

Given that, $0 < A < \dfrac{\pi}{4}$ and $0 < B < \dfrac{\pi}{4}$. Therefore,

$$0 < A + B < \frac{\pi}{2}$$

Also, $$-\frac{\pi}{4} < A - B < \frac{\pi}{4} \quad \text{and} \quad \sin(A-B) = \frac{1}{\sqrt{10}}$$

$\therefore$ $$0 < A - B < \frac{\pi}{4}$$

Now, $$\sin(A-B) = \frac{1}{\sqrt{10}}$$

$\Rightarrow$ $$\tan(A-B) = \frac{1}{3} \quad ...(ii)$$

$$\cos(A+B) = \frac{2}{\sqrt{29}}$$

$\Rightarrow$ $$\tan(A+B) = \frac{5}{2} \quad ...(iii)$$

From Eqs. (i), (ii) and (iii), we get

$$\tan 2A = \frac{\frac{5}{2} + \frac{1}{3}}{1 - \frac{5}{2}\times\frac{1}{3}} = \frac{17}{6} \times \frac{16}{1} = 17$$

Example 60. Prove that $(1+\tan 1^\circ)(1+\tan 2^\circ)\ldots(1+\tan 45^\circ)=2^{23}$.

Sol. $(1+\tan x^\circ)(1+\tan(45^\circ-x^\circ))$

$$=(1+\tan x^\circ)\left(1+\frac{1-\tan x^\circ}{1+\tan x^\circ}\right)=2$$

$\therefore (1+\tan 1^\circ)(1+\tan 44^\circ)$

$=(1+\tan 2^\circ)(1+\tan 43^\circ)$

$=(1+\tan 3^\circ)(1+\tan 42^\circ)$

...

...

$=(1+\tan 22^\circ)(1+\tan 23^\circ)$

$=2$

$(1+\tan 1^\circ)(1+\tan 2^\circ)\ldots(1+\tan 45^\circ)=2^{23}$

(as $1+\tan 45^\circ=2$)

Example 61. If $\cos(\beta-\gamma)+\cos(\gamma-\alpha)+\cos(\alpha-\beta)=-\frac{3}{2}$, Prove that

$\cos\alpha+\cos\beta+\cos\gamma=\sin\alpha+\sin\beta+\sin\gamma=0$

Sol. Given, $\cos(\beta-\gamma)+\cos(\gamma-\alpha)+\cos(\alpha-\beta)=-\frac{3}{2}$

or $3+2\cos(\beta-\gamma)+2\cos(\gamma-\alpha)+2\cos(\alpha-\beta)=0$

or $3+2(\cos\beta\cos\gamma+\sin\beta\sin\gamma)$

$+2(\cos\gamma\cos\alpha+\sin\gamma\sin\alpha)$

$+2(\cos\alpha\cos\beta+\sin\alpha\sin\beta)=0$

or $(\cos^2\alpha+\sin^2\alpha)+(\cos^2\beta+\sin^2\beta)+(\cos^2\gamma+\sin^2\gamma)$

$+2(\cos\beta\cos\gamma+\sin\beta\sin\gamma)+2(\cos\gamma\cos\alpha+\sin\gamma\sin\alpha)$

$+2(\cos\alpha\cos\beta+\sin\alpha\sin\beta)=0$

or $(\cos^2\alpha+\cos^2\beta+\cos^2\gamma+2\cos\alpha\cos\beta+2\cos\beta\cos\gamma$

$+2\cos\gamma\cos\gamma)+(\sin^2\alpha+\sin^2\beta+\sin^2\gamma$

$+2\sin\alpha\sin\beta+2\sin\beta\sin\gamma+2\sin\gamma\sin\alpha)=0$

or $(\cos\alpha+\cos\beta+\cos\gamma)^2+(\sin\alpha+\sin\beta+\sin\gamma)^2=0$

which is possible only when

$\cos\alpha+\cos\beta+\cos\gamma=0$ and $\sin\alpha+\sin\beta+\sin\gamma=0$

Example 62. Find the value of $\frac{\cos 25^\circ+\cot 55^\circ}{\tan 25^\circ+\tan 55^\circ}+\frac{\cot 55^\circ+\cot 100^\circ}{\tan 55^\circ+\tan 100^\circ}+\frac{\cot 100^\circ+\cot 25^\circ}{\tan 100^\circ+\tan 25^\circ}$.

Sol. $E=\frac{\cot 25^\circ+\cot 55^\circ}{\tan 25^\circ+\tan 55^\circ}+\frac{\cot 55^\circ+\cot 100^\circ}{\tan 55^\circ+\tan 100^\circ}+\frac{\cot 100^\circ+\cot 25^\circ}{\tan 100^\circ+\tan 25^\circ}$

$=\frac{1}{\tan 55^\circ\tan 100^\circ}+\frac{1}{\tan 55^\circ\tan 100^\circ}+\frac{1}{\tan 100^\circ\tan 25^\circ}$

$=\frac{\tan 25^\circ+\tan 55^\circ+\tan 100^\circ}{\tan 25^\circ\cdot\tan 55^\circ\cdot\tan 100^\circ}$

Since, $25^\circ+55^\circ+100^\circ=180^\circ$

$\tan 25^\circ+\tan 55^\circ+\tan 100^\circ=\tan 25^\circ\tan 55^\circ\tan 100^\circ$

$\Rightarrow \quad E=1$

Example 63. Prove that $\sum_{k=1}^{100}\sin(kx)\cos(101-k)x=50\sin(101x)$

Sol. Let $S=\sum_{k=1}^{100}\sin(kx)\cos(101-k)x$

$\Rightarrow \quad S=\sin x\cos 100x+\sin 2x\cos 99x$

$+\ldots+\sin 100x\cos x$ ...(i)

$S=\cos x\sin 100x+\cos 2x\sin 99x+\ldots+$

$\sin x\cos 100x$ ...(ii)

(on writing in reverse order)

On adding Eqs. (i) and (ii), we get

$2S=(\sin x\cos 100x+\cos x\sin 100x)$

$+(\sin 2x\cos 99x+\cos 2x\sin 99x)$

...

...

$+(\sin 100x\cos x+\sin x\cos 100x)$

$=\sin 101x+\sin 101x+\ldots+\sin 101x$ (100 times)

Hence, $S=50\sin(101x)$

Example 64. If $A=\frac{\pi}{5}$, then find the value of $\sum_{r=1}^{8}\tan(rA)\cdot\tan((r+1)A)$.

Sol. $\tan((r+1)A-(rA))=\frac{\tan(r+1)A-\tan(rA)}{1+\tan(r+1)A\cdot\tan(rA)}$

$\Rightarrow \quad S=\sum_{r=1}^{8}\tan(rA)\cdot\tan(r+1)A$

$=\sum_{r=1}^{8}(-1)+\frac{1}{\tan A}\sum_{r=1}^{8}(\tan(r+1)A-\tan(rA))$

$=-8+\frac{1}{\tan A}\cdot(\tan 9A-\tan A)$

Now, $\tan 9A=\tan\frac{9\pi}{5}$

$=\tan\left(2\pi-\frac{\pi}{5}\right)$

$=-\tan\frac{\pi}{5}$

$\Rightarrow \quad S=-8+\frac{1}{\tan A}(-2\tan A)$

$=-8-2=-10$

Example 65. Prove that
$\sin\theta\cdot\sec 3\theta+\sin 3\theta\cdot\sec 3^2\theta+\sin 3^2\theta\cdot\sec 3^3\theta+\ldots$
upto n terms $=\frac{1}{2}[\tan 3^n\theta-\tan\theta]$

Sol. $\sin\theta\cdot\sec 3\theta+\sin 3\theta\cdot\sec 3^2\theta+\sin 3^2\theta\cdot\sec 3^3\theta+\ldots$ upto n terms

$$=\sum_{r=1}^{n}\sin 3^{r-1}\theta\cdot\sec 3^r\theta$$

$$=\sum_{r=1}^{n}\frac{2\cos 3^{r-1}\theta\sin 3^{r-1}\theta}{2\cos 3^{r-1}\theta\cdot\cos 3^r\theta}$$

$$=\frac{1}{2}\sum_{r=1}^{n}\frac{\sin(2\cdot 3^{r-1}\theta)}{\cos 3^{r-1}\theta\cdot\cos 3^r\theta}$$

$$=\frac{1}{2}\sum_{r=1}^{n}\frac{\sin(3^r\theta-3^{r-1}\theta)}{\cos 3^{r-1}\theta\cdot\cos 3^r\theta}$$

$$\sin 3^r\theta\cdot\cos 3^{r-1}\theta$$

$$=\frac{1}{2}\sum_{r=1}^{n}\frac{-\cos 3^r\theta\cdot\sin 3^{r-1}\theta}{\cos 3^{r-1}\theta\cdot\cos 3^r\theta}$$

$$=\frac{1}{2}\sum_{r=1}^{n}(\tan 3^r\theta-\tan 3^{r-1}\theta)$$

$$=\frac{1}{2}[\tan 3^n\theta-\tan\theta]$$

Example 66. In a triangle ABC, if $\sin A\sin(B-C)=\sin C\sin(A-B)$, then prove that $\cot A$, $\cot B$, $\cot C$ are in AP.

Sol. $\sin A\sin(B-C)=\sin C\sin(A-B)$

$$\Rightarrow\quad\frac{\sin(B-C)}{\sin C\sin B}=\frac{\sin(A-B)}{\sin A\sin B}$$

$$\Rightarrow\frac{\sin B\cos C-\sin C\cos B}{\sin C\sin B}=\frac{\sin A\cos B-\sin B\cos A}{\sin A\sin B}$$

$$\Rightarrow\quad\cot C-\cot B=\cot B-\cot A$$

$$\Rightarrow\quad 2\cot B=\cot A+\cot C$$

$\therefore$ $\cot A$, $\cot B$, $\cot C$ are in A.P.

Example 67. If $0<\beta<\alpha<\pi/4$, $\cos(\alpha+\beta)=3/5$ and $\cos(\alpha-\beta)=4/5$, then evaluate $\sin 2\alpha$.

Sol. We know, $\sin(2\alpha)=\sin\{(\alpha+\beta)+(\alpha-\beta)\}$

$$=\sin(\alpha+\beta)\cdot\cos(\alpha-\beta)+\sin(\alpha-\beta)\cdot\cos(\alpha+\beta).$$

$$=\frac{4}{5}\cdot\frac{4}{5}+\frac{3}{5}\cdot\frac{3}{5}$$

[using $\cos(\alpha+\beta)=3/5$, $\cos(\alpha-\beta)=4/5$
$\Rightarrow\sin(\alpha+\beta)=4/5$, $\sin(\alpha-\beta)=3/5$]

$$=\frac{16+9}{25}=1\quad\Rightarrow\quad\sin 2\alpha=1$$

Example 68. If $\cos\alpha=\frac{1}{2}\left(x+\frac{1}{x}\right)$, $\cos\beta=\frac{1}{2}\left(y+\frac{1}{y}\right)$, then evaluate $\cos(\alpha-\beta)$.

Sol. $\cos\alpha=\frac{1}{2}\left(x+\frac{1}{x}\right)$

$$\Rightarrow x^2-2x\cos\alpha+1=0\ \Rightarrow\ x=\frac{2\cos\alpha\pm\sqrt{4\cos^2\alpha-4}}{2}$$

$$\Rightarrow\qquad x=\frac{2\cos\alpha\pm 2i\sin\alpha}{2}\qquad\{\text{as }\sqrt{-1}=i\}$$

$$\therefore\qquad x=\cos\alpha\pm i\sin\alpha$$

Similarly, $y=\cos\beta\pm i\sin\beta$

$$\therefore\qquad\frac{x}{y}=\frac{\cos\alpha\pm i\sin\alpha}{\cos\beta\pm i\sin\beta}=\cos(\alpha-\beta)\pm i\sin(\alpha-\beta)\qquad\ldots(i)$$

and $$\frac{y}{x}=\frac{\cos\beta\pm i\sin\beta}{\cos\alpha\pm i\sin\alpha}=\cos(\alpha-\beta)\mp i\sin(\alpha-\beta)\qquad\ldots(ii)$$

On adding Eqs. (i) and (ii), we get

$$\frac{x}{y}+\frac{y}{x}=2\cos(\alpha-\beta)$$

i.e. $$\cos(\alpha-\beta)=\frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}\right).$$

Example 69. If $2\sin\alpha\cos\beta\sin\gamma=\sin\beta\sin(\alpha+\gamma)$. Then, show $\tan\alpha$, $\tan\beta$ and $\tan\gamma$ are in harmonic progression.

Sol. We have, $2\sin\alpha\cos\beta\sin\gamma=\sin\beta\sin(\alpha+\gamma)$

or $2\sin\alpha\cos\beta\sin\gamma=\sin\beta\{\sin\alpha\cos\gamma+\cos\alpha\sin\gamma\}$

$\Rightarrow$ $2\sin\alpha\cos\beta\sin\gamma=\sin\alpha\sin\beta\cos\gamma+\cos\alpha\sin\beta\sin\gamma$

On dividing both sides by $\sin\alpha\sin\beta\sin\gamma$, we get

$$2\cot\beta=\cot\alpha+\cot\gamma\quad\text{or}\quad\frac{2}{\tan\beta}=\frac{1}{\tan\alpha}+\frac{1}{\tan\gamma}$$

i.e. $$\frac{1}{\tan\alpha},\frac{1}{\tan\beta},\frac{1}{\tan\gamma}\text{ are in AP}$$

or $$\tan\alpha,\tan\beta,\tan\gamma\text{ are in HP.}$$

Exercise for Session 6

1. If α lies in II quadrant, β lies in III quadrant and $\tan(\alpha+\beta)>0$, then $(\alpha+\beta)$ lies in ..... quadrants.
2. If $3\tan A\tan B=1$, then prove that $\dfrac{\cos(A-B)}{\cos(A+B)}=2$.
3. If $\tan\alpha=\dfrac{m}{m+1}$ and $\tan\beta=\dfrac{1}{2m+1}$, the find the value of $\alpha+\beta$.
4. If $\cos(\alpha+\beta)=\dfrac{4}{5}$, $\sin(\alpha-\beta)=\dfrac{5}{13}$ and $\alpha,\beta\in\left(0,\dfrac{\pi}{4}\right)$, then find the value of $\tan 2\alpha$.
5. If $\alpha+\beta=\dfrac{\pi}{2}$ and $\beta+\gamma=\alpha$, then find the value of $\tan\alpha$.
6. If $\cos(\theta-\alpha)=a$ and $\cos(\theta-\beta)=b$ then the value of $\sin^2(\alpha-\beta)+2ab\cos(\alpha-\beta)$.
7. If $2\cos A=x+\dfrac{1}{x}$, $2\cos\beta=y+\dfrac{1}{y}$ then show that $2\cos(A-B)=\dfrac{x}{y}+\dfrac{y}{x}$.
8. If $y=(1+\tan A)(1-\tan B)$, where $A-B=\dfrac{\pi}{4}$, then find the value of $(y+1)^{y+1}$.

Session 7

Sum of Sines/Cosines in Terms of Products

Converting Product into Sum/Difference and Vice-Versa

Product into Sum/Difference

1. $2\sin A\cos B=\sin(A+B)+\sin(A-B)$
2. $2\cos A\sin B=\sin(A+B)-\sin(A-B)$
3. $2\cos A\cos B=\cos(A+B)+\cos(A-B)$
4. $2\sin A\sin B=\cos(A-B)-\cos(A+B)$

Proof We know that

$$\sin A\cos B+\cos A\sin B=\sin(A+B) \quad ...(i)$$
$$\sin A\cos B-\cos A\sin B=\sin(A-B) \quad ...(ii)$$
$$\cos A\cos B-\sin A\sin B=\cos(A+B) \quad ...(iii)$$
$$\cos A\cos B+\sin A\sin B=\cos(A-B) \quad ...(iv)$$

Adding Eqs. (i) and (ii), we obtain

$$2\sin A\cos B=\sin(A+B)+\sin(A-B) \quad ...(v)$$

Subtracting Eqs. (ii) from (i), we get

$$2\cos A\sin B=\sin(A+B)-\sin(A-B) \quad ...(vi)$$

Adding Eqs. (iii) and (iv), we get

$$2\cos A\cos B=\cos(A+B)+\cos(A-B) \quad ...(vii)$$

Subtracting Eqs. (iii) from (iv), we get

$$2\sin A\sin B=\cos(A-B)-\cos(A+B) \quad ...(viii)$$

Above four formulas are used to convert product of two sines and cosines into the sum or difference of two sines and cosines.

Sum/Difference into Products

1. $\sin A+\sin B=2\sin\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)$
2. $\sin A-\sin B=2\cos\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{A-B}{2}\right)$
3. $\cos A+\cos B=2\cos\left(\dfrac{A-B}{2}\right)\cdot\cos\left(\dfrac{A+B}{2}\right)$
4. $\cos A-\cos B=2\sin\left(\dfrac{A+B}{2}\right)\cdot\sin\left(\dfrac{B-A}{2}\right)$

Proof (i) Let $A=C+D$ and $B=C-D$, then $C=\dfrac{A+B}{2}$

and $$D=\frac{A-B}{2}$$

$$\therefore \quad \text{L.H.S.}=\sin(C+D)+\sin(C-D)=2\sin C\cos D$$
$$=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}=\text{R.H.S}$$

Similarly we proof of (ii), (iii) and (iv).

Some other Useful Results

1. $\tan A + \tan B = \dfrac{\sin(A+B)}{\cos A \cos B}$, where $A, B \neq n\pi + \dfrac{\pi}{2}$
2. $\tan A - \tan B = \dfrac{\sin(A-B)}{\cos A \cos B}$, where $A, B \neq n\pi + \dfrac{\pi}{2}$
3. $\cot A + \cot B = \dfrac{\sin(A+B)}{\sin A \sin B}$, where $A, B \neq n\pi, n \in z$
4. $\cot A - \cot B = \dfrac{\sin(B-A)}{\sin A \sin B}$, where $A, B \neq n\pi, n \in z$

Example 70. Prove that $\cos 55^\circ + \cos 65^\circ + \cos 175^\circ = 0$.

Sol. L.H.S. $= \cos 55^\circ + \cos 65^\circ + \cos 175^\circ$

$$= 2\cos\frac{55^\circ + 65^\circ}{2}\cos\frac{55^\circ - 65^\circ}{2} + \cos 175^\circ$$

$$= 2\cos 60^\circ \cos(-5^\circ) + \cos 175^\circ$$

$$= 2 \times \frac{1}{2}\cos 5^\circ + \cos(180^\circ - 5^\circ)$$

$$= \cos 5^\circ - \cos 5^\circ = 0$$

Example 71. ***Prove that*** $\dfrac{\sin A + \sin 2A + \sin 4A + \sin 5A}{\cos A + \cos 2A + \cos 4A + \cos 5A} = \tan 3A$.

Sol. $\dfrac{\sin A + \sin 2A + \sin 4A + \sin 5A}{\cos A + \cos 2A + \cos 4A + \cos 5A}$

$$= \frac{(\sin 5A + \sin A) + (\sin 4A + \sin 2A)}{(\cos 5A + \cos A) + (\cos 4A + \cos 2A)}$$

$$= \frac{2\sin 3A \cos 2A + 2\sin 3A \cos A}{2\cos 3A \cos 2A + 2\cos 3A \cos A}$$

$$= \frac{2\sin 3A(\cos 2A + \cos A)}{2\cos 3A(\cos 2A + \cos A)} = \tan 3A$$

Example 72. Prove that $(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2 = 4\cos^2\left(\dfrac{\alpha - \beta}{2}\right)$.

Sol. L.H.S. $= (\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2$

$$= \left\{2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\right\}^2 + \left\{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\right\}$$

$$= 4\cos^2\left(\frac{\alpha-\beta}{2}\right)\left\{\cos^2\frac{\alpha+\beta}{2} + \sin^2\frac{\alpha+\beta}{2}\right\}$$

$$= 4\cos^2\left(\frac{\alpha-\beta}{2}\right) = \text{R.H.S.}$$

Example 73. ***If*** $\sin A = \sin B$ ***and*** $\cos A = \cos B$, ***then prove that*** $\sin\dfrac{A-B}{2} = 0$.

Sol. We have $\sin A = \sin B$ and $\cos A = \cos B$

or $\sin A - \sin B = 0$ and $\cos A - \cos B = 0$

or $2\sin\left(\dfrac{A-B}{2}\right)\sin\left(\dfrac{A+B}{2}\right) = 0$

and $-2\sin\left(\dfrac{A-B}{2}\right)\sin\left(\dfrac{A+B}{2}\right) = 0$

or $\sin\dfrac{A-B}{2} = 0$, which is common for both the equations.

Example 74. ***Prove that*** $\sin 20^\circ \sin 40^\circ \sin 80^\circ = \dfrac{\sqrt{3}}{8}$

Sol. L.H.S. $= \sin 20^\circ \sin 40^\circ \sin 80^\circ = \dfrac{1}{2}(2\sin 80^\circ \sin 40^\circ)\sin 20^\circ$

$$= \frac{1}{2}[\cos(80^\circ - 40^\circ) - \cos(80^\circ + 40^\circ)\sin 20^\circ$$

$$= \frac{1}{2}(\cos 40^\circ - \cos 120^\circ)\sin 20^\circ$$

$$= \frac{1}{4}(2\cos 40^\circ \sin 20^\circ - 2\cos 120^\circ \sin 20^\circ)$$

$$= \frac{1}{4}\left[\sin(40^\circ + 20^\circ) - \sin(40^\circ - 20^\circ) - 2\left(-\frac{1}{2}\right)\sin 20^\circ\right]$$

$$= \frac{1}{4}[\sin 60^\circ - \sin 20^\circ + \sin 20^\circ] = \frac{1}{4}\sin 60^\circ$$

$$= \frac{1}{4}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{8}$$

Example 75. Prove that $\sin A \cdot \sin(60^\circ - A) \cdot \sin(60^\circ + A) = \dfrac{1}{4}\sin 3A$

Sol. L.H.S. $= \sin A \cdot \sin(60^\circ - A) \cdot \sin(60^\circ + A)$

$$= \frac{1}{2}\sin A[2\sin(60^\circ + A)\cdot\sin(60^\circ - A)]$$

$$= \frac{1}{2}\sin A[\cos(60^\circ + A - 60^\circ + A) - \cos(60^\circ + A + 60^\circ - A)]$$

$$= \frac{1}{2}\sin A(\cos 2A - \cos 120^\circ)$$

$$= \frac{1}{4}(2\cos 2A \sin A - 2\cos 120^\circ \sin A)$$

$$= \frac{1}{4}\left[\sin(2A + A) - \sin(2A - A) - 2\left(-\frac{1}{2}\right)\sin A\right]$$

$$= \frac{1}{4}(\sin 3A - \sin A - \sin A) = \frac{1}{4}\sin 3A$$

Exercise for Session 7

1. Show that $\sin x + \sin 3x + \sin 5x + \sin 7x = 4 \sin 4x \cos 2x \cos x$.
2. Show that $\sin A \cdot \sin(B - C) + \sin B \cdot \sin(C - A) + \sin C \cdot \sin(A - B) = 0$.
3. Show that $\cos \alpha + \cos \beta + \cos \gamma + \cos(\alpha + \beta + \gamma) = 4 \cos \frac{\alpha + \beta}{2} \cdot \cos \frac{\beta + \gamma}{2} \cdot \cos \frac{\gamma + \alpha}{2}$.
4. If x and y are acute angles, such that $\cos x + \cos y = \frac{3}{2}$ and $\sin x + \sin y = \frac{3}{4}$, then the value of $\sin(x + y)$.
5. Find the value of expression $2 \cos \frac{\pi}{3} \cos \frac{9\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13}$.
6. Find the value of $\left(\frac{\cos A + \cos B}{\sin A - \sin B}\right)^n + \left(\frac{\sin A + \sin B}{\cos A - \cos B}\right)^n$ (where, n is an even)
7. Find the value of $\left(1 + \cos \frac{\pi}{8}\right)\left(1 + \cos \frac{3\pi}{8}\right)\left(1 + \cos \frac{5\pi}{8}\right)\left(1 + \cos \frac{7\pi}{8}\right)$.
8. In a triangle ABC, $\cos 3A + \cos 3B + \cos 3C = 1$, then find any one angle.

Session 8

Trigonometric Ratios of Multiples of an Angle

Trigonometric Ratios of Multiples of an Angle

Definition An angle of the form nA, where n is an integer is called a multiple angle, for example $2A$, $3A$, $4A$, ... etc. are multiple angles of A.

In this session we shall express trigonometrical ratios of multiple angles of A in terms of trigonometrical ratios of A.

Trigonometrical Ratios of $2A$ in term of Trigonometrical Ratio of A

1. $\sin 2A = 2 \sin A \cos A = \dfrac{2 \tan A}{1 + \tan^2 A}$
2. $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2 \sin^2 A$
$= 2 \cos^2 A - 1 = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$
3. $1 + \cos 2A = 2 \cos^2 A$, $1 - \cos 2A = 2 \sin^2 A$
or $\dfrac{1 + \cos 2A}{2} = \cos^2 A$, $\dfrac{1 - \cos 2A}{2} = \sin^2 A$
4. $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$, where $A \neq (2n + 1)\dfrac{\pi}{4}$

Proof $\sin 2A = \sin(A + A) = \sin A \cos A + \cos A \sin A$

[using the formula $\sin(A + B) = \sin A \cos B + \cos A \sin B$]

$$\boxed{= 2 \sin A \cos A}$$

$$\cos 2A = \cos(A + A) = \cos A \cos A - \sin A \sin A$$

$$\boxed{= \cos^2 A - \sin^2 A}$$

$$\tan 2A = \tan(A + A)$$

$$= \frac{\tan A + \tan A}{1 - \tan A \tan A} = \boxed{\frac{2 \tan A}{1 - \tan^2 A}}$$

Trigonometrical Ratios of $3A$ in terms of Trigonometrical Ratio of A

1. $\sin 3A = 3 \sin A - 4 \sin^3 A$
$= 4 \sin(60° - A) \cdot \sin A \cdot \sin(60° + A)$
2. $\cos 3A = 4 \cos^3 A - 3 \cos A$
$= 4 \cos(60° - A) \cos A \cos(60° + A)$
3. $\tan 3A = \dfrac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$

Proof

1. $\sin 3A = \sin(2A + A) = \sin 2A \cos A + \cos 2A \sin A$
$= 2 \sin A \cos A \cdot \cos A + (1 - 2\sin^2 A) \sin A$
$= 2 \sin A \cos^2 A + \sin A - 2 \sin^3 A$
$= 2 \sin A (1 - \sin^2 A) + \sin A - 2 \sin^3 A$
$= 2 \sin A - 2 \sin^3 A + \sin A - 2 \sin^3 A$
$= \boxed{3 \sin A - 4 \sin^3 A}$

2. $\cos 3A = \cos(2A + A) = \cos 2A \cdot A \cos A - \sin 2A \sin A$
$= (2 \cos^2 A - 1) \cos A - 2 \sin A \cos A \cdot \sin A$
$= 2 \cos^3 A - \cos A - 2 \cos A (1 - \cos^2 A)$
$= 2 \cos^3 A - \cos A - 2 \cos A + 2 \cos^3 A$
$= \boxed{4 \cos^3 A - 3 \cos A}$

3. $\tan 3A = \dfrac{\sin 3A}{\cos 3A} = \dfrac{3 \sin A - 4 \sin^3 A}{4 \cos^3 A - 3 \cos A}$
$= \dfrac{\sin A (3 - 4 \sin^2 A)}{\cos A (4 \cos^2 A - 3)} = \dfrac{\tan A (3 - 4 \sin^2 A)}{4 \cos^2 A - 3}$

On dividing by $\cos^2 A$ numerator and denominator

$= \dfrac{\tan A (3 \sec^2 A - 4 \tan^2 A)}{4 - 3 \sec^2 A}$
$= \dfrac{\tan A (3 + 3 \tan^2 A - 4 \tan^2 A)}{4 - 3 - 3 \tan^2 A}$
$= \dfrac{\tan A (3 - \tan^2 A)}{1 - 3 \tan^2 A} = \dfrac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$

Example 76. If $\sin A = \dfrac{3}{5}$, where $0° < A < 90°$, find the values of $\sin 2A$, $\cos 2A$, $\tan 2A$ and $\sin 4A$.

Sol. We have, $\sin A = \dfrac{3}{5}$, where $0° < A < 90°$

$\therefore \quad \cos^2 A = 1 - \sin^2 A$

$\Rightarrow \quad \cos A = + \sqrt{1 - \sin^2 A} = \sqrt{1 - \dfrac{9}{25}} = \dfrac{4}{5}$

$\therefore \quad \tan A = \dfrac{\sin A}{\cos A} = \dfrac{3}{4}$

$$\sin 2A = 2 \sin A \cos A = 2 \times \frac{3}{5} \times \frac{4}{5} = \frac{24}{25}$$

$$\cos 2A = 1 - 2 \sin^2 A = 1 - 2 \times \left(\frac{3}{5}\right)^2 = \frac{7}{25}$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A} = \frac{2 \times \frac{3}{4}}{1 - \left(\frac{3}{4}\right)^2} = \frac{\frac{6}{4}}{1 - \frac{9}{16}} = \frac{24}{7}$$

$$\left[\because \tan A = \frac{3}{4}\right]$$

$$\sin 4A = \sin 2A \cos 2A = 2 \times \frac{24}{25} \times \frac{7}{25} = \frac{336}{625}$$

$$\left[\because \sin 2A = \frac{24}{25} \text{ and } \cos 2A = \frac{7}{25}\right]$$

Example 77. Prove that $\dfrac{1 + \sin 2\theta}{1 - \sin 2\theta} = \left(\dfrac{1 + \tan \theta}{1 - \tan \theta}\right)^2$.

Sol. L.H.S. $= \dfrac{1 + \sin 2\theta}{1 - \sin 2\theta} = \dfrac{\sin^2 \theta + \cos^2 \theta + 2 \sin \theta \cos \theta}{\sin^2 \theta + \cos^2 \theta - 2 \sin \theta \cos \theta}$

$= \left(\dfrac{\sin \theta + \cos \theta}{\sin \theta - \cos \theta}\right)^2 = \left(\dfrac{1 + \tan \theta}{1 - \tan \theta}\right)^2$

[dividing numerator and denominator by $\cos \theta$]

Example 78. Prove that $\dfrac{1 - \tan^2\left(\dfrac{\pi}{4} - 4\right)}{1 + \tan^2\left(\dfrac{\pi}{4} - A\right)} = \sin 2A.$

Sol. $\dfrac{1 + \tan^2\left(\dfrac{\pi}{4} - A\right)}{1 + \tan^2\left(\dfrac{\pi}{4} - A\right)} = \dfrac{1 - \tan^2 \theta}{1 + \tan^2 \theta}$ $\left(\text{where } \dfrac{\pi}{4} - A = \theta\right)$

$= \cos 2\theta = \cos\left(\dfrac{\pi}{2} - 2A\right) = \sin 2A$

Example 79. ***Prove that*** $\dfrac{\sec 8\theta - 1}{\sec 4\theta - 1} = \dfrac{\tan 8\theta}{\tan 2\theta}$

Sol. We have, LHS $= \dfrac{\sec 8\theta - 1}{\sec 4\theta - 1}$

$\Rightarrow \quad \text{LHS} = \dfrac{\frac{1}{\cos 8\theta} - 1}{\frac{1}{\cos 4\theta} - 1} = \dfrac{1 - \cos 8\theta}{\cos 8\theta} \cdot \dfrac{\cos 4\theta}{1 - \cos \theta}$

$\Rightarrow \quad \text{LHS} = \dfrac{2 \sin^2 4\theta}{\cos 8\theta} \cdot \dfrac{\cos 4\theta}{2 \sin^2 2\theta}$

$$\left[\because 1 - \cos 8\theta = 2 \sin^2 \frac{8\theta}{2} = 2 \sin^2 4\theta \right.$$
$$\left. \text{and, } 1 - \cos 4\theta = 2 \sin^2 \frac{4\theta}{2} = 2 \sin^2 2\theta\right]$$

$\Rightarrow \quad \text{LHS} = \dfrac{(2 \sin 4\theta \cos 4\theta)}{\cos 8\theta} \times \dfrac{\sin 4\theta}{2 \sin^2 2\theta}$

$\Rightarrow \quad \text{LHS} = \left(\dfrac{2 \sin 4\theta \cos 4\theta}{\cos 8\theta}\right) \times \left(\dfrac{2 \sin 2\theta \cos 2\theta}{2 \sin^2 2\theta}\right)$

$$\Rightarrow \quad \text{LHS} = \left(\frac{\sin 2(4\theta)}{\cos 8\theta}\right) \times \left(\frac{\cos 2\theta}{\sin 2\theta}\right)$$

$$= \left(\frac{\sin 8\theta}{\cos 8\theta}\right) \times \left(\frac{\cos 2\theta}{\sin 2\theta}\right) = \tan 8\theta \cot 2\theta$$

$$\Rightarrow \quad \text{LHS} = \frac{\tan 8\theta}{\tan 2\theta} = \text{RHS}$$

Example 80. *Show that* $\sqrt{2+\sqrt{2+\sqrt{2+2\cos 8}}} = 2\cos\theta$

Sol. We have, $\text{LHS} = \sqrt{2+\sqrt{2+\sqrt{2(1+\cos 8\theta)}}}$

$$\Rightarrow \quad \text{LHS} = \sqrt{2+\sqrt{2+\sqrt{2(2\cos^2 4\theta)}}} \qquad \left[\because 1+\cos 8\theta = 2\cos^2\frac{8\theta}{2}\right]$$

$$\Rightarrow \quad \text{LHS} = \sqrt{2+\sqrt{2+\sqrt{(4\cos^2 4\theta)}}}$$

$$\Rightarrow \quad \text{LHS} = \sqrt{2+\sqrt{2+2\cos 4\theta}}$$

$$\Rightarrow \quad \text{LHS} = \sqrt{2+\sqrt{2(1+\cos 4\theta)}}$$

$$\Rightarrow \quad \text{LHS} = \sqrt{2+\sqrt{2(2\cos^2 2\theta)}} \qquad [\because 1+\cos 4\theta = 2\cos^2 2\theta]$$

$$\Rightarrow \quad \text{LHS} = \sqrt{2+2\cos 2\theta} = \sqrt{2(1+\cos 2\theta)}$$

$$= \sqrt{2(2\cos^2\theta)} = 2\cos\theta = \text{RHS}$$

Example 81. *Show that* $\sqrt{3}\,\text{cosec}\,20^\circ - \sec 20^\circ = 4$

Sol. We have, $\text{LHS} = \sqrt{3}\,\text{cosec}\,20^\circ - \sec 20^\circ$

$$\Rightarrow \quad \text{LHS} = \frac{\sqrt{3}}{\sin 20^\circ} - \frac{1}{\cos 20^\circ}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sqrt{3}\cos 20^\circ - \sin 20^\circ}{\sin 20^\circ \cos 20^\circ}$$

$$\Rightarrow \quad \text{LHS} = \frac{2\left\{\frac{\sqrt{3}}{2}\cos 20^\circ - \frac{1}{2}\sin 20^\circ\right\}}{\sin 20^\circ \cos 20^\circ}$$

$$\Rightarrow \quad \text{LHS} = \frac{2(\sin 60^\circ \cos 20^\circ - \cos 60^\circ \sin 20^\circ)}{\sin 20^\circ \cos 20^\circ}$$

$$\Rightarrow \quad \text{LHS} = \frac{2\sin(60^\circ - 20^\circ)}{\sin 20^\circ \cos 20^\circ} \Rightarrow \text{LHS} = \frac{2\sin 40^\circ}{\sin 20^\circ \cos 20^\circ}$$

$$= \frac{4\sin 40^\circ}{2\sin 20^\circ \cos 20^\circ} = \frac{4\sin 40^\circ}{\sin 40^\circ} = 4 = \text{RHS}$$

Example 82. Prove that $\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right) = \frac{1}{8}$.

Sol. We have,

$$\cos\frac{7\pi}{8} = \cos\left(\pi - \frac{\pi}{8}\right) = -\cos\frac{\pi}{8}$$

and $$\cos\frac{5\pi}{8} = \cos\left(\pi - \frac{3\pi}{8}\right) = -\cos\frac{3\pi}{8}$$

$$\therefore \text{LHS} = \left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1-\cos\frac{3\pi}{8}\right)\left(1-\cos\frac{\pi}{8}\right)$$

$$\Rightarrow \quad \text{LHS} = \left(1-\cos^2\frac{\pi}{8}\right)\left(1-\cos^2\frac{3\pi}{8}\right)$$

$$\Rightarrow \quad \text{LHS} = \sin^2\frac{\pi}{8}\sin^2\frac{3\pi}{8}$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{4}\left(3\sin^2\frac{\pi}{8}\right)\left(2\sin^2\frac{3\pi}{8}\right)$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{4}\left[\left(1-\cos\frac{\pi}{4}\right)\left(1-\cos\frac{3\pi}{4}\right)\right] \qquad \left[\because 1-\cos\theta = 2\sin^2\frac{\theta}{2}\right]$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{4}\left[\left(1-\frac{1}{\sqrt{2}}\right)\left(1+\frac{1}{\sqrt{2}}\right)\right] = \frac{1}{4}\left(1-\frac{1}{2}\right) = \frac{1}{8} = \text{RHS}$$

Example 83. If $\tan^2\theta = 2\tan^2\phi + 1$, prove that $\cos 2\theta + \sin^2\phi = 0$.

Sol. We have, $\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$

$$\Rightarrow \quad \cos 2\theta = \frac{1-(2\tan^2\phi+1)}{1+2\tan^2\phi+1} \qquad [\because \tan^2\theta = 2\tan^2\phi + 1]$$

$$\Rightarrow \quad \cos 2\theta = \frac{-2\tan^2\phi}{2+2\tan^2\phi} = \frac{-\tan^2\phi}{\sec^2\phi} = -\sin^2\phi$$

$$\therefore \quad \cos 2\theta + \sin^2\phi = 0$$

Example 84. Prove that $\tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cos 8\alpha = \cot\alpha$

Sol. We have, $\cot\theta - \tan\theta = \frac{1}{\tan\theta} - \tan\theta$

$$= \frac{1-\tan^2\theta}{\tan\theta} = 2\left\{\frac{1-\tan^2\theta}{2\tan\theta}\right\}$$

$$\Rightarrow \quad \cot\theta - \tan\theta = \frac{2}{\tan 2\theta}$$

$$\Rightarrow \quad \cot\theta - \tan\theta = 2\cot 2\theta \qquad \ldots(i)$$

We have to prove that

$$\tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha = \cot\alpha$$

or, $$\cot\alpha - \tan\alpha - 2\tan\alpha - 4\tan 4\alpha - 8\cot 8\alpha = 0$$

Now,

$$\text{LHS} = \cot\alpha - \tan\alpha - 2\tan 2\alpha - 4\tan 4\alpha - 8\cot 8\alpha$$

$$\Rightarrow \quad \text{LHS} = (\cot\alpha - \tan\alpha) - 2\tan 2\alpha - 4\tan 4\alpha - 8\cot 8\alpha$$

$$\Rightarrow \quad \text{LHS} = 2\cot 2\alpha - 2\tan 2\alpha - 4\tan 4\alpha - 8\cot 8\alpha \qquad \text{[using (i)]}$$

$$\Rightarrow \quad \text{LHS} = 2(\cot 2\alpha - \tan 2\alpha) - 4\tan 4\alpha - 8\cot 8\alpha$$

$\Rightarrow$ LHS $= 2(2\cot 4\alpha) - 4\tan 4\alpha - 8\cot 8\alpha$

[On replacing θ by 2α in Eq. (i)]

$\Rightarrow$ LHS $= 4\cot 4\alpha - 4\tan 4\alpha - 8\cot 8\alpha$

$\Rightarrow$ LHS $= 4(\cot 4\alpha - \tan 4\alpha) - 8\cot 8\alpha$

$\Rightarrow$ LHS $= 4(2\cot 8\alpha) - 8\cot 8\alpha$

[On replacing θ by 4α in Eq. (i)]

$\Rightarrow$ LHS $= 8\cot 8\alpha - 8\cot 8\alpha \Rightarrow$ LHS $= 0 =$ RHS

Example 85. Determine the smallest positive value of x (in degrees) for which $\tan(x+100^\circ) = \tan(x+50^\circ)\tan x \tan(x-50^\circ)$

Sol. We have, $\tan(x+100^\circ) = \tan(x+50^\circ)\tan x \tan(x-50^\circ)$

$$\Rightarrow \quad \frac{\tan(x+100^\circ)}{\tan(x+50^\circ)} = \tan(x+50^\circ)\tan x^\circ$$

$$\Rightarrow \quad \frac{\sin(x+100^\circ)\cos(x-50^\circ)}{\cos(x+100^\circ)\sin(x-50^\circ)} = \frac{\sin(x+50^\circ)\sin x}{\cos(x+50^\circ)\cos x}$$

$$\Rightarrow \frac{\sin(x+100^\circ)\cos(x-50^\circ)+\cos(x+100^\circ)\sin(x-50^\circ)}{\sin(x+100^\circ)\cos(x-50^\circ)-\cos(x+100^\circ)\sin(x-50^\circ)} = \frac{\sin(x+50^\circ)\sin x+\cos(x+50^\circ)\cos x}{\sin(x+50^\circ)\sin x-\cos(x+50^\circ)\cos x}$$

$$\Rightarrow \quad \frac{\sin(x+100^\circ+x-50^\circ)}{\sin(x+100^\circ-x+50^\circ)} = \frac{\cos(x+50^\circ-x)}{-\cos(x+50^\circ+x)}$$

$$\Rightarrow \quad \frac{\sin(2x+50^\circ)}{\sin 150^\circ} = \frac{\cos 50^\circ}{-\cos(2x+50^\circ)}$$

$$\Rightarrow \quad \sin(2x+50^\circ)\cos(2x+50^\circ) = -\sin 150^\circ \cos 50^\circ$$

$$\Rightarrow \quad 2\sin(2x+50^\circ)\cos(2x+50^\circ) = -2\cos 60^\circ \cos 50^\circ$$

$[\because \sin 150^\circ = \cos 60^\circ]$

$$\Rightarrow \quad \sin(4x+100^\circ) = \sin(270-50^\circ)$$

$$\Rightarrow \quad \sin(4x+100^\circ) = \sin 220^\circ$$

$$\Rightarrow \quad 4x+100^\circ = 220^\circ \Rightarrow x = 30^\circ$$

Example 86. Prove that

$$\frac{\sin x}{\cos 3x} + \frac{\sin 3x}{\cos 9x} + \frac{\sin 9x}{\cos 27x} = \frac{1}{2}(\tan 27x - \tan x)$$

Sol. We have, $\dfrac{\sin x}{\cos 3x} + \dfrac{\sin 3x}{\cos 9x} + \dfrac{\sin 9x}{\cos 27x}$

$$= \frac{1}{2}\left\{\frac{2\sin x\cos x}{\cos 3x\cos x} + \frac{2\sin 3x\cos 3x}{\cos 9x\cos 3x} + \frac{2\sin 9x\cos 9x}{\cos 27x\cos 9x}\right\}$$

$$= \frac{1}{2}\left\{\frac{\sin 2x}{\cos 3x\cos x} + \frac{\sin 6x}{\cos 9x\cos 3x} + \frac{\sin 18x}{\cos 27x\cos 9x}\right\}$$

$$= \frac{1}{2}\left\{\frac{\sin(3x-x)}{\cos 3x\cos x} + \frac{\sin(9x-3x)}{\cos 9x\cos 3x} + \frac{\sin(27x-9x)}{\cos 27x\cos 9x}\right\}$$

$$= \frac{1}{2}\{(\tan 3x - \tan x) + (\tan 9x - \tan 3x) + (\tan 27x - \tan 9x)\}$$

$$= \frac{1}{2}(\tan 27x - \tan x)$$

Exercise for Session 8

1. This question has statement which is true or false.

   If $\dfrac{\pi}{9} < \theta < \dfrac{\pi}{2}$, then the value of $\sqrt{1-\sin 2\theta} = \cos\theta - \sin\theta$.

2. If $\pi < \theta < \dfrac{3\pi}{2}$, then find the value of $\sqrt{\dfrac{1-\cos 2\theta}{1+\cos 2\theta}}$.

3. If $\tan x = -\dfrac{4}{3}$, x lies in II quadrant, then find the value of $\sin\dfrac{x}{2}$.

4. Prove that $\sin^4\dfrac{\pi}{8} + \sin^4\dfrac{3\pi}{8} + \sin^4\dfrac{5\pi}{8} + \sin^4\dfrac{7\pi}{8} = \dfrac{3}{2}$.

5. If $A = 2\sin^2\theta - \cos 2\theta$ and $A \in [\alpha, \beta]$, then find the values of α and β.

6. If $\sin x + \cos x = \dfrac{1}{5}$, then find the value of $\tan 2x$.

7. If $\tan 3A = \dfrac{3\tan A + k\tan^3 A}{1-3\tan^2 A}$, then k is equal to

8. If $\tan A + 2\tan 2A + 4\tan 4A + 8\cot 8A = k\cot A$, then find the value of k.

9. If $m^2\cos\dfrac{2\pi}{15}\cos\dfrac{4\pi}{15}\cos\dfrac{8\pi}{15}\cos\dfrac{14\pi}{15} = n^2$, then find the value of $\dfrac{m^2-n^2}{n^2}$.

10. If $(2^n+1)\theta = \pi$, then find the value of $2^n\cos\theta\cos 2\theta\cos 2^2\theta \ldots \cos 2^{n-1}\theta$.

Session 9

Trigonometric Ratios of Submultiple of an Angle

Definition

An angle of the form $\frac{A}{n}$, where n is an integer is called a submultiple angle of A.

For example $\frac{A}{2}, \frac{A}{3}, \frac{A}{4}$ ... etc., are submultiple angles of A.

In this session we shall express the trigonometric ratios of A in terms of the trigonometric ratios of submultiple angles $\frac{A}{2}, \frac{A}{3}$,... etc., and *vice-versa*.

Trigonometric Ratios of A in Terms of Trigonometric Ratios of $\frac{A}{2}$

(i) $\sin 2A = 2 \sin A \cos A$. Putting $\frac{A}{2}$ in place of A, we get

$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}$

(ii) $\cos 2A = \cos^2 A - \sin^2 A$. Putting $\frac{A}{2}$ in place of A, we get $\cos A = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}$

(iii) $\cos 2A = 2 \cos^2 A - 1$. Putting $\frac{A}{2}$ in place of A, we get

$\cos A = 2 \cos^2 \frac{A}{2} - 1$

(iv) $\cos 2A = 1 - 2 \sin^2 A$. Putting $\frac{A}{2}$ in place of A, we get

$\cos A = 1 - 2 \sin^2 \frac{A}{2}$

(v) $\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$

$\therefore \tan A = \frac{2 \tan \frac{A}{2}}{1 - \tan^2 \frac{A}{2}}$, putting $\frac{A}{2}$ in place of A

(vi) $\sin 2A = \frac{2 \tan A}{1 + \tan^2 A}$ $\therefore \sin A = \frac{2 \tan \frac{A}{2}}{1 + \tan^2 \frac{A}{2}}$, putting $\frac{A}{2}$ in place of A

(vii) $\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$ $\therefore \cos A = \frac{1 - \tan^2 \frac{A}{2}}{1 + \tan^2 \frac{A}{2}}$, putting $\frac{A}{2}$ in place of A

(viii) $\cot 2A = \frac{\cot^2 A - 1}{2 \cot A}$ $\therefore \cot A = \frac{\cot^2 \frac{A}{2} - 1}{2 \cot \frac{A}{2}}$, putting $\frac{A}{2}$ in place of A

Trigonometric Ratios of A in Terms of Trigonometric Ratios of $\frac{A}{3}$

(i) $\sin 3A = 3 \sin A - 4 \sin^3 A$. Putting $\frac{A}{3}$ in place of A, we get $\sin A = 3 \sin \frac{A}{3} - 4 \sin^3 \frac{A}{3}$

(ii) $\cos 3A = 4 \cos^3 A - 3 \cos A$

$\therefore \cos A = 4 \cos^3 \frac{A}{3} - 3 \cos \frac{A}{3}$, putting $\frac{A}{3}$ in place of A

(iii) $\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$, putting $\frac{A}{3}$ in place of A

$\therefore \tan A = \frac{3 \tan \frac{A}{3} - \tan^3 \frac{A}{3}}{1 - 3 \tan^2 \frac{A}{3}}$

Values of $\cos \frac{A}{2}$, $\sin \frac{A}{2}$ and $\tan \frac{A}{2}$ in Terms of $\cos A$

(i) $\cos^2 \frac{A}{2} = \frac{1 + \cos A}{2}$ $\therefore \cos \frac{A}{2} = \pm \sqrt{\frac{1 + \cos A}{2}}$

(ii) $\sin^2\frac{A}{2}=\frac{1-\cos A}{2}\ \therefore\ \sin\frac{A}{2}=\pm\sqrt{\frac{1-\cos A}{2}}$

(iii) $\tan^2\frac{A}{2}=\frac{1-\cos A}{1+\cos A}\ \therefore\ \tan\frac{A}{2}=\pm\sqrt{\frac{1-\cos A}{1+\cos A}}$

Note

If $\cos A$ is given, then there will be two values of $\cos\frac{A}{2}$, $\sin\frac{A}{2}$ and $\tan\frac{A}{2}$ but if A is given, then there will be only one value of $\cos\frac{A}{2}$, $\sin\frac{A}{2}$ and $\tan\frac{A}{2}$ because + sign or – sign before the radical sign can be fixed by knowing the quadrant in which $\frac{A}{2}$ lies.

Values of $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ in Terms of $\sin A$

$$\left(\cos\frac{A}{2}+\sin\frac{A}{2}\right)^2=\cos^2\frac{A}{2}+\sin^2\frac{A}{2}+2\cos\frac{A}{2}\sin\frac{A}{2}$$

$$=1+\sin A$$

$$\therefore\quad \cos\frac{A}{2}+\sin\frac{A}{2}=\pm\sqrt{1+\sin A}\qquad ...(i)$$

Similarly, $$\cos\frac{A}{2}-\sin\frac{A}{2}=\pm\sqrt{1-\sin A}\qquad ...(ii)$$

Adding Eqs. (i) and (ii), we get

$$\cos\frac{A}{2}=\pm\frac{1}{2}\sqrt{1+\sin A}\pm\frac{1}{2}\sqrt{1-\sin A}\qquad ...(iii)$$

Subtracting Eq. (ii) from Eq. (i), we get

$$\sin\frac{A}{2}=\pm\frac{1}{2}\sqrt{1+\sin A}\mp\frac{1}{2}\sqrt{1-\sin A}\qquad ...(iv)$$

Note

If $\sin A$ is given, then there will be 4 values of $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ but if A is given. there will be one and only one value of $\cos\frac{A}{2}$ and $\sin\frac{A}{2}$ because the + or – sign can be fixed before the radical sign in the following way

$$\cos\frac{A}{2}+\sin\frac{A}{2}=\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos\frac{A}{2}+\frac{1}{\sqrt{2}}\sin\frac{A}{2}\right)$$

$$=\sqrt{2}\left(\sin\frac{\pi}{4}\cos\frac{A}{2}+\cos\frac{\pi}{4}\sin\frac{A}{2}\right)=\sqrt{2}\sin\left(\frac{\pi}{4}+\frac{A}{2}\right)$$

Similarly, $\cos\frac{A}{2}-\sin\frac{A}{2}=\sqrt{2}\cos\left(\frac{\pi}{4}+\frac{\pi}{2}\right)$

Thus the sign of $\cos\frac{A}{2}+\sin\frac{A}{2}$ and $\cos\frac{A}{2}-\sin\frac{A}{2}$ can be fixed by knowing the quadrant in which $\frac{\pi}{4}+\frac{A}{2}$ lies.

Values of Trigonometric Ratios of Some Particular Angles

I. (i) **Value of sin 18°**

Let $\theta=18°$, then $5\theta=90°\ \therefore 2\theta+3\theta=90°$

or $2\theta=90°-3\theta\ \therefore\ \sin 2\theta=\sin(90°-3\theta)$

or $\sin 2\theta=\cos 3\theta$ or $2\sin\theta\cos\theta=4\cos^3\theta-3\cos\theta$

or $2\sin\theta=4\cos^2\theta-3$ [dividing by $\cos\theta$]

or $2\sin\theta=4(1-\sin^2\theta)-3=1-4\sin^2\theta$

or $4\sin^2\theta+2\sin\theta-1=0$

$$\therefore\sin\theta=\frac{-2\pm\sqrt{4+16}}{8}=\frac{-2\pm2\sqrt{5}}{8}=\frac{-1\pm\sqrt{5}}{4}$$

Thus $\sin\theta=\frac{-1+\sqrt{5}}{4},\frac{-1-\sqrt{5}}{4}$

$\because\ \theta=18°$

$\therefore\ \sin\theta=\sin 18°>0$, for $18°$ lies in the 1st quadrant

$\therefore\ \sin\theta$ i.e., $\sin 18°=\frac{\sqrt{5}-1}{4}$

(ii) **Value of cos 18°**

$$\cos^2 18°=1-\sin^2 18°=1-\left(\frac{\sqrt{5}-1}{4}\right)^2$$

$$=1-\frac{5+1-2\sqrt{5}}{16}=1-\frac{6-2\sqrt{5}}{16}$$

$$=\frac{16-6+2\sqrt{5}}{16}=\frac{10+2\sqrt{5}}{16}$$

$$\therefore\quad \cos 18°=\frac{1}{4}\sqrt{10+2\sqrt{5}}\qquad [\because\cos 18°>0]$$

(iii) **Value of tan 18°**

$$\tan 18°=\frac{\sin 18°}{\cos 18°}=\frac{\sqrt{5}-1}{4}\div\frac{\sqrt{10+2\sqrt{5}}}{4}$$

$$=\frac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}$$

(iv) **Value of cos 72° and sin 72°**

$$\text{(a)}\ \cos 72°=\cos(90°-18°)=\sin 18°=\frac{\sqrt{5}-1}{4}$$

$$\text{(b)}\ \sin 72°=\sin(90°-18°)=\cos 18°=\frac{1}{4}\sqrt{10+2\sqrt{5}}$$

II. (i) **Value of cos 36°**

$$\cos 36°=1-2\sin^2 18°=1-2\times\left(\frac{\sqrt{5}-1}{4}\right)^2$$

$$= 1 - 2 \times \frac{5 + 1 - 2\sqrt{5}}{16}$$

$$= 1 - \frac{3 - \sqrt{5}}{4}$$

$$= \frac{4 - 3 + \sqrt{5}}{4} = \frac{\sqrt{5} + 1}{4}$$

Thus, $\cos 36^\circ = \frac{\sqrt{5} + 1}{4}$

(ii) **Value of sin 36°**

$$\sin^2 36^\circ = 1 - \cos^2 36^\circ = 1 - \left(\frac{\sqrt{5} + 1}{4}\right)^2$$

$$= 1 - \frac{6 + 2\sqrt{5}}{16} = \frac{16 - 6 - 2\sqrt{5}}{16} = \frac{10 - 2\sqrt{5}}{16}$$

$\therefore \quad \sin 36^\circ = \frac{1}{4}\sqrt{10 - 2\sqrt{5}} \qquad [\because \sin 36^\circ > 0]$

(iii) **Values of sin 54° and cos 54°**

(a) $\sin 54^\circ = \sin(90^\circ - 36^\circ) = \cos 36^\circ = \frac{\sqrt{5} + 1}{4}$

(b) $\cos 54^\circ = \cos(90^\circ - 36^\circ) = \sin 36^\circ = \frac{1}{4}(\sqrt{10 - 2\sqrt{5}})$

III. (i) **Value of tan $7\frac{1^\circ}{2}$**

Let $\theta = 7\frac{1^\circ}{2}$, then $2\theta = 15^\circ$

Now, $\tan\theta = \frac{1 - \cos 2\theta}{\sin 2\theta}$

$[\because 1 - \cos 2\theta = 2\sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta]$

$$= \frac{1 - \cos 15^\circ}{\sin 15^\circ} = \frac{1 - \frac{\sqrt{3} + 1}{2\sqrt{2}}}{\frac{\sqrt{3} - 1}{2\sqrt{2}}}$$

$$= \frac{2\sqrt{2} - \sqrt{3} - 1}{\sqrt{3} - 1} = \frac{(2\sqrt{2} - \sqrt{3} - 1)(\sqrt{3} + 1)}{(\sqrt{3} - 1)(\sqrt{3} + 1)}$$

$$= \frac{2\sqrt{6} - 3 - \sqrt{3} + 2\sqrt{2} - \sqrt{3} - 1}{3 - 1}$$

$$= \frac{2\sqrt{6} - 2\sqrt{3} - 4 + 2\sqrt{2}}{2}$$

$$= \sqrt{6} - \sqrt{3} - 2 + \sqrt{2} = \sqrt{3}(\sqrt{2} - 1) - \sqrt{2}(\sqrt{2} - 1)$$

$$= (\sqrt{3} - \sqrt{2})(\sqrt{2} - 1)]$$

(ii) **Value of cot $82\frac{1^\circ}{2}$**

$$\cot 82\frac{1^\circ}{2} = \cot\left(90^\circ - 7\frac{1^\circ}{2}\right)$$

$$= \tan 7\frac{1^\circ}{2} = (\sqrt{3} - \sqrt{2})(\sqrt{2} - 1)$$

(iii) **Value of cot $7\frac{1^\circ}{2}$**

Let $\theta = 7\frac{1^\circ}{2}$, then $2\theta = 15^\circ$

Now, $\cot\theta = \frac{1 + \cos 2\theta}{\sin 2\theta} = \frac{1 + \cos 15^\circ}{\sin 15^\circ}$

$$= \frac{1 + \frac{\sqrt{3} + 1}{2\sqrt{2}}}{\frac{\sqrt{3} - 1}{2\sqrt{2}}} = \frac{2\sqrt{2} + \sqrt{3} + 1}{\sqrt{3} - 1}$$

$$= \frac{(2\sqrt{2} + \sqrt{3} + 1)(\sqrt{3} + 1)}{(\sqrt{3} - 1)(\sqrt{3} + 1)}$$

$$= \frac{2\sqrt{6} + 3 + \sqrt{3} + 2\sqrt{2} + \sqrt{3} + 1}{3 - 1}$$

$$= \frac{2\sqrt{6} + 2\sqrt{3} + 2\sqrt{2} + 4}{2} = \sqrt{6} + \sqrt{3} + \sqrt{2} + 2$$

$$= \sqrt{3}(\sqrt{2} + 1) + \sqrt{2}(\sqrt{2} + 1)$$

$$= (\sqrt{3} + \sqrt{2})(\sqrt{2} + 1)$$

$\therefore \quad \cot 7\frac{1^\circ}{2} = (\sqrt{3} + \sqrt{2})(\sqrt{2} + 1)$

(iv) **Value of tan $82\frac{1^\circ}{2}$**

$$\tan 82\frac{1^\circ}{2} = \tan\left(90^\circ - 7\frac{1^\circ}{2}\right)$$

$$= \cot 7\frac{1^\circ}{2} = (\sqrt{3} + \sqrt{2})(\sqrt{2} + 1)$$

IV. (i) **Value of cos $22\frac{1^\circ}{2}$**

Let $\theta = 22\frac{1^\circ}{2}$, then $2\theta = 45^\circ$

Now, $\cos^2 22\frac{1^\circ}{2} = \frac{1 + \cos 45^\circ}{2} = \frac{1 + \frac{1}{\sqrt{2}}}{2}$

$$= \frac{\sqrt{2} + 1}{2\sqrt{2}} = \frac{2 + \sqrt{2}}{4}$$

$$\therefore \quad \cos 22\frac{1^\circ}{2} = \frac{1}{2}\sqrt{2+\sqrt{2}} \qquad \left[\because \cos 22\frac{1^\circ}{2} > 0\right]$$

(ii) **Value of $\sin 22\frac{1^\circ}{2}$**

$$\sin^2 22\frac{1^\circ}{2} = \frac{1-\cos 45^\circ}{2} = \frac{1-\frac{1}{\sqrt{2}}}{2} = \frac{\sqrt{2}-1}{2\sqrt{2}} = \frac{2-\sqrt{2}}{4}$$

$$\therefore \quad \sin 22\frac{1^\circ}{2} = \frac{1}{2}\sqrt{2-\sqrt{2}} \qquad \left[\because \sin 22\frac{1^\circ}{2} > 0\right]$$

(iii) **Value of $\tan 22\frac{1^\circ}{2}$**

$$\tan 22\frac{1^\circ}{2} > 0$$

$$\therefore \quad \tan 22\frac{1^\circ}{2} = \sqrt{\frac{1-\cos 45^\circ}{1+\cos 45^\circ}} \qquad \left[\because \tan\theta = \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}\right]$$

$$= \sqrt{\frac{1-\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}}} = \sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}}$$

$$= \sqrt{\frac{(\sqrt{2}-1)^2}{(\sqrt{2}+1)(\sqrt{2}-1)}} = \sqrt{\frac{(\sqrt{2}-1)^2}{2-1}}$$

$$= \sqrt{(\sqrt{2}-1)^2} = \sqrt{2}-1$$

V. **Value of $\cos 67\frac{1^\circ}{2}$ and $\sin 67\frac{1^\circ}{2}$**

$$\cos 67\frac{1^\circ}{2} + \sin 67\frac{1^\circ}{2} = \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos 67\frac{1^\circ}{2} + \frac{1}{\sqrt{2}}\sin 67\frac{1^\circ}{2}\right)$$

$$= \sqrt{2}\left(\sin 45^\circ \cos 67\frac{1^\circ}{2} + \cos 45^\circ \sin 67\frac{1^\circ}{2}\right)$$

$$= \sqrt{2}\sin\left(45^\circ + 67\frac{1^\circ}{2}\right) = \sqrt{2}\sin 112\frac{1^\circ}{2} > 0$$

$$\therefore \cos 67\frac{1^\circ}{2} + \sin 67\frac{1^\circ}{2}$$

$$= \sqrt{\left(\cos 67\frac{1^\circ}{2} + \sin 67\frac{1^\circ}{2}\right)^2} = \sqrt{1+\sin 135^\circ}$$

$$= \sqrt{1+\frac{1}{\sqrt{2}}} = \sqrt{\frac{(\sqrt{2}+1)}{\sqrt{2}}} = \sqrt{\frac{4+2\sqrt{2}}{\sqrt{2}\cdot 2\sqrt{2}}}$$

$$= \sqrt{\frac{4+2\sqrt{2}}{4}} = \frac{1}{2}\sqrt{4+2\sqrt{2}} \qquad \ldots(i)$$

Again, $\cos 67\frac{1^\circ}{2} - \sin 67\frac{1^\circ}{2} = \sqrt{2}\sin\left(45^\circ - 67\frac{1^\circ}{2}\right)$

$$= -\sqrt{2}\sin 22\frac{1^\circ}{2} < 0$$

$$\therefore \cos 67\frac{1^\circ}{2} - \sin 67\frac{1^\circ}{2}$$

$$= -\sqrt{\left(\cos 67\frac{1^\circ}{2} - \sin 67\frac{1^\circ}{2}\right)^2}$$

$$= -\sqrt{1-\sin 135^\circ}$$

$$= \sqrt{1-\frac{1}{\sqrt{2}}} = -\sqrt{\frac{\sqrt{2}-1}{\sqrt{2}}} = \sqrt{\frac{4-2\sqrt{2}}{4}}$$

$$= -\frac{1}{2}\sqrt{4-2\sqrt{2}} \qquad \ldots(i)$$

Adding Eqs. (i) and (ii), we get

$$2\cos 67\frac{1^\circ}{2} = \frac{1}{2}\left(\sqrt{4+2\sqrt{2}} - \sqrt{4-2\sqrt{2}}\right)$$

$$\therefore \quad \cos 67\frac{1^\circ}{2} = \frac{1}{4}\left(\sqrt{4+2\sqrt{2}} - \sqrt{4-2\sqrt{2}}\right)$$

Similarly, subtracting Eq. (ii) from Eq. (i), we get

$$\sin 67\frac{1^\circ}{2} = \frac{1}{4}\left(\sqrt{4+2\sqrt{2}} + \sqrt{4-2\sqrt{2}}\right)$$

VI. **Value of $\sin 157\frac{1^\circ}{2}$**

Let $\theta = 157\frac{1^\circ}{2} \quad \therefore\ 2\theta = 315^\circ$

Now, $\sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}} \qquad \left[\because \sin 157\frac{1^\circ}{2} > 0\right]$

$$= \sqrt{\frac{1-\cos 315^\circ}{2}} = \sqrt{\frac{1-\cos 45^\circ}{2}} = \sqrt{\frac{1-\frac{1}{\sqrt{2}}}{2}}$$

Thus $\sin 157\frac{1^\circ}{2} = \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}} = \sqrt{\frac{2-\sqrt{2}}{4}}$

$$= \frac{1}{2}\sqrt{2-\sqrt{2}}$$

Similarly, $\cos 157\frac{1^\circ}{2} = -\sqrt{\frac{1+\cos 2\theta}{2}}$

$$= -\frac{1}{2}\sqrt{2+\sqrt{2}} \qquad \left[\because \cos 157\frac{1^\circ}{2} < 0\right]$$

All these values are tabulated as follows:

	7.5°	15°	18°	22.5°	36°	67.5°	75°
sin	$\frac{\sqrt{8-2\sqrt{6}-2\sqrt{5}}}{4}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$
cos	$\frac{\sqrt{8+2\sqrt{6}+2\sqrt{5}}}{4}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$
tan	$(\sqrt{3}-\sqrt{2})(\sqrt{2}-1)$	$2-\sqrt{3}$	$\sqrt{1-\frac{2}{\sqrt{5}}}$	$\sqrt{2}-1$	$\sqrt{5-2\sqrt{5}}$	$\sqrt{2}+1$	$2+\sqrt{3}$
cot	$(\sqrt{3}+\sqrt{2})(\sqrt{2}+1)$	$2+\sqrt{3}$	$\sqrt{(5+2\sqrt{5})}$	$\sqrt{2}+1$	$\sqrt{\left(1+\frac{2}{\sqrt{5}}\right)}$	$\sqrt{2}-1$	$2-\sqrt{3}$

Example 87. Show that

$$\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}=\frac{1+\sin\theta}{\cos\theta}=\tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right).$$

Sol. L.H.S. $=\dfrac{1+\dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}{1-\dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}=\dfrac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}$

$$=\frac{\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)^2}{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}$$

$$=\frac{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}=\frac{1+\sin\theta}{\cos\theta}$$

Again, $\dfrac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}=\dfrac{\tan\frac{\pi}{4}+\tan\frac{\theta}{2}}{1-\tan\frac{\pi}{4}\tan\frac{\theta}{2}}=\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$

Thus $\dfrac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}=\dfrac{1+\sin\theta}{\cos\theta}=\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=\text{RHS}$

Example 88. Prove that,

$$(\cos A+\cos B)^2+(\sin A+\sin B)^2=4\cos^2\frac{A-B}{2}.$$

Sol. L.H.S. $=(\cos^2 A+\cos^2 B+2\cos A\cos B)$
$+(\sin^2 A+\sin^2 B+2\sin A\sin B)$

$=(\cos^2 A+\sin^2 A)+(\cos^2 B+\sin^2 B)$
$+2(\cos A\cos B+\sin A\sin B)$

$=2+2\cos(A-B)=2[1+\cos(A-B)]$

$=2\times 2\cos^2\dfrac{A-B}{2}=4\cos^2\dfrac{A-B}{2}$

Example 89. Prove that,

$$\cos^4\frac{\pi}{8}+\cos^4\frac{3\pi}{8}+\cos^4\frac{5\pi}{8}+\cos^4\frac{7\pi}{8}=\frac{3}{2}$$

Sol. L.H.S. $=\left(\cos^4\dfrac{\pi}{8}+\cos^4\dfrac{7\pi}{8}\right)+\left(\cos^4\dfrac{3\pi}{8}+\cos^4\dfrac{5\pi}{8}\right)$

$=\left[\cos^4\dfrac{\pi}{8}+\cos^4\left(\pi-\dfrac{\pi}{8}\right)\right]$
$+\left[\cos^4\dfrac{3\pi}{8}+\cos^4\left(\pi-\dfrac{3\pi}{8}\right)\right]$

$=2\cos^4\dfrac{\pi}{8}+2\cos^4\dfrac{3\pi}{8}$ $\quad[\because\cos(\pi-\theta)=-\cos\theta]$

$=2\left[\left(\cos^2\dfrac{\pi}{8}\right)^2+\left(\cos^2\dfrac{3\pi}{8}\right)^2\right]$

$=2\left[\left(\dfrac{1+\cos\frac{\pi}{4}}{2}\right)^2+\left(\dfrac{1+\cos\frac{3\pi}{4}}{2}\right)^2\right]$

$=\dfrac{1}{2}\left[\left(1+\dfrac{1}{\sqrt{2}}\right)^2+\left(1-\dfrac{1}{\sqrt{2}}\right)^2\right]$

$=\dfrac{1}{2}\left[1+\dfrac{1}{2}+2\cdot\dfrac{1}{\sqrt{2}}+1+\dfrac{1}{2}-2\cdot\dfrac{1}{\sqrt{2}}\right]$

$=\dfrac{3}{2}=\text{R.H.S}$

Example 90. Find the value of $\tan\frac{\pi}{8}$.

Sol. Let $\theta=\dfrac{\pi}{8}$, then $2\theta=\dfrac{\pi}{4}$

Now, $\tan 2\theta=\dfrac{2\tan\theta}{1-\tan^2\theta}$

$$\therefore \quad \tan\frac{\pi}{4} = \frac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$$

$$\Rightarrow \quad 1 = \frac{2x}{1-x^2}, \text{ where } x = \tan\frac{\pi}{8}$$

$$\Rightarrow \quad 1 - x^2 = 2x$$

$$\Rightarrow \quad x^2 + 2x - 1 = 0$$

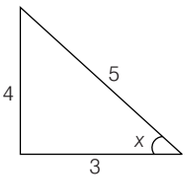
$$\Rightarrow \quad x = \frac{-2 \pm 2\sqrt{2}}{2} = -1 + \sqrt{2}, -1 - \sqrt{2}$$

$$\therefore \quad x = \tan\frac{\pi}{8} = \sqrt{2} - 1 \qquad \left[\because \tan\frac{\pi}{8} > 0\right]$$

Example 91. If $\tan x = -\frac{4}{3}, \frac{\pi}{2} < x < \pi$, then find the value of $\sin\frac{x}{2}, \cos\frac{x}{2}$ and $\tan\frac{x}{2}$.

Sol. Here $\frac{\pi}{2} < x < \pi \quad \therefore \frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}$

Hence $\sin\frac{x}{2}, \cos\frac{x}{2}, \tan\frac{x}{2}$ will be all positive.



Given, $\tan x = -\frac{4}{3}$ and $\frac{\pi}{2} < x < \pi$

$$\therefore \quad \cos x = -\frac{3}{5}$$

Now, $$\sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}} \qquad \left[\because \sin\frac{x}{2} > 0\right]$$

$$= \sqrt{\frac{1+\frac{3}{5}}{2}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

$$\cos\frac{x}{2} = \sqrt{\frac{1+\cos x}{2}} \qquad \left[\because \cos\frac{x}{2} > 0\right]$$

$$= \sqrt{\frac{1-\frac{3}{5}}{2}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5} \text{ and } \tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = 2$$

Example 92. Find the value of $\sin\frac{23\pi}{24}$.

Sol. $\sin\frac{23\pi}{24} = \sin\left(\pi - \frac{\pi}{24}\right) = \sin\frac{\pi}{24} = \sin 7\frac{1^\circ}{2}$

Let $\theta = 7\frac{1^\circ}{2}$, then $2\theta = 15^\circ$

Now, $\sin^2\theta = \frac{1-\cos 2\theta}{2} = \frac{1}{2}(1 - \cos 15^\circ)$

$$= \frac{1}{2}\left(1 - \frac{\sqrt{3}+1}{2\sqrt{2}}\right) = \frac{1}{2}\left(\frac{2\sqrt{2} - \sqrt{3} - 1}{2\sqrt{2}}\right)$$

$$= \frac{2\sqrt{2} - \sqrt{3} - 1}{4\sqrt{2}}$$

$$\therefore \quad \sin\theta = \sin 7\frac{1^\circ}{2} > 0$$

$$\therefore \quad \sin 7\frac{1^\circ}{2} = \sqrt{\frac{2\sqrt{2} - \sqrt{3} - 1}{4\sqrt{2}}} = \frac{1}{4}\sqrt{8 - 2\sqrt{6} - 2\sqrt{2}}$$

Example 93. If $\alpha = 112^\circ 30'$, find the value of $\sin\alpha$ and $\cos\alpha$.

Sol. Given, $\alpha = 112^\circ 30'$

$$\therefore \quad 2\alpha = 225^\circ$$

or $$\cos 2\alpha = \cos 225^\circ = \cos(180^\circ + 45^\circ)$$

$$= -\cos 45^\circ = -\frac{1}{\sqrt{2}}$$

Now, $$\sin^2\alpha = \frac{1-\cos 2\alpha}{2}$$

Since α lies in the 2nd quadrant $\therefore \sin\alpha$ is positive

$$\therefore \quad \sin\alpha = \sqrt{\frac{1-\cos 2\alpha}{2}} = \sqrt{\frac{1-\left(-\frac{1}{\sqrt{2}}\right)}{2}}$$

$$= \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}} = \frac{\sqrt{2+2\sqrt{2}}}{2}$$

Hence, $\sin\alpha = \frac{\sqrt{2+2\sqrt{2}}}{2}$

But $\cos\alpha$ is negative in 2nd quadrant

$$\therefore \quad \cos\alpha = -\sqrt{\frac{1+\cos 2\alpha}{2}}$$

$$= -\sqrt{\frac{1-\frac{1}{\sqrt{2}}}{2}} = -\sqrt{\frac{2-\sqrt{2}}{2}}$$

Example 94. If $\tan\frac{\theta}{2} = \sqrt{\frac{a-b}{a+b}}\tan\frac{\varphi}{2}$, prove that, $\cos\alpha = \frac{a\cos\varphi + b}{a + b\cos\varphi}$.

Sol. Given, $\tan\frac{\theta}{2} = \sqrt{\frac{a-b}{a+b}}\tan\frac{\varphi}{2}$

Now, $\cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{1-\frac{a-b}{a+b}\tan^2\frac{\varphi}{2}}{1+\frac{a-b}{a+b}\tan^2\frac{\varphi}{2}}$

$$= \frac{1 - \frac{a-b}{a+b} \cdot \frac{\sin^2 \frac{\varphi}{2}}{\cos^2 \frac{\varphi}{2}}}{1 + \frac{a-b}{a+b} \cdot \frac{\sin^2 \frac{\varphi}{2}}{\cos^2 \frac{\varphi}{2}}}$$

$$= \frac{(a+b)\cos^2 \frac{\varphi}{2} - (a-b)\sin^2 \frac{\varphi}{2}}{(a+b)\cos^2 \frac{\varphi}{2} + (a-b)\sin^2 \frac{\varphi}{2}}$$

$$= \frac{a\left(\cos^2 \frac{\varphi}{2} - \sin^2 \frac{\varphi}{2}\right) + b\left(\cos^2 \frac{\varphi}{2} + \sin^2 \frac{\varphi}{2}\right)}{a\left(\cos^2 \frac{\varphi}{2} + \sin^2 \frac{\varphi}{2}\right) + b\left(\cos^2 \frac{\varphi}{2} - \sin^2 \frac{\varphi}{2}\right)}$$

$$= \frac{a\cos\varphi + b}{a + b\cos\varphi}$$

Example 95. If $\cos\theta = \dfrac{\cos\alpha - \cos\beta}{1 - \cos\alpha\cos\beta}$, then prove that one of the values of $\tan\dfrac{\theta}{2}$ is $\tan\dfrac{\alpha}{2}\cot\dfrac{\beta}{2}$.

Sol. $\tan^2 \dfrac{\theta}{2} = \dfrac{1-\cos\theta}{1+\cos\theta} = \dfrac{1 - \frac{\cos\alpha - \cos\beta}{1 - \cos\alpha\, cos\beta}}{1 + \frac{\cos\alpha - \cos\beta}{1 - \cos\alpha\cos\beta}}$

$$= \frac{1 - \cos\alpha\cos\beta - \cos\alpha + \cos\beta}{1 - \cos\alpha\cos\beta + \cos\alpha - \cos\beta}$$

$$= \frac{(1-\cos\alpha) + \cos\beta(1-\cos\alpha)}{(1+\cos\alpha) - \cos\beta(1+\cos\alpha)}$$

$$= \frac{(1-\cos\alpha)(1+\cos\beta)}{(1+\cos\alpha)(1+\cos\beta)} = \tan^2 \frac{\alpha}{2}\cot^2 \frac{\beta}{2}$$

$\therefore \tan\dfrac{\theta}{2} = \pm \tan\dfrac{\alpha}{2}\cot\dfrac{\beta}{2}$

Hence one of the values of $\tan\dfrac{\theta}{2}$ is $\tan\dfrac{\alpha}{2}\cot\dfrac{\beta}{2}$.

Example 96. Prove that $\cos 6^\circ \cos 42^\circ \cos 66^\circ \cos 78^\circ = \dfrac{1}{16}$.

Sol. We have

$$\text{LHS} = \cos 6^\circ \cos 42^\circ \cos 66^\circ \cos 78^\circ$$

$$= \frac{1}{4}(2\cos 66^\circ \cos 6^\circ)(2\cos 78^\circ \cos 42^\circ)$$

$$= \frac{1}{4}[\cos(66^\circ + 6^\circ) + \cos(66^\circ - 6^\circ)] \times [\cos(78^\circ + 42^\circ) + \cos(78^\circ - 42^\circ)]$$

$$= \frac{1}{4}(\cos 72^\circ + \cos 60^\circ)(\cos 120^\circ + \cos 36^\circ)$$

$$= \frac{1}{4}\left(\sin 18^\circ + \frac{1}{2}\right)\left(-\frac{1}{2} + \cos 36^\circ\right)$$

$[\because \cos 72^\circ = \cos(90^\circ - 18^\circ) = \sin 18^\circ]$

$$= \frac{1}{4}\left[\frac{(\sqrt{5}-1)}{4} + \frac{1}{2}\right]\left[-\frac{1}{2} + \frac{(\sqrt{5}-1)}{4}\right]$$

$$\left[\because \sin 18^\circ = \frac{(\sqrt{5}-1)}{4} \text{ and } \cos 36^\circ = \frac{(\sqrt{5}+1)}{4}\right]$$

$$= \frac{1}{4} \cdot \frac{(\sqrt{5}+1)}{4} \cdot \frac{(\sqrt{5}-1)}{4} = \frac{(5-1)}{64}$$

$$= \frac{4}{64} = \frac{1}{16} = \text{RHS}$$

Example 97. Prove that $\sin\dfrac{\pi}{5}\sin\dfrac{2\pi}{5}\sin\dfrac{3\pi}{5}\sin\dfrac{4\pi}{5} = \dfrac{5}{16}$.

Sol. We have

$$\text{LHS} = \sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5}$$

$$= \sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\left(\pi - \frac{2\pi}{5}\right)\sin\left(\pi - \frac{\pi}{5}\right)$$

$$= \sin^2\frac{\pi}{5}\sin^2\frac{2\pi}{5} \qquad [\because \sin(\pi - \theta) = \sin\theta]$$

$$= (\sin 36^\circ)^2 \times (\sin 72^\circ)^2$$

$$= (\sin 36^\circ)^2 \times (\cos 18^\circ)^2$$

$[\because \sin 72^\circ = \sin(90^\circ - 18^\circ) = \cos 18^\circ]$

$$= \frac{(10 - 2\sqrt{5})}{16} \times \frac{(10 + 2\sqrt{5})}{16} = \frac{(100 - 20)}{(16 \times 16)}$$

$$\left[\because \sin 36^\circ = \frac{\sqrt{10 - 2\sqrt{5}}}{4} \text{ and } \cos 18^\circ = \frac{\sqrt{10 + 2\sqrt{5}}}{4}\right]$$

$$= \frac{80}{(16 \times 16)} = \frac{5}{16} = \text{RHS}$$

Example 98. Find the value of

(i) $\sin 22^\circ 30'$ (ii) $\cos 22^\circ 30'$

(iii) $\tan 22^\circ 30'$

Sol. (i) $\sin^2\theta = \dfrac{(1 - \cos 2\theta)}{2}$

$$\Rightarrow \quad \sin^2(22^\circ 30') = \frac{(1 - \cos 45^\circ)}{2} = \frac{\left(1 - \frac{1}{\sqrt{2}}\right)}{2} = \frac{\sqrt{2} - 1}{2\sqrt{2}}$$

$$\Rightarrow \quad \sin(22^\circ 30') = \sqrt{\frac{(\sqrt{2} - 1)}{2\sqrt{2}}}$$

(ii) $\cos^2\theta = \dfrac{(1+\cos 2\theta)}{2}$

$\Rightarrow \quad \cos^2(22°30') \dfrac{(1+\cos 45°)}{2} = \left(\dfrac{1+\dfrac{1}{\sqrt{2}}}{2}\right) = \dfrac{(\sqrt{2}+1)}{2\sqrt{2}}$

$\Rightarrow \quad \cos(20°30') = \sqrt{\dfrac{(\sqrt{2}+1)}{2\sqrt{2}}}$

(iii) $\tan^2(22°30') = \dfrac{\sin^2(22°30')}{\cos^2(22°30')} = \dfrac{(\sqrt{2}-1)}{(2\sqrt{2})} \times \dfrac{(2\sqrt{2})}{(\sqrt{2}+1)}$

$= \dfrac{\sqrt{2}-1}{\sqrt{2}+1} = \dfrac{(\sqrt{2}-1)}{(\sqrt{2}+1)} \times \dfrac{(\sqrt{2}-1)}{(\sqrt{2}-1)}$

$= (\sqrt{2}-1)^2$

$\Rightarrow \quad \tan(20°30') = (\sqrt{2}-1)$

Example 99. If $0 < x < \pi$ and $\cos x + \sin x = \dfrac{1}{2}$, then find the value of $\tan x$.

Sol. Given, $\cos x + \sin x = \dfrac{1}{2}$

$\Rightarrow \quad \dfrac{1-\tan^2\dfrac{x}{2}}{1+\tan^2\dfrac{x}{2}} + \dfrac{2\tan\dfrac{x}{2}}{1+\tan^2\dfrac{x}{2}} = \dfrac{1}{2}$

Let $\quad \tan\dfrac{x}{2} = t$, then $\dfrac{1-t^2}{1+t^2} + \dfrac{2t}{1+t^2} = \dfrac{1}{2}$

$\Rightarrow \quad 3t^2 - 4t - 1 = 0 \Rightarrow t = \dfrac{2 \pm \sqrt{7}}{3}$

$\Rightarrow \quad t = \tan\dfrac{x}{2} = \dfrac{2+\sqrt{7}}{3} \qquad \left[\because 0 < \dfrac{\pi}{2} < \dfrac{\pi}{2}, \tan\dfrac{x}{2}\right]$

Now, $\quad \tan x = \dfrac{2\tan\dfrac{x}{2}}{1-\tan^2\dfrac{x}{2}}$

$= \dfrac{2\left(\dfrac{2+\sqrt{7}}{2}\right)}{1-\left(\dfrac{2+\sqrt{7}}{3}\right)^2} = -\dfrac{3(2+\sqrt{7})}{1+2\sqrt{7}} \times \dfrac{1-2\sqrt{7}}{1-2\sqrt{7}}$

$\therefore \quad \tan x = -\left(\dfrac{4+\sqrt{7}}{3}\right)$

Exercise on Session 9

1. If $\tan\left(\dfrac{x}{2}\right) = \operatorname{cosec} x - \sin x$, then find the value of $\tan^2\left(\dfrac{x}{2}\right)$.
2. Find the value of $\cos^4\left(\dfrac{\pi}{8}\right)$.
3. Find the value of expression $\dfrac{1}{\cos 290°} + \dfrac{1}{\sqrt{3}\sin 250°}$.
4. If $x + \dfrac{1}{x} = 2\cos\theta$ then find the value of $x^n + \dfrac{1}{x^n}$.
5. Show that $\sin 47° + \sin 61° - \sin 11° - \sin 25° = \cos 7°$.
6. If α and β be two different roots of equation $a\cos\theta + b\sin\theta = c$, then show that $\sin(\alpha+\beta) = \dfrac{2ab}{a^2+b^2}$.
7. If $\sin\alpha + \sin\beta = a$ and $\cos\alpha + \cos\beta = b$, then show that $\tan\left(\dfrac{\alpha-\beta}{2}\right) = \pm\sqrt{\dfrac{4-a^2-b^2}{a^2+b^2}}$.
8. Show that $\cot 142\dfrac{1°}{2} = \sqrt{2} + \sqrt{3} - 2 - \sqrt{6}$.
9. If $\cos\theta = \dfrac{\cos\alpha + \cos\beta}{1+\cos\alpha\cos\beta}$, then prove that one of values of $\tan\dfrac{\theta}{2}$ is $\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}$.
10. Find the value $\tan\dfrac{\pi}{5} + 2\tan\dfrac{2\pi}{5} + 4\cot\dfrac{4\pi}{5}$.

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Session 10

Trigonometric Ratios of the Sum of Three or More Angles

Trigonometric Ratios of the Sum of Three or More Angles

$$e^{i(A+B+C)} = e^{iA} \cdot e^{iB} \cdot e^{iC}$$

$$\Rightarrow \cos(A+B+C) + i\sin(A+B+C)$$
$$= (\cos A + i\sin A)(\cos B + i\sin B)(\cos C + i\sin C)$$
$$\cos A\cos B\cos C + i\Sigma\cos A \cdot \cos B\sin C + i^2$$
$$\Sigma\cos A \cdot \sin B \cdot \sin C + i^3 \cdot \sin A \cdot \sin B \cdot \sin C$$
$$= \cos A \cdot \cos B \cdot \cos C \cdot$$
$$\{1 + i\Sigma\tan A + i^2\Sigma\tan A\tan B + i^3 \cdot \tan A\tan B\tan C\}$$

Now, equating real and imaginary parts, we get

$$\mathbf{\cos(A+B+C) = \text{`}cosA \cdot \cos B \cdot \cos C \cdot \{1 - \Sigma\tan A\tan B\}}$$

and $$\mathbf{\sin(A+B+C) = \text{`}\cos A\cos B\cos C \{\Sigma\tan A - \tan A\tan B\tan C\}}$$

Generalising Using this Method

$$\cos(A_1 + A_2 + \ldots + A_n) + i\sin(A_1 + A_2 + \ldots + A_n)$$
$$= \cos A_1 \cdot \cos A_2 \ldots \cos A_n \{1 + i \cdot \Sigma\tan A_1 + i^2$$
$$\Sigma\tan A_1\tan A_2 + i^3\Sigma\tan A_1\tan A_2\tan A_3 + \ldots i^n$$
$$\tan A_1 \cdot \tan A_2 \ldots \tan A_n\}$$

(i) On comparing real and imaginary parts, we get

$$\cos(A_1 + A_2 + \ldots + A_n)$$
$$= \cos A_1 \cdot \cos A_2 \ldots \cos A_n \cdot \{1 - \Sigma\tan A\tan A_2$$
$$+ \Sigma\tan A_1 \cdot \tan A_2\tan A_3 \cdot \tan A_4 \ldots\}$$

(ii) $(\sin(A_1 + A_2 + \ldots + A_n)$
$$= \cos A_1 \cdot \cos A_2 \ldots \cos A_n \cdot \{\Sigma\tan A_1 - \Sigma\tan A_1$$
$$\cdot \tan A_2 \cdot \tan A_3 + \ldots\}$$

(iii) $$\tan(A_1 + A_2 + \ldots + A_n) = \frac{S_1 - S_3 + S_5 \ldots}{1 - S_2 + S_4 - S_6 \ldots}$$

where, $S_1 = \Sigma\tan A_1$,
$S_2 = \Sigma\tan A_1 \cdot \tan A_2$;
$S_3 = \Sigma\tan A_1 . \tan A_2 . \tan A_3$

Example 100. If $\tan\theta_1, \tan\theta_2, \tan\theta_3, \tan\theta_4$ are the roots of the equation

$$x^4 - (\sin 2\beta)^3 + (\cos 2\beta)x^2 - (\cos\beta)x - \sin\beta = 0$$

Then, $\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4)$, is equal to

Sol. $$\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4) = \frac{S_1 - S_3}{1 - S_2 + S_4}$$
$$= \frac{\sin(2\beta) - \cos\beta}{1 - \cos 2\beta - \sin\beta} = \frac{\cos\beta(2\sin\beta - 1)}{\sin\beta(2\sin\beta - 1)} = \cot\beta$$

Multiple Angle Results in the General Form

Using the De-Moiver's theorem,

$$(\cos n\alpha + i\sin n\alpha) = (\cos\alpha + i\sin\alpha)^n$$
$$= {}^nC_0 \cdot \cos^n\alpha + {}^nC_1 . \cos^{n-1}\alpha(i\sin\alpha) +$$
$${}^nC_2\cos^{n-2}\alpha(i\sin\alpha)^2 + \ldots + {}^nC_n(i\sin\alpha)^n$$

$$\Rightarrow \cos(n\alpha) + i\sin(n\alpha) = ({}^nC_0\cos^n\alpha -$$
$${}^nC_2\cos^{n-2}\alpha\sin^2\alpha + {}^nC_4 . \cos^{n-4}\alpha \cdot \sin^4\alpha \ldots)$$
$$+ i({}^nC_1 \cdot \cos^{n-1}\alpha \cdot \sin\alpha - {}^nC_3\cos^{n-3}\alpha . \sin^3\alpha$$
$$+ {}^nC_5\cos^{n-5} \cdot \sin^5\alpha \ldots)$$

On comparing real and imaginary part, we get

$$\cos(n\alpha) = \cos^n\alpha(1 - {}^nC_2 \cdot \tan^2\alpha + {}^nC_4\tan^4\alpha \ldots) \ldots(i)$$

and $$\sin(n\alpha) = \cos^n\alpha({}^nC_1 \cdot \tan\alpha -$$
$${}^nC_3\tan^3\alpha + {}^nC_5\tan^5\alpha \ldots) \quad \ldots(ii)$$

On dividing Eq. (ii) by Eq. (i), we get,

$$\tan(n\alpha) = \frac{{}^nC_1 - {}^nC_3\tan^3\alpha + {}^nC_5\tan^5\alpha \ldots}{1 - {}^nC_2\tan^2\alpha + {}^nC_4\tan^4\alpha - {}^nC_6\tan^6\alpha \ldots}$$

On adding and subtracting Eq. (i) and Eq. (ii), we get

$$\sin(n\alpha) + \cos(n\alpha) = \cos^n\alpha\{1 + {}^nC_1\tan\alpha +$$
$${}^nC_2\tan^2\alpha - {}^nC_3\tan^3\alpha + {}^nC_4\tan^4\alpha + {}^nC_5\tan^5\alpha \ldots\}$$

and $$\sin(n\alpha) - \cos(n\alpha) = \cos^n\alpha\{-1 + {}^nC_1\tan\alpha$$
$$+ {}^nC_2\tan^2\alpha - {}^nC_3\tan^3\alpha - {}^nC_4\tan^4\alpha + \ldots\}$$

Example 101. Express $\sin^5\theta$ in term of $\sin(n\theta)$; $n \in N$.

Sol. We know,

$$\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right), \text{ using}$$

$$e^{i\theta} = \cos\theta + i\sin\theta = z$$

$$\therefore \quad \sin^5\theta = \left(\frac{1}{2i}\left(z - \frac{1}{z}\right)\right)^5$$

$$= \frac{1}{32i}\left\{{}^5C_0 \cdot z^5 - {}^5C_1 \cdot z^3 + {}^5C_2 \cdot z - {}^5C_3\left(\frac{1}{z}\right) + {}^5C_4\left(\frac{1}{z}\right)^3 - {}^5C_5\left(\frac{1}{z}\right)^5\right\}$$

$$= \frac{1}{32i}\left\{\left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)\right\}$$

$$= \frac{1}{32i}\{2i\sin 5\theta - 5(2i\sin 3\theta) + 10(2i\sin\theta)\}$$

$$= \frac{1}{16}\{\sin 5\theta - 5\sin 3\theta + 10\sin\theta\}$$

Summation of Series Containing sine and Cosine of Angles Forming an AP

(i) **Sine of angle forming an AP**

Let the series be,

$$S = \sin(\alpha) + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin(\alpha+(n-1)\beta)$$

On multiplying and dividing by $2\sin\frac{\beta}{2}$

$$= \frac{1}{2\sin\left(\frac{\beta}{2}\right)}\left\{2\sin\alpha \cdot \sin\left(\frac{\beta}{2}\right) + 2\sin(\alpha+\beta).\sin\left(\frac{\beta}{2}\right) + \ldots + 2\sin(\alpha+(n-1)\beta).\sin\left(\frac{\beta}{2}\right)\right\}$$

$$= \frac{1}{2\sin\left(\frac{\beta}{2}\right)} \cdot \left\{\left(\cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left(\alpha + \frac{\beta}{2}\right)\right) + \left(\cos\left(\alpha + \frac{\beta}{2}\right) - \cos\left(\alpha + \frac{3\beta}{2}\right)\right) + \ldots + \left(\cos\left(\alpha + \left(n - \frac{3}{2}\right)\beta\right) - \cos\left[\left(\alpha + \left(n - \frac{1}{2}\right)\beta\right]\right.\right.$$

$$= \frac{1}{2\sin\left(\frac{\beta}{2}\right)}\left\{\cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left(\alpha + \left(n - \frac{1}{2}\right)\beta\right)\right\}$$

$$= \frac{2\sin\left(\alpha + \left(\frac{n-1}{2}\right)\beta\right) \cdot \sin\left(n\frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}}$$

$$\therefore \ \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin[(\alpha+(n-1)\beta)]$$

$$= \frac{\sin\left(n\frac{\beta}{2}\right) \cdot \sin\left[\alpha + (n-1)\frac{\beta}{2}\right]}{\sin\left(\frac{\beta}{2}\right)}$$

(ii) **Cosine of angle forming an AP.**

Let,

$$S = \cos(\alpha) + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos[\alpha+(n-1)\beta]$$

On multiplying and dividing by $2\sin\left(\frac{\beta}{2}\right)$, we get

$$S = \frac{1}{2\sin\left(\frac{\beta}{2}\right)}\left\{2\sin\left(\frac{\beta}{2}\right) \cdot \cos(\alpha) + 2\sin\left(\frac{\beta}{2}\right) \cdot \cos(\alpha+\beta) + 2\sin\left(\frac{\beta}{2}\right) \cdot \cos(\alpha+2\beta) + \ldots + 2\sin\left(\frac{\beta}{2}\right)\cos(\alpha+(n+1)\beta\right\}$$

$$= \frac{1}{2\sin\left(\frac{\beta}{2}\right)}\left[\sin\left(\alpha + \frac{\beta}{2}\right) - \sin\left(\alpha - \frac{\beta}{2}\right)\right] + \left[\sin\left(\alpha + \frac{3\beta}{2}\right) - \sin\left(\alpha + \frac{\beta}{2}\right)\right] + \ldots + \left[\sin[\alpha + \left(n - \frac{3}{2}\right)\beta] - \sin\left[\alpha + \left(n - \frac{5}{2}\right)\beta\right]\right]$$

$$= \frac{1}{2\sin\left(\frac{\beta}{2}\right)} \cdot \left[\sin\left(\alpha + \left(n - \frac{1}{2}\right)\beta\right) - \sin\left(\alpha - \frac{\beta}{2}\right)\right]$$

$$= \frac{\cos\left(\alpha + (n-1)\left(\frac{\beta}{2}\right)\right) \cdot \sin\left(n\frac{\beta}{2}\right)}{1 - \left(\frac{\beta}{2}\right)}$$

$$\cos(\alpha) + \cos(\alpha+\beta) + \ldots + \cos(\alpha+(n-1)\beta)$$

$$= \frac{\sin\left(n\frac{\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)} \cdot \left\{\cos\left(\alpha + (n-1)\left(\frac{\beta}{2}\right)\right)\right\}$$

II Method

Let $S = \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin(\alpha+(n-1)\beta)$ ...(i)

$C = \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos(\alpha(n-1)\beta)$ ...(ii)

$C + iS = (\cos\alpha + i\sin\alpha) + (\cos(\alpha+\beta) + i\sin(\alpha+\beta)) + (\cos(\alpha+2\beta) + i\sin(\alpha+2\beta)) + \ldots + (\cos(\alpha+(n-1)\beta) + i\sin(\alpha+(n-1)\beta))$ $i(\alpha+(n-1)\beta)$

$= e^{i\alpha} + e^{i(\alpha+\beta)} + e^{i(\alpha+2\beta)} + \ldots + e^{i(\alpha+(n-1)\beta)}$

$= e^{i\alpha} \cdot \left(\frac{(e^{i\beta})^n - 1}{e^{i\beta} - 1}\right)$

$= e^{i\alpha} \left\{\frac{\cos(n\beta) - 1 + i\sin(n\beta)}{\cos\beta - 1 + i\sin\beta}\right\}$

$= e^{i\alpha} \left\{\frac{2i^2 \sin^2\left(n\frac{\beta}{2}\right) + 2i \cdot \sin\left(n\frac{\beta}{2}\right) \cdot \cos\left(n\frac{\beta}{2}\right)}{2i^2 \cdot \sin^2\left(\frac{\beta}{2}\right) + 2i \cdot \sin\left(\frac{\beta}{2}\right) \cdot \cos\left(\frac{\beta}{2}\right)}\right\}$

$= \frac{e^{i\alpha} \cdot 2i \sin\left(n\frac{\beta}{2}\right)}{2i \sin\left(\frac{\beta}{2}\right)} \left(\frac{\cos\left(n\frac{\beta}{2}\right) + i \cdot \sin\left(n\frac{\beta}{2}\right)}{\cos\left(\frac{\beta}{2}\right) + i \cdot \sin\left(\frac{\beta}{2}\right)}\right)$

$= \frac{e^{i\alpha} \sin\left(n\frac{\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)} \cdot e^{i\left(\frac{n-1}{2}\right)\beta} = \frac{\sin\left(n\frac{\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)} \cdot e^{i\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right]}$

$\therefore$ On comparing real and imaginary part, we get

$$C = \frac{\sin\left(n\frac{\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)} \cdot \left\{\cos\left(\alpha + (n-1)\left(\frac{\beta}{2}\right)\right)\right\}$$

and $$C = \frac{\sin\left(n\frac{\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)} \cdot \left\{\sin\left(\alpha + (n-1)\left(\frac{\beta}{2}\right)\right)\right\}$$

1. $\sin(A+B+C) = \sin A\cos B\cos C + \cos A\sin B\cos C + \cos A\cos B\sin C - \sin A\sin B\sin C$

   or $\sin(A+B+C) = \cos A\cos B\cos C(\tan A + \tan B + \tan C - \tan A\tan B\tan C)$

2. $\cos(A+B+C) = \cos A\cos B\cos C - \sin A\sin B\cos C - \sin A\cos B\sin C - \cos A\sin B\sin C$

   $\cos(A+B+C) = \cos A\cos B\cos C(1 - \tan A\tan B - \tan B\tan C - \tan C\tan A)$

3. $\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A}$

   In general;

4. $\sin(A_1 + A_2 + \ldots + A_n) = \cos A_1 \cos A_2 \ldots \cos A_n (S_1 - S_3 + S_5 - S_7 + \ldots)$

5. $\cos(A_1 + A_2 + \ldots + A_n) = \cos A_1 \cos A_2 \ldots \cos A_n (1 - S_2 + S_4 - S_6 + \ldots)$

6. $\tan(A_1 + A_2 + \ldots + A_n) = \frac{S_1 - S_3 + S_5 - S_7 + \ldots}{1 - S_2 + S_4 - S_6 + \ldots}$

   where;

   $S_1 = \tan A_1 + \tan A_2 + \ldots\ldots + \tan A_n$
   = tan sum of the tangents of the separate angles.

   $S_2 = \tan A_1 \tan A_2 + \tan A_1 \tan A_3 + \ldots$
   = the sum of the tangents taken two at a time.

   $S_3 = \tan A_1 \tan A_2 \tan A_3 + \tan A_2 \tan A_3 \tan A_4 + \ldots$
   = sum of the tangents three at a time and so on.

   If $A_1 = A_2 = \ldots = A_n = A$, then

   $S_1 = n\tan A$

   $S_2 = {}^nC_2 \tan^2 A$

   $S_3 = {}^nC_3 \tan^3 A, \ldots$

7. $\sin nA = \cos^n A({}^nC_1 \tan A - {}^nC_3 \tan^3 A + {}^nC_5 \tan^5 A \ldots)$

8. $\cos nA = \cos^n A(1 - {}^nC_2 \tan^2 A + {}^nC_4 \tan^4 A - \ldots)$

9. $\tan nA = \frac{{}^nC_1 \tan A - {}^nC_3 \tan^3 A + {}^nC_5 \tan^5 A \ldots}{1 - {}^nC_2 \tan^2 A + {}^nC_4 \tan^4 A - {}^nC_6 \tan^6 A + \ldots}$

10. $\sin nA + \cos nA = \cos^n A(1 + {}^nC_1 \tan A - {}^nC_2 \tan^2 A - {}^nC_3 \tan^3 A + {}^nC_4 \tan^4 A + {}^nC_5 \tan^5 A - {}^nC_6 \tan^6 A \ldots)$

11. $\sin nA - \cos nA = \cos^n A(-1 + {}^nC_1 \tan A + {}^nC_2 \tan^2 A - {}^nC_3 \tan^3 A - {}^nC_4 \tan^4 A + {}^nC_5 \tan^5 A + {}^nC_6 \tan^6 A \ldots)$

12. $\sin(\alpha) + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin(\alpha+(n-1)\beta) = \frac{\sin\{\alpha+(n-1)(\beta/2)\} \cdot \sin(n\beta/2)}{\sin(\beta/2)}$

13. $\cos(\alpha) + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos(\alpha+(n-1)\beta) = \frac{\cos(\alpha+(n-1)(\beta/2)) \cdot \sin(n(\beta/2))}{\sin(\beta/2)}$

Example 102. *Let n be an odd integer. In* $\sin n\theta = \sum_{r=0}^{n} b_r \sin^r \theta$, *for all real* θ. *Then, find* b_0 *and* b_1.

Sol. Here, $\sin n\theta = b_0 + b_1 \sin\theta + b_2 \sin^2\theta + \ldots$ ...(i)

Putting $\theta = 0$, we get

$$0 = b_0 \quad \ldots(ii)$$

Again, on differentiating Eq. (i) both sides w.r.t. θ, we get

$$n\cos n\theta = 0 + b_1 \cos\theta + b_2 2\sin\theta \cdot \cos\theta + \ldots$$

Again, putting $\theta = 0$, we get

$$n = b_1 \quad \ldots(iii)$$

$\therefore$ From Eqs. (ii) and (iii),

$$b_0 = 0 \text{ and } b_1 = n$$

Example 103. If $\cos 5\theta = a\cos^5\theta + b\cos^3\theta + c\cos\theta$. Then, find the value of c.

Sol. Here, $\cos 5\theta = a\cos^5\theta + b\cos^3\theta + c\cos\theta$ ...(i)

On differentiating Eq. (i) w.r.t. θ, we get

$$-5\sin 5\theta = a(5\cos^4\theta)(-\sin\theta) + b(3\cos^2\theta)(-\sin\theta) - c\sin\theta$$

Putting $\theta = \frac{\pi}{2}$ $\quad -5\sin\frac{5\pi}{2} = -c\sin\frac{\pi}{2} \quad \left[\text{as } \cos\frac{\pi}{2} = 0\right]$

$\therefore \quad c = 5$

Example 104. If $\sin^3 x \sin 3x = \sum_{m=0}^{n} c_m . \cos mx$ is an identity in x, where c_m's are constants, then find the value of n.

Sol. Here, $\sin^3 x \sin 3x = \frac{3\sin x - \sin 3x}{4} \cdot \sin 3x$

$$= \frac{3}{8}(2\sin x \sin 3x) - \frac{1}{4}(\sin^2 3x)$$

$$= \frac{3}{8}(\cos 2x - \cos 4x) - \frac{1}{8}(1 - \cos 6x)$$

$$= -\frac{1}{8} + \frac{3}{8}\cos 2x - \frac{3}{8}\cos 4x + \frac{1}{8}\cos 6x \quad \ldots(i)$$

Also, $\sum_{m=0}^{n} c_m . \cos mx = c_0 + c_1 \cos x + c_2 \cos 2x + \ldots + c_n \cos nx$ ...(ii)

On comparing Eqs. (i) and (ii), we get

$$n = 6.$$

Example 105. Evaluate $\sum_{r=1}^{n-1} \cos^2\left(\frac{r\pi}{n}\right)$.

Sol. Sum $= \frac{1}{2}\sum_{r=1}^{n-1}\left(1 + \cos\frac{2r\pi}{n}\right)$

$$= \frac{1}{2}(n-1) + \frac{1}{2}\left\{\cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \ldots + \cos\frac{(2n-2)\pi}{n}\right\}$$

$$= \frac{1}{2}(n-1) + \frac{1}{2}\left\{\frac{\sin(n-1)\frac{2\pi}{2n}}{\sin\frac{2\pi}{n.2}} \cdot \cos\left\{\frac{2\left(\frac{2\pi}{n}\right) + (n-2)\frac{2\pi}{n}}{2}\right\}\right\}$$

Using, $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos[\alpha + (n-1)\beta]$

$$= \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \cdot \cos\left\{\frac{2\alpha + (n-1)\beta}{2}\right\}$$

$$= \frac{1}{2}(n-1) + \frac{1}{2}\left\{\frac{\sin\frac{(n-1)\pi}{n} \cdot \cos\pi}{\sin(\pi/n)}\right\}$$

$$= \frac{1}{2}(n-1) + \frac{1}{2}\left\{\frac{\left(\sin\frac{\pi}{n}\right)(-1)}{\sin(\pi/n)}\right\}$$

$$= \frac{1}{2}(n-1) - \frac{1}{2} = \frac{n}{2} - 1$$

$\therefore \quad \sum_{r=1}^{n-1} \cos^2\left(\frac{r\pi}{n}\right) = \frac{n}{2} - 1$

Example 106. Evaluate $\sin\frac{\pi}{n} + \sin\frac{3\pi}{n} + \sin\frac{5\pi}{n} + \ldots$ to n terms.

Sol. $\sin\frac{\pi}{n} + \sin\frac{3\pi}{n} + \sin\frac{5\pi}{n} + \ldots$ to n terms

$$= \frac{\sin\frac{n2.\pi}{2.n}}{\sin\frac{2.\pi}{2.n}} \cdot \sin\left\{\frac{2\cdot\frac{\pi}{n} + (n-1)\cdot\frac{2\pi}{n}}{2}\right\}$$

Using, $\sin\alpha + \sin(\alpha+\beta) + \ldots + \sin(\alpha + (n-1)\beta)$

$$= \frac{\sin n\beta/2}{\sin\beta/2} \cdot \sin\left\{\frac{2\alpha + (n-1)\beta}{2}\right\}$$

$$= \frac{\sin\pi}{\sin\pi/n}\sin\left\{\frac{2\pi + 2n\pi - 2\pi}{2n}\right\}$$

$$= \frac{\sin^2\pi}{\sin\pi/n} = 0$$

$\therefore \quad \sin\frac{\pi}{n} + \sin\frac{3\pi}{n} + \sin\frac{5\pi}{n} + \ldots$ to n terms $= 0$

Exercise for Session 10

1. If $A + B + C = 180°$, then prove that $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$.
2. If $A + B + C = 180°$, then prove that $\tan^2 \frac{\theta}{2} = \tan \frac{B}{2} \tan \frac{C}{2}$. when $\cos \theta (\sin B + \sin C) = \sin A$.
3. If A, B, C are angles of a ΔABC, then prove that $\sin 2A + \sin 2B - \sin 2C = 4 \cos A \cos B \sin C$.
4. If in a ΔABC, $\frac{\sin A + \sin B + \sin C}{\sin A + \sin B - \sin C} = 2\lambda \cot \frac{A}{2} \cot \frac{B}{2}$, then find the value of λ.
5. If $A + B + C = 180°$, then find the value of $\frac{\cos A}{\sin B \sin C} + \frac{\cos B}{\sin C \sin A} + \frac{\cos C}{\sin A \sin B}$.
6. In ΔABC, show that $\frac{1 - \cos A + \cos B + \cos C}{1 - \cos C + \cos A + \cos B} = \tan \frac{A}{2} \cot \frac{C}{2}$.
7. In a ΔABC, if $\tan\left(\frac{B + C - A}{4}\right) \tan\left(\frac{C + A - B}{4}\right) \tan\left(\frac{A + B - C}{4}\right) = 1$, then find the value of $\cos A + \cos B + \cos C$.
8. If in a ΔABC, $\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \lambda \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$, then find the value of λ.
9. If $A + B + C = \frac{3\pi}{2}$, then show that $\cos 2A + \cos 2B + \cos 2C = 1 - 4 \sin A \sin B \sin C$.
10. If $\alpha + \beta + \gamma = 2\pi$, then show that $\tan \frac{\alpha}{2} + \tan \frac{\beta}{2} + \tan \frac{\gamma}{2} = \tan \frac{\alpha}{2} \tan \frac{\beta}{2} \tan \frac{\gamma}{2}$.

Session 11

Maximum and Minimum Values of Trigonometrical Functions

Conditional Trigonometrical Identities

We have certain trigonometric identities like, $\sin^2 \theta + \cos^2 \theta = 1$ and $1 + \tan^2 \theta = \sec^2 \theta$ etc. Such identities are identities in the sense that they hold for all value of the angles which satisfy the given condition among them and they are called conditional identities.

If A, B, C denote the angle of a ΔABC, then the relation $A + B + C = \pi$ enables us to establish many important identities involving trigonometric ratios of these angles.

(i) If $A + B + C = \pi$, then $A + B = \pi - C$,
$B + C = \pi - A$ and $C + A = \pi - \beta$

(ii) If $A + B + C = \pi$, then $\sin(A + B)$
$= \sin(\pi - C) = \sin C$
Similarly, $\sin(B + C) = \sin(\pi - A) = \sin A$
and $\sin(C + A) = \sin(\pi - B) = \sin B$

(iii) If $A + B + C = \pi$, then $\cos(A + B)$
$= \cos(\pi - C) = -\cos C$
Similarly, $\cos(B + C) = \cos(\pi - A) = -\cos A$
and $\cos(C + A) = \tan(\pi - B) = -\tan B$

(iv) If $A + B + C = \pi$, then $\tan(A + B)$
$= \tan(\pi - C) = -\tan C$
Similarly, $\tan(B + C) = \tan(\pi - A) = -\tan A$
and, $\tan(C + A) = \tan(\pi - B) = -\tan B$

(v) If $A + B + C = \pi$, then $\frac{A + B}{2} = \frac{\pi}{2} - \frac{C}{2}$
and $\frac{B + C}{2} = \frac{\pi}{2} - \frac{A}{2}$ and $\frac{C + A}{2} = \frac{\pi}{2} - \frac{B}{2}$
$\sin\left(\frac{A + B}{2}\right) = \sin\left(\frac{\pi}{2} - \frac{C}{2}\right) = \cos\left(\frac{C}{2}\right)$
$\cos\left(\frac{A + B}{2}\right) = \cos\left(\frac{\pi}{2} - \frac{C}{2}\right) = \sin\left(\frac{C}{2}\right)$

$$\tan\left(\frac{A+B}{2}\right)=\tan\left(\frac{\pi}{2}-\frac{C}{2}\right)=\cot\left(\frac{C}{2}\right)$$

All problems on conditional identities are broadly divided into the following four types :

(i) Identities involving sines and cosines of the multiple or sub-multiples of the angles involved.

(ii) Identities involving squares of sines and cosines of the multiple or sub-multiples of the angles involved.

(iii) Identities involving tangents and cotangents of the multiples or sub-multiples of the angles involved.

(iv) Identities involving cubes and higher powers of sines and cosines and some mixed identities.

TYPE I ***Identities involving sines and cosines of the multiple or submultiple of the angles involved***

Working Method

***Step* 1** Use C and D formulae.

***Step* 2** Use the given relation $(A+B+C=\pi)$ in the expression obtained in step 1 such that a factor can be taken common after using multiple angles formulae in the remaining term.

***Step* 3** Take the common factor outside.

***Step* 4** Again use the given relation $(A+B+C=\pi)$ within the bracket in such a manner so that we can apply C and D formulae.

Step 5 Find the result according to the given options.

TYPE II ***Identities involving squares of sines and cosines of multiple or sub-multiples of the angles involved.***

***Step* 1** Arrange the terms of the identify such that either $\sin^2 A-\sin^2 B=\sin(A+B).\sin(A-B)$

or $\cos^2 A-\sin^2 B=\cos(A+B)\cos(A-B)$ can be used.

***Step* 2** Take the common factor outside.

***Step* 3** Use the given relations $(A+B+C=\pi)$ within the bracket in such a manner so that we can apply C and D formulae.

***Step* 4** Find the result according to the given options.

Type III ***Identities for tan and cot of the angles***

Working Method

***Step* 1** Express the sum of the two angles in terms of third angle by using the given relation $(A+B+C=\pi)$

***Step* 2** Taking tangent or cotangent of the angles of both the sides.

***Step* 3** Use sum and difference formulae in the left hand side.

***Step* 4** Use cross-multiplication in the expression obtained in the step 3.

***Step* 5** Arrange the terms as per the result required.

Example 107. If $A+B+C=\pi$, then, find

$$\sin 2A+\sin 2B+\sin 2C.$$

Sol. $\sin 2A+\sin 2B+\sin 2C$

$$=2\sin\left(\frac{2A+2B}{2}\right)\cos\left(\frac{2A-2B}{2}\right)+\sin 2C$$

$$=2\sin(A+B)\cos(A-B)+\sin 2C$$

$$=2\sin(\pi-C)\cdot\cos(A-B)+\sin 2C$$

$[\because A+B+C=\pi,\ A+B=\pi-C$

$\therefore \sin(A+B)=\sin(\pi-C)=\sin C]$

$$=2\sin C\cos(A-B)+2\sin C\cos C$$

$$=2\sin C\,[\cos(A-B)+\cos C]$$

$$=2\sin C\,[\cos(A-B)-\cos(A+B)]$$

$[\because \cos(A-B)-\cos(A+B)=2\sin A\sin B,$ by C and D formula$]$

$$=2\sin C\,[2\sin A\sin B]$$

$$=4\sin A\sin B\sin C$$

Example 108. If $A+B+C=\pi$, then, find

$$\tan A+\tan B+B+\tan C$$

Sol. $A+B+C=\pi$

$$A+B=\pi-C\ \Rightarrow\ \tan(A+B)=\tan(\pi-C)$$

$$\Rightarrow\quad \frac{\tan A+\tan B}{1-\tan A\tan B}=-\tan C$$

$$\Rightarrow\quad \tan A+\tan B=-\tan C+\tan A\cdot\tan B\cdot\tan C$$

$$\tan A+\tan B+\tan C=\tan A\cdot\tan B\cdot\tan C$$

Maximum and Minimum Values of Trigonometrical Functions

As we have discussed in previous article that $-1\le\sin x\le 1$ and $-1\le\cos x\le 1$.

If there is a trigonometrical function of the form $a\sin x+b\cos x$, then by putting $a=r\cos\theta$, $b=r\sin\theta$, we have

$$a\sin x+b\cos x=r\cos\theta\sin x+r\sin\theta\cdot\cos x$$

$$=r(\cos\theta\sin x+\sin\theta\cos x)$$

$$=r\sin(x+\theta),\ \text{where } r=\sqrt{a^2+b^2},\ \tan\theta=\frac{b}{a}$$

Since, $-1 \le \sin(x+\theta) \le 1$

$\therefore \quad -r \le r\sin(x+\theta) \le r$

or $\quad -\sqrt{a^2+b^2} \le a\sin x + b\cos x \le \sqrt{a^2+b^2}$, for all x.

Hence, the maximum and minimum values of trigonometrical functions of the form $a\sin x + b\cos x$ are

$\sqrt{a^2+b^2}$ and $-\sqrt{a^2+b^2}$, respectively.

Note $|a\cos A + b\sin A| \le \sqrt{a^2+b^2}$.

Also, $\cos A \pm \sin A = \sqrt{2}\sin\left(\frac{\pi}{4} \pm A\right) = \sqrt{2}\cos\left(A \mp \frac{\pi}{4}\right)$.

Example 109. *Find the maximum and minimum value of* $3\sin 2x + 4\cos 2x + 3$.

Sol. As we know,

$$-\sqrt{a^2+b^2} \le a\sin A + b\cos A \le \sqrt{a^2+b^2}$$

$$\Rightarrow \quad -\sqrt{3^2+4^2} \le 3\sin 2x + 4\cos 2x \le \sqrt{3^2+4^2}$$

$$\Rightarrow \quad -5 \le 3\sin 2x + 4\cos 2x \le 5$$

$$\therefore \quad -5+3 \le 3\sin 2x + 4\cos 2x + 3 \le 5+3$$

$$\Rightarrow \quad (3\sin 2x + 4\cos 2x + 3) \in [-2, 8].$$

Example 110. *Find the maximum and minimum value of* $6\sin x\cos x + 4\cos 2x$.

Sol. We have, $6\sin x\cos x + 4\cos 2x$

$$\Rightarrow \quad 3\sin 2x + 4\cos 2x$$

$$\therefore \quad -\sqrt{3^2+4^2} \le 3\sin 2x + 4\cos 2x \le \sqrt{3^2+4^2}$$

$$\Rightarrow \quad 3\sin 2x + 4\cos 2x \in [-5, 5]$$

Hence, the maximum value is 5 and minimum value is -5.

Example 111. Prove that

$-4 \le 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3 \le 10$, for all values of θ.

Sol. We have, $5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right)$

$$\Rightarrow \quad 5\cos\theta + 3\cos\theta\cos\frac{\pi}{3} - 3\sin\theta\sin\frac{\pi}{3}$$

$$\Rightarrow \quad \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta \;\Rightarrow\; -7 \le \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta \le 7$$

$$\Rightarrow \quad -7 \le 5\cos\theta + 3\cos(\theta + \pi/3) \le 7, \text{ for all } \theta.$$

$$\Rightarrow \quad -7+3 \le 5\cos\theta + 3\cos(\theta + \pi/3) + 3 \le 7+3, \text{ for all } \theta.$$

$$\Rightarrow \quad -4 \le 5\cos\theta + 3\cos(\theta + \pi/3) + 3 \le 10, \text{ for all } \theta.$$

Hence proved.

Example 112. Find the maximum value of

$1 + \sin\left(\frac{\pi}{4} + \theta\right) + 2\cos\left(\frac{\pi}{4} - \theta\right)$ for all real value of θ.

Sol. We have, $1 + \sin\left(\frac{\pi}{4} + \theta\right) + 2\cos\left(\frac{\pi}{4} - \theta\right)$

$$= 1 + \frac{1}{\sqrt{2}}(\cos\theta + \sin\theta) + \sqrt{2}(\cos\theta + \sin\theta)$$

$$= 1 + \left(\frac{1}{\sqrt{2}} + \sqrt{2}\right)(\cos\theta + \sin\theta)$$

$$= 1 + \left(\frac{1}{\sqrt{2}} + \sqrt{2}\right)\cdot\sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right)$$

$\therefore$ The maximum value of $1 + \left(\frac{1}{\sqrt{2}} + \sqrt{2}\right)\cdot\sqrt{2} = 4$.

Example 113. *Find the maximum and minimum value of* $\cos^2\theta - 6\sin\theta\cos\theta + 3\sin^2\theta + 2$.

Sol. We have, $\cos^2\theta - 6\sin\theta\cos\theta + 3\sin^2\theta + 2$

$$= (1 - \sin^2\theta) - 3\sin 2\theta + 3\sin^2\theta + 2$$

$$= 2\sin^2\theta - 3\sin 2\theta + 3$$

$$= (1 - \cos 2\theta) - 3\sin 2\theta + 3$$

$$= 4 - (\cos 2\theta + 3\sin 2\theta) \quad \ldots(i)$$

As we have, $-\sqrt{10} \le \cos 2\theta + 3\sin 2\theta \le \sqrt{10}$

$$\therefore \quad -\sqrt{10} \le -(\cos 2\theta + 3\sin 2\theta) \le \sqrt{10}$$

or $\quad 4 - \sqrt{10} \le 4 - (\cos 2\theta + 3\sin 2\theta) \le 4 + \sqrt{10} \quad \ldots(ii)$

$\therefore$ From Eqs. (i) and (ii),

$$4 - \sqrt{10} \le \cos^2\theta - 6\sin\theta\cos\theta + 3\sin^2\theta + 2 \le 4 + \sqrt{10}$$

Example 114. *The minimum value of* $\cos 2\theta + \cos\theta$ *for all real values of* θ.

Sol. $\cos 2\theta + \cos\theta = 2\cos^2\theta - 1 + \cos\theta$

$$= -1 + 2\left(\cos^2\theta + \frac{1}{2}\cos\theta\right)$$

$$= -1 + 2\left(\cos^2\theta + \frac{1}{2}\cos\theta + \frac{1}{16} - \frac{1}{16}\right)$$

$$= -1 + 2\left(\cos\theta + \frac{1}{4}\right)^2 - \frac{1}{8}$$

$$= -\frac{9}{8} + 2\left(\cos\theta + \frac{1}{4}\right)^2 \ge -\frac{9}{8}$$

So, the minimum value of $\cos 2\theta + \cos\theta$ is $-\frac{9}{8}$.

Example 115. If $f(x) = \frac{\sin 3x}{\sin x}$, $x \ne n\pi$, then find range of $f(x)$.

Sol. $f(x) = \frac{\sin 3x}{\sin x} = \frac{3\sin x - 4\sin^3 x}{\sin x}$

$$\Rightarrow \quad f(x) = 3 - 4\sin^2 x. \quad \ldots(i)$$

We know, $0 < \sin^2 x \le 1 (\sin x \ne 0 \text{ as } x \ne n\pi)$

or $\quad -1 \le -\sin^2 x < 0$

or $\quad -4 \le -4\sin^2 x < 0 \quad$ or $\quad 3 - 4 \le 3 - 4\sin^2 x < 3$

or $\quad -1 \le 3 - 4\sin^2 x < 3 \quad \Rightarrow \quad -1 \le f(x) < 3$

Hence, range of $f(x) \in [-1, 3)$.

Application on Quadratic Equations

As we know, $ax^2 + bx + c = 0$, represents the quadratic equation whose,

$$\text{sum of roots} = \frac{-b}{a}.$$

$$\text{product of roots} = \frac{c}{a}.$$

$$(\alpha, \beta) \text{ roots} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

and if we want to form quadratic equation whose roots are given.

$\Rightarrow \quad x^2 - (\text{sum of the roots})x + (\text{product of the roots}) = 0.$

As above mentioned results are basics for quadratic equations, we discuss certain application on trigonometry.

Example 116. Find $\cos(\alpha + \beta)$, if $\tan\frac{\alpha}{2}$ and $\tan\frac{\beta}{2}$ are roots of the equations $8x^2 - 26x + 15 = 0$.

Sol. It is given $\tan\frac{\alpha}{2}$ and $\tan\frac{\beta}{2}$ are roots of $8x^2 - 26x + 15 = 0$.

$$\Rightarrow \quad \tan\frac{\alpha}{2} + \tan\frac{\beta}{2} = \frac{13}{4} \text{ and } \tan\frac{\alpha}{2}.\tan\frac{\beta}{2} = \frac{15}{8}$$

$$\therefore \quad \cos(\alpha + \beta) = \frac{1 - \tan^2\left(\frac{\alpha + \beta}{2}\right)}{1 + \tan^2\left(\frac{\alpha + \beta}{2}\right)},$$

$$\text{where } \tan\left(\frac{\alpha + \beta}{2}\right) = \frac{\tan\frac{\alpha}{2} + \tan\frac{\beta}{2}}{1 - \tan\frac{\alpha}{2}\tan\frac{\beta}{2}}$$

$$\text{or} \quad \tan\left(\frac{\alpha + \beta}{2}\right) = \frac{\frac{13}{4}}{1 - \frac{15}{8}} = \frac{-26}{7}$$

$$\Rightarrow \quad \cos(\alpha + \beta) = \frac{1 - \left(\frac{676}{49}\right)}{1 + \left(\frac{676}{49}\right)} = \frac{-627}{725}$$

$$\Rightarrow \quad \cos(\alpha + \beta) = \frac{-627}{725}$$

Example 117. If the solutions for θ from the equation $\sin^2\theta - 2\sin\theta + \lambda = 0$ lie in $\underset{n \in z}{\cup}\left(2n\pi - \frac{\pi}{6}, (2n+1)\pi + \frac{\pi}{6}\right)$. Then, find the possible set values of λ.

Sol. $\sin^2\theta - 2\sin\theta + \lambda = 0 \Rightarrow \sin\theta = \frac{2 \pm \sqrt{4 - 4\lambda}}{2} = 1 \pm \sqrt{1 - \lambda}.$

For real values, $1 - \lambda \ge 0$, i.e. $\lambda \le 1$.

As $-1 \le \sin\theta \le 1$, $\sin\theta = 1 - \sqrt{1 - \lambda}$. (neglecting $1 + \sqrt{1 - \lambda}$)

From question, $\sin\theta > -\frac{1}{2}$

Thus, $\quad -\frac{1}{2} < 1 - \sqrt{1 - \lambda} \le 1$

or $\quad -\frac{3}{2} < -\sqrt{1 - \lambda} \le 0 \Rightarrow \sqrt{1 - \lambda} < \frac{3}{2}$

$\Rightarrow \quad 1 - \lambda < \frac{9}{2} \Rightarrow \lambda > -\frac{5}{4}$

Example 118. If $ABCD$ is a convex quadrilateral such that $4\sec A + 5 = 0$, then find the quadratic equation whose roots are $\tan A$ and $\operatorname{cosec} A$.

Sol. $\sec A = -\frac{5}{4}$. So, $\frac{\pi}{2} < A < \pi$

Hence, $\tan A = -\frac{3}{4}$ and $\operatorname{cosec} A = \frac{5}{3}$

$\therefore$ Required quadratic equation is

$$x^2 - \left(-\frac{3}{4} + \frac{5}{3}\right)x + \left(-\frac{3}{4}\right) \times \frac{5}{3} = 0$$

$$x^2 - \frac{11}{12}x - \frac{5}{4} = 0 \text{ or } 12x^2 - 11x - 15 = 0$$

Example 119. If $\sec\alpha$ and $\operatorname{cosec}\alpha$ are the roots of $x^2 - px + q = 0$, then show $p^2 = q(q + 2)$.

Sol. Since, $\sec\alpha$ and $\operatorname{cosec}\alpha$ are the roots of $x^2 - px + q = 0$

$\therefore \quad \sec\alpha + \operatorname{cosec}\alpha = p$ and $\sec\alpha \operatorname{cosec}\alpha = q$

$\therefore \quad \sin\alpha + \cos\alpha = p\sin\alpha\cos\alpha$ and $\sin\alpha\cos\alpha = \frac{1}{q}$

$\therefore \quad \sin\alpha + \cos\alpha = \frac{p}{q}$

On squaring both sides, we get

$$\sin^2\alpha + \cos^2\alpha + 2\sin\alpha\cos\alpha = \frac{p^2}{q^2}$$

$$1 + 2\sin\alpha\cos\alpha = \frac{p^2}{q^2}$$

or $\quad 1 + \frac{2}{q} = \frac{p^2}{q^2} \quad \Rightarrow p^2 = q(q + 2)$

Example 120. Find the number of values of x in the interval $[0, 5\pi]$ satisfying the equation $3\sin^2 x - 7\sin x + 2 = 0$.

Sol. $3\sin^2 x - 7\sin x + 2 = 0$

$$\Rightarrow \quad \sin x = \frac{7 \pm \sqrt{49 - 24}}{6} = \frac{7 \pm 5}{6} = \frac{1}{3}, 2$$

$\therefore \quad \sin x = \frac{1}{3}$ (where, 2 is not possible).

$$\sin x = \frac{1}{3} = \sin\alpha; (0 < \alpha < \pi/2)$$

$\therefore$ $x = \alpha, \pi - \alpha, 2\pi + \alpha, 3\pi - \alpha, 4\pi + \alpha, 5\pi - \alpha$

Thus, the number of values of x is 6.

Example 121. $0 \le a \le 3$, $0 \le b \le 3$ ***and the equation,*** $x^2 + 4 + 3\cos(ax + b) = 2x$ ***has atleast one solution, then find the value of*** $(a + b)$.

Sol. $x^2 - 2x + 4 = -3\cos(ax + b)$

$\Rightarrow$ $(x-1)^2 + 3 = -3\cos(ax + b)$ ...(i)

As $-1 \le \cos(ax + b) \le 1$ and $(x-1)^2 \ge 0$

$\therefore$ Eq. (i) is only possible if,

$\cos(ax + b) = -1$ and $(x - 1) = 0$.

So, $a + b = \pi, 3\pi, 5\pi, \ldots$

and $3\pi > 6$

where $a + b \le 6$

$\Rightarrow$ $a + b = \pi$

Example 122. Find the values of p if it satisfy; $\cos\theta = x + \frac{p}{x}$, $x \in R$ for all real values of θ.

Sol. $x^2 - \cos\theta\, x + p = 0$

$\Rightarrow$ $x = \frac{\cos\theta \pm \sqrt{\cos^2\theta - 4p}}{2}$

For real x, $\cos^2\theta - 4p \ge 0. \Rightarrow 4p \le \cos^2\theta$

$4p \le \cos^2\theta \le 1.$

$\Rightarrow$ $p \le \frac{1}{4}$ for all values of θ.

Example 123. Find the set of values of $\lambda \in R$ such that $\tan^2\theta + \sec\theta = \lambda$ holds for some θ.

Sol. $\tan^2\theta + \sec\theta = \lambda \Rightarrow \sec^2\theta + \sec\theta - (\lambda + 1) = 0$

$\therefore$ $\sec\theta = \frac{-1 \pm \sqrt{1 + 4(\lambda + 1)}}{2}$

$= \frac{-1 \pm \sqrt{4\lambda + 5}}{2}$

For real $\sec\theta$, $4\lambda + 5 \ge 0$,

i.e. $\lambda \ge -\frac{5}{4}$ ...(i)

Also, $\sec\theta \ge 1$ or $\sec\theta \le -1$

$\therefore$ $\frac{-1 \pm \sqrt{4\lambda + 5}}{2} \ge 1$

or $\frac{-1 \pm \sqrt{4\lambda + 5}}{2} \le -1.$

$\Rightarrow$ $-1 \pm \sqrt{4\lambda + 5} \ge 2$

or $-1 \pm \sqrt{4\lambda + 5} \le -2.$

$\Rightarrow$ $4\lambda + 5 \ge 9$

or $4\lambda + 5 > 1$

$\Rightarrow$ $\lambda \ge 1$

or $\lambda \ge -1$ ...(ii)

From Eqs. (i) and (ii),

$\therefore$ $\lambda \ge -\frac{5}{4}$ and $\lambda \ge 1$

or $\lambda \ge -\frac{5}{4}$ and $\lambda \ge -1$

$\therefore$ $\lambda \ge 1$ or $\lambda \ge -1$

$\therefore$ $\lambda \in [-1, \infty)$.

Proving Trigonometric Inequality

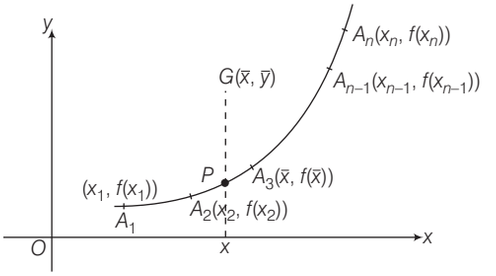
Jensen's Inequality

(i) Suppose that 'f' is a convex function on $[a, b] \in R$, for all $x_1, x_2, x_3 \ldots, x_n \in [a, b]$,

we have

$$f(x_1) + f(x_2) + \ldots + f(x_n) \ge n \cdot f\left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right)$$

Proof If $f(x)$ is concave up,



Here, $G\left(\frac{x_1 + x_2 + \ldots + x_n}{n}, \frac{f(x_1) + f(x_2) + \ldots + f(x_n)}{n}\right)$

and $P\left(\frac{x_1 + x_2 + \ldots + x_n}{n}, f\left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right)\right)$

From figure, ordinate of $G \ge$ ordinate of P.

$$\frac{f(x_1) + f(x_2) + \ldots + f(x_n)}{n} \ge f\left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right)$$

$\therefore$ $f(x_1) + f(x_2) + \ldots + f(x_n) \ge n \cdot f\left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right)$

(ii) Similarly, suppose that f is concave function on $[a, b] \in R$, for all $x_1, x_2, x_3, \ldots x_n \in [a, b]$, we have

$$f(x_1) + f(x_2) + \ldots + f(x_n) \le n \cdot f\left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right)$$

Example 124. If $A, B, C, \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then prove that $\cos A \cos B + \cos C \leq \frac{3}{2}$.

Sol. Since, for a function which is concave downwards

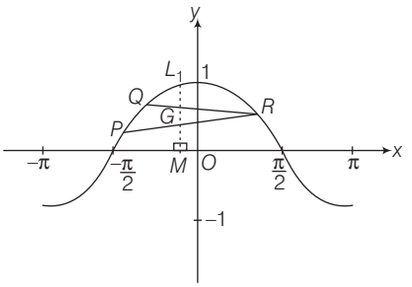
$$f\left(\frac{x_1 + x_2 + x_3}{3}\right) > \frac{f(x_1) + f(x_2) + f(x_3)}{3}$$

and we know that the graph of $y = \cos x$ is concave downwards for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Let $P(A, \cos A)$, $Q(B, \cos B)$ and $R(C, \cos C)$ be any three points on $y = \cos x$, then it is clear from the graph $GM \leq ML$

$$\Rightarrow \quad \frac{\cos A + \cos B + \cos C}{3} \leq \cos\left(\frac{A + B + C}{3}\right)$$

$$\therefore \quad \cos A + \cos B + \cos C \leq \frac{3}{2}, \text{ as at } B + C = \pi$$



Exercise for Session 11

1. Prove that the minimum value of $3\cos x + 4\sin x + 5$ is 0.
2. If $\sin\theta_1 + \sin\theta_2 + \sin\theta_3 = 3$, then find the value of $\cos\theta_1 + \cos\theta_2 + \cos\theta_3$.
3. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$, then prove that $x^2 + y^2 + z^2$ is independent of θ and ϕ.
4. Find the least value of $2\sin^2\theta + 3\cos^2\theta$.
5. α, β, γ are real numbers satisfying $\alpha + \beta + \gamma = \pi$. The find the minimum value of given expression $\sin\alpha + \sin\beta + \sin\gamma$.
6. If $A = \sin^2\theta + \cos^4\theta$, then find all real values of θ.
7. Find the minimum value of $\sec^2\theta + \text{cosec}^2\theta - 4$.
8. If $P = \cos(\cos x) + \sin(\cos x)$, then the least and greatest value of P respectively.
   (a) -1 and 1 (b) 0 and 2 (c) $-\sqrt{2}$ and $\sqrt{2}$ (d) 0 and $\sqrt{2}$
9. Let $\theta \in \left(0, \frac{\pi}{4}\right)$ and $t_1 = (\tan\theta)^{\tan\theta}$, $t_2 = (\tan\theta)^{\cot\theta}$, $t_3 = (\cot\theta)^{\tan\theta}$ and $t_4 = (\cot\theta)^{\cot\theta}$, then show that $t_4 > t_3 > t_1 > t_2$.
10. Find the ratio of greatest value of $2 - \cos x + \sin^2 x$ to its least value.

JEE Type Solved Examples :
Single Option Correct Type Questions

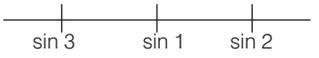
● **Ex. 1.** *In the inequality below, the value of the angle is expressed in radian measure. Which one of the inequalities below is true?*

(a) $\sin 1 < \sin 2 < \sin 3$ (b) $\sin 3 < \sin 2 < \sin 1$

(c) $\sin 2 < \sin 1 < \sin 3$ (d) $\sin 3 < \sin 1 < \sin 2$

Sol. (*d*) We have, $\sin 1 - \sin 2$

$$= -2\cos\frac{3}{2}\cdot\sin\frac{1}{2} < 0$$



$\therefore$ $\sin 1 < \sin 2$

Similarly $\sin 1 - \sin 3$ ...(i)

$= -2\cos 2 \sin 1 > 0$

$\Rightarrow$ $\sin 1 > \sin 3$ ...(ii)

From Eqs. (i) and (ii), we get

$\sin 3 < \sin 1 < \sin 2$

● **Ex. 2.** *In a ΔABC, $\angle B < \angle C$ and the values of B and C satisfy the equation $2\tan x - k(1+\tan^2 x) = 0$, where $(0 < k < 1)$. Then, the measure of $\angle A$ is*

(a) $\frac{\pi}{3}$ (b) $\frac{2\pi}{3}$

(c) $\frac{\pi}{2}$ (d) $\frac{3\pi}{4}$

Sol. (*c*) $\because k = \frac{2\tan x}{1+\tan^2 x} = \sin 2x$

$\Rightarrow$ $\sin 2C = \sin 2B$

But $\angle C > \angle B$

$\Rightarrow$ $2C = \pi - 2B \Rightarrow B + C = \frac{\pi}{2}$

$\therefore$ $\angle A = \frac{\pi}{2}$

● **Ex. 3.** *If M and m are maximum and minimum value of the function $f(x) = \frac{\tan^2 x + 4\tan x + 9}{1+\tan^2 x}$, then $(M+m)$ equals*

(a) 20 (b) 14

(c) 10 (d) 8

Sol. (*c*) Given, $f(x) = \frac{\tan^2 x + 4\tan x + 9}{1+\tan^2 x}$

$$= \frac{2(2\tan x)}{1+\tan^2 x} + 4\left(\frac{1-\tan^2 A}{1+\tan^2 A}\right) + 5$$

$$= 2\sin 2x + 4\cos 2x + 5$$

$\therefore$ $R_f = [\sqrt{5} - \sqrt{20}, 5 + \sqrt{20}]$

Hence, $(M+m) = 10$.

Alternate Method

$$f(x) = \frac{\tan^2 x}{1+\tan^2 x} + \frac{4\tan x}{1+\tan^2 x} + \frac{9}{1+\tan^2 x}$$

$$= \sin^2 x + 2\sin 2x + 9\cos^2 x$$

$$= 1 + 4(1+\cos 2x) + 2\sin 2x$$

$$= 5 + 2\sin 2x + 4\cos 2x$$

● **Ex. 4.** *The value of $4\cos\frac{\pi}{10} - 3\sec\frac{\pi}{10} - \tan\frac{\pi}{10}$ is equal to*

(a) 1

(b) $\sqrt{5} - 1$

(c) $\sqrt{5} + 1$

(d) zero

Sol. (*d*) We have, $4\cos 18° - \frac{3}{\cos 18°} - 2\tan 18°$

$$= \frac{4\cos^2 18° - 3 - 2\sin 18°}{\cos 18°}$$

$$= \frac{2(1+\cos 36°) - 2\sin 18° - 3}{\cos 18°}$$

$$= \frac{2(1+\cos 36° - \sin 18°) - 3}{\cos 18°}$$

$$= \frac{2\left(1+\frac{1}{2}\right) - 3}{\cos 18°} = 0$$

● **Ex. 5.** *For $0 < A < \frac{\pi}{2}$, the value of*

$$\log_{\frac{1}{2}}\left(\frac{1}{1+2\cos^2 A} + \frac{2}{\sec^2 A + 2}\right) \text{ is equal to}$$

(a) 1 (b) -1

(c) 2 (d) 0

Sol. (*d*) As, $\left(\frac{1}{1+2\cos^2 A} + \frac{2}{\sec^2 A + 2}\right)$

$$= \left(\frac{1}{1+2\cos^2 A} + \frac{2\cos^2 A}{1+2\cos^2 A}\right)$$

$$= \frac{(1+2\cos^2 A)}{(1+2\cos^2 A)} = 1$$

Hence, $\log_{\frac{1}{2}}(1) = 0$.

● **Ex. 6.** *The sum* $\frac{1}{\sin 45^\circ \sin 46^\circ} + \frac{1}{\sin 47^\circ \sin 48^\circ} + \frac{1}{\sin 49^\circ \sin 50^\circ} + \ldots + \frac{1}{\sin 133^\circ \sin 134^\circ}$ *is equal to*

(a) $\sec(1^\circ)$ (b) $\operatorname{cosec}(1^\circ)$

(c) $\cot(1^\circ)$ (d) None of these

Sol. (*b*) $T_1 = \frac{1}{\sin 1^\circ}\left[\frac{\sin(46^\circ - 45^\circ)}{\sin 45^\circ \sin 46^\circ}\right] = \frac{1}{\sin 1^\circ}[\cot 45^\circ - \cot 46^\circ]$

$$T_2 = \frac{1}{\sin 1^\circ}\left[\frac{\sin(48^\circ - 47^\circ)}{\sin 48^\circ \sin 47^\circ}\right]$$

$$= \frac{1}{\sin 1^\circ}[\cot 47^\circ - \cot 48^\circ]$$

$$T_l = \frac{1}{\sin 1^\circ}\left[\frac{\sin(133^\circ - 134^\circ)}{\sin 133^\circ \sin 134^\circ}\right]$$

$$= \frac{1}{\sin 1^\circ}[\cot 133^\circ - \cot 134^\circ]$$

On adding

$$\sum_{r=1}^{l} T_r = \frac{1}{\sin 1^\circ}[\{\cot 45^\circ + \cot 47^\circ + \cot 49^\circ + \ldots + \cot 133^\circ\} - \{\cot 46^\circ + \cot 48^\circ + \cot 50^\circ + \ldots + \cot 134^\circ\}]$$

$= \operatorname{cosec} 1^\circ$

[all terms cancelled except $\cot 45^\circ$ remains]

Ex. 7. *The range of k for which the inequality* $k\cos^2 x - k\cos x + 1 \geq 0 \;\forall\; x \in (-\infty, \infty)$, *is*

(a) $k < \frac{-1}{2}$ (b) $k < 4$

(c) $\frac{-1}{2} \leq k \leq 4$ (d) $\frac{1}{2} \leq k \leq 5$

Sol. (*c*) We have

$k\cos^2 x - k\cos x + 1 \geq 0 \;\forall\; x \in (-\infty, \infty)$

$\Rightarrow \quad k(\cos^2 x - \cos x) + 1 \geq 0$

But $\cos^2 x - \cos x = \left(\cos x - \frac{1}{2}\right)^2 - \frac{1}{4}$

$\Rightarrow \quad -\frac{1}{4} \leq \cos^2 x - \cos x \leq 2$

$\therefore$ We have, $2k + 1 \geq 0$ and $-\frac{k}{4} + 1 \geq 0$

Hence, $-\frac{1}{2} \leq k \leq 4.$

● **Ex. 8.** *If* $f(\theta) = \frac{1 - \sin 2\theta + \cos 2\theta}{2\cos 2\theta}$, *then value of* $f(11^\circ) \cdot f(34^\circ)$ *equals*

(a) $\frac{1}{2}$ (b) $\frac{3}{4}$ (c) $\frac{1}{4}$ (d) 1

Sol. (*a*) $f(\theta) = \frac{1 - \sin 2\theta + \cos 2\theta}{2\cos 2\theta}$

$$= \frac{(\cos\theta - \sin\theta)^2 + (\cos^2\theta - \sin^2\theta)}{2(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)}$$

$$= \frac{(\cos\theta - \sin\theta) + (\cos\theta + \sin\theta)}{2(\cos\theta + \sin\theta)}$$

$$= \frac{2\cos\theta}{2(\cos\theta + \sin\theta)} = \frac{1}{1 + \tan\theta}$$

$$f(11^\circ) \cdot f(34^\circ) = \frac{1}{(1 + \tan 11^\circ)} \cdot \frac{1}{(1 + \tan 34^\circ)}$$

$$= \frac{1}{(1 + \tan 11^\circ)} \cdot \frac{1}{(1 + \tan(45^\circ - 11^\circ))}$$

$$= \frac{1}{(1 + \tan 11^\circ)} \cdot \frac{1}{1 + \frac{1 - \tan 11^\circ}{1 + \tan 11^\circ}}$$

$$= \frac{1}{(1 + \tan 11^\circ)} \cdot \frac{1 + \tan 11^\circ}{2} = \frac{1}{2}$$

● **Ex. 9.** *The variable 'x' satisfying the equation* $|\sin x \cos x| + \sqrt{2 + \tan^2 x + \cot^2 x} = \sqrt{3}$, *belongs to the interval*

(a) $\left[0, \frac{\pi}{3}\right]$ (b) $\left[\frac{\pi}{3}, \frac{\pi}{2}\right]$

(c) $\left[\frac{3\pi}{4}, \pi\right]$ (d) Non-existent

Sol. (*d*) $|\sin x \cos x| + |\tan x + \cot x| = \sqrt{3}$

$\Rightarrow \quad |\sin x \cos x| + \frac{1}{|\sin x \cos x|} = \sqrt{3}$

but $\quad |\sin x \cos x| + \frac{1}{|\sin x \cos x|} \geq 2$

Hence, no solution.

● **Ex. 10.** *Let* α *be a real number such that* $0 \leq \alpha \leq \pi$. *If* $f(x) = \cos x + \cos(x + \alpha) + \cos(x + 2\alpha)$ *takes some constant number c for any* $x \in R$, *then the value of* $[c + \alpha]$ *is equal to*

Note $[y]$ denotes greatest integer less than or equal to y.

(a) 0 (b) 1 (c) -1 (d) 2

Sol. (*d*) $f(x) = \cos x + \cos(x + 2\alpha) + \cos(x + \alpha)$

$$= 2\cos(x + \alpha)\cos\alpha + \cos(x + \alpha)$$

$$= (2\cos\alpha + 1)\cos(x + \alpha)$$

As $\cos(x + \alpha)$ can take any real value from -1 to 1, $\forall x \in R$

$\therefore \quad f(x)$ is constant, so $(2\cos\alpha + 1) = 0$ must hold.

$\Rightarrow \quad \alpha = \frac{2\pi}{3}$ and $c = 0$

Hence, $[c + \alpha] = \left[0 + \frac{2\pi}{3}\right] = 2$

● **Ex. 11.** *In a ΔABC, if $4\cos A\cos B + \sin 2A + \sin 2B + \sin 2C = 4$, then ΔABC is*

(a) right angle but not isosceles
(b) isosceles but not right angled
(c) right angle isosceles
(d) obtuse angled

Sol. (*c*) We have, $4\cos A\cos B + 4\sin A\sin B\sin C = 4$

$$\Rightarrow \quad \sin C = \frac{1-\cos A\cos B}{\sin A\cos B} \le 1$$

$$\Rightarrow \quad 1 \le \sin A\sin B + \cos A\cos B$$

$$\Rightarrow \quad \cos(A-B) \ge 1$$

$$\Rightarrow \quad A = B \text{ and } \sin C = \frac{1-\cos^2 A}{\sin^2 A} = 1$$

$$\therefore \quad C = 90^\circ$$

and $\quad A = B = \frac{\pi}{4}$ (each).

Ex. 12. *For $\theta_1, \theta_2, \ldots, \theta_n \in \left(0, \frac{\pi}{2}\right)$, if $ln(\sec\theta_1 - \tan\theta_1) + ln(\sec\theta_2 - \tan\theta_2) + \ldots + ln(\sec\theta_n - \tan\theta_n) + ln\,\pi = 0$, then the value of $\cos((\sec\theta_1 + \tan\theta_1)(\sec\theta_2 + \tan\theta_2)\ldots\ldots(\sec\theta_n + \tan\theta_n))$ is equal to*

(a) $\cos\left(\frac{1}{\pi}\right)$ (b) -1
(c) 1 (d) 0

Sol. (*b*) $\ln\{(\sec\theta_1 - \tan\theta_1)(\sec\theta_2 - \tan\theta_2)\ldots(\sec\theta_n - \tan\theta_n)\} = \ln\left\{\frac{1}{\pi}\right\}$

[**Note** If $0 < x < \frac{\pi}{2}$, $\sec x - \tan x = \frac{1-\sin x}{\cos x} > 0$]

$$\therefore (\sec\theta_1 - \tan\theta_1)(\sec\theta_2 - \tan\theta_2)\ldots(\sec\theta_n - \tan\theta_n) = \frac{1}{\pi} \quad \ldots(i)$$

Let $(\sec\theta_1 + \tan\theta_1)(\sec\theta_2 + \tan\theta_2)\ldots(\sec\theta_n + \tan\theta_n) = x$ ...(ii)

On multiplying Eqs. (i) and (ii) we get

$$1 = \frac{x}{\pi}$$

$$\therefore \quad x = \pi$$

$$\therefore \quad \cos((\sec\theta_1 + \tan\theta_1)(\sec\theta_2 + \tan\theta_2)\ldots(\sec\theta_n + \tan\theta_n)) = \cos\pi = -1$$

● **Ex. 13.** *If A, B, C are interior angles of ΔABC such that $(\cos A + \cos B + \cos C)^2 + (\sin A + \sin B + \sin C)^2 = 9$, then number of possible triangles is*

(a) 0 (b) 1
(c) 3 (d) infinite

Sol. (*d*) $(\Sigma\cos A)^2 + (\Sigma\sin A)^2 = 9$

$$\Sigma(\cos^2 A + \sin^2 A) + 2(\Sigma\cos A\cos B + \sin A\sin B)$$

$$3 + 2\Sigma\cos(A-B) \le 3 + 2(3) = 9.$$

Equality holds if $A = B = C$

$\Rightarrow$ ΔABC is equilateral $\Rightarrow$ Infinite many equilateral

[**Note** We can vary side length of equilateral triangle]

● **Ex. 14.** *If $\operatorname{cosec}\frac{\pi}{32} + \operatorname{cosec}\frac{\pi}{16} + \operatorname{cosec}\frac{\pi}{8} + \operatorname{cosec}\frac{\pi}{4} + \operatorname{cosec}\frac{\pi}{2} = \cot\frac{\pi}{k}$, then the value of k is*

(a) 64 (b) 96 (c) 48 (d) 32

Sol. (*a*) $T_1 = \operatorname{cosec}\theta = \dfrac{\sin\left(\theta - \frac{\theta}{2}\right)}{\sin\frac{\theta}{2}\sin\theta}; \theta = \frac{\pi}{32}$

$$\left.\begin{aligned} T_1 &= \cot\frac{\theta}{2} - \cot\theta \\ T_2 &= \cot\theta - \cot 2\theta \\ T_3 &= \cot 2\theta - \cot 2^2\theta \\ T_4 &= \cot 2^2\theta - \cot 2^3\theta \\ T_5 &= 1 \end{aligned}\right\} \text{sum} = 1 + \cot\frac{\theta}{2} - \cot 8\theta$$

$$\text{Sum} = 1 + \cot\frac{\theta}{2} - \cot 8\theta$$

$$= 1 + \cot\frac{\pi}{64} - \cot\frac{\pi}{4} = \cot\frac{\pi}{64} = \cot\frac{\pi}{k} \quad \therefore \quad k = 64$$

● **Ex. 15.** *Let $S = \sum_{r=1}^{5}\cos(2r-1)\frac{\pi}{11}$ and $P = \prod_{r=1}^{4}\cos\left(2^r\frac{\pi}{15}\right)$, then*

(a) $\log_S P = -4$ (b) $P = 3S$
(c) $\operatorname{cosec} S > \operatorname{cosec} P$ (d) $\tan^{-1}P < \tan^{-1}S$

Sol. (d) We have, $\sum_{r=1}^{5}\cos(2r-1)\frac{\pi}{11}$

$$= \underbrace{\cos\frac{\pi}{11} + \cos\frac{3\pi}{11} + \cos\frac{5\pi}{11} + \cos\frac{7\pi}{11} + \cos\frac{9\pi}{11}}_{\text{cosine series}}$$

$$= \frac{2\cdot\cos\left(\frac{5\pi}{11}\right)\cdot\sin\left(\frac{5\pi}{11}\right)}{2\cdot\sin\frac{\pi}{11}} = \frac{\sin\left(\frac{10\pi}{11}\right)}{2\cdot\sin\left(\frac{\pi}{11}\right)} = \frac{1}{2}$$

Also, $\prod_{r=1}^{\pi}\cos\left(2^r\frac{\pi}{15}\right) = \cos\frac{2\pi}{15}\cdot\cos\frac{4\pi}{15}\cdot\cos\frac{8\pi}{15}\cdot\cos\frac{16\pi}{15}$

$$= \frac{\sin\left(\frac{32\pi}{15}\right)}{2^4\cdot\sin\left(\frac{2\pi}{15}\right)} = \frac{\sin\left(2\pi + \frac{2\pi}{15}\right)}{16\cdot\sin\left(\frac{2\pi}{15}\right)}$$

$$= \frac{\sin\left(\frac{2\pi}{15}\right)}{16 \cdot \sin\left(\frac{2\pi}{15}\right)} = \frac{1}{16}$$

Therefore, $\tan^{-1} P < \tan^{-1} S$.

● ***Ex. 16.*** *Set of values of x lying in* $[0, 2\pi]$ *satisfying the inequality* $|\sin x| > 2\sin^2 x$ *contains*

(a) $\left(0, \frac{\pi}{6}\right) \cup \left(\pi, \frac{7\pi}{6}\right)$ (b) $\left(0, \frac{7\pi}{6}\right)$

(c) $\frac{\pi}{6}$ (d) None of these

Sol. (*a*) $|\sin x| > 2\sin^2 x$

$\Rightarrow |\sin x|(2|\sin x| - 1) < 0$

$\Rightarrow 0 < |\sin x| < \frac{1}{2}$

$\Rightarrow x \in \left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, \pi\right) \cup \left(\pi, \frac{7\pi}{6}\right) \cup \left(\frac{11\pi}{6}, 2\pi\right)$

● ***Ex. 17.*** *The number of ordered pairs* (x, y)*, when* $x, y \in [0, 10]$ *satisfying* $\left(\sqrt{\sin^2 x - \sin x + \frac{1}{2}}\right) \cdot 2^{\sec^2 y} \le 1$ *is*

(a) 0 (b) 16

(c) infinite (d) 12

Sol. (*b*) $\sqrt{\sin^2 x - \sin x + \frac{1}{2}} = \sqrt{\left(\sin x - \frac{1}{2}\right)^2 + \frac{1}{2}} \ge \frac{1}{2}, \forall x$

and $\sec^2 y \ge 1, \forall y$, so $2^{\sec^2 y} \ge 2$. Hence, the above inequality holds only for those values of x and y for which $\sin x = \frac{1}{2}$

and $\sec^2 y = 1$.

Hence, $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$ and $y = 0, \pi, 2\pi, 3\pi$. Hence, required number of ordered pairs are 16.

● ***Ex. 18.*** *The least values of* $\text{cosec}^2 x + 25\sec^2 x$ *is*

(a) 0 (b) 26

(c) 28 (d) 36

Sol. (*d*) $\text{cosec}^2 x + 25\sec^2 x = 26 + \cot^2 x + 25\tan^2 x$

$= 26 + 10 + (\cot x - 5\tan x)^2 \ge 36$

● ***Ex. 19.*** *If* $x \sin a + y \sin 2a + z \sin 3a = \sin 4a$

$x \sin b + y \sin 2b + z \sin 3b = \sin 4b$

$x \sin c + y \sin 2c + z \sin 3c = \sin 4c$

Then, the roots of the equation

$t^3 - \left(\frac{z}{2}\right)t^2 - \left(\frac{y+2}{4}\right)t + \left(\frac{z-x}{8}\right) = 0$, *a, b, c* $\ne n\pi$, *are*

(a) $\sin a, \sin b, \sin c$

(b) $\cos a, \cos b, \cos c$

(c) $\sin 2a, \sin 2b, \sin 2c$

(d) $\cos 2a, \cos 2b, \cos 2c$

Sol. (*b*) Equation first can be written as

$x \sin a + y \times 2\sin a \cos a + z \times \sin a(3 - 4\sin^2 a)$

$= 2 \times 2 \sin a \cos a \cos 2a$

$\Rightarrow \quad x + 2y\cos a + z(3 + 4\cos^2 a - 4)$

$= 4\cos a(2\cos^2 a - 1)$ as $\sin a \ne 0$

$\Rightarrow \quad 8\cos^3 a - 4z\cos^2 a - (2y + 4)\cos a + (z - x) = 0$

$\Rightarrow \quad \cos^3 a - \left(\frac{z}{2}\right)\cos^2 a - \left(\frac{y+2}{4}\right)\cos a + \left(\frac{z-x}{8}\right) = 0$

which shows that $\cos a$ is a root of the equation

$$t^3 = \left(\frac{z}{2}\right)t^2 - \left(\frac{y+z}{4}\right)t + \left(\frac{z-x}{8}\right) = 0$$

Similarly, from second and third equation we can verify that $\cos b$ and $\cos c$ are the roots of the given equation.

● ***Ex. 20.*** *Let* α *and* β *be any two positive values of x for which* $2\cos x, |\cos x|$ *and* $1 - 3\cos^2 x$ *are in* GP. *The minium value of* $|\alpha + \beta|$ *is*

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{2}$ (d) None of these

Sol. (*d*) $\because 2\cos x, |\cos x|, 1 - 3\cos^2 x$ are in GP.

$\therefore \quad \cos^2 x = 2\cos x \cdot (1 - 3\cos^2 x)$

$\Rightarrow \quad 6\cos^3 x + \cos^2 x - 2\cos x = 0$

$\therefore \quad \cos x = 0, \frac{1}{2}, -\frac{2}{3}$

$\therefore \quad x = \frac{\pi}{2}, \frac{\pi}{3}, \cos^{-1}\left(-\frac{2}{3}\right)$ [$\because \alpha, \beta$ are positive]

If $\alpha = \frac{\pi}{2}, \beta = \frac{\pi}{3}$

Then, $|\alpha - \beta| = \frac{\pi}{6}$

● ***Ex. 21.*** *Let n be an odd integer. If* $\sin n\theta = \sum_{r=0}^{n} b_r \sin^r \theta$ *for all real* θ*, then*

(a) $b_0 = 1, b_1 = 3$ (b) $b_0 = 0, b_1 = n$

(c) $b_0 = -1, b_1 = n$ (d) $b_0 = 0, b_1 = n^2 - 3n - 3$

Sol. (*b*) Given, $\sin n\theta = \sum_{r=0}^{n} b_r \sin^r \theta = b_0 + b_1 \sin\theta + b_2 \sin^2 \theta + \ldots + b_n \sin^n \theta$ ...(i)

Putting $\theta = 0$ in Eq. (i), we get $0 = b_0$

Again, Eq. (i) can be written as $\sin n\theta = \sum_{r=0}^{n} b_r \sin^r \theta$

$$\frac{\sin n\theta}{\sin\theta} = \sum_{r=0}^{n} b_r \sin^{r-1}\theta$$

On taking limit as $\theta \to 0$, we get

$$\lim_{\theta\to 0}\frac{\sin n\theta}{\sin\theta} = b_1$$

$$\Rightarrow \quad \lim_{\theta\to 0} n\left(\frac{\sin\theta}{n\theta}\right)\left(\frac{\theta}{\sin\theta}\right) = b_1$$

$$\Rightarrow \quad n = b_1$$

Hence, $b_0 = 0; b_1 = n$

● ***Ex. 22.*** *The minimum and maximum values of* $ab\sin x + b\sqrt{(1-a^2)}\cos x + c\ (|a|<1, b>0)$ *respectively are*

(a) $\{b-c, b+c\}$ (b) $\{b+c, b-c\}$

(c) $\{c-b, b+c\}$ (d) None of these

Sol. (*c*) $ab\sin x + b\sqrt{(1-a^2)}\cos x$

Now, $\sqrt{(ab)^2 + (b\sqrt{(1-a^2)})^2}$

$$= \sqrt{a^2b^2 + b^2(1-a^2)}$$

$$= b\sqrt{(a^2+1-a^2)} = b$$

$$\Rightarrow \quad b\{(a\sin x + \sqrt{(1-a^2)}\cos x)\}$$

Let, $a = \cos\alpha$,

$\therefore \quad \sqrt{(1-a^2)} = \sin\alpha$

$\Rightarrow \quad b\sin(x+a)$

$\because \quad -1 \le \sin(x+\alpha) \le 1$

$\therefore \quad c-b \le b\sin(x+\alpha) + c \le b+c$

$\therefore \quad b\sin(x+\alpha) + c \in [c-b, c+b]$

● ***Ex. 23.*** *If* $\dfrac{\sin^3\theta - \cos^3\theta}{\sin\theta - \cos\theta} - \dfrac{\cos\theta}{\sqrt{(1+\cot^2\theta)}} - 2\tan\theta\cot\theta = -1, \theta \in [0, 2\pi]$, *then*

(a) $\theta \in \left(0, \dfrac{\pi}{2}\right) - \left\{\dfrac{\pi}{4}\right\}$ (b) $\theta \in \left(\dfrac{\pi}{2}, \pi\right) - \left\{\dfrac{3\pi}{4}\right\}$

(c) $\theta \in \left(\pi, \dfrac{3\pi}{2}\right) - \left\{\dfrac{5\pi}{4}\right\}$ (d) $\theta \in (0,\pi) - \left\{\dfrac{\pi}{4}, \dfrac{\pi}{2}\right\}$

Sol. (*d*) $\dfrac{\sin^3\theta - \cos^3\theta}{\sin\theta - \cos\theta} = \sin^2\theta + \cos^2\theta + \sin\theta\cos\theta,$

$\theta \ne \dfrac{\pi}{4}, \dfrac{5\pi}{4}$

$= 1 + \sin\theta\cos\theta$

and $\dfrac{\cos\theta}{\sqrt{(1+\cot^2\theta)}} = \dfrac{\cos\theta}{|\operatorname{cosec}\theta|} = \sin\theta\cos\theta\ \forall\ \theta \in (0,\pi)$

and $-2\tan\theta\cot\theta = -2, \theta \ne \dfrac{\pi}{2}$

Hence, LHS = RHS

But $\theta \ne \dfrac{\pi}{4}, \dfrac{5\pi}{4}, \dfrac{\pi}{2}$

Hence, $\theta \in (0,\pi) \sim \left\{\dfrac{\pi}{4}, \dfrac{\pi}{2}\right\}$

● ***Ex. 24.*** *If* $\cos x + \sin x = a\left(-\dfrac{\pi}{2} < x < -\dfrac{\pi}{4}\right)$, *then* $\cos 2x$ *is equal to*

(a) a^2 (b) $a\sqrt{(2-a)}$

(b) $a\sqrt{(2+a)}$ (d) $a\sqrt{(2-a^2)}$

Sol. (*d*) $\because -\dfrac{\pi}{2} < x < -\dfrac{\pi}{4}\left(-\pi < 2x < -\dfrac{\pi}{2}, i.e., \text{ in III quadrant}\right)$

$$\Rightarrow \quad \cos x + \sin x = a$$

Squaring both sides $\cos^2 x + \sin^2 x + 2\cos x\sin x = a^2$

$$\Rightarrow \quad \sin 2x = (a^2 - 1)$$

$$\cos 2x = \sqrt{1-(a^2-1)^2}$$

$$= \sqrt{a^2(2-a^2)}$$

$$= a\sqrt{(2-a^2)}$$

● ***Ex. 25.*** *If* $S = \cos^2\dfrac{\pi}{2} + \cos^2\dfrac{2\pi}{n} + \ldots + \cos^2\dfrac{(n-1)\pi}{n}$, *then* S *equals*

(a) $\dfrac{n}{2(n+1)}$ (b) $\dfrac{1}{2(n-1)}$

(c) $\dfrac{1}{2(n-2)}$ (d) $\dfrac{n}{2}$

Sol. (*c*) $S = \cos^2\dfrac{\pi}{n} + \cos^2\dfrac{2\pi}{n} + \ldots + \cos^2(n-1)\dfrac{\pi}{n}$

$$\frac{1}{2}\left[\begin{array}{l} 1+\cos\dfrac{2\pi}{n} + 1 + \cos\dfrac{4\pi}{n} + 1 + \cos\dfrac{6\pi}{n} \\ + \ldots + 1 + \cos 2(n-1)\dfrac{\pi}{n} \end{array}\right]$$

$$= \frac{1}{2}\left[n - 1 + \sum_{k=1}^{n-1}\cos\frac{2k\pi}{n}\right]$$

$$= \frac{1}{2}[n-1-1] = \frac{1}{2}(n-2)$$

● ***Ex. 26.*** *If* $\cos 5\theta = a\cos\theta + b\cos^3\theta + c\cos^5\theta + d$, *then*

(a) $a = 20$ (b) $b = -30$

(c) $a+b+c = 2$ (d) $a+b+c+d = 1$

Sol. (*d*) Put $\theta = \dfrac{\pi}{2}$ in the given inequality, we get $d = 0$

Put $\theta = 0$ in the given inequality, we get

$$a + b + c + d = 1 \qquad \ldots(i)$$

So, (d) is correct and (c) is not correct.

Now differentiate both sides with respect to θ, we get

$$-5\sin\theta = -a\sin\theta - 3b\cos^2\theta\sin\theta - 5c\cos^4\theta\sin\theta \quad ...(ii)$$

Put $\theta = \frac{\pi}{2}$, then $a = 5$ ...(iii)

Again putting $\theta = \frac{\pi}{4}$ in the given expression or in (2), we get

$$4a + 2b + c = -4 \quad ...(iv)$$

From (i), (iii) and (iv) we have $b = -20$ and $c = 16$

[**Note** We have found correct answer at the second step only however the complete solution is desired for better understanding of the solution.]

Alternates Solution

$$\cos 5\theta = \cos(2\theta + 3\theta) = \cos 2\theta\cos 3\theta - \sin 2\theta\sin 3\theta$$
$$= (2\cos^2\theta - 1)(4\cos^3\theta - 3\cos\theta) - (2\sin\theta\cos\theta)(3\sin\theta - 4\sin^3\theta)$$
$$= 8\cos^5\theta - 10\cos^3\theta + 3\cos\theta - 2(1-\cos^2\theta)\cos\theta\{3 - 4(1-\cos^2\theta)\}$$
$$= 8\cos^5\theta - 10\cos^3\theta + 3\cos\theta - 2(\cos\theta - \cos^3\theta)(4\cos^2\theta - 1) = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$$

● ***Ex. 27.*** *If A and B are acute positive angles satisfying the equations $3\sin^2 A + 2\sin^2 B = 1$ and $3\sin 2A - 2\sin 3B = 0$, then $A + 2B$ is equal to*

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$

(c) $\frac{3\pi}{4}$ (d) $\frac{2\pi}{3}$

Sol. (*b*) From the given relations, we have

$$\sin 2B = \left(\frac{3}{2}\right)\sin 2A \text{ and } 3\sin^2 A = 1 - 2\sin^2 B = \cos 2B$$

so that

$$\cos(A + 2B) = \cos A\cos 2B - \sin A\sin 2B$$
$$= \cos A\cdot 3\sin^2 A - \left(\frac{3}{2}\right)\sin A\sin 2A$$
$$= 3\cos A\sin^2 A - 3\sin^2 A\cos A = 0$$
$$A + 2B = \frac{\pi}{2}$$

● ***Ex. 28.*** *If* $A = \begin{bmatrix} \cos^2\alpha & \cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha & \sin^2\alpha \end{bmatrix}$ *and* $B = \begin{bmatrix} \cos^2\beta & \cos\beta\sin\beta \\ \cos\beta\sin\beta & \sin^2\beta \end{bmatrix}$

are two matrices such that AB is the null matrix, then

(a) $\alpha = \beta$ (b) $\cos(\alpha - \beta) = 0$

(c) $\sin(\alpha - \beta) = 0$ (d) None of these

Sol. (*b*) $AB = 0$

$$\Rightarrow \begin{bmatrix} \cos^2\alpha & \cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha & \sin^2\alpha \end{bmatrix}\begin{bmatrix} \cos^2\beta & \cos\beta\sin\beta \\ \cos\beta\sin\beta & \sin^2\beta \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \cos\alpha\cos\beta\cos(\alpha-\beta) & \cos\alpha\sin\beta\cos(\alpha-\beta) \\ \cos\beta\sin\alpha\cos(\alpha-\beta) & \sin\alpha\sin\beta\cos(\alpha-\beta) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \cos(\alpha - \beta) = 0$$

● ***Ex. 29.*** *If* $k_1 = \tan 27\theta - \tan\theta$ *and* $k_2 = \frac{\sin\theta}{\cos 3\theta} + \frac{\sin 3\theta}{\cos 9\theta} + \frac{\sin 9\theta}{\cos 27\theta}$ *then,*

(a) $k_1 = k_2$

(b) $k_1 = 2k_2$

(c) $k_1 + k_2 = 2$

(d) $k_2 = 2k_1$

Sol. (*b*) We can write

$$k_1 = \tan 27\theta - \tan 9\theta + \tan 9\theta - \tan 3\theta + \tan 3\theta - \tan\theta$$

But $\tan 3\theta - \tan\theta = \frac{\sin 3\theta\cos\theta - \cos 3\theta\sin\theta}{\cos 3\theta\cos\theta}$

$$= \frac{\sin 2\theta}{\cos 3\theta\cos\theta}$$
$$= \frac{2\sin\theta}{\cos 3\theta}$$

$$\therefore \quad k_1 = 2\left[\frac{\sin 9\theta}{\cos 27\theta} + \frac{\sin 3\theta}{\cos 9\theta} + \frac{\sin\theta}{\cos 3\theta}\right] = 2k_2$$

● ***Ex. 30.*** *If* $a^2 - 2a\cos x + 1 = 674$ *and* $\tan\left(\frac{x}{2}\right) = 7$ *then the integral value of a is*

(a) 25 (b) 49

(c) 67 (d) 74

Sol. (*a*) $674 = a^2 - 2a\frac{1 - \tan^2\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)} + 1$

$$= a^2 - 2a\times\frac{1-49}{1+49} + 1$$
$$= a^2 + 2a\times\frac{48}{50} + 1$$
$$\Rightarrow \quad 25a^2 + 48a - 673\times 25 = 0$$
$$\Rightarrow \quad (a - 25)(25a + 673) = 0$$
$$\Rightarrow \quad a = 25 \quad \text{(taking the integral value of } a\text{)}.$$

● **Ex. 31.** *The maximum value of* $(\cos\alpha_1)(\cos\alpha_2)\ldots$ $(\cos\alpha_n)$ *under the restriction* $0 \le \alpha_1, \alpha_2, \ldots, \alpha_n \le \frac{\pi}{2}$ *and* $(\cot\alpha_1)(\cot\alpha_2)\ldots(\cot\alpha_n) = 1$ *is*

(a) $\frac{1}{2^{\frac{n}{2}}}$ (b) $\frac{1}{2^n}$

(c) $\frac{1}{2n}$ (d) 1

Sol. (*a*) From the given relations we have

$$\prod_{i=1}^{n}(\cos\alpha_i) = \prod_{i=1}^{n}(\sin\alpha_i)$$

$$\Rightarrow \quad \prod_{i=1}^{n}(\cos^2\alpha_i) = \prod_{i=1}^{n}(\cos\alpha_i \sin\alpha_i) = \prod_{i=1}^{n}\left(\frac{\sin 2\alpha_i}{2}\right)$$

Since, $0 \le \alpha_i \le \frac{\pi}{2} \Rightarrow 0 \le 2\alpha_i \le \pi$

$\therefore \quad \prod_{i=1}^{n}(\cos^2\alpha_i) \le \frac{1}{2^n}$ as max. value of $\sin 2\alpha_i$ is 1 for all *i*.

$$\Rightarrow \quad \prod_{i=1}^{n}(\cos\alpha_i) \le \frac{1}{2^{\frac{n}{2}}}$$

So, the maximum value of the given expression is $\frac{1}{2^{\frac{n}{2}}}$.

● **Ex. 32.** *The value of expression* $\frac{\sin^3 x}{1+\cos x} + \frac{\cos^3 x}{1-\sin x}$ *is/are*

(a) $\sqrt{2}\cos\left[\frac{\pi}{4} - x\right]$ (b) $\sqrt{2}\cos\left[\frac{\pi}{4} + x\right]$

(c) $\sqrt{2}\sin\left[\frac{\pi}{4} - x\right]$ (d) None of these

Sol. (*a*) Let $\frac{\sin^3 x}{1+\cos x} + \frac{\cos^3 x}{1-\sin x} = A$, then

$$A = \frac{(\sin^3 x + \cos^3 x) + (\cos^4 x - \sin^4 x)}{(1+\cos x)(1-\sin x)}$$

$$\{(\sin^3 x + \cos^3 x)\}$$

or $$A = \frac{+\left\{\begin{matrix}(\cos x + \sin x)(\cos x - \sin x)\\ (\cos^2 x + \sin^2 x)\end{matrix}\right\}}{(1+\cos x)(1-\sin x)}$$

$$(\sin x + \cos x)\{(1 - \sin x\cos x)$$

or $$A = \frac{+(\cos x - \sin x)\}}{1 + \cos x - \sin x - \sin x\cos x}$$

or $$A = \sin x + \cos x$$

or $$A = \sqrt{2}\left[\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right] \quad \ldots(i)$$

or $$A = \sqrt{2}\left[\cos\frac{\pi}{4}\sin x + \sin\frac{\pi}{4}\cos x\right]$$

$$= \sqrt{2}\sin\left[\frac{\pi}{4} + x\right]$$

Again, by Eq. (i)

$$A = \sqrt{2}\left[\sin\frac{\pi}{4}\sin x + \cos\frac{\pi}{4}\cos x\right]$$

$$= \sqrt{2}\cos\left[\frac{\pi}{4} - x\right]$$

● **Ex. 33.** *Let* $0 \le \theta \le \frac{\pi}{2}$ *and* $x = X\cos\theta + Y\sin\theta$, $y = X\sin\theta - Y\cos\theta$ *such that* $x^2 + 2xy + y^2 = aX^2 + bY^2$, *where a and b are constant, then*

(a) $a = -1, b = -3$ (b) $\theta = \frac{\pi}{2}$

(c) $a = 3, b = -1$ (d) $\theta = \frac{\pi}{3}$

Sol. (*c*) $x^2 + y^2 = X^2 + Y^2$,

$$xy = (X^2 - Y^2)\sin\theta\cdot\cos\theta - XY(\cos^2\theta - \sin^2\theta)$$

$$x^2 + 4xy + y^2 = X^2 + Y^2 + 2(X^2 - Y^2)$$

$$\sin 2\theta - 2XY\cos 2\theta$$

$$= (1 + 2\sin 2\theta)X^2 + (1 - 2\sin 2\theta)Y^2 - 2\cos 2\theta\cdot XY$$

From the question,

$$a = 1 + 2\sin 2\theta,\ b = 1 - 2\sin 2\theta,\ \cos 2\theta = 0$$

$$\cos 2\theta = 0 \Rightarrow \theta = \frac{\pi}{4}, \text{ then}$$

$$a = 1 + 2\sin\frac{\pi}{2},\ b = 1 - 2\sin\frac{\pi}{2}$$

$\therefore \quad a = 3, b = -1$

● **Ex. 34.** *If* $0 < x < \frac{\pi}{2}$ *and* $\sin^n x + \cos^n x \ge 1$, *then*

(a) $n \in [2, \infty)$ (b) $n \in (-\infty, 2]$

(c) $n \in [-1, 1]$ (d) None of these

Sol. (*b*) Since, $0 < x < \frac{\pi}{2}$

$\therefore \quad 0 < \sin x < 1$ and $0 < \cos x < 1$

when $x = 2$, $\sin^n x + \cos^n x = 1$

when $n > 2$, both $\sin^n x$ and $\cos^n x$ will decreases and hence $\sin^n x + \cos^n x < 1$.

when $n > 2$, both $\sin^n x$ and $\cos^n x$ will increase and hence $\sin^n x + \cos^n x > 1$.

Thus, $\sin^n x + \cos^n x \ge 1$ for $n \le 2$.

● ***Ex. 35.*** *If* $a = \sin\frac{\pi}{18}\sin\frac{5\pi}{18}\sin\frac{7\pi}{18}$, *and* x *is the solution the equation* $y = 2[x] + 2$ *and* $y = 3[x-2]$, *where* $[x]$ *denotes the integral part of* x, *then* a *is equal to*

(a) $[x]$ (b) $\frac{1}{[x]}$

(c) $2[x]$ (d) $[x]^2$

Sol. (*b*) $a = \sin\frac{\pi}{18}\sin\frac{5\pi}{18}\sin\frac{7\pi}{18}$

$= \sin 10^\circ \sin 50^\circ \sin 70^\circ$

$= \frac{1}{2}[2\sin 70^\circ \sin 10^\circ]\sin 50^\circ$

$= \frac{1}{2}[\cos 60^\circ - \cos 80^\circ]\sin 50^\circ$

$= \frac{1}{4}\sin 50^\circ - \frac{1}{4}(2\cos 80^\circ \sin 50^\circ)$

$= \frac{1}{4}\sin 50^\circ - \frac{1}{4}(\sin 130^\circ - \sin 30^\circ)$

$= \frac{1}{4}\sin 50^\circ - \frac{1}{4}\sin 50^\circ + \frac{1}{4}\cdot\frac{1}{2} = \frac{1}{8}$

$y = 2[x] + 2$ and $y = 3[x-2]$

$\Rightarrow \quad 2[x] + 2 = 3[x-2]$

$= 3[x] + 3[-2] \Rightarrow [x] = 8$

$\therefore \quad a = \frac{1}{[x]}$

● ***Ex. 36.*** *If the mapping* $f(x) = ax + b$, $a < 0$ *and maps* $[-1, 1]$ *onto* $[0, 2]$, *then for all values of* θ, $A = \cos^2\theta + \sin^4\theta$ *is such that*

(a) $f\left(\frac{1}{4}\right) \le A \le f(0)$ (b) $f(0) \le A \le f(-2)$

(c) $f\left(\frac{1}{3}\right) \le A \le f(0)$ (d) $f(-1) < A \le f(-2)$

Sol. (*a*) Given, $f(x) = ax + b$

$\therefore \quad f'(x) = a$

Since, $a < 0$, $f(x)$ is a decreasing function

$\therefore \quad f(-1) = 2$ and $f(1) = 0$

$\Rightarrow \quad -a + b = 2$ and $a + b = 0$

$\therefore \quad a = -1$ and $b = 1$

Thus, $f(x) = -x + 1$

Clearly, $f(0) = 1$, $f\left(\frac{1}{4}\right) = \frac{3}{4}$, $f(-2) = 3$,

$f\left(\frac{1}{3}\right) = \frac{2}{3}$, $f(-1) = 2$

Also, $A = \frac{1+\cos 2\theta}{2} + \left(\frac{1-\cos 2\theta}{2}\right)^2$

$= \frac{1}{2} + \frac{1}{2}\cos 2\theta + \frac{1}{4} - \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos^2 2\theta$

$= \frac{3}{4} + \frac{1}{4}\left(\frac{1+\cos 4\theta}{2}\right) = \frac{7}{8} + \frac{1}{8}\cos 4\theta$

$\therefore \quad \frac{3}{4} \le A \le 1 \Rightarrow f\left(\frac{1}{4}\right) \le A \le f(0)$

● ***Ex. 37.*** *The value of* $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}$ *is equal to*

(a) 1 (b) -1

(b) $\frac{1}{2}$ (c) $-\frac{1}{2}$

Sol. (*d*) $\cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right)$

$= \text{Re}\left\{e^{\frac{2\pi i}{7}} + e^{\frac{4\pi i}{7}} + e^{\frac{6\pi i}{7}}\right\}$

$= \frac{e^{\frac{2\pi i}{7}} + e^{\frac{4\pi i}{7}} + e^{\frac{6\pi i}{7}} + e^{\frac{-4\pi i}{7}} + e^{\frac{-4\pi i}{7}} + e^{\frac{-6\pi i}{7}}}{2}$

$= \frac{-1 + \left(1 + e^{\frac{2\pi i}{7}} + e^{\frac{4\pi i}{7}} + e^{\frac{6\pi i}{7}} + e^{\frac{-2\pi i}{7}} + e^{\frac{-4\pi i}{7}} + e^{\frac{-6\pi i}{7}}\right)}{2}$

$= \frac{-1 + (\text{Sum of seven roots of unity})}{2}$

$= \frac{-1+0}{2} = -\frac{1}{2}$

● ***Ex. 38.*** *The number of integral value of* k *for which the equation* $7\cos x + 5\sin x = 2k + 1$ *has a solution is*

(a) 4 (b) 8

(c) 10 (d) 12

Sol. (*b*) Since, $-\sqrt{a^2+b^2} \le a\sin x + b\cos x \le \sqrt{a^2+b^2}$

$\therefore \quad -\sqrt{74} \le 7\cos x + 5\sin x \le \sqrt{74}$

So, $-\sqrt{74} < 2k + 1 < \sqrt{74}$

Therefore, $2k + 1 = \pm 8, \pm 7, \pm 6, ..., \pm 1, 0$

So, $k = -4, \pm 3, \pm 2, \pm 1, 0$, so, 8 values of k.

● ***Ex. 39.*** *If* $y = \frac{\sin^4 x - \cos^4 x + \sin^2 x\cos^2 x}{\sin^4 x + \cos^4 x + \sin^2 x\cos^2 x}$, $x \in \left(0, \frac{\pi}{2}\right)$, *then*

(a) $-\frac{3}{2} \le y \le \frac{1}{2}$ (b) $1 \le y \le \frac{1}{2}$

(c) $-\frac{5}{3} \le y \le 1$ (d) None of these

Sol. (*d*) $y = \frac{\sin^4 x - \cos^4 x + \sin^2 x\cos^2 x}{\sin^4 x + \cos^4 x + \sin^2 x\cos^2 x}$

$= \frac{(\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x) + \sin^2 x\cos^2 x}{(\sin^2 x + \cos^2 x)^2 - \sin^2 x\cos^2 x}$

$$= \frac{-\cos 2x + \frac{1}{4}\sin^2 2x}{1 - \frac{1}{4}\sin^2 2x}$$

$$= \frac{-4\cos 2x + 1 - \cos^2 2x}{4 - 1 + \cos^2 2x}$$

$$= \frac{1 - 4\cos 2x - \cos^2 2x}{3 + \cos^2 2x}$$

$\Rightarrow$ $(1+y)\cos^2 2x + 4\cos 2x + 3y - 1 = 0$

Since $\cos 2x$ is real, we have

$$16 - 4(3y-1)(1+y) > 0$$

or $$3y^2 + 2y - 5 \le 0$$

or $(3y+5)(y-1) \le 0 \Rightarrow -\frac{5}{3} \le y \le 1$

But $y = 1$ implies $\cos 2x = -1$ i.e. $x = \frac{\pi}{2}$ which is not permissible.

● ***Ex. 40.*** *The distance between the two parallel lines is 1 unit. A pont 'A' is chosen to lie between the lines at a distance 'd' from one of them. Triangle ABC is equilateral with B on one line and C on the other parallel line. The length of the side of the equilateral triangle is*

(a) $\frac{2}{3}\sqrt{d^2 + d + 1}$ (b) $2\sqrt{\frac{d^2 - d + 1}{3}}$

(c) $2\sqrt{d^2 - d + 1}$ (d) $\sqrt{d^2 - d + 1}$

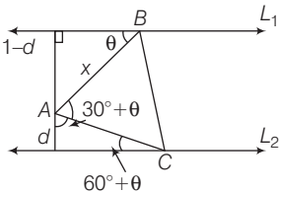
Sol. (*b*) From, figure

$$x\cos(\theta + 30^\circ) = d \quad \text{...(i)}$$

and $$x\sin\theta = 1 - d \quad \text{...(ii)}$$

Dividing $\sqrt{3}\cot\theta = \frac{1+d}{1-d}$, squaring Eq. (ii) and putting the value of $\cot\theta$, we get

$$x^2 = \frac{1}{3}(4d^2 - 4d + 4)$$



● ***Ex. 41.*** *If $a\sin x + b\cos(x+\theta) + b\cos(x-\theta) = d$, then the minimum value of $|\cos\theta|$ is equal to*

(a) $\frac{1}{2|b|}\sqrt{d^2 - a^2}$ (b) $\frac{1}{2|a|}\sqrt{d^2 - a^2}$

(c) $\frac{1}{2|d|}\sqrt{d^2 - a^2}$ (d) None of these

Sol. (*a*) $a\sin x + b\cos(x+\theta) + b\cos(x-\theta) = d$

$\Rightarrow$ $a\sin x + 2b\cdot\cos x\cdot\cos\theta = d$

$\Rightarrow$ $|d| \le \sqrt{a^2 + 4b^2\cdot\cos^2\theta}$

$\Rightarrow$ $\frac{d^2 - a^2}{4b^2} \le \cos^2\theta \Rightarrow |\cos\theta| \ge \frac{\sqrt{d^2 - a^2}}{2|b|}$

● ***Ex. 42.*** *The set of values of $\lambda \in R$ such that $\tan^2\theta + \sec\theta = \lambda$ holds for some θ is*

(a) $(-\infty, 1]$ (b) $(-\infty, -1]$

(c) ϕ (d) $[1, \infty)$

Sol. (*d*) $\because \tan^2\theta + \sec\theta = \lambda$

$\Rightarrow$ $\sec^2\theta + \sec\theta - 1 - \lambda = 0$

$\therefore$ $\sec\theta = \frac{-1 \pm \sqrt{(4\lambda + 5)}}{2}$

for real $\sec\theta$, $4\lambda + 5 \ge 0$ i.e, $\lambda \ge -\frac{5}{4}$

But we know that

$$\sec\theta \le -1 \text{ and } \sec\theta \ge 1$$

$\therefore$ $\frac{-1 \pm \sqrt{(4\lambda+5)}}{2} \le -1$ and $\frac{-1 \pm \sqrt{(4\lambda+5)}}{2} \ge 1$

$\Rightarrow$ $-1 - \sqrt{(4\lambda+5)} \le -2$ and $-1 + \sqrt{(4\lambda+5)} \ge 2$

$\Rightarrow$ $\sqrt{(4\lambda+5)} \ge 1$ and $\sqrt{(4\lambda+5)} \ge 3$

$\Rightarrow$ $\sqrt{4\lambda+5} \ge 3$

or $4\lambda + 5 \ge 9$ or $\lambda \ge 1$

$\therefore \lambda \in [1, \infty)$

● ***Ex. 43.*** *For $0 < \phi, \frac{\pi}{2}$, if $x = \sum_{n=0}^{\infty}\cos^{2n}\phi$, $y = \sum_{n=0}^{\infty}\sin^{2n}\phi$ and $z = \sum_{n=0}^{\infty}\cos^{2n}\phi\sin^{2n}\phi$, then*

(a) $xyz = xz + y$ (b) $xyz = xy + y$

(c) $xyz = x + y + z$ (d) $xyz = yz + x$

Sol. (*c*) We have,

$$x = \sum_{n=0}^{\infty}\cos^{2n}\phi = 1 + \cos^2\phi + \cos^4\phi + \ldots$$

$$= \frac{1}{1 - \cos^2\phi} = \frac{1}{\sin^2\phi}$$

Similarly, $y = \frac{1}{1 - \sin^2\phi} = \frac{1}{\cos^2\phi}$

and $z = \frac{1}{1 - \sin^2\phi\cos^2 x}$

$\therefore$ $z = \frac{1}{1 - \frac{1}{x}\cdot\frac{1}{y}} = \frac{xy}{xy - 1}$

$\Rightarrow$ $xyz = xy + z$

● ***Ex. 44.*** *If* $\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha=1, \frac{x}{a}\cos\beta+\frac{y}{b}\sin\beta=1$ *and* $\frac{\cos\alpha\cos\beta}{a^2}+\frac{\sin\alpha\sin\beta}{b^2}=0$, *then*

(a) $\tan\alpha\tan\beta=\frac{b^2(x^2+a^2)}{a^2(y^2+b^2)}$

(b) $x^2+y^2=a^2+b^2$

(c) $\tan\alpha\tan\beta=\frac{a^2}{b^2}$

(c) None of the above

Sol. (*b*) $\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha=1$ ...(i)

$$\frac{x}{a}\cos\beta+\frac{y}{b}\sin\beta=1 \quad ...(ii)$$

$$\frac{\cos\alpha\cdot\cos\beta}{a^2}+\frac{\sin\alpha\cdot\sin\beta}{b^2}=0 \quad ...(iii)$$

From Eqs. (i) and (ii), we get

$$x=a\frac{\sin\alpha-\sin\beta}{\sin(\alpha-\beta)}$$

and $$y=b\frac{\cos\beta-\cos\alpha}{\sin(\alpha-\beta)}$$

Now, $$x^2+y^2=\frac{a^2(\sin\alpha-\sin\beta)^2+b^2(\cos\beta-\cos\alpha)^2}{\sin^2(\alpha-\beta)}$$

$$=\frac{a^2(\sin^2\alpha\sin^2\beta)+b^2(\cos^2\alpha+\cos^2\beta)-2(a^2\sin\alpha\sin\beta+b^2\cos\alpha\cos\beta)}{\sin^2(\alpha-\beta)}$$

$$=\frac{a^2(\sin^2\alpha+\sin^2\beta)+b^2(\cos^2\alpha+\cos^2\beta)}{\sin^2(\alpha-\beta)}$$

[from Eq. (iii)]

$$=a^2+b^2+\frac{a^2\{\sin^2\alpha+\sin^2\beta-\sin^2(\alpha-\beta)\}+b^2\{\cos^2\alpha+\cos^2\beta-\sin^2(\alpha-\beta)\}}{\sin^2(\alpha-\beta)}$$

$$=a^2+b^2+\frac{1}{\sin^2(\alpha-\beta)}\left[a^2\begin{pmatrix}\sin^2\alpha+\sin^2\beta-\sin^2\alpha\cos^2\beta\\-\cos^2\alpha\sin^2\beta+2\sin\alpha\sin\beta\\\cos\alpha\cos\beta\end{pmatrix}+b^2\begin{Bmatrix}\cos^2\alpha(1-\sin^2\beta)\\+\cos^2\beta(1-\sin^2\alpha)\\+2\sin\alpha\sin\beta\sin\alpha\\\sin\beta\cos\alpha\cos\beta\end{Bmatrix}\right]$$

$$=a^2+b^2+\frac{2a^2\sin\alpha\sin\beta\cos(\alpha-\beta)+2b^2\cos\alpha\cos\beta\cos(\alpha-\beta)}{\sin^2(\alpha-\beta)}$$

$$=a^2+b^2+\frac{2a^2b^2\cos(\alpha-\beta)}{\sin^2(\alpha-\beta)}$$

$$\left[\frac{\sin\alpha\sin\beta}{b^2}+\frac{\cos\alpha\cos\beta}{a^2}\right] \text{ [from Eq. (iii)]}$$

$$=a^2+b^2+0=a^2+b^2$$

Thus, $x^2+y^2=a^2+b^2$

Now, $x^2+y^2=a^2+b^2$

$\Rightarrow$ $x^2-a^2=-(y^2-b^2)$

$\Rightarrow$ $\frac{x^2-a^2}{y^2-b^2}=-1$

$\Rightarrow$ $\frac{b^2(x^2-a^2)}{a^2(y^2-b^2)}=-\frac{b^2}{a^2}$

$=\tan\alpha\tan\beta$ [from Eq. (iii)]

Ex. 45. *If* α, β, γ *are acute angles and* $\cos\theta=\sin\beta/\sin\alpha$, $\cos\phi=\sin\gamma/\sin\alpha$ *and* $\cos(\theta-\phi)=\sin\beta\sin\gamma$, *then the value of* $\tan^2\alpha-\tan^2\beta-\tan^2\gamma$ *is equal to*

(a) -1 (b) 0

(c) 1 (d) 2

Sol. (*b*) From the third relation we get

$$\cos\theta\cos\phi+\sin\theta\sin\phi=\sin\beta\sin\gamma$$

$\Rightarrow$ $\sin^2\theta\sin^2\phi=(\cos\theta\cos\phi-\sin\beta\sin\gamma)^2$

$\Rightarrow$ $\left(1-\frac{\sin^2\beta}{\sin^2\alpha}\right)\left(1-\frac{\sin^2\gamma}{\sin^2\alpha}\right)=\left(\frac{\sin\beta\sin\gamma}{\sin^2\alpha}-\sin\beta\sin\gamma\right)^2$

$\Rightarrow$ $(\sin^2\alpha-\sin^2\beta)(\sin^2\alpha-\sin^2\gamma)=\sin^2\beta\sin^2\gamma(1-\sin^2\alpha)^2$

$\Rightarrow$ $\sin^4\alpha(1-\sin^2\beta\sin^2\gamma)-\sin^2\alpha(\sin^2\beta+\sin^2\gamma-2\sin^2\beta\sin^2\gamma)=0$

$\therefore$ $\sin^2\alpha=\frac{\sin^2\beta+\sin^2\gamma-2\sin^2\beta\sin^2\gamma}{1-\sin^2\beta\sin^2\gamma}$

and $\cos^2\alpha=\frac{1-\sin^2\beta-\sin^2\gamma+\sin^2\beta\sin^2\gamma}{1-\sin^2\beta\sin^2\gamma}$

$\Rightarrow$ $\tan^2\alpha=\frac{\sin^2\beta-\sin^2\beta\sin^2\gamma+\sin^2\gamma-\sin^2\beta\sin^2\gamma}{\cos^2\beta-\sin^2\gamma(1-\sin^2\beta)}$

$=\frac{\sin^2\beta\cos^2\gamma+\cos^2\beta\sin^2\gamma}{\cos^2\beta\cos^2\gamma}$

$=\tan^2\beta+\tan^2\gamma$

$\Rightarrow$ $\tan^2\alpha-\tan^2\beta-\tan^2\gamma=0$

● **Ex. 46.** *If* $\sqrt{2}\cos A = \cos B + \cos^3 B$, *and* $\sqrt{2}\sin A = \sin B - \sin^3 B$ *then* $\sin(A-B) =$

(a) ± 1 (b) $\pm \frac{1}{2}$

(c) $\pm \frac{1}{3}$ (d) $\pm \frac{1}{4}$

Sol. (*c*) $\sqrt{2}\cos A = \cos B + \cos^3 B$ ...(i)

and $\sqrt{2}\sin A = \sin B - \sin^3 B$ ...(ii)

$\Rightarrow \sqrt{2}\sin A\cos B - \sqrt{2}\cos A\sin B$

$= (\sin B - \sin^3 B)\cos B - (\cos B + \cos^3 B)\sin B$

$= -\sin B\cos B$

$\Rightarrow \sin(A-B) = \frac{-1}{2\sqrt{2}}\sin 2B$

Now squaring and adding Eqs. (i) and (ii), we get

$2 = \cos^2 B + \sin^2 B + \cos^6 B + \sin^6 B + 2(\cos^4 B - \sin^4 B)$

$\Rightarrow 1 = (\cos^2 A + \sin^2 A)^3 - 3\cos^2 A\sin^2 A(\cos^2 A + \sin^2 A) + 2\cos 2B$

$\Rightarrow 1 = 1 - \left(\frac{3}{4}\right)\sin^2 2B + 2\cos 2B$

$\Rightarrow -3\sin^2 2B + 8\cos 2B = 0$

$\Rightarrow 3\cos^2 2B + 8\cos 2B - 3 = 0$

$\Rightarrow \cos 2B = \frac{1}{3}$

$\Rightarrow \sin 2B = \pm\frac{2\sqrt{2}}{3}$

$\therefore \sin(A-B) = \pm\frac{1}{3}$

● **Ex. 47.** *If* x_1 *and* x_2 *are two distinct roots of the equation* $a\cos x + b\sin x = c$, *then* $\tan\frac{x_1 + x_2}{2}$ *is equal to*

(a) $\frac{a}{b}$ (b) $\frac{b}{a}$

(c) $\frac{c}{a}$ (d) $\frac{a}{c}$

Sol. (*b*) $a\cos x + b\sin x = c$

$\Rightarrow \frac{a\left(1-\tan^2\frac{x}{2}\right)}{1+\tan^2\frac{x}{2}} + \frac{2b\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = c$

$\Rightarrow (c+a)\tan^2\frac{x}{2} - 2b\tan\frac{x}{2} + c - a = 0$

$\Rightarrow \tan\frac{x_1}{2} + \tan\frac{x_2}{2} = \frac{2b}{c+a}$,

$\tan\frac{x_1}{2}\cdot\tan\frac{x_2}{2} = \frac{c-a}{c+a}$

Thus, $\tan\left(\frac{x_1+x_2}{2}\right) = \frac{\tan\frac{x_1}{2} + \tan\frac{x_2}{2}}{1 - \tan\frac{x_1}{2}\tan\frac{x_2}{2}}$

$= \frac{\frac{2b}{c+a}}{1-\left(\frac{c-a}{c+a}\right)} = \frac{b^2}{a}$

● **Ex. 48.** *The minimum value of the function*

$f(x) = \frac{\sin x}{\sqrt{1-\cos^2 x}} + \frac{\cos x}{\sqrt{1-\sec^2 x}} + \frac{\tan x}{\sqrt{\sec^2 x - 1}} + \frac{\cot x}{\sqrt{\operatorname{cosec}^2 x - 1}}$ *whenever it is defined is*

(a) 4 (b) -2

(c) 0 (d) 2

Sol. (*b*) $f(x) = \frac{\sin x}{\sqrt{1-\cos^2 x}} + \frac{\cos x}{\sqrt{1-\sin^2 x}} + \frac{\tan x}{\sqrt{\sec^2 x - 1}} + \frac{\cot x}{\sqrt{\operatorname{cosec}^2 x - 1}}$

$= \frac{\sin x}{|\sin x|} + \frac{\cos x}{|\cos x|} + \frac{\tan x}{|\tan x|} + \frac{\cot x}{|\cot x|}$

$= \begin{cases} 4, & x \in \text{1st quadrant} \\ -2, & x \in \text{2nd quadrant} \\ 0, & x \in \text{3rd quadrant} \\ -2, & x \in \text{4th quadrant} \end{cases}$

$f(x)_{\text{min.}} = -2$

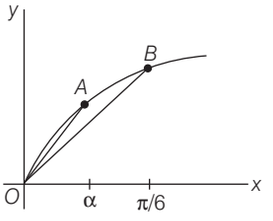
● **Ex. 49.** *If* $0 < \alpha < \frac{\pi}{6}$, *then* $\alpha(\operatorname{cosec}\alpha)$ *is*

(a) less than $\frac{\pi}{6}$ (b) greater than $\frac{\pi}{6}$

(c) less than $\frac{\pi}{3}$ (d) greater than $\frac{\pi}{3}$

Sol. (*c*) In the graph of $y = \sin x$. Let

$A \equiv (\alpha, \sin\alpha)$, $B = \left(\frac{\pi}{6}, \sin\frac{\pi}{6}\right)$



Clearly, slope of OA > slope of OB, so

$$\frac{\sin\alpha}{\alpha} > \frac{\sin\frac{\pi}{6}}{\frac{\pi}{6}} = \frac{3}{\pi} \Rightarrow \frac{\alpha}{\sin\alpha} < \frac{\pi}{3}.$$

● ***Ex. 50.*** *In which one of the following intervals the inequality* $\sin x < \cos x < \tan x < \cot x$ *can hold good?*

(a) $\left(\frac{7\pi}{4}, 2\pi\right)$ (b) $\left(\frac{3\pi}{4}, \pi\right)$

(c) $\left(\frac{5\pi}{4}, \frac{3\pi}{2}\right)$ (d) $\left(0, \frac{\pi}{4}\right)$

Sol. (d) In the second quadrant, $\sin x < \cos x$ is false, as $\sin x$ is positive and $\cos x$ is negative.

In the fourth quadrant, $\cos x < \tan x$ is false, as $\cos x$ is positive and $\tan x$ is negative.

In the third quadrant, i.e. $\left(\frac{5\pi}{4}, \frac{3\pi}{2}\right)$ if $\tan x < \cot x$ then $\tan^2 x < 1$, which is false.

Now, $\sin x < \cos x$ is true in $\left(0, \frac{\pi}{4}\right)$ and $\tan x < \cot x$ is also true.

Further, $\cos x < \tan x$, as $\tan x = \frac{(\sin x)}{(\cos x)}$ and $\cos x < 1$.

JEE Type Solved Examples :
More than One Correct Option Type Questions

● ***Ex. 51.*** *If* $x \in (0, \pi)$ *and* $\cos x + \sin x = \frac{1}{2}$, *then* $\tan x$ *is equal to*

(a) $\frac{4-\sqrt{7}}{3}$ (b) $\frac{4+\sqrt{7}}{3}$

(c) $\frac{-(4+\sqrt{7})}{3}$ (d) $\frac{-4+\sqrt{7}}{3}$

Sol. (c,d) Given, $\cos x + \sin x = \frac{1}{2}$

$$\Rightarrow \quad 1 + \sin 2x = \frac{1}{4}$$

$$\sin 2x = -\frac{3}{4} \quad \Rightarrow \quad 2x \in (\pi, 2\pi)$$

$$\Rightarrow \quad x \in \left(\frac{\pi}{2}, \pi\right) \quad \Rightarrow \quad \tan x < 0$$

$$\frac{2t}{1+t^2} = -\frac{3}{4} \quad \Rightarrow \quad 8t = -3 - 3t^2$$

$\Rightarrow 3t^2 + 8t + 3 = 0$, where $t = \tan x$

$$t = \frac{-8 \pm \sqrt{64-36}}{2\cdot 3};$$

$$t = \frac{-8 \pm \sqrt{28}}{2\cdot 3};$$

$$t = \frac{-(4+\sqrt{7})}{3}$$

or $$= \frac{-4+\sqrt{7}}{3}$$

● ***Ex. 52.*** *The value of the expression* $\tan\frac{\pi}{7} + 2\tan\frac{2\pi}{7} + 4\tan\frac{4\pi}{7} + 8\cot\frac{8\pi}{7}$ *is equal to*

(a) $\operatorname{cosec}\frac{2\pi}{7} + \cot\frac{2\pi}{7}$ (b) $\tan\frac{\pi}{14} - \cot\frac{\pi}{14}$

(c) $\frac{\sin\frac{2\pi}{7}}{1-\cos\frac{2\pi}{7}}$ (d) $\frac{1+\cos\frac{\pi}{7}+\cos\frac{2\pi}{7}}{\sin\frac{\pi}{7}+\sin\frac{2\pi}{7}}$

Sol. (a,c,d) $\tan\frac{\pi}{7} + 2\tan\frac{2\pi}{7} + 4\tan\frac{4\pi}{7} + 8\cot\frac{8\pi}{7} = \cot\frac{\pi}{7}$

$[\tan\theta + 2\tan 2\theta + 4\tan 4\theta + 8\cot 8\theta = \cot\theta \text{ when } \theta = \frac{\pi}{7}]$

(a) $\operatorname{cosec} 2\theta + \cot 2\theta = \frac{1+\cot 2\theta}{\sin 2\theta} = \cot\theta = \cot\frac{\pi}{7}$

(where, $\theta = \frac{\pi}{7}$)

(b) $\tan\frac{\pi}{14} - \cot\frac{\pi}{14} = -2\cot\frac{\pi}{7}$

(c) $\frac{\sin\frac{2\pi}{7}}{1-\cos\frac{2\pi}{7}} = \frac{2\sin\frac{\pi}{7}\cos\frac{\pi}{7}}{2\sin^2\frac{\pi}{7}} = \cot\frac{\pi}{7}$

(d) $\frac{\left(1+\cos\frac{2\pi}{7}\right)+\cos\frac{\pi}{7}}{2\sin\frac{\pi}{7}\cos\frac{\pi}{7}+\sin\frac{\pi}{7}} = \frac{2\cos^2\frac{\pi}{7}+\cos\frac{\pi}{7}}{2\sin\frac{\pi}{7}\left(\cos\frac{\pi}{7}+1\right)}$

$$= \cot\frac{\pi}{7}$$

● ***Ex. 53.*** *Two parallel chords are drawn on the same side of the centre of a circle of radius R. It is found that they subtend and angle of* θ *and* 2θ *at the centre of the circle. The perpendicular distance between the chords is*

(a) $2R\sin\frac{3\theta}{2}\sin\frac{\theta}{2}$

(b) $\left(1-\cos\frac{\theta}{2}\right)\left(1+2\cos\frac{\theta}{2}\right)R$

(c) $\left(1+\cos\frac{\theta}{2}\right)\left(1-2\cos\frac{\theta}{2}\right)R$

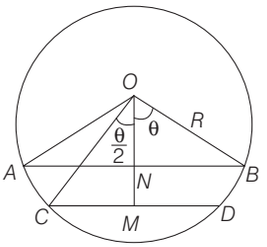
(d) $2R\sin\frac{3\theta}{4}\sin\frac{\theta}{4}$

Sol. (b,d) $OM = p_1 = R\cos\frac{\theta}{2}$

$ON = p_2 = R\cos\theta$

$MN = p_1 - p_2 = R\left(\cos\frac{\theta}{2} - \cos\theta\right)$

$= R\,2\sin\frac{3\theta}{4}\sin\frac{\theta}{4}$ (d)



Again convert $\cos\theta = 2\cos^2\frac{\theta}{2} - 1$ and factorise, we get

$= R(1-\cos\theta/2)^2\,(1+2\cos\theta/2)$

● ***Ex. 54.*** *If* $2x$ *and* $2y$ *are complementary angles and* $\tan(x+2y)=2$*, then which of the following is(are) correct ?*

(a) $\sin(x+y)=\frac{1}{2}$ (b) $\tan(x-y)=\frac{1}{7}$

(c) $\cot x + \cot y = 5$ (d) $\tan x\,\tan y = 6$

Sol. (b,c) We have, $2x + 2y = \frac{\pi}{2}$

$\Rightarrow \quad x + y = \frac{\pi}{4} \Rightarrow \sin(x+y) = \frac{1}{\sqrt{2}}$

Also, $y = \left(\frac{\pi}{4} - x\right)$,

So, $\tan(x+2y) = \tan\left(x + \frac{\pi}{2} - 2x\right)$

$= \tan\left(\frac{\pi}{2} - x\right) = \cot x$

$\therefore \quad 2 = \cot x \Rightarrow \tan x = \frac{1}{2}$

Similarly, $x = \left(\frac{\pi}{4} - y\right)$

So, $\tan(x+2y) = \tan\left(\frac{\pi}{4} + y\right) = \frac{1+\tan y}{1-\tan y}$

$\Rightarrow \quad 2 = \frac{1+\tan y}{1-\tan y} \Rightarrow \tan y = \frac{1}{3}$

$\cot y = 3$

Also, $\tan(x-y) - \frac{\tan x - \tan y}{1+\tan x\,\tan y} = \frac{\frac{1}{2}-\frac{1}{3}}{1+\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)}$

$= \left(\frac{1}{6}\right)\left(\frac{6}{7}\right) = \frac{1}{7}$

Now, verify alternatives.

● ***Ex. 55.*** *If* $2\cos\theta + 2\sqrt{2} = 3\sec\theta$*, where* $\theta \in (0, 2\pi)$*, then which of the following can be correct?*

(a) $\cos\theta = \frac{1}{\sqrt{2}}$ (b) $\tan\theta = 1$

(c) $\sin\theta = -\frac{1}{\sqrt{2}}$ (d) $\cot\theta = -1$

Sol. (a,b,c,d) $2\cos\theta + 2\sqrt{2} = 3\sec\theta$

$\therefore \quad 2\cos^2\theta + 2\sqrt{2}\cos\theta - 3 = 0$

$\cos\theta = \frac{-2\sqrt{2} \pm \sqrt{32}}{4} = \frac{-2\sqrt{2} \pm 4\sqrt{2}}{4}$

$\therefore \quad \cos\theta = \frac{1}{\sqrt{2}}$ or $\cos\theta = -\frac{3}{\sqrt{2}}$ (rejected)

$\therefore \quad \theta = \frac{\pi}{4}$ or $\frac{7\pi}{4} \Rightarrow \sin\theta = -\frac{1}{\sqrt{2}}$;

$\cot\theta = -1$; $\tan\theta = 1$

● ***Ex. 56.*** *The value of* x *in* $(0, \pi/2)$ *satisfying the equation,* $\frac{\sqrt{3}-1}{\sin x} + \frac{\sqrt{3}+1}{\cos x} = 4\sqrt{2}$ *is*

(a) $\frac{\pi}{12}$ (b) $\frac{5\pi}{12}$

(c) $\frac{7\pi}{24}$ (d) $\frac{11\pi}{36}$

Sol. (a,d) $\frac{\sqrt{3}-1}{2\sqrt{2}\sin x} + \frac{\sqrt{3}+1}{2\sqrt{2}\cos x} = 2$

$\sin\frac{\pi}{12}\cos x + \cos\frac{\pi}{12}\sin x = \sin 2x$

$\sin 2x = \sin\left(x + \frac{\pi}{12}\right)$

$\therefore \quad 2x = x + \frac{\pi}{12}$

or $\quad 2x = \pi - x - \frac{\pi}{12}$

$x = \frac{\pi}{12}$

or $\quad 3x = \frac{11\pi}{12}$

$\therefore \quad x = \frac{\pi}{12}$ or $\frac{11\pi}{36}$

● ***Ex. 57.*** *Which of the following statements are always correct? (where, Q denotes the set of rationals)*

(a) $\cos 2\theta \in Q$ and $\sin 2\theta \in Q \Rightarrow \tan\theta \in Q$ (if defined)

(b) $\tan\theta \in Q \Rightarrow \sin 2\theta$, $\cos 2\theta$ and $\tan 2\theta \in Q$ (if defined)

(c) If $\sin\theta \in Q$ and $\cos\theta \in Q \Rightarrow \tan 3\theta \in Q$ (if defined)

(d) If $\sin\theta \in Q \Rightarrow \cos 3\theta \in Q$

Sol. (a,b,c)

(a) $\tan\theta = \dfrac{1-\cos 2\theta}{\sin 2\theta} \Rightarrow$ (a) is correct

(b) $\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$; $\cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$;

$\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta} \Rightarrow$ (b) is correct

(c) $\tan 3\theta = \dfrac{\sin 3\theta}{\cos 3\theta} \Rightarrow$ (c) is correct

(d) $\sin\theta = \dfrac{1}{3}$ which is rational but

$\cos 3\theta = \cos\theta(4\cos^2\theta - 3)$ which is irrational $\Rightarrow$ (d) is incorrect.

● ***Ex. 58.*** *In* ΔABC, $\tan B + \tan C = 5$ *and* $\tan A \tan C = 3$, *then*

(a) ΔABC is an acute angled triangle

(b) ΔABC is an obtuse angled triangle

(c) sum of all possible values of $\tan A$ is 10

(d) sum of all possible values of $\tan A$ is 9

Sol. (a,c) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

$\Rightarrow \quad \tan A + 5 = 3\tan B$

$\Rightarrow \quad 5 + \tan A = 3(5 - \tan C)$

$\Rightarrow \quad 5 + \tan A = 15 - \dfrac{9}{\tan A}$

$\Rightarrow \quad \tan^2 A - 10\tan A + 9 = 0$

$\Rightarrow \quad \tan A = 1$ or $\tan A = 9$

$\Rightarrow \tan B$ and $\tan C$ are 2, 3 or $\dfrac{14}{3}, \dfrac{1}{3}$, respectively

$\Rightarrow \Delta ABC$ is always on acute angled triangle and sum of all possible values of $\tan A$ is 10.

● ***Ex. 59.*** $(m+2)\sin\theta + (2m-1)\cos\theta = 2m+1$, *if*

(a) $\tan\theta = \dfrac{3}{4}$ (b) $\tan\theta = \dfrac{4}{3}$

(c) $\tan\theta = \dfrac{2m}{(m^2-1)}$ (d) $\tan\theta = \dfrac{2m}{(m^2+1)}$

Sol. (b, c) The given relation can be written as

$(m+2)\tan\theta + (2m-1) = (2m+1)\sec\theta$

$\Rightarrow (m+2)^2\tan^2\theta + 2(m+2)(2m-1)\tan\theta + (2m-1)^2$

$= (2m+1)^2(1+\tan^2\theta)$

$\Rightarrow [(m+2^2) - (2m+1)^2]\tan^2\theta + 2(m+2)\tan\theta + (2m-1)^2$

$- (2m+1)^2 = 0$

$\Rightarrow 3(1-m^2)\tan^2\theta + (4m^2 + 6m - 4)\tan\theta - 8m = 0$

$\Rightarrow \quad (3\tan\theta - 4)[(1-m)^2\tan\theta + 2m] = 0$

which is true if $\tan\theta = \dfrac{4}{3}$ or $\tan\theta = \dfrac{2m}{(m^2-1)}$.

● ***Ex. 60.*** *If* $x\cos\alpha + y\sin\alpha = x\cos\beta + y\sin\beta = 2a\left(0 < \alpha, \beta < \dfrac{\pi}{2}\right)$, *then*

(a) $\cos\alpha + \cos\beta = \dfrac{4ax}{x^2+y^2}$

(b) $\cos\alpha\cos\beta = \dfrac{4a^2 - y^2}{x^2+y^2}$

(c) $\sin\alpha + \sin\beta = \dfrac{4ay}{x^2+y^2}$

(d) $\sin\alpha\sin\beta = \dfrac{4a^2 - x^2}{x^2+y^2}$

Sol. (a, b, c, d) We find out the given relations that α and β are the roots of the equation

$x\cos\theta + y\sin\theta = 2a$

$\Rightarrow \quad (x\cos\theta - 2a)^2 = (-y\sin\theta)^2$

$\Rightarrow \quad x^2\cos^2\theta - 4ax\cos\theta + 4a^2$

$= y^2\sin^2\theta = y^2(1-\cos^2\theta)$

$\Rightarrow \quad (x^2+y^2)\cos^2\theta - 4ax\cos\theta + 4a^2 - y^2 = 0$

which, being quadratic in $\cos\theta$, has two roots $\cos\alpha$ and $\cos\beta$, such that

$$\cos\alpha + \cos\beta = \frac{4ax}{x^2+y^2}$$

and $$\cos\alpha\cos\beta = \frac{4a^2 - y^2}{x^2+y^2}$$

Similarly, we can write (1) as a quadratic in $\sin\theta$, giving two values $\sin\alpha$ and $\sin\beta$, such that

$$\sin\alpha + \sin\beta = \frac{4ay}{x^2+y^2}$$

and $$\sin\alpha\sin\beta = \frac{4a^2 - x^2}{x^2+y^2}.$$

● ***Ex. 61.*** *Let* $y = \sin^2 x + \cos^4 x$. *Then, for all real x*

(a) the maximum value of y is 2

(b) the minimum value of y is $\dfrac{3}{4}$

(c) $y \le 1$

(d) $y \ge \dfrac{1}{4}$

Sol. (b, c) $y = \cos^4 x - \cos^2 x + 1$

$$= \left(\cos^2 x - \frac{1}{2}\right)^2 + \frac{3}{4}$$

$\therefore y_{\min} = \frac{3}{4}$ and y is maximum when $\left(\cos^2 x - \frac{1}{2}\right)^2$ is then maximum

$\therefore \quad y_{\max} = \frac{1}{4} + \frac{3}{4} = 1$

● **Ex. 62.** *If in* ΔABC, $\tan A + \tan B + \tan C = 6$ *and* $\tan A \tan B = 2$, *then* $\sin^2 A : \sin^2 B : \sin^2 C$ *is*

(a) 8 : 9 : 5 (b) 8 : 5 : 9
(c) 5 : 9 : 5 (d) 5 : 8 : 5

Sol. (*b, c*) $\tan A + \tan B + \tan C = 6$ ...(i)

$\Rightarrow \tan A \tan B \tan C = 6$

$2 \tan C = 6$

$\therefore \quad \tan C = 3$

$\therefore \quad \sin^2 C = \frac{\tan^2 C}{1 + \tan^2 C} = \frac{9}{1+9} = \frac{9}{10}$

From Eq. (i), $\tan A + \tan B = 3$ and $\tan A \tan B = 2$

$\tan A - \tan B$

$= \pm \sqrt{\{(\tan A + \tan B)^2 - 4 \tan A \tan B)\}}$

$= \pm 1$

we get, $\tan A = 2, 1$ and $\tan B = 1, 2$

$\therefore \quad \sin^2 A = \frac{4}{1+4}, \frac{1}{1+1}$ and $\sin^2 B = \frac{1}{1+1}, \frac{4}{1+4}$

$\Rightarrow \quad \sin^2 A = \frac{8}{10}, \frac{5}{10}$ and $\sin^2 B = \frac{5}{10}, \frac{8}{10}$

$\therefore \quad \sin^2 A : \sin^2 B : \sin^2 C = 8 : 5 : 9$ or $5 : 8 : 9$

● **Ex. 63.** *If* $0 \le x, y \le 180°$ *and* $\sin(x - y) = \cos(x + y) = \frac{1}{2}$, *then the values of* x *and* y *are given by*

(a) $x = 45°, y = 15°$ (b) $x = 45°, y = 135°$
(c) $x = 165°, y = 15°$ (d) $x = 165°, y = 135°$

Sol. (*a, d*) $\sin(x - y) = \frac{1}{2} \Rightarrow x - y = 30°$ or $150°$ (1)

and $\cos(x + 1) = \frac{1}{2} \Rightarrow x + y = 60°$ or $300°$ (2)

Since x and y lie between $0°$ and $180°$, (1) and (2) are simultaneously true when $x = 45°$, $y = 15°$, or $x = 165°$, $y = 135°$. But, for the values given by (b) or (c), (1) and (2) do not hold simultaneously.

● **Ex. 64.** *If* $\sin\alpha + \sin\beta = l, \cos\alpha \cos\beta = m$ *and* $\tan\left(\frac{\alpha}{2}\right) \tan\left(\frac{\beta}{2}\right) = n (\neq 1)$, *then*

(a) $\cos(\alpha - \beta) = \frac{l^2 + m^2 - 2}{2}$

(b) $\cos(\alpha + \beta) = \frac{m^2 - l^2}{m^2 + l^2}$

(c) $\frac{1+n}{1-n} = \frac{l^2 + m^2}{2n}$

(d) $\alpha + \beta = \frac{\pi}{2}$ if $l = m$

Sol. (*a, b, c, d*) Now, $l^2 = \sin^2\alpha + \sin^2\beta + 2\sin\alpha \sin\beta$ and $m^2 = \cos^2\alpha + \cos^2\beta + 2\cos\alpha \cos\beta$

$2\cos(\alpha - \beta) = l^2 + m^2 - 2$ (by adding)

$\Rightarrow \quad 2\cos 2\alpha + \cos 2\beta) = m^2 - l^2$ (by subtracting)

$\Rightarrow \quad 2\cos(\alpha + \beta)\cos(\alpha - \beta) + 2\cos(\alpha + \beta) = m^2 - l^2$

$\Rightarrow \quad \cos(\alpha + \beta) = \frac{m^2 - l^2}{m^2 + l^2}.$

● **Ex. 65.** *Let* $f(x) = ab \sin x + b\sqrt{1 - a^2} \cos x + c$, *where* $|a| < 1, b > 0$ *then*

(a) maximum value of $f(x)$ if b is $c = 0$
(b) difference of maximum and minimum values of $f(x)$ is $2b$
(c) $f(x) = c$ if $x = -\cos^{-1} a$
(d) $f(x) = c$ if $x = \cos^{-1} a$

Sol. (*a, b, c*) $f(x) = ab \sin x + b\sqrt{1 - a^2} \cos x + c$, where $|a| < 1, b < 0$

$f(x) = \sqrt{a^2b^2 + b^2 - b^2a^2} \sin(x + \alpha) + c$

$= b \sin(x + \alpha) + c$, where $\tan\alpha = \frac{b\sqrt{1-a^2}}{ab} = \frac{\sqrt{1-a^2}}{a}$

$= b \cos(x - \alpha) + c$, where $\tan\alpha = \frac{ab}{b\sqrt{1-a^2}}$

$= \frac{a}{\sqrt{1-a^2}}$

$f(x)_{\max} - f(x)_{\min} = c + b - (c - b) = 2b$

$f(x) = c$ if $x + \alpha = 0$

or $\quad x = -\alpha$ or $x = -\cos^{-1} a$

● **Ex. 66.** *If* $(x - a)\cos\theta + y \sin\theta = (x - a)\cos\phi + y \sin\phi = a$ *and* $\tan\left(\frac{\theta}{2}\right) - \tan\left(\frac{\phi}{2}\right) = 2b$, *then*

(a) $y^2 = 2ax - (1 - b^2)x^2$ (b) $\tan\frac{\theta}{2} = \frac{1}{x}(y + bx)$
(c) $y^2 = 2bx - (1 - a^2)x^2$ (d) $\tan\frac{\phi}{2} = \frac{1}{x}(y - bx)$

Sol. (*a, b*) Let, $\tan\left(\frac{\theta}{2}\right) = \alpha$ and $\tan\left(\frac{\phi}{2}\right) = \beta$, so that $\alpha - \beta = 2b$.

Also, $\quad \cos\theta = \frac{1 - \tan^2\left(\frac{\theta}{2}\right)}{1 + \tan^2\left(\frac{\theta}{2}\right)} = \frac{1 - \alpha^2}{1 + \alpha^2}$

And $\sin\theta = \dfrac{2\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)} = \dfrac{2\alpha}{1+\alpha^2}$

Similarly, $\cos\phi = \dfrac{1-\beta^2}{1+\beta^2}$ and $\sin\phi = \dfrac{2\beta}{1+\beta^2}$

Therefore, we have from the given relations

$$(x-a)\frac{1-\alpha^2}{1+\alpha^2} + y\left(\frac{2\alpha}{1+\alpha^2}\right) = a$$

$\Rightarrow \quad x\alpha^2 - 2y\alpha + 2a - x = 0$

Similarly $\quad x\beta^2 - 2y\beta + 2a = 0$

We see that α and β are roots of the equation

$$xz^2 - 2yz + 2a - x = 0,$$

So that $\alpha + \beta = \dfrac{2y}{x}$ and $\alpha\beta = \dfrac{(2a-x)}{x}$.

Now, from $(\alpha+\beta)^2 = (\alpha-\beta)^2 + 4\alpha\beta$, we get

$\Rightarrow \quad \left(\dfrac{2y}{x}\right)^2 = (2b)^2 + \dfrac{4(2a-x)}{x}$

$\Rightarrow \quad y^2 = 2ax - (1-b^2)x^2$

Also, from $\alpha + \beta = \dfrac{2y}{x}$ and $\alpha - \beta = 2b$, we get

$$\alpha = \frac{y}{x} + b \text{ and } \beta = \frac{y}{x} - b$$

$\Rightarrow \quad \tan\dfrac{\theta}{2} = \dfrac{1}{x}(y + bx)$

and $\quad \tan\dfrac{\phi}{2} = \dfrac{1}{x}(y - bx)$

● ***Ex. 67.*** *If* $\cos(\beta-\gamma) + \cos(\gamma-\alpha) + \cos(\alpha-\beta) = -\dfrac{3}{2}$ *then*

(a) $\Sigma\cos\alpha = 0$ (b) $\Sigma\sin\alpha = 0$

(c) $\Sigma\cos\alpha\sin\alpha = 0$ (d) $\Sigma(\cos\alpha + \sin\alpha) = 0$

Sol. (*a, b, d*) The given expression can be written as

$$2[\cos\beta\cos\gamma + \cos\gamma\cos\alpha + \cos\alpha\cos\beta] + 2[\sin\beta\sin\gamma + \sin\gamma\sin\alpha + \sin\alpha\sin\beta] + (\sin^2\alpha + \cos^2\alpha) + (\sin^2\beta + \cos^2\beta) + (\sin^2\gamma + \cos^2\gamma) = 0$$

$\Rightarrow (\cos\alpha + \cos\beta + \cos\gamma)^2 + (\sin\alpha + \sin\beta + \sin\gamma)^2 = 0$

$\Rightarrow \quad \Sigma\cos\alpha = 0$ and $\Sigma\sin\alpha = 0$

$\Rightarrow \quad \Sigma(\cos\alpha + \sin\alpha) = 0$

JEE Type Solved Examples :
Statement I and II Type Questions

● ***Ex. 68.*** **Statement I** $\tan 5\theta - \tan 3\theta - \tan 2\theta = \tan 5\theta \tan 3\theta \tan 2\theta$.

Statement II $x = y + z$

$\Rightarrow \tan x - \tan y - \tan z = \tan x \tan y \tan z$

(a) A (b) B

(c) C (d) D

Sol. (*a*) $\because 5\theta = 3\theta + 2\theta$

$$\Rightarrow \quad \tan 5\theta = \tan(3\theta + 2\theta) = \frac{\tan 3\theta + \tan 2\theta}{1 - \tan 3\theta \tan 2\theta}$$

$\Rightarrow \tan 5\theta - \tan 5\theta \tan 3\theta \tan 2\theta = \tan 3\theta + \tan 2\theta$

$\Rightarrow \quad \tan 5\theta - \tan 3\theta - \tan 2\theta = \tan 5\theta \tan 3\theta \tan 2\theta$

● ***Ex. 69.*** **Statement I** *The maximum value of* $\sin\theta + \cos\theta$ *is 2.*

Statement II *The maximum value of* $\sin\theta$ *is 1 and that of* $\cos\theta$ *is also 1.*

(a) A (b) B

(c) C (d) D

Sol. (*d*) $\because -\sqrt{2} \le \sin\theta + \cos\theta \le \sqrt{2}$

$\therefore$ Maximum value of $\sin\theta + \cos\theta$ is $\sqrt{2}$

But maximum value of $\sin\theta$ is 1 and that of $\cos\theta$ is also 1 which is always true.

● ***Ex. 70.*** **Statement I** *If* $a, b, c \in R$ *and not all equal, then* $\sec\theta = \dfrac{(bc+ca+ab)}{(a^2+b^2+c^2)}$,

Statement II $\sec\theta \le -1$ *and* $\sec\theta \ge 1$

(a) A (b) B

(c) C (d) D

Sol. (*d*) $\because a^2 + b^2 + c^2 - ab - bc - ca$

$$= \frac{1}{2}\{(a-b)^2 + (b-c)^2 + (c-a)^2\} > 0$$

$\Rightarrow \quad a^2 + b^2 + c^2 > ab + bc + ca$

or $\quad \dfrac{ab+bc+ca}{a^2+b^2+c^2} < 1$

$\Rightarrow \quad \sec\theta < 1$, which is false.

● ***Ex. 71.*** **Statement I** $\prod_{r=1}^{n}(1 + \sec 2^r\theta) = \tan 2^n\theta \cot\theta$

Statement II $\prod_{r=1}^{n}\cos(2^{r-1}\theta) = \dfrac{\sin(2^n\theta)}{2^n\sin\theta}$

(a) A (b) B

(c) C (d) D

Sol. (*a*) $\because \prod_{r=1}^{n}(1+\sec 2^r\theta)=\dfrac{\prod_{r=1}^{n}(1+\cos 2^r\theta)}{\prod_{r=1}^{n}\cos 2^r\theta}$

$$=\frac{\prod_{r=1}^{n}2\cos^2(2^{r-1}\theta)}{\prod_{r=1}^{n}\cos(2^r\theta)}$$

$$=\frac{2^n\cdot\prod_{r=1}^{n}\cos(2^{r-1}\theta)\prod_{r=1}^{n}\cos(2^{r-1}\theta)}{\frac{\cos(2^n\theta)}{\cos\theta}\prod_{r=1}^{n}\cos(2^{r-1}\theta)}$$

$$=\frac{\frac{2^n\cdot\sin(2^n\theta)}{2^n\sin\theta}\cdot\cos\theta}{\cos(2^n\theta)}$$

$$=\tan(2^n\theta)\cdot\cot\theta$$

- ***Ex. 72.*** **Statement I** $\cos 36^\circ > \sin 36^\circ$

Statement II $\cos 36^\circ > \tan 36^\circ$

(a) A (b) B
(c) C (d) D

Sol. (*b*) Since, $\cos\theta > \sin\theta$ for $0 \le \theta \le \dfrac{\pi}{4}$

So, Statement I is true.

Now, $\cos 36^\circ > \tan 36^\circ$

$\Rightarrow \cos 36^\circ > \dfrac{\sin 36^\circ}{\cos 36^\circ}$

$\Rightarrow \cos^2 36^\circ > \sin 36^\circ$

$\Rightarrow 1+\cos 72^\circ > 2\sin 36^\circ = 2\sin(30^\circ+6^\circ)$

$\Rightarrow 1+2\sin 9^\circ\cos 9^\circ > \cos 6^\circ + 2\cos 30^\circ\sin 6^\circ$

which is true

- ***Ex. 73.*** **Statement I** $\cos^3\alpha+\cos^3\left(\alpha+\dfrac{2\pi}{3}\right)+\cos^3\left(\alpha+\dfrac{4\pi}{3}\right)$

$$=3\cos\alpha\cos\left(\alpha+\frac{2\pi}{3}\right)\cos\left(\alpha+\frac{4\pi}{3}\right)$$

Statement II *If* $a+b+c=0 \Leftrightarrow a^3+b^3+c^3=3abc$

(a) A (b) B
(c) C (d) D

Sol. (*a*) $\because \cos\alpha+\cos\left(\alpha+\dfrac{2\pi}{3}\right)+\cos\left(\alpha+\dfrac{4\pi}{3}\right)$

$$=\cos\alpha+2\cos(\alpha+\pi)\cos\frac{\pi}{3}$$

$$=\cos\alpha+(-2\cos\alpha)\left(\frac{1}{2}\right)=0$$

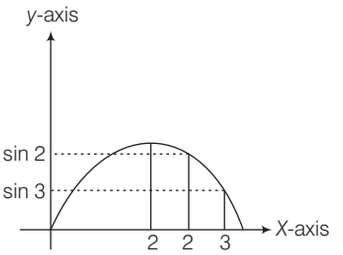
$$\therefore \cos^3\alpha+\cos^3\left(\alpha+\frac{2\pi}{3}\right)+\cos^3\left(\alpha+\frac{4\pi}{3}\right)$$

$$=3\cos\alpha\cos\left(\alpha+\frac{2\pi}{3}\right)\cos\left(\alpha+\frac{4\pi}{3}\right)$$

- ***Ex. 74.*** **Statement I** $\sin 2 > \sin 3$

Statement II *If* $x, y \in \left(\dfrac{\pi}{2}, \pi\right)$, $x < y$, *then* $\sin x > \sin y$

Sol. (*a*)



- ***Ex. 75.*** *Let* $\alpha, \beta, \gamma > 0$ *and* $\alpha+\beta+\gamma=\dfrac{\pi}{2}$

Statement I $\left|\tan\alpha\tan\beta-\frac{a!}{6}\right|+\left|\tan\beta\tan\gamma-\frac{b!}{2}\right|+\left|\tan\gamma\tan\alpha-\frac{c!}{3}\right| \le 0$, *where* $n! = 1.2 \ldots n$, *then* $\tan\alpha\tan\beta$, $\tan\beta\tan\gamma$, $\tan\gamma\tan\alpha$ *are in AP.*

Statement II $\tan\alpha\tan\beta+\tan\beta\tan\gamma+\tan\gamma\tan\alpha=1$

Sol. (*d*) Statement II $\alpha+\beta=\dfrac{\pi}{2}-\gamma$

$$\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{1}{\tan\gamma}$$

$\Rightarrow \Sigma\tan\alpha\tan\beta=1$

$\therefore$ Statement II is true.

Statement I $\tan\alpha\tan\beta=\dfrac{a!}{6}$,

$\tan\beta\tan\gamma=\dfrac{b!}{2}$

and $\tan\alpha\tan\beta=\dfrac{c!}{3}$

$$\frac{a!}{6}+\frac{b!}{2}+\frac{c!}{3}=1$$

$\Rightarrow a!=1\,b!=1\,c!=1$

$\Rightarrow \tan\alpha\tan\beta$, $\tan\gamma\tan\alpha$ and $\tan\beta\tan\gamma$ are not in AP.

$\therefore$ Statement I is false.

Hence, (d) is the correct answer.

- ***Ex. 76.*** **Statement I** *The triangle so obtained is an equilateral triangle.*

Statement II *If roots of the equation be* $\tan A$, $\tan B$ *and* $\tan C$, *then* $\tan A+\tan B+\tan C=3\sqrt{3}$

Sol. $(b)\ \tan A + \tan B + \tan C = 3\sqrt{3}$

and $\tan A \tan B \tan C = 3$

$\therefore \quad \tan A + \tan B + \tan C \neq \tan A \tan B \tan C$

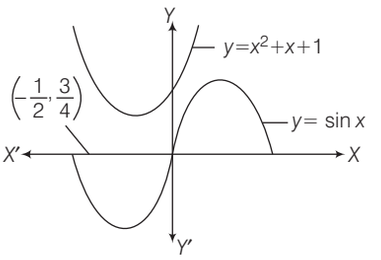
$\Rightarrow$ triangle does not exists.

- ***Ex. 77.*** *Let us define the function* $f(x) = x^2 + x + 1$

Statement I *The equation* $\sin x = f(x)$ *has no solution.*

Statement II *The curve* $y = \sin x$ *and* $y = f(x)$ *do not intersect each other when graph is observed.*

Sol. (a) Let $y = \sin x$ and $y = x^2 + x + 1$



Since, $-1 \le \sin x \le 1$ and $y = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$

It is clear from the graph that no two curves intersect each other.

JEE Type Solved Examples : Passage Based Questions

Passage I

(Ex. Nos. 78 to 80)

Consider, $f(x) = (x + 2a)(x + a - 4)\ (a \in R)$,

$g(x) = k(x^2 + x) + 3k + x (k \in R)$ and

$h(x) = (1 - \sin\theta)x^2 + 2(1 - \sin\theta)x - 3\sin\theta$

$\left(\theta \in R - (4n + 1)\frac{\pi}{2}, n \in I\right)$

- ***Ex. 78.*** *If* $f(x) < 0$ *for* $-1 \le x \le 1$*, then 'a' satisfies*

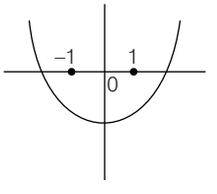
(a) $\frac{1}{2} < a < 3$ (b) $-\frac{1}{2} < a < \frac{1}{2}$

(c) $-3 < a < -\frac{1}{2}$ (d) $-3 < a < \frac{1}{2}$

Sol. (a) Given, $f(x) = (x + 2a)(x + a - 4)$

$= x^2 + (3a - 4)x + 2a(a - 4)$.

$\left.\begin{matrix} f(-1) < 0 \\ f(1) < 0 \end{matrix}\right\} n$



On solving, we get $\frac{1}{2} < a < 3$

- ***Ex. 79.*** *If* $g(x) > -3$ *for all real x, then the values of k are given by*

(a) $-1 < k < \frac{1}{11}$ (b) $-1 < k < 0$

(c) $0 < k < \frac{1}{11}$ (d) $k < \frac{1}{11}$

Sol. $(d)\ g(x) = k(x^2 + x) + 3k + x > -3\ \forall\ x$

$\Rightarrow \quad k(x^2 + x) + 3k + x + 3 > 0\ \forall\ x$

or $\quad kx^2 + (k + 1)x + (3k + 3) > 0\ \forall\ x$

$\left.\begin{matrix} k > 0 \\ D < 0 \end{matrix}\right\} n$

Here, $D = (k + 1)^2 - 4k.3(k + 1) < 0$

$\Rightarrow \quad k^2 + 2k + 1 - 12k^2 - 12 < 0$

$\Rightarrow \quad 11k^2 + 10k - 1 > 0$

$\Rightarrow \quad (k + 1)(11k - 1) > 0$

$\Rightarrow \quad k < -1 \quad \text{or} \quad k > \frac{1}{11}$

$\Rightarrow \quad k > \frac{1}{11} \qquad (\because k > 0)$

- ***Ex. 80.*** *If the quadratic equation* $h(x) = 0$ *has both roots complex, then* θ *belongs to*

(a) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (b) $\left(0, \frac{3\pi}{2}\right)$

(c) $\left(\frac{\pi}{6}, \frac{7\pi}{6}\right)$ (d) $\left(\frac{7\pi}{6}, \frac{11\pi}{6}\right)$

Sol. (d) Given, $(1 - \sin\theta)x^2 + 2(1 - \sin\theta)x - 3\sin\theta = 0$ has both roots complex, then $D < 0$

$(1 - \sin\theta)(1 + 2\sin\theta) < 0$

$\underbrace{(\sin\theta - 1)}_{(-)\text{ve number}}(2\sin\theta + 1) > 0$

$\Rightarrow \quad 2\sin\theta + 1 < 0$

$\sin\theta < -\frac{1}{2}$

$\Rightarrow \quad \theta \in \left(\frac{7\pi}{6}, \frac{11\pi}{6}\right)$

Passage II

(Ex. Nos. 81 to 83)

Let $f(\theta) = \sin\theta - \cos^2\theta - 1$, where $\theta \in R$ and $m \le f(\theta) \le M$.

● ***Ex. 81.*** *Let N denotes the number of solution of the equation $f(\theta) = 0$ in $[0, 4\pi]$ then the value of* $\log_{\sqrt{m^2}}(N) + \log_{\sqrt{m^2}}\left(\frac{1}{N+1}\right)$ *is equal to*

(a) $\frac{1}{2}$ (b) 1

(c) $\frac{-1}{2}$ (d) -1

Sol. (*c*) $f(\theta) = \sin\theta - (1 - \sin^2\theta) - 1$

$= \sin^2\theta + \sin\theta - 2$

$= \left(\sin\theta + \frac{1}{2}\right)^2 - 2 - \frac{1}{4}$

$= \left(\sin\theta + \frac{1}{2}\right)^2 - \frac{9}{4}$

$\therefore \quad f(\theta)_{\min} = 0 - \frac{9}{4} = \frac{-9}{4}$

$\Rightarrow \quad m = \frac{-9}{4}$

$f(\theta)_{\max} = \left(1 + \frac{1}{2}\right)^2 - \frac{9}{4} = \frac{9}{4} - \frac{9}{4} = 0$

$\Rightarrow \quad M = 0$

Hence, $m = \frac{-9}{4}$, $M = 0$

Now, $f(\theta) = 0$

$\Rightarrow \quad (\sin\theta + 2)(\sin\theta - 1) = 0$

$\Rightarrow \quad \sin\theta = 1$

$\Rightarrow \quad \theta = \frac{\pi}{2}$ and $\frac{5\pi}{2}$

Hence, $N = 2$, i.e. number of solution s of $\sin\theta = 1$ in $[0, 4\pi]$.

$\therefore \quad \log_{\sqrt{m^2}}(N) + \log_{\sqrt{m^2}}\left(\frac{1}{N+1}\right)$

$= \log_{|m|}\left(\frac{N}{N+1}\right)\text{v} = \log_{|m|}\left(\frac{2}{3}\right)$

$= \log_{\frac{9}{4}}\left(\frac{2}{3}\right) = \frac{-1}{2}.$

● ***Ex. 82.*** *The value of $(4m + 13)$ is equal to*

(a) 0 (b) 4

(c) 5 (d) 6

Sol. (*b*) As $m = \frac{-9}{4}$, so $(4m + 13) = 4$

● ***Ex. 83.*** *Sum of all values of x satisfying the equation* $x = \sqrt{\frac{1}{|m|} + \sqrt{\frac{1}{|m|} + \sqrt{\frac{1}{|m|} + \ldots\ldots\ldots \infty}}}$, is

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$

(c) $\frac{3}{3}$ (d) $\frac{4}{3}$

Sol. (*d*) $x = \sqrt{\frac{4}{9} + \sqrt{\frac{4}{9} + \sqrt{\frac{4}{9} + \ldots\ldots\ldots\ldots \infty}}}$

$\Rightarrow \quad x = \sqrt{\frac{4}{9} + x}$

$\Rightarrow \quad x^2 = \frac{4}{9} + x$

$9x^2 = 4 + 9x$

$\Rightarrow \quad 9x^2 - 9x - 4 = 0$

$\Rightarrow \quad 9x^2 - 12x + 3x - 4 = 0$

$\Rightarrow \quad (3x - 4)(3x + 1) = 0$

$\therefore \quad x = \frac{4}{3}$ and $x = \frac{-1}{3}$ (rejected)

Passage III

(Ex. Nos. 84 to 88)

The method of eliminating 'θ' from two given equations involving trigonometrical functions of 'θ'. By using given equations involving 'θ' and trigonometrical identities, we shall obtain an equation not involving 'θ'.

On the basis of above information answer the following questions.

● ***Ex. 84.*** *If $x\sin^3\theta + y\cos^3\theta = \sin\theta\cos\theta$ and $x\sin\theta - y\cos\theta = 0$ then (x, y) lie one*

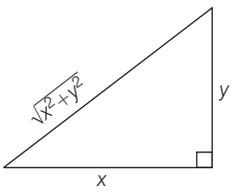
(a) a circle (b) a parabola

(c) an ellipse (d) a hyperbola

Sol. (*a*) We have, $x\sin^3\theta + y\cos^3\theta = \sin\theta\cos\theta$ ...(i)

and $x\sin\theta - y\cos\theta = 0$ ...(ii)

From Eq. (ii), $\tan\theta = \frac{y}{x}$



$\therefore \quad \sin\theta = \frac{y}{\sqrt{(x^2 + y^2)}}$ and $\cos\theta = \frac{x}{\sqrt{(x^2 + y^2)}}$

From Eq., (i) $x \times \frac{y^3}{(x^2+y^2)^{\frac{3}{2}}} + y \times \frac{x^3}{(x^2+y^2)^{\frac{3}{2}}}$

$$= \frac{xy}{(x^2+y^2)}$$

or $\frac{(x^2+y^2)}{(x^2+y^2)^{\frac{3}{2}}} = \frac{1}{(x^2+y^2)} \Rightarrow (x^2+y^2)^{\frac{1}{2}} = 1$

or $x^2 + y^2 = 1$ which is a circle

● ***Ex. 85.*** *If* $\frac{x}{a\cos\theta} = \frac{y}{b\sin\theta}$ ...(i)

and $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$, *then* (x, y) *lie on*

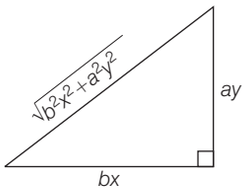
(a) a circle (b) a parabola

(c) an ellipse (d) a hyperbola

Sol. (*c*) $\because$ $\frac{x}{a\cos\theta} = \frac{y}{b\sin\theta}$ ...(i)

and $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$ ...(ii)

From Eq. (i), $\tan\theta = \frac{ay}{bx}$



From Eq. (ii),

$$\frac{ax}{\frac{bx}{\sqrt{(b^2x^2+a^2y^2)}}} - \frac{by}{\frac{ay}{\sqrt{(b^2x^2+a^2y^2)}}} = (a^2 - b^2)$$

$\Rightarrow$ $(a^2 - b^2)\sqrt{(b^2x^2 + a^2y^2)} = ab(a^2 - b^2)$

$\Rightarrow$ $b^2x^2 + a^2y^2 = a^2b^2$

$\therefore$ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which is an ellipse.

● ***Ex. 86.*** *If* $\tan\theta + \sin\theta = m$ *and* $\tan\theta - \sin\theta = n$, *then* $(m^2 - n^2)^2$ *is*

(a) $4\sqrt{mn}$ (b) $4mn$

(c) $16\sqrt{mn}$ (d) $16mn$

Sol. (*d*) $\because$ $m + n = 2\tan\theta$, $m - n = 2\sin\theta$ ...(i)

and $mn = \tan^2\theta - \sin^2\theta = \sin^2\theta(\sec^2\theta - 1)$

$$= \sin^2\theta \tan^2\theta$$

$$= \left(\frac{m-n}{2}\right)^2 \left(\frac{m+n}{2}\right)^2$$ [from Eq. (i)]

$\therefore$ $(m^2 - n^2)^2 = 16\,mn$

● ***Ex. 87.*** *If* $\sin\theta + \cos\theta = a$ *and* $\sin^3\theta + \cos^2\theta = b$, *then we get* $\lambda a^3 + \mu b + \nu a = 0$ *when* λ, μ, ν *are independent of* θ, *then the value of* $\lambda^3 + \mu^3 + \nu^3$ *is*

(a) -6 (b) -18 (c) -36 (d) -98

Sol. (*b*) $\sin\theta + \cos\theta = a$ ...(i)

$\sin^3\theta + \cos^3\theta = b$ ...(ii)

From Eq. (i),

$\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = a^2$

or $\sin\theta\cos\theta = \frac{a^2 - 1}{2}$ ...(iii)

From Eq. (ii),

$(\sin\theta + \cos\theta)^3 + 3\sin\theta\cos\theta(\sin\theta + \cos\theta) = b$

$\Rightarrow$ $a^3 - \frac{3(a^2-1)}{2}a = b$ [from Eqs. (i) and (iii)]

$\Rightarrow$ $2a^3 - 3a^3 + 3a = 2b \Rightarrow a^3 + 2b - 3a = 0$

On comparing, we get

$\lambda = 1, \mu = 2, \nu = -3$

$\therefore$ $\lambda + \mu + \nu = 0$

$\therefore$ $\lambda^3 + \mu^3 + \nu^3 = 3\lambda\mu\nu = 3(1)(2)(-3) = -18$

● ***Ex. 88.*** *After eliminating* 'θ' *from equations* $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ *and* $x\sin\theta - y\cos\theta = \sqrt{(a^2\sin^2\theta + b^2\cos^2\theta}$, *we get*

(a) $x^2 + y^2 = a^2 + b^2$ (b) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(c) $\frac{x^2}{a(a+b)} + \frac{y^2}{b(a+b)} = 1$ (d) $x^2 + y^2 = (a+b)^2$

Sol. (*c*) $\because$ $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ ...(i)

and $x\sin\theta - y\cos\theta = \sqrt{(a^2\sin^2\theta + b^2\cos^2\theta}$ ...(ii)

Squaring Eq. (i), we get

$$\frac{x^2}{a^2}\cos^2\theta + \frac{y^2}{b^2}\sin^2\theta + \frac{2xy}{ab}\sin\theta\cos\theta$$

$$= 1 = \sin^2\theta + \cos^2\theta$$

or $\left(\frac{x^2}{a^2} - 1\right)\cos^2\theta + \left(\frac{y^2}{b^2} - 1\right)\sin^2\theta$

$$+ \frac{2xy}{ab}\sin\theta\cos\theta = 0$$ ...(iii)

and squaring Eq. (ii), we get

$$x^2\sin^2\theta + y^2\cos^2\theta - 2xy\sin\theta\cos\theta$$

$$= a^2\sin^2\theta + b^2\cos^2\theta$$

$(x^2 - a^2)\sin^2\theta + (y^2 - b^2)\cos^2\theta - 2xy\sin\theta\cos\theta = 0$

$\Rightarrow$ $\left(\frac{x^2 - a^2}{ab}\right)\sin^2\theta + \left(\frac{y^2 - b^2}{ab}\right)\cos^2\theta$

$$-\frac{2xy}{ab}\sin\theta\cos\theta = 0 \qquad \text{...(iv)}$$

Adding Eqs. (iii) and (iv), we get

$$\left(\frac{x^2-a^2}{a}\right)\left(\frac{\sin^2\theta}{b}+\frac{\cos^2\theta}{a}\right)+\left(\frac{y^2-b^2}{b}\right)\left(\frac{\sin^2\theta}{b}+\frac{\cos^2\theta}{a}\right)=0$$

or $$\frac{x^2-a^2}{a}+\frac{y^2-b^2}{b}=0 \quad \text{or} \quad \frac{x^2}{a}+\frac{y^2}{b}=(a+b)$$

JEE Type Solved Examples : Matching Type Questions

● ***Ex. 89.*** *Match the statement of Column I with values of Column II.*

	Column-I		Column-II
(A)	The number of real roots of the equation $\cos^7 x + \sin^4 x = 1$ in $(-\pi, \pi)$ is	(p)	1
(B)	The value of $\sqrt{3}\,\text{cosec}\,20° - \sec 20°$ is	(q)	4
(C)	$4\cos 36° - 4\cos 72° + 4\sin 18°\cos 36°$ equals	(r)	3
(D)	The number of values of x where $x \in [-2\pi, 2\pi]$, which satisfy $\text{cosec}\,x = 1 + \cot x$	(s)	2

Sol. (A) → (r), (B) → (q), (C) → (r), (D) → (s)

(A) $\cos^7 x + \sin^4 x = 1$

$\cos^7 x = (1+\sin^2 x)\cos^2 x$

$\Rightarrow$ $\cos x = 0$ or $\cos^5 x = 1 + \sin^2 x$

$\cos x = 0 \Rightarrow x = -\frac{\pi}{2}, \frac{\pi}{2}$; $\cos^5 x = 1 + \sin^2 x$

$\Rightarrow$ $x = 0$ $[\because$ LHS ≤ 1 and RHS $\ge 1]$

$\therefore$ $x = -\frac{\pi}{2}, 0, \frac{\pi}{2}$.

(B) $\sqrt{3}\,\text{cosec}\,20° - \sec 20°$

$$= \frac{\sqrt{3}}{\sin 20°} - \frac{1}{\cos 20°} = \frac{\sqrt{3}\cos 20° - \sin 20°}{\sin 20°\cos 20°}$$

$$= \frac{2\left(\frac{\sqrt{3}}{2}\cos 20° - \frac{1}{2}\sin 20°\right)}{\sin 20°\cos 20°} = \frac{4\sin 40°}{\sin 40°} = 4$$

(C) $4\cos 36° - 4\cos 72° + 4\sin 18°\cdot\cos 36°$

$$= 4\left(\frac{\sqrt{5}+1}{4}\right) - 4\left(\frac{\sqrt{5}-1}{4}\right) + 4\left(\frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4}\right)$$

$$= \sqrt{5} + 1 - \sqrt{5} + 1 + 1 = 3$$

(D) $\text{cosec}\,x = 1 + \cot x$; $\frac{1}{\sin x} = \frac{\sin x + \cos x}{\sin x} \Rightarrow$

$\sin x + \cos x = 1$ and $\sin x \neq 0$

$$\cos\left(x - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$\Rightarrow$ $x - \frac{\pi}{4} = -2\pi + \frac{\pi}{4}, \frac{\pi}{4}$

$\left(\because x - \frac{\pi}{4} \in \left[-2\pi - \frac{\pi}{4}, 2\pi - \frac{\pi}{4}\right]\right)$

$\Rightarrow$ $x = -\frac{3\pi}{2}, \frac{\pi}{2}$

● ***Ex. 90.*** *Match the statement of Column I with values of Column II.*

	Column-I		Column-II
(A)	The number of solutions of the equation $\lvert\cot x\rvert = \cot x + \frac{1}{\sin x}$ $(0 < x < \pi)$	(p)	No solution
(B)	If $\sin\theta + \sin\phi = \frac{1}{2}$ and $\cos\theta + \cos\phi = 2$, then $\cot\left(\frac{\theta+\phi}{2}\right)$	(q)	$\frac{1}{3}$
(C)	$\sin^2\alpha + \sin\left(\frac{\pi}{3}-\alpha\right)\sin\left(\frac{\pi}{3}-\alpha\right)$	(r)	1
(D)	If $\tan\theta = 3\tan\phi$, then maximum value of $\tan^2(\theta-\phi)$ is	(s)	4

Sol. (A) → (r), (B) → (s), (C) → (p), (D) → (q)

(A) $\lvert\cot x\rvert = \cot x + \frac{1}{\sin x}$

If $0 < x < \frac{\pi}{2} \Rightarrow \cot x > 0$

So, $\cot x = \cot x + \frac{1}{\sin x} \Rightarrow \frac{1}{\sin x} = 0$ no solution

If $\frac{\pi}{2} < \cot x < \pi$, $-\cot x = \cot x + \frac{1}{\sin x}$

$\frac{2\cos x}{\sin x} + \frac{1}{\sin x} = 0$

$1 + 2\cos x = 0$ and $x \neq 0 \Rightarrow x = \frac{2\pi}{3}$

(B) Since, $\sin\phi + \sin\theta = \frac{1}{2}$ and $\cos\theta + \cos\phi = 2$ has no solution.

(C) $\sin^2\alpha + \sin\left(\frac{\pi}{3}-\alpha\right)\cdot\sin\left(\frac{\pi}{3}+\alpha\right)$

$$= \sin^2\alpha + \sin^2\frac{\pi}{3} - \sin^2\alpha = \frac{3}{4}$$

(D) $\tan\theta = 3\tan\phi$

$$\tan(\theta-\phi) = \frac{\tan\theta - \tan\phi}{1+\tan\theta\tan\phi} = \frac{2\tan\phi}{1+3\tan^2\phi}$$

$$= \frac{2}{\cot\phi + 3\tan\phi} \cdot \text{Max of } \tan\phi > 0$$

$$\frac{\cot\phi + 3\tan\phi}{2} \geq \sqrt{3} \qquad \text{(using AM} \geq \text{GM)}$$

$$\Rightarrow \quad (\cot\phi + 3\tan\phi)^2 \geq 12 \Rightarrow \tan^2(\theta-\phi) \leq \frac{1}{3}$$

● ***Ex. 91.*** *Match the statement of Column I with values of Column II.*

	Column-I		Column-II
(A)	The tangents of two acute angles are 3 and 2. The sine of twice their difference is	(p)	1
(B)	If $n = \frac{\pi}{4\alpha}$, then $\tan\alpha\tan 2\alpha\tan 3\alpha \ldots$ $\tan(2n-1)\alpha$ is equal to	(q)	0
(C)	If $x = y\cos\frac{2\pi}{3} = z\cos\frac{4\pi}{3}$, then $xy + yz + zx =$	(r)	$\frac{1}{2}$
(D)	The ratio of the greatest value of $2 - \cos x + \sin^2 x$ to its least value is	(s)	$\frac{7}{25}$
		(t)	$\frac{13}{4}$

Sol. (A) $\rightarrow$ (s), (B) $\rightarrow$ (p), (C) $\rightarrow$ (q), (D) $\rightarrow$ (t)

(A) Given, $\tan\alpha = 3$ and $\tan\beta = 2$

$$\therefore \quad \tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1-\tan\alpha\tan\beta} = \frac{3-2}{1+3\times 2} = \frac{1}{7}$$

$$\Rightarrow \quad \sin(\alpha-\beta) = \frac{1}{\sqrt{50}} \text{ and } \cos(\alpha-\beta) = \frac{7}{\sqrt{50}}$$

$$\therefore \quad \sin 2(\alpha-\beta) = 2\sin(\alpha-\beta)\cos(\alpha-\beta)$$

$$= 2\times\frac{1}{\sqrt{50}}\times\frac{7}{\sqrt{50}} = \frac{7}{25}$$

(B) We have, $\tan\alpha\cdot\tan(2n-1)\alpha = \tan\alpha\cdot\tan\left(\frac{\pi}{2\alpha}-1\right)\alpha$

$$= \tan\alpha\cdot\tan\left(\frac{\pi}{2}-\alpha\right) = \tan\alpha\cot\alpha = 1$$

$\therefore$ The given expression = 1.

(C) We have, $x = y\cos\frac{2\pi}{2} = z\cos\frac{4\pi}{3} = k$ (say)

$$\Rightarrow \quad \frac{1}{x} = \frac{1}{k}, \frac{1}{y} = \frac{\cos\frac{2\pi}{3}}{k}, \frac{1}{z} = \frac{\cos\frac{4\pi}{3}}{k}$$

$$\therefore \quad \frac{1}{x}+\frac{1}{y}+\frac{1}{z} = \frac{1}{k}\left(1+\cos\frac{2\pi}{3}+\cos\frac{4\pi}{3}\right)$$

$$= \frac{1}{k}\left(1-\frac{1}{2}-\frac{1}{2}\right) = 0$$

$$\Rightarrow \quad xy + xz + yz = 0$$

(D) We have, $2 - \cos x + \sin^2 x = 2 - \cos x + 1 - \cos^2 x$

$$= -(\cos^2 x + \cos x) + 3 = -\left[\left(\cos x + \frac{1}{2}\right)^2 - \frac{1}{4}\right] + 3$$

$$= \frac{13}{4} - \left(\cos x + \frac{1}{2}\right)^2$$

$\therefore$ Maximum value occurs at $\cos x = -\frac{1}{2}$ and it is 1

$\frac{13}{4}$ and minimum value occurs at $\cos x = 1$ and it is

$\therefore$ The required ratio is $\frac{13}{4}$.

● ***Ex. 92.*** *Match the statement of Column I with values of Column II*

	Column-I		Column-II
(A)	If α, β, γ and δ are four solutions of the equation $\tan\left(\theta+\frac{\pi}{4}\right) = 3\tan 3\theta$, no two of which have equal tangents, then the value of $\tan\alpha + \tan\beta + \tan\gamma + \tan\delta$ is	(p)	$\sqrt{2}$
(B)	If $\frac{\cos(\theta_1-\theta_2)}{\cos(\theta_1+\theta_2)} + \frac{\cos(\theta_3+\theta_4)}{\cos(\theta_3-\theta_4)} = 0$ then $\tan\theta_1\tan\theta_2\tan\theta_3\tan\theta_4 =$	(q)	$\sqrt{3}$
(C)	If $\sec(\alpha-\beta)$, $\sec\alpha$ and $\sec(\alpha+\beta)$ are in A.P. (with $\beta \neq 0$), then $\cos\alpha\sec\frac{\beta}{2} =$	(r)	-1
(D)	If $\cos\alpha = \frac{2\cos\beta - 1}{2-\cos\beta}$ $(0 < \alpha < \beta < \pi)$, then $\frac{\tan\frac{\alpha}{2}}{\tan\frac{\beta}{2}}$ is equal to	(s)	0

Sol. (A) $\rightarrow$ (s), (B) $\rightarrow$ (r), (C) $\rightarrow$ (p), (D) $\rightarrow$ (q)

(A) Using $\tan\left(\theta+\frac{\pi}{4}\right) = \frac{1+\tan\theta}{1-\tan\theta}$

and $\quad 3\tan 3\theta = \frac{3(3\tan\theta - \tan^3\theta)}{1-3\tan^2\theta}$

the given equation becomes

$$3\tan^4\theta - 6\tan^2\theta + 8\tan\theta - 1 = 0 \quad .$$

If $\tan\alpha$, $\tan\beta$, $\tan\gamma$ and $\tan\delta$ are the roots of this equation, then the sum of these roots, $\tan\alpha + \tan\beta + \tan\gamma + \tan\delta$ equals zero, since the coefficient of $\tan^3\theta$ is zero.

(B) The given equation can be written as

$$\Rightarrow \quad \frac{\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2}{\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2} + \frac{\cos\theta_3\cos\theta_4 - \sin\theta_3\sin\theta_4}{\cos\theta_3\cos\theta_4 + \sin\theta_3\sin\theta_4}$$

$$\Rightarrow \quad \frac{1+\tan\theta_1\tan\theta_2}{1-\tan\theta_1\tan\theta_2} + \frac{1-\tan\theta_3\tan\theta_4}{1+\tan\theta_3\tan\theta_4} = 0$$

$$\Rightarrow \quad \frac{2+2\tan\theta_1\tan\theta_2\tan\theta_3\tan\theta_4}{(1-\tan\theta_1\tan\theta_2)(1+\tan\theta_3\tan\theta_4)}=0$$

Showing that $\tan\theta_1\tan\theta_2\tan\theta_3\tan\theta_4=-1$.

(C) For the given A.P., we have

$$2\sec\alpha=\sec(\alpha-\beta)+\sec(\alpha+\beta), \text{ which gives}$$

$$\frac{2}{\cos\alpha}=\frac{1}{\cos(\alpha-\beta)}+\frac{1}{\cos(\alpha+\beta)}$$

$$=\frac{2\cos\alpha\cos\beta}{\cos^2\alpha-\sin^2\beta}$$

$$\Rightarrow \quad \cos^2\alpha-\sin^2\beta=\cos^2\alpha\cos\beta$$

$$\Rightarrow \quad \cos^2\alpha(1-\cos\beta)=\sin^2\beta$$

$$\Rightarrow \quad \cos^2\alpha\left(2\sin^2\frac{\beta}{2}\right)=4\sin^2\frac{\beta}{2}\cos^2\frac{\beta}{2}$$

$$\Rightarrow \quad \cos^2\alpha\sec^2\frac{\beta}{2}=2$$

(D) $1+\cos\alpha=1+\dfrac{2\cos\beta-1}{2-\cos\beta}$

$$=\frac{2-\cos\beta+2\cos\beta-1}{2-\cos\beta}=\frac{1+\cos\beta}{2-\cos\beta}$$

$$\Rightarrow \quad \cos^2\frac{\alpha}{2}=\frac{\cos^2\frac{\beta}{2}}{1+2\sin^2\frac{\beta}{2}} \qquad \text{...(i)}$$

$$\Rightarrow \quad 1-\cos^2\frac{\alpha}{2}=1-\frac{\cos^2\frac{\beta}{2}}{1+2\sin^2\frac{\beta}{2}}$$

$$=\frac{1+2\sin^2\frac{\beta}{2}-\cos^2\frac{\beta}{2}}{1+2\sin^2\frac{\beta}{2}}$$

$$=\frac{1+2\sin^2\frac{\beta}{2}-\left[1-\sin^2\frac{\beta}{2}\right]}{1+2\sin^2\frac{\beta}{2}}$$

$$\Rightarrow \quad \sin^2\frac{\alpha}{2}=\frac{3\sin^2\frac{\beta}{2}}{1+2\sin^2\frac{\beta}{2}} \qquad \text{...(ii)}$$

Dividing equation (ii) by (i), we get

$$\tan^2\frac{\alpha}{2}=3\tan^2\frac{\beta}{2}$$

$$\Rightarrow \quad \frac{\tan\frac{\alpha}{2}}{\tan\frac{\beta}{2}}=\sqrt{3}$$

JEE Type Solved Examples :
Single Integer Answer Type Questions

● ***Ex. 93.*** $\tan 46^\circ\tan 14^\circ-\tan 74^\circ\tan 14^\circ+\tan 74^\circ\tan 46^\circ$ *is equal to*

Sol. (3) $\dfrac{\tan 46^\circ+\tan 14^\circ}{1-\tan 46^\circ\tan 14^\circ}=\tan(46^\circ+14^\circ)=\sqrt{3}$ ...(i)

$$\frac{\tan 74^\circ-\tan 14^\circ}{1+\tan 74^\circ\tan 14^\circ}=\tan(74^\circ-14^\circ)$$

$$=\sqrt{3} \qquad \text{...(ii)}$$

$$\frac{\tan 74^\circ+\tan 46^\circ}{1-\tan 74^\circ\tan 46^\circ}=\tan(74^\circ+46^\circ)$$

$$=-\sqrt{3} \qquad \text{...(iii)}$$

From Eqs. (i), (ii) and (iii)

$$\tan 46^\circ\tan 14^\circ=1-\frac{\tan 46^\circ+\tan 14^\circ}{\sqrt{3}}$$

$$\tan 74^\circ\tan 14^\circ=\frac{\tan 74^\circ-\tan 14^\circ}{\sqrt{3}}-1$$

$$\tan 74^\circ\tan 46^\circ=1-\frac{\tan 74^\circ+\tan 46^\circ}{-\sqrt{3}}$$

$\therefore \tan 46^\circ\tan 14^\circ-\tan 74^\circ\tan 14^\circ+\tan 74^\circ\tan 46^\circ=3$

● ***Ex. 94.*** *Maximum value of the expression* $\log_3(9-2\cos^2\theta-4\sec^2\theta)$ *is equal to*

Sol. (1) For the expression $a\cos^2\theta+b\sec^2\theta$ if $b>a$, then minimum value attains at $\cos^2\theta=\sec^2\theta=1$

$\Rightarrow$ max of $\{9-(2\cos^2\theta+4\sec^2\theta)\}=3$

So, maximum of $\log_3(9-2\cos^2\theta+4\sec^2\theta))=1$

● ***Ex. 95.*** *Let* $x\in\left(0,\dfrac{\pi}{2}\right)$ *and* $\log_{24\sin x}(24\cos x)=\dfrac{3}{2}$, *then find the value of* $\operatorname{cosec}^2 x$.

Sol. (9) $(24\sin x)^{3/2}=24\cos x$

$$\Rightarrow \quad \sqrt{24}(\sin x)^{3/2}=\cos x$$

$$\Rightarrow \quad 24\sin^3 x=\cos^2 x=1-\sin^2 x$$

Put $\sin x=t$, we get

$$24t^3+t^2-1=0$$

$$\Rightarrow \quad (3t-1)\underbrace{(8t^2+3t+1)}_{>0}=0$$

$$\Rightarrow \qquad t = \frac{1}{3}$$

$$\therefore \qquad t = \frac{1}{3} \text{ i.e. } \sin x = \frac{1}{3}$$

$$\Rightarrow \qquad \operatorname{cosec} x = 3$$

$$\Rightarrow \qquad \operatorname{cosec}^2 x = 9$$

● ***Ex. 96.*** *If x and y are non-zero real numbers satisfying $xy(x^2 - y^2) = x^2 + y^2$, then find the minimum value of $x^2 + y^2$.*

Sol. (4) Put $x = r\cos\theta$ and $y = r\sin\theta$

Hence, we have to minimise r^2?

Now, $r^2 \cos\theta \sin\theta r^2(\cos^2\theta - \sin^2\theta) = r^2$

$$r^2 \sin 2\theta \cos 2\theta = 2$$

$$r^2 \frac{\sin 4\theta}{4} = 1$$

$$r^2 = \frac{4}{\sin 4\theta}$$

$$r^2 = 4\operatorname{cosec}^2 4\theta$$

$$\therefore \qquad r^2_{\min} = 4$$

● ***Ex. 97.*** *Using the identity*

$\sin^4 x = \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$ *or otherwise, if the value of* $\sin^4\left(\frac{\pi}{7}\right) + \sin^4\left(\frac{3\pi}{7}\right) + \sin^4\left(\frac{5\pi}{7}\right) = \frac{a}{b}$, *where a and b are coprime, find the value of $(a - b)$.*

Sol. (5) $\sin^4\left(\frac{\pi}{7}\right) = \frac{3}{8} - \frac{1}{2}\cos\left(\frac{2\pi}{7}\right) + \frac{1}{8}\cos\left(\frac{4\pi}{7}\right)$ ...(i)

and $\sin^4\left(\frac{3\pi}{7}\right) = \frac{3}{8} - \frac{1}{2}\cos\left(\frac{6\pi}{7}\right) + \frac{1}{8}\cos\left(\frac{12\pi}{7}\right)$

or $\sin^4\left(\frac{3\pi}{7}\right) = \frac{3}{8} + \frac{1}{2}\cos\left(\frac{\pi}{7}\right) - \frac{1}{8}\cos\left(\frac{5\pi}{7}\right)$ ...(ii)

Similarly, $\sin^4\left(\frac{5\pi}{7}\right) = \frac{3}{8} - \frac{1}{2}\cos\left(\frac{10\pi}{7}\right) + \frac{1}{8}\cos\left(\frac{20\pi}{7}\right)$

or $\sin^4\left(\frac{5\pi}{7}\right) = \frac{3}{8} + \frac{1}{2}\cos\left(\frac{3\pi}{7}\right) - \frac{1}{8}\cos\left(\frac{\pi}{7}\right)$ ...(iii)

On adding Eqs. (i), (ii) and (iii), we get

$$\sin^4\left(\frac{\pi}{7}\right) + \sin^4\left(\frac{3\pi}{7}\right) + \sin^4\left(\frac{5\pi}{7}\right)$$

$$= \frac{9}{8} + \frac{1}{2}\cos\left(\frac{5\pi}{7}\right) - \frac{1}{8}\cos\left(\frac{3\pi}{7}\right) + \frac{1}{2}\cos\left(\frac{\pi}{7}\right) - \frac{1}{8}\cos\left(\frac{5\pi}{7}\right) + \frac{1}{2}\cos\left(\frac{3\pi}{7}\right) - \frac{1}{8}\cos\left(\frac{\pi}{7}\right)$$

$$= \frac{9}{8} + \frac{3}{8}\underbrace{\left[\cos\left(\frac{\pi}{7}\right) + \cos\left(\frac{3\pi}{7}\right) + \cos\left(\frac{5\pi}{7}\right)\right]}_{\frac{1}{2}\left(\text{use} S = \frac{\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}\cos\left(\frac{\theta_1 + \theta_2}{2}\right)\right)}$$

$$= \frac{9}{8} + \frac{3}{16} = \frac{21}{16}$$

$$\therefore \qquad a - b = 21 - 16 = 5.$$

● ***Ex. 98.*** *In any triangle, if* $(\sin A + \sin B + \sin C)(\sin A + \sin B - \sin C) = 3\sin A \sin B$, *then the angle* $\frac{C}{10}$ *(in degree).*

Sol. (6) We have, $(\sin A + \sin B)^2 - \sin^2 C = 3\sin A \sin B$

$$\sin^2 A - \sin^2 C + \sin^2 B = \sin A \sin B$$

$$\sin(A + C)\sin(A - C) + \sin^2 B = \sin A \sin B$$

$$\sin B[\sin(A - C) + \sin(A + C)] = \sin A \sin B$$

[using, $\sin(A + C) = \sin B$]

$$2\sin A \cos C = \sin A \qquad (\sin B \neq 0)$$

$$\cos C = \frac{1}{2} \Rightarrow \frac{C}{10} = \frac{60^\circ}{10} = 6$$

● ***Ex. 99.*** *Find the exact value of the expression*

$$T = \frac{\sin 40^\circ}{\sin 80^\circ} + \frac{\sin 80^\circ}{\sin 20^\circ} - \frac{\sin 20^\circ}{\sin 40^\circ}$$

Sol. (3) We have, $\frac{1}{2\cos 40^\circ} + 4\cos 40^\circ \cdot \cos 20^\circ - \frac{1}{2\cos 20^\circ}$

$$= \frac{1}{2}\left[\frac{1}{\cos 40^\circ} - \frac{1}{\cos 20^\circ}\right] + 2(\cos 60^\circ + \cos 20^\circ)$$

$$= \frac{1}{2}\left[\frac{\cos 20^\circ - \cos 40^\circ}{\cos 40^\circ \cos 20^\circ}\right] + 1 + 2\cos 20^\circ$$

$$= \frac{2\sin 30^\circ \sin 10^\circ}{2\cos 40^\circ \cos 20^\circ} + 1 + 2\cos 20^\circ$$

$$= \frac{2\sin 10^\circ \sin 20^\circ}{\sin 80^\circ} + 1 + 2\cos 20^\circ$$

$$= \frac{2\sin 10^\circ \, 2\sin 10^\circ \cos 10^\circ}{\cos 10^\circ} + 2\cos 20^\circ + 1$$

$$= 2(1 - \cos 20^\circ) + 2\cos 20^\circ + 1 = 3$$

Alternatively $T_1 = \frac{2\sin 20^\circ \cos 20^\circ}{\cos 10^\circ} = 2 \cdot 2 \cdot \sin 10^\circ \cdot \cos 20^\circ$

$$= 2(\sin 30^\circ - \sin 10^\circ)$$

$$T_1 = 1 - 2\sin 10^\circ$$

$$T_2 = \frac{\sin 80^\circ}{\sin 20^\circ} = \frac{2\sin 40^\circ \cos 40^\circ}{\sin 20^\circ}$$

$$= 4\cos 20^\circ \cdot \cos 40^\circ$$

$$T_2 = 2[\cos 60^\circ + \cos 20^\circ] = 1 + 2\cos 20^\circ$$

$$T_2 = \frac{\sin 20^\circ}{\sin 40^\circ} = \frac{1}{2\cos 20^\circ}$$

$$\therefore\ T = T_1 + T_2 + T_3$$

$$= (1 - 2\sin 10^\circ) + (1 + 2\cos 20^\circ) - \frac{1}{2\cos 20^\circ}$$

$$= 2 + 2(\cos 20^\circ - \sin 10^\circ) - \frac{1}{2\cos 20^\circ}$$

$$= 2 + 2(\cos 20^\circ - \cos 80^\circ) - \frac{1}{2\cos 20^\circ}$$

$$= 2 + 2 \cdot 2\sin 50^\circ \cdot \sin 30^\circ - \frac{1}{2\cos 20^\circ}$$

$$= 2 + 2\sin 50^\circ - \frac{1}{2\cos 20^\circ}$$

$$T = 2 + \frac{4\sin 50^\circ \cos 20^\circ - 1}{2\cos 20^\circ}$$

$$= 2 + \frac{2[\sin 70^\circ + \sin 30^\circ] - 1}{2\cos 20^\circ}$$

$$= 2 + \frac{2\sin 70^\circ}{2\cos 20^\circ} = 2 + 1 = 3$$

● ***Ex. 100.*** *If* $\cot(\theta - \alpha), 3\cot\theta, \cot(\theta + \alpha)$ *are in* AP *(where,* $\theta \neq \frac{m\pi}{2}, \alpha \neq k\pi, n, k \in I$*), then* $\frac{2\sin^2\theta}{\sin^2\alpha}$ *is equal to*

Sol. (3) We have, $6\cot\theta = \cot(\theta - \alpha) + \cot(\theta + \alpha)$

$$\Rightarrow \quad \frac{6\cos\theta}{\sin\theta} = \frac{\sin 2\theta}{\sin(\theta - \alpha)\sin(\theta + \alpha)}$$

$$\Rightarrow \quad \frac{6\cos\theta}{\sin\theta} = \frac{2\sin\theta\cos\theta}{\sin^2\theta - \sin^2\alpha}$$

$$\Rightarrow \quad 3(\sin^2\theta - \sin^2\alpha) = \sin^2\theta$$

$$\text{or} \quad 2\sin^2\theta = 3\sin^2\alpha$$

$$\Rightarrow \quad \frac{2\sin^2\theta}{\sin^2\alpha} = 3$$

● ***Ex. 101.*** *If* $4\sin^2 x + \text{cosec}^2 x, a, \sin^2 y + 4\text{cosec}^2 y$ *are in* AP*, then minimum value of* $(2a)$ *is*

Sol. (9) $2a = 4\sin^2 x + \text{cosec}^2 x + \sin^2 y + 4\ \text{cosec}^2 y$

$$= (2\sin x - \text{cosec}\, x)^2 + 4 + (\sin y - \text{cosec}\, y)^2 + 3\ \text{cosec}^2 y + 2$$

$$= 6 + (2\sin x - \text{cosec}\ x)^2 + (\sin y - \text{cosec}\ y)^2 + 3\ \text{cosec}^2 y$$

$\therefore$ Minimum value of

$$2a = 6 + 3 = 9,$$

when $\quad 2\sin x = \text{cosec}\, x$

and $\quad \sin y = \text{cosec}\, y$

● ***Ex. 102.*** *If* $\sin\alpha, \sin\beta, \sin\gamma$ *are in* AP *and* $\cos\alpha, \cos\beta, \cos\gamma$ *are in* GP*, then the value of* $\frac{\cos^2\alpha + \cos^2\gamma + 4\cos\alpha\cos\gamma - 2\sin\alpha\sin\gamma - 2}{1 - 2\sin^2\beta}$*, where* $\beta \neq \frac{\pi}{4}$*, is equal to*

Sol. (4) Now, $\sin\alpha + \sin\gamma = 2\sin\beta$ and $\cos^2\beta = \cos\alpha \cdot \cos\gamma$

$$= \frac{\cos^2\alpha + \cos^2\gamma + 4\cos\alpha\cos\gamma - 2 - 2\sin\alpha\sin\gamma}{1 - 2\sin^2\beta}$$

$$= \frac{-\sin^2\alpha - \sin^2\gamma - 2\sin\alpha\sin\gamma + 4\cos\alpha\cos\gamma}{1 - 2\sin^2\beta}$$

$$= \frac{-(\sin\alpha + \sin\gamma)^2 + 4\cos\alpha\cos\gamma}{1 - 2\sin^2\beta}$$

$$= \frac{-4\sin^2\beta + 4\cos^2\beta}{\cos 2\beta} = 4$$

● ***Ex. 103.*** *Let*

$$\prod_{r=1}^{51} \tan\left(\frac{\pi}{3}\left(1 + \frac{3^r}{3^{50} - 1}\right)\right) = k \prod_{r=1}^{51} \cot\left[\frac{\pi}{3}\left(1 - \frac{3^r}{3^{50} - 1}\right)\right]$$

On solving equation, we get, $1 - 3\tan^2\left(\frac{\pi}{3^{50} - 1}\right) = \frac{a}{bk - 1}$*, (*$a, b \in I$*), then value of* $(a - b)$ *is equal to*

Sol. (5) We have,

$$\prod_{r=1}^{51} \tan\left(\frac{\pi}{3}\left(1 + \frac{3^r}{3^{50} - 1}\right)\right) = \prod_{r=1}^{51} \cot\left[\frac{\pi}{3}\left(1 - \frac{3^r}{3^{50} - 1}\right)\right]$$

Let $\frac{3^{r-1}\pi}{3^{50} - 1} = \theta_r$

$$\prod_{r-1}^{51} \tan\left(\frac{\pi}{3} + \theta_r\right)\tan\left(\frac{\pi}{3} + \theta_r\right) = k$$

$$\prod_{r-1}^{51} \frac{\tan 3\theta_r}{\tan\theta^r} = k$$

$$k = \frac{\tan\theta_2}{\tan\theta_1} \times \frac{\tan\theta_3}{\tan\theta_2} \times \ldots\ldots \frac{\tan\theta_{52}}{\tan\theta_{51}}$$

$$= \frac{\tan\theta_{52}}{\tan\theta} = \frac{\tan\left(\frac{3^{51}\pi}{3^{50} - 1}\right)}{\tan\left(\frac{\pi}{3^{50} - 1}\right)}$$

$$= \frac{\tan\left(3\pi + \frac{3\pi}{3^{50} - 1}\right)}{\tan\left(\frac{\pi}{3^{50} - 1}\right)}$$

$$= \frac{\tan\left(\frac{3\pi}{3^{50}-1}\right)}{\tan\left(\frac{\pi}{3^{50}-1}\right)}$$

Let, $\alpha = \frac{\pi}{3^{50}-1}$;

$k = \frac{\tan 3\alpha}{\tan\alpha}$;

$1 - 3\tan^2\alpha = \frac{8}{3k-1}$

So, $a = 8, b = 3$

$a - b = 5$

● ***Ex. 104.*** *If* $\sec A \tan B + \tan A \sec B = 91$*, then the value of* $(\sec A \sec B + \tan A \tan B)^2$ *is equal to*

Sol. (8282) $(\sec A \sec B + \tan A \tan B)^2$

$$- (\sec A \tan B + \tan A \sec B)^2$$

$$= \left[\frac{1 + \sin A \sin B}{\cos A \cos B}\right]^2 - \left[\frac{\sin B + \sin A}{\cos A \cos B}\right]^2$$

$$= \frac{1 + \sin^2 A \sin^2 B - \sin^2 B - \sin^2 A}{\cos^2 A \cos^2 B}$$

$$= \frac{1 - \sin^2 B \cos^2 A - \sin^2 A}{\cos^2 A \cos^2 B}$$

$$= \frac{\cos^2 A \cos^2 B}{\cos^2 A \cos^2 B} = 1$$

$\Rightarrow (\sec A \sec B + \tan A \tan B)^2 = (91)^2 + 1 = 8282.$

● ***Ex. 105.*** *If* $(25)^2 + a^2 + 50a\cos\theta = (31)^2 + b^2 + 62b\cos\theta = 1$ *and* $775 + ab + (31a + 25b)\cos\theta = 0$*, then the value of* $\text{cosec}^2\theta$ *is*

Sol. (1586) We can write $(a + 25\cos\theta)^2 + (25)^2 - (25\cos\theta)^2 = 1$ and

$\Rightarrow \quad (a + 25\cos\theta)^2 = 1 - (25\sin\theta)^2$

Similarly $\quad (b + 31\cos\theta)^2 = 1 - (31\sin\theta)^2$

Multiplying we get

$$[(a + 25\cos\theta)(b + 31\cos\theta)]^2 = [1 - (25\sin\theta)^2]$$

$$[1 - (31\sin\theta)^2]$$

$\Rightarrow \quad [ab + (31a + 25b)\cos\theta + 775\cos^2\theta]^2$

$$= 1 - (625 + 961)\sin^2\theta + (775\sin^2\theta)^2$$

$\Rightarrow \quad (-775 + 775\cos^2\theta)^2 = 1 - 1586\sin^2\theta + (775\sin^2\theta)^2$

$\Rightarrow \quad \text{cosec}^2\theta = 1586$

● ***Ex. 106.*** *If* $\sin x_1 + \sin x_2 + \sin x_3 + \ldots + \sin x_{2008} = 2008$*, then find the value of* $\sin^{2008} x_1 + \sin^{2008} x_2 + \sin^{2008} x_3 + \ldots + \sin^{2008} x_{2008}$.

Sol. (2008) We know that, $\sin x_i \le 1 \; \forall \; i$

$\Rightarrow \sin x_1 + \sin x_2 + \sin x_3 + \ldots + \sin x_{2008} \le 2008$

Thus, equality holds only when each of the terms is 1

i.e. $\quad \sin x_i = 1 \; \forall \; i = 1, 2, 3, \ldots, 2008.$

and consequently

$$\cos x_i = 0, \forall i = 1,2,3, \ldots, 2008$$

Now, $\sin^{2008} x_1 + \sin^{2008} x_2 + \sin^{2008} x_3 + \ldots \sin^{2008} x_{2008}$

$$= 1 + 1 + 1 + \ldots + 1 = 2008$$

● ***Ex. 107.*** *If* $4\sin 27° = \sqrt{\alpha} + \sqrt{\beta}$*, then the value of* $(\alpha + \beta - \alpha\beta + 2)^4$ *must be*

Sol. (400), We know $(\cos 27° + \sin 27°)^2$

$= 1 + \sin 54 = 1 + \cos 36°$

$\Rightarrow \quad \cos 27° + \sin 27° = \sqrt{(1 + \cos 36°)} \qquad [\because \text{LHS} > 0]$

Also, $\quad \cos 27° - \sin 27° = \sqrt{(1 - \cos 36°)}$

$[\because \cos 27° > \sin 27°]$

$\therefore \quad 2\sin 27° = \sqrt{(1 + \cos 36°)} - \sqrt{(1 - \cos 36°)}$

$$= \sqrt{\left(1 + \left(\frac{\sqrt{5}+1}{4}\right)\right)} - \sqrt{\left(1 - \left(\frac{\sqrt{5}+1}{4}\right)\right)}$$

$\therefore \quad 4\sin 27° = \sqrt{(5 + \sqrt{5})} - \sqrt{(3 - \sqrt{5})}$

On comparing, we get

$$\alpha = 5 + \sqrt{5}, \beta = 3 - \sqrt{5}$$

$\therefore \quad \alpha + \beta = 8, \alpha\beta = 10 - 2\sqrt{5}$

$\alpha + \beta - \alpha\beta + 2 = 2\sqrt{5}$

$\therefore (\alpha + \beta - \alpha\beta + 2)^4 = 400$

● ***Ex. 108.*** *If* $0 < A < \frac{\pi}{2}$ *and* $\sin A + \cos A + \tan A + \cot A + \sec A + \text{cosec } A = 7$ *and* $\sin A$ *and* $\cos A$ *are the roots of the equation* $4x^2 - 3x + a = 0$*, then the value of* $25a$ *must be*

Sol. (28) $\sin A$ and $\cos A$ are the roots of the equation $4x^2 - 3x + a = 0$, then

$$\sin A + \cos A = \frac{3}{4}, \sin A \cos A = \frac{a}{4} \qquad \ldots(i)$$

Also, $\sin A + \cos A + \tan A + \cot A + \sec A + \text{cosec } A = 7$

$$\Rightarrow \quad (\sin A + \cos A) + \left(\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}\right) + \left(\frac{1}{\cos A} + \frac{1}{\sin A}\right) = 7$$

$$\Rightarrow \quad (\sin A + \cos A) + \frac{1}{\sin A \cos A} + \frac{(\sin A + \cos A)}{\sin A \cos A} = 7$$

$$\Rightarrow \quad \frac{3}{4} + \frac{4}{a} + \frac{3}{a} = 7$$

$$\Rightarrow \quad \frac{3}{4} + \frac{7}{a} = 7$$

$$\Rightarrow \quad \frac{7}{a} = 7 - \frac{3}{4} = \frac{25}{4}$$

$$\therefore \quad 25a = 28$$

● ***Ex. 109.*** *Given that,* $f(n\theta) = \dfrac{2\sin\theta}{\cos 2\theta - \cos 4n\theta}$, *and* $f(\theta) + f(2\theta) + f(3\theta) + \ldots + f(n\theta) = \dfrac{\sin\lambda\theta}{\sin\theta \sin\mu\theta}$, *then the value of* $\mu - \lambda$ *is*

Sol. (1) $f(n\theta) = \dfrac{2\sin 2\theta}{\cos 2\theta - \cos 4n\theta}$

$$= \frac{2\sin 2\theta}{2\sin(2n+1)\theta \sin(2n-1)\theta}$$

$$= \frac{\sin((2n+1)\theta - (2n-1)\theta)}{\sin(2n+1)\theta \sin(2n-1)\theta}$$

$$= \frac{\sin(2n+1)\theta \cos(2n-1)\theta - \cos(2n+1)\theta \sin(2n-1)\theta}{\sin(2n+1)\theta \sin(2n-1)\theta}$$

$$= \cot(2n-1)\theta - \cot(2n+1)\theta$$

$\therefore f(\theta) + f(2\theta) + f(3\theta) + \ldots + f(n\theta)$

$$= \cot\theta - \cot(2n+1)\theta$$

$$= \frac{\sin(2n+1)\theta \cos\theta - \cos(2n+1)\theta \sin\theta}{\sin(2n+1)\theta \sin\theta}$$

$$= \frac{\sin 2n\theta}{\sin(2n+1)\theta \sin\theta}$$

$\therefore \quad \pi = 2n$ and $\mu = 2n+1$

Hence, $\mu - \lambda = 2n + 1 - 2n = 1$

● ***Ex. 110.*** *If* $\dfrac{1}{\cos 290^\circ} + \dfrac{1}{\sqrt{3}\sin 250^\circ} = \lambda$, *then the value of* $9\lambda^4 + 81\lambda^2 + 97$ *must be*

Sol. (785) Here, $\cos 290^\circ = \cos(270^\circ + 20^\circ) = \sin 20^\circ$ and $\sin 250^\circ = \sin(270^\circ - 20^\circ) = -\cos 20^\circ$

$$\therefore \text{The given expression} = \frac{1}{\sin 20^\circ} - \frac{1}{\sqrt{3}\cos 120^\circ} = \lambda$$

$$\Rightarrow \quad \frac{1}{\sin 20^\circ} - \frac{\cos 60^\circ}{\sin 60^\circ \cos 20^\circ} = \lambda$$

$$\Rightarrow \quad \frac{\sin 60^\circ \cos 20^\circ - \cos 60^\circ \sin 20^\circ}{\sin 20^\circ \cos 20^\circ \sin 60^\circ} = \lambda$$

$$\Rightarrow \quad \frac{\sin(60^\circ - 20^\circ)}{\frac{\sin 40^\circ}{2} \times \frac{\sqrt{3}}{2}} = \lambda$$

$$\therefore \quad \lambda = \frac{4}{\sqrt{3}}$$

$$\Rightarrow \quad \lambda^2 = \frac{16}{3}$$

Then, $9\lambda^4 + 81\lambda^2 + 97 = 9 \times \dfrac{256}{9} + 81 \times \dfrac{16}{3} + 97$

$$= 256 + 432 + 97 = 785$$

● ***Ex. 111.*** *If* $\log_{10}\sin x + \log_{10}\cos x = -1$ *and* $\log_{10}(\sin x + \cos x) = \dfrac{(\log_{10} n) - 1}{2}$, *then the value of 'n/3' is .....*

Sol. (4) Given, $\log_{10}\left(\dfrac{\sin 2x}{2}\right) = -1$

or $\quad \dfrac{\sin 2x}{2} = \dfrac{1}{10}$

or $\quad \sin 2x = \dfrac{1}{5}$

Also $\quad \log_{10}(\sin x + \cos x) = \dfrac{\log_{10}\left(\dfrac{n}{10}\right)}{2}$

or $\quad \log_{10}(\sin x + \cos x)^2 = \log_{10}\left(\dfrac{n}{10}\right)$

or $\quad 1 + \sin 2x = \dfrac{n}{10}$

or $\quad 1 + \dfrac{1}{5} = \dfrac{n}{10}$ or $\dfrac{6}{5} = \dfrac{n}{10}$

or $\quad \dfrac{n}{3} = 4$

● ***Ex. 112.*** *If* $498[16\cos x + 12\sin x] = 2k + 60$, *then the maximum value of k is*

Sol. (4950) $16\cos x + 12\sin x = \sqrt{16^2 + 12^2}\cos(x - \alpha)$, $\alpha = \tan^{-1}\left(\dfrac{3}{4}\right)$.

$$\Rightarrow \quad |2k + 60| \le 498 \times 20 \text{ as } |\cos(x - \alpha)| \le 1$$

$$\Rightarrow \quad k \le 4950$$

● ***Ex. 113.*** *If* $a\tan\alpha+\sqrt{(a^2-1)}\tan\beta+\sqrt{(a^2+1)}\tan\gamma=2a$, *where* a *is constant and* α,β,γ *are variable angles. Then the least value of* $2727(\tan^2\alpha+\tan^2\beta+\tan^2\gamma)$ *must be*

Sol. (3636) We have,

$$(a\tan\beta-\sqrt{(a^2-1)}\tan\alpha)^2+(\sqrt{(a^2-1)}\tan\gamma-\sqrt{(a^2+1)}\tan\beta)^2+(\sqrt{(a^2+1)}\tan\alpha-a\tan\gamma)^2\geq 0$$

$$\Rightarrow(a^2+a^2-1+a^2+1)(\tan^2\alpha+\tan^2\beta+\tan^2\gamma)-\{a\tan\alpha+\sqrt{(a^2-1)}\tan\beta+\sqrt{(a^2+1)}\tan\gamma\}^2\geq 0$$

(using Lagrange's identity)

$$\Rightarrow\ 3a^2(\tan^2\alpha+\tan^2\beta+\tan^2\gamma)-(2a)^2\geq 0$$

$$\therefore\quad 3(\tan^2\alpha+\tan^2\beta+\tan^2\gamma)\geq 4$$

Hence, $2727\quad(\tan^2\alpha+\tan^2\beta+\tan^2\gamma)\geq 3636$

$\therefore$ Least value is 3636.

● ***Ex. 114.*** *If* $\frac{\tan x}{2}=\frac{\tan y}{3}=\frac{\tan z}{5}$, $x+y+z=\pi$ *and* $\tan^2 x+\tan^2 y+\tan^2 z=\frac{38}{K}$ *then* $K=\ldots..$

Sol. (3) $\tan x=2t$, $\tan y=3t$, $\tan z=5t$

Also $\quad x+y+z=\pi$

$\therefore\tan x+\tan y+\tan z=\tan x\tan y\tan z$

$$\Rightarrow\quad t^2=\frac{1}{3}$$

$$\Rightarrow\quad \tan^2 x+\tan^2 y+\tan^2 z=t^2(4+9+25)=38t^2,$$

$$\therefore\quad K=3$$

● ***Ex. 115.*** *If* $\tan\left(\frac{3\pi}{11}\right)+4\sin\left(\frac{2\pi}{11}\right)=\lambda$, *then the value of* $1+\lambda^2+\lambda^4+\lambda^6$ *must be.*

Sol. (1464) Let $\lambda=\tan\left(\frac{3\pi}{11}\right)+4\sin\left(\frac{2\pi}{11}\right)$

$$=\frac{1}{\cos\left(\frac{3\pi}{11}\right)}\left\{\sin\left(\frac{3\pi}{11}\right)+4\sin\left(\frac{2\pi}{11}\right)\cos\left(\frac{3\pi}{11}\right)\right\}$$

$$\lambda\cos\left(\frac{3\pi}{11}\right)=\sin\left(\frac{3\pi}{11}\right)+4\sin\left(\frac{2\pi}{11}\right)\cos\left(\frac{3\pi}{11}\right)$$

$$\lambda^2\cos^2\left(\frac{3\pi}{11}\right)=\sin^2\left(\frac{3\pi}{11}\right)+16\sin^2\left(\frac{2\pi}{11}\right)\cos^2\left(\frac{3\pi}{11}\right)+8\sin\left(\frac{2\pi}{11}\right)\sin\left(\frac{3\pi}{11}\right)\cos\left(\frac{3\pi}{11}\right)$$

$$\lambda^2\left(2\cos^2\left(\frac{3\pi}{11}\right)\right)=2\sin^2\left(\frac{3\pi}{11}\right)+32\sin^2\left(\frac{2\pi}{11}\right)\cos^2\left(\frac{3\pi}{11}\right)+8\sin\left(\frac{2\pi}{11}\right)\sin\left(\frac{6\pi}{11}\right)$$

$$=\left(1-\cos\left(\frac{6\pi}{11}\right)\right)+8\left(1-\cos\frac{4\pi}{11}\right)\left(1+\cos\frac{6\pi}{11}\right)+4\left(\cos\frac{4\pi}{11}-\cos\frac{8\pi}{11}\right)$$

$$=9+7\cos\left(\frac{6\pi}{11}\right)-4\cos\left(\frac{4\pi}{11}\right)-8\cos\left(\frac{4\pi}{11}\right)\cos\left(\frac{6\pi}{11}\right)-4\cos\left(\frac{8\pi}{11}\right)$$

$$=9+7\cos\left(\frac{6\pi}{11}\right)-4\cos\left(\frac{4\pi}{11}\right)-4\left(\cos\left(\frac{10x}{11}\right)+\cos\left(\frac{2\pi}{11}\right)\right)-4\cos\left(\frac{8\pi}{11}\right)$$

$$=9+11\cos\left(\frac{6\pi}{11}\right)-4\left\{\cos\left(\frac{2\pi}{11}\right)+\cos\left(\frac{4\pi}{11}\right)+\cos\left(\frac{6\pi}{11}\right)+\cos\left(\frac{8\pi}{11}\right)+\cos\left(\frac{10\pi}{11}\right)\right\}$$

$$=9+11\cos\left(\frac{6\pi}{11}\right)-\frac{4\cdot\cos\left(\frac{6\pi}{11}\right)\sin\left(\frac{5\pi}{11}\right)}{\sin\left(\frac{\pi}{11}\right)}$$

$$=9+11\cos\left(\frac{6\pi}{11}\right)-\frac{2\left\{\sin\pi-\sin\left(\frac{\pi}{11}\right)\right\}}{\sin\left(\frac{\pi}{11}\right)}$$

$$=9+11\cos\left(\frac{6\pi}{11}\right)+2$$

$$=11\left(1+\cos\left(\frac{6\pi}{11}\right)\right)=11\left(2\cos^2\left(\frac{3\pi}{11}\right)\right)$$

$\therefore\quad \lambda^2=11$

Then, $1+\lambda^2+\lambda^4+\lambda^6=1+11+121+1331=1464$

Subjective Type Examples

● **Ex. 116.** *For all* θ *in* $[0, \pi/2]$, *show that* $\cos(\sin\theta) > \sin(\cos\theta)$.

Sol. We know,

$$\cos\theta + \sin\theta = \sqrt{2}\left[\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\right]$$

$$= \sqrt{2}\left[\sin\frac{\pi}{4}\cos\theta + \cos\frac{\pi}{4}\sin\theta\right]$$

$$= \sqrt{2}\sin\left(\frac{\pi}{4} + \theta\right)$$

$$\Rightarrow \quad \cos\theta + \sin\theta \le \sqrt{2} < \frac{\pi}{2} \qquad \{\text{as } \sqrt{2} = 1.414\}$$

$$\Rightarrow \quad \cos\theta + \sin\theta < \frac{\pi}{2} \qquad [\pi/2 = 1.57 \text{ approx}]$$

$$\Rightarrow \quad \cos\theta < \frac{\pi}{2} - \sin\theta$$

On taking sine both sides;

$$\sin(\cos\theta) < \sin\left(\frac{\pi}{2} - \sin\theta\right)$$

$$\Rightarrow \quad \sin(\cos\theta) < \cos(\sin\theta)$$

$$\therefore \quad \cos(\sin\theta) > \sin(\cos\theta)$$

Alternate Method

For $0 \le x \le \frac{\pi}{2}$

$$x \ge \sin x \qquad \ldots(i)$$

Replace x by $\cos\theta$, we get

$$\cos\theta \ge \sin(\cos\theta) \qquad \ldots(ii)$$

Also, we know $\cos\theta$ is decreasing for $0 \le \theta \le \frac{\pi}{2}$.

$$\text{As } \theta_1 < \theta_2 \Rightarrow \cos\theta_1 > \cos\theta_2 \text{ when } \theta_1, \theta_2 \in [0, \pi/2]$$

$\therefore$ Taking cos on both side of Eq. (i) and putting θ for x, we get

$$\cos\theta \le \cos(\sin\theta) \qquad \ldots(iii)$$

Using Eqs. (ii) and (iii),

$$\cos(\sin\theta) \ge \cos\theta \ge \sin(\cos\theta)$$

$$\Rightarrow \quad \cos(\sin\theta) > \sin(\cos\theta)$$

● **Ex. 117.** *Show that* $2^{\sin x} + 2^{\cos x} \ge 2^{1 - \frac{1}{\sqrt{2}}}$ *for all real* x.

Sol. Clearly, $2^{\sin x}$ and $2^{\cos x}$ are positive, so their AM $\ge$ GM

$$\therefore \quad \frac{2^{\sin x} + 2^{\cos x}}{2} \ge \sqrt{2^{\sin x}.2^{\cos x}} = \sqrt{2^{\sin x + \cos x}} \qquad \ldots(i)$$

As we know,

$$\sin x + \cos x \ge -\sqrt{2}$$

$$[\text{using } -\sqrt{a^2 + b^2} \le a\sin x + b\cos x \le \sqrt{a^2 + b^2}]$$

$$2^{\sin x + \cos x} \ge 2^{-\sqrt{2}}$$

$$\text{or} \quad \sqrt{2^{\sin x + \cos x}} \ge 2^{-\sqrt{2}/2}$$

$$\text{or} \quad \sqrt{2^{\sin x + \cos x}} \ge 2^{-1/\sqrt{2}} \qquad \ldots(ii)$$

From Eqs. (i) and (ii),

$$\frac{2^{\sin x} + 2^{\cos x}}{2} \ge \sqrt{2^{\sin x + \cos x}} \ge 2^{-1/\sqrt{2}}$$

$$\therefore \quad 2^{\sin x} + 2^{\cos x} \ge 2 \cdot 2^{-1/\sqrt{2}}$$

$$\text{or} \quad 2^{\sin x} + 2^{\cos x} \ge 2^{1 - 1\sqrt{2}} \text{ for all values of } x.$$

● **Ex. 118.** *Eliminate* θ *and* ϕ *if*

$$a\sin^2\theta + b\cos^2\theta = m$$

$$b\sin^2\phi + a\cos^2\phi = n$$

and $\quad a\tan\theta = b\tan\phi$

Sol. Dividing $a\sin^2\theta + b\cos^2\theta = m$ by $\cos^2\theta$, we get

$$a\tan^2\theta + b = m\sec^2\theta$$

$$\text{or} \quad (a - m)\tan^2\theta = (m - b) \qquad \ldots(i)$$

Dividing $b\sin^2\phi + a\cos^2\phi = n$ by $\cos^2\phi$, we get

$$b\tan^2\phi + a = n\sec^2\phi$$

$$\text{or} \quad (b - n)\tan^2\phi = (n - a) \qquad \ldots(ii)$$

On dividing Eq. (i) by Eq. (ii), we get

$$\frac{(a - m)}{(b - n)} \cdot \left(\frac{\tan\theta}{\tan\phi}\right)^2 = \frac{m - b}{n - a}$$

$$\Rightarrow \quad \left(\frac{a - m}{b - n}\right) \cdot \frac{b^2}{a^2} = \frac{m - b}{n - a} \qquad [\text{given, } a\tan\theta = b\tan\phi]$$

$$\text{or} \quad b^2(a - m)(n - a) = a^2(b - n)(m - b)$$

$$\text{or} \quad b^2\{(m + n)a - a^2 - mn\} = a^2\{(m + n)b - b^2 - mn\}$$

$$\text{or} \quad (m + n)(ab^2 - a^2b) + mn(a^2 - b^2) = 0$$

$$\text{or} \quad (m + n)ab(b - a) + mn(a - b)(a + b) = 0$$

$$\text{or} \quad (m + n)(ab) = mn(a + b)[a - b \ne 0]$$

● **Ex. 119.** *Let* $\cos A + \cos B + \cos C = \frac{3}{2}$ *in a* ΔABC, *show that the triangle is equilateral.*

Sol. In a triangle, $A + B + C = \pi$

$$\Rightarrow \quad \cos A + \cos B + \cos C = 2\cos\left(\frac{A + B}{2}\right).$$

$$\cos\left(\frac{A - B}{2}\right) + \cos C = \frac{3}{2}$$

$$\Rightarrow \quad 2\cos\left(\frac{\pi}{2} - \frac{C}{2}\right) \cdot \cos\left(\frac{A - B}{2}\right) + 1 - 2\sin^2\frac{C}{2} = \frac{3}{2}$$

$$\Rightarrow \quad 2\sin^2\frac{C}{2} - 2\sin\left(\frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right) - 1 + \frac{3}{2} = 0$$

$$\Rightarrow \quad 4\sin^2\frac{C}{2} - 4\sin\left(\frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right) + 1 = 0 \qquad ...(i)$$

Now, Eq. (i) is quadratic in $(\sin C/2)$ and is real.

$$\therefore \quad D \geq 0$$

$$\Rightarrow \quad 16\cos^2\left(\frac{A-B}{2}\right) - 16 \geq 0 \Rightarrow \cos^2\left(\frac{A-B}{2}\right) - 1 \geq 0$$

$$\Rightarrow \quad \cos^2\left(\frac{A-B}{2}\right) \geq 1$$

which is only possible if $\cos^2\left(\frac{A-B}{2}\right) = 1$

$$\Rightarrow \quad \cos^2\left(\frac{A-B}{2}\right) = 1$$

$$\frac{A-B}{2} = 0$$

$$\Rightarrow \quad A = B \qquad ...(ii)$$

Similarly, we can show $B = C$, $C = A$. Hence, the triangle is equilateral.

● ***Ex. 120.*** *If* $\frac{\tan 3A}{\tan A} = k$, *show that* $\frac{\sin 3A}{\sin A} = \frac{2k}{k-1}$ *and hence or otherwise prove that either* $k > 3$ *or* $k < \frac{1}{3}$.

Sol. $$\frac{\tan 3A}{\tan A} = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A} \cdot \frac{1}{\tan A} = k$$

$$\Rightarrow \quad \frac{3 - \tan^2 A}{1 - 3\tan^2 A} = k$$

$$\Rightarrow \quad (3 - \tan^2 A) = k(1 - 3\tan^2 A)$$

$$\Rightarrow \quad (3k - 1)\tan^2 A = k - 3$$

$$\Rightarrow \quad \tan^2 A = \left(\frac{k-3}{3k-1}\right) \qquad ...(i)$$

Now, $$\frac{\sin 3A}{\sin A} = \frac{3\sin A - 4\sin^3 A}{\sin A} = 3 - 4\sin^2 A$$

$$\Rightarrow \quad 3 - 4\sin^2 A = 3 - \frac{4}{\operatorname{cosec}^2 A} = 3 - \frac{4}{1 + \cot^2 A}$$

$$= 3 - \frac{4}{1 + \frac{1}{\tan^2 A}}$$

$$\Rightarrow \quad 3 - \frac{4}{1 + \left(\frac{3k-1}{k-3}\right)} \qquad \text{[using Eq. (i)]}$$

$$\Rightarrow \quad 3 - \frac{4(k-3)}{4(k-1)} = \frac{3k - 3 - k + 3}{k-1} = \frac{2k}{k-1} \qquad ...(ii)$$

Again from Eq. (i),

$$\tan^2 A = \frac{k-3}{3k-1} \qquad [\tan A \neq 0 \text{ and } \tan^2 A > 0]$$

$$\Rightarrow \quad \frac{k-3}{3k-1} > 0, \text{ using number line rule.}$$

which shows $k < 1/3$ or $k > 3$

● ***Ex. 121.*** *Let* A, B, C *be three angles such that* $A = \pi/4$ *and* $\tan B \tan C = p$. *Find all possible values of* p *such that* A, B, C *are the angles of triangles.*

Sol. Let us assume ΔABC.

$$\therefore \quad A + B + C = \pi$$

$$\Rightarrow \quad B + C = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \qquad ...(i) \ [\because A = \pi/4, \text{ given}]$$

Also, $$0 < B, C < 3\pi/4$$

$$\Rightarrow \quad \tan B \tan C = p$$

$$\Rightarrow \quad \frac{\sin B \cdot \sin C}{\cos B \cdot \cos C} = \frac{p}{1}$$

$$\Rightarrow \quad \frac{\sin B \cdot \sin C + \cos B \cdot \cos C}{\cos B \cdot \cos C - \sin B \cdot \sin C} = \frac{1+p}{1-p}$$

$$\Rightarrow \quad \frac{\cos(B-C)}{\cos(B+C)} = \frac{1+p}{1-p}$$

$$\Rightarrow \quad \cos(B-C) = \left(\frac{1+p}{1-p}\right)\cos\left(\frac{3\pi}{4}\right)$$

[using Eq. (i), $B + C = 3\pi/4$]

$$\Rightarrow \quad \cos(B-C) = \frac{1+p}{\sqrt{2}(p-1)} \qquad ...(ii)$$

Since, B or C can vary from 0 to $3\pi/4$.

$$\therefore \quad 0 \leq (B - C) < 3\pi/4$$

$$\Rightarrow \quad -\frac{1}{\sqrt{2}} < \cos(B-C) \leq 1 \qquad ...(iii)$$

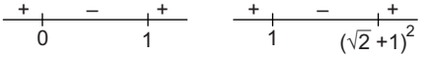
From Eqs. (ii) and (iii), we get

$$-\frac{1}{\sqrt{2}} < \frac{1+p}{\sqrt{2}(p-1)} \leq 1$$

$$\therefore \quad -\frac{1}{\sqrt{2}} < \frac{1+p}{\sqrt{2}(p-1)} \text{ and } \frac{1+p}{\sqrt{2}(p-1)} \leq 1$$

$$\Rightarrow \quad 0 < 1 + \frac{p+1}{p-1} \text{ and } \frac{(p+1) - \sqrt{2}(p-1)}{\sqrt{2}(p-1)} \leq 0$$

$$\Rightarrow \quad \frac{2p}{(p-1)} > 0 \quad \text{and} \quad \frac{[p - (\sqrt{2}+1)^2]}{(p-1)} \geq 0$$



$$\Rightarrow \quad p < 0 \text{ or } p > 1 \text{ and } p < 1 \text{ or } p \geq (\sqrt{2}+1)^2$$

The combining above expressions;

$$p < 0 \text{ or } p \geq (\sqrt{2}+1)^2$$

i.e. $$p \in (-\infty, 0) \cup [(\sqrt{2}+1)^2, \infty).$$

● **Ex. 122.** *If ABC is a triangle and* $\tan\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2}$ *are in HP, then find minimum value of* $\cot\frac{B}{2}$.

Sol. $A + B + C = \pi$

$$\Rightarrow \quad \frac{A}{2} + \frac{B}{2} = \frac{\pi}{2} - \frac{C}{2}$$

$$\Rightarrow \quad \cot\left(\frac{A}{2} + \frac{B}{2}\right) = \cot\left(\frac{\pi}{2} - \frac{C}{2}\right)$$

$$\Rightarrow \quad \frac{\cot\frac{A}{2}\cdot\cot\frac{B}{2} - 1}{\cot\frac{A}{2} + \cot\frac{B}{2}} = \tan\left(\frac{C}{2}\right) = \frac{1}{\cot\left(\frac{C}{2}\right)}$$

$$\Rightarrow \quad \cot\frac{A}{2}\cdot\cot\frac{B}{2}\cdot\cot\frac{C}{2} = \cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} \quad \ldots(i)$$

But $\tan\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2}$ are in HP

$$\Rightarrow \quad \cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2} \text{ are in AP}$$

$$\therefore \quad \cot\frac{A}{2} + \cot\frac{C}{2} = 2\cot\frac{B}{2} \quad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$\cot\frac{A}{2}\cdot\cot\frac{B}{2}\cot\frac{C}{2} = 3\cot\frac{B}{2}$$

$$\Rightarrow \quad \cot\frac{A}{2}\cdot\cot\frac{C}{2} = 3 \quad \ldots(iii)$$

As we know, AM ≥ GM

$$\Rightarrow \quad \frac{\cot\frac{A}{2} + \cot\frac{C}{2}}{2} \geq \sqrt{\cot\frac{A}{2}\cdot\cot\frac{C}{2}}$$

$$\Rightarrow \quad \frac{2\cot\frac{B}{2}}{2} \geq \sqrt{3} \qquad [\{\text{using Eq. (iii)}]$$

$$\Rightarrow \quad \cot\frac{B}{2} \geq \sqrt{3}$$

$\therefore$ Minimum value of $\cot\frac{B}{2}$ is $\sqrt{3}$.

● **Ex 123.** *(i) If* $\tan A - \tan B = x$ *and* $\cot B - \cot A = y$.

Prove that $\cot(A - B) = \frac{1}{x} + \frac{1}{y}$.

(ii) If $2\cos A = x + \frac{1}{x}$, $2\cos B = y + \frac{1}{y}$, *then show that*

$2\cos(A - B) = \frac{x}{y} + \frac{y}{x}$.

Sol. (i) If $\cot B - \cot A = y \Rightarrow \frac{\tan A - \tan B}{\tan A \tan B} = y$

$$\therefore \quad \frac{x}{y} = \tan A \tan B$$

Now, $\cot(A - B) = \frac{1}{\tan(A - B)}$

$$= \frac{1 + \tan A \tan B}{\tan A - \tan B}$$

$$= \frac{1 + \frac{x}{y}}{x} = \frac{1}{x} + \frac{1}{y} = \text{RHS}$$

(ii) $2\cos A = x + \frac{1}{x}$, since $4\sin^2 A = 4$

$$-4\cos^2 A = 4 - \left(x + \frac{1}{x}\right)^2$$

$$\therefore \quad 4\sin^2 A = -\left[\left(x + \frac{1}{x}\right)^2 - 4\right]$$

or $\quad 4\sin^2 A = i^2\left[\left(x - \frac{1}{x}\right)^2\right]$

$$\Rightarrow \quad 2\sin A = i\left(x - \frac{1}{x}\right) \quad \ldots(i)$$

Similarly, $\quad 2\cos B = y + \frac{1}{y}$

$$\Rightarrow \quad 2\sin B = i\left(y - \frac{1}{y}\right) \quad \ldots(ii)$$

Now, $2\cos(A - B) = 2[\cos A \cos B + \sin A \sin B]$

$$= \frac{2}{4}\left[\left(x + \frac{1}{x}\right)\left(y + \frac{1}{y}\right) + i^2\left(x - \frac{1}{x}\right)\left(y - \frac{1}{y}\right)\right)$$

$$= \frac{1}{2}\left[\left\{xy + \frac{1}{xy} + \frac{y}{x} + \frac{x}{y}\right\} - \left\{xy + \frac{1}{xy} - \frac{y}{x} - \frac{x}{y}\right\}\right]$$

$$= \frac{1}{2}\left[2\frac{y}{x} + 2\frac{x}{y}\right] = \frac{x}{y} + \frac{y}{x} = \text{RHS}$$

● **Ex. 124.** *If* $\tan\theta \tan\phi = \sqrt{\frac{a-b}{a+b}}$, *then prove that* $(a - b\cos 2\theta)(a - b\cos 2\phi)$ *is independent of* θ *and* ϕ.

Sol. Let us put,

$$\tan\theta = t_1 \text{ and } \tan\phi = t_2$$

$$\therefore \quad t_1^2 . t_2^2 = \frac{a-b}{a+b} \quad \ldots(i)$$

$$\left\{\text{given}, \tan\theta\tan\phi = \sqrt{\frac{a-b}{a+b}}\right\}$$

Also, $\quad \cos 2\theta = \frac{1 - \tan^2\theta}{1 + \tan^2\theta} = \frac{1 - t_1^2}{1 + t_1^2} \quad \ldots(ii)$

$$\cos 2\phi = \frac{1 - \tan^2\phi}{1 + \tan^2\phi} = \frac{1 - t_2^2}{1 + t_2^2} \quad \ldots(iii)$$

Now, $(a - b\cos 2\theta).(a - b\cos 2\phi)$

$$= \left\{a - b\left(\frac{1-t_1^2}{1+t_1^2}\right)\right\}\cdot\left\{a - b\left(\frac{1-t_2^2}{1+t_2^2}\right)\right\} \text{ [using Eqs. (ii) and (iii)]}$$

$$= \left\{\frac{(a-b)+(a+b)\,t_1^2}{1+t_1^2}\right\}\cdot\left\{\frac{(a-b)+(a+b)\,t_2^2}{1+t_2^2}\right\}$$

$$= \frac{(a+b)}{(1+t_1^2)}\left[\frac{a-b}{a+b}+t_1^2\right]\cdot\frac{(a+b)}{(1+t_2^2)}\left[\frac{a-b}{a+b}+t_2^2\right]$$

$$= \frac{(a+b)}{(1+t_1^2)}[t_1^2\,t_2^2+t_1^2]\cdot\frac{(a+b)}{(1+t_2^2)}[t_1^2t_2^2+t_2^2] \quad \text{[using Eq. (i)]}$$

$$= (a+b)^2\cdot(t_1^2.t_2^2) = (a+b)^2\cdot\frac{(a-b)}{(a+b)} = (a^2-b^2)$$

So, $(a - b\cos 2\theta)(a - b\cos 2\phi) = a^2 - b^2$, which is independent of θ and ϕ.

● ***Ex. 125.*** *Find all possible real values of x and y satisfying.*

$$\sin^2 x + 4\sin^2 y - \sin x - 2\sin y - 2\sin x\,\sin y + 1 = 0,$$
$$\forall\ x, y \in [0, \pi/2]$$

Sol. Given, equation can be rewritten as,

$$\sin^2 x - \sin x(1 + 2\sin y) + (4\sin^2 y - 2\sin y + 1) = 0$$

$$\Rightarrow \sin x = \frac{(1+2\sin y) \pm \sqrt{(1+2\sin y)^2 - 4(4\sin^2 y - 2\sin y + 1)}}{2}$$

$$= \frac{(1+2\sin y) \pm \sqrt{-3 - 12\sin^2 y + 12\sin y}}{2}$$

$$= \frac{(1+2\sin y) \pm \sqrt{-3(2\sin y - 1)^2}}{2} \qquad \ldots(i)$$

Since, $\sin x$ is real.

$\therefore$ From equation (i) is real only if,

$$2\sin y - 1 = 0 \text{ or } \sin y = \frac{1}{2} \quad \text{and} \quad \sin x = \frac{1+1}{2} = 1.$$

$$\Rightarrow \qquad y = \frac{\pi}{6} \text{ and } x = \frac{\pi}{2} \text{ as } x, y \in [0, \pi/2].$$

● **Ex. 126.** *Find the roots of the following cubic equations*
$2x^3 - 3x^2\cos(A-B) - 2x\cos^2(A+B) + \sin 2A$
$\sin 2B\cos(A-B) = 0.$

Sol. We know,

$$\sin 2A\sin 2B = \frac{1}{2}[\cos(2A-2B) - \cos(2A-2B)]$$

$$= \frac{1}{2}[2\cos^2(A-B) - 1 - 2\cos^2(A+B) + 1]$$

$$= \cos^2(A-B) - \cos^2(A+B)$$

$\therefore$ $\sin 2A.\sin 2B = \cos^2(A-B) - \cos^2(A+B)$ ...(i)

Now, $2x^3 - 3x^2\cos(A-B) - 2x\cos^2(A+B) + \sin 2A.\sin 2B.\cos(A-B) = 0$

$$\Rightarrow 2x^3 - 3x^2\cos(A-B) - 2x\cos^2(A+B) + \cos^3(A-B) - \cos^2(A+B).\cos(A-B) = 0$$

By inspection, we find that $x = -\frac{1}{2}\cos(A-B)$ because

$$\left(-\frac{1}{4} - \frac{3}{4} + 1\right)\cos^3(A-B) + \cos^2(A+B)\cos(A-B) - \cos^2(A+B)\cos(A-B) = 0$$

Hence, $2x + \cos(A-B)$ is factor of the given equation which when divided by it, given the other factor as,

$$x^2 - 2x\cos(A-B) + \cos^2(A-B) - \cos^2(A+B) = 0$$

$$2\cos(A-B) \pm$$

So, $x = \dfrac{\sqrt{4\cos^2(A-B) - 4\cos^2(A-B) + 4\cos^2(A+B)}}{2}$

$$x = \frac{2\cos(A-B) \pm 2\cos(A+B)}{2}$$

$x = \cos(A-B) + \cos(A+B)$ or $\cos(A-B) - \cos(A+B)$

Hence, the roots are,

$$2\cos A\cos B,\ 2\sin A\sin B \text{ and } -\frac{1}{2}\cos(A-B).$$

● ***Ex. 127.*** *If $m^2 + m'^2 + 2mm'\cos\theta = 1.$*

$$n^2 + n'^2 + 2nn'\cos\theta = 1$$

and $mn + m'n' + (mn' + m'n)\cos\theta = 0$, then prove that $m^2 + n^2 = \text{cosec}^2\theta$.

Sol. $m^2 + m'^2 + 2mm'\cos\theta = 1$

or $(m^2\cos^2\theta + m^2\sin^2\theta) + m'^2 + 2mm'\cos\theta = 1$

or $m^2\cos^2\theta + 2mm'\cos\theta + m'^2 = 1 - m^2\sin^2\theta$

or $(m\cos\theta + m')^2 = 1 - m^2\sin^2\theta$ ...(i)

Similarly, $n^2 + n'^2 + 2nn'\cos\theta = 1$

$\Rightarrow$ $(n\cos\theta + n')^2 = 1 - n^2\sin^2\theta$ ...(ii)

Finally, $mn + m'n' + (mn' + m'n)\cos\theta = 0$

$\Rightarrow$ $(mn\cos^2\theta + mn\sin^2\theta) + m'n' + mn'\cos\theta + m'n\cos\theta = 0$

$\Rightarrow$ $mn\cos^2\theta + m'n\cos\theta + m'n' + mn'\cos\theta = -mn\sin^2\theta$

$\Rightarrow$ $n\cos\theta(m\cos\theta + m') + n'(m' + m\cos\theta) = -mn\sin^2\theta$

$\Rightarrow$ $(m\cos\theta + m')(n\cos\theta + n') = -mn\sin^2\theta$

or $(m\cos\theta + m')^2(n\cos\theta + n')^2 = m^2n^2\sin^4\theta$

$\Rightarrow$ $(1 - m^2\sin^2\theta)(1 - n^2\sin^2\theta) = m^2n^2\sin^4\theta.$ [using Eqs. (i) and (ii)]

$\Rightarrow$ $1 - (m^2 + n^2)\sin^2\theta + m^2n^2\sin^4\theta = m^2n^2\sin^4\theta$

$\Rightarrow$ $(m^2 + n^2)\sin^2\theta = 1$

$\Rightarrow$ $m^2 + n^2 = \text{cosec}^2\theta$

● **Ex 128.** *Prove that from the equality* $\frac{\sin^4\alpha}{a}+\frac{\cos^4\alpha}{b}=\frac{1}{a+b}$ *follows the relation,*

$$\frac{\sin^8\alpha}{a^3}+\frac{\cos^8\alpha}{b^3}=\frac{1}{(a+b)^3} \text{ and}$$

$$\frac{\sin^{4n}x}{a^{2n-1}}+\frac{\cos^{4n}x}{b^{2n-1}}=\frac{1}{(a+b)^{2n-1}}.$$

Sol. Given condition can be rewritten as,

$$b(\sin^2\alpha)^2+a\cos^4\alpha=\frac{ab}{a+b}$$

$$\Rightarrow \quad b(1-\cos^2\alpha)^2+a\cos^4\alpha=\frac{ab}{a+b}$$

$$\Rightarrow \quad b(\cos^4\alpha-2\cos^2\alpha+1)+a\cos^4\alpha=\frac{ab}{a+b}$$

$$\Rightarrow \quad (a+b)^2\cos^4\alpha-2b(a+b)\cos^2\alpha+b(a+b)=ab$$

$$(a+b)^2\cos^4\alpha-2b(a+b)\cos^2\alpha+b^2=0$$

$$\Rightarrow \quad [(a+b)\cos^2\alpha-b]^2=0$$

$$\Rightarrow \quad \cos^2\alpha=\frac{b}{a+b} \Rightarrow \sin^2\alpha=\frac{a}{a+b} \qquad \text{...(i)}$$

$$\therefore \quad \frac{\sin^8\alpha}{a^3}+\frac{\cos^8\alpha}{b^3}=\frac{a^4}{a^3(a+b)^4}+\frac{b^4}{b^3(a+b)^4}$$

$$=\frac{a}{(a+b)^4}+\frac{b}{(a+b)^4}$$

$$=\frac{a+b}{(a+b)^4}=\frac{1}{(a+b)^3}$$

$$\therefore \quad \frac{\sin^8\alpha}{a^3}+\frac{\cos^8\alpha}{b^3}=\frac{1}{(a+b)^3}$$

Now, $\frac{\sin^{4n}x}{a^{2n-1}}+\frac{\cos^{4n}x}{b^{2n-1}}=\frac{a^{2n}}{a^{2n-1}(a+b)^{2n}}+\frac{b^{2n}}{b^{2n-1}(a+b)^{2n}}$

$$=\frac{a+b}{(a+b)^{2n}}=\frac{1}{(a+b)^{2n-1}}.$$

● **Ex 129.** *If* $a_{r+1}=\sqrt{\frac{1}{2}(1+a_r)}$, *the prove that*

$$\cos\left(\frac{\sqrt{1-a_0^2}}{a_1\cdot a_2\cdot a_3\ldots\text{to}\,\infty}\right)=a_0.$$

Sol. Let $a_0=\cos\theta$, then $a_{r+1}=\sqrt{\frac{1}{2}(1+a_r)}$ gives

$$a_1=\sqrt{\frac{1}{2}(1+a_0)}=\sqrt{\frac{1}{2}(1+\cos\theta)}=\cos\frac{\theta}{2}$$

$$a_2=\sqrt{\frac{1}{2}(1+a_1)}=\sqrt{\frac{1}{2}\left(1+\cos\frac{\theta}{2}\right)}=\cos\frac{\theta}{2^2}$$

$$a_3=\sqrt{\frac{1}{2}(1+a_2)}=\sqrt{\frac{1}{2}\left(1+\cos\frac{\theta}{2^2}\right)}=\cos\frac{\theta}{2^3},\ \text{... etc.}$$

$$\therefore \quad a_1\cdot a_2\cdot a_3\ldots a_n$$

$$\Rightarrow \quad \cos\frac{\theta}{2}\cdot\cos\frac{\theta}{2^2}\cdot\cos\frac{\theta}{2^3}\ldots\cos\frac{\theta}{2^n}$$

$$\Rightarrow \quad \frac{\cos\frac{\theta}{2}\cdot\cos\frac{\theta}{2^2}\cdot\cos\frac{\theta}{2^3}\ldots\cos\frac{\theta}{2^n}\cdot 2\sin\frac{\theta}{2^n}}{2\sin\frac{\theta}{2^n}}$$

$$\Rightarrow \quad \frac{\cos\frac{\theta}{2}\cdot\cos\frac{\theta}{2^2}\ldots\cos\frac{\theta}{2^{n-1}}\cdot\sin\frac{\theta}{2^{n-1}}}{2\sin\frac{\theta}{2^n}}$$

$$[\because 2\sin\alpha.\cos\alpha=\sin 2\alpha]$$

$$\Rightarrow \quad \frac{\cos\frac{\theta}{2}\cdot\cos\frac{\theta}{2^2}\ldots\cos\frac{\theta}{2^{n-2}}\cdot 2\sin\frac{\theta}{2^{n-2}}}{2^2\sin\frac{\theta}{2^n}}$$

$$=\ldots\ldots\ldots\ldots=\frac{\cos\frac{\theta}{2^{n-(n-1)}}\cdot\sin\frac{\theta}{2^{n-(n-1)}}}{2^{n-1}\cdot\sin\frac{\theta}{2^n}}$$

$$=\frac{\cos\frac{\theta}{2}\cdot\sin\frac{\theta}{2}}{2^{n-1}\cdot\sin\frac{\theta}{2^n}}=\frac{\sin\theta}{2^n.\sin\frac{\theta}{2^n}}$$

$$\therefore \quad a_1.a_2\ldots\text{to}\,\infty=\lim_{n\to\infty}\frac{\sin\theta}{2^n.\sin\frac{\theta}{2^n}}=\lim_{n\to\infty}\frac{\sin\theta}{\theta\cdot\frac{\sin(\theta/2^n)}{(\theta/2^n)}}=\frac{\sin\theta}{\theta}$$

$$\therefore \quad \frac{\sqrt{1-a_0^2}}{a_1a_2a_3\ldots\text{to}\,\infty}=\frac{\sqrt{1-\cos^2\theta}}{\frac{\sin\theta}{\theta}}=\theta$$

$$\therefore \quad \cos\left(\frac{\sqrt{1-a_0^2}}{\frac{a_1}{a_1a_2a_3\ldots\text{to}\,\infty}}\right)=\cos\theta=a_0.$$

● **Ex. 130.** *Evaluate* $\sum_{r=2}^{n}\sin r\alpha$, *where* $(n+2)\alpha=2\pi$ *(without using formula.)*

Sol. Let $S=\sum_{r=2}^{n}\sin r\alpha=\sin 2\alpha+\sin 3\alpha+\sin 4\alpha+\ldots+\sin n\alpha$

$$\therefore \quad 2\sin\frac{\alpha}{2}\cdot S=2\sin\alpha/2\sin 2\alpha+2\sin\frac{\alpha}{2}\cdot\sin 3\alpha+2\sin\frac{\alpha}{2}$$

$$\cdot\sin 4\alpha+\ldots+2\sin\alpha/2.\sin n\alpha$$

$$=\left\{\cos\frac{3\alpha}{2}-\cos\frac{5\alpha}{2}\right\}+\left\{\cos\frac{5\alpha}{2}-\cos\frac{7\alpha}{2}\right\}+\ldots+$$

$$\left\{\cos\left(n-\frac{1}{2}\right)\alpha-\cos\left(n+\frac{1}{2}\right)\alpha\right\}$$

$$=\cos\frac{3\alpha}{2}-\cos\left(n+\frac{1}{2}\right)\alpha$$

$$= 2\sin\left(\frac{\frac{3\alpha}{2} + \left(n+\frac{1}{2}\right)\alpha}{2}\right)\sin\left(\frac{\left(n+\frac{1}{2}\right)\alpha - \frac{3\alpha}{2}}{2}\right)$$

$$2\sin\frac{\alpha}{2}\cdot S = 2\sin\frac{(n+2)\alpha}{2}\cdot\sin\frac{(n-1)\alpha}{2}$$

$$\Rightarrow \quad S = \frac{\sin\frac{(n-1)\alpha}{2}}{\sin\frac{\alpha}{2}}\cdot\sin\frac{(n+2)\alpha}{2}$$

$$= \frac{\sin\frac{(n-1)\alpha}{2}}{\sin\frac{\alpha}{2}}\cdot\sin\frac{2\pi}{2} = 0$$

● ***Ex. 131.*** *Sum the series* $\sqrt{1+\cos\alpha} + \sqrt{1+\cos 2\alpha} + \sqrt{1+\cos 3\alpha} + \ldots$ *to n terms.*

Sol. We have,

$$\sqrt{1+\cos\alpha} + \sqrt{1+\cos 2\alpha} + \sqrt{1+\cos 3\alpha} + \ldots + \sqrt{1+\cos n\alpha}$$

$$= \sqrt{2\cos^2\alpha/2} + \sqrt{2\cos^2\alpha} + \sqrt{2\cos^2\frac{3\alpha}{2}} + \ldots \text{ to } n \text{ terms}$$

$$= \sqrt{2}\left\{\cos\frac{\alpha}{2} + \cos\frac{2\alpha}{2} + \cos\frac{3\alpha}{2} + \ldots + \text{to } n \text{ terms}\right\}$$

$$= \sqrt{2}\cdot\frac{\sin\frac{n\alpha}{4}}{\sin\frac{\alpha}{4}}\cdot\cos\left\{\frac{\alpha}{2} + (n-1)\frac{\alpha}{4}\right\} \qquad \{\text{using formula}\}$$

$$= \sqrt{2}\frac{\sin\frac{n\alpha}{4}}{\sin\frac{\alpha}{4}}.\cos\left\{(n+1)\frac{\alpha}{4}\right\}$$

● ***Ex. 132.*** *If* $A + B + C = \pi$, *show that*

$$\cot A + \cot B + \cot C - \operatorname{cosec} A \operatorname{cosec} B \cdot \operatorname{cosec} C = \cot A \cdot \cot B \cdot \cot C$$

Sol. $\text{LHS} = \dfrac{\cos A}{\sin A} + \dfrac{\cos A}{\sin B} + \dfrac{\cos C}{\sin C} - \dfrac{1}{\sin A\cdot\sin B\cdot\sin C}$

$$= \frac{\cos A\cdot\sin B\sin C + \cos B\sin A\sin C + \cos C\sin A\sin B - 1}{\sin A\sin B\sin C}$$

$$= \frac{\sin C(\cos A\sin B + \cos B\sin A) + \cos C\sin A\sin B - 1}{\sin A\sin B\sin C}$$

$$= \frac{\sin C\sin(A+B) + \cos C\sin A\sin B - 1}{\sin A\sin B\sin C}$$

$$= \frac{\sin^2 C + \cos C\sin A\sin B - 1}{\sin A\sin B\sin C}$$

$$[\text{using } \sin(A+B) = \sin(\pi - C) = \sin C]$$

$$= \frac{\cos C.\sin A\sin B - \cos^2 C}{\sin A\sin B\sin C}$$

$$= \frac{\cos C\{\sin A\sin B - \cos C\}}{\sin A\sin B\sin C}$$

$$[\because \cos C = \cos(\pi - (A+B)) = -\cos(A+B)]$$

$$= \frac{\cos C\{\sin A\sin B + \cos(A+B)\}}{\sin A\sin B\sin C}$$

$$= \frac{\cos C\{\sin A\sin B + \cos A\cos B - \sin A\sin B\}}{\sin A\sin B\sin C}$$

$$= \frac{\cos A\cos B\cos C}{\sin A\sin B\sin C} = \cot A\cdot\cot B\cdot\cot C = \text{RHS}$$

● ***Ex. 133.*** *In* ΔABC, *if* $\cot\theta = \cot A + \cot B + \cot C$, *prove that* $\sin^3\theta = \sin(A-\theta)\sin(B-\theta)\sin(C-\theta)$.

Sol. We have, $\cot\theta = \cot A + \cot B + \cot C$

$$\Rightarrow \quad \cot(\theta) - \cot(A) = \cot B + \cot C$$

$$\Rightarrow \quad \frac{\cos\theta}{\sin\theta} - \frac{\cos A}{\sin A} = \frac{\cos B}{\sin B} + \frac{\cos C}{\sin C}$$

$$\Rightarrow \quad \frac{\cos\theta\sin A - \cos A\sin\theta}{\sin\theta\sin A} = \frac{\cos B\sin C + \sin B\cos C}{\sin B\sin C}$$

$$\Rightarrow \quad \frac{\sin(A-\theta)}{\sin A\sin\theta} = \frac{\sin(B+C)}{\sin B\sin C}$$

$$\Rightarrow \quad \sin(A-\theta) = \frac{\sin^2 A\sin\theta}{\sin B\sin C} \qquad \ldots(\text{i})$$

Similarly, $\sin(B-\theta) = \dfrac{\sin^2 B\sin\theta}{\sin A\sin C}$ ...(ii)

and $\sin(C-\theta) = \dfrac{\sin^2 C\sin\theta}{\sin A\sin B}$ ...(iii)

Multiplying Eqs. (i), (ii) and (iii), we get

$$\sin(A-\theta)\sin(B-\theta)\sin(C-\theta) = \sin^3\theta.$$

● ***Ex. 134.*** *If A, B, C and D are angles of a quadrilateral and* $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{D}{2} = \frac{1}{4}$, *then prove that* $A = B = C = D = \pi/2$.

Sol. Now, $\left(2\sin\frac{A}{2}\cdot\sin\frac{B}{2}\right)\cdot\left(2\sin\frac{C}{2}\cdot\sin\frac{D}{2}\right) = 1$

$$\Rightarrow \left\{\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right\}\left\{\cos\left(\frac{C-D}{2}\right) - \cos\left(\frac{C+D}{2}\right)\right\} = 1$$

Since, $A + B = 2\pi - (C + D)$, the above equation becomes,

$$\left(\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right)\left(\cos\frac{C-D}{2} + \cos\frac{A+B}{2}\right) = 1$$

$$\Rightarrow \cos^2\left(\frac{A+B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\left\{\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{C-D}{2}\right\} + 1\right.$$

$$- \cos\left(\frac{A-B}{2}\right)\cos\left(\frac{C-D}{2}\right) = 0.$$

This is a quadratic equation in $\cos\left(\dfrac{A+B}{2}\right)$ which has real roots.

$$\Rightarrow \left\{\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{C-D}{2}\right)\right\}^2 - 4\left\{1-\cos\left(\frac{A-B}{2}\right)\cdot\cos\left(\frac{C-D}{2}\right)\right\}\geq 0$$

$$\Rightarrow \left(\cos\frac{A-B}{2}+\cos\frac{C-D}{2}\right)^2 \geq 4$$

$$\Rightarrow \cos\frac{A-B}{2}+\cos\frac{C-D}{2}\geq 2$$

Now, both $\cos\frac{A-B}{2}$ and $\cos\frac{C-D}{2}\leq 1$

$$\Rightarrow \cos\frac{A-B}{2}=1=\cos\frac{C-D}{2}$$

$$\Rightarrow \frac{A-B}{2}=0=\frac{C-D}{2}$$

$$\Rightarrow A=B, C=D$$

Similarly, $A=C, B=D$

$$\Rightarrow A=B=C=D=\pi/2$$

● ***Ex. 135.*** *If α, β are two different values of θ which satisfy is $bc\cos\theta\cos\phi + ac\sin\theta\sin\phi = ab$, then prove that $(b^2+c^2-a^2)\cos\alpha\cos\beta + ac\sin\alpha\sin\beta = a^2+b^2-c^2$.*

Sol. We have, $bc\cos\theta\cos\phi = ab - ac\sin\theta\sin\phi$.

$$\Rightarrow b^2c^2\cos^2\theta\cos^2\phi = a^2b^2 + a^2c^2\sin^2\theta\sin^2\phi - 2a^2bc\sin\theta\sin\phi$$

$$\Rightarrow (a^2c^2\sin^2\phi + b^2c^2\cos^2\phi)\sin^2\theta - 2a^2bc\sin\theta\sin\phi + a^2b^2 - b^2c^2\cos^2\phi = 0$$

$$\Rightarrow \sin\alpha\sin\beta = \frac{a^2b^2 - b^2c^2\cos^2\phi}{a^2c^2\sin^2\phi + b^2c^2\cos^2\phi} \quad \ldots(i)$$

Similarly, $ac\sin\theta\sin\phi = ab - bc\cos\phi\cos\theta$

$$\Rightarrow a^2c^2\sin^2\theta\sin^2\phi = a^2b^2 + b^2c^2\cos^2\theta\cos^2\phi - 2ab^2c\cos\theta\cos\phi$$

$$\therefore \cos\alpha\cos\beta = \frac{a^2b^2 - a^2c^2\sin^2\phi}{a^2c^2\sin^2\phi + b^2c^2\cos^2\phi} \quad \ldots(ii)$$

On substituting the value from Eqs. (i) and (ii) in $(b^2+c^2-a^2)\cos\alpha\cos\beta + ac\sin\alpha\sin\beta$, we get

$$\Rightarrow \frac{(b^2+c^2-a^2)(a^2b^2 - a^2c^2\sin^2\phi) + ac(a^2b^2 - b^2c^2\cos^2\phi)}{a^2c^2\sin^2\phi + b^2c^2\cos^2\phi}$$

$$\Rightarrow (a^2+b^2-c^2) = \text{RHS}$$

● ***Ex. 136.*** *Find all number pairs x, y that satisfy the equaion;*

$$\tan^4 x + \tan^4 y + 2\cot^2 x\cot^2 y = 3 + \sin^2(x+y);\ \forall\ x, y \in \left[0, \frac{\pi}{2}\right]$$

Sol. We know, $a^4 + b^4 \geq 2a^2b^2$ {AM ≥ GM}

$$\therefore \tan^4 x + \tan^4 y \geq 2\tan^2 x\tan^2 y \quad \ldots(i)$$

Equality occurring only when $\tan^2 x = \tan^2 y = 1$.

Also, $\tan^2 x\tan^2 y + \cot^2 x\cot^2 y \geq 2$ …(ii)

Since, $a + \frac{1}{a} \geq 2$ and equality occurring only when $a = 1$, i.e. $\tan^2 x\tan^2 y = 1$

From Eqs. (i) and (ii);

$$\tan^4 x + \tan^4 y + 2\cot^2 x\cot^2 y \geq 4 \quad \ldots(iii)$$

Also, $\text{RHS} = 3 + \sin^2(x+y) \leq 4$ …(iv)

From Eqs. (iii) and (iv),

$$\text{LHS} = \text{RHS} = 4$$

$$\Rightarrow \tan^2 x = \tan^2 y = \tan^2 x\tan^2 y = 1$$

$$\Rightarrow \tan x = \tan y = \pm 1$$

$$\Rightarrow \tan x = \tan y = 1 \quad \{\text{as } x, y \in [0, \pi/2]\}$$

$$\therefore x = y = \pi/4$$

Only one solution i.e. $(x = \pi/4, y = \pi/4)$.

● ***Ex. 137.*** *Prove that* $\tan\frac{3\pi}{11} + 4\sin\frac{2\pi}{11} = \sqrt{11}$.

Sol. Let $y = \tan\frac{3\pi}{11} + 4\sin\frac{2\pi}{11} = \frac{1}{\cos\frac{3\pi}{11}}\left(\sin\frac{3\pi}{11} + 4\sin\frac{2\pi}{11}\cos\frac{3\pi}{11}\right)$

$$y^2\cdot\cos^2\frac{3\pi}{11} = \sin^2\frac{3\pi}{11} + 16\sin^2\frac{2\pi}{11}\cdot\cos^2\frac{3\pi}{11} + 8\sin\frac{2\pi}{11}\cdot\cos\frac{3\pi}{11}\cdot\sin\frac{3\pi}{11}$$

$$\Rightarrow 2\cos^2\frac{3\pi}{11}y^2 = 2\sin^2\frac{3\pi}{11} + 32\sin^2\frac{2\pi}{11}\cdot\cos^2\frac{3\pi}{11} + 8\sin\frac{2\pi}{11}\cdot\sin\frac{6\pi}{11}$$

$$= \left(1-\cos\frac{6\pi}{11}\right) + 8\left(1-\cos\frac{4\pi}{11}\right)\cdot\left(1+\cos\frac{6\pi}{11}\right) + 4\left(\cos\frac{4\pi}{11} - \cos\frac{8\pi}{11}\right)$$

$$= 9 + 7\cos\frac{6\pi}{11} - 4\cos\frac{4\pi}{11} - 8\cos\frac{4\pi}{11}\cdot\cos\frac{6\pi}{11} - 4\cos\frac{8\pi}{11}$$

$$= 9 + 7\cos\frac{6\pi}{11} - 4\cos\frac{4\pi}{11} - 4\left(\cos\frac{10\pi}{11} + \cos\frac{2\pi}{11}\right) - 4\cos\frac{8\pi}{11}$$

$$= 9 + 11\cos\frac{6\pi}{11} - 4\left(\cos\frac{2\pi}{11} + \cos\frac{4\pi}{11} + \cos\frac{6\pi}{11} + \cos\frac{8\pi}{11} + \cos\frac{10\pi}{11}\right)$$

$$= 9 + 11\cos\frac{6\pi}{11} - 4\left(\frac{\cos\left(\frac{2\pi}{11} + 2\cdot\frac{2\pi}{11}\right)\cdot\sin\left(\frac{5\pi}{11}\right)}{\sin\pi/11}\right)$$

$$= 9 + 11\cos\frac{6\pi}{11} - \frac{4\cos\frac{6\pi}{11}\cdot\sin\frac{5\pi}{11}}{\sin\frac{\pi}{11}}$$

$$= 9 + 11\cos\frac{6\pi}{11} - \frac{2\sin\frac{12\pi}{11}}{\sin\frac{\pi}{11}}$$

$$= 9 + 11\cos\frac{6\pi}{11} + 2 = 11\left(1 + \cos\frac{6\pi}{11}\right)$$

$$2y^2 \cdot \left(\cos^2\frac{3\pi}{11}\right) = 22\cos^2\frac{3\pi}{11} \Rightarrow y^2 = 11$$

$$\Rightarrow \quad y = \sqrt{11} \qquad [\text{as } y > 0]$$

● ***Ex. 138.*** *Prove that* $\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7} = \frac{\sqrt{7}}{2}$.

Sol. Put, $7\theta = 2n\pi$, where n is any integer, then

$$4\theta = 2n\pi - 3\theta$$

$$\Rightarrow \quad \sin(4\theta) = \sin(2n\pi - 3\theta) = -\sin 3\theta \qquad \ldots(i)$$

This means $\sin\theta$ takes the values; $0, \pm\sin\frac{2\pi}{7}, \pm\sin\frac{4\pi}{7}$ and $\pm\sin\frac{8\pi}{7}$.

Since, $\sin\frac{6\pi}{7} = -\sin\left(\frac{8\pi}{7}\right)$

From Eq. (i), we now get $2\sin 2\theta.\cos 2\theta = 4\sin^3\theta - 3\sin\theta$

$$\Rightarrow \quad 4\sin\theta\cos\theta(1 - 2\sin^2\theta) = \sin\theta(4\sin^2\theta - 3)$$

$$\Rightarrow \quad 4\cos\theta(1 - 2\sin^2\theta) = 4\sin^2\theta - 3$$

$$\Rightarrow \quad 16\cos^2\theta(1 - 2\sin^2\theta)^2 = (4\sin^2\theta - 3)^2$$

$$\Rightarrow 16(1 - \sin^2\theta)(1 - 4\sin^2\theta + 4\sin^4\theta)$$

$$= 16\sin^4\theta - 24\sin^2\theta + 9$$

$$\Rightarrow \quad 64\sin^6\theta - 112\sin^4\theta + 56\sin^2\theta - 7 = 0$$

This is a cubic in $\sin^2\theta$ with the roots,

$$\sin^2\left(\frac{2\pi}{7}\right), \sin^2\left(\frac{4\pi}{7}\right), \sin^2\left(\frac{8\pi}{7}\right)$$

$\therefore$ Sum of the roots is

$$\sin^2\left(\frac{2\pi}{7}\right) + \sin^2\left(\frac{4\pi}{7}\right) + \sin^2\left(\frac{8\pi}{7}\right) = \frac{112}{64} = \frac{7}{4}$$

We already proved

$$\sin\frac{2\pi}{7}\cdot\sin\frac{4\pi}{7} + \sin\frac{4\pi}{7}\cdot\sin\frac{8\pi}{7} + \sin\frac{8\pi}{7}\sin\frac{\pi}{7} = 0$$

So, $\left(\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7}\right)^2 = \frac{7}{4}$

$$\Rightarrow \quad \sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7} = \frac{\sqrt{7}}{2}$$

Alternate Method

$$x^7 - 1 = 0$$

[assuming x as the seventh root of unity]

$$x^7 = 1 + 0.i = \cos(2k\pi) + i\sin(2k\pi)$$

$$x = \left(\cos\frac{2k\pi}{7} + i\sin\frac{2k\pi}{7}\right)$$

$$\Rightarrow \quad x = e^{i2k\pi/7} \qquad [\text{where, } k = 0, 1, 2, 3, 4, 5, 6]$$

$$\Rightarrow \quad \sum_{k=0}^{6} e^{i2k\pi/7} = 0$$

$$\Rightarrow \quad 1 + \sum_{k=1}^{6} e^{i2k\pi/7} = 0$$

$$\Rightarrow \quad 1 + \sum_{k=1}^{3} (e^{i2k\pi/7} + e^{-2k\pi/7}) = 0$$

$$\Rightarrow \quad 1 + \sum_{k=1}^{3} 2.\cos\frac{2k\,\pi}{7} = 0$$

$$\Rightarrow \quad 1 + 2\sum_{k=1}^{3}\left(1 - 2\sin^2\frac{k\,\pi}{7}\right) = 0$$

$$\Rightarrow \quad 1 + 2\left[3 - 2\left(\sin^2\frac{\pi}{7} + \sin^2\frac{2\pi}{7} + \sin^2\frac{3\pi}{7}\right)\right] = 0$$

$$\Rightarrow \quad \sin^2\frac{\pi}{7} + \sin^2\frac{2\pi}{7} + \sin^2\frac{3\pi}{7} = \frac{7}{4}$$

$$\Rightarrow \quad \sin^2\frac{8\pi}{7} + \sin^2\frac{2\pi}{7} + \sin^2\frac{4\pi}{7} = \frac{7}{4}$$

$$\Rightarrow \quad \sin^2\frac{2\pi}{7} + \sin^2\frac{4\pi}{7} + \sin^2\frac{8\pi}{7} = \frac{7}{4} \qquad \ldots(i)$$

and $\sin\frac{2\pi}{7}\cdot\sin\frac{4\pi}{7} + \sin\frac{4\pi}{7}\cdot\sin\frac{8\pi}{7} + \sin\frac{8\pi}{7}\cdot\sin\frac{2\pi}{7}$

$$= \frac{1}{2}\left[\cos\frac{2\pi}{7} - \cos\frac{6\pi}{7} + \cos\frac{4\pi}{7} - \cos\frac{12\pi}{7} + \cos\frac{6\pi}{7} - \cos\frac{10\pi}{7}\right]$$

$$= \frac{1}{2}\left[\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} - \cos\left(2\pi - \frac{2\pi}{7}\right) - \cos\left(2\pi - \frac{4\pi}{7}\right)\right]$$

$$= \frac{1}{2}\left[\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} - \cos\frac{2\pi}{7} - \cos\frac{4\pi}{7}\right] = 0 \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$\left(\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7}\right)^2 = \sin^2\frac{2\pi}{7} + \sin^2\frac{4\pi}{7} + \sin^2\frac{8\pi}{7}$$

$$= \frac{7}{4}$$

$$\Rightarrow \quad \sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7} = \frac{\sqrt{7}}{2}$$

● **Ex. 139.** *In a ΔABC, $\tan A + \tan B + \tan C = k$, then find the interval in which k should lie so that*

(A) there exists exactly one isosceles triangle ABC

(B) there exists exactly two isosceles triangle ABC

(C) can there exist three non-similar isosceles triangles for any real value of k.

Sol. Let $A = B$, then $2A + C = 180°$

and $2\tan A + \tan C = k$ ...(i)

Now, $2A + C = 180°$

$\Rightarrow \tan 2A = -\tan C$

Also, $2\tan A + \tan C = k$

$\Rightarrow 2\tan A + \tan(180 - 2A) = k$

$\Rightarrow 2\tan A - \tan 2A = k$

$\Rightarrow 2\tan A - \dfrac{2\tan A}{1 - \tan^2 A} = k$

$\Rightarrow 2\tan A(1 - \tan^2 A - 1) = k - k\tan^2 A$

$\Rightarrow 2\tan^3 A - k\tan^2 A + k = 0$

Let, $\tan A = x,\ x > 0$ (as $A < 90°$)

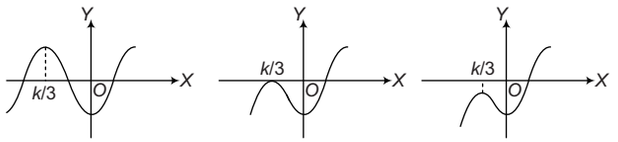
Then let, $f(x) = 2x^3 - kx^2 + k$ ...(ii)

$f'(x) = 6x^2 - 2kx = 0$

$\Rightarrow x = k/3, 0$

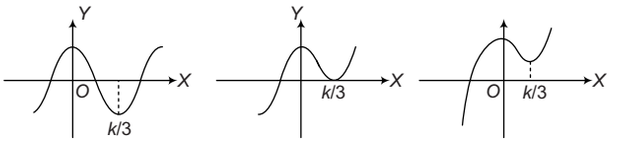
Following cases arises

(i) $k < 0$, three graphs of cubic equation (ii) are possible. Clearly, in all these case, only one triangle is possible and the condition for that triangle to be possible is $f(0) < 0 \Rightarrow k < 0$ so for $k < 0$ only one isosceles triangle is possible.



(ii) $k > 0$, three graphs of the cubic equation (ii) are possible. In fig. (i), two such triangle are possible. The condition is $f(k/3) < 0$.

$\Rightarrow k\left(1 - \dfrac{k^2}{27}\right) < 0 \Rightarrow k > 3\sqrt{3}$



In figure (ii), one such triangle is possible. The condition is $f(k/3) = 0$

$\Rightarrow k = 3\sqrt{3}.$

In figure (iii), no such triangle is possible. The condition is $f(k/3) > 0$

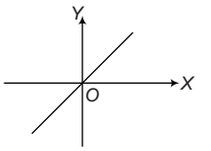
$\Rightarrow k\left(1 - \dfrac{k^2}{27}\right) > 0$

$\Rightarrow k < 3\sqrt{3}.$

(iii) $k = 0$, graph will be shown as, so no such triangle is possible. Hence, the solution for mentioned conditions;

$\therefore$ (A) either $k < 0$ or $k = 3\sqrt{3}$

(B) $k > 3\sqrt{3}$



(C) Clearly, there will never exists three or more than three non-similar isosceles triangle for any value of k.

Trigonometric Functions and Identities Exercise 1 :
Single Option Correct Type Questions

1. The value of $\sum_{n=1}^{10}\left(\sin\frac{2n\pi}{11}-\cos\frac{2n\pi}{11}\right)$ is equal to

(a) 2 (b) 1
(c) 0 (d) -1

2. Given, $a^2+2a+\text{cosec}^2\left(\frac{\pi}{2}(a+x)\right)=0$, then which of the following holds good?

(a) $a=1;\frac{x}{2}\in 1$

(b) $a=-1;\frac{x}{2}\in I$

(c) $a\in R;\ x\in\phi$

(d) a, x are finite but not possible to find

3. The minimum value of the function $f(x)=(3\sin x-4\cos x-10)(3\sin x+4\cos x-10)$, is

(a) 49 (b) $\frac{195-60\sqrt{2}}{2}$
(c) 84 (d) 48

4. The value of expression $\sum_{\theta=0}^{8}\frac{1}{1+\tan^3(10\theta)^\circ}$ equal is to

(a) 5 (b) $\frac{21}{4}$
(c) $\frac{14}{3}$ (d) $\frac{9}{2}$

5. The value of $\sqrt{1-\sin^2 110^\circ}\cdot\sec 110^\circ$ is equal to

(a) 2 (b) -1
(c) -2 (d) 1

6. If $\tan\alpha$, $\tan\beta$ are the roots of the equation $x^2+px+q=0$, then the value of

$\sin^2(\alpha+\beta)+p\sin(\alpha+\beta)\cos(\alpha+\beta)+q\cos^2(\alpha+\beta)$ is

(a) independent of p but dependent on q
(b) independent of q but dependent on p
(c) independent of both p and q
(d) dependent on both p and q

7. The value of the product

$\sin\left(\frac{\pi}{2^{2009}}\right)\cos\left(\frac{\pi}{2^{2009}}\right)\cos\left(\frac{\pi}{2^{2008}}\right)$

$\cos\left(\frac{\pi}{2^{2007}}\right)\cos\left(\frac{\pi}{2^{2006}}\right)\ldots\cos\left(\frac{\pi}{2^3}\right)\cos\left(\frac{\pi}{2^2}\right)$, is

(a) $\frac{1}{2^{2007}}$ (b) $\frac{1}{2^{2008}}$
(c) $\frac{1}{2^{2009}}$ (d) $\frac{1}{2^{2010}}$

8. If $\tan B=\frac{n\sin A\cos A}{1-n\cos^2 A}$, then $\tan(A+B)$ equals to

(a) $\frac{\sin A}{(1-n)\cos A}$ (b) $\frac{(n-1)\cos A}{\sin A}$
(c) $\frac{\sin A}{(n-1)\cos A}$ (d) $\frac{\sin A}{(n+1)\cos A}$

9. If $P=(\tan(3^{n+1}\theta)-\tan\theta)$ and $Q=\sum_{r=0}^{n}\frac{\sin(3^r\theta)}{\cos(3^{r+1}\theta)}$, then

(a) $P=2Q$ (b) $P=3Q$
(c) $2P=Q$ (d) $3P=Q$

10. The value of

$(\cos^4 1^\circ+\cos^4 2^\circ+\cos^4 3^\circ+\ldots+\cos^4 179^\circ)-$

$(\sin^4 1^\circ+\sin^4 2^\circ+\sin^4 3^\circ+\ldots+\sin^4 179^\circ)$ equals to

(a) $2\cos 1^\circ$ (b) -1
(c) $2\sin 1^\circ$ (d) 0

11. Suppose that 'a' is a non-zero real number for which $\sin x+\sin y=a$ and $\cos x+\cos y=2a$. The value of $\cos(x-y)$, is

(a) $\frac{3a^2-2}{2}$ (b) $\frac{7a^2-2}{2}$
(c) $\frac{9a^2-2}{2}$ (d) $\frac{5a^2-2}{2}$

12. Let $P(x)=\sqrt{(\cos x+\cos 2x+\cos 3x)^2+(\sin x+\sin 2x+\sin 3x)^2}$, then $P(x)$ is equal to

(a) $1+2\cos x$ (b) $1+\sin 2x$
(c) $1-2\cos x$ (d) None of these

13. If the maximum value of the expression $\frac{1}{5\sec^2\theta-\tan^2\theta+4\,\text{cosec}^2\theta}$ is equal to $\frac{p}{q}$ (where, p and q are coprime), then the value of $(p+q)$ is

(a) 14 (b) 15
(c) 16 (d) 18

14. Let $f_n(a)=\frac{\sin\alpha+\sin 3\alpha+\sin 5\alpha+\ldots+\sin(2n-1)\alpha}{\cos\alpha+\cos 3\alpha+\cos 5\alpha+\ldots+\cos(2n-1)\alpha}$.

Then, the value of $f_4\left(\frac{\pi}{32}\right)$ is equal to

(a) $\sqrt{2}+1$ (b) $\sqrt{2}-1$
(c) $2+\sqrt{3}$ (d) $2-\sqrt{3}$

15. The minimum value of $\left|\sin x+\cos x+\frac{\cos x+\sin x}{\cos^4 x-\sin^4 x}\right|$ is

(a) 2 (b) $\frac{3}{2}$ (c) $\sqrt{2}$ (d) 1

16. If $a = \cos(2012\,\pi)$, $b = \sec(2013\,\pi)$ and $c = \tan(2014\,\pi)$, then

(a) $a < b < c$ (b) $b < c < a$
(c) $c < b < a$ (d) $a = b < c$

17. In a ΔABC, the minimum value of $\sec^2\frac{A}{2} + \sec^2\frac{B}{2} + \sec^2\frac{C}{2}$ is equal to

(a) 3 (b) 4
(c) 5 (d) 6

18. The number of ordered pairs (x, y) of real numbers satisfying $4x^2 - 4x + 2 = \sin^2 y$ and $x^2 + y^2 \le 3$, is equal to

(a) 0 (b) 2
(c) 4 (d) 8

19. In a ΔABC, $3\sin A + 4\cos B = 6$ and $3\cos A + 4\sin B = 1$, then $\angle C$ can be

(a) 30° (b) 60°
(c) 90° (d) 150°

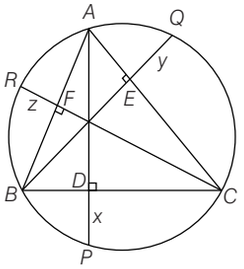
20. An equilateral triangle has side length 8. The area of the region containing all points outside the triangle but not more than 3 units from a point on the triangle is :

(a) $9(8 + \pi)$
(b) $8(9 + \pi)$
(c) $9\left(8 + \frac{\pi}{2}\right)$
(d) $8\left(9 + \frac{\pi}{2}\right)$

21. If $a\cos^3\alpha + 3a\cos\alpha\sin^2\alpha = m$ and $a\sin^3\alpha + 3a\cos^2\alpha\sin\alpha = n$. Then, $(m+n)^{2/3} + (m-n)^{2/3}$ is equal to

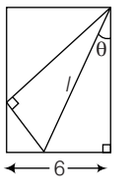
(a) $2a^2$ (b) $2a^{1/3}$
(c) $2a^{2/3}$ (d) $2a^3$

22. As shown in the figure, AD is the altitude on BC and AD produced meets the circumcircle of ΔABC at P where $DP = x$. Similarly, $EQ = y$ and $FR = z$. If a, b, c respectively denotes the sides BC, CA and AB, then $\frac{a}{2x} + \frac{b}{2y} + \frac{c}{2z}$ has the value equal to



(a) $\tan A + \tan B + \tan C$
(b) $\cot A + \cos B + \cot C$
(c) $\cos A + \cos B + \cos C$
(d) $\operatorname{cosec} A + \operatorname{cosec} B + \operatorname{cosec} C$

23. One side of a rectangular piece of paper is 6 cm, the adjacent sides being longer than 6 cm. One corner of the paper is folded so that is sets on the opposite longer side. If the length of the crease is l cm and it makes an angle θ with the long side as shown, then l is



(a) $\frac{3}{\sin\theta\cos^2\theta}$ (b) $\frac{6}{\sin^2\theta\cos\theta}$
(c) $\frac{3}{\sin\theta\cos\theta}$ (d) $\frac{3}{\sin^2\theta}$

24. The average of the numbers $n\sin n°$ for $n = 2, 4, 6, \ldots 180$

(a) 1 (b) $\cot 1°$
(c) $\tan 1°$ (d) $\frac{1}{2}$

25. A circle is inscribed inside a regular pentagon and another circle is circumscribed about this pentagon. Similarly, a circle is inscribed in a regular heptagon and another circumscribed about the heptagon. The area of the regions between the two circles in two cases are A_1 and A_2, respectively. If each polygon has a side length of 2 units, then which one of the following is true ?

(a) $A_1 = \frac{5}{7}A_2$ (b) $A_1 = \frac{25}{49}A_2$
(c) $A_1 = \frac{49}{25}A_2$ (d) $A_1 = A_2$

26. The value of $\sum_{r=1}^{18}\cos^2(5r)°$, where $x°$ denotes the x degrees, is equal to

(a) 0 (b) $\frac{7}{2}$
(c) $\frac{17}{2}$ (d) $\frac{25}{2}$

27. Minimum value of $4x^2 - 4x|\sin x| - \cos^2\theta$ is equal to

(a) -2 (b) -1
(c) $-\frac{1}{2}$ (d) 0

28. If in a triangle ABC, $\cos 3A + \cos 3B + \cos 3C = 1$, then one angle must be exactly equal to

(a) $\frac{\pi}{3}$ (b) $\frac{2\pi}{3}$ (c) π (d) $\frac{4\pi}{3}$

29. If $|\tan A| < 1$ and $|A|$ is acute, then $\frac{\sqrt{(1+\sin 2A)} + \sqrt{(1-\sin 2A)}}{\sqrt{(1+\sin 2A)} - \sqrt{(1-\sin 2A)}}$ is equal to

(a) $\tan A$ (b) $-\tan A$
(c) $\cot A$ (d) $-\cot A$

30. For any real θ, the maximum value of $\cos^2(\cos\theta) + \sin^2(\sin\theta)$ is

(a) 1 (b) $1 + \sin^2 1$
(c) $1 + \cos^2 1$ (d) does not exist

31. Minimum value of $27^{\cos 2x} \cdot 81^{\sin 2x}$ is

(a) -5 (b) $\frac{1}{5}$
(c) $\frac{1}{243}$ (d) $\frac{1}{27}$

32. $ABCD$ is a trapezium, such that AB and CD are parallel $BC \perp CD$. If $\angle ADB = \theta$, $BC = p$ and $CD = q$, then AB is equal to

(a) $\frac{(p^2 + q^2)\sin\theta}{p\cos\theta + q\sin\theta}$ (b) $\frac{p^2 + q^2\cos\theta}{p\cos\theta + q\sin\theta}$
(c) $\frac{p^2 + q^2}{p^2\cos\theta + q^2\sin\theta}$ (d) $\frac{(p^2 + q^2)\sin\theta}{(p\cos\theta + q\sin\theta)^2}$

33. If $4n\alpha = \pi$, then $\cot\alpha \cot 2\alpha \cot 3\alpha \ldots \cot(2n-1)\alpha$ is equal to

(a) 0 (b) 1
(c) n (d) None of these

34. If in a triangle ABC $(\sin A + \sin B + \sin C)$ $(\sin A + \sin B - \sin C) = 3\sin A \sin B$, then angle C is equal to

(a) 30° (b) 45°
(c) 60° (d) 75°

35. If α, β, γ are acute angles and $\cos\theta = \frac{\sin\beta}{\sin\alpha}$, $\cos\phi = \frac{\sin\gamma}{\sin\alpha}$ and $\cos(\theta - \phi) = \sin\beta\sin\gamma$, then $\tan^2\alpha - \tan^2\beta - \tan^2\gamma$ is equal to

(a) -1 (b) 0
(c) 1 (d) None of these

36. If $\tan\beta = \frac{n\sin\alpha\cos\alpha}{1 - n\sin^2\alpha}$, then $\tan(\alpha - \beta)$ is equal to

(a) $n\tan\alpha$ (b) $(1-n)\tan\alpha$
(c) $(1+n)\tan\alpha$ (d) None of these

37. If $\frac{\cos\theta}{a} = \frac{\sin\theta}{b}$, then $\frac{a}{\sec 2\theta} + \frac{b}{\operatorname{cosec} 2\theta}$ is equal to

(a) a (b) b
(c) $\frac{a}{b}$ (d) $a + b$

38. The graph of the function $\cos x\cos(x+2) - \cos^2(x+1)$ is

(a) a straight line passing through $(0, -\sin^2\theta)$ with slope 2
(b) a straight line passing through $(0, 0)$
(c) a parabola with vertex $(1, -\sin^2 1)$
(d) a straight line passing through the point $\left(\frac{\pi}{2}, -\sin^2 1\right)$ and parallel to the X-axis

39. $f(\theta) = |\sin\theta| + |\cos\theta|, \theta \in R$, then

(a) $f(\theta) \in [0, 2]$ (b) $f(\theta) \in [0, \sqrt{2}]$
(c) $f(\theta) \in [0, 1]$ (d) $f(\theta) \in [1, \sqrt{2}]$

40. If $A = \cos(\cos x) + \sin(\cos x)$ the least and greatest value of A are

(a) 0 and 2 (b) -1 and 1
(c) $-\sqrt{2}$ and $\sqrt{2}$ (d) 0 and $\sqrt{2}$

41. If $U_n = \sin n\theta \sec^n\theta$, $V_n = \cos n\theta \sec^n\theta \neq 1$, then $\frac{V_n - V_{n-1}}{U_{n-1}} + \frac{1}{n}\frac{U_n}{V_n}$ is equal to

(a) 0 (b) $\tan\theta$
(c) $-\tan\theta + \frac{\tan n\theta}{n}$ (d) $\tan\theta + \frac{\tan n\theta}{n}$

42. If $0 \le x \le \frac{\pi}{3}$ then range of $f(x) = \sec\left(\frac{\pi}{6} - x\right) + \sec\left(\frac{\pi}{6} + x\right)$ is

(a) $\left(\frac{4}{\sqrt{3}}, \infty\right)$ (b) $\left[\frac{4}{\sqrt{3}}, \infty\right)$
(c) $\left(0, \frac{4}{\sqrt{3}}\right]$ (d) $\left(0, \frac{4}{\sqrt{3}}\right)$

43. If $A = \sin^8\theta + \cos^{14}\theta$, then for all values of θ,

(a) $A \ge 1$ (b) $0 < A \le 1$
(c) $1 < 2a \le 3$ (d) None of these

44. The expression $3\left\{\sin^4\left(\frac{3\pi}{2} - \alpha\right) + \sin^4(3\pi + \alpha)\right\} - 2\left\{\sin^6\left(\frac{\pi}{2} + \alpha\right) + \sin^6(5\pi - \alpha)\right\}$ is equal to

(a) 0 (b) -1
(c) 1 (d) 3

45. The maximum value of $\sin\left(x + \frac{\pi}{6}\right) + \cos\left(x + \frac{\pi}{6}\right)$ in the interval $\left(0, \frac{\pi}{2}\right)$ is attained at

(a) $\frac{\pi}{12}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

46. If $\cot^2 x = \cot(x-y)\cdot\cot(x-z)$, then $\cot 2x$ is equal to $\left(x \neq \pm\frac{\pi}{4}\right)$

(a) $\frac{1}{2}(\tan y + \tan z)$ (b) $\frac{1}{2}(\cot y + \cot z)$
(c) $\frac{1}{2}(\sin y + \sin z)$ (d) None of these

47. The minimum value of the expression $\sin\alpha + \sin\beta + \sin\gamma$, where α, β, γ are real numbers satisfying $\alpha + \beta + \gamma = \pi$, is

(a) positive (b) zero
(c) negative (d) None of these

48. If $\cos x - \sin\alpha \cot\beta \sin x = \cos\alpha$, then $\tan\frac{x}{2}$ is equal to

(a) $\cot\frac{\alpha}{2}\tan\frac{\beta}{2}$ (b) $\cot\frac{\beta}{2}\tan\frac{\alpha}{2}$

(c) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}$ (d) None of these

49. If $\cos^4\theta\sec^2\alpha, \frac{1}{2}$ and $\sin^4\theta\,\text{cosec}^2\alpha$ are in AP, then $\cos^8\theta\sec^6\alpha, \frac{1}{2}$ and $\sin^8\theta\cdot\text{cosec}^6\alpha$ are in

(a) AP (b) GP

(c) HP (d) None of these

50. The maximum value of $\cos\alpha_1\cdot\cos\alpha_2\cdot\cos\alpha_3\cdot\ldots\cdot\cos\alpha_n$ under the restriction $0\le\alpha_1,\alpha_2,\ldots,\alpha_n\le\frac{\pi}{2}$ and $\cot\alpha_1\cdot\cot\alpha_2\cdot\ldots\cdot\cot\alpha_n = 1$ is

(a) $\frac{1}{2^{\frac{n}{2}}}$ (b) $\frac{1}{2^n}$

(c) $\frac{-1}{2^n}$ (d) 1

51. If $x\in(0,\pi)$ and $\sin x\cdot\cos^3 x > \cos x\cdot\sin^3 x$, then complete set of values of x is

(a) $x\in\left(0,\frac{\pi}{4}\right)\cup\left(\frac{\pi}{2},\frac{3\pi}{4}\right)$

(b) $x\in\left(\frac{\pi}{4},\frac{\pi}{2}\right)\cup\left(\frac{3\pi}{4},\pi\right)$

(c) $x\in\left(\frac{\pi}{4},\frac{\pi}{2}\right)$

(d) None of the above

52. If $u=\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}+\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$ then the difference between the maximum and minimum values of u^2 is given by

(a) $2(a^2+b^2)$ (b) $2\sqrt{a^2+b^2}$

(c) $(a+b)^2$ (d) $(a-b)^2$

53. For a positive integer n, let $f_n(\theta)=\frac{\tan\theta}{2}(1+\sec\theta)(1+\sec 2\theta)\ldots(1+\sec 2^n\theta)$, then

(a) $f_2\left(\frac{\pi}{16}\right)=0$ (b) $f_3\left(\frac{\pi}{32}\right)=-1$

(c) $f_4\left(\frac{\pi}{64}\right)=-1$ (d) $f_5\left(\frac{\pi}{128}\right)=1$

Trigonometric Functions and Identities Exercise 2 : More than One Option Correct Type Questions

54. Suppose $\cos x = 0$ and $\cos(x+z)=\frac{1}{2}$. Then, the possible value(s) of z is (are).

(a) $\frac{\pi}{6}$ (b) $\frac{5\pi}{6}$

(c) $\frac{7\pi}{6}$ (d) $\frac{11\pi}{6}$

55. Let $f_n(\theta)=2\sin\frac{\theta}{2}\sin\frac{3\theta}{2}+2\sin\frac{\theta}{2}\sin\frac{5\theta}{2}+2\sin\frac{\theta}{2}\sin\frac{7\theta}{2}+\ldots+2\sin\frac{\theta}{2}\sin(2n+1)\frac{\theta}{2}, n\in N$, then which of the following is/are correct?

(a) $f_9\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$ (b) $f_n\left(\frac{2\pi}{n}\right)=0, n\in N$

(c) $f_5\left(\frac{2\pi}{7}\right)=0$ (d) $f_9\left(\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}$

56. Let $P=\sin 25^\circ\sin 35^\circ\sin 60^\circ\sin 85^\circ$ and $Q=\sin 20^\circ\sin 40^\circ\sin 75^\circ\sin 80^\circ$. Which of the following relation(s) is (are) correct ?

(a) $P+Q=0$ (b) $P-Q=0$

(c) $P^2+Q^2=1$ (d) $P^2-Q^2=0$

57. For $0<\theta<\frac{\pi}{2}$, if $x=\sum_{n=0}^{\infty}\cos^{2n}\theta, y=\sum_{n=0}^{\infty}\sin^{2n}\theta$, $z=\sum_{n=0}^{\infty}\cos^{2n}\theta\sin^{2n}\theta$, then

(a) $xyz = xz + y$

(b) $xyz = xy + z$

(c) $xyz = x + y + z$

(d) $xyz = yz + x$

58. Let $P(x)=\cot^2 x\left(\frac{1+\tan x+\tan^2 x}{1+\cot x+\cot^2 x}\right)+\left(\frac{\cos x-\cos 3x+\sin 3x-\sin x}{2(\sin 2x+\cos 2x)}\right)^2$. Then, which of the following is (are) correct?

(a) The value of $P(18^\circ)+P(72^\circ)$ is 2.

(b) The value of $P(18^\circ)+P(72^\circ)$ is 3.

(c) The value of $P\left(\frac{4\pi}{3}\right)+P\left(\frac{7\pi}{6}\right)$ is 3.

(d) The value of $P\left(\frac{4\pi}{3}\right)+P\left(\frac{7\pi}{6}\right)$ is 2.

59. It is known that $\sin\beta = \frac{4}{5}$ and $0 < \beta < \pi$, then thes value

$$\sqrt{3}\sin(\alpha+\beta) - \frac{2}{\cos\frac{11\pi}{6}}\cos(\alpha+\beta)$$

of $\dfrac{}{\sin\alpha}$ is

(a) independent of α for all β in $(0, \pi)$

(b) $\frac{5}{\sqrt{3}}$ for $\tan\beta > 0$

(c) $\frac{\sqrt{3}(7 + 24\cot\alpha)}{15}$ for $\tan\beta < 0$

(d) zero for $\tan\beta > 0$

60. In cyclic quadrilateral $ABCD$, if $\cot A = \frac{3}{4}$ and $\tan B = \frac{-12}{5}$, then which of the following is (are) correct?

(a) $\sin D = \frac{12}{13}$ (b) $\sin(A + B) = \frac{16}{65}$

(c) $\cos D = \frac{-15}{13}$ (d) $\sin(C + D) = \frac{-16}{65}$

61. If the equation $2\cos^2 x + \cos x - a = 0$ has solutions, then a can be

(a) $\frac{-1}{4}$ (b) $\frac{-1}{8}$

(c) 2 (d) 5

62. If $A = \sin 44° + \cos 44°$, $B = \sin 45° + \cos 45°$ and $C = \sin 46° + \cos 46°$. Then, correct option(s) is/are

(a) $A < B < C$ (b) $C < B < A$

(c) $B > A$ (d) $A = C$

63. If $\tan(2\alpha+\beta) = x$ & $\tan(\alpha+2\beta) = y$, then $[\tan 3(\alpha+\beta)]$. $[\tan(\alpha-\beta)]$ is equal to (wherever defined)

(a) $\frac{x^2+y^2}{1-x^2y^2}$ (b) $\frac{x^2-y^2}{1+x^2y^2}$

(c) $\frac{x^2+y^2}{1+x^2y^2}$ (d) $\frac{x^2-y^2}{1-x^2y^2}$

64. If $x = \sec\phi - \tan\phi$ and $y = \operatorname{cosec}\phi + \cot\phi$, then

(a) $x = \frac{y+1}{y-1}$ (b) $x = \frac{y-1}{y+1}$

(c) $y = \frac{1+x}{1-x}$ (d) $xy + x - y + 1 = 0$

65. If $\tan\left(\frac{x}{2}\right) = \operatorname{cosec} x - \sin x$, then $\tan^2\left(\frac{x}{2}\right)$ is equal to

(a) $2 - \sqrt{5}$ (b) $\sqrt{5} - 2$

(c) $(9 - 4\sqrt{5})(2 + \sqrt{5})$ (d) $(9 + 4\sqrt{5})(2 - \sqrt{5})$

66 If $y = \frac{\sqrt{1-\sin 4A}+1}{\sqrt{1+\sin 4A}-1}$, then one of the value of y is

(a) $-\tan A$ (b) $\cot A$

(c) $\tan\left(\frac{\pi}{4}+A\right)$ (d) $-\cot\left(\frac{\pi}{4}+A\right)$

67. If $3\sin\beta = \sin(2\alpha+\beta)$, then

(a) $[\cot\alpha + \cot(\alpha+\beta)][\cot\beta - 3\cot(2\alpha+\beta)] = 6$

(b) $\sin\beta = \cos(\alpha+\beta)\sin\alpha$

(c) $2\sin\beta = \sin(\alpha+\beta)\cos\alpha$

(d) $\tan(\alpha+\beta) = 2\tan\alpha$

68. Let $P_n(u)$ be a polynomial is u of degree n. Then, for every positive integer n, $\sin 2nx$ is expressible is

(a) $P_{2n}(\sin x)$ (b) $P_{2n}(\cos x)$

(c) $\cos x\, P_{2n-1}(\sin x)$ (d) $\sin x\, P_{2n-1}(\cos x)$

69. If $\tan\theta = \frac{\sin\alpha - \cos\alpha}{\sin\alpha + \cos\alpha}$, then

(a) $\sin\alpha - \cos\alpha = \pm\sqrt{2}\sin\theta$

(b) $\sin\alpha + \cos\alpha = \pm\sqrt{2}\cos\theta$

(c) $\cos 2\theta = \sin 2\alpha$

(d) $\sin 2\theta + \cos 2\alpha = 0$

70. If $\cos 5\theta = a\cos\theta + b\cos^3\theta + c\cos^5\theta + d$, then

(a) $a = 20$

(b) $b = -20$

(c) $c = 16$

(d) $d = 5$

71. $x = \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha} = \sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha}$ then $x^2 = a^2 + b^2 + 2\sqrt{p(a^2+b^2) - p^2}$, where p is equal to

(a) $a^2\cos^2\alpha + b^2\sin^2\alpha$

(b) $a^2\sin^2\alpha + b^2\cos^2\alpha$

(c) $\frac{1}{2}[a^2 + b^2 + (a^2 - b^2)\cos 2\alpha]$

(d) $\frac{1}{2}[a^2 + b^2 - (a^2 - b^2)\cos 2\alpha]$

72. $\left(\frac{\cos A + \cos B}{\sin A - \sin B}\right)^n + \left(\frac{\sin A + \sin B}{\cos A - \cos B}\right)^n$ (n, even or odd) is equal to

(a) $2\tan^n\left(\frac{A-B}{2}\right)$ (b) $2\cot^n\left(\frac{A-B}{2}\right)$

(c) 0 (d) None of these

73. Let $P(k) = \left(1+\cos\frac{\pi}{4k}\right)\left(1+\cos\frac{(2k-1)\pi}{4k}\right)$ $\left(1+\cos\frac{(2k+1)\pi}{4k}\right)\left(1+\cos\frac{(4k-1)\pi}{4k}\right)$. Then

(a) $P(3) = \frac{1}{16}$ (b) $P(4) = \frac{2-\sqrt{2}}{16}$

(c) $P(5) = \frac{3-\sqrt{5}}{32}$ (d) $P(6) = \frac{2-\sqrt{3}}{16}$

74. If $x = a\cos^3\theta\sin^2\theta$, $y = a\sin^3\theta\cos^2\theta$ and $\frac{(x^2+y^2)^p}{(xy)^q}$ $(p, q \in N)$ is independent of θ, then

(a) $p = 4$ (b) $p = 5$

(c) $q = 4$ (d) $q = 5$

Trigonometric Functions and Identities Exercise 3: Statement I and II Type Questions

- This section contains 11 questions. Each question contains **Statement I** (Assertion) and **Statement II** (Reason). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are
  - (a) Both Statement I and Statement II are individually true and R is the correct explanation of Statement I.
  - (b) Both Statement I and Statement II are individually true but Statement II is not the correct explanation of Statement I.
  - (c) Statement I is true but Statement II is false.
  - (d) Statement I is false but Statement II is true.

75. **Statement I** $\tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\tan 8\alpha + 16\cot 16\alpha = \cot\alpha$

Statement II $\cot\alpha - \tan\alpha = 2\cot 2\alpha$

76. **Statement I** If $xy + yz + zx = 1$, then $\Sigma \frac{x}{(1+x^2)} = \frac{2}{\sqrt{\Pi(1+x^2)}}$.

Statement II In a ΔABC $\sin 2A + \sin 2B - \sin 2C = 4\cos A \cos B \sin C$

77. **Statement I** If α and β are two distinct solutions of the equation $a\cos x + b\sin x = c$, then $\tan\left(\frac{\alpha+\beta}{2}\right)$ is independent of c.

Statement II Solution of $a\cos x + b\sin x = c$ is possible, if $-\sqrt{(a^2+b^2)} \le c \le \sqrt{(a^2+b^2)}$

78. **Statement I** If A is obtuse angle in ΔABC, then $\tan B \tan C > 1$.

Statement II In ΔABC, $\tan A = \frac{\tan B + \tan C}{\tan B \tan C - 1}$

79. **Statement I** $\sin\left(\frac{2\pi}{7}\right) + \sin\left(\frac{4\pi}{7}\right) + \sin\left(\frac{8\pi}{7}\right) = -\frac{1}{2}$.

Statement II $\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$ is complex 7th root of unity.

80. **Statement I** The curve $y = 81^{\sin^2 x} + 81^{\cos^2 x} - 30$ intersects X-axis at eight points in the region $-\pi \le x \le \pi$.

Statement II The curve $y = \sin x$ or $y = \cos x$ intersects the X-axis at infinitely many points.

81. **Statement I** The numbers $\sin 18°$ and $-\sin 54°$ are the roots of a quadratic equation with integer coefficients.

Statement II If $x = 18°$, $\cos 3x = \sin 2x$ and if $y = -54°$ $\sin 2y = \cos 3y$.

82. **Statement I** The minimum value of the expression $\sin\alpha + \sin\beta + \sin\gamma$ where α, β, γ are real numbers such that $\alpha + \beta + \gamma = \pi$ is negative.

Statement II If $\alpha + \beta + \gamma = \pi$, then α, β, γ are the angles of a triangle.

83. **Statement I** If $2\sin\left(\frac{\theta}{2}\right) = \sqrt{1+\sin\theta} + \sqrt{1-\sin\theta}$ then $\frac{\theta}{2}$ lies between $2n\pi + \frac{\pi}{4}$ and $2n\pi + \frac{3\pi}{4}$.

Statement II If $\frac{\pi}{4} \le 0 \le \frac{3\pi}{4}$ then $\sin\frac{\theta}{2} > 0$.

84. **Statement I** If $2\cos\theta + \sin\theta = 1\left(\theta \ne \frac{\pi}{2}\right)$ then the value of $7\cos\theta + 6\sin\theta$ is 2.

Statement II If $\cos 2\theta - \sin\theta = \frac{1}{2}$, $0 < \theta < \frac{\pi}{2}$, then $\sin\theta + \cos 6\theta = 0$.

85. **Statement I** If $A > 0$, $B > 0$ and $A + B = \frac{\pi}{3}$, then the maximum value of $\tan A \tan B$ is $\frac{1}{3}$.

Statement II If $a_1 + a_2 + a_3 + \ldots + a_n = k$ (constant), then the value $a_1 a_2 a_3 \ldots a_n$ is greatest when $a_1 = a_2 = a_3 = \ldots = a_n$

Trigonometric Functions and Identities Exercise 4 : Passage Based Questions

Passage I

(Q. Nos. 86 and 87)

If a, b, c are the sides of ΔABC such that $3^{2a^2} - 2\cdot 3^{a^2+b^2+c^2} + 3^{2b^2+2c^2} = 0$, then

86. Triangle ABC is

(a) equilateral (b) right angled
(c) isosceles right angled (d) obtuse angled

87. If sides of ΔPQR are a, $b\sec C$, $c\,\text{cosec}\,C$. Then, area of ΔPQR is

(a) $\frac{\sqrt{3}}{4}a^2$ (b) $\frac{\sqrt{3}}{4}b^2$ (c) $\frac{\sqrt{3}}{4}c^2$ (d) $\frac{1}{2}abc$

Passage II

(Q. Nos. 88 to 90)

For $0 < x < \frac{\pi}{2}$, let $P_{mn}(x) = m\log_{\cos x}(\sin x) + n\log_{\cos x}(\cot x)$;

where $m, n \in \{1, 2, \ldots, 9\}$

[For example :

$P_{29}(x) = 2\log_{\cos x}(\sin x) + 9\log_{\cos x}(\cot x)$ and
$P_{77}(x) = 7\log_{\cos x}(\sin x) + 7\log_{\cos x}(\cot x)$]

On the basis of above information , answer the following questions :

88. Which of the following is always correct?

(a) $P_{mn}(x) \geq m\ \forall\ m \geq n$ (b) $P_{mn}(x) \geq n\ \forall\ m \geq n$
(c) $2P_{mn}(x) \leq n\ \forall\ m \leq n$ (d) $2P_{mn}(x) \leq m\ \forall\ m \leq n$

89. The mean proportional of numbers $P_{49}\left(\frac{\pi}{4}\right)$ and $P_{94}\left(\frac{\pi}{4}\right)$ is equal to

(a) 4 (b) 6
(c) 9 (d) 10

90. If $P_{34}(x) = P_{22}(x)$, then the value of $\sin x$ is expressed as $\left(\frac{\sqrt{q}-1}{p}\right)$, then $(p+q)$ equals

(a) 3 (b) 4
(c) 7 (d) 9

Note Mean proportional of a and b $(a > 0, b > 0)$ is $\sqrt{ab}$]

Passage III

(Q. Nos. 91 to 93)

If $7\theta = (2n+1)\pi$, where $n = 0, 1, 2, 3, 4, 5, 6$, then answer the following questions.

91. The equations whose roots are $\cos\frac{\pi}{7}, \cos\frac{3\pi}{7}, \cos\frac{5\pi}{7}$ is

(a) $8x^3 + 4x^2 + 4x + 1 = 0$
(b) $8x^3 - 4x^2 - 4x - 1 = 0$
(c) $8x^3 - 4x^2 - 4x - 1 = 0$
(d) $8x^3 + 4x^2 + 4x - 1 = 0$

92. The value of $\sec\frac{\pi}{7} + \sec\frac{3\pi}{7} + \sec\frac{5\pi}{7}$ is

(a) 4 (b) − 4
(c) 3 (d) − 3

93. The value of $\sec^2\frac{\pi}{7} + \sec^2\frac{3\pi}{7} + \sec^2\frac{5\pi}{7}$ is

(a) − 24 (b) 80 (c) 24 (d) − 80

Passage IV

(Q. Nos. 94 to 96)

If $1 + 2\sin x + 3\sin^2 x + 4\sin^3 x + \ldots$ upto infinite terms = 4 and number of solutions of the equation in $\left[\frac{-3\pi}{2}, 4\pi\right]$ is k.

94. The value of k is equal to

(a) 4 (b) 5 (c) 6 (d) 7

95. The value of $\left|\frac{\cos 2x - 1}{\sin 2x}\right|$ is equal to

(a) 1 (b) $\sqrt{3}$
(c) $2 - \sqrt{3}$ (d) $\frac{1}{\sqrt{3}}$

96. Sum of all internal angles of a k-sided regular polygon is

(a) 5π (b) 4π
(c) 3π (d) 2π

Passage V

(Q. Nos. 97 to 98)

Let α is a root of the equation $(2\sin x - \cos x)(1 + \cos x) = \sin^2 x$, β is a root of the equation $3\cos^2 x - 10\cos x + 3 = 0$ and γ is a root of the equation $1 - \sin 2x = \cos x - \sin x$, $0 \leq \alpha, \beta, \gamma \leq \frac{\pi}{2}$.

97. $\cos\alpha + \cos\beta + \cos\gamma$ can be equal to

(a) $\frac{3\sqrt{6} + 2\sqrt{2} + 6}{6\sqrt{2}}$ (b) $\frac{3\sqrt{3} + 8}{6}$
(c) $\frac{3\sqrt{3} + 2}{6}$ (d) None of these

98. $\sin(\alpha - \beta)$ is equal to

(a) 1 (b) 0
(c) $\frac{1 - 2\sqrt{6}}{6}$ (d) $\frac{\sqrt{3} - 2\sqrt{2}}{6}$

Trigonometric Functions and Identities Exercise 5: Matching Type Questions

99. Match the statement of Column I with values of Column II.

	Column I		Column II
(A)	If $\theta + \phi = \frac{\pi}{2}$, where θ and ϕ are positive, then $(\sin\theta + \sin\phi)\sin\left(\frac{\pi}{4}\right)$ is always less than	(p)	1
(B)	If $\sin\theta - \sin\phi = a$ and $\cos\theta + \cos\phi = b$, then $a^2 + b^2$ cannot exceed	(q)	2
(C)	If $3\sin\theta + 5\cos\theta = 5$, $(\theta \neq 0)$ then the value of $5\sin\theta - 3\cos\theta$ is	(r)	3
		(s)	4
		(t)	5

100. Match the statement of Column I with values of Column II.

	Column I		Column II
(A)	If maximum and minimum values of $\frac{7 + 6\tan\theta - \tan^2\theta}{(1 + \tan^2\theta)}$ for all real values of $\theta \sim \frac{\pi}{2}$ are λ and μ respectively, then	(p)	$\lambda + \mu = 2$
(B)	If maximum and minimum values of $5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3$ for all real values of θ are λ and μ respectively, then	(q)	$\lambda - \mu = 6$
(C)	If maximum and minimum values of $1 + \sin\left(\frac{\pi}{4} + \theta\right) + 2\cos\left(\frac{\pi}{4} - \theta\right)$ for all real values of θ and λ and μ respectively, then	(r)	$\lambda + \mu = 6$
		(s)	$\lambda - \mu = 10$
		(t)	$\lambda - \mu = 14$

101. Match the statement of Column I with values of Column II.

	Column I		Column II
(A)	The number of solutions of the equation $\lvert\cot x\rvert = \cot x + \frac{1}{\sin x}\ (0 < x < \pi)$ is	(p)	no solution
(B)	If $\sin\theta + \sin\phi = \frac{1}{2}$ and $\cos\theta + \cos\phi = 2$, then value of $\cot\left(\frac{\theta + \phi}{2}\right)$ is	(q)	$\frac{1}{3}$
(C)	The value of $\sin^2\alpha + \sin\left(\frac{\pi}{3} - \alpha\right)\sin\left(\frac{\pi}{3} + \alpha\right)$ is	(r)	1
(D)	If $\tan\theta = 3\tan\phi$, then maximum value of $\tan^2(\theta - \phi)$ is	(s)	2
		(t)	4

102. Match the statement of Column I with values of Column II.

	Column I		Column II
(A)	In a ΔABC, $\sin\left(\frac{A}{2}\right) + \sin\left(\frac{B}{2}\right) + \sin\left(\frac{C}{2}\right) =$	(p)	$-1 + 4\sin\left(\frac{\pi + A}{4}\right)\sin\left(\frac{\pi + B}{4}\right)\cos\left(\frac{\pi + C}{4}\right)$
(B)	In a ΔABC, $\sin\left(\frac{A}{2}\right) + \sin\left(\frac{B}{2}\right) - \sin\left(\frac{C}{2}\right) =$	(q)	$4\cos\left(\frac{\pi + A}{4}\right)\cos\left(\frac{\pi + B}{4}\right)\cos\left(\frac{\pi - C}{4}\right)$
(C)	In a ΔABC, $\cos\left(\frac{A}{2}\right) + \cos\left(\frac{B}{2}\right) - \cos\left(\frac{C}{2}\right)$	(r)	$1 + 4\sin\left(\frac{\pi - A}{4}\right)\sin\left(\frac{\pi - B}{4}\right)\sin\left(\frac{\pi - C}{4}\right)$
		(s)	$-1 + 4\cos\left(\frac{\pi - A}{4}\right)\cos\left(\frac{\pi - B}{4}\right)\sin\left(\frac{\pi - C}{4}\right)$
		(t)	$1 + 4\cos\left(\frac{\pi + A}{4}\right)\cos\left(\frac{\pi + B}{4}\right)\sin\left(\frac{\pi - C}{4}\right)$

Trigonometric Functions and Identities Exercise 6:
Single Integer Answer Type Questions

103. In a ΔABC, $\frac{1}{1+\tan^2\frac{A}{2}}+\frac{1}{1+\tan^2\frac{B}{2}}+\frac{1}{1+\tan^2\frac{C}{2}}=k$ $\left[1+\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right]$, then the value of k is

104. If $\frac{\sin\alpha}{\sin\beta}=\frac{\cos\gamma}{\cos\delta}$, then $\frac{\sin\left(\frac{\alpha-\beta}{2}\right)\cdot\cos\left(\frac{\alpha+\beta}{2}\right)\cdot\cos\delta}{\sin\left(\frac{\delta-\gamma}{2}\right)\cdot\sin\left(\frac{\delta+\gamma}{2}\right)\cdot\sin\beta}$ is equal to

105. Find the exact value of the expression
$\tan\frac{\pi}{20}-\tan\frac{3\pi}{20}+\tan\frac{5\pi}{20}-\tan\frac{7\pi}{20}+\tan\frac{9\pi}{20}$.

106. Let $x=\frac{\sum_{n=1}^{44}\cos n^\circ}{\sum_{n=1}^{44}\sin n^\circ}$, find the greatest integer that does not exceed.

107. Find θ (in degree) satisfying the equation,
$\tan 15^\circ\cdot\tan 25^\circ\cdot\tan 35^\circ=\tan\theta$, where $\theta\in(0, 45^\circ)$

108. Find the exact value of $\operatorname{cosec}10^\circ+\operatorname{cosec}50^\circ-\operatorname{cosec}70^\circ$.

109. If $\cos 5\alpha=\cos^5\alpha$, where $\alpha\in\left(0,\frac{\pi}{2}\right)$, then find the possible values of $(\sec^2\alpha+\operatorname{cosec}^2\alpha+\cot^2\alpha)$.

110. Compute the value of the expression
$\tan^2\frac{\pi}{16}+\tan^2\frac{2\pi}{16}+\tan^2\frac{3\pi}{16}+\ldots+\tan^2\frac{7\pi}{16}$.

111. Compute the square of the value of the expression
$\frac{4+\sec 20^\circ}{\operatorname{cosec}20^\circ}$

112. In ΔABC, if $\frac{\sin A}{3}=\frac{\cos B}{3}=\frac{\tan C}{2}$, then the value of $\left(\frac{\sin A}{\cot 2A}+\frac{\cos B}{\cot 2B}+\frac{\tan C}{\cot 2C}\right)$ is

113. Let f and g be function defined by $f(\theta)=\cos^2\theta$ and $g(\theta)=\tan^2\theta$, suppose α and β satisfy $2f(\alpha)-g(\beta)=1$, then value of $2f(\beta)-g(\alpha)$ is

114. If sum of the series $1+x\log_{\left|\frac{1-\sin x}{\cos x}\right|}\left(\frac{1+\sin x}{\cos x}\right)^{1/2}+x^2\log_{\left|\frac{1-\sin x}{\cos x}\right|}\left(\frac{1+\sin x}{\cos x}\right)^{1/4}+\ldots\infty$ (wherever defined) is equal to $\frac{k(1-x)}{(2-x)}$, then k is equal to

115. If $\frac{9x}{\cos\theta}+\frac{5y}{\sin\theta}=56$ and $\frac{9x\sin\theta}{\cos^2\theta}-\frac{5y\cos\theta}{\sin^2\theta}=0$ then the value of $\left[(9x)^{\frac{2}{3}}+(5y)^{\frac{2}{3}}\right]^3$ is

116. The angle A of the ΔABC is obtuse.
$x=2635-\tan B\tan C$, if $[x]$ denotes the greatest integer function, the value of $[x]$ is

117. If $4\cos 36^\circ+\cot\left(7\frac{1}{2}^\circ\right)=\sqrt{n_1}+\sqrt{n_2}+\sqrt{n_3}+\sqrt{n_4}+\sqrt{n_5}+\sqrt{n_6}$, then the value of $\sum_{i=1}^{6}n_i^2$ must be

118. If $\sin^2 A=x$ and $\prod_{r=1}^{4}\sin(rA)=ax^2+bx^3+cx^4+dx^5$, then the value of $10a-7b+15c-5d$ must be

119. If $x, y\in R$ satisfies $(x+5)^2+(y-12)^2=(14)^2$, then the minimum value of $\sqrt{x^2+y^2}$ is .....

120. The least degree of a polynomial with integer coefficient whose one of the roots may be $\cos 12^\circ$ is

121. If $A+B+C=180^\circ$, $\frac{\sin 2A+\sin 2B+\sin 2C}{\sin A+\sin B+\sin C}=k\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ then the value of $3k^3+2k^2+k+1$ is equal to

122. The value of $f(x)=x^4+4x^3+2x^2-4x+7$, when $x=\cot\frac{11\pi}{8}$ is .....

123. In any ΔABC, then minimum value of $2020\sum\frac{\sqrt{(\sin A)}}{(\sqrt{(\sin B)}+\sqrt{(\sin C)}-\sqrt{(\sin A)}}$ must be

124. If $\sin\theta+\sin^2\theta+\sin^3\theta=1$, then the value of $\cos^6\theta-4\cos^4\theta+8\cos^2\theta$ must be

125. $16\left(\cos\theta-\cos\frac{\pi}{8}\right)\left(\cos\theta-\cos\frac{3\pi}{8}\right)\left(\cos\theta-\cos\frac{5\pi}{8}\right)\left(\cos\theta-\cos\frac{7\pi}{8}\right)=\lambda\cos 4\theta$, then the value of λ is

.....

126. If $\frac{1}{\sin 20^\circ}+\frac{1}{\sqrt{3}\cos 20^\circ}=2k\cos 40^\circ$, then $18k^4+162k^2+369$ is equal to

Trigonometric Functions and Identities Exercise 7 :
Subjective Type Questions

127. Prove that $\tan 82\frac{1}{2}^\circ=(\sqrt{3}+\sqrt{2})(\sqrt{2}+1)$

or $\cot 7\frac{1}{2}^\circ=\sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$.

128. If $m\sin(\alpha+\beta)=\cos(\alpha-\beta)$, prove that

$$\frac{1}{1-m\sin 2\alpha}+\frac{1}{1-m\sin 2\beta}=\frac{2}{1-m^2}.$$

129. If $\alpha+\beta+\gamma=\pi$ and

$\tan\frac{1}{4}(\beta+\gamma-\alpha)\tan\frac{1}{4}(\gamma+\alpha-\beta)\tan\frac{1}{4}(\alpha+\beta-\gamma)=1$,

then prove that $1+\cos\alpha+\cos\beta+\cos\gamma=0$.

130. Find the value of a for which the equation $\sin^4 x+\cos^4 x=a$ has real solutions.

131. If a and b are positive quantities and $a\geq b$, then find the minimum positive values of $a\sec\theta-b\tan\theta$.

132. If a, b, c and k are constant quantities and α, β, γ are variable subjects to the relation $a\tan\alpha+b\tan\beta+c\tan\gamma=k$, then find the minimum value of $\tan^2\alpha+\tan^2\beta+\tan^2\gamma$.

133. If $\frac{x}{\tan(\theta+\alpha)}=\frac{y}{\tan(\theta+\beta)}=\frac{z}{\tan(\theta+\gamma)}$, prove that :

$$\Sigma\frac{x+y}{x-y}\sin^2(\alpha-\beta)=0.$$

134. Let $a_1, a_2, \ldots, a_n$ be real constants, x be a real variable and $f(x)=\cos(a_1+x)+\frac{1}{2}\cos(a_2+x)+\frac{1}{4}\cos(a_3+x)+\ldots+\frac{1}{2^{n-1}}\cos(a_n+x)$

Given that $f(x_1)=f(x_2)=0$, prove that $x_2-x_1=m\pi$ for some integer m.

135. Eliminate θ from the equations
$\tan(n\theta+\alpha)-\tan(n\theta+\beta)=x$ and
$\cot(n\theta+\alpha)-\cot(n\theta+\beta)=y$.

136. If $\{\sin(\alpha-\beta)+\cos(\alpha+2\beta)\sin\beta\}^2=4\cos\alpha\sin\beta\sin(\alpha+\beta)$. Then, prove that $\tan\alpha+\tan\beta=\frac{\tan\beta}{(\sqrt{2}\cos\beta-1)^2}$;

$\alpha, \beta\in(0, \pi/4)$.

137. If A, B, C are the angle of a triangle and $\begin{vmatrix}\sin A & \sin B & \sin C\\ \cos A & \cos B & \cos C\\ \cos^3 A & \cos^3 B & \cos^3 C\end{vmatrix}=0$, then show that ΔABC is an isosceles.

138. In any ΔABC, prove that

$$\Sigma\frac{\sqrt{\sin A}}{\sqrt{\sin B}+\sqrt{\sin C}-\sqrt{\sin A}}\geq 3$$

and the equality holds if and only if triangle is equilateral.

139. If $2(\cos(\alpha-\beta)+\cos(\beta-\gamma)+\cos(\gamma-\alpha)+3=0$, prove that $\frac{d\alpha}{\sin(\beta+\theta)\sin(\gamma+\theta)}+\frac{d\beta}{\sin(\alpha+\theta)\sin(\gamma+\theta)}+\frac{d\gamma}{\sin(\alpha+\theta)\sin(\beta+\theta)}=0$,

where, 'θ' is any real angle such that $\alpha+\theta, \beta+\theta, \gamma+\theta$ are not the multiple of π.

140. If the quadratic equation $4^{\sec^2\alpha}x^2+2x+\left(\beta^2-\beta+\frac{1}{2}\right)=0$ have real roots, then find all the possible values of $\cos\alpha+\cos^{-1}\beta$.

141. Four real constants a, b, A, B are given and $f(\theta)=1-a\cos\theta-b\sin\theta-A\cos 2\theta-B\sin 2\theta$. Prove that if $f(\theta)\geq 0, \forall\,\theta\in R$, then $a^2+b^2\leq 2$ and $A^2+B^2\leq 1$.

142. If $\frac{\cos\theta_1}{\cos\theta_2}+\frac{\sin\theta_1}{\sin\theta_2}=\frac{\cos\theta_0}{\cos\theta_2}+\frac{\sin\theta_0}{\sin\theta_2}=1$, where θ_1 and θ_0 do not differ by an even multiple of π, prove that

$$\frac{\cos\theta_1\cdot\cos\theta_0}{\cos^2\theta_2}+\frac{\sin\theta_1\cdot\sin\theta_0}{\sin^2\theta_2}=-1$$

143. Prove that
$\sum_{k=1}^{n-1} {}^{n}C_k [\cos k\,x \,.\cos(n+k)x + \sin(n-k)x \,.\sin(2n-k)x] =$
$(2^n - 2)\cos nx.$

144. Determine all the values of x in the interval $x \in [0, 2\pi]$ which satisfy the inequality
$2\cos x \le |\sqrt{1+\sin 2x} - \sqrt{1-\sin 2x}| \le \sqrt{2}.$

145. Find all the solutions of this equation
$x^2 - 3\left[\sin\left(x - \frac{\pi}{6}\right)\right] = 3$, where [.] represents the greatest integer function.

146. In a ΔABC, prove that
$\sum_{r=0}^{n} {}^{n}C_r a^r b^{n-r} \cos(r\,B - (n-r)A) = c^n.$

147. Resolve $z^5 + 1$ into linear and quadratic factors with real coefficients. Hence, or otherwise deduce that,
$4\sin\left(\frac{\pi}{10}\right)\cdot\cos\left(\frac{\pi}{5}\right) = 1.$

148. Prove that the roots of the equation
$8x^3 - 4x^2 - 4x + 1 = 0$ are $\cos\frac{\pi}{7}, \cos\frac{3\pi}{7}, \cos\frac{5\pi}{7}$ and hence, show that $\sec\frac{\pi}{7} + \sec\frac{3\pi}{7} + \sec\frac{5\pi}{7} = 4$ and deduce the equation whose roots are $\tan^2\frac{\pi}{7}, \tan^2\frac{3\pi}{7}, \tan^2\frac{5\pi}{7}$.

Trigonometric Functions and Identities Exercise 8 : Questions Asked in Previous 10 Years Exam

149. Let α and β be non-zero real numbers such that $2(\cos\beta - \cos\alpha) + \cos\alpha\cos\beta = 1$. Then which of the following is/are true?
[More than one correct option 2017 Adv.]
(a) $\sqrt{3}\tan\left(\frac{\alpha}{2}\right) - \tan\left(\frac{\beta}{2}\right) = 0$
(b) $\tan\left(\frac{\alpha}{2}\right) - \sqrt{3}\tan\left(\frac{\beta}{2}\right) = 0$
(c) $\tan\left(\frac{\alpha}{2}\right) + \sqrt{3}\tan\left(\frac{\beta}{2}\right) = 0$
(d) $\sqrt{3}\tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\beta}{2}\right) = 0$

150. Let $-\frac{\pi}{6} < \theta < -\frac{\pi}{12}$. Suppose α_1 and β_1 are the roots of the equation $x^2 - 2x\sec\theta + 1 = 0$, and α_2 and β_2 are the roots of the equation $x^2 + 2x\tan\theta - 1 = 0$. If $\alpha_1 > \beta_1$ and $\alpha_2 > \beta_2$, then $\alpha_1 + \beta_2$ equals to
[Single correct option 2016 Adv.]
(a) $2(\sec\theta - \tan\theta)$ (b) $2\sec\theta$
(c) $-2\tan\theta$ (d) 0

151. The value of $\sum_{k=1}^{13} \frac{1}{\sin\left(\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right)\sin\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)}$ is equal to
[Single correct option 2016 Adv.]
(a) $3 - \sqrt{3}$ (b) $2(3 - \sqrt{3})$ (c) $2(\sqrt{3} - 1)$ (d) $2(2 + \sqrt{3})$

152. Let $f : (-1, 1) \to R$ be such that $f(\cos 4\theta) = \frac{2}{2 - \sec^2\theta}$ for $\theta \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. Then, the value(s) of $f\left(\frac{1}{3}\right)$ is/are
[More than one correct option 2012]
(a) $1 - \sqrt{\frac{3}{2}}$ (b) $1 + \sqrt{\frac{3}{2}}$ (c) $1 - \sqrt{\frac{2}{3}}$ (d) $1 + \sqrt{\frac{2}{3}}$

153. The number of all possible values of θ, where $0 < \theta < \pi$, for which the system of equations
$(y+z)\cos 3\theta = (xyz)\sin 3\theta$
$x\sin 3\theta = \frac{2\cos 3\theta}{y} + \frac{2\sin 3\theta}{z}$,
and $(xyz)\sin 3\theta = (y+2z)\cos 3\theta + y\sin 3\theta$ have a solution (x_0, y_0, z_0) with $y_0 z_0 \neq 0$, is .....
[Integer Answer Type 2010]

154. For $0 < \theta < \frac{\pi}{2}$, the solution(s) of
$\sum_{m=1}^{6} \text{cosec}\left(\theta + \frac{(m-1)\pi}{4}\right)\text{cosec}\left(\theta + \frac{m\pi}{4}\right) = 4\sqrt{2}$ is/are
[More than one correct option 2009]
(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{12}$ (d) $\frac{5\pi}{12}$

155. If $\frac{\sin^4 x}{2} + \frac{\cos^4 x}{3} = \frac{1}{5}$, then
[Single correct option 2009]
(a) $\tan^2 x = \frac{2}{3}$ (b) $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{1}{125}$
(c) $\tan^2 x = \frac{1}{3}$ (d) $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{2}{125}$

156. Let $\theta \in \left(0, \frac{\pi}{4}\right)$ and $t_1 = (\tan\theta)^{\tan\theta}, t_2 = (\tan\theta)^{\cot\theta}$, $t_3 = (\cot\theta)^{\tan\theta}$ and $t_4 = (\cot\theta)^{\cot\theta}$, then
[Single correct option 2006]
(a) $t_1 > t_2 > t_3 > t_4$ (b) $t_4 > t_3 > t_1 > t_2$
(c) $t_3 > t_1 > t_2 > t_4$ (d) $t_2 > t_3 > t_1 > t_4$

157. The number of ordered pairs (α, β), where $\alpha, \beta \in (-\pi, \pi)$ satisfying $\cos(\alpha - \beta) = 1$ and $\cos(\alpha + \beta) = \frac{1}{e}$ is **[Single correct option 2005]**

(a) 0 (b) 1
(c) 2 (d) 4

II. JEE Mains and AIEEE

158. $5(\tan^2 x - \cos^2 x) = 2\cos 2x + 9$, then the value of $\cos 4x$ is **[2017 JEE Main]**

(a) $-\frac{3}{5}$ (b) $\frac{1}{3}$ (c) $\frac{2}{9}$ (d) $-\frac{7}{9}$

159. If $f_k(x) = \frac{1}{k}(\sin^k x + \cos^k x)$, where $x \in R, k \geq 1$, then $f_4(x) - f_6(x)$ is equal to **[2014 JEE Main]**

(a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) $\frac{1}{12}$

160. The expression $\frac{\tan A}{1 - \cot A} + \frac{\cot A}{1 - \tan A}$ can be written as **[2013 JEE Main]**

(a) $\sin A \cos A + 1$ (b) $\sec A \operatorname{cosec} A + 1$
(c) $\tan A + \cot A$ (d) $\sec A + \operatorname{cosec} A$

161. In a ΔPQR, if $3\sin P + 4\cos Q = 6$ and $4\sin Q + 3\cos P = 1$, then the angle R is equal to **[2012 AIEEE]**

(a) $\frac{5\pi}{6}$ (b) $\frac{\pi}{6}$
(c) $\frac{\pi}{4}$ (d) $\frac{3\pi}{4}$

162. If $A = \sin^2 x + \cos^4 x$, then for all real x **[2011 AIEEE]**

(a) $\frac{13}{16} \leq A \leq 1$ (b) $1 \leq A \leq 2$
(c) $\frac{3}{4} \leq A \leq \frac{13}{16}$ (d) $\frac{3}{4} \leq A \leq 1$

163. Let $\cos(\alpha + \beta) = \frac{4}{5}$ and $\sin(\alpha - \beta) = \frac{5}{13}$, where $0 \leq \alpha, \beta \leq \frac{\pi}{4}$. Then, $\tan 2\alpha$ is equal to **[2010 AIEEE]**

(a) $\frac{25}{16}$ (b) $\frac{56}{33}$ (c) $\frac{19}{12}$ (d) $\frac{20}{7}$

164. Let A and B denote the statements

$A: \cos\alpha + \cos\beta + \cos\gamma = 0$

$B: \sin\alpha + \sin\beta + \sin\gamma = 0$

If $\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = -\frac{3}{2}$, then **[2009 AIEEE]**

(a) A is true and B is false
(b) A is false and B is true
(c) Both A and B are true
(d) Both A and B are false

165. A triangular park is enclosed on two sides by a fence and on the third side by a straight river bank. The two sides having fence are of same length x. The maximum area enclosed by the park is **[2006 AIEEE]**

(a) $\sqrt{\frac{x^3}{8}}$ (b) $\frac{1}{2}x^2$ (c) πx^2 (d) $\frac{3}{2}x^2$

166. If $0 < x < \pi$ and $\cos x + \sin x = \frac{1}{2}$, then $\tan x$ is **[2006 AIEEE]**

(a) $\frac{(4 - \sqrt{7})}{3}$ (b) $-\frac{(4 + \sqrt{7})}{3}$
(c) $\frac{(1 + \sqrt{7})}{4}$ (d) $\frac{(1 - \sqrt{7})}{4}$

167. In a ΔPQR, $\angle R = \frac{\pi}{2}$. If $\tan\left(\frac{P}{2}\right)$ and $\tan\left(\frac{Q}{2}\right)$ are the roots of $ax^2 + bx + c = 0, a \neq 0$, then **[2005 AIEEE]**

(a) $b = a + c$ (b) $b = c$
(c) $c = a + b$ (d) $a = b + c$

Answers

Exercise for Session 1

1. 72°, 18° 2. $\frac{\pi}{2}$ cm 3. 70 m 4. 45° 5. 8π
6. $\left(\frac{11}{90}\right)^c$ 7. 252 cm 8. 880 cm/s 9. 1.7 cm 10. 7

Exercise for Session 2

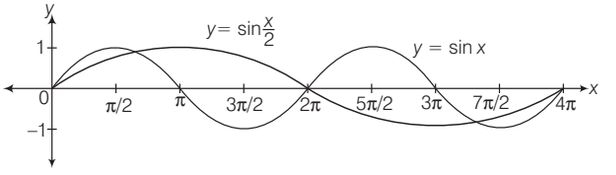
3. -3 4. $\pm\sqrt{a^2 + b^2 - c^2}$ 5. 13 6. 2 7. $\frac{1}{12}$
10. 4

Exercise for Session 3

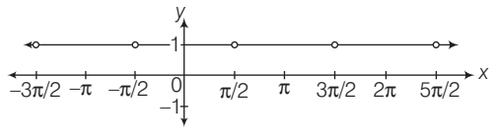
1. $\frac{2k}{k^2 + 1}$ 2. 1 4. $x = 1, y = 0$ 6. 0
7. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$ 9. 10 10. $\frac{7}{5}$

Exercise for Session 4

1. 28 3. -2
4.



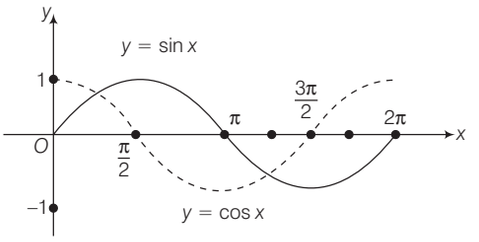
5.



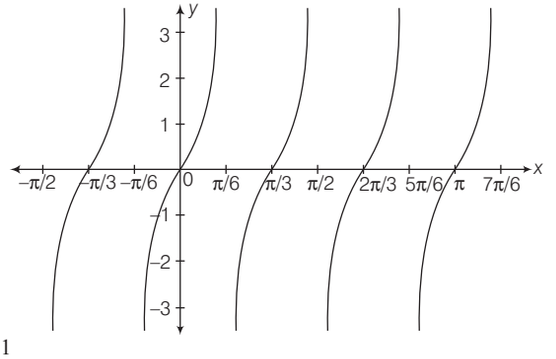
From the graph, the period of the function is π.

7. $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

8.



9.



10. $\frac{1}{4}$

Exercise for Session 5

1. $\sqrt{3}$ **2.** Negative **3.** -1 **4.** 1 **5.** 1 **6.** 1
7. 2 **8.** does not exists any real solutions **9.** 2
10. 0

Exercise for Session 6

1. I and IV **3.** $\frac{\pi}{4}$ **4.** $\frac{65}{33}$ **5.** $\tan\beta + 2\tan\gamma$
6. $a^2 + b^2$ **8.** 27

Exercise for Session 7

4. $\frac{4}{5}$ **5.** 0 **6.** $2\cot^n\left(\frac{A-B}{2}\right)$ **7.** $\frac{1}{8}$ **8.** $\frac{2\pi}{8}$

Exercise for Session 8

1. False **2.** $\tan\theta$ **3.** $\frac{2}{\sqrt{5}}$ **5.** $\alpha = -1, \beta = 3$ **6.** $\frac{24}{7}$
7. -1 **8.** 1 **9.** 15 **10.** 1

Exercise for Session 9

1. $-2 \pm \sqrt{5}$ **2.** $\frac{3+2\sqrt{2}}{8}$ **3.** $\frac{4}{\sqrt{3}}$ **4.** $2\cos n\theta$, **10.** $\cot\frac{\pi}{5}$

Exercise for Session 10

4. $\frac{1}{2}$ **5.** 2 **7.** -1 **8.** 1

Exercise for Session 11

2. 0 **4.** 2 **5.** positive **6.** $\frac{3}{4} \le A \le 1$ **7.** 4.
8. $-\sqrt{2}$ and $\sqrt{2}$ **10.** 13 : 4 or 1 : 4

Chapter Exercises

1. (b) **2.** (b) **3.** (a) **4.** (a) **5.** (b) **6.** (a)
7. (b) **8.** (a) **9.** (a) **10.** (b) **11.** (d) **12.** (d)
13. (d) **14.** (b) **15.** (a) **16.** (b) **17.** (b) **18.** (b)
19. (a) **20.** (a) **21.** (c) **22.** (a) **23.** (a) **24.** (b)
25. (d) **26.** (c) **27.** (b) **28.** (b) **29.** (c) **30.** (b)
31. (c) **32.** (a) **33.** (b) **34.** (c) **35.** (b) **36.** (b)
37. (a) **38.** (d) **39.** (d) **40.** (c) **41.** (c)
42. (b) **43.** (b) **44.** (c) **45.** (a) **46.** (b) **47.** (c)
48. (b) **49.** (a) **50.** (a) **51.** (a) **52.** (d) **53.** (a)
54. (a,b,c,d) **55.** (a,b,c) **56.** (b,d) **57.** (b,c) **58.** (b,c)
59. (b,c) **60.** (a,b,d) **61.** (b,c) **62.** (c,d) **63.** (d) **64.** (b, c, d)
65. (b, c) **66.** (a, b, c, d) **67.** (a, b, c, d) **68.** (c, d)
69. (a, b, c, d) **70.** (b, c) **71.** (a, b, c, d) **72.** (b, c)
73. (a, b, c, d) **74.** (b, c) **75.** (a) **76.** (b) **77.** (b)
78. (d) **79.** (d) **80.** (a) **81.** (a) **82.** (c) **83.** (b)
84. (b) **85.** (b) **86.** (b) **87.** (a) **88.** (b) **89.** (b)
90. (c) **91.** (b) **92.** (a) **93.** (c) **94.** (b) **95.** (d)
96. (c) **97.** (b) **98.** (c)
99. A—(p, q, r, s, t); B—(s, t); C—(r)
100. A—(r, s); B—(r, t); C—(p, q)
101. A—(r); B—(p); C—(p); D—(q)
102. A—(r, t); B—(p, s); C—q
103. (2) **104.** (1) **105.** (5) **106.** (2) **107.** (5) **108.** (6)
109. (5) **110.** (35) **111.** (3) **112.** (2) **113.** (1) **114.** (2)
115. (3136) **116.** (2634) **117.** (91) **118.** (3448)
119. (1) **120.** (4) **121.** (1673) **122.** (6) **123.** (6060)
124. (4) **125.** (2) **126.** (1745)
130. $\frac{1}{2} \le a \le 1$ **131.** Minimum value is $\sqrt{a^2 - b^2}$
132. Minimum value is $\left(\frac{k^2}{a^2 + b^2 + c^2}\right)$
135. $\cot(\alpha+\beta) = \frac{1}{x} - \frac{1}{y}$
140. $\cos\alpha - \cos^{-1}\beta = \begin{cases} \frac{\pi}{3} - 1, & \text{when } n \text{ is an even integer} \\ \frac{\pi}{3} + 1, & \text{when } n \text{ is an even integer} \end{cases}$
144. $x \in \left[\frac{\pi}{4}, \frac{7\pi}{4}\right]$
145. Only two solutions, $x = 0, \sqrt{3}$
148. Required equation is, $z^3 - 21z^2 + 35z - 7 = 0$ whose roots are $\tan^2\frac{\pi}{7}, \tan^2\frac{3\pi}{7}, \tan^2\frac{5\pi}{7}$
149. (b, c) **150.** (c) **151.** (c) **152.** (a,b) **153.** (3)
154. (c,d) **155.** (b) **156.** (b) **157.** (d) **158.** (d) **159.** (d)
160. (b) **161.** (b) **162.** (d) **163.** (b) **164.** (c) **165.** (b)
166. (b) **167.** (c)

Solutions

1. Given series

$$= \left(\sin\frac{2\pi}{11} - \cos\frac{2\pi}{11}\right) + \left(\sin\frac{4\pi}{11} - \cos\frac{4\pi}{11}\right) + \left(\sin\frac{6\pi}{11} - \cos\frac{6\pi}{11}\right) + \ldots + \left(\sin\frac{20\pi}{11} - \cos\frac{20\pi}{11}\right)$$

$$= \left(\sin\frac{2\pi}{11} + \sin\frac{4\pi}{11} + \ldots + \sin\frac{20\pi}{11}\right) - \left(\cos\frac{2\pi}{11} + \cos\frac{4\pi}{11} + \ldots + \cos\frac{20\pi}{11}\right)$$

$$= \frac{\sin\pi \cdot \sin\frac{10\pi}{11}}{\sin\frac{\pi}{11}} - \frac{\cos\pi \cdot \sin\frac{10\pi}{11}}{\sin\frac{\pi}{11}}$$

$$= 0 + \frac{\sin\left(\pi - \frac{\pi}{11}\right)}{\sin\frac{\pi}{11}} = 1$$

2. $(a+1)^2 + \operatorname{cosec}^2\left(\frac{\pi a}{2} + \frac{\pi x}{2}\right) - 1 = 0$

or $(a+1)^2 + \cot^2\left(\frac{\pi a}{2} + \frac{\pi x}{2}\right) = 0$

From option [b], if $a = -1$ and $\cot^2\left(\frac{-\pi}{2} + \frac{\pi x}{2}\right) = 0$

$\Rightarrow \quad \tan^2\left(\frac{\pi x}{2}\right) = 0$

$\Rightarrow \quad \frac{\pi}{2} = 1$

3. $f(x) = 9\sin^2 x - 16\cos^2 x - 10(3\sin x - 4\cos x) - 10(3\sin x + 4\cos x) + 100$

$= 25\sin^2 x - 60\sin x + 84$

$= (5\sin x - 6)^2 + 48$

$\therefore f(x)_{\min}$ occurs when $\sin x = 1$

Minimum value = 49

4. $S = \frac{1}{1+\tan^3 0^\circ} + \frac{1}{1+\tan^3 10^\circ} + \ldots + \frac{1}{1+\tan^3 80^\circ}$

Now, $\frac{1}{1+\tan^3\theta} + \frac{1}{1+\tan^3(90-\theta)}$

$= \frac{1}{1+\tan^3\theta} + \frac{1}{1+\cot^3\theta}$

$= \frac{1}{1+\tan^3\theta} + \frac{\tan^3\theta}{1+\tan^3\theta}$

$= \frac{1+\tan^3\theta}{1+\tan^3\theta} = 1$

Hence, $S = 1 + (1+1+1+1) = 5$

5. Clearly, $\sqrt{1-\sin^2 110^\circ} \cdot \sec 110^\circ$

$= |\cos 110^\circ| \sec 110^\circ$

$= -\cos 110^\circ \sec 110^\circ = -1$

6. $\tan\alpha + \tan\beta = -p$

$\tan\alpha \tan\beta = q$

$\tan(\alpha+\beta) = \frac{-p}{1-q} = \frac{p}{q-1}$

$\frac{1}{1+\tan^2(\alpha+\beta)}[\tan^2(\alpha+\beta) + p\tan(\alpha+\beta) + q]$

$= \frac{1}{1+\frac{p^2}{(q-1)^2}}\left[\frac{p^2}{(q-1)^2} + \frac{p^2}{(q-1)} + q\right]$

$= \frac{1}{(q-1)^2 + p^2}[p^2 + p^2(q-1) + q(q-1)^2]$

$= \frac{1}{p^2 + (q-1)^2}[p^2 q + q(q-1)^2]$

$= q\left[\frac{p^2 + (q-1)^2}{p^2 + (q-1)^2}\right] = q$

7. Let A be the expression. Multiplying A by 2^{2008} and using $2\sin\theta\cos\theta = \sin 2\theta$,

we have $\quad 2^{2008} A = \sin\frac{\pi}{2} = 1.\ A = \frac{1}{2^{2008}}$

Alternatively $\sin\left(\frac{\pi}{2^{2009}}\right)\cos\left(\frac{\pi}{2^{2009}}\right) = \frac{1}{2}\sin\left(\frac{\pi}{2^{2008}}\right)$

$= \frac{1}{2^2} \cdot 2\sin\left(\frac{\pi}{2^{2008}}\right)\cos\left(\frac{\pi}{2^{2008}}\right)$

$= \frac{1}{2^2}\sin\left(\frac{\pi}{2^{2007}}\right)$

Similarly, continued product upto,

$$\cos\left(\frac{\pi}{2^2}\right) = \frac{1}{2^{2008}}\sin\left(\frac{\pi}{2}\right) = \frac{1}{2^{2008}}$$

8. $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$= \frac{\tan A + \frac{n\sin A\cos A}{1 - n\cos^2 A}}{1 - \tan A \cdot \frac{n\sin A\cos A}{1 - n\cos^2 A}}$

$= \frac{\sin A(1 - n\cos^2 A) + n\sin A\cos^2 A}{\cos A(1 - n\cos^2 A) - n\sin^2 A\cos A}$

$= \frac{\sin A - 0}{\cos A(1 - n\cos^2 A - n\sin^2 A)}$

$= \frac{\sin A}{(1-n)\cos A}$

9. We have, $\quad Q = \sum_{r=0}^{n} \frac{\sin(3^r\theta)}{\cos(3^{r+1}\theta)}$

$= \frac{\sin\theta}{\cos 3\theta} + \frac{\sin 3\theta}{\cos 9\theta} + \frac{\sin 9\theta}{\cos 27\theta} + \ldots\ldots + \frac{\sin(3^n\theta)}{\cos(3^{n+1}\theta)}$

As, $\dfrac{\sin\theta}{\cos3\theta}=\dfrac{2\sin\theta\cos\theta}{2\cos\theta\cos3\theta}=\dfrac{\sin2\theta}{2\cos\theta\cos3\theta}$

$$=\frac{1}{2}\left[\frac{\sin(3\theta-\theta)}{\cos\theta\cos3\theta}\right]$$

$$=\frac{1}{2}(\tan3\theta-\tan\theta)$$

$\therefore$ $Q=\dfrac{1}{2}[(\tan3\theta-\tan\theta)+(\tan9\theta-\tan3\theta)+\ldots+\tan3^{n+1}\theta-\tan3^{n}\theta)]$

$\Rightarrow$ $Q=\dfrac{P}{2}\Rightarrow P=2Q$

10. Expression

$(\cos^4 1^\circ+\cos^4 2^\circ+\cos^4 3^\circ+\ldots+\cos^4 179^\circ)$

$-(\sin^4 1^\circ+\sin^4 2^\circ+\sin^4 3^\circ+\ldots+\sin^4 179^\circ)$

$=\cos2^\circ+\cos4^\circ+\cos6^\circ+\ldots+\cos(358^\circ)$

$$=\cos\frac{\left(\dfrac{2^\circ+358^\circ}{2}\right)\cdot\sin(179\times1^\circ)}{\sin1^\circ}$$

$=\cos(180^\circ)=-1.$

11. $\sin x+\sin y=a$ ...(i)

$\cos x+\cos y=2a$ ...(ii)

On squaring and adding Eqs. (i) and (ii), we get

$2+2\cos(x-y)=5a^2$

$$\cos(x-y)=\frac{5a^2-2}{2}$$

12. $P(x)=\sqrt{3+2(\cos x+\cos x+\cos2x)}$

$=\sqrt{3+2(2\cos x+2\cos x^2x-1}$

$=\sqrt{4\cos^2 x+4\cos^2 x+1}$

$=|2\cos x+1|$

13. Consider $y=5\,\text{secc}^2\theta-\tan^2\theta+4\text{cosec}^2\theta$

$\therefore$ $y=5+5\tan^2\theta-\tan^2\theta+4+\cot^2\theta$

$y=9+4(\tan^2+\cot^2)$

$=9+4[(\tan\theta-\cot\theta)^2+2]$

$\therefore$ $y_{\min}=9+8=17$

$\Rightarrow$ Maximum value of the expression is $\dfrac{1}{17}=\dfrac{p}{q}$

$\Rightarrow$ $p+q=1+17=18$

14. $f_n(\alpha)=\tan n\alpha$ and $f_n\left(\dfrac{\pi}{32}\right)=\tan\dfrac{\pi}{8}=\sqrt{2}-1$

15. Let $\sin x+\cos x=t$

$\therefore$ $y=\left|t+\dfrac{1}{t}\right|$

Hence, minimum value of y is 2.

16. $a=\cos(2012\,\pi)=1$

$b=\sec(2013\,\pi)=-1$

$c=\tan(2014\,\pi)=0$

$\therefore b<c<a$

17. InΔABC, $\sum\tan\dfrac{A}{2}\cdot\tan\dfrac{B}{2}=1$

$\therefore$ $\sum\tan^2\dfrac{A}{2}\geq\sum\tan\dfrac{A}{2}\tan\dfrac{B}{2}=1$

$[\because a^2+b^2+c^2-ab-bc-ca\geq0,\ \forall\ a,b,c\in R]$

$\therefore$ $3+\sum\tan^2\dfrac{A}{2}\geq4$

$\Rightarrow$ $3+\tan^2\dfrac{A}{2}+\tan^2\dfrac{B}{2}+\tan^2\dfrac{C}{2}\geq1+3$

$\Rightarrow$ $\sec^2\dfrac{A}{2}+\sec^2\dfrac{B}{2}+\sec^2\dfrac{C}{2}\geq4$

18. The given equation can be rewritten as

$(2x-1)^2+1=\sin^2 y$, which is possible only when $x=\dfrac{1}{2}$,

$\sin^2 y=1$

$\Rightarrow y=\dfrac{-\pi}{2},\dfrac{\pi}{2}$ [as $x^2+y^2\leq3$]

Thus, there are only two pairs (x, y) satisfying the given equation. They are $\left(\dfrac{1}{2},\dfrac{-\pi}{2}\right)$ and $\left(\dfrac{1}{2},\dfrac{\pi}{2}\right)$.]

19. Given,

$3\sin A+4\cos B=6$ ...(i)

$3\cos A+4\sin B=1$ ...(ii)

On squaring and adding Eqs. (i) and (ii), we get

$9+16+24\sin(A+B)=37$

$24\sin(A+B)=12$

$\sin(A+B)=\dfrac{1}{2}$

$\Rightarrow$ $\sin C=\dfrac{1}{2}$

$C=30^\circ$ or 150°

If $C=150^\circ$, then even of $B=0$ and $A=30^\circ$.

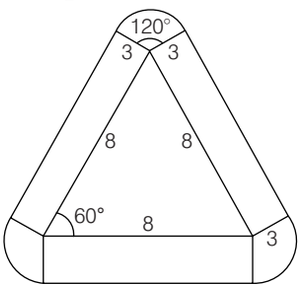
The quantity $3\sin A+4\cos B$

$$3\cdot\frac{1}{2}+4=5\frac{1}{2}<6$$

Hence, $C=150^\circ$ is not possible

$\Rightarrow$ $\angle C=30^\circ$ only

20. Area $=3\cdot(8\cdot3)+3\cdot\dfrac{1}{2}r^2\theta$



$=72+\dfrac{3}{2}\cdot9\cdot\dfrac{2\pi}{3}$

$=72+9\pi$

$=9(8+\pi)$

21. $m + n = a\{(\cos^3\alpha + \sin^3\alpha) + 3\cos\alpha\sin\alpha(\cos\alpha + \sin\alpha)\}$

$m + n = a\{\cos\alpha + \sin\alpha\}^3$

Similarly, $m - n = a\{\cos\alpha - \sin\alpha\}^3$

$(m+n)^{2/3} = a^{2/3}(\cos\alpha + \sin\alpha)^2$ ...(i)

Similarly, $(m-n)^{2/3} = a^{2/3}(\cos\alpha - \sin\alpha)^2$ ...(ii)

On adding Eqs. (i) and (ii), we get

$(m+n)^{2/3} + (m-n)^{2/3} = a^{2/3}(2)$

$\Rightarrow \quad = 2a^{2/3}$

22. $BD = x\tan C$ in ΔPDB

and $DC = x\tan B$ for ΔPDC

$\therefore \quad BD + DC = a = x(\tan B + \tan C)$

$$\frac{a}{x} = \tan B + \tan C$$

Similarly, $$\frac{b}{y} = \tan A + \tan C$$

$$\frac{c}{z} = \tan A + \tan B$$

$$\therefore \quad \frac{a}{2x} + \frac{b}{2y} + \frac{c}{2z} = \frac{1}{2}\left(\frac{a}{x} + \frac{b}{y} + \frac{c}{z}\right) \tan A + \tan B + \tan C$$

23. $\sin\theta = \dfrac{x}{l}$ ...(i)

Also, $\cos 2\theta = \dfrac{6-x}{x}$

$1 + \cos 2\theta = \dfrac{6}{x};$

$2\cos^2\theta = \dfrac{6}{l\sin\theta}$

[substituting $x = l\sin\theta$ from Eq. (i)]

$$l = \frac{3}{\sin\theta\cos^2\theta}$$

24. $S = 2\sin 2° + 4\sin 4° + \ldots + 178\sin 178 + 180°\sin 180°$

$S = 2[\sin 2° + 2\sin 4° + 3\sin 6° + \ldots + 89\sin 178°]$ ...(i)

$S = 2[89\sin 178° + 88\sin 176° + \ldots + 1\cdot\sin 2°]$ ...(ii)

On adding Eqs. (i) and (ii), we get

[converting in reverse order]

$2S = 2[90(\sin 2° + \sin 4° + \sin 6° + \ldots + \sin 178°)]$

$$S = 90\cdot\frac{\sin\left(\frac{n\theta}{2}\right)}{\sin\frac{\theta}{2}}\sin\left(\frac{(n+1)\theta}{2}\right)$$

$$= \frac{90\sin(89°)}{\sin 1°}\cdot\sin 90° \qquad [\theta = 2°]$$

$S = 90\cot 1°$

Average value $= \dfrac{90\cot 1°}{90} = \cot 1°$

25. In 1st case, $r = \cot\dfrac{\pi}{5}; R = \operatorname{cosec}\dfrac{\pi}{5}$

2nd case, $r_1 = \cot\dfrac{\pi}{7}; R_1 = \operatorname{cosec}\dfrac{\pi}{7}$

$$\therefore \quad A_1 = \pi(R^2 - r^2) = \pi\left(\operatorname{cosec}^2\frac{\pi}{5} - \cot^2\frac{\pi}{5}\right) = \pi$$

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M $A_2 = \pi(R_1^2 - r_1^2)$

$$= \pi\left(\operatorname{cosec}^2\frac{\pi}{7} - \cot^2\frac{\pi}{7}\right) = \pi$$

$\Rightarrow \quad A_1 = A_2$

26. $\sum_{r=1}^{18}\cos^2(5r)° = \cos^2 5° + \cos^2 10°$

$+ \cos^2 15° + \ldots + \cos^2 85° + \cos^2 90°$

$= (\cos^2 5° + \cos^2 85°) + (\cos^2 10° + \cos^2 80°)$

$+ (\cos^2 15° + \cos^2 75°) + \ldots + (\cos^2 40° + \cos^2 50°) + \cos^2 45°$

$= (\cos^2 5° + \sin^2 5°) + (\cos^2 10° + \sin^2 10°)$

$+ (\cos^2 15° + \sin^2 15°) + \ldots + (\cos^2 40° + \sin^2 40°) + \cos^2 45°$

$= 1 + 1 + 1 + \ldots + 1 + \dfrac{1}{2} = 8 + \dfrac{1}{2} = \dfrac{17}{2}$

27. $4x^2 - 4x|\sin\theta| - (1 - \sin^2\theta)$

$= -1 + (2x - |\sin\theta|)^2$

$\therefore$ Minimum value $= -1$

28. $\because \cos 3A + \cos 3B + \cos 3C = 1$

$\Rightarrow \quad \cos 3A + \cos 3B + \cos 3C - 1 = 0$

$\Rightarrow \quad \cos 3A + \cos 3B + \cos 3C + \cos 3\pi = 0$

$$\Rightarrow 2\cos\left(\frac{3A+3B}{2}\right)\cos\left(\frac{3A-3B}{2}\right) + 2\cos\left(\frac{3\pi+3C}{2}\right)\cos\left(\frac{3\pi-3C}{2}\right) = 0$$

$$\Rightarrow 2\cos\left(\frac{3\pi-3C}{2}\right)\left\{\cos\left(\frac{3A-3B}{2}\right) + \cos\left(\frac{3\pi+3C}{2}\right)\right\} = 0$$

$$\Rightarrow \quad 2\cos\left(\frac{3\pi}{2} - \frac{3C}{2}\right)\cdot 2\cos\left(\frac{3\pi+3C+3A-3B}{4}\right)\cdot\cos\left(\frac{3\pi+3C-3A+3B}{4}\right) = 0$$

$$\Rightarrow\ 2\cos\left(\frac{3\pi}{2}-\frac{3C}{2}\right)\cdot 2\cos\left(\frac{3\pi}{2}-\frac{3B}{2}\right)\cdot\cos\left(\frac{3\pi}{2}-\frac{3A}{2}\right)=0$$

$$\Rightarrow\quad -4\sin\left(\frac{3A}{2}\right)\sin\left(\frac{3B}{2}\right)\sin\left(\frac{3C}{2}\right)=0$$

$$\Rightarrow\quad \sin\left(\frac{3A}{2}\right)\sin\left(\frac{3B}{2}\right)\sin\left(\frac{3C}{2}\right)=0$$

$$\therefore\quad \frac{3A}{2}=\pi\ \text{ or }\ \frac{3B}{2}=\pi\ \text{ or }\ \frac{3C}{2}=\pi$$

$$\therefore\quad A=\frac{2\pi}{3}\text{ or }B=\frac{2\pi}{3}\text{ or }C=\frac{2\pi}{3}$$

29. $\because |\tan A|<1$

$$\Rightarrow\quad -1<\tan A<1\ \text{ and }\ 0\le|A|<\frac{\pi}{2}$$

$$\Rightarrow\quad -\frac{\pi}{2}<A<\frac{\pi}{2}$$

$$\because\quad \sqrt{(1+\sin 2A)}=\sqrt{1+\frac{2\tan A}{1+\tan^2 A}}$$

$$=\frac{|1+\tan A|}{\sqrt{(1+\tan^2 A)}}=\frac{(1+\tan A)}{\sqrt{(1+\tan^2 A)}}$$

$$\text{and }\sqrt{(1-\sin 2A)}=\sqrt{1-\left(\frac{2\tan A}{1+\tan^2 A}\right)}$$

$$=\frac{|1-\tan A|}{\sqrt{(1+\tan^2 A)}}$$

$$=\frac{(1-\tan A)}{\sqrt{(1-\tan^2 A)}}$$

$$\therefore\quad \frac{\sqrt{(1+\sin 2A)}+\sqrt{(1-\sin 2A)}}{\sqrt{(1+\sin 2A)}-\sqrt{(1-\sin 2A)}}=\frac{2}{2\tan A}=\cot A$$

30. Let $f(\theta)=\cos^2(\cos\theta)+\sin^2(\sin\theta)$

$\because\quad -1\le\cos\theta\le 1$ and $-1\le\sin\theta\le 1$

$\therefore\quad \cos 1\le\cos(\cos\theta)\le 1$ and $-\sin 1\le\sin(\sin\theta)\le\sin 1$

$\therefore\quad \cos^2 1\le\cos^2(\cos\theta)\le 1$ and $0\le\sin^2(\sin\theta)\le\sin^2 1$

$\therefore$ Maximum value of $f(\theta)=1+\sin^2 1$

31. Let $f(x)=27^{\cos 2x}\,81^{\sin 2x}=3^{3\cos 2x+4\sin 2x}$

$$=3^{5\left(\frac{3}{5}\cos 2x+\frac{4}{5}\sin 2x\right)}$$

Let $\frac{3}{5}=\sin\phi$ and $\frac{4}{5}=\cos\phi$

Thus, $f(x)=3^{5(\sin\phi\cos 2x+\cos\phi\sin 2x)}=3^{5[(\sin(\phi+2x)]}$

For minimum value of given function, $\sin(\phi+2x)$ will be minimum.

i.e. $\sin(\phi+2x)=-1$

$$\therefore\quad f(x)=3^{5(-1)}=\frac{1}{243}$$

Alternate Method

Let $f(x)=27^{\cos 2x}\,81^{\sin 2x}=3^{4\sin 2x}=3^{3\cos 2x+4\sin 2x}$

For minimum value of given function, $3\cos 2x+4\sin 2x$ will be minimum.

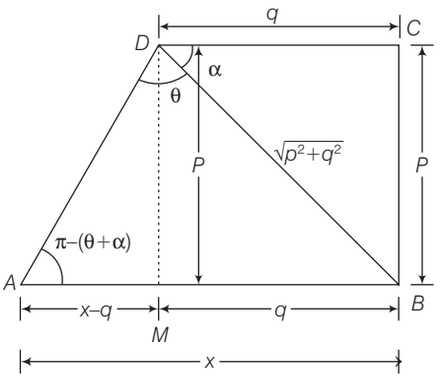
$$\therefore\quad -\sqrt{3^2+4^2}\le 3\cos 2x+4\sin 2x\le\sqrt{3^2+4^2}$$

$$\Rightarrow\quad -5\le 3\cos 2x+4\sin 2x\le 5$$

$\therefore$ Minimum of $3\cos 2x+4\sin 2x=-5$

So, $\min f(x)=3^{-5}=\frac{1}{243}$

32. Let $AB=x$ and $\angle BDC=\alpha$



In ΔDAM, $\tan(\pi-\theta-\alpha)=\frac{p}{x-q}$

$$\Rightarrow\quad \tan(\theta+\alpha)=\frac{p}{q-x}\Rightarrow q-x=p\cos(\theta+\alpha)$$

$$\Rightarrow\ x=q-p\cot(\theta+\alpha)=q-p\left(\frac{\cot\theta\cot\alpha-1}{\cot\alpha+\cot\theta}\right)$$

$$=\frac{q(\cot\alpha+\cot\theta)+p(\cot\theta\cot\alpha)+p}{\cot\alpha+\cot\theta}$$

$$=\frac{q\left(\frac{q}{p}+\frac{\cos\theta}{\sin\theta}\right)-p\left(\frac{q}{p}\frac{\cos\theta}{\sin\theta}\right)+p}{\frac{q}{p}+\frac{\cos\theta}{\sin\theta}}\qquad\left[\because\cot\alpha=\frac{q}{p}\right]$$

$$=\frac{\frac{q^2}{p}+\frac{q\cos\theta}{\sin\theta}-q\frac{\cos\theta}{\sin\theta}+p}{\frac{q\sin\theta+p\cos\theta}{p\sin\theta}}=\frac{(q^2+p^2)\sin\theta}{q\sin\theta+p\sin\theta}$$

33. Given $4n\alpha=\pi\ \Rightarrow\ 2n\alpha=\frac{\pi}{2}$

Now $\cot\alpha\cdot\cot(2n-1)\alpha=\cot\alpha\cot\left(\frac{\pi}{2}-\alpha\right)$

$$=\cot\alpha\cdot\tan\alpha=1$$

Similarly, $\cot 2\alpha\cot(2n-2)\alpha=1$,

$\cot 3\alpha\cdot\cot(2n-3)\alpha=1$, ..., $\cot(n-1)\alpha\cot(n+1)\alpha=1$

Thus $\cot\alpha\cot 2\alpha\cot 3\alpha\ldots\cot(2n-1)\alpha$

$=\{\cot\alpha\cot(2n-1)\alpha\}\{\cot 2\alpha\cot(2n-2)\alpha\}$

$\ldots\{\cot(n-1)\alpha\cot(n+1)\alpha\}\cdot\cot n\alpha$

$=1\cdot 1\cdot 1\ldots 1\cdot 1\qquad\left[\because\cot n\alpha=\cot\frac{\pi}{4}=1\right]$

$=1$

34. We have $(\sin A+\sin B+\sin C)(\sin A+\sin B-\sin C)$

$$=3\sin A\sin B$$

$$\Rightarrow\quad (\sin A+\sin B)^2-\sin^2 C=3\sin A\sin B$$

$$\Rightarrow\quad \sin^2 A+\sin^2 B-\sin^2 C=\sin A\sin B$$

$\Rightarrow \quad \sin^2 A + \sin(B+C)\sin(B-C) = \sin A \sin B$

$\Rightarrow \quad \sin A\,[\sin(B+C) + \sin(B-C)] = \sin A \sin B$

$[\because A + B + C = \pi]$

$\Rightarrow \quad \sin A(2 \sin B \cos C) = \sin A \sin B$

$\therefore \quad \cos C = \frac{1}{2} \Rightarrow C = 60°$

35. From the third relation we get

$\cos\theta \cos\phi + \sin\theta \sin\phi = \sin\beta \sin\gamma$

$\Rightarrow \sin^2\theta \sin^2\phi = (\cos\theta \cos\phi - \sin\beta \sin\gamma)^2$

$\Rightarrow \left(1 - \frac{\sin^2\beta}{\sin^2\alpha}\right)\left(1 - \frac{\sin^2\gamma}{\sin^2\alpha}\right) = \left(\frac{\sin\beta \sin\gamma}{\sin^2\alpha} - \sin\beta \sin\gamma\right)^2$

[from the first and second relations]

$\Rightarrow (\sin^2\alpha - \sin^2\beta)(\sin^2\alpha - \sin^2\gamma)$

$= \sin^2\beta \sin^2\gamma (1 - \sin^2\alpha)^2$

$\Rightarrow \sin^4\alpha(1 - \sin^2\beta \sin^2\gamma)$

$- \sin^2\alpha(\sin^2\beta + \sin^2\gamma - 2\sin^2\beta \sin^2\gamma) = 0$

$\therefore \quad \sin^2\alpha = \frac{\sin^2\beta - \sin^2\gamma - 2\sin^2\beta \sin^2\gamma}{1 - \sin^2\beta \sin^2\gamma} \qquad [\because \sin\alpha \neq 0]$

and $\cos^2\alpha = \frac{1 - \sin^2\beta - \sin^2\gamma + \sin^2\beta \sin^2\gamma}{1 - \sin^2\beta \sin^2\gamma}$

$\Rightarrow \quad \tan^2\alpha = \frac{\sin^2\beta - \sin^2\beta - \sin^2\gamma + \sin^2\gamma - \sin^2\beta \sin^2\gamma}{\cos^2\beta - \sin^2\gamma(1 - \sin^2\beta)}$

$= \frac{\sin^2\beta \cos^2\gamma + \cos^2\beta \sin^2\gamma}{\cos^2\beta \cos^2\gamma}$

$= \tan^2\beta + \tan^2\gamma$

$\Rightarrow \tan^2\alpha - \tan^2\beta - \tan^2\gamma = 0$

36. $\tan\beta = \frac{n \tan\alpha}{1 + \tan^2\alpha - n \tan^2\alpha}$

$= \frac{n \tan\alpha}{1 + (1-n)\tan^2\alpha}$

$\Rightarrow \tan(\alpha - \beta) = \frac{\tan\alpha - \frac{n \tan\alpha}{1 + (1-n)\tan^2\alpha}}{1 + \frac{\tan\alpha \cdot n \tan\alpha}{1 + (1-n)\tan^2\alpha}}$

$= \frac{\tan\alpha + (1-n)\tan^3\alpha - n \tan\alpha}{1 + (1-n)\tan^2\alpha + n \tan^2\alpha}$

$= \frac{(1-n)\tan\alpha(1 + \tan^2\alpha)}{1 + \tan^2\alpha} = (1-n)\tan\alpha$

37. Let $\frac{(\cos\theta)}{a} = \frac{(\sin\theta)}{b} = k$, so that $\cos\theta = ak$ and $\sin\theta = bk$. Then

$a \cos 2\theta + b \sin 2\theta = a(1 - 2\sin^2\theta) + 2b \sin\theta \cos\theta$

$= a - 2ab^2k^2 + 2b \cdot bk \cdot ak$

$= a - 2ab^2k^2 + 2ab^2k^2 = a$

38. Let $y = \cos x \cos(x+2) - \cos^2(x+1)$

$= \frac{1}{2}[\cos(2x+2) + \cos 2] - \cos^2(x+1)$

$= \frac{1}{2}[2\cos^2(x+1) - 1 + \cos 2] - \cos^2(x+1)$

$= -\frac{1}{2}(1 - \cos 2) = -\frac{1}{2}(2\sin^2 1) = -\sin^2 1$

This shows that $y = -\sin^2 1$ is a straight line which is parallel to X-axis and clearly passes through the point $\left(\frac{\pi}{2}, -\sin^2 1\right)$.

39. $f(\theta) = |\sin\theta| + |\cos\theta|, \forall\, \theta \in R$ Clearly, $f(\theta) > 0$.

Also, $f^2(\theta) = \sin^2\theta + \cos^2\theta + |2 \sin\theta \cdot \cos\theta|$

$= 1 + |\sin 2\theta|$

$0 \leq |\sin 2\theta| \leq 1$

$\Rightarrow \quad 1 \leq f^2(\theta) \leq 2 \Rightarrow 1 \leq f(\theta) \leq \sqrt{2}$

40. $A = \cos(\cos x) + \sin(\cos x)$

$= \sqrt{2}\left\{\cos(\cos x)\cos\frac{\pi}{4} + \sin(\cos x)\sin\frac{\pi}{4}\right\}$

$= \sqrt{2}\left\{\cos\left(\cos x - \frac{\pi}{4}\right)\right\}$

$\because \quad -1 \leq \cos\left(\cos x - \frac{\pi}{4}\right) \leq 1$

$\therefore \quad -\sqrt{2} \leq A \leq \sqrt{2}$

41. We have, $\frac{U_n}{V_n} = \tan n\theta$

and $\frac{V_n - V_{n-1}}{U_{n-1}} = \frac{\cos n\theta \sec^n\theta - \cos(n-1)\theta \sec^{n-1}\theta}{\sin(n-1)\theta \sec^{n-1}\theta}$

$= \frac{\cos n\theta \sec\theta - \cos(n-1)\theta}{\sin(n-1)\theta}$

$= \frac{\cos n\theta - \cos(n-1)\theta \cos\theta}{\cos\theta \sin(n-1)\theta}$

$\cos(n-1)\theta \cos\theta - \sin(n-1)\theta \sin\theta$

$= \frac{-\cos(n-1)\theta \cos\theta}{\cos\theta \sin(n-1)\theta}$

$= -\tan\theta$

So, that $\frac{V_n - V_{n-1}}{U_{n-1}} + \frac{1}{n}\frac{U_n}{V_n} = -\tan\theta + \frac{\tan n\theta}{n} \neq 0$

42. If $a, b > 0$

Using A.M. $\geq$ G.M., we get

$\frac{1}{a} + \frac{1}{b} \geq \frac{2}{\sqrt{ab}}$

$\Rightarrow \quad f(x) \geq \frac{2}{\sqrt{\cos\left(\frac{\pi}{6} - x\right)\cos\left(\frac{\pi}{6} + x\right)}}$

$= \frac{2}{\sqrt{\cos^2\frac{\pi}{6} - \sin^2 x}} = \frac{2}{\sqrt{\frac{3}{4} - \frac{1 - \cos 2x}{2}}}$

$= \frac{2}{\sqrt{\frac{1}{4} + \frac{\cos 2x}{2}}}$

Now for $0 \le x \le \frac{\pi}{3}$, $\frac{-1}{2} \le \cos 2x \le 1$

$$\Rightarrow \quad 0 \le \sqrt{\frac{1}{4} + \frac{\cos 2x}{2}} \le \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad f(x) \ge \frac{4}{\sqrt{3}}$$

Since 'f' is continuous range of 'f' is $\left[\frac{4}{\sqrt{3}}, \infty\right)$.

43. $\because 0 \le \sin^2\theta \le 1$ and $0 \le \cos^2\theta \le 1$

$$\Rightarrow \quad 0 \le \sin^8\theta \le \sin^2\theta \text{ and } 0 \le \cos^{14}\theta \le \cos^2\theta$$

$$\therefore \quad 0 < \sin^8\theta + \cos^{14}\theta \le \sin^2\theta + \cos^2\theta$$

Hence, $0 < A \le 1$

44. $\sin\left(\frac{3\pi}{2} - \alpha\right) = -\cos\alpha, \sin\left(\frac{\pi}{2} + \alpha\right) = \cos\alpha$

$$\sin(3\pi + \alpha) = -\sin\alpha$$

$$\sin(5\pi - \alpha) = -\sin\alpha$$

$$\therefore \quad 3\left\{\sin^4\left(\frac{3\pi}{2} - \alpha\right) + \sin^4(3\pi + \alpha)\right\} - 2\left[\sin^6\left(\frac{\pi}{2} + \alpha\right) + \sin^6(5\pi - \alpha)\right]$$

$$= 3\{\cos^4\alpha + \sin^4\alpha\} - 2\{\cos^6\alpha + \sin^6\alpha\}$$

$$= 3\{1 - 2\sin^2\alpha\cos^2\alpha\} - 2\{1 - 3\sin^2\alpha\cos^2\alpha\} = 1$$

45. $\sin\left(x + \frac{\pi}{6}\right) + \cos\left(x + \frac{\pi}{6}\right)$

$$= \sqrt{2}\left[\sin\left(x + \frac{\pi}{6} + \frac{\pi}{4}\right)\right]$$

$$= \sqrt{2}\sin\left(x + \frac{5\pi}{12}\right) \le \sqrt{2}$$

Equality holds when $x + \frac{5\pi}{12} = \frac{\pi}{2}$ ie, $x = \frac{\pi}{12}$

Therefore, maximum value of given expression is attained at

$$x = \frac{\pi}{12}$$

46. $\cot^2 x = \cot(x - y) \cdot \cot(x - z)$

$$\Rightarrow \quad \cot^2 x = \left(\frac{\cot x \cot y + 1}{\cot y - \cot x}\right)\left(\frac{\cot x \cdot \cot z + 1}{\cot z - \cot x}\right)$$

$$\Rightarrow \cot^2 x \cdot \cot y \cdot \cot z - \cot^3 x \cdot \cot y - \cot^3 x \cot z + \cot^4 x$$

$$= \cot^2 x \cdot \cot y \cdot \cot z + \cot x \cdot \cot y + \cot x \cdot \cot z + 1$$

$$\Rightarrow \cot x \cot y(1 + \cot^2 x) + \cot x \cot z(1 + \cot^2 x) + 1 - \cot^4 x = 0$$

$$\Rightarrow \quad \cot x(\cot y + \cot z)(1 + \cot^2 x) + (1 - \cot^2 x)(1 + \cot^2 x) = 0$$

$$\Rightarrow \quad \cot x(\cot y + \cot z) + (1 - \cot^2 x) = 0$$

$$\Rightarrow \quad \frac{\cot^2 x - 1}{2\cot x} = \frac{1}{2}(\cot y + \cot z)$$

$$\Rightarrow \quad \frac{1}{2}(\cot y + \cot z) = \cot 2x$$

47. Given that, $\alpha + \beta + \gamma = \pi$

Taking $\alpha = -\frac{\pi}{2}; \beta = -\frac{\pi}{2}$ and $\gamma = 2\pi$

$$\therefore \quad \sin\alpha + \sin\beta + \sin\gamma = -1 - 1 + 0 = -2$$

but $\sin\alpha + \sin\beta + \sin\gamma \ge -3$ for any α, β, γ

Hence, minimum value of $\sin\alpha + \sin\beta + \sin\gamma$ is negative.

48. $\cos x - \sin\alpha \cot\beta \sin x = \cos\alpha$

$$\Rightarrow \quad \sin\beta\cos x - \sin\alpha\cos\beta\sin x = \cos\alpha\sin\beta$$

$$\Rightarrow \sin\beta\left(1 - \tan^2\frac{x}{2}\right) - \sin\alpha\cos\beta \cdot 2\tan\frac{x}{2} = \cos\alpha\sin\beta\left(1 + \tan^2\frac{x}{2}\right)$$

$$\Rightarrow \tan^2\frac{x}{2}(-\sin\beta - \cos\alpha\sin\beta) - \sin\alpha\cos\beta \cdot 2\tan\frac{x}{2} + \sin\beta(1 - \cos\alpha) = 0$$

$$\Rightarrow \quad \tan\frac{x}{2} = \frac{-2\sin\alpha\cos\beta \pm \sqrt{4[\sin^2\alpha\cos^2\beta + \sin^2\beta(1 + \cos\alpha)(1 - \cos\alpha)]}}{2\sin\beta(1 + \cos\alpha)}$$

$$= \frac{-\sin\alpha\cos\beta \pm \sqrt{\sin^2\alpha(\sin^2\beta + \cos^2\beta)}}{\sin\beta(1 + \cos\alpha)}$$

$$= \frac{-\sin\alpha\cos\beta \pm \sin\alpha}{\sin\beta(1 + \cos\alpha)} = \frac{\sin\alpha(1 - \cos\beta \pm 1)}{\sin\beta(1 + \cos\alpha)}$$

$$= \tan\frac{\beta}{2}\tan\frac{\alpha}{2} \text{ or } -\tan\frac{\alpha}{2}\cot\frac{\beta}{2}$$

49. $\because \cos^4\theta\sec^4\alpha, \frac{1}{2}$ and $\sin^4\theta\,\mathrm{cosec}^2\alpha$ are in AP

$$1 = \cos^4\theta\sec^2\alpha + \sin^4\theta\,\mathrm{cosec}^2\alpha$$

$$\Rightarrow \quad 1 = \frac{\cos^4\theta}{\cos^2\alpha} + \frac{\sin^4\theta}{\sin^2\alpha}$$

$$\Rightarrow \quad (\sin^2\theta + \cos^2\theta)^2 = \frac{\cos^4\theta}{\cos^2\alpha} + \frac{\sin^4\theta}{\sin^2\alpha}$$

$$\Rightarrow \quad \cos^4\theta\left(\frac{1}{\cos^2\alpha} - 1\right) + \sin^4\theta\left(\frac{1}{\sin^2\alpha} - 1\right) - 2\sin^2\theta\cos^2\theta = 0$$

$$\Rightarrow \quad \sin^4\alpha\cos^4\theta + \sin^4\theta\cos^4\alpha - 2\sin^2\theta\cos^2\theta\sin^2\alpha\cos^2\alpha = 0$$

$$\Rightarrow \quad (\sin^2\alpha\cos^2\theta - \cos^2\alpha\sin^2\theta)^2 = 0$$

$$\Rightarrow \quad \tan^2\theta = \tan^2\alpha$$

$$\therefore \quad \theta = n\pi \pm \alpha, n \in I$$

Now, $\cos^8\theta\sec^6\alpha = \cos^8\alpha\sec^6\alpha = \cos^2\alpha$

and $\sin^8\theta\,\mathrm{cosec}^6\alpha = \sin^8\alpha \cdot \mathrm{cosec}^6\alpha = \sin^2\alpha$

Hence, $\cos^8\theta\sec^6\alpha, \frac{1}{2}, \sin^8\theta\,\mathrm{cosec}^6\alpha$

ie, $\cos^2\alpha, \frac{1}{2}, \sin^2\alpha$ are in AP.

50. Given, $(\cot\alpha_1)\cdot(\cot\alpha_2)\ldots(\cot\alpha_n)=1$

$$\Rightarrow \prod_{i=1}^{n}\cos\alpha_1=\prod_{i=1}^{n}\sin\alpha_1$$

$$\Rightarrow \prod_{i=1}^{n}\cos^2\alpha_1=\prod_{i=1}^{n}\sin\alpha_1\cos\alpha_1=\prod_{i=1}^{n}\frac{\sin 2\alpha_1}{2}\le\frac{1}{2^n}$$

$$\Rightarrow \prod_{i=1}^{n}\cos\alpha_1\le\frac{1}{2^{\frac{n}{2}}}$$

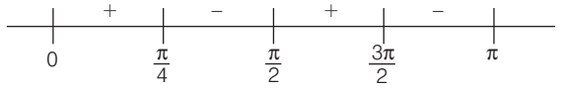
Hence, maximum value of $\prod_{i=1}^{n}\cos\alpha_1$ is $\frac{1}{2^{\frac{n}{2}}}$.

51. $\sin x\cdot\cos^3 x>\cos x\cdot\sin^3 x$

$$\Rightarrow \sin x\cdot\cos x(\cos^2 x-\sin^2 x)>0$$

$$\Rightarrow \sin x\cdot\cos x\cdot\cos 2x>0$$

$$\Rightarrow \cos x\cdot\cos 2x>0$$



$$\therefore x\in\left(0,\frac{\pi}{4}\right)\cup\left(\frac{\pi}{2},\frac{3\pi}{4}\right)$$

52. $u^2=a^2+b^2$

$$+2\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}\times\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$$

$$=a^2+b^2+2\sqrt{\sin^2\theta\cos^2\theta(a^4+b^4)+a^2b^2(\sin^4\theta+\cos^4\theta)}$$

$$=a^2+b^2+2\sqrt{a^2b^2(1-2\sin^2\theta\cos^2\theta)+(a^4+b^4)\sin^2\theta\cos^2\theta}$$

$$=(a^2+b^2)+2\sqrt{a^2b^2+(a^2-b^2)^2\sin^2\theta\cos^2\theta}$$

$$=(a^2+b^2)+2\sqrt{a^2b^2+\frac{(a^2-b^2)^2}{4}\sin^2 2\theta}$$

Max. $u^2=(a^2+b^2)+2\sqrt{a^2b^2+\frac{(a^2-b^2)^2}{4}}$

Min. $u^2=(a^2+b^2)+2ab$

$\Rightarrow$ Difference $2\sqrt{a^2b^2+\frac{(a^2-b^2)^2}{4}}-2ab$

$$=\sqrt{4a^2b^2+a^4+b^4-2a^2b^2}-2ab$$

$$=\sqrt{(a^2+b^2)^2}-2ab$$

$$=a^2+b^2-2ab=(a-b)^2$$

53. $\because \left(\tan\frac{\theta}{2}\right)(1+\sec\theta)=\tan\left(\frac{\theta}{2}\right)\left(\frac{1+\cos\theta}{\cos\theta}\right)$

$$=\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\cdot\frac{2\cos^2\frac{\theta}{2}}{\cos\theta}=\frac{2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}}{\cos\theta}$$

$$=\frac{\sin\theta}{\cos\theta}=\tan\theta \qquad \ldots(i)$$

$\therefore$ By repeated use of Eq. (i), we have

$$f_n(\theta)=\tan\theta\,(1+\sec 2\theta)\,(1+\sec 4\theta)\ldots(1+\sec 2^n\theta)$$

$$=\tan 2\theta\,(1+\sec 4\theta)\ldots(1+\sec 2^n\theta)$$

$$=\tan 4\theta\,(1+\sec 8\theta)\ldots(1+\sec 2^n\theta)$$

$$=\ldots=\tan 2^n\theta$$

Now, $f_2\left(\frac{\pi}{16}\right)=\tan\left(2^2\frac{\pi}{16}\right)=\tan\frac{\pi}{4}=1$

$$f_3\left(\frac{\pi}{32}\right)=\tan\left(2^3\frac{\pi}{32}\right)=\tan\frac{\pi}{4}=1$$

$$f_4\left(\frac{\pi}{64}\right)=\tan\left(2^4\frac{\pi}{64}\right)=\tan\frac{\pi}{4}=1$$

and $f_5\left(\frac{\pi}{128}\right)=\tan\frac{\pi}{4}=1$

54. $\cos x=0\Rightarrow x=(2n+1)\frac{\pi}{2}$

$$\Rightarrow \cos\left((2n+1)\frac{\pi}{2}+z\right)=\frac{1}{2}$$

$$\Rightarrow \sin z=\frac{1}{2}\text{ or }\sin z=-\frac{1}{2}$$

$$\Rightarrow z=\frac{\pi}{6},\frac{5\pi}{6},\frac{7\pi}{6},\frac{11\pi}{6}$$

55. $f_n(\theta)=\cos\theta-\cos2\theta+\cos2\theta-\cos3\theta+\ldots+\cos(n)\theta-\cos(n+1)\theta$

$$f_n(\theta)=\cos\theta-\cos(n+1)\theta$$

Now, check options.

56. $P=\sin25^\circ\sin35^\circ\sin60^\circ\sin85^\circ$

$$=\sin25^\circ\sin(60^\circ-25^\circ)\ \sin60^\circ\sin(60^\circ+25^\circ)$$

$$=\sin60^\circ\sin25^\circ\sin(60^\circ-25^\circ)\sin(60^\circ+25^\circ)$$

$$\therefore P=\sin60^\circ\times\frac{1}{4}\sin75^\circ \qquad \ldots(i)$$

$$Q=\sin20^\circ\sin40^\circ\sin75^\circ\sin80^\circ$$

$$=\sin20^\circ\sin(60^\circ-20^\circ)\sin75^\circ\sin(60^\circ+20^\circ)$$

$$=\sin75^\circ\sin20^\circ\sin(60^\circ-20^\circ)\sin(60^\circ+20^\circ)$$

$$\therefore Q=\sin75^\circ\times\frac{1}{4}\times\sin60^\circ \qquad \ldots(ii)$$

Hence, $P=Q$

57. $x=\frac{1}{1-\cos^2\theta}=\frac{1}{\sin^2\theta},y=\frac{1}{\cos^2\theta},$

$$z=\frac{1}{1-\sin^2\theta\cos^2\theta}$$

$$\Rightarrow \frac{1}{x}+\frac{1}{y}=1$$

$$\Rightarrow xy=x+y\ \Rightarrow\ \frac{1}{z}=1-\frac{1}{xy}$$

$$\Rightarrow xyz=xy+z=x+y+z$$

58. Given $P(x)=\cot^2 x\left(\frac{1+\tan x+\tan^2 x}{1+\cot x+\cot^2 x}\right)+\left(\frac{\cos x-\cos3x+\sin3x-\sin x}{2(\sin2x+\cos2x)}\right)^2$

$$=\frac{\cot^2+\cot x+1}{1+\cot x\ \cot^2 x}+\left(\frac{2\sin x\,(\sin2x+\cos2x)}{2(\sin2x+\cos2x)}\right)^2$$

$$= 1 + \sin^2 x$$

$\therefore \quad P(18°) = P(72°) = (1 + \sin^2 18°) + (1 + \sin^2 72°)$

$$= 1 + 1 + (\sin^2 18° + \cos^2 18°) = 3$$

59. $E = \dfrac{3\sin(\alpha + \beta) - 4\cos(\alpha + \beta)}{\sqrt{3}\sin\alpha}$

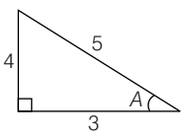
$$= \frac{3(\sin\alpha\cos\beta + \cos\alpha\sin\beta) - 4(\cos\alpha\cos\beta + \sin\alpha\sin\beta)}{\sqrt{3}\sin\alpha}$$

$$= \frac{5}{\sqrt{3}} \text{ for } 0 < \beta < \frac{\pi}{2}$$

and $E = \dfrac{\sqrt{3}(7 + 24\cot\alpha)}{15}$ for $\dfrac{\pi}{2} < \beta < \pi$.

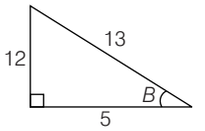
60. $\cot A = \dfrac{3}{4}$

$\Rightarrow \qquad \cot C = \dfrac{-3}{4}$



$\Rightarrow$ C is obtuse angle.

$\therefore \qquad \sin C = \dfrac{4}{5}, \cos C = -\dfrac{3}{5}$



$$\tan B = \frac{-12}{5}$$

$\Rightarrow \qquad \tan D = \dfrac{12}{5}$

$\Rightarrow D$ is an acute angle

$\therefore \qquad \sin D = \dfrac{12}{13} \quad \cos D = \dfrac{5}{13}$

Hence, $\sin(C + D) = \sin C \cdot \cos D + \cos C \cdot \sin D$

$$= \left(\frac{4}{5}\right)\left(\frac{5}{13}\right) + \left(\frac{-3}{5}\right)\left(\frac{12}{13}\right)$$

$$= \frac{20 - 36}{65} = \frac{-16}{65}$$

Also, $\sin(A + B) = \sin(2\pi - (C + D))$

$$= -\sin(C + D) = \frac{16}{65}.$$

61. $2\left(\cos^2 x + \dfrac{1}{2}\cos x\right) = a$

$$2\left(\cos x + \frac{1}{4}\right)^2 = a + \frac{1}{8}$$

$\therefore \qquad \left(\cos x + \dfrac{1}{4}\right)^2 = \dfrac{a}{2} + \dfrac{1}{16}$

$$\left(\cos x + \frac{1}{4}\right)^2 \in \left[0, \frac{25}{16}\right]$$

$\therefore \qquad \dfrac{8a + 1}{16} \in \left[0, \dfrac{25}{16}\right]$

$\Rightarrow \qquad 8a + 1 \in [0, 25]$

$\Rightarrow \qquad a \in \left[\dfrac{-1}{8}, 3\right]$.

62. $A = \sin 44° + \cos 44°$

$$= \cos 46° + \sin 46° = C$$

$B = \sin 45° + \cos 45° = \sqrt{2}[\sin 90°]$

$$A = \sqrt{2}\left[\frac{1}{\sqrt{2}}\sin 44° + \frac{1}{\sqrt{2}}\cos 44°\right]$$

$$= \sqrt{2}\,[\sin 44° \cdot \cos 45° + \cos 44° \cdot \sin 45°]$$

$$= \sqrt{2}\sin 89°$$

$\Rightarrow \quad B > A$

63. $\tan(2\alpha + \beta) = x$

$\tan(\alpha + 2\beta) = y$

$\Rightarrow \quad \tan(3(\alpha + \beta)).\ \tan(\alpha - \beta)$

$$= \tan[(2\alpha + \beta) + (\alpha + 2\beta)]\,.$$

$$\tan[2(\alpha + \beta) - (\alpha + 2\beta)]$$

$$= \frac{\tan(2\alpha + \beta) + \tan(\alpha + 2\beta)}{1 - \tan(2\alpha + \beta).\tan(\alpha + 2\beta)}\,.$$

$$\frac{\tan(2\alpha + \beta) - \tan(\alpha + 2\beta)}{1 + \tan(2\alpha + \beta).\tan(\alpha + 2\beta)}$$

$$= \frac{x + y}{1 - xy} \cdot \frac{x - y}{1 + xy} = \frac{x^2 - y^2}{1 - x^2y^2}$$

64. We have $x = \dfrac{1 - \sin\phi}{\cos\phi}, y = \dfrac{1 + \cos\phi}{\sin\phi}$

Multiplying, we get $xy = \dfrac{(1 - \sin\phi)(1 + \cos\phi)}{\cos\phi\sin\phi}$

$$1 - \sin\phi + \cos\phi - \sin\phi\cos\phi$$

$\Rightarrow \qquad xy + 1 = \dfrac{+ \sin\phi\cos\phi}{\cos\phi\sin\phi}$

$$= \frac{1 - \sin\phi + \cos\phi}{\cos\phi\sin\phi}$$

and $\quad x - y = \dfrac{(1 - \sin\phi)\sin\phi - \cos\phi(1 + \cos\phi)}{\cos\phi\sin\phi}$

$$= \frac{\sin\phi - \sin^2\phi - \cos\phi - \cos^2\phi}{\cos\phi\sin\phi}$$

$$= \frac{\sin\phi - \cos\phi - 1}{\cos\phi\sin\phi} = -(xy + 1)$$

Thus, $xy + x - y + 1 = 0$.

$\Rightarrow \qquad x = \dfrac{y - 1}{y + 1}$ and $y = \dfrac{1 + x}{1 - x}$.

65. The given relation can be written as

$$\tan\left(\frac{x}{2}\right) = \frac{1 - \sin^2 x}{\sin x} = \frac{\cos^2 x}{\sin x}$$

$$\Rightarrow \quad 2\sin^2\left(\frac{x}{2}\right)=\left[\cos^2\left(\frac{x}{2}\right)-\sin^2\left(\frac{x}{2}\right)\right]^2$$

$$\Rightarrow \quad 2\tan^2\left(\frac{x}{2}\right)=\left[1-\tan^2\left(\frac{x}{2}\right)\right]^2\Big/\left[1+\tan^2\left(\frac{x}{2}\right)\right]$$

$$\Rightarrow \quad 2y(1+y)=(1-y)^2 \qquad \left[\text{where } y=\tan^2\frac{x}{2}\right]$$

$$\Rightarrow \quad y^2+4y-1=0$$

$$\Rightarrow \quad y=\frac{-4\pm\sqrt{16+4}}{2}=-2\pm\sqrt{5}$$

Since $y>0$, we get

$$y=\sqrt{5}-2=\frac{(\sqrt{5}-2)^2}{\sqrt{5}+2}\cdot\frac{2+\sqrt{5}}{2+\sqrt{5}}$$

$$=(9-4\sqrt{5})(2+\sqrt{5})$$

66. $$y=\frac{\sqrt{(\cos 2A-\sin 2A)^2}+1}{\sqrt{(\cos 2A+\sin 2A)^2}-1}$$

$$\Rightarrow \quad y=\frac{\pm(\cos 2A-\sin 2A)+1}{\pm(\cos 2A+\sin 2A)-1}$$

which gives us four values of y, say y_1, y_2, y_3 and y_4 We have,

$$y_1=\frac{\cos 2A-\sin 2A+1}{\cos 2A+\sin 2A-1}=\frac{(1+\cos 2A)-\sin 2A}{(\cos 2A-1)+\sin 2A}$$

$$=\frac{2\cos^2 A-2\sin A\cos A}{-2\sin^2 A+2\sin A\cos A}$$

$$=\frac{\cos A(\cos A-\sin A)}{\sin A(\cos A-\sin A)}=\cot A$$

$$y_2=\frac{-(\cos 2A-\sin 2A)+1}{-(\cos 2A+\sin 2A)-1}$$

$$=\frac{(1-\cos 2A)+\sin 2A}{-(1+\cos 2A)-\sin 2A}$$

$$=\frac{2\sin^2 A+2\sin A\cos A}{-2\cos^2 A-2\sin A\cos A}=-\tan A$$

$$y_3=\frac{(\cos 2A-\sin 2A)+1}{-(\cos 2A+\sin 2A)-1}$$

$$=\frac{(1+\cos 2A)-\sin 2A}{-(1+\cos 2A)-\sin 2A}$$

$$=\frac{2\cos^2 A-2\sin A\cos A}{-2\cos^2 A-2\sin A\cos A}$$

$$=-\frac{\cos A-\sin A}{\cos A+\sin A}$$

$$=-\frac{1-\tan A}{1+\tan A}=-\tan\left(\frac{\pi}{4}-A\right)=-\cot\left(\frac{\pi}{4}+A\right)$$

$$y_4=\frac{-(\cos 2A-\sin 2A)+1}{(\cos 2A+\sin 2A)-1}$$

$$=\frac{(1-\cos 2A)+\sin 2A}{-(1-\cos 2A)+\sin 2A}$$

$$=\frac{2\sin^2 A-2\sin A\cos A}{-2\sin A+2\sin A\cos A}$$

67. $\because\ 3\sin\beta=\sin(2\alpha+\beta)$

$$\Rightarrow \quad 2\sin\beta=\sin(2\alpha+\beta)-\sin\beta$$

$$=2\cos(\alpha+\beta)\sin\alpha$$

$$\therefore \quad \sin\beta=\cos(\alpha+\beta)\sin\alpha \qquad \ldots(i)$$

$\therefore$ **Alternate** (b) is correct

Also, $\sin\beta=\sin(\alpha+\beta)-\alpha)$

$$=\sin(\alpha+\beta)\cos\alpha-\cos(\alpha+\beta)\sin\alpha \qquad \ldots(ii)$$

From Eqs. (i) and (ii)

$$\sin\beta=\sin(\alpha+\beta)\cos\alpha-\sin\beta$$

$$\therefore \quad 2\sin\beta=\sin(\alpha+\beta)\cos\alpha$$

($\therefore$ Alternate (c) is correct)

Alternate (a)

$$\text{LHS}=(\cot\alpha+\cos(\alpha+\beta))(\cot\beta-3\cot(2\alpha+\beta))$$

$$=\left(\frac{\sin(2\alpha+\beta)}{\sin\alpha\cdot\sin(\alpha+\beta)}\right)\left(\frac{\cos\beta}{\sin\beta}-\frac{3\cos(2\alpha+\beta)}{\sin(2\alpha+\beta)}\right)$$

$$=\left(\frac{3\sin\beta}{\sin\alpha\cdot\sin(\alpha+\beta)}\right)\left(\frac{\cos\beta}{\sin\beta}-\frac{3\cos(2\alpha+\beta)}{3\sin\beta}\right)$$

($\because 3\sin\beta=\sin(2\alpha+\beta)$)

$$=\left(\frac{3\sin\beta}{\sin\alpha\cdot\sin(\alpha+\beta)}\right)\left(\frac{\cos\beta-\cos(2\alpha+\beta)}{\sin\beta}\right)$$

$$=\left(\frac{3\sin\beta}{\sin\alpha\cdot\sin(\alpha+\beta)}\right)\left(\frac{2\sin(\alpha+\beta)\sin\alpha}{\sin\beta}\right)$$

$$=6$$

Alternate (d)

$\because \quad \tan(\alpha+\beta)=2\tan\alpha$

$$\Rightarrow \quad \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}=\frac{2\sin\alpha}{\cos\alpha}$$

$$\Rightarrow \quad \sin(\alpha+\beta)\cos\alpha=2\cos(\alpha+\beta)\sin\alpha$$

$$\Rightarrow \quad \sin(\alpha+\beta)\cos\alpha-\cos(\alpha+\beta)\sin\alpha=\cos(\alpha+\beta)\sin\alpha$$

$$\sin\beta=\cos(\alpha+\beta)\sin\alpha$$

[Alternate (b)]

68. $P_n(u)$ be a polynomial in u of degree n.

$$\therefore \quad \sin 2nx=2\sin nx\cos nx$$

$$=\sin x\,P_{2n-1}(\cos x) \text{ or } \cos x\,P_{2n-1}(\sin x)$$

69. $$\tan\theta=\frac{\sin\alpha-\cos\alpha}{\sin\alpha+\cos\alpha}$$

$$=\frac{\tan\alpha-1}{\tan\alpha+1}$$

$$\tan\theta=\tan\left(\alpha-\frac{\pi}{4}\right)$$

$$\Rightarrow \quad \theta=n\pi+\alpha-\frac{\pi}{4},\ n\in I$$

or $$2\theta=2n\pi+2\alpha-\frac{\alpha}{2}$$

$$\sin 2\theta=\sin\left(2\alpha-\frac{\pi}{2}\right)=-\cos 2\alpha$$

and $$\cos 2\theta=\cos\left(2\alpha-\frac{\pi}{2}\right)=\sin 2\alpha$$

and $\sin\alpha - \cos\alpha = \sqrt{2}\sin\left(\alpha - \frac{\pi}{4}\right)$

$= \sqrt{2}\sin\{\theta - n\pi\} = \pm\sqrt{2}\sin\theta$

and $\sin\alpha + \cos\alpha = \sqrt{2}\sin\left(\alpha + \frac{\pi}{4}\right)$

$= \sqrt{2}\sin\left\{\frac{\pi}{2} + \theta - n\pi\right\}$

$= \sqrt{2}\cos(\theta - n\pi) = \pm\sqrt{2}\cos\theta$

70. $\cos 5\theta = \cos(4\theta + \theta) = \cos 4\theta\cos\theta - \sin 4\theta\sin\theta$

$= (2\cos^2 2\theta - 1)\cos\theta - 2\sin 2\theta\cos 2\theta\sin\theta$

$= [2(2\cos^2\theta - 1)^2 - 1]\cos\theta - 2\cdot 2\cos\theta\sin^2\theta(2\cos^2\theta - 1)$

$= [2(4\cos^4\theta - 4\cos^2\theta + 1) - 1]\cos\theta - 4\cos\theta(2\cos^2\theta - 1)(1 - \cos^2\theta)$

$= \cos\theta(8\cos^4\theta - 8\cos^2\theta + 1) - 4\cos\theta(3\cos^2\theta - 2\cos^4\theta - 1)$

$= \cos\theta(16\cos^4\theta - 20\cos^2\theta + 5)$

71. $x = \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha} + \sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha}$

$\Rightarrow \quad x^2 = a^2 + b^2 + \sqrt{2(a^2\cos^2\alpha + b^2\sin^2\alpha)(a^2\sin^2\alpha + b^2\cos^2\alpha)}$

$a^2 + b^2 + 2k,$

where $k = \sqrt{[(a^2 + b^2) - (a^2\sin^2\alpha + b^2\cos^2\alpha)] \times (a^2\sin^2\alpha + b^2\cos^2\alpha)}$

$\therefore \quad x = a^2 + b^2 + 2\sqrt{(a^2 + b^2)p - p^2}$

where $p = a^2\sin^2\alpha + b^2\cos^2\alpha$

$= \frac{a^2}{2}(1 - \cos 2\alpha) + \frac{b^2}{2}(1 + \cos 2\alpha)$

72. $\left(\frac{\cos A + \cos B}{\sin A - \sin B}\right)^n + \left(\frac{\sin A + \sin B}{\cos A - \cos B}\right)^n$

$= \cos^n\left(\frac{A - B}{2}\right) + \cot^n\left(\frac{B - A}{2}\right)$

If n even, $2\cot^n\left(\frac{A - B}{2}\right)$, if n odd, O

73. $P(k) = \left(1 + \cos\frac{\pi}{4k}\right)\left(1 + \cos\left(\frac{\pi}{2} - \frac{\pi}{4k}\right)\right)\left(1 + \cos\left(\frac{\pi}{2} + \frac{\pi}{4k}\right)\right)\left(1 + \cos\left(\pi - \frac{\pi}{4k}\right)\right)$

$= \left(1 + \cos\frac{\pi}{4k}\right)\left(1 + \sin\frac{\pi}{4k}\right)\left(1 - \sin\frac{\pi}{4k}\right)\left(1 - \cos\frac{\pi}{4k}\right)$

$= \left(1 - \cos^2\frac{\pi}{4k}\right)\left(1 - \sin^2\frac{\pi}{4k}\right)$

$= \frac{4\sin^2\frac{\pi}{4k}\cdot\cos^2\frac{\pi}{4k}}{4}$

$P(k) = \frac{1}{4}\sin^2\left(\frac{\pi}{2k}\right)$

$\Rightarrow \quad P(3) = \frac{1}{4}\cdot\frac{1}{4} = \frac{1}{16}$

$\Rightarrow \quad P(4) = \frac{\pi}{4}\sin^2\frac{\pi}{2k} = \frac{1}{4}\sin^2\frac{\pi}{8}$

$= \frac{1}{8}\left(1 - \cos\frac{\pi}{4}\right) = \frac{2 - \sqrt{2}}{16}$

$\Rightarrow \quad P(5) = \frac{1}{4}\sin^2\frac{\pi}{10} = \frac{1}{8}\left(2\sin^2\frac{\pi}{10}\right) = \frac{1}{8}(1 - \cos 36^\circ)$

$= \frac{1}{8}\left(1 - \frac{\sqrt{5} + 1}{4}\right) = \frac{3 - \sqrt{5}}{32}$

$\Rightarrow \quad P(6) = \frac{1}{4}\sin^2\frac{\pi}{12} = \frac{1}{8}\left(2\sin^2\frac{\pi}{12}\right) = \frac{1}{8}\left(1 - \cos\frac{\pi}{6}\right)$

$= \frac{1}{8}\left(1 - \frac{\sqrt{3}}{2}\right) = \frac{2 - \sqrt{3}}{16}$

74. $x^2 + y^2 = a^2\sin^4\theta\cos^4\theta$

$xy = a^2\sin^5\theta\cos^5\theta$

$\therefore \quad \frac{(x^2 + y^2)^p}{(xy)^q} = \frac{a^{2p}(\sin\theta\cos\theta)^{4p}}{a^{2q}(\sin\theta\cos\theta)^{5q}}$

which is independent of θ if $4p = 5q$

i.e. $p = 5, q = 4.$

75. LHS

$= \tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\tan 8\alpha + 16\cot 16\alpha$

$= \cot\alpha - (\cot\alpha - \tan\alpha) + 2\tan 2\alpha + 4\tan 4\alpha + 8\tan 8\alpha + 16\cot 16\alpha$

$= \cot\alpha - 2(\cot 2\alpha - \tan 2\alpha) + 4\tan 4\alpha + 8\tan 8\alpha + 16\cot 16\alpha$

$(\because \cot\alpha - \tan\alpha = 2\cot 2\alpha)$

$= \cot\alpha - 4(\cot 4\alpha - \tan 4\alpha) + 8\tan 8\alpha + 16\cot 16\alpha$

$(\because \cot 2\alpha - \tan 2\alpha = 2\cot 4\alpha)$

$= \cot\alpha - 8(\cot 8\alpha - \tan 8\alpha) + 16\cot 16\alpha$

$= \cot\alpha - 16\cot 16\alpha + 16\cot 16\alpha$

$(\because \cot 8\alpha - \tan 8\alpha = 2\cot 16\alpha)$

$= \cot\alpha = \text{RHS}$

76. Let $x = \cot A, y = \cot B, z = \cot C$

$\therefore \quad \cot A\cot B + \cot B\cot C + \cot C\cot A = 1$

$\because \quad A + B + C = 180^\circ$

$\therefore \quad \Sigma\frac{x}{(1 + x^2)} = \Sigma\frac{\cot A}{(1 + \cot^2 A)}$

$= \frac{1}{2}\Sigma\frac{2\tan A}{(1 + \tan^2 A)} = \frac{1}{2}\Sigma\sin 2A$

$= \frac{1}{2}(\sin 2A + \sin 2B + \sin 2C)$

$= \frac{1}{2}(4\sin A\sin B\sin C) = 2\sin A\sin B\sin C$

$= \frac{2}{\sqrt{(1 + \cot^2 A)(1 + \cot^2 B)(1 + \cot^2 C)}}$

$= \frac{2}{\sqrt{(1 + x^2)(1 + y^2)(1 + z^2)}} = \frac{2}{\sqrt{\Pi(1 + x^2)}}$

and $\sin 2A + \sin 2B - \sin 2C$

$= 2\sin(A + B)\cos(A - B) - 2\sin C\cos C$

$= 2\sin C\{\cos(A - B) - \cos C\}$

$= 2\sin C\{\cos(A - C) + \cos(A + B)\}$

$= 2\sin C(2\cos A\cos B)$

$= 4\cos A\cos B\sin C$

77. $\because a\cos x + b\sin x = c$

$$\Rightarrow \quad a\left\{\frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}\right\} + b\left\{\frac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}\right\} = c$$

$$\Rightarrow \quad (a + c)\tan^2\left(\frac{x}{2}\right) - 2b\tan\left(\frac{x}{2}\right) + (c - a) = 0$$

$$\therefore \quad \tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\beta}{2}\right) = \frac{2b}{(a + c)}$$

and $$\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right) = \frac{c - a}{a + c}$$

Now, $$\tan\left(\frac{\alpha+\beta}{2}\right) = \frac{\tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\beta}{2}\right)}{1 - \tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)}$$

$$= \frac{\frac{2b}{a+c}}{1 - \left(\frac{c-a}{a+c}\right)} = \frac{b}{a} = \text{independent of } c$$

Also, $$-\sqrt{(a^2 + b^2)} \le a\cos x + b\sin x \le \sqrt{(a^2 + b^2)}$$

$$\therefore \quad -\sqrt{(a^2 + b^2)} \le c \le \sqrt{(a^2 + b^2)}$$

78. $\because A + B + C = 180°$

$$\Rightarrow \quad A = 180° - (B + C)$$

$$\therefore \quad \tan A = \tan(180° - (B + C))$$

$$= -\tan(B + C) = -\left\{\frac{\tan B + \tan C}{1 - \tan B\tan C}\right\}$$

$$= \left(\frac{\tan B + \tan C}{\tan B\tan C - 1}\right)$$

Now, $\because A$ is obtuse

$$\therefore \quad \tan A < 0,$$

then $$\tan B + \tan C > 0$$

$$\therefore \quad \tan B\tan C - 1 < 0$$

$$\Rightarrow \quad \tan B\tan C < 1$$

79. Let $S = \sin\left(\frac{2\pi}{7}\right) + \sin\left(\frac{4\pi}{7}\right) + \sin\left(\frac{8\pi}{7}\right)$

and $$C = \cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{8\pi}{7}\right)$$

$$\therefore \quad C + iS = \alpha + \alpha^2 + \alpha^4 \qquad \text{...(i)}$$

Where $\alpha = \cos\left(\frac{2\pi}{7}\right) + i\sin\left(\frac{2\pi}{7}\right)$ is complex 7th root of unity.

Then, $$C - iS = \bar{\alpha} + \bar{\alpha}^2 + \bar{\alpha}^4$$

$$= \alpha^6 + \alpha^5 + \alpha^3 \qquad \text{...(ii)}$$

Adding Eqs. (i) and (ii), then

$$2C = \alpha + \alpha^2 + \alpha^4 + \alpha^6 + \alpha^5 + \alpha^3 = -1$$

($\because$ sum of 7, 7th roots of unity is zero)

$$\therefore \quad C = -\frac{1}{2}$$

Also, multiplying Eqs. (i) and (ii), then $C^2 + S^2 = 2$

($\because \alpha^7 = 1$ and sum of 7, 7th roots of unity)

$$\Rightarrow \quad S^2 = 2 - \left(\frac{1}{2}\right)^2 = \frac{7}{4}$$

$$\therefore \quad S = \frac{\sqrt{7}}{2}$$

80. We observe that $y = 81^{\sin^2 x} + 81^{\cos^2 x} - 30 = 0$

$$\Rightarrow \quad 81^{\sin^2 x} + 81^{1 - \sin^2 x} - 30 = 0$$

$$\Rightarrow \quad 81^{2\sin^2 x} - 30.81^{\sin^2 x} + 81 = 0$$

$$\Rightarrow \quad (81^{\sin^2 x} - 3)(81^{\sin^2 x} - 27) = 0$$

$$\Rightarrow \quad \sin^2 x = \frac{1}{4} \text{ or } \frac{3}{4}$$

$$\Rightarrow \quad \sin x = \pm\frac{1}{2} \text{ or } \sin x = \pm\frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad x = \pm\frac{\pi}{6}, \pm\frac{5\pi}{6}, \text{ or } x = \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}$$

$\Rightarrow$ The graph $y = 81^{\sin^2 x} + 81^{\cos^2 x} - 30$

Intersects the X-axis at eight points in $(-\pi \le x \le \pi)$.

$\Rightarrow$ Statement-1 is true.

81. Statement-2 is correct, using it we have $\cos 3x = \sin 2x$

$$\Rightarrow \quad 4\cos^3 x - 3\cos x = 2\sin x\cos x$$

Similarly $4\cos^3 y - 3\cos y = 2\sin y\cos y$

So, $$4(1 - \sin^2 x) - 3 = 2\sin x$$

$$\Rightarrow \quad 4\sin^2 x + 2\sin x - 1 = 0$$

and $$4\sin^2 y + 2\sin y - 1 = 0$$

Hence, $\sin x = \sin 18°$ and $\sin y = \sin(-54°) = -\sin 54°$ are the roots of a quadratic equation with integer coefficients.

82. The minimum value of the sum can be -3 provided $\sin\alpha = \sin\beta = \sin\gamma = -1$

$$\Rightarrow \quad \alpha = (4l - 1)\frac{\pi}{2},\ \beta = (4m - 1)\frac{\pi}{2},\ \gamma = (4n - 1)\frac{\pi}{2}$$

Now $$\alpha + \beta + \gamma = \pi \ \Rightarrow\ [4(l + m + n) - 3]\frac{\pi}{2} = \pi$$

$\Rightarrow 4(l + m + n) = 5$ which is not possible as l, m, n are integers.

1. minimum value can not be -3.

But for $\alpha = \frac{3\pi}{2}, \beta = \frac{3\pi}{2}, \gamma = -2\pi, \alpha + \beta + \gamma = \pi$

and $$\sin\alpha + \sin\beta + \sin\gamma = 2$$

So, $\sin\alpha + \sin\beta + \sin\gamma$ can have negative values and thus the minimum value of the sum is negative proving that statement-1 is correct. But the statement-2 is false as $\alpha + \beta + \gamma = \pi$ for $\alpha = \beta = \frac{3\pi}{2}, \gamma = -2\pi$ which are not the angles of a triangle.

83. We have $2\sin\left(\frac{\theta}{2}\right)$

$$= \sqrt{\cos\left(\frac{\theta}{2}\right)+\sin\left(\frac{\theta}{2}\right)^2} + \sqrt{\left(\cos\left(\frac{\theta}{2}\right)-\sin\left(\frac{\theta}{2}\right)\right)^2}$$

$$= \left|\cos\left(\frac{\theta}{2}\right)+\sin\left(\frac{\theta}{2}\right)\right| + \left|\cos\left(\frac{\theta}{2}\right)-\sin\left(\frac{\theta}{2}\right)\right|$$

$$\Rightarrow \cos\left(\frac{\theta}{2}\right)+\sin\left(\frac{\theta}{2}\right)>0 \text{ and } \cos\left(\frac{\theta}{2}\right)-\sin\left(\frac{\theta}{2}\right)<0$$

$$\Rightarrow \sin\left(\frac{\theta}{2}+\frac{\pi}{4}\right)>0 \text{ and } \cos\left(\frac{\theta}{2}+\frac{\pi}{4}\right)<0$$

$$\Rightarrow 2n\pi+\frac{\pi}{2}<\frac{\theta}{2}+\frac{\pi}{4}>2n\pi+\pi$$

$$\Rightarrow 2n\pi+\frac{\pi}{4}<\frac{\theta}{2}<2n\pi+\frac{3\pi}{4}$$

So statement-1 is true but does not follow from statement-2 which is also true.

84. $2\cos\theta+\sin\theta=1$

$$\Rightarrow \frac{2\left(1-\tan^2\left(\frac{\theta}{2}\right)\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}+\frac{2\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}=1$$

$$\Rightarrow 3\tan^2\left(\frac{\theta}{2}\right)-2\tan\left(\frac{\theta}{2}\right)-1=0$$

$$\Rightarrow \tan\left(\frac{\theta}{2}\right)=-\frac{1}{3} \text{ as } \theta\neq\frac{\pi}{2}$$

Now $7\cos\theta+6\sin\theta$

$$=\frac{7\left(1-\tan^2\left(\frac{\theta}{2}\right)\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}+\frac{6\times2\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}$$

$$=\frac{7-7\tan^2\left(\frac{\theta}{2}\right)+12\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}=\frac{7-7\times\frac{1}{9}+12\left(-\frac{1}{3}\right)}{1+\frac{1}{9}}=2$$

Showing that statement-1 is true.

In statement-2

$$\cos 2\theta-\sin\theta=\frac{1}{2}$$

$$\Rightarrow 2(1-2\sin^2\theta)-\sin\theta=1$$

$$\Rightarrow 4\sin^2\theta+2\sin\theta-1=0$$

$$\Rightarrow \sin\theta=\frac{-2\pm\sqrt{4+16}}{8}=\frac{-1\pm\sqrt{5}}{4}$$

$$\Rightarrow \sin\theta=\frac{\sqrt{5}-1}{4} \Rightarrow \theta=18^\circ$$

$$\Rightarrow \cos 6\theta=\cos 108^\circ=\cos(90^\circ+18^\circ)$$
$$=-\sin 18^\circ$$

$$\Rightarrow \sin\theta+\cos 6\theta=0$$

So statement-2 is also true but does not lead to statement-1.

85. $\because A+B=\frac{\pi}{3}$

$$\therefore \tan(A+B)=\tan\left(\frac{\pi}{3}\right)=\sqrt{3}$$

$$\Rightarrow \frac{\tan A+\tan B}{1-\tan A\tan B}=\sqrt{3}$$

$$\Rightarrow \tan A\tan B=1-\frac{1}{\sqrt{3}}(\tan A+\tan B)$$

$\therefore \tan A\tan B$ will be maximum if $\tan A+\tan B$ is minimum.

But the minimum value of $\tan A+\tan B$ is obtained when $\tan A=\tan B$

$$\Rightarrow A=B=\frac{\pi}{6}$$

Hence, the maximum value of $\tan A\tan B$

$$=\tan\frac{\pi}{6}\cdot\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}}=\frac{1}{3}$$

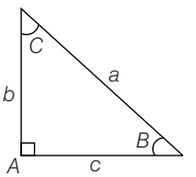
86. Let $3^{a^2}=A$ and $3^{b^2+c^2}=B$

$$\Rightarrow A^2-2A.B+B^2=0$$

$$\Rightarrow A=B$$

$$\Rightarrow a^2=b^2+c^2$$

87.



From figure, it is clear that $a=b\sec C=c\,\text{cosec}\,C$

$\Rightarrow$ Equilateral triangle $\Rightarrow$ Area $=\frac{\sqrt{3}}{4}a^2$

Sol. (Q. Nos. 88 to 90)

$7\theta=(2n+1)\pi$, $n=0$ to 6

$$4\theta=(2n+1)\pi-3\theta$$

$$\cos 4\theta=-\cos 3\theta$$

$$\Rightarrow 2\cos^2 2\theta-1=-(4\cos^3\theta-3\cos\theta)$$

$$\Rightarrow 2(2x^2-1)-1=-(4x^3-3x), \text{ where } x=\cos\theta$$

$$\Rightarrow 8x^4+4x^3-8x^2-3x+1=0$$

$$(x+1)(8x^3-4x^2-4x-1)=0$$

88. $P_{mn}=m\log_{\cos x}(\sin x)+n\log_{\cos x}(\cot x)$

$$\geq n(\log_{\cos x}(\sin x)+\log_{\cos x}(\cot x))\ \forall\ m\geq n$$
$$=n(\log_{\cos x}(\sin x\cdot\cot x))$$
$$=n\log_{\cos x}\cos x=n$$

Thus, $P_{mn}\geq n\ \forall\ m\geq n$

89. Clearly, $P_{49}\left(\frac{\pi}{4}\right)=4\log_{\frac{1}{\sqrt{2}}}\left(\frac{1}{\sqrt{2}}\right)+9\log_{\frac{1}{\sqrt{2}}}(1)=4$

Similarly $P_{94}=\left(\frac{\pi}{4}\right)=9$

Mean proportional of $P_{49}\left(\frac{\pi}{4}\right)$ and $P_{94}\left(\frac{\pi}{4}\right)$ is $\sqrt{9\times4}=6$

90. $P_{34}(x) = P_{22}(x)$

$\Rightarrow 3\log_{\cos x}(\sin x) + 4\log_{\cos x}(\cot x)$

$= 2(\log_{\cos x}(\sin x) + \log_{\cos x}(\cot x))$

$\Rightarrow 3(\log_{\cos x}(\sin x) + \log_{\cos x}(\cot x)) + \log_{\cos x}(\cot x) = 2$

$\Rightarrow 3 + \log_{\cos x}(\cot x) = 2$

$\Rightarrow \log_{\cos x}(\cot x) = -1$

$\Rightarrow \cot x = (\cos x)^{-1} \Rightarrow \dfrac{\cos x}{\sin x} = \dfrac{1}{\cos x}$

$\Rightarrow \cos^2 x = \sin x$

$1 - \sin^2 x = \sin x \Rightarrow \sin^2 x + \sin x - 1 = 0$

$\Rightarrow \sin x = \dfrac{-1 \pm \sqrt{5}}{2}$

$\Rightarrow \sin x = \dfrac{\sqrt{5} - 1}{2}$ $(\because \sin x \nless -1)$

Thus, $p + q = 7$

Sol. (Q. Nos. 91 to 93)

$7\theta = (2n + 1)\pi$, $n = 0$ to 6

$4\theta = (2n + 1)\pi - 3\theta$

$\cos 4\theta = -\cos 3\theta$

$\Rightarrow 2\cos^2 2\theta - 1 = -(4\cos^3\theta - 3\cos\theta)$

$\Rightarrow 2(2x^2 - 1) - 1 = -(4x^3 - 3x)$, where $x = \cos\theta$

$\Rightarrow 8x^4 + 4x^3 - 8x^2 - 3x + 1 = 0$

$(x + 1)(8x^3 - 4x^2 - 4x - 1) = 0$

91. The roots are $\cos\dfrac{\pi}{7}, \cos\dfrac{2\pi}{7} \ldots, \cos\dfrac{13\pi}{7}$

where $\cos\dfrac{\pi}{7} = \cos\dfrac{13\pi}{7}, \cos\dfrac{3\pi}{7} = \cos\dfrac{11\pi}{7}, \cos\dfrac{5\pi}{7} = \cos\dfrac{9\pi}{7}$

$\therefore$ The roots of $8x^3 - 4x^2 - 4x + 1 = 0$ are $\cos\dfrac{\pi}{7}, \cos\dfrac{3\pi}{7}, \cos\dfrac{5\pi}{7}$.

92. $\sec\dfrac{\pi}{7}, \sec\dfrac{3\pi}{7}, \sec\dfrac{5\pi}{7}$ are roots of $\dfrac{8}{x^3} - \dfrac{4}{x^2} - \dfrac{4}{x} + 1 = 0$

$\Rightarrow x^3 - 4x^2 - 4x + 8 = 0$

$\therefore \sec\dfrac{\pi}{7} + \sec\dfrac{3\pi}{7} + \sec\dfrac{5\pi}{7} = 4$

93. $\sec^2\dfrac{\pi}{7}, \sec^2\dfrac{3\pi}{7}, \sec^2\dfrac{5\pi}{7}$ are roots of $f(\sqrt{x}) = 0$

$\Rightarrow (\sqrt{x})^3 - 4(\sqrt{x})^2 - 4\sqrt{x} + 8 = 0$

$\Rightarrow x^3 - 24x^2 + 80x - 64 = 0$

$\therefore \sec^2\dfrac{\pi}{7} + \sec^2\dfrac{3\pi}{7} + \sec^2\dfrac{5\pi}{7} = 24$

Sol. (Q. Nos. 94 to 96)

Let $S = 1 + 2\sin x + 3\sin^2 x + 4\sin^3 x + \ldots$

$\Rightarrow \sin x.\, S = \sin x + 2\sin^2 x + 3\sin^3 x + \ldots$

$\therefore (1 - \sin x)\, S = 1 \ \sin x + \sin^2 x + \ldots$

$(1 - \sin x)\, S = \dfrac{1}{1 - \sin x}$

$\therefore S = \dfrac{1}{(1 - \sin x)^2}$

Given $S = 4 \Rightarrow \dfrac{1}{(1 - \sin x)^2} = 4$

$\Rightarrow \sin x = \dfrac{1}{2}$ or $\dfrac{3}{2}$ (rejected)

Number of solutions in $\left[\dfrac{-3\pi}{2}, 4\pi\right]$ is $k = 5$.

94. $k = 5$

95. $\left|\dfrac{\cos 2x - 1}{\sin 2x}\right| = \left|\dfrac{2\sin^2 x}{2\cos x \sin x}\right| = |\tan x| = \dfrac{1}{\sqrt{3}}$

96. Sum of interior angles $= (k - 2)\pi = 3\pi$

97. Now, $(2\sin x - \cos x)(1 + \cos x) = \sin^2 x$

$\Rightarrow (1 + \cos x)[2\sin x - \cos x - 1 + \cos x] = 0$

$\Rightarrow (1 + \cos x)(2\sin x - 1) = 0$

$\Rightarrow \cos x = -1$ or $\sin x = \dfrac{1}{2}$

So, $\sin\alpha = \dfrac{1}{2}$ $\left[\text{as } 0 \le \alpha \le \dfrac{\pi}{2}\right]$

$\therefore \cos\alpha = \dfrac{\sqrt{3}}{2}$

$3\cos^2 x - 10\cos x + 3 = 0$

$\Rightarrow \cos x = \dfrac{1}{3}, \cos x \ne 3$

$\Rightarrow \cos\beta = \dfrac{1}{3}, \sin\beta = \dfrac{2\sqrt{2}}{3}$

and $1 - \sin 2x = \cos x - \sin x$

$\Rightarrow \sin^2 x + \cos^2 x - 2\sin x \cos x = \cos x - \sin x$

$\Rightarrow (\cos x - \sin x)(\cos x - \sin x - 1) = 0$

$\Rightarrow \sin x = \cos x = \dfrac{1}{\sqrt{2}}$

or $\cos x - \sin x = 1$

$\Rightarrow \cos x = 1, \sin x = 0$

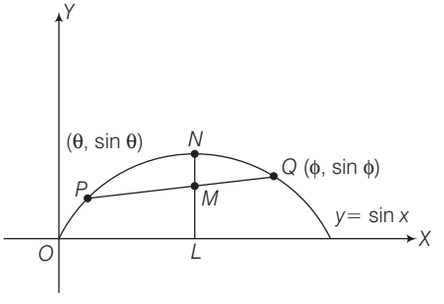
$\Rightarrow \cos\gamma = 1, \sin\gamma = 0$

$\therefore \cos\alpha + \cos\beta + \cos\gamma = \dfrac{\sqrt{3}}{2} + \dfrac{1}{3} + 1 = \dfrac{3\sqrt{3} + 8}{6}$

98. $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$= \dfrac{1}{2} \times \dfrac{1}{3} - \dfrac{\sqrt{3}}{2} \times \dfrac{2\sqrt{2}}{3} = \dfrac{1 - 2\sqrt{6}}{6}$

99. (A) If M is mid point of PQ, then $M = \left(\dfrac{\theta + \phi}{2}, \dfrac{\sin\theta + \sin\phi}{2}\right)$



Also, $N \equiv \left(\frac{\theta+\phi}{2}, \sin\left(\frac{\theta+\phi}{2}\right)\right)$

It is clear from the figure.

$ML \le NL$

$\Rightarrow \quad \frac{\sin\theta+\sin\phi}{2} \le \sin\left(\frac{\theta+\phi}{2}\right)$

$\Rightarrow \quad \sin\theta+\sin\phi \le 2\sin\left(\frac{\theta+\phi}{2}\right)$

$= 2\sin\left(\frac{\pi}{4}\right) = \sqrt{2}$

$\therefore \quad \sin\theta+\sin\phi \le \sqrt{2}$

and $(\sin\theta+\sin\phi)\sin\frac{\pi}{4} \le 1$ (p, q, r, s, t)

(B) $\because a^2+b^2 = (\sin\theta-\sin\phi)^2+(\cos\theta+\cos\phi)^2$

$= 2+2\cos(\theta+\phi)$

$= 4\cos^2\left(\frac{\theta+\phi}{2}\right) \le 4$ (s, t)

(C) $\because 3\sin\theta+5\cos\theta = 5$

$\Rightarrow \quad 3\sin\theta = 5(1-\cos\theta)$

Squaring both sides, then

$9\sin^2\theta = 25(1-\cos\theta)^2$

$\Rightarrow \; 9(1-\cos\theta)(1+\cos\theta) = 25(1-\cos\theta)^2$

$\Rightarrow \quad 9(1+\cos\theta) = 25(1-\cos\theta)\,(1-\cos\theta \ne 0)$

$\therefore \quad 34\cos\theta = 16$

$\cos\theta = \frac{8}{17}$, then $\sin\theta = \frac{15}{17}$

$\therefore \quad 5\sin\theta-3\cos\theta = \frac{75}{17}-\frac{24}{17} = 3$ (r)

Hence, $5\sin\theta-3\cos\theta = 3$

100. (A) Let $y = \frac{7+6\tan\theta-\tan^2\theta}{(1+\tan^2\theta)}$

$= 7\cos^2\theta+6\sin\theta\cos\theta-\sin^2\theta$

$= 7\left(\frac{1+\cos\theta}{2}\right)+3\sin 2\theta-\left(\frac{1-\cos 2\theta}{2}\right)$

$= 3\sin 2\theta+4\cos 2\theta+3$

$-\sqrt{(3^2+4^2)}+3 \le 3\sin 2\theta+4\cos 2\theta+3$

$\le \sqrt{(3^2+4^2)}+3$

$\therefore \quad -2 \le y \le 8 \;\Rightarrow\; \lambda = 8, \mu = -2$

$\Rightarrow \quad \lambda+\mu = 6, \lambda-\mu = 10$ (R, S)

(B) Let $y = 5\cos\theta+3\cos\left(\theta+\frac{\pi}{3}\right)+3$

$= 5\cos\theta+3\left(\frac{1}{2}\cos\theta-\frac{\sqrt{3}}{2}\sin\theta\right)+3$

$= \frac{13}{2}\cos\theta-\frac{3\sqrt{3}}{2}\sin\theta+3$

$\therefore \quad 3-\sqrt{\left(\frac{13}{2}\right)^2+\left(\frac{-3\sqrt{3}}{2}\right)^2} \le \frac{13}{2}\cos\theta-\frac{3\sqrt{3}}{2}\sin\theta+3$

$\le 3+\sqrt{\left(\frac{13}{2}\right)^2+\left(\frac{-3\sqrt{3}}{2}\right)^2}$

$\Rightarrow \quad 3-7 \le y \le 3+7$

$\Rightarrow \quad -4 \le y \le 10$

$\therefore \quad \lambda = 10, \mu = -4$

$\Rightarrow \quad \lambda+\mu = 6, \lambda-\mu = 14$ (R, T)

(C) Let $y = 1+\sin\left(\frac{\pi}{4}+\theta\right)+2\cos\left(\frac{\pi}{4}-\theta\right)$

$= 1+\cos\left(\frac{\pi}{2}-\left(\frac{\pi}{2}+\theta\right)\right)+2\cos\left(\frac{\pi}{4}-\theta\right)$

$= 1+\cos\left(\frac{\pi}{4}-\theta\right)+2\cos\left(\frac{\pi}{4}-\theta\right)$

$= 1+3\cos\left(\frac{\pi}{4}-\theta\right)$

$\therefore \quad -1 \le \cos\left(\frac{\pi}{4}-\theta\right) \le 1$

$\Rightarrow \quad -3 \le 3\cos\left(\frac{\pi}{4}-\theta\right) \le 3$

$\Rightarrow \quad 1-3 \le 1+3\cos\left(\frac{\pi}{4}-\theta\right) \le 1+3$

$\therefore \quad -2 \le y \le 4$

$\Rightarrow \quad \lambda = 4, \mu = -2$

$\therefore \quad \lambda+\mu = 2, \lambda-\mu = 6$ (P, Q)

101. (A) $|\cot x| = \cot x+\frac{1}{\sin x}$

If $0 < x < \frac{\pi}{2} \Rightarrow \cot x > 0$

So $\cot x = \cot x+\frac{1}{\sin x} \Rightarrow \frac{1}{\sin x} = 0$, no solution

If $\frac{\pi}{2} < \cot x < \pi, -\cot x = \cot x+\frac{1}{\sin x}$

$\Rightarrow \quad \frac{2\cos x}{\sin x}+\frac{1}{\sin x} = 0$

$\Rightarrow \quad 1+2\cos x = 0$ and $\sin x \ne 0 \Rightarrow x = \frac{2\pi}{3}$.

(B) since $\sin\phi+\sin\theta = \frac{1}{2}$ ...(i)

and $\cos\theta+\cos\phi = 2$ ...(ii)

(ii) is true only if $\theta = \phi = 0$ or 2π but $\theta = \phi = 0$ or 2π do not satisfy (i)

Hence given system of equation has no solution.

(C) $\sin^2\alpha+\sin\left(\frac{\pi}{3}-\alpha\right)\cdot\sin\left(\frac{\pi}{3}+\alpha\right)$

$= \sin^2\alpha+\sin^2\frac{\pi}{3}-\sin^2\alpha = \frac{3}{4}$.

(D) $\tan\theta = 3\tan\phi$

$\tan(\theta-\phi) = \frac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi} = \frac{2\tan\phi}{1+3\tan\phi}$

$= \frac{2}{\cot\phi+3\tan\phi}$ · Max if $\tan\phi > 0$

$$\frac{\cos\phi + 3\tan\phi}{2} \geq \sqrt{3} \qquad \text{(Using AM} \geq \text{GM)}$$

$$\Rightarrow \quad (\cot\phi + 3\tan\phi)^2 \geq 12 \Rightarrow \left[\frac{2}{\tan(\theta-\phi)}\right]^2 \geq 12$$

$$\Rightarrow \quad \tan^2(\theta-\phi) \leq \frac{1}{3}.$$

102. (A) $\sin\left(\frac{A}{2}\right) + \sin\left(\frac{B}{2}\right) + \sin\left(\frac{C}{2}\right)$

$$= 1 + \left(\sin\left(\frac{A}{2}\right) + \sin\left(\frac{B}{2}\right)\right) - \left(\sin\frac{\pi}{2} - \sin\frac{C}{2}\right)$$

$$= 1 + 2\sin\left(\frac{A+B}{4}\right)\cos\left(\frac{A-B}{4}\right) - 2\cos\left(\frac{\pi+C}{4}\right)\sin\left(\frac{\pi-C}{4}\right)$$

$$= 1 + 2\sin\left(\frac{\pi-C}{4}\right)\left\{\cos\left(\frac{A-B}{4}\right) - \cos\left(\frac{\pi+C}{4}\right)\right\}$$

$(\because A+B+C=\pi)$

$$= 1 + 2\sin\left(\frac{\pi-C}{4}\right)\left\{2\sin\left(\frac{\pi+C+A-B}{8}\right)\sin\left(\frac{\pi+C+B-A}{8}\right)\right\}$$

$$= 1 + 4\sin\left(\frac{\pi-C}{4}\right)\sin\left(\frac{\pi-B}{4}\right)\sin\left(\frac{\pi-A}{4}\right)$$

$$= 1 + 4\sin\left(\frac{\pi-A}{4}\right)\sin\left(\frac{\pi-B}{4}\right)\sin\left(\frac{\pi-C}{4}\right)$$

$$= 1 + 4\cos\left(\frac{\pi}{2} - \frac{\pi-A}{4}\right)\cos\left(\frac{\pi}{2} - \frac{\pi-B}{4}\right)\sin\left(\frac{\pi-C}{4}\right)$$

$$= 1 + 4\cos\left(\frac{\pi+A}{4}\right)\cos\left(\frac{\pi+B}{4}\right)\sin\left(\frac{\pi-C}{4}\right)$$

(B) $\sin\left(\frac{A}{2}\right) + \sin\left(\frac{B}{2}\right) - \sin\left(\frac{C}{2}\right)$

$$= -1 + \left(\sin\left(\frac{A}{2}\right) + \sin\left(\frac{B}{2}\right)\right) + \left(\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{C}{2}\right)\right)$$

$$= -1 + 2\sin\left(\frac{A+B}{4}\right)\cos\left(\frac{A-B}{4}\right) + 2\cos\left(\frac{\pi+C}{4}\right)\sin\left(\frac{\pi-C}{4}\right)$$

$$= -1 + 2\sin\left(\frac{\pi-C}{4}\right)\left\{\cos\left(\frac{A-B}{4}\right) + \cos\left(\frac{\pi+C}{4}\right)\right\}$$

$(\because A+B+C=\pi)$

$$= -1 + 2\sin\left(\frac{\pi-C}{4}\right)\left\{2\cos\left(\frac{\pi+C+A-B}{8}\right)\cos\left(\frac{\pi+C+B-A}{8}\right)\right\}$$

$$= -1 + 4\sin\left(\frac{\pi-C}{4}\right)\cos\left(\frac{\pi-B}{4}\right)\cos\left(\frac{\pi-A}{4}\right)$$

$$= -1 + 4\cos\left(\frac{\pi-A}{4}\right)\cos\left(\frac{\pi-B}{4}\right)\sin\left(\frac{\pi-C}{4}\right)$$

$$= -1 + 4\sin\left(\frac{\pi}{2} - \frac{\pi-A}{4}\right)\sin\left(\frac{\pi}{2} - \frac{\pi-B}{4}\right)\cos\left(\frac{\pi}{2} - \frac{\pi-C}{4}\right)$$

$$= -1 + 4\sin\left(\frac{\pi+A}{4}\right)\sin\left(\frac{\pi+B}{4}\right)\cos\left(\frac{\pi+C}{4}\right)$$

(C) $\cos\left(\frac{A}{2}\right) + \cos\left(\frac{B}{2}\right) - \cos\left(\frac{C}{2}\right)$

$$= \left(\cos\left(\frac{A}{2}\right) + \cos\left(\frac{B}{2}\right)\right) - \left(\cos\left(\frac{C}{2}\right) + \cos\left(\frac{\pi}{2}\right)\right)$$

$$= 2\cos\left(\frac{A+B}{4}\right)\cos\left(\frac{A-B}{4}\right) - \cos\left(\frac{\pi+C}{4}\right)\cos\left(\frac{\pi-C}{4}\right)$$

$$= 2\cos\left(\frac{\pi-C}{4}\right)\left\{\cos\left(\frac{A-B}{4}\right) - \cos\left(\frac{\pi+C}{4}\right)\right\}$$

$(\because A+B+C=\pi)$

$$= 2\cos\left(\frac{\pi-C}{4}\right)\left\{2\sin\left(\frac{\pi+C+A-B}{8}\right)\sin\left(\frac{\pi+C+B-A}{8}\right)\right\}$$

$$= 4\cos\left(\frac{\pi-C}{4}\right)\sin\left(\frac{\pi-B}{4}\right)\sin\left(\frac{\pi-A}{4}\right)$$

$$= 4\cos\left(\frac{\pi-C}{4}\right)\cos\left(\frac{\pi}{2} - \frac{\pi-B}{4}\right)\cos\left(\frac{\pi}{2} - \frac{\pi-A}{4}\right)$$

$$= 4\cos\left(\frac{\pi+A}{4}\right)\cos\left(\frac{\pi+B}{4}\right)\cos\left(\frac{\pi-C}{4}\right)$$

103. $\frac{1}{1+\tan^2\frac{A}{2}} + \frac{1}{1+\tan^2\frac{B}{2}} + \frac{1}{1+\tan^2\frac{C}{2}}$

$$= k\left[1 + \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right]$$

$$\Rightarrow \cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2}$$

$$= 2\left[1 + \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right] \qquad \text{[by using identity]}$$

104. $\frac{\sin\alpha}{\sin\beta} = \frac{\cos\gamma}{\cos\delta}$

$$\Rightarrow \quad \frac{\sin\alpha - \sin\beta}{\sin\beta} = \frac{\cos\gamma - \cos\delta}{\cos\delta} \qquad \text{(using dividendo)}$$

$$\Rightarrow \quad \frac{2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}{\sin\beta} = \frac{2\sin\left(\frac{\gamma+\delta}{2}\right)\sin\left(\frac{\delta-\gamma}{2}\right)}{\cos\delta}$$

105. Let $\frac{\pi}{20} = \theta \Rightarrow 10\theta = \frac{\pi}{2}$

$$\Rightarrow \quad 2\theta = 18^\circ \text{ or } \theta = 9^\circ$$

Now, $\tan\theta - \tan3\theta + \tan5\theta - \tan7\theta + \tan9\theta$

$\tan\theta - \tan3\theta + \tan5\theta - \cot3\theta + \cot\theta$

$(\tan\theta + \cot\theta) - (\tan3\theta + \cot3\theta) + \tan45^\circ$

[using $\tan5\theta = \tan45^\circ$]

$$E = \frac{2}{\sin2\theta} - \frac{2}{\sin6\theta} + 1$$

$$E = 2\left(\frac{1}{\sin 2\theta} - \frac{1}{\cos 4\theta}\right) + 1$$

$$E = 2\left(\frac{1}{\sin 18^\circ} - \frac{1}{\cos 36^\circ}\right) + 1$$

$$E = 2\left(\frac{4}{\sqrt{5}-1} - \frac{4}{\sqrt{5}+1}\right) + 1 = 5$$

Aliter $$E = 1 + 2\left(\frac{\sin 6\theta - \sin 2\theta}{\sin 2\theta \cdot \sin 6\theta}\right)$$

$$= 1 + 2\left(\frac{2\cos 4\theta \cdot \sin 2\theta}{\sin 2\theta \cdot \cos 4\theta}\right)$$

$$= 1 + 4 = 5$$

106. $$x = \frac{\cos 1^\circ + \cos 2^\circ + \ldots + \cos 44^\circ}{\sin 1^\circ + \sin 2^\circ + \ldots + \sin 44^\circ}$$

$$= \frac{\frac{\sin 22^\circ}{\sin(1/2)^\circ} \cdot \cos 22.5^\circ}{\frac{\sin 22^\circ}{\sin(1/2)^\circ} \cdot \sin 22.5^\circ} = \cot 22.5^\circ$$

[using the formula of sum of cos series]

$$S = \frac{\sin(n\theta/2)}{\sin(\theta/2)} \cos\frac{(n+1)\theta}{2},$$

for sine series, $$S = \frac{\sin(n\theta/2)}{\sin(\theta/2)} \sin\frac{(n+1)\theta}{2}$$

$$\cot\left(\frac{\pi}{8}\right) = \sqrt{2} + 1 = 2.414\ldots$$

$$\therefore \quad x = 2.414\ldots$$

Greatest integer = 2.

107. LHS $= \tan 15^\circ \cdot \tan(30^\circ - 5^\circ) \cdot \tan(30^\circ + 5^\circ)$

Let $t = \tan 30^\circ$ and $m = \tan 5^\circ$

$$\therefore \text{LHS} = \tan 15^\circ \cdot \frac{t-m}{1+tm} \cdot \frac{t+m}{1-tm} = \tan(3(5^\circ)) \cdot \frac{t^2 - m^2}{1 - t^2m^2}$$

$$= \frac{3m - m^3}{1 - 3m^2} \frac{1 - 3m^2}{3 - m^2}$$

$$= \frac{m(3-m^2)}{(1-3m^2)} \cdot \frac{(1-3m^2)}{3-m^2} = m = \tan 5^\circ$$

Hence, $\tan\theta = \tan 5^\circ$

$$\Rightarrow \quad \theta = 5^\circ.$$

108. We have, $$\frac{1}{\sin 10^\circ} + \frac{1}{\sin 50^\circ} - \frac{1}{\sin 70^\circ}$$

$$= \frac{1}{\cos 80^\circ} + \frac{1}{\cos 40^\circ} - \frac{1}{\cos 20^\circ}$$

$$= \frac{\cos 40^\circ \cos 20^\circ + \cos 80^\circ \cos 20^\circ - \cos 40^\circ \cos 80^\circ}{\cos 20^\circ \,.\, \cos 40^\circ \,.\, \cos 80^\circ}$$

$$= 8\left[\cos 20^\circ (\cos 40^\circ + \cos 80^\circ) - \cos 40^\circ \cos 80^\circ\right]$$

$$= 8[2\cos 20^\circ \cos 60^\circ \cos 20^\circ - \cos 40^\circ \cos 80^\circ]$$

$$= 4[2\cos^2 20^\circ - 2\cos 40^\circ \cos 80^\circ]$$

$$= 4[1 + \cos 40^\circ - (\cos 120^\circ + \cos 40^\circ)]$$

$$= 4 \cdot \frac{3}{2} = 6$$

109. We have, $\cos 5\alpha = \cos^5\alpha$

$$\cos 5\alpha = \cos(3\alpha + 2\alpha) = \cos 3\alpha \cos 2\alpha - \sin 3\alpha \sin 2\alpha$$

$$= (4\cos^3\alpha - 3\cos\alpha)(2\cos^2\alpha - 1) - (3\sin\alpha - 4\sin^3\alpha) 2\sin\alpha \cos\alpha$$

$$= (4\cos^3\alpha - 3\cos\alpha)(2\cos^2\alpha - 1) - (1 \cos^2\alpha)(3 - 4 + 4\cos^2\alpha) 2\cos\alpha$$

$$= (4\cos^3\alpha - 3\cos\alpha)(2\cos^2\alpha - 1) - (2 \cos^2\alpha - 2\cos^3\alpha)(4\cos^2\alpha - 1)$$

$$= 8 \cos^5\alpha - 4\cos^3\alpha - 6\cos^3\alpha + 3 \cos\alpha - [8 \cos^3\alpha - 2 \cos\alpha - 8\cos^5\alpha + 2\cos^3\alpha]$$

$$= 16 \cos^5\alpha - 20\cos^3\alpha + 5 \cos\alpha$$

$$\therefore \quad 16 \cos^5\alpha - 20 \cos^3\alpha + 5 \cos\alpha = \cos^5\alpha$$

$$15 \cos^5\alpha - 20 \cos^3\alpha + 5\cos\alpha = 0$$

$$5 \cos\alpha [3 \cos^4\alpha - 4 \cos^2\alpha + 1] = 0$$

Also $\cos\alpha = 0$

$$3 \cos^4\alpha - 3\cos^2\alpha - \cos^2\alpha + 1 = 0$$

$$3\cos^2\alpha(\cos^2\alpha - 1) - (\cos^2\alpha - 1) = 0$$

$$(3\cos^2\alpha - 1)(1 - \cos^2\alpha) = 0$$

$$\cos^2\alpha = 1$$

$$\cos^2\alpha = \frac{1}{3}$$

$$\Rightarrow \quad \sec^2\alpha = 3;\ \text{cosec}^2\alpha = \frac{3}{2};\ \cot^2\alpha = \frac{1}{2}$$

$$\therefore \quad (\sec^2\alpha + \text{cosec}^2\alpha + \cot^2\alpha) = 3 + \frac{3}{2} + \frac{1}{2} = 5$$

110. We have,

$$\tan^2\frac{\pi}{16} + \tan^2\frac{2\pi}{16} + \tan^2\frac{3\pi}{16} + \ldots + \tan^2\frac{7\pi}{16}$$

$$= \left(\tan^2\frac{\pi}{16} + \cot^2\frac{\pi}{16} + 2\right) + \left(\tan^2\frac{2\pi}{16} + \cot^2\frac{2\pi}{16} + 2\right) + \left(\tan^2\frac{3\pi}{16} + \cot^2\frac{3\pi}{16} + 2\right) + \tan^2\frac{4\pi}{16} - 6$$

$$\left[\text{If } A + B = \frac{\pi}{2}, \text{ then } \tan B = \tan\left(\frac{\pi}{2} - A\right) = \cot A\right.$$

$$\left.\text{So, } \tan^2\frac{7\pi}{16} = \tan^2\left(\frac{8\pi}{16} - \frac{\pi}{16}\right) = \cot^2\frac{\pi}{16} \text{ etc.}\right]$$

$$= \left(\tan\frac{\pi}{16} + \cot\frac{\pi}{16}\right)^2 + \left(\tan\frac{2\pi}{16} + \cot\frac{2\pi}{16}\right)^2 + \left(\tan\frac{3\pi}{16} + \cot\frac{3\pi}{16}\right)^2 - 5$$

$$= \frac{4}{\sin^2\frac{\pi}{8}} + \frac{4}{\sin^2\frac{3\pi}{8}} + \left(\frac{4}{\sin^2\frac{\pi}{4}} - 5\right)$$

$$= \frac{4\left(\sin^2\frac{3\pi}{8} + \sin^2\frac{\pi}{8}\right)}{\sin^2\frac{\pi}{8}\sin^2\frac{3\pi}{8}} + 3 = \frac{16}{\sin^2\frac{\pi}{4}} + 3 = 32 + 3 = 35$$

111. We have, $\dfrac{4+\sec 20^\circ}{\operatorname{cosec} 20^\circ}$

$$= \frac{\sin 20^\circ}{\cos 20^\circ}(4\cos 20^\circ + 1)$$
$$= \frac{2\sin 40^\circ + \sin 20^\circ}{\cos 20^\circ}$$
$$= \frac{\sin 40^\circ + (\sin 40^\circ + \sin 20^\circ)}{\cos 20^\circ}$$
$$= \frac{\sin 40^\circ + 2\sin 30^\circ \cos 10^\circ}{\cos 20^\circ}$$
$$= \frac{\sin 40^\circ + \sin 80^\circ}{\cos 20^\circ}$$
$$= \frac{2\sin 60^\circ \cos 20^\circ}{\cos 20^\circ} = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$$

Hence, square of the value of expression = 3

112. $A + B + C = \pi$ ...(i)

$$\frac{\sin A}{3} = \frac{\cos B}{3} = \frac{\tan C}{2} \quad \text{...(ii)}$$

$\Rightarrow \quad \sin A = \cos B \Rightarrow A + B = \dfrac{\pi}{2}$ (rejected)

Or $\quad A - B = \dfrac{\pi}{2}$ ...(iii)

$\Rightarrow \quad 2A + C = \dfrac{3\pi}{2}$ [from Eqs. (i) and (ii)]

Now $\quad \dfrac{\sin A}{\tan C} = \dfrac{3}{2}$ [from Eq. (ii)

$\Rightarrow \quad \dfrac{\sin A}{\tan\left(\dfrac{3\pi}{2} - 2A\right)} = \dfrac{3}{2}$ [from Eq. (iii)]

$\Rightarrow \quad 2\sin A = 3\cot 2A$

$\Rightarrow \quad 2\sin A = \dfrac{3\cdot(2\cos^2 A - 1)}{2\sin A \cos A}$

$\Rightarrow \quad 4\cos A(1 - \cos^2 A) = 3(2\cos^2 A - 1)$

$\Rightarrow \quad 4\cos^3 A + 6\cos^2 A - 4\cos A - 3 = 0$

Put $\cos A = -\dfrac{1}{2}$

$\Rightarrow \quad (2\cos A + 1)(2\cos^2 A + 2\cos A - 3) = 0$

$\Rightarrow \quad \cos A = -\dfrac{1}{2}$,

$$\cos A = \frac{-2 \pm \sqrt{4+24}}{4} = -1 \pm \sqrt{7} \text{ (rejected)}$$

$\Rightarrow \quad A = \dfrac{2\pi}{3}, B = \dfrac{\pi}{6}, C = \dfrac{\pi}{6}$

$\therefore \quad \dfrac{\sin A}{\cos 2A} + \dfrac{\cos B}{\cot 2B} + \dfrac{\tan C}{\cot 2C} = \dfrac{1}{2} + \dfrac{1}{2} + 1 = 2$

113. $f(\theta) = \dfrac{1}{1 + g(\theta)}$ ...(i)

Given, $2f(\alpha) - g(\beta) = 1$

$$2f(\alpha) = 1 + g(\beta) = \frac{1}{f(\beta)} \quad \text{[from Eq. (i)]}$$

$$2f(\alpha)\, f(\beta) = 1 \quad \text{...(ii)}$$

Now, $\quad 2f(\beta) - g(\alpha) = 2f(\beta) + 1 - (1 + g(\alpha))$

$$= 2f(\beta) + 1 - \frac{1}{f(\alpha)}$$
$$= \frac{2f(\alpha)\, f(\beta) - 1}{f(\alpha)} + 1 = 1 \quad \text{[from Eq. (ii)]}$$

114. As we know that $\dfrac{1-\sin x}{\cos x} = \dfrac{\cos x}{1+\sin x}$

$$\log_{\left|\frac{1-\sin x}{\cos x}\right|}\left|\frac{1+\sin x}{\cos x}\right| = -1$$

Now, series is

$$\text{Let } S = 1 - \frac{x}{2} - \frac{x^2}{4} - \ldots = 1 - \frac{\frac{x}{2}}{1 - \frac{x}{2}}$$
$$= 1 - \frac{-x}{2-x} = \frac{2-x-x}{2-x} = \frac{2(1-x)}{(2-x)} = \frac{k(1-x)}{(2-x)}$$

Thus, $k = 2$

115. From the second relation $9x\sin^3\theta = 5y\cos^3\theta$.

$\Rightarrow \quad \dfrac{\cos^3\theta}{9x} = \dfrac{\sin^3\theta}{5y} = k^3$ (say)

$\Rightarrow \quad \cos\theta = k(9x)^{\frac{1}{3}}$ and $\sin\theta = k(5y)^{\frac{1}{3}}$

Squaring and adding, we get

$$1 = \cos^2\theta + \sin^2\theta = k^2\left[(9x)^{\frac{2}{3}} + (5y)^{\frac{2}{3}}\right]$$

and $\quad \dfrac{9x}{k(9x)^{\frac{1}{3}}} + \dfrac{5y}{k(5y)^{\frac{1}{3}}} = 56$ (From Ist relation)

$\Rightarrow \quad (9x)^{\frac{2}{3}} + (5y)^{\frac{2}{3}} = 56k$

$\Rightarrow \quad \left[(9x)^{\frac{2}{3}} + (5y)^{\frac{2}{3}}\right]^2 = (56)^2k^2 = \dfrac{(56)^2}{(9x)^{\frac{2}{3}} + (5y)^{\frac{2}{3}}}$

$\Rightarrow \quad \left[(9x)^{\frac{2}{3}} + (5y)^{\frac{2}{3}}\right]^3 = (56)^2 = 3136.$

116. $A > \dfrac{\pi}{2} \Rightarrow B + C < \dfrac{\pi}{2}$

$\Rightarrow \quad \tan(B+C) > 0 \Rightarrow \dfrac{\tan B + \tan C}{1 - \tan B \tan C} > 0$

$\Rightarrow \quad \tan B \tan C < 1$ as $\tan B > 0, \tan C > 0$

$\Rightarrow \quad [x] = 2635 - 1 = 2634$

117. $\because \cot\left(7\dfrac{1^\circ}{2}\right) = \dfrac{1+\cos 15^\circ}{\sin 15^\circ}$

$$= \frac{1 + \dfrac{\sqrt{3}+1}{2\sqrt{2}}}{\dfrac{\sqrt{3}-1}{2\sqrt{2}}} = \frac{2\sqrt{2} + \sqrt{3} + 1}{\sqrt{3} - 1}$$
$$= \frac{(2\sqrt{2} + \sqrt{3} + 1)(\sqrt{3}+1)}{2}$$

$$= \frac{2\sqrt{6} + 2\sqrt{2} + 3 + \sqrt{3} + \sqrt{3} + 1}{2}$$

$$= \sqrt{6} + \sqrt{2} + 2 + \sqrt{3}$$

$$= \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{6}$$

and $4\cos 36° = 4\left(\frac{\sqrt{5}+1}{4}\right) = \sqrt{5} + 1 = \sqrt{5} + \sqrt{1}$

Hence, $4\cos 36° + \cot\left(7\frac{1°}{2}\right)$

$$= \sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5} + \sqrt{6}$$

$\therefore$ $n_1 = 1, n_2 = 2, n_3 = 3, n_4 = 4, n_5 = 5$ and $n_6 = 6$

$\therefore$ $\sum_{i=1}^{6} n_i^2 = n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^3 + n_6^2$

$$= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2$$

$$= 91$$

118. $\because \prod_{r=1}^{4} \sin(rA) = \sin A \sin 2A \sin 3A \sin 4A$

$$= \sin A \cdot 2\sin A \cos A \cdot (3\sin A - 4\sin^3 A \cdot 2\sin 2A \cos 2A)$$

$$= 2\sin^2 A \cos A \cdot \sin A(3 - 4\sin^2 A)$$

$$\cdot 4\sin A \cos A \cdot (1 - 2\sin^2 A)$$

$$= 8x^2(1-x)(3-4x)(1-2x)$$

$$= 24x^2 - 104x^3 + 144x^4 - 64x^5$$

On comparing, we get $a = 24, b = -104, c = 144, d = -64$

$10a - 7b + 15c - 5d$

$$= 10 \times 24 - 7 \times -104 + 15 \times 144 - 5 \times -64$$

$$= 240 + 728 + 2160 + 320 = 3448$$

119. Let $x + 5 = 14\cos\theta$ and $y - 12 = 14\sin\theta$

$\therefore x^2 + y^2 = (14\cos\theta - 5)^2 + (14\sin\theta + 12)^2$

$$= 196 + 25 + 144 + 28(12\sin\theta - 5\cos\theta)$$

$$= 365 + 28(12\sin\theta - 5\cos\theta)$$

$\therefore$ $\sqrt{x^2 + y^2}\Big|_{\min} = \sqrt{365 - 28 \times 13} = \sqrt{365 - 364} = 1$

120. $\because 12° \times 5 = 60°$

Let $12° = \theta$

$\therefore$ $5\theta = 60°$

$\Rightarrow$ $3\theta + 2\theta = 60°$

$\therefore$ $\cos(3\theta + 2\theta) = \cos 60°$

$\Rightarrow$ $\cos 3\theta \cos 2\theta - \sin 3\theta \sin 2\theta = \frac{1}{2}$

$\Rightarrow (4\cos^3\theta - 3\cos\theta)(2\cos^2\theta - 1) - (3\sin\theta - 4\sin^3\theta)$

$$2\sin\theta\cos\theta = \frac{1}{2}$$

Let $\cos\theta = x$

$\therefore$ $(4x^3 - 3x)(2x^2 - 1) - 2x(3 - 4(1 - x^2))(1 - x^2) = \frac{1}{2}$

$\Rightarrow$ $(8x^5 - 10x^3 + 3x) - (2x - 2x^3)(4x^2 - 1) = \frac{1}{2}$

$\Rightarrow (16x^5 - 20x^3 + 6x) - (4x - 4x^3)(4x^2 - 1) - 1 = 0$

$\Rightarrow$ $32x^5 - 40x^3 + 10x - 1 = 0$

$\Rightarrow$ $\left(x - \frac{1}{2}\right)(32x^4 + 16x^3 - 32x^2 - 16x + 2) = 0$

but $x \neq \frac{1}{2}$,

$\therefore$ $16x^4 + 8x^3 - 16x^2 - 8x + 1 = 0$

$\therefore$ Degree is 4.

121. From conditional identities, we have

$$\frac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C} = \frac{4\sin A \sin B \sin C}{4\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)}$$

$$= 8\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right) \Rightarrow k = 8$$

and $3k^3 + 2k^2 + k + 1 = 1536 + 128 + 8 + 1 = 1673$

122. $x = \cot\frac{11\pi}{8} = \cot\left(\pi + \frac{3\pi}{8}\right) = \cot\frac{3\pi}{8} = \sqrt{2} - 1$

$\Rightarrow$ $(x+1)^2 = 2$

$\therefore$ $x^2 + 2x - 1 = 0$

Now, $f(x) = x^4 + 4x^3 + 2x^2 - 4x + 7$

$$= x^2(x^2 + 2x - 1) + 2x^3 + 3x^2 - 4x + 7$$

$$= 0 + 2x^3 + 3x^2 - 4x + 7$$

$$= 2x(x^2 + 2x - 1) - x^2 - 2x + 7 = -x^2 - 2x + 7$$

$$= -(x^2 + 2x - 1) + 6 = 0 + 6 = 6$$

123. $\frac{\sqrt{(\sin A)}}{\sqrt{(\sin B)} + \sqrt{(\sin C)} - \sqrt{(\sin A)}} = \frac{\sqrt{a}}{\sqrt{b} + \sqrt{c} - \sqrt{a}}$

Now, $\sqrt{b} + \sqrt{c} - \sqrt{a} = \frac{(\sqrt{b} + \sqrt{c} - \sqrt{a})(\sqrt{b} + \sqrt{c} + \sqrt{a})}{(\sqrt{b} + \sqrt{c} + \sqrt{a})}$

$$= \frac{(\sqrt{b} + \sqrt{c})^2 - a}{(\sqrt{b} + \sqrt{c} + \sqrt{a})} = \frac{(b + c - a) + 2\sqrt{bc}}{(\sqrt{b} + \sqrt{c} + \sqrt{a})} > 0$$

Hence, $\sqrt{b} + \sqrt{c} - \sqrt{a} > 0$

Now, let $\sqrt{b} + \sqrt{c} - \sqrt{a} = x, \sqrt{c} + \sqrt{a} - \sqrt{b} = y$

and $\sqrt{a} + \sqrt{b} - \sqrt{c} = z$

$\therefore$ $\frac{\sqrt{\sin A}}{\sqrt{\sin B + \sqrt{\sin C} - \sqrt{\sin A}}} = \frac{y+z}{2x}$

$\Rightarrow$ $\sum \frac{\sqrt{(\sin A)}}{\sqrt{(\sin B)} + \sqrt{(\sin C)} - \sqrt{(\sin A)}}$

$$= \frac{1}{2}\left\{\frac{y}{x} + \frac{z}{x}\right\} + \frac{1}{2}\left\{\frac{z}{y} + \frac{x}{y}\right\} + \frac{1}{2}\left\{\frac{x}{z} + \frac{y}{z}\right\}$$

$$= \frac{1}{2}\left(\frac{x}{y} + \frac{y}{x}\right) + \frac{1}{2}\left(\frac{y}{z} + \frac{z}{y}\right) + \frac{1}{2}\left(\frac{z}{x} + \frac{x}{z}\right)$$

$$\geq 1 + 1 + 1 \quad (\because \text{AM} \geq \text{GM})$$

$2020 \sum \frac{\sqrt{(\sin A)}}{\sqrt{(\sin B)} + \sqrt{(\sin C)} - \sqrt{(\sin A)}} \leq 6060$

$\therefore$ Minimum value is 6060.

124. We have, $\sin\theta(1+\sin^2\theta)=1-\sin^2\theta$

$\Rightarrow \quad \sin\theta(2-\cos^2\theta)=\cos^2\theta$

Squaring both sides, we get

$\sin^2\theta(2-\cos^2\theta)^2=\cos^4\theta$

$\Rightarrow \quad (1-\cos^2\theta)(4-4\cos^2\theta+\cos^4\theta)=\cos^4\theta$

$\Rightarrow \quad -\cos^6\theta+5\cos^4\theta-8\cos^2\theta+4=\cos^4\theta$

$\therefore \quad \cos^6\theta-4\cos^4\theta+8\cos^2\theta=4$

125. $16\left(\cos\theta-\cos\frac{\pi}{8}\right)\left(\cos\theta-\cos\frac{3\pi}{8}\right)\left(\cos\theta-\cos\frac{5\pi}{8}\right)\left(\cos\theta-\cos\frac{7\pi}{8}\right)$

$=16\left(\cos\theta-\cos\frac{\pi}{8}\right)\left(\cos\theta-\cos\frac{7\pi}{8}\right)\times\left(\cos\theta-\cos\frac{3\pi}{8}\right)\left(\cos\theta-\cos\frac{5\pi}{8}\right)$

$=16\left(\cos\theta-\cos\frac{\pi}{8}\right)\left(\cos\theta+\cos\frac{\pi}{8}\right)\times\left(\cos\theta-\cos\frac{3\pi}{8}\right)\left(\cos\theta+\cos\frac{3\pi}{8}\right)$

$=16\left(\cos^2\theta-\cos^2\frac{\pi}{8}\right)\left(\cos^2\theta-\cos^2\frac{3\pi}{8}\right)$

$=16\left(\cos^2\theta-\cos^2\frac{\pi}{8}\right)\left(\cos^2\theta-\sin^2\frac{\pi}{8}\right)$

$=16\left(\cos^4\theta-\cos^2\theta+\sin^2\frac{\pi}{8}\cos^2\frac{\pi}{8}\right)$

$=16\left(\cos^4\theta-\cos^2\theta+\frac{1}{8}\right)$

$=16\left(-\cos^2\theta\sin^2\theta+\frac{1}{8}\right)=16\left(\frac{-\sin^2 2\theta}{4}+\frac{1}{8}\right)$

$=16\left(\frac{1-2\sin^2 2\theta}{8}\right)=\frac{16\cos 4\theta}{8}=2\cos 4\theta$

$\therefore \quad \lambda=2$

126. $2k\cos\cos 40^\circ=\frac{1}{\sin 20^\circ}+\frac{1}{\sqrt{3}\cos 20^\circ}$

$=\frac{\sqrt{3}\cos 20^\circ+\sin 20^\circ}{\sqrt{3}\sin 20^\circ\cos 20^\circ}$

$=\frac{\frac{\sqrt{3}}{2}\cos 20^\circ+\frac{1}{2}\sin 20^\circ}{\frac{\sqrt{3}}{4}\sin 40^\circ}$

$=\frac{\sin 60^\circ\cos 20^\circ+\cos 60^\circ\sin 20^\circ}{\left(\frac{\sqrt{3}}{4}\right)\sin 40^\circ}$

$=\left(\frac{4}{\sqrt{3}}\right)2\cos 40^\circ$

$\Rightarrow \quad 2k^2=16$

so $18k^4+162k^2+369=1745$

127. $\tan 82\frac{1}{2}^\circ=\cot 7\frac{1}{2}^\circ=\frac{\cos 7\frac{1}{2}^\circ}{\sin 7\frac{1}{2}^\circ}$

On multiplying numerator and denominator by $2\cos 7\frac{1}{2}^\circ$, we get

$\tan 82\frac{1}{2}^\circ=\frac{2\cos^2 7\frac{1}{2}^\circ}{2\sin 7\frac{1}{2}^\circ\cos 7\frac{1}{2}^\circ}=\frac{1+\cos 15^\circ}{\sin 15^\circ}$

$=\frac{1+\cos(45^\circ-30^\circ)}{\sin(45^\circ-30^\circ)}=\frac{1+\frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}}$

$=\frac{2\sqrt{2}+\sqrt{3}+1}{\sqrt{3}-1}\times\frac{\sqrt{3}+1}{\sqrt{3}+1}$

$=\frac{2\sqrt{2}(\sqrt{3}+1)+(\sqrt{3}+1)^2}{2}=\frac{2\sqrt{2}(\sqrt{3}+1)+(4+2\sqrt{3})}{2}$

$=\sqrt{2}(\sqrt{3}+1)+(2+\sqrt{3})=\sqrt{6}+\sqrt{2}+\sqrt{4}+\sqrt{3}$

$=\sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}=(\sqrt{3}+\sqrt{2})(\sqrt{2}+1)$

128. LHS $=\frac{2-m(\sin 2\alpha+\sin 2\beta)}{1-m(\sin 2\alpha+\sin 2\beta)+m^2\sin 2\alpha\sin 2\beta}$

$=\frac{2-2m\sin(\alpha+\beta)\cos(\alpha-\beta)}{1-2m\sin(\alpha+\beta).\cos(\alpha-\beta)+4m^2\sin\alpha\cos\alpha\sin\beta\cos\beta}$

$=\frac{2\{1-\cos^2(\alpha-\beta)\}}{1-2m\sin(\alpha+\beta).\cos(\alpha-\beta)+4m^2\sin\alpha\cos\alpha\sin\beta\cos\beta}$

[using $m\sin(\alpha+\beta)=\cos(\alpha-\beta)$]

$=\frac{2\sin^2(\alpha-\beta)}{1-2\cos^2(\alpha-\beta)+m^2[\sin(\alpha+\beta)+\sin(\alpha-\beta)][\sin(\alpha+\beta)-\sin(\alpha-\beta)]}$

$=\frac{2\sin^2(\alpha-\beta)}{1-2\cos^2(\alpha-\beta)+m^2\sin^2(\alpha+\beta)-m^2\sin^2(\alpha-\beta)}$

$=\frac{2\sin^2(\alpha-\beta)}{1-\cos^2(\alpha-\beta)-m^2\sin^2(\alpha-\beta)}$

$=\frac{2\sin^2(\alpha-\beta)}{\sin^2(\alpha-\beta)-m^2\sin^2(\alpha-\beta)}=\frac{2}{1-m^2}$

129. Given $\tan\frac{1}{4}(\beta+\gamma-2\alpha).\tan\frac{1}{4}(\gamma+\alpha-\beta)\tan\frac{1}{4}(\alpha+\beta-\gamma)=1$,

where $\alpha+\beta+\gamma=\pi$

$\Rightarrow \quad \tan\left(\frac{\pi-2\alpha}{4}\right)\tan\left(\frac{\pi-2\beta}{4}\right)\tan\left(\frac{\pi-2\gamma}{4}\right)=1$

$\Rightarrow \quad \left(1-\tan\frac{\alpha}{2}\right)\left(1-\tan\frac{\beta}{2}\right)\left(1-\tan\frac{\gamma}{2}\right)=\left(1+\tan\frac{\alpha}{2}\right)\left(1+\tan\frac{\beta}{2}\right)\left(1+\tan\frac{\gamma}{2}\right)$

$\Rightarrow \quad \tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}+\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}=0$

$\Rightarrow \quad \tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}=-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$ ...(i)

Also, $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\tan\frac{\gamma}{2}\tan\frac{\alpha}{2}=1$ ...(ii)

On squaring Eq. (i) and using Eq. (ii); $\left\{\because \frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}=\frac{\pi}{2}\right\}$

$$\tan^2\frac{\alpha}{2}+\tan^2\frac{\beta}{2}+\tan^2\frac{\gamma}{2}+2=\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}\tan^2\frac{\gamma}{2} \quad ...(iii)$$

The equation to be prove is

$$1+\cos\alpha+\cos\beta+\cos\gamma=0$$

$$\Rightarrow \quad 1+\frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}+\frac{1-\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}}+\frac{1-\tan^2\frac{\gamma}{2}}{1+\tan^2\frac{\gamma}{2}}=0$$

$$\Rightarrow \quad \frac{2}{1+\tan^2\frac{\alpha}{2}}+\frac{2\left(1-\tan^2\frac{\beta}{2}\tan^2\frac{\gamma}{2}\right)}{\left(1+\tan^2\frac{\beta}{2}\right)\left(1+\tan^2\frac{\gamma}{2}\right)}=0$$

$$\Rightarrow \left(1+\tan^2\frac{\alpha}{2}\right)\left(1-\tan^2\frac{\beta}{2}\tan^2\frac{\gamma}{2}\right)+\left(1+\tan^2\frac{\beta}{2}\right)\left(1+\tan^2\frac{\gamma}{2}\right)=0$$

$$\Rightarrow \left(1+\tan^2\frac{\alpha}{2}-\tan^2\frac{\beta}{2}\tan^2\frac{\gamma}{2}-\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}\tan^2\frac{\gamma}{2}\right)+\left(1+\tan^2\frac{\beta}{2}+\tan^2\frac{\gamma}{2}+\tan^2\frac{\beta}{2}\tan^2\frac{\gamma}{2}\right)=0$$

$$\Rightarrow \quad 2+\tan^2\frac{\alpha}{2}+\tan^2\frac{\beta}{2}+\tan^2\frac{\gamma}{2}=\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}\tan^2\frac{\gamma}{2} \quad ...(iv)$$

From Eq. (iii) and (iv)

$$1+\cos\alpha+\cos\beta+\cos\gamma=0$$

130. We have, $\sin^4 x+\cos^4 x\le\sin^2 x+\cos^2 x$,

as $|\sin x|\le 1$ and $|\cos x|\le 1$

$$\Rightarrow \quad a\le 1 \quad ...(i)$$

Next, $\sin^4 x+\cos^4 x=a$

$$\Rightarrow \quad (\sin^2 x+\cos^2 x)^2-2\sin^2 x\cos^2 x=a$$

$$\Rightarrow \quad 1-\frac{1}{2}\sin^2 2x=a$$

$$\Rightarrow \quad \frac{1}{2}\sin^2 2x=1-a$$

$$\Rightarrow \quad 1-a\le\frac{1}{2} \qquad \left[\because \frac{1}{2}\sin^2 x\le\frac{1}{2}\right]$$

$$\Rightarrow \quad a\ge\frac{1}{2} \quad ...(ii)$$

From Eqs. (i) and (ii),

$$\frac{1}{2}\le a\le 1$$

131. Let $a\sec\theta-b\tan\theta=x$

So, $a^2\sec^2\theta=(x+b\tan\theta)^2$

$$\Rightarrow \quad a^2(1+\tan^2\theta)=x^2+2bx\tan\theta+b^2\tan^2\theta$$

$$\Rightarrow \quad \tan^2\theta(a^2-b^2)-2bx\tan\theta+(a^2-x^2)=0$$

$$\Rightarrow \quad \left(\tan\theta-\frac{bx}{a^2-b^2}\right)^2=\frac{a^2(x^2+b^2-a^2)}{(a^2-b^2)^2}$$

Thus, $x^2+(b^2-a^2)\ge 0$

$$\Rightarrow \quad x^2\ge a^2-b^2$$

Thus, the minimum value of x is $\sqrt{a^2-b^2}$, which is attained at $\theta=\sin^{-1}\left(\frac{b}{a}\right)$.

132. We can write

$$(b\tan\gamma-c\tan\beta)^2+(c\tan\alpha-a\tan\gamma)^2+(a\tan\beta-b\tan\alpha)^2$$
$$=(a^2+b^2+c^2)(\tan^2\alpha+\tan^2\beta+\tan^2\gamma)-(a\tan\alpha+b\tan\beta+c\tan\gamma)^2$$

The minimum value of the LHS being zero, that of

$$(a^2+b^2+c^2)(\tan^2\alpha+\tan^2\beta+\tan^2\gamma)-k^2\ge 0$$

$$\Rightarrow \quad \tan^2\alpha+\tan^2\beta+\tan^2\gamma\ge\frac{k^2}{a^2+b^2+c^2}$$

Hence, minimum value of $\tan^2\alpha+\tan^2\beta+\tan\gamma$ is $\left(\frac{k^2}{a^2+b^2+c^2}\right)$.

133. Here, $\frac{x}{y}=\frac{\tan(\theta+\alpha)}{\tan(\theta+\beta)}$. By componendo and dividendo

$$\frac{x+y}{x-y}=\frac{\tan(\theta+\alpha)+\tan(\theta+\beta)}{\tan(\theta+\alpha)-\tan(\theta+\beta)}=\frac{\sin(2\theta+\alpha+\beta)}{\sin(\alpha-\beta)}$$

$$\therefore \quad \frac{x+y}{x-y}\sin^2(\alpha-\beta)=\sin(2\theta+\alpha+\beta)\cdot\sin(\alpha-\beta)$$

$$\frac{x+y}{x-y}\cdot\sin^2(\alpha-\beta)=\frac{1}{2}\{\cos2(\theta+\beta)-\cos2(\theta+\alpha)\} \quad ...(i)$$

Similarly,

$$\frac{y+z}{y-z}\cdot\sin^2(\beta-\gamma)=\frac{1}{2}\{\cos2(\theta+\gamma)-\cos2(\theta+\beta)\} \quad ...(ii)$$

and $$\frac{z+x}{z-x}\cdot\sin^2(\gamma-\alpha)=\frac{1}{2}\{\cos2(\theta+\alpha)-\cos2(\theta+\gamma)\} \quad ...(iii)$$

From Eqs. (i), (ii) and (iii), we get

$$\Sigma\frac{x+y}{x-y}\cdot\sin^2(\alpha-\beta)=0$$

134. $f(x)$ may be written as $f(x)=\sum_{k=1}^{n}\frac{1}{2^{k-1}}\cos(a_k+x)$

$$=\sum_{k=1}^{n}\frac{1}{2^{k-1}}(\cos a_k\cos x-\sin a_k\sin x)$$

$$=\left(\sum_{k=1}^{n}\frac{1}{2^{k-1}}\cdot\cos a_k\right)\cdot\cos x\left(-\sum_{k=1}^{n}\frac{1}{2^{k-1}}\cdot\sin a_k\right)\cdot\sin x$$

where, $A=\sum_{k=1}^{n}\frac{1}{2^{k-1}}\cos a_k$, $B=\sum_{k=1}^{n}\frac{1}{2^{k-1}}\sin a_k$. Now, A and B both cannot be zero, for if they were then $f(x)$ would vanish identically.

Now,

$$f(x_1)=A\cos x_1-B\sin x_1=0$$
$$f(x_2)=A\cos x_2-B\sin x_2=0$$

$$\Rightarrow \quad \tan x_1=\frac{A}{B} \text{ and } \tan x_2=\frac{A}{B}$$

$$\Rightarrow \quad \tan x_1=\tan x_2\Rightarrow \ x_2-x_1=m\pi.$$

135. $x = \tan(n\theta + \alpha) - \tan(n\theta + \beta)$

$$= \frac{\sin(n\theta + \alpha)}{\cos(n\theta + \alpha)} - \frac{\sin(n\theta + \beta)}{\cos(n\theta + \beta)}$$

$$= \frac{\sin(n\theta + \alpha - n\theta - \beta)}{\cos(n\theta + \alpha)\cos(n\theta + \beta)} = \frac{2\sin(\alpha - \beta)}{\cos(2n\theta + \alpha + \beta) + \cos(\alpha - \beta)}$$

$$\Rightarrow \cos(2n\theta + \alpha + \beta) + \cos(\alpha - \beta) = \frac{2\sin(\alpha - \beta)}{x} \quad \ldots(i)$$

Again $y = \cot(n\theta + \alpha) - \cot(n\theta + \beta)$

$$= \frac{\cos(n\theta + \alpha)}{\sin(n\theta + \alpha)} - \frac{\cos(n\theta + \beta)}{\sin(n\theta + \beta)} = \frac{\sin(n\theta + \beta - n\theta - \alpha)}{\sin(n\theta + \alpha)\sin(n\theta + \beta)}$$

$$\Rightarrow \quad y = \frac{\sin(\beta - \alpha)}{\cos(\alpha - \beta) - \cos(2n\theta + \alpha + \beta)}$$

$$\Rightarrow \cos(\alpha - \beta) - \cos(2n\theta + \alpha + \beta) = \frac{2\sin(\beta - \alpha)}{y} \quad \ldots(ii)$$

On adding Eqs. (i) and (ii), we get

$$2\cos(\alpha - \beta) = \frac{2\sin(\alpha - \beta)}{x} + \frac{2\sin(\beta - \alpha)}{y}$$

$$\Rightarrow \quad \cot(\alpha - \beta) = \frac{1}{x} - \frac{1}{y}$$

136. $\{\sin(\alpha - \beta) + \cos(\alpha + 2\beta)\cdot\sin\beta\}^2 = 4\cos\alpha\cdot\sin\beta\cdot\sin(\alpha + \beta)$

$$\Rightarrow \{\sin\alpha\cos\beta - \sin\beta\cos\alpha + (\cos\alpha\cos 2\beta - \sin\alpha\sin 2\beta)\sin\beta\}^2$$

$$= 4\cos\alpha\sin\beta\sin(\alpha + \beta)$$

$$\Rightarrow \{\tan\alpha - \tan\beta + \cos 2\beta\cdot\tan\beta - \sin 2\beta\cdot\tan\alpha\tan\beta\}^2$$

$$= 4\tan\beta(\tan\alpha + \tan\beta) \qquad \{\because \text{dividing by } \cos^2\alpha.\cos^2\beta\}$$

$\Rightarrow$

$$\{\tan\alpha\cdot\cos 2\beta - \tan\beta + \cos 2\beta\cdot\tan\beta\}^2 = 4\tan\beta\{\tan\alpha + \tan\beta\}$$

$$\Rightarrow \{(\tan\alpha + \tan\beta)\cdot\cos 2\beta - \tan\beta\}^2 = 4\tan\beta(\tan\alpha + \tan\beta) \ldots(i)$$

If $$(\tan\alpha + \tan\beta) = \frac{\tan\beta}{x} \quad \ldots(ii)$$

Eq. (i) becomes;

$$\left\{\frac{\tan\beta}{x}\cdot\cos 2\beta - \tan\beta\right\}^2 = 4\tan\beta\cdot\frac{\tan\beta}{x}$$

$$\Rightarrow \quad (\cos 2\beta - x)^2 = 4x$$

$$\Rightarrow \quad \cos^2 2\beta + x^2 - 2x\cos 2\beta = 4x$$

$$\Rightarrow \quad x^2 - 2x(\cos 2\beta + 2) + \cos^2 2\beta = 0$$

$$\Rightarrow \quad x = (\cos 2\beta + 2) \pm 2\sqrt{1 + \cos 2\beta}$$

$$\Rightarrow \quad x = \cos 2\beta + 2 \pm 2\sqrt{2\cos^2\beta}$$

$$\Rightarrow \quad x = 2\cos^2\beta - 1 + 2 \pm 2\sqrt{2}\cos\beta$$

$$= (\sqrt{2}\cos\beta \pm 1)^2$$

$$\Rightarrow \quad \tan\alpha + \tan\beta = \frac{\tan\beta}{x} = \frac{\tan\beta}{(\sqrt{2}\cos\beta - 1)^2} \qquad [\text{since, } x < 1]$$

137. Let $$\Delta = \begin{vmatrix} \sin A & \sin B & \sin C \\ \cos A & \cos B & \cos C \\ \cos^3 A & \cos^3 B & \cos^3 C \end{vmatrix}$$

$$= \cos A\cos B\cos C \begin{vmatrix} \tan A & \tan B & \tan C \\ 1 & 1 & 1 \\ \cos^2 A & \cos^2 B & \cos^2 C \end{vmatrix}$$

$$= \cos A\cos B\cos C \begin{vmatrix} \tan A & \tan B - \tan A & \tan C - \tan A \\ 1 & 0 & 0 \\ \cos^2 A & \cos^2 B - \cos^2 A & \cos^2 A - \cos^2 A \end{vmatrix}$$

$$\left[\text{since, } \tan B - \tan A = -\frac{\sin(A - B)}{\cos A\cos B},\right.$$

$\left.\cos^2 B - \cos^2 A = \sin(A - B)\sin(A + B)\right]$

$$\therefore \quad \Delta = -\cos A\cos B\cos C \begin{vmatrix} -\frac{\sin(A - B)}{\cos A\cos B} & -\frac{\sin(A - C)}{\cos A\cos C} \\ \sin(A - B)\sin(A + B) & \sin(A - C)\sin(A + C) \end{vmatrix}$$

$$= \cos A\cos B\cos C\cdot \frac{\sin(A - B).\sin(A - C)}{\cos A\cos B\cos C} \begin{vmatrix} \cos C & \cos B \\ \sin(A + B) & \sin(A + C) \end{vmatrix}$$

$$= -\sin(B - C)\sin(C - A)\sin(A - B) = 0$$

If $B = C$ or $C = A$ or $A = B$

Hence, ΔABC is an isosceles.

138. Here, $$\frac{\sqrt{\sin A}}{\sqrt{\sin B} + \sqrt{\sin C} - \sqrt{\sin A}} = \frac{\sqrt{a}}{\sqrt{b} + \sqrt{c} - \sqrt{a}}$$

Now, $$\sqrt{b} + \sqrt{c} - \sqrt{a} = \frac{(\sqrt{b} + \sqrt{c} - \sqrt{a})(\sqrt{b} + \sqrt{c} + \sqrt{a})}{(\sqrt{b} + \sqrt{c} + \sqrt{a})}$$

$$= \frac{b + c - a + 2\sqrt{bc}}{\sqrt{b} + \sqrt{c} + \sqrt{a}} > 0$$

Hence, $\sqrt{b} + \sqrt{c} - \sqrt{a} = 0$

Let $\sqrt{b} + \sqrt{c} - \sqrt{a} = x$, $\sqrt{c} + \sqrt{a} - \sqrt{b} = y$, $\sqrt{a} + \sqrt{b} - \sqrt{c} = z$

$$\therefore \quad \frac{\sqrt{\sin A}}{\sqrt{\sin B} + \sqrt{\sin C} - \sqrt{\sin A}} = \frac{y + z}{2x}$$

$$\Rightarrow \quad \Sigma = \frac{\sqrt{\sin A}}{\sqrt{\sin B} + \sqrt{\sin C} - \sqrt{\sin A}}$$

$$= \frac{1}{2}\left\{\frac{y}{x} + \frac{z}{x}\right\} + \frac{1}{2}\left\{\frac{y}{z} + \frac{z}{y}\right\} + \frac{1}{2}\left\{\frac{z}{x} + \frac{x}{y}\right\}$$

which is greater than or equal to 3, as each term $\left(\frac{1}{2}\left(\frac{x}{y} + \frac{y}{x}\right)\text{etc.}\right)$ is greater than or equal to 1.

(using AM $\geq$ GM)

Now, equality hold if and only if,

$$\frac{x}{y} = \frac{y}{x}, \frac{y}{z} = \frac{z}{y}$$

and $$\frac{z}{x} = \frac{x}{y} \quad \text{i.e. } x = y = z$$

$\Rightarrow$ $a = b = c$ i.e. triangle is equilateral.

139. $2(\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha)) + 3 = 0$

$$\Rightarrow \quad 2(\cos(\alpha + \theta - (\beta + \theta)) + \cos(\beta + \theta - (\gamma + \theta))$$

$$+ \cos(\gamma + \theta - (\alpha + \theta))) + 3 = 0$$

$$\Rightarrow \quad 2(\cos(\alpha + \theta).\cos(\beta + \theta) + \sin(\alpha + \theta).\sin(\beta + \theta) + \ldots + \ldots))$$

$$+ \{(\sin^2(\alpha + \theta) + \cos^2(\alpha + \theta)) + (\sin^2(\beta + \theta) + \cos^2(\beta + \theta))$$

$$+ (\sin^2(\gamma + \theta) + \cos^2(\gamma + \theta))\} = 0$$

$$\Rightarrow \quad (\sin(\gamma+\theta)+\sin(\beta+\theta)+\sin(\alpha+\theta))^2+(\cos(\alpha+\theta)+\cos(\beta+\theta)+\cos(\gamma+\theta))^2=0$$

which is only possible if;

$$\sin(\alpha+\theta)+\cos(\beta+\theta)+\sin(\gamma+\theta)=0 \quad \ldots(i)$$

$$\cos(\alpha+\theta)+\cos(\beta+\theta)+\cos(\gamma+\theta)=0 \quad \ldots(ii)$$

From Eq. (ii), we get

$$d(\cos(\alpha+\theta)+\cos(\beta+\theta)+\cos(\gamma+\theta))=0$$

$$\Rightarrow \sin(\alpha+\theta)\cdot d\alpha+\sin(\beta+\theta)\cdot d\beta+\sin(\gamma+\theta)\cdot d\gamma=0$$

$$\Rightarrow \frac{d\alpha}{\sin(\beta+\theta)\cdot\sin(\gamma+\theta)}+\frac{d\beta}{\sin(\alpha+\theta)\cdot\sin(\gamma+\theta)}+\frac{d\gamma}{\sin(\alpha+\theta)\cdot\sin(\beta+\theta)}=0$$

140. The quadratic equation,

$$4^{\sec^2\alpha}x^2+2x+\left(\beta^2-\beta+\frac{1}{2}\right)=0 \text{ have real roots}$$

$$\Rightarrow \text{Discriminant}=4-4\cdot4^{\sec^2\alpha}\left(\beta^2-\beta+\frac{1}{2}\right)\geq 0$$

$$\Rightarrow 4^{\sec^2\alpha}\left(\beta^2-\beta+\frac{1}{2}\right)\leq 1$$

$$\left[\text{but } 4^{\sec^2\alpha}\geq 4,\ \beta^2-\beta+\frac{1}{2}=\left(\beta+\frac{1}{2}\right)^2+\frac{1}{4}\geq\frac{1}{4}\right]$$

i.e. the equation will be satisfied only when $4^{\sec^2\alpha}=4$ and

$$\beta^2-\beta+\frac{1}{2}=\frac{1}{4}$$

$$\Rightarrow \sec^2\alpha=1 \text{ and } \left(\beta-\frac{1}{2}\right)^2=0$$

$$\Rightarrow \cos^2\alpha=1 \text{ and } \beta=\frac{1}{2}$$

$$\Rightarrow \alpha=n\pi \text{ and } \beta=\frac{1}{2}$$

$$\cos\alpha+\cos^{-1}\beta=\cos n\pi+\cos^{-1}\left(\frac{1}{2}\right)$$

$$=1+\frac{\pi}{3}, \text{ when } n \text{ is an even integer.}$$

$$=-1+\frac{\pi}{3}, \text{ when } n \text{ is an odd integer.}$$

i.e. values of $\cos\alpha+\cos^{-1}\beta$ is $\frac{\pi}{3}-1, \frac{\pi}{3}+1$.

141. $f(\theta)=1-(a\cos\theta+b\sin\theta)-(A\cos2\theta+B\sin2\theta)$

$$\Rightarrow \quad f(\theta)=1-\sqrt{a^2+b^2}\cos(\theta-\alpha)-\sqrt{A^2+B^2}\cos(2\theta-\beta)$$

Now, $f\left(\alpha+\frac{\pi}{4}\right)=1-\frac{\sqrt{a^2+b^2}}{\sqrt{2}}-\sqrt{A^2+B^2}\cos\left(\frac{\pi}{2}+2\alpha-\beta\right)$

$$=1-\frac{\sqrt{a^2+b^2}}{\sqrt{2}}+\sqrt{A^2+B^2}\sin(2\alpha-\beta) \quad \ldots(i)$$

and $f\left(\alpha-\frac{\pi}{4}\right)=1-\frac{\sqrt{a^2+b^2}}{\sqrt{2}}-\sqrt{A^2+B^2}\cos\left(2\alpha-\beta-\frac{\pi}{2}\right)$

$$=1-\frac{\sqrt{a^2+b^2}}{\sqrt{2}}-\sqrt{A^2+B^2}\sin(2\alpha-\beta) \quad \ldots(ii)$$

On adding Eqs. (i) and (ii),

$$f\left(\alpha+\frac{\pi}{4}\right)+f\left(\alpha-\frac{\pi}{4}\right)=2-2\frac{\sqrt{a^2+b^2}}{\sqrt{2}}\geq 0$$

$$\Rightarrow \quad \sqrt{a^2+b^2}\leq\sqrt{2}$$

$$\Rightarrow \quad a^2+b^2\leq 2$$

Similarly putting $\theta=\beta$ and $\beta+\pi$. We have,

$$f(\beta)+f(\beta+\pi)=2-2\sqrt{A^2+B^2}\geq 0$$

$$\Rightarrow \quad \sqrt{A^2+B^2}\leq 1 \Rightarrow A^2+B^2\leq 1$$

142. Clearly θ_1, θ_0 are roots of ; $\frac{\cos\theta}{\cos\theta_2}+\frac{\sin\theta}{\sin\theta_2}=1$

$$\Rightarrow \quad \frac{\cos\theta}{\cos\theta_2}=1-\frac{\sin\theta}{\sin\theta_2}\Rightarrow\frac{\cos^2\theta}{\cos^2\theta_2}=1+\frac{\sin^2\theta}{\sin^2\theta_2}-\frac{2\sin\theta}{\sin\theta_2}$$

$$\Rightarrow \sin^2\theta\left(\frac{1}{\sin^2\theta_2}+\frac{1}{\cos^2\theta_2}\right)-\frac{2\sin\theta}{\sin\theta_2}+\left(1-\frac{1}{\cos^2\theta_2}\right)=0$$

The roots of the equation are θ_0 and θ_1.

$$\Rightarrow \quad \sin\theta_0\cdot\sin\theta_1=\frac{1-\frac{1}{\cos^2\theta_2}}{\frac{1}{\sin^2\theta_2}+\frac{1}{\cos^2\theta_2}}$$

$$=(\cos^2\theta_2-1)\cdot\sin^2\theta_2=-\sin^4\theta_2$$

$$\Rightarrow \quad \frac{\sin\theta_0\cdot\sin\theta_1}{\sin^2\theta_2}=-\sin^2\theta_2 \quad \ldots(i)$$

Similarly, taking a quadratic in $\cos\theta$, we get

$$\Rightarrow \quad \frac{\cos\theta_0\cdot\cos\theta_1}{\sin^2\theta_2}=-\cos^2\theta_2 \quad \ldots(ii)$$

On adding Eqs. (i) and (ii), we get

$$\frac{\sin\theta_0\sin\theta_1}{\sin^2\theta_2}+\frac{\cos\theta_0\cos\theta_1}{\cos^2\theta_2}=-1$$

143. Let the given expression be E, then E can be written as,

$$E=\sum_{k=1}^{n-1}{}^nC_k\cos kx\cdot\cos(n+k)x+\sum_{k=1}^{n-1}{}^nC_k\sin(n-k)x.\sin(2n-k)x$$

or $$E=\sum_{k=1}^{n-1}{}^nC_k\cos kx\cos(n+k)x+\sum_{k=1}^{n-1}{}^nC_k\sin(k)x\cdot\sin(n+k)x$$

[replacing k by $(n-k)$ in the second]

Sum and using ${}^nC_k={}^nC_{n-k}$

$$E=\sum_{k=1}^{n-1}{}^nC_k(\cos kx\cos(n+k)x+\sin kx\cdot\sin(n+k)x)$$

$$=\sum_{k=1}^{n-1}{}^nC_k\cos nx$$

$$=\cos nx\{({}^nC_0+{}^nC_1+\ldots+{}^nC_n)-{}^nC_0-{}^nC_n\}$$

$$=\cos nx\{2^n-2\}$$

$$\therefore \quad E=(2^n-2)\cos nx$$

144. It is evident from the inequality that,

$$|\sqrt{1+\sin 2x}-\sqrt{1-\sin 2x}|\le\sqrt{2}\ \forall\ x\in[0,2\pi]$$

as $|\sqrt{1+\sin 2x}-\sqrt{1-\sin 2x}|\le\sqrt{1+\sin 2x}\le\sqrt{2}$

Now,

$2\cos x\le|\sqrt{1+\sin 2x}-\sqrt{1-\sin 2x}|$ holds for all x for which $\cos x\le 0$.

$$\Rightarrow\quad x\le\left[\frac{\pi}{2},\frac{3\pi}{2}\right] \qquad ...(i)$$

Now, if $\cos x>0$

Then, $4\cos^2 x\le 1+\sin 2x+1-\sin 2x-\sqrt{1-\sin^2 2x}$

$$\Rightarrow\quad 2+2\cos 2x\le 2-2|\cos x|$$

$$\Rightarrow\quad |\cos 2x|\le-\cos 2x$$

$$\Rightarrow\quad \cos 2x\le 0$$

$$\Rightarrow\quad x\in\left[\frac{\pi}{4},\frac{3\pi}{4}\right]\cup\left[\frac{5\pi}{4},\frac{7\pi}{4}\right] \qquad ...(ii)$$

Hence, from Eqs. (i) and (ii)

$$x\in\left[\frac{\pi}{4},\frac{7\pi}{4}\right]$$

145. The given equation can be rewritten as, $x^2-3=3\left[\sin\left(x-\frac{\pi}{6}\right)\right]$

Here, right hand side can take only the values –3, 0, 3.

Case I When $x^2-3=-3\ \Rightarrow x=0$

$$\text{At } x=0,\left[\sin\left(x-\frac{\pi}{6}\right)\right]=-1, \text{ so } x=0 \text{ is a solution.}$$

Case II When $x^2-3=0\Rightarrow x=\pm\sqrt{3}$

Now at $x=\sqrt{3}$, $\left[\sin\left(x-\frac{\pi}{6}\right)\right]=0\Rightarrow x=\sqrt{3}$

But at $x=-\sqrt{3}$, $\left[\sin\left(x-\frac{\pi}{6}\right)\right]=-1$, hence $x=-\sqrt{3}$ is not a solution.

Case III When $x^2-3=3\Rightarrow x=\pm\sqrt{6}$

$$\text{But}\left[\sin\left(\pm\sqrt{6}-\frac{\pi}{6}\right)\right]\neq 1\Rightarrow x=\pm\sqrt{6} \text{ is not a solution.}$$

Hence, the given equation has only two solutions $x=0$ and $\sqrt{3}$.

146. $\sum_{r=0}^{n} {}^nC_r a^r b^{n-r}\cos(rB-(n-r)A)$

$$=\text{real part of }\sum_{r=0}^{n} {}^nC_r a^r b^{n-r} e^{i\{rB-(n-r)A\}}$$

Now, $\sum_{r=0}^{n} {}^nC_r a^r b^{n-r} e^{i\{rB-(n-r)A\}}$

$$=\sum_{r=0}^{n} {}^nC_r (ae^{iB})^r (be^{-iA})^{n-r}=(ae^{iB}+be^{-iA})^n$$

$$=(a\cos B+ia\sin B+b\cos A-bi\sin A)^n$$

$$=\{(a\cos B+b\cos A)+i(a\sin B-b\sin A)\}^n$$

$$=\{C+i\cdot 0\}^n=C^n$$

147. Let $z^5+1=0\Rightarrow z^5=-1=(\cos(2r+1)+i\sin(2r+1)\pi$

$$\Rightarrow\quad z=e^{i\left(\frac{2r+1}{5}\right)\pi},\ r=0,1,2,3,4$$

$\Rightarrow$ Roots of $z^k+1=0$ are $e^{i\pi/5}, e^{i3\pi/5}, e^{i\pi}, e^{i7\pi/5}, e^{i9\pi/5}$. Clearly $e^{i7\pi/5}, e^{i9\pi/5}$ and $e^{i3\pi/5}, e^{i7\pi/5}$ are pairwise conjugate.

$$\Rightarrow\quad z^5+1=(z-e^{i\pi})(z-e^{i\pi/5})(z-e^{-i\pi/5})(z-e^{-i3\pi/5})(z-e^{i7\pi/5})$$

$$\Rightarrow\quad z^5+1=(z+1)\left(z^2-2z\cos\frac{\pi}{5}+1\right)\left(z^2-2z\cos\frac{3\pi}{5}+1\right) \qquad ...(i)$$

It is required factorisation of z^5+1.

Now, $$\frac{z^5+1}{z+1}=1-z+z^2-z^2+z^4 \qquad ...(ii)$$

$$\Rightarrow\quad 1-z+z^2-z^3+z^4=\left(z^2-2z\cos\frac{\pi}{5}+1\right)\left(z^2-2z\cos\frac{3\pi}{5}+1\right)$$

[using Eqs. (i) and (ii)}

On dividing both side by z^2

$$\left(z^2+\frac{1}{z^2}\right)-\left(z+\frac{1}{z}\right)+1=\left(z+\frac{1}{z}-2\cos\frac{\pi}{5}\right)\left(z+\frac{1}{z}-2\cos\frac{3\pi}{5}\right)$$

Let $z=e^{i\theta}$

$$\Rightarrow\quad z^2+\frac{1}{z^2}=2\cos 2\theta,\ z+\frac{1}{z}=2\cos\theta$$

$$\Rightarrow\quad 2\cos 2\theta-2\cos\theta+1=4\left(\cos\theta-\cos\frac{\pi}{5}\right)\left(\cos\theta-\cos\frac{3\pi}{5}\right)$$

Putting $\theta=0$, we get

$$\frac{1}{4}=\left(1-\cos\frac{\pi}{5}\right)\left(1-\cos\frac{3\pi}{5}\right)$$

$$\Rightarrow\quad \frac{1}{4}=2\sin^2\frac{\pi}{10}\cdot 2\sin^2\frac{3\pi}{10}$$

$$\Rightarrow\quad \sin\frac{\pi}{10}\cdot\sin\frac{3\pi}{10}=\frac{1}{4}$$

$$\Rightarrow\quad 4\sin\frac{\pi}{10}\cdot\cos\left(\frac{\pi}{2}-\frac{3\pi}{10}\right)=1$$

$$\Rightarrow\quad 4\sin\frac{\pi}{10}\cdot\cos\frac{\pi}{5}=1$$

148. Let $\theta=\frac{(2n+1)\pi}{7}$, where $n=0,1,2,3,4,5,6$.

Then, $4\theta=(2n+1)\pi-3\theta$

$$\Rightarrow\quad \cos 4\theta=-\cos 3\theta$$

$$\Rightarrow\quad 2\cos^2 2\theta-1=-(4\cos^3\theta-3\cos\theta)$$

$$\Rightarrow\quad 2(2\cos^2\theta-1)^2-1=-4\cos^3\theta+3\cos\theta$$

$$\Rightarrow\quad 2(2x^2-1)^2-1=-4x^3+3x \qquad [\text{put } x=\cos\theta]$$

$$\Rightarrow\quad 8x^4+4x^3-8x^2-3x+1=0$$

$$\Rightarrow\quad (x+1)(8x^3-4x^2-4x+1)=0$$

The roots of this equation are,

$$\cos\frac{\pi}{7},\cos\frac{3\pi}{7},\cos\frac{5\pi}{7},\cos\frac{7\pi}{7},\cos\frac{9\pi}{7},\cos\frac{11\pi}{7},\cos\frac{13\pi}{7}$$

$\therefore$ The roots of $8x^3-4x^2-4x+1=0$

are $$\cos\frac{\pi}{7},\cos\frac{3\pi}{7},\cos\frac{5\pi}{7} \qquad ...(i)$$

Put $x = \frac{1}{y}$ in Eq. (i) (i.e. $y = \sec\theta$), then

$\sec\frac{\pi}{7}, \sec\frac{3\pi}{7}, \sec\frac{5\pi}{7}$ are the roots of the equation.

$$\frac{8}{y^3} - \frac{4}{y^2} - \frac{4}{y} + 1 = 0$$

$$\Rightarrow \quad y^3 - 4y^2 - 4y + 8 = 0$$

$$\Rightarrow \quad \sec\frac{\pi}{7} + \sec\frac{3\pi}{7} + \sec\frac{5\pi}{7} = 4$$

Again putting $\frac{1}{x^2} = y$ in Eq. (i)

(i.e. $y = \sec^2\theta$)

$$\frac{8}{y^{3/2}} - \frac{4}{y} - \frac{4}{y^{1/2}} + 1 = 0$$

$$\Rightarrow \quad 8 - 4y^{1/2} - 4y + y^{3/2} = 0$$

$$\Rightarrow \quad y^{1/2}(y-4) = 4(y-2)$$

$$\Rightarrow \quad y(y-4)^2 = 16(y-2)^2$$

$$\Rightarrow \quad y^3 - 24y^2 + 80y - 64 = 0 \qquad \ldots(ii)$$

Hence, the roots are

$$\sec^2\frac{\pi}{7}, \sec^2\frac{3\pi}{7}, \sec^2\frac{5\pi}{7}$$

Now, putting $y = 1 + z$, (i.e. $z = \tan^2\theta$)

We have,

$$(1+z)^3 - 24(1+z)^2 + 80(1+z) - 64 = 0$$

$$\Rightarrow \quad z^3 - 21z^2 + 35z - 7 = 0$$

whose roots are $\tan^2\frac{\pi}{7}, \tan^2\frac{3\pi}{7}, \tan^2\frac{5\pi}{7}$.

149. We have, $2(\cos\beta - \cos\alpha) + \cos\alpha\cos\beta = 1$

or $4(\cos\beta - \cos\alpha) + 2\cos\alpha\cos\beta = 2$

$$\Rightarrow \quad 1 - \cos\alpha + \cos\beta - \cos\alpha\cos\beta$$
$$= 3 + 3\cos\alpha - 3\cos\beta - 3\cos\alpha\cos\beta$$

$$\Rightarrow \quad (1-\cos\alpha)(1+\cos\beta) = 3(1+\cos\alpha)(1-\cos\beta)$$

$$\Rightarrow \quad \frac{(1-\cos\alpha)}{(1+\cos\alpha)} = \frac{3(1-\cos\beta)}{1+\cos\beta}$$

$$\Rightarrow \quad \tan^2\frac{\alpha}{2} = 3\tan^2\frac{\beta}{2}$$

$$\therefore \quad \tan\frac{\alpha}{2} \pm \sqrt{3}\tan\frac{\beta}{2} = 0$$

150. Here, $x^2 - 2x\sec\theta + 1 = 0$ has roots α_1 and β_1.

$$\therefore \quad \alpha_1, \beta_1 = \frac{2\sec\theta \pm \sqrt{4\sec^2\theta - 4}}{2\times 1} = \frac{2\sec\theta \pm 2|\tan\theta|}{2}$$

Since, $\theta \in \left(-\frac{\pi}{6}, -\frac{\pi}{12}\right)$,

i.e. $\theta \in$ IV quadrant $= \frac{2\sec\theta \mp 2\tan\theta}{2}$

$\therefore \quad \alpha_1 = \sec\theta - \tan\theta$ and $\beta_1 = \sec\theta + \tan\theta$

$[\text{as } \alpha_1 > \beta_1]$

and $x^2 + 2x\tan\theta - 1 = 0$ has roots α_2 and β_2.

i.e. $$\alpha_2, \beta_2 = \frac{-2\tan\theta \pm \sqrt{4\tan^2\theta + 4}}{2}$$

$\therefore \quad \alpha_2 = -\tan\theta + \sec\theta$

and $\beta_2 = -\tan\theta - \sec\theta \qquad [\text{as } \alpha_2 > \beta_2]$

Thus, $\alpha_1 + \beta_2 = -2\tan\theta$

151. Here, $\sum_{k=1}^{13} \frac{1}{\sin\left\{\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right\}\sin\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)}$

Converting into differences, by multiplying and dividing by $\sin\left[\left(\frac{\pi}{4} + \frac{k\pi}{6}\right) - \left\{\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right\}\right]$, i.e. $\sin\left(\frac{\pi}{6}\right)$.

$$\therefore \sum_{k=1}^{13} \frac{\sin\left[\left(\frac{\pi}{4} + k\frac{\pi}{6}\right) - \left\{\frac{\pi}{4} + (k-1)\frac{\pi}{6}\right\}\right]}{\sin\frac{\pi}{6}\left\{\sin\left\{\frac{\pi}{4} + (k-1)\frac{\pi}{6}\right\}\sin\left(\frac{\pi}{4} + k\frac{\pi}{6}\right)\right\}}$$

$$= 2\sum_{k=1}^{13} \frac{\left[\begin{array}{l}\sin\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)\cos\left\{\frac{\pi}{4} + (k-1)\frac{\pi}{6}\right\} \\ -\sin\left\{\frac{\pi}{4} + (k-1)\frac{\pi}{6}\right\}\cos\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)\end{array}\right]}{\sin\left\{\frac{\pi}{4} + (k-1)\frac{\pi}{6}\right\}\sin\left(\frac{\pi}{4} + k\frac{\pi}{6}\right)}$$

$$= 2\sum_{k=1}^{13}\left[\cot\left\{\frac{\pi}{4} + (k-1)\frac{\pi}{6}\right\} - \cot\left(\frac{\pi}{4} + k\frac{\pi}{6}\right)\right]$$

$$= 2\left[\left\{\cot\left(\frac{\pi}{4}\right) - \cot\left(\frac{\pi}{4} + \frac{\pi}{6}\right)\right\} + \left\{\cot\left(\frac{\pi}{4} + \frac{\pi}{6}\right) - \cot\left(\frac{\pi}{4} + \frac{2\pi}{6}\right)\right\} + \ldots + \left\{\cot\left(\frac{\pi}{4} + 12\frac{\pi}{6}\right) - \cot\left(\frac{\pi}{4} + 13\frac{\pi}{6}\right)\right\}\right]$$

$$= 2\left\{\cot\frac{\pi}{4} - \cot\left(\frac{\pi}{4} + 13\frac{\pi}{6}\right)\right\}$$

$$= 2\left[1 - \cot\left(\frac{29\pi}{12}\right)\right] = 2\left[1 - \cot\left(2\pi + \frac{5\pi}{12}\right)\right]$$

$$= 2\left[1 - \cot\frac{5\pi}{12}\right] \qquad \left[\because \cot\frac{5\pi}{12} = (2-\sqrt{3})\right]$$

$$= 2(1 - 2 + \sqrt{3})$$

$$= 2(\sqrt{3} - 1)$$

152. $f(\cos 4\theta) = \frac{2}{2 - \sec^2\theta} \qquad \ldots(i)$

At $\cos 4\theta = \frac{1}{3} \Rightarrow 2\cos^2 2\theta - 1 = \frac{1}{3}$

$$\Rightarrow \quad \cos^2 2\theta = \frac{2}{3} \Rightarrow \cos 2\theta = \pm\sqrt{\frac{2}{3}} \qquad \ldots(ii)$$

$$\therefore \quad f(\cos 4\theta) = \frac{2\cdot\cos^2\theta}{2\cos^2\theta - 1} = \frac{1 + \cos 2\theta}{\cos 2\theta}$$

$$\Rightarrow \quad f\left(\frac{1}{3}\right) = 1 \pm \sqrt{\frac{3}{2}} \qquad [\text{from Eq. (ii)}]$$

153. Given equations can be written as

$$x\sin3\theta-\frac{\cos3\theta}{y}-\frac{\cos3\theta}{z}=0 \quad ...(i)$$

$$x\sin3\theta-\frac{2\cos3\theta}{y}-\frac{2\sin3\theta}{z}=0 \quad ...(ii)$$

and $x\sin3\theta-\frac{2}{y}\cos3\theta-\frac{1}{z}(\cos3\theta+\sin3\theta)=0$ ...(iii)

Eqs. (ii) and (iii), implies

$2\sin3\theta=\cos3\theta+\sin3\theta \Rightarrow \sin3\theta=\cos3\theta$

$\therefore \quad \tan3\theta=1$

$\Rightarrow \quad 3\theta=\frac{\pi}{4},\frac{5\pi}{4},\frac{9\pi}{4}$ or $\theta=\frac{\pi}{12},\frac{5\pi}{12},\frac{9\pi}{12}$

154. For $0<\theta<\frac{\pi}{2}$

$$\sum_{m=1}^{6}\operatorname{cosec}\left(\theta+\frac{(m-1)\pi}{4}\right)\operatorname{cosec}\left(\theta+\frac{m\pi}{4}\right)=4\sqrt{2}$$

$$\Rightarrow \quad \sum_{m=1}^{6}\frac{1}{\sin\left(\theta+\frac{(m-1)\pi}{4}\right)\sin\left(\theta+\frac{m\pi}{4}\right)}=4\sqrt{2}$$

$$\Rightarrow \quad \sum_{m=1}^{6}\frac{\sin\left[\theta+\frac{m\pi}{4}-\left(\theta+\frac{(m-1)\pi}{4}\right)\right]}{\sin\frac{\pi}{4}\left\{\sin\left(\theta+\frac{(m-1)\pi}{4}\right)\sin\left(\theta+\frac{m\pi}{4}\right)\right\}}=4\sqrt{2}$$

$$\Rightarrow \quad \sum_{m=1}^{6}\frac{\cot\left(\theta+\frac{(m-1)\pi}{4}\right)-\cot\left(\theta+\frac{m\pi}{4}\right)}{1/\sqrt{2}}=4\sqrt{2}$$

$$\Rightarrow \quad \sum_{m=1}^{6}\left[\cot\left(\theta+\frac{(m-1)\pi}{4}\right)-\cot\left(\theta+\frac{m\pi}{4}\right)\right]=4$$

$$\Rightarrow \quad \cot(\theta)-\cot\left(\theta+\frac{\pi}{4}\right)+\cot\left(\theta+\frac{\pi}{4}\right)-\cot\left(\theta+\frac{2\pi}{4}\right)$$

$$+\ldots+\cot\left(\theta+\frac{5\pi}{4}\right)-\cot\left(\theta+\frac{6\pi}{4}\right)=4$$

$$\Rightarrow \quad \cot\theta-\cot\left(\frac{3\pi}{2}+\theta\right)=4$$

$$\Rightarrow \quad \cot\theta+\tan\theta=4$$

$$\Rightarrow \quad \tan^2\theta-4\tan\theta+1=0$$

$$\Rightarrow \quad (\tan\theta-2)^2-3=0$$

$$\Rightarrow \quad (\tan\theta-2+\sqrt{3})(\tan\theta-2-\sqrt{3})=0$$

$$\Rightarrow \quad \tan\theta=2-\sqrt{3}$$

or $\quad \tan\theta=2+\sqrt{3}$

$$\Rightarrow \quad \theta=\frac{\pi}{12};\ \theta=\frac{5\pi}{12} \quad \left[\because\ \theta\in\left(0,\frac{\pi}{2}\right)\right]$$

155. $\frac{\sin^4 x}{2}+\frac{\cos^4 x}{3}=\frac{1}{5}$

$$\Rightarrow \quad \frac{\sin^4 x}{2}+\frac{(1-\sin^2 x)^2}{3}=\frac{1}{5}$$

$$\Rightarrow \quad \frac{\sin^4 x}{2}+\frac{1+\sin^4 x-2\sin^2 x}{3}=\frac{1}{5}$$

$$\Rightarrow \quad 5\sin^4 x-4\sin^2 x+2=\frac{6}{5}$$

$$\Rightarrow \quad 25\sin^4 x-20\sin^2 x+4=0$$

$$\Rightarrow \quad (5\sin^2 x-2)^2=0$$

$$\Rightarrow \quad \sin^2 x=\frac{2}{5}$$

$$\cos^2 x=\frac{3}{5},\ \tan^2 x=\frac{2}{3}$$

$\therefore \quad \frac{\sin^8 x}{8}+\frac{\cos^8 x}{27}=\frac{1}{125}$

156. As when $\theta\in\left(0,\frac{\pi}{4}\right)$, $\tan\theta<\cot\theta$

Since, $\quad \tan\theta<1$ and $\cot\theta>1$

$\therefore \quad (\tan\theta)^{\cot\theta}<1$ and $(\cot\theta)^{\tan\theta}>1$

$\therefore\ t_4>t_1$ which only holds in (b).

Therefore, (b) is the answer.

157. Since, $\quad \cos(\alpha-\beta)=1$

$\Rightarrow \quad \alpha-\beta=2n\pi$

But $\quad -2\pi<\alpha-\beta<2\pi \quad [\text{as } \alpha,\beta\in(-\pi,\pi)]$

$\therefore \quad \alpha-\beta=0$ ...(i)

Given, $\quad \cos(\alpha+\beta)=\frac{1}{e}$

$\Rightarrow \quad \cos2\alpha=\frac{1}{e}<1$, which is true for four values of α.

$[\text{as } -2\pi<2\alpha<2\pi]$

158. Given, $5(\tan^2 x-\cos^2 x)=2\cos2x+9$

$$\Rightarrow \quad 5\left(\frac{1-\cos2x}{1+\cos2x}-\frac{1+\cos2x}{2}\right)=2\cos2x+9$$

Put $\cos2x=y$, we have

$$5\left(\frac{1-y}{1+y}-\frac{1+y}{2}\right)=2y+9$$

$$\Rightarrow \quad 5(2-2y-1-y^2-2y)=2(1+y)(2y+9)$$

$$\Rightarrow \quad 5(1-4y-y^2)=2(2y+9+2y^2+9y)$$

$$\Rightarrow \quad 5-20y-5y^2=22y+18+4y^2$$

$$\Rightarrow \quad 9y^2+42y+13=0$$

$$\Rightarrow \quad 9y^2+3y+39y+13=0$$

$$\Rightarrow \quad 3y(3y+1)+13(3y+1)=0$$

$$\Rightarrow \quad (3y+1)(3y+13)=0$$

$$\Rightarrow \quad y=-\frac{1}{3},-\frac{13}{3}$$

$\therefore \quad \cos2x=-\frac{1}{3},-\frac{13}{3}$

$$\Rightarrow \quad \cos2x=-\frac{1}{3} \quad \left[\because\ \cos2x\neq-\frac{13}{3}\right]$$

Now, $\quad \cos4x=2\cos^2 2x-1=2\left(-\frac{1}{3}\right)^2-1$

$$=\frac{2}{9}-1=-\frac{7}{9}$$

159. $f_k(x) = \frac{1}{k}(\sin^k x + \cos^k x)$, where $x \in R$ and $k \geq 1$

Now, $f_4(x) - f_6(x)$

$$= \frac{1}{4}(\sin^4 x + \cos^4 x) - \frac{1}{6}(\sin^6 x + \cos^6 x)$$

$$= \frac{1}{4}(1 - 2\sin^2 x \cdot \cos^2 x) - \frac{1}{6}(1 - 3\sin^2 x \cdot \cos^2 x)$$

$$= \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$$

160. Given expression is

$$\frac{\tan A}{1 - \cot A} + \frac{\cot A}{1 - \tan A} = \frac{\sin A}{\cos A} \times \frac{\sin A}{\sin A - \cos A} + \frac{\cos A}{\sin A} \times \frac{\cos A}{\cos A - \sin A}$$

$$= \frac{1}{\sin A - \cos A}\left\{\frac{\sin^3 A - \cos^3 A}{\cos A \sin A}\right\}$$

$$= \frac{\sin^2 A + \sin A \cos A + \cos^2 A}{\sin A \cos A}$$

$$= \frac{1 + \sin A \cos A}{\sin A \cos A}$$

$$= 1 + \sec A \operatorname{cosec} A$$

161. **Given** A ΔPQR such that

$$3\sin P + 4\cos Q = 6 \quad \ldots(i)$$

$$4\sin Q + 3\cos P = 1 \quad \ldots(ii)$$

On squaring and adding Eqs. (i) and (ii) both sides we get

$$(3\sin P + 4\cos Q)^2 + (4\sin Q + 3\cos P)^2 = 36 + 1$$

$$\Rightarrow 9(\sin^2 P + \cos^2 P) + 16(\sin^2 Q + \cos^2 Q) + 2 \times 3 \times 4(\sin P \cos Q + \sin Q \cos P) = 37$$

$$\Rightarrow 24[\sin(P + Q)] = 37 - 25$$

$$\Rightarrow \sin(P + Q) = \frac{1}{2}$$

Since, P and Q are angles of ΔPQR, hence $0° < P, Q < 180°$.

$$\Rightarrow P + Q = 30° \text{ or } 150°$$

$$\Rightarrow R = 150° \text{ or } 30°$$

Hence, two cases arise here.

***Case* I** $R = 150°$

$R = 150° \Rightarrow P + Q = 30°$

$$\Rightarrow 0 < P,\ Q < 30°$$

$$\Rightarrow \sin P < \frac{1}{2}, \cos Q < 1$$

$$\Rightarrow 3\sin P + 4\cos Q < \frac{3}{2} + 4$$

$$\Rightarrow 3\sin P + 4\cos Q < \frac{11}{2} < 6$$

$\Rightarrow 3\sin P + 4\cos Q \Rightarrow 6$ is not possible.

***Case* II** $R = 30°$

Hence, $R = 30°$ is the only possibility.

162. $A = \sin^2 x + \cos^4 x$

$$\Rightarrow A = 1 - \cos^2 x + \cos^4 x$$

$$= \cos^4 x - \cos^2 x + \frac{1}{4} + \frac{3}{4}$$

$$= \left(\cos^2 x - \frac{1}{2}\right)^2 + \frac{3}{4} \quad \ldots(i)$$

where, $0 \leq \left(\cos^2 x - \frac{1}{2}\right)^2 \leq \frac{1}{4}$ $\ldots(ii)$

$$\therefore \frac{3}{4} \leq A \leq 1$$

163. $\cos(\alpha + \beta) = \frac{4}{5} \Rightarrow \alpha + \beta \in$ Ist quadrant

and $\sin(\alpha - \beta) = \frac{5}{13} \Rightarrow \alpha - \beta \in$ Ist quadrant

Now, $2\alpha = (\alpha + \beta) + (\alpha - \beta)$

$$\therefore \tan 2\alpha = \frac{\tan(\alpha + \beta) + \tan(\alpha - \beta)}{1 - \tan(\alpha + \beta)\tan(\alpha - \beta)} = \frac{\frac{3}{4} + \frac{5}{12}}{1 - \frac{3}{4} \cdot \frac{5}{12}} = \frac{56}{33}$$

164. $\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = -\frac{3}{2}$

$$\Rightarrow 2[\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta)] + 3 = 0$$

$$\Rightarrow 2[\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta)] + \sin^2\alpha + \cos^2\alpha + \sin^2\beta + \cos^2\beta + \sin^2\gamma + \cos^2\gamma = 0$$

$$\Rightarrow (\sin\alpha + \sin\beta + \sin\gamma)^2 + (\cos\alpha + \cos\beta + \cos\gamma)^2 = 0$$

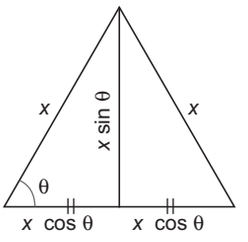
It is possible when,

$$\sin\alpha + \sin\beta + \sin\gamma = 0$$

and $$\cos\alpha + \cos\beta + \cos\gamma = 0$$

Hence, both statements A and B are true.

165. Area $= \frac{1}{2} \times$ Base $\times$ Altitude



$$= \frac{1}{2} \times (2x\cos\theta) \times (x\sin\theta) = \frac{1}{2}x^2 \sin 2\theta$$

[since, maximum value of $\sin 2\theta$ is 1]

$\therefore$ Maximum area $= \frac{1}{2}x^2$

166. Given, $\cos x + \sin x = \frac{1}{2}$

$$\therefore \frac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}} + \frac{2\tan\frac{x}{2}}{1 + \tan^2\frac{x}{2}} = \frac{1}{2}$$

Let $\tan\frac{x}{2} = t \Rightarrow \frac{1 - t^2}{1 + t^2} + \frac{2t}{1 + t^2} = \frac{1}{2}$

$$\Rightarrow \quad 2(1-t^2+2t) = 1+t^2 \Rightarrow 3t^2-4t-1=0$$

$$\Rightarrow \quad t = \frac{2\pm\sqrt{7}}{3}$$

As $\quad 0 < x < \pi \Rightarrow 0 < \frac{x}{2} < \frac{\pi}{2}$

So, $\tan\frac{x}{2}$ is positive.

$$\therefore \quad t = \tan\frac{x}{2} = \frac{2+\sqrt{7}}{3}$$

Now, $\quad \tan x = \dfrac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}} = \dfrac{2t}{1-t^2}$

$$\Rightarrow \quad \tan x = \frac{2\left(\frac{2+\sqrt{7}}{3}\right)}{1-\left(\frac{2+\sqrt{7}}{3}\right)^2}$$

$$\Rightarrow \quad \tan x = \frac{-3(2+\sqrt{7})}{1+2\sqrt{7}} \times \frac{1-2\sqrt{7}}{1-2\sqrt{7}}$$

$$\Rightarrow \quad \tan x = -\left(\frac{4+\sqrt{7}}{3}\right)$$

167. Since, $\tan\frac{P}{2}$ and $\tan\frac{Q}{2}$ are the roots of equation

$$ax^2+bx+c=0$$

$$\therefore \quad \tan\frac{P}{2}+\tan\frac{Q}{2} = -\frac{b}{a} \qquad \ldots(i)$$

and $\quad \tan\frac{P}{2}\tan\frac{Q}{2} = \frac{c}{a}$

Also, $\quad \frac{P}{2}+\frac{Q}{2}+\frac{R}{2} = \frac{\pi}{2} \qquad [\because P+Q+R=\pi]$

$$\Rightarrow \quad \frac{P+Q}{2} = \frac{\pi}{2} - \frac{R}{2}$$

$$\Rightarrow \quad \frac{P+Q}{2} = \frac{\pi}{4} \qquad \left[\because \angle R = \frac{\pi}{2} \text{ (given)}\right]$$

$$\Rightarrow \quad \tan\left(\frac{P}{2}+\frac{Q}{2}\right) = 1$$

$$\Rightarrow \quad \frac{\tan\frac{P}{2}+\tan\frac{Q}{2}}{1-\tan\frac{P}{2}\tan\frac{Q}{2}} = 1$$

$$\Rightarrow \quad \frac{-\frac{b}{a}}{1-\frac{c}{a}} = 1$$

$$\Rightarrow \quad -\frac{b}{a} = 1-\frac{c}{a} \qquad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad -b = a-c$$

$$\Rightarrow \quad c = a+b$$

Alternate Solution

Since, $\quad \angle R = \frac{\pi}{2}$

$$\Rightarrow \quad \angle P + \angle Q = \frac{\pi}{2}$$

$$\Rightarrow \quad \frac{\angle P}{2} = \frac{\pi}{4} - \frac{\angle Q}{2}$$

$$\therefore \quad \tan\frac{P}{2} = \tan\left(\frac{\pi}{4}-\frac{Q}{2}\right)$$

$$= \frac{\tan\frac{\pi}{4}-\tan\frac{Q}{2}}{1+\tan\frac{\pi}{4}\tan\frac{Q}{2}}$$

$$\Rightarrow \quad \tan\frac{P}{2}+\tan\frac{P}{2}\tan\frac{Q}{2} = 1-\tan\frac{Q}{2}$$

$$\Rightarrow \quad \tan\frac{P}{2}+\tan\frac{Q}{2} = 1-\tan\frac{P}{2}\tan\frac{Q}{2}$$

$$\Rightarrow \quad -\frac{b}{a} = 1-\frac{c}{a}$$

$$\Rightarrow \quad -b = a-c$$

$$\Rightarrow \quad c = a+b$$

CHAPTER

02

Trigonometric Equations and Inequations

Learning Part

Session 1

- Trigonometric Equations
- General Solution
- Principal Solution

Session 2

- Equation of the Form $a\cos\theta + b\sin\theta = c$
- Some Particular Equations

Session 3

- Solution of Simultaneous Trigonometric Equations
- Problems Based on Extreme Values of sin x and cos x

Session 4

- Trigonometric Inequality

Practice Part

- JEE Type Examples
- Chapter Exercises

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Session 1

Trigonometric Equations, Principal Solution and General Solution

Trigonometric Equations

The equations involving trigonometric functions of unknown angles are known as Trigonometric equations.

e.g. $\cos\theta = 0, \cos^2\theta - 4\cos\theta = 1,$

$\sin^2\theta + \sin\theta = 2, \cos^2\theta - 4\sin\theta = 1.$

A solution of a trigonometric equation is the value of the unknown angle that satisfies the equation.

e.g. $\sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$

or $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}, \ldots$

Thus, the trigonometric equation may have infinite number of solutions and can be classified as

(i) Principal solution (ii) General solution

Principal Solution

The solutions lying in the interval $[0, 2\pi]$ are called principal solutions.

General Solution

Since, trigonometric functions are periodic, a solution generalised by means of periodicity of the trigonometrical functions. The solution consisting of all possible solutions of a trigonometric equation is called its general solution.

We use following results for solving the trigonometric equations.

Result 1 $\sin\theta = 0 \Leftrightarrow \theta = n\pi, n \in I.$

We know that $\sin\theta = 0$ for all integral multiples of π.

$\therefore \quad \sin\theta = 0 \Leftrightarrow \theta = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

$\Rightarrow \quad \theta = n\pi, n \in I$

Thus, $\sin\theta = 0$

$\Leftrightarrow \quad \theta = n\pi, n \in I.$

Result 2 $\cos\theta = 0 \Leftrightarrow \theta = (2n+1)\frac{\pi}{2}, n \in I.$

We know that $\cos\theta = 0$ for all odd multiples of $\frac{\pi}{2}$.

$\therefore \quad \cos\theta = 0 \Leftrightarrow \theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

$\Rightarrow \quad \theta = (2n+1)\frac{\pi}{2}, n \in I$

Thus, $\cos\theta = 0$

$\Leftrightarrow \quad \theta = (2n+1)\frac{\pi}{2}, n \in I$

Result 3 $\tan\theta = 0 \Leftrightarrow \theta = n\pi, n \in I.$

We know that $\tan\theta = 0$ for all integral multiple of π.

$\therefore \quad \tan\theta = 0 \Leftrightarrow \theta = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

$\Rightarrow \quad \theta = n\pi, n \in I$

Thus, $\tan\theta = 0 \quad \Leftrightarrow \quad \theta = n\pi, n \in I$

Result 4 $\sin\theta = \sin\alpha \Leftrightarrow \theta = n\pi + (-1)^n \alpha$, where $n \in I$ and $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

We have, $\sin\theta = \sin\alpha$, where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Now, $\sin\theta - \sin\alpha = 0$

$\Leftrightarrow \quad 2\cos\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right) = 0$

$\Leftrightarrow \quad \cos\left(\frac{\theta+\alpha}{2}\right) = 0 \quad \text{or} \quad \sin\left(\frac{\theta-\alpha}{2}\right) = 0$

$\Leftrightarrow \quad \left(\frac{\theta+\alpha}{2}\right) = (2m+1)\frac{\pi}{2}, m \in I$

or $\left(\frac{\theta-\alpha}{2}\right) = m\pi, m \in I$

$\Leftrightarrow \quad (\theta+\alpha) = (2m+1)\pi, m \in I$

or $(\theta-\alpha) = 2m\pi, m \in I$

$\Leftrightarrow \quad \theta = (2m+1)\pi - \alpha, m \in I$

or $\theta = (2m\pi) + \alpha, m \in I$

$\Leftrightarrow \quad \theta = (\text{any odd multiple of } \pi) - \alpha$

or $\theta = (\text{any even multiple of } \pi) + \alpha$

$\Leftrightarrow \quad \theta = n\pi + (-1)^n \alpha$, where $n \in I$

Thus, $\sin\theta = \sin\alpha$

$\Leftrightarrow \quad \theta = n\pi + (-1)^n \alpha$, where $n \in I$ and $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Note $\sin\theta = 1 \Leftrightarrow \sin\theta = \sin\frac{\pi}{2} \Leftrightarrow \theta = n\pi + (-1)^n \frac{\pi}{2}$

$\Rightarrow \quad \theta = (4n+1)\frac{\pi}{2}, n \in N.$

Result 5 $\cos\theta = \cos\alpha \Leftrightarrow \theta = 2n\pi \pm \alpha, n \in I$ and $\alpha \in [0, \pi]$.

We have,

$$\cos\theta = \cos\alpha, \qquad \text{where } \alpha \in [0, \pi]$$

Now, $\cos\theta - \cos\alpha = 0$

$\Leftrightarrow \quad -2\sin\left(\frac{\theta+\alpha}{2}\right)\cdot\sin\left(\frac{\theta-\alpha}{2}\right) = 0$

$\Leftrightarrow \quad \sin\left(\frac{\theta+\alpha}{2}\right) = 0 \text{ or } \sin\left(\frac{\theta-\alpha}{2}\right) = 0$

$\Leftrightarrow \quad \frac{\theta+\alpha}{2} = n\pi \text{ or } \frac{\theta-\alpha}{2} = n\pi, n \in I$

$\Leftrightarrow \quad \theta + \alpha = 2n\pi \text{ or } \theta - \alpha = 2n\pi, n \in I$

$\Leftrightarrow \quad \theta = 2n\pi - \alpha \text{ or } \theta = 2n\pi + \alpha, n \in I$

$\Leftrightarrow \quad \theta = 2n\pi \pm \alpha, n \in I$

Thus, $\cos\theta = \cos\alpha$

$\Leftrightarrow \quad \theta = 2n\pi \pm \alpha, n \in I, \text{ where } \alpha \in [0, \pi]$

Note

(i) $\cos\theta = 1 \Leftrightarrow \cos\theta = \cos 0 \Leftrightarrow \theta = 2n\pi \pm 0 \Rightarrow \theta = 2n\pi$

(ii) $\cos\theta = -1 \Leftrightarrow \cos\theta = \cos\pi \Leftrightarrow \theta = 2n\pi \pm \pi$
$\Leftrightarrow \theta = (2n \pm 1)\pi \Rightarrow \theta = (2n+1)\pi$

(iii) $\sin\theta = \sin\alpha \text{ and } \cos\theta = \cos\alpha \Leftrightarrow \sin\left(\frac{\theta-\alpha}{2}\right) = 0$

$\Rightarrow \frac{\theta-\alpha}{2} = n\pi \Rightarrow \theta = 2n\pi + \alpha$

Result 6 $\tan\theta = \tan\alpha \Leftrightarrow \theta = n\pi + \alpha, n \in I$ where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

We have, $\tan\theta = \tan\alpha$, where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\Leftrightarrow \quad \frac{\sin\theta}{\cos\theta} = \frac{\sin\alpha}{\cos\alpha}$

$\Leftrightarrow \sin\theta\cos\alpha - \cos\theta\sin\alpha = 0$

$\Leftrightarrow \quad \sin(\theta - \alpha) = 0$

$\Leftrightarrow \quad \theta - \alpha = n\pi, n \in I$

$\Leftrightarrow \quad \theta = n\pi + \alpha, n \in I$

Thus, $\tan\theta = \tan\alpha$

$\Leftrightarrow \quad \theta = n\pi + \alpha \text{ where } \alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Result 7 $\sin^2\theta = \sin^2\alpha, \cos^2\theta = \cos^2\alpha, \tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha$

(i) $\sin^2\theta = \sin^2\alpha$

$\Leftrightarrow \quad \frac{1-\cos 2\theta}{2} = \frac{1-\cos 2\alpha}{2}$

$\Leftrightarrow \quad \cos 2\theta = \cos 2\alpha$

$\Leftrightarrow \quad 2\theta = 2n\pi \pm 2\alpha, n \in I$

$\Leftrightarrow \quad \theta = n\pi \pm \alpha, n \in I$

(ii) $\cos^2\theta = \cos^2\alpha$

$\Leftrightarrow \quad \frac{1+\cos 2\theta}{2} = \frac{1+\cos 2\alpha}{2}$

$\Leftrightarrow \quad \cos 2\theta = \cos 2\alpha$

$\Leftrightarrow \quad 2\theta = 2n\pi \pm 2\alpha, n \in I$

$\Leftrightarrow \quad \theta = n\pi \pm \alpha, n \in I$

(iii) $\tan^2\theta = \tan^2\alpha$

$\Leftrightarrow \quad \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}$

[applying componendo and dividendo]

$\Leftrightarrow \quad \cos 2\theta = \cos 2\alpha$

$\Leftrightarrow \quad 2\theta = 2n\pi \pm 2\alpha, n \in I$

$\Leftrightarrow \quad \theta = n\pi \pm \alpha, n \in I$

Summary of Above Results

1. $\sin\theta = 0 \Leftrightarrow \theta = n\pi$
2. $\cos\theta = 0 \Leftrightarrow \theta = (2n+1)\frac{\pi}{2}$
3. $\tan\theta = 0 \Leftrightarrow \theta = n\pi$
4. $\sin\theta = \sin\alpha \Leftrightarrow \theta = n\pi + (-1)^n \alpha, \text{ where } \alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
5. $\cos\theta = \cos\alpha \Leftrightarrow \theta = 2n\pi \pm \alpha, \text{ where } \alpha \in [0, \pi]$
6. $\tan\theta = \tan\alpha \Leftrightarrow \theta = n\pi + \alpha, \text{ where } \alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
7. $\sin^2\theta = \sin^2\alpha, \cos^2\theta = \cos^2\alpha, \tan^2\theta = \tan^2\alpha \Leftrightarrow \theta = n\pi \pm \alpha$
8. $\sin\theta = 1 \Leftrightarrow \theta = (4n+1)\frac{\pi}{2}$
9. $\cos\theta = 1 \Leftrightarrow \theta = 2n\pi$
10. $\cos\theta = -1 \Leftrightarrow \theta = (2n+1)\pi$
11. $\sin\theta = \sin\alpha \text{ and } \cos\theta = \cos\alpha \Leftrightarrow \theta = 2n\pi + \alpha$

Note

(i) In this chapter 'n' is taken as an integer, if not stated otherwise.

(ii) The general solution should be given unless the solution is required in a **specified interval or range.**

Example 1. If $\sin\alpha$, 1, $\cos 2\alpha$ are in GP, then find the general solution for α.

Sol. Since, $\sin\alpha$, 1, $\cos 2\alpha$ are in GP.

$$\therefore \quad 1 = \sin\alpha \cos 2\alpha$$
$$\Rightarrow \quad 1 = \sin\alpha(1 - 2\sin^2\alpha)$$
$$\Rightarrow \quad 2\sin^3\alpha - \sin\alpha + 1 = 0$$
$$\Rightarrow \quad (\sin\alpha + 1)(2\sin^2\alpha - 2\sin\alpha + 1) = 0$$
$$\Rightarrow \quad \sin\alpha + 1 = 0 \quad (\text{as } 2\sin^2\alpha - 2\sin\alpha + 1 \neq 0)$$
$$\therefore \quad \sin\alpha = -1$$
$$\Rightarrow \quad \sin\alpha = \sin\left(-\frac{\pi}{2}\right)$$
$$\Rightarrow \quad \alpha = n\pi + (-1)^n\left(-\frac{\pi}{2}\right), n \in Z$$
$$\Rightarrow \quad \alpha = n\pi + (-1)^{n+1}\left(\frac{\pi}{2}\right), n \in Z$$

Example 2. If $\frac{1}{6}\sin\theta$, $\cos\theta$ and $\tan\theta$ are in GP, then find the general solution for θ.

Sol. Since, $\frac{1}{6}\sin\theta$, $\cos\theta$, $\tan\theta$ are in GP.

$$\therefore \quad \cos^2\theta = \frac{1}{6}\sin\theta \cdot \tan\theta$$
$$\Rightarrow \quad 6\cos^3\theta + \cos^2\theta - 1 = 0$$

Note that $\cos\theta = \frac{1}{2}$ satisfies the equation (by trial),

$$\therefore \quad (2\cos\theta - 1)(3\cos^2\theta + 2\cos\theta + 1) = 0$$
$$\Rightarrow \quad \cos\theta = \frac{1}{2} \quad (\text{other values of } \cos\theta \text{ are imaginary})$$
$$\Rightarrow \quad \cos\theta = \cos\frac{\pi}{3}$$
$$\Rightarrow \quad \theta = 2n\pi \pm \frac{\pi}{3}, n \in Z$$

Example 3. Solve $\sin^2\theta - \cos\theta = \frac{1}{4}$ for θ and write the values of θ in the interval $0 \le \theta \le 2\pi$.

Sol. The given equation can be written as;

$$1 - \cos^2\theta - \cos\theta = \frac{1}{4}$$
$$\Rightarrow \quad \cos^2\theta + \cos\theta - \frac{3}{4} = 0$$
$$\Rightarrow \quad 4\cos^2\theta + 4\cos\theta - 3 = 0$$
$$\Rightarrow \quad (2\cos\theta - 1)(2\cos\theta + 3) = 0$$
$$\Rightarrow \quad \cos\theta = \frac{1}{2}, -\frac{3}{2}$$

Since, $\cos\theta = -\frac{3}{2}$ is not possible as; $-1 \le \cos\theta \le 1$

$$\therefore \quad \cos\theta = \frac{1}{2}$$
$$\Rightarrow \quad \cos\theta = \cos\frac{\pi}{3}$$
$$\Rightarrow \quad \theta = 2n\pi \pm \frac{\pi}{3}$$

For the given interval, $n = 0$ and $n = 1$.

$$\Rightarrow \quad \theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

Example 4. Solve $\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta = 0$.

Sol. We have,

$$\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta = 0$$
$$\Rightarrow \quad (\cos\theta + \cos 7\theta) + (\cos 3\theta + \cos 5\theta) = 0$$
$$\Rightarrow \quad 2\cos 4\theta \cdot \cos 3\theta + 2\cos 4\theta \cdot \cos\theta = 0$$
$$\Rightarrow \quad \cos 4\theta(\cos 3\theta + \cos\theta) = 0$$
$$\Rightarrow \quad \cos 4\theta(2\cos 2\theta \cos\theta) = 0$$

Either $\cos\theta = 0$

$$\Rightarrow \quad \theta = (2n + 1)\frac{\pi}{2}$$

or $\quad \cos 2\theta = 0 \Rightarrow \theta = (2n + 1)\frac{\pi}{4}$

or $\quad \cos 4\theta = 0 \Rightarrow \theta = (2n + 1)\frac{\pi}{8}$

$$\therefore \quad \theta = (2n + 1)\frac{\pi}{2}, (2n + 1)\frac{\pi}{4}, (2n + 1)\frac{\pi}{8}$$

Example 5. Find the number of solutions for,

$$\sin 5\theta \cdot \cos 3\theta = \sin 9\theta \cdot \cos 7\theta \text{ in } \left[0, \frac{\pi}{2}\right].$$

Sol. Here, $2\sin 5\theta \cdot \cos 3\theta = 2\sin 9\theta \cdot \cos 7\theta$

$$\Rightarrow \quad \sin 8\theta + \sin 2\theta = \sin 16\theta + \sin 2\theta$$
$$\Rightarrow \quad \sin 8\theta = \sin 16\theta \text{ or } \sin 16\theta = \sin 8\theta$$
$$\therefore \quad 16\theta = n\pi + (-1)^n 8\theta \qquad \ldots (i)$$

when n is even Eq. (i) becomes;

$$8\theta = n\pi \Rightarrow \theta = \frac{n\pi}{8} \qquad \ldots (ii)$$

when n is odd Eq. (i) becomes;

$$24\theta = n\pi \Rightarrow \theta = \frac{n\pi}{24} \qquad \ldots (iii)$$

$\therefore$ For the given interval $\left[0, \frac{\pi}{2}\right]$ Eq. (ii) and (iii) gives the solution as,

$$\theta = 0, \frac{\pi}{4}, \frac{\pi}{2} \text{ and } \frac{\pi}{24}, \frac{\pi}{8}, \frac{5\pi}{24}, \frac{7\pi}{24}, \frac{3\pi}{8}, \frac{11\pi}{24}$$

$\therefore$ Number of solutions is 9.

Example 6. Solve $\frac{\sin x + i\cos x}{1+i}$, $i=\sqrt{-1}$ when it is purely imaginary.

Sol. Here, $\frac{\sin x + i\cos x}{1+i} = \frac{(1-i)(\sin x + i\cos x)}{(1-i)(1+i)}$

$$= \frac{\sin x + \cos x + i(\cos x - \sin x)}{2}$$

which will be purely imaginary, if

$$\sin x + \cos x = 0$$

$$\Rightarrow \quad \tan x = -1 = \tan\left(-\frac{\pi}{4}\right)$$

$$\Rightarrow \quad x = n\pi - \frac{\pi}{4}, \text{ is the general solution.}$$

Example 7. Find the general solutions of

$$2^{1+|\cos x|+|\cos x|^2+|\cos x|^3+\ldots \text{ to } \infty} = 4$$

Sol. $2^{1+|\cos x|+|\cos x|^2+|\cos x|^3+\ldots \text{ to } \infty}$

$$\Rightarrow \quad \frac{1}{2^{1-|\cos x|}} = 2^2 \quad \Rightarrow \quad \frac{1}{1-|\cos x|} = 2$$

$$\Rightarrow \quad |\cos x| = \frac{1}{2} \quad \Rightarrow \quad \cos x = \pm\frac{1}{2}$$

$$\therefore \quad x = 2n\pi \pm \frac{\pi}{3}, 2n\pi \pm \left(\pi - \frac{\pi}{3}\right)$$

$$= 2n\pi \pm \frac{\pi}{3}, (2n \pm 1)\pi \mp \frac{\pi}{3}$$

$$\Rightarrow \quad x = n\pi \pm \frac{\pi}{3}$$

Example 8. If $x \neq \frac{n\pi}{2}$ and $(\cos x)^{\sin^2 x - 3\sin x + 2} = 1$, then find the general solutions of x.

Sol. As $x \neq \frac{n\pi}{2} \Rightarrow \cos x \neq 0, 1, -1$

So, $(\cos x)^{\sin^2 x - 3\sin x + 2} = 1$

$$\Rightarrow \quad \sin^2 x - 3\sin x + 2 = 0$$

$$\therefore \quad (\sin x - 2)(\sin x - 1) = 0$$

$$\Rightarrow \quad \sin x = 1, 2$$

where, $\sin x = 2$ is not possible and $\sin x = 1$ does not satisfy the equation.

$\therefore$ No general solution is possible.

Some Important Results

1. **While solving a trigonometric equation, squaring the equation at any step should be avoided as far as possible. If squaring is necessary, check the solution for extraneous values.**
2. **Never cancel terms containing unknown terms on the two sides, which are in product. It may cause loss of the genuine solution.**
3. **The answer should not contain such values of angles which make any of the terms undefined or infinite.**
4. **Domain should not change. If it changes, necessary corrections must be made.**
5. **Check that den**ominator is not zero at any stage while solving equations.

Example 9. *Find the set of values of x for which* $\frac{\tan 3x - \tan 2x}{1+\tan 3x \cdot \tan 2x} = 1$.

Sol. We have, $\frac{\tan 3x - \tan 2x}{1+\tan 3x \cdot \tan 2x} = 1 \quad \Rightarrow \quad \tan(3x - 2x) = 1$

$$\Rightarrow \quad \tan x = 1 \quad \Rightarrow \quad \tan x = \tan\frac{\pi}{4}$$

$$\Rightarrow \quad x = n\pi + \frac{\pi}{4} \quad [\text{using } \tan\theta = \tan\alpha \Leftrightarrow \theta = n\pi + \alpha]$$

But for this value of x,

$\tan 2x = \tan\left(2n\pi + \frac{\pi}{2}\right) = \infty$, which does not satisfy the given equation as it reduces to indeterminate form.

Hence, the solution set for x is ϕ.

Example 10. Solve $\sin x = 0$ and $\frac{\sin x}{\cos\frac{x}{2}\cos\frac{3x}{2}} = 0$ and show their solutions are different.

Sol. We have, $\sin x = 0 \Rightarrow x = n\pi$,

i.e. $x = 0, \pi, 2\pi, 3\pi, \ldots$ ...(i)

Where as,

$$\frac{\sin x}{\cos\frac{x}{2}\cos\frac{3x}{2}} = 0, \text{ where } \cos\frac{x}{2} \neq 0 \text{ and } \cos\frac{3x}{2} \neq 0$$

i.e. $\frac{x}{2} \neq \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$ and $\frac{3x}{2} \neq \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

i.e. $x \neq \pi, 3\pi, 5\pi, \ldots$ and $\frac{\pi}{3}, \pi, \frac{5\pi}{3}, \ldots$ ...(ii)

But $\frac{\sin x}{\cos\frac{x}{2}\cos\frac{3x}{2}} = 0 \Rightarrow \sin x = 0$

$\Rightarrow \quad x = \pi, 2\pi, 3\pi, 4\pi, \ldots$ ...(iii)

From Eqs. (ii) and (iii),

$x = 2\pi, 4\pi, 6\pi, \ldots$ ...(iv)

From Eqs. (i) and (iv);

The two equations are not equivalent. Since, some solutions of the first do not satisfy the second equation.

Example 11. Find number of solutions of $\tan x + \sec x = 2\cos x$ in $[0, 2\pi]$.

Sol. Here, $\tan x + \sec x = 2\cos x \Rightarrow \sin x + 1 = 2\cos^2 x$

$\Rightarrow \quad 2\sin^2 x + \sin x - 1 = 0 \Rightarrow \sin x = \frac{1}{2}, -1$

But $\sin x = -1 \Rightarrow x = \frac{3\pi}{2}$ which does not satisfy $\tan x + \sec x = 2\cos x$.

Thus, $\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$

Number of solutions of $\tan x + \sec x = 2\cos x$ is 2.

Example 12. Solve $\sec x - 1 = (\sqrt{2} - 1)\tan x$.

Sol. We have, $\sec x - 1 = (\sqrt{2} - 1)\tan x$

$\Rightarrow \quad 1 - \cos x = (\sqrt{2} - 1)\sin x$

$\Rightarrow \quad 2\sin^2 \frac{x}{2} - (\sqrt{2} - 1)\cdot 2\sin \frac{x}{2}\cos \frac{x}{2} = 0$

$\Rightarrow \quad 2\sin \frac{x}{2}\left\{\sin \frac{x}{2} - (\sqrt{2} - 1)\cos \frac{x}{2}\right\} = 0$

$\Rightarrow \quad \sin \frac{x}{2}\left[\sin \frac{x}{2} - (\sqrt{2} - 1)\cos \frac{x}{2}\right] = 0$

$\Rightarrow \quad \sin \frac{x}{2} = 0 \text{ or } \tan \frac{x}{2} = (\sqrt{2} - 1) = \tan \frac{\pi}{8}$

$\Rightarrow \quad \frac{x}{2} = n\pi \text{ or } \frac{x}{2} = n\pi + \frac{\pi}{8}$

$\therefore \quad x = 2n\pi, 2n\pi + \frac{\pi}{4}$

Note

$\theta \neq (2n+1)\frac{\pi}{2}$, otherwise the equation will be meaningless.

Example 13. ***Solve*** $\tan\theta + \tan 2\theta + \tan\theta \cdot \tan 2\theta = 1$.

Sol. We can re-write the given equation as;

$\tan\theta + \tan 2\theta = 1 - \tan\theta \cdot \tan 2\theta$

$\Rightarrow \quad \frac{\tan\theta + \tan 2\theta}{1 - \tan\theta \tan 2\theta} = 1 \Rightarrow \tan 3\theta = 1 = \tan \frac{\pi}{4}$

$\Rightarrow \quad 3\theta = n\pi + \frac{\pi}{4} \Rightarrow \theta = \frac{n\pi}{3} + \frac{\pi}{12}$

Example 14. ***Find the number of solutions of*** $|\cos x| = \sin x,\ 0 \le x \le 4\pi$.

Sol. **Case I** If $\cos x \ge 0$, i.e. $x \in \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, \frac{5\pi}{2}\right] \cup \left[\frac{7\pi}{2}, 4\pi\right]$

then, $\cos x = \sin x$.

$\Rightarrow \quad \tan x = 1 \Rightarrow x = n\pi + \frac{\pi}{4} = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$

$\therefore$ If $\cos x \ge 0$, the possible values of x are $\frac{\pi}{4}, \frac{9\pi}{4}$.

Case II If $\cos x < 0$, i.e. $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \cup \left(\frac{5\pi}{2}, \frac{7\pi}{2}\right)$, then

$-\cos x = \sin x$

$\Rightarrow \quad \tan x = -1 \Rightarrow x = n\pi - \frac{\pi}{4} = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}$

$\therefore$ If $\cos x < 0$, the possible values of x are $\frac{3\pi}{4}, \frac{11\pi}{4}$.

Thus, the possible number of solutions are 4.

Example 15. Solve $\cot\theta = \sin 2\theta$ by substituting $\sin 2\theta = \frac{2\tan\theta}{1 + \tan^2\theta}$ and again by substituting $\sin 2\theta = 2\sin\theta \cdot \cos\theta$ and check whether the two answer are same or not.

Sol. **Method I**

Put $\sin 2\theta = \frac{2\tan\theta}{1 + \tan^2\theta}$

$\therefore \quad \cot\theta = \sin 2\theta \Rightarrow \cot\theta = \frac{2\tan\theta}{1 + \tan^2\theta}$

$\Rightarrow \quad \frac{1}{\tan\theta} = \frac{2\tan\theta}{1 + \tan^2\theta}$

$\Rightarrow \quad 2\tan^2\theta = 1 + \tan^2\theta$

$\Rightarrow \quad \tan^2\theta = 1 \Rightarrow \tan^2\theta = (1)^2 = \tan^2 \frac{\pi}{4}$

$\Rightarrow \quad \theta = n\pi \pm \frac{\pi}{4}$ ...(i)

Method II Put $\sin 2\theta = 2\sin\theta\cos\theta$

$\therefore \quad \cot\theta = \sin 2\theta$

$\Rightarrow \quad \frac{\cos\theta}{\sin\theta} = 2\sin\theta\cos\theta$

$\Rightarrow \quad \cos\theta = 2\sin^2\theta\cos\theta$

$\Rightarrow \quad \cos\theta(1 - 2\sin^2\theta) = 0$

$\Rightarrow \quad \cos\theta = 0 \text{ or } \sin^2\theta = \frac{1}{2} = \left(\frac{1}{\sqrt{2}}\right)^2 = \sin^2 \frac{\pi}{4}$

$\Rightarrow \quad \theta = (2n+1)\frac{\pi}{2} \text{ or } \theta = n\pi \pm \frac{\pi}{4}$ ...(ii)

From Eqs. (i) and (ii), it is clear, first method gives less number of roots then the second method.

Note As far as possible, avoid the use of following formulae

$\sin 2x = \frac{2\tan x}{1 + \tan^2 x}, \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x},$

$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$ and $\tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$

As these formulae are not defined for some real values of x. Hence, in many cases the solution obtained with use of these formulae may not be the complete solutions set of the given equation.

Exercise for Session 1

1. Solve $\sin 5x = \cos 2x$.
2. Find the number of values of x in $[0, 5\pi]$ satisfying the equation $3\cos^2 x - 10\cos x + 7 = 0$.
3. If $2\tan^2 x - 5\sec x$ is equal to 1 for exactly 7 distinct values of $x \in \left[0, \frac{n\pi}{2}\right]$, $n \in N$, then find the greatest value of n.
4. Find the general solution of equation $\sec^2 x = \sqrt{2}(1 - \tan^2 x)$.
5. Solve $7\cos^2\theta + 3\sin^2\theta = 4$.
6. Find the general solution of the equation $\tan^2\alpha + 2\sqrt{3}\tan\alpha = 1$.
7. Find the number of solutions of $\sin^2 x - \sin x - 1 = 0$.
8. Find the general values of θ satisfying $\tan\theta + \tan\left(\frac{3\pi}{4} + \theta\right) = 2$.
9. Find the general solution of $\sin x + \sin 5x = \sin 2x + \sin 4x$.
10. Solve $\cos\theta\cos 2\theta\cos 3\theta = \frac{1}{4}$.
11. Solve $2\cot 2x - 3\cot 3x = \tan 2x$.
12. Find the roots of the equation $\cot x - \cos x = 1 - \cot x \cos x$.
13. If the equation $x^2 + 4x\sin\theta + \tan\theta = 0\left(0 < \theta < \frac{\pi}{2}\right)$ has repeated roots, then find the value of θ.
14. Find the number of solutions of the equation $2\sin^3 x + 6\sin^2 x - \sin x - 3 = 0$ in $(0, 2\pi)$.
15. Find the number of roots of the equation $16\sec^3\theta - 12\tan^2\theta - 4\sec\theta = 9$ in interval $(-\pi, \pi)$.

Session 2

Equation of the Form *a* cos θ + *b* sin θ = *c* and Some Particular Equations

Equation of the Form

$a\cos\theta + b\sin\theta = c$

To solve the equation $a\cos\theta + b\cos\theta = c$, put $a = r\cos\phi$ and $b = r\sin\phi$, where

$$r = \sqrt{a^2 + b^2} \text{ and } \phi = \tan^{-1}\frac{b}{a}$$

Substituting these values in the equation, we get,
$r\cos\phi\cos\theta + r\sin\phi\sin\theta = c$

$$\cos(\theta - \phi) = \frac{c}{r} \Rightarrow \cos(\theta - \phi) = \frac{c}{\sqrt{a^2 + b^2}}$$

If $|c| > \sqrt{a^2 + b^2}$, then the equation $a\cos\theta + b\sin\theta = c$ has no solution.

If $|c| \le \sqrt{a^2 + b^2}$, then put $\frac{|c|}{\sqrt{a^2 + b^2}} = \cos\alpha$, so that

$\cos(\theta - \phi) = \cos\alpha$

$\Rightarrow \quad (\theta - \phi) = 2n\pi \pm \alpha$

$\Rightarrow \quad \theta = 2n\pi \pm \alpha + \phi$, where $n \in I$

Working Rule

Step I Check whether $|c| \le \sqrt{a^2 + b^2}$ or not. If it is satisfied, no real solution exists.

Step II If the above condition is satisfied, divide both sides of the equation by $\sqrt{a^2 + b^2}$.

Example 16. ***Prove that the equation*** $p \cos x - q \sin x = r$ ***admits solution for*** x ***only if*** $-\sqrt{p^2 + q^2} < r < \sqrt{p^2 + q^2}$.

Sol. Here, $p \cos x - q \sin x = r$

On dividing both sides by $\sqrt{p^2 + q^2}$, we get

$$\frac{p}{\sqrt{p^2 + q^2}} \cos x - \frac{q}{\sqrt{p^2 + q^2}} \sin x = \frac{r}{\sqrt{p^2 + q^2}} \quad ...(i)$$

Put, $\frac{p}{\sqrt{p^2 + q^2}} = \cos \phi$, $\frac{q}{\sqrt{p^2 + q^2}} = \sin \phi$ in Eq. (i), we get

$$\cos \phi \cos x - \sin \phi \sin x = \frac{r}{\sqrt{p^2 + q^2}}$$

$$\Rightarrow \quad \cos (x + \phi) = \frac{r}{\sqrt{p^2 + q^2}}$$

As we know, $-1 \le \cos(x + \phi) \le 1$

$\therefore$ The above equation posses solution only if,

$$-1 \le \frac{r}{\sqrt{p^2 + q^2}} \le 1 \quad \text{or} \quad -\sqrt{p^2 + q^2} \le r \le \sqrt{p^2 + q^2}$$

Example 17. Solve $\sin x + \sqrt{3} \cos x = \sqrt{2}$.

Sol. Given, $\sqrt{3} \cos x + \sin x = \sqrt{2}$, dividing both sides by $\sqrt{a^2 + b^2} = \sqrt{(\sqrt{3})^2 + 1^2} = 2$, we get

$$\Rightarrow \quad \frac{\sqrt{3}}{2} \cos x + \frac{1}{2} \sin x = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \cos\left(x - \frac{\pi}{6}\right) = \cos\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \quad x - \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{4}$$

$$\Rightarrow \quad x = 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{6}$$

$$\Rightarrow \quad x = 2n\pi + \frac{5\pi}{12},\ 2n\pi - \frac{\pi}{12}, \text{ where } n \in I$$

Example 18. Find the number of distinct solutions of $\sec x + \tan x = \sqrt{3}$, where $0 \le x \le 3\pi$.

Sol. Here, we have $\sec x + \tan x = \sqrt{3}$

$$\Rightarrow \quad 1 + \sin x = \sqrt{3} \cos x$$

or $\quad \sqrt{3} \cos x - \sin x = 1$

On dividing both sides by $\sqrt{a^2 + b^2}$ i.e. $\sqrt{4} = 2$, we get

$$\Rightarrow \quad \frac{\sqrt{3}}{2} \cos x - \frac{1}{2} \sin x = \frac{1}{2}$$

$$\Rightarrow \quad \cos \frac{\pi}{6} \cos x - \sin \frac{\pi}{6} \sin x = \frac{1}{2}$$

$$\Rightarrow \quad \cos\left(x + \frac{\pi}{6}\right) = \cos\left(\frac{\pi}{3}\right)$$

$$\Rightarrow \quad x + \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{3}$$

or $\quad x = 2n\pi + \frac{\pi}{6}, 2n\pi - \frac{\pi}{2}$

Now, when $x = 2n\pi + \frac{\pi}{6}$, there are solution for $n = 0$, 1 and when $x = 2n\pi - \frac{\pi}{2}$, there are solution for $n = 1$.

Thus, total number of solutions are 3.

Example 19. Prove that the equation $k \cos x - 3 \sin x = k + 1$ is solvable only if $k \in (-\infty, 4]$.

Sol. Here,

$k \cos x - 3 \sin x = k + 1$, could be re-written as;

$$\frac{k}{\sqrt{k^2 + 9}} \cos x - \frac{3}{\sqrt{k^2 + 9}} \sin x = \frac{k + 1}{\sqrt{k^2 + 9}}$$

or $\quad \cos(x + \phi) = \frac{k + 1}{\sqrt{k^2 + 9}}$

which posses solution only if, $-1 \le \frac{k + 1}{\sqrt{k^2 + 9}} \le 1$

i.e. $\quad \left| \frac{k + 1}{\sqrt{k^2 + 9}} \right| \le 1$

i.e. $\quad (k + 1)^2 \le k^2 + 9$

i.e. $\quad k^2 + 2k + 1 \le k^2 + 9$

or $\quad k \le 4$

Thus, the interval in which, $k \cos x - 3 \sin x = k + 1$ admits solution for k is $(-\infty, 4]$.

Example 20. ***Let*** $[.]$ ***denotes the greatest integer less than or equal to*** x ***and*** $f(x) = \sin x + \cos x$***. Then, find the most general solution of*** $f(x) = \left[f\left(\frac{\pi}{10}\right) \right]$.

Sol. Here,

$f\left(\frac{\pi}{10}\right) = \sin 18^\circ + \cos 18^\circ = \sqrt{2} [\sin (45^\circ + 18^\circ)] = \sqrt{2} \sin 63^\circ$.

As $\sin 63^\circ > \sin 45^\circ = \frac{1}{\sqrt{2}}$ and $\sin 63^\circ < 1$

$\therefore \quad 1 < f\left(\frac{\pi}{10}\right) < \sqrt{2} \Rightarrow \left[f\left(\frac{\pi}{10}\right)\right] = 1$

So, the equation is $\sin x + \cos x = 1$.

$\therefore \quad \cos\left(x - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ or $x - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}$

$\Rightarrow \quad x = 2n\pi + \frac{\pi}{2}, 2n\pi$

Example 21. Find the number of solutions of

$$\cos x = |1 + \sin x|, \ 0 \le x \le 3\pi$$

Sol. As we know, $1 + \sin x \ge 0$, for all x

So, $\cos x = 1 + \sin x$, for all $x \Rightarrow \cos x - \sin x = 1$

On dividing both sides by $\sqrt{a^2 + b^2}$ i.e. by $\sqrt{2}$, we get

$$\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x = \frac{1}{\sqrt{2}}$$

$\Rightarrow \quad \cos\frac{\pi}{4}\cos x - \sin\frac{\pi}{4}\sin x = \cos\frac{\pi}{4}$

$\Rightarrow \quad \cos\left(x + \frac{\pi}{4}\right) = \cos\frac{\pi}{4} \Rightarrow x + \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}$

$\Rightarrow \quad x = 2n\pi, 2n\pi - \frac{\pi}{2}$, where $0 \le x \le 3\pi$

$\Rightarrow \quad x = 0, \frac{3\pi}{2}, 2\pi$ are the only solution.

Thus, number of solutions are 3.

Some Particular Equations

Equation of the Form

$a_0 \sin^n x + a_1 \sin^{n-1} x \cos x + a_2 \sin^{n-2} x \cos^2 x + \ldots + a_n \cos^n x = 0,$

where $a_0, a_1, \ldots, a_n$ are real number and the sum of the exponents of $\sin x$ and $\cos x$ in each term is equal to n, are said to be homogeneous with respect to $\sin x$ and $\cos x$. For $\cos x \ne 0$, above equation can be written as,

$$a_0 \tan^n x + a_1 \tan^{n-1} x + \ldots + a_n = 0$$

Example 22. Solve $3\cos^2\theta - 2\sqrt{3}\sin\theta\cos\theta - 3\sin^2\theta = 0$.

Sol. The given equation can be written as:

$3\tan^2\theta + 2\sqrt{3}\tan\theta - 3 = 0$

$\Rightarrow \quad \tan\theta = \frac{-2\sqrt{3} \pm \sqrt{12+36}}{6} = \frac{1}{\sqrt{3}}, -\sqrt{3}$

Either, $\tan\theta = \frac{1}{\sqrt{3}} = \tan\frac{\pi}{6}$

$\Rightarrow \quad \theta = n\pi + \frac{\pi}{6}$ ...(i)

or $\tan\theta = -\sqrt{3} = \tan\left(\frac{-\pi}{3}\right)$

$\Rightarrow \quad \theta = n\pi - \frac{\pi}{3}$ ...(ii)

$\therefore \quad \theta = n\pi + \frac{\pi}{6}$ or $\theta = n\pi - \frac{\pi}{3}$

Example 23. ***Solve the equation*** $5\sin^2 x - 7\sin x\cos x + 16\cos^2 x = 4$.

Sol. To solve this equation, we use the fundamental trigonometric identity, $\sin^2 x + \cos^2 x = 1$

Given equation can be written as

$4(\sin^2 x + \cos^2 x) + \sin^2 x - 7\sin x\cos x + 12\cos^2 x = 4$

$\Rightarrow \quad \sin^2 x - 7\sin x\cos x + 12\cos^2 x = 0$

On dividing by $\cos^2 x$ both sides, we get

$$\tan^2 x - 7\tan x + 12 = 0$$

Now, it can be factorised as;

$$(\tan x - 3)(\tan x - 4) = 0$$

$\Rightarrow \quad \tan x = 3, 4$

i.e. $\tan x = \tan(\tan^{-1} 3)$

or $\tan x = \tan(\tan^{-1} 4)$

$\Rightarrow \quad x = n\pi + \tan^{-1} 3$

or $x = n\pi + \tan^{-1} 4$

Equation of the Form

$R(\sin kx, \cos nx, \tan mx, \cot lx) = 0,$

where R is a rational function of the indicated arguments and k, l, m, n are natural numbers, can be reduced to a rational equation with respect to the arguments $\sin x$, $\cos x$, $\tan x$ and $\cot x$ by means of the formulae for trigonometric functions of the sum of angles (in particular, the formulas for double and triple angle) and then reduce the obtained equation to a rational equation with respect to the unknown, $t = \tan\frac{x}{2}$, by means of following formulae;

$$\sin x = \frac{2\tan\frac{x}{2}}{1 + \tan^2\frac{x}{2}}, \cos x = \frac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}},$$

$$\tan x = \frac{2\tan\frac{x}{2}}{1 - \tan^2\frac{x}{2}}, \cot x = \frac{1 - \tan^2\frac{x}{2}}{2\tan\frac{x}{2}}$$

Example 24. Solve the equation

$$(\cos x - \sin x)\left(2\tan x + \frac{1}{\cos x}\right) + 2 = 0$$

Sol. Using above formulae, we get

$$\left\{\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} - \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right\}\left\{\frac{4\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}} + \frac{1+\tan^2\frac{x}{2}}{1-\tan^2\frac{x}{2}}\right\} + 2 = 0$$

$$\left(\frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2}\right)\left(\frac{4t}{1-t^2} + \frac{1+t^2}{1-t^2}\right) + 2 = 0$$

(Taking $\tan\frac{x}{2} = t$)

$$\Rightarrow \qquad \frac{3t^4 + 6t^3 + 8t^2 - 2t - 3}{(t^2+1)(1-t^2)} = 0$$

Its roots are $t_1 = \frac{1}{\sqrt{3}}$ and $t_2 = -\frac{1}{\sqrt{3}}$.

Thus, the solution of the equation reduces to that of two elementary euqations,

$$\tan\frac{x}{2} = \frac{1}{\sqrt{3}} \text{ and } \tan\frac{x}{2} = -\frac{1}{\sqrt{3}} \Rightarrow \frac{x}{2} = n\pi \pm \frac{\pi}{6}$$

$$\Rightarrow \qquad x = 2n\pi \pm \frac{\pi}{3}, \text{ is required solution.}$$

Example 25. Solve the equation

$$\sin^{10} x + \cos^{10} x = \frac{29}{16}\cos^4 2x$$

Sol. Using half-angle formulae, we can represent the given equation in the form;

$$\left(\frac{1-\cos 2x}{2}\right)^5 + \left(\frac{1+\cos 2x}{2}\right)^5 = \frac{29}{16}\cos^4 2x$$

Put $\cos 2x = t$,

$$\left(\frac{1-t}{2}\right)^5 + \left(\frac{1+t}{2}\right)^5 = \frac{29}{16}t^4$$

$$\Rightarrow \qquad 24t^4 - 10t^2 - 1 = 0$$

whose only real root is, $t^2 = \frac{1}{2}$.

$$\therefore \qquad \cos^2 2x = \frac{1}{2} \Rightarrow 1 + \cos 4x = 1$$

$$\Rightarrow \qquad \cos 4x = 0 \Rightarrow 4x = (2n+1)\frac{\pi}{2}$$

$$\Rightarrow \qquad x = \frac{n\pi}{4} + \frac{\pi}{8};\ n \in I$$

Note

Some trigonometric equations can sometimes be simplified by lowering their degrees. If the exponent of the sines and cosines occurring into an equation are even, the lowering of the degree can be done by half angle formulas as in above example.

Equation of the Form

$R(\sin x + \cos x, \sin x \cdot \cos x) = 0$,

where R is a rational function of the arguments in brackets. To solve such equations, put $\sin x + \cos x = t$ ...(i)

and use the following identity

$$(\sin x + \cos x)^2 = \sin^2 x + \cos^2 x + 2\sin x \cos x$$
$$= 1 + 2\sin x \cos x$$

$$\Rightarrow \qquad \sin x \cos x = \frac{t^2 - 1}{2} \qquad \text{...(ii)}$$

Taking (i) and (ii) into account, we can reduce given equation into;

$$R\left(t, \frac{t^2-1}{2}\right) = 0$$

Similarly, by the substitution $(\sin x - \cos x) = t$, we can reduce the equation of the form;

$$R(\sin x - \cos x, \sin x \cos x) = 0$$

to an equation; $\qquad R\left(t, \frac{1-t^2}{2}\right) = 0$

Example 26. Solve the equation

$$\sin x + \cos x - 2\sqrt{2}\sin x \cos x = 0$$

Sol. Let $(\sin x + \cos x) = t$ and using the equation

$\sin x \cdot \cos x = \frac{t^2-1}{2}$, we get

$$t - 2\sqrt{2}\left(\frac{t^2-1}{2}\right) = 0$$

$$\Rightarrow \qquad \sqrt{2}t^2 - t - \sqrt{2} = 0$$

The numbers $t_1 = \sqrt{2}, t_2 = -\frac{1}{\sqrt{2}}$ are roots of this quadratic equation.

Thus, the solution of the given equation reduces to the solution of two trigonometric equations

	$\sin x + \cos x = \sqrt{2}$	and	$\sin x + \cos x = -\frac{1}{\sqrt{2}}$
or	$\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = 1$	and	$\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = -\frac{1}{2}$
or	$\sin x \cdot \cos\frac{\pi}{4} + \sin\frac{\pi}{4}\cos x = 1$	and	$\sin x \cos\frac{\pi}{4} + \sin\frac{\pi}{4}\cos x = -\frac{1}{2}$
$\Rightarrow$	$\sin\left(x + \frac{\pi}{4}\right) = 1$	and	$\sin\left(x + \frac{\pi}{4}\right) = -\frac{1}{2}$
$\Rightarrow$	$x + \frac{\pi}{4} = (4n+1)\frac{\pi}{2}$	and	$x + \frac{\pi}{4} = n\pi + (-1)^n \cdot \left(-\frac{\pi}{6}\right)$
$\Rightarrow$	$x = 2n\pi + \frac{\pi}{4}$	and	$x = n\pi + (-1)^{n+1}\frac{\pi}{6} - \left(\frac{\pi}{4}\right)$

Exercise for Session 2

1. Solve the equation $\sin x + \cos x = 1$.
2. Solve $\sqrt{3}\cos\theta - 3\sin\theta = 4\sin 2\theta\cos 3\theta$.
3. Solve $\cot\theta + \operatorname{cosec}\theta = \sqrt{3}$.
4. Solve $\sqrt{2}\sec\theta + \tan\theta = 1$.
5. Find the general solution of $(\sqrt{3}-1)\sin\theta + (\sqrt{3}+1)\cos\theta = 2$.
6. Find the number of integral values of k for which equation $7\cos x + 5\sin x = 2k+1$ has at least one solution.

   [Hint : $a\cos\theta + b\sin\theta = c$ has solution only when $|c| \le \sqrt{a^2+b^2}$].
7. Solve $2\sin^2 x - 5\sin x\cos x - 8\cos^2 x = -2$.
8. Solve the equation $(1-\tan\theta)(1+\sin 2\theta) = 1+\tan\theta$.

Session 3

Solution of Simultaneous Trigonometric Equations and Problems Based on Extreme Values of $\sin x$ and $\cos x$

Solution of Simultaneous Trigonometric Equations

Here, we discuss problems related to the solution of two equations satisfied simultaneously.

We may divide the problem into two categories :

(i) Two equations in one 'unknown' satisfied simultaneously.

(ii) Two equations in two 'unknowns' satisfied simultaneously.

Example 27. Find the most general values of θ which satisfies the equations $\sin\theta = -\frac{1}{2}$ and $\tan\theta = \frac{1}{\sqrt{3}}$.

Sol. First, find the values of θ lying between 0 and 2π and satisfying the two given equations separately. Select the value of θ which satisfies both the equations, then generalise it.

$$\sin\theta = -\frac{1}{2} \Rightarrow \theta = \frac{7\pi}{6} \text{ or } \frac{11\pi}{6}$$

$$\tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6} \text{ or } \frac{7\pi}{6} \text{ [value between 0 and } 2\pi]$$

Common value of $\theta = \frac{7\pi}{6}$

The required solution is, $\theta = 2n\pi + \frac{7\pi}{6}$

Example 28. If $\tan(A-B) = 1$ and $\sec(A+B) = \frac{2}{\sqrt{3}}$, then find the smallest positive values of A and B and their most general values.

Sol. For the smallest positive values, find $A+B$ and $A-B$ between 0 and 2π from the given equations.

Since, A and B are positive angles, $A+B > A-B$. Solve the two to get A and B.

For the most general values, find the general values of $A-B$ and $A+B$ by solving the given equations separately. Solve two to get A and B

$$\tan(A-B) = 1 \Rightarrow A - B = \frac{\pi}{4} \text{ or } \frac{5\pi}{4} \quad \text{...(i)}$$

Also, $\sec(A+B)=\frac{2}{\sqrt{3}} \Rightarrow A+B=\frac{\pi}{6}$ or $\frac{11\pi}{6}$

Since, $A+B>A-B$,

$\therefore \quad A+B=\frac{11\pi}{6}$ ...(ii)

On solving Eqs. (i) and (ii), we get

$$A=\frac{25\pi}{24}, B=\frac{19\pi}{24}$$

or $A=\frac{37\pi}{24}, B=\frac{7\pi}{24}$

Solve for the most general values

$$\tan(A-B)=1 \Rightarrow A-B=n\pi+\frac{\pi}{4} \quad ...(iii)$$

$$\sec(A+B)=\frac{2}{\sqrt{3}} \Rightarrow \cos(A+B)=\frac{\sqrt{3}}{2}$$

$\Rightarrow \quad A+B=2m\pi \pm \frac{\pi}{6}$ ...(iv)

On solving Eqs. (iii) and (iv), we get,

$$A=\frac{1}{2}\left[(2m+n)\pi+\frac{\pi}{4}\pm\frac{\pi}{6}\right]$$

$$B=\frac{1}{2}\left[(2m-n)\pi-\frac{\pi}{4}\pm\frac{\pi}{6}\right] \text{ where } m, n \in I$$

Example 29. Solve the system of equations

$$x+y=\frac{2\pi}{3} \text{ and } \frac{\sin x}{\sin y}=2$$

Sol. Let us reduce the second equation of the system to the form,

$$\sin x=2\sin y \quad ...(i)$$

Using $x+y=\frac{2\pi}{3}$ we get, $\sin x=2\sin\left(\frac{2\pi}{3}-x\right)$

$\Rightarrow \quad \sin x=2\left(\sin\frac{2\pi}{3}\cdot\cos x-\cos\frac{2\pi}{3}\cdot\sin x\right)$

$$=2\left(\frac{\sqrt{3}}{2}\cdot\cos x+\frac{1}{2}\cdot\sin x\right)$$

$\Rightarrow \quad \sin x=\sqrt{3}\cos x+\sin x$

$\Rightarrow \quad \cos x=0$

$\Rightarrow \quad x=(2n+1)\frac{\pi}{2}$

$\Rightarrow \quad x=\frac{\pi}{2}+n\pi$

Substituting in $x+y=\frac{2\pi}{3}$, we get

$$y=-n\pi+\frac{\pi}{6}$$

$\therefore \quad x=\frac{\pi}{2}+n\pi, y=\frac{\pi}{6}-n\pi$, where $n \in I$.

Example 30. If $r>0, -\pi \le \theta \le \pi$ and r, θ satisfy $r\sin\theta=3$ and $r=4(1+\sin\theta)$, then find the possible solutions of the pair (r, θ).

Sol. Here, $r=4(1+\sin\theta)$ and $r\sin\theta=3$

On eliminating θ from above equations; $r=4\left(1+\frac{3}{r}\right)$

$\Rightarrow \quad r^2-4r-12=0$

$\Rightarrow \quad (r-6)(r+2)=0$

$\Rightarrow \quad r=6$ or $r=-2$

$\therefore \quad r\sin\theta=3 \Rightarrow \sin\theta=\frac{1}{2}$ or $\sin\theta=-\frac{3}{2}$

Neglecting $\sin\theta=-\frac{3}{2}$, we get $\theta=\frac{\pi}{6}, \frac{5\pi}{6}$

$\therefore \quad (r, \theta)=\left(6, \frac{\pi}{6}\right)$ and $\left(6, \frac{5\pi}{6}\right)$ are the required pairs.

Problems Based on Extreme Values of sin *x* and cos *x*

Example 31. Solve $2\cos^2\frac{x}{2}\cdot\sin^2 x=x^2+\frac{1}{x^2}$, $0<x\le\frac{\pi}{2}$.

Sol. In this problem, terms on the two sides of the equation are different in nature.

LHS is in trigonometric form, whereas RHS is in algebraic form. Hence, we will used inequality method.

Here, LHS $=2\cos^2\frac{x}{2}\cdot\sin^2 x$

$$=(1+\cos x)\sin^2 x<2$$

$[\because 1+\cos x<2, \sin^2 x\le 1]$

and RHS $=x^2+\frac{1}{x^2}\ge 2$ $[\because$ A.M. $\ge$ GM$]$

Hence, LHS is never equal to RHS

$\therefore$ The given equation has no solution.

Example 32. ***Solve*** $\sin^6 x=1+\cos^4 3x$.

Sol. LHS $=\sin^6 x \le 1$

RHS $=1+\cos^4 3x \ge 1$

Hence, $\sin^6 x=1+\cos^4 3x$ is possible only when.

LHS = RHS = 1

$\Rightarrow \quad \sin^6 x=1$ and $1+\cos^4 3x=1$

$\Rightarrow \quad \sin^2 x=1$ and $\cos^4 3x=0$

$\Rightarrow \quad \cos^2 x=0$ and $\cos 3x=0$

$\Rightarrow \quad \cos x=0$ and $\cos 3x=0$

$\Rightarrow \quad x = (2m+1)\frac{\pi}{2}$

and $\quad 3x = (2n+1)\frac{\pi}{2}$, where $m, n \in I$

$\Rightarrow \quad x = (2m+1)\frac{\pi}{2}$ and $x = (2n+1)\frac{\pi}{6}$,

where $m, n \in I$

$\Rightarrow \quad x = (2m+1)\frac{\pi}{2}$

and $\quad x = (2n+1)\frac{\pi}{6}$

Common values of x is $(2n+1)\frac{\pi}{2}$, where $n \in I$. The required solution,

$$x = (2n+1)\frac{\pi}{2}, n \in I$$

Example 33. *Solve* $\sin^4 x = 1 + \tan^8 x$.

Sol. LHS $= \sin^4 x \le 1$

RHS $= 1 + \tan^8 x \ge 1$

$\Rightarrow$ LHS = RHS only when

$\sin^4 x = 1$ and $1 + \tan^8 x = 1$

$\Rightarrow \quad \sin^2 x = 1$ and $\tan^8 x = 0$

which in never possible, since $\sin x$ and $\tan x$ vanish simultaneously.

Therefore, the given equation has no solution.

Example 34. *Solve* $\sin^2 x + \cos^2 y = 2\sec^2 z$.

Sol. LHS $= \sin^2 x + \cos^2 y \le 2$

RHS $= 2\sec^2 z \ge 2$

Hence LHS = RHS only when,

$\sin^2 x = 1, \cos^2 y = 1$ and $\sec^2 z = 1$

$\Rightarrow \quad \cos^2 x = 0, \sin^2 y = 0$ and $\cos^2 z = 1$

$\Rightarrow \quad \cos x = 0, \sin y = 0, \sin z = 0$

$\Rightarrow x = (2m+1)\frac{\pi}{2}, y = n\pi, z = t\pi$,

where m, n, t are integer.

Example 35. Solve the equation $(\sin x + \cos x)^{1+\sin 2x} = 2$, when $-\pi \le x \le \pi$.

Sol. We know, $-\sqrt{a^2+b^2} \le a\sin\theta + b\cos\theta \le \sqrt{a^2+b^2}$ and $-1 \le \sin\theta \le 1$.

$\therefore (\sin x + \cos x)$ admits the maximum value as $\sqrt{2}$ and minimum value as $-\sqrt{2}$,

$(1 + \sin 2x)$ admits the maximum value as 2. Also, $(\pm\sqrt{2})^2 = 2$.

$\therefore$ The equation could hold only when,

$\sin x + \cos x = \pm\sqrt{2}$ and $1 + \sin 2x = 2$

Now, $\sin x + \cos x = \sqrt{2}$

$\Rightarrow \quad \cos\left(x - \frac{\pi}{4}\right) = 1$

$\Rightarrow \quad x = 2n\pi + \frac{\pi}{4}$ ...(i)

and $\quad \sin x + \cos x = -\sqrt{2}$

$\cos\left(x - \frac{\pi}{4}\right) = -1$

$x = (2n \pm 1)\pi + \frac{\pi}{4}$ ...(ii)

and $\quad 1 + \sin 2x = 2$

$\Rightarrow \quad \sin 2x = 1 = \sin\frac{\pi}{2}$

$\Rightarrow \quad 2x = n\pi + (-1)^n \frac{\pi}{2}$

$\Rightarrow \quad x = \frac{n\pi}{2} + (-1)^n \frac{\pi}{4}$ ...(iii)

The value in $[-\pi, \pi]$ satisfying Eqs. (i), (ii) and (iii) is $x = \frac{\pi}{4}, \frac{-3\pi}{4}$ (when $n = 0, -1$).

Example 36. Find the most general solutions for $2^{\sin x} + 2^{\cos x} = 2^{1-1/\sqrt{2}}$.

Sol. As we know, AM $\ge$ GM

$\therefore \quad \frac{2^{\sin x} + 2^{\cos x}}{2} \ge \sqrt{2^{\sin x} \cdot 2^{\cos x}}$ ...(i)

Now, Eq. (i) admits minimum value when

$\sin x + \cos x$ is $(-\sqrt{2})$

{using $-\sqrt{a^2+b^2} \le a\cos x + b\sin x \le \sqrt{a^2+b^2}$}

$\therefore \quad 2^{\sin x} + 2^{\cos x} \ge 2 \cdot \sqrt{2^{-\sqrt{2}}}$

or $\quad 2^{\sin x} + 2^{\cos x} \ge 2 \cdot 2^{-\frac{\sqrt{2}}{2}}$

or $\quad 2^{\sin x} + 2^{\cos x} \ge 2^{1-\frac{1}{\sqrt{2}}}$

Thus, the equation holds only when,

$\sin x + \cos x = -\sqrt{2}$

$\Rightarrow \quad \cos\left(x - \frac{\pi}{4}\right) = -1$

$\Rightarrow \quad x - \frac{\pi}{4} = 2n\pi \pm \pi$

$\Rightarrow \quad x = (2n \pm 1)\pi + \frac{\pi}{4}$

Example 37. Solve $\left|\sqrt{3}\cos x - \sin x\right| \ge 2$ for $x \in [0, 4\pi]$.

Sol. We know, $\left|\sqrt{3}\cos x - \sin x\right| \le \sqrt{3+1} = 2$ ...(i)

and $\left|\sqrt{3}\cos x - \sin x\right| \geq 2$ (given) ...(ii)

Thus, from Eqs. (i) and (ii), we must have

$$\left|\sqrt{3}\cos x - \sin x\right| = 2$$

$$\Rightarrow \left|\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x\right| = 1$$

$$\Rightarrow \left|\cos\frac{\pi}{6}\cos x - \sin\frac{\pi}{6}\cdot\sin x\right| = 1$$

$$\Rightarrow \left|\cos\left(x + \frac{\pi}{6}\right)\right| = 1$$

$$\Rightarrow \cos\left(x + \frac{\pi}{6}\right) = 1 \text{ or } \cos\left(x + \frac{\pi}{6}\right) = -1$$

$$\Rightarrow x + \frac{\pi}{6} = 0, 2\pi, 4\pi, \ldots \text{ or } \pi, 3\pi, 5\pi, \ldots$$

$$\Rightarrow x = \frac{11\pi}{6}, \frac{23\pi}{6}, \frac{5\pi}{6}, \frac{17\pi}{6}, \text{ for } x \in [0, 4\pi]$$

Hence, $x \in \left\{\frac{11\pi}{6}, \frac{23\pi}{6}, \frac{5\pi}{6}, \frac{17\pi}{6}\right\}$

Example 38. Show that the equation, $\sin x = [1 + \sin x] + [1 - \cos x]$ has no solution for $x \in R$. (where [.] represents greatest integers function).

Sol As we know that the period of $\sin x$ is 2π, we need to check the solution for $x \in [0, 2\pi]$.

Let us first check at those points on which $\sin x$ and $\cos x$ are integer and then for the values lying between them.

Case I (a) At $x = 0$

$[1 + \sin x] = 1$ and $[1 - \cos x] = 0$

$\therefore$ $\sin x = 1 + 0 = 1$ or $\sin 0 = 1$ (as $x = 0$)

$\Rightarrow$ $0 = 1$ (absurd)

(b) At $x = \frac{\pi}{2}$;

$[1 + \sin x] = 2$ and $[1 - \cos x] = 1$

$\therefore$ $\sin x = 2 + 1 = 3$ (absurd)

(c) At $x = \pi$

$[1 + \sin x] = 1$ and $[1 - \cos x] = 2$

$\therefore$ $\sin x = 1 + 2 = 3$ (absurd)

(d) At $x = \frac{3\pi}{2}$,

$[1 + \sin x] = 0$ and $[1 - \cos x] = 1$

$\therefore$ $\sin x = 0 + 1$ or $\sin\frac{3\pi}{2} = 1 \left(\text{as } x = \frac{3\pi}{2}\right)$

$\Rightarrow$ $-1 = 1$ (absurd)

(e) At $x = 2\pi$;

$[1 + \sin x] = 1$ and $[1 - \cos x] = 0$

$\therefore$ $\sin x = 1 + 0$

or $\sin 0 = 1$ (as $x = 0$)

$\Rightarrow$ $0 = 1$ (absurd)

$\therefore$ At $x = \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\right\}$

we do not have any solution.

Now, to check for the values lying between them.

Case II (a) When $x \in \left(0, \frac{\pi}{2}\right)$

$[1 + \sin x] = 1, [1 - \cos x] = 0$

$\Rightarrow$ $\sin x = 1 + 0$

But since $x \in \left(0, \frac{\pi}{2}\right)$, $\sin x \neq 1$

$\therefore$ $\sin x = 1$ is absurd when $x \in \left(0, \frac{\pi}{2}\right)$

(b) When $x \in \left(\frac{\pi}{2}, \pi\right)$

$[1 + \sin x] = 1, [1 - \cos x] = 1$

$\Rightarrow$ $\sin x = 1 + 1 = 2$ (absurd)

(c) When $x \in \left(\pi, \frac{3\pi}{2}\right)$

$[1 + \sin x] = 0, [1 - \cos x] = 1 \Rightarrow \sin x = 0 + 1 = 1$

But $x \in \left(\pi, \frac{3\pi}{2}\right)$ in which $\sin x \neq 1$

$\therefore$ $\sin x = 1$ is absurd, when $x \in \left(\pi, \frac{3\pi}{2}\right)$

(d) When $x \in \left(\frac{3\pi}{2}, 2\pi\right)$

$[1 + \sin x] = 0, [1 - \cos x] = 0$

$\therefore$ $\sin x = 0 + 0 = 0$

But $\sin x \neq 0$ when $x \in \left(\frac{3\pi}{2}, 2\pi\right)$

Thus, the given equation does not posses any solution for $x \in [0, 2\pi]$ or in general, $\sin x = [1 + \sin x] + [1 - \cos x]$ does not posses any solution for $x \in R$.

Exercise for Session 3

1. Find the general values of θ which satisfies the equations $\tan\theta = -1$ and $\cos\theta = \frac{1}{\sqrt{2}}$.
2. Find the most general solution of $\operatorname{cosec} x = -2$ and $\cot x = \sqrt{3}$.
3. Find the common roots of the equations $2\sin^2 x + \sin^2 2x = 2$ and $\sin 2x + \cos 2x = \tan x$.
4. Solve the equations, $\sqrt{3}\sin 2A = \sin 2B$ and $\sqrt{3}\sin^2 A + \sin^2 B = \frac{1}{2}(\sqrt{3}-1)$.
5. Find the number of solutions of $\sin^2 x \cos^2 x = 1 + \cos^2 x \sin^4 x$ in the interval $[0, \pi]$.
6. Solve: $1 + \sin x \sin^2 \frac{x}{2} = 0$.
7. Solve: $\cos^{50} x - \sin^{50} x = 1$.
8. Find the number of real solutions of the equation $(\cos x)^5 + (\sin x)^3 = 1$ in the interval $[0, 2\pi]$.
9. Find the number of solutions of the equation
$$1 + e^{\cot^2 x} = \sqrt{2|\sin x| - 1} + \frac{1-\cos 2x}{1+\sin^4 x} \text{ for } x \in (0, 5\pi).$$
10. Find the number of solutions as ordered pair (x, y) of the equation $2^{\sec^2 x} + 2^{\operatorname{cosec}^2 y} = 2\cos^2 x(1-\cos^2 2y)$ in $[0, 2\pi]$.

Session 4

Trigonometric Inequality

Trigonometric Inequality

An inequality involving trigonometric function of an unknown angle is called a trigonometric inequality.

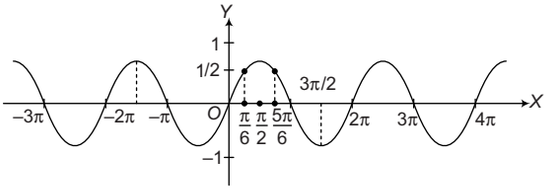
Solution of Trigonometric Inequality

To solve the trigonometric inequation of type $f(x) \le a$, or $f(x) \ge a$ where $f(x)$ is some trigonometric ratio, the following steps should be taken:

1. Draw the graph of $f(x)$ is an interval length equal to the fundamental period of $f(x)$.
2. Draw the line $y = a$.
3. Take the portion of the graph for which the inequalities satisfied.
4. To generalize, add $nT (n \in I)$ and take union over the set of integers, where T, is the fundamental period of $f(x)$.

Example 39. Find the solution set of inequality $\sin x > \frac{1}{2}$.

Sol. When $\sin x = \frac{1}{2}$, the two values of x between 0 and 2π are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$. From the graph of $y = \sin x$, it is obvious that $\sin x > \frac{1}{2}$ for $\frac{\pi}{6} < x < \frac{5\pi}{6}$



Hence, $\sin x > \frac{1}{2}$

$$\Rightarrow \qquad 2n\pi + \frac{\pi}{6} < x < 2n\pi + \frac{5\pi}{6}$$

The required solution set is $\underset{n \in I}{\cup}\left(2n\pi + \frac{\pi}{6}, 2n\pi + \frac{5\pi}{6}\right)$

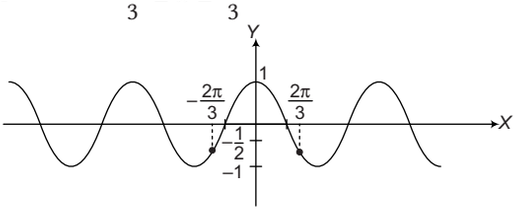
Note

We added $2n\pi$ to find the general solution as period of $\sin x$ is 2π.

Example 40. Find the solution set of inequality $\cos x \geq -\frac{1}{2}$.

Sol. From the graph of $y = \cos x$, it is obvious that $\cos x \geq -\frac{1}{2}$.

For $\qquad -\frac{2\pi}{3} \leq x \leq \frac{2\pi}{3}$



Hence, $\qquad \cos x \geq -\frac{1}{2}$

$$\Rightarrow \qquad 2n\pi - \frac{2\pi}{3} \leq x \leq 2n\pi + \frac{2\pi}{3}$$

The required solution set is

$$\underset{n \in I}{\cup}\left(2n\pi - \frac{2\pi}{3}, 2n\pi + \frac{2\pi}{3}\right)$$

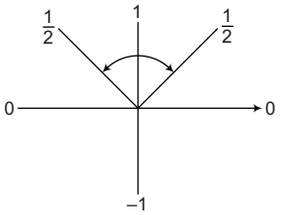
Example 41. Find the solution set for, $4\sin^2 x - 8\sin x + 3 \leq 0$ where $x \in [0, 2\pi]$.

Sol. Here, $\qquad 4\sin^2 x - 8\sin x + 3 \leq 0$

$$\Rightarrow \qquad (2\sin x - 1)(2\sin x - 3) \leq 0$$

Here, $2\sin x - 3$ is always negative.

$$\therefore \qquad 2\sin x - 1 \geq 0 \text{ i.e. } \sin x \geq \frac{1}{2}$$



$\therefore$ From the figure, $\frac{\pi}{6} \leq x \leq \frac{5\pi}{6}$

Example 42. Solve $2\cos^2\theta + \sin\theta \leq 2$, where $\pi/2 \leq \theta \leq \frac{3\pi}{2}$

Sol. $2\cos^2\theta + \sin\theta \leq 2$

or $\qquad 2(1 - \sin^2\theta) + \sin\theta \leq 2$

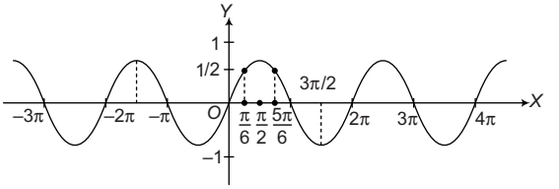
or $\qquad -2\sin^2\theta + \sin\theta \leq 0$

or $\qquad 2\sin^2\theta - \sin\theta \geq 0$

or $\qquad \sin\theta(2\sin\theta - 1) \geq 0$

or $\qquad \sin\theta(\sin\theta - 1/2) \geq 0$

which is possible if $\sin\theta \leq 0$ or $\sin\theta \geq 1/2$



From the graph

$$\sin\theta \geq 1/2 \Rightarrow \pi/2 \leq \theta \leq 5\pi/6$$

$$\sin\theta \leq 0 \Rightarrow \pi \leq \theta \leq 3\pi/2$$

Hence, the required values of θ are given by

$$\theta \in [\pi/2, 5\pi/6] \cup [\pi, 3\pi/2]$$

Example 43. Solve $\sin^2\theta > \cos^2\theta$

Sol. We have, $\qquad \sin^2\theta > \cos^2\theta \Rightarrow \cos 2\theta < 0$

$$\Rightarrow \qquad (\pi/2) < 2\theta < (3\pi/2) \text{ or } (\pi/4) < \theta < (3\pi/4)$$

Taking general values i.e., adding $2n\pi$, we get

$$2n\pi + \pi/2 < 2\theta < 2n\pi + 3\pi/2, n \in Z$$

or $\qquad n\pi + \pi/4 < \theta < n\pi + 3\pi/4$

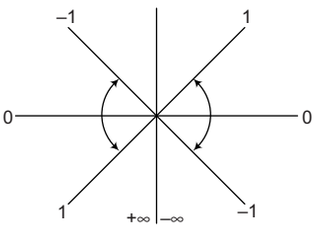
Example 44. Find the solution set for, $|\tan x| \leq 1$ when $x \in [-\pi, \pi]$.

Sol. Here, $|\tan x| \leq 1$

$\Rightarrow -1 \leq \tan x \leq 1$, the value scheme for this is shown below.

From the figure,

$$-\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$$



or $-\pi \le x \le -\frac{3\pi}{4}$

or $\frac{3\pi}{4} \le x \le \pi$

$\therefore \quad x \in \left[-\pi, -\frac{3\pi}{4}\right] \cup \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \pi\right]$

Example 45. ***Solve*** $\sin 2x > \sqrt{2}\sin^2 x + (2-\sqrt{2})\cos^2 x$

Sol. $\sin 2x > \sqrt{2}\sin^2 x + (2-\sqrt{2})\cos^2 x$

$\Rightarrow \quad 2\sin x \cos x > \sqrt{2}\sin^2 x + (2-\sqrt{2})\cos^2 x$

$\Rightarrow \quad \tan^2 x - \sqrt{2}\tan x + (\sqrt{2}-1) < 0$

$\Rightarrow \quad (\tan x - 1)(\tan x - (\sqrt{2}-1)) < 0$

$\Rightarrow \quad (\sqrt{2}-1) < \tan x < 1$

$\Rightarrow \quad \pi/8 < x < \pi/4$

$\therefore \quad x \in \left(n\pi + \frac{\pi}{8},\ n\pi + \frac{\pi}{4}\right)$ where $n \in Z$.

Example 46. Solve $\tan^3 x + 3 > 3\tan x + \tan^2 x$

Sol. $\tan^3 x - \tan^2 x + 3 - 3\tan x > 0$

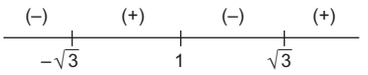
$\Rightarrow \quad \tan^2 x(\tan x - 1) - 3(\tan x - 1) > 0$

$\Rightarrow \quad (\tan x - 1)(\tan^2 x - 3) > 0$

$(\tan x - 1)(\tan x + \sqrt{3})(\tan x - \sqrt{3}) > 0$

$\Rightarrow \quad (y-1)(y+\sqrt{3})(y-\sqrt{3}) > 0$, where $\tan x = y$

Sign-scheme of above inequality is as follows:



$\therefore \quad -\sqrt{3} < y < 1$ or $y > \sqrt{3}$

$\Rightarrow \quad -\sqrt{3} < \tan x < 1$ or $\tan x > \sqrt{3}$

For $-\pi/2 < x < \pi/2$

$-\pi/3 < x < x/4$ or $\pi/3 < x < \pi/2$

$\therefore$ General solution is

$$x \in \left(n\pi + \frac{\pi}{3}, n\pi + \frac{\pi}{2}\right) \cup \left(n\pi - \frac{\pi}{3}, n\pi + \frac{\pi}{4}\right)$$

where $n \in Z$.

Exercise for Session 4

1. If $2\cos x < \sqrt{3}$ and $x \in [-\pi, \pi]$, then find the solution set for x.
2. Find the set of all x in the interval $[0, \pi]$ for which $2\sin^2 x - 3\sin x + 1 \ge 0$.
3. If $\cos x - \sin x \ge 1$ and $0 \le x \le 2\pi$, then find the solution set for x.
4. Solve $\sin\theta + \sqrt{3}\cos\theta \ge 1$, $-\pi < \theta \le \pi$.
5. Find the set of values of x, which satisfy $\sin x \cdot \cos^3 x > \cos x \cdot \sin^3 x$, $0 \le x \le 2\pi$.
6. Find the set of all x in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ which satisfy $|4\sin x - 1| < \sqrt{5}$.
7. Solve $\sin^4\left(\frac{x}{3}\right) + \cos^4\left(\frac{x}{3}\right) > \frac{1}{2}$.
8. Solve $\tan x - \tan^2 x > 0$ and $|2\sin x| < 1$.

JEE Type Solved Examples :
Single Option Correct Type Questions

● **Ex. 1.** *The number of solutions of equation* $8[x^2 - x] + 4[x] = 13 + 12[\sin x]$, $[.]$ *denotes GIF is*

(a) 0 (b) 2
(c) 4 (d) 6

Sol. (a) $8[x^2 - x] + 4[x] = 13 + 12[\sin x]$

LHS is always even and RHS is always odd. Hence, no solution.

● **Ex. 2.** *The number of ordered pairs* (x, y) *satisfying* $|x| + |y| = 2$ *and* $\sin\left(\frac{\pi x^2}{3}\right) = 1$ *is/are*

(a) 1 (b) 2
(c) 3 (d) 4

Sol. (d) $|x| + |y| = 2$

$\Rightarrow \quad |x|, |y| \in [0, 2]$

Also $\sin\left(\frac{\pi x^2}{3}\right) = 1$

$\Rightarrow \quad \frac{\pi x^2}{3} = (4n + 1)\frac{\pi}{2}$

$\Rightarrow \quad x^2 = (4n + 1)\frac{3}{2}$

$\because \quad |x| \in [0, 2]$, then only possible value of x^2 is $\frac{3}{2}$

$\therefore \quad |x| = \sqrt{\frac{3}{2}}, |y| = 2 - \sqrt{\frac{3}{2}}$

Hence, total number of ordered pairs is 4.

● **Ex. 3.** *Number of solutions of*

$$\cos^2\left[\frac{\pi}{4}(\sin x + \sqrt{2}\cos^2 x)\right] - \tan^2\left[x + \frac{\pi}{4}\tan^2 x\right] = 1,\ x \in [-2\pi, 2\pi] \text{ is}$$

(a) 1 (b) 2
(c) 4 (d) 8

Sol. (b) The solution is only possible, when

$$\cos^2\left(\frac{\pi}{4}(\sin x + \sqrt{2}\cos^2 x)\right) = 1 \qquad ...(i)$$

and $$\tan^2\left(x + \frac{\pi}{4}\tan^2 x\right) = 0 \qquad ...(ii)$$

Let's solved Eq. (i)

$$\frac{\pi}{4}(\sin x + \sqrt{2}\cos^2 x) = k\pi$$

$\Rightarrow \quad \sin x + \sqrt{2}\cos^2 x = 4k$

Now as $|\sin x + \sqrt{2}\cos^2 x| \le |\sin x| + \sqrt{2}|\cos^2 x| \le 1 + \sqrt{2} < 4$

$\therefore \ k = 0$ is only possible value.

$\Rightarrow \quad \sin x + \sqrt{2}\cos^2 x = 0$

$\Rightarrow \quad \sqrt{2}\sin^2 x - \sin x - \sqrt{2} = 0$

$\Rightarrow \quad \sin x = \frac{-1}{\sqrt{2}}, \sqrt{2}$

$\Rightarrow \quad \sin x = -\frac{1}{\sqrt{2}} \qquad (\because \sin x \ne \sqrt{2})$

$\Rightarrow \quad x = 2n\pi - \frac{\pi}{4}$ or $2n\pi + \frac{5\pi}{4}$

From Eq. (ii) we can say that the only solution possible is $x = 2n\pi - \frac{\pi}{4}$

Hence, for $x \in [-2\pi, 2\pi]$ we have 2 solutions.

● **Ex. 4.** *The general solution of* $\sin^2\theta\sec\theta + \sqrt{3}\tan\theta = 0$ *is*

(a) $\theta = m\pi + (-1)^{n+1}\frac{\pi}{3}, \theta = n\pi; n \in I$

(b) $\theta = m\pi, n \in I$

(c) $\theta = \frac{m\pi}{2}, n \in I$

(d) $\theta = m\pi + (-1)^{n+1}\frac{\pi}{3}, n \in I$

Sol. (b) $\sin^2\theta\sec\theta + \sqrt{3}\tan\theta = 0$

$\Rightarrow \quad \sin\theta(\sin\theta + \sqrt{3})\sec\theta = 0$

$\Rightarrow \quad \sin\theta = 0$

$\Rightarrow \quad \theta = n\pi, n \in I \qquad (\because \sin\theta \ne -\sqrt{3}, \sec\theta \ne 0)$

● **Ex. 5.** *The number of solutions of the equation* $\sin\left(\frac{\pi x}{2\sqrt{3}}\right) = x^2 - 2\sqrt{3}x + 4$ *is*

(a) 0 (b) 2
(c) more than 2 (d) 1

Sol. (d) We know that $-1 \le \sin\frac{\pi x}{2\sqrt{3}} \le 1$,

therefore, we must have

$$-1 \le x^2 - 2\sqrt{3}x + 4 \le 1$$

$\Rightarrow \quad -1 \le (x - \sqrt{3})^2 + 1 \le 1$

$\Rightarrow \quad -2 \le (x - \sqrt{3})^2 \le 0$

But, square of a real number cannot be negative, therefore, we must have $(x-\sqrt{3})^2=0$

$\Rightarrow \quad x=\sqrt{3}$

Note that $x=\sqrt{3}$ satisfies the given equation.

● ***Ex. 6.*** *x_1 and x_2 are two solutions of the equation $e^x \cos x=1$. The minimum number of the solutions of the equation $e^x \sin x=1$, lying between x_1 and x_2 can be*

(a) 0 (b) 1

(c) 3 (d) None of these

Sol. (b) We have $e^x \cos x=1$ or $\cos x=e^{-x}$

Consider the function, $f(x)=\cos x-e^{-x}$

$\Rightarrow \quad f(x_1)=0=f(x_2)$.

Clearly, $f(x)$ is continuous in $[x_1, x_2]$ and differentiable in (x_1, x_2).

Hence, by Rolle's theorem, there is atleast one $x\in(x_1,x_2)$ such that $f'(x)=-\sin x+e^{-x}=0$

$\Rightarrow \quad \sin xe^x=1$ has atleast one solution $\in(x_1, x_2)$.

● ***Ex. 7.*** *The product of common differences of all possible AP which are made from values of 'x' satisfying* $\cos^2\left(\frac{1}{2}\lambda x\right)+\cos^2\left(\frac{1}{2}\mu x\right)=1$

(a) $\frac{4\pi^2}{\lambda^2-\mu^2}$ (b) $\frac{4\pi}{\lambda-\mu}$

(c) $\frac{2\pi^2}{\lambda^2-\mu^2}$ (d) None of these

Sol. (a) $\frac{1+\cos(\lambda x)}{2}+\frac{1+\cos(\mu x)}{2}=1$

$\Rightarrow \quad \cos(\lambda x)+\cos(\mu x)=0$

$\Rightarrow \quad 2\cos\left(\frac{(\lambda+\mu)x}{2}\right)\cdot\cos\left(\frac{(\lambda-\mu)x}{2}\right)=0$

$\Rightarrow \quad \frac{(\lambda+\mu)x}{2}=(2n+1)\frac{\pi}{2}$

or $\quad \frac{(\lambda-\mu)x}{2}=(2n+1)\frac{\pi}{2}$

$x=\frac{(2n+1)\pi}{\lambda+\mu}$ or $x=\frac{(2n+1)\pi}{\lambda-\mu}$

Thus, common difference can be

$\frac{2\pi}{\lambda+\mu}$ or $\frac{2\pi}{\lambda-\mu}$

Now, product $=\frac{4\pi^2}{\lambda^2-\mu^2}$

● ***Ex. 8.*** *Number of solutions of the equation* $\cos^4 2x+2\sin^2 2x=17(\cos x+\sin x)^8, 0<x<2\pi$ *is*

(a) 4 (b) 8

(c) 10 (d) 16

Sol. (a) Let $\sin 2x=y$, then $1+y^4=17(1+y)^4$

Clearly, $\quad y=\sin 2x=-\frac{1}{2}$ is the only possibility

$\therefore \quad x=105°, 165°, 285°, 345°$.

● ***Ex. 9.*** *The number of values of θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ satisfying the equation $(\sqrt{3})^{\sec^2\theta}=\tan^4\theta+2\tan^2\theta$ is*

(a) 2 (b) 4

(c) 0 (d) 1

Sol. (a) $\tan^4\theta+2\tan^2\theta=(\tan^2\theta+1)^2-1$

$=(\sec^2\theta)^2-1=\sec^4\theta-1$

Put $\quad \sec^2\theta=t$

$\Rightarrow \quad (\sqrt{3})^t=t^2-1$

$\Rightarrow t=2$ is only solution as $t>1$

$\therefore$ There will be 2 values of θ in given interval.

● ***Ex. 10.*** *Number of solutions of the equation* $\cot(\theta)+\cot\left(\theta+\frac{\pi}{3}\right)+\cot\left(\theta-\frac{\pi}{3}\right)+\cot(3\theta)=0$, *where* $\theta\in\left(0,\frac{\pi}{2}\right)$

(a) Infinite (b) 0

(c) 1 (d) None of these

Sol. (c) $\cot(\theta)+\cot\left(\theta+\frac{\pi}{3}\right)+\cot\left(\theta-\frac{\pi}{3}\right)+\cot(3\theta)=0$

Put $\theta=\frac{\pi}{2}-\alpha$

$\tan\alpha+\tan\left(\alpha-\frac{\pi}{3}\right)+\tan\left(\alpha+\frac{\pi}{3}\right)+\tan 3\alpha=0$

$\tan\alpha+\frac{\tan\alpha-\tan\frac{\pi}{3}}{1+\tan\alpha\tan\frac{\pi}{3}}+\frac{\tan\alpha+\tan\frac{\pi}{3}}{1-\tan\alpha\tan\frac{\pi}{3}}+\tan 3\alpha=0$

$3\left(\frac{3\tan\alpha-\tan^3\alpha}{1-3\tan^2\alpha}\right)+\tan 3\alpha=0$

$4\tan 3\alpha=0$

$\Rightarrow \quad \tan 3\alpha=0$

$\Rightarrow \quad 3\alpha=n\pi$

$\Rightarrow \quad \alpha=\frac{\pi}{3}$

JEE Type Solved Examples :
More than One Correct Option Type Questions

● **Ex. 11.** *$0 < a < 2\pi$, $\sin^{-1}(\sin a) < x^2 - 2x$ for all $x \in I$ then $a \in$*

(a) $(0, \pi + 1)$ (b) $\left(\pi + 1, \frac{3\pi}{2}\right)$

(c) $\left(\frac{3\pi}{2}, 2\pi - 1\right)$ (d) $(2\pi - 1, 2\pi)$

Sol. (b,c) $\sin^{-1}(\sin a) < x^2 - 2x$ for all x

$\therefore \quad \sin^{-1}(\sin a) < \min(x^2 - 2x) = -1$

For $a \in \left(0, \frac{\pi}{2}\right)$, $a < -1 \Rightarrow a \in \phi$

For $a \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, $\sin^{-1}(\sin a) = \pi - a$

$\pi - a < -1$

$a > \pi + 1 \Rightarrow a \in \left(\pi + 1, \frac{3\pi}{2}\right)$

For $a \in \left(\frac{3\pi}{2}, 2\pi\right)$, $\sin^{-1}(\sin a) = a - 2\pi$

$a - 2\pi < -1$

$a < 2\pi - 1$

$a < 2\pi - 1 \Rightarrow a \in \left(\frac{3\pi}{2}, 2\pi - 1\right)$.

● **Ex. 12.** *If $\left(\cos^2 x + \frac{1}{\cos^2 x}\right)(1 + \tan^2 2y)(3 + \sin 3z) = 4$, then*

(a) x may be a multiple of π

(b) x cannot be an even multiple of π

(c) z can be a multiple of π

(d) y can be a multiple of $\frac{\pi}{2}$

Sol. (a,d) $\left(\cos^2 x + \frac{1}{\cos^2 x}\right)(1 + \tan^2 2y)(3 + \sin 3z) = 4$

Since, $\cos^2 x + \frac{1}{\cos^2 x} > 2, 1 + \tan^2 2y \geq 1, 2 \leq 3 + \sin 3z \leq 4$

so, the only possibility is

$\cos^2 x + \frac{1}{\cos^2 x} = 2, 1 + \tan^2 2y = 1, 3 + \sin 3z = 2$

$\Rightarrow \quad \cos x = \pm 1 \Rightarrow x = n\pi$

$\tan 2y = 0 \Rightarrow y = \frac{m\pi}{2}$

$\sin 3z = -1$

$\Rightarrow \quad z = (4k - 1)\frac{\pi}{6}; m, n, k \in I$

● **Ex. 13.** *The value of x in $\left(0, \frac{\pi}{2}\right)$ satisfying equation $\frac{\sqrt{5} - 1}{\sin x} + \frac{\sqrt{10 + 2\sqrt{5}}}{\cos x} = 8$ is*

(a) $\frac{\pi}{10}$ (b) $\frac{3\pi}{10}$

(c) $\frac{9\pi}{10}$ (d) $\frac{7\pi}{10}$

Sol. (a,b) $\frac{\sqrt{5} - 1}{4} \times \frac{1}{\sin x} + \frac{\sqrt{10 + 2\sqrt{5}}}{4} \times \frac{1}{\cos x} = 2$

$\frac{\sqrt{5} - 1}{4}\cos x + \frac{\sqrt{10 + 2\sqrt{5}}}{4} \times \sin x = 2\sin x \cdot \cos x$

$\sin\left[x + \frac{\pi}{10}\right] = \sin 2x$

$x + \frac{\pi}{10} = 2x, x = \frac{\pi}{10}$

or $\sin\left(x + \frac{\pi}{10}\right) = \sin(\pi - 2x)$

$\Rightarrow \quad x + \frac{\pi}{10} = \pi - 2x$

$\Rightarrow \quad 3x = \pi - \frac{\pi}{10} = \frac{9\pi}{10}$

$\Rightarrow \quad x = \frac{3\pi}{10}$

● **Ex. 14.** *Given $2\sin t = \frac{1 - 2x + 5x^2}{3x^2 - 2x - 1}$ the real values of $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ satisfying this lie in the interval*

(a) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (b) $\left[\frac{3\pi}{10}, \frac{\pi}{2}\right]$

(c) $\left[-\frac{\pi}{2}, \frac{\pi}{10}\right] \cup \left[\frac{3\pi}{10}, \frac{\pi}{2}\right]$ (d) All of these

Sol. (a,b,c,d) Given

$x^2(6\sin t - 5) + x(2 - 4\sin t) - (1 + 2\sin t) = 0$, since $x \in R, \Delta > 0$

$\Rightarrow \quad \sin t < \frac{1 - \sqrt{5}}{4}$

or $\sin t > \frac{1 + \sqrt{5}}{4}$

$\therefore \quad t \in \left[-\frac{\pi}{2}, \frac{\pi}{10}\right] \cup \left[\frac{3\pi}{10}, \frac{\pi}{2}\right]$

● ***Ex. 15.*** *The system of equations*

$\tan x = a\cot x,\ \tan 2x = b\cos y$

(a) Cannot have a solution if $a = 0$

(b) Cannot have a solution if $a = 1$

(c) Cannot have a solution if $2\sqrt{a} > |b(1-a)|$

(d) has a solution for all a and b

Sol. (b,c) If $a = 0$, then $\tan x = 0 \Rightarrow x = n\pi$ and then for any value of y such that $\cos y = 0$ the second equation satisfies option (a) is false.

If $a = 1$ then $\tan x = \cot x \Rightarrow \tan^2 x = 1$

$\Rightarrow$ $\tan 2x$ is not defined.

$\Rightarrow$ option (b) is true

Now from the first equation $\tan x = \sqrt{a}$

$\Rightarrow$ a must be positive

$$|\cos y| = \left|\frac{2\tan x}{b(1-\tan^2 x)}\right| = \left|\frac{2\sqrt{a}}{b(1-a)}\right| \le 1$$

$$\Rightarrow \quad 2\sqrt{a} \le |b(1-a)|$$

● ***Ex. 16.*** *If* $\dfrac{y+3}{2y+5} = \sin^2 x + 2\cos x + 1$, *then the value of y lie in the interval*

(a) $\left(-\infty, -\dfrac{8}{3}\right]$ (b) $\left[-\dfrac{12}{5}, \infty\right)$

(c) $\left[-\dfrac{8}{3}, -\dfrac{12}{5}\right]$ (d) $\left(-\infty, -\dfrac{8}{3}\right] \cup \left[-\dfrac{12}{5}, \infty\right)$

Sol. (a,b,d) $\cos^2 x - 2\cos x + 1 = 3 - \dfrac{y+3}{2y+5}$

$$(\cos x - 1)^2 = \frac{5y+12}{2y+5}$$

Also, $\quad -1 \le \cos x \le 1 \Rightarrow -2 \le \cos x - 1 \le 0$

$$\Rightarrow \quad 0 \le (\cos x - 1)^2 \le 4$$

$$\Rightarrow \quad 0 \le \frac{5y+12}{2y+5} \le 4$$

Now, $\quad \dfrac{5y+12}{2y+5} - 4 \le 0 \Rightarrow \dfrac{-3y-8}{2y+5} \le 0$

$$\Rightarrow \quad \frac{3y+8}{2y+5} \ge 0 \Rightarrow (3y+8)(2y+5) \ge 0$$

and $\quad \dfrac{5y+12}{2y+5} \ge 0 \Rightarrow (5y+12)(2y+5) \ge 0$

$$\Rightarrow \quad \left(-\infty, -\frac{8}{3}\right] \cup \left[-\frac{5}{2}, \infty\right)$$

and $\quad \left(-\infty, -\dfrac{5}{2}\right) \cup \left[-\dfrac{12}{5}, \infty\right)$

$$\therefore \quad y \in \left(-\infty, -\frac{8}{3}\right] \cup \left[-\frac{12}{5}, \infty\right)$$

● ***Ex. 17.*** *Which of the following set of values of 'x' satisfies the equation*

$$2^{\left(2\sin^2 x - 3\sin x + 1\right)} + 2^{\left(2 - 2\sin^2 x + 3\sin x\right)} = 9$$

(a) $x = n\pi \pm \dfrac{\pi}{6}, n \in I$ (b) $x = n\pi \pm \dfrac{\pi}{3}, n \in I$

(c) $x = n\pi, n \in I$ (d) $x = 2n\pi + \dfrac{\pi}{2}, n \in I$

Sol. (a,d) $2^{\left(2\sin^2 x - 3\sin x + 1\right)} + 2^{3-\left(2\sin^2 x - 3\sin x + 1\right)} = 9$

Let $2^{\left(2\sin^2 x - 3\sin x + 1\right)} = t$

$$\Rightarrow \quad t + \frac{8}{t} = 9 \Rightarrow t^2 - 9t + 8 = 0 \Rightarrow t = 1, 8$$

$$\Rightarrow \quad 2\sin^2 x - 3\sin x + 1 = 3$$

or $\quad 2\sin^2 x - 3\sin x + 1 = 0$

$$\Rightarrow \quad \sin x = -\frac{1}{2},$$

$$\sin x = \frac{1}{2},\ \sin x = 1$$

● ***Ex. 18.*** *For* $0 < \theta < \dfrac{\pi}{2}$, *the solution(s) of*

$$\sum_{m=1}^{6} \operatorname{cosec}\left(\theta + \frac{(m-1)\pi}{4}\right)\operatorname{cosec}\left(\theta + \frac{m\pi}{4}\right) = 4\sqrt{2}$$ *is (are)*

(a) $\dfrac{\pi}{4}$ (b) $\dfrac{\pi}{6}$

(c) $\dfrac{\pi}{12}$ (d) $\dfrac{5\pi}{12}$

Sol. (c, d) We have,

$$\sum_{m=1}^{6} \operatorname{cosec}\left[\theta + \frac{(m-1)\pi}{4}\right]\operatorname{cosec}\left[\theta + \frac{m\pi}{4}\right] = 4\sqrt{2}$$

$$\Rightarrow \quad \sum_{m=1}^{6} \frac{\sin\frac{\pi}{4}}{\sin\left[\theta + \frac{(m-1)\pi}{4}\right]\sin\left[\theta + \frac{m\pi}{4}\right]} = 4$$

$$\Rightarrow \quad \sum_{m=1}^{6} \frac{\sin\left[\left(\theta + \frac{m\pi}{4}\right) - \left(\theta + \frac{(m-1)\pi}{4}\right)\right]}{\sin\left(\theta + \frac{(m-1)\pi}{4}\right)\sin\left(\theta + \frac{m\pi}{4}\right)} = 4$$

$$\Rightarrow \sum_{m=1}^{6} \left[\sin\left(\theta + \frac{m\pi}{4}\right)\cos\left(\theta + \frac{(m-1)\pi}{4}\right) - \cos\left(\theta + \frac{m\pi}{4}\right)\sin\left(\theta + \frac{(m-1)\pi}{4}\right)\right] \Big/ \sin\left(\theta + \frac{(m-1)\pi}{4}\right)\sin\left(\theta + \frac{m\pi}{4}\right) = 4$$

$$\Rightarrow \quad \sum_{m=1}^{6}\left[\cot\left(\theta + \frac{(m-1)\pi}{4}\right) - \cot\left(\theta + \frac{m\pi}{4}\right)\right] = 4$$

$$\Rightarrow \left[\cot\theta - \cot\left(\theta + \frac{\pi}{4}\right)\right] + \left[\cot\left(\theta + \frac{\pi}{4}\right) - \cot\left(\theta + \frac{2\pi}{4}\right)\right] + \ldots + \left[\cot\left(\theta + \frac{5\pi}{4}\right) - \cot\left(\theta + \frac{6\pi}{4}\right)\right] = 4$$

$$\Rightarrow \quad \cot\theta - \cot\left(\theta + \frac{3\pi}{2}\right) = 4$$

$$\Rightarrow \quad \cot\theta + \tan\theta = 4$$

$$\Rightarrow \quad \cos^2\theta + \sin^2\theta = 4\sin\theta\cos\theta$$

$$\Rightarrow \quad \sin 2\theta = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$$

$$\Rightarrow \quad \theta = \frac{\pi}{12} \text{ or } \frac{5\pi}{12}$$

● ***Ex. 19.*** *If* $\frac{1+\sin 6^\circ}{\cos 6^\circ} = \tan A = \sqrt{\frac{1+\sin B}{1-\sin B}}$*; where A and* $B \in (0.90^\circ)$*, then*

(a) $A = 8B$ (b) $8A = B$

(c) $A - 7B = 6^\circ$ (d) $A + B = 54^\circ$

Sol. (a,c,d) $\frac{(\sin 3^\circ + \cos 3^\circ)^2}{(\cos^2 3^\circ - \sin^2 3^\circ)} = \tan A = \sqrt{\left(\frac{\sin\frac{B}{2} + \cos\frac{B}{2}}{\cos\frac{B}{2} - \sin\frac{B}{2}}\right)^2}$

$$\frac{\sin 3^\circ + \cos 3^\circ}{\cos 3^\circ - \sin 3^\circ} = \tan A = \frac{\sin\frac{B}{2} + \cos\frac{B}{2}}{\cos\frac{B}{2} - \sin\frac{B}{2}}$$

$$\frac{(\tan 3^\circ + 1)}{(-\tan 3^\circ + 1)} = \tan A = \frac{1 + \tan\frac{B}{2}}{1 - \tan\frac{B}{2}}$$

$$\tan(45^\circ + 3^\circ) = \tan A = \tan\left(45^\circ + \frac{B}{2}\right)$$

$$\Rightarrow \quad A = 48^\circ \text{ and } B = 6^\circ$$

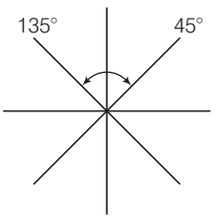
● ***Ex. 20.*** *If* $\sqrt{1+\sin A} - \sqrt{1-\sin A} = 2\cos\frac{A}{2}$*, then value of A can be*

(a) 110° (b) 260°

(c) 300° (d)190°

Sol. (a,b,d) $\sqrt{1+\sin A} - \sqrt{1-\sin A} = 2\cos\frac{A}{2}$

$$\left|\sin\frac{A}{2} + \cos\frac{A}{2}\right| - \left|\sin\frac{A}{2} - \cos\frac{A}{2}\right| = 2\cos\frac{A}{2}$$



So, $\sin\frac{A}{2} + \cos\frac{A}{2} > 0$ and $\cos\frac{A}{2} < \sin\frac{A}{2}$

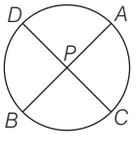
$$\Rightarrow \quad 45^\circ < \frac{A}{2} < 135^\circ \Rightarrow 90^\circ < A < 270^\circ$$

JEE Type Solved Examples : Passage Based Questions

Passage I

(Ex. Nos. 21 to 23)

Consider a circle, in which a point P is lying inside the circle such that $(PA)(PB) = (PC)(PD)$ (as shown in figure).



On the basis of above information, answer the following questions :

21. Let $PA = 4$, $PB = 3$ and CD is diameter of the circle having the length 8. If $PC > PD$, then $\frac{PC}{PD}$ is equal to

(a) 3 (b) 4

(c) 5 (d) 6

Sol. (a) (PA) (PB) = (PC) (PD)

$$4 \times 3 = x(2r - x)$$

$$\Rightarrow \quad 12 = x(8 - x)$$

$$x^2 - 8x + 12 = 0 \Rightarrow x = 6, 2$$

$$\frac{PC}{PD} = \frac{6}{2} = 3$$

22. If $PA = |\cos\theta + \sin\theta|$ and $PB = |\cos\theta - \sin\theta|$, then maximum value of $(PC)(PD)$, is equal to

(a) 1 (b) $2\sqrt{2}$

(c) $\sqrt{2}$ (d) 2

Sol. (a) $PC \cdot PD = (PA)(PB)$

$$= |\cos\theta + \sin\theta||\cos\theta - \sin\theta|$$

$$= |\cos^2\theta - \sin^2\theta| = \cos 2\theta$$

Maximum value = 1

23. If $\log_{PA} x = 2, \log_{PB} x = 3, \log_x PC = 4$, then $\log_{PD} x$ is equal to

(a) $\frac{7}{12}$ (b) $\frac{12}{7}$

(c) $-\frac{7}{12}$ (d) $-\frac{6}{19}$

Sol. (d) $(PA)(PB) = (PC)(PD)$

$$\log_x((PA)(PB)) = \log_x((PC)(PD))$$

$$\log_x PA + \log_x PB = \log_x PC + \log_x PD$$

$$\frac{1}{2} + \frac{1}{3} = 4 + \log_x PD$$

$$\log_x PD = \frac{5}{6} - 4 = -\frac{19}{6}$$

Passage II

(Ex. Nos. 24 to 26)

PA and PB are two tangents drawn from point P to circle of radius 5. A line is drawn from point P which cuts circle at C and D such that $PC = 5$ and $PD = 15$ and $\angle APB = \theta$.

On the basis of above information, answer the following questions :

24. Area of ΔAPB is

(a) $\frac{25\sqrt{3}}{2}$ (b) $25\sqrt{3}$

(c) $\frac{75\sqrt{3}}{2}$ (d) $\frac{75\sqrt{3}}{4}$

25. Value of $\sin 2\theta + \cos 4\theta + \sin 5\theta + \tan 7\theta + \cot 8\theta$ is equal to

(a) $\frac{4\sqrt{3}-1}{2}$ (b) $\frac{4-\sqrt{3}}{2\sqrt{3}}$

(c) $\frac{4+\sqrt{3}}{2\sqrt{3}}$ (d) $\frac{4\sqrt{3}+1}{2}$

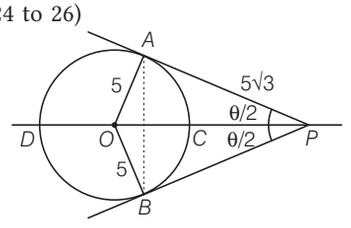
26. Number of solution(s) of the equation

$\log_{\cos\theta}(x+2) = 2 + 3\log_{(x+2)} \sin\left(\frac{5\theta}{2}\right)$ is

(a) 0 (b) 1

(c) 2 (d) 3

Sol. (Ex. Nos. 24 to 26)



$$OD = OC = 5, PD = DC + CP$$

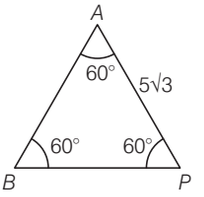
$$\Rightarrow \quad CP = 15 - 10 = 5, OP = 10$$

$$\Rightarrow \quad \sin\frac{\theta}{2} = \frac{OA}{OP} = \frac{5}{10} = \frac{1}{2} \Rightarrow \frac{\theta}{2} = 30^\circ$$

$$\Rightarrow \quad \theta = 60^\circ$$

24. (d) $\therefore$ APB is an equilateral triangle.

$$\text{Area} = \frac{\sqrt{3}}{4}(5\sqrt{3})^2 = \frac{75\sqrt{3}}{4}$$



25. (b) $\sin 2\theta + \cos 4\theta + \sin 5\theta + \tan 7\theta + \cot 8\theta$

$$= \sin 120^\circ + \cos 240^\circ + \sin 300^\circ + \tan 420^\circ + \cot 480^\circ$$

$$= \frac{\sqrt{3}}{2} - \frac{1}{2} - \frac{\sqrt{3}}{2} + \sqrt{3} - \frac{1}{\sqrt{3}} = \frac{4-\sqrt{3}}{2\sqrt{3}}$$

26. (c) Given $\log_{1/2}(x+2) = 2 + 3\log_{(x+2)}\left(\frac{1}{2}\right)$

Let $\log_{1/2}(x+2) = t$

$$t = 2 + \frac{3}{t}$$

$$\Rightarrow \quad t^2 - 2t - 3 = 0$$

$$\Rightarrow \quad t = 3 \text{ or } -1$$

$$\Rightarrow \quad \log_{1/2}(x+2) = 3$$

or $\quad \log_{1/2}(x+2) = -1$

$$\Rightarrow \quad x = -\frac{15}{8} \text{ or } x = 0$$

$\Rightarrow$ two solutions.

Passage III

(Ex. Nos. 27 and 28)

If $3\sin^2 x - 7\sin x + 2 = 0, x \in \left[0, \frac{\pi}{2}\right]$ and $f_n(\theta) = \sin^n\theta + \cos^n\theta$.

On the basis of above information, answer the following questions :

27. The value of $f_4(x)$ is

(a) $\frac{97}{81}$ (b) $\frac{57}{81}$ (c) $\frac{65}{81}$ (d) $\frac{73}{81}$

28. The value of $\frac{\sin 5x + \sin 4x}{1 + 2\cos 3x}$ is

(a) $\frac{3+2\sqrt{2}}{9}$ (b) $\frac{3+4\sqrt{2}}{9}$

(c) $\frac{4\sqrt{2}-2}{9}$ (d) $\frac{4\sqrt{2}-3}{9}$

Sol. (Ex. Nos. 27 to 28) $3\sin^2 x - 7\sin x + 2 = 0$

$$\Rightarrow \quad \sin x = \frac{1}{3} \text{ or } \sin x = 2 \text{ (reject)}$$

27. (c) $f_4(x) = \sin^4 x + \cos^4 x = 1 - 2(\sin x \cos x)^2$

$$= 1 - 2\left(\frac{1}{3} \cdot \frac{2\sqrt{2}}{3}\right)^2 = \frac{65}{81}$$

28. (b) $\dfrac{\sin 5x + \sin 4x}{1 + 2\cos 3x}$

$$= \frac{(\sin 5x + \sin 4x)\sin 3x}{(1 + 2\cos 3x)\sin 3x} = \frac{(\sin 5x + \sin 4x)\sin 3x}{\sin 3x + \sin 6x}$$

$$= \frac{2\sin\frac{9x}{2}\cos\frac{x}{2}\sin 3x}{2\sin\frac{9x}{2}\cos\frac{3x}{2}} = \frac{\cos\frac{x}{2} \cdot 2\sin\frac{3x}{2}\cos\frac{3x}{2}}{\cos\frac{3x}{2}}$$

$$= \sin 2x + \sin x$$

$$= \frac{4\sqrt{2}}{9} + \frac{1}{3} = \frac{4\sqrt{2} + 3}{9}$$

JEE Type Solved Examples :
Single Integer Answer Type Questions

● ***Ex. 29.*** *Number of integral solutions of the equation* $\log_{\sin x}\sqrt{\sin^2 x} + \log_{\cos x}\sqrt{\cos^2 x} = 2$, *where* $x \in [0, 6\pi]$ *is*

Sol. (4) $\log_{\sin x}\sqrt{\sin^2 x} + \log_{\cos x}\sqrt{\cos^2 x} = 2$

$\therefore$ $\sin x > 0$ and $\sin x \neq 1$

$\cos x > 0$ and $\cos x \neq 1$

Domain $x \in \left(0, \frac{\pi}{2}\right) \cup \left(2\pi, \frac{5\pi}{2}\right) \cup \left(4\pi, \frac{9\pi}{2}\right)$

$\Rightarrow$ Number of integers = 1 + 1 + 2 = 4

● ***Ex. 30.*** *If* $x_k = (\sec\theta)^{\frac{1}{2^k}} + (\tan\theta)^{\frac{1}{2^k}}$ *and* $y_k = (\sec\theta)^{\frac{1}{2^k}} - (\tan\theta)^{\frac{1}{2^k}}$, *then value of* $3y_n \prod_{k=0}^{n}(x_k)$ *is equal to*

Sol. (3) $x_k \cdot y_k = (\sec\theta)^{\frac{1}{2^{k-1}}} - (\tan\theta)^{\frac{1}{2^{k-1}}}$

$\Rightarrow$ $x_k \cdot y_k = y_{k-1}$

Now, $y_n \cdot \prod_{k=0}^{n} x_k = y_n \cdot \prod_{k=0}^{n} \frac{y_{k-1}}{y_k}$

$$= y_n \times \frac{y_{-1}}{y_0} \times \frac{y_0}{y_1} \times \ldots\ldots \times \frac{y_{n-1}}{y_n} = y_{-1}$$

$$= (\sec\theta)^2 - (\tan\theta)^2 = 1$$

● ***Ex. 31.*** *The number of ordered pairs* (α, β), *where* $\alpha, \beta \in [0, 2\pi]$ *satisfying* $\log_{2\sec x}(\beta^2 - 6\beta + 10) = \log_3 |\cos\alpha|$ *is*

Sol. (2) $\log_{2\sec x}(\beta^2 - 6\beta + 10) = \log_3 |\cos\alpha|$

it is only possible when

$\beta^2 - 6\beta + 10 = 1$ and $\cos\alpha = 1$

$\Rightarrow$ $\beta = 3$ and $\alpha = 0, 2\pi$

$\therefore$ Two ordered pairs (0, 3) and $(2\pi, 3)$.

● ***Ex. 32.*** *If* $\frac{\cos^3\theta}{(1 - \sin\theta)} + \frac{\sin^3\theta}{(1 + \cos\theta)} = 1 + \cos\theta$, *then number of possible values of* θ *is* (*where* $\theta \in [0, 2\pi]$).

Sol. (0) $\dfrac{\cos\theta\cos^2\theta}{(1 - \sin\theta)} + \dfrac{\sin\theta\sin^2\theta}{(1 + \cos\theta)} = \cos\theta + 1$

$$\cos\theta\frac{(1 - \sin^2\theta)}{(1 - \sin\theta)} + \frac{(1 - \cos^2\theta)\sin\theta}{(1 + \cos\theta)} = \cos\theta + 1$$

$$\cos\theta(1 + \sin\theta) + \sin\theta(1 - \cos\theta) = \cos\theta + 1$$

$$\sin\theta + \cos\theta = \cos\theta + 1 \Rightarrow \sin\theta = 1$$

$\theta = \frac{\pi}{2}$ which is not possible.

● ***Ex. 33.*** *If the sum of all values of x satisfying the system of equations*

$$\tan x + \tan y + \tan x \cdot \tan y = 5$$

$$\sin(x + y) = 4\cos x \cdot \cos y$$

is $\frac{k\pi}{2}$, *where* $x \in \left(0, \frac{\pi}{2}\right)$ *then find the values of k.*

Sol. (1) Given,

$\tan x + \tan y + \tan x \cdot \tan y = 5$ ...(i)

and $\sin(x + y) = 4\cos x \cdot \cos y$ ...(ii)

Now, from Eq. (ii), we get

$\sin x \cos y + \cos x \sin y = 4\cos x \cos y$

On dividing by $\cos x \cdot \cos y$, we get

$\tan x + \tan y = 4$ ...(iii)

$\therefore$ Eqs. (i) and (iii), $\tan x(4 - \tan x) = 1$

$\Rightarrow$ $\tan^2 x - 4\tan x + 1 = 0$

$\Rightarrow$ $\tan x = 2 + \sqrt{3}$ or $2 - \sqrt{3}$

$\Rightarrow$ $x = \frac{5\pi}{12}$ or $\frac{\pi}{12}$

$\therefore$ $\text{Sum} = \frac{6\pi}{12} = \frac{\pi}{2} = \frac{k\pi}{2} \Rightarrow k = 1$

JEE Type Solved Examples :
Statement I and II Type Questions

- This section contains **2** questions. Each question contains **Statement I (Assertion)** and **Statement II (Reason)**. Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are
  (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
  (b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement II.
  (c) Statement I is true, Statement II is false.
  (d) Statement I is false, Statement II true.

● ***Ex. 34.*** **Statement I** $x=\frac{k\pi}{13}, k\in I$ *does not represent the general solution of trigonometric equation.*

$\sin 13x-\sin 13x\cos 2x=0$

Statement II *Both* $x=r\pi, r\in I$ *and* $x=\frac{k\pi}{13}, k\in I$ *satisfies the trigonometric equation.*

$$\sin 13x-\sin 13x\cos 2x=0$$

Sol. (d) $\sin 13x(1-\cos 2x)=0$

$\Rightarrow \quad \sin 13x=0 \quad \text{or} \quad \cos 2x=1$

$\therefore \quad 13x=k\pi \quad \text{or} \quad \cos 2x=1$

$$x=\frac{k\pi}{13},\ k\in \text{Integer}.$$

$\Rightarrow \quad 2x=2n\pi \Rightarrow x=n\pi, n\in \text{Integer}$

$\Rightarrow$ Statement I is false and Statement II is true.

● ***Ex. 35.*** **Statement I** *Common value(s) of 'x' satisfying the equations.*

$\log_{\sin x}(\sec x+8)>0$ *and* $\log_{\sin x}\cos x+\log_{\cos x}\sin x=2$ *in* $(0,4\pi)$ *does not exist.*

Statement II *On solving above trigonometric equations we have to take intersection of trigonometric chains given by*

$$\sec x>1 \text{ and } x=n\pi+\frac{\pi}{4}, n\in I$$

Sol. (c) $\log_{\sin x}\cos x+\log_{\cos x}\sin x=2$ only

When $\sin x=\cos x$

$\Rightarrow \quad x=n\pi+\frac{\pi}{4}$ ...(i)

Also, $\log_{\sin x}(\sec x+8)>0$

$\Rightarrow \quad \sec x+8<1 \ \Rightarrow \ \sec x<7$ ...(ii)

Clearly, $x=\frac{\pi}{4}+n\pi$ satisfy Eq.(ii)

$\therefore$ Statement I is true and Statement II is false.

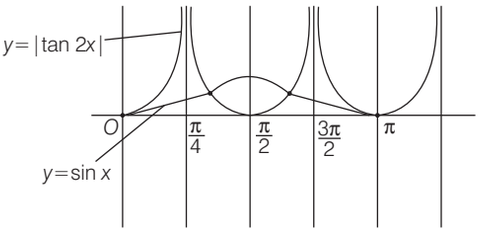
JEE Type Solved Examples :
Matching Type Questions

● ***Ex. 36.*** *Match the statement of Column I with the value of Column II.*

	Column I		Column II
A	The number of solutions of the equation $\lvert\tan 2x\rvert=\sin x;\ x\in[0,\pi]$	p	1
B	The value of $4\tan\frac{\pi}{16}-4\tan^3\frac{\pi}{16}+6\tan^2\frac{\pi}{16}-\tan^4\frac{\pi}{16}+1$	q	4
C	If the equation $\tan(p\cot x)=\cot p(\tan x)$ has a solution in $(0,x)-\left\{\frac{\pi}{2}\right\}$, then $\frac{4}{\pi}P_{\max}$ is	r	3
D	The value of $\frac{2x}{\pi}$ in $[0,2\pi]$, if $5^{\cos^2 2x+2\sin^2 x}+5^{2\cos^2 x+\sin^2 2x}=126$ has a solution	s	2

Sol. (A) $\rightarrow$(q); (B) $\rightarrow$ (s); (C) $\rightarrow$ (p); (D) $\rightarrow$ (p,r)

(A) Clearly, number of solutions of $\lvert\tan 2x\rvert=\sin x$ in $[0,\pi]$ are 4.



(B) $\tan 4A=\frac{2\tan 2A}{1-\tan^2 2A}=\frac{\frac{4\tan A}{1-\tan^2 A}}{1-\left(\frac{2\tan A}{1-\tan^2 A}\right)^2}$

$\Rightarrow \quad \tan 4A=\frac{4\tan A(1-\tan^2 A)}{1+\tan^4 A-6\tan^2 A}$

$=4\tan A-4\tan^3 A+(6\tan^2 A-\tan^2 A-1)\tan 4A=0$

If $A=\frac{\pi}{16}$

$\Rightarrow \quad 4\tan\frac{\pi}{16}-4\tan^3\frac{\pi}{6}+6\tan^2\frac{\pi}{16}-\tan^4\frac{\pi}{16}-1=0$

$\therefore$ Required value is 2.

(C) $\tan(p\cot x)=\cot(p\tan x)$

$$\tan(p\cot x)=\tan\left(\frac{\pi}{2}-p\tan x\right)$$

$$p\cot x=n\pi+\frac{\pi}{2}-p\tan x$$

$$p=\frac{n\pi+\frac{\pi}{2}}{\tan x+\cot x}=\frac{\pi}{2}\sin x\cos x \qquad (\because x\in[0,\pi])$$

$$P_{\max}=\frac{\pi}{4};\ \frac{4P_{\max}}{\pi}=1$$

(D) $5^{\cos^2 2x+2\sin^2 x}+5^{2\cos^2 x+\sin^2 2x}=126$

$$5^{\cos^2 2x+2\sin^2 x}+5^{2-2\sin^2 x+1-\cos^2 2x}$$

$$5^{\cos^2 2x+2\sin^2 x}+5^{3-\cos^2 2x+2\sin^2 x}$$

Put $\cos^2 2x+2\sin^2 x=y$, we get

$$5^y+5^{3-y}=126$$

$$5^y+\frac{125}{5^y}=126 \Rightarrow 5^y=125,1 \Rightarrow y=3,0$$

$$\cos^2 2x+2\sin^2 x=3 \Rightarrow x=\frac{\pi}{2},\frac{3\pi}{2}$$

$$\cos^2 2x+2\sin^2 x\neq 0\ ;\ \frac{2x}{\pi}=1,3$$

Subjective Type Examples

● ***Ex. 37.*** *If* $0\le x\le 3\pi, 0\le y\le 3\pi$ *and* $\cos x\cdot\sin y=1$, *then find the possible number of values of the ordered pair* (x,y).

Sol. Clearly, $\cos x\cdot\sin y=1$

$\Rightarrow \quad \cos x=1, \sin y=1$

or $\quad \cos x=-1, \sin y=-1$

Now, $\cos x=1$; $\sin y=1$

$\Rightarrow \quad x=0,\ 2\pi$ and $y=\frac{\pi}{2},\frac{5\pi}{2}$ ...(i)

and, $\cos x=-1, \sin y=-1$

$\Rightarrow \quad x=\pi, 3\pi$ and $y=\frac{3\pi}{2}$ ...(ii)

$\therefore$ From Eqs. (i) and (ii),

the required of ordered pairs are

$$\left(0,\frac{\pi}{2}\right)\left(0,\frac{5\pi}{2}\right)\left(2\pi,\frac{\pi}{2}\right)$$

$$\left(2\pi,\frac{5\pi}{2}\right)\left(\pi,\frac{3\pi}{2}\right)\left(3\pi,\frac{3\pi}{2}\right)$$

i.e. 6 solutions.

● ***Ex. 38.*** *If* $\theta\in[0,3\pi]$ *and* $r\in R$. *Then, find the pairs of* (r,θ) *satisfying* $2\sin\theta=r^4-2r^2+3$.

Sol. Here, $2\sin\theta=r^4-2r^2+3$

$\Rightarrow \quad 2\sin\theta=(r^2-1)^2+2$

$\Rightarrow \quad 2\sin\theta=(r^2-1)^2+2\ge 2$ ...(i)

But $\max(\sin\theta)=1$

$\Rightarrow \quad \max(2\sin\theta)=2$ ...(ii)

$\therefore$ From Eqs. (i) and (ii) ;

$\sin\theta=1$ and $r^2-1=0$

$\Rightarrow \quad \theta=\frac{\pi}{2},\frac{5\pi}{2}$ and $r=\pm 1$

$\therefore$ Required ordered pairs are

$$\left(\frac{\pi}{2},1\right)\left(\frac{\pi}{2},-1\right)\left(\frac{5\pi}{2},1\right)\left(\frac{5\pi}{2},-1\right)$$

i.e. 4 ordered pairs.

● ***Ex. 39.*** *Find all the values of* θ *satisfying the equation,* $\sin 7\theta=\sin\theta+\sin 3\theta$ *such that* $0\le\theta\le\pi$.

Sol. Given, $\sin 7\theta=\sin\theta+\sin 3\theta$

$\Rightarrow \quad \sin 7\theta-\sin\theta-\sin 3\theta=0$

$\therefore \quad 2\sin 3\theta\cdot\cos 4\theta-\sin 3\theta=0$

$\therefore \quad \sin 3\theta(2\cos 4\theta-1)=0$

$\Rightarrow \quad \sin 3\theta=0$ or $\cos 4\theta=\frac{1}{2}$

i.e. $\quad 3\theta=n\pi, n\in I$ or $4\theta=2n\pi\pm\frac{\pi}{3}, n\in I$

$\Rightarrow \quad \theta=\frac{n\pi}{3}$ or $\theta=\frac{n\pi}{2}\pm\frac{\pi}{12}$

put $n=0, \theta=0$ or $\theta=\pm\frac{\pi}{12}, \left(-\frac{\pi}{12}\text{ rejected}\right)$

put $\quad n=1, \theta=\frac{\pi}{3}$

or $\quad \theta=\frac{\pi}{2}+\frac{\pi}{12},\frac{\pi}{2}-\frac{\pi}{12}$

$\theta=\frac{7\pi}{12},\frac{5\pi}{12}$

put $\quad n=2, \theta=\frac{2\pi}{3}$

or $\theta = \pi \pm \frac{\pi}{12}\left(\pi + \frac{\pi}{12} \text{ rejected}\right)$

put $n = 3, \theta = \pi$

$\therefore$ Solutions are $0, \frac{\pi}{12}, \frac{\pi}{3}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{2\pi}{3}, \frac{11\pi}{12}$ and π.

● ***Ex. 40.*** *Solve* $\sin 3x + \cos 2x = -2$.

Sol. Since, $\sin 3x \geq -1$ and $\cos 2x \geq -1$. We have, $\sin 3x + \cos 2x \geq -2$.

Thus, the equality holds true if and only if;

$\sin 3x = -1$ and $\cos 2x = -1$

$\Rightarrow \quad 3x = n\pi + (-1)^n\left(-\frac{\pi}{2}\right)$ and $2x = 2n\pi \pm \pi$.

i.e. $\quad x = \frac{n\pi}{3} + (-1)^n\left(-\frac{\pi}{6}\right)$ and $x = n\pi \pm \frac{\pi}{2}, n \in I$

$\therefore$ Solution set is,

$$\left\{x \mid x = \frac{n\pi}{3} + (-1)^n\left(-\frac{\pi}{6}\right)\right\} \cap \left\{x \mid x = n\pi \pm \frac{\pi}{2}\right\}$$

Note

Here unlike all other problems the solution set consists of the intersection of two solution sets and not the union of the solution sets.

● ***Ex. 41.*** *Find all values of* θ *which satisfy,*

$\sin(3\theta + \alpha) + \sin(3\theta - \alpha) + \sin(\alpha - \theta) - \sin(\alpha + \theta) = \cos\alpha$ *given* $\cos\alpha \neq 0$.

Sol. Given, $\sin(3\theta + \alpha) + \sin(3\theta - \alpha) + \sin(\alpha - \theta) - \sin(\alpha + \theta) = \cos\alpha$

$\Rightarrow \quad 2\sin(3\theta)\cdot\cos(\alpha) + 2\sin(-\theta)\cos(\alpha) = \cos\alpha$

$\Rightarrow \quad 2(\sin 3\theta - \sin\theta)\cos\alpha = \cos\alpha$

$\Rightarrow \quad 2(\sin 3\theta - \sin\theta) = 1 \,(\text{as } \cos\alpha \neq 0 \text{ given})$

$\Rightarrow \quad 2\cdot 2\sin\theta\cdot\cos 2\theta = 1$

$\Rightarrow \quad 4\sin\theta(1 - 2\sin^2\theta) = 1$

$\Rightarrow \quad 4\sin\theta - 8\sin^3\theta = 1$

$\Rightarrow \quad 8\sin^3\theta - 4\sin\theta + 1 = 0$

$\Rightarrow \quad (2\sin\theta - 1)(4\sin^2\theta + 2\sin\theta - 1) = 0$

$\Rightarrow \quad \sin\theta = \frac{1}{2}$ or $\sin\theta = \frac{-2 \pm \sqrt{4+16}}{8} = \frac{-1 \pm \sqrt{5}}{4}$

$\Rightarrow \quad \theta = n\pi + (-1)^n\frac{\pi}{6}, n\pi + (-1)^n\sin^{-1}\left(\frac{\sqrt{5}-1}{4}\right)$

or $\quad n\pi + (-1)^n\sin^{-1}\left(\frac{-1-\sqrt{5}}{4}\right)$

$\Rightarrow \quad \theta = n\pi + (-1)^n\frac{\pi}{6}, \; n\pi + (-1)^n\frac{\pi}{10}$

or $\quad n\pi + (-1)^n\left(\frac{-3\pi}{10}\right), n \in I$

$\left[\because \sin\frac{\pi}{10} = \sin 18^\circ = \frac{\sqrt{5}-1}{4}, \sin\left(-\frac{3\pi}{10}\right) = \sin(-54^\circ) = \frac{-1-\sqrt{5}}{4}\right]$

$\therefore$ General solution set

$$= \left\{\theta \mid \theta = n\pi + (-1)^n\frac{\pi}{6}\right\} \cup \left\{\theta \mid \theta = n\pi + (-1)^n\frac{\pi}{10}\right\} \cup \left\{\theta \mid \theta = n\pi - (-1)^n\frac{3\pi}{10}\right\}$$

Ex. 42. *Solve the equation*

$$\sin^4 x + \cos^4 x = \frac{7}{2}\sin x\cdot\cos x$$

Sol. Given, $\sin^4 x + \cos^4 x = \frac{7}{2}\sin x\cdot\cos x$

$\Rightarrow \quad (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x = \frac{7}{2}\sin x\cos x$

$\Rightarrow \quad 1 - \frac{1}{2}(\sin 2x)^2 = \frac{7}{4}(\sin 2x)$

$\Rightarrow \quad 2\sin^2 2x + 7\sin 2x - 4 = 0$

$\Rightarrow \quad (2\sin 2x - 1)(\sin 2x + 4) = 0$

$\Rightarrow \quad \sin 2x = \frac{1}{2}$

or $\quad \sin 2x = -4 < -1$ (Rejected)

$\Rightarrow \quad 2x = n\pi + (-1)^n\frac{\pi}{6}$

i.e. $\quad x = \frac{n\pi}{2} + (-1)^n\frac{\pi}{12}$

● ***Ex. 43.*** *Find all the solutions of*

$$4\cos^2 x\sin x - 2\sin^2 x = 3\sin x$$

Sol. Given, $4\cos^2 x\sin x - 2\sin^2 x = 3\sin x$

$\Rightarrow \quad 4(1 - \sin^2 x)\sin x - 2\sin^2 x - 3\sin x = 0$

$\Rightarrow \quad 4\sin x - 4\sin^3 x - 2\sin^2 x - 3\sin x = 0$

$\Rightarrow \quad -4\sin^3 x - 2\sin^2 x + \sin x = 0$

$\Rightarrow \quad -\sin x(4\sin^2 x + 2\sin x - 1) = 0$

$\Rightarrow \quad \sin x = 0 \quad$ or $\quad 4\sin^2 x + 2\sin x - 1 = 0$

$\Rightarrow \quad \sin x = \sin 0 \quad$ or $\quad \sin x = \frac{-2 \pm \sqrt{4+16}}{2(4)}$

$\Rightarrow \quad x = n\pi \quad$ or $\quad \sin x = \frac{-1 \pm \sqrt{5}}{4} = \sin\frac{\pi}{10}, \sin\left(-\frac{3\pi}{10}\right)$

$\Rightarrow \quad x = n\pi, \; x = n\pi + (-1)^n\frac{\pi}{10}, \; x = n\pi + (-1)^n\left(-\frac{3\pi}{10}\right)$

$\therefore$ General solution set

$$= \{x \mid x = n\pi\} \cup \left\{x \mid x = n\pi + (-1)^n\frac{\pi}{10}\right\} \cup \left\{x \mid x = n\pi + (-1)^n\left(-\frac{3\pi}{10}\right)\right\}$$

● ***Ex. 44.*** *Solve the equation* $1+2\operatorname{cosec} x=\dfrac{-\sec^2\dfrac{x}{2}}{2}$.

Sol. Here, $1+2\operatorname{cosec} x=\dfrac{-\sec^2\dfrac{x}{2}}{2}$

$$\Rightarrow \quad 1+\frac{2}{\sin x}=-\frac{\left[1+\tan^2\frac{x}{2}\right]}{2}$$

i.e $\quad 2[2+\sin x]=-\left[1+\tan^2\dfrac{x}{2}\right]\sin x$

$$\therefore \quad 2\left[\frac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}+2\right]=-\left[1+\tan^2\frac{x}{2}\right]\cdot\left[\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right]$$

Put $\tan\dfrac{x}{2}=t$,

$$2\left[\frac{2t}{1+t^2}+2\right]=-(1+t^2)\left(\frac{2t}{1+t^2}\right)$$

$\therefore \quad 4t^2+4t+4+2t(1+t^2)=0$

$\therefore \quad 2t^3+4t^2+6t+4=0$

i.e. $\quad t^3+2t^2+3t+2=0$

$\therefore \quad (t+1)(t^2+t+2)=0$

$\Rightarrow \quad t=-1 \qquad [\because t^2+t+2\neq 0]$

or $\quad \tan\dfrac{x}{2}=t=-1$

$\Rightarrow \quad \dfrac{x}{2}=n\pi-\dfrac{\pi}{4}, n\in I$

Thus, $x=2n\pi-\dfrac{\pi}{2}, n\in I$ is the required solution.

● ***Ex. 45.*** *Find all values of* θ *lying between 0 and* 2π*, satisfying the equations*

$$r\sin\theta=\sqrt{3} \qquad ...(i)$$

$$r+4\sin\theta=2(\sqrt{3}+1) \qquad ...(ii)$$

Sol. We have to solve for θ,

$\therefore$ We shall eliminate r from Eqs. (i) and (ii),

From Eq. (i),

$$r=\frac{\sqrt{3}}{\sin\theta}$$

$\therefore$ From Eq. (ii), $\quad \dfrac{\sqrt{3}}{\sin\theta}+4\sin\theta=2(\sqrt{3}+1)$

$\Rightarrow \quad \sqrt{3}+4\sin^2\theta=2(\sqrt{3}+1)\cdot\sin\theta$

$\Rightarrow \quad 4\sin^2\theta-2\sqrt{3}\sin\theta-2\sin\theta+\sqrt{3}=0$

$\Rightarrow \quad 2\sin\theta(2\sin\theta-\sqrt{3})-1(2\sin\theta-\sqrt{3})=0$

$\Rightarrow \quad (2\sin\theta-1)(2\sin\theta-\sqrt{3})=0$

$\Rightarrow \quad \sin\theta=\dfrac{1}{2}$ or $\sin\theta=\dfrac{\sqrt{3}}{2}$

$\Rightarrow \quad \theta=n\pi+(-1)^n\dfrac{\pi}{6}$

or $\quad \theta=n\pi+(-1)^n\dfrac{\pi}{3}$

$\Rightarrow$ General solution set is,

$$\left\{\theta\,|\,\theta=n\pi+(-1)^n\frac{\pi}{6}\right\}\cup\left\{\theta\,|\,\theta=n\pi+(-1)^n\frac{\pi}{3}\right\}$$

put $n=0, \theta=\dfrac{\pi}{6}$ or $\dfrac{\pi}{3}$

put $n=1, \theta=\pi-\dfrac{\pi}{6}=\dfrac{5\pi}{6}$ or $\pi-\dfrac{\pi}{3}=\dfrac{2\pi}{3}$

$\therefore$ Solutions are $\dfrac{\pi}{6}, \dfrac{\pi}{3}, \dfrac{2\pi}{3}$ and $\dfrac{5\pi}{6}$.

● ***Ex. 46.*** *Solve the following system of equations.*

$\sin x+\cos y=1,\ \cos 2x-\cos 2y=1$

Sol. Given, $\quad \sin x+\cos y=1 \qquad ...(i)$

and $\quad (1-2\sin^2 x)-(2\cos^2 y-1)=1$

i.e. $\quad \sin^2 x+\cos^2 y=\dfrac{1}{2} \qquad ...(ii)$

Put $\sin x=u$ and $\cos y=v$ in Eqs. (i) and (ii),

$$u+v=1 \quad \text{and} \quad u^2+v^2=\frac{1}{2}$$

Solving above equations; $u=\dfrac{1}{2}$ and $v=\dfrac{1}{2} \Rightarrow \sin x=\dfrac{1}{2}$

$\Rightarrow \quad x=n\pi+(-1)^n\dfrac{\pi}{6}, n\in I$ and $\cos y=\dfrac{1}{2}$

$\Rightarrow \quad y=2m\pi\pm\dfrac{\pi}{3}, m\in I$

$\therefore$ The given equations have solutions,

$x=n\pi+(-1)^n\dfrac{\pi}{6}, n\in I$ and $y=2m\pi\pm\dfrac{\pi}{3}, m\in I$

● ***Ex. 47.*** *Find the coordinates of the points of intersection of the curves* $y=\cos x, y=\sin 3x$ *if* $-\dfrac{\pi}{2}\le x\le\dfrac{\pi}{2}$.

Sol. The point of intersection is given by

$$\sin 3x=\cos x=\sin\left(\frac{\pi}{2}-x\right)$$

$\Rightarrow \quad 3x=n\pi+(-1)^n\left(\dfrac{\pi}{2}-x\right)$

(i) Let n be even i.e. $n=2m$

$\Rightarrow \quad 3x=2m\pi+\left(\dfrac{\pi}{2}-x\right)$

$\Rightarrow \quad x=\dfrac{m\pi}{2}+\dfrac{\pi}{8} \qquad ...(i)$

(ii) Let n be odd i.e. $n = 2m + 1$

$$\Rightarrow \quad 3x = (2m+1)\pi - \left(\frac{\pi}{2} - x\right)$$

$$\Rightarrow \quad 3x = 2m\pi + \frac{\pi}{2} + x \Rightarrow 2x = 2m\pi + \frac{\pi}{2}$$

$$\Rightarrow \quad x = m\pi + \frac{\pi}{4} \qquad ...(ii)$$

Now, as $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$$\therefore \quad x = \frac{\pi}{8}, \frac{\pi}{4}, -\frac{3\pi}{8} \qquad \text{[From Eqs. (i) and (ii)]}$$

Thus, point of intersection are

$$\left(\frac{\pi}{8}, \cos\frac{\pi}{8}\right), \left(\frac{\pi}{4}, \cos\frac{\pi}{4}\right), \left(-\frac{3\pi}{8}, \cos\frac{3\pi}{8}\right)$$

● ***Ex. 48.*** *Find the range of y such that the equation in x, $y + \cos x = \sin x$ has a real solutions. For $y = 1$, find x such that $0 < x < 2\pi$.*

Sol. We have, $y = \sin x - \cos x = \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x - \frac{1}{\sqrt{2}}\cos x\right)$

$$= \sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$$

$$\Rightarrow \quad -\sqrt{2} \le y \le \sqrt{2} \qquad ...(i)$$

Now, for $y = 1$

$$\sin x - \cos x = 1$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}}\sin x - \frac{1}{\sqrt{2}}\cos x = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \sin\left(x - \frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \quad x - \frac{\pi}{4} = n\pi + (-1)^n \frac{\pi}{4}$$

$$\Rightarrow \quad x = n\pi + (-1)^n \frac{\pi}{4} + \frac{\pi}{4}$$

$$\therefore \quad x = \frac{\pi}{2}, \pi$$

● ***Ex. 49.*** *A triangle ABC is such that $\sin(2A + B) = \frac{1}{2}$. If A, B, C are in AP, then find the value of A, B and C.*

Sol. We have, $\sin(2A + B) = \frac{1}{2} = \sin\left(\frac{\pi}{6}\right)$

$$\Rightarrow \quad 2A + B = n\pi + (-1)^n \frac{\pi}{6} \qquad ...(i)$$

Also, we have, $A + B + C = \pi$ and $2B = A + C$

$$\Rightarrow \quad 3B = \pi \Rightarrow B = \frac{\pi}{3} \qquad ...(ii)$$

From Eq. (i), for $n = 1$

$$2A + B = \frac{5\pi}{6}$$

$$\Rightarrow \quad 2A = \frac{5\pi}{6} - \frac{\pi}{3} \qquad \text{[from Eq. (ii)]}$$

$$\Rightarrow \quad A = \frac{\pi}{4} \Rightarrow C = \frac{5\pi}{12}$$

● ***Ex. 50.*** *Find the number of values of θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, satisfying the equation, $(1 - \tan\theta)(1 + \tan\theta)\cdot\sec^2\theta + 2^{\tan^2\theta} = 0$. Also, find $\theta \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$, satisfying the given equation.*

Sol. Given, $(1 - \tan\theta)(1 + \tan\theta)\sec^2\theta + 2^{\tan^2\theta} = 0$

$$\Rightarrow \quad (1 - \tan^2\theta)(1 + \tan^2\theta) + 2^{\tan^2\theta} = 0$$

$$\Rightarrow \quad 1 - \tan^4\theta + 2^{\tan^2\theta} = 0$$

$$\Rightarrow \quad 1 + 2^{\tan^2\theta} = \tan^4\theta$$

By observation, we have $\tan^2\theta = 3$.

$$\Rightarrow \quad \theta = n\pi \pm \left(\frac{\pi}{3}\right)$$

Moreover there will be values of θ, satisfying, $3 < \tan^2\theta < 4$ and satisfying the given equation as if $f(x) = x^2 - 2^x - 1$, then $f(3^+)\, f(4^-) < 0$.

So the number of values of θ is 4.

And θ, lying in the interval $\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$ is $\pm\frac{\pi}{3}$.

● ***Ex. 51.*** *Find the general solutions of the equation $\left(\cos\frac{x}{4} - 2\sin x\right)\sin x + \left(1 + \sin\frac{x}{4} - 2\cos x\right)\cos x = 0$.*

Sol. Here,

$$\left(\cos\frac{x}{4} - 2\sin x\right)\sin x + \left(1 + \sin\frac{x}{4} - 2\cos x\right)\cos x = 0$$

$$\Rightarrow \cos\frac{x}{4}\cdot\sin x - 2\sin^2 x + \cos x + \sin\frac{x}{4}\cdot\cos x - 2\cos^2 x = 0$$

$$\Rightarrow \left(\sin x\cdot\cos\left(\frac{x}{4}\right) + \sin\left(\frac{x}{4}\right)\cdot\cos x\right) + \cos x - 2(\sin^2 x + \cos^2 x) = 0$$

$$\Rightarrow \quad \sin\left(x + \frac{x}{4}\right) + \cos x = 2$$

Since, the greatest value of $\sin\left(\frac{5x}{4}\right)$ and $\cos(x)$ is 1.

Therefore their sum is equal to 2 only if,

$\sin\left(\frac{5x}{4}\right) = 1$ and $\cos(x) = 1$.

$$\Rightarrow \quad \frac{5x}{4} = 2n\pi + \frac{\pi}{2} \text{ and } x = 2k\pi \ (n, k \in I)$$

Since, we have two choose those values of x which satisfy both of these equations, we have

$$2k\pi = \frac{8n\pi}{5} + \frac{2\pi}{5}$$

$\Rightarrow k = \dfrac{4n+1}{5}$, where both k and n are integers. We write,

$$k = n - \frac{(n-1)}{5}$$

for $\dfrac{n-1}{5} = m$, we have $n = 1 + 5m$ and $k = 1 + 4m$ $(m \in I)$

$\therefore \quad x = 2\pi + 8m\pi,\ m \in I.$

● ***Ex. 52.*** *Find all possible triplets (x, y, z) such that $(x+y)+(y+2z)\cos 2\theta + (z-x)\sin^2\theta = 0$, for all θ.*

Sol. We have,

$$(x+y)+(y+2z)\cos 2\theta + (z-x)\left(\frac{1-\cos 2\theta}{2}\right) = 0$$

$$\Rightarrow (2x+2y)+(2y+4z)\cos 2\theta + (z-x)-(z-x)\cos 2\theta = 0$$

$$\Rightarrow (2x+2y+z-x)+(2y+4z-z+x)\cos 2\theta = 0$$

$$\Rightarrow (x+2y+z)+(x+2y+3z)\cos 2\theta = 0$$

Which is zero for all values of θ, if

$$x+2y+z = 0 \text{ and } x+2y+3z = 0$$

$$\Rightarrow \quad x+2y+z = 0$$

and $\quad x+2y+3z = 0$

$$\frac{x}{6-2} = \frac{y}{1-3} = \frac{z}{2-2} = k$$

$$\Rightarrow \quad x = 4k,\ y = -2k,\ z = 0$$

or $\quad x = 2k,\ y = -k,\ z = 0$

i.e. $(2k, -k, 0)$ for any $k \in R$.

Hence, there are infinite number of triplets.

● ***Ex. 53.*** *For every real number find all the real solutions to equation* $\sin x + \cos(a+x) + \cos(a-x) = 2$.

Sol. Given, $\sin x + \cos(a+x) + \cos(a-x) = 2$

$$\Rightarrow \quad 1\cdot\sin x + 2\cdot\cos a\cdot\cos x = 2$$

Let $\quad 1 = r\sin\phi,\ 2\cos a = r\cos\phi$

$$\Rightarrow \quad r(\cos x\cos\phi + \sin x\cdot\sin\phi) = 2$$

$$\Rightarrow \quad r\cos(x-\phi) = 2, \begin{cases}\text{where } r^2 = 1+4\cos^2 a \\ \Rightarrow r = \sqrt{1+4\cos^2 a}\end{cases}$$

$$\Rightarrow \quad \cos(x-\phi) = \frac{2}{r}$$

$$\Rightarrow \quad \cos(x-\phi) = \frac{2}{\sqrt{1+4\cos^2 a}} \qquad \ldots(i)$$

This equation has real solution if,

$$\frac{2}{\sqrt{1+4\cos^2 a}} \le 1$$

or $\quad 2 \le \sqrt{1+4\cos^2 a}$ or $4\cos^2 a \ge 3$

or $\quad \cos^2 a \ge \dfrac{3}{4}$

$$-\cos^2 a \le -\frac{3}{4} \Rightarrow 1-\cos^2 a \le 1-\frac{3}{4}$$

$$\Rightarrow \quad \sin^2 a \le \frac{1}{4}$$

$$\Rightarrow \quad (\sin a + 1/2)(\sin a - 1/2) \le 0$$

$$\Rightarrow \quad -\frac{1}{2} \le \sin a \le \frac{1}{2}$$

$$\therefore \quad m\pi - \frac{\pi}{6} \le a \le m\pi + \frac{\pi}{6}, \text{ where } m \in I$$

Clearly, $x - \phi = 2n\pi \pm \cos^{-1}\left(\dfrac{2}{\sqrt{1+4\cos^2 a}}\right)$ [From Eq. (i)]

$$\Rightarrow \quad x = \phi + 2n\pi \pm \cos^{-1}\left(\frac{2}{\sqrt{1+4\cos^2 a}}\right)$$

where $\cos\phi = \dfrac{2\cos a}{\sqrt{1+4\cos^2 a}}$ or $\phi = \cos^{-1}\left(\dfrac{2\cos a}{\sqrt{1+4\cos^2 a}}\right)$

● ***Ex. 54.*** *Find the solutions of the equation $(\sin x + \cos x)\sin 2x = a(\sin^3 x + \cos^3 x)$ located between $\dfrac{\pi}{2}$ and π and for which values of 'a' does this equation have at most one solution satisfying the condition $\dfrac{\pi}{2} \le x \le \pi$.*

Sol. $(\sin x + \cos x)\sin 2x = a(\sin x + \cos x)[\sin^2 x + \cos^2 x - \sin x\cos x]$

$$\Rightarrow (\sin x + \cos x)\left[\sin 2x - a\left(1 - \frac{1}{2}\sin 2x\right)\right] = 0 \qquad \ldots(i)$$

Now, $\quad \sin x + \cos x = 0 \Rightarrow \tan x = -1 = \tan\dfrac{3\pi}{4}$

$$\therefore \quad x = \frac{3\pi}{4} \qquad \left[\because \frac{\pi}{2} \le x \le \pi\right]$$

Hence, there is always at least one root lying in $\dfrac{\pi}{2}$ and π for any value of the parameter a.

Now, $\quad 2\sin 2x - 2a + a\sin 2x = 0$ [from Eq. (i)]

$$\Rightarrow \quad \sin 2x = \frac{2a}{2+a} \qquad \ldots(ii)$$

Since, $\quad \dfrac{\pi}{2} < x < \pi$ or $\pi < 2x < 2\pi$

$$\Rightarrow \quad -1 \le \sin 2x < 0 \qquad \ldots(iii)$$

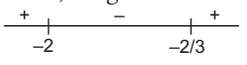
Now, from Eqs. (ii) and (iii), we have

$$-1 \le \frac{2a}{2+a} < 0, \text{ where } a \ne -2$$

$$\Rightarrow \quad -1 \le \frac{2a}{2+a} \text{ and } \frac{2a}{2+a} < 0$$

$$\Rightarrow \qquad 0 \le \frac{2a}{2+a} + 1 \text{ and } a(2+a) < 0$$

Using number line rules, we get



$$a < -2 \qquad \text{or} \qquad a \ge -\frac{2}{3} \qquad \ldots(A)$$

and $\qquad -2 < a \le 0 \qquad \ldots(B)$

From (A) and (B) $\qquad -\frac{2}{3} \le a \le 0 \qquad \ldots(iv)$

[i.e. common to both (A) and (B)]

Hence, for every value of 'a' satisfying the condition $-\frac{2}{3} \le a \le 0$ the equation, $\sin 2x = \frac{2a}{2+a}$ has the roots lying between $\frac{\pi}{2}$ and π.

Now, we have to find the solution of the equation

$\sin 2x = \frac{2a}{2+a}$, where $-\frac{2}{3} \le a \le 0$ and $\frac{\pi}{2} < x < \pi$

Clearly, $\qquad \pi < 2x < 2\pi$

Case I $\qquad \pi < 2x < \frac{3\pi}{2}$

$$0 < 2x - \pi < \frac{\pi}{2}$$

$\because \qquad \sin(2x) = \frac{2a}{2+a}$

$\therefore \qquad \sin(2x - \pi) = -\frac{2a}{2+a}; \; x = \frac{\pi}{2} - \frac{1}{2}\sin^{-1}\left(\frac{2a}{2+a}\right)$

Case II $\quad \frac{3\pi}{2} < 2x < 2\pi$

$$-\frac{\pi}{2} < 2x - 2\pi < 0$$

Since, $\qquad \sin 2x = \frac{2a}{2+a}$

$\therefore \qquad \sin(2x - 2\pi) = \frac{2a}{2+a}$

$$2x - 2\pi = \sin^{-1}\left(\frac{2a}{2+a}\right)$$

$$x = \pi + \frac{1}{2}\sin^{-1}\left(\frac{2a}{2+a}\right)$$

Thus, $\quad x = \begin{cases} \frac{3\pi}{4}, \text{ for } a \leftarrow (-\infty, \infty) \\ \frac{\pi}{2} - \frac{1}{2}\sin^{-1}\left(\frac{2a}{2+a}\right), & \text{for } \frac{-2}{3} \le a \le 0 \\ \pi + \frac{1}{2}\sin^{-1}\left(\frac{2a}{2+a}\right), & \text{for } \frac{-2}{3} \le a \le 0 \end{cases}$

● ***Ex. 55.*** *Solve the equation*

$$(\tan x)^{\cos^2 x} = (\cot x)^{\sin x}.$$

Sol. $(\tan x)^{\cos^2 x} = \left(\frac{1}{\tan x}\right)^{\sin x}$

$$\Rightarrow \qquad (\tan x)^{\cos^2 x + \sin x} = 1$$

Now, here two case arises,

Case I When $\tan x = 1$, the power $(\cos^2 x + \sin x)$ can take any value.

$$\therefore \qquad \tan x = 1$$

$$\Rightarrow \qquad x = n\pi + \frac{\pi}{4}$$

Case II When $\tan x \ne 1$, 0 and $\cos^2 x + \sin x = 0$

or $\qquad x \ne n\pi, n\pi + \frac{\pi}{4}$

and $\qquad 1 - \sin^2 x + \sin x = 0.$

$$\Rightarrow \qquad \sin^2 x - \sin x - 1 = 0$$

$$\therefore \qquad \sin x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{\sqrt{5}+1}{2}, \frac{1-\sqrt{5}}{2}$$

Here, $\frac{\sqrt{5}+1}{2} > 1$ which is not possible.

$$\sin x = \frac{1-\sqrt{5}}{2} = \sin\alpha$$

$$\therefore \; x = n\,\pi + (-1)^n \alpha, \text{ where } \alpha = \sin^{-1}\left(\frac{1-\sqrt{5}}{2}\right)$$

Thus, $x = n\pi + \frac{\pi}{4}$ or $x = n\pi + (-1)^n\alpha$,

where $\alpha = \sin^{-1}\left(\frac{1-\sqrt{5}}{2}\right)$

● ***Ex. 56.*** *Solve the equation*

$$\text{a cos x + cot x + 1 = cosec x.}$$

Sol. $a\cos x + \frac{\cos x}{\sin x} + 1 = \frac{1}{\sin x}$

$$\Rightarrow \qquad a\sin x\cos x + \cos x + \sin x = 1 \; (\sin x \ne 0)$$

$$\Rightarrow \qquad \sin x + \cos x = 1 - a\sin x\cos x$$

On squaring both sides, we get

$$1 + 2\sin x\cos x = 1 + a^2\sin^2 x\cos^2 x - 2a\sin x\cdot\cos x$$

$$a^2\sin^2 x\cos^2 x - 2(a+1)\sin x\cos x = 0$$

$$\sin x\cos x[a^2\sin x\cos x - 2(a+1)] = 0$$

$$\sin 2x[a^2\sin 2x - 4(a+1)] = 0$$

$\therefore \qquad \sin 2x = 0$ for any value of a.

$$2x = n\,\pi$$

$$x = \frac{n\pi}{2} \text{ for } a \in (-\infty, \infty)$$

or $\qquad a^2\sin 2x - 4(a+1) = 0$

$$\Rightarrow \quad \sin 2x = \frac{4(a+1)}{a^2}$$

$\because -1 \le \sin 2x \le 1$ for all values of x.

$$\therefore \quad -1 \le \frac{4(a+1)}{a^2} \le 1$$

$$-a^2 \le 4(a+1) \le a^2$$

i.e. $\quad -a^2 \le 4a + 4$ or $4a + 4 \le a^2$

$$\Rightarrow \quad a^2 + 4a + 4 \ge 0$$

or $\quad a^2 - 4a - 4 \ge 0$

$$\Rightarrow \quad (a+2)^2 \ge 0$$

or $\quad (a-2)^2 - 8 \ge 0$

which is true for all 'a' or $(a-2)^2 \ge 8$

$$\Rightarrow \quad a - 2 \ge 2\sqrt{2}$$

or $\quad a - 2 \le -2\sqrt{2}$

$$\Rightarrow \quad a \le 2 - 2\sqrt{2} \quad \text{or} \quad a \ge 2 + 2\sqrt{2}$$

or $\quad a \in (-\infty, 2 - 2\sqrt{2}] \cup [2 + 2\sqrt{2}, \infty)$

$$\Rightarrow \quad 2x = n\pi + (-1)^n \sin^{-1}\left(\frac{4(a+1)}{a^2}\right)$$

$$x = \frac{n\pi}{2} + \frac{(-1)^n}{2}\sin^{-1}\left(\frac{4(a+1)}{a^2}\right)$$

$$\Rightarrow \quad x = \begin{cases} \dfrac{n\,\pi}{2}; \text{ for } a \in (-\infty, \infty) \\ \dfrac{n\,\pi}{2} + \dfrac{(-1)^n}{2}\sin^{-1}\left(\dfrac{4(a+1)}{2}\right); \end{cases}$$

for $a \in (-\infty, 2 - 2\sqrt{2}] \cup [2 + 2\sqrt{2}, \infty)$

● ***Ex. 57.*** *Find the values of 'a' which the system of equations* $\sin x \cdot \cos y = a^2$ *and* $\sin y \cdot \cos x = a$ *have a solution.*

Sol. We have,

$$\sin x \cdot \cos y = a^2 \text{ and } \cos x \cdot \sin y = a$$

adding above equations,

$$\sin x \cos y + \cos x \sin y = a^2 + a$$

$$\Rightarrow \quad \sin(x+y) = a^2 + a \qquad \ldots(i)$$

As we know,

$$-1 \le \sin(x+y) \le 1$$

$$\therefore \quad -1 \le a^2 + a \le 1$$

i.e. $\quad -1 \le a^2 + a$ and $a^2 + a \le 1$

$$\Rightarrow \quad a^2 + a + 1 \ge 0 \text{ and } a^2 + a - 1 \le 0$$

$$\Rightarrow \quad \left(a + \frac{1}{2}\right)^2 + \frac{3}{4} \ge 0 \text{ and } \left(a + \frac{1}{2}\right)^2 - \frac{5}{4} \le 0$$

$$\Rightarrow \quad \text{true for all real '}a\text{' and } -\frac{\sqrt{5}}{2} \le a + \frac{1}{2} \le \frac{\sqrt{5}}{2}$$

$$\Rightarrow \quad -\left(\frac{\sqrt{5}+1}{2}\right) \le a \le \frac{\sqrt{5}-1}{2} \qquad \ldots(ii)$$

Again, on subtracting the two given equations,

$$\sin x \cos y - \cos x \sin y = a^2 - a$$

$$\Rightarrow \quad \sin(x-y) = a^2 - a \qquad \ldots(iii)$$

As we know,

$$-1 \le \sin(x-y) \le 1$$

$$\Rightarrow \quad -1 \le a^2 - a \le 1$$

$$\Rightarrow \quad -1 \le a^2 - a \quad \text{and} \quad a^2 - a \le 1$$

$$\Rightarrow \quad a^2 - a + 1 \ge 0 \text{ and } a^2 - a - 1 \le 0$$

$$\Rightarrow \quad \left(a - \frac{1}{2}\right)^2 + \frac{3}{4} \ge 0 \text{ and } \left(a - \frac{1}{2}\right)^2 \le \frac{5}{4}$$

$\Rightarrow$ True for all real 'a' and $-\frac{\sqrt{5}}{2} \le a - \frac{1}{2} \le \frac{\sqrt{5}}{2}$

$$\Rightarrow \quad \left(\frac{1-\sqrt{5}}{2}\right) \le a \le \frac{1+\sqrt{5}}{2} \qquad \ldots(iv)$$

From Eqs. (ii) and (iv) common solution is

$$\left(\frac{1-\sqrt{5}}{2}\right) \le a \le \left(\frac{\sqrt{5}-1}{2}\right)$$

or $\quad a \in \left[\frac{1-\sqrt{5}}{2}, \frac{\sqrt{5}-1}{2}\right]$

● ***Ex. 58.*** *Find all the numbers 'a' for which any root of the equation* $\sin 3x = a \sin x + (4 - 2|a|)\sin^2 x$ *is a root of the equation* $\sin 3x + \cos 2x = 1 + 2\sin x \cos 2x$ *and any root of the latter equation is a root of the former.*

Sol. The first equation of the system can be written as;

$$3\sin x - 4\sin^3 x = a\sin x + (4 - 2|a|)\sin^2 x$$

$$\sin x\{4\sin^2 x + (4 - 2|a|)\sin x + (a-3)\} = 0 \qquad \ldots(i)$$

The second equation is,

$$\sin 3x + \cos 2x = 1 + 2\sin x \cos 2x$$

$$\sin 3x + \cos 2x = 1 + \sin 3x - \sin x$$

$$\cos 2x = 1 - \sin x$$

$$1 - 2\sin^2 x = 1 - \sin x$$

$$\sin x(2\sin x - 1) = 0 \qquad \ldots(ii)$$

$\because$ Both equations have a common solution, therefore

$$\sin x = 0$$

Also, second root of Eq. (ii) i.e.

$$\sin x = \frac{1}{2}, \text{ satisfy Eq. (i).}$$

From Eq. (i), $4\left(\frac{1}{2}\right)^2 + (4 - 2|a|)\frac{1}{2} + (a-3) = 0$

$$\Rightarrow \quad 1 + 2 - |a| + (a-3) = 0$$

$$\Rightarrow \quad |a| = a \quad \text{or} \quad a \ge 0 \text{ for } \sin x = \frac{1}{2} \qquad \ldots(iii)$$

Again from Eq. (i);

$$4\sin^2 x + (4 - 2|a|)\sin x + (a-3) = 0$$

For real x, $\quad (4-2|a|)^2 - 4\cdot 4\cdot(a-3) \geq 0$

$\Rightarrow \quad (2-a)^2 - 4(a-3) \geq 0$

$\Rightarrow \quad 4 - 4a + a^2 - 4a + 12 \geq 0$

$\Rightarrow \quad a^2 - 8a + 16 \geq 0$

$\Rightarrow \quad (a-4)^2 \geq 0$

$\therefore \quad \sin x = \dfrac{-(4-2|a|) \pm 2\sqrt{(a-4)^2}}{2\cdot 4}$

$\Rightarrow \quad \sin x = \dfrac{|a| - 2 \pm \sqrt{(a-4)^2}}{4}$

$\Rightarrow \quad \sin x = \dfrac{|a| - 2 + a - 4}{4}$

or $\quad \sin x = \dfrac{|a| - 2 - a + 4}{4}$

$\Rightarrow \quad \sin x = \dfrac{2a-6}{4}$ or $\sin x = \dfrac{1}{2}$ [$\because |a| = a$ from Eq. (iii)]

$\Rightarrow \quad \sin x = \dfrac{a-3}{2}$ or $\sin x = \dfrac{1}{2}$

For real x the values of 'a' will be suitable in the following three cases (also $a \geq 0$).

(i) $\dfrac{a-3}{2} = 0 \Rightarrow a = 3$

(ii) $\dfrac{a-3}{2} = \dfrac{1}{2} \Rightarrow a = 4$

(iii) $\left|\dfrac{a-3}{2}\right| \leq 1$ or $|a-3| \leq 2$ or $-2 \leq a-3 \leq 2$

$\Rightarrow \quad 1 \leq a \leq 5$

$\therefore$ Hence, $a \in [1, 5]$.

● ***Ex. 59.*** *Solve the inequality*

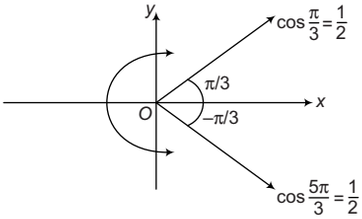
$$\frac{5}{4}\sin^2 x + \sin^2 x \cdot \cos^2 x > \cos 2x.$$

Sol. Re-writing the inequality in the form,

$$5(1 - \cos 2x) + 2(1 - \cos^2 2x) > 8\cos 2x$$

or $\quad 2\cos^2 2x + 13\cos 2x - 7 < 0$

Putting $y = \cos 2x$, we get



$$2y^2 + 13y - 7 < 0 \quad \text{or} \quad (2y-1)(y+7) < 0$$

or y lies between -7 and $\dfrac{1}{2}$ or $-7 < \cos 2x < \dfrac{1}{2}$

Since, $\quad -1 \leq \cos 2x$ for all x.

$\therefore \quad -1 \leq \cos 2x < \dfrac{1}{2}$,

Using figure, we get $2n\pi + \dfrac{\pi}{3} < 2x < 2n\pi + \dfrac{5\pi}{3}$

$\Rightarrow \quad n\pi + \dfrac{\pi}{6} < x < n\pi + \dfrac{5\pi}{6}$

● ***Ex. 60.*** *Solve the inequality,*

$$\sin x \cos x + \frac{1}{2}\tan x \geq 1$$

Sol. Left hand side is defined for all x except,

$$x = n\pi + \frac{\pi}{2}, \text{ where } n \in I$$

Now, we have $\dfrac{2\tan x}{1 + \tan^2 x} + \tan x \geq 2$ {from given equation}

Putting $y = \tan x$, we get

$$\frac{2y}{1+y^2} + y \geq 2 \quad \text{or} \quad \frac{2y + y(1+y^2) - 2(1+y^2)}{(1+y^2)} \geq 0$$

$\because \quad 1 + y^2 > 0$ for all y.

$\therefore \quad 2y + y(1+y^2) - 2(1+y^2) \geq 0$

$\Rightarrow \quad y^3 - 2y^2 + 3y - 2 \geq 0$

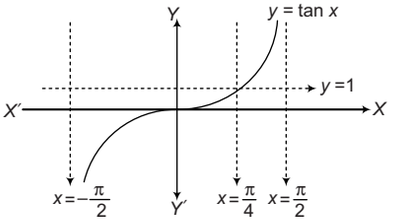
or $\quad y^2(y-1) - y(y-1) + 2(y-1) \geq 0$

or $\quad (y-1)(y^2 - y + 2) \geq 0$

where $y^2 - y + 2 = \left(y - \dfrac{1}{2}\right)^2 + \dfrac{7}{4} > 0$ for all y.

$\therefore \quad (y-1) \geq 0$ or $\tan x \geq 1$

From figure, we get



Hence, the solution of the inequality lies in the interval,

$$n\pi + \frac{\pi}{4} \leq x < n\pi + \frac{\pi}{2}, n \in I$$

i.e. $\quad x \in \left[n\pi + \dfrac{\pi}{4}, n\pi + \dfrac{\pi}{2}\right)$

● ***Ex. 61.*** *If $0 \leq x \leq 2\pi$, then solve the inequality,*

$$2^{\text{cosec}^2 x}\sqrt{\frac{1}{2}y^2 - y + 1} \leq \sqrt{2}.$$

Sol. The given inequality can be written as;

$$2^{\text{cosec}^2 x}\sqrt{(y-1)^2 + 1} \leq 2 \qquad \ldots(i)$$

Since, $\text{cosec}^2 x \geq 1$ for all real x.

$\therefore \quad 2^{\text{cosec}^2 x} \geq 2$ ...(ii)

Also, $(y-1)^2 + 1 \geq 1$

$\Rightarrow \quad \sqrt{(y-1)^2+1} \geq 1$ ...(iii)

From Eqs. (ii) and (iii),

$$2^{\text{cosec}^2 x}\sqrt{(y+1)^2+1} \geq 2 \quad \text{...(iv)}$$

$\therefore$ From Eqs. (i) and (iv), equality holds only when,

$$2^{\text{cosec}^2 x} = 2 \text{ and } \sqrt{(y-1)^2+1} = 1$$

or $\quad \text{cosec}^2 x = 1$

and $\quad (y-1)^2 + 1 = 1$

or $\quad \sin x = \pm 1$ and $y = 1$

or $\quad x = \frac{\pi}{2}, \frac{3\pi}{2}$ and $y = 1$

Hence, the solution of the given inequality is $x = \frac{\pi}{2}, \frac{3\pi}{2}$ and $y = 1$.

● ***Ex. 62.*** *Solve the inequality,*

$$5 + 2\cos 2x \leq 3\,|\,2\sin x - 1\,|.$$

Sol. The given inequality can be written as;

$$5 + 2(1 - 2\sin^2 x) \leq 3\,|\,2\sin x - 1\,|$$

or $\quad 7 - 4\sin^2 x \leq 3\,|\,2\sin x - 1\,|$

Putting $y = \sin x$,

$$7 - 4y^2 \leq 3\,|\,2y - 1\,| \quad \text{...(i)}$$

Now, consider the two cases

(i) $2y - 1 \geq 0$ or $y \geq \frac{1}{2}$

then, $\quad 7 - 4y^2 \leq 3(2y - 1)$

$[\because \text{ for } y \geq a, |\,y - a\,| = (y - a)]$

or $\quad 4y^2 + 6y - 10 \geq 0$ or $2y^2 + 3y - 5 \geq 0$

or $\quad (y-1)(2y+5) \geq 0$

Solving this inequality, we get $y \leq -\frac{5}{2}$ or $y \geq 1$.

But from the condition $y \geq \frac{1}{2}$, we have $y \geq 1$.

i.e. $\quad \sin x \geq 1$.

The inequality holds true only for x satisfying the equation $\sin x = 1$ that is

$$x = 2n\pi + \frac{\pi}{2} \text{ when } n \in I.$$

(ii) Let $2y - 1 < 0$ or $y < \frac{1}{2}$

then, $\quad 7 - 4y^2 \leq -3(2y - 1)$

$[\because \text{ for } y < a, |\,y - a\,| = -(y - a)]$

or $\quad 2y^2 - 3y - 2 \geq 0$

or $\quad (y-2)(2y+1) \geq 0$

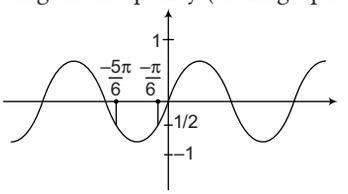
On solving the inequality, we get

$$y \leq -\frac{1}{2} \text{ or } y \geq 2.$$

But from condition $y < \frac{1}{2}$, we have $y \leq -\frac{1}{2}$.

or $\quad \sin x \leq -\frac{1}{2}$

On solving the inequality (from graph we have)



we get, $2n\pi - \frac{5\pi}{6} \leq x \leq 2n\pi - \frac{\pi}{6}$

Thus, $\quad x = 2n\pi + \frac{\pi}{2}$ or $x \in \left[2n\pi - \frac{5\pi}{6}, 2n\pi - \frac{\pi}{6}\right]$

Ex. 63. *Solve*

$$|\cos x - 2\sin 2x - \cos 3x\,| = 1 - 2\sin x - \cos 2x.$$

Sol. Here, LHS $= |\cos x - 2\sin 2x - \cos 3x|$

$= |(\cos x - \cos 3x) - 2\sin 2x|$

$= |2\sin 2x \cdot \sin x - 2\sin 2x|$

$= |2\sin 2x(\sin x - 1)|$

and RHS $= 1 - 2\sin x - \cos 2x$

$= 2\sin^2 x - 2\sin x = 2\sin x(\sin x - 1)$

Thus, $|\cos x - 2\sin 2x - \cos 3x\,| = 1 - 2\sin x - \cos 2x$ could be rewritten as,

$|\,2\sin 2x\,(\sin x - 1)| = 2\sin x(\sin x - 1)$

where $1 - \sin x \geq 0$, for all real x

$\therefore \quad |\,2\sin 2x\,|\,(1 - \sin x) = 2\sin x(\sin x - 1)$

$\Rightarrow \quad (|\,2\sin 2x\,| + 2\sin x)(1 - \sin x) = 0$

Either $\sin x = 1$

or $4\,|\sin x\,|\,|\cos x\,| + 2\sin x = 0$. So, two cases arises

Case I $\sin x = 1$

$\Rightarrow \quad x = 2n\,\pi + \frac{\pi}{2}, n \in I$

Case II $2\,|\sin x\,|\,|\cos x\,| + \sin x = 0$

If $\sin x > 0$

$\Rightarrow \quad 2\sin x\,|\cos x\,| + \sin x = 0$

or $\quad |\cos x\,| = -\frac{1}{2}$ (not possible)

Thus, no solution for

$\sin x > 0$.

Consider $\sin x \leq 0$

$\Rightarrow \quad -2\sin x\,|\cos x\,| + \sin x = 0$

$\Rightarrow \quad \sin x = 0$ i.e. $x = n\,\pi, n \in I$.

or $|\cos x| = \frac{1}{2}$

$\Rightarrow$ $\cos x = \pm\frac{1}{2}$ and $\sin x < 0$

$\Rightarrow$ Those values which lie in III or IV quadrant

$\therefore$ $x = \begin{cases} k\pi + \frac{\pi}{3}, & \text{for } k = (2m+1),\ m \in I \\ k\pi - \frac{\pi}{3}, & \text{for } k = (2m),\ m \in I \end{cases}$

Hence, $x = \begin{cases} 2n\pi + \frac{\pi}{2}, n \in I \\ (2n+1)\pi + \frac{\pi}{3}, n \in I \\ 2n\pi - \frac{\pi}{3}, n \in I \end{cases}$

● ***Ex. 64.*** *Prove that the equation* $2\sin x = |x| + a$ *has no solution for* $a \in \left(\frac{3\sqrt{3}-\pi}{3}, \infty\right)$.

Sol. We have, $\frac{3\sqrt{3}-\pi}{3} > 0$ and hence their arises three cases

Case I When $|x| \geq 2$, we have

$|x| + a > 2$, whereas $2\sin x \leq 2$

Hence the equation,

$2\sin x = |x| + a$, possesses no solution for $|x| \geq 2$

Case II When, $-2 \leq x \leq 0$, we have $0 \leq |x| \leq 2$

$\Rightarrow$ $2\sin x \leq 0$ and $|x| + a > 0$

$\therefore$ The equation,

$2\sin x = |x| + a$ has no solution.

Case III When, $0 < x < 2$, we have

In this case the given equation reduces to

$2\sin x = x + a$

Let $f(x) = 2\sin x - x$

$\Rightarrow$ $f'(x) = 2\cos x - 1 = 0$

$\Rightarrow$ $x = \frac{\pi}{3} \in (0, 2)$ is a critical point.

$f''(x) = -2\sin x < 0$ for $x = \frac{\pi}{3}$

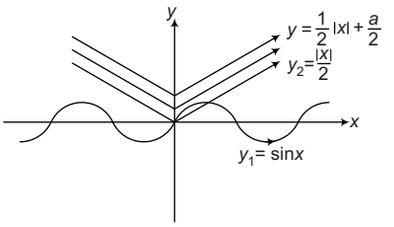
$\Rightarrow$ $x = \frac{\pi}{3}$ is a point of maxima.

$(f(x))_{\max} = 2\sin\frac{\pi}{3} - \frac{\pi}{3} = \frac{2\sqrt{3}}{2} - \frac{\pi}{3} = \frac{3\sqrt{3}-\pi}{3}$

$\therefore$ $a = 2\sin x - x$, cannot be greater $\frac{3\sqrt{3}-\pi}{3}$ for the equation to have a solution. Hence, the result.

Aliter We have, $\sin x = \frac{1}{2}|x| + \frac{a}{2}$.

Now, consider the graphs of $y_1 = \sin x$ and $y_2 = \frac{1}{2}|x|$.



The equation $\sin x = \frac{1}{2}|x| + \frac{a}{2}$ will have solution, if the line $y = \frac{1}{2}|x| + \frac{a}{2}$ (parallel to y_2) intersects or touches the curve $y_1 = \sin x$ at least one point. In this case we must have $\frac{dy_1}{dx} = \cos x = \frac{1}{2}$ (i.e. the slope of the line) $\Rightarrow$ $x = \frac{\pi}{3}$.

Hence, the solution exists if,

$$\frac{1}{2}\cdot\frac{\pi}{3} + \frac{1}{2}a \leq \sin\frac{\pi}{3} \Rightarrow a \leq \frac{3\sqrt{3}-\pi}{3}$$

● ***Ex. 65.*** *In* ΔABC, *prove that*

$$\operatorname{cosec}\frac{A}{2} + \operatorname{cosec}\frac{B}{2} + \operatorname{cosec}\frac{C}{2} \geq 6.$$

Sol. Since $\frac{A}{2}, \frac{B}{2}, \frac{C}{2}$ all are acute angles, we can use AM $\geq$ GM.

i.e. $$\frac{\operatorname{cosec}\frac{A}{2} + \operatorname{cosec}\frac{B}{2} + \operatorname{cosec}\frac{C}{2}}{3} \geq \left(\operatorname{cosec}\frac{A}{2}\cdot\operatorname{cosec}\frac{B}{2}\cdot\operatorname{cosec}\frac{C}{2}\right)^{1/3} \quad \ldots(i)$$

Consider, $\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2} = \frac{1}{2}\sin\frac{A}{2}\cdot\left(2\sin\frac{B}{2}\cdot\sin\frac{C}{2}\right)$

$$= \frac{1}{2}\sin\frac{A}{2}\left(\cos\left(\frac{B-C}{2}\right) - \cos\left(\frac{B+C}{2}\right)\right)$$

$$= \frac{1}{2}\sin\frac{A}{2}\left(\cos\left(\frac{B-C}{2}\right) - \sin\left(\frac{A}{2}\right)\right)$$

$$\leq \frac{1}{2}\sin\frac{A}{2}\left(1 - \sin\left(\frac{A}{2}\right)\right), \text{ as } \cos\left(\frac{B-C}{2}\right) \leq 1$$

$$\leq \frac{1}{2}\left(\sin\frac{A}{2} - \sin^2\frac{A}{2}\right)$$

$$\leq \frac{1}{2}\left(\frac{1}{4} - \left(\frac{1}{2} - \sin\frac{A}{2}\right)^2\right) \leq \frac{1}{8}$$

$\therefore$ $$\operatorname{cosec}\frac{A}{2}\cdot\operatorname{cosec}\frac{B}{2}\cdot\operatorname{cosec}\frac{C}{2} \geq 8 \quad \ldots(ii)$$

From Eqs. (i) and (ii),

$$\frac{\operatorname{cosec}\frac{A}{2} + \operatorname{cosec}\frac{B}{2} + \operatorname{cosec}\frac{C}{2}}{3} \geq (8)^{1/3}$$

$\Rightarrow$ $$\operatorname{cosec}\frac{A}{2} + \operatorname{cosec}\frac{B}{2} + \operatorname{cosec}\frac{C}{2} \geq 6$$

● **Ex. 66.** *If* $\frac{1}{\cos\alpha\cdot\cos\beta}+\tan\alpha\cdot\tan\beta=\tan\gamma$, *where* $0<\gamma<\frac{\pi}{2}$ *and* α,β *are positive acute angles, show that* $\frac{\pi}{4}<\gamma<\frac{\pi}{2}$.

Sol. Since, $\tan\gamma=\frac{1}{\cos\alpha\cos\beta}+\tan\alpha\cdot\tan\beta$, where $0<\gamma<\frac{\pi}{2}$

Now, $1-\tan^2\gamma=1-\left(\frac{1}{\cos\alpha\cos\beta}+\tan\alpha\cdot\tan\beta\right)^2$

$$=1-\left(\frac{1+\sin\alpha\sin\beta}{\cos\alpha\cos\beta}\right)^2$$

$$=\frac{\cos^2\alpha\cos^2\beta-\sin^2\alpha\sin^2\beta-1-2\sin\alpha\sin\beta}{\cos^2\alpha.\cos^2\beta}$$

$$=\frac{(1-\sin^2\alpha)(1-\sin^2\beta)-\sin^2\alpha\sin^2\beta-1-2\sin\alpha\sin\beta}{\cos^2\alpha\cos^2\beta}$$

$$=\frac{1-\sin^2\beta-\sin^2\alpha+\sin^2\alpha\sin^2\beta-\sin^2\alpha\sin^2\beta-1-2\sin\alpha\sin\beta}{\cos^2\alpha\cos^2\beta}$$

$$=-\frac{(\sin^2\alpha+\sin^2\beta+2\sin\alpha\sin\beta)}{\cos^2\alpha\cdot\cos^2\beta}=-\frac{(\sin\alpha+\sin\beta)^2}{\cos^2\alpha\cdot\cos^2\beta}$$

which is <0. ...(i)

Because, if it is equal to zero then $\sin\alpha+\sin\beta=0$

or $2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)=0$

$\Rightarrow$ either $\frac{\alpha-\beta}{2}=\frac{\pi}{2}$ or $\frac{\alpha+\beta}{2}=0$ which are impossible.

Thus, from Eq. (i);

$$1-\tan^2\gamma<0 \Rightarrow \tan^2\gamma>1$$

$$\Rightarrow \tan\gamma>1 \Rightarrow \gamma>\frac{\pi}{4}$$

$$\therefore \frac{\pi}{4}<\gamma<\frac{\pi}{2}.$$

● **Ex. 67.** *Find the quadrants of the coordinate planes such that for each point* (x,y) *on these quadrants (where* $x\neq0, y\neq0$*), the equation;*

$$\frac{\sin^4\theta}{x}+\frac{\cos^4\theta}{y}=\frac{1}{x+y}$$ *is solvable for* θ.

Sol. Here, $\frac{\sin^4\theta}{x}+\frac{\cos^4\theta}{y}=\frac{1}{x+y}$

$$\Rightarrow (x+y)y\sin^4\theta+(x+y)x\cos^4\theta=xy$$

$$\Rightarrow xy(\sin^4\theta+\cos^4\theta)+(x^2\cos^4\theta+y^2\sin^4\theta)=xy$$

$$\Rightarrow xy\{(\sin^2\theta+\cos^2\theta)^2-2\sin^2\theta\cos^2\theta)+(x\cos^2\theta-y\sin^2\theta)^2+2xy\cos^2\theta\sin^2\theta\}=xy$$

$$\Rightarrow xy(1-2\sin^2\theta\cos^2\theta)+\{(x\cos^2\theta-y\sin^2\theta)^2+2xy\cos^2\theta\sin^2\theta\}=xy$$

$$\Rightarrow xy-2xy\sin^2\theta\cos^2\theta+(x\cos^2\theta-y\sin^2\theta)^2+2xy\cos^2\theta\sin^2\theta=xy$$

$$\Rightarrow (x\cos^2\theta-y\sin^2\theta)^2=0 \Rightarrow \tan^2\theta=\frac{x}{y}$$

$\Rightarrow$ x and y must be of same sign, which is true in 1st and 3rd quadrant only.

Ex. 68. *For what values of 'b' does the equation* $\frac{b\cos x}{2\cos 2x-1}=\frac{b+\sin x}{(\cos^2 x-3\sin^2 x)\tan x}$ *possess solutions.*

Sol. The condition for the existence of solution are,

$$\left.\begin{array}{l}1.\ 2\cos2x-1\neq0 \text{ i.e. } \cos2x\neq\frac{1}{2} \text{ i.e. } x\neq\frac{\pi}{6}\\ 2.\ \tan x\neq0 \text{ i.e. } x\neq0,\pm\frac{\pi}{2}\\ 3.\ \cos^2x-3\sin^2x\neq0 \text{ i.e. } \tan^2x\neq\frac{1}{3} \text{ i.e. } x\neq\pm\frac{\pi}{6}\end{array}\right\} \quad ...(i)$$

Note $2\cos2x-1=2(\cos^2x-\sin^2x)-(\cos^2x+\sin^2x)$

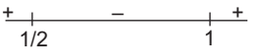
$$=\cos^2x-3\sin^2x$$

Subject to the above condition, the equation reduces to

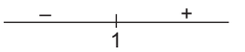
$$b\sin x=b+\sin x \Rightarrow \sin x=\frac{b}{b-1} \quad ...(ii)$$

which is only possible if ; $-1\leq\frac{b}{b-1}\leq1$

i.e. $\frac{b}{b-1}+1\geq0$ and $\frac{b}{b-1}-1\leq0$



and $\frac{1}{b-1}\leq0$



i.e. $b\leq\frac{1}{2}$ or $b>1$ and $b\leq1$ ...(iii)

From Eqs. (i), (ii) and (iii)

$$b\leq\frac{1}{2} \quad ...(iv)$$

When $b=\frac{1}{2}$, $x=\frac{-\pi}{2}$ (from Eq. (ii))

But from Eq. (i) $x\neq\frac{-\pi}{2}$

$$\Rightarrow b\neq\frac{1}{2} \quad ...(v)$$

From (iv) and (v) the equation possess solutions only when $b<\frac{1}{2}$.

Trigonometric Equations and Inequations Exercise 1 : Single Option Correct Type Questions

1. The equation $2\sin\frac{x}{2}\cos^2 x - 2\sin\frac{x}{2}\sin^2 x = \cos^2 x - \sin^2 x$ has a root for which the false statement is

(a) $\sin 2x = 1$ (b) $\cos x = \frac{1}{2}$

(c) $\cos 2x = -\frac{1}{2}$ (d) $\cos x = 1$

2. Let the smallest positive value of x for which the function $f(x) = \sin\frac{x}{3} + \sin\frac{x}{11}, (x \in R)$ achieves its maximum value be x_0. Express x_0 in degree i.e. $x_0 = \alpha^0$. Then, the sum of the digits in α is

(a) 15 (b) 17

(c) 16 (d) 18

3. The number of solutions of the equation $16(\sin^5 x + \cos^5 x) = 11(\sin x + \cos x)$ in the interval $[0, 2\pi]$ is

(a) 6 (b) 7

(c) 8 (d) 9

4. $x\sin a + y\sin 2a + z\sin 3a = \sin 4a$

$x\sin b + y\sin 2b + z\sin 3b = \sin 4b$

$x\sin c + y\sin 2c + z\sin 3c = \sin 4c.$

Then, the roots of the equation.

$t^3 - \frac{z}{2}t^2 - \frac{y+2}{4}t + \frac{z-x}{8} = 0 (a, b, c \neq n\pi)$ are

(a) $\sin a, \sin b, \sin c$

(b) $\cos a, \cos b, \cos c$

(c) sin2a, sin2b, sin2c

(d) $\cos 2a, \cos 2b, \cos 2c$

5. The least positive value of x satisfying

$$\frac{\sin^2 2x + 4\sin^4 x - 4\sin^2 x\cos^2 x}{4 - \sin^2 2x - 4\sin^2 x} = \frac{1}{9} \text{ is}$$

(a) $\pi/3$ (b) $\pi/6$

(c) $2\pi/3$ (d) $5\pi/6$

6. The maximum value of the expression $\left|\sqrt{\sin^2 x + 2a^2} - \sqrt{2a^2 - 1 - \cos^2 x}\right|$, where a and x are real number, is

(a) 1 (b) 2

(c) $\sqrt{2}$ (d) $\sqrt{3}$

7. The general solution of $8\tan^2\frac{x}{2} = 1 + \sec x$ is

(a) $x = 2n\pi \pm \cos^{-1}\left(\frac{-1}{3}\right)$ (b) $x = 2n\pi \pm \frac{\pi}{6}$

(c) $x = 2n\pi \pm \cos^{-1}\left(\frac{1}{3}\right)$ (d) None of these

8. The general solution of $\tan\theta + \tan 4\theta + \tan 7\theta = \tan\theta \cdot \tan 4\theta \cdot \tan 7\theta$

(a) $\theta = \frac{n\pi}{4}$ (b) $\theta = \frac{n\pi}{12}$

(c) $\theta = \frac{n\pi}{12}$ (d) None of these

9. The solution of the equation $e^{\sin x} - e^{-\sin x} - 4 = 0$ is

(a) $x = 0$

(b) $x = \sin^{-1}[\log(2 - \sqrt{5})]$

(c) no real solution

(d) None of the above

10. The number of the solution of the equation $\cos(\pi\sqrt{x-4}) \cdot \cos\pi\sqrt{x} = 1$ is

(a) >2 (b) 2

(c) 1 (d) 0

11. The number of real solution of the equation. $\sin(e^x) = 5^x + 5^{-x}$ is

(a) 0 (b) 1

(c) 2 (d) Infinitely many

12. ABC is a triangle such that $\sin(2A + B) = \sin(C - A) = -\sin(B + 2C) = 1/2$. If A, B and C are in AP, then the value of A, B and C are

(a) 45°, 60°, 75° (b) 30°, 60°, 90°

(c) 20°, 60°, 100° (d) None of these

13. Let $2\sin^2 x + 3\sin x - 2 > 0$ and $x^2 - x - 2 < 0$ (x is measured in radian). Then 'x' lies in the internal.

(a) $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$ (b) $\left(-1, \frac{5\pi}{6}\right)$

(c) $(-1, 2)$ (d) $\left(\frac{\pi}{6}, 2\right)$

14. The number of points of intersection of the two curves $y = 2\sin x$ and $y = 5x^2 + 2x + 3$ is

(a) 0 (b) 1

(c) 2 (d) ∞

15. The number of all possible triplets (a_1, a_2, a_3) such that $a_1 + a_2 \cos 2x + a_3 \sin^2 x = 0$ for all x is
(a) 0 (b) 1
(c) 3 (d) Infinite

16. The equation $\sin^4 x - (k+2)\sin^2 x - (k+3) = 0$ possesses a solution if
(a) $k > -3$ (b) $k < -2$
(c) $-3 \le k \le -2$ (d) k is any (+ve) value

17. In interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, the equation $\log_{\sin\theta}(\cos^2\theta) = 2$ has
(a) no solution (b) a unique solution
(c) two solution (d) infinitely many solution

18. If $\sum_{i=1}^{n} \cos\theta_i = n$, then $\sum_{i=1}^{n} \sin\theta_i$ is
(a) $n-1$ (b) 0
(c) n (d) $n+1$

19. If $0 < x < \pi/2$ and $\sin^n x + \cos^n x \ge 1$, then
(a) $n \in [2, \infty)$ (b) $(-\infty, 2]$
(c) $n \in [-1, 1]$ (d) None of these

20. The most general values of 'x' for which $\sin x + \cos x = \min_{a \in R} [1, a^2 - 4a + 6]$ are given by
(a) $2n\pi$ (b) $2n\pi + \frac{\pi}{2}$
(c) $n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{4}$ (d) None of these

21. Value of 'x' and 'y' satisfying the equation $\sin^7 y = |x^3 - x^2 - 9x + 9| + |x^3 - 4x - x^2 + 4| + \sec^2 2y + \cos^4 y$ are
(a) $x = 1, y = nx$ (b) $x = 1, y = 2n\pi + \frac{\pi}{2}$
(c) $x = 1, y = 2n\pi$ (d) None of these

22. If $\max\{5\sin\theta + 3\sin(\theta - \alpha)\} = 7$, then the set of possible value of α is $\theta \in R$
(a) $\left\{x : x = 2n\pi \pm \frac{\pi}{3}, n \in I\right\}$ (b) $\left\{x : x = 2n\pi \pm \frac{2\pi}{3}, n \in I\right\}$
(c) $\left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$ (d) None of these

23. The number of integral values of 'n' so that $\sin x(\sin x + \cos x) = n$ has atleast one solution, is
(a) 2 (b) 1
(c) 3 (d) zero

24. Total number of solution of $\sin\{x\} = \cos\{x\}$, where $\{\cdot\}$ denotes the fractional part, in $[0, 2\pi]$ is equal to
(a) 5 (b) 6
(c) 7 (d) None of these

25. If $a, b \in [0, \pi]$ and the equation $x^2 + 4 + 3\sin(ax + b) - 2x = 0$ has atleast one solution, then the value of $(a + b)$ can be:
(a) $\frac{7\pi}{2}$ (b) $\frac{3\pi}{2}$
(c) $\frac{9\pi}{2}$ (d) None of these

26. The value of a for which the equation $4\,\text{cosec}^2(\pi(a + x)) + a^2 - 4a = 0$ has a real solution, is
(a) $a = 1$ (b) $a = 2$
(c) $a = 10$ (d) None of these

27. If the equation $2\cos x + \cos 2\lambda x = 3$ has only one solution, then λ is
(a) 1 (b) A rational number
(c) An irrational number (d) None of these

28. Let n be positive integer such that $\sin\frac{\pi}{2n} + \cos\frac{\pi}{2n} = \frac{\sqrt{n}}{2}$. Then
(a) $6 \le n \le 8$ (b) $4 < n \le 8$
(c) $6 < n < 8$ (d) $4 < n < 8$

29. The number of solutions of the equation $5\sec\theta - 13 = 12\tan\theta$ in $[0, 2\pi]$ is
(a) 2 (b) 1
(c) 4 (d) 0

30. The number of solution of equation $x^3 + x^2 + 4x + 2\sin x = 0$ in $0 \le x \le 2\pi$ is
(a) Zero (b) One
(c) Two (d) Four

31. If $\tan\left(\frac{\pi}{2}\sin\theta\right) = \cot\left(\frac{\pi}{2}\cos\theta\right)$, then $\sin\theta + \cos\theta$ is equal to
(a) 0 (b) 1
(c) -1 (d) 1 or -1

32. The equation $\sin x + \sin y + \sin z = -3$ for $0 \le x \le 2\pi$, $0 \le y \le 2\pi$, $0 \le z \le 2\pi$ has
(a) one solution (b) two sets of solutions
(c) four sets of solutions (d) no solution

33. If $x = n\pi + (-1)^n \alpha, n \in I$ and $x = n\pi + (-1)^n \beta$ are the roots of $4\cos x - 3\sec x = \tan x$, then $4(\sin\alpha + \sin\beta)$ is
(a) -1 (b) 1
(c) 2 (d) None of these

34. If $\tan m\theta = \tan n\theta$ and general value of θ are in AP, then common difference is
(a) $\frac{1}{m-n}$ (b) $\frac{\pi}{m+n}$
(c) $\frac{\pi}{m-n}$ (d) None of these

35. If $\sin 3\alpha = 4\sin\alpha\sin(x+\alpha)\sin(x-\alpha)$, then

(a) $x = n\pi \pm \frac{\pi}{3}, n \in I$

(b) $x = n\pi \pm \frac{\pi}{6}, n \in I$

(c) $x = n\pi \pm \frac{\pi}{2}, n \in I$

(d) None of the above

36. $\lambda\cos x - 3\sin x = \lambda + 1$ is solvable only, if

(a) $\lambda \in [0,5]$ (b) $\lambda \in [4,5]$

(c) $\lambda \in (-\infty, 4]$ (d) None of these

37. $\cos 2x - 3\cos x + 1 = \dfrac{1}{(\cot 2x - \cot x)\sin(x-\pi)}$ holds, if

(a) $\cos x = 0$ (b) $\cos x = 1$

(c) $\cos x = \frac{5}{2}$ (d) for no value of x

38. If $\sec x \cos 5x = -1$ and $0 < x < \frac{\pi}{4}$, then x is equal to

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{4}$ (d) None of these

39. If $\sin^{100}\theta - \cos^{100}\theta = 1$, then θ is

(a) $2n\pi + \frac{\pi}{3}, n \in I$ (b) $n\pi + \frac{\pi}{2}, n \in I$

(c) $n\pi + \frac{\pi}{4}, n \in I$ (d) $2n\pi - \frac{\pi}{3}, n \in I$

40. If $\sqrt{3}\sin x - \cos x = \min_{\alpha \in R}\{2, e^2, \pi, \alpha^2 - 4\alpha + 7\}$, then

(a) $x = 2n\pi, n \in I$ (b) $x = 2n\pi + \frac{2\pi}{3}, n \in I$

(c) $x = n\pi + (-1)^n \frac{\pi}{4} + \frac{\pi}{6}, n \in I$ (d) $x = n\pi + (-1)^{n+1}\frac{\pi}{4} - \frac{\pi}{3}, n \in I$

41. The number of solutions of the equation $\cos 4x + 6 = 7\cos 2x$, when $x \in [315°, 317°]$ is

(a) 0 (b) 1

(c) 2 (d) 4

42. The number of solutions of $\cot(5\pi\sin\theta) = \tan(5\pi\cos\theta)$, $\forall\, \theta \in (0, 2\pi)$ is

(a) 7 (b) 14

(c) 21 (d) 28

43. If $\exp[(\sin^2 x + \sin^4 x + \sin^6 x + \ldots\infty)\ \ln 2]$ satisfies the equation $y^2 - 9y + 8 = 0$, then the value of $\dfrac{\cos x}{\cos x + \sin x}, 0 < x < \frac{\pi}{2}$, is

(a) $\sqrt{3} + 1$ (b) $\frac{\sqrt{3}-1}{2}$

(c) $\sqrt{3} - 1$ (d) None of these

44. Total number of solutions of $\cos x = \sqrt{1 - \sin 2x}$ in $[0, 2\pi]$, is equal to

(a) 2 (b) 3

(c) 5 (d) None of these

45. If the equation $\cos 3x \cos^3 x + \sin 3x \sin^3 x = 0$, then x is equal to

(a) $(2n+1)\frac{\pi}{4}$ (b) $(2n-1)\frac{\pi}{4}$

(c) $\frac{n\pi}{4}$ (d) None of these

46. Total number of solutions of $\sin x = \dfrac{|x|}{10}$ is equal to

(a) 4 (b) 6

(c) 7 (d) None of these

47. The number of all possible 5-tuples $(a_1, a_2, a_3, a_4, a_5)$ such that $a_1 + a_2\sin x + a_3\cos x + a_4\sin 2x + a_5\cos 2x = 0$ holds for all x is

(a) zero (b) 1

(c) 2 (d) infinite

48. x_1 and x_2 are two positive values of x for which $2\cos x, |\cos x|$ and $3\sin^2 x - 2$ are in GP. The minimum value of $|x_1 - x_2|$ is equal to

(a) $\frac{4\pi}{3}$ (b) $\frac{\pi}{3}$

(c) $2\cos^{-1}\left(\frac{2}{3}\right)$ (d) $\cos^{-1}\left(\frac{2}{3}\right)$

49. If $\cos x - \dfrac{\cot\beta\sin x}{2} = \frac{\sqrt{3}}{2}$, then the value of $\tan\frac{x}{2}$ is

(a) $\tan\frac{\beta}{2}\tan 15°$

(b) $\tan\frac{\beta}{2}$

(c) $\tan 15°$

(d) None of the above

50. The expression $n\sin^2\theta + 2n\cos(\theta+\alpha)\sin\alpha\sin\theta + \cos 2(\alpha+\theta)$ is independent of 'θ', the value of n is

(a) 1 (b) 2

(c) 3 (d) 4

51. The value of the determinant $\begin{vmatrix} 1 & a & a^2 \\ \cos(n-1)x & \cos nx & \cos(n+1)x \\ \sin(n-1)x & \sin nx & \sin(n+1)x \end{vmatrix}$ is zero if

(a) $x = n\pi$ (b) $x = n\pi/2$

(c) $x = (2n+1)\pi/2$ (d) $x = \frac{1+a^2}{2a} n \in I$

52. $\frac{\sin 3\alpha}{\cos 2\alpha} < 0$ if α lies in

(a) $(13\pi/48, 14\pi/48)$ (b) $(14\pi/48, 18\pi/48)$

(c) $(18\pi/48, 23\pi/48)$ (d) any of these intervals

53. If $f(x) = \begin{vmatrix} \sin^2\theta & \cos^2\theta & x \\ \cos^2\theta & x & \sin^2\theta \\ x & \sin^2\theta & \cos^2\theta \end{vmatrix}$ $\theta \in (0, \pi/2)$, then roots of $f(x) = 0$ are

(a) $1/2, -1$ (b) $1/2, -1, 0$

(c) $-1/2, 1, 0$ (d) $-1/2, -1, 0$

54. The equation $\sin x + \sin y + \sin z = -3$ for $0 \le x \le 2\pi$, $0 \le y \le 2\pi$, $0 \le z \le 2\pi$, has

(a) One solution (b) Two sets of solutions

(c) Four sets of solutions (d) No solutions

55. If $\sec x \cos 5x + 1 = 0$, where $0 < x < 2\pi$, then $x =$

(a) $\frac{\pi}{5}, \frac{\pi}{4}$ (b) $\frac{\pi}{5}$

(c) $\frac{\pi}{4}$ (d) None of these

56. If $|k| = 5$ and $0° \le \theta \le 360°$, then the number of different solutions of $3\cos\theta + 4\sin\theta = k$ is

(a) Zero (b) Two

(c) One (d) Infinite

57. If $\cot(\alpha + \beta) = 0$, then $\sin(\alpha + 2\beta) =$

(a) $\sin\alpha$ (b) $\cos\alpha$

(c) $\sin\beta$ (d) $\cos 2\beta$

58. If $\cot\theta + \cot\left(\frac{\pi}{4} + \theta\right) = 2$, then the general value of θ is

(a) $2n\pi \pm \frac{\pi}{6}$ (b) $2n\pi \pm \frac{\pi}{3}$

(c) $n\pi \pm \frac{\pi}{3}$ (d) $n\pi \pm \frac{\pi}{6}$

59. If $\cos 2\theta = (\sqrt{2} + 1)\left(\cos\theta - \frac{1}{\sqrt{2}}\right)$, then the value of θ is

(a) $2n\pi + \frac{\pi}{4}$ (b) $2n\pi \pm \frac{\pi}{4}$

(c) $2n\pi - \frac{\pi}{4}$ (d) None of these

60. If $\left|1 - \frac{|\sin x|}{1 + \sin x}\right| \ge \frac{2}{3}$, then $\sin x$ lies in

(a) $\left(-\infty, \frac{-1}{2}\right] \cup \left[\frac{1}{2}, \infty\right)$

(b) $\left(-\frac{1}{2}, \frac{1}{2}\right)$

(c) $\left[-\frac{1}{2}, \frac{1}{2}\right]$

(d) None of the above

Trigonometric Equations and Inequations Exercise 2 : More than One Option Correct Type Questions

61. The value of 't' which satisfies $(t - [|\sin x|])! = 3!\,5!\,7!$ is/are ..... where [.] is GIF

(a) 9 (b) 10

(c) 11 (d) 12

62. Let $f(x) = \cos(a_1 + x) + \frac{1}{2}\cos(a_2 + x) + \frac{1}{2^2}\cos(a_3 + x) + \ldots + \frac{1}{2^{n-1}}\cos(a_n + x)$

where $a_1, a_2, \ldots, a_n \in R$. If $f(x_1) = f(x_2) = 0$, then $|x_2 - x_1|$ may be equal to

(a) π (b) 2π

(c) 3π (d) $\frac{\pi}{2}$

63. Let α, β, γ be parametric angles of 3 points P, Q and R respectively lying on $x^2 + y^2 = 1$. If the lengths of chords AP, AQ and AR are in GP where A is $(1, 0)$, then [Given $\alpha, \beta, \gamma \in (0, 2\pi)$].

(a) $\sin\frac{\alpha+\gamma}{4}\cos\frac{\alpha-\gamma}{4} \ge \sin\frac{\beta}{2}$

(b) $\sin\left(\frac{\alpha+\gamma}{4}\right)\cos\left(\frac{\alpha-\gamma}{4}\right) \le \sin\frac{\beta}{2}$

(c) $\sin\frac{\alpha}{2}\sin\frac{\gamma}{2} \ge \sin\frac{\beta}{2}$

(d) $\sin\frac{\alpha}{2}\sin\frac{\gamma}{2} \le \sin\frac{\beta}{2}$

64. Let x, y, z be elements from interval $[0, 2\pi]$ satisfying the inequality $(4 + \sin 4x)(2 + \cot^2 y)(1 + \sin^4 z) \le 12\sin^2 z$, then

(a) the number of ordered pairs (x, y) is 5

(b) the number of ordered pairs (y, z) is 8

(c) the number of ordered pairs (z, x) is 8

(d) the number of pairs (y, z) such that $z = y$ is 2

65. The number of integral values of a for which the system of linear equations $x\sin\theta - 2y\cos\theta - az = 0$, $x + 2y + z = 0$, $-x + y + z = 0$ may have non- trivial solutions, then

(a) at $a = 2$ the given system will have finite solutions for $\theta \in R$

(b) number of possible integral values of a is 3

(c) for $a = 1$ the system will have infinite solutions

(d) for $a = 3$ the system will have unique solution

66. The equation $2\sin^3\theta + (2\lambda - 3)\sin^2\theta - (3\lambda + 2)\sin\theta - 2\lambda = 0$ has exactly three roots in $(0, 2\pi)$, then λ can be equal to

(a) 0 (b) $\frac{1}{2}$

(c) 1 (d) -1

67. If $x + y = 2\pi/3$ and $\sin x/\sin y = 2$, then the

(a) number of values of $x \in [0, 4\pi]$ are 4

(b) number of values of $x \in [0, 4\pi]$ are 2

(c) number of values of $y \in [0, 4\pi]$ are 2

(d) number of values of $y \in [0, 4\pi]$ are 8

68. If $0 \le x \le 2\pi$ and $|\cos x| \le \sin x$, then

(a) the set of all values of x is $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

(b) the number of solutions that are integral multiple of $\frac{\pi}{4}$ is four

(c) the sum of the largest and the smallest solution is π

(d) the set of all values of x is $x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right]$

69. If x and y are positive acute angles such that $(x + y)$ and $(x - y)$ satisfy the equation $\tan^2\theta - 4\tan\theta + 1 = 0$, then

(a) $x = \frac{\pi}{6}$ (b) $y = \frac{\pi}{4}$

(c) $y = \frac{\pi}{6}$ (d) $x = \frac{\pi}{4}$

70. If $x + y = \frac{4\pi}{3}$ and $\sin x = 2\sin y$, then

(a) $x = n\pi + \frac{\pi}{2}, n \in I$

(b) $y = \frac{5\pi}{6} - n\pi, n \in I$

(c) Both (a) and (b)

(d) None of the above

71. The number of solutions of the equations $y = \frac{1}{3}[\sin\theta + [\sin\theta + [\sin\theta]]]$ and $[y + [y]] = 2\cos\theta$ [where, [·] denotes the greatest integer function] is /are

(a) 0 (b) 1

(c) 2 (d) infinite

72. If $[\sin x] + [\sqrt{2}\cos x] = -3$, $x \in [0, 2\pi]$, (where, [·] denotes the greatest integer function), then

(a) $x \in \left(\pi, \frac{5\pi}{4}\right)$ (b) $x \in \left(\pi, \frac{7\pi}{6}\right)$

(c) $x \in \left[\pi, \frac{5\pi}{4}\right]$ (d) None of these

73. If $\alpha \in [-2\pi, 2\pi]$ and $\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2} = \sqrt{2}(\cos 36° - \sin 18°)$ then a values of α is

(a) $\frac{7\pi}{6}$ (b) $\frac{\pi}{6}$

(c) $-\frac{5\pi}{6}$ (d) $-\frac{\pi}{6}$

74. The number of values of α in the interval $[-\pi, 0]$ satisfying $\sin\alpha + \int_{\alpha}^{2\alpha}\cos 2x\,dx = 0$, then

(a) $\alpha = 0$

(b) $\alpha = 0, -\pi, -\frac{\pi}{3}$

(c) $\alpha = \frac{\pi}{6}, \frac{5\pi}{6}$

(d) None of the above

75. The solutions of $\theta \in [0, 2\pi]$ satisfying the equation $\log_{\sqrt{3}}\tan\theta\left(\sqrt{\log_{\tan\theta} 3 + \log_{\sqrt{3}} 3\sqrt{3}}\right) = -1$, then

(a) $\theta = \frac{\pi}{6}$ (b) $\frac{\pi}{3}, \frac{5\pi}{3}$

(c) has sum $\frac{4\pi}{3}$ (d) >2

76. If α and β are the solutions of $a\cos\theta + b\sin\theta = c$, then

(a) $\sin\alpha + \sin\beta = \frac{2bc}{a^2 + b^2}$

(b) $\sin\alpha\sin\beta = \frac{c^2 - a^2}{a^2 + b^2}$

(c) $\sin\alpha + \sin\beta = \frac{2ac}{b^2 + c^2}$

(d) $\sin\alpha + \sin\beta = \frac{c}{a^2 + b^2}$

77. The solution of the equation $\sin 2x + \sin 4x = 2\sin 3x$ is

(a) $x = \frac{n\pi}{3}$

(b) $x = n\pi$

(c) $x = 2n\pi$

(d) None of the above

78. The general solution of $4\sin^4 x + \cos^4 x = 1$ is

(a) $(2n + 1)\frac{\pi}{2}$ (b) $n\pi$

(c) $n\pi \pm \sin^{-1}\sqrt{\frac{2}{5}}$ (d) None of these

79. The values of $x, 0 \le x \le \frac{\pi}{2}$ whcih satisfy the equation $81^{\sin^2 x} + 81^{\cos^2 x} = 30$ are

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{6}$ (d) $\frac{7\pi}{18}$

80. All values of $x \in \left(0, \frac{\pi}{2}\right)$ such that $\frac{\sqrt{3}-1}{\sin x} + \frac{\sqrt{3}+1}{\cos x} = 4\sqrt{2}$ are

(a) $\frac{\pi}{15}$ (b) $\frac{\pi}{12}$

(c) $\frac{11\pi}{36}$ (d) $\frac{3\pi}{10}$

81. $\frac{\alpha^2}{1-\tan^2 x} = \frac{\sin^2 x + \alpha^2 - 2}{\cos 2x}$ has a solution if

(a) $\alpha \le -1$ (b) $\alpha \ge 1$

(c) $\alpha = 1/2$ (d) α is any real number

82. The equation $4\sin(x+\pi/3)\cos(x-\pi/6) = a^2 + \sqrt{3}\sin 2x - \cos 2x$ has a solution if the value of

(a) -2 (b) 0

(c) 2 (d) $a, a \in]-2, 2[$

83. Which of the following is/are correct.

(a) $(\tan x)^{\ln(\sin x)} > (\cot x)^{\ln(\sin x)}, \forall x \in (0, \pi/4)$

(b) $4^{\ln \operatorname{cosec} x} < 5^{\ln \operatorname{cosec} x}, \forall x \in (0, \pi/2)$

(c) $(1/2)^{\ln(\cos x)} < (1/3)^{\ln(\cos x)}, \forall x \in (0, \pi/2)$

(d) $2^{\ln(\tan x)} > 2^{\ln(\tan x)}, \forall x \in (0, \pi/2)$

84. The value of θ, lying between $\theta = 0$ and $\theta = \frac{\pi}{2}$ and satisfying the equation.

$$\begin{vmatrix} 1+\cos^2\theta & \sin^2\theta & 4\sin 4\theta \\ \cos^2\theta & 1+\sin^2\theta & 4\sin 4\theta \\ \cos^2\theta & \sin^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0, \text{ is}$$

(a) $\frac{11\pi}{24}$ (b) $\frac{7\pi}{24}$

(c) $\frac{5\pi}{24}$ (d) None of these

85. If $[x]$ denotes the greatest integer less than or equal to x then the equation $\sin x = [1+\sin x] + [1-\cos x]$ has no solution in

(a) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (b) $\left[\frac{\pi}{2}, \pi\right]$

(c) $\left[\pi, \frac{3\pi}{2}\right]$ (d) R

Trigonometric Equations and Inequations Exercise 3 : Passage Based Questions

Passage I

(Q. Nos. 86 to 88)

If number of solutions and sum of solutions of the equation $3\sin^2 x - 7\sin x + 2 = 0, x \in [0, 2\pi]$ are respectively N and S and $f_n(\theta) = \sin^n \theta + \cos^n \theta$. On the basis of above information, answer the following questions.

86. Value of N is

(a) 1 (b) 2 (c) 3 (d) 4

87. Value of S is

(a) $\frac{5\pi}{6}$ (b) $\frac{7\pi}{6}$ (c) 2π (d) π

88. If α is solution of equation $3\sin^2 x - 7\sin x + 2 = 0$, $x \in [0, 2\pi]$, then the value of $f_4(\alpha)$ is.

(a) $\frac{97}{81}$ (b) $\frac{57}{81}$ (c) $\frac{65}{81}$ (d) 0

Passage II

(Q. Nos. 89 to 90)

Let $\log_a N = \alpha + \beta$ where α is integer and $\beta = [0, 1)$. Then, On the basis of above information, answer the following questions.

89. The difference of largest and smallest integral value of N satisfying $\alpha = 3$ and $a = 5$, is

(a) 499 (b) 500 (c) 501 (d) 502

90. If N_1 is number of integers when $a = 2$ and $\alpha = 2$ and N_2 is number of integers when $\alpha = 1$ and $a = 3$, then the minimum value of $(N_1 \sec^2\theta + N_2 \operatorname{cosec}^2\theta)$

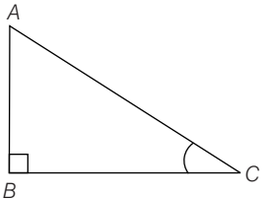
(a) $10 + 4\sqrt{6}$ (b) $10 + \sqrt{6}$ (c) 10 (d) 100

Passage III

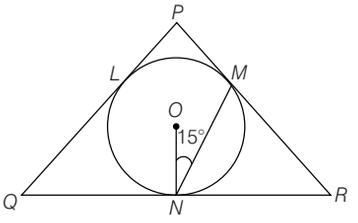
(Q. Nos. 91 to 93)

If an angle and a side of a right angle triangle is known, then rest of the sides and angles can be found as follows

In ΔABC (Figure 1), if $\angle B = 90°$, $\angle C = \theta$ and $BC = x$, then $AB = x\tan\theta$ and $AC = x\sec\theta$.



Now, consider an isosceles triangle PQR (Figure 2),



where $PQ = PR$ and $20N = \sqrt{3}$

On the basis of above information answer the following

91. The angle of triangle PQR are

(a) 150°, 15° 15° (b) 60°, 60°, 60°
(c) 120°, 30°, 30° (d) 75°, 52.5°, 52.5°

92. Area of circumcircle of quadrilateral $PLOM$ is

(a) π (b) $\frac{\pi}{4}$ (c) $\frac{3\pi}{4}$ (d) 3π

93. Length of the side QR is

(a) $\tan 15°$ (b) $\sqrt{3}\tan 15°$
(c) $\cot 15°$ (d) $\sqrt{3}\cot 15°$

Passage IV

(Q. Nos. 94 to 96)

α is a root of equation $(2\sin x - \cos x)(1+\cos x) = \sin^2 x$, β is a root of the equation $3\cos 2x - 10\cos x + 3 = 0$ and γ is a root of the equation $1 - \sin 2x = \cos x - \sin x : 0 \le \alpha, \beta, \gamma \le \pi/2$

94. $\cos\alpha + \cos\beta + \cos\gamma$ can be equal to

(a) $\frac{3\sqrt{6}+2\sqrt{2}+6}{6\sqrt{2}}$ (b) $\frac{3\sqrt{3}-8}{6}$
(c) $\frac{3\sqrt{3}+2}{6}$ (d) None of these

95. $\sin\alpha + \sin\beta + \sin\gamma$ can be equal to

(a) $\frac{14-3\sqrt{2}}{6\sqrt{2}}$ (b) $5/6$
(c) $\frac{3+4\sqrt{2}}{6}$ (d) $\frac{1+\sqrt{2}}{2}$

96. $\sin(\alpha - \beta)$ is equal to

(a) 1 (b) 0
(c) $\frac{1-2\sqrt{6}}{6}$ (d) $\frac{\sqrt{3}-2\sqrt{2}}{6}$

Passage V

(Q. Nos. 97 to 99)

Consider the equations

$$5\sin^2 x + 3\sin x\cos x - 3\cos^2 x = 2 \quad \ldots(i)$$
$$\sin^2 x - \cos 2x = 2 - \sin 2x \quad \ldots(ii)$$

97. If α is a root of (i) and β is a root of (ii), then $\tan\alpha + \tan\beta$ can be equal to

(a) $1 + \sqrt{69}/6$ (b) $-1 - \sqrt{69}/6$
(c) $\frac{-3+\sqrt{69}}{6}$ (d) $\frac{3-\sqrt{69}}{3}$

98. If $\tan\alpha$, $\tan\beta$ satisfy (i) and $\cos\gamma$, $\cos\delta$ satisfy (ii), then $\tan\alpha \cdot \tan\beta + \cos\gamma + \cos\delta$ can be equal to

(a) -1 (b) $-\frac{5}{3} + \frac{2}{\sqrt{13}}$
(c) $\frac{5}{3} - \frac{2}{\sqrt{13}}$ (d) $\frac{5}{3} + \frac{2}{\sqrt{13}}$

99. The number of solutions common to (i) and (ii) is

(a) 0 (b) 1
(c) finite (d) infinite

Trigonometric Equations and Inequations Exercise 4 : Single Integer Answer Type Questions

100. Let Δ_k be the area of triangle AP_kB which is inscribed in a circle of radius 2 units. If AB diameter of circle,

$\angle ABP_k = \frac{k\pi}{2n}$ and $\sum_{k=1}^{n+1} \Delta_k = 4\cot\frac{\pi}{32}$, then $\frac{n}{2}$ is equal to

101. If the sum of the roots of the equation $\cos 4x + 6 = 7\cos 2x$ in the interval $[0, 314]$ is $k\pi$, $k \in R$ Find $(k - 4948)$.

102. If equation $x^2\tan^2\theta - (2\tan\theta)x + 1 = 0$ and

$$\left(\frac{1}{1+\log_b ac}\right)x^2 + \left(\frac{1}{1+\log_c ab}\right)x + \left(\frac{1}{1+\log_a bc} - 1\right) = 0$$

(where $a, b, c, > 1$) have a common root and then 2nd equation has equal roots, then number of possible value of θ in $(0, \pi)$ is

103. Number of ordered pairs (x, y) which satisfies the relation $\frac{x^4+1}{8x^2} = \sin^2 y.\cos^2 y$, where $y \in [0, 2\pi]$.

104. The number of solutions for,

$$\left.\begin{array}{l}\sin\left(x - \frac{\pi}{4}\right) - \cos\left(x + \frac{3\pi}{4}\right) = 1 \\ \frac{2\cos 7x}{\cos 3 + \sin 3} > 2^{\cos 2x}\end{array}\right\} \text{ in } (0, 2\pi), \text{ is}$$

105. If $\cos A \sin\left(A-\frac{\pi}{6}\right)$ is maximum, when the values of A is equal to $\frac{\pi}{\lambda}$, then the value of λ is

106. Let $p, q \in N$ and $q > p$, the number of solutions of the equation $q|\sin\theta| = p|\cos\theta|$ in the interval $[0, 2\pi]$ is

107. If $\theta_1, \theta_2, \theta_3$ are three values lying in $[0, 2\pi]$ for which $\tan\theta = \lambda$, then $\left|\tan\frac{\theta_1}{3}\cdot\tan\frac{\theta_2}{3}+\tan\frac{\theta_2}{3}\cdot\tan\frac{\theta_3}{3}+\tan\frac{\theta_1}{3}\cdot\tan\frac{\theta_3}{3}\right|$ is equal to

108. If α be the smallest positive root of the equation $\sqrt{\sin(1-x)} = \sqrt{\cos x}$, then the approximate integral value of α must be

109. If x and y are the solutions of the equation $12\sin x + 5\cos x = 2y^2 - 8y + 21$, the value of $12\cot\left(\frac{xy}{2}\right)$ must be

110. If $\tan(\pi\cos\theta) = \cot(\pi\sin\theta)$, then $\cos^2(\theta - \pi/4)$ is equal to

111. If $3\sin x + 4\cos x = 5$, then the value of $90\tan^2(x/2) - 60\tan(x/2) + 10$ is equal to

Trigonometric Equations and Inequations Exercise 5 : Statement I and II Type Questions

- This section contains 6 questions. Each question contains **Statement I (Assertion)** and **Statement II (Reason)**. Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are
  - (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
  - (b) Statement I is true, Statement II is true, Statement II is not a correct explanation for Statement II.
  - (c) Statement I is true, Statement II is false.
  - (d) Statement I is false, Statement II is true.

112. **Statement I** $\sin x = a$, where $-1 < a < 0$, then for $x \in [0, n\pi]$ has $2(n-1)$ solutions $\forall\, n \in N$.

Statement II $\sin x$ takes value a exactly two times when we take one complete rotation covering all the quadrants starting from $x = 0$.

113. **Statement I** The number of solutions of the equation $|\sin x| = |x|$ is only one.

Statement II $|\sin x| \geq 0\, \forall\, x \in R$.

114. **Statement I** If $2\sin 2x - \cos 2x = 1$, $x \neq (2n+1)\pi/2$, n is the integer, then $\sin 2x + \cos 2x$ is equal to $1/5$.

Statement II $\sin 2x + \cos 2x = \frac{1 + 2\tan x - \tan^2 x}{1 + \tan^2 x}$

115. **Statement I** The system of linear equations

$$x + (\sin\alpha)y + (\cos\alpha)z = 0$$
$$x + (\cos\alpha)y + (\sin\alpha)z = 0$$
$$-x + (\sin\alpha)y - (\cos\alpha)z = 0$$

has a non trivial solution for only one value of α lying between 0 and π.

Statement II $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$

has no solution in the interval $-\pi/4 < x < \pi/4$.

116. Let $\theta \in (\pi/4, \pi/2)$, then

Statement I $(\cos\theta)^{\sin\theta} < (\cos\theta)^{\cos\theta} < (\sin\theta)^{\cos\theta}$

Statement II The equation $e^{\sin\theta} - e^{-\sin\theta} = 4$ has a unique solution.

117. **Statement I** If $\exp\{(\sin^2 x + \sin^4 x + \sin^6 x + \ldots \text{inf})\log_e 2\}$ satisfying the equation $x^2 - 9x + 8 = 0$, then the value of $\frac{\cos x}{\cos x + \sin x}$ is $\frac{\sqrt{3}-1}{2}\cdot(0 < x < \pi/2)$

Statement II $\sin^2 x + \sin^4 x + \sin^6 x + \ldots \text{inf} = \sec^2 x$

Trigonometric Equations and Inequations Exercise 6 : Matching Type Questions

- Math the statement of Column I with the value of Column II.

118.

	Column I		Column II
(A)	If α, β are the solutions of $\sin x = -\frac{1}{2}$ in $[0, 2\pi]$ and α, γ are the solutions of $\cos x = \frac{-\sqrt{3}}{2}$ in $[0, 2\pi]$, then	(p)	$\alpha - \beta = \pi$
(B)	If α, β are the solutions of $\cot x = -\sqrt{3}$ in $[0, 2\pi]$ and α, γ are the solutions of $\operatorname{cosec} x = -2$ in $[0, 2\pi]$, then	(q)	$\beta - \gamma = \pi$
(C)	If α, β are the solutions of $\sin x = -\frac{1}{2}$ in $[0, 2\pi]$ and α, γ are the solutions of $\tan x = \frac{1}{\sqrt{3}}$ in $[0, 2\pi]$, then	(r)	$\alpha - \gamma = \pi$
		(s)	$\alpha + \beta = 3\pi$
		(t)	$\beta + \gamma = 2\pi$

(a) A → (q, s) ; B → (p, t); C → (r, s, t)
(b) A → (q) ; B → (t); C → (r)
(c) A → (r, t) ; B → (t); C → (p, q)
(d) A → (p, q) ; B → (q, r) C → (r, s, t)

119.

	Column I		Column II
(A)	$2\sin\theta\lvert\cos\theta\rvert = \frac{1}{\sqrt{2}}$	(p)	$\theta = 3\pi/8$
(B)	$2\cos 2\theta \cos 4\theta + 2\cos^2 2\theta - 1 = 0$	(q)	$\theta = 7\pi/8$
(C)	$8\cos^2\theta\sin\theta - 4\cos^2\theta - 2\sin\theta + 1 = 0$	(r)	$\theta = 2\pi/3$
(D)	$\sin 4\theta = \pm 1$	(s)	$\theta = \pi/6$

(a) A → (p, q); B → (p, q, r); C → (r, s); D → (p, q)
(b) A → (r, s); B → (q, r); C → (r); D → (p, s)
(c) A → (p); B → (q); C → (r); D → (s)
(d) A → (s); B → (q, r); C → (r, s); D → (p, q)

120. If $f_n(\theta) = \dfrac{\sin\theta + \sin 3\theta + \sin 5\theta + \ldots + \sin((2n-1)\theta)}{\sin\theta + \cos\theta + \cos 5\theta + \ldots + \cos((2n-1)\theta)}$

	Column I		Column II
(A)	$f_3\left(\frac{\pi}{12}\right)$	(p)	$\sqrt{2} - 1$
(B)	$f_4\left(\frac{\pi}{32}\right)$	(q)	$2 - \sqrt{3}$
(C)	$f_6\left(\frac{\pi}{16}\right)$	(r)	$\sqrt{2} + 1$
(D)	$f_7\left(\frac{\pi}{84}\right)$	(s)	$2 + \sqrt{3}$
		(t)	1

(a) A → (p); B → (q); C → (r); D → (s,t)
(b) A → (t); B → (p); C → (r); D → (q)
(c) A → (q); B → (r); C → (s); D → (t)
(d) A → (r,t); B → (s); C → (p); D → (q)

Trigonometric Equations and Inequations Exercise 7 : Subjective Type Questions

121. Find the number of solutions of the equations;

(i) $|\cot x| = \cot x + \dfrac{1}{\sin x}$, when $x \in [0, 2\pi]$

(ii) $\sin^3 x \cos x + \sin^2 x \cdot \cos^2 x + \sin x \cdot \cos^3 x = 1$, when $x \in [0, 2\pi]$

(iii) $2^{\cos x} = |\sin x|$, when $x \in [-2\pi, 2\pi]$

(iv) $|\cos x| = [x]$, (where [.] denotes the greatest integer function).

(v) $x + 2\tan x = \dfrac{\pi}{2}$, when $x \in [0, 2\pi]$.

122. Find all value of α for which the equation $\sin^4 x + \cos^4 x + \sin 2x + \alpha = 0$ is valid. Also, find the general solution of the equation.

123. If $32\tan^8\theta = 2\cos^2\alpha - 3\cos\alpha$ and $3\cos 2\theta = 1$, then find the general value of α.

124. Solve for x and y $\quad 4^{\sin x} + 3^{\frac{1}{\cos y}} = 11$

$$5.16^{\sin x} - 2.3^{\frac{1}{\cos y}} = 2$$

125. Find all numbers x, y that satisfy the equation

$$\left(\sin^2 x + \frac{1}{\sin^2 x}\right)^2 + \left(\cos^2 x + \frac{1}{\cos^2 x}\right)^2 = 12 + \frac{1}{2}\sin y.$$

126. Find all the solutions of x, y in the equation

$$4\left(3\sqrt{4x - x^2}\sin^2\left(\frac{x+y}{2}\right) + 2\cos(x+y)\right) = 13 + 4\cos^2(x+y)$$

127. Solve for x and y, $1 - 2x - x^2 = \tan^2(x+y) + \cot^2(x+y)$.

128. Solve the system of equations

$$\tan^2 x + \cot^2 x = 2\cos^2 y$$

$$\cos^2 y + \sin^2 z = 1$$

129. Find all pairs of x, y that satisfy the equation

$$\cos x + \cos y + \cos(x+y) = -\frac{3}{2}$$

130. Solve the equation $\cot\left(\frac{\theta}{2}\right) - \text{cosec}\left(\frac{\theta}{2}\right) = \cot\theta$.

131. Find the general solution of $1 + \sin^3 x + \cos^3 x = \frac{3}{2}\sin 2x$.

132. Solve $\log_{(\sin x)} 2\log_{(\sin^2 x)} a = -1$ stating any condition on 'a' that may be required for the existence of the solution.

133. Find all the values of 'a' $(a \neq 0)$ for which the equation $\int_0^x (t^2 - 8t + 13)dt = x\sin\frac{a}{x}$ has a solution. Find the solution.

134. Find all values between 0 and π which satisfies the equation

$$\sin^8 x + \cos^8 x = \frac{17}{16}\cos^2 2x$$

135. Find all number pairs x, y that satisfy the equation

$$\tan^4 x + \tan^4 y + 2\cot^2 x.\cot^2 y = 3 + \sin^2(x+y).$$

136. Determine all values of 'a' for which the equation $\cos^4 x - (a+2)\cos^2 x - (a+3) = 0$, possesses solution. Find the solutions.

137. For $x \in (-\pi, \pi)$ find the value of x for which the given equation

$$(\sqrt{3}\sin x + \cos x)^{\sqrt{\sqrt{3}\sin 2x - \cos 2x + 2}} = 4 \text{ is satisfied.}$$

138. Show that the equation

$\sec\theta + \text{cosec}\,\theta = c$ has two roots between 0 and 2π, if $c^2 < 8$ and four root if $c^2 > 8$.

139. Solve the equation for x and y,

$|\sin x + \cos x|^{\sin^2 x - 1/4} = 1 + |\sin y|$ and

$\cos^2 y = 1 + \sin^2 y$.

Trigonometric Equations and Inequations Exercise 8 : Questions Asked in Previous 10 Years' Exam

(i) JEE Advanced & IIT-JEE

140. Let $S = \left\{x \in (-\pi, \pi): x \neq 0, \pm\frac{\pi}{2}\right\}$. The sum of all distinct solutions of the equation $\sqrt{3}\sec x + \text{cosec}\, x + 2(\tan x - \cot x) = 0$ in the set S is equal to **[Single Correct Option 2016 Adv.]**

(a) $-\frac{7\pi}{9}$ (b) $-\frac{2\pi}{9}$

(c) 0 (d) $\frac{5\pi}{9}$

141. The number of distinct solutions of the equation $\frac{5}{4}\cos^2 2x + \cos^4 x + \sin^4 x + \cos^6 x + \sin^6 x = 2$ in the interval $[0, 2\pi]$ is **[Integer Answer Type 2015 Adv.]**

142. For $x \in (0, \pi)$, the equation $\sin x + 2\sin 2x - \sin 3x = 3$ has

(a) infinitely many solutions **[Single Correct Option 2014 Adv.]**

(b) three solutions

(c) one solution

(d) no solution

143. Let $\theta, \phi \in [0, 2\pi]$ be such that $2\cos\theta(1 - \sin\phi) = \sin^2\theta$

$\left(\tan\frac{\theta}{2} + \cot\frac{\theta}{2}\right)\cos\phi - 1$, $\tan(2\pi - \theta) > 0$

and $-1 < \sin\theta < -\frac{\sqrt{3}}{2}$. Then, ϕ cannot satisfy **[More than One Correct Option 2012]**

(a) $0 < \phi < \frac{\pi}{2}$ (b) $\frac{\pi}{2} < \phi < \frac{4\pi}{3}$

(c) $\frac{4\pi}{3} < \phi < \frac{3\pi}{2}$ (d) $\frac{3\pi}{2} < \phi < 2\pi$

144. If $P = \{\theta : \sin\theta - \cos\theta = \sqrt{2}\cos\theta\}$ and $Q = \{\theta : \sin\theta + \cos\theta = \sqrt{2}\sin\theta\}$ be two sets. Then,

[Single Correct Option 2011]

(a) $P \subset Q$ and $Q - P \neq \phi$ (b) $Q \not\subset P$

(c) $P \not\subset Q$ (d) $P = Q$

145. The positive integer value of $n > 3$ satisfying the equation $\dfrac{1}{\sin\left(\dfrac{\pi}{n}\right)} = \dfrac{1}{\sin\left(\dfrac{2\pi}{n}\right)} + \dfrac{1}{\sin\left(\dfrac{3\pi}{n}\right)}$ is ...

[Integer Answer Type 2011]

146. The number of values of θ in the interval $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ such that $\theta \neq \dfrac{n\pi}{5}$ for $n = 0, \pm 1, \pm 2$ and $\tan\theta = \cot 5\theta$ as well as $\sin 2\theta = \cos 4\theta$ is..... **[Integer Answer Type 2010]**

147. The number of solutions of the pair of equations $2\sin^2\theta - \cos 2\theta = 0$ and $2\cos^2\theta - 3\sin\theta = 0$ in the interval $[0, 2\pi]$ is **[Single Correct Option 2007]**

(a) 0

(b) 1

(c) 2

(d) 4

148. The set of values of θ satisfying the inequation $2\sin^2\theta - 5\sin\theta + 2 > 0$, where $0 < \theta < 2\pi$, is

[Single Correct Option 2006]

(a) $\left(0, \dfrac{\pi}{6}\right) \cup \left(\dfrac{5\pi}{6}, 2\pi\right)$

(b) $\left[0, \dfrac{\pi}{6}\right] \cup \left[\dfrac{5\pi}{6}, 2\pi\right]$

(c) $\left[0, \dfrac{\pi}{3}\right] \cup \left[\dfrac{2\pi}{3}, 2\pi\right]$

(d) None of the above

(ii) JEE Main & AIEEE

149. If $0 \leq x < 2\pi$, then the number of real values of x, which satisfy the equation

$\cos x + \cos 2x + \cos 3x + \cos 4x = 0$, is

[2016 JEE Main]

(a) 3 (b) 5

(c) 7 (d) 9

150. The possible values of $\theta \in (0, \pi)$ such that $\sin(\theta) + \sin(4\theta) + \sin(7\theta) = 0$ are **[2011 AIEEE]**

(a) $\dfrac{2\pi}{9}, \dfrac{\pi}{4}, \dfrac{4\pi}{9}, \dfrac{\pi}{2}, \dfrac{3\pi}{4}, \dfrac{8\pi}{9}$ (b) $\dfrac{\pi}{4}, \dfrac{5\pi}{12}, \dfrac{\pi}{2}, \dfrac{2\pi}{3}, \dfrac{3\pi}{4}, \dfrac{8\pi}{9}$

(c) $\dfrac{2\pi}{9}, \dfrac{\pi}{4}, \dfrac{\pi}{2}, \dfrac{2\pi}{3}, \dfrac{3\pi}{4}, \dfrac{35\pi}{36}$ (d) $\dfrac{2\pi}{9}, \dfrac{\pi}{4}, \dfrac{\pi}{2}, \dfrac{2\pi}{3}, \dfrac{3\pi}{4}, \dfrac{8\pi}{9}$

151. The number of values of x in the interval $[0, 3\pi]$ satisfying the equation $2\sin^2 x + 5\sin x - 3 = 0$, is

(a) 6 (b) 1 **[2006 AIEEE]**

(c) 2 (d) 4

Answers

Exercise for Session 1

1. $x=(4n+1)\frac{\pi}{14},-(4n-1)\frac{\pi}{6}$ **2.** 3 **3.** 15 **4.** $n\pi\pm\frac{\pi}{8}$

5. $n\pi\pm\frac{\pi}{8}$ **6.** $(6n+1)\frac{\pi}{12}$ **7.** 4 **8.** $n\pi\pm\frac{\pi}{3}$

9. $\frac{n\pi}{3}$ **10.** $(2n+1)\frac{\pi}{8}$, $n\pi\pm\frac{\pi}{3}$ **11.** No solution

12. $n\pi+\frac{\pi}{4}$ **13.** $\frac{\pi}{12}$ or $\frac{5\pi}{12}$ **14.** 4 **15.** 0

Exercise for Session 2

1. $2n\pi$ or $(4n+1)\frac{\pi}{2}$ **2.** $\frac{n\pi}{3}+\frac{\pi}{18}$ or $\frac{-n\pi}{2}+\frac{\pi}{6}$ **3.** $2n\pi+\frac{\pi}{3}$

4. $2n\pi-\frac{\pi}{4}$ **5.** $2n\pi+\frac{\pi}{4}+\frac{\pi}{12}$ **6.** 8

7. $n\pi+\tan^{-1}2$ or $n\pi+\tan^{-1}\left(-\frac{3}{4}\right)$ **8.** $n\pi$, $n\pi-\frac{\pi}{4}$

Exercise for Session 3

1. $2n\pi+\frac{7\pi}{4}$ **2.** $2n\pi+\frac{7\pi}{6}$ **3.** $(2n+1)\frac{\pi}{4}$

4. $A=n\pi\pm\frac{\pi}{12}$; $B=n\pi\pm\frac{\pi}{6}$ **5.** 0 **6.** No solution

7. $n\pi$ **8.** 3 **9.** 5 **10.** 0

Exercise for Session 4

1. $\left[-\pi,-\frac{\pi}{6}\right)\cup\left(\frac{\pi}{6},\pi\right]$ **2.** $\left\{\frac{\pi}{2}\right\}\cup\left[0,\frac{\pi}{6}\right]\cup\left[\frac{5\pi}{6},\pi\right]$

3. $\left[\frac{3\pi}{2},2\pi\right]\cup\{0\}$ **4.** $\left[-\frac{\pi}{6},\frac{\pi}{2}\right]$

5. $\left(0,\frac{\pi}{4}\right)$ **6.** $\left(-\frac{\pi}{10},\frac{3\pi}{10}\right)$

7. $R-\left\{x:x=\frac{3n\pi}{2}\pm\frac{3\pi}{4},n\in Z\right\}$ **8.** $\left(n\pi,n\pi+\frac{\pi}{6}\right)$

Chapter Exercises

1. (d) **2.** (d) **3.** (a) **4.** (b) **5.** (b) **6.** (c)
7. (c) **8.** (b) **9.** (c) **10.** (c) **11.** (a) **12.** (a)
13. (d) **14.** (a) **15.** (d) **16.** (c) **17.** (b) **18.** (b)
19. (b) **20.** (c) **21.** (b) **22.** (a) **23.** (a) **24.** (b)
25. (b) **26.** (b) **27.** (c) **28.** (d) **29.** (a) 30. (b)
31. (d) **32.** (a) **33.** (a) **34.** (c) **35.** (a) **36.** (c)
37. (a) **38.** (a) **39.** (b) **40.** (b) **41.** (a) **42.** (b)
43. (b) **44.** (a) **45.** (a) **46.** (b) **47.** (b) **48.** (c)
49. (a) **50.** (b) **51.** (a) **52.** (a) **53.** (a) **54.** (a)
55. (c) **56.** (b) **57.** (a) **58.** (d) **59.** (b) **60.** (c)
61. (c,b) **62.** (a,b,c) **63.** (a,d) **64.** (c,d) **65.** (b,c,d)
66. (a,c,d) **67.** (a,c) **68.** (a,c) **69.** (c,d) **70.** (c)
71. (a) **72.** (a) **73.** (a,d) **74.** (a,b) **75.** (a,c) **76.** (a,b)
77. (a,c) **78.** (b,c) **79.** (a,c) **80.** (b,c) **81.** (a,b) **82.** (a,b,c,d)
83. (a,b,c,d) **84.** (a,b) **85.** (a,b,c,d)
86. (b) **87.** (d) **88.** (c) **89.** (a) **90.** (a) **91.** (c)
92. (b) **93.** (d) **94.** (a) **95.** (c) **96.** (c) **97.** (a)
98. (b) **99.** (a) **100.** (8) **101.** (2) **102.** (1) **103.** (8)
104. (1) **105.** (3) **106.** (4) **107.** (3) **108.** (2) **109.** (5)
110. (2) **111.** (0) **112.** (d) **113.** (b) **114.** (d) **115.** (b)
116. (c) **117.** (c) **118.** (a) **119.** (a) **120.** (b)

121. (i) $\rightarrow$ (2), (ii) No solution, (iii) 4, (iv) 0, (v) 3

122. $\alpha\in\left[\frac{-3}{2},\frac{1}{2}\right]$ and $x=\frac{nx}{2}+\frac{(-1)^n}{2}\sin^{-1}(1-\sqrt{2\alpha+3})$

123. $\alpha=2n\pi\pm\frac{2\pi}{3}$, $n\in Z$

124. $x=n\pi+(-1)^n\frac{\pi}{6}$ and $y=2m\pi\pm\frac{\pi}{3}$, $m,n\in Z$

125. $x=(2m+1)\frac{\pi}{4}$ and $y=2n\pi+\frac{\pi}{2}$, $n\in I$

126. $\left(2,\ 2n\pi\pm\frac{2\pi}{3}-2\right)$

127. $x=-1$ and $y=n\pi\pm\frac{\pi}{4}+1$, $n\in I$

128. $x=k\pi+\frac{\pi}{4}$, $y=m\pi$ and $z=n\pi$ where $k,m,n\in Z$

129. $x=2m\pi\pm\frac{2\pi}{3}$ and $y=2(m-n)\pi\pm\frac{2\pi}{3}$, $m,\ n\in I$

130. $\theta=4n\pi\pm\frac{2\pi}{3}$

131. $x=2n\pi\pm\frac{3\pi}{4}+\frac{\pi}{4}$

132. $x=n\pi+(-1)^n\sin^{-1}\{2^{-\sqrt{(-\log_2 a)/2}}\}$ and the condition is $0<a<1$

133. $a=3\pi(4n+1)$

134. $x=\frac{n\pi}{2}\pm(-1)^n\frac{\pi}{8}$

135. $x=$ $y=m\pi\pm\frac{\pi}{4}$, $n\in I$

136. $x=n\pi\pm\cos^{-1}\sqrt{a+3}$, where $n\in z$ and $a\in[-3,-2]$

137. $x=\frac{\pi}{3}$

139. $x=2m\pi+\frac{\pi}{2}$, $2m\pi$, $n\pi\pm\frac{\pi}{6}$ and $y=k\pi$; m, n, k, $\in I$

140. (c) **141.** (8) **142.** (d) **143.** (a,c,d) **144.** (d)
145. (7) **146.** (3) **147.** (c) **148.** (a) **149.** (c)
150. (a) **151.** (d)

Solutions

1. $(\cos^2 x - \sin^2 x)\left(2\sin\frac{x}{2} - 1\right) = 0$

$$\cos 2x = 0 \text{ or } \sin\frac{x}{2} = \frac{1}{2}$$

Hence, option (d) is false.

2. The maximum possible value is 2.

$\sin\left(\frac{x}{3}\right)$ takes the value 1 when

$$\frac{x}{3} = 2m\pi + \frac{\pi}{2}$$

i.e. $$\frac{x}{3} = 90 + 360m$$

$\sin\left(\frac{x}{11}\right)$ takes the value 1

when $$\frac{x}{11} = 2n\pi + \frac{\pi}{2}$$

i.e. $$\frac{x}{11} = 90 + 360n$$

We are looking for a common solution,

we have $3m - 11n = 2$.

Clearly, the smallest positive solution to this is $m = 8$, $n = 2$,

thus $x_0 = 8910°$, giving $\alpha = 8910$.

3. $16(\sin^5 x + \cos^5 x) - 11(\sin x + \cos x) = 0$

$\Rightarrow (\sin x + \cos x)\{16(\sin^4 x - \sin^3 x\cos x + \sin^2 x\cos^2 x - \sin x\cos^3 x + \cos^4 x) - 11\} = 0$

$\Rightarrow (\sin x + \cos x)\{16(1 - \sin^2 x\cos^2 x - \sin x\cos x) - 11\} = 0$

$\Rightarrow (\sin x + \cos x)(4\sin x\cos x - 1)(4\sin x\cos x + 5) = 0$

As $4\sin x\cos x + 5 \neq 0$, we have

$\sin x + \cos x = 0$, $4\sin x\cos x - 1 = 0$

The required values are $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{9\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{21\pi}{12}$.

There are 6 solutions in $[0, 2\pi]$.

4. a, b, c are roots of equation

$$x\sin\theta + y\sin 2\theta + + z\sin 3\theta = \sin 4\theta$$

$\Rightarrow x\sin\theta + y(2\sin\theta\cos\theta) + z(3\sin\theta - 4\sin^3\theta) = 4\sin\theta\cos\theta\cos 2\theta$

$\Rightarrow \cos^3\theta - \frac{z}{2}\cos^2\theta - \frac{y+2}{4}\cos\theta + \frac{z-x}{8} = 0$

5. $\frac{\sin^2 2x + 4\sin^4 x - 4\sin^2 x\cos^2 x}{4 - \sin^2 2x - 4\sin^2 x} = \frac{1}{9}$

$\Rightarrow \frac{4\sin^4 x}{4\cos^2 x - 4\sin^2 x\cos^2 x} = \frac{1}{9}$

$\Rightarrow \frac{\sin^4 x}{\cos^4 x} = \frac{1}{9} \Rightarrow \tan x = \pm\frac{1}{\sqrt{3}}$

$\Rightarrow x = \frac{\pi}{6}$

6. $\left|\sqrt{m} - \sqrt{n}\right| \le \sqrt{|m-n|}$

$\Rightarrow \left|\sqrt{\sin^2 x + 2a^2} - \sqrt{2a^2 - 1 - \cos^2 x}\right| \le \sqrt{\sin^2 x + 2a^2 - 2a^2 + 1 + \cos^2 x} \le \sqrt{2}$

7. Consider, $\tan^2\frac{x}{2} = 1 + \sec x$

$\Rightarrow 8\left(\frac{1-\cos x}{1+\cos x}\right) = 1 + \frac{1}{\cos x}$

$\Rightarrow 8\cos x - 8\cos^2 x = (1+\cos x)^2$

$\Rightarrow 8\cos x - 8\cos^2 x = 1 + \cos^2 x + 2\cos x$

$\Rightarrow 9\cos^2 x - 6\cos x + 1 = 0$

$\Rightarrow (3\cos x - 1)^2 = 0$

$\Rightarrow \cos x = \frac{1}{3} = \cos\alpha$

$\Rightarrow x = 2n\pi \pm \alpha$ where 'α' $= \cos^{-1}\frac{1}{3}$

[By using $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha$]

$\Rightarrow x = 2n\pi \pm \cos^{-1}\left(\frac{1}{3}\right)$

8. We have $\tan\theta + \tan 4\theta + \tan 7\theta = \tan\theta\cdot\tan 4\theta\cdot\tan 7\theta$

$\Rightarrow \tan\theta + \tan 4\theta = -\tan 7\theta + \tan\theta\cdot\tan 4\theta\cdot\tan 7\theta$

$\Rightarrow \tan\theta + \tan 4\theta = -\tan 7\theta(1 - \tan\theta\cdot\tan 4\theta)$

$\Rightarrow \frac{\tan\theta + \tan 4\theta}{1 - \tan\theta\cdot\tan 4\theta} = -\tan 7\theta$

$\Rightarrow \tan(\theta + 4\theta) = -\tan 7\theta$

$\Rightarrow \tan 5\theta = \tan(-7\theta)$

$\Rightarrow 5\theta = n\pi + (-7\theta)$

$\Rightarrow 12\theta = n\pi \Rightarrow \theta = \frac{n\pi}{12}$ (where $n \in I$)

9. Put $e^{\sin x} = t$

Given equation becomes $t^2 - 4t - 1 = 0$

$\Rightarrow t = 2 \pm \sqrt{5} \Rightarrow e^{\sin x} = 2 \pm \sqrt{5}$

Either $\sin x = \log_e(2 + \sqrt{5})$

or $\sin x = \log_e(2 - \sqrt{5})$

as $2 + \sqrt{5} > e$ or as $(2 - \sqrt{5})$ is $(-\text{ve})$ and log is not defined for $(-\text{ve})$ values.

$\Rightarrow \sin x > 1 \Rightarrow$ no solution.

10. For $\sqrt{x-4}$ to be real $x \ge 4$, for which $\sqrt{x}$ is also real.

Now, if $\cos(\pi\sqrt{x}) < 1$, then $\cos(\pi\sqrt{x-4}) > 1$

and, if $\cos(\pi\sqrt{x}) > 1$, then $\cos(\pi\sqrt{x-4}) < 1$

(since their product = 1)

But both of these are not possible as $\cos\theta$ cannot be greater than 1.

$\Rightarrow \cos(\pi\sqrt{x-4}) = 1$ and $\cos(\pi\sqrt{x}) = 1$

$\Rightarrow x - 4 = 0$ and $x = 0$

$\Rightarrow x = 4$ or $x = 0$

But $x = 0$ is not possible (as $x \ge 4$)

$\Rightarrow x = 4$ is only solution.

11. Consider R.H.S i.e., 5^x+5^{-x}

$\Rightarrow \quad \frac{5^x+5^{-x}}{2} \geq (5^x \cdot 5^{-x})^{1/2}$

$\Rightarrow \quad 5^x+5^{-x} \geq 2$ [By using A.M $\geq$ G.M]

$\Rightarrow$ From the given equation $\sin(e^x) \geq 2$

which is not possible for any real values of 'x'. Thus, the given equation has no solution.

12. We know that in a triangle $A+B+C=180°$

and $2B=A+C$ (As A, B, C are in A.P)

$\Rightarrow \quad 3B=180$ or $B=60°$

Now 'θ' be the common difference between A, B and C,

then $C-A=2\theta$ ...(i)

$\Rightarrow \quad \sin(C-A)=1/2$ (given)

$\Rightarrow \quad \sin(2\theta)=1/2$ [Using Eq. (i)]

$\Rightarrow \quad 2\theta=\frac{\pi}{6}$ or $\frac{5\pi}{6} \Rightarrow \theta=\frac{\pi}{12}$ or $\frac{5\pi}{12}$

Now, $A=B-\theta \Rightarrow \frac{\pi}{3}-\frac{\pi}{12}$ or $\frac{\pi}{3}-\frac{5\pi}{12}$

$\Rightarrow A=\frac{\pi}{4}$ as a cannot be less than '0' and $C=\frac{\pi}{3}+\frac{\pi}{12}=\frac{5\pi}{12}$

$\Rightarrow \quad A=45°, B=60°, C=75°$

13. Consider : Ist equation i.e. $2\sin^2 x+3\sin x-2>0$

$\Rightarrow \quad (2\sin x-1)(\sin x+2)>0$

$\Rightarrow \quad (2\sin x-1)>0$ [As $\sin x+2>0 \forall x \in R$]

$\Rightarrow \quad \sin x>\frac{1}{2} \Rightarrow x \in \left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$ ...(i)

Consider $x^2-x-2<0$

$\Rightarrow \quad (x-2)(x+1)<0 \Rightarrow -1<x<2$ ...(ii)

From Eqs. (i) and (ii) ,

Now, as, $2<\frac{5\pi}{6}$, we obtain that 'x' must lie in $\left(\frac{\pi}{6}, 2\right)$.

14. Consider : $y=5x^2+2x+3=5\left[x^2+\frac{2}{5}x+\frac{3}{5}\right]$

$$=5\left[\left(x+\frac{1}{5}\right)^2+\frac{3}{5}-\frac{1}{25}\right]=5\left(x+\frac{1}{5}\right)^2+\frac{14}{5}>2$$

As $y=2\sin x \leq 2$, so there cannot be any point of intersection.

15. We have $a_1+a_2\cos 2x+a_3\sin^2 x=0$

$\Rightarrow a_1+a_2\cos 2x+a_3(1-\cos 2x)/2=0$

$\Rightarrow$ which is zero for all value of 'x'.

If $a_1=-\frac{a_3}{2}=-a_2$ or $a_1=\frac{-k}{2}, a_2=\frac{k}{2}, a_3=k$

For any $k \in R$

Hence, the required number of triplets is infinite.

16. We have, $\sin^4 x-(k+2)\sin^2 x-(k+3)=0$

$\Rightarrow \quad \sin^2 x=\frac{(k+2)\pm\sqrt{(k+2)^2+4(k+3)}}{2}=\frac{(k+2)\pm(k+4)}{2}$

$\Rightarrow$ Either $\sin^2 x=k+3$ or $\sin^2 x=-1$

$\Rightarrow \quad 0 \leq \sin^2 x \leq 1$ or not possible

$\Rightarrow \quad 0 \leq k+3 \leq 1$

$\Rightarrow \quad -3 \leq k \leq -2$

17. $\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \Rightarrow -1 \leq \sin\theta \leq 1$

But here $0<\sin\theta<1$ [As $\log_a x$ is define for $a>0$ or $0<a<1$]

Now, $\log_{\sin\theta}\cos^2\theta=2$ [By using $\log_a b=c \Rightarrow b=a^c$]

$\Rightarrow \quad \cos^2\theta=\sin^2\theta \Rightarrow \tan^2\theta=1$

$\Rightarrow \quad \theta=n\pi+\frac{\pi}{4} \forall \theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow \quad \theta=\frac{\pi}{4}, \theta=\frac{-\pi}{4}$

(Reject as $\sin\left(\frac{-\pi}{4}\right)=\frac{-1}{\sqrt{2}}<0$)

$\Rightarrow$ The given equation has unique solution.

18. Consider $\sum_{i=1}^{n}\cos\theta_i=n$

$\cos\theta_1+\cos\theta_2+\cos\theta_3+\ldots+\cos\theta_n=\underbrace{1+1+1\ldots+n}_{n \text{ times}}$ is valid only when

$\cos\theta_1=1, \cos\theta_2=1, \cos\theta_3=1, \ldots, \cos\theta_n=1$

$\Rightarrow \quad \theta_1=\theta_2=\theta_3=\ldots=\theta_n=0$

$\Rightarrow \quad \sum_{i=1}^{n}\sin\theta_i=0$

19. **Case I** : For $n=2$, $\sin^2 x+\cos^2 x=1$.

Case II : If $n>2$, $\sin^n x$ and $\cos^n x$ both decrease then

$\sin^n x+\cos^n x<1$ (as $0<x<\pi/2$)

Case III : If $n<2$, $\sin^n x$ and $\cos^n x$ both increases then

$\sin^n x+\cos^n x>1$ (as $0<x<\pi/2$)

Then, $\sin^n x+\cos^n x \geq 1$ for $n \leq 2$

$\Rightarrow n \in (-\infty, 2]$

20. $\sin x+\cos x=\min_{a \in R}\{1, a^2-4a+6\}$ ...(i)

As, $a^2-4a+6=(a-2)^2+2>2$ for all a

$\Rightarrow$ (i) becomes $\sin x+\cos x=1$

$\Rightarrow \quad \sin\left(x+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$

$\Rightarrow \quad x+\frac{\pi}{4}=n\pi+(-1)^n\frac{\pi}{4}$

$\Rightarrow \quad x=n\pi+(-1)^n\frac{\pi}{4}-\frac{\pi}{4}$

21. $\sin^7 y=|x^3-x^2-9x+9|+|x^3-4x-x^2+4|+\sec^2 2y+\cos^4 y$

Now, for $x=1$ (According to the choices)

$\sin^7 y=\sec^2 2y+\cos^4 y$

$\Rightarrow \quad \sin^7 y \cdot \cos^2 2y=1+\cos^4 y \cdot \cos^2 2y$

Now, R.H.S ≥ 1 and L.H.S ≤ 1

$\Rightarrow$ L.H.S $=1$

$\Rightarrow \quad \sin^7 y \cdot \cos^2 2y=1$

$\Rightarrow \quad \sin^7 y=1$ and $\cos^2 2y=1$

$\Rightarrow \quad y=\frac{\pi}{2}$

General values of 'y' is $2n\pi+\frac{\pi}{2}$

Hence, $x=1$ and $y=2n\pi+\frac{\pi}{2}$

22. As : $5\sin\theta+3\sin(\theta-\alpha)=5\sin\theta+3$

$(\sin\theta\cos\alpha-\cos\theta\sin\alpha)=(5+3\cos\alpha)$

$\sin\theta-3\sin\alpha\cos\theta$

Now, $-\sqrt{(5+3\cos\alpha)^2+9\sin^2\alpha}\le 5\sin\theta$

$+3\sin(\theta-\alpha)\le\sqrt{(5+3\cos\alpha)^2+9\sin^2\alpha}$

$\Rightarrow \max_{\theta\in R}\{5\sin\theta+3\sin(\theta-\alpha)\}=\sqrt{34+30\cos\alpha}$

$\Rightarrow \sqrt{34+30\cos\alpha}=7\Rightarrow\cos\alpha=\dfrac{49-34}{30}$

$\Rightarrow \cos\alpha=\dfrac{1}{2}=\cos\dfrac{\pi}{3}\Rightarrow\alpha=2n\pi\pm\dfrac{\pi}{3}$

23. Consider : $\sin x(\sin x+\cos x)=n$

$\Rightarrow \sin^2 x+\sin x\cdot\cos x=n\Rightarrow\dfrac{1-\cos2x}{2}+\dfrac{\sin2x}{2}=n$

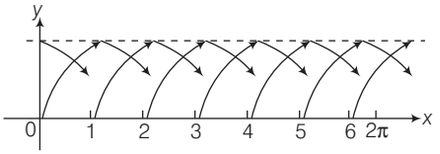
$\Rightarrow \sin2x-\cos2x=2n-1$

$\Rightarrow$ As $-\sqrt{2}\le\sin2x-\cos2x\le\sqrt{2}$

$\Rightarrow -\sqrt{2}\le 2n-1\le\sqrt{2}$

$\Rightarrow \dfrac{1-\sqrt{2}}{2}\le n\le\dfrac{1+\sqrt{2}}{2}\Rightarrow n=0,\ 1$

24. $\sin\{x\}=\cos\{x\}$ graph of $y=\sin\{x\}$ and $y\cos\{x\}$ meet exactly 6 times in $[0,\ 2\pi]$.



Points of intersection are at

$$x=\frac{\pi}{4},\ 1+\frac{\pi}{4},\ 2+\frac{\pi}{4},\ 3+\frac{\pi}{4},\ 5+\frac{\pi}{4}$$

25. $x^2+4-2x+3\sin(ax+b)=0$

$(x-1)^2+3+3\sin(ax+b)=0$

$\Rightarrow x=1$ and $\sin(ax+b)=-1$

$\Rightarrow \sin(a+b)=-1\ \Rightarrow\ a+b=\dfrac{3\pi}{2}$

26. Here, $4[1+\cot^2\pi(a+x)]+a^2-4a=0$

$\Rightarrow 4\cot^2\pi(a+x)+(a-2)^2=0$

$\Rightarrow a-2=0$ and $\cot^2\pi(a+x)=0\ \Rightarrow\ a=2$

27. As max $\cos\theta=1$, $2\cos x+\cos2\lambda x=3$ is possible only when $\cos x=1$ and $\cos2\lambda x=1$,

i.e. $\cos x=1$ and $\sin\lambda x=0$

Clearly, if λ is rational, say p/q, then $x=2q\pi$, $q\in I$, satisfies both the equations. Therefore, for exactly one solution, $x=0$, λ should be irrational.

28. $\sin\dfrac{\pi}{2n}+\cos\dfrac{\pi}{2n}=\sqrt{2}\sin\left(\dfrac{\pi}{2n}+\dfrac{\pi}{4}\right)$ or $\sin\left(\dfrac{\pi}{2n}+\dfrac{\pi}{4}\right)=\dfrac{\sqrt{n}}{2\sqrt{2}}$

Since $\dfrac{\pi}{4}<\dfrac{\pi}{2n}+\dfrac{\pi}{4}<\dfrac{3\pi}{4}$ for $n>1$

or $\dfrac{1}{\sqrt{2}}<\dfrac{\sqrt{n}}{2\sqrt{2}}\le 1$ or $2<\sqrt{n}\le 2\sqrt{2}$ or $4<n\le 8$.

If $n=1$, L.H.S $=1$, R.H.S $=1/2$

Similarly for $n=8$, $\sin\left(\dfrac{\pi}{16}+\dfrac{\pi}{4}\right)\ne 1$

$\therefore\quad 4<n<8$

29. $5\sec\theta-13=12\tan\theta$

or, $13\cos\theta+12\sin\theta=5$

or, $\dfrac{13}{\sqrt{13^2+12^2}}\cos\theta+\dfrac{12}{\sqrt{13^2+12^2}}\sin\theta$

$=\dfrac{5}{\sqrt{13^2+12^2}}$

or, $\cos(\theta-\alpha)=\dfrac{5}{\sqrt{313}}$, where $\cos\alpha=\dfrac{13}{\sqrt{313}}$

$\therefore\quad \theta=2n\pi\pm\cos^{-1}\dfrac{5}{\sqrt{313}}+\alpha$

$=2n\pi\pm\cos^{-1}\dfrac{5}{\sqrt{313}}+\cos^{-1}\dfrac{13}{\sqrt{313}}$

As $\cos^{-1}\dfrac{5}{\sqrt{313}}>\cos^{-1}\dfrac{13}{\sqrt{313}}$, then

$\theta\in[0,2\pi]$, when $n=0$ (One value, taking positive sign) and when $n=1$ (One value, taking negative sign).

30. Here, $x^3+(x+2)^2+2\sin x=4$.

Clearly, $x=0$ satisfies the equation.

If $0<x\le\pi$, $x^3+(x+2)^2+2\sin x>4$

If $\pi<x\le 2\pi$,

$x^3+(x+2)^2+2\sin x>27+25-2$

So, $x=0$ is the only solution.

31. If $\tan\left(\dfrac{\pi}{2}\sin\theta\right)=\cot\left(\dfrac{\pi}{2}\cos\theta\right)$

$\Rightarrow \tan\left(\dfrac{\pi}{2}\sin\theta\right)=\tan\left(\dfrac{\pi}{2}-\dfrac{\pi}{2}\cos\theta\right)$

$\Rightarrow \dfrac{\pi}{2}\sin\theta=n\pi+\dfrac{\pi}{2}-\dfrac{\pi}{2}\cos\theta, n\in I$

$\Rightarrow \sin\theta+\cos\theta=2n+1, n\in I$

$\Rightarrow \sin\theta+\cos\theta=2n+1,$

$n\in I$; but $-\sqrt{2}\le\sin\theta+\cos\theta\le\sqrt{2}$, therefore, $\sin\theta+\cos\theta=1$ or -1

32. We have, $\sin x+\sin y+\sin z=-3$

$0\le x\le 2\pi$

$0\le y\le 2\pi$

$0\le z\le 2\pi$

It is possible only, when $\sin x=\sin y=\sin z=-1$

$\therefore\quad x=y=z=\dfrac{3\pi}{2}$ for $x,y,z\in[0,2\pi]$

33. We have, $4\cos x-3\sec x=\tan x$

$\left(\cos x\ne 0$ i.e., x odd multiplied of $\dfrac{\pi}{2}\right)$

Then, $4\cos x-\dfrac{3}{\cos x}=\dfrac{\sin x}{\cos x}$

$4\cos^2 x-3=\sin x$

$4-4\sin^2 x-3=\sin x$

$4\sin^2 x+\sin x-1=0$

$\therefore \quad \sin x=\dfrac{-1+\sqrt{17}}{8}$

Either $\quad \sin x=\dfrac{-1+\sqrt{17}}{8}=\sin\alpha$ (let)

or $\quad \sin x=\dfrac{-1-\sqrt{17}}{8}=\sin\beta$ (let)

$\therefore \quad x=n\pi+(-1)^n\alpha$ or $x=n\pi+(-1)^n\beta$

$\therefore \quad \sin\alpha=\dfrac{-1+\sqrt{17}}{8}, \sin\beta=\dfrac{-1-\sqrt{17}}{8}$

So, $4(\sin\alpha+\sin\beta)=4\left[\dfrac{-1+\sqrt{17}}{8}+\dfrac{-1-\sqrt{17}}{8}\right]=4\left[\dfrac{-2}{8}\right]=-1$

34. We have, $\tan m\theta=\tan n\theta$

$m\theta=k\pi+n\theta, \forall k\in I$ (Using formula)

$m\theta-n\theta=k\pi, \forall k\in I$

$\theta=k\cdot\dfrac{\pi}{m-n}, \forall k\in I$ ($\because$ m and n are constants)

Then, if we put $k=1, 2, 3, \ldots$ we get

$\theta=\dfrac{\pi}{m-n}, \dfrac{2\pi}{m-n}, \dfrac{3\pi}{m-n}, \ldots$, which is AP.

Thus, common difference $=\dfrac{\pi}{m-n}$

35. $\because \sin3\alpha=4\sin\alpha\sin(x-\alpha)\sin(x+\alpha)$

$3\sin\alpha-4\sin^3\alpha=4\sin\alpha(\sin^2 x-\sin^2\alpha)$

On dividing both sides by $\sin\alpha$, we get

$3-4\sin^2\alpha=4\sin^2 x-4\sin^2\alpha$

$\Rightarrow \quad 4\sin^2 x=3$

$\Rightarrow \quad \sin^2 x=\dfrac{3}{4}$

$\Rightarrow \quad \sin^2 x=\left(\dfrac{\sqrt{3}}{2}\right)^2=\sin^2\dfrac{\pi}{3}$

$\therefore \quad x=n\pi\pm\dfrac{\pi}{3}, \forall n\in I$ (using formula)

36. We know that,

$a\sin\theta\pm b\cos\theta=c$ is solvable, if $|c|\le\sqrt{a^2+b^2}$.

Now, $\lambda\cos x-3\sin x=\lambda+1$ is solvable, if

$|\lambda+1|\le\sqrt{\lambda^2+9}$

$(\lambda+1)^2\le\lambda^2+9$

$\lambda^2+1+2\lambda\le\lambda^2+9$

$2\lambda\le9-1 \Rightarrow \lambda\le4$

$\therefore \quad \lambda\in(-\infty, 4]$

37. We have, $\cos2x-3\cos x+1=\dfrac{1}{(\cot2x-\cot x)\sin(x-\pi)}$

$2\cos^2 x-1-3\cos x+1=\dfrac{1}{\dfrac{\sin(x-2x)}{\sin2x\cdot\sin x}\cdot(-\sin x)}$

$\left[\text{using formula, } \cot A-\cot B=\dfrac{\sin(B-A)}{\sin A\cdot\sin B}\right]$

$2\cos^2 x-3\cos x=\dfrac{2\sin x\cdot\cos x\cdot\sin x}{\sin^2 x}$

$2\cos^2 x-3\cos x-2\cos x=0$

$2\cos^2 x-5\cos x=0$

$\cos x(2\cos x-5)=0$

Either $2\cos x-5=0$

$\Rightarrow \quad \cos x=\dfrac{5}{2}$ (which is not possible)

Then, $\quad \cos x=0$

38. $\because \sec x\cdot\cos5x=-1$ and $0<x<\dfrac{\pi}{4}$

$\dfrac{\cos5x}{\cos x}=-1$

$\Rightarrow \quad \cos5x=-\cos x$

$\cos5x+\cos x=0$

$\Rightarrow \quad 2\cos3x\cdot\cos2x=0$

Either $2\cos3x=0$ or $\cos2x=0$

$\cos3x=0$ or $2x=(2n+1)\dfrac{\pi}{2}$

$3x=(2n+1)\dfrac{\pi}{2}$ or $x=(2n+1)\dfrac{\pi}{4}, \forall n\in I$ (not possible)

$\therefore \quad x=(2n+1)\dfrac{\pi}{6}$

Put $n=0$, then $x=\dfrac{\pi}{6}$

$\therefore \quad x=\dfrac{\pi}{6}$

39. $\because \sin^{100}\theta-\cos^{100}\theta=1$

$\sin^{100}\theta=1+\cos^{100}\theta$

This equation is valid, if $\cos^{100}\theta=0$ and $\sin^{100}\theta=1$

($\because$ $0\le\cos^2\theta\le1$, $0\le\sin^2\theta\le1$)

$\therefore \quad \cos\theta=0$, then $\sin\theta=1$

$\theta=(2n+1)\dfrac{\pi}{2}, \forall n\in I$

$\theta=n\pi+\dfrac{\pi}{2}, \forall n\in I$

40. $\because \alpha^2-4\alpha+7=\alpha^2-4\alpha+4+3=(\alpha-2)^2+3\ge3$

and $\quad 2<e<3$ and $\pi=3.14$

$4<e^2<9$

Now, $\sqrt{3}\sin x-\cos x=\min\limits_{\alpha\in R}\{2, e^2, \pi, \alpha^2-4\alpha+7\}$

$\sqrt{3}\sin x-\cos x=2$

$\dfrac{\sqrt{3}}{2}\sin x-\dfrac{1}{2}\cos x=1$

$\sin x\cdot\cos\dfrac{\pi}{6}-\cos x\cdot\sin\dfrac{\pi}{6}=1$

$\sin\left(x-\dfrac{\pi}{6}\right)=1$

$\therefore \quad x-\dfrac{\pi}{6}=2n\pi+\dfrac{\pi}{2}, \forall n\in I$

$x=2n\pi+\dfrac{\pi}{2}+\dfrac{\pi}{6}, \forall n\in I$

$x=2n\pi+\dfrac{2\pi}{3}, \forall n\in I$

41. We have, $\cos 4x + 6 = 7\cos 2x$

$$2\cos^2 2x - 1 + 6 - 7\cos 2x = 0$$
$$2\cos^2 2x - 7\cos 2x + 5 = 0$$
$$(2\cos x - 5)(\cos 2x - 1) = 0$$

Thus, $\cos 2x = 1$

and $\cos 2x = \frac{5}{2}$ (which is not possible, $-1 \le \cos\theta \le 1$)

$\therefore$ $\cos 2x = 1$

$2x = 2n\pi, \forall n \in I$ (using formula)

$x = n\pi, \forall n \in I$

$x = \pi, 2\pi, 3\pi, \ldots\ldots$

i.e., $x = 180°, 360°, 540°, \ldots\ldots$

$x \notin [315°, 317°]$

So, $x = n\pi \notin [315°, 317°], \forall n \in I$

Hence, number of solutions is 0.

42. Let $A = 5\pi\sin\theta$ and $B = 5\pi\cos\theta$

Then, $\cot A - \tan B = 0$

$$\frac{\cos A}{\sin A} - \frac{\sin B}{\cos B} = 0 \Rightarrow \cos A \cdot \cos B - \sin A \cdot \sin B = 0$$

$\therefore$ $\cos(A + B) = 0$

$\Rightarrow$ $A + B = (2n + 1)\frac{\pi}{2}, \forall n \in I$ ...(i)

Now, $5\pi\sin\theta + 5\pi\cos\theta = (2n + 1)\frac{\pi}{2}, \forall n \in I$

$$\sin\theta + \cos\theta = \frac{2n + 1}{10}$$
$$\sin\theta \cdot \frac{1}{\sqrt{2}} + \cos\theta \cdot \frac{1}{\sqrt{2}} = \frac{2n + 1}{10\sqrt{2}}$$
$$\sin\left(\theta + \frac{\pi}{4}\right) = \frac{2n + 1}{10\sqrt{2}} \quad \ldots(ii)$$

$\because$ $-1 \le \sin\left(\theta + \frac{\pi}{4}\right) \le 1$

$\Rightarrow$ $-1 \le \frac{2n + 1}{10\sqrt{2}} \le 1$

$\Rightarrow$ $\frac{-10\sqrt{2} - 1}{2} \le n \le \frac{10\sqrt{2} - 1}{2}$

$-7.5 \le n \le 6.5$ $[\because \sqrt{2} = 1.4$ (let)$]$

$\therefore$ $n = -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6$

Hence, number of solutions is 14.

43. $\sin^2 x + \sin^4 x + \sin^6 x + \ldots\infty = \frac{\sin^2 x}{1 - \sin^2 x} = \tan^2 x$

$\Rightarrow$ $\exp^{[(\sin^2 x + \sin^4 x + \ldots)\ln 2]} = e^{\tan^2 x \ln} = e^{\ln 2^{\tan^2 x}}$

The given equation is $y^2 - 9y + 8 \Rightarrow (y-1)(y-8) = 0$

Either $y = 1 \Rightarrow 2^{\tan x} = 1 = 2^0 \Rightarrow \tan^2 x = 0$, but $x \in \left(0, \frac{\pi}{2}\right)$

$\therefore$ Neglecting $x = 0$ or $y = 2^3 \Rightarrow \tan^2 x = 3$

$\Rightarrow$ $\tan x = \pm\sqrt{3} \Rightarrow x = \frac{\pi}{3}$, as $0 < x < \frac{\pi}{2}$

$\Rightarrow$ $\frac{\cos x}{\cos x + \sin x} = \frac{1/2}{1/2 + \sqrt{3}/2} = \frac{1}{\sqrt{3} + 1} = \frac{\sqrt{3} - 1}{2}$

44. $\cos x = \sqrt{1 - \sin 2x} = |\sin x - \cos x|$

(i) $\sin x \le \cos x$

$\Rightarrow$ $\cos x = \cos x - \sin x \Rightarrow \sin x = 0$

where, $x \in \left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]$,

$\therefore$ $\sin x = 0$

$\Rightarrow$ $x = 2\pi$, neglecting $x = \pi$

(ii) $\sin x > \cos x \Rightarrow \tan x = 2$

where $x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$, $\therefore \tan x = 2 \Rightarrow x = \tan^{-1}(2)$

Thus, the given equation has two solutions.

45. We have, $\sin 3x = 3\sin x - 4\sin^3 x \Rightarrow \sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$

and $\cos 3x = 4\cos^3 x - 3\cos x$

$\Rightarrow$ $\cos^3 x = \frac{1}{4}(\cos 3x + 3\cos x)$

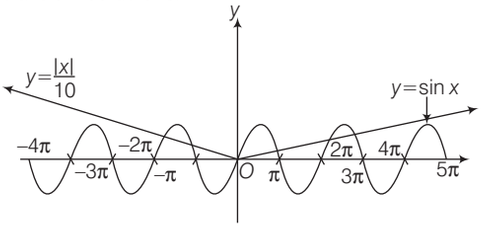
$\therefore$ $\cos 3x\cos^3 x + \sin 3x\sin^3 x$

$$= \frac{1}{4}[\cos^2 3x + 3\cos x\cos 3x + 3\sin x\sin 3x - \sin^2 3x]$$
$$= \frac{1}{4}[3\cos 2x + \cos 6x] = \cos^3 2x$$

$\Rightarrow$ $\cos 2x = 0 \Rightarrow 2x = (2n+1)\frac{\pi}{2}$

$\Rightarrow$ $x = (2n+1)\frac{\pi}{4}$

46. Graphs of $y = \sin x$ and $y = \frac{|x|}{10}$ meet exactly six times. Hence, there are six solutions.



47. Since, the equation $a_1 + a_2\sin x + a_3\cos x + a_4\sin 2x + a_5\cos 2x = 0$ holds for all values of x,

$a_1 + a_3 + a_5 = 0$ (on putting $x = 0$)

$a_1 - a_3 + a_5 = 0$ (on putting $x = \pi$)

$\Rightarrow$ $a_3 = 0$ and $a_1 + a_5 = 0$ ...(i)

Putting $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$, we get

$a_1 + a_2 - a_5 = 0$ and $a_1 - a_2 - a_5 = 0$

$\Rightarrow$ $a_2 = 0$ and $a_1 - a_5 = 0$ ...(ii)

Eqs. (i) and (ii) give $a_1 = a_2 = a_3 = a_5 = 0$

The given equation reduces to $a_4\sin 2x = 0$. This is true for all values of x, therefore $a_4 = 0$

Hence, $a_1 = a_2 = a_3 = a_4 = a_5 = 0$

Thus, the number of 5-tuples is one.

48. $\cos^2 x = 2\cos x(3\sin^2 x - 2)$

$\Rightarrow \cos x[\cos x - 2\{3(1 - \cos^2 x) - 2\}] = 0$

$\Rightarrow \cos x(6\cos^2 x - 2 + \cos x) \Rightarrow \cos x = 0$, which is not possible.

or $6\cos^2 x+\cos x-2=0 \Rightarrow \cos x=\frac{1}{2},-\frac{2}{3}$

$\Rightarrow x=\frac{\pi}{3},\frac{5\pi}{3}$ or $x=\pi-\cos^{-1}\left(\frac{2}{3}\right),\ \pi+\cos^{-1}\left(\frac{2}{3}\right)$

$\Rightarrow |x_1-x_3|=\frac{4\pi}{3}$ or $|x_1-x_2|=2\cos^{-1}\left(\frac{2}{3}\right)$

$\Rightarrow |x_1-x_2|_{\min}=2\cos^{-1}\left(\frac{2}{3}\right)$

49. $\cos x-\frac{\cot\beta\sin x}{2}=\frac{\sqrt{3}}{2}\Rightarrow\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}-\frac{\cot\beta\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}=\frac{\sqrt{3}}{2}$

$\Rightarrow 1-\tan^2\frac{x}{2}-\cot\beta\tan\frac{x}{2}=\frac{\sqrt{3}}{2}\left(1+\tan^2\frac{x}{2}\right)$

$\Rightarrow (2+\sqrt{3})\tan^2\frac{x}{2}+2\cot\beta\tan\frac{x}{2}+(\sqrt{3}-2)=0$

$\Rightarrow \tan\frac{x}{2}=\frac{-2\cot\beta\pm\sqrt{4\cot^2\beta+4}}{2(2+\sqrt{3})}=\frac{-2\cot\beta\pm 2\operatorname{cosec}\beta}{2(2+\sqrt{3})}$

$\Rightarrow \tan\frac{x}{2}=\frac{-\cot\beta+\operatorname{cosec}\beta}{(2+\sqrt{3})}$

or $\tan\frac{x}{2}=\frac{-\cot\beta-\operatorname{cosec}\beta}{(2+\sqrt{3})}$

$\Rightarrow \tan\frac{x}{2}=\tan\frac{\beta}{2}\tan 15^\circ$

50. $n\sin^2\theta+2n\cos(\theta+\alpha)\sin\alpha\sin\theta+\cos 2(\alpha+\theta)$

$=n\sin^2\theta+n\cos(\theta+\alpha)\{\cos(\theta-\alpha)-\cos(\theta+\alpha)\}+2\cos^2(\theta+\alpha)-1$

$=n\sin^2\theta+n(\cos^2\theta-\sin^2\alpha)-n\cos^2(\theta+\alpha)+2\cos^2(\alpha+\theta)-1$

$=n\sin^2\theta+n\cos^2\theta-n\sin^2\alpha+(2-n)\cos^2(\theta+\alpha)-1$

$=(n-1)-n\sin^2\alpha+(2-n)\cos^2(\theta+\alpha)\Rightarrow n=2$

51. Applying $C_1\to C_1-2\cos x C_2+C_3$ to the given determinant, we get

$$\begin{vmatrix}1-2a\cos x+a^2 & a & a^2\\ 0 & \cos nx & \cos(n+1)x\\ 0 & \sin nx & \sin(n+1)x\end{vmatrix}=(1-2a\cos x+a^2)\sin x=0$$

if $\sin x=0$ or $\cos x=(1+a^2)/2a$ i.e., if $n=n\pi,\ n\in I$

52. $\frac{\sin 3\alpha}{\cos 2\alpha}<0$ if $\sin 3\alpha>0$ and $\cos 2\alpha<0$ or $\sin 3\alpha<0$ and $\cos 2\alpha>0$

i.e., if $3\alpha\in(0,\pi)$ and $2\alpha\in(\pi/2,3\pi/2)$

or $3\alpha\in(\pi,2\pi)$ and $2\alpha\in(-\pi/2,\pi/2)$

i.e., if $\alpha\in(0,\pi/3)$ and $\alpha\in(\pi/4,3\pi/4)$

or $\alpha\in(\pi/3,2\pi/3)$ and $\alpha\in(-\pi/4,\pi/4)$ i.e., if $\alpha\in(\pi/4,\pi/3)$

since $(13\pi/48,14\pi/48)\subset(\pi/4,\pi/3)$,

option (a) is correct.

53. $f(x)=(\sin^2\theta+\cos^2\theta+x)\begin{vmatrix}1 & \cos^2\theta & x\\ 1 & x & \sin^2\theta\\ 1 & \sin^2\theta & \cos^2\theta\end{vmatrix}$

$$=(x+1)\begin{vmatrix}1 & \cos^2\theta & x\\ 0 & x-\cos^2\theta & \sin^2\theta-x\\ 0 & \sin^2\theta-\cos^2\theta & \cos^2\theta-x\end{vmatrix}$$

$=(x+1)[(x-\cos^2\theta)(\cos^2\theta-x)-(\sin^2\theta-\cos^2\theta)(\sin^2\theta-x)]$

$=(x+1)[-x^2-\cos^4\theta+2x\cos^2\theta$

$-x\cos^2\theta+x\sin^2\theta-\sin^4\theta+\sin^2\theta\cos^2\theta]$

$=(-1/2)(x+1)[(x-\sin^2\theta)^2+(x-\cos^2\theta)^2+(\sin^2\theta-\cos^2\theta)]^2$

So, $f(x)=0$, if $x=-1$ or $x=\sin^2\theta=\cos^2\theta$

$\sin^2\theta=\cos^2\theta$

$\Rightarrow \theta=\pi/4\Rightarrow x=1/2$

Hence, $x=-1,1/2$

54. Given, $\sin x+\sin y+\sin z=-3$ is satisfied only when $x=y=z=\frac{3\pi}{2}$, for $x,y,z\in[0,2\pi]$.

55. $\sec x\cos 5x=-1 \Rightarrow \cos 5x=-\cos x \Rightarrow 5x=2n\pi\pm(\pi-x)$

$\Rightarrow x=\frac{(2n+1)\pi}{6}$ or $\frac{(2n-1)\pi}{4}$

Hence, $x=\frac{\pi}{4},\frac{\pi}{2},\frac{3\pi}{4},\frac{5\pi}{6},\frac{5\pi}{6},\frac{7\pi}{6},\frac{7\pi}{4},\frac{9\pi}{6},\frac{11\pi}{6}$

56. $3\cos\theta+4\sin\theta=5\left[\frac{3}{5}\cos\theta+\frac{4}{5}\sin\theta\right]=5\cos(\theta-\alpha)$

where $\cos\alpha=3/5,\sin\alpha=4/5$

Now, $3\cos\theta+4\sin\theta=k$

$\therefore\quad 5\cos(\theta-\alpha)=\pm 5\Rightarrow\cos(\theta-\alpha)=\pm 1$

$\Rightarrow \theta-\alpha=0^\circ,180^\circ\Rightarrow\theta=\alpha,180^\circ+\alpha$

57. Given, $\cot(\alpha+\beta)=0$

$\Rightarrow \cos(\alpha+\beta)=0\Rightarrow \alpha+\beta=(2n+1)\frac{\pi}{2},n\in I$

$\therefore\quad \sin(a+2\beta)=\sin(2\alpha+2\beta-\alpha)=\sin[(2n+1)\pi-\alpha]$

$=\sin(2n\pi+\pi-\alpha)=\sin(\pi-\alpha)=\sin\alpha$

58. $\cot\theta+\cot\left(\frac{\pi}{4}+\theta\right)=2$

$\Rightarrow \frac{\cos\theta}{\sin\theta}+\frac{\cos\{(\pi/4)+\theta\}}{\sin\{(\pi/4)+\theta\}}=2$

$\Rightarrow \sin\left(\frac{\pi}{2}+2\theta\right)=2\sin\theta\sin\left(\frac{\pi}{4}+\theta\right)$

$\Rightarrow \sin\left(\frac{\pi}{4}+2\theta\right)+\cos\left(\frac{\pi}{4}+2\theta\right)=\frac{1}{\sqrt{2}}$

$\Rightarrow \cos 2\theta=\frac{1}{2}\ \Rightarrow\ 2\theta=2n\pi\pm\frac{\pi}{3}\ \Rightarrow\ \theta=n\pi\pm\frac{\pi}{6}.$

59. $2\cos^2\theta-(\sqrt{2}+1)\cos\theta-1+\frac{(\sqrt{2}+1)}{\sqrt{2}}=0$

$\Rightarrow \cos\theta=\frac{(\sqrt{2}+1)\pm\sqrt{(\sqrt{2}+1)^2-\frac{8}{\sqrt{2}}}}{4}$

$\Rightarrow \cos\theta=\cos\left(\frac{\pi}{4}\right)\ \Rightarrow\ \theta=2n\pi\pm\frac{\pi}{4}$

Trick : Since $\theta=\frac{\pi}{4}$ satisfies the equation and therefore the general value should be $2n\pi\pm\frac{\pi}{4}$.

60. $|\sin x|\geq 0\Rightarrow\frac{|\sin x|}{1+|\sin x|}<1\ \Rightarrow\ 1-\frac{|\sin x|}{1+|\sin x|}>0$

So, the given in equation becomes

$$1-\frac{|\sin x|}{1+|\sin x|}\geq\frac{2}{3}$$

$$\Rightarrow \quad \frac{1}{3}\geq\frac{|\sin x|}{1+|\sin x|} \Rightarrow |\sin x|\leq\frac{1}{2}$$

$$\Rightarrow \quad -\frac{1}{2}\leq\sin x\leq\frac{1}{2} \Rightarrow \sin x\in\left[-\frac{1}{2},\frac{1}{2}\right]$$

61. $(t-[|\sin x|])!=3!5!7!$

if $x=n\pi+\frac{\pi}{2}(n\in I)$

then $(t-1)!=3!\,5!\,7!$

$\Rightarrow \quad (t-1)!=10!$

$\Rightarrow \quad t-1=10$

$\Rightarrow \quad t=11$

If $x\neq n\pi+\frac{\pi}{2}(n\in I)$, then

$$(t-0)!=10!$$

$\Rightarrow \quad t=10$

62. $f(x)=\left(\cos a_1+\frac{\cos a_2}{2}+\cdots+\frac{\cos a_n}{2^{n-1}}\right)\cos x-\left(\frac{\sin a_1}{1}+\frac{\sin a_2}{2}+\cdots+\frac{\sin a_n}{2^{n-1}}\right)\sin x$

$\Rightarrow \quad f(x)=A\cos x-B\sin x$

Now, $f(x_1)=f(x_2)=0$

$\Rightarrow \quad \left.\begin{aligned}A\cos x_1-B\sin x_1=0\\ A\cos x_2-B\sin x_2=0\end{aligned}\right\}$

$\Rightarrow \quad \tan x_1=\tan x_2$

$\Rightarrow \quad x_1=n\pi+x_2$

$\Rightarrow \quad x_1-x_2=n\pi$

63. $AP=\sqrt{(1-\cos\alpha)^2+\sin^2\alpha}$

$=2|\sin\alpha/2|=2\sin\frac{\alpha}{2}$

Similarly $AQ=2\sin\frac{\beta}{2}$ and $AR=2\sin\frac{\gamma}{2}$

Now as AP, AQ, AR are in GP.

$\therefore \sin\frac{\alpha}{2},\sin\frac{\beta}{2},\sin\frac{\gamma}{2}$ are in GP.

$$\Rightarrow \quad \frac{\sin\frac{\alpha}{2}+\sin\frac{\gamma}{2}}{2}\geq\sin\frac{\beta}{2}$$

$$\Rightarrow \quad \sin\frac{\alpha+\gamma}{2}\cos\frac{\alpha-\gamma}{2}\geq\sin\frac{\beta}{2}$$

Also, $\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}\leq\sin\frac{\beta}{2}$

64. $\sin^2 z+\operatorname{cosec}^2 z\geq 2,\ 2+\cot^2 y\geq 2,\ 4+\sin 4x\geq 3$

$\Rightarrow \quad \sin^2 z=1,\ \cot^2 y=0,\ \sin 4x=-1$

$\Rightarrow \quad z\in\left\{\frac{\pi}{2},\frac{3\pi}{2}\right\}$,

$y\in\left\{\frac{\pi}{2},\frac{3\pi}{2}\right\}$, $x\in\left\{\frac{3\pi}{8},\frac{7\pi}{8},\frac{11\pi}{8},\frac{15\pi}{8}\right\}$.

65. For non-trivial solution

$$\begin{vmatrix}\sin\theta & -2\cos\theta & -a\\ -1 & 1 & 1\\ 1 & 2 & 1\end{vmatrix}=0$$

$\Rightarrow \quad \sin\theta+4\cos\theta=3a$

$$-\frac{\sqrt{17}}{3}\leq a\leq\frac{\sqrt{17}}{3}$$

So, 3 integral values.

66. The equation becomes $(\sin\theta-2)(\sin\theta+\lambda)(2\sin\theta+1)=0$

$\Rightarrow \quad \lambda=\pm 1,0$

67. $x+y=2\pi/3$ or $y=(2\pi/3)-x$

$\therefore \quad \sin x=2\sin\left(\frac{2\pi}{3}-x\right)$

$$=2\left[\left(\frac{\sqrt{3}}{2}\right)\cos x+\left(\frac{1}{2}\right)\sin x\right]$$

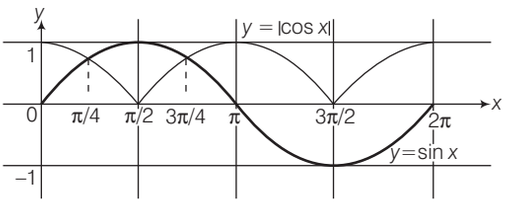
$$=\sqrt{3}\cos x+\sin x \Rightarrow \cos x=0$$

$\Rightarrow \quad x=n\pi+\frac{\pi}{2},n\in Z$

$\Rightarrow \quad y=\frac{2\pi}{3}-n\pi-\frac{\pi}{2}=\frac{\pi}{6}-n\pi$

Hence, for $x\in[0,4\pi]$, $x=\pi/2,3\pi/2,5\pi/2,7\pi/2$ and for $y\in[0,4\pi]$, $y=\pi/6,7\pi/6,13\pi/6,19\pi/6$

68. It is easier to solve the inequality using graphical method. The graphs of $y=|\cos x|$ and $y=\sin x$ are shown in the following figure.



From the figure, $|\cos x|\leq\sin x$ for $x\in\left[\frac{\pi}{4},\frac{3\pi}{4}\right]$

69. $(x+y)$ and $(x-y)$ satisfy the equation $\tan^2\theta-4\tan\theta+1=0$. Thus,

$$\tan(x+y)+\tan(x-y)=4$$

and $\tan(x+y)\tan(x-y)=1$

or $\tan 2x=\tan((x+y)+(x-y))$

or $\tan 2x=\frac{\tan(x+y)+\tan(x-y)}{1-\tan(x+y)\tan(x-y)}$

$\therefore \quad \tan 2x=\infty$ or $2x=90^\circ$ or $x=45^\circ=\frac{\pi}{4}$

$\therefore \quad y=\frac{\pi}{6}$

70. $\because x+y=\frac{4\pi}{3}$ ...(i)

and $\sin x=2\sin y$ (given)

$\sin x=2\sin\left(\frac{4\pi}{3}-x\right)$ [From Eq. (i)]

$$\sin x=2\left[\sin\frac{4\pi}{3}\cdot\cos x-\cos\frac{4\pi}{3}\cdot\sin x\right]$$

$$\sin x=2\left[\frac{-\sqrt{3}}{2}\cos x+\frac{1}{2}\sin x\right]$$

$$\sin x=-\sqrt{3}\cos x+\sin x$$

$$\cos x=0$$

$$x=(2n+1)\frac{\pi}{2},\forall n\in I$$

$$x=n\pi+\frac{\pi}{2}$$

Putting the value of x in Eq. (i) we get

$$y=\frac{4\pi}{3}-n\pi-\frac{\pi}{2}$$

$$y=\frac{5\pi}{6}-n\pi,\forall n\in I$$

$$\therefore\quad (x,y)=\left(n\pi+\frac{\pi}{2},\frac{5\pi}{6}-n\pi\right),\forall n\in I$$

71. We know that,

$$[x+I]=[x]+I, \text{ if } I \text{ is an integer.}$$

then, $$y=\frac{1}{3}[\sin\theta+[\sin\theta+[\sin\theta]]]$$

$$=\frac{1}{3}[\sin\theta+[\sin\theta]+[\sin\theta]]$$

$$=\frac{1}{3}([\sin\theta]+[\sin\theta]+[\sin\theta])$$

$$y=[\sin\theta] \quad \ldots(i)$$

$$[y+[y]]=2\cos\theta$$

$$[y]+[y]=2\cos\theta$$

$$[y]=\cos\theta \quad \ldots(ii)$$

$\because\quad -1\le\sin\theta\le 1$, then three cases arise:

Case I If $-1\le\sin\theta<0$, then $[\sin\theta]=-1$

$y=-1$ put in Eq. (ii), then

$$\cos\theta=[-1]=-1$$

$$\Rightarrow\quad \sin\theta=0$$

But $\quad -1\le\sin\theta<0$

Hence, this case is not possible.

Case II If $0\le\sin\theta<1$, then $[\sin\theta]=0$

and $\quad y=0 \quad$ [from Eq.(i)]

Put in Eq.(ii), $\cos\theta=0$

$$\Rightarrow\quad \sin\theta=1$$

But we have $0\le\sin\theta<1$, so this case is not possible.

Case III If $\sin\theta=1$, then $[\sin\theta]=1$

$y=1$ put in Eq. (ii), $\cos\theta=1\ \Rightarrow\ \sin\theta=0$

But, we have $\sin\theta=1$, so this case is not possible.

$\therefore$ Number of solutions is 0.

72. $\because [\sin x]+[\sqrt{2}\cos x]=-3,\ x\in[0,2\pi]$

It is possible only when $[\sin x]=-1$ and $[\sqrt{2}\cos x]=-2$

If $[\sin x]=-1\Rightarrow -1\le\sin x<0$,

$$\therefore\quad x\in(\pi,2\pi)$$

If $[\sqrt{2}\cos x]=-2$

$$\Rightarrow\quad -2\le\sqrt{2}\cos x<-1\ \Rightarrow\ -\sqrt{2}\le\cos x<-\frac{1}{\sqrt{2}}$$

$$\Rightarrow\quad -1\le\cos x<-\frac{1}{\sqrt{2}}\ \Rightarrow\ x\in\left[\pi,\frac{5\pi}{4}\right)$$

From Eqs. (i) and (ii), we get $x\in\left(\pi,\frac{5\pi}{4}\right)$

73. Here, $\cos\left(\frac{\alpha}{2}-\frac{\pi}{4}\right)=\frac{\sqrt{5}+1}{4}-\frac{\sqrt{5}-1}{4}=\frac{1}{2}\Rightarrow\frac{\alpha}{2}-\frac{\pi}{4}=2n\pi\pm\frac{\pi}{3}$

$$\therefore\quad \alpha=4n\pi+\frac{\pi}{2}\pm\frac{2\pi}{3}=4n\pi+\frac{7\pi}{6},\ 4n\pi-\frac{\pi}{6}$$

For, $\quad n=0,\ \alpha=\frac{7\pi}{6},-\frac{\pi}{6}$

74. $\sin\alpha+\left[\frac{\sin 2x}{2}\right]_{\alpha}^{2\alpha}=0\Rightarrow\sin\alpha(1+\cos 3\alpha)=0\Rightarrow\alpha=0,\ \cos 3\alpha=-1$

$3\alpha=-\pi,\ -3\pi\ \alpha=-\frac{\pi}{3},\ -\pi$

75. $\log_{\sqrt{3}}\tan\theta\left[\sqrt{\frac{\log_{\sqrt{3}}3}{\log_{\sqrt{3}}\tan\theta}+\frac{\log(\sqrt{3})^3}{\log_{\sqrt{3}}}}\right]=-1$

$$\Rightarrow\quad \log_{\sqrt{3}}\tan\theta\left[\sqrt{\frac{2}{\log_{\sqrt{3}}\tan\theta}+3}\right]=-1$$

Let $\log_{\sqrt{3}}\tan\theta=y\Rightarrow y\sqrt{\frac{2}{y}+3}=-1\Rightarrow\sqrt{\frac{2}{y}+3}=-\frac{1}{y}$

$$\Rightarrow\quad \frac{2}{y}+3=\frac{1}{y^2}\text{ or } y^2(2+3y)=y$$

$$\Rightarrow\quad y[3y^2+2y-1]=0$$

$\because\quad y<0$

$$y(3y-1)(y+1)=0$$

$$y=-1 \quad (\because y \text{ cannot be positive})$$

$$\Rightarrow\quad \log_{\sqrt{3}}\tan\theta=-1$$

$$\tan\theta=\frac{1}{\sqrt{3}}\ \ \therefore\ \ \theta=\frac{\pi}{6}\text{ and }\frac{7\pi}{6}$$

$\therefore$ There are two values of θ in $[0,2\pi]$

76. Consider $a\cos\theta+b\sin\theta=c$

$$\Rightarrow\quad a\cos\theta=c-b\sin\theta$$

$$\Rightarrow\quad a^2\cos^2\theta=(c-b\sin\theta)^2 \quad \text{(Squaring both sides)}$$

$$\Rightarrow\quad a^2(1-\sin^2\theta)=c^2-2bc\sin\theta+b^2\sin^2\theta$$

$$\Rightarrow\quad (a^2+b^2)\sin^2\theta-2bc\sin\theta+c^2-a^2=0$$

$\Rightarrow$ As α and β are values of 'θ' as given:

$\therefore$ roots of above equation are $\sin\alpha$ and $\sin\beta$.

$$\Rightarrow\quad \sin\alpha+\sin\beta=\text{sum of roots}=\frac{2bc}{a^2+b^2}$$

$$\sin\alpha\cdot\sin\beta=\text{products of roots}=\frac{c^2-a^2}{a^2+b^2}.$$

77. Consider $\sin 2x+\sin 4x=2\sin 3x$

$$\Rightarrow\quad (\sin 2x+\sin 4x)-2\sin 3x=0$$

$$\Rightarrow\quad 2\sin 3x\cdot\cos x-2\sin 3x=0$$

[By using $\sin A-\sin B=2\sin\frac{A+B}{2}\cdot\cos\frac{A-B}{2}$]

$\Rightarrow \quad 2\sin 3x(\cos x-1)=0$

$\Rightarrow$ Either $\sin 3x=0$ or $\cos x=1$

$\Rightarrow \quad 3x=n\pi \Rightarrow x=2n\pi$

$\Rightarrow \quad x=\frac{n\pi}{3}$ or $x=2n\pi$ [where $n\in I$]

78. We have, $4\sin^4 x+\cos^4 x=1$

$\Rightarrow \quad 4\sin^4 x=1-\cos^4 x=1\ (1-\cos^2 x)(1+\cos^2 x)$

$\Rightarrow \quad 4\sin^4 x-[(1-\cos^2 x)(1+\cos^2 x)]=0$

$\Rightarrow \quad 4\sin^4 x-[\sin^2 x(1+\cos^2 x)]=0$

$\Rightarrow \quad \sin^2 x[4\sin^2 x-(1+\cos^2 x)]=0$

$\Rightarrow \quad \sin^2 x[4\sin^2 x-(1+1-\sin^2 x)]=0$

$\Rightarrow \quad \sin^2 x[4\sin^2 x-1-(1-\sin^2 x)]=0$

$\Rightarrow \quad \sin^2 x(5\sin^2 x-2)=0$

Either $\sin x=0$ or $\sin x=\pm\sqrt{\frac{2}{5}}$

$\Rightarrow \quad x=n\pi$ or $x=n\pi\pm\alpha$

where $\alpha=\sin^{-1}\sqrt{\frac{2}{5}}$ and $n\in I$

79. Let $81^{\sin^2 x}=y$ ...(i)

then $81^{\cos^2 x}=81^{(1-\sin^2 x)}$

$=81.81^{-\sin^2 x}=81\cdot y^{-1}$ ...(ii)

So, the given can be written as

$y^2-30y+81=0 \Rightarrow y=3$ or $y=27$

By using Eqs. (i) and (ii)

$\Rightarrow$ Either $81^{\sin^2 x}=3$ or $81^{\sin^2 x}=27$

$\Rightarrow \quad 4\sin^2 x=1$ or $4\sin^2 x=3$

$\Rightarrow \quad \sin x=\frac{1}{2}$ or $\sin x=\frac{\sqrt{3}}{2}\left[\text{as } 0\le x\le\frac{\pi}{2}\right]$

$\Rightarrow \quad x=\frac{\pi}{6}$ or $x=\frac{\pi}{3}$ are only the solution.

80. The given equation can be written as

$$\frac{\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)}{\sin x}+\frac{\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)}{\cos x}=2$$

$$\Rightarrow \quad \frac{\sin\frac{\pi}{12}}{\sin x}+\frac{\cos\frac{\pi}{12}}{\cos x}=2$$

$$\Rightarrow \quad \sin\left(\frac{\pi}{12}+x\right)=\sin 2x$$

$$\therefore \quad \frac{\pi}{12}+x=2x \text{ or } \frac{\pi}{12}+x=\pi-2x$$

$$\Rightarrow \quad x=\frac{\pi}{12} \text{ or } \frac{11\pi}{36}$$

81. $\frac{\alpha^2}{1-\tan^2 x}=\frac{(\sin^2 x+\alpha^2-2)(1+\tan^2 x)}{1-\tan^2 x}$

$\Rightarrow \quad \alpha^2\cos^2 x=\sin^2 x+\alpha^2-2$

$\Rightarrow \quad 2=\sin^2 x(1+\alpha^2)$

$\Rightarrow \quad \sin^2 x=\frac{2}{1+\alpha^2}$

$\Rightarrow \quad 0\le\frac{2}{1+\alpha^2}\le 1$

$\Rightarrow \quad \alpha^2\ge 1\Rightarrow\alpha\le -1$ or $\alpha\ge 1$

82. $4[\sin x\cos\pi/3+\cos x\sin\pi/3]\times[\cos x\cos\pi/6+\sin x\sin\pi/6]$

$=a^2+\sqrt{3}\sin 2x-\cos 2x$

$$\Rightarrow \quad 4\left[\frac{1}{2}\sin x+\frac{\sqrt{3}}{2}\cos x\right]\left[\frac{\sqrt{3}}{2}\cos x+\frac{1}{2}\sin x\right]$$

$=a^2+\sqrt{3}\sin 2x-\cos 2x$

$\Rightarrow \quad \sqrt{3}\sin 2x+3\cos^2 x+\sin^2 x=a^2+\sqrt{3}\sin 2x-\cos 2x$

$\Rightarrow \quad \cos 2x+2=a^2-\cos 2x$

$\Rightarrow \quad \cos 2x=\frac{a^2-2}{2} \Rightarrow -1\le\frac{a^2-2}{2}\le 1$

$\Rightarrow \quad 0\le a^2\le 4 \Rightarrow -2\le a\le 2$

All values of a given in (a), (b), (c), (d) satisfy this relation.

83. (a) For $x\in\left(0,\frac{\pi}{4}\right)$, $\tan x<\cot x$

Also $\ln(\sin x)<0$

$\Rightarrow \quad (\tan x)^{\ln(\sin x)}>(\cot x)^{\ln(\sin x)}$

(b) For $x\in\left(0,\frac{\pi}{2}\right)$, $\operatorname{cosec} x\ge 1$

$\Rightarrow \quad \ln(\operatorname{cosec} x)\ge 0\Rightarrow 4^{\ln\operatorname{cosec} x}<5^{\ln(\operatorname{cosec} x)}$

(c) $x\in\left(0,\frac{\pi}{2}\right) \quad \Rightarrow \cos x\in(0,1)$

$\Rightarrow \ln(\cos x)<0$ Also, $\frac{1}{2}>\frac{1}{3}$

$\Rightarrow \quad \left(\frac{1}{2}\right)^{\ln(\cos x)}<\left(\frac{1}{3}\right)^{\ln(\cos x)}$

(d) For $x\in\left(0,\frac{\pi}{2}\right)$

Since, $\sin x<\tan x$, we get

$\ln(\sin x)<\ln(\tan x)$

$\Rightarrow \quad 2^{\ln(\sin x)}<2^{\ln(\tan x)}$

84. The given equation can be written as

$$\begin{vmatrix}1+\cos^2\theta & \sin^2\theta & 4\sin 4\theta\\ -1 & 1 & 0\\ 0 & -1 & 1\end{vmatrix}=0$$

[Applying $R_3\to R_3-R_2$ and $R_2\to R_2-R_1$]

$$\Rightarrow \quad \begin{vmatrix}2 & \sin^2\theta & 4\sin 4\theta\\ 0 & 1 & 0\\ -1 & -1 & 1\end{vmatrix}=0 \quad [\text{Applying } C_1\to C_1+C_2]$$

$\Rightarrow \quad 2+4\sin 4\theta=0\Rightarrow\sin 4\theta=-\frac{1}{2}$

$\Rightarrow \quad 4\theta=n\pi+(-1)^n\left(-\frac{\pi}{6}\right) \Rightarrow \theta=\frac{n\pi}{4}+(-1)^{n+1}\frac{\pi}{24}$

We have to choose values of θ s.t $0<\theta<\frac{\pi}{2}$

$\therefore \quad \theta=\frac{7\pi}{24},\frac{11\pi}{24}$

85. At $x=-\frac{\pi}{2}, \frac{3\pi}{2}$

$\Rightarrow \quad [1+\sin x]=0, [1-\cos x]=1$

$\therefore \quad \sin x=0+1 \Rightarrow -1=1$ (Absurd)

At $x=0$

$(1+\sin x)=1, (1-\cos x)=0$

$\therefore \quad \sin x=1+0 \Rightarrow 0=1$ (Absurd)

At $x=\frac{\pi}{2}$

$[1+\sin x]=2\cdot[1-\cos x]=1$

$\sin x=2+1=3$ (Absurd)

At $x=\pi$

$\sin x=1+2=3$ (Absurd)

In $\left(-\frac{\pi}{2}, 0\right), [1+\sin x]=0, [1-\cos x]=0$

$\therefore \quad \sin x=0+0=0$ (Absurd)

In $\left(0, \frac{\pi}{2}\right), [1+\sin x]=1, [1-\cos x]=0$

$\therefore \quad \sin x=1+0=1$ (Absurd)

In $\left(\frac{\pi}{2}, \pi\right), [1+\sin x]=1, [1-\cos x]=1$

$\therefore \quad \sin x=1+1=2$ (Absurd)

In $\left(\pi, \frac{3\pi}{2}\right), [1+\sin x]=0, [1-\cos x]=1$

$\therefore \quad \sin x=0+1=1$ (Absurd)

$\therefore$ All the four results hold.

Sol. (Q. Nos. 86 to 88)

$$3\sin^2 x-7\sin x+2=0$$

$$\sin x=\frac{1}{3} \text{ or } 2(\text{Reject})$$

86. $N=2$

One value in first quadrant and other lies in second quadrant.

87. Let $x+\alpha$, then two values α and $\pi-\alpha$.

$\Rightarrow \quad$ sum is π

88. $\because \quad \sin\alpha=\frac{1}{3} \Rightarrow \cos\alpha=\pm\frac{2\sqrt{2}}{3}$

$$f_4(\alpha)=\sin^4 x+\cos^4\alpha=1-2\sin^2\alpha\cos^2\alpha=\frac{65}{81}$$

Sol. (Q. Nos. 89 to 90)

89. If $\alpha=3$ and $a=5$, then $N\in[5^3, 5^4]$

largest value of N is $5^4-1=624$

smallest value of N is $5^3=125$

Difference of largest and smallest integral value of N
$=624-125=499$

90. If $\alpha=2$ and $a=2$, then

N_1 is $(2^3)-(2^2)=4$

If $\alpha=1$ and $a=3$ then N_2 is $3^2-3^1=6$

$$y=(N_1\sec^2\theta+N_2\operatorname{cosec}^2\theta)$$

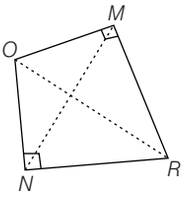
$$=(4\sec^2\theta+6\operatorname{cosec}^2\theta)$$

$\therefore$ The minimum values of y is $(2+\sqrt{6})^2=10+4\sqrt{6}$

Sol. (Q. Nos. 91 to 93)

91. $\because \quad \angle OMN=15^\circ=\angle ONM$

$\therefore \quad \angle MON=180^\circ-15^\circ-15^\circ=150^\circ$



Now, quadrilateral $ONRM$ is cyclic

$[\because \angle OMR=\angle ONR=90^\circ]$

$\therefore \quad \angle R=180^\circ-150^\circ=30^\circ=\angle Q \quad [\because PR=PQ]$

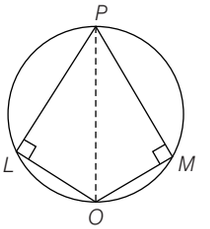
$\Rightarrow \quad \angle P=120^\circ$

92. $\angle PLO=\angle PMO=90^\circ$

$\therefore$ Quadrilateral $PLOM$ is cyclic and OP is diameter of circumcircle

$\Rightarrow \quad \angle LOM=180^\circ-\angle P=60^\circ$

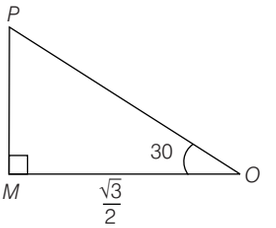
$\Rightarrow \quad \angle POM=30^\circ$



Now, is right angled Δ POM,

$$OP=\frac{\sqrt{3}}{2}\sec 30^\circ \quad [\because AC=x\sec\theta]$$

$$=\frac{\sqrt{3}}{2}\cdot\frac{2}{\sqrt{3}}=1$$

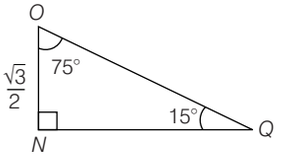


Radius of circumcircle $=\frac{1}{2}$

$\therefore \quad$ Area $=\pi\left(\frac{1}{2}\right)^2=\frac{\pi}{4}$

93. $\because \quad \angle OQN=15^\circ$

$$ON=\frac{\sqrt{3}}{2}$$



$\therefore$ In right angled ΔQON

$$NQ = ON \tan 75^\circ = \frac{\sqrt{3}}{2}\tan 75^\circ$$

$$\Rightarrow \quad QR = \sqrt{3}\tan 75^\circ = \sqrt{3}\cot 15^\circ$$

Sol. (Q, Nos. 94 to 96)

$$(2\sin x - \cos x)(1+\cos x) = \sin^2 x$$

$$\Rightarrow \quad (1+\cos x)[2\sin x - \cos x - 1 + \cos x] = 0$$

$$\Rightarrow \quad (1+\cos x)(2\sin x - 1) = 0$$

$$\Rightarrow \quad \cos x = -1 \text{ or } \sin x = 1/2$$

So $\quad \sin\alpha = 1/2 \ [\text{as } 0 \le \alpha \le \pi/2]$

$$\Rightarrow \quad \cos\alpha = \sqrt{3}/2$$

Next, $3\cos^2 x - 10\cos x + 3 = 0$ ...(i)

$$\Rightarrow \quad (3\cos x - 1)(\cos x - 3) = 0$$

$$\Rightarrow \quad \cos x = 1/3 \text{ as } \cos x \ne 3$$

So, $\quad \cos\beta = 1/3, \sin\beta = \frac{2\sqrt{2}}{3}$ ...(ii)

and $\quad 1 - \sin 2x = \cos x - \sin x$

$$\Rightarrow \ \sin^2 x + \cos^2 x - 2\sin x\cos x = \cos x - \sin x$$

$$\Rightarrow \quad (\cos x - \sin x)(\cos x - \sin x - 1) = 0$$

$\Rightarrow$ Either $\sin x = \cos x \Rightarrow \sin\gamma = \cos\gamma = 1/\sqrt{2}$ ...(iii)

or $\quad \cos x - \sin x = 1 \Rightarrow \cos x = 1, \sin x = 0$

$$\Rightarrow \quad \cos\gamma = 1, \sin\gamma = 0$$ ...(iv)

94. $\cos\alpha + \cos\beta + \cos\gamma$ can be equal to

$$\frac{\sqrt{3}}{2} + \frac{1}{3} + \frac{1}{\sqrt{2}} \text{ or } \frac{\sqrt{3}}{2} + \frac{1}{3} + 1$$

i.e., $\quad \frac{3\sqrt{6} + 2\sqrt{2} + 6}{6\sqrt{2}} \text{ or } \frac{3\sqrt{3} + 8}{6}$

95. $\sin\alpha + \sin\beta + \sin\gamma$ can be equal to

$$\frac{1}{2} + \frac{2\sqrt{2}}{3} + \frac{1}{\sqrt{2}} \text{ or } \frac{1}{2} + \frac{2\sqrt{2}}{3} + 0$$

i.e., $\quad \frac{3\sqrt{2} + 14}{6\sqrt{2}} \text{ or } \frac{3 + 4\sqrt{2}}{6}$

96. $\sin(\alpha - \beta)$ is equal to

$$\sin\alpha\cos\beta - \cos\alpha\sin\beta = \frac{1}{2}\times\frac{1}{3} - \frac{\sqrt{3}}{2}\times\frac{2\sqrt{2}}{3}$$

$$= \frac{1 - 2\sqrt{6}}{6}$$

97. $5\sin^2 x + 3\sin x\cos x - 3\cos^2 x = 2(\sin^2 x + \cos^2 x)$

$$\Rightarrow \ 3\tan^2 x + 3\tan x - 5 = 0 \ \Rightarrow \ \tan x = \frac{-3 \pm \sqrt{69}}{6}$$

and $\quad \sin^2 x - \cos 2x = 2 - \sin 2x$

$$\Rightarrow \quad 3\sin^2 x + 2\sin x\cos x = 3(\sin^2 x + \cos^2 x)$$

$$\Rightarrow \quad \cos x(2\sin x - 3\cos x) = 0$$

Either $\cos x = 0$ or $\tan x = \frac{3}{2} \ \Rightarrow \ \cos x = \pm\frac{2}{\sqrt{13}}$

$$\Rightarrow \quad \text{Taking } \tan\alpha = \frac{-3 \pm \sqrt{69}}{6}, \tan\beta = \frac{3}{2}$$

we get $\tan\alpha + \tan\beta = 1 \pm \sqrt{69}/6$

98. Taking $\tan\alpha = \frac{-3 + \sqrt{69}}{6}$,

$$\tan\beta = \frac{-3 - \sqrt{69}}{6}$$

$$\cos\gamma = 0, \cos\delta = \pm\frac{2}{\sqrt{13}}$$

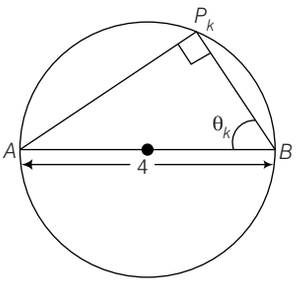
we get $\tan\alpha\tan\beta + \cos\gamma + \cos\delta = -\frac{5}{3} \pm \frac{2}{\sqrt{13}}$

99. (1) and (2) have no solution common.

100. Let $\theta_k = \frac{k\pi}{2n}$

$\because$ AB is diameter of circle.

$\therefore \Delta\, AP_kB$ is right angled triangle.



$$\Rightarrow \quad \Delta_k = \frac{1}{2}\cdot AB.\cos\theta_k \cdot AB\sin\theta_k$$

$$= 4\sin 2\theta_k$$ ...(i)

Now, $\sum_{k=1}^{n+1}\sin 2\theta_k = \sin 2\theta_1 + \sin 2\theta_2 + \ldots + \sin 2\theta_{n+1}$

$$= \sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \ldots + \sin\frac{(n+1)\pi}{n}$$

$$= \frac{\sin\left(n.\frac{\pi}{2n}\right)\sin\left(\frac{\pi}{n} + (n-1)\frac{\pi}{2n}\right)}{\sin\frac{\pi}{2n}}$$

$$= \frac{\sin\left(\frac{\pi(n+1)}{2n}\right)}{\sin\frac{\pi}{2n}} = \cot\frac{\pi}{2n} = \cot\frac{\pi}{32} \Rightarrow n = 16$$

$\therefore \quad \frac{n}{2} = 8$

101. $2\cos^2 2x - 7\cos 2x + 5 = 0$

$$\Rightarrow \quad \cos 2x = 1$$

$$\Rightarrow \quad x = n\pi$$

$$k = 1 + 2 + \ldots + 99$$

$$= \frac{99 \times 100}{2} = 4950$$

Now, $k - 4948 = 4950 - 4948 = 2$

102. In 2nd equation sum of coefficients is zero, hence its one root is 1 and second root is also 1 as it has equal roots.

Common root of first equation is 1.

$$\Rightarrow \quad \tan^2\theta - 2\tan\theta + 1 = 0$$

$$\Rightarrow \quad (\tan\theta - 1)^2 = 0$$

$\Rightarrow \tan\theta = 1$, hence one solution in $(0, \pi)$.

103. $\frac{x^4+1}{8x^2}=\frac{(2\sin y\cos y)^2}{4}$

$\Rightarrow \quad x^2+\frac{1}{x^2}=2\sin^2 2y$

LHS ≥ 2 and RHS ≤ 2

LHS = RHS will hold

only if LHS = 2 = RHS

i.e. $x^2=1$ and $\sin^2 2y=1$

$\Rightarrow \quad x=\pm 1$ and $2y=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},\frac{7\pi}{2}$

So, total number of ordered pairs are 8.

104. We have,

$$\sin\left(x-\frac{\pi}{4}\right)-\cos\left(x+\frac{3\pi}{4}\right)=1$$

$$\sin\left(x-\frac{\pi}{4}\right)+\sin\left(x+\frac{\pi}{4}\right)=1$$

$\Rightarrow \quad 2\sin x\cos\frac{\pi}{4}=1 \qquad [\because \sin\theta=\sin(\pi-\theta)]$

$\Rightarrow \quad \sin x=\frac{1}{\sqrt{2}}$

$\Rightarrow \quad x=\frac{\pi}{4},\frac{3\pi}{4}$

We know that 3 radians $\cong 171^\circ 22'$.

Therefore, $\sin 3>0$, $\cos 3<0$ and $|\cos 3|>\sin 3$

$\therefore \quad \cos 3+\sin 3<0$

Now, $\frac{2\cos 7x}{\cos 3+\sin 3}>2^{\cos 2x}$

$\Rightarrow \quad 2\cos 7x<2^{\cos 2x}(\cos 3+\sin 3) \qquad (\because \cos 3+\sin 3<0)$

$\Rightarrow \quad 2\cos 7x<0 \;\Rightarrow\; \cos 7x<0 \qquad (\therefore 2^{\cos x}>0)$

Clearly, $x=\frac{3\pi}{4}$ satisfies this equation.

Hence, $x=\frac{3\pi}{4}$ is the required solution.

Hence, only one solution.

105. $\because \cos A\sin\left(A-\frac{\pi}{6}\right)=\frac{1}{2}\left[2\cos A\sin\left(A-\frac{\pi}{6}\right)\right]$

$$=\frac{1}{2}\left[\sin\left(A+A-\frac{\pi}{6}\right)-\sin\left(A-A+\frac{\pi}{6}\right)\right]$$

$$=\frac{1}{2}\left[\sin\left(2A-\frac{\pi}{6}\right)-\sin\frac{\pi}{6}\right]$$

So, it is maximum, when $\sin\left(2A-\frac{\pi}{6}\right)$ is maximum.

i.e., $2A-\frac{\pi}{6}=\frac{\pi}{2}$

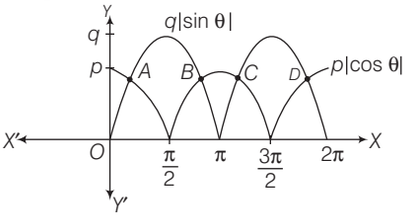
$$2A=\frac{\pi}{2}+\frac{\pi}{6}$$

$$A=\frac{\pi}{3}$$

but $A=\frac{\pi}{\lambda}$

Hence, $\lambda=3$

106. Draw the curves $p|\cos\theta|$ and $q|\sin\theta|$ and find the number of intersection points.



Hence, intersection points are A, B, C, D

$\therefore$ Number of solutions is 4.

107. $\tan\theta=\frac{3\tan\frac{\theta}{3}-\tan^3\frac{\theta}{3}}{1-3\tan^2\frac{\theta}{3}}$

$\Rightarrow \quad \tan^3\frac{\theta}{3}-3\lambda\tan\frac{\theta}{3}-3\tan\frac{\theta}{3}+\lambda=0$

$\therefore \quad \tan\frac{\theta_1}{3}\cdot\tan\frac{\theta_2}{3}+\tan\frac{\theta_2}{3}\cdot\tan\frac{\theta_3}{3}+\tan\frac{\theta_3}{3}\cdot\tan\frac{\theta_1}{3}=-3$

$\Rightarrow \quad \left|\tan\frac{\theta_1}{3}\cdot\tan\frac{\theta_2}{3}+\tan\frac{\theta_2}{3}\cdot\tan\frac{\theta_3}{2}+\tan\frac{\theta_1}{3}\cdot\tan\frac{\theta_3}{3}\right|=3$

108. $\sin(1-x)\geq 0$, $\cos x\geq 0$

$$\cos x=\cos\left\{\frac{\pi}{2}-(1-x)\right\}$$

$\therefore \quad x=2n\pi+\left(\frac{\pi}{2}-(1-x)\right)$

$\Rightarrow \quad x=n\pi-\frac{\pi}{4}+\frac{1}{2}$

Put $n=1$, $x=\frac{3\pi}{4}+\frac{1}{2}\Rightarrow[x]=2$

109. L.H.S ≤ 13 and R.H.S $=2(y-2)^2+13\geq 13$

Roots of eqn. exist if $y=2$ and $\sin\left\{x+\tan^2\frac{5}{12}\right\}=1$

Now, consider $\sin\left\{x+\tan^{-1}\frac{5}{12}\right\}=1$

$\Rightarrow \quad x+\tan^{-1}\frac{5}{12}=\sin^{-1}(1)$

$\Rightarrow \quad x+\tan^{-1}\frac{5}{12}=\frac{\pi}{2}$

$\Rightarrow \quad x=\frac{\pi}{2}-\tan^{-1}\frac{5}{12}$

$\Rightarrow \quad x=\cot^{-1}\frac{5}{12}$

$\Rightarrow \quad \cot x=\frac{5}{12}$

Hence, $12\cot\left(\frac{xy}{2}\right)=12\cot x=12\times\frac{5}{12}=5$

110. $\tan(\pi\cos\theta)=\tan(\pi/2-\pi\sin\theta)$

$\Rightarrow \quad \pi\cos\theta=n\pi+\pi/2-\pi\sin\theta\,(n\in I)$

$\Rightarrow \quad \pi(\sin\theta+\cos\theta)=(2n+1)\frac{\pi}{2}$

$\Rightarrow \quad \sin\theta+\cos\theta=\dfrac{2n+1}{2}$

$\Rightarrow \quad \cos(\pi/4-\theta)=\dfrac{2n+1}{2\sqrt{2}}$

Since $\quad -1\le\cos(\pi/4-\theta)\le 1$

$\Rightarrow \quad -1\le\dfrac{2n+1}{2\sqrt{2}}\le 1$

$\Rightarrow n=0$ or -1 as n is an integer

$\Rightarrow \quad \cos(\pi/4-\theta)=\pm(1/2\sqrt{2})$

$\Rightarrow \quad 8\cos^2(\pi/4-\theta)=1$

$\Rightarrow \quad 16\cos^2\left(\dfrac{\pi}{4}-\theta\right)=2$

111. From the given equation we have

$$3\frac{2\tan(x/2)}{1+\tan^2(x/2)}+4\frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}=5$$

$\Rightarrow \quad 6\tan(x/2)+4-4\tan^2(x/2)=5+5\tan^2(x/2)$

$\Rightarrow \quad 9\tan^2(x/2)-6\tan(x/2)+1=0$

$\Rightarrow \quad 90\tan^2(x/2)-60\tan(x/2)=-10$

$\because \quad 90\tan^2\dfrac{x}{2}-60\tan\dfrac{x}{2}+10=-10+10=0$

112. When $n=1$,. we have interval $[0,\pi]$, which covers only the first and second quadrants in which.

$\sin x=-1/2$ is not possible. Hence, the number of solutions zero. Also, from $2(n-1)$, we have zero solution when $n=1$.

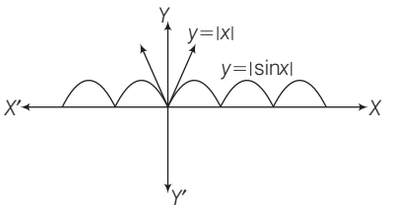
For $n=2$, we have interval $[0,2\pi]$ which covers all the quadrants only once. Hence, the number of solutions is two.

Also, from $2(n-1)$, we have two solutions, when $n=2$.

Fro $n=3$, we have interval $[0,3\pi]$, which covers the third and fourth quadrants only once. Hence, the number of solutions is two. But from $2(n-1)$, we have four solutions which contradict.

Hence, Statement I is false, and Statement II is true.

113. The graphs of $y=|\sin x|$ and $y=|x|$ is



$|\sin x|=|x|$ has only one solution $x=0$. But Statement II is not the only explanation of Statement I.

114. $\sin 2x=\dfrac{2\tan x}{1+\tan^2 x}$, $\cos 2x=\dfrac{1-\tan^2 x}{1+\tan^2 x}$

$\Rightarrow$ Statement II is correct.

In Statement I, we have

$$\cos^2 x=\sin 2x$$

$\Rightarrow \quad \cos x(\cos x-2\sin x)=0$

$\Rightarrow \quad \tan x=1/2$ as $\cos x\neq 0$

From Statement II we get

$$\sin 2x+\cos 2x=\frac{1+1-1/4}{1+1/4}=\frac{7}{5}$$

Statement I is false

115. In statement, $\begin{vmatrix}1 & \sin\alpha & \cos\alpha\\ 1 & \cos\alpha & \sin\alpha\\ -1 & \sin\alpha & -\cos\alpha\end{vmatrix}=0$

$\Rightarrow$ either $\sin\alpha=0$ or $\tan\alpha=1$

$\Rightarrow \quad \alpha=\pi/4$ (as $0<\alpha<\pi$)

and Statement II, $(\sin x+2\cos x)$

$$\begin{vmatrix}1 & \cos x & \cos\alpha\\ 1 & \sin x & \cos x\\ 1 & \cos x & \sin x\end{vmatrix}=0$$

$\Rightarrow \quad (\sin x+2\cos x)(\cos x-\sin x)^2=0$

$\Rightarrow \quad (\sin x+2\cos x)(\cos x-\sin x)^2=0$

which does not hold for any value of x as $-\pi/4<x<\pi/4$

116. For $(\pi/4,\pi/2)$, $0<\cos\theta<\dfrac{1}{\sqrt{2}}<\sin\theta<1$

So, $\quad (\cos\theta)^{\cos\theta}<(\sin\theta)^{\cos\theta}$

and $\quad (\cos\theta)^{\sin\theta}<(\sin\theta)^{\cos\theta}$

$\Rightarrow \quad (\cos\theta)^{\sin\theta}+(\cos\theta)^{\cos\theta}<(\sin\theta)^{\cos\theta}$

Showing that Statement I is true.

In Statement II let, $e^{\sin\theta}=t$

Then, $\quad t^2-4t-1=0$

$\Rightarrow \quad t=\dfrac{4\pm\sqrt{16+4}}{2}=2\pm\sqrt{5}$

$\Rightarrow \quad e^{\sin\theta}=2\pm\sqrt{5} \quad \Rightarrow\sin\theta=\log(2\pm\sqrt{5})$

Since, $2-\sqrt{5}<0$, $\sin\theta=\log(2+\sqrt{5})>\log_e$

$\Rightarrow \sin\theta>1$ which is not possible.

So, the give equation has no solution and the statement II is false.

117. $\sin^2 x+\sin^4 x+\sin^6 x+\ldots\text{inf}$

$$=\frac{\sin^2 x}{1-\sin^2 x}=\tan^2 x$$

$\Rightarrow$ Statement II is false.

Now, $\exp\{\tan^2 x\log_e 2\}=2^{\tan^2 x}$

So $2^{\tan^2 x}=1$ or $2^{\tan^2 x}=8$

$\Rightarrow \quad 2^{\tan^2 x}=8 \qquad [\because \tan x>0\Rightarrow 2^{\tan^2 x}>1]$

$\Rightarrow \quad \tan^2 x=3\Rightarrow\tan x=\sqrt{3}$

Now, $\quad \dfrac{\cos x}{\cos x+\sin x}=\dfrac{1}{1+\tan x}=\dfrac{1}{\sqrt{3}+1}=\dfrac{\sqrt{3}-1}{2}$

118. (A) $\sin x=-\dfrac{1}{2}$ and $\cos x=\dfrac{-\sqrt{3}}{2}$

$$\alpha=\frac{7\pi}{6},\beta=\frac{11\pi}{6}\ \alpha=\frac{7\pi}{6},\gamma=\frac{5\pi}{6}$$

$$\beta-\gamma-\frac{11\pi}{6}-\frac{5\pi}{6}=\pi$$

$$\alpha+\beta=\frac{7\pi}{6}+\frac{11\pi}{6}=3\pi$$

$\therefore \quad A\to q,s$

(B) $\cot x=-\sqrt{3}$ and $\operatorname{cosec} x=-2$

$$\alpha=\frac{11\pi}{6},\beta=\frac{5\pi}{6},\gamma=\frac{7\pi}{6}$$

$\therefore \quad \alpha-\beta=\pi$ and $\beta+\gamma=2\pi$

$\therefore \quad \beta \to p,t$

(C) $\sin x=-\frac{1}{2}$ and $\tan x=\frac{1}{\sqrt{3}}$

$\alpha=\frac{7\pi}{6}, \beta=\frac{11\pi}{6}, \gamma=\frac{\pi}{6}$

$\alpha-\gamma=\pi,\ \alpha+\beta=3\pi,\ \beta+\gamma=2\pi$

$\therefore \quad C \to r,s,t$

119. (A) $2\sin\cos\theta=1/\sqrt{2}$ if $\cos\theta>0$

$\Rightarrow \quad \sin2\theta=1/\sqrt{2} \Rightarrow \theta=\pi/8$ or $3\pi/8$

$\sin2\theta=-1/\sqrt{2}$ if $\cos\theta<0$

$\Rightarrow \quad \theta=5\pi/8$ or $7\pi/8$

$\therefore A \to p,q$

(B) $2\cos2\theta\cos4\theta+\cos4\theta=0$

$\Rightarrow \quad (2\cos2\theta+1)\cos4\theta=0$

Either $\cos4\theta=0$

$\Rightarrow \quad \theta=\frac{\pi}{8},\frac{3\pi}{8},\frac{5\pi}{8},\frac{7\pi}{8}$ or $\cos2\theta=-1/2$

$\Rightarrow \quad \theta=\pi/3$ or $2\pi/3$

$\therefore B \to p,q,r$

(C) $(4\cos^2\theta-1)(2\sin\theta-1)=0$

$\Rightarrow \quad \cos^2\theta=1/4 \Rightarrow \cos\theta=\pm1/2$

$\Rightarrow \quad \theta=\pi/3, 2\pi/3$ or $\sin\theta=1/2$

$\Rightarrow \quad \theta=\pi/6$

$\therefore C \to r,s$

(D) If $\sin4\theta=\pm1,\ \theta=\frac{\pi}{8},\frac{3\pi}{8},\frac{5\pi}{8},\frac{7\pi}{8}$

$\therefore D \to p,q$

120. $f_n(\theta)=\dfrac{\sin\theta+\sin3\theta+\sin5\theta+\ldots+\sin((2n-1)\theta)}{\cos\theta+\cos3\theta+\cos5\theta+\cdots+\cos[(2n-1)\theta]}$

$$=\frac{\sin\left[\theta+\frac{(n-1)2\theta}{2}\right]\sin\frac{n}{2}(2\theta)}{\sin\frac{2\theta}{2}}=\frac{\sin(n\theta)}{\cos(n\theta)}=\tan n\theta$$

$$\cos\frac{\left[\theta+\frac{(n-1)2\theta}{2}\right]\sin\frac{n}{2}(2\theta)}{\sin\frac{2\theta}{2}}$$

(A) $f_3\left(\frac{\pi}{12}\right)=\tan\left(\frac{\pi}{4}\right)=1$

(B) $f_4\left(\frac{\pi}{32}\right)=\tan\left(\frac{\pi}{8}\right)=\sqrt{2}-1$

(C) $f_6\left(\frac{\pi}{16}\right)=\tan\left(\frac{3\pi}{8}\right)=\sqrt{2}+1$

(D) $f_7\left(\frac{\pi}{84}\right)=\tan\left(\frac{\pi}{12}\right)=2-\sqrt{3}$

121. (i) If $\cot x>0$, then $\frac{1}{\sin x}=0$ which is not possible.

Now if $\cot x<0$,

then, $\quad -\cot x=\cot x+\frac{1}{\sin x}$

$\Rightarrow \quad \frac{2\cos x+1}{\sin x}=0 \Rightarrow \cos x=-\frac{1}{2}$

$\Rightarrow \quad x=2n\pi\pm\frac{2\pi}{3}, n\in I$ and $0\le x\le2\pi$.

$\therefore \quad x=\frac{2\pi}{3},\frac{4\pi}{3}$

(ii) We have, $\sin^3x\cos x+\sin^2x.\cos^2x+\sin x.\cos^3x=1$

$\Rightarrow \quad \sin x\cos x(\sin^2x+\sin x\cos x+\cos^2x)=1$

$\Rightarrow \quad \frac{\sin2x}{2}\left(1+\frac{\sin2x}{2}\right)=1$

$\Rightarrow \quad \sin2x(2+\sin2x)=4$

$\Rightarrow \quad \sin^2 2x+2\sin2x-4=0$

$\Rightarrow \quad \sin2x=\frac{-2\pm\sqrt{4+16}}{2}=-1\pm\sqrt{5}$

This is not possible, as $-1\le\sin2x\le1$.

Hence, the given equation has no solution.

(iii) We have, $2^{\cos x}=|\sin x|$

It is true only for $\cos x=0$ and $|\sin x|=1$

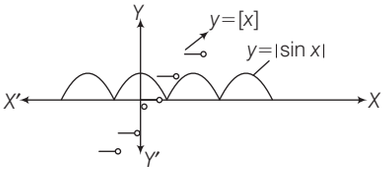
$\Rightarrow \cos x=\cos\frac{\pi}{2}$ and $\sin x=\pm1=\sin\left(\pm\frac{\pi}{2}\right)$

$\Rightarrow \quad x=2n\pi\pm\frac{\pi}{2}$

But, $x\in[-2\pi,2\pi]$

$x=-\frac{\pi}{2},\frac{\pi}{2},\frac{3\pi}{2},-\frac{3\pi}{2}$ Hence, number of solutions = 4.

(iv) We have, $\quad |\cos x|=[x]=y$ (say)

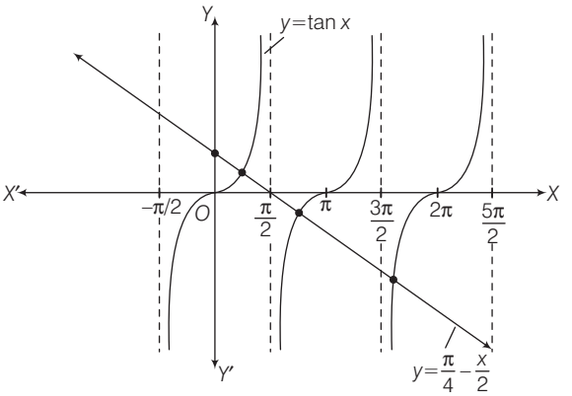


Graph of $|\cos x|$ and $[x]$ don't cut each other for any real value of x.

Hence, number of solutions is zero.

(v) We have $x+2\tan x=\frac{\pi}{2}$

or $\quad \tan x=\frac{\pi}{4}-\frac{x}{2}$



Now the graph of the curve $y = \tan x$ and $y = \frac{\pi}{4} - \frac{x}{2}$, in the interval $[0, 2\pi]$ intersect at three points.
Hence, number of solutions is three.

122. Here, $(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x.\cos^2 x + \sin 2x + \alpha = 0$

$$\Rightarrow \quad 1 - \frac{1}{2}\sin^2 2x + \sin 2x + \alpha = 0$$

$$\Rightarrow \quad \sin^2 2x - 2\sin 2x - 2(1 + \alpha) = 0$$

$$\Rightarrow \quad \sin 2x = \frac{2 \pm \sqrt{4 + 8(1 + \alpha)}}{2} = 1 \pm \sqrt{2\alpha + 3} \qquad \ldots(i)$$

$$\Rightarrow \quad \sin 2x = 1 - \sqrt{2\alpha + 3} \qquad [\because 1 + \sqrt{2\alpha + 3} \geq 1]$$

and $\sin 2x = 1$ is already included in the solution of $\sin 2x = 1 - \sqrt{2\alpha + 3}$

But $\sin 2x$ is real, so $2\alpha + 3 \geq 0$ i.e., $\alpha \geq -\frac{3}{2}$

Also, $-1 \leq \sin x \leq 1$

$$\therefore \quad -1 < 1 - 1\sqrt{2\alpha + 3} \leq 1$$

$$\Rightarrow \quad -2 \leq -\sqrt{2\alpha + 3} \leq 0$$

Squaring both sides,

$$0 \leq 2\alpha + 3 \leq 4$$

$$\Rightarrow \quad -3 \leq 2\alpha \leq 1$$

$$-\frac{3}{2} \leq \alpha \leq \frac{1}{2} \text{ or } \alpha \in \left[-\frac{3}{2}, \frac{1}{2}\right]_n$$

Also the general solution is, $x = \frac{n\pi}{2} + \frac{(-1)^n}{2}\sin^{-1}\{1 - \sqrt{2\alpha + 3}\}$

123. Here, $3\cos 2\theta = 1 \Rightarrow 3(2\cos^2\theta - 1) = 1$

$$\Rightarrow \quad 6\cos^2\theta = 4 \; ; \Rightarrow \cos^2\theta = \frac{2}{3}$$

Now, $\tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta} = \frac{1 - \cos^2\theta}{\frac{2}{3}} = \frac{1 - \frac{2}{3}}{\frac{2}{3}} = \frac{1}{2}$

$$\therefore \quad 32\tan^8\theta = 2\cos^2\alpha - 3\cos\alpha$$

$$\Rightarrow \quad 32\left(\frac{1}{2}\right)^4 = 2\cos^2\alpha - 3\cos\alpha$$

$$\Rightarrow \quad 2\cos^2\alpha - 3\cos\alpha - 2 = 0$$

$$\Rightarrow \quad (2\cos\alpha + 1)(\cos\alpha - 2) = 0$$

$$\Rightarrow \quad 2\cos\alpha + 1 = 0 \text{ or } \cos\alpha - 2 = 0$$

$$\Rightarrow \quad \cos\alpha = -\frac{1}{2} \text{ or } \cos\alpha = 2 \text{ (impossible)}$$

$$\Rightarrow \quad \alpha = 2n\pi \pm \left(\pi - \frac{\pi}{3}\right) = 2n\pi \pm \frac{2\pi}{3}, n \in Z$$

124. Let $4^{\sin x} = \lambda$ and $3^{1/\cos y} = \mu$

Then, the equation becomes, $\lambda + \mu = 11$

$5\lambda^2 - 2\mu = 2$, solving these equations we get,

$$2\lambda + 5\lambda^2 = 24$$

or $\quad 5\lambda^2 + 2\lambda - 24 = 0 \text{ or } (5\lambda + 12)(\lambda - 2) = 0$

So $\quad \lambda = 2, -\frac{12}{5}$

If $\quad \lambda = 2, \; 4^{\sin x} = 2, \; \therefore \; 2^{2\sin x} = 2^1.$

$$\therefore \quad \sin x = \frac{1}{2}$$

If $\lambda = -\frac{12}{5}$ then $4^{\sin x} = -\frac{12}{5}$, which is impossible as $4^{\sin x} > 0$

when $\lambda = 2$, we get $\mu = 11 - 2 = 9$

$$\Rightarrow \quad \frac{1}{3^{\cos y}} = 9 = 3^2$$

$$\Rightarrow \quad y = 2m\pi + \frac{\pi}{3}, m \in z \; \Rightarrow \; \cos y = \frac{1}{2}$$

Thus, $\quad x = n\pi + (-1)^n \frac{\pi}{6}$ and $y = 2m\pi \pm \frac{\pi}{3}$, where $m, n \in Z$.

125. The given equation is,

$$\sin y = 2\left(\sin^4 x + \frac{1}{\sin^4 x} + 2\right) + 2\left(\cos^4 x + \frac{1}{\cos^4 x} + 2\right) - 24$$

$$= 2(\sin^4 x + \cos^4 x) + 2\left(\frac{1}{\sin^4 x} + \frac{1}{\cos^4 x}\right) + 8 - 24$$

$$= 2(\sin^4 x + \cos^4 x)\left\{1 + \frac{1}{\sin^4 x \,.\, \cos^4 x}\right\} - 16$$

$$\Rightarrow \quad \sin y = 2\left(1 - \frac{1}{2}\sin^2 2x\right)\left(1 + \frac{16}{\sin^4 2x}\right) - 16$$

$$\Rightarrow \quad 16 + \sin y = (2 - \sin^2 2x)\left(1 + \frac{16}{\sin^4 2x}\right) \qquad \ldots(i)$$

Since, $\sin y \leq 1$

$$\Rightarrow \quad 16 + \sin y \leq 17$$

$\therefore$ L.H.S. is not greater than 17, on the other hand $0 \leq \sin^2 2x \leq 1$

$$\Rightarrow \quad 2 - \sin^2 2x \geq 1 \text{ and } \sin^4 2x \leq 1$$

$$\Rightarrow \quad \frac{16}{\sin^4 2x} \geq 16$$

$$\therefore \quad 1 + \frac{16}{\sin^4 2x} \geq 17$$

$$\Rightarrow \quad (2 - \sin^2 2x)\left(1 + \frac{16}{\sin^4 2x}\right) \geq 17$$

This shows that right member of equation (i) is not less than 17.
Thus the inequality holds only when

$$\sin y = 1 \text{ and } \sin^2 2x = 1$$

$$\Rightarrow \quad y = (4n + 1)\frac{\pi}{2}, n \in I$$

from second equation $\sin^2 2x = 1 = \sin^2\frac{\pi}{2}$

$$\Rightarrow \quad 2x = m\pi \pm \frac{\pi}{2}, m \in I$$

$$x = (2m + 1)\frac{\pi}{4}$$

Thus, the solution are $\left((2m \pm 1)\frac{\pi}{y}, (4n + 1)\frac{\pi}{2}\right)$

126. We have,

$$4\left[3\sqrt{4x - x^2}\left(\frac{1 - \cos(x + y)}{2}\right) + 2\cos(x + y)\right] = 13 + 4\cos^2(x + y)$$

$$\Rightarrow \; 6\sqrt{4x - x^2} - 6\sqrt{4x - x^2}\cos(x + y) + 8\cos(x + y)$$

$$= 13 + 4\cos^2(x+y)$$

$$\Rightarrow 4\cos^2(x+y) + (6\sqrt{4x-x^2} - 8)\cos(x+y) + (13 - 6\sqrt{4x-x^2}) = 0 \quad ...(i)$$

Let, $\cos(x+y) = t$ and $6\sqrt{4x-x^2} - 8 = a \Rightarrow 6\sqrt{4x-x^2} = 8 + a$

Clearly, $4x - x^2 \geq 0$ or $x(x-4) \leq 0$ or $0 \leq x \leq 4$ ...(ii)

Now, equation (i) reduces to;

$$4t^2 + at + (5-a) = 0$$

which is quadratic in t.

$$\therefore \quad D \geq 0$$

$$a^2 - 4.4(5-a) \geq 0$$

$$a^2 + 16a - 80 \geq 0$$

$$(a+20)(a-4) \geq 0$$

$$\Rightarrow \quad a \leq -20 \text{ or } a \geq 4 \quad ...(iii)$$

However, according to substitution;

$$a = 6\sqrt{4x-x^2} - 8$$

$$= 6\sqrt{4-(x-2)^2} - 8$$

$$-8 \leq a \leq 4 \quad ...(iv)$$

$$\Rightarrow \quad a = 4 \quad \{\text{from (iii) and (iv)}\}$$

Hence, $6\sqrt{4x-x^2} - 8 = 4$

$$\sqrt{4x-x^2} = 2$$

$$4x - x^2 = 4$$

$$x^2 - 4x + 4 = 0$$

$$\Rightarrow \quad (x-2)^2 = 0 \text{ or } x = 2$$

Now, equation (i) becomes;

$$\Rightarrow \quad 4\cos^2(2+y) + 4\cos(2+y) + 1 = 0$$

$$\Rightarrow \quad (2\cos(2+y) + 1)^2 = 0$$

$$\Rightarrow \quad 2\cos(2+y) + 1 = 0$$

$$\Rightarrow \quad \cos(2+y) = -\frac{1}{2} = \cos\left(\pi - \frac{\pi}{3}\right)$$

$$\Rightarrow \quad 2 + y = 2n\pi \pm \frac{2\pi}{3}$$

$$\Rightarrow \quad y = 2n\pi \pm \frac{2\pi}{3} - 2, n \in I$$

Thus, the solutions are $\left(2, 2n\pi \pm \frac{2\pi}{3} - 2\right)$

127. Rewriting the given equation as,

$\{\tan(x+y) - \cot(x+y)\}^2 = -(1 + 2x + x^2) = -(1+x)^2$

or $\{\tan(x+y) - \cot(x+y)\}^2 + (1+x)^2 = 0$

which is possible only when,

$\tan(x+y) - \cot(x+y) = 0$ and $1 + x = 0$

$$\tan^2(x+y) = 1 = \tan^2\frac{\pi}{4} \text{ and } x = -1$$

Now, $\tan^2(x+y) = \tan^2\frac{\pi}{4} \Rightarrow x + y = n\pi \pm \frac{\pi}{4}$

$$\Rightarrow \quad y = n\pi \pm \frac{\pi}{4} + 1, n \in I$$

$\therefore$ The required solutions are $\left(-1, n\pi \pm \frac{\pi}{4} + 1\right), n \in I.$

128. As we know, A.M. $\geq$ G.M.

$$\therefore \quad \frac{\tan^2 x + \cot^2 x}{2} \geq \sqrt{\tan^2 x \,.\, \cot^2 x}$$

$$\Rightarrow \quad \tan^2 x + \cot^2 x \geq 2$$

Now, from the first equation,

$$2\cos^2 y \geq 2$$

$\cos^2 y \geq 1$, which is only possible when,

$$\cos^2 y = 1$$

Putting $\cos^2 y = 1$ in second equation we get,

$$\sin^2 z = 0 \Rightarrow z = n\pi \quad ...(ii)$$

Similarly $\cos^2 y = 1 \Rightarrow y = m\pi$ ...(iii)

Alos, $\tan^2 x + \cot^2 x = 2\cos^2 y$

$$\Rightarrow \quad \tan^2 x + \cot^2 x = 2 \Rightarrow \tan^2 x = \cos^2 x = 1$$

$$\Rightarrow \quad x = k\pi \pm \frac{\pi}{4} \quad ...(iv)$$

Hence, the solutions are

$$x = k\pi \pm \frac{\pi}{4}, y = m\pi, z = n\pi \text{ where } k, m, n \in Z$$

129. The given equation can be rewrite as

$$2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) + 2\cos^2\left(\frac{x+y}{2}\right) - 1 = -\frac{3}{2}$$

or $4\cos^2\left(\frac{x+y}{2}\right) + 4\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) + 1 = 0$

or $\left(2\cos\left(\frac{x+y}{2}\right) + \cos\left(\frac{x-y}{2}\right)\right)^2 + \sin^2\left(\frac{x-y}{2}\right) = 0$

$$\Rightarrow \quad 2\cos\left(\frac{x+y}{2}\right) = -\cos\left(\frac{x-y}{2}\right) \text{ and } \sin^2\left(\frac{x-y}{2}\right) = 0$$

from second equation, we get,

$$x - y = 2n\pi, \text{ where } n \in I$$

or $y = x - 2n\pi$

Substituting the values of y in the first equation, we get

$$2\cos(x - n\pi) = -\cos n\pi$$

$$\Rightarrow \quad 2\cos x \,.\, \cos n\pi = -\cos n\pi$$

$$\Rightarrow \quad \cos x = -\frac{1}{2} = \cos\left(\pi - \frac{\pi}{3}\right) \Rightarrow x = 2m\pi \pm \frac{2\pi}{3}, m \in I$$

$\therefore$ Solutions are,

$$\left\{\left(2m\pi \pm \frac{2\pi}{3}\right), \left(2(m-n)\pi \pm \frac{2\pi}{3}\right)\right\}$$

130. Here $\cot\frac{\theta}{2} - \cot\theta = \text{cosec}\,\frac{\theta}{2}$

$$\Rightarrow \quad \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} - \frac{\cos\theta}{\sin\theta} = \text{cosec}\left(\frac{\theta}{2}\right)$$

$$\Rightarrow \quad \frac{2\cos^2\frac{\theta}{2} - \cos\theta}{\sin\theta} = \text{cosec}\,\frac{\theta}{2}$$

$$\Rightarrow \quad 2\cos^2\frac{\theta}{2} - \left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right) = \sin\theta \,.\, \text{cosec}\,\frac{\theta}{2}$$

$$\Rightarrow \quad 1 = \text{cosec}\,\frac{\theta}{2} \,.\, \sin\theta$$

$\Rightarrow \quad \sin\theta = \sin\frac{\theta}{2} \Rightarrow \sin\frac{\theta}{2}\left(2\cos\frac{\theta}{2} - 1\right) = 0$

$\Rightarrow \quad \sin\frac{\theta}{2} = 0 \text{ or } \cos\frac{\theta}{2} = \frac{1}{2}$

$\Rightarrow \quad \theta = 2n\pi \text{ or } \theta = 2\left(2n\pi \pm \frac{\pi}{3}\right)$

for $\theta = 2n\pi$, $\cot\frac{\theta}{2}$ and $\cot\theta$ are undefined,

Hence do not satisfy the given equation.

$\therefore$ The only solution is $\theta = 4n\pi \pm \frac{2\pi}{3}$

131. We know, $a + b + c = 0, \Leftrightarrow a^3 + b^3 + c^3 = 3abc$

Put $a = \sin x, b = \cos x, c = 1$

$$1 + \sin^3 x + \cos^3 x = 3\sin x \,.\, \cos x = \frac{3}{2}\sin 2x$$

$\Rightarrow \; a + b + c = 0 \Rightarrow 1 + \sin x + \cos x = 0$

$\Rightarrow \quad \sqrt{2}\cos\left(x - \frac{\pi}{4}\right) = -1$

$\Rightarrow \quad \cos\left(x - \frac{\pi}{4}\right) = \cos\frac{3\pi}{4}$

$\Rightarrow \quad x = 2n\pi \pm \frac{3\pi}{4} + \frac{\pi}{4}$

132. We have, $\log_{\sin x} 2 \,.\, \log_{\sin^2 x} a = -1$

$\Rightarrow \quad \log_{\sin x} 2 \,.\, \log_{\sin x} a = -2 \qquad \begin{pmatrix}\sin x > 0\\ a > 0\end{pmatrix} \ldots(i)$

$\Rightarrow \quad \log_{\sin x} 2 \,.\, \frac{\log_2 a}{\log_2 \sin x} = -2$

$\Rightarrow \quad \log_2 a = -2(\log_2 \sin x)^2$

$\Rightarrow \quad \log_2 \sin x = \pm\sqrt{\frac{(-\log_2 a)}{2}}$

$\left\{\text{as } \sin x < 1 \Rightarrow \log_2 \sin x < 0, \text{ so, reject } \log_2 \sin x = \sqrt{\frac{(-\log_2 a)}{2}}\right\}$

$\Rightarrow \quad \log_2 \sin x = -\sqrt{\frac{(-\log_2 a)}{2}}$

$\Rightarrow \quad \sin x = 2^{-\sqrt{(-\log_2 a)/2}}$

$\Rightarrow x = n\pi + (-1)^n \sin^{-1}\{2^{-\sqrt{(-\log_2 a)/2}}\}$ and the condition is $0 < a < 1$ so that $\log_2 a$ is defined and $\log_2 a < 0$.

133. We have, $\int_0^x (t^2 - 8t + 13)dt = x\sin\frac{a}{x}$

$\Rightarrow \quad \left(\frac{t^3}{3} - \frac{8t^2}{2} + 13t\right)_0^x = x\sin\frac{a}{x}$

$\Rightarrow \quad \frac{x^3}{3} - 4x^2 + 13x = x\sin\frac{a}{x}$

$\Rightarrow \quad x^2 - 12x + 39 = 3\sin\frac{a}{x} \qquad \{\because x \neq 0\}$

$\Rightarrow \quad (x-6)^2 + 3 = 3\sin\left(\frac{a}{x}\right)$

min. L.H.S. = 3 and max. R.H.S. = 3

$\Rightarrow \quad (x-6)^2 + 3 = 3 \text{ and } 3\sin\left(\frac{a}{x}\right) = 3$

$\Rightarrow \quad x = 6 \text{ and } \sin\frac{a}{x} = 1$

$\Rightarrow \quad x = 6 \text{ and } \frac{a}{x} = (4n+1)\frac{\pi}{2} \Rightarrow a = 3\pi(4n+1)$

134. The given equation can be rewritten as

$(\sin^4 x + \cos^4 x)^2 - 2\sin^4 x.\cos^4 x = \frac{17}{16}(1 - \sin^2 2x)$

Put $t = \sin x \,.\, \cos x$,

$$(1 - 2t^2)^2 - 2t^4 = \frac{17}{16}(1 - 4t^2)$$

$\Rightarrow \quad 32t^4 + 4t^2 - 1 = 0$

$\Rightarrow \quad 32t^4 + 8t^2 - 4t^2 - 1 = 0$

$\Rightarrow \quad (4t^2 + 1)(8t^2 - 1) = 0$

$\Rightarrow \quad t = \pm\frac{1}{2\sqrt{2}} \qquad [\because 4t^2 + 1 \neq 0]$

$\Rightarrow \quad \sin x \,.\, \cos x = \pm\frac{1}{2\sqrt{2}} \Rightarrow \sin 2x = \pm\frac{1}{\sqrt{2}}$

$\Rightarrow \quad 2x = n\pi \pm (-1)^n\left(\frac{\pi}{4}\right) \Rightarrow x = \frac{n\pi}{2} \pm (-1)^n\frac{\pi}{8}$

135. We have, $\tan^4 x + \tan^4 y + 2\cot^2 x\cot^2 y$

$$= 3 + \sin^2(x+y)$$

$\Rightarrow \tan^4 x + \tan^4 y + 2\cot^2 x\cot^2 y - 2 = 1 + \sin^2(x+y)$

$\Rightarrow (\tan^2 x - \tan^2 y) + 2(\tan x\tan y - \cot x\cot y)^2 = -1 + \sin^2(x+y)$

Clearly, L.H.S ≥ 0 and R.H.S ≤ 0

$\therefore \quad$ L.H.S = R.H.S = 0

$\Rightarrow \tan^2 x = \tan^2 y, \tan x\tan y = \cot x\cot y \text{ and } \sin^2(x+y) = 1$

$\Rightarrow \tan^2 x = \tan^2 y, \tan^2 x\tan^2 y = 1 \text{ and } \sin^2(x+y) = 1$

$\Rightarrow \tan^2 x = 1 \text{ and } x + y = n\pi \pm \frac{\pi}{2}, n, m, \in z$

$\Rightarrow \quad x = m\pi \pm \frac{\pi}{4} \text{ and } y = (n-m)\pi \mp \frac{\pi}{4}, n, m, \in z$

$\Rightarrow$ Thus, $x = y = p\pi \pm \frac{\pi}{4}, p \in z$

136. $\cos^2 x = \frac{(a+2) \pm \sqrt{(a+2)^2 + 4(a+3)}}{2}$

$$\cos^2 x = \frac{(a+2) \pm (a+4)}{2}$$

either, $\cos^2 x = -1$ (not possible)

or $\cos^2 x = a + 3$

Since, $0 \leq \cos^2 x \leq 1$

$\therefore \quad 0 \leq a + 3 \leq 1 \Rightarrow -3 \leq a \leq -2$

Also, $\cos^2 x = (a+3)$

$\Rightarrow \; x = n\pi \pm \cos^{-1}\sqrt{a+3}$, where $n \in I$ and $a \in [-3, -2]$

137. Here $\left[2\sin\left(x + \frac{\pi}{6}\right)\right]^{\sqrt{2\sin^2 x + 2\sqrt{3}\sin x\,.\,\cos x + 1}} = 4$

$\Rightarrow \left[2\sin\left(x + \frac{\pi}{6}\right)\right]^{\sqrt{3\sin^2 x + \cos^2 x + 2\sqrt{3}\sin x\,.\,\cos x}} = 4$

$\Rightarrow \left[2\sin\left(x + \frac{\pi}{6}\right)\right]^{|\sqrt{3}\sin x + \cos x|} = 4$

$$\Rightarrow \quad \left[2\sin\left(x+\frac{\pi}{6}\right)\right]^{|2\sin(x+\pi/6)|} = 4$$

$$\Rightarrow \quad \left\{\sin\left(x+\frac{\pi}{6}\right)\right\}^{|\sin(x+\pi/6)|} = 1$$

To make the R.H.S. well defined it is necessary that,

$$\sin\left(x+\frac{\pi}{6}\right) > 0 \quad \Rightarrow \quad x \in \left(-\frac{\pi}{6}, \frac{5\pi}{6}\right)$$

Now, we have $\left\{\sin\left(x+\frac{\pi}{6}\right)\right\}^{\sin(x+\pi/6)} = 1$

$$\Rightarrow \quad \sin\left(x+\frac{\pi}{6}\right) = 1 \quad \Rightarrow \quad x = \frac{\pi}{3}$$

138. We have,

$$\sec\theta + \operatorname{cosec}\theta = c$$

$$\sin\theta + \cos\theta = c\sin\theta\,.\,\cos\theta$$

Squaring both sides, we get

$$1 + 2\sin\theta\cos\theta = c^2\sin^2\theta\,.\,\cos^2\theta$$

$$1 + \sin 2\theta = \frac{c^2}{4}\sin^2 2\theta$$

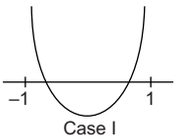
Put $\sin 2\theta = t$

$$c^2t^2 - 4t - 4 = 0,\ t \in [-1, 1]$$

Thus, the equation must have atleast one root $\in [-1, 1]$

Case I : $D \geq 0,\ f(-1) > 0,\ f(1) > 0$

$\Rightarrow\ 16 + 16c^2 \geq 0,\ c^2 + 4 - 4 > 0,\ c^2 - 4 - 4 > 0 \Rightarrow c^2 > 8$

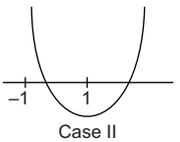


Case I

In this case we will get two distinct values of $\sin^z\theta$, resulting in 4 distinct values of $\theta \in (0, 2\pi)$ two of them would be repeated.

Case II : $f(1)f(-1) < 0$

$$\Rightarrow \quad (c^2 - 8)c^2 < 0 \Rightarrow c^2 < 8$$



Case II

In this case we will get one value of $\sin 2\theta$, resulting in 2 distinct values of $\theta \in (0, 2\pi)$.

139. $|\sin x + \cos x|^{\sin^2 x - 1/4} = 1 + |\sin y|$ ...(i)

and $\cos^2 y = 1 + \sin^2 y$ ...(ii)

$\Rightarrow\ 1 - \sin^2 y = 1 + \sin^2 y \Rightarrow \sin^2 y = 0,\ y = k\pi,\ k \in I$

$\Rightarrow\ |\sin x + \cos x|^{\sin^2 x - 1/4} = 1$

Then, either $\sin^2 x = \frac{1}{4}$ or $\sin x + \cos x = 1$

$$\Rightarrow \quad x = n\pi \pm \frac{\pi}{6} \text{ or } \cos\left(x - \frac{\pi}{4}\right) = \cos\frac{\pi}{4}$$

Now, $\cos\left(x - \frac{\pi}{4}\right) = \cos\frac{\pi}{4} \Rightarrow x - \frac{\pi}{4} = 2m\pi \pm \frac{\pi}{4}$

$$\Rightarrow \quad x = 2m\pi + \frac{\pi}{2} \text{ or } x = 2m\pi$$

$$\Rightarrow \quad x = 2m\pi \text{ or } 2m\pi + \frac{\pi}{2}$$

$$\Rightarrow \quad x = 2m\pi + \frac{\pi}{2}, 2m\pi,\ n\pi \pm \frac{\pi}{6} \text{ and } y = k\pi;\ m, n, k \in I$$

140. Given, $\sqrt{3}\sec x + \operatorname{cosec} x + 2(\tan x - \cot x) = 0$,

$$(-\pi < x < \pi) - \{0, \pm\pi/2\}$$

$$\Rightarrow \quad \sqrt{3}\sin x + \cos x + 2(\sin^2 x - \cos^2 x) = 0$$

$$\Rightarrow \quad \sqrt{3}\sin x + \cos x - 2\cos 2x = 0$$

Multiplying and dividing by $\sqrt{a^2 + b^2}$, i.e. $\sqrt{3+1} = 2$, we get

$$2\left(\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x\right) - 2\cos 2x = 0$$

$$\Rightarrow \quad \left(\cos x \cdot \cos\frac{\pi}{3} + \sin x \cdot \sin\frac{\pi}{3}\right) - \cos 2x = 0$$

$$\Rightarrow \quad \cos\left(x - \frac{\pi}{3}\right) = \cos 2x$$

$$\therefore \quad 2x = 2n\pi \pm \left(x - \frac{\pi}{3}\right)$$

$$[\because \cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha]$$

$$\Rightarrow \quad 2x = 2n\pi + x - \frac{\pi}{3} \text{ or } 2x = 2n\pi - x + \frac{\pi}{3}$$

$$\Rightarrow \quad x = 2n\pi - \frac{\pi}{3} \text{ or } 3x = 2n\pi + \frac{\pi}{3}$$

$$\Rightarrow \quad x = 2n\pi - \frac{\pi}{3} \text{ or } x = \frac{2n\pi}{3} + \frac{\pi}{9}$$

$$\therefore \quad x = \frac{-\pi}{3} \text{ or } x = \frac{\pi}{9}, \frac{-5\pi}{9}, \frac{7\pi}{9}$$

Now, sum of all distinct solutions $= \frac{-\pi}{3} + \frac{\pi}{9} - \frac{5\pi}{9} + \frac{7\pi}{9} = 0$

141. Here, $\frac{5}{4}\cos^2 2x + (\cos^4 x + \sin^4 x) + (\cos^6 x + \sin^6 x) = 2$

$$\Rightarrow \quad \frac{5}{4}\cot 2x + [(\cos^2 x + \sin^2 x)^2 - 2\sin^2 x\cos^2 x]$$

$$+ (\cos^2 x + \sin^2 x)[(\cos^2 x + \sin^2 x)^2 - 3\sin^2 x\cos^2 x] = 2$$

$$\Rightarrow \quad \frac{5}{4}\cos^2 2x + (1 - 2\sin^2 x\cos^2 x) + (1 - 3\cos^2 x\sin^2 x) = 2$$

$$\Rightarrow \quad \frac{5}{4}\cos^2 2x - 5\sin^2 x\cos^2 x = 0$$

$$\Rightarrow \quad \frac{5}{4}\cos^2 2x - \frac{5}{4}\sin^2 2x = 0$$

$$\Rightarrow \quad \frac{5}{4}\cos^2 2x - \frac{5}{4} + \frac{5}{4}\cos^2 2x = 0$$

$$\Rightarrow \quad \frac{5}{2}\cos^2 2x = \frac{5}{4}$$

$$\Rightarrow \quad \cos^2 2x = \frac{1}{2} \Rightarrow 2\cos^2 2x = 1$$

$$\Rightarrow \quad 1 + \cos 4x = 1$$

$$\Rightarrow \quad \cos 4x = 0, \text{ as } 0 \leq x \leq 2\pi$$

$$\therefore \quad 4x = \left\{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}, \frac{13\pi}{2}, \frac{15\pi}{2}\right\}$$

as $\quad 0 \le 4x \le 8\pi$

$$\therefore \quad x = \left\{\frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}\right\}$$

Hence, the total number of solutions are 8.

142. **Plan** For solving this type of questions, obtain the LHS and RHS in equation and examine, the two are equal or not for a given interval.

Given, trigonometrical equation

$$(\sin x - \sin 3x) + 2\sin 2x = 3$$

$$\Rightarrow \quad -2\cos 2x \sin x + 4\sin x \cos x = 3$$

$$\left[\because \sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right.$$
$$\left.\text{and } \sin 2\theta = 2\sin\theta\cos\theta\right]$$

$$\Rightarrow \quad 2\sin x(2\cos x - \cos 2x) = 3$$

$$\Rightarrow \quad 2\sin x(2\cos x - 2\cos^2 x + 1) = 3$$

$$\Rightarrow \quad 2\sin x\left[\frac{3}{2} - 2\left(\cos x - \frac{1}{2}\right)^2\right] = 3$$

$$\Rightarrow \quad 3\sin x - 3 = 4\left(\cos x - \frac{1}{2}\right)^2 \sin x$$

As $x \in (0, \pi)$ $\quad$ LHS ≤ 0 and RHS ≥ 0

For solution to exist, LHS = RHS = 0

Now, LHS = 0 $\Rightarrow$ $3\sin x - 3 = 0$

$$\Rightarrow \quad \sin x = 1 \Rightarrow x = \frac{\pi}{2}$$

For $\quad x = \dfrac{\pi}{2}$,

$$\text{RHS} = 4\left(\cos\frac{\pi}{2} - \frac{1}{2}\right)^2 \sin\frac{\pi}{2} = 4\left(\frac{1}{4}\right)(1) = 1 \neq 0$$

$\therefore$ No solution of the equation exists.

143. **Plan** It is based on range of $\sin x$,

i.e. $[-1,1]$ and the internal for $a < x < b$.

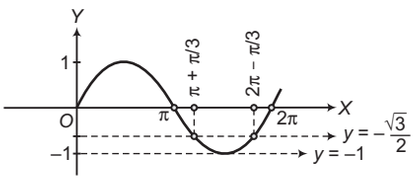
Description of Situation As $\theta, \phi \in [0, 2\pi]$ and

$$\tan(2\pi - \theta) > 0, \; -1 < \sin\theta < -\frac{\sqrt{3}}{2}$$

$$\tan(2\pi - \theta) > 0 \Rightarrow -\tan\theta > 0$$

$\therefore$ $\theta \in$ II or IV quadrant.

Also, $\quad -1 < \sin\theta < -\dfrac{\sqrt{3}}{2}$



$$\Rightarrow \quad \frac{4\pi}{3} < \theta < \frac{5\pi}{3} \text{ but } \theta \in \text{II or IV quadrant}$$

$$\Rightarrow \quad \frac{3\pi}{2} < \theta < \frac{5\pi}{3} \qquad \ldots(i)$$

Here, $\quad 2\cos\theta(1 - \sin\phi) = \sin^2\theta\left(\tan\dfrac{\theta}{2} + \cot\dfrac{\theta}{2}\right)\cos\phi - 1$

$$\Rightarrow \quad 2\cos\theta - 2\cos\theta\sin\phi = \sin^2\theta\left(\frac{\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}}{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)\cos\phi - 1$$

$$\Rightarrow \quad 2\cos\theta - 2\cos\theta\sin\phi = 2\sin^2\theta\left(\frac{1}{\sin\theta}\right)\cos\phi - 1$$

$$\Rightarrow \quad 2\cos\theta + 1 = 2\sin\phi\cos\theta + 2\sin\theta\cos\phi$$

$$\Rightarrow \quad 2\cos\theta + 1 = 2\sin(\theta + \phi) \qquad \ldots(ii)$$

From Eq. (i),

$$\frac{3\pi}{2} < \theta < \frac{5\pi}{3} \Rightarrow 2\cos\theta + 1 \in (1,2)$$

$$\therefore \quad 1 < 2\sin(\theta + \phi) < 2$$

$$\Rightarrow \quad \frac{1}{2} < \sin(\theta + \phi) < 1 \qquad \ldots(iii)$$

$$\Rightarrow \quad \frac{\pi}{6} < \theta + \phi < \frac{5\pi}{6} \quad \text{or} \quad \frac{13\pi}{6} < \theta + \phi < \frac{17\pi}{6}$$

$$\therefore \quad \frac{\pi}{6} - \theta < \phi < \frac{5\pi}{6} - \theta \quad \text{or} \quad \frac{13\pi}{6} - \theta < \phi < \left(\frac{17\pi}{6}\right) - \theta$$

$$\Rightarrow \quad \phi \in \left(-\frac{3\pi}{2}, -\frac{2\pi}{3}\right) \text{ or } \left(\frac{2\pi}{3}, \frac{7\pi}{6}\right), \text{ as } \theta \in \left(\frac{3\pi}{2}, \frac{5\pi}{3}\right)$$

144. $P = \{\theta : \sin\theta - \cos\theta = \sqrt{2}\cos\theta\}$

$$\Rightarrow \quad \cos\theta(\sqrt{2} + 1) = \sin\theta$$

$$\Rightarrow \quad \tan\theta = \sqrt{2} + 1$$

$$\Rightarrow \quad Q = \{\theta : \sin\theta + \cos\theta\} = \sqrt{2}\sin\theta$$

$$\Rightarrow \quad \sin\theta(\sqrt{2} - 1) = \cos\theta$$

$$\Rightarrow \quad \tan\theta = \frac{1}{\sqrt{2} - 1} \times \frac{\sqrt{2} + 1}{\sqrt{2} + 1} = (\sqrt{2} + 1)$$

$$\therefore \quad P = Q$$

145. Given, $n > 3 \in$ Integer

$$\text{and} \quad \frac{1}{\sin\left(\frac{\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)} + \frac{1}{\sin\left(\frac{3\pi}{n}\right)}$$

$$\Rightarrow \quad \frac{1}{\sin\frac{\pi}{n}} - \frac{1}{\sin\frac{3\pi}{n}} = \frac{1}{\sin\frac{2\pi}{n}}$$

$$\Rightarrow \quad \frac{\sin\frac{3\pi}{n} - \sin\frac{\pi}{n}}{\sin\frac{\pi}{n} \cdot \sin\frac{3\pi}{n}} = \frac{1}{\sin\frac{2\pi}{n}}$$

$$\Rightarrow \quad 2\cos\left(\frac{2\pi}{n}\right) \cdot \sin\frac{\pi}{n} = \frac{\sin\frac{\pi}{n} \cdot \sin\frac{3\pi}{n}}{\sin\frac{2\pi}{n}}$$

$$\Rightarrow \quad 2\sin\frac{2\pi}{n} \cdot \cos\frac{2\pi}{n} = \sin\frac{3\pi}{n}$$

$$\Rightarrow \quad \sin\frac{4\pi}{n} = \sin\frac{3\pi}{n} \Rightarrow \frac{4\pi}{n} = \pi - \frac{3\pi}{n}$$

$$\Rightarrow \quad \frac{7\pi}{n} = \pi \Rightarrow n = 7$$

146. Given, $\tan\theta = \cot 5\theta$

$\Rightarrow \quad \tan\theta = \tan\left(\frac{\pi}{2} - 5\theta\right) \Rightarrow \frac{\pi}{2} - 5\theta = n\pi + \theta$

$\Rightarrow \quad 6\theta = \frac{\pi}{2} - n\pi \Rightarrow \theta = \frac{\pi}{12} - \frac{n\pi}{6}$

Also, $\cos 4\theta = \sin 2\theta = \cos\left(\frac{\pi}{2} - 2\theta\right)$

$\Rightarrow \quad 4\theta = 2n\pi \pm \left(\frac{\pi}{2} - 2\theta\right)$

Taking positive sign,

$$6\theta = 2n\pi + \frac{\pi}{2} \Rightarrow \theta = \frac{n\pi}{3} + \frac{\pi}{12}$$

Taking negative sign,

$$2\theta = 2n\pi - \frac{\pi}{2} \Rightarrow \theta = n\pi - \frac{\pi}{4}$$

Above values of θ suggest that there are only 3 common solutions.

147. $2\sin^2\theta - \cos 2\theta = 0 \Rightarrow \sin^2\theta = \frac{1}{4}$

Also, $2\cos^2\theta = 3\sin\theta \Rightarrow \sin\theta = \frac{1}{2}$

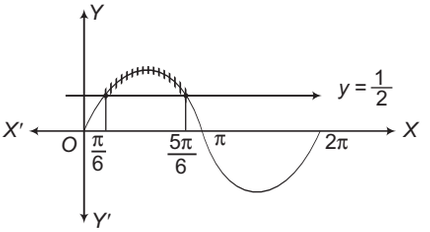
$\Rightarrow$ Two solutions exist in the interval $[0, 2\pi]$.

148. Since, $2\sin^2\theta - 5\sin\theta + 2 > 0$

$\Rightarrow \quad (2\sin\theta - 1)(\sin\theta - 2) > 0$

[where, $(\sin\theta - 2) < 0, \forall\, \theta \in R$]

$\therefore \quad (2\sin\theta - 1) < 0$



$\Rightarrow \quad \sin\theta < \frac{1}{2}$

$\therefore$ From the graph, $\theta \in \left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 2\pi\right)$

149. Given equation is

$\cos x + \cos 2x + \cos 3x + \cos 4x = 0$

$\Rightarrow \quad (\cos x + \cos 3x) + (\cos 2x + \cos 4x) = 0$

$\Rightarrow \quad 2\cos 2x \cos x + 2\cos 3x \cos x = 0$

$\Rightarrow \quad 2\cos x(\cos 2x + \cos 3x) = 0$

$\Rightarrow \quad 2\cos x\left(2\cos\frac{5x}{2}\cos\frac{x}{2}\right) = 0$

$\Rightarrow \quad \cos x \cdot \cos\frac{5x}{2} \cdot \cos\frac{x}{2} = 0$

$\Rightarrow \quad \cos x = 0$ or $\cos\frac{5x}{2} = 0$ or $\cos\frac{x}{2} = 0$

Now, $\cos x = 0$

$\Rightarrow \quad x = \frac{\pi}{2}, \frac{3\pi}{2} \quad [\because\ 0 \le x < 2\pi]$

$\cos\frac{5x}{2} = 0$

$\Rightarrow \quad \frac{5x}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}, \ldots,$

$\Rightarrow \quad x = \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5} \quad [\because\ 0 \le x < 2\pi]$

and $\cos\frac{x}{2} = 0 \Rightarrow \frac{x}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

$\Rightarrow \quad x = \pi \quad [\because\ 0 \le x < 2\pi]$

Hence, $x = \frac{\pi}{2}, \frac{3\pi}{2}, \pi, \frac{\pi}{5}, \frac{3\pi}{5}, \frac{7\pi}{5}, \frac{9\pi}{5}$

150. $\sin\theta + \sin 4\theta + \sin 7\theta = 0$

$\Rightarrow \quad \sin 4\theta + (\sin\theta + \sin 7\theta) = 0$

$\Rightarrow \quad \sin 4\theta + 2\sin 4\theta \cdot \cos 3\theta = 0$

$\Rightarrow \quad \sin 4\theta\{1 + 2\cos 3\theta\} = 0 \Rightarrow \sin 4\theta = 0, \cos 3\theta = -\frac{1}{2}$

As, $0 < \theta < \pi$

$\therefore \quad 0 < 4\theta < 4\pi$

$\therefore \quad 4\theta = \pi, 2\pi, 3\pi$

$\cos 3\theta = -\frac{1}{2}$

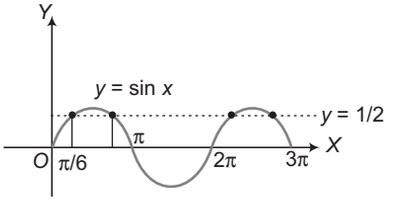
$0 < 3\theta < 3\pi \Rightarrow 3\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}$

$\Rightarrow \quad \theta = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}$

151. Given equation is $2\sin^2 x + 5\sin x - 3 = 0$.

$\Rightarrow \quad (2\sin x - 1)(\sin x + 3) = 0$

$\Rightarrow \quad \sin x = \frac{1}{2} \quad [\because \sin x \ne -3]$



It is clear from figure that the curve intersect the line at four points in the given interval.

Hence, number of solutions are 4.

CHAPTER

03

Properties and Solutions of Triangles

Learning Part

Session 1

- Basic Relation between the Sides and Angles of Triangle

Session 2

- Auxiliary Formulae

Session 3

- Circles Connected with Triangle

Session 4

- Orthocentre and its Distance from the Angular Points of a Triangle and Pedal Triangle
- Centroid of Triangle

Session 5

- Regular Polygons and Radii of the Inscribed and Circumscribing Circle a Regular Polygon

Session 6

- Quadrilaterals and Cyclic Quadrilaterals

Session 7

- Solution of Triangles

Session 8

- Height and Distance

Practice Part

- JEE Type Examples
- Chapter Exercises

Arihant on Your Mobile !

Exercises with the symbol can be practised on your mobile. See inside cover page to activate for free.

Session 1

Basic Relation between the Sides and Angles of Triangle

Basic Relation between the Sides and Angles of Triangle

In a ΔABC, the angles are denoted by capital letters A, B and C and the lengths of the sides opposite to these angles are denoted by small letters a, b and c, respectively.

Semi-perimeter of the triangle is written as :

$$s = \frac{a+b+c}{2},$$

and its area is denoted by Δ.

Some geometrical properties of A,B,C and a,b,c. In ΔABC

(i) $A + B + C = 180°$

(ii) $a + b > c, b + c > a, c + a > b$

(iii) $a > 0, b > 0, c > 0$

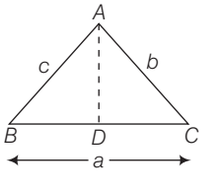
Sine Formula or Sine Rule

In any ΔABC, the sides are proportional to sines of the opposite angles,

i.e. $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Proof : *Case* I. When $\angle C$ is acute :

From A draw $AD \perp BC$



In ΔABD,

$$\sin B = \frac{AD}{AB} = \frac{AD}{c}$$

or $$AD = c \sin B \quad \ldots(i)$$

In ΔACD, $$\sin C = \frac{AD}{AC} = \frac{AD}{b}$$

or $$AD = b \sin C \quad \ldots(ii)$$

From Eqs. (i) and (ii), we get $c \sin B = b \sin C$ ...(1)

***Case* II. When $\angle C$ is obtuse :**

From A draw $AD \perp BC$

In ΔABD,

$$\sin B = \frac{AD}{AB} = \frac{AD}{c}$$

$\therefore$ $$AD = c \sin B \quad \ldots(iii)$$

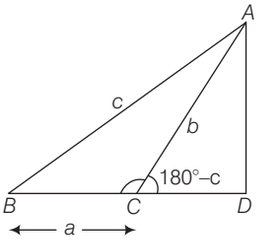
From ΔACD,

$$\sin(180° - C) = \frac{AD}{AC} = \frac{AD}{b}$$

$\therefore AD = b \sin(180° - C) = b \sin C$ ...(iv)

From Eqs. (iii) and (iv), we get

$$c \sin B = b \sin C \quad \ldots(2)$$



***Case* III. When $\angle C = 90°$**

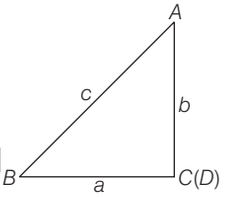
Draw $AD \perp BC$

In ΔABC, $\sin B = \frac{AD}{AB} = \frac{AD}{c}$

$\therefore$ $AD = c \sin B$

or $AC = c \sin B$

[$\because D$ and C are same point]



or $b = c \sin B$

or $$b \sin C = c \sin B \quad [\because C = 90°] \quad \ldots(3)$$

Thus from (1), (2) and (3), it follows that in all cases

$$b \sin C = c \sin B \text{ or } \frac{b}{\sin B} = \frac{c}{\sin C} \quad \ldots(4)$$

Similarly by drawing perpendicular from from C to AB, we can prove that

$$\frac{a}{\sin A} = \frac{b}{\sin C} \quad \ldots(5)$$

From (4) and (5), we get

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Cosine Formula or Cosine Rule

(i) $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ or $a^2 = b^2 + c^2 - 2ba \cos A$

(ii) $\cos B = \frac{c^2 + a^2 - b^2}{2ac}$

or $b^2 = c^2 + a^2 - 2ac \cos B$

(iii) $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

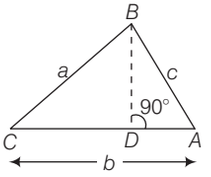
or $c^2 = a^2 + b^2 - 2ab \cos C$

Proof :

***Case* I. When $\angle A$ is acute**

Draw $BD \perp AC$

In ΔADB, $\sin A = \frac{BD}{AB} = \frac{BD}{c}$



$\therefore \quad BD = c \sin A$ ...(i)

and $\cos A = \frac{AD}{AB} = \frac{AD}{c}$

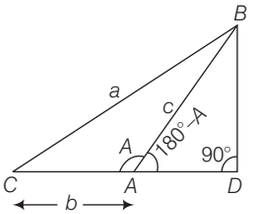
$\therefore \quad AD = c \cos A$ ...(ii)

Now $CD = AC - AD$

$= b - c \cos A$ ...(iii)

***Case* II. When $\angle A$ is obtuse**

In ΔADB,



$\sin(180° - A) = \frac{BD}{AB}$

$\therefore \quad \sin A + \frac{BD}{c}$

or $BD = c \sin A$ and $\cos(180° - A) = \frac{AD}{AB}$

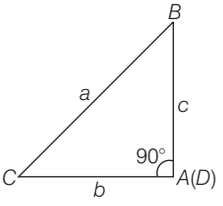
$\therefore \quad -\cos A = \frac{AD}{c}$ or $AD = -c \cos A$

Now, $CD = AC + AD = b - c \cos A$

***Case* III. When $\angle A = 90°$**

Here D and A are same points

In ΔACB, $BD = BA = c = c \sin A$



$[\because \angle A - 90° \therefore \sin A = \sin 90° = 1]$ ...(i)

and $AD = 0 = c \cos A \quad [\because \cos A = \cos 90° = 0]$ ...(ii)

$\therefore \quad CD = AC - AD = b - c \cos A$ ...(iii)

Thus in all cases $BD = c \sin A$ and $CD = b - c \cos A$

Now, in ΔBCD, $BC^2 = CD^2 + BD^2$

$\therefore \quad a^2 = (b - c \cos A)^2 + c^2 \sin^2 A$

$= b^2 + c^2 \cos^2 A - 2bc \cos A + c^2 \sin^2 A$

$= b^2 + c^2(\cos^2 A + \sin^2 A) - 2bc \cos A$

$= b^2 + c^2 - 2bc \cos A$ ...(1)

or $2bc \cos A = b^2 + c^2 - a^2$

$\therefore \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$

Also, from (1), $a^2 = b^2 + c^2 - 2bc \cos A$

Similarly, we can prove that $\cos B = \frac{c^2 + a^2 - b^2}{2ca}$

or $b^2 = a^2 + c^2 - 2ca \cos B$

and $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

or $c^2 = a^2 + b^2 - 2ab \cos C$

Projection Formulae

(i) $c = a \cos B + b \cos A$

(ii) $b = a \cos C + c \cos A$

(iii) $a = b \cos C + c \cos B$

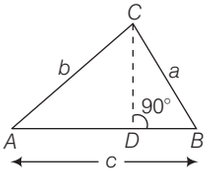
Proof :

***Case* I. When $\angle B$ is acute**

From ΔCBD,

$\cos B = \frac{BD}{BC} = \frac{BD}{a}$

$\therefore \quad BD = a \cos B$ ...(i)



In ΔADC,

$\cos A = \frac{AD}{AC} = \frac{AD}{b}$

$\therefore \quad AD = b \cos A$ ...(ii)

Now, $c = AB = AD + DB = b \cos A + a \cos B$

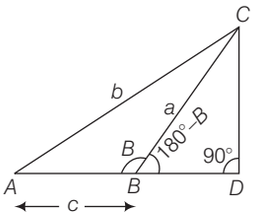
Thus, $c = b \cos A + a \cos B$

***Case* II. When $\angle B$ is obtuse**

From ΔCBD,

$\cos(180° - B) = \frac{BD}{BC}$

$$\therefore \quad -\cos B = \frac{BD}{a}$$



or $BD = -a\cos B$

In ΔCAD

$$\cos A = \frac{AD}{AC} = \frac{AD}{b}$$

or $AD = b\cos A$

Now, $c = AB = AD - BD = b\cos A + a\cos B$ ...(ii)

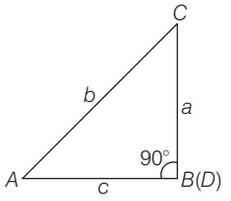
Case III. When $\angle B = 90°$

In ΔCAB,

$$\cos A = \frac{AB}{AC} = \frac{c}{b}$$

$$\therefore \quad c = b\cos A$$

$$= b\cos A + c\cos B$$



$[\because c\cos B = c\cos 90° = 0, \text{ as } \cos 90° = 0]$

Thus in all cases, $c = b\cos A + a\cos B$

Similarly, we can prove that

$$b = c\cos A + a\cos C$$

and $$a = b\cos C + c\cos B$$

Tangent Rule or Napier's Analogy

In any ΔABC, $\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\frac{C}{2}$

Proof: In ΔABC, we know

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k \text{ (say)} \qquad \text{[sine law]}$$

$$\therefore \quad a = k\sin A, b = k\sin B, c = k\sin C \qquad \text{...(i)}$$

Now, RHS

$$= \left(\frac{a-b}{a+b}\right)\cdot\cot\frac{C}{2}$$

$$= \left(\frac{k\sin A - k\sin B}{k\sin A + k\sin B}\right)\cdot\cot\frac{C}{2} \qquad \text{[using Eq(i)]}$$

$$= \frac{2\cos\left(\frac{A+B}{2}\right)\cdot\sin\left(\frac{A-B}{2}\right)}{2\sin\left(\frac{A+B}{2}\right)\cdot\cos\left(\frac{A-B}{2}\right)}\cdot\frac{\cos\frac{C}{2}}{\sin\frac{C}{2}}$$

$$= \frac{\sin\left(\frac{A-B}{2}\right)}{\cos\left(\frac{A-B}{2}\right)} \qquad \left[\begin{array}{l}\text{as, } \cos\left(\frac{A+B}{2}\right) = \sin\left(\frac{C}{2}\right)\\ \text{and } \sin\left(\frac{A+B}{2}\right) = \cos\left(\frac{C}{2}\right)\end{array}\right]$$

$$= \tan\left(\frac{A-B}{2}\right) = \text{LHS}$$

$$\Rightarrow \quad \tan\left(\frac{A-B}{2}\right) = \left(\frac{a-b}{a+b}\right)\cot\frac{C}{2}$$

Similarly it could be shown,

$$\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\frac{A}{2}, \tan\frac{C-A}{2} = \frac{c-a}{c+a}\cot\frac{B}{2}$$

Example 1. ***Find the angles of the triangles whose sides are*** $3+\sqrt{3}, 2\sqrt{3}$ ***and*** $\sqrt{6}$.

Sol. Let $a = 3+\sqrt{3}, b = 2\sqrt{3}, c = \sqrt{6}$

$$\Rightarrow \quad \cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{12+6-(9+3+6\sqrt{3})}{12\sqrt{2}}$$

$$= \frac{6-6\sqrt{3}}{12\sqrt{2}} = \frac{1-\sqrt{3}}{2\sqrt{2}}$$

$$\cos A = \cos(60° + 45°) \qquad \left[\text{as } \cos(60°+45°) = \frac{1-\sqrt{3}}{2\sqrt{2}}\right]$$

$$\therefore \quad A = 105°$$

Applying Sine formula $\frac{a}{\sin A} = \frac{b}{\sin B}$

$$\Rightarrow \quad \sin B = \frac{b}{a}\sin A = \frac{2\sqrt{3}}{3+\sqrt{3}}\sin(105°)$$

$$= \frac{2\sqrt{3}}{3+\sqrt{3}}\{\sin 60°.\cos 45° + \cos 60°\cdot\sin 45°\}$$

$$= \frac{2\sqrt{3}}{\sqrt{3}(\sqrt{3}+1)}\left\{\frac{\sqrt{3}+1}{2\sqrt{2}}\right\}$$

$$\sin B = \frac{1}{\sqrt{2}} = \sin 45° \quad [\because B \neq 180 - 45° \text{ as } B + A < 180°]$$

$$\Rightarrow \quad B = 45°$$

Here, $A = 105°, B = 45°$

$$\Rightarrow \quad C = 180° - (A + B) = 180° - (150°) = 30°$$

$$\therefore \quad \angle A = 105°, \angle B = 45° \text{ and } \angle C = 30°$$

Example 2. The sides of a triangle are 8 cm, 10 cm and 12 cm. Prove that the greatest angle is double of the smallest angle.

Sol. Let $a = 8$ cm, $b = 10$ cm and $c = 12$ cm. Hence, greatest angle is C and the smallest angle is A, {as we know greatest angle is opposite to greatest side and smallest angle is opposite to smallest side.} Here, we have to prove $C = 2A$, applying cosine law, we get

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{64 + 100 - 144}{2 \cdot 8 \cdot 10} = \frac{1}{8} \quad \ldots(i)$$

and $$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{100 + 144 - 64}{2 \cdot 10 \cdot 12} = \frac{3}{4} \quad \ldots(ii)$$

$$\cos 2A = 2\cos^2 A - 1 = 2 \cdot \frac{9}{16} - 1 \quad [\text{using Eq. (ii)}]$$

$$\therefore \quad \cos 2A = \frac{1}{8} \quad \ldots(iii)$$

From Eqs. (i) and (iii), we get

$$\cos 2A = \cos C$$

$$\Rightarrow \quad C = 2A.$$

Example 3. ***With usual notations, if in a*** ΔABC, $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$, ***then prove that***

$$\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}.$$

Sol. Let, $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13} = k$

$$\Rightarrow \quad 2(a + b + c) = 36k \quad \ldots(i)$$

$$b + c = 11k,\ c + a = 12k,\ a + b = 13k \quad \ldots(ii)$$

On solving Eqs. (i) and (ii), we get

$$a = 7k,\ b = 6k,\ c = 5k$$

Hence, $$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{36k^2 + 25k^2 - 49k^2}{60k^2}$$

$$= \frac{12}{60} = \frac{1}{5} = \frac{7}{35}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{49k^2 + 25k^2 - 36k^2}{70k^2}$$

$$= \frac{38}{70} = \frac{19}{35}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{49k^2 + 36k^2 - 25k^2}{84k^2}$$

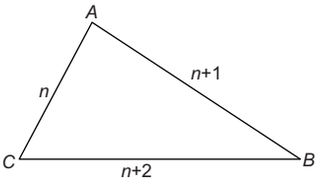
$$= \frac{60}{84} = \frac{5}{7}$$

$$= \frac{25}{35}$$

$$\Rightarrow \quad \frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$$

Example 4. The sides of a triangle are three consecutive natural numbers and its largest angle is twice the smallest one. Determine the sides of the triangle.

Sol. Let the sides be $n, n + 1, n + 2$



i.e. $AC = n,\ AB = n + 1,\ BC = n + 2$

Smallest angle is B and largest one is A.

Here, $A = 2B$

Also, $A + B + C = 180°$

$$\Rightarrow \quad 3B + C = 180° \Rightarrow C = 180° - 3B$$

Using, sine rule, $\frac{\sin A}{n+2} = \frac{\sin B}{n} = \frac{\sin C}{n+1}$

$$\Rightarrow \quad \frac{\sin 2B}{n+2} = \frac{\sin B}{n} = \frac{\sin(180° - 3B)}{n+1}$$

$$\Rightarrow \quad \underset{\text{(i)}}{\frac{\sin 2B}{n+2}} = \underset{\text{(ii)}}{\frac{\sin B}{n}} = \underset{\text{(iii)}}{\frac{\sin 3B}{n+1}}$$

From Eqs. (i) and (ii); $\frac{2\sin B\cos B}{n+2} = \frac{\sin B}{n}$

$$\cos B = \frac{n+2}{2n} \quad \ldots(iv)$$

and from Eqs. (ii) and (iii);

$$\frac{\sin B}{n} = \frac{3\sin B - 4\sin^3 B}{n+1}$$

$$\Rightarrow \quad \frac{\sin B}{n} = \frac{\sin B\{3 - 4\sin^2 B\}}{n+1}$$

$$\Rightarrow \quad \frac{n+1}{n} = 3 - 4(1 - \cos^2 B)$$

$$\frac{n+1}{n} = -1 + 4\cos^2 B \quad \ldots(v)$$

From Eqs. (iv) and (v), we get

$$\frac{n+1}{n} = -1 + 4\left(\frac{n+2}{2n}\right)^2$$

$$\frac{n+1}{n} + 1 = \left(\frac{n^2 + 4n + 4}{n^2}\right)$$

$$\frac{2n+1}{n} = \frac{n^2 + 4n + 4}{n^2}$$

$$2n^2 + n = n^2 + 4n + 4$$

$$\Rightarrow \quad n^2 - 3n - 4 = 0 \Rightarrow (n - 4)(n + 1) = 0$$

$$n = 4 \text{ or } -1 \qquad \text{where } n \neq -1$$

$\therefore$ $n = 4$. Hence, the sides are 4, 5, 6.

Example 5. Let O be a point inside a ΔABC, such that $\angle OAB = \angle OBC = \angle OCA = w$. Then, show that

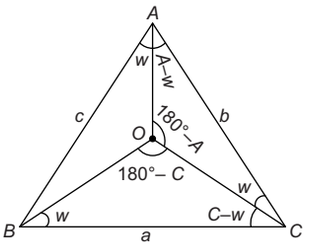
$$\cot w = \cot A + \cot B + \cot C.$$

Sol. In ΔOAC, Using sine rule,

$$\frac{\sin(A - w)}{OC} = \frac{\sin(180 - A)}{b}$$

$$\Rightarrow \quad \frac{\sin(A - w)}{OC} = \frac{\sin A}{b} \qquad \ldots(i)$$

Also, in ΔOBC



$$\frac{\sin w}{OC} = \frac{\sin(180 - C)}{a}$$

$$\Rightarrow \quad \frac{\sin w}{OC} = \frac{\sin C}{a} \qquad \ldots(ii)$$

On dividing Eqs. (i) and (ii); $\frac{\sin(A - w)}{\sin w} = \frac{a \sin A}{b \sin C}$

$$\left\{\text{as, we know, } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k\right\}$$

$$\Rightarrow \quad \frac{\sin(A - w)}{\sin w} = \frac{k \sin A \cdot \sin A}{k \sin B \cdot \sin C}$$

$$\Rightarrow \quad \frac{\sin A \cos w - \cos A \sin w}{\sin w} = \sin A \frac{\{\sin(\pi - (B + C))\}}{\sin B \sin C}$$

$$\Rightarrow \quad \sin A(\cot w) - \cos A = \sin A\left(\frac{\sin B \cos C + \cos B \sin C}{\sin B \sin C}\right)$$

$$\Rightarrow \quad \sin A(\cot w) - \cos A = \sin A(\cot C + \cot B)$$

$$\Rightarrow \quad \cot w - \cot A = \cot B + \cot C$$

$$\Rightarrow \quad \cot w = \cot A + \cot B + \cot C.$$

Example 6. Solve $b \cos^2 \frac{C}{2} + c \cos^2 \frac{B}{2}$ in terms of k, where k is perimeter of the ΔABC.

Sol. We have, $b \cos^2 \frac{C}{2} + c \cos^2 \frac{B}{2}$

$$\Rightarrow \quad \frac{b}{2}(1 + \cos C) + \frac{c}{2}(1 + \cos B)$$

$$\Rightarrow \quad \frac{b + c}{2} + \frac{1}{2}(b \cos C + c \cos B)$$

$$\Rightarrow \quad \frac{b + c}{2} + \frac{1}{2}a \qquad \text{[using projection formula]}$$

$$\Rightarrow \quad \frac{a + b + c}{2}$$

$$\therefore \quad b \cos^2 \frac{C}{2} + c \cos^2 \frac{B}{2} = \frac{k}{2} \quad \text{[where } k = a + b + c, \text{ given]}$$

Example 7. In a ΔABC, $c \cos^2 \frac{A}{2} + a \cos^2 \frac{C}{2} = \frac{3b}{2}$, then show a, b, c are in AP.

Sol. We have, $c \cos^2 \frac{A}{2} + a \cos^2 \frac{C}{2} = \frac{3b}{2}$

$$\Rightarrow \quad \frac{c}{2}(1 + \cos A) + \frac{a}{2}(1 + \cos C) = \frac{3b}{2}$$

$$\Rightarrow \quad a + c + (c \cos A + a \cos C) = 3b$$

$$\Rightarrow \quad a + c + b = 3b \qquad \text{[using projection formula]}$$

$$\Rightarrow \quad a + c = 2b$$

which shows a, b, c are in AP.

Example 8. In a ΔABC, $a = 2b$ and $|A - B| = \frac{\pi}{3}$. Determine the $\angle C$.

Sol. Given, $a = 2b$ ...(i)

$$\Rightarrow \quad \angle A > \angle B \quad \text{[as } a > b \text{ and we know greater angle is opposite to greater side]}$$

$$\Rightarrow \quad |A - B| = \frac{\pi}{3}$$

or $$A - B = \frac{\pi}{3} \qquad \ldots(ii) \text{ [as } A > B]$$

Using Napier's analogy, we have

$$\tan\left(\frac{A - B}{2}\right) = \frac{a - b}{a + b} \cot\left(\frac{C}{2}\right)$$

$$\Rightarrow \quad \tan\left(\frac{\pi}{6}\right) = \frac{2b - b}{2b + b} \cot\left(\frac{C}{2}\right)$$

[using Eqs. (i) and (ii)]

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{1}{3} \cot\left(\frac{C}{2}\right)$$

$$\Rightarrow \quad \cot\left(\frac{C}{2}\right) = \sqrt{3}$$

i.e. $$\frac{C}{2} = \frac{\pi}{6} \Rightarrow C = \frac{\pi}{3}$$

Example 9. In a ΔABC, the tangent of half the difference of two angles is one-third the tangent of half the sum of the angles. Determine the ratio of the sides opposite to the angles.

Sol. We have, $$\tan\left(\frac{A - B}{2}\right) = \frac{1}{3} \tan\left(\frac{A + B}{2}\right) \qquad \ldots(i)$$

Using Napier's analogy,

$$\tan\left(\frac{A - B}{2}\right) = \frac{a - b}{a + b} \cdot \cot\left(\frac{C}{2}\right) \qquad \ldots(ii)$$

From Eqs. (i) and (ii);

$$\frac{1}{3}\tan\left(\frac{A+B}{2}\right)=\frac{a-b}{a+b}\cdot\cot\left(\frac{C}{2}\right)$$

$$\Rightarrow \quad \frac{1}{3}\cot\left(\frac{C}{2}\right)=\frac{a-b}{a+b}\cdot\cot\left(\frac{C}{2}\right)$$

$$\left[\text{as } A+B+C=\pi \ \therefore \ \tan\left(\frac{B+C}{2}\right)=\tan\left(\frac{\pi}{2}-\frac{C}{2}\right)=\cot\frac{C}{2}\right]$$

$$\Rightarrow \quad \frac{a-b}{a+b}=\frac{1}{3} \quad \text{or} \quad 3a-3b=a+b$$

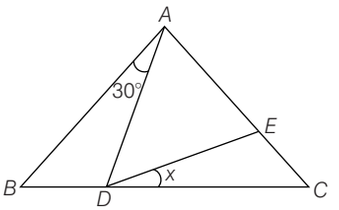
$$2a=4b$$

or $$\frac{b}{a}=\frac{1}{2}$$

Thus, the ratio of the sides opposite to the angles is $b:a=1:2$.

Exercise for Session 1

1. In the given figure, if $AB=AC$, $\angle BAD=30°$ and $AE=AD$, then x is equal to



(a) 15° (b) 10° (c) $12\frac{1}{2}$ (d) $7\frac{1}{2}$

2. In ΔABC, $a=4, b=12$ and $B=60°$, then the value of $\sin A$ is

(a) $\frac{1}{2\sqrt{3}}$ (b) $\frac{1}{3\sqrt{2}}$ (c) $\frac{2}{\sqrt{3}}$ (d) $\frac{\sqrt{3}}{2}$

3. Let ABC be a triangle such that $\angle A=45°$, $\angle B=75°$, then $a+c\sqrt{2}$ is equal to

(a) 0 (b) b (c) $2b$ (d) $-b$

4. Angles A, B and C of a ΔABC are in AP. If $\frac{b}{c}=\frac{\sqrt{3}}{\sqrt{2}}$, then $\angle A$ is equal to

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{5\pi}{12}$ (d) $\frac{\pi}{2}$

5. If $\cot\frac{A}{2}=\frac{b+c}{a}$, then ΔABC is

(a) Isosceles (b) Equilateral (c) Right angled (d) None of these

6. If in a ΔABC, $\frac{a^2-b^2}{a^2+b^2}=\frac{\sin(A-B)}{\sin(A+B)}$, then the triangle is

(a) Right angled or isosceles (b) Right angled and isosceles
(c) Equilateral (d) None of these

7. In any triangle ABC, $\frac{a^2\sin(B-C)}{\sin B+\sin C}+\frac{b^2\sin(C-A)}{\sin C+\sin A}+\frac{c^2\sin(A-B)}{\sin A+\sin B}=$

(a) $a+b+c$ (b) $a+b-c$ (c) $a-b+c$ (d) 0

8. In any ΔABC, if $2\cos B=\frac{a}{2}$, then the triangle is

(a) right angled (b) equilateral (c) isosceles (d) None of these

9. The expression $\frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4b^2c^2}$ is equal to

(a) $\cos^2 A$ (b) $\sin^2 A$ (c) $\cos A \cos B \cos C$ (d) None of these

10. In ΔABC, if $a \cos A = b \cos B$, then the triangle is

(a) Isosceles (b) Right angled
(c) Isosceles or right angled (d) Right angled isosceles

11. In a ΔABC, $(a+b+c)(b+c-a) = \lambda bc$, if

(a) $\lambda < 0$ (b) $\lambda > 0$
(c) $0 < \lambda < 4$ (d) $\lambda < 4$

12. If $a = 9, b = 8$ and $c = x$ satisfies $3\cos C = 2$, then

(a) $x = 5$ (b) $x = 6$ (c) $x = 4$ (d) $x = 7$

13. In ΔABC, if $\sin^2 A + \sin^2 B = \sin^2 C$, then the triangle is

(a) equilateral (b) isosceles (c) right angled (d) None of these

14. The sides of a triangle are $\alpha - \beta, \alpha + \beta$ and $\sqrt{3\alpha^2 + \beta^2}$, $(\alpha > \beta > 0)$. Its largest angle is

(a) $\frac{2\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{4}$ (d) $\frac{5\pi}{6}$

15. In any triangle, $\frac{1+\cos(A-B)\cos C}{1+\cos(A-C)\cos B} =$

(a) $\frac{a^2+b^2}{a^2+c^2}$ (b) $\frac{b^2+c^2}{b^2-c^2}$ (c) $\frac{c^2-a^2}{a^2+b^2}$ (d) None of these

16. If the sides of a ΔABC are in AP and a is the smallest side, then $\cos A$ equals

(a) $\frac{3c-4b}{2c}$ (b) $\frac{3c-4b}{2b}$ (c) $\frac{4c-3b}{2c}$ (d) None of these

17. In a ΔABC, $a^2 \cos 2B + b^2 \cos 2A + 2ab \cos(A-B) =$

(a) a^2 (b) c^2 (c) b^2 (d) $a^2 + b^2$

18. In any ΔABC, $2[bc \cos A + ca \cos B + ab \cos C] =$

(a) $a^2 + b^2 + c^2$ (b) $a^2 + b^2 - c^2$ (c) $a^2 - b^2 + c^2$ (d) None of these

19. In a ΔABC, $\tan\frac{1}{2}(A+B)\cdot\cot\frac{1}{2}(A-B)$ is equal to

(a) $\frac{a-b}{a+b}$ (b) $\frac{a+b}{c}$ (c) $\frac{a+b}{a-b}$ (d) $\frac{a-b}{2(a+b)}$

20. If in a ΔABC, $b = \sqrt{3}, c = 1$ and $B - C = 90°$, then $\angle A$ is

(a) 30° (b) 45° (c) 75° (d) 15°

Session 2

Auxiliary Formulae

Trigonometric Ratios of Half-angles

In any ΔABC, we have

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad \text{...(i) [using cosine law]}$$

$$2\sin^2\frac{A}{2} = 1 - \cos A \quad \text{...(ii)}$$

$$2\sin^2\frac{A}{2} = 1 - \left(\frac{b^2 + c^2 - a^2}{2bc}\right)$$

[using Eqs. (i) and (ii)]

$$= \frac{2bc - b^2 - c^2 + a^2}{2bc}$$

$$= \frac{a^2 - (b^2 + c^2 - 2bc)}{2bc} = \frac{a^2 - (b-c)^2}{2bc}$$

$$= \frac{(a + b - c)(a - b + c)}{2bc}$$

[we know $a + b + c = 2s$
$\Rightarrow a + b = 2s - c$ and $a + c = 2s - b$]

$$\therefore \quad 2\sin^2\frac{A}{2} = \frac{(2s - c - c)(2s - b - b)}{2bc}$$

$$2\sin^2\frac{A}{2} = \frac{4(s-c)(s-b)}{2bc}$$

$$\Rightarrow \quad \sin^2\frac{A}{2} = \frac{(s-b)(s-c)}{bc}$$

[since in a triangle, A is always less than 180°,
$\therefore \sin A/2$ is (+ve)]

or $$\boldsymbol{\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}} \quad \text{...(A)}$$

Similarly, it may be proved,

$$\boldsymbol{\sin\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}}} \quad \text{...(B)}$$

$$\boldsymbol{\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}} \quad \text{...(C)}$$

Again, $2\cos^2\frac{A}{2} = 1 + \cos A = 1 + \frac{b^2 + c^2 - a^2}{2bc}$

$$= \frac{2bc + b^2 + c^2 - a^2}{2bc}$$

$$= \frac{(b+c)^2 - a^2}{2bc}$$

$$= \frac{(b + c - a)(b + c + a)}{2bc}$$

[where $a + b + c = 2s$, $b + c = 2s - a$]

$$2\cos^2\frac{A}{2} = \frac{2s \times 2(s-a)}{2bc}; \quad \cos^2\frac{A}{2} = \frac{s(s-a)}{bc}$$

Since, $A/2$ is less than 90°,

$\therefore \quad \cos A/2 > 0.$

$$\therefore \quad \boldsymbol{\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}}$$

Similarly, $\boldsymbol{\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}}}$ and $\boldsymbol{\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}}$

Also, $$\tan\frac{A}{2} = \frac{\sin A/2}{\cos A/2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \times \sqrt{\frac{bc}{s(s-a)}}$$

$$\therefore \quad \boldsymbol{\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}}$$

Similarly, $\boldsymbol{\tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}}$

and $\boldsymbol{\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}}$

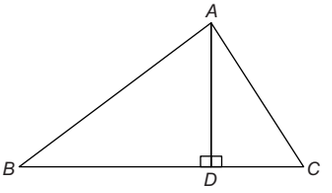
Area of Triangle

If Δ represents the area of a triangle ABC, then

area of $\Delta = \frac{1}{2} BC \cdot AD$, $\left[\because \Delta = \frac{1}{2}(\text{base})(\text{height})\right]$

$$= \frac{1}{2} a \cdot (c \sin B) \qquad \left[\text{as } \sin B = \frac{AD}{c}\right]$$

$$\Delta = \frac{1}{2} ac \cdot \sin B$$



Also, $$\sin C = \frac{AD}{b} \Rightarrow AD = b \sin C$$

$$\therefore \qquad \Delta = \frac{1}{2} a \cdot b \cdot \sin C$$

Similarly; $\Delta = \frac{1}{2} bc \cdot \sin A$

$$\therefore \qquad \Delta = \frac{1}{2} ab \sin C = \frac{1}{2} bc \sin A = \frac{1}{2} ca \sin B$$

Note

(i) Area of a triangle in terms of sides (Heron's formula) :

$$\Delta = \frac{1}{2} bc \sin A = \frac{1}{2} bc \cdot 2\sin\frac{A}{2} \cdot \cos\frac{A}{2}$$

$$= bc \sqrt{\frac{(s-b)(s-c)}{bc}} \sqrt{\frac{s(s-a)}{bc}}$$

$$\therefore \qquad \Delta = \sqrt{s(s-a)(s-b)(s-c)}.$$

(ii) Area of triangle in terms of one side and sine of three angles:

$$\Delta = \frac{1}{2} bc \sin A = \frac{1}{2} (k \sin B)(k \sin C) \cdot \sin A$$

$$= \frac{1}{2} k^2 \sin A \sin B \sin C = \frac{1}{2} \left(\frac{a}{\sin A}\right)^2 \cdot \sin A \cdot \sin B \cdot \sin C$$

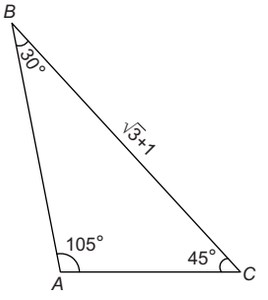
$$\therefore \qquad \Delta = \frac{a^2}{2} \cdot \frac{\sin B \sin C}{\sin A} = \frac{b^2}{2} \cdot \frac{\sin A \sin C}{\sin B} = \frac{c^2}{2} \cdot \frac{\sin A \sin B}{\sin C}$$

Example 10. If the angles of a triangle are 30° and 45°, and the included side is $(\sqrt{3}+1)$ cm, then prove that the area of the triangle is $\frac{1}{2}(\sqrt{3}+1)$.

Sol. We have, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

$$\Rightarrow \qquad \frac{\sin 105^\circ}{\sqrt{3}+1} = \frac{\sin 30^\circ}{b} = \frac{\sin 45^\circ}{c}$$

$$\Rightarrow \qquad b = \frac{\sqrt{3}+1}{2\sin 105^\circ}, \; c = \frac{\sqrt{3}+1}{\sqrt{2}\sin 105^\circ}$$



So, area of $\Delta ABC = \frac{1}{2} bc \sin A = \frac{1}{2} bc \sin 105^\circ$

$$= \frac{1}{2} \cdot \frac{(\sqrt{3}+1)^2}{2\sqrt{2}.\sin(60^\circ + 45^\circ)}$$

$$= \frac{(\sqrt{3}+1)^2}{4\sqrt{2}\left\{\frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}\right\}} = \frac{(\sqrt{3}+1)^2}{2(\sqrt{3}+1)} = \frac{1}{2}(\sqrt{3}+1)$$

Thus, area of Δ is $\frac{1}{2}(\sqrt{3}+1)$.

Aliter In above example we have $\angle A = 105^\circ$, $\angle B = 30^\circ$, $\angle C = 45^\circ$ and $a = \sqrt{3}+1$.

Thus, area of $\Delta = \frac{a^2}{2} \frac{\sin B \sin C}{\sin A}$ [using note (ii)]

$$= \frac{(\sqrt{3}+1)^2}{2} \cdot \frac{\sin 30^\circ . \sin 45^\circ}{\sin(105^\circ)} = \frac{1}{2}(\sqrt{3}+1)$$

Example 11. Consider the following statements concerning in ΔABC

(i) The sides a, b, c and area Δ are rational.

(ii) $a, \tan\frac{B}{2}, \tan\frac{C}{2}$ are rational.

(iii) $a, \sin A, \sin B, \sin C$ are rational.

Prove that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

Sol. a, b, c, Δ are rational (given).

$\Rightarrow$ $s, s-a, s-b, s-c$ are rational.

Now, $\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$

$$= \sqrt{\frac{s(s-a)(s-b)(s-c)}{s^2(s-b)^2}} = \frac{\Delta}{s(s-b)}$$

$\therefore$ $\tan\frac{B}{2}$ = rational [as $\Delta, s, (s-a)$ are rational]

Similarly, $\tan\frac{C}{2}$ is rational. Hence (i) $\Rightarrow$ (ii)

Now, $\sin B = \frac{2\tan\frac{B}{2}}{1+\tan^2\frac{B}{2}}$ is rational by (ii).

Similarly, $\sin C$ is also rational.

$$\tan\frac{A}{2} = \cot\frac{B+C}{2} = \frac{1 - \tan\frac{B}{2} \cdot \tan\frac{C}{2}}{\tan\frac{B}{2} + \tan\frac{C}{2}} = \text{rational by (ii)}$$

$\Rightarrow$ $\sin A$ is rational

Hence, (ii) $\Rightarrow$ (iii).

Now, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$, which is rational since 'a' and '$\sin A$' are rational.

$\Rightarrow \frac{b}{\sin B}$ and $\frac{c}{\sin C}$ are rational. But $\sin B$ and $\sin C$ are rational by (iii)

$\Rightarrow$ b and c are rational.

$\Rightarrow$ $\Delta = \frac{1}{2} bc \sin A$ is also rational.

Hence (iii) $\Rightarrow$ (i).

Exercise for Session 2

1. If in a ΔABC, $(s-a)(s-b)=s(s-c)$, then angle C is equal to

(a) 90° (b) 45° (c) 30° (d) 60°

2. In any ΔABC, if $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are in AP, then a, b, c are in

(a) AP (b) GP (c) HP (d) None of these

3. In any ΔABC, $\dfrac{\tan\frac{A}{2}-\tan\frac{B}{2}}{\tan\frac{A}{2}+\tan\frac{B}{2}}=$

(a) $\dfrac{a-b}{a+b}$ (b) $\dfrac{a-b}{c}$ (c) $\dfrac{a-b}{a+b+c}$ (d) $\dfrac{c}{a+b}$

4. In a ΔABC, $bc\cos^2\dfrac{A}{2}+ca\cos^2\dfrac{B}{2}+ab\cos^2\dfrac{C}{2}$ is equal to

(a) $(s-a)^2$ (b) $(s-b)^2$ (c) $(s-c)^2$ (d) s^2

5. In a ΔABC, if $\cos A+\cos C+4\sin^2\left(\dfrac{B}{2}\right)$, then a, b, c are in

(a) AP (b) GP (c) HP (d) None of these

6. If in a ΔABC, $3a=b+c$, then the value of $\cos\dfrac{B}{2}\cot\dfrac{C}{2}$ is

(a) 1 (b) $\sqrt{3}$ (c) 2 (d) None of these

7. In any ΔABC, $\left(\dfrac{b-c}{a}\right)\cos^2\left(\dfrac{A}{2}\right)+\left(\dfrac{c-a}{b}\right)\cos^2\left(\dfrac{b}{2}\right)+\left(\dfrac{a-b}{2}\right)\cos^2\left(\dfrac{C}{2}\right)$ is equal to

(a) $\dfrac{b^2-c^2}{a^2}$ (b) $\dfrac{c^2-a^2}{b^2}$ (c) $\dfrac{a^2-b^2}{c^2}$ (d) 0

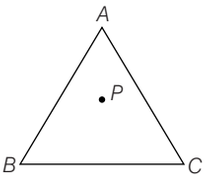
8. In a ΔABC , the tangent of half difference of two angles is one-third the tangent of half the sum of the two angles. The ratio of the sides opposite the angles is

(a) 2 : 3 (b) 1 : 3 (c) 2 : 1 (d) 3 : 4

9. If in a triangle, $a\cos^2\dfrac{C}{2}+c\cos^2\dfrac{A}{2}=\dfrac{3b}{2}$, then its sides will be in -

(a) AP (b) GP (c) HP (d) AGP

10. In the adjacent figure 'P' is any interior point of the equilateral triangle ABC of side length 2 unit



If x_a, x_b and x_c represent the distance of P from the sides BC, CA and AB respectively then $x_a+x_b+x_c$ is equal to

(a) 6 (b) $\sqrt{3}$ (c) $\dfrac{\sqrt{3}}{2}$ (d) $2\sqrt{3}x$

11. If $c^2=a^2+b^2$, then $4s(s-a)(s-b)(s-c)$ is equal to

(a) s^4 (b) b^2c^2 (c) c^2a^2 (d) a^2b^2

12. The number of possible $\angle ABC$ in which $BC=\sqrt{11}$ cm, $CA=\sqrt{13}$ cm and $A=60°$ is

(a) 0 (b) 1 (c) 2 (d) None of these

13. If two sides a, b and the $\angle A$ be such that two triangles are formed, then the sum of the two values of the third side is

(a) $b^2 - a^2$ (b) $2b \cos A$ (c) $2b \sin A$ (d) $\frac{b-c}{b+c}$

14. If in a ΔABC, $\sin A = \sin^2 B$ and $2\cos^2 A = 3\cos^2 B$, then the ΔABC is

(a) right angled (b) obtuse angled (c) isosceles (d) equilateral

15. If $a \cos A = b \cos B$, then the triangle is

(a) equilateral (b) right angled (c) isosceles (d) isosceles or right angled

16. Points D, E are taken on the side BC of a triangle ABC such that $BD = DE = EC$. If $\angle BAD = x$, $\angle DAE = y$, $\angle EAC = z$, then the value of $\frac{\sin(x+y)\sin(y+z)}{\sin x \sin z}$ is equal to

(a) 1 (b) 2 (c) 4 (d) None of these

17. If the base angles of a triangle are $22\frac{1^\circ}{2}$ and $112\frac{1^\circ}{2}$, then the height of the triangle is equal to

(a) half the base (b) the base (c) twice the base (d) four times the base

18. In a ΔABC, $a = 1$ and the perimeter is six times the AM of the sines of the angles. The measure of $\angle A$ is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{4}$

19. In a ΔABC, if median AD is perpendicular to AB, then $\tan A + 2\tan B$ is equal to

(a) 1 (b) 3 (c) 0 (d) $\frac{1}{2}$

20. If p is the product of the sines of angles of a triangle, and q the product of their cosines, then tangents of the angles are roots of the equation

(a) $qx^3 - px^2 + (1+q)x - p = 0$ (b) $px^3 - qx^2 + (1+p)x - q = 0$

(c) $(1+q)x^3 - px^2 + qx - p = 0$ (d) None of these

Session 3

Circles Connected with Triangle

Circles Connected with Triangle

Circumcircle of a Triangle

The circle which passes through the angular points of a ΔABC is called its Circumcircle. The centre of this circle is the point of intersection of perpendicular bisectors of the sides and called the Circumcentre. Its radius is always denoted by $\boldsymbol{R}$.

Note

1. Circumcentre of an a cute-angled triangle lies inside the triangle.
2. Circumcentre of an obtuse-angled triangle lies outside the triangle.
3. In a right angled triangle the circumcentre is the mid-point of hypotenuse.

Circum-radius

The radius of the circumcircle of a ΔABC is called the circum-radius given by;

(i) $R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C}$ (ii) $R = \frac{abc}{4\Delta}$

Proof

(i) Here, the perpendicular bisectors of the sides BC, CA and AB intersect at O.

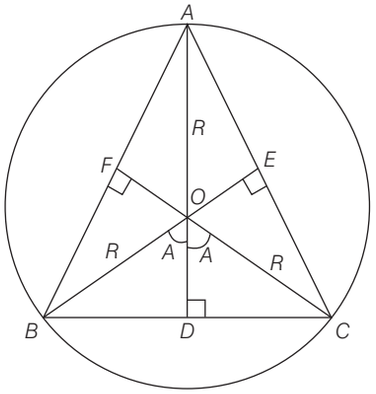
$\therefore$ O is the circumcentre such that,

$$OA = OB = OC = R$$

We have, $\angle BOC = 2\angle A$

$\therefore$ $\angle BOD = \angle COD = \angle A$

In ΔOBD, $\sin A = \frac{BD}{OB} = \frac{a/2}{R}$



$\Rightarrow \quad R = \frac{a}{2\sin A}$

Similarly, $R = \frac{b}{2\sin B}$ and $R = \frac{c}{2\sin C}$

Hence, $\boxed{R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C}}$

(ii) As discussed, Area of $\Delta = \frac{1}{2}bc\sin A$

$\Rightarrow \quad \sin A = \frac{2\Delta}{bc}$ ...(i)

Also, $R = \frac{a}{2\sin A}$ ...(ii)

$\therefore$ From Eqs. (i) and (ii);

$$R = \frac{a}{2\left(\frac{2\Delta}{bc}\right)} = \frac{abc}{4\Delta} \Rightarrow R = \frac{abc}{4\Delta}.$$

In-circle or Inscribed Circle of a Triangle

The circle that can be inscribed with in a triangle so as to touch each of its sides is called its inscribed circle or In-circle. The centre of this circle is the points of intersection of bisectors of the angles of the triangle. The radius of the circle is always denoted by 'r' and is equal to the length of perpendicular from its centre to any one of the sides of triangle.

In-radius The radius of the inscribed circle of a triangle is called the in-radius. It is denoted by 'r' and is given by

(i) $r = \frac{\Delta}{s}$

(ii) $r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}.$

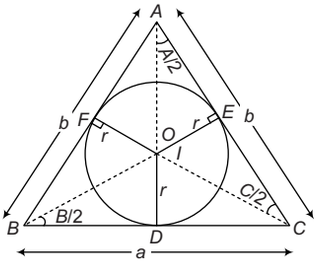
(iii) $r = \frac{a\sin B/2 \cdot \sin C/2}{\cos A/2}$

$r = \frac{b\sin C/2 \cdot \sin A/2}{\cos B/2} \Rightarrow r = \frac{c\sin A/2 . \sin B/2}{\cos C/2}$

(iv) $r = 4R\sin A/2 \sin B/2 \sin C/2$

Proof Let the internal bisectors of the angles of the triangle ABC meet at I. Suppose the circle touches the sides BC, CA and AB at D, E and F, respectively.

Then, ID, IE, IF are perpendicular to these sides and $ID = IE = IF = r$.



(i) We have, area of ΔABC = area of ΔIBC + area of ΔIAB + area of ΔICA

$$\Delta = \frac{1}{2}ar + \frac{1}{2}cr + \frac{1}{2}br$$

$$\Delta = \frac{1}{2}(a+b+c)r = sr \qquad \left[\text{as; } s = \frac{a+b+c}{2}\right]$$

$\Rightarrow \quad \Delta = sr$

or $\quad r = \frac{\Delta}{s}.$

(ii) Since, the lengths of the tangents to a circle from a given points are equal, therefore

$AE = AF$, $BD = BF$ and $CD = CE$. ...(I)

Now, $2s = a + b + c = BC + CA + AB$

$= (BD + DC) + (CE + EA) + (AF + FB)$

$= (BD + BF) + (AE + AF) + (CD + CE)$

$= 2(BD + AE + CD) = 2(BC + AE) = 2(a + AE)$

$\Rightarrow \quad s = a + AE$

$\Rightarrow \quad AE = (s - a)$

Now, in ΔIAE,

$$\tan\frac{A}{2} = \frac{r}{AE}$$

$\Rightarrow \quad r = AE\tan(A/2) = (s-a)\tan A/2$

$\therefore \quad r = (s-a)\tan A/2$

Similarly, $r = (s-b)\tan B/2$ and $r = (s-c)\tan C/2$

Hence, $r = (s-a)\tan A/2$

$= (s-b)\tan B/2 = (s-c)\tan C/2$

(iii) In ΔIBD and ΔICD, we have,

$$\tan B/2 = \frac{r}{BD} \text{ and } \tan\frac{C}{2} = \frac{r}{CD}$$

$$\therefore \quad BD = \frac{r}{\tan B/2} \text{ and } CD = \frac{r}{\tan C/2}$$

Now, $a = BD + CD$

$$\Rightarrow \quad a = \frac{r}{\tan\left(\frac{B}{2}\right)} + \frac{r}{\tan\left(\frac{C}{2}\right)}$$

$$\Rightarrow \quad a = r\left[\frac{\cos B/2}{\sin B/2} + \frac{\cos C/2}{\sin C/2}\right]$$

$$\Rightarrow \quad a = r\left[\frac{\cos B/2 \cdot \sin C/2 + \sin B/2 \cdot \cos C/2}{\sin B/2 \cdot \sin C/2}\right]$$

$$a = \frac{r\sin(B/2 + C/2)}{\sin B/2 \cdot \sin C/2} \qquad [\because A + B + C = \pi]$$

$$\therefore \quad \sin(B/2 + C/2) = \sin\left(\frac{\pi}{2} - \frac{A}{2}\right) = \cos A/2\,.$$

$$\Rightarrow \quad a = \frac{r\cos A/2}{\sin B/2 \cdot \sin C/2}$$

$$\therefore \quad r = \frac{a\sin B/2 \cdot \sin C/2}{\cos A/2}, r = \frac{b\sin A/2 \cdot \sin C/2}{\cos B/2} \text{ and}$$

$$r = \frac{c\sin A/2 \cdot \sin B/2}{\cos C/2}$$

(iv) We have, $r = \frac{a\sin B/2 \cdot \sin C/2}{\cos A/2}$ and $R = \frac{a}{2\sin A}$

$$\Rightarrow \quad r = \frac{2R\sin A \cdot \sin B/2 \cdot \sin C/2}{\cos A/2}$$

$$\Rightarrow \quad r = \frac{2R \cdot (2\sin A/2 \cdot \cos A/2) \cdot \sin B/2 \cdot \sin C/2}{\cos A/2}$$

$$\Rightarrow \quad r = 4R\sin A/2 \cdot \sin B/2 \cdot \sin C/2$$

Escribed Circles of a Triangle

The circle which touches the sides BC and two sides AB and AC produced of a triangle ABC is called the **Escribed circle opposite to the angle *A*.**

Its radius is denoted by r_1. Similarly, r_2 and r_3 denote the radii of the escribed circles opposite to the angles B and C, respectively. The centres of the escribed circles are called the ex-centres.

The centre of the escribed circles opposite to the angle A is the point of Intersection of external bisector of angles B and C. The internal bisector also passes through the same point. This centre is generally denoted by I_1.

Formulae for r_1, r_2, r_3

In any ΔABC, we have

(i) $r_1 = \frac{\Delta}{s-a}, r_2 = \frac{\Delta}{s-b}, r_3 = \frac{\Delta}{s-c}$

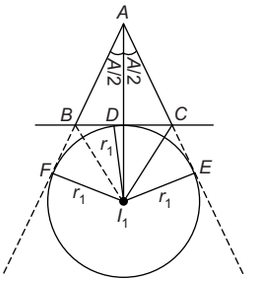
(ii) $r_1 = s\tan A/2, r_2 = s\tan B/2, r_3 = s\tan C/2$

(iii) $r_1 = \frac{a\cos B/2 \cdot \cos C/2}{\cos A/2}, r_2 = \frac{b\cos A/2 \cdot \cos C/2}{\cos B/2},$

$r_3 = \frac{c\cos A/2 \cdot \cos B/2}{\cos C/2}$

(iv) $r_1 = 4R\sin A/2\cos B/2.\cos C/2,$
$r_2 = 4R\cos A/2\sin B/2.\cos C/2,$
$r_3 = 4R\cos A/2\cos B/2.\sin C/2$

Proof (i) Let the ΔABC be as;



We have,

$$I_1D = I_1E = I_1F = r_1$$

Now, area of ΔABC = area of ΔI_1AC + area of ΔI_1AB − area of ΔI_1BC

$$\Rightarrow \quad \Delta = \frac{1}{2}I_1E \cdot AC + \frac{1}{2}I_1F \cdot AB - \frac{1}{2}I_1D \cdot BC$$

$$\Delta = \frac{1}{2}r_1b + \frac{1}{2}r_1c - \frac{1}{2}r_1a$$

$$\Delta = \frac{r_1}{2}(b + c - a)$$

$$\Delta = \frac{r_1}{2}(2s - 2a) \qquad [\text{using } a + b + c = 2s]$$

$$\Rightarrow \quad r_1 = \frac{\Delta}{s-a}$$

Similarly, $r_2 = \frac{\Delta}{s-b}$ and $r_3 = \frac{\Delta}{s-c}$

(ii) Since, the lengths of tangents to a circle from an external points are equal,

$$\therefore \quad AE = AF,\ BD = BF \text{and } CD = CE$$

Now, $AE + AF = (AC + CE) + (AB + BF)$

$$= (AC + CD) + (AB + BD)$$

$$= AC + AB + CD + BD$$

$$= AC + AB + BC$$
$$= a + b + c = 2s.$$
$$\Rightarrow \quad 2AF = 2s$$
$$\Rightarrow \quad AE = AF = s$$

In $\Delta I_1 AF$, $\tan A/2 = \frac{I_1F}{AF} = \frac{r_1}{AF}$

$$\Rightarrow \quad \tan A/2 = \frac{r_1}{s}$$
$$\Rightarrow \quad r_1 = s \tan A/2$$

Similarly, $r_2 = s \tan B/2$ and $r_3 = s \tan C/2$.

(iii) In $\Delta I_1 BD$, we have

$$\tan\left(\frac{B}{2}\right) = \frac{I_1D}{BD} = \frac{r_1}{BD}$$
$$\Rightarrow \quad BD = r_1 \tan\frac{B}{2}$$

Similarly, in $\Delta I_1 CD$, we have

$$CD = r_1 \tan\frac{C}{2}$$

Now, $a = BC = BD + CD = r_1 \tan\frac{B}{2} + r_1 \tan\frac{C}{2}$

$$= r_1\left(\tan\frac{B}{2} + \tan\frac{C}{2}\right) = r_1 \frac{\cos A/2}{\cos B/2 \cos C/2}$$
$$\Rightarrow \quad r_1 = \frac{a \cos B/2 \cos C/2}{\cos A/2}$$

Similarly,

$$r_2 = \frac{b \cos A/2 \cdot \cos C/2}{\cos B/2} \text{ and } r_3 = \frac{c \cos A/2 \cdot \cos B/2}{\cos C/2}$$

(iv) We have, $r_1 = \frac{a \cos B/2 \cdot \cos C/2}{\cos A/2}$ and $R = \frac{a}{2 \sin A}$

$$\Rightarrow \quad r_1 = \frac{2R \sin A \cdot \cos B/2 \cdot \cos C/2}{\cos A/2}$$
$$\Rightarrow \quad r_1 = \frac{4R \sin A/2 \cdot \cos A/2 \cdot \cos B/2 \cdot \cos C/2}{\cos A/2}$$
$$\therefore \quad r_1 = 4R \sin A/2 \cdot \cos B/2 \cdot \cos C/2$$

Similarly, $r_2 = 4R \cos A/2 \cdot \sin B/2 \cdot \cos C/2$

$$r_3 = 4R \cos A/2 \cdot \cos B/2 \cdot \sin C/2$$

Example 12. Show that $\frac{b-c}{r_1} + \frac{c-a}{r_2} + \frac{a-b}{r_3} = 0$.

Sol. LHS $\frac{(b-c)}{r_1} + \frac{(c-a)}{r_2} + \frac{(a-b)}{r_3}$

$$\Rightarrow \quad (b-c)\cdot\left(\frac{s-a}{\Delta}\right) + (c-a)\cdot\left(\frac{s-b}{\Delta}\right) + (a-b)\cdot\left(\frac{s-c}{\Delta}\right)$$
$$\Rightarrow \quad \frac{(s-a)(b-c) + (s-b)(c-a) + (s-c)(a-b)}{\Delta}$$
$$= \frac{s(b-c+c-a+a-b) - [ab - ac + bc - ba + ac - bc]}{\Delta}$$
$$= \frac{0}{\Delta} = 0 = \text{RHS}$$

Thus, $\frac{b-c}{r_1} + \frac{c-a}{r_2} + \frac{a-b}{r_3} = 0$

Example 13. ***If $r_1 = r_2 + r_3 + r$, then prove that the triangle is right angled.***

Sol. We have, $r_1 - r = r_2 + r_3$

$$\Rightarrow \quad \frac{\Delta}{s-a} - \frac{\Delta}{s} = \frac{\Delta}{s-b} + \frac{\Delta}{s-c}$$
$$\Rightarrow \quad \frac{s-s+a}{s(s-a)} = \frac{s-c+s-b}{(s-b)(s-c)}$$
$$\Rightarrow \quad \frac{a}{s(s-a)} = \frac{2s-(b+c)}{(s-b)(s-c)} \qquad [\text{as, } 2s = a+b+c]$$
$$\Rightarrow \quad \frac{a}{s(s-a)} = \frac{a}{(s-b)(s-c)}$$
$$\Rightarrow \quad s^2 - (b+c)s + bc = s^2 - as$$
$$\Rightarrow \quad s(-a+b+c) = bc$$
$$\Rightarrow \quad \frac{(b+c-a)(a+b+c)}{2} = bc$$
$$\Rightarrow \quad (b+c)^2 - (a)^2 = 2bc$$
$$\Rightarrow \quad b^2 + c^2 + 2bc - a^2 = 2bc$$
$$\Rightarrow \quad b^2 + c^2 = a^2$$
$$\therefore \quad \angle A = 90°$$

Example 14. Prove that $r \cot\frac{B}{2} \cdot \cot\frac{C}{2} = r_1$.

Sol. LHS $r \cot B/2 \cdot \cot C/2$

$$\Rightarrow \quad 4R \sin A/2 \cdot \sin B/2 \cdot \sin C/2 \cdot \frac{\cos B/2}{\sin B/2} \cdot \frac{\cos C/2}{\sin C/2}$$
$$[\text{as, } r = 4R \sin A/2 \cdot \sin B/2 \cdot \sin C/2]$$
$$\Rightarrow \quad 4R \cdot \sin A/2 \cdot \cos B/2 \cdot \cos C/2$$
$$\Rightarrow \quad r_1 = \text{RHS} \quad [\text{as, } r_1 = 4R \sin A/2 \cdot \cos B/2 \cdot \cos C/2]$$
$$\therefore \; r \cot B/2 \cdot \cot C/2 = r_1$$

Example 15. In a right angled triangle, prove that $r + 2R = s$.

Sol. In a right angled triangle, the circum centre lies on the hypotenuse.

$$\Rightarrow \quad R = \frac{a}{2} \qquad \text{...(i) } [\because \angle A = 90°]$$

Also, $r = (s-a)\tan A/2 = (s-a)\tan 45°$

$$r = (s-a) \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get $r = s - 2R$

$$\Rightarrow \quad r + 2R = s.$$

Example 16. The ex-radii r_1, r_2, r_3 of a ΔABC are in HP, show that its sides a, b, c are in AP.

Solution. r_1, r_2, r_3 are in HP.

$$\Rightarrow \quad \frac{2}{r_2} = \frac{1}{r_1} + \frac{1}{r_3}$$

$$\Rightarrow \quad \frac{2(s-b)}{\Delta} = \frac{(s-a)}{\Delta} + \frac{(s-c)}{\Delta}$$

$$\Rightarrow \quad 2s - 2b = 2s - (a + c)$$

$$\Rightarrow \quad 2b = a + c$$

Hence, a, b, c are in AP.

Example 17. If A, B, C are the angles of a triangle, then prove that

$$\cos A + \cos B + \cos C = 1 + \frac{r}{R}.$$

Sol. $\cos A + \cos B + \cos C$

$$= 2\cos\left(\frac{A+B}{2}\right) \cdot \cos\left(\frac{A-B}{2}\right) + \cos C$$

$$= 2\sin\frac{C}{2} \cdot \cos\left(\frac{A-B}{2}\right) + 1 - 2\sin^2\frac{C}{2}$$

$$= 1 + 2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \sin\left(\frac{C}{2}\right)\right]$$

$$= 1 + 2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right]$$

$$\left[\because \frac{C}{2} = 90° - \left(\frac{A+B}{2}\right)\right]$$

$$= 1 + 2\sin\frac{C}{2} \cdot 2\sin\frac{A}{2} \cdot \sin\frac{B}{2}$$

$$= 1 + 4\sin\frac{A}{2} \cdot \sin\frac{B}{2} \cdot \sin\frac{C}{2} = 1 + \frac{r}{R}$$

$$[\text{as, } r = 4R\sin A/2 \cdot \sin B/2 \cdot \sin C/2]$$

$$\Rightarrow \quad \cos A + \cos B + \cos C = 1 + \frac{r}{R}$$

Example 18. Find the ratio of the circum-radius and the inradius of ΔABC, whose sides are in the ratio $4 : 5 : 6$.

Sol. Here, $a = 4k,\ b = 5k,\ c = 6k$

$$\therefore \quad s = \frac{15k}{2} \qquad \ldots(i)$$

$$\therefore \quad \Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{\frac{15k}{2}\left(\frac{15k}{2} - 4k\right)\left(\frac{15k}{2} - 5k\right)\left(\frac{15k}{2} - 6k\right)}$$

$$= \frac{15\sqrt{7}}{4}k^2 \qquad \ldots(ii)$$

and $$R = \frac{abc}{4\Delta} = \frac{4k \cdot 5k \cdot 6k}{4 \cdot \frac{15}{4}\sqrt{7}k^2} \qquad [\text{using Eq. (ii)}]$$

$$\therefore \quad R = \frac{8}{\sqrt{7}}k \qquad \ldots(iii)$$

and $$r = \frac{\Delta}{s} = \frac{15\sqrt{7}}{4}k^2 \cdot \frac{2}{15k} \qquad [\text{using Eqs. (i) and (ii)}]$$

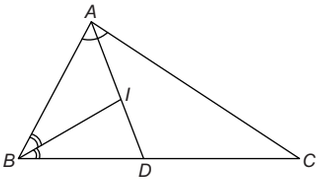
$$r = \frac{\sqrt{7}}{2}k \qquad \ldots(iv)$$

$$\therefore \quad \frac{R}{r} = \frac{8k/\sqrt{7}}{\sqrt{7}k/2} = \frac{16}{7} \qquad [\text{using Eqs. (iii) and (iv)}]$$

$$\Rightarrow \quad R : r = 16 : 7$$

Example 19. Find the ratio of $IA : IB : IC$, where I is the incentre of ΔABC.

Sol. Here, $BD : DC = c : b$



But $BD + DC = a$;

$$\therefore \quad BD = \frac{c}{b+c} \cdot a \qquad \ldots(i)$$

In ΔABD, $$\frac{BD}{\sin A/2} = \frac{AD}{\sin B}$$

$$\therefore \quad AD = \frac{ac}{b+c} \cdot \frac{\sin B}{\sin A/2} = \frac{2\Delta}{b+c} \cdot \operatorname{cosec} A/2 \qquad \ldots(ii)$$

Also, $$\frac{AI}{ID} = \frac{AB}{BD} = \frac{c}{\frac{ac}{b+c}} = \frac{b+c}{a} \qquad [\text{using Eq. (i)}]$$

or $$\frac{ID}{AI} = \frac{a}{b+c}$$

On adding '1', we get

$$\frac{ID}{AI} + 1 = \frac{a}{b+c} + 1 \Rightarrow \frac{ID + AI}{AI} = \frac{a+b+c}{b+c}$$

$$\Rightarrow \quad AI = \frac{b+c}{a+b+c} \cdot AD$$

$$\therefore \quad AI = \frac{b+c}{a+b+c} \cdot \frac{2\Delta}{b+c} \cdot \operatorname{cosec} A/2 = \frac{\Delta}{s} \cdot \operatorname{cosec} A/2$$

Similarly, $$BI = \frac{\Delta}{s} \cdot \operatorname{cosec} B/2$$

$$CI = \frac{\Delta}{s} \operatorname{cosec} C/2$$

$$\Rightarrow \quad IA : IB : IC = \frac{\Delta}{s} \cdot \operatorname{cosec} A/2 : \frac{\Delta}{s} \operatorname{cosec} B/2 : \frac{\Delta}{s} \operatorname{cosec} C/2$$

$$\therefore \quad IA : IB : IC = \operatorname{cosec} A/2 : \operatorname{cosec} B/2 : \operatorname{cosec} C/2$$

Note

Student are advised to remember the above result i.e.
$IA = r \operatorname{cosec} A/2$, $IB = r \operatorname{cosec} B/2$, $IC = r \operatorname{cosec} C/2$.

Example 20. If the sides of a triangle are in GP and the largest angle is twice the smallest angle, then find the relation for r.

Sol. Let the sides of Δ be $a, b = ar, c = ar^2$, where $r > 1$

Here, $c = 2A$ (given)

So, $B = \pi - A - C = \pi - 3A$

$$\therefore \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{1}{\sin A} = \frac{r}{\sin B} = \frac{r^2}{\sin C}$$

$$\Rightarrow \quad \frac{1}{\sin A} = \frac{r}{\sin 3A} = \frac{r^2}{\sin 2A}$$

$$\therefore \quad r^2 = 2\cos A \text{ and } r = \frac{\sin 3A}{\sin A} = 3 - 4\sin^2 A$$

$$r = 4\cos^2 A - 1$$

$$\therefore \quad r = r^4 - 1$$

Thus, the required relation is $r^4 - r - 1 = 0$.

Example 21. The equation $ax^2 + bx + c = 0$, where a, b, c are the sides of a ΔABC, and the equation $x^2 + \sqrt{2}x + 1 = 0$ have a common root. Find measure for $\angle C$.

Sol. Clearly, the roots of $x^2 + \sqrt{2}x + 1 = 0$ are non-real complex.

So, the one root common implies both roots are common.

So, $$\frac{a}{1} = \frac{b}{\sqrt{2}} = \frac{c}{1} = k$$

$$\therefore \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$= \frac{k^2 + 2k^2 - k^2}{2k.\sqrt{2}k} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \angle C = 45^\circ$$

Example 22. If in a ΔABC, the value of $\cot A, \cot B, \cot C$ are in AP show a^2, b^2, c^2 are in A.P.

Sol. Here; $\cot A$, $\cot B$, $\cot C$ are in AP.

$$\Rightarrow \quad \frac{b^2 + c^2 - a^2}{2bc \cdot \frac{a}{2R}}, \frac{c^2 + a^2 - b^2}{2ac \cdot \frac{b}{2R}}, \frac{a^2 + b^2 - c^2}{2ab \cdot \frac{c}{2R}} \text{ are in AP}$$

$$\Rightarrow \quad b^2 + c^2 - a^2, c^2 + a^2 - b^2, a^2 + b^2 - c^2 \text{ are in AP}$$

$$\left[\text{multiplying by } \frac{abc}{R}\right]$$

$$\Rightarrow \quad -2a^2, -2b^2, -2c^2 \text{ are in AP}$$

[subtracting $a^2 + b^2 + c^2$ from each]

$$\Rightarrow \quad a^2, b^2, c^2 \text{ are in AP.}$$

Aliter $2\cot B = \cot A + \cot C$

$$\Rightarrow \quad \frac{2(a^2 + c^2 - b^2)}{2ac \cdot kb} = \frac{b^2 + c^2 - a^2}{2bc \cdot ka} + \frac{a^2 + b^2 - c^2}{2ab \cdot kc}$$

[using sine and cosine law]

$$\Rightarrow \quad 2(a^2 + c^2 - b^2) = b^2 + c^2 - a^2 + a^2 + b^2 - c^2$$

$$\Rightarrow \quad 2(a^2 + c^2 - b^2) = 2b^2$$

$$\Rightarrow \quad a^2 + c^2 - b^2 = b^2 \quad \text{or} \quad a^2 + c^2 = 2b^2$$

i.e. a^2, b^2, c^2 are in AP.

Exercise for Session 3

1. The side of a triangle are 22 cm, 28 cm and 36 cm. So, find the area of the circumscribed circle.
2. If the lengths of the side of a triangle are 3, 4 and 5 units, then find the circum radius R.
3. In an equilateral triangle of side $2\sqrt{3}$ cm. The find circum-radius.
4. If $8R^2 = a^2 + b^2 + c^2$, then prove that the Δ is right angled.
5. In a ΔABC, show that $2R^2 \sin A \sin B \sin C = \Delta$.
6. In a ΔABC, show that $\frac{a\cos A + b\cos B + c\cos C}{a + b + c} = \frac{r}{R}$
7. If the sides of a triangles are 3 : 7 : 8, then find ratio $R : r$.
8. In an equilateral triangle show that the in-radius and the circum-radius are connected by $r = \frac{R}{2}$.
9. In any ΔABC, find $\sin A + \sin B + \sin C$.

10. In any ΔABC, show that $\cos A + \cos B + \cos C = \left(1 + \frac{r}{R}\right)$.

11. If the side be a, b and c, then show that $\frac{r_1 + r}{a} + \frac{r_2 - r}{b} = \frac{c}{r_3}$.

12. Show that $r_2 r_3 + r_3 r_1 + r_1 r_2 = s^2$

13. Show that $(r_1 + r_2)(r_2 + r_3)(r_3 + r_1) = 4Rs^2$

14. If $r_1 = r_2 + r_3 + r$, then show that Δ is right angled.

15. In an equilateral triangle, show that the in-radius, circumradius and one of the ex-radii are in the ratio 1 : 2 : 3.

16. Show that $\left(\frac{1}{r} - \frac{1}{r_1}\right)\left(\frac{1}{r} - \frac{1}{r_2}\right)\left(\frac{1}{r} - \frac{1}{r_3}\right) = \frac{16R}{r^2(\Sigma a)^2}$

17. If r_1, r_2, r_3 in a triangle be in HP, then show that the sides are in AP.

18. In a ΔABC, show that $r_1\, r_2\, r_3 = \Delta^2$.

19. If l_1, l_2, l_3 are respectively the perpendicular from the vertices of a triangle on the opposite side, then show that $l_1\, l_2\, l_3 = \frac{a^2 b^2 c^2}{8R}$.

20. If the angle of a triangle are in the ratio 1 : 2 : 3, then show that the sides opposite to the respective angle are in the ratio $1 : \sqrt{3} : 2$.

21. Show that, $4\, Rr \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = S$

22. If $(a - b)(s - c) = (b - c)(s - a)$, then show that r_1, r_2, r_3 are in HP.

23. To show that $\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r^2} = \frac{\Sigma a^2}{S^2}$

24. Show that $(r_1 - r)(r_2 - r)(r_3 - r) = 4Rr^2$

25. Show that $\left(\frac{1}{r_1} + \frac{1}{r_2}\right)\left(\frac{1}{r_2} + \frac{1}{r_3}\right)\left(\frac{1}{r_3} + \frac{1}{r_1}\right) = \frac{64R^3}{a^2 b^2 c^2}$

26. If the sides be a, b and c, then find the value of $(r + r_1)\tan\frac{B - C}{2} + (r + r_2)\tan\frac{C - A}{2} + (r + r_3)\tan\frac{A - B}{2}$

27. If the sides be a, b, c, then find value of $\frac{b - c}{r_1} + \frac{c - a}{r_2} + \frac{a - b}{r_3}$.

28. If the sides be a, b, c, then find $(r_1 - r)(r_2 + r_3)$.

29. If a, b, c are in AP, then show that r_1, r_2, r_3 are in HP.

30. Show that $\frac{r_1}{bc} + \frac{r_2}{ca} + \frac{r_3}{ab} = 2R - r$

31. Show that $r_1 + r_2 = c\cot\left(\frac{C}{2}\right)$

32. Show that $Rr(\sin A + \sin B + \sin C) = \Delta$

33. Show that $16R^2 r\, r_1\, r_2\, r_3 = a^2 b^2 c^2$

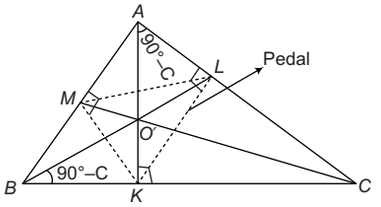
34. If $\frac{r}{r_1} = \frac{r_2}{r_3}$, then show that $c = 90^\circ$.

Session 4

Orthocentre and Its Distance from the Angular Points of a Triangle and Pedal Triangle and Centroid of Triangle

Orthocentre and Its Distance from the Angular Points of a Triangle and Pedal Triangle

Let ABC be any triangle, and let AK, BL and CM be the perpendicular from A, B and C upon the opposite sides of the triangle. These three perpendiculars meet at a point O' which is called the orthocentre of the triangle ABC. **The triangle *KLM*, formed by joining the feet of these perpendiculars is called the pedal triangle of *ABC*.**



In $\Delta O'BK$, $\tan(90^\circ - C) = \dfrac{O'K}{KB}$

$$\Rightarrow \quad O'K = KB.\tan(90^\circ - C) = KB\cot C$$

$$= AB\cos B\cot C \left[\because \text{from } \Delta ABK, \cos B = \frac{BK}{AB}\right]$$

$$= c\cdot\cos B\cdot\frac{\cos C}{\sin C}$$

$$\Rightarrow O'K = 2R\cos B\cos C \left[\because R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C}\right]$$

Similarly,

$$O'L = 2R\cos A\cos C \text{ and } O'M = 2R\cos A\cos B$$

In $\Delta AO'L$, $\cos(90^\circ - C) = \dfrac{AL}{AO'}$

$$\Rightarrow \quad AO' = AL\cdot\operatorname{cosec} C \Rightarrow AO' = c\cos A\cdot\operatorname{cosec} C$$

$$\left[\because \text{from } \Delta ALB, \cos A = \frac{AL}{AB}\right]$$

$$\Rightarrow \quad AO' = c\cdot\cos A\cdot\frac{1}{\sin C}$$

$$\Rightarrow \quad AO' = 2R\cos A$$

Similarly, $BO' = 2R\cos B$ and $CO' = 2R\cos C$

Thus, the distance of the orthocentre of the triangle from the angular points are,

$$AO' = 2R\cos A$$
$$BO' = 2R\cos B$$
$$CO' = 2R\cos C,$$

and its distance from the sides are,

$$O'K = 2R\cos B\cos C.$$
$$O'L = 2R\cos C\cos A.$$
$$O'M = 2R\cos A\cos B.$$

Some Relations between Orthocentre, Incentre, Escribed Circles, Centroid, Circum-centre and Pedal Triangle

(i) Orthocentre of the triangle is the incentre of the pedal triangle.

(ii) If I_1, I_2 and I_3 be the centres of escribed circles which are opposite to A, B and C respectively and I is the centre of incircle then ΔABC is the pedal triangle of the $\Delta I_1I_2I_3$ and I is the orthocentre of the $\Delta I_1I_2I_3$.

(iii) The centroid of the triangle lies on the line joining the circumcentre to the orthocentre and divides it in the ratio 1 : 2.

(iv) Circle circumscribing the pedal triangle of a given triangle bisects the sides of the given triangle and also the lines joining the vertices of the given triangle to the orthocentre of the given triangle. This circle is known as **nine point circle.**

(v) Circum-centre of the pedal triangle of a given triangle bisects the line joining the circumcentre of the triangle to the **orthocentre.**

Example 23. ***In ΔABC, a, b and c represents the sides, thus find the sides and angles of the pedal triangle.***

Sol. Let ΔABC be any triangle and let D, E, F be the feet of perpendicular from the angular points on the opposite

sides of the ΔABC, then the ΔDEF is known as **Pedal triangle of *ABC*.**

Here, $\angle HDC$ and $\angle HEC$ are 90° each.Thus, points H, D, C and E are concyclic,

$\therefore \quad \angle HDE = \angle HCE = 90° - A$ ...(i)

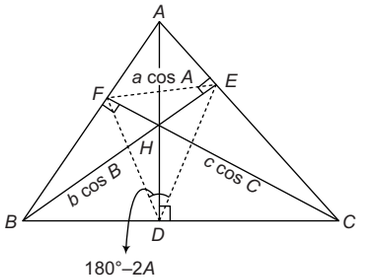
Similarly H, D, B, F are concyclic.

$\therefore \quad \angle HDF = \angle HBF = 90° - A$ ...(ii)

Hence, $\angle FDE = 180° - 2A$ [using Eqs. (i) and (ii)]

So, $\angle DEF = 180° - 2B$

and $\angle EFD = 180° - 2C$



Thus, angles of pedal triangle FDE are

$$180° - 2A, 180° - 2B, 180° - 2C.$$

Again in ΔBFD,

$$\angle FDB = 90° - \angle HDF = 90° - (90° - A) = A$$

$$\therefore \quad \frac{FD}{\sin B} = \frac{BF}{\sin A}$$

$$\therefore \quad FD = \frac{\sin B}{\sin A} \cdot BF$$

$$= \frac{\sin B}{\sin A} \cdot BC \cdot \cos B \quad \left[\because \text{In } \Delta BFC, \frac{BF}{BC} = \cos B\right]$$

$$= \frac{a \sin B \cdot \cos B}{\sin A} = 2R \sin B \cdot \cos B$$

$$FD = b \cos B$$

Similarly, $EF = a \cos A$ and $DE = c \cos C$

$\Rightarrow$ Sides of pedal triangle :

$a \cos A, b \cos B$ and $c \cos C$ or $R \sin 2A, R \sin 2B$ and $R \sin 2C$

Note

If given ΔABC is obtuse, then angles are represented by $2A, 2B, 2C - 180°$ and the sides are $a \cos A, b \cos B, -c \cos C$.

Example 24. Find the area, circum-radius and in-radius of the pedal triangle.

Sol. We know, area of $\Delta = \frac{1}{2}$ (product of the sides) $\times$ (sine of the included angle)

$$= \frac{1}{2}(R \sin 2B)(R \sin 2C) \cdot \sin(180° - 2A)$$

$$= \frac{1}{2} R^2 \cdot \sin 2A \cdot \sin 2B \cdot \sin 2C$$

The circum-radius $= \frac{EF}{2 \sin FDE} = \frac{R \sin 2A}{2 \sin(180° - 2A)} = \frac{R}{2}$

The in-radius of the pedal ΔDEF

$$= \frac{\text{ar } (\Delta DEF)}{\text{Semi - perimeter of } \Delta DEF}$$

$$= \frac{1}{2} \cdot \frac{R^2 \sin 2A \cdot \sin 2B \cdot \sin 2C}{2R \sin A \cdot \sin B \cdot \sin C}$$

$$= 2R \cos A . \cos B . \cos C$$

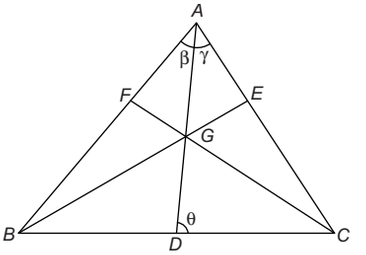
Thus, area of pedal $\Delta = \frac{1}{2} R^2 \sin 2A \cdot \sin 2B \sin 2C$

Circum-radius $= \frac{R}{2}$

In-radius $= 2R \cos A \cdot \cos B \cdot \cos C$

Centroid of Triangle

In ΔABC, the mid-points of the sides BC, CA and AB are D, E and F, respectively. The lines, AD, BE and CF are called medians of the triangle ABC, the points of concurrency of three medians is called centroid. Generally, it is represented by G.



By analytical geometry :

$$AG = \frac{2}{3} AD \; ; \; BG = \frac{2}{3} BE \text{ and } CG = \frac{2}{3} CF$$

Length of Medians and the Angles that the Median Makes with Sides

In above figure, $AD^2 = AC^2 + CD^2 - 2AC \cdot CD \cdot \cos C$

$$\therefore \quad AD^2 = b^2 + \frac{a^2}{4} - ab \cos C$$

$$\therefore \quad AD^2 = b^2 + \frac{a^2}{4} - ab \cdot \left(\frac{b^2 + a^2 - c^2}{2ab}\right)$$

$$AD^2 = \frac{2b^2 + 2c^2 - a^2}{4}$$

$$\Rightarrow \quad \boldsymbol{AD = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}}$$

or $$AD = \frac{1}{2}\sqrt{b^2 + c^2 + 2bc \cos A}$$

Similarly, $$\boldsymbol{BE = \frac{1}{2}\sqrt{2c^2 + 2a^2 - b^2}}$$

and $CF = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}$

Let $\angle BAD = \beta$ and $\angle CAD = \gamma$, we have

$$\frac{\sin\gamma}{\sin C} = \frac{DC}{AD} = \frac{a}{2 \cdot \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}}$$

$$\therefore \quad \sin\gamma = \frac{a\sin C}{\sqrt{2b^2 + 2c^2 - a^2}}$$

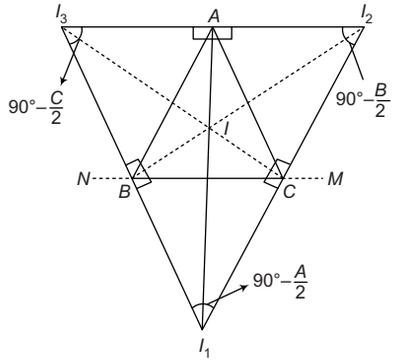
Similarly, $\sin\beta = \frac{a\sin B}{\sqrt{2b^2 + 2c^2 - a^2}}$

again $\frac{\sin\theta}{\sin C} = \frac{b}{\frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}}$

$$\therefore \quad \sin\theta = \frac{2b\sin C}{\sqrt{2b^2 + 2c^2 - a^2}}$$

Ex-central Triangle

Let ABC be a triangle and I be the centre of incircle. Let I_1, I_2 and I_3 be the centres of the escribed circles which are opposite to A, B, C respectively, then I_1, I_2, I_3 is called the Ex-central triangle of ΔABC.



By geometry IC bisects the $\angle ACB$ and I_2C bisects the $\angle ACM$.

$$\therefore \quad \angle ICI_2 = \angle ACI + \angle ACI_2$$

$$= \frac{1}{2}\angle ACB + \frac{1}{2}\angle ACM$$

$$= \frac{1}{2}(180°) = 90°$$

Similarly, $\angle ICI_1 = 90°$

Hence, I_1I_2 is perpendicular to IC.

Similarly, AI is perpendicular to I_2I_3 and BI is perpendicular to I_1I_3.

Hence, $I_1I_2I_3$ is a triangle, thus the triangle ABC is the pedal triangle of its ex-central triangle $I_1I_2I_3$.

Sides and Angles of the Ex-central Triangle

In above figure,

$$\angle BI_1C = \angle BI_1I + \angle CI_1I$$

$$= \angle BCI + \angle CBI$$

$$= \frac{C}{2} + \frac{B}{2} = 90° - \frac{A}{2}$$

or $\angle I_2I_1I_3 = 90° - \frac{A}{2}$

Thus the angles are,

$$90° - \frac{A}{2}, 90° - \frac{B}{2}, 90° - \frac{C}{2}$$

Again in right angled ΔI_1I_3C,

$$\frac{I_1C}{I_1I_3} = \cos\left(90° - \frac{A}{2}\right)$$

$$I_1C = I_1I_3 \sin\frac{A}{2} \qquad \ldots(i)$$

In ΔBI_1C, $\frac{I_1C}{\sin\angle I_1BC} = \frac{BC}{\sin\angle BI_1C}$

$$\Rightarrow \quad \frac{I_1I_3 \sin A/2}{\sin\left(\frac{180° - B}{2}\right)} = \frac{a}{\sin(90° - A/2)}$$

$$\Rightarrow \quad I_1I_3 = \frac{a\cos(B/2)}{\sin A/2 \cdot \cos A/2} = \frac{2R \cdot \sin A \cdot \cos(B/2)}{\frac{1}{2}\left[2\sin\left(\frac{A}{2}\right) \cdot \cos\left(\frac{A}{2}\right)\right]}$$

$$\therefore \quad \boldsymbol{I_1I_3 = 4R\cos\frac{B}{2}}$$

Similarly, $\boldsymbol{I_1I_2 = 4R\cos\frac{C}{2}}$

$$\boldsymbol{I_2I_3 = 4R\cos\frac{A}{2}}$$

Area and Circum-radius of the Ex-central Triangle

$\because$ Area of triangle $= \frac{1}{2}$ (Product of two sides) $\times$ (Sine of include angles)

$$= \frac{1}{2}(4R\cos B/2) \cdot (4R\cos C/2) \times \sin(90° - A/2)$$

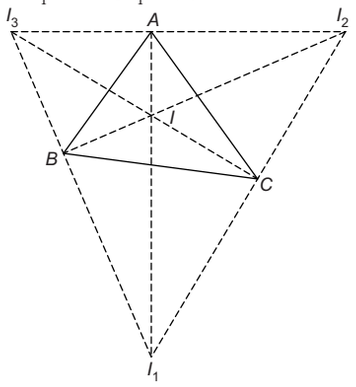
$$\Delta = 8R^2 \cos A/2 \cdot \cos B/2 \cdot \cos C/2$$

The circum-radius $\frac{I_2I_3}{2\sin I_2I_1I_3} = \frac{4R\cos A/2}{2\sin(90° - A/2)} = 2R.$

$\Rightarrow$ **Circum-radius $= 2R$.**

Distance between the In-centre and Ex-centres

Here, $\angle IBI_1 = \angle ICI_1 = 90°$



$\therefore$ II_1 is the diameter of the circumcircle of ΔBCI_1

$$\therefore \quad II_1 = \frac{BC}{\sin BI_1C} = \frac{a}{\sin(90° - A/2)} = \frac{2R\sin A}{\cos A/2}$$

$$II_1 = 4R \cdot \sin\left(\frac{A}{2}\right)$$

Similarly, $$II_2 = 4R\sin\left(\frac{B}{2}\right)$$

and $$II_3 = 4R\sin\left(\frac{C}{2}\right)$$

Further, $$\angle BI_1I = \angle BCI = \frac{C}{2}$$

$$\therefore \quad BI = II_1 \sin\frac{C}{2}$$

$$\Rightarrow \quad \boldsymbol{BI = 4R\sin A/2 \cdot \sin C/2}$$

Similarly $$\boldsymbol{AI = 4R\sin B/2 \cdot \sin C/2}$$

$$\boldsymbol{CI = 4R\sin A/2 \cdot \sin B/2}$$

Distance between an Ex-centre and Circum Centre

Let O be the circum centre and I be the in-centre, then AI produced passes through the ex-centre I_1.

Let AI, meet the circum-circle in D, join CI, BI, CD, BD, CI_1, BI_1.

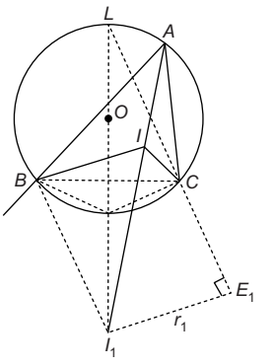
Draw I_1E_1 perpendicular to AC. Produce I_1O to meet the circle in L join CL.

The angle $\angle BI_1$ and $\angle CI_1$ are right angles, hence the circle on I_1I as diameter passes through B and C.

The chord BD and CD of the circum-centre subtend equal angles at A and are therefore, equal.

$\therefore$ $DB = DC = DI$

Hence D is the centre of the circle IBI_1C.



$$\therefore \quad DI_1 = DC = 4R\sin\frac{A}{2}$$

Now, $OI_1^2 - R^2 =$ Square of tangent from I_1. $= I_1D \cdot I_1A$

$$= 2R\sin\frac{A}{2} \cdot r_1\operatorname{cosec}\frac{A}{2}$$

$$\therefore \quad OI_1^2 = R^2 + 2Rr_1$$

or

$$\boldsymbol{OI_1 = R\sqrt{1 + 8\sin A/2 \cdot \cos B/2 \cdot \cos C/2}}$$

Similarly, $$OI_2 = \sqrt{R^2 + 2Rr_2}$$

$$\boldsymbol{OI_2 = R\sqrt{1 + 8\cos A/2 \cdot \sin B/2 \cdot \cos C/2}}$$

and $$OI_3 = \sqrt{R^2 + 2Rr_3}$$

$$\boldsymbol{OI_3 = R\sqrt{1 + 8\cos A/2 \cdot \cos B/2 \cdot \sin C/2}}$$

Example 25. Show that $II_1 \cdot II_2 \cdot II_3 = 16R^2r$.

Sol. Since, $II_1 = 4R\sin A/2$

$$II_2 = 4R\sin B/2$$

and $$II_3 = 4R\sin C/2$$

$$\therefore \quad II_1 \cdot II_2 \cdot II_3 = 64R^3 \cdot \sin\frac{A}{2} \cdot \sin\frac{B}{2} \cdot \sin\frac{C}{2}$$

Since $$r = 4R\sin A/2 \cdot \sin B/2 \cdot \sin C/2$$

$$\therefore \quad II_1 \cdot II_2 \cdot II_3 = 64R^3 \cdot \frac{r}{4R} = 16R^2r$$

Example 26. Prove that

$$\frac{II_1 \cdot I_2I_3}{\sin A} = \frac{II_2 \cdot I_3I_1}{\sin B}$$

Sol. LHS $$\frac{II_1 \cdot I_2I_3}{\sin A} = \frac{4R\sin A/2 \cdot 4R\cos A/2}{\sin A}$$

$$= \frac{16R^2\sin A/2 \cdot \cos A/2}{2\sin A/2 \cdot \cos A/2} = 8R^2$$

RHS $$\frac{II_2 \cdot I_3I_1}{\sin B} = \frac{4R\sin B/2 \cdot 4R\cos B/2}{\sin B}$$

$$= \frac{16R^2\sin B/2 \cdot \cos B/2}{2\sin B/2 \cdot \cos B/2}$$

$$= 8R^2$$

$\Rightarrow$ LHS = RHS

Example 27. If g, h, k denotes the side of a pedal triangle, then prove that

$$\frac{g}{a^2} + \frac{h}{b^2} + \frac{k}{c^2} = \frac{a^2 + b^2 + c^2}{2abc}$$

Sol. We have,

$$g = a\cos A, h = b\cos B, k = c\cos C \quad [\because \text{sides of pedal } \Delta]$$

$$\therefore \quad \frac{g}{a^2}+\frac{h}{b^2}+\frac{k}{c^2} = \frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}$$

$$= \frac{b^2+c^2-a^2}{2abc}+\frac{a^2+c^2-b^2}{2abc}+\frac{a^2+b^2-c^2}{2abc}$$

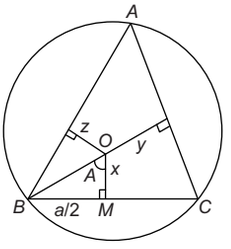
$$= \frac{a^2+b^2+c^2}{2abc}$$

$$\therefore \quad \frac{g}{a^2}+\frac{h}{b^2}+\frac{k}{c^2} = \frac{a^2+b^2+c^2}{2abc}$$

Example 28. If x, y, z are perpendicular from the circum centre of the sides of the ΔABC respectively.

Prove that $\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z} = \dfrac{abc}{4xyz}$

Sol. In ΔOBM, $\tan A = \dfrac{\frac{a}{2}}{x} = \dfrac{a}{2x}$



Similarly, $\tan B = \dfrac{b}{2y}$

and $\tan C = \dfrac{c}{2z}$

Since, $A + B = \pi - C$

$\Rightarrow \quad \tan(A+B) = \tan(\pi - C)$

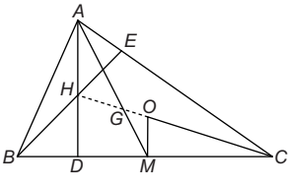
$\Rightarrow \quad \tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C$

$\Rightarrow \quad \dfrac{a}{2x}+\dfrac{b}{2y}+\dfrac{c}{2z} = \dfrac{abc}{8xyz}$

$\Rightarrow \quad \dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z} = \dfrac{abc}{4xyz}$

Example 29. If O, H and G represents circum centre, orthocentre and centroid respectively, then show $HG : GO = 2 : 1$. We have,

Sol. We have, two Δ's AGH and GMO are equiangular.



Also, $AH = 2R\cos A$

$OM = R\cos A$

$\therefore \quad \dfrac{AH}{OM} = \dfrac{2R\cos A}{R\cos A} = \dfrac{2}{1}$

Hence by similar Δ's,

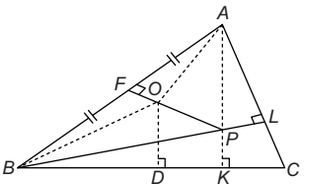
$$\frac{AH}{OM} = \frac{AG}{GM} = \frac{HG}{GO} = 2$$

$\therefore \Rightarrow G$ divides HO in the ratio of 2 : 1

or $HG : GO = 2 : 1$

Example 30. *Prove that the distance between the circumcentre and the orthocentre of a triangle* ABC *is* $R\sqrt{1-8\cos A\cos B\cos C}$.

Sol. Let O and P be the circumcentre and the orthocentre, respectively.



If OF is the perpendicular to AB, we have

$$\angle OAF = 90^\circ - \angle AOF = 90^\circ - C$$

Also, $\angle OAP = A - \angle OAF - \angle PAL$

$$= A - 2(90^\circ - C)$$

$$= A + 2C - 180^\circ$$

$$= A + 2C - (A + B + C)$$

$$= C - B$$

Also, $OA = R$ and $PA = 2R\cos A$

Now in ΔAOP, $OP^2 = OA^2 + PA^2 - 2OA.PA.\cos\angle OAP$

$$= R^2 + 4R^2\cos^2 A - 4R^2\cos A\cos(C-B)$$

$$= R^2 + 4R^2\cos A\{\cos A - \cos(C-B)\}$$

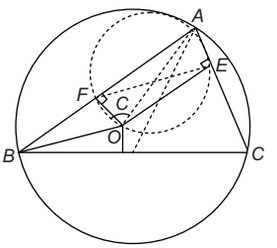
$$= R^2 - 4R^2\cos A\{\cos(B+C) + \cos(C-B)\}$$

$$= R^2 - 8R^2\cos A\cos B\cos C$$

Hence, $OP = R\sqrt{1-8\cos A\cos B\cos C}$

Example 31. *Find the distance between the circumcentre and the incentre of the* ΔABC.

Sol. Let O be circumcentre and OF be the perpendicular to AB. Let I be the incentre and IE be the perpendicular to AC.



Then, $\angle OAF = 90^\circ - C$

$\Rightarrow \quad \angle OAI = \angle IAF - \angle OAF$

$$= \frac{A}{2} - (90^\circ - C)$$

$$= \frac{A}{2} + C - \left(\frac{A+B+C}{2}\right) = \frac{C-B}{2}$$

Also, $AI = \dfrac{IE}{\sin A/2} = \dfrac{r}{\sin A/2}$

$$= 4R\sin B/2\sin C/2$$

Hence,

$$OI^2 = OA^2 + AI^2 - 2OA \cdot AI \cdot \cos \angle OAI$$
$$= R^2 + 16R^2 \sin^2 B/2 \cdot \sin^2 C/2 - 8R^2 \sin B/2 \sin C/2 \cos\left(\frac{C-B}{2}\right)$$

$$\Rightarrow \quad \frac{OI^2}{R^2} = 1 + 16\sin^2\frac{B}{2}\cdot\sin^2\frac{C}{2} - 8\sin\frac{B}{2}\sin\frac{C}{2}\left(\cos\frac{B}{2}\cos\frac{C}{2} + \sin\frac{B}{2}\cdot\sin\frac{C}{2}\right)$$
$$= 1 - 8\sin\frac{B}{2}\cdot\sin\frac{C}{2}\left(\cos\frac{B}{2}\cdot\cos\frac{C}{2} - \sin\frac{B}{2}\cdot\sin\frac{C}{2}\right)$$
$$= 1 - 8\sin\frac{B}{2}\cdot\sin\frac{C}{2}\cdot\cos\left(\frac{B+C}{2}\right)$$
$$= 1 - 8\sin\frac{B}{2}\cdot\sin\frac{C}{2}\cdot\sin\frac{A}{2}$$
$$\frac{OI}{R} = \sqrt{1 - 8\sin B/2 \cdot \sin C/2 \cdot \sin A/2}$$

or $$OI = R\sqrt{1 - \frac{2r}{R}}$$
$$OI = \sqrt{R^2 - 2Rr}.$$

Exercise for Session 4

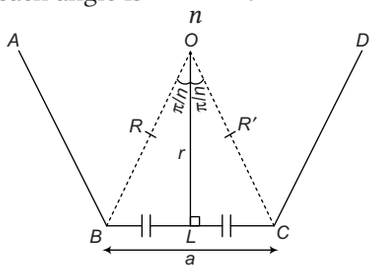
1. If H is the orthocentre of the ΔABC, then find AH.
2. A circle touches two of the smaller sides of a ΔABC $(a < b < c)$ and has its centre on the greatest side. Then, find the radius of the circles.
3. If the sides be a, b, c, then show that $a\cos A + b\cos B + c\cos C = 4R\sin A\sin B\sin C$
4. If the altitudes of a triangle be 3, 4, 6, then find its in-radius.
5. In a ΔABC, if $a = 3, b = 4, c = 5$, then find the distance between its incentre and circumcentre.
6. If p_1, p_2, p_3 are respectively the perpendicular from the vertices of a triangle to the opposite sides, then find the value of $p_1p_2p_3$.
7. Show that the distance between the circumcentre and the incentre of the triangle ABC is $\sqrt{R^2 - 2Rr}$.
8. Show that the distance between the circumcentre and the orthocentre of a triangle ABC is $R\sqrt{1 - 8\cos A\cos B\cos C}$.
9. If in a ΔABC, AD, BE and CF are the altitudes and R is the circumradius, then find the radius of the DEF.
10. If I, I_1, I_2 and I_3 be respectively the centre of the in-circle and the three escribed circles of a ΔABC, then find I_2I_3.

Session 5

Regular Polygons and Radii of the Inscribed and Circumscribing Circle a Regular Polygon

A regular polygon is a polygon which has all its sides as well as all its angle equal.

If the polygon has 'n' sides, Sum of the internal angles is $(n-2)\pi$ and each angle is $\frac{(n-2)\pi}{n}$.



Let AB, BC and CD be three consecutive sides of the regular polygon and n be the number of its sides. Let O be the point of intersection of the bisector of the angles $\angle ABC$ and $\angle BCD$.

The point O is both the incentre and circumcentre of polygon and so $BL = LC$. Hence we have,

$OB = OC = R$, the radius of the circumcircle and $OL = r$, the radius of the incircle.

It can be seen,

$$\boldsymbol{R = \frac{a}{2}\operatorname{cosec}\left(\frac{\pi}{n}\right)} \text{ and } \boldsymbol{r = \frac{a}{2}\cot\left(\frac{\pi}{n}\right)}$$

where 'a' is length of a side of the polygon.

The area of the polygon $= n$ area of ΔOBC

$$= n(OL)(BL) = n \cdot \frac{a}{2} \cdot \cot\left(\frac{\pi}{n}\right) \cdot \frac{a}{2}$$

$$= \boldsymbol{\frac{1}{4}na^2 \cdot \cot\left(\frac{\pi}{n}\right)}$$

Also, the area of the polygon.

$$= n(OL)(BL) = n(OL)(OL\tan\angle BOL)$$

$$= \boldsymbol{nr^2 \tan\left(\frac{\pi}{n}\right)}$$

Again, the area $= n\,(OL)(BL)$.

$$= n\,OB \cdot \cos\frac{\pi}{n} \cdot OB \cdot \sin\frac{\pi}{n}$$

$$= nR^2 \cdot \cos\frac{\pi}{n} \cdot \sin\frac{\pi}{n}$$

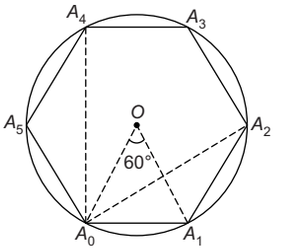
$$= \boldsymbol{\frac{n}{2}R^2 \sin\frac{2\pi}{n}}$$

Example 32. If A_0, A_1, A_2, A_3, A_4 and A_5 be the consecutive vertices of a regular hexagon inscribed in a unit circle. Then, find the product of length of A_0A_1, A_0A_2 and A_0A_4.

Sol. We know, in hexagon central angle is $\frac{360^\circ}{6} = 60^\circ$ and each

$$\text{angle} = \frac{(2n-4)\pi}{2n}$$

$$= \frac{(6-2)\times 180^\circ}{6} = 120^\circ$$



As the unit circle,

$\therefore$ radius $OA_0 = 1 = r$

In $\Delta A_0A_1A_2$,

$$\Rightarrow \quad \cos 120^\circ = \frac{A_0A_1^2 + A_1A_2^2 - A_0A_2^2}{2A_0A_1 \cdot A_1A_2}$$

$$= \frac{1+1-A_0A_2^2}{2\cdot 1\cdot 1}$$

$$\Rightarrow \quad A_0A_2 = \sqrt{3} \qquad \ldots(i)$$

Similarly in $\Delta A_0A_5A_4$, we have

$$A_0A_4 = \sqrt{3} \qquad \ldots(ii)$$

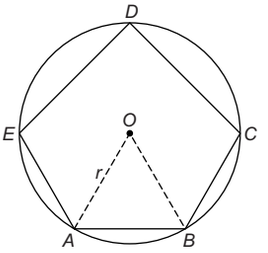
Thus, the value of,

$$(A_0A_1)\cdot(A_0A_2)\cdot(A_0A_4) = 1\cdot\sqrt{3}\cdot\sqrt{3}$$

$$= 3 \text{ square units}$$

Example 33. ***If the area of circle is A_1 and area of regular pentagon inscribed in the circle is A_2, then find the ratio of area of two.***

Sol. In ΔOAB, $OA = OB = r$ and $\angle AOB = \dfrac{360^\circ}{5} = 72^\circ$



$\therefore$ Area of $\Delta AOB = \dfrac{1}{2} \cdot r.r.\sin 72^\circ$

$\therefore$ Area of $\Delta AOB = \dfrac{1}{2} r^2 \cos 18^\circ$ ...(i)

Area of pentagon = 5 (area of ΔAOB)

$\Rightarrow$ $A_2 = 5\left\{\dfrac{1}{2} r^2 \cos 18^\circ\right\}$ ...(ii)

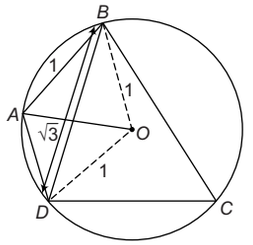
Also we know, Area of circle $= \pi r^2$

$\Rightarrow$ $A_1 = \pi r^2$ ...(iii)

Thus, $\dfrac{A_1}{A_2} = \dfrac{\pi r^2}{\frac{5}{2} r^2 \cos 18^\circ} = \dfrac{2\pi}{5} \sec\left(\dfrac{\pi}{10}\right)$

Example 34. If the area of cyclic quadrilateral $ABCD$ is $\left(\dfrac{3\sqrt{3}}{4}\right)$. The radius of the circle circumscribing it is 1. If $AB = 1$, $BD = \sqrt{3}$, then evaluate $BC \cdot CD$.

Sol. Here, In ΔBOD



$\Rightarrow$ $\angle BOD = 2C$

$\therefore$ $\cos 2C = \dfrac{1^2 + 1^2 - (\sqrt{3})^2}{2 \cdot 1 \cdot 1}$

or $\cos 2C = -\dfrac{1}{2}$

$\therefore$ $\angle C = 60^\circ$ ...(i)

Also, $\angle A + \angle C = 180^\circ$

[since $ABCD$ is cyclic quadrilateral]

$\Rightarrow$ $\angle A = 120^\circ$ ...(ii)

$\therefore$ In ΔABD,

$$\cos 120^\circ = \frac{1^2 + AD^2 - (\sqrt{3})^2}{2 \cdot AD \cdot 1}$$

$\Rightarrow$ $-\dfrac{1}{2} = \dfrac{AD^2 - 2}{2AD}$

$AD^2 + AD - 2 = 0$ or $AD = 1$ ...(iii)

Now, ar $(ABCD)$ = ar(ΔABD) + ar(ΔBCD)

$\Rightarrow$ $\dfrac{3\sqrt{3}}{4} = \dfrac{1}{2} \cdot 1 \cdot 1 \cdot \sin 120^\circ + \dfrac{1}{2} \cdot BC \cdot DC \cdot \sin 60^\circ$

$\Rightarrow$ $\dfrac{3\sqrt{3}}{4} = \dfrac{\sqrt{3}}{4} + \dfrac{1}{2} \cdot \dfrac{\sqrt{3}}{2} BC \cdot CD$

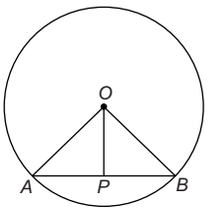
$\Rightarrow$ $\dfrac{2\sqrt{3}}{4} = \dfrac{\sqrt{3}}{4} BC \cdot CD$

or $BC \cdot CD = 2$

Example 35. A regular pentagon and a regular decagon have the same perimeter, prove that their areas are as $2 : \sqrt{5}$.

Sol. Let AB be one of the n sides of a regular polygon.

If $AB = a$,



area of polygon $= \dfrac{n}{2} \cdot AB \cdot OD$

$= \dfrac{na^2}{4} \cot \dfrac{\pi}{n}$ ...(i)

Let the perimeter of the pentagon and decagon be $10x$.

Then, each side of the pentagon is $2x$ and its area is

$5x^2 \cot \dfrac{\pi}{5}$ ...(ii)

[using Eq. (i) where $n = 5$ and $a = 2x$] again, each side of the decagon is x and its area is

$\dfrac{5}{2} x^2 \cot \dfrac{\pi}{10}$ ...(iii)

[using Eq. (i) where $n = 10$ and $a = x$]

$\dfrac{\text{Area of petagon}}{\text{Area of decagon}} = \dfrac{2\cot 36^\circ}{\cot 18^\circ} = \dfrac{2\cos 36^\circ \cdot \sin 18^\circ}{\sin 36^\circ \cdot \cos 18^\circ}$

$= \dfrac{2\cos 36^\circ \cdot \sin 18^\circ}{2\sin 18^\circ \cdot \cos 18^\circ \cdot \cos 18^\circ} = \dfrac{\cos 36^\circ}{\cos^2 18^\circ}$

$= \dfrac{2\cos 36^\circ}{2\cos^2 18^\circ} = \dfrac{2\cos 36^\circ}{1 + \cos 36^\circ}$

$= \dfrac{2(\sqrt{5} + 1)}{4\left\{1 + \dfrac{\sqrt{5} + 1}{4}\right\}}$

$= \dfrac{2(\sqrt{5} + 1)}{5 + \sqrt{5}} = \dfrac{2}{\sqrt{5}}$

Exercise for Session 5

1. Find the sum of the radii of the circles, which are respectively inscribed and circumscribed about a regular polygon of n sides.
2. Find the radius of the circumscribing circle of a regular polygon of n sides each of length is a.
3. If A, A_1, A_2, A_3 be the area of the in-circle and ex-circles, the show that $\frac{1}{\sqrt{A_1}}+\frac{1}{\sqrt{A_2}}+\frac{1}{\sqrt{A_3}}=\frac{1}{\sqrt{A}}$.
4. A regular polygon of nine sides, each of length 2, is inscribed in a circle, then find the radius of the circle.
5. Show that the area of the circle and the regular polygon of n-sides and of equal perimeter are in the ratio of $\tan\left(\frac{\pi}{n}\right):\frac{\pi}{n}$.
6. Let $A_1, A_2, A_3, \ldots, A_n$ be the vertices of an n-sided regular polygon such that $\frac{1}{A_1A_2}=\frac{1}{A_1A_3}+\frac{1}{A_1A_4}$. Find the value of n. Prove or disprove the converse of this result.
7. Let I_n is the area of n-sided regular polygon inscribed in a circle of unit radius and O_n be the area of the polygon circumscribing the given circle. Then, prove that $I_n=\frac{O_n}{2}\left(1+\sqrt{1-\left(\frac{2I_n}{n}\right)^2}\right)$

Session 6

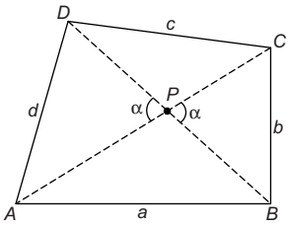
Quadrilaterals and Cyclic Quadrilaterals

Area of Quadrilateral

$ABCD$ is any quadrilateral where $AB=a$, $BC=b$, $CD=c$, $AD=d$ and $\angle DPA=\alpha$.

Let s denotes the area of quadrilateral, then

area of ΔDAC = area of ΔAPD + area of ΔDPC



$$=\frac{1}{2}DP\cdot AP\cdot\sin\alpha+\frac{1}{2}\cdot DP\cdot PC\cdot\sin(\pi-\alpha)$$

$$=\frac{1}{2}DP.(AP+PC)\sin\alpha$$

$$\text{Area of } \Delta DAC=\frac{1}{2}DP\cdot AC\cdot\sin\alpha \quad \ldots(i)$$

Similarly,

$$\text{area of } \Delta ABC=\frac{1}{2}BP\cdot AC\cdot\sin\alpha \quad \ldots(ii)$$

$\therefore$ $s=$ area of ΔDAC + area of ΔABC

$$=\frac{1}{2}DP\cdot AC\cdot\sin\alpha+\frac{1}{2}BP\cdot AC\cdot\sin\alpha$$

[using Eqs. (i) and (ii)]

$$=\frac{1}{2}(DP+BP)\cdot AC\cdot\sin\alpha$$

$$\Rightarrow\quad s=\frac{1}{2}BD\cdot AC\cdot\sin\alpha$$

$\therefore$ **Area of quadrilateral** $=\frac{1}{2}$ **(product of the diagonals) $\times$ (sine of included angle).**

Again,

We can express the area of Δ in terms of sides and the sum of two opposite angles :

In ΔABD, $BD^2 = a^2 + d^2 - 2ad\cos A$ ...(i)

In ΔBCD, $BD^2 = b^2 + c^2 - 2bc\cos C$ ...(ii)

From Eqs. (i) and (ii),

$$a^2 + d^2 - 2ad\cos A = b^2 + c^2 - 2bc\cos C$$

$$\therefore \quad a^2 + d^2 - b^2 - c^2 = 2ad\cos A - 2bc\cos C \quad ...(iii)$$

Also, $s = \Delta BAD + \Delta BCD$

$$= \frac{1}{2}ad\sin A + \frac{1}{2}bc\sin C$$

$$\Rightarrow \quad 4s = 2ad\sin A + 2bc\sin C \quad ...(iv)$$

On squaring and adding Eqs. (iii) and (iv) both sides, we get

$$16s^2 + (a^2 + d^2 - b^2 - c^2)^2$$

$$= 4a^2d^2 + 4b^2c^2 - 8abcd\cos(A + C) \quad ...(v)$$

Let $A + C = 2\alpha$, then

$$\cos(A + C) = \cos 2\alpha = 2\cos^2\alpha - 1$$

From Eq. (v), we get

$$16s^2 = 4a^2d^2 + 4b^2c^2 - 8abcd(2\cos^2\alpha - 1) - (a^2 + d^2 - b^2 - c^2)^2$$

$$= \{2(ad + bc)\}^2 - (a^2 + b^2 - c^2 - d^2)^2 - 16abcd\cos^2\alpha$$

$$= \{(a + d)^2 - (b + c)^2\}\{(b + c)^2 - (a - d)^2\} - 16abcd\cos^2\alpha$$

$$= (a + d + b - c)(a + d - b + c)(b + c + a - d)(b + c - a + d) - 16abcd\cos^2\alpha$$

Let, $2s = a + b + c + d$

$\therefore$

$$16s^2 = (2s - 2a)(2s - 2b)(2s - 2c)(2s - 2d) - 16abcd\cos^2\alpha$$

$$s^2 = (s - a)(s - b)(s - c)(s - d) - abcd\cos^2\alpha$$

$$\Rightarrow \quad s = \sqrt{(s - a)(s - b)(s - c)(s - d) - abcd\cos^2\alpha}$$

where $2\alpha = A + C$ and $2s = a + b + c + d$

Thus, area of quadrilateral;

$$\frac{1}{2}BD \cdot AC \cdot \sin\alpha, \text{ where } \angle DPA = \alpha$$

or $$\boldsymbol{A = \sqrt{(s - a)(s - b)(s - c)(s - d) - abcd\cos^2\alpha}},$$

where $2\alpha = A + C$

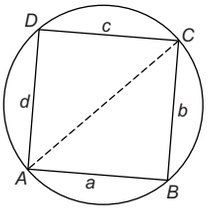
Area of Cyclic Quadrilateral

A quadrilateral is cyclic quadrilateral if its vertices lie on a circle.

Let $ABCD$ be a cyclic quadrilateral such that $AB = a$, $BC = b$, $CD = c$ and $DA = d$.

Then, $\angle B + \angle D = 180°$ and $\angle A + \angle C = 180°$.

Let $2s = a + b + c + d$ be the perimeter of the quadrilateral.



Now, Δ = area of cyclic quadrilateral $ABCD$

= area of ΔABC + area of ΔACD

$$= \frac{1}{2}ab\sin B + \frac{1}{2}cd\sin D$$

$$= \frac{1}{2}ab\sin B + \frac{1}{2}cd\sin(\pi - B) = \frac{1}{2}ab\sin B + \frac{1}{2}cd\sin B$$

$$\Rightarrow \quad \Delta = \frac{1}{2}(ab + cd)\sin B \quad ...(i)$$

Using cosine formula in a ΔABC and ΔACD, we have

$$AC^2 = AB^2 + BC^2 - 2AB \cdot BC \cdot \cos B$$

$$\Rightarrow \quad AC^2 = a^2 + b^2 - 2ab\cos B \quad ...(ii)$$

and $$AC^2 = AD^2 + CD^2 - 2AD \cdot CD \cdot \cos D$$

$$\Rightarrow \quad AC^2 = d^2 + c^2 - 2cd \cdot \cos(\pi - B) \quad ...(iii)$$

From Eqs. (ii) and (iii), we have

$$a^2 + b^2 - 2ab\cos B = d^2 + c^2 - 2cd\cos B$$

$$\Rightarrow \quad 2(ab + cd)\cos B = a^2 + b^2 - c^2 - d^2 \quad ...(iv)$$

$$\Rightarrow \quad 4(ab + cd)^2\cos^2 B = (a^2 + b^2 - c^2 - d^2)^2$$

$$\Rightarrow \quad 4(ab + cd)^2 \cdot (1 - \sin^2 B) = (a^2 + b^2 - c^2 - d^2)^2$$

$$\Rightarrow \quad 4(ab + cd)^2\sin^2 B = 4(ab + cd)^2 - (a^2 + b^2 - c^2 - d^2)^2$$

$$\Rightarrow \quad 4(ab + cd)^2 \cdot \sin^2 B = \{2(ab + cd) + (a^2 + b^2 - c^2 - d^2)\}\{2(ab + cd) - (a^2 + b^2 - c^2 - d^2)\}$$

$$\Rightarrow \quad 4(ab + cd)^2 \cdot \sin^2 B = \{(a + b)^2 - (c - d)^2\} \cdot \{(c + d)^2 - (a - b)^2\}$$

$$\Rightarrow \quad 4(ab + cd)^2 \cdot \sin^2 B = \{(a + b + c - d)(a + b - c + d)(c + d - a + b)(c + d + a - b)$$

$$\Rightarrow \quad 4(ab + cd)^2.\sin^2 B = (2s - 2d)(2s - 2c)(2s - 2b)(2s - 2a)$$

$$\Rightarrow \quad 16\Delta^2 = 16(s - a)(s - b)(s - c)(s - d) \quad \text{[using Eq. (i)]}$$

$$\Rightarrow \quad \Delta = \sqrt{(s - a)(s - b)(s - c)(s - d)} \quad ...(v)$$

$\therefore$ From Eqs. (i), (iv) and (v),

Area of cyclic quadrilateral;

$$\boldsymbol{\Delta = \frac{1}{2}(ab + cd) \cdot \sin B}, \boldsymbol{\Delta = \sqrt{(s - a)(s - b)(s - c)(s - d)}}$$

and $$\cos B = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}$$

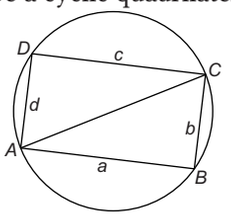
Ptolemy's Theorem

In a cyclic quadrilateral $ABCD$,

$$AC \cdot BD = AB \cdot CD + BC \cdot AD$$

i.e. in a cyclic quadrilateral the product of diagonals is equal to the sum of the products of the lengths of the opposite sides.

Proof: Let $ABCD$ be a cyclic quadrilateral, where



$$AC^2 = a^2 + b^2 - 2ab\cos B$$

and $$\cos B = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}$$

$$\Rightarrow \quad AC^2 = a^2 + b^2 - 2ab \cdot \left\{\frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}\right\}$$

$$\Rightarrow \quad AC^2 = \frac{(a^2 + b^2)(ab + cd) - ab(a^2 + b^2 - c^2 - d^2)}{(ab + cd)}$$

$$\Rightarrow \quad AC^2 = \frac{(a^2 + b^2) \cdot cd + ab(c^2 + d^2)}{ab + cd}$$

$$\Rightarrow \quad AC^2 = \frac{(ac + bd) \cdot (ad + bc)}{ab + cd} \qquad \ldots(i)$$

Similarly,

$$BD^2 = \frac{(ab + bd) \cdot (ac + bd)}{ad + bc} \qquad \ldots(ii)$$

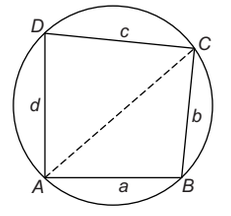
$$\Rightarrow \quad AC^2 \cdot BD^2 = (ac + bd)^2$$

$$\Rightarrow \quad AC \cdot BD = (ac + bd)$$

$$\Rightarrow \quad AC \cdot BD = AB \cdot CD + BC \cdot AD \text{ [Ptolemy's theorem]}$$

Circum-radius of a Cyclic Quadrilateral

Let $ABCD$ be a cyclic quadrilateral. Then the circum-circle of the quadrilateral $ABCD$ is also the circum circle of ΔABC.



Hence, the circum-radius of the cyclic quadrilateral

$$ABCD = R = \text{Circum-radius of } \Delta ABC = \frac{AC}{2\sin B}$$

$$= \frac{AC(ab + cd)}{4\Delta} \qquad [\text{using } \Delta = \frac{1}{2}(ab + cd)\sin B]$$

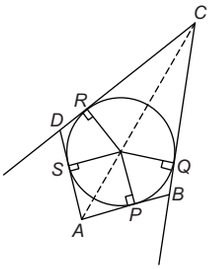
But, $$AC = \sqrt{\frac{(ac + bd)(ad + bc)}{(ab + cd)}} \qquad [\text{using Eq. (i)}]$$

Hence, $$\boldsymbol{R = \frac{1}{4\Delta}\sqrt{(ac + bd)(ad + bc)(ab + cd)}}$$

$\therefore$ $$\boldsymbol{R = \frac{1}{4}\sqrt{\frac{(ac + bd)(ad + bc)(ab + cd)}{(s - a)(s - b)(s - c)(s - d)}}}$$

Example 36. If the sides of a cyclic quadrilateral are 3, 3, 4, 4. Show that a circle can be inscribed in it.

Sol. By geometry,



$$AP = AS \qquad \ldots(i)$$

$$BP = BQ \qquad \ldots(ii)$$

$$DR = DS \qquad \ldots(iii)$$

$$CR = CQ \qquad \ldots(iv)$$

Adding all four equations, we get

$$AB + CD = AD + CB \qquad \ldots(v)$$

Now, the sides of cyclic quadrilateral 3, 3, 4, 4 inscribe the circle in it, if it satisfy (v).

i.e. let $AB + CD = 3 + 4 = 7$, $AD + CB = 3 + 4 = 7$

i.e. 3, 3, 4, 4 i.e. $3 = AB, 3 = BC, 4 = CD, 4 = CB$

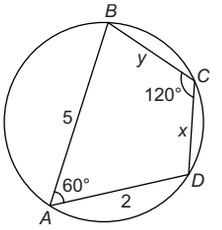
satisfy condition (v) or a circle can be inscribed.

Note

If sum of opposite side of a quadrilateral is equal, then and only then a circle can be inscribed in the quadrilateral.

Example 37. The two adjacent sides of a cyclic quadrilateral are 2 and 5 the angle between them is $60°$. If the area of the quadrilateral is $4\sqrt{3}$, then find the remaining two sides.

Sol. Let $AD = 2$, $AB = 5$, $\angle DAB = 60°$



Since, the quadrilateral is cyclic, $\angle BCD = 120°$

Area of

$$\Delta ABD = \frac{1}{2} \cdot 2 \cdot 5 \sin 60° = 5 \cdot \frac{\sqrt{3}}{2} \qquad \ldots(i)$$

Area of ΔBCD = Area of quadrilateral $ABCD$ − area of ΔABD

$$= 4\sqrt{3} - \frac{5\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$$

...(ii) [using Eq. (i)]

Let $CD = x$ and $BC = y$

Now, area of $\Delta BCD = \frac{1}{2}xy\sin 120°$

or $\frac{3\sqrt{3}}{2} = \frac{1}{2}xy\,\frac{\sqrt{3}}{2}$ ...(iii) [using Eq. (ii)]

$\Rightarrow$ $xy = 6$ ...(iv)

Applying cosine rule in ΔBAD, we get

$$\cos 60^\circ = \frac{AD^2 + AB^2 - BD^2}{2AD \cdot AB}$$

or $\frac{1}{2} = \frac{2^2 + 5^2 - BD^2}{2 \cdot 2 \cdot 5}$

$\Rightarrow$ $BD^2 = 19$

Applying cosine rule in ΔBCD, we get

$$\cos 120^\circ = \frac{x^2 + y^2 - 19}{2xy}$$

or $\frac{1}{2} = \frac{x^2 + y^2 - 19}{2xy}$

or $x^2 + y^2 + xy = 19$ [using $xy = 6$]

or $x^2 + y^2 = 13$

Now, $x^2 + y^2 + 2xy = 13 + 12 = 25$

$\Rightarrow$ $(x + y)^2 = 25$

$\Rightarrow$ $x + y = 5$

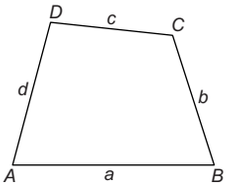
and $x^2 + y^2 - 2xy = 13 - 12$

$\Rightarrow$ $x - y = \pm 1$

Solving we get, $(x = 3, y = 2)$ or $(x = 2, y = 3)$

Example 38. If a, b, c, d are the sides of a quadrilateral, then find the minimum value of $\frac{a^2 + b^2 + c^2}{d^2}$.

Sol. Here, $AB = a$, $BC = b$, $CD = c$ and $AD = d$ are the sides of quadrilateral $ABCD$.



And we know, $(a - b)^2 + (b - c)^2 + (c - a)^2 \geq 0$

$\Rightarrow$ $2(a^2 + b^2 + c^2) \geq 2(ab + bc + ca)$

$\Rightarrow$ $3(a^2 + b^2 + c^2) \geq (a^2 + b^2 + c^2) + 2(ab + bc + a)$

[i.e., adding $(a^2 + b^2 + c^2)$ to both sides]

$\Rightarrow$ $3(a^2 + b^2 + c^2) \geq (a + b + c)^2$

[$\because$ sum of any three sides of quadrilateral is greater than fourth]

$\Rightarrow$ $3(a^2 + b^2 + c^2) \geq (a + b + c)^2 > d^2$ [$\because a + b + c > d$]

$\Rightarrow$ $\frac{a^2 + b^2 + c^2}{d^2} > \frac{1}{3}$

$\therefore$ Minimum value of $\frac{a^2 + b^2 + c^2}{d^2}$ is $\frac{1}{3}$.

Exercise for Session 6

1. The area of a cyclic quadrilateral $ABCD$ is $\frac{(3\sqrt{3})}{4}$. The radius of the circle circumscribing cyclic quadrilateral is 1. If $AB = 1, BD = \sqrt{3}$, then find $BC \cdot CD$.
2. If two adjacent sides of a cyclic quadrilateral are 2 and 5 and the angle between them is 60°. If the third side is 3, then find the remaining fourth side.
3. The ratio of the area of a regular polygon of n sides inscribed in a circle to that of the polygon of same number of sides circumscribing the same circle is 3 : 4. Then, the value of n is
4. A right angled trapezium is circumscribed about a circle. Find the radius of the circle. If the lengths of the bases (i.e. parallel sides) are equal to a and b.
5. If A, B, C, D are the angles of quadrilateral, then find $\frac{\Sigma \tan A}{\Sigma \cot A}$.

Session 7

Solution of Triangles

In a triangle, there are six variables. *viz.* three sides a, b, c and three angles A, B, C when any three of these six variables (except all the three angles) of a triangles are given, the triangle is known completely; that is the other three variables can be expressed in terms of the given variables and can be evaluated. This process is called the solution of triangles.

Solution of a Right Angled Triangle

***Case* I** When two sides are given.

Let the triangle be right angled at C, then we can determine the remaining variables as given in the following table :

Given	Required
(i) a, b	$\tan A = \frac{a}{b},\ B = 90° - A,\ C = \frac{a}{\sin A}$
(ii) a, c	$\sin A = \frac{a}{c},\ b = c\cos A,\ B = 90° - A$ where $\angle C = 90°$

***Case* II** When a side and an acute angle is given. In this case we can determine the remaining variables as given in the following table :

Given	Required
(i) a, A	$B = 90° - A,\ b = a\cot A,\ c = \frac{a}{\sin A}$
(ii) c, A	$B = 90° - A,\ a = c\sin A,\ b = c\cos A$

Solution of a Triangle in General

***Case* I** When three sides a, b and c are given.

In this case the remaining variables are determined by using the following table :

Given	Required
a, b, c	(i) Area of $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$, $2s = a + b + c$
	(ii) $\sin A = \frac{2\Delta}{bc},\ \sin B = \frac{2\Delta}{ac},\ \sin C = \frac{2\Delta}{ab}$
	(iii) $\tan\frac{A}{2} = \frac{\Delta}{s(s-a)},\ \tan\frac{B}{2} = \frac{\Delta}{s(s-b)},\ \tan\frac{C}{2} = \frac{\Delta}{s(s-c)}$

***Case* II** When two sides a, b and included $\angle C$ are given. In this case we use the following table :

Given	Required
a, b and $\angle C$	(i) Area of $\Delta = \frac{1}{2}ab\sin C$
	(ii) $\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cdot\cot\left(\frac{C}{2}\right)$
	(iii) $c = \frac{a\sin C}{\sin A}$

***Case* III** When one side a and two angles A and B are given. In this case we use the following table :

Given	Required
a and $\angle A, \angle B$	(i) $\angle C = 180° - (\angle A + \angle B)$
	(ii) $b = \frac{a\sin B}{\sin A}$ and $c = \frac{a\sin C}{\sin A}$
	(iii) $\Delta = \frac{1}{2}ac\sin B$

***Case* IV** When two sides a, b and $\angle A$ opposite to one side is given then,

$$\sin B = \frac{b}{a}\sin A \qquad \ldots(i)$$

$$\angle C = 180° - (\angle A + \angle B);\ c = \frac{a\sin C}{\sin A}$$

From Eq. (i), the following possibilities will arise :

(a) **When A is an acute angle and $a < b\sin A$** In this relation $\sin B = \frac{b}{a}\sin A$ gives that $\sin B > 1$ which is impossible.

Hence, no triangle is possible.

(b) **When A is an acute angle and $a = b\sin A$** In this case only one triangle is possible which is right angled at B.

$\therefore \quad a = b\sin A \Rightarrow \angle B = 90°$

(c) **When A is an acute angle and $a > b\sin A$** In this case there are two values of B given by $\sin B = \frac{b\sin A}{a}$, say B_1 and B_2 such that $B_1 + B_2 = 180°$ side 'c' can be obtained by using $c = \frac{a\sin C}{\sin A}$.

$\therefore \quad a > b\sin A \Rightarrow$ two triangle are possible.

Note

(i) In any right angled triangle, the orthocentre coincides with the vertex containing the right angled.

(ii) The mid-point of the hypotenuse of a right angled triangle is equidistant from the three vertices of the triangle.

(iii) The mid-point of the hypotenuse of the right angled triangle is the circum-centre of the triangle.

Example 39. In any ΔABC, the sides are 6 cm, 10 cm and 14 cm. Show that the triangle is obtuse-angled with the obtuse angle equal to 120°.

Sol. Let $a = 14$, $b = 10$ and $c = 6$ cm.

$$\Rightarrow \qquad s = \frac{14 + 10 + 6}{2} = 15 \text{ cm.}$$

As we know largest angle is opposite to the largest side,

$$\therefore \qquad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{100 + 36 - 196}{2(10)(6)} = -\frac{1}{2}$$

$$\Rightarrow \qquad A = 120^\circ$$

Example 40. If a, b and A are given in a triangle and C_1, C_2 are the possible values of the third side, prove that : $C_1^2 + C_2^2 - 2C_1C_2 \cos 2A = 4a^2 \cos^2 A$

Sol. $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$\Rightarrow c^2 - 2bc \cos A + b^2 - a^2 = 0$, which is quadratic is 'c'.

$$\therefore \qquad \left.\begin{aligned} C_1 + C_2 &= 2b\cos A \\ \text{and} \qquad C_1C_2 &= b^2 - a^2 \end{aligned}\right\} \qquad \text{...(i)}$$

$$\therefore \qquad C_1^2 + C_2^{\,2} - 2C_1C_2 \cos 2A$$

$$\Rightarrow \qquad (C_1 + C_2)^2 - 2C_1C_2 - 2C_1C_2 \cos 2A$$

[using Eq. (i)]

$$\Rightarrow \qquad (C_1 + C_2)^2 - 2C_1C_2(1 + \cos 2A)$$

$$\Rightarrow \qquad 4b^2 \cos^2 A - 2(b^2 - a^2) \cdot 2\cos^2 A$$

$$\Rightarrow \qquad 4a^2 \cos^2 A.$$

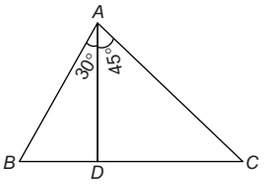
$$\therefore \qquad C_1^2 + C_2^2 - 2C_1C_2 \cos A = 4a^2 \cos A$$

Example 41. In a ΔABC, the median to the side BC is of length $\dfrac{1}{\sqrt{11 - 6\sqrt{3}}}$ and it divides the $\angle A$ into angles of 30° and 45°. Find the length of the side BC.

Sol. $\angle C = 180^\circ - (75^\circ + B)$

$$\angle C = 105^\circ - B$$

In ΔABD,



$$\frac{BD}{\sin 30^\circ} = \frac{AD}{\sin B} \Rightarrow BD = \frac{AD}{2\sin B} \qquad \text{...(i)}$$

In ΔADC,

$$\frac{CD}{\sin 45^\circ} = \frac{AD}{\sin C}$$

$$\Rightarrow \qquad CD = \frac{AD}{\sqrt{2}\sin(105^\circ - B)} \qquad \text{...(ii)}$$

Now, $\qquad BD = CD$

$$\Rightarrow \qquad \frac{AD}{2\sin B} = \frac{AD}{\sqrt{2}\sin(105^\circ - B)}$$

$$\Rightarrow \qquad \sqrt{2}\sin B = \sin(105^\circ - B)$$

$$\Rightarrow \qquad \sqrt{2}\sin B = \sin 105^\circ \cdot \cos B - \cos 105^\circ \cdot \sin B$$

$$\Rightarrow \qquad \sqrt{2}\sin B = \frac{(\sqrt{3} + 1)}{2\sqrt{2}}\cos B + \frac{(\sqrt{3} - 1)}{2\sqrt{2}}\sin B$$

$$\Rightarrow \qquad 4\sin B = (\sqrt{3} + 1)\cos B + (\sqrt{3} - 1)\sin B$$

$$\Rightarrow \qquad \cot B = 3\sqrt{3} - 4$$

$$\Rightarrow \qquad \sin B = \frac{1}{2\sqrt{11 - 6\sqrt{3}}}$$

Hence, $\qquad BC = 2BD = \dfrac{AD}{\sin B} = \dfrac{2\sqrt{11 - 6\sqrt{3}}}{\sqrt{11 - 6\sqrt{3}}} = 2$

Exercise for Session 7

1. In ΔABC, $a : b : c = (1 + x) : 1 : (1 - x)$, where $x \in (0, 1)$. If $\angle A = \dfrac{\pi}{2} + \angle C$, then find the value of x.

2. In a ΔABC, $2s$ = perimeter and R = circumradius. Then, find $\dfrac{s}{R}$.

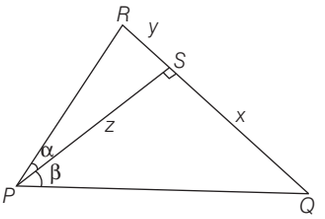
3. If in a ΔABC, $\angle C = 90^\circ$, then find the maximum value of $\sin A \sin B$.

4. If the area of a triangle is 81 square cm and its perimeter is 27 cm, then find its in-radius in centi-metres.

5. In a ΔABC, if $r_1 = 2r_2 = 3r_3$, then show that $\frac{a}{b} = \frac{5}{4}$.

6. Find in-radius of the triangle formed by the axes and the line $4x + 3y - 12 = 0$.

7. In a ΔPQR as shown in figure given that $x : y : z :: 2 : 3 : 6$, then find value of $\angle QPR$.



8. In a ΔABC, if $\frac{R}{r} \leq 2$, then show that the triangle is equilateral.

9. The angle of a right-angled triangle are in AP. Then, find the ratio of the in-radius and the perimeter.

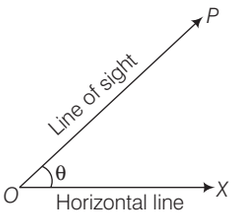
10. If in a triangle $\left(1 - \frac{r_1}{r_2}\right)\left(1 - \frac{r_1}{r_3}\right) = 2$, then show that the triangle is right angled.

Session 8

Height and Distance

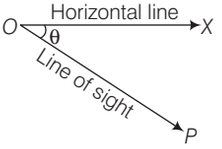
Angle of Elevation

If 'O' be the observer's eye and OX be the horizontal line through O. If object P is at a higher level than eye, then $\angle POX$ is called the angle of elevation.



Angle of Depression

If 'O' be the observer's eye and OX is a horizontal line object P is at a lower level than O, then the $\angle POX$ is called the angle of depression.



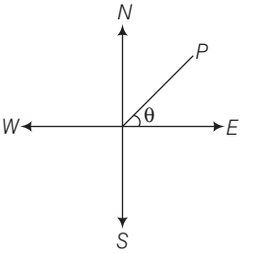
Note

(i) Angle of elevation and depression are always acute angle.

(ii) Angle of elevation of an object from an observer is same as angle O depression of an observer from the object.

Bearing

If the observer and the object are O and P be on the same level respectively, then bearings is defined. To measure the bearing the four standard direction East, West, North and South are taken as the cardinal directions. Angle between the line of observation. i.e., OP and any one standard direction is measured.



Thus, $\angle POE$ is called the bearing of the point P with respect to O measured from East to North.

In other words the bearing of P as seen from O is the direction in which P is seen from O.

Note

North-East means equally inclined to North and East. ENE means equally inclined to East and North-East.

Example 42. Two flagstaffs stand on a horizontal plane. A and B are two points on the line joining their feet and between them. The angles of elevation of the tops of the flagstaffs as seen from A are 30° and 60° and as seen from B are 60° and 45°. If AB is 30 m, the distance between the flagstaffs in metres is

(a) $30 + 15\sqrt{3}$ (b) $45 + 15\sqrt{3}$

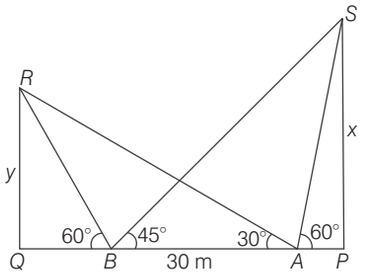
(c) $60 - 15\sqrt{3}$ (d) $60 + 15\sqrt{3}$

Sol. (d) Let x and y be the heights of the flagstaffs at P and Q respectively.

Then, $AP = x\cot 60^\circ = \dfrac{x}{\sqrt{3}}$, $AQ = y\cot 30^\circ = y\sqrt{3}$

$$BP = x\cot 45^\circ = x,\ BQ = y\cot 60^\circ = \frac{y}{\sqrt{3}}$$

$$\Rightarrow \qquad BP - AP = x - \frac{x}{\sqrt{3}} = AB$$



$$\Rightarrow \qquad 30\sqrt{3} = (\sqrt{3} - 1)x$$

$$\Rightarrow \qquad x = 15(3 + \sqrt{3})$$

Similarly, $\quad 30 = y\left(\sqrt{3} - \dfrac{1}{\sqrt{3}}\right) \Rightarrow y = 15\sqrt{3}$

So that, $\quad PQ = BP + BQ = x + \dfrac{y}{\sqrt{3}}$

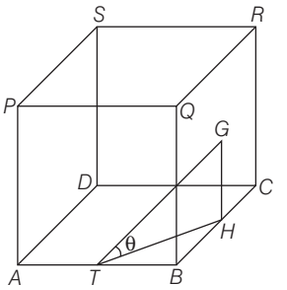
$$= 15(3 + \sqrt{13}) + 15 = (60 + 15\sqrt{3})\ \text{m}$$

Example 43. In a cubical hall $A\,B\,C\,D, P\,Q\,R\,S$ with each side 10 m, G is the centre of the wall $B\,C\,R\,Q$ and T is the mid-point of the side AB. The angle of elevation of G at the point T is

(a) $\sin^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$ (b) $\cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$

(c) $\tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$ (d) $\cot^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$

Sol. (a)



Let H be the mid-point of BC since

$$\angle TBH = 90^\circ,$$

$$(TH)^2 = (BT)^2 + (BH)^2$$

$$= 5^2 + 5^2 = 50$$

Also, $\quad \angle THG = 90^\circ,$

$$(TG)^2 = (TH)^2 + (GH)^2$$

$$= 50 + 25 = 75$$

Let θ be the required angle of elevation of G at T.

Then, $\quad \sin\theta = \dfrac{GH}{TG}$

$$= \frac{5}{5\sqrt{3}} = \frac{1}{\sqrt{3}}$$

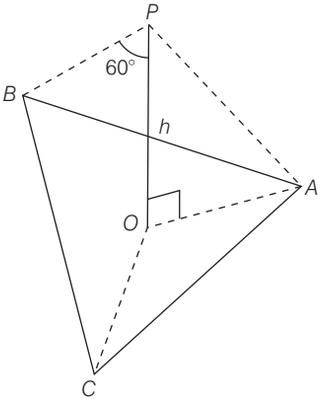
$$\Rightarrow \qquad \theta = \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

Example 44. Each side of an equilateral triangle subtends an angle of 60° at the top of a tower h m high located at the centre of the triangle. If a is the length of each side of the triangle, then

(a) $3a^2 = 2h^2$ (b) $2a^2 = 3h^2$

(b) $a^2 = 3h^2$ (d) $3a^2 = h^2$

Sol. (b)



Let O be the centre of the equilateral triangle ABC and OP the tower of height h. Then, each of the triangles PAB, PBC and PCA are equilateral.

Thus, $PA = PB = PC = a$.

Therefore, from right-angled triangle POA, we have $PA^2 = PO^2 + OA^2$.

$$\Rightarrow \quad a^2 = h^2 + \left(\frac{a}{2}\sec 30°\right)^2$$

$$= h^2 + \frac{a^2}{4}\cdot\frac{4}{3}$$

$$= h^2 + \frac{a^2}{3}$$

$$\Rightarrow \quad \frac{2}{3}a^2 = h^2$$

$$\Rightarrow \quad 2a^2 = 3h^2$$

Example 45. A vertical tower PQ subtends the same angle 30° at each of two places A and B, 60 m apart on the ground, AB subtends an angle 120° at the foot of the tower. If h is the height of the tower, then $9h^2 + h + 1$ is equal to

(a) 3121 (b) 2136

(c) 3600 (d) None of these

Sol. (d) $AP = PB = h\cot 30° = \sqrt{3}h$

$$-\frac{1}{2} = \cos 120° = \frac{3h^2 + 3h^2 - 60^2}{2\cdot 3h^2}$$

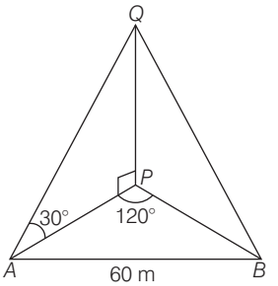
$$-3h^2 = 3h^2 + 3h^2 - 3600$$

$$9h^2 = 3600$$

$$h = 20$$

$$\therefore \quad 9h^2 + h + 1 = 3600 + 20 + 1$$

$$= 3681$$



Exercise for Session 8

1. If a tower subtends angles $\theta, 2\theta$ and 3θ at three points A, B and C respectively, lying on the same side of a horizontal line through the foot of the tower, show that $\frac{AB}{BC}\frac{\cot\theta - \cot 2\theta}{\cot 2\theta - \cot 3\theta}$.

2. A person stands at a point A due south of a tower of height h and observes that its elevation is 60°. He then walks westwards towards B, where the elevation is 45°. At a point C on AB produced, show that if he find it to be 30°. OA, OB, OC are in GP.

3. A train travelling on one of two intersecting railway lines, subtends at a certain station on the other line, an angle α when the front of the carriage reaches the junction and an angle β when the end of the carriage reaches it. Then, the two lines are inclined to each other at an angle θ, show that $2\cot\theta = \cot\alpha - \cot\beta$, $\cot\alpha + \cot\beta$

4. The angle of elevation of the top of the tower observed from each of the three points A, B, C on the ground, forming a triangle is the same angle α. If R is the circum-radius of the triangle ABC, then find the height of the tower $R\tan\alpha$.

5. The length of the shadow of a pole inclined at 10° to the vertical towards the sun is 2.05 metres, when the elevation of the sun is 38°. Then, find the length of the pole.

JEE Type Solved Examples :
Single Option Correct Type Questions

● **Ex. 1.** *If* $\tan^2\frac{\pi-A}{4}+\tan^2\frac{\pi-B}{4}+\tan^2\frac{\pi-C}{4}=1$, *then* ΔABC *is*

(a) equilateral (b) isosceles
(c) scalene (d) None of these

Sol. (a) Let $\alpha=\frac{\pi-A}{4},\beta=\frac{\pi-B}{4},\gamma=\frac{\pi-C}{4}$ ...(i)

$\Rightarrow \quad \alpha+\beta+\gamma=\frac{\pi}{2}$

$\Rightarrow \quad \Sigma\tan\alpha\cdot\tan\beta=1$

$\Sigma\tan^2\alpha=1=\Sigma\tan\alpha\tan\beta$

$\Rightarrow \quad \tan\alpha=\tan\beta=\tan\gamma$

$\therefore \quad \alpha=\beta=\gamma$

$\frac{\pi-A}{4}=\frac{\pi-B}{4}=\frac{\pi-C}{4}$ [From Eq. (i)]

$\therefore \quad A=B=C$

● **Ex. 2.** *In* ΔABC, $a^2+c^2=2002b^2$, *then* $\frac{\cot A+\cot C}{\cot B}$ *equals to*

(a) $\frac{1}{2001}$ (b) $\frac{2}{2001}$
(c) $\frac{3}{2001}$ (d) $\frac{4}{2001}$

Sol. (b) $\frac{\cot A+\cot C}{\cot B}=\frac{\sin(A+C)\sin B}{\sin A\sin C\sin B}$

$=\frac{\sin^2 B}{\sin A\cos B\sin C}=\frac{4R^2b^2}{4R^2\,ac\cos B}$

$=\frac{2b^2}{2ac\cos B}=\frac{2b^2}{a^2+c^2-b^2}$

$=\frac{2b^2}{2002\,b^2-b^2}=\frac{2}{2001}$

● **Ex. 3.** *A triangle has vertices A, B and C and the respective opposite sides have lengths a, b and c. This triangle is inscribed in a circle of radius R. If* $b=c=1$ *and the altitude from A to side BC has length* $\sqrt{\frac{2}{3}}$, *then R equals*

(a) $\frac{1}{\sqrt{3}}$ (b) $\frac{2}{\sqrt{3}}$
(c) $\frac{\sqrt{3}}{2}$ (d) $\frac{\sqrt{3}}{2\sqrt{2}}$

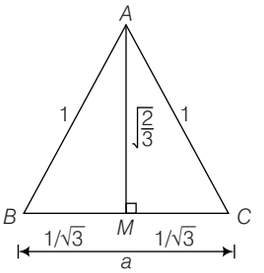
Sol. (d) $a=\frac{2}{\sqrt{3}}$ and $\Delta=\frac{1}{2}\,ah$

$=\frac{1}{2}\cdot\frac{2}{\sqrt{3}}\cdot\frac{\sqrt{2}}{\sqrt{3}}=\frac{\sqrt{2}}{3}$

$\therefore \quad R=\frac{abc}{4\Delta}$

$=\frac{(2)(1)(3)}{\sqrt{3}\,(4)\,\sqrt{2}}$

$=\frac{\sqrt{3}}{2\sqrt{2}}$



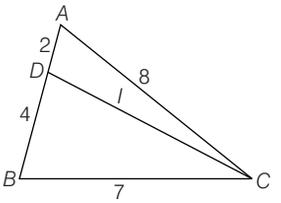
● **Ex. 4.** *In* ΔABC, *if* $AC=8$, $BC=7$ *and D lies between A and B such that* $AD=2$, $BD=4$, *then the length CD equals*

(a) $\sqrt{46}$ (b) $\sqrt{48}$ (c) $\sqrt{51}$ (d) $\sqrt{75}$

Sol. (c) $l^2=2^2+8^2-2\cdot2\cdot8\cos A$

$=4+64-32\cos A$

and $\cos A=\frac{6^2+8^2-7^2}{2\cdot6\cdot8}=\frac{51}{16\times6}=\frac{17}{32}$



$l^2=68-32\times\frac{17}{32}=51$

$l=\sqrt{51}$

● **Ex. 5.** *In a triangle, if*

$(a+b+c)(a+b-c)(b+c-a)$

$(c+a-b)=\frac{8a^2b^2c^2}{a^2+b^2+c^2}$, *then the triangle is*

(a) isosceles (b) right angled
(c) equilateral (d) obtuse angled

Sol. (b) We have, $s(s-a)(s-b)(s-c)=\frac{a^2b^2c^2}{2(a^2+b^2+c^2)}$

$\Rightarrow \quad a^2+b^2+c^2=\frac{a^2b^2c^2}{2\Delta^2}$ ...(i)

As, $\Delta=\frac{1}{2}bc\sin A=\frac{abc}{R}$

So, Eq. (i) becomes

$$a^2 + b^2 + c^2 = \frac{(a^2b^2c^2)}{2a^2 \cdot b^2 \cdot c^2} = 8R^2$$

$\Rightarrow \quad 4R^2(\sin^2 A + \sin^2 B + \sin^2 C) = 8R^2$

$\Rightarrow \quad \sin^2 A + \sin^2 B + \sin^2 C = 2$

or $\quad 2 + 2\cos A \cdot \cos B \cdot \cos C = 2$

$\Rightarrow \quad \cos A \cdot \cos B \cdot \cos C = 0$

$\therefore \Delta ABC$ must be right angled.

● ***Ex. 6.*** *Consider a ΔABC and let a, b and c denote the lengths of the sides opposite to vertices A, B and C, respectively. If $a = 1$, $b = 3$ and $C = 60°$, then $\sin^2 B$ is equal to*

(a) $\frac{27}{28}$ (b) $\frac{3}{28}$

(c) $\frac{81}{28}$ (d) $\frac{1}{3}$

Sol. (a) By Cosine law, $\cos 60° = \frac{1^2 + 3^2 - c^2}{2 \cdot 1 \cdot 3}$

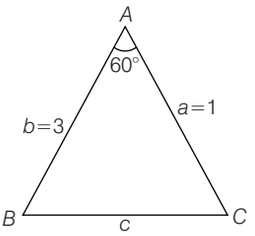
$\Rightarrow \quad c = \sqrt{7}$

Now, $\frac{b}{\sin B} = \frac{c}{\sin C}$ [By Sine law]

$\Rightarrow \quad \sin B = \frac{b \sin C}{c}$

$= \frac{3 \times \frac{\sqrt{3}}{2}}{\sqrt{7}}$

$= \frac{3\sqrt{3}}{2\sqrt{7}}$



Hence, $\sin^2 B = \frac{27}{28}$.

● ***Ex. 7.*** *In ΔABC, if $\cos A + \sin A - \frac{2}{\cos B + \sin B} = 0$, then $\frac{a+b}{c}$ is equal to*

(a) $\sqrt{2}$ (b) 1 (c) $\frac{1}{\sqrt{2}}$ (d) $2\sqrt{2}$

Sol. (a) We have,

$\Rightarrow \cos A \cos B + \sin A \sin B + \cos A \sin B + \sin A \cos B = 2$

$\Rightarrow \quad \cos(A - B) + \sin(A + B) = 2$

$\therefore \quad \cos(A - B) = 1$

and $\quad \sin(A + B) = 1$

$\Rightarrow \quad A = B$, so $a = b$

and $\quad \sin 2A = 1$

$\Rightarrow \quad A = 45°$

or $\quad A = 135°$ (Not possible)

Hence, $\frac{a+b}{c} = \frac{2a}{a\sqrt{2}} = \sqrt{2}$

● ***Ex. 8.*** *In ΔABC, if $\angle A = 30°$, $b = 10$ and $a = x$, then the values of x for which there are 2 possible triangles is given by (All symbol used have usual meaning in a triangle)*

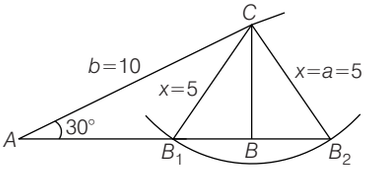
(a) $5 < x < 10$ (b) $x < \frac{5}{2}$

(c) $\frac{5}{3} < x < 10$ (d) $\frac{5}{2} < x < 10$

Sol. (a) If c is the third side, then the altitude to c has length $10 \sin 30° = 5$,

there are two triangles if x is greater than this value and less than the length of b.

If $x > 10$, point B will come to the left of A and $\angle A$ would be obtuse in the case.



Aliter We have, $\frac{\sqrt{3}}{2} = \cos 30°$

$= \frac{100 + c^2 - x^2}{2(10)(c)}$ [by using cosine rule]

$\Rightarrow \quad c^2 - 10\sqrt{3}c + (100 - x^2) = 0$

$c = \frac{10\sqrt{3} \pm \sqrt{4x^2 - 100}}{2}$

Now, for 2 distinct positive values of c, we must have

$10\sqrt{3} > \sqrt{4x^2 - 100}$

$\Rightarrow \quad 300 > 4x^2 - 100$

$\Rightarrow \quad x^2 < 100 \Rightarrow 5 < x < 10$

● ***Ex. 9.*** *In a ΔABC, $AB = AC$, P and Q are points on AC and AB respectively such that $CB = BP = PQ = QA$. If $\angle AQP = \theta$, then $\tan^2 \theta$ is a root of the equation*

(a) $y^3 + 21y^2 - 35y - 12 = 0$

(b) $y^3 - 21y^2 + 35y - 12 = 0$

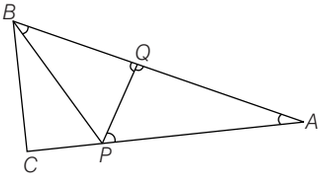
(c) $y^3 - 21y^2 + 35y - 7 = 0$

(d) $12y^3 - 35y^2 + 35y - 12 = 0$

Sol. (c) $\angle QAP = \angle QPA = 90 - \frac{\theta}{2}$

$\angle PQB = \angle PBQ = 180 - \theta$

$\angle BCA = \angle ABC = \angle BPC = 45 + \frac{\theta}{4}$



Now, $\left(90^\circ - \frac{\theta}{2}\right) + (2\theta - 180^\circ) + \left(45^\circ + \frac{\theta}{4}\right) = 180^\circ$

$\Rightarrow \quad \theta = \frac{5\pi}{7}$

$\Rightarrow \quad 7\theta = 5\pi$

$\Rightarrow \quad 4\theta = 5\pi - 3\theta$

$\Rightarrow \quad \tan 4\theta = -\tan 3\theta$

$\Rightarrow \quad \frac{2\tan 2\theta}{1 - \tan^2 2\theta} = -\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

$\Rightarrow \quad = \frac{2\left[\frac{2t}{1-t^2}\right]}{1 - \left(\frac{2t}{1-t^2}\right)^2} = \frac{3t - t^3}{1 - 3t^2}, \qquad$ where $t = \tan\theta$

$\Rightarrow \quad \frac{4(1-t^2)}{(1-t^2)^2 - 4t^2} = \frac{t^2 - 3}{1 - 3t^2}$

$\Rightarrow \quad 4(1 - 4t^2 + 3t^4) = (t^2 - 3)(1 - 6t^2 + t^4)$

$\Rightarrow \quad t^6 - 21t^4 + 35t^2 - 7 = 0$

$\therefore$ $\tan^2\theta$ is the root of the equation $y^3 - 21y^2 + 35y - 7 = 0$.

● ***Ex. 10.*** *The angle of elevation of tower from a point A due south of it is 30° and from a point B due west of it is 45°. If the height of the tower be 100 m, then AB=*

(a) 150 m (b) 200 m

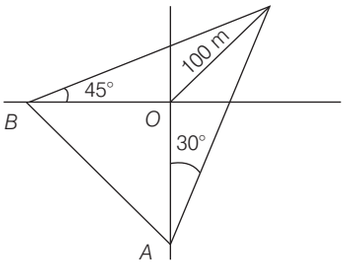
(c) 173.2 m (d) 141.4 m

Sol. (b) $OB = 100\cot 45^\circ$

$OA = 100 \cot 30^\circ$

$AB = \sqrt{(OA^2 + OB^2)}$

$AB = 200$ m



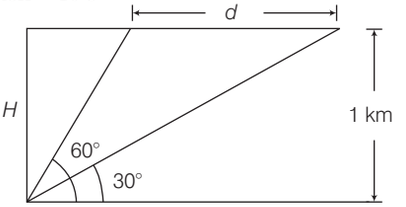
● ***Ex. 11.*** *An aeroplane flying horizontally 1 km above the ground is observed at an elevation of 60° and after 10 seconds the elevation is observed to be 30°. The uniform speed of the aeroplane in km/h is*

(a)240 (b) $240\sqrt{3}$

(c) $60\sqrt{3}$ (d) None of these

Sol. (b) $d = H \cot 30^\circ - H \cot 60^\circ$

Time taken = 10 s



Speed $= \frac{\cot 30^\circ - \cot 60^\circ}{10} \times 60 \times 60 = 240\sqrt{3}$

JEE Type Solved Examples :
More than One Correct Option Type Questions

● ***Ex. 12*** *In ΔABC, the ratio $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ is always equal to (All symbols used have usual meaning in a triangle.)*

(a) $2R$, where R is the circumradius

(b) $\frac{abc}{2\Delta}$, where Δ is the area of the triangle

(c) $\frac{2}{3}(a^2 + b^2 + c^2)^{\frac{1}{2}}$

(d) $\frac{(abc)^{\frac{2}{3}}}{(h_1h_2h_3)^{\frac{1}{3}}}$

Sol. (a,b,d) We know that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

and $R = \frac{abc}{4\Delta} \Rightarrow 2R = \frac{abc}{2\Delta}$

$\therefore$ a,b are true

Now, $\frac{2}{3}(a^2 + b^2 + c^2)^{1/2}$

If B true iff $a = b = c$

$\therefore$ 'c' is incorrect

Now, for option 'd'

We have, $\frac{(abc)^{2/3}}{(h_1h_2h_3)^{1/3}}$

We know, $\frac{1}{2}ah_1 = \frac{\pi}{2}bh_2 = \frac{1}{2}ch_3 = \Delta$

$$\Rightarrow \qquad h_1 = \frac{2\Delta}{a}, h_2 = \frac{2\Delta}{b}, h_3 = \frac{2\Delta}{c}$$

$$\because \qquad h_1 h_2 h_3 = \frac{8\Delta^3}{abc}$$

$$\therefore \qquad \frac{(abc)^{2/3}}{(h_1 h_2 h_3)^{1/3}} = \frac{(abc)^{2/3}}{\left(\frac{8\Delta^3}{abc}\right)^{1/3}} = \frac{(abc)^{2/3}(abc)^{1/3}}{2\Delta}$$

$$= \frac{abc}{2\Delta} = 2R$$

Hence option, *a*, *b* and *d* are correct.

● ***Ex. 13.*** *Let ABCD be a cyclic quadrilateral such that* $AB = 2$, $BC = 3$, $\angle B = 120°$ *and area of quadrilateral* $= 4\sqrt{3}$. *Which of the following is/are correct?*

(a) The value of $(AC)^2$ is equal to 19

(b) The sum of all possible values of product $AC \cdot BD$ is equal to 35

(c) The sum of all possible values of $(AD)^2$ is equal to 29

(d) The value of $(CD)^2$ can be 4

Sol. (a,c,d) Area of quadrilateral

$$= 4\sqrt{3} = \frac{1}{2} \times 2 \times 3 \sin 120° + \frac{1}{2} xy \sin 60°$$

$$4\sqrt{3} = \frac{\sqrt{3}}{2}\left[3 + \frac{xy}{2}\right]$$

$$16 = 6 + xy \Rightarrow xy = 10$$

$$AC^2 = 2^2 + 3^2 - 2 \cdot 2 \cdot 3 \times \cos 120°$$

$$= 4 + 9 + 6 = 19$$

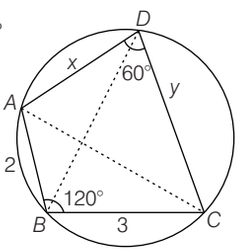
$$= x^2 + y^2 - 2xy \cos 60°$$

$$x^2 + y^2 - xy = 19$$

or $\quad x^2 + y^2 = 29$

$\Rightarrow \quad x = 5, y = 2$

or $\quad x = 2, y = 5$



● ***Ex. 14.*** *In a* ΔABC, *which of the following quantities denote the area of the triangle?*

(a) $\frac{a^2 - b^2}{2}\left(\frac{\sin A \sin B}{\sin(A - B)}\right)$

(b) $\frac{r_1 r_2 r_3}{\sqrt{\Sigma r_1 r_2}}$

(c) $\frac{a^2 + b^2 + c^2}{\cot A + \cot B + \cot C}$

(d) $r^2 \cot\frac{A}{2} \cdot \cot\frac{B}{2} \cdot \cot\frac{C}{2}$

Sol. (a,b,d) (a) $\frac{4R^2}{2}\left(\frac{\sin^2 A - \sin^2 B}{\sin(A - B)}\right) \cdot \frac{ab}{4R^2} = \frac{1}{2}\sin(A + B)\, ab$

$$= \frac{1}{2}(ab)\sin C = \frac{abc}{4R} = \Delta$$

(b) $r_1 r_2 r_3 = \frac{\Delta^3 . s}{s(s-a)(s-b)(s-c)} = \Delta \cdot s$

$$\sqrt{\Sigma r_1 r_2} = \sqrt{s^2\left(\Sigma \tan\frac{A}{2}\tan\frac{B}{2}\right)} = s \quad \left[\because \Sigma \tan\frac{A}{2}\tan\frac{B}{2} = 1\right]$$

$$\therefore \qquad \frac{r_1 r_2 r_3}{\sqrt{\Sigma r_1 r_2}} = \Delta$$

(c) Denominator $= \frac{(b^2 + c^2 - a^2)2R}{2bca} + \ldots + = \frac{R}{abc}$

$$[b^2 + c^2 - a^2 + c^2 + a^2 - b^2 + a^2 + b^2 - c^2]$$

$$= \frac{(a^2 + b^2 + c^2)R}{abc}$$

$$\therefore \qquad \frac{N^r}{D^r} = \frac{\left(\Sigma a^2\right) abc}{\left(\Sigma a^2\right) R} = \frac{abc}{R} = 4\Delta$$

$\Rightarrow \quad C$ is not correct.

(d) $\frac{\Delta^2}{s^2} \cdot \frac{s(s-a)}{\Delta} \cdot \frac{s(s-b)}{\Delta} \cdot \frac{s(s-c)}{\Delta}$

$$= \frac{s(s-a)(s-b)(s-c)}{\Delta} = \frac{\Delta^2}{\Delta} = \Delta$$

Hence, (*a*), (*b*) and (*d*) are correct.

● ***Ex. 15.*** *Consider the system of equations* $\sin x \cos 2y = (a^2 - 1)^2 + 1$ *and* $\cos x \sin 2y = a + 1$. *Which of the following ordered pairs* (x, y) *of real numbers can satisfy the given system of equations for permissible real values of a?*

(a) $\left(\frac{-\pi}{2}, \frac{-\pi}{2}\right)$ (b) $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

(c) $\left(\frac{3\pi}{2}, \frac{-\pi}{2}\right)$ (d) $\left(\frac{-\pi}{2}, \frac{3\pi}{2}\right)$

Sol. (a,c,d) *For permissible values of 'a', we must have*

$(a^2 - 1)^2 + 1 \le 1$ and $|a + 1| \le 1$

$\Rightarrow (a^2 - 1)^2 \le 0$ and $-1 \le a + 1 \le 1$

$\Rightarrow \quad -2 \le a \le 0 \Rightarrow a^2 - 1 = 0$

$\Rightarrow \quad a = 1$ or -1

$\therefore$ Permissible value of $a = -1$

Hence, the system of equations becomes $\sin x \cos 2y = 1$ and $\cos x \sin 2y = 0$.

Now, verify (a), (c), (d) alternatives.

● ***Ex. 16.*** *In a* ΔABC, *let* $2a^2 + 4b^2 + c^2 = 2a(2b + c)$, *then which of the following holds good?*

[Note All symbols used have usual meaning in a triangle.]

(a) $\cos B = \frac{-7}{8}$ (b) $\sin(A - C) = 0$

(c) $\frac{r}{r_1} = \frac{1}{5}$ (d) $\sin A : \sin B : \sin C = 1 : 2 : 1$

Sol. (b,c) We have, $2a^2 + 4b^2 + c^2 - 4ab - 2ac = 0$

$\Rightarrow \quad (a-2b)^2 + (a-c)^2 = 0$

$\Rightarrow \quad a = 2b,\ a = c$

$\Rightarrow \quad b = \frac{a}{2},\ c = a$

$\therefore \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$

$$= \frac{\frac{a^2}{4} + a^2 - a^2}{2\left(\frac{a}{2}\right)a} = \frac{1}{4}$$

Similarly, $\cos B = \frac{7}{8}$ and $\cos C = \frac{1}{4}$

Hence, option (a) is not correct.

As $a = c \Rightarrow A = C$

$\Rightarrow \quad A - C = 0 \Rightarrow \sin(A - C) = 0$

Hence, option (b) is correct.

As $\frac{r}{r_1} = \frac{(s-a)\tan\frac{A}{2}}{s\tan\frac{A}{2}} = \frac{(s-a)}{s}$

$$= 1 - \frac{a}{s} = 1 - \frac{a}{\frac{5a}{4}} = 1 - \frac{4}{5} = \frac{1}{5}$$

Hence, option (c) is correct.

Also, $a : b : c = a : \frac{a}{2} : a = 2 : 1 : 2 = \sin A : \sin B : \sin C$

Hence, option (d) is not correct.

● ***Ex. 17.*** *In ΔABC, angles A, B and C are in the ratio $1 : 2 : 3$, then which of the following is(are) correct? (All symbol used have usual meaning in a triangle).*

(a) Circum-radius of $\Delta ABC = c$

(b) $a : b : c = 1 : \sqrt{3} : 2$

(c) Perimeter of $\Delta ABC = 3 + \sqrt{3}$

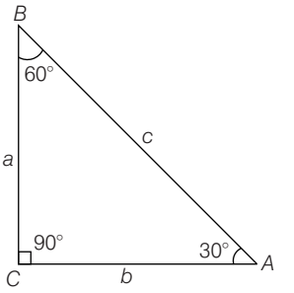
(d) Area of $\Delta ABC = \frac{\sqrt{3}}{8}c^2$

Sol. (b, d) Given, $A + 2A + 3A = 180°$,

$B = 60°$ and $C = 90°$

Now, $R = \frac{c}{2}$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2\left(\frac{c}{2}\right)$$



$$a = \frac{c}{2},\ b = \frac{\sqrt{3}c}{2}$$

So, $a : b : c = \frac{c}{2} : \frac{\sqrt{3}c}{2} : c = \frac{1}{2} : \frac{\sqrt{3}}{2} : 1 = 1 : \sqrt{3} : 2$

Perimeter $= (3 + \sqrt{3})k, (k \in R)$

Area of $\Delta ABC = \frac{1}{2}ab$

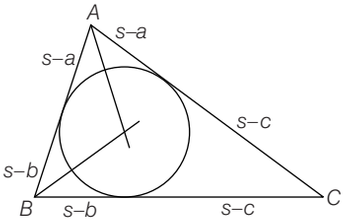
$$= \frac{1}{2}\left(\frac{c}{2}\right)\left(\frac{\sqrt{3}c}{2}\right) = \frac{\sqrt{3}}{8}c^2$$

● ***Ex. 18.*** *If the length of tangents from A, B, C to the incircle of ΔABC are 4, 6, 8, then which of the following is(are) correct? (All symbols used have usual meaning in a triangle.)*

(a) Area of ΔABC is $12\sqrt{6}$ (b) r_1, r_2, r_3 are in HP

(c) a, b, c are in AP (d) $r = \frac{4\sqrt{6}}{3}$

Sol. (b, c, d) $s - a = 4,\ s - b = 6,\ s - c = 8$



$\therefore \quad s = 18$

$\therefore \quad \Delta = \sqrt{18 \times 4 \times 6 \times 8} = 24\sqrt{6}$

$a = 14,\ b = 12$ and $c = 10$

$s - a, s - b, s - c$ are in AP.

$\therefore$ a, b, c are in AP.

$\frac{\Delta}{s-a}, \frac{\Delta}{s-b}, \frac{\Delta}{s-c}$ are in HP.

$\therefore r_1, r_2, r_3$ are in HP $r = \frac{\Delta}{s} = \frac{4\sqrt{6}}{3}$

● ***Ex. 19.*** *In ΔABC, let $b=10$, $c=10\sqrt{2}$ and $R=5\sqrt{2}$, then which of the following statement (s) is (are) correct?*

(a) Area of triangle ABC is 50.

(b) Distance between orthocentre and circumcentre is $5\sqrt{2}$.

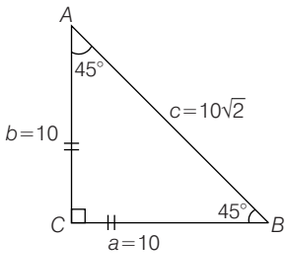
(c) Sum of circum-radius and in-radius of ΔABC is equal to 10.

(d) Length of internal angle bisector of $\angle ACB$ of ΔABC is $\frac{5}{2\sqrt{2}}$.

Sol. (a,b,c) We know, $\frac{b}{\sin B}=2R$

$$\Rightarrow \quad \sin B=\frac{10}{10\sqrt{2}}=\frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad B=45^\circ$$



Also, $\frac{c}{\sin C}=2R$

$$\Rightarrow \quad \sin C=\frac{10\sqrt{2}}{10\sqrt{2}}=1$$

$\therefore \quad A=45^\circ \Rightarrow C=90^\circ$

$\therefore \Delta ABC$ is isosceles right angled at C.

$\because \quad a=10, b=10, c=10\sqrt{2}$

Area of $\Delta ABC=\frac{1}{2}\times 10\times 10=50$

Distance between orthocentre and circumcentre

$$BD=DC=5\sqrt{2}$$

$$R+r=5\sqrt{2}+10-5\sqrt{2}=10 \qquad [\because r=10-5\sqrt{2}]$$

● ***Ex. 20.*** *Let 'ℓ' is the length of medians from the vertex A to the side BC of a ΔABC, then*

(a) $4\ell^2=2b^2+2c^2-a^2$

(b) $4\ell^2=b^2+c^2+2bc\cos A$

(c) $4\ell^2=a^2+4bc\cos A$

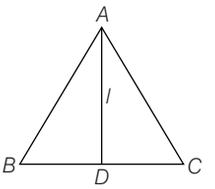
(d) $4\ell^2=(2s-a)^2-4bc\sin^2\frac{A}{2}$

Sol. (a, b, c, d) D is the mid-point of line BC.

$$AB^2+AC^2=2(AD^2+BD^2)$$

$$c^2+b^2=2\left[l^2+\left(\frac{a}{2}\right)^2\right]$$

$$\Rightarrow \quad c^2+b^2=2l^2+\frac{a^2}{2}$$



$$4l^2=2b^2+2c^2-a^2$$
$$=b^2+c^2+(b^2+c^2-a^2)$$
$$=b^2+c^2+2bc\cos A$$
$$=(b^2+c^2-a^2)+a^2+2bc\cos A$$
$$=2bc\cos A+a^2+2bc\cos A$$
$$=4bc\cos A+a^2$$

● ***Ex. 21.*** *If a right angled ΔABC of maximum area is inscribed within a circle of radius R, then (Δ represents area of ΔABC and r, r_1, r_2, r_3 represent in-radius and ex-radii, and s is the semi-perimeter of ΔABC, then*

(a) $\Delta=R^2$ (b) $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{\sqrt{2}+1}{R}$

(c) $r=(\sqrt{2}-1)R$ (d) $s=(1+\sqrt{2})R$

Sol. (a, b, c, d) For a right angled triangle inscribed in a circle of radius R the length of the hypotenuse is $2R$.

Then, area is maximum when its is isosceles triangle

With each side $=\sqrt{2}\,R$

$$\therefore \quad S=\frac{1}{2}(2\sqrt{2}+2)R=(\sqrt{2}+1)R$$

$$\Delta=\frac{1}{2}\sqrt{2}R\cdot\sqrt{2}R=R^2$$

$$r=\frac{\Delta}{S}=\frac{R^2}{(\sqrt{2}+1)R}$$

$$\Rightarrow \quad r=(\sqrt{2}-1)R$$

$$\Rightarrow \quad \frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{1}{r}=\frac{1}{(\sqrt{2}-1)R}$$

$$=\frac{\sqrt{2}+1}{R}$$

JEE Type Solved Examples :
Statement I and II Answer Type Questions

■ This section contains 3 questions. Each question contains **Statement I** (**Assertion**) and **Statement II** (**Reason**). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are

(a) Statement I is True, Statement II is True; Statement II is a correct explanation for Statement I.

(b) Statement I is True, Statement II is True; Statement II is NOT a correct explanation for Statement I.

(c) Statement I is True, Statement II is False.

(d) Statement I is False, Statement II is True.

● ***Ex. 22.*** **Statement I** *In a ΔABC, if $a < b < c$ and r is inradius and r_1, r_2, r_3 are the exradii opposite to angle A, B, C respectively, then $r < r_1 < r_2 < r_3$.*

Statement II *For, $\Delta ABC\ r_1r_2 + r_2r_3 + r_3r_1 = \dfrac{r_1r_2r_3}{r}$*

Sol. (b) **Statement I** $a < b < c$

$$s - a > s - b > s - c$$

$$s > s - a > s - b > s - c$$

$$\frac{\Delta}{s} < \frac{\Delta}{s-a} < \frac{\Delta}{s-b} < \frac{\Delta}{s-c}$$

$$r < r_1 < r_2 < r_3$$

Statement II $r_1 = \dfrac{\Delta}{s-a}, r_2 = \dfrac{\Delta}{s-b}, r_3 = \dfrac{\Delta}{s-c}, r = \dfrac{\Delta}{s}$

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{s}{\Delta} = \frac{1}{r}$$

● ***Ex. 23.*** **Statement I** *If the sides of a triangle are 13, 14, 15 then the radius of in circle = 4*

Statement II *In a ΔABC, $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \dfrac{a+b+c}{2}$ and $r = \dfrac{\Delta}{s}$*

Sol. (a) $s = 21$

$$\Delta = \sqrt{21 \cdot 8 \cdot 7 \cdot 6} = \sqrt{3 \cdot 7 \cdot 2^4 \cdot 7 \cdot 3} = 3 \cdot 7 \cdot 4 = 84$$

$$r = \frac{\Delta}{s} = \frac{84}{21} = 4$$

● ***Ex. 24.*** **Statement I** *In a ΔABC, $\Sigma \dfrac{\cos^2 \frac{A}{2}}{a}$ has the value equal to $\dfrac{s^2}{abc}$.*

Statement II *In a ΔABC, $\cos\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$*

$\cos\dfrac{\beta}{2} = \sqrt{\dfrac{(s-a)(s-c)}{ac}}, \cos\dfrac{c}{2} = \sqrt{\dfrac{(s-a)(s-b)}{ab}}$

Sol. (c) $\dfrac{\cos^2 \frac{A}{2}}{a} = \dfrac{s(s-a)}{abc}$

$$\therefore \quad \Sigma \frac{\cos^2 \frac{A}{2}}{a} \frac{s^2}{abc}$$

JEE Type Solved Examples :
Passage Based Questions

Passage I

(Ex. Nos. 25 to 27)

In a ΔABC, let $\tan A = 1$, $\tan B = 2$, $\tan C = 3$ and $c = 3$.

25. Area of the ΔABC is equal to

(a) $\dfrac{3\sqrt{2}}{2}$ (b) 3

(c) $2\sqrt{3}$ (d) $3\sqrt{2}$

26. The radius of the circle circumscribing the triangle ABC, is equal to

(a) $\dfrac{\sqrt{10}}{2}$ (b) $\sqrt{5}$ (c) $\sqrt{10}$ (d) $\dfrac{\sqrt{5}}{2}$

27. Let Δ denote the area of the ΔABC and Δp be the area of its pedal triangle. If $\Delta = k\,\Delta p$, then k is equal to

(a) $\sqrt{10}$ (b) $2\sqrt{5}$

(c) 5 (d) $2\sqrt{10}$

Sol. (Ex. Nos. 25 to 27) We have,

$\tan A = 1 \Rightarrow \sin A = \dfrac{1}{\sqrt{2}}$;

$\tan B = 2 \Rightarrow \sin B = \dfrac{2}{\sqrt{5}}$;

$\tan C = 3 \Rightarrow \sin C = \dfrac{3}{\sqrt{10}}$

Using Sine law,

$$a\sqrt{2} = \frac{b\sqrt{5}}{2} = \frac{c\sqrt{10}}{3}$$

$$\therefore \quad a\sqrt{2} = \frac{b\sqrt{5}}{2} = \sqrt{10} \qquad [\text{As } c = 3]$$

Now, $a = \sqrt{5}; b = 2\sqrt{2}; c = 3$

25. (b) $\therefore \Delta = \frac{1}{2} \cdot \sqrt{5} \cdot 2\sqrt{2} \sin C = \sqrt{10} \cdot \frac{3}{\sqrt{10}} = 3$

26. (a) $R = \frac{abc}{4\Delta} = \frac{\sqrt{5} \cdot 2\sqrt{2} \cdot 3}{4 \cdot 3} = \frac{\sqrt{10}}{2}$

27. (c) We know that, $\Delta_p = 2\Delta \cos A \cos B \cos C$

$$= 2 \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{5}} \times \frac{1}{\sqrt{10}} \times \Delta$$

$$= \frac{\Delta}{5}$$

$$\therefore \quad \Delta = 5\Delta_p$$

$$\Rightarrow \quad k = 5$$

Passage II

(Ex. Nos. 28 to 30)

Let ABC be any triangle and P be a point inside it such that $\angle PAB = \frac{\pi}{18}, \angle PBA = \frac{\pi}{9}, \angle PCA = \frac{\pi}{6}, \angle PAC = \frac{2\pi}{9}$. Let $\angle PCB = x$

28. x is equal to

(a) $\frac{\pi}{9}$ (b) $\frac{2\pi}{9}$

(c) $\frac{\pi}{3}$ (d) None of these

29. ΔABC is

(a) Equilateral (b) Isosceless

(c) Scalene (d) Right angled

30. Which of the following is true

(a) $BC > AC$ (b) $AC = AB$

(c) $AC > AB$ (d) $BC = AC$

Sol. (Ex. Nos 28 to 30)

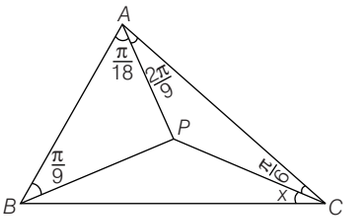
28. (a) In ΔPAB, $\frac{PA}{\sin 20} = \frac{PB}{\sin 10}$

$$\Rightarrow \quad \frac{PA}{PB} = \frac{\sin 20^\circ}{\sin 10^\circ}$$

Similarly, In ΔPBC and ΔPAC, $\frac{PB}{PC} = \frac{\sin x}{\sin(80^\circ - x)}$,

$$\frac{PC}{PA} = \frac{\sin 40^\circ}{\sin 30^\circ}.$$

$$\frac{PA}{PB} \times \frac{PB}{PC} \times \frac{PC}{PA} = 1$$



$$\Rightarrow \quad \frac{\sin 20^\circ}{\sin 10^\circ} \times \frac{\sin x}{\sin(80^\circ - x)} \times \frac{\sin 40^\circ}{\sin 30^\circ} = 1$$

$$\sin(80^\circ - x) = 4 \cos 10^\circ \sin 40^\circ \sin x = 2 \sin 50^\circ \sin x + \sin x$$

$$\Rightarrow \quad \sin(80^\circ - x) \sin x = 2 \sin 50^\circ \sin x$$

$$\Rightarrow \quad \sin(80^\circ - x) \cos 40^\circ = 2 \sin 50^\circ \sin x$$

$$\Rightarrow \quad \sin(40^\circ - x) = \sin x$$

$$\Rightarrow \quad x = 20^\circ$$

29. (b) $\angle A = \angle C = 50^\circ$

30. (c) $\angle ABC = 80^\circ$

$\therefore$ AC is longest side.

Passage III

(Ex. Nos. 31 to 33)

Let ΔABC be any triangle and D, E, F feet of perpendicular from vertices A, B, C on opposite side BC, CA, AB, respectively. Then, the ΔDEF is know as pedal Δ of ABC. H is orthocentre of the ΔABC. We note that $\angle HDC = \angle HEC = 90^\circ$, so the points H, D, C and E are concyclic.

In question ΔABC is assumed an acute angled triangle. In case ΔABC be obtuse angled with A as obtuse angle. The angle of pedal Δ will be $2A - 180^\circ, 2B, 2C$ and side will be represented by $-a \cos A, b \cos B, c \cos C$.

31. If l, m, n denote the side of a pedal triangle, then $\frac{l}{a^2} + \frac{m}{b^2} + \frac{n}{c^2}$ is equal to

(a) $\frac{a^2 + b^2 + c^2}{a^3 + b^3 + c^3}$ (b) $\frac{a^2 + b^2 + c^2}{2abc}$

(c) $\frac{a^3 + b^3 + c^3}{abc(a + b + c)}$ (d) $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$

32. If R be circum-radius of a Δ, then circum-radius of a pedal Δ is

(a) R (b) $\frac{2R}{3}$

(c) $\frac{R}{3}$ (d) $\frac{R}{2}$

33. The in-radius of pedal Δ of a ΔABC

(a) $\frac{R}{2}$ (b) $R \sin A \sin B \sin C$

(c) $2R \cos A \cos B \cos C$ (d) $4R \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$

Sol. (Ex. Nos 31 to 33)

31. (b) $l = -a \cos A, m = b \cos B, n = c \cos C$

$$\frac{l}{a^2} + \frac{m}{b^2} + \frac{n}{c^2} = \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$$

$$= \frac{a^2 + b^2 + c^2}{2abc}$$

32. (d) Circum-radius $= \dfrac{EF}{2\sin(\angle FDE)}$

$$= \frac{R\sin 2A}{2\sin(180^\circ - 2A)} = \frac{R}{2}$$

33. (c) Area of pedal $\Delta = \dfrac{1}{2}(DE)\,DF\sin(\angle EDF) = \dfrac{1}{2}(R\sin 2B)(R\sin 2C)\sin(180^\circ - 2A)$

$$= \frac{1}{2}R^2\sin 2A\sin 2B\sin 2C$$

Semi-perimeter of pedal Δ,

$$S = \frac{1}{2}R(\sin 2A + \sin 2B + \sin 2C)$$

In-radius $= \dfrac{\Delta}{S} = 2R\cos A\cos B\cos C.$

JEE Type Solved Examples : Matching Type Questions

● ***Ex. 34.*** *Match the statement of Column I with value of Column II.*

Column I		Column II	
(A)	In a triangle ABC if $a^4 - 2(b^2 + c^2)a^2 + b^2 + b^2c^2 + c^4 = 0$, then $\angle A$	p.	30°
(B)	In a triangle ABC, If $a^4 + b^4 + c^4 = a^2b^2 + 2b^2c^2 + 2c^2a^2$, then $\angle C$ is	q.	60°
(C)	In a triangle ABC, If $a^4 + b^4 + c^4 + 2a^2c^2 = 2a^2b^2 + 2b^2c^2$, then $\angle B$ is	r.	90°
		s.	120°
		t.	150°

Sol. (A) $\to (q, s)$; (B) $\to (p, t)$; C $\to (r)$

(A) $\because a^4 - 2(b^2 + c^2)a^2 + b^4 + b^2c^2 + c^4 = 0$

$\Rightarrow (b^2 + c^2 - a^2)^2 - (b^2 + c^2)^2 + b^4 + b^2c^2 + c^4 = 0$

$\Rightarrow (b^2 + c^2 - a^2)^2 = b^2c^2$

$\Rightarrow \left(\dfrac{b^2 + c^2 - a^2}{2bc}\right) = \dfrac{1}{4}$

$\Rightarrow \cos^2 A = \dfrac{1}{4}$

$\therefore \cos A = \pm\dfrac{1}{2}$

$\therefore A = 60^\circ$ or $120^\circ\,(q, s)$

(B) $\because \cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

or $a^2 + b^2 - c^2 + 2ab\cos C$

Squaring both sides, then

$a^4 + b^4 + c^4 + 2a^2b^2 - 2a^2c^2 - 2b^2c^2 = 4a^2b^2\cos^2 C$

$\Rightarrow 3a^2b^2 = 4a^2b^2\cos^2 C$

$(\because a^4 + b^4 + c^4 = a^2b^2 + 2b^2c^2 + 2c^2a^2)$

or $\cos^2 C = \dfrac{3}{4}$

$\cos C = \pm\dfrac{\sqrt{3}}{2}$

$\therefore C = 30^\circ$ or $150^\circ\,(p, t)$

(C) $\because a^4 + b^4 + c^4 + 2a^2c^2 = 2a^2b^2 + 2b^2c^2$

$\Rightarrow b^4 - 2(a^2 + c^2)b^2 + (a^2 + c^2)^2 = 0$

$\Rightarrow \{b^2 - (a^2 + c^2)\}^2 = 0$

$\Rightarrow\therefore a^2 + c^2 = b^2$

$\Rightarrow \cos B = 0$

$\therefore B = 90^\circ (r)$

JEE Type Solved Examples : Single Integer Answer Type Questions

● ***Ex. 35.*** *In a ΔABC, if $r_1 + r_3 + r = r_2$, then find the value of $(\sec^2 A + \cos^2 B - \cot^2 C)$.*

[**Note** All symbols used have usual meaning in a triangle.]

Sol. (1) We have, $r_1 + r_3 + r = r_2$

$\Rightarrow \left(\dfrac{\Delta}{s-a} + \dfrac{\Delta}{s-c}\right) = \left(\dfrac{\Delta}{s-b} - \dfrac{\Delta}{s}\right)$

$\Rightarrow \dfrac{2s - (a + c)}{(s-a)(s-c)} = \dfrac{b}{s(s-b)}$

$\Rightarrow (s-a)(s-c) = s(s-b)$

$\Rightarrow s^2 - (a + c)s + ac = s^2 - bs$

$\Rightarrow (a + c - b)s - ac = 0$

$\Rightarrow (a + c - b)\dfrac{(a + b + c)}{2} = ac$

$$\Rightarrow \quad (a+c)^2 - b^2 = 2ac$$
$$\Rightarrow \quad a^2 + c^2 + 2ac - b^2 = 2ac$$
$$\Rightarrow \quad a^2 + c^2 = b^2$$

So, $\angle B = \frac{\pi}{2}$ and $\angle A + \angle C = 90°$

Now,

$$\sec^2 A + \cos^2 B - \cot^2 C = \sec^2(90°-C) + \cos^2 90° - \cot^2 C$$
$$= \operatorname{cosec}^2 C + 0 - \cot^2 C$$
$$= 1$$

Hence, $(\sec^2 A + \cos^2 B - \cot^2 C) = 1$

Alternatively We have,

$$(r_1 + r_3) - (r_2 - r) = 0 \qquad ...(i)$$

As $r_1 + r_3 = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} + 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

$$= 4R\cos\frac{B}{2}\sin\left(\frac{A+C}{2}\right) = 4R\cos^2\frac{B}{2} \qquad ...(ii)$$

Also, $r_2 - r = 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2} - 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$$= 4R\sin\frac{B}{2}\cos\left(\frac{A+C}{2}\right) = 4R\sin^2\frac{B}{2} \qquad ...(iii)$$

$\therefore$ Using Eqs. (ii) and (iii) in Eq. (i), we get

$$4R\left(\cos^2\frac{B}{2} - \sin^2\frac{B}{2}\right) = 0$$
$$\Rightarrow \quad 4R\cos B = 0$$
$$\Rightarrow \quad \angle B = \frac{\pi}{2} \text{ and } \angle A + \angle C = 90°$$

Hence, $(\sec^2 A + \cos^2 B - \cot^2 C) = 1$

● ***Ex. 36.*** *In ΔABC, let $b = 6, c = 10$ and $r_1 = r_2 + r_3 + r$ then find area of ΔABC.*

[**Note** All symbols used have usual meaning in a triangle.]

Sol. (30) We have, $r_1 = r_2 + r_3 + r$ [given]

$$\Rightarrow \quad (r_1 - r) = (r_2 + r_3)$$
$$\Rightarrow \quad \frac{s-(s-a)}{s(s-a)} = \frac{2s-(b+c)}{(s-b)(s-c)}$$
$$\Rightarrow \quad \frac{(s-b)(s-c)}{s(s-a)} = 1$$
$$\Rightarrow \quad \tan^2\frac{A}{2} = 1 \Rightarrow \frac{A}{2} = 45°$$

Hence, $A = 90°$

Now, area of $\Delta ABC = \frac{1}{2}bc\sin A$

$$= \frac{1}{2}(6)(10)\sin 90°$$
$$= 30 \text{ sq units.}$$

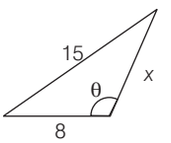
● ***Ex. 37.*** *Consider an obtuse angled triangles with side 8 cm, 15 cm and x cm (largest side being 15 cm). If x is an integer, then find the number of possible triangles.*

Sol. (5) Since, 15 be the largest side

$$\cos\theta = \frac{x^2 + 8^2 - 15^2}{16x}$$

for obtuse angled $-1 < \cos\theta < 0$

$$-16x < x^2 - 161 < 0$$



$$\Rightarrow \quad x^2 < 161$$
$$\Rightarrow \quad x < 13 \qquad ...(i)$$

and $x^2 + 16x - 161 > 0$

$$\Rightarrow \quad (x+23)(x-7) > 0$$

or $x > 7 \qquad ...(ii)$

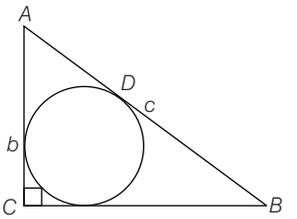
$$\therefore \quad 7 < x < 13$$
$$\Rightarrow \quad x \in \{8, 9, 10, 11, 12\}$$

Hence, number of possible values of x is 5.

● ***Ex. 38.*** *Let ABC be a right angled triangle at C. If the inscribed circle touches the side AB to D and $(AD)(BD) = 11$, then find the area of ΔABC.*

Sol. (11) We have,

$$(AD)(BD) = 11$$
$$\Rightarrow \quad (s-a)(s-b) = 11$$
$$\Rightarrow \quad (2s-2a)(2s-2b) = 44$$
$$\Rightarrow (b+c-a)(a+c-b) = 44$$
$$\Rightarrow \quad c^2 - (b-a)^2 = 44$$
$$\Rightarrow \quad 2ab = 44$$
$$[\text{As } c^2 = a^2 + b^2]$$
$$\Rightarrow \quad ab = 22$$



Now, area $(\Delta ABC) = \frac{1}{2}ab = \frac{1}{2}(22) = 11$

● ***Ex. 39.*** *Consider a ΔABC and let a, b and c denote the lengths of the sides opposite to vertices A, B and C, respecively. Suppose $a = 2, b = 3, c = 4$ and H be the orthocentre. Find $15(HA)^2$.*

Sol. (196) We know that, $HA = 2R\cos A$, where

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{9 + 16 - 4}{24}$$
$$= \frac{21}{24} = \frac{7}{8}$$
$$\therefore \quad \sin A = \sqrt{1 - \frac{49}{64}} = \frac{\sqrt{15}}{8}$$

Now, $\frac{a}{\sin A} = 2R$

$\Rightarrow \quad 2R = \frac{2\times 8}{\sqrt{15}} = \frac{16}{\sqrt{15}}$

$\therefore \quad HA = 2R\cos A = \frac{16}{\sqrt{15}} \times \frac{7}{8}$

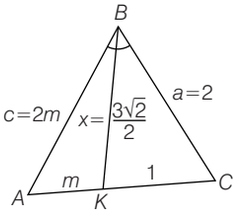
$= \frac{14}{\sqrt{15}}$

Hence, $15(AH)^2 = 196$.

● ***Ex. 40.*** *In a ΔABC, the internal angle bisector of $\angle ABC$ meets AC at K. If $BC = 2$, $CK = 1$ and $BK = \frac{3\sqrt{2}}{2}$, then find the length of side AB.*

Sol. (3) Using cosine law in ΔBKC,

$$\cos\frac{B}{2} = \frac{4 + \frac{18}{4} - 1}{2\cdot\frac{3\sqrt{2}}{2}\cdot 2}$$



$$= \frac{3 + \frac{9}{2}}{6\sqrt{2}} = \frac{15}{12\sqrt{2}} = \frac{5}{4\sqrt{2}}$$

Now, $x = \frac{2ac}{a+c}\cos\frac{B}{2} = \frac{(2\cdot 2\cdot 2\,\mathrm{m})}{2m+2}\cdot\frac{5}{4\sqrt{2}}$

$\Rightarrow \quad \frac{3\sqrt{2}}{2} = \frac{4m}{m+1}\cdot\frac{5}{4\sqrt{2}}$

$\Rightarrow \quad 3 = \frac{5m}{m+1}$

$\Rightarrow \quad 3m + 3 = 5m$

$m = \frac{3}{2}$

$\therefore \quad AB = 2m$

$\therefore \quad c = 3$

● ***Ex. 41.*** *In ΔABC has $AC = 13$, $AB = 15$ and $BC = 14$. Let O be the circumcentre of the ΔABC. If the length of perpendicular from the point 'O' on BC can be expressed as a rational $\frac{m}{n}$ in the lowest form, then find $(m+n)$.*

Sol. $(41)\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

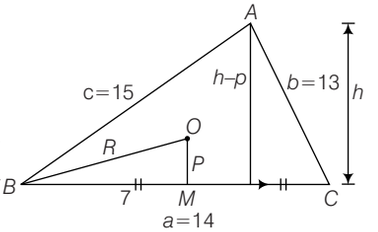
$= \sqrt{21\cdot 8\cdot 6\cdot 7} = \sqrt{84}$

$R = \frac{abc}{4\Delta} = \frac{(14)(13)(15)}{4\cdot 84} = \frac{2\cdot 13\cdot 15}{4\cdot 12} = \frac{65}{8}$

$\therefore \quad p = \sqrt{R^2 - 7^2}$

$= \sqrt{\left(\frac{65}{8}\right)^2 - 7^2}$

$= \frac{\sqrt{(65)^2 - (56)^2}}{8}$



$= \frac{\sqrt{121\cdot 9}}{8}$

$= \frac{33}{8} = \frac{m}{n}$

$\Rightarrow \quad m + n = 33 + 8 = 41.$

Subjective Type Examples

● ***Ex. 42.*** *Two sides of a triangle are given by the roots of the equation $x^2 - 2\sqrt{3}x + 2 = 0$. The angle between the sides is $\frac{\pi}{3}$. Find the perimeter of Δ.*

Sol. We have, two sides of a triangle are given by roots of the equation $x^2 - 2\sqrt{3}x + 2 = 0$.

$\Rightarrow \quad a + b = 2\sqrt{3}$

and $\quad ab = 2$ with $\angle C = \frac{\pi}{3}$ ...(i)

using cosine law, we have

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$\Rightarrow \quad \cos\frac{\pi}{3} = \frac{a^2 + b^2 - c^2}{2ab}$ [using Eq. (i)]

$\Rightarrow \quad \frac{1}{2} = \frac{a^2 + b^2 - c^2}{2ab}$

$\Rightarrow \quad a^2 + b^2 - c^2 = ab$

$\Rightarrow \quad (a+b)^2 - 2ab - c^2 = ab$ [using Eq. (i)]

$\Rightarrow \quad 12 - 4 - c^2 = 2$

$c^2 = 6;\ c = \sqrt{6}$

$\therefore$ The perimeter of Δ

$= a + b + c = 2\sqrt{3} + \sqrt{6}$

● ***Ex. 43.*** *If in ΔABC, $\angle A = 90°$ and c, $\sin B$, $\cos B$ are rational numbers, then show a and b are rational.*

Sol. Let AD be perpendicular from A to BC.

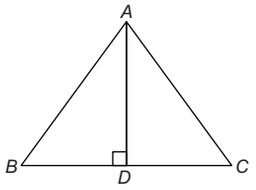
Then, $\cos B = \frac{BD}{c}$

$\Rightarrow$ $BD = c\cos B$

$\Rightarrow$ BD is rational.

Similarly, $AD = c\sin B$

$\Rightarrow$ AD is rational.



Now, $\sin C = \cos B = \frac{AD}{b}$

$\Rightarrow$ b is rational.

Since, $\cos C = \sin B = \frac{DC}{b}$

$\Rightarrow$ DC is rational.

Hence, $a = BD + DC$ is rational.

Thus, a is rational and b is rational.

● ***Ex. 44.*** *If the sides of a triangle ABC are in AP and 'a' is the smallest side, then express* $\cos A$ *in terms of b and c.*

Sol. Since sides of the triangle are in AP.

i.e. a, b, c are in AP and let $a < b < c$.

$\therefore$ $2b = a + c$ ...(i)

Now, $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

$= \frac{b^2 + c^2 - (2b - c)^2}{2bc}$ [using Eq. (i)]

$= \frac{b^2 + c^2 - 4b^2 - c^2 + 4bc}{2bc}$

$= \frac{4bc - 3b^2}{2bc}$

$\therefore$ $\cos A = \frac{4c - 3b}{2c}$

● ***Ex. 45.*** *If A, B and C are angles of a triangle such that* $\angle A$ *is obtuse, then show* $\tan B \tan C < 1$.

Sol. Since, A is obtuse angle,

then $90° < A < 180°$

$\Rightarrow$ $90° < 180 - (B + C) < 180°$

$\Rightarrow$ $-90° < -(B + C) < 0$

$\Rightarrow$ $90° > B + C > 0$

$\Rightarrow$ $B + C < 90°$

$\Rightarrow$ $B < 90° - C$

$\therefore$ $\tan B < \tan(90° - C)$

$\tan B < \cot C$

or $\tan B \cdot \tan C < 1$

If A is obtuse, then $\tan A \tan C < 1$.

● ***Ex. 46.*** *If A is the area and* $2s$ *the sum of the sides of a triangle, then show* $A \le \frac{s^2}{3\sqrt{3}}$.

Sol. We have, $2s = a + b + c$,

$A^2 = s(s - a)(s - b)(s - c)$

Now, AM $\ge$ GM

$\Rightarrow$ $\frac{s + (s - a) + (s - b) + (s - c)}{4}$

$\ge \{(s - a)(s - b)(s - c)\}^{1/4}$

$\Rightarrow$ $\frac{4s - 2s}{4} \ge (A^2)^{1/4} \Rightarrow \frac{s}{2} \ge A^{1/2}$

$\Rightarrow$ $A \le \frac{s^2}{4}$ ...(i)

Also, $\frac{(s - a) + (s - b) + (s - c)}{3}$

$\ge \{(s - a)(s - b)(s - c)\}^{1/3}$

or $\frac{3s - 2s}{3} \ge \left(\frac{A^2}{s}\right)^{1/3}$

or $\frac{s}{3} \ge \left(\frac{A^2}{s}\right)^{1/3}$ or $\frac{A^2}{s} \le \frac{s^3}{27}$

$\Rightarrow$ $A \le \frac{s^2}{3\sqrt{3}}$ ...(ii)

Thus, from Eqs. (i) and (ii);

$A \le \frac{s^2}{3\sqrt{3}}$

● ***Ex. 47.*** *In a triangle, if* $r_1 > r_2 > r_3$, *then show* $a > b > c$.

Sol. We have, $r_1 > r_2 > r_3$

$\Rightarrow$ $\frac{\Delta}{s - a} > \frac{\Delta}{s - b} > \frac{\Delta}{s - c}$

$\Rightarrow$ $\frac{s - a}{\Delta} < \frac{s - b}{\Delta} < \frac{s - c}{\Delta}$

$\Rightarrow$ $s - a < s - b < s - c$

$\Rightarrow$ $-a < -b < -c$ or $a > b > c$

● ***Ex. 48.*** *ABC is a triangle and D is the middle point of BC. If AD is perpendicular to AC, then prove that*

$$\cos A \cdot \cos C = \frac{2(c^2 - a^2)}{3ac}.$$

Sol. D is the mid-point of BC

$\Rightarrow$ $BD = DC = \frac{a}{2}$

Draw BE perpendicular to CA (produced).

$\angle DAC = \angle BEC = 90°$

In Δ's ADC and EBC,

$\angle BEC = \angle DAC = 90°$

and $\angle C$ is common.

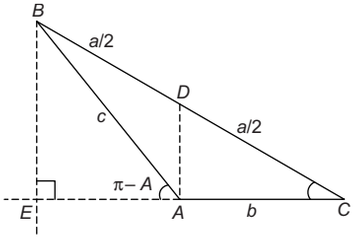
Hence, $\angle CDA = \angle CBE = 90° - C$

In Δ's ADC and EBC are similar

$\Rightarrow \quad \frac{DC}{BC} = \frac{DA}{BE} = \frac{CA}{CE} \Rightarrow EA = b$

and if $DA = y$, then $BE = 2y$

Now, $\cos C = \frac{b}{a/2} = \frac{2b}{a}$



and $\cos(\pi - A) = \frac{b}{c}$

$$\cos A = -\frac{b}{c}$$

$\therefore \quad \cos A \cdot \cos C = -\frac{2b^2}{ac}$ ...(i)

In ΔADC, $y^2 + b^2 = \frac{a^2}{4}$

$\Rightarrow \quad 4y^2 + 4b^2 = a^2$

In ΔBAE, $4y^2 + b^2 = c^2 \Rightarrow c^2 + 3b^2 = a^2$

$\Rightarrow \quad b^2 = \frac{a^2 - c^2}{3}$ ...(ii)

From Eqs. (i) and (ii), $\cos A \cdot \cos C = \frac{2(c^2 - a^2)}{3ac}$

● **Ex. 49.** *In a ΔABC, prove that*

$a^3 \cos(B - C) + b^3 \cos(C - A) + c^3 \cos(A - B) = 3abc$

Sol. First term of LHS $= a^3 \cos(B - C)$

$= a^2[a\cos(B - C)]$, where $a = 2R\sin A$

$= a^2 \cdot 2R\sin A \cdot \cos(B - C)$

$= a^2 2R\sin(B + C)\cos(B - C)$ [as $A + B + C = \pi$

$\therefore \sin A = \sin(\pi - (B + C)) = \sin(B + C)$]

$= a^2 \cdot R[2\sin(B + C) \cdot \cos(B - C)]$

$= a^2 \cdot R[\sin 2B + \sin 2C]$

$= a^2 \cdot R[2\sin B\cos B + 2\sin C\cos C]$

$= a^2[b\cos B + c\cos C]$ ...(i)

[as, $2R\sin B = b$ and $2R\sin C = c$]

Similarly,

$b^3 \cos(C - A) = b^2[a\cos A + c\cos C]$ ...(ii)

and $c^3 \cos(A - B) = c^2[a\cos A + b\cos B]$ ...(iii)

$\therefore$ From Eqs. (i), (ii) and (iii), we get,

$a^3 \cos(B - C) + b^3 \cos(C - A) + c^3 \cos(A - B)$

$= a^2 b\cos B + a^2 c\cos C + b^2 a\cos A +$

$b^2 c\cos C + c^2 a\cos A + c^2 b\cos B$

$= ab(a\cos B + b\cos A) + ac(a\cos C + c\cos A) +$

$bc(b\cos C + c\cos B)$

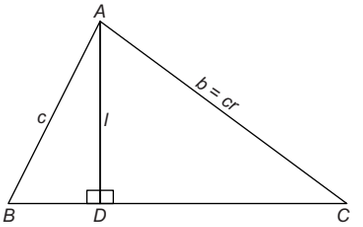
$= ab \cdot c + ac \cdot b + bc \cdot a$ [using projection formula]

$= 3abc$

$\therefore \quad a^3 \cos(B - C) + b^3(C - A) + c^3 \cos(A - B) = 3abc$

● **Ex. 50.** *In a triangle of base 'a', the ratio of the other sides is r (<1). Show that the altitude of the triangle is less than or equal to $\frac{ar}{1 - r^2}$.*

Sol. Let D be the foot of the altitude from A,



and $BC = a$, $AB = c$, $AC = cr$, $AD = l = C\sin B$

Now, $\frac{ar}{1 - r^2} = \frac{ac^2 r}{c^2 - r^2 c^2} = \frac{abc}{c^2 - b^2}$ [using Sine law]

$= \frac{a\sin B\sin C}{\sin^2 C - \sin^2 B}$

$= \frac{a\sin B\sin C}{\sin(C - B).\sin(C + B)}$

$= \frac{a\sin B.\sin C}{\sin A\sin(C - B)}$

$= \frac{c\sin B}{\sin(C - B)} = \frac{l}{\sin(C - B)}$

$\Rightarrow \quad l \le \frac{ar}{1 - r^2}$.

● **Ex. 51.** *Three circles touch one-another externally. The tangents at their points of contact meet at a point whose distance from a point of contact is 4. Find the ratio of the product of the radii to the sum of the radii of circles.*

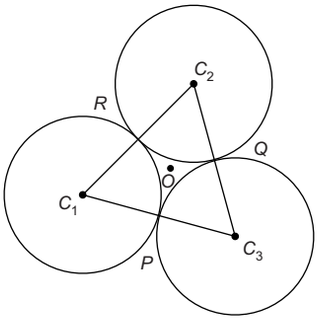
Sol. Let r_1, r_2 and r_3 be the radii of the three circles with centres at C_1, C_2 and C_3. Let the circles touch at P, Q and R.

Also, $C_1C_2 = r_1 + r_2$

$C_2C_3 = r_2 + r_3$

$C_3C_1 = r_3 + r_1$

Let O be the point whose distance from the points of contact is 4.



Then, O is the in-centre of $\Delta C_1C_2C_3$ with $OP = OQ = OR = 4$ being the radius of the in-circle.

Hence, $$4 = \frac{\Delta C_1C_2C_3}{\frac{1}{2}[C_1C_2 + C_2C_3 + C_3C_1]} = \frac{S}{s} \quad \ldots(i)$$

where, $s = r_1 + r_2 + r_3$,

$$S^2 = s(s - C_1C_2)(s - C_2C_3)(s - C_3C_1)$$
$$= s r_1 r_2 r_3.$$

$\therefore$ Eq. (i) gives, $$16 = \frac{S^2}{s^2} = \frac{s\, r_1r_2r_3}{s^2} = \frac{r_1r_2r_3}{r_1 + r_2 + r_3}$$

Hence, the ratio of the product of the radii to the sum of the radii = 16 : 1.

● ***Ex. 52.*** *The internal bisectors of the angles of a ΔABC meet the sides BC, CA, AB in D, E and F, respectively. Show that the area of the ΔDEF is equal to,*

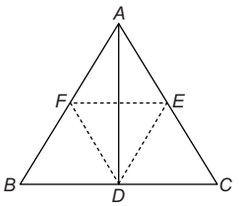
$$\frac{2\Delta abc}{(b+c)(c+a)(a+b)}$$

Sol. AD is the internal bisector of $\angle A$.

$$\Rightarrow \quad \frac{BD}{DC} = \frac{BA}{AC} = \frac{c}{b}$$

$$\Rightarrow \quad \frac{BD}{c} = \frac{DC}{b} = \frac{BD + DC}{c + b} = \frac{a}{b + c}$$

$$\Rightarrow \quad BD = \frac{ac}{b + c},\ DC = \frac{ab}{b + c}$$



Similarly, $$BF = \frac{ac}{a + b}$$

$$\Rightarrow \quad \text{Area of } \Delta BFD = \frac{1}{2}(BF \cdot BD) \cdot \sin B$$

$$\Rightarrow \quad \frac{\text{Area of } \Delta BFD}{\text{Area of } \Delta ABC} = \frac{(BF \cdot BD) \cdot \sin B}{ac \sin B}$$
$$= \frac{ac}{(a + b)(b + c)}$$

Now, area of $\Delta DEF = \Delta ABC - (\Delta BFD + \Delta DEC + \Delta AFE)$

$$\Rightarrow \quad \frac{\text{Area of } \Delta DEF}{\text{Area of } \Delta ABC} = 1 - \left(\frac{\Delta BFD + \Delta DEC + \Delta AFE}{\Delta ABC}\right)$$
$$= 1 - \left\{\frac{ac}{(a + b)(b + c)} + \frac{cb}{(c + a)(b + a)} + \frac{ab}{(a + c)(b + c)}\right\}$$
$$= \frac{2abc\,\Delta}{(a + b)(b + c)(c + a)}$$

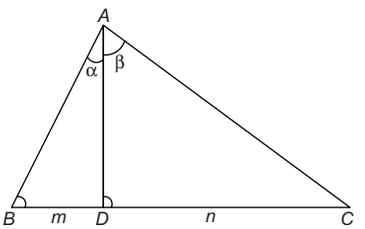
$$\Rightarrow \quad \text{Area of } \Delta DEF = \frac{2abc\ \Delta}{(a + b)(b + c)(c + a)}$$

● ***Ex. 53.*** *Let D be a point on the side BC of a ΔABC such that $BD : DC = m : n$ and $\angle ADC = \theta$, $\angle BAD = \alpha$ and $\angle DAC = \beta$. Prove that*

(i) $(m + n)\cot\theta = m\cot\alpha - n\cot\beta$

(ii) $(m + n)\cot\theta = n\cot B - m\cot C$

Sol. (i) Given, $$\frac{BD}{DC} = \frac{m}{n} \quad \text{and} \quad \angle ADC = \theta$$



Since, $\angle ADB = (180° - \theta)$, $\angle BAD = \alpha$

and $\angle DAC = \beta$

$\angle ABD = 180° - (\alpha + 180° - \theta) = \theta - \alpha$

and $\angle ACD = 180° - (\theta + \beta)$

From ΔABD,

$$\frac{BD}{\sin\alpha} = \frac{AD}{\sin(\theta - \alpha)} \quad \ldots(i)$$

From ΔADC,

$$\frac{DC}{\sin\beta} = \frac{AB}{\sin\{180° - (\theta + \beta)\}}$$

or $$\frac{DC}{\sin\beta} = \frac{AD}{\sin(\theta + \beta)} \quad \ldots(ii)$$

On dividing Eq. (i) by Eq. (ii), we get

$$\frac{BD\sin\beta}{DC\sin\alpha} = \frac{\sin(\theta + \beta)}{\sin(\theta - \alpha)}$$

or $$\frac{m}{n} \cdot \frac{\sin\beta}{\sin\alpha} = \frac{\sin\theta\cos\beta + \cos\theta\sin\beta}{\sin\theta\cos\alpha - \cos\theta\sin\alpha}$$

or $$m\sin\beta(\sin\theta\cos\alpha - \cos\theta\sin\alpha)$$
$$= n\sin\alpha(\sin\theta\cos\beta + \cos\theta\sin\beta)$$

or $$(m + n)\cot\theta = m\cot\alpha - n\cot\beta$$

(ii) Substitute $\alpha = \theta - B$ and $\beta = 180° - (\theta + C)$ in the above result,

Simplify and obtain $(m + n)\cot\theta = n\cot B - m\cot C$.

Note This is known as $m - n$ theorem.

● **Ex. 54.** *The base of a triangle is divided into three equal parts. If t_1, t_2, t_3 be the tangents of the angles subtended by these parts at the opposite vertex, prove that :*

$$\left(\frac{1}{t_1}+\frac{1}{t_2}\right)\left(\frac{1}{t_2}+\frac{1}{t_3}\right)=4\left(1+\frac{1}{t_2^2}\right)$$

Sol. Let the points P and Q divide the side BC in three equal parts

Such that $BP = PQ = QC = x$

Also let, $\angle BAP = \alpha, \angle PAQ = \beta, \angle QAC = \gamma$

and $\angle AQC = \theta$

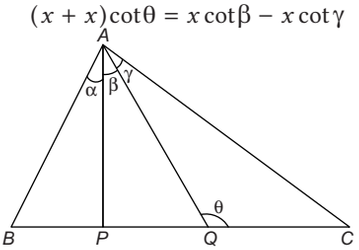
From question, $\tan\alpha = t_1, \tan\beta = t_2, \tan\gamma = t_3$.

Applying, $m : n$ rule in triangle ABC, we get

$$(2x + x)\cot\theta = 2x\cot(\alpha+\beta) - x\cot\gamma \quad ...(i)$$

From ΔAPC, we get

$$(x + x)\cot\theta = x\cot\beta - x\cot\gamma \quad ...(ii)$$



On dividing Eq. (i) and Eq. (ii), we get

$$\frac{3}{2}=\frac{2\cot(\alpha+\beta)-\cot\gamma}{\cot\beta-\cot\gamma}$$

or $3\cot\beta - \cot\gamma = \dfrac{4(\cot\alpha\cdot\cot\beta - 1)}{\cot\beta + \cot\alpha}$

or $3\cot^2\beta - \cot\beta\cot\gamma + 3\cot\alpha\cdot\cot\beta - \cot\alpha\cdot\cot\gamma = 4\cot\alpha\cdot\cot\beta - 4$

or $4 + 4\cot^2\beta = \cot^2\beta + \cot\alpha\cdot\cot\beta + \cot\beta\cdot\cot\gamma + \cot\gamma\cdot\cot\alpha$

or $4(1+\cot^2\beta) = (\cot\beta + \cot\alpha)(\cot\beta + \cot\gamma)$

or $4\left(1+\dfrac{1}{t_2^2}\right)=\left(\dfrac{1}{t_1}+\dfrac{1}{t_2}\right)\left(\dfrac{1}{t_2}+\dfrac{t}{t_3}\right)$

Hence, the result.

● **Ex. 55.** *In a ΔABC, if $\cot A + \cot B + \cot C = \sqrt{3}$. Prove that the triangle is equilateral.*

Sol. $\cot A + \cot B + \cot C = \sqrt{3}$

On squaring both sides, we get

$$\cot^2 A + \cot^2 B + \cot^2 C + 2\cot A\cdot\cot B + 2\cot B\cdot\cot C + 2\cot C\cdot\cot A = 3 \quad ...(i)$$

Also, $\cot A\cot B + \cot B\cot C + \cot C\cot A = 1 \quad ...(ii)$

From Eqs. (i) and Eqs. (ii), we have

$$\cot^2 A + \cot^2 B + \cot^2 C - \cot A\cot B - \cot C\cot B - \cot C\cot A = 0$$

or $\dfrac{1}{2}\{(\cot A - \cot B)^2 + (\cot B - \cot C)^2 + (\cot C - \cot A)^2\} = 0$

$\Rightarrow$ $\cot A = \cot B = \cot C$

[Since RHS is zero, thus each square must be zero]

$\Rightarrow$ $A = B = C$

i.e. the triangle is equilateral.

● **Ex. 56.** *In the ΔABC, if $(a^2 + b^2)\sin(A - B) = (a^2 - b^2)\sin(A + B)$. Prove that the triangle is either isosceles or right angled.*

Sol. Let $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}=K$ (say).

$\Rightarrow$ $a = K\sin A,\ b = K\sin B$

and $c = K\sin C$.

Now, the given relation is,

$$(a^2 + b^2)\sin(A - B) = (a^2 - b^2)\sin(A - B)$$

$\Rightarrow$ $K^2(\sin^2 A + \sin^2 B)\sin(A - B) = K^2(\sin^2 A - \sin^2 B)\sin(A + B)$

$\Rightarrow$ $(\sin^2 A + \sin^2 B)\sin(A - B) = \sin^2(A + B)\cdot\sin(A - B)$

or $\sin(A - B)[\sin^2 A + \sin^2 B - \sin^2 C] = 0$

Hence, either the first factor = 0;

or the second factor = 0

If $\sin(A - B) = 0 \Rightarrow A - B = 0 \Rightarrow A = B$

$\Rightarrow$ triangle is isosceles.

If $\sin^2 A + \sin^2 B - \sin^2 C = 0$

$\Rightarrow$ $\dfrac{a^2}{K^2}+\dfrac{b^2}{K^2}-\dfrac{c^2}{K^2}=0$ or $a^2 + b^2 = c^2$

$\Rightarrow$ the triangle is right angled.

Hence, the result.

● **Ex. 57.** *The sides of a ΔABC are in AP. If the $\angle A$ and $\angle C$ are the greatest and smallest angles respectively, prove that $4(1-\cos A)(1-\cos C) = \cos A + \cos C$*

Sol. Since A is the greatest and C is the smallest angle, 'a' is the greatest and 'c' is the smallest side.

$\Rightarrow$ a, b, c are in AP.

$\Rightarrow$ $2b = a + c$.

or $4R\sin B = 2R[\sin A + \sin C]$

or $2\cdot 2\sin\dfrac{B}{2}\cdot\cos\dfrac{B}{2}=2\sin\left(\dfrac{A+C}{2}\right)\cdot\cos\left(\dfrac{A-C}{2}\right)$

or $2\sin\dfrac{B}{2}=\cos\left(\dfrac{A-C}{2}\right)\quad\left[\because\dfrac{A+C}{2}=90^\circ-\dfrac{B}{2}\right]$

or $2\cos\left(\dfrac{A+C}{2}\right)=\cos\left(\dfrac{A-C}{2}\right)$ ...(i)

Now, LHS $= 4(1 - \cos A)(1 - \cos C)$

$= 4\cdot 2\sin^2 A/2\cdot 2\sin^2 C/2$

$= 4(2\sin A/2\cdot\sin C/2)^2$

$$= 4\left[\cos\frac{A-C}{2} - \cos\frac{A+C}{2}\right]^2 \left[\quad \text{[using Eq. (i)]}\right.$$

$$= \left[2\cos\frac{A-C}{2} - \cos\frac{A+C}{2}\right]^2 = 4\cos^2\left(\frac{A+C}{2}\right)$$

RHS $= \cos A + \cos C = 2\cos\frac{A+C}{2}\cdot\cos\frac{A-C}{2}$

$$= 2\cos\frac{A+C}{2}\cdot\cos\frac{A+C}{2} = 4\cos^2\frac{A+C}{2}$$

[using Eq. (i)]

Hence Proved.

● ***Ex. 58.*** *Perpendiculars are drawn from the angles A, B, C of an acute angled Δ on the opposite sides and produced to meet the circumscribing circle. If these produced parts be α, β, γ respectively, then show that*

$$\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma} = 2(\tan A + \tan B + \tan C)$$

Sol. Let AD be perpendicular from A on BC when AD is produced, it meets the circumscribing circle at E.

From question, $DE = \alpha$.

Since, angles in the same segment are equal,

$$\angle AEB = \angle ACB = \angle C$$

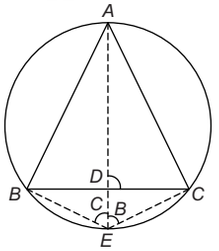
and $$\angle AEC = \angle ABC = \angle B$$

From the right angled triangle BDE,

$$\tan C = \frac{BD}{DE} \qquad \ldots(i)$$

From the right angled triangle CDE,

$$\tan B = \frac{CD}{DE} \qquad \ldots(ii)$$



On adding Eqs. (i) and (ii), we get

$$\tan B + \tan C = \frac{BD + CD}{DE}$$

$$= \frac{BC}{DE} = \frac{a}{\alpha} \qquad \ldots(iii)$$

Similarly, $$\tan C + \tan A = \frac{b}{\beta} \qquad \ldots(iv)$$

and $$\tan A + \tan B = \frac{c}{\gamma} \qquad \ldots(v)$$

On adding Eqs. (iii), (iv) and (v), we get

$$\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma} = 2(\tan A + \tan B + \tan C)$$

● ***Ex. 59.*** *In any ΔABC, if*

$$\cos\theta = \frac{a}{b+c}, \cos\phi = \frac{b}{a+c}, \cos\psi = \frac{c}{a+b}$$

where θ, φ and ψ lie between 0 and π, prove that

$$\tan^2\frac{\theta}{2} + \tan^2\frac{\phi}{2} + \tan^2\frac{\psi}{2} = 1$$

Sol. Given, $$\cos\theta = \frac{a}{b+c}$$

$$\Rightarrow \quad \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\phi}{2}} = \frac{a}{b+c}$$

$$\Rightarrow \quad \tan^2\frac{\theta}{2} = \frac{b+c-a}{a+b+c} \qquad \ldots(i)$$

Similarly, $$\tan^2\frac{\phi}{2} = \frac{a+c-b}{a+b+c} \qquad \ldots(ii)$$

and $$\tan^2\frac{\psi}{2} = \frac{a+b-c}{a+b+c} \qquad \ldots(iii)$$

On adding Eqs. (i), (ii) and (iii), we get

$$\tan^2\frac{\theta}{2} + \tan^2\frac{\phi}{2} + \tan^2\frac{\psi}{2} = \frac{a+b+c}{a+b+c}$$

$$\Rightarrow \quad \tan^2\frac{\theta}{2} + \tan^2\frac{\phi}{2} + \tan^2\frac{\psi}{2} = 1$$

● ***Ex. 60.*** *The product of the sines of the angles of a triangle is p and the product of their cosines is q. Show that the tangents of the angles are the roots of the equation;*

$$qx^2 - px^2 + (1+q)x - p = 0$$

Sol. From the question, $\sin A \cdot \sin B \cdot \cos C = p$

and $$\cos A \cdot \cos B \cdot \cos C = p$$

$$\Rightarrow \quad \tan A \cdot \tan B \cdot \tan C = \frac{p}{q} \qquad \ldots(i)$$

Also, $\tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C$

$$\Rightarrow \quad \tan A + \tan B + \tan C = \frac{p}{q} \qquad \ldots(ii)$$

Now, $\tan A \tan B + \tan B \tan C + \tan C \tan A$

$$= \frac{\sin A \cdot \sin B \cdot \cos C + \sin B \cdot \sin C \cdot \cos A + \sin C \cdot \sin A \cdot \cos B}{\cos A \cdot \cos B \cdot \cos C}$$

$$= \frac{1}{q}[\sin A \sin B \cos C + \sin C(\sin B \cdot \cos A + \cos B \cdot \sin A \sin C)]$$

$$= \frac{1}{q}[\sin A \sin B \cos C + \sin C \sin(A+B)]$$

$$= \frac{1}{q}[\sin A \cdot \sin B \cdot \cos C + \sin^2 C]$$

$$= \frac{1}{q}[1 - \cos^2 C + \sin A \sin B \cos C]$$

$$= \frac{1}{q}[1 + \cos C\{-\cos C + \sin A \sin B\}]$$

$$= \frac{1}{q}[1 + \cos C(\cos(A + B) + \sin A \sin B)]$$

$$= \frac{1}{q}[1 + \cos A \cos B \cos C] = \frac{1}{q}[1 + q] \quad \text{...(iii)}$$

The equation whose roots are $\tan A$, $\tan B$, $\tan C$ will be given by

$$x^3 - (\tan A + \tan B + \tan C)x^2 + (\tan A \tan B + \tan B \tan C + \tan C \tan A)x - \tan A \cdot \tan B \cdot \tan C = 0$$

or $$x^3 - \frac{p}{q}x^2 + \left(\frac{1+a}{q}\right)x - \frac{p}{q} = 0$$

[using Eqs. (i), (ii) and (iii)}

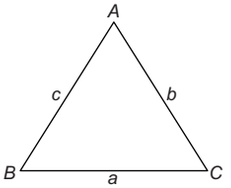
or $$qx^3 - px^2 + (1+q)x - p = 0$$ **Hence Proved.**

● ***Ex. 61.*** *Given the base 'a' of a triangle, the opposite angle A, and the product k^2 of the other two sides, show that it is not possible for 'a' to be less than $2k \sin \frac{A}{2}$.*

Sol. Given, $b \cdot c = k^2$

Now, $$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

or $$2k^2 \cos A = b^2 + \left(\frac{k^2}{b}\right)^2 - a^2$$



Since b^2 is real,

$$(a^2 + 2k^2 \cos A)^2 - 4k^4 \geq 0$$

$$\Rightarrow \quad (a^2 + 2k^2 \cos A + 2k^2)(a^2 + 2k^2 \cos A - 2k^2) \geq 0$$

$$\Rightarrow \quad (a^2 + 2k^2 \cdot 2\cos^2 A/2)(a^2 - 2k^2 \cdot 2\sin^2 A/2) \geq 0$$

$$\Rightarrow (a^2 + 4k^2 \cdot \cos^2 A/2)(a^2 - 4k^2 \sin^2 A/2) \geq 0$$

$$\Rightarrow a^2 - 4k^2 \sin^2 A/2 \geq 0$$

[as, $a^2 + 4k^2 \cos^2 A/2$ is always positive]

$$\Rightarrow (a + 2k \sin A/2)(a - 2k \sin A/2) \geq 0$$

$$\Rightarrow \quad a \leq -2k \sin A/2$$

or $$a \geq 2k \sin A/2$$ [since, $2k \sin(A/2)$ is real]

But 'a' must be positive.

$\Rightarrow$ $a \leq -2k \sin A/2$ is rejected

Hence, $a \geq 2k \sin A/2$

i.e. for the triangle to exist,

$$a \geq 2k \sin A/2$$

● ***Ex. 62.*** *In a ΔABC, having sides a, b, c if $C = 60°$, prove that*

$$\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$$

Sol. $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$ but $C = 60°$

$\therefore$ $$\frac{1}{2} = \frac{a^2 + b^2 - c^2}{2ab}$$

$\therefore$ $$ab = a^2 + b^2 - c^2$$

On adding $2ab$ both sides,

$\therefore$ $$3ab = a^2 + b^2 + 2ab - c^2$$

$\Rightarrow$ $$3ab = (a+b)^2 - c^2$$

$\Rightarrow$ $$3ab = (a+b+c)(a+b-c) \quad \text{...(i)}$$

Now, $$\frac{1}{a+c} + \frac{1}{b+c} = \frac{a+b+2c}{ab + c(a+b+c)}$$

On multiplying numerator and denominator by 3,

$$= \frac{3(a+b+2c)}{3ab + 3c(a+b+c)}$$

$$= \frac{3(a+b+2c)}{(a+b+c)(a+b-c) + 3c(a+b+c)}$$

[using Eq. (i) for $3ab$]

$$= \frac{3(a+b+2c)}{(a+b+c)(a+b+2c)}$$

$$= \frac{3}{(a+b+c)}$$

$\therefore$ $$\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{(a+b+c)}$$

● ***Ex. 63.*** *Let $1 < m < 3$. In a ΔABC, if $2b = (m+1)a$ and $\cos A = \frac{1}{2}\sqrt{\frac{(m-1)(m+3)}{m}}$, prove that there are two values to the third side, one of which is m times the other.*

Sol. From the formula for $\cos A$,

we can write $2bc \cos A = b^2 + c^2 - a^2$

but $$b = \left(\frac{m+1}{2}\right)a$$

$\therefore$ $$(m+1)ac \cos A = \left\{\frac{(m+1)^2}{4} - 1\right\}a^2 + c^2$$

$$= \left\{\frac{(m+1)^2 - 4}{4}\right\}a^2 + c^2$$

$$= \left\{\frac{(m-1)(m+3)}{4}\right\} a^2 + c^2 \qquad \ldots(i)$$

But from the given values of $\cos A$,

we can write

$$\frac{(m-1)(m+3)}{4} = m\cos^2 A,$$

$\therefore$ Eq. (i) gives as

$$(m+1)ac\cos A = ma^2\cos^2 A + c^2$$

or $\quad mac\cos A - c^2 = ma^2\cos^2 A - ac\cos A$

$\therefore \quad (ma\cos A - c) = a\cos A(ma\cos A - c)$

$\therefore \quad (c - a\cos A)(c - ma\cos A) = 0$

This implies 'c' has two values;

$a\cos A$ and $ma\cos A$, and the latter is m times the former.

Hence proved

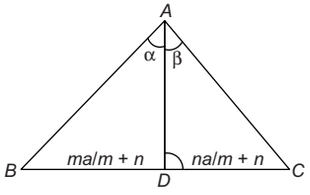
● ***Ex. 64.*** *In any ΔABC, if D be any points of the base BC, such that $\frac{BD}{DC} = \frac{m}{n}$ and $\angle BAD = \alpha$, $\angle DAC = \beta$, $\angle CDA = \theta$ and $AD = x$ then prove that*

$$(m+n)^2 \cdot x^2 = (m+n)(mb^2 + nc^2) - mna^2$$

Sol. In ΔADC,

$$AD^2 + DC^2 - 2AD \cdot DC \cdot \cos\theta = AC^2$$

i.e. $\quad x^2 + \left(\frac{na}{m+n}\right)^2 - 2x\left(\frac{na}{m+n}\right)\cdot\cos\theta = b^2 \qquad \ldots(i)$



In ΔABD, $\quad AD^2 + BD^2 - 2AD \cdot BD \cdot \cos(\pi - \theta) = AB^2$

i.e. $\quad x^2 + \left(\frac{ma}{m+n}\right)^2$

$$- 2x\left(\frac{ma}{m+n}\right)\cdot\cos\theta = c^2 \qquad \ldots(ii)$$

On adding m times Eq. (i) and n times Eq. (ii), we get

$$(m+n)x^2 + \frac{mn\,a^2}{m+n} = mb^2 + nc^2$$

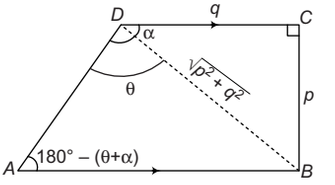
$$\Rightarrow \quad (m+n)^2 . x^2 = (m+n)(mb^2 + nc^2) - mna^2$$

● ***Ex. 65.*** *ABCD is a trapezium such that AB, DC are parallel and BC is perpendicular to them. If $\angle ADB = \theta$, $BC = p$ and $CD = q$, show that*

$$AB = \frac{(p^2 + q^2)\sin\theta}{p\cos\theta + q\sin\theta}$$

Sol. Let $\angle BDC = \alpha$, then $\angle DAB = 180° - (\theta + \alpha)$

applying sine formula to ΔABD, we get



$$\frac{AB}{\sin\theta} = \frac{BD}{\sin(\theta + \alpha)}$$

$\therefore \quad \frac{AB}{\sin\theta} = \frac{\sqrt{p^2 + q^2}}{\sin\theta\cos\alpha + \cos\theta\sin\alpha}$

$$\left[\begin{array}{l}\text{but in } \Delta BCD,\\ \sin\alpha = \frac{p}{\sqrt{p^2+q^2}} \text{ and } \cos\alpha = \frac{q}{\sqrt{p^2+q^2}}\end{array}\right]$$

$\therefore \quad AB = \frac{(p^2 + q^2)\sin\theta}{q\sin\theta + p\cos\theta}$ **Hence proved.**

● ***Ex. 66.*** *Prove that in ΔABC,*

$$\frac{(a+b+c)^2}{a^2+b^2+c^2} \Rightarrow \frac{\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}}{\cot A + \cot B + \cot C}$$

Sol. LHS (denominator) $= a^2 + b^2 + c^2$

$= [(b^2 + c^2 - a^2) + (c^2 + a^2 - b^2) + (a^2 + b^2 - c^2)]$

$= [2bc\cos A + 2ca\cos B + 2ab\cos C]$

$$= 2abc\left[\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}\right]$$

but, $\quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = K$

$\therefore$ Denominator (on LHS)

$$= \frac{2abc}{K}[\cot A + \cot B + \cot C] \qquad \ldots(i)$$

LHS (numerator) $= (a + b + c)^2$

$= K^2[\sin A + \sin B + \sin C]^2$

$= 16K^2\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2}$ $\left[\text{using } \Sigma\sin A = 4\Pi\cos\frac{A}{2}\right]$

On multiplying and dividing by $2K^3\sin A\sin B\sin C$, we get

$$= (2K^3\sin A\sin B\sin C)\frac{16K^2\cos^2\frac{A}{2}\cdot\cos^2\frac{B}{2}\cdot\cos^2\frac{C}{2}}{2K^3\sin A\cdot\sin B\cdot\sin C}$$

$$= (2abc)\frac{8}{K}\cdot\frac{\cos^2\frac{A}{2}\cdot\cos^2\frac{B}{2}\cdot\cos^2\frac{C}{2}}{8\left(\sin\frac{A}{2}\cos\frac{A}{2}\right)\left(\sin\frac{B}{2}\cos\frac{B}{2}\right)\left(\sin\frac{C}{2}\cos\frac{C}{2}\right)}$$

$$= \frac{2abc}{K}\left\{\cot\frac{A}{2}\cdot\cot\frac{B}{2}\cdot\cot\frac{C}{2}\right\} \qquad \ldots(ii)$$

Since A, B, C are angles of a triangle,

$$\cot\frac{A}{2}\cdot\cot\frac{B}{2}\cdot\cot\frac{C}{2}=\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2} \quad ...(iii)$$

From Eqs. (ii) and (iii)

$$\text{Numerator (on LHS)}=\frac{2abc}{K}[\Sigma\cot A/2] \quad ...(iv)$$

On dividing Eq. (iv) by Eq. (i), we get

$$\frac{\cot A/2+\cot B/2+\cot C/2}{\cot A+\cot B+\cot C}\Rightarrow\frac{(a+b+c)^2}{(a^2+b^2+c^2)}$$

● ***Ex. 67.*** *If the sides of a triangle are in AP, and its greatest angle exceeds the least angle by α, show that the sides are in the ratio $1+x:1:1-x$, where $x=\sqrt{\frac{1-\cos\alpha}{7-\cos\alpha}}$.*

Sol. Let ΔABC be the given Δ in which A is the greatest and C is the least angle.

Then, according to the hypothesis

$$A-C=\alpha \quad ...(i)$$

But $$A+C=\pi-B \quad ...(ii)$$

$\therefore$ From Eq. (i) and Eq. (ii), we get

$$A=\frac{\pi}{2}+\frac{\alpha-B}{2}$$

and $$C=\frac{\pi}{2}-\frac{\alpha+B}{2} \quad ...(iii)$$

Again by hypothesis,

$$a+c=2b$$

$$\therefore \quad \sin A+\sin C=2\sin B$$

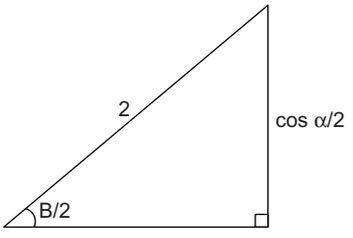
$$\Rightarrow 2\sin\left(\frac{A+C}{2}\right)\cos\left(\frac{A-C}{2}\right)=4\sin\frac{B}{2}\cdot\cos\frac{B}{2}$$

$$\left[\frac{A+C}{2}=\frac{\pi}{2}-\frac{B}{2}\right]$$

$$\Rightarrow \quad 2\cos\frac{B}{2}\cos\left(\frac{A-C}{2}\right)=4\sin\frac{B}{2}\cdot\cos\frac{B}{2}$$

$$\Rightarrow \quad \cos\left(\frac{\alpha}{2}\right)=2\sin\frac{B}{2} \quad \text{[using Eq. (i)]}$$

$$\therefore \quad \sin\frac{B}{2}=\frac{\cos\frac{\alpha}{2}}{2}$$



$$\sqrt{4-\cos^2\alpha/2}=\sqrt{4-\left(\frac{1+\cos\alpha}{2}\right)}=\sqrt{\frac{7-\cos\alpha}{2}}$$

But $$\frac{a}{c}=\frac{\sin A}{\sin C}=\frac{\cos\left(\frac{\alpha-B}{2}\right)}{\cos\left(\frac{\alpha+B}{2}\right)} \quad \text{[using Eq. (iii)]}$$

$$=\frac{\cos\alpha/2\cdot\cos B/2+\sin\alpha/2\cdot\sin B/2}{\cos\alpha/2\cdot\cos B/2-\sin\alpha/2\cdot\sin B/2}$$

On dividing numerator and denominator by $\cos\alpha/2.\cos B/2$

$$=\frac{1+\tan\alpha/2\,.\,\tan B/2}{1-\tan\alpha/2\,.\,\tan B/2}$$

Write $\tan\alpha/2\cdot\tan B/2$ from figure;

$$=\frac{1+\sqrt{\frac{2\sin^2\alpha/2}{7-\cos\alpha}}}{1-\sqrt{\frac{2\sin^2\alpha/2}{7-\cos\alpha}}}=\frac{1+\sqrt{\frac{1-\cos\alpha}{7-\cos\alpha}}}{1-\sqrt{\frac{1-\cos\alpha}{7-\cos\alpha}}}=\frac{1+x}{1-x}$$

$$\left[\text{given } x=\sqrt{\frac{1-\cos\alpha}{7-\cos\alpha}}\right]$$

$$\therefore \quad \frac{a}{1+x}=\frac{c}{1-x}=\frac{a+c}{2}=b$$

$$\therefore \quad \frac{a}{1+x}=\frac{b}{1}=\frac{c}{1-x}$$

Thus, the sides are in the ratio $1+x:1:1-x$.

● ***Ex. 68.*** *In a ΔABC, if $\tan A/2, \tan B/2, \tan C/2$ are in AP. Show that $\cos A, \cos B, \cos C$ are in AP.*

Sol. Given $\tan A/2, \tan B/2, \tan C/2$ are in AP.

$$\therefore \tan A/2-\tan B/2=\tan B/2-\tan C/2$$

$$\therefore \quad \frac{\sin A/2}{\cos A/2}-\frac{\sin B/2}{\cos B/2}=\frac{\sin B/2}{\cos B/2}-\frac{\sin C/2}{\cos C/2}$$

$$\Rightarrow \quad \frac{\sin A/2\cdot\cos B/2-\sin B/2\cdot\cos A/2}{\cos A/2\cdot\cos B/2}$$

$$=\frac{\sin B/2\cdot\cos C/2-\sin C/2\cdot\cos B/2}{\cos B/2\cdot\cos C/2}$$

$$\Rightarrow \quad \frac{\sin\left(\frac{A-B}{2}\right)}{\cos\frac{A}{2}}=\frac{\sin\left(\frac{B-C}{2}\right)}{\cos\frac{C}{2}} \quad ...(i)$$

but $$\cos\frac{A}{2}=\sin\frac{B+C}{2}$$

and $$\cos\frac{C}{2}=\sin\left(\frac{A+B}{2}\right) \quad ...(ii)$$

$$\Rightarrow \quad \sin\left(\frac{A-B}{2}\right)\cdot\sin\left(\frac{A+B}{2}\right)$$

$$=\sin\left(\frac{B-C}{2}\right)\cdot\sin\left(\frac{B+C}{2}\right) \quad \text{[from Eqs. (i) and (ii)]}$$

$$\Rightarrow \quad \cos B-\cos A=\cos C-\cos B$$

Hence, $\cos A, \cos B, \cos C$ are in AP.

● ***Ex. 69.*** *If a, b, c are in HP, then prove that $\sin^2\frac{A}{2}, \sin^2\frac{B}{2}, \sin^2\frac{C}{2}$ are also in HP.*

Sol. Given that $\frac{2}{b}=\frac{1}{a}+\frac{1}{c}$

i.e. $2ac = b(a+c)$ ...(i)

we want to prove

$$\frac{2}{\sin^2\frac{B}{2}}=\frac{1}{\sin^2\frac{A}{2}}+\frac{1}{\sin^2\frac{C}{2}}$$

i.e., $$\frac{2ac}{(s-a)(s-c)}=\frac{bc}{(s-b)(s-c)}+\frac{ab}{(s-a)(s-b)}$$

$$\left[\text{using } \sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}, \sin\frac{B}{2}=\sqrt{\frac{(s-a)(s-c)}{ac}}\right.$$
$$\left.\text{and } \sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}\right]$$

Consider LHS

$$\frac{2ac}{(s-a)(s-c)}=\frac{b(a+c)}{(s-a)(s-c)} \quad \text{[using Eq. (i)]}$$
$$=\frac{b(a+c)(s-b)}{(s-a)(s-b)(s-c)}=\frac{b[s(a+c)-b(a+c)]}{(s-a)(s-b)(s-c)}$$
$$=\frac{b(sa+sc-2ac)}{(s-a)(s-b)(s-c)} \quad \text{[using Eq. (i)]}$$
$$=\frac{b(sa-ac+sc-ac)}{(s-a)(s-b)(s-c)}$$
$$=\frac{b[a(s-c)+c(s-a)]}{(s-a)(s-b)(s-c)}$$
$$=\frac{ab}{(s-a)(s-b)}+\frac{bc}{(s-b)(s-c)}=\text{RHS}$$

● ***Ex. 70.*** *If r_1, r_2, r_3 are the ex-radii of ΔABC, then prove that*

$$\frac{bc}{r_1}+\frac{ca}{r_2}+\frac{ab}{r_3}=2R\left[\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{c}{a}+\frac{a}{c}\right)-3\right]$$

Sol. We know, $r_1=\frac{\Delta}{s-a}, r_2=\frac{\Delta}{s-b}, r_3=\frac{\Delta}{s-c}$

$$\therefore \quad \text{LHS} = \Sigma\frac{bc}{r_1}=\Sigma\frac{bc(s-a)}{\Delta}$$
$$=\frac{1}{\Delta}\Sigma bc(s-a)=\frac{1}{\Delta}[\Sigma bcs-\Sigma(bc)a]$$
$$=\frac{1}{\Delta}\left[\Sigma\frac{abc}{a}\cdot s-3abc\right]=\frac{1}{\Delta}\left[abc\,\Sigma\frac{s}{a}-3abc\right]$$
$$=\frac{abc}{\Delta}\left[\Sigma\frac{s}{a}-3\right] \qquad \left[\text{but } \frac{abc}{\Delta}=4R\right]$$
$$=4R\left[\Sigma\frac{s}{a}-3\right]=2R\left[\Sigma\frac{2s}{a}-6\right]$$
$$=2R\left[\Sigma\frac{a+b+c}{a}-6\right]$$
$$=2R\left[\Sigma\left(1+\frac{b+c}{a}\right)-6\right]$$
$$=2R\left[3+\Sigma\left(\frac{b+c}{a}\right)-6\right]$$
$$=2R\left[\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right)-3\right]$$
$$=2R\left[\frac{b}{a}+\frac{c}{a}+\frac{c}{b}+\frac{a}{b}+\frac{a}{c}+\frac{b}{c}-3\right]$$
$$=2R\left[\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{c}{a}+\frac{a}{c}\right)-3\right]$$

Hence proved.

● ***Ex. 71.*** *If r and R are radii of the incircle and circum-circle of ΔABC, then prove that :*

$$8rR\{\cos^2 A/2+\cos^2 B/2+\cos^2 C/2\}$$
$$=2bc+2ca+2ab-a^2-b^2-c^2.$$

Sol. $\text{LHS} = 8\left(\frac{\Delta}{s}\right)\cdot\left(\frac{abc}{4\Delta}\right)\{\Sigma\cos^2 A/2\}$

$$=\frac{abc}{s}\Sigma(2\cos^2 A/2)=\frac{abc}{s}\Sigma(1+\cos A)$$
$$=\frac{abc}{s}\Sigma\left(1+\frac{b^2+c^2-a^2}{2bc}\right)$$
$$=\frac{abc}{s}\Sigma\left(\frac{2bc+b^2+c^2-a^2}{2bc}\right)=\frac{abc}{s}\Sigma\left\{\frac{(b+c)^2-a^2}{2bc}\right\}$$
$$=\frac{(abc)}{s}\Sigma\frac{(a+b+c)(b+c-a)}{2bc}$$
$$=\frac{(abc)}{s}2s\Sigma\left(\frac{b+c-a}{2bc}\right), \qquad \text{where } a+b+c=2s$$
$$=\Sigma a(b+c-a)=\Sigma(ab+bc-a^2)$$
$$=2bc+2ca+2ab-a^2-b^2-c^2.$$

$\therefore$ $8rR\{\cos^2 A/2+\cos^2 B/2+\cos^2 C/2\}$
$$=2bc+2ab+2ca-a^2-b^2-c^2$$

● ***Ex. 72.*** *Prove that*

$r_1^2+r_2^2+r_3^3+r^2=16R^2-a^2-b^2-c^2.$

where r = in-radius, R = circumradius, r_1, r_2, r_3 are ex-radii.

Sol. $\text{LHS} = r^2+r_1^2+r_2^2+r_3^2$
$$=16R^2\sin^2 A/2\cdot\sin^2 B/2\sin^2 C/2+$$
$$16R^2\sin^2 A/2\cdot\cos^2 B/2\cdot\cos^2 C/2$$
$$+16R^2\cos^2 A/2\sin^2 B/2.\cos^2 C/2+16R^2\cos^2 A/2$$
$$\cdot\cos^2 B/2\cdot\sin^2 C/2$$

$$= 16R^2 \sin^2 A/2\,[\sin^2 B/2 \cdot \sin^2 C/2 + \cos^2 B/2 \cdot \cos^2 C/2]$$
$$+ 16R^2 \cos^2 A/2\,[\sin^2 B/2 \cdot \cos^2 C/2 + \cos^2 B/2 \cdot \sin^2 C/2]$$
$$= 4R^2 \sin^2 A/2\,[(2\sin^2 B/2)(2\sin^2 C/2) + (2\cos^2 B/2)(2\cos^2 C/2)]$$
$$+ 4R^2 \cos^2 A/2\,[(2\sin^2 B/2)(2\cos^2 C/2) + (2\cos^2 B/2)(2\cos^2 C/2)]$$
$$= 4R^2 \sin^2 A/2\,[(1-\cos B)(1-\cos C) + (1+\cos B)(1+\cos C)] + 4R^2 \cos^2 A/2\,[(1-\cos B)(1+\cos C) + (1+\cos B)(1-\cos C)]$$
$$= 4R^2 \sin^2 A/2\,[2 + 2\cos B \cos C] + 4R^2 \cos^2 A/2\,[2 - 2\cos B \cos C]$$
$$= 8R^2 + 8R^2 \cos B \cos C(\sin^2 A/2 - \cos^2 A/2)$$
$$= 8R^2 - 8R^2 \cos A.\cos B \cos C$$
$$= 8R^2 + 4R^2\,[\cos(A+B) + \cos(A-B)] \cdot \cos(A+B)$$
$$= 8R^2 + 4R^2\,[\cos^2 C + \cos^2 A - \sin^2 B]$$
$$= 8R^2 + 4R^2\,[1 - \sin^2 C + 1 - \sin^2 A - \sin^2 B]$$
$$= 16R^2 - 4R^2 \sin^2 A - 4R^2 \sin^2 B - 4R^2 \sin^2 C$$
$$= 16R^2 - a^2 - b^2 - c^2 = \text{RHS}$$

● ***Ex. 73.*** *Tangents are parallel to the three sides are drawn to the in-circle. If x, y, z are the lengths of the parts of the tangents with in the triangle, then prove that* $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

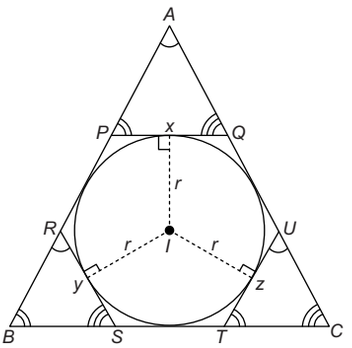
Sol. Let $PQ = x$, PQ parallel to BC.
$SR = y$, SR parallel to AC.
$TU = z$, TU parallel to AB.

Let I be the in-centre of the ΔABC.

Consider ΔAPQ,

By sine rule, $\frac{x}{\sin A} = \frac{AQ}{\sin B} = \frac{AP}{\sin C}$

$$\therefore \quad AQ = \frac{\sin B}{\sin A}x = \frac{b}{a}x$$
$$AP = \frac{\sin C}{\sin A}x = \frac{c}{a}x$$



Also 'r' will be ex-radius of ΔAPQ.

$$\therefore \quad r = \left(\frac{x + PA + AQ}{2}\right)\tan A/2$$
$$= \left[\frac{x + \frac{bx}{a} + \frac{cx}{a}}{2}\right]\tan A/2$$
$$= \frac{(a+b+c)}{2a}x \tan A/2 = \frac{sx}{a}\tan A/2.$$

Similarly in ΔBRS,

$$BR = \frac{c}{b}y;\ BS = \frac{a}{b}y \text{ and } r = \frac{sy}{b}\tan B/2$$
$$\left[\because BR + BS + y = \frac{c}{b}y + \frac{c}{b}y + y = \frac{sy}{b}\right]$$

and in ΔCTU, $CT = \frac{a}{c}z$, $CU = \frac{b}{c}z$ and $r = \frac{sz}{c}\tan\frac{C}{2}$

We also know that,

$$r = (s-a)\tan A/2 = (s-b)\tan B/2$$
$$= (s-c)\tan C/2$$

$$\therefore \quad \frac{sx}{a}\tan A/2 = (s-a)\tan A/2 \implies \frac{sx}{a} = s - a$$
$$\frac{sx}{b}\tan B/2 = (s-b)\tan B/2 \implies \frac{sx}{b} = s - b$$
$$\frac{sx}{c}\tan C/2 = (s-c)\tan C/2 \implies \frac{sx}{c} = s - c$$

On adding $s\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) = 3s - (a+b+c) = s$

$$\therefore \quad \left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) = 1$$

● ***Ex. 74.*** *In a ΔABC, if* $\cos A + 2\cos B + \cos C = 2$. *Prove that the sides of the triangle are in AP.*

Sol. $\cos A + 2\cos B + \cos C = 2$

or $\quad \cos A + \cos C = 2(1 - \cos B)$

$$\Rightarrow \quad 2\cos\left(\frac{A+C}{2}\right)\cdot\cos\left(\frac{A-C}{2}\right) = 4\sin^2 B/2$$

$$\Rightarrow \quad \cos\left(\frac{A-C}{2}\right) = 2\sin\frac{B}{2}$$
$$\left[\text{as } \cos\left(\frac{A+C}{2}\right) = \cos\left(\frac{\pi}{2} - \frac{B}{2}\right) = \sin\frac{B}{2}\right]$$

$$\Rightarrow \quad \cos\left(\frac{A-C}{2}\right) = 2\cos\left(\frac{A+C}{2}\right)$$

$$\Rightarrow \quad \cos\frac{A}{2}\cdot\cos\frac{C}{2} + \sin\frac{A}{2}\cdot\sin\frac{C}{2} = 2\cos\frac{A}{2}\cdot\cos\frac{C}{2} - 2\sin\frac{A}{2}\cdot\sin\frac{C}{2}$$

$$\Rightarrow \quad \cot\frac{A}{2}\cdot\cot\frac{C}{2} = 3$$

$$\Rightarrow \quad \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}\cdot\sqrt{\frac{s(s-c)}{(s-a)(s-b)}} = 3$$

$$\Rightarrow \quad \frac{s}{(s-b)} = 3 \quad \Rightarrow \quad s = 3s - 3b \quad \Rightarrow \quad 2s = 3b$$

$$\Rightarrow \quad a + c = 2b$$

$\therefore$ a, b, c are in AP.

● ***Ex. 75.*** *In a cyclic quadrilateral ABCD, prove that* $\tan^2\frac{B}{2} = \frac{(s-a)(s-b)}{(s-c)(s-d)}$, *$a, b, c$ and d being the lengths of sides AB, BC, CD and DA respectively and s is semi-perimeter of quadrilateral.*

Sol. From ΔABC, we get

$$AC^2 = a^2 + b^2 + 2ab\cos B \qquad \ldots(i)$$

and from ΔADC,

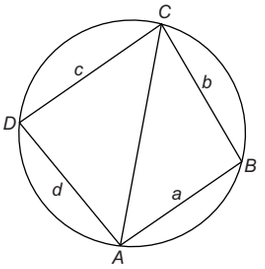
We have, $AC^2 = c^2 + d^2 + 2cd\cos D$

$$= c^2 + d^2 + 2cd\cos(180^\circ - B)$$

$$\Rightarrow \quad AC^2 = c^2 + d^2 - 2cd\cos B \qquad \ldots(ii)$$

From Eq. (i) and Eq. (ii)

$$\cos B = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}$$



Now since, $\tan^2\frac{B}{2} = \frac{1 - \cos B}{1 + \cos B}$

$$\tan^2\frac{B}{2} = \frac{2(ab + cd) - (a^2 + b^2 - c^2 - d^2)}{2(ab + cd) + (a^2 + b^2 - c^2 - d^2)}$$

$$= \frac{(c + d)^2 - (a - b)^2}{(a + b)^2 - (c - d)^2}$$

$$= \frac{(c + d + a - b)(c + d - a + b)}{(a + b - c + d)(a + b + c - d)}$$

$$= \frac{(2s - 2b)(2s - 2a)}{(2s - 2c)(2s - 2d)} = \frac{(s-a)(s-b)}{(s-d)(s-c)}$$

[since $a + b + c + d = 2s$]

$\therefore$ $\tan^2\frac{B}{2} = \frac{(s-a)(s-b)}{(s-d)(s-c)}$

● ***Ex. 76.*** *If x, y, z are the distances of the vertices of the ΔABC respectively from the orthocentre, then show that* $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{xyz}$.

Sol. Let H be the orthocentre. Then,

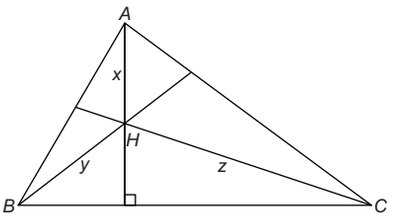
$$\angle BHC = 180^\circ - \angle HBC - \angle HCB$$

$$= 180^\circ - (90^\circ - C) - (90^\circ - B)$$

$$= B + C = \pi - A$$

So, $\text{ar}(\Delta HBC) = \frac{1}{2}BH \cdot CH \cdot \sin\angle BHC$

$$= \frac{1}{2}yz \cdot \sin(\pi - A) = \frac{1}{2}yz\sin A \qquad \ldots(i)$$



Similarly,

$$\text{ar}(\Delta CHA) = \frac{1}{2}zx\sin B \qquad \ldots(ii)$$

$$\text{ar}(\Delta AHB) = \frac{1}{2}xy\sin C \qquad \ldots(iii)$$

$\therefore$ $(\Delta ABC) = \frac{1}{2}yz\sin A + \frac{1}{2}zx\sin B + \frac{1}{2}xy\sin C$

[from Eqs. (i), (ii) and (iii)]

$$= \frac{1}{2}xyz\left\{\frac{\sin A}{x} + \frac{\sin B}{y} + \frac{\sin C}{z}\right\}$$

$$= \frac{1}{2}xyz \cdot \frac{1}{2R}\left\{\frac{a}{x} + \frac{b}{y} + \frac{c}{z}\right\} \qquad \ldots(iv)$$

$$\left[\text{as we know, } R = \frac{abc}{4\Delta}, \text{i.e. } \Delta = \frac{abc}{4R}\right]$$

$\therefore$ Eq. (iv) reduces to;

$$\frac{abc}{4}R = \frac{xyz}{4R}\left(\frac{a}{x} + \frac{b}{y} + \frac{c}{z}\right)$$

$$\Rightarrow \quad \frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{xyz}$$

● ***Ex. 77.*** *In the ΔABC, a similar $\Delta A'B'C'$ is inscribed so that $B'C' = \lambda BC$. If $B'C'$ is inclined at an angle θ with BC, then prove that* $\lambda\cos\theta = \frac{1}{2}$.

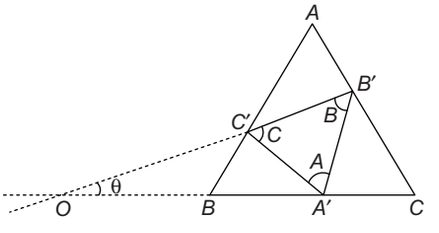
Sol. ΔABC and $\Delta A'B'C'$ are similar,

where $\angle B'A'C' = \angle BAC = A$

$\angle A'B'C' = \angle ABC = B$

$\angle B'C'A' = \Delta BCA = C$

In $\Delta AC'B'$, $\angle AB'C' = \theta + C$



So, using sine law on $\Delta AC'B'$,

$$\frac{AC'}{\sin(\theta + C)} = \frac{B'C'}{\sin A}$$

or $$\frac{AC'}{\sin(\theta + C)} = \frac{\lambda BC}{\sin A} = \frac{\lambda a}{\sin A} = 2\lambda R$$

In $\Delta BA'C'$, $\angle BA'C' = \angle A'C'B' - \angle A'OC' = C - \theta$

So, $$\frac{BC'}{\sin(C - \theta)} = \frac{A'C'}{\sin B}$$

$$\Rightarrow \quad \frac{BC'}{\sin(C - \theta)} = \frac{\lambda AC}{\sin B} = \frac{\lambda b}{\sin B} = 2\lambda R$$

$$\left[\because \frac{B'C'}{BC} = \frac{A'C'}{AC} = \lambda \text{ from similar } \Delta\right]$$

Thus we get, $AC' = 2\lambda R\sin(\theta + C)$

and $BC' = 2\lambda R\sin(C - \theta)$

$$\therefore \quad C = AB = AC' + BC' = 2\lambda R \ \{\sin(C + \theta) + \sin(C - \theta)\}$$
$$= 2\lambda R \cdot 2\sin C \cdot \cos\theta$$

$$\therefore \quad \cos\theta = \frac{C}{4\lambda R\sin C} = \frac{1}{4\lambda R} \cdot \frac{C}{\sin C} = \frac{1}{4\lambda R} \cdot 2R = \frac{1}{2\lambda}$$

$$\therefore \quad \lambda\cos\theta = \frac{1}{2}.$$

● ***Ex. 78.*** *The circle inscribed in the triangle ABC touches the side BC, CA and AB in the point A_1, B_1 and C_1 respectively. Similarly the circle inscribed in the $\Delta A_1B_1C_1$ touches the sides in A_2, B_2, C_2 respectively and so on. If $A_nB_nC_n$ be the nth Δ so formed, prove that its angle are $\frac{\pi}{3} - (2)^{-n}\left(A - \frac{\pi}{3}\right), \frac{\pi}{3} - (2)^{-n}\left(B - \frac{\pi}{3}\right)$ and $\frac{\pi}{3} - (2)^{-n}\left(C - \frac{\pi}{3}\right)$. Hence prove that the triangle so formed is ultimately equilateral.*

Sol. Let I be incentre of the in-circle, $\angle B_1A_1C_1 = \frac{1}{2}(\pi - A)$

$$A_1 = \frac{\pi}{3} - \frac{1}{2}\left(A - \frac{\pi}{3}\right)$$

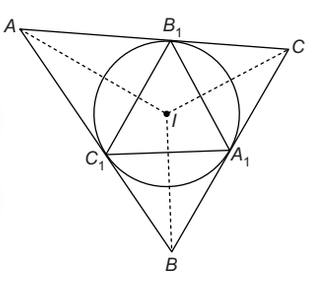
Similarly, $$A_2 = \frac{\pi}{3} - \frac{1}{2}\left(A_1 - \frac{\pi}{3}\right)$$

$$= \frac{\pi}{3} - \frac{1}{2}\left[\frac{\pi}{3} - \frac{1}{2}\left(A - \frac{\pi}{3}\right) - \frac{\pi}{3}\right]$$

$$= \frac{\pi}{3} + \left(\frac{1}{2}\right)^2\left(A - \frac{\pi}{3}\right)$$

$$A_3 = \frac{\pi}{3} - \left(\frac{1}{2}\right)^3\left(A - \frac{\pi}{3}\right)$$

$$A_n = \frac{\pi}{3} - \left(\frac{1}{2}\right)^n\left(A - \frac{\pi}{3}\right)$$



So on,

$$B_n = \frac{\pi}{3} - \left(\frac{1}{2}\right)^n\left(B - \frac{\pi}{3}\right)$$

$$C_n = \frac{\pi}{3} - \left(\frac{1}{2}\right)^n\left(C - \frac{\pi}{3}\right)$$

When $n \to \infty$ then $\left(\frac{1}{2}\right)^n$ becomes zeros or

$$A_n = B_n = C_n = \frac{\pi}{3}.$$

Hence, the triangle will be equilateral.

● ***Ex. 79.*** *In a ΔABC, prove that :*

$$2r \le \frac{a\cot A + b\cot B + c\cot C}{3} \le R$$

Sol. $$r = 4R\sin\frac{A}{2} \cdot \sin\frac{B}{2} \cdot \sin\frac{C}{2}$$

$$= 2R\left[\cos\left(\frac{A - B}{2}\right) - \cos\left(\frac{A + B}{2}\right)\right] \cdot \sin\frac{C}{2}$$

$$= R\left[2\cos\left(\frac{A + B}{2}\right) \cdot \cos\left(\frac{A - B}{2}\right) - 2\sin^2\frac{C}{2}\right]$$

$$= R[\cos A + \cos B + \cos C - 1]$$

$$\Rightarrow \quad 2r + 2R = 2R\,[\cos A + \cos B + \cos C]$$

$$\Rightarrow \quad a\cot A + b\cot B + c\cot C = 2r + 2R \ge 2r + 4r$$

$$\Rightarrow \quad 2r \le \frac{a\cot A + b\cot B + c\cot C}{3}$$

$$= \frac{2R}{3}(\cos A + \cos B + \cos C)$$

$$\le \frac{2}{3}R \cdot \frac{3}{2}$$

$$\Rightarrow \quad 2r \le \frac{a\cot A + b\cot B + \cot C}{3} \le R$$

● ***Ex. 80.*** *If A, B and C are angles of a triangle, then prove that* $E = \frac{\cos\left(\frac{B - C}{2}\right)}{\cos\left(\frac{B + C}{2}\right)} + \frac{\cos\left(\frac{C - A}{2}\right)}{\cos\left(\frac{C + A}{2}\right)} + \frac{\cos\left(\frac{A - B}{2}\right)}{\cos\left(\frac{A + B}{2}\right)} \ge 6$

Sol. Here, $A + B + C = \pi$

$$\Rightarrow E = \frac{\cos\left(\frac{B - C}{2}\right)}{\cos\left(\frac{B + C}{2}\right)} + \frac{\cos\left(\frac{C - A}{2}\right)}{\cos\left(\frac{C + A}{2}\right)} + \frac{\cos\left(\frac{A - B}{2}\right)}{\cos\left(\frac{A + B}{2}\right)}$$

$$= \frac{\cos\left(\frac{B - C}{2}\right)}{\sin\left(\frac{A}{2}\right)} + \frac{\cos\left(\frac{C - A}{2}\right)}{\sin\left(\frac{B}{2}\right)} + \frac{\cos\left(\frac{A - B}{2}\right)}{\sin\left(\frac{C}{2}\right)}$$

$$= \frac{2\cos\frac{A}{2}\cos\left(\frac{B-C}{2}\right)}{\sin A} + \frac{2\cos\frac{B}{2}\cos\left(\frac{C-A}{2}\right)}{\sin B} + \frac{2\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right)}{\sin C}$$

$$= \frac{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}{\sin A} + \frac{2\sin\left(\frac{A+C}{2}\right)\cos\left(\frac{C-A}{2}\right)}{\sin B} + \frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{\sin C}$$

$$= \left(\frac{\sin B + \sin C}{\sin A}\right) + \left(\frac{\sin C + \sin A}{\sin B}\right) + \left(\frac{\sin A + \sin B}{\sin C}\right)$$

$$= \left(\frac{\sin B}{\sin A} + \frac{\sin A}{\sin B}\right) + \left(\frac{\sin C}{\sin B} + \frac{\sin B}{\sin C}\right) + \left(\frac{\sin A}{\sin C} + \frac{\sin C}{\sin A}\right)$$

as A, B, C are angles of triangle

$\Rightarrow \quad 0 < A, B, C < \pi$

$\Rightarrow \quad \sin A, \sin B, \sin C > 0$

$\Rightarrow \quad E \geq 2 + 2 + 2 \quad \left[\text{as } x + \frac{1}{x} \geq 2, \text{if } x > 0\right]$

$\Rightarrow \quad E \geq 6$

● ***Ex. 81.*** *If Δ is the area of a Δ with side lengths a, b, c then show that $\Delta \leq \frac{1}{4}\sqrt{(a+b+c)abc}$. Also, show that equality occurs in the above inequality if and only if $a = b = c$.*

Sol. We know, $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$$= \sqrt{\frac{s}{8}(b+c-a)(c+a-b)(a+b-c)}$$

Since, sum of two sides is always greater than third side.

$\therefore \quad b + c - a, c + a - b,$

$a + b - c > 0.$

$\Rightarrow \quad (s-a)(s-b)(s-c) > 0$

let $(s-a) = x, (s-b) = y, (s-c) = z$

Now, $x + y = 2s - a - b = c, y + z = a$

and $z + x = b$

Since, AM $\geq$ GM, then $2\sqrt{xy} \leq x + y = c;$

$2\sqrt{yz} \leq y + z = a; 2\sqrt{zx} \leq z + x = b$

$\therefore \quad 8xyz \leq abc$

$\Rightarrow \quad (s-a)(s-b)(s-c) \leq \frac{1}{8}(abc)$

$\Rightarrow \quad s(s-a)(s-b)(s-c) \leq \frac{1}{8}s(abc) \leq \frac{1}{16}(a+b+c)(abc)$

$\Rightarrow \quad \Delta \leq \frac{1}{4}\sqrt{abc(a+b+c)}$ and equality holds if

$x = y = z \Rightarrow a = b = c$

Aliter RHS $= \frac{1}{4}\sqrt{abc(a+b+c)}$

$$= \frac{1}{4}\sqrt{2s \cdot 4R\,\Delta} = \frac{1}{4}\sqrt{8.\frac{\Delta}{r} \cdot R.\Delta}$$

$$= \Delta\sqrt{\frac{R}{2r}} = \frac{\Delta}{\sqrt{8\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}}} \quad \ldots(i)$$

Consider, $\sin\frac{A}{2} \cdot \sin\frac{B}{2} \cdot \sin\frac{C}{2}$

$$= \frac{1}{2}\left[\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right] \cdot \cos\left(\frac{A+B}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{4}\cos^2\left(\frac{A-B}{2}\right) - \left\{x - \frac{1}{2}\cos\left(\frac{A-B}{2}\right)\right\}^2\right]$$

$$\leq \frac{1}{8}\cos^2\left(\frac{A-B}{2}\right) \leq \frac{1}{8}$$

$\Rightarrow \quad \frac{1}{8\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}} \geq 1 \quad \ldots(ii)$

Thus, from Eqs. (i) and (ii); $\Delta \leq \frac{1}{4}\sqrt{(a+b+c)abc}$

and equality holds, $x = \frac{1}{2}\cos\left(\frac{A-B}{2}\right)$

and $\cos\left(\frac{A-B}{2}\right) = 1$

$\Rightarrow \quad A = B$ and $\cos\left(\frac{A+B}{2}\right) = \frac{1}{2}$

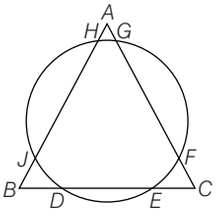
$\Rightarrow \quad \sin\frac{C}{2} = \frac{1}{2} \Rightarrow C = 60°$

$\therefore \quad A = B = 60°$

Thus, the equality holds of triangle is equilateral.

Properties and Solutions of Triangles Exercise 1 :
Single Option Correct Type Questions

1. In the adjoining figure, the circle meets the sides of an equilateral triangle at six points.



If $AG = 2$, $GF = 13$, $FC = 1$ and $HJ = 7$, then DE equals to
(a) $2\sqrt{22}$ (b) $7\sqrt{3}$
(c) 9 (d) 10

2. In a ΔABC, if $a = 13$, $b = 14$ and $c = 15$, then $\angle A$ is equal to (All symbols used have their usual meaning in a triangle.)
(a) $\sin^{-1}\frac{4}{5}$ (b) $\sin^{-1}\frac{3}{5}$ (c) $\sin^{-1}\frac{3}{4}$ (d) $\sin^{-1}\frac{2}{3}$

3. In a ΔABC, if $b = (\sqrt{3} - 1)a$ and $\angle C = 30°$, then the value of $(A - B)$ is equal to (All symbols used have usual meaning in the triangle.)
(a) 30° (b) 45°
(c) 60° (d) 75°

4. In a ΔABC, if $\angle C = 105°$, $\angle B = 45°$ and length of side $AC = 2$ units, then the length of the side AB is equal to
(a) $\sqrt{2}$ (b) $\sqrt{3}$
(c) $\sqrt{2} + 1$ (d) $\sqrt{3} + 1$

5. If P is a point on the altitude AD of the ΔABC such that $\angle CBP = \frac{B}{3}$, then AP is equal to
(a) $2a\sin\frac{C}{3}$ (b) $2b\sin\frac{A}{3}$
(c) $2c\sin\frac{B}{3}$ (d) $2c\sin\frac{C}{3}$

6. In ΔABC, if $2b = a + c$ and $A - C = 90°$, then $\sin B$ equals
[**Note** All symbols used have usual meaning in ΔABC.]
(a) $\frac{\sqrt{7}}{5}$ (b) $\frac{\sqrt{5}}{8}$ (c) $\frac{\sqrt{7}}{4}$ (d) $\frac{\sqrt{5}}{3}$

7. Let ABC be a right triangle with length of side $AB = 3$ and hyotenuse $AC = 5$. If D is a point on BC such that $\frac{BD}{DC} = \frac{AB}{AC}$, then AD is equal to
(a) $\frac{4\sqrt{3}}{3}$ (b) $\frac{3\sqrt{5}}{2}$ (c) $\frac{4\sqrt{5}}{3}$ (d) $\frac{5\sqrt{3}}{4}$

8. Two medians drawn from the acute angles of a right angled triangle intersect at an angle $\frac{\pi}{6}$. If the length of the hypotenuse of the triangle is 3 units, then the area of the triangle (in sq units) is $\sqrt{K}$, then K is
(a) 3 (b) $\frac{3\sqrt{5}}{2}$ (c) $\sqrt{3}$ (d) None of these

9. If in a right angle ΔABC, $4\sin A\cos B - 1 = 0$ and $\tan A$ is finite, then
(a) angles are in AP (b) angles are in GP
(c) angles are in HP (d) None of these

10. Let $A = \begin{bmatrix} a & b & c \\ p & q & r \\ 1 & 1 & 1 \end{bmatrix}$ and $B = A^2$

If $(a-b)^2 + (p-q)^2 = 25$, $(b-c)^2 + (q-r)^2 = 36$ and $(c-a)^2 + (r-p)^2 = 49$, then $\det B$ is
(a) 192 (b) 864
(c) 3456 (d) $25 \times 36 \times 47$

11. If in a ΔABC, the incircle passing through the point of intersection of perpendicular bisector of sides BC, AB, then $4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ equals to
(a) $\sqrt{2}$ (b) $\sqrt{2} - 1$
(c) $\sqrt{2} + 1$ (d) $\frac{1}{2}$

12. If two sides of a triangle are roots of the equation $x^2 - 7x + 8 = 0$ and the angle between these sides is $60°$, then the product of in-radius and circum-radius of the triangle is
(a) $\frac{8}{7}$ (b) $\frac{5}{3}$
(c) $\frac{5\sqrt{2}}{3}$ (d) 8

13. If median AD of a triangle ABC makes angle $\frac{\pi}{6}$ with side BC, then the value of $(\cot B - \cot C)^2$ is equal to
(a) 6 (b) 9
(c) 12 (d) 15

14. If the perimeter of the triangle formed by foot of altitudes of the ΔABC is equal to four times the circumradius of ΔABC, then ΔABC is
(a) isosceles triangle (b) equilateral triangle
(c) right angled triangle (d) None of these

15. In a triangle with one angle $\frac{2\pi}{3}$, the lengths of the sides form an A.P. If the length of the greatest side is 7 cm, the radius of the circumcircle of the triangle is

(a) $\frac{7\sqrt{3}}{3}$ cm (b) $\frac{5\sqrt{3}}{3}$ cm (c) $\frac{2\sqrt{3}}{3}$cm (d) $\sqrt{3}$ cm

16. Sides of a triangle ABC are in AP. If $a < \min\{b, c\}$, then $\cos A$ may be equal to

(a) $\frac{3c-4b}{2b}$ (b) $\frac{3c-4b}{2c}$

(c) $\frac{4c-3b}{2b}$ (d) $\frac{4c-3b}{2c}$

17. The product of the sines of the angles of a triangle is p and the product of their cosines is q. Then, the tangents of the angles are the roots of the equation

(a) $qx^3 - px^2 + (1+q)x - p = 0$

(b) $qx^3 - px^2 - (1-q)x - p = 0$

(c) $qx^3 - px^2 + (1+q)x + p = 0$

(d) None of the above

18. Let C be incircle of ΔABC. If the tangents of lengths t_1, t_2 and t_3 are drawn inside the given triangle parallel to sides a, b and c, respectively, then $\frac{t_1}{a} + \frac{t_2}{b} + \frac{t_3}{c}$ is equal to

(a) 0 (b) 1 (c) 2 (d) 3

19. If the sine of the angles of ΔABC satisfy the equation

$c^3x^3 - c^2(a+b+c)x^2 + lx + m = 0$

(where a, b, c are the sides of ΔABC), then ΔABC is

(a) always right angled for any l, m

(b) right angled only when

$l = c(ab + bc + ca) = c\,\Sigma ab,\ m = -abc$

(c) right angled only when

$l = \frac{c\Sigma ab}{4},\ m = -\frac{abc}{8}$

(d) never right angled

20. In a triangle ABC, medians AD and CE are drawn. If $AD = 5$, $\angle DAC = \frac{\pi}{8}$ and $\angle ACE = \frac{\pi}{4}$, the area of the triangle is

(a) $\frac{50}{9}$ (b) $\frac{25}{9}$ (c) $\frac{25}{3}$ (d) $\frac{25}{7}$

21. In a triangle ABC, $a \geq b \geq c$. If $\frac{a^3 + b^3 + c^3}{\sin^3 A + \sin^3 B + \sin^3 C} = 8$, then the maximum value of a is

(a) $\frac{1}{2}$ (b) 2 (c) 8 (d) 64

22. A triangle ABC exists such that

(a) $(b + c + a)(b + c - a) = 5bc$

(b) the sides are of lengths $\sqrt{19}, \sqrt{38}, \sqrt{116}$

(c) $\left(\frac{b^2-c^2}{a^2}\right) + \left(\frac{c^2-a^2}{b^2}\right) + \left(\frac{a^2-b^2}{c^2}\right) = 0$

(d) $\cos\left(\frac{B-C}{2}\right) = (\sin B + \sin C)\cos\left(\frac{B+C}{2}\right)$

23. If a ΔABC, a, b, A are given and b_1, b_2 are two values of the third sides b such that $b_2 = 2b_1$. Then, $\sin A$ is equal to

(a) $\sqrt{\frac{9a^2-c^2}{8a^2}}$ (b) $\sqrt{\frac{9a^2-c^2}{8c^2}}$

(c) $\sqrt{\frac{9a^2-c^2}{8b^2}}$ (d) None of these

24. In a triangle ABC, if $\cot A = (x^3 + x^2 + x)^{\frac{1}{2}}$, $\cot B = (x + x^{-1} + 1)^{\frac{1}{2}}$ and $\cot C = (x^{-3} + x^{-2} + x^{-1})^{\frac{-1}{2}}$, then the triangle is

(a) equilateral (b) isosceles

(c) right angled (d) obtuse angled

25. In a ΔABC, a, b, A are given and c_1, c_2 are two values of the third side c. The sum of the areas two triangles with sides a, b, c_1 and a, b, c_2 is

(a) $\frac{1}{2}b^2\sin 2A$

(b) $\frac{1}{2}a^2\sin 2A$

(c) $b^2\sin 2A$

(d) None of these

26. In ΔABC, if $a = 10$ and $b\cot B + c\cot C = 2(r + R)$, then the maximum area of ΔABC will be

(a) 50 (b) $\sqrt{50}$

(c) 25 (d) 5

27. Three circles touch one-another externally. The tangents at their point of contact meet at a point whose distance from a point contact is 4. Then, the ratio of the product of the radii to the sum of the radii of circles is

(a) 16 : 1 (b) 1 : 16

(c) 8 : 1 (d) None of these

28. Let a, b, c be the sides of a triangle. No two of them are equal and $\lambda \in R$. If the roots of the equation $x^2 + 2(a+b+c)x + 3\lambda(ab + bc + ca) = 0$ are real distinct, then

(a) $\lambda < \frac{4}{3}$ (b) $\lambda > \frac{5}{3}$

(c) $\lambda \in \left(\frac{1}{3}, \frac{5}{3}\right)$ (d) $\lambda \in \left(\frac{4}{3}, \frac{5}{3}\right)$

29. In triangle ABC, if P, Q, R divides sides BC, AC and AB, respectively, in the ratio $k:1$ (in order). If the ratio $\left(\frac{\text{area } \Delta PQR}{\text{area } \Delta ABC}\right)$ is $\frac{1}{3}$, then k is equal to

(a) $\frac{1}{3}$ (b) 2

(c) 3 (d) None of these

30. Let $f(x+y)=f(x)\cdot f(y)$ for all x and y and $f(1)=2$. If in a triangle ABC, $a=f(3)$, $b=f(1)+f(3)$, $c=f(2)+f(3)$,

(a) C (b) $2C$

(c) $3C$ (d) $4C$

31. Let a, b, c be given positive numbers, then values of x, y and $z \in R^+$ which satisfies equations $x+y+z=a+b+c$ and $4xyz=-(a^2x+b^2y+c^2z)=abc$ are respectively.

(a) $\frac{b+c}{2}, \frac{a+c}{2}, \frac{a+b}{2}$ (b) $\frac{a}{2}, \frac{b}{2}, \frac{c}{2}$

(c) $\frac{a+b}{2}, \frac{a+c}{2}, \frac{b+c}{2}$ (d) None of these

32. If 't_1', 't_2' and 't_3' are the lengths of the tangents drawn from centre of x-circle to the circumcircle of the ΔABC, then $\frac{1}{t_1^2}+\frac{1}{t_2^2}+\frac{1}{t_3^2}$ is equal to

(a) $\frac{abc}{a+b+c}$ (b) $\frac{a+b+c}{abc}$

(c) $\frac{a+b+c}{2abc}$ (d) $\frac{2abc}{a+b+c}$

33. In triangle ABC, $\angle A > \frac{\pi}{2}$. AA_1 and AA_2 are the median and altitude, espectively. If $\angle BAA_1 = \angle A_1AA_2 = \angle A_2AC$, then $\sin^3\frac{A}{3}\cdot\cos\frac{A}{3}$ is equal to

(a) $\frac{3a^3}{16b^2c}$ (b) $\frac{3a^3}{64b^2c}$

(c) $\frac{3a^2}{4b^2c}$ (d) $\frac{3a^3}{12b^2c}$

34. In an ambiguous case of solving a triangle when $a=\sqrt{5}$, $b=2$, $\angle A=\frac{\pi}{6}$ and the two possible values of third side are c_1 and c_2, then

(a) $|c_1-c_2|=2\sqrt{6}$ (b) $|c_1-c_2|=4\sqrt{6}$

(c) $|c_1-c_2|=4$ (d) $|c_1-c_2|=6$

35. If R_1 is the circumradius of the pedal triangle of a given triangle and R_2 is the circumradius of the pedal triangle of the pedal triangle formed, and so on $R_3, R_4 \ldots$, then the value of $\sum_{i=1}^{\infty} R_i$, where R (circumradius) of ΔABC is 5 is

(a) 8 (b) 10 (c) 12 (d) 15

36. If in a triangle $\left(1-\frac{r_1}{r_2}\right)\left(1-\frac{r_1}{r_3}\right)=2$, then the triangle is

(a) right angled (b) isosceles

(c) equilateral (d) None of these

37. If the median AD of a triangle ABC makes an angle θ with side AB, then $\sin(A-\theta)$ is equal to

(a) $\frac{b}{c}\sin\theta$ (b) $\frac{c}{b}\sin\theta$

(c) $\frac{c}{b}\cos\theta$ (d) None of these

38. In a ΔABC, angles A, B, C are in AP, then $\lim_{A\to C}\frac{\sqrt{3-4\sin A\sin C}}{|A-C|}$ is

(a) 1 (b) 2

(c) 3 (d) 4

39. In a triangle ABC, $(a+b+c)(b+c-a)=\lambda bc$ if

(a) $\lambda<0$ (b) $\lambda>6$

(c) $0<\lambda<4$ (d) $\lambda>4$

40. In the triangle ABC, if $(a^2+b^2)\sin(A-B)$ $=(a^2-b^2)\sin(A+B)$, then the triangle is

(a) either isosceles or right angled

(b) only right angled

(c) only isosceles triangle

(d) None of the above

41. In a ΔABC, sides a, b, c are in AP and $\frac{2}{1!9!}+\frac{2}{3!7!}+\frac{1}{5!5!}=\frac{8^a}{(2b)!}$, then the maximum value of $\tan A\tan B$ is equal to

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) $\frac{1}{5}$

42. If a, b, c be the sides of a triangle ABC and if roots of the equation $a(b-c)x^2+b(c-a)x+c(a-b)=0$ are equal, then $\sin^2\left(\frac{A}{2}\right), \sin^2\left(\frac{B}{2}\right), \sin^2\left(\frac{C}{2}\right)$ are in

(a) AP (b) GP

(c) HP (d) AGP

43. The ratio of the area of a regular polygon of n sides inscribed in a circle to that of the polygon of same number of sides circumscribing the same circle is 3 : 4. Then, the value of n is

(a) 6 (b) 4

(c) 8 (d) 12

44. In any ΔABC, $\Pi\left(\frac{\sin^2 A+\sin A+1}{\sin A}\right)$ is always greater than

(a) 9 (b) 3

(c) 27 (d) None of these

45. If the incircle of the triangle ABC, passes through it's circumcentre, then the $\cos A + \cos B + \cos C$ is
(a) -2 (b) $\sqrt{2}$
(c) $-\sqrt{2}$ (d) None of these

46. The perimeter of a triangle is 6 times the arithmetic mean of the sines of its angles. If the side a is 1, then A is equal to
(a) 30° (b) 60° (c) 90° (d) 120°

47. If there are only two linear functions f and g which map $[1, 2]$ on $[4, 6]$ and in a ΔABC, $c = f(1) + g(1)$ and a is the maximum value of r^2, where r is the distance of a variable point on the curve $x^2 + y^2 - xy = 10$ from the origin, then $\sin A : \sin C$ is
(a) 1 : 2 (b) 2 : 1
(c) 1 : 1 (d) None of these

48. A circle is inscribed in an equilateral triangle of side a. The area of any square inscribed in this circle is
(a) a^2 (b) $\frac{a^2}{4}$ (c) $\frac{a^2}{3}$ (d) $\frac{a^2}{6}$

49. In any triangle ABC, if $\sin A$, $\sin B$, $\sin C$ are in AP, then the maximum value of $\tan\frac{B}{2}$ is
(a) $-\frac{1}{\sqrt{3}}$ (b) $\frac{1}{\sqrt{3}}$
(c) $\frac{1}{3}$ (d) None of these

50. In a ΔABC, $2\cos A = \frac{\sin B}{\sin C}$ and $2^{\tan^2 B}$ is a solution of equation $x^2 - 9x + 8 = 0$, then ΔABC is
(a) equilateral (b) isosceles
(c) scalene (d) right angled

51. A triangle is inscribed in a circle. The vertices of the triangle divide the circle into three arcs of length 3, 4 and 5 unit, then the area of the triangle is equal to
(a) $\frac{9\sqrt{3}(1+\sqrt{3})}{\pi^2}$ sq unit (b) $\frac{9\sqrt{3}(\sqrt{3}-1)}{\pi^2}$ sq unit
(c) $\frac{9\sqrt{3}(1+\sqrt{3})}{2\pi^2}$ sq unit (d) $\frac{9\sqrt{3}(\sqrt{3}-1)}{2\pi^2}$ sq unit

52. If a, b and c are the sides of a triangle such that $b \cdot c = \lambda^2$, then the relation in a, λ and A is
(a) $c \geq 2\lambda\sin\left(\frac{C}{2}\right)$ (b) $b \geq 2\lambda\sin\left(\frac{A}{2}\right)$
(c) $a \geq 2\lambda\sin\left(\frac{A}{2}\right)$ (d) None of these

53. In ΔABC, AD is an altitude from A on side BC. If $b > c$, $\angle c = 23°$ and $AD = \frac{abc}{b^2 - c^2}$, then $\angle B$ is
(a) 110° (b) 113° (c) 120° (d) 130°

54. If in a triangle ABC, $a = 5$, $b = 4$ and $\cos(A - B) = \frac{31}{32}$, then the third side c is equal to
(a) 3 (b) 6
(c) 7 (d) 9

55. In ΔABC, if $AB = x$, $BC = x + 1$, $\angle C = \frac{\pi}{3}$, then the least integer value of x is
(a) 6 (b) 7
(c) 8 (d) None of these

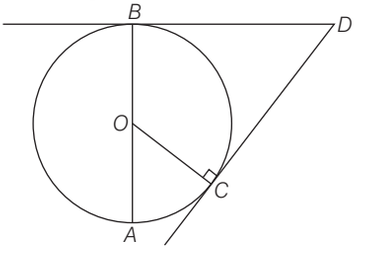
56. Three circular coins each of radii 1 cm are kept in an equilateral triangle so that all the three coins touch each other and also the sides of the triangle. Area of the triangle is
(a) $(4 + 2\sqrt{3})$ cm^2 (b) $\frac{1}{4}(12 + 7\sqrt{3})$ cm^2
(c) $\frac{1}{4}(48 + 7\sqrt{3})$ cm^2 (d) $(6 + 4\sqrt{3})$ cm^2

57. The sides of a triangle are in AP. If the angles A and C are the greatest and smallest angle respectively, then $4(1 - \cos A)(1 - \cos C)$ is equal to
(a) $\cos A - \cos C$ (b) $\cos A \cos C$
(c) $\cos A + \cos C$ (d) $\cos C - \cos A$

58. If in ΔABC, $c(a + b)\cos\frac{1}{2}B = b(a + c)\cos\frac{1}{2}C$, the triangle is
(a) isosceles
(b) equilateral
(c) right angled but not isosceles
(d) right angled and isosceles

59. In a triangle, the line joining the circumcentre to the incentre is parallel to BC, then $\cos B + \cos C$ is equal to
(a) $\frac{3}{2}$ (b) 1
(c) $\frac{3}{4}$ (d) $\frac{1}{2}$

60. In the given figure, AB is the diameter of the circle, centered at O. If $\angle COA = 60°$, $AB = 2r$, $AC = d$ and $CD = l$, then l is equal to



(a) $d\sqrt{3}$ (b) $\frac{d}{\sqrt{3}}$
(c) $3d$ (d) $\frac{\sqrt{3}d}{2}$

61. In a triangle ABC; AD, BE and CF are the altitudes and R is the circum radius, then the radius of the circle DEF is
(a) $2R$ (b) R
(c) $\frac{R}{2}$ (d) None of these

62. In a right angled triangle ABC, the bisector of the right angle C divides AB into segment x and y and $\tan\left(\frac{A-B}{2}\right)=t$, then $x:y$ is equal to
(a) $(1+t)=(1-t)$ (b) $(1-t):(1+t)$
(c) $1:(1+t)$ (d) $(1-t):1$

63. A variable triangle ABC is circumscribed about a fixed circle of unit radius. Side BC always touches the circle at D and has fixed direction. If B and C vary in such a way that $(BD)\cdot(CD)=2$, then locus of vertex A will be a straight line
(a) parallel to side BC
(b) right angle to side BC
(c) making and angle $\frac{\pi}{6}$ with BC
(d) making an angle $\sin^{-1}\left(\frac{2}{3}\right)$ with BC

65. A tower of height b subtends an angle at a point O on the level of the foot of the tower and at a distance a from the foot of the tower. If a pole mounted on the tower also subtends an equal angle at O, the height of the pole is
(a) $b\left(\frac{a^2-b^2}{a^2+b^2}\right)$ (b) $b\left(\frac{a^2+b^2}{a^2-b^2}\right)$
(c) $a\left(\frac{a^2-b^2}{a^2+b^2}\right)$ (d) $a\left(\frac{a^2+b^2}{a^2-b^2}\right)$

66. A balloon is observed simultaneously from three points A, B and C on a straight road directly under it. The angular elevation at B is twice and at C is thrice that of A. If the distance between A and B is 200 m and the distance between B and C is 100 m, then the height of balloon is given by
(a) 50 m (b) $50\sqrt{3}$ m
(c) $50\sqrt{2}$ m (d) None of these

67. A vertical pole (more than 100 m high) consists of two portions, the lower being one third of the whole. If the upper portion subtends an angle $\tan^{-1}\left(\frac{1}{2}\right)$ at a point in a horizontal plane through the foot of the pole and distance 40 ft from it, then the height of the pole is
(a) 100 ft (b) 120 ft
(c) 150 ft (d) None of these

Properties and Solutions of Triangles Exercise 2 : More than One Correct Type Questions

68. If area of ΔABC, Δ and $\angle C$, are given and if the side c opposite to given angles is minimum, then
(a) $a=\sqrt{\frac{2\Delta}{\sin C}}$ (b) $b=\sqrt{\frac{2\Delta}{\sin C}}$
(c) $a=\frac{4\Delta}{\sin C}$ (d) $b=\frac{4\Delta}{\sin^2 C}$

69. If Δ represents area of acute angled ΔABC, then $\sqrt{a^2b^2-4\Delta^2}+\sqrt{b^2c^2-4\Delta^2}+\sqrt{c^2a^2-4\Delta^2}$ equals to
(a) $a^2+b^2+c^2$
(b) $\frac{a^2+b^2+c^2}{2}$
(c) $ab\cos C+bc\cos A+ca\cos B$
(d) $ab\sin C+bc\sin A+ca\sin B$

70. In ΔABC, the value of $c\cos(A-\theta)+a\cos(C+\theta)$ cannot exceed $(\theta\in(0,2\pi))$ [Letters have usual meanings]
(a) a (b) b
(c) c (d) s

71. In ΔABC, if $ac=3$, $bc=4$ and $\cos(A-B)=\frac{3}{4}$, then
(a) measure of $\angle A$ is $\frac{\pi}{2}$
(b) measure of $\angle B$ is $\frac{\pi}{2}$
(c) $\cot\frac{C}{2}=\sqrt{7}$
(d) circumradius of ΔABC is $\frac{2}{7^{1/4}}$

72. In ΔABC, let $a=5$, $b=4$ and $\cos(A-B)=\frac{31}{32}$, then which of the following statement(s) is (are) correct?
[**Note** All symbols used have usual meaning in a triangle]
(a) The perimeter of ΔABC equals $\frac{15}{2}$
(b) The radius of circle inscribed in ΔABC equals $\frac{\sqrt{7}}{2}$
(c) The measure of $\angle C$ equals $\cos^{-1}\frac{1}{8}$
(d) The value of $R(b^2\sin 2C+c^2\sin 2B)$ equals 120

73. In which of the following situations, it is possible to have a ΔABC?

(All symbols used have usual meaning in a triangle)

(a) $(a + c - b)(a - c + b) = 4bc$

(b) $b^2 \sin 2C + \cos^2 \sin 2B = ab$

(c) $a = 3, b = 5, c = 7$ and $C = \frac{2\pi}{3}$

(d) $\cos\left(\frac{A - C}{2}\right) = \cos\left(\frac{A + C}{2}\right)$

74. In a ΔABC, let $BC = 1$, $AC = 2$ and measure of $\angle C$ is 30°. Which of the following statement(s) is(are) correct?

(a) $2\sin A = \sin B$

(b) Length of side AB equals $5 - 2\sqrt{3}$

(c) measure of $\angle A$ is less than 30°

(d) Circumradius of $\angle ABC$ is equal to length of side AB

75. Let one angle of a triangle be 60°, the area of triangle is $10\sqrt{3}$ and perimeter is 20 cm. If $a > b > c$ where a, b and c denote lengths of sides opposi9te to vertices A, B and C respectively, then which of the following is(are) correct?

(a) Inradius of triangle is $\sqrt{3}$

(b) Length of longest side of triangle is 7

(c) Circum-radius of triangles is $\frac{7}{\sqrt{3}}$

(d) Radius of largest escribed circle is $\frac{1}{12}$

76. In a ΔABC, if $a = 4$, $b = 8$ and $\angle C = 60°$, then which of the following relations is(are) correct?

[**Note** All symbols used have usual meaning in triangles ABC.]

(a) The area of ΔABC is $8\sqrt{3}$

(b) The value of $\Sigma \sin^2 A = 2$

(c) Inradius of triangle ABC is $\frac{2\sqrt{3}}{3 + \sqrt{3}}$

(d) The length of internal angle bisector of $\angle C$ is $\frac{4}{\sqrt{3}}$.

77. Given an isosceles triangle with equal sides of length b, base angle $\alpha < \frac{\pi}{4}$ and R, r the radii and O, I the centres of the circumcircle and incircle, respectively. Then

(a) $R = \frac{1}{2} b \operatorname{cosec} \alpha$

(b) $\Delta = 2b^2 \sin 2\alpha$

(c) $r = \frac{b \sin 2\alpha}{2(1 + \cos \alpha)}$

(d) $OI = \left| \frac{b \cos\left(\frac{3\alpha}{2}\right)}{2 \sin \alpha \cos\left(\frac{\alpha}{2}\right)} \right|$

78. There exist a triangle ABC satisfying

(a) $\tan A + \tan B + \tan C = 0$

(b) $\frac{\sin A}{2} = \frac{\sin B}{3} = \frac{\sin C}{7}$

(c) $(a + b)^2 = c^2 + ab$ and $\sqrt{2}(\sin A + \cos A) = \sqrt{3}$

(d) $\sin A + \sin B = \left(\frac{\sqrt{3} + 1}{2}\right) \cos A \cos B = \frac{\sqrt{3}}{4} = \sin A \sin B$

79. Let a, b, c be the sides of triangle whose perimeter is P and area is A, then

(a) $P^3 \le 27(b + c - a)(c + a - b)(a + b - c)$

(b) $P^2 \le 3(a^2 + b^2 + c^2)$

(c) $a^2 + b^2 + c^2 \ge 4\sqrt{3}A$

(d) $P^4 \le 25 < A$

80. If in ΔABC, $A = 90°$ and c, $\sin B$ and $\cos B$ are rational number, then

(a) a is rational

(b) a is irrational

(c) b is rational

(d) b is irrational

81. In ΔABC, which of the following statements are true

(a) maximum value of $\sin 2A + \sin 2B + \sin 2C$ is same as the maximum value of $\sin A + \sin B + \sin C$

(b) $R \ge 2r$, where R is circumradius and r is inradius

(c) $R^2 \ge \frac{abc}{(a + b + c)}$

(d) ΔABC is right angled if $r + 2R = s$, where s is semiperimeter

82. If l is the length of median from the the vertex A to the side BC of a ΔABC, then

(a) $4l^2 = 2b^2 + 2c^2 - a^2$

(b) $4l^2 = b^2 + c^2 + 2bc \cos A$

(c) $4l^2 = a^2 + 4bc \cos A$

(d) $4l^2 - (2s - a)^2 - 4bc \sin^2\left(\frac{A}{2}\right)$

83. If A, A_1, A_2, A_3 are the areas of the inscribed and escribed of a ΔABC, then

(a) $\sqrt{A_1} + \sqrt{A_2} + \sqrt{A_3} = \sqrt{\pi}(r_1 + r_2 + r_3)$

(b) $\frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}} = \frac{1}{\sqrt{A}}$

(c) $\frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}} = \frac{s^2}{\sqrt{\pi} r_1 r_2 r_3}$

(d) $\sqrt{A_1} + \sqrt{A_2} + \sqrt{A_3} = \sqrt{\pi}(4R + r)$

84. If a, b, A be given in a triangle and c_1 and c_2 two possible values of third side such that $c_1^2 + c_1 c_2 + c_2^2 = a^2$, then A is equal to

(a) 30° (b) 60°

(c) 90° (d) 120°

85. D, E and F are the middle points of the sides of the triangle ABC, then
(a) centroid of the triangle DEF is the same as that of ABC
(b) orthocentre of the triangle DEF is the circumcentre of ABC
(c) orthocentre of the triangle DEF is the incentre of ABC
(d) centroid of the triangle DEF is not the same as that of ABC

86. The sides of ΔABC satisfy the equation $2a^2 + 4b^2 + c^2 = 4ab + 2ac$. Then
(a) the triangle is isosceles
(b) the triangle is obtuse
(c) $B = \cos^{-1}\left(\frac{7}{8}\right)$ (d) $A = \cos^{-1}\left(\frac{1}{4}\right)$

87. If Δ represents the area of acute angled triangle ABC, then $\sqrt{a^2b^2 - 4\Delta^2} + \sqrt{b^2c^2 - 4\Delta^2} + \sqrt{c^2a^2 - 4\Delta^2}$ is equal to
(a) $a^2 + b^2 + c^2$ (b) $\frac{a^2 + b^2 + c^2}{2}$
(c) $ab\cos C + bc\cos A + ca\cos B$
(d) $ab\sin C + bc\sin A + ca\sin B$

88. In triangle, ABC, if $2a^2b^2 + 2b^2c^2 = a^4 + b^4 + c^4$, then angle B is equal to
(a) 45° (b) 135°
(c) 120° (d) 60°

89. If H is the orthocentre of triangle ABC, R = circumradius and $P = AH + BH + CH$, then
(a) $P = 2(R + r)$
(b) max. of P is $3R$
(c) min. of P is $3R$
(d) $P = 2(R - r)$

90. If inside a big circle exactly $n(n \geq 3)$ small circles, each of radius r, can be drawn in such a way that each small circle touches the big circle and also touches both its adjacent small circles, then the radius of big circle is
(a) $r\left(1 + \operatorname{cosec}\frac{\pi}{n}\right)$ (b) $\left(\frac{1 + \tan\frac{\pi}{n}}{\cos\frac{\pi}{n}}\right)$
(c) $r\left[1 + \operatorname{cosec}\frac{2\pi}{n}\right]$ (d) $\frac{r\left[\sin\frac{\pi}{2n} + \cos\frac{2\pi}{n}\right]^2}{\sin\frac{\pi}{n}}$

91. If in triangle ABC, a, b, c and angle A are given and $c\sin A < a < c$, then (b_1 and b_2 are values of b)
(a) $b_1 + b_2 = 2c\cos A$ (b) $b_1 + b_2 = c\cos A$
(c) $b_1b_2 = c^2 - a^2$ (d) $b_1b_2 = c^2 + a^2$

Properties and Solutions of Triangles Exercise 3 : Statement I and II Answer Type Questions

- This section contains 15 questions. Each question contains **Statement I (Assertion)** and **Statement II (Reason).** Each question has 4 choices (a), (b) and (d) out of which only one is correct. Choices are
  (a) Both Statement I and Statement II are correct and Statement II is the correct explanation of Statement I
  (b) Both Statement I and Statement II are correct and Statement II is not the correct explanation of Statement I
  (c) Statement I is correct but Statement II is incorrect
  (d) Statement II is correct but Statement I is incorrect

92. In a ΔABC,
$$a^3 + b^3 + c^3 = c^2(a + b + c)$$
(All symbol used have usual meaning in a triangle.)
Statement I The value of $\angle C = 60°$.
Statement II ΔABC must be equilateral.

93. In a ΔABC, let $a = 6$, $b = 3$ and $\cos(A - B) = \frac{4}{5}$.
Statement I $\angle B = \frac{\pi}{2}$
Statement II $\sin A = \frac{2}{\sqrt{5}}$

[**Note** All symbols used have usual meaning in a triangle.]

94. **Statement I** If in a triangle ABC $\sin^2 A + \sin^2 B + \sin^2 C = 2$, then one of the angles must be 90°.
Statement II In any triangles ABC
$\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A\cos b\cos C$

95. **Statement I** If A, B, C, D are angles of a cyclic quadrilateral then $\Sigma\sin A = 0$.
Statement II If A, B, C, D are angles of cyclic quadrilateral then, $\Sigma\cos A = 0$.

96. **Statement I** In any triangle ABC, the square of the length of the bisector AD is $bc\left(1 - \frac{a^2}{(b + c)^2}\right)$.
Statement II In any triangle ABC length of bisector AD is $\frac{2bc}{(b + c)}\cos\left(\frac{A}{2}\right)$.

97. **Statement I** If I is incentre of ΔABC and I_1 excentre opposite to A and P is intersection of II_1 and BC, then $IP \cdot I_1P = BP \cdot PC$

Statement II In a ΔABC, I is incentre and I_1 is excentre opposite to A, then IBI, I_1, C must be square.

98. All the notations used in statement I and statement II are usual.

Statement I In triangle ABC, if $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$, then value of $\frac{r_1 + r_2 + r_3}{r}$ is equal to 9.

Statement II If $\Delta ABC : \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} - 2R$, where R is circumradius.

99. **Statement I** In a triangle ABC if $\tan A : \tan B : \tan C = 1 : 2 : 3$, then $A = 45°$

Statement II If $p : q : r = 1 : 2 : 3$, then $p = 1$

100. **Statement I** In any right angled triangle $\frac{a^2 + b^2 + c^2}{R^2}$ is always equal to 8.

Statement II $a^2 = b^2 + c^2$

101. **Statement I** perimeter of a regular pentagon inscribed in a circle with centre O and radius a cm equals $10a \sin 36°$ cm.

Statement II Perimeter of a regular polygon inscribed in a circle with centre O and radius a cm equals $(3n - 5)\sin\left(\frac{360°}{2n}\right)$ cm, then it is n sided, where $n \geq 3$.

102. **Statement I** In any triangle ABC $a \cos A + b \cos B + c \cos C \leq s$.

Statement II In any triangle ABC $\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right) \leq \frac{1}{8}$

103. **Statement I** In a ΔABC, if $\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} = y\left(x^2 + \frac{1}{x^2}\right)$, then the maximum value of y is $\frac{9}{8}$.

Statement II In a ΔABC, $\sin\frac{A}{2} \cdot \sin\frac{B}{2} \cdot \sin\frac{C}{2} \leq \frac{1}{8}$

104. **Statement I** In any triangle $a \cos A + b \cos B + c \cos C \leq s$

Statement II In any triangle $\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right) \leq \frac{1}{8}$

105. **Statement I** In triangle ABC, $\frac{a^2 + b^2 + c^2}{\Delta} \geq 4\sqrt{3}$

Statement II If $a_i > 0, i = 1, 2, 3, \ldots, n$ which are not identical, then $\frac{a_1^m + a_2^m + \ldots + a_n^m}{n} > \left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)^m$, if $m < 0$ or $m > 1$.

106. **Statement I** AA_1, BB_1, CC_1 are the medians of triangle ABC whose centroid is G. If the points A, C_1, G and B are concyclic, then c^2, a^2, b^2 are in AP.

Statement II $BG \cdot CC_1 = BC_1 \cdot BA$

Properties and Solutions of Triangles Exercise 4 : Passage Based Questions

Passage I

(Q. Nos. 107 to 109)

R is circumradii of ΔABC, H is orthocentre. R_1, R_2, R_3 are circumradii of $\Delta AHB, \Delta AHC, \Delta BHC$. If AH produced meet the circumradii of ABC at M and intersect BC at L.

$$\angle AHB = 180° - C$$

$$\frac{c}{\sin(180° - C)} = 2R_1$$

$$\frac{c}{\sin C} = 2R_1$$

$$R_1 = R$$

107. $R_1R_2 + R_2R_3 + R_1R_3$ is equal to

(a) $2R^2$ (b) $3R^2$

(c) $5R^2$ (d) R^2

108. Area of ΔAHB

(a) $2R \cos A \cos B \cos C$

(b) $R^2 \cos A \cos B \cos C$

(c) $2R^2 \cos A \cos B \sin C$

(d) None of the above

109. Ratio of area of ΔAHB to ΔBML, is

(a) $\cos B : 2\cos A$ (b) $2 : 1$

(c) $\cos A : \cos B \cos C$ (d) None of these

Passage II

(Q. Nos. 110 to 112)

Let ABC be an acute triangle with $BC = a$, $CA = b$ and $AB = c$, where $a \neq b \neq c$. From any point 'p' inside ΔABC let D, E, F denote foot of perpendiculars from 'P' onto the sides BC, CA and AB, respectively. Now, answer the following questions.

110. All positions of point 'P' for which ΔDEF is isosceles lie on
(a) the incircle of ΔABC
(b) line of internal angle bisectors from A, B and C
(c) arcs of 3 circles
(d) None of the above

111. Let $A(7, 0)$, $B(4, 4)$ and $C(0, 0)$ and ΔDEF is isosceles with $DE = DF$. Then, the curve on which 'P' may lie
(a) $x = 4$ or $x + y = 7$ or $4x = 3y$
(b) $x = 4$ or $x^2 + y^2 = 4x + 4y$
(c) $3(x^2 + y^2) + 196 = 49(x + y)$
(d) None of the above

112. If ΔDEF is equilateral, then 'P'
(a) coincides with incentre of ΔABC
(b) coincides with orthocentre of ΔABC
(c) lies on pedal Δ of ABC
(d) None of the above

Passage III

(Q. Nos. 113 to 115)

In an acute angled ΔABC, let AD, BE and CF be the perpendicular from A, B and C upon the opposite sides of the triangle. (All symbols used have usual meaning in a triangle.)

113. The ratio of the product of the side lengths of the ΔDEF and ΔABC, is equal to
(a) $\dfrac{3(abc)^{\frac{1}{3}}}{4(a+b+c)}$
(b) $\dfrac{1}{4}$
(c) $\cos A \cos B \cos C$
(d) $\sin\left(\dfrac{A}{2}\right)\sin\left(\dfrac{B}{2}\right)\sin\left(\dfrac{C}{2}\right)$

114. The orthocentre of the ΔABC, is the
(a) centroid of the ΔDEF
(b) circum-centre of the ΔDEF
(c) incentre of the ΔDEF
(d) orthocentre of the ΔDEF

115. The circum-radius of the ΔDEF can be equal to
(a) $\dfrac{abc}{8\Delta}$ (b) $\dfrac{a}{4\sin A}$
(c) $\dfrac{R}{2}$ (d) $\dfrac{r}{8}\operatorname{cosec}\dfrac{A}{2}\operatorname{cosec}\dfrac{B}{2}\operatorname{cosec}\dfrac{C}{2}$

Passage IV

(Q.Nos. 116 to 118)

Let a, b, c are the sides opposite to angles A, B, C respectively in a ΔABC $\tan\dfrac{A-B}{2} = \dfrac{a-b}{a+b}\cot\dfrac{C}{2}$ and $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

If $a = 6$, $b = 3$ and $\cos(A - B) = \dfrac{4}{5}$

116. Angle C is equal to
(a) $\dfrac{\pi}{4}$ (b) $\dfrac{\pi}{2}$ (c) $\dfrac{3\pi}{4}$ (d) $\dfrac{2\pi}{3}$

117. Area of the triangle is equal to
(a) 8 (b) 9
(c) 10 (d) 11

118. Value of $\sin A$ is equal to
(a) $\dfrac{1}{\sqrt{5}}$ (b) $\dfrac{2}{\sqrt{5}}$
(c) $\dfrac{1}{2\sqrt{5}}$ (d) $\dfrac{1}{\sqrt{3}}$

Passage V

(Q. Nos. 119 to 123)

When any two sides and one of the opposite acute angle are given, under certain additional conditions two triangles are possible. The case when two triangles are possible is called the ambiguous case.

In fact when any two sides and the angle opposite to one of them are given either no triangle is possible or only one triangle is possible or two triangles are possible.

In the ambiguous case, let a, b and $\angle A$ are given and c_1, c_2 are two values of the third side c.

O*n the basis of above information, answer the following questions*

119. Two different triangles are possible when
(a) $b\sin A < a$ (b) $b\sin A < a$ and $b > a$
(c) $b\sin A < a$ and $b < a$ (d) $b\sin A < a$ and $a = b$

120. The difference between two values of c is
(a) $2\sqrt{(a^2 - b^2)}$ (b) $\sqrt{(a^2 - b^2)}$
(c) $2\sqrt{(a^2 - b^2\sin^2 A)}$ (d) $\sqrt{(a^2 - b^2\sin^2 A)}$

121. The value of $c_1^2 - 2c_1c_2\cos 2A + c_2^2$ is
(a) $4a\cos A$ (b) $4a^2\cos A$
(c) $4a\cos^2 A$ (d) $4a^2\cos^2 A$

122. If $\angle A = 45°$ and in ambiguous case (a, b, A are given) c_1, c_2 are two values of c and if θ be the angle between the two positions of the ambiguous side c then $\cos\theta$ is
(a) $\dfrac{c_1c_2}{c_1^2 + c_2^2}$ (b) $\dfrac{2c_1c_2}{c_1^2 + c_2^2}$
(c) $\dfrac{\sqrt{c_1c_2}}{(c_1 + c_2)}$ (d) $\dfrac{2\sqrt{c_1c_2}}{(c_1 + c_2)}$

123. If $2b = (m+1)a$ and $\cos A = \frac{1}{2}\sqrt{\left(\frac{(m-1)(m+3)}{m}\right)}$, where $1 < m < 3$, then $\frac{c_1}{c_2}$ is

(a) m or $\frac{1}{m}$ (b) $(m-1)$ or $\frac{1}{(m-1)}$

(c) $(m+1)$ or $\frac{1}{(m+1)}$ (d) $(m+3)$ or $\frac{1}{(m+3)}$

Passage VI

(Q.Nos. 124 to 126)

Consider a triangle ABC, where x, y, z are the length of perpendicular drawn from the vertices of the triangle to the opposite sides a, b, c respectively. Let the letters R, r, S, Δ denote the circumradius, inradius semi-perimeter and area of the triangle respectively.

124. If $\frac{bx}{c} + \frac{cy}{a} + \frac{az}{b} = \frac{a^2 + b^2 + c^2}{k}$, then the value of k is

(a) R (b) S (c) $2R$ (d) $\frac{3}{2}R$

125. If $\cot A + \cot B + \cot C = k\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right)$, then the value of k is

(a) R^2 (b) rR

(c) Δ (d) $a^2 + b^2 + c^2$

126. The value of $\frac{c\sin B + b\sin C}{x} + \frac{a\sin C + c\sin A}{y} + \frac{b\sin A + a\sin B}{z}$ is equal to

(a) $\frac{R}{r}$ (b) $\frac{S}{R}$ (c) 2 (d) 6

Passage VII

(Q.Nos. 127 to 131)

AL, BM and CN are perpendicular from angular points of a triangle ABC on the opposite sides BC, CA and AB respectively. Δ is the area of triangle ABC, (r) and R are the inradius and circumradius.

127. If perimeters of ΔLMN and ΔABC and λ and μ, then the value of $\frac{\lambda}{\mu}$ is

(a) $\frac{r}{R}$ (b) $\frac{R}{r}$

(c) $\frac{rR}{\Delta}$ (d) $\frac{\Delta}{rR}$

128. If areas of Δ's AMN, BNL and CLM are Δ_1, Δ_2 and Δ_3 respectively, then the value of $\Delta_1 + \Delta_2 + \Delta_3$ is

(a) $\Delta(2 + 2\cos A\cos B\cos C)$

(b) $\Delta(2 + 2\sin A\sin B\sin C)$

(c) $\Delta(1 - 2\cos A\cos B\cos C)$

(d) $\Delta(1 - 2\sin A\sin B\sin C)$

129. If area of ΔLMN is Δ', then the value of $\frac{\Delta'}{\Delta}$ is

(a) $2\sin A\sin B\sin C$ (b) $2\cos A\cos B\cos C$

(c) $\sin A\sin B\sin C$ (d) $\cos A\cos B\cos C$

130. Radius of the circum circle of ΔLMN is

(a) $2R$ (b) R

(c) $\frac{R}{2}$ (d) $\frac{R}{4}$

131. If radius of the incircle of ΔLMN is r', then the value of $r'\sec A\sec B\sec C$ is

(a) $4R$ (b) $3R$

(c) $2R$ (d) R

Properties and Solutions of Triangles Exercise 5 : Matching Type Questions

132. Match the statement of **Column I** with values of **Column II.**

	Column I	Column II
(A)	In a ΔABC, let $\angle C = \frac{\pi}{2}$, r = inradius, R = circumradius then $2(r+R)$	(p) $a+b+c$
(B)	If l, m, n are perpendicular drawn from the vertices of triangle having sides a, b and c then $\sqrt{2R\left(\frac{bl}{c} + \frac{cm}{a} + \frac{an}{b}\right) + 2ab + 2bc + 2ca}$	(q) $a-b$
(C)	In a ΔABC, $R(b^2\sin 2C + c^2\sin 2B)$ equals	(r) $a+b$
(D)	In a right angle triangle ABC, $\angle C = \frac{\pi}{2}$, then $4R\sin\frac{A+B}{2}\sin\frac{(A-B)}{2}$	(s) abc

133. Match the statement of **Column I** with the values of **Column II**.

	Column I		Column II
(A)	In a triangle ABC $(c-a)^2 = b^2 - ac$ and $\cos B + \sin C = \frac{3}{2}$	(p)	$A = 30°$
(B)	A, B, C are in A.P. and $C = 3A$	(q)	$B = 60°$
(C)	The length of the bisector of angle $B = \frac{\sqrt{3}ca}{(c+a)}$ and $a = b$	(r)	$C = 90°$
(D)	$\frac{1}{a+b} + \frac{1}{b+c} = \frac{3}{a+b+c}$	(s)	$A = B = C = 60°$

134. Match the statement of **Column I** with values of **Column II.**

	Column I		Column II
(A)	In a ΔABC, $(a+b+c)(b+c-a) = \lambda bc$, where $\lambda \in I$, then greatest value of λ is	(p)	3
(B)	In a ΔABC, $\tan A + \tan B + \tan C = 9$. If $\tan^2 A + \tan^2 B + \tan^2 C = k$, then least value of k satisfying is	(q)	$9(3)^{1/3}$
(C)	In a triangle ABC, then line joining the circumcentre to the incentre is parallel to BC, then value of $\cos B + \cos C$ is	(r)	1
(D)	If in a ΔABC, $a = 5$, $b = 4$ and $\cos(A - B) = \frac{31}{32}$, then the third side c is equal to	(s)	6

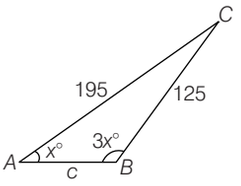
135. Match the statement of Column I with values of Column II

	Column I		Column II
(A)	In a ΔABC, if $2a^2 + b^2 + c^2 = 2ac + 2ab$, then	(p)	ΔABC is equilateral triangle
(B)	In a ΔABC, if $a^2 + b^2 + c^2 = \sqrt{2}b(c+a)$, then	(q)	ΔABC is right angled triangle
(C)	In a ΔABC, if $a^2 + b^2 + c^2 = bc + ca\sqrt{3}$, then	(r)	ΔABC is scalane triangle
		(s)	ΔABC is scalane right angled triangle
		(t)	Angles B, C, A are in AP

Properties and Solutions of Triangles Exercise 6 : Single Integer Answer Type Questions

136. If in ΔABC, $\angle C = \frac{\pi}{8}$, $a = \sqrt{2}$ and $b = \sqrt{2+\sqrt{2}}$, then find the sum of digits in the measure of angle A (in degree).

137. In the figure as shown, find the number of digits in the length of AB.



138. If $A = \frac{\pi}{7}$, $B = \frac{2\pi}{7}$ and $C = \frac{4\pi}{7}$ then in ΔABC, $\frac{a^2 + b^2 + c^2}{R^2}$ equals to

139. If A, B, C the angles of an acute angled ΔABC and

$$D = \begin{vmatrix} (\tan B + \tan C)^2 & \tan^2 A & \tan^2 A \\ \tan^2 B & (\tan A + \tan C)^2 & \tan^2 A \\ \tan^2 C & \tan^2 C & (\tan + \tan B)^2 \end{vmatrix},$$

then the least integral values of $\frac{D}{1000}$ is

140. In a ΔABC, P and Q are the mid-point or AB and AC, respectively. If O is the circum-centre of the ΔABC, then the value of $\left(\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta OPQ}\right) \cot B \cot C$ equals to

141. With usual notation in ΔABC, the numerical value of $\left(\frac{a+b+c}{r_1 + r_2 + r_3}\right)\left(\frac{a}{r_1} + \frac{b}{r_2} + \frac{c}{r_3}\right)$ is

142. In a ΔABC, $\cos A \cdot \cos C + \frac{\lambda(c^2 - a^2)}{3ca}$, where AD is the median through A and $AD \perp AC$, then the value of λ is ..... .

143. In a triangle ABC, medians AD and CE are drawn. If $AD = 5$, $\angle DAC = \frac{\pi}{8}$ and $\angle ACE = \frac{\pi}{4}$, then the area of the triangle ABC is equal to $\frac{5a}{b}$, then $a + b$ is equal to ..... .

144. In ΔABC, $\frac{r}{r_1} = \frac{1}{2}$, then the value of $16\left(\Sigma \tan\left(\frac{A}{2}\right)\right)$ must be.

145. In a ΔABC, the maximum value of $120\left(\frac{\Sigma a\cos^2\left(\frac{A}{2}\right)}{a+b+c}\right)$ must be

146. The sides of triangle are three consecutive natural numbers and its largest angle is twice the smaller one. The largest side of the triangle must be

147. In ΔABC, $\angle C = 2\angle A$, and $AC = 2BC$, then the value of $\frac{a^2+b^2+c^2}{R^2}$ (where R is circumradius of triangle) is ..... .

148. If a, b and A are given in a triangle and c_1, c_2 are the possible values of the third side, and $c_1^2 + c_2^2 - 2c_1c_2\cos A = \lambda a^2\cos^2 A$, then the value of λ is ..... .

149. In triangle ABC, $a = 5$, $b = 4$, $c = 3$. G is the centroid of triangle. If R_1 be the circumradius of triangle GAB then the value of $\frac{a}{65}R_1^2$ must be

150. A triangle ABC is inscribed in a circle of radius 1 and centre at O. The lines AO, BO, CO meet the opposite sides at D, E, F. Then $\frac{1}{AD}+\frac{1}{BE}+\frac{1}{CF}$ is equal to ..... .

151. In ΔABC, $a \geq b \geq c$ and if $\frac{a^3+b^3+c^3}{\sin^3 A+\sin^3 B+\sin^3 C} = 8$, than the maximum value of a is ..... .

152. In a cyclic quadrilateral $PQRS$, $PQ = 2$ units, $QR = 5$ units, $RS = 3$ units and $PQR = 60°$, then SP is ..... .

Properties and Solutions of Triangles Exercise 7 : Subjective Type Questions

153. In a ΔABC, the angles A and B are two values of θ satisfying $\sqrt{3}\cos\theta + \sin\theta = K$; where $|K| < 2$, then show triangles is obtuse angled.

154. If in an obtuse angled triangle the obtuse angle is $\frac{3\pi}{4}$ and the other two angles are equal to two values of θ satisfying $a\tan\theta + b\sec\theta = c$, where $|b| \leq \sqrt{a^2+c^2}$, then find the value of $a^2 - c^2$.

155. In a ΔABC, a, c, A are given and b_1, b_2 are two values of third side b such that $b_2 = 2b_1$. Then, the value of $\sin A$.

156. If P is a point on the altitude AD of the ΔABC, such that $\angle CBP = \frac{B}{3}$, then find the value of AP.

157. If R denotes circum-radius of ΔABC, evaluate $\frac{b^2-c^2}{2aR}$.

158. In ΔABC, $A = \frac{2\pi}{3}$, $b - c = 3\sqrt{3}$ cm and ar $(\Delta ABC) = \frac{9\sqrt{3}}{2}\text{cm}^2$. Solve for side a.

159. Find the value of $\tan A$, if area of ΔABC is $a^2 - (b-c)^2$.

160. In a ΔABC, $B = 90°$, $AC = h$ and the length of perpendicular from B to AC is p such that $h = 4p$. If $AB < BC$, then measure $\angle C$.

161. If in a ΔABC, $\sin^3 A + \sin^3 B + \sin^3 C = 3\sin A \cdot \sin B \cdot \sin C$, then find the value of determinant $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$.

162. In a ΔABC, the side a, b and c are such that they are the roots of $x^3 - 11x^2 + 38x - 40 = 0$. Then, the value of $\frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}$.

163. In a ΔABC the sides a, b and c are in AP. Evaluate $\left(\tan\frac{A}{2}+\tan\frac{C}{2}\right):\cot\frac{B}{2}$.

164. The sides of a Δ are in AP. and its area is $\frac{3}{5}\times$ (area of an equilateral triangle of the same perimeter). Find the ratio of its sides.

165. If AD, BE and CF are the medians of a ΔABC, then evaluate $(AD^2 + BE^2 + CF^2) : (BC^2 + CA^2 + AB^2)$.

166. AD is a median of the ΔABC. If AE and AF are medians of the ΔABD and ΔADC respectively, and $AD = m_1$, $AE = m_2$, $AF = m_3$, then find the value of $\frac{a^2}{8}$.

167. If $X = \tan\frac{B-C}{2}\cdot\tan\frac{A}{2}$; $Y = \tan\frac{A-B}{2}\cdot\tan\frac{C}{2}$; $Z = \tan\frac{C-A}{2}\cdot\tan\frac{B}{2}$, then find the value of $X + Y + Z + XYZ$.

168. Let ΔABC be equilateral on side BA produced, we choose a point P such that A lies between P and B. We now denote 'a' as the length of a side of ΔABC; r_1 as the radius of incircle of ΔPAC; and r_2 as the radius of the excircle of ΔPBC with respect to side BC. Determine the sum $(r_1 + r_2)$ as a function of 'a' alone.

169. A hexagon is inscribed in a circle of radius r. Two of its sides have length 1, two have length 2 and the last two have length 3. Prove that r is a root of the equation $2r^3 - 7r - 3 = 0$.

170. The base of a triangle is divided into three equal parts. If $\theta_1, \theta_2, \theta_3$ be the angles subtended by these parts at the vertex, then prove that

$$(\cot\theta_1 + \cot\theta_2)(\cot\theta_2 + \cot\theta_3) = 4\ \text{cosec}^2\theta_2$$

171. If the circum-radius of a Δ is $\frac{54}{\sqrt{1463}}$, and its sides are in GP with ratio $\frac{3}{2}$, then find the sides of the triangle.

172. Prove that $a^2 + b^2 + c^2 + 2abc < 2$, where a, b, c are the sides of triangle ABC such that $a + b + c = 2$.

173. Let points $P_1, P_2, P_3, \ldots, P_{n-1}$ divides the side BC of a ΔABC into n parts. Let $r_1, r_2, r_3, \ldots, r_n$ be the radii of inscribed circles and let $p_1, p_2, \ldots, p_n$ be the radii of excribed circles corresponding to vertex Δ for triangle ABP_1, AP_1P_2, …, $AP_{n-1}C$ and let r and P be the corresponding radii for the triangle ABC. Show that $\frac{r_1}{P_1}\cdot\frac{r_2}{P_2}\cdots\frac{r_n}{P_n} = \frac{r}{P}$.

174. A polygon of n sides, inscribed in a circle, is such that its sides subtend angles $2\alpha, 4\alpha, \ldots, 2n\alpha$ at the centre of the circles. Prove that its area A_1, is to the area A_2 of the regular polygon of n sides inscribed in the same circle, as $\sin n\alpha : n\sin\alpha$.

175. $A_1, A_2, A_3, \ldots, A_n$ is a regular polygon of n sides circumscribed about a circle of centre O and radius 'a'. P is any point distant 'c' from O. Show that the sum of the squares of the perpendiculars from P on the sides of the polygon is $n\left(a^2 + \frac{c^2}{2}\right)$.

176. Show that in any ΔABC, $a^3\cos 3B + 3a^2 b\cos(2B - A) + 3ab^2\cos(B - 2A) + b^3\cos 3A = c^3$

177. Let ABC be a Δ with altitudes h_1, h_2, h_3 and inradius r. Prove that $\frac{h_1 + r}{h_1 - r} + \frac{h_2 + r}{h_2 - r} + \frac{h_3 + r}{h_3 - r} \geq 6$.

178. If in a ΔABC, $\frac{a\cos A + b\cos B + c\cos C}{a\sin B + b\sin C + c\sin A} = \frac{a + b + c}{9R}$, then prove that Δ is equilateral.

179. In ΔABC, 'h' is the length of altitude drawn from vertex A on the side BC. Prove that : $2(b^2 + c^2) \geq 4h^2 + a^2$. Also, discuss the case when equality holds true.

180. Consider a ΔABC. A directly similar $\Delta A_1B_1C_1$ is inscribed in the ΔABC such that A_1, B_1 and C_1 are the interior points of the sides AC, AB and BC, respectively. Prove that $\frac{\text{Area}(\Delta A_1B_1C_1)}{\text{Area}(\Delta ABC)} \geq \frac{1}{\text{cosec}^2 A + \text{cosec}^2 B + \text{cosec}^2 C}$

181. Find the angle at the vertex of an isosceles triangle having the maximum area for the given length 'l' of the median to one of its equal sides.

182. Consider a ΔABC and points A_1 and B_1 on side BC such that $\angle BAA_1 = \angle B_1AC$. If incircle of ΔBAA_1 and B_1AC touch the sides BA_1 and B_1C at M and N respectively, prove that $\frac{1}{BM} + \frac{1}{MA_1} = \frac{1}{B_1N} + \frac{1}{NC}$.

183. An equilateral triangle PQR is circumscribed about a given ΔABC. Prove that the maximum area of ΔPQR is $2\Delta + \frac{a^2 + b^2 + c^2}{2\sqrt{3}}$. Where a, b, c are the sides of ΔABC and Δ is its area.

184. In a ΔABC, r_A, r_B, r_C are the radii of the circles which touch the incircle and the sides emanting from the vertices A, B, C respectively. Prove that, $\sqrt{r_Ar_B} + \sqrt{r_Br_C} + \sqrt{r_Cr_A} = r$

185. Find the points inside a Δ from which the sum of the squares of distance to the three sides is minimum. Also, find the minimum values of the sum of squares of distances.

186. In a scalene acute ΔABC, it is known that line joining circumcentre and orthocentre is parallel to BC. Prove that the angle $A \in \left(\frac{\pi}{3}, \frac{\pi}{2}\right)$.

187. Consider an acute angled ΔABC. Let AD, BE and CF be the altitudes drawn from the vertices to the opposite sides. Prove that : $\frac{EF}{a} + \frac{FD}{b} + \frac{DE}{c} = \frac{R + r}{R}$.

188. Two circle, the sum of whose radii is 'a' are placed in the same plane with their distance '$2a$' apart. An endless

string is fully stretched so as partly to surround the circle and to cross between them. Prove that length of string is $\left(\frac{4\pi}{3}+2\sqrt{3}\right)a$.

189. If Δ_0 is the area of Δ formed by joining the points of contact of incircle with the sides of the given triangle whose area is Δ. Similarly Δ_1, Δ_2 and Δ_3 are the corresponding area of the Δ formed by joining the points of contact of excircles with the sides. Prove that $\frac{\Delta_1}{\Delta}+\frac{\Delta_2}{\Delta}+\frac{\Delta_3}{\Delta}-\frac{\Delta_0}{\Delta}=2$.

190. Let P be the point inside the ΔABC. Such that $\angle APB=\angle BPC=\angle CPA$. Prove that $PA+PB+PC=\sqrt{\frac{a^2+b^2+c^2}{2}+2\sqrt{3}\Delta}$, where a, b, c, Δ are the sides and the area of ΔABC.

191. In an acute angled ΔABC, the points A', B' and C' are located such that A' is the point where altitude from A on BC meets the outward facing semi-circle drawn on BC as diameter, points B', C' are located similarly. Prove that $\{\text{ar}(BCA')\}^2+\{\text{ar}(CAB')\}^2+\{\text{ar}(ABC')\}^2=\{\text{ar}(ABC)^2\}$

Properties and Solutions of Triangles Exercise 8 : Questions Asked in Previous 10 Years' Exam

(i) JEE Advanced & IIT-JEE

192. In a ΔXYZ, let x, y, z be the lengths of sides opposite to the angles X, Y, Z respectively and $2s=x+y+z$. If $\frac{s-x}{4}=\frac{s-y}{3}=\frac{s-z}{2}$ and area of incircle of the ΔXYZ is $\frac{8\pi}{3}$, then **[More than one correct option 2016 Adv.]**

(a) area of the ΔXYZ is $6\sqrt{6}$

(b) the radius of circum-circle of the ΔXYZ is $\frac{35}{6}\sqrt{6}$

(c) $\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2}=\frac{4}{35}$

(d) $\sin^2\left(\frac{X+Y}{2}\right)=\frac{3}{5}$

193. In a triangle, the sum of two sides is x and the product of the same two sides is y. If $x^2-c^2=y$, where c is the third side of the triangle, then the ratio of the in-radius to the circum-radius of the triangle is **[Single correct option 2014 Adv.]**

(a) $\frac{3y}{2x(x+c)}$ (b) $\frac{3y}{2c(x+c)}$

(c) $\frac{3y}{4x(x+c)}$ (d) $\frac{3y}{4c(x+c)}$

194. Consider a ΔABC and let a, b and c denote the lengths of the sides opposite to vertices A, B and C, respectively. $a=6$, $b=10$ and the area of the triangle is $15\sqrt{3}$. If $\angle ACB$ is obtuse and if r denotes the radius of the incircle of the triangle, then r^2 is equal to..... **[Intger Answer Type 2013 Adv.]**

195. In a ΔPQR, P is the largest angle and $\cos P=\frac{1}{3}$. Further in circle of the triangle touches the sides PQ, QR and RP at N, L and M respectively, such that the lengths of PN, QL and RM are consecutive even integers. Then, possible length(s) of the side(s) of the triangle is (are) **[More than one correct option 2012]**

(a) 16 (b) 18 (c) 20 (d) 22

196. If ΔPQR is a triangle of area Δ with $a=2$, $b=\frac{7}{2}$ and $c=\frac{5}{2}$, where a, b and c are the lengths of the sides of the triangle opposite to the angles at P, Q and R, respectively. Then, $\frac{2\sin P-\sin 2P}{2\sin P+\sin 2P}$ equals **[More than one correct option 2012]**

(a) $\frac{3}{4\Delta}$ (b) $\frac{45}{4\Delta}$

(c) $\left(\frac{3}{4\Delta}\right)^2$ (d) $\left(\frac{45}{4\Delta}\right)^5$

197. If the angles A, B and C of a triangle are in an arithmetic progression and if a, b and c denote the lengths of the sides opposite to A, B and C respectively, then the value of the expression $\frac{a}{c}\sin 2C+\frac{c}{a}\sin 2A$ is **[Single correct option 2010]**

(a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) 1 (d) $\sqrt{3}$

198. Let ABC be a triangle such that $\angle ACB=\frac{\pi}{6}$. If a, b and c denote the lengths of the sides opposite to A, B and C, respectively. Then, the value(s) of x for which $a=x^2+x+1$, $b=x^2-1$ and $c=2x+1$ is (are) **[Single correct option 2010]**

(a) $-(2+\sqrt{3})$ (b) $1+\sqrt{3}$

(c) $2+\sqrt{3}$ (d) $4\sqrt{3}$

199. In a ΔABC with fixed base BC, the vertex A moves such that $\cos B + \cos C = 4\sin^2\frac{A}{2}$. If a, b and c denote the lengths of the sides of the triangle opposite to the angles A, B and C respectively, then **[More than one correct option 2009]**

(a) $b + c = 4a$
(b) $b + c = 2a$
(c) locus of point A is an ellipse
(d) locus of point A is a pair of straight line

200. Let ABC and ABC' be two non-congruent triangles with sides $AB = 4$, $AC = AC' = 2\sqrt{2}$ and $\angle B = 30°$. The absolute value of the difference between the areas of these triangles is **[Integer Answer Type 2009]**

201. A straight line through the vertex P of a ΔPQR intersects the side QR at the point S and the circum-circle of the ΔPQR at the point T. If S is not the centre of the circumcircle, then **[More than one correct option 2008]**

(a) $\frac{1}{PS} + \frac{1}{ST} < \frac{2}{\sqrt{QS \times SR}}$
(b) $\frac{1}{PS} + \frac{1}{ST} > \frac{2}{\sqrt{QS \times SR}}$
(c) $\frac{1}{PS} + \frac{1}{ST} < \frac{4}{QR}$
(d) $\frac{1}{PS} + \frac{1}{ST} > \frac{4}{QR}$

Passage

(Q.Nos. 202 to 204)

Consider the circle $x^2 + y^2 = 9$ and the parabola $y^2 = 8x$. They intersect at P and Q in the first and the fourth quadrants, respectively. Tangents to the circle at P and Q intersect the X-axis at R and tangents to the parabola at P and Q intersect the X-axis at S.

[One correct option 2007]

202. The radius of the in-circle of ΔPQR is

(a) 4 (b) 3
(c) $\frac{8}{3}$ (d) 2

203. The radius of the circum-circle of the ΔPRS is

(a) 5 (b) $3\sqrt{3}$
(c) $3\sqrt{2}$ (d) $2\sqrt{3}$

204. The ratio of the areas of ΔPQS and ΔPQR is

(a) $1 : \sqrt{2}$ (b) 1 : 2
(c) 1 : 4 (d) 1 : 8

205. Internal bisector of $\angle A$ of ΔABC meets side BC at D. A line drawn through D perpendicular to AD intersects the side AC at E and side AB at F. If a, b, c represent sides of ΔABC, then **[More than one correct option 2006]**

(a) AE is HM of b and c (b) $AD = \frac{2bc}{b+c}\cos\frac{A}{2}$
(c) $EF = \frac{4bc}{b+c}\sin\frac{A}{2}$ (d) ΔAEF is isosceles

206. In-radius of a circle which is inscribed in a isosceles triangle one of whose angle is $2\pi/3$ is $\sqrt{3}$, then area of triangle (in sq units) is **[Single correct option 2006]**

(a) $4\sqrt{3}$
(b) $12 - 7\sqrt{3}$
(c) $12 + 7\sqrt{3}$
(d) None of the above

207. In a ΔABC, among the following which one is true? **[Single correct option 2005]**

(a) $(b + c)\cos\frac{A}{2} = a\sin\left(\frac{B+C}{2}\right)$
(b) $(b + c)\cos\left(\frac{B+C}{2}\right) = a\sin\frac{A}{2}$
(c) $(b - c)\cos\left(\frac{B-C}{2}\right) = a\cos\left(\frac{A}{2}\right)$
(d) $(b - c)\cos\frac{A}{2} = a\sin\left(\frac{B-C}{2}\right)$

(ii) JEE Main & AIEEE

208. Let a vertical tower AB have its end A on the level ground. Let C be the mid-point of AB and P be a point on the ground such that $AP = 2AB$. If $\angle BPC = \beta$, then $\tan\beta$ is equal to **[2017 JEE Main]**

(a) $\frac{6}{7}$ (b) $\frac{1}{4}$
(c) $\frac{2}{9}$ (d) $\frac{4}{9}$

209. $ABCD$ is a trapezium such that AB and CD are parallel and $BC \perp CD$, if $\angle ADB = \theta$, $BC = p$ and $CD = q$, then AB is equal to **[2013 JEE Main]**

(a) $\frac{(p^2+q^2)\sin\theta}{p\cos\theta + q\sin\theta}$ (b) $\frac{p^2+q^2\cos\theta}{p\cos\theta + q\sin\theta}$
(c) $\frac{p^2+q^2}{p^2\cos\theta + q^2\sin\theta}$ (d) $\frac{(p^2+q^2)\sin\theta}{(p\cos\theta + q\sin\theta)^2}$

210. For a regular polygon, let r and R be the radii of the inscribed and the circumscribed circles. A false statement among the following is **[2010 AIEEE]**

(a) there is a regular polygon with $\frac{r}{R} = \frac{1}{2}$
(b) there is a regular polygon with $\frac{r}{R} = \frac{1}{\sqrt{2}}$
(c) there is a regular polygon with $\frac{r}{R} = \frac{2}{3}$
(d) there is a regular polygon with $\frac{r}{R} = \frac{\sqrt{3}}{2}$

211. In a ΔABC, let $\angle C = \frac{\pi}{2}$, if r is the in-radius and R is the circum-radius of the ΔABC, then $2(r + R)$ is equal to **[2005 AIEEE]**

(a) $c + a$ (b) $a + b + c$
(c) $a + b$ (d) $b + c$

212 If in ΔABC, the altitudes from the vertices A, B and C on opposite sides are in HP, then $\sin A$ $\sin B$ and $\sin C$ are in **[2005 AIEEE]**

(a) HP (b) AGP
(c) AP (d) GP

Answers

Exercise for Session 1

1. (a) **2.** (a) **3.** (c) **4.** (c) **5.** (c) **6.** (a)
7. (d) **8.** (c) **9.** (b) **10.** (c) **11.** (c) **12.** (d)
13. (c) **14.** (a) **15.** (a) **16.** (c) **17.** (b) **18.** (a)
19. (c) **20.** (a)

Exercise for Session 2

1. (a) **2.** (a) **3.** (b) **4.** (d) **5.** (a) **6.** (c)
7. (d) **8.** (c) **9.** (a) **10.** (b) **11.** (d) **12.** (c)
13. (b) **14.** (b) **15.** (d) **16.** (c) **17.** (a) **18.** (c)
19. (c) **20.** (a)

Exercise for Session 3

1. 1018.81 sq. cm **2.** 2.5 **3.** 2 cm **7.** 7:2 **9.** $\frac{S}{R}$
26. (1) **27.** 0 **28.** a^2b^2

Exercise for Session 4

1. $a\cot A$ **2.** $\frac{2A}{a+b}$ **4.** 4/3 **5.** $\frac{\sqrt{5}}{2}$ **6.** $2a^2b^2c^2$
9. $\frac{R}{2}$ **10.** $a\,\text{cosec}(A/2)$

Exercise for Session 5

1. $\frac{a}{2}\cot\frac{\pi}{2n}$ **2.** $\frac{a}{2}\text{cosec}\frac{\pi}{n}$ **4.** $\text{cosec}\,\frac{\pi}{9}$
6. 7, 3

Exercise for Session 6

1. 2 **2.** 2 **3.** 6 **4.** $\frac{ab}{a+b}$ **5.** $\pi\tan A$

Exercise for Session 7

1. $\frac{1}{\sqrt{7}}$ **2.** $\sin A + \sin B + \sin C$ **3.** $\frac{1}{2}$ **4.** 6 **6.** $r = 1$
7. $\frac{\pi}{4}$ **9.** $(2-\sqrt{3}):2\sqrt{3}$

Exercise for Session 8

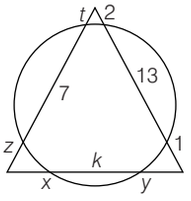
4. $R\tan\alpha$ **5.** $\frac{2.05\sin 38^\circ}{\sin 42^\circ}$

Chapter Exercises

1. (a) **2.** (a) **3.** (c) **4.** (c) **5.** (c) **6.** (c)
7. (b) **8.** (c **9.** (a) **10.** (b) **11.** (b) **12.** (b)
13. (c) **14.** (d) **15.** (a) **16.** (d) **17.** (a) **18.** (b)
19. (b) **20.** (c) **21.** (b) **22.** (d) **23.** (b) **24.** (c)
25. (a) **26.** (c) **27.** (a) **28.** (a) **29.** (b) **30.** (a)
31. (a) **32.** (b) **33.** (d) **34.** (c) **35.** (b) **36.** (a)
37. (b) **38.** (a) **39.** (c) **40.** (a) **41.** (b) **42.** (c)
43. (a) **44.** (c) **45.** (b) **46.** (a) **47.** (c) **48.** (d)
49. (b) **50.** (a) **51.** (a) **52.** (c) **53.** (b) **54.** (b)
55. (b) **56.** (d) **57.** (c) **58.** (a) **59.** (b) **60.** (a)
61. (c) **62.** (b) **63.** (a) **64.** (a) **65.** (b) **66.** (d)
67. (b) **68.** (a) **69.** (c)
70. (b,d) **71.** (b,c,d) **72.** (b,c,d)
73. (b,c) **74.** (a,c,d) **75.** (a,c)
76. (a,b) **77.** (a,c,d) **78.** (c,d)
79. (b,c) **80.** (a,c) **81.** (a,b,c,d)
82. (a,b,c,d) **83.** (a,b,c,d) **84.** (b,c)
85. (a,b) **86.** (a,c,d) **87.** (b,c)
88. (a,b) **89.** (a,b) **90.** (a,d)
91. (a,c)
92. (c) **93.** (d) **94.** (a) **95.** (d) **96.** (a) **97.** (c)
98. (a) **99.** (c) **100.** (a) **101.** (c) **102.** (a) **103.** (a)
104. (a) **105.** (a) **106.** (b) **107.** (b) **108.** (c) **109.** (c)
110. (c) **111.** (c) **112.** (d) **113.** (c) **114.** (c) **115.** (a,b,c,d)
116. (b) **117.** (b) **118.** (b) **119.** (b) **120.** (b) **121.** (d)
122. (b) **123.** (a) **124.** (c) **125.** (c) **126.** (d) **127.** (b)
128. (c) **129.** (d) **130.** (b) **131.** (a)
132. (A) $\to$ (r), (B) $\to$ (p), C $\to$ (s) , (D) $\to$ (q)
133. (A) $\to$ (p, q, r), (B) $\to$ (p, q, r), C $\to$ (q, s) , (D) $\to$ q
134. (A) $\to$ (p), (B) $\to$ (q), C $\to$ (r) , (D) $\to$ (s)
135. (A) $\to$ (p, t), (B) $\to$ (q, s), C $\to$ (q, r, t)
136. (9) **137.** (3) **138.** (7) **139.** (2) **140.** (4) **141.** (4)
142. (2) **143.** (8) **144.** (8) **145.** (9) **146.** (6) **147.** (8)
148. (4) **149.** (5) **150.** (2) **151.** (2) **152.** (2) **154.** $2ac$
155. $\sqrt{\frac{9a^2 - c^2}{8c^2}}$ **156.** $2c\sin\left(\frac{B}{3}\right)$ **157.** $(\sin(B - C))$
158. $(\sqrt{63} - 2\sqrt{3})$ **159.** $\left(\frac{8}{15}\right)$ **160.** (15°)
161. (0) **162.** $\left(\frac{9}{16}\right)$ **163.** $\left(\frac{2}{13}\right)$
164. (3 : 5 : 7) **165.** (3 : 4) **166.** $m_2^2 + m_3^2 - 2m_1^2$
167. (0) **168.** $\left(\frac{a\sqrt{3}}{2}\right)$ **171.** $\left(1, \frac{3}{2}, \frac{9}{4}\right)$
181. $\left(\frac{4}{5}\right)$ **185.** $\Delta_{\min} = \frac{4(s-a)(s-b)(s-c)s}{a^2 + b^2 + c^2}$
192.(a,c,d) **193.** (b) **194.** (3) **195.** (a, b)
196. (b,c,d) **197.** (d) **198.** (b) **199.** (b, c)
200. 4 sq units **201.** (d) **202.** (d) **203.** (b) **204.** (c)
205. (a,b,c,d) **206.** (c) **207.** (d) **208.** (c) **209.** (a)
210. (a,b,d) **211.** (c) **212.** (c)

Solutions

1.



$$x + k + y = 16$$

Also, $(t + 7) \times t = 2 \times (2 + 13)$

$\Rightarrow$ $t = 3$

Now, $x(x + k) = 6 \times (6 + 7)$

and $y(y + k) = 1 \times 14$

Solving, we get $x = 10 - \sqrt{22}$, $y = 6 - \sqrt{22}$ and $k = 2\sqrt{22}$.

2. We have, $s = \dfrac{a + b + c}{2} = 21$

No2, $\sin A = \dfrac{2\Delta}{bc}$

$$= \frac{2}{14 \cdot 15}\sqrt{21 \cdot 8 \cdot 7 \cdot 6} = \frac{4}{5}$$

$\Rightarrow$ $A = \sin^{-1}\dfrac{4}{5}$

Alternatively By using cosine rule in ΔABC, we get

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

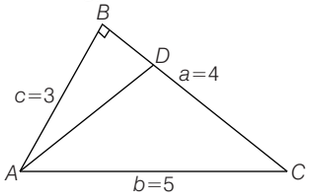
$$= \frac{(14)^2 + (15)^2 - (13)^2}{2(14)(15)}$$

$$= \frac{196 + 225 - 169}{420}$$

$$= \frac{252}{420} = \frac{3}{5}\left(0 < A < \frac{\pi}{2}\right)$$

Hence, $A = \sin^{-1}\dfrac{4}{5}$

3. Using Napier's Analogy



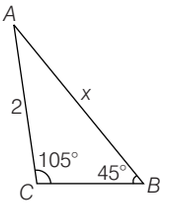
$$\tan\left(\frac{A - B}{2}\right) = \frac{a - b}{a + b}\cos\frac{C}{2}$$

$$= \frac{1 - (\sqrt{3} - 1)}{1 + (\sqrt{3} - 1)}(2 + \sqrt{3}) = \frac{1}{\sqrt{3}}$$

$\therefore$ $\dfrac{A - B}{2} = 30^\circ$

$\Rightarrow$ $A - B = 60^\circ$

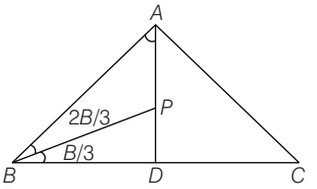
4. $\dfrac{x}{\sin 105^\circ} = \dfrac{2}{\sin 45^\circ}$



$$x = \frac{2\cos 15^\circ}{\sin 45^\circ} = \frac{2\sqrt{2}(\sqrt{3} + 1)}{2\sqrt{2}}$$

$$= \sqrt{3} + 1$$

5. $\dfrac{AP}{\sin\dfrac{2B}{3}} = \dfrac{c}{\sin\left(90^\circ + \dfrac{B}{3}\right)}$



$\Rightarrow$ $AP = 2c\sin\dfrac{B}{3}$

6. Given, $2b = a + c$

$\Rightarrow$ $2\sin B = \sin A + \sin C$

$\Rightarrow$ $2\left(2\sin\dfrac{B}{2}\cos\dfrac{B}{2}\right) = 2\sin\left(\dfrac{A + C}{2}\right)\cos\left(\dfrac{A - C}{2}\right)$

$\Rightarrow$ $2\sin\dfrac{B}{2} = \cos\left(\dfrac{90^\circ}{2}\right) = \dfrac{1}{\sqrt{2}}$

$\Rightarrow$ $\sin\dfrac{B}{2} = \dfrac{1}{2\sqrt{2}}$

$\Rightarrow$ $\cos\dfrac{B}{2} = \sqrt{1 - \dfrac{1}{8}} = \dfrac{\sqrt{7}}{2\sqrt{2}}$

Hence, $\sin B = 2\sin\dfrac{B}{2}\cos\dfrac{B}{2} = \dfrac{\sqrt{7}}{4}$

7. $\dfrac{BD}{DC} = \dfrac{AB}{AC}$

$\Rightarrow$ AD is the angle bisector

$\therefore$ $AD = \dfrac{2bc}{b + c}\cos\dfrac{A}{2}$

$$= \frac{2 \times 5 \times 3}{8}\sqrt{\frac{1 + \cos A}{2}}$$

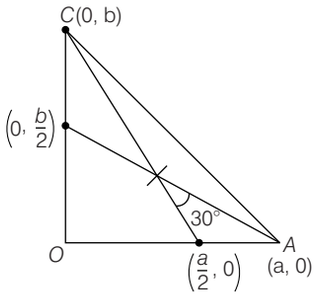
$$= \frac{15}{4}\sqrt{\frac{1 + \frac{3}{5}}{2}}$$

$$= \frac{15}{4} \times \frac{2}{\sqrt{5}} = \frac{3 \times \sqrt{5} \times \sqrt{5}}{2 \times \sqrt{5}}$$

$$= \frac{3\sqrt{5}}{2}$$

8. Slope of $GC = \dfrac{-2b}{a}$, slope of $AG = \dfrac{-b}{2a}$

$$\tan 30^\circ = \frac{\dfrac{3b}{2a}}{1 + \dfrac{b^2}{a^2}} \text{ and } a^2 + b^2 = 9$$



$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{3ab}{2(a^2 + b^2)}$$

$$\Rightarrow \quad \frac{1}{2}ab = \left(\frac{a^2 + b^2}{3\sqrt{3}}\right)$$

$$\Rightarrow \quad \sqrt{k} = \frac{9}{3\sqrt{3}} = \sqrt{3}$$

$$\Rightarrow \quad k = 3$$

9. $4\sin A \cos B = 1$, so A and B cannot be $\dfrac{\pi}{2}$

[as if $B = \dfrac{\pi}{2}$, then $\cos B = 0$ and if $A = \dfrac{\pi}{2}$, $\tan A$ is not defined)

$$\Rightarrow \quad C = \frac{\pi}{2},\ B = \frac{\pi}{2} - A$$

$$\Rightarrow \quad 4\sin A \cos\left(\frac{\pi}{2} - A\right) = 1$$

$$\Rightarrow \quad \sin^2 A = \frac{1}{4} \quad \Rightarrow \quad \sin A = \frac{1}{2}$$

$$\Rightarrow \quad A = \frac{\pi}{6} \quad \Rightarrow \quad B = \frac{\pi}{3}$$

So angles are in AP.

10. det A is twice the area of the triangle with vertices $(a, p)\ (b, q)\ (c, r)$ with sides 5, 6,7.

$$\Delta^2 = s(s-a)(s-b)(s-c)$$

$$\Rightarrow \quad 16\Delta^2 = 18 \cdot 8 \cdot 6 \cdot 4$$

$$\det B = (\det A)^2 = 4\Delta^2 = 18 \cdot 8 \cdot 6 = 864$$

11. Given that the circle passes through the circumcentre of ΔABC. Therefore, the distance between circumcenter, incentre $= \sqrt{R^2 - 2rR} = r$

$$\Rightarrow \quad r^2 + 2rR - R^2 = 0$$

$$\Rightarrow \quad \left(\frac{r}{R}\right)^2 + 2\frac{r}{R} - 1 = 0$$

$$\Rightarrow \quad \frac{r}{R} = \sqrt{2} - 1 \text{ and}$$

$$\cos A + \cos B + \cos C = 1 + \frac{r}{R} = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

12. Let a and b the roots of $x^2 - 7x + 8 = 0$. Then, $a + b = 7$, $ab = 8$

Also, $\quad C = 60^\circ$

Now $\quad \dfrac{1}{2} = \dfrac{a^2 + b^2 - c^2}{2ab}$

$$\Rightarrow \quad ab = (a+b)^2 - 2ab - c^2$$

$$\therefore \quad c^2 = (a+b)^2 - 3ab = `49 - 24 = 25$$

$$\Rightarrow \quad c = 5$$

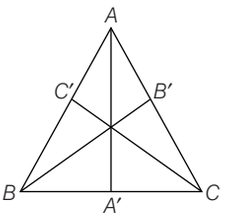
Thus, $\quad r.\ R = \dfrac{abc}{2(a+b+c)} = \dfrac{8 \times 5}{2(7+5)} = \dfrac{5}{3}$

13. Applying $m-n$ theorem

$$(BD + DC)\cot\frac{\pi}{6} = DC\cos B - BD\cot C$$

$$\Rightarrow \quad (\cot B - \cot C)^2 = 12$$

14. $A'C = b\cos C$, $B'C = a\cos C$



$$A'B' = c\cos C$$

Similarly, $A'C' = b\cos B$ and $B'C' = a\cos A$

Now, $\quad 4R = a\cos A + b\cos B + c\cos C$

$$\Rightarrow \quad \sin A \sin B \sin C = 1$$

$\Rightarrow$ This is only possible when $\angle A = \angle B = \angle C = \dfrac{\pi}{2}$, so triangle is not possible.

15. Let the sides of the triangle be $7, 7-d, 7-2d$.

Since, the given angle is the greatest (being obtuse) angle of the triangle, it is opposite to the greatest side of the triangle and we have,

$$7^2 = (7-d)^2 + (7-2d)^2 - 2(7-d)(7-2d)\cos\frac{2\pi}{3}$$

$$\Rightarrow \quad 7^2 = 2 \times 7^2 - 42d + 5d^2 - 2(7^2 - 21d + 2d^2)\left(\frac{-1}{2}\right)$$

$$\Rightarrow \quad d^2 - 9d + 14 = 0$$

$$\Rightarrow \quad (d-7)(d-2) = 0$$

$$\Rightarrow \quad d = 2 \qquad (d = 7 \text{ is not possible})$$

Therefore, the sides of the triangle are 7 cm, 5 cm, 3 cm.

Area of the triangle $\Delta = \dfrac{1}{2} \times 5 \times 3 \times \sin\dfrac{2\pi}{3}$

$= \dfrac{15\sqrt{3}}{4}\text{cm}^2$ and the radius of the circumcircle

$$R = \frac{7 \times 5 \times 3}{4\left(\dfrac{15\sqrt{3}}{4}\right)}$$

$$= \frac{7 \times 5 \times 3}{15\sqrt{3}} = \frac{7\sqrt{3}}{3}\text{ cm}$$

16. ∵ Sides are in AP and $a < \min\{b, c\}$

***Case* I** If $\min(b, c) = b$

Then, a, b, c are in AP.

i.e, $2b = a + c$

$$\therefore \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{b^2 + c^2 - (2b - c)^2}{2bc} = \frac{4bc - 3b^2}{2bc} = \frac{4c - 3b}{2c}$$

***Case* II** If $\min\{b, c\} = c$

Then a, c, b are in AP.

i.e, $2c = a + b$

$$\therefore \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{b^2 + c^2 - (2c - b)^2}{2bc} = \frac{4bc - 3c^2}{2bc} = \frac{4b - 3c}{2b}$$

17. Given, $\sin A \sin B \sin C = p$

and $\cos A \cos B \cos C = q$

$$\Rightarrow \quad \tan A \tan B \tan C = \frac{p}{q}$$

Also, $\tan A + \tan b + \tan C = \tan A \tan B \tan C$

$$\Rightarrow \quad \tan A + \tan B + \tan C = \frac{p}{q}$$

Now, $\tan A \tan B + \tan B \tan C + \tan C \tan A$

$$= \frac{\sin A \sin B \cos C + \sin B \sin C \cos A + \sin C \sin A \cos B}{\cos A \cos B \cos C}$$

$$= \frac{1}{q}[\sin A \sin B \cos C + \sin C(\sin B \cos A + \cos B \sin A)]$$

$$= \frac{1}{q}[\sin A \sin B \cos C + \sin C \sin(A + B)]$$

$$= \frac{1}{q}[\sin A \sin B \cos C + \sin^2 C]$$

$$= \frac{1}{q}[1 - \cos^2 C + \sin A \sin B \cos C]$$

$$= \frac{1}{q}[1 + \cos C(-\cos C + \sin A \sin B)]$$

$$= \frac{1}{q}[1 + \cos C(\cos(A + B) + \sin A \sin B)]$$

$$= \frac{1}{q}[1 + \cos A \cos B \cos C] = \frac{1}{q}[1 + q]$$

The equation whose roots are $\tan A, \tan B, \tan C$ will be give by

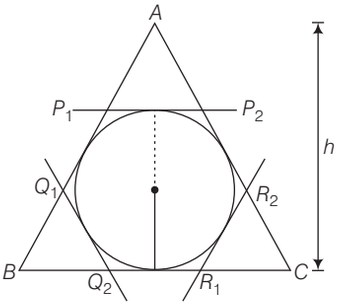
$x^3 - (\tan A + \tan B + \tan C)x^2 + (\tan A \tan B + \tan B \tan C + \tan C \tan A)x - \tan A \tan B \tan C = 0$

or $\quad x^3 - \frac{p}{q}x^2 + \frac{1 + q}{q}x - \frac{p}{q} = 0$

or $\quad qx^3 + px^2 + (1 + q)x - p = 0$

Hence, (a) is the correct answer.

18.



$\Delta AP_1P_2 \sim \Delta ABC$

$$\Rightarrow \quad \frac{t_1}{a} = \frac{h - 2r}{h} = 1 - \frac{2r}{h}$$

$$\Rightarrow \quad \frac{t_1}{a} = 1 - \frac{2\Delta}{sh} \qquad (\text{where } \Delta \equiv \text{ar}(\Delta ABC))$$

$$\Rightarrow \quad \frac{t_1}{a} = 1 - \frac{2\frac{1}{2}ah}{sh} \Rightarrow \frac{t_1}{a} = 1 - \frac{a}{s}$$

Similarly, $\quad \frac{t_2}{b} = 1 - \frac{b}{s}$ and $\frac{t_3}{c} = 1 - \frac{c}{s}$

$$\therefore \quad \frac{t_1}{a} + \frac{t_2}{b} + \frac{t_3}{c} = 3 - \frac{(a + b + c)}{s}$$

$= 3 - 2 = 1$

19. $\sin A, \sin B, \sin C$ are the roots of the equation $c^3x^3 - c^2(a + b + c)x^2 + lx + m = 0$

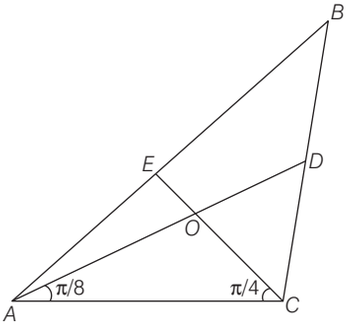
$$\therefore \sin A + \sin B + \sin C = \frac{c^2(a + b + c)}{c^3} = \frac{a + b + c}{c}$$

or $\quad \frac{a}{2R} + \frac{b}{2R} + \frac{c}{2R} = \frac{a + b + c}{c}$

Then, $\quad c = 2R, a + b + c \neq 0$

ie, $2R \sin C = 2R$.

20. Let O be the point of intersection of the medians of triangle ABC (Fig. 12.10). Then, the area of ΔABC is three times that of ΔAOC, O being the centroid of ΔABC, divides the median through B in the ratio 2 : 1.



And the height of ΔAOC is one-third that of ΔABC. Now, in ΔAOC, $AO = \left(\frac{2}{3}\right) AD = \frac{10}{3}$. Therefore, applying the sine rule to ΔAOC, we get

$$\frac{OC}{\sin\left(\frac{\pi}{8}\right)} = \frac{AO}{\sin\left(\frac{\pi}{4}\right)} \Rightarrow OC = \frac{10}{3} \cdot \frac{\sin\left(\frac{\pi}{8}\right)}{\sin\left(\frac{\pi}{4}\right)}$$

$$\text{Area of } \Delta AOC = \frac{1}{2} \cdot AO \cdot OC \cdot \sin \angle AOC$$

$$= \frac{1}{2} \cdot \frac{10}{3} \cdot \frac{10}{3} \cdot \frac{\sin\left(\frac{\pi}{8}\right)}{\sin\left(\frac{\pi}{4}\right)} \cdot \sin\left(\frac{\pi}{2} + \frac{\pi}{8}\right)$$

$$= \frac{50}{9} \cdot \frac{\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)}{\sin\left(\frac{\pi}{4}\right)} + \frac{50}{18} = \frac{25}{9}$$

$$\Rightarrow \text{Area of } \Delta ABC = 3 \cdot \frac{25}{9} = \frac{25}{3}$$

21. We have, $\frac{(2R)^2(\sin^3 A + \sin^3 B + \sin^3 C)}{\sin^3 A + \sin^3 + \sin^3 C} = 8$

$\Rightarrow R = 1$, the radius of the circumcircle is 1.

Greatest length of a side of a triangle inscribed in a circle can be equal to the diameter of the circle and hence the maximum value of the greatest side a is equal to 2.

22. (a) : $(b + c + a)(b + c - a) = 5bc$

$$\Rightarrow (b + c)^2 - a^2 = 5bc$$

$$\Rightarrow b^2 + c^2 + 2bc - a^2 = 5bc$$

$$\Rightarrow 2bc\cos A = 3bc$$

$$\cos A = \frac{3}{2} \text{ impossible.}$$

(b) : Let $a = \sqrt{19}, b = \sqrt{38}, c = \sqrt{116}$

$$\cos A = \frac{19 + 38 - 116}{2\sqrt{19}\sqrt{38}}$$

$$= -\frac{59}{2\sqrt{722}} = -\frac{29.5}{\sqrt{722}} < -1 \text{ impossible.}$$

(c) : $\because \frac{b^2 - c^2}{a^2} = \frac{\sin^2 B - \sin^2 C}{\sin^2 A}$

$$= \frac{\sin(B + C)\sin(B - C)}{\sin^2 A}$$

$$= \frac{\sin(B - C)}{\sin A}$$

$$= \frac{\sin(B - C)}{\sin(B + C)} \neq 0 \qquad (\because B \neq C)$$

$\because$ If $B > C$

Then, $B + C > B - C$

$$\frac{\sin(B + C)}{\sin(B - C)} > 1$$

and if $B < C$

$$C - B < C + B$$

or $$\frac{\sin(C - B)}{\sin(C + B)} < 1$$

or $$\frac{\sin(B - C)}{\sin(B + C)} > -1$$

Hence, $\frac{b^2 - c^2}{a^2} + \frac{c^2 - a^2}{b^2} + \frac{a^2 - b^2}{c^2} \neq 0$

(d) :

$$\cos\left(\frac{B - C}{2}\right) = (\sin B + \sin C)\cos\left(\frac{B + C}{2}\right)$$

$$\cos\left(\frac{B - C}{2}\right) = 2\sin\left(\frac{B + C}{2}\right)\cos\left(\frac{B - c}{2}\right)\cos\left(\frac{B + C}{2}\right)$$

$$\cos\left(\frac{B - C}{2}\right) \neq 0$$

$\therefore \sin(B + C) = 1$

$\therefore B + C = \frac{\pi}{2}$

Then, $\angle A = \frac{\pi}{2}$

23. We have, $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

$\Rightarrow b^2 - 2bc\cos A + (c^2 - a^2) = 0$

It is given that b_1 and b_2 are roots of this equation.

Therefore, $b_1 + b_2 = 2c\cos A$ and $b_1 b_2 = c^2 - a^2$

$\Rightarrow 3b_1 = 2c\cos A$ and $2b_1^2 = c^2 - a^2$

(since $b_2 = 2b_1$ given)

$$\Rightarrow 2\left(\frac{2c}{3}\cos A\right)^2 = c^2 - a^2$$

$$\Rightarrow 8c^2(1 - \sin^2 A) = 9c^2 - 9a^2$$

$$\Rightarrow \sin A = \sqrt{\frac{9a^2 - c^2}{8c^2}}$$

Hence, (b) is the correct answer.

24. We have, $\cot(A + B) = \frac{\cot A \cot B - 1}{\cot B + \cot A}$

$$= \frac{x^{\frac{1}{2}}(x^2 + x + 1)^{\frac{1}{2}} \cdot x^{\frac{-1}{2}}(x^2 + x + 1)^{\frac{1}{2}} - 1}{x^{\frac{-1}{2}}(x^2 + x + 1)^{\frac{1}{2}} + x^{\frac{1}{2}}(x^2 + x + 1)^{\frac{1}{2}}}$$

$$= \frac{x^2 + x + 1 - 1}{(x^2 + x + 1)^{\frac{1}{2}} + (x^{\frac{1}{2}} + x^{\frac{-1}{2}})}$$

$$= \frac{x(x + 1)x^{\frac{1}{2}}}{(x^2 + x + 1)^{\frac{1}{2}}(x + 1)} = x^{\frac{3}{2}}(x^2 + x + 1)^{\frac{-1}{2}}$$

$$= x^{-1} + x^{-2} + x^{-3})^{\frac{1}{2}} = \cot C$$

$\Rightarrow A + B = C$ and $A + B + C = \pi$

$\therefore C = \frac{\pi}{2}$

25. We have, $\cos A = \dfrac{c^2 + b^2 - a^2}{2bc}$

$\Rightarrow c^2 - 2bc\cos A + b^2 - a^2 = 0$

It is given that c_1 and c_2 are roots of this equation.

Therefore, $c_1 + c_2 = 2b\cos A$ and $c_1c_2 = b^2 - a^2$

$\Rightarrow \quad k(\sin C_1 + \sin C_2) = 2k\sin B\cos A$

$\Rightarrow \quad \sin C_1 + \sin C_2 = 2\sin B\cos A$

Now, sum of the area of two triangles

$$= \frac{1}{2}ab\sin C_1 + \frac{1}{2}ab\sin C_2$$

$$= \frac{1}{2}ab(\sin C_1 + \sin C_2)$$

$$= \frac{1}{2}ab(2\sin B\cos A)$$

$$= b\cdot b\sin A\cos A = \frac{1}{2}b^2\sin 2A$$

Hence, (a) is correct answer

26. $b\cot B + c\cot C = 2(r + R)$

$\Rightarrow \quad 2R\sin B\cdot\dfrac{\cos B}{\sin B} + 2R\sin C\cdot\dfrac{\cos C}{\sin C} = 2(r + R)$

$\Rightarrow \quad \cos B + \cos C = 1 + \dfrac{r}{R}$

$\Rightarrow \quad \cos B + \cos C = 1 + 4\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$

$\Rightarrow \quad \cos B + \cos C = \cos A + \cos B + \cos C$

$\therefore \quad \cos A = 0$

$\Rightarrow \quad A = \dfrac{\pi}{2}$

$\Rightarrow \quad a^2 = b^2 + c^2$

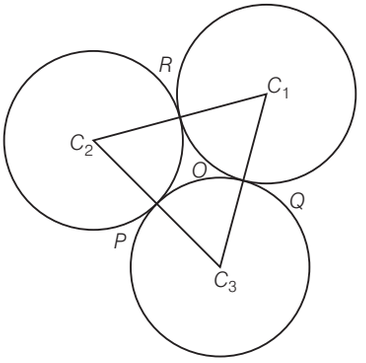
$\Rightarrow \quad b^2 + c^2 = 100$

Using A.M $\geq$ G.M., we get

$$\frac{b^2 + c^2}{2} \geq \sqrt{b^2c^2} \Rightarrow bc \leq 50$$

Hence, area of $\Delta ABC = \dfrac{1}{2}bc \leq 25$

27. Let r_1, r_2 and r_3 be the radii of the three circles with centres at C_1, C_2 and C_3. Let the circles touch at P, Q and R.



Also, $C_1C_2 = r_1 + r_2$, $C_2C_3 = r_2 + r_3$, $C_3C_1 = r_3 + r_1$

Let O be the point whose distance from the points of contact is 4.

Then, O is the incentre of the $\Delta C_1C_2C_3$ with $OP = OQ = OR = 4$, being the radius of the incircle

Hence, $4 = \dfrac{\Delta C_1C_2C_3}{\frac{1}{2}[C_1C_2 + C_2C_3 + C_3C_1]} = \dfrac{\Delta}{s}$ ...(i)

Where $s = r_1 + r_2 + r_3$,

$$\Delta^2 = s(s - C_1C_2)(s - C_2C_3)(s - C_3C_1) = s(r_1)(r_2)(r_3)$$

(i) gives $16 = \dfrac{\Delta^2}{s^2} = \dfrac{s(r_3)(r_1)(r_2)}{s^2} = \dfrac{r_1r_2r_3}{r_1 + r_2 + r_3}$

Hence, the ratio of the product of the radii to the sum of the radii = 16 : 1.

Hence, (a) is the correct answer.

28. $\because$ Roots are real and distinct.

$\therefore \quad \Delta > 0$

$\Rightarrow \quad 4(a + b + c)^2 - 12\lambda(ab + bc + ca) > 0$

$\Rightarrow \quad (a^2 + b^2 + c^2 + 2(ab + bc + ca)) - 3\lambda(ab + bc + ca) > 0$

$\Rightarrow \quad \dfrac{\Sigma a^2}{\Sigma ab} > (3\lambda - 2)$ ...(i)

Now, in a triangle

Difference of two sides < third side

ie, $\quad |a - b| < c, |b - c| < a$ and $|c - a| < b$

$\Rightarrow \quad (a - b)^2 + (b - c)^2 + (c - a)^2 < a^2 + b^2 + c^2$

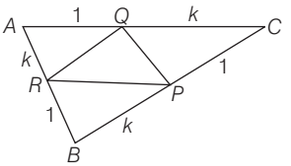
$\Rightarrow \quad a^2 + b^2 + c^2 < 2(ab + bc + ca)$

or $\quad \dfrac{\Sigma a^2}{\Sigma ab} < 2$ ...(ii)

From Eqs. (i) and (ii), we get

$$3\lambda - 2 < \frac{\Sigma a^2}{\Sigma ab} < 2 \Rightarrow 3\lambda - 2 < 2 \Rightarrow \lambda < \frac{4}{3}$$

29. Area of triangle



Area of ΔARQ,

$$= \frac{b}{(k+1)}\times\frac{kc}{k+1}\times\frac{1}{2}\sin A = \frac{k\Delta}{(k+1)^2}$$

Similarly, area of ΔBPR = Area of $\Delta PCQ = \dfrac{k\Delta}{(k+1)^2}$

Now, $\quad \dfrac{\Delta PQR}{\Delta ABC} = \dfrac{1}{3}$

or $\quad \dfrac{\Delta - \dfrac{3k}{(k+1)^2}\Delta}{\Delta} = \dfrac{1}{3}$

or $\quad 2k^2 - 5k + 2 = 0$

or $\quad k = \dfrac{1}{2}, 2$

30. $\therefore f(x+y) = f(x)\cdot f(y)$

for $x = y = 1$

$f(2) = 2^2$

for $x = 1, y = 2$

$f(3) = f(1)f(2) = 2^3$

$\therefore \quad f(n) = 2^n$

$\therefore \quad a = f(3) = 2^3 = 8$,

$b = f(1) + f(3) = 2 + 8 = 10$

and $c = f(2) + f(3) = 2^2 + 2^3 = 12$

$$\cos A = \frac{3}{4} \text{ and } \cos C = \frac{1}{8}$$

$$\cos 2A = 2\cos^2 A - 1$$

$$= 2 \times \frac{9}{16} - 1$$

$$= \frac{1}{8} = \cos C$$

$\therefore \quad 2A = C$

31. Given, $x + y + z = a + b + c$ ...(i)

and $a^2x + b^2y + c^2z + abc = 4xyz$ divide by $4xyz$, we get

$$\frac{a^2}{4yz} + \frac{b^2}{4xz} + \frac{c^2}{4xyz} = 1$$

$$\left(\frac{a}{2\sqrt{yz}}\right)^2 + \left(\frac{b}{2\sqrt{xz}}\right)^2 + \left(\frac{c}{2\sqrt{xy}}\right)^2$$

$$+ 2\left(\frac{a}{2\sqrt{yz}}\right)\left(\frac{b}{2\sqrt{xz}}\right)\left(\frac{c}{2\sqrt{xy}}\right) = 1$$

For using the trigonometric identify

$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos b \cos C$

$= 1,\ A + B + C = \pi$

Let $\frac{a}{2\sqrt{yz}} = \cos A, \frac{b}{2\sqrt{xz}} = \cos B, \frac{c}{2\sqrt{xy}} = \cos C$

Where A, B, C are acute angle's

From Eq. (i)

$\because \quad x + y + z = a + b + c$

$\Rightarrow \quad x + y + z = 2\sqrt{yz}\cos A + 2\sqrt{xz}\cos B + 2\sqrt{xy}\cos C$

$\Rightarrow \quad x + y + z - 2\sqrt{yz}\cos A - 2\sqrt{xz}$

$\cos B - 2\sqrt{xy}\cos(\pi - (A + B)) = 0$

$\Rightarrow \quad x + y + z - 2\sqrt{yz}\cos A - 2\sqrt{xz}$

$\cos B + 2\sqrt{xy}\cos(A + B) = 0$

$\Rightarrow \quad x + y + z - 2\sqrt{yz}\cos A - 2\sqrt{xz}$

$\cos B + 2\sqrt{xy}(\cos A \cos B - \sin A \sin B) = 0$

$\Rightarrow \quad x(\sin^2 B + \cos^2 B) + y(\sin^2 A + \cos^2 A)$

$+ z - 2\sqrt{yz}\cos A - 2\sqrt{xz}\cos B$

$\Rightarrow \quad 2\sqrt{xy}(\cos A \cos B - \sin A \sin B) = 0$

$\Rightarrow \quad (\sqrt{x}\sin B - \sqrt{y}\sin A)^2 + (\sqrt{x}\cos B + \sqrt{y}\cos A - \sqrt{z})^2 = 0$

$\Rightarrow \quad \sqrt{x}\sin B = \sqrt{y}\sin A$ and $\sqrt{x}\cos B + \sqrt{y}\cos A = \sqrt{z}$

$\Rightarrow \quad \sqrt{x}\frac{b}{2\sqrt{xz}} + \sqrt{y}\frac{a}{2\sqrt{yz}} = \sqrt{z}$

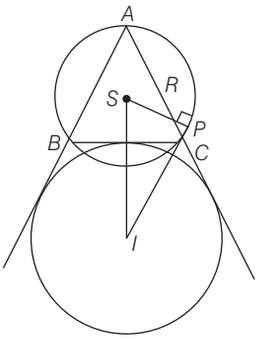
$\Rightarrow \quad \frac{b}{2\sqrt{z}} + \frac{a}{2\sqrt{z}} + \sqrt{z} \Rightarrow \frac{a+b}{2} = z$

$\therefore$ By symmetricity $x = \frac{b+c}{2}, y = \frac{a+c}{2}, z = \frac{a+b}{2}$

32. Let S and I_1 be respectively the centres of the circumcircle and the excircle touching BC.

It can be shown that

$$SI_1 = \sqrt{R^2 + 2Rr_1}$$



In $\Delta SI_1P, SI_1^2 = R^2 + t_1^2$

$R^2 = 2Rr_1 = R^2 + t_1^2, \frac{1}{t_1^2} = \frac{1}{2Rr_1}$

Similarly, $\frac{1}{t_2^2} = \frac{1}{2Rr_2}, \frac{1}{t_3^2} = \frac{1}{2Rr_3}$

$$\frac{1}{t_1^2} + \frac{1}{t_2^2} + \frac{1}{t_3^2} = \frac{1}{2R}\left[\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right]$$

$$= \frac{1}{2R}\left[\frac{s-a}{\Delta} + \frac{s-b}{\Delta} + \frac{s-c}{\Delta}\right]$$

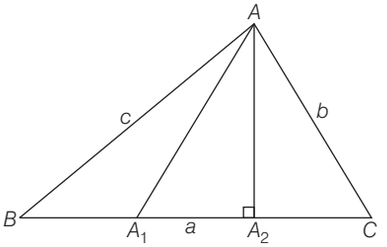
$$= \frac{1}{2R}\frac{s}{\Delta} = \frac{s}{2R\Delta} = \frac{a+b+c}{abc}$$

33. $\angle BAA_1 = \angle A_1AA_2 = \angle A_2AC$

Clearly, triangle AA_1C is isosceles

$\Rightarrow \quad AA_1 = AC = b$

$A_1A_2 = A_2C = \frac{a}{4}$



Now, $\sin^3\frac{A}{3}\cdot\cos\frac{A}{3} = \frac{1}{2}\sin^2\frac{A}{3}\cdot 2\sin\frac{A}{3}\cdot\frac{A}{3}$

$$= \frac{1}{2}\sin^2\frac{A}{3}\cdot\sin\frac{2A}{3} = \frac{1}{2}\left(\frac{A_2C}{b}\right)^2\cdot\frac{BA_2}{AB}$$

$$= \frac{1}{2}\cdot\frac{a^2}{16b^2}\cdot\frac{3a}{4}\cdot\frac{1}{c} = \frac{3a^3}{128b^2c}$$

34. $\because \cos A = \frac{b^2 + c^2 - a^2}{2bc}$

or $\quad c^2 - 2bc\cos A + b^2 - a^2 = 0$

$\therefore \quad c_1 + c_2 = 2b\cos A = 2 \times 2 \times \frac{\sqrt{3}}{2} = 2\sqrt{3}$

and $\quad c_1c_2 = b^2 - a^2 = 4 - 5 = -1$

$\therefore \quad |c_1 - c_2| = \sqrt{(c_1 + c_2)^2 - 4c_1c_2}$

$= \sqrt{12 + 4} = \sqrt{14} = 4$

35. Circumradius of triangle ABC, $R = 5$

$\therefore$ Circumradius of pedal triangle, $R_1 = \frac{5}{2}$, and so on.

Now, $\sum_{i=1}^{\infty} R_i = R_1 + R_2 + R_3 + \ldots\infty$

$$= 5 + \frac{5}{2} + \frac{5}{2^2} + \ldots\infty$$

$$= 5 \cdot \frac{5}{1 - \frac{1}{2}} = 10$$

36. $\left(1 - \frac{r_1}{r_2}\right)\left(1 - \frac{r_1}{r_3}\right) = 2$

$$\Rightarrow \left(1 - \frac{\frac{\Delta}{s-a}}{\frac{\Delta}{s-b}}\right)\left(1 - \frac{\frac{\Delta}{s-a}}{\frac{\Delta}{s-b}}\right) = 2$$

$$\Rightarrow \left(1 - \frac{s-b}{s-a}\right)\left(1 - \frac{s-c}{s-a}\right) = 2$$

$$\Rightarrow (b-a)(c-a) = 2(s-a)^2$$

$$\Rightarrow 2(bc - ab - ac + a^2) = (2s - 2a)^2$$

$$\Rightarrow 2bc - 2ab - 2ac + 2a^2 = (b + c - a)^2$$

$$\Rightarrow 2bc - 2ab - 2ac + 2a^2 = b^2 + c^2 + a^2 + 2bc - 2ac - 2ab$$

$$\Rightarrow b^2 + c^2 = a^2$$

$$\Rightarrow 2bc\cos A = 0$$

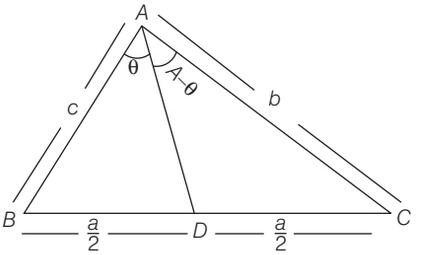
$$\therefore \angle A = 90^\circ$$

37. Prove that $\sin(A - \theta) = \frac{c}{b}\sin\theta$

In ΔABD,

$$\frac{BD}{\sin\theta} = \frac{AD}{\sin B}$$

$$\Rightarrow AD = BD\frac{\sin(B)}{\sin\theta} \quad \ldots(i)$$



In ΔACD,

$$\frac{CD}{\sin(A - \theta)} = \frac{AD}{\sin C}$$

$$\Rightarrow AD + CD \cdot \frac{\sin C}{\sin(A - \theta)} \quad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$(BD)\frac{\sin B}{\sin\theta} = \frac{\sin C}{\sin(A - \theta)}(CD)$$

$$\Rightarrow \frac{\sin B}{\sin C} = \frac{\sin\theta}{\sin(A - \theta)}$$

$$\frac{b}{c} = \frac{\sin(\theta)}{\sin(A - \theta)}$$

$$\Rightarrow \sin(A - \theta) = \frac{c}{b}\sin\theta$$

38. A, B, C are in A.P. $\Rightarrow B = 60^\circ$

$$\Rightarrow \cos B = \cos 60^\circ = \frac{1}{2} = \frac{c^2 + a^2 - b^2}{2ca}$$

$$\Rightarrow a^2 + c^2 = b^2 + ac$$

$$\Rightarrow (a - c)^2 = b^2 - ac$$

$$\Rightarrow |a - c| = \sqrt{b^2 - ac}$$

$$\Rightarrow |\sin A - \sin C| = \sqrt{\sin^2 B - \sin A \sin C}$$

$$\Rightarrow 2\cos\frac{A + C}{2}\left|\sin\frac{A - C}{2}\right| = \sqrt{\frac{3}{4} - \sin A \sin C}$$

$$\Rightarrow 2\left|\sin\frac{A - C}{2}\right| = \sqrt{3 - 4\sin A \sin C}$$

So, that $\lim_{A \to C} \frac{\sqrt{3 - 4\sin A \sin C}}{|A - C|} = \lim_{A \to C} \frac{2\sin\left|\frac{A - C}{2}\right|}{|A - C|} = 1$

39. We are given that $(a + b + c)(b + c - a) = \lambda bc$, or $(b + c)^2 - a^2 = \lambda bc$. That is, $b^2 + c^2 + 2bc - a^2 = \lambda bc$, or $b^2 + c^2 - a^2 = (\lambda - 2)bc$.

Therefore, $\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{\lambda - 2}{2}$

As A is the angle of a triangle, $-1 < \cos A < 1$.

Therefore,

$$-1 < \frac{(\lambda - 2)}{2} < 1$$

$$\Rightarrow 0 < \lambda < 4$$

40. Let $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$ (say)

$\Rightarrow a = k\sin A, b = k\sin B, c = k\sin C$

Now, the given relation is

$(a^2 + b^2)\sin(A - B) = (a^2 - b^2)\sin(A + B)$

or $k^2[\sin^2 A + \sin^2 B]\sin(A - B)$

$$= k^2[\sin^2 A - \sin^2 B]\sin(A + B)$$

$$\Rightarrow [\sin^2 A + \sin^2 B]\sin(A - B)$$

$$= \sin^2(A + B)\sin(A - B)$$

or $\sin(A - B)[\sin^2 A + \sin^2 B - \sin^2 C] = 0$

Hence, either the first factor = 0, or the second factor = 0

If $\sin(A - B) = 0$

$$\Rightarrow A - B = 0$$

$$\Rightarrow A = B$$

$\Rightarrow$ Triangle is isosceles.

If $\sin^2 A + \sin^2 B - \sin^2 C = 0$

$$\Rightarrow \quad \frac{a^2}{k^2}+\frac{b^2}{k^2}-\frac{c^2}{k^2}=0$$

or $$a^2+b^2-c^2=0$$

or $$a^2+b^2=c^2$$

$\Rightarrow$ The triangle is right angled.

41. $\because \frac{2}{1!9!}+\frac{2}{3!7!}+\frac{1}{5!5!}=\frac{8^a}{(2b)!}$

$$\Rightarrow \quad \frac{1}{1!9!}+\frac{1}{3!7!}+\frac{1}{5!5!}+\frac{1}{3!7!}+\frac{1}{9!1!}=\frac{8^a}{(2b)!}$$

$$\Rightarrow \quad \frac{1}{10!}\left(\frac{10!}{1!9!}+\frac{10!}{3!7!}+\frac{10!}{5!5!}+\frac{10!}{7!3!}+\frac{10!}{9!1!}\right)$$

$$=\frac{8^a}{(2b)!}$$

$$\frac{1}{10!}\left({}^{10}C_1+{}^{10}C_3+{}^{10}C_5+{}^{10}C_7+{}^{10}C_9\right)=\frac{8^a}{(2b)!}$$

$$\Rightarrow \quad \frac{2^9}{10!}=\frac{8^a}{(2b)!}=\frac{2^{3a}}{(2b)!}$$

$$\Rightarrow \quad a=3,\ b=5$$

Also, $2b+a+c \Rightarrow 10=3+c$

$$\Rightarrow \quad c=7$$

$$\therefore \quad a=3,\ b=5,\ c=7$$

$$\because \quad \frac{\tan A+\tan B}{2}\geq\sqrt{\tan A\tan B} \qquad \ldots(i)$$

Also, $$\cos C=\frac{a^2+b^2-c^2}{2ab}$$

$$=\frac{9+25-49}{30}=-\frac{1}{2}$$

$$\therefore \quad C=120^\circ \text{ and } A,\ B<60^\circ$$

$\tan A+\tan B+\tan C=\tan A\tan B\tan C$

$\Rightarrow \tan A+\tan B-\sqrt{3}=-\sqrt{3}\tan A\tan B$

$\Rightarrow \tan A+\tan B-\sqrt{3}=-\sqrt{3}\tan A\tan B$

$\therefore \tan A+\tan B=\sqrt{3}(1-\tan A\tan B)$ ...(ii)

Also, $\tan A+\tan B>0$

$\Rightarrow \sqrt{3}(1-\tan A\tan B)>0$

$\Rightarrow \tan A\tan B<1$ ...(iii)

From Eqs. (i) and (ii), we get

$$\frac{\sqrt{3}(1-\tan A\tan B)}{2}\geq\sqrt{(\tan A\tan B)}$$

Let $\tan A\tan b=\lambda$

$$\therefore \quad \sqrt{3}(1-\lambda)\geq 2\sqrt{\lambda}$$

$$\Rightarrow \quad 3\lambda^2-10\lambda+3\geq 0$$

$$\Rightarrow \quad (3\lambda-1)(\lambda-3)\geq 0$$

$$\because \quad \lambda-3<0 \qquad \text{[from Eq. (iii)]}$$

$$\therefore \quad 3\lambda-1\leq 0$$

$$\Rightarrow \quad \lambda\leq\frac{1}{3}$$

$$\tan A\tan b\leq\frac{1}{3}$$

42. $\because a(b-c)+b(c-a)+c(a-b)=0$

$\therefore \quad x=1$ is a root of the equation

$$a(b-c)x^2+b(c-a)x+c(a-b)=0$$

Then, other root = 1 ($\because$ roots are equal)

$$\therefore \quad |X|=\frac{c(a-b)}{a(b-c)}$$

$$\Rightarrow \quad ab-ac=ca-bc$$

$$\therefore \quad b+\frac{2ac}{a+c}$$

$\therefore a, b, c$ are in HP

Then, $\frac{1}{a},\frac{1}{b},\frac{1}{c}$ are in AP.

$$\Rightarrow \quad \frac{s}{a},\frac{s}{b},\frac{s}{c} \text{ are in AP.}$$

$$\Rightarrow \quad \frac{s}{a}-1,\frac{s}{b}-1,\frac{s}{c}-1 \text{ are in AP}$$

$$\Rightarrow \quad \frac{(s-a)}{a},\frac{(s-b)}{b},\frac{(s-c)}{c} \text{ in AP}$$

Multiplying in each by $\frac{abc}{(s-a)(s-b)(s-c)}$, then

$\frac{bc}{(s-b)(s-c)},\frac{ca}{(s-c)(s-a)},\frac{ab}{(s-a)(s-b)}$ are in AP.

$\Rightarrow \frac{(s-b)(s-c)}{bc},\frac{(s-c)(s-a)}{ca},\frac{(s-a)(s-b)}{ab}$ are in HP.

or $\sin^2\left(\frac{A}{2}\right),\sin^2\left(\frac{B}{2}\right),\sin^2\left(\frac{C}{2}\right)$ are in HP.

43. Let a be the radius of the circle, then the ratio of the area of regular polygons on n sides inscribed to circumscribing the same circle is given by

$$\frac{S_1}{S_2}=\frac{\frac{1}{2}na^2\sin\left(\frac{2\pi}{n}\right)}{na^2\tan\left(\frac{\pi}{n}\right)}=\frac{3}{4} \quad\Rightarrow\quad \cos^2\left(\frac{\pi}{n}\right)=\frac{3}{4}$$

or $$\cos\left(\frac{\pi}{n}\right)=\frac{\sqrt{3}}{2} \text{ or } \frac{\pi}{n}=\frac{\pi}{6}$$

$$\Rightarrow \quad n=6$$

44. $\Pi\left(\sin A+1+\frac{1}{\sin A}\right)$

$\because \quad \sin A+\frac{1}{\sin A}\geq 2$ (if $A=90^\circ$)

Then, $\sin B+\frac{1}{\sin B}>2$

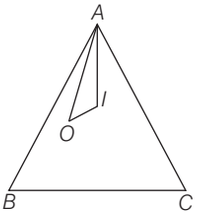
and $\sin C+\frac{1}{\sin C}>2$

$$\therefore\left(\sin A+\frac{1}{\sin A}+1\right)\left(\sin B+\frac{1}{\sin B}+1\right)\left(\sin C+\frac{1}{\sin C}+1\right)>27$$

$$\therefore \quad \Pi\left(\sin A+\frac{1}{\sin A}+1\right)>27$$

or $$\Pi\left(\frac{\sin A+\sin A+1}{\sin A}\right)>27$$

45. From the given information it is evident that distance between I (incentre) and O (circumcentre) should be equal to inradius of triangle.



If 'r' and 'R' be the inradius and circumradius respectively, then

$$AI = r\,\text{cosec}\,\frac{A}{2} = 4R\sin\frac{B}{2}\sin\frac{C}{2}$$

and $$AO = R,\ \angle OAI = \frac{A}{2} - \left(\frac{\pi}{2} - C\right)$$

$$= \frac{C-B}{2}$$

Now, $$IO^2 = OA^2 + AI^2 - 2(OA)(AI)\cos\frac{C-B}{2}$$

$$\Rightarrow \quad r^2 = R^2 + 16R^2\sin^2\frac{B}{2}\sin^2\frac{C}{2} - 8R^2\sin\frac{B}{2}\sin\frac{C}{2}\cos\left(\frac{C-B}{2}\right)$$

$$\Rightarrow r^2 = R^2\left(1 + 8\sin\frac{B}{2}\sin\frac{C}{2}\left(2\sin\frac{B}{2}\sin\frac{C}{2} - \cos\left(\frac{C-B}{2}\right)\right)\right)$$

$$= R^2\left(1 + 8\sin\frac{B}{2}\sin\frac{C}{2}\left(\sin\frac{B}{2}\sin\frac{C}{2} - \cos\frac{B}{2}\cos\frac{C}{2}\right)\right)$$

$$= R^2\left(1 + 8\sin\frac{B}{2}\sin\frac{C}{2}\left(\sin\frac{B}{2}\sin\frac{C}{2} - \cos\frac{B}{2}\cos\frac{C}{2}\right)\right)$$

$$= R^2\left(1 - 8\sin\frac{B}{2}\sin\frac{C}{2}\cos\left(\frac{B+C}{2}\right)\right)$$

$$= R^2\left(1 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$$

$$= R^2\left(1 - 8\cdot\frac{r}{4R}\right) = R^2 - 2rR$$

$$\Rightarrow \quad \left(\frac{r}{R}\right)^2 + 2\left(\frac{r}{R}\right) - 1 = 0$$

$$\Rightarrow \quad \frac{r}{R} = \frac{-2 \pm \sqrt{4+4}}{2} = \sqrt{2} - 1, \text{ as } \frac{r}{R} > 0$$

$$\Rightarrow \quad 1 + \frac{r}{R} = \sqrt{2}$$

$$\Rightarrow \quad \cos A + \cos B + \cos C = \sqrt{2}$$

Hence, (b) is the correct answer

46. $$a + b + c = 6\frac{(\sin A + \sin B + \sin C)}{3}$$

$$\Rightarrow \quad a + b + c = 2(\sin A + \sin B + \sin C)$$

$$\Rightarrow \quad 1 + \frac{\sin B}{\sin A} + \frac{\sin C}{\sin A} = 2(\sin A + \sin B + \sin C)$$

(using $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$)

$$\Rightarrow \quad \sin A + \sin B + \sin C = 2\sin A(\sin A + \sin B + \sin C)$$

$$\Rightarrow \quad \sin A = \frac{1}{2}$$

$$\Rightarrow \quad A = 30^\circ$$

47. Let linear function is $F(x) = Ax + B$

$\because \quad [1, 2] \to [4, 6]$

$\Rightarrow \quad F(1) = 4$

$\Rightarrow \quad A + B = 4$

and $\quad F(2) = 6$

$\therefore \quad A = 2, B = 2$

Then, one function is $F(x) = 2x + 2 = f(x)$ (say)

$$\left.\begin{aligned} F(1) = 6 \Rightarrow A + B = 6 \\ F(2) = 4 \Rightarrow 2A + B = 4 \end{aligned}\right\} A = -2,\ B = 8$$

Then, other function is $F(x) = -2x + 8 = g(x)$ (say)

$\therefore \quad c = f(1) + g(1) = 4 + 6 = 10$

Now, $x^2 + y^2 - xy = 10$

$$\Rightarrow \quad \frac{\left(x - \frac{y}{2}\right)^2}{\sqrt{(10)^2}} + \frac{y^2}{\left(\frac{\sqrt{10}}{\sqrt{3}}\right)^2} = 1$$

is an ellipse whose cente (0, 0).

Maximum distance from origin on any point on ellipse = Semi major axis = $\sqrt{10}$

$\therefore \quad r = \sqrt{10}$

Then, $\quad a = r^2 = 10$

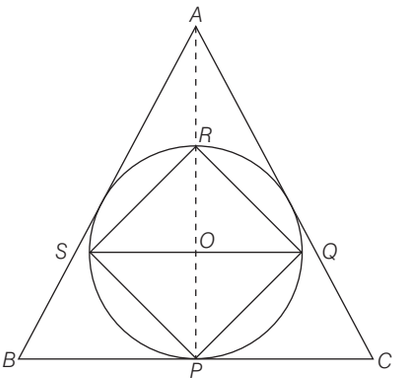
$\because \quad a = c = 10$

$\therefore \quad \sin A : \sin C = 1 : 1$

48. Let a be the length of each side of the equilateral triangle ABC.

Then r, the radius of the in-circle $= \left(\frac{1}{3}\right)AP$ (the altitude, median and the angle bisector of angle A)

$$\Rightarrow \quad r = \frac{1}{3}\sqrt{a^2 - \frac{a^2}{4}} = \frac{a}{2\sqrt{3}}$$



Area of the square $PQRS$ inscribed in this circle

$$= PQ^2 = OP^2 + OQ^2$$

$$= 2r^2 = 2 \times \frac{a^2}{4 \times 3} = \frac{a^2}{6}$$

49. $2\sin B = \sin A + \sin C$

$= \sin A + \sin(A + B)$

$= \sin A + \sin A\cos B + \cos A\sin B$

$\Rightarrow \quad \sin B(2 - \cos A) = \sin A(1 + \cos B)$

$\Rightarrow \quad \dfrac{\sin B}{1 + \cos B} = \dfrac{\sin A}{2 - \cos A}$

$\Rightarrow \quad \tan\dfrac{B}{2} = \dfrac{\sin A}{2 - \cos A} = k$ (suppose)

$\Rightarrow \quad \sin A + 2k - k\cos A$

$\Rightarrow \quad \sin A + k\cos A = 2k$

$\Rightarrow \quad \sin(A + \alpha) = \dfrac{2k}{\sqrt{1 + k^2}}$, where $\tan\alpha = \dfrac{k}{1}$

$\Rightarrow \quad \dfrac{2k}{\sqrt{1 + k^2}} \le 1 \;\Rightarrow\; 2k \le \sqrt{1 + k^2}$

$\Rightarrow \quad 4k^2 \le 1 + k^2 \;\Rightarrow\; 3k^2 - 1 \le 0$

$\Rightarrow \quad |k| \le \dfrac{1}{\sqrt{3}}$

$\Rightarrow$ Maximum value of $\tan\dfrac{B}{2}$ is $\dfrac{1}{\sqrt{3}}$.

50. $\because x^2 - 9x + 8 = 0$

$\therefore \quad x = 1, 8$

$\Rightarrow \quad 2^{\tan^2 B} = 1, 8 \;\Rightarrow\; 2^{\tan^2 B} = 2^0, 2^3$

$\tan^2 \ne 0$

$\therefore \quad \tan^2 B = 3 \;\Rightarrow\; \tan B = \sqrt{3}$

$\therefore \quad \angle B = 60°$

Also given, $2\cos A = \dfrac{\sin B}{\sin C}$

$\Rightarrow \quad 2\cos A\sin C = \sin B$

$\Rightarrow \quad \sin(A + C) - \sin(A - C) = \sin B$

$\Rightarrow \quad \sin(A - B) - \sin(A - C) = \sin B$

Then, $\sin(A - C) = 0$

$\Rightarrow \quad A = C$

$\Rightarrow \quad A + B + C = 180°$

$A + 60° + A = 180°$

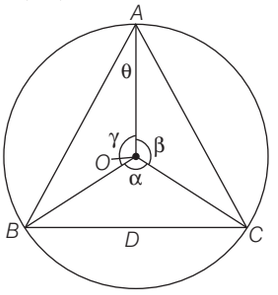
$A = 60° = C,\ B = 60°$

51. We have,

$\text{arc}(BC) = 3$

$\text{arc}(CA) = 4$

$\text{arc}(AB) = 5$



Let r be the radius of the circle, then

$3 = r\alpha,\ 4 = r\beta,\ 5 = r\gamma \qquad \left(\because \text{Angle} = \dfrac{\text{arc}}{\text{radius}}\right)$

Now, $3 + 4 + 5 = r(\alpha + \beta + \gamma) = r \cdot 2\pi$

$\therefore \quad r = \dfrac{6}{\pi}$

$\Delta ABC = \Delta OBC + \Delta OCA + \Delta OAB$

$= \dfrac{1}{2}r^2\sin\alpha + \dfrac{1}{2}r^2\sin\beta + \dfrac{1}{2}r^2\sin\gamma$

$= \dfrac{1}{2}r^2\left\{\sin\left(\dfrac{3}{r}\right) + \sin\left(\dfrac{4}{r}\right) + \sin\left(\dfrac{5}{r}\right)\right\}$

$= \dfrac{1}{2} \times \dfrac{36}{\pi^2}\left[\sin\dfrac{\pi}{2} + \sin\dfrac{2\pi}{3} + \sin\dfrac{5\pi}{6}\right]$

$= \dfrac{18}{\pi^2}\left\{1 + \dfrac{\sqrt{3}}{2} + \dfrac{1}{2}\right\}$

$= \dfrac{9\sqrt{3}(1 + \sqrt{3})}{\pi^2}$ sq unit

52. $\because \dfrac{b + c}{2} \ge \sqrt{bc}$

$\Rightarrow \quad \dfrac{b + c}{2} \ge \lambda$ ...(i)

and $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = \dfrac{b + c}{\sin B + \sin C}$

$= \dfrac{(b + c)}{2\sin\left(\dfrac{B + C}{2}\right)\cos\left(\dfrac{B - C}{2}\right)}$

$= \dfrac{(b + c)}{2\cos\dfrac{A}{2}\cos\left(\dfrac{B - C}{2}\right)}$

$\therefore \quad a = \left(\dfrac{b + c}{2}\right) \cdot \dfrac{\sin A}{\cos\dfrac{A}{2}\cos\left(\dfrac{B - C}{2}\right)}$

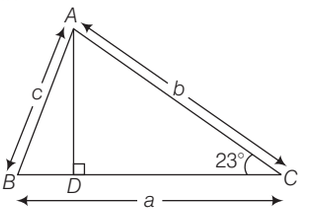
$= \left(\dfrac{b + c}{2}\right) \cdot \dfrac{2\sin\dfrac{A}{2}}{\cos\left(\dfrac{B - C}{2}\right)}$

$\ge \left(\dfrac{b + c}{2}\right) \cdot 2\sin\dfrac{A}{2} \qquad \left(\because \sec\left(\dfrac{B - C}{2}\right) \ge 1\right)$

$\ge 2\lambda\sin\dfrac{A}{2}$ [from Eq. (i)]

Hence, $a \ge 2\lambda\lim\dfrac{A}{2}$

53. $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$



$$= \frac{a}{2c} - \frac{b^2 - c^2}{2ac} = \frac{a}{2c} - \frac{b^2 - c^2}{abc} \cdot \frac{b}{2}$$

$$= \frac{a}{2c} - \frac{b}{2(AD)}$$

$$= \frac{\sin A}{2\sin C} - \frac{1}{2\sin C}$$

In ΔACD, $\cos B = \dfrac{\sin A - 1}{2\sin C}$

$$\frac{AD}{b} = \sin C \Rightarrow \frac{b}{AD} = \frac{1}{\sin C}$$

$\Rightarrow$ $2\cos B \sin C = \sin A - 1$

$\Rightarrow$ $\sin(B + C) - \sin(B - C) = \sin A - 1$

$\Rightarrow$ $\sin A - \sin(B - C) = \sin A - 1$

$\Rightarrow$ $\sin(B - C) = 1$

$$B - C = \frac{\pi}{2}$$

$$B = C + \frac{\pi}{2} = 23 + 90 = 113^\circ$$

$$\angle B = 113^\circ$$

54. We have, $\dfrac{31}{32} = \cos(A - B) = \dfrac{1 - \tan^2 \dfrac{A - B}{2}}{1 + \tan^2 \dfrac{A - B}{2}}$

$\Rightarrow$ $63 \tan^2 \dfrac{A - B}{2} = 1$

$\Rightarrow$ $\tan \dfrac{A - B}{2} = \dfrac{1}{\sqrt{63}}$

Now, $\tan \dfrac{A - B}{2} = \dfrac{a - b}{a + b} \cot \dfrac{C}{2}$

$\Rightarrow$ $\dfrac{1}{\sqrt{63}} = \dfrac{5 - 4}{5 + 4} \cot \dfrac{C}{2}$

$\Rightarrow$ $\tan \dfrac{C}{2} = \dfrac{\sqrt{63}}{9}$

Also $\cos C = \dfrac{1 - \tan^2\left(\dfrac{C}{2}\right)}{1 + \tan^2\left(\dfrac{C}{2}\right)}$

$$= \frac{1 - \frac{63}{81}}{1 + \frac{63}{81}} = \frac{18}{144} = \frac{1}{8}$$

$\therefore$ $c^2 = a^2 + b^2 - 2ab\cos C$

$$= 25 + 16 - 2 \cdot 5 \cdot 4 \cdot \left(\frac{1}{8}\right) = 36$$

Hence $c = 6$

55. Using cosine rule, we get

$$x^2 = (x + 1)^2 + b^2 - 2(x + 1)b\cos\frac{\pi}{3}$$

$\Rightarrow$ $0 = 2x + 1 + b^2 - (x + 1)b$

$\Rightarrow$ $b^2 - (x + 1)b + 2x + 1 = 0$

Since b is real, we have

$\Rightarrow$ $(x + 1)^2 - 4(2x + 1) \geq 0$

$\Rightarrow$ $x^2 - 6x - 3 \geq 0$

$\Rightarrow$ $x \geq 3 + \sqrt{12}$

The least integral value of x is 7.

56. Let $r = 1$ cm

$$a = BC = BD + DE + EC$$

$$= BD + C_1C_2 + BD$$

$$= 2BD + C_1C_2$$

$$= 2r\cot 30^\circ + 2r$$

$$= 2r(\sqrt{3} + 1)$$

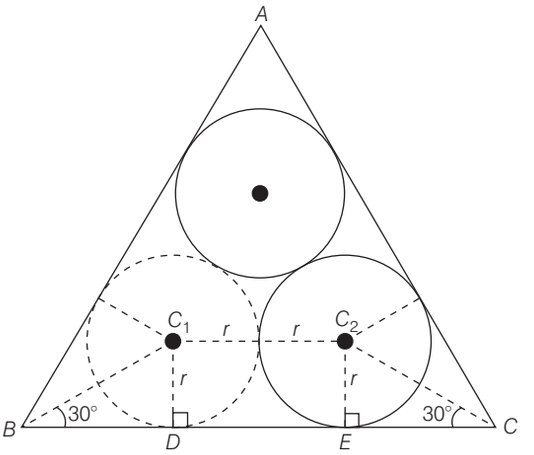
$$= 2(\sqrt{3} + 1) \text{ cm}$$

$\therefore$ Area of triangle

$$ABC = \frac{\sqrt{3}}{4}a^2 = \frac{\sqrt{3}}{4} \cdot 4(\sqrt{3} + 1)^2 \text{ cm}^2$$

$$= \sqrt{3}(4 + 2\sqrt{3}) \text{ cm}^2$$

$$= (6 + 4\sqrt{3}) \text{ cm}^2$$



57. We have, $2b = a + c$

$\therefore$ $2\sin B = \sin A + \sin C$

$$4\sin\frac{B}{2}\cos\frac{B}{2} = 2\sin\frac{A + C}{2}\cos\frac{A - C}{2}$$

$$= 2\cos\frac{B}{2}\cos\frac{A - C}{2}$$

$\therefore$ $2\sin\dfrac{B}{2} = \cos\dfrac{A - C}{2}$

i.e,. $2\cos\dfrac{A + C}{2} = \cos\dfrac{A - C}{2}$ ...(i)

Now, $\cos A + \cos C = 2\cos\dfrac{A + C}{2} \cdot \cos\dfrac{A - C}{2}$

$$= 2\cos\frac{A + C}{2}\left(2\cos\frac{A + C}{2}\right) \quad \text{[using Eq. (i).]}$$

$$= 4\cos^2\frac{A + C}{2} \quad \text{...(ii)}$$

$$4(1 - \cos A)(1 - \cos C) = 4 \cdot 2\sin^2\frac{A}{2} \cdot 2\sin^2\frac{C}{2}$$

$$= 4\left(2\sin\frac{A}{2}\sin\frac{C}{2}\right)^2 = 4\left\{\cos\frac{A-C}{2} - \cos\frac{A+C}{2}\right\}^2$$

$$= 4\left\{\cos\frac{A+C}{2} - \cos\frac{A+C}{2}\right\}^2 = 4\cos^2\frac{A+C}{2} \quad \text{...(iii)}$$

From Eqs. (ii) and (iii), we get

$2\cos A + \cos C = 4(1 - \cos A)(1 - \cos C)$

58. We have, $c(a+b)\cos\frac{1}{2}B = b(a+c)\cos\frac{1}{2}C$

$$\therefore\ c(a+b)\sqrt{\left\{\frac{s(s-b)}{ca}\right\}} = b(a+c)\sqrt{\left\{\frac{s(s-c)}{ab}\right\}}$$

or $(a+b)\sqrt{c(s-b)} = (a+c)\sqrt{b(s-c)}$

Squaring,

$(a+b)^2c(s-b) = (a+c)^2b(s-c)$

or $s[c(a+b)^2 - b(a+c)^2] - bc[(a+b)^2 - (a+c)^2] = 0$

or $s\{ca^2 + 2abc + cb^2 - ba^2 - 2abc - bc^2\}$

$$- bc(b-c)(2a+b+c) = 0$$

or $s\{bc(b-c) - a^2(b-c)\} - bc(b-c)(2a+b+c) = 0$

or $(b-c)\{s(bc-a^2) - bc(2a+b+c)\} = 0$

or $(b-c)\{s(bc-a^2) - bc(2s+a\} = 0$

or $-(b-c)\{s(bc+a^2) + abc\} = 0$

Since, a, b, c are all positive and so $s(bc+a^2) + abc \neq 0$

It follows that $b - c = 0$.

Hence ΔABC is isoceles.

59. M is the mid point of BC

$$MC = \frac{a}{2}$$

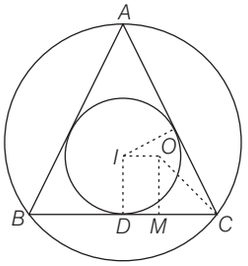
$$OC = R$$

$$OM = ID = r$$

$\therefore\ \Delta OMC$

$$R^2 = r^2 + \frac{a^2}{4}$$

$$R^2 = r^2 + \frac{(2R\sin A)^2}{4}$$



$$\Rightarrow \quad R^2\cos 2A = r^2$$

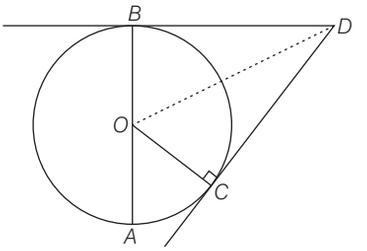
$$\Rightarrow \quad \cos A = \frac{r}{R} \Rightarrow r = R\cos A$$

$\because\ \cos A + \cos B + \cos C$

$$= 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 1 + \frac{r}{R} = 1 + \cos A$$

$$\therefore \quad \cos B + \cos C = 1$$

60.



$$AC = d, OA = OB = r, CD = BD = l, \angle COA = \frac{\pi}{3}$$

$$\therefore \quad AC^2 = OA^2 + OC^2 - 2AO\,OC\cos\frac{\pi}{3}$$

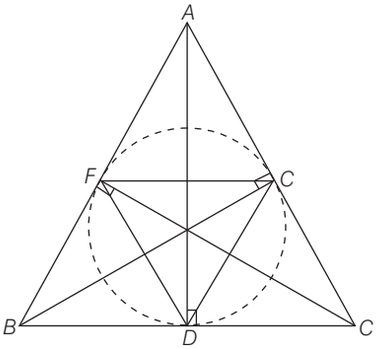
or $$d^2 = 2r^2 - 2r^2\frac{1}{2} = r^2$$

Also, $$\Delta BOD = \Delta = \frac{2\pi}{3\times 2} = \frac{\pi}{3}$$

or $$\tan\frac{\pi}{3} = \frac{BD}{OB} = \frac{1}{r}$$

$$\Rightarrow \quad l = r\sqrt{3} = d\sqrt{3}$$

61. In pedal ΔDEF,



$$EF = a\cos A$$

$$DE = c\cos C$$

$$DF = b\cos B$$

If circum radius of ΔDEF is R_1

Then, $R_1 = \dfrac{(a\cos A)(b\cos B)(c\cos C)}{4\cdot\frac{1}{2}\cdot DF\cdot DE\sin(\angle EDF)}$

(Here, $\Delta EDF = 180^\circ - 2A$)

$$= \frac{abc\cos A\cos B\cos C}{4\cdot\frac{1}{2}\cdot b\cos B\cdot c\cos C\cdot\sin(180^\circ - 2A)}$$

$$= \frac{a\cos A}{2\sin 2A} = \frac{2R\sin A\cos A}{4\sin A\cos A}$$

$$\therefore \quad R_1 = \frac{R}{2}$$

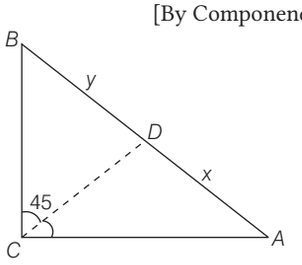
62. Let CD be the bisector of C, so that $AD = x$ and $BD = y$

in ΔACD, $\dfrac{AD}{\sin 45^\circ} = \dfrac{CD}{\sin A} \Rightarrow \sin A = \dfrac{CD}{\sqrt{2}x}$

Similarly in ΔBCD, $\dfrac{BD}{\sin 45^\circ} = \dfrac{CD}{\sin B} \Rightarrow \sin B = \dfrac{CD}{\sqrt{2}y}$

$$\frac{\sin A}{\sin B}=\frac{y}{x} \Rightarrow \frac{\sin A-\sin B}{\sin A+\sin B}=\frac{y-x}{y+x}$$

[By Componendo and Dividendo]



$$\Rightarrow \quad \frac{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}=\frac{y-x}{y+x}$$

$$\Rightarrow \quad \tan\frac{A-B}{2}=\frac{y-x}{y+x} \qquad [\text{As } A+B=90°]$$

$$\Rightarrow \quad \frac{y-x}{y+x}=\frac{t}{1} \qquad \text{(given)}$$

$\Rightarrow$ again by Componendo and Dividendo, $\frac{y}{x}=\frac{1+t}{1-t}$

$$\Rightarrow \quad x:y=(1-t):(1+t)$$

63. $BD=(s-b), CD=(s-c) \Rightarrow (s-b)(s-c)=2$

$$\Rightarrow \quad s(s-a)(s-b)(s-c)=2s(s-a)$$

$$\Rightarrow \quad \Delta^2=2s(s-a)$$

$$\Rightarrow \quad \frac{\Delta^2}{s^2}=\frac{2(s-a)}{s}=1 \quad \text{(radius of incircle of triangle } ABC)$$

$$\Rightarrow \quad \frac{a}{s}=\text{constant.}$$

Now, $\Delta=\frac{1}{2}aH_a$, where 'H_a' is the distance of 'A' from BC.

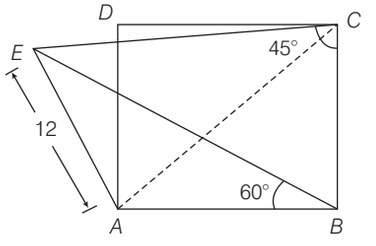
$$\Rightarrow \quad \frac{\Delta}{s}=\frac{1}{2}\frac{aH_a}{s}=1$$

$$\Rightarrow \quad H_a=\frac{2s}{a}=\text{constant}$$

$\Rightarrow$ Locus of 'A' will be a straight line parallel to side BC.

64. Let AE is vertical lamp-post.

Given, AE = 12 m



$$\tan 45°=\frac{AE}{AC}$$

$$AC=AE=12 \text{ m}$$

$$\tan 60°=\frac{AE}{AB}$$

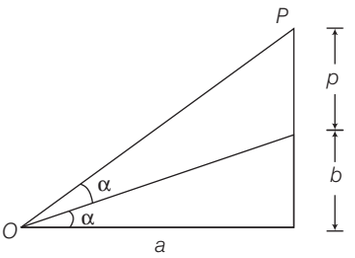
$$AB=\frac{AE}{\sqrt{3}}=4\sqrt{3}$$

$$BC=\sqrt{AC^2-AB^2}=\sqrt{144-48}$$

$$=\sqrt{96}=4\sqrt{6}$$

Area $=AB\times BC=4\sqrt{3}\times 4\sqrt{6}=48\sqrt{2}$ sq m.

65. $\tan\alpha=\frac{b}{a}, \tan 2\alpha=\frac{2(b/a)}{1-(b/a)^2}=\frac{p+b}{a}$



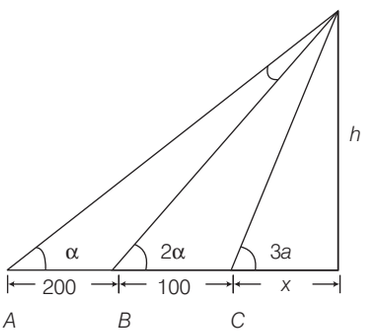
$$\Rightarrow \quad \frac{2ba}{a^2-b^2}=\frac{p+b}{a}$$

$$\Rightarrow \quad \frac{2ba^2-a^2b+b^3}{a^2-b^2}=p \Rightarrow p=\frac{b(a^2+b^2)}{(a^2-b^2)}$$

66. $x=h\cot 3\alpha$ ...(i)

$(x+100)=h\cot 2\alpha$ ... (ii)

$(x+300)=h\cot\alpha$ ...(iii)



From Eqs. (i) and (ii)

$$-100=h(\cot 3\alpha-\cot 2\alpha)$$

From Eqs. (ii) and (iii)

$$-200=h(\cot 2\alpha-\cot\alpha)$$

$$100=h\left(\frac{\sin\alpha}{\sin 3\alpha\sin 2\alpha}\right) \text{ and } 200=h\left(\frac{\sin\alpha}{\sin 2\alpha\sin\alpha}\right)$$

or $\quad \frac{\sin 3\alpha}{\sin\alpha}=\frac{200}{100} \Rightarrow \frac{\sin 3\alpha}{\sin\alpha}=2$

$$\Rightarrow \quad 3\sin\alpha-4\sin^3\alpha-2\sin\alpha=0$$

$$\Rightarrow \quad 4\sin^3\alpha-\sin\alpha=0 \Rightarrow \sin\alpha=0$$

or $\quad \sin^2\alpha=\frac{1}{4}=\sin^2\left(\frac{\pi}{6}\right)$

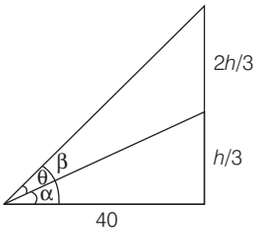
$$\Rightarrow \quad \alpha=\frac{\pi}{6}$$

Hence, $H=200\sin\frac{\pi}{3}=200\frac{\sqrt{3}}{2}=100\sqrt{3}$ [form Eq. (i)]

67. Obviously, from figure

$$\tan\alpha = \frac{h/3}{40} = \frac{h}{120} \quad \text{...(i)}$$

$$\tan\beta = \frac{h}{40} = \frac{3h}{120} \quad \text{...(ii)}$$



Therefore $\tan\theta = \tan(\beta - \alpha)$

$$\Rightarrow \quad \frac{1}{2} = \frac{\frac{3h}{120} - \frac{h}{120}}{1 + \frac{3h^2}{14400}} \Rightarrow h = 120, 40$$

But $h = 40$ cannot be taken according to the condition, therefore $h = 120$ ft.

68. $c^2 = a^2 + b^2 - 2ab\cos C = (a-b)^2 + 2ab(1-\cos C)$

$$= (a-b)^2 + \frac{4\Delta}{\sin C}(1-\cos C)$$

Hence, for c to be minimum $a = b$

Also $\quad \Delta = \frac{1}{2}ab\sin C \Rightarrow a^2 = \frac{2\Delta}{\sin C} = b^2$

69. $2\Delta = ab\sin C$

$$\Rightarrow \quad a^2b^2 - 4\Delta^2 = a^2b^2\cos^2 C$$

$$\therefore \quad \sqrt{a^2b^2 - 4\Delta^2} + \sqrt{b^2c^2 - 4\Delta^2} + \sqrt{c^2a^2 - 4\Delta^2}$$

$$= ab\cos C + bc\cos A + ca\cos B$$

70. $c\cos(A-\theta) + a\cos(C+\theta)$

$= \cos\theta(c\cos A + a\cos C) + \sin\theta(c\sin A - a\sin C) = b\cos\theta \geq b$

71. As $ac = 3$ and $bc = 4$

$$\therefore \quad \frac{b}{a} = \frac{4}{3}$$

$$\Rightarrow \quad \frac{b-a}{b+a} = \frac{4-3}{4+3} = \frac{1}{7} \qquad [\angle B > \angle A]$$

$$\tan\left(\frac{B-A}{2}\right) = \frac{b-a}{b+a}\cot\frac{C}{2}$$

$$\Rightarrow \quad \sqrt{\frac{1-\cos(B-A)}{1+\cos(B-A)}} = \frac{4-3}{4+3}\cot\frac{C}{2}$$

$$\Rightarrow \quad \sqrt{\frac{1-\frac{3}{4}}{1+\frac{3}{4}}} = \frac{1}{7}\cot\frac{C}{2}$$

$$\sqrt{\frac{1}{7}} = \frac{1}{7}\cot\frac{C}{2} \Rightarrow \cot\frac{C}{2} = \sqrt{7}$$

or $\quad \cos C = \frac{1-\tan^2\frac{C}{2}}{1+\tan^2\frac{C}{2}} = \frac{1-\frac{1}{7}}{1+\frac{1}{7}} = \frac{6}{8} = \frac{3}{4}$

$$\therefore \quad \cos C = \cos(A-B) \text{ or } \cos C = \cos(B-A)$$

$$\therefore \quad C = A - B \quad \text{or} \quad C = B - A$$

$$\pi - (A+B) = A - B \text{ or } \pi - (A+B) = B - A$$

$$\therefore \quad 2A = \pi \text{ or } 2B = \pi, \; A = \frac{\pi}{2} \text{ (Rejected) or } B = \frac{\pi}{2}$$

But $\quad \angle B = \frac{\pi}{2} \qquad [\text{As } \angle B > \angle A]$

But $\Rightarrow \frac{b}{a} = \frac{4}{3}\frac{b}{a} = \frac{4}{3} = k$ (let)

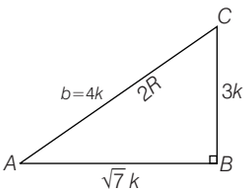
$$\Rightarrow \quad b = 4k, \; a = 3k$$

Now, $ac = 3$

$$\Rightarrow \quad 3\sqrt{7}\,k^2 = 3$$

$$\Rightarrow \quad k^2 = \frac{1}{\sqrt{7}}$$

$$\Rightarrow \quad k = \frac{1}{7^{1/4}}$$



$$\therefore \quad R = 2k = \frac{2}{7^{1/4}}$$

72. We have, $a = 5, b = 4$ and $\cos(A-B) = \frac{31}{32}$

$$\text{As, } \tan\left(\frac{A-B}{2}\right) = \left(\frac{a-b}{a+b}\right)\cot\frac{C}{2} = \frac{1}{9}\cot\frac{C}{2}$$

$$\Rightarrow \quad \cos(A-B) = \frac{1-\tan^2\left(\frac{A-B}{2}\right)}{1+\tan^2\left(\frac{A-B}{2}\right)}$$

$$\Rightarrow \quad \frac{31}{32} = \frac{1-\frac{1}{81}\cot^2\frac{C}{2}}{1+\frac{1}{81}\cot^2\frac{C}{2}}$$

$$\Rightarrow \quad 31 + \frac{31}{81}\cot^2\frac{C}{2} = 32 - \frac{32}{81}\cot^2\frac{C}{2}$$

$$\Rightarrow \quad \frac{7}{9}\cot^2\frac{C}{2} = 1 \Rightarrow \cot^2\frac{C}{2} = \frac{9}{7}$$

$$\text{As,} \quad \cos C = \frac{1-\tan^2\frac{C}{2}}{1+\tan^2\frac{C}{2}} = \frac{1-\frac{7}{9}}{1+\frac{7}{9}} = \frac{2}{16} = \frac{1}{8}$$

$$\text{Also,} \quad \cos C = \frac{a^2+b^2-c^2}{2ab}$$

$$\Rightarrow \quad \frac{25+16-c^2}{2(5)(4)} = \frac{1}{8} \Rightarrow c = 6$$

Now, verify alternatives.

(d) LHS $= R(b^2\sin 2C + c^2\sin 2B)$

$= R(2b^2\sin C\cos C + 2c^2\sin B\cos B)$

$= 2R[4R^2\sin^2 B\sin C\cos C + 4R^2\sin^2 C\cos B]$

$= 2R(4R^2)\sin B\sin C[\sin B\cos C + \cos B\sin C]$

$= 8R^2\sin A\sin B\sin C$

$= (2R\sin A)(2R\sin B)(2R\sin C) = abc$

$(5)(4)(6) = 120$

73. (a) $a^2-(c-b)^2=4bc$

$\Rightarrow a^2-(c^2+b^2-2bc)=4bc$

$a^2=b^2+c^2+2bc$

$\frac{-2bc}{2bc}=\frac{b^2+c^2-a^2}{2bc}$

$\Rightarrow \cos B=-1$ which is not possible.

(b) $2b^2\sin C\cos C+2c^2\sin B\cos B=ab$

$\frac{(b^2c\cos C+c^2b\cos B)}{R}=ab$

$\Rightarrow \frac{bc}{R}(b\cos C+c\cos B)=ab$

$\Rightarrow \frac{abc}{R}=ab, \Rightarrow \frac{c}{2R}=\frac{1}{2}$

$\Rightarrow \sin C=\frac{1}{2} \Rightarrow C=\frac{\pi}{6}$

(c) For $a=3, b=5, c=7$, we have

$\cos C=\frac{a^2+b^2-c^2}{2ab}=\frac{9+25-49}{2\times3\times5}$

$=\frac{-15}{30}=\frac{-1}{2}$

$\therefore C=\frac{2\pi}{3}$

(d) As $\cos\left(\frac{A-C}{2}\right)=\cos\left(\frac{A+C}{2}\right)$

$\Rightarrow 2\sin\frac{A}{2}\sin\frac{C}{2}=0$ which is not possible in triangle.

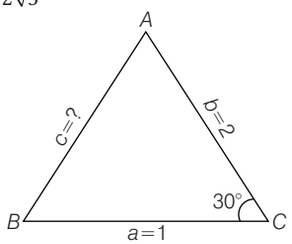
74. We have, $\frac{1}{\sin A}=\frac{2}{\sin B}=\frac{c}{\sin30^\circ}$

So, $2\sin A=\sin B$

Hence, option (a) is correct.

Also, $c^2=4+1-4\cos30^\circ$

$=5-2\sqrt{3}$



$\Rightarrow c=\sqrt{5-2\sqrt{3}}$

As $c>a \Rightarrow \angle A<\angle C$ and $\angle B$ is obtuse.

Also, $\Delta=\frac{1}{2}\times1\times2\times\frac{1}{2}=\frac{1}{2}$

So, $R=\frac{abc}{4\Delta}$

$=\frac{1\times2\times c}{4\Delta}=c=AB$

Hence, option (d) is correct.

75. $\Delta=10\sqrt{3}, s=10$

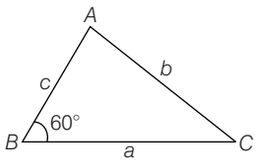
$\Rightarrow r=\frac{\Delta}{s}=\sqrt{3}$ and

$r=(s-b)\tan\frac{B}{2}$

$\Rightarrow s-b=3 \Rightarrow b=7$

$\Rightarrow a+c=13$ ...(i)

$\Rightarrow ac=40$ ...(ii)



From Eq. (ii), $a^2-13a+40=0$

$\Rightarrow a=5, a=8$

$\Rightarrow c=8$ and $c=5$.

Hence, $a=8, b=7$ and $c=5$ (As $a>b>c$)

$\Rightarrow r_1=\frac{\Delta}{s-a}, r_2=\frac{\Delta}{s-b}, r_3=\frac{\Delta}{s-c}$ will give $\frac{1}{12}:\frac{1}{13}:\frac{1}{15}$

Also, $\frac{b}{\sin B}=2R$

$\Rightarrow 2R=\frac{7}{\frac{\sqrt{3}}{2}} \Rightarrow R=\frac{7}{\sqrt{3}}$

76. $\cos C=\frac{1}{2}=\frac{16+64-c^2}{64}$

$\Rightarrow c^2=48 \Rightarrow c=4\sqrt{3}$

Also, $\frac{a}{\sin A}=\frac{c}{\sin C}$

$\Rightarrow \frac{4}{\sin A}=\frac{4\sqrt{3}}{\frac{\sqrt{3}}{2}}$

$\Rightarrow A=30^\circ$

As $\angle C=60^\circ, \angle A=30^\circ$

$\Rightarrow \angle B=90^\circ$

Now, $r=\frac{\Delta}{s}=\frac{\frac{1}{2}ac}{\frac{a+b+c}{2}}=\frac{ac}{a+b+c}$

$=\frac{4(4\sqrt{3})}{4+4\sqrt{3}+8}$

$=\frac{16\sqrt{3}}{12+4\sqrt{3}}=\frac{4\sqrt{3}}{3+\sqrt{3}}$

Now, verify alternatives.

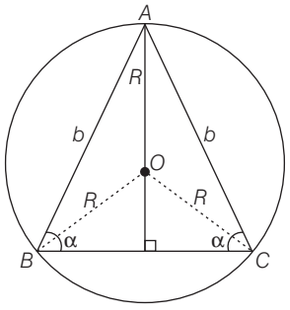
(c) $r=\frac{4\sqrt{3}}{3+\sqrt{3}}$

(d) T the length of internal angle bisector of $\angle C$ is $\frac{8}{\sqrt{3}}$.

Hence, options *a* and *b* are correct.

77. $R = \dfrac{b}{2\sin\alpha} = \dfrac{b}{2}\operatorname{cosec}\alpha$

$$\Delta = \frac{1}{2}b^2\sin(180^\circ - 2\alpha) = \frac{1}{2}b^2\sin 2\alpha$$



and $r = 4R\sin\dfrac{\alpha}{2}\cdot\sin\dfrac{\alpha}{2}\cdot\sin(90^\circ - \alpha)$

$$= \frac{2b}{\sin\alpha}\cdot\sin^2\frac{\alpha}{2}\cdot\cos\alpha\cdot\frac{\sin\alpha}{\sin\alpha}$$

$$= \frac{b(1-\cos\alpha)\sin 2\alpha}{2(1+\cos^2\alpha)} = \frac{b\sin 2\alpha}{2(1+\cos\alpha)}$$

and $OI = \sqrt{(R^2 - 2Rr)} = R\sqrt{\left(1 - \dfrac{2r}{R}\right)}$

$$= R\sqrt{1 - 4\cos\alpha + 4\cos^2\alpha} = R(2\cos\alpha - 1)$$

$$= R\left\{2\left(2\cos^2\frac{\alpha}{2} - 1\right) - 1\right\}$$

$$= R\left(4\cos^2\frac{\alpha}{2} - 3\right)$$

$$= \frac{R\left(4\cos^3\dfrac{\alpha}{2} - 3\cos\dfrac{\alpha}{2}\right)}{\cos\dfrac{\alpha}{2}}$$

$$= \frac{R\cos\left(\dfrac{3\alpha}{2}\right)}{\cos\dfrac{\alpha}{2}} = \left|\frac{b\cos\left(\dfrac{3\alpha}{2}\right)}{2\sin\alpha\cos\left(\dfrac{\alpha}{2}\right)}\right|$$

78. (a) Since, $\tan A + \tan B + \tan C = \tan A\tan B\tan C$

But here, $\tan A + \tan B + \tan C = 0$, impossible

(b)

$$\frac{\sin A}{2} = \frac{\sin B}{3} = \frac{\sin C}{7}$$

or $\dfrac{a}{2} = \dfrac{b}{3} = \dfrac{c}{7}$

$\Rightarrow$ $\dfrac{a+b}{5} = \dfrac{c}{7}$

or $\dfrac{a+b}{c} = \dfrac{5}{7} < 1$

$\therefore$ $a + b < c$ impossible.

(c) $(a+b)^2 = c^2 + ab$

$\Rightarrow$ $a^2 + b^2 - c^2 = -ab$

$\Rightarrow$ $\cos C = -\dfrac{1}{2}$

$\angle C = 120^\circ$

and $\sqrt{2}(\sin A + \cos A) = \sqrt{3}$

$$\sqrt{2}\cdot\sqrt{2}\sin\left(A + \frac{\pi}{4}\right) = \sqrt{3}$$

$$\sin\left(A + \frac{\pi}{4}\right) = \frac{\sqrt{3}}{2}$$

$$A + \frac{\pi}{4} = \frac{\pi}{3}$$

$\therefore$ $A = \dfrac{\pi}{12}$ possible

(d)

$\because$ $\sin A + \sin B = \dfrac{\sqrt{3}+1}{2}$ ...(i)

and $\cos A\cos B = \dfrac{\sqrt{3}}{4} = \sin A\sin B$

$\therefore$ $\cos A\cos B - \sin A\sin B = 0$

$\cos(A + B) = 0$

$A + B = \dfrac{\pi}{2}$

$\therefore$ $B = \dfrac{\pi}{2} - A$

From Eq. (i),

$$\sin A + \cos A = \frac{\sqrt{3}+1}{2}$$

$\Rightarrow$ $\sin\left(A + \dfrac{\pi}{4}\right) = \dfrac{\sqrt{3}+1}{2\sqrt{2}} = \sin\dfrac{5\pi}{12}$

$\Rightarrow$ $A + \dfrac{\pi}{4} = \dfrac{5\pi}{12}$

$\Rightarrow$ $A = \dfrac{\pi}{6}$

$\therefore$ $B = \dfrac{\pi}{3}$

Then, $C = \dfrac{\pi}{2}$ possible.

79. In ΔABC, $b + c - a > 0$, $c + a - b > 0$, $a + b - c > 0$, so

$$\frac{(b+c-a) + (c+a-b)(a+b-c)}{3} \geq \{(b+c-a)(c+a-b)(a+b-c)\}^{1/3}$$

$\Rightarrow$ $p^3 \geq 27\,(b+c-a)\,(c+a-b)\,(a+b-c)$

Also, $\dfrac{(a+b+c) + (b+c-a) + (c+a-b) + (a+b-c)}{4}$

$$> \{(a+b+c)(b+c-a)(c+a-b)(a+b-c)\}^{1/4}$$

$\Rightarrow$ $\dfrac{2P}{4} > (16A)^{1/4}$

$\Rightarrow$ $P > 4A^{1/4}$ or $P^4 = 256A$

For a given parameter, equilateral triangle has the largest area, so the area of triangle

$$A \le \frac{\sqrt{3}}{4}\left(\frac{P}{3}\right)^2$$

(for equilateral triangle, $a = b = c = \frac{P}{3}$)

Now, $P^2 = (a + b + c)^2$

$$= a^2 + b^2 + c^2 + 2(ab + bc + ca)$$

$$= 3(a^2 + b^2 + c^2) - 2(a^2 + b^2 + c^2 - ab - bc - ca \le 3(a^2 + b^2 + c^2)$$

(equality holds iff $a = b = c$)

Thus, $A \le \frac{\sqrt{3}}{4}\frac{a^2 + b^2 + c^2}{3}$

$\Rightarrow$ $a^2 + b^2 + c^2 \ge 4\sqrt{3}A$

80. Let AD be a perpendicular from$A$ on BC.

Then, $\frac{BD}{c} = \cos B$

$\Rightarrow BD$ is rational, similarly AD is rational.

Now, $\sin C = \cos B = \frac{AD}{b} \Rightarrow b$ is rational

Since, $\cos C = \sin B = \frac{DC}{b}$

$\Rightarrow$ DC is rational

Hence, $a = BD + DC$ is rational

Hence, both a and b are rational numbers.

81. (a) : $\because$ Maximum value of

$\sin 2A + \sin 2B + \sin 2C$

and $\sin A + \sin B + \sin C$ is same that is 3.

(b) : $\because \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \frac{1}{8}$

$\Rightarrow$ $\frac{r}{4R} \le \frac{1}{8} \Rightarrow R \ge 2r$

(c) $\because \frac{abc}{(a + b + c)} = \frac{4R\Delta}{2s} = 2R \cdot r = R(2r) \le R^2$

$\therefore$ $R^2 \ge \frac{abc}{(a + b + c)}$ $(\because \beta \ge 2r)$

(d)

$\angle B = 90^\circ$

$\therefore$ $r = (s - b)\tan\frac{B}{2} = s - b$

$R = \frac{b}{2\sin B} = \frac{b}{2}$

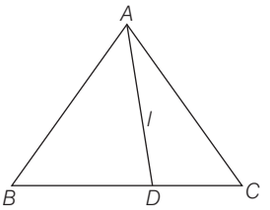
$\Rightarrow$ $2R = b$

$\therefore$ $r + 2R = s$

82. $\because D$ is the mid point

$\therefore$ $AB^2 + AC^2 = 2[AD^2 + BD^2]$

$$c^2 + b^2 = 2\left\{l^2 + \left(\frac{a}{2}\right)^2\right\}$$



$\Rightarrow$ $c^2 + b^2 = 2l^2 + \frac{a^2}{2}$

$\Rightarrow$ $4l^2 + 2b^2 + 2c^2 - a^2$

$$= b^2 + c^2 + (b^2 + c^2 - a^2)$$

$$= b^2 + c^2 + 2bc\cos A$$

$$= (b^2 + c^2 - a^2) + a^2 + 2bc\cos A$$

$$= 2bc\cos A + a^2 + 2bc\cos A$$

$$= 4bc\cos A + a^2$$

83. $\because A = \pi r^2$, $A_1 = \pi r_1^2$, $A_2 = \pi r_2^2$ and $A_3 = \pi r_3^2$

(a) :

$$\sqrt{A_1} + \sqrt{A_2} + \sqrt{A_3} = \sqrt{\pi}\,(r_1 + r_2 + r_3)$$

(b) :

$$\frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}}$$

$$= \frac{1}{\sqrt{\pi}}\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right) = \frac{1}{\sqrt{\pi}}\left(\frac{1}{r}\right)$$

$$= \frac{1}{\sqrt{\pi r^2}} = \frac{1}{\sqrt{A}}$$

(c) :

$\because$ $\frac{s^2}{\sqrt{\pi} r_1 r_2 r_3} = \frac{s^2}{\sqrt{\pi}\frac{\Delta^3}{(s-a)(s-b)(s-c)}}$

$$= \frac{s^2(s-a)(s-b)(s-c)}{\Delta^3\sqrt{\pi}} = \frac{s}{\Delta\sqrt{\pi}}$$

$$= \frac{1}{r\sqrt{\pi}} = \frac{1}{\sqrt{\pi r^2}} = \frac{1}{\sqrt{A}}$$

$$= \frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}}$$

(d) :

$\because$ $\sqrt{A_1} + \sqrt{A_2} + \sqrt{A_3}$

$$= \sqrt{\pi}(r_1 + r_2 + r_3) = \sqrt{\pi}(4R + r)$$

84. $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

$\Rightarrow$ $c^2 - 2bc\cos A + b^2 - a^2 = 0$

$\therefore$ $c_1 + c_2 = 2b\cos A, c_1c_2 = b^2 - a^2$

Given, $c_1^2 + c_1c_2 + c_2^2 = a^2$

$\Rightarrow$ $(c_1 + c_2)^2 - c_1c_2 = a^2$

$\Rightarrow$ $4b^2\cos^2 A - (b^2 - a^2) = a^2$

$\Rightarrow \quad 4\cos^2 A = 1$

$\Rightarrow \quad 2(1 + \cos 2A) = 1$

or $\quad \cos 2A = -\frac{1}{2} = \cos\frac{2\pi}{3}$

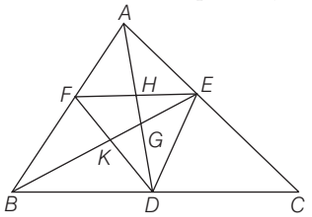
$\therefore \quad 2A = 2n\pi \pm \frac{2\pi}{3}$

$\therefore \quad A = n\pi \pm \frac{\pi}{3}, n \in I$

$\therefore \quad A = \frac{\pi}{3}$ or $\frac{2\pi}{3}$

i.e. $\quad A = 60°$ or $120°$

85. Let D, E and F be the mid points of the sides of ΔABC E and F are mid-points of AC andAB , respectively.



So, $EF \parallel BC$ and $EF = \frac{1}{2}BC$

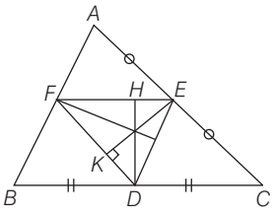
Thus, AD will also divide EF into equal parts.

Hence, DH is also median of ΔDEF.

Similarly BE, median of ΔABC, is also median of ΔDEF.

These two lines meet at G. So, the centroids of both triangles are same.

The orthocenter, O of ΔDEF is the point of intersection of the perpendiculars DH and EK drawn from D and E, respectively.



Since, $EF \parallel BC$, DH is perpendicular to BC also,

Similarly, EK is perpendicular to AC.

So, orthocenter of ΔDEF and circumentre of ΔABC is the same points.

86. $(a^2 - 2ac + c^2) + (a^2 - 4ab + 4b^2) = 0$

or $\quad (a - c)^2 + (a - 2b)^2 = 0$

$a = c$ and $a = 2b$

Therefore, the triangle is isoceles.

Also, $\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{7b^2}{8b^2} = \frac{7}{8}$

$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{1}{4}$

87. $2\Delta = ab\sin C$

$\Rightarrow \quad a^2b^2 - 4\Delta^2 = a^2b^2\cos^2 C$

$\Rightarrow \sqrt{a^2b^2 - 4\Delta^2} + \sqrt{b^2c^2 - 4\Delta^2} + \sqrt{c^2a^2 - 4\Delta^2}$

$= ab\cos C + bc\cos A + ca\cos B$

$= \frac{a^2 + b^2 + c^2}{2}$ (using cosine rule)

88. $2a^2b^2 + 2b^2c^2 = a^4 + b^4 + c^4$

Also, $\quad (a^2 - b^2 + c^2)^2 = a^4 + b^4 + c^4 - 2(a^2b^2 + b^2c^2 - c^2a^2)$

$\therefore \quad (a^2 - b^2 + c^2)^2 = 2c^2a^2$

$\therefore \quad \frac{a^2 - b^2 + c^2}{2ca} = \pm\frac{1}{\sqrt{2}} = \cos B$

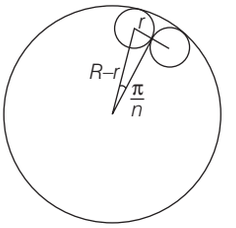
or $\quad B = 45°$ or $135°$

89. $AH = 2R\cos A$, $BH = 2R\cos B$, $CH = 2R\cos C$

$\therefore P = 2R(\cos A + \cos b + \cos C) = 2R\left(1 + \frac{r}{R}\right) = 2(R + r)$

We know that in any triangle, $r \le \frac{R}{2}$

90.



If R is radius of big circle, then $\sin\left(\frac{\pi}{n}\right) = \frac{r}{R - r}$

or $R = r\left(1 + \text{cosec}\left(\frac{\pi}{n}\right)\right) = \frac{r\left[\sin\frac{\pi}{2n} + \cos\frac{\pi}{2n}\right]^2}{\sin\frac{\pi}{n}}$

91. From the cosine formula, $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

or $\quad b^2 - (2c\cos A)b + (c^2 - a^2) = 0$

Which is s quadratic equation in b. Therefore,

$c\sin A < a < c$

Therefore, two triangles will be obtained. But this is possible when two values of the third side are also obtained. Clearly, two value of sides b will be b_1 and b_2. Let these be the roots of the above equation. Then,

Sum of roots $= b_1 + b_2 = 2c\cos A$ and $b_1b_2 = c^2 - a^2$

92. We have, $a^3 + b^3 + c^3 = c^2a + c^2b + c^3$

$\Rightarrow \quad a^3 + b^3 = c^2(a + b)$

$\Rightarrow \quad (a + b)(a^2 + b^2 - ab) = c^2(a + b)$

$\Rightarrow \quad \frac{a^2 + b^2 - c^2}{2ab} = \frac{1}{2} \Rightarrow \cos C = \frac{1}{2}$

$\angle C = 60°$

Hence, $A + B = 120°$

$\therefore$ ΔABC need not be equilateral.

Statement I is correct and statement II is false.

93. $S_1 : \cos(A-B) = \frac{4}{5}$

$$\Rightarrow \quad \frac{1-\tan^2\left(\frac{A-B}{2}\right)}{1+\tan^2\left(\frac{A-B}{2}\right)} = \frac{4}{5}$$

$$\Rightarrow \quad \frac{2\tan^2\left(\frac{A-B}{2}\right)}{2} = \frac{1}{9}$$

$$\Rightarrow \quad \tan\left(\frac{A-B}{2}\right) = \frac{1}{3} \qquad [\text{as } a > b \Rightarrow A > B]$$

Using, $\tan\left(\frac{A-B}{2}\right) = \left(\frac{a-b}{a+b}\right)\cot\frac{C}{2}$, we get

$$\frac{1}{3} = \frac{6-3}{6+3}\cot\frac{C}{2} \Rightarrow \cot\frac{C}{2} = 1$$

$\Rightarrow \quad \angle C = 90^\circ \Rightarrow$ Statement I is false.

S_2 : Using sine law in ΔABC, we get

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\Rightarrow \quad \frac{a}{\sin A} = \frac{\sqrt{a^2+b^2}}{\sin\frac{\pi}{2}}$$

$$\Rightarrow \quad \frac{6}{\sin A} = \sqrt{45} \Rightarrow \sin A = \frac{2}{\sqrt{5}}$$

$\Rightarrow$ Statement II is true.

94. Statement II can be proved (by using $A + B + C = \pi$) to be true. (conditional identities) From which we get

$3 - 2(\sin^2 A + \sin^2 B + \sin^2 C) = -1 - 4\cos A\cos B\cos C$

$\Rightarrow 3 - 2(2) = -1 - 4\cos A\cos B\cos C$

$\Rightarrow \cos A\cos B\cos C = 0$

$\Rightarrow$ one of the angles A, B, C is equal to 90°.

95. $\because A + C = 180^\circ, B + D = 180^\circ$

$\therefore \cos A = -\cos C, \cos B = -\cos D$

$\Rightarrow \cos A + \cos B + \cos C + \cos D$

$= (\cos A + \cos C) + (\cos B + \cos D)$

$= 0 + 0$

$= 0$

$\therefore \quad \Sigma\cos A = 0$

and $\quad \sin A = \sin C$

and $\quad \sin B = \sin D$

Then, $\quad \Sigma\sin A = \sin A \ \sin B + \sin C + \sin D$

$= 2(\sin C + \sin D) \neq 0$

96. Area of ΔABC = Area of ΔABD + Area of ΔACD

$$\Rightarrow \quad \frac{1}{2}bc\sin A = \frac{1}{2}c\cdot AD\sin\left(\frac{A}{2}\right) + \frac{1}{2}\cdot b\cdot AD\sin\left(\frac{A}{2}\right)$$

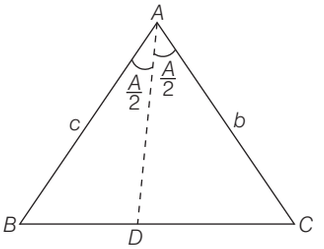
$$\Rightarrow \quad AD = \frac{bc\sin A}{(b+c)\sin\left(\frac{A}{2}\right)} = \frac{2bc\cos\left(\frac{A}{2}\right)}{b+c}$$

Also, $\quad (AD)^2 = \frac{4b^2c^2}{(b+c)^2} \times \frac{s(s-a)}{bc}$

$$= \frac{bc2s(2s-2a)}{(b+c)^2}$$

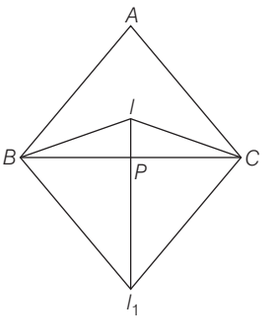
$$= bc\cdot\frac{(a+b+c)(b+c-a)}{(b+c)^2}$$

$$= \frac{bc\{(b+c)^2 - a^2\}}{(b+c)^2}$$



97. $ICI_1 = \frac{\pi}{2}$

$IBI_1 = \frac{\pi}{2}$,



$\therefore BICI_1$, is cyclic

Quadrilateral $BP \cdot PC = IP \cdot I_1P$

Hence, (c) is the correct answer.

98. Statement II is true.

Statement I $\tan A = \tan B = \tan C$

$A = B = C$ ie, $a = b = c$

$r_1 = r_2 = r_3$

$$\therefore \quad \frac{r_1 + r_2 + r_3}{r} = 3\cdot\frac{r_1}{r}$$

$$= 3\cdot\frac{\frac{\Delta}{s-a}}{\frac{\Delta}{s}} = 3\frac{a+b+c}{b+c-a} = 9$$

Hence, (a) is the correct answer.

99. Statement II is False. Because if $p = 2, q = 4, r = 6$, then $p : q : r = 1 : 2 : 3$ but $p \neq 1$

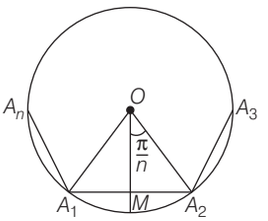
For statement I, let $\tan A = k$, $\tan B = 2k$, $\tan C = 3k$, then from $\tan A + \tan B + \tan C = \tan A\tan B\tan C$ (in a triangle) we get $6k = 6k^3 \Rightarrow k = 0, 1, -1$ but $k = 0, -1$ is not possible. So $k = 1 \Rightarrow \tan A = 1 \Rightarrow A = 45^\circ$. So, statement I is correct.

100. $a^2 = b^2 + c^2$, then, $\angle A = \dfrac{\pi}{2}$

$\therefore \quad a = 2R\sin A = 2R$

$\Rightarrow \quad \dfrac{a^2 + b^2 + c^2}{R^2} = \dfrac{a^2 + a^2}{R^2} = \dfrac{2a^2}{R^2} = \dfrac{2(2R)^2}{R^2} = 8$

101. $\because \angle A_1OA_2 = \dfrac{2\pi}{n}$



$\therefore \quad \angle A_1OM = \angle A_2OM = \dfrac{\pi}{n}$

$\therefore \quad A_1A_2 = 2MA_2$

$= 2a\sin\left(\dfrac{\pi}{n}\right)$ cm

$\therefore$ Perimeter $= 2an\ \text{in}\left(\dfrac{2\pi}{2n}\right)$ cm

$= 2an\sin\left(\dfrac{360^\circ}{2n}\right)$ cm

For, $n = 5$

Perimeter is $10a\sin(36^\circ)$ cm

102. Since L.H.S of the inequality in statement II is a symmetric function of sines of the angles of the triangle, its maximum value is attained. When $A = B = C = 60^\circ$ and hence the statement II is true. Statement I is true if

$2R(\sin A\cos A + \sin B\cos B + \sin C\cos C)$

$\le R(\sin A + \sin B + \sin C)$

or if $4\sin A\sin B\sin C \le 4\cos\left(\dfrac{A}{2}\right)\cos\left(\dfrac{B}{2}\right)\cos\left(\dfrac{C}{2}\right)$

[From conditional identities]

or if $\sin\left(\dfrac{A}{2}\right)\sin\left(\dfrac{B}{2}\right)\sin\left(\dfrac{C}{2}\right) \le \dfrac{1}{8}$

Which is true by statement II

So, statement I is correct

103 $\because \cos^2\dfrac{A}{2} + \cos^2\dfrac{B}{2} + \cos^2\dfrac{C}{2} = \dfrac{1}{2}(3 + \cos A + \cos B + \cos C)$

$\because \cos A + \cos B + \cos C = 1 + 4\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$ (from identity)

Then, $\cos^2\dfrac{A}{2} + \cos^2\dfrac{B}{2} + \cos^2\dfrac{C}{2}$

$= 2\left(1 + \sin\left(\dfrac{A}{2}\right)\sin\left(\dfrac{B}{2}\right)\sin\left(\dfrac{C}{2}\right)\right)$

$\because \quad 2\left(1 + \sin\left(\dfrac{A}{2}\right)\sin\left(\dfrac{B}{2}\right)\sin\left(\dfrac{C}{2}\right)\right)$

$= y\left(x^2 + \dfrac{1}{x^2}\right) \ge 2y \qquad (\because \text{AM} \ge \text{GM})$

$\therefore \quad y \le 1 + \sin\left(\dfrac{A}{2}\right)\sin\left(\dfrac{B}{2}\right)\sin\left(\dfrac{C}{2}\right) \le 1 + \dfrac{1}{8}$

i.e. $\quad y \le \dfrac{9}{8}$

$\therefore$ Maximum value of y is $\dfrac{9}{8}$

104. $\because a\cos A + b\cos B + c\cos C \le s$

$\Rightarrow 2R\sin A\cos A + 2R\sin B\cos B + 2R\sin C\cos C \le \left(\dfrac{a + b + c}{2}\right)$

$\Rightarrow R(\sin 2A + \sin 2B + \sin 2C) \le \dfrac{2}{2}(R\sin A + R\sin B + R\sin C)$

$\Rightarrow (\sin 2A + \sin 2B + \sin 2C) \le (\sin A + \sin B + \sin C)$

$\Rightarrow (4\sin A\sin B\sin C) \le \left\{4\cos\left(\dfrac{A}{2}\right)\cos\left(\dfrac{B}{2}\right)\cos\left(\dfrac{C}{2}\right)\right\}$

(from identities)

$\Rightarrow 4\left(2\sin\left(\dfrac{A}{2}\right)\cos\left(\dfrac{A}{2}\right)\right)\left(2\sin\left(\dfrac{B}{2}\right)\cos\left(\dfrac{B}{2}\right)\right)\left(2\sin\left(\dfrac{C}{2}\right)\cos\left(\dfrac{C}{2}\right)\right)$

$\le 4\cos\left(\dfrac{A}{2}\right)\cos\left(\dfrac{B}{2}\right)\cos\left(\dfrac{C}{2}\right)$

$\Rightarrow \therefore \sin\left(\dfrac{A}{2}\right)\sin\left(\dfrac{B}{2}\right)\sin\left(\dfrac{C}{2}\right) \le \dfrac{1}{8}$

105. We know that, in any triangle

$$\frac{s^2}{3\sqrt{3}} \ge \Delta \qquad \text{...(i)}$$

Now, $\dfrac{a^2 + b^2 + c^2}{3} \ge \left(\dfrac{a + b + c}{3}\right)^2 = \left(\dfrac{2s}{3}\right)^2$

$= \dfrac{4}{9}s^2 \ge \dfrac{4}{9} \times 3\sqrt{3}\Delta$ [from Eq. (i)]

$\therefore \quad \dfrac{a^2 + b^2 + c^2}{\Delta} \ge 4\sqrt{3}$

106. $\because A$, C_1, G and B_1 are concyclic then

$BG \cdot BB_1 = BC_1 \cdot BA$

$\Rightarrow \quad \dfrac{2}{3}BB_1 \cdot BB_1 = \dfrac{C}{2} \cdot C$

$\Rightarrow \quad \dfrac{2}{3}(BB_1)^2 = \dfrac{c^2}{2}$

$\Rightarrow \quad \dfrac{2}{3}\left(\dfrac{2a^2 + 2c^2 - b^2}{4}\right) = \dfrac{c^2}{2}$

$\Rightarrow \quad 2a^2 + 2c^2 - b^2 = 3c^2 \quad \text{or} \quad 2a^2 = b^2 + c^2$

$\therefore \quad c^2, a^2, b^2$ are in AP.

107. $R_1R_2 + R_2R_3 + R_1R_3 = 3R^2 \qquad [\because R_1 = R_2 = R_3 = R]$

108. $AH = 2R\cos A$

$BH = 2R\cos B$

$\Delta = \dfrac{1}{2}(AH)(BH)\sin(180^\circ - C)$

$\Delta = 2R^2\cos A\cos B\sin C$

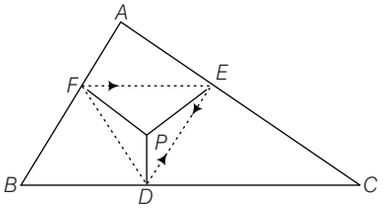
109. $2R^2 \cos A \cos B \sin C : 2R^2 \cos B \cos B \cos C \sin C$
$= \cos A : \cos B \cos C$

110. Let $DE = DF$ then

$BDPE$ is cyclic with BP as diameter

$$\Rightarrow \qquad BP = \frac{DE}{\sin B}$$

$$\Rightarrow \qquad DF = BP \sin B \text{ and } DE = CP \sin C$$



But $DE = DF$

$$\Rightarrow \qquad \frac{BP}{CP} = \frac{\sin C}{\sin B} = \frac{c}{b}$$

Hence, locus of P is arcs of 3 circles

111. $\frac{PB}{PC} = \frac{c}{b} = \frac{5}{7}$,

Hence, P will lie on circle (C).

112. For $DE = DF = EF$, P is a point(s) where the three arcs intersect

113. ΔDEF is the pedal triangle of ΔABC.

Sides of the Pedal triangle are
$a \cos A, b \cos B, c \cos C$ i.e. $R \sin 2A$, $R \sin 2B$, $R \sin 2C$.

$$\therefore \text{Required ratio } = \frac{R^3 \sin 2A \sin 2B \sin 2C}{8R^3 \sin A \sin B \sin C} = \cos A \cos B \cos C$$

114. Also, orthocentre of ΔABC is incentre of DEF.

115. Circum-radius of pedal triangle is $\frac{R}{2}$.

Sol. (Q.Nos. 116 to 118)

$$\cos(A - B) = \frac{4}{5} \Rightarrow \frac{1 - \tan^2 \frac{A-B}{2}}{1 + \tan^2 \frac{A-B}{2}} = \frac{4}{5}$$

$$\Rightarrow \qquad \frac{2 \tan^2 \frac{A-B}{2}}{2} = \frac{1}{9}$$

$$\Rightarrow \qquad \tan \frac{A-B}{2} = \frac{1}{3} \Rightarrow \frac{1}{3} = \frac{6-3}{6+3} \cot \frac{C}{2}$$

$$\Rightarrow \qquad \cos \frac{C}{2} = 1 \Rightarrow C = 90°$$

Area of triangle $= \frac{1}{2} ab \sin C \Rightarrow$ Area $= \frac{1}{2} \times 6 \times 3 \times 1 = 9$

$$\frac{a}{\sin A} = \frac{\sqrt{a^2 + b^2}}{1} \Rightarrow \frac{6}{\sin A} = \sqrt{45} \Rightarrow \sin A = \frac{2}{\sqrt{5}}$$

116. (b)

117. (b)

118. (b)

119. $\because \cos A = \frac{b^2 + c^2 - a^2}{2bc}$ or $c^2 - 2bc \cos A + b^2 - a^2 = 0$

For real roots , $D > 0$

$$\Rightarrow \qquad 4b^2 \cos^2 A - 4 \cdot 1 \cdot (b^2 - a^2) > 0$$

$$\Rightarrow \qquad a^2 > b^2 \sin^2 A$$

$$\therefore \qquad b \sin A < 0$$

Also, $c_1 + c_2 = 2b \cos A$ ...(i)

and $c_1 c_2 = b^2 - a^2$ ...(ii)

Then, $|c_1 - c_2| = \sqrt{(c_1 + c_2)^2 - 4c_1 c_2}$

$$= \sqrt{4b^2 \cos^2 A - 4(b^2 - a^2)}$$

$$= 2\sqrt{(a^2 - b^2 \sin^2 A)} \qquad \text{...(iii)}$$

Consider smaller root, say c_2

Then, $2c_2 = (c_1 + c_2) - (c_1 - c_2)$

$$= 2b \cos A - 2\sqrt{(a^2 - b^2 \sin^2 A)}$$

$$\Rightarrow \qquad c_2 = b \cos A - \sqrt{(a^2 - b^2 \sin^2 A)} > 0$$

$$\Rightarrow \qquad b \cos A > \sqrt{(a^2 - b^2 \sin^2 A)}$$

$$\Rightarrow \qquad b^2 \cos^2 A > a^2 - b^2 \sin A$$

$$\Rightarrow \qquad b^2 > a^2 \quad \text{or} \quad b > a$$

Hence, two different triangle are possible if $b \sin A < a$ and $b > a$

120. From Eq. (iii), $|c_1 - c_2| = 2\sqrt{(a^2 - b^2 \sin^2 A)}$

121. $c_1^2 - 2c_1 c^2 \cos 2A + c_2^2 = (c_1 + c_2)^2 - 2c_1 c_2 (1 + \cos 2A)$

$$= (c_1 + c_2)^2 - 4c_1 c_2 \cos^2 A$$

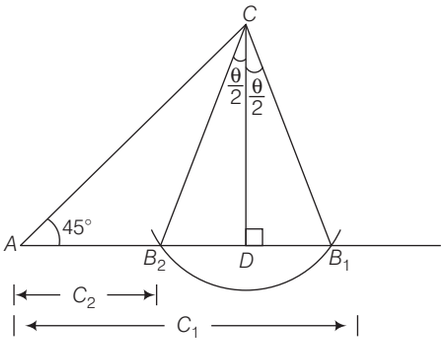
$$= (2b \cos A)^2 - 4(b^2 - a^2) \cos^2 A$$

[from Eqs. (i) and (ii)]

$$= 4 \cos^2 A \{b^2 - (b^2 - a^2)\}$$

$$= 4a^2 \cos^2 A$$

122. $B_1 D = B_2 D = \left(\frac{c_1 - c_2}{2}\right)$



$$\therefore \qquad AD = AB_2 + B_2 D$$

$$= c_2 + \left(\frac{c_1 - c_2}{2}\right)$$

$$\therefore \qquad AD = \left(\frac{c_1 + c_2}{2}\right)$$

Now, in ΔACD, $\tan 45^\circ = \dfrac{CD}{AD} = 1$

$\therefore \quad CD = AD = \left(\dfrac{c_1 + c_2}{2}\right)$

$\therefore$ In ΔB_2CD, $\tan\left(\dfrac{\theta}{2}\right) = \dfrac{B_2D}{CD} = \dfrac{\left(\dfrac{c_1 - c_2}{2}\right)}{\left(\dfrac{c_1 + c_2}{2}\right)} = \left(\dfrac{c_1 - c_2}{c_1 + c_2}\right)$

$\therefore \quad \cos\theta = \dfrac{1 - \tan^2\left(\dfrac{\theta}{2}\right)}{1 + \tan^2\left(\dfrac{\theta}{2}\right)} = \dfrac{1 - \left(\dfrac{c_1 - c_2}{c_1 + c_2}\right)^2}{1 + \left(\dfrac{c_1 - c_2}{c_1 + c_2}\right)^2}$

$= \dfrac{2c_1c_2}{(c_1^2 + c_2^2)}$

123. $\because \cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$c^2 - 2bc\cos A + b^2 - a^2 = 0$

$\Rightarrow c^2 - 2(m+1)\lambda c\cos A + (m^2 + 2m - 3)\lambda^2 = 0$

$\left(\because \dfrac{a}{2} = \dfrac{b}{m+1} = \lambda \text{ (say)}\right)$

$\therefore \quad c_1 + c_2 = 2(m+1)\lambda\cos A$

or $\quad (c_1 + c_2)^2 = 4(m+1)^2\lambda^2 \cdot \dfrac{(m-1)(m+3)}{4m}$

$= \dfrac{(m+1)^2(m-1)(m+3)}{m}$ ...(iv)

and $(c_1 - c_2)^2 = (c_1 + c_2)^2 - 4c_1c_2$

$= 4(m+1)^2\lambda^2\cos^2 A - 4(m^2 + 2m - 3)\lambda^2$

$= 4\lambda^2\left\{(m+1)^2 \cdot \dfrac{(m-1)(m+3)}{4} - (m+3)(m-1)\right\}$

$= 4\lambda^2\dfrac{(m-1)(m+3)}{4m}(m-1)^2$

$= \dfrac{\lambda^2(m-1)^2(m+3)}{m}$ ...(v)

From Eqs. (iv) and (v),

$\left(\dfrac{c_1 + c_2}{c_1 - c_2}\right)^2 = \left(\dfrac{m+1}{m-1}\right)^2$

$\Rightarrow \quad \dfrac{c_1 + c_2}{c_1 - c_2} = \pm\dfrac{m+1}{m-1}$

$= \dfrac{m+1}{m-1}$ or $\dfrac{m+1}{1-m}$

At last using componendo and dividendo rule, we get

$\dfrac{2c_1}{2c_2} = \dfrac{2m}{2}$ or $\dfrac{2}{m} = m$ or $\dfrac{1}{m}$

Alternate method :

We have, $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$\Rightarrow \quad 2bc\cos A = b^2 + c^2 - a^2$

$\Rightarrow \quad (m+1)ac\cos A = \dfrac{(m+1)^2}{4}a^2 + c^2 - a^2$

$(\because 2b = (m+1)a)$

$\Rightarrow \quad (m+1)ac\cos A = \{(m+1)^2 - 4\}\dfrac{a^2}{4} + c^2$

$mac\cos A + ac\cos A = (m-1)(m+3)\dfrac{a^2}{4} + c^2$

$= ma^2\cos^2 A + c^2$

$\left\{\because \dfrac{(m-1)(m+3)}{4} = m\cos^2 A\right\}$

$\Rightarrow \quad ma\cos A(c - a\cos A) - c(c - a\cos A) = 0$

$\Rightarrow \quad (c - a\cos A)(ma\cos A - c) = 0$

$\therefore \quad c = a\cos A$ and $c = ma\cos A$

If $\quad c_2 = a\cos A$ and $c_1 = ma\cos A$

$\therefore \quad \dfrac{c_1}{c_2} = m$

and if $c_1 = a\cos A$ and $c_2 = ma\cos A$

Then, $\quad \dfrac{c_1}{c_2} = \dfrac{1}{m}$

$\therefore \quad \dfrac{c_1}{c_2} = m$ or $\dfrac{1}{m}$

124. $\dfrac{bx}{c} + \dfrac{cy}{a} + \dfrac{az}{b} = b\sin B + c\sin C + a\sin A$

$= \dfrac{b^2 + c^2 + a^2}{2R}$

$\therefore \quad k = 2R$

Hence, (c) is the correct answer.

125. $\dfrac{1}{x^2} + \dfrac{1}{y^2} + \dfrac{1}{z^2} = \dfrac{a^2}{4\Delta^2} + \dfrac{b^2}{4\Delta^2} + \dfrac{c^2}{4\Delta^2} = \dfrac{a^2 + b^2 + c^2}{4\Delta^2}$

$\cot A + \cot B + \cot C = \dfrac{R}{abc}$

$(b^2 + c^2 - a^2 + c^2 + a^2 - b^2 + a^2 + b^2 - c^2)$

$= \dfrac{R}{abc}(b^2 + c^2 + a^2)$

$= \dfrac{R}{abc}\left(\dfrac{4\Delta^2}{x^2} + \dfrac{4\Delta^2}{y^2} + \dfrac{4\Delta^2}{z^2}\right)$

$= \dfrac{4\Delta^2 R}{abc}\left(\dfrac{1}{x^2} + \dfrac{1}{y^2} + \dfrac{1}{z^2}\right)$

$= \dfrac{4\Delta R}{abc} \cdot \Delta\left(\dfrac{1}{x^2} + \dfrac{1}{y^2} + \dfrac{1}{z^2}\right)$

$= \Delta\left(\dfrac{1}{x_2} + \dfrac{1}{y^2} + \dfrac{1}{z^2}\right)$

$\therefore \quad k = \Delta$

Hence, (c) is the correct answer.

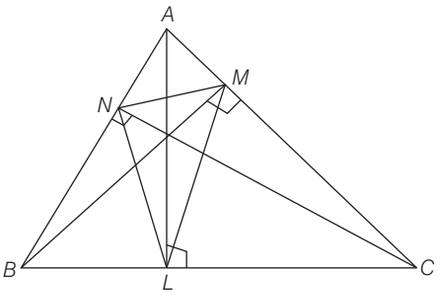
126. $\Sigma\dfrac{c\sin B + b\sin C}{x}, \Sigma\dfrac{x + x}{x} = 6$

Hence, (d) is the correct answer.

Sol. (Q.Nos 127 to 131)

Here, ΔMNL is pedal triangle, then

$MN = a\cos A$



$NL = b\cos B$

and $ML = c\cos C$

and $\angle MLN = \pi - 2A, \angle LMN = \pi - 2B$

and $\angle MNL = \pi - 2C$

127. $\lambda = MN + NL + LM$

$= a\cos A + b\cos B + c\cos C$

$= R(\sin 2A + \sin 2B + \sin 2C)$

$= 4R\sin A\sin B\sin C$

$= 4R\left(\frac{a}{2R}\right)\left(\frac{b}{2R}\right)\left(\frac{c}{2R}\right)$

$= \frac{abc}{2R^2} = \frac{4R\Delta}{2R^2} = \frac{2\Delta}{R}$

and $\mu = 2s = \frac{2\Delta}{r}$

$\therefore \quad \frac{\lambda}{\mu} = \frac{r}{R}$

128. In ΔABM,

$\cos A = \frac{AM}{AB} = \frac{AM}{c}$

$\therefore \quad AM = c\cos A$ similarly $AN = b\cos A$

$\therefore \quad \Delta_1 =$ Area of ΔAMN

$= \frac{1}{2}\cdot AM\cdot AN\cdot\sin A$

$= \frac{1}{2}(c\cos A)(b\cos A)(\sin A)$

$= \left(\frac{1}{2}bc\sin A\right)\cos^2 A = \Delta\cos^2 A$

$= \Delta\cos^2 A$

Similarly, $\Delta_2 = \Delta\cos^2 B$ and $\Delta_3 = \Delta\cos^2 C$

$\therefore \Delta_1 + \Delta_2 + \Delta_3 = \Delta(\cos^2 A + cos^2 B + \cos^2 C)$

$= \Delta(1 - 2\cos A\cos B\cos C)$ (from identity)

129. Δ' = Area of ΔLMN

$= \frac{1}{2}\cdot ML\cdot NL\cdot\sin(\pi - 2A)$

$= \frac{1}{2}\cdot(c\cos C)(b\cos B)(\sin 2A)$

$= \frac{1}{2}bc\cos B\cos C(2\sin A\cos A)$

$= \left(\frac{1}{2}bc\sin A\right)(2\cos A\cos B\cos C)$

$\therefore \quad \Delta' = \Delta(2\cos A\cos B\cos C)$

$\Rightarrow \quad \frac{\Delta'}{\Delta} = 2\cos A\cos B\cos C$

130. Let R' be the circumradius of ΔLMN, then $R' = \frac{MN}{2\sin(\angle MLN)}$

$= \frac{a\cos A}{2\sin(\pi - 2A)}$

$= \frac{2R\sin A\cos A}{2\sin 2A} = \frac{R\sin 2A}{2\sin 2A} = \frac{R}{2}$

131. $\because r' = 4R'\sin\left(\frac{L}{2}\right)\sin\left(\frac{M}{2}\right)\sin\left(\frac{N}{2}\right)$

Here, $L = \angle MLN = \pi - 2A,$

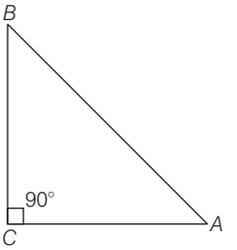
$M = \angle LMN = \pi - 2B$

and $N = \angle MNL = \pi - 2C$

and $R' = \frac{R}{2}$ (from Q. 4)

132. (A) $\to$ (r), (B) $\to$ (p), (C) $\to$ (s), (D) $\to$ (q)

(A) $2(r + R) = 2\left((s - c)\tan\frac{C}{2} + \frac{c}{2}\right)$



$2\left(s - \frac{c}{2}\right) = a + b$

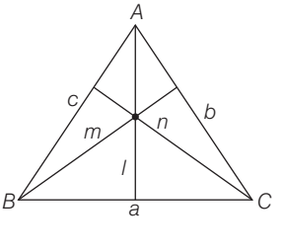
(B) $\sin C = \frac{l}{b}$ ...(i)

$\sin B = \frac{n}{a}$ ...(ii)

$\sin A = \frac{m}{c}$ ...(iii)

$2R\left(\frac{bl}{c}\right) + 2R\left(\frac{cm}{a}\right) + 2R\left(\frac{an}{b}\right)$

$\frac{c}{\sin c}\frac{bl}{c} + \frac{a}{\sin A}\frac{cm}{a} + \frac{b}{\sin B}\frac{an}{b}$



From Eqs. (i), (ii), (iii)

$$2R\left(\frac{bl}{c}+\frac{cm}{a}+\frac{an}{a}\right)=a^2+b^2+c^2$$

$$\therefore \quad \sqrt{2R\left(\frac{bl}{c}+\frac{cm}{a}+\frac{an}{b}\right)+2ab+2bc+2ac}=a+b+c$$

(C) $R=\dfrac{b}{2\sin B}=\dfrac{c}{2\sin C}$

Now, $Rb^2\sin 2C+Rc^2\sin 2B$

$=b^2c\cos C+Rc^2b\cos B$

$=bc(b\cos C+c\cos B)$

abc

(D) We have,

$$4e\sin\left(\frac{A+B}{2}\right)\sin\frac{A-B}{2}$$

$$=2R\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}\right)$$

$$=2R(\cos B-\cos A)=C\left(\frac{a}{c}-\frac{b}{c}\right) \qquad \left[\because R=\frac{C}{2}\right]$$

$$=a-b$$

133. (A) $\to$ (p, q, r); (B) $\to$ (p, q, r); (C) $\to$ (q, r); (D) $\to q$

(A) $c^2+a^2-2ac=b^2-ac$

$$\Rightarrow \quad \frac{c^2+a^2-b^2}{2ac}$$

$$=\frac{1}{2}\Rightarrow\cos B=\frac{1}{2}\Rightarrow B=60^\circ \text{ and } \cos B+\sin C=\frac{3}{2}$$

$\Rightarrow \sin C=1\Rightarrow C=90^\circ$ and $A=30^\circ$

(B) $A+B+C=180^\circ$, $B=\dfrac{A+C}{2}$

$\Rightarrow B=60^\circ$, $C+A=120^\circ$

$C=3A\Rightarrow A=30^\circ$, $C=90^\circ$

(C) Length of the bisector of angle B is

$$\frac{2ca}{c+a}\cos(B/2)$$

$$=\frac{\sqrt{3ca}}{c+a}$$

$$\Rightarrow \quad \cos\left(\frac{B}{2}\right)=\frac{\sqrt{3}}{2}=\cos 30^\circ$$

$\Rightarrow B=60^\circ$, $a=b\Rightarrow A=B=60^\circ=C$

(D) We have, $(a+b+c)(a+2b+c)$

$=3(a+b)(b+c)$

$$\Rightarrow \quad (a+c)^2+3b(a+c)+2b^2$$

$=3(ab+bc+ca+b^2)$

$$\Rightarrow \quad \frac{a^2+c^2-b^2}{ac}=1$$

$$\Rightarrow \quad \cos B=\frac{1}{2}$$

$$\Rightarrow \quad B=60^\circ$$

134. (A) $\to$ (p), (B) $\to$ (q), (C) $\to$ (r), (D) $\to$ (s)

(A) $(b+c)^2-a^2=\lambda bc$

or $b^2+c^2-a^2=(\lambda-2)bc$

$$\frac{(b^2+c^2-a^2)}{2bc}=\frac{\lambda-2}{2}$$

$$\cos A=\frac{\lambda-2}{2}<1$$

or $\lambda-2<2$

$\lambda<4$

or $\lambda=3$

(B) $\tan A+\tan B+\tan C=C=9$ in any triangle

$\tan A+\tan B+\tan C=\tan A\tan B\tan C$

$$\frac{\tan^2 A+\tan^2 B+\tan^2 C}{3}\geq 2(\tan A\tan B\tan C)^{2/3}$$

$k\geq 3(9)^{2/3}$

$k\geq 9\cdot 3^{1/3}$

(C) Since, the line joining the circumcentre to the incentre is parallel to BC

$\therefore \quad r=R\cos A$

$$\therefore \quad 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=R\cos A$$

$\therefore \quad -1+\cos A+\cos B+\cos C=\cos A$

$\therefore \quad \cos B+\cos C=1$

(D) $a=5$, $b=4$

$$\cos(A-B)=\frac{31}{32}$$

$$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}$$

$$=\frac{1}{9}\cot\frac{C}{2}$$

$$\cos(A-B)=\frac{1-\tan^2\frac{A-B}{2}}{1+\tan^2\frac{A-B}{2}}$$

$$\frac{31}{32}=\frac{1-\frac{1}{81}\cot^2\frac{C}{2}}{1+\frac{1}{81}\cot^2\frac{C}{2}}$$

$$31+\frac{31}{81}\cot^2\frac{C}{2}=32-\frac{32}{81}\cot^2\frac{C}{2}$$

$$\frac{7}{9}\cot^2\frac{C}{2}=1$$

$$\cot^2\frac{C}{2}=\frac{9}{7}=\tan^2\frac{C}{2}=\frac{7}{9}$$

$$\cos C=\frac{1-\tan^2\frac{C}{2}}{1+\tan^2\frac{C}{2}}=\frac{1-\frac{7}{9}}{1+\frac{7}{9}}=\frac{1}{8}$$

$$\because \quad C=\sqrt{a^2+b^2-2ab\cos C}$$

$=\sqrt{25+16-5}=6$

135. (A) $\to (p, t)$; (B) $\to (q, r)$; (C) $\to (q, r, t)$

(A) $\because 2a^2 + b^2 + c^2 = 2ac + 2ab$

$\Rightarrow \quad (a^2 + b^2 - 2ab) + (a^2 + c^2 - 2ac) = 0$

$$(a - b)^2 + (a - c)^2 = 0$$

Which is possible only when

$$a - b = 0, a - c = 0$$

$\therefore \quad a = b = c$

$\Rightarrow \Delta ABD$ is equilateral

$\therefore \quad \angle A = \angle B = \angle C = 60°$

(B) $\because a^2 + b^2 + c^2 = b\sqrt{2}(c + a)$

$\Rightarrow \quad a^2 + b^2 + c^2 - bc\sqrt{2} - ab\sqrt{2} = 0$

$\Rightarrow \quad 2a^2 + 2b^2 + 2c^2 - 2bc\sqrt{2} - 2ab\sqrt{2} = 0$

$(b^2 - 2bc\sqrt{2} + 2c^2) + (b^2 - 2ab\sqrt{2} + 2a^2) = 0$

$\Rightarrow \quad (b - c\sqrt{2})^2 + (b - a\sqrt{2})^2 = 0$

Which is the possible only when

$$b - c\sqrt{2} = 0, b - a\sqrt{2} = 0$$

$\therefore \quad b = c\sqrt{2}, b = a\sqrt{2}$

i.e. $\quad b^2 + b^2 = (c\sqrt{2})^2 + (a\sqrt{2})^2$

$\Rightarrow \quad c^2 + a^2 = b^2$

$\therefore \quad \cos B = \dfrac{c^2 + a^2 - b^2}{2ac} = 0$

$\therefore \quad \angle B = 90°$ also $c\sqrt{2} = a\sqrt{2}$

$\Rightarrow \quad c = a$

Then, $\quad \angle A = \angle C = 45°$

Hence, $\quad \angle A = \angle = 45°$

and $\quad \angle B = 90° (Q, S)$

(C) $\because \quad a^2 + b^2 + c^2 = bc + ca\sqrt{3}$

$\Rightarrow \quad a^2 + b^2 + c^2 - bc - ca\sqrt{3} = 0$

$\Rightarrow \quad \left(\dfrac{c\sqrt{3}}{2} - a\right)^2 + \left(\dfrac{c}{2} - b\right)^2 = 0$

Which is possible only when

$$\frac{c\sqrt{3}}{2} - a = 0 \text{ and } \frac{c}{2} - b = 0$$

$\therefore \quad a = \dfrac{c\sqrt{3}}{2}$ and $b = \dfrac{c}{2}$

Then, $\quad a^2 + b^2 = \dfrac{3c^2}{4} + \dfrac{c^2}{4} = c^2$

$\because \quad \cos C = \dfrac{a^2 + b^2 - c^2}{2ab} = 0$

$\because \quad \angle C = 90°$

Also, $\quad \dfrac{a}{\sin A} = \dfrac{c}{\sin C} \Rightarrow \dfrac{\frac{c\sqrt{3}}{2}}{\sin A} = \dfrac{c}{1}$

$\therefore \quad \sin A = \dfrac{\sqrt{3}}{2} = \sin 60°$

$\Rightarrow \quad \angle A = 60°, \angle B = 30°, \angle C = 90°$

$\because \quad 2C = A + B$

136. Applying cosine rule, $2ab \cos C = a^2 + b^2 - c^2$

$\Rightarrow \quad 2\sqrt{2}\sqrt{2 + \sqrt{2}} \cos\dfrac{\pi}{8} = 2 + 2 + \sqrt{2} - c^2$

Using $\cos\dfrac{\pi}{8} = \dfrac{\sqrt{2 + \sqrt{2}}}{2}$

$\Rightarrow \quad \dfrac{2\sqrt{2}(2 + \sqrt{2})}{2} = 4 + \sqrt{2} - c^2$

$\Rightarrow \quad c^2 = 2 - \sqrt{2} \Rightarrow c = \sqrt{2 - \sqrt{2}}$

Applying sine rule, $\dfrac{\sin C}{c} = \dfrac{\sin A}{a}$

$\Rightarrow \quad \dfrac{\sqrt{2 - \sqrt{2}}}{2\sqrt{2 - \sqrt{2}}} = \dfrac{\sin A}{\sqrt{2}}$

$\Rightarrow \quad \sin A = \dfrac{1}{\sqrt{2}} \Rightarrow A = 45°$

[$\because b$ is the longest side $\therefore \angle B$ will be the greatest $\therefore A \neq 135°$)]

Sum of the digits is $= 4 + 5 = 9$

137. Using sine law, $\dfrac{125}{\sin x} = \dfrac{195}{\sin 3x} = \dfrac{c}{\sin 4x}$

$$3 - 4\sin^2 x = \frac{195}{125} = \frac{39}{25}$$

$\Rightarrow \quad 4\sin^2 x = 3 - \dfrac{39}{25} = \dfrac{75 - 39}{25} = \dfrac{36}{25}$

$$\sin^2 x = \frac{9}{25} \Rightarrow \sin x = \frac{3}{5}$$

$\therefore \quad c = \dfrac{125 \sin 4x}{\sin x} = 125\dfrac{2\cos 2x\, 2\sin x \cos x}{\sin x}$

$= (125)(4)(\cos x)(1 - 2\sin^2 x)$

$= (500)\left(\dfrac{4}{5}\right)\left(1 - \dfrac{18}{25}\right) = 400\left(\dfrac{7}{25}\right)$

$= (16)(7) = 112$

$\therefore$ Number of digits in the length of side $= 3$

138. $a^2 + b^2 + c^2 = 2R^2(1 - \cos 2A + 1 - \cos 2B + 1 - \cos 2C)$

$= 2R^2\left[3 - (\cos\dfrac{2\pi}{7} + \cos\dfrac{4\pi}{7} + \cos\dfrac{6\pi}{7}\right]$

$= 2R^2\left(3 - \left(\dfrac{-1}{2}\right)\right) = 7R^2$

$\therefore \quad \dfrac{a^2 + b^2 + c^2}{R^2} = 7$

139. $D = 2\tan B \tan B \tan C (\tan A + \tan B + \tan C)^3$

$= 2(\tan A + \tan B + \tan C)^4$

$\geq 2(3\sqrt{3})^4 = 1458 = \dfrac{D}{1000} \geq 1.458$

$\therefore$ Least integer value is 2.

140. Clear $PQ = \dfrac{a}{2}, OA = R$

and $\quad \angle AOQ = B$

$OA \cos B = R \cos B$

now $\quad \angle AOQ = C$ since $PQ \,||\, BC$

$$\Rightarrow \quad |PQO = \frac{\pi}{2} - C$$

$$\Rightarrow \quad \text{Area of } \Delta OPQ = \frac{1}{2} PQ \cdot OQ \sin|PQO$$

$$= \frac{aR}{4} \cos B \cos C$$

$$\text{Area of } \Delta ABC = \frac{1}{2} ab \sin c = aR \sin B \sin C$$

$$\therefore \quad \frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta OPQ} = \frac{aR \sin B \sin C}{\frac{aR}{4} \cos B \cos C}$$

$$= 4 \tan B \tan C$$

$$\therefore \quad \left(\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta OPS}\right) \cot B \cdot \cot C = 4$$

141. $$\sum \frac{a}{r_1} = \sum \frac{2R \sin\frac{A}{2} \cdot 2\cos\frac{A}{2}}{4R \sin\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}} = \sum \frac{\sin\left(\frac{B}{2} + \frac{C}{2}\right)}{\cos\frac{B}{2} \cos\frac{C}{2}}$$

$$\sum \left(\tan\frac{B}{2} + \tan\frac{C}{2}\right) 2 \sum (\tan\frac{A}{c}) = 2 \sum \frac{r_1}{s} = 2\left[\frac{r_1 + r_2 + r_3}{s}\right]$$

$$= 2\left[\frac{r_1 + r_2 + r_3}{\frac{a+b+c}{2}}\right]$$

$$= \frac{a}{r_1} + \frac{b}{r_2} + \frac{c}{r_3} = 4\left(\frac{r_1 + r_2 + r_3}{a+b+c}\right)$$

$$\Rightarrow \left(\frac{a+b+c}{r_1 + r_2 + r_3}\right)\left(\frac{a}{r_1} + \frac{b}{r_2} + \frac{c}{r_3}\right) = 4$$

142. From the ΔABC,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \qquad \ldots(i)$$

From the ΔCAD,

$$\cos C = \frac{AC}{CD} = \frac{b}{a/2} = \frac{2b}{a} \qquad \ldots(ii)$$

From the ΔABD,

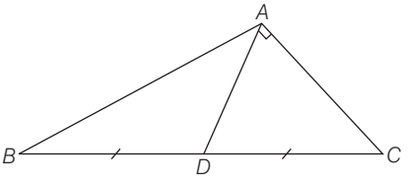
$$\frac{BD}{\sin(A - 90°)} = \frac{AB}{\sin \angle ADB}$$

or $$\frac{a/2}{-\cos A} = \frac{c}{\sin(90° + C)}$$

or $$\frac{a}{-2\cos A} = \frac{c}{\cos C}$$

$$\therefore \quad \cos A = \frac{a \cos C}{-2c}$$

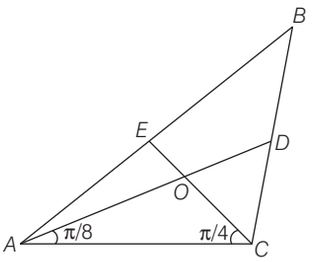
$$= \frac{a}{-2c} \cdot \frac{2b}{a} = -\frac{b}{c} \qquad \text{[From Eq. (ii)]}$$



$$\therefore \text{From Eq. (i)}, \frac{b^2 + c^2 - a^2}{2bc} = \frac{-b}{c}$$

or $$b^2 + c^2 - a^2 = -2b^2$$

or $$c^2 - a^2 = -3b^2 \qquad \ldots(iii)$$



$$\text{Now, } \cos A \cdot \cos C = \frac{b^2 + c^2 - a^2}{2bc} \cdot \frac{2b}{a}$$

$$= \frac{b^2 + c^2 - a^2}{ca}$$

$$= \frac{3b^3 + 3(c^2 - a^2)}{3ca}$$

$$= \frac{a^2 - c^2 + 3(c^2 - a^2)}{3ca} \qquad \text{[From Eq. (iii)]}$$

$$= \frac{2(c^2 - a^2)}{3ca}$$

$$\Rightarrow \quad \lambda = 2$$

143. Let O be the point of intersection of the medians of triangle ABC. Then, the area ofΔABC is three times that of ΔAOC. Now, in ΔAOC, $AO = \frac{2}{3} AD = \frac{10}{3}$. Therefore, applying the sine rule to ΔAOC, we get

$$\frac{OC}{\sin\left(\frac{\pi}{8}\right)} = \frac{AO}{\sin\left(\frac{\pi}{4}\right)}$$

$$\Rightarrow \quad OC = \frac{10}{3} \cdot \frac{\sin\left(\frac{\pi}{8}\right)}{\sin\left(\frac{\pi}{4}\right)}$$

$$\text{Area of } \Delta AOC = \frac{1}{2} \cdot AO \cdot OC \cdot \sin \angle AOC$$

$$= \frac{1}{2} \cdot \frac{10}{3} \cdot \frac{10}{3} \cdot \frac{\sin\left(\frac{\pi}{8}\right)}{\sin\left(\frac{\pi}{4}\right)} \cdot \sin\left(\frac{\pi}{2} + \frac{\pi}{8}\right)$$

$$= \frac{50}{9} \cdot \frac{\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)}{\sin\left(\frac{\pi}{4}\right)} = \frac{50}{18} = \frac{25}{9}$$

$$\therefore \text{Area of } \Delta ABC = 3 \cdot \frac{25}{9} = \frac{25}{3} = \frac{5a}{b}$$

$$\Rightarrow \quad a = 5 \text{ and } b = 3$$

$$\therefore \quad a + b = 5 + 3 = 8$$

144. $\dfrac{r}{r_1}=\dfrac{4R\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}}{4R\sin\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}}=\tan\frac{B}{2}\tan\frac{C}{2}=\frac{1}{2}$

Now, $\tan\frac{A}{2}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)$

$=\tan\frac{A}{2}\cdot\tan\frac{B}{2}+\tan\frac{A}{2}\cdot\tan\frac{C}{2}$

$=1-\tan\frac{B}{2}\cdot\tan\frac{C}{2}=1-\frac{1}{2}=\frac{1}{2}$

$\therefore\quad 16\tan\frac{A}{2}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)=\frac{1}{2}\times 16$

$=8$

145. $\dfrac{a\cos^2\frac{A}{2}+b\cos^2\frac{B}{2}+c\cos^2\frac{C}{2}}{a+b+c}$

$=\dfrac{a(1+\cos A)+b(1+\cos B)+c(1+\cos C)}{2(a+b+c)}$

$=\dfrac{(a+b+c)+(a\cos A+b\cos b+c\cos C)}{2(a+b+c)}$

$=\dfrac{(a+b+c)+(a\cos A+b\cos B+c\cos C)}{2\,(a+b+c)}$

$=\dfrac{1}{2}+\dfrac{a\cos A+b\cos B+c\cos C}{4s}$

$=\dfrac{1}{2}+\dfrac{R}{4s}(\sin 2A+\sin 2B+\sin 2C)$

$\left[\because \dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}=2R\right]$

$=\dfrac{1}{2}+\dfrac{R}{4s}(4\sin A\sin B\sin C)$

$=\dfrac{1}{2}+\dfrac{R}{4s}\times\left(4\cdot\dfrac{a}{2R}\cdot\dfrac{b}{2R}\cdot\dfrac{c}{2R}\right)$

$=\dfrac{1}{2}+\dfrac{abc}{8R^2s}=\dfrac{1}{2}+\dfrac{4R\Delta}{8R^2s}\ \dfrac{1}{2}+\dfrac{r}{2R}$

$=\dfrac{1}{2}\left(1+\dfrac{r}{R}\right)\le\dfrac{1}{2}\left(1+\dfrac{1}{2}\right)\qquad\left[\because \dfrac{r}{R}\le\dfrac{1}{2}\right]$

Hence, $\dfrac{a\cos^2\frac{A}{2}+b\cos^2\frac{B}{2}+c\cos^2\frac{C}{2}}{a+b+c}\le\dfrac{3}{4}$

$\therefore\quad 12\times\dfrac{\left(a\cos^2\frac{A}{2}+b\cos^2\frac{B}{2}+c\cos^2\frac{C}{2}\right)}{a+b+c}\le 12\times\dfrac{3}{4}=9$

146. Let $AB=n$, $AC=n+1$, $BC=n+2$

Further,

Let $A=2C$ (since AB is the smallest and BC is largest)

By sine rule, we have

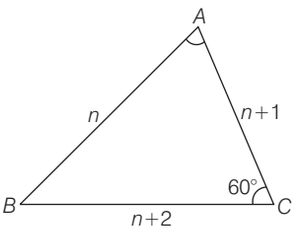
$\dfrac{\sin A}{n+2}=\dfrac{\sin B}{n+1}=\dfrac{\sin C}{n}$

$\because\quad A=2C$

and $\quad B=180^\circ-(A+3C)$

$\Rightarrow\quad B=180^\circ-3C$

$\therefore\quad \dfrac{\sin 2C}{n+2}=\dfrac{\sin 3C}{n+1}=\dfrac{\sin C}{n}$



$\Rightarrow\quad \dfrac{2\cos C}{n+2}=\dfrac{3-4\sin^2 C}{n+1}=\dfrac{1}{n}$

$\therefore\quad \cos C=\dfrac{n+2}{2n}$ and $\sin^2 C=\dfrac{2n-1}{4n}$

$\therefore\quad \left(\dfrac{n+2}{2n}\right)^2+\left(\dfrac{2n-1}{4n}\right)=1$

$\Rightarrow\quad n^2-3n-4=0$

$\therefore\quad n=4,\ n=-1$

$\Rightarrow\quad n=4,\ n\ne -1$

Then, sides are 4, 5, 6.

$\therefore$ Largest side is 6.

147. $A+B+C=\pi$

Given, $\quad C=2A$

$\Rightarrow\quad B=\pi-3A$

As $0<C<\pi\Rightarrow 0<2A<\pi\Rightarrow 0\ A<\dfrac{\pi}{2}$

By sine rule, $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}$

or $\quad \dfrac{a}{\sin A}=\dfrac{2a}{\sin(\pi-3A)}$

or $\quad \dfrac{a}{1}=\dfrac{2a}{3-4\sin^2 A}$

or $\quad 3-4\sin^2 A=2$

or $\quad \sin^2 A=\dfrac{1}{4}\ \text{ or }\ \sin A=\dfrac{1}{2}$

or $\quad A=\dfrac{\pi}{6}$ or $\dfrac{5\pi}{6}$

But $\quad 0<A<\dfrac{\pi}{2}$

$\Rightarrow\quad A=\dfrac{\pi}{6},\ \angle B=\dfrac{\pi}{2}$ and $\angle C=\dfrac{\pi}{3}$

$\Rightarrow\quad a^2+b^2+c^2$

$=4R^2[\sin^2 A+\sin^2 B+\sin^2 C]$

$=4R^2\left[\sin^2\dfrac{\pi}{6}+\sin^2\dfrac{\pi}{2}+\sin^2\dfrac{\pi}{3}\right]$

$=4R^2\left[\dfrac{1}{4}+1+\dfrac{3}{4}\right]=8R^2$

or $\quad \dfrac{a^2+b^2+c^2}{R^2}=8$

148. We have, $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$\Rightarrow c^2 - 2b\cos A + b^2 - a^2 = 0$, where is quadratic in 'c'

$$\left.\begin{aligned} \therefore\ c_1 + c_2 &= 2b\cos A \\ \text{and } c_1c_2 &= b^2 - a^2 \end{aligned}\right\} \quad \text{...(i)}$$

$\therefore \quad c_1^2 + c_2^2 - 2c_1c_2\cos 2A$

$\Rightarrow \quad (c_1 + c_2)^2 - 2c_1c_2 - 2c_1c_2\cos 2A$ [Using Eq. (i)]

$\Rightarrow \quad (c_1 + c_2)^2 - 2c_1c_2(1 + \cos 2A)$

$\Rightarrow \quad 4b^2\cos^2 A - 2(b^2 - a^2)\cdot 2\cos^2 A$

$= 4a^2\cos^2 A$

$\therefore \quad c_1^2 + c_2^2 - 2c_1c_2\cos A$

$= 4a^2\cos^2 A$

Hence, the value of $\lambda = 4$

149. $AG = \dfrac{2}{3}AA_1$, $BG = \dfrac{2}{3}BB_1$

$\Rightarrow \quad AG = \dfrac{2}{3}\sqrt{2b^2 + 2c^2 - a^2}$

$BG = \dfrac{2}{3}\sqrt{2a^2 + 2c^2 - b^2}$

$\Rightarrow \quad AG = \dfrac{2}{3}a$, $BG = \dfrac{2}{3}\sqrt{b^2 + 4c^2}$ as $a^2 = b^2 + c^2$

$\Rightarrow \quad AG = \dfrac{10}{3}$, $BG = \dfrac{2}{3}\sqrt{16 + 36} = \dfrac{4}{3}\sqrt{13}$

Also, $AB = c = 3$ and $\Delta_{GAB} = \dfrac{1}{3}\Delta_{ABC} = 2$

If R_1 be the circumradius of ΔGAB, then

$$R_1 = \frac{(AG)(BG)(AB)}{4\Delta_{GAB}}$$

$$= \frac{10}{3}\cdot\frac{4}{3}\sqrt{13}\cdot 3\cdot\frac{1}{4\cdot 2} = \frac{5\sqrt{13}}{3}\text{ unit}$$

$\therefore \quad \dfrac{9}{65}\times\dfrac{25\times 13}{9} = 5$

150. We have $\dfrac{1}{2}AD[c\cos C + b\cos B] = \Delta$

$\Rightarrow \quad \dfrac{1}{AD} + \dfrac{1}{BE} + \dfrac{1}{CF}$

$$= \frac{a\cos A + b\cos B + c\cos C}{\Delta} = 2$$

151. $\dfrac{a^3 + b^3 + c^3}{\sin^3 A + \sin^3 B + \sin^3 C} = 8$

$\Rightarrow \quad 8R^3 = 8 \Rightarrow R = 1$

Now, $a = 2R\sin A$

$a = 2\sin A$

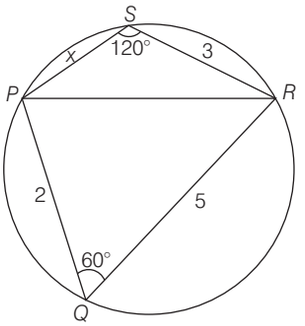
Hence, a is maximum when $\sin A$ is maximum.

152. From ΔPQR,

$$PR^2 = 2^2 + 5^2 - 2\cdot 5\cdot 2\cdot\frac{1}{2} = 19$$

In ΔPRS, $PR^2 = 3^2 + x^2 + 2\cdot 3x\cdot\dfrac{1}{2}$, where

$SP = x = x^2 + 3x + 9$



$\therefore \quad x^2 + 3x + 9 = 19$

$(x + 5)(x - 2) = 0$

$x + 5 = 0$ (not possible)

$\therefore \quad x = 2$

153. Since, A and B satisfying the given equation, therefore

$\sqrt{3}\cos A + \sin A = \sqrt{3}\cos B + \sin B$

$$\Rightarrow \quad \frac{\sin A - \sin B}{\cos B - \cos A} = \sqrt{3}$$

$$\Rightarrow \quad \frac{2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}{2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)} = \sqrt{3}$$

$$\Rightarrow \quad \tan\left(\frac{A+B}{2}\right) = \frac{1}{\sqrt{3}} = \tan\frac{\pi}{6}$$

$$\Rightarrow \quad \frac{A+B}{2} = \frac{\pi}{6}$$

$$\Rightarrow \quad A + B = \frac{\pi}{3}$$

Now, $C = \pi - \dfrac{\pi}{3} = \dfrac{2\pi}{3}$

Hence, triangle is obtuse angled.

154. We have, $a\tan\theta + b\sec\theta = c$

$\Rightarrow \quad a\sin\theta + b = c\cos\theta$

$\Rightarrow \quad a\cos\theta - a\sin\theta = b$ ...(i)

Now, let α and β are the other two angles of the triangle, then according to given condition we get

$c\cos\alpha - a\sin\alpha = c\cos\beta - a\sin\beta$

$\Rightarrow \quad c(\cos\alpha - \cos\beta) = a(\sin\alpha - \sin\beta)$

$$\Rightarrow \quad \frac{c}{a} = \frac{2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}{-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}$$

$$\Rightarrow \quad \frac{c}{a} = -\cot\left(\frac{\alpha+\beta}{2}\right)$$

$$\Rightarrow \quad \tan\left(\frac{\alpha+\beta}{2}\right) = -\frac{a}{c}$$

$$\therefore \quad \tan(\alpha+\beta)=\frac{2\tan\left(\frac{\alpha+\beta}{2}\right)}{1-\tan^2\left(\frac{\alpha+\beta}{2}\right)}=\frac{2\left(-\frac{a}{c}\right)}{1-\frac{a^2}{c^2}}=\frac{2ac}{a^2-c^2}$$

$$\Rightarrow \quad \tan(\pi-C)=\frac{2ac}{a^2-c^2}$$

$$\Rightarrow \quad \tan\left(\pi-\frac{3\pi}{4}\right)=\frac{2ac}{a^2-c^2} \qquad \left[\because c=\frac{3\pi}{4}\right]$$

$$\Rightarrow \quad \tan\frac{\pi}{4}=\frac{2ac}{a^2-c^2}$$

$$\Rightarrow \quad a^2-c^2=2ac$$

155. Here, the quadratic for third side b is given by

$$b^2-2bc\cos A+(c^2-a^2)=0$$

$$\therefore \quad b_1+b_2=2c\cos A \qquad \ldots(i)$$

and $$b_1 b_2=c^2-a^2 \qquad \ldots(ii)$$

also, $$b_2=2b_1 \qquad \ldots(iii)$$

From Eqs. (i) and (iii), $3b_1=2c\cos A$ $\ldots(iv)$

and from Eqs. (ii) and (iv),

$$2b_1^2=c^2-a^2 \qquad \ldots(v)$$

From Eqs. (iv) and (v), we have

$$\frac{2\cdot 4c^2\cos^2 A}{9}=c^2-a^2$$

$$\Rightarrow \quad 8c^2(1-\sin^2 A)=9c^2-9a^2$$

$$\Rightarrow \quad 1-\sin^2 A=\frac{9(c^2-a^2)}{8c^2}$$

$$\Rightarrow \quad \sin^2 A=1-\frac{9(c^2-a^2)}{8c^2}$$

$$\Rightarrow \quad \sin A=\sqrt{\frac{8c^2-9(c^2-a^2)}{8c^2}}$$

$$\Rightarrow \quad \sin A=\sqrt{\frac{8c^2-9c^2+9a^2}{8c^2}}$$

$$\therefore \quad \sin A=\sqrt{\frac{9a^2-c^2}{8c^2}}$$

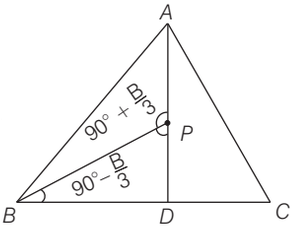
156. $\angle BPA=90^\circ+\left(\frac{B}{3}\right)$, $\angle ABP=\frac{2B}{3}$

In ΔABP, $\frac{AP}{\sin\left(\frac{2B}{3}\right)}=\frac{c}{\sin[90^\circ+B/3)]}=\frac{c}{\cos(B/3)}$ [by sine rule]

or $$AP=\frac{c\sin(2B/3)}{\cos(B/3)}$$

$$=\frac{2c\sin\left(\frac{B}{3}\right)\cos\left(\frac{B}{3}\right)}{\cos\left(\frac{B}{3}\right)}$$

$$=2c\sin\left(\frac{B}{3}\right)$$



157. Using $\frac{a}{2\sin A}=\frac{b}{2\sin B}=\frac{c}{2\sin C}=R$ we get,

$$\frac{b^2-c^2}{2aR}=\frac{4R^2(\sin^2 B-\sin^2 C)}{4R^2\sin A}$$

$$=\frac{\sin(B+C)\sin(B-C)}{\sin A}=\sin(B-C)$$

158. We know that,

or $$\text{ar}(\Delta ABC)=\frac{1}{2}bc\sin A$$

$$\Rightarrow \quad \frac{9\sqrt{3}}{2}=\frac{1}{2}bc\sin\frac{2\pi}{3}$$

$$\Rightarrow \quad 9\sqrt{3}=bc\times\frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad bc=18 \qquad \ldots(i)$$

Also, $$b-c=3\sqrt{3}$$

$$\Rightarrow \quad b=c+3\sqrt{3} \qquad \ldots(ii)$$

From Eq. (i) and Eq. (ii), we get

$$(c+3\sqrt{3})c=18$$

$$\Rightarrow \quad c^2+3\sqrt{3}c-18=0$$

$$\Rightarrow \quad c=\frac{-3\sqrt{3}\pm\sqrt{27+72}}{2}$$

$$=\frac{-3\sqrt{3}\pm\sqrt{99}}{2}=\frac{-3\sqrt{3}\pm3\sqrt{11}}{2}$$

$$\therefore \quad c=\frac{3\sqrt{11}-3\sqrt{3}}{2} \qquad [\because c>0]$$

$$\Rightarrow \quad b=\frac{3\sqrt{11}-3\sqrt{3}}{2}+3\sqrt{3}=\frac{3\sqrt{11}+3\sqrt{3}}{2}$$

$$\cos A=\frac{b^2+c^2-a^2}{2bc}$$

$$\Rightarrow \quad \frac{\sqrt{3}}{2}=\frac{b^2+c^2-a^2}{2bc}$$

$$\Rightarrow \quad \frac{\sqrt{3}}{2}=\frac{b^2+c^2-a^2}{2\times\frac{9}{4}(11-3)}$$

$$\Rightarrow \quad 2\sqrt{3}=63-a^2 \qquad [\because b^2+c^2=63]$$

$$\Rightarrow \quad a^2=63-2\sqrt{3}$$

$$\therefore \quad a=\sqrt{63-2\sqrt{3}}$$

159. We have, $$\Delta=a^2-(b-c)^2$$

$$=(a+b-c)(a-b+c)$$

$$\Rightarrow \quad \Delta^2=(a+b-c)^2(a-b+c)^2$$

$$=(2s-c-c)^2(2s-b-b)^2$$

$$=(2s-2c)^2(2s-2b)^2$$

$$=16(s-c)^2(s-b)^2$$

$$\therefore \quad s(s-a)(s-b)(s-c)=16(s-c)^2(s-b)^2$$

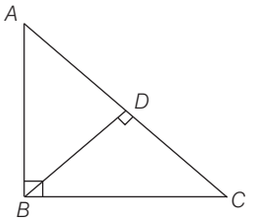
$$\Rightarrow \quad \frac{(s-b)(s-c)}{s(s-a)}=\frac{1}{16}$$

$$\Rightarrow \quad \tan^2\frac{A}{2}=\frac{1}{16}$$

$$\Rightarrow \quad \tan\frac{A}{2}=\frac{1}{4}$$

$$\because \quad \tan A=\frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}=\frac{2\times\frac{1}{4}}{1-\left(\frac{1}{4}\right)^2}=\frac{\left(\frac{1}{2}\right)}{\left(\frac{15}{16}\right)}=\frac{8}{15}$$

160. We have,



$\angle ABC=90°$, $\angle BDC=90°$, $AC=h$, $BD=p$ and $h=4p$

Let $\angle C=\theta \Rightarrow \angle A=90°-\theta$

Now, in ΔBDC

$$\frac{\sin C}{BD}=\frac{\sin D}{BC} \qquad \text{[using sine formula]}$$

$$\Rightarrow \quad \frac{\sin\theta}{p}=\frac{\sin 90°}{BC}$$

$$\Rightarrow \quad BC=\frac{p}{\sin\theta} \qquad \ldots(i)$$

Again, in ΔABC

$$\frac{\sin B}{AC}=\frac{\sin A}{BC} \Rightarrow \frac{\sin 90°}{h}=\frac{\sin(90°-\theta)}{BC}$$

$$\Rightarrow \quad BC=h\cos\theta \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we have

$$\frac{p}{\sin\theta}=h\cos\theta \Rightarrow \frac{p}{h}=\sin\theta\cos\theta$$

$$\Rightarrow \quad \frac{p}{4p}=\sin\theta\cos\theta$$

$$\Rightarrow \quad \sin\theta\cos\theta=\frac{1}{4} \Rightarrow \sin 2\theta=\frac{1}{2}$$

$$\therefore \quad 2\theta=30° \text{ or } 150°$$

$$\Rightarrow \quad \theta=15° \text{ or } 75°$$

Hence, $\angle C=15°$.

161. We have, $\sin^3 A+\sin^3 B+\sin^3 C=3\sin A\cdot\sin B\cdot\sin C$

$$\Rightarrow \quad k^3a^3+k^3b^3+\ +k^3c^3=3k^3abc$$

$$\left[\therefore \frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}=k\right]$$

$$\Rightarrow \quad a^3+b^3+c^3=3abc$$

$$\Rightarrow \quad a+b+c=0$$

$$[\because \text{if } a+b+c=0, \text{ then } a^3+b^3+c^3=3abc]$$

Now, $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

on apply, $C_1\to C_1+C_2+C_3$, we get-

$$=\begin{vmatrix} a+b+c & b & c \\ b+c+a & c & a \\ c+a+b & a & b \end{vmatrix}$$

$$=\begin{vmatrix} 0 & b & c \\ 0 & c & a \\ 0 & a & b \end{vmatrix} \qquad [\because a+b+c=0]$$

$$=0$$

162. We have,

$$a+b+c=11,\ ab+bc+ca=38 \text{ and } abc=40$$

$$\therefore \quad \frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}$$

$$=\frac{\left(\frac{b^2+c^2-a^2}{2bc}\right)}{a}+\frac{\left(\frac{c^2+a^2-b^2}{2ca}\right)}{b}+\frac{\left(\frac{a^2+b^2-c^2}{2ab}\right)}{c}$$

$$=\frac{b^2+c^2-a^2}{2abc}+\frac{c^2+a^2-b^2}{2abc}+\frac{a^2+b^2-c^2}{2abc}$$

$$=\frac{a^2+b^2+c^2}{2abc}=\frac{(a+b+c)^2-2(ab+bc+ca)}{2abc}$$

$$=\frac{(11)^2-2(38)}{2\times 40}=\frac{121-76}{80}$$

$$=\frac{45}{80}=\frac{9}{16}$$

163. We know that,

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

and $$\tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

$$\therefore \quad \tan\frac{A}{2}+\tan\frac{C}{2}=\sqrt{\frac{s-b}{s}}\left[\sqrt{\frac{s-c}{s-a}}+\sqrt{\frac{s-a}{s-c}}\right]$$

$$=\sqrt{\frac{s-b}{s}}\left[\frac{s-c+s-a}{\sqrt{s-a}\sqrt{s-c}}\right]$$

$$=\sqrt{\frac{s-b}{s}}\left[\frac{2s-a-c}{\sqrt{s-a}\sqrt{s-c}}\right]$$

$$=\sqrt{\frac{s-b}{s}}\left[\frac{b}{\sqrt{s-a}\sqrt{s-c}}\right]$$

$$=\frac{b}{s}\sqrt{\frac{s(s-b)}{(s-a)(s-c)}}$$

$$=\frac{b}{s}\cot\frac{B}{2} \qquad \left[\because \cot\frac{B}{2}=\sqrt{\frac{s(s-b)}{(s-a)(s-c)}}\right]$$

$$\therefore \quad \frac{\tan\frac{A}{2}+\tan\frac{C}{2}}{\cot\frac{B}{2}}=\frac{b}{s} \qquad \ldots(i)$$

Again, a, b, c are in AP

$$\therefore \quad a+c=2b$$

or $$a+b+c=3b \Rightarrow 2s=3b$$

$$\Rightarrow \quad \frac{b}{s} = \frac{2}{3} \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we have

$$\frac{\tan\frac{A}{2} + \tan\frac{C}{2}}{\cot\frac{B}{2}} = \frac{2}{3}$$

164. Let the sides of triangle be $a - d$, a, $a + d$.

$\therefore$ Perimeter $= a - d + a + a + d = 3a$

$\therefore$ Side of equilateral triangle having same perimeter will be a.

Semi-perimeter of given triangle

$$= \frac{a - d + a + a + d}{2} = \frac{3a}{2}$$

Now, according to question

$$\sqrt{\left(\frac{3a}{2}\right)\left(\frac{3a}{2} - a + d\right)\left(\frac{3a}{2} - a\right)\left(\frac{3a}{2} - a - d\right)}$$

$$= \frac{3}{5} \times \frac{\sqrt{3}}{4} a^2$$

$$\Rightarrow \quad \sqrt{\left(\frac{3a}{2}\right)\left(\frac{a}{2} + d\right)\left(\frac{a}{2}\right)\left(\frac{a}{2} - d\right)} = \frac{3\sqrt{3}}{20} a^2$$

$$\Rightarrow \quad = \frac{\sqrt{3}a}{2} \sqrt{\left(\frac{a^2}{4} - d^2\right)} = \frac{3\sqrt{3}}{20} a^2$$

$$\Rightarrow \quad \sqrt{\frac{a^2}{4} - d^2} = \frac{3a}{10}$$

$$\Rightarrow \quad \frac{a^2}{4} - d^2 = \frac{9a^2}{100}$$

$$\Rightarrow \quad d^2 = \frac{16}{100} a^2 \Rightarrow d = \frac{4}{10} a = \frac{2}{5} a$$

$\therefore$ Sides will be, $a - \frac{2}{5}a$, a, $a + \frac{2}{5}a$

or $\quad \frac{3a}{5}$, a, $\frac{7a}{5}$

$\therefore$ Required ration $= \frac{3a}{5} : a : \frac{7a}{5} = 3 : 5 : 7$

165. Let the sides of ΔABC are a, b, c.

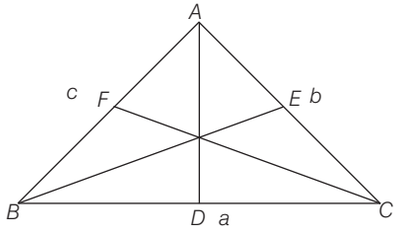
$\therefore \quad BC = a$, $CA = b$ and $AB = c$

Again, AD, BE and CF are the medians of ΔABC

$\therefore \quad AD = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$

$$BE = \frac{1}{2}\sqrt{2c^2 + 2a^2 - b^2}$$

and $\quad CF = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}$



$\therefore \quad AD^2 + BE^2 + CF^2 = \frac{1}{4}[2b^2 + 2c^2 - a^2 + 2c^2 + 2a^2 - b^2 + 2a^2 + 2b^2 - c^2]$

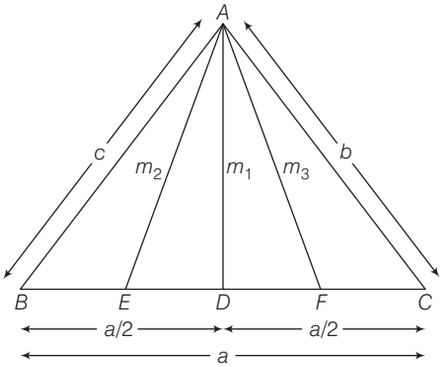
$$= \frac{1}{4}[3a^2 + 3b^2 + 3c^2]$$

$$= \frac{3}{4}[a^2 + b^2 + c^2]$$

$$= \frac{3}{4}[BC^2 + CA^2 + AB^2]$$

$\therefore \quad \frac{AD^2 + BE^2 + CF^2}{BC^2 + CA^2 + AB^2} = \frac{3}{4}$

166. We know that, in ΔABC length of median from the $\angle A$ is $\frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$



$\therefore \quad AD^2 = m_1^2 = \frac{c^2 + b^2}{2} - \frac{a^2}{4}$ ...(i)

Similarly, in ΔABD,

$$AE^2 = m_2^2 = \frac{c^2 + m_1^2}{2} - \frac{\left(\frac{a}{2}\right)^2}{4} \qquad \text{...(ii)}$$

and in ΔACD,

$$AF^2 = m_3^2 = \frac{b^2 + m_1^2}{2} - \frac{\left(\frac{a}{2}\right)^2}{4} \qquad \text{...(iii)}$$

Now, $\quad m_2^2 + m_3^2 = m_1^2 + \left(\frac{b^2 + c^2}{2}\right) - 2\left(\frac{a^2}{16}\right)$

$$= m_1^2 + m_1^2 + \frac{a^2}{4} - \frac{a^2}{8} \qquad \text{[from Eq. (i)]}$$

$$= 2m_1^2 + \frac{a^2}{8}$$

$$\Rightarrow \quad \frac{a^2}{8} = m_2^2 + m_3^2 - 2m_1^2$$

167. We know from Napier's analogy that

$$\tan\left(\frac{B - C}{2}\right) = \frac{b - c}{b + c} \cot\frac{A}{2}$$

$$\tan\left(\frac{A - B}{2}\right) = \frac{a - b}{a + b} \cot\frac{C}{2}$$

$$\tan\left(\frac{C - A}{2}\right) = \frac{c - a}{c + a} \cot\frac{B}{2}$$

So, $$X = \tan\left(\frac{B-C}{2}\right)\cdot\tan\frac{A}{2}$$

$$X = \frac{b-c}{b+c}\cdot\cot\frac{A}{2}\cdot\tan\frac{A}{2} = \frac{b-c}{b+c}$$

Similarly, $Y = \frac{a-b}{a+b}, Z = \frac{c-a}{c+a}$

Now, $X + Y + Z + XYZ$

$$= \frac{b-c}{b+c} + \frac{a-b}{a+b} + \frac{c-a}{c+a} + \left(\frac{b-c}{b+c}\right)\cdot\left(\frac{a-b}{a+b}\right)\cdot\left(\frac{c-a}{c+a}\right)$$

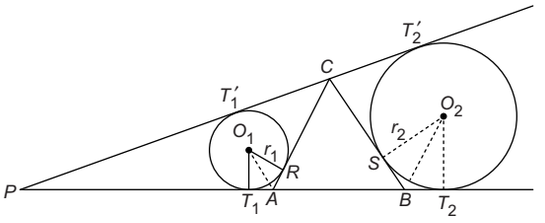
$$= \frac{(b-c)(a+b)(c+a) + (a-b)(b+c)(c+a) + (c-a)(b+c)(a+b)}{(b+c)(a+b)(c+a)} + \frac{(b-c)(a-b)(c-a)}{(b+c)(a+b)(c+a)}$$

After multiplication and solving, we get

$$X + Y + Z + XYZ = \frac{0}{(b+c)(b+a)(c+a)} = 0$$

168. Looking the figure, we see that $\angle T_1O_1R = 60°$ and it is the supplement of $\angle T_1AR = 120°$ {as an exterior angle for ΔABC}

Hence, $\angle AO_1R = 30°$



Similarly, we obtain $\angle BO_2S = 30°$

Since tangents drawn to a circle from external points are equal, we have

$$T_1T_2 = T_1A + AB + BT_2 = RA + SB + AB$$

$$= r_1\tan30° + a + r_2\tan30° = \frac{r_1 + r_2}{\sqrt{3}} + a.$$

and $$T_1'T_2' = T_2'C + CT_1'$$

$$= CR + CS = (a - RA) + (a - SB)$$

$$= 2a - \left(\frac{r_1 + r_2}{\sqrt{3}}\right)$$

Since, common external tangents to two circles are equal, $T_1T_2 = T_1'T_2'$

Hence, $$\frac{r_1 + r_3}{\sqrt{3}} + a = 2a - \frac{(r_1 + r_2)}{\sqrt{3}}$$

Hence, we find that $r_1 + r_2 = \frac{a\sqrt{3}}{2}$.

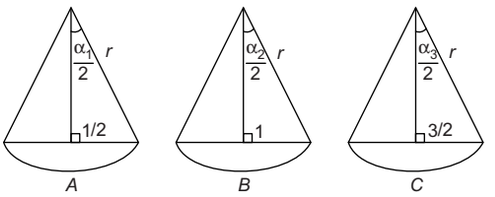
169. Equal chords subtend equal angles at the centre angles at the centre of circle; if each of sides of length i subtends an angles α_i $(i = 1, 2, 3)$ at the centre of the given circle, then $2\alpha_1 + 2\alpha_2 + 2\alpha_3 = 360°$

Hence $\frac{\alpha_1}{2} + \frac{\alpha_2}{2} = 90° - \frac{\alpha_3}{2}$,

and $\cos\left(\frac{\alpha_1}{2} + \frac{\alpha_2}{2}\right) = \cos\left(90° - \frac{\alpha_3}{2}\right) = \sin\frac{\alpha_3}{2}$

Next we apply the addition formula for the cosine;

$$\cos\frac{\alpha_1}{2}\cdot\cos\frac{\alpha_2}{2} - \sin\frac{\alpha_1}{2}\cdot\sin\frac{\alpha_2}{2} = \sin\frac{\alpha_3}{2}$$



where, $\sin\frac{\alpha_1}{2} = \frac{1/2}{r}, \cos\frac{\alpha_1}{2} = \frac{\sqrt{4r^2-1}}{2r}$ [from (A)]

$\sin\frac{\alpha_2}{2} = \frac{1}{r}, \cos\frac{\alpha_2}{2} = \frac{\sqrt{r^2-1}}{r}$ [from (B)]

$\sin\frac{\alpha_3}{2} = \frac{3/2}{r}$ [from (C)]

We substitute the expressions into Eq. (i) and obtain, after multiplying both sides by $2r^2$,

$$\sqrt{4r^2-1}.\sqrt{r^2-1} - 1 = 3r$$

Now, write it in the form;

$$\sqrt{(4r^2-1)(r^2-1)} = (3r+1),$$

and square, obtaining.

$$(4r^2-1)(r^2-1) = 9r^2 + 6r + 1$$

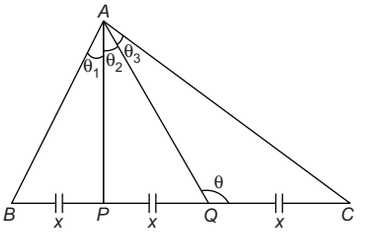
which is equivalent to,

$r(2r^3 - 7r - 3) = 0$, since $r \neq 0$ we have,

$2r^3 - 7r - 3 = 0.$

170. Let, $BP = PQ = QC = x$

Also, let



$\angle BAP = \theta_1, \angle PAQ = \theta_2, \angle QAC = \theta_3$

and let $\angle AQC = \theta$

Applying $m : n$ rule in ΔABC,

$$(2x + x)\cot\theta = 2x\cot(\theta_1 + \theta_2) - x\cot\theta_3$$

$$\Rightarrow \quad 3\cot\theta = 2\cot(\theta_1 + \theta_2) - \cot\theta_3 \qquad \ldots(i)$$

Applying $m : n$ rule in ΔAPC,

$$(x + x)\cot\theta = x\cot\theta_2 - x\cot\theta_3$$

$$2\cot\theta = \cot\theta_2 - \cot\theta_3 \quad \ldots(ii)$$

On dividing Eq. (i) by Eq. (ii), and on solving we, get

$$\frac{3}{2} = \frac{2\cot(\theta_1 + \theta_2) - \cot\theta_3}{\cot\theta_2 - \cot\theta_3}$$

$$\Rightarrow \quad 4(1 + \cot^2\theta_2) = (\cot\theta_1 + \cot\theta_2)(\cot\theta_2 + \cot\theta_3)$$

$$\Rightarrow \quad 4\,\text{cosec}^2\,\theta_2 = (\cot\theta_1 + \cot\theta_2)(\cot\theta_2 + \cot\theta_3)$$

171. Let the sides be $a, \frac{3a}{2}, \frac{9a}{4}$.

$$\text{Semi-perimeter} = \frac{a + \frac{3a}{2} + \frac{9a}{4}}{2} = \frac{19a}{8}$$

$$\text{Area of } \Delta = \sqrt{\frac{19a}{8}\cdot\frac{11a}{8}\cdot\frac{7a}{8}\cdot\frac{a}{8}} = \frac{a^2}{64}\sqrt{1463}.$$

$$\text{Now, circumradius} = \frac{\text{Product of sides}}{4 \times \text{area of } \Delta}$$

$$\Rightarrow \quad \frac{54}{\sqrt{1463}} = \frac{a \times \frac{3a}{2} \times \frac{9a}{4}}{4 \times \frac{a^2}{64} \times \sqrt{1463}}$$

$$\Rightarrow \quad a = 1.$$

Therefore, the sides of the triangles are $1, \frac{3}{2}$ & $\frac{9}{4}$.

172. Given $a + b + c = 2$

$$\Rightarrow \quad 1 - a + 1 - b + 1 - c = 1$$

$$\text{or} \quad x + y + z = 1$$

where $x = 1 - a, y = 1 - b, z = 1 - c$

Since, $a + b > c$

$$\Rightarrow \quad 0 < c < 1,$$

similarly $0 < a, b < 1$

hence $0 < x, y, z < 1.$

$$\text{Now, } a^2 + b^2 + c^2 + 2abc = (1 - x)^2 + (1 - y)^2 + (1 - z)^2 + 2(1 - x)(1 - y)(1 - z)$$

$$= 3 - 2(x + y + z) + (x^2 + y^2 + z^2) + 2[1 - (x + y + z) + (xy + yz + zx) - xyz]$$

$$= 1 + x^2 + y^2 + z^2 + 2(xy + yz + zx) - 2xyz$$

$$= 1 + (x + y + z)^2 - 2xyz$$

$$= 2 - 2xyz < 2 \text{ as } 0 < x, y, z < 1$$

173. Let s be the semi-perimeter and Δ be the area of ΔABC. Then

$$r = \frac{\Delta}{s},\ P = \frac{\Delta}{s - a}$$

$$\Rightarrow \quad \frac{r}{P} = \frac{s - a}{s}$$

$$\text{Also, } \tan\frac{B}{2}\cdot\tan\frac{C}{2} = \frac{s - a}{s}$$

$$\Rightarrow \quad \frac{r}{P} = \tan\frac{B}{2}\cdot\tan\frac{C}{2}$$

Let $\angle AP_1B = \alpha_1, \angle AP_2P_1 = \alpha_2, \angle AP_3P_2 = \alpha_3, \ldots,$

$$\angle AP_{n-1}P_{n-2} = \alpha_{n-1}$$

$$\Rightarrow \quad \frac{r_1}{P_1} = \tan\frac{B}{2}\cdot\tan\frac{\alpha_1}{2}$$

$$\frac{r_2}{P_2} = \tan\left(\frac{\pi - \alpha_1}{2}\right)\cdot\tan\frac{\alpha_2}{2}$$

$$\frac{r_3}{P_3} = \tan\left(\frac{\pi - \alpha_2}{2}\right)\cdot\tan\frac{\alpha_3}{2}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{r_n}{P_n} = \tan\left(\frac{\pi - \alpha_{n-1}}{2}\right)\tan\frac{C}{2}.$$

$$\Rightarrow \quad \frac{r_1}{P_1}\cdot\frac{r_2}{P_2}\ldots\frac{r_n}{P_n} = \tan\frac{B}{2}\cdot\tan\frac{C}{2} = \frac{r}{P}$$

174. Let r be the radius of the circle.

The sides of the polygon, which subtends angle 2α at centre, has length $2r\sin\alpha$.

Hence, the area of this polygon,

$$= \frac{1}{2}(2r\sin\alpha)\,.\,(r\cos\alpha) + \frac{1}{2}(2r\sin2\alpha)(r\cos2\alpha) + \ldots + \frac{1}{2}(2r\sin n\alpha)(r\cos n\alpha)$$

$$\Rightarrow \quad \frac{r^2}{2}[\sin2\alpha + \sin4\alpha + \ldots + \sin2n\alpha]$$

$$\Rightarrow \quad \frac{r^2}{4\sin\alpha}[(\cos\alpha - \cos3\alpha) + (\cos3\alpha - \cos5\alpha) + \ldots + (\cos(2n-1)\alpha - \cos(2n+1)\alpha)]$$

$$\Rightarrow \quad A_1 = \frac{r^2}{4\sin\alpha}[\cos\alpha - \cos(2n+1)\alpha]$$

$$= \frac{r^2}{4\sin(n)\alpha}[\sin\alpha.\sin(n+1)\alpha]$$

Also, $2\alpha + 4\alpha + \ldots + 2n\alpha = 2\pi \Rightarrow \alpha(1 + 2 + 3 \ldots + n) = \pi$

$$\text{or} \quad \frac{\alpha n(n+1)}{2} = \pi \ \text{ i.e. } (n+1)\alpha = \frac{2\pi}{n}$$

Also, A_2 = area of the regular polygon of n sides.

$$= \frac{1}{2}\left(2r\sin\frac{\pi}{4}\cdot r\cos\frac{\pi}{n}\right)n$$

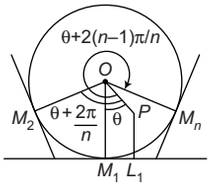
$$= \frac{r^2 n}{2}\left(\sin\frac{2\pi}{n}\right) = \frac{r^2}{2}n\cdot\sin(n+1)\alpha$$

Hence, $\frac{A_1}{A_2} = \frac{\sin n\alpha}{n\sin\alpha}$

175. Let $M_1, M_2, \ldots, M_n$ are the foot of the altitudes drawn from the centre O to the sides of the polygon and $L_1, L_2, \ldots, L_n$ that of P.

$$PL_1 = OM_1 - OP\cos\theta$$

$$PL_1^2 = OM_1^2 + OP^2\cos^2\theta - 2OM_1OP\cos\theta$$



$$= OM_1^2 + \frac{1}{2}OP^2 + \frac{1}{2}OP^2\cos 2\theta - 2OM_1 OP\cdot\cos\theta$$

$$\Sigma PL^2 = \Sigma OM_1^2 + \frac{1}{2}\Sigma OP^2 + \frac{1}{2}OP^2\Sigma\cos 2\theta - 2OM_1 OP\Sigma\cos\theta$$

$$= na^2 + \frac{n}{2}c^2 + \frac{1}{2}c^2(0) - 2ac(0)$$

$$= n\left(a^2 + \frac{c^2}{2}\right)$$

Here, $\Sigma\cos\theta = \cos\theta + \cos\left(\theta + \frac{2\pi}{n}\right) + \cos\left(\theta + \frac{4\pi}{n}\right) + \ldots + \cos\left(\theta + \frac{(2n-1)\pi}{n}\right)$

$$= \sum_{r=0}^{n-1}\cos\left(\theta + \frac{2r\pi}{n}\right) = \text{real part of } \sum_{r=0}^{n-1} i^{(\theta + 2\pi/n)}$$

$$= \text{real part of } e^{i\theta}\sum_{r=0}^{n-1} e^{2\pi i/n} = 0$$

Similarly $\Sigma\cos 2\theta = 0$.

176. Let $c = a^3\cos 3B + 3a^2b\cos(2B - A) + 3ab^2\cos(B - 2A) + b^3\cos 3A$ and $s = a^3\sin 3B + 3a^2b\sin(2B - A) + 3ab^2\sin(B - 2A) + b^3\sin(-3A)$

Now, $c + is = \{a^3\cos 3B + 3a^2b\cos(2B - A) + 3ab^2\sin(B - 2A) + b^3\cos 3A\} + i\{a^3\sin 3B + 3a^2b\sin(2B - A) + 3ab^2\sin(B - 2A) + b^3\sin(-3A)\}$

$= a^3e^{i3B} + 3a^2be^{i(2B-A)} + 3ab^2e^{i(B-2A)} + b^3e^{-i3A}\cdot(ae^{iB} + be^{-iA})^3$

$= (a\cos B + ai\sin B + b\cos A - bi\sin A)^3$

$= \{(a\cos B + b\cos A) + i(a\sin B - b\sin A)\}^3 = c^3$

$\Rightarrow \{a^3\cos 3B + 3a^2b\cos(2B - A) + 3ab^2\cos(B - 2A) + b^3\cos 3A\} = c^3$ [by equating real parts]

177. $\Delta = \frac{1}{2}ah_1$

$\Rightarrow h_1 = \frac{2\Delta}{a}$, similarly, $h_2 = \frac{2\Delta}{b}, h_3 = \frac{2\Delta}{c}$

So, $\frac{h_1 + r}{h_1 - r} + \frac{h_2 + r}{h_2 - r} + \frac{h_3 + r}{h_3 - r}$

$$= \frac{\frac{2\Delta}{a} + \frac{\Delta}{s}}{\frac{2\Delta}{a} - \frac{\Delta}{s}} + \frac{\frac{2\Delta}{b} + \frac{\Delta}{s}}{\frac{2\Delta}{b} - \frac{\Delta}{s}} + \frac{\frac{2\Delta}{c} + \frac{\Delta}{s}}{\frac{2\Delta}{c} - \frac{\Delta}{s}}$$

$$= \frac{2s + a}{2s - a} + \frac{2s + b}{2s - b} + \frac{2s + c}{2s - c}$$

$$= \frac{4s}{2s - a} + \frac{4s}{2s - b} + \frac{4s}{2s - c} - 3$$

$$= 3\left[\frac{1}{3}\left\{\frac{4s}{2s - a} + \frac{4s}{2s - b} + \frac{4s}{2s - c}\right\}\right] - 3$$

$$\geq \left(\frac{3}{\frac{2s - a}{4s} + \frac{2s - b}{4s} + \frac{2s - c}{c}}\right) - 3 \text{ [since A.M. } \geq \text{ H.M.]}$$

≥ 6

178. The given equation can be written as;

$$\frac{a\cos A + b\cos B + c\cos C}{a + b + c} = \frac{a\sin B + b\sin C + c\sin A}{9R}$$

$$\Rightarrow \frac{\sin A\cos A + \sin B\cos B + \sin C\cos C}{\sin A + \sin B + \sin C} = \frac{\frac{2\Delta}{c} + \frac{2\Delta}{a} + \frac{2\Delta}{b}}{9R}$$

$$\Rightarrow \frac{\sin 2A + \sin 2B + \sin 2C}{2(\sin A + \sin B + \sin C)} = \frac{2\Delta(ab + bc + ca)}{9abcR}$$

$$\Rightarrow \frac{2\sin C\{\cos(A - B) - \cos(A + B)\}}{4\cos\frac{C}{2}\left\{\cos\left(\frac{A + B}{2}\right) + \cos\left(\frac{A + B}{2}\right)\right\}} = \frac{2\Delta(ab + bc + ca)}{9abcR}$$

$$\Rightarrow \frac{4\sin A\sin B\sin C}{8\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$= \frac{2\Delta(ab + bc + ca)}{9abcR}$$

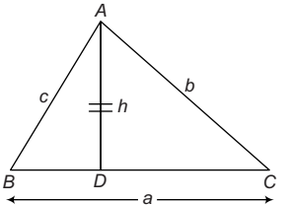
$$\Rightarrow \frac{9r}{2\Delta} = \frac{ab + bc + ca}{abc}$$

$$9abc = (a + b + c)(ab + bc + ca)$$

$$\Rightarrow a(b - c)^2 + b(c - a)^2 + c(a - b)^2 = 0 \Rightarrow a = b = c$$

Hence the result.

179. Area of ΔABC,



$$\Delta = \frac{1}{2}ah = \frac{1}{2}bc\sin A$$

$$\Rightarrow h = \frac{bc}{a}\sin A$$

$$\Rightarrow h^2 = \frac{b^2c^2}{a^2}(1 - \cos^2 A)$$

$$= \frac{b^2c^2}{a^2}(1 + \cos A)(1 - \cos A)$$

$$= \frac{b^2c^2}{a^2}\left(1 + \frac{b^2 + c^2 - a^2}{2bc}\right)\left(1 - \frac{b^2 + c^2 - a^2}{2bc}\right)$$

$$= \frac{1}{4a^2}((b + c)^2 + (a)^2)(a^2 - (b - c)^2)$$

$$\Rightarrow 4h^2 = \frac{1}{a^2}(a^2((b + c)^2 + (b - c)^2) - a^4 - (b^2 - c^2)^2)$$

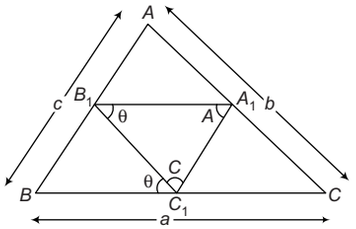
$$= 2(b^2 + c^2) - a^2 - \frac{(b^2 - c^2)^2}{a^2}$$

$$\Rightarrow 4h^2 + a^2 = 2(b^2 + c^2) - \frac{(b^2 - c^2)^2}{a^2}$$

$$\Rightarrow 4h^2 + a^2 \leq 2(b^2 + c^2)$$

Clearly equality holds if $b = c$ (i.e., isosceles Δ)

180. Let, $\frac{B_1C_1}{BC} = \frac{A_1C_1}{AC} = \frac{A_1B_1}{AB} = \lambda$



$\Rightarrow \quad B_1C_1 = \lambda a,\ A_1C_1 = \lambda b,\ A_1B_1 = \lambda c$

$\Rightarrow \quad \frac{\text{Area of } \Delta A_1B_1C_1}{\text{Area of } DABC} = \lambda^2$

Let $\angle B_1C_1B = \theta \Rightarrow \angle BB_1C_1 = \pi - (\theta + B)$

$\angle A_1C_1C = \pi - (\theta + C)$

$\angle C_1A_1C = \pi - (C + \pi - \theta - C) = \theta$

using Sine rule in ΔBB_1C_1, we get :

$$\frac{B_1C_1}{\sin B} = \frac{BC_1}{\sin(\theta + B)}$$

$$\Rightarrow \frac{\lambda a.\sin(\theta + B)}{\sin B} = BC_1$$

Similarly using Sine rule in ΔC_1A_1C, we get

$$\frac{A_1C_1}{\sin C} = \frac{CC_1}{\sin\theta} \Rightarrow CC_1 = \frac{\lambda b\cdot\sin\theta}{\sin C}$$

Now, $$a = BC_1 + CC_1 = \lambda\left(\frac{a\sin(\theta + B)}{\sin B} + \frac{b\sin\theta}{\sin C}\right)$$

$$\Rightarrow \lambda = \frac{\sin A\cdot\sin B\cdot\sin C}{\sin A\cdot\sin C\cdot\sin(\theta + B) + \sin\theta\cdot\sin^2 B}$$

$$= \frac{\sin A\sin B\sin C}{(\sin A\cdot\sin C\cdot\cos B + \sin^2 B)\sin\theta + \sin A\cdot\sin C\cdot\sin B\cdot\cos\theta}$$

$$\Rightarrow \lambda \geq \frac{\sin A\cdot\sin B\cdot\sin C}{\sqrt{(\sin A\cdot\sin C\cdot\cos B + \sin^2 B)^2 + (\sin A\cdot\sin B\cdot\sin B)^2}}$$

$$\lambda^2 \geq \frac{\sin^2 A\cdot\sin^2 B\cdot\sin^2 C}{\sin^2 A\cdot\sin^2 C + \sin^4 B + 2\sin A\cdot\sin C\cdot\cos B\,.\,\sin^2 B}$$

$$\geq \frac{\sin^2 A\cdot\sin^2 B\cdot\sin^2 C}{\sin^2 A\cdot\sin^2 C + \sin^2 B\{\sin^2 B + \cos(A - C)\cdot\cos B - \cos(A + C)\cdot\cos B\}}$$

$$\geq \frac{\sin^2 A\cdot\sin^2 B\cdot\sin^2 C}{\sin^2 A\cdot\sin^2 C + \sin^2 B\{1 - \cos^2 A + \sin^2 C\}}$$

$$\lambda^2_{\min} = \frac{\sin^2 A\cdot\sin^2 B\cdot\sin^2 C}{\sin^2 A\cdot\sin^2 C + \sin^2 B\cdot\sin^2 A + \sin^2 B\cdot\sin^2 C}$$

$$\therefore \quad \lambda^2_{\min} = \frac{1}{\text{cosec}^2 A + \text{cosec}^2 B + \text{cosec}^2 C}$$

Thus, $$\frac{\text{ar}(\Delta A_1B_1C_1)}{\text{ar}(\Delta ABC)} \geq \frac{1}{\text{cosec}^2 A + \text{cosec}^2 B + \text{cosec}^2 C}$$

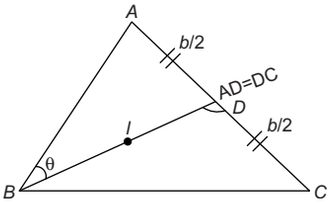
181. Let ABC be the triangle with D as the mid-point of AC.

$BD = l$ (given)

Area of $\Delta ABC = \frac{b^2}{c}\sin A$

Applying cosine rule to ΔABD, we get

$$\frac{b^2 + \frac{b^2}{4} - l^2}{b^2} = \cos A \Rightarrow b^2 = \frac{4l^2}{5 - 4\cos A}$$



Hence, $$\Delta = 2l^2\frac{\sin A}{5 - 4\cos A}$$

$$\Rightarrow \quad \Delta = 2l^2\cdot\frac{\frac{2t}{1 - t^2}}{5 - 4\frac{(1 - t^2)}{(1 + t^2)}}$$

$$\Rightarrow \quad \Delta = \frac{4l^2t}{9t^2 + 1} = \frac{4l^2}{9t + \frac{1}{t}}$$

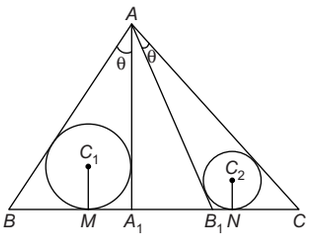
$$\Rightarrow \quad \Delta \leq \frac{2l^2}{3} \qquad \left[\because 9t + \frac{1}{t} \geq \left(\frac{1}{t}\cdot 9t\right)^{1/2} = 6\right]$$

Hence, maximum area $= \frac{2l^2}{3}$

Equality holds when, $9t = \frac{1}{t} \Rightarrow t = \frac{1}{3}$ [since $t > 0$]

$$\Rightarrow \quad \cos A = \frac{4}{5}$$

182. Let $\angle BAA_1 = \angle B_1AC = \theta$ and M and N be the points where incircles of Δ's BAA_1 and B_1AC touch the side BC.



If R_1 and r_1 be the circum-radius and inradius of ΔBAA_1, then

$$r = 4R_1\sin\frac{\theta}{2}\cdot\sin\frac{B}{2}\cdot\sin\left(\frac{\pi - (\theta + B)}{2}\right)$$

or $$r_1 = 4R_1\sin\frac{\theta}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{\alpha}{2},$$

where $\alpha = \pi - (\theta + B)$

Now, $$BM = r_1\cos\frac{B}{2} = 4R_1\sin\frac{\theta}{2}\cdot\sin\frac{\alpha}{2}\cdot\cos\frac{B}{2}$$

and $$MA_1 = r_1\cot\frac{\alpha}{2} = 4R_1\sin\frac{\theta}{2}\cdot\sin\frac{B}{2}\cdot\cos\frac{\alpha}{2}$$

Thus, $$\frac{1}{BM} + \frac{1}{MA_1} = \frac{1}{4R_1\sin\theta/2}$$

$$\left(\frac{1}{\sin\alpha/2\cdot\cos B/2} + \frac{1}{\sin B/2\cdot\cos\alpha/2}\right)$$

$$= \frac{1}{4R_1 \sin\theta/2} \cdot \frac{\sin\left\{\frac{\alpha + B}{2}\right\}}{\sin\alpha/2 \cdot \cos B/2 \cdot \sin B/2 \cdot \cos\alpha/2}$$

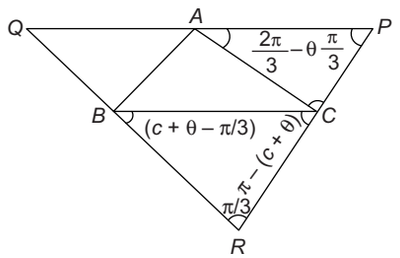
$$= \frac{\cos(\theta/2)}{R_1 \sin\theta/2 \sin\alpha \sin B} = \frac{2\cot\theta/2}{c \cdot \sin B}$$

Similarly, $\frac{1}{B_1N} + \frac{1}{NC} = \frac{2\cot\theta/2}{b\sin C}$

Hence, $\frac{1}{BM} + \frac{1}{MA_1} = \frac{1}{B_1N} + \frac{1}{NC}$ [as $b\sin C = c\sin B$]

183. Let the angle between PR and AC be θ

$\therefore$ In ΔAPC, from the sine rule



$$PC = \frac{b}{\sin\pi/3}\sin\left(\frac{2\pi}{3} - \theta\right)$$

In ΔBCR, from the sine rule.

$$CR = \frac{a\sin(c + \theta - \pi/3)}{\sin\pi/3}$$

$$\Rightarrow \quad PR = PC + CR = \frac{2}{\sqrt{3}}\left\{b\sin\left(\frac{\pi}{3} + \theta\right) + a\sin\left(c + \theta - \frac{\pi}{3}\right)\right\}$$

$$= \frac{2}{\sqrt{3}}$$

$$\left[\left(b\cos\frac{\pi}{3} + a\cos\left(c - \frac{\pi}{3}\right)\right)\sin\theta + \left(b\sin\frac{\pi}{3} + a\sin\left(c - \frac{\pi}{3}\right)\right)\cdot\cos\theta\right]$$

$\Rightarrow$ Max.

$$PR = \frac{2}{\sqrt{3}}\left[\sqrt{\left(b\cos\frac{\pi}{3} + a\cos\left(c - \frac{\pi}{3}\right)\right)^2 + \left(b\sin\frac{\pi}{3} + a\sin\left(c - \frac{\pi}{3}\right)\right)^2}\right]$$

$$= \frac{2}{\sqrt{3}}\sqrt{b^2 + a^2 + 2ab\cos\left(\frac{2\pi}{3} - c\right)}$$

Max. area of $\Delta PQR = \frac{\sqrt{3}}{4}[\text{max. } PR]^2$

$$= \frac{\sqrt{3}}{4} \cdot \frac{4}{3}[b^2 + a^2 - ab\cos C + \sqrt{3}ab\sin C]$$

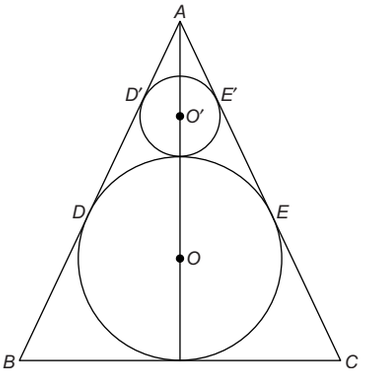
$$= \frac{2b^2 + 2a^2 - 2ab\cos C}{2\sqrt{3}} + ab\sin C$$

$$= \frac{a^2 + b^2 + c^2}{2\sqrt{3}} + 2\Delta \qquad [\because 2ab\cos C = a^2 + b^2 - c^2]$$

184. Let the circle of radius r_A touch the sides AB and AC at D and E, where as the incircle touches their sides at D and E. Let O and O' be centres of the inscribed and that the circle with radius r_A. O and O' lie on the bisector of angle A.

Also, $AO' = OD'\,\text{cosec}\,\frac{A}{2} = r_A\,\text{cosec}\,\frac{A}{2}$

and $AO = OD\,\text{cosec}\,\frac{A}{2} = r\,\text{cosec}\,\frac{A}{2}$



Hence, $OO' = r + r_A = AO - AO'$

$$= r\,\text{cosec}\,\frac{A}{2} - r_A\,\text{cosec}\frac{A}{2}$$

$$\Rightarrow \quad \frac{r_A}{r} = \frac{\text{cosec}\,\frac{A}{2} - 1}{\text{cosec}\,\frac{A}{2} + 1}$$

$$\Rightarrow \quad r_A = r\tan^2\left(\frac{\pi - A}{4}\right)$$

Similarly, $r_B = r\tan^2\left(\frac{\pi - B}{4}\right), r_C = r\tan^2\left(\frac{\pi - C}{4}\right)$

Hence, $\sqrt{r_Ar_B} + \sqrt{r_Br_C} + \sqrt{r_Cr_A}$

$$= \left\{\tan\left(\frac{\pi - A}{4}\right) + \tan\left(\frac{\pi - B}{4}\right) + \tan\left(\frac{\pi - B}{4}\right)\cdot\tan\left(\frac{\pi - C}{4}\right) + \tan\left(\frac{\pi - C}{4}\right)\cdot\tan\left(\frac{\pi - A}{4}\right)\right\}$$

$$= \frac{r}{\cos\left(\frac{\pi - A}{4}\right)\cdot\cos\left(\frac{\pi - B}{4}\right)\cdot\cos\left(\frac{\pi - C}{4}\right)}$$

$$\left\{\sin\left(\frac{\pi - A}{4}\right)\sin\left(\frac{\pi - B}{4}\right)\sin\left(\frac{\pi - C}{4}\right) + \ldots\right\}$$

$$= \frac{r}{\cos\left(\frac{\pi - A}{4}\right)\cdot\cos\left(\frac{\pi - B}{4}\right)\cdot\cos\left(\frac{\pi - C}{4}\right)}$$

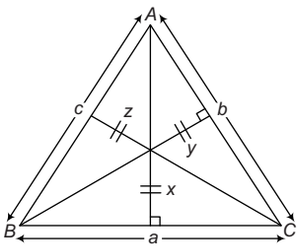
$$\left\{\cos\left(\frac{\pi + A}{4}\right)\cdot\cos\left(\frac{\pi - A}{4}\right) + \sin\frac{\pi - B}{4}\cdot\sin\frac{\pi - C}{4}\cdot\cos\frac{\pi - A}{4}\right\}$$

$$= \frac{r\cos\left(\frac{\pi - A}{4}\right)}{\cos\left(\frac{\pi - A}{4}\right)\cdot\cos\left(\frac{\pi - B}{4}\right)\cdot\cos\left(\frac{\pi - C}{4}\right)}$$

$$= \frac{r\cos\left(\frac{\pi - A}{4}\right)\cdot\cos\left(\frac{\pi - B}{4}\right)\cdot\cos\left(\frac{\pi - C}{4}\right)}{\cos\left(\frac{\pi - A}{4}\right)\cdot\cos\left(\frac{\pi - B}{4}\right)\cdot\cos\left(\frac{\pi - C}{4}\right)}$$

$\therefore \quad \sqrt{r_Ar_B} + \sqrt{r_Br_C} + \sqrt{r_Cr_A} = r$

185. If a,b,c are the lengths of the sides of the Δ and x,y,z are the lengths of perpendicular from the points on the sides BC, CA, AB respectively, we have to minimise,



$$\Delta = x^2 + y^2 + z^2 \qquad \ldots(i)$$

We have,

$$\frac{1}{2}ax + \frac{1}{2}by + \frac{1}{2}cz = \Delta$$

$$\Rightarrow \quad ax + by + cz = 2\Delta \qquad \ldots(ii)$$

where, Δ is the area of the ΔABC

we have the identity;

$$\Rightarrow \quad (x^2 + y^2 + z^2)(a^2 + b^2 + c^2) - (ax + by + cz)^2$$
$$= (ax - by)^2 + (by - cz)^2 + (cz - ax)^2$$

$$\Rightarrow \quad (x^2 + y^2 + z^2)(a^2 + b^2 + c^2) \geq (ax + by + cz)^2$$

$$\Rightarrow \quad (x^2 + y^2 + z^2)(a^2 + b^2 + c^2) \geq 4\Delta^2 \qquad \text{[using (ii)]}$$

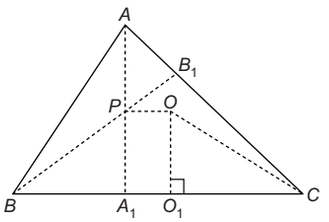
$$\Rightarrow \quad x^2 + y^2 + z^2 \geq \frac{4\Delta^2}{a^2 + b^2 + c^2} \text{ and equality holds only when}$$

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \frac{ax + by + cz}{a^2 + b^2 + c^2} = \frac{2\Delta}{a^2 + b^2 + c^2}$$

$\therefore$ The minimum value of Δ is $\dfrac{4\Delta^2}{a^2 + b^2 + c^2}$.

$$\Delta_{\min} = \frac{4(s-a)(s-b)(s-c)s}{a^2 + b^2 + c^2}.$$

186. In the adjacent figure AA_1 and BB_1 are altitudes drawn from A and B on the sides BC and AC, respectively. Let P, O be the orthocentre and circum-centre of the Δ.



We have been given that $PA_1 = OO_1$.

In ΔOO_1C, $\angle O_1OC = A$

and $\quad OC = R \Rightarrow OO_1 = R\cos A$

Also, $\quad BA_1 = AB\cos B = c\cos B$

and $\quad PA_1 = BA_1 \cot C$

$$PA_1 = c\cos B \cdot \cot C = \frac{c\cos B\cos C}{\sin C}$$

$$= 2R\cos B\cos C$$

Thus, $\quad R\cos A = 2R\cos B\cos C$

$$\Rightarrow \quad 2\cos B\cos C + \cos(B + C) = 0$$

$$\Rightarrow \quad \sin B\sin C = 3\cos B\cos C$$

or $\quad \tan B\tan C = 3$

Now in any $\quad \Delta, \Sigma\tan A = 3\tan A$

$$\Rightarrow \quad \tan A + \tan B + \tan C = 3\tan A$$

$$\Rightarrow \quad \tan B + \tan C = 2\tan A$$

Now,

$$(\tan B - \tan C)^2 > 0 \text{ as } \angle B \neq \angle C$$

$$\Rightarrow \quad (\tan B + \tan C)^2 > 4\tan B\tan C$$

$$\Rightarrow \quad 4\tan^2 A > 12$$

$$\Rightarrow \quad \tan A > \sqrt{3} \qquad \text{[as } \angle A \text{ is acute]}$$

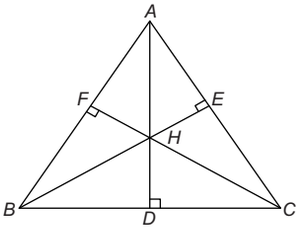
$$\Rightarrow \quad A \in \left(\frac{\pi}{3}, \frac{\pi}{2}\right)$$

187. $BD = AB\cos B = c\cos B$

also $\quad \angle BHD = \dfrac{\pi}{2} - \angle EBC = \dfrac{\pi}{2} - \left(\dfrac{\pi}{2} - C\right) = C$

$$\Rightarrow \quad BH = \frac{BD}{\sin C} = \frac{c\cos B}{\sin C} = 2R\cos B$$

Now, points H, D, B and F are concyclic and BH is the diameter of the circle passing through these four points. In fact this circle is also the circum-circle of ΔBFD.



$$\Rightarrow \quad \frac{FD}{\sin B} = BH = 2R\cos B$$

$$\Rightarrow \quad FD = 2R\sin B\cos B = b\cos B$$

$$\Rightarrow \quad \frac{FD}{b} = \cos B$$

Similarly, $\quad \dfrac{EF}{a} = \cos A \quad \text{and} \quad \dfrac{DE}{c} = \cos C$

Thus, $\quad \dfrac{EF}{a} + \dfrac{FD}{b} + \dfrac{DE}{c} = \cos A + \cos B + \cos C$

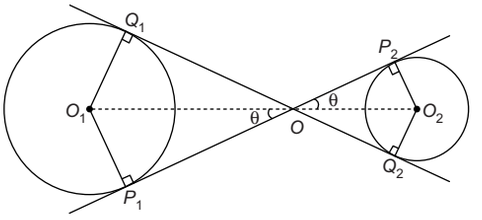
$$= 1 + 4\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}$$

$$= 1 + \frac{r}{R} = \frac{R + r}{R}$$

$\therefore \quad \dfrac{EF}{a} + \dfrac{FD}{b} + \dfrac{DE}{c} = \dfrac{R + r}{R}$

188. Let the centres of circle be O_1 and O_2, and their radii be r_1 and r_2 respectively.

We have,



$$r_1 + r_2 = a \text{ and } O_1O_2 = 2a$$

Let, $\angle O_1OP_1 = \theta \Rightarrow \angle P_2OO_2 = \theta$

Now, $O_1O_2 = O_1O + OO_2 = r_1\text{cosec}\,\theta + r_2\text{cosec}\,\theta$

$O_1O_2 = (r_1 + r_2)\,\text{cosec}\,\theta$

$\Rightarrow$ $\text{cosec}\,\theta = 2 \Rightarrow \theta = \dfrac{\pi}{6}$

Now, $P_1O = r_1 \cot\theta$ and $OP_2 = r_2 \cot\theta$

$\Rightarrow$ $P_1P_2 = (r_1 + r_2)\cot\theta = a\sqrt{3}$

Central angle, $\angle Q_1O_1P_1 = 2\left(\dfrac{\pi}{2} - \theta\right) = \pi - 2\theta$

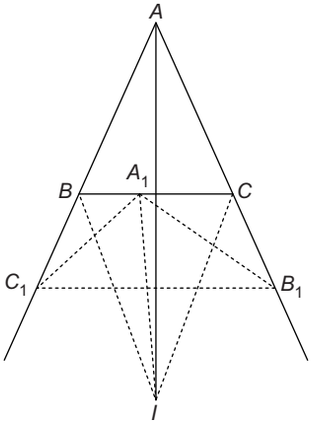
and $\angle P_2O_2Q_2 = \angle Q_1O_1P_1 = \pi - 2\theta$

$\Rightarrow$ Total length of string $= 2P_1P_2 + (r_1 + r_2)(2\pi - (\pi - 2\theta))$

$= 2a\sqrt{3} + a\left(\pi + \dfrac{\pi}{3}\right)$

$= \left(\dfrac{4\pi}{3} + 2\sqrt{3}\right)a$

189. Let A_1, B_1 and C_1 be the points of contact of the ex-circle opposite to vertex A, with the side AC, BC and AB, respectively.



$\Rightarrow$ $I_1A_1 = I_1B_1 = I_1C_1 = r_1$

$\angle B_1I_1A_1 = \pi - \angle B_1CA_1 = \pi - (\pi - C) = C$

Similarly, $\angle A_1I_1C_1 = B$

$\Rightarrow$ $\angle C_1I_1B_1 = (B + C)$

Now, $\Delta_1 = \Delta A_1I_1B_1 + \Delta A_1I_1C_1 - \Delta B_1I_1C_1$

$= \dfrac{1}{2}(r_1^2)(\sin C + \sin B - \sin A)$

$= \dfrac{1}{2}\dfrac{\Delta^2}{(s-a)^2}\left(\dfrac{c}{2R} + \dfrac{b}{2R} - \dfrac{a}{2R}\right)$

$= \dfrac{1}{2}\dfrac{\Delta^2}{(s-a)^2}\cdot\dfrac{(2s-2a)}{2R} = \dfrac{\Delta^2}{2R(s-a)} = \dfrac{r_1\Delta}{2R}$

$\Rightarrow$ $\dfrac{\Delta_1}{\Delta} = \dfrac{r_1}{2R}$

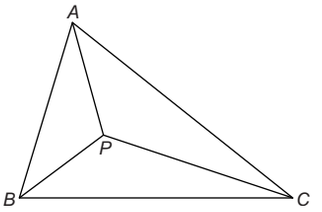
Similarly, $\dfrac{\Delta_2}{\Delta} = \dfrac{r_2}{2R}$, $\dfrac{\Delta_3}{\Delta} = \dfrac{r_3}{2R}$ and $\dfrac{\Delta_0}{\Delta} = \dfrac{r}{2R}$

Thus, $\dfrac{\Delta_1}{\Delta} + \dfrac{\Delta_2}{\Delta} + \dfrac{\Delta_3}{\Delta} - \dfrac{\Delta_0}{\Delta} = \dfrac{1}{2}R(r_1 + r_2 + r_3 - r)$

$= \dfrac{1}{2R}(4R) = 2$

190. Since, $\angle APB = \angle BPC = \angle CPA$

$\Rightarrow$ each of these angles is equal to $\dfrac{2\pi}{3}$



In ΔAPC we have,

$PA^2 + PC^2 - 2PA.PC\cos\dfrac{2\pi}{3} = b^2$

$\Rightarrow$ $b^2 = PA^2 + PC^2 + PA\cdot PC$

Similarly, in ΔBPC;

$a^2 = PB^2 + PC^2 + PB\cdot PC$

and in ΔAPB; $c^2 = AP^2 + PB^2 + PA\cdot PB$

Adding these results, we get

$a^2 + b^2 + c^2 = 2(PA^2 + PB^2 + PC^2) + PA\cdot PB + PB\cdot PC + PC\cdot PA$

$\Rightarrow$ $a^2 + b^2 + c^2 = 2((PA + PB + PC)^2 - 2\Sigma PA\cdot PB) + \Sigma PA\cdot PB$

Now, $\Delta = \Delta APC + \Delta BPC + \Delta APB$

$= \dfrac{1}{2}\sin\dfrac{2\pi}{B}(AP\cdot PC + BP\cdot PC + PA\cdot PB) = \dfrac{\sqrt{3}}{4}(\Sigma PA\cdot PB)$

Putting the value of $\Sigma PA\cdot PB$, we get

$a^2 + b^2 + c^2 = 2(PA + PB + PC)^2 - 3\cdot\dfrac{4\Delta}{\sqrt{3}}$

$\Rightarrow$ $(PA + PB + PC)^2 = \dfrac{a^2 + b^2 + c^2}{2} + 2\sqrt{3}\Delta$

$\Rightarrow$ $PA + PB + PC = \sqrt{\dfrac{a^2 + b^2 + c^2}{2} + 2\sqrt{3}\Delta}$

191. In right angled Δ's $A'BD$ and $CA'D$

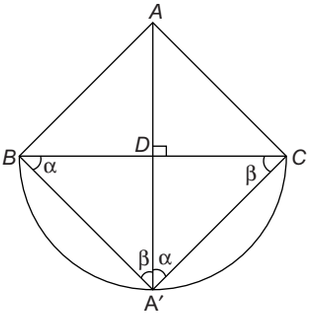
$\angle A'BD = \angle CA'D$

$\angle BDA' = \angle CDA' = 90°$

$\therefore$ The Δ's are equiangular and hence similar

So, $\dfrac{BD}{A'D} = \dfrac{A'D}{DC}$

So that $A'D^2 = BD.DC$



Since, $BD = AD\cot B$, $DC = AD\cot C$

$\therefore$ $A'D^2 = AD^2\cot B\cot C$

or $\{\text{ar}\,(BCA')\}^2 = \left(\dfrac{1}{2}BC\cdot A'D\right)^2$

$$= \frac{1}{4} BC^2 \cdot AD^2 \cdot \cot B \cdot \cot C$$

$$= \{\text{ar}(ABC)^2\} \cdot \cot B \cdot \cot C \quad \ldots(\text{i})$$

Similarly,

$$\{\text{ar}(CAB')\}^2 = \{\text{ar}(ABC)\}^2 \cdot \cot C \cdot \cot A \quad \ldots(\text{ii})$$

$$\{\text{ar}(ABC')\}^2 = \{\text{ar}(ABC)\}^2 \cdot \cot A \cdot \cot B \quad \ldots(\text{iii})$$

On adding Eqs. (i), (ii) and (iii), we get the desired result because in any Δ;

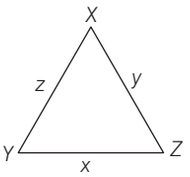
$$\cot A \cot B + \cot B \cot C + \cot C \cdot \cot A = 1$$

$$\Rightarrow \{\text{ar}(BCA')\}^2 + \{\text{ar}(CAB')\}^2 + \{\text{ar}(ABC')\}^2 = \{\text{ar}(ABC)\}^2$$

192. Given a ΔXYZ, where $2s = x + y + z$

and $$\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2}$$

$$\therefore \quad \frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2} = \frac{3s-(x+y+z)}{4+3+2} = \frac{s}{9}$$



or $$\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2} = \frac{s}{9} = \lambda \text{ (let)}$$

$$\Rightarrow \quad s = 9\lambda, s = 4\lambda + x, s = 3\lambda + y \text{ and } s = 2\lambda + z$$

$$\therefore \quad s = 9\lambda,\ x = 5\lambda,\ y = 6\lambda,\ z = 7\lambda$$

Now, $$\Delta = \sqrt{s(s-x)(s-y)(s-z)} \quad \text{[heron's formula]}$$

$$= \sqrt{9\lambda \cdot 4\lambda \cdot 3\lambda \cdot 2\lambda} = 6\sqrt{6}\lambda^2 \quad \ldots(\text{i})$$

Also, $$\pi r^2 = \frac{8\pi}{3} \Rightarrow r^2 = \frac{8}{3} \quad \ldots(\text{ii})$$

and $$R = \frac{xyz}{4\Delta} = \frac{(5\lambda)(6\lambda)(7\lambda)}{4 \cdot 6\sqrt{6}\lambda^2} = \frac{35\lambda}{4\sqrt{6}} \quad \ldots(\text{iii})$$

Now, $$r^2 = \frac{8}{3} = \frac{\Delta^2}{S^2} = \frac{216\lambda^4}{81\lambda^2} \Rightarrow \frac{8}{3} = \frac{8}{3}\lambda^2 \quad \text{[from Eq. (ii)]}$$

$$\Rightarrow \quad \lambda = 1$$

(a) $\Delta XYZ = 6\sqrt{6}\lambda^2 = 6\sqrt{6}$

$\therefore$ Option (a) is correct.

(b) Radius of circum-circle, $R = \frac{35}{4\sqrt{6}}\lambda = \frac{35}{4\sqrt{6}}$

$\therefore$ Option (b) is incorrect.

(c) Since, $$r = 4R \sin\frac{X}{2} \cdot \sin\frac{Y}{2} \cdot \sin\frac{Z}{2}$$

$$\Rightarrow \quad \frac{2\sqrt{2}}{\sqrt{3}} = 4 \cdot \frac{35}{4\sqrt{6}} \sin\frac{X}{2} \cdot \sin\frac{Y}{2} \cdot \sin\frac{Z}{2}$$

$$\Rightarrow \quad \frac{4}{35} = \sin\frac{X}{2} \cdot \sin\frac{Y}{2} \cdot \sin\frac{Z}{2}$$

$\therefore$ Option (c) is correct.

(d) $$\sin^2\left(\frac{X+Y}{2}\right) = \cos^2\left(\frac{Z}{2}\right), \text{ as } \frac{X+Y}{2} = 90° - \frac{Z}{2}$$

$$= \frac{s(s-z)}{xy} = \frac{9 \times 2}{5 \times 6} = \frac{3}{5}$$

$\therefore$ Option (d) is correct.

193. (i) $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

(ii) $R = \dfrac{abc}{4\Delta}$, $r = \dfrac{\Delta}{s}$

where, R, r, Δ denote the circum-radius, in-radius and area of triangle, respectively.

Let the sides of triangle be a, b and c.

Given, $$x = a + b$$

$$y = ab$$

$$x^2 - c^2 = y$$

$$\Rightarrow \quad a^2 + b^2 + 2ab - c^2 = ab$$

$$\Rightarrow \quad a^2 + b^2 - c^2 = -ab$$

$$\Rightarrow \quad \frac{a^2 + b^2 - c^2}{2ab} = -\frac{1}{2} = \cos 120°$$

$$\Rightarrow \quad \angle C = \frac{2\pi}{3}$$

$$\because \quad R = \frac{abc}{4\Delta},\ r = \frac{\Delta}{s}$$

$$\Rightarrow \quad \frac{r}{R} = \frac{4\Delta^2}{s(abc)} = \frac{4\left[\frac{1}{2} ab \sin\left(\frac{2\pi}{3}\right)\right]^2}{\frac{x+c}{2} \cdot y \cdot c}$$

$$\therefore \quad \frac{r}{R} = \frac{3y}{2c(x+c)}$$

194. We know $\Delta = \frac{1}{2} ab \sin C \Rightarrow 15\sqrt{3} = \frac{1}{2} \times 6 \times 10 \times \sin C$

$$\sin C = \frac{\sqrt{3}}{2} \text{ and } C \text{ is given to be obtuse.}$$

$$\Rightarrow \quad C = \frac{2\pi}{3} c \Rightarrow = \sqrt{a^2 + b^2 - 2ab\cos C}$$

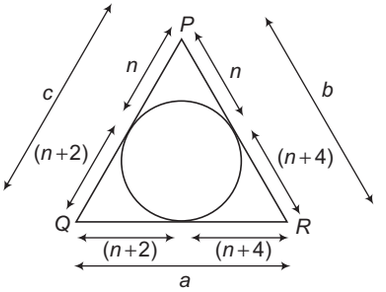
$$C = \sqrt{6^2 + 10^2 - 2 \times 6 \times 10 \times \cos\frac{2\pi}{3}} = 14$$

$$\therefore \quad r = \frac{\Delta}{s} \Rightarrow r^2 = \frac{225 \times 3}{\left(\frac{6+10+14}{2}\right)^2} = 3$$

195. Whenever cosine of angle and sides are given or to find out, we should always use cosine law.

i.e. $$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} \text{ and } \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$



$$\therefore \quad \cos P = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\Rightarrow \quad \frac{1}{3} = \frac{(2n+4)^2 + (2n+2)^2 - (2n+6)^2}{2(2n+4)(2n+2)} \qquad \left[\because \cos P = \frac{1}{3}, \text{ given}\right]$$

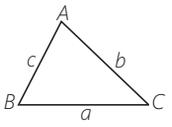
$$\Rightarrow \quad \frac{4n^2 - 16}{8(n+1)(n+2)} = \frac{1}{3} \Rightarrow \frac{n^2 - 4}{2(n+1)(n+2)} = \frac{1}{3}$$

$$\Rightarrow \quad \frac{(n-2)}{2(n+1)} = \frac{1}{3}$$

$$\Rightarrow \quad 3n - 6 = 2n + 2 \Rightarrow n = 8$$

$\therefore$ Sides are $(2n+2), (2n+4), (2n+6)$, i.e. 18, 20, 22.

196. If ΔABC has sides a, b, c.

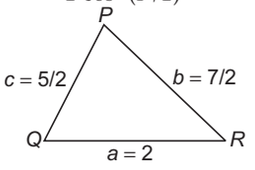


Then, $\tan(A/2) = \sqrt{\dfrac{(s-b)(s-a)}{s(s-a)}}$

where, $s = \dfrac{a+b+c}{2} = \dfrac{2 + \frac{7}{2} + \frac{5}{2}}{2} = 4$

$$\therefore \quad \frac{2\sin P - \sin 2P}{2\sin P + \sin 2P} = \frac{2\sin P(1 - \cos P)}{2\sin P(1 + \cos P)}$$

$$= \frac{2\sin^2(P/2)}{2\cos^2(P/2)} = \tan^2(P/2)$$



$$\Rightarrow \quad \frac{(s-b)(s-c)}{s(s-a)} \times \frac{(s-b)(s-c)}{(s-b)(s-c)}$$

$$= \frac{[(s-b)^2(s-c)^2]}{\Delta^2} = \frac{\left(4 - \frac{7}{2}\right)^2 \left(4 - \frac{5}{2}\right)^2}{\Delta^2}$$

$$= \left(\frac{3}{4\Delta}\right)^2$$

197. Since, A, B, C are in AP.

$$\Rightarrow \quad 2B = A + C \text{ i.e. } \angle B = 60^\circ$$

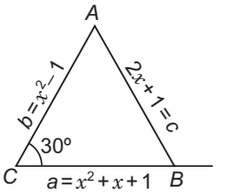
$$\therefore \quad \frac{a}{c}(2\sin C\cos C) + \frac{c}{a}(2\sin A\cos A)$$

$$= 2k(a\cos C + c\cos A) \qquad \left[\text{using, } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \frac{1}{k}\right]$$

$$= 2k(b) = 2\sin B \qquad [\text{using } b = a\cos C + c\cos A]$$

$$= \sqrt{3}$$

198. Using, $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$



$$\Rightarrow \quad \frac{\sqrt{3}}{2} = \frac{(x^2+x+1)^2 + (x^2-1)^2 - (2x+1)^2}{2(x^2+x+1)(x^2-1)}$$

$$\Rightarrow \quad (x+2)(x+1)(x-1)x + (x^2-1)^2 = \sqrt{3}(x^2+x+1)(x^2-1)$$

$$\Rightarrow \quad x^2 + 2x + (x^2-1) = \sqrt{3}(x^2+x+1)$$

$$\Rightarrow \quad (2-\sqrt{3})x^2 + (2-\sqrt{3})x - (\sqrt{3}+1) = 0$$

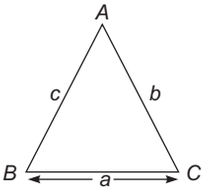
$$\Rightarrow \quad x = -(2+\sqrt{3}) \text{ and } x = 1+\sqrt{3}$$

But $\quad x = -(2+\sqrt{3})$

$\Rightarrow$ c is negative.

$\therefore$ $x = 1 + \sqrt{3}$ is the only solution.

199. Given, $\cos B + \cos C = 4\sin^2\dfrac{A}{2}$



$$\Rightarrow 2\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right) = 4\sin^2\frac{A}{2}$$

$$\Rightarrow 2\sin\frac{A}{2}\left[\cos\left(\frac{B-C}{2}\right) - 2\sin\frac{A}{2}\right] = 0$$

$$\Rightarrow \quad \cos\left(\frac{B-C}{2}\right) - 2\cos\left(\frac{B+C}{2}\right) = 0$$

As $\quad \sin\dfrac{A}{2} \neq 0$

$$\Rightarrow \quad -\cos\frac{B}{2}\cos\frac{C}{2} + 3\sin\frac{B}{2}\sin\frac{C}{2} = 0$$

$$\Rightarrow \quad \tan\frac{B}{2}\tan\frac{C}{2} = \frac{1}{3}$$

$$\Rightarrow \quad \sqrt{\frac{(s-a)(s-c)}{s(s-b)} \cdot \frac{(s-b)(s-a)}{s(s-c)}} = \frac{1}{3}$$

$$\Rightarrow \quad \frac{s-a}{s} = \frac{1}{3}$$

$$\Rightarrow \quad 2s = 3a \Rightarrow b + c = 2a$$

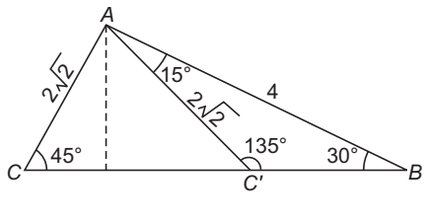
$\therefore$ Locus of A is an ellipse.

200. In ΔABC, by sine rule, $\dfrac{a}{\sin A} = \dfrac{2\sqrt{2}}{\sin 30^\circ} = \dfrac{4}{\sin C}$

$$\Rightarrow \quad C = 45^\circ, C' = 135^\circ$$

When, $\quad C = 45^\circ \Rightarrow A = 180^\circ - (45^\circ + 30^\circ) = 105^\circ$

When, $\quad C' = 135^\circ \Rightarrow A = 180^\circ - (135^\circ + 30^\circ) = 15^\circ$



$$\text{Area of } \Delta ABC = \frac{1}{2} AB \times AC \sin A = \frac{1}{2} \times 4 \times 2\sqrt{2} \sin (105°)$$

$$= 4\sqrt{2} \times \frac{\sqrt{3}+1}{2\sqrt{2}} = 2(\sqrt{3}+1) \text{ sq units}$$

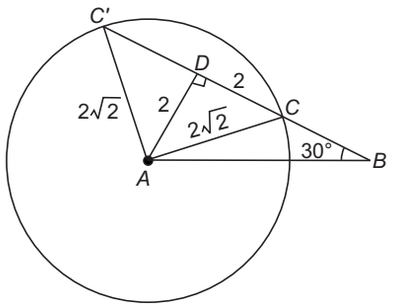
$$\text{Area of } \Delta ABC' = \frac{1}{2} AB \times AC \sin A$$

$$= \frac{1}{2} \times 4 \times 2\sqrt{2} \sin (15°) = 2(\sqrt{3}-1) \text{ sq units}$$

Difference of areas of triangle

$$= |2(\sqrt{3}+1) - 2(\sqrt{3}-1)| = 4 \text{ sq units}$$

Alternate Method



Here, $AD = 2$, $DC = 2$

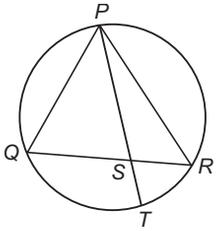
Difference of areas of ΔABC and $\Delta ABC'$

$$= \text{Area of } \Delta ACC'$$

$$= \frac{1}{2} AD \times CC' = \frac{1}{2} \times 2 \times 4 = 4 \text{ sq units}$$

201. Let a straight line through the vertex P of a given $\Delta\ PQR$ intersects the side QR at the point S and the circum-circle of $\Delta\ PQR$ at the point T.

Points P, Q, R, T are concyclic, then $PS \cdot ST = QS \cdot SR$



Now, $\dfrac{PS + ST}{2} > \sqrt{PS \cdot ST}$ [$\because$ AM > GM]

and $\dfrac{1}{PS} + \dfrac{1}{ST} > \dfrac{2}{\sqrt{PS \cdot ST}} = \dfrac{2}{\sqrt{QS \cdot SR}}$

Also, $\dfrac{SQ + QR}{2} > \sqrt{SQ \cdot SR}$

$\Rightarrow$ $\dfrac{QR}{2} > \sqrt{SQ \cdot SR}$

$\Rightarrow$ $\dfrac{1}{\sqrt{SQ \cdot SR}} > \dfrac{2}{QR} \Rightarrow \dfrac{2}{\sqrt{SQ \cdot SR}} > \dfrac{4}{QR}$

$\therefore$ $\dfrac{1}{PS} + \dfrac{1}{ST} > \dfrac{2}{\sqrt{QS \cdot SR}} > \dfrac{4}{QR}$

202. Radius of in-circle is, $r = \dfrac{\Delta}{s}$

Since, $\Delta = 16\sqrt{2}$

Now, $s = \dfrac{6\sqrt{2} + 6\sqrt{2} + 4\sqrt{2}}{2} = 8\sqrt{2}$

$\therefore$ $r = \dfrac{16\sqrt{2}}{8\sqrt{2}} = 2$

203. Equation of circum-circle of ΔPRS is

$$(x+1)(x-9) + y^2 + \lambda y = 0$$

It will pass through $(1, 2\sqrt{2})$, then $-16 + 8 + \lambda \cdot 2\sqrt{2} = 0$

$\Rightarrow$ $\lambda = \dfrac{8}{2\sqrt{2}} = 2\sqrt{2}$

$\therefore$ Equation of circum-circle is $x^2 + y^2 - 8x + 2\sqrt{2}y - 9 = 0$

Hence, its radius is $3\sqrt{3}$.

Alternate Solution

Let $\angle PSR = \theta \Rightarrow \sin\theta = \dfrac{2\sqrt{2}}{2\sqrt{3}}$

$\therefore$ $\sin\theta = \dfrac{PR}{2R}$

$\Rightarrow$ $PR = 6\sqrt{2} = 2R \cdot \sin\theta$

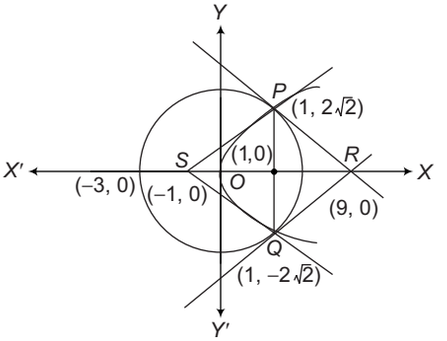
$\Rightarrow$ $R = 3\sqrt{3}$

204. Coordinates of P and Q are $(1, 2\sqrt{2})$ and $(1, -2\sqrt{2})$.

Now, $PQ = \sqrt{(4\sqrt{2})^2 + 0^2} = 4\sqrt{2}$

Area of $\Delta PQR = \dfrac{1}{2} \cdot 4\sqrt{2} \cdot 8 = 16\sqrt{2}$ sq units

Area of $\Delta PQS = \dfrac{1}{2} \cdot 4\sqrt{2} \cdot 2 = 4\sqrt{2}$ sq units

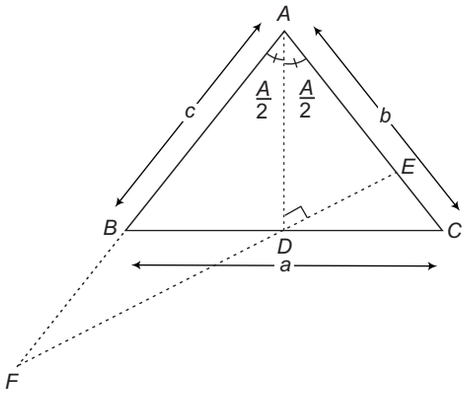


Ratio of areas of ΔPQS and ΔPQR is 1 : 4.

205. Since, $\Delta ABC = \Delta ABD + \Delta ACD$

$\Rightarrow$ $\dfrac{1}{2} bc \sin A = \dfrac{1}{2} c\ AD \sin\dfrac{A}{2} + \dfrac{1}{2} b\ AD \sin\dfrac{A}{2}$

$\Rightarrow$ $AD = \dfrac{2bc}{b+c} \cos\dfrac{A}{2}$



Again, $AE = AD \sec\frac{A}{2} = \frac{2bc}{b+c}$

$\Rightarrow$ AE is HM of b and c.

$$EF = ED + DF = 2DE = 2AD\tan\frac{A}{2}$$

$$= 2\frac{2bc}{b+c}\cos\frac{A}{2}\tan\frac{A}{2} = \frac{4bc}{b+c}\sin\frac{A}{2}$$

Since, $AD \perp EF$ and $DE = DF$ and AD is bisector.

$\Rightarrow$ ΔAEF is isosceles.

Hence, (a), (b), (c), (d) are correct answers.

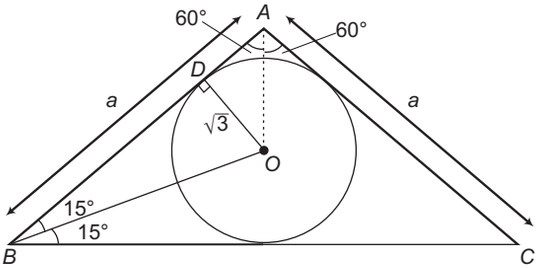
206. Let $AB = AC = a$ and $\angle A = 120°$.

$\therefore$ Area of triangle $= \frac{1}{2}a^2\sin 120°$

where, $a = AD + BD$

$$= \sqrt{3}\tan 30° + \sqrt{3}\cot 15°$$

$$= 1 + \frac{\sqrt{3}}{\tan(45° - 15°)}$$



$$\Rightarrow \quad a = 1 + \sqrt{3}\left(\frac{1 + \tan 45°\tan 30°}{\tan 45° - \tan 30°}\right)$$

$$= 1 + \sqrt{3}\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$$

$$= 4 + 2\sqrt{3}$$

$\therefore$ Area of a triangle $= \frac{1}{2}(4 + 2\sqrt{3})^2\left(\frac{\sqrt{3}}{2}\right)$

$$= (12 + 7\sqrt{3}) \text{ sq units}$$

207. Let a, b, c are the sides of ΔABC.

Now, $\frac{b+c}{a} = \frac{k(\sin B + \sin C)}{k\sin A}$

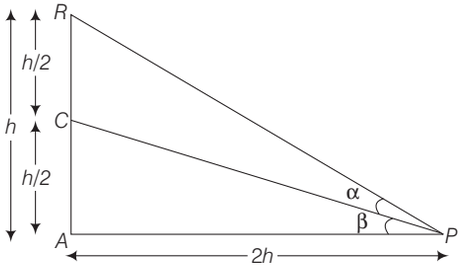
$$= \frac{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}{2\sin\frac{A}{2}\cos\frac{A}{2}}$$

$\Rightarrow$ $\frac{b+c}{a} = \frac{\cos\left(\frac{B-C}{2}\right)}{\sin\frac{A}{2}}$

Also, $\frac{b-c}{a} = \frac{\sin\left(\frac{B-C}{2}\right)}{\cos\frac{A}{2}}$

208. Let $AB = h$, then $AD = 2h$ and $AC = BC = \frac{h}{2}$

Again, let $\angle CPA = \alpha$



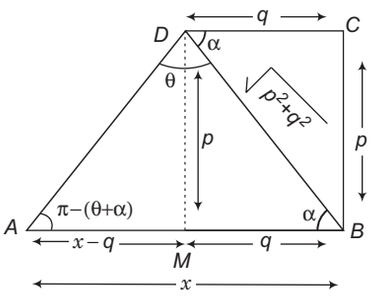
Now, in ΔABP, $\tan(\alpha+\beta) = \frac{AB}{AP} = \frac{h}{2h} = \frac{1}{2}$

Also, in ΔACP, $\tan\alpha = \frac{AC}{AP} = \frac{2}{2h} = \frac{1}{4}$

Now, $\tan\beta = \tan[(\alpha+\beta) - \alpha]$

$$= \frac{\tan(\alpha+\beta) - \tan\alpha}{1 + \tan(\alpha+\beta)\tan\alpha} = \frac{\frac{1}{2} - \frac{1}{4}}{1 + \frac{1}{2}\times\frac{1}{4}} = \frac{\frac{1}{4}}{\frac{9}{8}} = \frac{2}{9}$$

209. Let $AB = x$



In ΔDAM, $\tan(\pi - \theta - \alpha) = \frac{p}{x-q}$

$\Rightarrow$ $\tan(\theta+\alpha) = \frac{p}{q-x}$

$\Rightarrow$ $q - x = p\cot(\theta+\alpha)$

$\Rightarrow$ $x = q - p\cot(\theta+\alpha)$

$$= q - p\left(\frac{\cot\theta\cot\alpha - 1}{\cot\alpha + \cot\theta}\right) \qquad \left[\because \cot\alpha = \frac{q}{p}\right]$$

$$= q - p\left(\frac{\frac{q}{p}\cot\theta - 1}{\frac{q}{p} + \cot\theta}\right) = q - p\left(\frac{q\cot\theta - p}{q + p\cot\theta}\right)$$

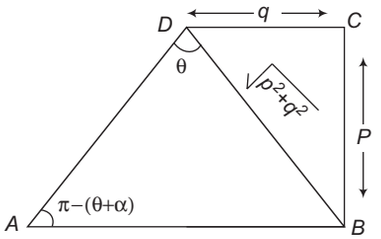
$$= q - p\left(\frac{q\cos\theta - p\sin\theta}{q\sin\theta + p\cos\theta}\right)$$

$$\Rightarrow \quad x = \frac{q^2\sin\theta + pq\cos\theta - pq\cos\theta + p^2\sin\theta}{p\cos\theta + q\sin\theta}$$

$$\Rightarrow \quad AB = \frac{(p^2 + q^2)\sin\theta}{p\cos\theta + q\sin\theta}$$

Alternate Solution

Applying sine rule in ΔABD,



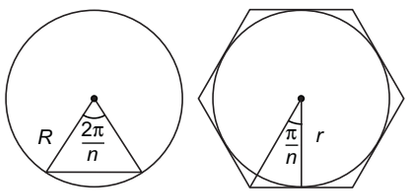
$$\frac{AB}{\sin\theta} = \frac{\sqrt{p^2 + q^2}}{\sin\{\pi - (\theta + \alpha)\}}$$

$$\Rightarrow \quad \frac{AB}{\sin\theta} = \frac{\sqrt{p^2 + q^2}}{\sin(\theta + \alpha)}$$

$$\Rightarrow \quad AB = \frac{\sqrt{p^2 + q^2}\sin\theta}{\sin\theta\cos\alpha + \cos\theta\sin\alpha}\left[\because \cos\alpha = \frac{q}{\sqrt{p^2 + q^2}}\right]$$

$$= \frac{(p^2 + q^2)\sin\theta}{p\cos\theta + q\sin\theta} \quad \text{and} \quad \sin\alpha = \frac{p}{\sqrt{p^2 + q^2}}$$

210.



By formula of regular polygon,

$$\frac{a}{2R} = \sin\frac{\pi}{n} \quad \text{and} \quad \frac{a}{2r} = \tan\frac{\pi}{n}$$

$$\therefore \quad \frac{r}{R} = \cos\frac{\pi}{n}$$

$n = 3$ gives $\frac{r}{R} = \frac{1}{2}$

$n = 4$ gives $\frac{r}{R} = \frac{1}{\sqrt{2}}$

$n = 6$ gives $\frac{r}{R} = \frac{\sqrt{3}}{2}$

211. We know that, $\frac{c}{\sin C} = 2R$

$$\Rightarrow \quad c = 2R \qquad \text{...(i)} \qquad [\because C = 90^\circ]$$

and $\tan\frac{C}{2} = \frac{r}{s - c}$

$$\Rightarrow \quad \tan\frac{\pi}{4} = \frac{r}{s - c}$$

$$\therefore \quad r = s - c$$

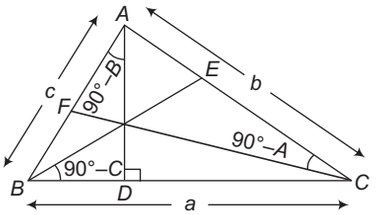
$$\Rightarrow \quad r = \frac{a + b + c}{2} - c$$

$$\Rightarrow \quad a + b - c = 2r \qquad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2(r + R) = a + b$$

212. In ΔBAD, $\cos(90^\circ - B) = \frac{AD}{c}$



$$\Rightarrow \quad AD = c\sin B$$

Similarly, $BE = a\sin C$

and $CF = b\sin A$

Since, AD, BE and CF are in HP.

So, $c\sin B$, $a\sin C$ and $b\sin A$ are in HP.

$$\Rightarrow \quad \frac{1}{\sin C\sin B}, \frac{1}{\sin A\sin C} \text{ and } \frac{1}{\sin B\sin A}$$

are in AP.

Hence, $\sin A$, $\sin B$ and $\sin C$ are in AP.

Alternate Method

$$\text{ar}(\Delta ABC) = \frac{1}{2} \times BC \times AD$$

$$\Rightarrow \quad \Delta = \frac{1}{2} \times a \times AD$$

$$\Rightarrow \quad AD = \frac{2\Delta}{a}$$

Similarly, $BE = \frac{2\Delta}{b}$

and $CF = \frac{2\Delta}{c}$

Since, AD, BE and CF are in HP.

So, $\frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ are in HP.

Hence, a, b and c are in AP.

$\therefore$ $\sin A$, $\sin B$ and $\sin C$ are in AP.

CHAPTER

04

Inverse Trigonometric Functions

Learning Part

Session 1

- Inverse Trigonometric Functions
- Inverse Function
- Domain and Range of Inverse Trigonometric Functions

Session 2

- Property I of Inverse Trigonometric Functions

Session 3

- Property II of Inverse Trigonometric Functions

Session 4

- Property III, IV and V of Inverse Trigonometric Functions

Session 5

- Property VI, VII and VIII of Inverse Trigonometric Functions

Session 6

- Property IX of Inverse Trigonometric Functions

Session 7

- Property X, XI, XII and XIII of Inverse Trigonometric Functions

Practice Part

- JEE Type Examples
- Chapter Exercises

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Session 1

Inverse Trigonometric Functions, Inverse Function, Domain and Range of Inverse Trigonometric Functions

Inverse Trigonometric Functions

Consider; $\sin\frac{\pi}{6}=\frac{1}{2}, \sin\frac{5\pi}{6}=\frac{1}{2}, \sin\frac{13\pi}{6}=\frac{1}{2},$

$\sin\frac{17\pi}{6}=\frac{1}{2}, \sin\left(-\frac{11\pi}{6}\right)=\frac{1}{2}, \sin\left(-\frac{7\pi}{6}\right)=\frac{1}{2}, \ldots$ etc.

Now, if the question is ''which is that angle or real number whose sine is $\frac{1}{2}$?''

Then there are infinite answers i.e. there is no unique answer.

Thus, the problem is, if the trigonometrical equation is $\sin y = x$, then for the given value of $x=\frac{1}{2}$, we have

$$y=\ldots,-\frac{7\pi}{6}, -\frac{11\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6},\ldots$$

i.e.

for given value of x, we get infinite value of y.

$\therefore$ Correspondence from the set

$\{x \mid x\in R; -1\le x\le 1\}$ to the set

$\{y \mid y\in R; \sin y = x\}$ is one to many correspondence

$\therefore$ It cannot be a function.

{$\because$ one-many and many-many are not function.}

However, if the question is ''which is the numerically smallest angle or real number whose sine is $\frac{1}{2}$?''

Then the answer is $\frac{\pi}{6}$.

This is one and only one answer i.e. $\frac{\pi}{6}$ is the unique answer.

In this case, the relation $\sin\frac{\pi}{6}=\frac{1}{2}$ is also written as;

$$\frac{\pi}{6}=\sin^{-1}\left(\frac{1}{2}\right) \text{ or arc } \sin\left(\frac{1}{2}\right) \quad \{\text{read as sine inverse } \frac{1}{2}\}$$

It must therefore, be noted that $\sin^{-1} x$ is an angle and denotes the smallest numerical angle, whose sine is x.

Similarly, $\cos^{-1} x$ and $\tan^{-1} x$, denotes an angle or real number; 'whose cosine is x' and 'whose tangent is x', respectively, provided that the answer given are numerically smallest available.

Note

(i) $\sin\frac{5\pi}{6}=\frac{1}{2}$ but $\frac{5\pi}{6}\neq \sin^{-1}(1/2)$

(ii) If there are two angles, one positive and the other negative having same numerical value. Then, we shall take the positive value.

e.g. $\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}$

and $\cos\left(-\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$

But we write $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)=\frac{\pi}{4}$, not $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)=-\frac{\pi}{4}$

Inverse Function

Definition If a function, say f is one to one and onto from A to B, then function g which associates each element $y\in B$ to one and only one element $x\in A$, such that $y=f(x)$, then g is called the inverse function of f, denoted by $x=g(y)$.

Usually we denote $g=f^{-1}$ (Read as f inverse)

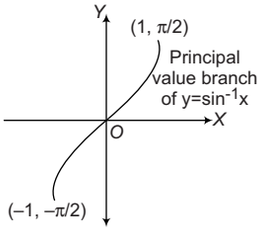
$\therefore$ $x=f^{-1}(y)$

Remark

If $y=f(x)$ and $x=g(y)$ are two functions such that $f\{g(y)\}=y$ and $g\{f(x)\}=x$ then f and g are said to be inverse functions of each other. Before we start with the definitions of $\sin^{-1}x, \cos^{-1}x$, $\tan^{-1}x$, etc., let us study the following discussion which will help us to define these inverse functions.

Domain and Range of Inverse Trigonometric Functions

1. If $\sin y = x$, then $y = \sin^{-1} x$, under certain conditions.



$\because$ $-1 \le \sin y \le 1$; but $\sin y = x$.

$\therefore$ $-1 \le x \le 1$

Also, $\sin y = -1 \Rightarrow y = -\pi/2$

and $\sin y = 1 \Rightarrow y = \pi/2$

Keeping in mind numerically smallest angles or real numbers, we have $-\pi/2 \le y \le \pi/2$

These restrictions on the values of x and y make the function $\sin y = x$, one-one and onto so that the inverse function exists, i.e. $y = \sin^{-1} x$ is meaningful.

Thus, Domain : $x \in [-1, 1]$

Range : $y \in [-\pi/2, \pi/2]$

Note

(i) We can restrict the values of y in any of the intervals $[-3\pi/2, -\pi/2]$, $[-\pi/2, \pi/2]$, $[\pi/2, 3\pi/2]$ etc. Corresponding to each such interval, we get a branch of the function $y = \sin^{-1} x$. The branch with range $[-\pi/2, \pi/2]$ is called the **principal value branch**, whereas other intervals as range give different branches of $\sin^{-1}$.

(ii) The numerically least angle is called the **principal value** of the function.

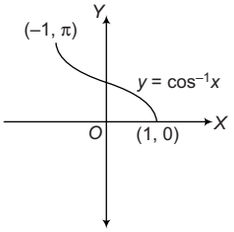
2. Let $\cos y = x$ then $y = \cos^{-1} x$, under certain conditions

$\because$ $-1 \le \cos y \le 1$.

$\therefore$ $-1 \le x \le 1$

Also, $\cos y = -1 \Rightarrow y = \pi$

and $\cos y = 1 \Rightarrow y = 0$



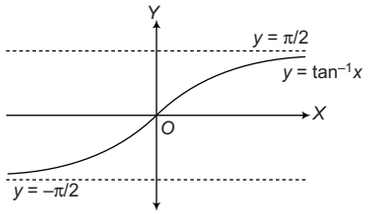
$\therefore$ $0 \le y \le \pi$

These restrictions on the values of x and y make the function $\cos y = x$ one-one and onto so that the inverse function exists i.e. $y = \cos^{-1} x$ is meaningful.

Thus, Domain : $x \in [-1, 1]$

Range : $y \in [0, \pi]$

3. If $\tan y = x$ then $y = \tan^{-1} x$, under certain conditions.



$\because$ $\tan y \in R$

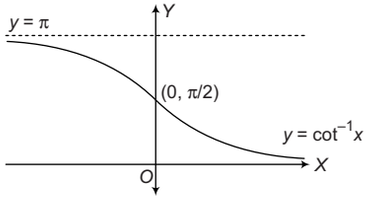
$\therefore$ $x \in R$

Also, $-\infty < \tan y < \infty \Rightarrow -\pi/2 < y < \pi/2$

These restrictions on x and y make the function, $\tan y = x$ one-one and onto so that the inverse function exists, i.e. $y = \tan^{-1} x$ is meaningful.

Thus, Domain : $x \in R$, Range : $y \in (-\pi/2, \pi/2)$

4. If $\cot y = x$, then $y = \cot^{-1} x$, under certain conditions.



$\because$ $\cot y \in R$ $\therefore$ $x \in R$;

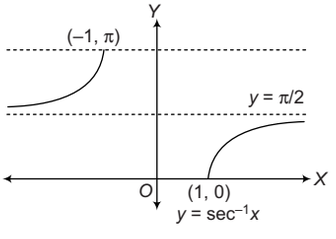
Also, $-\infty < \cot y < \infty \Rightarrow 0 < y < \pi$

These restrictions on x and y make the function, $\cot y = x$ one-one and onto so that the inverse function exists. i.e. $y = \cot^{-1} x$ is meaningful.

Thus, Domain : $x \in R$

Range : $y \in (0, \pi)$

5. If $\sec y = x$, then $y = \sec^{-1} x$, where $|x| \ge 1$ and $0 \le y \le \pi$, $y \ne \pi/2$

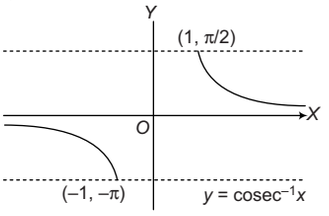


Here, Domain : $x \in R - (-1, 1)$

Range : $y \in [0, \pi] - \{\pi/2\}$

6. If $\operatorname{cosec} y = x$ then $y = \operatorname{cosec}^{-1} x$,

where $|x| \geq 1$ and $-\pi/2 \leq y \leq \pi/2,\ y \neq 0$



Here, Domain : $x \in R - (-1, 1)$

Range : $y \in [-\pi/2, \pi/2] - \{0\}$

Principal Values and Domains of Inverse Trigonometric/Circular Functions

	Function	Domain	Range (Principal Value Branch)
(i)	$y = \sin^{-1} x$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
(ii)	$y = \cos^{-1} x$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
(iii)	$y = \tan^{-1} x$	$x \in R$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
(iv)	$y = \operatorname{cosec}^{-1} x$	$x \leq -1$ or $x \geq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2},\ y \neq 0$
(v)	$y = \sec^{-1} x$	$x \leq -1$ or $x \geq 1$	$0 \leq y \leq \pi;\ y \neq \frac{\pi}{2}$
(vi)	$y = \cot^{-1} x$	$x \in R$	$0 < y < \pi$

Note

(i) 1st quadrant is common to the range of all the inverse trigonometric functions.

(ii) 3rd quadrant is not used in inverse trigonometric functions.

(iii) 4th quadrant is used in the clockwise direction i.e. $-\frac{\pi}{2} \leq y \leq 0$.

(iv) No inverse function is periodic.

(v) If no branch of an inverse trigonometric function is mentioned, then it means the principal value branch of the function.

Example 1 Find the value of $\tan\left[\cos^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right]$

Sol. Let $y = \tan\left[\cos^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right]$

$\because \quad = \tan\left[\frac{\pi}{3} + \left(-\frac{\pi}{6}\right)\right] = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$

Example 2 Find the value of $\cos\left[\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) + \frac{\pi}{6}\right]$

Sol. $\cos\left[\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) + \frac{\pi}{6}\right]$

$= \cos\left\{\frac{5\pi}{6} + \frac{\pi}{6}\right\} = \cos(\pi) = -1$

Example 3 Find domain of $\sin^{-1}(2x^2 - 1)$

Sol. Let $y = \sin^{-1}(2x^2 - 1)$

For y to be defined $-1 \leq (2x^2 - 1) \leq 1$

$\Rightarrow \ 0 \leq 2x^2 \leq 2 \ \Rightarrow \ 0 \leq x^2 \leq 1$

$\Rightarrow \ x \in [-1, 1]$

Exercise for Session 1

Find the value of the following

1. $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(\frac{1}{2}\right)\right]$

2. $\operatorname{cosec}[\sec^{-1}(-\sqrt{2}) + \cot^{-1}(-1)]$

Find the domain of the following

3. $y = \sec^{-1}(x^2 + 3x + 1)$

4. $y = \cos^{-1}\left(\frac{x^2}{1 + x^2}\right)$

5. $y = \tan^{-1}(\sqrt{x^2 - 1})$

Session 2

Property I of Inverse Trigonometric Functions

Property I

(i) $\sin^{-1}(\sin\theta)=\theta$; for all $\theta\in[-\pi/2,\pi/2]$

(ii) $\cos^{-1}(\cos\theta)=\theta$; for all $\theta\in[0,\pi]$

(iii) $\tan^{-1}(\tan\theta)=\theta$; for all $\theta\in(-\pi/2,\pi/2)$

(iv) $\text{cosec}^{-1}(\text{cosec}\,\theta)=\theta$; for all $\theta\in[-\pi/2,\pi/2],\theta\neq0$

(v) $\sec^{-1}(\sec\theta)=\theta$; for all $\theta\in[0,\pi],\theta\neq\pi/2$

(vi) $\cot^{-1}(\cot\theta)=\theta$; for all $\theta\in(0,\pi)$

Proof We know that, if $f:A\to B$ is a bijection, then

$f^{-1}:B\to A$ exists such that

$f^{-1}\ of(x)=f^{-1}(f(x))=x$ for all $x\in A$

Clearly, all these results are direct consequences of this property.

Aliter For any $\theta\in[-\pi/2,\pi/2]$, let $\sin\theta=x$. Then,

$\theta=\sin^{-1}x$

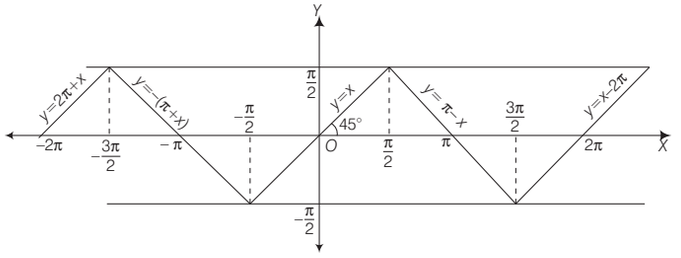
$\therefore\qquad \theta=\sin^{-1}(\sin\theta)$

Hence, $\sin^{-1}(\sin\theta)=\theta$ for all $\theta\in[-\pi/2,\pi/2]$

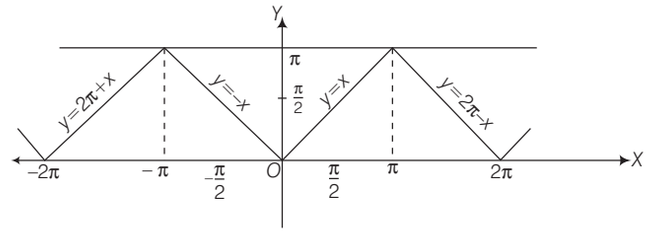
Similarly, we can prove other results.

Graphically they can be Shown As

(i) $y=\sin^{-1}(\sin x),\ x\in R,\ y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, is periodic with period 2π.

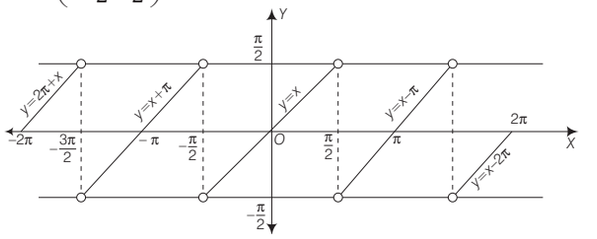


(ii) $y=\cos^{-1}(\cos x),\ x\in R,\ y\in[0,\pi]$, is periodic with period 2π.



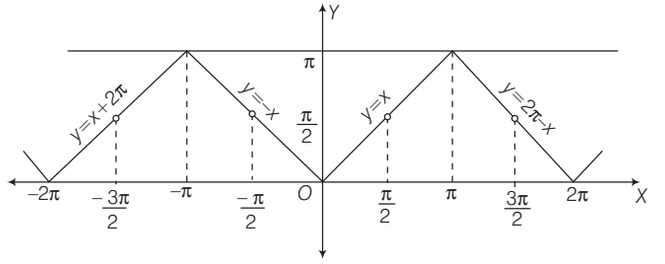
(iii) $y=\tan^{-1}(\tan x),\ x\in R-\left\{(2n-1)\frac{\pi}{2},n\in I\right\}$,

$y\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ is periodic with period π.

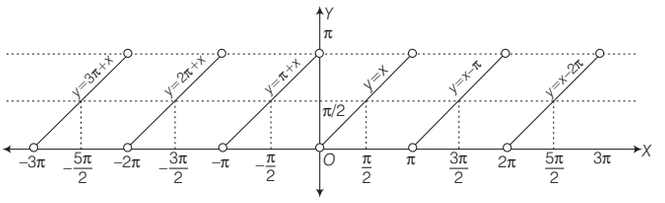


(iv) $y=\sec^{-1}(\sec x),\ x\in R-\left\{(2n-1)\frac{\pi}{2},n\in I\right\}$,

$y\in\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$, is periodic with period 2π.

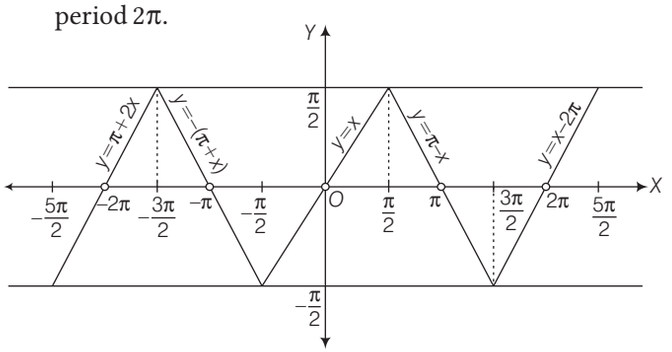


(v) $y=\cot^{-1}(\cot x),\ x\in R-\{n\pi,n\in I\},\ y\in(0,\pi)$, is periodic with period π.



(vi) $y = \text{cosec}^{-1}(\text{cosec}\, x)$,

$x \in R - \{n\pi, n \in I\}, y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$, is periodic with period 2π.



Remark

It should be noted that,

$\sin^{-1}(\sin\theta) \neq \theta$, if $\notin [-\pi/2, \pi/2]$. Infact, we have

$$\sin^{-1}(\sin\theta) = \begin{cases} -\pi - \theta, \text{ if } \theta \in [-3\pi/2, -\pi/2] \\ \theta, \text{ if } \theta \in [-\pi/2, \pi/2] \\ \pi - \theta, \text{ if } \theta \in [\pi/2, 3\pi/2] \\ -2\pi + \theta, \text{ if } \theta \in [3\pi/2, 5\pi/2] \text{ and so on} \end{cases}$$

$$\text{Similarly, } \cos^{-1}(\cos\theta) = \begin{cases} -\theta, \text{if } \theta \in [-\pi, 0] \\ \theta, \text{if } \theta \in [0, \pi] \\ 2\pi - \theta, \text{if } \theta \in [\pi, 2\pi] \\ -2\pi + \theta, \text{if } \theta \in [2\pi, 3\pi] \text{ and so on} \end{cases}$$

$$\tan^{-1}(\tan\theta) = \begin{cases} \theta + \pi, \text{if } \theta \in (-3\pi/2, -\pi/2) \\ \theta, \text{if } \theta \in (-\pi/2, \pi/2) \\ \theta - \pi, \text{if } \theta - \in (\pi/2, 3\pi/2) \\ \theta - 2\pi, \text{if } \theta \in (3\pi/2, 5\pi/2) \text{ and so on} \end{cases}$$

Example 4. Evaluate the following

(i) $\sin^{-1}(\sin\pi/4)$ (ii) $\cos^{-1}\left(\cos\frac{2\pi}{3}\right)$

(iii) $\tan^{-1}\left(\tan\frac{\pi}{3}\right)$ (iv) $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

(v) $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ (vi) $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)$

Sol. Recall that $\sin^{-1}(\sin\theta) = \theta$, if $-\pi/2 \leq \theta \leq \pi/2$

$\cos^{-1}(\cos\theta) = \theta$, if $0 \leq \theta \leq \pi$

and $\tan^{-1}(\tan\theta) = \theta$, if $-\pi/2 < \theta < \pi/2$

(i) $\sin^{-1}(\sin\pi/4) = \frac{\pi}{4}$ (ii) $\cos^{-1}\left(\cos\frac{2\pi}{3}\right) = \frac{2\pi}{3}$

(iii) $\tan^{-1}(\tan\pi/3) = \pi/3$

(iv) $\sin^{-1}\left(\sin\frac{2\pi}{3}\right) \neq \frac{2\pi}{3}$ as $\frac{2\pi}{3}$ does not lie between $-\pi/2$ and $\pi/2$.

Now, $\sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \sin^{-1}(\sin(\pi - \pi/3)) \left\{\because \frac{2\pi}{3} = \pi - \frac{\pi}{3}\right\}$

$= \sin^{-1}(\sin\pi/3) \quad \{\because \sin(\pi - \theta) = \sin\theta\}$

$= \frac{\pi}{3}$.

(v) $\cos^{-1}\left(\cos\frac{7\pi}{6}\right) \neq \frac{7\pi}{6}$, because $\frac{7\pi}{6}$ does not lie between 0 and π.

Now, $\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left(\cos\left(2\pi - \frac{5\pi}{6}\right)\right)$

$\left\{\because \frac{7\pi}{6} = 2\pi - \frac{5\pi}{6}\right\}$

$= \cos^{-1}\left(\cos\frac{5\pi}{6}\right) = \frac{5\pi}{6}$ $[\because \cos(2\pi - \theta) = \cos\theta]$

(vi) $\tan^{-1}\left(\tan\frac{2\pi}{3}\right) \neq \frac{2\pi}{3}$, because $\frac{2\pi}{3}$ does not lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

Now, $\tan^{-1}\left(\tan\frac{2\pi}{3}\right) = \tan^{-1}(\tan(\pi - \pi/3))$

$\left\{\because \frac{2\pi}{3} = \pi - \frac{\pi}{3}\right\}$

$= \tan^{-1}\left(-\tan\frac{\pi}{3}\right) \quad \{\because \tan(\pi - \theta) = -\tan\theta\}$

$= \tan^{-1}\left(\tan\left(\frac{-\pi}{3}\right)\right) \quad [\because \tan(-\theta) = -\tan\theta]$

$= -\frac{\pi}{3}$.

Example 5. Evaluate the following

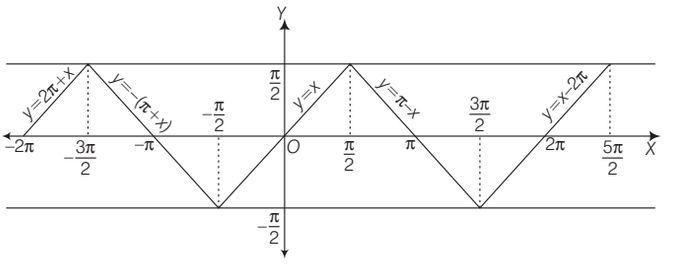
(i) $\sin^{-1}(\sin 7)$ (ii) $\sin^{-1}(\sin(-5))$

(iii) $\cos^{-1}(\cos 10)$ (iv) $\tan^{-1}(\tan(-6))$

Sol. (i) Let $y = \sin^{-1}(\sin 7)$

Note $\sin^{-1}(\sin 7) \neq 7$ as $7 \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\because \quad 2\pi < 7 < \frac{5\pi}{2}$



From the graph, we can see that if $2\pi \le x \le \frac{5\pi}{2}$, then

$y = \sin^{-1}(\sin x)$ can be written as

$$y = x - 2\pi$$

$\therefore \quad \sin^{-1}(\sin 7) = 7 - 2\pi$

Alternate Method

We know that, $\sin^{-1}(\sin\theta)=\theta$, if $-\pi/2 \le \theta \le \pi/2$

Here, $\theta = 7$ radians which does not lie between $-\pi/2$ and $\pi/2$.

But, $2\pi - 7$ and $7 - 2\pi$ both lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

Also, $\sin(7-2\pi)=\sin(-(2\pi-7)) = -\sin(2\pi-7) = \sin 7$

$\therefore \quad \sin^{-1}(\sin 7) = \sin^{-1}(\sin(7-2\pi)) = 7-2\pi$

(ii) Here, $\theta = -5$ radians. Clearly, it does not lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. But $2\pi - 5$ lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

Also, $\sin(2\pi-5) = -\sin 5 = \sin(-5)$

$\therefore \quad \sin^{-1}(\sin(-5)) = \sin^{-1}(\sin(2\pi-5)) = 2\pi - 5$

(iii) We know that, $\cos^{-1}(\cos\theta)=\theta$, if $0 \le \theta \le \pi$.

Here, $\theta = 10$ radians which does not lie between 0 and π. However, $(4\pi - 10)$ lies between 0 and π such that

$$\cos(4\pi-10) = \cos 10$$

$\therefore \quad \cos^{-1}(\cos 10) = \cos^{-1}(\cos(4\pi-10)) = 4\pi - 10$

(iv) We know that,

$\tan^{-1}(\tan\theta)=\theta$, if $-\pi/2 < \theta < \pi/2$.

Here, $\theta = -6$ radians which does not lie between $-\pi/2$ and $\pi/2$. But we find that $2\pi - 6$ lies between $-\pi/2$ and $\pi/2$ such that

$$\tan(2\pi-6) = -\tan 6 = \tan(-6)$$

$\therefore \quad \tan^{-1}(\tan(-6)) = \tan^{-1}(\tan(2\pi-6)) = 2\pi - 6$

Exercise for Session 2

1. Find the value of $\cos^{-1}\left(\cos\frac{2\pi}{3}\right) + \sin^{-1}\left(\sin\frac{2\pi}{3}\right)$.
2. Write the value of $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)$.
3. Find the value of $\sin^{-1}\left(\cos\frac{33\pi}{5}\right)$.
4. Find the value of $\cos^{-1}(\cos 13)$.
5. Find $\sin^{-1}(\sin\theta), \cos^{-1}(\cos\theta), \tan^{-1}(\tan\theta), \cot^{-1}(\cot\theta)$ for $\theta \in \left(\frac{5\pi}{2}, 3\pi\right)$.

Session 3

Property II of Inverse Trigonometric Functions

Property II

(i) $\sin(\sin^{-1} x) = x$, for all $x \in [-1,1]$

(ii) $\cos(\cos^{-1} x) = x$ for all $x \in [-1,1]$

(iii) $\tan(\tan^{-1} x) = x$ for all $x \in R$

(iv) $\operatorname{cosec}(\operatorname{cosec}^{-1} x) = x$ for all $x \in (-\infty,-1] \cup [1,\infty)$

(v) $\sec(\sec^{-1} x) = x$, for all $x \in (-\infty,-1] \cup [1,\infty)$

(vi) $\cot(\cot^{-1} x) = x$, for all $x \in R$

Proof We know that, if $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ exists such that $fof^{-1}(y) = f(f^{-1}(y)) = y$ for all $y \in B$.

Clearly, all these results are direct consequences of this property.

Aliter Let $\theta \in [-\pi/2, \pi/2]$ and $x \in [-1,1]$ such that $\sin\theta = x$.

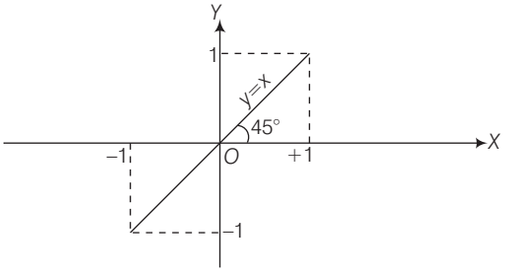
Then, $\theta = \sin^{-1} x \qquad \therefore \; x = \sin\theta = \sin(\sin^{-1} x)$

Hence, $\sin(\sin^{-1} x) = x$ for all $x \in [-1,1]$

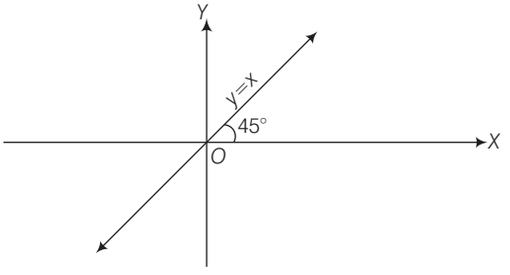
Similarly, we can prove other results.

Graphically they can be Shown As

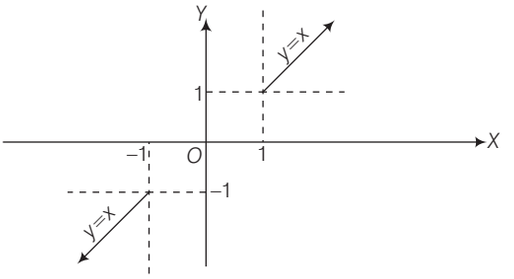
(i) $y = \sin(\sin^{-1} x) = \cos(\cos^{-1} x) = x$,
$x \in [-1, 1], y \in [-1, 1]$; y is aperiodic



(ii) $y = \tan(\tan^{-1} x) = \cot(\cot^{-1} x) = x, x \in R, y \in R$; y is aperiodic



(iii) $y = \text{cosec}\,(\text{cosec}^{-1} x) = \sec(\sec^{-1} x) = x$,
$|x| \geq 1, |y| \geq 1$; y is aperiodic



Example 6. Find the value of $\text{cosec}\left\{\cot\left(\cot^{-1}\frac{3\pi}{4}\right)\right\}$

Sol. Let $y = \text{cosec}\left\{\cot\left(\cot^{-1}\frac{3\pi}{4}\right)\right\}$ ...(i)

$\because \quad \cot(\cot^{-1} x) = x, \forall\ x \in R$

$\therefore \quad \cot\left(\cot^{-1}\frac{3\pi}{4}\right) = \frac{3\pi}{4}$

Now, from Eq. (i), we get $y = \text{cosec}\left(\frac{3\pi}{4}\right) = \sqrt{2}$

Example 7. Prove that
$\sec^2(\tan^{-1} 2) + \text{cosec}^2(\cot^{-1} 3) = 15$

Sol. $\sec^2(\tan^{-1}2) + \text{cosec}^2(\cot^{-1}3)$

$= \{1 + \tan^2(\tan^{-1}2)\} + \{1 + \cot^2(\cot^{-1}3)\}$

$= 1 + (\tan(\tan^{-1}2))^2 + 1 + (\cot(\cot^{-1}3))^2$

$= 2 + 2^2 + 3^2 = 15$

Exercise for Session 3

Evaluate the following :

1. $\cos\left\{\sin\left(\sin^{-1}\frac{\pi}{6}\right)\right\}$.
2. $\sin\left\{\cos\left(\cos^{-1}\frac{3\pi}{4}\right)\right\}$.
3. $\sin^2\left(\cos^{-1}\frac{1}{2}\right) + \cos^2\left(\sin^{-1}\frac{1}{3}\right)$.
4. $\tan^2(\sec^{-1}2) + \cot^2(\text{cosec}^{-1}3)$.
5. Find the solutions of the equation $\cos(\cos^{-1} x) = \text{cosec}\,(\text{cosec}^{-1} x)$.

Session 4

Property III, IV and V of Inverse Trigonometric Functions

Property III

(i) $\sin^{-1}(-x) = -\sin^{-1}(x)$, for all $x \in [-1,1]$

(ii) $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$, for all $x \in [-1,1]$

(iii) $\tan^{-1}(-x) = -\tan^{-1} x$, for all $x \in R$

(iv) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x$, for all $x \in (-\infty,-1] \cup [1,\infty)$

(v) $\sec^{-1}(-x) = \pi - \sec^{-1} x$, for all $x \in (-\infty,-1] \cup [1,\infty)$

(vi) $\cot^{-1}(-x) = \pi - \cot^{-1} x$ for all $x \in R$

Proof (i) Clearly, $-x \in [-1,1]$ for all $x \in [-1,1]$

Let $\sin^{-1}(-x) = \theta$ ...(i)

Then, $-x = \sin\theta$

$\Rightarrow$ $x = -\sin\theta$

$\Rightarrow$ $x = \sin(-\theta)$

$\Rightarrow$ $-\theta = \sin^{-1} x$

$\{\because x \in [-1,1]$ and $-\theta \in [-\pi/2, \pi/2]$ for all $\theta \in [-\pi/2, \pi/2]\}$

$\Rightarrow$ $\theta = -\sin^{-1} x$ ...(ii)

From Eqs. (i) and (ii), we get

$$\sin^{-1}(-x) = -\sin^{-1}(x)$$

(ii) Clearly, $-x \in [-1,1]$ for all $x \in [-1,1]$

Let $\cos^{-1}(-x) = \theta$ ...(i)

Then, $-x = \cos\theta$

$\Rightarrow$ $x = -\cos\theta$

$\Rightarrow$ $x = \cos(\pi - \theta)$

$\Rightarrow$ $\cos^{-1} x = \pi - \theta$

$\{\because x \in [-1,1]$ and $\pi - \theta \in [0,\pi]$ for all $\theta \in [0,\pi]\}$

$\Rightarrow$ $\theta = \pi - \cos^{-1} x$ ...(ii)

From Eqs. (i) and (ii), we get

$$\cos^{-1}(-x) = \pi - \cos^{-1} x$$

Similarly, we can prove other results.

Example 8. Find the value of $\cos^{-1}\{\sin(-5)\}$

Sol. Let $y = \cos^{-1}\{\sin(-5)\}$

$= \cos^{-1}(-\sin 5)$

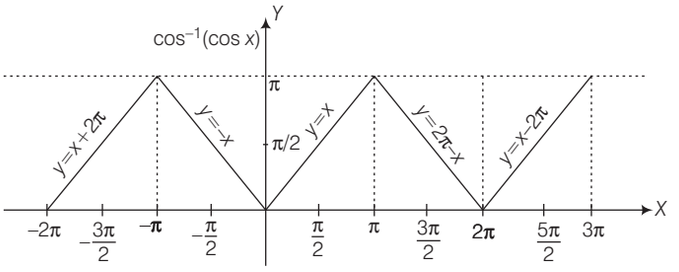
$\because \cos^{-1}(-x) = \pi - \cos^{-1} x, |x| \le 1$

$\therefore$ $y = \pi - \cos^{-1}(\sin 5)$

$= \pi - \cos^{-1}\left\{\cos\left(\frac{\pi}{2} - 5\right)\right\}$ ... (i)

Here, $-2\pi < \left(\frac{\pi}{2} - 5\right) < -\pi$

and the graph of $\cos^{-1}(\cos x)$ is as



From the graph, we can see that if $-2\pi \le x \le -\pi$, then $y = \cos^{-1}(\cos x)$ can be written as $y = x + 2\pi$.

$\therefore$ From the graph

$$\cos^{-1}\left\{\cos\left(\frac{\pi}{2} - 5\right)\right\} = \left(\frac{\pi}{2} - 5\right) + 2\pi = \left(\frac{5\pi}{2} - 5\right)$$

Now, from Eq. (i), we get

$$y = \pi - \left(\frac{5\pi}{2} - 5\right) \Rightarrow y = 5 - \frac{3\pi}{2}$$

Example 9. Evaluate the following

(i) $\sin^{-1}\left(\sin\left(\frac{-3\pi}{4}\right)\right)$ (ii) $\cot^{-1}(\cot(-4))$

Sol. (i) $\sin^{-1}\left(\sin\left(\frac{-3\pi}{4}\right)\right) = \sin^{-1}\left(-\sin^{-1}\left(\frac{3\pi}{4}\right)\right) = -\sin^{-1}\left(\sin\frac{3\pi}{4}\right)$

$= -\sin^{-1}\left(\sin\left(\pi - \frac{\pi}{4}\right)\right) = -\sin^{-1}\left(\sin\frac{\pi}{4}\right) = -\frac{\pi}{4}$

(ii) $\cot^{-1}(\cot(-4)) = \cot^{-1}(-\cot 4) = \pi - \cot^{-1}(\cot 4)$

$= \pi - \cot^{-1}(\cot(\pi + (4 - \pi)))$

$= \pi - \cot^{-1}(\cot(4 - \pi)) = 2\pi - 4$

Example 10. Evaluate the following

(i) $\sin\left(\frac{\pi}{2} - \sin^{-1}\left(\frac{-1}{2}\right)\right)$

(ii) $\sin\left(\frac{\pi}{2} - \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

Sol. (i) $\sin\left(\frac{\pi}{2}-\sin^{-1}\left(-\frac{1}{2}\right)\right)=\sin\left(\frac{\pi}{2}+\sin^{-1}\left(\frac{1}{2}\right)\right)$

$$=\sin\left(\frac{\pi}{2}+\frac{\pi}{6}\right)=\sin\frac{2\pi}{3}=\frac{\sqrt{3}}{2}$$

(ii) $\sin\left(\frac{\pi}{2}-\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)=\sin\left(\frac{\pi}{2}+\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$

$$=\sin\left(\frac{\pi}{2}+\frac{\pi}{3}\right)=\sin\frac{5\pi}{6}=\frac{1}{2}$$

Property IV

(i) $\sin^{-1}\left(\frac{1}{x}\right)=\text{cosec}^{-1}x$, for all $x\in(-\infty,-1]\cup[1,\infty)$

(ii) $\cos^{-1}\left(\frac{1}{x}\right)=\sec^{-1}x$, for all $x\in(-\infty,-1]\cup[1,\infty)$

(iii) $\tan^{-1}\left(\frac{1}{x}\right)=\begin{cases}\cot^{-1}x, \text{for } x>0\\ -\pi+\cot^{-1}x, \text{for } x<0\end{cases}$

Proof

(i) Let $\text{cosec}^{-1}x=\theta$ ...(i)

Then, $x=\text{cosec }\theta$

$\Rightarrow \frac{1}{x}=\sin\theta$

$\because x\in(-\infty,-1]\cup[1,\infty) \Rightarrow \frac{1}{x}\in[-1,1]-\{0\}$

$\text{cosec}^{-1}x=\theta \Rightarrow \theta\in[-\pi/2,\pi/2]-\{0\}$

$\Rightarrow \theta=\sin^{-1}\left(\frac{1}{x}\right)$ ...(ii)

From Eqs. (i) and (ii), we get

$$\sin^{-1}\left(\frac{1}{x}\right)=\text{cosec}^{-1}x$$

(ii) Let $\sec^{-1}x=\theta$ ...(i)

Then, $x\in(-\infty,-1]\cup[1,\infty)$ and $\theta\in[0,\pi]-\{\pi/2\}$

Now, $\sec^{-1}x=\theta$

$\Rightarrow x=\sec\theta \Rightarrow \frac{1}{x}=\cos\theta$

$\Rightarrow \theta=\cos^{-1}\left(\frac{1}{x}\right)$ ...(ii)

$\left\{\begin{array}{l}\because x=(-\infty,-1]\cup[1,\infty)\\ \Rightarrow \frac{1}{x}\in[-1,1]-\{0\} \text{ and } \theta\in[0,\pi]-\left\{\frac{\pi}{2}\right\}\end{array}\right.$

From Eqs. (i) and (ii), we get

$$\cos^{-1}\left(\frac{1}{x}\right)=\sec^{-1}(x)$$

(iii) Let $\cot^{-1}x=\theta$. Clearly, $x\in R$, $x\neq0$ and $\theta\in(0,\pi)$ ...(i)

Now, *two cases arises*

Case I When $x>0$

In this case, $\theta\in(0,\pi/2)$

$\therefore \cot^{-1}x=\theta$

$\Rightarrow x=\cot\theta$

$\Rightarrow \frac{1}{x}=\tan\theta$

$\theta=\tan^{-1}\left(\frac{1}{x}\right)$ ...(ii)

$\{\because \theta\in(0,\pi/2)\}$

From Eqs. (i) and (ii), we get

$$\tan^{-1}\left(\frac{1}{x}\right)=\cot^{-1}x, \text{ for all } x>0.$$

***Case* II** When $x<0$

In this case, $\theta\in(\pi/2,\pi)$ $\{\because x=\cot\theta<0\}$

Now, $\frac{\pi}{2}<\theta<\pi$

$\Rightarrow -\frac{\pi}{2}<\theta-\pi<0$

$\Rightarrow \theta-\pi\in(-\pi/2,0)$

$\therefore \cot^{-1}x=\theta$

$\Rightarrow x=\cot\theta$

$\Rightarrow \frac{1}{x}=\tan\theta$

$\Rightarrow \frac{1}{x}=-\tan(\pi-\theta)$ $\{\because \tan(\pi-\theta)=-\tan\theta\}$

$\Rightarrow \frac{1}{x}=\tan(\theta-\pi)$

$\Rightarrow \theta-\pi=\tan^{-1}\left(\frac{1}{x}\right)$ $\{\because \theta-\pi\in(-\pi/2,0)\}$

$\Rightarrow \tan^{-1}\left(\frac{1}{x}\right)=-\pi+\theta$ ...(iii)

From Eqs. (i) and (iii), we get

$$\tan^{-1}\left(\frac{1}{x}\right)=-\pi+\cot^{-1}x, \text{ if } x<0$$

Hence, $\tan^{-1}\left(\frac{1}{x}\right)=\begin{cases}\cot^{-1}x, \text{for } x>0\\ -\pi+\cot^{-1}x, \text{for } x<0\end{cases}$

Example 11. Evaluate the following

(i) $\sin\left(\text{cosec}^{-1}\frac{5}{3}\right)$ (ii) $\cot\left(\tan^{-1}\frac{3}{4}\right)$

Sol. (i) $\sin\left(\text{cosec}^{-1}\frac{5}{3}\right)=\sin\left(\sin^{-1}\frac{3}{5}\right)=\frac{3}{5}$

(ii) $\cot\left(\tan^{-1}\frac{3}{4}\right)=\cot\left(\cot^{-1}\frac{4}{3}\right)=\frac{4}{3}$

Example 12. Find the value of $\tan\left\{\cot^{-1}\left(\frac{-2}{3}\right)\right\}$

Sol. Let $y=\tan\left\{\cot^{-1}\left(\frac{-2}{3}\right)\right\}$ ...(i)

$\because \cot^{-1}(-x)=\pi-\cot^{-1}x,\ x\in R$

$\therefore$ Eq. (i) can be written as

$$y=\tan\left\{\pi-\cot^{-1}\left(\frac{2}{3}\right)\right\}$$

$$y=-\tan\left(\cot^{-1}\frac{2}{3}\right)$$

$\because \cot^{-1}x=\tan^{-1}\frac{1}{x}$ if $x>0$

$\therefore \quad y=-\tan\left(\tan^{-1}\frac{3}{2}\right)\Rightarrow y=-\frac{3}{2}$

Property V

(i) $\sin^{-1}x+\cos^{-1}x=\pi/2$, for all $x\in[-1,1]$

(ii) $\tan^{-1}x+\cot^{-1}x=\pi/2$, for all $x\in R$

(iii) $\sec^{-1}x+\text{cosec}^{-1}x=\frac{\pi}{2}$, for all $x\in(-\infty,-1]\cup[1,\infty)$

Proof

(i) Let, $\sin^{-1}x=\theta$ ...(i)

Then, $\theta\in[-\pi/2,\pi/2]$ $\{\because x\in[-1,1]\}$

$\Rightarrow -\pi/2\le\theta\le\pi/2$

$\Rightarrow -\pi/2\le-\theta\le\pi/2$

$\Rightarrow 0\le\frac{\pi}{2}-\theta\le\pi \Rightarrow \frac{\pi}{2}-\theta\in[0,\pi]$

Now, $\sin^{-1}x=\theta \Rightarrow x=\sin\theta$

$\Rightarrow x=\cos\left(\frac{\pi}{2}-\theta\right)$

$\Rightarrow \cos^{-1}x=\frac{\pi}{2}-\theta$

$\{\because x\in[-1,1]\text{ and }(\pi/2-\theta)\in[0,\pi]\}$

$\Rightarrow \theta+\cos^{-1}x=\pi/2$ ...(ii)

From Eqs. (i) and (ii), we get

$$\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$$

(ii) Let, $\tan^{-1}x=\theta$ ...(i)

Then, $\theta\in(-\pi/2,\pi/2)$ $\{\because x\in R\}$

$\Rightarrow -\frac{\pi}{2}<\theta<\frac{\pi}{2}$

$\Rightarrow -\frac{\pi}{2}<-\theta<\frac{\pi}{2}$

$\Rightarrow 0<\frac{\pi}{2}-\theta<\pi$

$\Rightarrow \left(\frac{\pi}{2}-\theta\right)\in(0,\pi)$

Now, $\tan^{-1}x=\theta$

$\Rightarrow x=\tan\theta$

$\Rightarrow x=\cot(\pi/2-\theta)$

$\Rightarrow \cot^{-1}x=\frac{\pi}{2}-\theta$ $\{\because \pi/2-\theta\in(0,\pi)\}$

$\Rightarrow \theta+\cot^{-1}x=\frac{\pi}{2}$ ...(ii)

From Eqs. (i) and (ii), we get

$$\tan^{-1}x+\cot^{-1}x=\pi/2$$

(iii) Let, $\sec^{-1}x=\theta$ ...(i)

Then, $\theta\in[0,\pi]-\{\pi/2\}$ $\{\because x\in(-\infty,-1]\cup[1,\infty)\}$

$\Rightarrow 0\le\theta\le\pi,\ \theta\ne\pi/2$

$\Rightarrow -\pi\le-\theta\le0,\ \theta\ne\pi/2$

$\Rightarrow -\frac{\pi}{2}\le\frac{\pi}{2}-\theta\le\frac{\pi}{2},\ \frac{\pi}{2}-\theta\ne0$

$\Rightarrow \left(\frac{\pi}{2}-\theta\right)\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right],\ \frac{\pi}{2}-\theta\ne0$

Now, $\sec^{-1}x=\theta$

$\Rightarrow x=\sec\theta$

$\Rightarrow x=\text{cosec}(\pi/2-\theta)$

$\Rightarrow \text{cosec}^{-1}x=\pi/2-\theta$

$\left\{\because\left(\frac{\pi}{2}-\theta\right)\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right],\frac{\pi}{2}-\theta\ne0\right\}$

$\Rightarrow \theta+\text{cosec}^{-1}x=\pi/2$ ...(ii)

From Eqs. (i) and (ii), we get

$$\sec^{-1}x+\text{cosec}^{-1}x=\pi/2$$

Example 13. ***Find the value of*** $\sin(2\cos^{-1}x + \sin^{-1}x)$ ***when*** $x = \frac{1}{5}$

Sol. Let $y = \sin[2\cos^{-1}x + \sin^{-1}x]$

$\because \quad \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, |x| \le 1$

$\therefore \quad y = \sin\left[2\cos^{-1}x + \frac{\pi}{2} - \cos^{-1}x\right]$

$= \sin\left[\frac{\pi}{2} + \cos^{-1}x\right] = \cos(\cos^{-1}x)$

$\because \quad x = \frac{1}{5}$

$\therefore \quad y = \cos\left(\cos^{-1}\frac{1}{5}\right)$ ... (i)

$\because \quad \cos(\cos^{-1}x) = x$ if $x \in [-1, 1]$

and $\frac{1}{5} \in [-1,1]$

$\therefore \quad \cos\left(\cos^{-1}\frac{1}{5}\right) = \frac{1}{5}$

Now, from Eq. (i), we get $y = \frac{1}{5}$

Example 14. ***Solve*** $\sin^{-1}x - \cos^{-1}x = \cos^{-1}\frac{\sqrt{3}}{2}$.

Sol. We have, $\sin^{-1}x - \cos^{-1}x = \cos^{-1}\frac{\sqrt{3}}{2}$

$\Rightarrow \quad \sin^{-1}x - \cos^{-1}x = \frac{\pi}{6}$ ...(i)

Also, $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ ...(ii)

On solving Eqs. (i) and (ii), we get $\sin^{-1}x = \frac{\pi}{3}$

and $\cos^{-1}x = \frac{\pi}{6}$

$\Rightarrow \quad x = \frac{\sqrt{3}}{2}$ is the only solution.

Exercise for Session 4

Evaluate the following

1. $\tan^{-1}\left\{\tan\left(-\frac{7\pi}{8}\right)\right\}$.
2. $\tan^{-1}\left\{\cot\left(-\frac{1}{4}\right)\right\}$.
3. $\sec\left(\cos^{-1}\left(\frac{2}{3}\right)\right)$.
4. $\operatorname{cosec}\left(\sin^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right)$.
5. $\cos\left[\cos^{-1}\left(\frac{-1}{3}\right) - \sin^{-1}\left(\frac{1}{3}\right)\right]$.
6. If $\sin^{-1}x = \frac{\pi}{5}$, for some $x \in (-1, 1)$, then find $\cos^{-1}x$.
7. If $\sec^{-1}x = \operatorname{cosec}^{-1}y$, then find the value of $\cos^{-1}\frac{1}{x} + \cos^{-1}\frac{1}{y}$.
8. Prove that $\tan^{-1}x + \tan^{-1}\frac{1}{x} = \begin{cases} \pi/2, \text{if } x > 0 \\ -\pi/2, \text{if } x < 0 \end{cases}$.

Solve the following

9. $5\tan^{-1}x + 3\cot^{-1}x = 2\pi$
10. $4\sin^{-1}x = \pi - \cos^{-1}x$

Session 5

Property VI, VII and VIII of Inverse Trigonometric Functions

Property VI

(i) $\tan^{-1} x + \tan^{-1} y$

$$= \begin{cases} \tan^{-1}\left(\dfrac{x+y}{1-xy}\right), \text{if } xy < 1 \\ \pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right), \text{if } x > 0, y > 0 \text{ and } xy > 1 \\ -\pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right), \text{if } x < 0, y < 0 \text{ and } xy > 1 \end{cases}$$

(ii) $\tan^{-1} x - \tan^{-1} y$

$$= \begin{cases} \tan^{-1}\left(\dfrac{x-y}{1+xy}\right), \text{if } xy > -1 \\ \pi + \tan^{-1}\left(\dfrac{x-y}{1+xy}\right), \text{if } x > 0, y < 0 \text{ and } xy < -1 \\ -\pi + \tan^{-1}\left(\dfrac{x-y}{1+xy}\right), \text{if } x < 0, y > 0 \text{ and } xy < -1 \end{cases}$$

Proof

(i) Let $\tan^{-1} x = A$ and $\tan^{-1} y = B$. Then,

$x = \tan A$ and $y = \tan B$ and $A, B \in (-\pi/2, \pi/2)$

$$\therefore \quad \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{x+y}{1-xy} \qquad \ldots(\text{i})$$

Now, *the following cases arises.*

Case I When $x > 0, y > 0$ and $xy < 1$

In this case, $x > 0, y > 0$ and $xy < 1 \Rightarrow \dfrac{x+y}{1-xy} > 0$

$$\Rightarrow \quad \tan(A+B) > 0$$

$\Rightarrow A + B$ lies in I quadrant or in III quadrant.

$$\Rightarrow \quad 0 < A + B < \pi/2$$

$$\left[\because \begin{matrix} x > 0 \Rightarrow 0 < A < \pi/2 \\ y > 0 \Rightarrow 0 < B < \pi/2 \end{matrix} \right\} \Rightarrow 0 < A + B < \pi\right]$$

$$\therefore \quad \tan(A+B) = \frac{x+y}{1-xy} \qquad \{\text{from Eq. (i)}\}$$

$$\Rightarrow \quad A + B = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\Rightarrow \quad \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

Case II When $x < 0, y < 0$ and $xy < 1$.

In this case;

$$x < 0, y < 0 \text{ and } xy < 1$$

$$\Rightarrow \quad \frac{x+y}{1-xy} < 0$$

$$\Rightarrow \quad \tan(A+B) < 0 \qquad \{\text{from Eq. (i)}\}$$

$\Rightarrow A + B$ lies in II quadrant or in IV quadrant.

$\Rightarrow A + B$ lies in IV quadrant.

$$\left[\because \begin{matrix} x < 0 \Rightarrow -\pi/2 < A < 0 \\ y < 0 \Rightarrow -\pi/2 < B < 0 \end{matrix} \right\} \Rightarrow -\pi < A + B < 0\right]$$

$$\Rightarrow \quad \frac{-\pi}{2} < A + B < 0$$

$$\therefore \quad \tan(A+B) = \frac{x+y}{1-xy} \qquad \{\text{from Eq. (i)}\}$$

$$\Rightarrow \quad A + B = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\Rightarrow \quad \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

Case III When $x > 0$ and $y < 0$ or $x < 0$ and $y > 0$

Consider,

$$x > 0 \text{ and } y < 0 \Rightarrow A \in (0, \pi/2)$$

and $\quad B \in (-\pi/2, 0)$

$$\Rightarrow \quad A + B \in (-\pi/2, \pi/2)$$

$$\therefore \quad \tan(A+B) = \frac{x+y}{1-xy} \qquad \{\text{from Eq. (i)}\}$$

$$\Rightarrow \quad A + B = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\Rightarrow \quad \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

Similarly, if $x < 0$ and $y > 0$, we have

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right);$$

It follows from above three cases that

$$\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right) \text{ if } xy<1$$

Case IV $x>0, y>0$ and $xy>1$.

In this case, we have

$$x>0, y>0 \text{ and } xy>1$$

$$\Rightarrow \quad \frac{x+y}{1-xy}<0$$

$$\Rightarrow \quad \tan(A+B)<0 \qquad \left\{\text{from Eq. (i), } \tan(A+B)=\frac{x+y}{1-xy}\right\}$$

$\Rightarrow A+B$ lies in II quadrant or in IV quadrant.

$\Rightarrow A+B$ lies in II quadrant.

$$\left\{\because x>0, y>0 \Rightarrow A, B\in(0,\pi/2) \Rightarrow A+B\in(0,\pi)\right.$$

$$\Rightarrow \quad \pi/2<A+B<\pi$$

$$\Rightarrow \quad \frac{\pi}{2}-\pi<(A+B)-\pi<0$$

$$\Rightarrow \quad -\pi/2<(A+B)-\pi<0$$

$$\therefore \tan(A+B)=\frac{x+y}{1-xy}$$

$$\Rightarrow \quad -\tan\{\pi-(A+B)\}=\frac{x+y}{1-xy}$$

$$\{\because \tan(\pi-(A+B))=-\tan(A+B)\}$$

$$\Rightarrow \quad \tan((A+B)-\pi)=\frac{x+y}{1-xy}$$

$$\Rightarrow \quad A+B-\pi=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\Rightarrow \quad A+B=\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\Rightarrow \quad \tan^{-1}x+\tan^{-1}y=\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

Case V When $x<0, y<0$ and $xy>1$

In this case, we have

$$x<0, y<0 \text{ and } xy>1$$

$$\Rightarrow \quad \frac{x+y}{1-xy}>0$$

$$\Rightarrow \quad \tan(A+B)>0 \qquad \left\{\text{from Eq. (i), } \tan(A+B)=\frac{x+y}{1-xy}\right\}$$

$\Rightarrow \tan(A+B)$ lies either in I quadrant or III quadrant.

$\{\because x<0, y<0 \Rightarrow A, B\in(-\pi/2,0) \Rightarrow A+B\in(-\pi,0)\}$

$\Rightarrow A+B$ lies in III quadrant.

$$\Rightarrow \quad -\pi<A+B<-\pi/2$$

$$\Rightarrow \quad \pi-\pi<\pi+(A+B)<\pi-\pi/2$$

$$\Rightarrow \quad 0<\pi+(A+B)<\pi/2$$

Now, $\tan(A+B)=\dfrac{x+y}{1-xy}$ {from Eq. (i)}

$$\Rightarrow \quad \tan(\pi+(A+B))=\frac{x+y}{1-xy}$$

$$\Rightarrow \quad \pi+A+B=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\Rightarrow \quad A+B=-\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\Rightarrow \quad \tan^{-1}x+\tan^{-1}y=-\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

(ii) This property can be proved by replacing y by $-y$ in above results.

Remark

If $x_1, x_2, x_3, \ldots, x_n \in R$, then

$$\tan^{-1}x_1+\tan^{-1}x_2+\ldots+\tan^{-1}x_n=\tan^{-1}\left(\frac{S_1-S_3+S_5\ldots}{1-S_2+S_4-S_6\ldots}\right)$$

where, S_k denotes the sum of the product of $x_1, x_2, \ldots, x_n$ taking k at a time.

Example 15. Prove that

(i) $\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{13}=\tan^{-1}\frac{2}{9}$

(ii) $\tan^{-1}2+\tan^{-1}3=\frac{3\pi}{4}$

(iii) $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

(iv) $\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{3}{5}-\tan^{-1}\frac{8}{19}=\frac{\pi}{4}$

(v) $\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

Sol. (i) $\text{LHS}=\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{13}=\tan^{-1}\left\{\dfrac{\frac{1}{7}+\frac{1}{13}}{1-\frac{1}{7}\times\frac{1}{13}}\right\}$

$$\left\{\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right); \text{if } xy<1\right\}$$

$$=\tan^{-1}\left(\frac{20}{90}\right)=\tan^{-1}\left(\frac{2}{9}\right)=\text{RHS}.$$

(ii) $\tan^{-1}2+\tan^{-1}3=\pi+\tan^{-1}\left(\frac{2+3}{1-2\times3}\right)$

$\left\{\text{using; } \tan^{-1}x+\tan^{-1}y=\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right), \text{if } xy>1\right\}$

$=\pi+\tan^{-1}(-1)$

$=\pi-\pi/4=3\pi/4$

(iii) $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}$

$=\left(\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{5}\right)+\tan^{-1}\frac{1}{8}$

$=\tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{5}}{1-\frac{1}{2}\times\frac{1}{5}}\right)+\tan^{-1}\left(\frac{1}{8}\right)$

$\left\{\text{using; } \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right), \text{if } xy<1\right\}$

$=\tan^{-1}\left(\frac{7}{9}\right)+\tan^{-1}\left(\frac{1}{8}\right)$

$=\tan^{-1}\left(\frac{\frac{7}{9}+\frac{1}{8}}{1-\frac{7}{9}\times\frac{1}{8}}\right)=\tan^{-1}\left(\frac{65}{65}\right)=\tan^{-1}(1)=\frac{\pi}{4}$

(iv) $\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{3}{5}-\tan^{-1}\frac{8}{19}$

$=\left(\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{3}{5}\right)-\tan^{-1}\frac{8}{19}$

$=\tan^{-1}\left(\frac{\frac{3}{4}+\frac{3}{5}}{1-\frac{3}{4}\times\frac{3}{5}}\right)-\tan^{-1}\left(\frac{8}{19}\right)$

$=\tan^{-1}\left(\frac{27}{11}\right)-\tan^{-1}\left(\frac{8}{19}\right)=\tan^{-1}\left(\frac{\frac{27}{11}-\frac{8}{19}}{1+\frac{27}{11}\times\frac{8}{19}}\right)$

$=\tan^{-1}\left(\frac{425}{425}\right)=\tan^{-1}1=\frac{\pi}{4}$

(v) $\left(\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}\right)+\left(\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}\right)$

$\Rightarrow \quad \tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\times\frac{1}{7}}\right)+\tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\times\frac{1}{8}}\right)$

$\Rightarrow \quad \tan^{-1}\left(\frac{6}{17}\right)+\tan^{-1}\left(\frac{11}{23}\right)$

$\Rightarrow \quad \tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\times\frac{11}{23}}\right)=\tan^{-1}\left(\frac{325}{325}\right)=\tan^{-1}(1)=\frac{\pi}{4}$

Example 16. Prove that

$\tan^{-1}\frac{yz}{xr}+\tan^{-1}\frac{zx}{yr}+\tan^{-1}\frac{xy}{zr}=\frac{\pi}{2}$, where

$x^2+y^2+z^2=r^2$

Sol. We know that,

$$\tan^{-1}x_1+\tan^{-1}x_2+\tan^{-1}x_3+\ldots+\tan^{-1}x_n$$
$$=\tan^{-1}\left(\frac{S_1-S_3+\ldots}{1-S_2+S_4\ldots}\right)$$

where, S_k denotes the sum of the product of $x_1, x_2, \ldots, x_n$ taking k at a time. Therefore,

$$\text{LHS}=\tan^{-1}\left(\frac{yz}{xr}\right)+\tan^{-1}\left(\frac{xz}{yr}\right)+\tan^{-1}\left(\frac{xy}{zr}\right)$$
$$=\tan^{-1}\left(\frac{\frac{yz}{xr}+\frac{xz}{yr}+\frac{xy}{zr}-\frac{xyz}{r^3}}{1-\left(\frac{x^2+y^2+z^2}{r^2}\right)}\right)$$
$$=\tan^{-1}\left(\frac{r^2\left(\frac{xy}{zr}+\frac{yz}{xr}+\frac{zx}{yr}-\frac{xyz}{r^3}\right)}{(x^2+y^2+z^2)-(x^2+y^2+z^2)}\right),$$

where $r^2=x^2+y^2+z^2$

$=\tan^{-1}(\infty)=\frac{\pi}{2}=\text{LHS}$

Example 17. Show that $(\tan^{-1}1+\tan^{-1}2+\tan^{-1}3)=\pi$

Sol. $\tan^{-1}2+\tan^{-1}3=\pi+\tan^{-1}\left(\frac{2+3}{1-2\times3}\right)$ {as (2) (3) >1}

$=\pi+\tan^{-1}(-1)=\pi-\tan^{-1}(1)$ ...(i)

$\therefore \quad \tan^{-1}(1)+\tan^{-1}(2)+\tan^{-1}(3)=\tan^{-1}(1)+\pi-\tan^{-1}(1)$ {using Eq. (i)}

$=\pi$

Example 18. Solve for x;

$\tan^{-1}(x+1)+\tan^{-1}x+\tan^{-1}(x-1)=\tan^{-1}3$

Sol. $\tan^{-1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}3-\tan^{-1}x$

$\Rightarrow \tan^{-1}\left(\frac{x+1+x-1}{1-(x+1)(x-1)}\right)=\tan^{-1}\left(\frac{3-x}{1+3x}\right)$, when $x^2-1<1$

and $3x<1$

$\Rightarrow \quad \tan^{-1}\left(\frac{2x}{2-x^2}\right)=\tan^{-1}\left(\frac{3-x}{1+3x}\right)$, when $-\sqrt{2}<x<\sqrt{2}$

and $\quad x<\frac{1}{3}$

$\Rightarrow \quad \frac{2x}{2-x^2}=\frac{3-x}{1+3x}$, when $-\sqrt{2}<x<\frac{1}{3}$

$\Rightarrow \quad 2x(1+3x)=(2-x^2)(3-x)$, when $-\sqrt{2}<x<\frac{1}{3}$

$\Rightarrow \quad 2x+6x^2=6-2x-3x^2+x^3$

$\Rightarrow \quad x^3-9x^2-4x+6=0$

$\Rightarrow \quad (x+1)(x^2-10x+6)=0$, when $-\sqrt{2}<x<\frac{1}{3}$

$\Rightarrow \quad x=-1$ and neglecting $x^2-10x+6=0$ as its roots does not $\in\left(-\sqrt{2},\frac{1}{3}\right)$

$\therefore \quad x=-1$

Example 19. Find the sum of the series $\sum_{m=1}^{\infty}\tan^{-1}\left(\frac{2m}{m^4+m^2+2}\right)$

Sol. We have, $\sum_{m=1}^{\infty}\tan^{-1}\left(\frac{2m}{m^4+m^2+2}\right)$

$$=\sum_{m=1}^{\infty}\tan^{-1}\left(\frac{(m^2+m+1)-(m^2-m+1)}{1+\{(m^2+1)^2-m^2\}}\right)$$

$$=\sum_{m=1}^{\infty}\tan^{-1}\left(\frac{(m^2+m+1)-(m^2-m+1)}{1+(m^2+1+m)(m^2+1-m)}\right)$$

$$=\sum_{m=1}^{\infty}\tan^{-1}(m^2+m+1)-\tan^{-1}(m^2-m+1)$$

$$=\{\tan^{-1}(3)-\tan^{-1}(1)\}+\{\tan^{-1}(7)-\tan^{-1}(3)\}+\{\tan^{-1}(13)-\tan^{-1}(7)\}+\ldots+\infty$$

$$=\tan^{-1}\infty-\tan^{-1}1=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$$

Example 20. If $\tan^{-1}2$, $\tan^{-1}3$ are two angles of a triangle. Then, find the third angle.

Sol. Let the angle of Δ be,

$A=\tan^{-1}2$, $B=\tan^{-1}3$ and angle C

$\therefore \quad C=\pi-(\tan^{-1}2+\tan^{-1}3)$

$$=\pi-\left\{\pi+\tan^{-1}\left(\frac{2+3}{1-2\cdot3}\right)\right\}$$

$$=\pi-\pi-\tan^{-1}(-1)=-\tan^{-1}(-1) \text{ or } C=\frac{\pi}{4}$$

$\therefore$ third angle is $\pi/4$

Example 21. Solve the following equations

(i) $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$

(ii) $\tan^{-1}2x+\tan^{-1}3x=\frac{\pi}{4}$

Sol. (i) $\tan^{-1}\left(\frac{x-1}{x-2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right)=\frac{\pi}{4}$

$\Rightarrow \quad \tan^{-1}\left(\frac{x-1}{x-2}\right)=\tan^{-1}(1)-\tan^{-1}\left(\frac{x+1}{x+2}\right)$

{where, $\tan^{-1}(1)=\pi/4$}

$\Rightarrow \quad \tan^{-1}\left(\frac{x-1}{x-2}\right)=\tan^{-1}\left(\frac{1-\frac{x+1}{x+2}}{1+\frac{x+1}{x+2}}\right)$

$\Rightarrow \quad \tan^{-1}\left(\frac{x-1}{x-2}\right)=\tan^{-1}\left(\frac{x+2-x-1}{x+2+x+1}\right)$

$\Rightarrow \quad \tan^{-1}\left(\frac{x-1}{x-2}\right)=\tan^{-1}\left(\frac{1}{2x+3}\right)$

$\Rightarrow \quad \frac{x-1}{x-2}=\frac{1}{2x+3}$

$\Rightarrow \quad (2x+3)(x-1)=(x-2)$

$\Rightarrow \quad 2x^2-2x+3x-3=(x-2)$

$\Rightarrow \quad 2x^2+x-3=x-2 \Rightarrow 2x^2=1$

$\Rightarrow \quad x=\pm\frac{1}{\sqrt{2}}$

(ii) $\tan^{-1}2x+\tan^{-1}3x=\pi/4$

$\Rightarrow \quad \tan^{-1}\left\{\frac{2x+3x}{1-6x^2}\right\}=\frac{\pi}{4}$, if $6x^2<1$

$\Rightarrow \quad \frac{5x}{1-6x^2}=1$

$\Rightarrow \quad 6x^2+5x-1=0$, if $x^2<\frac{1}{6}$

$\Rightarrow \quad (6x-1)(x+1)=0$ and $-\frac{1}{\sqrt{6}}<x<\frac{1}{\sqrt{6}}$

$\Rightarrow \quad x=1/6,-1$ and $-\frac{1}{\sqrt{6}}<x<\frac{1}{\sqrt{6}}$

$\Rightarrow \quad x=1/6$, neglecting $x=-1$ as $x\in\left(-\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}}\right)$

Property VII

(1) $\sin^{-1}x+\sin^{-1}y$

$$=\begin{cases}\sin^{-1}\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\}, \text{if } -1\le x,y\le1 \text{ and } x^2+y^2\le1 \text{ or if } xy<0 \text{ and } x^2+y^2>1\\ \pi-\sin^{-1}\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\}, \text{if } 0<x,y\le1 \text{ and } x^2+y^2>1\\ -\pi-\sin^{-1}\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\}, \text{if } -1\le x,y<0 \text{ and } x^2+y^2>1\end{cases}$$

(2) $\sin^{-1} x - \sin^{-1} y$

$$= \begin{cases} \sin^{-1}\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\}, \text{if } -1 \le x, y \le 1 \\ \quad \text{and } x^2 + y^2 \le 1 \text{ or if } xy > 0 \text{ and } x^2 + y^2 > 1 \\ \pi - \sin^{-1}\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\}, \text{if } 0 < x \le 1, \\ \qquad -1 \le y < 0 \text{ and } x^2 + y^2 > 1 \\ -\pi - \sin^{-1}\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\}, \text{if } -1 \le x < 0, \\ \qquad 0 < y < 1 \text{ and } x^2 + y^2 > 1 \end{cases}$$

Proof Let, $\sin^{-1} x = A$ and $\sin^{-1} y = B$. Then,

$x = \sin A,\ y = \sin B$ and $A,\ B \in [-\pi/2, \pi/2]$

$\Rightarrow$ $\cos A = \sqrt{1-x^2},\ \cos B = \sqrt{1-y^2}$

$\{\because \cos A, \cos B \ge 0 \text{ for } A,\ B \in [-\pi/2, \pi/2]\}$

$\therefore$ $\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\Rightarrow$ $\sin(A+B) = x\sqrt{1-y^2} + y\sqrt{1-x^2}$ ...(i)

$\sin(A-B) = x\sqrt{1-y^2} - y\sqrt{1-x^2}$ ...(ii)

$\cos(A+B) = \sqrt{1-x^2}\sqrt{1-y^2} - xy$ ...(iii)

$\cos(A-B) = \sqrt{1-x^2}\sqrt{1-y^2} + xy$ ...(iv)

Case I When $-1 \le x, y \le 1$ and $x^2 + y^2 \le 1$

In this case, we have

$x^2 + y^2 \le 1 \Rightarrow 1 - x^2 \ge y^2$ and $1 - y^2 \ge x^2$

$\Rightarrow$ $(1-x^2)(1-y^2) \ge x^2 y^2$

$\Rightarrow$ $\sqrt{1-x^2}\sqrt{1-y^2} - xy \ge 0$

$\Rightarrow$ $\cos(A+B) \ge 0$ {using Eq. (iii)}

$\Rightarrow A + B$ lies either in I quadrant or in IV quadrant.

$\Rightarrow A + B \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\left\{\because A, B \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Rightarrow -\pi \le A + B \le \pi\right\}$

$\therefore$ $\sin(A+B) = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

{from Eq. (i)}

$\Rightarrow$ $A + B = \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}$

$\{\because -\pi/2 \le A + B \le \pi/2\}$

$\Rightarrow \sin^{-1} x + \sin^{-1} y = \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}$

Case II When $xy < 0$ and $x^2 + y^2 > 1$.

In this case, we have

$xy < 0 \Rightarrow (x > 0 \text{ and } y < 0) \text{ or } (x < 0 \text{ and } y > 0)$

$\Rightarrow \left\{A \in \left(0, \frac{\pi}{2}\right] \text{ and } B \in \left[-\frac{\pi}{2}, 0\right)\right\}$ or $\left\{A \in \left[-\frac{\pi}{2}, 0\right)\right.$

and $\left.B \in \left(0, \frac{\pi}{2}\right]\right\} \Rightarrow -\frac{\pi}{2} \le A + B \le \frac{\pi}{2}$ ...(v)

and $x^2 + y^2 > 1$

$\Rightarrow 1 - x^2 < y^2$ and $1 - y^2 < x^2$

$\Rightarrow$ $(1-x^2)(1-y^2) < x^2 y^2$

$\Rightarrow$ $(\sqrt{1-x^2}\sqrt{1-y^2})^2 < (|xy|)^2$ $\{\because xy < 0\}$

$-|xy| < \sqrt{1-x^2}\sqrt{1-y^2} < |xy|$

$\Rightarrow$ $xy < \sqrt{1-x^2}\sqrt{1-y^2} < -xy$ $\{\because xy < 0\}$

$\Rightarrow$ $\sqrt{1-x^2}\sqrt{1-y^2} - xy > 0$ $\{\therefore |xy| = -xy\}$

$\Rightarrow$ $\cos(A+B) > 0$

$\Rightarrow A + B$ lies either in I quadrant or in IV quadrant.

$\Rightarrow$ $A + B \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ {using Eq. (v)}

$\therefore$ $\sin(A+B) = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

$\Rightarrow$ $A + B = \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}$

$\left\{\because A + B \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right\}$

$\Rightarrow$ $\sin^{-1} x + \sin^{-1} y = \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}$

Case III When $0 < x, y \le 1$ and $x^2 + y^2 > 1$.

In this case, we have

$0 < x, y \le 1 \Rightarrow A \in \left(0, \frac{\pi}{2}\right]$ and $B \in \left(0, \frac{\pi}{2}\right]$

$\Rightarrow$ $A + B \in (0, \pi]$ ...(vi)

and, $x^2 + y^2 > 1 \Rightarrow 1 - x^2 < y^2$ and $1 - y^2 < x^2$

$\Rightarrow$ $(1-x^2)(1-y^2) < x^2 y^2$

$\Rightarrow$ $\sqrt{1-x^2}\sqrt{1-y^2} < xy$ $\{\because xy > 0\}$

$\Rightarrow$ $\sqrt{1-x^2}\sqrt{1-y^2} - xy < 0$

$\Rightarrow$ $\cos(A+B) < 0$ {using Eq. (iii)}

$\Rightarrow (A + B)$ lies either in II quadrant or in III quadrant.

$\Rightarrow$ $\frac{\pi}{2} \le A + B \le \pi$ $\{\because A + B \in (0, \pi]$, from Eq. (vi)$\}$

$\Rightarrow$ $-\pi \le -(A+B) \le -\frac{\pi}{2} \Rightarrow 0 \le \pi - (A+B) \le \frac{\pi}{2}$

$\therefore$ $\sin(A+B) = x\sqrt{1-y^2} + y\sqrt{1-x^2}$ {from Eq. (i)}

$$\Rightarrow \quad \sin\{\pi-(A+B)\} = x\sqrt{1-y^2}+y\sqrt{1-x^2}$$

$$\{\because \sin(\pi-\theta)=\sin\theta\}$$

$$\Rightarrow \quad \pi-(A+B)=\sin^{-1}\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\}$$

$$\Rightarrow \quad A+B=\pi-\sin^{-1}\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\}$$

$$\Rightarrow \sin^{-1}x+\sin^{-1}y=\pi-\sin^{-1}\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\}$$

Case IV When $-1\le x, y<0$ and $x^2+y^2>1$

In this case, we have

$$-1\le x, y<0 \Rightarrow A\in\left[-\frac{\pi}{2},0\right) \text{ and } B\in\left[-\frac{\pi}{2},0\right)$$

$$\Rightarrow \quad A+B\in[-\pi,0) \qquad \ldots(\text{vii})$$

and $\quad x^2+y^2>1 \Rightarrow 1-x^2<y^2$

and $\quad 1-y^2<x^2 \Rightarrow (1-x^2)(1-y^2)<x^2y^2$

$$\Rightarrow \quad \sqrt{1-x^2}\sqrt{1-y^2}<xy \qquad \{\because xy>0\}$$

$$\Rightarrow \quad \sqrt{1-x^2}\sqrt{1-y^2}-xy<0$$

$$\Rightarrow \quad \cos(A+B)<0 \qquad \{\text{using Eq. (iii)}\}$$

$\Rightarrow$ $A+B$ lies either in II quadrant or in III quadrant.

$$\Rightarrow \quad -\pi\le A+B\le -\frac{\pi}{2} \qquad \{\text{using Eq. (vii)}\}$$

$$\Rightarrow \quad \frac{\pi}{2}\le -(A+B)\le\pi$$

$$\Rightarrow \quad -\frac{\pi}{2}\le -\pi-(A+B)\le 0$$

$$\therefore \quad \sin(A+B)=x\sqrt{1-y^2}+y\sqrt{1-x^2}$$

$$\Rightarrow \quad -\sin\{\pi+(A+B)\}=x\sqrt{1-y^2}+y\sqrt{1-x^2}$$

$$\Rightarrow \quad \sin\{-\pi-(A+B)\}=x\sqrt{1-y^2}+y\sqrt{1-x^2}$$

$$\Rightarrow \quad -\pi-(A+B)=\sin^{-1}\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\}$$

$$\Rightarrow \quad A+B=-\pi-\sin^{-1}\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\}$$

$$\Rightarrow \sin^{-1}x+\sin^{-1}y=-\pi-\sin^{-1}\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\}$$

(2) Do yourself.

Property VIII

(1) $\cos^{-1}x+\cos^{-1}y$

$$=\begin{cases}\cos^{-1}\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\}, \text{ if } -1\le x, \\ \qquad\qquad y\le 1 \text{ and } x+y\ge 0 \\ 2\pi-\cos^{-1}\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\}, \text{ if } -1\le x, \\ \qquad\qquad y\le 1 \text{ and } x+y\le 0\end{cases}$$

(2) $\cos^{-1}x-\cos^{-1}y$

$$=\begin{cases}\cos^{-1}\{xy+\sqrt{1-x^2}\sqrt{1-y^2}\}, \text{ if } -1\le x, \\ \qquad\qquad y\le 1 \text{ and } x\le y \\ -\cos^{-1}\{xy+\sqrt{1-x^2}\sqrt{1-y^2}\}, \text{ if } -1\le y\le 0, \\ \qquad\qquad 0<x\le 1 \text{ and } x\ge y\end{cases}$$

Proof (1) Let $\cos^{-1}x=A$ and $\cos^{-1}y=B$. Then,

$$x=\cos A,\ y=\cos B \text{ and } A, B\in[0,\pi]$$

$$\Rightarrow \quad \sin A=\sqrt{1-x^2} \text{ and } \sin B=\sqrt{1-y^2}$$

$$\{\because \sin A, \sin B\ge 0 \text{ for } A, B\in[0,\pi]\}$$

$$\therefore \quad \cos(A+B)=xy-\sqrt{1-x^2}\sqrt{1-y^2} \qquad \ldots(\text{i})$$

$$\cos(A-B)=xy+\sqrt{1-x^2}\sqrt{1-y^2} \qquad \ldots(\text{ii})$$

Case I When $-1\le x, y\le 1$ and $x+y\ge 0$

In this case,

$$-1\le x, y\le 1\Rightarrow A, B\in[0,\pi]$$

$$\Rightarrow \quad 0\le A+B\le 2\pi \qquad \ldots(\text{iii})$$

and $\quad x+y\ge 0$

$$\Rightarrow \quad \cos A+\cos B\ge 0$$

$$\Rightarrow \quad \cos A\ge -\cos B$$

$$\Rightarrow \quad \cos A\ge\cos(\pi-B)$$

$$\Rightarrow \quad A\le\pi-B \quad \{\because \cos\theta \text{ is decreasing on } [0,\pi]\}$$

$$\Rightarrow \quad A+B\le\pi \qquad \ldots(\text{iv})$$

From Eqs. (iii) and (iv), we get

$$0\le A+B\le\pi$$

$$\therefore \quad \cos(A+B)=xy-\sqrt{1-x^2}\sqrt{1-y^2}$$

$$\Rightarrow \quad A+B=\cos^{-1}\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\}$$

$$\Rightarrow \quad \cos^{-1}x+\cos^{-1}y=\cos^{-1}\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\}$$

Case II When $-1\le x, y\le 1$ and $x+y\le 0$

In this case,

$$-1\le x, y\le 1\Rightarrow A, B\in[0,\pi]$$

$$\Rightarrow \quad 0\le A+B\le 2\pi$$

and $\quad x+y\le 0\Rightarrow \cos A+\cos B\le 0 \qquad \ldots(\text{v})$

$$\Rightarrow \quad \cos A\le -\cos B$$

$$\Rightarrow \quad \cos A\le\cos(\pi-B)$$

$$\Rightarrow \quad A\ge\pi-B$$

$$\{\because \cos\theta \text{ is decreasing on } [0,\pi]\}$$

$$\Rightarrow \quad A+B\ge\pi \qquad \ldots(\text{vi})$$

From Eqs. (v) and (vi), we get $\pi\le A+B\le 2\pi$

$$\Rightarrow \quad -\pi\ge -(A+B)\ge -2\pi$$

$$\Rightarrow \quad \pi \geq 2\pi - (A+B) \geq 0$$

$$\Rightarrow \quad 0 \leq 2\pi - (A+B) \leq \pi$$

$$\therefore \quad \cos(A+B) = xy - \sqrt{1-x^2}\ \sqrt{1-y^2}$$

$$\Rightarrow \quad \cos\{2\pi - (A+B)\} = xy - \sqrt{1-x^2}\sqrt{1-y^2}$$

$$\Rightarrow \quad 2\pi - (A+B) = \cos^{-1}\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\}$$

$$\Rightarrow \quad A+B = 2\pi - \cos^{-1}\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\}$$

$$\Rightarrow \quad \cos^{-1}x + \cos^{-1}y = 2\pi - \cos^{-1}\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\}$$

(2) Do yourself.

Example 22. If $\cos^{-1}\frac{x}{a} + \cos^{-1}\frac{y}{b} = \alpha$, then show $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos\alpha = \sin^2\alpha$

Sol. We have, $\cos^{-1}\left(\frac{x}{a}\right) + \cos^{-1}\left(\frac{y}{b}\right) = \alpha$

$$\Rightarrow \quad \cos^{-1}\left(\frac{x}{a}\cdot\frac{y}{b} - \sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}\right) = \alpha$$

$$\Rightarrow \quad \frac{xy}{ab} - \sqrt{\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right)} = \cos\alpha$$

$$\Rightarrow \quad \left(\frac{xy}{ab} - \cos\alpha\right)^2 = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{x^2y^2}{a^2b^2}$$

$$\Rightarrow \quad \cos^2\alpha + \frac{x^2y^2}{a^2b^2} - \frac{2xy\cos\alpha}{ab} = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{x^2y^2}{a^2b^2}$$

$$\Rightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos\alpha = \sin^2\alpha$$

Example 23. If $\cos^{-1}\lambda + \cos^{-1}\mu + \cos^{-1}\gamma = 3\pi$, then find the value of $\lambda\mu + \mu\gamma + \gamma\lambda$.

Sol. We know, $0 \leq \cos^{-1}x \leq \pi$

Hence, from the question,

$$\cos^{-1}\lambda = \pi, \cos^{-1}\mu = \pi, \cos^{-1}\gamma = \pi$$

$$\therefore \quad \lambda = \mu = \gamma = -1$$

{$\because \cos^{-1}\lambda + \cos^{-1}\mu + \cos^{-1}\gamma = 3\pi$ only when each value attains its maximum}

$$\Rightarrow \quad \lambda\mu + \mu\gamma + \gamma\lambda = 3$$

Example 24. If $\sum_{i=1}^{2n}\cos^{-1}x_i = 0$, then find the value of $\sum_{i=1}^{2n}x_i$.

Sol. We know, $0 \leq \cos^{-1}x \leq \pi$.

$\therefore \cos^{-1}x_1, \cos^{-1}x_2, \cos^{-1}x_3, \ldots, \cos^{-1}x_{2n}$ are +ve and hence their sum is zero only when each value is zero.

i.e. $\cos^{-1}x_i = 0$ for all i.

$$\Rightarrow \quad x_i = 1 \text{ for all } i \qquad \ldots(i)$$

Now, $\sum_{i=1}^{2n}x_i = x_1 + x_2 + x_3 + \ldots + x_{2n}$

$$= \underbrace{\{1+1+1\ldots+1\}}_{2n \text{ times}} = 2n \qquad \{\text{using Eq. (i)}\}$$

$$\Rightarrow \quad \sum_{i=1}^{2n}x_i = 2n$$

Example 25. If $\sum_{i=1}^{2n}\sin^{-1}x_i = n\pi$, then find the value of $\sum_{i=1}^{2n}x_i$.

Sol. We know, $-\frac{\pi}{2} \leq \sin^{-1}x \leq \frac{\pi}{2}$

Hence, from the question, each value $\sin^{-1}x_1, \sin^{-1}x_2, \sin^{-1}x_3, \ldots, \sin^{-1}x_{2n}$ is equal to $\frac{\pi}{2}$

i.e. $\sin^{-1}x_i = \frac{\pi}{2}$ for all $i \Rightarrow x_i = 1$ for all i

Now, $\sum_{i=1}^{2n}x_i = x_1 + x_2 + \ldots + x_{2n} = \underbrace{\{1+1+\ldots+1\}}_{2n \text{ times}} = 2n$

$$\Rightarrow \quad \sum_{i=1}^{2n}x_i = 2n$$

Example 26. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$, then find the value of $\Sigma\frac{(x^{101}+y^{101})(x^{202}+y^{202})}{(x^{303}+y^{303})(x^{404}+y^{404})}$

Sol. We have, $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$

As we know that, $\sin^{-1}x \leq \frac{\pi}{2}$ for all $x \in [-1, 1]$

Hence, above result is possible only when

$$\sin^{-1}x = \frac{\pi}{2}, \sin^{-1}y = \frac{\pi}{2}, \sin^{-1}z = \frac{\pi}{2}$$

$$\Rightarrow \quad x = 1, y = 1, z = 1 \qquad \ldots(i)$$

$$\therefore \quad \Sigma\frac{(x^{101}+y^{101})(x^{202}+y^{202})}{(x^{303}+y^{303})(x^{404}+y^{404})}$$

$$= \frac{(x^{101}+y^{101})(x^{202}+y^{202})}{(x^{303}+y^{303})(x^{404}+y^{404})} + \frac{(y^{101}+z^{101})(y^{202}+z^{202})}{(y^{303}+z^{303})(y^{404}+z^{404})} + \frac{(z^{101}+x^{101})(z^{202}+x^{202})}{(z^{303}+x^{303})(z^{404}+x^{404})}$$ {expanding summation}

$$= \frac{(1+1)(1+1)}{(1+1)(1+1)} + \frac{(1+1)(1+1)}{(1+1)(1+1)} + \frac{(1+1)(1+1)}{(1+1)(1+1)}$$ {using Eq. (i)}

$$= 1+1+1 = 3$$

Example 27. If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$, prove that $x^2+y^2+z^2+2xyz=1$.

Sol. Here,

$$\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$$

$$\Rightarrow \quad \cos^{-1}x+\cos^{-1}y=\pi-\cos^{-1}z$$

$$\Rightarrow \quad \cos^{-1}x+\cos^{-1}y=\cos^{-1}(-z)$$

$$\{\because \cos^{-1}(-x)=\pi-\cos^{-1}x\}$$

$$\Rightarrow \quad \cos^{-1}\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\}=\cos^{-1}(-z)$$

$$\Rightarrow \quad xy-\sqrt{1-x^2}\sqrt{1-y^2}=-z$$

$$\Rightarrow \quad (xy+z)^2=(1-x^2)(1-y^2)$$

$$\Rightarrow \quad x^2y^2+z^2+2xyz=1-x^2-y^2+x^2y^2$$

$$\Rightarrow \quad x^2+y^2+z^2+2xyz=1.$$

Example 28. Find the sum of infinite series

$$s=\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)+\sin^{-1}\left(\frac{\sqrt{2}-1}{\sqrt{6}}\right)+\sin^{-1}\left(\frac{\sqrt{3}-\sqrt{2}}{2\sqrt{3}}\right)+\ldots+\sin^{-1}\left(\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n(n+1)}}\right)+\ldots\infty$$

Sol. Let, $T_n=\sin^{-1}\left(\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n(n+1)}}\right)$

$$=\sin^{-1}\left(\frac{1}{\sqrt{n}}\sqrt{1-\frac{1}{n+1}}-\frac{1}{\sqrt{n+1}}\sqrt{1-\frac{1}{n}}\right)$$

$$\Rightarrow \quad T_n=\sin^{-1}\left(\frac{1}{\sqrt{n}}\right)-\sin^{-1}\left(\frac{1}{\sqrt{n+1}}\right)$$

$$\therefore \quad s=\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)+T_2+T_3+T_4+\ldots\infty$$

$$=\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)+\left\{\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)-\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)\right\}+\left\{\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)-\sin^{-1}\left(\frac{1}{\sqrt{4}}\right)\right\}+\left\{\sin^{-1}\left(\frac{1}{\sqrt{4}}\right)-\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)\right\}+\ldots\infty$$

$$=2\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$=2\cdot\frac{\pi}{4}=\frac{\pi}{2}.$$

Exercise for Session 5

1. Show that $\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{15}{17}=\pi-\sin^{-1}\frac{84}{85}$
2. Evaluate $\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{16}{65}$
3. If $\tan^{-1}4+\tan^{-1}5=\cot^{-1}\lambda$, then find '$\lambda$'.
4. Prove that $\cos^{-1}\left(\frac{7}{2}(1+\cos 2x)+\sqrt{(\sin^2 x-48\cos^2 x)}\sin x\right)=x-\cos^{-1}(7\cos x)$, where $x\in\left(0,\frac{\pi}{2}\right)$.

Solve the following

5. $\sin^{-1}\frac{1}{3}+\sin^{-1}\frac{2}{3}=\sin^{-1}x$
6. $\sin^{-1}x+\sin^{-1}2x=\frac{2\pi}{3}$

Session 6

Property IX of Inverse Trigonometric Functions

Property IX

(i) For $0 < x < 1$,

$$\sin^{-1} x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

$$= \cot^{-1}\frac{\sqrt{1-x^2}}{x} = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$$

$$= \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$$

(ii) For $0 < x < 1$,

$$\cos^{-1} x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$$

$$= \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \sec^{-1}\left(\frac{1}{x}\right)$$

$$= \operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$$

(iii) For $x > 0$, $\tan^{-1} x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$

$$= \cot^{-1}\left(\frac{1}{x}\right) = \sec^{-1}(\sqrt{1+x^2})$$

$$= \operatorname{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)$$

Proof

(i) Let, $\sin^{-1} x = \theta$. Then $x = \sin\theta$

Now, $\cos\theta = \sqrt{1-\sin^2\theta}$

$\Rightarrow$ $\cos\theta = \sqrt{1-x^2}$

$\Rightarrow$ $\theta = \cos^{-1}\sqrt{1-x^2}$

$\Rightarrow$ $\sin^{-1} x = \cos^{-1}\sqrt{1-x^2}$

$$= \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) \qquad \{\because \theta = \sin^{-1} x\}$$

Again, $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$

$\Rightarrow$ $\tan\theta = \frac{x}{\sqrt{1-x^2}}$

$\Rightarrow$ $\theta = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

$\Rightarrow$ $\sin^{-1} x = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) \qquad \{\because \theta = \sin^{-1} x\}$

$\Rightarrow$ $\sin^{-1} x = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

$\{\because \tan^{-1} x = \cot^{-1}\frac{1}{x} \text{ for } x > 0\}$

Hence,

$$\sin^{-1} x = \cos^{-1}\sqrt{1-x^2} = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$$

$$= \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

$$= \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$$

$$= \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$$

Similarly, the other results can be proved.

Example 29. Evaluate the following

(i) $\sin\left(\tan^{-1}\frac{3}{4}\right)$ (ii) $\sin\left(\cos^{-1}\frac{3}{5}\right)$

(iii) $\cos\left(\tan^{-1}\frac{3}{4}\right)$ (iv) $\sin(\cot^{-1} x)$

Sol. (i) Let $y = \sin\left(\tan^{-1}\frac{3}{4}\right)$ ...(i)

Note

To find y we use $\sin(\sin^{-1}x) = x, -1, \le x \le 1$

For this we convert $\tan^{-1}x$ in $\sin^{-1}x$

Here, $\tan^{-1}\left(\frac{3}{4}\right) = \sin^{-1}\left(\frac{3/4}{\sqrt{1+(3/4)^2}}\right) = \sin^{-1}\left(\frac{3}{5}\right)$

Now, from Eq. (i), we get

or $y = \sin\left(\sin^{-1}\frac{3}{5}\right)$

$\therefore \quad y = \frac{3}{5}$

(ii) $\cos^{-1}\frac{3}{5} = \sin^{-1}\sqrt{1-\left(\frac{3}{5}\right)^2} = \sin^{-1}\frac{4}{5}.$

$\therefore \quad \sin\left(\cos^{-1}\frac{3}{5}\right) = \sin\left(\sin^{-1}\frac{4}{5}\right) = \frac{4}{5}$

(iii) $\tan^{-1}\left(\frac{3}{4}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1-\left(\frac{3}{4}\right)^2}}\right) = \cos^{-1}\frac{4}{5}$

$\therefore \quad \cos\left(\tan^{-1}\frac{3}{4}\right) = \cos\left(\cos^{-1}\frac{4}{5}\right) = \frac{4}{5}$

(iv) Let, $\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right) = \sin^{-1}\left(\frac{1/x}{\sqrt{1+1/x^2}}\right)$

$= \sin^{-1}\left(\frac{1}{\sqrt{x^2+1}}\right)$

$\therefore \quad \sin(\cot^{-1}x) = \sin\left(\sin^{-1}\frac{1}{\sqrt{x^2+1}}\right) = \frac{1}{\sqrt{1+x^2}}$

Example 30. Find the value of $\tan\left(\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right)$

Sol. Let $y = \tan\left(\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right)$ ...(i)

Let $\cos^{-1}\frac{\sqrt{5}}{3} = \theta \Rightarrow \theta \in \left(0, \frac{\pi}{2}\right)$ and $\cos\theta = \frac{\sqrt{5}}{3}$

$\therefore$ Eq. (i) becomes

$$y = \tan\left(\frac{\theta}{2}\right) \quad \text{... (ii)}$$

$\because \quad \tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta} = \frac{1-\frac{\sqrt{5}}{3}}{1+\frac{\sqrt{5}}{3}}$

$= \frac{3-\sqrt{5}}{3-\sqrt{5}} = \frac{(3-\sqrt{5})^2}{4}$

$\tan\frac{\theta}{2} = \pm\left(\frac{3-\sqrt{5}}{2}\right)$ ... (iii)

$\because \quad \theta \in \left(0, \frac{\pi}{2}\right) \Rightarrow \frac{\theta}{2} \in \left(0, \frac{\pi}{4}\right)$

$\therefore \quad \tan\frac{\theta}{2} > 0$

So, from Eq. (iii), we get

$$\tan\frac{\theta}{2} = \left(\frac{3-\sqrt{5}}{2}\right)$$

Now, from Eq. (ii), we get

$$y = \left(\frac{3-\sqrt{5}}{2}\right)$$

Example 31. Find the value of $\cos(2\cos^{-1}x + \sin^{-1}x)$ when $x = \frac{1}{5}$

Sol. Let $y = \cos[2\cos^{-1}x + \sin^{-1}x]$

$\because \quad \sin^{-1}x + \cos^{1}x = \frac{\pi}{2}, |x| \le 1$

$\therefore \quad y = \cos\left[2\cos^{-1}x + \frac{\pi}{2} - \cos^{-1}x\right]$

$= \cos\left[\frac{\pi}{2} + \cos^{-1}x\right] = -\sin(\cos^{-1}x)$

$\because \quad x = \frac{1}{5}$

$\therefore \quad y = -\sin\left(\cos^{-1}\frac{1}{5}\right)$

$\Rightarrow \quad y = -\sin\left(\sin^{-1}\sqrt{1-\left(\frac{1}{5}\right)^2}\right)$

$= -\sin\left(\sin^{-1}\frac{\sqrt{24}}{5}\right) = -\frac{\sqrt{24}}{5}$

Example 32. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$, prove that $x^4 + y^4 + z^4 + 4x^2y^2z^2 = 2(x^2y^2 + y^2z^2 + z^2x^2)$

Sol. We have

$\Rightarrow \quad \sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$

$\Rightarrow \quad \sin^{-1}x + \sin^{-1}y = \pi - \sin^{-1}z$

$\Rightarrow \quad \cos(\sin^{-1}x + \sin^{-1}y) = \cos(\pi - \sin^{-1}z)$

$\Rightarrow \quad \cos(\sin^{-1}x).\cos(\sin^{-1}y) - \sin(\sin^{-1}x).\sin(\sin^{-1}y)$

$= -\cos(\sin^{-1}z)$

$\Rightarrow \quad \sqrt{1-x^2}.\sqrt{1-y^2} - xy = -\sqrt{1-z^2}$

$\{\because \cos(\sin^{-1}x) = \cos\left(\cos^{-1}\sqrt{1-x^2}\right) = \sqrt{1-x^2}\}$

$$\Rightarrow \quad \sqrt{1-x^2}.\sqrt{1-y^2} = xy - \sqrt{1-z^2}$$

$$\Rightarrow \quad (1-x^2)(1-y^2) = x^2y^2 + 1 - z^2 - 2xy\sqrt{1-z^2}$$

{squaring both sides}

$$\Rightarrow \quad 1 - x^2 - y^2 + x^2y^2 = x^2y^2 + 1 - z^2 - 2xy\sqrt{1-z^2}$$

$$\Rightarrow \quad x^2 + y^2 - z^2 = 2xy\sqrt{1-z^2}$$

$$\Rightarrow \quad (x^2 + y^2 - z^2)^2 = 4x^2y^2(1-z^2)$$

$$\Rightarrow \quad x^4 + y^4 + z^4 - 2x^2z^2 - 2y^2z^2 + 2x^2y^2 = 4x^2y^2 - 4x^2y^2z^2$$

$$\Rightarrow \quad x^4 + y^4 + z^4 + 4x^2y^2z^2 = 2(x^2y^2 + y^2z^2 + z^2x^2)$$

Exercise for Session 6

Evaluate the following :

1. $\tan\left(\text{cosec}^{-1}\frac{\sqrt{41}}{4}\right)$
2. $\sec\left(\cot^{-1}\frac{16}{63}\right)$
3. $\sin^2\left[\tan^{-1}\frac{3}{4}\right]$
4. $\sin\left\{\frac{1}{2}\cot^{-1}\left(\frac{-3}{4}\right)\right\}$
5. Show that $\cot\left[\sin^{-1}\sqrt{\frac{13}{17}}\right] = \sin\left[\tan^{-1}\frac{2}{3}\right]$

Session 7

Property X, XI, XII and XIII of Inverse Trigonometric Functions

Property X

(i) $2\sin^{-1}x = \begin{cases} \sin^{-1}(2x\sqrt{1-x^2}), & \text{if } -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}} \\ \pi - \sin^{-1}(2x\sqrt{1-x^2}), & \text{if } \frac{1}{\sqrt{2}} \le x \le 1 \\ -\pi - \sin^{-1}(2x\sqrt{1-x^2}), & \text{if } -1 \le x \le -\frac{1}{\sqrt{2}} \end{cases}$

(ii) $3\sin^{-1}x = \begin{cases} \sin^{-1}(3x - 4x^3), & \text{if } -\frac{1}{2} \le x \le \frac{1}{2} \\ \pi - \sin^{-1}(3x - 4x^3), & \text{if } \frac{1}{2} < x \le 1 \\ -\pi - \sin^{-1}(3x - 4x^3), & \text{if } -1 \le x < -\frac{1}{2} \end{cases}$

Proof

(i) Let $\sin^{-1}x = \theta$. Then, $x = \sin\theta$

$$\Rightarrow \quad \cos\theta = \sqrt{1-x^2} \quad \{\because \cos\theta > 0 \text{ for } \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\}$$

$$\because \quad \sin 2\theta = 2\sin\theta\cos\theta$$

$$\therefore \quad \sin 2\theta = 2x\sqrt{1-x^2} \qquad \ldots(i)$$

Case I When $-\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$

$$-\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}} \Rightarrow -\frac{1}{\sqrt{2}} \le \sin\theta \le \frac{1}{\sqrt{2}} \Rightarrow -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$$

$$\Rightarrow \quad -\frac{\pi}{2} \le 2\theta \le \frac{\pi}{2}$$

Also, $-\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$

$$\Rightarrow \quad -1 \le 2x\sqrt{1-x^2} \le 1$$

$\therefore \quad \sin 2\theta = 2x\sqrt{1-x^2}$ {from Eq. (i)}

$\Rightarrow \quad 2\theta = \sin^{-1}(2x\sqrt{1-x^2})$

$\Rightarrow \quad 2\sin^{-1} x = \sin^{-1}(2x\sqrt{1-x^2})$

Case II When $\frac{1}{\sqrt{2}} \le x \le 1$

$\frac{1}{\sqrt{2}} \le x \le 1 \Rightarrow \frac{1}{\sqrt{2}} \le \sin\theta \le 1$

$\Rightarrow \quad \frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

$\Rightarrow \quad \frac{\pi}{2} \le 2\theta \le \pi$

$\Rightarrow \quad -\pi \le -2\theta \le -\frac{\pi}{2} \Rightarrow 0 \le \pi - 2\theta \le \frac{\pi}{2}$

Also, $\frac{1}{\sqrt{2}} \le x \le 1 \Rightarrow 0 \le 2x\sqrt{1-x^2} \le 1$

$\therefore \quad \sin 2\theta = 2x\sqrt{1-x^2}$ {from Eq. (i)}

$\Rightarrow \quad \sin(\pi - 2\theta) = 2x\sqrt{1-x^2}$

$\Rightarrow \quad \pi - 2\theta = \sin^{-1}(2x\sqrt{1-x^2})$

$\Rightarrow \quad \pi - 2\sin^{-1} x = \sin^{-1}(2x\sqrt{1-x^2})$

$\Rightarrow \quad 2\sin^{-1} x = \pi - \sin^{-1}(2x\sqrt{1-x^2})$

Case III When $-1 \le x \le -\frac{1}{\sqrt{2}}$

$-1 \le x \le -\frac{1}{\sqrt{2}} \Rightarrow -1 \le \sin\theta \le -\frac{1}{\sqrt{2}}$

$\Rightarrow \quad -\frac{\pi}{2} \le \theta \le -\frac{\pi}{4} \Rightarrow -\pi \le 2\theta \le -\frac{\pi}{2}$

$\Rightarrow \quad 0 \le \pi + 2\theta \le \frac{\pi}{2}$

Also, $-1 \le x \le -\frac{1}{\sqrt{2}} \Rightarrow -1 \le 2x\sqrt{1-x^2} \le 0$

$\therefore \quad \sin 2\theta = 2x\sqrt{1-x^2}$ {from Eq. (i)}

$\Rightarrow \quad -\sin(\pi + 2\theta) = 2x\sqrt{1-x^2}$

$\Rightarrow \quad \sin(-\pi - 2\theta) = 2x\sqrt{1-x^2}$

$\Rightarrow \quad -\pi - 2\theta = \sin^{-1}(2x\sqrt{1-x^2})$

$\Rightarrow \quad -\pi - 2\sin^{-1} x = \sin^{-1}(2x\sqrt{1-x^2})$

$\Rightarrow \quad 2\sin^{-1} x = -\pi - \sin^{-1}(2x\sqrt{1-x^2})$

(ii) Let, $\sin^{-1} x = \theta$. Then, $x = \sin\theta$

$\therefore \quad \sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\Rightarrow \quad \sin 3\theta = 3x - 4x^3$

Case I When $-\frac{1}{2} \le x \le \frac{1}{2}$

$-\frac{1}{2} \le x \le \frac{1}{2} \Rightarrow -\frac{1}{2} \le \sin\theta \le \frac{1}{2} \Rightarrow -\frac{\pi}{6} \le \theta \le \frac{\pi}{6}$

$\Rightarrow \quad -\frac{\pi}{2} \le 3\theta \le \frac{\pi}{2}$

Also, $-\frac{1}{2} \le x \le \frac{1}{2} \Rightarrow -1 \le 3x - 4x^3 \le 1$

$\therefore \quad \sin 3\theta = 3x - 4x^3$

$\Rightarrow \quad 3\theta = \sin^{-1}(3x - 4x^3)$

$\Rightarrow \quad 3\sin^{-1} x = \sin^{-1}(3x - 4x^3)$

Case II When $\frac{1}{2} < x \le 1$

$\frac{1}{2} < x \le 1 \Rightarrow \frac{1}{2} < \sin\theta \le 1$

$\Rightarrow \quad \frac{\pi}{6} < \theta \le \frac{\pi}{2}$

$\Rightarrow \quad \frac{\pi}{2} < 3\theta \le \frac{3\pi}{2}$

$\Rightarrow \quad -\frac{3\pi}{2} \le -3\theta < -\frac{\pi}{2}$

$\Rightarrow \quad -\frac{\pi}{2} \le \pi - 3\theta < \frac{\pi}{2}$

Also, $\frac{1}{2} < x \le 1 \Rightarrow -1 \le 3x - 4x^3 \le 1$

$\therefore \quad \sin 3\theta = (3x - 4x^3)$

$\Rightarrow \quad \sin(\pi - 3\theta) = (3x - 4x^3)$

$\Rightarrow \quad \pi - 3\theta = \sin^{-1}(3x - 4x^3)$

$\Rightarrow \quad \pi - 3\sin^{-1} x = \sin^{-1}(3x - 4x^3)$

$\Rightarrow \quad 3\sin^{-1} x = \pi - \sin^{-1}(3x - 4x^3)$

Case III When $-1 \le x < -\frac{1}{2}$

$-1 \le x < -\frac{1}{2}$

$\Rightarrow \quad -1 \le \sin\theta < -\frac{1}{2}$

$\Rightarrow \quad -\frac{\pi}{2} \le \theta < -\frac{\pi}{6}$

$\Rightarrow \quad -\frac{3\pi}{2} \le 3\theta < -\frac{\pi}{2}$

$$\Rightarrow \quad -\frac{\pi}{2} \le \pi + 3\theta < \frac{\pi}{2}$$

Also, $-1 \le x < -\frac{1}{2} \Rightarrow -1 \le 3x - 4x^3 \le 1$

$$\therefore \quad \sin 3\theta = 3x - 4x^3$$

$$\Rightarrow \quad -\sin(\pi + 3\theta) = 3x - 4x^3$$

$$\{\sin(\pi + 3\theta) = -\sin 3\theta\}$$

$$\Rightarrow \quad \sin(-\pi - 3\theta) = 3x - 4x^3$$

$$\Rightarrow \quad -\pi - 3\theta = \sin^{-1}(3x - 4x^3)$$

$$\Rightarrow \quad -\pi - 3\sin^{-1} x = \sin^{-1}(3x - 4x^3)$$

$$\Rightarrow \quad 3\sin^{-1} x = -\pi - \sin^{-1}(3x - 4x^3)$$

Property XI

$$\text{(i) } 2\cos^{-1} x = \begin{cases} \cos^{-1}(2x^2 - 1); & \text{if } 0 \le x \le 1 \\ 2\pi - \cos^{-1}(2x^2 - 1); & \text{if } -1 \le x \le 0 \end{cases}$$

$$\text{(ii) } 3\cos^{-1} x = \begin{cases} \cos^{-1}(4x^3 - 3x), & \text{if } \frac{1}{2} \le x \le 1 \\ 2\pi - \cos^{-1}(4x^3 - 3x), & \text{if } -\frac{1}{2} \le x \le \frac{1}{2} \\ 2\pi + \cos^{-1}(4x^3 - 3x), & \text{if } -1 \le x \le -\frac{1}{2} \end{cases}$$

Proof

(1) Let $\cos^{-1} x = \theta$. Then $x = \cos\theta$

$$\therefore \quad \cos 2\theta = 2\cos^2\theta - 1 = 2x^2 - 1$$

Case I When $0 \le x \le 1$

$$\Rightarrow \quad 0 \le \cos\theta \le 1 \Rightarrow 0 \le \theta \le \frac{\pi}{2}$$

$$\Rightarrow \quad 0 \le 2\theta \le \pi$$

Also, $0 \le x \le 1 \Rightarrow -1 \le 2x^2 - 1 \le 1$

$$\therefore \quad \cos 2\theta = 2x^2 - 1$$

$$\Rightarrow \quad 2\theta = \cos^{-1}(2x^2 - 1)$$

$$\Rightarrow \quad 2\cos^{-1} x = \cos^{-1}(2x^2 - 1)$$

Case II When $-1 \le x \le 0$

$$-1 \le x \le 0 \Rightarrow -1 \le \cos\theta \le 0$$

$$\Rightarrow \quad \frac{\pi}{2} \le \theta \le \pi \Rightarrow \pi \le 2\theta \le 2\pi$$

$$\Rightarrow \quad -2\pi \le -2\theta \le -\pi$$

$$\Rightarrow \quad 0 \le 2\pi - 2\theta \le \pi$$

Also, $-1 \le x \le 0 \Rightarrow -1 \le 2x^2 - 1 \le 1$

$$\therefore \quad \cos(2\pi - 2\theta) = 2x^2 - 1$$

$$\Rightarrow \quad 2\pi - 2\theta = \cos^{-1}(2x^2 - 1)$$

$$\Rightarrow \quad 2\pi - 2\cos^{-1} x = \cos^{-1}(2x^2 - 1)$$

$$\Rightarrow \quad 2\cos^{-1} x = 2\pi - \cos^{-1}(2x^2 - 1)$$

(ii) Let $\cos^{-1} x = \theta$. Then, $x = \cos\theta$

$$\therefore \quad \cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

$$\Rightarrow \quad \cos 3\theta = 4x^3 - 3x$$

Case I When $\frac{1}{2} \le x \le 1$

$$\frac{1}{2} \le x \le 1 \Rightarrow \frac{1}{2} \le \cos\theta \le 1$$

$$\Rightarrow \quad 0 \le \theta \le \frac{\pi}{3}$$

$$\Rightarrow \quad 0 \le 3\theta \le \pi$$

Also, $\frac{1}{2} \le x \le 1 \Rightarrow -1 \le 4x^3 - 3x \le 1$

$$\therefore \quad \cos 3\theta = 4x^3 - 3x$$

$$\Rightarrow \quad 3\theta = \cos^{-1}(4x^3 - 3x)$$

$$\Rightarrow \quad 3\cos^{-1} x = \cos^{-1}(4x^3 - 3x)$$

Case II When $-\frac{1}{2} \le x \le \frac{1}{2}$

$$-\frac{1}{2} \le x \le \frac{1}{2}$$

$$\Rightarrow \quad -\frac{1}{2} \le \cos\theta \le \frac{1}{2}$$

$$\Rightarrow \quad \frac{\pi}{3} \le \theta \le \frac{2\pi}{3}$$

$$\Rightarrow \quad \pi \le 3\theta \le 2\pi$$

$$\Rightarrow \quad -2\pi \le -3\theta \le -\pi$$

$$\Rightarrow \quad 0 \le 2\pi - 3\theta \le \pi$$

$$\therefore \quad \cos 3\theta = 4x^3 - 3x$$

$$\Rightarrow \quad \cos(2\pi - 3\theta) = 4x^3 - 3x$$

$$\Rightarrow \quad 2\pi - 3\theta = \cos^{-1}(4x^3 - 3x)$$

$$\Rightarrow \quad 3\theta = 2\pi - \cos^{-1}(4x^3 - 3x)$$

$$\Rightarrow \quad 3\cos^{-1} x = 2\pi - \cos^{-1}(4x^3 - 3x)$$

Case III When $-1 \le x \le -\frac{1}{2}$

$$-1 \le x \le -\frac{1}{2} \Rightarrow -1 \le \cos\theta \le -\frac{1}{2}$$

$$\Rightarrow \quad \frac{2\pi}{3} \le \theta \le \pi$$

$$\Rightarrow \quad 2\pi \le 3\theta \le 3\pi$$
$$\Rightarrow \quad -3\pi \le -3\theta \le -2\pi$$
$$\Rightarrow \quad -\pi \le 2\pi - 3\theta \le 0$$
$$\Rightarrow \quad 0 \le 3\theta - 2\pi \le \pi$$
$$\therefore \quad \cos 3\theta = 4x^3 - 3x$$
$$\Rightarrow \quad \cos(2\pi - 3\theta) = 4x^3 - 3x$$
$$\Rightarrow \quad \cos(3\theta - 2\pi) = 4x^3 - 3x$$
$$\Rightarrow \quad 3\theta - 2\pi = \cos^{-1}(4x^3 - 3x)$$
$$\Rightarrow \quad 3\theta = 2\pi + \cos^{-1}(4x^3 - 3x)$$
$$\Rightarrow \quad 3\cos^{-1} x = 2\pi + \cos^{-1}(4x^3 - 3x)$$

Property XII

$$\text{(i) } 2\tan^{-1} x = \begin{cases} \tan^{-1}\left(\dfrac{2x}{1-x^2}\right); & \text{if } -1 < x < 1 \\ \pi + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right); & \text{if } x > 1 \\ -\pi + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right); & \text{if } x < -1 \end{cases}$$

$$\text{(ii) } 3\tan^{-1} x = \begin{cases} \tan^{-1}\left(\dfrac{3x - x^3}{1-3x^2}\right); & \text{if } -\dfrac{1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}} \\ \pi + \tan^{-1}\left(\dfrac{3x - x^3}{1-3x^2}\right); & \text{if } x > \dfrac{1}{\sqrt{3}} \\ -\pi + \tan^{-1}\left(\dfrac{3x - x^3}{1-3x^2}\right); & \text{if } x < -\dfrac{1}{\sqrt{3}} \end{cases}$$

Proof

(i) Let $\tan^{-1} x = \theta$. Then, $x = \tan\theta$

$$\therefore \quad \tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

or $$\tan 2\theta = \frac{2x}{1 - x^2}$$

Case I When $-1 < x < 1$

$$-1 < x < 1 \Rightarrow -1 < \tan\theta < 1$$
$$\Rightarrow \quad -\frac{\pi}{4} < \theta < \frac{\pi}{4} \Rightarrow -\frac{\pi}{2} < 2\theta < \frac{\pi}{2}$$
$$\therefore \quad \tan 2\theta = \frac{2x}{1 - x^2}$$
$$\Rightarrow \quad 2\theta = \tan^{-1}\left(\frac{2x}{1 - x^2}\right)$$
$$\Rightarrow \quad 2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1 - x^2}\right)$$

Case II When $x > 1$

$$x > 1 \Rightarrow \tan\theta > 1$$
$$\Rightarrow \quad \frac{\pi}{2} > \theta > \frac{\pi}{4} \Rightarrow \pi > 2\theta > \frac{\pi}{2}$$
$$\Rightarrow \quad -\pi < -2\theta < -\frac{\pi}{2}$$
$$\Rightarrow \quad 0 < \pi - 2\theta < \frac{\pi}{2}$$
$$\Rightarrow \quad -\frac{\pi}{2} < -\pi + 2\theta < 0$$
$$\therefore \quad \tan 2\theta = \frac{2x}{1 - x^2}$$
$$\Rightarrow \quad -\tan(\pi - 2\theta) = \frac{2x}{1 - x^2}$$
$$\Rightarrow \quad \tan(-\pi + 2\theta) = \frac{2x}{1 - x^2}$$
$$\Rightarrow \quad -\pi + 2\theta = \tan^{-1}\left(\frac{2x}{1 - x^2}\right)$$
$$\Rightarrow \quad 2\theta = \pi + \tan^{-1}\left(\frac{2x}{1 - x^2}\right)$$
$$\Rightarrow \quad 2\tan^{-1} x = \pi + \tan^{-1}\left(\frac{2x}{1 - x^2}\right)$$

Case III When $x < -1$

$$x < -1 \Rightarrow \tan\theta < -1$$
$$\Rightarrow \quad -\frac{\pi}{2} < \theta < -\frac{\pi}{4}$$
$$\Rightarrow \quad -\pi < 2\theta < -\frac{\pi}{2}$$
$$\Rightarrow \quad 0 < \pi + 2\theta < \frac{\pi}{2}$$
$$\therefore \quad \tan 2\theta = \frac{2x}{1 - x^2}$$
$$\Rightarrow \quad \tan(\pi + 2\theta) = \frac{2x}{1 - x^2} \qquad \{\because \tan(\pi + \alpha) = \tan\alpha\}$$
$$\Rightarrow \quad \pi + 2\theta = \tan^{-1}\left(\frac{2x}{1 - x^2}\right)$$
$$\Rightarrow \quad 2\theta = -\pi + \tan^{-1}\left(\frac{2x}{1 - x^2}\right)$$
$$\Rightarrow \quad 2\tan^{-1} x = -\pi + \tan^{-1}\left(\frac{2x}{1 - x^2}\right)$$

(ii) Let $\tan^{-1} x = \theta$. Then $x = \tan\theta$

$$\therefore \quad \tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$
$$\Rightarrow \quad \tan 3\theta = \frac{3x - x^3}{1 - 3x^2}$$

Case I When $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

$$-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}} \Rightarrow -\frac{1}{\sqrt{3}} < \tan\theta < \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad -\frac{\pi}{6} < \theta < \frac{\pi}{6} \Rightarrow -\frac{\pi}{2} < 3\theta < \frac{\pi}{2}$$

$$\therefore \quad \tan 3\theta = \frac{3x - x^3}{1 - 3x^2} \Rightarrow 3\theta = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$$

$$\Rightarrow \quad 3\tan^{-1} x = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$$

Case II When $x > \frac{1}{\sqrt{3}}$

$$x > \frac{1}{\sqrt{3}} \Rightarrow \tan\theta > \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \frac{\pi}{2} > \theta > \frac{\pi}{6}$$

$$\Rightarrow \quad \frac{\pi}{2} < 3\theta < \frac{3\pi}{2}$$

$$\Rightarrow \quad -\frac{3\pi}{2} < -3\theta < -\frac{\pi}{2}$$

$$\Rightarrow \quad -\frac{\pi}{2} < \pi - 3\theta < \frac{\pi}{2}$$

$$\Rightarrow \quad -\frac{\pi}{2} < 3\theta - \pi < \frac{\pi}{2}$$

$$\therefore \quad \tan 3\theta = \frac{3x - x^3}{1 - 3x^2}$$

$$\Rightarrow \quad -\tan(\pi - 3\theta) = \frac{3x - x^3}{1 - 3x^2} \quad \{\because \tan(\pi - 3\theta) = -\tan 3\theta\}$$

$$\Rightarrow \quad 3\theta - \pi = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$$

$$\Rightarrow \quad 3\tan^{-1} x = \pi + \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$$

Case III When $x < -\frac{1}{\sqrt{3}}$

$$x < -\frac{1}{\sqrt{3}} \Rightarrow \tan\theta < -\frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad -\frac{\pi}{2} < \theta < -\frac{\pi}{6}$$

$$\Rightarrow \quad -\frac{3\pi}{2} < 3\theta < -\frac{\pi}{2}$$

$$\Rightarrow \quad -\frac{\pi}{2} < \pi + 3\theta < \frac{\pi}{2}$$

$$\therefore \quad \tan 3\theta = \frac{3x - x^3}{1 - 3x^2}$$

$$\Rightarrow \quad \tan(\pi + 3\theta) = \frac{3x - x^3}{1 - 3x^2} \qquad \{\because \tan(\pi + x) = \tan x\}$$

$$\Rightarrow \quad \pi + 3\theta = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$$

$$\Rightarrow \quad 3\tan^{-1} x = -\pi + \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$$

Property XIII

$$\text{(i) } 2\tan^{-1} x = \begin{cases} \sin^{-1}\left(\dfrac{2x}{1+x^2}\right); & \text{if } -1 \le x \le 1 \\ \pi - \sin^{-1}\left(\dfrac{2x}{1+x^2}\right); & \text{if } x > 1 \\ -\pi - \sin^{-1}\left(\dfrac{2x}{1+x^2}\right); & \text{if } x < -1 \end{cases}$$

$$\text{(ii) } 2\tan^{-1} x = \begin{cases} \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right); & \text{if } 0 \le x < \infty \\ -\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right); & \text{if } -\infty < x \le 0 \end{cases}$$

Proof

(i) Let $\tan^{-1} x = \theta$. Then, $x = \tan\theta$

$$\therefore \quad \sin 2\theta = \frac{2\tan\theta}{1 + \tan^2\theta} \Rightarrow \sin 2\theta = \frac{2x}{1 + x^2}$$

Case I When $-1 \le x \le 1$

$$-1 \le x \le 1 \Rightarrow -1 \le \tan\theta \le 1$$

$$\Rightarrow \quad -\frac{\pi}{4} \le \theta \le \frac{\pi}{4} \Rightarrow -\frac{\pi}{2} \le 2\theta \le \frac{\pi}{2}$$

$$\therefore \quad \sin 2\theta = \frac{2x}{1 + x^2} \Rightarrow 2\theta = \sin^{-1}\left(\frac{2x}{1 + x^2}\right)$$

$$\Rightarrow \quad 2\tan^{-1} x = \sin^{-1}\left(\frac{2x}{1 + x^2}\right)$$

Case II When $x > 1$

$$x > 1 \Rightarrow \tan\theta > 1$$

$$\Rightarrow \quad \frac{\pi}{4} < \theta < \frac{\pi}{2} \Rightarrow \frac{\pi}{2} < 2\theta < \pi$$

$$\Rightarrow \quad -\pi < -2\theta < -\frac{\pi}{2} \Rightarrow 0 < \pi - 2\theta < \frac{\pi}{2}$$

$$\therefore \quad \sin 2\theta = \frac{2x}{1 + x^2}$$

$$\Rightarrow \quad \sin(\pi - 2\theta) = \frac{2x}{1 + x^2}$$

$$\Rightarrow \quad \pi - 2\theta = \sin^{-1}\left(\frac{2x}{1 + x^2}\right)$$

$$\Rightarrow \quad \pi - 2\tan^{-1} x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \quad 2\tan^{-1} x = \pi - \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

Case III When $x < -1$

$$x < -1 \Rightarrow \tan\theta < -1$$

$$\Rightarrow \quad -\frac{\pi}{2} < \theta < -\frac{\pi}{4} \Rightarrow -\pi < 2\theta < -\frac{\pi}{2}$$

$$\Rightarrow \quad 0 < \pi + 2\theta < \frac{\pi}{2}$$

$$\Rightarrow \quad -\frac{\pi}{2} < -\pi - 2\theta < 0$$

$$\therefore \quad \sin 2\theta = \frac{2x}{1+x^2}$$

$$\Rightarrow \quad -\sin(\pi + 2\theta) = \frac{2x}{1+x^2}$$

$$\Rightarrow \quad \sin(-\pi - 2\theta) = \frac{2x}{1+x^2}$$

$$\Rightarrow \quad -\pi - 2\theta = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \quad -\pi - 2\tan^{-1} x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \quad 2\tan^{-1} x = -\pi - \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

(ii) Let $\tan^{-1} x = \theta$

Then $x = \tan\theta$

$$\therefore \quad \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta} \Rightarrow \cos 2\theta = \frac{1-x^2}{1+x^2}$$

Case I When $0 \le x < \infty$

$$0 \le x < \infty$$

$$\Rightarrow \quad 0 \le \theta < \frac{\pi}{2}$$

$$\Rightarrow \quad 0 \le 2\theta < \pi$$

$$\therefore \quad \cos 2\theta = \frac{1-x^2}{1+x^2} \Rightarrow 2\theta = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \quad 2\tan^{-1} x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

Case II When $-\infty < x \le 0$

$$-\infty < x \le 0$$

$$\Rightarrow \quad -\frac{\pi}{2} < \theta \le 0 \Rightarrow 0 \le -2\theta < \pi$$

$$\therefore \quad \cos 2\theta = \frac{1-x^2}{1+x^2}$$

$$\Rightarrow \quad \cos(-2\theta) = \frac{1-x^2}{1+x^2}$$

$$\Rightarrow \quad -2\theta = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \quad -2\tan^{-1} x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \quad 2\tan^{-1} x = -\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

Example 33. Define $y = \cos^{-1}(4x^3 - 3x)$ in terms of $\cos^{-1} x$ and also draw its graph.

Sol. We know, $3\cos^{-1} x = \begin{cases} \cos^{-1}(4x^3-3x), \text{ if } \frac{1}{2} \le x \le 1 \\ 2\pi - \cos^{-1}(4x^3-3x), \text{ if } -\frac{1}{2} \le x < \frac{1}{2} \\ 2\pi + \cos^{-1}(4x^3-3x), \text{ if } -1 \le x < -\frac{1}{2} \end{cases}$

$$\therefore \quad y = \cos^{-1}(4x^3 - 3x)$$

$$= \begin{cases} 3\cos^{-1} x & ; \quad \frac{1}{2} \le x \le 1 \\ 2\pi - 3\cos^{-1} x & ; \quad -\frac{1}{2} \le x < \frac{1}{2} \\ -2\pi + 3\cos^{-1} x & ; \quad -1 \le x < -\frac{1}{2} \end{cases}$$

Graph

For $y = \cos^{-1}(4x^3 - 3x)$

Domain : $[-1, 1]$

Range : $[0, \pi]$

(i) If $\frac{1}{2} \le x \le 1, y = 3\cos^{-1} x$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-3}{\sqrt{1-x^2}} = -3(1-x^2)^{-1/2} \qquad \text{... (i)}$$

$$\Rightarrow \quad \frac{dy}{dx} < 0 \text{ if } x \in \left[\frac{1}{2}, 1\right)$$

$\Rightarrow$ Decreasing if $x \in \left[\frac{1}{2}, 1\right)$

Again if we differentiate Eq. (i) w.r.t 'x', we get

$$\frac{d^2y}{dx^2} = -\frac{3x}{(1-x^2)^{3/2}}$$

$\Rightarrow \frac{d^2y}{dx^2} < 0$ if $x \in \left[\frac{1}{2}, 1\right)$

$\Rightarrow$ concavity downwards if $x \in \left[\frac{1}{2}, 1\right)$

(ii) If $-\frac{1}{2} \le x < \frac{1}{2}$, $y = 2\pi - 3\cos^{-1} x$

$\therefore \quad \frac{dy}{dx} = \frac{3}{\sqrt{1-x^2}} \Rightarrow \frac{dy}{dx} > 0$ if $x \in \left[-\frac{1}{2}, \frac{1}{2}\right)$

$\Rightarrow$ Increasing if $x \in \left[-\frac{1}{2}, \frac{1}{2}\right)$ and $\frac{d^2y}{dx^2} = \frac{3x}{(1-x^2)^{3/2}}$

(a) If $x \in \left[-\frac{1}{2}, 0\right)$ then $\frac{d^2y}{dx^2} < 0$

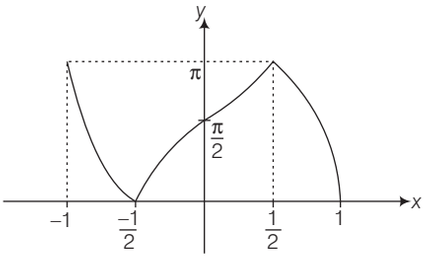
$\Rightarrow$ Concavity downwards if $x \in \left[-\frac{1}{2}, 0\right)$

(b) If $x \in \left[0, \frac{1}{2}\right)$ then $\frac{d^2y}{dx^2} > 0$

$\Rightarrow$ Concavity upwards if $x \in \left[0, \frac{1}{2}\right)$

(iii) Similarly, if $-1 \le x < -\frac{1}{2}$ then $\frac{dy}{dx} < 0$ and $\frac{d^2y}{dx^2} > 0$

$\therefore$ the graph of $y = \cos^{-1}(4x^3 - 3x)$ is as



Example 34. Prove that

$$4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99} = \frac{\pi}{4}$$

Sol. $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}$

$$= 2\left(2\tan^{-1}\frac{1}{5}\right) - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}$$

$$= 2\left\{\tan^{-1}\frac{2(1/5)}{1-(1/5)^2}\right\} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}$$

$$\left\{\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right), \text{ if } |x| < 1\right\}$$

$$= 2\tan^{-1}\frac{5}{12} - \left\{\tan^{-1}\frac{1}{70} - \tan^{-1}\frac{1}{99}\right\}$$

$$= \tan^{-1}\left(\frac{2(5/12)}{1-(5/12)^2}\right) - \tan^{-1}\left(\frac{\frac{1}{70}-\frac{1}{99}}{1+\frac{1}{70}\times\frac{1}{99}}\right)$$

$$\left\{\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right) \text{ if } |x| < 1 \text{ and } \tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)\right\}$$

$$= \tan^{-1}\left(\frac{120}{119}\right) - \tan^{-1}\left(\frac{29}{6931}\right)$$

$$= \tan^{-1}\left(\frac{120}{119}\right) - \tan^{-1}\left(\frac{1}{239}\right)$$

$$= \tan^{-1}\left(\frac{\frac{120}{119}-\frac{1}{239}}{1+\left(\frac{120}{119}\right)\left(\frac{1}{239}\right)}\right)$$

$$= \tan^{-1}(1) = \frac{\pi}{4}$$

Example 35. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$, prove that $x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$.

Sol. Let $\sin^{-1}x = A$, $\sin^{-1}y = B$ and $\sin^{-1}z = C$.

Then, $x = \sin A$, $y = \sin B$, $z = \sin C$

We have,

$$\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$$

$$\Rightarrow \quad A + B + C = \pi$$

$$\Rightarrow \quad \sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$$

$$\Rightarrow \quad 2\sin A \cdot \cos A + 2\sin B \cdot \cos B + 2\sin C \cdot \cos C = 4\sin A \sin B \sin C$$

$$\Rightarrow \quad x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$$

Example 36. ***Solve*** $\sin[2\cos^{-1}\{\cot(2\tan^{-1}x)\}] = 0$

Sol. We have,

$$\sin[2\cos^{-1}\{\cot(2\tan^{-1}x)\}] = 0$$

$$\Rightarrow \quad \sin\left[2\cos^{-1}\left\{\cot\left(\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right)\right\}\right] = 0$$

$$\left\{\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right\}$$

$$\Rightarrow \quad \sin\left[2\cos^{-1}\left\{\cot\left(\cot^{-1}\left(\frac{1-x^2}{2x}\right)\right)\right\}\right] = 0$$

$$\left\{\because \cot^{-1}x = \tan^{-1}\frac{1}{x}\right\}$$

$$\Rightarrow \quad \sin\left[2\cos^{-1}\left(\frac{1-x^2}{2x}\right)\right]=0$$

$$\Rightarrow \quad \sin\left[\sin^{-1}\left\{2\left(\frac{1-x^2}{2x}\right)\sqrt{1-\left(\frac{1-x^2}{2x}\right)^2}\right\}\right]=0$$

$$\{\because 2\cos^{-1}x=\sin^{-1}(2x\sqrt{1-x^2})\}$$

$$\Rightarrow \quad \left(\frac{1-x^2}{x}\right)\left[\sqrt{1-\left(\frac{1-x^2}{2x}\right)^2}\right]=0$$

$$\Rightarrow \quad \frac{1-x^2}{x}=0 \quad \text{or} \quad 1-\left(\frac{1-x^2}{2x}\right)^2=0$$

$$\Rightarrow \quad 1-x^2=0 \text{ or } \left(\frac{1-x^2}{2x}\right)^2=1$$

$$\Rightarrow \quad x=\pm 1 \text{ or } (1-x^2)^2=4x^2$$

Now, $(1-x^2)^2=4x^2$

$$\Rightarrow \quad (1-x^2)^2-(2x)^2=0$$

$$\Rightarrow \quad (1-x^2-2x)(1-x^2+2x)=0$$

$$\Rightarrow \quad x^2+2x-1=0 \text{ or } x^2-2x-1=0$$

$$\Rightarrow \quad x=-1\pm\sqrt{2}$$

or $\quad x=1\pm\sqrt{2}$

Hence, $x=\pm 1, -1\pm\sqrt{2}, 1\pm\sqrt{2}$ are the roots of the given equation.

Example 37. Find the value of

$$\tan\left\{\frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right)+\frac{1}{2}\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right\}$$

Sol. Let $x=\tan A$ and $y=\tan B$. Then,

$$\tan\left\{\frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right)+\frac{1}{2}\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right\}$$

$$=\tan\left\{\frac{1}{2}\sin^{-1}\left(\frac{2\tan A}{1+\tan^2 A}\right)+\frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2 B}{1+\tan^2 B}\right)\right\}$$

$$=\tan\left\{\frac{1}{2}\sin^{-1}(\sin 2A)+\frac{1}{2}\cos^{-1}(\cos 2B)\right\}$$

$$=\tan\left(\frac{2A}{2}+\frac{2B}{2}\right)=\tan(A+B)$$

$$=\frac{\tan A+\tan B}{1-\tan A\tan B}=\frac{x+y}{1-xy}$$

Aliter

$$\tan\left[\frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right)+\frac{1}{2}\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right]$$

$$=\tan\left[\frac{1}{2}\cdot 2\tan^{-1}x+\frac{1}{2}\cdot 2\tan^{-1}y\right]$$

$$=\tan(\tan^{-1}x+\tan^{-1}y)$$

$$=\tan\left(\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right)=\frac{x+y}{1-xy}$$

Exercise for Session 7

1. Define $y=\sin^{-1}(3x-4x^3)$ in terms of $\sin^{-1}x$ and also draw its graph.
2. Define $y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$ in terms of $\tan^{-1}x$ and also draw its graph.

Draw the graph of the following

3. $y=\cos^{-1}(2x^2-1)$
4. $y=\sin^{-1}(2x\sqrt{1-x^2})$
5. $y=\tan^{-1}\frac{2x}{1-x^2}$
6. $y=\sin^{-1}\frac{2x}{1+x^2}$
7. $y=\cos^{-1}\frac{1-x^2}{1+x^2}$

JEE Type Solved Examples :
Single Option Correct Type Questions

● **Ex. 1.** *Let* $f(x)=\sin x+\cos x+\tan x+\arcsin x+\arccos x+\arctan x$. *If M and m are maximum and minimum values of* $f(x)$, *then their arithmetic mean is equal to*

(a) $\frac{\pi}{2}+\cos 1$ (b) $\frac{\pi}{2}+\sin 1$

(c) $\frac{\pi}{4}+\tan 1+\cos 1$ (d) $\frac{\pi}{4}+\tan 1+\sin 1$

Sol. (*a*) Domain of f is $[-1,1]$; $f(x)=\sin x+\cos x+\tan x+\sin^{-1}x+\cos^{-1}x+\tan^{-1}x$

$$f'(x)=\cos x-\sin x+\underbrace{\sec^2 x}_{>1}+0+\underbrace{\frac{1}{1+x^2}}_{[1/2,1]}$$

Hence, $f'(x)>0\Rightarrow f$ is increasing

$\Rightarrow$ Range is $[f(-1),f(1)]$

$\therefore \quad f(x)|_{\min}=f(-1)=-\sin 1+\cos 1-\tan 1$

$$-\frac{\pi}{2}+\pi-\frac{\pi}{4}=\frac{\pi}{4}+\cos 1-\sin 1-\tan 1$$

and $f(x)|_{\max}=f(1)=\sin 1+\cos 1+\tan 1$

$$+\frac{\pi}{2}+\frac{\pi}{4}=\frac{3\pi}{4}+\cos 1+\sin 1+\tan 1$$

Now, $\quad \frac{M+m}{2}=\frac{\pi}{2}+\cos 1$

● **Ex. 2.** *The value of* $5\cdot\cot\left(\sum_{k=1}^{5}\cot^{-1}(k^2+k+1)\right)$ *is equal to*

(a) $\frac{5}{2}$ (b) 7

(c) -7 (d) $\frac{7}{2}$

Sol. (*b*) Consider

$$\sum_{k=1}^{5}\left(\tan^{-1}\left(\frac{(k+1)-k}{1+k(k+1)}\right)\right)=\sum_{k=1}^{5}\tan^{-1}(k+1)-\tan^{-1}k$$

Now, $\quad T_1=\tan^{-1}(2)-\tan^{-1}(1)$;

$T_2=\tan^{-1}(3)-\tan^{-1}(2)$ and so on

Hence, $\sum_{k=1}^{5}\cot^{-1}(k^2+k+1)=\tan^{-1}(6)-\tan^{-1}(1)$

$$=\tan^{-1}\left(\frac{5}{7}\right)$$

$$=\cot^{-1}\left(\frac{7}{5}\right)$$

$\therefore \quad 5\cot\left(\cot^{-1}\frac{7}{5}\right)=7$

● **Ex. 3.** *If the equation* $5\arctan(x^2+x+k)+3\operatorname{arc}\cot(x^2+x+k)=2\pi$, *has two distinct solutions, then the range of k, is*

(a) $\left(0,\frac{5}{4}\right]$ (b) $\left(-\infty,\frac{5}{4}\right)$ (c) $\left(\frac{5}{4},\infty\right)$ (d) $\left(-\infty,\frac{5}{4}\right]$

Sol. (*b*) We have $2\pi=\frac{3\pi}{2}+2\tan^{-1}(x^2+x+k)$

$$\left(\text{As, } \tan^{-1}\alpha+\cot^{-1}\alpha=\frac{\pi}{2}\forall\ \alpha\in R\right)$$

$\Rightarrow \quad \tan^{-1}(x^2+x+k)=\frac{\pi}{4}\Rightarrow x^2+x+k=1$

$\Rightarrow \quad x^2+x+(k-1)=0$

$\therefore$ For required condition, put $D>0$

$\Rightarrow \quad 1-4(k-1)>0\ \Rightarrow\ 5-4k>0\ \Rightarrow\ k<\frac{5}{4}$

● **Ex. 4.** *If* $f(x)=x^{11}+x^9-x^7+x^3+1$ *and* $f(\sin^{-1}(\sin 8))=\alpha$, α *is constant, then* $f(\tan^{-1}(\tan 8))$ *is equal to*

(a) α (b) $\alpha-2$

(c) $\alpha+2$ (d) $2-\alpha$

Sol. (*d*) $f(x)+f(-x)=2$

Now, $\quad (\sin^{-1}(\sin 8))=3\pi-8=y$ (say)

and $\quad (\tan^{-1}(\tan 8))=(8-3\pi)=-y$

Hence $\quad f(y)+f(-y)=2$

Given $\quad f(y)=\alpha\ \Rightarrow\ f(-y)=2-\alpha$

● **Ex. 5.** *If* $\sin^{-1}\left(x^2-\frac{x^4}{3}+\frac{x^6}{9}.....\right)+\cos^{-1}\left(x^4-\frac{x^8}{3}+\frac{x^{12}}{9}.....\right)=\frac{\pi}{2}$, *where* $0\le|x|<\sqrt{3}$, *then number of values of 'x' is equal to*

(a) 1 (b) 2 (c) 3 (d) 4

Sol. (*c*) $\sin^{-1}\underbrace{\left(x^2-\frac{x^4}{3}+\frac{x^6}{9}.....\right)}_{X}+\cos^{-1}\underbrace{\left(x^4-\frac{x^8}{3}+\frac{x^{12}}{9}.....\right)}_{Y}=\frac{\pi}{2}$

$\Rightarrow\ X=Y\ \Rightarrow\ \frac{x^2}{1+\frac{x^2}{3}}=\frac{x^4}{1+\frac{x^4}{3}}\ \Rightarrow\frac{3}{3+x^2}=\frac{3x^2}{3+x^4}$

$\Rightarrow \quad 9+3x^4=9x^2+3x^4\ \Rightarrow\ x^2=1$

Thus, $\quad x=0,1$ or -1

Hence, number of values is equal to 3.

● **Ex. 6.** *Suppose* $3\sin^{-1}(\log_2 x)+\cos^{-1}(\log_2 y)=\pi/2$ *and* $\sin^{-1}(\log_2 x)+2\cos^{-1}(\log_2 y)=11\pi/6$, *then the value of* $x^{-2}+y^{-2}$ equals

(a) 6 (b) 7

(c) 5 (d) $\frac{7}{2}$

Sol. (a) Let $\sin^{-1}(\log_2 x)=a$ and $\cos^{-1}(\log_2 y)=b$

$\therefore \quad 3a+b=\frac{\pi}{2}$ and $a+2b=\frac{11\pi}{6}$

$\Rightarrow \quad a=-\frac{\pi}{6}$ and $b=\pi$

Hence, $x=\frac{1}{\sqrt{2}}$ and $y=\frac{1}{2} \Rightarrow \frac{1}{x^2}+\frac{1}{y^2}=6$

● **Ex. 7.** *Range of* $f(x)=\sin^{-1}\log[x]+\log(\sin^{-1}[x])$, *where [ ] denotes GIF is*

(a) 1 (b) 2

(c) 0 (d) $\left\{\log\frac{\pi}{2}\right\}$

Sol. (*d*) Domain of $f(x)$ is [1, 2).

$\therefore$ Range is $\left\{\log\frac{\pi}{2}\right\}$.

● **Ex. 8.** $\sum_{n=1}^{5}\sin^{-1}(\sin(2n-1))$ *is*

(a) 1 (b) 2

(c) 3 (d) 4

Sol. (*a*) $\sum_{n=1}^{5}\sin^{-1}(\sin(2n-1))$

$=\sin^{-1}(\sin 1)+\sin^{-1}(\sin 3)+\sin^{-1}(\sin 5)$

$+\sin^{-1}(\sin 7)+\sin^{-1}(\sin 9)$

$=1+\pi-3+5-2\pi+7-2\pi+3\pi-9=1$

● **Ex. 9.** *If* α *and* $\beta(\alpha>\beta)$ *are roots of the equation* $x^2-\sqrt{2}x+\sqrt{3-2\sqrt{2}}=0$, *then the value of* $(\cos^{-1}\alpha+\tan^{-1}\alpha+\tan^{-1}\beta)$ *is equal to*

(a) $\frac{3\pi}{8}$ (b) $\frac{5\pi}{8}$

(c) $\frac{7\pi}{8}$ (d) $\frac{\pi}{3}$

Sol. (*a*) $x^2-\sqrt{2}x+\sqrt{3-2\sqrt{2}}=0 \Rightarrow x^2-\sqrt{2}x+\sqrt{2}-1=0$

$\Rightarrow \quad x^2-1-\sqrt{2}(x-1)=0 \Rightarrow (x-1)(x+1-\sqrt{2})=0$

$x=1,\sqrt{2}-1$

$\therefore \quad \alpha=1$ and $\beta=\sqrt{2}-1$

Hence, $\cos^{-1}\alpha+\tan^{-1}\alpha+\tan^{-1}\beta=0+\frac{\pi}{4}+\frac{\pi}{8}=\frac{3\pi}{8}$

● **Ex.10.** *If the mapping* $f(x)=mx+c, m>0$ *maps* $[-1,1]$ *onto* $[0,2]$, *then* $\tan\left(\tan^{-1}\frac{1}{7}+\cot^{-1}8+\cot^{-1}18\right)$ *is equal to*

(a) $f\left(\frac{2}{3}\right)$ (b) $f\left(\frac{1}{3}\right)$ (c) $f\left(\frac{-1}{3}\right)$ (d) $f\left(\frac{-2}{3}\right)$

Sol. (*d*) Clearly, $f(x)=x+1$ (As, $-1\le x\le 1$, $0\le x+1\le 2$) and $f(x)=mx+c$

Now, $\tan\left(\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}+\tan^{-1}\frac{1}{18}\right)$

$=\tan\left(\tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{7}\times\frac{1}{8}}\right)+\tan^{-1}\frac{1}{18}\right)$

$=\tan\left(\tan^{-1}\left(\frac{15}{55}\right)+\tan^{-1}\left(\frac{1}{18}\right)\right)$

$=\tan\left(\tan^{-1}\left(\frac{3}{11}\right)+\tan^{-1}\left(\frac{1}{18}\right)\right)$

$=\tan\left(\tan^{-1}\left(\frac{\frac{3}{11}+\frac{1}{18}}{1-\frac{3}{11}\times\frac{1}{18}}\right)\right)$

$=\tan\left(\tan^{-1}\frac{1}{3}\right)=\frac{1}{3}=f\left(\frac{-2}{3}\right)$

● **Ex.11.** *If* $(\sin^{-1}a)^2+(\cos^{-1}b)^2+(\sec^{-1}c)^2+(\operatorname{cosec}^{-1}d)^2=\frac{5\pi^2}{2}$, *then the value of* $(\sin^{-1}a)^2-(\cos^{-1}b)^2+(\sec^{-1}c)^2-(\operatorname{cosec}^{-1}d)^2$

(a) $-\pi^2$ (b) $-\frac{\pi^2}{2}$

(c) 0 (d) $\frac{\pi^2}{2}$

Sol. (*c*) As $0\le(\sin^{-1}a)^2\le\frac{\pi^2}{4}$, $0\le(\cos^{-1}b)^2\le\pi^2$,

$0\le(\sec^{-1}c)^2\le\pi^2\left(\text{except }\frac{\pi^2}{4}\right)$ and $0<(\operatorname{cosec}^{-1}d)^2\le\frac{\pi^2}{4}$

So, $0<(\sin^{-1}a)^2+(\cos^{-1}b)^2+(\sec^{-1}c)^2+(\operatorname{cosec}^{-1}d)^2\le\frac{5\pi^2}{2}$

$\therefore(\sin^{-1}a)^2+(\cos^{-1}b)^2+(\sec^{-1}c)^2+(\operatorname{cosec}^{-1}d)^2=\frac{5\pi^2}{2}$ (Given)

$\Rightarrow \quad (\sin^{-1}a)^2=\frac{\pi^2}{4}$

$(\cos^{-1}b)^2=\pi^2$

$(\sec^{-1}c)^2=\pi^2$

and $(\operatorname{cosec}^{-1}d)^2=\frac{\pi^2}{4}$

Hence, $(\sin^{-1}a)^2-(\cos^{-1}b)^2+(\sec^{-1}b)^2-(\operatorname{cosec}^{-1}d)^2=0$

● ***Ex. 12.*** *If* $f(x)=\sum_{r=1}^{n}\tan^{-1}\left(\frac{1}{x^2+(2r-1)x+(r^2-r+1)}\right)$, *then* $\left|\lim_{n\to\infty} f'(0)\right|$ *is*

(a) 1 (b) 2
(c) 3 (d) 4

Sol. (*a*) $f(x)=\sum_{r=1}^{n}(\tan^{-1}(x+r)-\tan^{-1}(x+r-1))$

$\Rightarrow \quad f(x)=\tan^{-1}(x+n)-\tan^{-1}(x)$

$\Rightarrow \quad f'(x)=\frac{1}{1+(x+n)^2}-\frac{1}{1+x^2}$

$\Rightarrow \quad f'(0)=\frac{1}{1+n^2}-1$

Now, $\lim_{n\to\infty} f'(0)=-1\Rightarrow\left|\lim_{n\to\infty} f'(0)\right|=1$

● ***Ex. 13.*** *The range of the function* $f(x)=\sec^{-1}(x)+\tan^{-1}(x)$, *is*

(a) $(0,\pi)$ (b) $\left(\frac{-\pi}{2},\frac{3\pi}{2}\right)$
(c) $\left(0,\frac{3\pi}{4}\right]$ (d) None of these

Sol. (*a*) $D_f=(-\infty,-1]\cup[1,\infty)$

Also, f is an increasing function.

For, $x\in(-\infty,-1], f(x)\in\left(0,\frac{3\pi}{4}\right]$ ... (i)

and for $x\in[1,\infty), f(x)\in\left[\frac{\pi}{4},\pi\right)$ ... (ii)

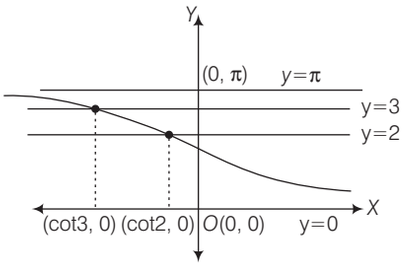
$\therefore$ For range of $f(x)$, (i) $\cup$ (ii) $\Rightarrow (0,\pi)$

● ***Ex.14.*** *The solution set of inequality* $(\cot^{-1}x)(\tan^{-1}x)+\left(2-\frac{\pi}{2}\right)\cot^{-1}x-3\tan^{-1}x-3\left(2-\frac{\pi}{2}\right)>0$, *is*

(a) $x\in(\tan 2,\tan 3)$
(b) $x\in(\cot 3,\cot 2)$
(c) $x\in(-\infty,\tan 2)\cup(\tan 3,\infty)$
(d) $x\in(-\infty,\cot 3)\cup(\cot 2,\infty)$

Sol. (*b*) Given,

$(\cot^{-1}x)(\tan^{-1}x)+\left(2-\frac{\pi}{2}\right)\cot^{-1}x-3\tan^{-1}x-3\left(2-\frac{\pi}{2}\right)>0$



$\Rightarrow \quad \cot^{-1}x\left(\tan^{-1}x+2-\frac{\pi}{2}\right)-3\left(\tan^{-1}x+2-\frac{\pi}{2}\right)>0$

$\Rightarrow \quad (\cot^{-1}x-3)(2-\cot^{-1}x)>0$

$\left(\text{As } \tan^{-1}x-\frac{\pi}{2}=-\cot^{-1}x\right)$

$\Rightarrow \quad (\cot^{-1}x-3)(\cot^{-1}x-2)<0$

$\Rightarrow \quad 2<\cot^{-1}x<3$

$\Rightarrow \cot 3<x<\cot 2$ (As $\cot^{-1}x$ is a decreasing function)

Hence, $x\in(\cot 3,\cot 2)$

● ***Ex.15.*** *Let* $f(x)=\sin(\sin^{-1}2x)+\text{cosec}(\text{cosec}^{-1}2x)+\tan(\tan^{-1}2x)$, *then which one of the following statement is/are incorrect?*

(a) $f(x)$ is odd function
(b) $f(x)$ is injective
(c) Range of $f(x)$ contains only two integers.
(d) The value of $f'\left(\frac{1}{2}\right)$ is equal to 6.

Sol. (*d*) Clearly $f(x)=6x$ and domain $=\left\{\frac{-1}{2},\frac{1}{2}\right\}$

JEE Type Solved Examples :
More than One Correct Option Type Questions

● ***Ex. 16.*** *If* $f(x)=\cos^{-1}(\cos(x+1))$ *and* $g(x)=\sin^{-1}(\sin(x+2))$, *then*

(a) $f(1)+g(1)=(\pi-1)$ (b) $f(1)>g(1)$
(c) $f(2)>g(2)$ (d) $f(2)<g(2)$

Sol. (a, b, c) $f(1)=\cos^{-1}(\cos 2)=2$

$g(1)=\sin^{-1}(\sin 3)=(\pi-3)$

$f(1)+g(1)=(\pi-1)$

and $f(1)>g(1)$

$f(2)=\cos^{-1}(\cos 3)=3$

$g(3)=\sin^{-1}(\sin 4)=(\pi-4)$

● ***Ex. 17.*** $\sum_{r=1}^{\infty} \tan^{-1}\left(\frac{2}{(r+2)^2}\right)$ *is*

(a) $\pi - (\tan^{-1}2 + \tan^{-1}3)$ (b) $\frac{\pi}{4}$

(c) $\frac{7\pi}{4}$ (d) $\pi + (\tan^{-1}2 + \tan^{-1}3)$

Sol. (a, b) $T_r = \tan^{-1}\left(\frac{2}{(r^2+4r+3)+1}\right) = \tan^{-1}\left(\frac{(r+3)-(r+1)}{1+(r+3)(r+1)}\right)$

$$T_r = \tan^{-1}(r+3) - \tan^{-1}(r+1)$$

$$\text{Sum} = \pi - (\tan^{-1}(2) + an^{-1}(3))$$

$$= \pi - (\pi + \tan^{-1}(-1)) = \frac{\pi}{4}$$

● ***Ex. 18.*** *If sides AB, BC and CA of a triangle ABC are represented by* $x + 2 = 0, 3x + y = 0$ *and* $x + 3y + 2 = 0$ *respectively, then identify the correct statement.*

(a) $\Sigma \tan A = \frac{4}{3}$

(b) $\Pi \tan A = -\frac{4}{3}$

(c) $\Sigma \tan A \tan B = -\frac{41}{9}$

(d) $\sin^2(A + B) + \cos^2 C = \frac{5}{4}$

Sol. (b, c)

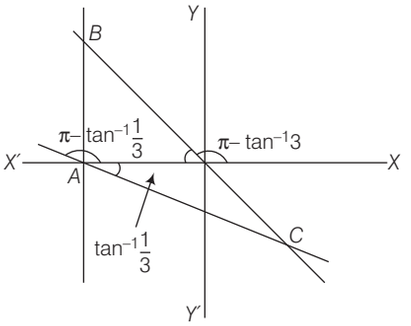
$$\angle A = \frac{\pi}{2} + \tan^{-1}\frac{1}{3}$$

$$\angle B = \frac{\pi}{2} - \tan^{-1}3$$

$$\angle C = \pi - (A + B)$$

$$= \tan^{-1}3 - \tan^{-1}\frac{1}{3}$$

$$= 2\tan^{-1}3 - \frac{\pi}{2}$$



$$\tan A + \tan B + \tan C = \tan\left(\frac{\pi}{2} + \tan^{-1}\frac{1}{3}\right) + \tan\left(\frac{\pi}{2} - \tan^{-1}3\right) + \tan\left(\tan^{-1}3 - \tan^{-1}\frac{1}{3}\right)$$

$$= -\cot\left(\tan^{-1}\frac{1}{3}\right) + \cot(\tan^{-1}3) + \tan\left(\tan^{-1}\left(\frac{3-\frac{1}{3}}{1+3.\frac{1}{3}}\right)\right)$$

$$= -3 + \frac{1}{3} + \frac{4}{3} = -\frac{4}{3}$$

$$\Sigma \tan A = \Pi \tan A = -\frac{4}{3}$$

$$\Sigma \tan A \tan B = -3.\frac{1}{3} + \frac{1}{3}.\frac{4}{3} + \frac{4}{3}(-3) = -5 + \frac{4}{9} = -\frac{41}{9}$$

$$\sin^2(A + B) + \cos^2 C = \sin^2(\pi - C) + \cos^2 C = 1$$

● ***Ex. 19.*** *Which of the following is/are correct?*

(a)

$$\cos(\cos(\cos^{-1}1)) < \sin(\sin^{-1}(\sin(\pi - 1))) < \sin(\cos^{-1}(\cos(2\pi - 2)))$$

(b) $\cos(\cos(\cos^{-1}1)) < \sin(\cos^{-1}(\cos(2\pi - 2))) < \sin(\sin^{-1}(\sin(\pi - 1))) < \tan(\cot^{-1}(\cot 1))$

(c) $\sum_{t=1}^{5000} \cos^{-1}(\cos(2t\pi - 1)) = \sum_{t=1}^{2500} \cot^{-1}(\cot(t\pi + 2))$, where $t \in I$

(d) $\cot^{-1}\cot\,\text{cosec}^{-1}\text{cosec}\,\sec^{-1}\sec\tan\tan^{-1}\cos\cos^{-1}\sin^{-1}\sin 4 = 4 - \pi$

Sol. (a,b,c,d) For (a) and (b)

$$\cos(\cos^{-1}1) = 1 \Rightarrow \cos(\cos(\cos^{-1}1)) = \cos 1$$

$$\sin^{-1}(\sin(\pi - 1)) = \pi - (\pi - 1) = 1$$

$$\Rightarrow \quad \sin(\sin^{-1}(\sin(\pi - 1))) = \sin 1$$

$$\cos^{-1}(\cos(2\pi - 2)) = \cos^{-1}(\cos 2) = 2$$

$$\Rightarrow \quad \sin(\cos^{-1}(\cos(2\pi - 2))) = \sin 2$$

$$\tan(\cot^{-1}(\cot 1)) = \tan 1$$

It is easy to compare $\cos 1, \sin 1, \sin 2, \tan 1$

$$\cos 1 < \sin 1 < \sin 2 < \tan 1 \Rightarrow \text{(a) is correct}$$

For (c)

$\because \cos^{-1}\cos x$ is periodic with period 2π

$$\therefore \quad \cos^{-1}\cos(2t\pi - 1) = \cos^{-1}(\cos 1) = 1 \ (t \in I)$$

$$\sum_{t=1}^{5000} \cos^{-1}\cos(2t\pi - 1) = 5000$$

Now, $\cot^{-1}\cot(t\pi + 2) = 2$ [$\cot^{-1}\cot x$ is periodic with period π]

$$\therefore \sum_{t=1}^{2500} \cot^{-1}\cot(t\pi + 2) = 5000 \Rightarrow \text{(c) is correct}$$

(d) $\sin^{-1}\sin 4 = \pi - 4$

$$\cos\cos^{-1}(\pi - 4) = 4 - \pi$$

$$\tan\tan^{-1}(4 - \pi) = \pi - 4$$

$$\sec^{-1}\sec(\pi - 4) = 4 - \pi$$

$$\text{cosec}^{-1}\text{cosec}(4 - \pi) = 4 - \pi$$

$$\cot^{-1}\cot(4 - \pi) = 4 - \pi$$

$\Rightarrow$ (d) is correct

● ***Ex. 20.*** *Let x_1 and $x_2\,(x_1 > x_2)$ be roots of the equation* $\sin^{-1}(\cos(\tan^{-1}(\text{cosec}(\cot^{-1}x)))) = \frac{\pi}{6}$, *then*

(a) $\sin^{-1}\frac{1}{x_1} + \cos^{-1}\frac{1}{x_2} = \pi$

(b) $\sin^{-1}\left(\frac{1}{x_1}\right) + \cos^{-1}\left(\frac{1}{x_2}\right) = 0$

(c) $\sin^{-1}\frac{1}{x_1} + \sin^{-1}\left(\frac{1}{x_2}\right) = 0$

(d) $\cos^{-1}\left(\frac{1}{x_1}\right) + \cos^{-1}\left(\frac{1}{x_2}\right) = \pi$

Sol. (a,c,d) Given, $\sin^{-1}(\cos(\tan^{-1}(\text{cosec}(\cot^{-1}x)))) = \frac{\pi}{6}$

$$\Rightarrow \quad \sin^{-1}(\cos(\tan^{-1}\sqrt{1+x^2})) = \frac{\pi}{6}$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{1}{\sqrt{x^2+2}}\right) = \frac{\pi}{6}$$

$$\Rightarrow \quad \sqrt{x^2+2} = 2$$

$$\Rightarrow \quad x^2 = 2$$

$$\Rightarrow \quad x = \pm\sqrt{2}$$

So, $x_1 = \sqrt{2}$ and $x_2 = -\sqrt{2}$

Now, verify alternatives.

JEE Type Solved Examples :
Passage Based Questions

Passage I

(Ex. Nos. 21 to 22)

Suppose f, g and h be three real valued function defined on R.

Let $f(x) = 2x + |x|$, $g(x) = \frac{1}{3}(2x - |x|)$ *and* $h(x) = f(g(x))$

● ***Ex. 21.*** *The range of the function* $k(x) = 1 + \frac{1}{\pi}(\cos^{-1}(h(x)) + \cot^{-1}(h(x)))$ *is equal to*

(a) $\left[\frac{1}{4}, \frac{7}{4}\right]$ (b) $\left[\frac{5}{4}, \frac{11}{4}\right]$

(c) $\left[\frac{1}{4}, \frac{5}{4}\right]$ (d) $\left[\frac{7}{4}, \frac{11}{4}\right]$

● ***Ex. 22.*** *The domain of definition of the function* $l(x) = \sin^{-1}(f(x) - g(x))$ *is equal to*

(a) $\left[\frac{3}{8}, \infty\right)$ (b) $(-\infty, 1]$

(c) $[-1, 1]$ (d) $\left(-\infty, \frac{3}{8}\right]$

Sol. (Ex. Nos. 21 to 22)

We have $f(x) = \begin{cases} 3x, & \geq 0 \\ x, & x < 0 \end{cases}$ and $g(x) \begin{cases} \frac{x}{3}, & x \geq 0 \\ x, & x < 0 \end{cases}$

Clearly, f and g are inverse of each other.

Now, $h(x) = f(g(x)) = \begin{cases} 3\left(\frac{x}{3}\right) = x, & x \geq 0 \\ x \quad , & x < 0 \end{cases}$

21. (b) As $h(x) = x, \forall\ x \in R$

$$\Rightarrow \quad k(x) = 1 + \frac{1}{\pi}(\cos^{-1}x + \cot^{-1}x)$$

Domain of $k(x) = [-1, 1]$ and $k(x)$ is decreasing function on $[-1, 1]$.

As $k(x)$ is continuous function on $[-1, 1]$.

Now, $k_{\text{min.}}(x = 1) = 1 + \frac{1}{\pi}(\cos^{-1}1 + \cot^{-1}1)$

$$= 1 + \frac{1}{\pi}\left(0 + \frac{\pi}{4}\right) = 1 + \frac{1}{4} = \frac{5}{4}$$

$$k_{\text{max.}}(x = -1) = 1 + \frac{1}{\pi}(\cos^{-1}(-1) + \cot^{-1}(-1))$$

$$= 1 + \frac{1}{\pi}\left[\pi + \frac{3\pi}{4}\right]$$

$$= 1 + \frac{7}{4} = \frac{11}{4}$$

$\Rightarrow$ Range of $k(x) = \left[\frac{5}{4}, \frac{11}{4}\right]$

22. (d) We have, $f(x) - g(x) = (2x + |x|) - \frac{1}{3}(2x - |x|)$

$$= \frac{4x}{3} + \frac{4}{3}|x|$$

$$= \begin{cases} \frac{8}{3}x; & x \geq 0 \\ 0\ ; & x < 0 \end{cases}$$

$\therefore$ For domain of function

$$0 \leq \frac{8x}{3} \leq 1 \Rightarrow 0 \leq x \leq \frac{3}{8}$$

$$\Rightarrow \quad \text{Domain of } l(x) = \left(-\infty, \frac{3}{8}\right]$$

(**Note** Range of function $l(x) = \left[0, \frac{\pi}{2}\right]$)

Passage II

(Ex. Nos. 23 to 24)

In ΔABC, if $\angle B = \sec^{-1}\left(\frac{5}{4}\right) + \operatorname{cosec}^{-1}\sqrt{5}$,

$$\angle C = \operatorname{cosec}^{-1}\left(\frac{25}{7}\right) + \cot^{-1}\left(\frac{9}{13}\right) \text{ and } c = 3$$

(All symbols used have their usual meaning in a triangle.)
On the basis of above information, answer the following questions.

● ***Ex. 23.*** $\tan A, \tan B, \tan C$ *are in*

(a) AP (b) GP
(c) HP (d) neither AP, GP nor HP

● ***Ex. 24.*** *The distance between orthocentre and centroid of triangle with sides* $a^2, b^{\frac{4}{3}}$ *and* c *is equal to*

(a) $\frac{5}{2}$ (b) $\frac{5}{3}$
(c) $\frac{10}{3}$ (d) $\frac{7}{2}$

Sol. (Ex. Nos. 23 to 24)

$$\angle B = \sec^{-1}\left(\frac{5}{4}\right) + \operatorname{cosec}^{-1}\sqrt{5}$$

$$= \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{1}{2}\right)$$

$$= \tan^{-1}\frac{\frac{3}{4}+\frac{1}{2}}{1-\frac{3}{4}.\frac{1}{2}} = \tan^{-1}2$$

$$\angle C = \operatorname{cosec}^{-1}\left(\frac{25}{7}\right) + \cot^{-1}\left(\frac{9}{13}\right)$$

$$= \tan^{-1}\left(\frac{7}{24}\right) + \tan^{-1}\left(\frac{13}{9}\right)$$

$$= \tan^{-1}\left(\frac{\frac{7}{24}+\frac{13}{9}}{1-\frac{7}{24}.\frac{13}{9}}\right) = \tan^{-1}3$$

$\tan B = 2$ and $\tan C = 3 \Rightarrow \tan A = 1$

$\because$ $\Sigma \tan A = \Pi \tan A$

$\therefore$ $\sin A = \frac{1}{\sqrt{2}}, \sin B = \frac{2}{\sqrt{5}}$ and $\sin C = \frac{3}{\sqrt{10}}$

$\therefore$ $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$\Rightarrow$ $\sqrt{2}\,a = \frac{\sqrt{5}b}{2} = \frac{3\sqrt{10}}{3}$

Hence, $a = \sqrt{5}$ and $b = 2\sqrt{2}, c = 3$

23. (*a*) $\tan A = 1, \tan B = 2, \tan C = 3$ are in AP

24. (*b*) The triangle sides $a^2, b^{\frac{4}{3}}$ and c will have side-length 5, 4 and 3 respectively.

$\therefore$ Distance between orthocentre and centroid

$$= \frac{2}{3}(\text{circumradius}) = \frac{\text{hypotenuse}}{3} = \frac{5}{3}$$

JEE Type Solved Examples :
Integer Answer Type Questions

● ***Ex. 25.*** *Let* $f(x) = x^2 - 2ax + a - 2$ *and* $g(x) = \left[2 + \sin^{-1}\frac{2x}{1+x^2}\right]$. *If the set of real values of 'a' for which* $f(g(x)) < 0, \forall\ x \in R$ *is* (k_1, k_2), *then find the value of* $(10k_1 - 3k_2)$.

[**Note** : $[k]$ denotes greatest integer less than or equal to k.]

Sol. (8) We have $g(x) = \left[2 + \sin^{-1}\frac{2x}{1+x^2}\right]$

$$= 2 + \left[\sin^{-1}\frac{2x}{1+x^2}\right]$$

As, $\sin^{1}\frac{2x}{1+x^2} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

$\therefore$ $\left[\sin^{-1}\frac{2x}{1+x^2}\right] = -2, -1, 0, 1$

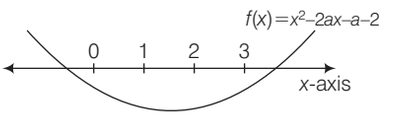
Range of $g(x) = \{0,1,2,3\}$ for $f(g(x)) < 0\ \forall\ x \in R$

$\Rightarrow$ $f(0) < 0$ and $(3) < 0$

Now, $f(0) < 0 \Rightarrow a - 2 < 0 \Rightarrow a < 2$

and $f(3) < 0 \Rightarrow 9 - 6a + a - 2 < 0$

$a > \frac{7}{5}$



$\therefore$ $a \in \left(\frac{7}{5}, 2\right)$

Hence, $k_1 = \frac{7}{5}, k_2 = 2$

$\therefore$ $(10k_1 - 3k_2) = 14 - 6 = 8$

● ***Ex. 26.*** *Let x_1, x_2, x_3 be the solution of*

$\tan^{-1}\left(\frac{2x+1}{x+1}\right)+\tan^{-1}\left(\frac{2x-1}{x-1}\right)=2\tan^{-1}(x+1)$ *where* $x_1<x_2<x_3$, *then* $2x_1+x_2+x_3^2$ *is equal to*

Sol. (1) Let $\alpha+\beta=2\gamma$ ...(i)

where $\tan\alpha=\frac{2x+1}{x+1}, \tan\beta=\frac{2x-1}{x-1}$ and

$\tan\gamma=x+1$

Taking tan in Eq. (i), we get

$$\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{2\tan\gamma}{1-\tan^2\gamma}$$

$$\Rightarrow \quad \frac{\frac{2x+1}{x+1}+\frac{2x-1}{x-1}}{1-\frac{4x^2-1}{x^2-1}}=\frac{2(x+1)}{1-(x+1)^2}$$

$$\Rightarrow \quad \frac{2x^2-x-1+2x^2+x-1}{x^2-1-4x^2+1}=\frac{2(x+1)}{-x^2-2x}$$

$$\Rightarrow \quad \frac{2(2x^2-1)}{-3x^2}=\frac{2(x+1)}{-(x^2+2x)}$$

$$x=0$$

or $2x^3+4x^2-x-2=3x^2+3x$

$\Rightarrow \quad 2x^3+x^2-4x-2=0$

$\Rightarrow \quad x^2(2x+1)-2(2x+1)=0$

$\Rightarrow \quad (x^2-2)(2x+1)=0$

$\Rightarrow \quad x=\sqrt{2}, -\sqrt{2}$ or $x=-\frac{1}{2}$

$x=-\sqrt{2}$ is rejected $\because$ it does not satisfy (i)

$\Rightarrow \quad x_1=-\frac{1}{2}, x_2=0$ and $x_3=\sqrt{2}$

$\Rightarrow \quad 2x_1+x_2+x_3^2=1$

● ***Ex. 27.*** *If the range of function* $f(x)=(\pi\sqrt{2}+\cos^{-1}\alpha)\,x^2+2(\cos^{-1}\beta)\,x+\pi\sqrt{2}-\cos^{-1}\alpha$ *is* $[0,\infty)$ *then find the value of* $|\alpha-\beta|+2\alpha\beta+1$

Sol. (3) Given, $f(x)=(\pi\sqrt{2}+\cos^{-1}\alpha)\,x^2+2(\cos^{-1}\beta)x+\pi\sqrt{2}-\cos^{-1}\alpha$

Clearly, graph of $f(x)$ is parabola opening upward.

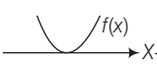
As, range of $f(x)$ is $[0,\infty)$, so discriminant $=0$

$\Rightarrow \quad b^2-4ac=0$

$\Rightarrow \quad 4(\cos^{-1}\beta)^2-4(\pi\sqrt{2}+\cos^{-1}\alpha)(\pi\sqrt{2}-\cos^{-1}\alpha)=0$

$\Rightarrow \quad 4(\cos^{-1}\beta)^2-4(2\pi^2-(\cos^{-1}\alpha)^2)=0$

$\Rightarrow \quad (\cos^{-1}\alpha)^2+(\cos^{-1}\beta)^2=2\pi^2$



$\Rightarrow \quad \cos^{-1}\alpha=\pi=\cos^{-1}\beta$

$\Rightarrow \quad \alpha=\beta=-1$

Hence, $|\alpha-\beta|+2\alpha\beta+1=0+2+1=3$

● ***Ex. 28.*** *Consider* $f(x)=\sin^{-1}[2x]+\cos^{-1}([x]-1)$ *(where [.] denotes greatest integer function.) If domain of* $f(x)$ *is* $[a,b)$ *and the range of* $f(x)$ *is* $\{c,d\}$, *then* $a+b+\frac{2d}{c}$ *is equal to (where* $c<d$*)*

Sol. (4) $f(x)=\sin^{-1}[2x]+\cos^{-1}([x]-1)$

$$-1\le[2x]\le1 \text{ and } -1\le[x]-1\le1$$

$\Rightarrow \quad -1\le2x<2$ and $0\le[x]\le2$

$\Rightarrow \quad -\frac{1}{2}<x<1$ and $0\le x<3$

$\Rightarrow \quad 0\le x<1$ Domain

$\Rightarrow \quad [x]=0$

$\Rightarrow \quad 0\le2x<2$

$\Rightarrow \quad [2x]=0$ or 1

Now, $f(x)=\sin^{-1}[2x]+\cos^{-1}(-1)$

$$=\left(0 \text{ or } \frac{\pi}{2}\right)+\pi=\pi \text{ or } \frac{3\pi}{2}$$

$\Rightarrow \quad a+b+\frac{2d}{c}=1+3=4$

● ***Ex. 29.*** *Let* $f(x)=\min(\tan^{-1}x, \cot^{-1}x)$ *and* $h(x)=f(x+2)-\pi/3$. *Let* x_1, x_2 *(where* $x_1<x_2$*) be the integers in the range of* $h(x)$, *then the value of* $(\cos^{-1}(\cos x_1)+\sin^{-1}(\sin x_2))$ *is equal to*

Sol. (1) $f(x)=\begin{cases}\tan^{-1}x; & x\le1\\ \cot^{-1}x; & x>1\end{cases}$

$$\Rightarrow \quad f(x+2)=\begin{cases}\tan^{-1}(x+2) & ; \quad x\le-1\\ \cot^{-1}(x+2) & ; \quad x>-1\end{cases}$$

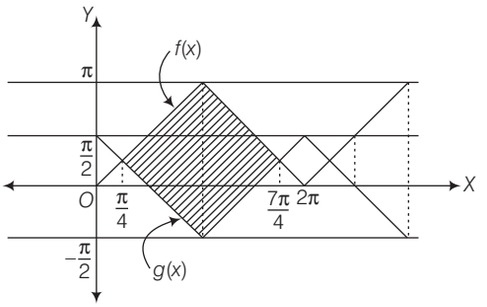
$\therefore$ Range of $h(x)$ is $\left(-\frac{5\pi}{6}, -\frac{\pi}{12}\right]$

$\therefore \quad \cos^{-1}\cos(-2)+\sin^{-1}\sin(-1)=1$

● ***Ex. 30.*** *If the area enclosed by the curves* $f(x)=\cos^{-1}(\cos x)$ *and* $g(x)=\sin^{-1}(\cos x)$ *in* $x\in\left[\frac{9\pi}{4}, \frac{15\pi}{4}\right]$ *is* $\frac{a\pi^2}{b}$ *(where, a and b coprime), then find* $(a-b)$.

Sol. (1) We have $g(x)=\sin^{-1}(\cos x)=\frac{\pi}{2}-\cos^{-1}(\cos x)$

Both the curves bound the regions of same area in $\left[\frac{\pi}{4}, \frac{7\pi}{4}\right], \left[\frac{9\pi}{4}, \frac{15\pi}{4}\right]$ and so on



∴ Required area = area of shaded square

$$=\frac{9\pi^2}{8}=\frac{a\pi^2}{b}$$

∴ $a=9$ and $b=8$

Hence, $a-b=1$

● ***Ex. 31.*** *Consider the curve* $y=\tan^{-1}x$ *and a point* $A\left(1,\frac{\pi}{4}\right)$ *on it. If the variable point* $P_i(x_i,y_i)$ *moves on the curve for* $i=1,2,3,...n(n\in N)$ *such that* $y_r=\sum_{m=1}^{r}\tan^{-1}\left(\frac{1}{2m^2}\right)$ *and* $B(x,y)$ *be the limiting position of variable point* P_n *as* $n\to\infty$, *then the value of reciprocal of the slope of AB will be*

Sol. (2) $y=\underset{n\to\infty}{\text{Lim}}\,y_n=\underset{n\to\infty}{\text{Lim}}\sum_{m=1}^{n}\tan^{-1}\left(\frac{1}{2m^2}\right)=$

$$\underset{n\to\infty}{\text{Lim}}\sum_{m=1}^{n}\tan^{-1}\left\{\frac{2}{1+(2m+1)(2m-1)}\right\}$$

$$=\underset{n\to\infty}{\text{Lim}}\sum_{m=1}^{n}\{\tan^{-1}(2m+1)-\tan^{-1}(2m-1)\}$$

$$=\underset{n\to\infty}{\text{Lim}}\{(\tan^{-1}3-\tan^{-1}1)+(\tan^{-1}5-\tan^{-1}3)+(\tan^{-1}7-\tan^{-1}5)....+\tan^{-1}(2n+1)-\tan^{-1}(2n-1)\}$$

$$=\underset{n\to\infty}{\text{Lim}}\{\tan^{-1}(2n+1)-\tan^{-1}1\}\to\frac{\pi}{4}$$

$\because B\to\left(1,\frac{\pi}{4}\right)$ i.e. coordinates of B approach, towards those of 'A'.

∴ Chord AB approches to be the tangent to $y=f(x)$ at A

$$\therefore (\text{slope of } AB)^{-1}=\left[\frac{d}{dx}\tan^{-1}x\right]_{\text{at } x=1}^{-1}$$

$$=(1+x^2)_{x=1}=2$$

● ***Ex. 32.*** *If* $\tan^{-1}x+\tan^{-1}\frac{\sqrt{1-y^2}}{y}=\frac{\pi}{3}$ *and* $\sin^{-1}y-\cos^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)=\frac{\pi}{6}$, *then* $\frac{5\sin^{-1}x}{\sin^{-1}y}$ *is*

Sol. (6) $\tan^{-1}x+\cos^{-1}y=\frac{\pi}{3}$ and $\sin^{-1}y-\cot^{-1}x=\frac{\pi}{6}$

$$\Rightarrow\quad \tan^{-1}x+\cos^{-1}y+\sin^{-1}y-\cot^{-1}x+\frac{\pi}{2}$$

or $\quad\tan^{-1}x=\cot^{-1}x\ \Rightarrow\ x=1$

Also, $\tan^{-1}x+\cos^{-1}y-\sin^{-1}y+\cot^{-1}x=\frac{\pi}{6}$

$$\Rightarrow\ \pi-2\sin^{-1}y=\frac{\pi}{6}\Rightarrow\ \sin^{-1}y=\frac{5\pi}{12}$$

$$\therefore\quad \frac{5\sin^{-1}x}{\sin^{-1}y}=6$$

● ***Ex. 33.*** *If* $A=\frac{1}{1}\cot^{-1}\left(\frac{1}{1}\right)+\frac{1}{2}\cot^{-1}\left(\frac{1}{2}\right)+\frac{1}{3}\cot^{-1}\left(\frac{1}{3}\right)$ *and* $B=1\cot^{-1}(1)+2\cot^{-1}(2)+3\cot^{-1}(3)$ *then* $|B-A|$ *is equal to* $\frac{a\pi}{b}+\frac{c}{d}\cot^{-1}(3)$ *where* $a,b,c,d\in N$ *are in their lowest form, find* $(b-a-c-d)$

Sol. (8) $B-A=(2\cot^{-1}(2)+3\cot^{-1}(3))-\left(\frac{1}{2}\cot^{-1}\left(\frac{1}{2}\right)+\frac{1}{3}\cot^{-1}\left(\frac{1}{3}\right)\right)$

$$=2(\cot^{-1}2+\cot^{-1}3)+\cot^{-1}3-\left(\frac{1}{3}\left(\cot^{-1}\frac{1}{2}+\cot^{-1}\frac{1}{3}\right)+\frac{1}{6}\cot^{-1}\frac{1}{2}\right)$$

$$=\frac{\pi}{2}+\cot^{-1}3-\left[\frac{\pi}{4}+\frac{1}{6}\tan^{-1}2\right]$$

$$=\frac{\pi}{4}+\cot^{-1}3-\frac{1}{6}\left[\frac{3\pi}{4}-\tan^{-1}3\right]$$

$$=\frac{\pi}{8}+\cot^{-1}3+\frac{1}{6}\tan^{-1}3$$

$$=\frac{\pi}{8}+\cot^{-1}3-\frac{1}{6}\left(\frac{\pi}{2}-\cot^{-1}3\right)$$

$$=\frac{\pi}{8}+\frac{\pi}{12}+\cot^{-1}3-\frac{1}{6}\cot^{-1}3$$

$$=\frac{5\pi}{24}+\frac{5}{6}\cot^{-1}3$$

Hence, $a=5;b=24;c=5;d=6$

$b-a-c-d=8$

JEE Type Solved Examples :
Statement I and II Type Questions

■ This section contains 2 questions. Each question contains **Statement I** (Assertion) and **Statement II** (Reason). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choice are

(a) Both Statement I and Statement II are correct and Statement II is the correct explanation of Statement I

(b) Both Statement I and Statement II are correct but Statement II is not the correct explanation of Statement I.

(c) Statement I is correct but Statement II is incorrect.

(d) Statement II is correct but Statement I is incorrect.

● ***Ex. 34.*** **Statement I** *If* α, β *are roots of* $6x^2+11x+3=0$, *then* $\cos^{-1}\alpha$ *exists but not* $\cos^{-1}\beta\,(\alpha>\beta)$.

Statement II *Domain of* $\cos^{-1}x$ *is* $[-1,1]$.

Sol. (*a*) Given, $6x^2+11x+3=0 \Rightarrow 6x^2+9x+2x+3=0$

$\Rightarrow \quad 3x(2x+3)+1(2x+3)=0$

$\Rightarrow \quad (2x+3)(3x+1)=0 \Rightarrow x=\frac{-3}{2},\frac{-1}{3}$

$\therefore \quad \beta=\frac{-3}{2}, \alpha=\frac{-1}{3} \qquad \left(\because \frac{-1}{3}>\frac{-3}{2}\right)$

$\therefore \quad \cos^{-1}\left(\frac{-1}{3}\right)$ exists $\quad \{\because$ Domain of $\cos^{-1}x$ is $[-1,1]\}$

● ***Ex. 35.*** **Statement I** *If* $\tan^{-1}x+\tan^{-1}y=\frac{\pi}{4}-\tan^{-1}z$ *and* $x+y+z=1$, *then arithmetic mean of odd powers of* x, y, z *is equal to* 1/3.

Statement II *For any* x, y, z *we have* $xyz-xy-yz-zx+x+y+z=1+(x-1)(y-1)(z-1)$

Sol. (*b*) We have, $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\frac{\pi}{4}$

Let $x=\tan A, y=\tan B$ and $z=\tan C$.

Then $\quad A+B+C=\frac{\pi}{4}$

Now, $\tan(A+B+C)=\frac{x+y+z-xyz}{1-(xy+yz+zx)}$

$\Rightarrow \quad 1=\frac{x+y+z-xyz}{1-(xy+yz+zx)}$

$\Rightarrow \quad 1-(xy+yz+zx)=x+y+z-xyz$

$\Rightarrow \quad (x-1)(y-1)(z-1)=0$

$\Rightarrow$ One of x,y,z is equal to 1

If $\quad z=1, x+y=0$

$\therefore \quad (x)^{odd}+(-x)^{odd}+1^{odd}=1$

Thus, AM of odd powers of x,y,z is equal to $\left(\frac{1}{3}\right)$.

JEE Type Solved Examples :
Matching Type Questions

● ***Ex. 36.*** *Match the principal values of* $\cos^{-1}(8x^4-8x^2+1)$ *given in column I with the corresponding intervals of x given in column II, for which it holds.*

	Column I		Column II
A	$4\cos^{-1}x$	p.	$0\le x\le\frac{1}{\sqrt{2}}$
B	$4\cos^{-1}x-2\pi$	q.	$\frac{1}{\sqrt{2}}\le x\le 1$
C	$2\pi-4\cos^{-1}x$	r.	$-1\le x\le-\frac{1}{\sqrt{2}}$
D	$4\pi-4\cos^{-1}x$	s.	$-\frac{1}{\sqrt{2}}\le x\le 0$

Sol. A $\to$ q, B $\to$ s, C $\to$ p, D $\to$ r

(A) $0\le\cos^{-1}(8x^4-8x^2+1)\le\pi$

$\Rightarrow \quad 0\le 4\cos^{-1}x\le\pi$

$\Rightarrow \quad 0\le\cos^{-1}x\le\frac{\pi}{4} \Rightarrow \frac{1}{\sqrt{2}}\le x\le 1$

(B) $0\le\cos^{-1}(8x^4-8x^2+1)\le\pi$

$\Rightarrow \quad 0\le 4\cos^{-1}x-2\pi\le\pi$

$\Rightarrow \quad 2\pi\le 4\cos^{-1}x\le 3\pi$

$\Rightarrow \quad \frac{\pi}{2}\le\cos^{-1}x\le\frac{3\pi}{4} \Rightarrow -\frac{1}{\sqrt{2}}\le x\le 0$

(C) $0\le 2\pi-4\cos^{-1}x\le\pi$

$\Rightarrow \quad -2\pi\le -4\cos^{-1}x\le-\pi$

$\Rightarrow \quad 2\pi\ge 4\cos^{-1}x\ge\pi$

$\Rightarrow \quad \frac{\pi}{2}\ge\cos^{-1}x\ge\frac{\pi}{4} \Rightarrow 0\le x\le\frac{1}{\sqrt{2}}$

(D) $0\le 4\pi-4\cos^{-1}x\le\pi$

$\Rightarrow \quad -4\pi\le -4\cos^{-1}x\le-3\pi$

$\Rightarrow \quad 4\pi\ge 4\cos^{-1}x\ge 3\pi$

$\Rightarrow \quad \pi\ge\cos^{-1}x\ge\frac{3\pi}{4}$

$\Rightarrow \quad -1\le x\le-\frac{1}{\sqrt{2}}$

Subjective Type Examples

● **Ex. 37.** *If* $A = 2\tan^{-1}(2\sqrt{2}-1)$ *and* $B = 3\sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{3}{5}\right)$, *then show* $A > B$.

Sol. We have,

$$A = 2\tan^{-1}(2\sqrt{2}-1) = 2\tan^{-1}(1.828)$$

$$\Rightarrow \quad A > 2\tan^{-1}(\sqrt{3})$$

$$\Rightarrow \quad A > \frac{2\pi}{3} \qquad \ldots(i)$$

Also we have,

$$\sin^{-1}\left(\frac{1}{3}\right) < \sin^{-1}\left(\frac{1}{2}\right) \Rightarrow \sin^{-1}\left(\frac{1}{3}\right) < \frac{\pi}{6}$$

$$\Rightarrow \quad 3\sin^{-1}\left(\frac{1}{3}\right) < \frac{\pi}{2}$$

Also, $\quad 3\sin^{-1}\left(\frac{1}{3}\right) = \sin^{-1}\left(3\cdot\frac{1}{3} - 4\left(\frac{1}{3}\right)^3\right)$

$$= \sin^{-1}\left(\frac{23}{27}\right) = \sin^{-1}(0.852)$$

$$\Rightarrow \quad 3\sin^{-1}\left(\frac{1}{3}\right) < \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow \quad 3\sin^{-1}\left(\frac{1}{3}\right) < \frac{\pi}{3}$$

Also, $\quad \sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}(0.6) < \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$$\Rightarrow \quad \sin^{-1}\left(\frac{3}{5}\right) < \frac{\pi}{3}$$

Hence, $\quad B = 3\sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{3}{5}\right) < \frac{2\pi}{3} \qquad \ldots(ii)$

From (i) and (ii), we have

$$A > B.$$

● **Ex. 38.** *Solve for* x *:* $(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = \frac{5\pi^2}{8}$.

Sol. We have $(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = \frac{5\pi^2}{8}$

$$\Rightarrow \quad (\tan^{-1}x + \cot^{-1}x)^2 - 2\tan^{-1}x.\cot^{-1}x = \frac{5\pi^2}{8}$$

$$\Rightarrow \quad \left(\frac{\pi}{2}\right)^2 - 2\tan^{-1}x.\left(\frac{\pi}{2} - \tan^{-1}x\right) = \frac{5\pi^2}{8}$$

$$\left\{\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2} \Rightarrow \cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x\right\}$$

$$\Rightarrow \quad \frac{\pi^2}{4} - 2\cdot\frac{\pi}{2}\cdot\tan^{-1}x + 2(\tan^{-1}x)^2 = \frac{5\pi^2}{8}$$

$$\Rightarrow \quad 2(\tan^{-1}x)^2 - \pi\tan^{-1}x - \frac{3\pi^2}{8} = 0$$

$$\Rightarrow \quad \tan^{-1}x = -\frac{\pi}{4}, \frac{3\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}x = -\frac{\pi}{4}$$

$$\Rightarrow \quad x = -1$$

$$\left\{\text{neglecting } \tan^{-1}x = \frac{3\pi}{4} \text{ as } \tan^{-1}x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\right\}$$

● **Ex. 39.** *Solve for* x *: If* $[\sin^{-1}\cos^{-1}\sin^{-1}\tan^{-1}x] = 1$, *where* $[.]$ *denotes the greatest integer function.*

Sol. We have, $[\sin^{-1}\cos^{-1}\sin^{-1}\tan^{-1}x] = 1$

$$\Rightarrow \quad 1 \le \sin^{-1}.\cos^{-1}.\sin^{-1}.\tan^{-1}x \le \frac{\pi}{2}$$

$$\Rightarrow \quad \sin 1 \le \cos^{-1}.\sin^{-1}.\tan^{-1}x \le 1$$

$$\Rightarrow \quad \cos\sin 1 \ge \sin^{-1}.\tan^{-1}x \ge \cos 1$$

$$\Rightarrow \quad \sin\cos\sin 1 \ge \tan^{-1}x \ge \sin\cos 1$$

$$\Rightarrow \quad \tan\sin\cos\sin 1 \ge x \ge \tan\sin\cos 1$$

Hence, $x \in [\tan\sin\cos 1, \tan\sin\cos\sin 1]$

● **Ex. 40.** *If* $\tan^{-1}y = 4\tan^{-1}x\left(|x| < \tan\frac{\pi}{8}\right)$, *find* y *as an algebraic function of* x *and hence prove that* $\tan\frac{\pi}{8}$ *is a root of the equation* $x^4 - 6x^2 + 1 = 0$.

Sol. We have,

$$\tan^{-1}y = 4\tan^{-1}x$$

$$\Rightarrow \quad \tan^{-1}y = 2\tan^{-1}\frac{2x}{1-x^2} \qquad (\text{as } |x| < 1)$$

$$= \tan^{-1}\frac{\frac{4x}{1-x^2}}{1-\frac{4x^2}{(1-x^2)^2}}$$

$$= \tan^{-1}\frac{4x(1-x^2)}{x^4-6x^2+1} \qquad \left(\text{as } \left|\frac{2x}{1-x^2}\right| < 1\right)$$

$$\Rightarrow \quad y = \frac{4x(1-x^2)}{x^4-6x^2+1}$$

If $\quad x = \tan\frac{\pi}{8}$

$$\Rightarrow \quad \tan^{-1}y = 4\tan^{-1}x = \frac{\pi}{2}$$

$$\Rightarrow \quad y = \infty \Rightarrow x^4 - 6x^2 + 1 = 0$$

● ***Ex. 41.*** *If x_1, x_2, x_3, x_4 are the roots of the equation*

$$x^4 - x^3\sin 2\beta + x^2\cos 2\beta - x\cos\beta - \sin\beta = 0$$

Then show :

$$\tan^{-1}x_1 + \tan^{-1}x_2 + \tan^{-1}x_3 + \tan^{-1}x_4 = \frac{\pi}{2} - \beta$$

Sol. We have, $\Sigma x_1 = \sin 2\beta$,

$$\Sigma x_1x_2 = \cos 2\beta,\ \Sigma x_1x_2x_3 = \cos\beta$$

and $x_1x_2x_3x_4 = -\sin\beta.$

$$\left\{\begin{array}{l}\text{using; } ax^4 + bx^3 + cx^2 + dx + e = 0 \text{ has four root } x_1, x_2, x_3, x_4\\ \Rightarrow \Sigma x_1 = x_1 + x_2 + x_3 + x_4 = -\dfrac{b}{a},\\ \qquad x_1x_2 + x_2x_3 + x_1x_3 + x_1x_4 + x_2x_4 + x_1x_4 = \Sigma x_1x_2 = \dfrac{c}{a}\\ \Sigma x_1x_2x_3 = x_1x_2x_3 + x_1x_2x_4 + x_2x_3x_4 + x_1x_3x_4 = -\dfrac{d}{a}\\ \qquad \text{and} \qquad x_1x_2x_3x_4 = \dfrac{e}{a}\end{array}\right\}$$

Let, $\alpha_1 = \tan^{-1}x_1$, $\alpha_2 = \tan^{-1}x_2$, $\alpha_3 = \tan^{-1}x_3$ and $\alpha_4 = \tan^{-1}x_4$

$\Rightarrow \quad \tan\alpha_1 = x_1$, $\tan\alpha_2 = x_2$, $\tan\alpha_3 = x_3$ and $\tan\alpha_4 = x_4$

$$\therefore \quad \tan(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) = \frac{(s_1 - s_3)}{1 - s_2 + s_4} = \frac{\Sigma x_1 - \Sigma x_1x_2x_3}{1 - \Sigma x_1x_2 + x_1x_2x_3x_4}$$

$$= \frac{\sin 2\beta - \cos\beta}{1 - \cos 2\beta - \sin\beta} = \frac{\cos\beta(2\sin\beta - 1)}{\sin\beta(2\sin\beta - 1)}$$

$$= \cot\beta = \tan\left(\frac{\pi}{2} - \beta\right)$$

$$\Rightarrow \qquad \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \frac{\pi}{2} - \beta$$

or $\tan^{-1}x_1 + \tan^{-1}x_2 + \tan^{-1}x_3 + \tan^{-1}x^4 = \dfrac{\pi}{2} - \beta$

● ***Ex. 42.*** *Find the number of positive integral solutions of the equation :*

$$\tan^{-1}x + \cos^{-1}\frac{y}{\sqrt{1-y^2}} = \sin^{-1}\frac{3}{\sqrt{10}}$$

Sol. Here,

$$\tan^{-1}x + \cos^{-1}\frac{y}{\sqrt{1-y^2}} = \sin^{-1}\frac{3}{\sqrt{10}}$$

$$\Rightarrow \qquad \tan^{-1}x + \tan^{-1}\left(\frac{1}{y}\right) = \tan^{-1}(3)$$

or $$\tan^{-1}\left(\frac{1}{y}\right) = \tan^{-1}3 - \tan^{-1}(x)$$

or $$\tan^{-1}\left(\frac{1}{y}\right) = \tan^{-1}\left(\frac{3-x}{1+3x}\right) \Rightarrow y = \frac{1+3x}{3-x}$$

As x, y are positive integers, $x = 1, 2$ and correspondingly $y = 2, 7$.

$\therefore$ Solutions are $(x, y) = (1, 2), (2, 7)$ i.e. two solution.

● ***Ex. 43.*** *If $\cot^{-1}\dfrac{n}{\pi} > \dfrac{\pi}{6}$, $n \in N$, then find the maximum value of n.*

Sol. Here, $\cot^{-1}\dfrac{n}{\pi} > \dfrac{\pi}{6}$

$$\Rightarrow \quad \cot\left(\cot^{-1}\frac{n}{\pi}\right) < \cot\frac{\pi}{6} \qquad \{\text{as } \cot x \text{ is decreasing in } [0, \pi/2]\}$$

$$\Rightarrow \qquad \frac{n}{\pi} < \sqrt{3}$$

$$\Rightarrow \qquad n < \sqrt{3}\pi = 5.5 \qquad \text{(approximately)}$$

So, the maximum value of n is 5, as $n \in N$.

● ***Ex. 44.*** *If $\cot^{-1}\left(\dfrac{n^2 - 10n + 21\cdot 6}{\pi}\right) > \dfrac{\pi}{6}$, $n \in N$, then find the minimum value of n.*

Sol. We have, $\cot^{-1}\left(\dfrac{n^2 - 10n + 21\cdot 6}{\pi}\right) > \dfrac{\pi}{6}$

$$\Rightarrow \qquad \frac{n^2 - 10n + 21\cdot 6}{\pi} < \cot\frac{\pi}{6}$$

$$\{\text{as } \cot x \text{ is decreasing for } 0 < x < \pi\}$$

$$\Rightarrow \qquad n^2 - 10n + 21.6 < \pi\sqrt{3}$$

$$\Rightarrow \quad n^2 - 10n + 25 + 21\cdot 6 - 25 < \pi\sqrt{3}$$

$$\Rightarrow \qquad (n-5)^2 < \pi\sqrt{3} + 3\cdot 4$$

$$\Rightarrow \qquad -\sqrt{\sqrt{3}\pi + 3\cdot 4} < n - 5 < \sqrt{\sqrt{3}\pi + 3.4} \qquad \ldots(i)$$

Since, $\sqrt{3}\pi = 5.5$ nearly

$$\therefore \qquad \sqrt{\sqrt{3}\pi + 3\cdot 4} \sim \sqrt{8\cdot 9} \sim 2\cdot 9$$

$$\Rightarrow \qquad 2\cdot 1 < n < 7\cdot 9$$

$$\therefore \qquad n = 3, 4, 5, 6, 7 \qquad \{\text{as } n \in N\}$$

or minimum value of $n = 3$.

● ***Ex. 45.*** *Find the set of values of k for which $x^2 - kx + \sin^{-1}(\sin 4) > 0$ for all real x.*

Sol. We know,

$$\sin^{-1}(\sin 4) = \sin^{-1}(\sin(\pi - 4)) = \pi - 4$$

$$\left\{\because -\frac{\pi}{2} < \pi - 4 < \frac{\pi}{2}\right\}$$

$\therefore$ We have $x^2 - kx + \pi - 4 > 0$ for all $x \in R$

$$\therefore \qquad D < 0, \text{ i.e. } k^2 - 4(\pi - 4) < 0$$

or $$k^2 + 4(4 - \pi) < 0$$

which is not true for any real k. $\{\text{as } k^2 + 4(4-\pi) > 0\}$

● **Ex. 46.** *Find the greatest and least value of;* $(\sin^{-1}x)^3+(\cos^{-1}x)^3$.

Sol. We have, $(\sin^{-1}x)^3+(\cos^{-1}x)^3=(\sin^{-1}x+\cos^{-1}x)$
$[(\sin^{-1}x+\cos^{-1}x)^2-3\sin^{-1}x.\cos^{-1}x]$

$$=\frac{\pi}{2}\left[\frac{\pi^2}{4}-3\sin^{-1}x\left(\frac{\pi}{2}-\sin^{-1}x\right)\right]$$

$$=\frac{3\pi}{2}\left[(\sin^{-1}x)^2-\frac{\pi}{2}\sin^{-1}x+\frac{\pi^2}{12}\right]$$

$$=\frac{3\pi}{2}\left[\left(\sin^{-1}x-\frac{\pi}{4}\right)^2+\frac{\pi^2}{12}-\frac{\pi^2}{16}\right]$$

$$=\frac{3\pi}{2}\left[\left(\sin^{-1}x-\frac{\pi}{4}\right)^2+\frac{\pi^2}{48}\right]$$

So, the least value is; $\frac{3\pi}{2}\cdot\frac{\pi^2}{48}=\frac{\pi^3}{32}\left\{\text{when}\left(\sin^{-1}x-\frac{\pi}{4}\right)=0\right\}$

Also, $\left(\sin^{-1}x-\frac{\pi}{4}\right)^2\le\left(\frac{3\pi}{4}\right)^2$

$\therefore$ The greatest value $=\frac{3\pi}{2}\left[\frac{9\pi^2}{16}+\frac{\pi^2}{48}\right]=\frac{7\pi^3}{8}$.

● **Ex. 47.** *If* x_r *is given by,* $x_{r+1}=\sqrt{\frac{1}{2}(1+x_r)}$. *Then, show :* $\cos^{-1}x_0=\frac{\sqrt{1-x_0^2}}{x_1x_2x_3}$ *... up to infinity.*

Sol. Given : $x_{r+1}=\sqrt{\frac{1}{2}(1+x_r)}$

Let $x_0=\cos\theta$

$$x_1=\sqrt{\frac{1}{2}(1+x_0)}=\sqrt{\frac{1}{2}(1+\cos\theta)}=\cos\left(\frac{\theta}{2}\right)$$

$$x_2=\sqrt{\frac{1}{2}(1+x_1)}=\sqrt{\frac{1}{2}\left(1+\cos\frac{\theta}{2}\right)}=\cos\left(\frac{\theta}{2^2}\right)$$

Similarly, $x_3=\cos\left(\frac{\theta}{2^3}\right)$

.....

.....

$$x_n=\cos\left(\frac{\theta}{2^n}\right)$$

$$\Rightarrow\quad x_1x_2x_3\ldots x_n=\cos\frac{\theta}{2}.\cos\frac{\theta}{2^2}.\cos\frac{\theta}{2^3}\ldots\cos\frac{\theta}{2^n}$$

$$=\frac{\sin\theta}{2^n.\sin\frac{\pi}{2^n}} \qquad \ldots(i)$$

$$\therefore\quad x_1x_2\ldots x_n\ldots\infty=\lim_{n\to\infty}(x_1x_2\ldots x_n)$$

$$=\lim_{n\to\infty}\left(\frac{\sin\theta}{2^n.\sin\frac{\theta}{2^n}}\right) \qquad \{\text{using Eq. (i)}\}$$

$$=\lim_{n\to\infty}\frac{\sin\theta}{\theta\cdot\frac{\sin\frac{\theta}{2^n}}{\frac{\theta}{2^n}}}=\frac{\sin\theta}{\theta} \qquad \ldots(ii)$$

$$\therefore\quad \frac{\sqrt{1-x_0^2}}{x_1x_2x_3\ldots x_\infty}=\frac{\sqrt{1-\cos^2\theta}}{\frac{\sin\theta}{\theta}} \qquad \{\text{using Eq. (ii)}\}$$

$$=\theta=\cos^{-1}(x_0) \qquad \{\because\ \cos\theta=x_0\}$$

$$\Rightarrow\quad \frac{\sqrt{1-x_0^2}}{x_1x_2x_3\ldots\infty}=\cos^{-1}x_0.$$

● **Ex. 48.** *Express the equation;*

$$\cot^{-1}\frac{y}{\sqrt{1-x^2-y^2}}=2\tan^{-1}\sqrt{\frac{3-4x^2}{4x^2}}-\tan^{-1}\sqrt{\frac{3-4x^2}{x^2}}$$

as a rational integral equation in x *and* y.

Sol. $\cot^{-1}\left(\frac{y}{\sqrt{1-x^2-y^2}}\right)=\tan^{-1}\left(\frac{\sqrt{1-x^2-y^2}}{y}\right)$

Also, $2\tan^{-1}\sqrt{\frac{3-4x^2}{4x^2}}=\tan^{-1}\frac{2\cdot\sqrt{\frac{3-4x^2}{4x^2}}}{1-\left(\frac{3-4x^2}{4x^2}\right)}$

$$=\tan^{-1}\left(\frac{4x\sqrt{3-4x^2}}{8x^2-3}\right)$$

Hence, the given equation is

$$\tan^{-1}\frac{\sqrt{1-x^2-y^2}}{y}=\tan^{-1}\left(\frac{4x\sqrt{3-4x^2}}{8x^2-3}\right)-\tan^{-1}\sqrt{\frac{3-4x^2}{x^2}}$$

$$=\tan^{-1}\left[\frac{\frac{4x\sqrt{3-4x^2}}{8x^2-3}-\sqrt{\frac{3-4x^2}{x^2}}}{1+\frac{4x\sqrt{3-4x^2}}{8x^2-3}\cdot\sqrt{\frac{3-4x^2}{x^2}}}\right]$$

$$=\tan^{-1}\left[\frac{(3-4x^2)^{3/2}}{9x-8x^3}\right]$$

$$\therefore\quad \frac{\sqrt{1-x^2-y^2}}{y}=\frac{(3-4x^2)^{3/2}}{9x-8x^3}$$

Squaring and simplifying

$$\frac{1-x^2}{y^2}=1+\frac{(3-4x^2)^3}{(9x-8x^3)^2}$$

$$=\frac{(9x-8x^3)^2+(3-4x^2)^3}{(9x-8x^3)^2}=\frac{27-27x^2}{(9x-8x^3)^2}$$

$$\therefore\quad y^2=\frac{x^2(9-8x^2)^2}{27}$$

● ***Ex. 49.*** *Prove that*
$\cos^{-1}\left(\dfrac{\cos x+\cos y}{1+\cos x\cdot\cos y}\right)=2\tan^{-1}\left(\tan\dfrac{x}{2}\tan\dfrac{y}{2}\right)$.

Sol. Let, $\tan\dfrac{x}{2}\cdot\tan\dfrac{y}{2}=\tan\theta$ ...(i)

Consider $\dfrac{\cos x+\cos y}{1+\cos x.\cos y}=\dfrac{\dfrac{1-\tan^2 x/2}{1+\tan^2 x/2}+\dfrac{1-\tan^2 y/2}{1+\tan^2 y/2}}{1+\dfrac{(1-\tan^2 x/2)}{(1+\tan^2 x/2)}\cdot\dfrac{(1-\tan^2 y/2)}{(1+\tan^2 y/2)}}$

$=\dfrac{(1-\tan^2 x/2)(1+\tan^2 y/2)+(1+\tan^2 x/2)(1-\tan^2 y/2)}{(1+\tan^2 x/2)(1+\tan^2 y/2)+(1-\tan^2 x/2).(1-\tan^2 y/2)}$

$=\dfrac{\begin{array}{l}1-\tan^2 x/2+\tan^2 y/2-\tan^2 x/2\tan^2 y/2+1-\tan^2 y/2\\+\tan^2 x/2-\tan^2 x/2\tan^2 y/2\end{array}}{\begin{array}{l}1+\tan^2 x/2+\tan^2 y/2+\tan^2 x/2\tan^2 y/2+1-\tan^2 x/2\\-\tan^2 y/2+\tan^2 x/2.\tan^2 y/2\end{array}}$

$=\dfrac{2-2\tan^2 x/2\tan^2 y/2}{2+2\tan^2 x/2\tan^2 y/2}$

$=\dfrac{1-\tan^2\theta}{1+\tan^2\theta}$ {from Eq. (i)}

$=\cos 2\theta$

$\therefore$ LHS $=\cos^{-1}(\cos 2\theta)=2\theta$

$=2\left\{\tan^{-1}\left(\tan\dfrac{x}{2}\cdot\tan\dfrac{y}{2}\right)\right\}$

● ***Ex. 50.*** *If* $x=\text{cosec}\,(\tan^{-1}(\cos(\cot^{-1}(\sec(\sin^{-1}a)))))$ *and* $y=\sec(\cot^{-1}\sin(\tan^{-1}(\text{cosec}(\cos^{-1}a))))$, *where* $a\in[0,1]$. *Find the relationship between* x *and* y *in terms of 'a'.*

Sol. Here,

$x=\text{cosec}\,(\tan^{-1}(\cos(\cot^{-1}(\sec(\sin^{-1}a)))))$

$\Rightarrow\quad x=\text{cosec}\left(\tan^{-1}\left(\cos\left(\cot^{-1}\left(\dfrac{1}{\sqrt{1-a^2}}\right)\right)\right)\right)$

$\Rightarrow\quad x=\text{cosec}\left(\tan^{-1}\left(\dfrac{1}{\sqrt{2-a^2}}\right)\right)$

$\Rightarrow\quad x=\sqrt{3-a^2}$

and $y=\sec(\cot^{-1}\sin(\tan^{-1}(\text{cosec}(\cos^{-1}a))))$

$\Rightarrow\quad y=\sec\left(\cot^{-1}\left(\sin\left(\tan^{-1}\left(\dfrac{1}{\sqrt{1-a^2}}\right)\right)\right)\right)$

$\Rightarrow\quad y=\sec\left(\cot^{-1}\left(\dfrac{1}{\sqrt{2-a^2}}\right)\right)$

$\Rightarrow\quad y=\sqrt{3-a^2}$ ...(ii)

From Eqs. (i) and (ii), $x=y=\sqrt{3-a^2}$.

● ***Ex. 51.*** *Show that* $\tan^{-1}\left\{\tan\dfrac{\alpha}{2}\tan\left(\dfrac{\pi}{4}-\dfrac{\beta}{2}\right)\right\}$
$=\tan^{-1}\left(\dfrac{\sin\alpha\cos\beta}{\cos\alpha+\sin\beta}\right)$ *where* $\alpha,\beta\in\left(0,\dfrac{\pi}{2}\right)$.

Sol. Since, $0<\alpha<\dfrac{\pi}{2}$

$\therefore\quad 0<\dfrac{\alpha}{2}<\dfrac{\pi}{4}\Rightarrow 0<\tan\dfrac{\alpha}{2}<1$ ...(i)

Similarly, $0<\beta<\dfrac{\pi}{2}$

$\Rightarrow\quad 0<\dfrac{\beta}{2}<\dfrac{\pi}{4}\Rightarrow -\dfrac{\pi}{4}<-\dfrac{\beta}{2}<0$

$\Rightarrow\quad 0<\dfrac{\pi}{4}-\dfrac{\beta}{2}<\dfrac{\pi}{4}\Rightarrow 0<\tan\left(\dfrac{\pi}{4}-\dfrac{\beta}{2}\right)<1$ ...(ii)

From Eqs. (i) and (ii), we get

$0<\tan\dfrac{\alpha}{2}\tan\left(\dfrac{\pi}{4}-\dfrac{\beta}{2}\right)<1.$

Since, $2\tan^{-1}x=\tan^{-1}\dfrac{2x}{1-x^2}$, when $x\in(0,1)$

$\therefore\quad 2\tan^{-1}\left\{\tan\dfrac{\alpha}{2}\tan\left(\dfrac{\pi}{4}-\dfrac{\beta}{2}\right)\right\}$

$=\tan^{-1}\dfrac{2\tan\dfrac{\alpha}{2}\tan\left(\dfrac{\pi}{4}-\dfrac{\beta}{2}\right)}{1-\tan^2\dfrac{\alpha}{2}.\tan^2\left(\dfrac{\pi}{4}-\dfrac{\beta}{2}\right)}$

$=\tan^{-1}\dfrac{2\tan\dfrac{\alpha}{2}\cdot\dfrac{1-\tan\left(\dfrac{\beta}{2}\right)}{1+\tan\left(\dfrac{\beta}{2}\right)}}{1-\tan^2\dfrac{\alpha}{2}\left[\dfrac{1-\tan\dfrac{\beta}{2}}{1+\tan\dfrac{\beta}{2}}\right]^2}$

$=\tan^{-1}\dfrac{2\tan\dfrac{\alpha}{2}\left(1-\tan^2\dfrac{\beta}{2}\right)}{\left(1+\tan\dfrac{\beta}{2}\right)^2-\tan^2\dfrac{\alpha}{2}\left(1-\tan\dfrac{\beta}{2}\right)^2}$

$=\tan^{-1}\dfrac{2\tan\dfrac{\alpha}{2}\left(1-\tan^2\dfrac{\beta}{2}\right)}{\left(1+\tan^2\dfrac{\beta}{2}\right)\left(1-\tan^2\dfrac{\alpha}{2}\right)+2\tan\dfrac{\beta}{2}\left(1+\tan^2\dfrac{\alpha}{2}\right)}$

$=\tan^{-1}\dfrac{\dfrac{2\tan\dfrac{\alpha}{2}}{1+\tan^2\dfrac{\alpha}{2}}\cdot\dfrac{1-\tan^2\dfrac{\beta}{2}}{1+\tan^2\dfrac{\beta}{2}}}{\dfrac{1-\tan^2\dfrac{\alpha}{2}}{1+\tan^2\dfrac{\alpha}{2}}+\dfrac{2\tan\dfrac{\beta}{2}}{1+\tan^2\dfrac{\beta}{2}}}$

$=\tan^{-1}\left(\dfrac{\sin\alpha.\cos\beta}{\cos\alpha+\sin\beta}\right)$

● ***Ex. 52.*** *Solve :*

$$3\sin^{-1}\frac{2x}{1+x^2}-4\cos^{-1}\frac{1-x^2}{1+x^2}+2\tan^{-1}\frac{2x}{1-x^2}=\frac{\pi}{3}$$

Sol. Let $\tan^{-1}x=\theta$ for $x\geq 0$

Case I When $0\leq x<1$, then $0\leq\theta<\frac{\pi}{4}$ and so $0\leq 2\theta<\frac{\pi}{2}$

$$\therefore\quad \sin^{-1}\frac{2x}{1+x^2}=\sin^{-1}(\sin 2\theta)=2\theta=2\tan^{-1}x$$

$$\cos^{-1}\frac{1-x^2}{1+x^2}=\cos^{-1}(\cos 2\theta)=2\theta=2\tan^{-1}x$$

$$\tan^{-1}\frac{2x}{1-x^2}=\tan^{-1}(\tan 2\theta)=2\theta=2\tan^{-1}x$$

Thus, $3\left(\sin^{-1}\frac{2x}{1+x^2}\right)-4\left(\cos^{-1}\frac{1-x^2}{1+x^2}\right)+2\left(\tan^{-1}\frac{2x}{1-x^2}\right)=\frac{\pi}{3}$

$$\Rightarrow\ 3(2\tan^{-1}x)-4(2\tan^{-1}x)+2(2\tan^{-1}x)=\frac{\pi}{3}.$$

$$\Rightarrow\quad 2\tan^{-1}x=\frac{\pi}{3}$$

$$\Rightarrow\quad \tan^{-1}x=\frac{\pi}{6}\Rightarrow\ x=\frac{1}{\sqrt{3}}$$

as $\quad 0\leq\frac{1}{\sqrt{3}}<1,\ x=\frac{1}{\sqrt{3}}$ is a solution

Case II When $x=1$, $\tan^{-1}\frac{2x}{1-x^2}$ is not defined.

$\therefore\quad x=1$, cannot be a solution.

Case III If $x>1$, then $\frac{\pi}{4}<\theta<\frac{\pi}{2}$ and so $\frac{\pi}{2}<2\theta<\pi$

$$\therefore\quad \sin^{-1}\left(\frac{2x}{1+x^2}\right)=\sin^{-1}(\sin 2\theta)=\pi-2\theta=\pi-2\tan^{-1}x.$$

$$\cos^{-1}\left(\frac{2x}{1+x^2}\right)=\cos^{-1}(\cos 2\theta)=2\theta=2\tan^{-1}x$$

$$\text{and}\ \tan^{-1}\left(\frac{2x}{1-x^2}\right)=\tan^{-1}(\tan 2\theta)=2\theta-\pi=2\tan^{-1}x-\pi$$

Thus, the given equation becomes

$$3(\pi-2\tan^{-1}x)-4(2\tan^{-1}x)+2(2\tan^{-1}x-\pi)=\frac{\pi}{3}$$

$$\Rightarrow\quad \pi-10\tan^{-1}x=\frac{\pi}{3}$$

or $\quad \tan^{-1}x=\frac{\pi}{15}$

i.e. $\quad x=\tan\frac{\pi}{15}<\tan\frac{\pi}{4}<1$

$$\therefore\quad x=\tan\frac{\pi}{15}\ \text{is not a solution.}$$

Thus, $x=\frac{1}{\sqrt{3}}$ is the only solution for the given equation for $x\geq 0$.

● ***Ex. 53.*** *Obtain the integral values of p for which the following system of equations possesses real solutions :*

$$\cos^{-1}x+(\sin^{-1}y)^2=\frac{p\pi^2}{4}\ \textit{and}\ (\cos^{-1}x)(\sin^{-1}y)^2=\frac{\pi^4}{16}$$

Also, find these solution.

Sol. Let $\cos^{-1}x=a\Rightarrow a\in[0,\pi]$

and $\quad \sin^{-1}y=b\Rightarrow b\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

We have $\quad a+b^2=\frac{p\pi^2}{4}$ ...(i)

and $\quad ab^2=\frac{\pi^4}{16}$ ...(ii)

Since $\quad b^2\in\left[0,\frac{\pi^2}{4}\right]\Rightarrow a+b^2\in\left[0,\pi+\frac{\pi^2}{4}\right]$

So, Eq. (i) $\Rightarrow 0\leq\frac{p\pi^2}{4}\leq\pi+\frac{\pi^2}{4}$

i.e. $\quad 0\leq p\leq\frac{4}{\pi}+1$

Since $\quad p\in Z$, so $p=0,1$ or 2

But, if $p=0$, then $a=b=0$.

$\Rightarrow$ Equation (ii) will not be satisfied.

Now, substituting the value of b^2 from Eq. (i) in the Eq. (ii), we get

$$a\left(\frac{p\pi^2}{4}-a\right)=\frac{\pi^4}{16}\Rightarrow 16a^2-4p\pi^2a+\pi^4=0 \qquad \text{...(iii)}$$

Since, $\quad a\in R\ \Rightarrow\ D\geq 0$

i.e. $\quad 16p^2\pi^4-64\pi^4\geq 0\ \Rightarrow\ p^2\geq 4\ \Rightarrow\ p\geq 2$

Thus, we conclude that the only value of p that satisfies all conditions is $p=2$. Substituting $p=2$ in Eq. (iii), we get

$$16a^2-8\pi^2a+\pi^4=0$$

$$\Rightarrow\quad (4a-\pi^2)^2=0$$

$$\Rightarrow\quad a=\frac{\pi^2}{4}=\cos^{-1}x$$

$$\Rightarrow\quad x=\cos\frac{\pi^2}{4}$$

From Eq. (ii), we get $\frac{\pi^2}{4}\cdot b^2=\frac{\pi^4}{16}\Rightarrow b=\pm\frac{\pi}{2}=\sin^{-1}y$

$$\Rightarrow\quad y=\pm 1.$$

● ***Ex. 54.*** *Solve the equation* $2(\sin^{-1}x)^2-(\sin^{-1}x)-6=0$

Sol. Let, $\sin^{-1}x=y$, we get

$$2y^2-y-6=0$$

$$2y^2-4y+3y-6=0$$

$$y=2\ \text{and}\ y=-1.5$$

$\therefore\quad \sin^{-1}x=2\ \text{and}\ \sin^{-1}x=-1.5$

Since $2>\frac{\pi}{2}$ and $|-1.5|<\frac{\pi}{2}$, the only solution is $x=\sin(-1.5)$.

● ***Ex. 55.*** *Solve the equation* $\sin^{-1}6x+\sin^{-1}6\sqrt{3}x=-\dfrac{\pi}{2}$

Sol. Let us transfer $\sin^{-1}6\sqrt{3}x$ into the right hand side of the equation and calculate the sine of the both sides of the resulting equation

$$\sin(\sin^{-1}6x)=\sin\left(-\sin^{-1}6\sqrt{3}x-\frac{\pi}{2}\right)$$

$$\Rightarrow \quad 6x=-\sin(\sin^{-1}6\sqrt{3}x+\sin^{-1}1)$$

$$\{\text{using } \sin(-\theta)=-\sin(\theta)\}$$

$$\Rightarrow \quad 6x=-\sin(\sin^{-1}\sqrt{1-108x^2})$$

$$\{\text{using } \sin^{-1}x+\sin^{-1}y=\sin^{-1}\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\}\}$$

$$\Rightarrow \quad 6x=-\sqrt{1-108x^2} \qquad \ldots(i)$$

Squaring both sides, we get

$$36x^2=1-108x^2 \Rightarrow 144x^2=1$$

whose roots are $x=\dfrac{1}{12}$ and $x=-\dfrac{1}{12}$.

Let us verify :

Substituting $x=\dfrac{-1}{12}$ in the given equation, we get

$$\sin^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$

$$=-\frac{\pi}{6}-\frac{\pi}{3}=-\frac{\pi}{2}$$

Thus, $x=\dfrac{-1}{12}$ is the root of given equation. But, when substituting $x=\dfrac{1}{12}$ in Eq. (i), we get

LHS $\qquad 6x=\dfrac{1}{2}$

RHS $\qquad -\sqrt{1-108x^2}=-1/2$

i.e. LHS ≠ RHS of Eq. (i).

Hence, $x=-\dfrac{1}{12}$ is a root of the given equation as it satisfy both given and Eq. (i).

● ***Ex. 56.*** *Solve the equation :* $2\tan^{-1}(2x-1)=\cos^{-1}x$.

Sol. Here, $\qquad 2\tan^{-1}(2x-1)=\cos^{-1}x$

or $\qquad \cos(2\tan^{-1}(2x-1))=x \qquad \left\{\text{We know } \cos2\theta=\dfrac{1-\tan^2\theta}{1+\tan^2\theta}\right\}$

$$\Rightarrow \quad \frac{1-(2x-1)^2}{1+(2x-1)^2}=x$$

$$\Rightarrow \quad \frac{-2x^2+2x}{1-2x+2x^2}=x \Rightarrow 2x^3-x=0$$

$$\Rightarrow \quad x=0,\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}.$$

Now, to verify which of the following satisfy given equation,

Case I For $x=0$;

$$2\tan^{-1}(-1)=\cos^{-1}(0)$$

or $\qquad -\dfrac{\pi}{2}=\dfrac{\pi}{2}$

$\therefore\ x=0$ is not a solution of given equation.

Case II For $x=\dfrac{\sqrt{2}}{2}$

The right hand and the left hand side of the equation are equal to $\dfrac{\pi}{4}$ and $2\tan^{-1}(\sqrt{2}-1)$.

But $\qquad \tan^{-1}(\sqrt{2}-1)=\dfrac{\pi}{8}$.

$\therefore\ x=\dfrac{\sqrt{2}}{2}$ is a root of the given equation.

Case III For $x=-\dfrac{\sqrt{2}}{2}$

The left hand side of the equation is negative and the right hand side is positive.

Consequently, $x=-\dfrac{\sqrt{2}}{2}$ is not a root of the given equation.

Thus from above;

$x=\dfrac{\sqrt{2}}{2}$ is the only solution.

Inverse Trigonometric Functions Exercise 1 :
Single Option Correct Type Questions

1. $\cot^{-1}\left(\sqrt{\frac{1-x^2}{1+x^2}}\right)$ is equal to

(a) $\cos^{-1}(x^2)$ (b) $\frac{\pi}{2}-\frac{1}{2}\cos^{-1}(x^2)$

(c) $\frac{\pi}{3}-\frac{1}{2}\cos^{-1}(x^2)$ (d) None of these

2. The value of $\cos\left(\frac{1}{2}\cos^{-1}\frac{1}{8}\right)$ is equal to

(a) 3/4 (b) $-3/4$

(c) 1/16 (d) 1/4

3. The inequality $\sin^{-1}(\sin 5) > x^2 - 4x$ holds if

(a) $x = 2-\sqrt{9-2\pi}$

(b) $x = 2+\sqrt{9-2\pi}$

(c) $x \in (2-\sqrt{9-2\pi}, 2+\sqrt{9-2\pi})$

(d) $x > 2+\sqrt{9-2\pi}$

4. The value of $\sin^{-1}\left\{\left(\sin\frac{\pi}{3}\right)\frac{x}{\sqrt{(x^2+k^2-kx)}}\right\}$

$-\cos^{-1}\left\{\left(\cos\frac{\pi}{6}\right)\frac{x}{\sqrt{(x^2+k^2-kx)}}\right\}$, where

$\left(\frac{k}{2} < x < 2k, k > 0\right)$ is

(a) $\tan^{-1}\left(\frac{2x^2+xk-k^2}{x^2-2xk+k^2}\right)$

(b) $\tan^{-1}\left(\frac{x^2+2xk-2k^2}{x^2-2xk+k^2}\right)$

(c) $\tan^{-1}\left(\frac{x^2+2xk-2k^2}{2x^2-2xk+2k^2}\right)$

(d) None of the above

5. If $a \le \tan^{-1}\left(\frac{1-x}{1+x}\right) \le b$ where $0 \le x \le 1$, then $(a, b) =$

(a) $\left(0, \frac{\pi}{4}\right)$ (b) $\left(0, \frac{\pi}{2}\right)$

(c) $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ (d) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

6. Sum of infinite terms of the series

$\cot^{-1}\left(1^2+\frac{3}{4}\right)+\cot^{-1}\left(2^2+\frac{3}{4}\right)+\cot^{-1}\left(3^2+\frac{3}{4}\right)+\ldots$ is

(a) $\pi/4$ (b) $\tan^{-1}2$

(c) $\tan^{-1}3$ (d) $\tan^{-1}4$

7. Solution of equation $\cot^{-1}x + \sin^{-1}\frac{1}{\sqrt{5}} = \frac{\pi}{4}$ is

(a) $x = 3$ (b) $x = 1/\sqrt{5}$

(c) $x = 0$ (d) None of these

8. Solution set of the inequality

$(\cot^{-1}x)^2 - (5\cot^{-1}x) + 6 > 0$ is

(a) $(\cot 3, \cot 2)$

(b) $(-\infty, \cot 3) \cup (\cot 2, \infty)$

(c) $(\cot 2, \infty)$

(d) None of the above

9. Sum to infinite terms of the series $\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{2}{9}\right)$

$+\tan^{-1}\left(\frac{4}{33}\right)+\ldots+\tan^{-1}\left(\frac{2^{n-1}}{1+2^{2n-1}}\right)+\ldots$

(a) $\pi/4$ (b) $\pi/2$

(c) π (d) None of these

10. If $x+\frac{1}{x}=2$, the principal value of $\sin^{-1}x$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$

(c) π (d) $\frac{3\pi}{2}$

11. If $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then the value of

$\tan^{-1}\left(\frac{\tan x}{4}\right)+\tan^{-1}\left(\frac{3\sin 2x}{5+3\cos 2x}\right)$ is

(a) $x/2$ (b) $2x$

(c) $3x$ (d) x

12. If $\sin^{-1}x + \sin^{-1}y = \frac{2\pi}{3}$, then $\cos^{-1}x + \cos^{-1}y$

(a) $\frac{2\pi}{3}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{6}$ (d) π

13. $\cot\left(\frac{\pi}{4}-2\cot^{-1}3\right)$ is

(a) 1 (b) 7

(c) -1 (d) None of these

14. $\sin\left[\tan^{-1}\frac{1-x^2}{2x}+\cos^{-1}\frac{1-x^2}{1+x^2}\right]$ is

(a) 1 (b) 0

(c) -1 (d) None of these

15. If $\cos^{-1}\left(\frac{1-a^2}{1+a^2}\right)-\cos^{-1}\left(\frac{1-b^2}{1+b^2}\right)=2\tan^{-1}x$, then x is

(a) $\frac{a-b}{1+ab}$ (b) $\frac{b-a}{1+ab}$

(c) $\frac{a+b}{1-ab}$ (d) None of these

16. If $\left|\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right|<\frac{\pi}{3}$, then

(a) $x\in\left[-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right]$ (b) $x\in\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$

(c) $x\in\left[0,\frac{1}{\sqrt{3}}\right]$ (d) None of these

17. The value of

$$\cos^{-1}\left[\cot\left(\sin^{-1}\sqrt{\frac{2-\sqrt{3}}{4}}+\cos^{-1}\left(\frac{\sqrt{12}}{4}\right)+\sec^{-1}\sqrt{2}\right)\right]\text{ is}$$

(a) 0 (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{6}$ (d) $\frac{\pi}{2}$

18. If $\tan^{-1}\frac{x}{\pi}<\frac{\pi}{3}$, $x\in N$, then the maximum value of x is

(a) 2 (b) 5

(c) 7 (d) None of these

19. If $\tan^{-1}\frac{\sqrt{(1+x^2)}-\sqrt{(1-x^2)}}{\sqrt{(1+x^2)}+\sqrt{(1-x^2)}}=\alpha$, then x^2 is

(a) $\cos 2\alpha$ (b) $\sin 2\alpha$

(c) $\tan 2\alpha$ (d) $\cot 2\alpha$

20. The number of positive integral solutions of $\tan^{-1}x+\cot^{-1}y=\tan^{-1}3$ is

(a) one (b) two

(c) three (d) four

21. If $\text{cosec}^{-1}(\text{cosec}\,x)$ and $\text{cosec}(\text{cosec}^{-1}x)$ are equal functions, then the maximum range of value of x is

(a) $\left[-\frac{\pi}{2},-1\right]\cup\left[1,\frac{\pi}{2}\right]$

(b) $\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$

(c) $(-\infty,-1]\cup[1,\infty)$

(d) $[-1,0)\cup(0,1]$

22. The value of $\lim\limits_{|x|\to\infty}\cos(\tan^{-1}(\sin(\tan^{-1}x)))$ is equal to

(a) -1 (b) $\sqrt{2}$

(c) $-\frac{1}{\sqrt{2}}$ (d) $\frac{1}{\sqrt{2}}$

23. Complete solution set of $[\cot^{-1}x]+2[\tan^{-1}x]=0$, where [.] denotes the greatest integer function, is equal to

(a) $(0,\cot 1)$ (b) $(0,\tan 1)$

(c) $(\tan 1,\infty)$ (d) $(\cot 1,\tan 1)$

24. If $\sin^{-1}:[-1,1]\to\left[\frac{\pi}{2},\frac{3\pi}{2}\right]$ and $\cos^{-1}:[-1,1]\to[0,\pi]$ be two bijective function, respectively inverses of bijective functions $\sin:\left[\frac{\pi}{2},\frac{3\pi}{2}\right]\to[-1,1]$ and $\cos:[0,\pi]\to[-1,1]$, then $\sin^{-1}x+\cos^{-1}x$ is

(a) $\frac{\pi}{2}$ (b) π

(c) $\frac{3\pi}{2}$ (d) not a constant

25. If $a\sin^{-1}x-b\cos^{-1}x=c$, then $a\sin^{-1}x+b\cos^{-1}x$ is equal to

(a) 0 (b) $\frac{\pi ab+c(b-a)}{a+b}$

(c) $\frac{\pi}{2}$ (d) $\frac{\pi ab+c(a-b)}{a+b}$

26. The number of integer x satisfying $\sin^{-1}|x-2|+\cos^{-1}(1-|3-x|)=\frac{\pi}{2}$ is

(a) 1 (b) 2

(c) 3 (d) 4

27. The value of α such that $\sin^{-1}\frac{2}{\sqrt{5}},\sin^{-1}\frac{3}{\sqrt{10}},\sin^{-1}\alpha$ are the angles of a triangle is

(a) $\frac{-1}{\sqrt{2}}$ (b) $\frac{1}{2}$

(c) $\frac{1}{\sqrt{3}}$ (d) $\frac{1}{\sqrt{2}}$

28. Let $\begin{vmatrix}\tan^{-1}x & \tan^{-1}2x & \tan^{-1}3x\\ \tan^{-1}3x & \tan^{-1}x & \tan^{-1}2x\\ \tan^{-1}2x & \tan^{-1}3x & \tan^{-1}x\end{vmatrix}=0$, then the number of values of x satisfying the equation is

(a) 1 (b) 2

(c) 3 (d) 4

29. If the equation $x^3+bx^2+cx+1=0$, $(b<c)$, has only one real root α, then the value of $2\tan^{-1}(\text{cosec}\,\alpha)+\tan^{-1}(2\sin\alpha\sec^2\alpha)$ is

(a) $-\pi$ (b) $-\frac{\pi}{2}$

(c) $\frac{\pi}{2}$ (d) π

30. Let $u = \cot^{-1}\sqrt{\cos 2\theta} - \tan^{-1}\sqrt{\cos 2\theta}$, then the value of sinu is

(a) $\cos 2\theta$ (b) $\sin 2\theta$
(c) $\tan^2\theta$ (d) $\cot^2\theta$

31. Let $f(x) = 2\tan^{-1} x = \begin{cases} \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), & x \geq 0 \\ -\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), & x < 0 \end{cases}$

The function $f(x)$ is continuous everywhere but not differentiable at x equals to

(a) 1 (b) -1
(c) 0 (d) $\dfrac{1}{\sqrt{2}}$

32. Let $f(x) = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) \forall\ x \in R$. The function $f(x)$ is continuous everywhere but not differentiable at x is/are

(a) 0, 1 (b) -1, 1
(c) -1, 0 (d) 0, 2

33. Let $f(x) = \tan^{-1}(x^2 - 18x + a) > 0 \forall\ x \in R$. Then the value of a lies in

(a) $(81, \infty)$ (b) $[81, \infty)$
(c) $(-\infty, 81)$ (d) $(-\infty, 81]$

34. Let $f(x) = \sin^{-1} 2x + \cos^{-1} 2x + \sec^{-1} 2x$. Then the sum of the maximum and minimum values of $f(x)$ is

(a) π (b) 2π
(c) 3π (d) $\dfrac{\pi}{2}$

35. If $\tan^{-1}\dfrac{b}{c+a} + \tan^{-1}\dfrac{c}{a+b} = \dfrac{\pi}{4}$, where a, b, c are the sides of ΔABC, then ΔABC is

(a) Acute-angled triangle
(b) Obtuse-angled triangle
(c) Right-angled triangle
(d) Equilateral triangle

36. Solutions of $\sin^{-1}(\sin x) = \sin x$ are, if $x \in (0, 2\pi)$

(a) 4 real roots
(b) 2 positive real roots
(c) 2 negative real roots
(d) 5 real roots

37. The equation $e^{\frac{2\sin^{-1} x}{\pi}} = \dfrac{y}{\log y}$, has

(a) Unique solution
(b) Infinite many solution
(c) $x = 1$
(d) $y = e$

38. Let $f(x) = 1 + 2\sin\left(\dfrac{e^x}{e^x + 1}\right)$, $x \geq 0$, then $f^{-1}(x)$ is equal to (assuming f is bijective)

(a) $\log\left(\dfrac{\sin^{-1}\left(\frac{x-1}{2}\right)}{1-\sin^{-1}\left(\frac{x-1}{2}\right)}\right)$ (b) $\log\left(\dfrac{\sin\left(\frac{x-1}{2}\right)}{1-\sin\left(\frac{x-1}{2}\right)}\right)$

(c) $e^{\frac{\sin^{-1}\left(\frac{x-1}{2}\right)}{1-\sin^{-1}\left(\frac{x-1}{2}\right)}}$ (d) $e^{\frac{\sin\left(\frac{x-1}{2}\right)}{1-\sin\left(\frac{x-1}{2}\right)}}$

39. $\cos^{-1}(\cos(2\cot^{-1}(\sqrt{2}-1)))$ is equal to

(a) $\sqrt{2}-1$ (b) $\dfrac{\pi}{4}$
(c) $\dfrac{3\pi}{4}$ (d) None of these

40. The maximum value of $f(x) = \tan^{-1}\left(\dfrac{(\sqrt{12}-2)x^2}{x^4 + 2x^2 + 3}\right)$ is

(a) 18° (b) 36°
(c) 22.5° (d) 15°

41. If $\tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x} = 4°$, then

(a) $x = \tan 2°$ (b) $x = \tan 4°$
(c) $x = \tan(1/4)°$ (d) $x = \tan 8°$

42. If $\tan^{-1}(\sin^2\theta - 2\sin\theta + 3) + \cot^{-1}(5^{\sec^2 y} + 1) = \dfrac{\pi}{2}$, then the value of $\cos^2\theta - \sin\theta$ is equal to

(a) 0
(b) -1
(c) 1
(d) None of the above

43. The number of solutions of the equation $|\tan^{-1}|x|| = \sqrt{(x^2+1)^2 - 4x^2}$ is

(a) 1 (b) 2
(c) 3 (d) 4

44. For any real number $x \geq 1$, the expression $\sec^2(\tan^{-1} x) - \tan^2(\sec^{-1} x)$ is equal to

(a) 1 (b) 2
(c) $2x^2$ (d) $2\sqrt{2}$

45. Let $f : R \to \left[0, \dfrac{\pi}{2}\right)$ be defined by $f(x) = \tan^{-1}(3x^2 + 6x + a)$. If $f(x)$ is an onto function, then the value of a is

(a) 1 (b) 2
(c) 3 (d) 4

46. The value of expression
$\tan^{-1}\left(\frac{\sqrt{2}}{2}\right)+\sin^{-1}\left(\frac{\sqrt{5}}{5}\right)-\cos^{-1}\left(\frac{\sqrt{10}}{10}\right)$ is

(a) $\cot^{-1}\left(\frac{1+\sqrt{2}}{1-\sqrt{2}}\right)$ (b) $\cot^{-1}\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$

(c) $-\pi+\cot^{-1}\left(\frac{1+\sqrt{2}}{1-\sqrt{2}}\right)$ (d) $\pi-\cot^{-1}\left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)$

47. The value of $\sec\left(2\cot^{-1}2+\cos^{-1}\frac{3}{5}\right)$ is equal to

(a) $\frac{25}{24}$ (b) $-\frac{24}{7}$

(c) $\frac{25}{7}$ (d) $-\frac{25}{7}$

48. Which one of the following statement is meaningless?

(a) $\cos^{-1}\left(\ln\left(\frac{2e+4}{3}\right)\right)$ (b) $\text{cosec}^{-1}\left(\frac{\pi}{3}\right)$

(c) $\cot^{-1}\left(\frac{\pi}{2}\right)$ (d) $\sec^{-1}(\pi)$

49. The value of $\sec\left[\sin^{-1}\left(-\sin\frac{50\pi}{9}\right)+\cos^{-1}\cos\left(-\frac{31\pi}{9}\right)\right]$ is equal to

(a) $\sec\frac{10\pi}{9}$ (b) $\sec\frac{\pi}{9}$

(c) 1 (d) -1

50. The number k is such that $\tan\{\arctan(2)+\arctan(20k)\}=k$. The sum of all possible values of k is

(a) $-\frac{19}{40}$ (b) $-\frac{21}{40}$

(c) 0 (d) $\frac{1}{5}$

51. The value of $\sum_{r=2}^{\infty}\tan^{-1}\left(\frac{1}{r^2-5r+7}\right)$, is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$

(c) $\frac{3\pi}{4}$ (d) $\frac{5\pi}{4}$

52. If $x=\tan^{-1}1-\cos^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\frac{1}{2}$;
$y=\cos\left(\frac{1}{2}\cos^{-1}\left(\frac{1}{8}\right)\right)$, then

(a) $x=\pi y$
(b) $y=\pi x$
(c) $\tan x=-(4/3)y$
(d) $\tan x=(4/3)y$

53. The value of
$\tan^{-1}\left(\frac{1}{2}\tan 2A\right)+\tan^{-1}(\cot A)+\tan^{-1}(\cot^3 A)$ for $0<A<(\pi/4)$ is

(a) $4\tan^{-1}(1)$ (b) $2\tan^{-1}(2)$

(c) 0 (d) None of these

54. The sum $\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{4n}{n^4-2n^2+2}\right)$ is equal to

(a) $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{2}{3}$ (b) $4\tan^{-1}1$

(c) $\frac{\pi}{2}$ (d) $\sec^{-1}(-\sqrt{2})$

55. Number of solution(s) of the equations
$\cos^{-1}(1-x)-2\cos^{-1}x=\frac{\pi}{2}$ is

(a) 3 (b) 2

(c) 1 (d) 0

56. There exists a positive real number x satisfying $\cos(\tan^{-1}x)=x$. The number value of $\cos^{-1}\left(\frac{x^2}{2}\right)$ is

(a) $\frac{\pi}{10}$ (b) $\frac{\pi}{5}$

(c) $\frac{2\pi}{5}$ (d) $\frac{4\pi}{5}$

57. The range of values of p for which the equation $\sin\cos^{-1}(\cos(\tan^{-1}x))=p$ has a solution is

(a) $\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right]$ (b) $[0,1)$

(c) $\left[\frac{1}{\sqrt{2}},1\right)$ (d) $(-1,1)$

58. Number of solutions of the equation
$\log_{10}(\sqrt{5\cos^{-1}x-1})+\frac{1}{2}\log_{10}(2\cos^{-1}x+3)$
$+\log_{10}\sqrt{5}=1$ is

(a) 0
(b) 1
(c) more than one but finite
(d) infinite

59. Which of the following is the solution set of the equations $\sin^{-1}x=\cos^{-1}x+\sin^{-1}(3x-2)$?

(a) $\left\{\frac{1}{2},1\right\}$ (b) $\left[\frac{1}{2},1\right]$

(c) $\left[\frac{1}{3},1\right]$ (d) $\left\{\frac{1}{3},1\right\}$

60. The set of values of x, satisfying the equation $\tan^2(\sin^{-1}x) > 1$ is

(a) $[-1,1]$ (b) $\left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$

(c) $(-1, 1) - \left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$ (d) $[-1,1] - \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

61. The solution set of the equation

$$\sin^{-1}\sqrt{1-x^2} + \cos^{-1}x = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) - \sin^{-1}x$$

(a) $[-1,1] - \{0\}$
(b) $(0,1] \cup \{-1\}$
(c) $[-1,0) \cup \{1\}$
(d) $[-1,1]$

62. The value of the angle $\tan^{-1}(\tan 65° - 2\tan 40°)$ in degrees is equal to

(a) $-20°$ (b) $20°$ (c) $25°$ (d) $40°$

63. If $\cos^{-1}\frac{x}{a} - \sin^{-1}\frac{y}{b} = \theta\,(a, b, \neq 0)$, then the maximum value of $b^2x^2 + a^2y^2 + 2ab\,xy\sin\theta$ equals

(a) ab (b) $(a+b)^2$

(c) $2(a+b)^2$ (d) a^2b^2

64. The value of $\sum_{r=1}^{\infty}\tan^{-1}\left(\frac{1}{r^2+5r+7}\right)$ is equal to

(a) $\tan^{-1}3$ (b) $\frac{\pi}{4}$

(c) $\sin^{-1}\frac{1}{\sqrt{10}}$ (d) $\cot^{-1}2$

65. The range of the function,

$$f(x) = \tan^{-1}\left(\frac{1+x}{1-x}\right) - \tan^{-1}x \text{ is}$$

(a) $\{\pi/4\}$ (b) $\{-(\pi/4), 3\pi/4\}$
(c) $\{\pi/4, -(3\pi/4)\}$ (d) $\{3\pi/4\}$

66. Let $g : R \to \left(0, \frac{\pi}{3}\right]$ is defined by $g(x) = \cos^{-1}\left(\frac{x^2-k}{1+x^2}\right)$

Then the possible values of 'k' for which g is surjective function, is

(a) $\left\{\frac{1}{2}\right\}$ (b) $\left(-1, -\frac{1}{2}\right]$

(c) $\left\{-\frac{1}{2}\right\}$ (d) $\left[-\frac{1}{2}, 1\right)$

67. Number of values of x satisfying simultaneously $\sin^{-1}x = 2\tan^{-1}x$ and

$\tan^{-1}\sqrt{x(x-1)} + \text{cosec}^{-1}\sqrt{1+x-x^2} = \frac{\pi}{2}$, is

(a) 0 (b) 1
(c) 2 (d) 3

68. Number of values of x satisfying the equation $\cos(3\arccos(x-1)) = 0$ is equal to

(a) 0 (b) 1
(c) 2 (d) 3

69. Which one of the following function contains only one integer in its range?

[**Note** $\text{sgn}(k)$ denotes the signum function of k.]

(a) $f(x) = \frac{1}{2}\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

(b) $g(x) = \text{sgn}\left(x + \frac{1}{x}\right)$

(c) $h(x) = \sin^2 x + 2\sin x + 2$

(d) $k(x) = \cos^{-1}(x^2 - 2x + 2)$

70. If range of the function $f(x) = \tan^{-1}(3x^2 + bx + 3)$, $x \in R$ is $\left[0, \frac{\pi}{2}\right)$, then square of sum of all possible values of b will be

(a) 0 (b) 18
(c) 72 (d) None of these

Inverse Trigonometric Functions Exercise 2 : More than One Correct Type Questions

71. Let $\theta = \tan^{-1}\left(\tan\frac{5\pi}{4}\right)$ and $\phi = \tan^{-1}\left(-\tan\frac{2\pi}{3}\right)$ then

(a) $\theta > \phi$ (b) $4\theta - 3\phi = 0$

(c) $\theta + \phi = \frac{7\pi}{12}$ (d) None of these

72. Let $f(x) = e^{\cos^{-1}\sin(x+\pi/3)}$ then

(a) $f\left(\frac{8\pi}{9}\right) = e^{5\pi/18}$ (b) $f\left(\frac{8\pi}{9}\right) = e^{13\pi/18}$

(c) $f\left(-\frac{7\pi}{4}\right) = e^{\pi/12}$ (d) $f\left(-\frac{7\pi}{4}\right) = e^{11\pi/12}$

73. If the numerical value of $\tan\left\{\cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{2}{3}\right)\right\}$ is $\frac{a}{b}$, HCF $(a, b)=1$, then

(a) $a+b=23$ (b) $a-b=11$

(c) $3b=a+1$ (d) $2a=3b$

74. Let $f(x)=\sin^{-1}x+\cos^{-1}x$. Then $\frac{\pi}{2}$ is equal to

(a) $f\left(-\frac{1}{2}\right)$ (b) $f(k^2-2k+3), k\in R$

(c) $f\left(\frac{1}{1+k^2}\right), k\in R$ (d) $f(-2)$

75. $\cos^{-1}x=\tan^{-1}x$, then

(a) $x^2=\frac{\sqrt{5}-1}{2}$ (b) $x^2=\frac{\sqrt{5}+1}{2}$

(c) $\sin(\cos^{-1}x)=\frac{\sqrt{5}-1}{2}$ (d) $\tan(\cos^{-1}x)=\frac{\sqrt{5}-1}{2}$

76. The value(s) of x satisfying the equation $\sin^{-1}|\sin x|=\sqrt{\sin^{-1}|\sin x|}$ is/are given by (n is any integer)

(a) $n\pi-1$ (b) $n\pi$

(c) $n\pi+1$ (d) $2n\pi+1$

77. If $(\sin^{-1}x+\sin^{-1}w)(\sin^{-1}y+\sin^{-1}z)=\pi^2$, then

$$D=\begin{vmatrix} x^{N_1} & y^{N_2} \\ z^{N_3} & w^{N_4} \end{vmatrix} (N_1, N_2, N_3, N_4\in N)$$

(a) has a maximum value of 2

(b) has a minimum value of 0

(c) 16 different D are possible

(d) has a minimum value of -2

78. Indicate the relation which can hold in their respective domain for infinite values of x.

(a) $\tan|\tan^{-1}x|=|x|$

(b) $\cot|\cot^{-1}x|=|x|$

(c) $\tan^{-1}|\tan x|=|x|$

(d) $\sin|\sin^{-1}x|=|x|$

79. To the equation $2^{2\pi/\cos^{-1}x}-\left(a+\frac{1}{2}\right).2^{\pi/\cos^{-1}x}-a^2=0$ has only one real root, then

(a) $1\le a\le 3$ (b) $a\ge 1$

(c) $a\le -3$ (d) $a\ge 3$

80. $\sin^{-1}(\sin 3)+\sin^{-1}(\sin 4)+\sin^{-1}(\sin 5)$ when simplified reduces to

(a) an irrational number

(b) a rational number

(c) an even prime

(d) a negative integer

81. If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$, then

(a) $x^2+y^2+z^2+2xyz=1$

(b) $2(\sin^{-1}x+\sin^{-1}y+\sin^{-1}z)=\cos^{-1}x+\cos^{-1}y+\cos^{-1}z$

(c) $xy+yz+zx=x+y+z-1$

(d) $\left(x+\frac{1}{x}\right)+\left(y+\frac{1}{y}\right)+\left(z+\frac{1}{z}\right)\ge 6$

82. $2\tan(\tan^{-1}(x)+\tan^{-1}(x^3))$ where $x\in R-\{-1,1\}$ is equal to

(a) $\frac{2x}{1-x^2}$

(b) $\tan(2\tan^{-1}x)$

(c) $\tan(\cot^{-1}(-x)-\cot^{-1}(x))$

(d) $\tan(2\cot^{-1}x)$

83. Let $f(x)=\sin^{-1}|\sin x|+\cos^{-1}(\cos x)$. Which of the following statement(s) is/are TRUE?

(a) $f(f(3))=\pi$

(b) $f(x)$ is periodic with fundamental period 2π

(c) $f(x)$ is neither even nor odd

(d) Range of $f(x)$ is $[0,2\pi]$

84. If $f(x)=\sin^{-1}x\,.\cos^{-1}x\,.\tan^{-1}x\,.\cot^{-1}x\,.\sec^{-1}x\,.\operatorname{cosec}^{-1}x$, then which of the following statement(s) hold(s) good?

(a) The graph of $y=f(x)$ does not lie above x axis

(b) The non-negative difference between maximum minimum value of the function $y=f(x)$ is $\frac{3\pi^6}{64}$

(c) The function $y=f(x)$ is not injective.

(d) Number of non-negative integers in the domain of $f(x)$ is two.

85. Let $\alpha=3\cos^{-1}\left(\frac{5}{\sqrt{28}}\right)+3\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$ and $\beta=4\sin^{-1}\left(\frac{7\sqrt{2}}{10}\right)-4\tan^{-1}\left(\frac{3}{4}\right)$,

then which of the following does not hold(s) good?

(a) $\alpha<\pi$ but $\beta>\pi$

(b) $\alpha>\pi$ but $\beta<\pi$

(c) Both α and β are equal

(d) $\cos(\alpha+\beta)=0$

86. Let function $f(x)$ be defined as $f(x)=|\sin^{-1}x|+\cos^{-1}\left(\frac{1}{x}\right)$. Then which of the following is/are TRUE.

(a) $f(x)$ is injective in its domain.

(b) $f(x)$ is many-one in its domain.

(c) Range of f is singleton set

(d) $\operatorname{sgn}(f(x))=1$, where sgn x denotes signum function of x.

87. Which of the following pairs(s) of function is(are) identical?

(a) $f(x) = \sin(\tan^{-1} x)$, $g(x) = \dfrac{x}{\sqrt{1+x^2}}$

(b) $f(x) = \text{sgn}\,(\cot^{-1} x)$, $g(x) = \sec^2 x - \tan^2 x$, where sgn x denotes signum function of x.

(c) $f(x) = e^{\ln\left(\cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)\right)}$, $g(x) = \cos^{-1}\left(\dfrac{x^2-1}{x^2+1}\right)$

(d) $f(x) = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$, $g(x) = 2\tan^{-1} x$

88. The value of $\sum_{n=1}^{\infty} \cot^{-1}(n^2+n+1)$ is also equal to

(a) $\cot^{-1}(-1) + \text{sec}^{-1}(1) - \text{cosec}^{-1}(1)$

(b) $\cot^{-1}(2) + \cot^{-1}(3)$

(c) minimum value of the function $f(x) = \tan^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

(d) $\cos^{-1}\left(\cos 41\,\dfrac{\pi}{4}\right)$

89. Let $f : I - \{-1, 0, 1\} \to [-\pi, \pi]$ be defined as $f(x) = 2\tan^{-1} x - \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$, then which of the following statements(s) is (are) correct?

(a) $f(x)$ is bijective

(b) $f(x)$ is injective but not surjective

(c) $f(x)$ is neither injective nor surjective

(d) $f(x)$ is an odd function

[**Note** : I denotes the set of integers]

90. If $\log x = \dfrac{-1}{3}$, $\log y = \dfrac{2}{5}$ and $P = \log(\sin(\text{arc}\cos\sqrt{1-x^2}))$, $Q = \log\left(\cos\left(\text{arc}\tan\dfrac{\sqrt{1-x^2y^2}}{xy}\right)\right)$, then

(a) $P = \dfrac{-1}{9}$ (b) $P + Q = \dfrac{-4}{15}$

(c) $P - Q = \dfrac{-2}{5}$ (d) $\dfrac{P}{Q} = -5$

Inverse Trigonometric Functions Exercise 3 : Passage Based Questions

Passage I

(Q. Nos. 91 to 93)

Let S denotes the set consisting of four functions and $S = \{[x], \sin^{-1} x, |x|, \{x\}\}$ where, $\{x\}$ denotes fractional part and $[x]$ denotes greatest integer function. Let A, B, C are subsets of S.

Suppose

A : consists of odd function(s)

B : consists of discontinuous function(s)

and C : consists of non-decreasing function(s) or increasing function(s).

If $f(x) \in A \cap C$; $g(x) \in B \cap C$; $h(x) \in B$ but not C and $l(x) \in$ neither A nor B nor C.

Then, answer the following.

91. The function $l(x)$ is

(a) periodic

(b) even

(c) odd

(d) neither odd nor even

92. The range of $g(f(x))$ is

(a) $\{-1, 0, 1\}$ (b) $\{-1, 0\}$

(c) $\{0, 1\}$ (d) $\{-2, -1, 0, 1\}$

93. The range of $f(h(x))$ is

(a) $\left(0, \dfrac{\pi}{2}\right)$ (b) $\left[0, \dfrac{\pi}{2}\right)$

(c) $\left(0, \dfrac{\pi}{2}\right]$ (d) $\left[0, \dfrac{\pi}{2}\right]$

Passage II

(Q. Nos. 94 to 96)

Let f be a real-valued function defined on R (the set of real numbers) such that $f(x) = \sin^{-1}(\sin x) + \cos^{-1}(\cos x)$

94. The value of $f(10)$ is equal to

(a) $6\pi - 20$ (b) $7\pi - 20$

(c) $20 - 7\pi$ (d) $20 - 6\pi$

95. The area bounded by curve $y = f(x)$ and x-axis from $\dfrac{\pi}{2} \le x \le \pi$ is equal to

(a) $\dfrac{\pi^2}{4}$ (b) $\dfrac{\pi^2}{2}$ (c) π^2 (d) $\dfrac{\pi^2}{8}$

96. Number of values of x in interval (0, 3) so that $f(x)$ is an integer, is equal to

(a) 1 (b) 2

(c) 3 (d) 0

Passage III

(Q. Nos. 97 to 98)

Consider a real-valued function

$$f(x)=\sqrt{\sin^{-1}x+2}+\sqrt{1-\sin^{-1}x}$$

97. The domain of definition of $f(x)$ is

(a) $[-1,1]$ (b) $[\sin 1,1]$ (c) $[-1,\sin 1]$ (d) $[-1,0]$

98. The range of $f(x)$ is

(a) $[0,\sqrt{3}]$ (b) $[1,\sqrt{3}]$ (c) $[1,\sqrt{6}]$ (d) $[\sqrt{3},\sqrt{6}]$

Passage IV

(Q. Nos. 99 to 101)

Given that,

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\begin{cases}2\tan^{-1}x, |x|\le 1\\ -\pi+2\tan^{-1}x, x>1\\ \pi+2\tan^{-1}x, x<-1\end{cases}$$

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right)=\begin{cases}2\tan^{-1}x, |x|\le 1\\ \pi-2\tan^{-1}x,\ x>1 \text{ and}\\ -(\pi+2\tan^{-1}x), x<-1\end{cases}$$

$$\sin^{-1}x+\cos^{-1}x=\pi/2 \text{ for } -1\le x\le 1$$

99. $\sin^{-1}\left(\frac{4x}{x^2+4}\right)+2\tan^{-1}\left(-\frac{x}{2}\right)$ is independent of x, then

(a) $x\in[-3,4]$ (b) $x\in[-2,2]$
(c) $x\in[-1,1]$ (d) $x\in[1,\infty)$

100. If $\cos^{-1}\frac{6x}{1+9x^2}=-\frac{\pi}{2}+2\tan^{-1}3x$, then $x\in$

(a) $\left(\frac{1}{3},\infty\right)$ (b) $(-1,\infty)$
(c) $(-\infty, m-1)$ (d) None of these

101. If $(x-1)(x^2+1)>0$, then $\sin\left(\frac{1}{2}\tan^{-1}\frac{2x}{1-x^2}-\tan^{-1}x\right)$ is equal to

(a) 1 (b) $\frac{1}{\sqrt{2}}$
(c) -1 (d) None of these

Passage V

(Q. Nos. 102 to 104)

For $x, y, z, t\in R$, $\sin^{-1}x+\cos^{-1}y+\sec^{-1}z\ge t^2-\sqrt{2\pi}t+3\pi$

102. The value of $x+y+z$ is equal to

(a) 1 (b) 0
(c) 2 (d) -1

103. The principal value of $\cos^{-1}(\cos 5t^2)$ is

(a) $\frac{3\pi}{2}$ (b) $\frac{\pi}{2}$
(c) $\frac{\pi}{3}$ (d) $\frac{2\pi}{3}$

104. The value of $\cos^{-1}(\min\{x,y,z\})$ is

(a) 0 (b) $\frac{\pi}{2}$ (c) π (d) $\frac{\pi}{3}$

Inverse Trigonometric Functions Exercise 4 : Single Integer Answer Type Questions

105. Let $f(x)=\tan^{-1}\left(\frac{(x-2)}{x^2+2x+2}\right)$, then $26f'(1)$ is

106. Let $f(x)=(\arctan x)^3+(\operatorname{arc\,cot} x)^3$. If the range of $f(x)$ is $[a,b)$, then find the value of $\frac{b}{7a}$.

107. If $\sum_{n=0}^{\infty}2\operatorname{arc\,cot}\left(\frac{n^2+n+4}{2}\right)=k\pi$, then find the value of k.

108. Find the number of solutions of the equation $\tan\left(\sum_{r=1}^{5}\cot^{-1}(2r^2)\right)=\frac{5x+6}{6x+5}$.

109. $\lim_{z\to 0}\left[\{\max(\sin^{-1}x+\cos^{-1}x)^2, \min(x^2+4x+7))\}.\frac{\sin^{-1}z}{z}\right]$ is equal to (where [.] denotes greatest integer function)

110. If $\sin(30^\circ+\arctan x)=\frac{13}{14}$ and $0<x<1$, the value of x is $\frac{a\sqrt{3}}{b}$, where a and b are positive integers with no common factors. Find the value of $\left(\frac{a+b}{2}\right)$.

111. Let $f: R\to\left(0,\frac{2\pi}{3}\right]$ defined as $f(x)=\cot^{-1}(x^2-4x+\alpha)$. Find the smallest integral value of α such that $f(x)$ is into function.

112. Let L denotes the number of subjective functions $f : A \to B$, where set A contains 4 elements and set B contains 3 elements, M denotes number of elements in the range of the function $f(x) = \sec^{-1}(\text{sgn } x) + \text{cosec}^{-1}(\text{sgn } x)$, where sgn x denotes signum function of x and N denotes coefficient of t^5 in $(1+t^2)^5\ (1+t^3)^8$.

Find the value of $(N - LM)$

113. Number of solution(s) of the equations $\cos^{-1}(\cos x) = x^2$ is

114. If $\cos^{-1}(x) + \cos^{-1}(y) + \cos^{-1}(z) = \pi(\sec^2(u) + \sec^4(v) + \sec^6(w))$, where u, v, w are least non-negative angles such that $u < v < w$, then the value of $x^{2000} + y^{2002} + z^{2004} + \dfrac{36\pi}{u+v+w}$ is .....

115. Let $f(x) = \cos(\tan^{-1}(\sin(\cot^{-1} x)))$. The simplest form of $f(x)$ can be written as $\left(\dfrac{x^2+A}{x^2+B}\right)^{1/2}$. Then the value of $(A + B)$ is .....

116. Find the value of λ for which $\tan^{-1}\left\{\dfrac{\cos 2\alpha \sec 2\beta + \cos 2\beta \sec 2\alpha}{\lambda}\right\} = \tan^{-1}\{\tan^2(\alpha+\beta)\tan^2(\alpha-\beta)\} + \tan^{-1} 1$

117. The least value of n for which $(n-2)x^2 + 8x + n + 4 > \sin^{-1}(\sin 12) + \cos^{-1}(\cos 12)$, $\forall x \in R$, where $n \in N$, is .....

118. If $0 < \cos^{-1} x < 1$ and $1 + \sin(\cos^{-1} x) + \sin^2(\cos^{-1} x) + \sin^3(\cos^{-1} x) + \ldots \infty = 2$, then the value of $12x^2$ is .....

119. The number of real solutions of the equation $\sqrt{1+\cos 2x} = \sqrt{2}\sin^{-1}(\sin x)$, $-\pi \le x \le \pi$, is

Inverse Trigonometric Functions Exercise 5 : Statement I and II Type Questions

- This section contain 6 questions. Each question contains **Statement I** (Assertion) and **Statement II** (Reason). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are
  - (a) Statement I is True; Statement II is True; Statement II is a correct explanation for Statement I;
  - (b) Statement I is True; Statement II is True; Statement II is NOT a correct explanation for Statement I;
  - (c) Statement I is True; Statement II is False;
  - (d) Statement I is False; Statement II is True.

120. **Statement I** $y = \tan^{-1}(\tan x)$ and $y = \cos^{-1}(\cos x)$ does not have any solution, if $x \in \left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$

Statement II $y = \tan^{-1}(\tan x) = x - \pi$, $x \in \left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$ and

$$y = \cos^{-1}(\cos x) = \begin{cases} 2\pi - x, & x \in \left[\pi, \dfrac{3\pi}{2}\right] \\ x, & x \in \left[\dfrac{\pi}{2}, \pi\right] \end{cases}$$

121. **Statement I** $\sin^{-1}\left(\dfrac{1}{\sqrt{e}}\right) > \tan^{-1}\left(\dfrac{1}{\sqrt{\pi}}\right)$

Statement II $\sin^{-1} x > \tan^{-1} y$ for $x > y$, $\forall\ x, y \in (0, 1)$

122. **Statement I** $\text{cosec}^{-1}\left(\dfrac{1}{2} + \dfrac{1}{\sqrt{2}}\right) > \sec^{-1}\left(\dfrac{1}{2} + \dfrac{1}{\sqrt{2}}\right)$

Statement II $\text{cosec}^{-1} x > \sec^{-1} x$, if $1 \le x < \sqrt{2}$

123. Let $f(x) = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

Statement I $f'(2) = -\dfrac{2}{5}$ and

Statement II $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right) = \pi - 2\tan^{-1} x$, $\forall\ x > 1$

124. **Statement I** $\sin^{-1} 2x + \sin^{-1} 3x = \dfrac{\pi}{3}$

$\Rightarrow x = \sqrt{\dfrac{3}{76}}$ only.

and

Statement II Sum of two negative angles cannot be positive.

125. **Statement I** Number of roots of the equation $\cot^{-1} x \cos^{-1} 2x + \pi = 0$ is zero.

Statement II Range of $\cot^{-1} x$ and $\cos^{-1} x$ is $(0, \pi)$ and $[0, \pi]$, respectively.

Inverse Trigonometric Functions Exercise 6 :
Matching Type Questions

126. Let $t_1 = (\sin^{-1} x)^{\sin^{-1} x}$, $t_2 = (\sin^{-1} x)^{\cos^{-1} x}$, $t_3 = (\cos^{-1} x)^{\sin^{-1} x}$, $t_4 = (\cos^{-1} x)^{\cos^{-1} x}$,

Match the following items of **Column I** with **Column II**

	Column I		Column II
A.	$x \in (0, \cos 1)$	(p)	$t_1 > t_2 > t_4 > t_3$
B.	$x \in \left(\cos 1, \frac{1}{\sqrt{2}}\right)$	(q)	$t_4 > t_3 > t_1 > t_2$
C.	$x \in \left(\frac{1}{\sqrt{2}}, \sin 1\right)$	(r)	$t_2 > t_1 > t_4 > t_3$
D.	$x \in (\sin 1, 1)$	(s)	$t_3 > t_4 > t_1 > t_2$

127. Match the following items of **Column I** with **Column II**

	Column I		Column II
A.	$\sin^{-1} x + x > 0$, for	(p)	$x < 0$
B.	$\cos^{-1} x - x \geq 0$, for	(q)	$x \in (0, 1]$
C.	$\tan^{-1} x + x < 0$, for	(r)	$x \in [-1, 0)$
D.	$\cot^{-1} x + x > 0$, for	(s)	$x > 0$

Inverse Trigonometric Functions Exercise 7 :
Subjective Type Questions

128. Solve $\sin^{-1}\left(\frac{5}{x}\right) + \sin^{-1}\left(\frac{12}{x}\right) = \frac{\pi}{2}$

129. Solve $\tan^{-1}\left(\frac{x+1}{x-1}\right) + \tan^{-1}\left(\frac{x-1}{x}\right) = \tan^{-1}(-7)$

130. If a, b, c are positive, show that

$$\tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}} + \tan^{-1}\sqrt{\frac{b(a+b+c)}{ac}} + \tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}} = \pi$$

131. Find the sum to the n term of the series

$\operatorname{cosec}^{-1}\sqrt{10} + \operatorname{cosec}^{-1}\sqrt{50} + \operatorname{cosec}^{-1}\sqrt{170}$

$+ \ldots + \operatorname{cosec}^{-1}\sqrt{(n^2+1)(n^2+2n+2)}$

132. If $x_i \in [0, 1]\ \forall i = 1, 2, 3, \ldots, 28$ then find the maximum value of

$$\sqrt{\sin^{-1} x_1}\sqrt{\cos^{-1} x_2} + \sqrt{\sin^{-1} x_2}\sqrt{\cos^{-1} x_3} + \sqrt{\sin^{-1} x_3}\sqrt{\cos^{-1} x_4} + \ldots + \sqrt{\sin^{-1} x_{28}}\sqrt{\cos^{-1} x_1}$$

133. Find the value of $\sum_{r=1}^{10}\sum_{s=1}^{10} \tan^{-1}\left(\frac{r}{s}\right)$.

134. Find the value

$$\lim_{n\to\infty} \sum_{k=2}^{n} \cos^{-1}\left(\frac{1+\sqrt{(k-1)\,k(k+1)(k+2)}}{k(k+1)}\right)$$

135. If $\frac{m\tan(\alpha-\theta)}{\cos^2\theta} = \frac{n\tan\theta}{\cos^2(\alpha-\theta)}$

Prove that : $\theta = \frac{1}{2}\left[\alpha - \tan^{-1}\left(\frac{n-m}{n+m}\tan\alpha\right)\right]$

136. Prove that :

$\tan^{-1}(e^{i\theta}) = \frac{n\pi}{2} + \frac{\pi}{4} - \frac{i}{2}\operatorname{In}\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$, where '$n$' is an integer.

137. If the quadratic equation;

$4^{\sec^2\alpha}x^2 + 2x + \left(\beta^2 - \beta + \frac{1}{2}\right) = 0$ have meal roots, then find all the possible value of $\cos\alpha + \cos^{-1}\beta$.

Inverse Trigonometric Functions Exercise 8 : Questions Asked in Previous 10 Years Exam

(i) JEE Advanced & IIT-JEE

138. If $\alpha = 3\sin^{-1}\left(\frac{6}{11}\right)$ and $\beta = 3\cos^{-1}\left(\frac{4}{9}\right)$, where the inverse trigonometric functions take only the principal values, then the correct option(s) is/are **[More than one correct option, JEE 2015 Adv.]**

(a) $\cos\beta > 0$ (b) $\sin\beta < 0$
(c) $\cos(\alpha+\beta) > 0$ (d) $\cos\alpha < 0$

139. If $0 < x < 1$, then $\sqrt{1+x^2}\,[\{x\cos(\cot^{-1}x) + \sin(\cot^{-1}x)\}^2 - 1]^{1/2}$ is equal to **[Single correct option, IIT-JEE 2008 3M]**

(a) $\frac{x}{\sqrt{1+x^2}}$ (b) x
(c) $x\sqrt{1+x^2}$ (d) $\sqrt{1+x^2}$

(ii) JEE Main/AIEEE

140. If $\tan^{-1}y = \tan^{-1}x + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, where $|x| < \frac{1}{\sqrt{3}}$. Then, the value of y is **[2015 JEE Main]**

(a) $\frac{3x-x^3}{1-3x^2}$ (b) $\frac{3x+x^3}{1-3x^2}$
(c) $\frac{3x-x^3}{1+3x^2}$ (d) $\frac{3x+x^3}{1+3x^2}$

141. The value of $\cot\left\{\sum_{n=1}^{23}\cot^{-1}\left(1+\sum_{k=1}^{n}2k\right)\right\}$ is **[2013 JEE Main]**

(a) $\frac{23}{25}$ (b) $\frac{25}{23}$ (c) $\frac{23}{24}$ (d) $\frac{24}{23}$

142. If x, y and z are in AP and $\tan^{-1}x$, $\tan^{-1}y$ and $\tan^{-1}z$ are also in AP, then **[2013 JEE Main]**

(a) $x = y = z$ (b) $2x = 3y = 6z$
(c) $6x = 3y = 2z$ (d) $6x = 4y = 3z$

143. The value of $\cot\left(\text{cosec}^{-1}\frac{5}{3} + \tan^{-1}\frac{2}{3}\right)$ is **[AIEEE 2008]**

(a) $\frac{5}{17}$ (b) $\frac{6}{17}$
(c) $\frac{3}{17}$ (d) $\frac{4}{17}$

144. If $\sin^{-1}\left(\frac{x}{5}\right) + \text{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$, then the value of x is **[2007 AIEEE]**

(a) 1 (b) 3
(c) 4 (d) 5

145. If $\cos^{-1}x - \cos^{-1}\frac{y}{2} = \alpha$, then $4x^2 - 4xy\cos\alpha + y^2$ is equal to **[2005 AIEEE]**

(a) $-4\sin^2\alpha$ (b) $4\sin^2\alpha$
(c) 4 (d) $2\sin 2\alpha$

Answers

Exercise for Session 1

1. 1/2 **2.** –1 **3.** $(-\infty, -3]\cup[-2, -1]\cup[0, \infty)$
4. R **5.** $(-\infty, -1]\cup[1, \infty)$

Exercise for Session 2

1. π **2.** $\frac{2\pi}{5}$ **3.** $\frac{-\pi}{10}$
4. $13 - 4\pi$
5. $\sin^{-1}(\sin\theta) = 3\pi - \theta$; $\cos^{-1}(\cos\theta) = \theta - 2\pi$
$\tan^{-1}(\tan\theta) = \theta - 3\pi$; $\cot^{-1}(\cot\theta) = \theta - 2\pi$

Exercise for Session 3

1. $\frac{\sqrt{3}}{2}$ **2.** not defined **3.** $\frac{59}{36}$ **4.** 11 **5.** $x = \pm 1$

Exercise for Session 4

1. $\frac{\pi}{8}$ **2.** $\left(\frac{1}{4} - \frac{\pi}{2}\right)$ **3.** $\frac{3}{2}$ **4.** $-\sqrt{3}$
5. 0 **6.** $\frac{3\pi}{10}$ **7.** $\frac{\pi}{2}$ **9.** $x = 1$ **10.** $x = \frac{1}{2}$

Exercise for Session 5

2. $\frac{\pi}{2}$ **3.** $\lambda = \frac{-19}{9}$ **5.** $x = \frac{\sqrt{5}+4\sqrt{2}}{9}$ **6.** $x = \frac{1}{2}$

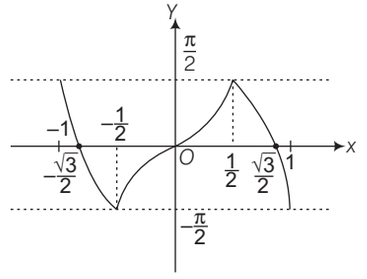
Exercise for Session 6

1. $\frac{4}{5}$ **2.** $\frac{65}{16}$ **3.** $\frac{9}{25}$ **4.** $\frac{2\sqrt{5}}{5}$

Exercise for Session 7

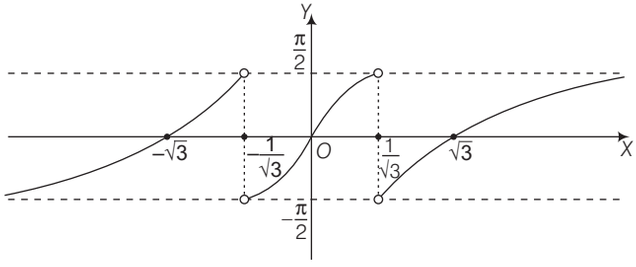
1. $y = \sin^{-1}(3x - 4x^3) = \begin{cases} 3\sin^{-1}x & ; \quad -\frac{1}{2} \le x \le \frac{1}{2} \\ \pi - 3\sin^{-1}x & ; \quad \frac{1}{2} < x \le 1 \\ -\pi - 3\sin^{-1}x & ; \quad -1 \le x < -\frac{1}{2} \end{cases}$

Graph of $y = \sin^{-1}(3x - 4x^3)$



2. $y = \tan^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right) = \begin{cases} 3\tan^{-1}x & ; \ -\dfrac{1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}} \\ \pi + 3\tan^{-1}x & ; \ -\infty < x < -\dfrac{1}{\sqrt{3}} \\ -\pi + 3\tan^{-1}x & ; \ \dfrac{1}{\sqrt{3}} < x < \infty \end{cases}$

Graph of $y = \tan^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right)$



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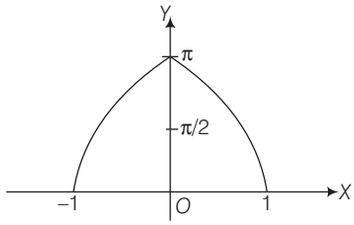

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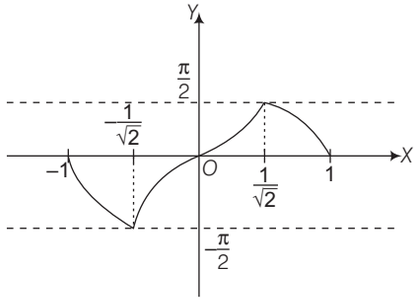

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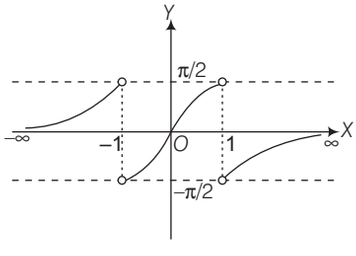

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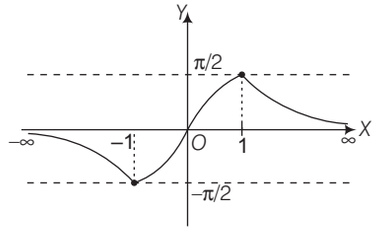

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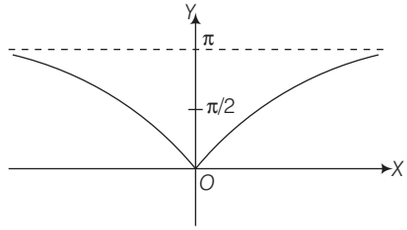


Chapter Exercises

1. (b)	**2.** (a)	**3.** (c)	**4.** (d)	**5.** (a)	**6.** (b)
7. (a)	**8.** (b)	**9.** (a)	**10.** (b)	**11.** (d)	**12.** (b)
13. (b)	**14.** (a)	**15.** (a)	**16.** (b)	**17.** (d)	**18.** (b)
19. (b)	**20.** (b)	**21.** (a)	**22.** (d)	**23.** (d)	**24.** (d)
25. (d)	**26.** (b)	**27.** (d)	**28.** (a)	**29.** (a)	**30.** (c)
31. (c)	**32.** (b)	**33.** (a)	**34.** (b)	**35.** (c)	**36.** (d)
37. (b)	**38.** (a)	**39.** (c)	**40.** (d)	**41.** (d)	**42.** (c)
43. (d)	**44.** (b)	**45.** (c)	**46.** (c)	**47.** (d)	**48.** (a)
49. (d)	**50.** (a)	**51.** (c)	**52.** (c)	**53.** (a)	**54.** (a)
55. (c)	**56.** (c)	**57.** (b)	**58.** (b)	**59.** (a)	**60.** (c)
61. (c)	**62.** (c)	**63.** (d)	64. (c)	**65.** (c)	**66.** (c)
67. (c)	**68.** (d)	**69.** (d)	**70.** (a)	**71.** (b,c)	**72.** (b,c)
73. (a,b,c)	**74.** (a,c)		**75.** (a,c)	**76.** (a,b,c)	**77.** (a,c,d)
78. (a,b,c,d)		**79.** (b,c)	**80.** (b,d)	**81.** (a,b)	
82. (a,b,c)		**83.** (a,b)	**84.** (a,b)		
85. (a,b,d)	**86.** (a,d)	**87.** (a,c)	**88.** (a,b,d)	**89.** (c,d)	**90.** (b,c,d)
91. (b)	**92.** (d)	**93.** (b)	**94.** (b)	**95.** (b)	**96.** (c)
97. (c)	**98.** (d)	**99.** (b)	**100.** (a)	**101.** (c)	**102.** (d)
103. (b)	**104.** (c)	**105.** (9)	**106.** (4)	**107.** (1)	**108.** (0)
109. (3)	**110.** (8)	**111.** (4)	**112.** (4)	**113.** (3)	**114.** (9)
115. (3)	**116.** (2)	**117.** (5)	**118.** (9)	**119.** (2)	**120.** (a)
121. (a)	**122.** (a)	**123.** (a)	**124.** (a)	**125.** (a)	

126. A → q; B → s; C → r; D → p;

127. A → q; B → r; C → p,r; D → q,r,s;

128. $x = 13$ **129.** No Solution

131. $\tan^{-1}(n+1) - \dfrac{\pi}{4}$ **132.** 7π **133.** 25π **134.** $\dfrac{\pi}{6}$

137. $\dfrac{\pi}{3} - 1$ and $\dfrac{\pi}{3} + 1$

138. (b,c,d)	**139.** (c)	**140.** (a)	**141.** (b)	**142.** (a)
143. (b)	**144.** (b)	**145.** (b)		

Solutions

1. Substituting $x^2 = \cos 2\theta$, we obtain

$$\cot^{-1}\left(\sqrt{\frac{1-x^2}{1+x^2}}\right) = \cot^{-1}\left(\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}\right) = \cot^{-1}(\tan\theta)$$

$$= \cot^{-1}\left(\cot\left(\frac{\pi}{2}-\theta\right)\right)$$

$$= \frac{\pi}{2}-\theta$$

2. Let $\cos^{-1}\frac{1}{8} = \theta$, where $0 < \theta < \frac{\pi}{2}$. Then

$$\frac{1}{2}\cos^{-1}\frac{1}{8} = \frac{1}{2}\theta$$

$$\Rightarrow \quad \cos\left(\frac{1}{2}\cos^{-1}\frac{1}{8}\right) = \cos\frac{1}{2}\theta$$

Now $$\cos^{-1}\frac{1}{8} = \theta$$

$$\Rightarrow \quad \cos\theta = \frac{1}{8} \Rightarrow 2\cos^2\frac{\theta}{2} - 1 = \frac{1}{8}$$

$$\Rightarrow \quad \cos^2\frac{\theta}{2} = \frac{9}{16}$$

$$\Rightarrow \quad \cos\frac{\theta}{2} = \frac{3}{4}$$

$$\left[\because 0 < \frac{\theta}{2} < \frac{\pi}{4}, \text{ so } \cos\frac{\theta}{2} \neq -\frac{3}{4}\right]$$

3. $\sin^{-1}(\sin 5) > x^2 - 4x$

$$\Rightarrow \quad \sin^{-1}(\sin(5-2\pi) > x^2 - 4x$$

$$\Rightarrow \quad 5 - 2\pi > x^2 - 4x$$

$$\Rightarrow \quad 9 - 2x > (x-2)^2$$

$$\Rightarrow \quad (x-2)^2 < 9 - 2\pi$$

$$\Rightarrow \quad -\sqrt{9-2\pi} < x - 2 < \sqrt{9-2\pi}$$

$$\Rightarrow \quad 2 - \sqrt{9-2\pi} < x < 2 + \sqrt{9-2\pi}$$

4. We have, $\sin^{-1}\left\{\left(\sin\frac{\pi}{3}\right)\frac{x}{\sqrt{(x^2+k^2-kx)}}\right\}$

$$- \cos^{-1}\left\{\left(\cos\frac{\pi}{6}\right)\frac{x}{\sqrt{(x^2+k^2-kx)}}\right\}$$

$$= \sin^{-1}\left\{\frac{\sqrt{3}x}{2\sqrt{x^2+k^2-kx}}\right\} - \cos^{-1}\left\{\frac{\sqrt{3}x}{2\sqrt{x^2+k^2-kx}}\right\}$$

$$= \frac{\pi}{2} - 2\cos^{-1}\left\{\frac{\sqrt{3}x}{\sqrt{4x^2-4kx+4k^2}}\right\}$$

$$= \frac{\pi}{2} - \cos^{-1}\left\{\frac{6x^2}{4x^2-4kx+4k^2} - 1\right\}$$

$$= \sin^{-1}\left\{\frac{2x^2+4xk-4k^2}{4x^2-4xk+4k^2}\right\}$$

$$= \sin^{-1}\left\{\frac{x^2+2xk-2k^2}{2x^2-2xk+2k^2}\right\}$$

5. $0 \le x \le 1 \Rightarrow 0 \le \tan^{-1}x \le \frac{\pi}{4}$

$$\Rightarrow \quad \frac{-\pi}{4} \le -\tan^{-1}x \le 0$$

$$\Rightarrow \quad 0 \le \frac{\pi}{4} - \tan^{-1}x \le \frac{\pi}{4}$$

6. $T_n = \cot^{-1}\left(n^2+\frac{3}{4}\right) = \tan^{-1}\left(\frac{1}{n^2+\frac{3}{4}}\right)$

$$= \tan^{-1}\left(n+\frac{1}{2}\right) - \tan^{-1}\left(n-\frac{1}{2}\right)$$

$$S_n = \sum_{n=1}^{n} t_n = \tan^{-1}\left(n+\frac{1}{2}\right) - \tan^{-1}\frac{1}{2}$$

$$S_\infty = \tan^{-1}(\infty) - \tan^{-1}\frac{1}{2} = \frac{\pi}{2} - \tan^{-1}\frac{1}{2} = \cot^{-1}\frac{1}{2} = \tan^{-1}2$$

7. We have, $\cot^{-1}x + \sin^{-1}\frac{1}{\sqrt{5}} = \frac{\pi}{4}$

$$\Rightarrow \quad \tan^{-1}\frac{1}{x} + \tan^{-1}\frac{1/\sqrt{5}}{\sqrt{1-\frac{1}{5}}} = \frac{\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}\frac{1}{x} + \tan^{-1}\frac{1}{2} = \tan^{-1}1$$

$$\Rightarrow \quad \tan^{-1}\frac{1}{x} = \tan^{-1}1 - \tan^{-1}\frac{1}{2}$$

$$\Rightarrow \quad \tan^{-1}\frac{1}{x} = \tan^{-1}\left(\frac{1-\frac{1}{2}}{1+1\cdot\frac{1}{2}}\right)$$

$$\Rightarrow \quad \tan^{-1}\frac{1}{x} = \tan^{-1}\left(\frac{1}{3}\right)$$

$$\Rightarrow \quad x = 3$$

8. $(\cot^{-1}x)^2 - (5\cot^{-1}x) + 6 > 0$

$$(\cot^{-1}x - 3)(\cot^{-1}x - 2) > 0$$

$$x > \cot 2 \text{ and } x < \cot 3$$

$$\therefore \quad x \in (-\infty, \cot 3) \cup (\cot 2, \infty)$$

9. $\lim_{n\to\infty}\sum_{r=1}^{n}\tan^{-1}\left(\frac{2^{r-1}}{1+2^{2r-1}}\right) = \lim_{n\to\infty}\sum_{r=1}^{n}\tan^{-1}\left(\frac{2^r-2^{r-1}}{1+2^r\cdot 2^{r-1}}\right)$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\{\tan^{-1}(2^r) - \tan^{-1}(2^{r-1})\}$$

$$= \lim_{n\to\infty}(\tan^{-1}2^n - \tan^{-1}2^\circ)$$

$$= \tan^{-1}2^\infty - \tan^{-1}1$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

10. $x+\frac{1}{x}=2$

$$\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)=0$$

$$\frac{x-1}{\sqrt{x}}=0$$

$$x=1$$

$\therefore \quad \sin^{-1}x=\frac{\pi}{2}$

11. $\tan^{-1}\left(\frac{\tan x}{4}\right)+\tan^{-1}\left(\frac{\frac{6\tan x}{1+\tan^2 x}}{5+\frac{3(1-\tan^2 x)}{1+\tan^2 x}}\right)$

$\Rightarrow \tan^{-1}\left(\frac{\tan x}{4}\right)+\tan^{-1}\left(\frac{3\tan x}{4+\tan^2 x}\right)$

$$=\tan^{-1}\left(\frac{\frac{\tan x}{4}+\frac{3\tan x}{4+\tan^2 x}}{1-\frac{3\tan^2 x}{4(4+\tan^2 x)}}\right)$$

$$=\tan^{-1}\left(\frac{16\tan x+\tan^3 x}{16+\tan^2 x}\right)$$

$$=\tan^{-1}(\tan x)=x$$

12. $\left(\frac{\pi}{2}-\cos^{-1}x\right)+\left(\frac{\pi}{2}-\cos^{-1}y\right)=\frac{2\pi}{3}$

$\Rightarrow \quad \cos^{-1}x+\cos^{-1}y=\frac{\pi}{3}$

13. $\cot\left[\frac{\pi}{4}-\cot^{-1}\frac{3^2-1}{2\times 3}\right]$

$$=\cot\left[\frac{\pi}{4}-\cot^{-1}\frac{4}{3}\right]$$

$$=\frac{\cot\frac{\pi}{4}\cdot\frac{4}{3}+1}{\frac{4}{3}-\cot\frac{\pi}{4}}=\frac{\frac{4}{3}+1}{\frac{4}{3}-1}=7$$

14. $\sin\left[\tan^{-1}\frac{1-\tan^2\theta}{2\tan\theta}+\cos^{-1}\frac{1-\tan^2\theta}{1+\tan^2\theta}\right]$

[Put $x=\tan\theta$]

$\Rightarrow \sin[\tan^{-1}(\cot 2\theta)+\cos^{-1}(\cos 2\theta)]$

$$=\sin\left[\tan^{-1}\left(\tan\left(\frac{\pi}{2}-\theta\right)\right)+2\theta\right]$$

$$=\sin\left(\frac{\pi}{2}-2\theta+2\theta\right)=1$$

15. $2\tan^{-1}x=\cos^{-1}\frac{1-x^2}{1+x^2}$

$\Rightarrow 2\tan^{-1}a-2\tan^{-1}b=2\tan^{-1}x$

$\Rightarrow \quad \frac{a-b}{1+ab}=x$

16. $\frac{-\pi}{3}<\cos^{-1}\frac{1-x^2}{1+x^2}<\frac{\pi}{3}$

$\Rightarrow \quad 0\le\cos^{-1}\frac{1-x^2}{1+x^2}<\frac{\pi}{3}$

$\Rightarrow \quad \frac{1}{2}<\frac{1-x^2}{1+x^2}\le 1$

$\Rightarrow \quad 0\le x^2<\frac{1}{3}$

$\Rightarrow \quad x\in\left(\frac{-1}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}}\right)$

17. $\cos^{-1}\left[\cot\left(\sin^{-1}\sqrt{\frac{2-\sqrt{3}}{4}}+\cos^{-1}\frac{\sqrt{3}}{2}+\cos^{-1}\frac{1}{\sqrt{2}}\right)\right]$

$\Rightarrow \cos^{-1}\left[\cot\left(\sin^{-1}\sqrt{\frac{2-\sqrt{3}}{4}}+\cos^{-1}\sqrt{\frac{2-\sqrt{3}}{4}}\right)\right]$

$$=\cos^{-1}\left(\cot\frac{\pi}{2}\right)=\frac{\pi}{2}$$

18. $\tan\left(\tan^{-1}\frac{x}{\pi}\right)<\tan\frac{\pi}{3}$

$\Rightarrow \quad \frac{x}{\pi}<\sqrt{3}$

$\Rightarrow \quad x<\sqrt{3}\pi=5.5$. Maximum value is 5.

19. $\frac{\sqrt{(1+x^2)}-\sqrt{(1-x^2)}}{\sqrt{(1+x^2)}+\sqrt{(1-x^2)}}=\frac{\tan\alpha}{1}$

Now, use 'C' and 'D' $\Rightarrow\frac{1+x^2}{1-x^2}=\frac{\tan^2\left(\frac{\pi}{4}+\alpha\right)}{1}$

Again, use 'C' and 'D' $=x^2=-\cos\left(\frac{\pi}{2}+2\alpha\right)=\sin 2\alpha$

20. $\tan^{-1}x+\cot^{-1}y=\tan^{-1}3\Rightarrow\cot^{-1}y=\tan^{-1}3-\tan^{-1}x$

$\Rightarrow \quad \tan^{-1}\left(\frac{1}{y}\right)=\tan^{-1}\frac{3-x}{1+3x}$

$\Rightarrow \quad y=\frac{1+3x}{3-x}$

Hence $x=1, 2$, and $y=2\cdot 7$

21. $\text{cosec}^{-1}(\text{cosec}\ x)=x\ \forall\ x\in R=-(-1,1)$

Also, range of $\text{cosec}^{-1}(\text{cosec}\ x)\in\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$

So, combining these two, we get

$$x\in\left[-\frac{\pi}{2},-1\right]\cup\left[1,\frac{\pi}{2}\right]$$

22. $\lim\limits_{|x|\to\infty}\cos(\tan^{-1}(\sin(\tan^{-1}x)))$

$$=\cos(\tan^{-1}(\sin(\tan^{-1}\infty)))=\cos\left(\tan^{-1}\left(\sin\left(\frac{\pi}{2}\right)\right)\right)$$

$$=\cos(\tan^{-1}(1))=\cos\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$

23. $[\cot^{-1}x] + 2[\tan^{-1}x] = 0$

$\Rightarrow \quad [\cot^{-1}x] = 0, [\tan^{-1}x] = 0$

or $\quad [\cot^{-1}x] = 2, [\tan^{-1}x] = -1$

Now, $\quad [\cot^{-1}x] = 0 \Rightarrow x \in (\cot 1, \infty)$

$[\tan^{-1}x] = 0 \Rightarrow x \in (0, \tan 1)$

Therefore, $\quad [\cot^{-1}x] = [\tan^{-1}x] = 0,\ x \in (\cot 1, \tan 1)$

$[\cot^{-1}x] = 2 \Rightarrow x \in (\cot 3, \cot 2]$

$[\tan^{-1}x] = -1 \Rightarrow x \in [-\tan 1, 0)$

Hence, no such x exists.

Thus, the solution set is $(\cot 1, \tan 1)$

24. Let $\sin^{-1}x = \theta \Rightarrow x = \sin\theta, \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$

Now, $\cos^{-1}x = \cos^{-1}(\sin\theta) = \cos^{-1}\left(-\cos\left(\frac{3\pi}{2} - \theta\right)\right)$

$$= \pi - \cos^{-1}\left(\cos\left(\frac{3\pi}{2} - \theta\right)\right)$$

$$= \pi - \left(\frac{3\pi}{2} - \theta\right), \text{ as } 0 \le \frac{3\pi}{2} - \theta \le \pi$$

$$= \theta - \frac{\pi}{2} = \sin^{-1}x - \frac{\pi}{2}$$

Hence, $\sin^{-1}x + \cos^{-1}x = 2\sin^{-1}x - \frac{\pi}{2}$

25. $a\sin^{-1}x - b\cos^{-1}x = c$

We have, $b\sin^{-1}x + b\cos^{-1}x = \frac{b\pi}{2}$

Adding $(a + b)\sin^{-1}x = \frac{b\pi}{2} + c$

or $\quad \sin^{-1}x = \frac{\left(\frac{b\pi}{2}\right) + c}{a + b} = \frac{b\pi + 2c}{2(a + b)}$

$\therefore \quad \cos^{-1}x = \frac{\pi}{2} - \frac{b\pi + 2c}{2(a + b)} = \frac{\pi a - 2c}{2(a + b)}$

$\Rightarrow \ a\sin^{-1}x + b\cos^{-1}x = \frac{\pi ab + c(a - b)}{a + b}$

26. $\sin^{-1}|x - 2| + \cos^{-1}(1 - |3 - x|) = \frac{\pi}{2}$

or $\quad |x - 2| = 1 - |3 - x|$

$|x - 2| + |3 - x| = 1$

or $\quad |x - 2| + |3 - x| = |(x - 2) + (3 - x)|$

or $\quad (x - 2)(3 - x) \ge 0$

or $\quad 2 \le x \le 3$

27. According to the question,

$$\sin^{-1}\frac{2}{\sqrt{5}} + \sin^{-1}\frac{3}{\sqrt{10}} + \sin^{-1}\alpha = \pi$$

$$\Rightarrow \quad \tan^{-1}2 + \tan^{-1}3 + \tan^{-1}\left(\frac{\alpha}{\sqrt{1 - \alpha^2}}\right) = \pi$$

$$\Rightarrow \quad \pi + \tan^{-1}\frac{2 + 3}{1 - (2)(3)} + \tan^{-1}\left(\frac{\alpha}{\sqrt{1 - \alpha^2}}\right) = \pi$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{\alpha}{\sqrt{1 - \alpha^2}}\right) = \frac{\pi}{4}$$

$$\Rightarrow \quad \frac{\alpha}{\sqrt{1 - \alpha^2}} = 1 \Rightarrow \alpha = \frac{1}{\sqrt{2}}$$

28. Expanding, we have

$(\tan^{-1}x)^3 + (\tan^{-1}2x)^3 + (\tan^{-1}3x)^3$

$= 3\tan^{-1}x\tan^{-1}2x\tan^{-1}3x$

$\Rightarrow \quad x = 0$

29. Let $f(x) = x^3 + bx^2 + cx + 1$

$f(0) = 1 > 0,\ f(-1) = b - c < 0$

So, $\quad \alpha \in (-1, 0)$

So, $2\tan^{-1}(\text{cosec}\,\alpha) + \tan^{-1}(2\sin\alpha\sec^2\alpha)$

$$= 2\tan^{-1}\left(\frac{1}{\sin\alpha}\right) + \tan^{-1}\left(\frac{2\sin\alpha}{1 - \sin^2\alpha}\right)$$

$$= 2\left[\tan^{-1}\left(\frac{1}{\sin\alpha}\right) + \tan^{-1}(\sin\alpha)\right]$$

$$= 2\left(-\frac{\pi}{2}\right) = -\pi \qquad (\text{as } \sin\alpha < 0)$$

30. Given, $u = \cot^{-1}\sqrt{\cos 2\theta} - \tan^{-1}\sqrt{\cos 2\theta}$,

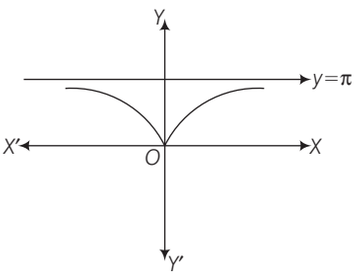
Put $\quad \cos 2\theta = \tan^2\theta$

$\Rightarrow \quad u = \cot^{-1}(\tan\theta) - \tan^{-1}(\tan\theta)$

$$= \frac{\pi}{2} - \theta - \theta = \frac{\pi}{2} - 2\theta$$

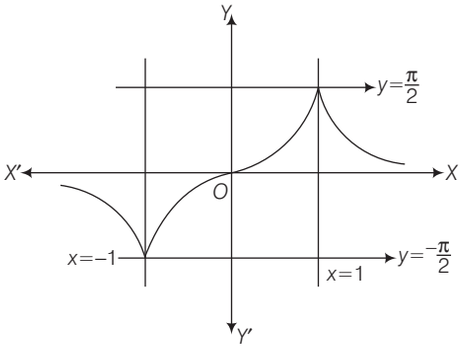
$\Rightarrow \quad \sin u = \sin\left(\frac{\pi}{2} - 2\theta\right) = \cos 2\theta = \tan^2\theta$

31.



Thus, $f(x)$ is not differentiable at $x = 0$

32.



From the graph, it is clear that, $f(x)$ is not differentiable at $x = 1, -1$

33. Given, $f(x) = \tan^{-1}(x^2 - 18x + a) > 0$

$\Rightarrow \quad \tan^{-1}(x^2 - 18x + a) > 0$

$\Rightarrow \quad x^2 - 18x + a > 0 \Rightarrow 18^2 - 4a < 0$

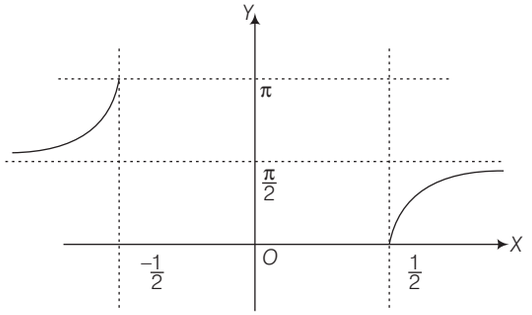
$\Rightarrow \quad a > \dfrac{18^2}{4} = \dfrac{18 \times 18}{4} = 81$

$\Rightarrow \quad a > 81 \Rightarrow a \in (81, \infty)$

34. $f(x) = \sin^{-1}2x + \cos^{-1}2x + \sec^{-1}2x$

$$f(x) = \frac{\pi}{2} + \sec^{-1}2x$$

Graph of $\sec^{-1}2x$ is as following



$(f(x))_{\text{minimum}} = \dfrac{\pi}{2} + 0 = \dfrac{\pi}{2}$

$(f(x))_{\text{maximum}} = \dfrac{\pi}{2} + \pi = \dfrac{3\pi}{2}$

$$\text{Sum} = \frac{\pi}{2} + \frac{3\pi}{2} = 2\pi$$

35. Given, $\tan^{-1}\dfrac{\left[\dfrac{b}{c+a} + \dfrac{c}{a+b}\right]}{1 - \dfrac{b}{c+a}\cdot\dfrac{c}{a+b}} = \dfrac{\pi}{4}$

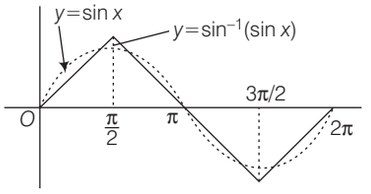
$\Rightarrow \quad \dfrac{ab + b^2 + c^2 + ac}{ac + bc + a^2 + ab - bc} = 1$

$\Rightarrow \quad ab + b^2 + c^2 + ac = ac + a^2 + ab$

$$b^2 + c^2 = a^2$$

$\therefore \Delta ABC$ is right angled at A

36. By graph, clearly it has 5 real roots



37. $-1 \le x \le 1, y > 0$

Both will be equated for infinite values of x and y. Therefore infinite many solutions.

38. $y = 1 + 2\sin\left(\dfrac{e^x}{e^x + 1}\right), x \ge 0$

$\Rightarrow \quad y - 1 = 2\sin\left(\dfrac{e^x}{e^x + 1}\right)$

$\Rightarrow \quad \dfrac{e^x}{e^x + 1} = \sin^{-1}\left(\dfrac{y-1}{2}\right)$

$\Rightarrow \quad e^x\left(\sin^{-1}\left(\dfrac{y-1}{2}\right) - 1\right) = -\sin^{-1}\left(\dfrac{y-1}{2}\right)$

$\Rightarrow \quad e^x = \dfrac{\sin^{-1}\left(\dfrac{y-1}{2}\right)}{1 - \sin^{-1}\left(\dfrac{y-1}{2}\right)}$

$$x = \log\left(\frac{\sin^{-1}\left(\dfrac{y-1}{2}\right)}{1 - \sin^{-1}\left(\dfrac{y-1}{2}\right)}\right)$$

$$f^{-1}(x) = \log\left(\frac{\sin^{-1}\left(\dfrac{x-1}{2}\right)}{1 - \sin^{-1}\left(\dfrac{x-1}{2}\right)}\right)$$

39. $\cos^{-1}(\cos(2\cot^{-1}(\sqrt{2} - 1)))$

$= \cos^{-1}(\cos(2(67 \cdot 5°)))$

$= \cos^{-1}(\cos(135°))$

$= 135° = \dfrac{3\pi}{4}$

40. $f(x) = \tan^{-1}\left(\dfrac{(\sqrt{12} - 2)x^2}{x^4 + 2x^2 + 3}\right) = \tan^{-1}\left(\dfrac{2(\sqrt{3} - 1)}{x^2 + \dfrac{3}{x^2} + 2}\right)$

As $x^2 + \dfrac{3}{x^2} \ge 2\sqrt{3}$ [using AM $\ge$ GM]

$\Rightarrow \quad x^2 + \dfrac{3}{x^2} + 2 \ge 2 + 2\sqrt{3}$

$\therefore \quad (f(x))_{\max} = \tan^{-1}\left(\dfrac{2(\sqrt{3} - 1)}{2(\sqrt{3} + 1)}\right) = \dfrac{\pi}{12}$

41. $\tan^{-1}\dfrac{\sqrt{1 + x^2} - 1}{x} = 4°$ $\qquad (x \ne 0)$

$\Rightarrow \quad \dfrac{\sqrt{1 + x^2} - 1}{x} = \tan 4°$

$\Rightarrow \quad \sqrt{1 + x^2} = 1 + x\tan 4°$

$\Rightarrow \quad 1 + x^2 = 2x\tan 4° + 1 + x^2\tan^2 4°$

$\Rightarrow \quad x = 0 \text{ or } x = \dfrac{2\tan 4°}{1 - \tan^2 4°} = \tan 8°$

Since $x \ne 0$, we have $x = \tan 8°$

42. From the given equation $\sin^2\theta - 2\sin\theta + 3 = 5^{\sec^2 y} + 1$,

we get $\qquad (\sin\theta - 1)^2 + 2 = 5^{\sec^2 y} + 1$

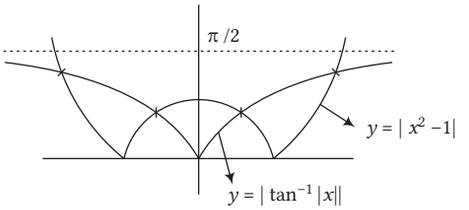
LHS ≤ 6, RHS ≥ 6

Possible solution is $\sin\theta = -1$ when L.H.S. = R.H.S.

$\Rightarrow \quad \cos^2\theta = 0$

$\Rightarrow \quad \cos^2\theta - \sin\theta = 1$

43. $y = |x^2 - 1| = |\tan^{-1}|x||$



From graph, clearly it has 4 solutions.

44. $1 + \tan^2(\tan^{-1}x) - (\sec^2(\sec^{-1}x) - 1)$

$= 1 + (\tan(\tan^{-1}x))^2 - (\sec(\sec^{-1}x))^2 + 1$

$= 1 + x^2 - x^2 + 1 = 2$

45. The equation $3x^2 + 6x + a = 0$ must have equal roots

So, $D = 0$

$\Rightarrow \quad 36 - 12a = 0 \Rightarrow a = 3$

46. $\tan^{-1}\frac{1}{\sqrt{2}} + \sin^{-1}\frac{1}{\sqrt{5}} - \cos^{-1}\frac{1}{\sqrt{10}}$

$= \tan^{-1}\frac{1}{\sqrt{2}} + \tan^{-1}\frac{1}{2} - \tan^{-1}3$

$= \tan^{-1}\frac{1}{\sqrt{2}} - \left[\tan^{-1}3 - \tan^{-1}\frac{1}{2}\right]$

$\Rightarrow \quad \tan^{-1}\frac{1}{\sqrt{2}} - \left[\tan^{-1}\frac{3-\frac{1}{2}}{1+\frac{3}{2}}\right] = \tan^{-1}\frac{1}{\sqrt{2}} - \tan^{-1}1$

$= -\left[\tan^{-1}1 - \tan^{-1}\frac{1}{\sqrt{2}}\right]$

$= -\left[\tan^{-1}\frac{1-\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}}\right] = -\tan^{-1}\frac{\sqrt{2}-1}{\sqrt{2}+1}$

$= -\cot^{-1}\frac{\sqrt{2}+1}{\sqrt{2}-1} = -\left[\cot^{-1} - \left(\frac{1+\sqrt{2}}{1-\sqrt{2}}\right)\right]$

$= -\left[\pi - \cot^{-1}\frac{1+\sqrt{2}}{1-\sqrt{2}}\right] = -\pi + \cot^{-1}\frac{1+\sqrt{2}}{1-\sqrt{2}}$

47. $\alpha = 2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{4}{3}$

$= \tan^{-1}\frac{1}{1-\frac{1}{4}} + \tan^{-1}\frac{4}{3} = 2\tan^{-1}\frac{4}{3}$

(using $2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$)

Let $\theta = \tan^{-1}\frac{4}{3} \Rightarrow \tan\theta = \frac{4}{3}$.

Now, $\sec\alpha = \sec 2\theta = \frac{1+\tan^2\theta}{1-\tan^2\theta} = \frac{1+(16/9)}{1-(16/9)} = \frac{25}{-7}$

Hence, $\sec\theta = -\frac{25}{7}$

48. $\frac{2e+4}{3} > e \Rightarrow \ln\left(\frac{2e+4}{3}\right) > 1$

49. $\sin^{-1}\left(-\sin\frac{50\pi}{9}\right) = -\sin^{-1}\sin\left(\frac{50\pi}{9}\right) = -\sin^{-1}\sin\left(\frac{14\pi}{9}\right)$

$= -\sin^{-1}\sin\left(2\pi - \frac{4\pi}{9}\right)$

$= -\sin^{-1}\left(-\sin\left(\frac{4\pi}{9}\right)\right) = \frac{4\pi}{9}$

$\cos^{-1}\cos\left(-\frac{31\pi}{9}\right) = \cos^{-1}\cos\left(\frac{31\pi}{9}\right)$

$= \cos^{-1}\cos\left(4\pi - \frac{5\pi}{9}\right) = \cos^{-1}\cos\frac{5\pi}{9} = \frac{5\pi}{9}$

Hence, $\sec\left(\frac{4\pi}{9} + \frac{5\pi}{9}\right) = \sec\pi = -1$

50. $\tan\left\{\underbrace{\text{arc}\tan(2)}_{A} + \underbrace{\text{arc}\tan(20k)}_{B}\right\} = k;$

$\Rightarrow \quad \frac{\tan A + \tan B}{1 - \tan A\tan B} = k \Rightarrow \frac{2+20k}{1-(2)(20k)} = k$

or $\quad 40k^2 + 19k + 2 = 0$

Now, sum of solutions, $k_1 + k_2 = -\frac{19}{40}$

51. $\sum_{r=2}^{\infty}\tan^{-1}\left(\frac{(r-2)-(r-3)}{1+(r-3)(r-2)}\right) = \sum_{r=2}^{\infty}(\tan^{-1}(r-2) - \tan^{-1}(r-3))$

Now, $\quad T_2 = \tan^{-1}0 - \tan^{-1}(-1)$

$T_3 = \tan^1 1 - \tan^{-1}0$

$T_4 = \tan^{-1}2 - \tan^{-1}1$

$\vdots \qquad \vdots$

$T_n = \tan^{-1}(n-2) - \tan^{-1}(n-3)$

$\Rightarrow \quad S_n = \tan^{-1}(n-2) + \frac{\pi}{4}$

$\therefore \quad S_\infty = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$

52. $x = -\pi/4;\ y = \cos\frac{\theta}{2}$; where $\cos\theta = \frac{1}{8}$

and $\quad \cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}} = \frac{3}{4}$

53. $\tan^{-1}\left(\frac{1}{2}\tan 2A\right) + \tan^{-1}(\cot A) + \tan^{-1}(\cot^3 A)$

$= \tan^{-1}\left(\frac{1}{2}\tan 2A\right) + \tan^{-1}\left(\frac{\cot A + \cot^3 A}{1 - \cot^4 A}\right) + \pi$

$\left(0 < A < \frac{\pi}{4} \Rightarrow \cot A > 1\right)$

$= \tan^{-1}\left(\frac{\tan A}{1-\tan^2 A}\right) + \pi + \tan^{-1}\frac{\cot A(1+\cot^2 A)}{(1-\cot^2 A)(1+\cot^2 A)}$

$= \pi + \tan^{-1}\left(\frac{\tan A}{1-\tan^2 A}\right) + \tan^{-1}\left(\frac{\cot A}{1-\cot^2 A}\right)$

$= \pi = 4\tan^{-1}(1)$

54. $T_n = \tan^{-1}\left(\dfrac{(n+1)^2-(n-1)^2}{1+(n^2-1)^2}\right) = \tan^{-1}(n+1)^2 - \tan^{-1}(n-1)^2$

$S_n = \tan^{-1}(n+1)^2 + \tan^{-1}n^2 - \tan^{-1}1^2$

$\Rightarrow S_\infty = \dfrac{\pi}{2}+\dfrac{\pi}{2}-\dfrac{\pi}{4} = \pi - \dfrac{\pi}{4}$

$= \pi - \sec^{-1}(\sqrt{2})$

$= \sec^{-1}(-\sqrt{2})$

55. $\dfrac{\pi}{2} - \cos^{-1}(1-x) + 2\cos^{-1}x = 0$

$\sin^{-1}(1-x) + 2\cos^{-1}x = 0$

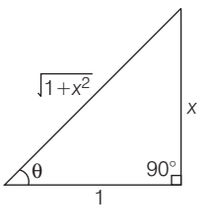
Domain is $[0, 1]$

Now, in $[0, 1]$, $\sin^{-1}(1-x) \in \left[0, \dfrac{\pi}{2}\right]$ and $2\cos^{-1}x \in [0, \pi]$

$\left.\begin{array}{l}\text{Hence, } \sin^{-1}(1-x) = 0\\ \text{and } \quad \cos^{-1}x = 0\end{array}\right] \Rightarrow x = 1$

56. Let $\tan^{-1}(x) = \theta \Rightarrow x = \tan\theta$

$\cos\theta = x$ (given)



$\dfrac{1}{\sqrt{1+x^2}} = x$

$x^2(1+x^2) = 1$

$\Rightarrow \quad x^2 = \dfrac{-1 \pm \sqrt{5}}{2}$

$\Rightarrow \quad x^2 = \dfrac{\sqrt{5}-1}{2}$ (x^2 can not be $-$ ve) $\Rightarrow \dfrac{x^2}{2} = \dfrac{\sqrt{5}-1}{4}$

$\cos^{-1}\left(\dfrac{\sqrt{5}-1}{4}\right) = \cos^{-1}\left(\sin\dfrac{\pi}{10}\right) = \cos^{-1}\left(\cos\dfrac{2\pi}{5}\right) = \dfrac{2\pi}{5}$

57. $\sin\cos^{-1}(\cos(\tan^{-1}x)) = p$

For $x \in R$ $\tan^{-1}x \in (-\pi/2, \pi/2)$

$\cos(\tan^{-1}x) \in (0,1]$

$\cos^{-1}\cos(\tan^{-1}x) \in [0, \pi/2)$

$\sin(\cos^{-1}(\cos(\tan^{-1}x))) \in [0,1)$

58. $\cos^{-1}x = t \Rightarrow x \in [-1,1]$ and $t \in [0, \pi]$

Now, we have $\log_{10}\sqrt{5t-1} + \dfrac{1}{2}\log_{10}(2t+3) + \dfrac{1}{2}\log_{10}5 = 1$; $\left(t > \dfrac{1}{5} \text{ and } t > -\dfrac{3}{2}\right)$

$\Rightarrow \quad \log_{10}((5t-1)(2t+3).5) = 2$

$\Rightarrow \quad (5t-1)(2t+3).5 = 100$

$\Rightarrow \quad (5t-1)(2t+3) = 20$

$10t^2 + 13t - 3 = 20$

$\Rightarrow \quad 10t^2 + 23t - 10t - 23 = 0$

$t(10t+23) - (10t+23) = 0$

$(t-1)(10t+23) = 0 \Rightarrow t = 1$ or $t = -\dfrac{23}{10}$ (rejected)

$\cos^{-1}x = 1 \Rightarrow x = \cos 1$

59. $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$ ...(i)

and $\sin^{-1}x - \cos^{-1}x = \sin^{-1}(3x-2)$ ...(ii)

From Eqs. (i) and (ii), we get

$2\cos^{-1}x = \cos^{-1}(3x-2)$

Note that $(3x-2) \in [-1,1]$ i.e. $-1 \le 3x-2 \le 1$

$\Rightarrow \quad x \in \left[\dfrac{1}{3}, 1\right]$

Now, $2\cos^{-1}x = \cos^{-1}(3x-2) \Rightarrow \cos^{-1}(2x^2-1) = \cos^{-1}(3x-2)$

$\Rightarrow \quad 2x^2 - 3x + 1 = 0$

$\Rightarrow \quad x = 1$ or $\dfrac{1}{2}$

Hence, $x = 1$ or $\dfrac{1}{2}$

60. $\tan^2(\sin^{-1}x) > 1 \Rightarrow \tan(\sin^{-1}x) > 1$ or $\tan(\sin^{-1}x) < -1$

$\sin^{-1}x > \dfrac{\pi}{4}$ or $\sin^{-1}x < \dfrac{\pi}{4}$; $x \in (-1,1) - \left[-\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right]$

Note that domain is $(-1,1)$

61. $\sin^{-1}\sqrt{1-x^2} + \cos^{-1}x = \cot^{-1}\dfrac{\sqrt{1-x^2}}{x} - \sin^{-1}x$

or $\dfrac{\pi}{2} + \sin^{-1}\sqrt{1-x^2} = \cot^{-1}\dfrac{\sqrt{1-x^2}}{x}$

$\Rightarrow \tan^{-1}\dfrac{\sqrt{1-x^2}}{x} + \sin^{-1}\sqrt{1-x^2} = 0$

$\Rightarrow \quad -1 \le x < 0 \cup \{1\}$

62. Consider $\tan 65^\circ - 2\tan 40^\circ$

$= \tan(45^\circ + 20^\circ) - 2\tan 40^\circ$

$= \dfrac{1+\tan 20^\circ}{1-\tan 20^\circ} - \dfrac{4\tan 20^\circ}{1-\tan^2 20^\circ}$

$= \dfrac{(1+\tan 20^\circ)^2 - 4\tan 20^\circ}{(1-\tan 20^\circ)(1+\tan 20^\circ)} = \dfrac{(1-\tan 20^\circ)(1-\tan 20^\circ)}{(1-\tan 20^\circ)(1+\tan 20^\circ)}$

$= \dfrac{(1-\tan 20^\circ)}{(1+\tan 20^\circ)} = \tan(45^\circ - 20^\circ) = \tan 25^\circ$

$\therefore \quad \tan^{-1}(\tan 25^\circ) = 25^\circ$

63. We have $\cos^{-1}\dfrac{x}{a} + \cos^{-1}\dfrac{y}{b} = \dfrac{\pi}{2} + \theta$

$\Rightarrow \quad \dfrac{xy}{ab} - \sqrt{1-\dfrac{x^2}{a^2}}\sqrt{1-\dfrac{y^2}{b^2}} = -\sin\theta$

$\Rightarrow \quad \dfrac{xy}{ab} + \sin\theta = \sqrt{1-\dfrac{x^2}{a^2}}\sqrt{1-\dfrac{y^2}{b^2}}$

On squaring both sides, we get

$$\frac{x^2y^2}{a^2b^2}+\sin^2\theta+\frac{2xy}{ab}\sin\theta=1-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{x^2y^2}{a^2b^2}$$

$$\Rightarrow \quad b^2x^2+a^2y^2+2ab\ xy\sin\theta=a^2b^2\cos^2\theta\le a^2b^2$$

64. $a_n=\tan^{-1}\left(\frac{1}{1+(n+3)(n+2)}\right)=\tan^{-1}\left(\frac{(n+3)-(n+2)}{1+(n+3)(n+2)}\right)$

$$=\tan^{-1}(n+3)-\tan^{-1}(n+2)$$

$$\therefore \quad S_n=f(n)-f(1)=\tan^{-1}(n+3)-\tan^{-1}3$$

$$\Rightarrow \quad S_\infty=\frac{\pi}{2}-\tan^{-1}3=\cot^{-1}3=\sin^{-1}\frac{1}{\sqrt{10}}$$

65. $\because \tan^{-1}\left(\frac{1+x}{1-x}\right)=\begin{cases}\tan^{-1}(1)+\tan^{-1}x, & \text{if } x<1\\ -\pi+\tan^{-1}(1)+\tan^{-1}x, & \text{if } x>1\end{cases}$

Now, $f(x)=\begin{cases}\tan^{-1}(1), & \text{if } x<1\\ -\pi+\tan^{-1}(1), & \text{if } x>1\end{cases}$

$\therefore$ Range $f=\left\{\frac{\pi}{4},\frac{-3\pi}{4}\right\}$

66. We have $\frac{1}{2}\le\frac{x^2-k}{1+x^2}<1\Rightarrow\frac{1}{2}\le1-\frac{(k+1)}{x^2+1}<1,\ \forall\ x\in R$

$$\Rightarrow \quad k+1>0 \quad \text{and} \quad \frac{k+1}{x^2+1}\le\frac{1}{2}$$

So, $\quad k>-1$ and $x^2+1\ge2k+2$

$\Rightarrow\ x^2-(2k+1)\ge0,\ \forall\ x\in R\Rightarrow4(2k+1)\le0$

$$\therefore \quad k\le-\frac{1}{2}$$

Hence, $k\in\left(-1,\frac{-1}{2}\right]$

67. Domain is $x\in[-1,1]$

Given, $\sin^{-1}x=2\tan^{-1}x$

$$\Rightarrow \quad \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

$$\Rightarrow \quad \frac{x}{\sqrt{1-x^2}}=\frac{2x}{1-x^2}$$

$$\therefore \quad x=0 \text{ or } (1-x^2)^2=4(1-x^2)$$

$$\Rightarrow \quad (1-x^2)(3+x^2)=0$$

$$\Rightarrow \quad x=-1,\ 0,\ 1 \qquad \ldots(i)$$

$$\tan^{-1}\sqrt{x(x-1)}+\operatorname{cosec}^{-1}\sqrt{1+x-x^2}=\frac{\pi}{2}$$

$$x(x-1)\ge0\cap x-x^2\ge0\Rightarrow x(x-1)=0$$

$$\Rightarrow \quad x=0,\ 1 \qquad \ldots(ii)$$

Now, Eqs. (i) $\cap$ (ii) gives $x=0,1$

Hence, number of common solution are 2.

68. Let $\theta=\text{arc}\cos(x-1)$

Now, $\quad \cos3\theta=4\cos^3\theta-3\cos\theta$

So, $\quad 4y^3-3y=0,$ where $y=x-1$

$$\therefore \quad y=\pm\frac{\sqrt{3}}{2},\ 0 \ \Rightarrow\ x=1\pm\frac{\sqrt{3}}{2},\ 1$$

Hence, three values of x.

Aliter :

$$\cos(3\cos^{-1}(x-1))=0$$

$$\Rightarrow\ 3\cos^{-1}(x-1)=(2n+1)\frac{\pi}{2},\ n\in Z$$

$$\Rightarrow \quad \cos^{-1}(x-1)=(2n+1)\frac{\pi}{6},\ n\in Z$$

$$\Rightarrow \quad \cos^{-1}(x-1)=\frac{\pi}{6},\frac{\pi}{2},\frac{5\pi}{6}$$

$$\Rightarrow \quad x-1=\frac{\sqrt{3}}{2},\ 0,\ \frac{-\sqrt{3}}{2}$$

$$\therefore \quad x=1+\frac{\sqrt{3}}{2},1,1-\frac{\sqrt{3}}{2}$$

69. (a) $f(x)=\frac{1}{2}\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$$D_f=R$$

As, $0\le\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)<\pi$

$$\therefore \quad R_f=\left[0,\frac{\pi}{2}\right)$$

(b) $g(x)=\text{sgn}\left(x+\frac{1}{x}\right)$

$D_g=(-\infty,0)\cup(0,\infty)$

$R_g=\{-1,1\}$

(c) $h(x)=\sin^2x+2\sin x+2$

$D_h=R$

Also, $h(x)=(\sin x+1)^2+1$

$$\therefore \quad R_h=[1,5]$$

(d) $k(x)=\cos^{-1}(x^2-2x+2)=\cos^{-1}((x-1)^2+1)$

$$\therefore \quad D_k=\{1\}$$

$$R_k=\{0\}$$

70. $\therefore 0\le\tan^{-1}(3x^2+bx+3)<\frac{\pi}{2}$

$$\Rightarrow \quad 0\le3x^2+bx+3<\infty$$

Thus, range of $3x^2+bx+3$ is $[0,\infty)$

Now, $D=b^2-4\cdot3\cdot3=0\ \Rightarrow\ b^2=36\Rightarrow b=\pm6$

Sum of values of $b=0$

$\therefore$Square of sum of values of $b=0$

71. $\theta=\tan^{-1}\left(\tan\frac{5\pi}{4}\right)$ and $\phi=\tan^{-1}\left(-\tan\frac{2\pi}{3}\right)$

$$=\tan^{-1}\tan\left(\pi+\frac{\pi}{4}\right) \text{ and } \phi=\tan^{-1}\left(-\tan\left(\pi-\frac{\pi}{3}\right)\right)$$

$= \tan^{-1}\tan\frac{\pi}{4}$ and $\phi = \tan^{-1}\tan\frac{\pi}{3}$

$= \frac{\pi}{4}$ and $\phi = \frac{\pi}{3}$

$\therefore \quad 4\theta - 3\phi = 0$ and $\theta + \phi = \frac{7\pi}{12}$

72. $f\left(\frac{8\pi}{9}\right) = e^{\cos^{-1}\sin\left(\frac{8\pi}{9}+\frac{\pi}{3}\right)} = e^{\cos^{-1}\sin\frac{11\pi}{9}}$

$= e^{\cos^{-1}\cos\frac{13\pi}{18}} = e^{13\pi/18}$

and $f\left(-\frac{7\pi}{4}\right) = e^{\cos^{-1}\sin\left(\frac{-7\pi}{4}+\frac{\pi}{3}\right)}$

$= e^{\cos^{-1}\sin\left(\frac{-17\pi}{12}\right)}$

$= e^{\cos^{-1}\cos\left(\frac{\pi}{12}\right)} = e^{\pi/12}$

73. Let $\cos^{-1}\left(\frac{4}{5}\right) = \alpha$, that is $\cos\alpha = \frac{4}{5}$, so that

$$\tan\alpha = \sqrt{\left\{\left(\frac{5}{4}\right)^2 - 1\right\}} = \frac{3}{4}$$

and $\tan\left(\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{2}{3}\right) = \frac{\tan\alpha + \frac{2}{3}}{1 - \tan\alpha\cdot\frac{2}{3}}$

$$= \frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{2}{3}\cdot\frac{3}{4}} = \frac{17}{6} = \frac{a}{b} \qquad \text{(given)}$$

so, $a = 17, b = 6, a + b = 23$

$a - b = 11$ and $3b = a + 1$

74. $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, \forall\ x \in [-1, 1]$

75. Given, $\cos^{-1}x = \tan^{-1}x$

$\Rightarrow \quad x = \cos\theta = \tan\theta$

$\Rightarrow \quad \cos^2\theta = \sin\theta$

$\Rightarrow \sin^2\theta + \sin\theta - 1 = 0$

$\Rightarrow \quad \sin\theta = \frac{-1 \pm \sqrt{1+4}}{2}$

$\Rightarrow \quad \sin\theta = \frac{\sqrt{5}-1}{2}$

$\Rightarrow \quad x^2 = \cos^2\theta = \frac{\sqrt{5}-1}{2}$

$\therefore$ Option (a) is correct.

and $\sin(\cos^{-1}x) = \sin\theta = \frac{1}{2}(\sqrt{5}-1)$

Also, option (c) is correct

and $\tan(\cos^{-1}x) = \tan\theta \neq \frac{1}{2}(\sqrt{5}-1)$

$\therefore$ Option (d) is not correct.

76. The solution of $y = \sqrt{y}$ is $y = 0$ or $y = 1$

If $\sin^{-1}|\sin x| = 1 \Rightarrow x = 1$ or $x = \pi - 1$

But $y = \sin^{-1}|\sin x|$ is periodic with period π, so $x = n\pi + 1$ or $n\pi - 1$.

Again if $\sin^{-1}|\sin x| = 0$, $x = n\pi$

77. $(\sin^{-1}x + \sin^{-1}w)(\sin^{-1}y + \sin^{-1}z) = \pi^2$

$\therefore \quad \sin^{-1}x + \sin^{-1}w = \sin^{-1}y + \sin^{-1}z = \pi$

or $\quad \sin^{-1}x + \sin^{-1}w = \sin^{-1}y + \sin^{-1}z = -\pi$

$\therefore \quad x = y = z = w = 1$ or $x = y = z = w = -1$

Hence, the maximum value of $\begin{vmatrix} x^{N_1} & y^{N_2} \\ z^{N_3} & w^{N_4} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$ and minimum value $\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2$ Also, there are 16 different determinants as each place value is either 1 or -1.

78. Since, $|\tan^{-1}x| = \begin{cases} \tan^{-1}x, & \text{if } x \geq 0 \\ -\tan^{-1}x, & \text{if } x < 0 \end{cases}$

$\Rightarrow \quad |\tan^{-1}x| = \tan^{-1}|x|\ \forall\ x \in R$

$\Rightarrow \quad \tan|\tan^{-1}x| = \tan\tan^{-1}|x| = |x|$

Also, $\quad |\cot^{-1}x| = \cot^{-1}x;\ \forall\ x \in R$

$\Rightarrow \quad \cot|\cot^{-1}x| = x,\ \forall\ x \in R$

$$\tan^{-1}|\tan x| = \begin{cases} x, & \text{if } 0 < x < \frac{\pi}{2} \\ -x, & \text{if } -\frac{\pi}{2} < x < 0 \end{cases}$$

$$\sin|\sin^{-1}x| = \begin{cases} x, & x \in [0, 1] \\ -x, & x \in [-1, 0) \end{cases}$$

79. $1 \leq \frac{\pi}{\cos^{-1}x} < \infty \Rightarrow 2 \leq \frac{\pi}{2^{\cos^{-1}x}} < \infty$

Hence, 2 should lie between or on the roots of

$t^2 - \left(a + \frac{1}{2}\right)t - a^2 = 0$, where $t = 2^{\pi/\cos^{-1}x}$

$\Rightarrow \quad f(2) \leq 0 \ \Rightarrow\ a^2 + 2a - 3 \geq 0$

$\Rightarrow a \in (-\infty, -3] \cup [1, \infty)$

80. $E = \pi - 3 + \pi - 4 + 5 - 2\pi = -2$

81. $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$

$\Rightarrow \quad \sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{\pi}{2}$

Also, $\quad \cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$

$\Rightarrow \quad \cos^{-1}x + \cos^{-1}y = \cos^{-1}(-z)$

$\Rightarrow \quad xy - \sqrt{1-x^2}\sqrt{1-y^2} = -z$

$\Rightarrow \quad x^2 + y^2 + z^2 + 2xyz = 1$

Option (d) can be true only if $x, y, z > 0$; for (c) put

$x = y = z = 1/2$

82. Let $\tan^{-1}x = \alpha$ and $\tan^{-1}x^3 = \beta$

$\tan\alpha = x$ and $\tan\beta = x^3$

$\therefore 2\tan(\alpha+\beta)=\dfrac{2(\tan\alpha+\tan\beta)}{1-\tan\alpha\tan\beta}=2\left[\dfrac{x+x^3}{1-x^4}\right]=\dfrac{2x}{1-x^2}$

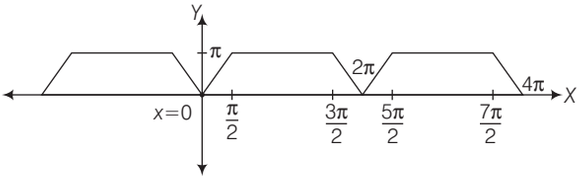
Also, $\dfrac{2x}{1-x^2}=\dfrac{2\tan\alpha}{1-\tan^2\alpha}=\tan 2\alpha=\tan(2\tan^{-1}x)$

$=\tan\left(2\left(\dfrac{\pi}{2}-\cot^{-1}x\right)\right)=\tan(\pi-\cot^{-1}x-\cot^{-1}x)$

$=\tan(\cot^{-1}(-x)-\cot^{-1}(x))$

83. $f(x)=\begin{cases} 2x & ; \quad 0\le x\le \dfrac{\pi}{2} \\ \pi & ; \quad \dfrac{\pi}{2}<x\le\dfrac{3\pi}{2} \\ 4\pi-2x & ; \quad \dfrac{3\pi}{2}<x\le 2\pi \end{cases}$

Clearly, $f(x)$ is periodic function with period 2π. The graph of $f(x)$ is shown below.



84. Domain of $\sin^{-1}x$ and $\cos^{-1}x$, each is $[-1,1]$ and that of $\sec^{-1}x$ and $\operatorname{cosec}^{-1}x$, each is $(-\infty,-1]\cup[1,\infty)$

$\therefore$ Domain of $f(x)$ must be $\{-1,1\}$

$\therefore$ Range of $f(x)$ will be $\{f(-1),f(1)\}$

where, $f(-1)=\sin^{-1}(-1)\cdot\cos^{-1}(-1)\cdot\tan^{-1}(-1)\cot^{-1}(-1)$

$\cdot\sec^{-1}(-1)\cdot\operatorname{cosec}^{-1}(-1)$

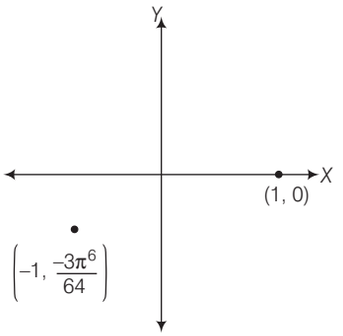
$=\left(\dfrac{-\pi}{2}\right).(\pi).\left(\dfrac{-\pi}{4}\right).\left(\dfrac{3\pi}{4}\right).(\pi).\left(\dfrac{-\pi}{2}\right)=\dfrac{-3\pi^6}{64}$

and $\quad f(1)=0$ {as $\cos^{-1}1=0$}

(i) Thus, the graph of $f(x)$ is a two point graph which doesn't lies above X-axis.

(ii) $f(x)_{\max}=0$ and $f(x)_{\min}=\dfrac{-3\pi^6}{64}$

Hence, $|f(x)_{\max}-f(x)_{\min}|=\dfrac{3\pi^6}{64}$



(iii) $f(x)$ is one-one. Hence, injective.

(iv) Domain is $\{-1,1\}$

$\therefore$ Number of non-negative integers in the domain of $f(x)$ is one.

85. $\alpha=3\tan^{-1}\left(\dfrac{\sqrt3}{5}\right)+3\tan^{-1}\left(\dfrac{\sqrt3}{2}\right)=3\left[\tan^{-1}\dfrac{\dfrac{\sqrt3}{5}+\dfrac{\sqrt3}{2}}{1-\dfrac{3}{10}}\right]=3\tan^{-1}\left(\dfrac{7\sqrt3}{7}\right)=\pi$

$\beta=4\left[\tan^{-1}7-\tan^{-1}\dfrac{3}{4}\right]=4\left[\tan^{-1}\dfrac{7-\dfrac{3}{4}}{1+\dfrac{21}{4}}\right]$

$=4\tan^{-1}\left(\dfrac{25}{25}\right)=\pi$

86. $f(x)=|\sin^{-1}x|+\cos^{-1}\left(\dfrac{1}{x}\right)$

Domain of $f(x)$ is $\{-1,1\}$

$f(1)=\dfrac{\pi}{2},\ f(-1)=\dfrac{3\pi}{2}$

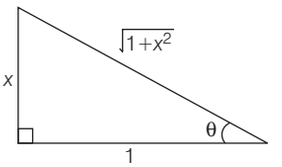
So, function $f(x)$ is injective .

$\operatorname{sgn}(f(x))=1\,(f(x)>0)$

Range of $\quad f(x)=\left\{\dfrac{\pi}{2},\dfrac{3\pi}{2}\right\}$

87. (a) $f(x)=\sin(\tan^{-1}x)$

Put $\tan^{-1}x=\theta\Rightarrow x=\tan\theta$



$\Rightarrow\ f(x)=\dfrac{x}{\sqrt{1+x^2}}=g(x)\ \Rightarrow$ identical functions

(b) $f(x)=\operatorname{sgn}(\cot^{-1}x)=1,\forall\in R$

But $g(x)=\sec^2x-\tan^{-1}x\neq 1,\forall\ x\in R$. (Think ? Domain of $g(x)$)

(c) As $\dfrac{x^2-1}{x^2+1}=\dfrac{x^2+1-2}{x^2+1}=1-\dfrac{2}{x^2+1}\in[-1,1)$

So, $\cos^{-1}\left(\dfrac{x^2-1}{x^2+1}\right)\in(0,\pi]$

So, $f(x)$ and $g(x)$ are identical functions.

(d) $f(x)=\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)=\begin{cases} -(\pi+2\tan^{-1}x) & , \quad x\le -1 \\ 2\tan^{-1}x & , \quad -1<x<1 \\ \pi-2\tan^{-1}x & , \quad x\ge 1 \end{cases}$

So, $f(x)$ and $g(x)$ are not identical functions.

88. $T_n=\cot^{-1}(n^2+n+1)=\tan^{-1}\left(\dfrac{(n+1)-n}{1+n(n+1)}\right)$

$=(\tan^{-1}(n+1)-\tan^{-1}n)$

So, $\sum\limits_{n=1}^{n}T_n=(\tan^{-1}2-\tan^{-1}1)+$

$(\tan^{-1}3-\tan^{-1}2)+.....+(\tan^{-1}(n+1)-\tan^{-1}n)$

$$= \tan^{-1}(n+1) - \tan^{-1}1$$

$$= \tan^{-1}\left[\frac{n+1-1}{1+(n+1)\cdot 1}\right] = \tan^{-1}\left(\frac{n}{n+2}\right)$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} T_n = \tan^{-1}1 = \frac{\pi}{4}$$

(a) $\cot^{-1}(-1) + \sec^{-1}1 - \operatorname{cosec}^{-1}1 = \frac{3\pi}{4} + 0 - \frac{\pi}{2}$

$$= \frac{3\pi - 2\pi}{4} = \frac{\pi}{4}$$

(b) $\cot^{-1}2 + \cot^{-1}3 = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{1}\left(\frac{1}{3}\right) = \frac{\pi}{4}$

(c) $\because \quad -1 < \frac{1-x^2}{1+x^2} \le 1$

$$\therefore \quad \frac{-\pi}{4} < \tan^{-1}\left(\frac{1-x^2}{1+x^2}\right) \le \frac{\pi}{4}$$

Hence, minimum value of $f(x)$ does not exist and maximum value of $f(x)$ is $\frac{\pi}{4}$

(d) As $\frac{41\pi}{4} = \left(10\pi + \frac{\pi}{4}\right)$

So, $\cos^{-1}\left(\cos\frac{41\pi}{4}\right) = \cos^{-1}\left(\cos\frac{\pi}{4}\right) = \frac{\pi}{4}$

89. We know that, $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$

$$= \begin{cases} \pi + 2\tan^{-1}x & , \quad x < -1 \\ 2\tan^{-1}x & , \quad -1 < x \le 1 \\ -(\pi - 2\tan^{-1}x) & , \quad x > 1 \end{cases}$$

$$\therefore \quad f(x) = \begin{cases} -\pi, & x < -1 \\ \pi, & x > 1 \end{cases}$$

So, $R_f = \{-\pi, \pi\}$

90. $P = \frac{-5}{15}$ and $Q = \frac{1}{15}$

Sol. (Q.Nos. 91-93)

$A = \{\sin^{-1}x\}$; $B = \{[x], \{x\}\}$; $C = \{[x], \sin^{-1}x\}$

$$f(x) = A \cap C = \sin^{-1}x$$

$$g(x) = B \cap C = [x]$$

$$h(x) = \{x\}$$

$|x|$ is a function which is neither odd, nor discontinuous, nor non-decreasing.

$\therefore \quad l(x)$ is $|x|$

91. $l(x) = |x|$ is an even function

92. $g(f(x)) = [\sin^{-1}x]$

$\because$ Range of $\sin^{-1}x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\therefore$ Range of $g(f(x)) = \{-2, -1, 0, 1\}$

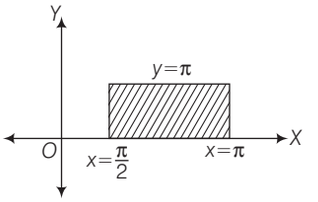
93. $f(h(x)) = \sin^{-1}\{x\}$

Domain is R and range is $\left[0, \frac{\pi}{2}\right)$.

94. We have, $f(10) = \sin^{-1}(\sin 10) + \cos^{-1}(\cos 10)$

$$= (3\pi - 10) + (4\pi - 10) = (7\pi - 20)$$

95. Clearly, $f(x) = (\pi - x) + x = \pi, \forall\ x \in \left[\frac{\pi}{2}, \pi\right]$



So, area $= \frac{\pi}{2} \times \pi = \frac{\pi^2}{2}$

96. As $\sin^{-1}(\sin x) = \begin{cases} x, & x \in \left[0, \frac{\pi}{2}\right) \\ \pi - x, & x \in \left[\frac{\pi}{2}, \pi\right) \end{cases}$

Also, $\cos^{-1}(\cos x) = x, \ x \in [0, \pi)$

Now, $f(x) = \begin{cases} 2x, & x \in \left[0, \frac{\pi}{2}\right) \\ \pi, & x \in \left[\frac{\pi}{2}, \pi\right) \end{cases}$

As, $f(x) \in I$ in $x \in (0,3) \Rightarrow x = \frac{1}{2}, 1, \frac{3}{2}$

So, number of values of x are three.

97. Given, $f(x) = \sqrt{\sin^{-1}x + 2} + \sqrt{1 - \sin^{-1}x}$

Clearly, for domain of $f(x)$, $1 - \sin^{-1}x \ge 0$

(As, $\sin^{-1}x + 2 > 0, \forall\ x \in [-1,1]$)

$\Rightarrow \quad \sin^{-1}x \le 1 \Rightarrow x \le \sin 1$

$\therefore \quad D_f = [-1, \sin 1]$

98. Given, $f(x) = \sqrt{\sin^{-1}x + 2} + \sqrt{1 - \sin^{-1}x}$, where $x \in [-1, \sin 1]$

Let $y = \sqrt{\sin^{-1}x + 2} + \sqrt{1 - \sin^{-1}x}$

Then, $y > 0, \forall\ x \in [-1, \sin 1]$

Now, $y^2 = (\sin^{-1}x + 2) + (1 - \sin^{-1}x) + 2\sqrt{(\sin^{-1}x + 2)(1 - \sin^{-1}x)}$

$$\Rightarrow \quad y^2 = 3 + 2\sqrt{\frac{9}{4} - \left(\sin^{-1}x + \frac{1}{2}\right)^2}$$

Clearly, $y^2_{\max}\left(\sin^{-1}x = \frac{-1}{2}\right) = 3 + 2\sqrt{\frac{9}{4} - 0} = 3 + 3 = 6$

$$\Rightarrow \quad y_{\max}\left(x = -\sin\frac{1}{2}\right) = \sqrt{6}$$

Also, $y^2_{\min}(\sin^{-1}x = 1) = 3 + 2\sqrt{\frac{9}{4} - \frac{9}{4}} = 3$

$$\Rightarrow \quad y_{\min}(x = \sin 1) = \sqrt{3}$$

Hence, range of $f = [\sqrt{3}, \sqrt{6}]$

99. $\sin^{-1}\left(\frac{4x}{x^2+4}\right)+2\tan^{-1}\left(-\frac{x}{2}\right)$

$$=\sin^{-1}\left(\frac{2\cdot\frac{x}{2}}{\left(\frac{x}{2}\right)^2+1}\right)-2\tan^{-1}\frac{x}{2}$$

$$=2\tan^{-1}\frac{x}{2}-2\tan^{-1}\frac{x}{2}=0$$

Here, $\left|\frac{x}{2}\right|\le 1$

$|x|\le 2 \Rightarrow -2\le x\le 2$

100. $\cos^{-1}\frac{6x}{1+9x^2}=-\frac{\pi}{2}+2\tan^{-1}3x$

$$\Rightarrow \quad \frac{\pi}{2}-\sin^{-1}\frac{6x}{1-9x^2}=\frac{-\pi}{2}+2\tan^{-1}3x$$

$$\Rightarrow \quad \sin^{-1}\frac{6x}{1+9x^2}=\pi-2\tan^{-1}3x$$

$$\Rightarrow \quad \sin^{-1}\frac{2\cdot 3x}{1+(3x)^2}=\pi-2\tan^{-1}3x$$

Above is true when $3x>1$

$\Rightarrow x>\frac{1}{3}$

$\Rightarrow x\in\left(\frac{1}{3},\infty\right)$

101. $(x-1)(x^2+1)>0 \Rightarrow x>1$

$$\therefore \quad \sin\left[\frac{1}{2}\tan^{-1}\left(\frac{2x}{1-x^2}\right)-\tan^{-1}x\right]$$

$$=\sin\left[\frac{1}{2}(-\pi+2\tan^{-1}x)-\tan^{-1}x\right]$$

$$=\sin\left(-\frac{\pi}{2}\right)=-1$$

Sol. (Q. Nos. 102 to 104)

$\sin^{-1}x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$\cos^{-1}y\in[0,\pi]$

$\sec^{-1}z\in\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$

$\Rightarrow \sin^{-1}x+\cos^{-1}y+\sec^{-1}z\le\frac{\pi}{2}+\pi+\pi=\frac{5\pi}{2}$

Also, $t^2-\sqrt{2\pi}t+3\pi=t^2-2\sqrt{\frac{\pi}{2}}t+\frac{\pi}{2}-\frac{\pi}{2}+3\pi$

$$=\left(t-\sqrt{\frac{\pi}{2}}\right)^2+\frac{5\pi}{2}\ge\frac{5\pi}{2}$$

The given inequation exists if equality holds, i.e.

$\text{LHS}=\text{RHS}=\frac{5\pi}{2}$

$\Rightarrow \quad x=1, y=-1, z=-1$ and $t=\sqrt{\frac{\pi}{2}}$

$\Rightarrow \quad \cos^{-1}(\cos 5t^2)=\cos^{-1}\left(\cos\left(\frac{5\pi}{2}\right)\right)=\frac{\pi}{2}$

$\cos^{-1}(\min\{x,y,z\})=\cos^{-1}(-1)=\pi$

105. $f(x)=\tan^{-1}\left(\frac{\left(\frac{x}{2}-1\right)}{1+x\left(\frac{x}{2}+1\right)}\right)$

$$=\tan^{-1}(x)-\tan^{-1}\left(\frac{x}{2}+1\right)$$

$$f'(x)=\frac{1}{(1+x^2)}-\frac{1}{\left(1+\left(\frac{x}{2}+1\right)^2\right)}\times\frac{1}{2}$$

$$f'(1)=\frac{1}{2}-\frac{1}{2\left(1+\frac{9}{7}\right)}=\frac{1}{2}-\frac{2}{13}=\frac{9}{26}$$

$\Rightarrow \quad 26f'(1)=9$

106. We have, $f(x)=(\tan^{-1}x)^3+(\cot^{-1}x)^3=(\tan^{-1}x+\cot^{-1}x)((\tan^{-1}x)^2-(\tan^{-1}x)(\cot^{-1}x)+(\cot^{-1}x)^2)$

$$=\frac{\pi}{2}\left((\tan^{-1}x)^2-(\tan^{-1}x)\left(\frac{\pi}{2}-\tan^{-1}x\right)+\left(\frac{\pi}{2}-\tan^{-1}1x\right)^2\right)$$

$$\left(\text{Using } \cot^{-1}x=\frac{\pi}{2}-\tan^{-1}x\right)$$

$$=\frac{3\pi}{2}\left(\left(\tan^{-1}x-\frac{\pi}{4}\right)^2+\frac{\pi^2}{48}\right)$$

Clearly, $f(x)$ will be minimum when $\left(\tan^{-1}x-\frac{\pi}{4}\right)^2=0$

and $f(x)$ will be maximum when $\left(\tan^{-1}x-\frac{\pi}{4}\right)^2=\left(-\frac{\pi}{2}-\frac{\pi}{4}\right)^2$

$$\therefore \quad a=f(x)_{\min}=\frac{3\pi}{2}\left(0+\frac{\pi^2}{48}\right)=\frac{\pi^3}{32}$$

and $\quad b=f(x)_{\max}=\frac{3\pi}{2}\left(\left(\frac{-3\pi}{4}\right)^2+\frac{\pi^2}{48}\right)=\frac{7\pi^3}{8}$

Hence, $\quad \frac{b}{a}=\frac{\frac{7\pi^3}{8}}{\frac{7\pi^3}{32}}=4$

107. We have, $T_n=2\operatorname{arc\,cot}\left(\frac{n^2+n+4}{2}\right)=2\tan^{-1}\left(\frac{2}{n^2+n+4}\right)$

$$T_n=2\tan^{-1}\left(\frac{\frac{1}{2}}{1+\frac{n(n+1)}{2\,.\,2}}\right)=2\left(\tan^{-1}\left(\frac{n+1}{2}\right)-\tan^{-1}\left(\frac{n}{2}\right)\right)$$

Hence, $S_n=\sum_{n=0}^{n}T_n=2\sum_{n=0}^{n}\left(\tan^{-1}\left(\frac{n+1}{2}\right)-\tan^{-1}\left(\frac{n}{2}\right)\right)$

$$S_n = \tan^{-1}\left(\frac{1}{2}\right) - \tan^{-1}(0)$$

$$\tan^{-1}(1) - \tan^{-1}\left(\frac{1}{2}\right)$$

$$\tan^{-1}\left(\frac{n+1}{2}\right) - \tan^{-1}\left(\frac{n}{2}\right)$$

$$\Rightarrow \quad S_n = 2\tan^{-1}\left(\frac{n+1}{2}\right)$$

$$\therefore \quad \lim_{n\to\infty} S_n = 2\cdot\left(\frac{\pi}{2}\right) = \pi = k\pi \text{ (given)} \Rightarrow k = 1$$

108. As, $\sum_{r=1}^{5}\cot^{-1}(2r^2) = \sum_{r=1}^{5}\tan^{-1}\left(\frac{2}{4r^2}\right)$

$$= \sum_{r=1}^{5}\tan^{-1}\left(\frac{(2r+1)(2r-1)}{1+(2r+1)(2r-1)}\right)$$

$$= \sum_{r=1}^{5}(\tan^{-1}(2r+1) - \tan^{-1}(2r-1))$$

$$= \tan^{-1}11 - \tan^{-1}1 = \tan^{-1}\left(\frac{11-1}{1+11\times 1}\right)$$

$$= \tan^{-1}\left(\frac{10}{12}\right) = \tan^{-1}\left(\frac{5}{6}\right)$$

$$\therefore \quad \tan\left(\sum_{r=1}^{5}\cot^{-1}(2r^2)\right) = \tan\left(\tan^{-1}\frac{5}{6}\right) = \frac{5}{6}$$

Now, $\tan\left(\sum_{r=1}^{5}\cot^{-1}(2r^2)\right) = \frac{5x+6}{6x+5} \Rightarrow \frac{5}{6} = \frac{5x+6}{6x+5}$

$\Rightarrow$ The given equations has no real solution.

109. $\min(x^2 + 4x + 7) = 3$

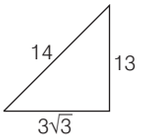
$$\max\left\{\frac{\pi^2}{4}, 3\right\} = 3$$

$$\lim_{z\to 0}\left[3\cdot\frac{\sin^{-1}z}{z}\right] = 3$$

110. $\sin\left(\frac{\pi}{6} + \tan^{-1}x\right) = \frac{13}{14}$

$$\frac{\pi}{6} + \tan^{-1}x = \sin^{-1}\left(\frac{13}{14}\right) = \tan^{-1}\left(\frac{13}{3\sqrt{3}}\right)$$

$$\tan^{-1}x = \tan^{-1}\left(\frac{13}{3\sqrt{3}}\right) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$



$$= \tan^{-1}\left[\frac{\frac{13}{3\sqrt{3}} - \frac{1}{\sqrt{3}}}{1+\frac{13}{9}}\right] = \tan^{-1}\left[\frac{10}{3\sqrt{3}}\left(\frac{9}{22}\right)\right]$$

$$\tan^{-1}x = \tan^{-1}\left[\frac{5\sqrt{3}}{11}\right]$$

$$\therefore \quad x = \frac{5\sqrt{3}}{11} \Rightarrow \frac{a\sqrt{3}}{b}$$

$$\Rightarrow \quad a + b = 16 \Rightarrow \left(\frac{a+b}{2}\right) = 8$$

Aliter : $\sin(30^\circ + \tan^{-1}x) = \frac{13}{14}$

$$\Rightarrow \frac{1}{2}\cos(\tan^{-1}x) + \frac{\sqrt{3}}{2}\sin(\tan^{-1}x) = \frac{13}{14}$$

$$\Rightarrow \quad 7(1+\sqrt{3}x) = 13\sqrt{1+x^2}$$

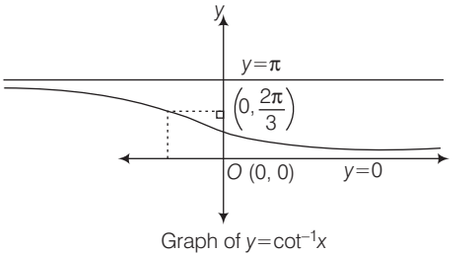
$$\Rightarrow \quad 11x^2 - 49\sqrt{3}x + 60 = 0 \qquad \text{(On squaring)}$$

$$\therefore \quad x = \frac{49\sqrt{3} \pm 30\sqrt{3}}{22}$$

$$= \frac{5\sqrt{3}}{11}, 4\sqrt{3} \qquad \text{(Reject)}$$

$$\therefore \quad x = \frac{5\sqrt{3}}{11}$$

111. Clearly $x^2 - 4x + \alpha > \text{or} \geq \frac{-1}{\sqrt{3}}, \forall\ x \in R$



Graph of $y=\cot^{-1}x$

$$\Rightarrow \quad x^2 - 4x + \alpha + \frac{1}{\sqrt{3}} > 0, \forall\ x \in R$$

So, discriminant $< 0 \Rightarrow 16 - 4\left(\alpha + \frac{1}{\sqrt{3}}\right) < 0$

$$4 - \alpha - \frac{1}{\sqrt{3}} < 0 \Rightarrow \alpha > 4 - \frac{1}{\sqrt{3}}$$

$$\therefore \quad \alpha \in \left(4 - \frac{1}{\sqrt{3}}, \infty\right)$$

Hence, minimum integral $\alpha = 4$

112. L : $3^4 - [{}^3C_1(2^4 - 2) + {}^3C_2] = 36$

M : If $x > 0$, Sgn $(x) = 1$

$$f(x) = 0 + \frac{\pi}{2} = \frac{\pi}{2}$$

For $x = 0$ $f(x)$ is not defined

$$\therefore \quad \text{for } x < 0,\ f(x) = \pi - \frac{\pi}{2} = \frac{\pi}{2}$$

$$\therefore \quad M = 1$$

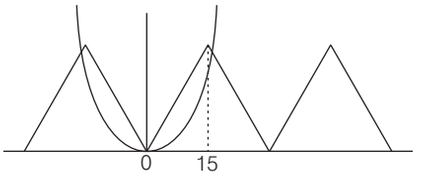
N : Coefficient of t^5 = coefficient of t^2 in$(1 + t^2)^5 \times$ coefficient of t^3 in $(1 + t^3)^8$

$$= 5 \times 8 = 40$$

Hence $\quad L = 36;\ M = 1$ and $N = 40$

$$\Rightarrow \quad N - LM = 40 - 36 = 4$$

113.



Number of solutions are three.

114. $\sec^2 u, \sec^4 v, \sec^6 w \in [1, \infty)$

$\therefore \quad \sec^2(u) + \sec^4(v) + \sec^6(w) \in [3, \infty)$

$\therefore \quad \pi(\sec^2 u + \sec^4 v + \sec^6 w) \in [3\pi, \infty)$

But $\quad \cos^{-1} x + \cos^{-1} y + \cos^{-1} z \in [0, 3\pi]$

So equation is possible of LHS = RHS = 3π

$\therefore \quad \cos^{-1} x = \cos^{-1} y = \cos^{-1} z = \pi$

$\therefore \quad x = y = z = -1$

and $\quad \sec^2 u = \sec^4 v = \sec^6 w = 1$

$\therefore \quad u = \pi, v = 2\pi, w = 3\pi$

$$\therefore x^{2000} + y^{2002} + z^{2004} + \frac{36\pi}{u+v+w} = 1 + 1 + 1 + \frac{36\pi}{6\pi} = 9$$

115. Given, $f(x) = \cos(\tan^{-1}(\sin(\cot^{-1} x)))$

$$= \cos\left(\tan^{-1}\left(\sin^{-1}\frac{1}{\sqrt{1+x^2}}\right)\right)$$

$$= \cos\left(\tan^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\right)$$

$$= \cos\left(\cos^{-1}\left(\frac{\sqrt{1+x^2}}{\sqrt{2+x^2}}\right)\right)$$

$$= \frac{\sqrt{1+x^2}}{\sqrt{2+x^2}} = \left(\frac{x^2+1}{x^2+2}\right)^{1/2}$$

$\Rightarrow \quad A = 1, B = 2$

$\Rightarrow \quad A + B = 1 + 2 = 3$

116. We have,

$$\text{RHS} = \tan^{-1}\{\tan^2(\alpha+\beta)\cdot\tan^2(\alpha-\beta)\} + \tan^{-1} 1$$

$$= \tan^{-1}\left\{\frac{\tan^2(\alpha+\beta)\tan^2(\alpha-\beta)+1}{1-\tan^2(\alpha+\beta)\tan^2(\alpha-\beta)}\right\}$$

$$= \tan^{-1}\left\{\frac{\sin^2(\alpha+\beta)\sin^2(\alpha-\beta)+\cos^2(\alpha+\beta)\cos^2(\alpha-\beta)}{\cos^2(\alpha+\beta)\cos^2(\alpha-\beta)-\sin^2(\alpha+\beta)\sin^2(\alpha-\beta)}\right\}$$

$$= \tan^{-1}\left\{\frac{\{2\sin(\alpha+\beta)\sin(\alpha-\beta)\}^2}{\{2\cos(\alpha+\beta)\cos(\alpha-\beta)\}^2}\right\}$$

$$\frac{+\{2\cos(\alpha+\beta)\cos(\alpha-\beta)\}^2}{-\{2\sin(\alpha+\beta)\sin(\alpha-\beta)\}^2}$$

$$= \tan^{-1}\left\{\frac{(\cos2\beta-\cos2\alpha)^2+(\cos2\alpha+\cos2\beta)^2}{(\cos2\alpha+\cos2\beta)^2-(\cos2\beta-\cos2\alpha)^2}\right\}$$

$$= \tan^{-1}\left\{\frac{\cos2\alpha+\cos^2 2\beta}{2\cos2\alpha\cos2\beta}\right\}$$

$$= \tan^{-1}\left\{\frac{\cos2\alpha\sec2\beta+\cos2\beta\sec2\alpha)}{2}\right\}$$

Hence, the value of λ is equal to 2.

117. We have, $\sin^{-1}(\sin 12) + \cos^{-1}(\cos 12)$

$$= \sin^{-1}(\sin(4\pi-(4\pi-12))) + \cos^{-1}(\cos(4\pi-(4\pi-12)))$$

$$= -(4\pi-12) + (4\pi-12) = 0$$

So that, $(n-2)x^2 + 8x + n + 4 > 0, \forall\ x \in R$

$\Rightarrow \quad n - 2 > 0 \Rightarrow n \geq 3$

and $\quad 8^2 - 4(n-2)(n+4) < 0$

or, $\quad n^2 + 2n - 24 > 0 \Rightarrow n = 4$

$\Rightarrow \quad n \geq 5 \Rightarrow n = 5$

118. $1 + \sin(\cos^{-1} x) + \sin^2(\cos^{-1} x) + \ldots \infty = 2$

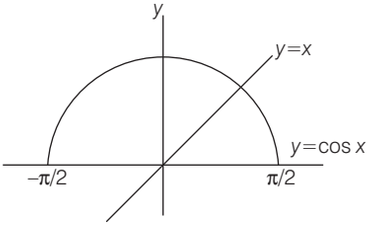
$$\Rightarrow \quad \frac{1}{1-\sin(\cos^{-1} x)} = 2$$

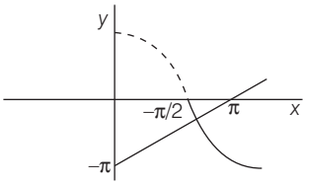
or $\quad \frac{1}{2} = 1 - \sin(\cos^{-1} x)$

or $\quad \sin(\cos^{-1} x) = \frac{1}{2}$ or $\cos^{-1} x = \frac{\pi}{6}$

or $\quad x = \frac{\sqrt{3}}{2}$ or $12x^2 = 9$

119. Here $|\cos x| = \sin^{-1}(\sin x)$





If $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ then $\cos x = x$

In the case there is one solution obtained graphically.

If $\frac{\pi}{2} < x \leq \pi$ then $-\cos x$

$$= \sin^{-1}\{\sin(\pi - x)\} = \pi - x$$

$\therefore \quad \cos x = x - \pi$

In this case there is one solution, obtained graphically.

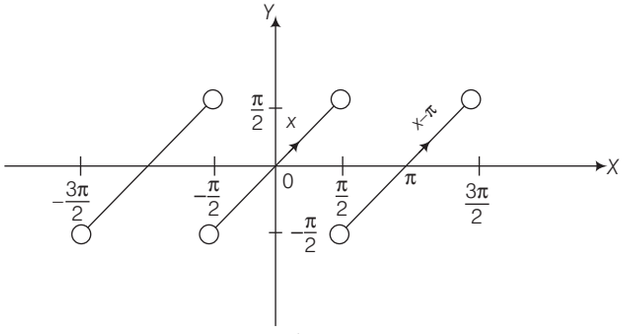
If $-\pi \leq x < -\frac{\pi}{2}$ then

$$-\cos x = \sin^{-1}\{\sin(-\pi - x)\} = -x - \pi$$

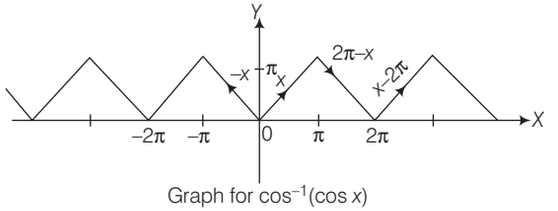
i.e., $\quad \cos x = x + \pi.$

This gives no solution as can be seen from then graphs.

120. From graph



Graph for $\tan^{-1}(\tan x)$



Graph for $\cos^{-1}(\cos x)$

Statement II is true and correct explanation for Statement 1.

121. $\sin^{-1}x = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}} > \tan^{-1}x > \tan^{-1}y$

$$\left\{\because x > y, \frac{x}{\sqrt{1-x^2}} > x\right\}$$

$\therefore$ Statement II is true

$$e < \pi$$

$$\frac{1}{\sqrt{e}} > \frac{1}{\sqrt{\pi}}$$

by Statement II

$$\sin^{-1}\left(\frac{1}{\sqrt{e}}\right) > \tan^{-1}\left(\frac{1}{\sqrt{e}}\right) > \tan^{-1}\left(\frac{1}{\sqrt{\pi}}\right)$$

Statement I is true

122. $\operatorname{cosec}^{-1}x > \sec^{-1}x$

$$\operatorname{cosec}^{-1}x > \frac{\pi}{4} - \operatorname{cosec}^{-1}x$$

$$\operatorname{cosec}^{-1}x > \frac{\pi}{4}$$

$$1 \le x < \sqrt{2}$$

and $\left(\dfrac{1}{2} + \dfrac{1}{\sqrt{2}}\right) \in [1, \sqrt{2}]$

Statement II is true and explains Statement I.

123. $f(x) = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) = \pi - 2\tan^{-1}x,\ x \ge 1$

$$f'(x) = -\frac{2}{1+x^2}$$

$$\Rightarrow \quad f'(2) = -\frac{2}{5}$$

Statement I is true, Statement II is true, Statement II is a correct explanation for Statement I.

124. $\sin^{-1}2x + \sin^{-1}3x = \dfrac{\pi}{3}$

$$\therefore \quad \frac{\pi}{2} - \cos^{-1}2x + \frac{\pi}{2} - \cos^{-1}3x = \frac{\pi}{3}$$

$$\therefore \quad \cos^{-1}2x + \cos^{-1}3x = \frac{2\pi}{3}$$

$$6x^2 - \sqrt{1 - 13x^2 + 36x^4} = -\frac{1}{2}$$

$$\Rightarrow \quad \left\{6x^2 + \frac{1}{2}\right\}^2 = 1 - 13x^2 + 36x^4$$

$$\Rightarrow \quad 19x^2 = \frac{3}{4} \Rightarrow x = \pm\sqrt{\frac{3}{76}}$$

But sum of two negative angles cannot be $\dfrac{\pi}{3}$

$\therefore\ x = \sqrt{\dfrac{3}{76}}$ is only solution.

125. Statement 2 is correct, from which we can say $\cot^{-1}x + \cos^{-1}2x = -\pi$ is not possible. Hence, both the statements are correct, and Statement 2 is the correct explanation of Statement 1.

126. (A) In $(0, \cos 1)$, we have $\cos^{-1}x > \sin^{-1}x$

Also, $\cos^{-1}x > 1$ and $\sin^{-1}x < 1$

The greatest is $(\cos^{-1}x)^{\cos^{-1}x} = t_4$ and the least is

$$(\sin^{-1}x)^{\cos^{-1}x} = t_2$$

and $(\sin^{-1}x)^{\sin^{-1}x} < (\cos^{-1}x)^{\sin^{-1}x}$

$\Rightarrow \quad t_1 < t_3$

So, $t_4 > t_3 > t_1 > t_2$

(B) Similarly, in $\cos 1 < x < \dfrac{1}{\sqrt{2}}$

$\cos^{-1}x > \sin^{-1}x$ and both are less than 1 So, greatest is t_3 and least is t_2 and $t_4 > t_1$.

Hence, $t_3 > t_4 > t_1 > t_2$

(C) For $\dfrac{1}{\sqrt{2}} < x < \sin 1$

We have, $1 > \sin^{-1}x > \cos^{-1}x$

So, greatest is t_2 and least is t_3 also $t_1 > t_4$

Hence, $t_2 > t_1 > t_4 > t_3$

(D) For $\sin 1 < x < 1$, we have

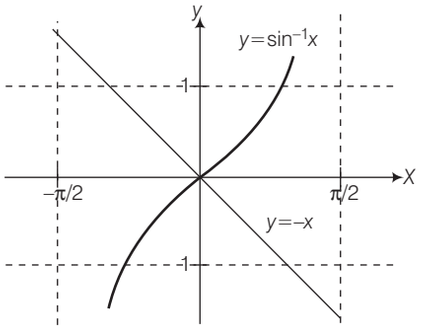
$$\sin^{-1}x > 1 > \cos^{-1}x$$

So, the greatest is t_1 and least is t_3 and $t_2 > t_4$

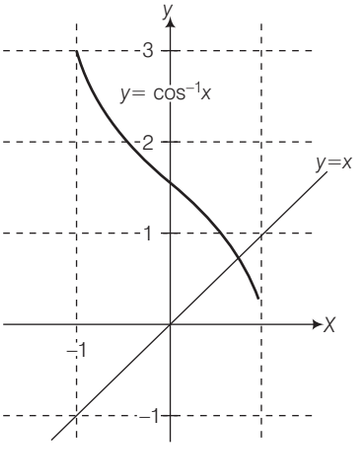
Hence, $t_1 > t_2 > t_4 > t_3$

127. Follow the following graph for solution

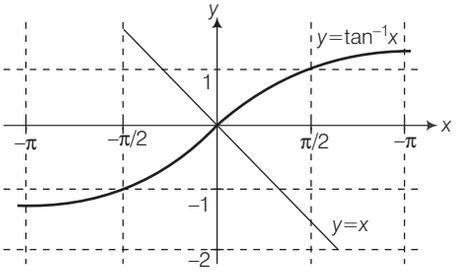
(A)



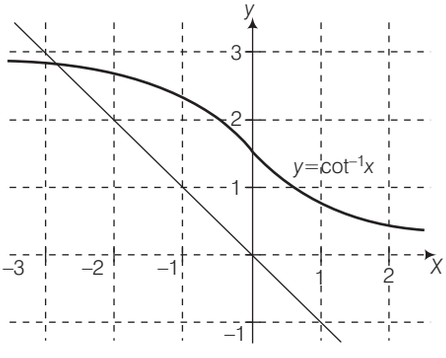
(B)



(A)



(C)



128. $\sin^{-1}\left(\frac{5}{x}\right)+\sin^{-1}\left(\frac{12}{x}\right)=\frac{\pi}{2}$

$\Leftrightarrow \quad x>12$ and $\cos^{-1}\left(\frac{\sqrt{x^2-25}}{x}\right)+\sin^{-1}\left(\frac{12}{x}\right)=\frac{\pi}{2}$

$\Leftrightarrow \quad x>12$ and $\cos^{-1}\left(\frac{\sqrt{x^2-25}}{x}\right)=\cos^{-1}\left(\frac{12}{x}\right)$

$\Leftrightarrow \quad x>12$ and $\frac{\sqrt{x^2-25}}{x}=\frac{12}{x}$

$\Leftrightarrow \quad x>12$ and $x^2-25=12^2$

$\Leftrightarrow \quad x>12$ and $x^2=169$

$\Leftrightarrow \quad x>12$ and $x=\pm 13$

$\therefore\ x=13$ is only solution.

129. Taking tan on both sides

$\frac{\frac{x+1}{x-1}+\frac{x-1}{x}}{1-\frac{x+1}{x-1}\cdot\frac{x-1}{x}}=-7$ on simplification, we get $x=2$

Substituting, we get

$\tan^{-1}3+\tan^{-1}\frac{1}{2}=\tan^{-1}(-7)$ but L.H.S is

$\pi+\tan^{-1}(-7)$. Hence, no solution.

130. Let $X=\sqrt{\frac{a(a+b+c)}{bc}}$, $Y=\sqrt{\frac{b(a+b+c)}{ac}}$, $Z=\sqrt{\frac{c(a+b+c)}{ab}}$

$$XY=\sqrt{\frac{a(a+b+c)}{bc}}\cdot\frac{b(a+b+c)}{ac}=\frac{a+b+c}{c}$$

$$=\left(1+\frac{a+b}{c}\right)>1\,(a,b,c>0)$$

$$\therefore\quad \tan^{-1}X+\tan^{-1}Y=\pi+\tan^{-1}\left[\frac{X+Y}{1-XY}\right]$$

$$=\pi+\tan^{-1}\left(\frac{\sqrt{\frac{a(a+b+c)}{bc}}+\sqrt{\frac{b(a+b+c)}{ac}}}{1-\frac{a+b+c}{c}}\right)$$

$$=\pi+\tan^{-1}\left(\frac{\sqrt{\frac{a+b+c}{c}}+\frac{\sqrt{a}}{\sqrt{b}}+\frac{\sqrt{b}}{\sqrt{a}}}{-\frac{(a+b)}{c}}\right)$$

$$=\pi+\tan^{-1}\left(-\sqrt{\frac{a+b+c}{c}}\left(\frac{a+b}{\sqrt{ab}}\right)\left(\frac{c}{a+c}\right)\right)$$

$$=\pi-\tan^{-1}\frac{\sqrt{c(a+b+c)}}{ab}=\pi-\tan^{-1}Z$$

$\therefore\quad \tan^{-1}X+\tan^{-1}Y+\tan^{-1}Z=\pi$

131. Let $\theta=\text{cosec}^{-1}\sqrt{(n^2+1)(n^2+2n+2)}$

$$\text{cosec}^2\theta=(n^2+1)(n^2+2n+2)$$

$$=(n^2+1)^2+2n(n^2+1)+n^2+1=(n^2+n+1)^2+1$$

$$\cot^2\theta=(n^2+n+1)^2$$

$$\tan\theta=\frac{1}{n^2+n+1}=\frac{(n+1)-n}{1+(n+1)n}$$

$$\theta=\tan^{-1}\left[\frac{(n+1)-n}{1+(n+1)n}\right]$$

$$= \tan^{-1}(n+1) - \tan^{-1}n$$

Thus, sum of n terms of the given series

$$= (\tan^{-1}2 - \tan^{-1}1) + (\tan^{-1}3 - \tan^{-1}2) + (\tan^{-1}4 - \tan^{-1}3) + \ldots + (\tan^{-1}(n+1) - \tan^{-1}n)$$

$$= \tan^{-1}(n+1) - \pi/4$$

132. We have

$$E = \sqrt{\sin^{-1}x_1}\sqrt{\cos^{-1}x_2} + \sqrt{\sin^{-1}x_2}\sqrt{\cos^{-1}x_3} + \sqrt{\sin^{-1}x_3}\sqrt{\cos^{-1}x_4} + \ldots + \sqrt{\sin^{-1}x_{28}}\sqrt{\cos^{-1}x_1}$$

$x_1 \in [0, 1]\ \forall i = 1, 2, 3, \ldots, 28$

$\therefore \sin^{-1}x_1 > 0$

Now using A.M. $\geq$ G.M. we have $\frac{a^2+b^2}{2} \geq ab$, where $a, b > 0$

$$\therefore \quad \sqrt{\sin^{-1}x_1}\sqrt{\cos^{-1}x_2} \leq \left(\frac{\sin^{-1}x_1 + \cos^{-1}x_2}{2}\right)$$

$$\sqrt{\sin^{-1}x_2}\sqrt{\cos^{-1}x_3} \leq \left(\frac{\sin^{-1}x_2 + \cos^{-1}x_3}{2}\right)$$

$$\sqrt{\sin^{-1}x_{28}}\sqrt{\cos^{-1}x_1} \leq \left(\frac{\sin^{-1}x_{28} + \sin^{-1}x_1}{2}\right)$$

On adding all, we get

$$E \leq \sum_{i=1}^{28} \frac{\sin^{-1}x_i + \cos^{-1}x_i}{2}$$

$$\therefore \quad E \leq \frac{28\left(\frac{\pi}{2}\right)}{2}$$

$$\therefore \quad E_{\max} = 7\pi$$

133. $S = \sum_{r=1}^{10}\sum_{s=1}^{10} \tan^{-1}\left(\frac{r}{s}\right)$

$$\therefore \quad S = \sum_{r=1}^{10}\sum_{s=1}^{10} \tan^{-1}\left(\frac{s}{r}\right) \text{ (As } r \text{ and } s \text{ are independent)}$$

On adding, we get

$$2S = \sum_{r=1}^{10}\sum_{s=1}^{10}\left(\tan^{-1}\left(\frac{r}{s}\right) + \tan^{-1}\left(\frac{s}{r}\right)\right)$$

$$\Rightarrow \quad 2S = \sum_{r=1}^{10}\sum_{s=1}^{10}\frac{\pi}{2} = \frac{\pi}{2}\sum_{r=1}^{10}10 = \frac{100\pi}{2}$$

$$S = 25\pi$$

134. $S_{n-1} = \sum_{k=2}^{n} \cos^{-1}\left(\frac{1 + \sqrt{(k-1)\,k(k+1)\,(k+2)}}{k(k+1)}\right)$

$$= \cos^{-1}\left(\frac{1}{k}\cdot\frac{1}{(k+1)} + \frac{1 + \sqrt{(k-1)\,k(k+1)\,(k+2)}}{k(k+1)}\right)$$

$$= \cos^{-1}\left(\frac{1}{k}\cdot\frac{1}{k+1} + \sqrt{1-\frac{1}{k^2}}\sqrt{1-\frac{1}{(k+1)^2}}\right)$$

$$= \cos^{-1}\frac{1}{k+1} - \cos^{-1}\frac{1}{k}$$

Substituting $k = 2, 3, 4\ldots$, we get

$$t_2 = \cos^{-1}\left(\frac{1}{3}\right) - \cos^{-1}\left(\frac{1}{2}\right)$$

$$t_3 = \cos^{-1}\left(\frac{1}{4}\right) - \cos^{-1}\left(\frac{1}{3}\right)$$

$$t_n = \cos^{-1}\left(\frac{1}{n+1}\right) - \cos^{-1}\left(\frac{1}{n}\right)$$

$$S_{n-1} = \cos^{-1}\left(\frac{1}{n+1}\right) - \cos^{-1}\left(\frac{1}{2}\right)$$

$$\therefore \quad S = \lim_{n\to\infty}\left[\cos^{-1}\left(\frac{1}{n+1}\right) - \cos^{-1}\left(\frac{1}{2}\right)\right]$$

$$= \cos^{-1}(0) - \cos^{-1}\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$$

135. Given, $\frac{m\tan(\alpha-\theta)}{\cos^2\theta} = \frac{n\tan\theta}{\cos^2(\alpha-\theta)}$

$$\Rightarrow \quad \frac{n}{m} = \frac{\tan(\alpha-\theta).\cos^2(\alpha-\theta)}{\cos^2\theta.\tan\theta}$$

$$= \frac{2\sin(\alpha-\theta).\cos(\alpha-\theta)}{2\sin\theta.\cos\theta}$$

$$\Rightarrow \quad \frac{n}{m} = \frac{\sin(2\alpha-2\theta)}{\sin(2\theta)}$$

using componendo and dividendo; we get.

$$\Rightarrow \quad \frac{n-m}{n+m} = \frac{\sin(2\alpha-2\theta) - \sin2\theta}{\sin(2\alpha-2\theta) + \sin2\theta} = \frac{2\sin(\alpha-2\theta).\cos\alpha}{2\sin\alpha\cos(\alpha-2\theta)}$$

$$\Rightarrow \quad \left(\frac{n-m}{n+m}\right)\tan\alpha = \tan(\alpha-2\theta)$$

$$\Rightarrow \quad \alpha - 2\theta = \tan^{-1}\left\{\left(\frac{n-m}{n+m}\right)\tan\alpha\right\}$$

$$\Rightarrow \quad 2\theta = \alpha - \tan^{-1}\left\{\left(\frac{n-m}{n+m}\right)\tan\alpha\right\}$$

$$\Rightarrow \quad \theta = \frac{1}{2}\left[\alpha - \tan^{-1}\left\{\left(\frac{n-m}{n+m}\right)\tan\alpha\right\}\right]$$

136. Let $\tan^{-1}(e^{i\theta}) = A + iB$ or $e^{i\theta} = \tan(A + iB)$

$$\Rightarrow \quad \tan^{-1}(e^{-i\theta}) = A - iB. \text{ or } e^{-i\theta} = \tan(A - iB)$$

Now, $\cos2A = \cos(A + iB + A - iB)$

$$\therefore \quad \cos2A = \cos(A+iB).\cos(A-iB) - \sin(A+iB)\sin(A-iB)$$

$$= \cos(A+iB).\cos(A-iB)\{1 - \tan(A+iB).\tan(A-iB)\}$$

$$= \cos(A+iB).\cos(A-iB)\{1 - e^{i\theta}.e^{-i\theta}\}$$

$$= \cos(A+iB).\cos(A-iB)\{1-1\} = 0$$

$$\Rightarrow \quad 2A = (2n+1)\frac{\pi}{2} \text{ or } A = \frac{n\pi}{2} + \frac{\pi}{4}$$

Also $\tan2iB = \tan(A + iB - A + iB)$

$$= \frac{\tan(A+iB) - \tan(A-iB)}{1 + \tan(A+iB).\tan(A-iB)}$$

$$= \frac{e^{i\theta} - e^{-i\theta}}{1 + e^{i\theta}.e^{-i\theta}} = i\sin\theta$$

$$\Rightarrow \quad \frac{e^{-2B}-e^{2B}}{i(e^{-2B}+e^{2B})}=i\sin\theta$$

or $$\frac{e^{2B}-e^{-2B}}{e^{2B}+e^{-2B}}=\sin\theta$$

or $$\frac{e^{2B}}{e^{-2B}}=\frac{1+\sin\theta}{1-\sin\theta}=\frac{1+\cos\left(\frac{\pi}{2}-\theta\right)}{1-\cos\left(\frac{\pi}{2}-\theta\right)}$$

$$e^{4B}=\frac{\cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{\sin^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}$$

$$\Rightarrow \quad e^{2B}=\cot\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

$$\Rightarrow \quad B=-\frac{1}{2}\ln\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

137. The quadratic equation $4^{\sec^2\alpha}.x^2+2x+\left(\beta^2-\beta+\frac{1}{2}\right)=0$ have real roots.

$\Rightarrow$ discriminant $=4-4.4^{\sec^2\alpha}\left(\beta^2-\beta+\frac{1}{2}\right)\geq 0$

$$\Rightarrow \quad 4^{\sec^2\alpha}\left(\beta^2-\beta+\frac{1}{2}\right)\leq 1 \qquad \text{...(i)}$$

But $4^{\sec^2\alpha}\geq 4,\ \beta^2-\beta+\frac{1}{2}=\left(\beta-\frac{1}{2}\right)^2+\frac{1}{4}\geq\frac{1}{4}$

i.e. equation will be satisfied only when

$$4^{\sec^2\alpha}=4 \quad \text{and}\ \beta^2-\beta+\frac{1}{2}=\frac{1}{4}$$

$$\Rightarrow \quad \sec^2\alpha=1 \text{ and } \left(\beta-\frac{1}{2}\right)^2=0$$

$$\Rightarrow \quad \cos^2\alpha=1 \text{ and } \beta=\frac{1}{2}$$

$$\Rightarrow \quad \alpha=n\pi \text{ and } \beta=\frac{1}{2}$$

$$\therefore \quad \cos\alpha+\cos^{-1}\beta=\cos n\pi+\cos^{-1}\frac{1}{2}$$

$$=\begin{cases}1+\frac{\pi}{3}, & \text{when } n \text{ is even integer}\\ -1+\frac{\pi}{3}, & \text{when } n \text{ is odd integer}\end{cases}$$

i.e. the value of;

$$\cos\alpha+\cos^{-1}\beta \text{ is } \frac{\pi}{3}-1 \text{ and } \frac{\pi}{3}+1.$$

138. Here, $\alpha=3\sin^{-1}\left(\frac{6}{11}\right)$ and $\beta=3\cos^{-1}\left(\frac{4}{9}\right)$ as $\frac{6}{11}>\frac{1}{2}$

$$\Rightarrow \quad \sin^{-1}\left(\frac{6}{11}\right)>\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$$

$$\therefore \quad \alpha=3\sin^{-1}\left(\frac{6}{11}\right)>\frac{\pi}{2} \ \Rightarrow\ \cos\alpha<0$$

Now, $\beta=3\cos^{-1}\left(\frac{4}{9}\right)$

As $\frac{4}{9}<\frac{1}{2} \ \Rightarrow\ \cos^{-1}\left(\frac{4}{9}\right)>\cos^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}$

$$\therefore \quad \beta=3\cos^{-1}\left(\frac{4}{9}\right)>\pi$$

$\therefore$ $\cos\beta<0$ and $\sin\beta<0$

Now, $\alpha+\beta$ is slightly greater than $\frac{3\pi}{2}$.

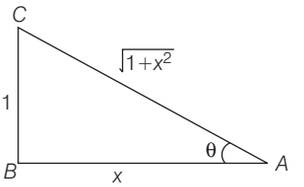
$\therefore$ $\cos(\alpha+\beta)>0$

139. We have, $0<x<1$

Let $\cot^{-1}x=\theta \ \Rightarrow\ \cot\theta=x$

$$\Rightarrow \quad \sin\theta=\frac{1}{\sqrt{1+x^2}}=\sin(\cot^{-1}x)$$

and $$\cos\theta=\frac{x}{\sqrt{1+x^2}}=\cos(\cot^{-1}x)$$



Now, $\sqrt{1+x^2}\ [\{x\cos(\cot^{-1}x)+\sin(\cot^{-1}x)\}^2-1]^{1/2}$

$$=\sqrt{1+x^2}\left[\left(x\frac{x}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+x^2}}\right)^2-1\right]^{1/2}$$

$$=\sqrt{1+x^2}\left[\left(\frac{1+x^2}{\sqrt{1+x^2}}\right)^2-1\right]^{1/2}$$

$$=\sqrt{1+x^2}[1+x^2-1]^{1/2}=x\sqrt{1+x^2}$$

140. Given, $\tan^{-1}y=\tan^{-1}x+\tan^{-1}\left(\frac{2x}{1-x^2}\right)$, where $|x|<\frac{1}{\sqrt{3}}$

$$\Rightarrow \quad \tan^{-1}y=\tan^{-1}\left\{\frac{x+\frac{2x}{1-x^2}}{1-x\left(\frac{2x}{1-x^2}\right)}\right\}$$

$$\left[\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right),\right.$$

$$\left.\text{where } x>0,\ y>0 \text{ and } xy<1\right]$$

$$=\tan^{-1}\left(\frac{x-x^3+2x}{1-x^2-2x^2}\right)$$

$$\tan^{-1}y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

$$\Rightarrow \quad y=\frac{3x-x^3}{1-3x^2}$$

Aliter

$$|x|<\frac{1}{\sqrt{3}} \ \Rightarrow\ -\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}$$

Let $x = \tan\theta$

$\Rightarrow \quad -\frac{\pi}{6} < \theta < \frac{\pi}{6}$

$\therefore \quad \tan^{-1} y = \theta + \tan^{-1}(\tan 2\theta)$

$= \theta + 2\theta = 3\theta$

$\Rightarrow \quad y = \tan 3\theta$

$\Rightarrow \quad y = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

$\Rightarrow \quad y = \frac{3x - x^3}{1 - 3x^2}$

141. We have, $\cot\left[\sum_{n=1}^{23} \cot^{-1}\left(1 + \sum_{k=1}^{n} 2k\right)\right]$

$\Rightarrow \cot\left[\sum_{n=1}^{23} \cot^{-1}(1 + 2 + 4 + 6 + 8 + \ldots + 2n)\right]$

$\Rightarrow \cot\left[\sum_{n=1}^{23} \cot^{-1}\{1 + n(n+1)\}\right]$

$\Rightarrow \quad \cot\left[\sum_{n=1}^{23} \tan^{-1}\frac{1}{1 + n(n+1)}\right]$

$\Rightarrow \quad \cot\left[\sum_{n=1}^{23} \tan^{-1}\left\{\frac{(n+1) - n}{1 + n(n+1)}\right\}\right]$

$\Rightarrow \quad \cot\left[\sum_{n=1}^{23} (\tan^{-1}(n+1) - \tan^{-1} n)\right]$

$\Rightarrow \cot[(\tan^{-1}2 - \tan^{-1}1) + (\tan^{-1}3 - \tan^{-1}2)$

$+ (\tan^{-1}4 - \tan^{-1}3)] + \ldots + (\tan^{-1}24 - \tan^{-1}23)]$

$\Rightarrow \cot(\tan^{-1}24 - \tan^{-1}1)$

$\Rightarrow \cot\left(\tan^{-1}\frac{24-1}{1 + 24\cdot(1)}\right) = \cot\left(\tan^{-1}\frac{23}{25}\right)$

$= \cot\left(\cot^{-1}\frac{25}{23}\right) = \frac{25}{23}$

142. Since, x, y and z are in an AP.

$\therefore \quad 2y = x + z$

Also, $\tan^{-1}x$, $\tan^{-1}y$ and $\tan^{-1}z$ are in an AP.

$\therefore \quad 2\tan^{-1}y = \tan^{-1}x + \tan^{-1}(z)$

$\Rightarrow \quad \tan^{-1}\left(\frac{2y}{1-y^2}\right) = \tan^{-1}\left(\frac{x+z}{1-xz}\right)$

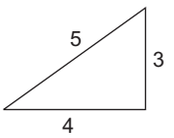
$\Rightarrow \quad \frac{x+z}{1-y^2} = \frac{x+z}{1-xz}$

$\Rightarrow \quad y^2 = xz$

Since, x, y and z are in an AP as well as in a GP.

$\therefore \quad x = y = z$

143. Since, $\operatorname{cosec}^{-1}\left(\frac{5}{3}\right) = \tan^{-1}\left(\frac{3}{4}\right)$



$\therefore \quad \cot\left(\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{2}{3}\right)$

$= \cot\tan^{-1}\left[\frac{\frac{3}{4} + \frac{2}{3}}{1 - \frac{1}{2}}\right] = \cot\tan^{-1}\left[\frac{\left(\frac{17}{12}\right)}{\left(\frac{1}{2}\right)}\right]$

$= \cot\left[\tan^{-1}\left(\frac{17}{6}\right)\right] = \frac{6}{17}$

144. We have, $\sin^{-1}\left(\frac{x}{5}\right) + \operatorname{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$

$\Rightarrow \quad \sin^{-1}\left(\frac{x}{5}\right) + \sin^{-1}\left(\frac{4}{5}\right) = \frac{\pi}{2}$

$\Rightarrow \quad \sin^{-1}\left(\frac{x}{5}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{4}{5}\right)$

$\Rightarrow \quad \sin^{-1}\left(\frac{x}{5}\right) = \cos^{-1}\left(\frac{4}{5}\right)$

$\Rightarrow \quad \sin^{-1}\left(\frac{x}{5}\right) = \sin^{-1}\left(\frac{3}{5}\right)$

$\therefore \quad x = 3$

145. Given that, $\cos^{-1}x - \cos^{-1}\frac{y}{2} = \alpha$

$\Rightarrow \quad \cos^{-1}\left(\frac{xy}{2} + \sqrt{1-x^2}\sqrt{1-\frac{y^2}{4}}\right) = \alpha$

$\Rightarrow \quad \frac{xy}{2} + \sqrt{1-x^2}\sqrt{1-\frac{y^2}{4}} = \cos\alpha$

$\Rightarrow \quad 2\sqrt{1-x^2}\sqrt{1-\frac{y^2}{4}} = 2\cos\alpha - xy$

On squaring both sides, we get

$\frac{4(1-x^2)(4-y^2)}{4} = 4\cos^2\alpha + x^2y^2 - 4xy\cos\alpha$

$\Rightarrow \quad 4 - 4x^2 - y^2 + x^2y^2 = 4\cos^2\alpha + x^2y^2 - 4xy\cos\alpha$

$\therefore \quad 4x^2 - 4xy\cos\alpha + y^2 = 4\sin^2\alpha$

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PREFACE

"YOU CAN DO ANYTHING IF YOU SET YOUR MIND TO IT, I TEACH GEOMETRY TO JEE ASPIRANTS BUT BELIEVE THE MOST IMPORTANT FORMULA IS ***COURAGE + DREAMS = SUCCESS****"*

It is a matter of great pride and honour for me to have received such an overwhelming response to the previous editions of this book from the readers. In a way, this has inspired me to revise this book thoroughly as per the changed pattern of JEE Main & Advanced. I have tried to make the contents more relevant as per the needs of students, many topics have been re-written, a lot of new problems of new types have been added in etc. All possible efforts are made to remove all the printing errors that had crept in previous editions. The book is now in such a shape that the students would feel at ease while going through the problems, which will in turn clear their concepts too.

A Summary of changes that have been made in Revised & Enlarged Edition

- Theory has been completely updated so as to accommodate all the changes made in JEE Syllabus & Pattern in recent years.
- The most important point about this new edition is, now the whole text matter of each chapter has been divided into small sessions with exercise in each session. In this way the reader will be able to go through the whole chapter in a systematic way.
- Just after completion of theory, Solved Examples of all JEE types have been given, providing the students a complete understanding of all the formats of JEE questions & the level of difficulty of questions generally asked in JEE.
- Along with exercises given with each session, a complete cumulative exercises have been given at the end of each chapter so as to give the students complete practice for JEE along with the assessment of knowledge that they have gained with the study of the chapter.
- Last 10 Years questions asked in JEE Main & Adv, IIT-JEE & AIEEE have been covered in all the chapters.

However I have made the best efforts and put my all teaching experience in revising this book. Still I am looking forward to get the valuable suggestions and criticism from my own fraternity i.e. the fraternity of JEE teachers.

I would also like to motivate the students to send their suggestions or the changes that they want to be incorporated in this book. All the suggestions given by you all will be kept in prime focus at the time of next revision of the book.

Amit M. Agarwal

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SYLLABUS

JEE MAIN

Three Dimensional Geometry

Coordinates of a point in space, distance between two points, section formula, direction ratios and direction cosines, angle between two intersecting lines. Skew lines, the shortest distance between them and its equation. Equations of a line and a plane in different forms, intersection of a line and a plane, coplanar lines.

Vector Algebra

Vectors and scalars, addition of vectors, components of a vector in two dimensions and three dimensional space, scalar and vector products, scalar and vector triple product.

JEE ADVANCED

Locus Problems

Three Dimensions Direction cosines and direction ratios, equation of a straight line in space, equation of a plane, distance of a point from a plane.

Vectors

Addition of vectors, scalar multiplication, scalar products, dot and cross products, scalar triple products and their geometrical interpretations.

CHAPTER

01

Vector Algebra

Learning Part

Session 1

- Scalar and Vector Quantities
- Representation of Vectors
- Position Vector of a Point in Space
- Direction Cosines
- Rectangular Resolution of a Vector in 2D and 3D Systems

Session 2

- Addition & Subtraction of Vectors
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Session 1

Scalar and Vector Quantities, Representation of Vectors, Position Vector of a Point in Space, Direction Cosines, Rectangular Resolution of a Vector in 2D and 3D Systems

Vectors represent one of the most important mathematical systems, which is used to handle certain types of problems in Geometry, Mechanics and other branches of Applied Mathematics, Physics and Engineering.

Scalar and Vector Quantities

Physical quantities are divided into two categories-Scalar quantities and Vector quantities. Those quantities which have only magnitude and which are not related to any fixed direction in space are called scalar quantities or briefly scalars. Examples of scalars are mass, volume, density, work, temperature etc.

A scalar quantity is represented by a real number along with a suitable unit. Second kind of quantities are those which have both magnitude and direction, such quantities are called vectors. Displacement, velocity, acceleration, momentum, weight, force etc., are examples of vector quantities.

Example 1. ***Classify the following measures as scalars and vectors***

(i) 20 m north-west ***(ii) 10 Newton***

(iii) 30 km/h

(iv) 50m/s towards north

(v) 10^{-19} ***coloumb***

Sol. (i) Directed distance -Vector

(ii) Force-Vector

(iii) Speed-Scalar

(iv) Velocity-Vector

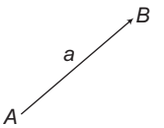
(v) Electric charge-Scalar

Representation of Vectors

Geometrically, a vector is represented by a directed line segment.

For example, $\mathbf{a} = \mathbf{AB}$. Here, A is called the initial point and B is called the terminal point or tip.

A directed line segment with initial point A and terminal point B is denoted by **AB** or $\overrightarrow{AB}$. Vectors are also denoted by small letters with an arrow above it or by small bold letters, e.g. **a, b, c** etc. or $\vec{a}, \vec{b}, \vec{c}$ etc.



Here, in the figure $\mathbf{a} = \mathbf{AB}$ and magnitude or modulus of **a** is expressed as $|\mathbf{a}| = |\mathbf{AB}| = AB$ (Distance between initial and terminal points).

Remarks

1. The magnitude of a vector is always a non-negative real number.
2. Every vector **AB** has the following three characteristics

Length The length of **AB** will be denoted by $|\mathbf{AB}|$ or AB.

Support The line of unlimited length of which AB is a segment is called the support of the vector **AB**.

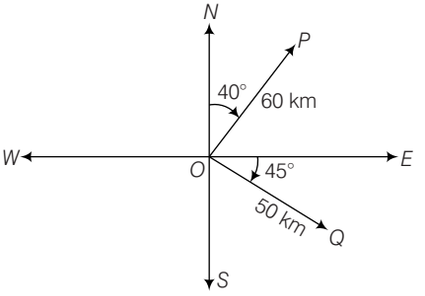
Sense The sense of **AB** is from A to B and that of **BA** is from B to A. Thus, the sense of a directed line segment is from its initial point to the terminal point.

Example 2. ***Represent graphically***

(i) A displacement of 60 km, 40° east of north

(ii) A displacement of 50 km south-east

Sol. (i) The vector **OP** represent the required vector.



(ii) The vector **OQ** represent the required vector.

Types of Vectors

1. **Zero or null vector** A vector whose magnitude is zero is called zero or null vector and it is represented by 0.

   The initial and terminal points of the directed line segment representing zero vector are coincident and its direction is arbitrary.

2. **Unit vector** A vector whose modulus is unity, is called a unit vector. The unit vector in the direction of a vector **a** is denoted by $\hat{\mathbf{a}}$, read as ''*a cap*'. Thus, $|\hat{\mathbf{a}}| = 1$.

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\text{Vector}}{\text{Magnitude of } a}$$

3. **Like and unlike vectors** Vectors are said to be like when they have the same sense of direction and unlike when they have opposite directions.
4. **Collinear or parallel vectors** Vectors having the same of parallel supports are called collinear vectors.
5. **Coinitial vectors** Vectors having the same initial point are called coinitial vectors.
6. **Coplanar vectors** A system of vectors is said to be coplanar, if they lie in the same plane or their supports are parallel to the same plane.
7. **Coterminous vectors** Vectors having the same terminal points are called coterminous vectors.
8. **Negative of a vector** The vector which has the same magnitude as the given vector **a** but opposite direction, is called the negative of **a** and is denoted by $-\mathbf{a}$. Thus, if $\mathbf{PQ} = \mathbf{a}$, then $\mathbf{QP} = -\mathbf{a}$.
9. **Reciprocal of a vector** A vector having the same direction as that of a given vector a but magnitude equal to the reciprocal of the given vector is known as the reciprocal of a and is denoted by $\mathbf{a}^{-1}$. Thus, if $|\mathbf{a}| = a$, then $|\mathbf{a}^{-1}| = 1/a$.

Remark

A unit vector is self reciprocal.

10. **Localised vector** A vector which is drawn parallel to a given vector through a specified point in space is called a localised vector. For example, a force acting on a rigid body is a localised vector as its effect depends on the line of action of the force.
11. **Free vectors** If the value of a vector depends only on its length and direction and is independent of its position in the space, it is called a free vector.

Remark

Unless otherwise stated all vectors will be considered as free vectors.

12. **Equality of vectors** Two vectors **a** and **b** are said to be equal, if

    (i) $|\mathbf{a}| = |\mathbf{b}|$

    (ii) they have the same or parallel support.

    (iii) they have the same sense.

    Two unit vectors may not be equal unless they have the same direction.

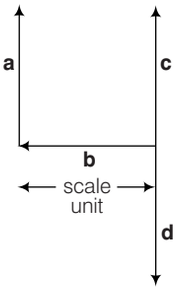
Example 3. ***In the following figure, which of the vectors are:***

(i) Collinear

(ii) Equal

(iii) Co-initial

(iv) Collinear but not equal



Sol. (i) ***a***,***c*** and ***d*** are collinear vectors.

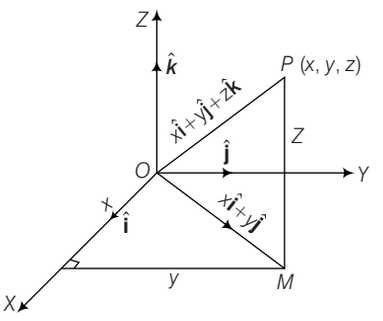
(ii) ***a*** and ***c*** are equal vectors

(iii) ***b***,***c*** and ***d*** are co-initial vectors

(iv) ***a*** and ***d*** are collinear but they are not equal, as their directions are not same.

Position Vector of a Point in Space

Let O be the fixed point in space and $X'OX$, $Y'OY$ and $Z'OZ$ be three lines perpendicular to each other at O. Then, these three lines called X-axis, Y-axis and Z-axis which constitute the rectangular coordinate system. The planes XOY, YOZ and ZOX, called respectively, the XY-plane, the YZ-plane and the ZX-plane.



Now, let P be any point in space. Then, position of P is given by triad (x, y, z) where x, y, z are perpendicular distance from YZ-plane, ZX-plane and XY-plane respectively.

The vector **OP** is called the position vector of point P with respect to the origin O and written as

$$\mathbf{OP} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are unit vectors parallel to X-axis, Y-axis and Z-axis. We usually denote position vector by $\mathbf{r}$.

Remarks

1. If A and B are any two points in space having coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively, then distance between the points A and $B = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.
2. Using distance formula, the magnitude of **OP** (or $\mathbf{r}$) is given by $|\mathbf{OP}| = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = \sqrt{x^2 + y^2 + z^2}$
3. Two vectors are equal if they have same components. i.e. if $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$ are equal, then $a_1 = b_1$, $a_2 = b_2$ and $a_3 = b_3$.

Example 4. ***Find a unit vector parallel to the vector*** $-3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$.

Sol. Let $\mathbf{a} = -3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$

Then, $|\mathbf{a}| = \sqrt{(-3)^2 + (4)^2} = 5$

$\therefore$ Unit vector parallel to $\mathbf{a} = \hat{\mathbf{a}} = \dfrac{1}{|\mathbf{a}|} \cdot \mathbf{a}$

$$= \frac{-3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}}{5} = \frac{-3}{5}\hat{\mathbf{i}} + \frac{4}{5}\hat{\mathbf{j}}$$

Example 5. ***Let*** $\mathbf{a} = 12\hat{\mathbf{i}} + n\hat{\mathbf{j}}$ ***and*** $|\mathbf{a}| = 13$***, find the value of*** n.

Sol. Here, $\mathbf{a} = 12\hat{\mathbf{i}} + n\hat{\mathbf{j}}$

$$\Rightarrow \quad |\mathbf{a}| = \sqrt{12^2 + n^2} = 13$$

$$\Rightarrow \quad 144 + n^2 = 169$$

$$\Rightarrow \quad n^2 = 25 \quad \text{or} \quad n = \pm 5$$

Example 6. ***Write two vectors having same magnitude.***

Sol. Let $\mathbf{a} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

Then, $|\mathbf{a}| = |\mathbf{b}| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$

Example 7. ***If one side of a square be represented by the vectors*** $3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$***, then the area of the square is***

(a) 12 (b) 13
(c) 25 (d) 50

Sol. (d) Let $\mathbf{a} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$ then $|\mathbf{a}|$

$$= \sqrt{3^2 + 4^2 + 5^2} = \sqrt{9 + 16 + 25} = 5\sqrt{2}$$

Thus, the length of a side of square $= 5\sqrt{2}$

Hence, area of square $= (5\sqrt{2})^2 = 25 \times 2 = 50$

Direction Cosines

Let $\mathbf{r}$ be the position vector of a point $P(x, y, z)$. Then, direction cosines of $\mathbf{r}$ are the cosines of angles α, β and γ that the vector $\mathbf{r}$ makes with the positive direction of X, Y and Z-axes respectively. We usually denote direction cosines by l, m and n respectively.

In the figure, we may note that ΔOAP is right angled triangle and in it we have

$$\cos\alpha = \frac{x}{r} \ (r \text{ stands for } |\mathbf{r}|)$$

Similarly, from the right angled triangles OBP and OCP, we get

$$\cos\beta = \frac{y}{r} \quad \text{and} \quad \cos\gamma = \frac{z}{r}$$

Thus, we have the following

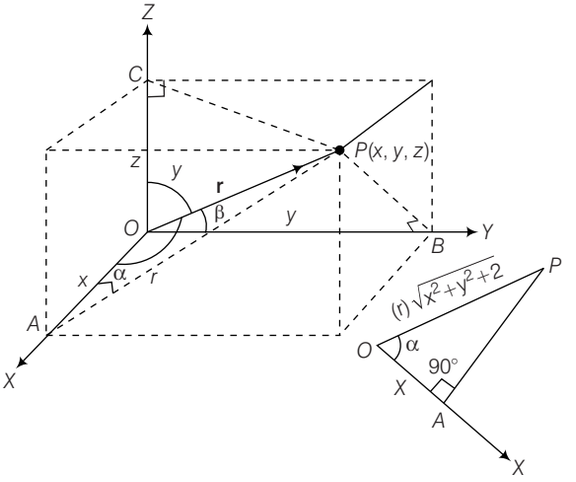
$$\cos\alpha = l = \frac{x}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{|\mathbf{r}|} = \frac{x}{r}$$

$$\cos\beta = m = \frac{y}{\sqrt{x^2 + y^2 + z^2}} = \frac{y}{|\mathbf{r}|} = \frac{y}{r}$$

and $\cos\gamma = n = \dfrac{z}{\sqrt{x^2 + y^2 + z^2}} = \dfrac{z}{|\mathbf{r}|} = \dfrac{z}{r}$

Clearly, $\boldsymbol{l^2 + m^2 + n^2 = 1}$.

Here, $\alpha = \angle POX, \beta = \angle POY, \gamma = \angle POZ$ and $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are the unit vectors along OX, OY and OZ respectively.



Remarks

1. The coordinates of point P may also be expressed as (lr, mr, nr).
2. The numbers l, m and n, proportional to the direction cosines, are called the direction ratios of vector $\mathbf{r}$ and are denoted by a, b and c respectively.
3. If $\mathbf{r} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$, then a, b and c are direction ratios of the given vector.
   Also, if $a^2 + b^2 + c^2 = 1$, then a, b and c will be direction cosines of given vector.

Example 8. ***The direction cosines of the vector*** $3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$ ***are***

(a) $\frac{3}{5}, \frac{-4}{5}, \frac{1}{5}$ (b) $\frac{3}{5\sqrt{2}}, \frac{-4}{5\sqrt{2}}, \frac{1}{\sqrt{2}}$

(c) $\frac{3}{\sqrt{2}}, \frac{-4}{\sqrt{2}}, \frac{1}{\sqrt{2}}$ (d) $\frac{3}{5\sqrt{2}}, \frac{4}{5\sqrt{2}}, \frac{1}{\sqrt{2}}$

Sol. (b) $\mathbf{r} = 3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$

$\Rightarrow \quad |\mathbf{r}| = \sqrt{3^2 + (-4)^2 + 5^2} = 5\sqrt{2}$

Hence, direction cosines are $\frac{3}{5\sqrt{2}}, \frac{-4}{5\sqrt{2}}, \frac{5}{5\sqrt{2}}$

i.e. $\frac{3}{5\sqrt{2}}, \frac{-4}{5\sqrt{2}}, \frac{1}{\sqrt{2}}$.

Example 9. ***Show that the vector*** $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ ***is equally inclined to the axes*** OX, OY and OZ.

Sol. Let $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$

If $\mathbf{a}$ makes angles α, β, γ with X, Y and Z-axes respectively, then

$$\cos\alpha = \frac{1}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}}$$

$$\cos\beta = \frac{1}{\sqrt{3}}$$

and $$\cos\gamma = \frac{1}{\sqrt{3}}$$

Thus, we have $\cos\alpha = \cos\beta = \cos\gamma$, i.e. $\alpha = \beta = \gamma$

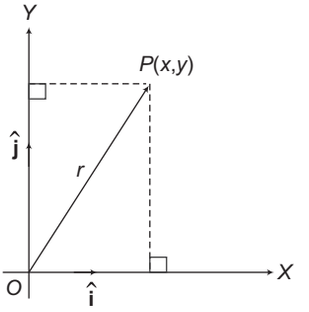
Hence, $\mathbf{a}$ is equally inclined to the axes.

Rectangular Resolution of a Vector in 2D and 3D Systems

In Two Dimensional System

Any vector $\mathbf{r}$ in two dimensional system can be expressed as $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$. The vectors $x\hat{\mathbf{i}}$ and $y\hat{\mathbf{j}}$ are called the perpendicular component vectors of $\mathbf{r}$.

The scalars x and y are called the components or resolved parts of $\mathbf{r}$ in the directions of X-axis and Y-axis, respectively and the ordered pair (x, y) is known as coordinates of point whose position vector is $\mathbf{r}$.

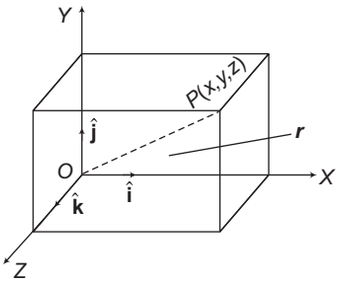


Also, the magnitude of $\mathbf{r} = \sqrt{x^2 + y^2}$ and if θ is the inclination of $\mathbf{r}$ with the X-axis, then $\theta = \tan^{-1}\left(\frac{y}{x}\right)$.

In three Dimensional System

Any vector $\mathbf{r}$ in three dimensional system can be expressed as

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$



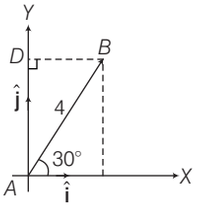
The vectors $x\hat{\mathbf{i}}$, $y\hat{\mathbf{j}}$ and $z\hat{\mathbf{k}}$ are called the right angled components of $\mathbf{r}$.

The scalars x, y and z are called the components or resolved parts of $\mathbf{r}$ in the directions of X-axis, Y-axis and Z-axis, respectively and ordered triplet (x, y, z) is known as coordinates of P whose position vector is $\mathbf{r}$. Also, the magnitude or modulus of

$$\mathbf{r} = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}.$$

Example 10. ***Let*** AB ***be a vector in two dimensional plane with the magnitude 4 units and making an angle of 30° with*** X***-axis and lying in the first quadrant.***
Find the components of AB ***along the two axes of coordinates. Hence, represent*** AB ***in terms of unit vectors*** $\hat{\mathbf{i}}$ ***and*** $\hat{\mathbf{j}}$.

Sol. Let us consider A as origin. From the diagram, it can be seen that the component of **AB** along X-axis



$$= AB\cos 30° = 4\cos 30°$$

$$= 4 \times \frac{\sqrt{3}}{2} = 2\sqrt{3}$$

and the component of **AB** along Y-axis

$$= AB\sin 30° = 4 \times \frac{1}{2} = 2$$

Hence, $\mathbf{AB} = 2\sqrt{3}\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$

Exercise for Session 1

1. Classify the following measures as scalars and vector :

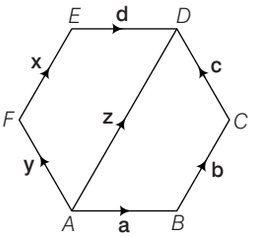
(i) 20 kg weight (ii) 45°

(iii) 10 m south-east (iv) 50 m/sec^2

2. Represent the following graphically:

(i) A displacement of 40 km, 30° west of south (ii) a displacement of 70 km, 40° north of west

3. In the given figure, *ABCDEF* is a regular hexagon, which vectors are:



(i) Collinear (ii) Equal

(iii) Coinitial (iv) Collinear but not equal

4. Answer the following as true or false

(i) **a** and **a** are collinear.

(ii) Two collinear vectors are always equal in magnitude.

(iii) Zero vector is unique.

(iv) Two vectors having same magnitude are collinear.

5. Find the perimeter of a triangle with sides $3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$, $4\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$ and $7\hat{\mathbf{i}} + \hat{\mathbf{j}}$.

6. Find the angle of vector $\mathbf{a} = 6\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$ with X-axis.

7. Write the direction ratios of the vector $\mathbf{r} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and hence calculate its direction cosines.

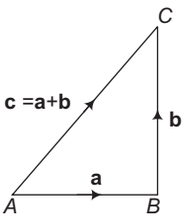
Session 2

Addition & Subtraction of Vectors, Multiplication of Vector by Scalar, Section Formula

Addition of Vectors
(Resultant of Vectors)

1. Triangle Law of Addition

If two vectors are represented by two consecutive sides of a triangle, then their sum is represented by the third side of the triangle, but in opposite direction. This is known as the triangle law of addition of vectors. Thus, if $\mathbf{AB} = \mathbf{a}$, $\mathbf{BC} = \mathbf{b}$ and $\mathbf{AC} = \mathbf{c}$, then $\mathbf{AB} + \mathbf{BC} = \mathbf{AC}$ i.e. $\mathbf{a} + \mathbf{b} = \mathbf{c}$.

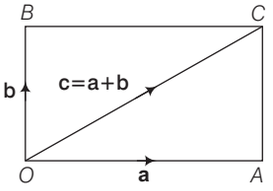


2. Parallelogram Law of Addition

If two vectors are represented by two adjacent sides of a parallelogram, then their sum is represented by the diagonal of the parallelogram whose initial point is the same as the initial point of the given vectors. This is known as parallelogram law of vector addition.

Thus, if $\mathbf{OA} = \mathbf{a}, \mathbf{OB} = \mathbf{b}$ and $\mathbf{OC} = \mathbf{c}$

Then, $\mathbf{OA} + \mathbf{OB} = \mathbf{OC}$ i.e. $\mathbf{a} + \mathbf{b} = \mathbf{c}$, where *OC* is a diagonal of the parallelogram *OACB*.

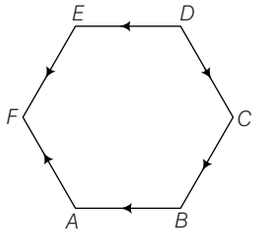


Remarks

1. The magnitude of $\mathbf{a} + \mathbf{b}$ is not equal to the sum of the magnitudes of $\mathbf{a}$ and $\mathbf{b}$.
2. From the figure, we have $\mathbf{OA} + \mathbf{AC} = \mathbf{OC}$ (By triangle law of vector addition)
   or $\mathbf{OA} + \mathbf{OB} = \mathbf{OC}$ ($\because \mathbf{AC} = \mathbf{OB}$), which is the parallelogram law. Thus, we may say that the two laws of vector addition are equivalent to each other.

3. Polygon law of addition

If the number of vectors are represented by the sides of a polygon taken in order, the resultant is represented by the closing side of the polygon taken in the reverse order.



In the figure, $\mathbf{AB} + \mathbf{BC} + \mathbf{CD} + \mathbf{DE} + \mathbf{EF} = \mathbf{AF}$

4. Addition in Component Form

If the vectors are defined in terms of $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$, i.e. if $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$, then their sum is defined as
$\mathbf{a} + \mathbf{b} = (a_1 + b_1)\hat{\mathbf{i}} + (a_2 + b_2)\hat{\mathbf{j}} + (a_3 + b_3)\hat{\mathbf{k}}$.

Properties of Vector Addition

Vector addition has the following properties

(i) **Closure** The sum of two vectors is always a vector.

(ii) **Commutativity** For any two vectors $\mathbf{a}$ and $\mathbf{b}$,
$\Rightarrow \quad \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

(iii) **Associativity** For any three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$,
$\Rightarrow \quad \mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$

(iv) **Identity** Zero vector is the identity for addition. For any vector $\mathbf{a}$.
$\Rightarrow \quad \mathbf{0} + \mathbf{a} = \mathbf{a} = \mathbf{a} + \mathbf{0}$

(v) **Additive inverse** For every vector $\mathbf{a}$ its negative vector $-\mathbf{a}$ exists such that $\mathbf{a} + (-\mathbf{a}) = (-\mathbf{a}) + \mathbf{a} = \mathbf{0}$
i.e. $(-\mathbf{a})$ is the additive inverse of the vector $\mathbf{a}$.

Example 11. ***Find the unit vector parallel to the resultant vector of*** $2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$ ***and*** $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$.

Sol. Resultant vector, $\mathbf{r} = (2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 5\hat{\mathbf{k}}) + (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$
$= 3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$

Unit vector parallel to $\mathbf{r} = \frac{1}{|\mathbf{r}|}\mathbf{r}$

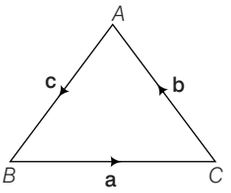
$$= \frac{1}{\sqrt{3^2 + 6^2 + (-2)^2}}(3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$$

$$= \frac{1}{7}(3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$$

Example 12. ***If*** $\mathbf{a}, \mathbf{b}$ ***and*** $\mathbf{c}$ ***are the vectors represented by the sides of a triangle, taken in order, then prove that*** $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$.

Sol. Let ABC be a triangle such that

$\mathbf{BC} = \mathbf{a}$, $\mathbf{CA} = \mathbf{b}$ and $\mathbf{AB} = \mathbf{c}$



Then, $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{BC} + \mathbf{CA} + \mathbf{AB}$

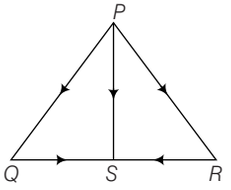
$= \mathbf{BA} + \mathbf{AB}$ $(\because \mathbf{BC} + \mathbf{CA} = \mathbf{BA})$

$= -\mathbf{AB} + \mathbf{AB}$

$\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$ **Hence proved.**

Example 13. ***If*** S ***is the mid-point of side*** QR ***of a*** ΔPQR***, then prove that*** $\mathbf{PQ} + \mathbf{PR} = 2\mathbf{PS}$.

Sol. Clearly, by triangle law of addition, we have



$$\mathbf{PQ} + \mathbf{QS} = \mathbf{PS} \quad \text{...(i)}$$

and $\mathbf{PR} + \mathbf{RS} = \mathbf{PS}$ ... (ii)

On adding Eqs. (i) and (ii), we get

$$(\mathbf{PQ} + \mathbf{QS}) + (\mathbf{PR} + \mathbf{RS}) = 2\mathbf{PS}$$

$$\Rightarrow \quad (\mathbf{PQ} + \mathbf{PR}) + (\mathbf{QS} + \mathbf{RS}) = 2\mathbf{PS}$$

$$\Rightarrow \quad \mathbf{PQ} + \mathbf{PR} + 0 = 2\mathbf{PS}$$

$[\because S$ is the mid-point of $\mathbf{QR} \therefore \mathbf{QS} = -\mathbf{RS}]$

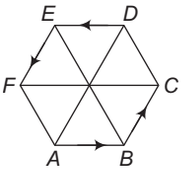
Hence, $\mathbf{PQ} + \mathbf{PR} = 2\mathbf{PS}$ **Hence proved.**

Example 14. ***If*** $ABCDEF$ ***is a regular hexagon, prove that*** $\mathbf{AD} + \mathbf{EB} + \mathbf{FC} = 4\mathbf{AB}$.

Sol. We have,

$$\mathbf{AD} + \mathbf{EB} + \mathbf{FC} = (\mathbf{AB} + \mathbf{BC} + \mathbf{CD}) + (\mathbf{ED} + \mathbf{DC} + \mathbf{CB}) + \mathbf{FC}$$

$$= \mathbf{AB} + (\mathbf{BC} + \mathbf{CB}) + (\mathbf{CD} + \mathbf{DC}) + \mathbf{ED} + \mathbf{FC}$$



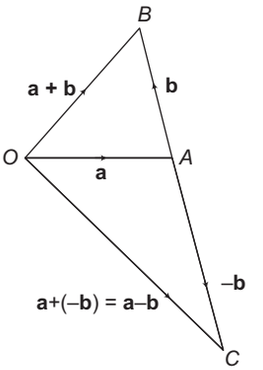
$= \mathbf{AB} + \mathbf{O} + \mathbf{O} + \mathbf{AB} + 2\mathbf{AB} = 4\mathbf{AB}$

$(\because \mathbf{ED} = \mathbf{AB}, \mathbf{FC} = 2\mathbf{AB})$

Hence proved.

Subtraction of Vectors

If $\mathbf{a}$ and $\mathbf{b}$ are two vectors, then their subtraction $\mathbf{a} - \mathbf{b}$ is defined as $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$, where $-\mathbf{b}$ is the negative of $\mathbf{b}$ having magnitude equal to that of $\mathbf{b}$ and direction opposite to $\mathbf{b}$.



If $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$

and $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$

Then, $\mathbf{a} - \mathbf{b} = (a_1 - b_1)\hat{\mathbf{i}} + (a_2 - b_2)\hat{\mathbf{j}} + (a_3 - b_3)\hat{\mathbf{k}}$

Properties of Vector Subtraction

(i) $\mathbf{a} - \mathbf{b} \neq \mathbf{b} - \mathbf{a}$

(ii) $(\mathbf{a} - \mathbf{b}) - \mathbf{c} \neq \mathbf{a} - (\mathbf{b} - \mathbf{c})$

(iii) Since, any one side of a triangle is less than the sum and greater than the difference of the other two sides, so for any two vectors a and b, we have

(a) $|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$ (b) $|\mathbf{a} + \mathbf{b}| \geq |\mathbf{a}| - |\mathbf{b}|$

(c) $|\mathbf{a} - \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$ (d) $|\mathbf{a} - \mathbf{b}| \geq |\mathbf{a}| - |\mathbf{b}|$

Remark

If A and B are two points in space having coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2), then

$\mathbf{AB}$ = Position Vector of B – Position Vector of A

$= (x_2\hat{\mathbf{i}} + y_2\hat{\mathbf{j}} + z_2\hat{\mathbf{k}}) - (x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}} + z_1\hat{\mathbf{k}})$

$= (x_2 - x_1)\hat{\mathbf{i}} + (y_2 - y_1)\hat{\mathbf{j}} + (z_2 - z_1)\hat{\mathbf{k}}$

Example 15. ***If*** $A=(0,1), B=(1,0)$, $C=(1,2)$ ***and*** $D=(2,1)$, ***prove that vector*** **AB** ***and*** **CD** ***are equal.***

Sol. Here, $\mathbf{AB}=(1-0)\hat{\mathbf{i}}+(0-1)\hat{\mathbf{j}}=\hat{\mathbf{i}}-\hat{\mathbf{j}}$

and $\mathbf{CD}=(2-1)\hat{\mathbf{i}}+(1-2)\hat{\mathbf{j}}=\hat{\mathbf{i}}-\hat{\mathbf{j}}$

Clearly, $\mathbf{AB}=\mathbf{CD}$ **Hence proved.**

Example 16. ***If the position vectors of*** **A** ***and*** **B** ***respectively*** $\hat{\mathbf{i}}+3\hat{\mathbf{j}}-7\hat{\mathbf{k}}$ ***and*** $5\hat{\mathbf{i}}-2\hat{\mathbf{j}}+4\hat{\mathbf{k}}$, ***then find*** **AB.**

Sol. Let O be the origin, then we have

$\mathbf{OA}=\hat{\mathbf{i}}+3\hat{\mathbf{j}}-7\hat{\mathbf{k}}$

and $\mathbf{OB}=5\hat{\mathbf{i}}-2\hat{\mathbf{j}}+4\hat{\mathbf{k}}$

Now, $\mathbf{AB}=\mathbf{OB}-\mathbf{OA}=(5\hat{\mathbf{i}}-2\hat{\mathbf{j}}+4\hat{\mathbf{k}})-(\hat{\mathbf{i}}+3\hat{\mathbf{j}}-7\hat{\mathbf{k}})$

$=4\hat{\mathbf{i}}-5\hat{\mathbf{j}}+11\hat{\mathbf{k}}$

Example 17. ***Vectors drawn from the origin*** O ***to the points*** A, B ***and*** C ***are respectively*** $\mathbf{a}, \mathbf{b}$ ***and*** $4\mathbf{a}-3\mathbf{b}$. ***Find*** **AC** ***and*** **BC.**

Sol. We have, $\mathbf{OA}=\mathbf{a}$, $\mathbf{OB}=\mathbf{b}$ and $\mathbf{OC}=4\mathbf{a}-3\mathbf{b}$

Clearly, $\mathbf{AC}=\mathbf{OC}-\mathbf{OA}=(4\mathbf{a}-3\mathbf{b})-(\mathbf{a})$

$=3\mathbf{a}-3\mathbf{b}$

and $\mathbf{BC}=\mathbf{OC}-\mathbf{OB}=(4\mathbf{a}-3\mathbf{b})-(\mathbf{b})=4\mathbf{a}-4\mathbf{b}$

Example 18. ***Find the direction cosines of the vector joining the points*** $A(1,2,-3)$ ***and*** $B(-1,-2,1)$, ***directed from*** A ***to*** B.

Sol. Clearly,

$\mathbf{AB}=(-1-1)\hat{\mathbf{i}}+(-2-2)\hat{\mathbf{j}}+(1+3)\hat{\mathbf{k}}=-2\hat{\mathbf{i}}-4\hat{\mathbf{j}}+4\hat{\mathbf{k}}$

Now, $|\mathbf{AB}|=\sqrt{(-2)^2+(-4)^2+(4)^2}=\sqrt{36}=6$

$\therefore$ Unit vector along $\mathbf{AB}=\dfrac{\mathbf{AB}}{|\mathbf{AB}|}=\dfrac{-2\hat{\mathbf{i}}-4\hat{\mathbf{j}}+4\hat{\mathbf{k}}}{6}$

$=-\dfrac{1}{3}\hat{\mathbf{i}}-\dfrac{2}{3}\hat{\mathbf{j}}+\dfrac{2}{3}\hat{\mathbf{k}}$

Example 19. ***Let*** α, β ***and*** γ ***be distinct real numbers. The points with position vectors*** $\alpha\hat{\mathbf{i}}+\beta\hat{\mathbf{j}}+\gamma\hat{\mathbf{k}}$, $\beta\hat{\mathbf{i}}+\gamma\hat{\mathbf{j}}+\alpha\hat{\mathbf{k}}$ and $\gamma\hat{\mathbf{i}}+\alpha\hat{\mathbf{j}}+\beta\hat{\mathbf{k}}$

(a) are collinear

(b) form an equilateral triangle

(c) form a scalene triangle

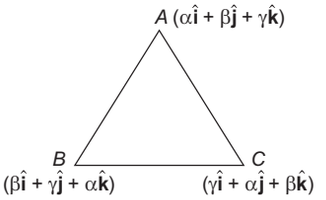
(d) form a right angled triangle

Sol. (b) Let the given points be A, B and C with position vectors $\alpha\hat{\mathbf{i}}+\beta\hat{\mathbf{j}}+\gamma\hat{\mathbf{k}}$, $\beta\hat{\mathbf{i}}+\gamma\hat{\mathbf{j}}+\alpha\hat{\mathbf{k}}$ and $\gamma\hat{\mathbf{i}}+\alpha\hat{\mathbf{j}}+\beta\hat{\mathbf{k}}$.

As, α, β and γ are distinct real numbers, therefore ABC form a triangle.

Clearly, $\mathbf{AB}=\mathbf{OB}-\mathbf{OA}=(\beta\hat{\mathbf{i}}+\gamma\hat{\mathbf{j}}+\alpha\hat{\mathbf{k}})-(\alpha\hat{\mathbf{i}}+\beta\hat{\mathbf{j}}+\gamma\hat{\mathbf{k}})$

$=(\beta-\alpha)\hat{\mathbf{i}}+(\gamma-\beta)\hat{\mathbf{j}}+(\alpha-\gamma)\hat{\mathbf{k}}$



Now, $|\mathbf{AB}|=\sqrt{(\beta-\alpha)^2+(\gamma-\beta)^2+(\alpha-\gamma)^2}$

Similarly, $\mathbf{BC}=\mathbf{CA}=\sqrt{(\beta-\alpha)^2+(\gamma-\beta)^2+(\alpha-\gamma)^2}$

$\therefore \Delta ABC$ is an equilateral triangle.

Example 20. ***If the position vectors of the vertices of a triangle be*** $2\hat{\mathbf{i}}+4\hat{\mathbf{j}}-\hat{\mathbf{k}}$, $4\hat{\mathbf{i}}+5\hat{\mathbf{j}}+\hat{\mathbf{k}}$ ***and*** $3\hat{\mathbf{i}}+6\hat{\mathbf{j}}-3\hat{\mathbf{k}}$, ***then the triangle is***

(a) right angled (b) isosceles

(c) equilateral (d) None of these

Sol. (a, b) Let A, B, C be the vertices of given triangle with position vectors, $2\hat{\mathbf{i}}+4\hat{\mathbf{j}}-\hat{\mathbf{k}}$, $4\hat{\mathbf{i}}+5\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $3\hat{\mathbf{i}}+6\hat{\mathbf{j}}-3\hat{\mathbf{k}}$ respectively.

Then, we have

$\mathbf{OA}=2\hat{\mathbf{i}}+4\hat{\mathbf{j}}-\hat{\mathbf{k}}$, $\mathbf{OB}=4\hat{\mathbf{i}}+5\hat{\mathbf{j}}+\hat{\mathbf{k}}$

and $\mathbf{OC}=3\hat{\mathbf{i}}+6\hat{\mathbf{j}}-3\hat{\mathbf{k}}$

Clearly, $\mathbf{AB}=\mathbf{OB}-\mathbf{OA}=2\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}}$

$\mathbf{BC}=-\hat{\mathbf{i}}+\hat{\mathbf{j}}-4\hat{\mathbf{k}}$

and $\mathbf{AC}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}-2\hat{\mathbf{k}}$

Now, $\mathbf{AB}=|\mathbf{AB}|=\sqrt{2^2+1^2+2^2}=3$

$\mathbf{BC}=|\mathbf{BC}|=\sqrt{(-1)^2+(1)^2+(-4)^2}=3\sqrt{2}$

and $\mathbf{AC}=|\mathbf{AC}|=\sqrt{1^2+2^2+(-2)^2}=3$

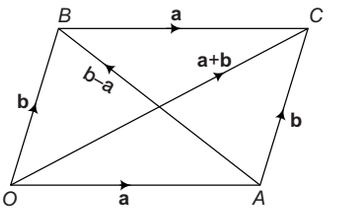
$\because$ $AB=AC$ and $BC^2=AB^2+AC^2$

$\therefore$ The triangle is isosceles and right angled.

Example 21. ***The two adjacent sides of a parallelogram are*** $2\hat{\mathbf{i}}+4\hat{\mathbf{j}}-5\hat{\mathbf{k}}$ ***and*** $\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}}$. ***Find the unit vectors along the diagonals of the parallelogram.***

Sol. Let $OABC$ be the given parallelogram and let the adjacent sides OA and OB be represented by $a=2\hat{\mathbf{i}}+4\hat{\mathbf{j}}-5\hat{\mathbf{k}}$ and $b=\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}}$ respectively.

Now, the vectors along the two diagonals are



$\mathbf{d}_1=\mathbf{a}+\mathbf{b}=3\hat{\mathbf{i}}+6\hat{\mathbf{j}}-2\hat{\mathbf{k}}$

The required unit vectors are

$$\hat{\mathbf{n}}_1 = \frac{\mathbf{d}_1}{|\mathbf{d}_1|} = \frac{3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}}}{\sqrt{3^2 + 6^2 + (-2)^2}}$$

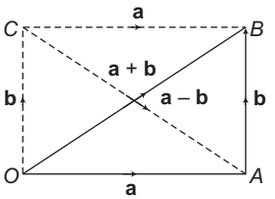
$$= \frac{3}{7}\hat{\mathbf{i}} + \frac{6}{7}\hat{\mathbf{j}} - \frac{2}{7}\hat{\mathbf{k}}$$

and $$\hat{\mathbf{n}}_2 = \frac{d_2}{|d_2|} = \frac{-\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 8\hat{\mathbf{k}}}{\sqrt{(-1)^2 + (-2)^2 + 8^2}}$$

$$= \frac{-1}{\sqrt{69}}\hat{\mathbf{i}} - \frac{2}{\sqrt{69}}\hat{\mathbf{j}} + \frac{8}{\sqrt{69}}\hat{\mathbf{k}}$$

Example 22. ***If*** $\mathbf{a}$ ***and*** $\mathbf{b}$ ***are any two vectors, then give the geometrical interpretation of the relation*** $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}|$.

Sol. Let $\mathbf{OA} = \mathbf{a}$ and $\mathbf{AB} = \mathbf{b}$. Completing the parallelogram $OABC$.



Then, $\mathbf{OC} = \mathbf{b}$ and $\mathbf{CB} = \mathbf{a}$

From ΔOAB, we have

$$\mathbf{OA} + \mathbf{AB} = \mathbf{OB} \Rightarrow \mathbf{a} + \mathbf{b} = \mathbf{OB} \quad \ldots(i)$$

From ΔOCA, we have

$$\mathbf{OC} + \mathbf{CA} = \mathbf{OA}$$

$$\Rightarrow \quad \mathbf{b} + \mathbf{CA} = \mathbf{a} \Rightarrow \mathbf{CA} = \mathbf{a} - \mathbf{b} \quad \ldots(ii)$$

Clearly, $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}| \Rightarrow |\mathbf{OB}| = |\mathbf{CA}|$

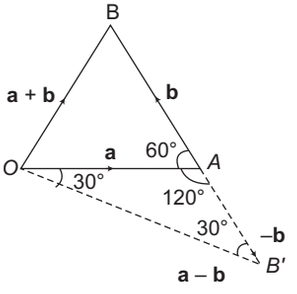
Diagonals of parallelogram $OABC$ are equal.

$OABC$ is a rectangle.

$$\Rightarrow \quad \mathbf{OA} \perp \mathbf{OC} \Rightarrow \mathbf{a} \perp \mathbf{b}$$

Example 23. ***If the sum of two unit vectors is a unit vector, prove that the magnitude of their difference is*** $\sqrt{3}$.

Sol. Let $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ be two unit vectors represented by sides OA and AB of a ΔOAB.



Then, $\mathbf{OA} = \hat{\mathbf{a}}$, $\mathbf{AB} = \hat{\mathbf{b}}$

$$\mathbf{OB} = \mathbf{OA} + \mathbf{AB} = \hat{\mathbf{a}} + \hat{\mathbf{b}}$$

(using triangle law of vector addition)

It is given that, $|\hat{\mathbf{a}}| = |\hat{\mathbf{b}}| = |\hat{\mathbf{a}} + \hat{\mathbf{b}}| = 1$

$$\Rightarrow \quad |\mathbf{OA}| + |\mathbf{AB}| = |\mathbf{OB}| = 1$$

ΔOAB is equilateral triangle.

Since, $|\mathbf{OA}| = |\hat{\mathbf{a}}| = 1 = |-\hat{\mathbf{b}}| = |\mathbf{AB}'|$

Therefore, $\Delta OAB'$ is an isosceles triangle.

$$\Rightarrow \quad \angle AB'O = \angle AOB' = 30°$$

$$\Rightarrow \quad \angle BOB' = \angle BOA + \angle AOB' = 60° + 30° = 90°$$

(since, $\Delta BOB'$ is right angled)

$\therefore$ In $\Delta BOB'$, we have

$$|BB'|^2 = |OB|^2 + |OB'|^2$$

$$= |\hat{\mathbf{a}} + \hat{\mathbf{b}}|^2 + |\hat{\mathbf{a}} - \hat{\mathbf{b}}|^2$$

$$2^2 = 1^2 + |\hat{\mathbf{a}} - \hat{\mathbf{b}}|^2$$

$$|\hat{\mathbf{a}} - \hat{\mathbf{b}}| = \sqrt{3}$$

Hence proved.

Multiplication of a Vector by a Scalar

If $\mathbf{a}$ is a vector and m is a scalar (i.e. a real number), then $m\,\mathbf{a}$ is a vector whose magnitude is m times that of $\mathbf{a}$ and whose direction is the same as that of $\mathbf{a}$, if m is positive and opposite to that of $\mathbf{a}$, if m is negative.

$\therefore$ Magnitude of $m\mathbf{a} = |m\mathbf{a}| \Rightarrow m$ (magnitude of $\mathbf{a}$) $= m\,|\mathbf{a}|$

Again, if $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$,

then $m\,\mathbf{a} = (ma_1)\,\hat{\mathbf{i}} + (ma_2)\,\hat{\mathbf{j}} + (ma_3)\,\hat{\mathbf{k}}$

Properties of Multiplication of Vectors by a Scalar

The following are properties of multiplication of vectors by scalars, for vectors $\mathbf{a}$, $\mathbf{b}$ and scalars m, n

(i) $m(-\mathbf{a}) = (-m)\,\mathbf{a} = -(m\mathbf{a})$

(ii) $(-m)(-\mathbf{a}) = m\,\mathbf{a}$

(iii) $m(n\mathbf{a}) = (mn)\,\mathbf{a} = n(m\mathbf{a})$

(iv) $(m+n)\,\mathbf{a} = m\mathbf{a} + n\mathbf{a}$

(v) $m(\mathbf{a} + \mathbf{b}) = m\,\mathbf{a} + m\,\mathbf{b}$

Example 24. ***If*** $\mathbf{a}$ ***is a non-zero vector of modulus*** a ***and*** m, ***is a non-zero scalar, then*** $m\mathbf{a}$ ***is a unit vector, if***

(a) $m = \pm 1$ (b) $m = |\mathbf{a}|$

(c) $m = \dfrac{1}{|\mathbf{a}|}$ (d) $m = \pm 2$

Sol. (c) Since, $m\mathbf{a}$ is a unit vector, $|m\mathbf{a}| = 1$

$$\Rightarrow \quad |m|\,|\mathbf{a}| = 1$$

$$\Rightarrow \quad |m| = \frac{1}{|\mathbf{a}|} \Rightarrow m = \pm\frac{1}{|\mathbf{a}|}$$

Example 25. ***For a non-zero vector*** **a**, ***the set of real numbers, satisfying*** $|(5-x)\,\mathbf{a}| < |2\mathbf{a}|$ ***consists of all*** x ***such that***

(a) $0 < x < 3$

(b) $3 < x < 7$

(c) $-7 < x < -3$

(d) $-7 < x < 3$

Sol. (b) We have, $|(5-x)\,\mathbf{a}| < |2\mathbf{a}|$

$$|5-x|\,|\mathbf{a}| < 2|\,\mathbf{a}|$$

$$\Rightarrow \quad |5-x| < 2$$

$$\Rightarrow \quad -2 < 5-x < 2$$

$$\Rightarrow \quad 3 < x < 7$$

Example 26. ***Find a vector of magnitude (5/2) units which is parallel to the vector*** $3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$.

Sol. Here, $\mathbf{a} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$

Then, $|\mathbf{a}| = \sqrt{3^2 + 4^2} = 5$

$\therefore$ A unit vector parallel to

$$\mathbf{a} = \hat{\mathbf{a}} = \frac{a}{|\mathbf{a}|} \cdot = \frac{1}{5}(3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}) \qquad \ldots(i)$$

Hence, the required vector of magnitude (5/2) units and parallel to **a**

$$= \frac{5}{2} \cdot \hat{\mathbf{a}} = \frac{5}{2} \cdot \frac{1}{5}(3\hat{\mathbf{i}} + 4\hat{\mathbf{j}})$$

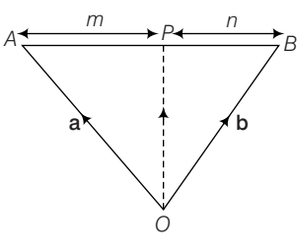
$$= \frac{1}{2}(3\hat{\mathbf{i}} + 4\hat{\mathbf{j}})$$

Section Formula

Let A and B be two points with position vectors **a** and **b** respectively. Let P be a point on AB dividing it is the ratio $m : n$.

Internal Division

If P divides AB internally in the ratio $m : n$. Then the position vector of P is given by



$$\mathbf{OP} = \frac{m\mathbf{b} + n\mathbf{a}}{m+n}$$

Proof

Let O be the origin. Then $\mathbf{OA} = \mathbf{a}$ and $\mathbf{OB} = \mathbf{b}$. Let r be the position vector of P which divides AB internally is the ratio $m : n$. Then

$$\frac{AP}{PB} = \frac{m}{n}$$

or $$n\mathbf{AP} = m\mathbf{PB}$$

or $$n(\text{PV of } P - \text{PV of } A) = m\,(\text{PV of } B - \text{PV of } P)$$

or $$n(\mathbf{r} - \mathbf{a}) = m(\mathbf{b} - \mathbf{r})$$

or $$n\mathbf{r} - n\mathbf{a} = m\mathbf{b} - m\mathbf{r}$$

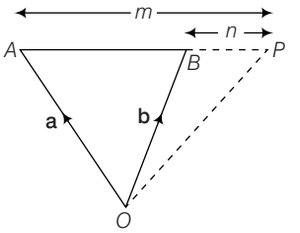
or $$\mathbf{r}(n+m) = m\mathbf{b} + n\mathbf{a}$$

or $$r = \frac{m\mathbf{b} + n\mathbf{a}}{m+n}$$

or $$\mathbf{OP} = \frac{m\mathbf{b} + n\mathbf{a}}{m+n}$$

External Division

If P divides AB externally in the ratio $m : n$. Then, the position vector of P is given by



$$\mathbf{OP} = \frac{m\mathbf{b} - n\mathbf{a}}{m-n}$$

Proof

Let O be the origin. Then $\mathbf{OA} = \mathbf{a}$, $\mathbf{OB} = \mathbf{b}$. Let $\mathbf{r}$ be the position vector of point P dividing AB externally in the ratio $m : n$.

Then, $$\frac{AP}{BP} = \frac{m}{n}$$

or $$nAP = mBP$$

or $$n\mathbf{AP} = m\mathbf{BP}$$

or $$n\,(\text{PV of } P - \text{PV of } A) = m\,(\text{PV of } P - \text{PV of } B)$$

or $$n(\mathbf{r} - \mathbf{a}) = m(\mathbf{r} - \mathbf{b})$$

or $$n\mathbf{r} - n\mathbf{a} = m\mathbf{r} - m\mathbf{b}$$

or $$r(m-n) = m\mathbf{b} - n\mathbf{a}$$

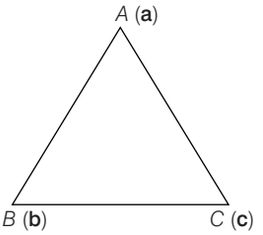
or $$\mathbf{r} = \frac{m\mathbf{b} - n\mathbf{a}}{m-n}$$

or $$\mathbf{OP} = \frac{m\mathbf{b} - n\mathbf{a}}{m-n}$$

Remarks

1. Position vector of mid-point of AB is $\frac{\mathbf{a}+\mathbf{b}}{2}$.

2. In ΔABC, having vertices $A(\mathbf{a})$, $B(\mathbf{b})$ and $C(\mathbf{c})$



(i) Position vector of centroid is $\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}$.

(ii) Position vector of incentre is
$$\frac{BC\mathbf{a}+AC\mathbf{b}+AB\mathbf{c}}{AB+BC+AC}.$$

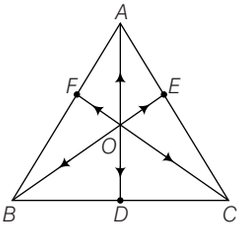
(iii) Position vector of orthocentre is
$$\frac{\tan A\,\mathbf{a}+\tan B\,\mathbf{b}+\tan C\,\mathbf{c}}{\tan A+\tan B+\tan C}.$$

(iv) Position vector of circumcentre is
$$\frac{\sin 2A\,\mathbf{a}+\sin 2B\,\mathbf{b}+\sin 2C\,\mathbf{c}}{\sin 2A+\sin 2B+\sin 2C}.$$

Example 27. *If D, E and F are the mid-points of the sides BC, CA and AB respectively of the ΔABC and O be any point, then prove that*

$$\mathbf{OA}+\mathbf{OB}+\mathbf{OC}=\mathbf{OD}+\mathbf{OE}+\mathbf{OF}$$

Sol. Since, D is the mid-point of BC, therefore by section formula, we have



$$\mathbf{OD}=\frac{\mathbf{OB}+\mathbf{OC}}{2}$$

$$\Rightarrow \quad \mathbf{OB}+\mathbf{OC}=2\mathbf{OD} \qquad \ldots(i)$$

Similarly, $\quad \mathbf{OC}+\mathbf{OA}=2\mathbf{OE} \qquad \ldots(ii)$

and $\quad \mathbf{OB}+\mathbf{OA}=2\mathbf{OF} \qquad \ldots(iii)$

On adding Eqs. (i), (ii) and (iii), we get

$$2(\mathbf{OA}+\mathbf{OB}+\mathbf{OC})=2(\mathbf{OD}+\mathbf{OE}+\mathbf{OF})$$

$$\Rightarrow \quad \mathbf{OA}+\mathbf{OB}+\mathbf{OC}=\mathbf{OD}+\mathbf{OE}+\mathbf{OF}$$

Hence proved.

Example 28. *Find the position vectors of the points which divide the join of points $A(2\mathbf{a}-3\mathbf{b})$ and $B(3\mathbf{a}-2\mathbf{b})$ internally and externally in the ratio $2:3$.*

Sol. Let P be a point which divide AB internally in the ratio $2:3$. Then, by section formula, position vector of P is given by

$$\mathbf{OP}=\frac{2(3\mathbf{a}-2\mathbf{b})+3(2\mathbf{a}-3\mathbf{b})}{2+3}$$

$$=\frac{6\mathbf{a}-4\mathbf{b}+6\mathbf{a}-9\mathbf{b}}{5}=\frac{12}{5}\mathbf{a}-\frac{13}{5}\mathbf{b}$$

Similarly, the position vector of the point (P') which divides AB externally in the ratio $2:3$ is given by

$$\mathbf{OP}'=\frac{2(3\mathbf{a}-2\mathbf{b})-3(2\mathbf{a}-3\mathbf{b})}{2-3}$$

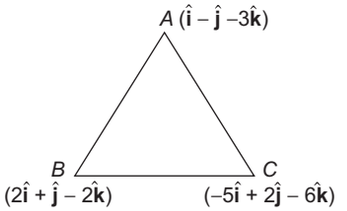
$$=\frac{6\mathbf{a}-4\mathbf{b}-6\mathbf{a}+9\mathbf{b}}{-1}=\frac{5\mathbf{b}}{-1}=-5\mathbf{b}$$

Example 29. *The position vectors of the vertices A, B and C of a triangle are $\hat{\mathbf{i}}-\hat{\mathbf{j}}-3\hat{\mathbf{k}}$, $2\hat{\mathbf{i}}+\hat{\mathbf{j}}-2\hat{\mathbf{k}}$ and $-5\hat{\mathbf{i}}+2\hat{\mathbf{j}}-6\hat{\mathbf{k}}$, respectively. The length of the bisector AD of the $\angle BAC$, where D is on the segment BC, is*

(a) $\frac{3}{4}\sqrt{10}$ (b) $\frac{1}{4}$

(c) $\frac{11}{2}$ (d) None of these

Sol. (b)



$$(a)\ |\mathbf{AB}|=|(2\hat{\mathbf{i}}+\hat{\mathbf{j}}-2\hat{\mathbf{k}})-(\hat{\mathbf{i}}-\hat{\mathbf{j}}-3\hat{\mathbf{k}})|$$

$$=|\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}}|$$

$$=\sqrt{1^2+2^2+1^2}=\sqrt{6}$$

$$|\mathbf{AC}|=|(-5\hat{\mathbf{i}}+2\hat{\mathbf{j}}-6\hat{\mathbf{k}})-(\hat{\mathbf{i}}-\hat{\mathbf{j}}-3\hat{\mathbf{k}})|$$

$$=|-6\hat{\mathbf{i}}+3\hat{\mathbf{j}}-3\hat{\mathbf{k}}|$$

$$=\sqrt{(-6)^2+3^2+(-3)^2}=\sqrt{54}=3\sqrt{6}$$

$$\mathbf{BD}:\mathbf{DC}=\mathbf{AB}:\mathbf{AC}=\frac{\sqrt{6}}{3\sqrt{6}}=\frac{1}{3}$$

$$\therefore \text{ Position vector of } D=\frac{1(-5\hat{\mathbf{i}}+2\hat{\mathbf{j}}-6\hat{\mathbf{k}})+3(2\hat{\mathbf{i}}+\hat{\mathbf{j}}-2\hat{\mathbf{k}})}{1+3}$$

$$=\frac{1}{4}(\hat{\mathbf{i}}+5\hat{\mathbf{j}}-12\hat{\mathbf{k}})$$

$\therefore$ $\mathbf{AD}$ = Position vector of D – Position vector of A

$$\mathbf{AD} = \frac{1}{4}(\hat{\mathbf{i}} + 5\hat{\mathbf{j}} - 12\hat{\mathbf{k}}) - (\hat{\mathbf{i}} - \hat{\mathbf{j}} - 3\hat{\mathbf{k}}) = \frac{1}{4}(-3\hat{\mathbf{i}} + 9\hat{\mathbf{j}})$$

$$= \frac{3}{4}(-\hat{\mathbf{i}} + 3\hat{\mathbf{j}})$$

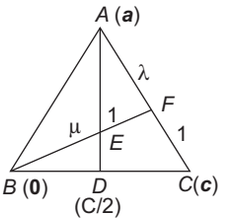
$$|\mathbf{AD}| = \frac{3}{4}\sqrt{(-1)^2 + 3^2} = \frac{3}{4}\sqrt{10}$$

Example 30. ***The median*** AD ***of the*** ΔABC ***is bisected at*** E.BE ***meets*** AC ***in*** F***. Then,*** $AF : AC$ ***is equal to***

(a) 3/4 (b) 1/3

(c) 1/2 (d) 1/4

Sol. (b) Let position vector of A w.r.t. B is $\mathbf{a}$ and that of C w.r.t. B is $\mathbf{c}$.



Position vector of D w.r.t.

$$B = \frac{0 + \mathbf{c}}{2} = \frac{\mathbf{c}}{2}$$

Position vector of

$$E = \frac{\mathbf{a} + \frac{\mathbf{c}}{2}}{2} = \frac{\mathbf{a}}{2} + \frac{\mathbf{c}}{4} \quad \ldots(i)$$

Let $AF : FC = \lambda : 1$ and $BE : EF = \mu : 1$

Position vector of $F = \dfrac{\lambda\mathbf{c} + \mathbf{a}}{1 + \lambda}$

Now, position vector of

$$E = \frac{\mu\left(\frac{\lambda\mathbf{c} + \mathbf{a}}{1 + \lambda}\right) + 1 \cdot 0}{\mu + 1} \quad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$\frac{\mathbf{a}}{2} + \frac{\mathbf{c}}{4} = \frac{\mu}{(1 + \lambda)(1 + \mu)}\mathbf{a} + \frac{\lambda\mu}{(1 + \lambda)(1 + \mu)}\mathbf{c}$$

$$\Rightarrow \quad \frac{1}{2} = \frac{\mu}{(1 + \lambda)(1 + \mu)}$$

and $$\frac{1}{4} = \frac{\lambda\mu}{(1 + \lambda)(1 + \mu)}$$

$$\Rightarrow \quad \lambda = \frac{1}{2}$$

$$\therefore \quad \frac{AF}{AC} = \frac{AF}{AF + FC} = \frac{\lambda}{1 + \lambda} = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3}$$

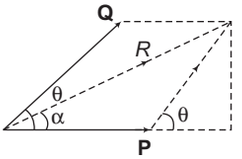
Magnitude of Resultant of Two Vectors

Let $\mathbf{R}$ be the resultant of two vectors $\mathbf{P}$ and $\mathbf{Q}$. Then,

$$\mathbf{R} = \mathbf{P} + \mathbf{Q}$$

$$|\mathbf{R}| = R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$$

where, $|\mathbf{P}| = P, |\mathbf{Q}| = Q$, $\tan\alpha = \dfrac{Q\sin\theta}{P + Q\cos\theta}$



Deduction When $|\mathbf{P}| = |\mathbf{Q}|$, i.e. $P = Q$

$$\tan\alpha = \frac{P\sin\theta}{P + P\cos\theta}$$

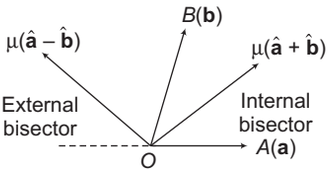
$$= \frac{\sin\theta}{1 + \cos\theta} = \tan\frac{\theta}{2}$$

$\therefore$ $$\alpha = \frac{\theta}{2}$$

Hence, the angular bisector of two unit vectors $\mathbf{a}$ and $\mathbf{b}$ is along the vector sum $\mathbf{a} + \mathbf{b}$.

Remarks

1. The internal bisector of the angle between any two vectors is along the vector sum of the corresponding unit vectors.
2. The external bisector of the angle between two vectors is along the vector difference of the corresponding unit vectors.



Example 31. ***The sum of two forces is 18 N and resultant whose direction is at right angles to the smaller force is 12 N. The magnitude of the two forces are***

(a) 13, 5 (b) 12, 6

(c) 14, 4 (d) 11, 7

Sol. (a) We have, $|\mathbf{P}| + |\mathbf{Q}| = 18\text{N}; |\mathbf{R}| = |\mathbf{P} + \mathbf{Q}| + 12\,\text{N}$

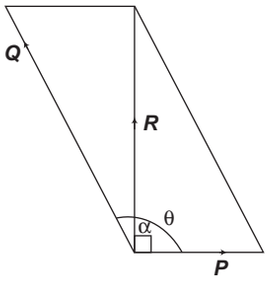
$$\alpha = 90^\circ$$

$$P + Q\cos\theta = 0$$

$$\Rightarrow \quad Q\cos\theta = -P$$

Now, $R^2 = P^2 + Q^2 + 2PQ\cos\theta$

$$\Rightarrow \quad R^2 = P^2 + Q^2 + 2P(-P) = Q^2 - P^2$$



$\Rightarrow \quad 12^2 = (P+Q)(Q-P) = 18(Q-P)$

$\Rightarrow \quad Q - P = 8 \quad \text{and} \quad Q + P = 18$

$\Rightarrow \quad Q = 13, P = 5$

$\therefore$ Magnitude of two forces are 5 N and 13 N.

Example 32. ***The length of longer diagonal of the parallelogram constructed on*** $5\mathbf{a}+2\mathbf{b}$ ***and*** $\mathbf{a}-3\mathbf{b}$, ***when it is given that*** $|\mathbf{a}| = 2\sqrt{2}, |\mathbf{b}| = 3$ ***and angle between*** $\mathbf{a}$ ***and*** $\mathbf{b}$ ***is*** $\frac{\pi}{4}$, ***is***

(a) 15 (b) $\sqrt{113}$

(c) $\sqrt{593}$ (d) $\sqrt{369}$

Sol. (c) Length of the two diagonals will be

$$d_1 = |(5\mathbf{a}+2\mathbf{b}) + (\mathbf{a}-3\mathbf{b})|$$

and $\quad d_2 = |(5\mathbf{a}+2\mathbf{b}) - (\mathbf{a}-3\mathbf{b})|$

$\Rightarrow \quad d_1 = |6\mathbf{a} - \mathbf{b}|, d_2 = |4\mathbf{a} + 5\mathbf{b}|$

Thus,

$$d_1 = \sqrt{|6\mathbf{a}|^2 + |-\mathbf{b}|^2 + 2|6\mathbf{a}||-\mathbf{b}|\cos(\pi - \pi/4)}$$

$$= \sqrt{36(2\sqrt{2})^2 + 9 + 12\cdot 2\sqrt{2}\cdot 3\cdot\left(-\frac{1}{\sqrt{2}}\right)} = 15$$

$$d_2 = \sqrt{|4\mathbf{a}|^2 + |5\mathbf{b}|^2 + 2|4\mathbf{a}||5\mathbf{b}|\cos\frac{\pi}{4}}$$

$$= \sqrt{16\times 8 + 25\times 9 + 40\times 2\sqrt{2}\times 3\times\frac{1}{\sqrt{2}}}$$

$$= \sqrt{593}$$

$\therefore$ Length of the longer diagonal $= \sqrt{593}$

Example 33. ***The vector*** $\mathbf{c}$, ***directed along the internal bisector of the angle between the vectors*** $\mathbf{a} = 7\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$ ***and*** $\mathbf{b} = -2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ ***with*** $|\mathbf{c}| = 5\sqrt{6}$, ***is***

(a) $\frac{5}{3}(\hat{\mathbf{i}} - 7\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$ (b) $\frac{5}{3}(5\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

(c) $\frac{5}{3}(\hat{\mathbf{i}} + 7\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$ (d) $\frac{5}{3}(-5\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

Sol. (a) Let $\mathbf{a} = 7\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$

and $\quad \mathbf{b} = -2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

Now, required vector $\mathbf{c} = \lambda\left(\frac{\mathbf{a}}{|\mathbf{a}|} + \frac{\mathbf{b}}{|\mathbf{b}|}\right)$

$$= \lambda\left(\frac{7\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - 4\hat{\mathbf{k}}}{9} + \frac{-2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}}{3}\right)$$

$$= \frac{\lambda}{9}(\hat{\mathbf{i}} - 7\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$$

$$|\mathbf{c}|^2 = \frac{\lambda^2}{81}\times 54 = 150$$

$\Rightarrow \quad \lambda = \pm 15$

$\Rightarrow \quad \mathbf{c} = \pm\frac{5}{3}(\hat{\mathbf{i}} - 7\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

Exercise for Session 2

1. If $\mathbf{a} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $\mathbf{b} = -\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$, then find $\mathbf{a} + \mathbf{b}$. Also, find a unit vector along $\mathbf{a} + \mathbf{b}$.
2. Find a unit vector in the direction of the resultant of the vectors $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, $-\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $3\hat{\mathbf{i}} + \hat{\mathbf{j}}$.
3. Find the direction cosines of the resultant of the vectors $(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}), (-\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}), (\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}})$ and $(\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}})$.
4. In a regular hexagon *ABCDEF*, show that $\mathbf{AE}$ is equal to $\mathbf{AC} + \mathbf{AF} - \mathbf{AB}$
5. Prove that $3\mathbf{OD} + \mathbf{DA} + \mathbf{DB} + \mathbf{DC}$ is equal to $\mathbf{OA} + \mathbf{OB} - \mathbf{OC}$.
6. In a regular hexagon *ABCDEF*, prove that $\mathbf{AB} + \mathbf{AC} + \mathbf{AD} + \mathbf{AE} + \mathbf{AF} = 3\mathbf{AD}$.
7. *ABCDE* is a pentagon, prove that $\mathbf{AB} + \mathbf{BC} + \mathbf{CD} + \mathbf{DE} + \mathbf{EA} = 0$.
8. The position vectors of *A*, *B*, *C*, *D* are $\mathbf{a}$, $\mathbf{b}$, $2\mathbf{a} + 3\mathbf{b}$ and $\mathbf{a} - 2\mathbf{b}$, respectively. Show that $\mathbf{DB} = 3\mathbf{b} - \mathbf{a}$ and $\mathbf{AC} = \mathbf{a} + 3\mathbf{b}$.
9. If $P(-1, 2)$ and $Q(3, -7)$ are two points, express the vector $\mathbf{PQ}$ in terms of unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$. Also, find distance between point *P* and *Q*. What is the unit vector in the direction of $\mathbf{PQ}$?
10. If $\mathbf{OP} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\mathbf{OQ} = 3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, find the modulus and direction cosines of $\mathbf{PQ}$.
11. Show that the points *A*, *B* and *C* with position vectors $\mathbf{a} = 3\hat{\mathbf{j}} - 4\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$, $\mathbf{b} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{c} = \hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$ respectively, form the vertices of a right angled triangle.
12. If $\mathbf{a} = 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $|x\mathbf{a}| = 1$, then find x.
13. If $\mathbf{p} = 7\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and $\mathbf{q} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}} + 5\hat{\mathbf{k}}$, then find the magnitude of $\mathbf{p} - 2\mathbf{q}$.
14. Find a vector in the direction of $5\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, which has magnitude 8 units.
15. If $\mathbf{a} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $\mathbf{b} = 3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, then find a vector in the direction of $\mathbf{a}$ and having magnitude as $|\mathbf{b}|$.
16. Find the position vector of a point *P* which divides the line joining two points *A* and *B* whose position vectors are $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $-\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ respectively, in the ratio $2 : 1$.
    (i) internally (ii) externally
17. If the position vector of one end of the line segment *AB* be $2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and the position vector of its middle point be $3(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$, then find the position vector of the other end

Session 3

Linear Combination of Vectors, Theorem on Coplanar & Non-coplanar Vectors, Linear Independence and Dependence of Vectors

Linear Combination of Vectors

A vector $\mathbf{r}$ is said to be a linear combination of vectors **a, b** and **c**... etc., if there exist scalars x, y and z etc., such that $\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c} + \ldots$

For examples Vectors $\mathbf{r}_1 = 2\mathbf{a} + \mathbf{b} + 3\mathbf{c}$ and $\mathbf{r}_2 = \mathbf{a} + 3\mathbf{b} + \sqrt{2}\mathbf{c}$ are linear combinations of the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Collinearity and Coplanarity of Vectors

Relation between Two Collinear Vectors (or Parallel Vectors)

Let $\mathbf{a}$ and $\mathbf{b}$ be two collinear vectors and let $\hat{\mathbf{x}}$ be the unit vector in the direction of $\mathbf{a}$. Then, the unit vector in the direction of $\mathbf{b}$ is $\hat{\mathbf{x}}$ or $-\hat{\mathbf{x}}$ according as $\mathbf{a}$ and $\mathbf{b}$ are like or unlike parallel vectors. Now, $\mathbf{a} = |\mathbf{a}|\,\hat{\mathbf{x}}$ and $\mathbf{b} = \pm|\mathbf{b}|\,\hat{\mathbf{x}}$.

$$\therefore \qquad \mathbf{a} = \left(\frac{|\mathbf{a}|}{|\mathbf{b}|}\right)|\mathbf{b}|\,\hat{\mathbf{x}} \Rightarrow \mathbf{a} = \pm\left(\frac{|\mathbf{a}|}{|\mathbf{b}|}\right)\mathbf{b}$$

$$\Rightarrow \qquad \mathbf{a} = \lambda\mathbf{b}, \text{ where } \lambda = \pm\frac{|\mathbf{a}|}{|\mathbf{b}|}$$

Thus, if $\mathbf{a}$ and $\mathbf{b}$ are collinear vectors, then $\mathbf{a} = \lambda\mathbf{b}$ or $\mathbf{b} = \lambda\mathbf{a}$ for some scalar λ i.e, there exist two non-zero scalar quantities x and y so that $x\mathbf{a} + y\mathbf{b} = \mathbf{O}$

An Important Theorem

Theorem : Vectors $\mathbf{a}$ and $\mathbf{b}$ are two non-zero, non-collinear vectors and x, y are two scalars such that

$$x\mathbf{a} + y\mathbf{b} = 0$$

Then, $\qquad x = 0, y = 0$

Proof It is given that $\qquad x\mathbf{a} + y\mathbf{b} = 0 \qquad \ldots(i)$

Suppose that $x \neq 0$, then dividing both sides of (i) by the scalar x, we get

$$\mathbf{a} = -\frac{y}{x}\mathbf{b} \qquad \ldots(ii)$$

Now, $\frac{y}{x}$ is a scalar, because x and y are scalars.

Hence, Eq. (ii) expresses $\mathbf{a}$ as product of $\mathbf{b}$ by a scalar, so that $\mathbf{a}$ and $\mathbf{b}$ are collinear. Thus, we arrive at a contradiction because $\mathbf{a}$ and $\mathbf{b}$ are given to be non-collinear.

Thus our supposition that $x \neq 0$, is wrong.

Hence, $x = 0$. Similarly, $y = 0$

Remarks

1. $x\mathbf{a} + y\mathbf{b} = 0 \Rightarrow \begin{cases} \mathbf{a} = 0, \mathbf{b} = 0 \\ \text{or} \\ x = 0, y = 0 \\ \text{or} \\ \mathbf{a} \,||\, \mathbf{b} \end{cases}$

2. If $\mathbf{a}$ and $\mathbf{b}$ are two non-collinear (or non-parallel) vectors, then $x_1\mathbf{a} + y_1\mathbf{b} = x_2\mathbf{a} + y_2\mathbf{b}$

$\Rightarrow \qquad x_1 = x_2$ and $y_1 = y_2$

Proof $x_1\mathbf{a} + y_1\mathbf{b} = x_2\mathbf{b} + y_2\mathbf{b}$

$\Rightarrow \qquad (x_1 - x_2)\mathbf{a} + (y_1 - y_2)\mathbf{b} = 0$

$\Rightarrow \qquad x_1 - x_2 = 0$ and $y_1 - y_2 = 0$ [$\because \mathbf{a}$ and $\mathbf{b}$ are non-collinear]

$\Rightarrow \qquad x_1 = x_2$ and $y_1 = y_2$

If $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $b = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$, then $\mathbf{a} \,||\, \mathbf{b}$

$\Rightarrow \qquad \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$

Test of Collinearity of Three Points

(i) Three points A, B and C are collinear, if $\mathbf{AB} = \lambda\mathbf{BC}$

(ii) Three points with position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are collinear iff there exist scalars x, y and z not all zero such that $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = 0$, where $x + y + z = 0$

Proof Let us suppose that points A, B and C are collinear and their position vectors are **a, b** and **c** respectively. Let C divide the join of $\mathbf{a}$ and $\mathbf{b}$ in the ratio $y : x$. Then,

$$\mathbf{c} = \frac{x\mathbf{a} + y\mathbf{b}}{x + y}$$

or $\qquad x\mathbf{a} + y\mathbf{b} - (x + y)\mathbf{c} = 0$

or $\qquad x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = 0$, where $z = -(x + y)$

Also, $x+y+z=x+y-(x+y)=0$

Conversely, let $x\mathbf{a}+y\mathbf{b}+z\mathbf{c}=0$, where $x+y+z=0$. Therefore,

$$x\mathbf{a}+y\mathbf{b}=-z\mathbf{c}=(x+y)\mathbf{c} \quad (\because x+y=-z)$$

or $$\mathbf{c}=\frac{x\mathbf{a}+y\mathbf{b}}{x+y}$$

This relation shows that $\mathbf{c}$ divides the join of $\mathbf{a}$ and $\mathbf{b}$ in the ratio $y:x$. Hence, the three points A, B and C are collinear.

(iii) If $\mathbf{a}=a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}$, $\mathbf{b}=b_1\hat{\mathbf{i}}+b_2\hat{\mathbf{j}}$ and $\mathbf{c}=c_1\hat{\mathbf{i}}+c_2\hat{\mathbf{j}}$, then the points with position vector **a, b** and **c** will be collinear iff $\begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix}=0.$

Proof The points with position vector $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ will be collinear iff there exist scalars x, y and z not all zero such that,

$x(a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}})+y(b_1\hat{\mathbf{i}}+b_2\hat{\mathbf{j}})+z(c_1\hat{\mathbf{i}}+c_2\hat{\mathbf{j}})=0$ and $x+y+z=0$

$$\Rightarrow \quad xa_1+yb_1+zc_1=0$$
$$xa_2+yb_2+zc_2=0$$
$$x+y+z=0$$

Thus, the points will be collinear iff the above system of equation's have non-trivial solution

Hence, the points will be collinear

iff $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{vmatrix}=0$ or $\begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix}=0.$

Example 34. *Show that the vectors* $2\hat{\mathbf{i}}-3\hat{\mathbf{j}}+4\hat{\mathbf{k}}$ *and* $-4\hat{\mathbf{i}}+6\hat{\mathbf{j}}-8\hat{\mathbf{k}}$ *are collinear.*

Sol. Let $\mathbf{a}=2\hat{\mathbf{i}}-3\hat{\mathbf{j}}+4\hat{\mathbf{k}}$ and $\mathbf{b}=-4\hat{\mathbf{i}}+6\hat{\mathbf{j}}-8\hat{\mathbf{k}}$

Consider, $\mathbf{b}=-4\hat{\mathbf{i}}+6\hat{\mathbf{j}}-8\hat{\mathbf{k}}=-2(2\hat{\mathbf{i}}-3\hat{\mathbf{j}}+4\hat{\mathbf{k}})=-2\mathbf{a}$

$\therefore$ The vectors $\mathbf{a}$ and $\mathbf{b}$ are collinear.

Example 35. *Show that the points* $A(1,2,3), B(3,4,7)$ *and* $C(-3,-2,-5)$ *are collinear. Find the ratio in which point* C *divides* AB.

Sol. Clearly, $\mathbf{AB}=(3-1)\hat{\mathbf{i}}+(4-2)\hat{\mathbf{j}}+(7-3)\hat{\mathbf{k}}$

$$=2\hat{\mathbf{i}}+2\hat{\mathbf{j}}+4\hat{\mathbf{k}}$$

and $\mathbf{BC}=(-3-3)\hat{\mathbf{i}}+(-2-4)\hat{\mathbf{j}}+(-5-7)\hat{\mathbf{k}}$

$$=6\hat{\mathbf{i}}-6\hat{\mathbf{j}}-12\hat{\mathbf{k}}$$
$$=-3(2\hat{\mathbf{i}}+2\hat{\mathbf{j}}+4\hat{\mathbf{k}})=-3\mathbf{AB}$$

$\because \quad \mathbf{BC}=-3\mathbf{AB}$

$\therefore$ A, B and C are collinear.

Now, let C divide AB in the ratio $k:1$, then

$$\mathbf{OC}=\frac{k\mathbf{OB}+1\cdot\mathbf{OA}}{k+1}$$

$$\Rightarrow \quad -3\hat{\mathbf{i}}-2\hat{\mathbf{j}}-5\hat{\mathbf{k}}=\frac{k(3\hat{\mathbf{i}}+4\hat{\mathbf{j}}+7\hat{\mathbf{k}})+(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}})}{k+1}$$

$$\Rightarrow \quad -3\hat{\mathbf{i}}-2\hat{\mathbf{j}}-5\hat{\mathbf{k}}=\left(\frac{3k+1}{k+1}\right)\hat{\mathbf{i}}+\left(\frac{4k+2}{k+1}\right)\hat{\mathbf{j}}+\left(\frac{7k+3}{k+1}\right)\hat{\mathbf{k}}$$

$$\Rightarrow \quad \frac{3k+1}{k+1}=-3;\ \frac{4k+2}{k+1}=-2 \text{ and } \frac{7k+3}{k+1}=-5$$

From, all relations, we get $k=\frac{-2}{3}$

Hence, C divides AB externally in the ratio $2:3$.

Example 36. *If the position vectors of* A, B, C *and* D *are* $2\hat{\mathbf{i}}+\hat{\mathbf{j}}$, $\hat{\mathbf{i}}-3\hat{\mathbf{j}}$, $3\hat{\mathbf{i}}+2\hat{\mathbf{j}}$ *and* $\hat{\mathbf{i}}+\lambda\hat{\mathbf{j}}$, *respectively and* $\mathbf{AB}\,||\,\mathbf{CD}$, *then* λ *will be*

(a) –8 (b) –6
(c) 8 (d) 6

Sol. (b) $\mathbf{AB}=(\hat{\mathbf{i}}-3\hat{\mathbf{j}})-(2\hat{\mathbf{i}}+\hat{\mathbf{j}})=-\hat{\mathbf{i}}-4\hat{\mathbf{j}}$;

$$\mathbf{CD}=(\hat{\mathbf{i}}+\lambda\hat{\mathbf{j}})-(3\hat{\mathbf{i}}+2\hat{\mathbf{j}})=-2\hat{\mathbf{i}}+(\lambda-2)\hat{\mathbf{j}};$$

$\mathbf{AB}\,||\,\mathbf{CD} \Rightarrow \mathbf{AB}=x\,\mathbf{CD}$

$$-\hat{\mathbf{i}}-4\hat{\mathbf{j}}=x\{-2\hat{\mathbf{i}}+(\lambda-2)\hat{\mathbf{j}}\}$$
$$\Rightarrow \quad -1=-2x,\ -4=(\lambda-2)x$$
$$\Rightarrow \quad x=\frac{1}{2} \text{ and } \lambda=-6$$

Example 37. *The points with position vectors* $60\hat{\mathbf{i}}+3\hat{\mathbf{j}}$, $40\hat{\mathbf{i}}-8\hat{\mathbf{j}}$ *and* $a\hat{\mathbf{i}}-52\hat{\mathbf{j}}$ *are collinear, if* a *is equal to*

(a) –40 (b) 40
(c) 20 (d) None of these

Sol (a) The three points are collinear if

$$\begin{vmatrix} 60 & 3 & 1 \\ 40 & -8 & 1 \\ a & -52 & 1 \end{vmatrix}=0$$

$$\Rightarrow \quad 60(-8+52)-3(40-a)+(-2080+8a)=0$$
$$\Rightarrow \quad 2640-120+3a-2080+8a=0$$
$$11a=-440$$
$$\Rightarrow \quad a=-40$$

Example 38. *Let* $\mathbf{a}, \mathbf{b}$ *and* $\mathbf{c}$ *be three non-zero vectors such that no two of these are collinear. If the vector* $\mathbf{a}+2\mathbf{b}$ *is collinear with* $\mathbf{c}$ *and* $\mathbf{b}+3\mathbf{c}$ *is collinear with* a *(*λ *being some non-zero scalar), then* $\mathbf{a}+2\mathbf{b}+6\mathbf{c}$ *is equal to*

(a) $\mathbf{0}$ (b) $\lambda\mathbf{b}$
(c) $\lambda\mathbf{c}$ (d) $\lambda\mathbf{a}$

Sol. (a) As $\mathbf{a}+2\mathbf{b}$ and $\mathbf{c}$ are collinear $\mathbf{a}+2\mathbf{b}=\lambda\mathbf{c}$ ...(i)

Again, $\mathbf{b}+3\mathbf{c}$ is collinear with $\mathbf{a}$.

$\therefore \quad \mathbf{b}+3\mathbf{c}=\mu\mathbf{a}$ ...(ii)

Now, $\mathbf{a}+2\mathbf{b}+6\mathbf{c}=(\mathbf{a}+2\mathbf{b})+6\mathbf{c}=\lambda\mathbf{c}+6\mathbf{c}$

$=(\lambda+6)\mathbf{c}$ ...(iii)

Also, $\mathbf{a}+2\mathbf{b}+6\mathbf{c}=\mathbf{a}+2(\mathbf{b}+3\mathbf{c})=\mathbf{a}+2\mu\mathbf{a}$

$=(2\mu+1)\mathbf{a}$ ...(iv)

From Eqs. (iii) and (iv), we get

$(\lambda+6)\mathbf{c}=(2\mu+1)\mathbf{a}$

But $\mathbf{a}$ and $\mathbf{c}$ are non-zero, non-collinear vectors,

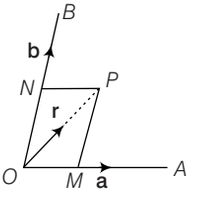
$\therefore \quad \lambda+6=0=2\mu+1$

Hence, $\mathbf{a}+2\mathbf{b}+6\mathbf{c}=0$

Theorem of Coplanar Vectors

Let $\mathbf{a}$ and $\mathbf{b}$ be two non-zero, non-collinear vectors. Then any vector $\mathbf{r}$ coplanar with $\mathbf{a}$ and $\mathbf{b}$ can be uniquely expressed as a linear combination $x\mathbf{a}+y\mathbf{b}$; x and y being scalars.

Proof Let $\mathbf{a}$ and $\mathbf{b}$ be any two non-zero, non-collinear vectors and $\mathbf{r}$ be any vector coplanar with $\mathbf{a}$ and $\mathbf{b}$. We take any point O in the plane of $\mathbf{a}$ and $\mathbf{b}$



Let $\mathbf{OA}=\mathbf{a}$, $\mathbf{OB}=\mathbf{b}$ and $\mathbf{OP}=\mathbf{r}$

Clearly, OA, OB and OP are coplanar.

Through P, we draw lines PM and PN, parallel to OB and OA respeetively meeting OA and OB at M and N respectively.

We have, $\mathbf{OP}=\mathbf{OM}+\mathbf{MP}$

$=\mathbf{OM}+\mathbf{ON}$ [$\because MP=ON$ and $MP \,||\, ON$] ...(i)

Now, $\mathbf{OM}$ and $\mathbf{OA}$ are collinear vectors

$\mathbf{OM}=x\,\mathbf{OA}=x\mathbf{a}$, where x is scalar.

Similarly, $\mathbf{ON}=y\mathbf{OB}=y\mathbf{b}$, where y is a scalar.

Hence, from Eq. (i), $\mathbf{OP}=x\mathbf{a}+y\mathbf{b}$ or $\mathbf{r}=x'\mathbf{a}+y'\mathbf{b}$

Uniqueness: If possible, let $\mathbf{r}=x\mathbf{a}+y\mathbf{b}$ and $\mathbf{r}=x'\mathbf{a}+y'\mathbf{b}$ be two different ways of representing r.

Then, we have $x\mathbf{a}+y\mathbf{b}=x'\mathbf{a}+y'\mathbf{b}$

$\Rightarrow \quad (x-x')\mathbf{a}+(y-y')\mathbf{b}=0$

But $\mathbf{a}$ and $\mathbf{b}$ are non-collinear vectors

$\therefore \quad x-x'=0$ and $y-y'=0$

$\Rightarrow \quad x'=x$ and $y'=y$

Thus, the uniqueness in established.

Test of Coplanarity of Three Vectors

(i) Three vectors **a, b, c** are coplanar iff any one of them is a linear combination of the remaining two, i.e. iff $\mathbf{a}=x\mathbf{b}+y\mathbf{c}$ where x and y are scalars.

(ii) If three points with position vectors $\mathbf{a}=a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}+a_3\hat{\mathbf{k}}$, $\mathbf{b}=b_1\hat{\mathbf{i}}+b_2\hat{\mathbf{j}}+b_3\hat{\mathbf{k}}$ and $\mathbf{c}=c_1\hat{\mathbf{i}}+c_2\hat{\mathbf{j}}+c_3\hat{\mathbf{k}}$ are coplanar,

then $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=0.$

If vectors $\mathbf{a}$,$\mathbf{b}$ and $\mathbf{c}$ are coplanar, then there exist scalars x and y such that $\mathbf{c}=x\mathbf{a}+y\mathbf{b}$.

Hence, $c_1\hat{\mathbf{i}}+c_2\hat{\mathbf{j}}+c_3\hat{\mathbf{k}}=x(a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}+a_3\hat{\mathbf{k}})$

$+y(b_1\hat{\mathbf{i}}+b_2\hat{\mathbf{j}}+b_3\hat{\mathbf{k}})$

Now, $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are non-coplanar and hence independent.

Then, $c_1=xa_1+yb_1, c_2=xa_2+yb_2$

and $c_3=xa_3+yb_3$

The above system of equations in terms of x and y is consistent. Thus,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0 \text{ or } \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=0.$$

Remark

If vectors $x_1\mathbf{a}+y_1\mathbf{b}+z_1\mathbf{c}$, $x_2\mathbf{a}+y_2\mathbf{b}+z_2\mathbf{c}$ and $x_3\mathbf{a}+y_3\mathbf{b}+z_3\mathbf{c}$ are coplanar(where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are non-coplanar).

Then, $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}=0$

Test of Coplanarity of Four Points

(i) To prove that four points $A(\mathbf{a})$, $B(\mathbf{b})$, $C(\mathbf{c})$ and $D(\mathbf{d})$ are coplanar, it is just sufficient to prove that vectors $\mathbf{AB}$, $\mathbf{AC}$ and $\mathbf{AD}$ and coplanar.

(ii) Four points with position vectors **a, b , c** and **d** are coplanar iff there exist scalars x, y, z and u not all zero such that $x\,\mathbf{a}+y\,\mathbf{b}+z\,\mathbf{c}+u\,\mathbf{d}=0$, where $x+y+z+u=0$.

(iii) Four points with position vectors

$\mathbf{a}=a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}+a_3\hat{\mathbf{k}}$,

$\mathbf{b}=b_1\,\hat{\mathbf{i}}+b_2\hat{\mathbf{j}}+b_3\hat{\mathbf{k}}$

$\mathbf{c}=c_1\hat{\mathbf{i}}+c_2\hat{\mathbf{j}}+c_3\hat{\mathbf{k}}$

and $\mathbf{d}=d_1\hat{\mathbf{i}}+d_2\hat{\mathbf{j}}+d_3\hat{\mathbf{k}}$

will be coplanar, iff $\begin{vmatrix} a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \\ d_1 & d_2 & d_3 & 1 \end{vmatrix} = 0$

or $\begin{vmatrix} d_1 - a_1 & d_2 - a_2 & d_3 - a_3 \\ b_1 - a_1 & b_2 - a_2 & b_3 - a_3 \\ c_1 - a_1 & c_2 - a_2 & c_3 - a_3 \end{vmatrix} = 0$

Theorem on Non-coplanar Vectors

Theorem 1

If **a, b, c**, are three non-zero, non-coplanar vectors and x, y, z are three scalars such that

$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = 0.$$

Then $x = y = z = 0.$

Proof It is given that $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = 0$ ...(i)

Suppose that $x \neq 0$

Then Eq. (i) can be written as

$$x\mathbf{a} = -y\mathbf{b} - z\mathbf{c}$$

$$\Rightarrow \quad \mathbf{a} = -\frac{y}{x}\mathbf{b} - \frac{z}{x}\mathbf{c} \qquad \text{...(ii)}$$

Now, $\frac{y}{x}$ and $\frac{z}{x}$ are scalars because x, y and z are scalars.

Thus, Eq. (ii) expresses **a** as a linear combination of **b** and **c**.

Hence, **a** is coplanar with **b** and **c** which is contrary to our hypothesis because **a**,**b** and **c** are given to be non-coplanar.

Thus, our supposition that $x \neq 0$ is wrong.

Hence, $x = 0$

Similarly, we can prove that $y = 0$ and $z = 0$

Theorem 2

If **a**,**b** and **c** are non-coplanar vectors, then any vector **r** can be uniquely expressed as a linear combination $x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$; x, y and z being scalars.

or

Any vector in space can be expressed as a linear combination of three non-coplanar vectors.

Proof Take any point O.

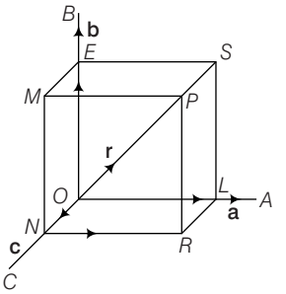
Let **a, b, c** be any three non-coplanar vectors and **r** be any vector in space.

Let $\mathbf{OA} = \mathbf{a}, \mathbf{OB} = \mathbf{b},$

$\mathbf{OC} = \mathbf{c}, \mathbf{OP} = \mathbf{r}$

Here, the three lines OA, OB, OC are not coplanar. Hence, they determine three different planes BOC, COA and AOB when taken in pairs.

Through P, draw planes parallel to these planes BOC, COA and AOB meeting OA, OB and OC in L, E and N respectively. Thus we obtain a parallelopiped with OP as diagonal and three coterminous edges OL, OE and ON along OA, OB and OC, respectively.



$\therefore$ **OL** is collinear with **OA**.

$\therefore$ $\mathbf{OL} = x\mathbf{OA} = x\mathbf{a}$, where x is a scalar.

Similarly, $\mathbf{OE} = y\,\mathbf{b}$ and $\mathbf{ON} = z\mathbf{c}$,

where y and z are scalars.

Now, $\mathbf{OP} = \mathbf{OR} + \mathbf{RP} = (\mathbf{ON} + \mathbf{NR}) + \mathbf{RP}$

$= \mathbf{ON} + \mathbf{OL} + \mathbf{OE}$ [$\because \mathbf{NR} = \mathbf{OL}$ and $\mathbf{RP} = \mathbf{OE}$]

$= \mathbf{OL} + \mathbf{OE} + \mathbf{ON} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$

Thus, $\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$

Hence, **r** can be expressed as a linear combination of **a**, **b** and **c**.

Uniqueness If possible let

$$\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$$

and $$\mathbf{r} = x'\mathbf{a} + y'\mathbf{b} + z'\mathbf{c}$$

be two different ways of representing **r**, then we have

$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = x'\mathbf{a} + y'\mathbf{b} + z'\mathbf{c}$$

$$\Rightarrow \quad (x - x')\mathbf{a} + (y - y')\mathbf{b} + (z - z')\mathbf{c} = 0$$

Now **a**, **b** and **c** are non-coplanar vectors

$\therefore \quad x - x' = 0, y - y' = 0$ and $z - z' = 0$

$\Rightarrow \quad x = x', y = y'$ and $z = z'$

Hence, the uniqueness is established.

Remark

If **a**, **b**, **c** are any three non-coplanar vectors in space, then

$$x_1\mathbf{a} + y_1\mathbf{b} + z_1\mathbf{c} = x_2\mathbf{a} + y_2\mathbf{b} + z_2\mathbf{c}$$

$$\Rightarrow \quad x_1 = x_2, y_1 = y_2, z_1 = z_2$$

Proof $x_1\mathbf{a} + y_1\mathbf{b} + z_1\mathbf{c} = x_2\mathbf{a} + y_2\mathbf{b} + z_2\mathbf{c}$

$$\Rightarrow \quad (x_1 - x_2)\mathbf{a} + (y_1 - y_2)\mathbf{b} + (z_1 - z_2)\mathbf{c} = 0$$

$$\Rightarrow \quad x_1 - x_2 = 0, y_1 - y_2 = 0 \text{ and } z_1 - z_2 = 0$$

$$\Rightarrow \quad x_1 = x_2, y_1 = y_2 \text{ and } z_1 = z_2$$

Example 39. ***Check whether the given three vectors are coplanar or non-coplanar.***
$-2\hat{\mathbf{i}}-2\hat{\mathbf{j}}+4\hat{\mathbf{k}}, -2\hat{\mathbf{i}}+4\hat{\mathbf{j}}-2\hat{\mathbf{k}}, 4\hat{\mathbf{i}}-2\hat{\mathbf{j}}-2\hat{\mathbf{k}}$

Sol. Let $\mathbf{a} = -2\hat{\mathbf{i}}-2\hat{\mathbf{j}}+4\hat{\mathbf{k}}$

$\mathbf{b} = -2\hat{\mathbf{i}}+4\hat{\mathbf{j}}-2\hat{\mathbf{k}}$ and $\mathbf{c} = 4\hat{\mathbf{i}}-2\hat{\mathbf{j}}-2\hat{\mathbf{k}}$

Now, consider

$$\begin{vmatrix} -2 & -2 & 4 \\ -2 & 4 & -2 \\ 4 & -2 & -2 \end{vmatrix} = -2(-8-4)+2(4+8)+4(4-16)$$

$$= 24 + 24 - 48 = 0$$

$\therefore$ The vectors are coplanar.

Example 40. ***If the vectors*** $4\hat{\mathbf{i}}+11\hat{\mathbf{j}}+m\hat{\mathbf{k}}$, $7\hat{\mathbf{i}}+2\hat{\mathbf{j}}+6\hat{\mathbf{k}}$ ***and*** $\hat{\mathbf{i}}+5\hat{\mathbf{j}}+4\hat{\mathbf{k}}$ ***are coplanar, then*** m ***is equal to***

(a) 38 (b) 0
(c) 10 (d) -10

Sol. (c) Since the three vectors are coplanar, one will be a linear combination of the other two.

$\therefore \quad 4\hat{\mathbf{i}}+11\hat{\mathbf{j}}+m\hat{\mathbf{k}} = x(7\hat{\mathbf{i}}+2\hat{\mathbf{j}}+6\hat{\mathbf{k}}) + y(\hat{\mathbf{i}}+5\hat{\mathbf{j}}+4\hat{\mathbf{k}})$

$\Rightarrow \quad 4 = 7x + y$ ...(i)

$11 = 2x + 5y$ ...(ii)

$m = 6x + 4y$ ...(iii)

From Eqs. (i) and (ii), we get

$$x = \frac{3}{11} \text{ and } y = \frac{23}{11}$$

From Eq. (iii), we get

$$m = 6 \times \frac{3}{11} + 4 \times \frac{23}{11} = 10$$

Trick Since, vectors $4\hat{\mathbf{i}}+11\hat{\mathbf{j}}+m\hat{\mathbf{k}}$, $7\hat{\mathbf{i}}+2\hat{\mathbf{j}}+6\hat{\mathbf{k}}$ and $\hat{\mathbf{i}}+5\hat{\mathbf{j}}+4\hat{\mathbf{k}}$ are coplanar.

$\therefore$
$$\begin{vmatrix} 4 & 11 & m \\ 7 & 2 & 6 \\ 1 & 5 & 4 \end{vmatrix} = 0$$

$\Rightarrow \quad 4(8-30) - 11(28-6) + m(35-2) = 0$

$\Rightarrow \quad -88 - 11 \times 22 + 33m = 0$

$\Rightarrow \quad -8 - 22 + 3m = 0$

$\Rightarrow \quad 3m = 30 \Rightarrow m = 10$

Example 41. ***If*** $\mathbf{a}, \mathbf{b}$ ***and*** $\mathbf{c}$ ***are non-coplanar vectors, prove that*** $3\mathbf{a}-7\mathbf{b}-4\mathbf{c}$, $3\mathbf{a}-2\mathbf{b}+\mathbf{c}$ ***and*** $\mathbf{a}+\mathbf{b}+2\mathbf{c}$ ***are coplanar.***

Sol. Let $\alpha = 3\mathbf{a}-7\mathbf{b}-4\mathbf{c}, \beta = 3\mathbf{a}-2\mathbf{b}+\mathbf{c}$

and $\gamma = \mathbf{a}+\mathbf{b}+2\mathbf{c}$

Also, let $\alpha = x\beta + y - \gamma$

$\Rightarrow \quad 3\mathbf{a}-7\mathbf{b}-4\mathbf{c} = x(3\mathbf{a}-2\mathbf{b}+\mathbf{c}) + y(\mathbf{a}+\mathbf{b}+2\mathbf{c})$

$= (3x+y)\mathbf{a} + (-2x+y)\mathbf{b} + (x+2y)\mathbf{c}$

Since, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are non-coplanar vectors.

Therefore,

$3x + y = 3, -2x + y = -7$

and $x + 2y = -4$

Solving first two, we find that $x = 2$ and $y = -3$. These values of x and y satisfy the third equation as well.

So, $x + 2$ and $y = -3$ is the unique solution for the above system of equation.

$\Rightarrow \quad \alpha = 2\beta - 3\gamma$

Hence, the vectors α, β and γ are coplanar, because α is uniquely written as linear combination of other two.

Trick For the vectors α, β, γ to be coplanar, we must have

$$\begin{vmatrix} 3 & -7 & -4 \\ 3 & -2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 0, \text{ which is true}$$

Hence, α, β, γ are coplanar.

Example 42. ***The value of*** λ ***for which the four points*** $2\hat{\mathbf{i}}+3\hat{\mathbf{j}}-\hat{\mathbf{k}}$, $\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}}$, $3\hat{\mathbf{i}}+4\hat{\mathbf{j}}-2\hat{\mathbf{k}}$ ***and*** $\hat{\mathbf{i}}-\lambda\hat{\mathbf{j}}+6\hat{\mathbf{k}}$ ***are coplanar***

(a) 8 (b) 0
(c) -2 (d) 6

Sol. (c) The given four points are coplanar.

$$\begin{vmatrix} 2 & 1 & 3 & 1 \\ 3 & 2 & 4 & \lambda \\ -1 & 3 & -2 & 6 \\ 1 & 1 & 1 & 0 \end{vmatrix} = 0$$

$\Rightarrow$
$$\begin{vmatrix} 2 & 1 & 3 & 1 \\ 0 & 0 & 0 & -(\lambda+2) \\ -1 & 3 & -2 & 6 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$$

Operating $(R_2 \to R_2 - R_1 - R_4)$

$\Rightarrow \quad -(\lambda+2)\begin{vmatrix} 2 & 1 & 3 \\ -1 & 3 & -2 \\ 1 & 1 & 1 \end{vmatrix} = 0 \Rightarrow \lambda = -2$

Example 43. ***Show that the points*** $P(\mathbf{a}+2\mathbf{b}+\mathbf{c})$, $Q(\mathbf{a}-\mathbf{b}-\mathbf{c})$, $R(3\mathbf{a}+\mathbf{b}+2\mathbf{c})$ ***and*** $S(5\mathbf{a}+3\mathbf{b}+5\mathbf{c})$ ***are coplanar given that*** $\mathbf{a}, \mathbf{b}$ ***and*** $\mathbf{c}$ ***are non-coplanar.***

Sol. To show that P, Q, R, S are coplanar, we will show that $\mathbf{PQ}$, $\mathbf{PR}$, $\mathbf{PS}$ are coplanar.

$\mathbf{PQ} = -3\mathbf{b} - 2\mathbf{c}$

$\mathbf{PR} = 2\mathbf{a} - \mathbf{b} + \mathbf{c}$

$\mathbf{PS} = 4\mathbf{a} + \mathbf{b} + 4\mathbf{c}$

Let $\mathbf{PQ} = x\mathbf{PR} + y\mathbf{PS}$

$\Rightarrow \quad -3\mathbf{b} - 2\mathbf{c} = x(2\mathbf{a} - \mathbf{b} + \mathbf{c}) + y(4\mathbf{a} + \mathbf{b} + 4\mathbf{c})$

$\Rightarrow \quad -3\mathbf{b} - 2\mathbf{c} = (2x + 4y)\mathbf{a} + (-x + y)\mathbf{b} + (x + 4y)\mathbf{c}$

As the vectors **a, b, c** are non-coplanar, we can equate their coefficients.

$\Rightarrow \quad 0 = 2x + 4y$

$\Rightarrow \quad -3 = -x + y$

$\Rightarrow \quad -2 = x + 4y$

$x = 2, y = -1$ is the unique solution for the above system of equations.

$\Rightarrow \quad \mathbf{PQ} = 2\mathbf{PR} - \mathbf{PS}$

PQ, **PR**, **PS** are coplanar because **PQ** is a linear combination of **PR** and **PS**

$\Rightarrow$ The points **P, Q, R, S** are also coplanar.

Trick For the vectors **PQ, PR** and **PS** to be coplanar, we must have $\begin{vmatrix} 0 & -3 & -2 \\ 2 & -1 & 1 \\ 4 & 1 & 4 \end{vmatrix} = 0$ which is true

$\therefore$ The **PQ, PR, PS** are coplanar.

Hence, the points *P*, *Q*, *R*, *S* are also coplanar.

Linear Independence and Dependence of Vectors

1. Linearly Independent Vectors

A set of non-zero vectors $\mathbf{a}_1, \mathbf{a}_2 ...., \mathbf{a}_n$ is said to be linearly independent, if

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + ... + x_n\mathbf{a}_n = 0$$

$\Rightarrow \quad x_1 = x_2 = ... = x_n = 0.$

2. Linearly Dependence Vectors

A set of vector $\mathbf{a}_1, \mathbf{a}_1 ...., \mathbf{a}_n$ is said to be linearly dependent, if there exist scalars $x_1, x_2, ..., x_n$ not all zero such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + ...+ x_n\mathbf{a}_n = 0$

Properties of Linearly Independent and Dependent Vectors

(i) A super set of a linearly dependent set of vectors is linearly dependent.

(ii) A subset of a linearly independent set of vectors is linearly independent.

(iii) Two non-zero, non-collinear vectors are linearly independent.

(iv) Any two collinear vectors are linearly dependent.

(v) Any three non-coplanar vectors are linearly independent.

(vi) Any three coplanar vectors are linearly dependent.

(vii) Three vectors $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$, $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$ and $\mathbf{c} = c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}} + c_3\hat{\mathbf{k}}$ will be linearly dependent vectors iff $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$

(viii) Any four vectors in 3-dimensional space are linearly dependent.

Example 44. ***Show that the vectors*** $\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}, 2\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - \hat{\mathbf{k}}$ ***and*** $3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$ ***and linearly independent.***

Sol. Let $\alpha = \hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

$\beta = 2\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - \hat{\mathbf{k}}$

and $\gamma = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$

Also, let $x\alpha + y\beta + z\gamma = 0$

$\therefore \quad x(\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) + y(2\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - \hat{\mathbf{k}}) + z(3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}) = 0$

or $(x + 2y + 3z)\hat{\mathbf{i}} + (-3x - 4y + 2z)\hat{\mathbf{j}} + (2x - y - z)\hat{\mathbf{k}} = 0$

Equating the coefficient of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$, we get

$$x + 2y + 3z = 0$$
$$-3x - 4y + 2z = 0$$
$$2x - y - z = 0$$

Now, $\begin{vmatrix} 1 & 2 & 3 \\ -3 & -4 & 2 \\ 2 & -1 & -1 \end{vmatrix} = 1(4 + 2) - 2(3 - 4) + 3(3 + 8) = 41 \neq 0$

$\therefore$ The above system of equations have only trivial solution.

Thus, $x = y = z = 0$

Hence, the vectors α, β and γ are linearly independent.

Trick Consider the determinant of coefficients of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$

i.e. $\begin{vmatrix} 1 & -3 & 2 \\ 2 & -4 & -1 \\ 3 & 2 & -1 \end{vmatrix} = 1(4 + 2) + 3(-2 + 3) + 2(4 + 12)$

$$= 6 + 3 + 32 = 41 \neq 0$$

$\therefore$ The given vectors are non-coplanar. Hence, the vectors are linearly independent.

Example 45. ***If*** $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{b} = 4\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ ***and*** $\mathbf{c} = \hat{\mathbf{i}} + \alpha\hat{\mathbf{j}} + \beta\hat{\mathbf{k}}$ ***are linearly dependent vectors and*** $|c| = \sqrt{3}$***, then***

(a) $\alpha = 1, \beta = -1$ (b) $\alpha = 1, \beta = \pm 1$

(c) $\alpha = -1, \beta = \pm 1$ (d) $\alpha = \pm 1, \beta = 1$

Sol. (d) The given vectors are linearly dependent, hence there exist scalars x, y and z not all zero, such that

$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = 0$$

i.e. $x(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) + y(4\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}) + z(\hat{\mathbf{i}} + \alpha\hat{\mathbf{j}} + \beta\hat{\mathbf{k}}) = 0$

i.e. $(x+4y+z)\hat{\mathbf{i}}+(x+3y+\alpha z)\hat{\mathbf{j}}+(x+4y+\beta z)\hat{\mathbf{k}}=0$

$\Rightarrow \quad x+4y+z=0,\ x+3y+\alpha z=0,\ x+4y+\beta z=0$

For non-trivial solution $\begin{vmatrix}1 & 4 & 1\\ 1 & 3 & \alpha\\ 1 & 4 & \beta\end{vmatrix}=0\Rightarrow\beta=1$

$$|c|^2=3\Rightarrow 1+\alpha^2+\beta^2=3$$

$\Rightarrow \quad \alpha^2=2-\beta^2=2-1=1$

$\therefore \quad \alpha=\pm 1$

Trick $|c|=\sqrt{1+\alpha^2+\beta^2}=\sqrt{3}$

$\Rightarrow \quad \alpha^2+\beta^2=2$

$\because$ $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly dependent, hence $\begin{vmatrix}1 & 1 & 1\\ 4 & 3 & 4\\ 4 & \alpha & \beta\end{vmatrix}=0$

$\Rightarrow \quad \beta=1$

$\therefore \quad \alpha^2=1 \quad \Rightarrow \quad \alpha=\pm 1$

Exercise for Session 3

1. Show that the points $A(1, 3, 2), B(-2, 0, 1)$ and $C(4, 6, 3)$ are collinear.
2. If the position vectors of the points A, B and C be $\mathbf{a}$, $\mathbf{b}$ and $3\mathbf{a}-2\mathbf{b}$ respectively, then prove that the points A, B and C are collinear.
3. The position vectors of four points P, Q, R and S are $2\mathbf{a}+4\mathbf{c}$, $5\mathbf{a}+3\sqrt{3}\mathbf{b}+4\mathbf{c}$, $-2\sqrt{3}\mathbf{b}+\mathbf{c}$ and $2\mathbf{a}+\mathbf{c}$ respectively, prove that PQ is parallel to RS.
4. If three points A, B and C have position vectors $(1, x, 3)$, $(3, 4, 7)$ and $(y, -2, -5)$, respectively and if they are collinear, then find (x, y).
5. Find the condition that the three points whose position vectors, $\mathbf{a}=a\hat{\mathbf{i}}+b\hat{\mathbf{j}}+c\hat{\mathbf{k}}$, $\mathbf{b}=\hat{\mathbf{i}}+c\hat{\mathbf{j}}$ and $\mathbf{c}=-\hat{\mathbf{i}}-\hat{\mathbf{j}}$ are collinear.
6. Vectors $\mathbf{a}$ and $\mathbf{b}$ are non-collinear. Find for what values of x vectors $\mathbf{c}=(x-2)\mathbf{a}+\mathbf{b}$ and $\mathbf{d}=(2x+1)\mathbf{a}-\mathbf{b}$ are collinear?
7. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are three vectors of which every pair is non-collinear. If the vectors $\mathbf{a}+\mathbf{b}$ and $\mathbf{b}+\mathbf{c}$ are collinear with $\mathbf{c}$ and $\mathbf{a}$ respectively, then find $\mathbf{a}+\mathbf{b}+\mathbf{c}$.
8. Show that the vectors $\hat{\mathbf{i}}-\hat{\mathbf{j}}-\hat{\mathbf{k}}$, $2\hat{\mathbf{i}}+3\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $7\hat{\mathbf{i}}+3\hat{\mathbf{j}}-4\hat{\mathbf{k}}$ are coplanar.
9. If the vectors $2\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}}$, $\hat{\mathbf{i}}+2\hat{\mathbf{j}}-3\hat{\mathbf{k}}$ and $3\hat{\mathbf{i}}+a\hat{\mathbf{j}}+5\hat{\mathbf{k}}$ are coplanar, then prove that $a=4$.
10. Show that the vectors $\mathbf{a}-2\mathbf{b}+3\mathbf{c}$, $-2\mathbf{a}+3\mathbf{b}-4\mathbf{c}$ and $-\mathbf{b}+2\mathbf{c}$ are coplanar vector, where $\mathbf{a}$, $\mathbf{b}$,$\mathbf{c}$ are non-coplanar vectors.
11. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are non-coplanar vectors, then prove that the four points $2\mathbf{a}+3\mathbf{b}-\mathbf{c}$, $\mathbf{a}-2\mathbf{b}+3\mathbf{c}$, $3\mathbf{a}+4\mathbf{b}-2\mathbf{c}$ and $\mathbf{a}-6\mathbf{b}+6\mathbf{c}$ are coplanar.

JEE Type Solved Examples :
Single Option Correct Type Questions

● **Ex. 1** *The non-zero vectors* $\mathbf{a}, \mathbf{b}$ *and* $\mathbf{c}$ *are related by* $\mathbf{a} = 8\mathbf{b}$ *and* $\mathbf{c} = -7\mathbf{b}$ *angle between* $\mathbf{a}$ *and* $\mathbf{c}$ *is*

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$

(c) π (d) 0

Sol. (c) $\mathbf{a}$ and $\mathbf{b}$ vectors are in the same direction, $\mathbf{b}$ and $\mathbf{c}$ are in the opposite direction.

$\Rightarrow$ $\mathbf{a}$ and $\mathbf{c}$ are in opposite directions.

$\therefore$ Angle between $\mathbf{a}$ and $\mathbf{c}$ is π.

● **Ex. 2** *A unit vector* $\mathbf{a}$ *makes an angle* $\frac{\pi}{4}$ *with Z-axis. If* $\mathbf{a} + \hat{\mathbf{i}} + \hat{\mathbf{j}}$ *is a unit vector, then* $\mathbf{a}$ *is equal to*

(a) $\frac{\hat{\mathbf{i}}}{2} + \frac{\hat{\mathbf{j}}}{2} + \frac{\hat{\mathbf{k}}}{\sqrt{2}}$ (b) $\frac{\hat{\mathbf{i}}}{2} + \frac{\hat{\mathbf{j}}}{2} - \frac{\hat{\mathbf{k}}}{\sqrt{2}}$

(c) $-\frac{\hat{\mathbf{i}}}{2} - \frac{\hat{\mathbf{j}}}{2} + \frac{\hat{\mathbf{k}}}{\sqrt{2}}$ (d) None of these

Sol. (c) Let $\mathbf{a} = l\hat{\mathbf{i}} + m\hat{\mathbf{j}} + n\hat{\mathbf{k}}$, where $l^2 + m^2 + n^2 = 1$. $\mathbf{a}$ makes an angle $\frac{\pi}{4}$ with Z-axis.

$$\therefore \quad n = \frac{1}{\sqrt{2}}, \; l^2 + m^2 = \frac{1}{2} \qquad \ldots(i)$$

$$\therefore \quad \mathbf{a} = l\hat{\mathbf{i}} + m\hat{\mathbf{j}} + \frac{\hat{\mathbf{k}}}{\sqrt{2}}$$

$$\mathbf{a} + \hat{\mathbf{i}} + \hat{\mathbf{j}} = (l+1)\hat{\mathbf{i}} + (m+1)\hat{\mathbf{j}} + \frac{\hat{\mathbf{k}}}{\sqrt{2}}$$

Its magnitude is 1, hence $(l+1)^2 + (m+1)^2 = \frac{1}{2}$ ...(ii)

From Eqs. (i) and (ii), we get

$$2lm = \frac{1}{2} \Rightarrow l = m = -\frac{1}{2}$$

Hence, $\mathbf{a} = -\frac{\hat{\mathbf{i}}}{2} - \frac{\hat{\mathbf{j}}}{2} + \frac{\hat{\mathbf{k}}}{\sqrt{2}}$

● **Ex. 3** *If the resultant of two forces of magnitudes* P *and* Q *acting at a point at an angle of* 60° *is* $\sqrt{7}Q$*, then* P/Q *is*

(a) 1 (b) $\frac{3}{2}$

(c) 2 (d) 4

Sol. (c) $R^2 = P^2 + Q^2 + 2PQ\cos\theta$

$\Rightarrow (\sqrt{7}Q)^2 = P^2 + Q^2 + 2PQ\cos 60^\circ$

$\Rightarrow 7Q^2 = P^2 + Q^2 + PQ$

$\Rightarrow P^2 + PQ - 6Q^2 = 0$

$\Rightarrow P^2 + 3PQ - 2PQ - 6Q^2 = 0$

$\Rightarrow P(P+3Q) - 2Q(P+3Q) = 0$

$\Rightarrow (P-2Q)(P+3Q) = 0$

$\Rightarrow P - 2Q = 0$ or $P + 3Q = 0$

From $P - 2Q = 0 \Rightarrow \frac{P}{Q} = 2$

● **Ex. 4** *A vector* $\mathbf{a}$ *has the components* $2p$ *and* 1 *w.r.t. a rectangular cartesian system. This system is rotated through a certain angle about the origin in the counter clockwise sense. If with respect to a new system,* $\mathbf{a}$ *has components* $(p+1)$ *and* 1*, then*

(a) $p = 0$ (b) $p = 1$ or $p = -\frac{1}{3}$

(c) $p = -1$ or $p = \frac{1}{3}$ (d) $p = 1$ or $p = -1$

Sol. (b) We have, $\mathbf{a} = 2p\hat{\mathbf{i}} + \hat{\mathbf{j}}$

On rotation, let $\mathbf{b}$ be the vector with components $(p+1)$ and 1 so that,

$$\mathbf{b} = (p+1)\hat{\mathbf{i}} + \hat{\mathbf{j}}$$

Now, $|\mathbf{a}| = |\mathbf{b}| \Rightarrow a^2 = b^2$

$\Rightarrow 4p^2 + 1 = (p+1)^2 + 1 \Rightarrow 4p^2 = (p+1)^2$

$\Rightarrow 2p = \pm(p+1) \Rightarrow 3p = -1$ or $p = 1$

$\therefore \quad p = -\frac{1}{3}$ or $p = 1$

● **Ex. 5** *ABC is an isosceles triangle right angled at A. Forces of magnitude* $2\sqrt{2}$, 5 *and* 6 *act along* **BC**, **CA** *and* **AB** *respectively. The magnitude of their resultant force is*

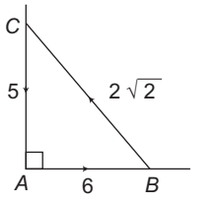
(a) 4 (b) 5

(c) $11 + 2\sqrt{2}$ (d) 30

Sol. (b) $R\cos\theta = 6\cos 0^\circ + 2\sqrt{2}\cos(180^\circ - B) + 5\cos 270^\circ$

$$R\cos\theta = 6 - 2\sqrt{2}\cos B \qquad \ldots(i)$$

$$R\sin\theta = 6\sin 0^\circ + 2\sqrt{2}\sin(180^\circ - B) + 5\sin 270^\circ$$



$$R\sin\theta = 2\sqrt{2}\sin B - 5 \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$R^2 = 36 + 8\cos^2 B - 24\sqrt{2}\cos B + 8\sin^2 B + 25 - 20\sqrt{2}\sin B$$

$$= 61 + 8(\cos^2 B + \sin^2 B) - 24\sqrt{2}\cos B - 20\sqrt{2}\sin B$$

$\because$ ABC is a right angled isosceles triangle.

i.e. $\angle B = \angle C = 45°$

$\therefore$ $R^2 = 61 + 8(1) - 24\sqrt{2}\,\frac{1}{\sqrt{2}} - 20\sqrt{2}\cdot\frac{1}{\sqrt{2}} = 25$

$\therefore$ $R = 5$

● **Ex. 6** *A line segment has length* 63 *and direction ratios are* 3, − 2 *and* 6. *The components of line vector are*

(a) −27, 18, 54 (b) 27, − 18, 54

(c) 27, − 18, − 54 (d) −27, − 18, − 54

Sol. (b) Let the components of line segment on axes are x, y and z.

So, $x^2 + y^2 + z^2 = 63^2$

Now, $\frac{x}{3} = \frac{y}{-2} = \frac{z}{6} = k$

$\because$ $(3k)^2 + (-2k)^2 + (6k)^2 = 63^2$

$k = \pm\frac{63}{7} = \pm 9$

$\therefore$ Components are $(27, -18, 54)$ or $(-27, 18, -54)$.

● **Ex. 7** *If the vectors* $6\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, $2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 6\hat{\mathbf{k}}$ *and* $3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ *form a triangle, then it is*

(a) right angled (b) obtuse angled

(c) equilateral (d) isosceles

Sol. (b) $\mathbf{AB}$ = Position vectors of B Position vector of A

$= (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 6\hat{\mathbf{k}}) - (6\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) = -4\hat{\mathbf{i}} + 5\hat{\mathbf{j}} - 9\hat{\mathbf{k}}$

$\Rightarrow$ $|\mathbf{AB}| = \sqrt{16 + 25 + 81} = \sqrt{122}$

$\mathbf{BC} = \hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$

$\Rightarrow$ $|\mathbf{BC}| = \sqrt{1 + 9 + 16} = \sqrt{26}$ and $\mathbf{AC} = -3\hat{\mathbf{i}} + 8\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$

$\Rightarrow$ $|\mathbf{AC}| = \sqrt{98}$

Therefore, $AB^2 = 122$, $BC^2 = 26$ and $AC^2 = 98$

$\Rightarrow$ $AB^2 + BC^2 = 26 + 122 = 148$

Since, $AC^2 < AB^2 + BC^2$, therefore ΔABC is an obtuse angled triangle.

● **Ex. 8** *The position vectors of the points A, B and C are* $(2\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}})$, $(3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}})$ *and* $(\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 3\hat{\mathbf{k}})$ *respectively. These points.*

(a) form an isosceles triangle

(b) form a right angled triangle

(c) are collinear

(d) form a scalene triangle

Sol. (c) $\mathbf{AB} = (3-2)\hat{\mathbf{i}} + (-2-1)\hat{\mathbf{j}} + (1+1)\hat{\mathbf{k}}$

$= \hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

$\mathbf{BC} = (1-3)\hat{\mathbf{i}} + (4+2)\hat{\mathbf{j}} + (-3-1)\hat{\mathbf{k}}$

$= -2\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$

$\mathbf{CA} = (2-1)\hat{\mathbf{i}} + (1-4)\hat{\mathbf{j}} + (-1+3)\hat{\mathbf{k}}$

$= \hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$

$|\mathbf{AB}| = \sqrt{1 + 9 + 4} = \sqrt{14}$

$|\mathbf{BC}| = \sqrt{4 + 36 + 16} = \sqrt{56} = 2\sqrt{14}$

$|\mathbf{CA}| = \sqrt{1 + 9 + 4} = \sqrt{14}$

So, $|\mathbf{AB}| + |\mathbf{AC}| = |\mathbf{BC}|$ and angle between AB and BC is 180°.

So, points A, B and C cannot form an isosceles triangle.

Hence, A, B and C are collinear.

● **Ex. 9** *The position vector of a point C with respect to B is* $\hat{\mathbf{i}} + \hat{\mathbf{j}}$ *and that of B with respect to A is* $\hat{\mathbf{i}} - \hat{\mathbf{j}}$. *The position vector of C with respect to A is*

(a) $2\hat{\mathbf{i}}$ (b) $2\hat{\mathbf{j}}$

(c) $-2\hat{\mathbf{j}}$ (d) $-2\hat{\mathbf{i}}$

Sol. (a) Since, position vectors of a point C with respect to B is

$\mathbf{BC} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$ ...(i)

Similarly, $\mathbf{AB} = \hat{\mathbf{i}} - \hat{\mathbf{j}}$ ...(ii)

Now, by Eqs. (i) and (ii),

$\mathbf{AC} = \mathbf{AB} + \mathbf{BC} = 2\hat{\mathbf{i}}$

● **Ex. 10** *In a* ΔABC, *if* $2\mathbf{AC} = 3\mathbf{CB}$, *then* $2\mathbf{OA} + 3\mathbf{OB}$ *is equal to*

(a) $5\mathbf{OC}$ (b) $-\mathbf{OC}$

(c) $\mathbf{OC}$ (d) None of these

Sol. (a) $2\mathbf{OA} + 3\mathbf{OB} = 2(\mathbf{OC} + \mathbf{CA}) + 3(\mathbf{OC} + \mathbf{CB})$

$= 5\mathbf{OC} + 2\mathbf{CA} + 3\mathbf{CB} = 5\mathbf{OC}$ ($\because$ $2\mathbf{CA} = -3\mathbf{CB}$)

● **Ex. 11** *If* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *and* $\mathbf{d}$ *be the position vectors of the points A, B, C and D respectively, referred to same origin O such that no three of these points are collinear and* $\mathbf{a} + \mathbf{c} = \mathbf{b} + \mathbf{d}$, *then quadrilateral ABCD is a*

(a) square (b) rhombus

(c) rectangle (d) parallelogram

Sol. (d) Given, $\mathbf{a} + \mathbf{c} = \mathbf{b} + \mathbf{d}$

$\Rightarrow$ $\frac{1}{2}(\mathbf{a} + \mathbf{c}) = \frac{1}{2}(\mathbf{b} + \mathbf{d})$

Here, mid-points of $\mathbf{AC}$ and $\mathbf{BD}$ coincide, where $\mathbf{AC}$ and $\mathbf{BD}$ are diagonals. In addition, we know that, diagonals of a parallelogram bisect each other.

Hence, quadrilateral is parallelogram.

● **Ex. 12** *P is a point on the side BC of the* ΔABC *and Q is a point such that* $\mathbf{PQ}$ *is the resultant of* $\mathbf{AP}$, $\mathbf{PB}$ *and* $\mathbf{PC}$. *Then, ABQC is a*

(a) square

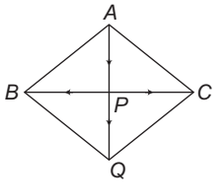
(b) rectangle

(c) parallelogram

(d) trapezium

Sol. (c) $\mathbf{AP}+\mathbf{PB}+\mathbf{PC}=\mathbf{PQ}$ or $\mathbf{AP}+\mathbf{PB}=\mathbf{PQ}+\mathbf{CP}$

$\Rightarrow \quad \mathbf{AB}=\mathbf{CQ}$



Hence, it is a parallelogram.

● ***Ex. 13*** *If ABCD is a parallelogram and the position vectors of A, B and C are* $\hat{\mathbf{i}}+3\hat{\mathbf{j}}+5\hat{\mathbf{k}}$, $\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ *and* $7\hat{\mathbf{i}}+7\hat{\mathbf{j}}+7\hat{\mathbf{k}}$, *then the position vector of D will be*

(a) $7\hat{\mathbf{i}}+5\hat{\mathbf{j}}+3\hat{\mathbf{k}}$ (b) $7\hat{\mathbf{i}}+9\hat{\mathbf{j}}+11\hat{\mathbf{k}}$

(c) $9\hat{\mathbf{i}}+11\hat{\mathbf{j}}+13\hat{\mathbf{k}}$ (d) $8\hat{\mathbf{i}}+8\hat{\mathbf{j}}+8\hat{\mathbf{k}}$

Sol. (b) Let position vector of D is $x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}$, then $\mathbf{AB}=\mathbf{DC}$.

$\Rightarrow \quad -2\hat{\mathbf{j}}-4\hat{\mathbf{k}}=(7-x)\hat{\mathbf{i}}+(7-y)\hat{\mathbf{j}}+(7-z)\hat{\mathbf{k}}$

$\Rightarrow \quad x=7,\ y=9$ and $z=11$

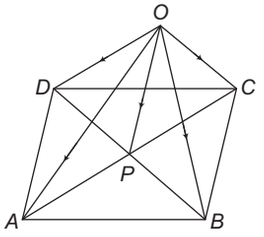
Hence, position vector of D will be $7\hat{\mathbf{i}}+9\hat{\mathbf{j}}+11\hat{\mathbf{k}}$.

● ***Ex. 14*** *P is the point of intersection of the diagonals of the parallelogram ABCD. If O is any point, then* $\mathbf{OA}+\mathbf{OB}+\mathbf{OC}+\mathbf{OD}$ *is equal to*

(a) $\mathbf{OP}$ (b) $2\mathbf{OP}$

(c) $3\mathbf{OP}$ (d) $4\mathbf{OP}$

Sol. (d) We know that, P will be the mid-point of AC and BD.



$\therefore \quad \mathbf{OA}+\mathbf{OC}=2\mathbf{OP}$ ...(i)

and $\mathbf{OB}+\mathbf{OD}=2\mathbf{OP}$ ...(ii)

On adding Eqs. (i) and (ii), we get

$$\mathbf{OA}+\mathbf{OB}+\mathbf{OC}+\mathbf{OD}=4\mathbf{OP}$$

● ***Ex. 15*** *If C is the middle point of AB and P is any point outside AB, then*

(a) $\mathbf{PA}+\mathbf{PB}=\mathbf{PC}$

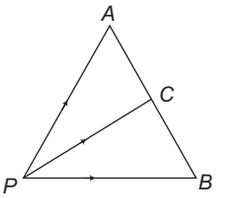
(b) $\mathbf{PA}+\mathbf{PB}=2\mathbf{PC}$

(c) $\mathbf{PA}+\mathbf{PB}+\mathbf{PC}=0$

(d) $\mathbf{PA}+\mathbf{PB}+2\mathbf{PC}=0$

Sol. (b) $\mathbf{PA}+\mathbf{PB}=(\mathbf{PA}+\mathbf{AC})+(\mathbf{PB}+\mathbf{BC})-(\mathbf{AC}+\mathbf{BC})$

$=\mathbf{PC}+\mathbf{PC}-(\mathbf{AC}-\mathbf{CB})=2\mathbf{PC}-0$ $(\because \mathbf{AC}=\mathbf{CB})$



$\therefore \quad \mathbf{PA}+\mathbf{PB}=2\mathbf{PC}$

● ***Ex. 16*** *If O be the circumcentre and O′ be the orthocentre of the* ΔABC, *then* $\mathbf{O'A}+\mathbf{O'B}+\mathbf{O'C}$ *is equal to*

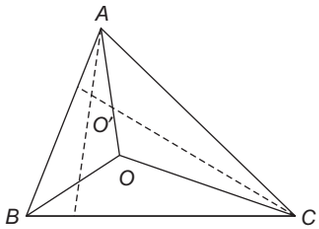
(a) $\mathbf{OO'}$ (b) $2\mathbf{O'O}$ (c) $2\mathbf{OO'}$ (d) 0

Sol. (b) $\mathbf{O'A}=\mathbf{O'O}+\mathbf{OA}$

$\mathbf{O'B}=\mathbf{OO'}+\mathbf{OB}$

$\mathbf{O'C}=\mathbf{O'O}+\mathbf{OC}$

$\Rightarrow \ \mathbf{O'A}+\mathbf{O'B}+\mathbf{O'C}=3\mathbf{O'O}+\mathbf{OA}+\mathbf{OB}+\mathbf{OC}$



Since, $\mathbf{OA}+\mathbf{OB}+\mathbf{OC}=\mathbf{OO'}=-\mathbf{O'O}$

$\therefore \quad \mathbf{O'A}+\mathbf{O'B}+\mathbf{O'C}=2\mathbf{O'O}$

● ***Ex. 17*** *Five points given by A, B, C, D and E are in a plane. Three forces* $\mathbf{AC}$, $\mathbf{AD}$ *and* $\mathbf{AE}$ *act at A and three forces* $\mathbf{CB}$, $\mathbf{DB}$ *and* $\mathbf{EB}$ *act B. Then, their resultant is*

(a) $2\mathbf{AC}$ (b) $3\mathbf{AB}$

(c) $3\mathbf{DB}$ (d) $2\mathbf{BC}$

Sol. (b) Points A, B, C, D and E are in a plane.

$\text{Resultant}=(\mathbf{AC}+\mathbf{AD}+\mathbf{AE})+\mathbf{CB}+\mathbf{BD}+\mathbf{EB})$

$=(\mathbf{AC}+\mathbf{CB})+(\mathbf{AD}+\mathbf{DB})+(\mathbf{AE}+\mathbf{EB})$

$=\mathbf{AB}+\mathbf{AB}+\mathbf{AB}=3\mathbf{AB}$

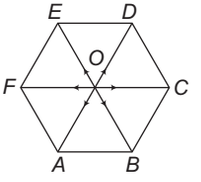
● ***Ex. 18*** *If the vectors represented by the sides AB and BC of the regular hexagon ABCDEF be* $\mathbf{a}$ *and* $\mathbf{b}$, *then the vector represented by* $\mathbf{AE}$ *will be*

(a) $2\mathbf{b}-\mathbf{a}$ (b) $\mathbf{b}-\mathbf{a}$

(c) $2\mathbf{a}-\mathbf{b}$ (d) $\mathbf{a}+\mathbf{b}$

Sol. (a) As in figure, $\mathbf{AB}=\mathbf{a}$, $\mathbf{BC}=\mathbf{b}$,

So, $\mathbf{AD}=2\mathbf{b}$ and $\mathbf{ED}=\mathbf{a}$



Now, $\mathbf{AE}+\mathbf{ED}=\mathbf{AD}$

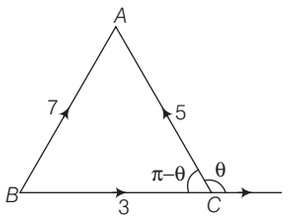
$\Rightarrow \quad \mathbf{AE}=\mathbf{AD}-\mathbf{ED}=2\mathbf{b}-\mathbf{a}$

● ***Ex. 19*** *If* $\mathbf{a}+\mathbf{b}+\mathbf{c}=0$ *and* $|\mathbf{a}|=3, |\mathbf{b}|=5, |\mathbf{c}|=7$, *then the angle between* $\mathbf{a}$ *and* $\mathbf{b}$ *is*

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

Sol. (b)



Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$. Then, $\angle C = \pi - \theta$.

$$\therefore \quad \cos(\pi-\theta)=\frac{3^2+5^2-7^2}{2(3)(5)}$$

$$\therefore \quad -\cos\theta=\frac{-1}{2}$$

$$\therefore \quad \theta=60^\circ=\frac{\pi}{3}$$

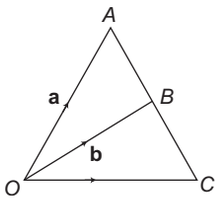
● ***Ex. 20*** *If* $\mathbf{a}$ *and* $\mathbf{b}$ *are the position vectors of A and B respectively, then the position vector of a point C on AB produced such that* $\mathbf{AC}=3\mathbf{AB}$ *is*

(a) $3\mathbf{a}-\mathbf{b}$ (b) $3\mathbf{b}-\mathbf{a}$

(c) $3\mathbf{a}-2\mathbf{b}$ (d) $3\mathbf{b}-2\mathbf{a}$

Sol. (d) Since, given that $\mathbf{AC}=3\mathbf{AB}$. It means that point C divides AB externally.

Thus, $\mathbf{AC}:\mathbf{BC}=3:2$



Hence, $\mathbf{OC}=\frac{3\cdot\mathbf{b}-2\cdot\mathbf{a}}{3-2}=3\mathbf{b}-2\mathbf{a}$

● ***Ex. 21*** *Let A and B be points with position vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *with respect to the origin O. If the point C on OA is such that* $2AC=CO$, *CD is parallel to OB and* $|\mathbf{CD}|=3|\mathbf{OB}|$, *then* $\mathbf{AD}$ *is equal to*

(a) $3\mathbf{b}-\frac{\mathbf{a}}{2}$ (b) $3\mathbf{b}+\frac{\mathbf{a}}{2}$

(c) $3\mathbf{b}-\frac{\mathbf{a}}{3}$ (d) $3\mathbf{b}+\frac{\mathbf{a}}{3}$

Sol. (c) Since, $\mathbf{OA}=\mathbf{a}$, $\mathbf{OB}=\mathbf{b}$ and $2\mathbf{AC}=\mathbf{CO}$

By section formula, $\mathbf{OC}=\frac{2}{3}\mathbf{a}$

Therefore, $|\mathbf{CD}|=3|\mathbf{OB}|$

$$\Rightarrow \quad \mathbf{CD}=3\mathbf{b}$$

$$\Rightarrow \quad \mathbf{OD}=\mathbf{OC}+\mathbf{CD}=\frac{2}{3}\mathbf{a}+3\mathbf{b}$$

Hence, $\mathbf{AD}=\mathbf{OD}-\mathbf{OA}=\frac{2}{2}\mathbf{a}+3\mathbf{b}-\mathbf{a}$

$$=3\mathbf{b}-\frac{1}{3}\mathbf{a}$$

● ***Ex. 22*** *If position vectors of a point A is* $\mathbf{a}+2\mathbf{b}$ *and* $\mathbf{a}$ *divides AB in the ratio* 2 : 3, *then the position vector of B is*

(a) $2\mathbf{a}-\mathbf{b}$ (b) $\mathbf{b}-2\mathbf{a}$

(c) $\mathbf{a}-3\mathbf{b}$ (d) $\mathbf{b}$

Sol. (c) If x be the position vector of B, then $\mathbf{a}$ divides AB in the ratio 2 : 3.

$$\therefore \quad \mathbf{a}=\frac{2x+3(\mathbf{a}+2\mathbf{b})}{2+3}$$

$$\Rightarrow \quad 5\mathbf{a}-3\mathbf{a}-6\mathbf{b}=2x$$

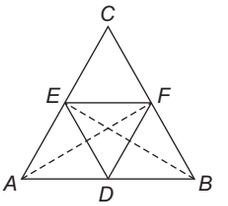
$$\Rightarrow \quad x=\mathbf{a}-3\mathbf{b}$$

● ***Ex. 23*** *If D, E and F are respectively, the mid-points of AB, AC and BC in* ΔABC, *then* $\mathbf{BE}+\mathbf{AF}$ *is equal to*

(a) $\mathbf{DC}$ (b) $\frac{1}{2}\mathbf{BF}$

(c) $2\mathbf{BF}$ (d) $\frac{3}{2}\mathbf{BF}$

Sol. (a) $\mathbf{BE}+\mathbf{AF}=\mathbf{OE}-\mathbf{OB}+\mathbf{OF}-\mathbf{OA}$



$$=\frac{\mathbf{OA}+\mathbf{OC}}{2}-\mathbf{OB}+\frac{\mathbf{OB}+\mathbf{OC}}{2}-\mathbf{OA}$$

$$=\mathbf{OC}-\frac{\mathbf{OA}+\mathbf{OB}}{2}=\mathbf{OC}-\mathbf{OD}=\mathbf{DC}$$

● ***Ex. 24*** *In a quadrilateral PQRS,* $\mathbf{PQ}=a, \mathbf{QR}=b$, $\mathbf{SP}=a-b$. *If M is the mid-point of QR and X is a point of SM such that,* $SX=\frac{4}{5}SM$, *then*

(a) $\mathbf{PX}=\frac{1}{5}\mathbf{PR}$

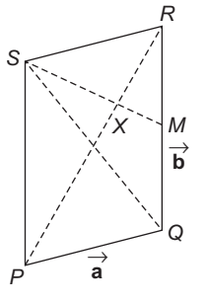
(b) $\mathbf{PX}=\frac{3}{5}\mathbf{PR}$

(c) $\mathbf{PX}=\frac{2}{5}\mathbf{PR}$

(d) None of the above

Sol. (b) If we take point P as the origin, the position vectors of Q and S are $\mathbf{a}$ and $\mathbf{b}-\mathbf{a}$ respectively.

In ΔPQR, we have



$$\mathbf{PR}=\mathbf{PQ}+\mathbf{QR} \Rightarrow \mathbf{PR}=\mathbf{a}+\mathbf{b}$$

$\therefore$ Position vector of $R=\mathbf{a}+\mathbf{b}$

$$\Rightarrow \quad \text{PV of } M=\frac{\mathbf{a}+(\mathbf{a}+\mathbf{b})}{2}=\left(\mathbf{a}+\frac{1}{2}\mathbf{b}\right)$$

Now, $$SX=\frac{4}{5}SM$$

$$\Rightarrow \qquad XM=SM-SX=SM-\frac{4}{5}SM=\frac{1}{5}SM$$

$$\therefore \qquad SX:XM=4:1$$

$$\Rightarrow \quad \text{PV of } X=\frac{4\left(\mathbf{a}+\frac{1}{2}\mathbf{b}\right)+1(\mathbf{b}-\mathbf{a})}{4+1}$$

$$=\frac{3\mathbf{a}+2\mathbf{b}}{5} \Rightarrow \mathbf{PX}=\frac{3}{5}(\mathbf{a}+\mathbf{b})$$

$$\Rightarrow \qquad \mathbf{PX}=\frac{3}{5}\mathbf{PR}$$

● ***Ex. 25*** *Orthocentre of an equilateral triangle ABC is the origin O. If* $\mathbf{OA}=\mathbf{a}, \mathbf{OB}=\mathbf{b}, \mathbf{OC}=\mathbf{c}$, *then* $\mathbf{AB}+2\mathbf{BC}+3\mathbf{CA}$ *is equal to*

(a) $3\mathbf{c}$ (b) $3\mathbf{a}$

(c) $\mathbf{0}$ (d) $3\mathbf{b}$

Sol. (b) For an equilateral triangle, centroid is the same as orthocentre

$$\therefore \qquad \frac{\mathbf{OA}+\mathbf{OB}+\mathbf{OC}}{3}=0$$

$$\therefore \qquad \mathbf{OA}+\mathbf{OB}+\mathbf{OC}=0$$

Now, $$\mathbf{AB}+2\mathbf{BC}+3\mathbf{CA}$$

$$=\mathbf{OB}-\mathbf{OA}+2\mathbf{OC}-2\mathbf{OB}+3\mathbf{OA}-3\mathbf{OC}$$

$$=-\mathbf{OB}+2\mathbf{OA}-\mathbf{OC}$$

$$=-(\mathbf{OB}+\mathbf{OA}+\mathbf{OC})+3\mathbf{OA}=3\mathbf{OA}=3\mathbf{a}$$

● ***Ex. 26*** *If* $\mathbf{a}, \mathbf{b}$, *and* $\mathbf{c}$ *are position vector of A, B and C respectively of* ΔABC *and if* $|\mathbf{a}-\mathbf{b}|=4, |\mathbf{b}-\mathbf{c}|=2$, $|\mathbf{c}-\mathbf{a}|=3$, *then the distance between the centroid and incentre of* ΔABC *is*

(a) 1 (b) $\frac{1}{2}$

(c) $\frac{1}{3}$ (d) $\frac{2}{3}$

Sol. (c) Let G be centroid and I be incenter.

$$|\mathbf{GI}|=|\mathbf{OI}-\mathbf{OG}|=\left|\frac{2\mathbf{a}+3\mathbf{b}+4\mathbf{c}}{9}-\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}\right|$$

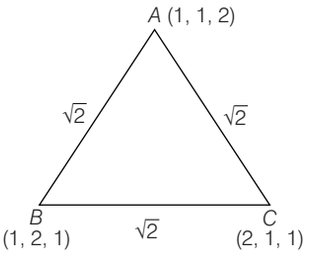
$$=\left|\frac{-\mathbf{a}+\mathbf{c}}{9}\right|=\frac{3}{9}=\frac{1}{3}$$

● ***Ex. 27*** *Let position vector of points A, B and C of triangle* ΔABC *respectively be* $\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}}$, $\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}}$ *and* $2\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$. *Let* l_1, l_2 *and* l_3 *be the lengths of perpendiculars drawn from the orthocenter 'O' on the sides AB, BC and CA, then* $(l_1+l_2+l_3)$ *equals*

(a) $\frac{2}{\sqrt{6}}$ (b) $\frac{3}{\sqrt{6}}$

(c) $\frac{\sqrt{6}}{2}$ (d) $\frac{\sqrt{6}}{3}$

Sol. (c)



Clearly, triangle formed by the given points $\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}}$, $\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $2\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ is equilateral as $AB=BC=AC=\sqrt{2}$.

$\therefore$ Distance of orthcentre 'O' from the sides is equal to inradius of the triangle.

$$\therefore l_1=l_2=l_3=\text{inradius}=r=\frac{\Delta}{s}=\frac{\frac{\sqrt{3}}{4}(\sqrt{2})^2}{\frac{3}{2}(\sqrt{2})}=\frac{1}{\sqrt{6}}$$

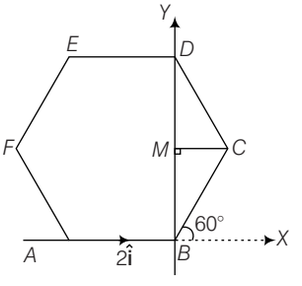
$$\Rightarrow \qquad (l_1+l_2+l_3)=\frac{3}{\sqrt{6}}=\frac{\sqrt{6}}{2}$$

● ***Ex. 28*** *ABCDEF is a regular hexagon in the XY-plane with vertices in the anticlockwise direction. If* $\mathbf{AB}=2\hat{\mathbf{i}}$, *then* $\mathbf{CD}$ *is*

(a) $\hat{\mathbf{i}}+3\hat{\mathbf{j}}$ (b) $\hat{\mathbf{i}}+2\hat{\mathbf{j}}$

(c) $-\hat{\mathbf{i}}+3\hat{\mathbf{j}}$ (d) None of these

Sol.



AB is along the X-axis and BD is along the Y-axis.

$\mathbf{AB} = 2\hat{\mathbf{i}} \Rightarrow AB = BC = CD = \ldots\ldots\ldots = 2$

From the figure, $BM = BC\sin 60° = 2\sin 60° = \sqrt{3}$

$\therefore \quad \mathbf{BD} = 2\sqrt{3}\hat{\mathbf{j}}$

$\mathbf{BC} = BC\cos 60°\hat{\mathbf{i}} + BC\sin 60°\hat{\mathbf{j}} = \hat{\mathbf{i}} + \sqrt{3}\hat{\mathbf{j}}$

$\mathbf{CD} = \mathbf{BD} - \mathbf{BC} = 2\sqrt{3}\hat{\mathbf{j}} - (\hat{\mathbf{i}} + \sqrt{3}\mathbf{j}) = -\hat{\mathbf{i}} + \sqrt{3}\hat{\mathbf{j}}$

● ***Ex. 29*** *The vertices of triangle are $A(1, 1, 2), B(4, 3, 1)$ and $C(2, 3, 5)$. A vector representing the internal bisector of the $\angle A$ is*

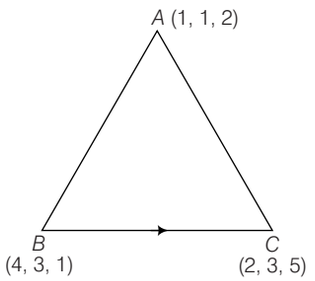
(a) $\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ (b) $2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$

(c) $2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ (d) None of these

Sol. (c) From the figure, we have

$\mathbf{b} = \mathbf{AC} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$

and $\mathbf{c} = \mathbf{AB} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$



$\therefore$ Unit vector along the bisector of $\angle A$ is given by

$$= \frac{\mathbf{b} + \mathbf{c}}{2} = \frac{(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) + (3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}})}{\sqrt{14}}$$

$$= \frac{2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{14}}$$

$\therefore$ Any vector along the angle bisector of

$\angle A = 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$

● ***Ex. 30*** *Let $\mathbf{a} = (1, 1, -1), \mathbf{b} = (5, -3, -3)$ and $\mathbf{c} = (3, -1, 2)$. If $\mathbf{r}$ is collinear with $\mathbf{c}$ and has length $\frac{|\mathbf{a} + \mathbf{b}|}{2}$, then $\mathbf{r}$ equals*

(a) $\pm 3\mathbf{c}$ (b) $\pm\frac{3}{2}\mathbf{c}$

(c) $\pm\mathbf{c}$ (d) $\pm\frac{2}{3}\mathbf{c}$

Sol. (c) Let $\mathbf{r} = \lambda\mathbf{c}$

Given $|\mathbf{r}| = |\lambda||\mathbf{c}|$

$$\therefore \quad \frac{|\mathbf{a} + \mathbf{b}|}{2} = |\lambda|\,|\mathbf{c}|$$

$$\therefore \quad |6\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 4\hat{\mathbf{k}}| = 2|\lambda||3\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}|$$

$$\therefore \quad \sqrt{56} = 2|\lambda|\sqrt{14}$$

$$\therefore \quad \lambda = \pm 1$$

$$\therefore \quad \mathbf{r} = \pm\mathbf{c}$$

● ***Ex. 31*** *In a trapezium, the vector $\mathbf{BC} = \lambda\mathbf{AD}$. We will then find that $\mathbf{p} = \mathbf{AC} + \mathbf{BD}$ is collinear with $\mathbf{AD}$. If $\mathbf{p} = \mu\,\mathbf{AD}$, then*

(a) $\mu = \lambda + 1$ (b) $\lambda = \mu + 1$

(c) $\lambda + \mu = 1$ (d) $\mu = 2 + \lambda$

Sol. (a) We have, $\mathbf{p} = \mathbf{AC} + \mathbf{BD} = \mathbf{AC} + \mathbf{BC} + \mathbf{CD}$

$= \mathbf{AC} + \lambda\mathbf{AD} + \mathbf{CD}$

$= \lambda\mathbf{AD} + (\mathbf{AC} + \mathbf{CD}) = \lambda\,\mathbf{AD} + \mathbf{AD} = (\lambda + 1)\mathbf{AD}$

Therefore, $\mathbf{p} = \mu\mathbf{AD} \Rightarrow \mu = \lambda + 1$

● ***Ex. 32*** *If the position vectors of the points A, B and C be $\hat{\mathbf{i}} + \hat{\mathbf{j}}$, $\hat{\mathbf{i}} - \hat{\mathbf{j}}$ and $a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$ respectively, then the points A, B and C are collinear, if*

(a) $a = b = c = 1$

(b) $a = 1$, b and c are arbitrary scalars

(c) $a = b = c = 0$

(d) $c = 0$, $a = 1$ and b is arbitrary scalars

Sol. (d) Here, $\mathbf{AB} = -2\hat{\mathbf{j}}$, $\mathbf{BC} = (a - 1)\hat{\mathbf{i}} + (b + 1)\hat{\mathbf{j}} + c\hat{\mathbf{k}}$

The points are collinear, then $\mathbf{AB} = k(\mathbf{BC})$

$$-2\hat{\mathbf{j}} = k\{(a - 1)\hat{\mathbf{i}} + (b + 1)\hat{\mathbf{j}} + c\hat{\mathbf{k}}\}$$

On comparing, $k(a - 1) = 0, k(b + 1) = -2, kc = 0$

Hence, $c = 0$, $a = 1$ and b is arbitrary scalar.

● ***Ex. 33*** *Let a, b and c be distinct non-negative numbers and the vectors $a\hat{\mathbf{i}} + a\hat{\mathbf{j}} + c\hat{\mathbf{k}}$, $\hat{\mathbf{i}} + \hat{\mathbf{k}}$, $c\hat{\mathbf{i}} + c\hat{\mathbf{j}} + b\hat{\mathbf{k}}$ lie in a plane, then the quadratic equation $ax^2 + 2cx + b = 0$ has*

(a) real and equal roots

(b) real and unequal roots

(c) unreal roots

(d) both roots real and positive

Sol. (a) $a\hat{\mathbf{i}} + a\hat{\mathbf{j}} + c\hat{\mathbf{k}}$, $\hat{\mathbf{i}} + \hat{\mathbf{k}}$ and $c\hat{\mathbf{i}} + c\hat{\mathbf{j}} + b\hat{\mathbf{k}}$ are coplanar

$$\therefore \quad \begin{vmatrix} a & a & c \\ 1 & 0 & 1 \\ c & c & b \end{vmatrix} = 0 \Rightarrow c^2 - ab = 0$$

For, equation $ax^2 + 2cx + b = 0$

$$D = 4c^2 - 4ab = 0$$

So, roots are real and equal.

● ***Ex. 34*** *The number of distinct real values of λ for which the vectors $\lambda^3\hat{\mathbf{i}} + \hat{\mathbf{k}}$, $\hat{\mathbf{i}} - \lambda^3\hat{\mathbf{j}}$ and $\hat{\mathbf{i}} + (2\lambda - \sin\lambda)\hat{\mathbf{j}} - \lambda\hat{\mathbf{k}}$ are coplanar is*

(a) 0 (b) 1

(c) 2 (d) 3

Sol. (a) Put $\Delta = 0 \Rightarrow \lambda^7 + \lambda^3 + 2\lambda - \sin\lambda = 0$

Let $f(\lambda) = \lambda^7 + \lambda^3 + 2\lambda - \sin\lambda$

$\Rightarrow f'(\lambda) = (7\lambda^6 + 3\lambda^2 + 2 - \cos\lambda) > 0, \forall \in R$

$\therefore \quad f(\lambda) = 0$ has only one real solution $\lambda = 0$.

● ***Ex. 35*** *The points $A(2-x, 2, 2)$, $B(2, 2-y, 2)$, $C(2, 2, 2-z)$ and $D(1, 1, 1)$ are coplanar, then locus of $P(x, y, z)$ is*

(a) $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$ (b) $x+y+z=1$

(c) $\frac{1}{1-x}+\frac{1}{1-y}+\frac{1}{1-z}=1$ (d) None of these

Sol. (a) Here, $\mathbf{AB}=x\hat{\mathbf{i}}-y\hat{\mathbf{j}}$

$$\mathbf{AC}=x\hat{\mathbf{i}}-z\hat{\mathbf{k}};\ \mathbf{AD}=(x-1)\hat{\mathbf{i}}-\hat{\mathbf{j}}-\hat{\mathbf{k}}$$

As, these vectors are coplanar

$$\Rightarrow \begin{vmatrix} x & -y & 0 \\ x & 0 & -z \\ x-1 & -1 & -1 \end{vmatrix}=0 \Rightarrow \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$$

● ***Ex. 36*** *$\mathbf{p}=2\mathbf{a}-3\mathbf{b}$, $\mathbf{q}=\mathbf{a}-2\mathbf{b}+\mathbf{c}$ and $\mathbf{r}=-3\mathbf{a}+\mathbf{b}+2\mathbf{c}$, where $\mathbf{a}, \mathbf{b}, \mathbf{c}$ being non-zero non-coplanar vectors, then the vector $-2\mathbf{a}+3\mathbf{b}-\mathbf{c}$ is equal to*

(a) $\mathbf{p}-4\mathbf{q}$ (b) $\frac{-7\mathbf{q}+\mathbf{r}}{5}$

(c) $2\mathbf{p}-3\mathbf{q}+\mathbf{r}$ (d) $4\mathbf{p}-2\mathbf{r}$

Sol. (b) Let $-2\mathbf{a}+3\mathbf{b}-\mathbf{c}=x\mathbf{p}+y\mathbf{q}+z\mathbf{r}$

$$\Rightarrow -2\mathbf{a}+3\mathbf{b}-\mathbf{c}=(2x+y-3z)\mathbf{a}+(-3x-2y+z)\mathbf{b}+(y+2z)\mathbf{c}$$

$$\therefore \quad 2x+y-3z=-2, -3x-2y+z=3$$

and $$y+2z=-1$$

On solving these, we get $x=0, y=-\frac{7}{5}, z=\frac{1}{5}$

$$\therefore \quad -2\mathbf{a}+3\mathbf{b}-c=\frac{(-7\mathbf{q}+\mathbf{r})}{5}$$

Trick Check alternates one-by-one

i.e. (a) $\mathbf{p}-4\mathbf{q}=-2\mathbf{a}+5\mathbf{b}-4\mathbf{c}$

(b) $\frac{-7\mathbf{q}+\mathbf{r}}{5}=-2\mathbf{a}+3\mathbf{b}-\mathbf{c}$

● ***Ex. 37*** *If a_1 and a_2 are two values of a for which the unit vector $a\hat{\mathbf{i}}+b\hat{\mathbf{j}}+\frac{1}{2}\hat{\mathbf{k}}$ is linearly dependent with $\hat{\mathbf{i}}+2\hat{\mathbf{j}}$ and $\hat{\mathbf{j}}-2\hat{\mathbf{k}}$, then $\frac{1}{a_1}+\frac{1}{a_2}$ is equal to*

(a) 1 (b) $\frac{1}{8}$ (c) $\frac{-16}{11}$ (d) $\frac{-11}{16}$

Sol. (c) $a\hat{\mathbf{i}}+b\hat{\mathbf{j}}+\frac{1}{2}\hat{\mathbf{k}}=l(\hat{\mathbf{i}}+2\hat{\mathbf{j}})+m(\hat{\mathbf{j}}-2\hat{\mathbf{k}})$

$$\Rightarrow \quad a=l,\ b=2l+m \text{ and } m=\frac{-1}{4}$$

$a\hat{\mathbf{i}}+b\hat{\mathbf{j}}+\frac{1}{2}\hat{\mathbf{k}}$ is unit vector

$$\therefore \quad a^2+b^2=\frac{3}{4} \Rightarrow 5a^2-a-\frac{11}{16}=0$$

a_1 and a_2 are roots of above equation

$$\Rightarrow \quad \frac{1}{a_1}+\frac{1}{a_2}=\frac{a_1+a_2}{a_1a_2}=-\frac{16}{11}$$

JEE Type Solved Examples :
More than One Correct Option Type Questions

● ***Ex. 38*** *The vector $\hat{\mathbf{i}}+x\hat{\mathbf{j}}+3\hat{\mathbf{k}}$ is rotated through an angle θ and is doubled in magnitude. It now becomes $4\hat{\mathbf{i}}+(4x-2)\hat{\mathbf{j}}+2\hat{\mathbf{k}}$. The values of x are*

(a) 1 (b) $\frac{-2}{3}$

(c) 2 (d) $\frac{4}{3}$

Sol. (b,c) Let $\alpha=\hat{\mathbf{i}}+x\hat{\mathbf{j}}+3\hat{\mathbf{k}}$,

$$\beta=4\hat{\mathbf{i}}+(4x-2)\hat{\mathbf{j}}+2\hat{\mathbf{k}}$$

Given, $2|\alpha|=|\beta|$

$$\Rightarrow \quad 2\sqrt{10+x^2}=\sqrt{20+4(2x-1)^2}$$

$$\Rightarrow \quad 10+x^2=5+(4x^2-4x+1)$$

$$\Rightarrow \quad 3x^2-4x-4=0$$

$$\Rightarrow \quad x=2, -\frac{2}{3}$$

● ***Ex. 39*** *$\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are three coplanar unit vectors such that $\mathbf{a}+\mathbf{b}+\mathbf{c}=0$. If three vectors $\mathbf{p}, \mathbf{q}$ and $\mathbf{r}$ are parallel to $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ respectively, and have integral but different magnitudes, then among the following options, $|\mathbf{p}+\mathbf{q}+\mathbf{r}|$ can take a value equal to*

(a) 1 (b) 0 (c) $\sqrt{3}$ (d) 2

Sol. (c,d) Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ lie in the XY-plane.

Let $\mathbf{a}=\hat{\mathbf{i}}$, $\mathbf{b}=-\frac{1}{2}\hat{\mathbf{i}}+\frac{\sqrt{3}}{2}\hat{\mathbf{j}}$ and $\mathbf{c}=-\frac{1}{2}\hat{\mathbf{i}}-\frac{\sqrt{3}}{2}\hat{\mathbf{j}}$

Therefore, $|\mathbf{p}+\mathbf{q}+\mathbf{r}|=|\lambda\mathbf{a}+\mu\mathbf{b}+\nu\mathbf{c}|$

$$=\left|\lambda\hat{\mathbf{i}}+\mu\left(-\frac{1}{2}\hat{\mathbf{i}}+\frac{\sqrt{3}}{2}\hat{\mathbf{j}}\right)+\nu\left(-\frac{1}{2}\hat{\mathbf{i}}-\frac{\sqrt{3}}{2}\hat{\mathbf{j}}\right)\right|$$

$$=\left|\left(\lambda-\frac{\mu}{2}-\frac{\nu}{2}\right)\hat{\mathbf{i}}+\frac{\sqrt{3}}{2}(\mu-\nu)\hat{\mathbf{j}}\right|$$

$$=\sqrt{\left(\lambda-\frac{\mu}{2}-\frac{\nu}{2}\right)^2+\frac{3}{4}(\mu-\nu)^2}$$

$$= \sqrt{\lambda^2 + \mu^2 + \nu^2 - \lambda\mu - \lambda\nu - \mu\nu}$$

$$= \frac{1}{\sqrt{2}}\sqrt{(\lambda-\mu)^2 + (\mu-\nu)^2 + (\nu-\lambda)^2}$$

$$= \frac{1}{\sqrt{2}}\sqrt{1+1+4} = \sqrt{3}$$

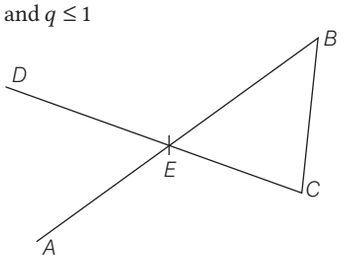
$\Rightarrow$ $|\mathbf{p}+\mathbf{q}+\mathbf{r}|$ can take a value equal to $\sqrt{3}$ and 2.

● ***Ex. 40*** *A, B, C and D are four points such that* $\mathbf{AB} = m(2\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 2\hat{\mathbf{k}}), \mathbf{BC} = (\hat{\mathbf{i}} - 2\hat{\mathbf{j}})$ *and* $\mathbf{CD} = n(-6\hat{\mathbf{i}} + 15\hat{\mathbf{j}} - 3\hat{\mathbf{k}})$. *If CD intersects AB at some point E, then*

(a) $m \geq \frac{1}{2}$ (b) $n \geq \frac{1}{3}$ (c) $m = n$ (d) $m < n$

Sol. (a, b) Let $\mathbf{EB} = p\ \mathbf{AB}$ and $\mathbf{CE} = q\,\mathbf{CD}$

Then $0 < p$ and $q \leq 1$



Since, $\mathbf{EB} + \mathbf{BC} + \mathbf{CE} = \mathbf{0}$

$$pm(2\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) + (\hat{\mathbf{i}} - 2\hat{\mathbf{j}}) + qn(-6\hat{\mathbf{i}} + 15\hat{\mathbf{j}} - 3\hat{\mathbf{k}}) = \mathbf{0}$$

$$\Rightarrow (2pm + 1 - 6qn)\hat{\mathbf{i}} + (-6pm - 2 + 15qn)\hat{\mathbf{j}} + (2pm - 6qn)\hat{\mathbf{k}} = \mathbf{0}$$

$$\Rightarrow \quad 2pm - 6qn + 1 = \mathbf{0},$$
$$-6pm - 2 + 15qn = \mathbf{0}$$
$$2pm - 6qn = \mathbf{0}$$

Solving these, we get

$$p = \frac{1}{(2m)} \quad \text{and} \quad q = \frac{1}{(3n)}$$

$$\therefore \quad 0 < \frac{1}{(2m)} \leq 1 \quad \text{and} \quad 0 < \frac{1}{(3n)} \leq 1$$

$$\Rightarrow \quad m \geq \frac{1}{2} \quad \text{and} \quad n \geq \frac{1}{3}$$

● ***Ex. 41*** *If non-zero vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *are equally inclined to coplanar vector* $\mathbf{c}$, *then* $\mathbf{c}$ *can be*

(a) $\frac{|\mathbf{a}|}{|\mathbf{a}| + 2|\mathbf{b}|}\mathbf{a} + \frac{|\mathbf{b}|}{|\mathbf{a}| + |\mathbf{b}|}\mathbf{b}$

(b) $\frac{|\mathbf{b}|}{|\mathbf{a}| + |\mathbf{b}|}\mathbf{a} + \frac{|\mathbf{a}|}{|\mathbf{a}| + |\mathbf{b}|}\mathbf{b}$

(c) $\frac{|\mathbf{a}|}{|\mathbf{a}| + 2|\mathbf{b}|}\mathbf{a} + \frac{|\mathbf{b}|}{|\mathbf{a}| + 2|\mathbf{b}|}\mathbf{b}$

(d) $\frac{|\mathbf{b}|}{2|\mathbf{a}| + |\mathbf{b}|}\mathbf{a} + \frac{|\mathbf{a}|}{2|\mathbf{a}| + |\mathbf{b}|}\mathbf{b}$

Sol. (b,d) Since, $\mathbf{a}$ and $\mathbf{b}$ are equally inclined to $\mathbf{c}$, therefore $\mathbf{c}$ must be of the form $t\left(\frac{\mathbf{a}}{|\mathbf{a}|} + \frac{\mathbf{b}}{|\mathbf{b}|}\right)$

Now, $\frac{|\mathbf{b}|}{|\mathbf{a}| + |\mathbf{b}|}\mathbf{a} + \frac{|\mathbf{a}|}{|\mathbf{a}| + |\mathbf{b}|}\mathbf{b} = \frac{|\mathbf{a}||\mathbf{a}|}{|\mathbf{a}| + |\mathbf{a}|}\left(\frac{\mathbf{a}}{|\mathbf{a}|} + \frac{\mathbf{b}}{|\mathbf{b}|}\right)$

Also, $\frac{|\mathbf{b}|}{2|\mathbf{a}| + |\mathbf{b}|}\mathbf{a} + \frac{|\mathbf{a}|}{2|\mathbf{a}| + |\mathbf{b}|}\mathbf{b} = \frac{|\mathbf{a}||\mathbf{b}|}{2|\mathbf{a}| + |\mathbf{b}|}\left(\frac{\mathbf{a}}{|\mathbf{a}|} + \frac{\mathbf{b}}{|\mathbf{b}|}\right)$

Other two vectors cannot be written in the from $t\left(\frac{\mathbf{a}}{|\mathbf{a}|} + \frac{\mathbf{b}}{|\mathbf{b}|}\right)$

● ***Ex. 42*** *The vectors* $x\hat{\mathbf{i}} + (x+1)\hat{\mathbf{j}} + (x+2)\hat{\mathbf{k}}$, $(x+3)\hat{\mathbf{i}} + (x+4)\hat{\mathbf{j}} + (x+5)\hat{\mathbf{k}}$ *and* $(x+6)\hat{\mathbf{i}} + (x+7)\hat{\mathbf{j}} + (x+8)\hat{\mathbf{k}}$ *are coplanar if x is equal to*

(a) 1 (b) −3 (c) 4 (d) 0

Sol. (a, b, c, d)

$x\hat{\mathbf{i}} + (x+1)\hat{\mathbf{j}} + (x+2)\hat{\mathbf{k}}$, $(x+3)\hat{\mathbf{i}} + (x+4)\hat{\mathbf{j}} + (x+5)\hat{\mathbf{k}}$ and $(x+6)\hat{\mathbf{i}} + (x+7)\hat{\mathbf{j}} + (x+8)\hat{\mathbf{k}}$ are coplanar. We have determinant of their coefficients as $\begin{vmatrix} x & x+1 & x+2 \\ x+3 & x+4 & x+5 \\ x+6 & x+7 & x+8 \end{vmatrix}$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we have

$\begin{vmatrix} x & 1 & 2 \\ x+3 & 1 & 2 \\ x+6 & 1 & 2 \end{vmatrix} = 0$. Hence, $x \in R$.

● ***Ex. 43*** *Given three vectors* $\mathbf{a}, \mathbf{b}$, *and* $\mathbf{c}$ *are non-zero and non-coplanar vectors. Then which of the following are coplanar.*

(a) $\mathbf{a}+\mathbf{b}, \mathbf{b}+\mathbf{c}, \mathbf{c}+\mathbf{a}$ (b) $\mathbf{a}-\mathbf{b}, \mathbf{b}+\mathbf{c}, \mathbf{c}+\mathbf{a}$

(c) $\mathbf{a}+\mathbf{b}, \mathbf{b}-\mathbf{c}, \mathbf{c}+\mathbf{a}$ (d) $\mathbf{a}+\mathbf{b}, \mathbf{b}+\mathbf{c}, \mathbf{c}-\mathbf{a}$

Sol. (b, c, d) $\mathbf{c}+\mathbf{a} = (\mathbf{b}+\mathbf{c}) + (\mathbf{a}-\mathbf{b})$

$\mathbf{a}+\mathbf{b} = (\mathbf{b}-\mathbf{c}) + (\mathbf{c}+\mathbf{a})$

$\mathbf{a}+\mathbf{c} = (\mathbf{a}+\mathbf{b}) + (\mathbf{c}-\mathbf{a})$

So, vectors in options (b), (c) and (d) are coplanar.

● ***Ex. 44*** *In a four-dimensional space where unit vectors along the axes are* $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ *and* $\hat{\mathbf{i}}$, *and* $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ *are four non-zero vectors such that no vector can be expressed as a linear combination of others and* $(\lambda-1)(\mathbf{a}_1 - \mathbf{a}_2) + \mu(\mathbf{a}_2 + \mathbf{a}_3) + \gamma(\mathbf{a}_3 + \mathbf{a}_4 - 2\mathbf{a}_2) + \mathbf{a}_3 + \delta\mathbf{a}_4 = 0$, *then*

(a) $\lambda = 1$ (b) $\mu = -\frac{2}{3}$ (c) $\gamma = \frac{2}{3}$ (d) $\delta = \frac{1}{3}$

Sol. (a, b, d)

$(\lambda-1)(\mathbf{a}_1 - \mathbf{a}_2) + \mu(\mathbf{a}_2 + \mathbf{a}_3) + \gamma(\mathbf{a}_3 + \mathbf{a}_4 - 2\mathbf{a}_2) + \mathbf{a}_3 + \delta\mathbf{a}_4 = \mathbf{0}$

i.e. $(\lambda-1)\mathbf{a}_1 + (1-\lambda+\mu-2\gamma)\mathbf{a}_2 + (\mu+\gamma+1)\mathbf{a}_3 + (\gamma+\delta)\mathbf{a}_4 = \mathbf{0}$

Since, $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ and $\mathbf{a}_4$ are linearly independent, we have

$$\lambda - 1 = 0, 1 - \lambda + \mu - 2\gamma = 0,$$
$$\mu + \gamma + 1 = 0 \quad \text{and} \quad \gamma + \delta = 0$$

i.e. $\lambda = 1, \mu = 2\gamma, \mu + \gamma + 1 = 0, \gamma + \delta = 0$

Hence, $\lambda = 1, \mu = -\frac{2}{3}, \gamma = -\frac{1}{3}, \delta = \frac{1}{3}$

JEE Type Solved Examples :
Statement I and II Type Questions

Directions (Ex. Nos. 45-51) *This section is based on* ***Statement*** I *and* ***Statement*** II. *Select the correct answer from the codes given below.*

(a) Both Statement I and Statement II are correct and Statement II is the correct explanation of Statement I
(b) Both Statement I and Statement II are correct but Statement II is not the correct explanation of Statement I
(c) Statement I is correct but Statement II is incorrect
(d) Statement II is correct but Statement I is incorrect

● ***Ex. 45*** **Statement I** *If* $|\mathbf{a}| = 3, |\mathbf{b}| = 4$ *and* $|\mathbf{a} + \mathbf{b}| = 5$, *then* $|\mathbf{a} - \mathbf{b}| = 5$.

Statement II *The length of the diagonals of a rectangle is the same.*

Sol. (a) We have, adjacent sides of triangle $|\mathbf{a}| = 3, |\mathbf{b}| = 4$

The length of the diagonal is $|\mathbf{a} + \mathbf{b}| = 5$

Since, it satisfies the Pythagoras theorem, $\mathbf{a} \perp \mathbf{b}$

So, the parallelogram is a rectangle.

Hence, the length of the other diagonal is $|\mathbf{a} - \mathbf{b}| = 5$.

● ***Ex. 46*** **Statement I** *If* $|\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$, *then* $\mathbf{a}$ *and* $\mathbf{b}$ *are perpendicular to each other.*

Statement II *If the diagonals of a parallelogram are equal in magnitude, then the parallelogram is a rectangle.*

Sol. (a) $\mathbf{a} + \mathbf{b} = \mathbf{a} - \mathbf{b}$ are the diagonals of a parallelogram whose sides are $\mathbf{a}$ and $\mathbf{b}$.

$$|\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$$

Thus, diagonals of the parallelogram have the same length.

So, the parallelogram is a rectangle, i.e. $\mathbf{a} \perp \mathbf{b}$.

● ***Ex. 47*** **Statement I** *If I is the incentre of* ΔABC, *then* $|\mathbf{BC}|\mathbf{IA} + |\mathbf{CA}|\mathbf{IB} + |\mathbf{AB}|\mathbf{IC} = 0$

Statement II *The position vector of centroid of* ΔABC *is* $\frac{\mathbf{OA} + \mathbf{OB} + \mathbf{OC}}{3}$.

Sol. (b) We know that,

$$\mathbf{OI} = \frac{|\mathbf{CB}|\mathbf{OA} + |\mathbf{CA}|\mathbf{OB} + |\mathbf{AB}|\mathbf{OC}}{|\mathbf{BC}| + |\mathbf{CA}| + |\mathbf{AB}|}$$

and $$\mathbf{OG} = \frac{\mathbf{OA} + \mathbf{OB} + \mathbf{OC}}{3}$$

● ***Ex. 48*** **Statement I** *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are unit vectors inclined at an angle* α *and* $\mathbf{x}$ *is a unit vector bisecting the angle between them, then* $\mathbf{x} = \frac{\mathbf{u} + \mathbf{v}}{2\sin\frac{\alpha}{2}}$.

Statement II *If ABC is an isosceles triangles with* $AB = AC = 1$, *then vectors representing bisector of angle A is given by* $\mathbf{AB} = \frac{\mathbf{AB} + \mathbf{AC}}{2}$.

Sol. (d) We know that the unit vector along bisector of unit vectors $\mathbf{u}$ and $\mathbf{v}$ is $\frac{\mathbf{u} + \mathbf{v}}{2\cos\frac{\theta}{2}}$, where θ is the angle between vectors $\mathbf{u}$ and $\mathbf{v}$.

Also, in an isosceles ΔABC in which $AB = AC$, the median and bisector from A must be same line.

● ***Ex. 49*** **Statement I** *If* $\mathbf{a} = 2\hat{\mathbf{i}} + \hat{\mathbf{k}}, \mathbf{b} = 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ *and* $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$ *are coplanar, then* $\mathbf{c} = 4\mathbf{a} - \mathbf{b}$.

Statement II *A set vectors* $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \ldots, \mathbf{a}_n$ *is said to be linearly independent, if every relation of the form* $l_1\mathbf{a}_1 + l_2\mathbf{a}_2 + l_3\mathbf{a}_3 + \ldots + l_n\mathbf{a}_n = 0$ *implies that* $l_1 = l_2 = l_3 = \ldots = l_n = 0$ *(scalar).*

Sol. (b) $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b} \Rightarrow \lambda = 4$ and $\mu = -1$

● ***Ex. 50*** **Statement I** *Let* $A(\mathbf{a}), B(\mathbf{b})$ *and* $C(\mathbf{c})$ *be three points such that* $\mathbf{a} = 2\hat{\mathbf{i}} + \hat{\mathbf{k}}, \mathbf{b} = 3\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ *and* $\mathbf{c} = -\hat{\mathbf{i}} + 7\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$. *Then, OABC is a tetrahedron.*

Statement II *Let* $A(\mathbf{a}), B(\mathbf{b})$ *and* $C(\mathbf{c})$ *be three points such that vectors* $\mathbf{a}, \mathbf{b}$ *and* $\mathbf{c}$ *are non-coplanar. Then OABC is a tetrahedron.*

Sol. (a) Given vectors are non-coplanar.

Hence, the answer is (a).

● ***Ex. 51*** **Statement I** *Let* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *and* $\mathbf{a}$ *be the position vectors of four points A, B, C and D and* $3\mathbf{a} - 2\mathbf{b} + 5\mathbf{c} - 6\mathbf{d} = 0$. *Then points A, B, C and D are coplanar.*

Statement II *Three non-zero linearly dependent co-initial vectors (*$\mathbf{PQ}, \mathbf{PR}$ *and* $\mathbf{PS}$*) are coplanar. Then* $\mathbf{PQ} = \lambda\mathbf{PR} + \mu\mathbf{PS}$, *where* λ *and* μ *are scalars.*

Sol. (a)

$$3\mathbf{a} - 2\mathbf{b} + 5\mathbf{c} - 6\mathbf{d} = (2\mathbf{a} - 2\mathbf{b}) + (-5\mathbf{a} + 5\mathbf{c}) + (6\mathbf{a} - 6\mathbf{d})$$
$$= -2\mathbf{AB} + 5\mathbf{AC} - 6\mathbf{AD} = 0$$

Therefore, $\mathbf{AB}$, $\mathbf{AC}$ and $\mathbf{AD}$ are linearly dependent.

Hence, by Statement II, Statement I is true.

JEE Type Solved Examples :
Passage Based Questions

Passage I
(Ex. Nos. 52 to 54)

ABCD is a parallelogram. L is a point on BC which divides BC in the ratio 1 : 2. *AL intersects BD at P. M is a point on DC which divides DC in the ratio* 1 : 2 *and AM intersects BD in Q.*

● ***Ex. 52*** *Points P divides AL in the ratio*

(a) 1 : 2 (b) 1 : 3
(c) 3 : 1 (d) 2 : 1

● ***Ex. 53*** *Point Q divides DB in the ratio*

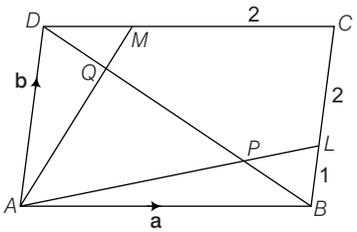
(a) 1 : 2 (b) 1 : 3
(c) 3 : 1 (d) 2 : 1

● ***Ex. 54*** *PQ : DB is equal to*

(a) $\frac{2}{3}$ (b) $\frac{1}{3}$
(c) $\frac{1}{2}$ (d) $\frac{3}{4}$

Sol. (Ex. Nos. 52-54)

52. (c)



$$\mathbf{BL} = \frac{1}{3}\mathbf{b}$$

$$\therefore \quad \mathbf{AL} = \mathbf{a} + \frac{1}{3}\mathbf{b}$$

Let $\mathbf{AP} = \lambda\mathbf{AL}$ and P divides DB in the ratio $\mu : 1 - \mu$

Then, $$\mathbf{AP} = \lambda\mathbf{a} + \frac{\lambda}{3}\mathbf{b} \quad \ldots(i)$$

Also, $$\mathbf{AP} = \mu\mathbf{a} + (1-\mu)\mathbf{b} \quad \ldots(ii)$$

From Eqs. (i) and (ii),

$$\lambda\mathbf{a} + \frac{\lambda}{3}\mathbf{b} = \mu\mathbf{a} + (1-\mu)\mathbf{b}$$

$$\therefore \quad \lambda = \mu$$

and $$\frac{\lambda}{3} = 1 - \mu$$

$$\therefore \quad \lambda = \frac{3}{4}$$

53. (b) Hence, P divides AL in the ratio 3 : 1 and P divides DB in the ratio 1 : 3 Similarly Q divides DB in the ratio 1 : 3.

Thus, $$DQ = \frac{1}{4}DB$$

and $$PB = \frac{1}{4}DB$$

54. (b) $\therefore$ $$PQ = \frac{1}{2}DB,$$

i.e. $$PQ : DB = 1 : 2$$

Passage II
(Ex. Nos. 55 to 56)

Let A, B, C, D, E represent vertices of a regular pentagon ABCDE. Given the position vector of these vertices be $\mathbf{a}, \mathbf{a} + \mathbf{b}, \mathbf{b}, \lambda\mathbf{a}$ *and* $\lambda\mathbf{b}$ *respectively.*

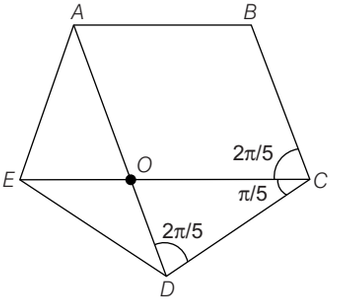
● ***Ex. 55*** *The ratio* $\frac{AD}{BC}$ *is equal to*

(a) $1 - \cos\frac{3\pi}{5} : \cos\frac{3\pi}{5}$ (b) $1 + 2\cos\frac{2\pi}{5} : \cos\frac{\pi}{5}$
(c) $1 + 2\cos\frac{\pi}{5} : 2\cos\frac{\pi}{5}$ (d) None of these

● ***Ex. 56*** *AD divides EC in the ratio*

(a) $\cos\frac{2\pi}{5} : 1$ (b) $\cos\frac{3\pi}{5} : 1$
(c) $1 : 2\cos\frac{\pi}{5}$ (d) 1 : 2

Sol. (Ex. Nos. 55-56) Given *ABCDE* is a regular pentagon



Let position vector point A and C be $\mathbf{a}$ and $\mathbf{b}$, respectively.

AD is parallel to BC and AB is parallel to EC.

Therefore,

$AOCB$ is a parallelogram and position vector of B is $\mathbf{a} + \mathbf{b}$.

The position vectors of E and D are $\lambda\mathbf{b}$ and $\lambda\mathbf{a}$ respectively.

Also, $OA = BC = AB = OC = 1$ (let)

Therefore, $AOCB$ is rhombus.

$$\angle ABC = \angle AOC = \frac{3\pi}{5}$$

and $$\angle OAB = \angle BCO = \pi - \frac{3\pi}{5} = \frac{2\pi}{5}$$

Further, $OA = AE = 1$ and $OC = CD = 1$

Thus, ΔEAO and ΔOCD are isosceles.

In ΔOCD, using sine rule we get.

$$\frac{OC}{\sin\frac{2\pi}{5}} = \frac{OD}{\sin\frac{\pi}{5}}$$

$$\Rightarrow \quad OD = \frac{1}{2\cos\frac{\pi}{5}} = OE$$

$$\Rightarrow \quad AD = OA + OD = 1 + \frac{1}{2\cos\frac{\pi}{5}}$$

55. (c) $\dfrac{AD}{BC} = 1 + \dfrac{1}{2\cos\frac{\pi}{5}} = \dfrac{1 + 2\cos\frac{\pi}{5}}{2\cos\frac{\pi}{5}}$

56. (c) $\dfrac{OE}{OC} = \dfrac{1}{2\cos\frac{\pi}{5}}$

Passage III

(Ex. Nos. 57 to 58)

In a parallelogram OABC vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *respectively, the position vectors of vertices* A, B, C *with reference to* O *as origin. A point* E *is taken on the side* BC *which divides it in the ratio of* $2:1$. *Also, the line segment* AE *intersects the line bisecting the angle* $\angle AOC$ *internally at point* P. *If* CP *when extended meets* AB *in point* F, *then*

● ***Ex. 57*** *The position vector of point P is*

(a) $\dfrac{|\mathbf{a}||\mathbf{c}|}{3|\mathbf{c}| + 2|\mathbf{a}|}\left(\dfrac{\mathbf{a}}{|\mathbf{a}|} + \dfrac{\mathbf{c}}{|\mathbf{c}|}\right)$

(b) $\dfrac{3|\mathbf{a}||\mathbf{c}|}{3|\mathbf{c}| + 2|\mathbf{a}|}\left(\dfrac{\mathbf{a}}{|\mathbf{a}|} + \dfrac{\mathbf{c}}{|\mathbf{c}|}\right)$

(c) $\dfrac{2|\mathbf{a}||\mathbf{c}|}{3|\mathbf{c}| + 2|\mathbf{a}|}\left(\dfrac{\mathbf{a}}{|\mathbf{a}|} + \dfrac{\mathbf{c}}{|\mathbf{c}|}\right)$

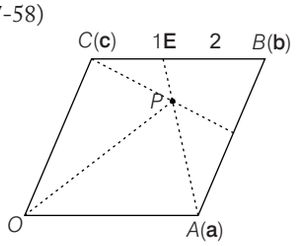
(d) None of the above

● ***Ex. 58*** *The ratio in which F divides AB is*

(a) $\dfrac{2|\mathbf{a}|}{\|\mathbf{a}| - 3|\mathbf{c}\|}$ (b) $\dfrac{|\mathbf{a}|}{\|\mathbf{a}| - 3|\mathbf{c}\|}$

(c) $\dfrac{3|\mathbf{a}|}{\|\mathbf{a}| - 3|\mathbf{c}\|}$ (d) $\dfrac{3|\mathbf{c}|}{3\|\mathbf{c}| - |\mathbf{a}\|}$

Sol. (Ex. Nos. 57-58)



Let the position vector of A and C be $\mathbf{a}$ and $\mathbf{c}$ respectively. Therefore,

Position vector of

$$\boldsymbol{B} = \mathbf{b} = \mathbf{a} + \mathbf{c} \qquad \ldots(i)$$

Also, position vector of

$$E = \frac{\mathbf{b} + 2\mathbf{c}}{3} = \frac{\mathbf{a} + 3\mathbf{c}}{3} \qquad \ldots(ii)$$

Now, point P lies on angle bisector of $\angle AOC$. Thus,

Position vector of point

$$P = \lambda\left(\frac{\mathbf{a}}{|\mathbf{a}|} + \frac{\mathbf{b}}{|\mathbf{b}|}\right) \qquad \ldots(iii)$$

Also, let P divides EA in ration $\mu : 1$. Therefore, Position vector of P

$$= \frac{\mu\mathbf{a} + \dfrac{\mathbf{a} + 3\mathbf{c}}{3}}{\mu + 1} = \frac{(3\mu + 1)\mathbf{a} + 3\mathbf{c}}{3(\mu + 1)} \qquad \ldots(iv)$$

Comparing Eqs. (iii) and (iv), we get

$$\lambda\left(\frac{\mathbf{a}}{|\mathbf{a}|} + \frac{\mathbf{c}}{|\mathbf{c}|}\right) = \frac{(3\mu + 1)\mathbf{a} + 3\mathbf{c}}{3(\mu + 1)}$$

$$\Rightarrow \quad \frac{\lambda}{|\mathbf{a}|} = \frac{3\mu + 1}{3(\mu + 1)} \text{ and } \frac{\lambda}{|\mathbf{c}|} = \frac{1}{\mu + 1}$$

$$\Rightarrow \quad \frac{3|\mathbf{c}| - |\mathbf{a}|}{3|\mathbf{a}|} = \mu$$

$$\Rightarrow \quad \frac{\lambda}{|\mathbf{c}|} = \frac{1}{\dfrac{3|\mathbf{c}| - |\mathbf{a}|}{3|\mathbf{a}|} + 1} \Rightarrow \lambda = \frac{3|\mathbf{a}||\mathbf{c}|}{3|\mathbf{c}| + 2|\mathbf{a}|}$$

57. (b) So, position vector of P is $\dfrac{3|\mathbf{a}||\mathbf{c}|}{3|\mathbf{c}| + 2|\mathbf{a}|}\left(\dfrac{\mathbf{a}}{|\mathbf{a}|} + \dfrac{\mathbf{c}}{|\mathbf{c}|}\right)$.

58. (d) Let F divides AB in ratio $t:1$, then position vector of F is $\dfrac{t\mathbf{b} + \mathbf{a}}{t + 1}$

Now, points C, P, F are collinear, Then, $\mathbf{CF} = m\mathbf{CP}$

$$\Rightarrow \quad \frac{t(\mathbf{a} + \mathbf{c})}{t + 1} - \mathbf{c} = m\left\{\frac{3|\mathbf{a}||\mathbf{c}|}{3|\mathbf{c}| + 2|\mathbf{a}|}\left(\frac{\mathbf{a}}{|\mathbf{a}|} + \frac{\mathbf{c}}{|\mathbf{c}|}\right) - \mathbf{c}\right\}$$

Comparing coefficients, we get

$$\frac{t}{t + 1} = m\frac{3|\mathbf{c}|}{3|\mathbf{c}| + 2|\mathbf{a}|}$$

and $$\frac{-1}{t + 1} = m\frac{|\mathbf{a}| - 3|\mathbf{c}|}{3|\mathbf{c}| + 2|\mathbf{a}|}$$

$$t = \frac{3|\mathbf{c}|}{3|\mathbf{c}| - |\mathbf{a}|}$$

JEE Type Solved Examples :
Matching Type Questions

● ***Ex. 59*** *In the Cartesian plane, a man starts at origin and walks a distance of 3 units of the North-East direction and reaches a point P. From P, he walks a distance of 4 units in the North-West direction to reach a point Q. Construct the parallelogram OPQR with PO and PQ as adjacent sides. Let M be the mid-point of PQ.*

	Column I		Column II
A.	The position vector of P is	(p)	$\frac{3}{\sqrt{2}}(\hat{\mathbf{i}}+\hat{\mathbf{j}})$
B.	The position vector of R is	(q)	$\frac{1}{\sqrt{2}}(\hat{\mathbf{i}}+5\hat{\mathbf{j}})$
C.	The position vector of M is	(r)	$2\sqrt{2}(-\hat{\mathbf{i}}+\hat{\mathbf{j}})$
D.	If the line OM meets the diagonal PR in the point T, then **OT** equals	(s)	$\frac{\sqrt{2}}{3}(\hat{\mathbf{i}}+5\hat{\mathbf{j}})$

Sol. A $\to p$, B $\to r$, C $\to q$, D $\to s$

(**A**) Let $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ be the unit vectors along **OX** and **OY** respectively.

Now, $OP = 3$ and $\angle XOP = 45°$ implies that

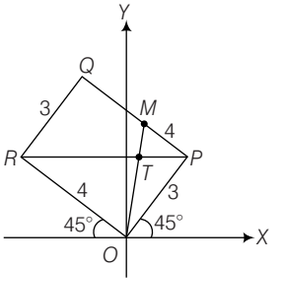
$$\mathrm{OP} = (3\cos 45°)\hat{\mathbf{i}} + (3\sin 45°)\hat{\mathbf{j}} = \frac{3}{\sqrt{2}}(\hat{\mathbf{i}}+\hat{\mathbf{j}})$$

(**B**) Again, $\angle X\mathrm{OR} = 135°$ and $OR = 4$ implies that

$$\mathbf{OR} = \frac{4}{\sqrt{2}}(-\hat{\mathbf{i}}+\hat{\mathbf{j}}) = 2\sqrt{2}(-\hat{\mathbf{i}}+\hat{\mathbf{j}})$$

(**C**) The position vector of Q is given by

$$\mathbf{OP}+\mathbf{PQ} = \mathbf{OP}+\mathbf{OR} = \frac{1}{\sqrt{2}}(-\hat{\mathbf{i}}+7\hat{\mathbf{j}})$$



$$\therefore\quad \mathbf{OM} = \frac{\left(\frac{3}{\sqrt{2}}\right)(\hat{\mathbf{i}}+\hat{\mathbf{j}}) + \left(\frac{1}{\sqrt{2}}\right)(-\hat{\mathbf{i}}+7\hat{\mathbf{j}})}{2}$$

$$= \frac{2\hat{\mathbf{i}}+10\hat{\mathbf{j}}}{2\sqrt{2}} = \frac{\hat{\mathbf{i}}+5\hat{\mathbf{j}}}{\sqrt{2}}$$

(**D**) Now, $PT : RT = 1 : 2$

Therefore, $$\mathbf{OT} = \frac{1(\mathbf{OR})+2(OP)}{3}$$

$$= \frac{\left(\frac{4}{\sqrt{2}}\right)(-\hat{\mathbf{i}}+\hat{\mathbf{j}}) + 2\left[\left(\frac{3}{\sqrt{2}}\right)(\hat{\mathbf{i}}+\hat{\mathbf{j}})\right]}{3}$$

$$= \frac{\sqrt{2}}{3}(\hat{\mathbf{i}}+5\hat{\mathbf{j}})$$

JEE Type Solved Examples :
Single Integer Answer Type Questions

● ***Ex. 60*** *P and Q have position vectors* **a** *and* **b** *relative to the origin O and X, Y divide* **PQ** *internally and externally respectively in the ratio* 2 : 1*. Vector* **XY** *is* $\lambda\mathbf{a}+\mu\mathbf{b}$*, then the value of* $|\lambda+\mu|$ *is*

Sol. (0) Since, X and Y divide **PQ** internally and externally in the ratio 2 : 1, then $X = \frac{2\mathbf{b}+\mathbf{a}}{3}$ and $y = 2\mathbf{b}-\mathbf{a}$

$\therefore$ XY = Position vector of y-Position vector of x

$$= 2\mathbf{b}-\mathbf{a}-\frac{2\mathbf{b}+\mathbf{a}}{3} = \frac{4\mathbf{b}}{3}-\frac{4\mathbf{a}}{3}$$

On comparing it with $\lambda\mathbf{a}+\mu\mathbf{b}$, we get

$$\lambda = -\frac{4}{3} \text{ and } \mu = \frac{4}{3}$$

$$\therefore\quad |\lambda+\mu| = \left|\frac{-4}{3}+\frac{4}{3}\right| = 0$$

● ***Ex. 61*** *If* $A(1,-1,-3)$, $B(2,1,-2)$ *and* $C(-5,2,-6)$ *are the position vectors of the vertices of* ΔABC*. The length of the bisector of its internal angle at A is* $\frac{\lambda\sqrt{10}}{4}$*, then value of* λ *is*

Sol. (3) We have, $\mathbf{AB} = \hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}}$, $\mathbf{AC} = -6\hat{\mathbf{i}}+3\hat{\mathbf{j}}-3\hat{\mathbf{k}}$

$$\Rightarrow\quad |\mathbf{AB}| = \sqrt{6} \text{ and } |\mathbf{AC}| = 3\sqrt{6}$$

Clearly, point D divides BC in the ratio $AB : AC$, i.e. 1 : 3

$$\therefore \text{Position vector of } D = \frac{(-5\hat{\mathbf{i}}+2\hat{\mathbf{j}}-6\hat{\mathbf{k}})+3(2\hat{\mathbf{i}}+\hat{\mathbf{j}}-2\hat{\mathbf{k}})}{1+3}$$

$$= \frac{1}{4}(\hat{\mathbf{i}}+5\hat{\mathbf{j}}-12\hat{\mathbf{k}})$$

$$\therefore\quad \mathbf{AD} = \frac{1}{4}(\hat{\mathbf{i}}+5\hat{\mathbf{j}}-12\hat{\mathbf{k}})-(\hat{\mathbf{i}}-\hat{\mathbf{j}}-3\hat{\mathbf{k}}) = \frac{3}{4}(-\hat{\mathbf{i}}+3\hat{\mathbf{j}})$$

$$\Rightarrow\quad |\mathbf{AD}| = AD = \frac{3}{4}\sqrt{10}$$

$$\therefore\quad \lambda = 3$$

● ***Ex. 62*** *Let ABC be a triangle whose centroid is G, orthocentre is H and circumcentre is the origin 'O'. If D is any point in the plane of the triangle such that no three of O, A, C and D are collinear satisfying the relation* $\mathbf{AD}+\mathbf{BD}+\mathbf{CH}+3\mathbf{HG}=\lambda\mathbf{HD}$, *then what is the value of the scalar 'Δ'*

Sol. LHS $=\mathbf{d}-\mathbf{a}+\mathbf{d}-\mathbf{b}+\mathbf{h}-\mathbf{c}+3(\mathbf{g}-\mathbf{h})$

$$=2\mathbf{d}-(\mathbf{a}+\mathbf{b}+\mathbf{c})+3\frac{(\mathbf{a}+\mathbf{b}+\mathbf{c})}{3}-2\mathbf{h}$$

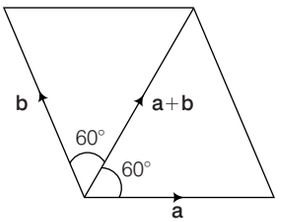
$$=2\mathbf{d}-2\mathbf{h}=2(\mathbf{d}-\mathbf{h})=2\mathbf{HD}\Rightarrow\lambda=2$$

● ***Ex. 63*** *Let* $\mathbf{a},\mathbf{b}$ *and* $\mathbf{c}$ *be unit vectors such that* $\mathbf{a}+\mathbf{b}-\mathbf{c}=0$. *If the area of triangle formed by vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *is A, then what is the value of* $16A^2$ *?*

Sol. (3) Given $\mathbf{a}+\mathbf{b}=\mathbf{c}$

Now, vector $\mathbf{c}$ is along the diagonal of the parallelogram which has adjacent side vectors $\mathbf{a}$ and $\mathbf{b}$. Since, $\mathbf{c}$ is also a unit vector, triangle formed by vectors $\mathbf{a}$ and $\mathbf{b}$ is an equilateral triangle.

Then, Area of triangle $=\frac{\sqrt{3}}{4}\Rightarrow A^2=\frac{3}{10}\Rightarrow 16A^2=3$



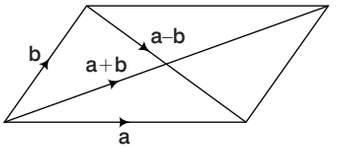
● ***Ex. 64*** *Find the least positive integral value of x for which the angle between vectors* $\mathbf{a}=x\hat{\mathbf{i}}-3\hat{\mathbf{j}}-\hat{\mathbf{k}}$ *and* $\mathbf{b}=2x\hat{\mathbf{i}}+x\hat{\mathbf{j}}-\hat{\mathbf{k}}$ *is acute.*

Sol. (2) Let $\mathbf{a}=x\hat{\mathbf{i}}-3\hat{\mathbf{j}}-\hat{\mathbf{k}}$ and $\mathbf{b}=2x\hat{\mathbf{i}}+x\hat{\mathbf{j}}-\hat{\mathbf{k}}$ be the adjacent sides of the parallelogram.

Now angle between $\mathbf{a}$ and $\mathbf{b}$ is acute, i.e. $|\mathbf{a}+\mathbf{b}|>|\mathbf{a}-\mathbf{b}|$

$$\Rightarrow\left|3x\hat{\mathbf{i}}+(x-3)\hat{\mathbf{j}}-2\hat{\mathbf{k}}\right|^2>\left|-x\hat{\mathbf{i}}-(x+3)\hat{\mathbf{j}}\right|^2$$

or $$9x^2+(x-3)^2+4>x^2+(x+3)^3$$



or $$8x^2-12x+4>0 \text{ or } 2x^2-3x+1>0$$

or $$(2x-1)(x-1)>0\Rightarrow x<\frac{1}{2}\text{ or }x>1$$

Hence, the least positive integral value is 2.

● ***Ex. 65*** *If the points* $a(\cos\alpha+\hat{\mathbf{i}}\sin\gamma)$, $b(\cos\beta+\hat{\mathbf{i}}\sin\beta)$ *and* $c(\cos\gamma+\hat{\mathbf{i}}\sin\gamma)$ *are collinear, then the value of* $|z|$ *is ...* *(where* $z=bc\sin(\beta-\gamma)+ca\sin(\gamma-\alpha)+ab\sin(\alpha+\beta)+3\hat{\mathbf{i}}$*)*

Sol. (3) $\begin{vmatrix} a\cos\alpha & a\sin\alpha & 1\\ b\cos\beta & b\sin\beta & 1\\ c\cos\gamma & c\sin\gamma & 10\end{vmatrix}=0$

$$\Rightarrow bc\sin(\gamma-\beta)+a\sin(\alpha-\gamma)+ab\sin(\beta-\alpha)=0$$

$$\Rightarrow |z|=3$$

Subjective Type Questions

● ***Ex. 66*** *A particle in equilibrium is subjected to four forces*

$$\mathbf{F}_1=-10\hat{\mathbf{k}},\qquad \mathbf{F}_2=u\left(\frac{4}{13}\hat{\mathbf{i}}-\frac{12}{13}\hat{\mathbf{j}}+\frac{3}{13}\hat{\mathbf{k}}\right)$$

$$\mathbf{F}_3=v\left(-\frac{4}{13}\hat{\mathbf{i}}-\frac{12}{13}\hat{\mathbf{j}}+\frac{3}{13}\hat{\mathbf{k}}\right)$$

and $\mathbf{F}_4=w(\cos\theta\,\hat{\mathbf{i}}+\sin\theta\,\hat{\mathbf{j}})$

Find the values of u, v and w in terms of θ.

Sol. Since, the particle is in equilibrium.

$$\mathbf{F}_1+\mathbf{F}_2+\mathbf{F}_3+\mathbf{F}_4=0$$

$$-10\hat{\mathbf{k}}+u\left(\frac{4}{13}\hat{\mathbf{i}}-\frac{12}{13}\hat{\mathbf{j}}+\frac{3}{13}\hat{\mathbf{k}}\right)+v\left(-\frac{4}{13}\hat{\mathbf{i}}-\frac{12}{13}\hat{\mathbf{j}}+\frac{3}{13}\hat{\mathbf{k}}\right)+w(\cos\theta\hat{\mathbf{i}}+\sin\theta\hat{\mathbf{j}})=0$$

$$\Rightarrow\left(\frac{4u}{13}-\frac{4v}{13}+w\cos\theta\right)\hat{\mathbf{i}}+\left(\frac{-12}{13}u-\frac{12}{13}v+w\sin\theta\right)\hat{\mathbf{j}}+\left(\frac{3}{13}u+\frac{3}{13}v-10\right)\hat{\mathbf{k}}=0$$

$$\Rightarrow \frac{4u}{13}-\frac{4v}{13}+w\cos\theta=0 \qquad ...(i)$$

$$-\frac{12}{13}u-\frac{12}{13}v+w\sin\theta=0 \qquad ...(ii)$$

$$\frac{3}{13}u+\frac{3}{13}v-10=0 \qquad ...(iii)$$

From Eq. (iii), we get $u+v=\frac{130}{3}$

From Eq. (ii), we get

$$-\frac{12}{13}(u+v)+w\sin\theta=0$$

$$\Rightarrow -\frac{12}{13}\left(\frac{130}{3}\right)+w\sin\theta=0$$

$$\Rightarrow w=\frac{40}{\sin\theta}=40\text{ cosec }\theta$$

On substituting the value of w in Eqs. (i) and (ii), we get

$$u-v=-130\cot\theta$$

and $$u+v=\frac{130}{3}$$

On solving, we get

$$u + \frac{65}{3} - 65\cot\theta$$

$$v + \frac{65}{3} + 65\cot\theta \quad \text{and} \quad w = 40 \operatorname{cosec}\theta$$

● ***Ex. 67*** *Find all values of 'λ' such that $x, y, z \neq (0,0,0)$ and $(\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}})\, x + (3\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}})\, y + (-4\hat{\mathbf{i}} + 5\hat{\mathbf{j}})z = \lambda(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$, where $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are unit vectors along the coordinate axes.*

Sol. Here,

$$(\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}})x + (3\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}})y + (-4\hat{\mathbf{i}} + 5\hat{\mathbf{j}})z = \lambda(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$$

On comparing the coefficients of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$, we get

$$x + 3y - 4z = \lambda x$$

$$\Rightarrow \quad (1-\lambda)x + 3y - 4z = 0 \qquad \ldots(i)$$

$$x - 3y + 5z = \lambda y$$

$$\Rightarrow \quad x - (3+\lambda)y + 5z = 0 \qquad \ldots(ii)$$

$$3x + y = \lambda z$$

$$\Rightarrow \quad 3x + y - \lambda z = 0 \qquad \ldots(iii)$$

The Eqs. (i), (ii) and (iii) will have a non-trivial solution, if

$$\begin{vmatrix} 1-\lambda & 3 & -4 \\ 1 & -(3+\lambda) & 5 \\ 3 & 1 & -\lambda \end{vmatrix} = 0$$

$$[\because \ (x, y, z) \neq (0, 0, 0) \therefore \Delta = 0]$$

$$\Rightarrow (1-\lambda)\{\lambda(3+\lambda) - 5\} - 3\{-\lambda - 15\} - 4\{1 + 3(\lambda + 3)\} = 0$$

$$\Rightarrow \quad (1-\lambda)\{\lambda^2 + 3\lambda - 5\} - 3\{-\lambda - 15\} - 4\{3\lambda + 10\} = 0$$

$$\Rightarrow \quad \lambda^3 + 2\lambda^2 + \lambda = 0$$

$$\Rightarrow \quad \lambda(\lambda^2 + 2\lambda + 1) = 0$$

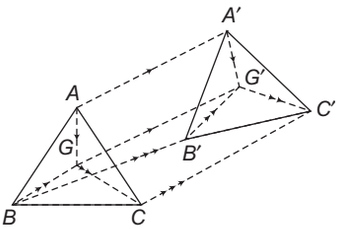
$$\Rightarrow \quad \lambda(\lambda + 1)^2 = 0$$

$$\therefore \quad \lambda = 0 \quad \text{or} \quad \lambda = -1$$

● ***Ex. 68*** *If G is the centroid of the ΔABC and if G' is the centroid of another $\Delta A'B'C'$, then prove that $AA' + BB' + CC' = 3GG'$.*

Sol. Here,

G is centroid of ΔABC and G' is centroid of $\Delta A'B'C'$, shown as in figure.



Clearly, $\mathbf{AA'} = \mathbf{AG} + \mathbf{GG'} + \mathbf{G'A'}$ (polygon law)

$\mathbf{BB'} = \mathbf{BG} + \mathbf{GG'} + \mathbf{G'B'}$

$\mathbf{CC'} = \mathbf{CG} + \mathbf{CG'} + \mathbf{G'C'}$

On adding these

$$\mathbf{AA'} + \mathbf{BB'} + \mathbf{CC'} = 3\mathbf{GG'} + (\mathbf{AG} + \mathbf{BG} + \mathbf{CG}) + (\mathbf{G'A'} + \mathbf{G'B'} + \mathbf{G'C'})$$

$$= 3\mathbf{GG'} + (\mathbf{AG} + 2\mathbf{DG}) + (\mathbf{G'A'} + 2\mathbf{G'D'})$$

(using $\mathbf{AD}$ and $\mathbf{A'D'}$ as the medians of ΔABC and $\Delta A'B'C'$, respectively)

$$= 3\mathbf{GG'} + (\mathbf{AG} + \mathbf{GA}) + \mathbf{G'A'} + \mathbf{A'G'}$$

$$= 3\mathbf{GG'} + O + O$$

$$\therefore \ \mathbf{AA'} + \mathbf{BB'} + \mathbf{CC'} = 3\mathbf{GG'}$$

Aliter

We know by triangle law

$$\mathbf{AA'} = \mathbf{OA'} - \mathbf{OA}$$

$$\mathbf{BB'} = \mathbf{OB'} - \mathbf{OB}$$

$$\mathbf{CC'} = \mathbf{OC'} - \mathbf{OC}$$

$$\Rightarrow \quad \mathbf{AA'} + \mathbf{BB'} + \mathbf{CC'} = (\mathbf{OA'} + \mathbf{OB'} + \mathbf{OC'}) - (\mathbf{OA} + \mathbf{OB} + \mathbf{OC})$$

$$= 3\mathbf{OG'} - 3\mathbf{OG'} = 3\mathbf{GG'}$$

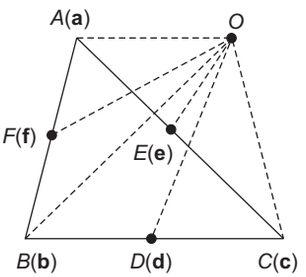
● ***Ex. 69*** *If D, E and F are the mid-points of the sides BC, CA and AB, respectively of a ΔABC and O is any point, show that*

(i) $\mathbf{AD} + \mathbf{BE} + \mathbf{CF} = 0$

(ii) $\mathbf{OE} + \mathbf{OF} + \mathbf{DO} = \mathbf{OA}$

(iii) $\mathbf{AD} + \frac{2}{3}\mathbf{BE} + \frac{1}{3}\mathbf{CF} = \frac{1}{2}\mathbf{AC}$

Sol. Consider the point O as origin, we have,



(i) $\mathbf{AD} + \mathbf{BE} + \mathbf{CF} = = (\mathbf{d} - \mathbf{a}) + (\mathbf{e} - \mathbf{b}) + (\mathbf{f} - \mathbf{c})$

$$= (\mathbf{d} + \mathbf{e} + \mathbf{f}) - (\mathbf{a} + \mathbf{b} + \mathbf{c}) = 0 \quad \text{[using Eq. (i)]}$$

$$\Rightarrow \quad \mathbf{AD} + \mathbf{BE} + \mathbf{CF} = 0$$

(ii) Here, $\mathbf{OE} + \mathbf{OF} + \mathbf{OD} = \mathbf{e} + \mathbf{f} - \mathbf{d}$

$$= \frac{\mathbf{c} + \mathbf{a}}{2} + \frac{\mathbf{a} + \mathbf{b}}{2} - \frac{\mathbf{b} + \mathbf{c}}{2} = \mathbf{a} = \mathbf{OA}$$

$$\therefore \quad \mathbf{OE} + \mathbf{OF} + \mathbf{OD} = \mathbf{OA}$$

(iii) Here, $\mathbf{AD} + \frac{2}{3}\mathbf{BE} + \frac{1}{3}\mathbf{CF} = (\mathbf{d} - \mathbf{a}) + \frac{2}{3}(\mathbf{e} - \mathbf{b}) + \frac{1}{3}(\mathbf{f} - \mathbf{c})$

$$= \frac{\mathbf{b} + \mathbf{c}}{2} - \mathbf{a} + \frac{2}{3}\left(\frac{\mathbf{c} + \mathbf{a}}{2} - \mathbf{b}\right) + \frac{1}{3}\left(\frac{\mathbf{a} + \mathbf{b}}{2} - \mathbf{c}\right)$$

$$= \mathbf{a}\left(-1 + \frac{1}{3} + \frac{1}{6}\right) + \mathbf{b}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{6}\right) + \mathbf{c}\left(\frac{1}{2} + \frac{1}{3} - \frac{1}{3}\right)$$

$$= -\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{c} = \frac{1}{2}(\mathbf{c} - \mathbf{a})$$

$$= \frac{1}{2}\mathbf{AC}$$

$$\mathbf{AD} + \frac{2}{3}\mathbf{BE} + \frac{1}{3}\mathbf{CF} = \frac{1}{2}\mathbf{AC}$$

● ***Ex. 70*** *If* $\mathbf{A}$ *and* $\mathbf{B}$ *be two vectors and k be any scalar quantity greater than zero, then prove that*

$$|\mathbf{A} + \mathbf{B}|^2 \le (1+k)|\mathbf{A}|^2 \left(1 + \frac{1}{k}\right)|\mathbf{B}|^2$$

Sol. We know, $(1 + k)|\mathbf{A}|^2 + \left(1 + \frac{1}{k}\right)|\mathbf{B}|^2$

$$= |\mathbf{A}|^2 + k|\mathbf{A}|^2 + |\mathbf{B}|^2 + \frac{1}{k}|\mathbf{B}|^2 \quad \ldots(i)$$

Also, $$k|\mathbf{A}|^2 + \frac{1}{k}|\mathbf{B}|^2 \ge 2\left(k|\mathbf{A}|^2 \cdot \frac{1}{k}|\mathbf{B}|^2\right)^{\frac{1}{2}} = 2|\mathbf{A}||\mathbf{B}| \quad \ldots(ii)$$

(since, Arithmetic mean ≥ Geometric mean)

So, $(1 + k)|\mathbf{A}|^2 + \left(1 + \frac{1}{k}\right)|\mathbf{B}|^2 \ge |\mathbf{A}|^2 + |\mathbf{B}|^2 + 2|\mathbf{A}| \cdot |\mathbf{B}|$

$= (|\mathbf{A}| + |\mathbf{B}|)^2$ [using Eqs. (i) and (ii)]

And also, $|\mathbf{A}| + |\mathbf{B}| \ge |\mathbf{A} + \mathbf{B}|$

Hence, $(1 + k)|\mathbf{A}|^2 + \left(1 + \frac{1}{k}\right)|\mathbf{B}|^2 \ge |\mathbf{A} + \mathbf{B}|^2$

● ***Ex. 71*** *If O is the circumcentre and O' the orthocenter of* ΔABC *prove that*

(i) $\mathbf{SA} + \mathbf{SB} + \mathbf{SC} = 3\mathbf{SG}$, where S is any point in the plane of ΔABC.

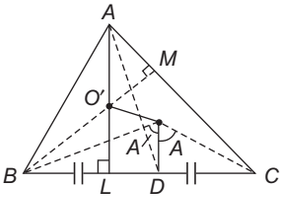
(ii) $\mathbf{OA} + \mathbf{OB} + \mathbf{OC} = \mathbf{OO'}$

(iii) $\mathbf{O'A} + \mathbf{O'B} + \mathbf{O'C} = 2\mathbf{O'O}$

(iv) $\mathbf{AO'} + \mathbf{O'B} + \mathbf{O'C} = \mathbf{AP}$

where, $\mathbf{AP}$ is diameter of the circumcircle.

Sol. Let G be the centroid of ΔABC, first we shall show that circumcentre O, orthocenter O' and centroid G are collinear and $O'G = 2OG$.



Let AL and BM be perpendiculars on the sides BC and CA, respectively. Let AD be the median and OD be the perpendicular from O on side BC. If R is the circumradius of circumcircle of ΔABC, then $OB = OC = R$.

In ΔOBD, we have $OD = R\cos A$ ...(i)

In ΔABM, $AM = AB\cos A = c\cos A$ ...(ii)

Form $\Delta AO'M$, $AO' = AM\sec(90° - C)$

$= c\cos A \operatorname{cosec} C$

$= \frac{c}{\sin C} \cdot \cos A = 2R\cos A$

$\left(\because \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R\right)$

$AO' = 2(OD)$ ...(iii)

Now, $\Delta AGO'$ and ΔOGD are similar.

$\therefore$ $\frac{OG}{O \cdot G} = \frac{GD}{GA} = \frac{OD}{AO'} = \frac{1}{2}$ [using Eq. (iii)]

$\Rightarrow$ $2OG = O'G$

(i) We have, $\mathbf{SA} + \mathbf{SB} + \mathbf{SC} = \mathbf{SA} + (\mathbf{SB} + \mathbf{SC})$

$= \mathbf{SA} + 2\mathbf{SD}$ ($\because$ D is the mid-point of $\mathbf{BC}$)

$= (1 + 2)\mathbf{SG} = 3\mathbf{SG}$

(ii) On replacing S by O in Eq. (i), we get

$\mathbf{OA} + \mathbf{OB} + \mathbf{OC} = 3\mathbf{OG}$

$= 2\mathbf{O}G + \mathbf{OG} = \mathbf{GO'} + \mathbf{OG}$

$= \mathbf{OG} + \mathbf{GO'} = \mathbf{OO'}$

(iii) $\mathbf{O'A} + \mathbf{O'B} + \mathbf{O'C} = 3\mathbf{O'G}$ [from Eq. (i)]

$= 2\mathbf{O'G} + \mathbf{O'G}$

$= 2\mathbf{O'G} + 2\mathbf{GO}$ ($\because$ $2\mathbf{OG} = \mathbf{O'G}$)

$= 2\mathbf{O'O}$

(iv) $\mathbf{AO} + \mathbf{O'B} + \mathbf{O'C} = 2\mathbf{AO'} + (\mathbf{O'A} + \mathbf{O'B} + \mathbf{O'C})$

$= 2\mathbf{AO'} + 2\mathbf{O'O}$ [From Eq. (iii)]

$= 2(\mathbf{AO'} + \mathbf{O'O}) = 2\mathbf{AO} = \mathbf{AP}$

($\because$ $\mathbf{AO}$ is the circumradius of ΔABC)

● ***Ex. 72*** *If* $\mathbf{c} = 3\mathbf{a} + 4\mathbf{b}$ *and* $2\mathbf{c} = \mathbf{a} - 3\mathbf{b}$*, show that,*

(i) $\mathbf{c}$ *and* $\mathbf{a}$ *have the same direction and* $|\mathbf{c}| > |\mathbf{a}|$.

(ii) $\mathbf{c}$ *and* $\mathbf{b}$ *have opposite direction and* $|\mathbf{c}| > |\mathbf{b}|$.

Sol. We have,

$\mathbf{c} = 3\mathbf{a} + 4\mathbf{b}$ and $2\mathbf{c} = \mathbf{a} - 3\mathbf{b}$

$\Rightarrow$ $2(3\mathbf{a} + 4\mathbf{b}) = \mathbf{a} - 3\mathbf{b}$

$\Rightarrow$ $5\mathbf{a} = -11\mathbf{b}$

$\Rightarrow$ $\mathbf{a} = -\frac{11}{5}\mathbf{b}$ and $\mathbf{b} = -\frac{5}{11}\mathbf{a}$

(i) $\mathbf{c} = 3\mathbf{a} + 4\mathbf{b} = 3\mathrm{a} + 4\left(-\frac{5}{11}\mathrm{a}\right)$ $\left(\text{using } \mathbf{b} = -\frac{5}{11}\mathbf{a}\right)$

$= 3\mathbf{a} - \frac{20}{11}\mathbf{a} = \frac{13}{11}\mathbf{a}$

which shows that $\mathbf{c}$ and $\mathbf{a}$ have the same direction.

And $\mathbf{c} = \frac{13}{11}\mathbf{a}$

$\Rightarrow$ $|\mathbf{c}| = \frac{13}{11}|\mathbf{a}| \Rightarrow |\mathbf{c}| > |\mathbf{a}|$

(ii) We have, $\mathbf{c} = 3\mathbf{a} + 4\mathbf{b}$ and $\mathbf{a} = -\frac{11}{5}\mathbf{b}$

$\mathbf{c} = 3\left(-\frac{11}{5}\mathbf{b}\right) + 4\mathbf{b} = -\frac{33}{5}\mathbf{b} + 4\mathbf{b}$

$\mathbf{c} = -\frac{13}{5}\mathbf{b}$

This shows **c** and **b** have opposite directions.

Also, $|\mathbf{c}| = \left|-\frac{13}{5}\mathbf{b}\right| = \frac{13}{5}|\mathbf{b}| \Rightarrow |\mathbf{c}| > |\mathbf{b}|$

● ***Ex. 73*** *A transversal cuts the sides OL, OM and diagonal ON of a parallelogram at A, B and C respectively. Prove that* $\frac{OL}{OA} + \frac{OM}{OB} = \frac{ON}{OC}$.

Sol. We have,

$$\mathbf{ON} = \mathbf{OL} + \mathbf{LN} = \mathbf{OL} + \mathbf{OM} \quad \text{...(i)}$$

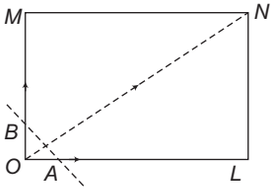
Let $\mathbf{OL} = x\mathbf{OA}, \mathbf{OM} = y\mathbf{OB}$ ...(ii)

and $\mathbf{ON} = z\mathbf{OC}$

So, $|\mathbf{OL}| = x|\mathbf{OA}|, |\mathbf{OM}| = y|\mathbf{OB}|$ and $|\mathbf{ON}| = z|\mathbf{OC}|$

$\therefore \quad x = \frac{OL}{OA}, y = \frac{OM}{OB}$ and $z = \frac{ON}{OC}$

∴From Eqs. (i) and (ii), we have



$$z\mathbf{OC} = x\mathbf{OA} + y\mathbf{OB}$$

$\Rightarrow \quad x\mathbf{OA} + y\mathbf{OB} - z\mathbf{OC} = \mathbf{0}$

∴Points *A*, *B* and *C* are collinear, the sum of the coefficients of their *PV* must be zero.

$\Rightarrow \quad x + y - z = 0$

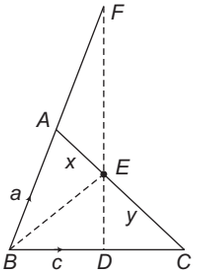
i.e. $\frac{OL}{OA} + \frac{OM}{OB} = \frac{ON}{OC}$

● ***Ex. 74*** *If D, E and F be three points on the sides BC, CA and AB, respectively of a ΔABC. such that the points D, E and F are collinear then prove that* $\frac{\mathbf{BD}}{\mathbf{CD}} \cdot \frac{\mathbf{CE}}{\mathbf{AE}} \cdot \frac{\mathbf{AF}}{\mathbf{BF}} = 1$ *(Menelau's theorem)*

Sol. Here, *D*, *E* and *F* be the points on the sides **BC**, **CA** and **AB** respectively of Δ*ABC*. Such that points D, E and F are collinear, be Shawn as the adjoining figuece.

Let *B* as the origin, $\mathbf{BA} = \mathbf{a}$ and $\mathbf{BC} = \mathbf{c}$

Then, $\mathbf{BF} = k\mathbf{a}$ and $\mathbf{BD} = l\mathbf{c}$



where, *k* and *l* are scalars.

$\therefore \quad \frac{BD}{BC} = l$ and $\frac{BF}{BA} = k$ ...(i)

i.e. $BC : BD = 1 : l$

$\Rightarrow \quad \frac{BC}{BD} - 1 = \frac{1}{l} - 1 \Rightarrow \frac{DC}{BD} = \frac{1-l}{l}$

$\Rightarrow \quad \frac{BD}{DC} = \frac{l}{1-l}$ and $\frac{BA}{BF} = \frac{1}{k}$

$\Rightarrow \quad 1 - \frac{BA}{BF} = 1 - \frac{1}{k} \Rightarrow \frac{AF}{BF} = \frac{k-1}{k}$ ...(ii)

Now, let *E* divide the line *AC* in the ratio of $x : y$

So, that $\mathbf{BE} = \frac{x\mathbf{c} + y\mathbf{a}}{x+y} = \frac{x \cdot \frac{\mathbf{BD}}{l} + y.\frac{\mathbf{BF}}{k}}{x+y}$ ...(iii)

$\Rightarrow \quad \mathbf{BE} - \frac{x}{l(x+y)}\mathbf{BD} - \frac{y}{k(x+y)}\mathbf{BF} = 0$

Since, *D*, *E* and *F* are collinear.

Sum of coefficients must be zero.

Hence, $1 - \frac{x}{l(x+y)} - \frac{y}{k(x+y)} = 0$

$\Rightarrow \quad (x+y) - \frac{x}{l} - \frac{y}{k} = 0 \Rightarrow x + y = \frac{x}{l} + \frac{y}{k}$

$\Rightarrow \quad x\left(1 - \frac{1}{l}\right) = y\left(\frac{1}{k} - 1\right) \Rightarrow x\left(\frac{l-1}{l}\right) = y\left(\frac{1-k}{k}\right)$

$\Rightarrow \quad \frac{l}{l-1} \cdot \frac{y}{x} \cdot \frac{k-1}{k} = 1$

$\Rightarrow \quad \frac{BD}{DC} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = 1$ [using Eqs. (i), (ii) and (iii)]

● ***Ex. 75*** *Let* $\mathbf{A}(t) = f_1(i)\hat{\mathbf{i}} + f_2(t)\hat{\mathbf{j}}$ *and* $\mathbf{B}(t) = g_1(t)\hat{\mathbf{i}} + g_2(t)\hat{\mathbf{j}}\ t \in [0,1]$, *where* f_1, f_2, g_1 *and* g_2 *are continuous functions. Then show that* $\mathbf{A}(t)$ *and* $\mathbf{B}(t)$ *are parallel for some t.*

Sol. If $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are non-zero vectors for all *t* and $\mathbf{A}(0) = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$, $\mathbf{A}(1) = 6\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$, $\mathbf{B}(0) = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$, and $\mathbf{B}(1) = 2\hat{\mathbf{i}} + 6\hat{\mathbf{j}}$.

In order to prove that $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are parallel vectors for some values of *t*. It is sufficient to show that $\mathbf{A}(t) = \lambda\mathbf{B}(t)$ for some λ.

$\Leftrightarrow \quad \{f_1(t)\hat{\mathbf{i}} + f_2(t)\hat{\mathbf{j}}\} = \lambda\{g_1(t)\hat{\mathbf{i}} + g_2(t)\hat{\mathbf{j}}\}$

$\Leftrightarrow \quad f_1(t) = \lambda g_1 t$ and $f_2(t) = \lambda g_2(t)$

$\Leftrightarrow \quad \frac{f_1(t)}{f_2(t)} = \frac{g_1(t)}{g_2(t)}$

$\Leftrightarrow \quad f_1(t)g_2(t) - f_2(t)g_1 t = 0$ for some $t \in [0, 1]$

Let $f(t) = f_1(t)g_2(t) - f_2(t)g_1(t), t \in [0, 1]$

Since, f_1, f_2, g_1 and g_2 are continuous functions.

∴$F(t)$ is also a continuous function.

Also, $f(0) = f_1(0)g_2(0) - g_1(0)f_2(0)$

$= 2 \times 2 - 3 \times 3 = 4 - 9 = -5 < 0$

and $f(1) = f_1(1)g_2(1) - g_1(1)f_2(1)$

$$= 6 \times 6 - 2 \times 2 = 32 > 0$$

Thus, $F(t)$ is a continuous function on $[0, 1]$ such that $F(0) \cdot F(1) < 0$.

$\therefore$ By intermediate value theorem, there exists some $t \in (0, 1)$ such that

$$f(t) = 0$$

$$\Rightarrow \quad f_1(t)g_2(t) - f_2(t)g_1t = 0$$

$$\Rightarrow \quad \mathbf{A}(t) = \lambda\mathbf{B}(t) \text{ for some } \lambda.$$

Hence, $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are parallel vectors.

● ***Ex. 76*** *Prove that if* $\cos\alpha \neq 1, \cos\beta \neq 1$ *and* $\cos\gamma \neq 1$*, then the vectors* $\mathbf{a} = \hat{\mathbf{i}}\cos\alpha + \hat{\mathbf{j}} + \hat{\mathbf{k}}, \mathbf{b} = \hat{\mathbf{i}} + \hat{\mathbf{j}}\cos\beta + \hat{\mathbf{k}}$, $\mathbf{c} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}\cos\gamma$ *can never be coplanar.*

Sol. Suppose that, a, b and c are coplanar.

$$\Rightarrow \quad \begin{vmatrix} \cos\alpha & 1 & 1 \\ 1 & \cos\beta & 1 \\ 1 & 1 & \cos\gamma \end{vmatrix} = 0$$

On applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\Rightarrow \quad \begin{vmatrix} \cos\alpha & 1 & 1 \\ 1-\cos\alpha & \cos\beta - 1 & 0 \\ 1-\cos\alpha & 0 & \cos\gamma - 1 \end{vmatrix} = 0$$

$$\Rightarrow \cos\alpha(\cos\beta - 1)(\cos\gamma - 1) - (1 - \cos\alpha)(\cos\gamma - 1) - (1 - \cos\alpha)(\cos\beta - 1) = 0$$

On dividing throughout by $(1 - \cos\alpha)(1 - \cos\beta)(1 - \cos\gamma)$, we get

$$\frac{\cos\alpha}{1-\cos\alpha} + \frac{1}{1-\cos\beta} + \frac{1}{1-\cos\gamma} = 0$$

$$\Rightarrow \quad \frac{-(1-\cos\alpha)+1}{1-\cos\alpha} + \frac{1}{1-\cos\beta} + \frac{1}{1-\cos\gamma} = 0$$

$$\Rightarrow \quad -1 + \frac{1}{(1-\cos\alpha)} + \frac{1}{(1-\cos\beta)} + \frac{1}{(1-\cos\gamma)} = 0$$

$$\frac{1}{1-\cos\alpha} + \frac{1}{1-\cos\beta} + \frac{1}{1-\cos\gamma} = 1$$

$$\Rightarrow \quad \operatorname{cosec}^2\frac{\alpha}{2} + \operatorname{cosec}^2\frac{\beta}{2} + \operatorname{cosec}^2\frac{\gamma}{2} = 2, \text{ which is not possible.}$$

As, $\quad \operatorname{cosec}^2\dfrac{\alpha}{2} \geq 1, \operatorname{cosec}^2\dfrac{\beta}{2} \geq 1$

and $\quad \operatorname{cosec}^2\dfrac{\gamma}{2} \geq 1$

$\because$ They cannot be coplanar.

● ***Ex. 77*** *If the vectors* $x\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\hat{\mathbf{i}} + y\hat{\mathbf{j}} + \hat{\mathbf{k}}$ *and* $\hat{\mathbf{i}} + \hat{\mathbf{j}} + z\hat{\mathbf{k}}$ *are coplanar where,* $x \neq 1$, $y \neq 1$ *and* $z \neq 1$*, then prove that* $\dfrac{1}{1-x} + \dfrac{1}{1-y} + \dfrac{1}{1-z} = 1$

Sol. The vectors are coplanar, if we can find two scalars λ and μ such that

$$(x\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) = \lambda(\hat{\mathbf{i}} + y\hat{\mathbf{j}} + \hat{\mathbf{k}}) + \mu(\hat{\mathbf{i}} + \hat{\mathbf{j}} + z\hat{\mathbf{k}})$$

$$\Rightarrow \quad x = \lambda + \mu, 1 = \lambda y + \mu, 1 = \lambda + \mu z$$

$$\Rightarrow \quad x = \lambda + \mu, y = \frac{1-\mu}{\lambda}, z = \frac{1-\lambda}{\mu}$$

$$\Rightarrow \quad 1 - x = 1 - \lambda - \mu, 1 - y = \frac{\lambda - 1 + \mu}{\lambda},$$

$$1 - z = \frac{\mu - 1 + \lambda}{\mu}$$

$$\therefore \quad \frac{1}{1-x} + \frac{1}{1-y} + \frac{1}{1-z} = \frac{1}{1-\lambda-\mu} + \frac{\lambda}{\lambda+\mu-1} + \frac{\mu}{\lambda+\mu-1}$$

$$= \frac{-1+\lambda+\mu}{\lambda+\mu-1} = 1$$

$$\therefore \quad \frac{1}{1-x} + \frac{1}{1-y} + \frac{1}{1-z} = 1$$

Aliter

Thus, above problem could also be solved as

$$\begin{vmatrix} x & 1 & 1 \\ 1 & y & 1 \\ 1 & 1 & z \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x-1 & 0 & 1-z \\ 0 & y-1 & 1-z \\ 1 & 1 & z \end{vmatrix} = 0$$

(using $R_1 \to R_1 - R_3$ and $R_2 \to R_2 - R_3$)

$$\Rightarrow (x-1)(y-1)(z-1)\begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ \dfrac{-1}{1-x} & \dfrac{-1}{1-y} & \dfrac{-z}{1-z} \end{vmatrix} = 0$$

$$\left(\text{using } R_1 \to \frac{1}{x-1}R_1, R_2 \to \frac{1}{y-1}R_2, R_3 \to \frac{1}{z-1}R_3\right)$$

$$\Rightarrow \quad -\frac{1}{(1-x)}(1) + \frac{1}{(1-y)}(-1) - \frac{z}{(1-z)}(1) = 0$$

(expanding along R_3)

$$\Rightarrow \quad \frac{-1}{(1-x)} - \frac{1}{(1-y)} + \frac{(1-z)-1}{(1-z)} = 0$$

$$\Rightarrow \quad \frac{1}{1-x} + \frac{1}{1-y} + \frac{1}{1-z} = 1$$

● ***Ex. 78*** *If* $\mathbf{a}, \mathbf{b}$ *and* $\mathbf{c}$ *be any three non-coplanar vectors, then prove that the points* $l_1\mathbf{a} + m_1\mathbf{b} + n_1\mathbf{c}$, $l_2\mathbf{a} + m_2\mathbf{b} + n_2\mathbf{c}$, $l_3\mathbf{a} + m_3\mathbf{b} + n_3\mathbf{c}$ *and* $l_4\mathbf{a} + m_4\mathbf{b} + n_4\mathbf{c}$ *are coplanar, if* $\begin{vmatrix} l_1 & m_1 & n_1 & 1 \\ l_2 & m_2 & n_2 & 1 \\ l_3 & m_3 & n_3 & 1 \\ l_4 & m_4 & n_4 & 1 \end{vmatrix} = 0$

Sol. We know that, four points having position vectors, $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ are coplanar, if there exists scalars x, y, z and t such that

$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} + t\mathbf{d} = 0 \quad \text{where, } x + y + z + t = 0$$

So, the given points will be coplanar, if there exists scalars x, y, z and t such that

$$x(l_1\mathbf{a} + m_1\mathbf{b} + n_1\mathbf{c}) + y(l_2\mathbf{a} + m_2\mathbf{b} + n_2\mathbf{c}) + z(l_3\mathbf{a} + m_3\mathbf{b} + n_3\mathbf{c}) + t(l_4\mathbf{a} + m_4\mathbf{b} + n_4\mathbf{c}) = 0$$

where, $x + y + z + t = 0$

$$\Rightarrow (l_1x + l_2y + l_3z + l_4t)\mathbf{a} + (m_1x + m_2y + m_3z + m_4t)\mathbf{b} + (n_1x + n_2y + n_3z + n_4t)\mathbf{c} = 0$$

where, $x + y + z + t = 0$

$$l_1x + l_2y + l_3z + l_4t = 0 \quad \text{...(i)}$$

$$m_1x + m_2y + m_3z + m_4t = 0 \quad \text{...(ii)}$$

$$n_1x + n_2y + n_3z + n_4t = 0 \quad \text{...(iii)}$$

and $x + y + z + t = 0$ ...(iv)

Eliminating x, y, z and t from above equations, we get

$$\begin{vmatrix} l_1 & l_2 & l_3 & l_4 \\ m_1 & m_2 & m_3 & m_4 \\ n_1 & n_2 & n_3 & n_4 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$$

● **Ex. 79** *If $\mathbf{r}_1, \mathbf{r}_2$ and $\mathbf{r}_3$ are the position vectors of three collinear points and scalars l and m exists such that $\mathbf{r}_3 = l\,\mathbf{r}_1 + m\mathbf{r}_2$, then show that $l + m = 1$.*

Sol. Let A, B and C be the three points whose position vectors referred to O are $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$, respectively.

$$\mathbf{AB} = \mathbf{OB} - \mathbf{OA} = \mathbf{r}_2 - \mathbf{r}_1$$

$$\mathbf{BC} = \mathbf{OC} - \mathbf{OB} = \mathbf{r}_3 - \mathbf{r}_2$$

Now, if A, B and C are collinear points, then $\mathbf{AB}$ and $\mathbf{AC}$ are in the same line and $\mathbf{BC} = \lambda\,(\mathbf{AC})$

$$\Rightarrow \quad (\mathbf{r}_3 - \mathbf{r}_2) = \lambda(\mathbf{r}_2 - \mathbf{r}_1)$$

$$\Rightarrow \quad \mathbf{r}_3 = -\lambda\mathbf{r}_1 + (\lambda + 1)\,\mathbf{r}_2$$

$$\Rightarrow \quad \mathbf{r}_3 = -\lambda\mathbf{r}_1 + m\mathbf{r}_2$$

where, $l = -\lambda$ and $m = \lambda + 1$

$$\Rightarrow \quad l + m = -\lambda + (\lambda + 1) = 1$$

● **Ex. 80** *Show that points with position vectors $\mathbf{a} - 2\mathbf{b} + 3\mathbf{c}, -2\mathbf{a} + 3\mathbf{b} - \mathbf{c}$ and $4\mathbf{a} - 7\mathbf{b} + 7\mathbf{c}$ are collinear. It is given that vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ and non-coplanar.*

Sol. The three points are collinear, if we can find λ_1, λ_2 and λ_3, such that

$\lambda_1\,(\mathbf{a} - 2\mathbf{b} + 3\mathbf{c}) + \lambda_2\,(-2\mathbf{a} + 3\mathbf{b} - \mathbf{c}) + \lambda_3$

$(4\mathbf{a} - 7\mathbf{b} + 7\mathbf{c}) = 0$ with $\lambda_1 + \lambda_2 + \lambda_3 = 0$

On equating the coefficients $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ separately to zero, we get $\lambda_1 - 2\lambda_2 + 4\lambda_3 = 0, -2\lambda_1 + 3\lambda_2 - 7\lambda_3 = 0$ and $3\lambda_1 - \lambda_2 + 7\lambda_3 = 0$

On solving we get $\lambda_1 = -2, \lambda_2 = 1, \lambda_3 = 1$

So that, $\lambda_1 + \lambda_2 + \lambda_3 = 0$

Hence, the given vectors are collinear.

Vector Algebra Exercise 1 :
Single Option Correct Type Questions

1. If $\mathbf{a} = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{b} = 2\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$ and $\mathbf{c} = -\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, then $\mathbf{a} + \mathbf{b} + \mathbf{c}$ is
(a) $3\hat{\mathbf{i}} - 4\hat{\mathbf{j}}$ (b) $3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$
(c) $4\hat{\mathbf{i}} - 4\hat{\mathbf{j}}$ (d) $4\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$

2. What should be added in vector $\mathbf{a} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ to get its resultant a unit vector $\hat{\mathbf{i}}$?
(a) $-2\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ (b) $-2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$
(c) $2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ (d) None of these

3. If $\mathbf{a} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 8\hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$, then the magnitude of $\mathbf{a} + \mathbf{b}$ is equal to
(a) 13 (b) $\frac{13}{3}$
(b) $\frac{3}{13}$ (d) $\frac{4}{13}$

4. If $\mathbf{a} = 2\hat{\mathbf{i}} + 5\hat{\mathbf{j}}$ and $\mathbf{b} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}}$, then the unit vector along $\mathbf{a} + \mathbf{b}$ will be
(a) $\frac{\hat{\mathbf{i}} - \hat{\mathbf{j}}}{\sqrt{2}}$ (b) $\hat{\mathbf{i}} + \hat{\mathbf{j}}$
(c) $\sqrt{2}(\hat{\mathbf{i}} + \hat{\mathbf{j}})$ (d) $\frac{\hat{\mathbf{i}} + \hat{\mathbf{j}}}{\sqrt{2}}$

5. The unit vector parallel to the resultant vector of $2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$ and $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ is
(a) $\frac{1}{7}(3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$
(b) $\frac{\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{3}}$
(c) $\frac{\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}}{\sqrt{6}}$
(d) $\frac{1}{\sqrt{69}}(-\hat{\mathbf{i}} - \hat{\mathbf{j}} + 8\hat{\mathbf{k}})$

6. If $\mathbf{a} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, $\mathbf{b} = -\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{c} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}}$, then the unit vector along its resultant is
(a) $3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ (b) $\frac{3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 4\hat{\mathbf{k}}}{50}$
(c) $\frac{3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 4\hat{\mathbf{k}}}{5\sqrt{2}}$ (d) None of these

7. If $\mathbf{a} = (2, 5)$ and $\mathbf{b} = (1, 4)$, then the vector parallel to $(\mathbf{a} + \mathbf{b})$ is
(a) (3, 5) (b) (1, 1)
(c) (1, 3) (d) (8, 5)

8. In the ΔABC, $\mathbf{AB} = \mathbf{a}$, $\mathbf{AC} = \mathbf{c}$ and $\mathbf{BC} = \mathbf{b}$, then
(a) $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$ (b) $\mathbf{a} + \mathbf{b} - \mathbf{c} = 0$
(c) $\mathbf{a} - \mathbf{b} + \mathbf{c} = 0$ (d) $-\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$

9. If O is the origin and the position vector of A is $4\hat{\mathbf{i}} + 5\hat{\mathbf{j}}$, then a unit vector parallel to $\mathbf{OA}$ is
(a) $\frac{4}{\sqrt{41}}\hat{\mathbf{i}}$ (b) $\frac{5}{\sqrt{41}}\hat{\mathbf{i}}$
(c) $\frac{1}{\sqrt{41}}(4\hat{\mathbf{i}} + 5\hat{\mathbf{j}})$ (d) $\frac{1}{\sqrt{41}}(4\hat{\mathbf{i}} - 5\hat{\mathbf{j}})$

10. The position vectors of the points A, B and C are $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$, $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, respectively. If A is chosen as the origin, then the position vectors of B and C are
(a) $\hat{\mathbf{i}} + 2\hat{\mathbf{k}}$, $\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ (b) $\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, $\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$
(c) $-\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, $\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ (d) $-\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, $\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$

11. The position vectors of P and Q are $5\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + a\hat{\mathbf{k}}$ and $-\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$, respectively. If the distance between them is 7, then the value of a will be
(a) –5, 1 (b) 5, 1
(c) 0, 5 (d) 1, 0

12. If position vector of points A, B and C are respectively $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ and $AB = CX$, then position vector of point X is
(a) $-\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ (b) $\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$
(c) $\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$ (d) $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$

13. The position vectors of A and B are $2\hat{\mathbf{i}} - 9\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$ and $6\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 8\hat{\mathbf{k}}$ respectively, then the magnitude of $\mathbf{AB}$ is
(a) 11 (b) 12
(c) 13 (d) 14

14. If the position vectors of P and Q are $(\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 7\hat{\mathbf{k}})$ and $(5\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}})$, then $|\mathbf{PQ}|$ is
(a) $\sqrt{158}$ (b) $\sqrt{160}$
(c) $\sqrt{161}$ (d) $\sqrt{162}$

15. If the position vectors of P and Q are $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 7\hat{\mathbf{k}}$ and $5\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ respectively, the cosine of the angle between $\mathbf{PQ}$ and Z-axis is
(a) $\frac{4}{\sqrt{162}}$ (b) $\frac{11}{\sqrt{162}}$
(c) $\frac{5}{\sqrt{162}}$ (d) $\frac{-5}{\sqrt{162}}$

16. If the position vectors of A and B are $\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 7\hat{\mathbf{k}}$ and $5\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$, then the direction cosine of $\mathbf{AB}$ along Y-axis is
(a) $\frac{4}{\sqrt{162}}$ (b) $-\frac{5}{\sqrt{162}}$
(c) -5 (d) 11

17. The direction cosines of vector $\mathbf{a} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$ in the direction of positive axis of X, is

(a) $\pm\frac{3}{\sqrt{50}}$ (b) $\frac{4}{\sqrt{50}}$

(c) $\frac{3}{\sqrt{50}}$ (d) $-\frac{4}{\sqrt{50}}$

18. The direction cosines of the vector $3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$ are

(a) $\frac{3}{5}, -\frac{4}{5}, \frac{1}{5}$ (b) $\frac{3}{5\sqrt{2}}, \frac{-4}{5\sqrt{2}}, \frac{1}{\sqrt{2}}$

(c) $\frac{3}{\sqrt{2}}, \frac{-4}{\sqrt{2}}, \frac{1}{\sqrt{2}}$ (d) $\frac{3}{5\sqrt{2}}, \frac{4}{5\sqrt{2}}, \frac{1}{\sqrt{2}}$

19. The point having position vectors $2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$, $3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $4\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ are the vertices of

(a) right angled triangle
(b) isosceles triangle
(c) equilateral triangle
(d) collinear

20. If the position vectors of the vertices A, B and C of a ΔABC are $7\hat{\mathbf{j}} + 10\mathbf{k}$, $-\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$ and $-4\hat{\mathbf{i}} + 9\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$, respectively. The triangle is

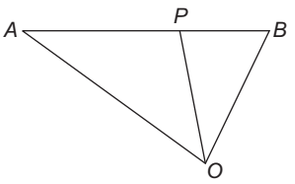
(a) equilateral
(b) isosceles
(c) scalene
(d) right angled and isosceles also

21. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the position vectors of the vertices A, B and C of the ΔABC, then the centroid of ΔABC is

(a) $\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}$ (b) $\frac{1}{2}\left(\mathbf{a} + \frac{\mathbf{b}+\mathbf{c}}{2}\right)$

(c) $\mathbf{a} + \frac{\mathbf{b}+\mathbf{c}}{2}$ (d) $\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{2}$

22. If in the given figure, $\mathbf{OA} = \mathbf{a}$, $\mathbf{OB} = \mathbf{b}$ and $AP : PB = m : n$, then $\mathbf{OP}$ is equal to



(a) $\frac{m\mathbf{a}+n\mathbf{b}}{m+n}$ (b) $\frac{n\mathbf{a}+m\mathbf{b}}{m+n}$

(c) $m\mathbf{a} - n\mathbf{b}$ (d) $\frac{m\mathbf{a}-n\mathbf{b}}{m-n}$

23. If $\mathbf{a}$ and $\mathbf{b}$ are position vector of two points A, B and C divides AB in ratio $2:1$, then position vector of C is

(a) $\frac{\mathbf{a}+2\mathbf{b}}{3}$ (b) $\frac{2\mathbf{a}+\mathbf{b}}{3}$

(c) $\frac{\mathbf{a}+2}{3}$ (d) $\frac{\mathbf{a}+\mathbf{b}}{2}$

24. The position vector of the points which divides internally in the ratio $2:3$ the join of the points $2\mathbf{a} - 3\mathbf{b}$ and $3\mathbf{a} - 2\mathbf{b}$, is

(a) $\frac{12}{5}\mathbf{a} + \frac{13}{5}\mathbf{b}$ (b) $\frac{12}{5}\mathbf{a} - \frac{13}{5}\mathbf{b}$

(c) $\frac{3}{5}\mathbf{a} - \frac{2}{5}\mathbf{b}$ (d) None of these

25. If O is origin and C is the mid-point of $A(2, -1)$ and $B(-4, 3)$. Then, value of $\mathbf{OC}$ is

(a) $\hat{\mathbf{i}} + \hat{\mathbf{j}}$ (b) $\hat{\mathbf{i}} - \hat{\mathbf{j}}$
(c) $-\hat{\mathbf{i}} + \hat{\mathbf{j}}$ (d) $-\hat{\mathbf{i}} - \hat{\mathbf{j}}$

26. If the position vectors of the points A and B are $\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $3\hat{\mathbf{i}} - \hat{\mathbf{j}} - 3\hat{\mathbf{k}}$, then what will be the position vector of the mid-point of AB

(a) $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$ (b) $2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$

(c) $2\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$ (d) $\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$

27. The position vectors of A and B are $\hat{\mathbf{j}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $3\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$. The position vector of the middle point of the line AB is

(a) $\frac{1}{2}\hat{\mathbf{i}} - \frac{1}{2}\hat{\mathbf{j}} + \hat{\mathbf{k}}$ (b) $2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \frac{5}{2}\hat{\mathbf{k}}$

(c) $\frac{3}{2}\hat{\mathbf{i}} - \frac{1}{2}\hat{\mathbf{j}} + \frac{3}{2}\hat{\mathbf{k}}$ (d) None of these

28. If the vector $\mathbf{b}$ is collinear with the vector $\mathbf{a} = (2\sqrt{2}, -1, 4)$ and $|\mathbf{b}| = 10$, then

(a) $\mathbf{a} \pm \mathbf{b} = 0$ (b) $\mathbf{a} \pm 2\mathbf{b} = 0$
(c) $2\mathbf{a} \pm \mathbf{b} = 0$ (d) None of these

29. If $\mathbf{a} = (1, -1)$ and $\mathbf{b} = (-2, m)$ are two collinear vectors, then m is equal to

(a) 4 (b) 3
(c) 2 (d) 0

30. The points with position vectors $10\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$, $12\hat{\mathbf{i}} - 5\hat{\mathbf{j}}$ and $a\hat{\mathbf{i}} + 11\hat{\mathbf{j}}$ are collinear, if a is equal to

(a) -8 (b) 4
(c) 8 (d) 12

31. The vectors $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, $\lambda\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$, $-3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$ are collinear, if λ is equal to

(a) 3 (b) 4
(c) 5 (d) 6

32. If the points $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$ and $\mathbf{a} + k\mathbf{b}$ be collinear, then k is equal to

(a) 0 (b) 2
(c) -2 (d) Any real number

33. If the position vectors of A, B, C and D are $2\hat{\mathbf{i}} + \hat{\mathbf{j}}$, $\hat{\mathbf{i}} - 3\hat{\mathbf{j}}$, $3\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$ and $\hat{\mathbf{i}} + \lambda\hat{\mathbf{j}}$ respectively and $\mathbf{AB} \| \mathbf{CD}$, then λ will be

(a) -8 (b) -6
(c) 8 (d) 6

34. If the vectors $3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $6\hat{\mathbf{i}} - 4x\hat{\mathbf{j}} + y\hat{\mathbf{k}}$ are parallel, then the value of x and y will be
(a) $-1, -2$ (b) $1, -2$
(c) $-1, 2$ (d) $1, 2$

35. If $\mathbf{a}$ and $\mathbf{b}$ are two non-collinear vectors and $x\mathbf{a} + y\mathbf{b} = 0$
(a) $x = 0$, but y is not necessarily zero
(b) $y = 0$, but x is not necessarily zero
(c) $x = 0, y = 0$
(d) None of the above

36. Four non-zero vectors will always be
(a) linearly dependent
(b) linearly independent
(c) either (a) or (b)
(d) None of the above

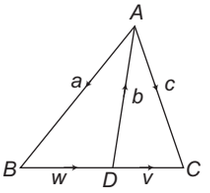
37. The vectors **a, b** and $\mathbf{a} + \mathbf{b}$ are
(a) collinear (b) coplanar
(c) non-coplanar (d) None of these

38. If $(x, y, z) \neq (0, 0, 0)$ and $(\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}})x + (3\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}})y + (-4\hat{\mathbf{i}} + 5\hat{\mathbf{j}})z = \lambda(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$, then the value of λ will be
(a) $-2, 0$ (b) $0, -2$
(c) $-1, 0$ (d) $0, -1$

39. The number of integral values of p for which $(p+1)\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + p\hat{\mathbf{i}}, p\hat{\mathbf{i}} + (p+1)\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$ and $-3\hat{\mathbf{i}} + p\hat{\mathbf{j}} + (p+1)\hat{\mathbf{k}}$ are linearly dependent vectors is
(a) 0 (b) 1
(c) 2 (d) 3

40. The vectors $\mathbf{AB} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{k}}$ and $\mathbf{AC} = 5\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ are the sides of a ΔABC. The length of the median through A is
(a) $\sqrt{18}$ (b) $\sqrt{72}$
(c) $\sqrt{33}$ (d) $\sqrt{288}$

41. In the figure, a vectors x satisfies the equation $\boldsymbol{x} - \boldsymbol{w} = \boldsymbol{v}$. Then, x is equal to



(a) $2\mathbf{a} + \mathbf{b} + \mathbf{c}$ (b) $\mathbf{a} + 2\mathbf{b} + \mathbf{c}$
(c) $\mathbf{a} + \mathbf{b} + 2\mathbf{c}$ (d) $\mathbf{a} + \mathbf{b} + \mathbf{c}$

42. Vectors $\mathbf{a} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, $\mathbf{b} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{c} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ are so placed that the end point of one vector is the starting point of the next vector. Then the vectors are
(a) not coplanar
(b) coplanar but cannot form a triangle
(c) coplanar and form a triangle
(d) coplanar and can form a right angled triangle

43. If $OP = 8$ and **OP** makes angles $45°$ and $60°$ with OX-axis and OY-axis respectively, then **OP** is equal to
(a) $8(\sqrt{2}\hat{\mathbf{i}} + \hat{\mathbf{j}} \pm \hat{\mathbf{k}})$ (b) $4(\sqrt{2}\hat{\mathbf{i}} + \hat{\mathbf{j}} \pm \hat{\mathbf{k}})$
(c) $\frac{1}{4}(\sqrt{2}\hat{\mathbf{i}} + \hat{\mathbf{j}} \pm \hat{\mathbf{k}})$ (d) $\frac{1}{8}(\sqrt{2}\hat{\mathbf{i}} + \hat{\mathbf{j}} \pm \hat{\mathbf{k}})$

44. Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be three units vectors such that $3\mathbf{a} + 4\mathbf{b} + 5\mathbf{c} = 0$. Then which of the following statements is true?
(a) $\mathbf{a}$ is parallel to $\mathbf{b}$
(b) $\mathbf{a}$ is perpendicular to $\mathbf{b}$
(c) $\mathbf{a}$ is neither parallel nor perpendicular to $\mathbf{b}$
(d) None of the above

45. A, B, C, D and E are five coplanar points, then $\mathbf{DA} + \mathbf{DB} + \mathbf{DC} + \mathbf{AE} + \mathbf{BE} + \mathbf{CE}$ is equal to
(a) **DE** (b) 3**DE**
(c) 2**DE** (d) 4**ED**

46. If the vectors $\mathbf{a}$ and $\mathbf{b}$ are linearly independent satisfying $(\sqrt{3}\tan\theta + 1)\mathbf{a} + (\sqrt{3}\sec\theta - 2)\mathbf{b} = 0$, then the most general value of θ are
(a) $n\pi - \frac{\pi}{6}, n \in Z$ (b) $2n\pi \pm \frac{11\pi}{6}, n \in Z$
(c) $n\pi \pm \frac{\pi}{6}, n \in Z$ (d) $2n\pi + \frac{11\pi}{6}, n \in Z$

47. The unit vector bisecting **OY** and **OZ** is
(a) $\frac{\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{3}}$ (b) $\frac{\hat{\mathbf{j}} - \hat{\mathbf{k}}}{\sqrt{2}}$
(c) $\frac{\hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{2}}$ (d) $\frac{-\hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{2}}$

48. A line passes through the points whose position vectors are $\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ and $\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$. The position vector of a point on it at unit distance from the first point is
(a) $\frac{1}{5}(5\hat{\mathbf{i}} + \hat{\mathbf{j}} - 7\hat{\mathbf{k}})$ (b) $\frac{1}{5}(4\hat{\mathbf{i}} + 9\hat{\mathbf{j}} - 15\hat{\mathbf{k}})$
(c) $(\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$ (d) $\frac{1}{5}(\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$

49. If D, E and F are the middle points of the sides BC, CA and AB of the ΔABC, then $\mathbf{AD} + \mathbf{BE} + \mathbf{CF}$ is
(a) a zero vector (b) a unit vector
(c) 0 (d) None of these

50. If P and Q are the middle points of the sides BC and CD of the parallelogram $ABCD$, then $\mathbf{AP} + \mathbf{AQ}$ is equal to
(a) **AC** (b) $\frac{1}{2}$**AC**
(b) $\frac{2}{3}$**AC** (d) $\frac{3}{2}$**AC**

51. The figure formed by the four points $\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$, $2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$, $3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ and $\hat{\mathbf{k}} - \hat{\mathbf{j}}$ is
(a) rectangle (b) parallelogram
(c) trapezium (d) None of these

52. A and B are two points. The position vector of $\mathbf{A}$ is $6\mathbf{b} - 2\mathbf{a}$. A point P divides the line AB in the ratio 1 : 2. If $\mathbf{a} - \mathbf{b}$ is the position vector of P, then the position vector of B is given by
(a) $7\mathbf{a} - 15\mathbf{b}$
(b) $7\mathbf{a} + 15\mathbf{b}$
(c) $15\mathbf{a} - 7\mathbf{b}$
(d) $15\mathbf{a} + 7\mathbf{b}$

53. If three points A, B and C are collinear, whose position vectors are $\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 8\hat{\mathbf{k}}$, $5\hat{\mathbf{i}} - 2\hat{\mathbf{k}}$ and $11\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$ respectively, then the ratio in which B divides AC is
(a) 1 : 2 (b) 2 : 3
(c) 2 : 1 (d) 1 : 1

54. If in a triangle, $\mathbf{AB} = \mathbf{a}$, $\mathbf{AC} = \mathbf{b}$ and D, E are the mid-points of AB and AC respectively, then $\mathbf{DE}$ is equal to
(a) $\frac{\mathbf{a}}{4} - \frac{\mathbf{b}}{4}$ (b) $\frac{\mathbf{a}}{2} - \frac{\mathbf{b}}{2}$
(c) $\frac{\mathbf{b}}{4} - \frac{\mathbf{a}}{4}$ (d) $\frac{\mathbf{b}}{2} - \frac{\mathbf{a}}{2}$

55. If $ABCD$ is parallelogram, $\mathbf{AB} = 2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$ and $\mathbf{AD} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, then the unit vectors in the direction of BD is
(a) $\frac{1}{\sqrt{69}}(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 8\hat{\mathbf{k}})$ (b) $\frac{1}{69}(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 8\hat{\mathbf{k}})$
(c) $\frac{1}{\sqrt{69}}(-\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 8\hat{\mathbf{k}})$ (d) $\frac{1}{69}(-\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 8\hat{\mathbf{k}})$

56. If A, B and C are the vertices of a triangle whose position vectors are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ and G is the centroid of the ΔABC, then $\mathbf{GA} + \mathbf{GB} + \mathbf{GC}$ is
(a) 0 (b) $\mathbf{A} + \mathbf{B} + \mathbf{C}$
(c) $\frac{\mathbf{a} + \mathbf{b} + \mathbf{c}}{3}$ (d) $\frac{\mathbf{a} + \mathbf{b} - \mathbf{c}}{3}$

57. If $ABCDEF$ is regular hexagon, then $\mathbf{AD} + \mathbf{EB} + \mathbf{FC}$ is equal to
(a) 0 (b) $2\mathbf{AB}$
(c) $3\mathbf{AB}$ (d) $4\ \mathbf{AB}$

58. $ABCDE$ is a pentagon. Forces $\mathbf{AB}$, $\mathbf{AE}$, $\mathbf{DC}$ and $\mathbf{ED}$ act at a point. Which force should be added to this system to make the resultant $2\mathbf{AC}$?
(a) $\mathbf{AC}$ (b) $\mathbf{AD}$
(c) $\mathbf{BC}$ (d) $\mathbf{BD}$

59. If $ABCDEF$ is a regular hexagon and $\mathbf{AB} + \mathbf{AC} + \mathbf{AD} + \mathbf{AE} + \mathbf{AF} = \lambda\mathbf{AD}$, then λ is equal to
(a) 2 (b) 3
(c) 4 (d) 6

60. Let us define the length of a vector $a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$ as $|a| + |b| + |c|$. This definition coincides with the usual definition of length of a vector $a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$ if and only if
(a) $a = b = c = 0$
(b) any two of a, b and c are zero
(c) any one of a, b and c is zero
(d) $a + b + c = 0$

61. If $\mathbf{a}$ and $\mathbf{b}$ are two non-zero and non-collinear vectors, then $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are
(a) linearly dependent vectors
(b) linearly independent vectors
(c) linearly dependent and independent vectors
(d) None of the above

62. If $|\mathbf{a} + \mathbf{b}| < |\mathbf{a} - \mathbf{b}|$, then the angle between $\mathbf{a}$ and $\mathbf{b}$ can lie in the interval.
(a) $(-\pi/2,\ \pi/2)$ (b) $(0,\ \pi)$
(c) $(\pi/2,\ 3\pi/2)$ (d) $(0,\ 2\pi)$

63. The magnitudes of mutually perpendicular forces $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are 2, 10 and 11 respectively. Then the magnitude of its resultant is
(a) 12 (b) 15
(c) 9 (d) None of these

64. If $\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$ bisects the angle between $\mathbf{a}$ and $-\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, where $\mathbf{a}$ is a unit vector, then
(a) $\mathbf{a} = \frac{1}{105}(41\hat{\mathbf{i}} + 88\hat{\mathbf{j}} - 40\hat{\mathbf{k}})$
(b) $\mathbf{a} = \frac{1}{105}(41\hat{\mathbf{i}} + 88\hat{\mathbf{j}} + 40\hat{\mathbf{k}})$
(c) $\mathbf{a} = \frac{1}{105}(-41\hat{\mathbf{i}} + 88\hat{\mathbf{j}} - 40\hat{\mathbf{k}})$
(d) $\mathbf{a} = \frac{1}{105}(41\hat{\mathbf{i}} - 88\hat{\mathbf{j}} - 40\hat{\mathbf{k}})$

65. Let $\mathbf{a} = \hat{\mathbf{i}}$ be a vector which makes an angle of 120° with a unit vector $\mathbf{b}$. Then, the unit vector $(\mathbf{a} + \mathbf{b})$ is
(a) $-\frac{1}{2}\hat{\mathbf{i}} + \frac{\sqrt{3}}{2}\hat{\mathbf{j}}$ (b) $-\frac{\sqrt{3}}{2}\hat{\mathbf{i}} + \frac{1}{2}\hat{\mathbf{j}}$
(c) $\frac{1}{2}\hat{\mathbf{i}} + \frac{\sqrt{3}}{2}\hat{\mathbf{j}}$ (d) $\frac{\sqrt{3}}{2}\hat{\mathbf{i}} - \frac{1}{2}\hat{\mathbf{j}}$

66. Given three vectors $\mathbf{a} = 6\hat{\mathbf{i}} - 3\hat{\mathbf{j}}$, $\mathbf{b} = 2\hat{\mathbf{i}} - 6\hat{\mathbf{j}}$ and $\mathbf{c} = -2\hat{\mathbf{i}} + 21\hat{\mathbf{j}}$ such that $\alpha = \mathbf{a} + \mathbf{b} + \mathbf{c}$. Then, the resolution of the vector α into components with respect to $\mathbf{a}$ and $\mathbf{b}$ is given by
(a) $3\mathbf{a} - 2\mathbf{b}$ (b) $3\mathbf{b} - 2\mathbf{a}$
(c) $2\mathbf{a} - 3\mathbf{b}$ (d) $\mathbf{a} - 2\mathbf{b}$

67. 'I' is the incentre of ΔABC whose corresponding sides are a, b, c respectively. $a\,\mathbf{IA} + b\,\mathbf{IB} + c\,\mathbf{IC}$ is always equal to
(a) $\mathbf{0}$ (b) $(a + b + c)\mathbf{BC}$
(c) $(\mathbf{a} + \mathbf{b} + \mathbf{c})\mathbf{AC}$ (d) $(a + b + c)\mathbf{AB}$

68. If $\mathbf{x}$ and $\mathbf{y}$ are two non-collinear vectors and ABC is a triangle with side lengths a, b and c satisfying $(20a - 15b)\mathbf{x} + (15b - 12c)\mathbf{y} + (12c - 20a)(\mathbf{x} \times \mathbf{y}) = \mathbf{0}$, then ΔABC is
(a) an acute angled triangle (b) an obtuse angled triangle
(c) a right angled triangle (d) a scalene triangle

69. If $\mathbf{x}$ and $\mathbf{y}$ are two non-collinear vectors and a, b and c represent the sides of a ΔABC satisfying $(a-b)\mathbf{x}+(b-c)\mathbf{y}+(c-a)(\mathbf{x}\times\mathbf{y})=0$, then ΔABC is (where $\mathbf{x}\times\mathbf{y}$ is perpendicular to the plane of $\mathbf{x}$ and $\mathbf{y}$)
(a) an acute angled triangle
(b) an obtuse angled triangle
(c) a right angled triangle
(d) a scalene triangle

70. If the resultant of two forces is of magnitude P and equal to one of them and perpendicular to it, then the other force is
(a) $P\sqrt{2}$ (b) P
(c) $P\sqrt{3}$ (d) None of these

71. If $\mathbf{b}$ is a vector whose initial point divides the join of $5\hat{\mathbf{i}}$ and $5\hat{\mathbf{j}}$ in the ratio $k:1$ and whose terminal point in the origin and $|\mathbf{b}|\leq\sqrt{37}$, then k lies in the interval
(a) $[-6,-1/6]$ (b) $[-\infty,-6]\cup[-1/6,\infty]$
(c) $[0,6]$ (d) None of these

72. If $4\hat{\mathbf{j}}+7\hat{\mathbf{j}}+8\hat{\mathbf{k}}$, $2\hat{\mathbf{i}}+3\hat{\mathbf{j}}+4\hat{\mathbf{k}}$ and $2\hat{\mathbf{i}}+5\hat{\mathbf{j}}+7\hat{\mathbf{k}}$ are the position vectors of the vertices A, B and C respectively of ΔABC. The position vector of the point where the bisector of $\angle A$ meets BC is
(a) $\frac{1}{3}(6\hat{\mathbf{i}}+13\hat{\mathbf{j}}+18\hat{\mathbf{k}})$ (b) $\frac{2}{3}(6\hat{\mathbf{i}}+12\hat{\mathbf{j}}-8\hat{\mathbf{k}})$
(c) $\frac{1}{3}(-6\hat{\mathbf{i}}-8\hat{\mathbf{j}}-9\hat{\mathbf{k}})$ (d) $\frac{2}{3}(-6\hat{\mathbf{i}}-12\hat{\mathbf{j}}+8\hat{\mathbf{k}})$

73. If $\mathbf{a}$ and $\mathbf{b}$ are two unit vectors and θ is the angle between them, then the unit vector along the angular bisector of $\mathbf{a}$ and $\mathbf{b}$ will be given by
(a) $\frac{\mathbf{a}-\mathbf{b}}{2\cos(\theta/2)}$ (b) $\frac{\mathbf{a}+\mathbf{b}}{2\cos(\theta/2)}$
(c) $\frac{\mathbf{a}-\mathbf{b}}{\cos(\theta/2)}$ (d) None of these

74. A, B, C and D have position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$, respectively, such that $\mathbf{a}-\mathbf{b}=2(\mathbf{d}-\mathbf{c})$. Then,
(a) AB and CD bisect each other
(b) BD and AC bisect each other
(c) AB and CD trisect each other
(d) BD and AC trisect each other

75. On the xy plane where O is the origin, given points, $A(1,0)$, $B(0,1)$ and $C(1,1)$. Let P, Q and R be moving point on the line OA, OB, OC respectively such that $\mathbf{OP}=45t(\mathbf{OA})$, $\mathbf{OQ}=60t(\mathbf{OB})$, $\mathbf{OR}=(1-t)(\mathbf{OC})$ with $t>0$. If the three points P, Q and R are collinear, then the value of t is equal to
(a) $\frac{1}{106}$ (b) $\frac{7}{187}$
(c) $\frac{1}{100}$ (d) None of these

76. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three non-coplanar vectors such that $\mathbf{a}+\mathbf{b}+\mathbf{c}=\alpha\mathbf{d}$ and $\mathbf{b}+\mathbf{c}+\mathbf{d}=\beta\mathbf{a}$, then $\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}$ is equal to
(a) 0 (b) $\alpha\mathbf{a}$
(c) $\beta\mathbf{b}$ (d) $(\alpha+\beta)\mathbf{c}$

77. The position vectors of the points P and Q with respect to the origin O are $\mathbf{a}=\hat{\mathbf{i}}+3\hat{\mathbf{j}}-2\hat{\mathbf{k}}$ and $\mathbf{b}=3\hat{\mathbf{i}}-\hat{\mathbf{j}}-2\hat{\mathbf{k}}$, respectively. If M is a point on PQ, such that OM is the bisector of POQ, then OM is
(a) $2(\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}})$ (b) $2\hat{\mathbf{i}}+\hat{\mathbf{j}}-2\hat{\mathbf{k}}$
(c) $2(-\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}})$ (b) $2(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})$

78. $ABCD$ is a quadrilateral. E is the point of intersection of the line joining the mid-points of the opposite sides. If O is any point and $\mathbf{OA}+\mathbf{OB}+\mathbf{OC}+\mathbf{OD}=x\mathbf{OE}$, then x is equal to
(a) 3 (b) 9
(c) 7 (d) 4

79. In the ΔOAB, M is the mid-point of AB, C is a point on OM, such that $2\mathbf{OC}=\mathbf{CM}$. X is a point on the side OB such that $\mathbf{OX}=2\mathbf{XB}$. The line XC is produced to meet OA in Y. Then, $\frac{OY}{YA}$ is equal to
(a) $\frac{1}{3}$ (b) $\frac{2}{7}$
(c) $\frac{3}{2}$ (d) $\frac{2}{5}$

80. Points X and Y are taken on the sides QR and RS, respectively of a parallelogram $PQRS$, so that $\mathbf{QX}=4\mathbf{XR}$ and $\mathbf{RY}=4\mathbf{YS}$. The line XY cuts the line PR at Z. Then, $\mathbf{PZ}$ is
(a) $\frac{21}{25}\mathbf{PR}$ (b) $\frac{16}{25}\mathbf{PR}$
(c) $\frac{17}{25}\mathbf{PR}$ (d) None of these

81. Find the value of λ so that the points P, Q, R and S on the sides OA, OB, OC and AB, respectively, of a regular tetrahedron $OABC$ are coplanar. It is given that $\frac{OP}{OA}=\frac{1}{3}, \frac{OQ}{OB}=\frac{1}{2}, \frac{OR}{OC}=\frac{1}{3}$ and $\frac{OS}{AB}=\lambda$.
(a) $\lambda=\frac{1}{2}$ (b) $\lambda=-1$
(c) $\lambda=0$ (d) for no value of λ

82. $OABCDE$ is a regular hexagon of side 2 units in the XY-plane. O being the origin and OA taken along the X-axis. A point P is taken on a line parallel to Z-axis through the centre of the hexagon at a distance of 3 units from O. Then, the vector $\mathbf{AP}$ is
(a) $-\hat{\mathbf{i}}+3\hat{\mathbf{j}}+\sqrt{5}\hat{\mathbf{k}}$ (b) $\hat{\mathbf{i}}-\sqrt{3}\hat{\mathbf{j}}+5\hat{\mathbf{k}}$
(c) $-\hat{\mathbf{i}}+\sqrt{3}\hat{\mathbf{j}}+\sqrt{5}\hat{\mathbf{k}}$ (d) $\hat{\mathbf{i}}+\sqrt{3}\hat{\mathbf{j}}+\sqrt{5}\hat{\mathbf{k}}$

Vector Algebra Exercise 2 :
More than One Option Correct Type Questions

83. If the vectors $\hat{\mathbf{i}} - \hat{\mathbf{j}}, \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{a}$ form a triangle, then $\mathbf{a}$ may be
(a) $-\hat{\mathbf{i}} - \hat{\mathbf{k}}$ (b) $\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$
(c) $2\hat{\mathbf{j}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ (d) $\hat{\mathbf{i}} + \hat{\mathbf{k}}$

84. If the resultant of three forces $\mathbf{F}_1 = p\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}, \mathbf{F}_2 = 6\hat{\mathbf{i}} - \hat{\mathbf{k}}$ and $\mathbf{F}_3 = -5\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ acting on a particle has a magnitude equal to 5 units, then the value of p is
(a) -6 (b) -4
(c) 2 (d) 4

85. Let ABC be a triangle, the position vectors of whose vertices are $7\hat{\mathbf{j}} + 10\hat{\mathbf{k}}, -\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$ and $-4\hat{\mathbf{i}} + 9\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$. Then, ΔABC is
(a) isosceles (b) equilateral
(c) right angled (d) None of these

86. The sides of a parallelogram are $2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$ and $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$. The unit vector parallel to one of the diagonals is
(a) $\frac{1}{7}(3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$ (b) $\frac{1}{7}(3\hat{\mathbf{i}} - 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$
(c) $\frac{1}{\sqrt{69}}(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 8\hat{\mathbf{k}})$ (d) $\frac{1}{\sqrt{69}}(-\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 8\hat{\mathbf{k}})$

87. If $A(-4, 0, 3)$ and $B(14, 2, -5)$, then which one of the following points lie on the bisector of the angle between **OA** and **OB** (O is the origin of reference)?
(a) (2, 2, 4) (b) (2, 11, 5)
(c) $(-3, -3, -6)$ (d) (1, 1, 2)

88. If point $\hat{\mathbf{i}} + \hat{\mathbf{j}}, \hat{\mathbf{i}} - \hat{\mathbf{j}}$ and $p\hat{\mathbf{i}} + q\hat{\mathbf{j}} + r\hat{\mathbf{k}}$ are collinear, then
(a) $p = 1$ (b) $r = 0$
(c) $q \in R$ (d) $q \neq 1$

89. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are non-coplanar vectors and λ is a real number, then the vectors $\mathbf{a} + 2\mathbf{b} + 3\mathbf{c}$, $\lambda\mathbf{b} + \mu\mathbf{c}$ and $(2\lambda - 1)\mathbf{c}$ are coplanar when
(a) $\mu \in R$ (b) $\lambda = \frac{1}{2}$
(c) $\lambda = 0$ (d) no value of λ

Vector Algebra Exercise 3 :
Statement I and II Type Questions

- **Directions** (Q. Nos. 90-92) This section is based on Statement I and Statement II. Select the correct answer from the codes given below.
  (a) Both Statement I and Statement II are correct and Statement II is the correct explanation of Statement I
  (b) Both Statement I and Statement II are correct but Statement II is not the correct explanation of Statement I
  (c) Statement I is correct but Statement II is incorrect
  (d) Statement II is correct but Statement I is incorrect

90. **Statement I** In ΔABC, $\mathbf{AB} + \mathbf{BC} + \mathbf{CA} = 0$
Statement II If $\mathbf{OA} = \mathbf{a}, \mathbf{OB} = \mathbf{b}$, then $\mathbf{AB} = \mathbf{a} + \mathbf{b}$

91. **Statement I** $\mathbf{a} = \hat{\mathbf{i}} + p\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $\mathbf{b} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + q\hat{\mathbf{k}}$ are parallel vectors, if $p = \frac{3}{2}$ and $q = 4$.
Statement II $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$ are parallel $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$.

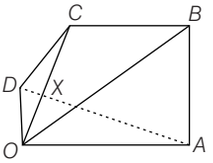
92. **Statement I** If three points P, Q and R have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively, and $2\mathbf{a} + 3\mathbf{b} - 5\mathbf{c} = 0$, then the points P, Q and R must be collinear.
Statement II If for three points A, B and C, $\mathbf{AB} = \lambda\,\mathbf{AC}$, then points A, B and C must be collinear.

Vector Algebra Exercise 4 : Passage based Type Questions

Passage I

(Q. Nos. 93 and 94)

- Let $OABCD$ be a pentagon in which the sides OA and CB are parallel and the sides OD and AB are parallel. Also, $OA : CB = 2 : 1$ and $OD : AB = 1 : 3$.



93. The ratio $\frac{OX}{XC}$ is

(a) 3/4 (b) 1/3
(c) 2/5 (d) 1/2

94. The ratio $\frac{AX}{XD}$ is

(a) 5/2 (b) 6
(c) 7/3 (d) 4

Passage II

(Q. Nos. 95 and 96)

- Consider the regular hexagon $ABCDEF$ with centre at O (origin).

95. $\mathbf{AD} + \mathbf{EB} + \mathbf{FC}$ is equal to

(a) $2\mathbf{AB}$ (b) $3\mathbf{AB}$
(c) $4\mathbf{AB}$ (d) None of these

96. Five forces $\mathbf{AB}$, $\mathbf{AC}$, $\mathbf{AD}$, $\mathbf{AE}$, $\mathbf{AF}$ act at the vertex A of a regular hexagon $ABCDEF$. Then, their resultant is

(a) $3\mathbf{AO}$ (b) $2\mathbf{AO}$
(c) $4\mathbf{AO}$ (d) $6\mathbf{AO}$

Passage III

(Q. Nos. 97 to 99)

- Three points A, B and C have position vectors $-2\mathbf{a} + 3\mathbf{b} + 5\mathbf{c}$, $\mathbf{a} + 2\mathbf{b} + 3\mathbf{c}$ and $7\mathbf{a} - \mathbf{c}$ with reference to an origin O. Answer the following questions.

97. Which of the following is true?

(a) $\mathbf{AC} = 2\mathbf{AB}$ (b) $\mathbf{AC} = -3\mathbf{AB}$
(c) $\mathbf{AC} = 3\mathbf{AB}$ (d) None of these

98. Which of the following is true?

(a) $2\mathbf{OA} - 3\mathbf{OB} + \mathbf{OC} = \mathbf{0}$
(b) $2\mathbf{OA} + 7\mathbf{OB} + 9\mathbf{OC} = \mathbf{0}$
(c) $\mathbf{OA} + \mathbf{OB} + \mathbf{OC} = \mathbf{0}$
(d) None of the above

99. B divided AC in ratio

(a) 2 :1 (b) 2 : 3
(c) 2 : −3 (d) 1 : 2

Passage IV

(Q. Nos. 100 and 101)

- If two vectors $\mathbf{OA}$ and $\mathbf{OB}$ are there, then their resultant $\mathbf{OA} + \mathbf{OB}$ can be found by completing the parallelogram $OACB$ and $\mathbf{OC} = \mathbf{OA} + \mathbf{OB}$. Also, If $|\mathbf{OA}| = |\mathbf{OB}|$, then the resultant will bisect the angle between them.

100. A vector $\mathbf{C}$ directed along internal bisector of angle between vectors $\mathbf{A} = 7\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$ and $\mathbf{B} = -2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ with $|\mathbf{C}| = 5\sqrt{6}$ is

(a) $\frac{5}{3}(\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}})$ (b) $\frac{5}{3}(\hat{\mathbf{i}} - 7\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$
(c) $\frac{5}{3}(5\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$ (d) $\frac{5}{3}(-5\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

101. If internal and external bisectors of $\angle A$ of ΔABC meet the base BC at D and E respectively, then (D and E lie on same side of B)

(a) $BC = \frac{BD + BE}{4}$ (b) $BC^2 = BD \times DE$
(c) $\frac{2}{BC} = \frac{1}{BD} + \frac{1}{BE}$ (d) None of these

Passage V

(Q. Nos. 102 and 103)

- Let $C : r(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$ be a differentiable curve, i.e. $\lim\limits_{x \to 0} \frac{r(t+h) - r(h)}{h}$ exist for all t,

$\therefore \quad r'(t) = x'(t)\,\hat{\mathbf{i}} + y'(t)\,\hat{\mathbf{j}} + z'(t)\,\hat{\mathbf{k}}$

If $r'(t)$, is tangent to the curve C at the point $P[x(t), y(t), z(t)]$ and $r'(t)$ points in the direction of increasing t.

102. The point P on the curve $r(t) = (1 - 2t)\hat{\mathbf{i}} + t^2\hat{\mathbf{j}} + 2e^{2(t-1)}\hat{\mathbf{k}}$ at which the tangent vector $r'(t)$ is parallel to the radius vector $r(t)$ is

(a) (− 1, 1, 2) (b) (1, − 1, 2)
(c) (− 1, 1, − 2) (d) (1, 1, 2)

103. The tangent vector to $r(t) = 2t^2\hat{\mathbf{i}} + (1 - t)\hat{\mathbf{j}} + (3t^2 + 2)\hat{\mathbf{k}}$ at (2, 0, 5) is

(a) $4\hat{\mathbf{i}} + \hat{\mathbf{j}} - 6\hat{\mathbf{k}}$ (b) $4\hat{\mathbf{i}} - \hat{\mathbf{j}} + 6\hat{\mathbf{k}}$
(c) $2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 6\hat{\mathbf{k}}$ (d) $2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 6\hat{\mathbf{k}}$

Vector Algebra Exercise 5 :
Matching Type Question

104. **a** and **b** form the consecutive sides of a regular hexagon *ABCDEF*.

Column I	Column II
a. If $\mathbf{CD} = x\mathbf{a} + y\mathbf{b}$, then	p. $x = -2$
b. If $\mathbf{CE} = x\mathbf{a} + y\mathbf{b}$, then	q. $x = -1$
c. If $\mathbf{AE} = x\mathbf{a} + y\mathbf{b}$, then	r. $y = 1$
d. If $\mathbf{AD} = -x\mathbf{b}$, then	s. $y = 2$

Vector Algebra Exercise 6 :
Single Integer Answer Type Questions

105. If the resultant of three forces $\mathbf{F}_1 = p\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$, $\mathbf{F}_2 = -5\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $\mathbf{F}_3 = 6\hat{\mathbf{i}} - \hat{\mathbf{k}}$ acting on a particle has a magnitude equal to 5 units. Then, what is difference in the values of p?

106. Vectors along the adjacent sides of parallelogram are $\mathbf{a} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{b} = 2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + \hat{\mathbf{k}}$. Find the length of the longer diagonal of the parallelogram.

107. If vectors $\mathbf{a} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}, \mathbf{b} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{c} = \lambda\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ are coplanar, then find the value of $(\lambda - 4)$.

108. If $\mathbf{a} + \mathbf{b}$ is along the angle bisector of **a** and **b**, where $|\mathbf{a}| = \lambda |\mathbf{b}|$, then the number of digits in value of λ is

109. Let **p** be the position vector of orthocentre and **g** is the position vector of the centroid of ΔABC, where circumcentre is the origin. If $\mathbf{p} = k\mathbf{g}$, then the value of k is

110. In a ΔABC, a line is drawn passing through centroid dividing AB internally in ratio 2 : 1 and AC in $\lambda : 1$ (internally). The value of λ is

111. A vector **a** has component $2p$ and 1 with respect to a rectangular cartesian system. The system is rotated through a certain angle about the origin in the counter clockwise sense. If with to the new system, **a** has components $(p + 1)$ and 1, where p take the values p_1 and p_2. Then, the value of $3|p_1 + p_2|$ is

Vector Algebra Exercise 7 :
Subjective Type Questions

112. A vector **a** has components a_1, a_2 and a_3 in a right handed rectangular cartesian system $OXYZ$. The coordinate system is rotated about Z-axis through angle $\frac{\pi}{2}$. Find components of **a** in the new system.

113. Find the magnitude and direction of $\mathbf{r}_1 - \mathbf{r}_2$ when $|\mathbf{r}_1| = 5$ and points North-East while $|\mathbf{r}_2| = 5$ but points North-West.

114. Let $OACB$ be a parallelogram with O at the origin and **OC** a diagonal. Let D be the mid-point of **OA**. Using vector methods prove that **BD** and **CO** intersects in the same ratio. Determine this ratio.

115. ΔABC is a triangle with the point P on side BC such that $3\mathbf{BP} = 2\mathbf{PC}$, the point Q is on the line **CA** such that $4\mathbf{CO} = \mathbf{QA}$. Find the ratio in which the line joining the common point R of **AP** and **BQ** and the point S divides **AB**.

116. In a ΔABC internal angle bisectors AI, BI and CI are produced to meet opposite sides in A', B' and C', respectively. Prove that the maximum value of $\frac{AI \cdot BI \cdot CI}{AA' \cdot BB' \cdot CC'}$ is $\frac{8}{27}$.

117. Let $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \ldots, \mathbf{r}_n$ be the position vectors of points $P_1, P_2, P_3, \ldots, P_n$ relative to an origin O. Show that if the vectors equation $a_1\mathbf{r}_1 + a_2\mathbf{r}_2 + \ldots + a_n\mathbf{r}_n = 0$ holds, then a similar equation will also hold good with respect to any other origin O', if $a_1 + a_2 + a_3 + \ldots + a_n = 0$.

118. Let $OABCD$ be a pentagon in which the sides OA and CB are parallel and the sides OD and AB are parallel as shown in figure. Also, $OA : CB = 2 : 1$ and $OD : AB = 1 : 3$. If the diagonals OC and AD meet at x, find $OX : OC$.

119. If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is a linearly independent system of vectors, examine the system $\mathbf{p}, \mathbf{q}$ and $\mathbf{r}$, where
$\mathbf{p} = (\cos a)\mathbf{u} + (\cos b)\mathbf{v} + (\cos c)\mathbf{w}$
$\mathbf{q} = (\sin a)\mathbf{u} + (\sin b)\mathbf{v} + (\sin c)\mathbf{w}$
$\mathbf{r} = \sin(x + a)\mathbf{u} + \sin(x + b)\mathbf{v} + \sin(x + c)\mathbf{w}$ for linearly dependent.

Vector Algebra Exercise 8 : Questions Asked in Previous Years Exam

120. If the vectors $\mathbf{AB} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{k}}$ and $\mathbf{AC} = 5\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ are the sides of a ΔABC, then the length of the median through A is **[JEE Main 2013, 2003]**
(a) $\sqrt{18}$ (b) $\sqrt{72}$
(c) $\sqrt{33}$ (d) $\sqrt{45}$

121. Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be three non-zero vectors which are pairwise non-collinear. If $\mathbf{a} + 3\mathbf{b}$ is collinear with $\mathbf{c}$ and $\mathbf{b} + 2\mathbf{c}$ is collinear with $\mathbf{a}$, then $\mathbf{a} + 3\mathbf{b} + 6\mathbf{c}$ is **[AIEEE 2011]**
(a) $\mathbf{a} + \mathbf{c}$ (b) $\mathbf{a}$
(c) $\mathbf{c}$ (d) 0

122. The non-zero vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are related by $\mathbf{a} = 8\mathbf{b}$ and $\mathbf{c} = -7\mathbf{b}$. Then, the angle between a and c is **[AIEEE 2008]**
(a) π (b) 0
(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$

123. If C is the mid-point of AB and P is any point outside AB, then **[AIEEE 2005]**
(a) $\mathbf{PA} + \mathbf{PB} + \mathbf{PC} = 0$ (b) $\mathbf{PA} + \mathbf{PB} + 2\mathbf{PC} = 0$
(c) $\mathbf{PA} + \mathbf{PB} = \mathbf{PC}$ (d) $\mathbf{PA} + \mathbf{PB} = 2\mathbf{PC}$

124. If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are three non-zero vectors such that no two of these are collinear. If the vector $\mathbf{a} + 2\mathbf{b}$ is collinear with $\mathbf{c}$ and $\mathbf{b} + 3\mathbf{c}$ is collinear with $\mathbf{a}$ (λ being some non-zero scalar), then $\mathbf{a} + 2\mathbf{b} + 6\mathbf{c}$ is equal to **[AIEEE 2004]**
(a) $\lambda\mathbf{a}$ (b) $\lambda\mathbf{b}$
(c) $\lambda\mathbf{c}$ (d) 0

125. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are non-coplanar vectors and λ is a real number, then the vectors $\mathbf{a} + 2\mathbf{b} + 3\mathbf{c}$, $\lambda\mathbf{b} + 4\mathbf{c}$ and $(2\lambda - 1)\mathbf{c}$ are non-coplanar for **[AIEEE 2004]**
(a) all values of λ
(b) all except one value of λ
(c) all except two values of λ
(d) no value of λ

126. Consider points A, B, C and D with position vectors $7\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$, $\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 10\hat{\mathbf{k}}$, $-\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ and $5\hat{\mathbf{i}} - \hat{\mathbf{j}} + 5\hat{\mathbf{k}}$, respectively. Then, $ABCD$ is a **[AIEEE 2003]**
(a) square (b) rhombus
(c) rectangle (d) None of these

127. If $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$ and vectors $(1, a, a^2)$, $(1, b, b^2)$ and $(1, c, c^2)$ are non-coplanar, then the product abc equal to **[AIEEE 2003]**
(a) 2 (b) -1
(c) 1 (d) 0

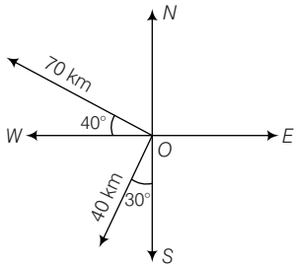
128. The vector $\hat{\mathbf{i}} + x\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ is rotated through an angle θ and doubled in magnitude, then it becomes $4\hat{\mathbf{i}} + (4x - 2)\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$. The value of x are **[AIEEE 2002]**
(a) $\left\{-\frac{2}{3}, 2\right\}$ (b) $\left(\frac{1}{3}, 2\right)$
(c) $\left\{\frac{2}{3}, 0\right\}$ (d) $\{2, 7\}$

Answers

Exercise for Session 1

1. (i) vector (ii) scalar (iii) vector (iv) scalar

2.



3. (i) **a**, **d**; **b**, **x**, **z**; **c**, **y** (ii) **b**, **x**; **a**, **d**; **c**, **y**
(iii) **a**, **y**, **z** (iv) **b**, **z**; **x**, **z**

4. (i) True (ii) False
(iii) False (iv) False

5. $\sqrt{450}$ **6.** $\cos^{-1}\frac{6}{7}$

7. Direction ratios are 1, – 1, 2 and Direction cosines are $\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$.

Exercise for Session 2

1. $\hat{\mathbf{i}}+\hat{\mathbf{k}}$; $\frac{1}{\sqrt{2}}\hat{\mathbf{i}}+\frac{1}{\sqrt{2}}\hat{\mathbf{k}}$ **2.** $\frac{3}{5\sqrt{2}}\hat{\mathbf{i}}+\frac{5}{5\sqrt{2}}\hat{\mathbf{j}}+\frac{4}{5\sqrt{2}}\hat{\mathbf{k}}$

3. $\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$

9. $4\hat{\mathbf{i}}-9\hat{\mathbf{j}}, \sqrt{97}, \frac{4}{\sqrt{97}}\hat{\mathbf{i}}-\frac{9}{\sqrt{97}}\hat{\mathbf{j}}$

10. $\sqrt{59}$; $\frac{1}{\sqrt{59}}\hat{\mathbf{i}}-\frac{7}{\sqrt{59}}\hat{\mathbf{j}}+\frac{3}{\sqrt{59}}\hat{\mathbf{k}}$ **12.** $\pm\frac{1}{3}$

13. $\sqrt{66}$ **14.** $\frac{8}{\sqrt{30}}(5\hat{\mathbf{i}}-\hat{\mathbf{j}}+2\hat{\mathbf{k}})$

15. $7/3$, $(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\hat{\mathbf{k}})$

16. (i) $-\frac{1}{3}\hat{\mathbf{i}}+\frac{4}{3}\hat{\mathbf{j}}+\frac{1}{3}\hat{\mathbf{k}}$ (ii) $-3\hat{\mathbf{i}}+3\hat{\mathbf{k}}$

17. $4\hat{\mathbf{i}}+3\hat{\mathbf{j}}+7\hat{\mathbf{k}}$

Exercise for Session 3

4. (2,–3) **5.** $a-2b=1$

6. $x=\frac{1}{3}$ **7.** 0

Chapter Exercises

1. (c)	**2.** (a)	**3.** (a)	**4.** (d)	**5.** (a)	**6.** (c)
7. (c)	**8.** (b)	**9.** (c)	**10.** (d)	**11.** (a)	**12.** (a)
13. (d)	**14.** (d)	**15.** (b)	**16.** (b)	**17.** (c)	**18.** (b)
19. (c)	**20.** (d)	**21.** (a)	**22.** (b)	**23.** (a)	**24.** (b)
25. (c)	**26.** (b)	**27.** (b)	**28.** (c)	**29.** (c)	**30.** (c)
31. (a)	**32.** (d)	**33.** (b)	**34.** (a)	**35.** (c)	**36.** (a)
37. (b)	**38.** (d)	**39.** (b)	**40.** (c)	**41.** (b)	**42.** (b)
43. (b)	**44.** (d)	**45.** (b)	**46.** (d)	**47.** (c)	**48.** (a)
49. (a)	**50.** (d)	**51.** (c)	**52.** (a)	**53.** (b)	**54.** (d)
55. (c)	**56.** (a)	**57.** (d)	**58.** (c)	**59.** (b)	**60.** (b)
61. (b)	**62.** (c)	**63.** (b)	**64.** (d)	**65.** (c)	**66.** (c)
67. (a)	**68.** (c)	**69.** (a)	**70.** (a)	**71.** (b)	**72.** (a)
73. (b)	**74.** (d)	**75.** (b)	**76.** (a)	**77.** (b)	**78.** (d)
79. (b)	**80.** (a)	**81.** (b)	**82.** (c)	**83.** (a,b,d)	**84.** (b,c)
85. (a,c)	**86.** (a,d)	**87.** (a,c,d)	**88.** (a,b,d)	**89.** (a,b,c)	**90.** (c)
91. (a)	**92.** (a)	**93.** (c)	**94.** (b)	**95.** (c)	**96.** (d)
97. (c)	**98.** (a)	**99.** (d)	**100.** (b)	**101.** (c)	**102.** (a)

103. (b) **104.** a $\rightarrow$ q, r ; b $\rightarrow$ p, r

105. (2, – 4) **106.** (7) **107.** (2)

108. (1) **109.** (3) **110.** (2) **111.** (2)

112. $(a_2, -a_1, a_3)$ **113.** $5\sqrt{5}$, West to East

114. 2 :1 **115.** 6 : 1 **118.** 2 : 5

120. (c) **121.** (d) **122.** (a) **123.** (d) **124.** (d) **125.** (c)

126. (d) **127.** (b) **128.** (a)

Solutions

1. $\mathbf{a}+\mathbf{b}+\mathbf{c}=(3+2-1)\hat{\mathbf{i}}+(-2-4+2)\hat{\mathbf{j}}+(1-3+2)\hat{\mathbf{k}}$

$=4\hat{\mathbf{i}}-4\hat{\mathbf{j}}$

2. Let $\mathbf{b}$ should be added, then $\mathbf{a}+\mathbf{b}=\hat{\mathbf{i}}$

$\Rightarrow \quad \mathbf{b}=\hat{\mathbf{i}}-\mathbf{a}=\hat{\mathbf{i}}-(3\hat{\mathbf{i}}+4\hat{\mathbf{j}}-2\hat{\mathbf{k}})$

$=-2\hat{\mathbf{i}}-4\hat{\mathbf{j}}+2\hat{\mathbf{k}}$

3. $|\mathbf{a}+\mathbf{b}|=|3\hat{\mathbf{i}}+4\hat{\mathbf{j}}-12\hat{\mathbf{k}}|$

$=|\sqrt{3^2+4^2+(12)^2}|=13$

4. $\mathbf{a}+\mathbf{b}=4\hat{\mathbf{i}}+4\hat{\mathbf{j}}$,

Therefore, unit vector $=\dfrac{4(\hat{\mathbf{i}}+\hat{\mathbf{j}})}{\sqrt{32}}=\dfrac{\hat{\mathbf{i}}+\hat{\mathbf{j}}}{\sqrt{2}}$

5. Resultant vector $=(2\hat{\mathbf{i}}+4\hat{\mathbf{j}}-5\hat{\mathbf{k}})+(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}})$

$=3\hat{\mathbf{i}}+6\hat{\mathbf{j}}-2\hat{\mathbf{k}}$

Unit vector $=\dfrac{3\hat{\mathbf{i}}+6\hat{\mathbf{j}}-2\hat{\mathbf{k}}}{\sqrt{9+36+4}}=\dfrac{1}{7}(3\hat{\mathbf{i}}+6\hat{\mathbf{j}}-2\hat{\mathbf{k}})$

6. $\mathbf{R}=3\hat{\mathbf{i}}+5\hat{\mathbf{j}}+4\hat{\mathbf{k}}$

$\Rightarrow \quad \mathbf{R}=\dfrac{3\hat{\mathbf{i}}+5\hat{\mathbf{j}}+4\hat{\mathbf{k}}}{5\sqrt{2}}$

7. $\mathbf{a}+\mathbf{b}=3\hat{\mathbf{i}}+9\hat{\mathbf{j}}=3(\hat{\mathbf{i}}+3\hat{\mathbf{j}})$. Hence, it is parallel to (1, 3).

8. $\mathbf{AB}+\mathbf{BC}+\mathbf{CA}=0$

$\Rightarrow \quad \mathbf{a}+\mathbf{b}-\mathbf{c}=0$

9. Unit vector parallel to $\mathbf{OA}=\dfrac{4\hat{\mathbf{i}}+5\hat{\mathbf{j}}}{\sqrt{16+25}}=\dfrac{1}{\sqrt{41}}(4\hat{\mathbf{i}}+5\hat{\mathbf{j}})$

10. $\mathbf{OA}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}-\hat{\mathbf{k}}, \mathbf{OB}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$

and $\mathbf{OC}=2\hat{\mathbf{i}}+3\hat{\mathbf{j}}+2\hat{\mathbf{k}}$

Position vector of B w.r.t origin at A at

$\mathbf{AB}=\mathbf{OB}-\mathbf{OA}=-\hat{\mathbf{j}}+2\hat{\mathbf{k}}$

Position vector of C w.r.t. origin at A is

$\mathbf{AC}=\mathbf{OC}-\mathbf{OA}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+3\hat{\mathbf{k}}$

11. $7=\sqrt{(5+1)^2+(4-2)^2+(a+2)^2}$

$\Rightarrow \quad a+2=\pm 3 \Rightarrow a=-5, 1$

12. $\mathbf{AB}=\mathbf{CX}\Rightarrow \hat{\mathbf{j}}-\hat{\mathbf{i}}=$ position vector of point $X-\hat{\mathbf{k}}$

$\therefore$ Position vector of point $X=-\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$

13. $\mathbf{AB}=(6-2)\hat{\mathbf{i}}+(-3+9)\hat{\mathbf{j}}+(8+4)\hat{\mathbf{k}}$

$=4\hat{\mathbf{i}}+6\hat{\mathbf{j}}+12\hat{\mathbf{k}}$

$|\mathbf{AB}|=\sqrt{16+25+144}=14$

14. $\mathbf{PQ}=(5-1)\hat{\mathbf{i}}+(-2-3)\hat{\mathbf{j}}+(4+7)\hat{\mathbf{k}}$

$=4\hat{\mathbf{i}}-5\hat{\mathbf{j}}+11\hat{\mathbf{k}}$

$|\mathbf{PQ}|=\sqrt{16+25+121}=\sqrt{162}$

15. $\mathbf{PQ}=\mathbf{OQ}-\mathbf{OP}=4\hat{\mathbf{i}}-5\hat{\mathbf{j}}+11\hat{\mathbf{k}}$

$\therefore \quad \dfrac{\mathbf{PQ}}{|\mathbf{PQ}|}=\dfrac{4}{\sqrt{162}}\hat{\mathbf{i}}-\dfrac{5}{\sqrt{162}}\hat{\mathbf{j}}+\dfrac{11}{\sqrt{162}}\hat{\mathbf{k}}$

$\therefore \quad \cos\gamma=\dfrac{11}{\sqrt{162}}$, where γ is the angle of $\mathbf{PQ}$ with Z-axis.

16. $\mathbf{AB}=4\hat{\mathbf{i}}-5\hat{\mathbf{j}}+11\hat{\mathbf{k}}$

Direction cosine along Y-axis $=\dfrac{-5}{\sqrt{16+25+121}}=\dfrac{-5}{\sqrt{162}}$

17. $\dfrac{3}{\sqrt{3^2+4^2+5^2}}=\dfrac{3}{\sqrt{50}}$

18. Vector $\mathbf{A}=3\hat{\mathbf{i}}-4\hat{\mathbf{j}}+5\hat{\mathbf{k}}$. We know that, direction cosines of

$\mathbf{A}=\dfrac{3}{\sqrt{3^2+4^2+5^2}}, \dfrac{-4}{\sqrt{3^2+4^2+5^2}}, \dfrac{5}{\sqrt{3^2+4^2+5^2}}$

$=\dfrac{3}{5\sqrt{2}}, \dfrac{-4}{5\sqrt{2}}, \dfrac{1}{\sqrt{2}}$

19. Here, $\mathbf{OA}=2\hat{\mathbf{i}}+3\hat{\mathbf{j}}+4\hat{\mathbf{k}}$

$\mathbf{OB}=3\hat{\mathbf{i}}+4\hat{\mathbf{j}}+2\hat{\mathbf{k}}$

and $\mathbf{OC}=4\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}}$

So, $\mathbf{AB}=\hat{\mathbf{i}}+\hat{\mathbf{j}}-2\hat{\mathbf{k}}$, $\mathbf{BC}=\hat{\mathbf{i}}-2\hat{\mathbf{j}}+\hat{\mathbf{k}}$, $\mathbf{CA}=2\hat{\mathbf{i}}-\hat{\mathbf{j}}-\hat{\mathbf{k}}$

Clearly, $|AB|=|BC|=|CA|=\sqrt{6}$

So, these points are vertices of an equilateral triangle.

20. Given, position vectors of A, B and C are $7\hat{\mathbf{j}}+10\hat{\mathbf{k}}$, $-\hat{\mathbf{i}}+6\hat{\mathbf{j}}+6\hat{\mathbf{k}}$ and $-4\hat{\mathbf{i}}+9\hat{\mathbf{j}}+6\hat{\mathbf{k}}$, respectively.

$\therefore \quad |\mathbf{AB}|=|-\hat{\mathbf{i}}-\hat{\mathbf{j}}-4\hat{\mathbf{k}}|=\sqrt{18}$

$|\mathbf{BC}|=|-3\hat{\mathbf{i}}+3\hat{\mathbf{j}}|=\sqrt{18}$

$|\mathbf{AC}|=|-4\hat{\mathbf{i}}+2\hat{\mathbf{j}}-4\hat{\mathbf{k}}|=\sqrt{36}$

Clearly, $AB=BC$ and $(AC)^2=(AB)^2+(BC)^2$

Hence, triangle is right angled isosceles.

24. Position vectors of the points which divides internally is $\dfrac{3(2\mathbf{a}-3\mathbf{b})+2(3\mathbf{a}-2\mathbf{b})}{5}=\dfrac{12\mathbf{a}-13\mathbf{b}}{5}$

25. Coordinate of C is $\left(\dfrac{2-4}{2}, \dfrac{-1+3}{2}\right)=(-1, 1)$

$\therefore \quad \mathbf{OC}=-\hat{\mathbf{i}}+\hat{\mathbf{j}}$

26. $\dfrac{3\hat{\mathbf{i}}-\hat{\mathbf{j}}-3\hat{\mathbf{k}}+\hat{\mathbf{j}}+3\hat{\mathbf{j}}-\hat{\mathbf{k}}}{2}=2\hat{\mathbf{i}}+\hat{\mathbf{j}}-2\hat{\mathbf{k}}$

27. $\dfrac{\mathbf{a}+\mathbf{b}}{2}=2\hat{\mathbf{i}}-\hat{\mathbf{j}}+\dfrac{5}{2}\hat{\mathbf{k}}$

28. It is given that $\mathbf{b}$ is collinear with the vector $\mathbf{a}$.

$\therefore \quad \mathbf{b}=\lambda\mathbf{a}$ ...(i)

$=2\sqrt{2}\lambda\hat{\mathbf{i}}-\lambda\hat{\mathbf{j}}+4\lambda\hat{\mathbf{k}}$

Also, $|\mathbf{b}|=10$

$\Rightarrow \quad \sqrt{(2\sqrt{2}\lambda)^2+(-\lambda)^2+(4\lambda)^2}=10$

$\Rightarrow \quad 25\lambda^2=100$

$\Rightarrow \quad \lambda=\pm 2$ ... (ii)

From Eqs. (i) and (ii), we have

$\mathbf{b}=\pm 2\mathbf{a} \Rightarrow 2\mathbf{a}+\mathbf{b}=0$

29. Condition for collinearity, $\mathbf{b} = \lambda\ \mathbf{a}$

$\Rightarrow \quad (-2\hat{\mathbf{i}} + m\hat{\mathbf{j}}) = \lambda\ (\hat{\mathbf{i}} - \hat{\mathbf{j}})$

Comparison of coefficient, we get

$\Rightarrow \quad \lambda = -2$ and $-\lambda = m$

So, $\quad m = 2$

30. If given points be A, B and C, then $\mathbf{AB} = k(\mathbf{BC})$ or

$$2\hat{\mathbf{i}} - 8\hat{\mathbf{j}} = k[(a-12)\hat{\mathbf{i}} + 16\hat{\mathbf{j}}]$$

$\Rightarrow \quad k = -\frac{1}{2}$

Also, $\quad 2 = k\,(a-12)$

$\Rightarrow \quad a = 8$

31. $\begin{vmatrix} 1 & 2 & 3 \\ \lambda & 4 & 7 \\ -3 & -2 & -5 \end{vmatrix} = 0$

$\Rightarrow \quad \lambda = 3$

32. $(\mathbf{a} - \mathbf{b}) - (\mathbf{a} + \mathbf{b}) = [(\mathbf{a} + k\mathbf{b}) - (\mathbf{a} - \mathbf{b})]$

$\Rightarrow \quad -2\,\mathbf{b} = (k+1)\,\mathbf{b}$

Hence, $k \in R$

33. $\mathbf{AB} = -\hat{\mathbf{i}} - 4\hat{\mathbf{j}}, \mathbf{CD} = -2\hat{\mathbf{i}} + (\lambda - 2)\hat{\mathbf{j}}$

$\because \quad \mathbf{AB} \,||\, \mathbf{CD}$

So, $\quad \frac{-1}{-2} = \frac{-4}{\lambda - 2} \Rightarrow \lambda - 2 = -8$

$\Rightarrow \quad \lambda = -6$

34. Obviously, $\frac{3}{6} = \frac{2}{-4x} = -\frac{1}{y}$

$\Rightarrow \quad x = -1$ and $y = -2$

35. If $\mathbf{a}$ and $\mathbf{b}$ are two non-zero, non-collinear vectors and x and y are two scalars such that $x\mathbf{a} + y\,\mathbf{b} = 0$ then $x = 0$ and $y = 0$ because one will be a scalar multiple of the other and hence collinear which is a contradiction.

36. Four or more than four non-zero vectors are always linearly dependent.

37. These are coplanar because $1\,(\mathbf{a}) + 1\,(\mathbf{b}) = \mathbf{a} + \mathbf{b}$

38. Comparing the coefficients of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$, and the corresponding equations are

$$x + 3y - 4z = \lambda x \quad \text{or} \quad (1-\lambda)x + 3y - 4z = 0 \qquad \ldots(\text{i})$$

$$x - (\lambda + 3)y + 5z = 0 \qquad \ldots(\text{ii})$$

$$3x + y - \lambda z = 0 \qquad \ldots(\text{iii})$$

These Eqs. (i), (ii) and (iii) have a non-trivial solution, if

$$\begin{vmatrix} (1-\lambda) & 3 & -4 \\ 1 & -(\lambda+3) & 5 \\ 3 & 1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda = 0, -1$$

39. The vectors are linearly dependent

$$\Rightarrow \quad \begin{vmatrix} p+1 & -3 & p \\ p & p+1 & -3 \\ -3 & p & p+1 \end{vmatrix} = 0$$

$$\Rightarrow \quad (2p-2)\begin{vmatrix} 1 & -3 & p \\ 1 & p+1 & -3 \\ 1 & p & P+1 \end{vmatrix} = 0$$

$$\Rightarrow \quad 2(p-1)\begin{vmatrix} 1 & -3 & p \\ 0 & p+4 & -3-p \\ 0 & p+3 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad 2(p-1)(p+4+(p+3)^2) = 0$$

$$\Rightarrow \quad (p-1)(p^2+7p+13) = 0$$

Roots of $p^2 + 7p + 13 = 0$ are (imaginary)

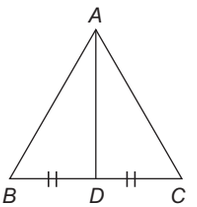
$\therefore \quad p = 1$

Only integral value of p is 1.

40. PV of $\mathbf{AD} = \frac{(3+5)\hat{\mathbf{i}} + (0-2)\hat{\mathbf{j}} + (4+4)\hat{\mathbf{k}}}{2}$

$$= 4\hat{\mathbf{i}} - \hat{\mathbf{j}} + 4\hat{\mathbf{k}}$$

$$|\mathbf{AD}| = \sqrt{16+16+1} = \sqrt{33}$$



41. $\mathbf{v} = \mathbf{b} + \mathbf{c}$

$\mathbf{w} = \mathbf{b} + \mathbf{a}$

We have, $\quad \mathbf{x} = \mathbf{v} + \mathbf{w} = \mathbf{a} + 2\,\mathbf{b} + \mathbf{c}$

42. Note that $\mathbf{a} + \mathbf{b} = \mathbf{c}$

43. Here is the only vector $4(\sqrt{2}\hat{\mathbf{i}} + \hat{\mathbf{j}} \pm \hat{\mathbf{k}})$, whose length is 8.

44. $3\mathbf{a} + 4\mathbf{b} + 5\mathbf{c} = 0$

Hence, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar.

No other conclusion can be derived from it.

45. A, B, C, D and E are five coplanar points.

$$\mathbf{DA} + \mathbf{DB} + \mathbf{AE} + \mathbf{BE} + \mathbf{CE}$$

$$= (\mathbf{DA} + \mathbf{AE}) + (\mathbf{DB} + \mathbf{BE}) + (\mathbf{DC} + \mathbf{CE})$$

$$= \mathbf{DE} + \mathbf{DE} + \mathbf{DE} = 3\mathbf{DE}$$

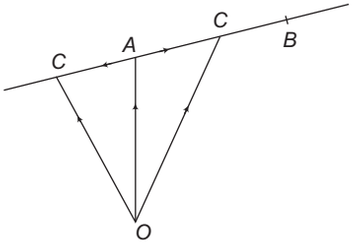
46. $\sqrt{3}\tan\theta + 1 = 0$ and $\sqrt{3}\sec\theta - 2 = 0$

$\Rightarrow \quad \theta = \frac{11\pi}{6}$

$\Rightarrow \quad \theta = 2n\pi + \frac{11\pi}{6}, n \in Z$

47. $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are unit vectors along Y and Z-axes, then unit vector bisecting $\mathbf{OY}$ and $\mathbf{OZ}$ is $\frac{\hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{2}}$.

48.



We have, $A(\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}})$ and $B(\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}})$

On line AB points C and C' are at distance 1 unit from A.

$\mathbf{OC} = \mathbf{OA} + \mathbf{AC}$, where $\mathbf{AC}$ is unit vector in direction of $\mathbf{AB}$

$$\therefore \quad \mathbf{OC} = \mathbf{OA} + \frac{\mathbf{AB}}{|\mathbf{AB}|}$$

Similarly, $$\mathbf{OC'} = \mathbf{OA} - \frac{\mathbf{AB}}{|\mathbf{AB}|}$$

49. $$\mathbf{AD} = \mathbf{OD} - \mathbf{OA}$$

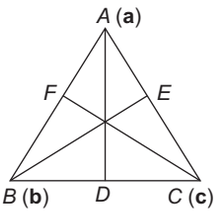
$$= \frac{\mathbf{b} + \mathbf{c}}{2} - \mathbf{a} = \frac{\mathbf{b} + \mathbf{c} - 2\mathbf{a}}{2}$$

[where, O is the origin for reference]

Similarly, $$\mathbf{BE} = \mathbf{OE} - \mathbf{OB} = \frac{\mathbf{c} + \mathbf{a}}{2} - \mathbf{b}$$

$$= \frac{\mathbf{c} + \mathbf{a} - 2\mathbf{b}}{2}$$

and $$\mathbf{CF} = \frac{\mathbf{a} + \mathbf{b} - 2\mathbf{b}}{2}$$

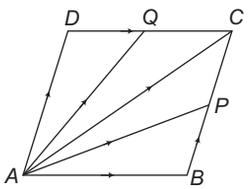


Now, $\mathbf{AD} + \mathbf{BE} + \mathbf{CF} = \frac{\mathbf{b} + \mathbf{c} - 2\mathbf{a}}{2}$

$$+ \frac{\mathbf{c} + \mathbf{a} - 2\mathbf{b}}{2} + \frac{\mathbf{a} + \mathbf{b} - 2\mathbf{c}}{2} = \mathbf{0}$$

50. $\mathbf{AP} = \mathbf{AB} + \mathbf{BP} = \mathbf{AB} + \frac{1}{2}\mathbf{BC} = \mathbf{AB} + \frac{1}{2}\mathbf{AD}$ ...(i)

$\mathbf{AQ} = \mathbf{AD} + \mathbf{DQ} = \mathbf{AD} + \frac{1}{2}\mathbf{DC} = \mathbf{AD} + \frac{1}{2}\mathbf{AB}$ ... (ii)



By Eqs. (i) and (ii), we get

$$\mathbf{AP} + \mathbf{AQ} = \frac{3}{2}(\mathbf{AB} + \mathbf{AD})$$

$$= \frac{3}{2}(\mathbf{AB} + \mathbf{BC}) = \frac{3}{2}\mathbf{AC}$$

51. Let $A \equiv (1, 1, -1)$, $B \equiv (2, 3, 0)$, $C \equiv (3, 5, -2)$ and $D \equiv (0, -1, 1)$.

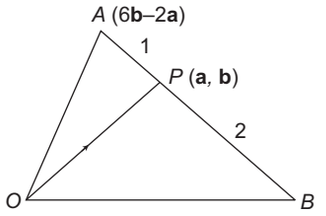
So, $\mathbf{AB} = (1, 2, 1)$, $\mathbf{BC} = (1, 2, -2)$, $\mathbf{CD} = (-3, -6, 3)$

and $\mathbf{DA} = (1, 2, -2)$.

Clearly, $\mathbf{BC} \,||\, \mathbf{DA}$ but AB in not parallel to CD.

So, it is a trapezium.

52. $$\mathbf{OP} = \frac{1\,(\mathbf{OB}) + 2\,(6\mathbf{b} - 2\mathbf{a})}{1 + 2}$$



$$\Rightarrow \quad 3(\mathbf{a} - \mathbf{b}) = \mathbf{OB} + 12\mathbf{b} - 4\mathbf{a}$$

$$\Rightarrow \quad \mathbf{OB} = 7\mathbf{a} - 15\mathbf{b}$$

53. Let the B divide AC in ratio $\lambda : 1$, then

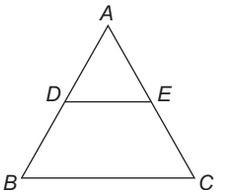
$$5\hat{\mathbf{i}} - 2\hat{\mathbf{k}} = \frac{\lambda(11\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 7\hat{\mathbf{k}}) + \hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 8\hat{\mathbf{k}}}{\lambda + 1}$$

$$\Rightarrow \quad 3\lambda - 2 = 0$$

$$\Rightarrow \quad \lambda = \frac{2}{3}, \text{ i.e. ratio} = 2 : 3$$

54. We know by fundamental theorem of proportionality that,

$\mathbf{DE} = \frac{1}{2}\mathbf{BC}$



In triangle, $\mathbf{BC} = \mathbf{b} - \mathbf{a}$

Hence, $\mathbf{DE} = \frac{1}{2}(\mathbf{b} - \mathbf{a})$

55. Since, $\mathbf{AB} + \mathbf{BD} = \mathbf{AD}$

$$\mathbf{BD} = \mathbf{AD} - \mathbf{AB}$$

$$\Rightarrow \quad = (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) - (2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 5\hat{\mathbf{k}})$$

$$= -\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 8\hat{\mathbf{k}}$$

Hence, unit vector in the direction of $\mathbf{BD}$ is

$$\frac{-\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 8\hat{\mathbf{k}}}{|-\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 8\hat{\mathbf{k}}|} = \frac{-\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 8\hat{\mathbf{k}}}{\sqrt{69}}$$

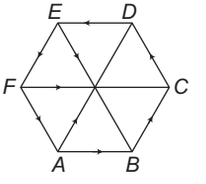
56. Position vectors of vertices A, B and C of the $\Delta ABC = \mathbf{a}, \mathbf{b}$ and $\mathbf{c}$. We know that, position vector of centroid of the triangle, $\mathbf{G} = \frac{\mathbf{a} + \mathbf{b} + \mathbf{c}}{3}$.

Therefore, $\mathbf{GA} + \mathbf{GB} + \mathbf{GC}$

$$= \left(\mathbf{a} - \frac{\mathbf{a} + \mathbf{b} + \mathbf{c}}{3}\right) + \left(\mathbf{b} - \frac{\mathbf{a} + \mathbf{b} + \mathbf{c}}{3}\right) + \left(\mathbf{c} - \frac{\mathbf{a} + \mathbf{b} + \mathbf{c}}{3}\right)$$

$$= \frac{1}{3}(2\mathbf{a} - \mathbf{b} - \mathbf{c} + 2\mathbf{b} - \mathbf{a} - \mathbf{c} + 2\mathbf{c} - \mathbf{a} - \mathbf{b}) = 0$$

57. A regular hexagon $ABCDEF$.



We know from the hexagon that **AD** is parallel to **BC** or **AD** = 2**BC** is parallel to **FA** or **EB** = 2**FA** and **FC** is parallel to **AB** or **FC** = 2**AB**.

Thus, $\mathbf{AD}+\mathbf{EB}+\mathbf{FC}=2\mathbf{BC}+2\mathbf{FA}+2\mathbf{AB}$

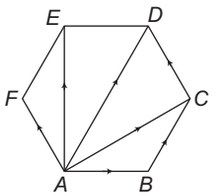
$=2(\mathbf{FA}+\mathbf{AB}+\mathbf{BC})=2(\mathbf{FC})=2(2\mathbf{AB})=4\mathbf{AB}$

58. $\mathbf{AE}+\mathbf{ED}+\mathbf{DC}+\mathbf{AB}$

$=\mathbf{AD}+\mathbf{DC}+\mathbf{AB}=\mathbf{AC}+\mathbf{AB}$

Obviously, if **BC** is added to this system, then it will be $\mathbf{AC}+\mathbf{AB}+\mathbf{BC}=\mathbf{AC}+\mathbf{AC}=2\mathbf{AC}$.

59. By triangle law, $\mathbf{AB}=\mathbf{AD}-\mathbf{BD}, \mathbf{AC}=\mathbf{AD}-\mathbf{CD}$



Therefore,

$\mathbf{AB}+\mathbf{AC}+\mathbf{AD}+\mathbf{AE}+\mathbf{AF}$

$=3\mathbf{AD}+(\mathbf{AE}-\mathbf{BD})+(\mathbf{AF}-\mathbf{CD})=3\mathbf{AD}$

Hence, $\lambda=3$ $(\because \mathbf{AE}=\mathbf{BD}, \mathbf{AF}=\mathbf{CD})$

60. $|a|+|b|+|c|=\sqrt{a^2+b^2+c^2}$

$\Leftrightarrow\ 2|ab|+2|bc|+2|ca|=0$

$\Leftrightarrow\ ab=bc=ca=0 \Leftrightarrow$ any two of a, b and c are zero

61. Since, **a** and **b** are non-collinear, so **a** + **b** and **a** − **b** will also be non-collinear.

Hence, **a** + **b** and **a** − **b** are linearly independent vectors.

62. $|\mathbf{a}+\mathbf{b}|<|\mathbf{a}-\mathbf{b}|$

$\Rightarrow \quad \frac{\pi}{2}<\theta<\frac{3\pi}{2}$

63. $R=\sqrt{4+100+121}=15$

64. We must have $\lambda(\hat{\mathbf{i}}-3\hat{\mathbf{j}}+5\hat{\mathbf{k}})=\mathbf{a}+\frac{2\hat{\mathbf{k}}+2\hat{\mathbf{j}}-\hat{\mathbf{i}}}{3}$

Therefore, $3\mathbf{a}=3\lambda(\hat{\mathbf{i}}-3\hat{\mathbf{j}}+5\hat{\mathbf{k}})-(2\hat{\mathbf{k}}+2\hat{\mathbf{j}}-\hat{\mathbf{i}})$

$=\hat{\mathbf{i}}(3\lambda+1)-\hat{\mathbf{j}}(2+9\lambda)+\hat{\mathbf{k}}(15\lambda-2)$

or $3|\mathbf{a}|=\sqrt{(3\lambda+1)^2+(2+9\lambda)^2+(15\lambda-2)^2}$

or $9=(3\lambda+1)^2+(2+9\lambda)^2+(15\lambda-2)^2$

or $315\lambda^2-18\lambda=0 \Rightarrow \lambda=0, \frac{2}{35}$

If $\lambda=0, \mathbf{a}=\hat{\mathbf{i}}-2\hat{\mathbf{j}}-2\hat{\mathbf{k}}$ (not acceptable)

For $\lambda=\frac{2}{35}, \mathbf{a}=\frac{41}{105}\hat{\mathbf{i}}-\frac{88}{105}\hat{\mathbf{j}}-\frac{40}{105}\hat{\mathbf{k}}$

65. $\mathbf{b}=\cos 120^\circ\hat{\mathbf{i}}+\sin 120^\circ\hat{\mathbf{j}}$

or $\mathbf{b}=-\frac{1}{2}\hat{\mathbf{i}}+\frac{\sqrt{3}}{2}\hat{\mathbf{j}}$

Therefore, $\mathbf{a}+\mathbf{b}=\hat{\mathbf{i}}-\frac{1}{2}\hat{\mathbf{i}}+\frac{\sqrt{3}}{2}\hat{\mathbf{j}}=\frac{1}{2}\hat{\mathbf{i}}+\frac{\sqrt{3}}{2}\hat{\mathbf{j}}$

66. $\alpha=\mathbf{a}+\mathbf{b}+\mathbf{c}=6\hat{\mathbf{i}}+12\hat{\mathbf{j}}$

Let $\alpha=x\mathbf{a}+y\mathbf{b} \Rightarrow 6x+2y=6$

and $-3x-6y=12$

$\therefore \quad x=2, y=-3$

$\therefore \quad \alpha=2\mathbf{a}-3\mathbf{b}$

67. Let the incentre be at the origin and be $A(\mathbf{p}), B(\mathbf{q})$ and $C(\mathbf{r})$. Then,

$\mathbf{IA}=\mathbf{p}, \mathbf{IB}=\mathbf{q}$ and $\mathbf{IC}=\mathbf{r}$

Incentre I is $\frac{ap+bq+cr}{a+b+c}$, where $p=BC$, $q=AC$ and $r=AB$

Incentre is at the origin. Therefore,

$\frac{a\mathbf{p}+b\mathbf{q}+c\mathbf{r}}{a+b+c}=\mathbf{0}$,

or $a\mathbf{b}+b\mathbf{q}+c\mathbf{r}=\mathbf{0}$

$\Rightarrow \quad a\mathbf{IA}+b\mathbf{IB}+c\mathbf{IC}=\mathbf{0}$

68. Since **x**, **y** and **x** × **y** are linearly independent, we have

$20a-15b=15b-12c=12c-20a=0$

$\Rightarrow \quad \frac{a}{3}=\frac{b}{4}=\frac{c}{5} \Rightarrow\ c^2=a^2+b^2$

Hence, ΔABC is right angled.

69. As **x**, **y** and **x** × **y** are non-collinear vectors, vectors are linearly independent.

Hence,

$a-b=0=b-c=c-a$

or $a=b=c$

Therefore, the triangle is equilateral.

70. $|\mathbf{AB}|=|Q|=\sqrt{P^2+P^2}=P\sqrt{2}$

71. The point the divides $5\hat{\mathbf{i}}$ and $5\hat{\mathbf{j}}$ in th ratio of

$k:1$ is $\frac{(5\hat{\mathbf{j}})k+(5\hat{\mathbf{i}})1}{k+1}$

$\therefore \quad \mathbf{b}=\frac{5\hat{\mathbf{i}}+5k\hat{\mathbf{j}}}{k+1}$

Also $|\mathbf{b}|\le\sqrt{37}$

$\Rightarrow \quad \frac{1}{k+1}\sqrt{25+25k^2}\le\sqrt{37}$

or $5\sqrt{1+k^2}\le\sqrt{37}(k+1)$

On Squaring both sides, we get

$25(1+k^2)\le 37\,(k^2+2k+1)$

or $6k^2+37k+6\ge 0$ or $(6k+1)(k+6)\ge 0$

$k\in(-\infty,-6)\cup\left[-\frac{1}{6},\infty\right]$

72. Let the bisector of $\angle A$ meets BC at D, then AD divides BC in the ratio $AB:AC$.

$\therefore$ Position vectors of D

$=\frac{|\mathbf{AB}|(2\hat{\mathbf{i}}+5\hat{\mathbf{j}}+7\hat{\mathbf{k}})+|\mathbf{AC}|(2\hat{\mathbf{i}}+3\hat{\mathbf{j}}+4\hat{\mathbf{k}})}{|\mathbf{AB}|+|\mathbf{AC}|}$

Here, $|\mathbf{AB}|=|-2\hat{\mathbf{i}}-4\hat{\mathbf{j}}-4\hat{\mathbf{k}}|=6$

and $|\mathbf{AC}| = |-2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - \hat{\mathbf{k}}| = 3$

$\therefore$ Position vector of D

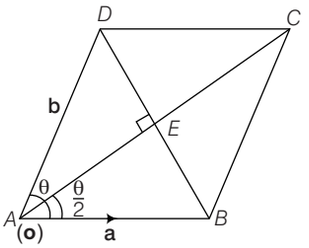
$$= \frac{6(2\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 7\hat{\mathbf{k}}) + 3(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}})}{6+3}$$

$$= \frac{18\hat{\mathbf{i}} + 39\hat{\mathbf{j}} + 54\hat{\mathbf{k}}}{9}$$

$$= \frac{1}{3}(6\hat{\mathbf{i}} + 13\hat{\mathbf{j}} + 18\hat{\mathbf{k}})$$

73. Vector in the direction of angular bisector of $\mathbf{a}$ and $\mathbf{b}$ is $\dfrac{\mathbf{a}+\mathbf{b}}{2}$.

Unit vector in this direction is $\dfrac{\mathbf{a}+\mathbf{b}}{|\mathbf{a}+\mathbf{b}|}$



From the figure, position vector of E is $\dfrac{\mathbf{a}+\mathbf{b}}{2}$.

Now in triangle AEB, $AE = AB\cos\dfrac{\theta}{2}$

$$\Rightarrow \qquad \left|\frac{\mathbf{a}+\mathbf{b}}{2}\right| = \cos\frac{\theta}{2}$$

Hence, unit vector along the bisector is $\dfrac{\mathbf{a}+\mathbf{b}}{2\cos(\theta/2)}$.

74. $\mathbf{a} - \mathbf{b} = 2(\mathbf{d} - \mathbf{c})$

$$\therefore \qquad \frac{\mathbf{a}+2\mathbf{c}}{2+1} = \frac{\mathbf{b}+2\mathbf{d}}{2+1}$$

Hence, AC and BD trisect each other as LHS is the position vector of a point trisecting A an C, and RHS that of B and D.

75. Again, it is given that the point P, Q and R are collinear.

$$\Rightarrow \qquad \mathbf{PQ} = \lambda\mathbf{QR}$$

$$\Rightarrow \qquad 15t(4\hat{\mathbf{j}} - 3\hat{\mathbf{i}}) = \lambda[(1-t)(\hat{\mathbf{i}} + \hat{\mathbf{j}}) - 60t\hat{\mathbf{j}}]$$

$$\Rightarrow \qquad = \lambda[(1-t)\hat{\mathbf{i}} + (1-61t)\hat{\mathbf{j}}]$$

$$\Rightarrow \qquad \frac{45t}{t-1} = \frac{60t}{1-61t}$$

$$\Rightarrow \qquad \frac{3t}{t-1} = \frac{4t}{1-61t}$$

$$\Rightarrow \qquad 3(1-61t) = 4(t-1)$$

$$\Rightarrow \qquad 3 - 183t = 4t - 4 \Rightarrow 187t = 7$$

$$\therefore \qquad t = \frac{7}{187}$$

76. We have, $\mathbf{a} + \mathbf{b} + \mathbf{c} = \alpha\mathbf{d}$

and $\mathbf{b} + \mathbf{c} + \mathbf{d} = \beta\mathbf{a}$

$\therefore$ $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} = (\alpha+1)\mathbf{d}$

and $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} = (\beta+1)\mathbf{a}$

$\Rightarrow$ $(\alpha+1)\mathbf{d} = (\beta+1)\mathbf{a}$

If $\alpha \neq -1$, then $(\alpha+1)\mathbf{d} = (\beta+1)\mathbf{a}$

$$\Rightarrow \qquad \mathbf{d} = \frac{\beta+1}{\alpha+1}\mathbf{a}$$

$$\Rightarrow \qquad \mathbf{a} + \mathbf{b} + \mathbf{c} = \alpha\mathbf{d}$$

$$\Rightarrow \qquad \mathbf{a} + \mathbf{b} + \mathbf{c} = \alpha\left(\frac{\beta+1}{\alpha+1}\right)\mathbf{a}$$

$$\Rightarrow \qquad \left[1 - \frac{\alpha(\beta+1)}{\alpha+1}\right]\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$$

$\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are coplanar which is contradiction to the given condition.

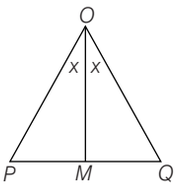
$\therefore$ $\alpha = -1$

and so $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} = 0$

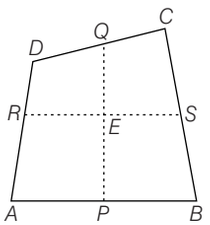
77. Since $|\mathbf{OP}| = |\mathbf{OQ}| = \sqrt{14}$, ΔOPQ is an isosceles.

Hence, the internal bisector OM is perpendicular to PQ and M is the mid-point of P and Q. Therefore,

$$\mathbf{OM} = \frac{1}{2}(\mathbf{OP} + \mathbf{OQ}) = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$$



78.



Let $\mathbf{OA} = \mathbf{a}, \mathbf{OB} = \mathbf{b}, \mathbf{OC} = \mathbf{c}$

and $\mathbf{OD} = \mathbf{d}$

Therefore, $\mathbf{OA} + \mathbf{OB} + \mathbf{OC} + \mathbf{OD}$

$= \mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}$

P, the mid-point of $\mathbf{AB}$, is $\dfrac{\mathbf{a}+\mathbf{b}}{2}$.

Q, the mid-point of $\mathbf{CD}$, is $\dfrac{\mathbf{c}+\mathbf{d}}{2}$.

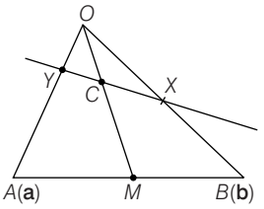
Therefore, the mid-point of $\mathbf{PQ}$ is $\dfrac{\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}}{4}$.

Similarly, the mid-point of $\mathbf{RS}$ is $\dfrac{\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}}{4}$

i.e. $\mathbf{OE} = \dfrac{\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}}{4}$

$\Rightarrow$ $x = 4$

79.



$\mathbf{OA} = \mathbf{a}$, $\mathbf{OB} = \mathbf{b}$

$\therefore \quad \mathbf{OM} = \dfrac{\mathbf{a}+\mathbf{b}}{2}$

$\therefore \quad \mathbf{OC} = \dfrac{\mathbf{a}+\mathbf{b}}{6}$

$\mathbf{OX} = \dfrac{2}{3}\mathbf{b}$

Let $\dfrac{OY}{YA} = \lambda$

$\therefore \quad \mathbf{OY} = \dfrac{\lambda}{\lambda+1}\mathbf{a}$

Now points Y, C and X are collinear.

$\therefore \quad \mathbf{YC} = m\mathbf{CX}$

$\therefore \quad \dfrac{\mathbf{a}+\mathbf{b}}{6} - \dfrac{\lambda}{\lambda+1}\mathbf{a} = m\dfrac{2\mathbf{b}}{3} - m\dfrac{\mathbf{a}+\mathbf{b}}{6}$

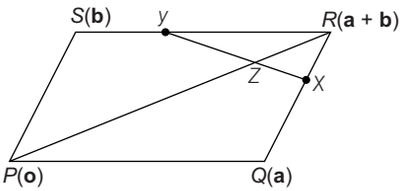
Comparing coefficients of $\mathbf{a}$ and $\mathbf{b}$

$\therefore \quad \dfrac{1}{6} - \dfrac{\lambda}{\lambda+1} = -\dfrac{m}{6}$

and $\dfrac{1}{6} = \dfrac{2m}{3} - \dfrac{m}{6}$

$\therefore \quad m = \dfrac{1}{3}$ and $\lambda = \dfrac{2}{7}$

80. $\dfrac{4\left(\dfrac{5\mathbf{a}+4\mathbf{b}}{3}\right)+\dfrac{\mathbf{a}+5\mathbf{b}}{5}}{4+1} = \dfrac{21(\mathbf{a}+\mathbf{b})}{25} = \dfrac{21}{25}\mathbf{PR}$



PV of X is $\dfrac{4(\mathbf{a}+\mathbf{b})+\mathbf{a}}{5} = \dfrac{5\mathbf{a}+4\mathbf{b}}{5}$

PV of Y is $\dfrac{4\mathbf{b}+\mathbf{a}+\mathbf{b}}{5} = \dfrac{\mathbf{a}+5\mathbf{b}}{5}$

Now, $\mathbf{PZ} = m\mathbf{PR}$

$\mathbf{PZ} = m(\mathbf{a}+\mathbf{b})$

Let Z divided YX in the ratio $\lambda : 1$

PV of $Z = \dfrac{\lambda\mathbf{OX}+\mathbf{OY}}{\lambda+1}$

$\therefore \quad \mathbf{PZ} = \dfrac{\left(\dfrac{5\mathbf{a}+4\mathbf{b}}{5}\right)+\dfrac{\mathbf{a}+5\mathbf{b}}{5}}{\lambda+1} = m(\mathbf{a}+\mathbf{b})$

Comparing coefficients of $\mathbf{a}$ and $\mathbf{b}$

$m = \dfrac{5\lambda+1}{5(\lambda+1)}$

and $m = \dfrac{4\lambda+5}{5(\lambda+1)}$

$\therefore \quad \lambda = 4$

$\therefore \quad \mathbf{PZ} = \dfrac{4\left(\dfrac{5\mathbf{a}+4\mathbf{b}}{5}\right)+\dfrac{\mathbf{a}+5\mathbf{b}}{5}}{4+1}$

$= \dfrac{21(\mathbf{a}+\mathbf{b})}{25} = \dfrac{21}{25}\mathbf{PR}$

81. Let $\mathbf{OA} = \mathbf{a}, \mathbf{OB} = \mathbf{b}$ and $\mathbf{OC} = \mathbf{c}$,

then $\mathbf{AB} = \mathbf{b} - \mathbf{a}$ and $\mathbf{OP} = \dfrac{1}{3}\mathbf{a}$.

$OQ = \dfrac{1}{2}\mathbf{b}$, $\mathbf{OR} = \dfrac{1}{3}\mathbf{c}$.

Since P, Q, R and S are coplanar, then

$\mathbf{PS} = \alpha\mathbf{PQ} + \beta\mathbf{PR}$

($\mathbf{PS}$ can be written as a linear combination of $\mathbf{PQ}$ and $\mathbf{PR}$)

$= \alpha(\mathbf{OQ} - \mathbf{OP}) + \beta(\mathbf{OR} - \mathbf{OP})$

i.e. $\mathbf{OS} - \mathbf{OP} = -(\alpha+\beta)\dfrac{\mathbf{a}}{3} + \dfrac{\alpha}{2}\mathbf{b} + \dfrac{\beta}{3}\mathbf{c}$

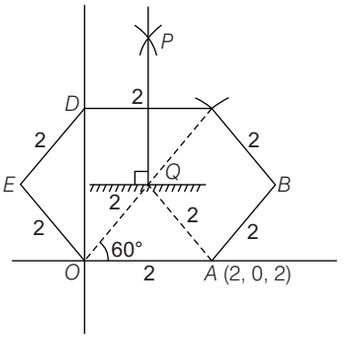
$\Rightarrow \quad \mathbf{OS} = (1-\alpha-\beta)\dfrac{\mathbf{a}}{3} + \dfrac{\alpha}{2}\mathbf{b} + \dfrac{\beta}{3}\mathbf{c}$ ... (i)

Given $\mathbf{OS} = \lambda\mathbf{AB} = \lambda(\mathbf{b}-\mathbf{a})$ ...(ii)

From Eq. (i) and Eq. (ii), $\beta = 0$, $\dfrac{1-\alpha}{3} = -\lambda$ and $\dfrac{\alpha}{2} = \lambda$

$\Rightarrow \quad 2\lambda = 1 + 3\lambda$ or $\lambda = -1$

82.



Here, coordinate of Q are $(2\cos 60°, 2\sin 60°)$

$\Rightarrow \quad Q(1, \sqrt{3}, 0)$

$\therefore \quad P(1, \sqrt{3}, z)$

$OP = 3$

$\Rightarrow \quad \sqrt{1+3+z^2} = 3$ or $z^2 = 5$

$z = \sqrt{5}$

$\therefore \quad P(1, \sqrt{3}, \sqrt{5}) \Rightarrow \mathbf{OP} = \hat{\mathbf{i}} + \sqrt{3}\hat{\mathbf{j}} + \sqrt{5}\hat{\mathbf{k}}$

Now, $\mathbf{AP} = \mathbf{OP} - \mathbf{OA} = \hat{\mathbf{i}} + \sqrt{3}\hat{\mathbf{j}} + \sqrt{5}\hat{\mathbf{k}} - 2\hat{\mathbf{i}}$

$= -\hat{\mathbf{i}} + \sqrt{3}\hat{\mathbf{j}} + \sqrt{5}\hat{\mathbf{k}}$

83. $\mathbf{a} = [\pm(\hat{\mathbf{i}} - \hat{\mathbf{j}}) \pm (\hat{\mathbf{j}} + \hat{\mathbf{k}})]$

$= \pm(\hat{\mathbf{i}} + \hat{\mathbf{k}}), \pm(\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - \hat{\mathbf{k}})$

84. Let $\mathbf{R}$ be the resultant. Then

$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = (p+1)\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$

Given, $|\mathbf{R}| = 5$. Therefore, $(p+1)^2 + 16 = 25$

or $p + 1 = \pm 3$ or $p = 2, -4$

85. We have, $\mathbf{AB} = -\hat{\mathbf{i}} - \hat{\mathbf{j}} - 4\hat{\mathbf{k}}, \mathbf{BC} = -3\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$

and $\mathbf{CA} = 4\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$.

Therefore, $|\mathbf{AB}| = |\mathbf{BC}| = 3\sqrt{2}$ and $|\mathbf{CA}| = 6$

Clearly, $|\mathbf{AB}|^2+|\mathbf{BC}|^2=|\mathbf{AC}|^2$

Hence, the triangle is right angled isosceles triangle.

86. Let $\mathbf{a}=2\hat{\mathbf{i}}+4\hat{\mathbf{j}}-5\hat{\mathbf{k}}$ and $\mathbf{b}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}}$.

Then, the diagonals of the parallelogram are

$\mathbf{p}=\mathbf{a}+\mathbf{b}$

and $\mathbf{q}=\mathbf{b}-\mathbf{a}$,

i.e. $\mathbf{p}=3\hat{\mathbf{i}}+6\hat{\mathbf{j}}-2\hat{\mathbf{k}}$, $\mathbf{q}=-\hat{\mathbf{i}}-2\hat{\mathbf{j}}+8\hat{\mathbf{k}}$

So, unit vectors along the diagonals are

$$\frac{1}{7}(3\hat{\mathbf{i}}+6\hat{\mathbf{j}}-2\hat{\mathbf{k}}) \text{ and } \frac{1}{\sqrt{69}}(-\hat{\mathbf{i}}-2\hat{\mathbf{j}}+8\hat{\mathbf{k}})$$

87. $\mathbf{OA}=-4\hat{\mathbf{i}}+3\hat{\mathbf{k}}$; $\mathbf{OB}=14\hat{\mathbf{i}}+2\hat{\mathbf{j}}-5\hat{\mathbf{k}}$

$$\mathbf{a}=\frac{-4\hat{\mathbf{i}}+3\hat{\mathbf{k}}}{5};\ \mathbf{b}=\frac{14\hat{\mathbf{i}}+2\hat{\mathbf{j}}-5\hat{\mathbf{k}}}{15}$$

$$\mathbf{r}=\frac{\lambda}{15}[-12\hat{\mathbf{i}}+9\hat{\mathbf{j}}+14\hat{\mathbf{i}}+2\hat{\mathbf{j}}-5\hat{\mathbf{k}}]$$

$$=\frac{\lambda}{15}[2\hat{\mathbf{i}}+2\hat{\mathbf{j}}+4\hat{\mathbf{k}}]=\frac{2\lambda}{15}[\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}}]$$

88. Points $A(\hat{\mathbf{i}}+\hat{\mathbf{j}})$, $B(\hat{\mathbf{i}}-\hat{\mathbf{j}})$ and $C(p\hat{\mathbf{i}}+q\hat{\mathbf{j}}+r\hat{\mathbf{k}})$ are collinear

Now, $\mathbf{AB}=-2\hat{\mathbf{j}}$

and $\mathbf{BC}=(p-1)\hat{\mathbf{i}}+(q+1)\hat{\mathbf{j}}+r\hat{\mathbf{k}}$

Vectors **AB** and **BC** must be collinear

$\Rightarrow$ $p=1$, $r=0$ and $q\neq -1$

89. For coplanar vectors, $\begin{vmatrix}1 & 2 & 3\\ 0 & \lambda & \mu\\ 0 & 0 & 2\lambda-1\end{vmatrix}=0$

or $(2\lambda-1)\lambda=0$ or $\lambda=0,\frac{1}{2}$

90. In ΔABC, $\mathbf{AB}+\mathbf{BC}=\mathbf{AC}=-\mathbf{CA}$

or $\mathbf{AB}+\mathbf{BC}+\mathbf{CA}=\mathbf{0}$

$\mathbf{OA}+\mathbf{AB}=\mathbf{OB}$ is the triangle law of addition.

Hence, Statement 1 is true and Statement 2 is is false.

91. $\frac{1}{2}=\frac{p}{3}=\frac{2}{q}\Rightarrow p=\frac{3}{2}$ and $q=4$

92. $2\mathbf{a}+3\mathbf{b}-5\mathbf{c}=0$

$3(\mathbf{b}-\mathbf{a})=5(\mathbf{c}-\mathbf{a})$

$\Rightarrow$ $\mathbf{AB}=\frac{5}{3}\mathbf{AC}$

Hence, **AB** and **AC** must be parallel since there is a common point A. The points A, B and C must be collinear.

Solutions (Q.Nos. 93-94)

Let the position vectors of A, B, C and D be $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$, respectively. Then,

$OA:CB=2:1$

$\Rightarrow$ $\mathbf{OA}=2\mathbf{CB}\Rightarrow \mathbf{a}=2(\mathbf{b}-\mathbf{c})$ ... (i)

and $OD:AB=1:3$

$3\mathbf{OD}=\mathbf{AB}$

$\Rightarrow$ $3\mathbf{d}=(\mathbf{b}-\mathbf{a})=\mathbf{b}-2(\mathbf{b}-\mathbf{c})$ [using Eq. (i)]

$=-\mathbf{b}+2\mathbf{c}$ ... (ii)

Let $OX:XC=\lambda:1$ and $AX:XD=\mu:1$

Now, X divides OC in the ratio $\lambda:1$. Therefore,

$$\text{PV of } X=\frac{\lambda\mathbf{c}}{\lambda+1} \quad \text{...(iii)}$$

X also divided AD in the ratio $\mu:1$. Therefore,

$$\text{PV of } X=\frac{\mu\mathbf{d}+\mathbf{a}}{\mu+1} \quad \text{...(iv)}$$

From Eqs. (iii) and (iv), we get

$$\frac{\lambda\mathbf{c}}{\lambda+1}=\frac{\mu\mathbf{d}+\mathbf{a}}{\mu+1}$$

or $$\left(\frac{\lambda}{\lambda+1}\right)\mathbf{c}=\left(\frac{\mu}{\mu+1}\right)\mathbf{d}+\left(\frac{1}{\mu+1}\right)\mathbf{a}$$

or $$\left(\frac{\lambda}{\lambda+1}\right)\mathbf{c}=\left(\frac{\mu}{\mu+1}\right)\left(\frac{-\mathbf{b}+2\mathbf{c}}{3}\right)+\left(\frac{1}{\mu+1}\right)2(\mathbf{b}-\mathbf{c})$$

[using Eqs. (i) and (ii)]

or $$\left(\frac{\lambda}{\lambda+1}\right)\mathbf{c}=\left(\frac{6-\mu}{3(\mu+1)}\right)\mathbf{b}+\left(\frac{2\mu}{3(\mu+1)}-\frac{2}{\mu+1}\right)\mathbf{c}$$

or $$\left(\frac{\lambda}{\lambda+1}\right)\mathbf{c}=\left(\frac{6-\mu}{3(\mu+1)}\right)\mathbf{b}+\left(\frac{2\mu-6}{3(\mu+1)}\right)\mathbf{c}$$

or $$\left(\frac{6-\mu}{3(\mu+1)}\right)\mathbf{b}+\left(\frac{2\mu-6}{3(\mu+1)}-\frac{\lambda}{\lambda+1}\right)\mathbf{c}=0$$

or $$\frac{6-\mu}{3(\mu+1)}=0$$

and $$\frac{2\mu-6}{3(\mu+1)}-\frac{\lambda}{\lambda+1}=0$$

(as **b** and **c** are non-collinear)

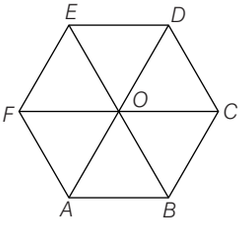
or $\mu=6$, $\lambda=\frac{2}{5}$

Hence, $OX:XC=2:5$ and $AX/XD=\frac{\mu}{1}=\frac{6}{1}$

93. (c) **94.** (b)

Solutions (Q.Nos. 95-96)

Consider the regular hexagon $ABCDEF$ with centre at O (origin).



$\mathbf{AD}+\mathbf{EB}+\mathbf{FC}=2\mathbf{AO}+2\mathbf{OB}+2\mathbf{OC}$

$=2(\mathbf{AO}+\mathbf{OB})+2\mathbf{OC}$

$=2\mathbf{AB}+2\mathbf{AB}$ [$\because$ $\mathbf{OC}=\mathbf{AB}$]

$=4\mathbf{AB}$

$\mathbf{R}=\mathbf{AB}+\mathbf{AC}+\mathbf{AD}+\mathbf{AE}+\mathbf{AF}$

$=\mathbf{ED}+\mathbf{AC}+\mathbf{AD}+\mathbf{AE}+\mathbf{CD}$

[$\because$ $\mathbf{AB}=\mathbf{ED}$ and $\mathbf{AF}=\mathbf{CD}$]

$=(\mathbf{AC}+\mathbf{CD})+(\mathbf{AE}+\mathbf{ED})+\mathbf{AD}$

$=\mathbf{AD}+\mathbf{AD}+\mathbf{AD}=3\mathbf{AD}=6\mathbf{AO}$

95. (c) **96.** (d)

Solutions (Q.Nos. 97-99)

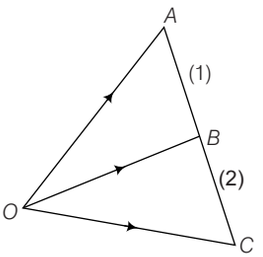
97. $\mathbf{AB} = \mathbf{OB} - \mathbf{OA} = 3\mathbf{a} - \mathbf{b} - 2\mathbf{c}$

$\mathbf{AC} = \mathbf{OC} - \mathbf{OA} = 9\mathbf{a} - 3\mathbf{b} - 6\mathbf{c} = 3\mathbf{AB}$

98. $2\mathbf{OA} - 3\mathbf{OB} + \mathbf{OC}$

$= 2(-2\mathbf{a} + 3\mathbf{b} + 5\mathbf{c}) - 3(\mathbf{a} + 2\mathbf{b} + 3\mathbf{c}) + (7\mathbf{a} - \mathbf{c}) = \mathbf{0}$

99. $\because \quad 2\mathbf{OA} - 3\mathbf{OB} + \mathbf{OC} = \mathbf{0}$



$\Rightarrow \qquad \mathbf{OB} = \dfrac{2\mathbf{OA} + \mathbf{OC}}{3}$

$\Rightarrow$ B Divides AC in 1 : 2.

Solutions (Q.Nos. 100-101)

100. Here, $\mathbf{c} = t(\hat{\mathbf{a}} + \hat{\mathbf{b}}) = t\left(\dfrac{7\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - 4\hat{\mathbf{k}}}{9} + \dfrac{(-2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}}{3}\right)$

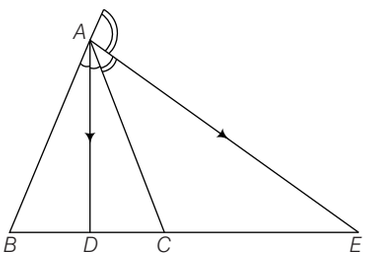
$\Rightarrow \quad \mathbf{c} = t\left(\dfrac{\hat{\mathbf{i}} - 7\hat{\mathbf{j}} + 2\hat{\mathbf{k}}}{9}\right)$

Also, $\qquad |\mathbf{c}| = 5\sqrt{6} \Rightarrow \dfrac{t}{9}.\sqrt{1 + 49 + 4} = 5\sqrt{6}$

$\therefore \qquad t = 15 \Rightarrow \mathbf{c} = \dfrac{15}{9}(\hat{\mathbf{i}} - 7\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

or $\qquad = \dfrac{5}{3}(\hat{\mathbf{i}} - 7\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

101. Here, $\dfrac{AB}{AC} = \dfrac{BD}{DC}$ and $\dfrac{AB}{AC} = \dfrac{BE}{CE}$



$\Rightarrow \qquad \dfrac{BD}{DC} = \dfrac{BE}{CE} \Rightarrow \dfrac{BD}{BC - BD} = \dfrac{BE}{BE - BC}$

$\Rightarrow \quad BD \cdot BE - BD \cdot BC = BC \cdot BE - BD \cdot BE$

$\Rightarrow \qquad 2BD \cdot BE = (BD + BE) \cdot BC$

or $\qquad \dfrac{2}{BC} = \dfrac{1}{BD} + \dfrac{1}{BE}$

Solutions (Q.Nos. 102-103)

102. $r'(t) = -2\hat{\mathbf{i}} + 2t\hat{\mathbf{j}} + 4e^{2(t-1)}\hat{\mathbf{k}}$

Since, $r'(t)$ is parallel to $r(t)$,

so $\qquad r(t) = \alpha r'(t)$

$1 - 2t = -2\alpha, t^2 = \alpha t, 2e^{2(t-1)}$

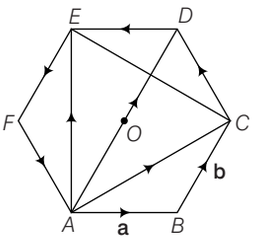
$= 4\alpha e^{2(t-1)}, \alpha = \dfrac{1}{2}$

The only value of t which satisfies all three equations is $t = 1$.
So, $r(1)$ is the required point $(-1, 1, 2)$.

103. $(2, 0, 5)$ corresponding to $r(1)$ and $r'(t) = 4t\hat{\mathbf{i}} - \hat{\mathbf{j}} + 6t\hat{\mathbf{k}}$

So, the required tangent vector is $r'(1) = 4\hat{\mathbf{i}} - \hat{\mathbf{j}} + 6\hat{\mathbf{k}}$.

104.



$\mathbf{AB} = \mathbf{a}, \mathbf{BC} = \mathbf{b}$

$\therefore \qquad \mathbf{AC} = \mathbf{AB} + \mathbf{BC} = \mathbf{a} + \mathbf{b}$ ... (i)

$\mathbf{AD} = 2\mathbf{BC} = 2\mathbf{b}$ ... (ii)

(because AD is parallel to BC and twice is length)

$\mathbf{CD} = \mathbf{AD} - \mathbf{AC} = 2\mathbf{b} - (\mathbf{a} + \mathbf{b}) = \mathbf{b} - \mathbf{a}$

$\mathbf{FA} = -\mathbf{CD} = \mathbf{a} - \mathbf{b}$ ... (iii)

$\mathbf{DE} = -\mathbf{AB} = -\mathbf{a}$ ... (iv)

$\mathbf{EF} = -\mathbf{BC} = -\mathbf{b}$ ... (v)

$\mathbf{AE} = \mathbf{AD} + \mathbf{DE} = 2\mathbf{b} - \mathbf{a}$ ... (vi)

$\mathbf{CE} = \mathbf{CD} + \mathbf{DE} = \mathbf{b} - \mathbf{a} - \mathbf{a} = \mathbf{b} - 2\mathbf{a}$ ... (vii)

105. Let $\mathbf{R}$ be the resultant. Then,

$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = (p + 1)\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$

Given, $|\mathbf{R}| = 5$, Therefore $R^2 = 25$

$\because \quad (p + 1)^2 + 16 = 25 \quad \text{or} \quad p + 1 = \pm 3 \quad \text{or} \quad p = 2, -4$

106. Vectors along to sides are $\mathbf{a} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{b} = 2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + \hat{\mathbf{k}}$

Clearly the vector along the longer diagonal is

$\mathbf{a} + \mathbf{b} = 3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

Hence, length of the longer diagonal is

$|\mathbf{a} + \mathbf{b}| = |3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + 2\hat{\mathbf{k}}| = 7$

107. Vector $\mathbf{a} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$, $\mathbf{b} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{c} = \lambda\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ are coplanar.

$\Rightarrow \quad \begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ \lambda & 1 & 2 \end{vmatrix} = 0 \quad \text{or} \quad \lambda - 3 + 2(-5) = 0 \quad \text{or} \quad \lambda = 13$

Number of digits in value of λ is 2.

108. Since, angle bisector of $\mathbf{a}$ and $\mathbf{b}$

$\Rightarrow \qquad h(\hat{\mathbf{a}} + \hat{\mathbf{b}}) = h\left(\dfrac{\mathbf{a}}{|\mathbf{a}|} + \dfrac{\mathbf{b}}{|\mathbf{b}|}\right)$ ... (i)

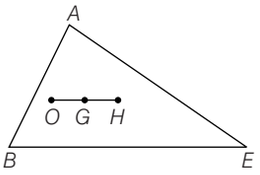
given, $\mathbf{a} + \mathbf{b}$ is along angle bisector.

$\Rightarrow \qquad \mu\left(\dfrac{\mathbf{a}}{|\mathbf{a}|} + \dfrac{\mathbf{b}}{|\mathbf{b}|}\right) = \mathbf{a} + \mathbf{b}$

only, when $|\mathbf{a}| = |\mathbf{b}| = \mu$

$\therefore \qquad |\mathbf{a}| = |\mathbf{b}| \Rightarrow \lambda = 1$

109.



Here, O is circum centre $= \mathbf{0}$, G is centroid $= \mathbf{g}$

H is orthocentre $= \mathbf{p}$

Since, $\dfrac{\mathbf{OG}}{\mathbf{GH}} = \dfrac{1}{2}$

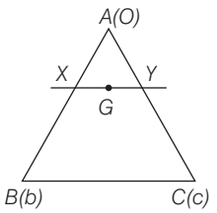
$\Rightarrow \quad \dfrac{\mathbf{g}}{\mathbf{p}-\mathbf{g}} = \dfrac{1}{2} \Rightarrow 2\mathbf{g} = \mathbf{p} - \mathbf{g}$

or $\mathbf{p} = 3\mathbf{g}$

$\therefore \quad k = 3$

110. $\mathbf{XG} = k\mathbf{GY}$

$$\frac{\mathbf{b}+\mathbf{c}}{3} - \frac{2\mathbf{b}}{3} = k\left(\frac{\lambda\mathbf{c}}{1+\lambda} - \frac{\mathbf{b}+\mathbf{c}}{3}\right), k = 1$$



$$\frac{2\mathbf{c}}{3} = \frac{\lambda}{1+\lambda}\,\mathbf{c}$$

$\Rightarrow \quad \dfrac{\lambda}{1+\lambda} = \dfrac{2}{3}$

$\Rightarrow \quad \lambda = 2$

111. We have, $\mathbf{a} = 2p\hat{\mathbf{i}} + \hat{\mathbf{j}}$

Let $\mathbf{b}$ be the vector obtained from $\mathbf{a}$ by rotating the axes. Then, the components of $\mathbf{b}$ are $p+1$ and 1. Therefore,

$\mathbf{b} = (p+1)\,\hat{\alpha} + \hat{\beta}$

where $\hat{\alpha}$ and $\hat{\beta}$ are unit vectors along the new axes.

But $|\mathbf{b}| = |\mathbf{a}|$

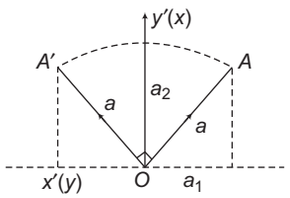
$\Rightarrow \quad 4p^2 + 1 = (p+1)^2 + 1$

$\Rightarrow \quad 3p^2 - 2p - 1 = 0 \Rightarrow p = 1, -\dfrac{1}{3}$

$\Rightarrow \quad p_1 = 1$ and $p_2 = -\dfrac{1}{3}$

$\therefore \quad 3\,|p_1 + p_2| = 3\left|1 - \dfrac{1}{3}\right| = 2$

112. Here, $\mathbf{a}$ is rotated about Z-axis, the Z-component of $\mathbf{a}$ will remain unchanged namely a_3

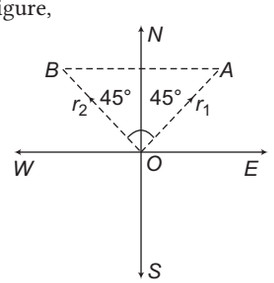


Now, if it is turned through an angle $\dfrac{\pi}{2}$. As shown in adjoining figure.

$\therefore$ Now components are $(a_2, -a_1, a_3)$.

113. Here, $\mathbf{r}_1 = \mathbf{OA}$ pointing North-East and $\mathbf{r}_2 = \mathbf{OB}$ pointing North-West. Where $|\mathbf{OA}| = |\mathbf{OB}| = 5$.

As shown in figure,



$\therefore \quad \angle BOA = 90°$

$\Rightarrow \quad \mathbf{r}_1 - \mathbf{r}_2 = \mathbf{BA}$ (using triangle law)

Clearly, $\angle BOA$ is right angled at O.

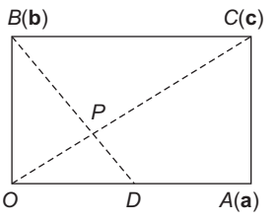
$\therefore \quad BA^2 = OA^2 + OB^2 = 5^2 + 5^2 = 50$

$\Rightarrow \quad |BA| = 5\sqrt{5}$

or $|\mathbf{r}_1 - \mathbf{r}_2| = 5\sqrt{5}$

i.e. $\mathbf{r}_1 - \mathbf{r}_2$ has magnitude $5\sqrt{5}$ and points from West to East.

114. Let $\mathbf{OACB}$ be a parallelogram shown as



Here, $\mathbf{OD} = \dfrac{1}{2}\mathbf{BC}$

$\Rightarrow \quad \mathbf{OP} + \mathbf{PD} = \dfrac{1}{2}(\mathbf{BP} + \mathbf{PC})$ [using Δ law]

$\Rightarrow \quad 2\mathbf{OP} + 2\mathbf{PD} = \mathbf{BP} + \mathbf{PC}$

$-2\mathbf{PO} + 2\mathbf{PD} = -\mathbf{PB} + \mathbf{PC}$

$\Rightarrow \quad \mathbf{PB} + 2\mathbf{PD} = \mathbf{PC} + 2\mathbf{PO}$

$\Rightarrow \quad \dfrac{\mathbf{PB} + 2\mathbf{PD}}{1+2} = \dfrac{\mathbf{PC} + 2\mathbf{PO}}{1+2}$

The common point P of BD and CO divides each in the ratio 2 : 1.

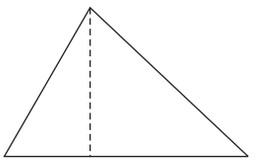
115. Let S be the point of intersection of $\mathbf{AB}$ and $\mathbf{CR}$. Let A be the origin and the position vectors of the points B, C, P, Q, R and S be $\mathbf{b}, \mathbf{c}, \mathbf{p}, \mathbf{q}, \mathbf{r}$ and $\mathbf{s}$ respectively.

$\therefore \quad \mathbf{p} = \dfrac{3\mathbf{b} + 2\mathbf{c}}{5}$

and $\mathbf{q} = \dfrac{4\mathbf{c}}{5}$ ...(i)

$\Rightarrow \quad \dfrac{5\mathbf{p} - 3\mathbf{b}}{2} = \dfrac{5\mathbf{q}}{4} \Rightarrow 10\mathbf{p} - 6\mathbf{b} = 5\mathbf{q}$

i.e. $10\mathbf{p} = 5\mathbf{q} + 6\mathbf{b} \Rightarrow \frac{10\mathbf{p}}{11} = \frac{5\mathbf{q} + 6\mathbf{b}}{11} = \mathbf{r}$



$\Rightarrow \frac{11\mathbf{r}}{10} = \mathbf{p} = \frac{3\mathbf{b} + 2\mathbf{c}}{5}$ [using Eq. (ii)]

$11\mathbf{r} = 6\mathbf{b} + 4\mathbf{c}$

$11\mathbf{r} - 4\mathbf{c} = 6\mathbf{b}$

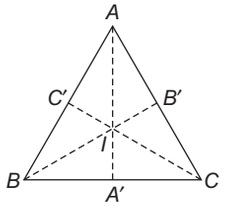
$\frac{11\mathbf{r} - 4\mathbf{c}}{7} = \frac{6}{7}\mathbf{b} = \mathbf{s}$, thus s divides AB in the ratio $6 : 1$.

116. Since, angle bisectors divides opposite side in the ratio of sides containing the angle.

$\Rightarrow BA' = \frac{ac}{b+c}$ and $CA' = \frac{ab}{a+c}$

Now, BI is also angle bisector of $\angle B$ for $\Delta ABA'$.

$\Rightarrow \frac{AI}{AI'} = \frac{b+c}{a} \Rightarrow \frac{AI}{AA'} = \frac{b+c}{a+b+c}$



Similarly, $\frac{BI}{BB'} = \frac{a+c}{a+b+c}$

and $\frac{CI}{CC'} = \frac{a+b}{a+b+c}$

$\Rightarrow \frac{\mathbf{AI} \cdot \mathbf{BI} \cdot \mathbf{CI}}{\mathbf{AA'} \cdot \mathbf{BB'} \cdot \mathbf{CC'}} = \frac{(b+c)(a+c)(a+b)}{(a+b+c)(a+b+c)(a+b+c)}$ ...(i)

As we know $\mathbf{AM} \geq \mathbf{GM}$, we get

$$\frac{\frac{b+c}{a+b+c} + \frac{c+a}{a+b+c} + \frac{a+b}{a+b+c}}{3} \geq \left[\frac{(a+b)(b+c)(c+a)}{(a+b+c)^3}\right]^{\frac{1}{3}}$$

$\Rightarrow \frac{2(a+b+c)}{3(a+b+c)} \geq \frac{[(a+b)(b+c)(c+a)]^{1/3}}{a+b+c}$

$\Rightarrow \frac{(a+b)(b+c)(c+a)}{(a+b+c)} \leq \frac{8}{27}$ ...(ii)

From Eqs. (i) and (ii), we get

$$\frac{AI \cdot BI \cdot CI}{AA' \cdot BB' \cdot CC'} \leq \frac{8}{27}$$

117. Let the position vector of O' with reference to O as the origin be α.

Then, $\mathbf{OO'} = \alpha$

Now, $\mathbf{O'}P_i$ = Position vector

or P_i – Position vector of $O' = \mathbf{r}_i - \alpha$

$i = 1, 2, \ldots\ldots, n$

Let the position vectors of P_1, P_2, P_3,....., P_n with respect to O' as the origin be $\mathbf{R}_1, \mathbf{R}_2$....., $\mathbf{R}_n$ respectively. Then, $\mathbf{R}_i = \mathbf{O'}P_i = \mathbf{r}_i - \alpha$, $i = 1, 2, \ldots\ldots, n$ [using Eq. (i)]

Now, $a_1\mathbf{R}_1 + a_2\mathbf{R}_2 + \ldots\ldots + a_n\mathbf{R}_n = 0$

$\Rightarrow \sum_{i=1}^{n} a_i\mathbf{R}_i = 0 \Rightarrow \sum_{i=1}^{n} \mathbf{a}_i(\mathbf{r}_i - \alpha) = 0$

$\Rightarrow \sum_{i=1}^{n} a_i\mathbf{r}_i - \sum_{i=1}^{n} a_i\,\alpha = 0$

$\Rightarrow 0 - \alpha\left(\sum_{i=1}^{n} a_i\right) = 0 \quad \left[\because \sum_{i=1}^{n} a_i\mathbf{r}_i = 0 \text{ (given)}\right]$

$\Rightarrow \sum_{i=1}^{n} a_i = 0$

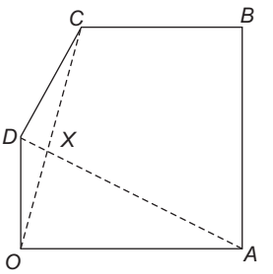
Thus, $\sum_{i=1}^{n} a_i\mathrm{R}_i = 0$ will hold good, if $\sum_{i=1}^{n} a_i = 0$.

118. Let O be the origin of reference.

Let the position vectors of A, B, C and D be $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$, respectively.

Then, $OA : CB = 2 : 1$

$\Rightarrow \frac{OA}{CB} = \frac{2}{1} \Rightarrow OA + 2CB$



$\Rightarrow \mathbf{OA} = 2\mathbf{CB}$

$\Rightarrow \mathbf{a} = 2(\mathbf{b} - \mathbf{c})$ ...(i)

and $OD : AB = 1 : 3$

$\Rightarrow \frac{OD}{AB} = \frac{1}{3} \Rightarrow 3OD = AB$

$\Rightarrow 3\mathbf{OD} = \mathbf{AB}$

$\Rightarrow 3\mathbf{d} = (\mathbf{b} - \mathbf{a}) = \mathbf{b} - 2(\mathbf{b} - \mathbf{c})$ [using Eq. (i)]

$\Rightarrow 3\mathbf{d} = -\mathbf{b} + 2\mathbf{c}$ ...(ii)

Let $OX : XC = \lambda : 1$ and $AX : XD = \mu : 1$

Now, X divides OC in the ratio $\lambda : 1$. Therefore,

$$\text{PV of } X = \frac{\lambda c}{\lambda + 1} \quad \text{...(iii)}$$

X also divides AD in the ratio $\mu : 1$

$$\text{PV of } X = \frac{\mu\mathbf{d} + \mathbf{a}}{\mu + 1}$$

From Eqs. (iii) and (iv), we get

$$\frac{\lambda c}{\lambda + 1} = \frac{\mu\mathbf{d} + \mathbf{a}}{\mu + 1}$$

$\Rightarrow \left(\frac{\lambda}{\lambda + 1}\right)\mathbf{c} = \left(\frac{\mu}{\mu + 1}\right)\mathbf{d} + \left(\frac{1}{\mu + 1}\right)\mathbf{a}$

$$\Rightarrow \quad \left(\frac{\lambda}{\lambda+1}\right)\mathbf{c} = \left(\frac{\mu}{\mu+1}\right)\left(\frac{-\mathbf{b}+2\mathbf{c}}{3}\right) + \left(\frac{1}{\mu+1}\right)2(\mathbf{b}-\mathbf{c})$$

[using Eqs. (i) and (iv)]

$$\Rightarrow \quad \left(\frac{\lambda}{\lambda+1}\right)\mathbf{c} = \left(\frac{2}{\mu+1} - \frac{\mu}{3(\mu+1)}\right)\mathbf{b} + \left(\frac{2\mu}{3(\mu+1)} - \frac{2}{\mu+1}\right)\mathbf{c}$$

$$\Rightarrow \quad \left(\frac{\lambda}{\lambda+1}\right)\mathbf{c} = \left(\frac{6-\mu}{3(\mu+1)}\right)\mathbf{b} + \left(\frac{2\mu-6}{3(\mu+1)}\right)\mathbf{c}$$

$$\Rightarrow \quad \left(\frac{6-\mu}{3(\mu+1)}\right)\mathbf{b} + \left(\frac{2\mu-6}{3(\mu+1)} - \frac{\lambda}{\lambda+1}\right)\mathbf{c} = 0$$

$$\Rightarrow \quad \frac{6-\mu}{3(\mu+1)} = 0 \text{ and } \frac{2\mu-6}{3(\mu+1)} - \frac{\lambda}{\lambda+1} = 0$$

(since, $\mathbf{b}$ and $\mathbf{c}$ are non-collinear)

$$\Rightarrow \quad \mu = 6 \text{ and } \lambda = \frac{2}{5}$$

Hence, $OX : XC = 2 : 5$

119. let l, m and n be scalars such that

$$l\mathbf{p} + m\mathbf{q} + n\mathbf{r} = 0$$

$$\Rightarrow \{l(\cos a)\mathbf{u} + (\cos b)\mathbf{v} + (\cos c)\mathbf{w}\} + m\{(\sin a)\mathbf{u} + (\sin b)\mathbf{v} + (\sin c)\mathbf{w}\} + n\{\sin(x+a)\mathbf{u} + \sin(x+b)\mathbf{v} + \sin(x+c)\mathbf{w}\} = 0$$

$$\Rightarrow \{l\cos a + m\sin a + n\sin(x+a)\}\mathbf{u} + \{l\sin b + m\sin(x+b)\}\mathbf{v} + \{l\cos c + m\sin c + n\sin(x+c)\}\mathbf{w} = 0$$

$$\Rightarrow \quad l\cos a + m\sin a + n\sin(x+a) = 0 \quad \ldots(i)$$

$$l\cos b + m\sin b + n\sin(x+b) = 0 \quad \ldots(ii)$$

$$l\cos c + m\sin c + n\sin(x+c) = 0 \quad \ldots(iii)$$

This is a homogeneous system of linear equations in l, m and n. The determinant of the coefficient matrix is

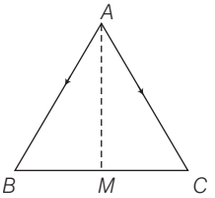
$$\Delta = \begin{vmatrix} \cos a & \sin a & \sin(x+a) \\ \cos b & \sin b & \sin(x+b) \\ \cos c & \sin c & \sin(x+c) \end{vmatrix} = \begin{vmatrix} \cos a & \sin a & 0 \\ \cos b & \sin b & 0 \\ \cos c & \sin c & 0 \end{vmatrix} = 0$$

(using $C_3 \to C_3 - \sin x\, C_1 - \cos x\, C_2$)

$\Rightarrow$ So, the above system of equations has non-trivial solutions also. This means that l, m and n may attain non-zero values also.

Hence, the given system of vectors is a linearly dependent system of vectors.

120. We know that, the sum of three vectors of a triangle is zero.



$$\therefore \quad \mathbf{AB} + \mathbf{BC} + \mathbf{CA} = 0$$

$$\Rightarrow \quad \mathbf{BC} = \mathbf{AC} - \mathbf{AB} \qquad [\because \mathbf{AC} = -\mathbf{CA}]$$

$$\Rightarrow \quad \mathbf{AB} = \frac{\mathbf{AC} - \mathbf{AB}}{2} \qquad [\because M \text{ is a mid-point of } BC]$$

Also, $\mathbf{AB} + \mathbf{BM} + \mathbf{MA} = 0$

[by properties of a triangle]

$$\Rightarrow \quad \mathbf{AB} + \frac{\mathbf{AC} - \mathbf{AB}}{2} = \mathbf{AM} \qquad [\because \mathbf{AM} = -\mathbf{MA}]$$

$$\Rightarrow \quad \mathbf{AM} = \frac{\mathbf{AB} + \mathbf{AC}}{2}$$

$$= \frac{3\,\hat{\mathbf{i}} + 4\hat{\mathbf{k}} + 5\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}}{2}$$

$$= 4\,\hat{\mathbf{i}} - \hat{\mathbf{j}} + 4\hat{\mathbf{k}}$$

$$\Rightarrow \quad |\mathbf{AM}| = \sqrt{4^2 + 1^2 + 4^2} = \sqrt{33}$$

121. As, $\mathbf{a} + 3\mathbf{b}$ is collinear with $\mathbf{c}$.

$$\therefore \quad \mathbf{a} + 3\mathbf{b} = \lambda\mathbf{c} \quad \ldots(i)$$

Also, $\mathbf{b} + 2\mathbf{c}$ is collinear with $\mathbf{a}$.

$$\Rightarrow \quad \mathbf{b} + 2\mathbf{c} = \mu\mathbf{a} \quad \ldots(ii)$$

From Eq. (ii), we get

$$\mathbf{a} + 3\mathbf{b} + 6\mathbf{c} = (\lambda + 6)\mathbf{c} \quad \ldots(iii)$$

From Eq. (ii), we get

$$\mathbf{a} + 3\mathbf{b} + 6\mathbf{c} = (1 + 3\mu)\mathbf{a} \quad \ldots(iv)$$

From Eqs. (iii) and (iv), we get

$$\therefore \quad (\lambda + 6)\mathbf{c} = (1 + 3\mu)\mathbf{a}$$

Since, $\mathbf{a}$ is not collinear with $\mathbf{c}$.

$$\Rightarrow \quad \lambda + 6 = 1 + 3\mu = 0$$

From Eq. (iv), we get

$$\mathbf{a} + 3\mathbf{b} + 6\mathbf{c} = 0$$

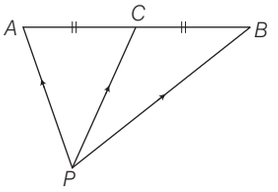
122. Since, $\mathbf{a} = 8\,\mathbf{b}$ and $\mathbf{c} = -7\mathbf{b}$

So, $\mathbf{a}$ is parallel to $\mathbf{b}$ and $\mathbf{c}$ is anti-parallel to $\mathbf{b}$.

$\Rightarrow$ $\mathbf{a}$ and $\mathbf{c}$ are anti-parallel.

So, the angle between $\mathbf{a}$ and $\mathbf{c}$ is π.

123. Let P be the origin outside of AB and C is mid-point of AB, then



$$\mathbf{PC} = \frac{\mathbf{PA} + \mathbf{PB}}{2} \Rightarrow 2\mathbf{PC} = \mathbf{PA} + \mathbf{PB}$$

124. If $\mathbf{a} + 2\mathbf{b}$ is collinear with $\mathbf{c}$, then $\mathbf{a} + 2\mathbf{b} = t\mathbf{c}$ $\ldots(i)$

Also, $\mathbf{b} + 3\mathbf{c}$ is collinear with $\mathbf{a}$, then

$$\mathbf{b} + 3\mathbf{c} = \lambda\mathbf{a}$$

$$\Rightarrow \quad \mathbf{b} = \lambda\mathbf{a} - 3\mathbf{c} \quad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$\mathbf{a} + 2(\lambda\mathbf{a} - 3\mathbf{c}) = t\mathbf{c}$$

$$\Rightarrow \quad (\mathbf{a} - 6\mathbf{c}) = t\mathbf{c} - 2\lambda a$$

On comparing the coefficients of $\mathbf{a}$ and $\mathbf{c}$, we get

$$1 = -2\lambda \Rightarrow \lambda = -\frac{1}{2}$$

and $\quad -6 = t \Rightarrow t = -6$

From Eq. (i), we get

$$\mathbf{a} + 2\mathbf{b} = -6\mathbf{c}$$

$$\Rightarrow \quad \mathbf{a} + 2\mathbf{b} + 6\mathbf{c} = 0$$

125. The three vectors $(\mathbf{a} + 2\mathbf{b} + 3\mathbf{c})$, $(\lambda\mathbf{b} + 4\mathbf{c})$ and $(2\lambda - 1)\mathbf{c}$ are non-coplanar, if

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & \lambda & 4 \\ 0 & 0 & 2\lambda - 1 \end{vmatrix} \neq 0$$

$$\Rightarrow \quad (2\lambda - 1)(\lambda) \neq 0$$

$$\Rightarrow \quad \lambda \neq 0, \frac{1}{2}$$

So, these three vectors are non-coplanar for all except two values of λ.

126. Given that, $\mathbf{OA} = 7\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$

$$\mathbf{OB} = \hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 10\hat{\mathbf{k}}$$

$$\mathbf{OC} = -\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$$

$$\mathbf{OD} = 5\hat{\mathbf{i}} - \hat{\mathbf{j}} + 5\hat{\mathbf{k}}$$

Now, $AB = \sqrt{(7-1)^2 + (-4+6)^2 + (7-10)^2}$

$$= \sqrt{36 + 4 + 9}$$

$$= \sqrt{49} = 7$$

$$BC = \sqrt{(1+1)^2 + (-6+3)^2 + (10-4)^2}$$

$$= \sqrt{4 + 9 + 36}$$

$$= \sqrt{49} = 7$$

$$CD = \sqrt{(-1-5)^2 + (-3+1)^2 + (4-5)^2}$$

$$= \sqrt{36 + 4 + 1}$$

$$= \sqrt{41}$$

and $DA = \sqrt{(5-7)^2 + (-1+4)^2 + (5-7)^2}$

$$= \sqrt{4 + 9 + 4}$$

$$= \sqrt{17}$$

Hence, option (d) is correct.

127. Since, $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = 0$

$$\Rightarrow \quad \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = abc \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad (1 + abc) \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = 0 \qquad \left[\because \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} \neq 0 \right]$$

$$\Rightarrow \quad 1 + abc = 0$$

$$\Rightarrow \quad abc = -1$$

128. Since, the vector $\hat{\mathbf{i}} + x\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ is doubled in magnitude, then it becomes

$$4\hat{\mathbf{i}} + (4x-2)\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$$

$$\therefore \quad 2\,|\hat{\mathbf{i}} + x\hat{\mathbf{j}} + 3\hat{\mathbf{k}}\,| = 4\hat{\mathbf{i}} + (4x-2)\hat{\mathbf{j}} + 2\mathbf{k}\,|$$

$$\Rightarrow \quad 2\sqrt{1 + x^2 + 9} = \sqrt{16 + (4x-2)^2 + 4}$$

$$\Rightarrow \quad 40 + 4x^2 = 20 + (4x-2)^2$$

$$\Rightarrow \quad 3x^2 - 4x - 4 = 0$$

$$\Rightarrow \quad (x-2)(3x+2) = 0$$

$$\Rightarrow \quad x = 2, -\frac{2}{3}$$

CHAPTER

02

Product of Vectors

Learning Part

Session 1

- Product of Two Vectors
- Components of a Vector Along and Perpendicular to Another Vector
- Application of Dot Product in Mechanics

Session 2

- Vector or Cross Product of Two Vectors
- Area of Parallelogram and Triangle
- Moment of a Force and Couple
- Rotation About an Axis

Session 3

- Scalar Triple Product

Session 4

- Vector Triple Product

Practice Part

- JEE Type Examples
- Chapter Exercises

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Session 1

Product of Two Vectors, Components of a Vector Along and Perpendicular to Another Vector, Application of Dot Product in Mechanics

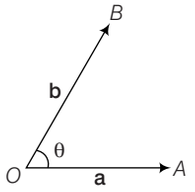
Product of Two Vectors

Product of two vectors is processed by two methods. When the product of two vectors results in a scalar quantity, then it is called **scalar product**. It is also known as **dot product** because we denote it by putting a **dot** (.) between two vectors.

When the product of two vectors results in a vector quantity, then this product is called **vector product**. It is also known as **cross product** because we denote it by putting a **cross** ($\times$) between two vectors.

Scalar or Dot Product of Two Vectors

If $\mathbf{a}$ and $\mathbf{b}$ are two non-zero vectors and θ be the angle between them, then their scalar product (or dot product) is denoted by $\mathbf{a} \cdot \mathbf{b}$ and is defined as the scalar $|\mathbf{a}||\mathbf{b}|\cos\theta$, where $|\mathbf{a}|$ and $|\mathbf{b}|$ are modulii of a and b respectively and $0 \le \theta \le \pi$.

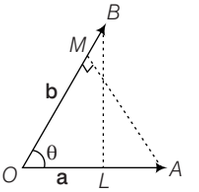


Remarks

1. $\mathbf{a} \cdot \mathbf{b} \in R$
2. $\mathbf{a} \cdot \mathbf{b} \le |\mathbf{a}|\,|\mathbf{b}|$
3. $\mathbf{a} \cdot \mathbf{b} > 0 \Rightarrow$ Angle between $\mathbf{a}$ and $\mathbf{b}$ is acute.
4. $\mathbf{a} \cdot \mathbf{b} < 0 \Rightarrow$ Angle between $\mathbf{a}$ and $\mathbf{b}$ is obtuse.
5. The dot product of a zero and non-zero vector is a scalar zero.

Geometrical interpretation of scalar product

Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors represented by **OA** and **OB** respectively. Let θ be the angle between **OA** and **OB**. Draw $BL \perp OA$ and $AM \perp OB$.



From triangles OBL and OAM, we have $OL = OB\cos\theta$ and $OM = OA\cos\theta$. Here, OL and OA are known as projection of $\mathbf{b}$ on $\mathbf{a}$ and $\mathbf{a}$ on $\mathbf{b}$ respectively.

Now, $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$

$= |\mathbf{a}|\,(OB\cos\theta) = |\mathbf{a}|\,(OL)$

= (Magnitude of $\mathbf{a}$) (Projection of $\mathbf{b}$ on $\mathbf{a}$) ...(i)

Again, $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$

$= |\mathbf{b}|(|\mathbf{a}|\cos\theta) = |\mathbf{b}|\,(OA\cos\theta) = |\mathbf{b}|\,(OM)$

$\mathbf{a} \cdot \mathbf{b}$ = (Magnitude of $\mathbf{b}$)

(Projection of $\mathbf{a}$ on $\mathbf{b}$) ...(ii)

Thus, geometrically interpreted, **the scalar product of two vectors is the product of modulus of either vector and the projection of the other in its direction.**

Remarks

1. Projection of $\mathbf{a}$ on $\mathbf{b} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} = \mathbf{a} \cdot \mathbf{b}$
2. Projection of $\mathbf{b}$ on $\mathbf{a} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \mathbf{a} \cdot \mathbf{b}$
3. Angle between two vectors it $\mathbf{a}$ and $\mathbf{b}$ be two vectors inclined at an angle θ, then $\mathbf{a}, \mathbf{b} = |\mathbf{a}| \cdot |\mathbf{b}| \cos\theta$

$$\Rightarrow \quad \cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| \cdot |\mathbf{b}|}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| \cdot |\mathbf{b}|}\right)$$

If $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$

$$\theta = \cos^{-1}\left(\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}\right)$$

Properties of Scalar Product

(i) **Commutativity** The scalar product of two vector is commutative i.e., $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$.

(ii) **Distributivity of scalar product over vector addition** The scalar product of vectors is distributive over vector addition i.e.,

(a) $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$ (Left distributivity)

(b) $(\mathbf{b}+\mathbf{c})\cdot\mathbf{a}=\mathbf{b}\cdot\mathbf{a}+\mathbf{c}\cdot\mathbf{a}$ (right distributivity)

(iii) Let $\mathbf{a}$ and $\mathbf{b}$ be two non-zero vectors $\mathbf{a}\cdot\mathbf{b}=0\Leftrightarrow\mathbf{a}\perp\mathbf{b}$.

As $\hat{\mathbf{i}},\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are mutually perpendicular unit vectors along the coordinate axes, therefore

$\hat{\mathbf{i}}\cdot\hat{\mathbf{j}}=\hat{\mathbf{j}}\cdot\hat{\mathbf{i}}=0,\ \hat{\mathbf{j}}\cdot\hat{\mathbf{k}}=\hat{\mathbf{k}}\cdot\hat{\mathbf{j}}=0;\ \hat{\mathbf{k}}\cdot\hat{\mathbf{i}}=\hat{\mathbf{i}}\cdot\hat{\mathbf{k}}=0$

(iv) For any vector $\mathbf{a},\mathbf{a}\cdot\mathbf{a}=|\mathbf{a}|^2$.

As $\hat{\mathbf{i}},\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are unit vectors along the coordinate axes, therefore $\hat{\mathbf{i}}\cdot\hat{\mathbf{i}}=|\hat{\mathbf{i}}|^2=1,\ \hat{\mathbf{j}}\cdot\hat{\mathbf{j}}=|\hat{\mathbf{j}}|^2=1$ and $\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}=|\hat{\mathbf{k}}|^2=1$.

(v) If m is a scalar and $\mathbf{a}$ and $\mathbf{b}$ be any two vectors then $(m\mathbf{a})\cdot\mathbf{b}=m(\mathbf{a}\cdot\mathbf{b})=\mathbf{a}\cdot(m\mathbf{b})$.

(vi) If m and n are scalars and $\mathbf{a}$ and $\mathbf{b}$ be two vectors, then $(m\mathbf{a})\cdot n\mathbf{b}=mn(\mathbf{a}\cdot\mathbf{b})=(mn\mathbf{a})\cdot\mathbf{b}=\mathbf{a}\cdot(mn\mathbf{b})$

(vii) For any vectors $\mathbf{a}$ and $\mathbf{b}$, we have

(a) $\mathbf{a}\cdot(-\mathbf{b})=-(\mathbf{a}\cdot\mathbf{b})=(-\mathbf{a})\cdot\mathbf{b}$

(b) $(-\mathbf{a})\cdot(-\mathbf{b})=\mathbf{a}\cdot\mathbf{b}$

(viii) For any two vectors $\mathbf{a}$ and $\mathbf{b}$, we have

(a) $|\mathbf{a}+\mathbf{b}|^2=|\mathbf{a}|^2+|\mathbf{b}|^2+2\mathbf{a}\cdot\mathbf{b}$

(b) $|\mathbf{a}-\mathbf{b}|=|\mathbf{a}|^2+|\mathbf{b}|^2-2\mathbf{a}\cdot\mathbf{b}$

(c) $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})=|\mathbf{a}|^2-|\mathbf{b}|^2$

(d) $|\mathbf{a}+\mathbf{b}|=|\mathbf{a}|+|\mathbf{b}|\ \Rightarrow\ \mathbf{a}\,||\,\mathbf{b}$

(e) $|\mathbf{a}+\mathbf{b}|^2=|\mathbf{a}|^2+|\mathbf{b}|^2\ \Rightarrow\ \mathbf{a}\perp\mathbf{b}$

(f) $|\mathbf{a}+\mathbf{b}|=|\mathbf{a}-\mathbf{b}|\Rightarrow\mathbf{a}\perp\mathbf{b}$

Scalar Product in Terms of Components

If $\mathbf{a}=a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}+a_3\hat{\mathbf{k}}$ and $\mathbf{b}=b_1\hat{\mathbf{i}}+b_2\hat{\mathbf{j}}+b_3\hat{\mathbf{k}}$, then $\mathbf{a}\cdot\mathbf{b}=a_1b_1+a_2b_2+a_3b_3$.

Thus, scalar product of two vectors is equal to the sum of the products of their corresponding components. In particular, $\mathbf{a}\cdot\mathbf{a}=|\mathbf{a}|^2=a_1^2+a_2^2+a_3^2$.

Example 1. If θ is the angle between the vectors $\mathbf{a}=2\hat{\mathbf{i}}+2\hat{\mathbf{j}}-\hat{\mathbf{k}}$ and $\mathbf{b}=6\hat{\mathbf{i}}-3\hat{\mathbf{j}}+2\hat{\mathbf{k}}$, then

(a) $\cos\theta=\frac{4}{21}$ (b) $\cos\theta=\frac{3}{19}$

(c) $\cos\theta=\frac{2}{19}$ (d) $\cos\theta=\frac{5}{21}$

Sol. (a) Angle between $\mathbf{a}$ and $\mathbf{b}$ is given by

$$\cos\theta=\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}=\frac{(2\hat{\mathbf{i}}+2\hat{\mathbf{j}}-\hat{\mathbf{k}})\cdot(6\hat{\mathbf{i}}-3\hat{\mathbf{j}}+2\hat{\mathbf{k}})}{\sqrt{2^2+2^2+(-1)^2}\cdot\sqrt{6^2+(-3)^2+2^2}}=\frac{12-6-2}{3\cdot7}=\frac{4}{21}$$

Example 2. $(\mathbf{a}\cdot\hat{\mathbf{i}})\,\hat{\mathbf{i}}+(\mathbf{a}\cdot\hat{\mathbf{j}})\,\hat{\mathbf{j}}+(\mathbf{a}\cdot\hat{\mathbf{k}})\,\hat{\mathbf{k}}$ is equal to

(a) $\mathbf{a}$ (b) $2\mathbf{a}$

(c) $3\mathbf{a}$ (d) $\mathbf{0}$

Sol. (a) Let $\mathbf{a}=a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}+a_3\hat{\mathbf{k}}$

$\therefore\quad \mathbf{a}\cdot\hat{\mathbf{i}}=(a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}+a_3\hat{\mathbf{k}})\,\hat{\mathbf{i}}=a_1,$

$\mathbf{a}\cdot\hat{\mathbf{j}}=a_2,\mathbf{a}\cdot\hat{\mathbf{k}}=a_3$

So, $(\mathbf{a}\cdot\hat{\mathbf{i}})\hat{\mathbf{i}}+(\mathbf{a}\cdot\hat{\mathbf{j}})\,\hat{\mathbf{j}}+(\mathbf{a}\cdot\hat{\mathbf{k}})\,\hat{\mathbf{k}}$

$=a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}+a_3\hat{\mathbf{k}}=\mathbf{a}$

Example 3. If $|\mathbf{a}|=3,|\mathbf{b}|=4$, then a value of λ for which $\mathbf{a}+\lambda\mathbf{b}$ is perpendicular to $\mathbf{a}-\lambda\mathbf{b}$.

(a) 9 / 16 (b) 3 / 4

(c) 3 / 2 (d) 4 / 3

Sol. (b) $\mathbf{a}+\lambda\mathbf{b}$ is perpendicular to $\mathbf{a}-\lambda\mathbf{b}$.

$\therefore\quad (\mathbf{a}+\lambda\mathbf{b})\cdot(\mathbf{a}-\lambda\mathbf{b})=0$

$\Rightarrow\quad |\mathbf{a}^2|-\lambda(\mathbf{a}\cdot\mathbf{b})+\lambda(\mathbf{b}\cdot\mathbf{a})-\lambda^2|\mathbf{b}|^2=0$

$\Rightarrow\quad |\mathbf{a}|^2-\lambda^2|\mathbf{b}|^2=0$

$\Rightarrow\quad \lambda=\pm\frac{|\mathbf{a}|}{|\mathbf{b}|}=\pm\frac{3}{4}$

Example 4. The projection of $\mathbf{a}=2\hat{\mathbf{i}}+3\hat{\mathbf{j}}-2\hat{\mathbf{k}}$ on $\mathbf{b}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}}$ is

(a) $\frac{1}{\sqrt{14}}$ (b) $\frac{2}{\sqrt{14}}$

(c) $\sqrt{14}$ (d) $\frac{-2}{\sqrt{14}}$

Sol. (b) Projection of $\mathbf{a}$ on $\mathbf{b}$

$$=\mathbf{a}\cdot\frac{\mathbf{b}}{|\mathbf{b}|}=\frac{(2\hat{\mathbf{i}}+3\hat{\mathbf{j}}-2\hat{\mathbf{k}})\cdot(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}})}{|\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}}|}=\frac{2+6-6}{\sqrt{14}}=\frac{2}{\sqrt{14}}$$

Example 5. If $\mathbf{a}=5\hat{\mathbf{i}}-\hat{\mathbf{j}}+7\hat{\mathbf{k}}$ and $\mathbf{b}=\hat{\mathbf{i}}-\hat{\mathbf{j}}+\lambda\hat{\mathbf{k}}$, then find λ such that $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$ are orthogonal.

Sol. Clearly, $\mathbf{a}+\mathbf{b}=(5\hat{\mathbf{i}}-\hat{\mathbf{j}}+7\hat{\mathbf{k}})+(\hat{\mathbf{i}}-\hat{\mathbf{j}}+\lambda\hat{\mathbf{k}})$

$=6\hat{\mathbf{i}}-2\hat{\mathbf{j}}+(7+\lambda)\hat{\mathbf{k}}$

and $\mathbf{a}-\mathbf{b}=(5\hat{\mathbf{i}}-\hat{\mathbf{j}}+7\hat{\mathbf{k}})-(\hat{\mathbf{i}}-\hat{\mathbf{j}}+\lambda\hat{\mathbf{k}})$

$=4\hat{\mathbf{i}}+(7-\lambda)\hat{\mathbf{k}}$

Since, $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$ are orthogonal

$\therefore \quad (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})=0$

$\Rightarrow \quad [6\hat{\mathbf{i}}-2\hat{\mathbf{j}}+(7+\lambda)\hat{\mathbf{k}}](4\hat{\mathbf{i}}+(7-\lambda)\mathbf{k})=0$

$\Rightarrow \quad 6\times 4-2\times 0+(7+\lambda)(7-\lambda)=0$

$\Rightarrow \quad 24+49-\lambda^2=0$

$\Rightarrow \quad \lambda^2=73$

$\Rightarrow \quad \lambda=\pm\sqrt{73}$

Example 6. If $\mathbf{a},\mathbf{b}$ and $\mathbf{c}$ are unit vectors such that $\mathbf{a}+\mathbf{b}+\mathbf{c}=0$, then find the value of $\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}$.

Sol. Consider, $\mathbf{a}+\mathbf{b}+\mathbf{c}=0$

On squaring both sides, we get

$(\mathbf{a}+\mathbf{b}+\mathbf{c})^2=0^2$

$\Rightarrow \quad (\mathbf{a}+\mathbf{b}+\mathbf{c})\cdot(\mathbf{a}+\mathbf{b}+\mathbf{c})=0\cdot 0$

$\Rightarrow \quad |\mathbf{a}|^2+|\mathbf{b}|^2+|\mathbf{c}|^2+2\mathbf{a}\cdot\mathbf{b}+2\mathbf{b}\cdot\mathbf{c}+2\mathbf{c}\cdot\mathbf{a}=0$

$\Rightarrow \quad 1+1+1+2(\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a})=0$

$\Rightarrow \quad \mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}=-\frac{3}{2}$

Example 7. If $\mathbf{a},\mathbf{b}$ and $\mathbf{c}$ are mutually perpendicular vectors of equal magnitudes, then find the angle between the vectors $\mathbf{a}$ and $\mathbf{a}+\mathbf{b}+\mathbf{c}$.

Sol. Let θ be the angle between the vectors $\mathbf{a}$ and $\mathbf{a}+\mathbf{b}+\mathbf{c}$.

Then, $\cos\theta=\dfrac{\mathbf{a}\cdot(\mathbf{a}+\mathbf{b}+\mathbf{c})}{|\mathbf{a}|\,|\mathbf{a}+\mathbf{b}+\mathbf{c}}$

$=\dfrac{\mathbf{a}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}}{|\mathbf{a}|\,|\mathbf{a}+\mathbf{b}+\mathbf{c}|}$

$=\dfrac{|\mathbf{a}|^2}{|\mathbf{a}|\,|\mathbf{a}+\mathbf{b}+\mathbf{c}|}$

$[\because \mathbf{a}\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{c}=0$ as $\mathbf{a},\mathbf{b},\mathbf{c}$ are mutually perpendicular vectors$]$

$=\dfrac{|\mathbf{a}|}{|\mathbf{a}+\mathbf{b}+\mathbf{c}|}$ ...(i)

Now consider,

$|\mathbf{a}+\mathbf{b}+\mathbf{c}|^2=(\mathbf{a}+\mathbf{b}+\mathbf{c})\cdot(\mathbf{a}+\mathbf{b}+\mathbf{c})$

$=|\mathbf{a}|^2+|\mathbf{b}|^2+|\mathbf{c}|^2+2(\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a})$

$=3|\mathbf{a}|^2+2(0)=3|\mathbf{a}|^2$

$[\because |\mathbf{a}|=|\mathbf{b}|=|\mathbf{c}|$ and $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{c}=\mathbf{c}\cdot\mathbf{a}=0]$

$|\mathbf{a}+\mathbf{b}+\mathbf{c}|=\sqrt{3}\,|\mathbf{a}|$

From Eq. (i), we get

$\cos\theta=\dfrac{1}{\sqrt{3}}$

$\Rightarrow \quad \theta=\cos^{-1}\dfrac{1}{\sqrt{3}}$

Example 8. If the vectors $\mathbf{a}=(c\log_2 x)\,\hat{\mathbf{i}}-6\hat{\mathbf{j}}+3\hat{\mathbf{k}}$ and $\mathbf{b}=(\log_2 x)\,\hat{\mathbf{i}}+2\hat{\mathbf{j}}+(2c\log_2 x)\,\hat{\mathbf{k}}$ make an obtuse angle for any $x\in(0,\infty)$. Then, determine the interval to which c belongs.

Sol. For the vectors $\mathbf{a}$ and $\mathbf{b}$ to be inclined at an obtuse angle, we must have

$\mathbf{a}\cdot\mathbf{b}<0,\ \forall\ x\in(0,\infty)$

$\Rightarrow \quad c(\log_2 x)^2-12+6c\log_2 x<0,\ \forall\ x\in(0,\infty)$

$\Rightarrow \quad cy^2+6cy-12<0,\forall\ y\in R$, where $y=\log_2 x$

$\Rightarrow \quad c<0$ and $36c^2+48c<0$,

(using $ax^2+bx+c<0$, $\forall\ x\in R$ iff $a<0$ and $D<0$)

$\Rightarrow \quad c<0$ and $c(3c+4)<0\ \Rightarrow c\in\left(-\frac{4}{3},0\right)$

Example 9. If $a+2b+3c=4$, then find the least value of $a^2+b^2+c^2$.

Sol. Consider vectors $\mathbf{p}=a\hat{\mathbf{i}}+b\hat{\mathbf{j}}+c\hat{\mathbf{k}}$ and $\mathbf{q}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}}$

Now, $\cos\theta=\dfrac{a+2b+3c}{\sqrt{a^2+b^2+c^2}\,\sqrt{1^2+2^2+3^2}}$

$\Rightarrow \quad \cos^2\theta=\dfrac{(a+2b+3c)^2}{14\,(a^2+b^2+c^2)}\le 1 \qquad [\because \cos^2\theta\le 1]$

$\Rightarrow \quad a^2+b^2+c^2\ge\dfrac{(a+2b+3c)^2}{14}=\dfrac{16}{14}=\dfrac{8}{7}$

Hence, least value of $a^2+b^2+c^2$ is $\frac{8}{7}$.

Example 10. Find the unit vector which makes an angle of 45° with the vector $2\hat{\mathbf{i}}+2\hat{\mathbf{j}}-\hat{\mathbf{k}}$ and an angle of 60° with the vector $\hat{\mathbf{j}}-\hat{\mathbf{k}}$.

Sol. Let the unit vector be $\mathbf{c}=c_1\hat{\mathbf{i}}+c_2\hat{\mathbf{j}}+c_3\hat{\mathbf{k}}$ so that; it makes an angle of 45° with $2\hat{\mathbf{i}}+2\hat{\mathbf{j}}-\hat{\mathbf{k}}$.

$\Rightarrow \quad \dfrac{2c_1+2c_2-c_3}{3}=\cos 45^\circ \qquad (\because |\hat{\mathbf{c}}|=1)$

$\Rightarrow \quad 2c_1+2c_2-c_3=\dfrac{3}{\sqrt{2}}$ ...(i)

Also, it makes an angle of 60° with $\hat{\mathbf{j}}-\hat{\mathbf{k}}$.

$\Rightarrow \quad \dfrac{c_2-c_3}{\sqrt{2}}=\cos 60^\circ \quad (\because |\hat{\mathbf{j}}-\hat{\mathbf{k}}|=\sqrt{2}$ and $|\hat{\mathbf{c}}|=1)$

$\Rightarrow \quad c_2-c_3=\dfrac{\sqrt{2}}{2}$ ...(ii)

$\Rightarrow \quad c_1^2+c_2^2+c_3^2=1$ ...(iii)

(using $|\mathbf{c}|=|c_1\hat{\mathbf{i}}+c_2\hat{\mathbf{j}}+c_3\hat{\mathbf{k}}|=1$)

From Eq. (ii), $c_2=\dfrac{1}{\sqrt{2}}+c_3$ and from Eq. (i), $c_1=\dfrac{-c_3}{2}+\dfrac{1}{2\sqrt{2}}$

On substituting in Eq. (iii), we get

$$\frac{c_3^2}{4}+\frac{1}{2}+\frac{2c_3}{\sqrt{2}}+\frac{1}{8}-\frac{c_3}{2\sqrt{2}}+c_3^2+c_3^2=1$$

$$\Rightarrow \quad \frac{9}{4}c_3^2+\frac{3c_3}{2\sqrt{2}}+\frac{5}{8}=1 \Rightarrow c_3=-\frac{1}{\sqrt{2}},\frac{1}{3\sqrt{2}}$$

Hence, the required vectors are $\left(\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{3\sqrt{2}},\frac{4}{3\sqrt{2}},\frac{1}{3\sqrt{2}}\right)$.

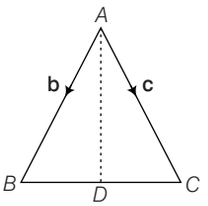
Example 11. Show that the median to the base of an isosceles triangle is perpendicular to base.

Sol. Let ABC be an isosceles triangle in which $AB = AC$.

Let A be the origin of reference and let

$$\mathbf{AB}=\mathbf{b},\mathbf{AC}=\mathbf{c}$$

Let D be the middle point of **BC**.



Then, $\mathbf{AD}=\dfrac{\mathbf{b}+\mathbf{c}}{2}$

Now, $\mathbf{BC}=\mathbf{c}-\mathbf{b}$

$$\therefore \quad \mathbf{AD}\cdot\mathbf{BC}=\left(\frac{\mathbf{b}+\mathbf{c}}{2}\right)\cdot(\mathbf{c}-\mathbf{b})$$

$$=\frac{1}{2}(|\mathbf{c}|^2-|\mathbf{b}|^2)=\frac{1}{2}(|\mathbf{AC}|^2-|\mathbf{AB}|^2)$$

$$=\frac{1}{2}(0)=0$$

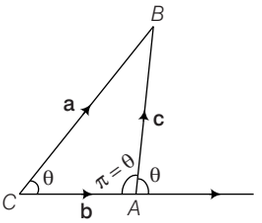
Hence, median to the base of an isosceles triangle is perpendicular to base.

Example 12. Using vector method, prove that in a triangle, $a^2=b^2+c^2-2bc\cos A$ (cosine law).

Sol. In a ΔABC,

Let $\mathbf{AB}=\mathbf{c},\mathbf{BC}=\mathbf{a},\mathbf{CA}=\mathbf{b}$

$\because \quad \mathbf{a}+\mathbf{b}+\mathbf{c}=0$



We have, $\mathbf{a}=-(\mathbf{b}+\mathbf{c})$

$$\therefore \quad |\mathbf{a}|=|-(\mathbf{b}+\mathbf{c})| \Rightarrow |\mathbf{a}|^2=|\mathbf{b}+\mathbf{c}|^2$$

$$\Rightarrow \quad |\mathbf{a}|^2=|\mathbf{b}|^2+|\mathbf{c}|^2+2\mathbf{b}\cdot\mathbf{c}$$

$$\Rightarrow \quad |\mathbf{a}^2|=|\mathbf{b}|^2+|\mathbf{c}|^2+2|\mathbf{b}||\mathbf{c}|\cos(\pi-A)|$$

Since, angle between **b** and **c** = The angle between CA produced and AB

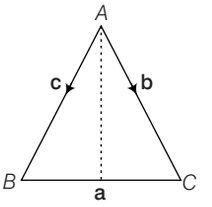
$$\therefore \quad a^2=b^2+c^2-2bc\cos A$$

Example 13. Using vector method, prove that in a triangle, $a=b\cos C+c\cos B$ (projection formula).

Sol. In a ΔABC,

Let $\mathbf{AB}=\mathbf{c},\mathbf{BC}=\mathbf{a},\mathbf{CA}=\mathbf{b}$

$\because \quad \mathbf{a}+\mathbf{b}+\mathbf{c}=0$



We have, $|\mathbf{a}|=|-(\mathbf{b}+\mathbf{c})|$

$$\therefore \quad \mathbf{a}\cdot\mathbf{a}=-(\mathbf{b}+\mathbf{c})\cdot\mathbf{a}$$

or $$|\mathbf{a}|^2=-\mathbf{b}\cdot\mathbf{a}-\mathbf{c}\cdot\mathbf{a}$$

$$=-|\mathbf{b}||\mathbf{a}|\cos(\pi-C)-|\mathbf{c}||\mathbf{a}|\cos(\pi-B)$$

Since, the angle between **b** and $\mathbf{a}=(\pi-C)$ and angle between **c** and $\mathbf{a}=(\pi-\mathbf{B})$

$$\therefore \quad a^2=ab\cos C+ac\cos B$$

$$\Rightarrow \quad a=b\cos C+c\cos B$$

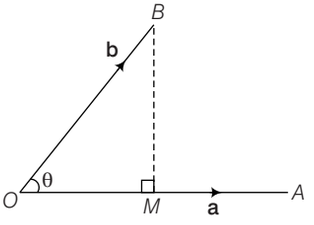
Components of a Vector Along and Perpendicular to Another Vector

Let **a** and **b** be two vectors represented by **OA** and **OB** and let θ be the angle between **a** and **b**. Draw $MB\perp OA$, shown as

$$\therefore \quad \mathbf{b}=\mathbf{OM}+\mathbf{MB}$$

$$\Rightarrow \quad \mathbf{OM}=(OM)\mathbf{a}=(OB\cos\theta)\,\mathbf{a}=(|\mathbf{b}|\cos\theta)\,\mathbf{a}$$

$$=\left(|\mathbf{b}|\frac{(\mathbf{a}\cdot\mathbf{b})}{|\mathbf{a}|^2}\right)\mathbf{a}$$



$$=\left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\mathbf{a}=\frac{\mathbf{a}\cdot\mathbf{b}\,\mathbf{a}}{|\mathbf{a}||\mathbf{a}|}=\left(\frac{\mathbf{a}-\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$$

$$\mathbf{b}=\mathbf{OM}+\mathbf{MB}$$

$$\mathbf{MB}=\mathbf{b}-\mathbf{OM}=\mathbf{b}-\left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$$

Thus, the components of $\mathbf{b}$ along and perpendicular to $\mathbf{a}$ are $\left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$ and $\mathbf{a}-\left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$ respectively.

Example 14. If $\mathbf{a}=4\hat{\mathbf{i}}+6\hat{\mathbf{j}}$ and $\mathbf{b}=3\hat{\mathbf{j}}+4\hat{\mathbf{k}}$, then find the component of $\mathbf{a}$ along $\mathbf{b}$.

Sol. The component of vector $\mathbf{a}$ along $\mathbf{b}$ is $\frac{(\mathbf{a}\cdot\mathbf{b})}{|\mathbf{b}|^2}$

$$=\frac{18}{25}(3\hat{\mathbf{j}}+4\hat{\mathbf{k}})$$

Example 15. Express the vector $\mathbf{a}=5\hat{\mathbf{i}}-2\hat{\mathbf{j}}+5\hat{\mathbf{k}}$ as sum of two vectors such that one is parallel to the vector $\mathbf{b}=3\hat{\mathbf{i}}+\hat{\mathbf{k}}$ and the other is perpendicular to $\mathbf{b}$.

Sol. Required vectors are

$$\left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b} \text{ and } \mathbf{a}-\left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b}$$

Clearly, $\left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b}=2(3\hat{\mathbf{i}}+\hat{\mathbf{k}})=6\hat{\mathbf{i}}+2\hat{\mathbf{k}}$ and so,

$$\mathbf{a}-\left(\frac{\mathbf{a}-\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b}=(5\hat{\mathbf{i}}-2\hat{\mathbf{j}}+5\hat{\mathbf{k}})-(6\hat{\mathbf{i}}+2\hat{\mathbf{k}})$$

$$=-\hat{\mathbf{i}}-2\hat{\mathbf{j}}+3\hat{\mathbf{k}}$$

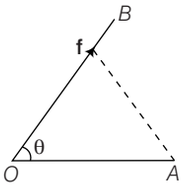
Note that $(6\hat{\mathbf{i}}+2\hat{\mathbf{k}})+(-\hat{\mathbf{i}}-2\hat{\mathbf{j}}+3\hat{\mathbf{k}})$

$$=5\hat{\mathbf{i}}-2\hat{\mathbf{j}}+5\hat{\mathbf{k}}=\mathbf{a}$$

Application of Dot Product in Mechanics (Work done)

A force acting on a particle is said to do work, if the particle is displaced in a direction which is not perpendicular to force.

Let a particle be placed at O and a force f represented by **OB** be acting on the particle at O. Due to the application of force $\mathbf{f}$, the particle is displaced in the direction of **OA**. Let **OA** be the displacement.



Then, the component of **OA** in the the direction of force $\mathbf{f}$ is, $|\mathbf{OA}|\cos\theta$

Work done $=|\mathbf{f}||\mathbf{OA}|\cos\theta=\mathbf{f}\cdot\mathbf{OA}$

$=\mathbf{f}\cdot\mathbf{d}$, where $\mathbf{d}=\mathbf{OA}$

$\therefore$ Work done = (Force) · (Displacement)

Remarks

1. The work done by the resultant of a number of forces $\mathbf{f}_1\mathbf{f}_2, \mathbf{f}_3, \ldots\mathbf{f}_n$ in a displacement $\mathbf{d}$ of a particle is equal to the sum of work done by the forces separately
   i.e. Work done $=\mathbf{f}_1\cdot\mathbf{d}+\mathbf{f}_2\cdot\mathbf{d}+\ldots.+\mathbf{f}_n\cdot\mathbf{d}$
   $=(\mathbf{f}_1+\mathbf{f}_2+\ldots+\mathbf{f}_n)\cdot\mathbf{d}$
   $=\mathbf{R}\cdot\mathbf{d}$ where, $\mathbf{R}=\mathbf{f}_1+\mathbf{f}_2+\ldots+\mathbf{f}_n$
2. The work done by a force $\mathbf{f}$ when its point of application experiences a number of consecutive displacements $\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\ldots,\mathbf{d}_n$, is equal to the work done by the forces in single displacement from the beginning to end.
   i.e., Work done $=\mathbf{f}\cdot(\mathbf{d}_1+\mathbf{d}_2+\ldots.+\mathbf{d}_n)$
   = The work done by the force $\mathbf{f}$ in the single displacement from the beginning to end

Example 16. Two forces $\mathbf{f}_1=3\hat{\mathbf{i}}-2\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $\mathbf{f}_2=\hat{\mathbf{i}}+3\hat{\mathbf{j}}-5\hat{\mathbf{k}}$ acting on a particle at A move it to B. Find the work done if the position vector of A and B are $-2\hat{\mathbf{i}}+5\hat{\mathbf{k}}$ and $3\hat{\mathbf{i}}-7\hat{\mathbf{j}}+2\hat{\mathbf{k}}$.

Sol. Let $\mathbf{R}$ be the resultant of two forces $\mathbf{f}_1$ and $\mathbf{f}_2$ and $\mathbf{d}$ be the displacement.

Then, $\mathbf{R}=(3\hat{\mathbf{i}}-2\hat{\mathbf{j}}+\hat{\mathbf{k}})+(\hat{\mathbf{i}}+3\hat{\mathbf{j}}-5\hat{\mathbf{k}})$

$=4\hat{\mathbf{i}}+\hat{\mathbf{j}}-4\hat{\mathbf{k}}$

and $\mathbf{d}=(3\hat{\mathbf{i}}-7\hat{\mathbf{j}}+2\hat{\mathbf{k}})-(-2\hat{\mathbf{i}}+5\hat{\mathbf{k}})=5\hat{\mathbf{i}}-7\hat{\mathbf{j}}-3\hat{\mathbf{k}}$

$\therefore$ The total work done = The work done by resultant

$=\mathbf{R}\cdot\mathbf{d}=(4\hat{\mathbf{i}}+\hat{\mathbf{j}}-4\hat{\mathbf{k}})\cdot(5\hat{\mathbf{i}}-7\hat{\mathbf{j}}-3\hat{\mathbf{k}})$

$=20-7+12=25$ units

Example 17. Forces of magnitudes 5 and 3 units acting in the directions $6\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}}$ and $3\hat{\mathbf{i}}-2\hat{\mathbf{j}}+6\hat{\mathbf{k}}$, respectively act on a particle which is displaced from the point $(2, 2, -1)$ and $(4, 3, 1)$. Find the work done by the forces.

Sol. Let $\mathbf{R}$ be the resultant of two forces and $\mathbf{d}$ be the displacement.

Then,

$$\mathbf{R}=5\frac{(6\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}})}{\sqrt{36+4+9}}+3\frac{(3\hat{\mathbf{i}}-2\hat{\mathbf{j}}+6\hat{\mathbf{k}})}{\sqrt{9+4+36}}$$

$$=\frac{1}{7}(39\hat{\mathbf{i}}+4\hat{\mathbf{j}}+33\hat{\mathbf{k}})$$

and $\mathbf{d}=(4\hat{\mathbf{i}}+3\hat{\mathbf{j}}+\hat{\mathbf{k}})-(2\hat{\mathbf{i}}+2\hat{\mathbf{j}}-\hat{\mathbf{k}})=2\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}}$

$\therefore$ Total work done $=\mathbf{R}\cdot\mathbf{d}=\frac{1}{7}(78+4+66)$

$$=\frac{148}{7}\text{ units}$$

Exercise for Session 1

1. Find the angle between the vectors $\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and $3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$.
2. Find and angle between two vectors $\mathbf{a}$ and $\mathbf{b}$ with magnitudes $\sqrt{3}$ and 2 respectively such that $\mathbf{a} \cdot \mathbf{b} = \sqrt{6}$
3. Show that the vectors $2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$ are at right angles.
4. If $\mathbf{r} \cdot \hat{\mathbf{i}} = \mathbf{r} \cdot \hat{\mathbf{j}} = \mathbf{r} \cdot \hat{\mathbf{k}}$ and $|\mathbf{r}| = 3$, then find vector $\mathbf{r}$.
5. Find the angle between the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$, if $\mathbf{a} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and $\mathbf{b} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$.
6. Find the projection of the vector $\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$ on the vector $7\hat{\mathbf{i}} - \hat{\mathbf{j}} + 8\hat{\mathbf{k}}$.
7. If the projection of vector $x\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ on vector $2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 5\hat{\mathbf{k}}$ is $\frac{1}{\sqrt{30}}$, then find the value of x.
8. If $|\mathbf{a}| + |\mathbf{b}| = |\mathbf{c}|$ and $\mathbf{a} + \mathbf{b} = \mathbf{c}$, then find the angle between $\mathbf{a}$ and $\mathbf{b}$.
9. If three unit vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ satisfy $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$, then find the angle between $\mathbf{a}$ and $\mathbf{b}$.
10. If $\mathbf{a} = x\hat{\mathbf{i}} + (x - 1)\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{b} = (x + 1)\hat{\mathbf{i}} + \hat{\mathbf{j}} + a\hat{\mathbf{k}}$ make an acute angle, $\forall\ x \in R$, then find the values of $\mathbf{a}$.
11. Find the component of $\hat{\mathbf{i}}$ in the direction of the vector $\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$.
12. Find the vector components of a vector $2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$ along and perpendicular to the non-zero vector $2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$.
13. A particle is acted upon by constant forces $4\hat{\mathbf{i}} + \hat{\mathbf{j}} - 3\hat{\mathbf{k}}$ and $3\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$ which displace it from a point $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ to the point $5\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + \hat{\mathbf{k}}$. Find the work done by the forces in standard units.

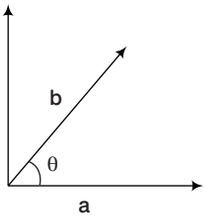
Session 2

Vector or Cross Product of Two Vectors, Area of Parallelogram and Triangle, Moment of a Force and Couple, Rotation About an Axis

Vector or Cross Product of Two Vectors

Let $\mathbf{a}$ and $\mathbf{b}$ be two non-zero, non- parallel vectors. Then the vector product $\mathbf{a} \times \mathbf{b}$, in that order, is defined as a vector whose magnitude is

$$|\mathbf{a}||\mathbf{b}|\sin\theta$$



where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, whose direction is perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$ in such a way that $\mathbf{a}$, $\mathbf{b}$ and this direction constitute a right handed system.

In other words, $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\, \boldsymbol{n}$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, $\hat{\mathbf{n}}$ is a unit vector perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}$, $\mathbf{b}$ and n form a right handed system.

Properties of Vector Product

(i) Vector product is not commutative i.e., if $\mathbf{a}$ and $\mathbf{b}$ are any two vectors, then $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$, however $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$.

(ii) Vector product is not associative, i.e. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$

(iii) If $\mathbf{a}$ and $\mathbf{b}$ are two vectors and m is a scalar, then

$$m\mathbf{a} \times \mathbf{b} = m(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times m\mathbf{b}$$

(iv) If $\mathbf{a}$ and $\mathbf{b}$ are two vectors and m and n are scalars, then $m\mathbf{a} \times n\mathbf{b} = mn(\mathbf{a} \times \mathbf{b}) = m(\mathbf{a} \times \mathbf{b}) = n(m\mathbf{a} \times \mathbf{b})$

(v) Distributivity of vector product over vector addition. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be any three vectors. Then,

(a) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ (left distributivity)

(b) $(\mathbf{b} + \mathbf{c}) \times \mathbf{a} = \mathbf{b} \times \mathbf{a} + \mathbf{c} \times \mathbf{a}$ (right distributivity)

(vi) For any three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, we have $\mathbf{a} \times (\mathbf{b} - \mathbf{c}) = \mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{c}$.

(vii) The vector product of any vector (zero or non-zero) with zero vector is a zero vector i.e.

$$\mathbf{a} \times \mathbf{0} = \mathbf{0} \times \mathbf{a} = \mathbf{0}$$

(viii) The vector product of two non-zero vectors is zero vector iff they are parallel (collinear) i.e. $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ $\Leftrightarrow \mathbf{a} \,||\, \mathbf{b}$, $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors.
It follows from the above property that $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for every non-zero vector $\mathbf{a}$, which in turn implies that $\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = \mathbf{0}$.

(ix) Vector product of orthonormal triad of unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ using the definition of the vector product obtain

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}},\ \hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}},\ \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}},$$
$$\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}},\ \hat{\mathbf{k}} \times \hat{\mathbf{j}} = -\hat{\mathbf{i}},\ \hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}$$

(x) **Lagrange's identity** If $\mathbf{a}$, $\mathbf{b}$ are any two vectors, then

$$|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$$

or $$|\mathbf{a} \times \mathbf{b}|^2 + (\mathbf{a} \cdot \mathbf{b})^2 = |\mathbf{a}|^2 |\mathbf{b}|^2$$

Vector Product in Terms of Components

If $\mathbf{a} = a_1\,\hat{\mathbf{i}} + a_2\,\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $\mathbf{b} = b_1\,\hat{\mathbf{i}} + b_2\,\hat{\mathbf{j}} + b_3\,\hat{\mathbf{k}}$

Then, $a \times b = (a_2b_3 - a_3b_2)\hat{\mathbf{i}} - (a_1b_3 - a_3b_1)\hat{\mathbf{j}}$

$$+(a_1b_2 - a_2b_1)\hat{\mathbf{k}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example 18. If $\mathbf{a} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$ and $\mathbf{b} = m\hat{\mathbf{i}} + n\hat{\mathbf{j}} + 12\hat{\mathbf{k}}$ and $\mathbf{a} \times \mathbf{b} = \mathbf{0}$. Then, find the values of m and n.

Sol. Clearly, $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 3 & -5 \\ m & n & 12 \end{vmatrix}$

$$= \hat{\mathbf{i}}(36 + 5n) - \hat{\mathbf{j}}(24 + 5m) + \hat{\mathbf{k}}(2n - 3m)$$

Since, $\mathbf{a} \times \mathbf{b} = \mathbf{0}$

$\therefore (36 + 5n)\,\hat{\mathbf{i}} - (24 + 5m)\,\hat{\mathbf{j}} + (2n - 3m)\,\hat{\mathbf{k}} = 0\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$

On comparing the coefficients of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$, we get

$$36 + 5n = 0,\ -(24 + 5m) = 0 \text{ and } 2n - 3m = 0$$

$$\Rightarrow \quad n = -\frac{36}{5} \text{ and } m = -\frac{24}{5}$$

Example 19. Show that $(\mathbf{a}-\mathbf{b})\times(\mathbf{a}+\mathbf{b})=2(\mathbf{a}\times\mathbf{b})$

Sol. Consider, $(\mathbf{a}-\mathbf{b})\times(\mathbf{a}+\mathbf{b})=(\mathbf{a}-\mathbf{b})\times\mathbf{a}+(\mathbf{a}-\mathbf{b})\times\mathbf{b}$

[By distributivity of vector product over vector addition]

$=\mathbf{a}\times\mathbf{a}-\mathbf{b}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}-\mathbf{b}\times\mathbf{b}$

[Again, by distributivity of vector product over vector addition]

$=0+\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{b}-0 \quad [\because \mathbf{a}\times\mathbf{b}=-(\mathbf{b}\times\mathbf{a})]$

$=2(\mathbf{a}\times\mathbf{b})$ **Hence Proved**

Example 20. ***If*** $\mathbf{a}$ ***is any vector, then*** $(\mathbf{a}\times\hat{\mathbf{i}})^2+(\mathbf{a}\times\hat{\mathbf{j}})^2+(\mathbf{a}\times\hat{\mathbf{k}})^2$ ***is equal to***

(a) $|\mathbf{a}|^2$ (b) 0

(c) $3|\mathbf{a}|^2$ (d) $2|\mathbf{a}|^2$

Sol. (d) Let $\mathbf{a}=a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}+a_3\hat{\mathbf{k}}$

$\therefore\ \mathbf{a}\times\hat{\mathbf{i}}=(a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}+a_3\hat{\mathbf{k}})\times\hat{\mathbf{i}}=-a_2\hat{\mathbf{k}}+a_3\hat{\mathbf{j}}$

$(\mathbf{a}\times\hat{\mathbf{i}})^2=(\mathbf{a}\times\hat{\mathbf{i}})\cdot(\mathbf{a}\times\hat{\mathbf{i}})$

$=(-a_2\hat{\mathbf{k}}+a_3\hat{\mathbf{j}})\cdot(-a_2\hat{\mathbf{k}}+a_3\hat{\mathbf{j}})=a_2^2+a_3^2$

Similarly, $(\mathbf{a}\times\hat{\mathbf{j}})^2=a_3^2+a_1^2$

and $(\mathbf{a}\times\hat{\mathbf{k}})^2=a_1^2+a_2^2$

$\therefore\ (\mathbf{a}\times\hat{\mathbf{i}})^2+(\mathbf{a}\times\hat{\mathbf{j}})^2+(\mathbf{a}\times\hat{\mathbf{k}})^2$

$=2(a_1^2+a_2^2+a_3^2)=2|\mathbf{a}|^2$

Example 21. If $\mathbf{a}\cdot\mathbf{b}=0$ and $\mathbf{a}\times\mathbf{b}=0$, prove that $\mathbf{a}=0$ or $\mathbf{b}=0$.

Sol. Given, $\mathbf{a}\cdot\mathbf{b}=0$ and $\mathbf{a}\times\mathbf{b}=0$

Now, $\mathbf{a}\cdot\mathbf{b}=0\Rightarrow\mathbf{a}=0$ or $\mathbf{b}=0$ or $\mathbf{a}\perp\mathbf{b}$

and $\mathbf{a}\times\mathbf{b}=0\Rightarrow\mathbf{a}=0$ or $\mathbf{b}=0$ or $\mathbf{a}\,||\,\mathbf{b}$

Since, $\mathbf{a}\perp\mathbf{b}$ and $\mathbf{a}\,||\,\mathbf{b}$ can never hold simultaneously.

$\therefore\quad \mathbf{a}\cdot\mathbf{b}=0$ and $\mathbf{a}\times\mathbf{b}=0$

$\Rightarrow\quad \mathbf{a}=0$ or $\mathbf{b}=0$

Example 22. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are vectors such that $\mathbf{a}\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{c}$, $\mathbf{a}\times\mathbf{b}=\mathbf{a}\times\mathbf{c}$ and $\mathbf{a}\neq0$, then show that $\mathbf{b}=\mathbf{c}$

Sol. $\mathbf{a}\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{c}$ and $\mathbf{a}\neq0$

$\Rightarrow\quad \mathbf{a}\cdot\mathbf{b}-\mathbf{a}\cdot\mathbf{c}=0$ and $\mathbf{a}\neq0$

$\Rightarrow\quad \mathbf{a}\cdot(\mathbf{b}-\mathbf{c})=0$ and $\mathbf{a}\neq0$

$\Rightarrow\quad \mathbf{a}\perp(\mathbf{b}-\mathbf{c})$ or $\mathbf{b}=\mathbf{c}$ ...(i)

Again, $\mathbf{a}\times\mathbf{b}=\mathbf{a}\times\mathbf{c}$ and $\mathbf{a}\neq0$

$\Rightarrow\quad \mathbf{a}\times(\mathbf{b}-\mathbf{c})=0$ and $\mathbf{a}\neq0$

$\Rightarrow\quad \mathbf{a}\,||\,(\mathbf{b}-\mathbf{c})=0$ and $\mathbf{b}=\mathbf{c}$ ...(ii)

$\therefore$ From Eqs. (i) and (ii), we have

$\mathbf{b}=\mathbf{c}$

[as $\mathbf{a}$ cannot be both parallel and perpendicular to $(\mathbf{b}-\mathbf{c})$]

Example 23. If $\mathbf{a},\mathbf{b}$ and $\mathbf{c}$ are three non-zero vectors such that $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=0$ and $\mathbf{b}$ and $\mathbf{c}$ are not parallel vectors, prove that $\mathbf{a}=\lambda\mathbf{b}+\mu\mathbf{c}$ where λ and μ are scalar.

Sol. We have, $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=0$

$\Rightarrow\quad \mathbf{a}=0$ or $\mathbf{b}\times\mathbf{c}=0$ or $\mathbf{a}\perp(\mathbf{b}\times\mathbf{c})$

$\Rightarrow\quad \mathbf{a}=0$ or $\mathbf{b}\,||\,\mathbf{c}$ or $\mathbf{a}\perp(\mathbf{b}\times\mathbf{c})$

But $\mathbf{a}\neq0$ and $\mathbf{b}\neq\mathbf{c}$

$\therefore\quad \mathbf{a}\perp(\mathbf{b}\times\mathbf{c})$

$\Rightarrow\quad \mathbf{a}$ lies in the plane of $\mathbf{b}$ and $\mathbf{c}$.

$\Rightarrow\quad \mathbf{a},\mathbf{b}$ and $\mathbf{c}$ are coplanar.

$\Rightarrow\quad a=\lambda b+\mu c$

Example 24. ***If*** $\mathbf{a}\times\mathbf{b}=\mathbf{a}\times\mathbf{c}$, $\mathbf{a}\neq0$, ***show that*** $\mathbf{b}=\mathbf{c}+t\mathbf{a}$ ***for some scalar*** t.

Sol. We have, $\mathbf{a}\times\mathbf{b}=\mathbf{a}\times\mathbf{c}$

$\Rightarrow\quad \mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{c}=0$

$\Rightarrow\quad \mathbf{a}\times(\mathbf{b}-\mathbf{c})=0$

$\Rightarrow\quad \mathbf{a}=0$ or $(\mathbf{b}-\mathbf{c})=0$ or $\mathbf{a}\,||\,(\mathbf{b}-\mathbf{c})$

$\Rightarrow\quad \mathbf{a}\,||\,(\mathbf{b}-\mathbf{c})$ $(\because \mathbf{a}\neq0$ and $\mathbf{b}\neq0)$

$\Rightarrow\quad \mathbf{b}-\mathbf{c}=t\mathbf{a}$ (for some scalar t)

$\Rightarrow\quad \mathbf{b}=\mathbf{c}+t\mathbf{a}$

Example 25. ***For any two vector*** $\mathbf{u}$ ***and*** $\mathbf{v}$, ***prove that***

(i) $(\mathbf{u}\cdot\mathbf{v})^2+|\mathbf{u}\times\mathbf{v}|^2=|\mathbf{u}|^2|\mathbf{v}|^2$

(ii) $(1+|\mathbf{u}|^2)(1+|\mathbf{v}|^2)=(1-\mathbf{u}\cdot\mathbf{v})^2+|\mathbf{u}+\mathbf{v}+(\mathbf{u}\times\mathbf{v})|^2$

Sol. (i) To show $(\mathbf{u}\cdot\mathbf{v})^2+|\mathbf{u}\times\mathbf{v}^2|=|\mathbf{u}|^2|\mathbf{v}|^2$

Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$.

$\Rightarrow\quad \mathbf{u}\cdot\mathbf{v}=uv\cos\theta$

and $|\mathbf{u}\times\mathbf{v}|=uv\sin\theta$

$\Rightarrow\quad (\mathbf{u}\cdot\mathbf{v})^2+|\mathbf{u}\times\mathbf{v}|^2=u^2v^2\cos^2\theta+u^2v^2\sin^2\theta$

$\Rightarrow\quad (\mathbf{u}\cdot\mathbf{v})^2+|\mathbf{u}\times\mathbf{v}|^2=u^2v^2$

$\Rightarrow\quad (\mathbf{u}\cdot\mathbf{v})^2+|\mathbf{u}\times\mathbf{v}|^2=|\mathbf{u}|^2|\mathbf{v}|^2$

(ii) Taking RHS $(1-\mathbf{u}\cdot\mathbf{v})^2+|\mathbf{u}+\mathbf{v}+(\mathbf{u}\times\mathbf{v})|^2$

$\Rightarrow 1+(\mathbf{u}\cdot\mathbf{v})^2-2\mathbf{u}\cdot\mathbf{v}+|[\mathbf{u}+\mathbf{v}+(\mathbf{u}\times\mathbf{v})]\cdot[\mathbf{u}+\mathbf{v}+(\mathbf{u}\times\mathbf{v})]|$

$\Rightarrow 1+|\mathbf{u}|^2|\mathbf{v}|^2\cos^2\theta-2|\mathbf{u}||\mathbf{v}|\cos\theta+\mathbf{u}\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{v}+\mathbf{u}(\mathbf{u}\times\mathbf{v})+\mathbf{v}\cdot\mathbf{u}+\mathbf{v}\cdot\mathbf{v}+\mathbf{v}\cdot(\mathbf{u}\times\mathbf{v})+(\mathbf{u}\times\mathbf{v})+(\mathbf{u}\times\mathbf{v})\cdot\mathbf{u}+(\mathbf{u}\times\mathbf{v})\cdot\mathbf{v}+(\mathbf{u}\times\mathbf{v})\cdot(\mathbf{u}\times\mathbf{v})$

$\Rightarrow 1+|\mathbf{u}|^2|\mathbf{v}|^2\cos^2\theta-2|\mathbf{u}||\mathbf{v}|\cos\theta+|\mathbf{u}|^2+|\mathbf{u}||\mathbf{v}|\cos\theta+0+|\mathbf{u}||\mathbf{v}|\cos\theta+|\mathbf{v}^2+0+0+0+|\mathbf{u}+\mathbf{v}|^2$

$\Rightarrow 1+|\mathbf{u}|^2|\mathbf{v}|^2\cos^2\theta+|\mathbf{u}|^2+|\mathbf{v}|^2+|\mathbf{u}|^2|\mathbf{v}|^2\sin^2\theta$

$\Rightarrow 1+|\mathbf{u}|^2|\mathbf{v}|^2+|\mathbf{u}|^2+|\mathbf{v}|^2$

$=(1+|\mathbf{u}|^2)(1+|\mathbf{v}|^2)=\text{LHS}$

Angle between Two Vectors

If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, then $\sin\theta = \dfrac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}||\mathbf{b}|}$.

Expression for $\sin\theta$

If $\mathbf{a} = a_1\,\hat{\mathbf{i}} + a_2\,\hat{\mathbf{j}} + a_3\,\hat{\mathbf{k}}$, $\mathbf{b} = b_1\,\hat{\mathbf{i}} + b_2\,\hat{\mathbf{j}} + b_3\,\hat{\mathbf{k}}$ and θ be angle between $\mathbf{a}$ and $\mathbf{b}$, then

$$\sin^2\theta = \frac{(a_2b_3 - a_3b_2)^2 + (a_1b_3 - a_3b_1)^2 + (a_1b_2 - a_2b_1)^2}{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)}$$

Example 26. The sine of the angle between the vector $\mathbf{a} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{b} = 2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ is

(a) $\sqrt{\dfrac{74}{99}}$ (b) $\sqrt{\dfrac{25}{99}}$

(c) $\sqrt{\dfrac{37}{99}}$ (d) $\dfrac{5}{\sqrt{41}}$

Sol. (a) $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & 1 & 1 \\ 2 & -2 & 1 \end{vmatrix} = 3\hat{\mathbf{i}} - \hat{\mathbf{j}} - 8\hat{\mathbf{k}}$

$$\sin\theta = \frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}||\mathbf{b}|} = \frac{\sqrt{74}}{\sqrt{11}\cdot\sqrt{9}} = \sqrt{\frac{74}{99}}$$

Example 27. If $|\mathbf{a}| = 2, |\mathbf{b}| = 5$ and $|\mathbf{a}\times\mathbf{b}| = 8$, then find the value of $\mathbf{a}\cdot\mathbf{b}$.

Sol. We have, $|\mathbf{a}| = 2, |\mathbf{b}| = 5$

and $|\mathbf{a}\times\mathbf{b}| = 8$.

Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$.

Then, $\sin\theta = \dfrac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}||\mathbf{b}|} = \dfrac{8}{2\times5} = \dfrac{4}{5}$

Now, $\cos\theta = \pm\sqrt{1 - \sin^2\theta}$

$$= \pm\sqrt{1 - \frac{16}{25}} = \pm\frac{3}{5}$$

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta = \pm\left(2\cdot5\cdot\frac{3}{5}\right) = \pm6$$

Vector Normal to the Plane of Two Given Vectors

If $\mathbf{a}$ and $\mathbf{b}$ are two non-zero, non-parallel vectors and let θ be the angle between them. $\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a unit vector perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}$, $\mathbf{b}$, $\hat{\mathbf{n}}$ form a right handed system.

$$\Rightarrow \quad (\mathbf{a}\times\mathbf{b}) = |\mathbf{a}\times\mathbf{b}|\,\hat{\mathbf{n}}$$

$$\Rightarrow \quad \hat{\mathbf{n}} = \frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}$$

Thus, $\hat{\mathbf{n}} = \dfrac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}$ is a unit vector perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$. Note that $-\dfrac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}$ is also a unit vector perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$. Vectors of magnitude 'λ' normal to the plane of $\mathbf{a}$ and $\mathbf{b}$ are given by

$$\pm\frac{\lambda(\mathbf{a}\times\mathbf{b})}{|\mathbf{b}\times\mathbf{b}|}$$

Example 28. The unit vector perpendicular to the vectors $6\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and $3\hat{\mathbf{i}} - 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$, is

(a) $\dfrac{2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}}{7}$ (b) $\dfrac{2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 6\hat{\mathbf{k}}}{7}$

(c) $\dfrac{2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 6\hat{\mathbf{k}}}{7}$ (d) $\dfrac{2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}}{7}$

Sol. (c) Let $\mathbf{a} = 6\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and $\mathbf{b} = 3\hat{\mathbf{i}} - 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 6 & 2 & 3 \\ 3 & -6 & -2 \end{vmatrix}$$

$$= 14\hat{\mathbf{i}} + 21\hat{\mathbf{j}} - 42\hat{\mathbf{k}} = 7(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 6\hat{\mathbf{k}})$$

$$|\mathbf{a}\times\mathbf{b}| = 7\,|2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 6\hat{\mathbf{k}}| = 7\cdot7$$

$$\therefore \quad \frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|} = \frac{1}{7}(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 6\hat{\mathbf{k}})$$

which is a unit vector perpendicular to $\mathbf{a}$ and $\mathbf{b}$.

Example 29. Find unit vectors perpendicular to the plane determined by the points

$P(1, -1, 2)$, $Q(2, 0, -1)$ and $R(0, 2, 1)$

Sol. Clearly, required unit vector is a unit vector perpendicular to the plane of $\mathbf{PQ}$ and $\mathbf{PR}$.

Now,

$$\mathbf{PQ} = (2\hat{\mathbf{i}} - \hat{\mathbf{k}}) - (\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}})$$

$$= \hat{\mathbf{i}} + \hat{\mathbf{j}} - 3\hat{\mathbf{k}}$$

$$\mathbf{PR} = (2\hat{\mathbf{j}} + \hat{\mathbf{k}}) - (\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}})$$

$$= -\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$$

and $\mathbf{PQ}\times\mathbf{PR} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & -3 \\ -1 & 3 & -1 \end{vmatrix} = 8\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$

$\therefore$ Required unit vectors

$$= \pm\frac{\mathbf{PQ}\times\mathbf{PR}}{|\mathbf{PQ}\times\mathbf{PR}|} = \pm\frac{(8\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 4\hat{\mathbf{k}})}{4\sqrt{6}}$$

$$= \pm\frac{1}{\sqrt{6}}(2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$$

Example 30. *Let* $\mathbf{A}, \mathbf{B}$ *and* $\mathbf{C}$ *be unit vectors. Suppose* $\mathbf{A}\cdot\mathbf{B} = \mathbf{A}\cdot\mathbf{C} = 0$ *and the angle between* $\mathbf{B}$ *and* $\mathbf{C}$ *is* $\frac{\pi}{4}$*. Then,*

(a) $\mathbf{A} = \pm 2\,(\mathbf{B}\times\mathbf{C})$ (b) $\mathbf{A} = \pm\sqrt{2}\,(\mathbf{B}\times\mathbf{C})$

(c) $\mathbf{A} = \pm 3\,(\mathbf{B}+\mathbf{C})$ (d) $\mathbf{A} = \pm\sqrt{3}\,(\mathbf{B}\times\mathbf{C})$

Sol. (b) Since, $\mathbf{A}\cdot\mathbf{B} = 0$

$$\Rightarrow \quad \mathbf{A}\perp\mathbf{B} \text{ and } \mathbf{A}\cdot\mathbf{C} = 0$$

$$\Rightarrow \quad \mathbf{A}\perp\mathbf{C}$$

$$\therefore \quad \mathbf{A} = \pm\frac{\mathbf{B}\times\mathbf{C}}{|\mathbf{B}\times\mathbf{C}|}$$

[$\because$ $\mathbf{A}$ is a unit vector perpendicular to both $\mathbf{B}$ and $\mathbf{C}$]

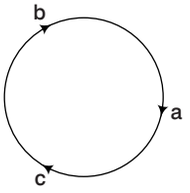
Here, $|\mathbf{B}\times\mathbf{C}| = |\mathbf{B}||\mathbf{C}|\sin\frac{\pi}{4}$

$$= 1\cdot 1\cdot\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

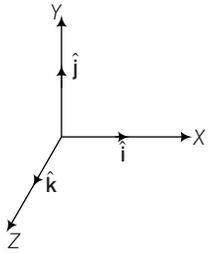
So, $\mathbf{A} = \pm\dfrac{(\mathbf{B}\times\mathbf{C})}{\frac{1}{\sqrt{2}}} = \pm\sqrt{2}\,(\mathbf{B}\times\mathbf{C})$

Right Handed System and Left Handed System of Vectors

(i) **Right handed system of vectors** Three mutually perpendicular vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ from a right handed system of vector iff $\mathbf{a}\times\mathbf{b} = \mathbf{c}, \mathbf{b}\times\mathbf{c} = \mathbf{a}, \mathbf{c}\times\mathbf{a} = \mathbf{b}$.

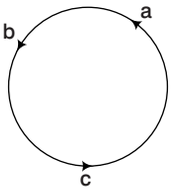


For example, the unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ form a right handed system, $\hat{\mathbf{i}}\times\hat{\mathbf{j}} = \hat{\mathbf{k}}, \hat{\mathbf{j}}\times\hat{\mathbf{k}} = \hat{\mathbf{i}}, \hat{\mathbf{k}}\times\hat{\mathbf{i}} = \hat{\mathbf{j}}$



(ii) **Left handed system of vectors** The vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ mutually perpendicular to one another form a left handed system of vectors iff

$$\mathbf{c}\times\mathbf{b} = \mathbf{a}, \mathbf{a}\times\mathbf{c} = \mathbf{b}, \mathbf{b}\times\mathbf{a} = \mathbf{c}.$$



Example 31. *The vectors* $\mathbf{c}, \mathbf{a} = x\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ *and* $\mathbf{b} = \hat{\mathbf{j}}$ *are such that* $\mathbf{a}, \mathbf{c}$ *and* $\mathbf{b}$ *form a right handed system, then* $\mathbf{c}$ *is*

(a) $z\,\hat{\mathbf{i}} - x\,\hat{\mathbf{k}}$ (b) 0

(c) $y\,\hat{\mathbf{j}}$ (d) $-z\hat{\mathbf{i}} + x\,\hat{\mathbf{k}}$

Sol. (a) $\mathbf{a}, \mathbf{c}$ and $\mathbf{b}$ form a right handed system.

Hence, $\mathbf{b}\times\mathbf{a} = \mathbf{c}$

$$\Rightarrow \quad \mathbf{c} = \hat{\mathbf{j}}\times(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}})$$

$$= -x\,\hat{\mathbf{k}} + z\,\hat{\mathbf{i}} = z\,\hat{\mathbf{i}} - x\,\hat{\mathbf{k}}$$

Example 32. If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are three non-zero vectors such that $\mathbf{a}\times\mathbf{b} = \mathbf{c}$ and $\mathbf{b}\times\mathbf{c} = \mathbf{a}$, prove that $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are mutually at right angles and $|\mathbf{b}| = 1$ and $|\mathbf{c}| = |\mathbf{a}|$.

Sol. $\mathbf{a}\times\mathbf{b} = \mathbf{c}$ and $\mathbf{a} = \mathbf{b}\times\mathbf{c}$

$\Rightarrow$ $\mathbf{c}\perp\mathbf{a}$, $\mathbf{c}\perp\mathbf{b}$ and $\mathbf{a}\perp\mathbf{b}$, $\mathbf{a}\perp\mathbf{c}$ $\Rightarrow$ $\mathbf{a}\perp\mathbf{b}$, $\mathbf{b}\perp\mathbf{c}$ and $\mathbf{c}\perp\mathbf{a}$

$\Rightarrow$ $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are mutually perpendicular vectors.

Again, $\mathbf{a}\times\mathbf{b} = \mathbf{c}$ and $\mathbf{b}\times\mathbf{c} = \mathbf{a}$

$$\Rightarrow \quad |\mathbf{a}\times\mathbf{b}| = |\mathbf{c}| \text{ and } |\mathbf{b}\times\mathbf{c}| = |\mathbf{a}|$$

$$\Rightarrow \quad |\mathbf{a}||\mathbf{b}|\sin\frac{\pi}{2} = |\mathbf{c}| \text{ and } |\mathbf{b}||\mathbf{c}|\sin\frac{\pi}{2} = |\mathbf{a}|$$

$$\Rightarrow \quad |\mathbf{a}||\mathbf{b}| = |\mathbf{c}|$$

and $|\mathbf{b}||\mathbf{c}| = |\mathbf{a}|$ ($\because \mathbf{a}\perp\mathbf{b}$ and $\mathbf{b}\perp\mathbf{c}$)

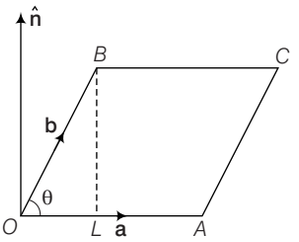
$$\Rightarrow \quad |\mathbf{b}|^2|\mathbf{c}| = |\mathbf{c}|$$

$$\Rightarrow \quad |\mathbf{b}|^2 = 1 \Rightarrow |\mathbf{b}| = 1$$

On putting in $|\mathbf{a}||\mathbf{b}| = |\mathbf{c}| \Rightarrow |\mathbf{a}| = |\mathbf{c}|$

Geometrical Interpretation of Vector Product

If $\mathbf{a}$ and $\mathbf{b}$ are two non-zero, non-parallel vectors represented by **OA** and **OB** respectively and let θ be the angle between them. Complete the parallelogram *OACB*. Draw $BL \perp OA$.



In ΔOBL, $\sin\theta = \frac{BL}{OB}$

$\Rightarrow$ $BL = OB\sin\theta = |\mathbf{b}|\sin\theta$

Now, $\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\mathbf{n}$

$= (OA)(BA)\,\mathbf{n}$

$=$ (Base $\times$ Height) $\mathbf{n}$ $=$ (Area of parallelogram $OACB$) $\mathbf{n}$

$=$ Vector area of the parallelogram $OACB$

Thus, $\mathbf{a}\times\mathbf{b}$ is a vector whose magnitude is equal to the area of the parallelogram having $\mathbf{a}$ and $\mathbf{b}$ as its adjacent sides and whose direction $\mathbf{n}$ is perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{n}$ form a right handed system. Hence, $\mathbf{a}\times\mathbf{b}$ represents the vector area of the parallelogram having adjacent sides along $\mathbf{a}$ and $\mathbf{b}$.

Area of Parallelogram and Triangle

(i) The area of a parallelogram with adjacent sides $\mathbf{a}$ and is $|\mathbf{a}\times\mathbf{b}|$.

(ii) The area of a parallelogram with diagonals $\mathbf{d}_1$ and $\mathbf{d}_2$ is $\frac{1}{2}|\mathbf{d}_1\times\mathbf{d}_2|$.

(iii) The area of a plane quadrilateral $ABCD$ is $\frac{1}{2}|\mathbf{AC}\times\mathbf{BD}|$, where AC and BD are its diagonals.

(iv) The area of a triangle with adjacent sides $\mathbf{a}$ and $\mathbf{b}$ is $\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$.

(v) The area of a ΔABC is $\frac{1}{2}|\mathbf{AB}\times\mathbf{AC}|$

or $\frac{1}{2}|\mathbf{BC}\times\mathbf{BA}|$

or $\frac{1}{2}|\mathbf{CB}\times\mathbf{CA}|$

(vi) If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are position vectors of a ΔABC, then its area $= \frac{1}{2}|(\mathbf{a}\times\mathbf{b})+(\mathbf{b}\times\mathbf{c})+(\mathbf{c}\times\mathbf{a})|$

Remark

Three points with position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are collinear, if $(\mathbf{a}\times\mathbf{b})+(\mathbf{b}\times\mathbf{c})+(\mathbf{c}\times\mathbf{a}) = 0$

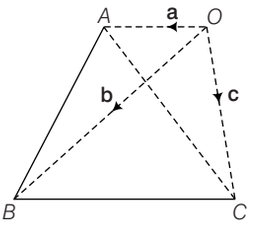
Example 33. If $\mathbf{a}$,$\mathbf{b}$ and $\mathbf{c}$ are position vectors of the vertices A,B and C of ΔABC, show that the area of ΔABC is $\frac{1}{2}|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}|$.

Deduce the condition for points $\mathbf{a}$,$\mathbf{b}$ and $\mathbf{c}$ to be collinear.

Sol. Area of $\Delta ABC = \frac{1}{2}|\mathbf{AB}\times\mathbf{AC}|$

Now, $\mathbf{AB}$ = Position vector of B − Position vector of A

$\mathbf{AB} = \mathbf{b}-\mathbf{a}$



$\mathbf{AC}$ = Position vector of C − Position vector of A

$\mathbf{AC} = \mathbf{c}-\mathbf{a}$

$\mathbf{AB}\times\mathbf{AC} = (\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})$

$= \mathbf{b}\times\mathbf{c}-\mathbf{b}\times\mathbf{a}-\mathbf{a}\times\mathbf{c}+\mathbf{a}\times\mathbf{a}$ $(\because \mathbf{a}\times\mathbf{a}=0)$

$= \mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}$

Hence, area of $\Delta ABC = \frac{1}{2}|\mathbf{AB}\times\mathbf{AC}|$

$= \frac{1}{2}|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}| = 0$

If the points A, B and C are collinear, then area of $\Delta ABC = 0$

$\Rightarrow$ $\frac{1}{2}|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}| = 0$

$\Rightarrow$ $|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}| = 0$

$\Rightarrow$ $\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a} = 0$

Thus, $\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a} = 0$

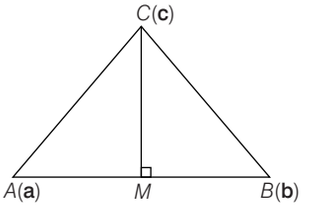
is the required condition of collinearity of three points $\mathbf{a}$,$\mathbf{b}$ and $\mathbf{c}$.

Example 34. Show that the perpendicular distance of the point $\mathbf{c}$ from to the joining $\mathbf{a}$ and $\mathbf{b}$ is $\frac{|\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}|}{|\mathbf{b}-\mathbf{a}|}$

Sol. Let ABC be a triangle and let $\mathbf{a}$,$\mathbf{b}$ and $\mathbf{c}$ be the position of its vertices A, B and C respectively. Let CM be the perpendicular from C on AB.

Then, area of $\Delta ABC = \frac{1}{2}(AB)\cdot(CM) = \frac{1}{2}|\mathbf{AB}||\mathbf{CM}|$

Also, area of $\Delta ABC = \frac{1}{2}|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}|$



$\therefore$ $\frac{1}{2}|\mathbf{AB}||\mathbf{CM}| = \frac{1}{2}|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}|$

$\Rightarrow$ $\mathbf{CM} = \frac{|\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}|}{|\mathbf{b}-\mathbf{a}|}$

Example 35.

(i) Find the area of the quadrilateral whose diagonals are given by

$$3\hat{\mathbf{i}}+\hat{\mathbf{j}}-2\hat{\mathbf{k}},\ \hat{\mathbf{i}}-3\hat{\mathbf{j}}+4\hat{\mathbf{k}}$$

(ii) $A_1, A_2, \ldots, A_n$ are the vertices of a regular plane polygon with n sides. O is the centre. Show that

$$\sum_{i=1}^{n-1}(\mathbf{OA}_i \times \mathbf{OA}_{i+1}) = (1-n)(\mathbf{OA}_2 \times \mathbf{OA}_1)$$

Sol. (i) Area of the quadrilateral $= \frac{1}{2}|d_1 \times d_2|$

$$= \frac{1}{2}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = \frac{1}{2}|-2\hat{\mathbf{i}}-14\hat{\mathbf{j}}-10\hat{\mathbf{k}}|$$

$$= \frac{1}{2}\sqrt{4+196+100} = \frac{10\sqrt{3}}{2} = 5\sqrt{3}$$

(ii) $A_1, A_2, \ldots, A_n$ are the vertices of a regular plane polygon of n sides and centre O.

Let $|\mathbf{OA}_i| = k, \forall\ i = 1 < 2, 3, \ldots, n$

Let $\hat{\mathbf{e}}_i$ be the unit vector along $\mathbf{OA}_i$

$$\mathbf{OA}_i = k\hat{\mathbf{e}}_i$$

$$\mathbf{OA}_i \times \mathbf{OA}_{i+1} = k\hat{\mathbf{e}}_i \times k\hat{\mathbf{e}}_{i+1} = k^2\hat{\mathbf{x}}_i$$

where $\hat{\mathbf{x}}_i$ is a unit vector in the direction perpendicular to the plane of the polygon and $\hat{\mathbf{x}}_i = \hat{\mathbf{x}}_{i+1}$ for $i = 1, 2, 3, \ldots, n-1$

$$\therefore \quad \text{LHS} = \sum_{i=1}^{n-1}(\mathbf{OA}_i \times \mathbf{OA}_{i+1}) = k^2\sum_{r=1}^{n-1}\hat{\mathbf{x}}_i$$

$$= k^2(n-1)\hat{\mathbf{x}}_i = (n-1)(\mathbf{OA}_i \times \mathbf{OA}_2)$$

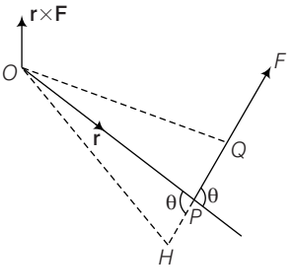
$$= (1-n)(\mathbf{OA}_2 \times \mathbf{OA}_1)$$

Moment of a Force and Couple

Moment of a Force

(i) **About a point** Let a force $\mathbf{F}$ be applied at a point P of a rigid body then, the moment of $\mathbf{F}$ about a point O measures the tendency of $\mathbf{F}$ to turn the body about point O. If this tendency of rotation about O is in anti-clockwise direction, the moment is positive, otherwise it is negative.

Let $\mathbf{r}$ be the position vector of P relative to O. Then, the moment or torque of $\mathbf{F}$ about the point O is defined as the vector $\mathbf{M} = \mathbf{r} \times \mathbf{F}$.



If several forces are acting through the same point P then the vector sum of the moment of the separate forces about O is equal to the moment of their resultant force about O.

Remark

Moment of a force $\mathbf{F}$ about a point $\mathbf{A} = \mathbf{AB} \times \mathbf{F}$, where $\mathbf{B}$ is any point on $\mathbf{F}$.

(ii) **About a line** The moment of a force $\mathbf{F}$ acting at a point P about a line L is a scalar given by $(\mathbf{r} \times \mathbf{F}) \cdot \hat{\mathbf{a}}$, where $\mathbf{a}$ is a unit vector in the direction of the line and $\mathbf{OP} = \mathbf{r}$, where O is any point on the line.

Thus, the moment of a force $\mathbf{F}$ about a line is the resolved part (component) along this line, of the moment of $\mathbf{F}$ about any point on the line.

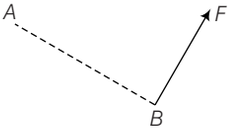
Remark

The moment of a force about a point is a vector while the moment about a straight line is a scalar quantity.

Example 36. Find the moment about $(1, -1, -1)$ of the force $3\hat{\mathbf{i}}+4\hat{\mathbf{j}}-5\hat{\mathbf{k}}$ acting at $(1, 0, -2)$.

Sol. Let $A \equiv (1, -1, -1), B \equiv (1, 0, -2)$

and $\mathbf{F} = 3\hat{\mathbf{i}}+4\hat{\mathbf{j}}-5\hat{\mathbf{k}}$



Then, moment of force $\mathbf{F}$ about A is given by $\mathbf{AB} \times \mathbf{F}$.

Here, $\mathbf{AB} = (\hat{\mathbf{i}}-2\hat{\mathbf{k}}) - (\hat{\mathbf{i}}-\hat{\mathbf{j}}-\hat{\mathbf{k}}) = \hat{\mathbf{j}}-\hat{\mathbf{k}}$

$$\therefore \quad \mathbf{AB} \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 1 & -1 \\ 3 & 4 & -5 \end{vmatrix}$$

$$= \hat{\mathbf{i}}(-5+4) - \hat{\mathbf{j}}(0+3) + \hat{\mathbf{k}}(0-3)$$

$$= -\hat{\mathbf{i}}-3\hat{\mathbf{j}}-3\hat{\mathbf{k}}$$

Example 37. Three forces $\hat{\mathbf{i}}+2\hat{\mathbf{j}}-3\hat{\mathbf{k}}, 2\hat{\mathbf{i}}+3\hat{\mathbf{j}}+4\hat{\mathbf{k}}$ and $\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}}$ acting on a particle at the point (0, 1, 2). The magnitude of the moment of the forces about the point $(1,-2,0)$ is

(a) $2\sqrt{35}$ (b) $6\sqrt{10}$

(c) $4\sqrt{7}$ (d) None of these

Sol. (b) Total force $\mathbf{F}=(\hat{\mathbf{i}}+2\hat{\mathbf{j}}-3\hat{\mathbf{k}})+(2\hat{\mathbf{i}}+3\hat{\mathbf{j}}+4\hat{\mathbf{k}})+(\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}})=4\hat{\mathbf{i}}+4\hat{\mathbf{j}}+2\hat{\mathbf{k}}$

Moment of the forces about

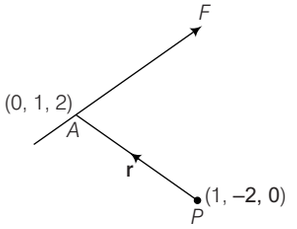
$$P=\mathbf{r}\times\mathbf{F}=\mathbf{PA}\times\mathbf{F}$$

$$\mathbf{PA}=(0-1)\hat{\mathbf{i}}+(1+2)\hat{\mathbf{j}}+(2-0)\hat{\mathbf{k}}$$

$$=-\hat{\mathbf{i}}+3\hat{\mathbf{j}}+2\hat{\mathbf{k}}$$

$\therefore$ Moment about $P=(-\hat{\mathbf{i}}+3\hat{\mathbf{j}}+2\hat{\mathbf{k}})\times(4\hat{\mathbf{i}}+4\hat{\mathbf{j}}+2\hat{\mathbf{k}})$

$$=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ -1 & 3 & 2\\ 4 & 4 & 2\end{vmatrix}=-2\hat{\mathbf{i}}+10\hat{\mathbf{j}}-16\hat{\mathbf{k}}$$



Magnitude of the moment

$$=|-2\hat{\mathbf{i}}+10\hat{\mathbf{j}}-16\hat{\mathbf{k}}|$$

$$=2\sqrt{1^2+5^2+8^2}=2\sqrt{90}=6\sqrt{10}$$

Example 38. Find the moment about a line through the origin having the direction of $2\hat{\mathbf{i}}-2\hat{\mathbf{j}}+\hat{\mathbf{k}}$ due to 30 kg force acting at a point $(-4,2,5)$ in the direction of $12\hat{\mathbf{i}}-4\hat{\mathbf{j}}-3\hat{\mathbf{k}}$.

Sol. Let $\mathbf{F}$ be the force. Then,

$$\mathbf{F}=\frac{30(12\hat{\mathbf{i}}-4\hat{\mathbf{j}}-3\hat{\mathbf{k}})}{\sqrt{144+16+9}}=\frac{30}{13}(12\hat{\mathbf{i}}-4\hat{\mathbf{j}}-3\hat{\mathbf{k}})$$

Suppose the force $\mathbf{F}$ acts at point $P(-4,2,5)$ the moment of $\mathbf{F}$ acting at P about a line in the direction $2\hat{\mathbf{i}}-2\hat{\mathbf{j}}+\hat{\mathbf{k}}$ is equal to the resolve part along the line of moment of $\mathbf{F}$ about a point on the line.

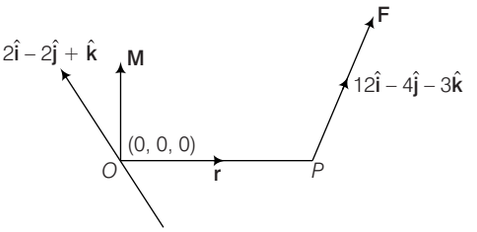
$\therefore$ $\mathbf{r}=\mathbf{OP}=(-4\hat{\mathbf{i}}+2\hat{\mathbf{j}}+5\hat{\mathbf{k}})-(0)$

$$=-4\hat{\mathbf{i}}+2\hat{\mathbf{j}}+5\hat{\mathbf{k}}$$

Let $\mathbf{M}$ be the moment $\mathbf{F}$ about O. Then,

$$\mathbf{M}=\mathbf{r}\times\mathbf{F}=\frac{30}{13}\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ -4 & 2 & 5\\ 12 & -4 & -3\end{vmatrix}$$

$$=\frac{30}{13}(4\hat{\mathbf{i}}+48\hat{\mathbf{j}}-8\hat{\mathbf{k}})$$



Let $\mathbf{a}$ be unit vector in the direction of $2\hat{\mathbf{i}}-2\hat{\mathbf{j}}+\hat{\mathbf{k}}$. Then,

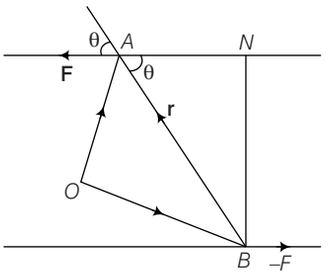
$$\mathbf{a}=\frac{2\hat{\mathbf{i}}-2\hat{\mathbf{j}}+\hat{\mathbf{k}}}{\sqrt{4+4+1}}=\frac{1}{3}(2\hat{\mathbf{i}}-2\hat{\mathbf{j}}+\hat{\mathbf{k}})$$

Thus, the moment of $\mathbf{F}$ about the given line

$$=\mathbf{M}\cdot\mathbf{a}=\frac{30}{13}(14\hat{\mathbf{i}}+48\hat{\mathbf{j}}-8\hat{\mathbf{k}})\cdot\frac{1}{3}(2\hat{\mathbf{i}}-2\hat{\mathbf{j}}+\hat{\mathbf{k}})=-\frac{760}{13}$$

Moment of a Couple

A system consisting of a pair of equal unlike parallel forces is called a couple. The vector sum to two forces of a couple is always zero vector.



The moment of a couple is a vector perpendicular to the plane of couple and its magnitude is the product of the magnitude of either force with perpendicular distance between the lines of the forces.

$$\mathbf{M}=\mathbf{r}\times\mathbf{F}, \text{ where } \mathbf{r}=\mathbf{BA}$$

$$|\mathbf{M}|=|\mathbf{BA}\times\mathbf{F}|=|\mathbf{F}||\mathbf{BA}|\sin\theta$$

where θ is the angle between $\mathbf{BA}$ and $\mathbf{F}$

$$=|\mathbf{F}|(BN)=|\mathbf{F}|\alpha$$

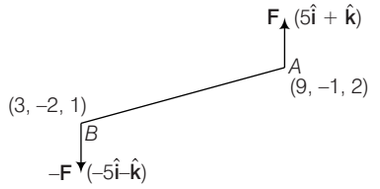
where, $\alpha=BN$ is the arm of the couple and $+ve$ or $-ve$ sign is to be taken accordingly as the forces indicate a counter clockwise rotation or clockwise rotation.

Example 39. The moment of the couple formed by the forces $5\hat{\mathbf{i}}+\hat{\mathbf{k}}$ and $-5\hat{\mathbf{i}}-\hat{\mathbf{k}}$ acting at the points $(9,-1,2)$ and $(3,-2,1)$ respectively, is

(a) $-\hat{\mathbf{i}}+\hat{\mathbf{j}}+5\hat{\mathbf{k}}$ (b) $\hat{\mathbf{i}}-\hat{\mathbf{j}}-5\hat{\mathbf{k}}$

(c) $2\hat{\mathbf{i}}-2\hat{\mathbf{j}}-10\hat{\mathbf{k}}$ (d) $-2\hat{\mathbf{i}}+2\hat{\mathbf{j}}+10\hat{\mathbf{k}}$

Sol. (b) Moment of the couple,



$$= \mathbf{BA} \times \mathbf{F} = \{(9-3)\,\hat{\mathbf{i}} + (-1+2)\,\hat{\mathbf{j}} + (2-1)\,\hat{\mathbf{k}}\} \times (5\hat{\mathbf{i}} + \hat{\mathbf{k}})$$

$$= (6\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) \times (5\hat{\mathbf{i}} + \hat{\mathbf{k}}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 6 & 1 & 1 \\ 5 & 0 & 1 \end{vmatrix}$$

$$= \hat{\mathbf{i}} - \hat{\mathbf{j}} - 5\hat{\mathbf{k}}$$

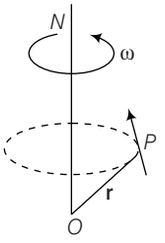
Rotation About an Axis

When a rigid body rotates about a fixed axis *ON* with an angular velocity ω, then velocity v of a particle *P* is given by

$$v = \omega \times \mathbf{r},$$

were, $\mathbf{r} = \mathbf{OP}$

and $\omega = |\omega|$ (unit vector along *ON*)



Example 40. A particle has an angular speed of 3 rad/s and the axis of rotation passes through the points (1, 1, 2) and (1, 2, – 2). Find the velocity of the particle at point *P* (3, 6, 4).

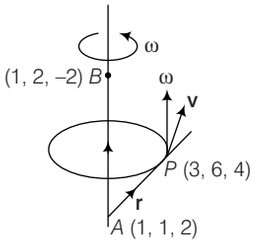
Sol. Clearly, $\mathbf{OA} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

$$\mathbf{OB} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$$

$$\mathbf{AB} = \hat{\mathbf{j}} - 4\hat{\mathbf{k}} = |\mathbf{AB}| = \sqrt{17}$$

and $\mathbf{AP} = (3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + 4\hat{\mathbf{k}}) - (\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

$$= 2\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$$



$$\therefore \quad \omega = \frac{3}{\sqrt{17}}(\hat{\mathbf{j}} - 4\hat{\mathbf{k}})\,2 \text{ and } \mathbf{r}$$

Now, $\mathbf{v} = \omega \times \mathbf{r} = \frac{3}{\sqrt{17}}(\hat{\mathbf{j}} - 4\hat{\mathbf{k}}) \times (2\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

$$= \frac{3}{\sqrt{17}}(22\hat{\mathbf{i}} - 8\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$$

Example 41. A rigid body is spinning about a fixed point $(3, -2, -1)$ with an angular velocity of 4 rad/s, the axis of rotation being in the direction of $(1, 2, -2)$. Find the velocity of the particle at the point (4, 1, 1).

Sol. $\omega = 4\left(\frac{\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}}{\sqrt{1+4+4}}\right) = \frac{4}{3}(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$

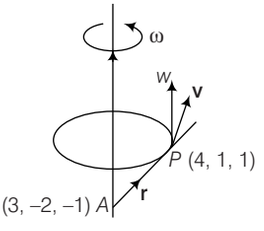
$$\mathbf{r} = \mathbf{OP} - \mathbf{OA}$$

$$= (4\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) - (3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - \hat{\mathbf{k}})$$

$$= \hat{\mathbf{i}} + 3\hat{\mathbf{i}} + 2\hat{\mathbf{k}}$$

$$\mathbf{v} = \omega \times \mathbf{r} = \frac{4}{3}(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}) \times (\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$$

$$\Rightarrow \quad \frac{4}{3}(10\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + \hat{\mathbf{k}})$$



Exercise for Session 2

1. Find $|\mathbf{a}\times\mathbf{b}|$, if $\mathbf{a}=\hat{\mathbf{i}}-7\hat{\mathbf{j}}+7\hat{\mathbf{k}}$ and $\mathbf{b}=3\hat{\mathbf{i}}-2\hat{\mathbf{j}}-2\hat{\mathbf{k}}$.
2. Find the values of λ and μ for which $(2\hat{\mathbf{i}}+6\hat{\mathbf{j}}+27\hat{\mathbf{k}})\times(\hat{\mathbf{i}}+\lambda\hat{\mathbf{j}}+\mu\hat{\mathbf{k}})=0$
3. If $\mathbf{a}=2\hat{\mathbf{i}}+3\hat{\mathbf{j}}-\hat{\mathbf{k}}$, $\mathbf{b}=-\hat{\mathbf{i}}+2\hat{\mathbf{j}}-4\hat{\mathbf{k}}$, $\mathbf{c}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$, then find the value of $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{c})$.
4. Prove that $(\mathbf{a}\cdot\hat{\mathbf{i}})(\mathbf{a}\times\hat{\mathbf{i}})+(\mathbf{a}\cdot\hat{\mathbf{j}})(\mathbf{a}\times\hat{\mathbf{j}})+(\mathbf{a}\times\hat{\mathbf{k}})(\mathbf{a}\times\hat{\mathbf{k}})=0$.
5. If $\mathbf{a}\times\mathbf{b}=\mathbf{c}\times\mathbf{d}$ and $\mathbf{a}\times\mathbf{c}=\mathbf{b}\times\mathbf{d}$, then show that $\mathbf{a}-\mathbf{d}$ is parallel to $\mathbf{b}-\mathbf{c}$.
6. If $(\mathbf{a}\times\mathbf{b})^2+(\mathbf{a}\cdot\mathbf{b})^2=144$ and $|\mathbf{a}|=4$, then find the value of $|\mathbf{b}|$.
7. If $|\mathbf{a}|=2$, $|\mathbf{b}|=7$ and $(\mathbf{a}\times\mathbf{b})=3\hat{\mathbf{i}}+2\hat{\mathbf{j}}+6\hat{\mathbf{k}}$, find the angle between $\mathbf{a}$ and $\mathbf{b}$.
8. Let the vectors $\mathbf{a}$ and $\mathbf{b}$ be such that $|\mathbf{a}|=3$, $|\mathbf{b}|=\frac{\sqrt{2}}{3}$ and $\mathbf{a}\times\mathbf{b}$ is a unit vector, then find the angle between $\mathbf{a}$ and $\mathbf{b}$.
9. If $|\mathbf{a}|=\sqrt{26}$, $|\mathbf{b}|=7$, and $|\mathbf{a}\times\mathbf{b}|=35$, find $\mathbf{a}\cdot\mathbf{b}$.
10. Find a unit vector perpendicular to the plane of two vectors $\mathbf{a}=\hat{\mathbf{i}}-\hat{\mathbf{j}}+2\hat{\mathbf{k}}$ and $\mathbf{b}=2\hat{\mathbf{i}}+3\hat{\mathbf{j}}-\hat{\mathbf{k}}$.
11. Find a vector of magnitude 15, which is perpendicular to both the vectors $4\hat{\mathbf{i}}-\hat{\mathbf{j}}+8\hat{\mathbf{k}}$ and $-\hat{\mathbf{j}}+\hat{\mathbf{k}}$.
12. Let $\mathbf{a}=\hat{\mathbf{i}}+4\hat{\mathbf{j}}+2\hat{\mathbf{k}}$, $\mathbf{b}=3\hat{\mathbf{i}}-2\hat{\mathbf{j}}+7\hat{\mathbf{k}}$ and $\mathbf{c}=2\hat{\mathbf{i}}-\hat{\mathbf{j}}+4\hat{\mathbf{k}}$.
    Find a vector $\mathbf{d}$ which is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ and $\mathbf{c}\cdot\mathbf{d}=15$.
13. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be unit vectors such that $\mathbf{a}\cdot\mathbf{b}=0=\mathbf{a}\cdot\mathbf{c}$. If the angle between $\mathbf{b}$ and $\mathbf{c}$ is $\frac{\pi}{6}$, then find $\mathbf{a}$.
14. Find the area of the triangle whose adjacent sides are determined by the vectors $\mathbf{a}=-2\hat{\mathbf{i}}-5\hat{\mathbf{k}}$ and $\mathbf{b}=\hat{\mathbf{i}}-2\hat{\mathbf{j}}-\hat{\mathbf{k}}$.
15. Find the area of the parallelogram whose adjacent sides are represented by the vectors $3\hat{\mathbf{i}}+\hat{\mathbf{j}}-2\hat{\mathbf{k}}$ and $\hat{\mathbf{i}}-3\hat{\mathbf{j}}+4\hat{\mathbf{k}}$.
16. Show that the area of the parallelogram having diagonals $3\hat{\mathbf{i}}+\hat{\mathbf{j}}-2\hat{\mathbf{k}}$ and $\hat{\mathbf{i}}-3\hat{\mathbf{j}}+4\hat{\mathbf{k}}$ is $5\sqrt{3}$.
17. A force $\mathbf{F}=2\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}}$ acts at point A whose position vector is $2\hat{\mathbf{i}}-\hat{\mathbf{j}}$. Find the moment of force $\mathbf{F}$ about the origin.
18. Find the moment of $\mathbf{F}$ about point $(2,-1,3)$, when force $\mathbf{F}=3\hat{\mathbf{i}}+2\hat{\mathbf{j}}-4\hat{\mathbf{k}}$ is acting on point $(1,-1,2)$.
19. Forces $2\hat{\mathbf{i}}+\hat{\mathbf{j}}$, $2\hat{\mathbf{i}}-3\hat{\mathbf{j}}+6\hat{\mathbf{k}}$ and $-\hat{\mathbf{i}}+2\hat{\mathbf{j}}-\hat{\mathbf{k}}$ act at a point P, with position vector $4\hat{\mathbf{i}}-3\hat{\mathbf{j}}-\hat{\mathbf{k}}$. Find the moment of the resultant of these force about the point Q whose position vector is $6\hat{\mathbf{i}}+\hat{\mathbf{j}}-3\hat{\mathbf{k}}$.

Session 3

Scalar Triple Product

Scalar Triple Product

The scalar triple product is defined for three vectors and it is defined as the dot product of one of the vectors with the cross product of the other two.

If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are any three vector, then their scalar product is defined as $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$.

We denote it by $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$.

It is also called the mixed or box product.

Remark

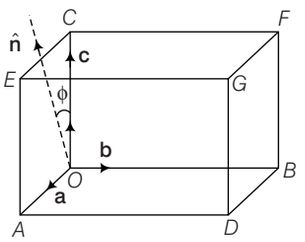
Result of scalar triple product is always a scalar.

Geometrical Interpretation of Scalar Triple Product

Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be three vectors. Consider a parallelopiped having coterminous edges **OA**, **OB** and **OC** such that $\mathbf{OA} = a, \mathbf{OB} = b$ and $\mathbf{OC} = c$. Then, $\mathbf{a} \times \mathbf{b}$ is a vector perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$. Let ϕ be the angle between $\mathbf{c}$ and $\mathbf{a} \times \mathbf{b}$.

If $\mathbf{n}$ is a unit vector along $\mathbf{a} \times \mathbf{b}$, then ϕ is the angle between $\mathbf{n}$ and $\mathbf{c}$.

Now, $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$



$= (\text{Area of parallelogram } OADB)\ \mathbf{n} \cdot \mathbf{c}$

$= (\text{Area of parallelogram } OADB)\ (\mathbf{n} \cdot \mathbf{c})$

$= (\text{Area of parallelogram } OADB)\ (|\mathbf{c}||\mathbf{n}|\cos\phi)$

$= (\text{Area of parallelogram } OADB)\ (|\mathbf{c}|\cos\phi)$

$= (\text{Area of parallelogram } OADB)\ (OL)$

$= \text{Area of base} \times \text{height}$

$= \text{Volume of parallelopiped}$

Height of parallelopiped

$$= \frac{\text{Volume of parallelopiped}}{\text{Area of base}}$$

Properties of Scalar Triple Product

(i) If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are given by

$$\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$$

$$\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$$

$$\mathbf{c} = c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}} + c_3\hat{\mathbf{k}}$$

Then, $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

(ii) $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ i.e. position of dot and cross can be interchanged without altering product. Hence, it is also represented by $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$.

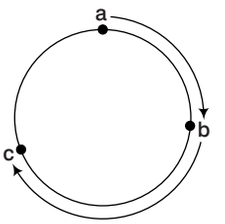
(iii) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = [\mathbf{b}\ \mathbf{c}\ \mathbf{a}] = [\mathbf{c}\ \mathbf{a}\ \mathbf{b}]$

(iv) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = -[\mathbf{b}\ \mathbf{a}\ \mathbf{c}]$

(v) $[k\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = k[\mathbf{a}\ \mathbf{b}\ \mathbf{c}], [k_1\mathbf{a}\ k_2\mathbf{b}\ k_3\mathbf{c}], = k_1k_2k_3\ [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

(vi) $[\mathbf{a} + \mathbf{b}\ \mathbf{c}\ \mathbf{d}] = [\mathbf{a}\ \mathbf{c}\ \mathbf{d}] + [\mathbf{b}\ \mathbf{c}\ \mathbf{d}]$

(vii) $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in that order form a right handed system, if $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] > 0$.



(viii) The necessary and sufficient condition for three non-zero, non-collinear vector $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ to be coplanar is that $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$ i.e., $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar $\Leftrightarrow [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$.

(ix) $[x_1\mathbf{a} + y_1\mathbf{b} + z_1\mathbf{c},\ x_2\mathbf{a} + y_2\mathbf{b} + z_2\mathbf{c},\ x_3\mathbf{a} + y_3\mathbf{b} + z_3\mathbf{c}]$

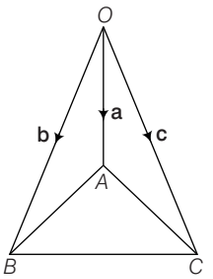
$$= \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} [\mathbf{a}\,\mathbf{b}\,\mathbf{c}]$$

Remarks

1. Four points **A**, **B**, **C**, **D** are coplanar if [**AB**, **AC**, **AD**] = 0
2. Four points **a**, **b**, **c** and **d** are coplanar, if
   [**d b c**] + [**d c a**] + [**d a b**] = [**a b c**]
   or [**a b c**] + [**a c d**] + [**a d b**] = [**d b c**]
3. [**a a b**] = [**b b a**] = [**c c b**] = 0
   i.e., if any two vectors are same, then vectors are coplanar.

Volume of Tetrahedron (A pyramid having a triangular base)

If *OABC* is a tetrahedron as shown in figure, where $\mathbf{OA}=\mathbf{a}$, $\mathbf{OB}=\mathbf{b}$, and $\mathbf{OC}=\mathbf{c}$, then volume of tetrahedron $=\frac{1}{6}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$



Remarks

1. The six mid-points of the six edges of a tetrahedron lie in a sphere, if the pair of opposite edges are perpendicular to each other.
2. Centre of the sphere is the centroid of the tetrahedron.
3. $GA^2 + GB^2 + GC^2 + GO^2 = 12r^2$, *G* being the centroid.
4. The angle between any two plane faces of a regular tetrahedron is $\cos^{-1}\frac{1}{3}$.
5. Angle between the any edge and a face not containing the angle is $\cos^{-1}\sqrt{\frac{1}{3}}$ (for regular tetrahedron).
6. Any two edges of regular tetrahedron are perpendicular to each other.
7. The distance of any vertex from the opposite face of regular tetrahedron is $\sqrt{\frac{2}{3}}k$, *k* being the length of any edge.

Example 42. ***Find the volume of the parallelopiped whose edges are represented by*** $\mathbf{a}=2\hat{\mathbf{i}}-3\hat{\mathbf{j}}+4\hat{\mathbf{k}}$, $\mathbf{b}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}-\hat{\mathbf{k}}$ ***and*** $\mathbf{c}=3\hat{\mathbf{i}}-\hat{\mathbf{j}}+2\hat{\mathbf{k}}$.

Sol. Here, $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=\begin{vmatrix}2&-3&4\\1&2&-1\\3&-1&2\end{vmatrix}=2(4-1)+3(2+3)+4(-1-6)$

$=6+15-28=-7$

$\therefore$ The volume of the parallelopiped $=\left|[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\right|=7$.

Example 43. Let $\mathbf{a}=x\hat{\mathbf{i}}+12\hat{\mathbf{j}}-\hat{\mathbf{k}}$, $\mathbf{b}=2\hat{\mathbf{i}}+2x\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $\mathbf{c}=\hat{\mathbf{i}}+\hat{\mathbf{k}}$. If $\mathbf{b},\mathbf{c},\mathbf{a}$ in that order form a left handed system, then find the value of *x*.

$$[x_1\mathbf{a}+y_1\mathbf{b}+z_1\mathbf{c},\ x_2\mathbf{a}+y_2\mathbf{b}+z_2\mathbf{c},\ x_3\mathbf{a}+y_3\mathbf{b}+z_3\mathbf{c}]=\begin{vmatrix}x_1&y_1&z_1\\x_2&y_2&z_2\\x_3&y_3&z_3\end{vmatrix}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$

Sol. Since, $\mathbf{b},\mathbf{c},\mathbf{a}$ form a left handed system, three fore $[\mathbf{b},\mathbf{c},\mathbf{a}]<0$

$$\Rightarrow \begin{vmatrix}2&2x&-1\\1&0&1\\x&12&-1\end{vmatrix}<0$$

$$\Rightarrow 2(0-12)-2x(-1-x)+1(12-0)<0$$

$$\Rightarrow -24+2x+2x^2+12<0$$

$$\Rightarrow 2x^2+2x-12<0 \Rightarrow x^2+x-6<0$$

$$\Rightarrow (x-2)(x+3)<0 \Rightarrow x\in(-3,2)$$

Example 44. For any three vectors $\mathbf{a},\mathbf{b}$ and $\mathbf{c}$ prove that $[\mathbf{a}+\mathbf{b}\ \mathbf{b}+\mathbf{c}\ \mathbf{c}+\mathbf{a}]=2[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

Sol. We have, $[\mathbf{a}+\mathbf{b}\ \mathbf{b}+\mathbf{c}\ \mathbf{c}+\mathbf{a}]$

$$=\{(\mathbf{a}+\mathbf{b})\times(\mathbf{b}\times\mathbf{c})\}\cdot(\mathbf{c}+\mathbf{a})$$

$$=\{\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}\}\cdot(\mathbf{c}+\mathbf{a}) \qquad \{\because \mathbf{b}\times\mathbf{b}=0\}$$

$$=\{\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}\}\cdot(\mathbf{c}+\mathbf{a})$$

$$=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}+(\mathbf{a}\times\mathbf{c})\cdot\mathbf{c}+(\mathbf{b}\times\mathbf{c})\cdot\mathbf{c}+(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}+(\mathbf{a}\times\mathbf{c})\cdot\mathbf{a}+(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}$$

$$=[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]+0+0+0+0+[\mathbf{b}\ \mathbf{c}\ \mathbf{a}] \qquad [\because [\mathbf{a}\ \mathbf{c}\ \mathbf{c}]=0,[\mathbf{b}\ \mathbf{c}\ \mathbf{c}]=0,\ [\mathbf{a}\ \mathbf{b}\ \mathbf{a}]=0,[\mathbf{a}\ \mathbf{c}\ \mathbf{a}]=0]$$

$$=[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]+[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=2[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$

Example 45. If $\mathbf{a},\mathbf{b}$ and $\mathbf{c}$ are coplanar show $[\mathbf{a}+\mathbf{b}\ \mathbf{b}+\mathbf{c}\ \mathbf{c}+\mathbf{a}]$ are coplanar.

Sol. Since, $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ are coplanar

$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=0 \qquad \ldots(i)$$

and shown in above example

$$[\mathbf{a}+\mathbf{b}\ \mathbf{b}+\mathbf{c}\ \mathbf{c}+\mathbf{a}]=2[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=0 \qquad [\text{using Eq. (i)}]$$

which shows $[\mathbf{a}+\mathbf{b}\ \mathbf{b}+\mathbf{c}\ \mathbf{c}+\mathbf{a}]$ are coplanar, if $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ are coplanar.

Example 46. For any three vectors $\mathbf{a},\mathbf{b}$ and $\mathbf{c}$ prove that $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2=\begin{vmatrix}\mathbf{a}\cdot\mathbf{a}&\mathbf{a}\cdot\mathbf{b}&\mathbf{a}\cdot\mathbf{c}\\\mathbf{b}\cdot\mathbf{a}&\mathbf{b}\cdot\mathbf{b}&\mathbf{b}\cdot\mathbf{c}\\\mathbf{c}\cdot\mathbf{a}&\mathbf{c}\cdot\mathbf{b}&\mathbf{c}\cdot\mathbf{c}\end{vmatrix}$

Sol. Let $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$

$\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$

$\mathbf{c} = c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}} + c_3\hat{\mathbf{k}}$

Then, $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

On multiplying row-by-row, we get

$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2 =$

$$\begin{vmatrix} a_1a_1 + a_2a_2 + a_3a_3 & a_1b_1 + a_2b_2 + a_3b_3 & a_1c_1 + a_2c_2 + a_3c_3 \\ b_1a_1 + b_2a_2 + b_3a_3 & b_1b_1 + b_2b_2 + b_3b_3 & b_1c_1 + b_2c_2 + b_3c_3 \\ c_1a_1 + c_2a_2 + c_3a_3 & c_1b_1 + c_2b_2 + c_3b_3 & c_1c_1 + c_2c_2 + c_3c_3 \end{vmatrix}$$

$$= \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix}$$

Example 47. If $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{l}$ and $\mathbf{m}$ are vectors, prove that

$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,(\mathbf{l}\times\mathbf{m}) = \begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{a}\cdot\mathbf{l} & \mathbf{b}\cdot\mathbf{l} & \mathbf{c}\cdot\mathbf{l} \\ \mathbf{a}\cdot\mathbf{m} & \mathbf{b}\cdot\mathbf{m} & \mathbf{c}\cdot\mathbf{m} \end{vmatrix}$$

Sol. Let $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$, $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$,

$\mathbf{c} = c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}} + c_3\hat{\mathbf{k}}$, $\mathbf{l} = l_1\hat{\mathbf{i}} + l_2\hat{\mathbf{j}} + l_3\hat{\mathbf{k}}$

and $\mathbf{m} = m_1\hat{\mathbf{i}} + m_2\hat{\mathbf{j}} + m_3\hat{\mathbf{k}}$.

$$\Rightarrow [\mathbf{a}\ \mathbf{b}\ \mathbf{c}](\mathbf{l}\times\mathbf{m}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{vmatrix}$$

On multiplying row-by-row, we get

$$= \begin{vmatrix} a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}} & a_1l_1 + a_2l_2 + a_3l_3 & a_1m_1 + a_2m_2 + a_3m_3 \\ b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}} & b_1l_1 + b_2l_2 + b_3l_3 & b_1m_1 + b_2m_2 + b_3m_3 \\ c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}} + c_3\hat{\mathbf{k}} & b_1l_1 + b_2l_2 + b_3l_3 & c_1m_1 + c_2m_2 + c_3m_3 \end{vmatrix}$$

$$= \begin{vmatrix} \mathbf{a} & \mathbf{a}\cdot\mathbf{l} & \mathbf{a}\cdot\mathbf{m} \\ \mathbf{b} & \mathbf{b}\cdot\mathbf{l} & \mathbf{b}\cdot\mathbf{m} \\ \mathbf{c} & \mathbf{c}\cdot\mathbf{l} & \mathbf{c}\cdot\mathbf{m} \end{vmatrix} = \begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{a}\cdot\mathbf{l} & \mathbf{b}\cdot\mathbf{l} & \mathbf{c}\cdot\mathbf{l} \\ \mathbf{a}\cdot\mathbf{m} & \mathbf{b}\cdot\mathbf{m} & \mathbf{c}\cdot\mathbf{m} \end{vmatrix}$$

Example 48. If $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-collinear vectors, then show that $\mathbf{a}\times\mathbf{b} = [\mathbf{a}\ \mathbf{b}\ \mathbf{i}]\,\hat{\mathbf{i}} + [\mathbf{a}\ \mathbf{b}\ \mathbf{j}]\,\hat{\mathbf{j}} + [\mathbf{a}\ \mathbf{b}\ \mathbf{k}]\,\hat{\mathbf{k}}$

Sol. Let $\mathbf{a}\times\mathbf{b} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ ...(i)

$(\mathbf{a}\times\mathbf{b})\cdot\hat{\mathbf{i}} = (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})\cdot\hat{\mathbf{i}}$

$(\mathbf{a}\times\mathbf{b})\cdot\hat{\mathbf{i}} = x$

Also, $(\mathbf{a}\times\mathbf{b})\cdot\hat{\mathbf{j}} = y$ ...(ii)

$(\mathbf{a}\times\mathbf{b})\cdot\hat{\mathbf{k}} = z$

$\mathbf{a}\times\mathbf{b} = [\mathbf{a}\ \mathbf{b}\ \mathbf{i}]\,\hat{\mathbf{i}} + [\mathbf{a}\ \mathbf{b}\ \mathbf{j}]\,\hat{\mathbf{j}} + [\mathbf{a}\ \mathbf{b}\ \mathbf{k}]\,\hat{\mathbf{k}}$

Example 49. If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are any three vectors in space, then show that $(\mathbf{c}+\mathbf{b})\times(\mathbf{c}+\mathbf{a})\cdot(\mathbf{c}+\mathbf{b}+\mathbf{a}) = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

Sol. Here, $(\mathbf{c}+\mathbf{b})\times(\mathbf{c}+\mathbf{a})\cdot(\mathbf{c}+\mathbf{b}+\mathbf{a})$

$\Rightarrow$ $(\mathbf{c}\times\mathbf{c} + \mathbf{c}\times\mathbf{a} + \mathbf{b}\times\mathbf{c} + \mathbf{b}\times\mathbf{a})\cdot(\mathbf{c}+\mathbf{b}+\mathbf{a})$

$\Rightarrow$ $(\mathbf{c}\times\mathbf{a} + \mathbf{b}\times\mathbf{c} + \mathbf{b}\times\mathbf{a})\cdot(\mathbf{c}+\mathbf{b}+\mathbf{a})$ $(\because \mathbf{c}\times\mathbf{c} = 0)$

$\Rightarrow (\mathbf{c}\times\mathbf{a})\cdot\mathbf{c} + (\mathbf{c}\times\mathbf{a})\cdot\mathbf{b} + (\mathbf{c}\times\mathbf{a})\cdot\mathbf{a} + (\mathbf{b}\times\mathbf{c})\cdot\mathbf{c} + (\mathbf{b}\times\mathbf{c})\cdot\mathbf{b}$
$+ (\mathbf{b}\times\mathbf{c})\cdot\mathbf{a} + (\mathbf{b}\times\mathbf{a})\cdot\mathbf{c} + (\mathbf{b}\times\mathbf{a})\cdot\mathbf{b} + (\mathbf{b}\times\mathbf{a})\cdot\mathbf{a}$

$\Rightarrow$ $0 + [\mathbf{c}\ \mathbf{a}\ \mathbf{b}] + 0 + 0 + 0 + [\mathbf{b}\ \mathbf{c}\ \mathbf{a}] + [\mathbf{b}\ \mathbf{a}\ \mathbf{c}] + 0 + 0$

$\Rightarrow$ $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] + [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] - [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ $(\because \mathbf{c}\ \mathbf{a}\ \mathbf{b} = \mathbf{a}\ \mathbf{b}\ \mathbf{c})$

$\Rightarrow$ $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ $(\because [\mathbf{b}\ \mathbf{a}\ \mathbf{c}] = -[\mathbf{a}\ \mathbf{b}\ \mathbf{c}])$

Example 50. *If* $\mathbf{u}, \mathbf{v}$ *and* $\mathbf{w}$ *are three non-coplanar vectors, then* $(\mathbf{u}+\mathbf{v}-\mathbf{w})\cdot[(\mathbf{u}-\mathbf{v})\times(\mathbf{v}-\mathbf{w})]$ *is equal to*

(a) 0 (b) $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$

(c) $\mathbf{u}\cdot(\mathbf{w}\times\mathbf{v})$ (d) $3\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$

Sol. (b) $(\mathbf{u}+\mathbf{v}-\mathbf{w})\cdot[\mathbf{u}-\mathbf{v}\times(\mathbf{v}-\mathbf{w})]$

$\Rightarrow$ $(\mathbf{u}+\mathbf{v}-\mathbf{w})\cdot[(\mathbf{u}\times\mathbf{v})\times(\mathbf{u}\times\mathbf{w}) - 0 + (\mathbf{v}\times\mathbf{w})]$

$= [\mathbf{u}\ \mathbf{u}\ \mathbf{v}] + [\mathbf{v}\ \mathbf{u}\ \mathbf{v}] - [\mathbf{w}\ \mathbf{u}\ \mathbf{v}] - [\mathbf{u}\ \mathbf{u}\ \mathbf{w}] - [\mathbf{v}\ \mathbf{u}\ \mathbf{w}]$
$+ [\mathbf{w}\ \mathbf{u}\ \mathbf{w}] + [\mathbf{u}\ \mathbf{v}\ \mathbf{w}] + [\mathbf{v}\ \mathbf{v}\ \mathbf{w}] - [\mathbf{w}\ \mathbf{v}\ \mathbf{w}]$

$= 0 + 0 - [\mathbf{u}\ \mathbf{v}\ \mathbf{w}] - 0 + [\mathbf{u}\ \mathbf{v}\ \mathbf{w}] + 0 + [\mathbf{u}\ \mathbf{v}\ \mathbf{w}] + 0 - 0$

$= [\mathbf{u}\ \mathbf{v}\ \mathbf{w}] = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$

Example 51. *If* $\mathbf{a}, \mathbf{b}$ *and* $\mathbf{c}$ *are non-coplanar vectors and* λ *is a real number, then the vector* $\mathbf{a} + 2\mathbf{b} + 3\mathbf{c}$, $\lambda\mathbf{b} + 4\mathbf{c}$ *and* $(2\lambda - 1)\mathbf{c}$ *are non-coplanar for*

(a) no value of λ

(b) all except one value of λ

(c) all except two values of λ

(d) all values of λ

Sol. (c) Since, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are non-coplanar vectors.

$\therefore$ $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \neq 0$

Now, $\mathbf{a} + 2\mathbf{b} + 3\mathbf{c}$, $\lambda\mathbf{b} + 4\mathbf{c}$ and $(2\lambda - 1)\mathbf{c}$ will be non-coplanare iff

$(\mathbf{a} + 2\mathbf{b} + 3\mathbf{c})\cdot\{(\lambda\mathbf{b} + 4\mathbf{c})\times(2\lambda - 1)\mathbf{c}\} \neq 0$

i.e., $(\mathbf{a} + 2\mathbf{b} + 3\mathbf{c})\cdot\{\lambda(2\lambda - 1)(\mathbf{b}\times\mathbf{c}) \neq 0$

i.e., $\lambda(2\lambda - 1)[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\} \neq 0$

$\therefore$ $\lambda \neq 0, \frac{1}{2}$

Thus, given vectors will be non-coplanar for all values of λ except two values 0 and $\frac{1}{2}$.

Example 52. *If* x, y *and* z *are distinct scalars such that* $[x\mathbf{a} + y\mathbf{b} + z\mathbf{c},\ x\mathbf{b} + y\mathbf{c} + z\mathbf{a},\ x\mathbf{c} + y\mathbf{a} + z\mathbf{b}] = 0]$ *where* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{c}$ *are non-coplanar vectors, then*

(a) $x + y + z = 0$ (b) $xy + yz + zx = 0$

(c) $x^3 + y^3 + z^3 = 0$ (d) $x^2 + y^2 + z^2 = 0$

Sol. (a) $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are non-coplanar.

$\therefore$ $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \neq 0$

Now, consider $[x\mathbf{a}+y\mathbf{b}+z\mathbf{c},\ x\mathbf{b}+y\mathbf{c}+z\mathbf{a},\ x\mathbf{c}+y\mathbf{a}+z\mathbf{b}]=0$

$$\Rightarrow \begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix} [\mathbf{a}\,\mathbf{b}\,\mathbf{c}]=0 \Rightarrow \begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix}=0 \quad [\because [\mathbf{a}\,\mathbf{b}\,\mathbf{c}]\neq 0]$$

$$\Rightarrow (x+y+z)(x^2+y^2+z^2-xy-yz-zx)=0$$

$$\Rightarrow \frac{1}{2}(x+y+z)\{(x-y)^2\}+(y-z)^2+(z-x)^2\}=0$$

$$\Rightarrow x+y+z=0 \quad \text{or} \quad x=y=z$$

But x, y and z are distinct

$$\therefore x+y+z=0.$$

Example 53. If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are three non-coplanar uni-modular vectors, each inclined with other at an angle 30°, then volume of tetrahedron whose edges are $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ is

(a) $\dfrac{\sqrt{3\sqrt{3}-5}}{12}$ (b) $\dfrac{3\sqrt{3}-5}{12}$

(c) $\dfrac{5\sqrt{2}+3}{12}$ (d) None of these

Sol. (a) Since, the volume of tetrahedron with edges $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is $[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]$.

Where, $\mathbf{a}\cdot\mathbf{a}=\mathbf{b}\cdot\mathbf{b}=\mathbf{c}\cdot\mathbf{c}=1$

and $\mathbf{a}\cdot\mathbf{c}=\mathbf{b}\cdot\mathbf{c}=\mathbf{a}\cdot\mathbf{c}=\dfrac{\sqrt{3}}{2}$ (given)

$$\therefore V=\frac{1}{6}[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]$$

$$\Rightarrow V^2=\frac{1}{36}[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]^2=\frac{1}{36}\begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix}$$

$$=\frac{1}{36}\begin{vmatrix} 1 & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 1 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 1 \end{vmatrix}=\frac{1}{36}\left(\frac{3\sqrt{3}}{4}-\frac{5}{4}\right)$$

$$\therefore V=\frac{1}{12}\sqrt{3\sqrt{3}-5}$$

Exercise for Session 3

1. If $\mathbf{a}$ and $\mathbf{b}$ are two vectors such that $|\mathbf{a}\times\mathbf{b}|=2$, then find the value of $[\mathbf{a}\,\mathbf{b}\,\mathbf{a}\times\mathbf{b}]$.
2. If the vectors $2\hat{\mathbf{i}}-3\hat{\mathbf{j}}$, $\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}}$ and $3\hat{\mathbf{i}}-\hat{\mathbf{k}}$ form three concurrent edges of a parallelopiped , the find the volume of the parallelopiped.
3. If the volume of a parallelopiped whose adjacent edges are $\mathbf{a}=2\hat{\mathbf{i}}+3\hat{\mathbf{j}}+4\hat{\mathbf{k}}$, $\mathbf{b}=\hat{\mathbf{i}}+\alpha\hat{\mathbf{j}}+2\hat{\mathbf{k}}$, $\mathbf{c}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\alpha\hat{\mathbf{k}}$ is 15, then find the value of α, where $\alpha>0$.
4. The position vector of the four angular points of a tetrahedron are $\mathbf{A}(\hat{\mathbf{j}}+2\hat{\mathbf{k}})$, $\mathbf{B}(3\hat{\mathbf{i}}+\hat{\mathbf{k}})$, $\mathbf{C}(4\hat{\mathbf{i}}+3\hat{\mathbf{j}}+6\hat{\mathbf{k}})$ and $\mathbf{D}=(2\hat{\mathbf{i}}+3\hat{\mathbf{j}}+2\hat{\mathbf{k}})$. Find the volume of the tetrahedron *ABCD*.
5. Find the altitude of a parallelopiped whose three coterminous edges are vectors $\mathbf{A}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$, $\mathbf{B}=2\hat{\mathbf{i}}+4\hat{\mathbf{j}}-\hat{\mathbf{k}}$ and $\mathbf{C}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+3\hat{\mathbf{k}}$ with $\mathbf{A}$ and $\mathbf{B}$ as the sides of the base of the parallelopiped.
6. Examine whether the vectors $\mathbf{a}=2\hat{\mathbf{i}}+3\hat{\mathbf{j}}+2\hat{\mathbf{k}}$, $\mathbf{b}=\hat{\mathbf{i}}-\hat{\mathbf{j}}+2\hat{\mathbf{k}}$ and $\mathbf{c}=3\hat{\mathbf{i}}+2\hat{\mathbf{j}}+4\hat{\mathbf{k}}$ from a left handed or a right handed system.
7. Prove that the four points $4\hat{\mathbf{i}}+5\hat{\mathbf{j}}+\hat{\mathbf{k}}$, $-(\hat{\mathbf{j}}+\hat{\mathbf{k}})$, $(3\hat{\mathbf{i}}+9\hat{\mathbf{j}}+4\hat{\mathbf{k}})$ and $4(-\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})$ are coplanar.
8. Prove that $[\mathbf{a}\,\mathbf{b}\,\mathbf{c}][\mathbf{u}\,\mathbf{v}\,\mathbf{w}]=\begin{vmatrix} \mathbf{a}\cdot\mathbf{u} & \mathbf{b}\cdot\mathbf{u} & \mathbf{c}\cdot\mathbf{u} \\ \mathbf{a}\cdot\mathbf{v} & \mathbf{b}\cdot\mathbf{v} & \mathbf{c}\cdot\mathbf{v} \\ \mathbf{a}\cdot\mathbf{w} & \mathbf{b}\cdot\mathbf{w} & \mathbf{c}\cdot\mathbf{w} \end{vmatrix}$
9. If $[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]=2$, then find the value of $[(\mathbf{a}+2\mathbf{b}-\mathbf{c})(\mathbf{a}-\mathbf{b})(\mathbf{a}-\mathbf{b}-\mathbf{c})]$.
10. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three non-coplanar vectors, then find the value of

$$\frac{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}{\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})}+\frac{\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})}{\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})}+\frac{\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})}{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}$$

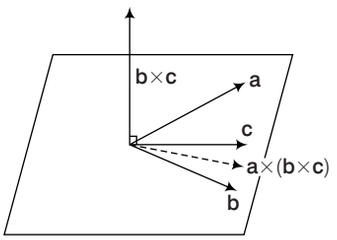
Session 4

Vector Triple Product

Vector Triple Product

It is defined for three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ as the vector $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$.

This vector being perpendicular to $\mathbf{a}$ and $\mathbf{b}\times\mathbf{c}$. But $\mathbf{b}\times\mathbf{c}$ is a vector perpendicular to the plane of $\mathbf{b}$ and $\mathbf{c}$.



$\therefore\ \mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ lie in the plane of $\mathbf{b}$ and $\mathbf{c}$, i.e., it is coplanar with $\mathbf{b}$ and $\mathbf{c}$.

i.e., $$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = l\mathbf{b} + m\mathbf{c} \qquad \ldots(\text{i})$$

Taking the scalar product of this equation with $\mathbf{a}$, we get

$$0 = l(\mathbf{a}\cdot\mathbf{b}) + m(\mathbf{a}\cdot\mathbf{c}) \quad \left[\because \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \text{ is } \perp \text{ to } \mathbf{a} \;\Rightarrow\; \therefore (\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a} = 0\right]$$

$$\Rightarrow \quad l(\mathbf{a}\cdot\mathbf{b}) = -m(\mathbf{a}\cdot\mathbf{c})$$

$$\Rightarrow \quad \frac{l}{\mathbf{a}\cdot\mathbf{c}} = -\frac{m}{\mathbf{a}\cdot\mathbf{b}} = \lambda \qquad \text{(say)}$$

$$\Rightarrow \quad l = \lambda(\mathbf{a}\cdot\mathbf{c})$$

and $$m = -\lambda(\mathbf{a}\cdot\mathbf{b})$$

Substituting the value of l and m in Eq. (i), we get

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \lambda\,[(\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}]$$

Here, the value of λ can be determined by taking specific values of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

If we choose the coordinate axes in such a way that,

$$\mathbf{a} = a_1\hat{\mathbf{i}},\ \mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}}$$

and $$\mathbf{c} = c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}} + c_3\hat{\mathbf{k}},$$

it is easy to show that $\lambda = 1$.

Hence, $$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}$$

and $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\times\mathbf{c}$, if some of all $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are zero vectors or $\mathbf{a}$ and $\mathbf{c}$ are collinear.

Remarks

1. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is a linear combination of those two vectors which are with in brackets.
2. If $\mathbf{r} = \mathbf{a}\times(\mathbf{b}\times\mathbf{c})$, then $\mathbf{r}$ is perpendicular to $\mathbf{a}$ and lie in the plane of $\mathbf{b}$ and $\mathbf{c}$.
3. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}$
   $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{c}\cdot\mathbf{a})\,\mathbf{b} - (\mathbf{c}\cdot\mathbf{b})\mathbf{a}$
   Aid to memory
   $$\text{I}\times(\text{II}\times\text{III}) = (\text{I}\cdot\text{III})\,\text{II} - (\text{I}\cdot\text{II})\,\text{III}$$
4. $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = ((\mathbf{a}\times\mathbf{b})\cdot\mathbf{d})\,\mathbf{c} - ((\mathbf{a}\times\mathbf{b})\cdot\mathbf{c})\,\mathbf{d}$
   $= [\mathbf{a}\,\mathbf{b}\,\mathbf{d}]\,\mathbf{c} - [\mathbf{a}\,\mathbf{b}\,\mathbf{c}]\,\mathbf{d}$
   $\Rightarrow$ The vector $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$ lies in the plane of $\mathbf{c}$ and $\mathbf{d}$.
   Also, $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = -(\mathbf{c}\times\mathbf{d})\times(\mathbf{a}\times\mathbf{b})$
   $= -\{((\mathbf{c}\times\mathbf{d})\cdot\mathbf{b})\,\mathbf{a} - ((\mathbf{c}\times\mathbf{d})\cdot(\mathbf{a}\,\mathbf{b})\}$
   $= -[\mathbf{c}\,\mathbf{d}\,\mathbf{b}]\,\mathbf{a} + [\mathbf{c}\,\mathbf{d}\,\mathbf{a}]\,\mathbf{b}$
   Which shows that $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$ lies in the plane of $\mathbf{a}$ and $\mathbf{b}$.
   Thus, the vector $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$ lies along the common section of the plane of $\mathbf{c}$,$\mathbf{d}$ and that of the plane of $\mathbf{a}$, $\mathbf{b}$.

Lagrange's Identity

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = \begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{a}\cdot\mathbf{d} \\ \mathbf{b}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix}$$

Proof $\text{LHS} = (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = \mathbf{u}\cdot(\mathbf{c}\times\mathbf{d})$

where $\mathbf{u} = \mathbf{a}\times\mathbf{b} = (\mathbf{u}\times\mathbf{c})\cdot\mathbf{d}$

$= ((\mathbf{a}\times\mathbf{b})\times\mathbf{c})\cdot\mathbf{d}$

$= ((\mathbf{c}\cdot\mathbf{a})\cdot\mathbf{b} - (\mathbf{c}\cdot\mathbf{a})\cdot\mathbf{a})\cdot\mathbf{d}$

$= (\mathbf{c}\cdot\mathbf{a})\,(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{c}\cdot\mathbf{b})\,(\mathbf{a}\cdot\mathbf{d})$

$= (\mathbf{a}\cdot\mathbf{c})\,(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{b}\cdot\mathbf{c})\,(\mathbf{a}\cdot\mathbf{d})$

$$= \begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{a}\cdot\mathbf{d} \\ \mathbf{b}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix}$$

Example 54. If $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}, \mathbf{b} = \hat{\mathbf{i}} + \hat{\mathbf{j}}, \mathbf{c} = \hat{\mathbf{i}}$ and $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$, then $\lambda + \mu$ is equal to

(a) 0 (b) 1
(c) 2 (d) 3

Sol. (a) $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$

$\Rightarrow \quad (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\,\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{b}$

$\Rightarrow \quad \lambda = -\mathbf{b}\cdot\mathbf{c}, \mu = \mathbf{a}\cdot\mathbf{c}$

$\therefore \quad \lambda + \mu = \mathbf{a}\cdot\mathbf{c} - \mathbf{b}\cdot\mathbf{c} = (\mathbf{a} - \mathbf{b})\cdot\mathbf{c}$

$= \{(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) - (\mathbf{i} + \hat{\mathbf{j}})\} = \hat{\mathbf{k}}\cdot\hat{\mathbf{i}} = 0$

Example 55. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three non-parallel unit vectors such that $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\frac{1}{2}\mathbf{b}$, then find the angles which $\mathbf{a}$ makes with $\mathbf{b}$ and $\mathbf{c}$.

Sol. We have, $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\frac{1}{2}\mathbf{b}$

$\Rightarrow \quad (\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}=\frac{1}{2}\mathbf{b}$

$\Rightarrow \quad \mathbf{a}\cdot\mathbf{c}=\frac{1}{2}$ and $\mathbf{a}\cdot\mathbf{b}=0$ (comparing $\mathbf{c}$ and $\mathbf{b}$)

$\Rightarrow \quad \mathbf{a}\cdot\mathbf{c}=\frac{1}{2}$ and $\mathbf{a}\perp\mathbf{b}$

Suppose $\mathbf{a}$ makes angle θ with $\mathbf{c}$. Then, $\mathbf{a}\cdot\mathbf{c}=\frac{1}{2}$

$\Rightarrow \quad |\mathbf{a}||\mathbf{c}|\cos\theta=\frac{1}{2} \Rightarrow \cos\theta=\frac{1}{2} \quad (\because |\mathbf{a}||\mathbf{c}|\neq 0)$

$\Rightarrow \quad \theta=\frac{\pi}{3}$

Thus, $\mathbf{a}$ is perpendicular to $\mathbf{b}$ and makes an angle $\frac{\pi}{3}$ with $\mathbf{c}$.

Example 56. If $\mathbf{a}=-\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $\mathbf{b}=2\hat{\mathbf{i}}+\hat{\mathbf{k}}$, then find the vector x satisfying the conditions.

(i) that it is coplanar with $\mathbf{a}$ and $\mathbf{b}$.

(ii) it is perpendicular to $\mathbf{b}$.

(iii) $\mathbf{a}\cdot\mathbf{x}=7$

Sol. Since, x is in the plane of $\mathbf{a}$ and $\mathbf{b}$ and is perpendicular to $\mathbf{b}$.

$\therefore \quad x=\lambda\{\mathbf{b}\times(\mathbf{a}\times\mathbf{b})\}$

$\Rightarrow \quad x=\lambda\{(\mathbf{b}\cdot\mathbf{b})\mathbf{a}-(\mathbf{b}\cdot\mathbf{a})\mathbf{b}\}$

$=\lambda\{5(-\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})-(-1)(2\hat{\mathbf{i}}+\hat{\mathbf{k}})\}$

$=\lambda\{-5\hat{\mathbf{i}}+5\hat{\mathbf{j}}+5\hat{\mathbf{k}}+2\hat{\mathbf{i}}+\hat{\mathbf{k}}\}$

$=\lambda\{-3\hat{\mathbf{i}}+5\hat{\mathbf{j}}+6\hat{\mathbf{k}}\}$

Now, $\mathbf{a}\cdot\mathbf{x}=7$

$\Rightarrow \quad -3\lambda+5\lambda+6\lambda=7$

$\Rightarrow \quad 8\lambda=7 \Rightarrow \lambda=\frac{7}{8}$

Hence, $x=\frac{7}{8}(-3\hat{\mathbf{i}}+5\hat{\mathbf{j}}+6\hat{\mathbf{k}})$

Example 57. Prove that $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})+\mathbf{b}\times(\mathbf{c}\times\mathbf{a})+\mathbf{c}\times(\mathbf{a}\times\mathbf{b})=0$

Sol. We have, $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})+\mathbf{b}\times(\mathbf{c}\times\mathbf{a})+\mathbf{c}\times(\mathbf{a}\times\mathbf{b})$

$=\{(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}\}+\{(\mathbf{b}\cdot\mathbf{a})\mathbf{c}-(\mathbf{b}\cdot\mathbf{c})\mathbf{a}\}+\{(\mathbf{c}\cdot\mathbf{b})\mathbf{a}-(\mathbf{c}\cdot\mathbf{a})\mathbf{b}\}$

$=[(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}+(\mathbf{a}\cdot\mathbf{b})\mathbf{c}-(\mathbf{b}\cdot\mathbf{c})\mathbf{a}+(\mathbf{b}\cdot\mathbf{c})\mathbf{a}-(\mathbf{a}\cdot\mathbf{c})\mathbf{b}]=0$

Example 58. ***Show that the vectors*** $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$, $\mathbf{b}\times(\mathbf{c}\times\mathbf{a})$ and $\mathbf{c}\times((\mathbf{a}\times\mathbf{b})$ are coplanar.

Sol. Let $\mathbf{p}=\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$, $\mathbf{q}=\mathbf{b}\times(\mathbf{c}\times\mathbf{a})$ and $\mathbf{r}=\mathbf{c}\times(\mathbf{a}\times\mathbf{b})$, then

$\mathbf{p}+\mathbf{q}+\mathbf{r}=\mathbf{a}\times(\mathbf{b}\times\mathbf{c})+\mathbf{b}\times(\mathbf{c}\times\mathbf{a})+\mathbf{c}\times(\mathbf{a}\times\mathbf{b})=0$

$\Rightarrow \quad \mathbf{p}=(-1),\ \mathbf{q}=(-1)\,\mathbf{r}$

which shows $\mathbf{p}$ is linear combination of $\mathbf{q}$ and $\mathbf{r}$.

So, $\mathbf{p}$, $\mathbf{q}$ are coplanar.

Hence, $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$, $\mathbf{b}\times(\mathbf{c}\times\mathbf{a})$ and $\mathbf{c}\times(\mathbf{a}\times\mathbf{b})$ are coplanar.

Example 59. Prove that $[\mathbf{a}\times\mathbf{b}\ \mathbf{b}\times\mathbf{c}\ \mathbf{c}\times\mathbf{a}]=[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$

Sol. We have, $[\mathbf{a}\times\mathbf{b}\ \mathbf{b}\times\mathbf{c}\ \mathbf{c}\times\mathbf{a}]$

$=\{(\mathbf{a}\times\mathbf{b})\times(\mathbf{b}\times\mathbf{c})\}\cdot(\mathbf{c}\times\mathbf{a})$

$=\{\mathbf{d}\times(\mathbf{b}\times\mathbf{c})\}\cdot(\mathbf{c}\times\mathbf{a})$ [where, $\mathbf{d}=(\mathbf{a}\times\mathbf{b})$]

$=[(\mathbf{d}\cdot\mathbf{c})\mathbf{b}-(\mathbf{d}\cdot\mathbf{b})\mathbf{c}]\cdot(\mathbf{c}\times\mathbf{a})$

$=[\{(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c})\,\mathbf{b}-(\mathbf{a}\times\mathbf{b})\,\mathbf{b}\}\,\mathbf{c}]\cdot(\mathbf{c}\times\mathbf{a})$

$=\{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,\mathbf{b}-0\}.(\mathbf{c}\times\mathbf{a})$ $[\because (\mathbf{a}\ \mathbf{b}\ \mathbf{b})=0]$

$=\{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,\{\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})\}$

$=[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,[\mathbf{b}\ \mathbf{c}\ \mathbf{a}]$ $\{\because [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=[\mathbf{b}\ \mathbf{c}\ \mathbf{a}]\}$

$=[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$

Example 60. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar show $[\mathbf{a}\times\mathbf{b}\ \mathbf{b}\times\mathbf{c}\ \mathbf{c}\times\mathbf{a}]$ are coplanar.

Sol. Since, $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ are coplanar.

$\Rightarrow \quad [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=0$

and $[\mathbf{a}\times\mathbf{b}\ \mathbf{b}\times\mathbf{c}\ \mathbf{c}\times\mathbf{a}]=[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2=0$

$\therefore [\mathbf{a}\times\mathbf{b}\ \mathbf{b}\times\mathbf{c}\ \mathbf{c}\times\mathbf{a}]$ are coplanar, if $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar.

Example 61. If $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are vectors such that $|\mathbf{B}|=|\mathbf{C}|$, prove that $\{(\mathbf{A}+\mathbf{B})\times(\mathbf{A}+\mathbf{C})\}\times(\mathbf{B}\times\mathbf{C})\cdot(\mathbf{B}+\mathbf{C})=0$.

Sol. Let $\mathbf{R}_1=\mathbf{A}+\mathbf{B}$, $\mathbf{R}_2=\mathbf{A}+\mathbf{C}$, $\mathbf{R}_3=\mathbf{B}+\mathbf{C}$

$\therefore$ LHS $=\{(\mathbf{A}+\mathbf{B})\times(\mathbf{A}+\mathbf{C})\}\times(\mathbf{B}\times\mathbf{C})\}\times(\mathbf{B}\times\mathbf{C})\cdot(\mathbf{B}+\mathbf{C})$

$\Rightarrow \quad \{(\mathbf{R}_1\times\mathbf{R}_2)\times(\mathbf{B}\times\mathbf{C})\}\cdot\mathbf{R}_3$

$\Rightarrow \quad [\{\mathbf{R}_1\cdot(\mathbf{B}\times\mathbf{C})\}\,\mathbf{R}_2-\{\mathbf{R}_2\cdot(\mathbf{B}\times\mathbf{C})\,\mathbf{R}_1\}]\cdot\mathbf{R}_3$

$\Rightarrow \quad [\mathbf{A}+\mathbf{B}\ \mathbf{B}\ \mathbf{C}]\,[\mathbf{R}_2\cdot\mathbf{R}_3]-[\mathbf{A}+\mathbf{C}\ \mathbf{B}\ \mathbf{C}]\,(\mathbf{R}_1\cdot\mathbf{R}_3)$

$\Rightarrow \quad \{[\mathbf{A}\ \mathbf{B}\ \mathbf{C}]+[\mathbf{B}\ \mathbf{B}\ \mathbf{C}]\}\,[(\mathbf{A}+\mathbf{C})\cdot(\mathbf{B}+\mathbf{C})]-\{[\mathbf{A}\ \mathbf{B}\ \mathbf{C}]+[\mathbf{C}\ \mathbf{B}\ \mathbf{C}]\}\,[(\mathbf{A}+\mathbf{B})\cdot(\mathbf{B}+\mathbf{C})]$

$\Rightarrow \quad [\mathbf{A}\ \mathbf{B}\ \mathbf{C}](\mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}+\mathbf{C}\cdot\mathbf{B}+\mathbf{C}\cdot\mathbf{C})-[\mathbf{A}\ \mathbf{B}\ \mathbf{C}](\mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}+\mathbf{B}\cdot\mathbf{B}+\mathbf{B}\cdot\mathbf{C})$

$\Rightarrow \quad [\mathbf{A}\ \mathbf{B}\ \mathbf{C}](\mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}+\mathbf{C}\cdot\mathbf{B}+|\mathbf{C}|^2-\mathbf{A}\cdot\mathbf{B}-\mathbf{A}\cdot\mathbf{C}+\mathbf{B}\cdot\mathbf{B}+\mathbf{B}\cdot\mathbf{C})$

$\Rightarrow \quad [\mathbf{A}\ \mathbf{B}\ \mathbf{C}](|\mathbf{C}|^2-|\mathbf{B}|^2)$

$\Rightarrow \quad [\mathbf{A}\ \mathbf{B}\ \mathbf{C}](0)$ $(\because |\mathbf{B}|=|\mathbf{C}|)$

$\Rightarrow \quad 0=\text{RHS}$

Example 62. *If* $\mathbf{b}$ *and* $\mathbf{c}$ *are two non-collinear vector such that* $\mathbf{a} \parallel (\mathbf{b} \times \mathbf{c})$*, then prove that* $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c})$ *is equal to* $|\mathbf{a}|^2 (\mathbf{b} \cdot \mathbf{c})$.

Sol. Since, $\mathbf{a} \parallel (\mathbf{b} \times \mathbf{c})$, therefore $\mathbf{a} \perp \mathbf{b}$ and $\mathbf{a} \perp \mathbf{c}$

$$\Rightarrow \quad \mathbf{a} \cdot \mathbf{b} = 0 \text{ and } \mathbf{a} \cdot \mathbf{c} = 0$$

Now, consider $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c}) = \begin{vmatrix} \mathbf{a} \cdot \mathbf{a} & \mathbf{a} \cdot \mathbf{c} \\ \mathbf{b} \cdot \mathbf{a} & \mathbf{b} \cdot \mathbf{c} \end{vmatrix} = \begin{vmatrix} \mathbf{a} \cdot \mathbf{a} & 0 \\ 0 & \mathbf{b} \cdot \mathbf{c} \end{vmatrix}$

$$= (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{c}) = |\mathbf{a}^2| (\mathbf{b} \cdot \mathbf{c}).$$

Reciprocal System of Vectors

The two system of vectors are called reciprocal system of vectors if by taking dot product, we get unity.

Thus, if $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three non-coplanar vectors and if.

$$\mathbf{a}' = \frac{\mathbf{b} \times \mathbf{c}}{|\mathbf{a}\ \mathbf{b}\ \mathbf{c}|},\ \mathbf{b}' = \frac{\mathbf{c} \times \mathbf{a}}{|\mathbf{a}\ \mathbf{b}\ \mathbf{c}|} \text{ and } \mathbf{c}' = \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a}\ \mathbf{b}\ \mathbf{c}|}$$

Then $\mathbf{a}'$, $\mathbf{b}'$ and $\mathbf{c}'$ are said to be reciprocal system of vectors for the vector $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Remarks

1. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ are reciprocal system of vectors, then
$$\mathbf{a} \cdot \mathbf{a}' = \frac{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})}{(\mathbf{a}\ \mathbf{b}\ \mathbf{c})} = \frac{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} = 1$$
Similarly, $\mathbf{b} \cdot \mathbf{b}' = \mathbf{c} \cdot \mathbf{c}' = 1$
2. $\mathbf{a} \cdot \mathbf{b}' = \mathbf{a} \cdot \mathbf{c}' = \mathbf{b} \cdot \mathbf{a}' = \mathbf{b} \cdot \mathbf{a}' = \mathbf{c} \cdot \mathbf{a}' = \mathbf{c} \cdot \mathbf{b}' = 0$
3. $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \cdot [\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}'] = 1$
Proof : We have,
$$[\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}'] = \left[\frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} = \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} = \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}\right]$$
$$= \frac{1}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^3} [\mathbf{b} \times \mathbf{c}\ \mathbf{c} \times \mathbf{a}\ \mathbf{a} \times \mathbf{b}] = \frac{1}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^3} [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$$
$$= \frac{1}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$$
$\therefore [\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}'] \cdot [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 1$
4. The orthogonal triad of vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ is self reciprocal.
Let $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ be the system of vectors reciprocal to the system $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ then, we have,
$$\hat{\mathbf{i}} = \frac{\hat{\mathbf{j}} \times \hat{\mathbf{k}}}{[\hat{\mathbf{i}}\ \hat{\mathbf{j}}\ \hat{\mathbf{k}}]} = \hat{\mathbf{j}}$$
Similarly, $\hat{\mathbf{j}}' = \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}' = \hat{\mathbf{k}}$
5. $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are non-coplanar iff $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ are non-coplanar.
$$\because \quad [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\ [\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}'] = 1$$
$$\therefore \quad [\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}'] = \frac{1}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$$
So, $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \neq 0 \Leftrightarrow [\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}'] \neq 0$
Thus, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are non-coplanar iff $[\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}']$ are non-coplanar.
6. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are non-coplanar vectors, then
$$\mathbf{r} = \frac{[\mathbf{x}\ \mathbf{b}\ \mathbf{c}]\ \mathbf{a} + [\mathbf{x}\ \mathbf{c}\ \mathbf{a}]\ \mathbf{b} + [\mathbf{x}\ \mathbf{a}\ \mathbf{b}]\ \mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$$

Example 63. *Find the set of vectors reciprocal to the set of vectors* $2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}, \hat{\mathbf{i}} - \hat{\mathbf{j}} - 2\hat{\mathbf{k}}, -\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

Sol. Let the given vector be $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.

Now, $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \begin{vmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ -1 & 2 & 2 \end{vmatrix}$

$$= 2(-2 + 4) - 3(2 - 2) - 1(2 - 1)$$
$$= 4 - 1 = 3$$

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & -2 \\ -1 & 2 & 2 \end{vmatrix} = 2\hat{\mathbf{i}} + \hat{\mathbf{k}}$$

$$\mathbf{c} \times \mathbf{a} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & 2 & 2 \\ 2 & 3 & -1 \end{vmatrix} = -8\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 7\hat{\mathbf{k}}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 3 & -1 \\ 1 & -1 & -2 \end{vmatrix} = -7\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$$

Hence, $\mathbf{a}' = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} = \frac{2\hat{\mathbf{i}} + \hat{\mathbf{k}}}{3}$

$$\mathbf{b}' = \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} = \frac{-8\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 7\hat{\mathbf{k}}}{3}$$

and $\mathbf{c}' = \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} = \frac{-7\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}}}{3}$

Example 64. Find a set of vector reciprocal to the vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$.

Sol. Let the given vectors be denoted by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ where $\mathbf{c} = \mathbf{a} \times \mathbf{b}$.

$$\therefore \quad [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \times \mathbf{b})^2 \quad \text{...(i)}$$

and let the reciprocal system of vector be $\mathbf{a}'$, $\mathbf{b}'$ and $\mathbf{c}'$.

$$\therefore \quad \mathbf{a}' = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} = \frac{\mathbf{b} \times (\mathbf{a} \times \mathbf{b})}{[\mathbf{a} \times \mathbf{b}]^2}$$

$$\mathbf{b}' = \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} = \frac{(\mathbf{a} \times \mathbf{b}) \times \mathbf{a}}{[\mathbf{a} \times \mathbf{b}]^2}$$

$$\mathbf{c}' = \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} = \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{a} \times \mathbf{b}]^2}$$

$\therefore \mathbf{a}'$, $\mathbf{b}'$ and $\mathbf{c}'$ are required reciprocal system of vectors for $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$.

Example 65. If $\mathbf{a}' = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}, \mathbf{b}' = \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}, \mathbf{c}' = \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$, then show that
$\mathbf{a} \times \mathbf{a}' + \mathbf{b} \times \mathbf{b}' + \mathbf{c} \times \mathbf{c}' = 0$, where $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are non-coplanar.

Sol. Here, $\mathbf{a}\times\mathbf{a}' = \dfrac{\mathbf{a}\times(\mathbf{b}\times\mathbf{c})}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$

$$\mathbf{a}\times\mathbf{a}' = \frac{(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} \quad ...(i)$$

Similarly,

$$\mathbf{b}\times\mathbf{b}' = \frac{(\mathbf{b}\cdot\mathbf{a})\mathbf{c}-(\mathbf{b}\cdot\mathbf{c})\mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} \quad ...(ii)$$

$$\mathbf{c}\times\mathbf{c}' = \frac{(\mathbf{c}\cdot\mathbf{b})\mathbf{a}-(\mathbf{c}\cdot\mathbf{a})\mathbf{b}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} \quad ...(iii)$$

$$\mathbf{a}\times\mathbf{a}'+\mathbf{b}\times\mathbf{b}'+\mathbf{c}\times\mathbf{c}'$$

$$= \frac{(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}+(\mathbf{b}\cdot\mathbf{a})\mathbf{c}-(\mathbf{b}\cdot\mathbf{c})\mathbf{a}+(\mathbf{c}\cdot\mathbf{b})\mathbf{a}-(\mathbf{c}\cdot\mathbf{a})\mathbf{b}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$$

$(\because \mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a})$

$= 0$

Example 66. If (e_1, e_2, e_3) and (e_1', e_2', e_3') are two sets of non-coplanar vectors such that $\hat{\mathbf{i}} = 1, 2, 3$, we have $e_{\mathbf{i}}\cdot e_{\mathbf{j}}' = \begin{cases}1, & \text{if } \hat{\mathbf{i}}=\hat{\mathbf{j}}\\ 0, & \text{if } \hat{\mathbf{i}}\neq\hat{\mathbf{j}}\end{cases}$, then show that $[e_1, e_2, e_3], [e_1'\ e_2'\ e_3'] = 1$.

Sol. We have, $e_1\cdot e_2' = 0, e_1\cdot e_3' = 0$

$\Rightarrow$ $e_1 \perp e_2'$ and $e_1 \perp e_3'$

$\therefore$ $e_1 \parallel (e_2'\times e_3')$

$\therefore$ $e_1 = \lambda\,(e_2'\times e_3')$ ...(i)

$$e_1\cdot e_1' = \lambda(e_2'\times e_3')\cdot e_1'$$

$$1 = \lambda\,[e_1'\ e_2'\ e_3'] \qquad (\because e_1 e_1' = 1, \text{ given})$$

$$\lambda = \frac{1}{[e_1'\ e_2'\ e_3']}$$

From Eq. (i),

$$e_1 = \frac{e_2'\times e_3'}{[e_1'\ e_2'\ e_3']}$$

Similarly, $e_2 = \dfrac{e_3'\times e_1'}{[e_1'\ e_2'\ e_3']}$

and $e_3 = \dfrac{e_1'\times e_2'}{[e_1'\ e_2'\ e_3']}$

$\therefore$ $[e_1\ e_2\ e_3] = \dfrac{[e_2'\times e_3'\ e_3'\times e_1'\ e_1'\times e_2']}{[e_1'\ e_2'\ e_3']^3}$

$\Rightarrow$ $[e_1, e_2, e_3][e_1'\ e_2'\ e_3']^3 = [e_2'\times e_3'\ e_3'\times e_1'\ e_1'\times e_2']$ ...(ii)

Now, $[e_2'\times e_3'\ e_3'\times e_1'\ e_1'\times e_2'] = [e_1'\ e_2'\ e_3']^2$ ...(iii)

$\therefore$ From Eqs. (ii) and (iii), we get

$$[e_1\ e_2\ e_3][e_1'\ e_2'\ e_3']^3 = [e_1'\ e_2'\ e_3']^2$$

$$[e_1\ e_2\ e_3][e_1'\ e_2'\ e_3'] = 1$$

Solving of Vector Equations

Solving a vector equation means determining an unknown vector (or a number of vectors satisfying the given conditions)

Generally, to solve vector equations we express the unknown as the linear combination of three non-coplanar vector as; $\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z(\mathbf{a}\times\mathbf{b})$; as $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a}\times\mathbf{b}$ are non-coplanar and find x, y and z using given conditions.

Sometimes, we can directly solve the given condition it would be more clear from some examples.

Example 67. Solve the vector equation $\mathbf{r}\times\mathbf{b} = \mathbf{a}\times\mathbf{b}$, $\mathbf{r}\cdot\mathbf{c} = 0$ provided that $\mathbf{c}$ is not perpendicular to $\mathbf{b}$.

Sol. We are given,

$$\mathbf{r}\times\mathbf{b} = \mathbf{a}\times\mathbf{b}$$

$\Rightarrow$ $(\mathbf{r}-\mathbf{a})\times\mathbf{b} = 0$

Hence, $(\mathbf{r}-\mathbf{a})$ and $\mathbf{b}$ are parallel.

$\Rightarrow$ $\mathbf{r}-\mathbf{a} = t\mathbf{b}$

and we known $\mathbf{r}\cdot\mathbf{c} = 0$, ...(i)

$\therefore$ Taking dot product of Eq. (i) by $\mathbf{c}$, we get

$$\mathbf{r}\cdot\mathbf{a}-\mathbf{a}\cdot\mathbf{c} = t(\mathbf{b}\cdot\mathbf{c})$$

$\Rightarrow$ $0-\mathbf{a}\cdot\mathbf{c} = t(\mathbf{b}\cdot\mathbf{c})$

$\Rightarrow$ $t = -\left(\dfrac{\mathbf{a}\cdot\mathbf{c}}{\mathbf{b}\cdot\mathbf{c}}\right)$ ...(ii)

$\therefore$ From Eqs. (i) and (ii) solution of $\mathbf{r}$ is

$$\mathbf{r} = \mathbf{a} - \left(\frac{\mathbf{a}\cdot\mathbf{c}}{\mathbf{b}\cdot\mathbf{c}}\right)\mathbf{b}$$

Example 68. Solve for $\mathbf{x}$, such that $\mathbf{A}\cdot\mathbf{X} = \mathbf{C}$ and $\mathbf{A}\times\mathbf{X} = \mathbf{B}$ with $C \neq 0$.

Sol. We have, $\mathbf{A}\times\mathbf{X} = \mathbf{B}$

Taking vector product of both sides with $\mathbf{A}$, we get

$$\mathbf{A}\times\mathbf{B} = \mathbf{A}\times(\mathbf{A}\times\mathbf{X})$$

$$= (\mathbf{A}\cdot\mathbf{X})\mathbf{A}-(\mathbf{A}\cdot\mathbf{A})\mathbf{X}$$

$$= \mathbf{C}\mathbf{A}-|\mathbf{A}|^2\mathbf{X}$$

(using $\mathbf{A}\cdot\mathbf{X} = \mathbf{C}$ and $\mathbf{A}\cdot t\mathbf{A} = |\mathbf{A}|^2$)

$\Rightarrow$ $|\mathbf{A}|^2\mathbf{X} = \mathbf{C}\mathbf{A}-\mathbf{A}\times\mathbf{B}$

or $\mathbf{X} = \dfrac{\mathbf{C}\mathbf{A}+\mathbf{B}\times\mathbf{A}}{|\mathbf{A}|^2}$

Example 69. Solve the vector equation $\mathbf{r}\times\mathbf{a}+k\mathbf{r} = \mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are two given vector and k is any scalar.

Sol. Since, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a}\times\mathbf{b}$ are two non-coplanar vectors.

$$\mathbf{r} = x\mathbf{a}+y\mathbf{b}+z(\mathbf{a}\times\mathbf{b}) \quad ...(i)$$

(where, x, y and z scalars)

On putting $\mathbf{r}$ in $\mathbf{r}\times\mathbf{a}+k\mathbf{r}=\mathbf{b}$, we get

$$\{x\mathbf{a}+y\mathbf{b}+z(\mathbf{a}\times\mathbf{b})\}\times\mathbf{a}+k\{x\mathbf{a}+y\mathbf{b}+z(\mathbf{a}\times\mathbf{b})\}=\mathbf{b}$$

$$\Rightarrow y(\mathbf{b}\times\mathbf{a})+z\{(\mathbf{a}\cdot\mathbf{a})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{a}\}+k\{x\mathbf{a}+y\mathbf{b}+z(\mathbf{a}\times\mathbf{b})\}=\mathbf{b}$$

$$\Rightarrow \{kx-z(\mathbf{a}\cdot\mathbf{b})\}\mathbf{a}+\{ky+z(\mathbf{a}\cdot\mathbf{a})\}\mathbf{b}+\{(y+zk)\}(\mathbf{a}\times\mathbf{b})=\mathbf{b}$$

$$\Rightarrow kx-z(\mathbf{a}\cdot\mathbf{b})=0,\ ky+z(\mathbf{a}\cdot\mathbf{a})=1$$

$$\Rightarrow -y+zk=0$$

On solving these equations, we get

$$z=\frac{1}{k^2+|\mathbf{a}|^2},$$

$$x=\frac{\mathbf{a}\cdot\mathbf{b}}{k(|\mathbf{a}|^2+k^2)}$$

and

$$y=\frac{k}{k^2+|\mathbf{a}|^2}$$

On putting theses value in Eq. (i), we get the solution,

$$\mathbf{r}=\frac{(\mathbf{a}\cdot\mathbf{b})}{k(k^2+|\mathbf{a}|^2)}+\frac{k}{k^2+|\mathbf{a}|^2}(\mathbf{b})+\frac{1}{k^2+|\mathbf{a}|^2}(\mathbf{a}\times\mathbf{b})$$

$\mathbf{r}=\dfrac{1}{k^2+|\mathbf{a}|^2}\left[\dfrac{(\mathbf{a}\cdot\mathbf{b})\,\mathbf{a}}{k}+(k)\,\mathbf{b}+(\mathbf{a}\times\mathbf{b})\right]$ is required solution.

Example 70. Solve for vectors $\mathbf{A}$ and $\mathbf{B}$, where $\mathbf{A}+\mathbf{B}=\mathbf{a}$, $\mathbf{A}\times\mathbf{B}=\mathbf{b}$, $\mathbf{A}\cdot\mathbf{a}=1$

Sol. We have, $\mathbf{A}+\mathbf{B}=\mathbf{a}$

$$\Rightarrow \mathbf{A}\cdot\mathbf{a}+\mathbf{B}\cdot\mathbf{a}=\mathbf{a}\cdot\mathbf{a}$$

$$\Rightarrow 1+\mathbf{B}\cdot\mathbf{a}=\mathbf{a}^2 \qquad (\text{given } \mathbf{A}\cdot\mathbf{a}=1)$$

$$\Rightarrow \mathbf{B}\cdot\mathbf{a}=\mathbf{a}^2-1 \qquad \ldots(\text{i})$$

Also, $\mathbf{A}\times\mathbf{B}=\mathbf{b}$

$$\Rightarrow \mathbf{a}\times(\mathbf{A}\times\mathbf{B})=\mathbf{a}\times\mathbf{b}$$

$$\Rightarrow (\mathbf{a}\cdot\mathbf{B})\mathbf{A}-(\mathbf{a}\cdot\mathbf{A})\mathbf{B}=\mathbf{a}\times\mathbf{b}$$

$$\Rightarrow (\mathbf{a}^2-1)\mathbf{A}-\mathbf{B}=\mathbf{a}\times\mathbf{b}$$

[using Eq. (i) and $\mathbf{a}\cdot\mathbf{A}=1$] ...(ii)

and $\mathbf{A}+\mathbf{B}=\mathbf{a}$

From Eqs. (i) and (ii), we get

$$\mathbf{A}=\frac{(\mathbf{a}\times\mathbf{b})+\mathbf{a}}{\mathbf{a}^2} \text{ and } \mathbf{B}=\mathbf{a}-\left\{\frac{(\mathbf{a}\times\mathbf{b})+\mathbf{a}}{\mathbf{a}^2}\right\}$$

$$\Rightarrow \mathbf{B}=\frac{(\mathbf{b}\times\mathbf{a})+\mathbf{a}(\mathbf{a}^2-1)}{\mathbf{a}^2}$$

Thus, $\mathbf{A}=\dfrac{(\mathbf{a}\times\mathbf{b})+\mathbf{a}}{\mathbf{a}^2}$ and $\mathbf{B}=\dfrac{(\mathbf{b}\times\mathbf{a})\ \mathbf{a}(\mathbf{a}^2-1)}{\mathbf{a}^2}$

Exercise for Session 4

1. Find the value of $\alpha\times(\beta\times\gamma)$, where $\alpha=2\hat{\mathbf{i}}-10\hat{\mathbf{j}}+2\hat{\mathbf{k}}$, $\beta=3\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}}$, $\gamma=2\hat{\mathbf{i}}+\hat{\mathbf{j}}+3\hat{\mathbf{k}}$.
2. Find the vector of length 3 unit which is perpendicular to $\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and lies in the plane of $\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $2\hat{\mathbf{i}}-3\hat{\mathbf{j}}$.
3. Show that $(\mathbf{b}\times\mathbf{c}).\,\mathbf{a}\times\mathbf{d})+(\mathbf{c}\times\mathbf{a})(\mathbf{b}\times\mathbf{d})+(\mathbf{a}\times\mathbf{b})(\mathbf{c}\times\mathbf{d})=0$
4. Prove that $\hat{\mathbf{i}}\times(\mathbf{a}+\hat{\mathbf{i}})+\hat{\mathbf{j}}\times(\mathbf{a}\times\hat{\mathbf{j}})+\hat{\mathbf{k}}\times(\mathbf{a}\times\hat{\mathbf{k}})=2\mathbf{a}$.
5. Prove that $[\mathbf{a}\times\mathbf{b}\ \ \mathbf{a}\times\mathbf{c}\ \ \mathbf{d}]=(\mathbf{a}\cdot\mathbf{d})[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$
6. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are non-coplanar unit vector such that $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\dfrac{\mathbf{b}+\mathbf{c}}{\sqrt{2}}$, $\mathbf{b}$ and $\mathbf{c}$ are non-parallel, then prove that the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\dfrac{3\pi}{4}$.
7. Find a set of vectors reciprocal to the set of vectors $-\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$, $\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}}$, $\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$.
8. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ are reciprocal system of vectors, then prove that $\mathbf{a}'\times\mathbf{b}'+\mathbf{b}'\times\mathbf{c}'+\mathbf{c}'\times\mathbf{a}'=\dfrac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$.
9. Solve $\mathbf{r}\times\mathbf{b}=\mathbf{a}$, where $\mathbf{a}$ and $\mathbf{b}$ are given vectors such that $\mathbf{a}\cdot\mathbf{b}=0$.
10. Find vector $\mathbf{r}$, if $\mathbf{r}\cdot\mathbf{a}=\mathbf{m}$ and $\mathbf{r}\times\mathbf{b}=\mathbf{c}$, where $\mathbf{a}\cdot\mathbf{b}\neq0$.

JEE Type Solved Examples :
Only One Option Correct Type Questions

● **Ex. 1** *If* $|\mathbf{a}|=5, |\mathbf{a}-\mathbf{b}|=8$ *and* $|\mathbf{a}+\mathbf{b}|=10$, *then* $|\mathbf{b}|$ *is equal to*

(a) 1 (b) $\sqrt{57}$

(c) 3 (d) None of these

Sol. (b) We know that for any two vectors

$$|\mathbf{a}+\mathbf{b}|^2+|\mathbf{a}-\mathbf{b}|^2=2(|\mathbf{a}|^2+|\mathbf{b}|^2)$$

$$\Rightarrow \quad (10)^2+(8)^2=2[(5)^2+|\mathbf{b}|^2]$$

$$\Rightarrow \quad 100+64=50+2|\mathbf{b}|^2 \Rightarrow |\mathbf{b}|^2=57$$

$$\therefore \quad |\mathbf{b}|=\sqrt{57}$$

● **Ex. 2** *Angle between diagonals of a parallelogram whose sides are represented by* $\mathbf{a}=2\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ *and* $\mathbf{b}=\hat{\mathbf{i}}-\hat{\mathbf{j}}-\hat{\mathbf{k}}$

(a) $\cos^{-1}\left(\frac{1}{3}\right)$ (b) $\cos^{-1}\left(\frac{1}{2}\right)$

(c) $\cos^{-1}\left(\frac{4}{9}\right)$ (d) $\cos^{-1}\left(\frac{5}{9}\right)$

Sol. (a) Let $\mathbf{c}$ and $\mathbf{d}$ be the diagonals of parallelogram.

Then, $\mathbf{c}=\mathbf{a}+\mathbf{b}$ and $\mathbf{d}=\mathbf{a}-\mathbf{b}$

$$\Rightarrow \quad \mathbf{c}=3\hat{\mathbf{i}} \text{ and } \mathbf{d}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\hat{\mathbf{k}}$$

Let θ be the angle between $\mathbf{c}$ and $\mathbf{d}$.

Then, $$\cos\theta=\frac{\mathbf{c}\cdot\mathbf{d}}{|\mathbf{c}||\mathbf{d}|}=\frac{3\hat{\mathbf{i}}\cdot(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\hat{\mathbf{k}})}{\sqrt{(3)^2}\sqrt{1^2+2^2+2^2}}$$

$$=\frac{3}{3\times 3}=\frac{1}{3}$$

$$\therefore \quad \theta=\cos^{-1}\left(\frac{1}{3}\right)$$

● **Ex. 3** *Let* $\mathbf{a},\mathbf{b},\mathbf{c}$, *be vectors of length* 3, 4, 5 *respectively and* $\mathbf{a}$ *be perpendicular to* $(\mathbf{b}+\mathbf{c})$, $\mathbf{b}$ *to* $(\mathbf{c}+\mathbf{a})$ *and* $\mathbf{c}$ *to* $(\mathbf{a}+\mathbf{b})$, *then the value of* $(\mathbf{a}+\mathbf{b}+\mathbf{c})$ *is*

(a) $2\sqrt{5}$ (b) $2\sqrt{2}$

(c) $10\sqrt{5}$ (d) $5\sqrt{2}$

Sol. (d) We have, $|\mathbf{a}|=3, |\mathbf{b}|=4$ and $|\mathbf{c}|=5$. It is given that

$$\mathbf{a}\perp(\mathbf{b}+\mathbf{c}), \mathbf{b}\perp(\mathbf{c}+\mathbf{a}) \text{ and } \mathbf{c}\perp(\mathbf{a}+\mathbf{b})$$

$$\Rightarrow \quad \mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=0, \mathbf{b}\cdot(\mathbf{c}+\mathbf{a})=0 \text{ and } \mathbf{c}\cdot(\mathbf{a}+\mathbf{b})=0$$

$$\Rightarrow \quad \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}=\mathbf{b}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{a}=\mathbf{c}\cdot\mathbf{a}+\mathbf{c}\cdot\mathbf{b}=0$$

or $\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}=\mathbf{0}$ (adding all the above equations)

Now, $$|\mathbf{a}+\mathbf{b}+\mathbf{c}|^2=|\mathbf{a}|^2+|\mathbf{b}|^2+|\mathbf{c}|^2+2(\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a})$$

$$=3^2+4^2+5^2=50$$

$$\therefore \quad |\mathbf{a}+\mathbf{b}+\mathbf{c}|=5\sqrt{2}$$

● **Ex. 4** *Let* $a,b>0$ *and* $\alpha=\frac{\hat{\mathbf{i}}}{a}+\frac{4\hat{\mathbf{j}}}{b}+b\hat{\mathbf{k}}$ *and* $\beta=b\hat{\mathbf{i}}+a\hat{\mathbf{j}}+\frac{1}{b}\hat{\mathbf{k}}$, *then the maximum value of* $\frac{10}{5+\alpha\cdot\beta}$ *is*

(a) 1 (b) 2

(c) 4 (d) 8

Sol. (a) $$\alpha\cdot\beta=\frac{b}{a}+\frac{4a}{b}+1\geq 5 \qquad (\because \text{AM} \geq \text{GM})$$

So, $$\left(\frac{10}{5+\alpha\cdot\beta}\right)_{\max}=1.$$

● **Ex. 5** *If the unit vectors* $\mathbf{e}_1$ *and* $\mathbf{e}_2$ *are inclined at an angle* 2θ *and* $|\mathbf{e}_1-\mathbf{e}_2|<1$, *then for* $\theta\in[0,\pi]$, θ *may lie in the interval*

(a) $\left[0,\frac{\pi}{6}\right)$ (b) $\left[\frac{\pi}{6},\frac{\pi}{2}\right]$

(c) $\left(\frac{5\pi}{6},\pi\right]$ (d) $\left[\frac{\pi}{2},\frac{5\pi}{6}\right]$

Sol. (a) It is given that $\mathbf{e}_1$ and $\mathbf{e}_2$ are two unit vectors inclined at an angle 2θ and $|\mathbf{e}_1-\mathbf{e}_2|<1$.

$$\therefore \quad |\mathbf{e}_1-\mathbf{e}_2|<1 \Rightarrow |\mathbf{e}_1-\mathbf{e}_2|^2<1$$

$$\Rightarrow \quad 4\sin^2\theta<1 \qquad [\because |\mathbf{e}_1-\mathbf{e}_2|^2=4\sin^2\theta]$$

$$\Rightarrow \quad \sin^2\theta<\frac{1}{4}$$

$$\therefore \quad \theta\in\left[0,\frac{\pi}{6}\right)$$

● **Ex. 6** *If* $\mathbf{a}=3\hat{\mathbf{i}}-\hat{\mathbf{j}}+5\hat{\mathbf{k}}$ *and* $\mathbf{b}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}-3\hat{\mathbf{k}}$ *are given vector. A vector* $\mathbf{c}$ *which is perpendicular to Z-axis satisfying* $\mathbf{c}\cdot\mathbf{a}=9$ *and* $\mathbf{c}\cdot\mathbf{b}=-4$. *If inclination of* $\mathbf{c}$ *with X-axis and Y-axis is* α *and* β *respectively, then which of the following is not true?*

(a) $\alpha>\frac{\pi}{4}$ (b) $\beta>\frac{\pi}{2}$

(c) $\alpha>\frac{\pi}{2}$ (d) $\beta<\frac{\pi}{2}$

Sol. (c) $\mathbf{c}$ lies in XY-plane

$$\therefore \quad \mathbf{c}=x\hat{\mathbf{i}}+y\hat{\mathbf{j}}$$

From the given conditions $3x-y=9$

and $x+2y=-4$

Solving, we get $\mathbf{c}=2\hat{\mathbf{i}}-3\hat{\mathbf{j}}$

$$\therefore \quad \alpha=\cos^{-1}\left(\frac{2}{\sqrt{13}}\right), \ \beta=\cos^{-1}\left(\frac{-3}{\sqrt{13}}\right)$$

● ***Ex. 7*** *If A is 3×3 matrix and $\mathbf{u}$ is a vector. If $A\mathbf{u}$ and $\mathbf{u}$ are orthogonal for all real $\mathbf{u}$, then matrix A is a*

(a) singular (b) non-singular

(c) symmetric (d) skew-symmetric

Sol. (a) $A\mathbf{u}\cdot\mathbf{u}=0$

$\Rightarrow \quad |A||\mathbf{u}|^2=0$, Since $|\mathbf{u}|\neq 0 \Rightarrow |A|=0$

$\therefore$ A is singular.

● ***Ex. 8*** *Let the cosine of angle between the vectors $\mathbf{p}$ and $\mathbf{q}$ be λ such that $2\mathbf{p}+\mathbf{q}=\hat{\mathbf{i}}+\hat{\mathbf{j}}$ and $\mathbf{p}+2\mathbf{q}=\hat{\mathbf{i}}-\hat{\mathbf{j}}$, then λ is equal to*

(a) $\frac{5}{9}$ (b) $-\frac{4}{5}$

(c) $\frac{3}{9}$ (d) $\frac{7}{9}$

Sol. (b) It is given that $2\mathbf{p}+\mathbf{q}=\hat{\mathbf{i}}+\hat{\mathbf{j}}$

and $\mathbf{p}+2\mathbf{q}=\hat{\mathbf{i}}-\hat{\mathbf{j}}$

$\Rightarrow \quad \mathbf{p}=\frac{1}{3}\hat{\mathbf{i}}+\hat{\mathbf{j}}$ and $\mathbf{q}=\frac{1}{3}\hat{\mathbf{i}}-\hat{\mathbf{j}}$

Let θ be the angle between $\mathbf{p}$ and $\mathbf{q}$. Then

$$\cos\theta=\frac{\mathbf{p}\cdot\mathbf{q}}{|\mathbf{p}||\mathbf{q}|}=\frac{\left(\frac{1}{3}\hat{\mathbf{i}}+\hat{\mathbf{j}}\right)\cdot\left(\frac{1}{3}\hat{\mathbf{i}}-\hat{\mathbf{j}}\right)}{\sqrt{\left(\frac{1}{3}\right)^2+(1)^2}\sqrt{\left(\frac{1}{3}\right)^2+(-1)^2}}$$

$$=\frac{\frac{1}{9}-1}{\sqrt{\frac{1}{9}+1}\sqrt{\frac{1}{9}+1}}=\frac{-\frac{8}{9}}{\frac{10}{9}}=-\frac{8}{10}=-\frac{4}{5}$$

$\therefore \quad \lambda=\cos\theta=-\frac{4}{5}$

● ***Ex. 9*** *Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be vectors with magnitudes 3, 4 and 5, respectively and $\mathbf{a}+\mathbf{b}+\mathbf{c}=0$, then the values of $\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}$ is*

(a) 47 (b) 25

(c) 50 (d) -25

Sol. (d) We observe, $|\mathbf{a}|^2+|\mathbf{b}|^2=3^2+4^2=5^2=|\mathbf{c}|^2$

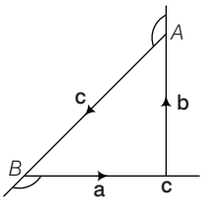
$\mathbf{a}\cdot\mathbf{b}=0$

$$\mathbf{b}\cdot\mathbf{c}=|\mathbf{b}||\mathbf{c}|\cdot\cos\left(\pi-\cos^{-1}\frac{4}{5}\right)$$

$$=4\times5\left\{-\cos\left(\cos^{-1}\frac{4}{5}\right)\right\}$$

$$=4\times5\times\left(-\frac{4}{5}\right)=-16$$

$$\mathbf{c}\cdot\mathbf{a}=|\mathbf{c}||\mathbf{a}|\cdot\cos\left(\pi-\cos^{-1}\frac{3}{5}\right)$$



$$=5\cdot3\left\{-\cos\left(\cos^{-1}\frac{3}{5}\right)\right\}=5\cdot3\cdot\left(-\frac{3}{5}\right)=-9$$

$\therefore \quad \mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}=0-16-9=-25$

Trick $\because \mathbf{a}+\mathbf{b}+\mathbf{c}=0$

On squaring both the sides, we get $|\mathbf{a}+\mathbf{b}+\mathbf{c}|^2=0$

$\Rightarrow \quad |\mathbf{a}|^2+|\mathbf{b}|^2+|\mathbf{c}|^2+2(\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a})=0$

$\Rightarrow \quad 2(\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a})=-(9+16+25)$

$\Rightarrow \quad \mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}=-25$

● ***Ex. 10*** *Let $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ be such that $|\mathbf{u}|=1, |\mathbf{v}|=2$, $|\mathbf{w}|=3$. If the projection $\mathbf{v}$ along $\mathbf{u}$ is equal to that of $\mathbf{w}$ along $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{w}$ are perpendicular to each other, then $|\mathbf{u}-\mathbf{v}+\mathbf{w}|$ equals.*

(a) $\sqrt{14}$ (b) $\sqrt{7}$

(c) 2 (d) 14

Sol. (a) We have,

Projection of $\mathbf{v}$ and $\mathbf{u}$ = Projection of $\mathbf{w}$ along $\mathbf{u}$

$\Rightarrow \quad \frac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{u}|}=\frac{\mathbf{w}\cdot\mathbf{u}}{|\mathbf{u}|} \Rightarrow \mathbf{v}\cdot\mathbf{u}=\mathbf{w}\cdot\mathbf{u}$ ... (i)

Also, $\mathbf{v}$ and $\mathbf{w}$ are perpendicular to each other

$\therefore \quad \mathbf{v}\cdot\mathbf{w}=0$ ... (ii)

Now, $|\mathbf{u}-\mathbf{v}+\mathbf{w}|^2=|\mathbf{u}|^2+|\mathbf{v}|^2+|\mathbf{w}|^2-2(\mathbf{u}\cdot\mathbf{v})$

$-2(\mathbf{v}\cdot\mathbf{w})+2(\mathbf{u}\cdot\mathbf{w})$

$\Rightarrow \quad |\mathbf{u}-\mathbf{v}+\mathbf{w}|^2=1+4+9$

$\Rightarrow \quad |\mathbf{u}-\mathbf{v}+\mathbf{w}|=\sqrt{14}$

● ***Ex. 11*** *If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are unit vectors, then $|\mathbf{a}-\mathbf{b}|^2+|\mathbf{b}-\mathbf{c}|^2+|\mathbf{c}-\mathbf{a}|^2$ does not exceed*

(a) 4 (b) 9

(c) 8 (d) 6

Sol. (b) We have, $|\mathbf{a}-\mathbf{b}|^2+|\mathbf{b}-\mathbf{c}|^2+|\mathbf{c}-\mathbf{a}|^2$

$=|\mathbf{a}|+|\mathbf{b}|^2-2(\mathbf{a}\cdot\mathbf{b})+|\mathbf{b}|^2+|\mathbf{c}|^2-2(\mathbf{b}\cdot\mathbf{c})+|\mathbf{c}|^2$

$+|\mathbf{a}|^2-2(\mathbf{c}\cdot\mathbf{a})$

$=2[|\mathbf{a}|^2+|\mathbf{b}|^2+|\mathbf{c}|^2-(\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a})]$

$=2[3-(\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a})]$

$=6-2(\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a})$ ...(i)

Now, $|\mathbf{a}+\mathbf{b}+\mathbf{c}|^2\geq0$

$\Rightarrow \quad |\mathbf{a}|^2+|\mathbf{b}|^2+|\mathbf{c}|^2+2(\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a})\geq0$

$$\Rightarrow \quad 3 + 2(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}) \geq 0$$

$$\Rightarrow \quad \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a} \geq -\frac{3}{2}$$

$$\Rightarrow \quad -2(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}) \leq 3 \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we obtain

$$|\mathbf{a}-\mathbf{b}|^2 + |\mathbf{b}-\mathbf{c}|^2 + |\mathbf{c}-\mathbf{a}|^2 \leq 6 + 3$$

$$\Rightarrow \quad |\mathbf{a}-\mathbf{b}|^2 + |\mathbf{b}-\mathbf{c}|^2 + |\mathbf{c}-\mathbf{a}|^2 \leq 9$$

● ***Ex. 12*** *The vectors* $\mathbf{a} = 2\lambda^2\hat{\mathbf{i}} + 4\lambda\hat{\mathbf{j}} + \hat{\mathbf{k}}$ *and* $\mathbf{b} = 7\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \lambda\hat{\mathbf{k}}$ *make an obtuse angle whereas the angle between* $\mathbf{b}$ *and* $\hat{\mathbf{k}}$ *is acute and less than* $\pi/6$,

(a) $0 < \lambda < \frac{1}{2}$ (b) $\lambda > \sqrt{159}$

(c) $-\frac{1}{2} < \lambda < 0$ (d) null set

Sol. (d) As angle between $\mathbf{a}$ and $\mathbf{b}$ is obtuse, $\mathbf{a}\cdot\mathbf{b} < 0$

$$\Rightarrow \quad (2\lambda^2\hat{\mathbf{i}} + 4\lambda\hat{\mathbf{j}} + \hat{\mathbf{k}})\cdot(7\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \lambda\hat{\mathbf{k}}) < 0$$

$$\Rightarrow \quad 14\lambda^2 - 8\lambda + \lambda < 0$$

$$\Rightarrow \quad \lambda(2\lambda - 1) < 0$$

$$\Rightarrow \quad 0 < \lambda < \frac{1}{2} \qquad \ldots(i)$$

Angle between $\mathbf{b}$ and $\mathbf{k}$ is acute and less than $\frac{\pi}{6}$.

$$\mathbf{b}\cdot\mathbf{k} = |\mathbf{b}|\,|\mathbf{k}|\cos\theta$$

$$\Rightarrow \quad \lambda = \sqrt{53 + \lambda^2}\,1\cdot\cos\theta$$

$$\Rightarrow \quad \cos\theta = \frac{\lambda}{\sqrt{53 + \lambda^2}}$$

$$\theta < \frac{\pi}{6} \Rightarrow \cos\theta > \cos\frac{\pi}{6}$$

$$\Rightarrow \quad \cos\theta > \frac{\sqrt{3}}{2} \Rightarrow \frac{\lambda}{\sqrt{53 + \lambda^2}} > \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad 4\lambda^2 - 3(53 + \lambda^2) > 0$$

$$\Rightarrow \quad \lambda^2 > 159$$

$$\Rightarrow \quad \lambda < -\sqrt{159} \qquad \ldots(ii)$$

From Eqs. (i) and (ii), $\lambda = \phi$

$\therefore$ Domain of λ is null set.

● ***Ex. 13*** *The locus of a point equidistant from two given points whose position vectors are* $\mathbf{a}$ *and* $\mathbf{b}$ *is equal to*

(a) $\left[\mathbf{r} - \frac{1}{2}(\mathbf{a}+\mathbf{b})\right]\cdot(\mathbf{a}+\mathbf{b}) = 0$

(b) $\left[\mathbf{r} - \frac{1}{2}(\mathbf{a}+\mathbf{b})\right]\cdot(\mathbf{a}-\mathbf{b}) = 0$

(c) $\left[\mathbf{r} - \frac{1}{2}(\mathbf{a}+\mathbf{b})\right]\cdot\mathbf{a} = 0$

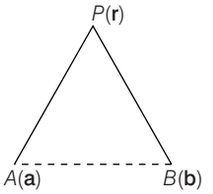
(d) $[\mathbf{r} - (\mathbf{a}+\mathbf{b})]\cdot\mathbf{b} = 0$

Sol. (b) Let $P(\mathbf{r})$ be a point on the locus.

$$\therefore \quad AP = BP$$

$$\Rightarrow \quad |\mathbf{r}-\mathbf{a}| = |\mathbf{r}-\mathbf{b}| \Rightarrow |\mathbf{r}-\mathbf{a}|^2 = |\mathbf{r}-\mathbf{b}|^2$$

$$\Rightarrow \quad (\mathbf{r}-\mathbf{a})\cdot(\mathbf{r}-\mathbf{a}) = (\mathbf{r}-\mathbf{b})\cdot(\mathbf{r}-\mathbf{b})$$



$$\Rightarrow \quad 2\mathbf{r}\cdot(\mathbf{a}-\mathbf{b}) = \mathbf{a}\cdot\mathbf{a} - \mathbf{b}\cdot\mathbf{b}$$

$$\Rightarrow \quad \mathbf{r}\cdot(\mathbf{a}-\mathbf{b}) = \frac{1}{2}(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})$$

$$\therefore \quad \left[\mathbf{r} - \frac{1}{2}(\mathbf{a}+\mathbf{b})\right]\cdot(\mathbf{a}-\mathbf{b}) = 0$$

This is the locus of P.

● ***Ex. 14*** *In cartesian coordinates the point A is* (x_1, y_1), *where* $x_1 = 1$ *on the curve* $y = x^2 + x + 10$. *Then tangent at A cuts the X-axis at B. The value of the dot product* $\mathbf{OA}\cdot\mathbf{AB}$ *is*

(a) $-\frac{520}{3}$ (b) -148

(c) 140 (d) 12

Sol. (b) Given curve is $y = x^2 + x + 10$ ...(i)

When $x = 1$, $y = 1^2 + 1 + 10 = 12$

$$\therefore \quad A = (1, 12)$$

$$\therefore \quad \mathbf{OA} = \hat{\mathbf{i}} + 12\hat{\mathbf{j}}$$

From Eq. (i), $\frac{dy}{dx} = 2x + 1$

Equation of tangent at A is $y - 12 = \left(\frac{dy}{dx}\right)_{(1,12)}(x-1)$

$$\Rightarrow \quad y - 12 = (2\times 1 + 1)(x-1)$$

$$\Rightarrow \quad y - 12 = 3x - 3$$

$$\therefore \quad y = 3(x+3)$$

This tangent cuts X-axis (i.e. $y = 0$) at $(-3, 0)$

$$\therefore \quad B = (-3, 0)$$

$$\mathbf{OB} = -3\hat{\mathbf{i}} + 0\cdot\hat{\mathbf{j}} = -3\hat{\mathbf{i}},\ \mathbf{OA}\cdot\mathbf{AB} = \mathbf{OA}\cdot(\mathbf{OB} - \mathbf{OA})$$

$$(\hat{\mathbf{i}} + 12\hat{\mathbf{j}})\cdot(-3\hat{\mathbf{i}} - \hat{\mathbf{i}} - 12\hat{\mathbf{j}}) = (\hat{\mathbf{i}} + 12\hat{\mathbf{j}})\cdot(-4\hat{\mathbf{i}} - 12\hat{\mathbf{j}})$$

$$= -4 - 144 = -148$$

● ***Ex. 15*** *In a tetrahedron OABC, the edges are of lengths,* $|OA| = |BC| = a, |OB| = |AC| = b, |OC| = |AB| = c$. *Let* G_1 *and* G_2 *be the centroids of the triangle ABC and AOC such that* $OG_1 \perp BG_2$, *then the value of* $\frac{a^2 + c^2}{b^2}$ *is*

(a) 2 (b) 3

(c) 6 (d) 9

Sol. (b) $OG_1 \cdot BG_2 = 0$.

$$\Rightarrow \quad \frac{a+b+c}{3} \cdot \frac{a+c-3b}{3} = 0$$

$$\Rightarrow \quad a^2 + c^2 - 3b^2 + 2a \cdot c - 2b \cdot c - 2a \cdot b = 0$$

Now, $\quad |c-a|^2 = b^2, |c-b|^2 = a^2$ and $|a-b|^2 = c^2$

$$\therefore \quad 2a \cdot c = a^2 + c^2 - b^2,\ 2b \cdot c = b^2 + c^2 - a^2,$$

$$2a \cdot b = a^2 + b^2 - c^2$$

Putting in the above result, we get $2a^2 + 2c^2 - 6b^2 = 0$

$$\Rightarrow \quad \frac{a^2 + c^2}{b^2} = 3$$

● ***Ex. 16*** *If OABC is a tetrahedron such that* $\mathbf{OA}^2 + \mathbf{BC}^2 = \mathbf{OB}^2 + \mathbf{CA}^2 = \mathbf{OC}^2 + \mathbf{AB}^2$, *then which of the following is not true?*

(a) $OA \perp BC$ (b) $OB \perp AC$

(c) $OC \perp AB$ (d) $AB \perp AC$

Sol. (d) Let $\mathbf{OA} = \mathbf{a}$, OB = b, $\mathbf{OC} = \mathbf{c}$

Then from the given conditions

$$\mathbf{a} \cdot \mathbf{a} + (\mathbf{b} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{c}) = \mathbf{b} \cdot \mathbf{b} + (\mathbf{c} - \mathbf{a}) \cdot (\mathbf{c} - \mathbf{a})$$

$$\Rightarrow \quad -2\mathbf{b} \cdot \mathbf{c} = -2\mathbf{c} \cdot \mathbf{a}$$

$$\Rightarrow \quad \mathbf{c} \cdot (\mathbf{b} - \mathbf{a}) = 0 \Rightarrow \mathbf{BA} \cdot \mathbf{OC} = 0$$

Hence $AB \perp OC$. Similarly,

$BC \perp OA$ and $CA \perp OB$

● ***Ex. 17*** *If* a, b, c *and* $A, B, C \in R - \{0\}$ *such that* $aA + bB + cC + \sqrt{(a^2 + b^2 + c^2)(A^2 + B^2 + C^2)} = 0$, *then value of* $\frac{aB}{bA} = \frac{bC}{cB} + \frac{cA}{aC}$ *is*

(a) 3 (b) 4

(c) 5 (d) 6

Sol. (a) Let $\mathbf{r}_1 = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$ and $\mathbf{r}_2 = A\hat{\mathbf{i}} + B\hat{\mathbf{j}} + C\hat{\mathbf{k}}$

$$\Rightarrow \quad \mathbf{r}_1 \cdot \mathbf{r}_2 = aA + bB + cC$$

$$\Rightarrow \quad |\mathbf{r}_1||\mathbf{r}_2| = \sqrt{(a^2 + b^2 + c^2)(A^2 + B^2 + C^2)}$$

$$\therefore \quad \mathbf{r}_1\mathbf{r}_2 = -|\mathbf{r}_1||\mathbf{r}_2|$$

$\Rightarrow$ $\mathbf{r}_1$ and $\mathbf{r}_2$ are anti-parallel

$$\Rightarrow \quad \frac{a}{A} = \frac{b}{B} = \frac{c}{C} = k, \text{ where } k \text{ is any constant}$$

$$\Rightarrow \quad \frac{aB}{bA} + \frac{bC}{cB} + \frac{cA}{aC} = 3$$

● ***Ex. 18*** *The unit vector in ZOX plane making angles* 45° *and* 60° *respectively, with* $a = 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$ *and* $\mathbf{b} = \hat{\mathbf{j}} - \hat{\mathbf{k}}$, *is*

(a) $\frac{1}{\sqrt{2}}(-\hat{\mathbf{i}} + \hat{\mathbf{k}})$ (b) $\frac{1}{\sqrt{2}}(\hat{\mathbf{i}} - \hat{\mathbf{k}})$

(c) $\frac{\sqrt{3}}{2}(\hat{\mathbf{i}} + \hat{\mathbf{k}})$ (d) None of these

Sol. (b) Let the required vector be $\mathbf{r} = x\hat{\mathbf{i}} + z\hat{\mathbf{k}}$, since $\mathbf{r}$ is a unit vector.

$$\therefore \quad x^2 + y^2 = 1$$

It is given that $\mathbf{r}$ makes 45° and 60° angles with $\mathbf{a}$ and $\mathbf{b}$ respectively.

$$\therefore \quad \cos 45° = \frac{\mathbf{r} \cdot \mathbf{a}}{|\mathbf{a}||\mathbf{r}|} \text{ and } \cos 60° = \frac{\mathbf{r} \cdot \mathbf{b}}{|\mathbf{r}||\mathbf{b}|}$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}} = \frac{2x - y}{3} \text{ and } \frac{1}{2} = -\frac{-y}{\sqrt{2}}$$

$$\Rightarrow \quad 2x - y = \frac{3}{\sqrt{2}} \text{ and } y = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad x = \frac{1}{\sqrt{2}}$$

$$\therefore \quad \mathbf{r} = \frac{1}{\sqrt{2}}(\hat{\mathbf{i}} - \hat{\mathbf{k}})$$

● ***Ex. 19*** *A unit vector perpendicular to the vector* $-\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ *and making equal angles with X and Y-axes can be*

(a) $\frac{1}{3}(2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}})$ (b) $\frac{1}{3}(2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - \hat{\mathbf{k}})$

(c) $\frac{1}{3}(2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}})$ (d) $\frac{1}{3}(2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}})$

Sol. (a) Let the required vector be $\mathbf{r} = l\hat{\mathbf{i}} + m\hat{\mathbf{j}} + n\hat{\mathbf{k}}$, where l, m, n are the direction cosines of $\mathbf{r}$ such that $l = m$.

It is given that $\mathbf{r}$ is perpendicular to $-\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$. Therefore,

$$\mathbf{r} \cdot (-\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) = 0$$

$$\Rightarrow \quad -l + 2m + 2n = 0$$

$$\Rightarrow \quad l + 2n = 0 \qquad [\because l = m]$$

$$\Rightarrow \quad l = -2n$$

Now, $\quad l^2 + m^2 + n^2 = 1$

$$\Rightarrow \quad 4n^2 + 4n^2 + n^2 = 1$$

$$\Rightarrow \quad n = \pm\frac{1}{3}$$

$$\therefore \quad l = \mp\frac{2}{3}, m = \mp\frac{2}{3}, n = \mp\frac{1}{3}$$

Hence, $\quad \mathbf{r} = \mp\frac{1}{3}(2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}})$

● ***Ex. 20*** *If* $(\mathbf{a} + 3\mathbf{b}) \cdot (7\mathbf{a} - 5\mathbf{b}) = 0$ *and* $(\mathbf{a} - 4\mathbf{b}) \cdot (7\mathbf{a} - 2\mathbf{b}) = 0$. *Then, the angle between* $\mathbf{a}$ *and* $\mathbf{b}$ *is*

(a) 60° (b) 30°

(c) 90° (d) None of these

Sol. (a) We have, $(\mathbf{a} + 3\mathbf{b}) \cdot (7\mathbf{a} - 5\mathbf{b}) = 0$

$$\Rightarrow \quad 7|\mathbf{a}|^2 + 16\mathbf{a} \cdot \mathbf{b} - 15|\mathbf{b}|^2 = 0 \qquad \ldots(i)$$

and $\quad (\mathbf{a} - 4\mathbf{b}) \cdot (7\mathbf{a} - 2\mathbf{b}) = 0$

$$\Rightarrow \quad 7|\mathbf{a}|^2 - 30\mathbf{a} \cdot \mathbf{b} + 8|\mathbf{b}|^2 = 0 \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$\mathbf{a}\cdot\mathbf{b}=\frac{15|\mathbf{b}|^2-7|\mathbf{a}|^2}{16}$$

$$\mathbf{a}\cdot\mathbf{b}=\frac{1}{30}(7|\mathbf{a}|^2+8|\mathbf{b}|^2)$$

$$\Rightarrow \quad \frac{15|\mathbf{b}|^2-7|\mathbf{a}|^2}{8}=\frac{1}{15}(7|\mathbf{a}|^2+8|\mathbf{b}|^2)$$

$$\Rightarrow \quad 15(15|\mathbf{b}|^2-7|\mathbf{a}|^2)=8(7|\mathbf{a}|^2+8|\mathbf{b}|^2)$$

$$\Rightarrow \quad 225|\mathbf{b}|^2-105|\mathbf{a}|^2=56|\mathbf{a}|^2+64|\mathbf{b}|^2$$

$$\Rightarrow \quad 161|\mathbf{b}|^2=161|\mathbf{a}|^2$$

$$\Rightarrow \quad |\mathbf{b}|^2=|\mathbf{a}|^2$$

From Eq. (i), we get

$$16\mathbf{a}\cdot\mathbf{b}=15|\mathbf{b}|^2-7|\mathbf{a}|^2=15|\mathbf{b}|^2-7|\mathbf{b}|^2$$

$$\Rightarrow \quad 16\mathbf{a}\cdot\mathbf{b}=8|\mathbf{b}|^2$$

$$\Rightarrow \quad \mathbf{a}\cdot\mathbf{b}=\frac{1}{2}|\mathbf{b}|^2$$

$$\Rightarrow \quad |\mathbf{a}||\mathbf{b}|\cos\theta=\frac{1}{2}|\mathbf{b}|^2$$

$$\Rightarrow \quad \cos\theta=\frac{1}{2}$$

$$\therefore \quad \theta=60^\circ$$

● ***Ex. 21*** *Let two non-collinear vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *inclined at an angle* $\frac{2\pi}{3}$ *be such that* $|\mathbf{a}|=3$ *and* $|\mathbf{b}|=2$. *If a point P moves so that at any time t its position vector* $\mathbf{OP}$ *(where O is the origin) is given as* $\mathbf{OP}=\left(t+\frac{1}{t}\right)\mathbf{a}+\left(t-\frac{1}{t}\right)\mathbf{b}$, *then least distance of P from the origin is*

(a) $\sqrt{2\sqrt{133}-10}$ (b) $\sqrt{2\sqrt{133}+10}$

(c) $\sqrt{5+\sqrt{133}}$ (d) None of these

Sol. (b) We have, $|\mathbf{OP}|^2=\left(t+\frac{1}{t}\right)^2|\mathbf{a}|^2+\left(t-\frac{1}{t}\right)^2|\mathbf{b}|^2+2\left(t^2-\frac{1}{t^2}\right)|\mathbf{a}||\mathbf{b}|\cos\left(\frac{2\pi}{3}\right)$

$$\therefore \quad |\mathbf{OP}|^2=9\left(t+\frac{1}{t}\right)^2+4\left(t-\frac{1}{t}\right)^2+2\left(t^2-\frac{1}{t^2}\right)3\cdot2\cdot\left(\frac{-1}{2}\right)$$

$$=9\left(t^2+\frac{1}{t^2}+2\right)+4\left(t^2+\frac{1}{t^2}-2\right)-6\left(t^2-\frac{1}{t^2}\right)$$

$$=7t^2+\frac{19}{t^2}+10$$

$$\Rightarrow \quad |\mathbf{OP}|^2\geq 2\cdot\sqrt{7t^2\cdot\frac{19}{t^2}}+10 \qquad (\because \text{AM}\geq\text{GM})$$

$\therefore$ Minimum value of $|\mathbf{OP}|=\sqrt{10+2\sqrt{133}}$

● ***Ex. 22*** *If* $\mathbf{a},\mathbf{b},\mathbf{c}$ *be non-zero vectors such that* $\mathbf{a}$ *is perpendicular to* $\mathbf{b}$ *and* $\mathbf{c}$ *and* $|\mathbf{a}|=1,|\mathbf{b}|=2,|\mathbf{c}|=1,\mathbf{b}\cdot\mathbf{c}=1$ *and there is a non-zero vector* $\mathbf{d}$ *coplanar with* $\mathbf{a}+\mathbf{b}$ *and* $2\mathbf{b}-\mathbf{c}$ *and* $\mathbf{d}\cdot\mathbf{a}=1$, *then minimum value of* $|\mathbf{d}|$ *is*

(a) $\frac{2}{\sqrt{13}}$ (b) $\frac{3}{\sqrt{13}}$

(c) $\frac{4}{\sqrt{5}}$ (d) $\frac{4}{\sqrt{13}}$

Sol. (d) $\mathbf{a}\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{c}=0,|\mathbf{a}|=|\mathbf{c}|=1,|\mathbf{b}|=2$ and $\mathbf{b}\cdot\mathbf{c}=1$

let $\quad \mathbf{d}=x(\mathbf{a}+\mathbf{b})+y(2\mathbf{b}-\mathbf{c})$

But $\quad \mathbf{d}\cdot\mathbf{a}=1$

$$\Rightarrow \quad x(1+0)+0=1$$

$$\Rightarrow \quad x=1$$

$$\Rightarrow \quad \mathbf{d}=\mathbf{a}+\mathbf{b}+y(2\mathbf{b}-\mathbf{c})$$

$$\Rightarrow \quad |\mathbf{d}|^2=|\mathbf{a}|^2+|\mathbf{b}|+2\mathbf{a}\cdot\mathbf{b}+y^2(2\mathbf{b}-\mathbf{c})^2+2y(\mathbf{a}+\mathbf{b})\cdot(2\mathbf{b}-\mathbf{c})$$

$$\Rightarrow \quad |\mathbf{d}|^2=1+4+y^2(16+1-4)+2y(8-1)$$

$$=13y^2-14y+5$$

$$\therefore \quad |\mathbf{d}|_{\min}=\sqrt{\frac{4\times13\times5-14\times14}{4\times13}}=\frac{4}{\sqrt{13}}$$

● ***Ex. 23*** *A groove is in the form of a broken line ABC and the position vectors of the three points are respectively* $2\hat{\mathbf{i}}-3\hat{\mathbf{j}}+2\hat{\mathbf{k}}$, $3\hat{\mathbf{i}}+2\hat{\mathbf{j}}-\hat{\mathbf{k}}$ *and* $\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$. *A force of magnitude* $24\sqrt{3}$ *acts on a particle of unit mass kept at the point A and moves it along the groove to the point C. If the line of action of the force is parallel to the vector* $\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}}$ *all along, the number of units of work done by the force is*

(a) $144\sqrt{2}$ (b) $144\sqrt{3}$

(c) $72\sqrt{2}$ (d) $72\sqrt{3}$

Sol. (c) $\mathbf{F}=(24\sqrt{3})\frac{\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}}}{|\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}}|}=\frac{24\sqrt{3}}{\sqrt{6}}(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}})$

$$=12\sqrt{2}(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}})$$

Displacement, $\mathbf{r}=$ Position Vector of $C-$ Position Vector of A

$$=(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})-(2\hat{\mathbf{i}}-3\hat{\mathbf{j}}+2\hat{\mathbf{k}})$$

$$=(-\hat{\mathbf{i}}+4\hat{\mathbf{j}}-\hat{\mathbf{k}})$$

Work done by the force

$$W=\mathbf{r}\cdot\mathbf{F}=(-\hat{\mathbf{i}}+4\hat{\mathbf{j}}-\hat{\mathbf{k}})\cdot12\sqrt{2}(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}})$$

$$=12\sqrt{2}(-1+8-1)=72\sqrt{2}$$

● ***Ex. 24*** *For any vectors* $\mathbf{a},\mathbf{b};|\mathbf{a}\times\mathbf{b}|^2+(\mathbf{a}\cdot\mathbf{b})^2$ *is equal to*

(a) $|\mathbf{a}|^2|\mathbf{b}|^2$ (b) $|\mathbf{a}+\mathbf{b}|$

(c) $|\mathbf{a}|^2-|\mathbf{b}|^2$ (d) 0

Sol. (a) We have, $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$

$$\Rightarrow \quad |\mathbf{a}\times\mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2\sin^2\theta$$
$$= |\mathbf{a}|^2|\mathbf{b}|^2(1-\cos^2\theta)$$
$$= |\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a}|^2|\mathbf{b}|^2\cos^2\theta$$
$$= |\mathbf{a}|^2|\mathbf{b}|^2 - (|\mathbf{a}||\mathbf{b}|\cos\theta)^2$$
$$= |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2$$

$$\therefore |\mathbf{a}\times\mathbf{b}|^2 + (\mathbf{a}\cdot\mathbf{b})^2 = |\mathbf{a}|^2|\mathbf{b}|^2$$

● ***Ex. 25*** *If* $\mathbf{a} = \hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}, \mathbf{b} = \hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}}$, *then vectors perpendicular to* $\mathbf{a}$ *and* $\mathbf{b}$ *is/are*

(a) $\lambda(\hat{\mathbf{i}}+\hat{\mathbf{j}})$ (b) $\lambda(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})$

(c) $\lambda(\hat{\mathbf{i}}-\hat{\mathbf{j}})$ (d) None of these

Sol. (c) Any vector perpendicular to both $\mathbf{a}$ and $\mathbf{b} = \lambda(\mathbf{a}\times\mathbf{b})$

$$\text{Now, } \mathbf{a}\times\mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$
$$= -2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} = -2(\hat{\mathbf{i}}-\hat{\mathbf{j}})$$
$$= -2(\hat{\mathbf{i}}-\hat{\mathbf{j}})$$

$\therefore$ Required vector $= \lambda(\hat{\mathbf{i}}-\hat{\mathbf{j}})$

● ***Ex. 26*** *If* $\mathbf{a}\times\mathbf{b} = \mathbf{b}\times\mathbf{c} \neq 0$, *then the correct statement is*

(a) $\mathbf{b} \,||\, \mathbf{c}$ (b) $\mathbf{a} \,||\, \mathbf{b}$

(c) $(\mathbf{a}+\mathbf{c}) \,||\, \mathbf{b}$ (d) None of these

Sol. (c) We have,

$$\mathbf{a}\times\mathbf{b} = \mathbf{b}\times\mathbf{c}$$
$$\Rightarrow \quad \mathbf{a}\times\mathbf{b} - \mathbf{b}\times\mathbf{c} = 0$$
$$\Rightarrow \quad \mathbf{a}\times\mathbf{b} + \mathbf{c}\times\mathbf{b} = 0$$
$$\Rightarrow \quad (\mathbf{a}+\mathbf{c})\times\mathbf{b} = 0$$
$$\therefore \quad (\mathbf{a}+\mathbf{c}) \,||\, \mathbf{b}$$

[$\because$ if vector product of two vectors is zero, then both vectors are parallel to each other]

● ***Ex. 27*** *If* $\mathbf{a} = \hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}}, \mathbf{b} = -\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}}$ *and* $\mathbf{c} = 3\hat{\mathbf{i}}+\hat{\mathbf{j}}$. *If* $(\mathbf{a}+\mathbf{b}) \perp \mathbf{c}$, *then* t *is equal to*

(a) 5 (b) 4

(c) 3 (d) 2

Sol. (a) We have, $\mathbf{a} = \hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}}$

$$\mathbf{b} = -\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}}$$

and $\mathbf{c} = 3\hat{\mathbf{i}}+\hat{\mathbf{j}}$

Since, $(\mathbf{a}+t\mathbf{b})$ is perpendicular to $\mathbf{c}$

$$\therefore \quad (\mathbf{a}+t\mathbf{b})\cdot\mathbf{c} = 0$$
$$[(1-t)\hat{\mathbf{i}} + (2+2t)\hat{\mathbf{j}} + (3+t)\hat{\mathbf{k}}]\cdot(3\hat{\mathbf{i}}+\hat{\mathbf{j}}) = 0$$
$$\Rightarrow \quad 3(1-t) + (2+2t) = 0$$
$$\Rightarrow \quad t = 5$$

● ***Ex. 28*** *If* $\mathbf{a} = 2\hat{\mathbf{i}}-3\hat{\mathbf{j}}+\hat{\mathbf{k}}, \mathbf{b} = -\hat{\mathbf{i}}+\hat{\mathbf{k}}, \mathbf{c} = 2\hat{\mathbf{j}}-\hat{\mathbf{k}}$, *then the area (in sq units) of parallelogram with diagonals* $\mathbf{a}+\mathbf{b}$ *and* $\mathbf{b}+\mathbf{c}$ *will be*

(a) $\sqrt{21}$ (b) $2\sqrt{21}$

(c) $\frac{1}{2}\sqrt{21}$ (d) None of these

Sol. (c) We have, $\mathbf{a} = 2\hat{\mathbf{i}}-3\hat{\mathbf{j}}+\hat{\mathbf{k}}$

$$\mathbf{b} = -\hat{\mathbf{i}}+\hat{\mathbf{k}} \text{ and } \mathbf{c} = 2\hat{\mathbf{j}}-\hat{\mathbf{k}}$$

Since, $(\mathbf{a}+\mathbf{b})$ and $(\mathbf{b}+\mathbf{c})$ are the diagonals of the parallelogram

Now, $\mathbf{a}+\mathbf{b} = \hat{\mathbf{i}}-3\hat{\mathbf{j}}+2\hat{\mathbf{k}}$ and $\mathbf{b}+\mathbf{c} = -\hat{\mathbf{i}}+2\hat{\mathbf{j}}$

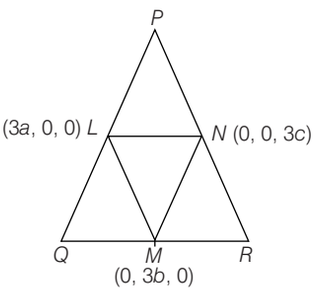
$$\therefore \text{ Area of parallelogram} = \frac{1}{2}|(\mathbf{a}+\mathbf{b})\times(\mathbf{b}+\mathbf{c})|$$
$$= \frac{1}{2}|(\hat{\mathbf{i}}-3\hat{\mathbf{j}}+2\hat{\mathbf{k}})\times(-\hat{\mathbf{i}}+2\hat{\mathbf{j}})|$$
$$= \left|\frac{1}{2}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -3 & 2 \\ -1 & 2 & 0 \end{vmatrix}\right| = \left|\frac{1}{2}(-4\hat{\mathbf{i}}-2\hat{\mathbf{j}}-\hat{\mathbf{k}})\right|$$
$$= \frac{1}{2}\sqrt{(-4)^2(-2)^2+(-1)^2} = \frac{\sqrt{21}}{2} \text{ sq units.}$$

● ***Ex. 29*** *The coordinates of the mid-points of the sides of* ΔPQR *are* $(3a,0,0), (0,3b,0)$ *and* $(0,0,3c)$ *respectively, then the area of* ΔPQR *is*

(a) $18\sqrt{b^2c^2+c^2a^2+a^2b^2}$ (b) $9\sqrt{b^2c^2+c^2a^2+a^2b^2}$

(c) $\frac{9}{2}\sqrt{b^2c^2+c^2a^2+a^2b^2}$ (d) $18\sqrt{ab+bc+ca}$

Sol. (a) Let L, M, N be the mid-points of the sides of ΔPQR.



$$\text{Area of } \Delta LMN = \frac{1}{2}|\mathbf{MN}\times\mathbf{ML}|$$
$$= \frac{1}{2}|(-3b\hat{\mathbf{j}}+3c\hat{\mathbf{k}})\times(3a\hat{\mathbf{i}}-3b\hat{\mathbf{j}})|$$
$$= \frac{1}{2}|9(bc\hat{\mathbf{i}}+ca\hat{\mathbf{j}}+ab\hat{\mathbf{k}})|$$
$$= \frac{9}{2}\sqrt{(bc)^2+(ca)^2+(ab)^2}$$

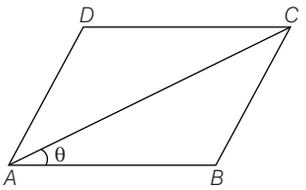
$$\text{Now, area of } \Delta PQR = 4\times(\text{Area of } \Delta LMN)$$
$$= 4\times\frac{9}{2}\sqrt{b^2c^2+c^2a^2+a^2b^2}$$
$$= 18\sqrt{b^2c^2+c^2a^2+a^2b^2}$$

● ***Ex. 30*** *In a parallelogram ABCD,* $\mathbf{AB}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ *and diagonal* $\mathbf{AC}=\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}}$ *and area of parallelogram is* $\sqrt{8}$ *sq units, then* $\angle BAC$ *is equal to*

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$

(c) $\sin^{-1}\left(\frac{\sqrt{8}}{3}\right)$ (d) $\cos^{-1}\left(\frac{\sqrt{8}}{3}\right)$

Sol. (c) We have, $\mathbf{AB}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $\mathbf{AC}=\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}}$



Let θ be the $\angle BAC$. Then,

$$\sin\theta=\frac{|\mathbf{AB}\times\mathbf{AC}|}{|\mathbf{AB}||\mathbf{AC}|}$$

Now, $$\mathbf{AB}\times\mathbf{AC}=\begin{vmatrix}\hat{\mathbf{i}}&\hat{\mathbf{j}}&\hat{\mathbf{k}}\\1&1&1\\1&-1&1\end{vmatrix}=2\hat{\mathbf{i}}-2\hat{\mathbf{k}}$$

$\therefore \quad |\mathbf{AB}\times\mathbf{AC}|=\sqrt{8}$

Hence, $$\sin\theta=\frac{\sqrt{8}}{\sqrt{3}\times\sqrt{3}}$$

$\Rightarrow \quad \theta=\sin^{-1}\left(\frac{\sqrt{8}}{3}\right)$

● ***Ex. 31*** *Let* ΔABC *be a given triangle. If* $|\mathbf{BA}-t\mathbf{BC}|\geq|\mathbf{AC}|$ *for any* $t\in R$, *then* ΔABC *is*

(a) Equilateral (b) Right angled

(c) Isosceles (d) None of these

Sol. (b) $|\mathbf{BA}|^2+t^2|\mathbf{BC}|^2-2\mathbf{BA}\cdot\mathbf{BC}\,t-|\mathbf{AC}|^2\geq 0,\ \forall\ t\in R$ ...(i)

Discriminante of the quadratic equation ≤ 0

$\Rightarrow \quad 4(\mathbf{BA}\cdot\mathbf{BC})^2-|\mathbf{BC}|^2|\mathbf{BA}|^2+4|\mathbf{BC}|^2|\mathbf{AC}|^2\leq 0$ ...(ii)

Using $(\mathbf{BA}\cdot\mathbf{BC})^2-|\mathbf{BC}|^2|\mathbf{BA}|^2$

$$=-|\mathbf{BA}\times\mathbf{BC}|^2$$

$$=-|(\mathbf{BC}+\mathbf{CA})\times\mathbf{BC}|^2$$

$$=-|\mathbf{CA}\times\mathbf{BC}|^2$$

Using Eq. (ii) in Eq. (i).

$$|\mathbf{BC}|^2|\mathbf{AC}|^2\leq|\mathbf{AC}\times\mathbf{BC}|^2$$

But $\quad |\mathbf{AC}\times\mathbf{BC}|=|\mathbf{AC}||\mathbf{BC}|\sin C$

$\Rightarrow \quad \sin^2 C\geq 1$

$\Rightarrow \quad \sin C=\pm 1$

$\Rightarrow \quad \angle C=\frac{\pi}{2}$

● ***Ex. 32*** *If* $a^2+b^2+c^2=1$ *where* $a,b,c\in R$, *then the maximum value of* $(4a-3b)^2+(5b-4c)^2+(3c-5a)^2$ *is*

(a) 25 (b) 50

(c) 144 (d) None of these

Sol. (b) Let $\mathbf{r}_1=a\hat{\mathbf{i}}+b\hat{\mathbf{j}}+c\hat{\mathbf{k}},\ \mathbf{r}_2=3\hat{\mathbf{i}}+4\hat{\mathbf{j}}+5\hat{\mathbf{k}}$ ...(i)

$$|\mathbf{r}_1\times\mathbf{r}_2|\leq|\mathbf{r}_1|^2|\mathbf{r}_2|^2$$

Now, $$\mathbf{r}_1\times\mathbf{r}_2=\begin{vmatrix}\hat{\mathbf{i}}&\hat{\mathbf{j}}&\hat{\mathbf{k}}\\a&b&c\\3&4&5\end{vmatrix}$$

$$=\hat{\mathbf{i}}(5b-4c)+\hat{\mathbf{j}}(3c-5a)+\hat{\mathbf{k}}(4a-3b)$$

So, from Eq. (i), we get

$$(5b-4c)^2+(3c-5a)^2+(4a-3b)^2\leq 50$$

● ***Ex. 33*** *If* $\mathbf{a},\mathbf{b}$ *and* $\mathbf{c}$ *are pth, qth, rth terms of HP and* $\mathbf{u}=(q-r)\hat{\mathbf{i}}+(r-p)\hat{\mathbf{j}}+(p-q)\hat{\mathbf{k}},\ \mathbf{v}=\frac{\hat{\mathbf{i}}}{a}+\frac{\hat{\mathbf{j}}}{b}+\frac{\hat{\mathbf{k}}}{c}$, *then*

(a) $\mathbf{u}$ and $\mathbf{v}$ are parallel vectors

(b) $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors

(c) $\mathbf{u}\cdot\mathbf{v}=1$

(d) $\mathbf{u}\times\mathbf{v}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$

Sol. (b) Let A be the first term and D be the common difference of the corresponding AP. Then,

$$\frac{1}{a}=A+(p-1)D,\ \frac{1}{b}=A+(q-1)D,\ \frac{1}{c}=A+(r-1)D$$

$\Rightarrow \quad a^{-1}(q-r)+b^{-1}(r-p)+c^{-1}(p-q)=0$

$\Rightarrow \quad \mathbf{v}\cdot\mathbf{u}=0$

$\Rightarrow \quad \mathbf{u}\perp\mathbf{v}$

Hence, $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors.

● ***Ex. 34*** *If the vector product of a constant vector* $\mathbf{OA}$ *with variable vector* $\mathbf{OB}$ *in a fixed plane OAB be a constant vector, then the locus of B is*

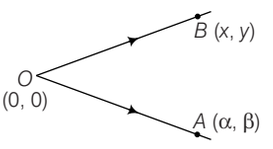
(a) a straight line perpendicular to $\mathbf{OA}$

(b) a circle with centre O radius equal to $|\mathbf{OA}|$

(c) a straight line parallel to $\mathbf{OA}$

(d) None of the above

Sol. (c) Let $A(\alpha,\beta)$ point be given and O be taken as the origin



We have, $\quad \mathbf{OA}=\alpha\hat{\mathbf{i}}+\beta\hat{\mathbf{j}}$

and $\quad \mathbf{OB}=x\hat{\mathbf{i}}+y\hat{\mathbf{j}}$

Now, $|\mathbf{OA}\times\mathbf{OB}|=|(\alpha y-\beta x)\,\hat{\mathbf{k}}|=$ constant

$\Rightarrow \quad \alpha y - \beta x = \text{constant}$

$\therefore$ Locus of $B(x, y)$ is a line parallel to OA because slope of

$$OA = \frac{\beta}{\alpha}$$

● ***Ex. 35*** *Unit vector perpendicular to the plane of ΔABC with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ of the vertices A, B, C is*

(a) $\dfrac{\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}}{\Delta}$

(b) $\dfrac{\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}}{2\Delta}$

(c) $\dfrac{\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}}{4\Delta}$

(d) None of the above

Sol. (b) The required vector is given by

$$\hat{\mathbf{n}} = \frac{\mathbf{AB}\times\mathbf{AC}}{|\,\mathbf{AB}\times\mathbf{AC}\,|}$$

$$\begin{aligned}\mathbf{AB}\times\mathbf{AC} &= (\mathbf{b}\times\mathbf{a})\times(\mathbf{c}-\mathbf{a})\\ &= \mathbf{b}\times\mathbf{c}-\mathbf{b}\times\mathbf{a}-\mathbf{a}\times\mathbf{c}+\mathbf{a}\times\mathbf{a}\\ &= \mathbf{b}\times\mathbf{c}+\mathbf{a}\times\mathbf{b}+\mathbf{c}\times\mathbf{a} \qquad [\because \mathbf{a}\times\mathbf{a}=0]\end{aligned}$$

We also know that,

$$\text{Area of } \Delta ABC = \frac{1}{2}|\,\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}|$$

$$\therefore \quad \hat{\mathbf{n}} = \frac{\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}}{|\,\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}\,|} = \frac{\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}}{2\Delta}$$

$$\left[\because \Delta = \frac{1}{2}|\,\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}\,|\right]$$

● ***Ex. 36*** *The vector $\mathbf{r}$ satisfying the conditions that*

I. *it is perpendicular to $3\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\hat{\mathbf{k}}$ and $18\hat{\mathbf{i}}-22\hat{\mathbf{j}}-5\hat{\mathbf{k}}$*

II. *it makes an obtuse angle with Y-axis.*

III. $|\,\mathbf{r}\,| = 14$

(a) $2(-2\hat{\mathbf{i}}-3\hat{\mathbf{j}}+6\hat{\mathbf{k}})$

(b) $2(2\hat{\mathbf{i}}-3\hat{\mathbf{j}}+6\hat{\mathbf{k}})$

(c) $4\hat{\mathbf{i}}+6\hat{\mathbf{j}}-12\hat{\mathbf{k}}$

(d) None of the above

Sol. (a) Let $\mathbf{a} = 3\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\hat{\mathbf{k}}$ and $\mathbf{b} = 18\hat{\mathbf{i}}-22\hat{\mathbf{j}}-5\hat{\mathbf{k}}$

Then, the required vector $\mathbf{r}$ is given by

$$\mathbf{r} = \lambda(\mathbf{a}\times\mathbf{b})$$

$$\Rightarrow \quad \mathbf{r} = \lambda(34\hat{\mathbf{i}}+51\hat{\mathbf{j}}-102\hat{\mathbf{k}}) = 17\lambda(2\hat{\mathbf{i}}+3\hat{\mathbf{j}}-6\hat{\mathbf{k}})$$

Now, $\quad |\,\mathbf{r}\,| = 14 \Rightarrow 119\,|\,\lambda\,| = 14$

$$\Rightarrow \quad |\,\lambda\,| = \frac{2}{17}$$

Since, $\mathbf{r}$ makes an obtuse angle with Y-axis. Therefore,

$$\lambda = -\frac{2}{17}$$

Hence, $\quad \mathbf{r} = -2(2\hat{\mathbf{i}}+3\hat{\mathbf{j}}-6\hat{\mathbf{k}})$

or $\quad \mathbf{r} = 2(-2\hat{\mathbf{i}}-3\hat{\mathbf{j}}+6\hat{\mathbf{k}})$

● ***Ex. 37*** *Let a, b, c denote the lengths of the sides of a triangle such that*

$$(a-b)\,\mathbf{u} + (b-c)\,\mathbf{v} + (c-a)\,(\mathbf{u}\times\mathbf{v}) = 0$$

for any two non-collinear vectors $\mathbf{u}$ and $\mathbf{v}$, then the triangle is

(a) right angled (b) equilateral

(c) isosceles (d) scalene

Sol. (b) Since, $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{u}\times\mathbf{b}$ are non-coplanar vectors.

$\therefore \quad (a-b)\,\mathbf{u} + (b-c)\mathbf{v} + (c-a)\,(\mathbf{u}\times\mathbf{v}) = 0$

$\Rightarrow \quad a-b = 0 = b-c = c-a$

$\Rightarrow \quad a = b = c$

So, the triangle is equilateral.

● ***Ex. 38*** *The value of $\hat{\mathbf{i}}\cdot(\hat{\mathbf{j}}\times\hat{\mathbf{k}}) + \hat{\mathbf{j}}\cdot(\hat{\mathbf{k}}\times\hat{\mathbf{i}}) + \hat{\mathbf{k}}\cdot(\hat{\mathbf{i}}\times\hat{\mathbf{j}})$ is*

(a) 3 (b) 2

(c) 1 (d) 0

Sol. (a) We have, $\hat{\mathbf{i}}\cdot(\hat{\mathbf{j}}\times\hat{\mathbf{k}}) + \hat{\mathbf{j}}\cdot(\hat{\mathbf{k}}\times\hat{\mathbf{i}}) + \hat{\mathbf{k}}\cdot(\hat{\mathbf{i}}\times\hat{\mathbf{j}})$

$= \hat{\mathbf{i}}\cdot\hat{\mathbf{i}} + \hat{\mathbf{j}}\cdot\hat{\mathbf{j}} + \hat{\mathbf{k}}\cdot\hat{\mathbf{k}}$ $[\because \hat{\mathbf{i}}\times\hat{\mathbf{j}} = \hat{\mathbf{k}},\ \hat{\mathbf{j}}\times\hat{\mathbf{k}} = \hat{\mathbf{i}},\ \hat{\mathbf{k}}\times\hat{\mathbf{i}} = \hat{\mathbf{j}}]$

$= |\,\hat{\mathbf{i}}\,|^2 + |\,\hat{\mathbf{j}}\,|^2 + |\,\hat{\mathbf{k}}\,|^2 = 1+1+1 = 3$ $[\because \hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are unit vectors$]$

● ***Ex. 39*** *For non-zero vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$; $|\,(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}\,| = |\,\mathbf{a}\,||\,\mathbf{b}\,||\,\mathbf{c}\,|$ holds if and only if*

(a) $\mathbf{a}\cdot\mathbf{b} = 0, \mathbf{b}\cdot\mathbf{c} = 0$ (b) $\mathbf{b}\cdot\mathbf{c} = 0, \mathbf{c}\cdot\mathbf{a} = 0$

(c) $\mathbf{c}\cdot\mathbf{a} = 0, \mathbf{a}\cdot\mathbf{b} = 0$ (d) $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{c} = \mathbf{c}\cdot\mathbf{a} = 0$

Sol. (d) We have, $|\,(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}\,| = |\,\mathbf{a}\,||\,\mathbf{b}\,||\,\mathbf{c}\,|$

$\Rightarrow \quad ||\,\mathbf{a}\,||\,\mathbf{b}\,||\,\mathbf{c}\,|\sin\theta\cos\alpha\,| = |\,\mathbf{a}\,||\,\mathbf{b}\,||\,\mathbf{c}\,|$

$\Rightarrow \quad |\sin\theta\,||\cos\alpha\,| = 1$

$\Rightarrow \quad \theta = \dfrac{\pi}{2}$ and $\alpha = 0$

$\Rightarrow \quad \mathbf{a}\perp\mathbf{b}$ and $\mathbf{c}\,||\,\hat{\mathbf{n}}$

$\Rightarrow \quad \mathbf{a}\perp\mathbf{c}$ and $\mathbf{c}\perp$ both $\mathbf{a}$ and $\mathbf{b}$

$\Rightarrow$ $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are mutually perpendicular.

$\Rightarrow \quad \mathbf{a}\cdot\mathbf{c} = \mathbf{b}\cdot\mathbf{c} = \mathbf{c}\cdot\mathbf{a} = 0$

● ***Ex. 40*** *The position vectors of three vertices A, B, C of a tetrahedron $OABC$ with respect to its vertex O are $\hat{\mathbf{i}}, 6\hat{\mathbf{j}}, \hat{\mathbf{k}}$, then its volume (in cu units) is*

(a) 3 (b) $\dfrac{1}{3}$

(c) $\dfrac{1}{6}$ (d) 6

Sol. (d) We have, $A(6\hat{i} + 0\hat{j} + 0\hat{k})$, $B(0\hat{i} + 6\hat{j} + 0\hat{k})$, $C(0\hat{i} + 0\hat{j} + \hat{k})$ and $O(0\hat{i} + 0\hat{j} + 0\hat{k})$

$\therefore$ $\mathbf{OA} = 6\hat{i} + 0\hat{j} + 0\hat{k}$

$\mathbf{OB} = 0\hat{i} + 6\hat{j} + 0\hat{k}$

and $\mathbf{OC} = 0\hat{i} + 0\hat{j} + \hat{k}$

$\therefore$ Volume of tetrahedron

$$= \frac{1}{6}[\mathbf{OA}\ \mathbf{OB}\ \mathbf{OC}] = \frac{1}{6}[6\hat{i}\ \ 6\hat{j}\ \ \hat{k}]$$

$$= \frac{1}{6}\begin{vmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{6}[6(6-0)] = 6 \text{ cu units}$$

● ***Ex. 41*** *A parallelopiped is formed by planes drawn parallel to coordinate axes through the points $A = (1, 2, 3)$ and $B = (9, 8, 5)$. The volume of that parallelopiped is equal to (in cubic units)*

(a) 192 (b) 48

(c) 32 (d) 96

Sol. (d) Translating the axes through $A(1, 2, 3)$.

A changes to $(0, 0, 0)$, B changes to $(8, 6, 2)$.

$\therefore$ Coterminous edges are of lengths 8, 6, 2.

Volume of parallelopiped $= 8 \cdot 6 \cdot 2 = 96$ cu units.

● ***Ex. 42*** *If $|\mathbf{a}| = 1, |\mathbf{b}| = 3$ and $|\mathbf{c}| = 5$, then the value of $[\mathbf{a} - \mathbf{b}\ \mathbf{b} - \mathbf{c}\ \mathbf{c} - \mathbf{a}]$ is*

(a) 0 (b) 1

(c) -1 (d) None of these

Sol. (a) $[\mathbf{a} - \mathbf{b}\ \mathbf{b} - \mathbf{c}\ \mathbf{c} - \mathbf{a}] + \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}[\mathbf{abc}]$

$$= [1(1-0) + 1(0-1) + 0(0+1)]\,[\mathbf{abc}]$$

$$= 0 \times [\mathbf{abc}] = 0$$

● ***Ex. 43*** *If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are three non-coplanar vectors, then $3\mathbf{a} - 7\mathbf{b} - 4\mathbf{c}$, $3\mathbf{a} - 2\mathbf{b} + \mathbf{c}$ and $\mathbf{a} + \mathbf{b} + \lambda\mathbf{c}$ will be coplanar, if λ is*

(a) -1 (b) 1

(c) 3 (d) 2

Sol. (d) Let $\alpha = 3\mathbf{a} - 7\mathbf{b} - 4\mathbf{c}$, $\beta = 3\mathbf{a} - 2\mathbf{b} + \mathbf{c}$ and $\gamma = \mathbf{a} + \mathbf{b} + \lambda\mathbf{c}$

For α, β and γ to be coplanar $[\alpha\ \beta\ \gamma] = 0$

$$\Rightarrow \begin{vmatrix} 3 & -7 & -4 \\ 3 & -2 & 1 \\ 1 & 1 & \lambda \end{vmatrix}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$$

$$[3(-2\lambda - 1) + 7(3\lambda - 1) - 4(3 + 2)\,(abc)] = 0$$

$$\Rightarrow (15\lambda - 30)\,[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$$

Since, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are non-coplanar

$\therefore$ $15\lambda - 30 = 0 \Rightarrow \lambda = 2$

● ***Ex. 44.*** *Let $\mathbf{r} = (\mathbf{a} \times \mathbf{b})\sin x + (\mathbf{b} \times \mathbf{c})\cos y + (\mathbf{c} \times \mathbf{a})$, where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are non-zero non-coplanar vectors. If $\mathbf{r}$ is orthogonal to $3\mathbf{a} + 5\mathbf{b} + 2\mathbf{c}$, then the value of $\sec^2 y + \text{cosec}^2 x + \sec y\ \text{cosec}\ x$ is*

(a) 3 (b) 4

(c) 5 (d) 6

Sol. (a) $\mathbf{r} \cdot (3\mathbf{a} + 5\mathbf{b} + 2\mathbf{c}) = 0$

$$\Rightarrow \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})[2\sin x + 3\cos y + 5] = 0$$

$$\Rightarrow 2\sin x + 3\cos y + 5 = 0$$

$$\Rightarrow 2\sin x + 3\cos y = -5 \qquad [\because \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \neq 0]$$

$$\Rightarrow \sin x = -1, \cos y = 1$$

$$\Rightarrow \text{cosec}\ x = -1, \sec y = -1$$

● ***Ex. 45*** *Let a, b, c be distinct non-negative numbers. If the vectors $a\hat{i} + a\hat{j} + c\hat{k}$, $\hat{i} + \hat{k}$ and $c\hat{i} + c\hat{j} + b\hat{k}$ lie in a plane, then c is*

(a) HM of a and b (b) 0

(c) AM of a and b (d) GM of a and b

Sol. (d) Since, the given points lie in a plane

$$\therefore \begin{vmatrix} a & a & c \\ 1 & 0 & 1 \\ c & c & b \end{vmatrix} = 0$$

On applying $C_1 \to C_1 - C_2$

$$\Rightarrow \begin{vmatrix} 0 & a & c \\ 1 & 0 & 1 \\ 0 & c & b \end{vmatrix} = 0$$

$$\Rightarrow -1(ab - c^2) = 0$$

$$\Rightarrow c^2 = ab$$

Hence, c is GM of a and b.

● ***Ex. 46*** *If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are non-coplanar vectors and λ is a real number, then $[\lambda(\mathbf{a} + \mathbf{b})\,|\,\lambda^2\mathbf{b}\,|\,\lambda\mathbf{c}] = [a\ \mathbf{a} + \mathbf{c}\ \mathbf{b}]$ for*

(a) exactly two values of λ

(b) exactly one value of λ

(c) no value of λ

(d) exactly three values of λ

Sol. (c) Given, $[\lambda(\mathbf{a} + \mathbf{b})\lambda^2\mathbf{b}\lambda\mathbf{c}] = [\mathbf{a}\,\mathbf{b} + \mathbf{c}\,\mathbf{b}]$

$$\Rightarrow \begin{vmatrix} \lambda(a_1 + b_1) & \lambda(a_2 + b_2) & \lambda(a_3 + b_3) \\ \lambda^2 b_1 & \lambda^2 b_2 & \lambda^2 b_3 \\ \lambda c_1 & \lambda c_2 & \lambda c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\Rightarrow \lambda^4 \begin{vmatrix} a_1 + b_1 & a_2 + b_2 & a_3 + b_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

[applying $R_1 \to R_1 - R_2$ in LHS and $R_2 \to R_2 - R_3$ in RHS]

$$\Rightarrow \quad \lambda^4 \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = - \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\Rightarrow \quad \lambda^4 = -1$$

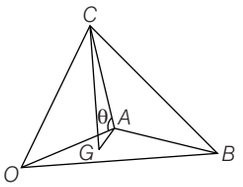
Hence, no real value of λ exists.

● ***Ex. 47*** *In a regular tetrahedron, let* θ *be angle between any edge and a face not containing the edge. The value of* $\cos^2\theta$ *is*

(a) 1/6 (b) 1/9

(c) 1/3 (d) None of these

Sol. (c) Let $OABC$ be the tetrahedron. Let G be the centroid of the face OAB, then $GA = \frac{1}{\sqrt{3}} AC$.



Then, $\cos\theta = \frac{GA}{CA} = \frac{1}{\sqrt{3}}$

$\therefore \quad \cos^2\theta = \frac{1}{3}$

● ***Ex. 48*** *DABC be a tetrahedron such that AD is perpendicular to the base ABC and* $\angle ABC = 30°$. *The volume of tetrahedron is* 18. *If value of* $AB + BC + AD$ *is minimum, then the length of AC is*

(a) $6\sqrt{2-\sqrt{3}}$ (b) $3(\sqrt{6}-\sqrt{2})$

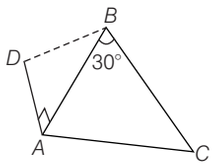
(c) $6\sqrt{2+\sqrt{3}}$ (d) $3(\sqrt{6}+\sqrt{2})$

Sol. (a) $\text{Volume} = \frac{1}{3} AD\left(\frac{1}{2} AB \cdot BC \sin 30°\right)$

$$\Rightarrow \quad 18 = \frac{1}{12}(AD \cdot AB \cdot BC)$$

$$\Rightarrow \quad AD \cdot AB \cdot BC = 216$$

Now, $AB + BC + AD \geq 3(AD \cdot AB \cdot BC)^{1/3}$



$$\Rightarrow \quad AB + BC + AD \geq 18$$

Minimum value occurs when $AB = BC = AD = 6$

Hence, $AC = \sqrt{AB^2 + BC^2 - 2AB \cdot BC \cdot \cos 30°}$

$$= 6\sqrt{2-\sqrt{3}}$$

● ***Ex. 49*** *If* $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}, \mathbf{b} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}, \mathbf{c} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$, *then the value of* $\begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix}$ *is*

(a) 2 (b) 4

(c) 16 (d) 64

Sol. (c) We have, $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$

$\mathbf{b} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{c} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$

We know that, $\begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix} = [\mathbf{a}\,\mathbf{b}\,\mathbf{c}]^2$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix}^2$$

$$= [1(1-2) - 1(-1-1) + 1(2+1)]^2$$

$$= [-1 + 2 + 3]^2 = [4]^2 = 16$$

● ***Ex. 50*** *The value of a so that the volume of parallelopiped formed by* $\hat{\mathbf{i}} + a\hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\hat{\mathbf{j}} + a\hat{\mathbf{k}}$ *and* $a\hat{\mathbf{i}} + \hat{\mathbf{k}}$ *becomes minimum is*

(a) -3 (b) 3

(c) $1/\sqrt{3}$ (d) $\sqrt{3}$

Sol. (c) Volume of the parallelopiped

$$V = [\hat{\mathbf{i}} + a\hat{\mathbf{j}} + \hat{\mathbf{k}}\hat{\mathbf{j}} + a\hat{\mathbf{k}}\, a\hat{\mathbf{i}} + \hat{\mathbf{k}}]$$

$$= (\hat{\mathbf{i}} + a\hat{\mathbf{j}} + \hat{\mathbf{k}}) \cdot \{(\hat{\mathbf{j}} + a\hat{\mathbf{k}}) \times (a\hat{\mathbf{i}} + \hat{\mathbf{k}})\}$$

$$= (\hat{\mathbf{i}} + a\hat{\mathbf{j}} + \hat{\mathbf{k}}) \cdot (\hat{\mathbf{i}} + a^2\hat{\mathbf{j}} - a\hat{\mathbf{k}}) = 1 + a^3 - a$$

$$\frac{dV}{da} = 3a^2 - 1, \frac{d^2V}{da^2} = 6a, \frac{dV}{da} = 0$$

$$\Rightarrow \quad 3a^2 - 1 = 0 \quad \Rightarrow \quad a = \pm\frac{1}{\sqrt{3}}$$

At $a = \frac{1}{\sqrt{3}}, \frac{d^2V}{da^2} = \frac{6}{\sqrt{3}} > 0$

$\therefore V$ is minimum at $a = \frac{1}{\sqrt{3}}$

● ***Ex. 51*** *If* $\mathbf{a}, \mathbf{b}$ *and* $\mathbf{c}$ *be any three non-zero and non-coplanar vectors, then any vector* $\mathbf{r}$ *is equal to*

(a) $z\mathbf{a} + x\mathbf{b} + y\mathbf{c}$ (b) $x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$

(c) $y\mathbf{a} + z\mathbf{b} + x\mathbf{c}$ (d) None of these

where, $x = \frac{[\mathbf{r}\,\mathbf{b}\,\mathbf{c}]}{[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]}, y = \frac{[\mathbf{r}\,\mathbf{c}\,\mathbf{a}]}{[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]}, z = \frac{[\mathbf{r}\,\mathbf{a}\,\mathbf{b}]}{[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]}$

Sol. (b) Since, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three non-coplanar vectors, we may assume $\mathbf{r} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$

$$[\mathbf{r}\ \mathbf{b}\ \mathbf{c}] = (\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c})\cdot(\mathbf{b}\times\mathbf{c}) = \alpha\{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})\}$$
$$= \alpha[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$
$$\Rightarrow \quad \alpha = \frac{[\mathbf{r}\ \mathbf{b}\ \mathbf{c}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$$

But $\quad x = \dfrac{[\mathbf{r}\ \mathbf{b}\ \mathbf{c}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$

$$\therefore \quad \alpha = x$$

Similarly, $\beta = y, \gamma = z$

$$\therefore \quad \mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$$

● ***Ex. 52*** *The position vectors of vertices of ΔABC are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{a}\cdot\mathbf{a} = \mathbf{b}\cdot\mathbf{b} = \mathbf{c}\cdot\mathbf{c} = 3$. If $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$, then the position vectors of the orthocentre of ΔABC is*

(a) $\mathbf{a}+\mathbf{b}+\mathbf{c}$ (b) $\dfrac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$

(c) 0 (d) None of these

Sol. (a) Hence, $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$

So, the points O, A, B and C are coplanar. Also, $OA = OB = OC = \sqrt{3}$, hence origin O is the circumcentre.

Position vector of the centroid G is $\dfrac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}$.

Now, orthocentre divides OG in the ratio of 3 : 2 externally. So, position vectors of orthocentre is $\mathbf{a}+\mathbf{b}+\mathbf{c}$.

● ***Ex. 53*** *If α and β are two mutually perpendicular unit vectors $\{r\alpha + r\beta + s(\alpha\times\beta\}, [\alpha + (\alpha\times\beta)]$ and $\{s\alpha + s\beta + t(\alpha\times\beta)\}$ are coplanar, then s is equal to*

(a) AM of r and t (b) GM of r and t

(c) HM of r and t (d) None of these

Sol. Since, α and β are two mutually perpendicular vectors and $(r\alpha + r\beta + s(\alpha\times\beta), [\alpha + (\alpha\times\beta)], \{s\alpha + s\beta + t(\alpha\times\beta)\}$ are coplanar

$$\Rightarrow \quad \begin{vmatrix} r & r & s \\ 1 & 0 & 1 \\ s & s & t \end{vmatrix} = 0$$

$$\Rightarrow \quad s^2 = rt$$

● ***Ex. 54*** *Let $\mathbf{b} = -\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$ and $\mathbf{c} = 2\hat{\mathbf{i}} - 7\hat{\mathbf{j}} - 10\hat{\mathbf{k}}$. If $\mathbf{a}$ be a unit vector and the scalar triple product $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ has the greatest value, then $\mathbf{a}$ is equal to*

(a) $\dfrac{1}{\sqrt{3}}(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$ (b) $\dfrac{1}{\sqrt{5}}(\sqrt{2}\hat{\mathbf{i}} - \hat{\mathbf{j}} - \sqrt{2}\hat{\mathbf{k}})$

(c) $\dfrac{1}{3}(2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}})$ (d) $\dfrac{1}{\sqrt{59}}(3\hat{\mathbf{i}} - 7\hat{\mathbf{j}} - \hat{\mathbf{k}})$

Sol. (c) $\mathbf{b}\times\mathbf{c} = 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$

$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \mathbf{a}\cdot(2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}) = 1\cdot 3\cdot\cos\theta \le 3$$

The greatest value of $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 3$, which is obtained when $\theta = 0$.

So, $\quad \mathbf{a} = \dfrac{\mathbf{b}\times\mathbf{c}}{|\mathbf{b}\times\mathbf{c}|} = \dfrac{2}{3}\hat{\mathbf{i}} + \dfrac{2}{3}\hat{\mathbf{j}} - \dfrac{1}{3}\hat{\mathbf{k}}$

● ***Ex. 55*** *The vectors*

$$\mathbf{u} = (al + a_1l_1)\hat{\mathbf{i}} + (am + a_1m_1)\hat{\mathbf{j}} + (an + a_1n_1)\hat{\mathbf{k}}$$
$$\mathbf{v} = (bl + b_1l_1)\hat{\mathbf{i}} + (bm + b_1m_1)\hat{\mathbf{j}} + (bn + b_1n_1)\hat{\mathbf{k}} \text{ and}$$
$$\mathbf{w} = (cl + c_1l_1)\hat{\mathbf{i}} + (cm + c_1m_1)\hat{\mathbf{j}} + (cn + c_1n_1)\hat{\mathbf{k}}$$

(a) form an equilateral triangle

(b) are coplanar

(c) are collinear

(d) are mutually perpendicular

Sol. (b) We have,

$$[\mathbf{u}\ \mathbf{v}\ \mathbf{w}] = \begin{vmatrix} al + a_1l_1 & am + a_1m_1 & an + a_1n_1 \\ bl + b_1l_1 & bm + b_1m_1 & bn + b_1n_1 \\ cl + c_1l_1 & cm + c_1m_1 & cn + c_1n_1 \end{vmatrix}$$

$$\Rightarrow \quad [\mathbf{u}\ \mathbf{v}\ \mathbf{w}] = \begin{vmatrix} a & a_1 & 0 \\ b & b_1 & 0 \\ c & c_1 & 0 \end{vmatrix} \begin{vmatrix} l & l_1 & 0 \\ m & m_1 & 0 \\ n & n_1 & 0 \end{vmatrix} = 0$$

Hence, the given vectors are coplanar.

● ***Ex. 56*** *Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three vectors such that $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 2$. If $\mathbf{r} = l(\mathbf{b}\times\mathbf{c}) + m(\mathbf{c}\times\mathbf{a}) + n(\mathbf{a}\times\mathbf{b})$ is perpendicular to $\mathbf{a}+\mathbf{b}+\mathbf{c}$, then the value of $(l+m+n)$ is*

(a) 2 (b) 1

(c) 0 (d) None of these

Sol. (c) It is given that $\mathbf{r}$ perpendicular $(\mathbf{a}+\mathbf{b}+\mathbf{c})$

$$\therefore \quad \mathbf{r}\cdot(\mathbf{a}+\mathbf{b}+\mathbf{c}) = 0$$
$$\Rightarrow \quad l[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] + m[\mathbf{c}\ \mathbf{a}\ \mathbf{b}] + n[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$$
$$\Rightarrow \quad 2(l + m + n) = 0 \qquad [\because [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 2]$$
$$\Rightarrow \quad l + m + n = 0$$

● ***Ex. 57*** *If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are three mutually perpendicular vectors, then the projection of the vectors $l\dfrac{\mathbf{a}}{|\mathbf{a}|} + m\dfrac{\mathbf{b}}{|\mathbf{b}|} + n\dfrac{(\mathbf{a}\times\mathbf{b})}{|\mathbf{a}\times\mathbf{b}|}$ along the angle bisector of the vectors $\mathbf{a}$ and $\mathbf{b}$ is*

(a) $\dfrac{l+m}{\sqrt{2}}$ (b) $\sqrt{l^2 + m^2 + n^2}$

(c) $\dfrac{\sqrt{l^2 + m^2}}{\sqrt{l^2 + m^2 + b^2}}$ (d) None of these

Sol. (a) A vector parallel to the bisector of the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$ is

$$\frac{\mathbf{a}}{|\mathbf{a}|} + \frac{\mathbf{b}}{|\mathbf{b}|} = \hat{\mathbf{a}} + \hat{\mathbf{b}}$$

$\therefore$Units vector along the bisector

$$= \frac{\hat{\mathbf{a}} + \hat{\mathbf{b}}}{|\hat{\mathbf{a}} + \hat{\mathbf{b}}|} = \frac{1}{\sqrt{2}}(\hat{\mathbf{a}} + \hat{\mathbf{b}})$$

$$(\hat{\mathbf{a}} + \hat{\mathbf{b}})^2 = |\mathbf{a}|^2 + |\hat{\mathbf{b}}| + 2\hat{\mathbf{a}} \cdot \hat{\mathbf{b}} = 1 + 1 + 0 = 2$$

$\therefore$ Required projection

$$= \left[l\frac{\mathbf{a}}{|\mathbf{a}|} + m\frac{\mathbf{b}}{|\mathbf{b}|} + n\frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|} \right] \cdot \frac{1}{\sqrt{2}}(\hat{\mathbf{a}} + \hat{\mathbf{b}})$$

$$= \frac{1}{\sqrt{2}}(l + m)$$

$$\left[\therefore \frac{\mathbf{a} \cdot \hat{\mathbf{a}}}{|\mathbf{a}|} = \frac{\mathbf{b} \cdot \hat{\mathbf{b}}}{|\mathbf{b}|} = 1 \text{ and } \hat{\mathbf{a}} \cdot \hat{\mathbf{b}} = \hat{\mathbf{a}} \cdot (\mathbf{a} \times \mathbf{b}) = \hat{\mathbf{b}} \cdot (\hat{\mathbf{a}} \times \mathbf{b}) = 0 \right]$$

● ***Ex. 58*** *If the volume of parallelopiped formed by the vectors* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *as three coterminous edges is 27 cu units, then the volume of the parallelopiped have* $\alpha = \mathbf{a} + 2\mathbf{b} - \mathbf{c}$, $\beta = \mathbf{a} - \mathbf{b}$ *and* $\gamma = \mathbf{a} - \mathbf{b} - \mathbf{c}$ *as three coterminous edges is*

(a) 27 (b) 9

(c) 81 (d) None of these

Sol. (c) We have, $|[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]| = 27$ cu units

Now, $$[\alpha\ \beta\ \gamma] = \begin{vmatrix} 1 & 2 & -1 \\ 1 & -1 & 0 \\ 1 & -1 & -1 \end{vmatrix} [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 3[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$

$\therefore$ Required volume $= |[\alpha\ \beta\ \gamma]|$

$$= 3|[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]|$$

$$= 3 \times 27 = 81 \text{ cu units}$$

● ***Ex. 59*** *If* V *is the volume of the parallelopiped having three coterminous edges as* $\mathbf{a}, \mathbf{b}$ *and* $\mathbf{c}$, *then the volume of parallelopiped having three coterminous edges as* $\alpha = (\mathbf{a} \cdot \mathbf{a})\mathbf{a} + (\mathbf{a} \cdot \mathbf{b})\mathbf{b} + (\mathbf{a} \cdot \mathbf{c})\mathbf{c}$, $\beta = (\mathbf{a} \cdot \mathbf{b})\mathbf{a} + (\mathbf{b} \cdot \mathbf{b})\mathbf{b} + (\mathbf{b} \cdot \mathbf{c})\mathbf{c}$ *and* $\gamma = (\mathbf{a} \cdot \mathbf{c})\ \mathbf{a} + (\mathbf{b} \cdot \mathbf{c})\ \mathbf{b} + (\mathbf{c} \cdot \mathbf{c})\ \mathbf{c}$ *is*

(a) V^3 (b) $3V$

(c) V^2 (d) $2V$

Sol. (a) We have, $|[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]| = V$

Let V_1 be the volume of the parallelopiped formed by the vectors α, β and γ. Then,

$$V_1 = |[\alpha\ \beta\ \gamma]|$$

Now, $$[\alpha\ \beta\ \gamma] = \begin{vmatrix} \mathbf{a} \cdot \mathbf{a} & \mathbf{a} \cdot \mathbf{b} & \mathbf{a} \cdot \mathbf{c} \\ \mathbf{a} \cdot \mathbf{b} & \mathbf{b} \cdot \mathbf{b} & \mathbf{b} \cdot \mathbf{c} \\ \mathbf{a} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{c} & \mathbf{c} \cdot \mathbf{c} \end{vmatrix} [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$

$$\Rightarrow \quad [\alpha\ \beta\ \gamma] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2\ [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$

$$\Rightarrow \quad [\alpha\ \beta\ \gamma] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^3$$

$\because$ $$V_1 = |[\alpha\ \beta\ \gamma]$$

$$= |[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^3| = V^3$$

● ***Ex. 60*** *Let* $\mathbf{r}, \mathbf{a}, \mathbf{b}$ *and* $\mathbf{c}$ *be four non-zero vectors such that* $\mathbf{r} \cdot \mathbf{a} = 0$, $|\mathbf{r} \times \mathbf{b}| = |\mathbf{r}||\mathbf{b}|, |\mathbf{r} \times \mathbf{c}| = |\mathbf{r}||\mathbf{c}|$, *then* $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ *is*

(a) $|\mathbf{a}||\mathbf{b}||\mathbf{c}|$ (b) $-|\mathbf{a}||\mathbf{b}|\mathbf{c}|$

(c) 0 (d) None of these

Sol. (c) Given, $\mathbf{r} \cdot \mathbf{a} = 0$

$$|\mathbf{r} \times \mathbf{b}| = |\mathbf{r}||\mathbf{b}|$$

and $$|\mathbf{r} \times \mathbf{c}| = |\mathbf{r}||\mathbf{c}|$$

This shows $\mathbf{r}$ is perpendicular to both $\mathbf{b}$ and $\mathbf{c}$.

$\Rightarrow$ $\mathbf{r}$ is perpendicular to $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$

$\therefore$ $$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$$

● ***Ex. 61*** *If* $\mathbf{a}, \mathbf{b}$ *and* $\mathbf{c}$ *are any three vector forming a linearly independent system, then* $\theta \in R$, $[\mathbf{a}\cos\theta + \mathbf{b}\sin\theta + \mathbf{c}\cos 2\theta, \mathbf{a}\cos\left(\frac{2\pi}{3} + \theta\right) + \mathbf{b}\sin\left(\frac{2\pi}{3} + \theta\right) + \mathbf{c}\cos 2\left(\frac{2\pi}{3} + \theta\right)$ $\mathbf{a}\cos\left(\theta - \frac{2\pi}{3}\right) + \mathbf{b}\sin\left(\theta - \frac{2\pi}{3}\right) + \mathbf{c}\cos 2\left(\theta - \frac{2\pi}{3}\right)$ *is*

(a) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\cos\theta$

(b) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\cos 2\theta$

(c) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\cos 3\theta$

(d) None of the above

Sol. (d) Since $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly independent,

$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \neq 0$$

We know that,

$$[\mathbf{a}x_1 + \mathbf{b}x_2 + \mathbf{c}x_3 \mathbf{a}y_1 + \mathbf{b}y_2 + \mathbf{c}y_3\, \mathbf{a}z_1 + \mathbf{b}z_2 + \mathbf{c}z_3]$$

$$= [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \cdot \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

Hence, the given system can be written as

$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \begin{vmatrix} \cos\theta & \sin\theta & \cos 2\theta \\ \cos\left(\frac{2\pi}{3} + \theta\right) & \sin\left(\frac{2\pi}{3} + \theta\right) & \cos 2\left(\theta + \frac{2\pi}{3}\right) \\ \cos\left(\theta - \frac{2\pi}{3}\right) & \sin\left(\theta - \frac{2\pi}{3}\right) & \cos 2\left(\theta - \frac{2\pi}{3}\right) \end{vmatrix}$$

On applying $R_3 \to R_3 + R_2 + R_1$, we get

$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \begin{vmatrix} \cos\theta & \sin\theta & \cos 2\theta \\ \cos\left(\frac{2\pi}{3} + \theta\right) & \sin\left(\frac{2\pi}{3} + \theta\right) & \cos 2\left(\theta + \frac{2\pi}{3}\right) \\ 0 & 0 & 0 \end{vmatrix}$$

$\Rightarrow [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,(0) = 0$

● **Ex. 62** *Let* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *be three non-coplanar vectors and* $\mathbf{d}$ *be a non-zero vector, which is perpendicular to* $\mathbf{a}+\mathbf{b}+\mathbf{c}$. *Now, if* $\mathbf{d}=(\sin x)(\mathbf{a}\times\mathbf{b})+(\cos y)(\mathbf{b}\times\mathbf{c})+2(\mathbf{c}\times\mathbf{a})$, *then the minimum value of* (x^2+y^2) *is*

(a) π^2 (b) $\frac{\pi^2}{2}$

(c) $\frac{\pi^2}{4}$ (d) $\frac{5\pi^2}{4}$

Sol. (d) Given, $\mathbf{d}\cdot(\mathbf{a}+\mathbf{b}+\mathbf{c})=0$

and $\mathbf{d}=\sin x\,(\mathbf{a}\times\mathbf{b})+\cos y\,(\mathbf{b}\times\mathbf{c})+2\,(\mathbf{c}\times\mathbf{a})$ ...(i)

$\because$ $\mathbf{a}\cdot\mathbf{d}=\cos y[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ ... (ii)

$\mathbf{b}\cdot\mathbf{d}=2\,[\mathbf{b}\ \mathbf{c}\ \mathbf{a}]$ ... (iii)

$\mathbf{c}\cdot\mathbf{d}=\sin x\,[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ ... (iv)

On adding Eqs. (ii), (iii) and (iv), we get

$\mathbf{a}\cdot\mathbf{d}+\mathbf{b}\cdot\mathbf{d}+\mathbf{c}\cdot\mathbf{d}=(\cos y+2+\sin x)[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

$\because$ $\sin x+\cos y+2=0$

$\Rightarrow$ $\sin x+\cos y=-2$

$\Rightarrow$ $\sin x=-1$

$[\because -1<\sin x\le 1$ and $-1\le\cos y\le 1]$

and $\cos y=-1$

Since, we have to find the minimum value of

$x^2+y^2;\ x=-\frac{\pi}{2}, y=\pi$

$\therefore$ $x^2+y^2=\frac{\pi^2}{4}+\pi^2=\frac{5\pi^2}{4}$

● **Ex. 63** *Let* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *be three vectors of magnitude* 1, 1 *and* 2 *respectively. If* $\mathbf{a}\times(\mathbf{a}\times\mathbf{c})+\mathbf{b}=0$, *then the acute angle between* $\mathbf{a}$ *and* $\mathbf{c}$ *is*

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{6}$ (d) None of these

Sol. (c) Given, $|\mathbf{a}|=1, |\mathbf{b}|=1,$ and $|\mathbf{c}|=2$

Also, $\mathbf{a}\times(\mathbf{a}\times\mathbf{c})+\mathbf{b}=0$

$\Rightarrow$ $(\mathbf{a}\cdot\mathbf{c})\,\mathbf{a}-(\mathbf{a}\cdot\mathbf{a})\mathbf{c}+\mathbf{b}=0$

$\Rightarrow$ $(\mathbf{a}\cdot\mathbf{c})\mathbf{a}-\mathbf{c}+\mathbf{b}=0$ $[\because \mathbf{a}\cdot\mathbf{a}=|\mathbf{a}|^2=1]$

$\Rightarrow$ $(\mathbf{a}\cdot\mathbf{c})\mathbf{a}-\mathbf{c}=-\mathbf{b}$

$\Rightarrow$ $|(\mathbf{a}\cdot\mathbf{c})\mathbf{a}-\mathbf{c}|=|-\mathbf{b}|$

$\Rightarrow$ $|(\mathbf{a}\cdot\mathbf{c})\mathbf{a}-\mathbf{c}|^2=|\mathbf{b}|^2$

$\Rightarrow$ $|(\mathbf{a}\cdot\mathbf{c})\mathbf{a}|^2+|\mathbf{c}|^2-2\{(\mathbf{a}\cdot\mathbf{c})\mathbf{a}\cdot\mathbf{c}\}=|\mathbf{b}|^2$

$\Rightarrow$ $(\mathbf{a}\cdot\mathbf{c})\,|\mathbf{a}|^2+|\mathbf{c}|^2-2(\mathbf{a}\cdot\mathbf{c})\,(\mathbf{a}\cdot\mathbf{c})=|\mathbf{b}|^2$

$\Rightarrow$ $(\mathbf{a}\cdot\mathbf{c})^2\,\{|\mathbf{a}|^2-2\}+|\mathbf{c}|^2=|\mathbf{b}|^2$

$\Rightarrow$ $-(\mathbf{a}\cdot\mathbf{c})^2+4=1$

$\Rightarrow$ $\mathbf{a}\cdot\mathbf{c}=\pm\sqrt{3}$

$\Rightarrow$ $|\mathbf{a}|\,|\mathbf{c}|\cos\theta=\sqrt{3}$

where, θ is an acute angle between $\mathbf{a}$ and $\mathbf{c}$.

$\Rightarrow$ $\cos\theta=\frac{\sqrt{3}}{2}$

$\therefore$ $\theta=\frac{\pi}{6}$

● **Ex. 64** *Let* $\mathbf{a}=2\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}, \mathbf{b}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}-\hat{\mathbf{k}}$ *and* $\mathbf{c}$ *is a unit vector coplanar to them. If* $\mathbf{c}$ *is perpendicular to* $\mathbf{a}$, *then* $\mathbf{c}$ *is equal to*

(a) $\frac{1}{\sqrt{2}}(-\hat{\mathbf{j}}+\hat{\mathbf{k}})$ (b) $-\frac{1}{\sqrt{3}}(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})$

(c) $\frac{1}{\sqrt{5}}(\hat{\mathbf{i}}-2\hat{\mathbf{j}})$ (d) $\frac{1}{\sqrt{3}}(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})$

Sol. (a) $\mathbf{a}\times(\mathbf{a}\times\mathbf{b})=(\mathbf{a}\cdot\mathbf{c})\,\mathbf{a}-(\mathbf{a}\cdot\mathbf{a})\mathbf{b}$

$=3(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})-6(\hat{\mathbf{i}}+2\hat{\mathbf{j}}-\hat{\mathbf{k}})=-9\hat{\mathbf{j}}+9\hat{\mathbf{k}}$

$\therefore$ Required unit vector $=\frac{\mathbf{a}\times(\mathbf{a}\times\mathbf{b})}{|\mathbf{a}\times(\mathbf{a}\times\mathbf{b})|}$

$=\pm\frac{1}{\sqrt{2}}(-\hat{\mathbf{j}}+\hat{\mathbf{k}})$

● **Ex. 65** *Let* $\mathbf{a}=2\hat{\mathbf{i}}+\hat{\mathbf{j}}-2\hat{\mathbf{k}}$ *and* $\mathbf{b}=\hat{\mathbf{i}}+\hat{\mathbf{j}}$. *If* $\mathbf{c}$ *is a vector such that* $\mathbf{a}\cdot\mathbf{c}=|\mathbf{c}|, |\mathbf{c}-\mathbf{a}|=2\sqrt{2}$ *and the angle between* $\mathbf{a}\times\mathbf{b}$ *and* $\mathbf{c}$ *is* 30°, *then* $|(\mathbf{a}\times\mathbf{b})\times\mathbf{c}|$ *is equal to*

(a) $\frac{2}{3}$ (b) $\frac{3}{2}$

(c) 2 (d) 3

Sol. (b) $|\mathbf{c}-\mathbf{a}|=2\sqrt{2}$

$\Rightarrow$ $|\mathbf{c}|^2+|\mathbf{a}|^2-2\mathbf{a}\cdot\mathbf{c}=8$

$\Rightarrow$ $|\mathbf{c}|^2+(\sqrt{9})^2-2|\mathbf{c}|=8$

$\Rightarrow$ $|\mathbf{c}|^2-2|\mathbf{c}|+1=0$

$\Rightarrow$ $(|\mathbf{c}|-1)^2=0\ \Rightarrow\ |\mathbf{c}|=1$

Now, $\mathbf{a}\times\mathbf{b}=2\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}}$

$\therefore$ $|\mathbf{a}\times\mathbf{b}|=\sqrt{4+4+1}=3$

$\therefore$ $|(\mathbf{a}\times\mathbf{b})\times\mathbf{c}|=|\mathbf{a}\times\mathbf{b}|\,|\mathbf{c}|\sin 30°$

$=3\times1\times\frac{1}{2}=\frac{3}{2}$

● **Ex. 66** *Let* $\hat{\mathbf{a}}$ *and* $\hat{\mathbf{b}}$ *be two unit vectors such that* $\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}=\frac{1}{3}$ *and* $\hat{\mathbf{a}}\times\hat{\mathbf{b}}=\hat{\mathbf{c}}$. *Also* $\bar{F}=\alpha\hat{\mathbf{a}}+\beta\hat{\mathbf{b}}+\gamma\hat{\mathbf{c}}$, *where* α, β, γ *are scalars. If* $\alpha=k_1(\hat{\mathbf{F}}\cdot\hat{\mathbf{a}})-k_2(\hat{\mathbf{F}}\cdot\hat{\mathbf{b}})$, *then the value of* $2(k_1+k_2)$ *is*

(a) $2\sqrt{3}$ (b) $\sqrt{3}$

(c) 3 (d) 1

Sol. (c) $\bar{\mathbf{F}}=\alpha\hat{\mathbf{a}}+\beta\hat{\mathbf{b}}+\gamma\hat{\mathbf{c}}$

$\therefore$ $\bar{F}\cdot(\hat{\mathbf{b}}\times\hat{\mathbf{c}})=\alpha\ [\hat{\mathbf{a}}\ \hat{\mathbf{b}}\ \hat{\mathbf{c}}]$

$$\therefore \quad \alpha = \frac{\hat{\mathbf{F}}\cdot(\hat{\mathbf{b}}\times\hat{\mathbf{c}})}{[\hat{\mathbf{a}}\,\hat{\mathbf{b}}\,\hat{\mathbf{c}}]}$$

Now, $\hat{\mathbf{b}}\times\hat{\mathbf{c}} = \hat{\mathbf{b}}\times(\hat{\mathbf{a}}\times\hat{\mathbf{b}}) = (\hat{\mathbf{b}}\cdot\hat{\mathbf{b}})\hat{\mathbf{a}} - (\hat{\mathbf{b}}\cdot\hat{\mathbf{a}})\hat{\mathbf{b}}$

$$= \hat{\mathbf{a}} - \frac{1}{3}\hat{\mathbf{b}}$$

$$[\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}}] = (\hat{\mathbf{a}}\times\hat{\mathbf{b}})^2 = |\hat{\mathbf{a}}|^2\,|\hat{\mathbf{b}}|^2 - (\hat{\mathbf{a}}\cdot\hat{\mathbf{b}})^2$$

$$= 1 - \frac{1}{9} = \frac{8}{9}$$

$$\therefore \quad \alpha = \frac{9}{8}\left\{\hat{\mathbf{F}}\cdot\hat{\mathbf{a}} - \frac{1}{3}\hat{\mathbf{F}}\cdot\hat{\mathbf{b}}\right\}$$

$$\therefore \quad k_1 = \frac{9}{8}, k_2 = \frac{3}{8}$$

$$\Rightarrow \quad k_1 + k_2 = \frac{12}{8} = \frac{3}{2}$$

$$\therefore \quad 2(k_1 + k_2) = 3$$

● ***Ex. 67*** *Let* $\mathbf{a} = \hat{\mathbf{i}} - \hat{\mathbf{j}}$, $\mathbf{b} = \hat{\mathbf{j}} - \hat{\mathbf{k}}$ *and* $\mathbf{c} = \hat{\mathbf{k}} - \hat{\mathbf{i}}$. *If* $\mathbf{d}$ *is a unit vector such that* $\mathbf{a}\cdot\mathbf{d} = 0 = (\mathbf{b}\ \mathbf{c}\ \mathbf{d})$, *then* $\mathbf{d}$ *is equal to*

(a) $\pm\dfrac{(\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}})}{\sqrt{6}}$ (b) $\pm\dfrac{(\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}})}{\sqrt{3}}$

(c) $\pm\dfrac{(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})}{\sqrt{3}}$ (d) $\pm\hat{\mathbf{k}}$

Sol. (a) We have, $\mathbf{a}\cdot\hat{\mathbf{d}} = 0$ and $[\mathbf{b}\ \mathbf{c}\ \hat{\mathbf{d}}] = 0$

$\Rightarrow \mathbf{a} \perp \hat{\mathbf{d}}$ and $\mathbf{b}$, $\mathbf{c}$, $\hat{\mathbf{d}}$ are coplanar.

$\Rightarrow \hat{\mathbf{d}} \perp \mathbf{a}$ and $\hat{\mathbf{d}}$ lies in the plane of $\mathbf{b}$ and $\mathbf{c}$, we know that the vector $\mathbf{r} = \mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is perpendicular to $\mathbf{a}$ and lies in the plane of $\mathbf{b}$ and $\mathbf{c}$

$$\therefore \quad \hat{\mathbf{d}} = \pm\frac{\mathbf{r}}{|\mathbf{r}|}$$

Now, $\mathbf{r} = \mathbf{a}\times(\mathbf{b}\times\mathbf{c})$

$\Rightarrow \mathbf{r} = (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

$\Rightarrow \mathbf{r} = -(\hat{\mathbf{j}} - \hat{\mathbf{k}}) + (\hat{\mathbf{k}} - \hat{\mathbf{i}})$

$= -\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

$$\therefore \quad \hat{\mathbf{d}} = \pm\frac{\mathbf{r}}{|\mathbf{r}|} = \pm\frac{(-\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}})}{\sqrt{1+1+4}}$$

$$= \pm\frac{\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}}{\sqrt{6}}$$

● ***Ex. 68*** *If* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{c}$ *are non-coplanar unit vectors such that* $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \dfrac{1}{\sqrt{2}}(\mathbf{b} + \mathbf{c})$, *then the angle between* $\mathbf{a}$ *and* $\mathbf{b}$ *is*

(a) $\dfrac{3\pi}{4}$ (b) $\dfrac{\pi}{4}$

(c) $\dfrac{\pi}{2}$ (d) π

Sol. (a) We have, $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \dfrac{\mathbf{b}\times\mathbf{c}}{\sqrt{2}}$

$$\Rightarrow \quad (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} = \frac{1}{\sqrt{2}}\mathbf{b} + \frac{1}{\sqrt{2}}\mathbf{c}$$

$$\Rightarrow \quad (\mathbf{a}\cdot\mathbf{c}) - \frac{1}{\sqrt{2}} = 0 \qquad [\because \mathbf{a}, \mathbf{b}, \mathbf{c} \text{ are non-coplanar}]$$

and $\mathbf{a}\cdot\mathbf{b} = -\dfrac{1}{\sqrt{2}}$

$$\Rightarrow \quad |\mathbf{a}|\,|\mathbf{b}|\cos\theta = -\frac{1}{\sqrt{2}} \Rightarrow \cos\theta = -\frac{1}{\sqrt{2}}$$

$$\therefore \quad \theta = \frac{3\pi}{4}$$

● ***Ex. 69*** *The unit vector which is orthogonal to the vector* $3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$ *and is coplanar with the vectors* $2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ *and* $\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ *is*

(a) $\dfrac{2\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{41}}$ (b) $\dfrac{2\hat{\mathbf{i}} - 3\hat{\mathbf{j}}}{\sqrt{13}}$

(c) $\dfrac{3\hat{\mathbf{j}} - \hat{\mathbf{k}}}{\sqrt{10}}$ (d) $\dfrac{4\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 3\hat{\mathbf{k}}}{\sqrt{34}}$

Sol. (c) Let $\mathbf{a} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$, $\mathbf{b} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{c} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$

Then, by definition, a vector orthogonal to $\mathbf{a}$ and coplanar to $\mathbf{b}$ and $\mathbf{c}$ is given by

$\Rightarrow \quad \mathbf{a}\times(\mathbf{b}\times\mathbf{c})$

$\Rightarrow \ \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

$= 7(2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) - (14)(\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}) = 21\hat{\mathbf{j}} - 7\hat{\mathbf{k}}$

Hence, a unit vector $= \dfrac{\mathbf{a}\times(\mathbf{b}\times\mathbf{c})}{|\mathbf{a}\times(\mathbf{b}\times\mathbf{c})|} = \dfrac{3\hat{\mathbf{j}} - \hat{\mathbf{k}}}{\sqrt{10}}$

● ***Ex. 70*** *Let* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{c}$ *be non-zero vectors such that* $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \dfrac{1}{3}|\mathbf{b}|\,|\mathbf{c}|\,\mathbf{a}$. *If* θ *is the acute angle between the vector* $\mathbf{b}$ *and* $\mathbf{c}$, *then* $\sin\theta$ *is equal to*

(a) $\dfrac{2\sqrt{2}}{3}$ (b) $\dfrac{\sqrt{2}}{3}$

(c) $\dfrac{2}{3}$ (d) $\dfrac{1}{3}$

Sol. (a) $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \dfrac{1}{3}|\mathbf{b}||\mathbf{c}|\mathbf{a}$

$$\Rightarrow \quad (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\,\mathbf{a} = \frac{1}{3}|\mathbf{b}||\mathbf{c}|\mathbf{a}$$

$$\Rightarrow \quad (\mathbf{a}\cdot\mathbf{c})\mathbf{b} = \{(\mathbf{b}\cdot\mathbf{c}) + \frac{1}{3}|\mathbf{b}||\mathbf{c}|\}\mathbf{a}$$

$$\Rightarrow \quad (\mathbf{a}\cdot\mathbf{c})\mathbf{b} = |\mathbf{b}||\mathbf{c}|\left\{\cos\theta + \frac{1}{3}\right\}\mathbf{a}$$

As $\mathbf{a}$ and $\mathbf{b}$ are not parallel, $\mathbf{a}\cdot\mathbf{c} = 0$ and $\cos\theta + \dfrac{1}{3} = 0$

$$\Rightarrow \quad \cos\theta = -\frac{1}{3} \Rightarrow \sin\theta = \frac{2\sqrt{2}}{3}$$

● ***Ex. 71*** *The value for* $[\mathbf{a}\times(\mathbf{b}+\mathbf{c}),\ \mathbf{b}\times(\mathbf{c}-2\mathbf{a}),\ \mathbf{c}\times(\mathbf{a}+3\mathbf{b})]$ *is equal to*

(a) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$

(b) $7[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$

(c) $-5[\mathbf{a}\times\mathbf{b}\ \mathbf{b}\times\mathbf{c}\ \mathbf{c}\times\mathbf{a}]$

(d) None of the above

Sol. (b) Let $\mathbf{a}\times\mathbf{b}=l$, $\mathbf{b}\times\mathbf{c}=m$ and $\mathbf{c}\times\mathbf{a}=n$,

$$\therefore\quad [\mathbf{a}\times(\mathbf{b}+\mathbf{c}),\ \mathbf{b}\times(\mathbf{c}-2\mathbf{a}),\ \mathbf{c}\times(\mathbf{a}+3\mathbf{b})]$$
$$=[l-n,\ m+2l,\ n-3m]$$
$$=\begin{vmatrix}1 & 0 & -1\\ 2 & 1 & 1\\ 0 & -3 & 1\end{vmatrix}[lmn]$$
$$=7[lmn]=7[\mathbf{a}\times\mathbf{b}\ \mathbf{b}\times\mathbf{c}\ \mathbf{c}\times\mathbf{a}]$$
$$=7[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$$

● ***Ex. 72*** *If* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *and* $\mathbf{p}, \mathbf{q}, \mathbf{r}$ *are reciprocal system of vectors, then* $\mathbf{a}\times\mathbf{p}+\mathbf{b}\times\mathbf{q}+c\times\mathbf{r}$ *is equal to*

(a) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ (b) $[\mathbf{p}+\mathbf{q}+\mathbf{r}]$

(c) 0 (d) $\mathbf{a}+\mathbf{b}+\mathbf{c}$

Sol. (c) $\mathbf{p}=\dfrac{\mathbf{b}\times\mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]},\ \mathbf{q}=\dfrac{\mathbf{c}\times\mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]},\ \mathbf{r}=\dfrac{\mathbf{a}\times\mathbf{b}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$

$$\mathbf{a}\times\mathbf{p}=\mathbf{a}\times\frac{(\mathbf{b}\times\mathbf{c})}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}=\frac{(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$$

Similarly, $\mathbf{b}\times\mathbf{q}=\dfrac{(\mathbf{a}\cdot\mathbf{b})\mathbf{c}-(\mathbf{b}\cdot\mathbf{a})\mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$

and $\mathbf{c}\times\mathbf{r}=\dfrac{(\mathbf{b}\cdot\mathbf{c})\mathbf{a}-(\mathbf{a}\cdot\mathbf{c})\mathbf{b}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$

$$\therefore\quad \mathbf{a}\times\mathbf{p}+\mathbf{b}\times\mathbf{q}+\mathbf{c}\times\mathbf{r}$$
$$=\frac{1}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}\{(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}+(\mathbf{a}\cdot\mathbf{b})\mathbf{c}-(\mathbf{b}\cdot\mathbf{c})\mathbf{a}$$
$$+(\mathbf{b}\cdot\mathbf{c})\mathbf{a}-(\mathbf{a}\cdot\mathbf{c})\mathbf{b}\}$$
$$=\frac{1}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}\times\mathbf{0}=\mathbf{0}$$

● ***Ex. 73*** *Solve* $\mathbf{a}\cdot\mathbf{r}=x$, $\mathbf{b}\cdot\mathbf{r}=y$, $\mathbf{c}\cdot\mathbf{r}=z$, *where* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *are given non-coplanar vectors.*

Sol. Given $\mathbf{a}\cdot\mathbf{r}=x$, $\mathbf{b}\cdot\mathbf{r}=y$, $\mathbf{c}\cdot\mathbf{r}=z$

Let $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$, be the reciprocal vectors of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, respectively.

Then, $\mathbf{a}=\dfrac{\mathbf{b}\times c}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]},\ \mathbf{b}'=\dfrac{\mathbf{c}\times\mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]},\ \mathbf{c}=\dfrac{\mathbf{a}\times\mathbf{b}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$

Now, $\mathbf{r}=(\mathbf{r}\cdot\mathbf{a})\mathbf{a}'+(\mathbf{r}\cdot\mathbf{b})\mathbf{b}'+(\mathbf{r}\cdot\mathbf{c})\mathbf{c}'$

$=x\mathbf{a}'+y\mathbf{b}'+z\mathbf{c}'$

JEE Type Solved Examples :
More than One Option Correct Type Questions

● ***Ex. 74*** *If* $\mathbf{z}_1=a\hat{\mathbf{i}}+b\hat{\mathbf{j}}$ *and* $\mathbf{z}_2=c\hat{\mathbf{i}}+d\hat{\mathbf{j}}$ *are two vectors in* $\hat{\mathbf{i}}$ *and* $\hat{\mathbf{j}}$ *system, where* $|\mathbf{z}_1|=|\mathbf{z}_2|=r$ *and* $\mathbf{z}_1\cdot\mathbf{z}_2=0$, *then* $\mathbf{w}_1=a\hat{\mathbf{i}}+c\hat{\mathbf{j}}$ *and* $\mathbf{w}_2=b\hat{\mathbf{i}}+d\hat{\mathbf{j}}$ *satisfy*

(a) $|\mathbf{w}_1|=r$

(b) $|\mathbf{w}_2|=r$

(c) $\mathbf{w}_1\cdot\mathbf{w}_2=0$

(d) None of the above

Sol. (a, b, c) $|z_1|=|z_2|=r$ and $z_1, z_2=0$

$$\Rightarrow\quad a^2+b^2=c^2+d^2=r \qquad \text{... (i)}$$

and $ac+bd=0$

as, $ac=-bd$

$$\Rightarrow\quad \frac{a}{b}=\frac{c}{-d}=\lambda \qquad \text{...(ii)}$$

From Eqs. (i) and (ii),

$$a^2(1+\lambda^2)=d^2(1+\lambda^2)$$
$$\Rightarrow\quad a^2=d^2 \text{ and } b^2=c^2$$

Now, $|w_1|=a^2+c^2=a^2+b^2=h=|w_2|$

$$w_1\cdot w_2=ab+bd=0$$

● ***Ex. 75*** *If unit vectors* $\hat{\mathbf{i}}$ *and* $\hat{\mathbf{j}}$ *are at right angles to each other and* $\mathbf{p}=3\hat{\mathbf{i}}+4\hat{\mathbf{j}}$, $\mathbf{q}=5\mathbf{i}$, $4\mathbf{r}=\mathbf{p}+\mathbf{q}$ *and* $2\mathbf{s}=\mathbf{p}-\mathbf{q}$, *then*

(a) $|\mathbf{r}+k\mathbf{s}|=|\mathbf{r}-k\mathbf{s}|$ for all real k

(b) $\mathbf{r}$ is perpendicular to $\mathbf{s}$

(c) $\mathbf{r}+\mathbf{s}$ is perpendicular to $\mathbf{r}-\mathbf{s}$

(d) $|\mathbf{r}|=|\mathbf{s}|=|\mathbf{p}|=|\mathbf{q}|$

Sol. (a, b, d) We have, $\mathbf{p}=3\hat{\mathbf{i}}+4\hat{\mathbf{j}}$ and $\mathbf{p}=5\hat{\mathbf{i}}$

Also, $4\mathbf{r}=\mathbf{p}+\mathbf{q}=3\hat{\mathbf{i}}+4\hat{\mathbf{j}}+5\hat{\mathbf{i}}$

$$=8\hat{\mathbf{i}}+4\hat{\mathbf{j}}\ \Rightarrow\ \mathbf{r}=2\hat{\mathbf{i}}+\hat{\mathbf{j}}$$

and $2\mathbf{s}=\mathbf{p}-\mathbf{q}=3\hat{\mathbf{i}}+4\hat{\mathbf{j}}+5\hat{\mathbf{i}}=-2\hat{\mathbf{i}}+4\hat{\mathbf{j}}$

$$\Rightarrow\quad \mathbf{s}=-\hat{\mathbf{i}}+2\hat{\mathbf{j}}$$

Now, $|\mathbf{r}+k\mathbf{s}|=|\mathbf{r}-k\mathbf{s}|$

$$\Rightarrow\quad |2\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}}\hat{\mathbf{i}}+2k\hat{\mathbf{j}}|^2=|2\hat{\mathbf{i}}+\hat{\mathbf{j}}+k\hat{\mathbf{i}}-2k\hat{\mathbf{j}}|^2$$
$$\Rightarrow\quad (2-k)^2+(1+2k)^2=(2+k)^2+(1-2k)^2$$

Which is true for all values of k.

Now, $\mathbf{r}\cdot\hat{\mathbf{s}}=(2\hat{\mathbf{i}}+\hat{\mathbf{j}})(-\hat{\mathbf{i}}+2\hat{\mathbf{j}})$

$$=-2+2=0$$

$$\therefore\quad \mathbf{r}\perp\mathbf{s}$$

Also, $(\mathbf{r}+\mathbf{s})\cdot(\mathbf{r}-\mathbf{s}) = (\hat{\mathbf{i}}+3\hat{\mathbf{j}})(3\hat{\mathbf{i}}-\hat{\mathbf{j}}) = 3-3=0$

$\therefore \quad (\mathbf{r}+\mathbf{s}) \perp (\mathbf{r}-\mathbf{s})$

Also, $|\mathbf{r}| = \sqrt{(2)^2+1} = \sqrt{5}$

$|\mathbf{s}| = \sqrt{(-1)^2+(2)^2} = \sqrt{5}$

$|\mathbf{p}| = \sqrt{3^2+4^2} = 5$

$|\mathbf{q}| = \sqrt{5^2} = 5$

$\therefore \quad |\mathbf{r}| = |\mathbf{s}|$ and $|\mathbf{p}| = |\mathbf{q}|$

● ***Ex. 76*** $\mathbf{a}, \mathbf{b}$ *and* $\mathbf{c}$ *are three vectors such that* $\mathbf{a}\cdot\mathbf{a} = \mathbf{b}\cdot\mathbf{b} = \mathbf{c}\cdot\mathbf{c} = 3$ *and* $|\mathbf{a}-\mathbf{b}|^2 + |\mathbf{b}-\mathbf{c}|^2 + |\mathbf{c}-\mathbf{a}|^2 = 27$, *then*

(a) $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are necessarily coplanar.

(b) $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ represent sides of a triangle in magnitude and direction

(c) $\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}$ has the least value $-9/2$

(d) $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ represent orthogonal triad of vectors

Sol. (a, b, c) Here,

$|\mathbf{a}-\mathbf{b}|^2 + |\mathbf{b}-\mathbf{c}|^2 + |\mathbf{c}-\mathbf{a}|^2$

$= 2(|\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 - \mathbf{a}\cdot\mathbf{b} - \mathbf{b}\cdot\mathbf{c} - \mathbf{c}\cdot\mathbf{c})$

$\Rightarrow \quad \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a} = -\frac{9}{2}$

Now, $|\mathbf{a}+\mathbf{b}+\mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a})$

$= 3+3+3-2\left(\frac{9}{2}\right) = 0$

$\therefore \quad \mathbf{a}+\mathbf{b}+\mathbf{c} = 0$ ...(i)

Also, $|\mathbf{a}+\mathbf{b}+\mathbf{c}|^2 \geq 0$

$\Rightarrow \quad \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a} \geq -\frac{9}{2}$ ...(ii)

Thus, least value is $-9/2$

● ***Ex. 77*** *If* $\mathbf{a}$ *and* $\mathbf{b}$ *are non-zero vectors such that* $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-2\mathbf{b}|$, *then*

(a) $2\,\mathbf{a}\cdot\mathbf{b} = |\mathbf{b}|^2$

(b) $\mathbf{a}\cdot\mathbf{b} = |\mathbf{b}|^2$

(c) least value of $\mathbf{a}\cdot\mathbf{b} + \frac{1}{|\mathbf{b}|^2+2}$ is $\sqrt{2}$

(d) least value of $\mathbf{a}\cdot\mathbf{b} + \frac{1}{|\mathbf{b}|+2}$ is $\sqrt{2}-1$

Sol. (a, d) $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-2\mathbf{b}|$

$\Rightarrow \quad \mathbf{a}\cdot\mathbf{b} = \frac{|\mathbf{b}|^2}{2}$

Also, $\mathbf{a}\cdot\mathbf{b} + \frac{1}{|\mathbf{b}|^2+2} = \frac{|\mathbf{b}|^2+2}{2} + \frac{1}{|\mathbf{b}|^2+2} - 1$

$\geq \sqrt{2}-1$ (using AM $\geq$ GM)

● ***Ex. 78*** *If vectors* $\mathbf{b} = (\tan\alpha, -1, 2\sqrt{\sin\alpha/2})$ *and* $\mathbf{c} = \left(\tan\alpha, \tan\alpha, -\frac{3}{\sqrt{\sin\alpha/2}}\right)$ *are orthogonal and vectors* $\mathbf{a} = (1, 3, \sin 2\alpha)$ *makes an obtuse angle with the Z-axis, then the value of* α *is*

(a) $\alpha = (4n+1)\pi + \tan^{-1}2$

(b) $\alpha = (4n+1)\pi - \tan^{-1}2$

(c) $\alpha = (4n+2)\pi + \tan^{-1}2$

(d) $\alpha = (4n+2)\pi - \tan^{-1}2$

Sol. (b, d), Since, $\mathbf{a} = (1, 3, \sin 2\alpha)$ makes an obtuse angle with the Z-axis, its z-component is negative.

Thus, $-1 \leq \sin 2\alpha < 0$ ...(i)

But $\mathbf{b}\cdot\mathbf{c} = 0$ ($\because$ orthogonal)

$\tan^2\alpha - \tan\alpha - 6 = 0$

$\therefore \quad (\tan\alpha - 3)(\tan\alpha + 2) = 0$

$\Rightarrow \quad \tan\alpha = 3, -2$

Now, $\tan\alpha = 3$.

Therefore, $\sin 2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha} = \frac{6}{1+9} = \frac{3}{5}$

(not possible as $\sin 2\alpha < 0$)

Now, if $\tan\alpha = -2$,

$\Rightarrow \quad \sin 2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha} = \frac{-4}{1+4} = \frac{-4}{5}$

$\Rightarrow \quad \tan 2\alpha > 0$

Hence, 2α is the third quadrant. Also, $\sqrt{\sin\alpha/2}$ is meaningful. If $0 < \sin\alpha/2 < 1$, the

$\alpha = (4n+1)\pi - \tan^{-1}2$

and $\alpha = (4n+2)\pi - \tan^{-1}2$

● ***Ex. 79*** *If* $\mathbf{a}$ *and* $\mathbf{b}$ *are any two unit vectors, then the possible integers in the range of* $\frac{3|\mathbf{a}+\mathbf{b}|}{2} + 2|\mathbf{a}-\mathbf{b}|$, *is/are*

(a) 2 (b) 3

(c) 4 (d) 5

Sol. (b, c, d) We have, $|\mathbf{a}| = |\mathbf{b}| = 1$

Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$.

$|\mathbf{a}+\mathbf{b}| = 2\cos\frac{\theta}{2}$

and $|\mathbf{a}-\mathbf{b}| = 2\sin\frac{\theta}{2}$

$\because \quad 3\cos\frac{\theta}{2} + 4\sin\frac{\theta}{2} : \theta \in [0, \pi]$

$-5 \leq 3\cos\frac{\theta}{2} + 4\sin\frac{\theta}{2} \leq 5$

The possible range are 3, 4 or 5.

● **Ex. 80** *Which of the following expressions are meaningful?*

(a) $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$ (b) $(\mathbf{u}\cdot\mathbf{v})\cdot\mathbf{w}$

(c) $(\mathbf{u}\cdot\mathbf{v})\,\mathbf{w}$ (d) $\mathbf{u}\times(\mathbf{v}\cdot\mathbf{w})$

Sol. (a, c) (i) Since, $\mathbf{v}\times\mathbf{w}$ is a vector, therefore, $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$ is a scalar quantity.

$\therefore$ (a) is meaningful.

(ii) $(\mathbf{u}\cdot\mathbf{v})$ is scalar.

$\therefore(\mathbf{u}\cdot\mathbf{v})\cdot\mathbf{w}$ is not meaningful.

(iii) $(\mathbf{u}\cdot\mathbf{v})$ is a scalar.

So, $(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$ is a scalar multiple of $\mathbf{w}$.

$\therefore(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$ is meaningful.

(iv) $(\mathbf{v}\cdot\mathbf{w})$ is a scalar.

So, $\mathbf{u}\times(\mathbf{v}\cdot\mathbf{w})$ is not meaningful as cross product is taken for two vector quantity and not for a vector and scalar.

● **Ex. 81** *If* $\mathbf{a}+2\mathbf{b}+3\mathbf{c}=0$, *then* $\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}=$

(a) $2(\mathbf{a}\times\mathbf{b})$ (b) $6(\mathbf{b}\times\mathbf{c})$

(c) $3(\mathbf{c}\times\mathbf{a})$ (d) 0

Sol. (a, b, c) $\mathbf{a}=-(2\mathbf{b}+3\mathbf{c})$

$$\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}$$
$$=-(2\mathbf{b}+3\mathbf{c})\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\{-(2\mathbf{b}+3\mathbf{c})\}$$
$$=-3\mathbf{c}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}-2\mathbf{c}\times\mathbf{b}=6(\mathbf{b}\times\mathbf{c})$$

Similarly, putting the values of $\mathbf{b}$ and $\mathbf{c}$ in terms of $\mathbf{a}$ and $\mathbf{a}$, $\mathbf{b}$ respectively in $\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}$, we get the desired results.

● **Ex. 82** *Let* $\alpha=a\hat{\mathbf{i}}+b\hat{\mathbf{j}}+c\hat{\mathbf{k}}$, $\beta=b\hat{\mathbf{i}}+c\hat{\mathbf{j}}+a\hat{\mathbf{k}}$ *and* $\gamma=c\hat{\mathbf{i}}+a\hat{\mathbf{j}}+b\hat{\mathbf{k}}$ *be three coplanar vectors with* $a\neq b$ *and* $\mathbf{v}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$. *Then* $\mathbf{v}$ *is perpendicular to*

(a) α (b) β

(c) γ (d) None of these

Sol. (a, b, c) It is given that α,β and γ are coplanar vectors.

Therefore, $[\alpha\ \beta\ \gamma]=0\Rightarrow\begin{vmatrix}a&b&c\\b&c&a\\c&a&b\end{vmatrix}=0$

or $3abc-a^3-b^3-c^3=0$

or $a^3+b^3+c^3-3abc=0$

or $(a+b+c)(a^2+b^2+c^2-ab-bc-ca)=0$

or $a+b+c=0\quad[\because a^2+b^2+c^2-ab-bc-ca\neq 0]$

$\Rightarrow\ \mathbf{v}\cdot\alpha=\mathbf{v}\cdot\beta=\mathbf{v}\cdot\gamma=0$

Hence, $\mathbf{v}$ is perpendicular to α,β and γ

● **Ex. 83** *If* $\mathbf{a}$ *is perpendicular to* $\mathbf{b}$ *and* p *is non-zero scalar such that* $p\mathbf{r}+(\mathbf{r}\cdot\mathbf{b})\mathbf{a}=\mathbf{c}$, *then* $\mathbf{r}$ *satisfy*

(a) $[\mathbf{r}\ \mathbf{a}\ \mathbf{c}]=0$ (b) $p^2\mathbf{r}=p\mathbf{a}-(\mathbf{c}\cdot\mathbf{a})\,\mathbf{b}$

(c) $p^2\mathbf{r}=p\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$ (d) $p^2\mathbf{r}=p\mathbf{c}-(\mathbf{b}\cdot\mathbf{c})\,\mathbf{a}$

Sol. (a, d) Given, $\mathbf{a}\cdot\mathbf{b}=0$

and $p\mathbf{r}+(\mathbf{r}\cdot\mathbf{b})\,\mathbf{a}=\mathbf{c}$ ... (i)

On taking dot product by $\mathbf{b}$, we get

$$p(\mathbf{r}\cdot\mathbf{b})+(\mathbf{r}\cdot\mathbf{b})\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{c}$$
$$\Rightarrow\quad p(\mathbf{r}\cdot\mathbf{b})=\mathbf{b}\cdot\mathbf{c}$$
$$\Rightarrow\quad p\left(\frac{\mathbf{c}-p\mathbf{r}}{\mathbf{a}}\right)=\mathbf{b}\cdot\mathbf{c}$$
$$\Rightarrow\quad p\mathbf{c}-p^2\mathbf{r}=(\mathbf{b}\cdot\mathbf{c})\mathbf{a}$$
$$\therefore\quad p^2r=p\mathbf{c}-(\mathbf{b}\cdot\mathbf{c})\mathbf{a}$$
$$\Rightarrow\quad \mathbf{r}=\frac{\mathbf{c}}{p}-\frac{(\mathbf{b}\cdot\mathbf{c})}{p^2}\mathbf{a}$$
$$\Rightarrow\quad [\mathbf{r}\ \mathbf{a}\ \mathbf{c}]=0$$

● **Ex. 84** *If* $\alpha(\mathbf{a}\times\mathbf{b})+\beta(\mathbf{b}\times\mathbf{c})+\gamma(\mathbf{c}\times\mathbf{a})=0$, *then*

(a) $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are coplanar if all of $\alpha,\beta,\gamma\neq 0$

(b) $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are coplanar if any one of $\alpha,\beta,\gamma\neq 0$

(c) $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are non-coplanar for any $\alpha,\beta,\gamma\neq 0$

(d) None of the above

Sol. (a, b) We have, $\alpha(\mathbf{a}\times\mathbf{b})+\beta(\mathbf{b}\times\mathbf{c})+\gamma(\mathbf{c}\times\mathbf{a})=0$

Taking dot product with $\mathbf{c}$, we have

$$\alpha[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=0$$

Similarly, taking dot product with b and c, we have

$$\gamma[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=0,\beta[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=0$$

Now, even if one of $\alpha,\beta,\gamma\neq 0$, then we have $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=0$

$\Rightarrow$ $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are coplanar.

● **Ex. 85** *If* $\mathbf{a}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ *and* $\mathbf{b}=\hat{\mathbf{i}}-\hat{\mathbf{j}}$, *then the vectors* $(\mathbf{a}\cdot\hat{\mathbf{i}})\hat{\mathbf{i}}+(\mathbf{a}\cdot\hat{\mathbf{j}}+(\mathbf{a}\cdot\hat{\mathbf{k}})\hat{\mathbf{k}},(\mathbf{b}\cdot\hat{\mathbf{i}})\hat{\mathbf{i}}+(\mathbf{b}\cdot\hat{\mathbf{j}})\hat{\mathbf{j}}+(\mathbf{b}\cdot\hat{\mathbf{k}})\hat{\mathbf{k}}$ *and* $\hat{\mathbf{i}}+\hat{\mathbf{j}}-2\hat{\mathbf{k}}$

(a) are mutually perpendicular

(b) are coplanar

(c) form a parallelopiped of volume 6 units

(d) form a parallelopiped of volume 3 units

Sol. (a, c) Given $\mathbf{a}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $\mathbf{b}=\hat{\mathbf{i}}-\hat{\mathbf{j}}$

$(\mathbf{a}\cdot\hat{\mathbf{i}})\hat{\mathbf{i}}+(\mathbf{a}\cdot\hat{\mathbf{j}})\hat{\mathbf{j}}+(\mathbf{a}\cdot\hat{\mathbf{k}})\hat{\mathbf{k}}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}=\mathbf{x}$ (say)

and $(b\cdot\hat{\mathbf{i}})\hat{\mathbf{i}}+(b\cdot\hat{\mathbf{j}})\hat{\mathbf{j}}+(b\cdot\hat{\mathbf{k}})\hat{\mathbf{k}}=\hat{\mathbf{i}}-\hat{\mathbf{j}}=\mathbf{y}$ (say)

and $\hat{\mathbf{i}}+\hat{\mathbf{j}}-2\hat{\mathbf{k}}=\hat{\mathbf{z}}$ (say)

Clearly, $\mathbf{x}\cdot\mathbf{y}=\mathbf{y}\mathbf{z}=\mathbf{z}\cdot\mathbf{x}=0$

$\therefore$ $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are mutually perpendicular

$$\text{Volume of parallelopiped}=\begin{vmatrix}1&1&1\\1&-1&0\\1&1&-2\end{vmatrix}$$
$$=1(2-0)-1(-2-0)+1(1+1)$$
$$=2+2+2=6$$

$\because$ $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{x}$ are not coplanar, i.e., $[\mathbf{x}\ \mathbf{y}\ \mathbf{z}]\neq 0$

$\therefore$Volume of parallelopiped formed by $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ is 6 cu units.

● ***Ex. 86*** *The volume of the parallelopiped whose coterminous edges are represented by the vectors* $2\mathbf{b}\times\mathbf{c}$, $3\mathbf{c}\times\mathbf{a}$ *and* $4\mathbf{a}\times\mathbf{b}$ *where* $\mathbf{a}=(1+\sin\theta)\hat{\mathbf{i}}+\cos\theta\hat{\mathbf{j}}+\sin 2\theta\hat{\mathbf{k}}$

$$\mathbf{b}=\sin\left(\theta+\frac{2\pi}{3}\right)\hat{\mathbf{i}}+\cos\left(\theta+\frac{2\pi}{3}\right)\hat{\mathbf{j}}+\sin\left(2\theta+\frac{4\pi}{3}\right)\hat{\mathbf{k}},$$

$$\mathbf{c}=\sin\left(\theta-\frac{2\pi}{3}\right)\hat{\mathbf{i}}+\cos\left(\theta-\frac{2\pi}{3}\right)\hat{\mathbf{j}}+\sin\left(2\theta-\frac{4\pi}{3}\right)\hat{\mathbf{k}}$$ *is* 18 *cubic units, then the value of* θ *in the interval* $\left(0,\frac{\pi}{2}\right)$, *is/are*

(a) $\frac{\pi}{9}$ (b) $\frac{2\pi}{9}$

(c) $\frac{\pi}{3}$ (d) $\frac{4\pi}{9}$

Sol. (a, b, d) Volume $=|[2\mathbf{b}\times\mathbf{c}\ \ 3\mathbf{c}\times\mathbf{a}\ \ 4\mathbf{a}\times\mathbf{b}]|=18$

$$\Rightarrow\quad 24[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2=18$$

$$\Rightarrow\quad |[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]|=\frac{\sqrt{3}}{2}$$

$$\text{Now, }[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=\begin{vmatrix}(1+\sin\theta) & \cos\theta & \sin 2\theta\\ \sin\left(\theta+\frac{2\pi}{3}\right) & \cos\left(\theta+\frac{2\pi}{3}\right) & \sin\left(\theta+\frac{4\pi}{3}\right)\\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta-\frac{4\pi}{3}\right)\end{vmatrix}$$

Applying $R_1\to R_1+R_2+R_3$ and expanding

$$|[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]|=\sqrt{3}\,|\cos 3\theta|=\frac{\sqrt{3}}{2}$$

$$\Rightarrow\quad \cos 3\theta=\pm\frac{1}{2}\Rightarrow 3\theta=\frac{\pi}{3},\frac{2\pi}{3},\frac{4\pi}{3}$$

$$\Rightarrow\quad \theta=\frac{\pi}{9},\frac{2\pi}{9},\frac{4\pi}{9}$$

● ***Ex. 87*** *If* $\mathbf{a}=x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}$, $\mathbf{b}=y\hat{\mathbf{i}}+z\hat{\mathbf{j}}+x\hat{\mathbf{k}}$ *and* $\mathbf{c}=z\hat{\mathbf{i}}+x\hat{\mathbf{j}}+y\hat{\mathbf{k}}$, *then* $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ *is/are*

(a) parallel to $(y-z)\hat{\mathbf{i}}+(z-x)\hat{\mathbf{j}}+(x-y)\hat{\mathbf{k}}$

(b) orthogonal to $\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$

(c) orthogonal to $(y+z)\hat{\mathbf{i}}+(z+x)\hat{\mathbf{j}}+(x+y)\hat{\mathbf{k}}$

(d) parallel to $\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$

Sol. (a, b, c) $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

$$=(xz+yx+yz)(y\hat{\mathbf{i}}+z\hat{\mathbf{j}}+x\hat{\mathbf{k}})-(xy+yz+zx)(z\hat{\mathbf{i}}+x\hat{\mathbf{j}}+y\hat{\mathbf{k}})$$

$$=(xy+yz+zx)[(y-z)\hat{\mathbf{i}}+(z-x)\hat{\mathbf{j}}+(x-y)\hat{\mathbf{k}}]$$

Clearly, 1 to $\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and also to $(y+z)\hat{\mathbf{i}}+(z+x)\hat{\mathbf{j}}+(x+y)\hat{\mathbf{k}}$, as dot product are zero, clearly parallel to

$$(y-z)\hat{\mathbf{i}}+(z-x)\hat{\mathbf{j}}+(x-y)\hat{\mathbf{k}}$$

● ***Ex. 88*** *If* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{c}$ *are three non-zero vectors, then which of the following statement(s) is/are true?*

(a) $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$, $\mathbf{b}\times(\mathbf{c}\times\mathbf{a})$, $\mathbf{c}\times(\mathbf{a}\times\mathbf{b})$ from a right handed system.

(b) $\mathbf{c}$, $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$, $\mathbf{a}\times\mathbf{b}$ from a right handed system

(c) $\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}<0$, if $\mathbf{a}+\mathbf{b}+\mathbf{c}=0$

(d) $\frac{(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{b}\times\mathbf{c})}{(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\times\mathbf{c})}=-1$, if $\mathbf{a}+\mathbf{b}+\mathbf{c}=0$

Sol. (b, c, d)

(a) $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})+\mathbf{b}\times(\mathbf{c}\times\mathbf{a})+\mathbf{c}\times(\mathbf{a}\times\mathbf{b})=0$

$\Rightarrow$ vector are coplanar, so do not form RHS.

(b) $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$, $\mathbf{a}\times\mathbf{b}$, $\mathbf{c}$ in that order form RHS.

$\Rightarrow$ $\mathbf{c}$, $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$, $\mathbf{a}\times\mathbf{b}$ also form RHS as they are in same cyclic order.

(c) $\mathbf{a}+\mathbf{b}+\mathbf{c}=0\Rightarrow|\mathbf{a}+\mathbf{b}+\mathbf{c}|^2=0$

$$\Rightarrow\quad a^2+b^2+c^2=-2(\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a})$$

$$\Rightarrow\quad \mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}\cdot<0$$

(d) $a+b+c=0$

$$\Rightarrow\quad \mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}=\mathbf{c}\times\mathbf{a}$$

● ***Ex. 89*** *Let the unit vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *be perpendicular and unit vector* $\mathbf{c}$ *is inclined at angle* α *to* $\mathbf{a}$ *and* $\mathbf{b}$. *If* $\mathbf{c}=l\mathbf{a}+m\mathbf{b}+n(\mathbf{a}\times\mathbf{b})$, *then*

(a) $l=m$ (b) $n^2=1-2l^2$

(c) $n^2=-\cos 2\alpha$ (d) $m^2=\frac{1+\cos 2\alpha}{2}$

Sol. (a, b, c, d)

$$\mathbf{a}\cdot\mathbf{b}=0,\ \mathbf{c}\cdot\mathbf{a}=\mathbf{c}\cdot\mathbf{b}=\cos\alpha$$

Take dot products with $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively.

$$\Rightarrow\quad l=m,\ l^2+m^2+n^2=1$$

$$n^2=-\cos 2\alpha,$$

$$m^2=\frac{1+\cos 2\alpha}{2}$$

● ***Ex. 90*** *If* $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ *is perpendicular to* $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$, *we may have*

(a) $(\mathbf{a}\cdot\mathbf{c})\,|\mathbf{b}|^2=(\mathbf{a}\cdot\mathbf{b})(\mathbf{b}\cdot\mathbf{c})$ (b) $\mathbf{a}\cdot\mathbf{b}=0$

(c) $\mathbf{a}\cdot\mathbf{c}=0$ (d) $\mathbf{b}\cdot\mathbf{c}=0$

Sol. (a, c)

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

and $$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=-(\mathbf{c}\cdot\mathbf{b})\mathbf{a}+(\mathbf{a}\cdot\mathbf{c})\mathbf{b}$$

We have been given

$$(\mathbf{a}\times(\mathbf{b}\times\mathbf{c}))\cdot((\mathbf{a}\times\mathbf{b})\times\mathbf{c})=0$$

$\therefore$ $((\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c})\cdot((\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{c}\cdot\mathbf{b})\mathbf{a})=0$

or $(\mathbf{a}\cdot\mathbf{c})^2|\mathbf{b}|^2-(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{b})$

$$-(\mathbf{a}\cdot\mathbf{b})(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{c})+(\mathbf{a}\cdot\mathbf{b})(\mathbf{b}\cdot\mathbf{c})(\mathbf{c}\cdot\mathbf{a})=0$$

or $$(\mathbf{a}\cdot\mathbf{c})^2|\mathbf{b}|^2=(\mathbf{a}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{b})(\mathbf{b}\cdot\mathbf{c})$$

or $(\mathbf{a}\cdot\mathbf{c})((\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{b})-(\mathbf{a}\cdot\mathbf{b})(\mathbf{b}\cdot\mathbf{c}))=0$

$\mathbf{a}\cdot\mathbf{c}=0$

or $(\mathbf{a}\cdot\mathbf{c})|\mathbf{b}|^2=(\mathbf{a}\cdot\mathbf{b})(\mathbf{b}\cdot\mathbf{c})$

● ***Ex. 91*** *If* $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})\cdot(\mathbf{a}\times\mathbf{d})=0$*, then which of the following may be true?*

(a) $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ are necessarily coplanar

(b) $\mathbf{a}$ lies in the plane of $\mathbf{c}$ and $\mathbf{d}$

(c) $\mathbf{b}$ lies in the plane of $\mathbf{a}$ and $\mathbf{d}$

(d) $\mathbf{c}$ lies in the plane of $\mathbf{a}$ and $\mathbf{d}$

Sol. (b, c, d) $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})\cdot(\mathbf{a}\times\mathbf{d})=0$

or $[\mathbf{a}\ \mathbf{c}\ \mathbf{d}]\,\mathbf{b}-[\mathbf{b}\ \mathbf{c}\ \mathbf{d}]\mathbf{a})\cdot(\mathbf{a}\times\mathbf{d})=0$

or $[\mathbf{a}\ \mathbf{c}\ \mathbf{d}]\,[\mathbf{d}\ \mathbf{a}\ \mathbf{d}]=0$

Hence, either $\mathbf{c}$ or $\mathbf{b}$ must lie in the plane of $\mathbf{a}$ and $\mathbf{d}$.

● ***Ex. 92*** *The angles of a triangle, two of whose sides are represented by vectors* $\sqrt{3}(\hat{\mathbf{a}}\times\hat{\mathbf{b}})$ *and* $\hat{\mathbf{b}}-(\mathbf{a}\cdot\hat{\mathbf{b}})\hat{\mathbf{a}}$*, where* $\hat{\mathbf{b}}$ *is a non-zero vector and* $\hat{\mathbf{a}}$ *is a unit vector in the direction of* $\hat{\mathbf{a}}$*, are*

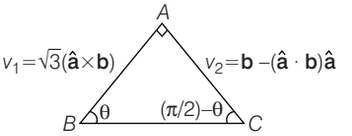
(a) $\tan^{-1}(\sqrt{3})$ (b) $\tan^{-1}(1/\sqrt{3})$

(c) $\cot^{-1}(0)$ (d) $\tan^{-1}(1)$

Sol. (a, b, c) Consider $\mathbf{V}_1\cdot\mathbf{V}_2=0$

$\Rightarrow$ $A=90^\circ$

Using the sine law, $\left|\dfrac{\mathbf{b}-(\hat{\mathbf{a}}\cdot\mathbf{b})\hat{\mathbf{a}}}{\sin\theta}\right|=\dfrac{\sqrt{3}\,|\,\hat{\mathbf{a}}\times\hat{\mathbf{b}}\,|}{\cos\theta}$



or $\tan\theta=\dfrac{1}{\sqrt{3}}\dfrac{|\,\mathbf{b}-(\hat{\mathbf{a}}\cdot\hat{\mathbf{b}})\hat{\mathbf{a}}\,|}{|\,\hat{\mathbf{a}}\times\hat{\mathbf{b}}\,|}$

$=\dfrac{1}{\sqrt{3}}\dfrac{|\,(\hat{\mathbf{a}}\times\hat{\mathbf{b}})\times\hat{\mathbf{a}}\,|}{|\,\hat{\mathbf{a}}\times\hat{\mathbf{b}}\,|}$

$=\dfrac{1}{\sqrt{3}}\dfrac{|\,\hat{\mathbf{a}}\times\hat{\mathbf{b}}\,||\,\hat{\mathbf{a}}\,|\sin 90^\circ}{|\,\hat{\mathbf{a}}\times\hat{\mathbf{b}}\,|}=\dfrac{1}{\sqrt{3}}$

or $\theta=\dfrac{\pi}{6}$

JEE Type Solved Examples : Statement Type I & II Questions

■ **Directions** (Q. Nos. 93-96) This section is based on Statement I and Statement II. Select the correct answer from the codes given below.

(a) Both Statement I and Statement II are correct and Statement II is the correct explanation of Statement I

(b) Both Statement I and Statement II are correct but Statement II is not the correct explanation of Statement I

(c) Statement I is correct but Statement II is incorrect

(d) Statement II is correct but Statement I is incorrect

● ***Ex. 93*** *Let the vectors* $\mathbf{PQ}, \mathbf{OR}, \mathbf{RS}, \mathbf{ST}, \mathbf{TU}$ *and* $\mathbf{UP}$ *represent the sides of a regular hexagon.*

Statement I $\mathbf{PQ}\times(\mathbf{RS}+\mathbf{ST})\neq 0$

Statement II $\mathbf{PQ}\times\mathbf{RS}=0$ *and* $\mathbf{PQ}\times\mathbf{ST}\neq 0$

Sol. (c) Clearly, $\mathbf{RS}+\mathbf{ST}=\mathbf{RT}$, which is not parallel to $\mathbf{PQ}$.

$\therefore$ $\mathbf{PQ}\times(\mathbf{RS}+\mathbf{ST})\neq 0$

So, Statement I is correct.

Also, $\mathbf{PQ}$ is not parallel to $\mathbf{RS}$.

$\therefore$ $\mathbf{PQ}\times\mathbf{RS}\neq 0$

So, Statement II is not correct.

● ***Ex. 94*** $\mathbf{p}, \mathbf{q}$ *and* $\mathbf{r}$ *are three vectors defined by* $\mathbf{p}=\mathbf{a}\times(\mathbf{b}+\mathbf{c}), \mathbf{q}=\mathbf{b}\times(\mathbf{c}+\mathbf{a})$ *and* $\mathbf{r}=\mathbf{c}\times(\mathbf{a}+\mathbf{b})$

Statement I $\mathbf{p}, \mathbf{q}$ *and* $\mathbf{r}$ *are coplanar.*

Statement II *Vectors* $\mathbf{p}, \mathbf{q}, \mathbf{r}$ *are linearly independent.*

Sol. (c) **Statement I** $\mathbf{p}+\mathbf{q}+\mathbf{r}=\mathbf{a}\times(\mathbf{b}+\mathbf{c})+\mathbf{b}\times(\mathbf{c}+\mathbf{a})+\mathbf{c}\times(\mathbf{a}+\mathbf{b})$

$=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}+\mathbf{b}\times\mathbf{a}+\mathbf{c}\times\mathbf{a}+\mathbf{c}\times\mathbf{b}$

$=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}-\mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{c}-\mathbf{b}\times\mathbf{c}$

$\therefore \mathbf{p}=-\mathbf{q}-\mathbf{r}$ (a linear combination of $\mathbf{q}$ and $\mathbf{r}$)

Therefore, $\mathbf{p}, \mathbf{q}, \mathbf{r}$ are coplanar and hence statement I is true.

Statement II $\mathbf{p}+\mathbf{q}+\mathbf{r}=0\Rightarrow\mathbf{p}, \mathbf{q}, \mathbf{r}$ are not linearly independent.

Therefore, statement II is not true.

● ***Ex. 95*** **Statement I** *If in a* $\Delta ABC, \mathbf{BC}=\dfrac{\mathbf{p}}{|\mathbf{p}|}-\dfrac{\mathbf{q}}{|\mathbf{q}|}$ *and* $\mathbf{AC}=\dfrac{2\mathbf{p}}{|\mathbf{p}|}$*;* $|\mathbf{p}|\neq|\mathbf{q}|$*, then the value of* $\cos 2A+\cos 2B+\cos 2C$ *is* -1.

Statement II *If in* $\Delta ABC, \angle C=90^\circ$*, then* $\cos 2A+\cos 2B+\cos 2C=-1$

Sol. (b) **Statement II** In $\Delta ABC, \angle C = 90°$

$\therefore \quad \cos 2A + \cos 2B + \cos 2C$

$= 2\cos(A+B)\cos(A-B) + \cos 180°$

$= \cos(180° - C)\cos(A-B) - 1$

$= -2\cos C \cos(A-B) - 1$

$= 0 - 1 = -1 \qquad [\because \cos C = \cos 90° = 0]$

Therefore, Statement II is true.

Statement I $\mathbf{BC} = \hat{p} - \hat{q}, \mathbf{AC} = 2\hat{p}$

$\therefore \qquad \mathbf{AB} = \mathbf{AC} + \mathbf{CB} = 2\hat{p} - (\hat{p} - \hat{q}) = \hat{p} + \hat{q}$

Now, $\mathbf{AB} \cdot \mathbf{BC} = (\hat{p} + \hat{q}).(\hat{p} - \hat{q}) = |\hat{p}|^2 - |\hat{q}|^2 = 1 - 1 = 0$

$\therefore \qquad \angle B = 90°$

Now, $\cos 2A + \cos 2B + \cos 2C$

$= \cos 2A + \cos 2C + \cos 2B$

$= 2\cos(A+C)\cos(A-C) + \cos 180°$

$= 2\cos(180° - B)\cos(A-C) - 1$

$= -2\cos B \cos(A-C) - 1 = -1 \qquad (\because B = 90°)$

Therefore, Statement I is also true.

Thus both Statements are true but Statement II is not the correct explanation of Statement I.

● ***Ex. 96*** **Statement I** *If* $\mathbf{a}$ *is perpendicular to* $\mathbf{b}$ *and* $\mathbf{c}$*, then* $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = 0$

Statement II *If* $\mathbf{b}$ *is perpendicular to* $\mathbf{c}$*, then* $\mathbf{b} \times \mathbf{c} = 0$

Sol. (c) If $\mathbf{a}$ is perpendicular to $\mathbf{b}$ and $\mathbf{c}$, then $\mathbf{a} \parallel (\mathbf{b} \times \mathbf{c})$

$\therefore \qquad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = 0$

Therefore, Statement I is true.

But, if $\mathbf{b} \perp \mathbf{c}$, then $\mathbf{b} \times \mathbf{c} \neq 0$

Therefore, Statement II is not true.

JEE Type Solved Examples : Passage Based Type Questions

Passage I

(Ex. Nos. 97-99)

Let $\mathbf{a} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 6\hat{\mathbf{k}}$, $\mathbf{b} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$ and $\mathbf{c} = -2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$. Let $\mathbf{a}_1$ be the projection of $\mathbf{a}$ on $\mathbf{b}$ and $\mathbf{a}_2$ be the projection of $\mathbf{a}_1$ and $\mathbf{c}$. Then

● ***Ex. 97*** $\mathbf{a}_2$ *is equal to*

(a) $\frac{943}{49}(2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 6\hat{\mathbf{k}})$ (b) $\frac{943}{49^2}(2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 6\hat{\mathbf{k}})$

(c) $\frac{943}{49}(-2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}})$ (d) $\frac{943}{49^2}(-2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}})$

Sol. (b)

$$\mathbf{a}_1 = \left[(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 6\hat{\mathbf{k}}).\frac{(2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}})}{7}\right]\frac{2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}}{7}$$

$$= \frac{-41}{49}(2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}})$$

$$\mathbf{a}_2 = \frac{-41}{49}\left((2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}).\frac{(-2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}})}{7}\right) \times \frac{(-2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}})}{7}$$

$$= \frac{-41}{(49)^2}(-4 - 9 + 36)(-2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}})$$

$$= \frac{943}{49^2}(2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 6\hat{\mathbf{k}})$$

● ***Ex. 98*** $\mathbf{a}_1 . \mathbf{b}$ *is equal to*

(a) -41 (b) $-\frac{41}{7}$

(c) 41 (d) 287

Sol. (a) $\mathbf{a}_1 \cdot \mathbf{b} = -\frac{41}{49}(2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}) \cdot (2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}) = -41$

● ***Ex. 99.*** *Which of the following is true?*

(a) $\mathbf{a}$ and $\mathbf{a}_2$ are collinear

(b) $\mathbf{a}_1$ and $\mathbf{c}$ are collinear

(c) $\mathbf{a}$, $\mathbf{a}_1$ and $\mathbf{b}$ are coplanar

(d) $\mathbf{a}$, $\mathbf{a}_1$ and $\mathbf{a}_2$ are coplanar

Sol. (c) $\mathbf{a}$, $\mathbf{a}_1$ and $\mathbf{b}$ are coplanar because $\mathbf{a}_1$ and $\mathbf{b}$ are collinear.

Passage II

(Ex. Nos. 100-102)

Let $\mathbf{a}$, $\mathbf{b}$ be two vectors perpendicular to each other and $|\mathbf{a}| = 2$, $|\mathbf{b}| = 3$ and $\mathbf{c} \times \mathbf{a} = \mathbf{b}$.

● ***Ex. 100*** *The least value of* $|\mathbf{c} - \mathbf{a}|$ *is*

(a) 1 (b) $\frac{1}{2}$

(c) $\frac{1}{4}$ (d) $\frac{3}{2}$

● ***Ex. 101*** *When* $|\mathbf{c} - \mathbf{a}|$ *is least the value of* α *(when* α *is angle between* $\mathbf{a}$ *and* $\mathbf{c}$*) equals*

(a) $\tan^{-1}\left(\frac{1}{2}\right)$ (b) $\tan^{-1}\left(\frac{3}{4}\right)$

(c) $\cos^{-1}\left(\frac{2}{3}\right)$ (d) None of these

● ***Ex. 102*** *When* $|\mathbf{c}-\mathbf{a}|$ *attains least value, then the value of* $|\mathbf{c}|$ *is*

(a) $\frac{1}{2}$ (b) $\frac{7}{2}$

(c) $\frac{5}{2}$ (d) 4

Sol. For (Ex. Nos. 100-102)

Here, $|\mathbf{a}|=2, |\mathbf{b}|=3$

$$\mathbf{c}\times\mathbf{a}=\mathbf{b} \Rightarrow |\mathbf{c}||\mathbf{a}|\sin\alpha=|\mathbf{b}|$$

$$\Rightarrow \quad |\mathbf{c}|=\frac{3}{2}\operatorname{cosec}\alpha$$

Consider, $|\mathbf{c}-\mathbf{a}|^2=|\mathbf{c}|^2-2(\mathbf{a}\,.\,\mathbf{c})+|\mathbf{a}|^2$

$$=\frac{9}{4}\operatorname{cosec}^2\alpha+4-2(2)\,.\left(\frac{3}{2}\right).\operatorname{cosec}\alpha\,.\cos\alpha$$

$$=\frac{25}{4}+\frac{9}{4}\,.\cot^2\alpha-6\cot\alpha$$

$$=\frac{25}{4}+\left(\frac{3}{2}\cot\alpha-2\right)^2-4$$

$$=\frac{9}{4}+\left(\frac{3}{2}\cot\alpha-2\right)^2\geq\frac{9}{4}$$

$|\mathbf{c}-\mathbf{a}|\geq\frac{3}{2}$ and least possible when

$\frac{3}{2}\cot\alpha=2 \Rightarrow \tan\alpha=\frac{3}{4} \Rightarrow \alpha=\tan^{-1}\frac{3}{4}$

Also, $|\mathbf{c}|=\frac{3}{2\sin\alpha}=\frac{3}{2\left(\frac{3}{5}\right)}=\frac{5}{2}$

100. (d) ***101.*** (b) ***102.*** (c)

Passage III

(Ex. Nos. 103-105)

Consider a triangular pyramid *ABCD* the position vectors of whose angular points are $A(3,0,1)$, $B(-1,4,1)$, $C(5,2,3)$ and $D(0,-5,4)$. Let *G* be the point of intersection of the medians of triangle *BCD*.

● ***Ex. 103*** *The length of vector* **AG** *is*

(a) $\sqrt{17}$ (b) $\frac{\sqrt{51}}{3}$

(c) $\frac{3}{\sqrt{6}}$ (d) $\frac{\sqrt{59}}{4}$

● ***Ex. 104*** *Area of triangle ABC in sq units is*

(a) 24 (b) $8\sqrt{6}$

(c) $4\sqrt{6}$ (d) None of these

● ***Ex. 105*** *The length of the perpendicular from vertex D on the opposite face is*

(a) $\frac{14}{\sqrt{6}}$ (b) $\frac{2}{\sqrt{6}}$

(c) $\frac{3}{\sqrt{6}}$ (d) None of these

Sol. For (Ex. Nos. 103-105)

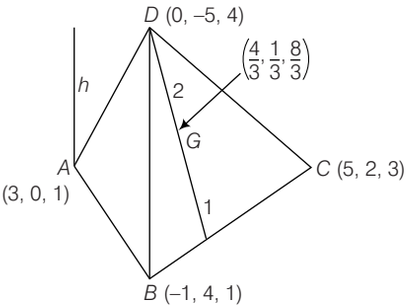
Point *G* is $\left(\frac{4}{3},\frac{1}{3},\frac{8}{3}\right)$. Therefore,

$$|\mathbf{AG}|^2=\left(\frac{5}{3}\right)^2+\frac{1}{9}+\left(\frac{5}{3}\right)^2=\frac{51}{9}$$

or $$|\mathbf{AG}|=\frac{\sqrt{51}}{3}$$

$$\mathbf{AB}=-4\hat{\mathbf{i}}+4\hat{\mathbf{j}}+0\hat{\mathbf{k}}$$

$$\mathbf{AC}=2\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\hat{\mathbf{k}}$$



$$\therefore \quad \mathbf{AB}\times\mathbf{AC}=-8\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & -1 & 0\\ 1 & 1 & 1\end{vmatrix}$$

$$=8(\hat{\mathbf{i}}+\hat{\mathbf{j}}-2\hat{\mathbf{k}})$$

Area of $\Delta ABC=\frac{1}{2}|\mathbf{AB}\times\mathbf{AC}|=4\sqrt{6}$

$$\mathbf{AD}=-3\hat{\mathbf{i}}-5\hat{\mathbf{j}}+3\hat{\mathbf{k}}$$

The length of the perpendicular from the vertex *D* on the opposite face

$$=|\text{ Projection of }\mathbf{AD}\text{ on }\mathbf{AB}\times\mathbf{AC}|$$

$$=\left|\frac{(-3\hat{\mathbf{i}}-5\hat{\mathbf{j}}+3\hat{\mathbf{k}})\,(\hat{\mathbf{i}}+\hat{\mathbf{j}}-2\hat{\mathbf{k}})}{\sqrt{6}}\right|$$

$$=\left|\frac{-3-5-6}{\sqrt{6}}\right|$$

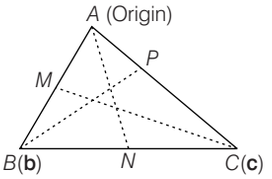
$$=\frac{14}{\sqrt{6}}$$

103. (b) ***104.*** (c) ***105.*** (a)

Passage IV

(Ex. Nos. 106-108)

Let A, B, C represent the vertices of a triangle, where A is the origin and B and C have position vectors $\mathbf{b}$ and $\mathbf{c}$ respectively. Points **M**, **N** and **P** are taken on sides **AB**, **BC** and **CA** respectively, such that $\frac{\mathbf{AM}}{\mathbf{AB}} = \frac{\mathbf{BN}}{\mathbf{BC}} = \frac{\mathbf{CP}}{\mathbf{CA}} = \alpha$



[Now answer the following questions]

● ***Ex. 106.*** $\mathbf{AN} + \mathbf{BP} + \mathbf{CM}$ *is*

(a) $3\alpha\,(\mathbf{b} + \mathbf{c})$ (b) $\alpha(\mathbf{b} + \mathbf{c})$

(c) $(1 - \alpha)\,(\mathbf{b} + \mathbf{c})$ (d) 0

Sol. (d) Since $\frac{\mathbf{AM}}{\mathbf{AB}} = \alpha$,

$\therefore$ $\mathbf{P.V.}$ of $\mathbf{M} = \alpha\ \mathbf{b}$

Since, $\frac{\mathbf{BN}}{\mathbf{BC}} = \alpha$

$\therefore$ $\frac{\mathbf{BN}}{\mathbf{NC}} = \frac{\alpha}{1-\alpha}$

$\therefore$ Position vector of $\mathbf{N} = (1 - \alpha)\mathbf{b} + \alpha\mathbf{c}$

Since, $\frac{\mathbf{CP}}{\mathbf{CA}} = \alpha$

$\therefore$ $\frac{\mathbf{AP}}{\mathbf{PC}} = \frac{1-\alpha}{\alpha}$

$\therefore$ P.V. of $\mathbf{p} = (1 - \alpha)\mathbf{c}$

Now, $\mathbf{AN} = (1 - \alpha)\mathbf{b} + \alpha\mathbf{c}$

$\mathbf{BP} = (1 - \alpha)\mathbf{c} - \mathbf{b}$

$\mathbf{CM} = \alpha\mathbf{b} - \mathbf{c}$

$\therefore$ AN + BP + CM = 0

● ***Ex. 107.*** *The vectors* $\mathbf{AN}, \mathbf{BP}$ *and* $\mathbf{CM}$ *are*

(a) concurrent (b) sides of a triangle

(c) non-coplanar (d) None of these

Sol. (b) Since $\mathbf{AN} + \mathbf{BP} + \mathbf{CM} = 0$

Hence, **AN**, **BP** and **CM** from the sides of a triangle.

● ***Ex. 108.*** *If* Δ *represents the area enclosed by the three vectors* $\mathbf{AN}, \mathbf{BP}$ *and* $\mathbf{CM}$*, then the value of* α *for which* Δ *is least*

(a) does not exist (b) $\frac{1}{2}$

(c) $\frac{1}{4}$ (d) None of these

Sol. (b) $\Delta = \frac{1}{2}|\mathbf{AN} \times \mathbf{BP}| = \frac{1}{2}|\{(1-\alpha)\mathbf{b} + \alpha\mathbf{c}\} \times \{(1-\alpha)\mathbf{c} - \mathbf{b}\}|$

$= \frac{1}{2}|(1-\alpha)^2(\mathbf{b} \times \mathbf{c}) + \alpha(\mathbf{b} \times \mathbf{c})|$

$= \frac{1}{2}|\mathbf{b} \times \mathbf{c}|(\alpha^2 - \alpha + 1)$

$= \frac{1}{2}|\mathbf{b} \times \mathbf{c}|\left\{\left(\alpha - \frac{1}{2}\right)^2 + \frac{3}{4}\right\}$

$\therefore \Delta$ is least, if $\alpha = \frac{1}{2}$

Passage V

(Ex. Nos. 109-110)

If AP, BQ and CR are the altitudes of acute ΔABC and $9AP + 4BQ + 7CR = 0$.

● ***Ex. 109.*** $\angle ACB$ *is equal to*

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$

(c) $\cos^{-1}\left(\frac{1}{3\sqrt{7}}\right)$ (d) $\cos^{-1}\left(\frac{1}{\sqrt{7}}\right)$

● ***Ex. 110.*** $\angle ABC$ *is equal to*

(a) $\cos^{-1}\left(\frac{2}{\sqrt{7}}\right)$ (b) $\frac{\pi}{2}$

(c) $\cos^{-1}\left(\frac{\sqrt{7}}{3}\right)$ (d) $\frac{\pi}{3}$

Sol. For (Ex. Nos. 109-110)

Since, sum of three vectors $9\mathbf{AP}$, $4\mathbf{BQ}$ and $7\mathbf{CR}$ is zero, there is a Δ whose sides have lengths $9|\mathbf{AP}|$, $4|\mathbf{BQ}|$, $7|\mathbf{CR}|$ and are parallel to the corresponding vectors.

H is orthocentre.

$\Rightarrow$ $\angle BHP = 90° - \angle QBC = \angle ACB$

$\Rightarrow$ Angle between **AP** and **BQ** equal to $\angle ACB$, similarly angle between **BQ** and **CR** be $\angle BAC$.

$\Rightarrow$ $\frac{AB}{7|\mathbf{CR}|} = \frac{BC}{9|\mathbf{AP}|} = \frac{AC}{4|\mathbf{BQ}|}$

2. ar. $(\Delta ABC) = |\mathbf{AB} \times \mathbf{CR}| = |\mathbf{BC} \times \mathbf{AP}| = |\mathbf{CA} \times \mathbf{BQ}|$

$\Rightarrow$ $\frac{c^2}{7} = \frac{a^2}{9} = \frac{b^2}{4}$

$\therefore$ $a : b : c = 3 : 2 : \sqrt{7}$

$\Rightarrow$ $\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{1}{2}$

$\Rightarrow$ $\angle C = 60° = \frac{\pi}{3}$

and $\cos B = \frac{a^2 + b^2 - c^2}{2ac} = \frac{2}{\sqrt{7}}$

109. (b) ***110.*** (a)

Passage VI

(Ex. Nos. 111 to 113)

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are non-zero unit vectors inclined pairwise with the same angle θ. p, q, r are non-zero scalars satisfying $\mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{c} = p\mathbf{a} + q\mathbf{b} + r\mathbf{c}$

● ***Ex. 111*** *Volume of parallelopiped with edges* $\mathbf{a}, \mathbf{b}$ *and* $\mathbf{c}$ *is*

(a) $p + (q + r)\cos\theta$ (b) $(p + q + r)\cos\theta$

(c) $2p - (q + r)\cos\theta$ (d) None of these

● ***Ex. 112*** *The value of* $\left(\frac{q}{p} + 2\cos\theta\right)$ *is*

(a) 1 (b) 0

(c) $2[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ (d) None of these

● ***Ex. 113*** *The value of* $|(p+q)\cos\theta + r|$ *is*

(a) $(1+\cos\theta)\sqrt{1-2\cos\theta}$

(b) $2\sin^2\frac{\theta}{2}|\sqrt{1+2\cos\theta}|$

(c) $(1-\sin\theta)\sqrt{1+2\cos\theta}$

(d) None of the above

Sol. For (Ex. Nos. 111-113)

Volume of parallelopiped $= [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$

Now, we have

$$\mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{c} = p\mathbf{a} + q\mathbf{b} + r\mathbf{c}$$

$\Rightarrow \mathbf{a}\cdot(\mathbf{a}\times\mathbf{b}) + \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = p(\mathbf{a}\cdot\mathbf{a}) + q(\mathbf{a}\cdot\mathbf{b}) + r(\mathbf{a}\cdot\mathbf{c})$

$\Rightarrow [\mathbf{a}\ \mathbf{a}\ \mathbf{b}] + [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = p|\mathbf{a}|^2 + q|\mathbf{a}||\mathbf{b}|\cos\theta + r|\mathbf{a}||\mathbf{c}|\cos\theta$

$\Rightarrow [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = p + q\cos\theta + r\cos\theta = p + (q + r)\cos\theta$

$\therefore$ Volume of parallelopiped $= p + (q + r)\cos\theta$

Taking dot products with $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ respectively with given equation

$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = p + (q + r)\cos\theta \quad ...(i)$$

$$0 = (p + r)\cos\theta + q \quad ...(ii)$$

$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = (p + q)\cos\theta + r \quad ...(iii)$$

Also, $$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2 = \begin{vmatrix} 1 & \cos\theta & \cos\theta \\ \cos\theta & 1 & \cos\theta \\ \cos\theta & \cos\theta & 1 \end{vmatrix}$$

$\Rightarrow [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2 = (1-\cos\theta)^2(1+2\cos\theta)$

$\therefore v = |[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]| = |1-\cos\theta|\sqrt{1+2\cos\theta}|$

$$= 2\sin^2\frac{\theta}{2}\cdot|\sqrt{1+2\cos\theta}|$$

From Eqs. (i) and (iii), $p = r$ substituting in Eq. (ii), we get

$2p\cos\theta + q = 0$

$\Rightarrow \frac{q}{p} + 2\cos\theta = 0$

111. (a) ***112.*** (c) ***113.*** (b)

JEE Type Solved Examples : Matching Type Questions

● ***Ex. 114*** *Match the items of Column I with items of Column II.*

	Column I		Column II
A.	If $\|\mathbf{a}+\mathbf{b}\| = \|\mathbf{a}+2\mathbf{b}\|$, then angle between $\mathbf{a}$ and $\mathbf{b}$ is	p.	90°
B.	If $\|\mathbf{a}+\mathbf{b}\| = \|\mathbf{a}-2\mathbf{b}\|$, then angle between $\mathbf{a}$ and $\mathbf{b}$ is	q.	obtuse
C.	If $\|\mathbf{a}+\mathbf{b}\| = \|\mathbf{a}-\mathbf{b}\|$ then angle between $\mathbf{a}$ and $\mathbf{b}$ is	r.	0°
D.	Angle between $\mathbf{a}\times\mathbf{b}$ and a vector perpendicular to the vector $\mathbf{c}\times(\mathbf{a}\times\mathbf{b})$ is	s.	acute

Sol. A → q; B → s; C → p; D → r

(A) $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}+2\mathbf{b}|$

$$a^2 + b^2 + 2\mathbf{a}\cdot\mathbf{b} = a^2 + 4b^2 + 4\mathbf{a}\cdot\mathbf{b}$$

or $$2\mathbf{a}\cdot\mathbf{b} = -3b^2 < 0$$

Hence, angle between $\mathbf{a}$ and $\mathbf{b}$ is obtuse.

(B) $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-2\mathbf{b}|$

or $a^2 + b^2 + 2\mathbf{a}\cdot\mathbf{b} = a^2 + 4b^2 - 4\mathbf{a}\cdot\mathbf{b}$

or $$6\mathbf{a}\cdot\mathbf{b} = 3b^2 > 0$$

Hence, angle between $\mathbf{a}$ and $\mathbf{b}$ is acute.

(C) $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}|$

$\Rightarrow \mathbf{a}\cdot\mathbf{b} = 0$

Hence, $\mathbf{a}$ is perpendicular to $\mathbf{b}$.

(D) $\mathbf{c}\times(\mathbf{a}\times\mathbf{b})$ lies in the plane of vectors $\mathbf{a}$ and $\mathbf{b}$.

A vector perpendicular to this plane is parallel to $\mathbf{a}\times\mathbf{b}$

Hence, angle is 0°.

● ***Ex. 115*** *Match the items of Column I with items of Column II.*

	Column I		Column II
A.	Let $\|\mathbf{a}\| = \|\mathbf{b}\| = 2$, $\mathbf{x} = \mathbf{a} + \mathbf{b}$, $\mathbf{y} = \mathbf{a} - \mathbf{b}$. If $\|\mathbf{x} \times \mathbf{y}\| = 2\{\lambda - (\mathbf{a} \cdot \mathbf{b})^2\}^{\frac{1}{2}}$, then the value of λ is	p.	4
B.	The non-zero value of λ for which angle between $\lambda\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\hat{\mathbf{i}} + \lambda\hat{\mathbf{j}} + \hat{\mathbf{k}}$ is $\frac{\pi}{3}$, is	q.	42
C.	If $\|\mathbf{a}\| = \|\mathbf{b}\| = 1$ and $\|\mathbf{c}\| = 2$, then the maximum value of $\|\mathbf{a} - 2\mathbf{b}\|^2 + \|\mathbf{b} - 2\mathbf{c}\|^2 + \|\mathbf{c} - 2\mathbf{a}\|^2$ is	r.	16
		s.	7

Sol. A → (r), B → (p), C → (q)

(A) $\mathbf{x} \times \mathbf{y} = (\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} - \mathbf{b})$

$= \mathbf{a} \times \mathbf{a} - \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{a} - \mathbf{b} \times \mathbf{b} = -2(\mathbf{a} \times \mathbf{b})$

$\text{LHS} = |\mathbf{x} \times \mathbf{y}| = |-2(\mathbf{a} \times \mathbf{b})| = 2|\mathbf{a}||\mathbf{b}|\sin\theta$

$\text{RHS} = 2(\lambda - (\mathbf{a} \cdot \mathbf{b})^2)^{\frac{1}{2}} = 2(\lambda - |\mathbf{a}|^2|\mathbf{b}|^2\cos^2\theta)^{\frac{1}{2}}$

$\because \quad 8\sin\theta = 2(\lambda - 16\cos^2\theta)^{\frac{1}{2}}$

$\Rightarrow \quad 16\sin^2\theta = \lambda - 16\cos^2\theta$

$\Rightarrow \quad 16(\sin^2\theta + \cos^2\theta) = \lambda$

$\therefore \quad \lambda = 16$

(B) We have, $\cos\frac{\pi}{3} = \frac{1}{2}$

$$= \frac{(\lambda\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) \cdot (\hat{\mathbf{i}} + \lambda\hat{\mathbf{j}} + \hat{\mathbf{k}})}{\sqrt{\lambda^2 + 2}\sqrt{\lambda^2 + 2}}$$

$\because \quad \frac{1}{2} = \frac{\lambda + \lambda + 1}{\lambda^2 + 2}$

$\Rightarrow \quad \lambda^2 + 2 = 2(2\lambda + 1)$

$\Rightarrow \quad \lambda^2 - 4\lambda = 0$

$\Rightarrow \quad \lambda(\lambda - 4) = 0$

$\Rightarrow \quad \lambda = 0 \text{ and } 4$

$\therefore \quad \lambda = 4$

(C) We have, $|\mathbf{a} - 2\mathbf{b}|^2 + |\mathbf{b} - 2\mathbf{c}|^2 + |\mathbf{c} - 2\mathbf{a}|^2$

$$= |\mathbf{a}|^2 + 4|\mathbf{b}|^2 - 4\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2 + 4|\mathbf{c}|^2 - 4\mathbf{b} \cdot \mathbf{c} + |\mathbf{c}|^2 + 4|\mathbf{a}|^2 - 4\mathbf{c} \cdot \mathbf{a}$$

$\Rightarrow \quad 1 + 4 + 1 + 16 + 4 + 4 - 4(\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a})$

$\Rightarrow \quad 30 - 4(\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a})$

We know that,

$$(\mathbf{a} + \mathbf{b} + \mathbf{c})^2 = \Sigma\mathbf{a}^2 + 2\Sigma\mathbf{a} \cdot \mathbf{b} \geq 0$$

i.e. $1 + 1 + 4 + 2(\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a}) \geq 0$

$\Rightarrow \quad \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a} \geq -3$

$\Rightarrow \quad \geq 30 - 4(-3) \geq 30 + 12 \geq 42$

● ***Ex. 116*** *Match the items of Column I with items of Column II.*

	Column I		Column II
A.	Given two vectors $\mathbf{a} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$, $\mathbf{b} = -2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\lambda = \frac{\text{Projection of } \mathbf{a} \text{ on } \mathbf{b}}{\text{Projection of } \mathbf{b} \text{ on } \mathbf{a}}$, then the value of 3λ is	p.	0
B.	If $\mathbf{a} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, $\mathbf{b} = -\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{c} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\mathbf{a} + p\mathbf{b}$ is normal to $\mathbf{c}$, then p is	q.	7
C.	Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three non-zero vectors such that $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$, then $\lambda(\mathbf{a} \times \mathbf{b}) + \mathbf{c} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} = 0$, where λ is equal to	r.	5
D.	The points whose position vectors are $p\hat{\mathbf{i}} + q\hat{\mathbf{j}} + r\hat{\mathbf{k}}$, $q\hat{\mathbf{i}} + r\hat{\mathbf{j}} + p\hat{\mathbf{k}}$ and $r\hat{\mathbf{i}} + p\hat{\mathbf{j}} + q\hat{\mathbf{k}}$ are collinear, then the value of $(p^2 + q^2 + r^2 - pq - qr - rp)$ is	s.	2

Sol. A → (q), B → (r), C → (s), D → (p)

(A) Given, $\mathbf{a} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$

and $\mathbf{b} = -2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$

$$\text{Now, } \lambda = \frac{\text{Projection of } \mathbf{a} \text{ on } \mathbf{b}}{\text{Projection of } \mathbf{b} \text{ on } \mathbf{a}} = \frac{\left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}\right)}{\left(\frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}|}\right)} = \frac{|\mathbf{a}|}{|\mathbf{b}|}$$

$$= \frac{\sqrt{2^2 + (-3)^2 + 6^2}}{(-2)^2 + 2^2 + (-1)^2} = \frac{\sqrt{4 + 9 + 36}}{\sqrt{4 + 4 + 1}} = \frac{7}{3}$$

$\therefore \quad 3\lambda = 7$

(B) Given, $\mathbf{a} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$,

$\mathbf{b} = -\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{c} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}}$

Since, $\mathbf{a} + p\mathbf{b}$ is normal to $\mathbf{c}$

$\Rightarrow \quad (\mathbf{a} + p\mathbf{b}) \cdot \mathbf{c} = 0$

$\Rightarrow \quad [(1 - p)\hat{\mathbf{i}} + (2 + 2p)\hat{\mathbf{j}} + (3 + p)\hat{\mathbf{k}}] \cdot (3\hat{\mathbf{i}} + \hat{\mathbf{j}}) = 0$

$\Rightarrow \quad 3(1 - p) + 2 + 2p = 0$

$\Rightarrow \quad 3 - 3p + 2 + 2p = 0$

$\therefore \quad p = 5$

(C) Given, $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$

$\Rightarrow \quad \mathbf{a} \times \mathbf{a} + \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} = \mathbf{a} \times 0$

$\Rightarrow \quad \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} = 0 \quad \ldots(i)$

Also, $\mathbf{b} \times \mathbf{a} + \mathbf{b} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} = \mathbf{b} \times 0$

$\Rightarrow \quad \mathbf{b} \times \mathbf{a} + \mathbf{b} \times \mathbf{c} = 0$

$\Rightarrow \quad -\mathbf{a} \times \mathbf{b} - \mathbf{c} \times \mathbf{b} = 0$

$\Rightarrow \quad \mathbf{a} \times \mathbf{b} + \mathbf{c} \times \mathbf{b} = 0 \quad \ldots(ii)$

On adding Eqs. (i) and (ii), we get

$$2(\mathbf{a} \times \mathbf{b}) + \mathbf{c} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} = 0$$

On comparing, we get $\lambda = 2$

(D) $\begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix} = 0$

$\Rightarrow (p+q+r)(p^2+q^2+r^2-pq-qr-rp)=0$

$\therefore \quad p^2+q^2+r^2-pq-qr-rp=0$

● ***Ex. 117*** *Match the items of Column I with items of Column II.*

	Column I		Column II
A.	If $\mathbf{a}$ and $\mathbf{b}$ are two unit vectors inclined at $\frac{\pi}{3}$, then $16[\mathbf{a}\ \mathbf{b}+\mathbf{a}\times\mathbf{b}\ \mathbf{b}]$ is	p.	-12
B.	If $\mathbf{b}$ and $\mathbf{c}$ are orthogonal unit vectors and $\mathbf{b}\times\mathbf{c}=\mathbf{a}$, then $[\mathbf{a}+\mathbf{b}+\mathbf{c}\ \ \mathbf{a}+\mathbf{b}\ \ \mathbf{b}+\mathbf{c}]$ is	q.	1
C.	If $\lvert\mathbf{a}\rvert=\lvert\mathbf{b}\rvert=\lvert\mathbf{c}\rvert=2$ and $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{c}=\mathbf{c}\cdot\mathbf{a}=2$, then $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\cos 45°$ is equal to	r.	3
D.	$\mathbf{a}=2\hat{\mathbf{i}}+3\hat{\mathbf{j}}-\hat{\mathbf{k}}$, $\mathbf{b}=-\hat{\mathbf{i}}+2\hat{\mathbf{j}}-4\hat{\mathbf{k}}$, $\mathbf{c}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $\mathbf{d}=3\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}}$, then $\frac{1}{7}(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})$ is equal to	s.	4

Sol. A → (p); B → (q); C → (s); D → (r)

(A) Given $\mathbf{a}$ and $\mathbf{b}$ are two unit vectors, i.e., $|\mathbf{a}|=|\mathbf{b}|=1$ and angle between them is $\frac{\pi}{3}$.

$$\sin\theta=\frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}||\mathbf{b}|} \Rightarrow \sin\frac{\pi}{3}=|\mathbf{a}\times\mathbf{b}|$$

$$\frac{\sqrt{3}}{2}=|\mathbf{a}\times\mathbf{b}|$$

Now, $[\mathbf{a}\ \mathbf{b}+\mathbf{a}\times\mathbf{b}\ \mathbf{b}]=[\mathbf{a}\ \mathbf{b}\ \mathbf{b}]+[\mathbf{a}\ \mathbf{a}\times\mathbf{b}\ \mathbf{b}]$

$=0+[\mathbf{a}\ \mathbf{a}\times\mathbf{b}\ \mathbf{b}]$

$=(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{b}\times\mathbf{a})=-(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b})$

$=-|\mathbf{a}\times\mathbf{b}|^2=-\frac{3}{4}$

(B) If $\mathbf{b}$ and $\mathbf{c}$ are orthogonal $\mathbf{b}\cdot\mathbf{c}=0$

Also, it is given that $\mathbf{b}\times\mathbf{c}=\mathbf{a}$.

Now $[\mathbf{a}+\mathbf{b}+\mathbf{c}\ \mathbf{a}+\mathbf{b}\ \mathbf{b}+\mathbf{c}]$

$=[\mathbf{a}\ \mathbf{a}+\mathbf{b}\ \mathbf{b}+\mathbf{c}]+[\mathbf{b}+\mathbf{c}\ \mathbf{a}+\mathbf{b}\ \mathbf{b}+\mathbf{c}]$

$=[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$

$=\mathbf{a}\cdot\mathbf{a}=|\mathbf{a}|^2=1$ (because $\mathbf{a}$ is a unit vector)

(C) We know that, $[\mathbf{a}\times\mathbf{b}\ \mathbf{b}\times\mathbf{c}\ \mathbf{c}\times\mathbf{a}]=[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$

and $=[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2=\begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix}=\begin{vmatrix} 4 & 2 & 1 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{vmatrix}=32$

$\therefore \quad [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=4\sqrt{2}$

(D) $(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{d})=21$

JEE Type Solved Examples :
Single Integer Answer Type Questions

● ***Ex. 118*** *Given that* $\mathbf{u}=\hat{\mathbf{i}}-2\hat{\mathbf{j}}+3\hat{\mathbf{k}}$; $\mathbf{v}=2\hat{\mathbf{i}}+\hat{\mathbf{j}}+4\hat{\mathbf{k}}$; $\mathbf{w}=\hat{\mathbf{i}}+3\hat{\mathbf{j}}+3\hat{\mathbf{k}}$ *and* $(\mathbf{u}\cdot\mathbf{R}-15)\hat{\mathbf{i}}+(\mathbf{v}\cdot\mathbf{R}-30)\hat{\mathbf{j}}+(\mathbf{w}\cdot\mathbf{R}-20)\hat{\mathbf{k}}=\mathbf{0}$. *Then, the greatest integer less than or equal to* $|\mathbf{R}|$ *is*

Sol. (6) Let $\mathbf{R}=x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}$

$\mathrm{u}=\hat{\mathbf{i}}-2\hat{\mathbf{j}}+3\hat{\mathbf{k}}$; $\mathrm{v}=2\hat{\mathbf{i}}+\hat{\mathbf{j}}+4\hat{\mathbf{k}}$; $\mathrm{w}=\hat{\mathbf{i}}+3\hat{\mathbf{j}}+3\hat{\mathbf{k}}$

$(\mathbf{u}\cdot\mathbf{R}-15)\hat{\mathbf{i}}+(\mathbf{v}\cdot\mathbf{R}-30)\hat{\mathbf{j}}+(\mathbf{w}\cdot\mathbf{R}-20)\hat{\mathbf{k}}=\mathbf{0}$ (given)

So, $\mathbf{u}\cdot\mathbf{R}=15\Rightarrow x-2y+3z=15$ ...(i)

$\mathbf{v}\cdot\mathbf{R}=30\Rightarrow 2x+y+4z=30$ ...(ii)

$\mathbf{w}\cdot\mathbf{R}=25\Rightarrow x+3y+3z=25$ ...(iii)

Solving, we get

$x=4$

$y=2$

$z=5$

Now, $|\mathbf{R}|=\sqrt{4^2+2^2+5^2}=\sqrt{45}$

$[|\mathbf{R}|]=[\sqrt{45}]=6$

● ***Ex. 119*** *The position vector of a point P is* $\mathbf{r}=x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}$, *where* $x, y, z\in N$ *and* $\mathbf{a}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}}$. *If* $\mathbf{r}\cdot\mathbf{a}=20$ *and the number of possible of P is* 9λ, *then the value of* λ *is*

Sol. (9) $\mathbf{r}\cdot\mathbf{a}=20 \Rightarrow x+2y+z=20,\ x, y, z\in N$

$\therefore$ Number of non-negative integer solution are

$\Rightarrow {}^{17}C_1+{}^{15}C_1+{}^{13}C_1+\ldots+{}^{1}C_1$

$=81=9\lambda \Rightarrow \lambda=9$

● ***Ex. 120*** *Let* $\mathbf{u}$ *be a vector on rectangular coordinate system with sloping angle* 60°. *Suppose that* $|\mathbf{u}-\hat{\mathbf{i}}|$ *is geometric mean of* $|\mathbf{u}|$ *and* $|\mathbf{u}-2\hat{\mathbf{i}}|$, *where* $\hat{\mathbf{i}}$ *is the unit vector along the X-axis. Then, the value of* $(\sqrt{2}+1)|\mathbf{u}|$ *is*

Sol. (1) Since, angle between $\mathbf{u}$ and $\hat{\mathbf{i}}$ is 60°, we have

$$\mathbf{u}\cdot\hat{\mathbf{i}}=|\mathbf{u}||\hat{\mathbf{i}}|\cos 60°=\frac{|\mathbf{u}|}{2}$$

Given that, $|\mathbf{u}|, |\mathbf{u}-\hat{\mathbf{i}}|, |\mathbf{u}-2\hat{\mathbf{i}}|$ are in GP.

So, $|\mathbf{u}-\hat{\mathbf{i}}|^2=|\mathbf{u}||\mathbf{u}-2\hat{\mathbf{i}}|$

Squaring both sides,

$$[|\mathbf{u}|^2 + |\hat{\mathbf{i}}|^2 - 2\mathbf{u}\cdot\hat{\mathbf{i}}]^2 = |\mathbf{u}|^2[|\mathbf{u}|^2 + 4|\hat{\mathbf{i}}|^2 - 4\mathbf{u}\cdot\hat{\mathbf{i}}]$$

$$\left(|\mathbf{u}|^2 + 1 - \frac{2|\mathbf{u}|}{2}\right)^2 = |\mathbf{u}|^2\left(|\mathbf{u}|^2 + 4 - \frac{|\mathbf{u}|}{2}\right)$$

or $\quad |\mathbf{u}| + 2|\mathbf{u}| - 1 = 0$

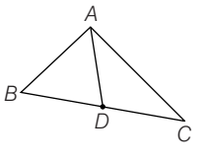
$\Rightarrow \quad |\mathbf{u}|^2 = -\dfrac{2 \pm 2\sqrt{2}}{2}$

or $\quad |\mathbf{u}| = \sqrt{2} - 1$

$\Rightarrow \quad (\sqrt{2}+1)|\mathbf{u}| = (\sqrt{2}+1)(\sqrt{2}-1) = 2 - 1 = 1$

● ***Ex. 121*** *Let* $A(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 5\hat{\mathbf{k}})$, $B(-\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$ *and* $C(\lambda\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + \mu\hat{\mathbf{k}})$ *are vertices of a triangle and its median through A is equally inclined to the positive directions of the axes, the value of* $2\lambda - \mu$ *is*

Sol. (4) Median through A is



$$\mathbf{AD} = \frac{\mathbf{AB} + \mathbf{AC}}{2}$$

$$\mathbf{AB} = -3\hat{\mathbf{i}} - 3\hat{\mathbf{k}}$$

$$\mathbf{AC} = (\lambda - 2)\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + (\mu - 5)\hat{\mathbf{k}}$$

$$\mathbf{AD} = \frac{1}{2}[(\lambda - 5)\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + (\mu - 8)\hat{\mathbf{k}}]$$

We have, **AD** is equally inclined to the positive direction of axes

$$\cos\theta = \frac{\frac{1}{2}[(\lambda-5)\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + (\mu-8)\hat{\mathbf{k}}]\cdot\hat{\mathbf{i}}}{|\mathbf{AD}||\hat{\mathbf{i}}|}$$

$$= \frac{\frac{1}{2}[(\lambda-5)\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + (\mu-8)\hat{\mathbf{k}}]\cdot\hat{\mathbf{j}}}{|\mathbf{AD}||\hat{\mathbf{j}}|}$$

$$= \frac{\frac{1}{2}[(\lambda-5)\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + (\mu-8)\hat{\mathbf{k}}]\cdot\hat{\mathbf{k}}}{|\mathbf{AD}||\hat{\mathbf{k}}|}$$

$\Rightarrow \quad \dfrac{\lambda - 5}{2} = 1 = \dfrac{\mu - 8}{2}$

$\Rightarrow \quad \lambda = 7$ and $\mu = 10$

$\therefore \quad 2\lambda - \mu = 14 - 10 = 4$

● ***Ex. 122*** *Three vectors* $\mathbf{a}(|\mathbf{a}| \neq 0)$, $\mathbf{b}$ *and* $\mathbf{c}$ *are such that* $\mathbf{a}\times\mathbf{b} = 3\mathbf{a}\times\mathbf{c}$, *also* $|\mathbf{a}| = |\mathbf{b}| = 1$ *and* $|\mathbf{c}| = \dfrac{1}{3}$. *If the angle between* $\mathbf{b}$ *and* $\mathbf{c}$ *is* $60°$ *and* $|\mathbf{b} - 3\mathbf{c}| = \lambda|\mathbf{a}|$, *then the value of* λ *is*

Sol. (1) We have, $\quad \mathbf{a}\times\mathbf{b} = 3\mathbf{a}\times\mathbf{c}$

$\Rightarrow \quad \mathbf{a}\times(\mathbf{b} - 3\mathbf{c}) = 0$

$\Rightarrow \mathbf{a}$ is parallel to $\mathbf{b} - 3\mathbf{c}$.

Now, $\quad \mathbf{b} - 3\mathbf{c} = \lambda\mathbf{a}$

$\Rightarrow \quad |\mathbf{b} - 3\mathbf{c}|^2 = \lambda^2|\mathbf{a}|^2$

$\Rightarrow \quad |\mathbf{b}|^2 + 9|\mathbf{c}|^2 - 6(\mathbf{b}\cdot\mathbf{c}) = \lambda^2|\mathbf{a}|^2$

$\Rightarrow \quad 1 + 9\times\dfrac{1}{9} - 6\times 1\times\dfrac{1}{3}\times\dfrac{1}{2} = \lambda^2(1)^2$

$\Rightarrow \quad 1 + 1 - 1 = \lambda^2 \Rightarrow \lambda^2 = 1$

$\therefore \quad \lambda = \pm 1$

● ***Ex. 123*** *If* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *are unit vectors such that* $\mathbf{a}\cdot\mathbf{b} = 0 = \mathbf{a}\cdot\mathbf{c}$ *and the angle between* $\mathbf{b}$ *and* $\mathbf{c}$ *is* $\dfrac{\pi}{3}$, *then the value of* $|\mathbf{a}\times\mathbf{b} - \mathbf{a}\times\mathbf{c}|$.

Sol. $\mathbf{a}\,.\,\mathbf{b} = 0 \Rightarrow \mathbf{a}\perp\mathbf{b}$

$\mathbf{a}\,.\,\mathbf{c} = 0 \Rightarrow \mathbf{a}\perp\mathbf{c}$

$\Rightarrow \quad \mathbf{a}\perp\mathbf{b} - \mathbf{c}$

$\therefore \quad |\mathbf{a}\times\mathbf{b} - \mathbf{a}\times\mathbf{c}| = |\mathbf{a}\times(\mathbf{b} - \mathbf{c})|$

$= |\mathbf{a}||\mathbf{b} - \mathbf{c}| = |\mathbf{b} - \mathbf{c}|$

Now, $\quad |\mathbf{b} - \mathbf{c}|^2 = |\mathbf{b}|^2 + |\mathbf{c}|^2 - 2|\mathbf{b}||\mathbf{c}|\cos\dfrac{\pi}{3}$

$= 2 - 2\times\dfrac{1}{2} = 1$

$|\mathbf{b} - \mathbf{c}| = 1$

● ***Ex. 124*** *If the area of the triangle whose vertices are* $A(-1, 1, 2)$; $B(1, 2, 3)$ *and* $C(t, 1, 1)$ *is minimum, then the absolute value of parameter t is*

Sol. $\mathbf{AB} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{AC} = (t+1)\hat{\mathbf{i}} + 0\hat{\mathbf{j}} - \hat{\mathbf{k}}$

$$\mathbf{AB}\times\mathbf{AC} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 1 & 1 \\ t+1 & 0 & -1 \end{vmatrix}$$

$$= -\hat{\mathbf{i}} + (t+3)\hat{\mathbf{j}} - (t+1)\hat{\mathbf{k}}$$

$$= \sqrt{1 + (t+3)^2 + (t+1)^2}$$

$$= \sqrt{2t^2 + 8t + 11}$$

Area of $\Delta ABC = \dfrac{1}{2}|\mathbf{AB}\times\mathbf{AC}|$

$$= \frac{1}{2}\sqrt{2t^2 + 8t + 1}$$

Let $f(t) = \Delta^2 = \dfrac{1}{4}(2t^2 + 8t + 1)$

$f(t) = 0 \Rightarrow t = -2$

At $\quad t = -2$, $f''(t) > 0$

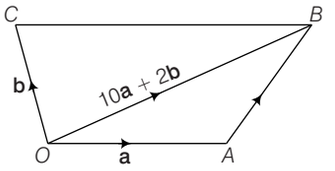
So, Δ is minimum at $t = -2$

● ***Ex. 125*** *Let* $\mathbf{OA}=\mathbf{a}, \mathbf{OB}=10\mathbf{a}+2\mathbf{b}$ *and* $\mathbf{OC}=\mathbf{b}$*, where O, A and C are non-collinear points. Let p denote the area of quadrilateral OACB, and let q denote the area of parallelogram with OA and OC as adjacent sides. If p = k q, then k is equal to*

Sol. Here, $\mathbf{OA}=\mathbf{a}$, $\mathbf{OB}=10\mathbf{a}+2\mathbf{b}$, $\mathbf{OC}=\mathbf{b}$

q = Area of parallelogram with OA and OC as adjacent side.

$\therefore \quad q=|\mathbf{a}\times\mathbf{b}|$ ...(i)



p = Area of quadrilateral $OABC$

$=$ Area of ΔOAB + Area of ΔOBC

$$=\frac{1}{2}|\mathbf{a}\times(10\mathbf{a}+2\mathbf{b})|+\frac{1}{2}|(10\mathbf{a}+2\mathbf{b})\times\mathbf{b}|$$

$$=|\mathbf{a}\times\mathbf{b}|+5|\mathbf{a}\times\mathbf{b}|$$

$\therefore \quad p=6|\mathbf{a}\times\mathbf{b}|$

or $\quad p=6q$ [From Eq. (i)]

$\therefore \quad k=6$

● ***Ex. 126.*** *If* $\mathbf{x}, \mathbf{y}$ *are two non-zero and non-collinear vectors satisfying* $[(a-2)\alpha^2+(b-3)\alpha+c]\mathbf{x}+[(a-2)\beta^2+(b-3)\beta+c]\mathbf{y}+[(a-2)\gamma^2+(b-3)\gamma+c](\mathbf{x}\times\mathbf{y})=0$ *where* α, β, γ *are three distinct real numbers, then find the value of* $(a^2+b^2+c^2-4)$.

Sol. (9) Since, $\mathbf{x}$ and $\mathbf{y}$ are non-collinear vectors, therefore $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{x}\times\mathbf{y}$ are non-coplanar vectors.

So, $[(a-2)\alpha^2+(b-3)\alpha+c]\mathbf{x}+[(a-2)\beta^2+(b-3)\beta+c]\mathbf{y}+[(a-2)\gamma^2+(b-3)\gamma+c](\mathbf{x}\times\mathbf{y})=0$

$\Rightarrow$ Coefficient of each vector $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{x}\times\mathbf{y}$ is zero.

$$(a-2)\alpha^2+(b-3)\alpha+c=0$$

$$(a-2)\beta^2+(b-3)\beta+c=0$$

$$(a-2)\gamma^2+(b-3)\gamma+c=0$$

The above three equations will satisfy if the coefficients of α, β and γ are zero because α, β and γ are three distinct real numbers

$$a-2=0 \text{ or } a=2,$$

$$b-3=0 \text{ or } b=3 \text{ and } c=0$$

$\therefore \quad a^2+b^2+c^2=2^2+3^2+0^2=4+9=13$

● ***Ex. 127*** *Let* $\mathbf{v}=2\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}}$ *and* $\mathbf{w}=\hat{\mathbf{i}}+3\hat{\mathbf{k}}$*. If* $\hat{\mathbf{u}}$ *is unit vector and the maximum value of* $[\mathbf{u}\ \mathbf{v}\ \mathbf{w}]=\sqrt{\lambda}$*, then the value of* $(\lambda-51)$ *is*

Sol. (8) We have, $[\mathbf{u}\ \mathbf{v}\ \mathbf{w}]=\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$

$\Rightarrow \quad [\mathbf{u}\ \mathbf{v}\ \mathbf{w}]\le|\mathbf{u}||\mathbf{v}\times\mathbf{w}|$ $[\because \mathbf{a}\cdot\mathbf{b}\le|\mathbf{a}||\mathbf{b}|]$

$\Rightarrow \quad [\mathbf{u}\ \mathbf{v}\ \mathbf{w}]\le|\mathbf{v}\times\mathbf{w}|$ $[\because |\mathbf{u}|=1]$

Now, $\quad \mathbf{v}\times\mathbf{w}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 2 & 1 & -1\\ 1 & 0 & 3\end{vmatrix}=3\hat{\mathbf{i}}-7\hat{\mathbf{j}}-\hat{\mathbf{k}}$

$\therefore \quad |\mathbf{v}\times\mathbf{w}|=\sqrt{9+49+1}=\sqrt{59}$

Hence, maximum value of $[\mathbf{u}\ \mathbf{v}\ \mathbf{w}]=\sqrt{59}$

On comparing, we get $\lambda=59$

$\therefore \quad \lambda-51=8$

● ***Ex. 128*** *Let* $\mathbf{a}=\alpha\hat{\mathbf{i}}+2\hat{\mathbf{j}}-3\hat{\mathbf{k}}, \mathbf{b}=\hat{\mathbf{i}}+2\alpha\hat{\mathbf{j}}-2\hat{\mathbf{k}}$ *and* $\mathbf{c}=2\hat{\mathbf{i}}-\alpha\hat{\mathbf{j}}+\hat{\mathbf{k}}$*. Then the value of* 6α*, such that* $\{(\mathbf{a}\times\mathbf{b})\times(\mathbf{b}\times\mathbf{c})\}\times(\mathbf{c}\times\mathbf{a})=0$*, is*

Sol. (4) $\mathbf{a}=\alpha\hat{\mathbf{i}}+2\hat{\mathbf{j}}-3\hat{\mathbf{k}}$, $\mathbf{b}=\hat{\mathbf{i}}+2\alpha\hat{\mathbf{j}}-2\hat{\mathbf{k}}$, $\mathbf{c}=2\hat{\mathbf{i}}-\alpha\hat{\mathbf{j}}+\hat{\mathbf{k}}$

$\{(\mathbf{a}\times\mathbf{b})\times(\mathbf{b}\times\mathbf{c})\}\times(\mathbf{c}\times\mathbf{a})=\mathbf{0}$

or $\quad \{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\mathbf{b}-[\mathbf{a}\ \mathbf{b}\ \mathbf{b}]\mathbf{c}\}\times(\mathbf{c}\times\mathbf{a})=\mathbf{0}$

or $\quad [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\mathbf{b}\times(\mathbf{c}\times\mathbf{a})=\mathbf{0}$

or $\quad [\mathbf{a}\ \mathbf{b}\ \mathbf{c}](\mathbf{a}\cdot\mathbf{b})\mathbf{c}-(\mathbf{b}\cdot\mathbf{c})\mathbf{a})=\mathbf{0}$

or $\quad [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=0$ $(\because \mathbf{a}$ and $\mathbf{c}$ are not collinear)

$\Rightarrow \quad \begin{vmatrix}\alpha & 2 & -3\\ 1 & 2\alpha & -2\\ 2 & -\alpha & 1\end{vmatrix}=0$

or $\quad \alpha(2\alpha-2\alpha)-2(1+4)-3(-\alpha-4\alpha)=0$ or $10-15\alpha=0$

$\therefore \quad \alpha=\frac{2}{3}$

$$6\alpha=\frac{6\times2}{3}=4$$

Subjective Type Questions

● ***Ex. 129*** *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be two unit vectors such that* $\hat{\mathbf{a}}+\hat{\mathbf{b}}$ *is also a unit vector. Then, find the angle between* $\hat{\mathbf{a}}$ *and* $\hat{\mathbf{b}}$.

Sol. Since, $\hat{\mathbf{a}}+\hat{\mathbf{b}}$ is a unit vector.

$$\Rightarrow \quad |\hat{\mathbf{a}}+\hat{\mathbf{b}}|=1 \Rightarrow |\hat{\mathbf{a}}+\hat{\mathbf{b}}|^2=1$$

$$\Rightarrow \quad (\hat{\mathbf{a}}+\hat{\mathbf{b}})\cdot(\hat{\mathbf{a}}+\hat{\mathbf{b}})=1$$

$$\Rightarrow \quad \hat{\mathbf{a}}\cdot\hat{\mathbf{a}}+\hat{\mathbf{b}}\cdot\hat{\mathbf{b}}+2\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}=1 \Rightarrow 1+1+2\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}=1$$

$$\Rightarrow \quad \hat{\mathbf{a}}\cdot\hat{\mathbf{b}}=-\frac{1}{2} \Rightarrow |\hat{\mathbf{a}}|\,|\hat{\mathbf{b}}|\cos\theta=-\frac{1}{2}$$

$$\Rightarrow \quad \cos\theta=-\frac{1}{2} \qquad (\because |\hat{\mathbf{a}}|=|\hat{\mathbf{b}}|=1)$$

$$\Rightarrow \quad \theta=120^\circ$$

Hence, the angle between $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}=120^\circ$.

● ***Ex. 130*** *Determine the values of c so that for all real x, the vectors* $cx\hat{\mathbf{i}}-6\hat{\mathbf{j}}+3\hat{\mathbf{k}}$ *and* $x\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2cx\hat{\mathbf{k}}$ *make an obtuse angle with each other.*

Sol. Given, $\mathbf{a}=cx\hat{\mathbf{i}}-6\hat{\mathbf{j}}+3\hat{\mathbf{k}}$ and $\mathbf{b}=x\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2cx\hat{\mathbf{k}}$

$\because$ $\mathbf{a}$ and $\mathbf{b}$ make an obtuse angle with each other.

$$\therefore \quad \cos\theta=\frac{\mathbf{a}\,.\,\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}<0$$

i.e., $$\frac{cx^2-12+6cx}{\sqrt{c^2x^2+36+9}\sqrt{x^2+4+4c^2x^2}}<0$$

$$\Rightarrow \quad cx^2+6cx-12<0 \qquad \ldots(i)$$

Now, two cases are possible.

Case I $\quad c\neq 0$

$\Rightarrow cx^2+6cx-12$ is a quadratic equation which has real solution

''iff $A<0$ and $B^2-4AC<0$''

i.e. if $c<0$ and $36c^2+48c<0$

i.e. if $c<0$ and $12c(3c+4)<0$

$$\Rightarrow \quad 3c+4>0 \qquad [\because c>0]$$

$$\Rightarrow \quad -\frac{4}{3}<c<0 \qquad \ldots(ii)$$

Case II $\quad c=0$

$\Rightarrow -12<0$ which is an identity.

$\therefore$ $\quad c=0$ satisfy Eq. (i) $\qquad \ldots(iii)$

$\therefore$ From Eqs. (ii) and (iii), we get $-\frac{4}{3}<c\leq 0$

● ***Ex. 131*** *A, B, C and D are four points in space. Using vector methods, prove that* $AC^2+BD^2+AD^2+BC^2 \geq AB^2+CD^2$ *what is the implication of the sign of equality.*

Sol. Let the position vector of A, B, C and D be $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$, respectively.

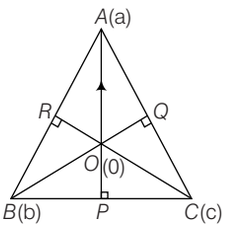
Then, $AC^2+BD^2+AD^2+BC^2$

$$\begin{aligned}
&=(\mathbf{c}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{a})+(\mathbf{d}-\mathbf{b})\cdot(\mathbf{d}-\mathbf{b})\\
&\quad+(\mathbf{d}-\mathbf{a})\,.\,(\mathbf{d}-\mathbf{a})+(\mathbf{c}-\mathbf{b})\,.\,(\mathbf{c}-\mathbf{b})\\
&=|\mathbf{c}|^2+|\mathbf{a}|^2-2\mathbf{a}\cdot\mathbf{c}+|\mathbf{d}|^2+|\mathbf{b}|^2-2\mathbf{d}\cdot\mathbf{b}\\
&\quad+|\mathbf{d}|^2+|\mathbf{a}|^2-2\mathbf{a}\,.\,\mathbf{d}+|\mathbf{c}|^2+|\mathbf{b}|^2-2\mathbf{b}\,.\,\mathbf{c}\\
&=|\mathbf{a}|^2+|\mathbf{b}|^2-2\mathbf{a}\cdot\mathbf{b}+|\mathbf{c}|^2+|\mathbf{d}|^2-2\mathbf{c}\cdot\mathbf{d}\\
&\quad+|\mathbf{a}|^2+|\mathbf{b}|^2+|\mathbf{c}|^2+|\mathbf{d}|^2-2\mathbf{a}\cdot\mathbf{b}+2\mathbf{c}\cdot\mathbf{d}\\
&\quad-2\mathbf{a}\cdot\mathbf{c}-2\mathbf{b}\,.\,\mathbf{d}-2\mathbf{a}\cdot\mathbf{d}-2\mathbf{b}\cdot\mathbf{c}\\
&=(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})+(\mathbf{c}-\mathbf{d})\cdot(\mathbf{c}\cdot\mathbf{d})+\\
&\quad(\mathbf{a}+\mathbf{b}-\mathbf{c}-\mathbf{d})\cdot(\mathbf{a}+\mathbf{b}-\mathbf{c}-\mathbf{d})\\
&=AB^2+CD^2+(\mathbf{a}+\mathbf{b}-\mathbf{c}-\mathbf{d})\cdot(\mathbf{a}+\mathbf{b}\\
&\quad-\mathbf{c}-\mathbf{d})\geq AB^2+CD^2
\end{aligned}$$

$\therefore$ $\quad AC^2+BD^2+AD^2+BC^2\geq AB^2+CD^2$

● ***Ex. 132*** *Using vector method, prove that the altitudes of a triangle are concurrent.*

Sol. Let the point of intersection O of two altitudes BQ and CR be taken as origin and the position vectors of the vertices A, B, C be $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ respectively. Let AO produced meet BC at P. We will show that AP is perpendicular to BC , showing there by that the three altitudes are concurrent.



$$\therefore \quad \mathbf{OB}=\mathrm{b},\ \mathbf{BQ}=\mu\mathbf{b}$$

as its collinear with $\mathbf{OB}$.

Similarly, since $\quad \mathbf{OC}=\mathbf{C}$

$$\therefore \quad \mathbf{CR}=\nu\mathbf{C}$$

Now, $\quad \mathbf{AC}=\mathbf{c}-\mathbf{a}$ and $\mathbf{AB}=\mathbf{b}-\mathbf{a}$

Since, $BQ\perp AC$, we have $\mu\mathbf{b}\,.\,(\mathbf{c}-\mathbf{a})$ and so $\mathbf{a}\,.\,\mathbf{b}=\mathbf{b}\,.\,\mathbf{c}$

Again, since $CR\perp AB$, $\nu\mathbf{c}\,.\,(\mathbf{b}-\mathbf{a})=0$

$$\therefore \quad \mathbf{b}\,.\,\mathbf{c}=\mathbf{c}\,.\,\mathbf{a}$$

$$\therefore \quad \mathbf{a}\,.\,\mathbf{b}=\mathbf{b}\,.\,\mathbf{c}=\mathbf{c}\,.\,\mathbf{a}$$

or $\quad \mathbf{a}\,.\,(\mathbf{c}-\mathbf{b})=0$

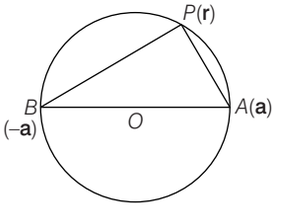
$$\Rightarrow \quad \lambda\mathbf{a}\,.\,(\mathbf{c}-\mathbf{b})=0$$

$$\therefore \quad \mathbf{AP}\,.\,\mathbf{BC}=0 \Rightarrow AP\perp BC$$

● ***Ex. 133*** *Using vector method, prove that the angle in a semi-circle is a right angle.*

Sol. Take the centre O as origin and AB is the diameter, so that $OA=OB$.

If the point A is a, then B is $-a$ and $|a|=r=$ radius.



Let P be any point r on the circumference, so that $|r| = OP = r$

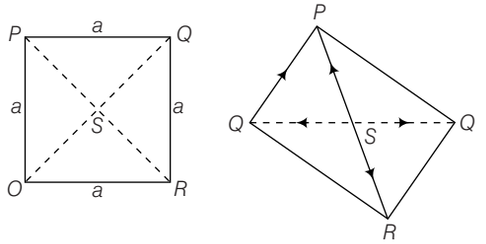
Then, $\mathbf{AP}$ = Position vector of P − Position vector of $A = r - a$

and $\mathbf{BP}$ = Position vector of P − Position vector of $B = r + a$

$\therefore \quad \mathbf{AP} \cdot \mathbf{BP} = (r-a)\cdot(r+a) = r^2 - a^2 = r^2 - r^2 = 0$

● ***Ex. 134*** *The corner P of the square OPQR is folded up so that the plane OPQ is perpendicular to the plane OQR, find the angle between OP and QR.*

Sol. After folding OPQ, $PS \perp SR$.



Here, $SQ \perp SR$, $SQ \perp PS$

Let $\mathbf{SR} = \frac{a}{\sqrt{2}}\hat{\mathbf{j}}$, $\mathbf{SQ} = \frac{a}{\sqrt{2}}\hat{\mathbf{i}}$, $\mathbf{SP} = \frac{a}{\sqrt{2}}\hat{\mathbf{j}}$

$$\mathbf{OP} = -\mathbf{SO} + \mathbf{SP} = \frac{a}{\sqrt{2}}\hat{\mathbf{j}} + \frac{a}{\sqrt{2}}\hat{\mathbf{i}} = \frac{a}{\sqrt{2}}(\hat{\mathbf{j}} + \hat{\mathbf{i}})$$

$$\mathbf{QR} = \mathbf{SR} - \mathbf{SQ} = \frac{a}{\sqrt{2}}(\hat{\mathbf{k}} - \hat{\mathbf{i}})$$

$$|\mathbf{OP}| = \frac{a}{\sqrt{2}}\sqrt{2} = a \;\Rightarrow\; |\mathbf{QR}| = a$$

Cosine of angle between $\mathbf{OP}$ and

$$\mathbf{QR} = \frac{\mathbf{OP}\cdot\mathbf{QR}}{|\mathbf{OP}||\mathbf{QR}|} = \frac{a^2}{2}\frac{(\hat{\mathbf{i}}+\hat{\mathbf{j}})\cdot(\hat{\mathbf{k}}-\hat{\mathbf{i}})}{a^2}$$

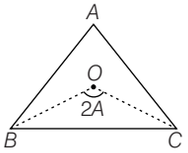
$$\Rightarrow \qquad \cos\theta = \frac{1}{2}(-1) = -\frac{1}{2} \;\Rightarrow\; \theta = \frac{2\pi}{3}$$

● ***Ex. 135*** *In a ΔABC, prove by vector method that*

$$\cos 2A + \cos 2B + \cos 2C \geq -\frac{3}{2}$$

Sol. As we know, $(\mathbf{OA} + \mathbf{OB} + \mathbf{OC})^2 \geq 0$ ...(i)

and $|\mathbf{OA}|^2 = |\mathbf{OB}|^2 = |\mathbf{OC}|^2 = R^2$ ...(ii)



$\therefore$ Using Eq. (i),

$$|\mathbf{OA}|^2 + |\mathbf{OB}|^2 + |\mathbf{OC}|^2 + 2(\mathbf{OA}.\mathbf{OB} + \mathbf{OB}.\mathbf{OC} + \mathbf{OC}.\mathbf{OA}) \geq 0$$

$$\Rightarrow\; 3R^2 + 2R^2(\cos 2A + \cos 2B + \cos 2C) \geq 0$$

$$\Rightarrow \qquad \cos 2A + \cos 2B + \cos 2C \geq -\frac{3}{2}$$

● ***Ex. 136*** *Let $\beta = 4\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ and γ be two vectors perpendicular to each other in the XY-plane. Find all the vectors in the same plane having the projections 1, 2 along β and γ, respectively.*

Sol. Here, $\beta = 4\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$

Since, γ is perpendicular to β i.e. $\beta \cdot \gamma = 0$

$\therefore$ We can choose $\gamma = 3\lambda\hat{\mathbf{i}} - 4\lambda\hat{\mathbf{j}}$ for all values of λ.

Let the required vector be $\alpha = l\,\hat{\mathbf{i}} + m\,\hat{\mathbf{j}}$.

Now, projection of α along $\beta = \frac{\alpha\cdot\beta}{|\beta|}$

$$1 = \frac{4l + 3m}{5} \;\Rightarrow\; 4l + 3m = 5 \qquad \text{...(i)}$$

Similarly, projection of α along $\gamma = \frac{\alpha\cdot\gamma}{|\gamma|}$

$$\Rightarrow \qquad 2 = \frac{3\lambda l - 4\lambda m}{5\lambda}$$

$$\Rightarrow \qquad 3l - 4m = 10 \qquad \text{...(ii)}$$

On solving Eqs. (i) and (ii), we get

$$l = 2 \text{ and } m = -1$$

$$\alpha = 2\hat{\mathbf{i}} - \hat{\mathbf{j}}$$

● ***Ex. 137*** *If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are three coplanar vectors. If $\mathbf{a}$ is not parallel to $\mathbf{b}$, show that*

$$\mathbf{c} = \frac{\begin{vmatrix}\mathbf{c}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b}\\ \mathbf{c}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{b}\end{vmatrix}\mathbf{a} + \begin{vmatrix}\mathbf{a}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{a}\\ \mathbf{a}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{b}\end{vmatrix}\mathbf{b}}{\begin{vmatrix}\mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b}\\ \mathbf{a}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{b}\end{vmatrix}}$$

Sol. Since, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar, we may write

$$\mathbf{c} = \lambda_1\mathbf{a} + \lambda_2\mathbf{b}$$

$$\Rightarrow \qquad \mathbf{a}\cdot\mathbf{c} = \lambda_1\mathbf{a}\cdot\mathbf{a} + \lambda_2\mathbf{a}\cdot\mathbf{b} \qquad \text{...(i)}$$

and $\mathbf{b}\cdot\mathbf{c} = \lambda_1\mathbf{b}\cdot\mathbf{a} + \lambda_2\mathbf{b}\cdot\mathbf{b}$ ...(ii)

On solving Eqs. (i) and (ii), by Cramer's rule, we find that

$$\lambda_1 = \frac{\begin{vmatrix}\mathbf{a}\cdot\mathbf{c} & \mathbf{a}\cdot\mathbf{b}\\ \mathbf{b}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{b}\end{vmatrix}}{\begin{vmatrix}\mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b}\\ \mathbf{a}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{b}\end{vmatrix}} \quad \text{and} \quad \lambda_2 = \frac{\begin{vmatrix}\mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{c}\\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{c}\end{vmatrix}}{\begin{vmatrix}\mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b}\\ \mathbf{a}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{b}\end{vmatrix}}$$

On substituting λ_1 and λ_2, we get

$$\mathbf{c} = \frac{\begin{vmatrix}\mathbf{c}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b}\\ \mathbf{c}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{b}\end{vmatrix}\mathbf{a} + \begin{vmatrix}\mathbf{a}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{a}\\ \mathbf{a}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{b}\end{vmatrix}\mathbf{b}}{\begin{vmatrix}\mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b}\\ \mathbf{a}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{b}\end{vmatrix}}$$

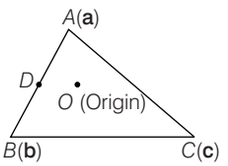
● ***Ex. 138*** *In ΔABC, D is the mid-point of side AB and E is the centroid of ΔCDA. If $\mathbf{OE}\cdot\mathbf{CD}=0$, where O is the circumcentre of ΔABC, using vectors prove that $AB = AC$.*

Sol. Let us take O to be the origin and position vector of the vertices A, B and C be $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively.

We have,

$$|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}|$$

Now, $\mathbf{D} = \dfrac{\mathbf{a}+\mathbf{b}}{2}$ ($\because$ mid-point of AB)



$$\mathbf{E} = \frac{3\mathbf{a}+\mathbf{b}+2\mathbf{c}}{2}$$

$\therefore$ $\mathbf{CD} = \dfrac{\mathbf{a}+\mathbf{b}}{2} = \dfrac{\mathbf{a}+\mathbf{b}-2\mathbf{c}}{2}$

and $\mathbf{OE} = \dfrac{3\mathbf{a}+\mathbf{b}+2\mathbf{c}}{2}$

$\therefore$ $\mathbf{OE}\cdot\mathbf{CD} = 0$

$\Rightarrow$ $\dfrac{1}{4}(3\mathbf{a}+\mathbf{b}+2\mathbf{c})\cdot(\mathbf{a}+\mathbf{b}-2\mathbf{c}) = 0$

$$3|\mathbf{a}|^2 + |\mathbf{b}|^2 - 4|\mathbf{c}|^2 + 4\mathbf{a}\cdot\mathbf{b} - 4\mathbf{a}\cdot\mathbf{c} = 0$$

$\Rightarrow$ $\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{c}$

$\Rightarrow$ $3|\mathbf{a}|^2 + |\mathbf{b}|^2 - 4|\mathbf{c}|^2 = 0$

$\Rightarrow$ $|\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|^2 + |\mathbf{c}| - 2\mathbf{a}\cdot\mathbf{c}$ ($\because |\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}|$)

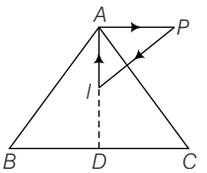
$\Rightarrow$ $|\mathbf{a}-\mathbf{b}|^2 = |\mathbf{a}-\mathbf{c}|^2$

$\Rightarrow$ $|\mathbf{AB}|^2 = |\mathbf{AC}|^2$

$\Rightarrow$ $|\mathbf{AB}| = |\mathbf{AC}|$

● ***Ex. 139*** *Let I be the incentre of ΔABC. Using vectors prove that for any point P $a(PA)^2 + b(PB)^2 + c(PC)^2 = a(IA)^2 + b(IB)^2 + c(IC)^2 + (a+b+c)(IP)^2$ where a, b and c have usual meanings.*

Sol. We have, $\mathbf{IP} + \mathbf{IA} = \mathbf{PA}$



$\Rightarrow$ $|\mathbf{PA}|^2 = |\mathbf{PI}|^2 + |\mathbf{IA}|^2 + 2\mathbf{PI}\cdot\mathbf{IA}$

$$a|\mathbf{PA}|^2 = a|\mathbf{PI}|^2 + a|\mathbf{IA}|^2 + 2\mathbf{PI}\cdot(a\mathbf{IA}) \quad \ldots(i)$$

Similarly, $b|\mathbf{PB}|^2 = b|\mathbf{PI}|^2 + b|\mathbf{IB}|^2 + 2\mathbf{PI}\cdot(b\,\mathbf{IB})$ ...(ii)

$$c|\mathbf{PC}|^2 = c|\mathbf{PI}|^2 + c|\mathbf{IC}|^2 + 2\mathbf{PI}\cdot(c\,\mathbf{IC})$$

On adding Eqs. (i), (ii) and (iii), we get

$$a|\mathbf{PA}|^2 + b|\mathbf{PB}|^2 + c|\mathbf{PC}|^2 = (a+b+c)|\mathbf{PI}|^2 + a|\mathbf{IA}|^2 + b|\mathbf{IB}|^2 + c|\mathbf{IC}|^2 + 2\mathbf{PI}\cdot|a\mathbf{IA} + b\mathbf{IB} + c\mathbf{IC}|^2$$

$\Rightarrow$ $a|\mathbf{PA}|^2 + b|\mathbf{PB}|^2 + c|\mathbf{PC}|^2 = (a+b+c)|\mathbf{PI}|^2 + a|\mathbf{IA}|^2 + b|\mathbf{IB}|^2 + c|\mathbf{IC}|^2$

($\because$ $a\,\mathbf{IA} + b\,\mathbf{IB} + c\,\mathbf{IC} = 0$) shown as, since D be point of intersection of AI with side BC, we have $BD : DC = c : b$ and

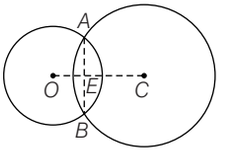
$$\mathbf{AI} : \mathbf{ID} = b + c : a$$

$\Rightarrow$ $\mathbf{ID} = \dfrac{c\,\mathbf{IC} + b\,\mathbf{IB}}{b+c}$ and $a\,\mathbf{AI} = (\mathbf{b}+\mathbf{c})\,\mathbf{ID}$

$a\,\mathbf{AI} = c\,\mathbf{IC} + b\,\mathbf{IB} \Rightarrow a\,\mathbf{IA} + b\,\mathbf{IB} + c\,\mathbf{IC} = 0$

● ***Ex. 140*** *If two circles intersect, prove by using vector method, that the line joining their centres is perpendicular to their common chord.*

Sol. Let O be the centre of the first circle and C be the centre of second. Let a and b be the radii of the two circles. Position vector of C is $\mathbf{c}$ and AB be point of intersection of two circles.



If $\mathbf{r}$ is the position vector of A.

$\Rightarrow$ $\mathbf{CA} = \mathbf{OA} - \mathbf{OC} = \mathbf{r} - \mathbf{c}$ ...(i)

($\because$ $\mathbf{OA} = \mathbf{r}$ and $\mathbf{OC} = \mathbf{c}$)

Also, $\mathbf{r}\cdot\mathbf{r} = a^2$ and $(\mathbf{r}-\mathbf{c})\cdot(\mathbf{r}-\mathbf{c}) = b^2$ ...(ii)

Hence, at the point of intersection of two circle

$$a^2 - 2\mathbf{r}\cdot\mathbf{c} + |\mathbf{c}|^2 = b^2 \Rightarrow \mathbf{r}\cdot\mathbf{c} = \frac{1}{2}[b^2 - a^2 - |\mathbf{c}|^2]$$

If E is the point of intersection of OC and AB, then

$$\mathbf{OA} = \mathbf{OE} + \mathbf{EA} = \lambda c + k_1\mathbf{AB}$$

$$\mathbf{OB} = \mathbf{OE} + \mathbf{EB} = \lambda c + k_2\mathbf{AB}$$

$\Rightarrow$ $2\mathbf{OA}\cdot\mathbf{c} = 2\mathbf{r}\cdot\mathbf{c} = 2\ [\lambda\mathbf{c} + k_1\mathbf{AB}]\cdot\mathbf{c} = \mathbf{a}^2 - \mathbf{b}^2 + |\mathbf{c}|^2$

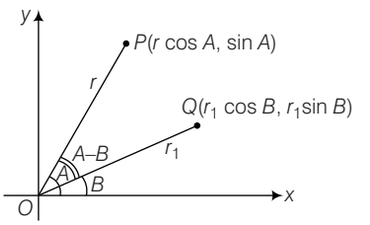
and $2\mathbf{OB}\cdot\mathbf{c} = 2\mathbf{r}\cdot\mathbf{c} = 2\{\lambda\mathbf{c} + k_2\mathbf{AB}\}\cdot\mathbf{c} = \mathbf{a}^2 - \mathbf{b}^2 + |\mathbf{c}|^2$

$\Rightarrow$ $2[\lambda\mathbf{c} - k_1\mathbf{AB}]\cdot\mathbf{c} = 2\ [\lambda\mathbf{c} + k_2\mathbf{AB}]\ \mathbf{c} \Rightarrow \mathbf{AB}\cdot\mathbf{c} = 0$

Hence, AB is perpendicular to OC.

● ***Ex. 141*** *Using vector method prove that $\cos(A-B) = \cos A\cos B + \sin A\sin B$.*

Sol. Let OX and OY be two lines perpendicular to each other and $\angle POX = A$, $\angle QOX = B$. So that, $\angle POQ = A - B$ shown as,



Let $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ denote unit vectors along OX and OY so that,

$$\hat{\mathbf{i}}\cdot\hat{\mathbf{i}}=\hat{\mathbf{j}}\cdot\hat{\mathbf{j}}=1 \text{ and } \hat{\mathbf{i}}\cdot\hat{\mathbf{j}}=\hat{\mathbf{j}}\cdot\hat{\mathbf{i}}=0$$

Also, let $OP=r$ and $OQ=r_1$

$\therefore P(r\cos A, r\sin A)$ and $Q(r_1\cos B, r_1\sin B)$

$$\therefore \quad \mathbf{OP}=(r\cos A)\,\hat{\mathbf{i}}+(r\sin A)\,\hat{\mathbf{j}} \qquad ...(i)$$

and $\mathbf{OQ}=(r_1\cos B)\,\hat{\mathbf{i}}+(r_1\sin B)\,\hat{\mathbf{j}}$

By definition

$$\mathbf{OP}\cdot\mathbf{OQ}=|\mathbf{OP}|\,|\mathbf{OQ}|\cos\angle POQ=r_1 r\cos(A-B)$$

$$\therefore \quad \mathbf{OP}\cdot\mathbf{OQ}=rr_1\cos(A-B) \qquad ...(ii)$$

Also, from Eq. (i)

$$\mathbf{OP}\cdot\mathbf{OQ}=rr_1\cos A\cos B+rr_1\sin A\sin B$$

$$=rr_1(\cos A\cos B+\sin A\sin B) \qquad ...(iii)$$

From Eqs. (ii) and (iii), we get

$rr_1\cos(A-B)=rr_1(\cos A\cos B+\sin A\sin B)$

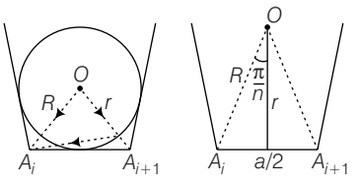
$\Rightarrow \cos(A-B)=\cos A\cos B+\sin A\sin B$

● ***Ex. 142*** *A circle is inscribed in an n-sided regular polygon $A_1, A_2\ldots, A_n$ having each side a unit for any arbitrary point P on the circle, prove that*

$$\sum_{i=1}^{n}(PA_i)^2=n\frac{a^2}{4}\left(\frac{1+\cos^2\pi/n}{\sin^2\pi/n}\right)$$

Sol. Let the centre of the incircle be the reference point.

Then, $\mathbf{PA}_i=\mathbf{OA}_i-\mathbf{OP}$



$$\mathbf{PA}_i\cdot\mathbf{PA}_i=(\mathbf{OA}_i-\mathbf{OP})\cdot(\mathbf{OA}_i-\mathbf{OP})$$

$$(\mathbf{PA}_i)^2=(|\mathbf{OA}_i|)^2+(|\mathbf{OP}|)^2-2\mathbf{OA}_i\cdot\mathbf{OP}$$

$$\sum_{i=1}^{n}(\mathbf{PA}_i)^2=\sum_{i=1}^{n}(|\mathbf{OA}_i|)^2+(|\mathbf{OP}|^2)-2\mathbf{OA}_i\cdot\mathbf{OP}$$

$$=nR^2+nr^2-2\mathbf{OP}\cdot\sum_{i=1}^{n}\mathbf{OA}_i \qquad ...(i)$$

$$=n(R^2+r^2)-2\,OP\cdot(0)$$

Now, $$R=\frac{a}{2}\operatorname{cosec}\frac{\pi}{n}, r=\frac{a}{2}\cot\frac{\pi}{n} \qquad ...(ii)$$

$$\therefore \quad R^2+r^2=\frac{a^2}{4}\left(\operatorname{cosec}^2\frac{\pi}{n}+\cot^2\frac{\pi}{n}\right)$$

$$=\frac{a^2}{4}\left(\frac{1+\cos^2\pi/n}{\sin^2\pi/n}\right) \qquad ...(iii)$$

$\therefore$ From Eqs. (i) and (iii), we get

$$\Rightarrow \quad \sum_{i=1}^{n}(PA_i)^2=n\frac{a^2}{4}\left(\frac{1+\cos^2\pi/n}{\sin^2\pi/n}\right)$$

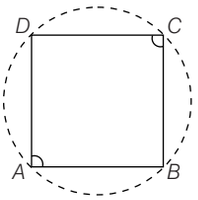
● ***Ex. 143*** *If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ are the position vector of the vertices of a cyclic quadrilateral ABCD, prove that*

$$\frac{\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{d}+\mathbf{d}\times\mathbf{a}|}{(\mathbf{b}-\mathbf{a})\cdot(\mathbf{d}-\mathbf{a})}+\frac{|\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{d}+\mathbf{d}\times\mathbf{b}|}{(\mathbf{b}-\mathbf{c})\cdot(\mathbf{d}\cdot\mathbf{a})}=0$$

Sol. Consider, $\dfrac{|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{d}+\mathbf{d}\times\mathbf{a}|}{(\mathbf{b}-\mathbf{a})\cdot(\mathbf{d}-\mathbf{a})}$

$$=\frac{(\mathbf{a}-\mathbf{d})\times(\mathbf{b}-\mathbf{a})}{(\mathbf{b}-\mathbf{a})\cdot(\mathbf{d}-\mathbf{a})}=\frac{|\mathbf{a}-\mathbf{d}|\,|\mathbf{b}-\mathbf{a}|\sin A}{|\mathbf{b}-\mathbf{c}|\,|\mathbf{d}-\mathbf{c}|\cos A}$$

$$=\tan A \qquad ...(i)$$



Again, $\dfrac{|\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{d}+\mathbf{d}\times\mathbf{b}|}{(\mathbf{b}-\mathbf{c})\cdot(\mathbf{d}-\mathbf{c})}=\dfrac{|(\mathbf{b}-\mathbf{c})\times(\mathbf{c}-\mathbf{d})|}{(\mathbf{b}-\mathbf{c})\cdot(\mathbf{d}-\mathbf{c})}$

$$=\frac{|\mathbf{b}-\mathbf{c}|\,|\mathbf{c}-\mathbf{d}|\sin C}{|\mathbf{b}-\mathbf{c}|\,|\mathbf{d}-\mathbf{c}|\cos C}=\tan C$$

As cyclic quadrilateral

$$A=180^\circ-C$$

$$\Rightarrow \quad \tan A=\tan(180^\circ-C)$$

$$\Rightarrow \quad \tan A+\tan C=0$$

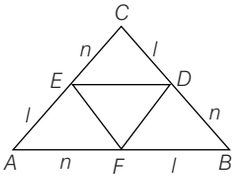
$$\Rightarrow \quad \frac{|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{d}+\mathbf{d}\times\mathbf{a}|}{(\mathbf{b}-\mathbf{a})\cdot(\mathbf{d}-\mathbf{a})}+\frac{|\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{d}+\mathbf{d}\times\mathbf{b}|}{(\mathbf{b}-\mathbf{c})\cdot(\mathbf{d}-\mathbf{c})}=0$$

● ***Ex. 144*** *In ΔABC, points D, E and F are taken on the sides BC, CA and AB, respectively such that*

$$\frac{BD}{DC}=\frac{CE}{EA}=\frac{AF}{AB}=n.$$

Prove that, $\Delta DEF=\dfrac{n^2-n+1}{(n+1)^2}\Delta ABC.$

Sol. Take A is the origin and let the position vectors of the points B and C be $\mathbf{b}$ and $\mathbf{c}$, respectively.



$\therefore$The position vector of D, E and F are

$$\frac{n\mathbf{c}+\mathbf{b}}{\mathbf{n}+1}, \frac{\mathbf{c}}{n+1}, \frac{n\mathbf{b}}{n+1}$$

$$\mathbf{FD}=\mathbf{AD}-\mathbf{AF}=\frac{n\mathbf{c}+\mathbf{b}-n\mathbf{b}}{n+1}=\frac{n\mathbf{c}+(1-n)\,\mathbf{b}}{n+1}$$

and $$\mathbf{EF}=\mathbf{AF}-\mathbf{AE}=\frac{n\mathbf{b}-\mathbf{c}}{n+1}$$

Now, vector area of $\Delta ABC = \frac{1}{2}(\mathbf{b} \times \mathbf{c})$ and vector area of ΔDEF

$$= \frac{1}{2}(\mathbf{FD} \times \mathbf{FE})$$

$$= \frac{1}{2(n+1)^2}\{(n\mathbf{b} - \mathbf{c}) \times n\mathbf{c} + (1-n)\mathbf{b}\}$$

$$= \frac{1}{2(n+1)^2}\{n^2\mathbf{b} \times \mathbf{c} + (1-n)\,\mathbf{b} \times \mathbf{c}\}$$

$$= \frac{1}{2(n+1)^2}[(n^2 - n + 1)(\mathbf{b} \times \mathbf{c})]$$

$$= \frac{n^2 - n + 1}{2(n+1)^2}\,\Delta ABC$$

$\therefore$ Area of $\Delta\, DEF = \frac{n^2 - n + 1}{2(n+1)^2}$ area of $\Delta\, ABC$

● ***Ex. 145*** *Let the area of a given Δ ABC be Δ. Points A_1, B_1 and C_1 are the mid-points of the sides BC, CA and AB, respectively. Point A_2 is the mid-point of CA, lines C_1A_1 and $A\,A_2$ meet the median BB_1 at E and D, respectively. If Δ is the area of the quadrilateral A_1A_2DE, using vectors prove that* $\frac{\Delta_1}{\Delta} = \frac{11}{56}$.

Sol. Let the position verctor of A, B and C be **a**, **b** and **c**, respectively.

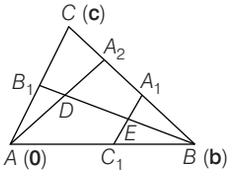
We have, $\mathbf{AC}_1 = \frac{b}{2}$, $\mathbf{AB}_1 = \frac{c}{2}$

$$\mathbf{AA}_2 = \frac{\mathbf{b} + \mathbf{c}}{2},\ \mathbf{AA}_2 = \frac{3\mathbf{c} + \mathbf{b}}{4}$$

Equation of the lines BB_1, AA_2 and C_1A_1 are

$$\mathbf{r} = \mathbf{b} + \lambda_1\left(\frac{\mathbf{c}}{2} - \mathbf{b}\right)$$

$$\mathbf{r} = \lambda_2\frac{3\mathbf{c} + \mathbf{b}}{4} \quad \text{and} \quad \mathbf{r} = \frac{\mathbf{b}}{2} + \lambda_3\left(\frac{\mathbf{c}}{2}\right)$$



For the point D, we have

$$\mathbf{b} + \lambda_1\left(\frac{\mathbf{c}}{2} - \mathbf{b}\right) = \lambda_2\left(\frac{3\mathbf{c} + \mathbf{b}}{4}\right)$$

$$\mathbf{b}\left(1 - \lambda_1 - \frac{\lambda_2}{4}\right) + \frac{\mathbf{c}}{4}(2\lambda_1 - 3\lambda_2) = 0$$

$$\Rightarrow \qquad \lambda_1 = \frac{6}{7},\ \lambda_2 = \frac{4}{7}$$

$$\therefore \qquad \mathbf{AD} = \frac{3\mathbf{c} + \mathbf{b}}{7}$$

For the point E, we have $\mathbf{b} + \lambda_1\left(\frac{\mathbf{c}}{2} - \mathbf{b}\right) = \frac{\mathbf{b}}{2} + \frac{\lambda_3}{2}\,\mathbf{c}$

$$\Rightarrow \qquad \mathbf{b}\left(\frac{1}{2} - \lambda_1\right) + \frac{\mathbf{c}}{2}(\lambda_1 - \lambda_3) = 0$$

$$\Rightarrow \qquad \lambda_1 = \lambda_3 = \frac{1}{2}$$

$$\therefore \qquad \mathbf{AE} = \frac{2\mathbf{b} + \mathbf{c}}{4}$$

Now, $\mathbf{EA}_2 = \frac{3\mathbf{c} + \mathbf{b} - 2\mathbf{b} - \mathbf{c}}{4} = \frac{2\mathbf{c} - \mathbf{b}}{7}$

$$\mathbf{DA}_1 = \frac{\mathbf{b} + \mathbf{c}}{2} - \frac{3\mathbf{c} + \mathbf{b}}{7} = \frac{5\mathbf{b} + \mathbf{c}}{14}$$

Area of quadrilateral $EA_1A_2\,D = \frac{1}{2}|\mathbf{EA}_2 \times \mathbf{DA}_1|$

$$= \frac{1}{112}|(2\mathbf{c} - \mathbf{b}) \times (5\mathbf{b} + \mathbf{c})|$$

$$= \frac{1}{112}|10\mathbf{c} \times \mathbf{b} - \mathbf{b} \times \mathbf{c}|$$

$$= \frac{11}{112}|\mathbf{c} \times \mathbf{b}| = \frac{11}{56} \cdot \frac{1}{2}|\mathbf{c} \times \mathbf{b}| = \frac{11}{56}$$

Thus, required ratio is $\frac{11}{56}$.

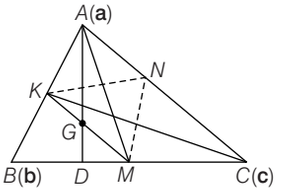
● ***Ex. 146*** *Let ABC be an acute angled triangle with centriod G and the internal bisectors of angles A, B and C meets BC, CA and AB in M, N and K respectively using vectors, prove that if G lies on one of the sides of Δ MNK, then one of the altitudes of Δ ABC equals the sum of other two.*

Sol. Let G be on MK.

Let the position vectors of B and C with reference to origin A be **b** and **c**, respectively.

$\therefore$ $BC = a$, $CA = b$ and $AB = c$

Now, $\frac{BM}{MC} = \frac{c}{b}$



$\therefore$ Position vector of $M = \frac{b\mathbf{b} + c\mathbf{c}}{b + c}$

Similarly, position vector of $K = \frac{b\,\mathbf{b}}{a + b}$

$$\mathbf{MK} = \mathbf{PV} \text{ of } K - \mathbf{PV} \text{ of } M = \frac{b\mathbf{b}}{a+b} - \frac{b\mathbf{b} + c\mathbf{c}}{b + c}$$

PV of $G = \frac{\mathbf{b} + \mathbf{c}}{3}$

$$\mathbf{GK} = \mathbf{PV} \text{ of } \mathbf{K} - \mathbf{PV} \text{ of } \boldsymbol{G} = \frac{b\mathbf{b}}{a+b} - \frac{\mathbf{b} + \mathbf{c}}{3}$$

Since, G lies on MK, $\mathbf{MK} \times \mathbf{GK} = 0$

$$\Rightarrow \left(\frac{b\mathbf{b}}{a+b} - \frac{b\mathbf{b}+c\mathbf{c}}{b+c}\right) \times \left(\frac{b\mathbf{b}}{a+b} - \frac{\mathbf{b}+\mathbf{c}}{3}\right) = 0$$

$$\Rightarrow -\frac{b\mathbf{b}}{a+b} \times \frac{\mathbf{b}+c}{3} - \frac{b\mathbf{b}\times c\mathbf{c}}{a+c} \times \frac{b\mathbf{b}}{a+b} + \frac{b\mathbf{b}+c\mathbf{c}}{b+c} \times \frac{\mathbf{b}+\mathbf{c}}{3} = 0$$

$$\Rightarrow -\frac{b(\mathbf{b}\times\mathbf{c})}{3(a+b)} - \frac{\mathbf{bc}\,(\mathbf{c}\times\mathbf{b})}{(b+c)(a+b)} + \frac{\mathbf{b}\,(\mathbf{b}\times\mathbf{c})}{3(b+c)} + \frac{\mathbf{b}\,(\mathbf{c}\times\mathbf{b})}{3} = 0$$

$$\left[\frac{b}{3(a+b)} - \frac{bc}{(b+c)(a+b)} + \frac{b}{3(b+c)} + \frac{c}{3(b+c)}\right](\mathbf{c}\times\mathbf{b}) = 0$$

$$\Rightarrow \frac{b}{3(a+b)} - \frac{bc}{(b+c)(a+b)} - \frac{b}{3(b+c)} + \frac{c}{3(b+c)} = 0$$

$$\Rightarrow b(b+c) - 3bc - b(a+b) + c(a+b) = 0$$

$$\Rightarrow b^2 + bc - 3bc - ab - b^2 - ac + bc$$

$$\Rightarrow ac = ab + bc$$

$$\Rightarrow \frac{1}{b} = \frac{1}{c} + \frac{1}{a}$$

$$\Rightarrow \frac{2\Delta}{b} = \frac{2\Delta}{c} + \frac{2\Delta}{a}$$ (where, Δ denotes area of ΔABC)

$\Rightarrow P_b = P_a + P_c$ denotes the altitudes drawn through A, B and C, respectively.

Aliter

$$g = \alpha\,\mathbf{m} + (1-\alpha)\,\mathbf{k}$$

$$\therefore \quad \frac{\mathbf{b}+\mathbf{c}}{3} = \frac{\alpha\,(b\,\mathbf{b}+c\mathbf{c})}{b+c} + \frac{(1-\alpha)\,b\,\mathbf{b}}{a+b}$$

On comparing coefficients of $\mathbf{b}$ and $\mathbf{c}$, we get

$$\frac{1}{3} = \frac{\alpha c}{b+c}$$

$$\Rightarrow \alpha = \frac{b+c}{3c}$$

and $\frac{\alpha b}{b+c} + \frac{(1-\alpha)\,b}{a+b} = \frac{1}{3}$ substituting α, we get

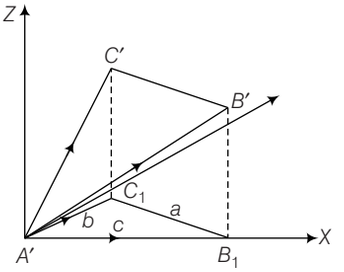
$$ca = ab + bc$$

$$\Rightarrow \frac{1}{b} = \frac{1}{c} + \frac{1}{a}$$

● ***Ex. 147*** *Three poles of height x, $x+y$ and $x+z$ are posted at the vertices A, B and C of a triangular park of sides a, b and c, respectively. A plane sheet is mounted at the tops of the poles. If the plane of the sheet is inclined at an angle θ to the horizontal plane, prove using vector*

$$\theta = \tan^{-1}\left\{\frac{\sqrt{\frac{y^2}{c^2} + \frac{z^2}{b^2} - \frac{2yz}{bc}\cos A}}{\sin A}\right\}$$

Sol. Let A', B' and C' be the tops of the poles at A, B and C, respectively. Through A' draw a $\Delta A'B_1C_1$ congruent to ΔABC and parallel to the horizontal plane of the park. Take $A'B_1$ as the X-axis and a line perpendicular to it as the Y-axis (in the plane of $\Delta A'B_1C_1$) and a line through A' and perpendicular to the plane $A'B_1C_1$ as the Z-axis.



If $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are the unit vectors along these axes, then

$$\mathbf{A'B_1} = c\,\hat{\mathbf{i}}$$
$$\mathbf{A'C_1} = (b\cos A)\hat{\mathbf{i}} + (b\sin A)\hat{\mathbf{j}}$$
$$\mathbf{A'B'} = c\hat{\mathbf{i}} + y\,\hat{\mathbf{k}}$$
$$\mathbf{A'C'} = (b\cos A)\hat{\mathbf{i}} + (b\sin A)\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

Since, the planes $A'B'C'$ is inclined at an angle θ to the plane $A'B_1C_1$, angle between the normals to the planes is $(\pi - \theta)$.

Obviously, the unit vector normal to the plane $A'B_1C_1$ is $\mathbf{k}$ and the normal vector to $A'B'C'$ is

$$[(b\cos A)\,\hat{\mathbf{i}} + (b\sin A)\,\hat{\mathbf{j}} + (z)\,\hat{\mathbf{k}}] \times (c\,\hat{\mathbf{i}} + y\,\hat{\mathbf{k}})$$
$$= (yb\sin A)\,\hat{\mathbf{i}} - (yb\cos A - zc)\hat{\mathbf{j}} - (bc\sin A)\hat{\mathbf{k}}$$

$$\cos(\pi - \theta) = \frac{\{(yb\sin A)\hat{\mathbf{i}} - (yb\cos A - zc)\hat{\mathbf{j}} - (bc\sin A)\hat{\mathbf{k}}\}\cdot\hat{\mathbf{k}}}{\sqrt{y^2b^2\sin^2 A + y^2b^2\cos^2 A + z^2c^2 - 2yzbc\cos A + b^2c^2\sin^2 A}}$$

$$\Rightarrow \cos\theta = \frac{\sin A}{\sqrt{\frac{y^2}{c^2} + \frac{z^2}{b^2} - \frac{2yz}{bc}\cos A + \sin^2 A}}$$

$$\Rightarrow \tan\theta = \frac{\sqrt{\frac{y^2}{c^2} + \frac{z^2}{b^2} - \frac{2yz}{bc}\cos A}}{\sin A}$$

● ***Ex. 148*** *If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are three vectors such that $\mathbf{a}\times\mathbf{b} = \mathbf{c}$, $\mathbf{b}\times\mathbf{c} = \mathbf{a}$ and $\mathbf{c}\times\mathbf{a} = \mathbf{b}$, then prove that $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}|$*

Sol. Here, $\mathbf{a}\times\mathbf{b} = \mathbf{c}$ (given)

$$\Rightarrow (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \mathbf{c}\cdot\mathbf{c}$$

$$\Rightarrow [\mathbf{a}\,\mathbf{b}\,\mathbf{c}] = |\mathbf{c}|^2 \quad \ldots(i)$$

Also, $\mathbf{b}\times\mathbf{c} = \mathbf{a}$ (given)

$$(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a} = \mathbf{a}\cdot\mathbf{a}$$

$$\Rightarrow [\mathbf{b}\,\mathbf{c}\,\mathbf{a}] = |\mathbf{a}|^2 \quad \ldots(ii)$$

and $\mathbf{c}\times\mathbf{a} = \mathbf{b}$

$$\Rightarrow (\mathbf{c}\times\mathbf{a})\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{b}$$ (given)

$$\Rightarrow [\mathbf{c}\,\mathbf{a}\,\mathbf{b}] = |\mathbf{b}|^2$$ (given)

Since, $[\mathbf{a}\,\mathbf{b}\,\mathbf{c}] = [\mathbf{b}\,\mathbf{c}\,\mathbf{a}] = [\mathbf{c}\,\mathbf{a}\,\mathbf{b}]$ ...(iii)

∴From Eqs. (i), (ii) and (iii), we get

$$|\mathbf{c}|^2 = |\mathbf{a}|^2 = |\mathbf{b}|^2 \Rightarrow |\mathbf{c}| = |\mathbf{a}| = |\mathbf{b}|$$

● ***Ex. 149*** *If* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *and* $\mathbf{d}$ *are four coplanar points, then show that* $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = [\mathbf{b}\ \mathbf{c}\ \mathbf{d}] + [\mathbf{c}\ \mathbf{a}\ \mathbf{d}] + [\mathbf{a}\ \mathbf{b}\ \mathbf{d}]$

Sol. Since, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are coplanar points.

We have, $\mathbf{b}-\mathbf{a}$, $\mathbf{c}-\mathbf{a}$ and $\mathbf{d}-\mathbf{a}$ are coplanar.

$$\Rightarrow \quad [\mathbf{b}-\mathbf{a}\ \mathbf{c}-\mathbf{a}\ \mathbf{d}-\mathbf{a}] = 0$$

$$\Rightarrow \quad \{(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})\}\cdot(\mathbf{d}-\mathbf{a}) = 0$$

$$\Rightarrow \quad (\mathbf{b}\times\mathbf{c})\cdot\mathbf{d} - (\mathbf{b}\times\mathbf{c})\cdot\mathbf{a} - (\mathbf{b}\times\mathbf{c})\cdot\mathbf{a} = 0$$

$$-(\mathbf{a}\times\mathbf{c})\cdot\mathbf{d} + (\mathbf{a}\times\mathbf{c})\cdot\mathbf{a} = 0$$

$$\Rightarrow \quad [\mathbf{b}\ \mathbf{c}\ \mathbf{d}] - [\mathbf{b}\ \mathbf{c}\ \mathbf{a}] - [\mathbf{b}\ \mathbf{a}\ \mathbf{d}] - [\mathbf{a}\ \mathbf{c}\ \mathbf{d}] = 0$$

$$\Rightarrow \quad [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = [\mathbf{b}\ \mathbf{c}\ \mathbf{d}] + [\mathbf{a}\ \mathbf{b}\ \mathbf{d}] + [\mathbf{c}\ \mathbf{a}\ \mathbf{d}]$$

● ***Ex. 150*** *Let* $\hat{\mathbf{u}}$ *and* $\hat{\mathbf{v}}$ *be unit vectors. If* $\mathbf{w}$ *is a vector such that* $\mathbf{w}+(\mathbf{w}\times\mathbf{u}) = \mathbf{v}$, *then prove that* $|(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}| \le \frac{1}{2}$ *and that the equality holds if and only if* $\mathbf{u}$ *is perpendicular to* $\mathbf{v}$.

Sol. $\mathbf{w}+(\mathbf{w}\times\mathbf{u}) = \mathbf{v}$

$$\Rightarrow \quad \mathbf{w}\times\mathbf{u} = v - \mathbf{w} \qquad \ldots(\text{i})$$

$$\Rightarrow \quad (\mathbf{w}\times\mathbf{u})^2 = v^2 + w^2 - 2\mathbf{v}\cdot\mathbf{w}$$

$$\Rightarrow \quad 2\mathbf{v}\cdot\mathbf{w} = 1 + w^2 - (\mathbf{u}\times\mathbf{w})^2 \qquad \ldots(\text{ii})$$

Also, taking dot product of Eq. (i) with $\mathbf{v}$, we get

$$\mathbf{w}\cdot\mathbf{v} + (\mathbf{w}\times\mathbf{u})\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{v}$$

$$\Rightarrow \quad \mathbf{v}\cdot(\mathbf{w}\times\mathbf{u}) = 1 - \mathbf{w}\cdot\mathbf{v} \qquad \ldots(\text{iii})\ (\because \mathbf{v}\cdot\mathbf{v} = |\mathbf{v}|^2 = 1)$$

Now, $\mathbf{v}\cdot(\mathbf{w}\times\mathbf{u}) = 1 - \frac{1}{2}[1 + w^2 - (\mathbf{u}\times\mathbf{w})^2]$ [using Eqs. (ii) and (iii)]

$$= \frac{1}{2} - \frac{w^2}{2} + \frac{(\mathbf{u}\times\mathbf{w})^2}{2} \qquad (\because 0 \le \cos^2\theta \le 1)$$

$$= \frac{1}{2}(1 - w^2 + w^2\sin^2\theta) \qquad \ldots(\text{iv})$$

As we know, $0 \le w^2\cos^2\theta \le w^2$

$$\therefore \quad \frac{1}{2} \ge \frac{1 - w^2\cos^2\theta}{2} \ge \frac{1 - w^2}{2}$$

$$\Rightarrow \quad \frac{1 - w^2\cos^2\theta}{2} \le \frac{1}{2} \qquad \ldots(\text{v})$$

From Eqs. (iv) and (v), we get $|\mathbf{v}\cdot(\mathbf{w}\times\mathbf{u})| \le \frac{1}{2}$

Equality holds only when $\cos^2\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

i.e., $\mathbf{u} \perp \mathbf{w} = 0 \Rightarrow \mathbf{u}\cdot\mathbf{w} = 0$

$$\mathbf{w} + (\mathbf{w}\times\mathbf{u}) = \mathbf{v}$$

$$\mathbf{u}\cdot\mathbf{w} + \mathbf{u}\cdot(\mathbf{w}\times\mathbf{u}) = \mathbf{u}\cdot\mathbf{v}$$

$$0 + 0 = \mathbf{u}\cdot\mathbf{v} \ \Rightarrow \ \mathbf{u}\cdot\mathbf{v} = 0 \ \Rightarrow \ \mathbf{u} \perp \mathbf{v}$$

● ***Ex. 151*** *Prove that*

$$\mathbf{R} + \frac{[\mathbf{R}.(\beta\times(\beta\times\alpha)]\alpha}{|\alpha\times\beta|^2} + \frac{\mathbf{R}.(\alpha\times(\alpha\times\beta)]\beta}{|\alpha\times\beta|^2} = \frac{[\mathbf{R}\,\alpha\,\beta\,](\alpha\times\beta)}{|\alpha\times\beta|^2}$$

Sol. α, β and $\alpha\times\beta$ are three non-coplanar vectors. Any vector $\mathbf{R}$ can be represented as a linear combination of these vectors.

$$\Rightarrow \quad \mathbf{R} = k_1\alpha + k_2\beta + k_3(\alpha\times\beta) \qquad \ldots(\text{i})$$

$$\Rightarrow \quad \mathbf{R}.(\alpha\times\beta) = k_3(\alpha\times\beta).(\alpha\times\beta) = k_3(\alpha\times\beta)^2$$

$$\Rightarrow \quad k_3 = \frac{\mathbf{R}.(\alpha\times\beta)}{|\alpha\times\beta|^2} = \frac{[\mathbf{R}\alpha\beta]}{|\alpha\times\beta|^2}$$

On taking dot product of Eq. (i) with $\alpha\times(\alpha\times\beta)$

$$\Rightarrow \quad \mathbf{R}.\alpha\times(\alpha\times\beta) = k_2(\alpha\times(\alpha\times\beta)).\beta$$

$$k_2[(\alpha.\beta)\alpha - (\alpha.\alpha)\beta].\beta = k_2[(\alpha.\beta)^2 - \alpha^2\beta^2]$$

$$= -k_2|\alpha\times\beta|^2$$

$$\Rightarrow \quad k_2 = \frac{-[\mathbf{R}.(\alpha\times(\alpha\times\beta))]}{|\alpha\times\beta|^2}$$

Similarly, $k_1 = -\frac{[\mathbf{R}.(\beta\times(\beta\times\alpha))]}{|\alpha\times\beta|^2}$

$$\Rightarrow \quad \mathbf{R} = \frac{-[\mathbf{R}.(\beta\times(\beta\times\alpha))]\alpha}{|\alpha\times\beta|^2} - \frac{[\mathbf{R}.(\alpha\times(\alpha\times\beta))]\beta}{|\alpha\times\beta|^2} + \frac{[\mathbf{R}.(\alpha\times\beta)](\alpha\times\beta)}{|\alpha\times\beta|^2}$$

$$\Rightarrow \quad \mathbf{R} + \frac{[\mathbf{R}.(\beta\times(\beta\times\alpha))]\alpha}{|\alpha\times\beta|^2} + \frac{[\mathbf{R}.(\alpha\times(\alpha\times\beta))]\beta}{|\alpha\times\beta|^2} = \frac{[\mathbf{R}.(\alpha\times\beta)](\alpha\times\beta)}{|\alpha\times\beta|^2}$$

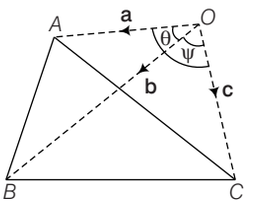
● ***Ex. 152*** *If* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{c}$ *represents the sides of tetrahedron and* θ *be an angle between* $\mathbf{a}$ *and* $\mathbf{b}$, ϕ *be an angle between* $\mathbf{a}$ *and* $\mathbf{c}$, ψ *be an angle between* $\mathbf{b}$ *and* $\mathbf{c}$, *then prove that the volume of the tetrahedron is given by*

$$v^2 = \frac{a^2b^2c^2}{36}\begin{vmatrix} 1 & \cos\theta & \cos\theta \\ \cos\theta & 1 & \cos\psi \\ \cos\phi & \cos\psi & 1 \end{vmatrix}$$

Sol. *OABC* represent a tetrahedron, where

$\mathbf{OA} = \mathbf{a}, \mathbf{OB} = \mathbf{b}, \mathbf{OC} = \mathbf{c}$ relative to O

Volume of tetrahedron $(v) = \frac{1}{6}[\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]$



Also, $v^2 = \frac{1}{36}[\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]^2 = \frac{1}{36}\begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix}$

$$= \frac{1}{36}\begin{vmatrix} a^2 & ab\cos\theta & ac\cos\phi \\ ab\cos\theta & b^2 & bc\cos\psi \\ ac\cos\phi & bc\cos\psi & c^2 \end{vmatrix}$$

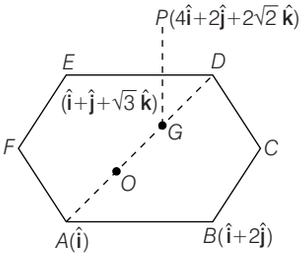
$$= \frac{1}{36}a^2b^2c^2\begin{vmatrix} 1 & \cos\theta & \cos\phi \\ \cos\theta & 1 & \cos\psi \\ \cos\phi & \cos\psi & 1 \end{vmatrix}$$

● ***Ex. 153*** *A pyramid with vertex at the point P, whose position vector is* $4\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\sqrt{3}\,\hat{\mathbf{k}}$ *has a regular hexagonal base ABCDEF. Position vectors of points A and B are* $\hat{\mathbf{i}}$ *and* $\hat{\mathbf{i}}+2\hat{\mathbf{j}}$*, respectively. Centre of the base has the position vector* $\hat{\mathbf{i}}+\hat{\mathbf{j}}+\sqrt{3}\hat{\mathbf{k}}$*. Altitude drawn from P on the base meets the diagonal AD at point G. Find all possible position vectors of G. It is given that volume of the pyramid is* $6\sqrt{3}$ *cu units.*

Sol. Let the centre of base be (0).

$$\mathbf{AB}=2\hat{\mathbf{j}} \Rightarrow |\mathbf{AB}| = 2$$

$$\Delta OAB = \frac{1}{4}\cdot 4\sqrt{3} = \sqrt{3}$$



$\Rightarrow$ Base are a $= 6\sqrt{3}$ sq unit.

Let height of the pyramid be *h*.

$$\Rightarrow \quad \frac{1}{3}\cdot 6\sqrt{3}h = 6\sqrt{3}$$

$$\Rightarrow \quad h = 3 \text{ units}$$

$$\mathbf{AP} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\sqrt{3}\hat{\mathbf{k}}$$

$$\Rightarrow \quad |\mathbf{AP}| = \sqrt{9+4+12} = 5 \text{ units}$$

$$\Rightarrow \quad \mathbf{AP} = \sqrt{9+4+12} = 5 \text{ units}$$

$$\Rightarrow \quad \mathbf{AG} = \sqrt{25-9} = 4 \text{ units}$$

$$|\mathbf{AG}| = 4 \text{ units}$$

Now, **AQ** and **AO** are collinear.

$$\Rightarrow \quad \mathbf{AG} = \lambda\mathbf{AO}$$

$$\Rightarrow \quad |\mathbf{AG}| = |\lambda||\mathbf{AO}|$$

$$\Rightarrow \quad 2|\lambda| = 4$$

$$\Rightarrow \quad |\lambda| = 2$$

$$\Rightarrow \quad \mathbf{AG} = \pm(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\sqrt{3}\hat{\mathbf{k}})$$

$$\Rightarrow \quad \mathbf{G} = \pm 2(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\sqrt{3}\hat{\mathbf{k}}) + \hat{\mathbf{i}}$$

$$= -(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\sqrt{3}\hat{\mathbf{k}}), -3\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\sqrt{3}\hat{\mathbf{k}}$$

● ***Ex. 154*** *Let* $\hat{\mathbf{a}}, \hat{\mathbf{b}}$ *and* $\hat{\mathbf{c}}$ *be the non-coplanar unit vectors. The angle between* $\hat{\mathbf{b}}$ *and* $\hat{\mathbf{c}}$ *be* α *and angle between* $\hat{\mathbf{c}}$ *and* $\hat{\mathbf{a}}$ *be* β *and between* $\hat{\mathbf{a}}$ *and* $\hat{\mathbf{b}}$ *be* γ*. If* $A(\hat{\mathbf{a}}\cos\alpha, 0)$*,* $B(\hat{\mathbf{b}}\cos\beta, 0)$ *and* $C(\hat{\mathbf{c}}\cos\gamma, 0)$*, then show that in* ΔABC*.*

$$\frac{|\hat{\mathbf{a}}\times(\hat{\mathbf{b}}\times\hat{\mathbf{c}})|}{\sin A} = \frac{|\hat{\mathbf{b}}\times(\hat{\mathbf{c}}\times\hat{\mathbf{a}})|}{\sin B} = \frac{|\hat{\mathbf{c}}\times(\hat{\mathbf{a}}\times\hat{\mathbf{b}})|}{\sin C}$$

$$= \frac{\pi|\hat{\mathbf{a}}(\hat{\mathbf{b}}\times\hat{\mathbf{c}})|}{|\Sigma\sin\alpha\cos\beta\cos\gamma\,\hat{\eta}|}$$

where, $\hat{\eta}_1 = \dfrac{\hat{\mathbf{b}}\times\hat{\mathbf{c}}}{|\hat{\mathbf{b}}\times\hat{\mathbf{c}}|}$, $\hat{\eta}_2 = \dfrac{\hat{\mathbf{c}}\times\hat{\mathbf{a}}}{|\hat{\mathbf{c}}\times\hat{\mathbf{a}}|}$ *and* $\hat{\eta}_3 = \dfrac{\hat{\mathbf{a}}\times\hat{\mathbf{b}}}{|\hat{\mathbf{a}}\times\hat{\mathbf{b}}|}$

Sol. We know from sine rule,

$$\frac{AB}{\sin C} = \frac{AC}{\sin B} = \frac{BC}{\sin A}$$

$$= 2R = \frac{(AB)(BC)(CA)}{2(\Delta ABC)} \qquad \ldots(i)$$

$$BC = |\mathbf{BC}|\,| = |\hat{\mathbf{c}}\cos\gamma - \hat{\mathbf{b}}\cos\beta|$$

$$= |(\hat{\mathbf{a}}\cdot\hat{\mathbf{b}})\hat{\mathbf{c}}\cdot(\hat{\mathbf{c}}\cdot\hat{\mathbf{a}})\hat{\mathbf{b}}\,| = |\hat{\mathbf{a}}\times(\hat{\mathbf{b}}\times\hat{\mathbf{c}})|$$

Similarly, $AC = |\mathbf{AC}| = |\hat{\mathbf{b}}\times(\hat{\mathbf{c}}\times\hat{\mathbf{a}})|$

and $AB = |\mathbf{AB}| = |\hat{\mathbf{c}}\times(\hat{\mathbf{a}}\times\hat{\mathbf{b}})|$

Also, $\Delta ABC = \frac{1}{2}|\mathbf{BC}\times\mathbf{BA}|$

$$= \frac{1}{2}|(\hat{\mathbf{c}}\cos\gamma - \hat{\mathbf{b}}\cos\beta)\times(\hat{\mathbf{a}}\cos\alpha - \hat{\mathbf{b}}\cos\beta)|$$

$$= \frac{1}{2}|(\hat{\mathbf{c}}\times\hat{\mathbf{a}})\cos\alpha\cos\gamma + (\hat{\mathbf{b}}\times\hat{\mathbf{c}})\cos\gamma\cos\beta + (\hat{\mathbf{a}}\times\hat{\mathbf{b}})\cos\beta\cos\alpha|$$

$$= \frac{1}{2}|\hat{\eta}_1\sin\alpha\cos\beta\cos\gamma + \hat{\eta}_2\sin\beta\cos\alpha\cos\gamma + \hat{\eta}_3\sin\gamma\cos\alpha\cos\beta|$$

$\Rightarrow \quad 2\Delta ABC = |\Sigma\hat{\eta}_1\sin\alpha\cos\beta\cos\gamma|$

$\therefore$Eq. (i) reduces to

$$\frac{|\hat{\mathbf{a}}\times(\hat{\mathbf{b}}\times\hat{\mathbf{c}})|}{\sin A} = \frac{|\hat{\mathbf{b}}\times(\hat{\mathbf{c}}\times\hat{\mathbf{a}})|}{\sin B} = \frac{|\hat{\mathbf{c}}\times(\hat{\mathbf{a}}\times\hat{\mathbf{b}})|}{\sin C}$$

$$= \frac{\pi|\hat{\mathbf{a}}\times(\hat{\mathbf{b}}\times\hat{\mathbf{c}})|}{|\Sigma\sin\alpha\cos\beta\cos\gamma\,\hat{\eta}|}$$

● ***Ex. 155*** *Let a and b be given non-zero and non-collinear vectors, such that* $\mathbf{c}\times\mathbf{a} = \mathbf{b}-\mathbf{c}$*. Express c in terms of a, b and* $\mathbf{a}\times\mathbf{b}$

Sol. Let $\mathbf{c} = x_1\mathbf{a} + x_2\mathbf{b} + x_3(\mathbf{a}\times\mathbf{b})$

$$\Rightarrow \quad \mathbf{c}\times\mathbf{a} = x_2(\mathbf{b}\times\mathbf{a}) - x_3\mathbf{a}\times(\mathbf{a}\times\mathbf{b})$$

$$= x_2(\mathbf{b}\times\mathbf{a}) - x_3(\mathbf{a}\cdot\mathbf{b})\mathbf{a} + x_3|\mathbf{a}|^2\mathbf{b}$$

We have been given, $\mathbf{c}\times\mathbf{a} = \mathbf{b}-\mathbf{c}$

$$\Rightarrow \quad \mathbf{b} - x_1\mathbf{a} - x_2\mathbf{b} - x_3(\mathbf{a}\times\mathbf{b}) = -x_2(\mathbf{a}\times\mathbf{b}) - x_3(\mathbf{a}\cdot\mathbf{b})\mathbf{a} + x_3|\mathbf{a}|^2\mathbf{b}$$

$$\Rightarrow x_3\{(\mathbf{a}\cdot\mathbf{b}) - x_1\}\mathbf{a} + (1 - x_2 - x_3|\mathbf{a}|^2)\mathbf{b} + (x_2 - x_3)(\mathbf{a}\times\mathbf{b}) = 0$$

Now, **a**, **b** and $\mathbf{a}\times\mathbf{b}$ are linearly independent.

Hence, $x_3(\mathbf{a}\cdot\mathbf{b}) = x_1,\ 1 = x_2 + x_3|\mathbf{a}|^2,\ x_2 = x_3$

$$x_2 = x_3 = \frac{1}{1+|\mathbf{a}|^2} \text{ and } x_1 = \frac{\mathbf{a}\cdot\mathbf{b}}{1+|\mathbf{a}|^2}$$

$$\mathbf{c} = \frac{\mathbf{a}\cdot\mathbf{b}}{(1+|\mathbf{a}|^2)}\mathbf{a} + \frac{1}{(1+|\mathbf{a}|^2)}[\mathbf{b}+(\mathbf{a}\times\mathbf{b})]$$

Product of Vectors Exercise 1 :
Single Correct Type Questions

1. If $\mathbf{a}$ has magnitude 5 and points North-East and vector $\mathbf{b}$ has magnitude 5 and points North-West, then $|\mathbf{a}-\mathbf{b}|$ is equal to
(a) 25 (b) 5
(c) $7\sqrt{3}$ (d) $5\sqrt{2}$

2. If $|\mathbf{a}+\mathbf{b}|>|\mathbf{a}-\mathbf{b}|$, then the angle between $\mathbf{a}$ and $\mathbf{b}$ is
(a) acute (b) obtuse
(c) $\pi/2$ (d) π

3. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three vectors such that $\mathbf{a}=\mathbf{b}+\mathbf{c}$ and the angle between $\mathbf{b}$ and $\mathbf{c}$ is $\frac{\pi}{2}$, then
(a) $a^2=b^2+c^2$ (b) $b^2=c^2+a^2$
(c) $c^2=a^2+b^2$ (d) $2a^2-b^2=c^2$
Note Here, $a=|\mathbf{a}|$, $b=|\mathbf{b}|$ and $c=|\mathbf{c}|$

4. If the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$ be θ and $\mathbf{a}\cdot\mathbf{b}=\cos\theta$, then the true statement is
(a) $\mathbf{a}$ and $\mathbf{b}$ are equal vectors
(b) $\mathbf{a}$ and $\mathbf{b}$ are like vectors
(c) $\mathbf{a}$ and $\mathbf{b}$ are unlike vectors
(d) $\mathbf{a}$ and $\mathbf{b}$ are unit vectors

5. If the vector $\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ makes angles α, β and γ with vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ respectively, then
(a) $\alpha=\beta\neq\gamma$ (b) $\alpha=\gamma\neq\beta$
(c) $\beta=\gamma\neq\alpha$ (d) $\alpha=\beta=\gamma$

6. $(\mathbf{r}\cdot\hat{\mathbf{i}})^2+(\mathbf{r}\cdot\hat{\mathbf{j}})^2+(\mathbf{r}\cdot\hat{\mathbf{k}})^2$ is equal to
(a) $3r^2$ (b) r^2
(c) 0 (d) None of these

7. Let $\mathbf{a}$ and $\mathbf{b}$ be two unit vectors inclined at an angle θ, then $\sin(\theta/2)$ is equal to
(a) $\frac{1}{2}|\mathbf{a}-\mathbf{b}|$ (b) $\frac{1}{2}|\mathbf{a}+\mathbf{b}|$
(c) $|\mathbf{a}-\mathbf{b}|$ (d) $|\mathbf{a}+\mathbf{b}|$

8. If $\mathbf{a}=4\hat{\mathbf{i}}+6\hat{\mathbf{j}}$ and $\mathbf{b}=3\hat{\mathbf{j}}+4\hat{\mathbf{k}}$, then the component of $\mathbf{a}$ along $\mathbf{b}$ is
(a) $\frac{18}{10\sqrt{3}}(3\hat{\mathbf{j}}+4\hat{\mathbf{k}})$ (b) $\frac{18}{25}(3\hat{\mathbf{j}}+4\hat{\mathbf{k}})$
(c) $\frac{18}{\sqrt{3}}(3\hat{\mathbf{j}}+4\hat{\mathbf{k}})$ (d) $(3\hat{\mathbf{j}}+4\hat{\mathbf{k}})$

9. If vector $\mathbf{a}=2\hat{\mathbf{i}}-3\hat{\mathbf{j}}+6\hat{\mathbf{k}}$ and vector $\mathbf{b}=-2\hat{\mathbf{i}}+2\hat{\mathbf{j}}-\hat{\mathbf{k}}$, then $\dfrac{\text{Projection of vector } \mathbf{a} \text{ on vector } \mathbf{b}}{\text{Projection of vector } \mathbf{b} \text{ on vector } \mathbf{a}}$ is equal to
(a) $\frac{3}{7}$ (b) $\frac{7}{3}$
(c) 3 (d) 7

10. If $\mathbf{a}$ and $\mathbf{b}$ are two vectors, then $(\mathbf{a}\times\mathbf{b})^2$ is equal to
(a) $\begin{vmatrix}\mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{a}\\ \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{a}\end{vmatrix}$ (b) $\begin{vmatrix}\mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b}\\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b}\end{vmatrix}$
(c) $\begin{vmatrix}\mathbf{a}\cdot\mathbf{b}\\ \mathbf{b}\cdot\mathbf{a}\end{vmatrix}$ (d) None of these

11. The moment of the force $\mathbf{F}$ acting at a point P, about the point C is
(a) $\mathbf{F}\times\mathbf{CP}$
(b) $\mathbf{CP}\cdot\mathbf{F}$
(c) a vector having the same direction as $\mathbf{F}$
(d) $\mathbf{CP}\times\mathbf{F}$

12. The moment of a force represented by $\mathbf{F}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}}$ about the point $2\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}}$ is equal to
(a) $5\hat{\mathbf{i}}-5\hat{\mathbf{j}}+5\hat{\mathbf{k}}$ (b) $5\hat{\mathbf{i}}+5\hat{\mathbf{j}}-6\hat{\mathbf{k}}$
(c) $-5\hat{\mathbf{i}}-5\hat{\mathbf{j}}+5\hat{\mathbf{k}}$ (d) $-5\hat{\mathbf{i}}-5\hat{\mathbf{j}}+2\hat{\mathbf{k}}$

13. A force of magnitude 6 acts along the vector $(9,6,-2)$ and passes through a point $A(4,-1,-7)$. Then moment of force about the point $O(1,-3,2)$ is
(a) $\frac{150}{11}(2\hat{\mathbf{i}}-3\hat{\mathbf{j}})$ (b) $\frac{6}{11}(50\hat{\mathbf{i}}-75\hat{\mathbf{j}}+36\hat{\mathbf{k}})$
(c) $150\,(2\hat{\mathbf{i}}-3\hat{\mathbf{j}})$ (d) $6(50\hat{\mathbf{i}}-75\hat{\mathbf{j}}+36\hat{\mathbf{k}})$

14. A force $\mathbf{F}=2\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}}$ acts at a point A, whose position vector is $2\hat{\mathbf{i}}-\hat{\mathbf{j}}$. The moment of $\mathbf{F}$ about the origin is
(a) $\hat{\mathbf{i}}+2\hat{\mathbf{j}}-4\hat{\mathbf{k}}$ (b) $\hat{\mathbf{i}}-2\hat{\mathbf{j}}-4\hat{\mathbf{k}}$
(c) $\hat{\mathbf{i}}+2\hat{\mathbf{j}}+4\hat{\mathbf{k}}$ (d) $\hat{\mathbf{i}}-2\hat{\mathbf{j}}+4\hat{\mathbf{k}}$

15. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are any three vectors and their inverse are $\mathbf{a}^{-1}$, $\mathbf{b}^{-1}$ and $\mathbf{c}^{-1}$ and $[\mathbf{abc}]\neq 0$, then $[\mathbf{a}^{-1}\mathbf{b}^{-1}\mathbf{c}^{-1}]$ will be
(a) zero (b) one
(c) non-zero (d) $[\mathbf{abc}]$

16. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three non-coplanar vectors, that $\dfrac{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}{\mathbf{c}\times\mathbf{a}\cdot\mathbf{b}}+\dfrac{\mathbf{b}\cdot\mathbf{a}\times\mathbf{c}}{\mathbf{c}\cdot\mathbf{a}\times\mathbf{b}}$ is equal to
(a) 0 (b) 2
(c) -2 (d) None of these

17. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is coplanar with
(a) $\mathbf{b}$ and $\mathbf{c}$ (b) $\mathbf{c}$ and $\mathbf{a}$
(c) $\mathbf{a}$ and $\mathbf{b}$ (d) $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$

18. If $\mathbf{u}=\hat{\mathbf{i}}\times(\mathbf{a}\times\hat{\mathbf{i}})+\hat{\mathbf{j}}\times(\mathbf{a}\times\hat{\mathbf{j}})+\hat{\mathbf{k}}\times(\mathbf{a}\times\hat{\mathbf{k}})$, then
(a) $\mathbf{u}=0$
(b) $\mathbf{u}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$
(c) $\mathbf{u}=2\mathbf{a}$
(d) $\mathbf{u}=\mathbf{a}$

19. If $\mathbf{a} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$, $\mathbf{b} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{c} = \hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$, then $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is equal to
(a) $20\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$
(b) $20\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 7\hat{\mathbf{k}}$
(c) $20\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 7\hat{\mathbf{k}}$
(d) None of the above

20. If $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = 0$, then
(a) $|\mathbf{a}| = |\mathbf{b}| \cdot |\mathbf{c}| = 1$ (b) $\mathbf{b} \parallel \mathbf{c}$
(c) $\mathbf{a} \parallel \mathbf{b}$ (d) $\mathbf{b} \perp \mathbf{c}$

21. A vector whose modulus is $\sqrt{51}$ and makes the same angle with $\mathbf{a} = \dfrac{\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}}{3}$, $\mathbf{b} = \dfrac{-4\hat{\mathbf{i}} - 3\hat{\mathbf{k}}}{5}$ and $\mathbf{c} = \hat{\mathbf{j}}$, will be
(a) $5\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + \hat{\mathbf{k}}$
(b) $5\hat{\mathbf{i}} + \hat{\mathbf{j}} - 5\hat{\mathbf{k}}$
(c) $5\hat{\mathbf{i}} + \hat{\mathbf{j}} + 5\hat{\mathbf{k}}$
(d) $\pm(5\hat{\mathbf{i}} - \hat{\mathbf{j}} - 5\hat{\mathbf{k}})$

22. The horizontal force and the force inclined at an angle 60° with the vertical, whose resultant is in vertical direction of P kg, are
(a) $P, 2P$ (b) $P, P\sqrt{3}$
(c) $2P, P\sqrt{3}$ (d) None of these

23. If $x + y + z = 0$, $|\mathbf{x}| = |\mathbf{y}| = |\mathbf{z}| = 2$ and θ is angle between $\mathbf{y}$ and $\mathbf{z}$, then the value of $\text{cosec}^2\,\theta + \cot^2\theta$ is equal to
(a) 4 / 3 (b) 5 / 3
(c) 1 / 3 (d) 1

24. The value of x for which the angle between the vectors $\mathbf{a} = -3\hat{\mathbf{i}} + x\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{b} = x\,\hat{\mathbf{i}} + 2x\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$ is acute and the angle between $\mathbf{b}$ and X-axis lies between $\pi/2$ and π satisfy
(a) $x > 0$ (b) $x < 0$
(c) $x > 1$ only (d) $x < -1$ only

25. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are non-coplanar vectors and $\mathbf{d} = \lambda\mathbf{a} + \mu\mathbf{b} + \nu\,\mathbf{c}$, then λ is equal to
(a) $\dfrac{[\mathbf{dbc}]}{[\mathbf{bac}]}$ (b) $\dfrac{[\mathbf{bcd}]}{[\mathbf{bca}]}$
(c) $\dfrac{[\mathbf{bdc}]}{[\mathbf{abc}]}$ (d) $\dfrac{[\mathbf{cbd}]}{[\mathbf{abc}]}$

26. If the vectors $3\mathbf{p} + \mathbf{q}$, $5\mathbf{p} - 3\mathbf{q}$ and $2\mathbf{p} + \mathbf{q}$, $4\mathbf{p} - 2\mathbf{q}$ are pairs of mutually perpendicular vectors, then $\sin(\mathbf{pq})$ is
(a) $\sqrt{55}$ / 4 (b) $\sqrt{55}$ / 8
(c) 3/16 (d) $\sqrt{247}$ / 16

27. Let $\mathbf{u} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$, $\mathbf{v} = \hat{\mathbf{i}} - \hat{\mathbf{j}}$ and $\mathbf{w} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$. If $\hat{\mathbf{n}}$ is a unit vector such that $\mathbf{u} \cdot \hat{\mathbf{n}} = 0$ and $\mathbf{v} \cdot \hat{\mathbf{n}} = 0$, then $|\mathbf{w} \cdot \hat{\mathbf{n}}|$ is equal to
(a) 1 (b) 2 (c) 3 (d) 0

28. Given a parallelogram $ABCD$. If $|\mathbf{AB}| = a$, $|\mathbf{AD}| = b$ and $|\mathbf{AC}| = c$, then $\mathbf{DB} \cdot \mathbf{AB}$ has the value
(a) $\dfrac{3a^2 + b^2 - c^2}{2}$
(b) $\dfrac{a^2 + 3b^2 - c^2}{2}$
(c) $\dfrac{a^2 - b^2 + 3c^2}{2}$
(d) None of the above

29. For two particular vectors $\mathbf{A}$ and $\mathbf{B}$, it is known that $\mathbf{A} \times \mathbf{B} = \mathbf{B} \times \mathbf{A}$. What must be true about the two vectors?
(a) Atleast one of the two vectors must be the zero vector
(b) $\mathbf{A} \times \mathbf{B} = \mathbf{B} \times \mathbf{A}$ is true for any two vectors
(c) One of the two vectors is a scalar multiple of the other vector
(d) The two vectors must be perpendicular to each other

30. For some non-zero vector $\mathbf{V}$, if the sum of $\mathbf{V}$ and the vector obtained from $\mathbf{V}$ by rotating it by $\angle 2\alpha$ equals to the vector obtained from $\mathbf{V}$ by rotating it by $\angle\alpha$, then the value of α, is
(a) $2n\pi \pm \dfrac{\pi}{3}$ (b) $n\pi \pm \dfrac{\pi}{3}$
(c) $2n\pi \pm \dfrac{2\pi}{3}$ (d) $n\pi \pm \dfrac{2\pi}{3}$

31. In the isosceles ΔABC, $|\mathbf{AB}| = |\mathbf{BC}| = 8$, a point E divides AB internally in the ratio 1 : 3, then the cosine of the angle between $\mathbf{CE}$ and $\mathbf{CA}$ is (where, $|\mathbf{CA}| = 12$)
(a) $-\dfrac{3\sqrt{7}}{8}$ (b) $\dfrac{3\sqrt{8}}{17}$
(c) $\dfrac{3\sqrt{7}}{8}$ (d) $\dfrac{-3\sqrt{8}}{17}$

32. Given an equilateral ΔABC with side length equal to a. Let M and N be two points respectively, on the side AB and AC such that $\mathbf{AN} = k\mathbf{AC}$ and $\mathbf{AM} = \dfrac{\mathbf{AB}}{3}$. If $\mathbf{BN}$ and $\mathbf{CM}$ are orthogonal, then the value of k is
(a) $\dfrac{1}{5}$ (b) $\dfrac{1}{4}$
(c) $\dfrac{1}{3}$ (d) $\dfrac{1}{2}$

33. In a quadrilateral $ABCD$, $\mathbf{AC}$ is the bisector of the $(\mathbf{AB}, \mathbf{AD})$ which is $\dfrac{2\pi}{3}$, $15|\mathbf{AC}| = 3|\mathbf{AB}| = 5|\mathbf{AD}|$, then $\cos(\mathbf{BA}, \mathbf{CD})$ is equal to
(a) $-\dfrac{\sqrt{14}}{7\sqrt{2}}$ (b) $-\dfrac{\sqrt{21}}{7\sqrt{3}}$
(c) $\dfrac{2}{\sqrt{7}}$ (d) $\dfrac{2\sqrt{7}}{14}$

34. If the distance from the point $P(1, 1, 1)$ to the line passing through the points $Q(0, 6, 8)$ and $R(-1, 4, 7)$ is expressed in the form $\sqrt{p/q}$, where p and q are co-prime, then the value of $\frac{(p+q)(p+q-1)}{2}$ is equal to

(a) 4950 (b) 5050
(c) 5150 (d) None of these

35. Given the vectors

$$\mathbf{u} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}}$$
$$\mathbf{v} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$$
$$\mathbf{w} = \hat{\mathbf{i}} - \hat{\mathbf{k}}$$

If the volume of the parallelopiped having $-c\mathbf{u}, \mathbf{v}$ and $c\mathbf{w}$ as concurrent edges, is 8, then c is equal to

(a) ± 2 (b) 4
(c) 8 (d) Cannot be determined

36. The vector $\mathbf{c}$ is perpendicular to the vectors $\mathbf{a} = (2, -3, 1)$, $\mathbf{b} = (1, -2, 3)$ and satisfies the condition $\mathbf{c}\cdot(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 7\hat{\mathbf{k}}) = 10$. Then, the vector $\mathbf{c}$ is equal to

(a) (7, 5, 1) (b) $(-7, -5, -1)$
(c) (1, 1, -1) (d) None of these

37. Let $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$, $\mathbf{b} = \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b}$. If the vectors, $\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$, $3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\mathbf{c}$ are coplanar, then $\frac{\alpha}{\beta}$ is equal to

(a) 1 (b) 2
(c) 3 (d) -3

38. A rigid body rotates about an axis through the origin with an angular velocity $10\sqrt{3}$ rad/s. If ω points in the direction of $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, then the equation to the locus of the points having tangential speed 20 m/s is

(a) $x^2 + y^2 + z^2 - xy - yz - zx - 1 = 0$
(b) $x^2 + y^2 + z^2 - 2xy - 2yz - 2zx - 1 = 0$
(c) $x^2 + y^2 + z^2 - xy - yz - zx - 2 = 0$
(d) $x^2 + y^2 + z^2 - 2xy - 2yz - 2zx - 2 = 0$

39. A rigid body rotates with constant angular velocity ω about the line whose vector equation is, $\mathbf{r} = \lambda(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$. The speed of the particle at the instant it passes through the point with position vector $(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 5\hat{\mathbf{k}})$ is equal to

(a) $\omega\sqrt{2}$ (b) 2ω
(c) $\omega/\sqrt{2}$ (d) None of these

40. Consider ΔABC with $A = (\mathbf{a})$, $B = (\mathbf{b})$ and $C = (\mathbf{c})$. If $\mathbf{b}\cdot(\mathbf{a} + \mathbf{c}) = \mathbf{b}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$, $|\mathbf{b} - \mathbf{a}| = 3$ and $|\mathbf{c} - \mathbf{b}| = 4$, then the angle between the medians **AM** and **BD** is

(a) $\pi - \cos^{-1}\left(\frac{1}{5\sqrt{13}}\right)$ (b) $\pi - \cos^{-1}\left(\frac{1}{13\sqrt{5}}\right)$
(c) $\cos^{-1}\left(\frac{1}{5\sqrt{13}}\right)$ (d) $\cos^{-1}\left(\frac{1}{13\sqrt{5}}\right)$

41. Given unit vectors $\mathbf{m}$, $\mathbf{n}$ and $\mathbf{p}$ such that angle between $\mathbf{m}$ and $\mathbf{n}$ = Angle between $\mathbf{p}$ and $(\mathbf{m} \times \mathbf{n}) = \frac{\pi}{6}$, then $[\mathbf{npm}]$ is equal to

(a) $\sqrt{3}/4$ (b) 3/4
(c) 1/4 (d) None of these

42. If $\mathbf{a}$ and $\mathbf{b}$ are unit vectors, then the vector defined as $\mathbf{V} = (\mathbf{a} + \mathbf{b}) \times (\mathbf{a} + \mathbf{b})$ is collinear to the vector

(a) $\mathbf{a} + \mathbf{b}$ (b) $\mathbf{b} - \mathbf{a}$
(c) $2\mathbf{a} - \mathbf{b}$ (d) $\mathbf{a} + 2\mathbf{b}$

43. If $\mathbf{a}$ and $\mathbf{b}$ are orthogonal unit vectors, then for any non-zero vector $\mathbf{r}$, the vector $(\mathbf{r} \times \mathbf{a})$ is equal to

(a) $[\mathbf{r}\ \hat{\mathbf{a}}\ \hat{\mathbf{b}}](\hat{\mathbf{a}} + \hat{\mathbf{b}})$
(b) $[\mathbf{r}\ \hat{\mathbf{a}}\ \hat{\mathbf{b}}]\hat{\mathbf{a}} + (\mathbf{r}\cdot\hat{\mathbf{a}})(\hat{\mathbf{a}} \times \hat{\mathbf{b}})$
(c) $[\mathbf{r}\ \hat{\mathbf{a}}\ \hat{\mathbf{b}}]\hat{\mathbf{b}} + (\mathbf{r}\cdot\hat{\mathbf{b}})(\hat{\mathbf{b}} \times \hat{\mathbf{a}})$
(d) $[\mathbf{r}\ \hat{\mathbf{a}}\ \hat{\mathbf{b}}]\hat{\mathbf{b}} + (\mathbf{r}\cdot\hat{\mathbf{a}})(\hat{\mathbf{a}} \times \hat{\mathbf{b}})$

44. If vector $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ is rotated through an angle of 90°, so as to cross the positive direction of Y-axis, then the vector in the new position is

(a) $-\frac{2}{\sqrt{5}}\hat{\mathbf{i}} + \sqrt{5}\hat{\mathbf{j}} - \frac{4}{\sqrt{5}}\hat{\mathbf{k}}$ (b) $\frac{2}{\sqrt{5}}\hat{\mathbf{i}} - \sqrt{5}\hat{\mathbf{j}} + \frac{4}{\sqrt{5}}\hat{\mathbf{k}}$
(c) $4\hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}}$ (d) None of these

45. 10 different vectors are lying on a plane out of which four are parallel with respect to each other. Probability that three vectors chosen from them will satisfy the equation $\lambda_1\mathbf{a} + \lambda_2\mathbf{b} + \lambda_3\mathbf{c} = 0$, where λ_1, λ_2 and $\lambda_3 \neq 0$ is

(a) $\frac{{}^6C_2 \times {}^4C_1}{{}^{10}C_3}$ (b) $\frac{({}^6C_3 \times {}^4C_1) + {}^6C_3}{{}^{10}C_3}$
(c) $\frac{({}^6C_2 \times {}^4C_1) + ({}^4C_3)}{{}^{10}C_3}$ (d) $\frac{({}^6C_2 + {}^4C_1) + ({}^6C_2 \times {}^4C_1)}{{}^{10}C_3}$

46. If $\hat{\mathbf{a}}$ is a unit vector and projection of $\mathbf{x}$ along $\hat{\mathbf{a}}$ is 2 units and $(\hat{\mathbf{a}} \times \mathbf{x}) + \mathbf{b} = \mathbf{x}$, then $\mathbf{x}$ is equal to

(a) $\frac{1}{2}[\hat{\mathbf{a}} - \mathbf{b} + (\hat{\mathbf{a}} \times \mathbf{b})]$
(b) $\frac{1}{2}[2\hat{\mathbf{a}} - \mathbf{b} + (\hat{\mathbf{a}} \times \mathbf{b})]$
(c) $[\hat{\mathbf{a}} + (\hat{\mathbf{a}} \times \mathbf{b})]$
(d) None of the above

47. If $\mathbf{a}\cdot\mathbf{b}$ and $\mathbf{c}$ are any three non-zero vectors, then the component of $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ perpendicular to $\mathbf{b}$ is

(a) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \frac{(\mathbf{a} \times \mathbf{b})\cdot(\mathbf{c} \times \mathbf{a})}{|b|^2}\mathbf{b}$
(b) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \frac{(\mathbf{a} \times \mathbf{c})\cdot(\mathbf{a} \times \mathbf{b})}{|b|^2}\mathbf{b}$
(c) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \frac{(\mathbf{b} \times \mathbf{c})\cdot(\mathbf{b} \times \mathbf{a})}{|b|^2}\mathbf{b}$
(d) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \frac{(\mathbf{a} \times \mathbf{b})\cdot(\mathbf{b} \times \mathbf{c})}{|b|^2}\mathbf{b}$

48. The position vector of a point P is $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, where $x, y, z \in N$ and $\mathbf{a} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$. If $\mathbf{r}\cdot\mathbf{a} = 20$, then the number of possible position of P is

(a) 81 (b) 49
(c) 100 (d) 36

49. Let $a, b > 0$ and $\alpha = \dfrac{\hat{\mathbf{i}}}{a} + \dfrac{4\hat{\mathbf{j}}}{b} + b\hat{\mathbf{k}}$ and $\beta = b\hat{\mathbf{i}} + a\hat{\mathbf{j}} + \dfrac{1}{b}\hat{\mathbf{k}}$, then the maximum value of $\dfrac{10}{5 + \alpha\cdot\beta}$ is

(a) 1 (b) 2
(c) 4 (d) 8

50. If **a, b** and **c** are any three vectors forming a linearly independent system, then $\forall\ \theta \in R$

$$\left[\mathbf{a}\cos\theta + \mathbf{b}\sin\theta + \mathbf{c}\cos 2\theta, \mathbf{a}\cos\left(\frac{2\pi}{3} + \theta\right) + \mathbf{b}\sin\left(\frac{2\pi}{3} + \theta\right) + \mathbf{c}\cos 2\left(\frac{2\pi}{3} + \theta\right), \mathbf{a}\cos\left(\theta - \frac{2\pi}{3}\right) + \mathbf{b}\sin\left(\theta - \frac{2\pi}{3}\right) + \mathbf{c}\cos 2\left(\theta - \frac{2\pi}{3}\right)\right]$$ equals

(a) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\cos\theta$
(b) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\cos 2\theta$
(c) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\cos 3\theta$
(d) None of the above

51. Two adjacent sides of a parallelogram $ABCD$ are given by $\mathbf{AB} = 2\hat{\mathbf{i}} + 10\hat{\mathbf{j}} + 11\hat{\mathbf{k}}$ and $\mathbf{AD} = -\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$. The side AD is rotated by an acute angle α in the plane of parallelogram so that AD becomes AD'. If AD' makes a right angle with the side AB, then the cosine of angle α is given by

(a) $\dfrac{8}{9}$ (b) $\dfrac{\sqrt{17}}{9}$
(c) $\dfrac{1}{9}$ (d) $\dfrac{4\sqrt{5}}{9}$

52. If in a ΔABC, $\mathbf{BC} = \dfrac{\mathbf{e}}{|\mathbf{e}|} - \dfrac{\mathbf{f}}{|\mathbf{f}|}$ and $\mathbf{AC} = \dfrac{2\mathbf{e}}{|\mathbf{e}|}$; $|\mathbf{e}| \neq |\mathbf{f}|$, then the value of $\cos 2A + \cos 2B + \cos 2C$ must be

(a) -1 (b) 0
(c) 2 (d) $\dfrac{-3}{2}$

53. **a, b, c** are three unit of vectors, **a** and **b** are perpendicular to each other and vector **c** is equally inclined to both **a** and **b** at an angle θ. If $\mathbf{c} = \alpha\,\mathbf{a} + \beta\,\mathbf{b} + \gamma(\mathbf{a}\times\mathbf{b})$, where α, β, γ are constants , then

(a) $\alpha = \beta = -\cos\theta,\ \gamma^2 = \cos 2\theta$
(b) $\alpha = \beta = \cos\theta, \gamma^2 = \cos 2\theta$
(c) $\alpha = \beta = \cos\theta, \gamma^2 = -\cos 2\theta$
(d) $\alpha = \beta = -\cos\theta, \gamma^2 = -\cos 2\theta$

54. The ΔABC is such that the mid-points of the sides BC, CA, AB are $(l, 0, 0), (0, m, 0), (0, 0, n)$ respectively. Then, $\dfrac{AB^2 + BC^2 + CA^2}{l^2 + m^2 + n^2}$ is equal to

(a) 2 (b) 4
(c) 8 (d) 16

55. The angle between the lines whose direction cosines are given by $2l - m + 2n = 0,\ lm + mn + nl = 0$ is

(a) $\dfrac{\pi}{6}$ (b) $\dfrac{\pi}{4}$
(c) $\dfrac{\pi}{3}$ (d) $\dfrac{\pi}{2}$

56. A line makes an angle θ both with X and Y-axes. A possible range of θ is

(a) $\left[0, \dfrac{\pi}{4}\right]$ (b) $\left[0, \dfrac{\pi}{2}\right]$
(c) $\left[\dfrac{\pi}{4}, \dfrac{\pi}{2}\right]$ (d) $\left[\dfrac{\pi}{6}, \dfrac{\pi}{3}\right]$

57. Let **a**, **b** and **c** be the three vectors having magnitudes 1, 5 and 3 respectively such that the angle between **a** and **b** is θ and $\mathbf{a}\times(\mathbf{a}\times\mathbf{b}) = \mathbf{c}$, then $\tan\theta$ is equal to

(a) 0 (b) $\dfrac{2}{3}$
(c) $\dfrac{3}{5}$ (d) $\dfrac{3}{4}$

58. The perpendicular distance of a corner of a unit cube from a diagonal not passing through it is

(a) $\sqrt{\dfrac{3}{2}}$ (b) $\sqrt{\dfrac{2}{3}}$
(c) $\sqrt{\dfrac{3}{4}}$ (d) $\sqrt{\dfrac{4}{3}}$

59. If **p, q** are two non-collinear vectors such that $(b-c)\,\mathbf{p}\times\mathbf{q} + (c-a)\mathbf{p} + (a-b)\mathbf{q} = \mathbf{0}$ where a, b, c are lengths of sides of a triangle, then the triangle is

(a) right angled (b) obtuse angled
(c) equilateral (d) right angled isosceles triangle

60. Let $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{b} = -\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{c} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{d} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$. Then, the line of intersection of planes one determined by **a, b** and other determined by **c, d** is perpendicular to

(a) X-axis (b) Y-axis
(c) both X-axis and Y-axis (d) both Y-axis and Z-axis

61. A parallelopiped is formed by planes drawn parallel to coordinate axes through the points $A = (1, 2, 3)$ and $B = (9, 8, 5)$. The volume of that parallelopiped is equal to (in cubic units)

(a) 192 (b) 48
(c) 32 (d) 96

62. Let **a, b** and **c** be three non-coplanar vectors and **d** be a non-zero vector, which is perpendicular to $\mathbf{a}+\mathbf{b}+\mathbf{c}$. Now, if $\mathbf{d}=(\sin x)(\mathbf{a}\times\mathbf{b})+(\cos y)(\mathbf{b}\times\mathbf{c})+2(\mathbf{c}\times\mathbf{a})$, then minimum value of x^2+y^2 is equal to

(a) π^2 (b) $\frac{\pi^2}{2}$

(c) $\frac{\pi^2}{4}$ (d) $\frac{5\pi^2}{4}$

63. If $\alpha(\mathbf{a}\times\mathbf{b})+\beta(\mathbf{b}\times\mathbf{c})+\gamma(\mathbf{c}\times\mathbf{a})=0$, then

(a) *a, b, c* are coplanar if all of $\alpha, \beta, \gamma \neq 0$

(b) *a, b, c* are coplanar if any one $\alpha, \beta, \gamma = 0$

(c) *a, b c* are non-coplanar for any α, β, γ

(d) None of the above

64. Let area of faces,

$\Delta OAB=\lambda_1, \Delta OAC=\lambda_2, \Delta OBC=\lambda_3, \Delta ABC=\lambda_4$

and h_1, h_2, h_3, h_4 be perpendicular height from 0 to face ΔABC, A to the face ΔOBC, B to the face ΔOAC, C to the face ΔOAB, then the face

$$\frac{1}{3}\lambda_1 h_4\cdot\frac{1}{3}\lambda_2 h_3+\frac{1}{3}\lambda_3 h_2+\frac{1}{3}\lambda_4 h_1$$

(a) $\frac{2}{3}|[\mathbf{AB}\ \mathbf{AC}\ \mathbf{OA}]|$ (b) $\frac{1}{3}|[\mathbf{AB}\ \mathbf{AC}\ \mathbf{OA}]|$

(c) $\frac{1}{3}|[\mathbf{OA}\ \mathbf{OB}\ \mathbf{OC}]|$ (d) None of these

65. Given four vectors **a**, **b**, **c** and **d**. The vectors **a, b, c** are coplanar but not collinear pair by pair and the vector **d** is not coplanar with the vecotrs **a, b** and **c**. If it is known that the angle between **a** and **b** is equal to that between **b** and **c**, each being equal to 60°. The angle between **d** and **a** is α and between **d** and **b** is β. Then, the angle between the vectors **d** and **c**.

(a) $\cos^{-1}(\cos\beta-\cos\alpha)$

(b) $\sin^{-1}(\cos\beta-\cos\alpha)$

(c) $\sin^{-1}(\sin\beta-\sin\alpha)$

(d) $\cos^{-1}(\tan\beta-\tan\alpha)$

66. The shortest distance between a diagonal of a unit cube and a diagonal of a face skew to it is

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$

(c) $\frac{1}{\sqrt{3}}$ (d) $\frac{1}{\sqrt{6}}$

67. Let $\mathbf{V}=2\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $\mathbf{W}=\hat{\mathbf{i}}+3\hat{\mathbf{k}}$. If U is a unit vector, then the maximum value of the scalar triple product $[\mathbf{U}\ \mathbf{V}\ \mathbf{W}]$ is

(a) -1 (b) $\sqrt{35}$

(c) $\sqrt{59}$ (d) $\sqrt{60}$

68. The length of the edge of the regular tetrahedron $ABCD$ is 'a'. Points E and F are taken on the edges AD and BD respectively such that 'E' divides **DA** and 'F' divides **BD** in the ratio 2 : 1 each. Then, area of ΔCEF is

(a) $\frac{5a}{12\sqrt{3}}$ sq units (b) $\frac{a}{12\sqrt{3}}$ sq units

(c) $\frac{a^2}{12\sqrt{3}}$ sq units (d) $\frac{5a^2}{12\sqrt{3}}$ sq units

69. If two adjacent sides of two rectangles are represented by the vectors $\mathbf{p}=5\mathbf{a}-3\mathbf{b}$, $\mathbf{q}=-\mathbf{a}-2\mathbf{b}$ and $\mathbf{r}=-4\mathbf{a}-\mathbf{b}; \mathbf{s}=-\mathbf{a}+\mathbf{b}$ respectively, then the angle between the vectors $\mathbf{x}=\frac{1}{3}(\mathbf{p}+\mathbf{r}+\mathbf{s})$ and $\mathbf{y}=\frac{1}{5}(\mathbf{r}+\mathbf{s})$

(a) $\pi-\cos^{-1}\left(\frac{19}{5\sqrt{43}}\right)$ (b) $\cos^{-1}\left(\frac{19}{5\sqrt{43}}\right)$

(c) $-\cos^{-1}\left(\frac{19}{5\sqrt{43}}\right)$ (d) $\pi-\cos^{-1}\left(\frac{19}{\sqrt{43}}\right)$

70. Let **a**, **b**, **c** are three vectors along the adjacent edges of a tetrahedron, if $|\mathbf{a}|=|\mathbf{b}|=|\mathbf{c}|=2$ and $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{c}=\mathbf{c}\cdot\mathbf{a}=2$, then volume of tetrahedron is

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{2}{\sqrt{3}}$

(c) $\frac{\sqrt{3}}{2}$ (d) $\frac{2\sqrt{2}}{3}$

71. The angle θ between two non-zero vectors **a** and **b** satisfies the relation

$\cos\theta=(\mathbf{a}\times\hat{\mathbf{i}})\cdot(\mathbf{b}\times\hat{\mathbf{i}})+(\mathbf{a}\times\hat{\mathbf{j}})\cdot(\mathbf{b}\times\hat{\mathbf{j}})+(\mathbf{a}\times\hat{\mathbf{k}})\cdot(\mathbf{b}\times\hat{\mathbf{k}})$,

then the least value of $|\mathbf{a}|+|\mathbf{b}|$ is equal to (where $\theta\neq 90°$)

(a) $\frac{1}{2}$ (b) 2

(c) $\sqrt{2}$ (d) 4

72. If the angle between the vectors $\mathbf{a}=\hat{\mathbf{i}}+(\cos x)\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $\mathbf{b}=(\sin^2 x-\sin x)\hat{\mathbf{i}}-(\cos x)\hat{\mathbf{j}}+(3-4\sin x)\hat{\mathbf{k}}$ is obtuse and $x\in\left(0,\frac{\pi}{2}\right)$, then the exhaustive set of values of 'x' is equal to

(a) $x\in\left(0,\frac{\pi}{6}\right)$ (b) $x\in\left(\frac{\pi}{6},\frac{\pi}{2}\right)$

(c) $x\in\left(\frac{\pi}{6},\frac{\pi}{3}\right)$ (d) $x\in\left(\frac{\pi}{3},\frac{\pi}{2}\right)$

73. If position vectors of the points A, B and C are **a**, **b**, **c** respectively and the points D and E divides line segments AC and AB in the ratio 2 : 1 and 1 : 3, respectively. Then, point of intersection of BD and EC divides EC in the ratio

(a) 2 : 1 (b) 1 : 3

(c) 1 : 2 (d) 3 : 2

Product of Vectors Exercise 2 :
More than One Option Correct Type Questions

74. If vectors $\mathbf{a}$ and $\mathbf{b}$ are non-collinear, than $\frac{\mathbf{a}}{|\mathbf{a}|}+\frac{\mathbf{b}}{|\mathbf{b}|}$ is
(a) a unit vector
(b) in the plane of $\mathbf{a}$ and $\mathbf{b}$
(c) equally inclined to $\mathbf{a}$ and $\mathbf{b}$
(d) perpendicular to $\mathbf{a}\times\mathbf{b}$

75. If $\mathbf{a}\times\mathbf{b}(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$, then
(a) $(\mathbf{c}\times\mathbf{a})\times\mathbf{b}=\mathbf{0}$ (b) $\mathbf{c}\times(\mathbf{a}\times\mathbf{b})=\mathbf{0}$
(c) $\mathbf{b}\times(\mathbf{c}\times\mathbf{a})=\mathbf{0}$ (d) $(\mathbf{c}\times\mathbf{a})\times\mathbf{b}=\mathbf{b}\times(\mathbf{c}\times\mathbf{a})=\mathbf{0}$

76. Let $\mathbf{a}$ and $\mathbf{b}$ be two non-collinear unit vectors. If $\mathbf{u}=\mathbf{a}-(\mathbf{a}\,.\,\mathbf{b})\,\mathbf{b}$ and $\mathbf{v}=\mathbf{a}\times\mathbf{b}$, then $|\mathbf{v}|$ is
(a) $|\mathbf{u}|$ (b) $|\mathbf{u}|+|\mathbf{u}.\mathbf{a}|$
(c) $|\mathbf{u}|+|\mathbf{u}.\mathbf{b}|$ (d) $|\mathbf{u}|+\mathbf{u}.(\mathbf{a}+\mathbf{b})$

77. The scalars l and m such $l\,\mathbf{a}+m\,\mathbf{b}=\mathbf{c}$, where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are give vectors, are equal to
(a) $l=\frac{(\mathbf{c}\times\mathbf{b}).(\mathbf{a}\times\mathbf{b})}{(\mathbf{a}\times\mathbf{b})^2}$ (b) $l=\frac{(\mathbf{c}\times\mathbf{a}).(\mathbf{b}\times\mathbf{a})}{(\mathbf{b}\times\mathbf{a})^2}$
(c) $m=\frac{(\mathbf{c}\times\mathbf{a}).(\mathbf{b}\times\mathbf{a})}{(\mathbf{b}\times\mathbf{a})^2}$ (d) $m=\frac{(\mathbf{c}\times\mathbf{a}).(\mathbf{b}\times\mathbf{a})}{(\mathbf{b}\times\mathbf{a})^2}$

78. Let $\hat{\mathbf{r}}$ be a unit vector satisfying $\hat{\mathbf{r}}\times\mathbf{a}=\mathbf{b}$, where $|\mathbf{a}|=\sqrt{3}$ and $|\mathbf{b}|=\sqrt{2}$. Then,
(a) $\hat{\mathbf{r}}=\frac{2}{3}(\mathbf{a}+\mathbf{a}\times\mathbf{b})$ (b) $\hat{\mathbf{r}}=\frac{1}{3}(\mathbf{a}+\mathbf{a}\times\mathbf{b})$
(c) $\hat{\mathbf{r}}=\frac{2}{3}(\mathbf{a}-\mathbf{a}\times\mathbf{b})$ (d) $\hat{\mathbf{r}}=\frac{1}{3}(-\mathbf{a}+\mathbf{a}\times\mathbf{b})$

79. $a_1, a_2, a_3 \in R-\{0\}$ and $a_1+a_2\cos 2x+a_3\sin^2 x=0$ for all $x\in R$, then
(a) vectors $\mathbf{a}=a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}+a_3\hat{\mathbf{k}}$ and $\mathbf{b}=4\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}}$ are perpendicular to each other
(b) vectors $\mathbf{a}=a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}+a_3\hat{\mathbf{k}}$ and $\mathbf{b}=-\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}}$ are perpendicular to each other
(c) if vectors $\mathbf{a}=a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}+a_3\hat{\mathbf{k}}$ is of length $\sqrt{6}$ units, then one of the ordered triplet $(a_1, a_2, a_3)=(1,-1,-2)$
(d) if vectors $2a_1+3a_2+6a_3$, then $|a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}+a_3\hat{\mathbf{k}}|$ is $2\sqrt{6}$

80. If $\mathbf{a}$ and $\mathbf{b}$ are two vectors and angles between them is θ, then
(a) $|\mathbf{a}\times\mathbf{b}|^2+(\mathbf{a}\cdot\mathbf{b})^2=|\mathbf{a}|^2|\mathbf{b}|^2$
(b) $|\mathbf{a}\times\mathbf{b}|=(\mathbf{a}\cdot\mathbf{b})$, if $\theta=\pi/4$
(c) $\mathbf{a}\times\mathbf{b}=(\mathbf{a}\cdot\mathbf{b})\,\hat{\mathbf{n}}$ (where $\hat{\mathbf{n}}$ is a normal unit vector), if $\theta=\pi/4$
(d) $|\mathbf{a}\times\mathbf{b}|\cdot(\mathbf{a}+\mathbf{b})=0$

81. If unit vectors $\mathbf{a}$ and $\mathbf{b}$ are inclined at an angle 2θ such that $|\mathbf{a}-\mathbf{b}|<1$ and $0\le\theta<=\pi$, then θ lies in the interval
(a) $[0,\pi/6)$ (b) $(5\pi/6\ \pi]$
(c) $[\pi/6, \pi/2)$ (d) $(\pi/2, 5\pi/6]$

82. $\mathbf{b}$ and $\mathbf{c}$ are non-collinear if $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})+(\mathbf{a}\cdot\mathbf{b})\,\mathbf{b}=(4-2x-\sin y)\,\mathbf{b}+(x^2-1)\,\mathbf{c}$ and $(\mathbf{c}\cdot\mathbf{c})\,\mathbf{a}=\mathbf{c}$. Then,
(a) $x=1$ (b) $x=-1$
(c) $y=(4n+1)\frac{\pi}{2}, n\in I$ (d) $y=(2n+1)\frac{\pi}{2}, n\in I$

83. If in triangle ABC, $\mathbf{AB}=\frac{\mathbf{u}}{|\mathbf{u}|}-\frac{\mathbf{v}}{|\mathbf{v}|}$ and $\mathbf{AC}=\frac{2\mathbf{u}}{|\mathbf{u}|}$, where $|\mathbf{u}|\neq|\mathbf{v}|$, then
(a) $1+\cos 2A+\cos 2B+\cos 2C=0$
(b) $\sin A=\cos C$
(c) projection of AC on BC is equal to BC
(d) projection of AB on BC is equal to AB

84. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three non-zero vectors satisfying the condition $\mathbf{a}\times\mathbf{b}=\mathbf{c}$ and $\mathbf{b}\times\mathbf{c}=\mathbf{a}$, then which of the following always hold(s) good?
(a) $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are orthogonal in pairs.
(b) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=|\mathbf{b}|$
(c) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=|\mathbf{c}|^2$
(d) $|\mathbf{b}|=|\mathbf{c}|$

85. Given the following information about the non-zero vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$
(i) $(\mathbf{A}\times\mathbf{B})\times\mathbf{A}=0$ (ii) $\mathbf{B}\cdot\mathbf{B}=4$
(iii) $\mathbf{A}\cdot\mathbf{B}=-6$ (iv) $\mathbf{B}\cdot\mathbf{C}=6$
Which one of the following holds good?
(a) $\mathbf{A}\times\mathbf{B}=0$ (b) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=0$ (c) $\mathbf{A}\cdot\mathbf{A}=8$ (d) $\mathbf{A}\cdot\mathbf{C}=-9$

86. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are non-zero vectors such that they are not orthogonal pairwise and such that $\mathbf{V}_1=\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ and $\mathbf{V}_2=(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$, then which of the following hold(s) good?
(a) $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (b) $\mathbf{a}$ and $\mathbf{c}$ are collinear
(c) $\mathbf{b}$ and $\mathbf{c}$ are orthogonal (d) $\mathbf{b}=\lambda\,(\mathbf{a}\times\mathbf{c})$ when λ is a scalar

87. Given three vectors
$\mathbf{U}=2\hat{\mathbf{i}}+3\hat{\mathbf{j}}-6\hat{\mathbf{k}}$, $\mathbf{V}=6\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}}$ and $\mathbf{W}=3\hat{\mathbf{i}}-6\hat{\mathbf{j}}-2\hat{\mathbf{k}}$
which of the following hold good for the vectors $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$?
(a) $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$ are linearly dependent
(b) $(\mathbf{U}\times\mathbf{V})\times\mathbf{W}=\mathbf{0}$
(c) $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$ form a triplet of mutually perpendicular vectors
(d) $(\mathbf{U}\times(\mathbf{V}\times\mathbf{W})=0$

88. Let $\mathbf{a}=2\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}}$, $\mathbf{b}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}-\hat{\mathbf{k}}$ and $\mathbf{c}=\hat{\mathbf{i}}+\hat{\mathbf{j}}-2\hat{\mathbf{k}}$ be three vectors. A vector in the plane of $\mathbf{b}$ and $\mathbf{c}$ whose projection on $\mathbf{a}$ is of magnitude $\sqrt{\frac{2}{3}}$ is
(a) $2\hat{\mathbf{i}}+3\hat{\mathbf{j}}-3\hat{\mathbf{k}}$ (b) $2\hat{\mathbf{i}}+3\hat{\mathbf{j}}+3\hat{\mathbf{k}}$
(c) $-2\hat{\mathbf{i}}-\hat{\mathbf{j}}+5\hat{\mathbf{k}}$ (d) $2\hat{\mathbf{i}}+\hat{\mathbf{j}}+5\hat{\mathbf{k}}$

89. Three vectors $\mathbf{a}$ ($|\mathbf{a}| \neq 0$), $\mathbf{b}$ and $\mathbf{c}$ are such that $\mathbf{a} \times \mathbf{b} = 3\mathbf{a} \times \mathbf{c}$. Also, $|\mathbf{a}| = |\mathbf{b}| = 1$ and $|\mathbf{c}| = \frac{1}{3}$. If the angle between $\mathbf{b}$ and $\mathbf{c}$ is 60°, then

(a) $\mathbf{b} = 3\mathbf{c} + \mathbf{a}$ (b) $\mathbf{b} = 3\mathbf{c} - \mathbf{a}$

(c) $\mathbf{a} = 6\mathbf{c} + 2\mathbf{b}$ (d) $\mathbf{a} = 6\mathbf{c} - 2\mathbf{b}$

90. Let **a, b, c** be non-zero vectors and $|\mathbf{a}| = 1$ and $\mathbf{r}$ is a non-zero vector such that $\mathbf{r} \times \mathbf{a} = \mathbf{b}$ and $\mathbf{r} \cdot \mathbf{c} = 1$, then

(a) $\mathbf{a} \perp \mathbf{b}$ (b) $\mathbf{r} \perp \mathbf{b}$

(c) $\mathbf{r} \cdot \mathbf{a} = \dfrac{1 - [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}{\mathbf{a} \cdot \mathbf{b}}$ (d) $[\mathbf{r}\,\mathbf{a}\,\mathbf{b}] = 0$

91. If $\mathbf{a}$ and $\mathbf{b}$ are two unit vectors perpendicular to each other and $\mathbf{c} = \lambda_1\mathbf{a} + \lambda_2\mathbf{b} + \lambda_3(\mathbf{a} \times \mathbf{b})$, then the following is (are) true

(a) $\lambda_1 = \mathbf{a} \cdot \mathbf{c}$

(b) $\lambda_2 = |\vec{\mathbf{b}} \times \vec{\mathbf{a}}|$

(c) $\lambda_3 = |(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}|$

(d) $\lambda_1 + \lambda_2 + \lambda_3 = (\mathbf{a} + \mathbf{b} + \mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$

92. Given three non-coplanar vectors $\mathbf{OA} = \mathbf{a}, \mathbf{OB} = \mathbf{b}, \mathbf{OC} = \mathbf{c}$.
Let S be the centre of the sphere passing through the points, O, A, B, C if $\mathbf{OS} = \mathbf{x}$, then

(a) $\mathbf{x}$ must be linear combination of $\mathbf{a}, \mathbf{b}, \mathbf{c}$

(b) $\mathbf{x}$ must be linear combination of $\mathbf{b} \times \mathbf{c}$, $\mathbf{c} \times \mathbf{a}$ and $\mathbf{a} \times \mathbf{b}$

(c) $\mathbf{x} = \dfrac{a^2(\mathbf{b} \times \mathbf{c}) + b^2(\mathbf{c} \times \mathbf{a}) + c^2(\mathbf{a} \times \mathbf{b})}{2[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]}$, $a = |\mathbf{a}|, b = |\mathbf{b}|, c = |\mathbf{c}|$

(d) $\mathbf{x} = \mathbf{a} + \mathbf{b} + \mathbf{c}$

93. If $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{i}} - \hat{\mathbf{j}}$, then the vectors $(\mathbf{a} \cdot \hat{\mathbf{i}})\hat{\mathbf{i}} + (\mathbf{a} \cdot \hat{\mathbf{j}})\hat{\mathbf{j}} + (\mathbf{a} \cdot \hat{\mathbf{k}})\hat{\mathbf{k}}, (\mathbf{b} \cdot \hat{\mathbf{i}})\hat{\mathbf{i}} + (\mathbf{b} \cdot \hat{\mathbf{j}})\hat{\mathbf{j}} + (\mathbf{b} \cdot \hat{\mathbf{j}})\hat{\mathbf{k}}$ and $\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$

(a) are mutually perpendicular

(b) are coplanar

(c) form a parallelopiped of volume 6 units

(d) form a parallelopiped of volume 3 units

94. If $\mathbf{a} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}, \mathbf{b} = y\hat{\mathbf{i}} + z\hat{\mathbf{j}} + x\hat{\mathbf{k}}, \mathbf{c} = z\hat{\mathbf{i}} + x\hat{\mathbf{j}} + y\hat{\mathbf{k}}$, then $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is

(a) parallel to $(y - z)\hat{\mathbf{i}} + (z - x)\hat{\mathbf{j}} + (x - y)\hat{\mathbf{k}}$

(b) orthogonal to $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$

(c) orthogonal to $(y + z)\hat{\mathbf{i}} + (z + x)\hat{\mathbf{j}} + (x + y)\hat{\mathbf{k}}$

(d) parallel to $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$

95. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are three non-zero vectors, then which of the following statement (s) is/are true?

(a) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}), \mathbf{b} \times (\mathbf{c} \times \mathbf{a}), \mathbf{c} \times (\mathbf{a} \times \mathbf{b})$ form a right handed system

(b) $\mathbf{c}, (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}, \mathbf{a} \times \mathbf{b}$ form a right handed system

(c) $\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a} < 0$, if $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$

(d) $\dfrac{(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{b} \times \mathbf{c})}{(\mathbf{b} \times \mathbf{c}) \cdot (\mathbf{a} \times \mathbf{c})} = -1$, if $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$

96. Let the unit vectors $\mathbf{a}$ and $\mathbf{b}$ be perpendicular and unit vector $\mathbf{c}$ is inclined at angle α to $\mathbf{a}$ and $\mathbf{b}$. If $\mathbf{c} = l\mathbf{a} + m\mathbf{b} + n(\mathbf{a} \times \mathbf{b})$, then

(a) $l = m$ (b) $n^2 = 1 - 2l^2$

(c) $n^2 = -\cos 2\alpha$ (d) $m^2 = \dfrac{1 + \cos 2\alpha}{2}$

97. If **a, b, c** are three non-zero vectors, then which of the following statement(s) is/are true?

(a) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}), \mathbf{b} \times (\mathbf{c} \times \mathbf{a}), \mathbf{c} \times (\mathbf{a} \times \mathbf{b})$ form a right handed system

(b) $\mathbf{c}, (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}, \mathbf{a} \times \mathbf{b}$ form a right handed system

(c) $\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a} < \mathbf{0}$ if $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$

(d) $\dfrac{(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{b} \times \mathbf{c})}{(\mathbf{b} \times \mathbf{c}) \cdot (\mathbf{a} \times \mathbf{c})} = -1$ if $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$

98. Let $\mathbf{a}$ and $\mathbf{b}$ be two given perpendicular vectors, which are non-zero. A vector $\mathbf{r}$ satisfying the equation $\mathbf{r} \times \mathbf{b} = \mathbf{a}$, can be .....

(a) $\mathbf{b} - \dfrac{\mathbf{a} \times \mathbf{b}}{|\mathbf{b}|^2}$ (b) $2\mathbf{b} - \dfrac{(\mathbf{a} \times \mathbf{b})}{|\mathbf{b}|^2}$

(c) $|\mathbf{a}|\mathbf{b} - \dfrac{\mathbf{a} \times \mathbf{b}}{|\mathbf{b}|^2}$ (d) $|\mathbf{b}|\mathbf{b} - \dfrac{(\mathbf{a} \times \mathbf{b})}{|\mathbf{b}|^2}$

99. If $\mathbf{a}$ and $\mathbf{b}$ are any two vectors, then possible integers(s) in the range of $\dfrac{3|\mathbf{a} + \mathbf{b}|}{2} + 2|\mathbf{a} - \mathbf{b}|$ is

(a) 2 (b) 3

(c) 4 (d) 5

100. If $\mathbf{a}$ is perpendicular to $\mathbf{b}$ and p is non-zero scalar such that $p\,\mathbf{r} + (\mathbf{r} \cdot \mathbf{b})\,\mathbf{a} = \mathbf{c}$, then $\mathbf{r}$

(a) $[\mathbf{r}\ \mathbf{a}\ \mathbf{c}] = 0$

(b) $p^2\mathbf{r} = p\mathbf{a} - (\mathbf{c} \cdot \mathbf{a})\mathbf{b}$

(c) $p^2\mathbf{r} = p\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

(d) $p^2\mathbf{r} = p\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$

101. In a four-dimensional space where unit vectors along axes are $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ and $\hat{\mathbf{l}}$ and $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ are four non-zero vectors such that no vector can be expressed as linear combination of others and $(\lambda - 1)(\mathbf{a}_1 - \mathbf{a}_2) + \mu(\mathbf{a}_2 + \mathbf{a}_3) + \gamma(\mathbf{a}_3 + \mathbf{a}_4 - 2\mathbf{a}_2) + \mathbf{a}_3 + \delta\mathbf{a}_4 = \mathbf{0}$, then

(a) $\lambda = 1$ (b) $\mu = -\dfrac{2}{3}$

(c) $\lambda = \dfrac{2}{3}$ (d) $\delta = \dfrac{1}{3}$

102. A vector $(\mathbf{d})$ is equally inclined to three vectors $\mathbf{a} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}, \mathbf{b} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\mathbf{c} = 3\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$. Let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ be three vectors in the plane of $\mathbf{a}, \mathbf{b}$; $\mathbf{b}, \mathbf{c}$; $\mathbf{c}, \mathbf{a}$ respectively, then

(a) $\mathbf{x} \cdot \mathbf{d} = 14$ (b) $\mathbf{y} \cdot \mathbf{d} = 3$

(c) $\mathbf{z} \cdot \mathbf{d} = 0$

(d) $\mathbf{r} \cdot \mathbf{d} = 0$, where $\mathbf{r} = \lambda\mathbf{x} + \mu\mathbf{y} + \delta\mathbf{z}$

103. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are non-zero, non-collinear vectors such that a vctors such that a vector $\mathbf{p} = ab\cos(2\pi - (\mathbf{a}, \mathbf{c}))\,\mathbf{c}$ and $\mathbf{a}\,\mathbf{q} = \mathbf{ac}\ \cos(\pi - (\mathbf{a}\ \mathbf{c}))$ then $\mathbf{b} + \mathbf{q}$ is
(a) parallel to $\mathbf{a}$ (b) perpendicular to $\mathbf{a}$
(c) coplanar with $\mathbf{b}$ and $\mathbf{c}$ (d) coplanar with $\mathbf{a}$ and $\mathbf{c}$

104. Given three vevtors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ such that they are non-zero , non-coplanar vectors, then which of the following are coplanar.
(a) $\mathbf{a}+\mathbf{b}, \mathbf{b}+\mathbf{c}, \mathbf{c}+\mathbf{a}$ (b) $\mathbf{a}-\mathbf{b}, \mathbf{b}+\mathbf{c}, \mathbf{c}+\mathbf{a}$
(c) $\mathbf{a}+\mathbf{b}, \mathbf{b}-\mathbf{c}, \mathbf{c}+\mathbf{a}$ (d) $\mathbf{a}+\mathbf{b}, \mathbf{b}+\mathbf{c}, \mathbf{c}-\mathbf{a}$

105. If $\mathbf{r} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \lambda(2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 4\hat{\mathbf{k}})$ and $\mathbf{r}\cdot(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}) = 3$ are the equations of a line and a plane respectively, then which of the following is incorrect?
(a) line is perpendicular to the plane
(b) line lies in the plane
(c) line is parallel to the plane but does not lie in the plane
(d) line cuts the plane obliquely

106. If vectors $\mathbf{a}$ and $\mathbf{b}$ are two adjacent sides of a parallelogram, then the vector representing the altitude of the parallelogram which is perpendicular to $\mathbf{a}$ is
(a) $\mathbf{b} + \dfrac{\mathbf{b}\times\mathbf{a}}{|\mathbf{a}|^2}$ (b) $\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}$
(c) $\mathbf{b} - \dfrac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a}$ (d) $\dfrac{\mathbf{a}\times(\mathbf{b}\times\mathbf{a})}{|\mathbf{a}|^2}$

107. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three vectors such that each of them are non-collinear, $\mathbf{a}+\mathbf{b}$ and $\mathbf{b}+\mathbf{c}$ are collinear with $\mathbf{c}$ and $\mathbf{a}$ respectively and $\mathbf{a}+\mathbf{b}+\mathbf{c} = \mathbf{k}$. Then, $(|\mathbf{k}|, |\mathbf{k}|)$ lies on
(a) $y^2 = 4ax$ (b) $x^2 + y^2 - ax - by = 0$
(c) $x^2 - y^2 = 1$ (d) $|x| + |y| = 1$

108. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are non-coplanar unit vectors also $\mathbf{b}, \mathbf{c}$ are non-collinear and $2\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}+\mathbf{c}$, then
(a) angle between $\mathbf{a}$ and $\mathbf{c}$ is 60°
(b) angle between $\mathbf{b}$ and $\mathbf{c}$ is 30°
(c) angle between $\mathbf{a}$ and $\mathbf{b}$ is 120°
(d) $\mathbf{b}$ is perpendicular to $\mathbf{c}$

109. If $\mathbf{a} = \dfrac{1}{7}(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}); \mathbf{b} = \dfrac{1}{7}(6\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}})$;
$\mathbf{c} = c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}} + c_3\hat{\mathbf{k}}$ and matrix $A = \begin{bmatrix} \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & \frac{-3}{7} \\ c_1 & c_2 & c_3 \end{bmatrix}$
and $AA^T = I$, then $\mathbf{c}$
(a) $\dfrac{3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + 2\hat{\mathbf{k}}}{7}$ (b) $\dfrac{1}{7}(3\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$
(c) $\dfrac{1}{7}(-3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$ (d) $-\dfrac{1}{7}(3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

Product of Vectors Exercise 3 : Statement I & II Type Questions

■ **Directions** (Q. Nos. 110 to 121) Each of these questions contains two statements.

Statement I (Assertion) and **Statement II** (Reason) Each of these questions also has four alternatives choices, only one of which is the correct answer. You have to select the correct choice, as given below.

(a) Statement I is true, Statement II is true and Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true but Statement II is not a correct explanation for Statement I.
(c) Statement I is true, Statement II is false.
(d) Statement I is false, Statement II is true.

110. **Statement I** A component of vector $\mathbf{b} = 4\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ in the direction perpendicular to the direction of vector $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ is $\hat{\mathbf{i}} - \hat{\mathbf{j}}$.

Statement II A component of vector in the direction $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ is $2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$.

111. **Statement I** $a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}, b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$, and $c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}} + c_3\hat{\mathbf{k}}$ are three mutually perpendicular unit vector, then $a_1\hat{\mathbf{i}} + b_1\hat{\mathbf{j}} + c_1\hat{\mathbf{k}}, a_2\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + c_2\hat{\mathbf{k}}$, and $a_3\hat{\mathbf{i}} + b_3\hat{\mathbf{j}} + c_3\hat{\mathbf{k}}$, may be mutually perpendicular unit vectors.

Statement II Value of determinant and its transpose are the same.

112. Consider the vector $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$.

Statement I $\mathbf{a}\times\mathbf{b} = (\hat{\mathbf{i}}\times\mathbf{b}).(\mathbf{b})\,\hat{\mathbf{i}} + (\hat{\mathbf{j}}\times\mathbf{a}).(\mathbf{b}\,\hat{\mathbf{j}}\times(\hat{\mathbf{k}}\times\mathbf{a}).\mathbf{b})\,\hat{\mathbf{k}}$

Statement II $\mathbf{c} = (\hat{\mathbf{i}}\,.\,\mathbf{c})\hat{\mathbf{i}} + (\hat{\mathbf{j}}\,.\,\mathbf{c})\hat{\mathbf{j}} + (\hat{\mathbf{k}}\,.\,\mathbf{c})\,\hat{\mathbf{k}}$

113. **Statement I** Distance of point $D(1, 0, -1)$ from the plane of points $A(1, -2, 0)$, $B(3, 1, 2)$ and $C(-1, 1, -1)$ is $\dfrac{8}{\sqrt{229}}$.

Statement II Volume of tetrahedron formed by the points A, B, C and D is $\dfrac{\sqrt{229}}{2}$.

114. **Statement I** $\mathbf{A} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}, \mathbf{B} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ and $\mathbf{C} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$, then $|\mathbf{A}\times(\mathbf{A}\times(\mathbf{A}\times\mathbf{B})).\mathbf{C}| = 243$

Statement II $|\mathbf{A}\times(\mathbf{A}\times(\mathbf{A}\times\mathbf{B})).\mathbf{C}| = |\mathbf{A}|^2\,|[\mathbf{ABC}]|$

115. **Statement I** The number of vectors of unit length and perpendicular to both the vectors $\hat{\mathbf{i}}+\hat{\mathbf{j}}$ and $\hat{\mathbf{j}}+\hat{\mathbf{k}}$ is zero.

Statement II $\mathbf{a}$ and $\mathbf{b}$ are two non-zero and non-parallel vectors it is true that $\mathbf{a}\times\mathbf{b}$ is perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b}$.

116. **Statement I** (S_1) : If $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ are non-collinear points. Then, every point (x, y) in the plane of ΔABC, can be expressed in the form $\left(\frac{kx_1+lx_2+mx_3}{k+l+m}, \frac{ky_1+ly_2+my_3}{k+l+m}\right)$

Statement II (S_2) The condition for coplanarity of four points $A(\mathbf{a}), B(\mathbf{b}), C(\mathbf{c}), D(\mathbf{d})$ is that there exists scalars, l, m, n, p not all zeros such that

$$l\mathbf{a}+m\mathbf{b}+n\mathbf{c}+p\mathbf{d}=\mathbf{0}$$

where $l+m+n+p=0$

117. If $\mathbf{a}, \mathbf{b}$ are non-zero vectors such that $|\mathbf{a}+\mathbf{b}|=|\mathbf{a}-2\mathbf{b}|$, then

Statement I Least value of $\mathbf{a}\cdot\mathbf{b}+\frac{4}{|\mathbf{b}|^2+2}$ is $2\sqrt{2}-1$

Statement II The expression $\mathbf{a}\cdot\mathbf{b}+\frac{4}{|\mathbf{b}|^2+2}$ is least when magnitude of $\mathbf{b}$ is $\sqrt{2\tan\left(\frac{\pi}{8}\right)}$

118. **Statement I** If $\mathbf{a}=3\hat{\mathbf{i}}-3\hat{\mathbf{j}}+\hat{\mathbf{k}}$, $\mathbf{b}=-\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $\mathbf{c}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $\mathbf{d}=2\hat{\mathbf{i}}-\hat{\mathbf{j}}$, then there exist real numbers α, β, γ such that $\mathbf{a}=\alpha\mathbf{b}+\beta\mathbf{c}+\gamma\mathbf{d}$

Statement II $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are four vectors in a 3-dimensional space. If $\mathbf{b}, \mathbf{c}, \mathbf{d}$ are non-coplanar, then there exist real numbers α, β, γ such that $\mathbf{a}=\alpha\mathbf{b}+\beta\mathbf{c}+\gamma\mathbf{d}$.

119. **Statement I** Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ are position vector four points A, B, C and D and $3\mathbf{a}-2\mathbf{b}+5\mathbf{c}-6\mathbf{d}=\mathbf{0}$, then points A, B, C and D are coplanar.

Statement II Three non-zero, linearly dependent coinitial vectors (**PQ**, **PR** and **PS**) are coplanar.

120. If $\mathbf{a}=\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}}$, $\mathbf{b}=2\hat{\mathbf{i}}+\hat{\mathbf{j}}-3\hat{\mathbf{k}}$ and $\mathbf{r}$ is a vector satisfying $2\mathbf{r}+\mathbf{r}\times\mathbf{a}=\mathbf{b}$.

Statement I $\mathbf{r}$ can be expressed in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a}\times\mathbf{b}$.

Statement II $\mathbf{r}=\frac{1}{7}(7\hat{\mathbf{i}}+5\hat{\mathbf{j}}-9\hat{\mathbf{k}}+\mathbf{a}\times\mathbf{b})$

121. Let $\hat{\mathbf{a}}$ and $\hat{\mathbf{c}}$ be units vectors at an angle $\frac{\pi}{3}$ with each other. If $(\hat{\mathbf{a}}\times(\hat{\mathbf{b}}\times\hat{\mathbf{c}}))\cdot(\hat{\mathbf{a}}\times\hat{\mathbf{c}})=5$ then

Statement I $[\hat{\mathbf{a}}\ \hat{\mathbf{b}}\ \hat{\mathbf{c}}]=10$

because

Statement II $[\mathbf{x}\ \mathbf{y}\ \mathbf{z}]=0$, if $\mathbf{x}=\mathbf{y}$ or $\mathbf{y}=\mathbf{z}$ or $\mathbf{x}=\mathbf{z}$

Product of Vectors Exercise 4 :
Passage Based Questions

Passage I

(Q. Nos. 122-124)

Consider three vectors $\mathbf{p}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$, $\mathbf{q}=2\hat{\mathbf{i}}+4\hat{\mathbf{j}}-\hat{\mathbf{k}}$ and $\mathbf{r}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+3\hat{\mathbf{k}}$ and let $\mathbf{s}$ be a unit vector, then

122. $\mathbf{p}, \mathbf{q}$ and $\mathbf{r}$ are

(a) linearly dependent
(b) can form the sides of a possible triangle
(c) such that the vectors $(\mathbf{q}-\mathbf{r})$ is orthogonal to $\mathbf{p}$
(d) such that each one of these can be expressed as a linear combination of the other two

123. If $(\mathbf{p}\times\mathbf{q})\times\mathbf{r}=u\,\mathbf{p}+v\,\mathbf{q}+w\,\mathbf{r}$, then $(u+v+w)$ is equal to

(a) 8 (b) 2
(c) -2 (d) 4

124. The magnitude of the vector $(\mathbf{p}\cdot\mathbf{s})(\mathbf{q}\times\mathbf{r})+(\mathbf{q}\cdot\mathbf{s})(\mathbf{r}\times\mathbf{p})+(\mathbf{r}\cdot\mathbf{s})(\mathbf{p}\times\mathbf{q})$ is

(a) 4 (b) 8
(c) 18 (d) 2

Passage II

(Q. Nos. 125-127)

Consider the three vectors $\mathbf{p}, \mathbf{q}$ and $\mathbf{r}$ such that $\mathbf{p}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $\mathbf{q}=\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}}$; $\mathbf{p}\times\mathbf{r}=\mathbf{q}+c\mathbf{p}$ and $\mathbf{p}\cdot\mathbf{r}=2$

125. The value of $[\mathbf{p}\ \mathbf{q}\ \mathbf{r}]$ is

(a) $-\frac{5\sqrt{2}c}{|\mathbf{r}|}$ (b) $-\frac{8}{3}$
(c) 0 (d) greater than 0

126. If $\mathbf{x}$ is a vector such that $[\mathbf{p}\ \mathbf{q}\ \mathbf{r}]\,\mathbf{x}=(\mathbf{p}\times\mathbf{q})\times\mathbf{r}$, then $\mathbf{x}$ is

(a) $c\,(\hat{\mathbf{i}}-2\hat{\mathbf{j}}+\hat{\mathbf{k}})$ (b) a unit vector
(c) indeterminate, as $[\mathbf{p}\ \mathbf{q}\ \mathbf{r}]$ (d) $-(\hat{\mathbf{i}}-2\hat{\mathbf{j}}+\hat{\mathbf{k}})/2$

127. If y is a vector satisfying $(1+c)\,\mathbf{y}=\mathbf{p}\times(\mathbf{q}\times\mathbf{r})$, then the vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{r}$

(a) are collinear (b) are coplanar
(c) represent the coterminus edges of a tetrahedron whose volume is c cu units
(d) represent the coterminus edge of a paralloepiped whose volume is c cu units

Passage III

(Q. Nos. 128-130)

Let P and Q are two points on the curve

$y = \log_{1/2}(x - 0.5) + \log_2 \sqrt{4x^2 - 4x + 1}$

and P is also on the circle $x^2 + y^2 = 10$, Q lies inside the given circle such that its abscissa is an integer.

128. The coordinates of P are given by

(a) (1, 2) (b) (2, 4) (c) (3, 1) (d) (3, 5)

129. $\mathbf{OP} \cdot \mathbf{OQ}$, O being the origin is

(a) 4 or 7 (b) 4 or 2 (c) 2 or 3 (d) 7 or 8

130. Max $\{|\mathbf{PQ}|\}$ is

(a) 1 (b) 4 (c) 0 (d) 2

Passage IV

(Q. Nos. 131 to 134)

If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are three given non-coplanar vectors and any arbitrary vector $\mathbf{r}$ is in space, where

$$\Delta_1 = \begin{vmatrix} \mathbf{r}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{a} \\ \mathbf{r}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{b} \\ \mathbf{r}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix};\ \Delta_2 = \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{r}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{a} \\ \mathbf{a}\cdot\mathbf{b} & \mathbf{r}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{b} \\ \mathbf{a}\cdot\mathbf{c} & \mathbf{r}\cdot\mathbf{c} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{a} & \mathbf{r}\cdot\mathbf{a} \\ \mathbf{a}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{b} & \mathbf{r}\cdot\mathbf{b} \\ \mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c} & \mathbf{r}\cdot\mathbf{c} \end{vmatrix};\quad \Delta = \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{a} \\ \mathbf{a}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{b} \\ \mathbf{a}\cdot\mathbf{c} & \mathbf{c}\cdot\mathbf{c} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix}$$

131. The vector $\mathbf{r}$ is expressible in the form

(a) $\mathbf{r} = \frac{\Delta_1}{2\Delta}\mathbf{a} + \frac{\Delta_2}{2\Delta}\mathbf{b} + \frac{\Delta_3}{2\Delta}\mathbf{c}$

(b) $\mathbf{r} = \frac{2\Delta_1}{\Delta}\mathbf{a} + \frac{2\Delta_2}{\Delta}\mathbf{b} + \frac{2\Delta_3}{\Delta}\mathbf{c}$

(c) $\mathbf{r} = \frac{\Delta}{\Delta_1}\mathbf{a} + \frac{\Delta}{\Delta_2}\mathbf{b} + \frac{\Delta}{\Delta_3}\mathbf{c}$

(d) $\mathbf{r} = \frac{\Delta_1}{\Delta}\mathbf{a} + \frac{\Delta_2}{\Delta}\mathbf{b} + \frac{\Delta_3}{\Delta}\mathbf{c}$

132. The vector $\mathbf{r}$ is expressible as

(a) $\mathbf{r} = \frac{[\mathbf{r}\,\mathbf{b}\,\mathbf{c}]}{2[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]}\mathbf{a} + \frac{[\mathbf{r}\mathbf{c}\mathbf{a}]}{2[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]}\mathbf{b} + \frac{[\mathbf{r}\,\mathbf{a}\,\mathbf{b}]}{2[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]}\mathbf{c}$

(b) $\mathbf{r} = \frac{2[\mathbf{r}\,\mathbf{b}\,\mathbf{c}]}{[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]}\mathbf{a} + \frac{2[\mathbf{r}\,\mathbf{c}\,\mathbf{b}]}{[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]}\mathbf{b} + \frac{2[\mathbf{r}\,\mathbf{a}\,\mathbf{b}]}{[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]}\mathbf{c}$

(c) $\mathbf{r} = \frac{1}{[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]}([\mathbf{r}\,\mathbf{b}\,\mathbf{c}]\mathbf{a} + [\mathbf{r}\,\mathbf{c}\,\mathbf{a}]\mathbf{b} + [\mathbf{r}\,\mathbf{a}\,\mathbf{b}]\,\mathbf{c})$

(d) None of the above

133. If vector is expressible as $\mathbf{r} = x\,\mathbf{a} + y\,\mathbf{b} + g\,\mathbf{c}$, then

(a) $\mathbf{a} = \frac{1}{[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]}[(\mathbf{a}\,.\,\mathbf{a})(\mathbf{b}\times\mathbf{c}) + (\mathbf{b}\,.\,\mathbf{b})(\mathbf{c}\times\mathbf{a}) + (\mathbf{c}\,.\,\mathbf{c})(\mathbf{a}\times\mathbf{b})]$

(b) $\mathbf{a} = \frac{1}{[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]}[(\mathbf{a}\,.\,\mathbf{a})(\mathbf{b}\times\mathbf{c}) + (\mathbf{b}\,.\,\mathbf{a})(\mathbf{c}\times\mathbf{a}) + (\mathbf{a}\,.\,\mathbf{a})(\mathbf{a}\times\mathbf{b})]$

(c) $\mathbf{a} = (\mathbf{a}\,.\,\mathbf{a})(\mathbf{b}\times\mathbf{c}) + (\mathbf{a}\,.\,\mathbf{b})(\mathbf{c}\times\mathbf{a}) + (\mathbf{c}\,.\,\mathbf{a})(\mathbf{a}\times\mathbf{b})$

(d) None of the above

134. The value of $\begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{a}\,.\,\mathbf{p} & \mathbf{b}\,.\,\mathbf{p} & \mathbf{c}\,.\,\mathbf{p} \\ \mathbf{a}\,.\,\mathbf{q} & \mathbf{b}\,.\,\mathbf{q} & \mathbf{c}\,.\,\mathbf{q} \end{vmatrix}$ is

(a) $(\mathbf{p}\times\mathbf{q})\,[\mathbf{a}\times\mathbf{b}\;\mathbf{b}\times\mathbf{c}\;\mathbf{c}\times\mathbf{a}]$

(b) $2(\mathbf{p}\times\mathbf{q})\,[\mathbf{a}\times\mathbf{b}\;\mathbf{b}\times\mathbf{c}\;\mathbf{c}\times\mathbf{a}]$

(c) $4(\mathbf{p}\times\mathbf{q})[\mathbf{a}\times\mathbf{b}\;\mathbf{b}\times\mathbf{c}\;\mathbf{c}\times\mathbf{a}]$

(d) $(\mathbf{p}\times\mathbf{q})\,\sqrt{|[\mathbf{a}\times\mathbf{b}\;\mathbf{b}\times\mathbf{c}\;\mathbf{c}\times\mathbf{a}]|}$

Passage V

(Q. Nos. 135-136)

Let $g(x) = \int_0^x (3t^2 + 2t + 9)\,dt$ and $f(x)$ be a decreasing function, $\forall\ x \geq 0$ such that $\mathbf{AB} = f(x)\hat{\mathbf{i}} + g(x)\hat{\mathbf{j}}$ and $\mathbf{AC} = g(x)\hat{\mathbf{i}} + f(x)\hat{\mathbf{j}}$ are the two smallest sides of a ΔABC whose circumcentre lies outside the triangle, $\forall\ x > 0$.

135. Which of the following is true (for $x > 0$)?

(a) $f(x) > 0,\ g(x) < 0$

(b) $f(x) < 0,\ g(x) < 0$

(c) $f(x) > 0,\ g(x) > 0$

(d) $f(x) < 0, g(x) > 0$

136. $\lim\limits_{t\to 0}\ \lim\limits_{x\to\infty}\left(\cot\left(\frac{\pi}{4}(1-t^2)\right)\right)^{f(x)\cdot g(x)}$ is equal to

(a) 0 (b) 1

(c) e (d) does not exist

Passage VI

(Q. Nos 137-139)

Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be the vector , such that $|\mathbf{x}| = |\mathbf{y}| = |\mathbf{z}| = \sqrt{2}$ and $\mathbf{x}, \mathbf{y}, \mathbf{z}$ make angles of 60° with each other also,

$\mathbf{x}\times(\mathbf{y}\times\mathbf{z}) = \mathbf{a}$

$\mathbf{y}\times(\mathbf{z}\times\mathbf{x}) = \mathbf{b}$ and $\mathbf{x}\times\mathbf{y} = \mathbf{c}$, then

137. The value of $\mathbf{x}$ is

(a) $(\mathbf{a}+\mathbf{b})\times\mathbf{c} - (\mathbf{a}+\mathbf{b})$

(b) $(\mathbf{a}+\mathbf{b}) - (\mathbf{a}+\mathbf{b})\times\mathbf{c}$

(c) $\frac{1}{2}\{(\mathbf{a}+\mathbf{b})\times\mathbf{c} - (\mathbf{a}+\mathbf{b})\}$

(d) None of the above

138. The value of $\mathbf{y}$ is

(a) $\frac{1}{2}[(\mathbf{a}+\mathbf{b}) + (\mathbf{a}+\mathbf{b})\times\mathbf{c}]$ (b) $2[(\mathbf{a}+\mathbf{b}) + (\mathbf{a}+\mathbf{b})\times\mathbf{c}]$

(c) $4[(\mathbf{a}+\mathbf{b}) + (\mathbf{a}+\mathbf{b})\times\mathbf{c}]$ (d) None of these

139. The value of $\mathbf{z}$ is

(a) $\frac{1}{2}[(\mathbf{b}-\mathbf{a})\times\mathbf{c} + (\mathbf{a}+\mathbf{b})]$

(b) $\frac{1}{2}[(\mathbf{b}-\mathbf{a}) + (\mathbf{a}+\mathbf{b})\times\mathbf{c}]$

(c) $(\mathbf{b}-\mathbf{a})\times\mathbf{c} + (\mathbf{a}+\mathbf{b})$

(d) None of the above

Passage VII
(Q. Nos. 140-142)

a, b, c are non-zero unit vectors inclined pairwise with the same angle θ. p, q, r are non-zero scalars satisfying $\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}=p\mathbf{a}+q\mathbf{b}+r\mathbf{c}$. Now, answer the following questions.

140. Volume of parallelopiped with edges **a, b** and **c** is equal to

(a) $p+(q+r)\cos\theta$ (b) $(p+q+r)\cos\theta$
(c) $2p-(r+q)\cos\theta$ (d) None of these

141. $\frac{q}{p}+2\cos\theta$ is equal to

(a) 1 (b) 2 **[a b c]**
(c) 0 (d) None of these

142. $|(p+q)\cos\theta+r|$ is equal to

(a) $(1+\cos\theta)\sqrt{1-2\cos\theta}$

(b) $2\sin^2\frac{\theta}{2}|\sqrt{1+2\cos\theta}|$

(c) $(1-\sin\theta)|\sqrt{1+2\cos\theta}|$

(d) None of the above

Product of Vectors Exercise 5 : Matching Type Questions

143. Given two vectors $\mathbf{a}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\hat{\mathbf{k}}$ and $\mathbf{b}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}}$. Match the Column I with Column II and mark the correct option from the codes given below.

	Column I		Column II
A.	A vector coplanar with **a** and **b**	p.	$-3\hat{\mathbf{i}}+3\hat{\mathbf{j}}+4\hat{\mathbf{k}}$
B.	A vector which is perpendicular to both **a** and **b**	q.	$2\hat{\mathbf{i}}-2\hat{\mathbf{j}}+3\hat{\mathbf{k}}$
C.	A vector which is equally inclined to **a** and **b**	r.	$\hat{\mathbf{i}}+\hat{\mathbf{j}}$
D.	A vector which forms a triangle with **a** and **b**	s.	$\hat{\mathbf{i}}-\hat{\mathbf{j}}+5\hat{\mathbf{k}}$

144. Volume of parallelepiped formed by vectors $\mathbf{a}\times\mathbf{b}$, $\mathbf{b}\times\mathbf{c}$ and $\mathbf{c}\times\mathbf{a}$ is 36 sq units.

	Column I		Column II
A.	Volume of parallelopiped formed by vectors **a**, **b** and **c** is	p.	0 sq units
B.	Volume of tetrahedron formed by vectors **a**, **b** and **c** is	q.	12 sq units
C.	Volume of parallelopiped formed by vectors $\mathbf{a}+\mathbf{b}$, $\mathbf{b}+\mathbf{c}$ and $\mathbf{c}+\mathbf{a}$ is	r.	6 sq units
D.	Volume of parallelopiped formed by vectors $\mathbf{a}-\mathbf{b}$, $\mathbf{b}-\mathbf{c}$ and $\mathbf{c}-\mathbf{a}$ is	s.	1 sq units

145. Match the statement of Column I with values of Column II.

	Column I		Column II
A.	Let O be an interior point of ΔABC such that $\mathbf{OA}+2\mathbf{OB}+3\mathbf{OC}=0$, then the ratio of the area of ΔABC to the area of ΔAOC, with O as the origin	p.	0
B.	$\mathbf{A}\cdot\mathbf{B}=\mathbf{A}\cdot\mathbf{C}=0$, $\mathbf{B}\cdot\mathbf{C}=3/2$ $\mathbf{A}\cdot\mathbf{A}=\mathbf{B}\cdot\mathbf{B}=\mathbf{C}\cdot\mathbf{C}=1$, **[A B C]** =	q.	$\frac{1}{2}$
C.	If **a**, **b**, **c** and **d** are non-zero vectors such that no three of them are in the same plane and no two are orthogonal, then the value of the scalar $\frac{(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\times\mathbf{d})+(\mathbf{c}\times\mathbf{a})\cdot(\mathbf{b}\times\mathbf{d})}{(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{d}\times\mathbf{c})}$ is	r.	2

146. Match the statement of Column I with values of Column II.

	Column I		Column II
A.	$\lvert\mathbf{a}\rvert=\lvert\mathbf{b}\rvert=2$, $\mathbf{x}=\mathbf{a}+\mathbf{b}$, $\mathbf{y}=\mathbf{a}-\mathbf{b}$ If $\lvert\mathbf{x}\times\mathbf{y}\rvert=2\{\lambda-(\mathbf{a}\cdot\mathbf{b})^2\}^{1/2}$, then value of λ is	p.	4
B.	The non-zero value of λ for which angle between $\lambda\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $\hat{\mathbf{i}}+\lambda\hat{\mathbf{j}}+\hat{\mathbf{k}}$ is $\frac{\pi}{3}$	q.	42

	Column I		Column II
C.	The non-zero value of k for which the lines $kx - 4y + 7z + 16 = 0 = 4x + 3y - 2z + 3$ and $x - 3y + 4z + 6 = 0 = x - y + z + 1$ are coplanar is	r.	16
D.	If $\|\mathbf{a}\| = \|\mathbf{b}\| = 1$ and $\|\mathbf{c}\| = 2$, then maximum value of $\|\mathbf{a} - 2\mathbf{b}\|^2 + \|\mathbf{b} - 2\mathbf{c}\|^2 + \|\mathbf{c} - 2\mathbf{a}\|^2$ is	s.	7
		t.	5

147. Match the statement of Column I with values of Column II.

	Column I		Column II
A.	Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be the three vectors such that $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}+\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}+\mathbf{b}) = 0$ and $\|\mathbf{a}\| = 1, \|\mathbf{b}\| = 4, \|\mathbf{c}\| = 8$, then $\|\mathbf{a}+\mathbf{b}+\mathbf{c}\|$ is equal to	p.	$\frac{5}{2}$
B.	If $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are vectors reciprocal to the non-coplanar vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ then $[\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3][\mathbf{b}_1\ \mathbf{b}_2\ \mathbf{b}_3]$ is equal to	q.	9
C.	$ABCD$ is a quadrilateral with $AB = \mathbf{a}$, $AD = \mathbf{b}$ and $AC = 2\mathbf{a} + 3\mathbf{b}$. If its area is α times the area of the parallelogram with AB, AD as its adjacent sides, then α is equal to	r.	8
D.	If $\mathbf{d} = x(\mathbf{a}\times\mathbf{b}) + y(\mathbf{b}\times\mathbf{c}) + z(\mathbf{c}\times\mathbf{a})$ and $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \frac{1}{8}$, then $x + y + z = R(\mathbf{a}+\mathbf{b}+\mathbf{c})\cdot\mathbf{d}$, where R = adjacent sides, then α is equal to	s.	1

Product of Vectors Exercise 6 :
Integer Type Questions

148. Let $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$ and $\hat{\mathbf{w}}$ are three unit vectors, the angle between $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ is twice that of the angle between $\hat{\mathbf{u}}$ and $\hat{\mathbf{w}}$ and $\hat{\mathbf{v}}$ and $\hat{\mathbf{w}}$, then $[\hat{\mathbf{u}}\ \hat{\mathbf{v}}\ \hat{\mathbf{w}}]$ is equal. to

149. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three vectors such that $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 1$, then find the value of $[\mathbf{a}+\mathbf{b}\ \mathbf{b}+\mathbf{c}\ \mathbf{c}+\mathbf{a}] + [\mathbf{a}\times\mathbf{b}\ \mathbf{b}\times\mathbf{c}\ \mathbf{c}\times\mathbf{a}] + [\mathbf{a}+(\mathbf{b}\times\mathbf{c})\ \mathbf{b}\times(\mathbf{c}\times\mathbf{a})\ \mathbf{c}\times(\mathbf{a}\times\mathbf{b})]$.

150. If $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ are the three unit vectors and α, β and γ are scalars such that $\hat{\mathbf{c}} = \alpha\hat{\mathbf{a}} + \beta\hat{\mathbf{b}} + \gamma(\hat{\mathbf{a}}\times\hat{\mathbf{b}})$. If is given that $\hat{\mathbf{a}}\cdot\hat{\mathbf{b}} = 0$ and $\hat{\mathbf{c}}$ makes equal angle with both $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$, then evaluate $\alpha^2 + \beta^2 + \gamma^2$.

151. The three vectors $\hat{\mathbf{i}} + \hat{\mathbf{j}}$, $\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\hat{\mathbf{k}} + \hat{\mathbf{i}}$ taken two at a time form three planes. If V be the volume of the tetrahedron having adjacent sides as the three unit vectors drawn perpendicular to those three planes, then find the value of $9\sqrt{3}V$.

152. Let $\hat{\mathbf{c}}$ be a unit vector coplanar with $\mathbf{a} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $\mathbf{b} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ such that $\hat{\mathbf{c}}$ is perpendicular to $\mathbf{a}$. If P be the projection of $\hat{\mathbf{c}}$ along $\mathbf{b}$, where $P = \frac{\sqrt{11}}{k}$ then find k.

153. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three vectors having magnitudes 1, 2 and 3, respectively satisfy the relation $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 6$. If $\hat{\mathbf{d}}$ is a unit vector coplanar with $\mathbf{b}$ and $\mathbf{c}$ such that $\mathbf{b}\cdot\hat{\mathbf{d}} = 1$, then evaluate $|(\mathbf{a}\times\mathbf{c})\cdot\hat{\mathbf{d}}|^2 + |(\mathbf{a}\times\mathbf{c})\times\hat{\mathbf{d}}|^2$.

154. Let $A(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 5\hat{\mathbf{k}})$, $B(-\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$ and $C(\lambda\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + \mu\hat{\mathbf{k}})$ are vertices of a triangle and its median through A is equally inclined to the positive directions of the axes. The value of $2\lambda - \mu$ is equal to

155. If V is the volume of the parallelopiped having three coterminous edges as $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the volume of the parallelopiped having three coterminous edges as $\alpha = (\mathbf{a}\cdot\mathbf{a})\mathbf{a} + (\mathbf{a}\cdot\mathbf{b})\mathbf{b} + (\mathbf{a}\cdot\mathbf{c})\mathbf{c}$, $\beta = (\mathbf{b}\cdot\mathbf{a})\mathbf{a} + (\mathbf{b}\cdot\mathbf{b})\mathbf{b} + (\mathbf{b}\cdot\mathbf{c})\mathbf{c}$ and $\gamma = (\mathbf{c}\cdot\mathbf{a})\mathbf{a} + (\mathbf{c}\cdot\mathbf{b})\mathbf{b} + (\mathbf{c}\cdot\mathbf{c})\mathbf{c}$ is V^λ, then $\lambda =$

156. If $\mathbf{a}$, $\mathbf{b}$ are vectors perpendicular to each other and $|\mathbf{a}| = 2, |\mathbf{b}| = 3, \mathbf{c}\times\mathbf{a} = \mathbf{b}$, then the least value of $2|\mathbf{c} - \mathbf{a}|$ is

157. If M and N are the mid-point of the diagonals AC and BD, respectively of a quadrilateral $ABCD$, then $AB + AD + CB + CD = kMN$, where $k =$ ……. .

158. If $\mathbf{a}\times\mathbf{b} = \mathbf{c}$, $\mathbf{b}\times\mathbf{c} = \mathbf{a}$, $\mathbf{c}\times\mathbf{a} = \mathbf{b}$. If vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are forming a right handed system, then the volume of tetrahedron formed by vectors $3\mathbf{a} - 2\mathbf{b} + 2\mathbf{c}$, $-a - 2\mathbf{c}$ and $2\mathbf{a} - 3\mathbf{b} + 4\mathbf{c}$ is

159. Let $\mathbf{a}$ and $\mathbf{c}$ be unit vectors inclined at $\frac{\pi}{3}$ with each other. If $(\mathbf{a}\times(\mathbf{b}\times\mathbf{c}))\cdot(\mathbf{a}\times\mathbf{c}) = 5$, then $-[\mathbf{abc}] - 1 =$

160. Volume of parallelopiped formed by vectors $\mathbf{a}\times\mathbf{b}$, $\mathbf{b}\times\mathbf{c}$ and $\mathbf{c}\times\mathbf{a}$ is 36 sq units, then the volume of the parallelopiped formed by the vectors.

161. If α and β are two perpendicular unit vectors such that $\mathbf{x} = \hat{\beta} - (\alpha \times \mathbf{x})$; then the value of $4|\mathbf{x}|^2$ is.

162. The volume of the tetrahedron whose vertices are the points with position vectors $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $-\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$, $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 7\hat{\mathbf{k}}$ and $3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + \lambda\hat{\mathbf{k}}$ is 22, then the digit at unit place of λ is.

163. Volume of a tetrahedron whose coterminous edges are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is 3 and volume of a parallelopiped whose coterminous edges are $\mathbf{a} + \mathbf{b} - \mathbf{c}$, $\mathbf{a} - \mathbf{b}$, $\mathbf{b} - \mathbf{c}$ is V. Then, units digit of V is.

Product of Vectors Exercise 7 :
Subjective Type Questions

164. Prove Cauchy-Schwartz inequality $(\mathbf{a} \,.\, \mathbf{b})^2 \le |\mathbf{a}|^2 \cdot |\mathbf{b}|^2$

165. Two points P and Q are given in the rectangular cartesian coordinates in the curve $y = 2^{x+2}$, such that $\mathbf{OP} \,.\, \hat{\mathbf{i}} = -1$ and $\mathbf{OQ} \cdot \hat{\mathbf{i}} = 2$, where $\hat{\mathbf{i}}$ is a unit vector along the X-axis. Find the magnitude of $\mathbf{OQ} - 4\mathbf{OP}$.

166. O is the origin and A is a fixed point on the circle of radius $\mathbf{a}$ with centre O. The vector $\mathbf{OA}$ is denoted by $\mathbf{a}$. A variable point P lie on the tangent at A and $\mathbf{OP} = \mathbf{r}$. Show that $\mathbf{a} \,.\, \mathbf{r} = a^2$. Hence, if $P(x, y)$ and $A(x_1, y_1)$ deduce the equation of tangent at A to this circle.

167. If a is real constant and A, B and C are variable angles and $\sqrt{a^2 - 4}\tan A + a\tan B + \sqrt{a^2 + 4}\tan C = 6a$, then find the least value of $\tan^2 A + \tan^2 B + \tan^2 C$.

168. Given, the angles A, B and C of ΔABC. Find $\cos \angle BAM$, where M is mid-point of BC.

169. Find the perpendicular distance of $A(1, 4, -2)$ from the segment BC, where $B(2, 1, -2)$ and $C(0, -5, 1)$.

170. Given angles, A, B and C of $\Delta\ ABC$. Let M be the mid-point of segment $\mathbf{AB}$ and let D be the foot of the bisector of $\angle C$. Find the ratio of $\dfrac{\text{Area of } \Delta CDM}{\text{Area of } \Delta ABC}$ and also $\cos\phi = \cos \angle DCM$.

171. In the ΔABC a point P is taken on the side AB such that $AP : BP = 1 : 2$ and a point Q is taken on the side BC such that $CQ : BQ = 2 : 1$. If R be the point of intersection of lines AQ and CP, using vector find the area of $\Delta\ ABC$, if it is known that area of ΔABC is one unit.

172. If one diagonal of a quadrilateral bisects the other, then it also bisects the quadrilateral.

173. Two forces $F_1 = \{2, 3\}$ and $F_2 = \{4, 1\}$ are specified relative to a general cartesian form. Their points of application are respectively, $A = (1, 1)$ and $B = (2, 4)$. Find the coordinates of the resultant and the equation of the straight line l containing it.

174. A non-zero vector $\mathbf{a}$ is parallel to the line of intersection of the plane determined by the vectors, $\hat{\mathbf{i}}$, $\hat{\mathbf{i}} + \hat{\mathbf{j}}$ and the plane determined by the vectors $\hat{\mathbf{i}} - \hat{\mathbf{j}}$ and $\hat{\mathbf{i}} + \hat{\mathbf{k}}$. Find the angle between $\mathbf{a}$ and $\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

175. The vector $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ turns through a right angle while passing through the positive X-axis on the way. Find the vector in its new position.

176. Let $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ are unit vectors and $\mathbf{w}$ is a vector such that $\hat{\mathbf{u}} \times \hat{\mathbf{v}} + \hat{\mathbf{u}} = \mathbf{w}$ and $\mathbf{w} \times \hat{\mathbf{u}} = \hat{\mathbf{v}}$, then find the value of $[\hat{\mathbf{u}}\ \hat{\mathbf{v}}\ \mathbf{w}]$.

177. $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are three vectors given by $2\hat{\mathbf{i}} + \hat{\mathbf{k}}$, $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $4\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$ Then, find $\mathbf{R}$, which satisfies the relation $\mathbf{R} \times \mathbf{B} = \mathbf{C} \times \mathbf{B}$ and $\mathbf{R} \cdot \mathbf{A} = 0$.

178. If $\mathbf{x} \cdot \mathbf{a} = 0$, $\mathbf{x} \cdot \mathbf{b} = 1$, $[\mathbf{x}\ \mathbf{a}\ \mathbf{b}] = 1$ and $\mathbf{a} \cdot \mathbf{b} \ne 0$, then find $\mathbf{x}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

179. Let $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and z be unit vectors such that $\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}} = \mathbf{a}$, $\hat{\mathbf{x}} \times (\hat{\mathbf{y}} \times \hat{\mathbf{z}}) = \mathbf{b}, (\hat{\mathbf{x}} \times \hat{\mathbf{y}}) \times \hat{\mathbf{z}} = \mathbf{c}, \mathbf{a} \cdot \hat{\mathbf{x}} = \dfrac{3}{2}, \mathbf{a} \cdot \hat{\mathbf{y}} = \dfrac{7}{4}$ and $|\mathbf{a}| = 2$. Find $\mathbf{x}$, $\mathbf{y}$ and z in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

180. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three mutually perpendicular vectors of equal magnitude. If the vector $\mathbf{x}$ satisfies the equation. $\mathbf{a} \times \{(\mathbf{x} - \mathbf{b}) \times \mathbf{a}\} + \mathbf{b} \times \{(\mathbf{x} - \mathbf{c}) \times \mathbf{b}\} + \mathbf{c} \times \{(\mathbf{x} - \mathbf{a}) \times \mathbf{c}\} = 0$ then find $\mathbf{x}$.

181. Given vectors $\mathbf{CB} = \mathbf{a}$, $\mathbf{CA} = \mathbf{b}$ and $\mathbf{CO} = \mathbf{x}$, where O is the centre of circle circumscribed about $\Delta\ ABC$, then find vector $\mathbf{x}$.

Product of Vectors Exercise 8 : Questions Asked in Previous Years' Exam

(i) JEE Advanced & IIT-JEE

182. Let O be the origin and let PQR be an arbitrary triangle. The point S is such that

$$\mathbf{OP}\cdot\mathbf{OQ}+\mathbf{OR}\cdot\mathbf{OS}=\mathbf{OR}\cdot\mathbf{OP}+\mathbf{OQ}\cdot\mathbf{OS}$$
$$=\mathbf{OQ}\cdot\mathbf{OR}+\mathbf{OP}\cdot\mathbf{OS}$$

Then the triangle PQR has S as its **[Single Correct Type, 2017 Adv.]**

(a) centroid (b) orthocentre
(c) incentre (d) circumcentre

Passage

(Q. Nos. 183-184)

Let O be the origin and OX, OY, OZ be three unit vectors in the directions of the sides, QR, RP, PQ respectively of a ΔPQR. **[Passage Type Question, 2017 Adv.]**

183. If the triangle PQR varies, then the minimum value of $\cos(P+Q)+\cos(Q+R)+\cos(R+P)$ is

(a) $-\frac{3}{2}$ (b) $\frac{3}{2}$ (c) $\frac{5}{3}$ (d) $-\frac{5}{3}$

184. $|\mathbf{OX}\times\mathbf{OY}|=$

(a) $\sin(P+Q)$ (b) $\sin(P+R)$
(c) $\sin(Q+R)$ (d) $\sin 2R$

185. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three unit vectors such that $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\frac{\sqrt{3}}{2}(\mathbf{b}+\mathbf{c})$. If $\mathbf{b}$ is not parallel to $\mathbf{c}$, then the angle between $\mathbf{a}$ and $\mathbf{b}$ is **[Single Correct Type, 2016 Adv.]**

(a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\frac{2\pi}{3}$ (d) $\frac{5\pi}{6}$

186. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three non-zero vectors such that no two of them are collinear and $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=\frac{1}{3}|\mathbf{b}||\mathbf{c}|\mathbf{a}$. If θ is the angle between vectors $\mathbf{b}$ and $\mathbf{c}$, then a value of $\sin\theta$ is **[Single Correct Type, 2015 Adv.]**

(a) $\frac{2\sqrt{2}}{3}$ (b) $\frac{-\sqrt{2}}{3}$ (c) $\frac{2}{3}$ (d) $\frac{-2\sqrt{3}}{3}$

187. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are unit vectors satisfying $|\mathbf{a}-\mathbf{b}|^2+|\mathbf{b}-\mathbf{c}|^2+|\mathbf{c}-\mathbf{a}|^2=9$, then $|2\mathbf{a}+5\mathbf{b}+5\mathbf{c}|$ is equal to **[Subjective Type Question, 2012]**

188. The vector(s) which is/are coplanar with vectors $\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}}$ and $\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}}$, are perpendicular to the vector $\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ is/are **[More than One Option Correct Type, 2011]**

(a) $\hat{\mathbf{j}}-\hat{\mathbf{k}}$ (b) $-\hat{\mathbf{i}}+\hat{\mathbf{j}}$
(c) $\hat{\mathbf{i}}-\hat{\mathbf{j}}$ (d) $-\hat{\mathbf{j}}+\hat{\mathbf{k}}$

189. Let $\mathbf{a}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$, $\mathbf{b}=\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $\mathbf{c}=\hat{\mathbf{i}}-\hat{\mathbf{j}}-\hat{\mathbf{k}}$ be three vectors. A vector $\mathbf{v}$ in the plane of $\mathbf{a}$ and $\mathbf{b}$, whose projection on $\mathbf{c}$ is $\frac{1}{\sqrt{3}}$, is given by **[Single Correct Type, 2011 Adv.]**

(a) $\hat{\mathbf{i}}-3\hat{\mathbf{j}}+3\hat{\mathbf{k}}$ (b) $-3\hat{\mathbf{i}}-3\hat{\mathbf{j}}-\hat{\mathbf{k}}$
(c) $3\hat{\mathbf{i}}-\hat{\mathbf{j}}+3\hat{\mathbf{k}}$ (d) $\hat{\mathbf{i}}+3\hat{\mathbf{j}}-3\hat{\mathbf{k}}$

190. Two adjacent sides of a parallelogram $ABCD$ are given by $\mathbf{AB}=2\hat{\mathbf{i}}+10\hat{\mathbf{j}}+11\hat{\mathbf{k}}$ and $\mathbf{AD}=-\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\hat{\mathbf{k}}$. The side AD is rotated by an acute angle α in the plane of the parallelogram so that AD becomes AD'. If AD' makes a right angle with the side AB, then the cosine of the angle α is given by **[Single Correct Type, 2010 Adv.]**

(a) $\frac{8}{9}$ (b) $\frac{\sqrt{17}}{9}$
(c) $\frac{1}{9}$ (d) $\frac{4\sqrt{5}}{9}$

191. Let P, Q, R and S be the points on the plane with position vectors $-2\hat{\mathbf{i}}-\hat{\mathbf{j}}$, $4\hat{\mathbf{i}}$, $3\hat{\mathbf{i}}+3\hat{\mathbf{j}}$ and $-3\hat{\mathbf{i}}+2\hat{\mathbf{j}}$, respectively. The quadrilateral $PQRS$ must be a **[Single Correct Type, IITJEE 2010]**

(a) parallelogram, which is neither a rhombus nor a rectangle
(b) square
(c) rectangle, but not a square
(d) rhombus, but not a square

192. If $\mathbf{a}$ and $\mathbf{b}$ are vectors in space given by $\mathbf{a}=\frac{\hat{\mathbf{i}}-2\hat{\mathbf{j}}}{\sqrt{5}}$ and $\mathbf{b}=\frac{2\hat{\mathbf{i}}+\hat{\mathbf{j}}+3\hat{\mathbf{k}}}{\sqrt{14}}$, then the value of $(2\mathbf{a}+\mathbf{b})\cdot[(\mathbf{a}\times\mathbf{b})\times(\mathbf{a}-2\mathbf{b})]$ is **[Integer Type Question, 2010]**

193. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are the unit vectors such that $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=1$ and $\mathbf{a}\cdot\mathbf{c}=\frac{1}{2}$, then **[More than One Option Correct Type, 2009]**

(a) $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are non-coplanar
(b) $\mathbf{a}$, $\mathbf{b}$, $\mathbf{d}$ are non-coplanar
(c) $\mathbf{b}$, $\mathbf{d}$ are non-parallel
(d) $\mathbf{a}$, $\mathbf{d}$ are parallel and $\mathbf{b}$, $\mathbf{c}$ are parallel

194. The edges of a parallelopiped are of unit length and are parallel to non-coplanar unit vector $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, $\hat{\mathbf{c}}$ such that $\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}=\hat{\mathbf{b}}\cdot\hat{\mathbf{c}}=\hat{\mathbf{c}}\cdot\hat{\mathbf{a}}=\frac{1}{2}$. Then, the volume of the parallelopiped is **[Single Correct Type, IIT-JEE 2008]**

(a) $\frac{1}{\sqrt{2}}$ cu unit (b) $\frac{1}{2\sqrt{2}}$ cu unit
(c) $\frac{\sqrt{3}}{2}$ cu unit (d) $\frac{1}{\sqrt{3}}$ cu unit

195. Let two non-collinear unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ form an acute angle. A point P moves, so that at any time t the position vector **OP** (where, O is the origin) is given by $\hat{\mathbf{a}}\cos t + \hat{\mathbf{b}}\sin t$. When P is farthest from origin O, let M be the length of **OP** and $\hat{\mathbf{u}}$ be the unit vector along **OP**. Then, **[Single Correct Type, IIT-JEE 2008]**

(a) $\hat{\mathbf{u}} = \frac{\hat{\mathbf{a}}+\hat{\mathbf{b}}}{|\hat{\mathbf{a}}+\hat{\mathbf{b}}|}$ and $M = (1+\hat{\mathbf{a}}\cdot\hat{\mathbf{b}})^{1/2}$

(b) $\hat{\mathbf{u}} = \frac{\hat{\mathbf{a}}-\hat{\mathbf{b}}}{|\hat{\mathbf{a}}-\hat{\mathbf{b}}|}$ and $M = (1+\hat{\mathbf{a}}\cdot\hat{\mathbf{b}})^{1/2}$

(c) $\hat{\mathbf{u}} = \frac{\hat{\mathbf{a}}+\hat{\mathbf{b}}}{|\hat{\mathbf{a}}+\hat{\mathbf{b}}|}$ and $M = (1+2\hat{\mathbf{a}}\cdot\hat{\mathbf{b}})^{1/2}$

(d) $\hat{\mathbf{u}} = \frac{\hat{\mathbf{a}}-\hat{\mathbf{b}}}{|\hat{\mathbf{a}}-\hat{\mathbf{b}}|}$ and $M = (1+2\hat{\mathbf{a}}\cdot\hat{\mathbf{b}})^{1/2}$

196. Let the vectors **PQ**, **QR**, **RS**, **ST**, **TU** and **UP** represent the sides of a regular hexagon.

Statement I $\mathbf{PQ}\times(\mathbf{RS}+\mathbf{ST}) \neq \mathbf{0}$, because

Statement II $\mathbf{PQ}\times\mathbf{RS} = \mathbf{0}$ and $\mathbf{PQ}\times\mathbf{ST} \neq \mathbf{0}$

[Single Correct Type, 2007, 3M]

(a) Statement I is true, Statement II is true and Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true but Statement II is not a correct explanation for Statement I.
(c) Statement I is true, Statement II is false.
(d) Statement I is false, Statement II is true.

197. The number of distinct real values of λ, for which the vectors $-\lambda^2\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$, $\hat{\mathbf{i}}-\lambda^2\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $\hat{\mathbf{i}}+\hat{\mathbf{j}}-\lambda^2\hat{\mathbf{k}}$ are coplanar, is **[Single Correct Type, IIT-JEE 2007]**

(a) 0 (b) 1
(c) $\pm\sqrt{2}$ (d) 3

198. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be unit vectors such that $\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0}$. Which one of the following is correct? **[Single Correct Type, IIT-JEE 2007]**

(a) $\mathbf{a}\times\mathbf{b} = \mathbf{b}\times\mathbf{c} = \mathbf{c}\times\mathbf{a} = \mathbf{0}$
(b) $\mathbf{a}\times\mathbf{b} = \mathbf{b}\times\mathbf{c} = \mathbf{c}\times\mathbf{a} \neq \mathbf{0}$
(c) $\mathbf{b}\times\mathbf{b} = \mathbf{b}\times\mathbf{c} = \mathbf{a}\times\mathbf{c} = \mathbf{0}$
(d) $\mathbf{a}\times\mathbf{b}$, $\mathbf{b}\times\mathbf{c}$, $\mathbf{c}\times\mathbf{a}$ are mutually perpendicular

199. Let **A** be vector parallel to line of intersection of planes P_1 and P_2 through origin. P_1 is parallel to the vectors $2\hat{\mathbf{j}}+3\hat{\mathbf{k}}$ and $4\hat{\mathbf{j}}-3\hat{\mathbf{k}}$ and P_2 is parallel to $\hat{\mathbf{j}}-\hat{\mathbf{k}}$ and $3\hat{\mathbf{i}}+3\hat{\mathbf{j}}$, then the angle between vector **A** and $2\hat{\mathbf{i}}+\hat{\mathbf{j}}-2\hat{\mathbf{k}}$ is **[More than One Option Correct Type, 2006, 5M]**

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{6}$ (d) $\frac{3\pi}{4}$

200. Let, $\mathbf{a} = \hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}}$, $\mathbf{c} = \hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}}$. A vector coplanar to $\mathbf{a}$ and $\mathbf{b}$ has a projection along $\mathbf{c}$ of magnitude $\frac{1}{\sqrt{3}}$, then the vector is **[Single Correct Type, IIT-JEE 2006]**

(a) $4\hat{\mathbf{i}}-\hat{\mathbf{j}}+4\hat{\mathbf{k}}$ (b) $4\hat{\mathbf{i}}+\hat{\mathbf{j}}-4\hat{\mathbf{k}}$
(c) $2\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ (d) None of these

201. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are three non-zero, non-coplanar vectors and

$$\mathbf{b}_1 = \mathbf{b}-\frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a},\ \mathbf{b}_2 = \mathbf{b}+\frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a},$$

$$\mathbf{c}_1 = \mathbf{c}-\frac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a}-\frac{\mathbf{c}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b},\ \mathbf{c}_2 = \mathbf{c}-\frac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a}-\frac{\mathbf{c}\cdot\mathbf{b}_1}{|\mathbf{b}|^2}\mathbf{b}_1,$$

$$\mathbf{c}_3 = \mathbf{c}-\frac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a}-\frac{\mathbf{a}\cdot\mathbf{b}_2}{|\mathbf{b}_2|^2}\mathbf{b}_2,\ \mathbf{c}_4 = \mathbf{a}-\frac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a}.$$

Then, which of the following is a set of mutually orthogonal vectors? **[Single Correct Type, IIT-JEE 2005]**

(a) $\{\mathbf{a},\mathbf{b}_1,\mathbf{c}_1\}$ (b) $\{\mathbf{a},\mathbf{b}_1,\mathbf{c}_2\}$
(c) $\{\mathbf{a},\mathbf{b}_2,\mathbf{a}_3\}$ (d) $\{\mathbf{a},\mathbf{b}_2,\mathbf{c}_4\}$

202. The unit vector which is orthogonal to the vector $3\hat{\mathbf{i}}+2\hat{\mathbf{j}}+6\hat{\mathbf{k}}$ and is coplanar with the vectors $2\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}}$ is **[Single Correct Type, IIT-JEE 2004]**

(a) $\frac{2\hat{\mathbf{i}}-6\hat{\mathbf{j}}+\hat{\mathbf{k}}}{\sqrt{41}}$ (b) $\frac{2\hat{\mathbf{i}}-3\hat{\mathbf{j}}}{\sqrt{13}}$
(c) $\frac{3\hat{\mathbf{j}}-\hat{\mathbf{k}}}{\sqrt{10}}$ (d) $\frac{4\hat{\mathbf{i}}+3\hat{\mathbf{j}}-3\hat{\mathbf{k}}}{\sqrt{34}}$

203. The value of a, so that the volume of parallelopiped formed by $\hat{\mathbf{i}}+a\hat{\mathbf{j}}+\hat{\mathbf{k}}$, $\hat{\mathbf{j}}+a\hat{\mathbf{k}}$ and $a\hat{\mathbf{i}}+\hat{\mathbf{k}}$ become minimum, is **[Single Correct Type, IIT-JEE 2003]**

(a) -3 (b) 3
(c) $1/\sqrt{3}$ (d) $\sqrt{3}$

204. If $\mathbf{a} = (\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})$, $\mathbf{a}\cdot\mathbf{b} = 1$ and $\mathbf{a}\times\mathbf{b} = \hat{\mathbf{j}}-\hat{\mathbf{k}}$, then $\mathbf{b}$ is equal to **[Single Correct Type, IIT-JEE 2003]**

(a) $\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}}$ (b) $2\hat{\mathbf{j}}-\hat{\mathbf{k}}$
(c) $\hat{\mathbf{i}}$ (d) $2\hat{\mathbf{i}}$

205. If $\mathbf{V} = 2\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}}$ and $\mathbf{W} = \hat{\mathbf{i}}+3\hat{\mathbf{k}}$. If $\mathbf{U}$ is a unit vector, then the maximum value of the scalar triple product $[\mathbf{U}\ \mathbf{V}\ \mathbf{W}]$ is **[Single Correct Type, IIT-JEE 2002]**

(a) -1 (b) $\sqrt{10}+\sqrt{6}$
(c) $\sqrt{59}$ (d) $\sqrt{60}$

206. If $\mathbf{a}$ and $\mathbf{b}_1$ are two unit vectors such that $\mathbf{a}+2\mathbf{b}$ and $5\mathbf{a}-4\mathbf{b}$, are perpendicular to each other, then the angle between $\mathbf{a}$ and $\mathbf{b}$ is **[Single Correct Type, 2002, 1M]**

(a) 45° (b) 60°
(c) $\cos^{-1}\left(\frac{1}{3}\right)$ (d) $\cos^{-1}\left(\frac{2}{7}\right)$

(ii) JEE Main & AIEEE

207. Let $\mathbf{a} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\mathbf{c}$ be a vector such that $|\mathbf{c} - \mathbf{a}| = 3$, $|(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}| = 3$ and the angle between $\mathbf{c}$ and $\mathbf{a} \times \mathbf{b}$ is 30°. Then, $\mathbf{a} \cdot \mathbf{c}$ is equal to **[JEE Main 2017]**

(a) $\frac{25}{8}$ (b) 2

(c) 5 (d) $\frac{1}{8}$

208. If $[\mathbf{a} \times \mathbf{b}\ \mathbf{b} \times \mathbf{c}\ \mathbf{c} \times \mathbf{a}] = \lambda\ [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$, then λ is equal to

(a) 0 (b) 1 **[JEE Main 2014]**

(c) 2 (d) 3

209. Let $\mathbf{a}$ and $\mathbf{b}$ be two unit vectors. If the vectors $\mathbf{c} = \mathbf{a} + 2\mathbf{b}$ and $\mathbf{d} = 5\mathbf{a} - 4\mathbf{a}$ are perpendicular to each other, then the angle between $\mathbf{a}$ and $\mathbf{b}$ is **[AIEEE 2012]**

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{2}$

(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$

210. Let $ABCD$ be a parallelogram such that $\mathbf{AB} = \mathbf{q}$, $\mathbf{AD} = \mathbf{p}$ and $\angle BAD$ be an acute angle. If $\mathbf{r}$ is the vector that coincides with the altitude directed from the vertex B to the side AD, then $\mathbf{r}$ is given by **[AIEEE 2012]**

(a) $\mathbf{r} = 3\mathbf{q} + \frac{3(\mathbf{p} \cdot \mathbf{q})}{(\mathbf{p} \cdot \mathbf{p})}\mathbf{p}$ (b) $\mathbf{r} = -\mathbf{q} + \left(\frac{\mathbf{p} \cdot \mathbf{q}}{\mathbf{p} \cdot \mathbf{p}}\right)\mathbf{p}$

(c) $\mathbf{r} = \mathbf{q} - \left(\frac{\mathbf{p} \cdot \mathbf{q}}{\mathbf{p} \cdot \mathbf{p}}\right)\mathbf{p}$ (d) $\mathbf{r} = -3\mathbf{q} + \frac{3(\mathbf{p} \cdot \mathbf{q})}{(\mathbf{p} \cdot \mathbf{p})}\mathbf{p}$

211. If $\mathbf{a} = \frac{1}{\sqrt{10}}(3\hat{\mathbf{i}} + \hat{\mathbf{k}})$ and $\mathbf{b} = \frac{1}{7}(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 6\hat{\mathbf{k}})$, then the value of $(2\mathbf{-b}) \cdot [(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} + 2\mathbf{b})]$ is **[AIEEE 2011]**

(a) –3 (b) 5

(c) 3 (d) –5

212. The vectors $\mathbf{a}$ and $\mathbf{b}$ are not perpendicular and $\mathbf{c}$ and $\mathbf{d}$ are two vectors satisfying $\mathbf{b} \times \mathbf{c} = \mathbf{b} \times \mathbf{d}$ and $\mathbf{a} \cdot \mathbf{b} = 0$. Then, the vector $\mathbf{d}$ is equal to **[AIEEE 2011]**

(a) $\mathbf{c} + \left(\frac{\mathbf{a} \cdot \mathbf{c}}{\mathbf{a} \cdot \mathbf{b}}\right)\mathbf{b}$ (b) $\mathbf{b} + \left(\frac{\mathbf{b} \cdot \mathbf{c}}{\mathbf{a} \cdot \mathbf{b}}\right)\mathbf{c}$

(c) $\mathbf{c} - \left(\frac{\mathbf{a} \cdot \mathbf{c}}{\mathbf{a} \cdot \mathbf{b}}\right)\mathbf{b}$ (d) $\mathbf{b} - \left(\frac{\mathbf{b} \cdot \mathbf{c}}{\mathbf{a} \cdot \mathbf{b}}\right)\mathbf{c}$

213. If the vectors $p\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\hat{\mathbf{i}} + q\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\hat{\mathbf{i}} + \hat{\mathbf{j}} + r\hat{\mathbf{k}}$ (Where, $p \neq q \neq r \neq 1$) are coplanar, then the value of $pqr - (p + q + r)$ is **[AIEEE 2011]**

(a) – 2 (b) 2

(c) 0 (d) –1

214. Let $\mathbf{a} = \hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\mathbf{a} = \hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}}$. Then, the vector $\mathbf{b}$ satisfying $\mathbf{a} \times \mathbf{b} + \mathbf{c} = 0$ and $\mathbf{a} \cdot \mathbf{b} = 3$, is **[AIEEE 2010]**

(a) $-\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ (b) $2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

(c) $\hat{\mathbf{i}} - \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ (d) $\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$

215. If the vectors $\mathbf{a} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, $\mathbf{b} = 2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{c} = \lambda\hat{\mathbf{i}} + \hat{\mathbf{j}} + \mu\hat{\mathbf{k}}$ are mutually orthogonal, then (λ, μ) is equal to **[AIEEE 2010]**

(a) (– 3, 2) (b) (2, – 3)

(c) (– 2, 3) (d) (3, – 2)

216. If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are non-coplanar vectors and p, q are real numbers, then the equality $[3\mathbf{u}\ p\mathbf{v}\ p\mathbf{w}] - [p\mathbf{v}\ \mathbf{w}\ q\mathbf{u}] - [2\mathbf{w}\ q\mathbf{v}\ q\mathbf{u}] = 0$ holds for **[AIEEE 2009]**

(a) exactly two values of (p, q)

(b) more than two but not all values of (p, q)

(c) all values of (p, q)

(d) exactly one value of (p, q)

217. The vector $\mathbf{a} = \alpha\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \beta\hat{\mathbf{k}}$ lies in the plane of the vectors $\mathbf{b} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\mathbf{c} = \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and bisects the angle between $\mathbf{b}$ and $\mathbf{c}$. Then, which one of the following gives possible values of α and β? **[AIEEE 2008]**

(a) $\alpha = 1, \beta = 1$ (b) $\alpha = 2, \beta = 2$

(c) $\alpha = 1, \beta = 2$ (d) $\alpha = 2, \beta = 1$

218. If $\mathbf{u}$ and $\mathbf{v}$ are unit vectors and θ is the acute angle between them, then $2\mathbf{u} \times 3\mathbf{v}$ is a unit vector for **[AIEEE 2007]**

(a) exactly two values of θ

(b) more than two values of θ

(c) no value of θ

(d) exactly one value of θ

219. Let $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $\mathbf{c} = x\hat{\mathbf{i}} + (x-2)\hat{\mathbf{j}} - \hat{\mathbf{k}}$. If the vector $\mathbf{c}$ lies in the plane of $\mathbf{a}$ and $\mathbf{b}$, then x equal to

(a) 0 (b) 1 **[AIEEE 2007]**

(c) – 4 (d) –2

220. If $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$, where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are any three vectors such that $\mathbf{a} \cdot \mathbf{b} \neq 0$, $\mathbf{b} \cdot \mathbf{c} \neq 0$, then $\mathbf{a}$ and $\mathbf{c}$ are

(a) inclined at an angle of $\frac{\pi}{6}$ between them **[AIEEE 2006]**

(b) perpendicular

(c) parallel

(d) inclined at an angle of $\frac{\pi}{3}$ between them

221. The value of a, for which the points, A, B, C with position vectors $2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$ and $a\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$ respectively are the vertices of a right angled triangle with $C = \frac{\pi}{2}$ are **[AIEEE 2006]**

(a) –2 and –1 (b) –2 and 1

(c) 2 and –1 (d) 2 and 1

222. The distance between the line $\mathbf{r} = 2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}} + \lambda(\hat{\mathbf{i}} - \hat{\mathbf{j}} + 4\hat{\mathbf{k}})$ and the plane $\mathbf{r} \cdot (\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + \hat{\mathbf{k}}) = 5$ is **[AIEEE 2005]**

(a) $\frac{10}{3}$ (b) $\frac{3}{10}$ (c) $\frac{10}{3\sqrt{3}}$ (d) $\frac{10}{9}$

223. For any vector $\mathbf{a}$, the value of $(\mathbf{a}\times\hat{\mathbf{i}})^2+(\mathbf{a}\times\hat{\mathbf{j}})^2+(\mathbf{a}\times\hat{\mathbf{k}})^2$ is equal to **[AIEEE 2005]**

(a) $4\mathbf{a}^2$ (b) $2\mathbf{a}^2$
(c) $\mathbf{a}^2$ (d) $3\mathbf{a}^2$

224. If $\mathbf{a},\mathbf{b},\mathbf{c}$ are non-coplanar vectors and λ is a real number, then $[\lambda(\mathbf{a}+\mathbf{b})\quad\lambda^2\mathbf{b}\quad\lambda\mathbf{c}]=[\mathbf{a}\ \mathbf{b}+\mathbf{c}\ \mathbf{b}]$ for **[AIEEE 2005]**

(a) exactly two values of λ
(b) exactly three values of λ
(c) no value of λ
(d) exactly one value of λ

225. Let $\mathbf{a}=\hat{\mathbf{i}}-\hat{\mathbf{k}},\mathbf{b}=x\hat{\mathbf{i}}+\hat{\mathbf{j}}+(1-x)\hat{\mathbf{k}}$ and $\mathbf{c}=y\hat{\mathbf{i}}+x\hat{\mathbf{j}}+(1+x-y)\hat{\mathbf{k}}$. Then, $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ depends on **[AIEEE 2005]**

(a) Neither x nor y
(b) Both x and y
(c) Only x
(d) Only y

226. Let $\mathbf{u},\mathbf{v},\mathbf{w}$ be such that $|\mathbf{u}|=1,|\mathbf{v}|=2,|\mathbf{w}|=3$. If the projection $\mathbf{v}$ along $\mathbf{u}$ is equal to that of $\mathbf{w}$ along $\mathbf{u}$ and $\mathbf{v},\mathbf{w}$ are perpendicular to each other, then $|\mathbf{u}-\mathbf{v}+\mathbf{w}|$ equal to **[AIEEE 2004]**

(a) 2 (b) $\sqrt{7}$
(c) $\sqrt{14}$ (d) 14

227. Let $\mathbf{a},\mathbf{b}$ and $\mathbf{c}$ be non-zero vectors such that $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=\frac{1}{3}|\mathbf{b}||\mathbf{c}|\mathbf{a}$. If θ is an acute angle between the vectors $\mathbf{b}$ and $\mathbf{c}$, then $\sin\theta$ is equal to **[AIEEE 2004]**

(a) $\frac{1}{3}$ (b) $\frac{\sqrt{2}}{3}$
(c) $\frac{2}{3}$ (d) $\frac{2\sqrt{2}}{3}$

228. A particle is acted upon by constant forces $4\hat{\mathbf{i}}+\hat{\mathbf{j}}-3\hat{\mathbf{k}}$ and $3\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}}$ which displace it from a point $\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}}$ to the point $5\hat{\mathbf{i}}+4\hat{\mathbf{j}}+\hat{\mathbf{k}}$. The work done in standard units by the forces is given by **[AIEEE 2004]**

(a) 40 units (b) 30 units
(c) 25 units (d) 15 units

229. If $\mathbf{u},\mathbf{v}$ and $\mathbf{w}$ are three non-coplanar vectors, then $(\mathbf{u}+\mathbf{v}-\mathbf{w})\cdot[(\mathbf{u}-\mathbf{v})\times(\mathbf{v}-\mathbf{w})]$ equal to **[AIEEE 2003]**

(a) 0 (b) $\mathbf{u}\cdot\mathbf{v}\times\mathbf{w}$
(c) $\mathbf{u}\cdot\mathbf{w}\times\mathbf{v}$ (d) $3\mathbf{u}\cdot\mathbf{v}\times\mathbf{w}$

230. $\mathbf{a},\mathbf{b},\mathbf{c}$ are three vectors, such that $\mathbf{a}+\mathbf{b}+\mathbf{c}=0,|\mathbf{a}|=1,|\mathbf{b}|=2,|\mathbf{c}|=3$, then $\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}$ is equal to

(a) 0 (b) -7 **[AIEEE 2003]**
(c) 7 (d) 1

231. A tetrahedron has vertices at $O(0,0,0)$, $A(1,2,1)$, $B(2,1,3)$ and $C(-1,1,2)$. Then, the angle between the faces OAB and ABC will be **[AIEEE 2003]**

(a) $\cos^{-1}\left(\frac{19}{35}\right)$ (b) $\cos^{-1}\left(\frac{17}{31}\right)$
(c) $30°$ (d) $90°$

232. Let $\mathbf{u}=\hat{\mathbf{i}}+\hat{\mathbf{j}},\mathbf{v}=\hat{\mathbf{i}}-\hat{\mathbf{j}}$ and $\mathbf{w}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}}$. If $\mathbf{n}$ is a unit vector such that $\mathbf{u}\cdot\mathbf{n}=0$ and $\mathbf{v}\cdot\mathbf{n}=0$, then $|\mathbf{w}\cdot\mathbf{n}|$ is equal to **[AIEEE 2003]**

(a) 0 (b) 1
(c) 2 (d) 3

233. Given, two vectors are $\hat{\mathbf{i}}-\hat{\mathbf{j}}$ and $\hat{\mathbf{i}}+2\hat{\mathbf{j}}$, the unit vector coplanar with the two vectors and perpendicular to first is **[AIEEE 2002]**

(a) $\frac{1}{\sqrt{2}}(\hat{\mathbf{i}}+\hat{\mathbf{j}})$ (b) $\frac{1}{\sqrt{5}}(2\hat{\mathbf{i}}+\hat{\mathbf{j}})$
(c) $\pm\frac{1}{\sqrt{2}}(\hat{\mathbf{i}}+\hat{\mathbf{j}})$ (d) None of these

Answers

Exercise for Session 1

1. $\cos^{-1}\left(\frac{5}{7}\right)$ 2. $\frac{\pi}{4}$ 4. $\mathbf{r} = \pm\sqrt{3}(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$
5. $\frac{\pi}{2}$ 6. $\frac{60}{\sqrt{114}}$ 7. $-\frac{5}{2}$
8. $0°$ 9. $\frac{2\pi}{3}$ 10. $a > 2$
11. $\frac{1}{6}(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$ 12. $\frac{19}{9}(2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}})$; $\frac{1}{9}(-20\hat{\mathbf{i}} + 8\hat{\mathbf{j}} + 16\hat{\mathbf{k}})$
13. 40

Exercise for Session 2

1. $19\sqrt{2}$ 2. $\lambda = 3$ and $\mu = \frac{27}{2}$
3. -74 6. 3 7. $\frac{\pi}{6}$
8. $\frac{\pi}{4}$ 9. ± 7 10. $\frac{-\hat{\mathbf{i}}}{\sqrt{3}} + \frac{\hat{\mathbf{j}}}{\sqrt{3}} + \frac{\mathbf{k}}{\sqrt{3}}$
11. $\frac{5}{3}(7\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - 4\hat{\mathbf{k}})$ 12. $\frac{1}{3}(160\,\hat{\mathbf{i}} - 4\hat{\mathbf{i}} - 70\hat{\mathbf{k}})$
13. $\pm 2\,(\mathbf{b} \times \mathbf{c})$ 14. $\frac{1}{2}\sqrt{65}$ sq. units
15. $\frac{\sqrt{61}}{2}$ sq. units. 17. $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$
18. $2\hat{\mathbf{i}} - 7\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ 19. $-20\hat{\mathbf{i}} + 16\hat{\mathbf{j}} + 12\hat{\mathbf{k}}$

Exercise for Session 3

1. 4 2. 4 cubic unit 3. $\frac{9}{2}$
4. 4 cubic unit 5. $\frac{2\sqrt{38}}{19}$
6. **a, b, c** form a right handed system. 9. 6 10. 1

Exercise for Session 4

1. 0 2. $3\frac{(-7\hat{\mathbf{i}} + 8\hat{\mathbf{j}} - \mathbf{k})}{\sqrt{114}}$
7. $\mathbf{a}' = \frac{1}{2}(-\mathbf{i} + \mathbf{k})$, $\mathbf{b}' = \frac{1}{2}(-\hat{\mathbf{j}} + \hat{\mathbf{k}})$ and $\mathbf{c}' = \frac{1}{2}(\mathbf{i} + \mathbf{j})$
9. $\mathbf{r} = y\,\mathbf{b} = \frac{1}{\mathbf{b}^2}(\mathbf{a} \times \mathbf{b})$ 10. $\mathbf{r} = \frac{1}{\mathbf{a} \cdot \mathbf{b}}(\mathbf{a} \times \mathbf{c} + \mathbf{m}\mathbf{b})$

Chapter Exercises

1. (d) 2. (a) 3. (a) 4. (d) 5. (d) 6. (b)
7. (a) 8. (b) 9. (b) 10. (d) 11. (d) 12. (d)
13. (a) 14. (c) 15. (c) 16. (a) 17. (a) 18. (c)
19. (a) 20. (b) 21. (d) 22. (c) 23. (b) 24. (b)
25. (b) 26. (b) 27. (c) 28. (a) 29. (c) 30. (a)
31. (c) 32. (a) 33. (c) 34. (a) 35. (a) 36. (b)
37. (d) 38. (c) 39. (a) 40. (a) 41. (a) 42. (b)
43. (c) 44. (a) 45. (d) 46. (b) 47. (d) 48. (a)
49. (a) 50. (d) 51. (b) 52. (a) 53. (c) 54. (c)
55. (d) 56. (c) 57. (d) 58. (b) 59. (c) 60. (d)
61. (d) 62. (d) 63. (a) 64. (a) 65. (a) 66. (a)
67. (b) 68. (d) 69. (b) 70. (d) 71. (c) 72. (b)
73. (d) 74. (b,c,d) 75. (a,c,d) 76. (a,c) 77. (a,c) 78. (b,d)
79. (a,b,c,d) 80. (a,b,c,d) 81. (a,b) 82. (a,c)
83. (a,b,c) 84. (a,c) 85. (a,b,c) 86. (b,d)
87. (b,c,d) 88. (a,c) 89. (a,b)
90. (a,b,c) 91. (a,d) 92. (a,b,c) 93. (a,c) 94. (a,b,c)
95. (b,c,d) 96. (a,b,c,d) 97. (c,d) 98. (a,b,c,d)
99. (b,c,d) 100. (a,d) 101. (a,b,d)
102. (c,d) 103. (b,c) 104. (b,c,d) 105. (a,c,d)
106. (c,d) 107. (a,b)
108. (a,c) 109. (b,c) 110. (c) 111. (a) 112. (a) 113. (d)
114. (d) 115. (d) 116. (a) 117. (a) 118. (b) 119. (a)
120. (a) 121. (b) 122. (c) 123. (b) 124. (a) 125. (b)
126. (d) 127. (c) 128. (c) 129. (a) 130. (d) 131. (d)
132. (d) 133. (c) 134. (b) 135. (d) 136. (a) 137. (a)
138. (a) 139. (b) 140. (a) 141. (c) 142. (b)
143. (A) → (p,r), (B) → (q), (C) → (s), (D) → (p)
144. (A) → (r), (B) → (s), (C) → (q), (D) → (p)
145. (A) → (s), (B) → (r), (C) → (q)
146. (A) → (r), (B) → (p), (C) → (s), (D) → (q)
147. (A) → (q), (B) → (s), (C) → (p), (D) → (r)
148. (0) 149. (3) 150. (1) 151. (2) 152. (6) 153. (9)
154. (2) 155. (3) 156. (3) 157. (4) 158. (2) 159. (9)
160. (5) 161. (2) 162. (3) 163. (8) 165. (10)
166. $xx_1 + 2yy_1 = a$ 167. (12)
168. $\frac{\sin C + \sin B \cdot \sin A}{\sqrt{\sin^2 B + \sin^2 C + 2\sin B \cdot \sin C \cdot \cos A}}$
169. $\frac{3\sqrt{26}}{7}$ 170. $\frac{\sin A - \sin B}{2(\sin A + \sin B)}$ 171. $\frac{49}{28}$ sq units
173. $\{6, 4\}$ and $4x - 6y + 13 = 0$ 174. $\theta = \frac{\pi}{4}$
175. $\left(2\sqrt{2}\,\hat{\mathbf{i}} - \frac{1}{\sqrt{2}}\hat{\mathbf{j}} - \frac{1}{\sqrt{2}}\hat{\mathbf{k}}\right)$ 176. 1
177. $-\hat{\mathbf{i}} - 8\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ 178. $\frac{\mathbf{a} \cdot \mathbf{b}}{(\mathbf{a} \cdot \mathbf{b})^2 - \mathbf{a}^2\,\mathbf{b}^2}$
179. $\hat{\mathbf{x}} = \frac{1}{3}(3\mathbf{a} + 4\mathbf{b} + 8\mathbf{c})$, $\hat{\mathbf{y}} = -4\mathbf{c}$, $\hat{\mathbf{z}} = \frac{4}{3}(c - b)$
180. $x = \frac{a + b + c}{2}$
181. $x = \frac{1}{2}\frac{\mathbf{a}^2\mathbf{b}^2 - \mathbf{b}^2(\mathbf{a} \cdot \mathbf{b})}{2(\mathbf{a}^2\,\mathbf{b}^2) - (\mathbf{a} \cdot \mathbf{b})^2}\mathbf{a} + \frac{1}{2}\frac{\mathbf{a}^2\mathbf{b}^2 - \mathbf{a}^2(\mathbf{a} \cdot \mathbf{b})}{(\mathbf{a}^2\,\mathbf{b}^2) - (\mathbf{a} \cdot \mathbf{b})^2} \cdot \mathbf{b}$
182. (b) 183. (a) 184. (a) 185. (d) 186. (a) 187. (3)
188. (a) 189. (c) 190. (b) 191. (a) 192. (5) 193. (c)
194. (a) 195. (a) 196. (c) 197. (c) 198. (b) 199. (b,d)
200. (a) 201. (b) 202. (c) 203. (c) 204. (c) 205. (c)
206. (b) 207. (b) 208. (b) 209. (c) 210. (b) 211. (d)
212. (c) 213. (a) 214. (d) 215. (a) 216. (d) 217. (d)
218. (d) 219. (d) 220. (c) 221. (d) 222. (c) 223. (b)
224. (c) 225. (a) 226. (c) 227. (d) 228. (a) 229. (b)
230. (b) 231. (a) 232. (d) 233. (a)

Solutions

1. Since, $\mathbf{a} \perp \mathbf{b} \Rightarrow \mathbf{a}\cdot\mathbf{b} = 0$

$|\mathbf{a}-\mathbf{b}|^2 = (\mathbf{a}-\mathbf{b})^2 = \mathbf{a}^2 + \mathbf{b}^2 - 2\,\mathbf{a}\cdot\mathbf{b} = 25 + 25$

$\Rightarrow \quad |\mathbf{a}-\mathbf{b}| = 5\sqrt{2}$

2. $|\mathbf{a}+\mathbf{b}| > |\mathbf{a}-\mathbf{b}|$

On squaring both sides, we get

$\mathbf{a}^2 + \mathbf{b}^2 + 2\mathbf{a}\cdot\mathbf{b} > \mathbf{a}^2 + \mathbf{b}^2 - 2\mathbf{a}\cdot\mathbf{b}$

$\Rightarrow \quad 4\mathbf{a}\cdot\mathbf{b} > 0 \Rightarrow \cos\theta > 0$

Hence, $\theta < 90°$ (acute)

3. Given that, $\mathbf{a} = \mathbf{b} + \mathbf{c}$ and angle between $\mathbf{b}$ and $\mathbf{c}$ is $\frac{\pi}{2}$.

So, $\mathbf{a}^2 = \mathbf{b}^2 + \mathbf{c}^2 + 2\mathbf{b}\cdot\mathbf{c}$

$\Rightarrow \quad \mathbf{a}^2 = \mathbf{b}^2 + \mathbf{c}^2 + 2|\,\mathbf{b}\,||\,\mathbf{c}\,|\cos\frac{\pi}{2}$

$\Rightarrow \quad \mathbf{a}^2 = \mathbf{b}^2 + \mathbf{c}^2 + 0$

$\therefore \quad \mathbf{a}^2 = \mathbf{b}^2 + \mathbf{c}^2$

i.e., $\mathbf{a}^2 = \mathbf{b}^2 + \mathbf{c}^2$

4. Obviously, $\mathbf{a}$ and $\mathbf{b}$ are unit vectors.

5. Angle between $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\hat{\mathbf{i}}$ is equal to

$$\cos^{-1}\left\{\frac{(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})\cdot\hat{\mathbf{i}}}{|\,\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}|\,|\,\hat{\mathbf{i}}\,|}\right\} \Rightarrow a = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

Similarly, angle between $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\hat{\mathbf{j}}$ is

$$\beta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \text{ and between } \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}} \text{ and } \hat{\mathbf{k}} \text{ is}$$

$$\gamma = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

Hence, $a = \beta = \gamma$

6. Let $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{i}} \Rightarrow \mathbf{r}\cdot\hat{\mathbf{i}} = x, \mathbf{r}\cdot\hat{\mathbf{j}} = y, \mathbf{r}\cdot\hat{\mathbf{k}} = z$

$\Rightarrow \quad (\mathbf{r}\cdot\hat{\mathbf{i}})^2 + (\mathbf{r}\cdot\hat{\mathbf{j}})^2 + (\mathbf{r}\cdot\hat{\mathbf{k}})^2 = x^2 + y^2 + z^2 = r^2$

7. $|\mathbf{a}-\mathbf{b}| = \sqrt{1^2 + 1^2 - 2\cdot 1^2\cos\theta} = \sqrt{2(1-\cos\theta)}$

$$= \sqrt{2}\times\sqrt{2}\,\sin\frac{\theta}{2} = 2\sin\frac{\theta}{2} \Rightarrow \sin\frac{\theta}{2} = \frac{|\,\mathbf{a}-\mathbf{b}\,|}{2}$$

8. The component of vector $\mathbf{a}$ along $\mathbf{b}$ is

$$\frac{(\mathbf{a}\cdot\mathbf{b})\mathbf{b}}{|\,\mathbf{b}\,|^2} = \frac{18}{25}(3\hat{\mathbf{j}} + 4\hat{\mathbf{k}})$$

9. Required value $= \dfrac{\mathbf{b}\cdot\mathbf{a}}{|\,\mathbf{b}\,|} \Big/ \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\,\mathbf{a}\,|} = \dfrac{|\,\mathbf{a}\,|}{|\,\mathbf{b}\,|} = \dfrac{7}{3}$

10. $(\mathbf{a}\times\mathbf{b})^2 = a^2b^2 - (\mathbf{a}\,.\,\mathbf{b})^2 = \begin{vmatrix}\mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b}\\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b}\end{vmatrix}$

11. Torque $= \mathbf{r}\times\mathbf{F}$ or $\mathbf{CP}\times\mathbf{F}$

12. $\mathbf{r}\times\mathbf{F} = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 2 & -1 & 1\\ 1 & 2 & 3\end{vmatrix} = -5\hat{\mathbf{i}} - 5\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$

13. $\mathbf{OA} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 9\hat{\mathbf{k}}, \mathbf{F} = (9\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}})\times\frac{6}{11}$

$$\therefore \quad \text{Moment} = \mathbf{OA}\times\mathbf{F} = \frac{6}{11}\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 3 & 2 & -9\\ 9 & 6 & -2\end{vmatrix}$$

$$= \frac{6}{11}(50\hat{\mathbf{i}} - 75\hat{\mathbf{j}}) = \frac{150}{11}(2\hat{\mathbf{i}} - 3\hat{\mathbf{j}})$$

14. Force $(\mathbf{F}) = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$ and its position vector $= 2\hat{\mathbf{i}} - \hat{\mathbf{j}}$. We know that the position vector of a force about origin $(\mathbf{r}) = (2\hat{\mathbf{i}} - \hat{\mathbf{j}}) - (0\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 0\hat{\mathbf{k}})$ or $\mathbf{r} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}}$ Therefore, moment of the force about origin

$$= \mathbf{r}\times\mathbf{F} = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 2 & -1 & 0\\ 2 & 1 & -1\end{vmatrix} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}.$$

15. $\mathbf{a}^{-1} = \dfrac{\mathbf{b}\times\mathbf{c}}{[\mathbf{abc}]}, \mathbf{c}^{-1} = \dfrac{\mathbf{a}\times\mathbf{b}}{[\mathbf{abc}]}, \mathbf{b}^{-1} = \dfrac{\mathbf{c}\times\mathbf{a}}{[\mathbf{abc}]}$

$$\Rightarrow [\mathbf{a}^{-1}\ \mathbf{b}^{-1}\ \mathbf{c}^{-1}] = \frac{(\mathbf{b}\times\mathbf{c})}{[\mathbf{abc}]}\cdot\left(\frac{(\mathbf{c}\times\mathbf{a})}{[\mathbf{abc}]}\times\frac{(\mathbf{a}\times\mathbf{b})}{[\mathbf{abc}]}\right)$$

$$= \frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{abc}]}\cdot\frac{\mathbf{a}}{[\mathbf{abc}]} = \frac{1}{[\mathbf{abc}]} \neq 0$$

16. $\dfrac{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}{\mathbf{c}\times\mathbf{a}\cdot\mathbf{b}} + \dfrac{\mathbf{b}\cdot\mathbf{a}\times\mathbf{c}}{\mathbf{c}\cdot\mathbf{a}\times\mathbf{b}} = \dfrac{[\mathbf{abc}]}{[\mathbf{cab}]} + \dfrac{[\mathbf{bac}]}{[\mathbf{cab}]}$

$$= \frac{[\mathbf{abc}]}{[\mathbf{cab}]} - \frac{[\mathbf{abc}]}{[\mathbf{cab}]} = 0$$

17. $\mathbf{b}\times\mathbf{c}$ is a vector perpendicular to $\mathbf{b}$, $\mathbf{c}$. Therefore, $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is a vector again in plane of $\mathbf{b}$, $\mathbf{c}$.

18. Let $\mathbf{a} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$

$\mathbf{u} = \hat{\mathbf{i}}\times(\mathbf{a}\times\hat{\mathbf{i}}) + \hat{\mathbf{j}}\times(\mathbf{a}\times\hat{\mathbf{i}}) + \hat{\mathbf{k}}\times(\mathbf{a}\times\hat{\mathbf{k}})$

$= (\hat{\mathbf{i}}\cdot\hat{\mathbf{i}})\mathbf{a} - \hat{\mathbf{i}}(\mathbf{a}\cdot\hat{\mathbf{i}}) + (\hat{\mathbf{j}}\cdot\hat{\mathbf{j}})\mathbf{a} - \hat{\mathbf{j}}(\mathbf{a}\cdot\hat{\mathbf{j}})$

$+ (\hat{\mathbf{k}}\cdot\hat{\mathbf{k}})\mathbf{a} - \hat{\mathbf{k}}(\mathbf{a}\cdot\hat{\mathbf{k}}) = 3\mathbf{a} - \mathbf{a} = 2\mathbf{a}$

19. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{a}\times\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 2 & -1 & 1\\ 1 & 3 & -1\end{vmatrix} = \mathbf{a}\times(-2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 7\hat{\mathbf{k}})$

$$= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & 2 & -2\\ -2 & 3 & 7\end{vmatrix} = 20\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$$

20. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = 0$

$\Rightarrow \quad \mathbf{a}\,||\,(\mathbf{b}\times\mathbf{c})$ or $\mathbf{b}\times\mathbf{c} = 0$

i.e., $\mathbf{b}\,||\,\mathbf{c}$ or $\mathbf{a} = 0$

21. Let the required vector be $\alpha = d_1\hat{\mathbf{i}} + d_2\hat{\mathbf{j}} + d_3\hat{\mathbf{k}}$

where, $d_1^2 + d_2^2 + d_3^2 = 51$ (given) ...(i)

Now, each of the given vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is a unit vectors.

$$\cos\theta = \frac{\mathbf{d}\cdot\mathbf{a}}{|\,\mathbf{d}\,||\,\mathbf{a}\,|} = \frac{\mathbf{d}\cdot\mathbf{b}}{|\,\mathbf{d}\,||\,\mathbf{b}\,|} = \frac{\mathbf{d}\cdot\mathbf{c}}{|\,\mathbf{d}\,||\,\mathbf{c}\,|}$$

or $\mathbf{d}\cdot\mathbf{a} = \mathbf{d}\cdot\mathbf{b} = \mathbf{d}\cdot\mathbf{c}$

$|\,\mathbf{d}\,| = \sqrt{51}$ cancels out and $|\,\mathbf{a}\,| = |\,\mathbf{b}\,| = |\,\mathbf{c}\,| = 1$

Hence, $\frac{1}{3}(d_1 - 2d_2 + 2d_3) = \frac{1}{5}(-4d_1 + 0d_2 - 3d_3) = d_2$

$\Rightarrow \quad d_1 - 5d_2 + 2d_3 = 0$

and $\quad 4d_1 + 5d_2 + 3d_3 = 0$

On solving, we get $\quad \frac{d_1}{5} = \frac{d_2}{-1} = \frac{d_3}{-5} = \lambda$ (say)

Putting d_1, d_2 and d_3 in Eq. (i), we get $\lambda = \pm 1$

Hence, the required vectors are $\pm (5\hat{\mathbf{i}} - \hat{\mathbf{j}} - 5\hat{\mathbf{k}})$

22. Let $\mathbf{OA} = P_1\hat{\mathbf{i}}$, $\mathbf{CB} = -P_1\hat{\mathbf{i}}$ and $\mathbf{OB} = -P_1\hat{\mathbf{i}} + P\hat{\mathbf{j}}$

$$\frac{\mathbf{OB}\cdot\hat{\mathbf{j}}}{OB} = \cos 60^\circ$$

$$\Rightarrow \quad \frac{(-P_1\hat{\mathbf{i}} + P\hat{\mathbf{j}})\cdot\hat{\mathbf{j}}}{\sqrt{P_1^2 + P^2}} = \frac{1}{2}$$

$$\Rightarrow \quad 2P = \sqrt{P^2 + P_1^2}$$

$$\Rightarrow \quad P_1 = P\sqrt{3}$$

$$|\mathbf{OB}| = \sqrt{P^2 + P_1^2} = \sqrt{P^2 + 3P^2} = 2P$$

23. $\mathbf{x} + \mathbf{y} + \mathbf{z} = 0 \Rightarrow \mathbf{x} = -(\mathbf{y} + \mathbf{z})$

$$|\mathbf{x}|^2 = (\mathbf{y} + \mathbf{z})\cdot(\mathbf{y} + \mathbf{z})$$

$$\Rightarrow \quad |\mathbf{x}|^2 = |\mathbf{y}|^2 + |\mathbf{z}|^2 + 2\mathbf{y}\cdot\mathbf{z}$$

$$\Rightarrow \quad |\mathbf{x}|^2 = |\mathbf{y}|^2 + |\mathbf{z}|^2 + 2|\mathbf{y}||\mathbf{z}|\cos\theta$$

$$\Rightarrow \quad 4 = 4 + 4 + 2\times 2\times 2\cos\theta$$

$$\Rightarrow \quad \cos\theta = \frac{-1}{2} \Rightarrow \theta = 120^\circ$$

$\therefore \operatorname{cosec}^2 120^\circ + \cot^2 120^\circ$

$$= \left(\frac{2}{\sqrt{3}}\right)^2 + \left(-\frac{1}{\sqrt{3}}\right)^2 = \frac{4}{3} + \frac{1}{3} = \frac{5}{3}$$

24. For acute angle $\mathbf{a}\cdot\mathbf{b} > 0$

i.e., $\quad -3x + 2x^2 + 1 > 0$

$\Rightarrow \quad (x-1)(2x-1) > 0$

For obtuse angle between $\mathbf{b}$ and X-axis $\mathbf{b}\cdot\hat{\mathbf{i}} < 0 \Rightarrow x < 0$

25. Since, $\mathbf{d} = \lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c}$

$$\therefore \quad \mathbf{d}\cdot(\mathbf{b}\times\mathbf{c}) = \lambda\mathbf{a}\cdot(\mathbf{a}\times\mathbf{c}) + \mu\mathbf{b}\cdot(\mathbf{b}\times\mathbf{c}) + \nu\mathbf{c}\cdot(\mathbf{b}\times\mathbf{c})$$

$$= \lambda\,[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$

$$\Rightarrow \quad \lambda = \frac{[\mathbf{dbc}]}{[\mathbf{abc}]} = \frac{[\mathbf{bcd}]}{[\mathbf{bca}]}$$

26. $(3\mathbf{p} + \mathbf{q})\cdot(5\mathbf{p} - 3\mathbf{q}) = 0$

or $\quad 15\mathbf{p}^2 - 3\mathbf{q}^2 = 4\mathbf{p}\cdot\mathbf{q}$ ...(i)

$(2\mathbf{p} + \mathbf{q})\cdot(4\mathbf{p} - 2\mathbf{q}) = 0$ or $8\mathbf{p}^2 = 2\mathbf{q}^2$

$$\Rightarrow \quad \mathbf{q}^2 = 4\mathbf{p}^2 \quad \text{...(ii)}$$

Now, $\quad \cos\theta = \frac{\mathbf{p}\cdot\mathbf{q}}{|\mathbf{p}||\mathbf{q}|}$

On substituting $\mathbf{q}^2 = 4\mathbf{p}^2$ in Eq. (i), we get

$$\Rightarrow \quad 3\mathbf{p}^2 = 4\mathbf{p}\cdot\mathbf{q}$$

$$\cos\theta = \frac{3}{4}\cdot\frac{\mathbf{p}^2}{|\mathbf{p}|2|\mathbf{p}|} = \frac{3}{8} \Rightarrow \sin\theta = \frac{\sqrt{55}}{8}$$

27. $\hat{\mathbf{n}} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$,

where, $a_1^2 + a_2^2 + a_3^2 = 1$

$$\mathbf{u}\cdot\hat{\mathbf{n}} = 0 \Rightarrow a_1 + a_2 = 0$$

Also, $\quad \mathbf{v}\cdot\hat{\mathbf{n}} = 0 \Rightarrow a_1 - a_2 = 0$

Hence, $\quad a_1 = a_2 = 0$

$\therefore \quad a_3 = 1$ or -1

$\therefore \quad \hat{\mathbf{n}} = \hat{\mathbf{k}}$ or $-\hat{\mathbf{k}}$

$$|\mathbf{w}\cdot\hat{\mathbf{n}}| = 3$$

28. To find $(\mathbf{a} - \mathbf{b})\cdot\mathbf{a}$

i.e., $\quad |\mathbf{a}|^2 - \mathbf{a}\cdot\mathbf{b}$ ...(i)

Now, $\quad \mathbf{a} + \mathbf{b} = \mathbf{c}$

$\Rightarrow \quad |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a}\cdot\mathbf{b} = |\mathbf{c}|^2$ ...(ii)

On substituting the value of $\mathbf{a}\cdot\mathbf{b}$ from Eq. (ii) in Eq. (i), we get

$$a^2 - \frac{1}{2}(c^2 - a^2 - b^2)$$

$$\Rightarrow \quad \frac{3a^2 + b^2 - c^2}{2}$$

29. $\mathbf{A}\times\mathbf{B} = -\mathbf{A}\times\mathbf{B}$

$$\mathbf{A}\times\mathbf{B} = 0$$

either $\quad \mathbf{A} = 0$

or $\quad \mathbf{B} = 0$

or $\mathbf{A}$ and $\mathbf{B}$ are collinear

30. Given, $\quad \mathbf{V} + \mathbf{V}_1 = \mathbf{V}_2$

Also, $\quad \mathbf{V}\cdot\mathbf{V}_1 = 2\alpha$

$$(\mathbf{V}_1)^2 = (\mathbf{V}_2 - \mathbf{V})^2 \quad \text{and} \quad \mathbf{V}\cdot\mathbf{V}_2 = \alpha$$

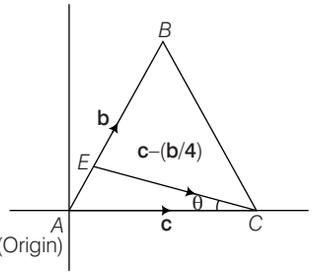
Also, $\quad |\mathbf{V}| = |\mathbf{V}_1| = |\mathbf{V}_2| = \lambda$ (say)

Hence, $\quad \lambda^2 = 2\lambda^2 - 2\lambda^2\cos\alpha$

$$\Rightarrow \quad \cos\alpha = \frac{1}{2}$$

31. Given, $\quad |\mathbf{b}| = |\mathbf{b} - \mathbf{c}| = 8$ and $|\mathbf{c}| = 12$ ...(i)

$$\mathbf{AE} = \frac{\mathbf{b}}{4} \text{ and } \mathbf{EC} = \mathbf{c} - \frac{\mathbf{b}}{4}$$



$$\cos\theta = \frac{\mathbf{c}\cdot\left(\mathbf{c} - \frac{\mathbf{b}}{4}\right)}{|\mathbf{c}|\left|\mathbf{c} - \frac{\mathbf{b}}{4}\right|} = \frac{\mathbf{c}^2 - \frac{\mathbf{c}\cdot\mathbf{b}}{4}}{12\left|\mathbf{c} - \frac{\mathbf{b}}{4}\right|} \quad \text{...(ii)}$$

From Eq. (i), $|\mathbf{b}| = 8, |\mathbf{c}| = 12$

$$\therefore \quad |\mathbf{b} - \mathbf{c}|^2 = |\mathbf{b}|^2$$

$$\Rightarrow \quad |\mathbf{b}|^2 + |\mathbf{c}|^2 - 2\mathbf{b}\cdot\mathbf{c} = |\mathbf{b}|^2$$

$$\Rightarrow \quad \mathbf{b}\cdot\mathbf{c} = 72 \qquad \ldots(iii)$$

and $$\left|\mathbf{c}-\frac{\mathbf{b}}{4}\right|^2 = |\mathbf{c}|^2 + \frac{|\mathbf{b}|^2}{|16|} - \frac{\mathbf{b}\cdot\mathbf{c}}{2}$$

$$= 144 + 4 - 36 = 112$$

$$\therefore \quad \left|\mathbf{c}-\frac{\mathbf{b}}{4}\right| = 4\sqrt{7} \qquad \ldots(iv)$$

From Eqs. (ii), (iii) and (iv)

$$\cos\theta = \frac{144-10}{12\times 4\sqrt{7}} = \frac{3\sqrt{7}}{8}$$

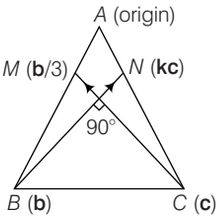
32. $\mathbf{BN}\cdot\mathbf{CM} = 0$

$$(k\mathbf{c}-\mathbf{b})\cdot\left(\frac{\mathbf{b}}{3}-\mathbf{c}\right) = 0$$

$$\frac{k}{3}\cdot\frac{a^2}{2} - ka^2 - \frac{a^2}{3} + \frac{a^2}{2} = 0$$

$$\frac{k}{6} - k + \frac{1}{6} = 0$$

$$\frac{5k}{6} = \frac{1}{6} \Rightarrow k = \frac{1}{5}$$



33. Given, $15|\mathbf{AC}| = 3|\mathbf{AB}| = 5|\mathbf{AD}|$

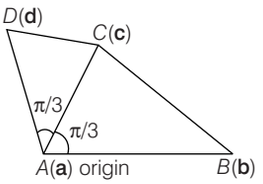
Let $|\mathbf{AC}| = \lambda > 0$

$\therefore \quad |\mathbf{AB}| = 5\lambda$

$|\mathbf{AD}| = 3\lambda$

Now, $\cos(\mathbf{BA}\cdot\mathbf{CD}) = \dfrac{\mathbf{BA}\cdot\mathbf{CD}}{|\mathbf{BA}||\mathbf{CD}|}$

$$= \frac{\mathbf{b}\cdot(\mathbf{d}-\mathbf{c})}{|\mathbf{b}||\mathbf{d}-\mathbf{c}|} \qquad \ldots(i)$$



Now, numerator of Eq. (i), we get

$$\mathbf{b}\cdot\mathbf{c} - \mathbf{b}\cdot\mathbf{d} = |\mathbf{b}||\mathbf{c}|\cos\frac{\pi}{3} - |\mathbf{b}||\mathbf{d}|\cos\frac{2\pi}{3}$$

$$= (5\lambda)(\lambda)\frac{1}{2} + 5\lambda(3\lambda)\frac{1}{2}$$

$$= \frac{5\lambda^2 + 15\lambda^2}{2} = 10\lambda^2$$

Denominator of Eq. (i)

$$= |\mathbf{b}||\mathbf{d}-\mathbf{c}|$$

Now, $|\mathbf{d}-\mathbf{c}|^2 = \mathbf{d}^2 + \mathbf{c}^2 - 2\mathbf{c}\cdot\mathbf{d}$

$$= 9\lambda^2 + \lambda^2 - 2(\lambda)(3\lambda)\frac{1}{2}$$

$$= 10\lambda^2 - 3\lambda^2 = 7\lambda^2$$

$\therefore \quad |\mathbf{d}-\mathbf{c}| = \sqrt{7}\lambda$

Denominator of Eq. (i)

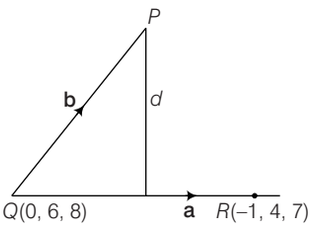
$$= (5\lambda)(7\lambda) = 5\sqrt{7}\lambda^2$$

$$\therefore \quad \cos(\mathbf{BA}\cdot\mathbf{CD}) = \frac{10\lambda^2}{5\sqrt{7}\lambda^2} = \frac{2}{\sqrt{7}}$$

34. $\mathbf{a} = -\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$

$$\mathbf{b} = \hat{\mathbf{i}} - 5\hat{\mathbf{j}} - 7\hat{\mathbf{k}}$$

$$|\mathbf{d}| = \frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}|};\ |\mathbf{a}| = \sqrt{6}$$



$$|\mathbf{a}\times\mathbf{b}|^2 = \mathbf{a}^2\mathbf{b}^2 - (\mathbf{a}\cdot\mathbf{b})^2 = (6)(75)$$

$$-(-1+10+7)^2 = 450 - 256 = 194$$

$$|\mathbf{a}\times\mathbf{b}| = \sqrt{194}$$

$$\therefore \quad d = \sqrt{\frac{194}{6}} = \sqrt{\frac{97}{3}}$$

$\therefore \quad p + q = 100$

$$\Rightarrow \quad \frac{(p+q)(p+q-1)}{2} = \frac{100\times 99}{2} = 4950$$

35. $$V = -c^2[\mathbf{u}\,\mathbf{v}\,\mathbf{w}] = -c^2\begin{vmatrix} 2 & -1 & -1 \\ 1 & -1 & 2 \\ 1 & 0 & -1 \end{vmatrix}$$

$$= -c^2[2(1-0) - 1(1) + (-2-1)]$$

$$= -c^2[2-1-3] = 8$$

$\therefore \quad 2c^2 = 8 \Rightarrow c = 2 \text{ or } -2$

36. Let $\mathbf{c} = \lambda(\mathbf{a}\times\mathbf{b})$

Hence, $\lambda(\mathbf{a}\times\mathbf{b})\cdot(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 7\hat{\mathbf{k}}) = 10$

$$\lambda\begin{vmatrix} 2 & -3 & 1 \\ 1 & -2 & 3 \\ 1 & 2 & -7 \end{vmatrix} = 10$$

$$\Rightarrow \quad \lambda = -1 \Rightarrow \mathbf{c} = -(\mathbf{a}\times\mathbf{b})$$

$\mathbf{a} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & -3 & 1 \\ 1 & -2 & 3 \end{vmatrix} = (-9+2)\hat{\mathbf{i}} - (5)\hat{\mathbf{j}} + (-4+3)\hat{\mathbf{k}}$$

$$\Rightarrow \quad (-7, -5, -1)$$

37. $\mathbf{V}_1 = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$

$\mathbf{V}_2 = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$

$\mathbf{V}_3 = \mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b} = \alpha(\hat{\mathbf{i}} + \hat{\mathbf{j}}) + \beta(\hat{\mathbf{j}} + \hat{\mathbf{k}})$

$= \alpha\hat{\mathbf{i}} + (\alpha+\beta)\hat{\mathbf{j}} + \beta\hat{\mathbf{k}} = \mathbf{c}$

Since, $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_3$ are coplanar.

Now, $\begin{vmatrix} 1 & -2 & 1 \\ 3 & 2 & -1 \\ \alpha & \alpha+\beta & \beta \end{vmatrix} = 0$, using $C_2 \to C_2 - (C_1 + C_3)$, we get

$$\begin{vmatrix} 1 & -4 & 1 \\ 3 & 0 & -1 \\ \alpha & 0 & \beta \end{vmatrix} = 0, \text{ hence } 4(3\beta + \alpha) = 0$$

$$\Rightarrow \quad 3\beta + \alpha = 0$$

$$\Rightarrow \quad \frac{\alpha}{\beta} = -3$$

38. $\hat{\mathbf{n}} = \dfrac{\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{3}}$;

$$\omega = |\,\omega\,|\,\hat{\mathbf{n}} = 10(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$$

Now, $\mathbf{v} = \omega \times \mathbf{r} = 10\,(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) \times (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$,

where $\mathbf{r}$ is the position vector of the point whose locus is to be determined.

Hence, $\mathbf{v} = 10\,[(z-y)\hat{\mathbf{i}} - (z-x)\hat{\mathbf{j}} + (y-x)\hat{\mathbf{k}}]$

$\therefore$ $|\,\mathbf{v}\,| = 10\sqrt{(x-y)^2 + (y-z)^2 + (z-x)^2}$

Hence, $2(x^2 + y^2 + z^2 - xy - yz - zx) = 4$

$\Rightarrow$ $x^2 + y^2 + z^2 - xy - yz - zx - 2 = 0$

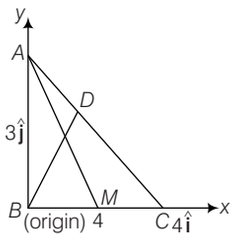
which is the equation of a cylinder.

39. $\hat{\mathbf{n}} = \dfrac{\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}}{3}$; $\omega = \dfrac{\omega}{3}(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

$$\mathbf{v} = \omega \times \mathbf{r} = \frac{\omega}{3}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & 2 \\ 2 & 3 & 5 \end{vmatrix} = \frac{\omega}{3}(4\hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}})$$

$\therefore$ $|\,\mathbf{v}\,| = \dfrac{\omega}{3}\sqrt{18} = \omega\sqrt{2}$

40. $\mathbf{b}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{c} = \mathbf{b}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$



or $\mathbf{b}\cdot(\mathbf{a} - \mathbf{b}) - \mathbf{c}\cdot(\mathbf{a} - \mathbf{b}) = 0$

or $(\mathbf{b} - \mathbf{c})\cdot(\mathbf{a} - \mathbf{b}) = 0$

$\Rightarrow$ **BC** and **AB** are perpendicular.

Now, find angle between **AM** and **BD**.

where, $\mathbf{AM} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}}$,

$$\mathbf{BD} = \frac{4\hat{\mathbf{i}} + 3\hat{\mathbf{j}}}{2}$$

$\therefore$ $\cos\theta = \dfrac{\mathbf{AM}\cdot\mathbf{BD}}{|\,\mathbf{AM}\,||\,\mathbf{BD}\,|} = \dfrac{-1}{5\sqrt{13}}$

$\Rightarrow$ $\theta = \pi - \cos^{-1}\left(\dfrac{1}{5\sqrt{13}}\right)$

41. $[\mathbf{npm}] = \sin\theta\,\cos\phi = \sin\dfrac{\pi}{6}\cdot\cos\dfrac{\pi}{6} = \dfrac{1}{2}\cdot\dfrac{\sqrt{3}}{2} = \dfrac{\sqrt{3}}{4}$

42. $\mathbf{V} = (\mathbf{a}\times\mathbf{b})\times\mathbf{a} + (\mathbf{a}\times\mathbf{b})\times\mathbf{b}$

$= \mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{a} + (\mathbf{a}\cdot\mathbf{b})\mathbf{b} - \mathbf{a} = (\mathbf{b} - \mathbf{a}) + (\mathbf{b} - \mathbf{a})(\mathbf{a}\cdot\mathbf{b})$

$\mathbf{V} = (\mathbf{b} - \mathbf{a})(1 + \mathbf{a}\cdot\mathbf{b}) = \lambda(\mathbf{b} - \mathbf{a})$

43. Since, $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, hence $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a}\times\mathbf{b}$ are non-coplanar. Hence, any vector say $(\mathbf{r}\times\mathbf{a})$ can be expressed as

$\therefore$ $\mathbf{r}\times\mathbf{a} = x\mathbf{a} + y\,\mathbf{b} + z(\mathbf{a}\times\mathbf{b})$ ...(i)

dot with $\mathbf{a}$ $\quad 0 = x + 0 + 0 \;\Rightarrow\; x = 0$

dot with $\mathbf{b}$ $\quad [\mathbf{r}\;\mathbf{a}\;\mathbf{b}] = 0 + y + 0 \;\Rightarrow\; y = [\mathbf{r}\;\mathbf{a}\;\mathbf{b}]$

dot with $\mathbf{r}$ $\quad 0 = x\mathbf{a}\cdot\mathbf{r} + y\,\mathbf{r}\cdot\mathbf{b} + z[\mathbf{r}\;\mathbf{a}\;\mathbf{b}]$

$$0 = [\mathbf{r}\;\mathbf{a}\;\mathbf{b}](\mathbf{r}\cdot\mathbf{b}) + z[\mathbf{r}\;\mathbf{a}\;\mathbf{b}] \;\Rightarrow\; z = -(\mathbf{r}\cdot\mathbf{b})$$

Hence, $\mathbf{r}\times\mathbf{a} = [\mathbf{r}\;\mathbf{a}\;\mathbf{b}]\,\mathbf{b} - (\mathbf{r}\cdot\mathbf{b})(\mathbf{a}\times\mathbf{b})$

$$\mathbf{r}\times\mathbf{a} = [\mathbf{r}\;\mathbf{a}\;\mathbf{b}]\,\mathbf{b} + (\mathbf{r}\cdot\mathbf{b})(\mathbf{b}\times\mathbf{a})$$

44. Since, $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ is rotated so as to cross Y-axis, the vector in new position. Let the required vector be $x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$

where, $x^2 + y^2 + z^2 = 9$ ...(i)

$x + 2y + 2z = 0$ ...(ii)

and $\begin{vmatrix} x & y & z \\ 1 & 2 & 2 \\ 0 & 1 & 0 \end{vmatrix} \;\Rightarrow\; 2x - z = 0$ ...(iii)

On solving Eqs. (i), (ii) and (iii), we get

$$x = -\frac{2}{\sqrt{5}},\; y = \sqrt{5},\; z = \frac{-4}{\sqrt{5}}$$

$\therefore$ Required vector, is $\dfrac{-2}{\sqrt{5}}\hat{\mathbf{i}} + \sqrt{5}\hat{\mathbf{j}} - \dfrac{4}{\sqrt{5}}\hat{\mathbf{k}}$

45.

Set A	$\rightarrow$	Set B
(Parallel)		(Non-parallel)
4		6
ways $\rightarrow$		

(i) 3 from $B \rightarrow {}^6C_3$

(ii) 2 from B, 1 from $A \rightarrow {}^4C_1 \times {}^6C_2$

(iii) 3 from $A \rightarrow {}^4C_3$

Total number of ways

$$= {}^6C_3 + ({}^4C_1 \times {}^6C_2) + {}^4C_1$$

$\therefore$ $n(E) = {}^6C_{13} + {}^4C_1 + ({}^4C_1 \times {}^6C_{12})$

and $n(S) = {}^{10}C_3$

$\Rightarrow$ $P(E) = \dfrac{{}^6C_3 + {}^4C_1 + ({}^6C_2 \times {}^4C_1)}{{}^{10}C_3}$

46. $(\hat{\mathbf{a}}\times\mathbf{x}) + \mathbf{b} = \mathbf{x}$

$\Rightarrow$ $\hat{\mathbf{a}}\times(\hat{\mathbf{a}}\times\mathbf{x}) + (\hat{\mathbf{a}}\times\mathbf{b}) = \hat{\mathbf{a}}\times\mathbf{x}$

$$(\hat{\mathbf{a}}\cdot\mathbf{x})\,\hat{\mathbf{a}} - (\hat{\mathbf{a}}\cdot\hat{\mathbf{a}})\,\mathbf{x} + (\hat{\mathbf{a}}\times\mathbf{b}) = \mathbf{x} - \mathbf{b}$$

Projection of $\mathbf{x}$ along $\hat{\mathbf{a}}$ is 2 units

$\Rightarrow$ $\dfrac{(\hat{\mathbf{a}}\cdot\mathbf{x})}{|\,\hat{\mathbf{a}}\,|} = 2 \quad\Rightarrow\; \hat{\mathbf{a}}\cdot\mathbf{x} = 2$

So, $\mathbf{x} = \dfrac{1}{2}[2\hat{\mathbf{a}} - \mathbf{b} + (\hat{\mathbf{a}}\times\mathbf{b})]$

47. We know, $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}$

Component of $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ along $\mathbf{b}$ is

$$\left[\left\{\frac{(\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}}{|\,\mathbf{b}\,|^2}\right\}\cdot\mathbf{b}\right]\mathbf{b} = \left(\frac{(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{b}) - (\mathbf{a}\cdot\mathbf{b})(\mathbf{b}\cdot\mathbf{c})}{|\,\mathbf{b}\,|^2}\right)\mathbf{b}$$

So, component of $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ perpendicular to $\mathbf{b}$ is

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) - \left(\frac{(\mathbf{a}\cdot\mathbf{c}\,(\mathbf{b}\cdot\mathbf{b}) - (\mathbf{a}\cdot\mathbf{b})(\mathbf{b}\cdot\mathbf{c})}{|\,\mathbf{b}\,|^2}\right)\mathbf{b}$$

$$= \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) + \left(\frac{(\mathbf{a}\cdot\mathbf{b})(\mathbf{b}\cdot\mathbf{c}) - (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{b})}{|\,\mathbf{b}\,|^2}\right)\mathbf{b}$$

$$= \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) + \frac{(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{b}\times\mathbf{c})}{|\,\mathbf{b}\,|^2}\,\mathbf{b}$$

48. $\mathbf{r}\cdot\mathbf{a}=20, \Rightarrow x+2y+z=20, x,y,z\in N$

The number of non-negative integral solution are

$$^{17}C_1+{}^{15}C_1+\ldots{}^{1}C_1=81$$

49. $\alpha\cdot\beta=\dfrac{b}{a}+\dfrac{4a}{b}+1$

as $\dfrac{b}{a}+\dfrac{4a}{b}+1\geq 5$

So, $\left(\dfrac{10}{5+\alpha\cdot\beta}\right)_{\max}=1$

50. The system of vectors is coplanar.

$\because$ Their sum is zero.

51.

$$\cos\theta=\frac{\mathbf{AB}\cdot\mathbf{AD}}{|\mathbf{AB}|\,|\mathbf{AD}|}=\frac{8}{9}$$

$\therefore\quad \cos^{-1}\left(\dfrac{8}{9}\right)+\alpha=\dfrac{\alpha}{2}$ by hypothesis

$\therefore\quad \sin\alpha=\dfrac{8}{9}$

$\therefore\quad \cos\alpha=\sqrt{1-\dfrac{64}{81}}=\dfrac{\sqrt{17}}{9}$

52. $\mathbf{BA}+\mathbf{AC}=\mathbf{BC}$

$\Rightarrow\quad \mathbf{BA}=\mathbf{BC}-\mathbf{AC}$

$\Rightarrow\quad \dfrac{\mathbf{e}}{|\mathbf{e}|}-\dfrac{\mathbf{f}}{|\mathbf{f}|}-\dfrac{2\mathbf{e}}{|\mathbf{e}|}=-\left(\dfrac{\mathbf{e}}{|\mathbf{e}|}+\dfrac{\mathbf{f}}{|\mathbf{f}|}\right)$

Now, $\mathbf{BA}\cdot\mathbf{BC}=\left(\dfrac{\mathbf{e}}{|\mathbf{e}|}+\dfrac{\mathbf{f}}{|\mathbf{f}|}\right)\left(\dfrac{\mathbf{e}}{|\mathbf{e}|}-\dfrac{\mathbf{f}}{|\mathbf{e}|}\right)=0$

$\Rightarrow\quad \angle B=90^\circ$

$\Rightarrow\quad \cos 2B=-1$

and $\cos 2A+\cos 2C=2\cos(A+C)\cos(A-C)=0$

$(\because A+C=90^\circ)$

$\Rightarrow \cos 2A+\cos 2B+\cos 2C=-1$

53. We have, $\mathbf{c}\cdot\mathbf{a}=\mathbf{c}\cdot\mathbf{b}=\cos\theta, \mathbf{a}\cdot\mathbf{b}=0$

Now, $\mathbf{c}=\alpha\mathbf{a}+\beta\mathbf{b}+\gamma(\mathbf{a}\times\mathbf{b})$

Taking the dot product of both sides with $\mathbf{a}$, we get

$$\mathbf{c}\cdot\mathbf{a}=\alpha=\cos\theta \qquad (\because |\mathbf{a}|^2=1, \mathbf{a}\cdot\mathbf{b}=0)$$

Similarly, $\beta=\cos\theta$

Now, taking the dot product with $\mathbf{a}\times\mathbf{b}$, we get

$$[\mathbf{c}\ \mathbf{a}\ \mathbf{b}]=\gamma|\mathbf{a}\times\mathbf{b}|^2=\gamma$$

Now, $[\mathbf{c}\ \mathbf{a}\ \mathbf{b}]^2=[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$

$$=\begin{vmatrix}\mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c}\\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c}\\ \mathbf{c}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c}\end{vmatrix}=\begin{vmatrix}1 & 0 & \cos\theta\\ 0 & 1 & \cos\theta\\ \cos\theta & \cos\theta & 1\end{vmatrix}$$

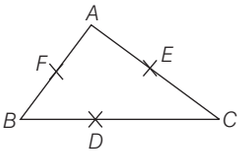
$$=1-\cos^2\theta+\cos\theta(-\cos\theta)$$

Thus, $\alpha=\beta=\cos\theta, \gamma^2=-\cos 2\theta.$

54. The mid-points of sides are $D(1, 0, 0)$, $F(0, 0, n)$.

$$EF^2=\frac{BC^2}{4}\ \Rightarrow\ BC^2=4(m^2+n^2)$$

$$\frac{AB^2+BC^2+CA^2}{l^2+m^2+n^2}=8$$



55. Eliminating m,

$$2(l+n)^2+nl=0$$

or $(2l+n)(l+2n)=0$

$n=-2l \ \Rightarrow m=-2l$

or $l=-2n \ \Rightarrow m=-2n$

The d.r's $1, -2, -2$ and $-2, -2, 1$. The lines are perpendicular.

56. $\cos^2\theta+\cos^2\theta+\cos^2\gamma=1$

$\Rightarrow\quad \cos^2\gamma=-\cos 2\theta$

$\Rightarrow\quad \cos 2\theta\leq 0 \qquad \Rightarrow\ \theta\in\left[\dfrac{\pi}{4},\dfrac{\pi}{2}\right]$

57. $\mathbf{a}\times(\mathbf{a}\times\mathbf{b})=\mathbf{c}\Rightarrow|\mathbf{a}|\,|\mathbf{a}\times\mathbf{b}|=|\mathbf{c}|$

$1(1\times 5)\sin\theta=3$

$\sin\theta=\dfrac{3}{5}$ gives $\tan\theta=\dfrac{3}{4}$.

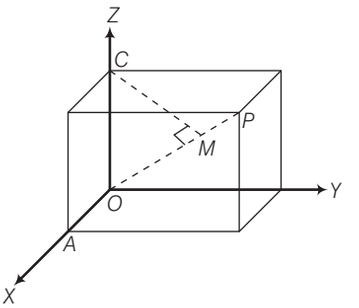
58. From the figure the vector equation of OP is $r=\lambda(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})$

$\therefore\quad OM=$ projection of $\mathbf{OC}$ on $\mathbf{OP}$

$$=\mathbf{OC}\cdot\mathbf{OP}=\frac{1}{\sqrt{3}}$$

Now, $CM^2=OC^2-OM^2=1-\dfrac{1}{3}=\dfrac{2}{3}$

$\therefore\quad CM=\sqrt{\dfrac{2}{3}}$



59. $\mathbf{p}\times\mathbf{q}, \mathbf{p}, \mathbf{q}$ are non-coplanar vectors

$\Rightarrow\ b-c=0, c-a=0, a-b=0$

$\Rightarrow\ a=b=c$

$\Rightarrow\ \Delta$ is equilateral.

60. $(\mathbf{a}\times\mathbf{b})\times(\vec{\mathbf{c}}\times\mathbf{d})=[\mathbf{abd}]\mathbf{c}-[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\mathbf{d}$

$=4\mathbf{c}-4\mathbf{d}$

$=-8\hat{\mathbf{i}}$ perpendicular to Y-axis , Z-axis

61. Translating the axes through $A(1, 2, 3)$.

A changes to $(0, 0, 0)$ B changes to $(8, 6, 2)$.

$\therefore$ Coterminous edges are of lengths 8, 6, 2.

Volume of parallelopiped $= 8 \cdot 6 \cdot 2 = 96$ cu units

62. $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are non-coplanar $\Rightarrow [\mathbf{a}, \mathbf{b}, \mathbf{c}] \neq 1$ Also, $\mathbf{a} \times \mathbf{b}$, $\mathbf{b} \times \mathbf{c}$, $\mathbf{c} \times \mathbf{a}$ are non-coplanar given

$$\mathbf{d} = \sin x\,(\mathbf{a} \times \mathbf{b}) + \cos y\,(\mathbf{b} \times \mathbf{c}) + 2(\mathbf{c} \times \mathbf{a}).$$

Taking dot product with $\mathbf{a} + \mathbf{b} + \mathbf{c}$, we get

$$O = \sin x[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] + \cos y[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] + 2[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$

$\Rightarrow \sin x + \cos y + 2 = 0$

$\Rightarrow \quad \sin x + \cos y = -2$

$$\Rightarrow \qquad x = (4n-1)\frac{\pi}{2}, y = (2n-1)\pi, n \in z$$

for least value of $x^2 + y^2$, $x = \dfrac{-\pi}{2}$, $y = \pi$ and least value is $\dfrac{5\pi^2}{4}$.

63. We have, $\alpha(\mathbf{a} \times \mathbf{b}) + \beta(\mathbf{b} \times \mathbf{c}) + \gamma(\mathbf{c} \times \mathbf{a}) = 0$

Taking dot product with c, we have

$$\alpha[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] + \beta[\mathbf{b}\ \mathbf{c}\ \mathbf{c}] + \gamma[\mathbf{c}\ \mathbf{a}\ \mathbf{c}] = 0$$

i.e. $$\alpha[\mathbf{abc}] + 0 + 0 = 0$$

$$\alpha[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$$

Similarly, taking dot product with b and c, we have

$$\gamma\,[\mathbf{abc}] = 0, \beta\,[\mathbf{abc}] = 0$$

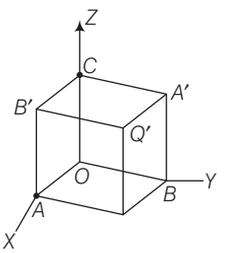
Now, even if one of $\alpha, \beta, \gamma \neq 0$, then we have $[\mathbf{abc}] = 0$

$\Rightarrow$ **a, b, c** are coplanar.

64. $\dfrac{1}{3}\lambda_1 h_4 + \dfrac{1}{3}\lambda_2 h_3 + \dfrac{1}{3}\lambda_3 h_2 + \dfrac{1}{3}\lambda_4 h_1 = 4$ area the tetrahedron $OABC$.

65. $\theta = \cos^{-1}(\cos\beta - \cos\alpha)$

66.



Equation of $OO' \Rightarrow \dfrac{x}{1} = \dfrac{y}{1} = \dfrac{z}{1} \Rightarrow \mathbf{r} = \mathbf{a} + t\mathbf{b}$

Equation of $\mathbf{AB}: \dfrac{x-1}{-1} = \dfrac{y}{1} = \dfrac{z}{0} \Rightarrow (\alpha)\mathbf{r} = \mathbf{c} + s\mathbf{d}$,

where $\mathbf{a} = 0$, $\mathbf{b} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{c} = \hat{\mathbf{i}}$, $\mathbf{d} = -\hat{\mathbf{i}} + \hat{\mathbf{j}}$

Shortest distance $= \dfrac{|(\mathbf{c}-\mathbf{a})\cdot(\mathbf{b}\times\mathbf{d})|}{|\mathbf{b}\times\mathbf{d}|} = \dfrac{1}{\sqrt{6}}$

67. $v \times w = 3\hat{\mathbf{i}} - 5\hat{\mathbf{j}} - \hat{\mathbf{k}}$

Maximum value of $[uvw] = |u|\,|v \times w| = 1 \cdot \sqrt{35} = \sqrt{35}$

68. $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}|$

$|\mathbf{b} - \mathbf{a}| = |\mathbf{b} - \mathbf{c}| = |\mathbf{c} - \mathbf{a}| = a$

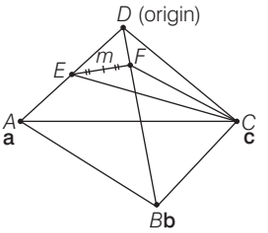
$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{c} = \mathbf{c}\cdot\mathbf{a} = \mathbf{AB}\cdot\mathbf{AC} = \mathbf{CA}\cdot\mathbf{CB} = \mathbf{BA}\cdot\mathbf{BC}$$

$$= a^2\cos\frac{\pi}{3}$$

$\Rightarrow \qquad (\mathbf{EF})^2 = \dfrac{a^2}{\sqrt{3}} \Rightarrow \mathbf{EF} = \dfrac{a}{\sqrt{3}}$

$$|\mathbf{CF}| = |\mathbf{CE}| = \frac{\sqrt{7}\mathbf{a}}{3} \quad \text{and} \quad |\mathbf{CM}| = \frac{5a}{6}$$

where 'm' is middle point of **EF**.



Area of $\Delta CEF = \dfrac{1}{2}|\mathbf{EF}|\,|\mathbf{CM}|$

$$= \frac{1}{2} \times \frac{a}{\sqrt{3}} \times \frac{5a}{6} = \frac{5a^2}{12\sqrt{3}} \text{ sq units}$$

69. $\mathbf{p}\cdot\mathbf{q} = 0$ and $\mathbf{r}\cdot\mathbf{s} = 0$

$\Rightarrow \qquad (5\mathbf{a} - 3\mathbf{b})\cdot(-\mathbf{a} - 2\mathbf{b}) = 0$

$$6\mathbf{b}^2 - 7\mathbf{a}\cdot\mathbf{b} - 5\mathbf{a}^2 = 0 \qquad \text{...(i)}$$

$\Rightarrow \qquad (-4\mathbf{a} - \mathbf{b})\cdot(-\mathbf{a} + \mathbf{b}) = 0$

$$4\mathbf{a}^2 - \mathbf{b}^2 - 3\mathbf{a}\cdot\mathbf{b} = 0 \qquad \text{...(ii)}$$

Now, $$\mathbf{x} = \frac{1}{3}(\mathbf{p} + \mathbf{r} + \mathbf{s})$$

$$\mathbf{x} = \frac{1}{3}(5\mathbf{a} - 3\mathbf{b} - 4\mathbf{a} - \mathbf{b} - \mathbf{a} + \mathbf{b})$$

$\therefore \qquad \mathbf{x} = -\mathbf{b}, \mathbf{y} = \dfrac{1}{5}(\mathbf{r} + \mathbf{s}) = \dfrac{1}{5}(-5\mathbf{a}) = -\mathbf{a}$

Angle between $\mathbf{x}$ and $\mathbf{y}$ i.e $\cos\theta = \dfrac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}||\mathbf{y}|} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}$

From Eqs. (i) and (ii), we get

$$|\mathbf{a}| = \sqrt{\frac{25}{19}}\cdot\sqrt{\mathbf{a}\ \mathbf{b}} \quad \text{and} \quad |\mathbf{b}| = \sqrt{\frac{43}{19}}\sqrt{\mathbf{a}\ \mathbf{b}}$$

$$|\mathbf{a}|\,|\mathbf{b}| = \frac{\sqrt{25\times 43}}{19}(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}) = \frac{19}{5\sqrt{43}} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}$$

$\Rightarrow \qquad \cos\theta = \dfrac{19}{5\sqrt{43}}$

$\therefore \qquad \theta = \cos^{-1}\left(\dfrac{19}{5\sqrt{43}}\right)$

70. Volume of tetrahedron $= \dfrac{1}{6}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

Now, $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2 = \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix} = \begin{vmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{vmatrix}$

$$= 4(12) + 2(-4) + 2(-4)$$

$$\text{Volume} = \frac{1}{6} \times 4\sqrt{2} = \frac{2\sqrt{2}}{3}$$

71. Given, $\cos\theta = (\mathbf{a} \times \hat{\mathbf{i}})\cdot(\mathbf{b} \times \hat{\mathbf{i}}) + (\mathbf{a} + \hat{\mathbf{j}})\cdot(\mathbf{b} \times \hat{\mathbf{j}})$

$$+ (\mathbf{a} \times \hat{\mathbf{k}})\cdot(\mathbf{b} \times \hat{\mathbf{k}}) \qquad \text{...(i)}$$

Consider, $(\mathbf{a} \times \hat{\mathbf{i}})\cdot(\mathbf{b} \times \hat{\mathbf{i}}) = [(\mathbf{a} \times \hat{\mathbf{i}})\mathbf{b}\ \hat{\mathbf{i}}] = ((\mathbf{a} \times \hat{\mathbf{i}}) \times \mathbf{b})\cdot\hat{\mathbf{i}}$

$$((\mathbf{a}\cdot\mathbf{b})\hat{\mathbf{i}})-(\hat{\mathbf{i}}\cdot\mathbf{b})\mathbf{a})\hat{\mathbf{i}}=(\mathbf{a}\cdot\mathbf{b})(\hat{\mathbf{i}}\cdot\hat{\mathbf{i}})-(\hat{\mathbf{i}}\cdot\hat{\mathbf{b}})(\mathbf{a}\cdot\hat{\mathbf{i}})$$
$$=\mathbf{a}\cdot\mathbf{b}-\mathbf{a}_1\mathbf{b}_1$$

Similarly, $(\mathbf{a}\times\hat{\mathbf{j}})\cdot(\mathbf{b}\times\hat{\mathbf{j}})=\mathbf{a}\cdot\mathbf{b}-\mathbf{a}_2\mathbf{b}_2$

and $(\mathbf{a}\times\hat{\mathbf{k}})(\mathbf{b}\times\hat{\mathbf{k}})=\mathbf{a}\cdot\mathbf{b}-a_3b_3$

$\therefore$ From Eq. (i), we get

$$\cos\theta=3\mathbf{a}\cdot\mathbf{b}-(\mathbf{a}_1\mathbf{b}_1+\mathbf{a}_2\mathbf{b}_2+\mathbf{a}_3\mathbf{b}_3)$$
$$=3\mathbf{a}\cdot\mathbf{b}-\mathbf{a}\cdot\mathbf{b}$$

$$\Rightarrow\quad\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}=2\mathbf{a}\cdot\mathbf{b}\ \Rightarrow\ |\mathbf{a}||\mathbf{b}|=\frac{1}{2}$$

Now, use AM $\geq$ GM on $|\mathbf{a}|,|\mathbf{b}|$

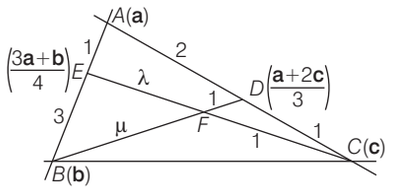
$$\therefore\quad\frac{|\mathbf{a}|+|\mathbf{b}|}{2}\geq(|\mathbf{a}|\cdot|\mathbf{b}|)^{\frac{1}{2}}\ \Rightarrow\ |\mathbf{a}|+|\mathbf{b}|\geq\frac{2}{\sqrt{2}}$$

$$\Rightarrow\quad|\mathbf{a}|+|\mathbf{b}|\geq\sqrt{2}$$

72. $\because\ \mathbf{a}\cdot\mathbf{b}<0$

$$\Rightarrow\ (\sin^2x-\sin x)-\cos^2x+3-4\sin x<0$$
$$\Rightarrow\quad2\sin^2x-5\sin x+2<0$$
$$\Rightarrow\quad\underbrace{(\sin x-2)}_{(-)\text{ve}}(2\sin x-1)<0$$
$$\Rightarrow\quad\sin x>\frac{1}{2}\ \Rightarrow\ x\in\left(\frac{\pi}{6},\frac{\pi}{2}\right)$$

73.



Positive vector of point $E=\dfrac{3\mathbf{a}+\mathbf{b}}{4}$

Position vector of point $D=\dfrac{\mathbf{a}+2\mathbf{c}}{3}$

Let point F divides EC in $\lambda:1$ and BD in $\mu:1$,

then $$\frac{\mathbf{b}+\frac{\mu}{3}(\mathbf{a}+2\mathbf{c})}{\mu+1}=\frac{\lambda\mathbf{c}+\frac{3\mathbf{a}+\mathbf{b}}{4}}{\lambda+1}$$

$$\left[\mathbf{b}+\frac{\mu}{3}(\mathbf{a}+2\mathbf{c})\right](\lambda+1)=\left(\lambda\mathbf{c}+\frac{3\mathbf{a}+\mathbf{b}}{4}\right)(\mu+1)$$

Comparing the coefficient of $\mathbf{a}$ $\mathbf{b}$ and $\mathbf{c}$.

$$\Rightarrow\quad\frac{\mu(\lambda+1)}{3}=\frac{3(\mu+1)}{4}\qquad\ldots(\text{i})$$

$$\Rightarrow\quad\lambda+1=\frac{\mu+1}{4}\qquad\ldots(\text{ii})$$

$$\frac{2\mu}{3}(\lambda+1)=\lambda(\mu+1)\qquad\ldots(\text{iii})$$

On solving, we get $\quad\lambda=\dfrac{3}{2}$

74. Obviously, $\dfrac{\mathbf{a}}{|\mathbf{a}|}+\dfrac{\mathbf{b}}{|\mathbf{b}|}$ is a vector in the plane of $\mathbf{a}$ and $\mathbf{b}$ and hence perpendicular to $\mathbf{a}\times\mathbf{b}$. It is also equally inclined to $\mathbf{a}$ and $\mathbf{b}$ as it is along the angle bisector.

75. $$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$$
or $\quad(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}=(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}-(\mathbf{c}\cdot\mathbf{b})\,\mathbf{a}$
or $\quad(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}-(\mathbf{c}\cdot\mathbf{b})\,\mathbf{a}=\mathbf{0}$
or $\quad\mathbf{b}\times(\mathbf{c}\times\mathbf{a})=\mathbf{0}$
or $\quad(\mathbf{c}\times\mathbf{a})\times\mathbf{b}=\mathbf{0}$
or $\quad\mathbf{b}\times(\mathbf{c}\times\mathbf{a})=(\mathbf{c}\times\mathbf{a})\times\mathbf{b}=\mathbf{0}$

76. Let angle between $\mathbf{a}$ and $\mathbf{b}$ be θ

$$\mathbf{v}=\mathbf{a}\times\mathbf{b}=|\,\mathbf{a}\,||\,\mathbf{b}\,|\sin\theta\,\hat{\mathbf{n}}$$

$$\therefore\quad|\,\mathbf{v}\,|=\sin\theta,\qquad\left(\because|\,\mathbf{a}\,|=1,|\,\mathbf{b}\,|=1,\ \mathbf{n}=\frac{(\mathbf{a}\times\mathbf{b})}{|\,\mathbf{a}\times\mathbf{b}\,|}=\frac{\mathbf{v}}{|\,\mathbf{v}\,|}\right)$$

$$\mathbf{u}=\mathbf{a}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{b}=\mathbf{a}-\cos\theta\,\mathbf{b}$$
$$(\because\mathbf{a}\cdot\mathbf{b}=|\,\mathbf{a}\,||\,\mathbf{b}\,|\cos\theta=\cos\theta)$$

$$\mathbf{u}\cdot\mathbf{u}=|\,\mathbf{u}\,|^2=1+\cos^2\theta-2\cos\theta=\sin^2\theta$$

$$\therefore\quad|\,\mathbf{u}\,|=\sin\theta$$

$$\mathbf{u}\cdot\mathbf{a}=\mathbf{a}\cdot\mathbf{a}=-\cos\theta\,\mathbf{a}\cdot\mathbf{b}=1-\cos^2\theta=\sin^2\theta$$

$$\mathbf{u}\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{b}-\cos\theta\,\mathbf{b}\cdot\mathbf{b}=\cos\theta-\cos\theta=0$$

$$\mathbf{u}\cdot(\mathbf{a}+\mathbf{b})=(\mathbf{a}-\cos\theta\,\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})$$
$$=1+\cos\theta-\cos^2\theta-\cos\theta$$
$$=1-\cos^2\theta=\sin^2\theta$$

77. Here, $(l\mathbf{a}+m\mathbf{b})\times\mathbf{b}=\mathbf{c}\times\mathbf{b}$

$$\Rightarrow\quad l\mathbf{a}\times\mathbf{b}=\mathbf{c}\times\mathbf{b}$$
$$\Rightarrow\quad l(\mathbf{a}\times\mathbf{b})^2=(\mathbf{c}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b})$$
$$\Rightarrow\quad l=\frac{(\mathbf{c}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b})}{(\mathbf{a}\times\mathbf{b})^2}$$

Similarly, $\quad m=\dfrac{(\mathbf{c}\times\mathbf{a})\cdot(\mathbf{b}\times\mathbf{a})}{(\mathbf{b}\times\mathbf{a})^2}$

78. $\mathbf{a}\times(\mathbf{r}\times\mathbf{a})=\mathbf{a}\times\mathbf{b}$

$$3\,\mathbf{r}-(\mathbf{a}\cdot\mathbf{r})\,\mathbf{a}=\mathbf{a}\times\mathbf{b}$$

Also, $\quad|\,\mathbf{r}\times\mathbf{a}\,|=|\mathbf{b}|$

$$\Rightarrow\quad\sin^2\theta=\frac{2}{3}$$
or $\quad(1-\cos^2\theta)=\dfrac{2}{3}$
or $\quad\dfrac{1}{3}=\cos^2\theta\ \Rightarrow\ \mathbf{a}\cdot\mathbf{r}=\pm1$
$$\Rightarrow\quad3\mathbf{r}\pm\mathbf{a}=\mathbf{a}\times\mathbf{b}$$
or $\quad\mathbf{r}=\dfrac{1}{3}(\mathbf{a}\times\mathbf{b}\pm\mathbf{a})$

79. $a_1+a_2\cos2x+a_3\sin^2x=0,\ \forall\ x\in R$

or $\quad(a_1+a_2)+\sin^2x(a_3-2a_2)=0$

$$\Rightarrow\quad a_1+a_2=0\text{ and }a_3-2a_2=0$$
$$\frac{a_1}{-1}=\frac{a^2}{1}=\frac{a_3}{2}=\lambda(\neq0)$$
$$\Rightarrow\quad a^1=-\lambda,a_2=\lambda,a_3=2\lambda$$

80. $\mathbf{a}\times\mathbf{b}=|\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}$

or $\quad|\mathbf{a}\times\mathbf{b}|=|\mathbf{a}||\mathbf{b}|\sin\theta$

or $\quad\sin\theta=\dfrac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}|\times|\mathbf{b}|}\qquad\ldots(\text{i})$

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

$$\Rightarrow \quad \cos\theta = \frac{|\mathbf{a}\cdot\mathbf{b}|}{|\mathbf{a}||\mathbf{b}|} \qquad \ldots(ii)$$

From Eqs. (i) and (ii),

$$\sin^2\theta + \cos^2\theta = 1$$

$$\Rightarrow \quad |\mathbf{a}\times\mathbf{b}|^2 + (\mathbf{a}\cdot\mathbf{b})^2 = |\mathbf{a}|^2|\mathbf{b}|^2$$

If $\theta = \pi/4$, then $\sin\theta = \cos\theta = 1/\sqrt{2}$. Therefore,

$$|\mathbf{a}\times\mathbf{b}| = \frac{|\mathbf{a}||\mathbf{b}|}{\sqrt{2}} \text{ and } \mathbf{a}\cdot\mathbf{b} = \frac{|\mathbf{a}||\mathbf{b}|}{\sqrt{2}}$$

$$|\mathbf{a}\times\mathbf{b}| = \mathbf{a}\cdot\mathbf{b}$$

$$\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}} = \frac{|\mathbf{a}||\mathbf{b}|}{\sqrt{2}}\,\hat{\mathbf{n}}$$

$$= (\mathbf{a}\cdot\mathbf{b})\,\hat{\mathbf{n}}$$

81. We have, $|\mathbf{a}-\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2(\mathbf{a}\cdot\mathbf{b})$

or $|\mathbf{a}-\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos 2\theta$

or $|\mathbf{a}-\mathbf{b}|^2 = 2 - \cos 2\theta \qquad (\because |\mathbf{a}| = |\mathbf{b}| = 1)$

$$= 4\sin^2\theta \quad \text{or} \quad |\mathbf{a}-\mathbf{b}| = 2|\sin\theta|$$

Now, $|\mathbf{a}-\mathbf{b}| < 1$

$$\Rightarrow \quad 2|\sin\theta| < 1 \text{ or } |\sin\theta| < \frac{1}{2}$$

$$\Rightarrow \quad \theta \in [0, \pi/6) \text{ or } \theta \in (5\pi/6, \pi)$$

82. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) + (\mathbf{a}\cdot\mathbf{b})\,\mathbf{b}$

$$= (4 - 2x - \sin y)\,\mathbf{b} + (x^2 - 1)\,\mathbf{c}$$

or $(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c} + (\mathbf{a}\cdot\mathbf{b})\,\mathbf{b}$

$$= (4 - 2x - \sin y)\,\mathbf{b} + (x^2 - 1)\,\mathbf{c}$$

Now, $(\mathbf{c}\cdot\mathbf{c})\,\mathbf{a} = \mathbf{c}$.

Therefore, $(\mathbf{c}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{c}) = (\mathbf{c}\cdot\mathbf{c})$ or $\mathbf{a}\,.\,\mathbf{c} = 1$

$$\Rightarrow \quad 1 + \mathbf{a}\cdot\mathbf{b} = 4 - 2x - \sin y,\ x^2 - 1 = -(\mathbf{a}\cdot\mathbf{b})$$

or $1 = 4 - 2x - \sin y + x^2 - 1$

or $\sin y = x^2 - 2x + 2 = (x-1)^2 + 1$

But, $\sin y \le 1 \ \Rightarrow \ x = 1, \sin y = 1 \ \Rightarrow \ y = (4n+1)\dfrac{\pi}{2},\ n \in I$

83. $\mathbf{AB} + \mathbf{BC} = \mathbf{AC}$

$$\mathbf{BC} = \frac{2\mathbf{u}}{|\mathbf{u}|} - \frac{\mathbf{u}}{|\mathbf{u}|} + \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{u}}{|\mathbf{u}|} + \frac{\mathbf{v}}{|\mathbf{v}|}$$

$$\mathbf{AB}\cdot\mathbf{BC} = \left(\frac{\mathbf{u}}{|\mathbf{u}|} - \frac{\mathbf{v}}{|\mathbf{v}|}\right)\left(\frac{\mathbf{u}}{|\mathbf{u}|} + \frac{\mathbf{v}}{|\mathbf{v}|}\right)$$

$$= (\hat{\mathbf{u}} - \hat{\mathbf{v}})\cdot(\hat{\mathbf{u}} + \hat{\mathbf{v}}) = 1 - 1 = 0$$

$$\Rightarrow \quad \angle B = 90^\circ$$

$$\Rightarrow \quad 1 + \cos 2A + \cos 2B + \cos 2C = 0$$

84. Clearly, $\mathbf{a}\cdot\mathbf{c} = 0$ and $\mathbf{b}\cdot\mathbf{c} = 0$. Also, $\mathbf{a}\cdot\mathbf{b} = 0$

$$\mathbf{a}\times\mathbf{b} = \mathbf{c}$$

dot with $\mathbf{b} \Rightarrow \mathbf{b}\cdot\mathbf{c} = 0$

Similarly, $\mathbf{b}\times\mathbf{c} = \mathbf{a}$

dot with $\mathbf{b} \Rightarrow \mathbf{a}\cdot\mathbf{b} = 0$

dot with $\mathbf{c} \Rightarrow \mathbf{a}\cdot\mathbf{c} = 0$

$$\Rightarrow \quad \mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{c} = \mathbf{c}\cdot\mathbf{a} = 0$$

Again, $\dfrac{|\mathbf{a}||\mathbf{b}|}{|\mathbf{b}||\mathbf{c}|} = \dfrac{|\mathbf{c}|}{|\mathbf{a}|}$

$$\Rightarrow \quad \frac{|\mathbf{a}|}{|\mathbf{c}|} = \frac{|\mathbf{c}|}{|\mathbf{a}|}$$

$$\Rightarrow \quad |\mathbf{a}| = |\mathbf{c}| \text{ and } |\mathbf{b}| = 1$$

$$\Rightarrow \quad \mathbf{a}\times\mathbf{b}\cdot\mathbf{c} = |\mathbf{a}||\mathbf{b}||\mathbf{c}| = |\mathbf{a}|^2 = |\mathbf{c}|^2$$

(children will assume $\mathbf{a} = \hat{\mathbf{i}}$; $\mathbf{b} = \hat{\mathbf{j}}$ and $\mathbf{c} = \hat{\mathbf{k}}$ but in this case all the four will be correct which will be wrong).

85. Given, $|\mathbf{A}||\mathbf{B}|\cos\theta = -6; |\mathbf{B}| = 2$ (given)

$$\mathbf{B}\cdot\mathbf{C} = |\mathbf{B}||\mathbf{C}|\cos\phi = 6$$

and $(\mathbf{A}\times\mathbf{B})\times\mathbf{A} = 0$

$$(\mathbf{A}\cdot\mathbf{A})\mathbf{B} - (\mathbf{B}\cdot\mathbf{A})\mathbf{A} = 0$$

$$(\mathbf{A}\cdot\mathbf{A}) = -6\mathbf{A} \qquad \ldots(i)$$

$\therefore$ $\mathbf{A}$ and $\mathbf{B}$ are collinear and θ between $\mathbf{A}$ and $\mathbf{B}$ is π.

$$\Rightarrow \quad \mathbf{A}\times\mathbf{B} = 0$$

$\Rightarrow$ (a) is correct.

$$\Rightarrow \quad \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} = 0$$

$\Rightarrow$ (b) is correct.

Also, $\mathbf{A}\cdot\mathbf{B} = -6$ and $|\mathbf{B}| = 2$

$\therefore$ $|\mathbf{A}||\mathbf{B}|\cos\pi = -6|\mathbf{A}|\cdot(2) = 6$

$$\Rightarrow \quad |\mathbf{A}| = 3 \ \Rightarrow \ \mathbf{A}\cdot\mathbf{A} = 9$$

$\Rightarrow$ (c) is not correct.

Again, $\mathbf{A}\cdot\mathbf{C} = ?$

dot with $\mathbf{C}$ in the Eq. (i)

$$9(\mathbf{B}\cdot\mathbf{C}) = -6\mathbf{A}\cdot\mathbf{C}$$

$$9(6) = -6(\mathbf{A}\cdot\mathbf{C}) \ \Rightarrow \ \mathbf{A}\cdot\mathbf{C} = -9$$

$\Rightarrow$ (d) is correct.

86. $\mathbf{V}_1 = \mathbf{V}_2$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\times\mathbf{c}$$

$$(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\,\mathbf{a}$$

$\therefore$ $(\mathbf{a}\cdot\mathbf{b})\mathbf{c} = (\mathbf{b}\cdot\mathbf{c})\,\mathbf{a}$

$\Rightarrow$ Either $\mathbf{c}$ and $\mathbf{a}$ are collinear or $\mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{c}$

$$\Rightarrow \quad \mathbf{b} = \lambda\,(\mathbf{a}\times\mathbf{c})$$

87. It may be observed that

$$[\mathbf{U}\,\mathbf{V}\,\mathbf{W}] = \begin{vmatrix} 2 & 3 & -6 \\ 6 & 2 & 3 \\ 3 & -6 & -2 \end{vmatrix} = 343 \neq 0$$

$\Rightarrow$ $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$ are non-coplanar, hence linearly independent

Further $\mathbf{U}\times\mathbf{V} = \mathbf{W}$ and $\mathbf{V}\times\mathbf{W} = \mathbf{U}$

They form a right handed triplet of mutually perpendicular vectors and of course!

$$\Rightarrow \ (\mathbf{U}\times\mathbf{V})\times\mathbf{W} = 0 \text{ and } \mathbf{U}\times(\mathbf{V}\times\mathbf{W})$$

88. Let the required vector be $\mathbf{d} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$. For this to be coplanar wih $\mathbf{b}$ and $\mathbf{c}$, we must have

$$\begin{vmatrix} x & y & z \\ 1 & 2 & -1 \\ 1 & 1 & -2 \end{vmatrix} = 0$$

$$\Rightarrow \quad x(-4+1) + y(-1+2) + z(1-2) = 0$$

$$-3x + y - z = 0$$

The projection of $\mathbf{d}$ on $\mathbf{a}$ is $\dfrac{|\mathbf{a}\cdot\mathbf{d}|}{|\mathbf{a}|}$.

So, $$\sqrt{\frac{2}{3}} = \frac{1}{\sqrt{6}}|2x - y + z|$$

$\Rightarrow$ $$2x - y + z = \pm 2$$

The choices (a) and (c) satisfy the Eqs. (i) and (ii).

89. $$\mathbf{a} \times (\mathbf{b} - 3\mathbf{c}) = \mathbf{0}$$

$\Rightarrow$ $$\mathbf{b} - 3\mathbf{c} = \lambda\mathbf{a}$$

$\Rightarrow$ $$|\mathbf{b} - 3\mathbf{c}| = |\lambda\mathbf{a}|$$

$\Rightarrow$ $$1 + 1 - 6.1 \cdot \frac{1}{3} \cdot \frac{1}{2} = |\lambda| \quad \Rightarrow \lambda \pm 1.$$

$\therefore$ $$\mathbf{b} - 3\mathbf{c} = \pm\mathbf{a}$$

90. $(\mathbf{a} \times \mathbf{c}) \cdot (\mathbf{r} \times \mathbf{a}) = (\mathbf{a} \times \mathbf{c}) \cdot \mathbf{b}$

91. (a) is proved if we take dot product of both sides with $\mathbf{a}$.

(b) If we take dot product with $\mathbf{b}$, we get

$$\lambda_2 = \mathbf{b} \cdot \mathbf{c}$$

$\Rightarrow$ Option (b) is not true.

(c) If we take dot product of both sides with $\mathbf{a} \times \mathbf{b}$, we get

$$[\mathbf{c}\ \mathbf{b}\ \mathbf{a}] = \lambda_3[\mathbf{a} \times \mathbf{b}]^2$$

$\Rightarrow$ $$\lambda_3 = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \text{ or } \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

$\Rightarrow$ Option (c) is wrong.

(d) is correct since $\lambda_1 + \lambda_2 + \lambda_3 = \mathbf{c} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{c} + [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$.

92. (a) Since, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, are non-coplanar, option (a) is true.

Since, $\mathbf{b} \times \mathbf{c}$, $\mathbf{c} \times \mathbf{a}$, $\mathbf{a} \times \mathbf{b}$ are also non-coplanar.

(b) is also correct.

Since, $\mathbf{x} = \lambda(\mathbf{b} \times \mathbf{c}) + \mu(\mathbf{c} \times \mathbf{a}) + \nu(\mathbf{a} \times \mathbf{b})$

We have, $\lambda = \dfrac{\mathbf{a} \cdot \mathbf{x}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$, (on taking dot product with $\mathbf{a}$)

μ and ν have similar values.

Also, $$|\mathbf{x}| = |\mathbf{a} - \mathbf{x}|$$

$\Rightarrow$ $$\mathbf{a} \cdot \mathbf{x} = \frac{a^2}{2}, \text{ etc.}$$

$\Rightarrow$ Option (c) is correct.

If (c) is correct (d) is ruled out.

93. $\alpha = (\mathbf{a} \cdot \hat{\mathbf{i}})\hat{\mathbf{i}} + (\mathbf{a} \cdot \hat{\mathbf{j}})\hat{\mathbf{j}} + (\mathbf{a} \cdot \hat{\mathbf{k}})\hat{\mathbf{k}} = \mathbf{a} = (1, 1, 1)$

$\beta = (\mathbf{b} \cdot \hat{\mathbf{i}})\hat{\mathbf{i}} + (\mathbf{b} \cdot \hat{\mathbf{j}})\hat{\mathbf{j}} + (\mathbf{b} \cdot \hat{\mathbf{k}})\hat{\mathbf{k}} = \mathbf{b} = (1, -1, 0)$

$\gamma = (1, 1, -2)$

$\therefore$ $$\alpha \cdot \beta = \beta \cdot \gamma = \gamma \cdot \alpha = 0$$

$\Rightarrow \alpha, \beta, \gamma$ are mutually perpendicular $\alpha, \beta, \gamma = 6$

$\Rightarrow \alpha, \beta, \gamma$ form a parallelopiped of volume 6 units.

94. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

$$= (xz + yx + yz)(y\hat{\mathbf{i}} + z\hat{\mathbf{j}} + x\hat{\mathbf{k}}) - (xy + yz + zx)(z\hat{\mathbf{i}} + x\hat{\mathbf{j}} + y\hat{\mathbf{k}})$$

$$= (xz + yx + yz)((y - z)\hat{\mathbf{i}} + (z - x)\hat{\mathbf{j}} + (x - y)\hat{\mathbf{k}})$$

Clearly, perpendicular to $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and also to $(y + z)\hat{\mathbf{i}} + (z + x)\hat{\mathbf{j}} + (x + y)\hat{\mathbf{k}}$ as dot products are zeros.

Clearly, parallel to $(y - z)\hat{\mathbf{i}} + (z - x)\hat{\mathbf{j}} + (x - y)\hat{\mathbf{k}}$

95. $A \rightarrow \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{O}$

$\Rightarrow$ Vectors are coplanar, so do not form RHS

$B \rightarrow (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}, \mathbf{a} \times \mathbf{b}, \vec{\mathbf{c}} = \mathbf{0}$ in that order form RHS

$\Rightarrow \mathbf{c}, (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}, \mathbf{a} \times \mathbf{b}$ also form RHS as they are in same cyclic order.

$$C \rightarrow \mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0} \ \Rightarrow (\mathbf{a} + \mathbf{b} + \mathbf{c})^2 = \mathbf{0}$$

$\Rightarrow$ $$\mathbf{a}^2 + \mathbf{b}^2 + \mathbf{c}^2 = -2(\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a})$$

Hence, $$\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a} < 0$$

$$D \rightarrow \mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$$

$\Rightarrow$ $$\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}$$

Using this we get result.

96. $\mathbf{a} \cdot \mathbf{b} = 0, \mathbf{c} \cdot \mathbf{a} = \mathbf{c} \cdot \mathbf{b} = \cos\alpha$

Take dot products with $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$, respectively.

$$l = m, n^2 + l^2 + m^2 = 1$$

$$n^2 = -\cos 2\alpha, m^2 = \frac{1 + \cos 2\alpha}{2}$$

97. $A \rightarrow \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b}(\mathbf{c} \times \mathbf{a}) + \mathbf{c}(\mathbf{a} \times \mathbf{b}) = \mathbf{0}$

$\Rightarrow$ Vectors are coplanar, so do not form RHS

$B \rightarrow (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}, \mathbf{a} \times \mathbf{b}, \mathbf{c}$ in that order form RHS

$\Rightarrow \mathbf{c}, (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}, \mathbf{a} \times \mathbf{b}$ also form RHS as they are in same cyclic order

$$C \rightarrow \mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$$

$\Rightarrow$ $$(\mathbf{a} + \mathbf{b} + \mathbf{c})^2 = 0$$

$\Rightarrow$ $$\mathbf{a}^2 + \mathbf{b}^2 + \mathbf{c}^2 = -2(\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a})$$

Hence, $\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a} < 0$

$$D \rightarrow \mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$$

$\Rightarrow$ $$\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}$$

Using this we get result.

98. Since $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a} \times \mathbf{b}$ are non-coplanar,

$$\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z(\mathbf{a} \times \mathbf{b}), \mathbf{r} \times \mathbf{b} = \mathbf{a}$$

$\Rightarrow$ $$x\mathbf{a} \times \mathbf{b} + z(\mathbf{a} \times \mathbf{b}) \times \mathbf{b} = \mathbf{a}$$

$\Rightarrow$ $$x(\mathbf{a} \times \mathbf{b}) + z(\mathbf{b} \cdot \mathbf{a})\mathbf{b} - (\mathbf{b} \cdot \mathbf{b})\mathbf{a} = \mathbf{a}$$

$\Rightarrow$ $$x(\mathbf{a} \times \mathbf{b}) - \mathbf{a}(1 + |\mathbf{b}|^2 z) = 0$$

$\Rightarrow$ $$x = 0, z = -\frac{1}{|\mathbf{b}|^2}$$

$\therefore$ $\mathbf{r} = y\mathbf{b} - \dfrac{1}{|\mathbf{b}|^2}(\mathbf{a} \times \mathbf{b})$, where y is any scalar.

99. Let angle between $\mathbf{a}$ and $\mathbf{b}$ be θ.

We have, $$|\mathbf{a}| = |\mathbf{b}| = 1$$

Now, $|\mathbf{a} + \mathbf{b}| = 2\cos\dfrac{\theta}{2}$ and $|\mathbf{a} - \mathbf{b}| = 2\sin\dfrac{\theta}{2}$

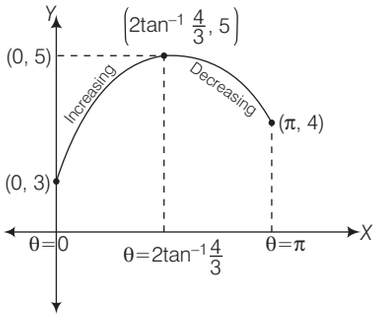
Consider, $F\theta = \dfrac{3}{2}\left(2\cos\dfrac{\theta}{2}\right) + 2\left(2\sin\dfrac{\theta}{2}\right)$

$\therefore$ $$F(\theta) = 3\cos\frac{\theta}{2} + 4\sin\frac{\theta}{2}, \theta \in [0, \pi]$$

$$F(\theta) = \frac{-3}{2}\sin\frac{\theta}{2} + 2\cos\frac{\theta}{2}$$

Now, $F(\theta) = 0$

$\Rightarrow$ $$\tan\frac{\theta}{2} = \frac{4}{3}$$



Clearly, $F(\theta) = 3$

$$F\left(\theta = 2\tan^{-1}\frac{4}{3}\right) = 3\left(\frac{3}{5}\right) + 4\left(\frac{4}{5}\right) = \frac{9}{5} + \frac{16}{5} = \frac{25}{5} = 5$$

$$F(\theta = \pi) = 4$$

$\therefore$ Range $= [3, 5]$

Hence, possible integer(s) in the range of $F(\theta)$ in $[\,0, \pi\,]$ are 3 *viz*, 3, 4 and 5.

100. Let $\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z(\mathbf{a} \times \mathbf{b})$ with $\mathbf{b}$

$$\mathbf{r} \cdot \mathbf{b} = 0 + y|\mathbf{b}|^2$$

$$\mathbf{a}(\mathbf{r} \cdot \mathbf{b}) = y(\mathbf{b})^2\mathbf{a}$$

$\Rightarrow$ $$\mathbf{c} - p\mathbf{r} = y|\mathbf{b}|^2\mathbf{a}$$

$$\mathbf{r} = \frac{1}{p}\mathbf{c} - \frac{y|b|^2}{p}\mathbf{a}$$

$\therefore$ $[\mathbf{r}\ \mathbf{ac}] = 0$

Now, $$\mathbf{r} \cdot \mathbf{b} = \frac{1}{p}\mathbf{c} \cdot \mathbf{b}$$

$\therefore$ $$y|b|^2 = \frac{\mathbf{b} \cdot \mathbf{c}}{p}$$

$\therefore$ $$\mathbf{r} = \frac{1}{p}\mathbf{c} - \frac{1}{p^2}(\mathbf{b} \cdot \mathbf{c})\mathbf{a}$$

101. $(\lambda - 1)(\mathbf{a}_1 - \mathbf{a}_2) + \mu(\mathbf{a}_2 + \mathbf{a}_3) + \gamma(\mathbf{a}_3 + \mathbf{a}_4 - 2\mathbf{a}_2) + \mathbf{a}_3 + \delta\mathbf{a}_4 = 0$

i.e. $(\lambda - 1)\mathbf{a}_1 + (1 - \lambda + \mu - 2\gamma)\mathbf{a}_2 + (\mu + \gamma + 1)\mathbf{a}_3 + (\gamma + \delta)\mathbf{a}_4 = 0$

Since, $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ are linearly independent

$\therefore$ $\lambda - 1 = 0, 1 - \lambda + \mu - 2\gamma = 0, \mu + \lambda + 1 = 0$

$\gamma + \delta = 0$

i.e. $\lambda = 1, \mu = 2\gamma, \mu + \gamma + 1 = 0, \gamma + \delta = 0$

i.e. $\lambda = 1, \mu = -\frac{2}{3}, \gamma = -\frac{1}{3}, \delta = \frac{1}{3}$

102. Since $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$

$\therefore$**a, b** and **c** are complanar vectors

Further since **d** is equally inclined to **a**, **b** and **c**

$\therefore \mathbf{d} \cdot \mathbf{a} = \mathbf{d} \cdot \mathbf{b} = \mathbf{d} \cdot \mathbf{c} = 0$

$\therefore \mathbf{d} \cdot \mathbf{r} = 0$

103. $\mathbf{p} = ab\cos(2\pi - \theta)\mathbf{c}$, where θ is the angle between **a** and **b** and $\mathbf{q} = ac\cos(\pi - \phi)\mathbf{b}$ where ϕ is the angle between **a** and **b** now $\mathbf{p} + \mathbf{q} = (ab\cos\theta)\mathbf{c} - ac\cos\phi\mathbf{b} = (\mathbf{a} \cdot \mathbf{b})\mathbf{c} - (\mathbf{a} \cdot \mathbf{c})\mathbf{b} = \mathbf{a} \times (\mathbf{c} \times \mathbf{b})$

$\Rightarrow B$ and C

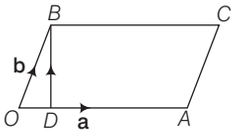
104. Verify $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_3$ in order to quickly answer

105. Since, $(2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 4\hat{\mathbf{k}}) \cdot (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}) = 0$

and $(\hat{\mathbf{i}} + \hat{\mathbf{j}}) \cdot (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}), 1 + 2 = 3$

$\Rightarrow$ Line lies in the plane

106. $$\mathbf{OD} = \frac{(\mathbf{a} \cdot \mathbf{b})\mathbf{a}}{|\mathbf{a}|^2}$$



$\Rightarrow$ $$\mathbf{DB} = \mathbf{b} - \mathbf{OD} = \mathbf{b} - \frac{(\mathbf{a} \cdot \mathbf{b})\mathbf{a}}{|\mathbf{a}|^2}$$

$$= \frac{(\mathbf{a} \cdot \mathbf{a})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{a}}{|\mathbf{a}|^2} = \frac{\mathbf{a} \times (\mathbf{b} \times \mathbf{a})}{|\mathbf{a}|^2}$$

107. $\mathbf{a} + \mathbf{b} = \lambda\mathbf{c}$; $\mathbf{b} + \mathbf{c} = \mu\mathbf{a}$

$$\mathbf{a} - \mathbf{c} = \lambda\mathbf{c} - \mu\mathbf{a}$$

$$\mathbf{a}(1 + \mu) = \mathbf{c}(1 + \lambda)$$

but **a** and **c** are non-collinear $\Rightarrow \mu = -1, \lambda = -1$

$\therefore$ $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0 = \hat{\mathbf{k}} \quad \Rightarrow |\mathbf{k}| = 0$

$\Rightarrow (k, k) \equiv (0, 0)$ all the given curves pass through (0, 0)

108. $(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} = \frac{\mathbf{b} + \mathbf{c}}{2}$; $\vec{\mathbf{b}}\left(\vec{\mathbf{a}} \cdot \vec{\mathbf{c}} - \frac{1}{2}\right) - \mathbf{c}\left(\mathbf{a} \cdot \mathbf{b} + \frac{1}{2}\right) = 0$

$\Rightarrow$ $$\mathbf{a} \cdot \mathbf{c} = \frac{1}{2} \text{ and } \mathbf{a} \cdot \mathbf{c} = -\frac{1}{2}$$

109. $AA^T = I \Rightarrow \mathbf{a}, \mathbf{b}, \mathbf{c}$ are orthogonal unit vectors

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = \frac{1}{49}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 3 & 6 \\ 6 & 2 & -3 \end{vmatrix} = \frac{1}{7}(-3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$$

$\Rightarrow$ $\mathbf{c} = \pm\frac{1}{7}(3\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

110. Component of vector $\mathbf{b} = 4\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ in the direction of $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ is $\frac{\mathbf{a} \cdot \mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}|\,|\mathbf{a}|}$ or $3\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$. Then, component in the direction perpendicular to the direction of $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ is $\mathbf{b} - 3\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 3\hat{\mathbf{k}} = \hat{\mathbf{i}} - \hat{\mathbf{j}}$

111. Le the three given unit vector be $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$. Since, they are mutually perpendicular, $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 1$.

Therefore, $$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 1$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 1$$

Hence, $a_1\hat{\mathbf{i}} + b_1\hat{\mathbf{j}} + c_1\hat{\mathbf{k}}$, $a_2\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + c_2\hat{\mathbf{k}}$ and $a_3\hat{\mathbf{i}} + b_3\hat{\mathbf{j}} + c_3\hat{\mathbf{k}}$ may be mutually perpendicular.

112. Statement II is true (see properties of dot product)

Also, $(\hat{\mathbf{i}} \times \mathbf{b}) \cdot \mathbf{b} = \hat{\mathbf{i}} \cdot (\mathbf{a} \times \mathbf{b})$

$\Rightarrow$ $\mathbf{a} \times \mathbf{b} = (\hat{\mathbf{i}}\,.\,(\mathbf{a} \times \mathbf{b}))\,\hat{\mathbf{i}} + (\hat{\mathbf{j}} \cdot (\mathbf{a} \times \mathbf{b}))\,\hat{\mathbf{j}} + (\hat{\mathbf{k}} \cdot (\mathbf{a} \times \mathbf{b}))\,\hat{\mathbf{k}}$

113. $\mathbf{AD} = 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$, $\mathbf{BD} = -2\hat{\mathbf{i}} - \hat{\mathbf{j}} - 3\hat{\mathbf{k}}$ and $\mathbf{CD} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}}$

Volume of tetrahedron $= \frac{1}{6}[\mathbf{AD}\ \mathbf{BD}\ \mathbf{CD}]$

$$= \frac{1}{6}\begin{Vmatrix} 0 & 2 & -1 \\ -2 & -1 & -3 \\ 2 & -1 & 0 \end{Vmatrix} = \frac{8}{3}$$

Also, area of the triangle $ABC = \frac{1}{2}|\mathbf{AB} \times \mathbf{AC}|$

$$= \frac{1}{2}\begin{Vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 3 & 2 \\ -2 & 3 & -1 \end{Vmatrix} = \frac{1}{2}|-9\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 12\hat{\mathbf{k}}| = \frac{\sqrt{299}}{2}$$

Then, $\frac{8}{3} = \frac{1}{3} \times$(Distance of D from base ABC) $\times$(Area of triangle ABC)

Distance of D from base $ABC = 16 / \sqrt{229}$

114. $\mathbf{A} \times (\mathbf{A} \cdot \mathbf{B})\ \mathbf{A} - (\mathbf{A} \cdot \mathbf{A})\ \mathbf{B}) \cdot \mathbf{C}$

$$= \left(\underbrace{\mathbf{A} \times (\mathbf{A} \cdot \mathbf{B})\ \mathbf{A}}_{\text{zero}} - (\mathbf{A} \cdot \mathbf{A})\ \mathbf{A} \times \mathbf{B} \right) \cdot \mathbf{C} = -|\mathbf{A}|^2\ [\mathbf{A}\ \mathbf{B}\ \mathbf{C}]$$

Now, $|\mathbf{A}|^2 = 4 + 9 + 36 = 49$

$$[\mathbf{A}\ \mathbf{B}\ \mathbf{C}] = \begin{vmatrix} 2 & 3 & 6 \\ 1 & 1 & -2 \\ 1 & 2 & 1 \end{vmatrix} = 2(1 + 4) - 1(3 - 12) + 1(-6 - 6)$$

$$= 10 + 6 - 12 = 7 = 10 + 9 - 12 = 7$$

$$\therefore \quad |-|\mathbf{A}|^2\ [\mathbf{A}\ \mathbf{B}\ \mathbf{C}]| = 49 \times 7 = 343$$

115. $\pm \frac{\mathbf{a} \times \mathbf{b}}{\mathbf{a} \times \mathbf{b}} \perp \mathbf{a}$ and $\mathbf{b}$

$\therefore$ There are two such vectors

116. $\frac{k}{k+l+m} + \frac{l}{k+l+m} + \frac{m}{k+l+m} = 1$

$\Rightarrow$ Point lies in the plane of $\Delta\ ABC$.

117. $a^2 + b^2 + 2\mathbf{a} \cdot \mathbf{b} = a^2 + 4b^2 - 4\mathbf{a} \cdot \mathbf{b} \Rightarrow 6ab\cos\theta = 3b^2$

$$\Rightarrow \quad \mathbf{a} \cdot \mathbf{b} = \frac{1}{2}b^2$$

$$GE = \frac{b^2}{2} + \frac{4}{b^2+2} = \frac{b^2+2}{2} + \frac{4}{b^2+2} - 1$$

$$= \sqrt{2}\left[\frac{b^2+2}{2\sqrt{2}} + \frac{2\sqrt{2}}{b^2+2}\right] - 1 = \sqrt{2}(\geq 2) - 1 \geq \sqrt{2} - 1$$

It is least when $\frac{b^2+2}{2\sqrt{2}} = 1$

$$\Rightarrow \qquad \mathbf{b} = \sqrt{2(\sqrt{2}-1)} = \sqrt{2\tan\pi/8}$$

118. Both the statements are true and statement II is the not correct explanation of statement I. Because $\mathbf{b}, \mathbf{c}, \mathbf{d}$ in statement I are coplanar.

119. $3\mathbf{a} - 2\mathbf{b} + 5\mathbf{c} - 6\mathbf{d} = (2\mathbf{a} - 2\mathbf{b})$

$$+ (-5\mathbf{a} + 5\mathbf{c}) + (6\mathbf{a} - 6\mathbf{d}) = -2\mathbf{AB} + 5\mathbf{AC} - 6\mathbf{AD} = 0$$

$\therefore \mathbf{AB}$, $\mathbf{AC}$ and $\mathbf{AD}$ are linearly dependent, hence by statement II, the statement I is true.

120. $\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z(\mathbf{a} \times \mathbf{b})$

121. $(\hat{\mathbf{a}} \times (\mathbf{b} \times \hat{\mathbf{c}})) \cdot (\hat{\mathbf{a}} \times \hat{\mathbf{c}}) = (\hat{\mathbf{a}} \cdot \hat{\mathbf{c}})[\mathbf{b}\hat{\mathbf{a}}\hat{\mathbf{c}}] = 5$

$$\Rightarrow \qquad [\mathbf{b}\hat{\mathbf{a}}\hat{\mathbf{c}}] = 10 \Rightarrow [\hat{\mathbf{a}}\mathbf{b}\hat{\mathbf{c}}] = -10.$$

Solutions (Q.Nos. 122-124)

122. $\underbrace{(\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 4\hat{\mathbf{k}})}_{(\mathbf{q}-\mathbf{r})} \cdot (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) = 1 + 3 - 4 = 0$

Since, $[\mathbf{q}\ \mathbf{p}\ \mathbf{r}] = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & -1 \\ 1 & 1 & 3 \end{vmatrix}$

$$= (12 + 1) - 1(6 + 1) + 1(2 - 4) = 13 - 7 - 2 = 4$$

$\Rightarrow$ (a) (b) (d) are wrong.

123. $(\mathbf{p} \times \mathbf{r}) \times \mathbf{r} = (\mathbf{p} \cdot \mathbf{r})\ \mathbf{q} - (\mathbf{q} \cdot \mathbf{r})\ \mathbf{p}$

$$\Rightarrow \qquad u = -(\mathbf{q} \cdot \mathbf{r}) = -(2 + 4 - 3) = -3$$

$$v = \mathbf{p} \cdot \mathbf{r} = 1 + 1 + 3 = 5 \text{ and } w = 0$$

Hence, $\quad u = -3,\ v = 5,\ w = 0 \ \Rightarrow\ u + v + w = 2$

124. $\because \mathbf{p}, \mathbf{q}$ and $\mathbf{r}$ are non-coplanar, therefore

$\mathbf{q} \times \mathbf{r}$, $\mathbf{r} \times \mathbf{p}$ and $\mathbf{p} \times \mathbf{q}$ are also non-coplanar

Hence, $\quad \mathbf{s} = l(\mathbf{q} \times \mathbf{r}) + w(\mathbf{r} \times \mathbf{p}) + n(\mathbf{p} \times \mathbf{q})$

$$\therefore \qquad l = \frac{\mathbf{s} \cdot \mathbf{p}}{[\mathbf{p}\ \mathbf{q}\ \mathbf{r}]},\ w = \frac{\mathbf{s} \cdot \mathbf{q}}{[\mathbf{p}\ \mathbf{q}\ \mathbf{r}]},\ n = \frac{\mathbf{s} \cdot \mathbf{r}}{[\mathbf{p}\ \mathbf{q}\ \mathbf{r}]}$$

Hence, $\mathbf{s}[\mathbf{p}\ \mathbf{q}\ \mathbf{r}] = (\mathbf{s} \cdot \mathbf{p})\ (\mathbf{q} \times \mathbf{r}) + (\mathbf{s} \cdot \mathbf{q})$

$$(\mathbf{r} \times \mathbf{p}) + (\mathbf{s} \cdot \mathbf{r})\ (\mathbf{p} \times \mathbf{q})$$

$$|(\mathbf{s} \cdot \mathbf{p})\ (\mathbf{q} \times \mathbf{r}) + (\mathbf{s} \cdot \mathbf{q})\ (\mathbf{r} \times \mathbf{p}) + (\mathbf{s} \cdot \mathbf{r})\ (\mathbf{p} \times \mathbf{q})$$

$$= |\mathbf{s}\ [\mathbf{p}\ \mathbf{q}\ \mathbf{r}]| = [\mathbf{p}\ \mathbf{q}\ \mathbf{r}]\ (\text{as}|\mathbf{s}| = 1)$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & -1 \\ 1 & 1 & 3 \end{vmatrix} = (12 + 1) - 1(6 + 1) + 1(2 - 4) = 13 - 7 - 2 = 4$$

Solutions (Q.Nos. 125-127)

125. Given, $\quad \mathbf{p} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}};\ \mathbf{q} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$

$$\mathbf{p} \times \mathbf{r} = \mathbf{q} + c\mathbf{p} \text{ and } \mathbf{p} \cdot \mathbf{r} = 2$$

$\therefore \quad \mathbf{p} \times (\mathbf{p} \times \mathbf{r}) = \mathbf{p} \times \mathbf{q}\ (\mathbf{q} + c\mathbf{p})$ and $\mathbf{p} \cdot (\mathbf{p} \times \mathbf{r}) = \mathbf{p} \cdot (\mathbf{q} + c\mathbf{p})$

$\therefore \quad (\mathbf{p} \cdot \mathbf{r})\mathbf{p} - (\mathbf{p} \cdot \mathbf{p})\mathbf{r} = \mathbf{p} \times \mathbf{q} + c\mathbf{0}$

$$0 = \mathbf{p} \cdot \mathbf{q} + \mathbf{c}\ (\mathbf{p} \cdot \mathbf{q})$$

$$\therefore \qquad (\mathbf{p} \cdot \mathbf{p})\mathbf{r} = (\mathbf{p} \cdot \mathbf{r})\mathbf{p} - \mathbf{p} \times \mathbf{q} \qquad \ldots(\text{i})$$

$$c = -\frac{\mathbf{p} \cdot \mathbf{q}}{\mathbf{p} \cdot \mathbf{p}} \qquad \ldots(\text{ii})$$

But $\qquad \mathbf{p} \cdot \mathbf{p} = |\mathbf{p}| = \mathbf{p}^2 = 3 \qquad \ldots(\text{iii})$

$$\mathbf{p} \cdot \mathbf{p} = 1 - 1 + 1 = 1 \qquad \ldots(\text{iv})$$

$$\mathbf{p} \times \mathbf{q} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 2\hat{\mathbf{i}} - 2\hat{\mathbf{k}} \qquad \ldots(\text{v})$$

Using Eqs. (iii), (iv) in Eq. (i) and Eq. (ii), we get

$$3\mathbf{r} = 2\mathbf{p} - 2\hat{\mathbf{i}} + 2\hat{\mathbf{k}} \text{ and } c = -\frac{1}{3} \qquad \ldots(\text{vi})$$

$$\therefore \qquad \mathbf{r} = \frac{1}{3}[2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}} - 2\hat{\mathbf{i}} + 2\hat{\mathbf{k}}]$$

$\therefore \quad \mathbf{r} = \frac{2}{3}(\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$ ...(vii)

Now, $[\mathbf{p}\ \mathbf{q}\ \mathbf{r}] = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & \frac{1}{2} & \frac{4}{3} \end{vmatrix} = \left(\frac{-4}{3} - \frac{2}{3}\right) - 1\left(\frac{4}{3}\right) + 1\left(\frac{2}{3}\right)$

$\therefore \quad [\mathbf{p}\ \mathbf{q}\ \mathbf{r}] = -2 - \frac{2}{5} = -\frac{8}{3}$ ...(viii)

126. $[\mathbf{p}\ \mathbf{q}\ \mathbf{r}]\,\mathbf{x} = (\mathbf{p} \times \mathbf{q}) \times \mathbf{r} \Rightarrow \frac{8}{3}\mathbf{x} = (\mathbf{p} \cdot \mathbf{r})\,\mathbf{q} - (\mathbf{q} \cdot \mathbf{r})\,\mathbf{p}$

$\Rightarrow \quad \left(-\frac{8}{3}\right)\mathbf{x} = 2\mathbf{q} - \frac{2\mathbf{p}}{3}$ $\left(\mathbf{q} \cdot \mathbf{r} = \frac{2}{3}\text{, verify yourself}\right)$

$\therefore \quad \mathbf{x} = -\frac{3}{8} \cdot \frac{2}{3}(3\mathbf{q} - \mathbf{p}) = \frac{1}{4}(3\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 3\hat{\mathbf{k}} - \hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}})$

$= -\frac{1}{4}(2\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

$\therefore \quad \mathbf{x} = -\frac{1}{2}(\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}})$

127. As $c = -\frac{1}{3}$ from Eq. (vi)

$\therefore \quad \left(1 - \frac{1}{3}\right)\mathbf{y} = (\mathbf{p} \cdot \mathbf{r})\,\mathbf{q} - (\mathbf{p} \cdot \mathbf{q})\,\mathbf{r}$

$\frac{2}{3}\mathbf{y} = 2\mathbf{q} - (1)\mathbf{r}$ [As $\mathbf{p} \cdot \mathbf{q} = 1$ from Eq. (iv)]

$\therefore \quad \mathbf{y} = \frac{3}{2}\left(2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}} - \frac{2}{3}\hat{\mathbf{j}} - \frac{4}{3}\hat{\mathbf{k}}\right) = \frac{3}{2}\frac{(6\hat{\mathbf{i}} - 8\hat{\mathbf{j}} + \hat{\mathbf{k}})}{3}$

$\mathbf{y} = 3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + \hat{\mathbf{k}}$

$\therefore \quad [\mathbf{x}\ \mathbf{y}\ \mathbf{r}] = \begin{vmatrix} -\frac{1}{2} & 1 & -\frac{1}{2} \\ 3 & -4 & 1 \\ 0 & \frac{2}{3} & \frac{4}{3} \end{vmatrix} = \frac{1}{2}\left(-\frac{16}{3} - \frac{2}{3}\right) - 1(4) - \frac{1}{2}(2)$

$\therefore \quad [\mathbf{x}\ \mathbf{y}\ \mathbf{r}] = 3 - 4 - 1 = -2$

$\therefore \quad \left|\frac{1}{6}[\mathbf{x}\ \mathbf{y}\ \mathbf{r}]\right| = \left|-\frac{1}{3}\right| = |\mathbf{c}|$

$\Rightarrow \mathbf{x}, \mathbf{y}, \mathbf{r}$ are the coterminous edges of a tetrahedron whose volume is $|\mathbf{c}|$.

Solutions (Q.Nos. 128-130)

128. $y = \log_{1/2}\left(x - \frac{1}{2}\right) + \log_2 \sqrt{(2x-1)^2}$

But, $x > \frac{1}{2} = \log_{1/2}\left(x - \frac{1}{2}\right) + \log_2(2x - 1)$

$y = 1$

$P \equiv (3, 1)$

129. $\mathbf{OP} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}}$

$\mathbf{Q} \equiv (1, 1)$ or $(2, 1)$

$\mathbf{OQ} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $2\hat{\mathbf{i}} + \hat{\mathbf{j}}$

$\mathbf{OP} \cdot \mathbf{OQ} = 3 + 1 = 4$ and $6 + 1 = 7$

130. $\mathbf{PQ} = \mathbf{OQ} - \mathbf{OP} = -2\hat{\mathbf{i}}$ or $\hat{\mathbf{i}}$

$|\mathbf{PQ}| = 2$ or 1

Solutions (Q.Nos. 131-134)

131. Since, $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are non-coplanar vectors, then

$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \neq 0 \Rightarrow [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2 \neq 0$

$\Rightarrow \quad [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2 = \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{a} \\ \mathbf{a}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{b} \\ \mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix} \neq 0$

Since, any vector $\mathbf{r}$ in space can be expressed as a linear combination of three non-coplanar vectors.

So, let $\mathbf{r} = l\mathbf{a} + m\mathbf{b} + n\mathbf{c}$ ...(i)

taking dot product by $\mathbf{a}, \mathbf{b}, \mathbf{c}$ successively, we get

$\mathbf{r} \cdot \mathbf{a} = l\,\mathbf{a} \cdot \mathbf{a} + m\,\mathbf{b} \cdot \mathbf{a} + n\,\mathbf{c} \cdot \mathbf{a}$ ...(ii)

$\mathbf{r} \cdot \mathbf{b} = l\,\mathbf{a} \cdot \mathbf{b} + m\,\mathbf{b} \cdot \mathbf{b} + n\,\mathbf{c} \cdot \mathbf{b}$ ...(iii)

$\mathbf{r} \cdot \mathbf{c} = l\,\mathbf{a} \cdot \mathbf{c} + m\,\mathbf{b} \cdot \mathbf{c} + n\,\mathbf{c} \cdot \mathbf{c}$ ...(iv)

Now, eliminating l, m and n from above 4 relations, we get

$$\begin{vmatrix} \mathbf{r} & \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{r}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{a} \\ \mathbf{r}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{b} \\ \mathbf{r}\cdot\mathbf{c} & \mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix} = 0$$

Now, expanding along first row, we get

$$r = \left(\frac{\Delta_1}{\Delta}\right)\mathbf{a} + \left(\frac{\Delta_2}{\Delta}\right)\mathbf{b} + \left(\frac{\Delta_3}{\Delta}\right)\mathbf{c}$$

132. Since, $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are three non-coplanar vectors, then

On three exists scalars x, y, z, such that

$\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$ ...(i)

Taking dot product by $\mathbf{b} \times \mathbf{c}$, $\mathbf{c} \times \mathbf{a}$ and $\mathbf{a} \times \mathbf{b}$ successively, we get

$\mathbf{r} \cdot (\mathbf{b} \times \mathbf{c}) = (x\mathbf{a} + y\mathbf{b} + z\mathbf{c}) \cdot (\mathbf{b} \times \mathbf{c}) = x[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

$\mathbf{r} \cdot (\mathbf{c} \times \mathbf{a}) = y[\mathbf{b}\ \mathbf{c}\ \mathbf{a}]$

$\mathbf{r} \cdot (\mathbf{a} \times \mathbf{b}) = z[\mathbf{c}\ \mathbf{a}\ \mathbf{b}]$

$\because \quad x = \frac{[\mathbf{r}\ \mathbf{b}\ \mathbf{c}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]},\ y = \frac{[\mathbf{r}\ \mathbf{c}\ \mathbf{a}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$ and $z = \frac{[\mathbf{r}\ \mathbf{a}\ \mathbf{b}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$

On substituting the values of x, y, z in Eq. (i), we get

$$r = \left\{\frac{[\mathbf{r}\ \mathbf{b}\ \mathbf{c}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}\right\}\mathbf{a} + \left\{\frac{[\mathbf{r}\ \mathbf{c}\ \mathbf{a}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}\right\}\mathbf{b} + \left\{\frac{[\mathbf{r}\ \mathbf{a}\ \mathbf{b}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}\right\}\mathbf{c}$$

or $\mathbf{r} = \frac{1}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}\{[\mathbf{r}\ \mathbf{b}\ \mathbf{c}]\mathbf{a} + [\mathbf{r}\ \mathbf{c}\ \mathbf{a}]\,\mathbf{b} + [\mathbf{r}\ \mathbf{a}\ \mathbf{b}]\,\mathbf{c}\}$

133. We know that,

$[\mathbf{a} \times \mathbf{b}\ \mathbf{b} \times \mathbf{c}\ \mathbf{c} \times \mathbf{a}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$

Clearly, $[\mathbf{a} \times \mathbf{b}\ \mathbf{b} \times \mathbf{c}\ \mathbf{c} \times \mathbf{a}] \neq 0\,\{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \neq 0\}$

$\Rightarrow \mathbf{a} \times \mathbf{b}\ \mathbf{b} \times \mathbf{c}\ \mathbf{c} \times \mathbf{a}$ are non-coplanar.

We also know that any vector in space can be expressed as a linear combination of any three non-coplanar vectors, so let.

$\mathbf{a} = l(\mathbf{b} \times \mathbf{c}) + m(\mathbf{c} \times \mathbf{a}) + n(\mathbf{a} \times \mathbf{b})$ ...(i)

On taking dot product on both sides by $\mathbf{a}, \mathbf{b}, \mathbf{c}$ successively, we get

$\mathbf{a} \cdot \mathbf{a} = l\,[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

$\mathbf{a} \cdot \mathbf{b} = m\,[\mathbf{c}\ \mathbf{a}\ \mathbf{b}]$

$\mathbf{c} \cdot \mathbf{a} = n\,[\mathbf{c}\ \mathbf{a}\ \mathbf{b}]$

$\because \quad l = \frac{\mathbf{a} \cdot \mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]},\ m = \frac{\mathbf{a} \cdot \mathbf{b}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$ and $n = \frac{\mathbf{a} \cdot \mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$

On substituting these values in Eq. (i), we get

$$\mathbf{a}=\frac{\mathbf{a}\cdot\mathbf{a}}{[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]}(\mathbf{b}\times\mathbf{c})+\frac{\mathbf{a}\cdot\mathbf{b}}{[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]}(\mathbf{c}\times\mathbf{a})+\frac{\mathbf{a}\cdot\mathbf{c}}{[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]}(\mathbf{a}\times\mathbf{b})$$

or $\mathbf{a}=\dfrac{1}{[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]}\{\mathbf{a}\cdot\mathbf{a}\,(\mathbf{b}\times\mathbf{c})+\mathbf{a}\cdot\mathbf{b}\,(\mathbf{c}\times\mathbf{a})+\mathbf{a}\cdot\mathbf{c}\,(\mathbf{a}\times\mathbf{b})\}$

134. Let $\mathbf{a}=a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}+a_3\hat{\mathbf{k}}$, $\mathbf{b}=b_1\hat{\mathbf{i}}+b_2\hat{\mathbf{j}}+b_3\hat{\mathbf{k}}$,

$\mathbf{c}=c_1\hat{\mathbf{i}}+c_2\hat{\mathbf{j}}+c_3\hat{\mathbf{k}}$, $\mathbf{p}=p_1\hat{\mathbf{i}}+p_2\hat{\mathbf{j}}+p_3\hat{\mathbf{k}}$,

and $\mathbf{q}=q_1\hat{\mathbf{i}}+q_2\hat{\mathbf{j}}+q_3\hat{\mathbf{k}}$,

Then, $\begin{vmatrix}\mathbf{a}&\mathbf{b}&\mathbf{c}\\ \mathbf{a}\cdot\mathbf{p}&\mathbf{b}\cdot\mathbf{p}&\mathbf{c}\cdot\mathbf{p}\\ \mathbf{a}\cdot\mathbf{q}&\mathbf{b}\cdot\mathbf{q}&\mathbf{c}\cdot\mathbf{q}\end{vmatrix}$

$$\begin{vmatrix}a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}+a_3\hat{\mathbf{k}} & b_1\hat{\mathbf{i}}+b_2\hat{\mathbf{j}}+b_3\hat{\mathbf{k}} & c_1\hat{\mathbf{i}}+c_2\hat{\mathbf{j}}+c_3\hat{\mathbf{k}}\\ a_1p_1+a_2p_2+a_3p_3 & b_1p_1+b_2p_2+b_3p_3 & c_1p_1+c_2p_2+c_3p_3\\ a_1q_1+a_2q_2+a_3q_3 & b_1q_1+b_2q_2+b_3q_3 & c_1q_1+c_2q_2+c_3q_3\end{vmatrix}$$

$$\begin{vmatrix}\hat{\mathbf{i}}&\hat{\mathbf{j}}&\hat{\mathbf{k}}\\ p_1&p_2&p_3\\ q_1&q_2&q_3\end{vmatrix}\cdot\begin{vmatrix}a_1&a_2&a_3\\ b_1&b_2&b_3\\ c_1&c_2&c_3\end{vmatrix}$$

$$=(\mathbf{p}\times\mathbf{q})\,[\mathbf{abc}]$$
$$=\sqrt{[\mathbf{abc}]^2}(\mathbf{p}\times\mathbf{q})$$
$$=\sqrt{[\mathbf{a}\times\mathbf{bb}\times\mathbf{cc}\times\mathbf{a}]}\,(\mathbf{p}\times\mathbf{q})$$

Solutions (Q.Nos. 135-136)

135. $\because g'(x)=3x^2+2x+0>0,\ \forall\ x\ge 0$

$\Rightarrow g(x)$ is an $\uparrow$ ing function.

If circumcentre lies outside, then triangle is obtuse angle triangle and angle containing the given sides is obtuse angle.

Therefore,

$$(f(x)\hat{\mathbf{i}}+g(x)\hat{\mathbf{j}})\cdot(g(x)\hat{\mathbf{i}}+f(x)\hat{\mathbf{j}})<0$$

$\Rightarrow$ $f(x)\cdot g(x)<0$ ...(i)

$\Rightarrow$ $g(x)\uparrow$ for $x\ge 0$

$\Rightarrow$ $g(x)>g(0)\forall x>0$, Also, $g(0)=0$

$\Rightarrow$ $g(x)>0\forall x>0$, (i) $\Rightarrow$ $f(x)<0$

$\therefore$ $f(x)<0$ and $g(x)>0\ \forall\ x>0$

136. If $x\to\infty$ then $g(x)\to\infty$ and $f(x)$ is some negative number, then

$$\lim_{t\to 0}\lim_{x\to\infty}\Bigg(\underbrace{\cot\Bigg(\underbrace{\frac{\pi}{4}\underbrace{(1-t^2)}_{1^-}}_{}\Bigg)}_{1^+}\Bigg)^{\overbrace{f(x)\cdot g(x)}^{\to\infty}}=0$$

Solutions (Q.Nos. 137 to 139)

We have $|\mathbf{x}|=|\mathbf{y}|=|\mathbf{z}|=\sqrt{2}$ and $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ make angle of 60° with each other.

$\therefore$ $\mathbf{x}\cdot y=|\mathbf{x}|\,|\mathbf{y}|\cos 60^\circ=\sqrt{2}(\sqrt{2})\cdot\dfrac{1}{2}=1$

$$\mathbf{y}\cdot\mathbf{z}=|\mathbf{y}|\,|\mathbf{z}|\cos 60^\circ=\sqrt{2}(\sqrt{2})\left(\frac{1}{2}\right)=1$$

and $\mathbf{x}\cdot\mathbf{z}=|\mathbf{x}|\,|\mathbf{z}|\cos 60^\circ=\sqrt{2}(\sqrt{2})\left(\dfrac{1}{2}\right)=1$

$$\mathbf{x}\cdot\mathbf{x}=|\mathbf{x}|^2=2$$

$$\mathbf{y}\cdot\mathbf{y}=|\mathbf{y}|^2=2\ \text{ and }\ \mathbf{z}\cdot\mathbf{z}=|z|^2=2$$

Now, $\mathbf{x}\times(\mathbf{y}\times\mathbf{z})=\mathbf{a}$ and $\mathbf{y}\times(\mathbf{z}\times\mathbf{x})=\mathbf{b}$ (given)

$\Rightarrow$ $(\mathbf{x}\cdot\mathbf{z})\,\mathbf{y}-(\mathbf{x}\cdot\mathbf{y})\mathbf{z}=\mathbf{a}$ and $(\mathbf{y}\cdot\mathbf{x})\mathbf{z}-(\mathbf{y}\cdot\mathbf{z})\mathbf{x}=\mathbf{b}$

$\Rightarrow$ $\mathbf{y}-\mathbf{z}=\mathbf{a}$ and $\mathbf{z}-\mathbf{x}=\mathbf{b}$

$\Rightarrow$ $\mathbf{y}-\mathbf{x}=\mathbf{a}+\mathbf{b}$

Thus, we have

$$\mathbf{y}-\mathbf{z}=\mathbf{a} \quad ...(i)$$
$$\mathbf{z}-\mathbf{x}=\mathbf{b} \quad ...(ii)$$
$$\mathbf{y}-\mathbf{x}=\mathbf{a}+\mathbf{b} \quad ...(iii)$$

Now, $\mathbf{x}\times\mathbf{y}=\mathbf{c}$ (given)

$\Rightarrow$ $\mathbf{x}\times(\mathbf{x}\times\mathbf{y})=\mathbf{x}\times\mathbf{c}$ (taking cross-product with $\mathbf{x}$)

$\Rightarrow$ $(\mathbf{x}\cdot\mathbf{y})\times\mathbf{x}-(\mathbf{x}\cdot\mathbf{x})\mathbf{y}=\mathbf{x}\times\mathbf{c}$

$\Rightarrow$ $\mathbf{x}-2\mathbf{y}=\mathbf{x}\times\mathbf{c}$ ...(iv)

Again, $\mathbf{x}\times\mathbf{y}=\mathbf{c}$

$\Rightarrow$ $\mathbf{y}\times(\mathbf{x}\times\mathbf{y})=\mathbf{y}\times\mathbf{c}$ (taking cross product with $\mathbf{y}$)

$\Rightarrow$ $(\mathbf{y}\cdot\mathbf{y})\mathbf{x}-(\mathbf{y}\cdot\mathbf{x})\mathbf{y}=\mathbf{y}\times\mathbf{c}$

$\Rightarrow$ $2\mathbf{x}-\mathbf{y}=\mathbf{y}\times\mathbf{c}$ ...(iv)

On subtracting Eqs. (iv) and (v), we get

$$\mathbf{x}-\mathbf{y}=(\mathbf{y}\times\mathbf{c})-(\mathbf{x}\times\mathbf{c})$$

$\Rightarrow$ $\mathbf{x}\times\mathbf{y}=(\mathbf{y}-\mathbf{x})\times\mathbf{c}$

$\Rightarrow$ $\mathbf{x}+\mathbf{y}=(\mathbf{a}+\mathbf{b})\times\mathbf{c}$ ...(vi)

Adding Eqs. (iii) and (vi), we get

$$2\mathbf{y}=(\mathbf{a}+\mathbf{b})+(\mathbf{a}+\mathbf{b})\times\mathbf{c},\ \mathbf{y}=\frac{1}{2}[(\mathbf{a}+\mathbf{b})\,(\mathbf{a}+\mathbf{b})\times\mathbf{c}$$

Substituting the value of $\mathbf{y}$ in Eq. (iii) in Eq. (i), we get

$$\mathbf{x}=\frac{1}{2}[(\mathbf{a}+\mathbf{b})+(\mathbf{a}+\mathbf{b})\times\mathbf{c}]-(\mathbf{a}+\mathbf{b})$$

$\Rightarrow$ $\mathbf{x}=\dfrac{1}{2}[(\mathbf{a}+\mathbf{b})\times\mathbf{c}-(\mathbf{a}+\mathbf{b})]$

$$\mathbf{z}=\frac{1}{2}[(\mathbf{a}+\mathbf{b})+(\mathbf{a}+\mathbf{b})\times\mathbf{c}]-\mathbf{a}$$

$$\mathbf{z}=\frac{1}{2}[(\mathbf{b}-\mathbf{a})+(\mathbf{a}+\mathbf{b})\times\mathbf{c}].$$

137. (a) ***138.*** (a) ***139.*** (b)

Solutions (Q.Nos. 140 to 142)

Taking dot products with $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ respectively with given equation

$$[\mathbf{abc}]=p+(q+r)\cos\theta \quad ...(i)$$
$$0=(p+r)\cos\theta+q \quad ...(ii)$$
$$[\mathbf{abc}]=(p+q)\cos\theta+r \quad ...(iii)$$

Also, $[\mathbf{abc}]^2=\begin{vmatrix}1&\cos\theta&\cos\theta\\ \cos\theta&1&\cos\theta\\ \cos\theta&\cos\theta&1\end{vmatrix}$

$$=2\cos^2\theta-3\cos^3\theta+1=(1-\cos\theta)^2\,(1+2\cos\theta)$$

$\therefore$ $v=|[\mathbf{abc}]|=|1-\cos\theta|\sqrt{1+2\cos\theta}|$

$$=2\sin^2\frac{\theta}{2}|\sqrt{1+2\cos\theta}|$$

From Eqs. (i) and (iii) $p = r$; substituting in Eq. (ii), we get

$$\therefore \quad 2p\cos\theta + q = 0 \Rightarrow \frac{q}{p} + 2\cos\theta = 0$$

140. (a) **141.** (c) **142.** (b)

143. (A) We know that 3 vectors are coplanar, if $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = 0$
Clearly, $-3\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ and $\hat{\mathbf{i}} + \hat{\mathbf{j}}$ are two vectors lie in the plane ($\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$).

(B) $\mathbf{a} \times \mathbf{b}$ is a vector which perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

$$\therefore \quad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & 2 & 2 \\ -2 & 1 & 2 \end{vmatrix} = \hat{\mathbf{i}}(4-2) - \hat{\mathbf{j}}(-2+4) + \hat{\mathbf{k}}(-1+4)$$

$$= 2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$$

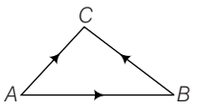
(C) $\mathbf{c}$ is a vector which is equally inclined to $\mathbf{a}$ and $\mathbf{b}$

$$\therefore \quad \mathbf{c}\cdot\mathbf{a} = \mathbf{c}\cdot\mathbf{b}$$

Clearly $\hat{\mathbf{i}} - \hat{\mathbf{j}} + 5\hat{\mathbf{k}}$ satisfies the condition.

(D) $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are from the triangle

$$\Rightarrow \quad \mathbf{AB} + \mathbf{BC} = \mathbf{AC}$$



144. Given, $[\mathbf{a}\times\mathbf{b}\ \mathbf{b}\times\mathbf{c}\ \mathbf{c}\times\mathbf{a}] = 36$

$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 6$$

$\Rightarrow$ Volume of tetrahedron from by vectors

$\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is $\frac{1}{6}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]] = 1$

$$[\mathbf{a}+\mathbf{b}\ \mathbf{b}+\mathbf{c}\ \mathbf{c}+\mathbf{a}] = 2[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 12$$

$\mathbf{a}-\mathbf{b}$, $\mathbf{b}-\mathbf{c}$ and $\mathbf{c}-\mathbf{a}$ are coplanar

$$\Rightarrow \quad [\mathbf{a}-\mathbf{b}\ \mathbf{b}-\mathbf{c}\ \mathbf{c}-\mathbf{a}] = 0$$

145. (A) $\dfrac{\text{Area of } \Delta ABC}{\text{Area of } \Delta AOC} = \dfrac{\frac{1}{2}|\mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{a}|}{\frac{1}{2}|\mathbf{a}\times\mathbf{c}|}$

Now, $\mathbf{a} + 2\mathbf{b} + 3\mathbf{c} = 0$

Cross with $\mathbf{b}$, $\mathbf{a}\times\mathbf{b} + 3\mathbf{c}\times\mathbf{b} = 0$

$$\Rightarrow \quad \mathbf{a}\times\mathbf{b} = 3(\mathbf{b}\times\mathbf{c})$$

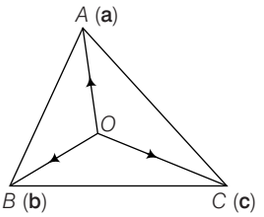
Cross with $\mathbf{a}$, $2\mathbf{a}\times\mathbf{b} + 3\mathbf{a}\times\mathbf{c} = 0$

$$\Rightarrow \quad \mathbf{a}\times\mathbf{b} = \frac{3}{2}(\mathbf{c}\times\mathbf{a})$$

$$\therefore \quad \mathbf{a}\times\mathbf{b} = \frac{3}{2}(\mathbf{c}\times\mathbf{a}) = 3(\mathbf{b}\times\mathbf{c})$$

Let $(\mathbf{c}\times\mathbf{a}) = \mathbf{p}$

$$\mathbf{a}\times\mathbf{b} = \frac{3\mathbf{p}}{2} \ ; \ \mathbf{b}\times\mathbf{c} = \frac{\mathbf{p}}{2}$$



$$\therefore \quad \text{Ratio} = \frac{|\mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{a}|}{|\mathbf{c}\times\mathbf{a}|} = \frac{\left|\frac{3\mathbf{p}}{2} + \frac{\mathbf{p}}{2} + \mathbf{p}\right|}{|\mathbf{p}|} = \frac{3|\mathbf{p}|}{|\mathbf{p}|} = 3$$

(B) $[\mathbf{ABC}]^2 = \begin{vmatrix} \mathbf{A}\cdot\mathbf{A} & \mathbf{A}\cdot\mathbf{B} & \mathbf{A}\cdot\mathbf{C} \\ \mathbf{B}\cdot\mathbf{A} & \mathbf{B}\cdot\mathbf{B} & \mathbf{B}\cdot\mathbf{C} \\ \mathbf{C}\cdot\mathbf{A} & \mathbf{C}\cdot\mathbf{B} & \mathbf{C}\cdot\mathbf{C} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & 1 \end{vmatrix}$

$$1 = \left(1 - \frac{3}{4}\right) = \frac{1}{4}$$

$$[\mathbf{ABC}] = \frac{1}{2}$$

(C) $(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\times\mathbf{d}) = \begin{vmatrix} \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{a} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{d} \end{vmatrix}$

Similarly, compute others which gives (i).

146. (A) $\mathbf{x}\cdot\mathbf{y} = |\mathbf{a}|^2 - |\mathbf{b}|^2 = 0$

$\mathbf{x}$ is perpendicular to $\mathbf{y}$.

$$|\mathbf{x}\times\mathbf{y}|^2 = |\mathbf{x}|^2|\mathbf{y}|^2$$

$$\{|\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a}\cdot\mathbf{b}|\}\{|\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a}\cdot\mathbf{b}\}$$

$$= 64 - 4(\mathbf{a}\cdot\mathbf{b})^2 = 4\left\{16 - (\mathbf{a}\cdot\mathbf{b})^{\frac{1}{2}}\right\}$$

$$\lambda = 16$$

(B) $\dfrac{2\lambda + 1}{\sqrt{\lambda^2 + 2}\sqrt{\lambda^2 + 2}} = \dfrac{1}{2}$

$$2(2\lambda + 1) = \lambda^2 + 2$$

$$\lambda^2 - 4\lambda = 0, \ \lambda = 0 \text{ or } 4$$

$\lambda = 4$ is non-zero value.

(C) If the lines are coplanar, all the 4 planes will have a common point.

Solving $4x + 3y - 2z + 3 = 0$

$$x - 3y + 4z + 6 = 0$$

$$x - y + z + 1 = 0$$

We get $x = -\frac{1}{3}$, $y = -3$, $z = \frac{-11}{3}$

Substituting in $kx - 4y + 7z + 16 = 0$

We get $k = 7$

(D) $E = |\mathbf{a} - 2\mathbf{b}|^2 + |\mathbf{b} - 2\mathbf{c}|^2 + |\mathbf{c} - 2\mathbf{a}|^2$

$$= 5(|\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2) - 4[\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}]$$

$$= 5\cdot 6 - 4[\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}]$$

$$\Rightarrow \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a} = \frac{30 - E}{4}$$

Also, $|\mathbf{a} + \mathbf{b} + \mathbf{c}|^2 \geq 0$

$$6 + 2[\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}] \geq 0$$

$$6 + 2\left[\frac{30 - E}{4}\right] \geq 0$$

$$12 + 30 - E \geq 0$$

$$42 \geq E$$

$$E \geq 42$$

147. (A) $\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}=\mathbf{b}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{a}=0$

$\Rightarrow \mathbf{b}\cdot\mathbf{c}=\mathbf{a}\cdot\mathbf{c}=-\mathbf{a}\cdot\mathbf{b}=-\dfrac{17}{2}$

$\Rightarrow\ |\mathbf{a}+\mathbf{b}+\mathbf{c}|=\sqrt{a^2+b^2+c^2-2(\mathbf{a}\cdot\mathbf{b})}=9$

(B) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$, write in terms of $(\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{b}_1\mathbf{b}_2\mathbf{b}_3)$

$$[\mathbf{def}]=\left[\frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{abc}]}\ \frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{abc}]}\ \frac{\mathbf{a}\times\mathbf{b}}{[\mathbf{abc}]}\right]=\frac{1}{[\mathbf{abc}]}$$

$\Rightarrow\ [\mathbf{abc}]\,[\mathbf{def}]=1$

(C) $\alpha=\dfrac{\text{ar}(ABCD)}{\text{ar(parallelogram)}}=$

$$\left(\frac{\frac{1}{2}|\mathbf{a}\times(\mathbf{a}+3\mathbf{b})|+\frac{1}{2}|(2\mathbf{a}+3\mathbf{b})\times\mathbf{b}|}{|\mathbf{a}\times\mathbf{b}|}\right)=\frac{5}{2}$$

(D) $x=\dfrac{\mathbf{d}\cdot\mathbf{c}}{[\mathbf{abc}]},\ y=\dfrac{\mathbf{d}\cdot\mathbf{a}}{(\mathbf{a}\ \mathbf{b}\ \mathbf{c})},\ z=\dfrac{\mathbf{d}\cdot\mathbf{b}}{[\mathbf{abc}]}$

$$x+y+z=\frac{\mathbf{d}\cdot(\mathbf{a}+\mathbf{b}+\mathbf{c})}{[\mathbf{abc}]}$$

$$\Rightarrow\qquad R=\frac{4}{[\mathbf{abc}]}=8$$

148. Let the angle between $\mathbf{u}$ and $\mathbf{v}$ is θ and $\mathbf{w}$ and $\mathbf{u}$ is θ

$$[\hat{\mathbf{u}}\ \hat{\mathbf{v}}\ \hat{\mathbf{w}}]^2=\begin{bmatrix}\hat{\mathbf{u}}\cdot\hat{\mathbf{u}} & \hat{\mathbf{u}}\cdot\hat{\mathbf{v}} & \hat{\mathbf{u}}\cdot\hat{\mathbf{w}}\\ \hat{\mathbf{v}}\cdot\hat{\mathbf{u}} & \hat{\mathbf{v}}\cdot\hat{\mathbf{v}} & \hat{\mathbf{v}}\cdot\hat{\mathbf{w}}\\ \hat{\mathbf{w}}\cdot\hat{\mathbf{u}} & \hat{\mathbf{w}}\cdot\hat{\mathbf{v}} & \hat{\mathbf{w}}\cdot\hat{\mathbf{w}}\end{bmatrix}$$

$$=\begin{bmatrix}1 & \cos2\theta & \cos\theta\\ \cos2\theta & 1 & \cos\theta\\ \cos\theta & \cos\theta & 1\end{bmatrix}=0$$

149. $2[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]+[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2+0=2\times1+1^2=3$

150. $\hat{\mathbf{a}}\cdot\hat{\mathbf{c}}=\hat{\mathbf{b}}\cdot\hat{\mathbf{c}}=\cos\theta$

$\hat{\mathbf{c}}=\alpha\hat{\mathbf{a}}+\beta\hat{\mathbf{b}}+\gamma(\hat{\mathbf{a}}+\hat{\mathbf{b}})$

Taking dot product with $\hat{\mathbf{a}}$ both sides $\cos\theta=\alpha$

Taking dot product with $\hat{\mathbf{b}}$ both sides $\cos\theta=\beta$

Taking dot product with $\mathbf{c}$ both sides

$$1=\alpha\cos\alpha+\beta\cos\theta+\gamma\,[\hat{\mathbf{a}}\ \hat{\mathbf{b}}\ \hat{\mathbf{c}}]$$

But $$[\hat{\mathbf{a}}\ \hat{\mathbf{b}}\ \hat{\mathbf{c}}]^2=\begin{bmatrix}1 & 0 & \cos\theta\\ 0 & 1 & \cos\theta\\ \cos\theta & \cos\theta & 1\end{bmatrix}$$

$$=1-2\cos^2\theta$$

So, $$1=\cos^2\theta+\cos^2\theta+\gamma\sqrt{1-2\cos^2\theta}$$

$$\Rightarrow\qquad \gamma=\sqrt{1-2\cos^2\theta}$$

So, $\alpha^2+\beta^2+\gamma^2=1$

151. The three adjacent sides of tetrahedron is given by

$$\frac{(\hat{\mathbf{i}}+\hat{\mathbf{j}})\times(\hat{\mathbf{j}}+\hat{\mathbf{k}})}{|(\hat{\mathbf{i}}+\hat{\mathbf{j}})\times(\hat{\mathbf{j}}+\hat{\mathbf{k}})|},\ \frac{(\hat{\mathbf{j}}+\hat{\mathbf{k}})(\hat{\mathbf{k}}+\hat{\mathbf{i}})}{|(\hat{\mathbf{j}}+\hat{\mathbf{k}})\times(\hat{\mathbf{k}}+\hat{\mathbf{i}})|},\ \frac{(\hat{\mathbf{k}}+\hat{\mathbf{i}})(\hat{\mathbf{i}}+\hat{\mathbf{j}})}{|(\hat{\mathbf{k}}+\hat{\mathbf{i}})\times(\hat{\mathbf{i}}+\hat{\mathbf{j}})|}$$

i.e., $$\frac{\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}}{\sqrt{3}},\ \frac{\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}}}{\sqrt{3}},\ \frac{-\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}}{\sqrt{3}}$$

$$V=\frac{1}{6}\times\frac{1}{3\sqrt{3}}\begin{Vmatrix}1 & -1 & 1\\ 1 & 1 & -1\\ -1 & 1 & 1\end{Vmatrix}=\frac{2}{9\sqrt{3}}$$

So, $$9\sqrt{3}V=2$$

152. Let $\hat{\mathbf{c}}=x\mathbf{a}+y\,\mathbf{b}$, where x and y are scalars.

$\Rightarrow\quad \hat{\mathbf{c}}=x(\hat{\mathbf{i}}-\hat{\mathbf{j}}+2\hat{\mathbf{k}})+y(2\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}})$

$\Rightarrow\quad \hat{\mathbf{c}}=\hat{\mathbf{i}}(x+2y)+\hat{\mathbf{j}}(-x-y)+\hat{\mathbf{k}}(2x+y)$

But, $$\hat{\mathbf{c}}\cdot\mathbf{a}=0$$

$$6x+5y=0\ \Rightarrow\ y=-\frac{6x}{5}$$

So, $$\hat{\mathbf{c}}=\frac{-7x}{5}\hat{\mathbf{i}}+\frac{x}{5}\hat{\mathbf{j}}+\frac{4x}{5}\hat{\mathbf{k}}$$

We have, $\dfrac{49x^2+x^2+16x^2}{25}=1\ \Rightarrow\ x^2=\dfrac{25}{66}$

$\therefore$ $$\hat{\mathbf{c}}=\pm\frac{5}{\sqrt{66}}\left(\frac{-7}{5}\hat{\mathbf{i}}+\frac{1}{5}\hat{\mathbf{j}}+\frac{4}{5}\hat{\mathbf{k}}\right)$$

$$p=|\hat{\mathbf{c}}\cdot\mathbf{b}|=\frac{\sqrt{11}}{6}$$

So, $\dfrac{\sqrt{11}}{p}=6\ \Rightarrow\ k=6$

143. Let the angle between $\mathbf{a}$ and $\mathbf{b}$ is α and $\mathbf{a}\times\mathbf{b}$ and $\mathbf{c}$ is β.

$\therefore\quad |[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]|=6$

$\Rightarrow\quad \sin\alpha\cos\beta=1\ \Rightarrow\ \sin\alpha=1,\cos\beta=1$

$\Rightarrow\quad \alpha=90°,\beta=0°$

$\Rightarrow$ $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are mutually perpendicular.

Again, $[\mathbf{b}\ \mathbf{c}\ \hat{\mathbf{d}}]=0$

$$\Rightarrow\qquad \begin{vmatrix}4 & 0 & 1\\ 0 & 9 & \mathbf{c}\cdot\hat{\mathbf{d}}\\ 1 & \mathbf{c}\cdot\hat{\mathbf{d}} & \mathbf{1}\end{vmatrix}=0$$

$$\Rightarrow\qquad \mathbf{c}\cdot\hat{\mathbf{d}}=\pm\frac{3\sqrt{3}}{2}$$

We have, $$\mathbf{a}\cdot\mathbf{b}=0$$

$$|\mathbf{a}\times\mathbf{c}\cdot\hat{\mathbf{d}}|^2=\begin{vmatrix}1 & 0 & 0\\ 0 & 9 & \frac{3\sqrt{3}}{2}\\ 0 & \frac{3\sqrt{3}}{2} & 1\end{vmatrix}$$

$$=9-\frac{27}{4}=\frac{9}{4}$$

$$|(\mathbf{a}\times\mathbf{c})\times\hat{\mathbf{d}}|^2=|(\mathbf{a}\cdot\hat{\mathbf{d}})\ \mathbf{c}-(\mathbf{c}\cdot\hat{\mathbf{d}})\mathbf{a}|^2$$

$$=\left|\frac{3\sqrt{3}}{2}\mathbf{a}\right|^2=\frac{27}{4}$$

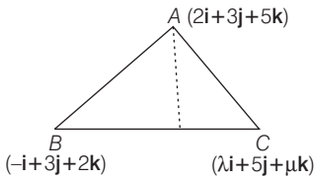
So, $$|\mathbf{a}\times\mathbf{c}\cdot\hat{\mathbf{d}}|^2+|(\mathbf{a}\times\mathbf{c})\times\hat{\mathbf{d}}|^2=\frac{36}{4}=9$$

154. P.V. of $D = \frac{\lambda-1}{2}\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + \frac{\mu+2}{2}\hat{\mathbf{k}}$

D.R. of AD $= \frac{\lambda-4}{2}, 1, \frac{\mu-8}{2}$

But directions of AD should be $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

$$\Rightarrow \quad \frac{\lambda-4}{2} = 1 = \frac{\mu-8}{2}$$



$$\lambda = 6, \mu = 10$$
$$2\lambda - \mu = 2$$

155. $v = [\mathbf{abc}]$

$$[\alpha\ \beta\ \gamma] = \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix} [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$
$$= [\mathbf{abc}]\,[\mathbf{abc}]\,[\mathbf{abc}] = v^3$$

$\therefore \quad \lambda = 3$

156. $\mathbf{c}\times\mathbf{a} = \mathbf{b} \Rightarrow |\mathbf{c}\times\mathbf{a}| = |\mathbf{b}|$

$$\Rightarrow \quad |\mathbf{c}|\,|\mathbf{a}|\sin\theta = 3,$$
$$|\mathbf{c}| = \frac{3}{2\sin\theta}$$
$$|\mathbf{c}-\mathbf{a}|^2 = |\mathbf{c}|^2 + |\mathbf{a}|^2 - 2\mathbf{c}\cdot\mathbf{a}$$
$$= |\mathbf{c}|^2 + 4 - 2|\mathbf{c}||\mathbf{a}|\cos\theta$$
$$= \frac{9}{4\sin^2\theta} + 4 - 2\cdot\frac{3}{2\sin\theta}\cdot 2\cdot\cos\theta$$
$$= 4 + \frac{9}{4}\operatorname{cosec}^2\theta - 6\cot\theta$$
$$= \frac{9}{4} + \left(\frac{3}{2}\cot\theta - 2\right)^2$$
$$|\mathbf{c}-\mathbf{a}|^2 \geq \frac{9}{4}$$
$$\Rightarrow \quad |\mathbf{c}-\mathbf{a}| \geq \frac{3}{2}$$
$$\Rightarrow \quad 2|\mathbf{c}-\mathbf{a}| \geq 3$$

$\therefore$ Min. of $2|\mathbf{c}-\mathbf{a}| = 3$

157. In ΔABD, N is the mid-point of BD.

$$\therefore \quad AB + AD = 2AN \qquad \ldots(i)$$

In ΔCBD, N is the mid-point of BD.

$$\therefore \quad \mathbf{CB} + \mathbf{CD} = 2\mathbf{CN} \qquad \ldots(ii)$$

Adding Eqs. (i) and (ii), we have

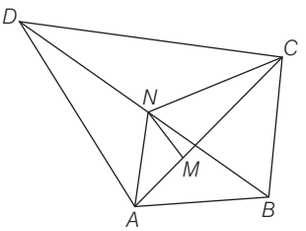
$$AB + AD + CB + CD = 2(AN + CN) \qquad \ldots(iii)$$

In Δ ANC, M is the mid-point of AC

$$\therefore \quad AN + CN = 2MN$$

From Eq. (iii), we get

$$AB + AD + CB + CD = 2\,(2MN) = 4MN$$



158. $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = 1; [\mathbf{abc}] = 1$

Volume of the tetrahedron $= \frac{1}{6}$

$$= \frac{1}{6}\left\| \begin{matrix} 3 & -2 & 2 \\ -1 & 0 & -2 \\ 2 & -3 & 4 \end{matrix} \right| [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \Bigg| = 2$$

159. $[\mathbf{a}\times(\mathbf{b}\times\mathbf{c})]\cdot(\mathbf{a}\times\mathbf{c}) = 5$

$$\Rightarrow\ [(\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}]\cdot(\mathbf{a}\times\mathbf{c}) = 5 \ \Rightarrow\ (\mathbf{a}\cdot\mathbf{c})\,[\mathbf{bac}] = 5$$
$$\Rightarrow \quad [\mathbf{abc}] = -10$$
$$\Rightarrow \quad -[\mathbf{abc}] - 1 = +10 - 1 = 9$$

160. $\begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix} = [\mathbf{abc}]^2 = [\mathbf{a}\times\mathbf{b}\ \mathbf{b}\times\mathbf{c}\ \mathbf{c}\times\mathbf{a}]^2 = 36$

$$= 4\times 9 = 2^2\times 3^2$$

$\therefore \quad P + q = 5$

161. $\hat{\alpha}\cdot\hat{x} = \hat{\alpha}\cdot\hat{\beta} - \hat{\alpha}\cdot(\hat{\alpha}\times\mathbf{x}) = 0$

Also, $\hat{\alpha}\times\hat{x} = \hat{\alpha}\times\hat{\beta} - \hat{\alpha}\times(\hat{\alpha}\times\mathbf{x})$

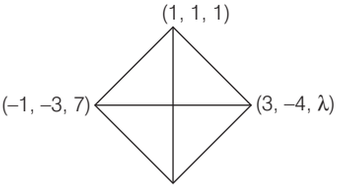
$$= \alpha\times\hat{\beta} - (\hat{\alpha}\cdot\mathbf{x})\hat{\alpha} + |\alpha|^2\mathbf{x}$$

$$\Rightarrow \quad \hat{\beta} - \mathbf{x} = \alpha\times\hat{\beta} + x \quad \text{or} \quad 2\mathbf{x} = \hat{\beta} - \alpha\times\hat{\beta}$$

$$\Rightarrow \quad 4|\mathbf{x}|^2 = |\hat{\beta}|^2 + |\alpha\times\hat{\beta}|^2 - 2\hat{\beta}\cdot(\hat{\alpha}\times\hat{\beta}) = 2$$

$$\Rightarrow \quad |\mathbf{x}|^2 = \frac{1}{2} \qquad \Rightarrow\ 4|\mathbf{x}|^2 = 2$$

162. $\frac{1}{6}\begin{vmatrix} -2 & -4 & 6 \\ 0 & 1 & -8 \\ 2 & -5 & \lambda-1 \end{vmatrix} = 22 \ \Rightarrow\ \lambda = 133$



163. $\frac{1}{6}[\mathbf{abc}] = 3 \ \Rightarrow\ [\mathbf{abc}] = 18$

$$V = [\mathbf{a}+\mathbf{b}-\mathbf{c}\ \mathbf{a}-\mathbf{b}\ \mathbf{b}-\mathbf{c}]$$
$$= (\mathbf{a}+\mathbf{b}-\mathbf{c})\cdot(\mathbf{a}-\mathbf{b})\times(\mathbf{b}-\mathbf{c}) = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = [\mathbf{abc}] = 18$$

164. Let θ be the angle between vectors $\mathbf{a}$ and $\mathbf{b}$. Then,

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$
$$\Rightarrow \quad (\mathbf{a}\cdot\mathbf{b})^2 = |\mathbf{a}|^2|\mathbf{b}|^2\cos^2\theta$$

Now, $\cos^2\theta \leq 1 \ \Rightarrow\ |\mathbf{a}|^2|\mathbf{b}|^2\cos^2\theta \leq |\mathbf{a}|^2|\mathbf{b}|^2$

$$\therefore \quad (\mathbf{a}\cdot\mathbf{b})^2 \leq |\mathbf{a}|^2|\mathbf{b}|^2$$

165. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the two points on $y = 2^{x+2}$

$\mathbf{OP} \cdot \hat{\mathbf{i}}$ = Projection of $\mathbf{OP}$ on the X-axis

$\Rightarrow \quad x_1 = -1 \qquad (\because \mathbf{OP} \,.\, \hat{\mathbf{i}} = -1)$

Also, (x_1, y_1) lies on $y = 2^{x+2}$

$\therefore \quad y_1 = 2^{x_1+2} \Rightarrow y_1 = 2$

Also, $\mathbf{OQ} \cdot \hat{\mathbf{i}}$ = Projection of $\mathbf{OQ}$ on X-axis.

$\Rightarrow \quad x_2 = 2 \qquad (\text{given } \mathbf{OQ} \,.\, \hat{\mathbf{i}} = 2)$

As (x_2, y_2) lies on $y = 2^{x+2}$

$$y_2 = 2^{x_2+2};\ y_2 = 16$$

Thus, $\mathbf{OP} = x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}} = -\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$

and $\mathbf{OQ} = x_2\hat{\mathbf{i}} + y_2\hat{\mathbf{j}} = 2\hat{\mathbf{i}} + 16\hat{\mathbf{j}}$

$\Rightarrow \quad \mathbf{OQ} - 4\mathbf{OP} = 6\hat{\mathbf{i}} + 8\hat{\mathbf{j}}$

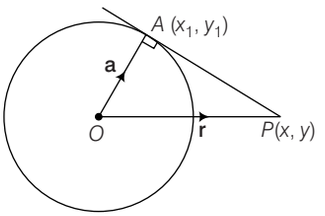
$\Rightarrow \quad |\mathbf{OQ} - 4\mathbf{OP}| = \sqrt{36 + 64} = 10$

166. Let $A(x_1, y_1)$ in XY-plane.

$\therefore \quad \mathbf{OA} = \mathbf{a} = x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}}$

$\mathbf{OP} = \mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$

$\because$ Point P lies on the tangent to the circle.



$\therefore \mathbf{OA}$ is perpendicular to $\mathbf{AP}$.

$\Rightarrow \quad \mathbf{OA} \cdot \mathbf{AP} = 0 \Rightarrow \mathbf{a} \cdot (\mathbf{r} - \mathbf{a}) = 0$

i.e., $\quad \mathbf{a} \cdot \mathbf{r} - \mathbf{a} \cdot \mathbf{a} = 0$

or $\quad \mathbf{a} \cdot \mathbf{r} = \mathbf{a} \cdot \mathbf{a}$

$\Rightarrow \quad \mathbf{a} \cdot \mathbf{r} = a^2$

$\Rightarrow \quad (x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}}) \cdot (x\hat{\mathbf{i}} + y\hat{\mathbf{j}}) = a^2$

$\Rightarrow \quad xx_1 + yy_1 = a^2$

which is the equation of the tangent to the circle at the point A.

167. The given relation can be rewritten as,

$(\sqrt{a^2 - 4}\hat{\mathbf{i}} + a\hat{\mathbf{j}} + \sqrt{a^2 + 4}\hat{\mathbf{k}}) \,.\, (\tan A\,\hat{\mathbf{i}} + \tan B\,\hat{\mathbf{j}} + \tan C\,\hat{\mathbf{k}}) = 6a$

$\Rightarrow \quad \sqrt{(a^2 - 4) + a^2 + (a^2 + 4)}$

$\sqrt{\tan^2 A + \tan^2 B + \tan^2 C} \,.\, \cos\theta = 6a \quad (\because a \,.\, b = |a||b|\cos\theta)$

$\sqrt{3}a \,.\, \sqrt{\tan^2 A + \tan^2 B + \tan^2 C} \,.\, \cos\theta = 6a$

$\Rightarrow \quad \tan^2 A + \tan^2 B + \tan^2 C = 12\sec^2\theta \qquad \ldots(i)$

Also, $\quad 12\sec^2\theta \geq 12 \qquad (\because \sec^2\theta \geq 1) \ldots(ii)$

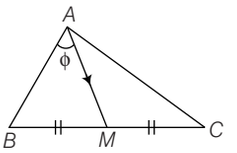
From Eqs. (i) and (ii), $\tan^2 A + \tan^2 B + \tan^2 C \geq 12$

$\therefore$ Least value of $\tan^2 A + \tan^2 B + \tan^2 C = 12$

168. Here, M is the mid-point of BC.

$\therefore \quad \mathbf{AM} = \mathbf{AB} + \frac{1}{2}(\mathbf{AB} + \mathbf{AC}) \quad (\text{using } \mathbf{AB} + \mathbf{BC} = \mathbf{AC})$

$\mathbf{AB} \,||\, (\mathbf{AB} + \mathbf{AC})$



Let angle $BAM = \phi$

$$\therefore \quad \cos\phi = \frac{\mathbf{AB} \cdot (\mathbf{AB} + \mathbf{AC})}{|\mathbf{AB}||\mathbf{AB} + \mathbf{AC}|} = \frac{|\mathbf{AB}|^2 + \mathbf{AB} \cdot \mathbf{AC}}{c\sqrt{b^2 + c^2 + 2bc\cos A}}$$

$$= \frac{c^2 + c \cdot b\cos A}{c\sqrt{c^2 + b^2 + 2bc\cos A}} = \frac{c + b\cos A}{\sqrt{b^2 + c^2 + 2b\cos A}}$$

and $\quad \frac{b}{c} = \frac{\sin B}{\sin C} \Rightarrow \cos\phi$

$$= \frac{\sin C + \sin B \sin A}{\sqrt{\sin^2 B + \sin^2 C + 2\sin B \sin C \cos A}}$$

169. Let $\mathbf{p} = \mathbf{BA}$ and $\mathbf{q} = \mathbf{BC}$

Now, required perpendicular distance

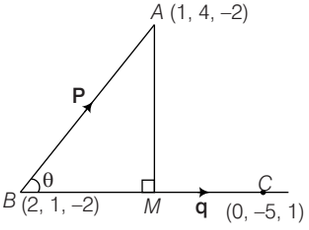
$= AM = (BA)\sin\theta$

$= |\mathbf{p}| \sin\theta \qquad \ldots(i)$

Consider , $|\mathbf{q} \times \mathbf{p}| = |\mathbf{q}||\mathbf{p}| \sin\theta$

On dividing by $|\mathbf{q}|$

$$\frac{|\mathbf{q} \times \mathbf{p}|}{|\mathbf{q}|} = |\mathbf{p}| \sin\theta \qquad \ldots(ii)$$



From Eqs. (i) and (ii), required perpendicular distance

$$= \frac{|\mathbf{q} \times \mathbf{p}|}{|\mathbf{q}|} \qquad \ldots(iii)$$

where, $\quad \mathbf{q} = \mathbf{BC} = \mathbf{OC} - \mathbf{OB} = -2\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$

$\mathbf{P} = \mathbf{AB} = \mathbf{OA} - \mathbf{OB} = -\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$

$\therefore \quad |\mathbf{q}| = \sqrt{4 + 36 + 9} = 7 \qquad \ldots(iv)$

and $\quad \mathbf{q} \times \mathbf{p} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -2 & -6 & 3 \\ -1 & 3 & 0 \end{vmatrix} = -9\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 12\hat{\mathbf{k}}$

$$|\mathbf{q} \times \mathbf{p}| = \sqrt{81 + 9 + 144} = 3\sqrt{26} \qquad \ldots(v)$$

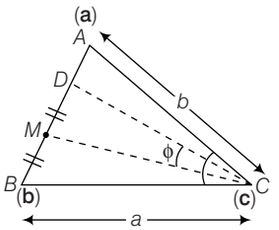
From Eqs. (iii), (iv) and (v), we get

Perpendicular distance $= \frac{3\sqrt{26}}{7}$

170. Here, $AM = MD$ and CD is angle bisector of $\angle C$.

$$\mathbf{CD} = \frac{a\mathbf{b} + b\mathbf{a}}{a + b}$$

and $\quad \mathbf{CM} = \frac{\mathbf{a} + \mathbf{b}}{2}$



Where, $\mathbf{a} = \mathbf{CB}$ and $\mathbf{b} = \mathbf{CA}$

Consequently,

$$\text{Area of } \Delta CDM = \frac{1}{2}(\mathbf{CD} \times \mathbf{CM})$$

$$= \frac{1}{2}\frac{(a\mathbf{b} + b\mathbf{a}) \times (\mathbf{a} + \mathbf{b})}{1(a + b)}$$

$$= \frac{a(\mathbf{b} \times \mathbf{a}) + a(\mathbf{b} \times \mathbf{b}) + b(\mathbf{a} \times \mathbf{a}) + b(\mathbf{a} \times \mathbf{b})}{4(a + b)}$$

$$= \frac{(b - a)(\mathbf{a} \times \mathbf{b})}{4(a + b)}$$

(using $\mathbf{a} \times \mathbf{a} = \mathbf{b} \times \mathbf{b} = 0$ and $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$)

$$= \frac{b - a}{2(a + b)}\left(\frac{\mathbf{a} \times \mathbf{b}}{2}\right)$$

$$= \frac{b - a}{2(a + b)}(\text{Area of } \Delta ABC)$$

$$\therefore \frac{\text{Area of } \Delta CDM}{\text{Area of } \Delta ABC} = \frac{(a - b)}{2(a + b)} = \frac{\sin A - \sin B}{2(\sin A + \sin B)}$$

Also, $\mathbf{CD} \,||\, (a\mathbf{b} + b\mathbf{a})$ and $\mathbf{CM} \,||\, (\mathbf{a} + \mathbf{b})$

$$\cos\phi = \frac{(a\mathbf{b} + b\mathbf{a}) \cdot (\mathbf{a} + \mathbf{b})}{|a\mathbf{b} + b\mathbf{a}|\,|\mathbf{a} + \mathbf{b}|}$$

$$= \frac{a|\mathbf{b}|^2 + b|\mathbf{a}|^2 + |\mathbf{a}|\,|\mathbf{b}|\,(a + b)\cos C}{\sqrt{2a^2b^2 + 2a^2b^2\cos C}\sqrt{a^2 + b^2 + 2ab\cos CA}}$$

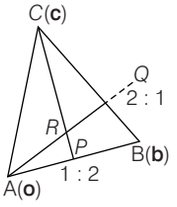
[Where, $|a| = a$ and $|b| = b$]

$$= \frac{(a + b)\cos(C/2)}{\sqrt{a^2 + b^2 + 2ab\cos C}}$$

$$= \frac{(\sin A + \sin B)\cos(C/2)}{\sqrt{\sin^2 A + \sin^2 B + 2\sin A \sin B \cos C}}$$

171. Let $A(\mathbf{O})$, $B(\mathbf{b})$, $C(\mathbf{c})$, $P(\mathbf{p})$, $Q(\mathbf{q})$, $R(\mathbf{r})$

We have, $p = \frac{b}{3}$



and $$\mathbf{Q} = \frac{2\mathbf{b} + \mathbf{c}}{3}$$

Equation of the line AQ, $\mathbf{r} = \lambda_1\left(\frac{2\mathbf{b} + \mathbf{c}}{3}\right)$

Equation of the line CP, $\mathbf{r} = \mathbf{c} + \lambda_2\left(\frac{\mathbf{b}}{3} - \mathbf{c}\right)$

R is the point of intersection of AQ and CP.

$\Rightarrow$ For point R, we have

$$\lambda_1\left(\frac{2\mathbf{b} + \mathbf{c}}{3}\right) = \mathbf{c} + \lambda_2\left(\frac{\mathbf{b}}{3} - \mathbf{c}\right)$$

$$\Rightarrow \frac{2\lambda_2}{3} = \frac{\lambda_2}{3} \text{ (comparing coefficients of } \mathbf{b} \text{ and } \mathbf{c})$$

and $$\frac{\lambda_1}{3} = 1 - \lambda_2$$

On solving, we get

$$\lambda_1 = 3/7, \lambda_2 = 6/7$$

$$\Rightarrow \mathbf{R} = \frac{1}{7}(2\mathbf{b} + \mathbf{c})$$

Now, $$\mathbf{RB} = \mathbf{b} - \frac{1}{7}\cdot(2\mathbf{b} + \mathbf{c}) = \frac{5\mathbf{b} - \mathbf{c}}{7}$$

$$\mathbf{RC} = \mathbf{c} - \frac{1}{7}(2\mathbf{b} + \mathbf{c}) = \frac{(6\mathbf{c} - 2\mathbf{b})}{7}$$

$$\therefore \quad |\mathbf{RB} \times \mathbf{RC}| = \frac{1}{49}(5\mathbf{b} - \mathbf{c}) \times (6\mathbf{c} - 2\mathbf{b})$$

$$= \frac{1}{49}(30\mathbf{b} \times \mathbf{c} + 2\mathbf{c} \times \mathbf{b}) = \frac{28}{49}(\mathbf{b} \times \mathbf{c})$$

$$\Rightarrow \quad |\mathbf{RB} \times \mathbf{RC}| = \frac{28}{49}|\mathbf{b} \times \mathbf{c}| = \frac{28}{49} \qquad [(\text{area of } \Delta ABC)\cdot 2]$$

($\because$ area of $\Delta BRC = 1$)

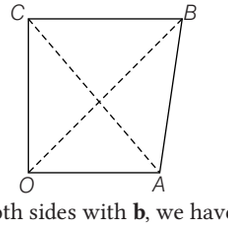
$$\Rightarrow \text{Area of } \Delta ABC = \frac{1}{2}|\mathbf{RB} \times \mathbf{RC}| \cdot \frac{49}{28} = (\text{Area of } \Delta BRC)\cdot\frac{49}{28}$$

$\therefore$ Area of $\Delta ABC = \frac{49}{28}$ sq units.

172. Let $OABC$ be a given quadrilateral such that its diagonal **OB** bisects the diagonal **AC** let $\mathbf{OA} = \mathbf{a}$, $\mathbf{OB} = \mathbf{b}$, $\mathbf{OC} = \mathbf{c}$.

Since, the mid-point $\frac{\mathbf{a} + \mathbf{c}}{2}$ of **AC** lies on **OB**, there exits a scalar t such that,

Area of $\Delta ABC = \frac{\mathbf{a} + \mathbf{c}}{2} = t\mathbf{b} \;\Rightarrow\; \mathbf{a} + \mathbf{c} = 2t\,\mathbf{b}$



On multiplying both sides with **b**, we have

$$(\mathbf{a} + \mathbf{c}) \times \mathbf{b} = 2t\,\mathbf{b} \times \mathbf{b}$$

$$\Rightarrow \quad \mathbf{a} \times \mathbf{b} + \mathbf{c} \times \mathbf{b} = 0$$

$$\Rightarrow \quad \mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c}$$

$$\Rightarrow \quad \frac{1}{2}|\mathbf{a} \times \mathbf{b}| = \frac{1}{2}|\mathbf{b} \times \mathbf{c}|$$

$\Rightarrow$ Area of ΔOAB = Area of ΔOBC

Hence, the diagonal OB bisects the quadrilateral.

173. The coordinates of the resulting force $F = F_1 + F_2 = \{6, 4\}$ i.e., resultant F are 6 and 4. Now, let $M(a, y)$ be a arbitrary point of l. Then, the moment of the resultant about point M is equal to zero.

This moment is equal to sum of the moments $\mathbf{MA} \times \mathbf{F}_1$ and $\mathbf{MB} \times \mathbf{F}_2$ of component forces (the cross product of vectors is distributive.)

Since, $\mathbf{MA} = (1 - x, 1 - y)$, $\mathbf{MB} = \{2 - x, 4 - y\}$, if follows that

$$(\mathbf{MA} \times \mathbf{F}_1)(\hat{\mathbf{i}} + \hat{\mathbf{j}}) = 3(1 - x) - 2(1 - y)$$
$$= 1 - 3x + 2y$$
$$(\mathbf{MB} \times \mathbf{F}_2) \cdot (\hat{\mathbf{i}} + \hat{\mathbf{j}}) = (2 - x) - 4(4 - y)$$
$$= -14 - x + 4y$$

Hence, the equation of straight line l is

$$(1 - 3x + 2y) + (-14 - x + 4y) = 0$$
$$\Rightarrow \quad -4x + 6y - 13 = 0$$
$$\Rightarrow \quad 4x - 6y + 13 = 0$$

174. Let $\mathbf{a} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$

Now, $\mathbf{a}$, $\hat{\mathbf{i}}$ and $\hat{\mathbf{i}} + \hat{\mathbf{j}}$ are coplanar and $\mathbf{a}$, $\hat{\mathbf{i}} - \hat{\mathbf{j}}$ and $\hat{\mathbf{i}} + \hat{\mathbf{k}}$ are coplanar.

$$\Rightarrow \quad [\mathbf{a}\ \hat{\mathbf{i}}\ \hat{\mathbf{i}} + \hat{\mathbf{j}}] = 0 \text{ and } [\mathbf{a}\ \hat{\mathbf{i}} - \hat{\mathbf{j}}\ \hat{\mathbf{i}} + \hat{\mathbf{k}}] = 0$$
$$\Rightarrow \quad \begin{vmatrix} x & y & z \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 0 \text{ and } \begin{vmatrix} x & y & z \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 0$$
$$\therefore \quad z = 0 \text{ and } -x - y + z = 0$$
$$\Rightarrow \quad z = 0 \text{ and } x + y = 0$$
$$\Rightarrow \quad y = -x$$
$$\therefore \quad \mathbf{a} = x\hat{\mathbf{i}} - x\hat{\mathbf{j}}$$
$$\Rightarrow \quad \mathbf{a} = \frac{\hat{\mathbf{i}} - \hat{\mathbf{j}}}{\sqrt{2}}$$

Let the angle between $\mathbf{a}$ and $\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ be θ.

$$\cos\theta = \mathbf{a} \cdot \frac{\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}}{|\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}|}$$
$$= \frac{(\hat{\mathbf{i}} - \hat{\mathbf{j}})}{\sqrt{2}} \cdot \frac{(\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}})}{\sqrt{1 + 4 + 4}}$$
$$= \frac{(1 - \hat{\mathbf{j}})(\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}})}{\sqrt{2} \cdot 3} = \frac{1 + 2}{3\sqrt{2}} = \frac{1}{\sqrt{2}}$$
$$\therefore \quad \theta = \frac{\pi}{4}$$

175. In the new position, let the vector be $x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$. Since, it is perpendicular to the given vector.

$$(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) = 0$$
$$\Rightarrow \quad x + 2y + 2z = 0 \qquad \ldots(i)$$

The magnitude is the new position which also remains the same.

$$\Rightarrow \quad x^2 + y^2 + z^2 = 1 + 4 + 4 = 9 \qquad \ldots(ii)$$

The given vector, the vector in new position and the X-axis are coplanar.

$$\Rightarrow \quad \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 2 \\ x & y & z \end{vmatrix} = 0$$
$$\Rightarrow \quad y = z \text{ and } x = -4y \text{ (using } x + 2y + 22 = 0)$$

Hence, $\quad x^2 + y^2 + z^2 = 9$

$$\Rightarrow \quad 16y^2 + y^2 + y^2 = 9$$
$$\Rightarrow \quad y^2 = \frac{1}{2},\ y = \pm\frac{1}{\sqrt{2}}$$
$$\Rightarrow \quad x = \pm 2\sqrt{2}$$

It is given that the vector passes through the positive X-axis.

$$\Rightarrow \quad x = 2\sqrt{2} \text{ and } y = 0 - \frac{1}{\sqrt{2}} = z$$

Hence, required vector is $\left(2\sqrt{2}\,\hat{\mathbf{i}} - \frac{1}{\sqrt{2}}\hat{\mathbf{j}} - \frac{1}{\sqrt{2}}\hat{\mathbf{k}}\right)$.

176. $\hat{\mathbf{u}} + \hat{\mathbf{v}} + \hat{\mathbf{u}} + \mathbf{w}$ and $\mathbf{w} \times \hat{\mathbf{u}} = \hat{\mathbf{v}}$

$$\Rightarrow \quad (\hat{\mathbf{u}} + \hat{\mathbf{v}} + \hat{\mathbf{u}}) \times \hat{\mathbf{u}} = \mathbf{w} \times \hat{\mathbf{u}}$$
$$\Rightarrow \quad (\hat{\mathbf{u}} + \hat{\mathbf{v}}) \times \hat{\mathbf{u}} + \hat{\mathbf{u}} \times \hat{\mathbf{u}} = \hat{\mathbf{v}}$$
$$\Rightarrow \quad (\hat{\mathbf{u}} \cdot \hat{\mathbf{u}})\hat{\mathbf{v}} - (\hat{\mathbf{v}} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}} + \hat{\mathbf{u}} \times \hat{\mathbf{u}} = \hat{\mathbf{v}}$$
(using $\hat{\mathbf{u}} \cdot \hat{\mathbf{u}} = 1$ and $\hat{\mathbf{u}} \times \hat{\mathbf{u}} = 0$, since unit vectors)
$$\Rightarrow \quad \hat{\mathbf{v}} - (\hat{\mathbf{v}} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}} = \hat{\mathbf{v}}$$
$$\Rightarrow \quad (\hat{\mathbf{u}} \cdot \hat{\mathbf{v}})\hat{\mathbf{u}} = 0 \ \Rightarrow\ \hat{\mathbf{u}} \cdot \hat{\mathbf{v}} = 0$$

Now, $[\hat{\mathbf{u}}\ \hat{\mathbf{v}}\ \mathbf{w}]$

$$\Rightarrow \quad \hat{\mathbf{u}} \cdot (\hat{\mathbf{v}} \times \mathbf{w}) \ \Rightarrow \hat{\mathbf{u}} \cdot [\hat{\mathbf{v}} \times (\hat{\mathbf{u}} \times \hat{\mathbf{v}} + \hat{\mathbf{u}})]$$
(given $\mathbf{w} = \hat{\mathbf{u}} \times \hat{\mathbf{v}} + \hat{\mathbf{u}}$)
$$\Rightarrow \quad \hat{\mathbf{u}} \cdot [\hat{\mathbf{v}} \times (\hat{\mathbf{u}} \times \hat{\mathbf{v}}) + \hat{\mathbf{v}} \times \hat{\mathbf{u}}]$$
$$\Rightarrow \quad \hat{\mathbf{u}} \cdot [(\hat{\mathbf{v}} \cdot \hat{\mathbf{v}})\hat{\mathbf{u}} - (\hat{\mathbf{v}} \cdot \hat{\mathbf{u}})\,\hat{\mathbf{v}} + \hat{\mathbf{v}} \times \hat{\mathbf{u}}]$$
[$\because \hat{\mathbf{u}} \cdot \hat{\mathbf{v}} = 0$ from Eq. (i)]
$$\Rightarrow \quad |\hat{\mathbf{v}}|^2 (\hat{\mathbf{u}} \cdot \hat{\mathbf{u}}) - \hat{\mathbf{u}} \cdot (\hat{\mathbf{v}} \times \hat{\mathbf{u}})$$
$$\Rightarrow \quad |\hat{\mathbf{v}}|^2 |\hat{\mathbf{u}}|^2 - 0 \qquad (\because [\hat{\mathbf{u}}\ \hat{\mathbf{v}}\ \mathbf{w}] = 0)$$
$$(\because |\hat{\mathbf{u}}| = |\hat{\mathbf{v}}| = 1)$$
$$\Rightarrow \quad [\hat{\mathbf{u}}\ \hat{\mathbf{v}}\ \mathbf{w}] = 1$$

177. We have,

$$\mathbf{R} \times \mathbf{B} = \mathbf{C} \times \mathbf{B} \text{ and } \mathbf{R} \cdot \mathbf{A} = 0$$
$$\Rightarrow \quad \mathbf{A} \times (\mathbf{R} \times \mathbf{B}) = \mathbf{A} \times (\mathbf{C} \times \mathbf{B})$$
$$\Rightarrow \quad (\mathbf{A} \cdot \mathbf{B})\mathbf{R} - (\mathbf{A} \cdot \mathbf{R})\mathbf{B} = (\mathbf{A} \cdot \mathbf{B})\mathbf{C} - (\mathbf{A} \cdot \mathbf{C})\mathbf{B} \qquad \ldots(i)$$

where, $\quad \mathbf{A} = 2\hat{\mathbf{i}} + \hat{\mathbf{k}},\ \mathbf{B} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$

and $\quad \mathbf{C} = 4\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$

$$\therefore \quad \mathbf{A} \cdot \mathbf{B} = 2 + 1 = 3,\ \mathbf{A} \cdot \mathbf{C} = 8 + 7 = 15$$

Hence, Eq. (i) reduces to

$$3\mathbf{R} - 0 \cdot \mathbf{B} = 3\mathbf{C} - 15\mathbf{B}$$

or $\quad \mathbf{R} = \mathbf{C} - 5\mathbf{B} = (4\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 7\hat{\mathbf{k}}) - 5(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$

$$\therefore \quad \mathbf{R} = -\hat{\mathbf{i}} - 8\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$$

178. Since, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$ are non-coplanar vectors.

Let $\quad \mathbf{x} = \lambda + \mu\mathbf{b} + \gamma(\mathbf{a} \times \mathbf{b}) \qquad \ldots(i)$

$$\mathbf{x} \cdot \mathbf{a} = \lambda\mathbf{a} \cdot \mathbf{a} + \mu\mathbf{b} \cdot \mathbf{a} + \gamma(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b}$$
$$\Rightarrow \quad 0 \times \lambda|\mathbf{a}|^2 + \mu\mathbf{a} \cdot \mathbf{b} \qquad \ldots(ii)$$

Again from Eq. (i),

$$\mathbf{x} \cdot \mathbf{b} = \lambda\mathbf{a} \cdot \mathbf{b} + \mu\mathbf{b} \cdot \mathbf{b} + \gamma(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b}$$
$$1 = \lambda\mathbf{a} \cdot \mathbf{b} + \mu|\mathbf{b}|^2 \qquad \ldots(iii)$$

From Eq. (i)

$$\mathbf{x} \cdot (\mathbf{a} \times \mathbf{b}) = \lambda\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) + \mu\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) + \gamma(\mathbf{a} \times \mathbf{b})^2$$
$$[\mathbf{x}\ \mathbf{a}\ \mathbf{b}] = \lambda[\mathbf{a}\ \mathbf{a}\ \mathbf{b}] + \mu[\mathbf{b}\ \mathbf{a}\ \mathbf{b}] + \gamma(\mathbf{a} \times \mathbf{b})^2$$
$$1 = \gamma(\mathbf{a} \times \mathbf{b})^2 \qquad \ldots(iv)$$

From Eq. (ii), $\mu(\mathbf{a} \cdot \mathbf{b}) = -\lambda|\mathbf{a}|^2 \ \Rightarrow\ \mu = -\lambda\frac{|\mathbf{a}|^2}{\mathbf{a} \cdot \mathbf{b}}$

From Eq. (iii),

$$1 = \lambda \mathbf{a}\cdot\mathbf{b} - \lambda\frac{|\mathbf{a}|^2|\mathbf{b}|^2}{\mathbf{a}\cdot\mathbf{b}}$$

$$\Rightarrow \quad 1 = \lambda\left(\frac{(\mathbf{a}\cdot\mathbf{b})^2 - |\mathbf{a}|^2|\mathbf{b}|^2}{\mathbf{a}\cdot\mathbf{b}}\right)$$

$$\Rightarrow \quad \lambda = \frac{\mathbf{a}\cdot\mathbf{b}}{(\mathbf{a}\cdot\mathbf{b})^2 - \mathbf{a}^2\mathbf{b}^2}$$

179. $$\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}} = \mathbf{a} \qquad ...(i)$$

$$\mathbf{a}\cdot\hat{\mathbf{x}} + \mathbf{a}\cdot\hat{\mathbf{y}} + \mathbf{a}\cdot\hat{\mathbf{z}} = \mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2 = 4$$

$$\frac{3}{2} + \frac{7}{4} + \mathbf{a}\cdot\hat{\mathbf{z}} = 4 \Rightarrow \mathbf{a}\cdot\hat{\mathbf{z}} = \frac{3}{4} \qquad ...(ii)$$

From Eq. (i), $\hat{\mathbf{x}}\cdot(\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}) = \hat{\mathbf{x}}\cdot\mathbf{a} = \mathbf{a}\cdot\hat{\mathbf{x}} = \frac{3}{2}$

$$\Rightarrow \quad \hat{\mathbf{x}}\cdot\hat{\mathbf{x}} + \hat{\mathbf{x}}\cdot\hat{\mathbf{y}} + \hat{\mathbf{x}}\cdot\hat{\mathbf{z}} = \frac{3}{2}$$

$$\Rightarrow \quad 1 + \hat{\mathbf{x}}\cdot\hat{\mathbf{y}} + \hat{\mathbf{x}}\cdot\hat{\mathbf{z}} = \frac{3}{2}$$

$$\Rightarrow \quad \hat{\mathbf{x}}\cdot\hat{\mathbf{y}} + \hat{\mathbf{x}}\cdot\hat{\mathbf{z}} = \frac{1}{2} \qquad ...(iii)$$

From Eq. (i), $\hat{\mathbf{y}}\cdot(\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}) = \hat{\mathbf{y}}\cdot\mathbf{a} = \frac{7}{4}$

$$\hat{\mathbf{x}}\cdot\hat{\mathbf{y}} + 1 + \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = \frac{7}{4}$$

$$\hat{\mathbf{x}}.\hat{\mathbf{y}} + \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = \frac{3}{4} \qquad ...(iv)$$

From Eq. (i), $(\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}})^2 = (\mathbf{a})^2$

$$\Rightarrow \quad \hat{\mathbf{x}}\cdot\hat{\mathbf{x}} + \hat{\mathbf{y}}\cdot\hat{\mathbf{y}} + \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} + 2(\hat{\mathbf{x}}\cdot\hat{\mathbf{y}} + \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} + \hat{\mathbf{z}}\cdot\hat{\mathbf{x}}) = |\mathbf{a}|^2$$

$$3 + 2(\hat{\mathbf{x}}\cdot\hat{\mathbf{y}} + \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} + \hat{\mathbf{z}}\cdot\hat{\mathbf{x}}) = 4 \qquad ...(v)$$

From Eqs. (iii), (iv) and (v), we get

$$\hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = 0,\ \hat{\mathbf{x}}\cdot\hat{\mathbf{z}} = -\frac{1}{4}, \hat{\mathbf{x}}\cdot\hat{\mathbf{y}} = \frac{3}{4}$$

Now, $$\hat{\mathbf{x}}\times(\hat{\mathbf{y}}\times\hat{\mathbf{z}}) = \mathbf{b}$$

$$(\hat{\mathbf{x}}\cdot\hat{\mathbf{z}})\hat{\mathbf{y}} - (\hat{\mathbf{x}}\cdot\hat{\mathbf{y}})\hat{\mathbf{z}} = \mathbf{b}$$

$$-\frac{1}{4}\hat{\mathbf{y}} - \frac{3}{4}\hat{\mathbf{z}} = \mathbf{b} \qquad ...(vi)$$

Again, $$(\hat{\mathbf{x}}\times\hat{\mathbf{y}})\times\hat{\mathbf{z}} = \mathbf{c}$$

$$(\hat{\mathbf{x}}\cdot\hat{\mathbf{z}})\hat{\mathbf{y}} - (\hat{\mathbf{y}}\cdot\hat{\mathbf{z}})\hat{\mathbf{x}} = \hat{\mathbf{c}} \quad \Rightarrow \quad -\frac{1}{4}\hat{\mathbf{y}} = \mathbf{c}$$

$$\hat{\mathbf{y}} = -4\mathbf{c} \qquad [\text{from Eq. (vi)}]$$

$$\hat{\mathbf{z}} = \frac{4}{3}(\mathbf{c} - \mathbf{b})$$

From Eq. (i), $\hat{\mathbf{x}} = \mathbf{a} - \hat{\mathbf{y}} - \hat{\mathbf{z}}$

$$\hat{\mathbf{x}} = \frac{1}{3}(3\mathbf{a} + 4\mathbf{b} + 8\mathbf{c})$$

$$\hat{\mathbf{x}} = \frac{1}{3}(3\mathbf{a} + 4\mathbf{b} + 8\mathbf{c})$$

$$\hat{\mathbf{y}} = -4\mathbf{c};\ \hat{\mathbf{z}} = \frac{4}{3}(\mathbf{c} - \mathbf{b})$$

180. Here,

$$\mathbf{a}\times\{(\mathbf{x} - \mathbf{b})\times a\} + \mathbf{b}\times\{(\mathbf{x} - \mathbf{c})\times\mathbf{b}\} + \mathbf{c}\times\{(\mathbf{x} - \mathbf{a})\times\mathbf{c}\} = 0$$

$$(\mathbf{a}\cdot\mathbf{a})(\mathbf{x} - \mathbf{b}) - \{\mathbf{a}\cdot(\mathbf{x} - \mathbf{b})\}\mathbf{a} + (\mathbf{b}\cdot\mathbf{b})(\mathbf{x} - \mathbf{c})$$

$$-\{\mathbf{b}\cdot(\mathbf{x} - \mathbf{c})\}\mathbf{b} + (\mathbf{c}\cdot\mathbf{c})(\mathbf{x} - \mathbf{a}) - \{\mathbf{c}\cdot(\mathbf{x} - a)\}\mathbf{c} = 0$$

$$\Rightarrow \quad \lambda^2(\mathbf{x} - \mathbf{b}) - \{\mathbf{a}\cdot\mathbf{x} - 0\}\mathbf{a} + \lambda^2(\mathbf{x} - \mathbf{c}) - \{\mathbf{b}\cdot\mathbf{x} - 0\}\mathbf{b}$$

$$+ \lambda^2(\mathbf{x} - \mathbf{a}) - (\mathbf{c}\cdot\mathbf{x} - 0)\} = \mathbf{c} = 0$$

(using $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{c} = \mathbf{c}\cdot\mathbf{a} = 0$ and $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = \lambda$)

$$\Rightarrow \quad \lambda^2\{\mathbf{x} - \mathbf{b} + \mathbf{x} - \mathbf{c} + \mathbf{x} - \mathbf{a}\}$$

$$= \{(\mathbf{a}\cdot\mathbf{x})\mathbf{a} + (\mathbf{b}\cdot\mathbf{x})\mathbf{b} + (\mathbf{c}\cdot\mathbf{x})\mathbf{c}\}$$

Let $\mathbf{x} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$

($\mathbf{x}$ is linear combination of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$)

$$\Rightarrow \quad \mathbf{a}\cdot\mathbf{x} = \alpha\mathbf{a}\cdot\mathbf{a} \Rightarrow \mathbf{a}\cdot\mathbf{x} = \alpha\lambda^2$$

$$\mathbf{b}\cdot\mathbf{x} = \beta\lambda^2$$

$$\mathbf{c}\cdot\mathbf{x} = \gamma\lambda^2$$

From Eq. (i), $\lambda^2\{3\mathbf{x} - (\mathbf{a} + \mathbf{b} + \mathbf{c})\}(\mathbf{a}\cdot\mathbf{x})$

$$\mathbf{a} + (\mathbf{b}\cdot\mathbf{x})\mathbf{b} + (\mathbf{c}\cdot\mathbf{x})\mathbf{c}$$

and from Eq. (iii)

$$\mathbf{a}\cdot\mathbf{x} = \lambda^2\alpha,\ \mathbf{b}\cdot\mathbf{x} = \lambda^2\beta,\ \mathbf{c}\cdot\mathbf{x} = \lambda^2\gamma$$

Above equation reduces to

$$\lambda^2(3\mathbf{x} - (\mathbf{a} + \mathbf{b} + \mathbf{c})\} = \lambda^2(\mathbf{a}\alpha + \mathbf{b}\beta + \mathbf{c}\gamma)$$

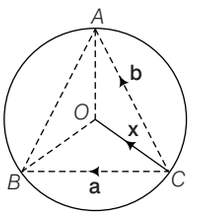
$$\Rightarrow \quad 3\mathbf{x} - (\mathbf{a} - \mathbf{b} + \mathbf{c})$$

$$\Rightarrow \quad 2\mathbf{x} = \mathbf{a} + \mathbf{b} + \mathbf{c}$$

$$\Rightarrow \quad \mathbf{x} = \frac{\mathbf{a} + \mathbf{b} + \mathbf{c}}{2}$$

181. Here, $\mathbf{OC} = \mathbf{x}$, $\mathbf{CA} = \mathbf{b}$, $\mathbf{CB} = \mathbf{a}$

$OA = (\mathbf{b} - x)$ and $OB = \mathbf{a} - x$



Now, $$\mathbf{OA}^2 = \mathbf{OB}^2 = \mathbf{OC}^2$$

$$\mathbf{x}^2 = (\mathbf{a} - \mathbf{x})^2 = (\mathbf{b} - \mathbf{x})^2$$

$$\Rightarrow \quad \mathbf{x}\cdot\mathbf{x} = (\mathbf{a} - \mathbf{x})\cdot(\mathbf{a} - \mathbf{x}) = (\mathbf{b} - \mathbf{x})(\mathbf{b} - \mathbf{x})$$

$$\Rightarrow \quad \mathbf{x}\cdot\mathbf{x} = \mathbf{a}\cdot\mathbf{a} - 2\mathbf{a}\cdot\mathbf{x} + \mathbf{x}\cdot\mathbf{x} = \mathbf{b}\cdot\mathbf{b} - 2\mathbf{b}\cdot\mathbf{x} + \mathbf{x}\cdot\mathbf{x}$$

$$\mathbf{a}\cdot\mathbf{x} = \frac{\mathbf{a}^2}{2} \text{ and } \mathbf{b}\cdot\mathbf{x} = \frac{\mathbf{b}^2}{2}$$

Now, if we take $\mathbf{x} = \lambda\mathbf{a} + \mu\mathbf{b}$, then from Eq. (i)

$$\lambda\mathbf{a}^2 + \mu\cdot\mathbf{a}\cdot\mathbf{b} = \frac{\mathbf{a}^2}{2}$$

and $$\lambda\mathbf{a}\cdot\mathbf{b} + \mu\mathbf{b}^2 = \frac{b^2}{2}$$

∴On solving Eqs. (ii) and (iii),

$$\lambda = \frac{\mathbf{a}^2\mathbf{b}^2 - \mathbf{b}^2(\mathbf{a}\cdot\mathbf{b})}{2(\mathbf{a}^2\mathbf{b}^2) - (\mathbf{a}\cdot\mathbf{b})^2}$$

and $$\mu = \frac{\mathbf{a}^2\mathbf{b}^2 - a^2(\mathbf{a}\cdot\mathbf{b})}{2(\mathbf{a}^2\mathbf{b}^2) - (\mathbf{a}\cdot\mathbf{b})^2}$$

$$\Rightarrow \quad \mathbf{x} = \frac{1}{2}\frac{\mathbf{a}^2\mathbf{b}^2 - \mathbf{b}^2(\mathbf{a}\cdot\mathbf{b})}{2(\mathbf{a}^2\mathbf{b}^2) - (\mathbf{a}\cdot\mathbf{b})^2}\mathbf{a} + \frac{1}{2}\frac{\mathbf{a}^2\mathbf{b}^2 - \mathbf{a}^2(\mathbf{a}\cdot\mathbf{b})}{(\mathbf{a}^2\mathbf{b}^2) - (\mathbf{a}\cdot\mathbf{b})^2}\cdot\mathbf{b}$$

182. $\mathbf{OP}\cdot\mathbf{OQ}+\mathbf{OR}\cdot\mathbf{OS}=\mathbf{OR}\cdot\mathbf{OP}+\mathbf{OQ}\cdot\mathbf{OS}$

$\Rightarrow \quad \mathbf{OP}(\mathbf{OQ}-\mathbf{OR})+\mathbf{OS}(\mathbf{OR}-\mathbf{OQ})=0$

$\Rightarrow \quad (\mathbf{OP}-\mathbf{OS})(\mathbf{OQ}-\mathbf{OR})=0$

$\Rightarrow \quad \mathbf{SP}\cdot\mathbf{RQ}=0$

Similarly $\mathbf{SR}\cdot\mathbf{PQ}=0$ and $\mathbf{SQ}\cdot\mathbf{PR}=0$

$\therefore$ S is orthocentre.

183. $\cos(P+Q)+\cos(Q+R)+\cos(R+P)$

$$=-(\cos R+\cos P+\cos Q)$$

Max. of $\cos P+\cos Q+\cos R=\dfrac{3}{2}$

Min. of $\cos(P+Q)+\cos(Q+R)+\cos(R+P)$ is

$$=-\frac{3}{2}$$

184. $\sin R=\sin(P+Q)$

185. Given, $|\hat{\mathbf{a}}|=|\hat{\mathbf{b}}|=|\hat{\mathbf{c}}|=1$

and $\quad \hat{\mathbf{a}}\times(\hat{\mathbf{b}}\times\hat{\mathbf{c}})=\dfrac{\sqrt{3}}{2}(\hat{\mathbf{b}}+\hat{\mathbf{c}})$

Now, consider $\hat{\mathbf{a}}\times(\hat{\mathbf{b}}\times\hat{\mathbf{c}})=\dfrac{\sqrt{3}}{2}(\hat{\mathbf{b}}+\hat{\mathbf{c}})$

$\Rightarrow \quad (\hat{\mathbf{a}}\cdot\hat{\mathbf{c}})\,\hat{\mathbf{b}}-(\hat{\mathbf{a}}\cdot\hat{\mathbf{b}})\,\hat{\mathbf{c}}=\dfrac{\sqrt{3}}{2}\hat{\mathbf{b}}+\dfrac{\sqrt{3}}{2}\hat{\mathbf{c}}$

On comparing, we get

$$\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}=-\frac{\sqrt{3}}{2}\Rightarrow|\hat{\mathbf{a}}||\hat{\mathbf{b}}|\cos\theta=-\frac{\sqrt{3}}{2}$$

$\Rightarrow \quad \cos\theta=-\dfrac{\sqrt{3}}{2} \qquad [\because |\hat{\mathbf{a}}|=|\hat{\mathbf{b}}|=1]$

$\Rightarrow \quad \cos\theta=\cos\left(\pi-\dfrac{\pi}{6}\right)\Rightarrow\theta=\dfrac{5\pi}{6}$

186. Given, $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=\dfrac{1}{3}|\mathbf{b}|\,|\mathbf{c}|\,\mathbf{a}$

$\Rightarrow \quad -\mathbf{c}\times(\mathbf{a}\times\mathbf{b})=\dfrac{1}{3}|\mathbf{b}||\mathbf{c}|\,\mathbf{a}$

$\Rightarrow \quad -(\mathbf{c}\cdot\mathbf{b})\cdot\mathbf{a}+(\mathbf{c}\cdot\mathbf{a})\,\mathbf{b}=\dfrac{1}{3}|\mathbf{b}|\,|\mathbf{c}|\,\mathbf{a}$

$$\left[\frac{1}{3}|\mathbf{b}||\mathbf{c}|+(\mathbf{c}\cdot\mathbf{b})\right]\mathbf{a}=(\mathbf{c}\cdot\mathbf{a})\mathbf{b}$$

Since, $\mathbf{a}$ and $\mathbf{b}$ are not collinear.

$$\mathbf{c}\cdot\mathbf{b}+\frac{1}{3}|\mathbf{b}||\mathbf{c}|=0 \text{ and } \mathbf{c}\cdot\mathbf{a}=0$$

$\Rightarrow \quad |\mathbf{c}||\mathbf{b}|\cos\theta+\dfrac{1}{3}|\mathbf{b}||\mathbf{c}|=0$

$\Rightarrow \quad |\mathbf{b}||\mathbf{c}|\left(\cos\theta+\dfrac{1}{3}\right)=0$

$\Rightarrow \quad \cos\theta+\dfrac{1}{3}=0 \qquad (\because |\mathbf{b}|\neq0,|\mathbf{c}|\neq0)$

$\Rightarrow \quad \cos\theta=-\dfrac{1}{3}\ ;\ \sin\theta=\dfrac{\sqrt{8}}{3}=\dfrac{2\sqrt{2}}{3}$

187. If a, b, c are any three vectors

Then, $\quad |\mathbf{a}+\mathbf{b}+\mathbf{c}|^2\geq0$

$\Rightarrow \quad |\mathbf{a}|^2+|\mathbf{b}|^2+|\mathbf{c}|^2+2(\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a})\geq0$

$\therefore \quad \mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}\geq\dfrac{-1}{2}(|\mathbf{a}|^2+|\mathbf{b}|^2+|\mathbf{c}|^2)$

Given, $|\mathbf{a}-\mathbf{b}|^2+|\mathbf{b}-\mathbf{c}|^2+|\mathbf{c}-\mathbf{a}|^2=9$

$\Rightarrow |\mathbf{a}|^2+|\mathbf{b}|^2-2\mathbf{a}\cdot\mathbf{b}+|\mathbf{b}|^2+|\mathbf{c}|^2-2\mathbf{b}\cdot\mathbf{c}+|\mathbf{c}|^2+|\mathbf{a}|^2-2\mathbf{c}\cdot\mathbf{a}=9$

$\Rightarrow \quad 6-2(\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a})=9 \qquad [\because|\mathbf{a}|=|\mathbf{b}|=|\mathbf{c}|=1]$

$\Rightarrow \quad \mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}=-\dfrac{3}{2}$ ...(i)

Also, $\quad \mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}\geq\dfrac{-1}{2}(|\mathbf{a}|^2+|\mathbf{b}|^2+|\mathbf{c}|^2)$

$$\geq-\frac{3}{2}$$ ...(ii)

From Eqs. (i) and (ii), $|\mathbf{a}+\mathbf{b}+\mathbf{c}|=0$

as $\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}$ is minimum when $|\mathbf{a}+\mathbf{b}+\mathbf{c}|=0$

$\Rightarrow \quad \mathbf{a}+\mathbf{b}+\mathbf{c}=0$

$\therefore \quad |2\mathbf{a}+5\mathbf{b}+5\mathbf{c}|=|2\mathbf{a}+5(\mathbf{b}+\mathbf{c})|=|2\mathbf{a}-5\mathbf{a}|=3$

188. Let $\mathbf{a}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}}$, $\mathbf{b}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}}$

and $\mathbf{c}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$

$\therefore$ A vector coplanar to $\mathbf{a}$ and $\mathbf{b}$ and perpendicular to $\mathbf{c}$

$=\lambda(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=\lambda\{(\mathbf{a}\cdot\mathbf{c})\,\mathbf{v}-(\mathbf{b}\cdot\mathbf{c})\,\mathbf{a}\}$

$=\lambda\{(1+1+4)(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}})-(1+2+1)(\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}})\}$

$=\lambda\{6\,\hat{\mathbf{i}}+12\,\hat{\mathbf{j}}+6\,\hat{\mathbf{k}}-6\hat{\mathbf{i}}-6\hat{\mathbf{j}}-12\hat{\mathbf{k}}\}$

$=\lambda\{6\,\hat{\mathbf{j}}-6\,\hat{\mathbf{k}}\}=6\lambda\{\hat{\mathbf{j}}-\hat{\mathbf{k}}\}$

For, $\lambda=\dfrac{1}{6}\ \Rightarrow$ Option (a) is correct.

and for $\lambda=-\dfrac{1}{6}\ \Rightarrow$ Option (d) is correct.

189. Let $\mathbf{v}=\mathbf{a}+\lambda\mathbf{b}$

$\mathbf{v}=(1+\lambda)\,\hat{\mathbf{i}}+(1-\lambda)\,\hat{\mathbf{j}}\,(1+\lambda)\,\hat{\mathbf{k}}$

Projection of $\mathbf{v}$ on $\mathbf{c}=\dfrac{1}{\sqrt{3}}$

$\Rightarrow \quad \dfrac{\mathbf{v}\cdot\mathbf{c}}{|\mathbf{c}|}=\dfrac{1}{\sqrt{3}}$

$\Rightarrow \quad \dfrac{(1+\lambda)-(1-\lambda)-(1+\lambda)}{\sqrt{3}}=\dfrac{1}{\sqrt{3}}$

$\Rightarrow \quad 1+\lambda-1+\lambda-1-\lambda=1$

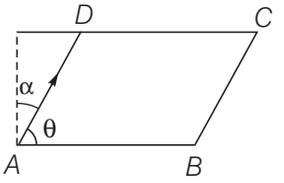
$\Rightarrow \quad \lambda-1=1$

$\Rightarrow \quad \lambda=2$

$\therefore \quad \mathbf{v}=3\,\hat{\mathbf{i}}-\hat{\mathbf{j}}+3\hat{\mathbf{k}}$

190. $\mathbf{AB}=2\hat{\mathbf{i}}+10\hat{\mathbf{j}}+11\hat{\mathbf{k}}$

$\mathbf{AD}=-\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\hat{\mathbf{k}}$



Angle 'θ' between $\mathbf{AB}$ and $\mathbf{AD}$ is

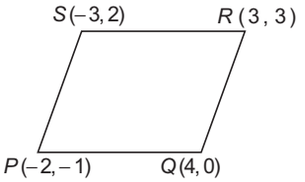
$$\cos(\theta)=\left|\frac{\mathbf{AB}\cdot\mathbf{AD}}{|\mathbf{AB}||\mathbf{AD}|}\right|$$

$$= \left|\frac{-2+20+22}{(15)(3)}\right| = \frac{8}{9} \Rightarrow \sin(\theta) = \frac{\sqrt{17}}{9}$$

Since, $\alpha + \theta = 90°$

$\therefore \quad \cos(\alpha) = \cos(90° - \theta) = \sin(\theta) = \frac{\sqrt{17}}{9}$

191. $m_{PQ} = \frac{1}{6}, m_{SR} = \frac{1}{6}, m_{RQ} = -3, m_{SP} = -3$



$\Rightarrow$ Parallelogram, but neither

$PR = SQ$ nor $PR \perp SQ$.

$\therefore$ So, it is a parallelogram, which is neither a rhombus nor a rectangle.

192. From the given information, it is clear that $\mathbf{a} = \frac{\hat{\mathbf{i}} - 2\hat{\mathbf{j}}}{\sqrt{5}}$

$\Rightarrow \quad |\mathbf{a}| = 1, |\mathbf{b}| = 1, \mathbf{a}\cdot\mathbf{b} = 0$

Now, $(2\mathbf{a} + \mathbf{b})\cdot[(\mathbf{a}\times\mathbf{b})\times(\mathbf{a} - 2\mathbf{b})]$

$= (2\mathbf{a} + \mathbf{b})\cdot[a^2\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\cdot\mathbf{a} + 2\,\mathbf{b}^2\cdot\mathbf{a} - 2(\mathbf{b}\cdot\mathbf{a})\cdot b]$

$= [2\mathbf{a} + \mathbf{b}]\cdot[\mathbf{b} + 2\mathbf{a}] = 4\mathbf{a}^2 + \mathbf{b}^2$

$= 4\cdot 1 + 1 = 5 \qquad [\text{as } \mathbf{a}\cdot\mathbf{b} = 0]$

193. Let angle between $\mathbf{a}$ and $\mathbf{b}$ be θ_1, $\mathbf{c}$ and $\mathbf{d}$ be θ_2 and $\mathbf{a}\times\mathbf{b}$ and $\mathbf{b}\times\mathbf{d}$ be θ.

Since, $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = 1 \Rightarrow \sin\theta_1\cdot\sin\theta_2\cdot\cos\theta = 1$

$\Rightarrow \qquad \theta_1 = 90°, \theta_2 = 90°, \theta = 0°$

$\Rightarrow \ \mathbf{a}\perp\mathbf{b}, \mathbf{c}\perp\mathbf{d}, (\mathbf{a}\times\mathbf{b}) \| (\mathbf{c}\times\mathbf{d})$

So, $\mathbf{a}\times\mathbf{b} = k(\mathbf{c}\times\mathbf{d})$ and $\mathbf{a}\times\mathbf{b} = k(\mathbf{c}\times\mathbf{d})$

$\Rightarrow \quad (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = k(\mathbf{c}\times\mathbf{d})\cdot\mathbf{c}$ and $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{d} = k(\mathbf{c}\times\mathbf{d})\cdot\mathbf{d}$

$\Rightarrow \quad [\mathbf{a}\,\mathbf{b}\,\mathbf{c}] = 0$ and $[\mathbf{a}\,\mathbf{b}\,\mathbf{d}] = 0$

$\Rightarrow$ $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{a}$, $\mathbf{b}$, $\mathbf{d}$ are coplanar vectors, so options (a) and (b) are incorrect.

Let $\mathbf{b} \| \mathbf{d} \qquad \Rightarrow \mathbf{b} = \pm\mathbf{d}$

As $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = 1 \Rightarrow (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{b}) = \pm 1$

$\Rightarrow \quad [\mathbf{a}\times\mathbf{b}\ \mathbf{c}\ \mathbf{b}] = \pm 1 \Rightarrow [\mathbf{c}\,\mathbf{b}\,\mathbf{a}\times\mathbf{b}] = \pm 1$

$\Rightarrow \quad \mathbf{c}\cdot[\mathbf{b}\times(\mathbf{a}\times\mathbf{b})] = \pm 1 \Rightarrow \mathbf{c}\cdot[\mathbf{a} - (\mathbf{b}\cdot\mathbf{a})\mathbf{b}] = \pm 1$

$\Rightarrow \quad \mathbf{c}\cdot\mathbf{a} = \pm 1 \qquad [\because \mathbf{a}\cdot\mathbf{b} = 0]$

Which is a contradiction, so option (c) is correct.

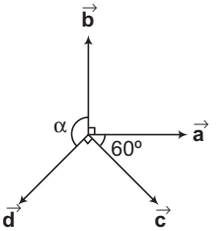
Let option (d) is correct.

$\Rightarrow \quad \mathbf{d} = \pm\mathbf{a}$ and $\mathbf{c} = \pm\mathbf{b}$

As $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = 1$

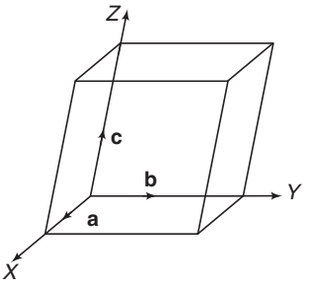
$\Rightarrow \quad (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{b}\times\mathbf{a}) = \pm 1$

Which is a contradiction, so option (d) is incorrect.



Alternatively options (c) and (d) may be observed from the above figure.

194. The volume of the parallelopiped with coterminous edges as $\hat{\mathbf{a}}, \hat{\mathbf{b}}, \hat{\mathbf{c}}$ is given by $[\hat{\mathbf{a}}\,\hat{\mathbf{b}}\,\hat{\mathbf{c}}] = \hat{\mathbf{a}}\cdot(\hat{\mathbf{b}}\times\hat{\mathbf{c}})$



Now, $[\hat{\mathbf{a}}\,\hat{\mathbf{b}}\,\hat{\mathbf{c}}]^2 = \begin{vmatrix} \hat{\mathbf{a}}\cdot\hat{\mathbf{a}} & \hat{\mathbf{a}}\cdot\hat{\mathbf{b}} & \hat{\mathbf{a}}\cdot\hat{\mathbf{c}} \\ \hat{\mathbf{b}}\cdot\hat{\mathbf{a}} & \hat{\mathbf{b}}\cdot\hat{\mathbf{b}} & \hat{\mathbf{b}}\cdot\hat{\mathbf{c}} \\ \hat{\mathbf{c}}\cdot\hat{\mathbf{a}} & \hat{\mathbf{c}}\cdot\hat{\mathbf{b}} & \hat{\mathbf{c}}\cdot\hat{\mathbf{c}} \end{vmatrix} = \begin{vmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{vmatrix}$

$\Rightarrow \quad [\hat{\mathbf{a}}\,\hat{\mathbf{b}}\,\hat{\mathbf{c}}]^2 = 1\left(1 - \frac{1}{4}\right) - \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right) + \frac{1}{2}\left(\frac{1}{4} - \frac{1}{2}\right) = \frac{1}{2}$

Thus, the required volume of the parallelopiped

$= \frac{1}{\sqrt{2}}$ cu unit

195. Given, $\mathbf{OP} = \hat{\mathbf{a}}\cos t + \hat{\mathbf{b}}\sin t$

$\Rightarrow |\mathbf{OP}| = \sqrt{(\hat{\mathbf{a}}\cdot\hat{\mathbf{a}})\cos^2 t + (\hat{\mathbf{b}}\cdot\hat{\mathbf{b}})\sin^2 t + 2\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}\sin t\cos t}$

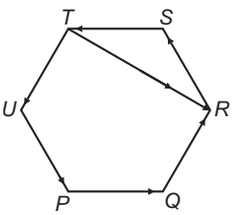
$\Rightarrow |\mathbf{OP}| = \sqrt{1 + \hat{\mathbf{a}}\cdot\hat{\mathbf{b}}\sin 2t}$

$\Rightarrow \quad |\mathbf{OP}|_{\max} = M = \sqrt{1 + \hat{\mathbf{a}}\cdot\hat{\mathbf{b}}}$ at $\sin 2t = 1 \Rightarrow t = \frac{\pi}{4}$

At $t = \frac{\pi}{4}$, $\mathbf{OP} = \frac{1}{\sqrt{2}}(\hat{\mathbf{a}} + \hat{\mathbf{b}})$

Unit vector along $\mathbf{OP}$ at $\left(t = \frac{\pi}{4}\right) = \frac{\hat{\mathbf{a}} + \hat{\mathbf{b}}}{|\hat{\mathbf{a}} + \hat{\mathbf{b}}|}$

196. Since, $\mathbf{PQ}$ is not parallel to $\mathbf{TR}$.



$\because$ $\mathbf{TR}$ is resultant of $\mathbf{RS}$ and $\mathbf{ST}$ vectors.

$\Rightarrow \qquad \mathbf{PQ}\times(\mathbf{RS} + \mathbf{ST}) \neq \mathbf{0}$.

But for Statement II, we have $\mathbf{PQ}\times\mathbf{RS} = \mathbf{0}$

which is not possible as $\mathbf{PQ}$ not parallel to $\mathbf{RS}$.

Hence, Statement I is true and Statement II is false.

197. Since, given vectors are coplanar

$\therefore \qquad \begin{vmatrix} -\lambda^2 & 1 & 1 \\ 1 & -\lambda^2 & 1 \\ 1 & 1 & -\lambda^2 \end{vmatrix} = 0$

$\Rightarrow \qquad \lambda^6 - 3\lambda^2 - 2 = 0$

$\Rightarrow (1 + \lambda^2)^2(\lambda^2 - 2) = 0 \Rightarrow \lambda = \pm\sqrt{2}$

198. Since, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are unit vectors and $\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0}$, then $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ represent an equilateral triangle.

$\therefore \quad \mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}=\mathbf{c}\times\mathbf{a}\neq\mathbf{0}.$

199. Let vector $\mathbf{AO}$ be parallel to line of intersection of planes P_1 and P_2 through origin.

Normal to plane p_1 is

$$\mathbf{n}_1=[(2\hat{\mathbf{j}}+3\hat{\mathbf{k}})\times(4\hat{\mathbf{j}}-3\hat{\mathbf{k}})]=-18\hat{\mathbf{i}}$$

Normal to plane p_2 is

$$\mathbf{n}_2=(\hat{\mathbf{j}}-\hat{\mathbf{k}})\times(3\hat{\mathbf{i}}+3\hat{\mathbf{j}})=3\hat{\mathbf{i}}-3\hat{\mathbf{j}}-3\hat{\mathbf{k}}$$

So, $\mathbf{OA}$ is parallel to $\pm(\mathbf{n}_1\times\mathbf{n}_2)=54\,\hat{\mathbf{j}}-54\hat{\mathbf{k}}$.

$\therefore$ Angle between $54\,(\hat{\mathbf{j}}-\hat{\mathbf{k}})$ and $(2\hat{\mathbf{i}}+\hat{\mathbf{j}}-2\hat{\mathbf{k}})$ is

$$\cos\theta=\pm\left(\frac{54+108}{3\cdot54\cdot\sqrt{2}}\right)=\pm\frac{1}{\sqrt{2}}$$

$$\therefore \quad \theta=\frac{\pi}{4},\frac{3\pi}{4}$$

Hence, (b) and (d) are correct answers.

200. Let vector $\mathbf{r}$ be coplanar to $\mathbf{a}$ and $\mathbf{b}$.

$$\therefore \quad \mathbf{r}=\mathbf{a}+t\,\mathbf{b}$$

$$\Rightarrow \quad \mathbf{r}=(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}})+t\,(\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}})$$

$$=(1+t)\hat{\mathbf{i}}+(2-t)\hat{\mathbf{j}}+(1+t)\hat{\mathbf{k}}$$

The projection of $\mathbf{r}$ on $\mathbf{c}=\frac{1}{\sqrt{3}}\cdot$ [given]

$$\Rightarrow \quad \frac{\mathbf{r}\cdot\mathbf{c}}{|\,\mathbf{c}\,|}=\frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \frac{|\,1\cdot(1+t)+1\cdot(2-t)-1\cdot(1+t)\,|}{\sqrt{3}}=\frac{1}{\sqrt{3}}$$

$\Rightarrow (2-t)=\pm1 \Rightarrow t=1$ or 3

When, $t=1$, we have $\mathbf{r}=2\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}}$

When, $t=3$, we have $\mathbf{r}=4\hat{\mathbf{i}}-\hat{\mathbf{j}}+4\hat{\mathbf{k}}$

201. Since, $\mathbf{b}_1=\mathbf{b}-\frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a}$, $\mathbf{b}_1=\mathbf{b}+\frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a}$

and $\mathbf{c}=\mathbf{c}-\frac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a}-\frac{\mathbf{c}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}$, $\mathbf{c}_2=\mathbf{c}-\frac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a}-\frac{\mathbf{c}\cdot\mathbf{b}_1}{|\mathbf{b}|^2}\mathbf{b}_1$

$$\mathbf{c}_3=\mathbf{c}-\frac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a}-\frac{\mathbf{c}\cdot\mathbf{b}_2}{|\mathbf{b}|^2}\mathbf{b}_2,\ \mathbf{c}_4=\mathbf{a}-\frac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a}.$$

which shows $\mathbf{a}\cdot\mathbf{b}_1=0=\mathbf{a}\cdot\mathbf{c}=\mathbf{b}_1\cdot\mathbf{c}_2$

So, $\{\mathbf{a}, \mathbf{b}_1, \mathbf{c}_2\}$ are mutually orthogonal vectors.

202. As we know that, a vector coplanar to $\mathbf{a},\mathbf{b}$ and orthogonal to $\mathbf{c}$ is $\lambda\,\{(\mathbf{a}\times\mathbf{b})\times\mathbf{c}\}$.

$\therefore$ A vector coplanar to $(2\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})$, $(\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}})$ and orthogonal to $3\hat{\mathbf{i}}+2\hat{\mathbf{j}}+6\hat{\mathbf{k}}$

$$=\lambda\,[\{(2\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})\times(\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}})\}\times(3\hat{\mathbf{i}}+2\hat{\mathbf{j}}+6\hat{\mathbf{k}})]$$

$$=\lambda\,[(2\hat{\mathbf{i}}-\hat{\mathbf{j}}-3\hat{\mathbf{k}})\times(3\hat{\mathbf{i}}+2\hat{\mathbf{j}}+6\hat{\mathbf{k}})]=\lambda\,(21\hat{\mathbf{j}}-7\hat{\mathbf{k}})$$

$$\therefore \text{ Unit vector } =+\frac{(21\hat{\mathbf{j}}-7\hat{\mathbf{k}})}{\sqrt{(21)^2+(7)^2}}$$

$$=+\frac{(3\hat{\mathbf{j}}-\hat{k})}{\sqrt{10}}$$

203. We know that, volume of parallelopiped whose edges are $\mathbf{a},\mathbf{b},\mathbf{c}$ $=[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]$.

$$\therefore \quad [\mathbf{a}\,\mathbf{b}\,\mathbf{c}]=\begin{vmatrix}1 & a & 1\\ 0 & 1 & a\\ a & 0 & 1\end{vmatrix}=1+a^3-a$$

Let $f(a)=a^3-a+1$

$$\Rightarrow \quad f'(a)=3a^2-1 \ \Rightarrow\ f''(a)=6a$$

For maximum or minimum, put $f'(a)=0$

$\Rightarrow a=\pm\frac{1}{\sqrt{3}}$, which shows $f(a)$ is minimum at $a=\frac{1}{\sqrt{3}}$ and maximum at $a=-\frac{1}{\sqrt{3}}$.

204. We know that, $\mathbf{a}\times(\mathbf{a}\times\mathbf{b})=(\mathbf{a}\cdot\mathbf{b})\mathbf{a}-(\mathbf{a}\cdot\mathbf{a})\mathbf{b}$

$$\therefore \quad (\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})\times(\hat{\mathbf{j}}-\hat{\mathbf{k}})=(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})-(\sqrt{3})^2\mathbf{b}$$

$$\Rightarrow \quad -2\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}-3\mathbf{b} \ \Rightarrow\ 3\mathbf{b}=+3\hat{\mathbf{i}}$$

$$\therefore \quad \mathbf{b}=\hat{\mathbf{i}}$$

205. Given, $\mathbf{V}=2\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}}$ and $\mathbf{W}=\hat{\mathbf{i}}+3\hat{\mathbf{k}}$

$$[\mathbf{U}\,\mathbf{V}\,\mathbf{W}]=\mathbf{U}\cdot[(2\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}})\times(\hat{\mathbf{i}}+3\hat{\mathbf{k}})]$$

$$=\mathbf{U}\cdot(3\hat{\mathbf{i}}-7\hat{\mathbf{j}}-\hat{\mathbf{k}})=|\mathbf{U}||3\hat{\mathbf{i}}-7\hat{\mathbf{j}}-\hat{\mathbf{k}}|\cos\theta$$

Which is maximum, if angle between $\mathbf{U}$ and $3\hat{\mathbf{i}}-7\hat{\mathbf{j}}-\hat{\mathbf{k}}$ is 0 and maximum value

$$=|3\hat{\mathbf{i}}-7\hat{\mathbf{j}}-\hat{\mathbf{k}}|=\sqrt{59}$$

206. Since, $(\mathbf{a}+2\mathbf{b})\cdot(5\mathbf{a}-4\mathbf{a})=0$

$$\Rightarrow \quad 5|\mathbf{a}|^2+6\mathbf{a}\cdot\mathbf{b}-8|\mathbf{b}|^2=0$$

$$\Rightarrow \quad 6\mathbf{a}\cdot\mathbf{b}=3 \qquad [\because|\mathbf{a}|=|\mathbf{b}|=1]$$

$$\Rightarrow \quad \cos\theta=\frac{1}{2} \ \Rightarrow\ \theta=60^\circ$$

207. We have, $\mathbf{a}=2\hat{\mathbf{i}}+\hat{\mathbf{j}}-2\hat{\mathbf{k}}$

$$\Rightarrow\ |\,\mathbf{a}\,|=\sqrt{4+1+4}=3$$

and $\mathbf{b}=\hat{\mathbf{i}}+\hat{\mathbf{j}}\Rightarrow|\,\mathbf{b}\,|=\sqrt{1+1}=\sqrt{2}$

Now, $|\,\mathbf{c}-\mathbf{a}\,|=3\Rightarrow|\,\mathbf{c}-\mathbf{a}\,|^2=9$

$$\Rightarrow \quad (\mathbf{c}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{a})=9$$

$$\Rightarrow \quad |\mathbf{c}|^2+|\mathbf{a}|^2-2\mathbf{c}\cdot\mathbf{a}=9 \qquad \text{...(i)}$$

Again, $|\,\mathbf{a}\times\mathbf{b}\,|\times\mathbf{c}\,|=3$

$$\Rightarrow \quad |\,\mathbf{a}\times\mathbf{b}\,||\,\mathbf{c}\,|\sin30^\circ=3\Rightarrow|\,\mathbf{c}\,|=\frac{6}{|\,\mathbf{a}\times\mathbf{b}\,|}$$

But $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 2 & 1 & -2\\ 1 & 1 & 0\end{vmatrix}=2\hat{\mathbf{i}}-2\hat{\mathbf{j}}+\hat{\mathbf{k}}$

$$\therefore \quad |\,\mathbf{c}\,|\frac{6}{\sqrt{4+4+1}}=2 \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$(2)^2+(3)^2-2\mathbf{c}\cdot\mathbf{a}=9 \ \Rightarrow\ 4+9-2\mathbf{c}\cdot\mathbf{a}=9 \ \Rightarrow\ \mathbf{c}\cdot\mathbf{a}=2$$

208. Use the formulae, $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}$,

$$[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]=[\mathbf{b}\,\mathbf{c}\,\mathbf{a}]=[\mathbf{c}\,\mathbf{a}\,\mathbf{b}]$$

and $[\mathbf{a}\,\mathbf{a}\,\mathbf{b}]=[\mathbf{a}\,\mathbf{b}\,\mathbf{b}]=[\,\mathbf{a}\,\mathbf{c}\,\mathbf{c}]=0$

Further, simplify it and get the result.

Now, $[\mathbf{a}\times\mathbf{b}\ \mathbf{b}\times\mathbf{c}\ \mathbf{c}\times\mathbf{a}]$
$= \mathbf{a}\times\mathbf{b}\cdot((\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a}))$
$= \mathbf{a}\times\mathbf{b}\cdot((\mathbf{k}\times\mathbf{c}\times\mathbf{a}))$ [Here, $\mathbf{k}=\mathbf{b}\times\mathbf{c}$]
$= \mathbf{a}\times\mathbf{b}\cdot[(\mathbf{k}\cdot\mathbf{a})\,\mathbf{c}-(\mathbf{k}\cdot\mathbf{c})\,\mathbf{a}]$
$= (\mathbf{a}\times\mathbf{b})\cdot((\mathbf{b}\times\mathbf{c}\cdot\mathbf{a})\,\mathbf{c}-(\mathbf{b}\times\mathbf{c}\cdot\mathbf{c})\mathbf{a})$
$= (\mathbf{a}\times\mathbf{b})\cdot([\mathbf{b}\,\mathbf{c}\,\mathbf{a}]\,\mathbf{c})-0$ [$\because[\mathbf{b}\times\mathbf{c}\cdot\mathbf{c}]=0$]
$= \mathbf{a}\times\mathbf{b}\cdot\mathbf{c}\,[\mathbf{b}\,\mathbf{c}\,\mathbf{a}]=[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]\,[\mathbf{b}\,\mathbf{c}\,\mathbf{a}]$
$= [\mathbf{a}\,\mathbf{b}\,\mathbf{c}]^2$ {$\because[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]=[\mathbf{b}\,\mathbf{c}\,\mathbf{a}]$}

Hence, $[\mathbf{a}\times\mathbf{b}\ \mathbf{b}\times\mathbf{c}\ \mathbf{c}\times\mathbf{a}]=\lambda\,[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]^2$

$\Rightarrow$ $[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]^2=\lambda\,[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]^2$

$\Rightarrow$ $\lambda=1$

209. Given that,

(i) $\mathbf{a}$ and $\mathbf{b}$ are unit vectors,
i.e. $|\mathbf{a}|=|\mathbf{b}|=1$

(ii) $\mathbf{c}=\mathbf{a}+2\mathbf{b}$ and $\mathbf{d}=5\mathbf{a}-4\mathbf{b}$

(iii) $\mathbf{c}$ and $\mathbf{d}$ are perpendicular to each other.
i.e. $\mathbf{c}\cdot\mathbf{d}=0$

To find Angle between $\mathbf{a}$ and $\mathbf{b}$.

Now, $\mathbf{c}\cdot\mathbf{d}=0\ \Rightarrow\ (\mathbf{a}+2\,\mathbf{b})\cdot(5\,\mathbf{a}-4\,\mathbf{b})=0$

$\Rightarrow$ $5\,\mathbf{a}\cdot\mathbf{a}-4\,\mathbf{a}\cdot\mathbf{b}+10\,\mathbf{b}\cdot\mathbf{a}-8\,\mathbf{b}\cdot\mathbf{b}=0$

$\Rightarrow$ $6\,\mathbf{a}\cdot\mathbf{b}=3$

$\Rightarrow$ $\mathbf{a}\cdot\mathbf{b}=\frac{1}{2}$

So, the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\frac{\pi}{3}$.

210. Given,

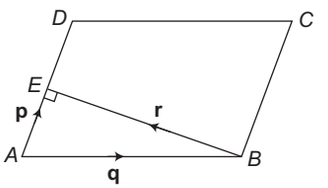
(i) A parallelogram $ABCD$ such that $\mathbf{AB}=\mathbf{q}$ and $\mathbf{AD}=\mathbf{p}$.

(ii) The altitude from vertex B to side AD coincides with a vector $\mathbf{r}$.

To find The vector $\mathbf{r}$ in terms of $\mathbf{p}$ and $\mathbf{q}$.

Let E be the foot of perpendicular from B to side AD.

AE = Projection of vector $\mathbf{q}$ on $\mathbf{p}=\mathbf{q}\cdot\mathbf{p}=\frac{\mathbf{q}\cdot\mathbf{p}}{|\mathbf{p}|}$



$\mathbf{AE}$ = Vector along AE of length AE

$=|\mathbf{AE}|\,\mathbf{AE}=\left(\frac{\mathbf{q}\cdot\mathbf{p}}{|\mathbf{p}|}\right)\mathbf{p}=\frac{(\mathbf{q}\cdot\mathbf{p})\mathbf{p}}{|\mathbf{p}|^2}$

Now, applying triangles law in ΔABE, we get

$\mathbf{AB}+\mathbf{BE}=\mathbf{AE}$

$\Rightarrow$ $\mathbf{q}+\mathbf{r}=\frac{(\mathbf{q}\cdot\mathbf{p})\mathbf{p}}{|\mathbf{p}|^2}\ \Rightarrow\ \mathbf{r}=\frac{(\mathbf{q}\cdot\mathbf{p})\mathbf{p}}{|\mathbf{p}|^2}-\mathbf{q}$

$\Rightarrow$ $\mathbf{r}=-\mathbf{q}+\left(\frac{\mathbf{q}\cdot\mathbf{p}}{\mathbf{p}\cdot\mathbf{p}}\right)\mathbf{p}$

211. $\mathbf{a}=\frac{1}{\sqrt{10}}(3\hat{\mathbf{i}}+\hat{\mathbf{k}})$ and $\mathbf{b}=\frac{1}{7}(2\,\hat{\mathbf{i}}+3\,\hat{\mathbf{j}}-6\,\hat{\mathbf{k}})$

$\therefore$ $(2\mathbf{a}-\mathbf{b})\cdot\{(\mathbf{a}\times\mathbf{b})\times(\mathbf{a}+2\mathbf{b})\}$
$=(2\mathbf{a}-\mathbf{b})\cdot\{(\mathbf{a}\times\mathbf{b})\times\mathbf{a}+(\mathbf{a}\times\mathbf{b})\times2\mathbf{b}\}$
$=(2\mathbf{a}-\mathbf{b})\cdot\{(\mathbf{a}\cdot\mathbf{a})\,\mathbf{b}-(\mathbf{b}\cdot\mathbf{a})\,\mathbf{a}$
$+2\,(\mathbf{a}\cdot\mathbf{b})\,\mathbf{b}-2\,(\mathbf{b}\cdot\mathbf{b})\,\mathbf{a}\}$
$=(2\mathbf{a}-\mathbf{b})\cdot\{1\,(\mathbf{b})-(0)\,\mathbf{a}+2\,(0)\,\mathbf{b}-2\,(1)\,\mathbf{a}\}$
[as $\mathbf{a}\cdot\mathbf{b}=0$ and $\mathbf{a}\cdot\mathbf{a}=\mathbf{b}\cdot\mathbf{b}=1$]
$=(2\mathbf{a}-\mathbf{b})\,(\mathbf{b}-2\mathbf{a})$
$=-(4\,|\mathbf{a}|^2-4\,\mathbf{a}\cdot\mathbf{b}+|\mathbf{b}|^2)=-\{4-0+1\}=-5$

212. Given, $\mathbf{a}\cdot\mathbf{b}\neq0,\ \mathbf{a}\cdot\mathbf{d}=0$ ...(i)

and $\mathbf{b}\times\mathbf{c}=\mathbf{b}\times\mathbf{d}$

$\Rightarrow$ $\mathbf{b}\times(\mathbf{c}-\mathbf{d})=0$

$\therefore$ $\mathbf{b}\,||\,(\mathbf{c}-\mathbf{d})$

$\Rightarrow$ $\mathbf{c}-\mathbf{d}=\lambda\mathbf{b}$

$\Rightarrow$ $\mathbf{d}=\mathbf{c}-\lambda\mathbf{b}$ ...(ii)

Taking dot product with $\mathbf{a}$, we get

$\mathbf{a}\cdot\mathbf{d}=\mathbf{a}\cdot\mathbf{c}-\lambda\mathbf{a}\cdot\mathbf{b}$

$\Rightarrow$ $0=\mathbf{a}\cdot\mathbf{c}-\lambda\,(\mathbf{a}\cdot\mathbf{b})$

$\therefore$ $\lambda=\frac{\mathbf{a}\cdot\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}}$ ...(iii)

$\therefore$ $\mathbf{d}=\mathbf{c}-\frac{(\mathbf{a}\cdot\mathbf{c})}{(\mathbf{a}\cdot\mathbf{b})}\mathbf{b}$

213. Given, $\mathbf{a}=p\,\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$, $\mathbf{b}=\hat{\mathbf{i}}+q\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $\mathbf{c}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+r\hat{\mathbf{k}}$ are coplanar and $p\neq q\neq r\neq1$.

Since, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar.

$\Rightarrow$ $[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]=0$

$\Rightarrow$ $\begin{vmatrix} p & 1 & 1\\ 1 & q & 1\\ 1 & 1 & r\end{vmatrix}=0$

$\Rightarrow$ $p\,(qr-1)-1\,(r-1)+1\,(1-q)=0$

$\Rightarrow$ $pqr-p-r+1+1-q=0$

$\therefore$ $pqr-(p+q+r)=-2$

214. We have, $\mathbf{a}\times\mathbf{b}+\mathbf{c}=0$

$\Rightarrow$ $\mathbf{a}\times(\mathbf{a}\times\mathbf{b})+\mathbf{a}\times\mathbf{c}=0$

$\Rightarrow$ $(\mathbf{a}\cdot\mathbf{b})\mathbf{a}-(\mathbf{a}\cdot\mathbf{a})\mathbf{b}+\mathbf{a}\times\mathbf{c}=0$

$\Rightarrow$ $3\mathbf{a}-2\mathbf{b}+\mathbf{a}\times\mathbf{c}=0$

$\Rightarrow$ $2\mathbf{b}=3\mathbf{a}+\mathbf{a}\times\mathbf{c}$

$\Rightarrow$ $2\mathbf{b}=3\hat{\mathbf{j}}-3\hat{\mathbf{k}}-2\hat{\mathbf{i}}-\hat{\mathbf{j}}-\hat{\mathbf{k}}=-2\hat{\mathbf{i}}+2\hat{\mathbf{j}}-4\hat{\mathbf{k}}$

$\therefore$ $\mathbf{b}=-\hat{\mathbf{i}}+\hat{\mathbf{j}}-2\,\hat{\mathbf{k}}$

215. Since, the given vectors are mutually orthogonal, therefore

$\mathbf{a}\cdot\mathbf{b}=2-4+2=0$

$\mathbf{a}\cdot\mathbf{c}=\lambda-1+2\mu=0$ ...(i)

and $\mathbf{b}\cdot\mathbf{c}=2\lambda+4+\mu=0$ ...(ii)

On solving Eqs. (i) and (ii), we get

$\mu=2$ and $\lambda=-3$

Hence, $(\lambda,\mu)=(-3,2)$

216. Since, $[3\mathbf{u}\ p\mathbf{v}\ p\mathbf{w}]-[p\mathbf{v}\ \mathbf{w}\ q\mathbf{u}]-[2\mathbf{w}\ q\mathbf{v}\ q\mathbf{u}]=0$

$\therefore 3p^2[\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})]-pq\,[\mathbf{v}\cdot(\mathbf{w}\times\mathbf{u})]$
$-2q^2\,[\mathbf{w}\cdot(\mathbf{v}\times\mathbf{u})]=0$

$\Rightarrow \quad (3p^2 - pq + 2q^2)\,[\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})] = 0$

But $\quad [\mathbf{u}\ \mathbf{v}\ \mathbf{w}] \neq 0$

$\Rightarrow \quad 3p^2 - pq + 2q^2 = 0$

$\therefore \quad p = q = 0$

217. Given that, $\mathbf{b} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\mathbf{c} = \hat{\mathbf{j}} + \hat{\mathbf{k}}$.

The equation of bisector of $\mathbf{b}$ and $\mathbf{c}$ is

$$\mathbf{r} = \lambda(\mathbf{b} + \mathbf{c}) = \lambda\left(\frac{\hat{\mathbf{i}} + \hat{\mathbf{j}}}{\sqrt{2}} + \frac{\hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{2}}\right)$$

$$= \frac{\lambda}{\sqrt{2}}(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}) \qquad \ldots(\text{i})$$

Since, vector $\mathbf{a}$ lies in plane of $\mathbf{b}$ and $\mathbf{c}$.

$\therefore \quad \mathbf{a} = \mathbf{b} + \mu\mathbf{c}$

$\Rightarrow \quad \frac{\lambda}{\sqrt{2}}(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}) = (\hat{\mathbf{i}} + \hat{\mathbf{j}}) + \mu(\hat{\mathbf{j}} + \hat{\mathbf{k}})$

On equating the coefficient of $\hat{\mathbf{i}}$ both sides, we get

$$\frac{\lambda}{\sqrt{2}} = 1 \ \Rightarrow \ \lambda = \sqrt{2}$$

On putting $\lambda = \sqrt{2}$ in Eq. (i), we get

$$\mathbf{r} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$$

Since, the given vector a represents the same bisector equation $\hat{\mathbf{r}}$.

$\therefore \quad \alpha = 1 \quad \text{and} \quad \beta = 1$

218. Since, $(2\mathbf{u} \times 3\mathbf{v})$ is a unit vector.

$\Rightarrow \quad |2\mathbf{u} \times 3\mathbf{v}| = 1$

$\Rightarrow \quad 6|\mathbf{u}\,||\,\mathbf{v}||\sin\theta| = 1$

$\Rightarrow \quad \sin\theta = \frac{1}{6} \qquad [\because |\mathbf{u}| = |\mathbf{v}| = 1]$

Since, θ is an acute angle, then there is exactly one value of θ for which $(2\mathbf{u} \times 3\mathbf{v})$ is a unit vector.

219. Since, given vectors $\mathbf{v}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar.

$$\therefore \quad \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ x & x-2 & -1 \end{vmatrix} = 0$$

$\Rightarrow \quad 1\{1 - 2(x-2)\} - 1(-1 - 2x) + 1(x - 2 + x) = 0$

$\Rightarrow \quad 1 - 2x + 4 + 1 + 2x + 2x - 2 = 0$

$\Rightarrow \quad 2x = -4 \ \Rightarrow \ x = -2$

220. Since, $\quad (\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \mathbf{a}\times(\mathbf{b}\times\mathbf{c})$

$\therefore \quad (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\,\mathbf{a} = (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}$

$\Rightarrow \quad (\mathbf{b}\cdot\mathbf{c})\mathbf{a} = (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

$\Rightarrow \quad \mathbf{a} = \frac{(\mathbf{a}\cdot\mathbf{b})}{(\mathbf{b}\cdot\mathbf{c})}\cdot\mathbf{c}$

Hence, $\mathbf{a}$ is parallel to $\mathbf{c}$.

221. Since, position vectors of A, B, C are $2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$ and $a\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$, respectively.

Now, $\quad \mathbf{AC} = (a\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}) - (2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}})$

$$= (a-2)\hat{\mathbf{i}} - 2\hat{\mathbf{j}}$$

and $\quad \mathbf{BC} = (a\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}) - (\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}})$

$$= (a-1)\hat{\mathbf{i}} + 6\hat{\mathbf{k}}$$

Since, the ΔABC is right angled at C, then

$$\mathbf{AC}\cdot\mathbf{BC} = 0$$

$\Rightarrow \quad \{(a-2)\hat{\mathbf{i}} - 2\hat{\mathbf{j}}\}\cdot\{(a-1)\hat{\mathbf{i}} + 6\hat{\mathbf{k}}\} = 0$

$\Rightarrow \quad (a-2)(a-1) = 0$

$\therefore \quad a = 1 \quad \text{and} \quad a = 2$

222. Line is parallel to plane as

$(\hat{\mathbf{i}} - \hat{\mathbf{j}} + 4\hat{\mathbf{k}})\cdot(\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + \hat{\mathbf{k}}) = 1 - 5 + 4 = 0$

General point on the line is

$(\lambda + 2, -\lambda - 2, 4\lambda + 3)$.

For $\lambda = 0$, a point on this line is $(2, -2, 3)$ and distance from

$\mathbf{r}\cdot(\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + \hat{\mathbf{k}}) = 5$ or $x + 5y + z = 5$ is

$$d = \left|\frac{2 + 5(-2) + 3 - 5}{\sqrt{1 + 25 + 1}}\right| \Rightarrow \ d = \left|\frac{-10}{3\sqrt{3}}\right| = \frac{10}{3\sqrt{3}}$$

223. Let $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$

Then, $\quad \mathbf{a}\times\hat{\mathbf{i}} = -a_2\hat{\mathbf{k}} + a_3\hat{\mathbf{j}}$

$\mathbf{a}\times\hat{\mathbf{j}} = a_1\hat{\mathbf{k}} - a_3\hat{\mathbf{i}}$

$\mathbf{a}\times\hat{\mathbf{k}} = -a_1\hat{\mathbf{j}} + a_2\hat{\mathbf{i}}$

$\therefore \quad (\mathbf{a}\times\hat{\mathbf{i}})^2 + (\mathbf{a}\times\hat{\mathbf{j}})^2 + (\mathbf{a}\times\hat{\mathbf{k}})^2$

$= a_2^2 + a_3^2 + a_1^2 + a_3^2 + a_1^2 + a_2^2$

$= 2(a_1^2 + a_2^2 + a_3^2) = 2a^2$

224. Given that, $[\lambda(\mathbf{a} + \mathbf{b}) \quad \lambda^2\mathbf{b} \quad \lambda\,\mathbf{c}] = [\mathbf{a} \quad \mathbf{b} + \mathbf{c} \quad \mathbf{b}]$

$$\therefore \begin{vmatrix} \lambda(a_1 + b_1) & \lambda(a_2 + b_2) & \lambda(a_3 + b_3) \\ \lambda^2 b_1 & \lambda^2 b_2 & \lambda^2 b_3 \\ \lambda c_1 & \lambda c_2 & \lambda c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\Rightarrow \quad \lambda^4 \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$\Rightarrow \quad \lambda^4 = -1$

So, no real value of λ exists.

225. Given, vectors are

$$\mathbf{a} = \hat{\mathbf{i}} - \hat{\mathbf{k}}, \quad \mathbf{b} = x\hat{\mathbf{i}} + \hat{\mathbf{j}} + (1-x)\hat{\mathbf{k}}$$

and $\quad \mathbf{c} = y\hat{\mathbf{i}} + x\hat{\mathbf{j}} + (1 + x - y)\hat{\mathbf{k}}$

$$\therefore \quad [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \begin{vmatrix} 1 & 0 & -1 \\ x & 1 & 1-x \\ y & x & 1+x-y \end{vmatrix}$$

Applying $C_3 \to C_3 + C_1$, we get

$$= \begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ y & x & 1+x \end{vmatrix} = 1(1 + x) - x = 1$$

Thus, $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ depends upon neither x nor y.

226. Since, $\quad |\mathbf{u}| = 1, |\mathbf{v}| = 2, |\mathbf{w}| = 3$

The projection of $\mathbf{v}$ along $\mathbf{u} = \frac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{u}|}$

and the projection of $\mathbf{w}$ along $\mathbf{u} = \frac{\mathbf{w}\cdot\mathbf{u}}{|\mathbf{u}|}$

According to given condition,

$$\frac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{u}|}=\frac{\mathbf{w}\cdot\mathbf{u}}{|\mathbf{u}|}$$

$\Rightarrow \quad \mathbf{v}\cdot\mathbf{u}=\mathbf{w}\cdot\mathbf{u}$ ...(i)

Since, $\mathbf{v},\mathbf{w}$ are perpendicular to each other.

$\therefore \quad \mathbf{v}\cdot\mathbf{w}=0$ ...(ii)

Now, $|\mathbf{u}-\mathbf{v}+\mathbf{w}|^2=|\mathbf{u}|^2+|\mathbf{v}|^2+|\mathbf{w}|^2 -2\,\mathbf{u}\cdot\mathbf{v}-2\,\mathbf{v}\cdot\mathbf{w}+2\,\mathbf{u}\cdot\mathbf{w}$

$\Rightarrow \quad |\mathbf{u}-\mathbf{v}+\mathbf{w}|^2=1+4+9-2\mathbf{u}\cdot\mathbf{v}+2\mathbf{v}\cdot\mathbf{u}$ [from Eqs. (i) and (ii)]

$\Rightarrow \quad |\mathbf{u}-\mathbf{v}+\mathbf{w}|^2=1+4+9$

$\Rightarrow \quad |\mathbf{u}-\mathbf{v}+\mathbf{w}|=\sqrt{14}$

227. Given that, $\frac{1}{3}|\mathbf{b}||\mathbf{c}|\mathbf{a}=(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$

We know that,

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}-(\mathbf{b}\cdot\mathbf{c})\mathbf{a}$$

$\therefore \quad \frac{1}{3}|\mathbf{b}||\mathbf{c}|\mathbf{a}=(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}-(\mathbf{b}\cdot\mathbf{c})\mathbf{a}$

On comparing the coefficients of $\mathbf{a}$ and $\mathbf{b}$, we get

$$\frac{1}{3}|\mathbf{b}||\mathbf{c}|=-\mathbf{b}\cdot\mathbf{c} \text{ and } \mathbf{a}\cdot\mathbf{c}=0$$

$\Rightarrow \quad \frac{1}{3}|\mathbf{b}||\mathbf{c}|=-|\mathbf{b}||\mathbf{c}|\cos\theta$

$\Rightarrow \quad \cos\theta=-\frac{1}{3} \quad \Rightarrow 1-\sin^2\theta=\frac{1}{9}$

$\Rightarrow \quad \sin^2\theta=\frac{8}{9}$

$\therefore \quad \sin\theta=\frac{2\sqrt{2}}{3}$ $\left[\because\ 0\le\theta\le\frac{\pi}{2}\right]$

228. Total force, $\mathbf{F}=(4\hat{\mathbf{i}}+\hat{\mathbf{j}}-3\hat{\mathbf{k}})+(3\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}})$

$\therefore \quad \mathbf{F}=7\hat{\mathbf{i}}+2\hat{\mathbf{j}}-4\hat{\mathbf{k}}$

The particle is displaced from $A(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}})$ to $B(5\hat{\mathbf{i}}+4\hat{\mathbf{j}}+\hat{\mathbf{k}})$.

Now, displacement,

$$\mathbf{AB}=(5\hat{\mathbf{i}}+4\hat{\mathbf{j}}+\hat{\mathbf{k}})-(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}})=4\hat{\mathbf{i}}+2\hat{\mathbf{j}}-2\hat{\mathbf{k}}$$

$\therefore$ Work done $=\mathbf{F}\cdot\mathbf{AB}$

$$=(7\hat{\mathbf{i}}+2\hat{\mathbf{j}}-4\hat{\mathbf{k}})\cdot(4\hat{\mathbf{i}}+2\hat{\mathbf{j}}-2\hat{\mathbf{k}})$$

$$=28+4+8=40 \text{ units}$$

229. $(\mathbf{u}+\mathbf{v}-\mathbf{w})\cdot[(\mathbf{u}-\mathbf{v})\times(\mathbf{v}-\mathbf{w})]$

$$=(\mathbf{u}+\mathbf{v}-\mathbf{w})\cdot[\mathbf{u}\times\mathbf{v}-\mathbf{u}\times\mathbf{w}-\mathbf{v}\times\mathbf{v}+\mathbf{v}\times\mathbf{w}]$$

$$=\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v})-\mathbf{u}\cdot(\mathbf{u}\times\mathbf{w})+\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})+\mathbf{v}\cdot(\mathbf{u}\times\mathbf{v})$$
$$-\mathbf{v}\cdot(\mathbf{u}\times\mathbf{w})+\mathbf{v}\cdot(\mathbf{v}\times\mathbf{w})-\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})+\mathbf{w}\cdot(\mathbf{u}\times\mathbf{w})-\mathbf{w}\cdot(\mathbf{v}\times\mathbf{w})$$

$$=\mathbf{u}\cdot\mathbf{v}\times\mathbf{w}-\mathbf{v}\cdot\mathbf{u}\times\mathbf{w}-\mathbf{w}\cdot\mathbf{u}\times\mathbf{v} \qquad \{[\mathbf{a},\mathbf{a}\ \mathbf{b}]=0\}$$

$$=\mathbf{u}\cdot\mathbf{v}\times\mathbf{w}+\mathbf{w}\cdot\mathbf{u}\times\mathbf{v}-\mathbf{w}\cdot\mathbf{u}\times\mathbf{v}=\mathbf{u}\cdot\mathbf{v}\times\mathbf{w}$$

230. Given that, $|\mathbf{a}|=1,|\mathbf{b}|=2,|\mathbf{c}|=3$

and $\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0}$

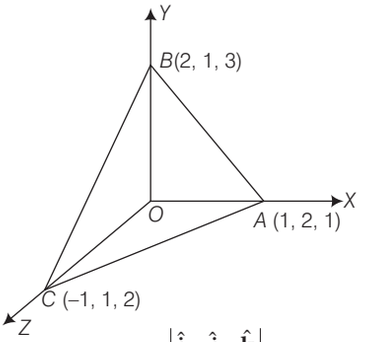
Now, $(\mathbf{a}+\mathbf{b}+\mathbf{c})^2=|\mathbf{a}|^2+|\mathbf{b}|^2+|\mathbf{c}|^2+2(\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a})$

$\Rightarrow \quad 0=1^2+2^2+3^2+2(\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a})$

$\Rightarrow \quad 2(\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a})=-14$

$\Rightarrow \quad \mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}=-7$

231. Vector perpendicular to face OAB is $\mathbf{n}_1$.



$$=\mathbf{OA}\times\mathbf{OB}=\begin{vmatrix}\hat{\mathbf{i}}&\hat{\mathbf{j}}&\hat{\mathbf{k}}\\1&2&1\\2&1&3\end{vmatrix}=5\hat{\mathbf{i}}-\hat{\mathbf{j}}-3\hat{\mathbf{k}}$$

Vector perpendicular to face ABC is $\mathbf{n}_2$

$$=\mathbf{AB}\times\mathbf{AC}=\begin{vmatrix}\hat{\mathbf{i}}&\hat{\mathbf{j}}&\hat{\mathbf{k}}\\1&-1&2\\-2&-1&1\end{vmatrix}=\hat{\mathbf{i}}-5\hat{\mathbf{j}}-3\hat{\mathbf{k}}$$

Since, angle between faces is equal to the angle between their normals.

$$\therefore\ \cos\theta=\frac{\mathbf{n}_1\cdot\mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}=\frac{5\times1+(-1)\times(-5)+(-3)\times(-3)}{\sqrt{5^2+(-1)^2+(-3)^2}\,\sqrt{1^2+(-5)^2+(-3)^2}}$$

$$=\frac{5+5+9}{\sqrt{35}\,\sqrt{35}}=\frac{19}{35} \Rightarrow \theta=\cos^{-1}\left(\frac{19}{35}\right)$$

232. Given that, $\mathbf{u}=\hat{\mathbf{i}}+\hat{\mathbf{j}}$, $\mathbf{v}=\hat{\mathbf{i}}-\hat{\mathbf{j}}$, $\mathbf{w}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}}$,

$\mathbf{u}\cdot\mathbf{n}=0$ and $\mathbf{v}\cdot\mathbf{n}=0$

i.e. $\mathbf{n}=\frac{\mathbf{u}\times\mathbf{v}}{|\mathbf{u}\times\mathbf{v}|}$

Now, $\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\hat{\mathbf{i}}&\hat{\mathbf{j}}&\hat{\mathbf{k}}\\1&1&0\\1&-1&0\end{vmatrix}=0\hat{\mathbf{i}}-0\hat{\mathbf{j}}-2\hat{\mathbf{k}}=-2\hat{\mathbf{k}}$

$\therefore \quad |\mathbf{w}\cdot\mathbf{n}|=\frac{|\mathbf{w}\cdot\mathbf{u}\times\mathbf{v}|}{|\mathbf{u}\times\mathbf{v}|}=\frac{|-6\hat{\mathbf{k}}|}{|-2\hat{\mathbf{k}}|}=3$

$[\because \mathbf{w}\cdot(\mathbf{u}\times\mathbf{w})=(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}})\cdot(-2\hat{\mathbf{k}})=-6\hat{\mathbf{k}}]$

Hence, $|\mathbf{w}\cdot\mathbf{n}|=3$

233. Given two vectors lie in XY-plane. So, a vector coplanar with them is $\mathbf{a}=x\hat{\mathbf{i}}+y\hat{\mathbf{j}}$

Since, $\mathbf{a}\perp(\hat{\mathbf{i}}-\hat{\mathbf{j}})$

$\Rightarrow \quad (x\hat{\mathbf{i}}+y\hat{\mathbf{j}})\cdot(\hat{\mathbf{i}}-\hat{\mathbf{j}})=0$

$\Rightarrow \quad x-y=0$

$\Rightarrow \quad x=y$

$\therefore \quad \mathbf{a}=x\hat{\mathbf{i}}+x\hat{\mathbf{j}}$

and $4|\mathbf{a}|=\sqrt{x^2+x^2}=x\sqrt{2}$

$\therefore$ Required unit vector

$$=\frac{\mathbf{a}}{|\mathbf{a}|}=\frac{x(\hat{\mathbf{i}}+\hat{\mathbf{j}})}{x\sqrt{2}}=\frac{1}{\sqrt{2}}(\hat{\mathbf{i}}+\hat{\mathbf{j}})$$

CHAPTER

03

Three Dimensional Coordinate System

Learning Part

Session 1

- Introduction
- Position Vector of a Point in Space
- Shifting of Origin • Distance Formula • Section Formula
- Direction Cosines and Direction Ratios of a Vector
- Projection of the Line Segment Joining Two Points on a Given Line

Session 2

- Equation of a Straight Line in Space
- Angle between Two Lines • Perpendicular Distance of a Point from a Line
- Shortest Distance between Two Lines

Session 3

- Plane
- Equation of Plane in Various Form
- Angles between Two Planes • Family of Planes
- Two Sides of a Plane • Distance of a Point from a Plane
- Equation of Planes Bisecting the Angle between Two Planes
- Line and Plane

Session 4

- Sphere

Practice Part

- JEE Type Examples
- Chapter Exercises

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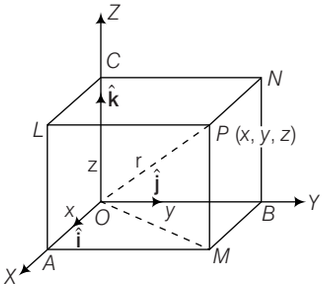
Session 1

Introduction, Position Vector of a Point in Space, Shifting of Origin, Distance Formula, Section Formula,Direction Cosines and Direction Ratios of a Vector, Projection of the Line Segment Joining Two Points on a Given Line

Introduction

Let OY and OZ be two perpendicular lines which intersect at O and let a third straight line OX be perpendicular to the plane in which they lie. The three mutually perpendicular lines form a set of coordinate axis. They determine three mutually perpendicular planes called coordinate planes.



Remarks

1. The axes to coordinate form a right handed set (in the figure) i.e. a right handed screw, driven from O to X would rotate in the sense from OY to OZ.
2. The points A, B and C are the orthogonal projections of P on the X, Y and Z-axes.
3. Points L, M and N are $(x, 0, z)$, $(x, y, 0)$, $(0, y, z)$ and A, B and C are $(x, 0, 0)$, $(0, y, 0)$, $(0, 0, z)$, respectively.

Position Vector of a Point in Space

Let $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ be unit vector (called base vector) along OX, OY and OZ, respectively.

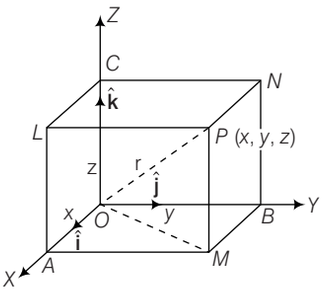
Let $P(x, y, z)$ be a point in space, let the position of P be $\mathbf{r}$.

Then, $\mathbf{r} = \mathbf{OP} = \mathbf{OM} + \mathbf{MP}$

$= (\mathbf{OA} + \mathbf{AM}) + \mathbf{MP} = \mathbf{OA} + \mathbf{OB} + \mathbf{OC}$

$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$

Thus, the position vector of a point P is, $x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$.



Signs of Coordinates of a Point in Various Octants

Octant/ Coordinates	$OXYZ$	$OX'YZ$	$OXY'Z$	$OXYZ'$	$OX'YZ'$	$OX'YZ'$	$OXY'Z'$	$OX'Y'Z'$
x	+	−	+	+	−	−	+	−
y	+	+	−	+	−	+	−	−
z	+	+	+	−	+	−	−	−

Note

Any point on X-axis $= (x, 0, 0)$

Y-axis $= (0, y, 0)$

Z-axis $= (0, 0, z)$

XY-plane $= (x, y, 0)$

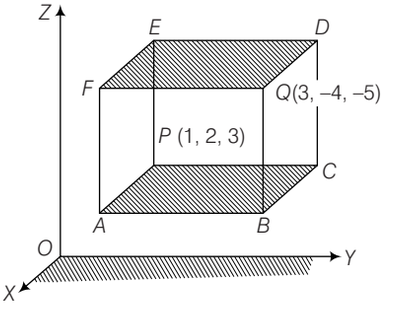
YZ-plane $= (0, y, z)$

ZX-plane $= (x, 0, z)$

$\therefore \quad OP = \sqrt{x^2 + y^2 + z^2}$

Example 1. ***Planes are drawn parallel to the coordinate planes through the points*** $(1, 2, 3)$ ***and*** $(3, -4, -5)$. ***Find the lengths of the edges of the parallelopiped so formed.***

Sol. Let $P = (1, 2, 3)$, $Q = (3, -4, -5)$ through which planes are drawn parallel to the coordinate planes shown as,



$\therefore PE$ = Distance between parallel planes $ABCP$ and $FQDE$, i.e. (along Z-axis)

$$= |-5-3| = 8$$

PA = Distance between parallel planes $ABQF$ and $PCDE$, i.e. (along X-axis)

$$= |3-1| = 2$$

PC = Distance between parallel planes $BCDQ$ and $APEF$, i.e. (along Y-axis)

$$= |-4-2| = 6$$

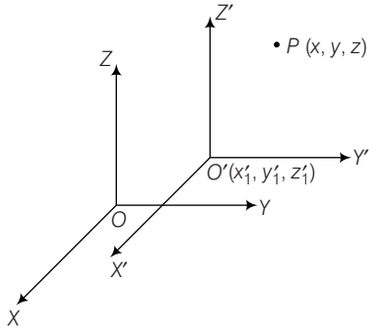
$\therefore$ Lengths of edges of the parallelopiped are 2, 6, 8.

Shifting of Origin

Shifting the origin to another point without changing the directions of the axes is called the translation of axes.

Let the origin $O(0, 0, 0)$ be shifted to another point $O'(x', y', z')$ without changing the direction of axes. Let the new coordinate frame be $O'X'Y'Z'$. Let $P(x, y, z)$ be a point with respect to the coordinate frame $OXYZ$. Then, coordinate of point P with respect to new coordinate frame $O'X'Y'Z'$ is (x_1, y_1, z_1), where $x_1 = x - x'$,

$$y_1 = y - y', z_1 = z - z'$$



Example 2. If the origin is shifted $(1, 2, -3)$ without changing the directions of the axis, then find the new coordinates of the point $(0, 4, 5)$ with respect to new frame.

Sol. In the new frame $x' = x - x_1, y' = y - y_1, z' = z - z_1$, where (x_1, y_1, z_1) is shifted origin.

$$\Rightarrow \quad x' = 0 - 1 = -1,$$

$$y' = 4 - 2 = 2, z' = 5 + 3 = 8$$

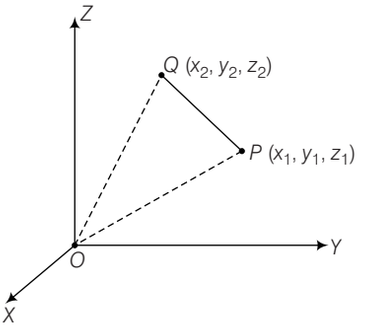
Hence, the coordinates of the point with respect to the new coordinates frame are $(-1, 2, 8)$.

Distance Formula

The distance between the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Proof. Let O be the origin and let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two given points.



The, $\quad \mathbf{OP} = x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}} + z_1\hat{\mathbf{k}}$

$$\mathbf{OQ} = x_2\hat{\mathbf{i}} + y_2\hat{\mathbf{j}} + z_2\hat{\mathbf{k}}$$

Now, $\mathbf{PQ}$ = Position vector of Q − Position vector of P

$$= \mathbf{OQ} - \mathbf{OP}$$

$$= (x_2\hat{\mathbf{i}} + y_2\hat{\mathbf{j}} + z_2\hat{\mathbf{k}}) - (x_1\hat{\mathbf{i}} + y_2\hat{\mathbf{j}} + z_1\hat{\mathbf{k}})$$

$$= (x_2 - x_1)\hat{\mathbf{i}} + (y_2 - y_1)\hat{\mathbf{j}} + (z_2 - z_1)\hat{\mathbf{k}}$$

$\therefore \quad PQ = |\mathbf{PQ}|$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Hence, $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Example 3. Find the distance between the points $P(-2, 4, 1)$ and $Q(1, 2, -5)$.

Sol. We have, $PQ = \sqrt{(1+2)^2 + (2-4)^2 + (-5-1)^2}$

$$PQ = \sqrt{3^2 + (-2)^2 + (-6)^2}$$

$$= \sqrt{9 + 4 + 36}$$

$$= \sqrt{49} = 7$$

Example 4. Prove by using distance formula that the points $P(1, 2, 3)$, $Q(-1, -1, -1)$ and $R(3, 5, 7)$ are collinear.

Sol. We have,

$$PQ = \sqrt{(-1-1)^2 + (-1-2)^2 + (-1-3)^2}$$

$$= \sqrt{4 + 9 + 16} = \sqrt{29}$$

$$QR = \sqrt{(3+1)^2 + (5+1)^2 + (7+1)^2}$$

$$= \sqrt{16 + 36 + 64}$$

$$= \sqrt{116} = 2\sqrt{29}$$

and $\quad PR = \sqrt{(3-1)^2 + (5-2)^2 + (7-3)^2}$

$$= \sqrt{4 + 9 + 16} = \sqrt{29}$$

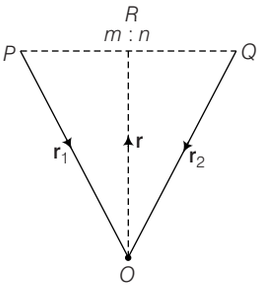
Since, $\quad QR = QP + PR$

Therefore, the given points are collinear.

Section Formula

1. Section Formula for Internal Division

Let P and Q be two points whose position vectors are $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively. Let R be a point on PQ dividing it in the ratio $m : n$. Then, the position vector of R is given by $\mathbf{r} = \dfrac{m\mathbf{r}_2 + n\mathbf{r}_1}{m + n}$.



Proof. Let O be the origin. Then, $\mathbf{OP} = \mathbf{r}_1$, $\mathbf{OQ} = \mathbf{r}_2$ and $\mathbf{OR} = \mathbf{r}$

Now, $$\frac{PR}{RQ} = \frac{m}{n}$$

$$\Rightarrow \quad n\mathbf{PR} = m\mathbf{RQ}$$

$$\Rightarrow \quad n(\mathbf{OR} - \mathbf{OP}) = m(\mathbf{OQ} - \mathbf{OR})$$

$$\Rightarrow \quad n(\mathbf{r} - \mathbf{r}_1) = m(\mathbf{r}_2 - \mathbf{r})$$

$$\Rightarrow \quad (m + n)\mathbf{r} = m\mathbf{r}_2 + n\mathbf{r}_1$$

$$\Rightarrow \quad \mathbf{r} = \frac{m\mathbf{r}_2 + n\mathbf{r}_1}{m + n}$$

Corollary

Mid-point formula Let P and Q be two points whose position vectors are given by $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively. Then, the position vector of the mid-point R of PQ is given by, $\mathbf{r} = \dfrac{\mathbf{r}_1 + \mathbf{r}_2}{2}$.

2. Section Formula for External Division

Let P and Q be the points whose position vectors are $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively. Let R be a point on PQ dividing it externally in the ratio $m : n$. Then, the position vector of R is given by,

$$\mathbf{r} = \frac{m\mathbf{r}_2 - n\mathbf{r}_1}{m - n}$$

Proof. Let O be the origin. Then, $\mathbf{OP} = \mathbf{r}_1$, $\mathbf{OQ} = \mathbf{r}_2$ and $\mathbf{OR} = \mathbf{r}$

Now, $$\frac{PR}{QR} = \frac{m}{n}$$

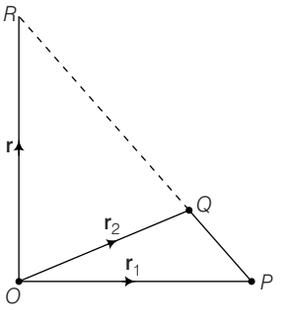
$$\Rightarrow \quad n\mathbf{PR} = m\mathbf{QR}$$

$$\Rightarrow \quad n(\mathbf{OR} - \mathbf{OP}) = m(\mathbf{OR} - \mathbf{OQ})$$

$$\Rightarrow \quad n(\mathbf{r} - \mathbf{r}_1) = m(\mathbf{r} - \mathbf{r}_2)$$

$$\Rightarrow \quad (m - n)\mathbf{r} = m\mathbf{r}_2 - n\mathbf{r}_1$$

$$\Rightarrow \quad \mathbf{r} = \left(\frac{m\mathbf{r}_2 - n\mathbf{r}_1}{m - n}\right)$$



Corollary 1. If $R(x, y, z)$ is a point dividing the join of $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio $m : n$.

Then, $x = \dfrac{mx_2 + nx_1}{m + n}$, $y = \dfrac{my_2 + ny_1}{m + n}$, $z = \dfrac{mz_2 + nz_1}{m + n}$

Corollary 2. The coordinates of the mid-point of the joint of $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}, \dfrac{z_1 + z_2}{2}\right)$.

Corollary 3. The coordinates of a point R which divides the join of $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ externally in the ratio $m : n$ are

$$\left(\frac{mx_2 - nx_1}{m - n}, \frac{my_2 - ny_1}{m - n}, \frac{mz_2 - nz_1}{m - n}\right)$$

Corollary 4. The coordinate of centroid of triangle with vertices $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$ is

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right).$$

Corollary 5. Centroid of tetrahedron with vertices $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4)$ is

$$\left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4}\right).$$

Example 5. Find the ratio in which $2x + 3y + 5z = 1$ divides the line joining the points $(1, 0, -3)$ and $(1, -5, 7)$.

Sol. Let $2x + 3y + 5z = 1$ divides $(1, 0, -3)$ and $(1, -5, 7)$ in the ratio of $k : 1$ at point P.

Then, $P=\left(\frac{k+1}{k+1}, \frac{-5k}{k+1}, \frac{7k-3}{k+1}\right)$ which must satisfy

$2x+3y+5z=1$

$\Rightarrow \quad 2\left(\frac{k+1}{k+1}\right)+3\left(\frac{-5k}{k+1}\right)+5\left(\frac{7k-3}{k+1}\right)=1$

$\Rightarrow \qquad 2k+2-15k+35k-15=k+1$

$\Rightarrow \qquad 21k=14 \Rightarrow k=\frac{2}{3}$

$\therefore$ $2x+3y+5z=1$, divides $(1, 0, -3)$ and $(1, -5, 7)$ in the ratio of $2:3$.

Example 6. If $A(3,2,-4)$, $B(5,4,-6)$ and $C(9,8,-10)$ are three collinear points, then find the ratio in which point C divides AB.

Sol. Let C divide AB in the ratio $\lambda:1$. Then,

$$C \equiv \left(\frac{5\lambda+3}{\lambda+1}, \frac{4\lambda+2}{\lambda+1}, \frac{-6\lambda-4}{\lambda+1}\right)=(9, 8, -10)$$

Comparing, $5\lambda+3=9\lambda+9$ or $4\lambda=-6$

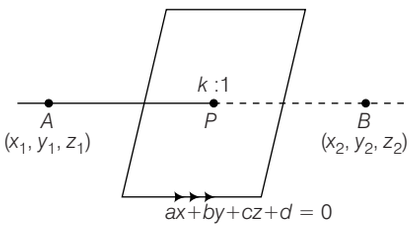
$\therefore \qquad \lambda=-\frac{3}{2}$

Also, from $4\lambda+2=8\lambda+8$ and $-6\lambda-4=-10\lambda-10$, we get the same value of λ.

$\therefore$ C divides AB in the ratio $3:2$ externally.

Example 7. Show that the plane $ax+by+cz+d=0$ divides the line joining (x_1, y_1, z_1) and (x_2, y_2, z_2) in the ratio of $\left(-\frac{ax_1+ay_1+cz_1+d}{ax_2+by_2+cz_2+d}\right)$.

Sol. Let the plane $ax+by+cz+d=0$ divides the line joining (x_1, y_1, z_1) and (x_2, y_2, z_2) in the ratio $k:1$ as shown in figure.



$\therefore$ Coordinates of $P\left(\frac{kx_2+x_1}{k+1}, \frac{ky_2+y_1}{k+1}, \frac{kz_2+z_1}{k+1}\right)$

must satisfy $ax+by+cz+d=0$

i.e., $a\left(\frac{kx_2+x_1}{k+1}\right)+b\left(\frac{ky_2+y_1}{k+1}\right)+c\left(\frac{kz_2+z_1}{k+1}\right)+d=0$

$\Rightarrow a(kx_2+x_1)+b(ky_2+y_1)+c(kz_2+z_1)+d(k+1)=0$

$\Rightarrow \quad k(ax_2+by_2+cz_2+d)+(ax_1+by_1+cz_1+d)=0$

$\Rightarrow \qquad k=-\frac{(ax_1+by_1+cz_1+d)}{(ax_2+by_2+cz_2+d)}$

Remark

Students are advised to learn above result as a formula i.e. $ax+by+cz+d=0$ divides join of (x_1, y_1, z_1) and (x_2, y_2, z_2) in ratio of $-\frac{(ax_1+by_1+cz_1+d)}{(ax_2+by_2+cz_2+d)}$.

Example 8. Find the ratio in which the join of $(2,1,5)$, $(3,4,3)$ is divided by the plane $2x+2y-2z-1=0$.

Sol. Using above result,

$$\text{Required ratio}=\frac{\{(2(2)+2(1)-2(5)-1\}}{\{2(3)+2(4)-2(3)-1\}}$$

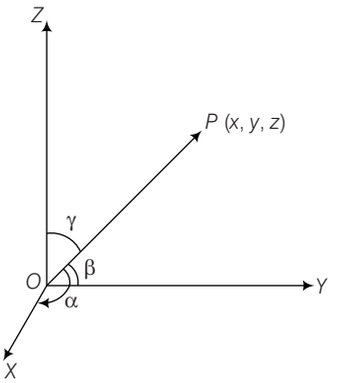
$$=\frac{\{6-11\}}{\{14-7\}}=\frac{-5}{7}$$

$\Rightarrow 2x+2y-2z-1=0$ divides $(2,1,5)$ and $(3, 4, 5)$ externally in ratio of $5:7$.

Direction Cosines and Direction Ratios of a Vector

1. Direction Cosines (DC's)

If α, β and γ are the angles which a vector **OP** makes with the positive directions of the coordinate axes OX, OY and OZ respectively. Then $\cos\alpha, \cos\beta$ and $\cos\gamma$ are known as direction cosines of **OP** and are generally denoted by letters l, m and n, respectively.



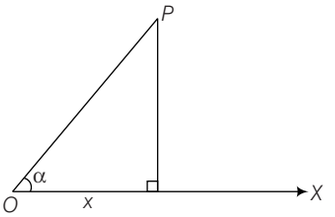
Thus, $l=\cos\alpha; m=\cos\beta; n=\cos\gamma$. The angles α, β and γ are known as direction angles and they satisfy the condition $0 \le \alpha, \beta, \gamma \le \pi$.

It can be seen form the figure

$$\cos\alpha=\frac{x}{OP}$$

Similarly, $$\cos\beta=\frac{y}{OP}$$

$$\cos\gamma=\frac{z}{OP}$$



Where, OP is the modulus of positive vector of P.

Clearly, $OP = \sqrt{x^2 + y^2 + z^2}$

So, $l^2 + m^2 + n^2 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma$

$$= \frac{x^2 + y^2 + z^2}{OP^2} = \frac{x^2 + y^2 + z^2}{x^2 + y^2 + z^2} = 1$$

$\therefore \quad l^2 + m^2 + n^2 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\therefore$ If $\mathbf{OP} = \mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$

Then, $\hat{\mathbf{r}} = l\,\hat{\mathbf{i}} + m\,\hat{\mathbf{j}} + n\,\hat{\mathbf{k}}$

By definition it follows that the direction cosine of the axis x are $\cos 0°, \cos 90°, \cos 90°$, i.e. $(1, 0, 0)$.

Similarly, direction cosine of the axes Y and Z are $(0, 1, 0)$ and $(0, 0, 1)$, respectively.

2. Direction Ratios (DR's)

Let l, m and n be the direction cosines of a vector $\mathbf{r}$ and a, b and c be three numbers such that a, b and c are proportional to l, m and n

i.e. $\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = k$

or $(l, m, n) = (ka, kb, kc)$

$\Rightarrow (a, b, c)$ are direction ratios.

If $\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ are direction cosines of a vector $\mathbf{r}$, then its direction ratios are $(1, -1, 1)$ or $(-1, 1, -1)$ or $(2, -2, 2)$ or $(\lambda, -\lambda, \lambda)$... etc.

It is evident from the above definition that to obtain direction ratios of a vector from its direction cosines, we just multiply them by a common number.

"That shows there can be infinitely many direction ratios for a given vector but the direction cosines are unique".

To obtain direction cosines from direction ratios.

Let a, b and c be direction ratios of a vector $\mathbf{r}$ having direction cosines l, m and n. Then,

$l = \lambda a, m = \lambda b, n = \lambda c$ (by definition)

$\therefore \quad l^2 + m^2 + n^2 = 1$

$\Rightarrow \quad a^2\lambda^2 + b^2\lambda^2 + c^2\lambda^2 = 1$

$\Rightarrow \quad \lambda = \pm\frac{1}{\sqrt{a^2 + b^2 + c^2}}$

So, $l = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}}$,

$m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}}$,

$n = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}$

For example, let the direction ratios of a point be $(3, 1, -2)$.

$\Rightarrow$ Direction cosines are

$$\left(\frac{3}{\sqrt{3^2 + 1^2 + (-2)^2}}, \frac{1}{\sqrt{3^2 + 1^2 + (-2)^2}}, \frac{-2}{\sqrt{3^2 + 1^2 + (-2)^2}}\right)$$

$$\Rightarrow \left(\frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}\right)$$

3. Angle between Two Vectors in Terms of Direction Cosines and Direction Ratios

Let $\mathbf{a}$ and $\mathbf{b}$ be two given vectors with direction cosines l_1, m_1, n_1 and l_2, m_2, n_2 respectively. Then,

$\mathbf{a} = l_1\hat{\mathbf{i}} + m_1\hat{\mathbf{j}} + n_1\hat{\mathbf{k}}$ and $\mathbf{b} = l_2\hat{\mathbf{i}} + m_2\hat{\mathbf{j}} + n_2\hat{\mathbf{k}}$

$\therefore \cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\cdot|\mathbf{b}|}$, where θ be the angle between $\mathbf{a}$ and $\mathbf{b}$.

$$\Rightarrow \quad \cos\theta = \frac{|l_1\hat{\mathbf{i}} + m_1\hat{\mathbf{j}} + n_1\hat{\mathbf{k}}| \cdot |l_2\hat{\mathbf{i}} + m_2\hat{\mathbf{j}} + n_2\hat{\mathbf{k}}|}{|l_1\hat{\mathbf{i}} + m_1\hat{\mathbf{j}} + n_1\hat{\mathbf{k}}|\,|l_2\hat{\mathbf{i}} + m_2\hat{\mathbf{j}} + n_2\hat{\mathbf{k}}|}$$

$$\Rightarrow \quad \cos\theta = \frac{l_1l_2 + m_1m_2 + n_1n_2}{\sqrt{l_1^2 + m_1^2 + n_1^2}\,\sqrt{l_2^2 + m_2^2 + n_2^2}}$$

$\Rightarrow \quad \cos\theta = l_1l_2 + m_1m_2 + n_1n_2 \qquad [\because l^2 + m^2 + n^2 = 1]$

Also, $\sin^2\theta = 1 - \cos^2\theta$

$$= (l_1^2 + m_1^2 + n_1^2)(l_1^2 + m_2^2 + n_2^2) - (l_1l_2 + m_1m_2 + n_1n_2)^2$$

$\Rightarrow \sin^2\theta = (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2$

$$\Rightarrow \quad \sin\theta = \pm\sqrt{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2}$$

Remarks

1. Acute angle θ between the two lines having direction cosines l_1, m_1, n_1 and l_2, m_2, n_2 is given by

$$\cos\theta = |\,l_1l_2 + m_1m_2 + n_1n_2\,|$$

$$\sin\theta = \sqrt{(l_1m_2 - l_2m_1)^2 + (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2}$$

2. If a_1, b_1, c_1 and a_2, b_2, c_2 are the direction ratios of two lines, then the acute angle θ between them is given by

$$\cos\theta = \frac{|\, a_1a_2 + b_1b_2 + c_1c_2 \,|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$\sin\theta = \frac{\sqrt{(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2}}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

3. The two lines with direction cosines l_1, m_1, n_1 and l_2, m_2, n_2 are perpendicular to each other if $\theta = \frac{\pi}{2}$

$$\Rightarrow \quad \cos\theta = l_1l_2 + m_1m_2 + n_1n_2 = 0$$

4. The two lines with direction cosines l_1, m_1, n_1 and l_2, m_2, n_2 are parallel to each other if $\theta = 0$

or $\pi \Rightarrow \sin\theta = 0$

$$\Rightarrow (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2 = 0$$

$$\Rightarrow \quad \frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$$

5. The angle between two lines having direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 is given by

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}\sqrt{a_2^2 + b_2^2 + c_2^2)}}$$

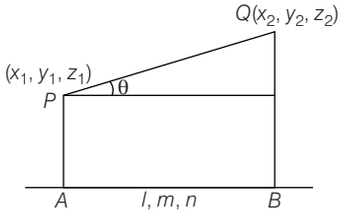
Thus, the two straight lines are perpendicular, if

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

The two straight lines are parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Projection of the Line Segment Joining Two Points on a Given Line

The projection of the line segment joining two given points (x_1, y_1, z_1) and (x_2, y_2, z_2) on the line having direction cosines l, m, n is given by $l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$, which is clear from the vector.



Clearly, $\mathbf{PQ} = (x_2 - x_1)\hat{\mathbf{i}} + (y_2 - y_1)\hat{\mathbf{j}} + (z_2 - z_1)\hat{\mathbf{k}}$

and the line $\mathbf{AB} = l\hat{\mathbf{i}} + m\hat{\mathbf{j}} + n\hat{\mathbf{k}}$

The projection of $\mathbf{PQ}$ on $\mathbf{AB}$

$$= \frac{\mathbf{PQ} \cdot \mathbf{AB}}{|\, \mathbf{AB} \,|} = \frac{l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)}{\sqrt{l^2 + m^2 + n^2}}$$

$$= l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$$

Example 9. What are the direction cosines of a line which is equally inclined to the coordinate axes?

Sol. If α, β and γ are the angles that a line makes with the coordinate axes, then if they are equally inclined.

$$\Rightarrow \quad \alpha = \beta = \gamma$$

Also, $l^2 + m^2 + n^2 = 1$

$$\Rightarrow \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\Rightarrow \quad \cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$$

$$\Rightarrow \quad 3\cos^2\alpha = 1$$

$$\Rightarrow \quad \cos\alpha = \pm\frac{1}{\sqrt{3}} = \cos\beta = \cos\gamma$$

$\therefore$ Direction cosines are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

or $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$.

Example 10. If a line makes angles α, β and γ with the coordinates axes, prove that $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$.

Sol. Let l, m and n be the direction cosines of the given vector.

Then, $l = \cos\alpha, m = \cos\beta, n = \cos\gamma$

Now, $l^2 + m^2 + n^2 = 1$

$$\Rightarrow \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\Rightarrow \quad 1 - \sin^2\alpha + 1 - \sin^2\beta + 1 - \sin^2\gamma = 1$$

$$\Rightarrow \quad \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$$

Example 11. A line OP through origin O is inclined at 30° and 45° to OX and OY, respectively. Find the angle at which it is inclined to OZ.

Sol. Let l, m and n be the direction cosines of the given vector.

$$l^2 + m^2 + n^2 = 1$$

where, $\alpha = 30°, \beta = 45°$

$$\therefore \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\Rightarrow \quad \cos^2 30° + \cos^2 45° + \cos^2\gamma = 1$$

$$\Rightarrow \quad \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cos^2\gamma = 1$$

$$\Rightarrow \quad \cos^2\gamma = 1 - \frac{3}{4} - \frac{1}{2}$$

$$\Rightarrow \quad \cos^2\gamma = \frac{4 - 3 - 2}{4}$$

$$\Rightarrow \quad \cos^2\gamma = -\frac{1}{4}$$ which is not possible.

$\therefore$ There exists no point which is inclined to 30° to X-axis and 45° to Y-axis.

Example 12. Find the direction cosines of a vector $\mathbf{r}$ which is equally inclined to OX, OY and OZ. If $|\mathbf{r}|$ is given, find the total number of such vectors.

Sol. Let l, m and n be the direction cosines of $\mathbf{r}$.

Since, $\mathbf{r}$ is equally inclined with X, Y and Z-axes.

$$\therefore \quad l^2 + m^2 + n^2 = 1$$

$$\Rightarrow \quad 3l^2 = 1 \qquad (\because l = m = n)$$

$$\Rightarrow \quad l = \pm \frac{1}{\sqrt{3}}$$

$\therefore$ Direction cosines of $\mathbf{r}$ are $\pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}$.

Now, $\mathbf{r} = |\mathbf{r}|(l\hat{\mathbf{i}} + m\hat{\mathbf{j}} + n\hat{\mathbf{k}})$

$$\Rightarrow \quad \mathbf{r} = |\mathbf{r}| \left(\pm \frac{1}{\sqrt{3}} \hat{\mathbf{i}} \pm \frac{1}{\sqrt{3}} \hat{\mathbf{j}} \pm \frac{1}{\sqrt{3}} \hat{\mathbf{k}} \right)$$

Since, '+' and '–' signs can be arranged at three planes.

There are eight vectors (i.e. $2 \times 2 \times 2$) which are equally inclined to axes.

Example 13. If the points $(0, 1, -2)$, $(3, \lambda, -1)$ and $(\mu, -3, -4)$ are collinear, verify whether the point $(12, 9, 2)$ is also on the same line.

Sol. Let the points be A, B and C, whose coordinates are $(0, 1, -2)$, $(3, \lambda, -1)$ and $(\mu, -3, -4)$ respectively.

Let $D = (12, 9, 2)$

$\Rightarrow$ DR's of $AB = (3 - 0, \lambda - 1, -1 + 2)$

$= (3, \lambda - 1, 1)$

DR's of $AC = (\mu - 0, -3 - 1, -4 - (-2))$

$= (\mu, -4, -2)$

Since, A, B and C are collinear.

$$\Rightarrow \quad \frac{3}{\mu} = \frac{\lambda - 1}{-4} = \frac{1}{-2}$$

$$\Rightarrow \quad \mu = -6, \lambda = 3$$

$\therefore$ Direction ratios of AB are $(3, 2, 1)$.

Now, direction ratios of AD are $(12 - 0, 9 - 1, 2 - (2))$ or $(12, 8, 4)$

Here, $\frac{3}{12} = \frac{2}{8} = \frac{1}{4}$

$\therefore \quad AB \parallel AD$

Since, AB and AD lie on same straight line.

Hence, the point $(12, 9, 2)$ is on the same line.

Example 14. A vector $\mathbf{r}$ has length 21 and direction ratios $2, -3, 6$. Find the direction cosines and components of $\mathbf{r}$, given that $\mathbf{r}$ makes an obtuse angle with X-axis.

Sol. Here, direction ratio's are $2, -3, 6$.

$\therefore$ Direction cosines can be written as $(2\lambda, -3\lambda, 6\lambda)$.

where, $(2\lambda)^2 + (-3\lambda)^2 + (6\lambda)^2 = 1 \qquad (\because l^2 + m^2 + n^2 = 1)$

as $\quad 49\lambda^2 = 1$

$$\Rightarrow \quad \lambda = \mp \frac{1}{7}$$

$\therefore$ Direction cosines are $\left(\pm \frac{2}{7}, \mp \frac{3}{7}, \pm \frac{6}{7} \right)$.

But it makes obtuse angle with X-axis $\Rightarrow \; l < 0$.

$\therefore$ Direction cosines are $\left(-\frac{2}{7}, \frac{3}{7}, -\frac{6}{7} \right)$

Also, $\mathbf{r} = |\mathbf{r}|(l\hat{\mathbf{i}} + m\hat{\mathbf{j}} + n\hat{\mathbf{k}})$

$$\Rightarrow \quad \mathbf{r} = 21\left(-\frac{2}{7}\hat{\mathbf{i}} + \frac{3}{7}\hat{\mathbf{j}} - \frac{6}{7}\hat{\mathbf{k}} \right) \qquad (\text{given}, |\mathbf{r}| = 21)$$

$$\mathbf{r} = 3(-2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 6\hat{\mathbf{k}})$$

So, the component of $\mathbf{r}$ along X, Y and Z-axes are $-6\hat{\mathbf{i}}$, $9\hat{\mathbf{j}}$ and $-18\hat{\mathbf{k}}$, respectively.

Example 15. Find the angle between the lines whose direction cosines are $\left(-\frac{\sqrt{3}}{4}, \frac{1}{4}, -\frac{\sqrt{3}}{2} \right)$ and $\left(-\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2} \right)$.

Sol. Let θ be the required angle, then

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

$$= \frac{3}{16} + \frac{1}{16} - \frac{3}{4} = -\frac{1}{2}$$

$$\therefore \quad \cos\theta = -\frac{1}{2} \Rightarrow \theta = 120^\circ$$

Example 16.

(i) Find the angle between the lines whose direction ratios are 1, 2, 3 and -3, 2, 1.

(ii) Find the acute angle between two diagonal of a cube.

Sol. (i) Let θ be the required angle, then

$$\cos\theta = \frac{1 \times -3 + 2 \times 2 + 3 \times 1}{\sqrt{1 + 4 + 9}\sqrt{1 + 4 + 9}} = \frac{4}{14} = \frac{2}{7}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{2}{7} \right)$$

(ii) From the figure given below, the direction ratios of the diagonals OP and CD of a given cube are given by

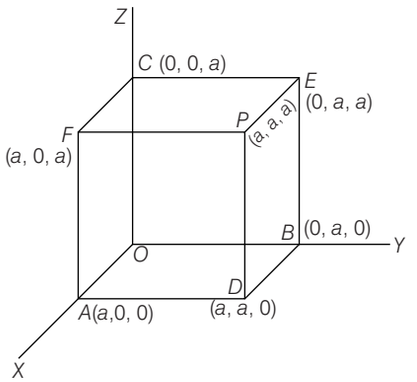
$$a - 0, a - 0, a - 0$$

and $\quad a - 0, a - 0, 0 - a$

and hence their respective direction cosines are

$$\frac{a}{\sqrt{a^2 + a^2 + a^2}}, \frac{a}{\sqrt{a^2 + a^2 + a^2}}, \frac{-a}{\sqrt{a^2 + a^2 + a^2}}$$

i.e. $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$



and $\frac{a}{\sqrt{a^2+a^2+a^2}}, \frac{a}{\sqrt{a^2+a^2+a^2}}, \frac{-a}{\sqrt{a^2+a^2+a^2}}$

i.e. $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$

Let θ be the angle between these diagonals, then

$$\cos\theta = \frac{1}{\sqrt{3}}\times\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}\times\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}\times\frac{-1}{\sqrt{3}} = \frac{1}{3}+\frac{1}{3}-\frac{1}{3} = \frac{1}{3}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{1}{3}\right)$$

Example 17. Find the angle between the lines whose direction cosines are given by $l+m+n=0$ and $2l^2+2m^2-n^2=0$.

Sol. $l^2+m^2+n^2=1$

$$l+m+n=0 \quad \ldots(i)$$

$$2l^2+2m^2-n^2=0 \quad \ldots(ii)$$

$$2(l^2+m^2)-n^2=0$$

$$2(1-n^2)=n^2 \Rightarrow 3n^2=2 \Rightarrow n=\pm\sqrt{\frac{2}{3}} \quad \ldots(iii)$$

$$2(l^2+m^2)=n^2=(-(l+m))^2 \quad \ldots(iv)$$

$$\Rightarrow \quad 2l^2+2m^2=l^2+m^2+2lm$$

$$\Rightarrow \quad l^2+m^2-2lm=0$$

$$\Rightarrow \quad (l-m)^2=0 \Rightarrow l=m$$

$$\Rightarrow \quad l+m=\mp\sqrt{\frac{2}{3}}$$

$$\Rightarrow \quad 2l=\mp\sqrt{\frac{2}{3}}$$

$$\therefore \quad l=\pm\frac{1}{\sqrt{6}}, m=\pm\frac{1}{\sqrt{6}}$$

Direction cosines are $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}\right)$ and $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\sqrt{\frac{2}{3}}\right)$

or $\left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}\right)$ and $\left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\sqrt{\frac{2}{3}}\right)$

The angle between these lines in both the cases is $\cos^{-1}\left(-\frac{1}{3}\right)$.

Example 18. If the direction cosines of a variable line in two adjacent points be l, m, n and $l+\delta l$, $m+\delta m$, $n+\delta n$, show that the small angle $\delta\theta$ between the two positions, is given by $\delta\theta^2=\delta l^2+\delta m^2+\delta n^2$.

Sol. We have, $l^2+m^2+n^2=1$

and $\quad (l+\delta l)^2+(m+\delta m)^2+(n+\delta n)^2=1$

$$\Rightarrow l^2+m^2+n^2+(\delta l)^2+(\delta m)^2+(\delta n)^2+2(l\delta l+m\delta m+n\delta n)=1$$

$$\Rightarrow \quad 1+(\delta l)^2+(\delta m)^2+(\delta n)^2+2(l\delta l+m\delta m+n\delta n)=1$$

$$\Rightarrow \quad (\delta l)^2+(\delta m)^2+(\delta n)^2=-2(l\delta l+m\delta m+n\delta n) \quad \ldots(i)$$

Let $\delta\theta$ be angle between the two positions.

$$\cos\delta\theta = l(l+\delta l)+m(m+\delta m)+n(n+\delta n)$$

$$\Rightarrow \quad 1-2\sin^2\frac{\delta\theta}{2}=1+l\delta l+m\delta m+n\delta n \quad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$(\delta l)^2+(\delta m)^2+(\delta n)^2=4\sin^2\frac{\delta\theta}{2}$$

$$\Rightarrow \quad 4\left(\frac{\delta\theta}{2}\right)^2=l\delta l+m\delta m+n\delta n$$

$$\Rightarrow \quad l\delta l+m\delta m+n\delta n=(\delta\theta)^2,$$

$$\left(\text{since, } \sin\frac{\delta\theta}{2}\to\frac{\delta\theta}{2} \text{ as } \delta\theta \text{ is very small}\right)$$

Example 19. If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two mutually perpendicular lines, shows that the direction cosines of the line perpendicular to both of them are $m_1n_2-m_2n_1$; $n_1l_2-n_2l_1$; $l_1m_2-l_2m_1$.

Sol. Let l, m and n be the direction cosines of the line perpendicular to both the given lines.

$$\therefore \quad ll_1+mm_1+nn_1=0 \quad \text{and} \quad ll_2+mm_2+nn_2=0$$

Solving them, we get $\quad \frac{l}{\begin{matrix}m_1 & n_1\\ m_2 & n_2\end{matrix}} = \frac{m}{\begin{matrix}n_1 & l_1\\ n_2 & l_2\end{matrix}} = \frac{n}{\begin{matrix}l_1 & m_1\\ l_2 & m_2\end{matrix}}$

$$\Rightarrow \quad \frac{l}{m_1n_2-m_2n_1}=\frac{m}{n_1l_2-n_2l_1}=\frac{n}{l_1m_2-l_2m_1}=k$$

$$\therefore \quad l=k(m_1n_2-m_2n_1),\ m=k(n_1l_2-n_2l_1),\ n=k(l_1m_2-l_2m_1)$$

On squaring and adding, we get

$$l^2+m^2+n^2=k^2\{(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2\}+(l_1m_2-l_2m_1)^2$$

$$\Rightarrow \quad 1=k^2\{\sin^2\theta\}$$

where, θ is the angle between the given lines as we know,

$$\sin\theta=\sqrt{(a_1b_2-a_2b_1)^2+(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2}$$

where, a_1, b_1, c_1 and a_2, b_2, c_2 are direction cosines.

$$\Rightarrow \quad 1=k^2\cdot 1 \qquad (\because \theta=90°, \text{given})$$

$$\Rightarrow \quad k=1$$

Hence, direction cosines of a line perpendicular to both of them are $m_1n_2-m_2n_1, n_1l_2-n_1l_2, l_1m_2-l_2m_1$.

Example 20. Find the direction cosines of the line which is perpendicular to the lines with direction cosines proportional to $(1, -2, -2)$ and $(0, 2, 1)$.

Sol. If l, m and n are the direction cosines of the line perpendicular to the given line, then

$$l \cdot (1) + m \cdot (-2) + n \cdot (-2) = 0$$

$$\Rightarrow \quad l - 2m - 2n = 0 \qquad \text{...(i)}$$

and $$l \cdot 0 + m \cdot 2 + n \cdot 1 = 0$$

$$0 + 2m + n = 0 \qquad \text{...(ii)}$$

Then, from Eqs. (i) and (ii) by cross multiplication, we get

$$\frac{l}{2} = \frac{m}{-1} = \frac{n}{2}$$

$$\Rightarrow \quad \frac{l}{2} = \frac{m}{-1} = \frac{n}{2}$$

$$= \frac{\sqrt{l^2 + m^2 + n^2}}{\sqrt{4 + 1 + 4}} = \frac{1}{3} \qquad (\because\ l^2 + m^2 + n^2 = 1)$$

$$\Rightarrow \quad l = \frac{2}{3}, m = -\frac{1}{3},$$

$$n = \frac{2}{3}$$

Example 21. Let $A(-1, 2, 1)$ and $B(4, 3, 5)$ be two given points. Find the projection of AB on a line which makes angle 120° and 135° with Y and Z-axes respectively, and an acute angle with X-axis.

Sol. Let α be an acute angle that the given line make with X-axis. Then, $\cos^2\alpha + \cos^2 120^\circ + \cos^2 135^\circ = 1$

$$\Rightarrow \quad \cos^2\alpha = 1 - \frac{1}{4} - \frac{1}{2} = \frac{4 - 2 - 1}{4} = \frac{1}{4}$$

$$\Rightarrow \quad \cos\alpha = \pm\frac{1}{2} \text{ but } \alpha \text{ is acute}$$

$$\therefore \quad \cos\alpha = +\text{ve}$$

$$\Rightarrow \quad \cos\alpha = \frac{1}{2} = \cos 60^\circ \Rightarrow \alpha = 60^\circ$$

Thus, the direction cosines of the given straight line are $\cos 60^\circ, \cos 120^\circ, \cos 135^\circ$, i.e. $\frac{1}{2}, -\frac{1}{2}, -\frac{1}{\sqrt{2}}$

Hence the projection of AB on the line

$$= \frac{1}{2}(4 + 1) - \frac{1}{2}(3 - 2) - \frac{1}{\sqrt{2}}(5 - 1) = \frac{5}{2} - \frac{1}{2} - 2\sqrt{2}$$

$$= (2 - 2\sqrt{2}) \text{ units}$$

Exercise for Session 1

1. In how many disjoint parts does the three dimensional rectangular cartesian coordinate system divide the space.
2. Find the distance between the points $(k, k + 1, k + 2)$ and $(0, 1, 2)$.
3. Show that the points $(1, 2, 3)$, $(-1, -2, -1)$, $(2, 3, 2)$ and $(4, 7, 6)$ are the vertices of a parallelogram.
4. If the mid-points of the sides of a triangle are $(1, 5, -1)$, $(0, 4, -2)$ and $(2, 3, 4)$. Find its vertices.
5. Find the maximum distance between the points $(3\sin\theta, 0, 0)$ and $(4\cos\theta, 0, 0)$.
6. If $A = (1, 2, 3)$, $B = (4, 5, 6)$, $C = (7, 8, 9)$ and D, E, F are the mid-points of the triangle ABC, then find the centroid of the triangle DEF.
7. A line makes angles α, β and γ with the coordinate axes. If $\alpha + \beta = 90^\circ$, then find γ.
8. If α, β and γ are angles made by a line with positive direction of X-axis, Y-axis and Z-axis respectively, then find the value of $\cos 2\alpha + \cos 2\beta + \cos 2\gamma$.
9. If $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosine of a line, then find the value of $\cos^2\alpha + (\cos\beta + \sin\gamma)(\cos\beta - \sin^2\gamma)$.
10. A line makes angles α, β, γ, δ with the four diagonals of a cube, then prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$.
11. Find the direction cosine of line which is perpendicular to the lines with direction ratio $[1, -2, -2]$ and $[0, 2, 1]$.
12. The projection of a line segment on the axis 1, 2, 3 respectively. Then find the length of line segment.

Session 2

Equation of a Straight Line in Space, Angle between Two Lines, Perpendicular Distance of a Point from a Line, Shortest Distance between Two Lines

Equation of a Straight Line in Space

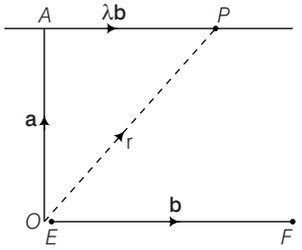
A straight line in space is specified basically in two ways

(i) A line passing through a given point and parallel to a given vector.

(ii) A line passing through any two given points.

1. Vector Equation of a Line Passing Through a Given Point and Parallel to a Given Vector

To find the vector equation of a straight line which passes through a given point and is parallel to a given vector.



Let A be the given point and let EF be the given line, then through A draw AP parallel to given line EF.

Let $\mathbf{b}$ any vector parallel to the given line. Take any point O as the origin of reference. Let a the position vector of the given point A.

Let P be any point on the AP and let its position vector be $\mathbf{r}$. Then, we have

$$\mathbf{r} = \mathbf{OP} = \mathbf{OA} + \mathbf{AP} = \mathbf{a} + \lambda\mathbf{b} \qquad (\text{where, } \mathbf{AP} = \lambda\mathbf{b})$$

Hence, the vector equation of straight line

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} \qquad \ldots(\text{i})$$

Remarks

1. Here, $\mathbf{r}$ is the position vector of any point $P(x, y, z)$ on the line

$\therefore \quad \mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$.

2. In particular, the equation of the straight line through origin and parallel to $\mathbf{b}$ is $\mathbf{r} = \lambda\mathbf{b}$.

2. Cartesian Equation of a Line Passing Through a Given Point and Given Direction Ratios

Let the coordinates of the given point A be (x_1, y_1, z_1) and the direction ratios of the line be a, b and c. Consider the coordinate of any point P be (x, y, z). Then,

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}};$$

$$\mathbf{a} = x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}} + z_1\hat{\mathbf{k}}$$

and $$\mathbf{b} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$$

Substituting these values in (i) and equating the coefficients of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$, we get

$$x = x_1 + \lambda a$$
$$y = y_1 + \lambda b$$
$$z = z_1 + \lambda c$$

These are parametric equations of the line.

Eliminating the parameter λ, we get

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

Remarks

1. Parametric equation of straight line

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \lambda$$

$\Rightarrow \quad x = x_1 + \lambda a,\ y = y_1 + \lambda b,\ z = z_1 + \lambda c$

(where, λ being parameter)

2. Since, X, Y and Z-axes pass through origin and have direction cosines (1, 0, 0), (0, 1, 0) and (0, 0, 1).

$\therefore$ Their equations are

Equation of X-axis, $\frac{x - 0}{1} = \frac{y - 0}{0} = \frac{z - 0}{0}$

$\Rightarrow \quad y = 0$ and $z = 0$

Equation of Y-axis, $\frac{x - 0}{0} = \frac{y - 0}{1} = \frac{z - 0}{0}$

$\Rightarrow \quad x = 0$ and $z = 0$

Equation of Z-axis, $\frac{x - 0}{0} = \frac{y - 0}{0} = \frac{z - 0}{1}$

$\Rightarrow \quad x = 0$ and $y = 0$

Example 22. Find the equation of straight line parallel to $2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and passing through the point $(5, -2, 4)$.

Sol. **Vector form** Let $P = (5, -2, 4)$, then $\mathbf{OP} = 5\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}} = \mathbf{a}$

Also, $\mathbf{b} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$

So, equation of straight line passing through $\mathbf{a}$ and parallel to straight line whose direction ratios are $\mathbf{b}$ is given as

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$$

$$\Rightarrow \quad \mathbf{r} = (5\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}) + \lambda(2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}})$$

Cartesian form Here, $(x_1, y_1, z_1) = (5, -2, 4)$ and parallel to straight line whose DR's are $(2, -1, 3)$, so equation of the straight line is $\frac{x-5}{2} = \frac{y+2}{-1} = \frac{z-4}{3}$.

Example 23. Find the vector equation of a line passing through $(2, -1, 1)$ and parallel to the line whose equation is $\frac{x-3}{2} = \frac{y+1}{7} = \frac{z-2}{-3}$

Sol. Since, the required line is parallel to

$$\frac{x-3}{2} = \frac{y+1}{7} = \frac{z-2}{-3}$$

it follows that the required line passing through $A(2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}})$ has the direction of $2\hat{\mathbf{i}} + 7\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$. Hence, the vector equation of the required line is $\mathbf{r} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}} + \lambda(2\hat{\mathbf{i}} + 7\hat{\mathbf{j}} - 3\hat{\mathbf{k}})$ where λ is a parameter.

Example 24. The cartesian equation of a line are $6x - 2 = 3y + 1 = 2z - 2$. Find its direction ratios and also find the vector equation of the line.

Sol. We know that,

$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$ is cartesian equation of straight line.

$$\therefore \quad 6x - 2 = 3y + 1 = 2z - 2$$

$$\Rightarrow \quad 6\left(x - \frac{1}{3}\right) = 3\left(y + \frac{1}{3}\right) = 2(z - 1)$$

$$\Rightarrow \quad \frac{x - \frac{1}{3}}{\frac{1}{6}} = \frac{y + \frac{1}{3}}{\frac{1}{3}} = \frac{z-1}{\frac{1}{2}}$$

$$\Rightarrow \quad \frac{x - \frac{1}{3}}{1} = \frac{y + \frac{1}{3}}{2} = \frac{z-1}{3}$$

which shows given line passes through $\left(\frac{1}{3}, -\frac{1}{3}, 1\right)$ and has direction ratios $(1, 2, 3)$.

$\therefore$ Its vector equation is

$$\mathbf{r} = \left(\frac{1}{3}\hat{\mathbf{i}} - \frac{1}{3}\hat{\mathbf{j}} + \hat{\mathbf{k}}\right) + \lambda(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$$

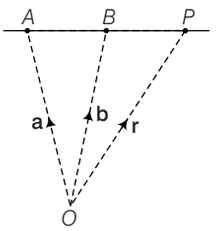
3. Vector Equation of a Line Passing Through Two Given Points

The vector equation of a line passing through two points whose position vectors $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{r} = a + \lambda(\mathbf{b} - \mathbf{a})$$

Let O be the origin and A and B be the given points with position vectors $\mathbf{a}$ and $\mathbf{b}$, respectively.

Then, $\mathbf{OP} = \mathbf{r}$, $\mathbf{OA} = \mathbf{a}$ and $\mathbf{OB} = \mathbf{b}$



Since, $\mathbf{AP}$ is collinear with $\mathbf{AB}$.

$$\therefore \quad \mathbf{AP} = \lambda\ \mathbf{AB} \text{ for some scalar } \lambda$$

$$\Rightarrow \quad \mathbf{OP} - \mathbf{OA} = \lambda(\mathbf{OB} - \mathbf{OA})$$

$$\Rightarrow \quad \mathbf{r} - \mathbf{a} = \lambda(\mathbf{b} - \mathbf{a})$$

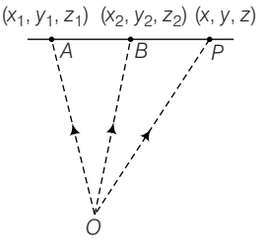
$$\Rightarrow \quad \mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$$

$\therefore$ Equation of straight line passing through $\mathbf{a}$ and $\mathbf{b}$.

$$\Rightarrow \quad \mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$$

4. Cartesian Equation of a Line Passing Through Two Given Points

Equation of straight line passing through (x_1, y_1, z_1), (x_2, y_2, z_2).



The direction ratios of

$$AB = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

The direction ratios of

$$AP = (x - x_1, y - y_1, z - z_1)$$

Since, they are proportional

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$$

$$= \frac{z - z_1}{z_2 - z_1}$$

Example 25. Find the vector equation of line passing through $A(3,4,-7)$ and $B(1,-1,6)$. Also, find its cartesian equations.

Sol. Since, the line passes through $A(3\hat{\mathbf{i}}+4\hat{\mathbf{j}}-7\hat{\mathbf{k}})$ and $B(\hat{\mathbf{i}}-\hat{\mathbf{j}}+6\hat{\mathbf{k}})$, its vector equation is

$$\mathbf{r}=3\hat{\mathbf{i}}+4\hat{\mathbf{j}}-7\hat{\mathbf{k}}+\lambda[(\hat{\mathbf{i}}-\hat{\mathbf{j}}+6\hat{\mathbf{k}})-(3\hat{\mathbf{i}}+4\hat{\mathbf{j}}-7\hat{\mathbf{k}})]$$

or $\mathbf{r}=3\hat{\mathbf{i}}+4\hat{\mathbf{j}}-7\hat{\mathbf{k}}-\lambda(2\hat{\mathbf{i}}+5\hat{\mathbf{j}}-13\hat{\mathbf{k}})$ ...(i)

where λ is a parameter.

The cartesian equivalent of (i) is $\frac{x-3}{2}=\frac{y-4}{5}=\frac{z+7}{-13}$.

Example 26. Find the equation of a line which passes through the point $(2,3,4)$ and which has equal intercepts on the axes.

Sol. Since, lines has equal intercepts on axes, it is equally inclined to axes.

$\Rightarrow$ line is along the vector $a(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})$

$\Rightarrow$ Equation of line is $\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{1}$

Angle between Two Lines

Vector Form

Let $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$ ...(i)

and $\mathbf{r}=\mathbf{a}'+\mu\mathbf{b}'$ ...(ii)

be two straight line in space.

Clearly, Eqs. (i) and (ii) are straight line in the directions of $\mathbf{b}$ and $\mathbf{b}'$, respectively.

Let θ be the between the straight lines (i) and (ii).

Then, θ is the angle between the vectors $\mathbf{b}$ and $\mathbf{b}'$ also

$$\mathbf{b}\cdot\mathbf{b}'=|\mathbf{b}||\mathbf{b}'|\cos\theta$$

$$\Rightarrow \quad \cos\theta=\frac{\mathbf{b}\cdot\mathbf{b}'}{|\mathbf{b}||\mathbf{b}'|}$$

Cartesian Form

Let $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ ...(i)

and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_1}=\frac{z-z_2}{c_2}$ ...(ii)

be two straight lines. Then, $\mathbf{b}=a_1\hat{\mathbf{i}}+b_1\hat{\mathbf{j}}+c_1\hat{\mathbf{k}}$

$$\mathbf{b}'=a_2\hat{\mathbf{i}}+b_2\hat{\mathbf{j}}+c_2\hat{\mathbf{k}}$$

So, that $\mathbf{b}\cdot\mathbf{b}'=a_1a_2+b_1b_2+c_1c_2$

and $|\mathbf{b}|=\sqrt{a_1^2+b_1^2+c_1^2}\ ;\ |\mathbf{b}'|=\sqrt{a_2^2+b_2^2+c_2^2}$

$$\therefore \quad \cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

Condition for Perpendicularity

The lines are perpendicular, then

$$\mathbf{b}\cdot\mathbf{b}'=0$$

$$\Rightarrow \quad a_1a_2+b_1b_2+c_1c_2=0$$

Condition for Parallelism

The lines are parallel, then $\mathbf{b}=(\mathbf{b}')\,\lambda$, for some scalar λ

$$\Rightarrow \quad \frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$$

Example 27. Find the angle between the pair of lines

$$\mathbf{r}=3\hat{\mathbf{i}}+2\hat{\mathbf{j}}-4\hat{\mathbf{k}}+\lambda\,(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\hat{\mathbf{k}})$$

and $\mathbf{r}=5\,\hat{\mathbf{i}}-2\hat{\mathbf{k}}+\mu\,(3\hat{\mathbf{i}}+2\hat{\mathbf{j}}+6\hat{\mathbf{k}})$

Sol. Given line are

$$\mathbf{r}=(3\hat{\mathbf{i}}+2\hat{\mathbf{j}}-4\hat{\mathbf{k}})+\lambda(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\hat{\mathbf{k}})$$

and $\mathbf{r}=(5\hat{\mathbf{i}}-2\hat{\mathbf{k}})+\mu(3\hat{\mathbf{i}}+2\hat{\mathbf{j}}+6\hat{\mathbf{k}})$

We know that, angle between $\mathbf{r}=\mathbf{a}_1+\lambda\mathbf{b}_1$ and $\mathbf{r}=\mathbf{a}_2+\mu\mathbf{b}_2$,

$$\cos\theta=\frac{\mathbf{b}_1\cdot\mathbf{b}_2}{|\mathbf{b}_1||\mathbf{b}_2|}$$

$$\therefore \quad \cos\theta=\frac{(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\hat{\mathbf{k}})\cdot(3\hat{\mathbf{i}}+2\hat{\mathbf{j}}+6\hat{\mathbf{k}})}{\sqrt{1^2+2^2+2^2}\sqrt{3^2+2^2+6^2}}=\frac{3+4+12}{\sqrt{9}\cdot\sqrt{49}}=\frac{19}{21}$$

$$\therefore \quad \theta=\cos^{-1}\left(\frac{19}{21}\right)$$

Example 28. Prove that the line $x=ay+b,\ z=cy+d$ and $x=a'y+b',\ z=dy+d'$ are perpendicular, if $aa'+cc'+1=0$.

Sol. We can write the equations of straight line as

$$\frac{x-b'}{a'}=y,\ y=\frac{z-d'}{c'}$$

$$\Rightarrow \quad \frac{x-b'}{a'}=\frac{y-0}{1}=\frac{z-d'}{c'} \qquad \text{...(i)}$$

and $\frac{x-b}{a}=y,\ y=\frac{z-d}{c}$

$$\Rightarrow \quad \frac{x-b}{a}=\frac{y-0}{1}=\frac{z-d_2}{c} \qquad \text{...(ii)}$$

We know that, $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$

and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$

are perpendicular, if $a_1a_2+b_1b_2+c_1c_2=0$

$\therefore$ For the straight lines given by Eqs. (i) and (ii), to be perpendicular.

$$a'a+1\cdot1+c'c=0$$

$$\Rightarrow \quad aa'+cc'+1=0$$

Perpendicular Distance of a Point from a Line

1. Foot of Perpendicular from a Point on the Given Line

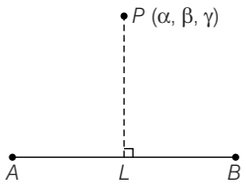
(i) **Cartesian Form** Here, the equation of line AB is

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$$

Let L be the foot of the perpendicular drawn from $P(\alpha,\beta,\gamma)$ on the line $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$.

Let the coordinates of L be $(x_1+a\lambda, y_1+b\lambda, z_1+c\lambda)$.

Then, the direction ratios of PL are $(x_1+a\lambda-\alpha, y_1+b\lambda-\beta, z_1+c\lambda-\gamma)$.



Direction ratios of AB are (a,b,c).

Since PL is perpendicular to AB.

$$a(x_1+a\lambda-\alpha)+b(y_1+b\lambda-\beta)+c(z_1+c\lambda-\gamma)=0$$

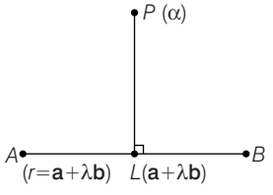
$$\lambda=\frac{a(\alpha-x_1)+b(\beta-y_1)+c(\gamma+z_1)}{a^2+b^2+c^2}$$

Putting the value of λ in $(x_1+a\lambda, y_1+b\lambda, z_1+c\lambda)$, we get the foot of the perpendicular. Now, we can get distance PL using distance formula.

(ii) **Vector Form** Let L be the foot of the perpendicular drawn from $P(\alpha)$ on the line $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$.

Since, $\mathbf{r}$ denotes the position vector of any point on the line $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$, the position vector of L will be $(\mathbf{a}+\lambda\mathbf{b})$

Directions ratios of $PL=\mathbf{a}-\alpha+\lambda\mathbf{b}$



Since, **PL** is perpendicular to **b**,

$$(\mathbf{a}-\alpha+\lambda\mathbf{b})\cdot\mathbf{b}=0$$

$$\Rightarrow\quad (\mathbf{a}-\alpha)\cdot\mathbf{b}+\lambda\mathbf{b}\cdot\mathbf{b}=0$$

$$\Rightarrow\quad \lambda=\frac{-(\mathbf{a}-\alpha)\cdot\mathbf{b}}{|\mathbf{b}|^2}$$

$\Rightarrow$ Position vector of L is $\mathbf{a}-\left(\frac{(\mathbf{a}-\alpha)\cdot\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b}$, which is the foot of the perpendicular.

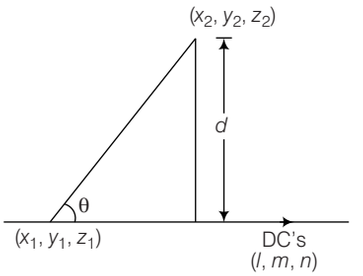
(iii) The distance of the point (x_2,y_2,z_2) from the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$, (where l,m and n are direction cosines of the line), is

$$[(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2-\{l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)\}^2]^{1/2}$$

Let $\mathbf{r}_1=(x_2-x_1)\hat{\mathbf{i}}+(y_2-y_1)\hat{\mathbf{j}}+(z_2-z_1)\hat{\mathbf{k}}$

$\mathbf{r}_2=l\hat{\mathbf{i}}+m\hat{\mathbf{j}}+n\hat{\mathbf{k}}$

$$\therefore\quad \cos\theta=\frac{\mathbf{r}_2\cdot\mathbf{r}_1}{|\mathbf{r}_2|\cdot|\mathbf{r}_1|}$$



Also, $d=|\mathbf{r}_1|\sin\theta$

$$d^2=|\mathbf{r}_1|^2\sin^2\theta$$

$$=|\mathbf{r}_1|^2(1-\cos^2\theta)$$

$$=|\mathbf{r}_1|^2\left(1-\frac{(\mathbf{r}_1\cdot\mathbf{r}_2)^2}{|\mathbf{r}_1|^2\cdot|\mathbf{r}_2|^2}\right)$$

$$d^2=|\mathbf{r}_1|^2-(\mathbf{r}_1\cdot\mathbf{r}_2)^2 \qquad (\text{where}, |\mathbf{r}_2|=1)$$

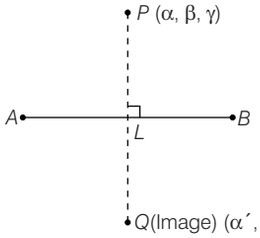
$$\Rightarrow\quad d=\sqrt{|\mathbf{r}_1|^2-(\mathbf{r}_1\cdot\mathbf{r}_2)^2}$$

$$d=\sqrt{\{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2-\{l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)\}^2}$$

2. Reflection or Image of a Point in a Straight Line

(i) **Cartesian Form** To find the reflection or image of a point in a straight line in cartesian form.

Let $P(\alpha,\beta,\gamma)$ be the point and $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$ be the given line.



Let L be the foot of perpendicular from P to AB and let Q be the image of the point in the given line, where $PL = LQ$.

Let the coordinates of L be

$$(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$$

Then, direction ratios of PL are

$$(x_1 + a\lambda - \alpha, y_1 + b\lambda - \beta, z_1 + c\lambda - \gamma)$$

Since, PL is perpendicular to the given line, whose direction ratios are a, b and c.

$$\therefore (x_1 + a\lambda - \alpha)\cdot a + (y_1 + b\lambda - \beta)\cdot b + (z_1 + c\lambda - \gamma)\cdot c = 0$$

$$\Rightarrow \qquad \lambda = \frac{\{a(\alpha - x_1) + b(\beta - y_1) + c(\gamma - z_1)}{a^2 + b^2 + c^2}$$

Substituting λ, we get L, (foot of perpendicular)

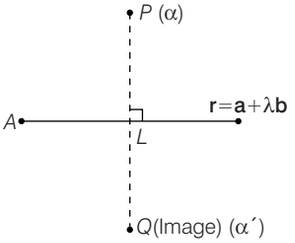
Let coordinates of $Q(\alpha', \beta', \gamma')$ be image.

$\therefore$ Mid-point of PQ is L.

$$\therefore \frac{\alpha + \alpha'}{2} = x_1 + a\lambda, \frac{\beta + \beta'}{2} = y_1 + b\lambda, \frac{\gamma + \gamma'}{2} = z_1 + c\lambda$$

$$\therefore \qquad \alpha' = 2(x_1 + a\lambda) - \alpha, \beta' = 2(y_1 + b\lambda) - \beta,$$
$$\gamma' = 2(z_1 + c\lambda) - \gamma$$

(ii) **Vector Form** To find the reflection or image of a point in a straight line in vector form. Let $P(\alpha)$ be the given point and $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ be the given line.



Let Q be the image of P in $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$

$$\therefore \qquad \mathbf{PL} = \mathbf{a} + \lambda\mathbf{b} - \alpha$$

Since, $\mathbf{PL}$ is perpendicular to the given line,

$$\therefore \qquad \mathbf{PL} \perp \mathbf{b}$$

$$\Rightarrow \qquad \mathbf{PL}\cdot\mathbf{b} = 0$$

$$\Rightarrow \qquad (\mathbf{a} + \lambda\mathbf{b} - \alpha)\cdot\mathbf{b} = 0$$

$$\Rightarrow \qquad \lambda = -\frac{[(\mathbf{a} - \alpha)\cdot\mathbf{b}]}{|\mathbf{b}|^2}$$

$\therefore$ Position vector of L,

$$\mathbf{a} + \lambda\mathbf{b} = \mathbf{a} - \left(\frac{(\mathbf{a} - \alpha)\cdot\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b}$$

Let Q be the image of point P and α' be the position vector.

Since, L is mid-point of PQ.

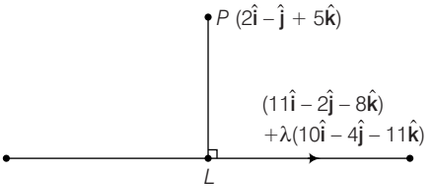
$$\Rightarrow \qquad \frac{\alpha + \alpha'}{2} = \mathbf{a} - \left(\frac{(\mathbf{a} - \alpha)\cdot\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b}$$

$$\Rightarrow \qquad \alpha' = 2\mathbf{a} - \left(\frac{2(\mathbf{a} - \alpha)\cdot\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b} - \alpha$$

which is image of P on $\mathbf{r}$.

Example 29. Find the foot of perpendicular drawn from the point $2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 5\hat{\mathbf{k}}$ to the line $\mathbf{r} = (11\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 8\hat{\mathbf{k}}) + \lambda(10\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - 11\hat{\mathbf{k}})$. Also, find the length of the perpendicular.

Sol. Let L be the foot of the perpendicular drawn from $P(2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 5\hat{\mathbf{k}})$ on the line



$$\mathbf{r} = (11\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 8\hat{\mathbf{k}}) + \lambda(10\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - 11\hat{\mathbf{k}})$$

Let the position vector of L is

$$(11\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 8\hat{\mathbf{k}}) + \lambda(10\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - 11\hat{\mathbf{k}})$$
$$= (11 + 10\lambda)\hat{\mathbf{i}} + (-2 - 4\lambda)\hat{\mathbf{j}} + (-8 - 11\lambda)\hat{\mathbf{k}}$$

$\therefore \mathbf{PL}$ = Position vector of L − Position vector of P

$$= (9 + 10\lambda)\hat{\mathbf{i}} + (-1 - 4\lambda)\hat{\mathbf{j}} + (-13 - 11\lambda)\hat{\mathbf{k}}$$

Since, PL is perpendicular to the given line and parallel to

$$\mathbf{b} = 10\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - 11\hat{\mathbf{k}}. \quad \Rightarrow \quad \mathbf{PL}\cdot\mathbf{b} = 0$$

$$\Rightarrow \{(9 + 10\lambda)\hat{\mathbf{i}} + (-1 - 4\lambda)\hat{\mathbf{j}} + (-13 - 11\lambda)\hat{\mathbf{k}}\}\cdot(10\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - 11\hat{\mathbf{k}}) = 0$$

$$\Rightarrow 10(9 + 10\lambda) - 4(-1 - 4\lambda) - 11(-13 - 11\lambda) = 0$$

$$\Rightarrow \qquad \lambda = -1$$

On putting $\lambda = -1$, we get L as $(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$

Now, $\mathbf{PL} = (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) - (2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 5\hat{\mathbf{k}})$

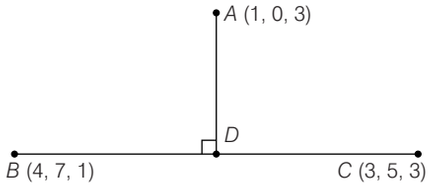
$$= (-\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$$

Hence, the length of perpendicular from P on the given line

$$= |\mathbf{PL}| = \sqrt{1 + 9 + 4} = \sqrt{14}$$

Example 30. Find the coordinates of the foot of the perpendicular drawn from point $A(1, 0, 3)$ to the join of points $B(4, 7, 1)$ and $C(3, 5, 3)$.

Sol. Let D be the foot of the perpendicular and let it divide BC in the ratio $\lambda : 1$. Then, the coordinates of D are $\frac{3\lambda + 4}{\lambda + 1}$, $\frac{5\lambda + 7}{\lambda + 1}$ and $\frac{3\lambda + 1}{\lambda + 1}$.



Now, $\mathbf{AD} \perp \mathbf{BC} \Rightarrow \mathbf{AD} \cdot \mathbf{BC} = 0$

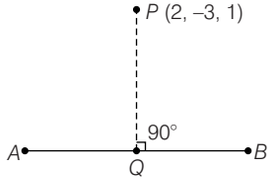
$\Rightarrow \quad (2\lambda + 3) + 2(5\lambda + 7) + 4 = 0 \Rightarrow \lambda = -\frac{7}{4}$

$\Rightarrow$ Coordinates of D are $\frac{5}{3}, \frac{7}{3}$ and $\frac{17}{3}$.

Example 31. Find the length of perpendicular from $P(2, -3, 1)$ to the line $\frac{x+1}{2} = \frac{y-3}{3} = \frac{z+2}{-1}$

Sol. Given line is $\frac{x+1}{2} = \frac{y-3}{3} = \frac{z+2}{-1} = r$ ...(i)

and $P(2, -3, 1)$



Coordinates of any point on line (i) may be taken as

$(2r - 1, 3r + 3, -r - 2)$

Let $Q = (2r - 1, 3r + 3, -r - 2)$

Direction ratio's of PQ are $(2r - 3, 3r + 6, -r - 3)$.

Direction ratio's of AB are $(2, 3, -1)$.

Since, $PQ \perp AB$

$2(2r - 3) + 3(3r + 6) - 1(-r - 3) = 0$

$\Rightarrow \quad r = -\frac{15}{14}$

$\therefore \quad Q = \left(-\frac{22}{7}, -\frac{3}{14}, -\frac{13}{14}\right)$

$$PQ^2 = \left(2 + \frac{22}{7}\right)^2 + \left(-3 + \frac{3}{14}\right)^2 + \left(1 + \frac{13}{14}\right)^2 = \frac{531}{14}$$

$$PQ = \sqrt{\frac{531}{14}} \text{ units}$$

Example 32. ***Find the length of the perpendicular drawn from point (2, 3, 4) to line*** $\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$.

Sol. Let P be the foot of the perpendicular from $A(2, 3, 4)$ to the given line l whose equation is

$$\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$$

or $\quad \frac{x-4}{-2} = \frac{y}{6} = \frac{z-1}{-3} = k$ (say). ...(i)

Therefore, $x = 4 - 2k, y = 6k, z = 1 - 3k$

As P lies on (i), coordinates of P are $(4 - 2k, 6k, 1 - 3k)$ for some value of k.

The direction ratios of AP are

$(4 - 2k - 2, 6k - 3, 1 - 3k - 4)$

or $\quad (2 - 2k, 6k - 3, -3 - 3k)$.

Also, the direction ratios of l are -2, 6 and -3.

Since, $AP \perp l$

$\Rightarrow \quad -2(2 - 2k) + 6(6k - 3) - 3(-3 - 3k) = 0$

$\Rightarrow \quad -4 + 4k + 36k - 18 + 9 + 9k = 0$

or $\quad 49k - 13 = 0$ or $k = \frac{13}{49}$

We have, $AP^2 = (4 - 2k - 2)^2 + (6k - 3)^2 + (1 - 3k - 4)^2$

$= (2 - 2k)^2 + (6k - 3)^2 + (-3 - 3k)^2$

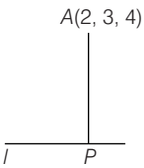
$= 4 - 8k + 4k^2 + 36k^2 - 36k + 9 + 9 + 18k + 9k^2$

$= 22 - 26k + 49k^2$

$= 22 - 26\left(\frac{13}{49}\right) + 49\left(\frac{13}{49}\right)^2$

$= \frac{22 \times 49 - 26 + 13 + 13^2}{49} = \frac{909}{49}$

$AP = \frac{3}{7}\sqrt{101}$



Aliter

We know that the distance of the point (x_2, y_2, z_2) from the line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - (l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1))^2}$$

Here, (x_2, y_2, z_2) are $(2, 3, 4)$ and (x_1, y_1, z_1) are $(4, 0, 1)$ and $(l, m, n) = \left(\frac{-2}{7}, \frac{6}{7}, \frac{-3}{7}\right)$.

$$\therefore \quad d = \sqrt{(2-4)^2+(3-0)^2+(4-1)^2 - \left[\frac{-2}{7}(2-4)+\frac{6}{7}(3-0)-\frac{3}{7}(4-1)\right]^2}$$

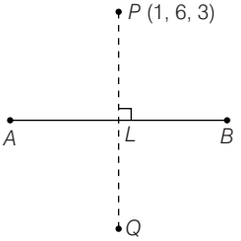
$$= \sqrt{4+9+9-\left(\frac{4+18-9}{7}\right)^2}$$

$$= \sqrt{22-\frac{169}{49}} = \sqrt{\frac{1078-169}{49}}$$

$$= \frac{\sqrt{909}}{7} = \frac{3}{7}\sqrt{101}$$

Example 33. Find the image of the point (1, 6, 3) in the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$.

Sol. Let P be the given point and let L be the foot of perpendicular from P to the given line.



The coordinates of a general point on the given line are given by

$$\frac{x-0}{1} = \frac{y-1}{2} = \frac{z-2}{3} = \lambda$$

i.e. $\quad x = \lambda,\ y = 2\lambda + 1,\ z = 3\lambda + 2$

Let the coordinates of L be

$$(\lambda, 2\lambda + 1, 3\lambda + 2) \qquad \text{...(i)}$$

So, direction ratios of PL are

$$(\lambda - 1, 2\lambda - 5, 3\lambda - 1)$$

Direction ratios of the given line are (1, 2, 3) which is perpendicular to PL

$$(\lambda - 1)\cdot 1 + (2\lambda - 5)\cdot 2 + (3\lambda - 1)\cdot 3 = 0$$

$$\Rightarrow \quad \lambda = 1$$

So, coordinates of L are (1, 3, 5).

Let $Q(x_1, y_1, z_1)$ be the image of $P(1, 6, 3)$ on given line.

Since, L is mid-point of PQ.

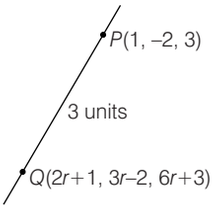
$$\therefore \quad 1 = \frac{x_1+1}{2}, 3 = \frac{y_1+6}{2}, 5 = \frac{z_1+3}{2}$$

$$\Rightarrow \quad x_1 = 1,\ y_1 = 0,\ z_1 = 7$$

$\therefore$ Image of $P(1, 6, 3)$ in the given line is (1, 0, 7).

Example 34. Find the coordinates of those points on the line $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{6}$ which are at a distance of 3 units from points $(1, -2, 3)$.

Sol. Here, $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{6}$,



is the given straight line ...(i)

Let $P = (1, -2, 3)$ on the straight line.

Here, direction ratios of line (i) are (2, 3, 6).

$\therefore$ Direction cosines of line (i) are $\frac{2}{7}, \frac{3}{7}, \frac{6}{7}$.

Equation of line (i) may be written as

$$\frac{x-1}{2/7} = \frac{y+2}{3/7} = \frac{z-3}{6/7} \qquad \text{...(ii)}$$

Coordinates of any point on the line (ii) may be taken as

$$\left(\frac{2}{7}r + 1, \frac{3}{7}r - 2, \frac{6}{7}r + 3\right)$$

Let $Q\left(\frac{2}{7}r + 1, \frac{3}{7}r - 2, \frac{6}{7}r + 3\right)$

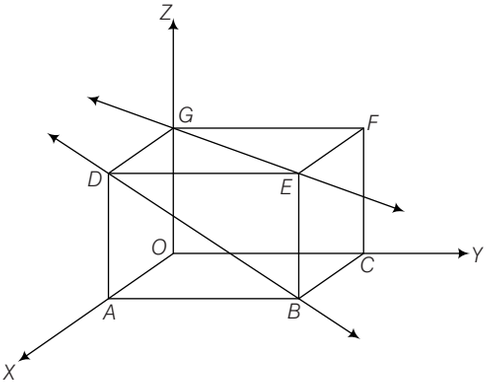
Given, $\quad |\mathbf{r}| = 3$

$\therefore \quad r = \pm 3$

Putting the value of r, we have $Q = \left(\frac{13}{7}, \frac{-5}{7}, \frac{39}{7}\right)$ or $\left(\frac{1}{7}, \frac{-23}{7}, \frac{3}{7}\right)$

Shortest Distance between Two Lines

If two lines in space intersect at a point, then the shortest distance between them is zero. Also, if two lines in space are parallel, then the shortest distance between them will be the perpendicular distance, i.e. the length of the perpendicular drawn from any point on one line onto the other line. Further, in a space, there are lines which are neither intersecting nor parallel. In fact, such pair of lines are non-coplanar and are called skew lines.



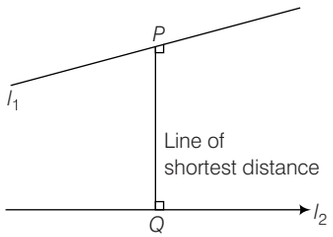
Line *GE* goes diagonally across the ceiling and line *DB* passes through one corner of the ceiling directly above *A* and goes diagonally down the wall. These lines are skew because they are not parallel and also never meet.

By the shortest distance between two lines, we mean the join of a point in one line with one point on the other line so that the length of the segment so obtained is the smallest.

1. Shortest Distance between Two Skew Straight Lines

Line of Shortest Distance

If l_1 and l_2 are two skew lines, then there is one and only one line perpendicular to each of lines l_1 and l_2 which is known as the line of shortest distance.



Here, distance *PQ* is called to be shortest distance.

Vector Form

Let l_1 and l_2 be two lines whose equations are

$\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{b}_2$, respectively.

Clearly, l_1 and l_2 pass through the points A and B with $\mathbf{a}_1$ and $\mathbf{a}_2$, respectively and are parallel to the vectors $\mathbf{b}_1$ and $\mathbf{b}_2$, respectively.

Since, $\mathbf{PQ}$ is perpendicular to both l_1 and l_2 which are parallel to $\mathbf{b}_1$ and $\mathbf{b}_2$.

$\therefore$ $\mathbf{PQ}$ is parallel to $\mathbf{b}_1 \times \mathbf{b}_2$.

Let $\hat{\mathbf{n}}$ be a unit vector along $\mathbf{PQ}$, then $\hat{\mathbf{n}} = \pm\dfrac{\mathbf{b}_1 \times \mathbf{b}_2}{|\mathbf{b}_1 \times \mathbf{b}_2|}$

$\therefore$ $PQ =$ Projection of $\mathbf{AB}$ on $\mathbf{PQ}$

$\Rightarrow$ $\mathbf{PQ} = \mathbf{AB} \cdot \hat{\mathbf{n}}$

$$= \pm(\mathbf{a}_2 - \mathbf{a}_1) \cdot \frac{(\mathbf{b}_1 \times \mathbf{b}_2)}{|\mathbf{b}_1 \times \mathbf{b}_2|}$$

$$= \pm\frac{(\mathbf{b}_1 \times \mathbf{b}_2)(\mathbf{a}_2 - \mathbf{a}_1)}{|\mathbf{b}_1 \times \mathbf{b}_2|}$$

Hence, distance $PQ = \left|\dfrac{(\mathbf{b}_1 \times \mathbf{b}_2) \cdot (\mathbf{a}_2 - \mathbf{a}_1)}{|\mathbf{b}_1 \times \mathbf{b}_2|}\right|$

$$= \frac{[\mathbf{b}_1 \mathbf{b}_2 (\mathbf{a}_2 - \mathbf{a}_1)]}{|\mathbf{b}_1 \times \mathbf{b}_2|}$$

Condition for Lines to Intersecting

The two lines are intersecting, if

$$\left|\frac{(\mathbf{b}_1 \times \mathbf{b}_2) \cdot (\mathbf{a}_2 - \mathbf{a}_1)}{|\mathbf{b}_1 \times \mathbf{b}_2|}\right| = 0$$

$\Rightarrow$ $(\mathbf{b}_1 \times \mathbf{b}_2) \cdot (\mathbf{a}_2 - \mathbf{a}_1) = 0$

$\Rightarrow$ $[\mathbf{b}_1 \mathbf{b}_2 (\mathbf{a}_2 - \mathbf{a}_1)] = 0$

Cartesian Form

Let the two skew lines be

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$$

and $$\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$$

Vector equations for these two lines are

$$\mathbf{r} = (x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}} + z_1\hat{\mathbf{k}}) + \lambda(a_1\hat{\mathbf{i}} + b_1\hat{\mathbf{j}} + c_1\hat{\mathbf{k}})$$

and $$\mathbf{r} = (x_2\hat{\mathbf{i}} + y_2\hat{\mathbf{j}} + z_2\hat{\mathbf{k}}) + \mu(a_2\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + c_2\hat{\mathbf{k}})$$

$\because$ Shortest distance $n = \left|\dfrac{(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{b}_1 \times \mathbf{b}_2)}{|\mathbf{b}_1 \times \mathbf{b}_2|}\right|$

$\Rightarrow$ $$d = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - a_1c_2)^2 + (a_1b_2 - a_2b_1)^2}}$$

Conditions for Lines to Intersect

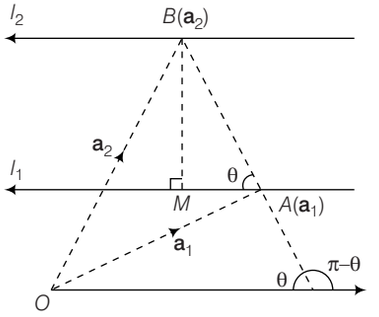
The lines are intersecting, if shortest distance $= 0$

$\Rightarrow$ $$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

2. Shortest Distance between Parallel Lines

Let l_1 and l_2 be two parallel lines whose equations are

$\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}$ or $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{b}$, respectively.



Clearly, l_1 and l_2 pass through the points A and B with position vectors $\mathbf{a}_1$ and $\mathbf{a}_2$, respectively and both are parallel to the vector $\mathbf{b}$, where BM is the shortest distance between l_1 and l_2.

Let θ be the angle between AB and l_1.

$$\therefore \quad \sin\theta = \frac{BM}{AB}$$

$$\Rightarrow \quad BM = AB\sin\theta = |\mathbf{AB}|\sin\theta$$

Now, $|\mathbf{AB}\times\mathbf{b}| = |\mathbf{AB}||\mathbf{b}|\sin(\pi-\theta)$

$$|\mathbf{AB}||\mathbf{b}|\sin\theta$$

$$= (|\mathbf{AB}|\sin\theta)|\mathbf{b}| = BM|\mathbf{b}|$$

$$\therefore \quad BM = \frac{|\mathbf{AB}\times\mathbf{b}|}{|\mathbf{b}|} = \frac{|(\mathbf{a}_2-\mathbf{a}_1)\times\mathbf{b}|}{|\mathbf{b}|}$$

$\therefore$ Shortest distance between parallel lines

$\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{b}$ is

$$d = \frac{|(\mathbf{a}_2-\mathbf{a}_1)\times\mathbf{b}|}{|\mathbf{b}|}$$

Example 35. Show that the two lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = z$, intersect. Also, find the point of intersection of these lines.

Sol. Here, $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ ...(i)

and $\frac{x-4}{5} = \frac{y-1}{2} = \frac{z-0}{1}$ ...(ii)

Any point on line (i) is $P(2r+1, 3r+2, 4r+3)$ and any point on the line (ii) is $Q(5\lambda+4, 2\lambda+1, \lambda)$.

They intersect if and only if

$$2r+1 = 5\lambda+4,\ 3r+2 = 2\lambda+1,\ 4r+3 = \lambda$$

On solving, $r = -1, \lambda = -1$

Clearly, for these values of λ and r $P(-1, -1, -1)$

Hence, lines (i) and (ii) intersect at $(-1, -1, -1)$.

Example 36. Find the shortest distance between the lines $\mathbf{r} = (4\hat{\mathbf{i}} - \hat{\mathbf{j}}) + \lambda(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}})$ and $\mathbf{r} = (\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) + \mu(2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 5\hat{\mathbf{k}})$.

Sol. We know, the shortest distance between the lines

$\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}_2$

$$\Rightarrow \quad d = \left|\frac{(\mathbf{a}_2-\mathbf{a}_1)\cdot(\mathbf{b}_1\times\mathbf{b}_2)}{|\mathbf{b}_1\times\mathbf{b}_2|}\right|$$

On comparing the given equation with the equations $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}_2$ respectively, we have

$\mathbf{a}_1 = 4\hat{\mathbf{i}} - \hat{\mathbf{j}}$, $\mathbf{a}_2 = \hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, $\mathbf{b}_1 = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$ and $\mathbf{b}_2 = 2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$

Now, $\mathbf{a}_2 - \mathbf{a}_1 = -3\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

and $$\mathbf{b}_1\times\mathbf{b}_2 = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & -3 \\ 2 & 4 & -5 \end{vmatrix} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 0\hat{\mathbf{k}}$$

$$\therefore \quad (\mathbf{a}_2-\mathbf{a}_1)\cdot(\mathbf{b}_1\times\mathbf{b}_2) = (-3\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 2\hat{\mathbf{k}})\cdot(2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 0\hat{\mathbf{k}}) = -6$$

and $|\mathbf{b}_1\times\mathbf{b}_2| = \sqrt{4+1+0} = \sqrt{5}$

$\therefore$ Shortest distance, $d = \left|\frac{(\mathbf{a}_2-\mathbf{a}_1)\cdot(\mathbf{b}_1\times\mathbf{b}_2)}{|\mathbf{b}_1\times\mathbf{b}_2|}\right| = \left|\frac{-6}{\sqrt{5}}\right| = \frac{6}{\sqrt{5}}$

Example 37. ***Find the shortest distance between the lines***

$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$.

Sol. Given lines are

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \quad \text{...(i)}$$

and $$\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5} \quad \text{...(ii)}$$

Here, $x_1 = 1, y_1 = 2, z_3 = 3;\ x_2 = 2, y_2 = 4, z_2 = 5$

$l_1 = 2, m_1 = 3, n_1 = 4;\ l_2 = 3, m_2 = 4, n_2 = 5$

Shortest distance between the lines (i) and (ii) are modulus of

$$= \frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}{\sqrt{(l_1m_2 - l_2m_1) + (m_1n_2 - m_2n_1)^2 + (l_1n_2 - l_2n_1)^2}} \quad \text{...(iii)}$$

Now, $= \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$

$$= 1(15-16) - 2(10-12) + 2(8-9) = 1$$

Also, $(l_1m_2 - l_2m_1)^2 + (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2$

$$= (8-9)^2 + (15-16)^2 + (10-12)^2$$

$$= 6$$

From Eq. (iii) shortest distance between lines (i) (ii), we get

$$= \left|\frac{1}{\sqrt{6}}\right| = \frac{1}{\sqrt{6}}$$

Example 38. Find the shortest distance and the vector equation of the line of shortest distance between the lines given by

$$\mathbf{r} = (3\hat{\mathbf{i}} + 8\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) + \lambda(3\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}})$$

and $$\mathbf{r} = (-3\hat{\mathbf{i}} - 7\hat{\mathbf{j}} + 6\hat{\mathbf{k}}) + \mu(-3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}})$$

Sol. Given lines are

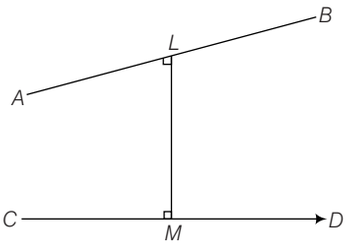
$$\mathbf{r} = (3\hat{\mathbf{i}} + 8\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) + \lambda(3\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}) \quad \text{...(i)}$$

and $\mathbf{r} = (-3\hat{\mathbf{i}} - 7\hat{\mathbf{j}} + 6\hat{\mathbf{k}}) + \mu(-3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}})$ ...(ii)

Equation of lines (i) and (iii) in cartesian form,

$$AB: \frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1} = \lambda \quad \text{...(iii)}$$

and $$CD: \frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4} = \mu \quad \text{...(iv)}$$



Let $L(3\lambda + 3), -\lambda + 8, \lambda + 3), M(-3\mu - 3, 2\mu - 7, 4\mu + 6)$

Direction ratios of LM are

$$(3\lambda + 3\mu + 6, -\lambda - 2\mu + 15, \lambda - 4\mu - 3)$$

Since, $LM \perp AB$

$$\therefore \quad 3(3\lambda + 3\mu + 6) - 1(-\lambda - 2\mu + 15) + 1(\lambda - 4\mu - 3) = 0$$

or $$11\lambda + 7\mu = 0 \quad \text{...(v)}$$

Again, $LM \perp CD$

$$-3(3\lambda + 3\mu + 6) - 2(-\lambda - 2\mu + 15) + 4(\lambda - 4\mu - 3) = 0$$

or $$-7\lambda - 29\mu = 0 \quad \text{...(vi)}$$

Solving Eqs. (v) and (vi), we get

$$\lambda = 0 = \mu$$

$$\therefore \quad L \equiv (3, 8, 3),$$

$$M \equiv (-3, -7, 6)$$

Hence, the shortest distance,

$$LM = \sqrt{(3+3)^2 + (8+7)^2 + (3-6)^2} = 3\sqrt{30} \text{ units}$$

$\therefore$ Vector equation of LM is

$$\mathbf{r} = 3\hat{\mathbf{i}} + 8\hat{\mathbf{j}} + 3\hat{\mathbf{k}} + t(6\hat{\mathbf{i}} + 15\hat{\mathbf{j}} - 3\hat{\mathbf{k}})$$

Also, the cartesian equation of LM is

$$\frac{x-3}{6} = \frac{y-8}{15} = \frac{z-3}{-3}$$

Example 39. Find the shortest distance between lines $\mathbf{r} = (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}) + \lambda(2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}})$ and $\mathbf{r} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}} + \mu(2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}})$.

Sol. Here lines (i) and (ii) are passing through the points $\mathbf{a}_1 = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{a}_2 = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}}$, respectively, and are parallel to the vector $\mathbf{b} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$.

Hence, the distance between the lines using the formula

$$\frac{|\mathbf{b} \times (\mathbf{a}_2 - \mathbf{a}_1)|}{|\mathbf{b}|} = \frac{\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 1 & 2 \\ 1 & -3 & -2 \end{vmatrix}}{3}$$

$$= \frac{|4\hat{\mathbf{i}} - 6\hat{\mathbf{j}} - 7\hat{\mathbf{k}}|}{3} = \frac{\sqrt{16 + 36 + 49}}{3} = \sqrt{\frac{101}{3}}$$

Example 40. Find the equation of a line which passes through the point (1, 1, 1) and intersects the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x+2}{1} = \frac{y-3}{2} = \frac{z+1}{4}$.

Sol. Any line passing through the point (1, 1, 1) is

$$\frac{x-1}{a} = \frac{y-1}{b} = \frac{z-1}{c} \quad \text{...(i)}$$

This line intersects the line $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$.

If $a : b : c \neq 2 : 3 : 4$ and $\begin{vmatrix} 1-1 & 2-1 & 3-1 \\ a & b & c \\ 2 & 3 & 4 \end{vmatrix} = 0$

$$\Rightarrow \quad a - 2b + c = 0$$

Again, line (i) intersects line

$$\frac{x-(-2)}{1} = \frac{y-3}{2} = \frac{z-(-1)}{4} \quad \text{...(ii)}$$

If $a : b : c \neq 1 : 2 : 4$ and $\begin{vmatrix} -2-1 & 3-1 & -1-1 \\ a & b & c \\ 1 & 2 & 4 \end{vmatrix} = 0$

$$\Rightarrow \quad 6a + 5b - 4c = 0 \quad \text{...(iii)}$$

From (ii) and (iii) by cross multiplication, we have

$$\frac{a}{8-5} = \frac{b}{6+4} = \frac{c}{5+12}$$

$$\Rightarrow \quad \frac{a}{3} = \frac{b}{10} = \frac{c}{17}$$

So, the required lines is $\frac{x-1}{3} = \frac{y-1}{10} = \frac{z-1}{17}$

Example 41. If the straight lines $x = -1 + s$, $y = 3 - \lambda s$, $z = 1 + \lambda s$ and $x = \frac{t}{2}$, $y = 1 + t$, $z = 2 - t$, with parameters s and t, respectively, are coplanar, then find λ.

Sol. The given lines $\frac{x+1}{1} = \frac{y-3}{-\lambda} = \frac{z-1}{\lambda} = s$

$\frac{x-0}{y_2} = \frac{y-1}{1} = \frac{y-2}{-1} = t$ are coplanar if

$$\begin{vmatrix} 0+1 & 1-3 & 2-1 \\ 1 & -\lambda & \lambda \\ 1/2 & 1 & -1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & -2 & 1 \\ 1 & -\lambda & \lambda \\ 1/2 & 1 & -1 \end{vmatrix} = 0$$

$$\Rightarrow \quad 1(\lambda - \lambda) + 2\left(1 - \frac{\lambda}{2}\right) + 1\left(1 + \frac{\lambda}{2}\right) = 0$$

$$\Rightarrow \quad \lambda = -2$$

Exercise for Session 2

1. The cartesian equation of a line is $\frac{x-3}{2}=\frac{y+1}{-2}=\frac{z-3}{5}$. Find the vector equation of the line.

2. A line passes through the point with position vector $2\hat{\mathbf{i}}-3\hat{\mathbf{j}}+4\hat{\mathbf{k}}$ and is in the direction of $3\hat{\mathbf{i}}+4\hat{\mathbf{j}}-5\hat{\mathbf{k}}$. Find the equation of the line is vector and cartesian forms.

3. Find the coordinates of the point where the line through $(3, 4, 1)$ and $(5, 1, 6)$ crosses *XY*-plane.

4. Find the angle between the pairs of line $\mathbf{r}=3\hat{\mathbf{i}}+2\hat{\mathbf{j}}-4\hat{\mathbf{k}}+\lambda(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\hat{\mathbf{k}})$, $\hat{\mathbf{r}}=5\hat{\mathbf{i}}-2\hat{\mathbf{j}}+\mu(3\hat{\mathbf{i}}+2\hat{\mathbf{j}}+6\hat{\mathbf{k}})$

5. Show that the two lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-4}{5}=\frac{y-1}{2}=z$ intersect. Find also the point of intersection of these lines.

6. Find the magnitude of the shortest distance between the lines $\frac{x}{2}=\frac{y}{-3}=\frac{z}{1}$ and $\frac{x-2}{3}=\frac{y-1}{-5}=\frac{z+2}{2}$.

7. Find the perpendicular distance of the point $(1, 1, 1)$ from the line $\frac{x-2}{2}=\frac{y+3}{2}=\frac{z}{-1}$.

8. Find the equation of the line drawn through the point $(1, 0, 2)$ to meet at right angles the line $\frac{x+1}{3}=\frac{y-2}{-2}=\frac{z+1}{-1}$.

9. Find the equation of line through $(1, 2, -1)$ and perpendicular to each of the lines $\frac{x}{1}=\frac{y}{0}=\frac{z}{-1}$ and $\frac{x}{3}=\frac{y}{4}=\frac{z}{5}$.

10. Find the image of the point (1, 2, 3) in the line $\frac{x-6}{3}=\frac{y-7}{2}=\frac{z-7}{-2}$.

Session 3

Plane, Equation of Plane in Various Form, Angle between Two Planes, Family of Planes, Two Sides of a Plane, Distance of a Point from a Plane, Equation of Planes Bisecting the Angle between Two Planes, Line and Plane

Plane

A plane is a surface such that if any two points are taken on it, the line segment joining them lies completely on the surface.

General Form General equation of the first degree in x, y, z always represents a plane.

The general equation of plane is $ax + by + cz + d = 0$.

Proof. Let first degree equation in x, y and z be

$$ax + by + cz + d = 0 \quad \text{...(i)}$$

In order to prove that Eq. (i) is the equation of plane, it is sufficient to show that every point on the line joining two points lies on the surface represented by Eq. (i).

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points on the surface represented by Eq. (i).

Then, $$ax_1 + by_1 + cz_1 + d = 0 \quad \text{...(ii)}$$

and $$ax_2 + by_2 + cz_2 + d = 0 \quad \text{...(iii)}$$

Let R be any arbitrary point on the line segment joining P and Q. Suppose R divides PQ in the ratio $\lambda : 1$.

$\therefore$ R is $\left(\frac{x_1 + \lambda x_2}{1+\lambda}, \frac{y_1 + \lambda y_2}{1+\lambda}, \frac{z_1 + \lambda z_2}{1+\lambda}\right)$

We are to show that R lies on the surface represented by Eq. (i) for all values of λ. For this, it is sufficient to prove that R satisfy Eq. (i)

On putting this value of R in LHS of Eq. (i), we obtain

$$a\left(\frac{x_1 + \lambda x_2}{\lambda + 1}\right) + b\left(\frac{y_1 + \lambda y_2}{\lambda + 1}\right) + c\left(\frac{z_1 + \lambda z_2}{\lambda + 1}\right) + d$$

$$= \frac{1}{\lambda + 1}\{(ax_1 + by_1 + cz_1) + \lambda(ax_2 + by_2 + cz_2)\}$$

$$= \frac{1}{\lambda + 1}[0 + 0] \qquad \text{[using Eqs. (ii) and (iii)]}$$

$$= 0$$

which shows that the point R lies on Eq. (i). Since, R is an arbitrary point on the line segment joining P and Q.

$\therefore$ Every point on PQ lies on the surface represented by Eq. (i).

Hence, $ax + by + cz + d = 0$ is equation of plane.

Equation of a Plane Passing Through a Given Point

The general equation of a plane passing through a given point (x_1, y_1, z_1) is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$, where a, b and c are constants.

Proof. The general equation of plane is

$$ax + by + cz + d = 0 \quad \text{...(i)}$$

If it passes through (x_1, y_1, z_1)

$$\Rightarrow \quad ax_1 + by_1 + cz_1 + d = 0 \quad \text{...(ii)}$$

On subtracting Eq. (i) from Eq. (ii), we get

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

which is the equation of a plane passing through (x_1, y_1, z_1).

Example 42. Show that the four points $(0, -1, -1)$, $(-4, 4, 4)$, $(4, 5, 1)$ and $(3, 9, 4)$ are coplanar. Find the equation of the plane containing them.

Sol. We shall find the equation of a plane passing through any three out of the given four points and show that the fourth point satisfies the equation.

Now, any plane passing through $(0, -1, -1)$ is

$$a(x - 0) + b(y + 1) + c(z + 1) = 0 \quad \text{...(i)}$$

If it passes through $(-4, 4, 4)$, we have

$$a(-4) + b(5) + c(5) = 0 \quad \text{...(ii)}$$

Also, if plane passes through $(4, 5, 1)$,we have

$$a(4) + b(6) + c(2) = 0$$

$$\Rightarrow \quad 2a + 3b + c = 0 \quad \text{...(iii)}$$

On solving Eqs. (ii) and (iii) by cross multiplication method, we obtain

$$\frac{a}{-5} = \frac{b}{7} = \frac{c}{-11} = k$$

On putting in Eq. (i), we get

$$-5kx + 7k(y + 1) - 11k(z + 1) = 0$$

$$\Rightarrow \quad -5x + 7y - 11z - 4 = 0$$

(required equation of plane)

Clearly, the fourth point namely (3, 9, 4) satisfies this equation. Hence, the given points are coplanar and the equation of plane containing those points, is $5x - 7y + 11z + 4 = 0$

Equation of Plane in Various Form

A plane is determined uniquely if

(i) The normal to the plane and its distance from the origin is given, i.e. the equation of a plane in normal form.

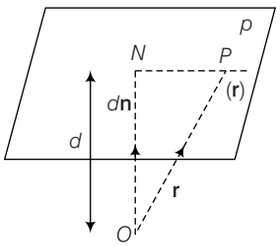
(ii) It passes through a point and is perpendicular to a given direction.

(iii) It passes through three given non-collinear points.

Equation of Plane in Normal Form

Vector Form

The vector equation of a plane normal to unit vector $\hat{\mathbf{n}}$ and at a distance d from the origin is $\mathbf{r} \cdot \hat{\mathbf{n}} = d$.



Proof. Let O be the origin and let ON be the perpendicular from O to the given plane π such that $\mathbf{ON} = d\hat{\mathbf{n}}$, where d is perpendicular distance of plane from origin.

Let P be a point on the plane, with position vector $\mathbf{r}$ so that $\mathbf{OP} = \mathbf{r}$

Now, $\mathbf{NP} \perp \mathbf{ON}$

$$\Rightarrow \quad \mathbf{NP} \cdot \mathbf{ON} = 0 \qquad \ldots(i)$$

$$\Rightarrow \quad (\mathbf{OP} - \mathbf{ON}) \cdot \mathbf{ON} = 0$$

$$\Rightarrow \quad (\mathbf{r} - d\hat{\mathbf{n}}) \cdot d\hat{\mathbf{n}} = 0$$

$$\Rightarrow \quad \mathbf{r} \cdot d\hat{\mathbf{n}} - d^2 \hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = 0$$

$$\Rightarrow \quad d\mathbf{r} \cdot \hat{\mathbf{n}} - d^2 |\mathbf{n}|^2 = 0 \qquad (\because d \neq 0)$$

$$\Rightarrow \quad \mathbf{r} \cdot \hat{\mathbf{n}} - d = 0 \qquad (\because |\hat{\mathbf{n}}| = 1)$$

$$\Rightarrow \quad \mathbf{r} \cdot \hat{\mathbf{n}} = d \qquad \ldots(ii)$$

Thus, the required equation of the plane is $\mathbf{r} \cdot \hat{\mathbf{n}} = d$.

Cartesian Form

Equation (ii) gives the vector equation of a plane, where $\hat{\mathbf{n}}$ is the unit vector normal to the plane. Let $P(x, y, z)$ be any point on the plane. Then

$$\mathbf{OP} = \mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

Let l, m and n be the direction cosines of $\hat{\mathbf{n}}$.

Then, $\hat{\mathbf{n}} = (l\hat{\mathbf{i}} + m\hat{\mathbf{j}} + n\hat{\mathbf{k}})$

Therefore, (ii) gives

$$(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot (l\hat{\mathbf{i}} + m\hat{\mathbf{j}} + n\hat{\mathbf{k}}) = d$$

or $\quad lx + my + nz = d \qquad \ldots(iii)$

This is the cartesian equation of the plane in the normal form.

Note

Equation (iii) shows that if $\mathbf{r} \cdot (a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}) = d$ is the vector equation of a plane, then $ax + by + cz = d$ is the cartesian equation of the plane, where a, b and c are the direction ratios of the normal to the plane.

The equation $\mathbf{r} \cdot \mathbf{n} = d$ is in normal form, if $\mathbf{n}$ is a unit vector and d is the distance of the plane from the origin. If $\mathbf{n}$ is not a unit vector, then to reduce the equation $\mathbf{r} \cdot \mathbf{n} = d$ to normal form, we reduce the equation $\mathbf{r} \cdot \mathbf{n} = d$ to normal form by dividing both sides by $|\mathbf{n}|$, we get

$$\frac{\mathbf{r} \cdot \mathbf{n}}{|\mathbf{n}|} = \frac{d}{|\mathbf{n}|} \Rightarrow \mathbf{r} \cdot \mathbf{n} = \frac{d}{|\mathbf{n}|} = p \text{ (distance from the origin)}$$

Example 43. Find the vector equation of plane which is at a distance of 8 units from the origin and which is normal to the vector $2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$.

Sol. Here, $d = 8$ and $n = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

$$\mathbf{n} = \frac{\mathbf{n}}{|\mathbf{n}|} = \frac{2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}}{3}$$

Hence, the required equation of plane is, $\mathbf{r} \cdot \mathbf{n} = d$

$$\Rightarrow \quad \mathbf{r} \cdot \left(\frac{2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}}{3} \right) = 8$$

$$\Rightarrow \quad \mathbf{r} \cdot (2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) = 24$$

Example 44. Reduce the equation $\mathbf{r} \cdot (3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 12\hat{\mathbf{k}}) = 5$ to normal form and hence find the length of perpendicular from the origin to the plane.

Sol. The given equation of plane is

$$\mathbf{r} \cdot (3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 12\hat{\mathbf{k}}) = 5 \text{ or } \mathbf{r} \cdot \mathbf{n} = 5$$

where, $\mathbf{n} = 3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 12\hat{\mathbf{k}}$

Since, $|\mathbf{n}| = \sqrt{9 + 16 + 144} = 13 \neq 1$, therefore the given equation is not the normal form. To reduce to normal form we

divide both sides by $|\mathbf{n}|$ i.e. $\frac{\mathbf{r}\cdot\mathbf{n}}{|\mathbf{n}|}=\frac{5}{|\mathbf{n}|}$ or $\mathbf{r}\cdot\left(\frac{3}{13}\hat{\mathbf{i}}-\frac{4}{13}\hat{\mathbf{j}}+\frac{12}{13}\hat{\mathbf{k}}\right)=\frac{5}{13}$. This is the normal form of the equation of given plane and length perpendicular $=\frac{5}{13}$.

Example 45. Find the distance of the plane $2x-y-2z-9=0$ from the origin.

Sol. The plane can be put in vector form as $\mathbf{r}\cdot(2\hat{\mathbf{i}}-\hat{\mathbf{j}}-2\hat{\mathbf{k}})=9$.

Here, $\mathbf{n}=2\hat{\mathbf{i}}-\hat{\mathbf{j}}-2\hat{\mathbf{k}}$

$\Rightarrow \quad \frac{\mathbf{n}}{|\mathbf{n}|}=\frac{2\hat{\mathbf{i}}-\hat{\mathbf{j}}-2\hat{\mathbf{k}}}{3}$

Dividing equation throughout by 3, we have equation of plane in normal form as $\mathbf{r}\cdot\frac{(2\hat{\mathbf{i}}-\hat{\mathbf{j}}-2\hat{\mathbf{k}})}{3}=3$, in which 3 is the distance of the plane from the origin.

Example 46. Find the vector equation of a line passing through $3\hat{\mathbf{i}}-5\hat{\mathbf{j}}+7\hat{\mathbf{k}}$ and perpendicular to the plane $3x-4y+5z=8$.

Sol. The given plane $3x-4y+5z=8$.

or $\quad (3\hat{\mathbf{i}}-4\hat{\mathbf{j}}+5\hat{\mathbf{k}})(x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}})=8$.

This shows that $\mathbf{d}=3\hat{\mathbf{i}}-4\hat{\mathbf{j}}+5\hat{\mathbf{k}}$ is normal to the given plane.

Therefore, the required line is parallel to $3\hat{\mathbf{i}}-4\hat{\mathbf{j}}+5\hat{\mathbf{k}}$.

Since, the required line passes through $3\hat{\mathbf{i}}-5\hat{\mathbf{j}}+7\hat{\mathbf{k}}$, its equation is given by $\mathbf{r}=3\hat{\mathbf{i}}-5\hat{\mathbf{j}}+7\hat{\mathbf{k}}+\lambda(3\hat{\mathbf{i}}-4\hat{\mathbf{j}}+5\hat{\mathbf{k}})$, where λ is a parameter.

Example 47. Find the unit vector perpendicular to the plane $\mathbf{r}\cdot(2\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}})=5$.

Sol. Vector normal to the plane is $\mathbf{n}=2\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}}$

Hence, unit vector perpendicular to the plane is

$$\frac{\mathbf{n}}{|\mathbf{n}|}=\frac{2\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}}}{\sqrt{2^2+1^2+2^2}}$$

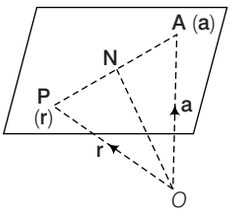
$$=\frac{1}{3}(2\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}})$$

Vector Equation of a Plane Passing Through a Given Point and Normal to a Given Vector

The vector equation of a plane passing through a point having position vector $\mathbf{a}$ and normal to vector $\mathbf{n}$ is $(\mathbf{r}-\mathbf{a})\cdot\mathbf{n}=0$.

Proof Suppose the planer π passes through a point having position vector $\mathbf{a}$ and is normal to the vector $\mathbf{n}$. Let O be the origin and $\mathbf{r}$ be the position vector of any point P on the plane π. Then, $\mathbf{OP}=\mathbf{r}$.

Since, $\mathbf{AP}$ lies in the plane and $\mathbf{n}$ is a normal to the plane π.



$\therefore \qquad \mathbf{AP}\perp\mathbf{n}$

$\Rightarrow \qquad \mathbf{AP}\cdot\mathbf{n}=0 \Rightarrow (\mathbf{r}\cdot\mathbf{a})\cdot\mathbf{n}=0 \qquad (\because \mathbf{AP}=\mathbf{r}-\mathbf{a})$

Hence, the required equation of the plane is

$$(\mathbf{r}-\mathbf{a})\cdot\mathbf{n}=0$$

Note

The above equation can be written as $\mathbf{r}\cdot\mathbf{n}=d$, where $\mathbf{d}=\mathbf{a}\cdot\mathbf{n}$ (known as scalar product form of plane).

Cartesian Form

If $\mathbf{r}=x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}$, $\mathbf{a}=x_1\hat{\mathbf{i}}+y_1\hat{\mathbf{j}}+z_1\hat{\mathbf{k}}$ and $\mathbf{n}=a\hat{\mathbf{i}}+b\hat{\mathbf{j}}+c\hat{\mathbf{k}}$, then $(\mathbf{r}-\mathbf{a})=(x-x_1)\hat{\mathbf{i}}+(y-y_1)\hat{\mathbf{j}}+(z-z_1)\hat{\mathbf{k}}$

Then equation of the plane can be written as

$(x-x_1)\hat{\mathbf{i}}+(y-y_1)\hat{\mathbf{j}}+(z-z_1)\hat{\mathbf{k}}\cdot(a\hat{\mathbf{i}}+b\hat{\mathbf{j}}+c\hat{\mathbf{k}})=0$

$\Rightarrow a(x-x_1)+b(y-y_1)+c(z-z_1)=0$

Thus, the coefficients of x, y and z in the cartesian equation of a plane are the direction ratios of the normal to the plane.

Example 48. Find the equation of the plane passing through the point (2, 3, 1) having (5, 3, 2) as the direction ratios of the normal to the plane.

Sol. The equation of the plane passing through (x_1, y_1, z_1) and perpendicular to the line with direction ratios a, b and c is given by $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$.

Now, since the plane passes through (2, 3, 1) and is perpendicular to the line having direction ratios (5, 3, 2), the equation of the plane is given by $5(x-2)+3(y-3)+2(z-1)=0$ or $5x+3y+2z=21$.

Example 49. The foot of the perpendicular drawn from the origin to a plane is $(12,-4,3)$. Find the equation of the plane.

Sol. Since $P(12,-4,3)$ is the foot of the perpendicular from the origin to the plane OP is normal to the plane π. Thus, the direction ratios of narmal to the plane are 12, –4 and 3. Now, since the plane passes through $(12,-4,3)$, its equation is given by

$$12(x-12)-4(y+4)+3(z-3)=0$$

or $\quad 12x-4y+3z-169=0$

Example 50. A vector **n** of magnitude 8 units is inclined to *X*-axis at 45°, *Y*-axis at 60° and an acute angle with *Z*-axis. If a plane passes through a point $(\sqrt{2}, -1, 1)$ and is normal to **n**, then find its equation in vector form.

Sol. Let γ be the angle made by **n** with Z-axis, then direction cosines of **n** are

$$l = \cos 45° = \frac{1}{\sqrt{2}},\ m = \cos 60° = \frac{1}{2} \text{ and } n = \cos\gamma$$

$$\therefore \quad l^2 + m^2 + n^2 = 1 \Rightarrow \frac{1}{2} + \frac{1}{4} + n^2 = 1$$

$$\Rightarrow \qquad n^2 = \frac{1}{4}$$

$$n = \frac{1}{2} \text{ (neglecting } n = -\frac{1}{2} \text{ as } \gamma \text{ is acute : } n > 0)$$

We have, $|\mathbf{n}| = 8$

$$\mathbf{n} = |\mathbf{n}|(l\hat{\mathbf{i}} + m\hat{\mathbf{j}} + n\hat{\mathbf{k}})$$

$$\mathbf{n} = 8\left(\frac{1}{\sqrt{2}}\hat{\mathbf{i}} + \frac{1}{2}\hat{\mathbf{j}} + \frac{1}{2}\hat{\mathbf{k}}\right) = 4\sqrt{2}\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$$

The required plane passes through the point $(\sqrt{2}, -1, 1)$ having position vector

$$\mathbf{a} = \sqrt{2}\,\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$$

So, its vector equation is $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$

$$\Rightarrow \qquad \hat{\mathbf{r}} \cdot \hat{\mathbf{n}} = \hat{\mathbf{a}} \cdot \hat{\mathbf{n}}$$

$$\Rightarrow \mathbf{r} \cdot (4\sqrt{2}\,\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 4\hat{\mathbf{k}}) = (\sqrt{2}\,\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}) \cdot (4\sqrt{2}\,\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 4\hat{\mathbf{k}})$$

$$\Rightarrow \mathbf{r} \cdot (4\sqrt{2}\,\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 4\hat{\mathbf{k}}) = 8$$

$$\Rightarrow \quad \mathbf{r} \cdot (\sqrt{2}\,\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) = 2$$

Example 51. Find the equation of the plane such that image of point (1, 2, 3) in it is (– 1, 0, 1).

Sol. Since, the image of $A(1, 2, 3)$ in the plane is $B(-1, 0, 1)$, the plane passes through the mid-point (0, 1, 2) of AB and is normal to the vector $\mathbf{AB} = -2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$.

Hence, the equation of the plane is $-2(x - 0) - 2(y - 1)$

$$-2(z - 2) = 0$$

or $\qquad x + y + z = 3.$

Equation of a Plane Passing through Three Given Points

Cartesian Form

Let the plane be passing through points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$.

Let $P(x, y, z)$ be any point on the plane.

Then, vectors **PA**, **BA** and **CA** are coplanar.

$$[\mathbf{PA}\ \mathbf{BA}\ \mathbf{CA}] = 0$$

$$\Rightarrow \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0, \text{ which is the required}$$

equation of the plane

Vector Form

Vector form of the equation of the plane passing through three points *A*, *B* and *C* having position vectors **a**, **b** and **c**, respectively.

Let **r** be the position vector of any point *P* in the plane.

Hence, vector $\mathbf{AP} = \mathbf{r} - \mathbf{a}$ $\mathbf{AB} = \mathbf{b} - \mathbf{a}$ and $\mathbf{AC} = \mathbf{c} - \mathbf{a}$ are coplanar.

Hence, $(\mathbf{r} - \mathbf{a}) \cdot \{(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})\} = 0$

$$\Rightarrow \qquad (\mathbf{r} - \mathbf{a}) \cdot (\mathbf{b} \times \mathbf{c} - \mathbf{b} \times \mathbf{a} - \mathbf{a} \times \mathbf{c} + \mathbf{a} \times \mathbf{a}) = 0$$

$$\Rightarrow \qquad (\mathbf{r} - \mathbf{a}) \cdot (\mathbf{b} \times \mathbf{c} + \mathbf{a} \times \mathbf{b} + \mathbf{c} \times \mathbf{a}) = 0$$

$$\Rightarrow \qquad \mathbf{r} \cdot (\mathbf{b} \times \mathbf{c} + \mathbf{a} \times \mathbf{b} + \mathbf{c} \times \mathbf{a})$$

$$= \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) + \mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) + \mathbf{a} \cdot (\mathbf{c} \times \mathbf{a})$$

$$\Rightarrow \qquad [\mathbf{r}\ \mathbf{b}\ \mathbf{c}] + [\mathbf{r}\ \mathbf{a}\ \mathbf{b}] + [\mathbf{r}\ \mathbf{c}\ \mathbf{a}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$

which is the required equation of the plane.

Note

1. If *p* is the length of perpendicular from the origin on this plane, then $p = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] / n$, where $n = |\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}|$.
2. Four points **a, b, c** and **d** are coplanar if **d** lies on the plane containing **a, b** and **c**.

   or $\qquad \mathbf{d} \cdot [\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

   or $\qquad [\mathbf{d}\ \mathbf{a}\ \mathbf{b}] + [\mathbf{d}\ \mathbf{b}\ \mathbf{c}] + [\mathbf{d}\ \mathbf{c}\ \mathbf{a}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

Example 52. Find the equation of the plane passing through $A(2, 2, -1)$, $B(3, 4, 2)$ and $C(7, 0, 6)$. Also find a unit vector perpendicular to this plane.

Sol. Here, $(x_1, y_1, z_1) \equiv (2, 2, -1)$, $(x_2, y_2, z_2) \equiv (3, 4, 2)$ and $(x_3, y_3, z_3) \equiv (7, 0, 6)$.

Then, the equation of the plane is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

or $\qquad \begin{vmatrix} x - 2 & y - 2 & z - (-1) \\ 3 - 2 & 4 - 2 & 2 - (-1) \\ 7 - 2 & 0 - 2 & 6 - (-1) \end{vmatrix} = 0$

or $\qquad 5x + 2y - 3z = 17$

A normal vector to this plane is $\mathbf{d} = 5\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$ ...(i)

Therefore, a unit vector normal to (i) is given by

$$\hat{\mathbf{n}} = \frac{\mathbf{d}}{|\mathbf{d}|} = \frac{5\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}}{\sqrt{25 + 4 + 9}}$$

$$= \frac{1}{\sqrt{38}}(5\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}})$$

Example 53. Find equation of plane passing through the points $P(1, 1, 1)$, $Q(3, -1, 2)$ and $R(-3, 5, -4)$.

Sol. Let the equation of plane passing through $(1, 1, 1)$ be $a(x-1)+b(y-1)+c(z-1)=0$, as it passes through the points Q and R.

$\therefore \quad 2a-2b+c=0$

and $\quad -4a+4b-5c=0$

Hence, solving by cross multiplication method, we get

$$\frac{a}{10-4}=\frac{b}{-4+10}=\frac{c}{8-8}=k$$

$\therefore \quad a=6k,\ b=6k,\ c=0$

On substituting in Eq. (i), we get

$$6(x-1)+6(y-1)+0=0$$

i.e. $x+y=2$; which is the required equation.

Aliter Equation of plane passing through (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is

$$=\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$$

i.e. $\begin{vmatrix} x-1 & y-1 & z-1 \\ 3-1 & -1-1 & 2-1 \\ -3-1 & 5-1 & -4-1 \end{vmatrix}=0$

On solving, we get $x+y=2$

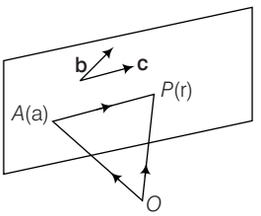
Equation of a Plane Passing Through a Given Point and Parallel to Two Given Vectors

Let a plane pass through $A(\mathbf{a})$ and is parallel to the plane formed by two vectors $\mathbf{b}$ and $\mathbf{c}$. Since, $\mathbf{AP}$ lies in the plane and $\mathbf{b}$ and $\mathbf{c}$ are two non-collinear vectors,

$$\mathbf{AP}=\lambda\mathbf{b}+\mu\mathbf{c}$$

$$\Rightarrow \quad \mathbf{r}-\mathbf{a}=\lambda\mathbf{b}+\mu\mathbf{c}$$

$$\Rightarrow \quad \mathbf{r}=\mathbf{a}+\lambda\mathbf{b}+\mu\mathbf{c}$$



Here, λ and μ are arbitrary scalars.

This form is also called the parametric form of the plane. It can also be written in the non-parametric form as

$$(\mathbf{r}-\mathbf{a})\cdot(\mathbf{b}\times\mathbf{c})=0$$

or $\quad [\mathbf{r}\ \mathbf{b}\ \mathbf{c}]=[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

Cartesian Form

From $(\mathbf{r}-\mathbf{a})\cdot(\mathbf{b}\times\mathbf{c})=0$, we have $[\mathbf{r}-\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

$$\Rightarrow \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}=0,$$ which is the required equation of the plane, where $\mathbf{b}=x_2\hat{\mathbf{i}}+y_2\hat{\mathbf{j}}+z_2\hat{\mathbf{k}}$ and $\mathbf{c}=x_3\hat{\mathbf{i}}+y_3\hat{\mathbf{j}}+z_3\hat{\mathbf{k}}$.

Example 54. Find the vector equation of the following planes in cartesian form $\mathbf{r}=\hat{\mathbf{i}}-\hat{\mathbf{j}}+\lambda(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})+\mu(\hat{\mathbf{i}}-2\hat{\mathbf{j}}+3\hat{\mathbf{k}})$.

Sol. The equation of the plane is

$$\mathbf{r}=\hat{\mathbf{i}}-\hat{\mathbf{j}}+\lambda(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})+\mu(\hat{\mathbf{i}}-2\hat{\mathbf{j}}+3\hat{\mathbf{k}}).$$

Let $\quad \mathbf{r}=x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}$

Hence, the equation is

$$(x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}})-(\hat{\mathbf{i}}-\hat{\mathbf{j}})=\lambda(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})+\mu(\hat{\mathbf{i}}-2\hat{\mathbf{j}}+3\hat{\mathbf{k}})$$

Thus, vectors $(x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}})-(\hat{\mathbf{i}}-\hat{\mathbf{j}})$, $\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$, $\hat{\mathbf{i}}-2\hat{\mathbf{j}}+3\hat{\mathbf{k}}$ are coplanar.

Therefore, the equation of the plane is

$$\begin{vmatrix} x-1 & y-(-1) & z-0 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix}=0$$

or $\quad 5x-2y-3z-7=0$

Intercept Form of a Plane

The equation of a plane having intercepting lengths a, b and c with X-axis, Y-axis and Z-axis, respectively is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

Proof Let O be the origin and let OX, OY and OZ be the coordinate axes.

Let the plane meets the coordinate axes at the points P, Q and R, respectively such that

$OP=a$, $OQ=b$ and $P=c$. Then, the coordinates of the points are $P(a, 0, 0)$, $Q(0, b, 0)$ and $R(0, 0, c)$.

Let the equation of plane be

$$Ax+By+Cz+D=0 \qquad \text{...(i)}$$

Since, Eq. (i) passes through $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$, we have

$$Aa+D=0 \ \Rightarrow\ A=\frac{-D}{a}$$

$$Bb+D=0 \ \Rightarrow\ B=\frac{-D}{b}$$

$$Cc + D = 0 \Rightarrow C = \frac{-D}{b}$$

On putting these values in Eq. (i),we get required equation of plane as

$$\frac{-D}{a}x - \frac{D}{b}y - \frac{D}{c}z = -D$$

$$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Example 55. A plane meets the coordinates axes in A, B and C such that the centroid of the ΔABC is the point (p, q, r), show that the equation of the plane is $\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 3$

Sol. Let the required equation of plane be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad \text{...(i)}$$

Then, the coordinates of A, B and C are $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$, respectively. So, the centroid of the ΔABC,

$$\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$$

But the coordinate of the centroid are (p, q, r).

$$\therefore \quad \frac{a}{3} = p, \frac{b}{3} = q, \frac{c}{3} = r$$

On putting the values of a, b and c in Eq. (i), we get

The required plane as

$$\frac{x}{3p} + \frac{y}{3q} + \frac{z}{3r} = 1$$

$$\Rightarrow \quad \frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 3$$

Example 56. A variable plane moves in such a way that the sum of the reciprocals of its intercepts on the three coordinate axes is constant. Show that the plane passes through the fixed point.

Sol. Let the equation of the plane be $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. Then, the intercepts made by the plane with axes are$a$, b and c.

$$\therefore \quad \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \text{constant}\,(k) \qquad \text{...(i) (given)}$$

$$\Rightarrow \quad \frac{1}{ak} + \frac{1}{bk} + \frac{1}{ck} = 1 \text{ comparing with } \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

$$x = \frac{1}{k}, y = \frac{1}{k}$$

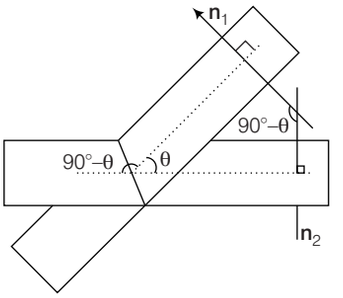
and $$z = \frac{1}{k}$$

This shows Eq. (i) passes through the fixed point $\left(\frac{1}{k'}, \frac{1}{k'}, \frac{1}{k'}\right)$.

Angle between Two Planes

Vector Form

The angle between two planes is defined as the angle between their normals.



Let θ be the angle between planes $\mathbf{r} \cdot \mathbf{n}_1 = d_1$ and $\mathbf{r} \cdot \mathbf{n}_2 = d_2$ then $\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n_1}||\mathbf{n_2}|}$

Condition for Perpendicularity

If the planes $\mathbf{r} \cdot \mathbf{n}_1 = d_1$ and $\mathbf{r} \cdot \mathbf{n}_2 = \mathbf{d}_2$ are perpendicular, then $\mathbf{n}_1$and $\mathbf{n}_2$ are perpendicular. Therefore, $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$

Condition for Parallelism

If the planes $\mathbf{r} \cdot \mathbf{n}_1 = d_1$ and $\mathbf{r} \cdot \mathbf{n}_2 = d_2$ are parallel, there exists the scalar λ such that $\mathbf{n}_1 = \lambda \mathbf{n}_2$.

Cartesian Form

If the planes are $a_1x + b_1y + c_1z + d = 0$

and $$a_2x + b_2y + c_2z + d_2 = 0$$

$$\Rightarrow \quad \cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Condition for parallelism

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \lambda$$

Condition for perpendicularity

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

Example 57. Find the angle between the planes $2x + y - 2x + 3 = 0$ and $\mathbf{r} \cdot (6\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) = 5$.

Sol. Normals along the given planes are $2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ and $6\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

Then angle between planes,

$$\theta = \cos^{-1}\frac{(2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}) \cdot (6\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}})}{\sqrt{(2)^2 + (1)^2 + (-2)^2}\sqrt{(6)^2 + (3)^2 + (2)^2}} = \cos^{-1}\frac{11}{21}$$

Example 58. Show that $ax+by+r=0$, $by+cz+p=0$ and $cz+ax+q=0$ are perpendicular to XY, YZ and ZX planes, respectively.

Sol. The planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ are perpendicular to each other if and only if $a_1a_2+b_1b_2+c_1c_2=0$.

The equation of XY, YZ and ZX planes are $z=0$, $x=0$ and $y=0$, respectively.

Now, we have to show that $z=0$ is perpendicular to

$$ax+by+r=0.$$

It follows immediately, since $a(0)+b(0)+(0)(1)=0$, other parts can be done similarly.

Family of Planes

Plane Parallel to a Given Plane

Since parallel planes have the same normal vector, so equation of a plane parallel to $\mathbf{r}\cdot\mathbf{n}=d_1$ is of the form $\mathbf{r}\cdot\mathbf{n}=d_2$, where d_2 is determined by the given conditions.

In cartesian form, if $ax+by+cz+d=0$ be the given plane then the plane parallel to this plane is $ax+by+cz+k=0$.

Example 59. Find the equation of the plane through the point $(1,4,-2)$ and parallel to the plane $-2x+y-3z=7$.

Sol. Let the equation of a plane parallel to the plane $-2x+y-3z=7$ be

$$-2x+y-3z+k=0 \quad ...(i)$$

This passes through (1, 4, – 2), therefore

$$(-2)(1)+4-3(-2)+k=0$$

$$\Rightarrow \quad -2+4+6+k=0 \Rightarrow k=-8$$

Putting $k=-8$ in Eq. (i), we obtain

$$-2x+y-3z-8=0 \text{ or } -2x+y-3z=8$$

This is the equation of the required plane.

Example 60. Find the equation of the plane passing through $(3,4,-1)$, which is parallel to the plane $\mathbf{r}\cdot(2\hat{\mathbf{i}}-3\hat{\mathbf{j}}+5\hat{\mathbf{k}})+7=0$.

Sol. The equation of any plane which is parallel to $\mathbf{r}\cdot(2\hat{\mathbf{i}}-3\hat{\mathbf{j}}+5\hat{\mathbf{k}})+7=0$ is

$$\mathbf{r}\cdot(2\hat{\mathbf{i}}-3\hat{\mathbf{j}}+5\hat{\mathbf{k}})+\lambda=0$$

or $$2x-3y+5z+\lambda=0$$

Further (i) will pass through (3, 4, – 1)

if $(2)(3)+(-3)(4)+5(-1)+\lambda=0$

or $$-11+\lambda=0 \Rightarrow \lambda=11$$

Thus, equation of the required plane is

$$\mathbf{r}\cdot(2\hat{\mathbf{i}}-3\hat{\mathbf{j}}+5\hat{\mathbf{k}})+11=0$$

Equation of any Plane Passing Through the Line of Intersection of Two Plane

The equation of the plane passing through the line of intersection of the planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ is $(a_1x+b_1y+c_1z+d_1)+k(a_2x+b_2y+c_2z+d_2)=0$

Proof Let the given plane be

$$a_1x+b_1y+c_1z+d_1=0 \quad ...(i)$$

and $$a_2x+b_2y+c_2z+d_2=0 \quad ...(ii)$$

$\therefore$ Required plane is $(a_1x+b_1y+c_1z+d_1)$

$$+k(a_2x+b_2y+c_2z+d_2)=0 \quad ...(iii)$$

Clearly, plane (iii) represents the equation of plane.

Let (α,β,γ) be a point on the line of intersection of planes (i) and (ii), then P lies on planes (i) and (ii).

$\therefore$ $$a_1\alpha+b_1\beta+c_1\gamma+d_1=0 \quad ...(iv)$$

and $$a_2\alpha+b_2\beta+c_2\gamma+d_2=0 \quad ...(v)$$

Now, multiply by k in plane (v) and then adding planes (iv) and (v), we get

$$(a_1\alpha+b_1\beta+c_1\gamma+d_1)$$
$$+k(a_2\alpha+b_2\beta+c_2\gamma+d_2)=0$$

$\Rightarrow P(\alpha,\beta,\gamma)$ lies on plane (iii).

Hence, plane (iii) passes through each point on the line of intersection of planes (i) and (ii).

Thus, plane (iii) is the equation of plane passing through the line of intersection of planes (i) and (ii).

Vector Form

Equation of planes passing through the line of intersection of planes

$\mathbf{r}\cdot\mathbf{n}_1=d_1$ and $\mathbf{r}\cdot\mathbf{n}_2=d_2$ is

$$(\mathbf{r}\cdot\mathbf{n}_1-d_1)+k(\mathbf{r}\cdot\mathbf{n}_2-d_2)=0$$

or $\mathbf{r}\cdot(\mathbf{n}_1+k\mathbf{n}_2)=d_1+kd_2$, k being any scalar.

Example 61. Find the equation of the plane containing the line of intersection of the plane $x+y+z-6=0$ and $2x+3y+4z+5=0$ and passing through the points $(1,1,1)$.

Sol. The equation of a plane through the line of intersection of the given plane is

$$(x+y+z-6)+\lambda(2x+3y+4z+5)=0 \quad ...(i)$$

If line (i) passes through (1, 1, 1), we have

$$-3+14\lambda=0$$

$$\Rightarrow \quad \lambda=\frac{3}{14}.$$

Putting $\lambda = \frac{3}{14}$ in line (i), we obtain the equation of the required plane as

$$(x + y + z - 6) + \frac{3}{14}(2x + 3y + 4z + 5) = 0$$

$$\Rightarrow \quad 20x + 23y + 26z - 69 = 0$$

Example 62. Find the planes passing through the intersection of planes $\mathbf{r}\cdot(2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}) = 1$ and $\mathbf{r}\cdot(\hat{\mathbf{i}} - \hat{\mathbf{j}}) + 4 = 0$ and perpendicular to planes $\mathbf{r}\cdot(2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}) = -8$.

Sol. The equation of any plane through the line of intersection of the given planes is

$$\{\mathbf{r}\cdot(2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}) - 1\} + \lambda\{\mathbf{r}\cdot(\hat{\mathbf{i}} - \hat{\mathbf{j}}) + 4\} = 0$$

$$\mathbf{r}\cdot\{(2 + \lambda)\hat{\mathbf{i}} - (3 + \lambda)\hat{\mathbf{j}} + 4\hat{\mathbf{k}}\} = 1 - 4\lambda \qquad \text{...(i)}$$

If it is perpendicular to $\mathbf{r}\cdot(2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}) + 8 = 0$, then

$$\{(2 + \lambda)\hat{\mathbf{i}} - (3 + \lambda)\hat{\mathbf{j}} + 4\hat{\mathbf{k}}\}\cdot(2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}) = 0$$

$$2(2 + \lambda) + (3 + \lambda) + 4 = 0$$

$$\lambda = \frac{-11}{3}$$

Putting $\lambda = -\frac{11}{3}$ in line (i), we obtain the equation of the required plane as $\mathbf{r}\cdot(-5\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 12\hat{\mathbf{k}}) = 47$

Two Sides of a Plane

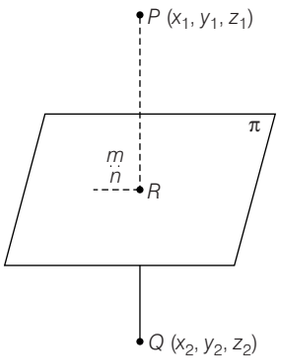
Let $ax + by + cz + d = 0$ be the plane, then the points (x_1, y_1, z_1) and (x_2, y_2, z_2) lie on the same side or opposite side according as

$$\frac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d} > 0 \text{ or } < 0$$

Proof Here equation of plane is,

$$ax + by + cz + d = 0 \qquad \text{...(i)}$$

Let Eq. (i) divide the line segment joining P and Q at R internally in the ratio $m : n$.



Then, $R\left(\frac{mx_2 + nx_1}{m+n}, \frac{my + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$

Since, R lies on the plane (i).

$$\therefore a\left(\frac{mx_2 + nx_1}{m+n}\right) + b\left(\frac{my_2 + ny_1}{m+n}\right) + c\left(\frac{mz_2 + nz_1}{m+n}\right) + d = 0$$

$$\Rightarrow \quad a(mx_2 + nx_1) + b(my_2 + ny_1) + c(mz_2 + nz_1) + d(m + n) = 0$$

$$\Rightarrow \quad m(ax_2 + by_2 + cz_2 + d) + n(ax_1 + by_1 + cz_1 + d) = 0$$

$$\Rightarrow \quad \frac{m}{n} = -\frac{(ax_1 + by_1 + cz_1 + d)}{(ax_2 + by_2 + cz_2 + d)} \qquad \text{...(ii)}$$

Now, if $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$

are of same sign $\frac{m}{n} < 0$ (external division)

are of opposite sign $\frac{m}{n} > 0$ (internal division)

$\therefore$ If $\frac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d} > 0$ (same side)

$\frac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d} < 0$ (opposite side)

Example 63. Find the interval of α for which $(\alpha, \alpha^2, \alpha)$ and $(3, 2, 1)$ lies on same side of $x + y - 4z + 2 = 0$.

Sol. $(\alpha, \alpha^2, \alpha)$ and $(3, 2, 1)$ lies on same side of $x + y - 4z + 2 = 0$

$$\therefore (\alpha + \alpha^2 - 4\alpha + 2)(3 + 2 - 4 + 2) > 0$$

$$\Rightarrow \quad \alpha^2 - 3\alpha + 2 > 0$$

$$(\alpha - 1)(\alpha - 2) > 0 \Rightarrow \alpha \in (-\infty, 1] \cup (2, \infty]$$

Distance of a Point from a Plane

Vector Form

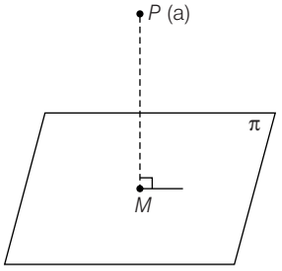
The length of the perpendicular from a point having position vector $\mathbf{a}$ to the plane $\mathbf{r}\cdot\mathbf{n} = d$ is given by

$$P = \frac{|\mathbf{a}\cdot\mathbf{n} - d|}{|\mathbf{n}|}$$

Proof. Let π be the given plane and $P(\mathbf{a})$ be the given point. Let PM be the length of perpendicular from P to the plane π.

Since, line PM passes through $P(\mathbf{a})$ and is parallel to the vector $\mathbf{n}$ which is normal to the plane π. So, vector equation of line PM is

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{n} \qquad \text{...(i)}$$



Point M is the intersection of Eq. (i) and the given plane π.

$\therefore \quad (\mathbf{a}+\lambda\mathbf{n})\cdot\mathbf{n}=d$

$\Rightarrow \quad \mathbf{a}\cdot\mathbf{n}+\lambda\mathbf{n}\cdot\mathbf{n}=d \Rightarrow \lambda=\dfrac{d-(\mathbf{a}\cdot\mathbf{n})}{|\mathbf{n}|^2}$

On putting the value of λ in Eq. (i), we obtain the position vector of M given by

$$\mathbf{r}=\mathbf{a}+\left(\frac{d-\mathbf{a}\cdot\mathbf{n}}{|\mathbf{n}|^2}\right)\mathbf{n}$$

$$\mathbf{PM}=\text{Position vector of } M-\text{Position vector of } P$$

$$=\mathbf{a}+\left(\frac{d-(\mathbf{a}\cdot\mathbf{n})}{|\mathbf{n}|^2}\right)\mathbf{n}-\mathbf{a}$$

$$\mathbf{PM}=\left(\frac{d-(\mathbf{a}\cdot\mathbf{n})}{|\mathbf{n}|^2}\right)\mathbf{n}$$

$\Rightarrow \quad PM=|\mathbf{PM}|=\left|\dfrac{(d-\mathbf{a}\cdot\mathbf{n})\,\mathbf{n}}{|\mathbf{n}|^2}\right|$

$$=\frac{|d-(\mathbf{a}\cdot\mathbf{n})||\mathbf{n}|}{|\mathbf{n}|^2}=\frac{|d-(\mathbf{a}\cdot\mathbf{n})|}{|\mathbf{n}|}$$

Thus, the length of perpendicular from a point having position vector $\mathbf{a}$ on the plane $\mathbf{r}\cdot\mathbf{n}=d$ is $\dfrac{|\mathbf{a}\cdot\mathbf{n}-d|}{|\mathbf{n}|}$

Cartesian Form

The length of perpendicular from a point $P(x_1, y_1, z_1)$ to the plane $ax+by+cz+d=0$. Then, the equation of PM is

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c} \qquad \text{...(i)}$$

The coordinates of any point on this line are

$$(x_1+ar, y_1+br, z_1+cr)$$

Thus, the point coincides with M iff it lies on plane.

i.e. $a(x_1+ar)+b(y_1+br)+c(z_1+cr)+d=0$

i.e. $$r=-\frac{(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2} \qquad \text{...(ii)}$$

Now, $$PM=\sqrt{(x_1+ar-x_1)^2+(y_1+br-y_1)^2+(z_1+cr-z_1)^2}$$

$$=\sqrt{(a^2+b^2+c^2)\,r^2}=\sqrt{a^2+b^2+c^2}\,|r|$$

$$=\sqrt{a^2+b^2+c^2}\left|\frac{-(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}\right|$$

[from Eq. (ii)]

$\therefore \quad PM=\dfrac{|(ax_1+by_1+cz_1+d)|}{\sqrt{a^2+b^2+c^2}}$

Example 64. Find the distance of the point (2, 1, 0) from the plane $2x+y+2z+5=0$.

Sol. We know that the distance of the point (x_1, y_1, z_1) from the plane $ax+by+cz+d=0$ is

$$\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$

So, required distance $=\dfrac{|2\times2+1+2\times0+5|}{\sqrt{2^2+1^2+2^2}}=\dfrac{10}{3}$.

Distance between the Parallel Planes

The distance between two parallel planes

$$ax+by+cz+d_1=0$$

and $$ax+by+cz+d_2=0$$

is given by $$d=\left|\frac{(d_2-d_1)}{\sqrt{a^2+b^2+c^2}}\right|$$

Proof. Let $d=$ Difference of the length of perpendicular from origin to the two planes.

$$=\left|\frac{|d_1|}{\sqrt{a^2+b^2+c^2}}-\frac{|d_2|}{\sqrt{a^2+b^2+c^2}}\right|.$$

if d_1 and d_2 are of same side $=\left|\dfrac{d_1-d_2}{\sqrt{a^2+b^2+c^2}}\right|$

Vector Form

The distance between two parallel plane $\mathbf{r}\cdot\mathbf{n}=d_1$ and $\mathbf{r}\cdot\mathbf{n}=d_2$ is given by

$$d=\frac{|d_1-d_2|}{|\mathbf{n}|}$$

Example 65. Find the distance between the parallel planes $x+2y-2z+1=0$ and $2x+4y-4z+5=0$.

Sol. We know that, distance between parallel planes $ax+by+cz+d_1=0$ and $ax+by+cz+d_2=0$ is,

$$\frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$$

$\therefore$ Distance between $x+2y-2z+1=0$

and $x+2y-2z+\frac{5}{2}=0$ is

$$\frac{\left|\frac{5}{2}-1\right|}{\sqrt{1+4+4}}=\frac{1}{2}$$

Equation of Planes Bisecting the Angle between Two Planes

Equation of the planes bisecting the angle between the planes.

$a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ is

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=\pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

Proof. Given planes are

$$a_1x+b_1y+c_1z+d_1=0 \quad ...(i)$$

and $$a_2x+b_2y+c_2z+d_2=0 \quad ...(ii)$$

Let $P(x,y,z)$ be a point on the plane bisecting the angle between planes (i) and (ii).

Let PL and PM be the length of perpendiculars from P to planes (i) and (ii).

$\therefore \quad PL=PM$

$$\Rightarrow \quad \left|\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}\right|=\left|\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=\pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

This is equation of planes bisecting the angles between the planes (i) and (ii).

Vector Form

Equation of planes bisecting the angle between planes $\mathbf{r}\cdot\mathbf{n}_1=d_1$ and $\mathbf{r}\cdot\mathbf{n}_2=d_2$ are

$$\left|\frac{\mathbf{r}\cdot\mathbf{n}_1-d_1}{\mathbf{n}_1}\right|=\left|\frac{\mathbf{r}\cdot\mathbf{n}_2-d_2}{\mathbf{n}_2}\right|$$

$$\Rightarrow \quad \frac{\mathbf{r}\cdot\mathbf{n}_1-d_1}{\mathbf{n}_1}=\pm\frac{\mathbf{r}\cdot\mathbf{n}_2-d_2}{\mathbf{n}_2}$$

$$\Rightarrow \quad \mathbf{r}\cdot\frac{\mathbf{n}_1}{|\mathbf{n}_1|}\pm\mathbf{r}\cdot\frac{\mathbf{n}_2}{|\mathbf{n}_1|}=\frac{d_1}{|\mathbf{n}_1|}\pm\frac{d_2}{|\mathbf{n}_2|}$$

$$\Rightarrow \quad \mathbf{r}\cdot(\hat{\mathbf{n}}_1\pm\hat{\mathbf{n}}_2)=\frac{d_1}{|\mathbf{n}_1|}\pm\frac{d_2}{|\mathbf{n}_2|}$$

Bisector of the Angle between the Two Planes Containing the Origin

Let the equation of the two planes be

$$a_1x+b_1y+c_1z+d_1=0 \quad ...(i)$$

and $$a_2x+b_2y+c_2z+d_2=0 \quad ...(ii)$$

where, d_1 and d_2 are positive, then equation of the bisector of the angle between the planes (i) and (ii) containing the origin is

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

Bisector of the Acute and Obtuse Angles between Two Planes

Let the two planes be

$$a_1x+b_1y+c_1z+d_1=0 \quad ...(i)$$

and $$a_2x+b_2y+c_2z+d_2=0 \quad ...(ii)$$

where, d_1 and $d_2>0$

(a) If $a_1a_2+b_1b_2+c_1c_2>0$, the origin lies in the obtuse angle between the two planes and the equation of bisector of the acute angle is,

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=-\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

(b) If $a_1a_2+b_1b_2+c_1c_2<0$, then origin lies in the acute angle between the two planes and the equation of bisector of the acute angle between two planes is

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=+\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

Example 66. Find the equation of the bisector planes of the angles between the planes $2x-y+2z+3=0$ and $3x-2y+6z+8=0$ and specify the plane which bisects the acute angle and the plane which bisects the obtuse angle.

Sol. The two given planes are

$$2x-y+2z+3=0 \text{ and } 3x-2y+6z+8=0$$

where, d_1 and $d_2>0$

and $$a_1a_2+b_1b_2+c_1c_2=6+2+12>0$$

$\therefore$ $$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=-\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

(obtuse angle bisector)

and $$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

(acute angle bisector)

i.e. $\frac{2x-y+2z+3}{\sqrt{4+1+4}} = \pm \frac{3x-2y+6z+8}{\sqrt{9+4+36}}$

$\Rightarrow (14x-7y+14z+21) = \pm (9x-6y+18z+24)$

Taking positive sign on the right hand side, we get

$5x-y-4z-3=0$ (obtuse angle bisector)

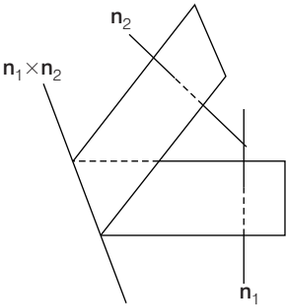
and taking negative sign on the right hand side, we get

$23x-13y+32z+45=0$ (acute angle bisector)

Line and Plane

Line of Intersection of Two Planes

Let two non-parallel planes are $\mathbf{r}\cdot\mathbf{n}_1 = d_1$ and $\mathbf{r}\cdot\mathbf{n}_2 = d_2$



Now line of intersection of planes is perpendicular to vector $\mathbf{n}_1$ and $\mathbf{n}_2$.

$\therefore$ Line of intersection is parallel to vector $\mathbf{n}_1 \times \mathbf{n}_2$.

If we wish to find the equation of line of intersection of planes $a_1x+b_1y+c_1z-d_1=0$ and $a_2x+b_2y+c_2z-d_2=0$, then we find any point on the line by putting $z=0$ (say), then we can find corresponding values of x and y be solving equations $a_1x+b_1y-d_1=0$ and $a_2x+b_2y-d_2=0$. Thus, by fixing the value of $z=\lambda$, we can find the corresponding value of x and y in terms of λ. After getting x, y and z in terms of λ, we can find the equation of line in symmetric form.

Example 67. Reduce the equation of line $x-y+2z=5$ and $3x+y+z=6$ in symmetrical form.
Or
Find the line of intersection of planes $x-y+2z=5$ and $3x+y+z=6$.

Sol. Given $x-y+2z=5, 3x+y+z=6$.

Let $z=\lambda$

Then, $x-y=5-2\lambda$

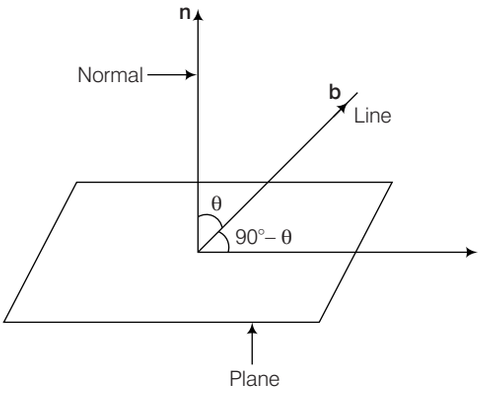
and $3x+y=6-\lambda$.

Solving these two equations, $4x=11-3\lambda$

and $4y=4x-20+8\lambda=-9+5\lambda$

The equation of the line is $\frac{4x-11}{-3}=\frac{4y+9}{5}=\frac{z-0}{1}$.

Angle between a Line and a Plane



The angle between a line and a plane is the complement of the angle between the line and the normal to the plane.

If the equation of the line is $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$ and that of the plane is $\mathbf{r}\cdot\mathbf{n}=d$, then angle θ between the line and the normal to the plane is $\cos\theta = \left|\frac{\mathbf{b}\cdot\mathbf{n}}{|\mathbf{b}||\mathbf{n}|}\right|$.

So, the angle ϕ between the line and the plane is given by $90^\circ-\theta$.

$\sin\phi = \left|\frac{\mathbf{b}\cdot\mathbf{n}}{|\mathbf{b}||\mathbf{n}|}\right|$ or $\phi = \sin^{-1}\left|\frac{\mathbf{b}\cdot\mathbf{n}}{|\mathbf{b}||\mathbf{n}|}\right|$

Line $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$ and plane $\mathbf{r}\cdot\mathbf{n}=d$ are perpendicular if $\mathbf{b}=\lambda\mathbf{n}$ or $\mathbf{b}\times\mathbf{n}=\mathbf{0}$ and parallel if $\mathbf{b}\perp\mathbf{n}$ or $\mathbf{b}\cdot\mathbf{n}=0$.

Example 68. Find the angle between the line $\mathbf{r}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}-\hat{\mathbf{k}}+\lambda(\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}})$ and the plane $\mathbf{r}\cdot(2\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}})=4$.

Sol. We know that if θ is the angle between the lines $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$ and $\mathbf{r}\cdot\mathbf{n}=p$, then $\sin\theta = \left|\frac{\mathbf{b}\cdot\mathbf{n}}{|\mathbf{b}||\mathbf{n}|}\right|$

Therefore, if θ is the angle between $\mathbf{r}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}-\hat{\mathbf{k}}+\lambda(\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}})$ and $\mathbf{r}\cdot(2\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}})=4$, then

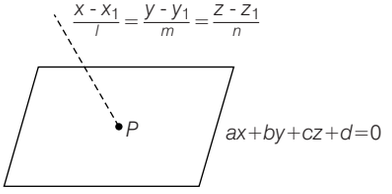
$$\sin\theta = \left|\frac{(\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}})\cdot(2\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}})}{|\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}}||2\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}}|}\right| = \frac{2+1+1}{\sqrt{1+1+1}\sqrt{4+1+1}} = \frac{4}{\sqrt{3}\sqrt{6}} = \frac{4}{3\sqrt{2}}$$

$\Rightarrow \quad \theta = \sin^{-1}\left(\frac{4}{3\sqrt{2}}\right)$

Intersection of a Line and a Plane

To find the point of intersection of the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ and the plane

$ax + by + cz + d = 0.$



Let $\dfrac{x-x_1}{l}=\dfrac{y-y_1}{m}=\dfrac{z-z_1}{n}=r$

$\therefore \quad (x=rl+x_1, y=mr+y_1, z=nr+z_1)$ ...(i)

be a point in the plane say P.

It must satisfy the equation of plane.

$\therefore \quad a(x_1+lr)+b(y_1+mr)+c(z_1+nr)+d=0$

$\Rightarrow \quad (ax_1+by_1+cz_1+d)+r(al+bm+cn)=0$

$\Rightarrow \quad r=-\dfrac{(ax_1+by_1+cz_1+d)}{al+bm+cn}$

On substituting the value of r Eq. (i), we get the coordinates of the required point of intersection.

(i) Condition for a Line to be Parallel to a Plane

Let line $\dfrac{x-x_1}{l}=\dfrac{y-y_1}{m}=\dfrac{z-z_1}{n}$ be parallel to plane $ax+by+cz+d=0$ iff;

$\theta=0$ or π or $\sin\theta=0 \Rightarrow al+bm+cm=0$

(ii) Condition for a Line to Lie in the Plane

Condition for $\dfrac{x-x_1}{l}=\dfrac{y-y_1}{m}=\dfrac{z-z_1}{n}$ to lie in the plane $ax+by+cz+d=0$ are

$al+bm+cn=0$ and $ax_1+by_1+cz_1+d=0$

Note

A line will be in a plane iff

(i) the normal to the plane is perpendicular to the line.

(ii) a point on the line in the plane.

Example 69. Find the distance between the point with position vector $-\hat{\mathbf{i}}-5\hat{\mathbf{j}}-10\hat{\mathbf{k}}$ and the point of intersection of the line $\dfrac{x-2}{3}=\dfrac{y+1}{4}=\dfrac{z-2}{12}$ with the plane $x-y+z=5$.

Sol. The coordinates of any point on the line

$\dfrac{x-2}{3}=\dfrac{y+1}{4}=\dfrac{z-2}{2}=r$ (say) are $(3r+2, 4r-1, 12r+2)$

If it lies on the plane $x-y+z=5$, then

$3r+2-4r+1+12r+2=5 \Rightarrow 11r=0 \Rightarrow r=0.$

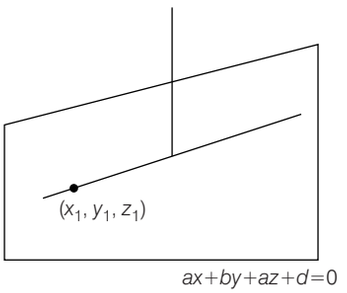
Putting $r=0$ in (i), we obtain $(2,-1,2)$ as the coordinates of the point of intersection of the given line and plane.

Required distance = distance between points $(-1,-5,-10)$ and $(2,-1,2)=\sqrt{(2+1)^2+(-1+5)^2+(2+10)^2}$

$=\sqrt{9+16+144}=\sqrt{169}=13.$

Coplanarity of Two Lines

The straight line $\dfrac{x-x_1}{l}=\dfrac{y-y_1}{m}=\dfrac{z-z_1}{n}$ lies in a given plane $ax+by+cz+d=0$ if $ax_1+by_1+cz_1+d=0$ and $al+bm+cn=0$



Thus, the general equation of the plane containing a straight line

$\dfrac{x-x_1}{l}=\dfrac{y-y_1}{m}=\dfrac{z-z_1}{n}$ is

$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$

where, $al+bm+cn=0$

The equation of the plane containing a straight line $\dfrac{x-x_1}{l}=\dfrac{y-y_1}{m}=\dfrac{z-z_1}{n}$ and parallel to the straight line $\dfrac{x-x_2}{l_1}=\dfrac{y-y_2}{m_1}=\dfrac{z-z_2}{n_1}$ is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l & m & n \\ l_1 & m_1 & n_1 \end{vmatrix}=0$$

Hence, the equation of the plane containing two given straight lines

$\dfrac{x-x_1}{l}=\dfrac{y-y_1}{m}=\dfrac{z-z_1}{n}$

and $\dfrac{x-x_2}{l_1}=\dfrac{y-y_2}{m_1}=\dfrac{z-z_2}{n_1}$

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l & m & n \\ l_1 & m_1 & n_1 \end{vmatrix}=0$$

or $$\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l & m & n \\ l_1 & m_1 & n_1 \end{vmatrix} = 0$$

If the lines $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + \lambda\mathbf{b}_2$ are coplanar, then

$$[\mathbf{a}_1\ \mathbf{b}_1\ \mathbf{b}_2] = [\mathbf{a}_2\ \mathbf{b}_1\ \mathbf{b}_2]$$

and the equation of plane containing them is

$$[\mathbf{r}\ \mathbf{b}_1\ \mathbf{b}_2] = [\mathbf{a}_1\ \mathbf{b}_1\ \mathbf{b}_2]$$

or $$[\mathbf{r}\ \mathbf{b}_1\ \mathbf{b}_2] = [\mathbf{a}_2\mathbf{b}_1\ \mathbf{b}_2]$$

Example 70. Find the equation of plane passing through the point (0, 7, –7) and containing the line $\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}$.

Sol. Let the equation of the plane passing through the point $(0, 7, -7)$ be $a(x-0) + b(y-7) + c(z+7) = 0$ ...(i)

The line $\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}$ passes through the point $(-1, 3, -2)$ and has direction ratios $-3, 2, 1$. If (i) contains this line, it must pass through $(-1, 3, -2)$ and must be parallel to the line. Therefore,

$$a(-1) + b(3-7) + c(-2+7) = 0$$

i.e. $$a(-1) + b(-4) + c(5) = 0 \quad \text{...(ii)}$$

and $$-3a + 2b + 1c = 0 \quad \text{...(iii)}$$

On solving Eqs. (ii) and (iii) by cross multiplication, we get

$$\frac{a}{-14} = \frac{b}{-14} = \frac{c}{-14} \Rightarrow \frac{a}{1} = \frac{b}{1} = \frac{c}{1} = \lambda \quad \text{(say)}$$

$$\Rightarrow \quad a = \lambda, b = \lambda, c = \lambda$$

Putting the values of a, b, c in (i), we obtain

$$\lambda(x-0) + \lambda(y-7) + \lambda(z+7) = 0$$

$$\Rightarrow \quad x + y + z = 0$$

This is the equation of the required plane.

Example 71. Prove that the lines $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$ and $\frac{x-2}{1} = \frac{y-4}{4} = \frac{z-6}{7}$ are coplanar. Also, find the plane containing these two lines.

Sol. We know that, the line $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$

and $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$ are coplanar if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

and the equation of the plane containing these two lines is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

Here, $x_1 = -1, y_1 = -3, z_1 = -5,$

$x_2 = 2, y_2 = 4, z_2 = 6, l_1 = 3,$

$m_1 = 5, n_1 = 7, l_2 = 1, m_2 = 4, n_2 = 7.$

$$\therefore \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = \begin{vmatrix} 3 & 7 & 11 \\ 3 & 5 & 7 \\ 1 & 4 & 7 \end{vmatrix} = 0$$

so, the given lines are coplanar.

The equation of the plane containing the lines is

$$\begin{vmatrix} x+1 & y+3 & z+5 \\ 3 & 5 & 7 \\ 1 & 4 & 7 \end{vmatrix} = 0$$

ar $(x+1)(35-28) - (y+3)(21-7) + (z+5)(12-5) = 0$

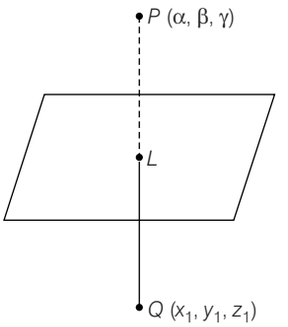
or $$x - 2y + z = 0.$$

Image of a Point in a Plane

To find the image of the point (α, β, γ) in the plane

$$ax + by + cz + d = 0 \quad \text{...(i)}$$

Let $Q(x_1, y_1, z_1)$ be the image of point P in the plane (i).



Let PQ meet plane (i) at L, direction ratios of normal to plane (i) are (a, b, c), since PQ perpendicular of plane (i).

So, direction ratios of PQ are a, b, c.

$\Rightarrow$ Equation of line PQ is,

$$\frac{x-\alpha}{a} = \frac{y-\beta}{b} = \frac{z-\gamma}{c} = r \quad \text{(say)}$$

Coordinate of any point on line PQ may be taken as

$$(ar+\alpha, br+\beta, cr+\gamma)$$

Let $$Q(ar+\alpha, br+\beta, cr+\gamma)$$

Since, L is the middle point of PQ

$$\therefore \quad L = \left(\alpha + \frac{ar}{2}, \beta + \frac{br}{2}, \gamma + \frac{cr}{2}\right)$$

Since, L lies on plane (i), we get

$$a\left(\frac{ar}{2}+\alpha\right)+b\left(\frac{br}{2}+\beta\right)+c\left(\frac{cr}{2}+\gamma\right)+d=0$$

$$\Rightarrow \quad (a^2+b^2+c^2)\frac{r}{2}=-(a\alpha+b\beta+c\gamma+d)$$

$$\Rightarrow \quad r=\frac{-2(a\alpha+b\beta+c\gamma+d)}{a^2+b^2+c^2}$$

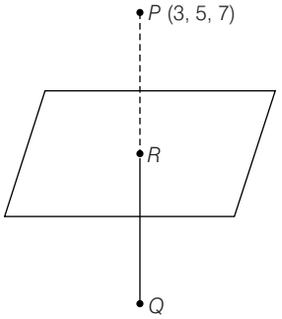
Example 72. Find the image of the point $P(3,5,7)$ in the plane $2x+y+z=0$.

Sol. Given plane is $2x+y+z=0$ ...(i)

and the point $P(3, 5, 7)$

DR's of normal to the plane (i) are 2, 1, 1.

Let Q be the image of a point P in plane (i).



$\therefore$ Equation of line PR is $\frac{x-3}{2}=\frac{y-5}{1}=\frac{z-7}{1}=r$

Let $R(2r+3, r+5, r+7)$

Since, R lies on plane (i).

$\therefore \quad 2(2r+3)+(r+5)+(r+7)=0\ ;\ 6r+18=0$

$\therefore \quad r=-3 \ \ \therefore \quad R\equiv(-3,2,4)$

Let $Q\equiv(\alpha,\beta,\gamma)$

Since, R is mid-point of PQ.

$\therefore \quad -3=\frac{\alpha+3}{2} \ \Rightarrow\ \alpha=-9$

$$2=\frac{\beta+5}{2} \ \Rightarrow\ \beta=-1$$

$$4=\frac{\gamma+7}{2} \ \Rightarrow\ \gamma=1$$

$\therefore \quad Q\equiv(-9,1,1)$

Example 73. Find the length and the foot of the perpendicular from the point $(7,14,5)$ to the plane $2x+4y-z=2$.

Sol. The required length $=\frac{2(7)+4(14)-(5)-2}{\sqrt{2^2+4^2+1^2}}$

$$=\frac{14+56-5-2}{\sqrt{4+16+1}}=\frac{63}{\sqrt{21}}$$

Let the coordinates of the foot of the perpendicular from the point $P(7, 14, 5)$ be $M(\alpha, \beta, \gamma)$.

Then, the direction ratios of PM are $\alpha-7, \beta-14$ and $\gamma-5$.

Therefore, the direction ratios of the normal to the plane are $\alpha-7, \beta-14$ and $\gamma-5$.

But the direction ratios of normal to the given plane $2x+4y-z=2$ are 2, 4 and -1.

Hence, $\quad \frac{\alpha-7}{2}=\frac{\beta-14}{4}=\frac{\gamma-5}{-1}=k$

$\therefore \quad \alpha=2k+7, \beta=4k+14$ and $\gamma=-k+5$ ...(i)

Since, α, β and γ lie on the plane $2x+4y-z=2$,

$2\alpha+4\beta-\gamma=2$

$\Rightarrow\ 2(7+2k)+4(14+4k)-(5-k)=2$

$\Rightarrow \quad 14+4k+56+16k-5+k=2$

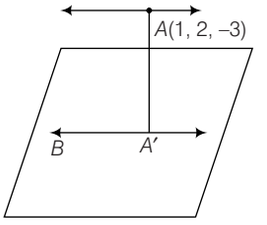
$\Rightarrow \quad 21k=-63 \ \Rightarrow\ k=-3$

Now, putting $k=-3$ in (i), we get $\alpha=1, \beta=2, \gamma=8$

Hence, the foot of the perpendicular is (1, 2, 8).

Example 74. Find the image of the line $\frac{x-1}{9}=\frac{y-2}{-1}=\frac{z+3}{-3}$ in the plane $3x-3y+10z-26=0$.

Sol.



$$\frac{x-1}{9}=\frac{y-2}{-1}=\frac{z+3}{-3} \qquad ...(i)$$

$$3x-3y+10z-26=0 \qquad ...(ii)$$

The direction ratios of the line are 9, -1 and -3 and direction ratios of the normal to the given plane are 3, -3 and 10.

Since, $9\cdot3+(-1)(-3)+(-3)\,10=0$ and the point $(1, 2, -3)$ of line (i) does not lie in plane (ii) for $3\cdot1-3\cdot2+10\cdot(-3)-26\neq0$, line (i) is parallel to plane (ii). Let A' be the image of point $A(1, 2, -3)$ in plane (ii). Then the image of the line (i) in the plane (ii) is the line through A' and parallel to the line (i).

Let point A' be (p, q, r). Then

$$\frac{p-1}{3}=\frac{q-2}{-3}=\frac{r+3}{10}$$

$$=-\frac{(3(1)-3(2)+10(-3)-26)}{9+9+100}=\frac{1}{2}$$

The point $A'\left(\frac{5}{2},\frac{1}{2},2\right)$

The equation of line BA' is $\frac{x-\left(\frac{5}{2}\right)}{9}=\frac{y-\left(\frac{1}{2}\right)}{-1}=\frac{z-2}{-3}$

Exercise for Session 3

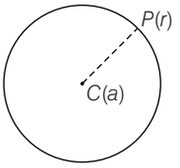
1. Find the equation of plane passing through the point (1, 2, 3) and having the vector $\mathbf{r} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ normal to it.
2. Find a unit vector normal to the plane through the points (1, 1, 1), (– 1, 2, 3) and (2, – 1, 3).
3. Show that the four points (0, – 1, 0), (2, 1, – 1), (1, 1, 1) and (3, 3, 0) are coplaner. Also, find equation of plane through them.
4. Find the equation of plane passing through the line of intersection of planes $3x + 4y - 4 = 0$ and $x + 7y + 3z + z = 0$ and also through origin.
5. Find equation of angle bisector of plane $x + 2y + 3z - z = 0$ and $2x - 3y + z + 4 = 0$.
6. Find image of point $(1, 3, 4)$ in the plane $2x - y + z + 3 = 0$.
7. Find the angle between the line $\frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6}$ and the plane $3x + y + z = 7$.
8. Find the equation of plane which passes through the point (1, 2, 0) and which is perpendicular to the plane $x - y + z = 3$ and $2x + y - z + 4 = 0$.
9. Find the distance of the point $(-1, -5, -10)$ from the point of intersection of the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and plane $x - y + z = 5$.
10. Find the equation of a plane containing the lines $\frac{x-5}{4} = \frac{y+7}{4} = \frac{z+3}{-5}$ and $\frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3}$.
11. Find the equation of the plane which passes through the point $(3, 4, -5)$ and contains the line $\frac{x+1}{2} = \frac{y-1}{3} = \frac{z+2}{-1}$.
12. Find the equation of the planes parallel to the plane $x - 2y + 2z - 3 = 0$. Which is at a unit distance from the point $(1, 2, 3)$.
13. Find the equation of the bisector planes of the angles between the plane $x + 2y + 2z = 19$ and $4x - 3y + 12z + 3 = 0$ and specify the plane which bisects the acute angle and the plane which bisects the obtuse angle.
14. Find the equation of the image of the plane $x - 2y + 2z = 3$ in the plane $x + y + z = 1$.
15. Find the equation of a plane which passes through the point (1, 2, 3) and which is at the maximum distance from the point (– 1, 0, 2).

Session 4

Sphere

Sphere

A sphere is the locus of a point which moves in space in such a way that its distance from a fixed point always remains constant. The fixed point is called the centre of the sphere and the fixed distance is called the radius of sphere. Shown as in adjoining figure.

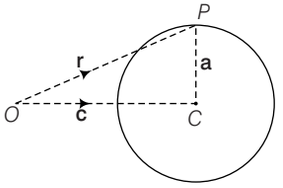


Equation of Sphere whose Centre *c* and Radius is *a*

Let O be the origin of reference and $\mathbf{C}$ be the centre of sphere whose position vector is $\mathbf{c}$. Let P be any point on the surface of the sphere whose position vector is $\mathbf{r}$.

Thus, $\mathbf{OP} = \mathbf{r}$ and $\mathbf{OC} = \mathbf{c}$

$\therefore$ $\mathbf{CP} = \mathbf{OP} - \mathbf{OC} = \mathbf{r} - \mathbf{c}$



Now, $|\mathbf{r} - \mathbf{c}| = \mathbf{a}$ [radius of sphere]

$\Rightarrow$ $|\mathbf{r} - \mathbf{c}|^2 = \mathbf{a}^2$

$\Rightarrow$ $(\mathbf{r} - \mathbf{c}) \cdot (\mathbf{r} - \mathbf{c}) = \mathbf{a}^2$

$\Rightarrow$ $r^2 - 2\mathbf{r} \cdot \mathbf{c} + c^2 = \mathbf{a}^2$

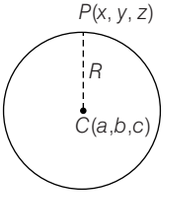
$\Rightarrow$ $r^2 - 2r \cdot \mathbf{c} + (c^2 - a^2) = 0$

which is the required equation of sphere.

Cartesian Equation of a Sphere

The equation of sphere with centre (a, b, c) and radius R is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2$$



Proof. Let C be the centre of the sphere.

Then, coordinates of C are (a, b, c). Let $P(x, y, z)$ be any point on the sphere, then

$$CP = R$$

$\Rightarrow$ $$CP^2 = R^2$$

$\Rightarrow$ $$(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2$$

Since, $P(x, y, z)$ is an arbitrary point on the sphere, therefore required equation of the sphere is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2$$

Remarks

1. The above equation is called the central form of a sphere. If the centre is at the origin, then equation of sphere is,
$$x^2 + y^2 + z^2 = R^2$$
(known as the standard form of the sphere)
2. Above equation can also be written as
$x^2 + y^2 + z^2 - 2ax - 2by - 2cz + (a^2 + b^2 + c^2 - R^2) = 0$
which has the following characteristics of the equation of sphere
   (i) It is a second degree equation in x, y and z.
   (ii) The coefficient of x^2, y^2 and z^2 are all equal.
   (iii) The term containing the product of xy, yz and zx are absent.

Example 75. Find the vector equation of a sphere with centre having the position vector $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and radius $\sqrt{3}$.

Sol. We know that equation of sphere is

$$|\mathbf{r} - \mathbf{c}| = a \quad \text{(vector form)}$$

$\Rightarrow$ $$|\mathbf{r} - (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})| = \sqrt{3}$$

which is the required equation of sphere.

Example 76. Find the equation of sphere whose centre is (5, 2, 3) and radius is 2 in cartesian form.

Sol. The required equation of the sphere is

$$(x-5)^2 + (y-2)^2 + (z-3)^2 = 2^2$$

$\Rightarrow$ $$x^2 + y^2 + z^3 - 10x - 4y - 6z + 34 = 0$$

Example 77. Find the equation of a sphere whose centre is (3, 1, 2) and radius is 5.

Sol. The equation of the sphere whose centre is (3, 1, 2) and radius is 5, is

$$(x-3)^2+(y-1)^2+(z-2)^2=5^2$$

$$x^2+y^2+z^2-6x-2y-4z-11=0$$

General Equation of Sphere

The equation $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ represents a sphere with centre $(-u,-v,-w)$ i.e.

$$\left(-\frac{1}{2}\text{ coefficient of } x,-\frac{1}{2}\text{ coefficient of } y,-\frac{1}{2}\text{ coefficient of } z\right)$$

and radius $=\sqrt{u^2+v^2+w^2-d}$.

Note

The equation $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ represents a real sphere, if $u^2+v^2+w^2-d>0$. If $u^2+v^2+w^2-d=0$, then it represents a point sphere. The sphere is imaginary, if $u^2+v^2+w^2-d<0$.

Example 78. Find the centre and radius of the sphere $2x^2+2y^2+2z^2-2x-4y+2z+3=0$.

Sol. The given equation

$$x^2+y^2+z^2-x-2y+z+\frac{3}{2}=0;$$

where cetnre is

$$\left(-\frac{1}{2}\text{ coefficient of } x,-\frac{1}{2}\text{ coefficient of } y,-\frac{1}{2}\text{ coefficient of } z\right)$$

$$\therefore \quad \text{Centre}=\left(\frac{1}{2},-1,-\frac{1}{2}\right)$$

and $$\text{Radius}=\sqrt{\left(\frac{1}{2}\right)^2+(-1)^2+\left(-\frac{1}{2}\right)^2-\frac{3}{2}}$$

$$=\sqrt{\frac{1}{4}+1+\frac{1}{4}-\frac{3}{2}}=0$$

$\therefore$ Given sphere represents a point sphere $\left(\frac{1}{2},-1,-\frac{1}{2}\right)$.

Example 79. Find the equation of the sphere passing through (0, 0, 0), (1, 0, 0), (0, 1, 0) and (0, 0, 1).

Sol. Let the equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \qquad ...(i)$$

As (i) passes through (0, 0, 0), (1, 0, 0), (0,1, 0) and (0, 0, 1), we must have $d=0, 1+2u+d=0$

$$1+2v+d=0 \text{ and } 1+2w+d=0$$

Since, $d=0$, we get $2u=2v=2w=-1$

Thus, the equation of the required sphere is

$$x^2+y^2+z^2-x-y-z=0.$$

Example 80. Find the equation of a sphere which passes through (1, 0, 0), (0, 1, 0) and (0, 0, 1) and has radius as small as possible.

Sol. Let the equation of the required sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \qquad ...(i)$$

As the sphere passes through (1, 0, 0), (0,1, 0) and (0, 0, 1), we get

$$1+2u+d=0, 1+2v+d=0 \text{ and } 1+2w+d=0$$

$$\Rightarrow \qquad u=v=w=-\frac{1}{2}(d+1)$$

If R is the radius of the sphere, then $R^2=u^2+v^2+w^2-d$

$$\Rightarrow \qquad R^2=\frac{3}{4}(d+1)^2-d$$

$$=\frac{3}{4}\left[d^2+2d+1-\frac{4}{3}d\right]$$

$$=\frac{3}{4}\left[d^2+\frac{2}{3}d+1\right]$$

$$=\frac{3}{4}\left[\left(d+\frac{1}{3}\right)^2+1-\frac{1}{9}\right]$$

$$=\frac{3}{4}\left[\left(d+\frac{1}{3}\right)^2+\frac{8}{9}\right]$$

The last equation shows that R^2 (and thus R) will be the least if an only if $d=-\frac{1}{3}$.

Therefore, $u=v=w=-\frac{1}{2}\left(1-\frac{1}{3}\right)=-\frac{1}{3}$

Hence, the equation of the required sphere is $x^2+y^2+z^2-\frac{2}{3}$

$(x+y+z)-\frac{1}{3}=0$ or $3(x^2+y^2+z^2)-2(x+y+z)-1=0$.

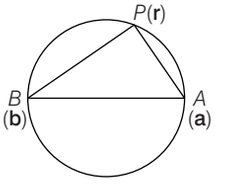
Diameter Form of the Equation of a Sphere

If the position vectors of the extremities of a diameter of a sphere are $\mathbf{a}$ and $\mathbf{b}$, then its equation is

$$(\mathbf{r}-\mathbf{a})\cdot(\mathbf{r}-\mathbf{b})=0$$

$$\Rightarrow \qquad |\mathbf{r}|^2-\mathbf{r}\cdot(\mathbf{a}+\mathbf{b})+\mathbf{a}\cdot\mathbf{b}=0$$

Proof. Let $\mathbf{a}$ and $\mathbf{b}$ be the position vectors of the extremities A and B of a diameter AB of sphere. Let $\mathbf{r}$ be the position vector of any point P on the sphere. Then,



$$\mathbf{AP}=\mathbf{r}-\mathbf{a} \text{ and } \mathbf{BP}=\mathbf{r}-\mathbf{b}$$

Since, the diameter of a sphere subtends a right at any point on the sphere, therefore

$$\Rightarrow \quad \angle \mathbf{APB} = \frac{\pi}{2}$$

$$\Rightarrow \quad \mathbf{AP} \cdot \mathbf{BP} = 0$$

$$\Rightarrow \quad (\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{b}) = 0$$

$$\mathbf{r} \cdot \mathbf{r} - \mathbf{r} \cdot \mathbf{b} - \mathbf{r} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} = 0$$

$$|\mathbf{r}|^2 - (\mathbf{a} + \mathbf{b}) \cdot \mathbf{r} + \mathbf{a} \cdot \mathbf{b} = 0$$

This is the required equation of sphere.

Vector Form

If the position vectors of the extremities of a diameter of a sphere are **a** and **b**, then its equation is

$$|\mathbf{r} - \mathbf{a}|^2 + |\mathbf{r} - \mathbf{b}|^2 = |\mathbf{a} - \mathbf{b}|^2$$

Proof. Let **a** and **b** be the position vectors of the extremities A and B of a diameter of a sphere. Let **r** be the position vector of any point P on the sphere, then

$$\mathbf{AP} = \mathbf{r} - \mathbf{a}$$

and $$\mathbf{BP} = \mathbf{r} - \mathbf{b}$$

Since, ΔAPB is a right angled triangle.

$$\therefore \quad AP^2 + BP^2 = AB^2$$

$$\Rightarrow \quad |\mathbf{AP}|^2 + |\mathbf{BP}|^2 = |\mathbf{AB}|^2$$

$$\Rightarrow \quad |\mathbf{r} - \mathbf{a}|^2 + |\mathbf{r} - \mathbf{b}|^2 = |\mathbf{a} - \mathbf{b}|^2$$

This is the required equation of the sphere.

Cartesian Form

If (x_1, y_1, z_1) and (x_2, y_2, z_2) are the coordinates of the extremities of a diameter of a sphere, then its equation is,

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$$

Example 81. Find the equation of the sphere described on the joint of points A and B having Position position vectors $2\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 7\hat{\mathbf{k}}$ and $-2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$, respectively, as the diameter. Find the centre and the radius of the sphere.

Sol. If point P with position vector $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ is any point on the sphere, then $\mathbf{AP} \cdot \mathbf{BP} = 0$

$$(x - 2)(x + 2) + (y - 6)(y - 4) + (z + 7)(z + 3) = 0$$

$$\Rightarrow (x^2 - 4) + (y^2 - 10y + 24) + (z^2 + 10z + 21) = 0$$

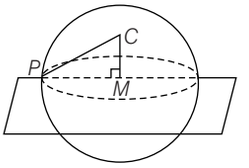
$$\Rightarrow \quad x^2 + y^2 + z^2 - 10y + 10z + 41 = 0$$

The centre of this sphere is $(0, 5, -5)$ and its radius is

$$\sqrt{5^2 + (-5)^2 - 41} = \sqrt{9} = 3$$

Section of a Sphere by a Plane

Consider a sphere intersected by a plane. The set of points common to both sphere and plane is called a plane section of a sphere.



It can be easily seen the plane section of sphere is a circle.

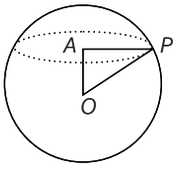
Let C be the centre of the sphere and M be the foot of the perpendicular from C on the plane. Then, M is the centre of the circle and radius of circle is given by PM.

i.e. $$PM = \sqrt{CP^2 - CM^2}$$

The centre M of the circle is the point of intersection of the plane and line CM, which passes through C and is perpendicular to given plane.

Example 82. ***Find the radius of the circular section in which the sphere*** $|\mathbf{r}| = 5$ ***is cut by the plane*** $\mathbf{r} \cdot (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) = 3\sqrt{3}$.

Sol. Let A be the foot of the perpendicular from the centre O to the plane $\mathbf{r} \cdot (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) - 3\sqrt{3} = 0$



Then, $|OA| = \left| \dfrac{0 \cdot (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) - 3\sqrt{3}}{|\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}|} \right| = \dfrac{3\sqrt{3}}{\sqrt{3}} = 3$ (perpendicular distance of a point from the plane)

If P is any point on the circle, then P lies on the plane as well as on the sphere. Therefore, OP = radius of the sphere = 5

Now, $$AP^2 = OP^2 - OA^2 = 5^2 - 3^2 = 16$$

$$\Rightarrow \quad AP = 4$$

Example 83. ***Find the centre of the circle given by*** $\mathbf{r} \cdot (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) = 15$ ***and*** $|\mathbf{r} - (\hat{\mathbf{j}} + 2\hat{\mathbf{k}})| = 4$.

Sol. The equation of a line through the centre $\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and normal to the given plane is

$$\mathbf{r} = (\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) + \lambda(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) \quad \text{...(i)}$$

This meets the plane at a point for which we must have

$$[(\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) + \lambda(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}})] \cdot (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) = 15$$

$$\Rightarrow \quad 6 + 9\lambda = 15$$

$$\Rightarrow \quad \lambda = 1$$

On putting $\lambda = 1$ in Eq. (i), we obtain the position vectors of the centre as $\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$, Hence, the coordinates of the centre of the circle are (1, 3, 4).

Condition of Tengency of a Plane to a Sphere

A plane touches a given sphere, if the perpendicular distance from the centre of the sphere to the planes is equal to the radius of the sphere.

Vector Form

The plane $\mathbf{r}\cdot\mathbf{n}=d$ touches the sphere $|\mathbf{r}-\mathbf{a}|=R$,

if $\dfrac{|\mathbf{a}\cdot\mathbf{n}-d|}{|\mathbf{n}|}=R$

Cartesian Form

The plane $lx+my+nz=p$ touches the sphere

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$,

if $(ul+vm+wn+p)^2=(l^2+m^2+n^2)(u^2+v^2+w^2-d)$

Example 84. Show that the plane $2x-2y+z+12=0$ touches the sphere $x^2+y^2+z^2-2z-4y+2z-3=0$.

Sol. The given plane will touch the given sphere if the perpendicular distance from the centre of the sphere to the plane is equal to the radius of the sphere. The centre of the given sphere $x^2+y^2+z^2-2x-4y+2z-3=0$ is $(1, 2, -1)$ and its radius is $\sqrt{1^2+2^2+(-1)^2-(-3)}=3$.

Length of the perpendicular from $(1, 2, -1)$ to the plane $2x-2y+z+12=0$ is

$$\left|\frac{2(1)-2(2)+(-1)+12}{\sqrt{2^2+(-2)^2+1^2}}\right|=\frac{9}{3}=3$$

Thus, the given plane touches the given sphere.

Example 85. ***Find the equation of the sphere whose centre has the position vector*** $3\hat{\mathbf{i}}+6\hat{\mathbf{j}}-4\hat{\mathbf{k}}$ ***and which touches the plane*** $\mathbf{r}\cdot(2\hat{\mathbf{i}}-2\hat{\mathbf{j}}-\hat{\mathbf{k}})=10$.

Sol. Let the radius of the required sphere be R. Then, its equation is

$$|\mathbf{r}-(3\hat{\mathbf{i}}+6\hat{\mathbf{j}}-4\hat{\mathbf{k}})|=R \qquad \text{...(i)}$$

Since, the plane $\mathbf{r}\cdot(2\hat{\mathbf{i}}-2\hat{\mathbf{j}}-\hat{\mathbf{k}})=10$ touches the sphere (i), therefore length of perpendicular from the centre to the plane $\mathbf{r}\cdot(2\hat{\mathbf{i}}-2\hat{\mathbf{j}}-\hat{\mathbf{k}})=10$ is equal to R.

i.e. $\dfrac{|(3\hat{\mathbf{i}}-6\hat{\mathbf{j}}-4\hat{\mathbf{k}})\cdot(2\hat{\mathbf{i}}-2\hat{\mathbf{j}}-\hat{\mathbf{k}})-10|}{|2\hat{\mathbf{i}}-2\hat{\mathbf{j}}-\hat{\mathbf{k}}|}=R \;\Rightarrow\; R=4$

On putting $R=4$ in Eq. (i), we obtain $|\mathbf{r}-(3\hat{\mathbf{i}}+6\hat{\mathbf{j}}-4\hat{\mathbf{k}})|=4$ as the equation of the required sphere.

Example 86. A variable plane passes through a fixed point (a,b,c) and cuts the coordinate axes at points A, B and C. Show that the locus of the centre of the sphere $OABC$ is $\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}=2$.

Sol. Let (α, β, γ) be any point on the locus. Then according to the given condition, (α, β, γ) is the centre of the sphere through the origin. Therefore, its equation is given by

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=(0-\alpha)^2+(0-\beta)^2+(0-\gamma)^2$$

$$x^2+y^2+z^2-2\alpha x-2\beta y-2\gamma z=0$$

To obtain its point of intersection with the X-axis, we put $y=0$ and $z=0$, so that

$$x^2-2\alpha x=0$$

$$\Rightarrow \quad x(x-2\alpha)=0$$

$$\Rightarrow \quad x=0 \quad \text{or} \quad x=2\alpha$$

Thus, the plane meets X-axis at $O(0, 0, 0)$ and $A(2\alpha, 0, 0)$. Similarly, it meets Y-axis at $O(0, 0, 0)$ and $B(0, 2\beta, 0)$, and Z-axis at $O(0, 0, 0)$ and $C(0, 0, 2\gamma)$.

The equation of the plane through A, B and C is

$$\frac{x}{2\alpha}+\frac{y}{2\beta}+\frac{z}{2\gamma}=1 \qquad \text{(intercept form)}$$

Since, it passes through (a, b, c), we get

$$\frac{a}{2\alpha}+\frac{b}{2\beta}+\frac{c}{2\gamma}=1$$

or

$$\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}=2$$

Hence, locus of (α, β, γ) is $\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}=2$.

Example 87. A sphere of constant radius k passes through the origin and meets the axis at A, B and C. Prove that the centroid of triangle ABC lies on the sphere $9(x^2+y^2+z^2)=4k^2$.

Sol. Let the equation of any sphere passing through the origin and having radius k be

$$x^2+y^2+z^2+2ux+2vy+2wz=0$$

As the radius of the sphere is k, we get

$$u^2+v^2+w^2=k^2$$

Note that (i) meets the X-axis at $O(0, 0, 0)$ and $A(-2u, 0, 0)$; Y-axis at $O(0, 0, 0)$ and $B(0, -2v, 0)$, and Z-axis at $O(0, 0, 0)$ and $C(0, 0, -2w)$.

Let the centroid of the triangle ABC be (α, β, γ), Then

$$\alpha=-\frac{2u}{3}, \beta=-\frac{2v}{3}, \gamma=-\frac{2w}{3}$$

$$\Rightarrow \quad u=-\frac{3\alpha}{2}, v=-\frac{3\beta}{2}, w=-\frac{3v}{2}$$

Putting this in (ii), we get

$$\left(\frac{-3}{2}\alpha\right)^2+\left(\frac{-3}{2}\beta\right)^2+\left(\frac{-3}{2}\gamma\right)^2=k^2$$

$$\Rightarrow \quad \alpha^2+\beta^2+\gamma^2=\frac{4}{9}k^2$$

This shows that the centroid of triangle ABC lies on

$$x^2+y^2+z^2=\frac{4}{9}k^2.$$

Exercise for Session 4

1. Find the centre and radius of sphere $2(x-5)(x+1)+2(y+5)(y-1)+2(z-2)(z+2)=7$
2. Obtain the equation of the sphere with the points $(1,-1,1)$ and $(3,-3,3)$ as the extremities of a diametre and find the coordinates of its centre.
3. Find the equation of sphere which passes through (1, 0, 0) and has its centre on the positive direction of *Y*-axis and has radius 2.
4. Find the equation of sphere if it touches the plane $\mathbf{r}\cdot(2\hat{\mathbf{i}}-2\hat{\mathbf{j}}-\hat{\mathbf{k}})=0$ and the position vector of its centre is $3\hat{\mathbf{i}}+6\hat{\mathbf{j}}-4\hat{\mathbf{k}}$.
5. Find the value of λ for which the plane $x+y+z=\sqrt{3}\lambda$ touches the sphere $x^2+y^2+z^2-2x-2y-2z=6$.
6. Find the equation of sphere concentric with sphere $2x^2+2y^2+2z^2-6x+2y-4z=1$ and double its radius.
7. A sphere has the equation $|\mathbf{r}-\mathbf{a}|^2+|\mathbf{r}-\mathbf{b}|^2=72$, where $\mathbf{a}=\hat{\mathbf{i}}+3\hat{\mathbf{j}}-6\hat{\mathbf{k}}$ and $\mathbf{b}=2\hat{\mathbf{i}}+4\hat{\mathbf{j}}+2\hat{\mathbf{k}}$

   Find

   (i) The centre of sphere

   (ii) The radius of sphere

   (iii) Perpendicular distance from the centre of the sphere to the plane $\mathbf{r}\cdot(2\hat{\mathbf{i}}+2\hat{\mathbf{j}}-\hat{\mathbf{k}})+3=0$.

JEE Type Solved Examples :
Single Option Correct Type Questions

● **Ex. 1** *If a line makes angle* α, β *and* γ *with the coordinates axes, then*

(a) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma - 1 = 0$

(b) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma - 2 = 0$

(c) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma + 1 = 0$

(d) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma + 2 = 0$

Sol. (c) If $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are the DC's of a line, then

$$2\cos^2\alpha + 2\cos^2\beta + 2\cos^2\gamma = 2$$

$$\Rightarrow \quad 1 + \cos 2\alpha + 1 + \cos 2\beta + 1 + \cos 2\gamma = 2$$

$$\Rightarrow \quad \cos 2\alpha + \cos 2\beta + \cos 2\gamma + 1 = 0$$

● **Ex. 2** *The points* $(5, -4, 2)\,(4, -3, 1), (7, -6, 4)$ *and* $(8, -7, 5)$ *are the vertices of*

(a) a rectangle (b) a square

(c) a parallelogram (d) None of these

Sol. (c) Let $A(5, -4, 2)$, $B(4, -3, 1)$, $C(7, -6, 4)$ and $D(8, -7, 5)$

$$\mathbf{AB} = -\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$$

$$\mathbf{BC} = 3\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$$

$$\mathbf{CD} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$$

and $\mathbf{DA} = -3\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$

Clearly $\mathbf{AB} \parallel \mathbf{CD}$ and $\mathbf{BC} \parallel \mathbf{DA}$

Also, $\mathbf{AB} \cdot \mathbf{BC} = -9 \neq 0$

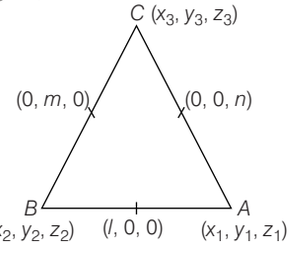
$\therefore$ $ABCD$ is a parallelogram.

● **Ex. 3** *In* ΔABC *the mid-point of the sides AB, BC and CA are respectively* $(l, 0, 0), (0, m, 0)$ *and* $(0, 0, n)$*. Then,* $\dfrac{AB^2 + BC^2 + CA^2}{l^2 + m^2 + n^2}$ *is equal to*

(a) 2 (b) 4

(c) 8 (d) 16

Sol. (c) From the figure,



$$x_1 + x_2 = 2l,\ y_1 + y_2 = 0,\ z_1 + z_2 = 0$$

$$x_2 + x_3 = 0,\ y_2 + y_3 = 2m,\ z_2 + z_3 = 0$$

and $x_1 + x_3 = 0,\ y_1 + y_3 = 0,\ z_1 + z_3 = 2n$

On solving, we get

$$x_1 = l,\ x_2 = l,\ x_3 = -l$$

$$y_1 = -m,\ y_2 = m,\ y_3 = m$$

and $z_1 = n,\ z_2 = -n,\ z_3 = n$

$\therefore$ Coordinates are $A(l, -m, n)$, $B(l, m, -n)$ and $C(-l, m, n)$

$$\therefore \quad \frac{AB^2 + BC^2 + CA^2}{l^2 + m^2 + n^2}$$

$$= \frac{(4m^2 + 4n^2) + (4l^2 + 4n^2) + (4l^2 + 4m^2)}{l^2 + m^2 + n^2} = 8$$

● **Ex. 4** *The angle between a line with direction ratios proportional to* 2, 2, 1 *and a line joining* $(3, 1, 4)$ *to* $(7, 2, 12)$*, is*

(a) $\cos^{-1}\left(\frac{2}{3}\right)$ (b) $\cos^{-1}\left(-\frac{2}{3}\right)$

(c) $\tan^{-1}\left(\frac{2}{3}\right)$ (d) None of these

Sol. (a) A line with direction ratios proportional to 2, 2, 1 is parallel to the vector $\mathbf{a} = 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$.

Line joining $P(3, 1, 4)$ to $Q(7, 2, 12)$ is parallel to the vector

$$\mathbf{PQ} = 4\hat{\mathbf{i}} + \hat{\mathbf{j}} + 8\hat{\mathbf{k}}.$$

Let θ be the required angle. Then,

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{PQ}}{|\mathbf{a}|\,|\mathbf{PQ}|} = \frac{8 + 2 + 8}{\sqrt{4 + 4 + 1}\,\sqrt{16 + 1 + 64}}$$

$$\Rightarrow \quad \cos\theta = \frac{18}{3 \times 9} = \frac{2}{3} \ \Rightarrow\ \theta = \cos^{-1}\left(\frac{2}{3}\right)$$

● **Ex. 5** *The angle between the lines* $2x = 3y = -z$ *and* $6x = -y = -4z$ *is*

(a) 30° (b) 45°

(c) 60° (d) 90°

Sol. (d) Given, equation of lines can be rewritten as

$$\frac{x}{1/2} = \frac{y}{1/3} = \frac{z}{-1}$$

and $\dfrac{x}{1/6} = \dfrac{y}{-1} = \dfrac{z}{-1/4}$

$$\therefore \quad \cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\,\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$= \frac{\frac{1}{2} \times \frac{1}{6} + \frac{1}{3} \times (-1) - 1 \times \left(-\frac{1}{4}\right)}{\sqrt{\frac{1}{4} + \frac{1}{9} + 1}\,\sqrt{\frac{1}{36} + 1 + \frac{1}{16}}}$$

$$= \frac{\frac{1}{12} - \frac{1}{3} + \frac{1}{4}}{\sqrt{\frac{1}{4} + \frac{1}{9} + 1}\,\sqrt{\frac{1}{36} + 1 + \frac{1}{16}}}$$

$$\Rightarrow \quad \cos\theta = 0 \ \Rightarrow\ \theta = 90°$$

● ***Ex. 6*** *A line makes the same angle* θ *with X-axis and Z-axis. If the angle* β*, which it makes with Y-axis, is such that* $\sin^2\beta = 3\sin^2\theta$*, then the value of* $\cos^2\theta$ *is*

(a) $\frac{1}{5}$ (b) $\frac{2}{5}$

(c) $\frac{3}{5}$ (d) $\frac{2}{3}$

Sol. (c) Since, $\cos^2\theta + \cos^2\beta + \cos^2\theta = 1$ $[\because l^2 + m^2 + n^2 = 1]$

$\Rightarrow$ $2\cos^2\theta + 1 - 3\sin^2\theta = 1$ $[\because \sin^2\beta = 3\sin^2\theta]$

$\Rightarrow$ $2\cos^2\theta - 3(1 - \cos^2\theta) = 0$

$\Rightarrow$ $5\cos^2\theta = 3 \Rightarrow \cos^2\theta = \frac{3}{5}$

● ***Ex. 7*** *The projection of a line segment on the coordinate axes are 2, 3, 6. Then, the length of the line segment is*

(a) 7 (b) 5

(c) 1 (d) 11

Sol. (a) Let the length of the line segment be r and its direction cosines be l, m, n. Then, its projections on the coordinate axes are lr, mr, nr.

$\therefore$ $lr = 2, mr = 3$ and $nr = 6$

$\Rightarrow$ $l^2r^2 + m^2r^2 + n^2r^2 = 4 + 9 + 36$

$\Rightarrow$ $r^2(l^2 + m^2 + n^2) = 49$

$\Rightarrow$ $r^2 = 49 \Rightarrow r = 7$ $[\because l^2 + m^2 + n^2 = 1]$

● ***Ex. 8*** *The equation of the straight line through the origin and parallel to the line* $(b+c)x + (c+a)y + (a+b)z = k = (b-c)x + (c-a)y + (a-b)z$ *are*

(a) $\frac{x}{b^2 - c^2} = \frac{y}{c^2 - a^2} = \frac{z}{a^2 - b^2}$

(b) $\frac{x}{b} = \frac{y}{b} = \frac{z}{a}$

(c) $\frac{x}{a^2 - bc} = \frac{y}{b^2 - ca} = \frac{z}{c^2 - ab}$

(d) None of the above

Sol. (c) Equations of straight line through the origin are

$$\frac{x-0}{l} = \frac{y-0}{m} = \frac{z-0}{n}$$

where, $l(b+c) + m(c+a) + n(a+b) = 0$

and $l(b-c) + m(c-a) + n(a-b) = 0$

On solving, $\frac{1}{2(a^2 - bc)} = \frac{m}{2(b^2 - ca)}$

$$= \frac{n}{2(c^2 - ab)}$$

Equations of the straight line are

$$\frac{x}{a^2 - bc} = \frac{y}{b^2 - ca} = \frac{z}{c^2 - ab}.$$

● ***Ex. 9*** *The coordinates of the foot of the perpendicular drawn from the point* $A(1, 0, 3)$ *to the join of the points* $B(4, 7, 1)$ *and* $C(3, 5, 3)$ *are*

(a) $\left(\frac{5}{3}, \frac{7}{3}, \frac{17}{3}\right)$ (b) $(5, 7, 17)$

(c) $\left(\frac{5}{7}, -\frac{7}{3}, \frac{17}{3}\right)$ (d) $\left(-\frac{5}{3}, \frac{7}{3}, -\frac{17}{3}\right)$

Sol. (a) Let D be the foot of the perpendicular and let it divide BC in the ratio $\lambda : 1$. Then, the coordinates of D are

$$\left(\frac{3\lambda + 4}{\lambda + 1}, \frac{5\lambda + 7}{\lambda + 1}, \frac{3\lambda + 1}{\lambda + 1}\right)$$

Now, $\mathbf{AD} \perp \mathbf{BC} \Rightarrow \mathbf{AD} \cdot \mathbf{BC} = 0$

$\Rightarrow$ $-(2\lambda + 3) - 2(5\lambda + 7) - 4 = 0 \Rightarrow \lambda = -\frac{7}{4}$

So, the coordinates of D are $\left(\frac{5}{3}, \frac{7}{3}, \frac{17}{3}\right)$

● ***Ex. 10*** *A mirror and a source of light are situated at the origin O and at a point on OX, respectively. A ray of light from the source strikes the mirror and is reflected. If the direction ratios of the normal to the plane are proportional to* $1, -1, 1$*, then direction cosines of the reflected ray are*

(a) $\frac{1}{3}, \frac{2}{3}, \frac{2}{3}$ (b) $-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}$

(c) $-\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}$ (d) $-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}$

Sol. (d) Let the source of light be situated at $A(a, 0, 0)$, where, $a \neq 0$.

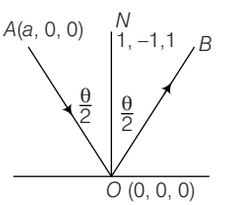
Let OA be the incident ray, OB be the reflected ray and ON be the normal to the mirror at O.

$\therefore$ $\angle AON = \angle NOB = \frac{\theta}{2}$ (say)

Direction ratios of **OA** are proportional to $a, 0, 0$ and so its direction cosines are $1, 0, 0$.

Direction cosines of ON are $\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

$\therefore$ $\cos\frac{\theta}{2} = \frac{1}{\sqrt{3}}$



Let l, m, n be the direction cosines of the reflected ray OB.

Then, $\frac{l+1}{2\cos\frac{\theta}{2}} = \frac{1}{\sqrt{3}}, \frac{m+0}{2\cos\frac{\theta}{2}} = -\frac{1}{\sqrt{3}}$

and $\frac{n+0}{2\cos\frac{\theta}{2}} = \frac{1}{\sqrt{3}}$

$\Rightarrow \quad l = \frac{2}{3} - 1, m = -\frac{2}{3}, n = \frac{2}{3}$

$\Rightarrow \quad l = -\frac{1}{3}, m = -\frac{2}{3}, n = \frac{2}{3}$

Hence, direction cosines of the reflected ray are $-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}$.

● ***Ex. 11*** *Equation of plane passing through the points* $(2, 2, 1), (9, 3, 6)$ *and perpendicular to the plane* $2x + 6y + 6z - 1 = 0$, *is*

(a) $3x + 4y + 5z = 9$

(b) $3x + 4y - 5z + 9 = 0$

(c) $3x + 4y - 5z - 9 = 0$

(d) None of the above

Sol. (c) Equation of a plane passing through (2, 2, 1) is

$$a(x-2) + b(y-2) + c(z-1) = 0 \quad \text{...(i)}$$

This passes through (9, 3, 6) and is perpendicular to

$$2x + 6y + 6z - 1 = 0$$

$\therefore \quad 7a + b + 5c = 0$ and $2a + 6b + 6c = 0$

Solving these two by cross-multiplication, we get

$$\frac{a}{-24} = \frac{b}{-32} = \frac{c}{40}$$

$\Rightarrow \quad \frac{a}{-3} = \frac{b}{-4} = \frac{c}{5}$

Substituting the values of a, b, c in Eq. (i), we get $3x + 4y - 5z - 9 = 0$ as the required plane.

● ***Ex. 12*** *If the position vectors of the points A and B are* $3\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ *and* $\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$ *respectively, then the equation of the plane through B and perpendicular to AB is*

(a) $2x + 3y + 6z + 28 = 0$

(b) $3x + 2y + 6z = 28$

(c) $2x - 3y + 6z + 28 = 0$

(d) $3x - 2y + 6z = 28$

Sol. (a) We have, $\mathbf{AB} = -2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 6\hat{\mathbf{k}}$

So, vector equation of the plane is

$$|\mathbf{r} - (\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 4\hat{\mathbf{k}})| \cdot \mathbf{AB} = 0$$

$\Rightarrow \quad \mathbf{r} \cdot (-2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 6\hat{\mathbf{k}}) = (\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 4\hat{\mathbf{k}}) \cdot (-2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 6\hat{\mathbf{k}})$

$\Rightarrow \quad -2x - 3y - 6z = -2 + 6 + 24$

$\Rightarrow \quad 2x + 3y + 6z + 28 = 0$

● ***Ex. 13*** *A straight line 'L' cuts the lines AB, AC and AD of a parallelogram ABCD at points* B_1, C_1 *and* D_1, *respectively. If* $\mathbf{AB}_1 = \lambda_1 \mathbf{AB}$, $\mathbf{AD}_1 = \lambda_2 \mathbf{AD}$ *and* $\mathbf{AC}_1 = \lambda_3 \mathbf{AC}$, *then* $\frac{1}{\lambda_3}$ *is equal to*

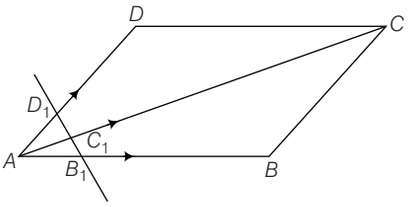
(a) $\frac{1}{\lambda_1} + \frac{1}{\lambda_2}$ (b) $\frac{1}{\lambda_1} - \frac{1}{\lambda_2}$

(c) $-\lambda_1 + \lambda_2$ (d) $\lambda_1 + \lambda_2$

Sol. (a) Let $\mathbf{AB} = \mathbf{a}$, $\mathbf{AD} = \mathbf{b}$, then $\mathbf{AC} = \mathbf{a} + \mathbf{b}$

Given, $\mathbf{AB}_1 = \lambda_1 \mathbf{a}$, $\mathbf{AD}_1 = \lambda_2 \mathbf{b}$, $\mathbf{AC}_1 = \lambda_3(\mathbf{a} + \mathbf{b})$

$\mathbf{B}_1\mathbf{D}_1 = \mathbf{AD}_1 - \mathbf{AB}_1 = \lambda_2 \mathbf{b} - \lambda_1 \mathbf{a}$



Since, vectors $\mathbf{D}_1\mathbf{C}_1$ and $\mathbf{B}_1\mathbf{D}_1$ are collinear, we have $\mathbf{D}_1\mathbf{C}_1 = k\, \mathbf{B}_1\mathbf{D}_1$ for some $k \in R$.

$\Rightarrow \quad \mathbf{AC}_1 - \mathbf{AD}_1 = k \cdot \mathbf{B}_1\mathbf{D}_1$

$\Rightarrow \quad \lambda_3(\mathbf{a} + \mathbf{b}) - \lambda_2 \mathbf{b} = k \cdot (\lambda_2 \mathbf{b} - \lambda_1 \mathbf{a})$

$\Rightarrow \quad \lambda_3 \mathbf{a} + (\lambda_3 - \lambda_2)\mathbf{b} = k \cdot \lambda_2 \mathbf{b} - k \cdot \lambda_1 \mathbf{a}$

Thus, $\lambda_3 = -k\lambda_1$ and $\lambda_3 - \lambda_2 = k\lambda_2$

$\Rightarrow \quad k = \frac{-\lambda_3}{\lambda_1} = \frac{\lambda_3 - \lambda_2}{\lambda_2} \Rightarrow \lambda_1\lambda_2 = \lambda_1\lambda_3 + \lambda_2\lambda_3$

$\Rightarrow \quad \frac{1}{\lambda_3} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$

● ***Ex. 14*** *If the direction cosines of two lines are such that* $l + m + n = 0, l^2 + m^2 - n^2 = 0$, *then the angle between them is*

(a) π (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

Sol. (b) If l, m, n are direction cosines of two lines are such that

$$l + m + n = 0 \quad \text{...(i)}$$

and $\quad l^2 + m^2 - n^2 = 0 \quad \text{...(ii)}$

$\Rightarrow \quad l^2 + m^2 - (-l - m)^2 = 0$

$\Rightarrow \quad 2lm = 0 \Rightarrow l = 0$ or $m = 0$

If $l = 0$, then $n = -m$

$\Rightarrow \quad l : m : n = 0 : 1 : -1$

and if $m = 0$, then $n = -l$

$\Rightarrow \quad l : m : n = 1 : 0 : -1$

$\therefore \quad \cos\theta = \frac{0 + 0 + 1}{\sqrt{0 + 1 + 1}\sqrt{0 + 1 + 1}} = \frac{1}{2}$

$\Rightarrow \quad \theta = \frac{\pi}{3}$

● ***Ex. 15*** *The equation of the plane passing through the mid-point of the line points* $(1, 2, 3)$ *and* $(3, 4, 5)$ *and perpendicular to it is*

(a) $x + y + z = 9$ (b) $x + y + z = -9$

(c) $2x + 3y + 4z = 9$ (d) $2x + 3y + 4z = -9$

Sol. (a) The DR's of the joining of the points (1,2, 3) and (3, 4, 5) and $(3 - 1, 4 - 2, 5 - 3)$, ie. (2, 2, 2)

Also, the mid-point of the join of the points (1, 2, 3) and (3, 4, 5) is (2, 3, 4).

$\therefore$ Equation of plane which passes through (2, 3, 4) and the DR's of its normal are (2, 2, 2) is

$$2(x-2)+2(y-3)+2(z-4)=0$$
$$\Rightarrow \quad x+y+z-9=0$$
$$\Rightarrow \quad x+y+z=9$$

● ***Ex. 16*** *Equation of the plane that contains the lines* $\mathbf{r}=(\hat{\mathbf{i}}+\hat{\mathbf{j}})+\lambda(\hat{\mathbf{i}}+2\hat{\mathbf{j}}-\hat{\mathbf{k}})$ *and,* $\mathbf{r}=(\hat{\mathbf{i}}+\hat{\mathbf{j}})+\mu(-\hat{\mathbf{i}}+\hat{\mathbf{j}}-2\hat{\mathbf{k}})$ *is*

(a) $\mathbf{r}\cdot(2\hat{\mathbf{i}}+\hat{\mathbf{j}}-3\hat{\mathbf{k}})=-4$

(b) $\mathbf{r}\times(-\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})=0$

(c) $\mathbf{r}\cdot(-\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})=0$

(d) None of the above

Sol. (c) The lines are parallel to the vectors $\mathbf{b}_1=\hat{\mathbf{i}}+2\hat{\mathbf{j}}-\hat{\mathbf{k}}$ and $\mathbf{b}_2=-\hat{\mathbf{i}}+\hat{\mathbf{j}}-2\hat{\mathbf{k}}$. Therefore, the plane is normal to the vector

$$\mathbf{n}=\mathbf{b}_1\times\mathbf{b}_2=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & 2 & -1\\ -1 & 1 & -2\end{vmatrix}=-3\hat{\mathbf{i}}+3\hat{\mathbf{j}}+3\hat{\mathbf{k}}$$

The required plane passes through $(\hat{\mathbf{i}}+\hat{\mathbf{j}})$ and is normal to the vector $\mathbf{n}$. Therefore, its equation is

$\mathbf{r}\cdot\mathbf{n}=\mathbf{a}\cdot\mathbf{n}$

$\Rightarrow \mathbf{r}\cdot(-3\hat{\mathbf{i}}+3\hat{\mathbf{j}}+3\hat{\mathbf{k}})=(\hat{\mathbf{i}}+\hat{\mathbf{j}})\cdot(-3\hat{\mathbf{i}}+3\hat{\mathbf{j}}+3\hat{\mathbf{k}})$

$\Rightarrow \mathbf{r}\cdot(-3\hat{\mathbf{i}}+3\hat{\mathbf{j}}+3\hat{\mathbf{k}})=-3+3$

$\Rightarrow \mathbf{r}\cdot(-\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})=0$

● ***Ex. 17*** *The line* $\frac{x-2}{3}=\frac{y+1}{2}=\frac{z-1}{-1}$ *intersects the curve* $xy=c^2, z=0$, *if c is equal to*

(a) ± 1 (b) $\pm\frac{1}{3}$

(c) $\pm\sqrt{5}$ (d) None of these

Sol. (c) At the point on the line where it intersects the given curve, we have $z=0$, so that

$$\frac{x-2}{3}=\frac{y+1}{2}=\frac{0-1}{-1}$$
$$\Rightarrow \quad \frac{x-2}{3}=1 \text{ and } \frac{y+1}{2}=1$$
$$\Rightarrow \quad x=5 \text{ and } y=1.$$

Putting these values of x and y in $xy=c^2$, we get

$c^2=5\Rightarrow c=\pm\sqrt{5}$.

● ***Ex. 18*** *The distance between the line* $\mathbf{r}=2\hat{\mathbf{i}}-2\hat{\mathbf{j}}+3\hat{\mathbf{k}}+\lambda(\hat{\mathbf{i}}-\hat{\mathbf{j}}+4\hat{\mathbf{k}})$ *and the plane* $\mathbf{r}\cdot(\hat{\mathbf{i}}+5\hat{\mathbf{j}}+\hat{\mathbf{k}})=5$, *is*

(a) $\frac{10}{9}$ (b) $\frac{10}{3\sqrt{3}}$

(c) $\frac{10}{3}$ (d) None of these

Sol. (b) Clearly, the given line passes through the point $\mathbf{a}=2\hat{\mathbf{i}}-2\hat{\mathbf{j}}+3\hat{\mathbf{k}}$ and is parallel to the vector $\mathbf{b}=\hat{\mathbf{i}}-\hat{\mathbf{j}}+4\hat{\mathbf{k}}$.

The plane is normal to the vector $\mathbf{n}=\hat{\mathbf{i}}+5\hat{\mathbf{j}}+\hat{\mathbf{k}}$.

We have, $\mathbf{b}\cdot\mathbf{n}=1-5+4=0$

So, the line is parallel to the plane.

$\therefore$Required distance

= Length of the perpendiculars from a point on the line to the given plane.

= Length of the perpendicular from $(2\hat{\mathbf{i}}-2\hat{\mathbf{j}}+3\hat{\mathbf{k}})$ to the given plane.

$$=\left|\frac{(2\hat{\mathbf{i}}-2\hat{\mathbf{j}}+3\hat{\mathbf{k}})\cdot(\hat{\mathbf{i}}+5\hat{\mathbf{j}}+\hat{\mathbf{k}})-5}{\sqrt{1+25+1}}\right|$$
$$=\left|\frac{2-10+3-5}{3\sqrt{3}}\right|=\frac{10}{3\sqrt{3}}$$

● ***Ex. 19*** *If the plane* $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=1$, *cuts the coordinate axes in A, B, C, then the area of* ΔABC *is*

(a) $\sqrt{29}$ sq. units (b) $\sqrt{41}$ sq. units

(c) $\sqrt{61}$ sq. units (d) None of these

Sol. (c) The given plane cuts the coordinate axes in $A(2, 0, 0)$, $B(0, 3, 0)$ and $C(0, 0, 4)$.

$\therefore$ Area of $\Delta ABC=\frac{1}{2}\,AB\times AC\times\sin\angle BAC$

Now, $AB=\sqrt{4+9+0}=\sqrt{13}$, $AC=\sqrt{4+0+16}=\sqrt{20}$.

$$\cos\angle BAC=\frac{\mathbf{AB}\cdot\mathbf{AC}}{|\,\mathbf{AB}\,||\,\mathbf{AC}\,|}=\frac{(-2\hat{\mathbf{i}}+3\hat{\mathbf{j}})\cdot(-2\hat{\mathbf{i}}+4\hat{\mathbf{k}})}{\sqrt{4+9}\,\sqrt{4+16}}$$
$$\Rightarrow \cos\angle BAC=\frac{4+0+0}{\sqrt{13}\sqrt{20}}=\frac{4}{\sqrt{13}\,\sqrt{20}}=\frac{2}{\sqrt{65}}$$
$$\Rightarrow \sin\angle BAC=\sqrt{1-\frac{4}{65}}=\sqrt{\frac{61}{65}}$$

Hence, Area of $\Delta ABC=\frac{1}{2}\times\sqrt{13}\times\sqrt{20}\times\sqrt{\frac{61}{65}}=\sqrt{61}$ sq. units.

● ***Ex. 20*** *The distance of the point* $(1,-2,3)$ *from the plane* $x-y+z=5$ *measured parallel to the line* $\frac{x}{2}=\frac{y}{3}=\frac{z-1}{-6}$ *is*

(a) 1 (b) 2

(c) 4 (d) None of these

Sol. (a) The equation of the line passing through $P(1,-2,3)$ and parallel to the given line is

$$\frac{x-1}{2}=\frac{y+2}{3}=\frac{z-3}{-6}$$

Suppose it meets the plane $x-y+z=5$ at the point Q given by

$$\frac{x-1}{2}=\frac{y+2}{3}=\frac{z-3}{-6}$$
$$=\lambda \text{ i.e. } (2\lambda+1, 3\lambda-2, -6\lambda+3)$$

This lies on $x - y + z = 5$. Therefore,

$$2\lambda + 1 - 3\lambda + 2 - 6\lambda + 3 = 5$$

$$\Rightarrow \quad -7\lambda = -1 \Rightarrow \lambda = \frac{1}{7}$$

So, the coordinates of Q are $\left(\frac{9}{7}, -\frac{11}{7}, \frac{15}{7}\right)$.

Hence, required distance $= PQ = \sqrt{\frac{4}{49} + \frac{9}{49} + \frac{36}{49}} = 1$.

● ***Ex. 21*** *The length of the perpendicular from the origin to the plane passing through the point* $\mathbf{a}$ *and containing the line* $\mathbf{r} = \mathbf{b} + \lambda\mathbf{c}$ *is*

(a) $\dfrac{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}{|\mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{a}|}$

(b) $\dfrac{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}{|\mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{c}|}$

(c) $\dfrac{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}{|\mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{a}|}$

(d) $\dfrac{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}{|\mathbf{a}\times\mathbf{b} + \mathbf{c}\times\mathbf{a}|}$

Sol. (c) The plane passing through $\mathbf{a}$ and containing the line $\mathbf{r} = \mathbf{b} + \lambda\mathbf{c}$ also passes through the point $\mathbf{b}$ and is parallel to the vector $\mathbf{c}$. So, it is normal to the vector $(\mathbf{a} - \mathbf{b}) \times \mathbf{c}$.

Thus, the equation of the plane is

$$(\mathbf{r} - \mathbf{a}) \cdot |(\mathbf{a} - \mathbf{b}) \times \mathbf{c}| = 0$$

$$\Rightarrow \quad (\mathbf{r} - \mathbf{a}) \cdot (\mathbf{a}\times\mathbf{c} - \mathbf{b}\times\mathbf{c}) = 0$$

$$\Rightarrow \quad \mathbf{r} \cdot (\mathbf{a}\times\mathbf{c} - \mathbf{b}\times\mathbf{c}) = \mathbf{a} \cdot (\mathbf{a}\times\mathbf{c} - \mathbf{b}\times\mathbf{c})$$

$$\Rightarrow \quad \mathbf{r} \cdot (\mathbf{a}\times\mathbf{c} - \mathbf{b}\times\mathbf{c}) = -\mathbf{a} \cdot (\mathbf{b}\times\mathbf{c})$$

$$\Rightarrow \quad \mathbf{r} \cdot (\mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{a}) - [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$$

$\therefore$ Length of the perpendicular from the origin to this plane

$$= \left| \frac{\mathbf{0} \cdot (\mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{a}) - [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}{|\mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{a}|} \right|$$

$$= \frac{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}{|\mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{a}|}$$

● ***Ex. 22*** *If* $P(0, 1, 0)$ *and* $Q(0, 0, 1)$ *are two points, then the projection of* $\mathbf{PQ}$ *on the plane* $x + y + z = 3$ *is*

(a) 2 (b) 3

(c) $\sqrt{2}$ (d) $\sqrt{3}$

Sol. (c) The projection of PQ on the given plane is $PQ\cos\theta$, where θ is the angle between PQ and the plane.

Let $\mathbf{n}$ be a vector normal to the plane.

We have, $\mathbf{PQ} = -\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{n} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$

$$\therefore \quad \sin\theta = \frac{\mathbf{PQ} \cdot \mathbf{na}}{|\mathbf{PQ}|\,|\mathbf{n}|} = 0$$

$\Rightarrow \mathbf{PQ}$ is parallel to the plane.

Hence, projection of $\mathbf{PQ}$ on the given plane

$$= |\mathbf{PQ}|\cos\theta$$

$$= |\mathbf{PQ}| = \sqrt{1 + 1} = \sqrt{2}$$

● ***Ex. 23*** *The equation of the plane through the intersection of the planes* $x + y + z = 1$ *and* $2x + 3y - z + 4 = 0$ *and parallel to X-axis, is*

(a) $y - 3z + 6 = 0$ (b) $3y - z + 6 = 0$

(c) $y + 3z + 6 = 0$ (d) $3y - 2z + 6 = 0$

Sol. (a) The equation of the plane through the intersection of the planes $x + y + z = 1$ and $2x + 3y - z + 4 = 0$ is

$$(x + y + z - 1) + \lambda(2x + 3y - z + 4) = 0$$

or, $$(2\lambda + 1)x + (3\lambda + 1)y + (1 - \lambda)z + 4\lambda - 1 = 0 \quad \text{...(i)}$$

It is parallel to X-axis, i.e. $\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$

$$\therefore \quad 1(2\lambda + 1) + 0 \times (3\lambda + 1) + 0(1 - \lambda) = 0$$

$$\Rightarrow \quad \lambda = -\frac{1}{2}$$

Substituting $\lambda = -\frac{1}{2}$ in Eq. (i), we get

$y - 3z + 6 = 0$ as the equation of the required plane.

● ***Ex. 24*** *A plane passes through the point* $(1, 1, 1)$. *If b, c, a are the direction ratios of a normal to the plane where a, b, c* $(a < b < c)$ *are the prime factors of* 2001, *then the equation of the plane* II *is*

(a) $29x + 31y + 3z = 63$

(b) $23x + 29y - 29z = 23$

(c) $23x + 29y + 3z = 55$

(d) $31x + 37y + 3z = 71$

Sol. (c) The equation of the plane is

$$b(x - 1) + c(y - 1) + a(z - 1) = 0 \quad \text{...(i)}$$

Now, $2001 = 3 \times 23 \times 29$

$$\therefore \quad a < b < c \Rightarrow a = 3,\ b = 23 \text{ and } c = 29.$$

Substituting the values of a, b, c in Eq. (i), we obtain $23x + 29y + 3z = 55$ as the equation of the required plane.

● ***Ex. 25*** *If the direction ratios of two lines are given by* $a + b + c = 0$ *and* $2ab + 2ac - bc = 0$, *then the angle between the lines is*

(a) π (b) $\dfrac{2\pi}{3}$

(c) $\dfrac{\pi}{2}$ (d) $\dfrac{\pi}{3}$

Sol. (b) We have,

$$a + b + c = 0 \text{ and } 2ab + 2ac - bc = 0$$

$$\Rightarrow \quad a = -(b + c) \text{ and } 2a(b + c) - bc = 0$$

$$\Rightarrow \quad -2(b + c)^2 - bc = 0$$

$$\Rightarrow \quad 2b^2 + 5bc + 2c^2 = 0$$

$$\Rightarrow \quad (2b + c)(b + 2c) = 0$$

$$\Rightarrow \quad 2b + c = 0 \text{ or, } b + 2c = 0$$

If $2b + c = 0$, then $a = -(b + c) \Rightarrow a = b$

$$\therefore \quad a = b \text{ and } c = -2b \Rightarrow \frac{a}{1} = \frac{b}{1} = \frac{c}{-2}$$

If $b + 2c = 0$, then $a = -(b + c) \Rightarrow a = c$

$\therefore$ $a = c$ and $b = -2c \Rightarrow \frac{a}{1} = \frac{b}{-2} = \frac{c}{1}$

Thus, the direction ratios of two lines are proportional to 1, 1, -2 and 1, -2, 1, respectively. So, the angle θ between them is given by

$$\cos\theta = \frac{1-2-2}{\sqrt{1+1+4}\sqrt{1+4+1}} = \frac{-1}{2} \Rightarrow \theta = \frac{2\pi}{3}$$

● ***Ex. 26*** *A tetrahedron has vertices at $O(0,0,0)$, $A(1,2,1)$, $B(2,1,3)$ and $C(-1,1,2)$. Then, the angle between the faces OAB and ABC will be*

(a) $90°$ (b) $\cos^{-1}\left(\frac{19}{35}\right)$

(c) $\cos^{-1}\left(\frac{17}{31}\right)$ (d) $30°$

Sol. (b) Let $\mathbf{n}_1$ and $\mathbf{n}_2$ be the vectors normal to the faces OAB and ABC. Then,

$$\mathbf{n}_1 = \mathbf{OA} \times \mathbf{OB} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 5\hat{\mathbf{i}} - \hat{\mathbf{j}} - 3\hat{\mathbf{k}}$$

and $\mathbf{n}_2 = \mathbf{AB} \times \mathbf{AC} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & 2 \\ -2 & -1 & 1 \end{vmatrix} = \hat{\mathbf{i}} - 5\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$

If θ is the angle between the faces OAB and ABC, then

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}$$

$$\Rightarrow \quad \cos\theta = \frac{5+5+9}{\sqrt{25+1+9}\sqrt{1+25+9}} = \frac{19}{35}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{19}{35}\right)$$

● ***Ex. 27*** *The vector equation of the plane through the point $(2,1,-1)$ and passing through the line of intersection of the plane $\mathbf{r}\cdot(\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}) = 0$ and $\mathbf{r}\cdot(\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) = 0$, is*

(a) $\mathbf{r}\cdot(\hat{\mathbf{i}} + 9\hat{\mathbf{j}} + 11\hat{\mathbf{k}}) = 0$ (b) $\hat{\mathbf{r}}\cdot(\hat{\mathbf{i}} + 9\hat{\mathbf{j}} + 11\hat{\mathbf{k}}) = 6$

(c) $\hat{\mathbf{r}}\cdot(\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 13\hat{\mathbf{k}}) = 0$ (d) None of these

Sol. (a) The vector equation of a plane through the line of intersection of the plane $\mathbf{r}\cdot(\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}) = 0$ and $\mathbf{r}\cdot(\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) = 0$ can be written as

$$\mathbf{r}\cdot(\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + -\hat{\mathbf{k}}) + \lambda\{\mathbf{r}\cdot(\hat{\mathbf{j}} + 2\hat{\mathbf{k}})\} = 0 \quad \text{...(i)}$$

This passes through $2\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$.

$$\therefore \quad (2\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}})\cdot(\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}) + \lambda(2\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}})\cdot(\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) = 0$$

$$\Rightarrow \quad (2+3+1) + \lambda(0+1-2) = 0 \Rightarrow \lambda = 6.$$

Putting the value of λ in Eq. (i), we get the equation of the required plane as

$$\mathbf{r}\cdot(\hat{\mathbf{i}} + 9\hat{\mathbf{j}} + 11\hat{\mathbf{k}}) = 0$$

● ***Ex. 28*** *The vector equation of the plane through the point $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and perpendicular to the line of intersection of the plane $\mathbf{r}\cdot(3\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}) = 1$ and $\mathbf{r}\cdot(\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 2\hat{\mathbf{k}}) = 2$, is*

(a) $\mathbf{r}\cdot(2\hat{\mathbf{i}} + 7\hat{\mathbf{j}} - 13\hat{\mathbf{k}}) = 1$ (b) $\mathbf{r}\cdot(2\hat{\mathbf{i}} - 7\hat{\mathbf{j}} - 13\hat{\mathbf{k}}) = 1$

(c) $\mathbf{r}\cdot(2\hat{\mathbf{i}} + 7\hat{\mathbf{j}} + 13\hat{\mathbf{k}}) = 0$ (d) None of these

Sol. (b) The line of intersection of the planes

$$\mathbf{r}\cdot(3\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}) = 1$$

and $$\mathbf{r}\cdot(\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 2\hat{\mathbf{k}}) = 2$$

is common to both the planes. Therefore, it is perpendicular to normals to the two planes, i.e.

$$\mathbf{n}_1 = 3\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$$

and $$\mathbf{n}_2 = \hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$$

Hence, it is parallel to the vector $\mathbf{n}_1 \times \mathbf{n}_2 = -2\hat{\mathbf{i}} + 7\hat{\mathbf{j}} + 13\hat{\mathbf{k}}$.

Thus, we have to find the equation of the plane passing through $\mathbf{a} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and normal to the vector $\mathbf{n} = \mathbf{n}_1 \times \mathbf{n}_2$.

The equation of the required plane is

$$(\mathbf{r} - \mathbf{a})\cdot\mathbf{n} = 0$$

$$\Rightarrow \quad \mathbf{r}\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n}$$

$$\Rightarrow \quad \mathbf{r}\cdot(-2\hat{\mathbf{i}} + 7\hat{\mathbf{j}} + 13\hat{\mathbf{k}}) = (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}})\cdot(-2\hat{\mathbf{i}} + 7\hat{\mathbf{j}} + 13\hat{\mathbf{k}})$$

$$\Rightarrow \quad \mathbf{r}\cdot(2\hat{\mathbf{i}} - 7\hat{\mathbf{j}} - 13\hat{\mathbf{k}}) = 1$$

● ***Ex. 29*** *The cartesian equation of the plane $\mathbf{r} = (1 + \lambda - \mu)\hat{\mathbf{i}} + (2 - \lambda)\hat{\mathbf{j}} + (3 - 2\lambda + 2\mu)\hat{\mathbf{k}}$, is*

(a) $2x + y = 5$ (b) $2x - y = 5$

(c) $2x + z = 5$ (d) $2x - z = 5$

Sol. (c) We have,

$$\mathbf{r} = (1 + \lambda - \mu)\hat{\mathbf{i}} + (2 - \lambda)\hat{\mathbf{j}} + (3 - 2\lambda + 2\mu)\hat{\mathbf{k}}$$

$$\Rightarrow \quad \mathbf{r} = (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) + \lambda(\hat{\mathbf{i}} - \hat{\mathbf{j}} - 2\hat{\mathbf{k}}) + \mu(-\hat{\mathbf{i}} + 2\hat{\mathbf{k}})$$

which is a plane passing through $\mathbf{a} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and parallel to the vectors $\mathbf{b} = \hat{\mathbf{i}} - \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ and $\mathbf{c} = -\hat{\mathbf{i}} + 2\hat{\mathbf{k}}$.

Therefore, it is normal to the vector

$$\mathbf{n} = \mathbf{b} \times \mathbf{c} = -2\hat{\mathbf{i}} - \hat{\mathbf{k}}$$

Hence, its vector equation is

$$(\mathbf{r} - \mathbf{a})\cdot\mathbf{n} = 0$$

$$\Rightarrow \quad \mathbf{r}\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n}$$

$$\Rightarrow \quad \mathbf{r}\cdot(-2\hat{\mathbf{i}} - \hat{\mathbf{k}}) = -2 - 3$$

$$\Rightarrow \quad \mathbf{r}\cdot(2\hat{\mathbf{i}} + \hat{\mathbf{k}}) = 5$$

So, the cartesian equation of the plane is

$$(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})\cdot(2\hat{\mathbf{i}} + \hat{\mathbf{k}}) = 5 \Rightarrow 2x + z = 5$$

● ***Ex. 30*** *A variable plane is at a distance, k from the origin and meets the coordinates axis in A, B, C. Then, the locus of the centroid of ΔABC is*

(a) $x^{-2} + y^{-2} + z^{-2} = k^{-2}$

(b) $x^{-2} + y^{-2} + z^{-2} = 4k^{-2}$

(c) $x^{-2} + y^{-2} + z^{-2} = 16k^{-2}$

(d) $x^{-2} + y^{-2} + z^{-2} = 9k^{-2}$

Sol. (d) Let the equation of the variable plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

This meets the coordinates axes at $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.

Let (α, β, γ) be the coordinates of the centroid of ΔABC. Then,

$$\alpha=\frac{a}{3}, \beta=\frac{b}{3}, \gamma=\frac{c}{3}$$

$$\Rightarrow \quad a=3\alpha, b=3\beta, c=3\gamma \qquad \ldots(i)$$

The plane is at a distance, k from the origin.

$$\therefore \quad \left|\frac{\frac{0}{a}+\frac{0}{b}+\frac{0}{c}-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}\right|=k$$

$$\Rightarrow \quad \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{k^2}$$

$$\Rightarrow \quad \alpha^{-2}+\beta^{-2}+\gamma^{-2}=9k^{-2}$$

Hence, the locus of (α, β, γ) is $x^{-2}+y^{-2}+z^{-2}=9k^{-2}$

JEE Type Solved Examples :
More than One Correct Option Type Questions

● ***Ex. 31*** *The direction ratios of the line* $x-y+z-5=0=x-3y-6$ *are*

(a) 3, 1, − 2 (b) 2, − 4, 1

(c) $\frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}$ (d) $\frac{2}{\sqrt{21}}, \frac{-4}{\sqrt{21}}, \frac{1}{\sqrt{21}}$

Sol. (a, c) Let the DR's of line are a, b and c.

As the line is perpendicular to both the planes

$$\Rightarrow \quad a-b+c=0$$

$$a-3b+0\cdot c=0$$

$$\frac{a}{3}=\frac{b}{1}=\frac{c}{-2}$$

Hence, (a) and (c) are correct answers.

● ***Ex. 32*** *The equation of the line* $x+y+z-1=0$, $4x+y-2z+2=0$ *written in the symmetrical form is*

(a) $\frac{x+1}{1}=\frac{y-2}{-2}=\frac{z-0}{1}$

(b) $\frac{x}{1}=\frac{y}{-2}=\frac{z-1}{1}$

(c) $\frac{\frac{x+1}{2}}{1}=\frac{y-1}{-2}=\frac{\frac{z-1}{2}}{1}$

(d) $\frac{x-1}{2}=\frac{y+2}{-1}+\frac{z-2}{2}$

Sol. (a, b, c, d) $x+y+z-1=0$

$$4x+y-2z+2=0$$

$\therefore$ Direction ratios of the line are $(-3, 6, -3)$.

i.e. $< 1, -2, 1 >$

Let $z=k$, then $x=k-1$, $y=2-2k$

i.e. $(k-1, 2-2k, k)$ is any point on the line.

$\therefore$ $(-1, 2, 0)$, $(0, 0, 1)$, $\left(-\frac{1}{2}, 1, \frac{1}{2}\right)$ and $(1, -2, 2)$ are points on the line.

Hence, (a), (b), (c) and (d) are the correct answers.

● ***Ex. 33*** *The direction cosines of the lines bisecting the angle between the line whose direction cosines are* l_1, m_1, n_1 *and* l_2, m_2, n_2 *and the angle between these lines is* θ, *are*

(a) $\frac{l_1+l_2}{\cos\frac{\theta}{2}}, \frac{m_1+m_2}{\cos\frac{\theta}{2}}, \frac{n_1+n_2}{\cos\frac{\theta}{2}}$

(b) $\frac{l_1+l_2}{2\cos\frac{\theta}{2}}, \frac{m_1+m_2}{2\cos\frac{\theta}{2}}, \frac{n_1+n_2}{2\cos\frac{\theta}{2}}$

(c) $\frac{l_1+l_2}{\sin\frac{\theta}{2}}, \frac{m_1+m_2}{\sin\frac{\theta}{2}}, \frac{n_1+n_2}{\sin\frac{\theta}{2}}$

(d) $\frac{l_1-l_2}{2\sin\frac{\theta}{2}}, \frac{m_1-m_2}{2\sin\frac{\theta}{2}}, \frac{n_1-n_2}{2\sin\frac{\theta}{2}}$

Sol. (b, d) Distance ratio of the bisector are

$$< l_1+l_2, m_1+m_2, n_1+n_2 >$$

$$=\sqrt{(l_1+l_2)^2+(m_1+m_2)^2+(n_1+n_2)^2}$$

$$=\sqrt{2+2(l_1l_2+m_1m_2+n_1n_2)}$$

$$=\sqrt{2+2\cos\theta}=2\cos\frac{\theta}{2}$$

$\therefore$ Direction cosines are $\left\langle \frac{l_1+l_2}{2\cos\frac{\theta}{2}}, \frac{m_1+m_2}{2\cos\frac{\theta}{2}}, \frac{n_1+n_2}{2\cos\frac{\theta}{2}} \right\rangle$

Distance ratio of the other bisector are

$$< l_1-l_2, m_1-m_2, n_1-n_2 > \sqrt{(l_1-l_2)^2+(m_1-m_2)^2+(n_1-n_2)^2}$$

$$=2\sin\frac{\theta}{2}$$

$\therefore$ Direction cosines of the bisector are

$$\left\langle \frac{l_1-l_2}{2\sin\frac{\theta}{2}}, \frac{m_1-m_2}{2\sin\frac{\theta}{2}}, \frac{n_1-n_2}{2\sin\frac{\theta}{2}} \right\rangle$$

Hence, (b) and (d) are correct answers.

● ***Ex. 34*** *Consider the planes* $3x-6y+2z+5=0$ *and* $4x-12y+3z=3$. *The planes* $67x+162y+47z+44=0$ *bisects the angle between the planes which*

(a) contains origin (b) is acute

(c) is obtuse (d) None of these

Sol. (a,b) $3x-6y+2z+5=0$ ...(i)

$-4x+12y-3z+3=0$ ...(ii)

$$\frac{3x-6y+2z+5}{\sqrt{9+36+4}}=\frac{-4x+12y-3z+3}{\sqrt{16+144+9}}$$

Bisects the angle between the planes that contains the origin.

$13(3x-6y+2z+5)=7(-4x+12y-3z+3)$

$39x-78y+26z+65=-28x+84y-21z+21$

$67x-162y+47z+44=0$ ...(iii)

Let θ be the angle between Eqs. (i) and (iii), then find cos θ and then we obtain $|\tan\theta|<1$.

Hence, (a) and (b) are the correct answer.

● ***Ex. 35*** *Consider the equation of line AB is* $\frac{x}{2}=\frac{y}{-3}=\frac{z}{6}$. *Through a point* $P(1,2,5)$ *line PN is drawn perpendicular to AB and line PQ is drawn parallel to the plane* $3x+4y+5z=0$ *to meet AB is Q. Then,*

(a) coordinate of N are $\left(\frac{52}{49},-\frac{78}{49},\frac{156}{49}\right)$

(b) the coordinates of Q are $\left(3,-\frac{9}{2},9\right)$

(c) the equation of PN is $\frac{x-1}{3}=\frac{y-2}{-176}=\frac{z-5}{-89}$

(d) coordinates of N are $\left(\frac{156}{49},\frac{52}{49}-\frac{78}{49}\right)$

Sol. (a,b,c) Equation of line AB is $\frac{x}{2}=\frac{y}{-3}=\frac{z}{6}$

Its DR's are $<2,-3,6>$

Let the coordinates be $<2r,-3r,6r>$

DR's of PN are $<2r-1,-3r-2,6r-5>$

It is perpendicular to AB

$\therefore\quad 2(2r-1)-3(-3r-2)+6(6r-5)=0$

$4r-2+9r+6+36r-30=0$

$49r=26$ i.e. $r=\frac{26}{49}$

(a) $\therefore$ Coordinates of N are $\left(\frac{52}{49},-\frac{78}{49},\frac{156}{49}\right)$

(b) Let the coordinates of Q be $(2r,-3r,6r)$, then DR's of PQ are $<2r-1,-3r-2,6r-5>$. Since, PQ is parallel to the plane.

$\therefore\quad 3(2r-1)+4(-3r-2)+5(6r-5)=0$

$6r-3-12r-8+30r-25=0$

$24r=36,\ r=\frac{3}{2}$

$\therefore$ Coordinates of Q are $\left(3,-\frac{9}{2},9\right)$.

Equation of PN is $\frac{x-1}{3}=\frac{y-2}{-176}=\frac{z-5}{-89}$.

● ***Ex. 36*** *The equation of a plane is* $2x-y-3z=5$ *and* $A(1,1,1)$, $B(2,1,-3)$, $C(1,-2,-2)$ *and* $D(-3,1,2)$ *are four points. Which of the following line segment are intersected by the plane?*

(a) AD (b) AB

(c) AC (d) BC

Sol. (b, c) For $A(1,1,1)$, $2x-y-3z-5=2-1-3-5<0$

For $B(2,1,-3)$, $2x-y-3z-5=0-1+9-5>0$

For $C(1,-2,-2)$, $2x-y-3z-5=2+2+6-5>0$. A, D

For $D(-3,1,2)$, $2x-y-3z-5=-6-1-6-5=-18<0$

are on one side of the plane and B, C are on the other side, the line segments AB, AC, BD, CD intercept the plane.

● ***Ex. 37*** *The coordinates of a point on the line* $\frac{x-1}{2}=\frac{y+1}{-3}=z$ *at a distance* $4\sqrt{14}$ *from the point* $(1,-1,0)$ *are*

(a) $(9,-13,4)$

(b) $(8\sqrt{14}+1,-12\sqrt{14}-1,4\sqrt{14})$

(c) $(-7,11,-4)$

(d) $(-8\sqrt{14}+1,12\sqrt{14}-1,-4\sqrt{14})$

Sol. (a, c) The coordinates of any point on the given line are $(2r+1,-3r-1,r)$

The distance of this point from the point $(1,-1,0)$ is given to be $4\sqrt{14}$.

$\Rightarrow\quad (2r)^2+(-3r)^2+(r)^2=(4\sqrt{14})^2$

$\Rightarrow\quad 14r^2=16\times 14$

$\Rightarrow\quad r=\pm 4$

So, the coordinate of the required point are

$(9,-13,4)$ or $(-7,11,-4)$.

● ***Ex. 38*** *The line whose vector equation are*

$$\mathbf{r}=2\hat{\mathbf{i}}-3\hat{\mathbf{j}}+7\hat{\mathbf{k}}+\lambda(2\hat{\mathbf{i}}+p\hat{\mathbf{j}}+5\hat{\mathbf{k}})$$

and

$$\mathbf{r}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}}+\mu(3\hat{\mathbf{i}}-p\hat{\mathbf{j}}+p\hat{\mathbf{k}})$$

are perpendicular for all values of λ and μ if p equals to

(a) -1 (b) 2

(c) 5 (d) 6

Sol. (a, d) The given lines are perpendicular for all values of λ and μ if the vectors.

$2\hat{\mathbf{i}}+p\hat{\mathbf{j}}+5\hat{\mathbf{k}}$ and $3\hat{\mathbf{i}}-p\hat{\mathbf{j}}+p\hat{\mathbf{k}}$ are perpendicular

$\Rightarrow\quad 2\times 3+p(-p)+5p=0$

$\Rightarrow\quad p^2-5p-6=0$

$\Rightarrow\quad p=-1$ or 6

● ***Ex. 39*** *Equation of a plane passing through the lines* $2x-y+z-3=0, 3x+y+z-5=0$ *and which is at a distance of* $\frac{1}{\sqrt{6}}$ *from the point* $(2,1,-1)$ *is*

(a) $2x-y+z-3=0$

(b) $3x+y+z-5=0$

(c) $62x+29y+19z-105=0$

(d) $x+2y-2=0$

Sol. (a,c) Equation of a plane through the given line is

$2x-y+z-3+\lambda(3x+y+z-5)=0$

$\Rightarrow \quad (2+3\lambda)x+(\lambda-1)y+(\lambda+1)z-(3+5\lambda)=0$

So, $$\frac{1}{\sqrt{6}}=\frac{2(2+3\lambda)+(\lambda-1)-(\lambda+1)-3-5\lambda}{\sqrt{(2+3\lambda)^2+(\lambda-1)^2+(\lambda+1)^2}}$$

$\Rightarrow \quad 11\lambda^2+12\lambda+6=6(\lambda-1)^2$

$=6(\lambda^2-2\lambda+1)$

$\Rightarrow \quad 5\lambda^2+24\lambda=0$

$\Rightarrow \quad \lambda=0 \text{ or } \lambda=-\frac{24}{5}.$

Thus, the equation of the required planes are

$2x-y+z-3=0$ or $62x+29y+19z-105=0.$

● ***Ex. 40*** *The plane passing through the point* $(-2,-2,2)$ *and containing the line joining the points* $(1,1,1)$ *and* $(1,-1,2)$ *makes intercepts of lengths a, b, c respectively on the axes of x, y and z respectively, then*

(a) $a=3b$ (b) $b=2c$

(c) $a+b+c=12$ (d) $a+2b+2c=0$

Sol. (a,b,c) Equation of any plane passing through $(-2,-2,2)$ is

$$A(x+2)+B(y+2)+C(z-2)=0$$

Since it contains the line joining $(1,1,1)$ and $(1,-1,2)$ these points also lie on this plane.

$\Rightarrow \quad 3A+3B-C=0$ and $3A+B+0=0$

$\Rightarrow \quad \frac{A}{1}=\frac{B}{-3}=\frac{C}{-6}.$

So, the equation of the plane is

$$(x+2)-3(y+2)-6(z-2)=0$$

or $$x-3y-6z+8=0$$

or $$\frac{x}{-8}+\frac{y}{\frac{8}{3}}+\frac{z}{\frac{8}{6}}=1 \Rightarrow a=8, b=\frac{8}{3}, c=\frac{8}{6}$$

$\Rightarrow \quad a=3b, b=2c, a+b+c=12$

and $$a+2b+2c=16$$

JEE Type Solved Examples :
Statement I and II Type Questions

■ **Directions** (Q.Nos. 41-45) For the following questions, choose the correct answers from the codes (a), (b), (c) and (d) defined as follows:

(a) Statement I is true, Statement II is also true; Statement II is the correct explanation of Statement I.

(b) Statement I is true, Statement II is also true; Statement II is not the correct explanation of Statement I.

(c) Statement I is true, Statement II is false.

(d) Statement I is false, Statement II is true.

● ***Ex. 41*** **Statement I** *A line L is perpendicular to the plane* $3x-4y+5z=10.$

Statement II *Direction cosines of L be* $<\frac{3}{5\sqrt{2}},-\frac{4}{5\sqrt{2}},\frac{1}{\sqrt{2}}>.$

Sol. (a) $lx+my+nz=P$ be the equation of a plane in the normal form.

$\therefore$ DR of the plane

$$3x-4y+5z=10 \text{ be } <3,-4,5>.$$

$\Rightarrow$ Direction cosines

$$<\frac{3}{5\sqrt{2}},\frac{-4}{5\sqrt{2}},\frac{1}{\sqrt{2}}>.$$

● ***Ex. 42*** *The equation of two straight line are* $\frac{x-1}{2}=\frac{y+3}{1}=\frac{z-2}{-3}$ *and* $\frac{x-2}{1}=\frac{y-1}{-3}=\frac{z+3}{2}$

Statement I *The given lines are coplanar.*

Statement II *The equation* $2x_1-y_1=1, x_1+3y_1=4, 3x_1+2y_1=5$ *are consistent.*

Sol. (a) Any point on the first line is $(2x_1+1, x_2-3, -3x_1-2)$.

Any point on the second line is $(y_1-2, 3y_1+1, 2y_1-3)$.

If two lines are coplanar, then $2x_1-y_1=1$, $x_1+3y_1=4$, $3x_1+2y_1=5$ are consistent.

● ***Ex. 43*** **Statement I** *The distance between the planes* $4x-5y+3z=5$ *and* $4x-5y+3z+2=0$ *is* $\frac{3}{5\sqrt{2}}.$

Statement II *The distance between* $ax+by+cz+d_1=0$ *and* $ax+by+cz+d_2=0$ *is* $\left|\frac{d_1-d_2}{\sqrt{a^2+b^2+c^2}}\right|.$

Sol. (d) Distance $=\left|\frac{5+2}{\sqrt{50}}\right|=\frac{7}{5\sqrt{2}}$

● ***Ex. 44*** *Given the line* $L: \frac{x-1}{3}=\frac{y+1}{2}=\frac{z-3}{-1}$ *and the plane* $\pi: x-2y-z=0$

Statement I *L lies in* π.

Statement II *L is parallel to* π.

Sol. (c) $x=1+3r, y=-1+2r, t=3-r$

$\therefore \quad 1+3r-2(-1+2r)-3+r=0$

$3\times1-2\times2+1\times1=0$

Hence, L is parallel to π.

● ***Ex. 45*** **Statement I** *Line* $\frac{x-1}{3}=\frac{y-2}{11}=\frac{z+1}{11}$ *lies in the plane* $11x-3z-14=0$.

Statement II *A straight line lies in a plane, if the line is parallel to plane and a point of the line in the plane.*

Sol. (a) Statement I $(1, 2, -1)$ is a point on the line and

$11+3-14=0$.

$\therefore$ The point lies on the plane $11x-3z-14=0$.

Further $3\times11+11(-3)=0$.

$\therefore$ The line lies in the plane.

Statement II is also true.

JEE Type Solved Examples : Passage Based Questions

Passage I

(Ex. Nos. 46 to 48)

Two line whose equation are $\frac{x-3}{2}=\frac{y-2}{3}=\frac{z-1}{\lambda}$ *and* $\frac{x-2}{3}=\frac{y-3}{2}=\frac{z-2}{3}$ *lie in the same plane, then.*

● ***Ex. 46*** *The value of* $\sin^{-1}\sin\lambda$ *is equal to*

(a) 3 (b) $\pi-3$

(c) 4 (d) $\pi-4$

● ***Ex. 47*** *Point of intersection of the lines lies on*

(a) $3x+y+z=20$ (b) $2x+y+z=25$

(c) $3x+2y+z=24$ (d) $x=y=z$

● ***Ex. 48*** *Angle between the plane containing both the lines and the plane* $4x+y+2z=0$ *is equal to*

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$

(c) $\frac{\pi}{6}$ (d) $\cos^{-1}\frac{2}{\sqrt{186}}$

Sol. (Ex. 46-48)

46. (d) Both lines and coplanar

$$\begin{vmatrix}2 & 3 & \lambda\\ 3 & 2 & 3\\ 1 & -1 & -1\end{vmatrix}=0$$

$\Rightarrow \quad 2(-2+3)+3(3+3)+\lambda(-3-2)=0$

$\Rightarrow \quad \lambda=4$

$\sin^{-1}\sin4=\sin^{-1}\sin(\pi-4)=\pi-4$

47. (d) Let $\frac{x-3}{2}=\frac{y-2}{3}=\frac{z-1}{4}=r_1$

$\Rightarrow \quad x=3+2r_1$

$y=2+3r_1$

$z=1+4r_1$

It will lie on $\frac{x-2}{3}=\frac{y-3}{2}=\frac{z-2}{3} \Rightarrow r_1=1$

So, point of intersection is (5, 5, 5).

48. (b) Equation of plane contains both lines

$$\begin{vmatrix}x-3 & y-2 & z-1\\ 2 & 3 & 4\\ 3 & 2 & 3\end{vmatrix}=0$$

$(x-3)(1)+(y-2)(12-6)+(z-1)(4-9)=0$

$x+6y-5z=10$

Thus, the angle is $\frac{\pi}{2}$.

Passage II

(Ex. Nos. 49 to 51)

Let $a_1x+b_1y+c_1z+d_1=0$ *and* $a_2x+b_2y+c_2z=d_2=0$ *be two planes, where* $d_1, d_2>0$. *Then, origin lies in acute angle, if* $a_1a_2+b_1b_2+c_1c_2<0$ *and origin lies in obtuse angle, if* $a_1a_2+b_1b_2+c_1c_2>0$.

Further point (x_1, y_1, z_1) *and origin both lie either in acute angle or in obtuse angle. If* $(a_1x_1+b_1y_1+c_1z_1+d_1)(a_2x_1+b_2y_1+c_2z_1+d_2)>0$.

One of (x_1, y_1, z_1) *and origin lie in acute and the other in obtuse angle; if* $(a_1x_1+b_1y_1+c_1z_1+d_1)(a_2x_1+b_2y_1+c_2z_1+d_2)<0$.

● ***Ex. 49*** *Given that planes* $2x+3y-4z+7=0$ *and* $x-2y+3z-5=0$. *If a point P is* $(1, -2, 3)$, *then*

(a) *O* and *P* both lie in acute angle between the planes

(b) *O* and *P* both lie in obtuse angle

(c) *O* lies in acute angle, *P* lies in obtuse angle

(d) *O* lies in obtuse angle, *P* lies an acute angle

● **Ex. 50** *Given the planes* $x+2y-3z+5=0$ *and* $2x+y+3z+1=0$. *If a point P is* $(2,-1,2)$, *then*

(a) *O* and *P* both lie in acute angle between the planes
(b) *O* and *P* both lie in obtuse angle
(c) *O* lies in acute angle, *P* lies in obtuse angle
(d) *O* lies in obtuse angle, *P* lies an acute angle

● **Ex. 51** *Given the planes* $x+2y-3z+2=0$ *and* $x-2y+3z+7=0$, *if the point P is* $(1,2,2)$ *then*

(a) *O* and *P* both lie in acute angle between the planes
(b) *O* and *P* both lie in obtuse angle
(c) *O* lies in acute angle, *P* lie in obtuse angle
(d) *O* lies in obtuse angle, *P* lies in acute angle

Sol. (Ex. 49-51)

49. (b) Equation of the second plane is $-x+2y-3z+5=0$

$2(-1)+3\cdot 2+(-4)(-3)>0$

$\therefore$ Origin lies in obtuse angle.

$(2\times 1+3(-2)-4\times 3+7)(-1+2(-2)-3\times 3+5)$

$=(2-6-12+7)(-1-4-9+5)>0$

$\therefore P$ lies in obtuse angle.

50. (c) $1\times 2+2\times 1-3\times 3<0$

$\therefore$ Origin lies in acute angle.

Also, $(2+2(-1)-3(2)+5)(2\times 2-1+3\times 2+1)=(-1)(10)<0$

$\therefore$ P lies in obtuse angle.

51. (a) $1-4-9<0$

$\therefore$ Origin lies in acute angle.

Further $(1+4-6+2)(1-4+6+7)>0$

$\therefore$ The point P lies in acute angle.

Passage III

(Ex. Nos. 52 to 54)

In a parallelogram OABC with position vectors of A is $3\hat{\mathbf{i}}+4\hat{\mathbf{j}}$ *and C is* $4\hat{\mathbf{i}}+3\hat{\mathbf{j}}$ *with reference to O as origin. A point E is taken on the side BC which divides it in the ratio of* $2:1$. *Also, the line segment AE intersects the line bisecting the* $\angle AOC$ *internally at P. CP when extended meets AB at point F.*

● **Ex. 52** *The position vector of P is*

(a) $\hat{\mathbf{i}}+\hat{\mathbf{j}}$ (b) $\frac{2}{3}(\hat{\mathbf{i}}+\hat{\mathbf{j}})$

(c) $\frac{13}{3}(\hat{\mathbf{i}}+\hat{\mathbf{j}})$ (d) $\frac{21}{5}(\hat{\mathbf{i}}+\hat{\mathbf{j}})$

● **Ex. 53** *The equation of line parallel to* **CP** *and passing through* $(2,3,4)$ *is*

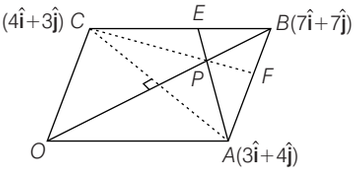
(a) $\frac{x-2}{1}=\frac{y-3}{5}, z=4$ (b) $\frac{x-2}{1}=\frac{y-3}{6}, z=4$

(c) $\frac{x-2}{2}=\frac{y-3}{5}, z=3$ (d) $\frac{x-2}{3}=\frac{y-3}{5}, z=3$

● **Ex. 54** *The equation of plane containing line AC and at a maximum distance from B is*

(a) $\mathbf{r}\cdot(\hat{\mathbf{i}}+\hat{\mathbf{j}})=7$ (b) $\mathbf{r}\cdot(\hat{\mathbf{i}}-\hat{\mathbf{j}})=7$

(c) $\mathbf{r}\cdot(2\hat{\mathbf{i}}-\hat{\mathbf{j}})=7$ (d) $\mathbf{r}\cdot(3\hat{\mathbf{i}}+4\hat{\mathbf{j}})=7$



Sol. (Ex 52-54)

52. (d) $\mathbf{OB}=7\hat{\mathbf{i}}+7\hat{\mathbf{j}}$, $\mathbf{OE}=5\hat{\mathbf{i}}+\frac{13}{3}\hat{\mathbf{j}}$, $\mathbf{OP}=\frac{21}{5}(\hat{\mathbf{i}}+\hat{\mathbf{j}})$

53. (b) Direction ratio of **CP** is (1, 6, 0), then equation of line passing through (2, 3, 4) and parallel to **CP** is

$$\frac{x-2}{1}=\frac{y-3}{6}=\frac{z-4}{0}$$

54. (a) The plane containing line AC and at a maximum distance from B must be perpendicular to the plane $OABC$.

Since, $OABC$ is rhombus, so **OB** must normal to the plane. So, equation of required plane is

$$[\mathbf{r}-4\hat{\mathbf{i}}-3\hat{\mathbf{j}}]\cdot(\hat{\mathbf{i}}+\hat{\mathbf{j}})=0$$

$\Rightarrow$ $$(\hat{\mathbf{i}}+\hat{\mathbf{j}})=7$$

Passage IV

(Ex. Nos. 55 to 57)

A ray of light comes along the line $L=0$ *and strikes the plane mirror kept along the plane* $P=0$ *at B.* $A(2,1,6)$ *is a point on the line* $L=0$ *whose image about* $P=0$ *is A′. It is given that* $L=0$ *is* $\frac{x-2}{3}=\frac{y-1}{4}=\frac{z-6}{5}$ *and* $P=0$ *is* $x+y-2z=3$.

● **Ex. 55** *The coordinates of A′ are*

(a) (6, 5, 2) (b) (6, 5, − 2)
(c) (6, − 5, 2) (d) None of these

● **Ex. 56** *The coordinates of B are*

(a) (5, 10, 6) (b) (10, 15, 11)
(c) (− 10, − 15, − 14) (d) None of these

● **Ex. 57** *If* $L_1=0$ *is the reflected ray, then its equation is*

(a) $\frac{x+10}{4}=\frac{y-5}{4}=\frac{z+2}{3}$

(b) $\frac{x+10}{3}=\frac{y+15}{5}=\frac{z+14}{5}$

(c) $\frac{x+10}{4}=\frac{y+15}{5}=\frac{z+14}{3}$

(d) None of the above

Sol. (Ex 55-57)

55. (b) Let $Q(x_2, y_2, z_2)$ be the image of $A(2, 1, 6)$ about mirror $x + y - 2z = 3$. Then,

$$\frac{x_2 - 2}{1} = \frac{y_2 - 1}{1} = \frac{z_2 - 6}{-2}$$

$$= \frac{-2(2 + 1 - 12 - 3)}{1^2 + 1^2 + 2^2} = 4$$

$$\Rightarrow \quad (x_2, y_2, z_2) \equiv (6, 5, -2)$$

56. (c) Let $\frac{x-2}{3} = \frac{y-1}{4} = \frac{z-6}{5} = \lambda$

$x = 2 + 3\lambda, y = 1 + 4\lambda, z = 6 + 5\lambda$ lies on plane $x + y - 2z = 3$

$$\Rightarrow \quad 2 + 3\lambda + 1 + 4\lambda - 2(6 + 5\lambda) = 3$$

$$\Rightarrow \quad 3 + 7\lambda - 12 - 10\lambda = 3$$

$$\Rightarrow \quad -3\lambda = 12$$

$$\Rightarrow \quad \lambda = -4$$

Point $B \equiv (-10, -15, -14)$

57. (c) The equation of the reflected ray $L_1 = 0$ is the line joining $Q(x_2, y_2, z_2)$ and $B(-10, -15, -14)$.

$$\frac{x + 10}{16} = \frac{y + 15}{20} = \frac{z + 14}{12}$$

or $$\frac{x + 10}{4} = \frac{y + 15}{5} = \frac{z + 14}{3}$$

Passage V

(Ex. Nos. 58 to 60)

The line of greatest slope on an inclined plane P_1 is the line in the plane P_1 which is perpendicular to the line of intersection of the plane P_1 and a horizontal plane P_2.

● ***Ex. 58*** *Assuming the plane $4x - 3y + 7z = 0$ to be horizontal, the direction cosines of the line of greatest slope in the plane $2x + y - 5z = 0$ are*

(a) $\frac{3}{\sqrt{11}}, \frac{-1}{\sqrt{11}}, \frac{1}{\sqrt{11}}$ (b) $\frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{-1}{\sqrt{11}}$

(c) $\frac{-3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}}$ (d) None of these

● ***Ex. 59*** *The equation of a line of greatest slope can be*

(a) $\frac{x}{3} = \frac{y}{1} = \frac{z}{-1}$ (b) $\frac{x}{3} = \frac{y}{-1} = \frac{z}{1}$

(c) $\frac{x}{-3} = \frac{y}{1} = \frac{z}{1}$ (d) $\frac{x}{1} = \frac{y}{3} = \frac{z}{-1}$

● ***Ex. 60*** *The coordinates of a point on the plane $2x + y - 5z = 0$, $2\sqrt{11}$ unit away from the line of intersection of $2x + y - 5z = 0$ and $4x - 3y + 7z = 0$ are*

(a) $(6, 2, -2)$ (b) $(3, 1, -1)$

(c) $(6, -2, 2)$ (d) $(1, 3, -1)$

Sol. (Ex. 58-60)

58. (a) Plane P_1 is of the form $\mathbf{r} \cdot \mathbf{n}_1 = 0$, where $\mathbf{n}_1 = (4, -3, 7)$

Plane P_2 is of the form $\mathbf{r} \cdot \mathbf{n}_2 = 0$,

where $\mathbf{n}_2 = (2, 1, -5)$

The vector $\mathbf{b}$ along the line of intersection of planes is

$$\mathbf{n}_1 \times \mathbf{n}_1 = (4, 17, 5) = \mathbf{n}_3$$

Since the line of greatest slope is perpendicular to $\mathbf{n}_3$ and $\mathbf{n}_2$ the vector along the line of greatest slope

$$= \mathbf{n}_2 \times \mathbf{n}_3 = (3, -1, 1) = \mathbf{n}_4$$

and the unit vector

$$\mathbf{n}_4 = \hat{\mathbf{n}} = \left(\frac{3}{\sqrt{11}}, \frac{-1}{\sqrt{11}}, \frac{1}{\sqrt{11}}\right)$$

59. (b) Since, $(0, 0, 0)$ is a point on both planes, it lies on the line of intersection.

Hence, the equation a line of greatest slope can be

$$\frac{x}{3} = \frac{y}{-1} = \frac{z}{1}$$

60. (c) The point on the line $\frac{x}{3} = \frac{y}{-1} = \frac{z}{1}$ at a distance $2\sqrt{11}$ unit from the origin is given by

$$\frac{x}{\frac{3}{\sqrt{11}}} = \frac{y}{\frac{-1}{\sqrt{11}}} = \frac{z}{1} = 2\sqrt{11}$$

The point is $(6, -2, 2)$.

JEE Type Solved Examples : Matching Type Questions

● ***Ex. 61*** *Match the entries between following two columns.*

	Column I		Column II
A.	If the line $\frac{x-1}{1}=\frac{y+1}{-2}=\frac{z+1}{\lambda}$ lies in the plane $3x-2y+5z=0$, then λ is equal to	p.	$\sin^{-1}\sqrt{\frac{6}{25}}$
B.	If $(3,\lambda,\mu)$ is a point on the line $2x+y+z-3=0=x-2y+z-1$, then $\lambda+\mu$ is equal to	q.	$-\frac{7}{5}$
C.	The angle between the line $x=y=z$ and the plane $4x-3y+5z=2$ is	r.	-3
D.	The angle between the planes $x+y+z=0$ and $3x-4y+5z=0$ is	s.	$\cos^{-1}\sqrt{\frac{8}{75}}$

Sol. (A) → (q), (B) → (r), (C) → (p), (D) → (s)

(A) $3\cdot1-2(-2)+5(\lambda)=0 \Rightarrow \lambda=-\frac{7}{5}$

(B) Point $(3,\lambda,\mu)$ lies on $2x+y+z-3=x-2y+z-1$

$\Rightarrow 3\cdot2+\lambda+\mu-3=0$ and $3-2\lambda+\mu-1=0$

$\Rightarrow \lambda+\mu+3=0$ and $2\lambda-\mu-2=0$

So, $\lambda+\mu=-3$

(C) $\sin\theta=\frac{1\cdot4+1(-3)+1\cdot5}{\sqrt{1^2+1^2+1^2}\sqrt{16+9+25}}=\frac{6}{\sqrt{3}\sqrt{50}}$

$\theta=\sin^{-1}\sqrt{\frac{6}{25}}$

(D) $\cos\theta=\frac{1\cdot3+1(-4)+1\cdot5}{\sqrt{3}\sqrt{16+9+25}}=\frac{4}{\sqrt{3}\sqrt{50}}$

$\theta=\cos^{-1}\sqrt{\frac{8}{75}}$

● ***Ex. 62*** *Match the following*

	Column I		Column II
A.	$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-1}{3}=\frac{y-3}{4}=\frac{z-5}{5}$ are	p.	coincident
B.	$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-3}{2}=\frac{y-5}{3}=\frac{z-7}{4}$ are	q.	parallel and different
C.	$\frac{x-2}{5}=\frac{y+3}{4}=\frac{5-z}{2}$ and $\frac{x-7}{5}=\frac{y-1}{4}=\frac{z-2}{-2}$ are	r.	skew
D.	$\frac{x-3}{2}=\frac{y+2}{3}=\frac{z-4}{5}$ and $\frac{x-3}{3}=\frac{y-2}{2}=\frac{z-7}{5}$ are	s.	intersecting in a point

Sol. (A) → (s), (B) → (p), (C) → (q), (D) → (r)

(A) Both the lines pass through the point (7, 11, 15).

(B) < 2, 3, 4 > are direction ratios of both the lines. Also, the point (1, 2, 3) is common to both.

∴The lines are coincident.

(c) < 5, 4, − 2 > are direction ratios of both the lines.

∴The lines are parallel.

Also, $x=2+5\lambda, y=-3+4\lambda, z=5-2\lambda$

∴ $\frac{2+5\lambda-7}{5}=\frac{-3+4\lambda-1}{4}=\frac{5-2\lambda-2}{-2}$

i.e. $\lambda-1=\lambda-1=\frac{3-2\lambda}{-2}$

∴No value of λ.

Thus, the lines are parallel and different.

(D) < 2, 3, 5 > and <3, 2, 5 > are direction ratios of first and second line, respectively.

● ***Ex. 63*** *Match the followings*

	Column I		Column II
A.	The coordinates of a point on the line $x=4y+5$, $z=3y-6$ at a distance 3 from the point (5, 3, − 6) is/are	p.	$(-1,-2,0)$
B.	The plane containing the lines $\frac{x-2}{3}=\frac{y+3}{5}=\frac{z+5}{7}$ and parallel to $\hat{\mathbf{i}}+4\hat{\mathbf{j}}+7\hat{\mathbf{k}}$ has the point	q.	$(5,0,-6)$
C.	A line passes through two points $A(2,-3,-1)$ and $B(8,-1,2)$. The coordinates of a point on this line nearer to the origin and at a distance of 14 units from A is/are	r.	$(2,5,7)$
D.	The coordinates of the foot of the perpendicular from the point $(3,-1,11)$ on the line $\frac{x}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ is/are	s.	$(14,1,5)$

Sol. (A) → (q), (B) → (p), (C) → (s), (D) → (r)

(A) The given line is $x=4y+5, z=3y-6$,

or $\frac{x-5}{4}=y,\ \frac{z+6}{3}=y$

or $\frac{x-5}{4}=\frac{y}{1}=\frac{z+6}{3}=\lambda$ (say)

Any point on the line is of the form $(4\lambda+5, \lambda, 3\lambda-6)$.

The distance between $(4\lambda+5, \lambda, 3\lambda-6)$ and (5, 3, − 6) is 3 units (given).

Therefore, $(4\lambda+5-5)^2+(\lambda-3)^2+(3\lambda-6+6)^2=9$

$\Rightarrow 16\lambda^2+\lambda^2+9-6\lambda+9\lambda^2=9$

$\Rightarrow 26\lambda^2-6\lambda=0$

$\Rightarrow \lambda=0, \frac{3}{13}$

The point is (5, 0, − 6)

(B) The equation of the plane containing the lines $\frac{x-2}{3}=\frac{y+3}{5}=\frac{z+5}{7}$ and parallel to $\hat{\mathbf{i}}+4\hat{\mathbf{j}}+7\hat{\mathbf{k}}$.

$$\begin{vmatrix} x-2 & y+3 & z+5 \\ 1 & 4 & 7 \\ 3 & 5 & 7 \end{vmatrix} = 0 \Rightarrow x-2y+z-3=0$$

Point $(-1,-2,0)$ lies on this plane.

(C) The line passing through points $A(2,-3,-1)$

and $B(8,-1,2)$ is $\dfrac{x-2}{8-2}=\dfrac{y+3}{-1+3}=\dfrac{z+1}{2+1}$

or $\dfrac{x-2}{6}=\dfrac{y+3}{2}=\dfrac{z+1}{3}=\lambda$ (say)

Any point on this line is of the form $P(6\lambda+2, 2\lambda-3, 3\lambda-1)$, whose distance from point $A(2,-3,-1)$ is 14 units. Therefore,

$\Rightarrow$ $PA=14 \Rightarrow PA^2=(14)^2$

$\Rightarrow$ $(6\lambda)^2+(2\lambda)^2+(3\lambda)^2=196$

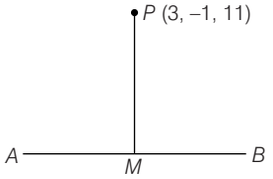
$\Rightarrow$ $49\lambda^2=196 \Rightarrow \lambda^2=4 \Rightarrow \lambda=\pm 2$

Therefore, the required points are $(14, 1, 5)$ and $(-10,-7,-7)$. The point nearer to the origin is $(14, 1, 5)$.

(D) Any point on line AB, $\dfrac{x}{2}=\dfrac{y-2}{3}=\dfrac{z-3}{4}=\lambda$ is

$M(2\lambda, 3\lambda+2, 4\lambda+3)$. Therefore, the direction ratios of PM are $2\lambda-3, 3\lambda+3$ and $4\lambda-8$.

But $PM \perp AB$



$\therefore$ $2(2\lambda-3)+3(3\lambda+3)+4(4\lambda-8)=0$

$4\lambda-6+9\lambda+9+16\lambda-32=0$

$29\lambda-29=0;\ \lambda=1$

Therefore, foot of the perpendicular is $M(2, 5, 7)$.

● *Ex. 64* Match the followings

	Column I	Column II
A.	Image of the point $(3, 5, 7)$ in the plane $2x+y+z=-18$ is	p. $(-1,-1,-1)$
B.	The point of intersection of the line $\dfrac{x-2}{-3}=\dfrac{y-1}{-2}=\dfrac{z-3}{2}$ and the plane $2x+y-z=3$ is	q. $(-21,-7,-5)$
C.	The foot of the perpendicular from the point $(1,1,2)$ to the plane $2x-2y+4z+5=0$ is	r. $\left(\dfrac{5}{2},\dfrac{2}{3},\dfrac{8}{3}\right)$
D.	The intersection point of the lines $\dfrac{x-1}{2}=\dfrac{y-2}{3}=\dfrac{z-3}{4}$ and $\dfrac{x-4}{5}=\dfrac{y-1}{2}=z$ is	s. $\left(-\dfrac{1}{12},\dfrac{25}{12},\dfrac{2}{12}\right)$

Sol. (A) $\rightarrow$ (q), (B) $\rightarrow$ (r), (C) $\rightarrow$ (s), (D) $\rightarrow$ (p)

(A) If the required image is (x, y, z), then

$$\frac{x-3}{2}=\frac{y-5}{1}=\frac{z-7}{1}=-\frac{2(6+5+7+18)}{2^2+1^2+1^2}$$

$=-12$ or $(-21,-7,-5)$.

(B) Any point on the line $\dfrac{x-2}{-3}=\dfrac{y-1}{2}=\dfrac{z-3}{2}=\lambda$ is

$(-3\lambda+2, 2\lambda+1, 2\lambda+3)$, which lies on plane $2x+y-z=3$. Therefore,

$$-6\lambda+4+2\lambda+1-2\lambda-3=3$$

$$-6\lambda=1$$

$$\lambda=-\frac{1}{6}$$

Therefore, the point is $\left(\dfrac{5}{2},\dfrac{2}{3},\dfrac{8}{3}\right)$

(C) If (x, y, z) is required foot of the perpendicular, then

$$\frac{x-1}{2}=\frac{y-1}{-2}=\frac{z-2}{4}=\frac{(2-2+8+5)}{2^2+(-2)^2+4^2}$$

or $(x, y, z)\equiv\left(\dfrac{-1}{12},\dfrac{25}{12},\dfrac{-2}{12}\right)$

(D) Any point on the line $\dfrac{x-1}{2}=\dfrac{y-2}{3}=\dfrac{z-3}{4}=\lambda$ is

$P(2\lambda+1, 3\lambda+2, 4\lambda+3)$, which satisfies the line

$$\frac{x-4}{5}=\frac{y-1}{2}=\frac{z}{1}$$

or $\dfrac{2\lambda+1-4}{5}=\dfrac{3\lambda+2-1}{2}=\dfrac{4\lambda+3}{1}$

$\Rightarrow$ $\lambda=-1$

The required point is $(-1,-1,-1)$.

● *Ex. 65* $\dfrac{x-2}{3}=\dfrac{y-3}{4}=\dfrac{z-4}{5}$

	Column I	Column II
A.	Point on the line at a distance $10\sqrt{2}$ from $(2, 3, 4)$	p. $(-1,-1,-1)$
B.	Point on the line common to the plane $x+y+z+3=0$	q. $(2, 3, 4)$
C.	Point on the line at a distance $\sqrt{29}$ from the origin.	r. $(8, 11, 14)$
D.	Point on the line common to the plane $x+y-z+3=0$	s. $(-4,-5,-6)$

Sol. (A) $\rightarrow$ (r, s), (B) $\rightarrow$ (p), (C) $\rightarrow$ (q), (D) $\rightarrow$ (s)

Any point on the line is $(3r+2, 4r+3, 5r+4$

(A) $(3r+2-2)^2+(4r+3-3)^2+(5r+4-4)^2=200$

$\Rightarrow$ $(9+16+25)r^2=200 \Rightarrow r=\pm 2$

For $r=2$, the point is $(8, 11, 14)$, For $r=-2$ it is $(-4,-5,-6)$

(B) $3r+2+4r+3+5r+4+3=0$

$\Rightarrow$ $12r+12=0 \Rightarrow r=-1$

and the point on the line common to the plane is $(-1,-1,-1)$.

(C) $(3r+2)^2+(4r+3)^2+(5r+4)^2=29$

$$50r^2+76r=0 \Rightarrow r=0, r=-\frac{76}{50}$$

For $r=0$, the point is $(2, 3, 4)$.

(D) $3r+2+4r+3-5r-4+3=0$

$\Rightarrow$ $2r+4=0 \Rightarrow r=-2$

$\therefore$ The point on the line common to the plane is $(-4,-5,-6)$

JEE Type Solved Examples :
Single Integer Answer Type Questions

● **Ex. 66** *If the perpendicular distance of the point (6, 5, 8) from the Y-axis is* 5λ *unit, then* λ *is equal to*

Sol. (2) Foot of perpendicular from (6, 5, 8) on Y-axis is (0, 5, 0).

Required distance $= \sqrt{(6-0)^2 + (5-5)^2 + (8-0)^2}$

$= 10$ unit

$\Rightarrow \qquad 5\lambda = 10 \Rightarrow \lambda = \dfrac{10}{5} = 2$

● **Ex. 67** *A parallelopiped is formed by planes drawn through the points (2, 4, 5) and (5, 9, 7) parallel to the coordinate planes. The length of the diagonal of the parallelopiped is*

Sol. (7) The length of the edges are given by $a = 5-2 = 3$, $b = 9-3 = 6$, $c = 7-5 = 2$, so length of the diagonal

$= \sqrt{a^2 + b^2 + c^2}$

$= \sqrt{9 + 36 + 4}$

$= 7$ units

● **Ex. 68** *If the shortest distance between the lines* $\dfrac{x-3}{3} = \dfrac{y-8}{-1} = \dfrac{z-3}{1}$ *and* $\dfrac{x+3}{-3} = \dfrac{y+7}{2} = \dfrac{z-6}{4}$ *is* $\lambda\sqrt{30}$ *unit, then the value of* λ *is*

Sol. (3) Given, lines are

$\mathbf{r} = 3\hat{\mathbf{i}} + 8\hat{\mathbf{j}} + 3\hat{\mathbf{k}} + \lambda(3\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}})$

$\mathbf{r} = (-3\hat{\mathbf{i}} - 7\hat{\mathbf{j}} + 6\hat{\mathbf{k}}) + \mu(-3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}})$

where λ, μ are parameters.

Shortest distance

$$= \frac{[(-3-3)\hat{\mathbf{i}} + (7-8)\hat{\mathbf{j}} + (6-3)\hat{\mathbf{k}}]}{[(3\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}) \times (-3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}})]}$$

$$= \frac{(-6\hat{\mathbf{i}} - 15\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) \times (-6\hat{\mathbf{j}} - 15\hat{\mathbf{k}} + 3\hat{\mathbf{k}})}{\sqrt{36 + 225 + 9}}$$

$= \sqrt{270} = 3\sqrt{30}$ unit

● **Ex. 69** *If the planes* $x - cy - bz = 0$, $cx - y + az = 0$ *and* $bx + ay - z = 0$ *pass through a line, then the value of* $a^2 + b^2 + c^2 + 2abc$ *is*

Sol. (1) Given, planes are

$$x - cy - bz = 0 \qquad \text{...(i)}$$

$$cx - y + az = 0 \qquad \text{...(ii)}$$

$$bx + ay - z = 0 \qquad \text{(iii)}$$

Equation of planes passing through the line of intersection of planes (i) and (ii) may be taken as

$$(x - cy - bz) + \lambda(cx - y + az) = 0 \qquad \text{...(iv)}$$

Now, planes (iii) and (iv) are same

$\therefore \qquad \dfrac{1 + c\lambda}{b} = \dfrac{-(c + \lambda)}{a} = \dfrac{-b + a\lambda}{-1}$

By eliminating λ, we get $a^2 + b^2 + c^2 + 2abc = 1$

● **Ex. 70** *If the line* $\dfrac{x-4}{1} = \dfrac{y-2}{1} = \dfrac{z-k}{2}$ *lies exactly on the plane* $2x - 4y + z = 7$, *the value of* k *is*

Sol. (7) The point (4, 2, k) must satisfy the plane.

So, $\qquad 8 - 8 + k = 7 \Rightarrow k = 7$

Subjective Type Questions

● **Ex. 71** *The equation of motion of rockets are*

$$x = 2t,\ y = -4t,\ z = 4t,$$

where the time 't' is given in second and the coordinates of a moving point in kilometres.
What is the path of the rocket? At what distance will the rocket be from the starting point $O(0, 0, 0)$ *in 10s.*

Sol. Eliminating 't' from the given equations, we get the equation of the path,

$$\frac{x}{2} = \frac{y}{-4} = \frac{z}{4}$$

or $\qquad \dfrac{x}{1} = \dfrac{y}{-2} = \dfrac{z}{2}$

Thus, the path of the rocket represents a straight line passing through the origin.

For $\qquad t = 10$ s

We have, $x = 20,\ y = -40,\ z = 40$

and $\qquad |\mathbf{r}| = |\mathbf{OM}| = \sqrt{x^2 + y^2 + z^2}$

$= \sqrt{400 + 1600 + 1600} = 60$ km

● **Ex. 72** *Write the equation of a tangent to the curve* $x = t$, $y = t^2$, $z = t^3$ *at its point* $M(1, 1, 1)$; $(t = 1)$.

Sol. Here, $\mathbf{r} = t\hat{\mathbf{i}} + t^2\hat{\mathbf{j}} + t^3\hat{\mathbf{k}}$

$$\frac{dr}{dt} = \hat{\mathbf{i}} + 2t\hat{\mathbf{j}} + 3t^2\hat{\mathbf{k}}$$

Hence, the direction of the tangent at the point M is determined by the vector.

$$\left(\frac{dr}{dt}\right)_M = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$$

Thus, the equation of the desired tangent is,

$$\frac{x-1}{1} = \frac{y-1}{2} = \frac{z-1}{3}$$

● ***Ex. 73*** *Find the locus of a point, the sum of squares of whose distances from the planes $x - z = 0$, $x - 2y + z = 0$ and $x + y + z = 0$ is 36.*

Sol. Given planes are $x - z = 0$, $x - 2y + z = 0$

and $x + y + z = 0$.

Let the point whose locus is required be $P(\alpha, \beta, \gamma)$. According to question,

$$\frac{|\alpha + \gamma|^2}{2} + \frac{|\alpha - 2\beta + \gamma|^2}{6} + \frac{|\alpha + \beta + \gamma|^2}{3} = 36$$

or $3(\alpha^2 + \gamma^2 - 2\alpha\gamma) + \alpha^2 + 4\beta^2 + \gamma^2 - 4\alpha\beta - 4\beta\gamma + 2\alpha\gamma$

$$+ 2(\alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\beta\gamma + 2\alpha\gamma) = 36 \times 6$$

or $6\alpha^2 + 6\beta^2 + 6\gamma^2 = 36 \times 6$

or $\alpha^2 + \beta^2 + \gamma^2 = 36$

Hence, the required equation of locus is

$$x^2 + y^2 + z^2 = 36$$

● ***Ex. 74*** *The plane $ax + by = 0$ is rotated through an angle α about its line of intersection with the plane $z = 0$. Show that the equation to the plane in new position is $ax + by \pm z\sqrt{a^2 + b^2}\tan\alpha = 0$*

Sol. Given planes are

$$ax + by = 0 \quad \text{...(i)}$$

and $$z = 0 \quad \text{...(ii)}$$

∴Equation of any plane passing through the line of intersection of planes (i) and (ii) may be taken as,

$$ax + by + kz = 0 \quad \text{...(iii)}$$

The direction cosines of a normal to the plane (iii) are

$$\frac{a}{\sqrt{a^2 + b^2 + k^2}}, \frac{b}{\sqrt{a^2 + b^2 + k^2}}, \frac{k}{\sqrt{a^2 + b^2 + k^2}}$$

The direction cosines of a normal to the plane (i) are

$$\frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}}, 0$$

Since, the angle between the planes (i) and (iii) is α.

$$\therefore \quad \cos\alpha = \frac{a \cdot a + b \cdot b + k \cdot 0}{\sqrt{a^2 + b^2 + k^2}\sqrt{a^2 + b^2}}$$

$$= \sqrt{\frac{a^2 + b^2}{a^2 + b^2 + k^2}}$$

$$k^2 \cos^2\alpha = \alpha^2(1 - \cos^2\alpha) + b^2(1 - \cos^2\alpha)$$

$$k^2 = \frac{(a^2 + b^2)\sin^2\alpha}{\cos^2\alpha}$$

On putting in Eq. (iii) $k = \pm\sqrt{a^2 + b^2}\tan\alpha$, we get equation of plane as,

$$ax + by \pm z\sqrt{a^2 + b^2}\tan\alpha = 0.$$

● ***Ex. 75*** *Assuming the plane $4x - 3y + 7z = 0$ to be horizontal, find the equation of the line of greatest slope through the point $(2, 1, 1)$ in the plane $2x + y - 5z = 0$.*

Sol. The required line passing through the point (2, 1, 1) in the plane $2x + y - 5z = 0$ and is having greatest slope, so it must be perpendicular to the line of intersection of the planes

$$2x + y - 5z = 0 \quad \text{...(i)}$$

and $$4x - 3y + 7z = 0 \quad \text{...(ii)}$$

Let the DR's of the line of intersection of Eqs. (i) and (ii) are a, b, c.

$$2a + b - 5c = 0$$

and $$4a - 3b + 7c = 0$$

(as DR's of straight line (a, b, c) is perpendicular to DR's of normal to both the planes)

$$\frac{a}{4} = \frac{b}{17} = \frac{c}{5}$$

Now, let the direction ratio of required line be proportional to l, m and n, then its equation be

$$\frac{x-2}{l} = \frac{y-1}{m} = \frac{z-1}{n}$$

where, $2l + m - 5n = 0$ and $4l + 17m + 5n = 0$

So, $$\frac{l}{3} = \frac{m}{-1} = \frac{n}{1}$$

Thus, the required line is $\frac{x-2}{3} = \frac{y-1}{-1} = \frac{z-1}{1}$

● ***Ex. 76*** *Does $\frac{a}{x-y} + \frac{b}{y-z} + \frac{c}{z-x} = 0$ represents a pair of planes?*

Sol. Here, given equation is $\frac{a}{x-y} + \frac{b}{y-z} + \frac{c}{z-x} = 0$

$$\Rightarrow a(y-z)(z-x) + b(x-y)(z-x) + c(x-y)(y-z) = 0$$

$$\Rightarrow -axy + ayz - az^2 + axz + bxz - bx^2 - byz$$

$$+ byx + cxy - cxz - cy^2 + cyz = 0$$

$$\Rightarrow bx^2 + cy^2 + az^2 - (b + c - a)xy -$$

$$(c + a - b)yz - (a + b - c)zx = 0$$

∴Value of determinant;

$$\begin{vmatrix} b & -\frac{1}{2}(b + c - a) & -\frac{1}{2}(a + b - c) \\ -\frac{1}{2}(b + c - a) & c & -\frac{1}{2}(c + a - b) \\ -\frac{1}{2}(a + b - c) & \frac{1}{2}(c + a - b) & a \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 \\ -\frac{1}{2}(b+c-a) & c & -\frac{1}{2}(c+a-b) \\ -\frac{1}{2}(a+b-c) & -\frac{1}{2}(c+a-b) & a \end{vmatrix}$$

$[R_1 \to R_1 + R_2 + R_3]$

$= 0$

Hence, the given equation represents a pair of planes.

● **Ex. 77** *If the straight line* $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ *intersect the curve* $ax^2 + by^2 = 1, z = 0$, *then prove that* $a(\alpha n - \gamma l)^2 + b(\beta n - \gamma m)^2 = n^2$.

Sol. Here, $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = \lambda$

$\therefore$ Any point on the given line

$$(\lambda l + \alpha, \lambda m + \beta + \lambda n + y)$$

If it lies on the curve $ax^2 + by^2 = 1, z = 0$

(as the point of intersection)

$$a(\alpha + l\lambda)^2 + b(\beta + m\lambda)^2 = 1 \quad ...(i)$$

and $$\lambda n + \gamma = 0 \quad ...(ii)$$

From Eq. (ii), $\lambda = \frac{-\gamma}{n}$ must satisfy Eq. (i), we get

$$a\left(\alpha - \frac{l\gamma}{n}\right)^2 + b\left(\beta - \frac{m\gamma}{n}\right)^2 = 1$$

$$a(n\alpha - l\gamma)^2 + b(n\beta - m\gamma)^2 = n^2$$

● **Ex. 78** *Prove that the three lines from O with direction cosines* $l_1, m_1, n_1; l_2, m_2, n_2$ *and* l_3, m_3, n_3 *are coplanar, if* $l_1(m_2n_3 - n_2m_3) + m_1(n_2l_3 - l_2n_3) + n_1(l_2m_3 - l_3m_2) = 0$.

Sol. Here, three given lines are coplanar, if they have common perpendicular.

Let DC's of common perpendicular be l, m and n.

$$ll_1 + mm_1 + nn_1 = 0 \quad ...(i)$$

$$ll_2 + mm_2 + nn_2 = 0 \quad ...(ii)$$

and $$ll_3 + mm_3 + nn_3 = 0 \quad ...(iii)$$

Solving Eqs. (ii) and (iii) by cross multiplication method, we get

$$\frac{l}{m_2 \quad n_2} = \frac{m}{n_2 \quad l_2} = \frac{n}{l_2 \quad m_2}$$

$$m_3 \quad n_3 \quad l_3 \quad m_3$$

$$\Rightarrow \frac{l}{m_2n_3 - n_2m_3} = \frac{m}{n_2l_3 - n_3l_2} = \frac{n}{l_2m_3 - l_3m_2} = k$$

$$\Rightarrow l = k(m_2n_3 - n_2m_3),\ m = k(n_2l_3 - n_3l_2),\ n = k(l_2m_3 - l_3m_2)$$

Substituting in Eq. (i), we get

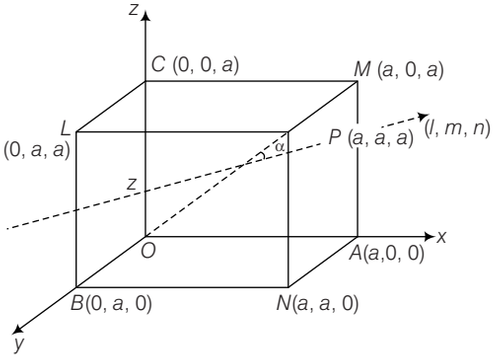
$$k(m_2n_3 - n_2m_3)l_1 + k(n_2l_3 - n_3l_2)m_1 + k(l_2m_3 - l_3m_2)n_2 = 0$$

$$\Rightarrow l_1(m_2n_3 - n_2m_2) + m_1(n_2l_3 - n_3l_2) + n_1(l_2m_3 - l_3m_2)$$

● **Ex. 79** *A line makes angle,* α, β, γ *and* δ *with the four diagonals of cube, prove that*

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}.$$

Sol. Let the cube be shown in the figure, where four diagonals are OP, AL, BM and CN and $A(a, 0, 0)$, $B(0, a, 0)$, $C(0, 0, a)$, $L(0, a, a)$, $M(a, 0, a)$, $N(a, a, 0)$ and $P(a, a, a)$, hence direction cosines of OP are



$$\frac{a}{\sqrt{a^2+a^2+a^2}}, \frac{a}{\sqrt{a^2+a^2+a^2}}, \frac{a}{\sqrt{a^2+a^2+a^2}}$$

$$= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

The DC's of AL are $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

The DC's of BM are $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

The DC's of CN are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$.

Let the DC's of required line be (l, m, n).

$$\therefore \quad \cos\alpha = \frac{l+m+n}{\sqrt{3}}, \cos\beta = \frac{-l+m+n}{\sqrt{3}},$$

$$\cos\gamma = \frac{l-m+n}{\sqrt{3}} \text{ and } \cos\delta = \frac{l+m-n}{\sqrt{3}}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$$

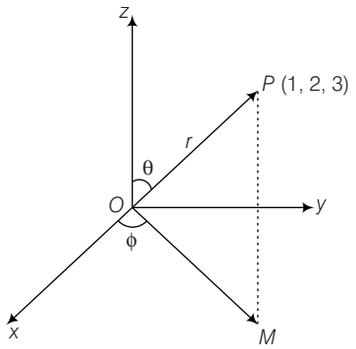
$$= \frac{1}{3}\{(l+m+n)^2 + (-l+m+n)^2 + (l-m+n)^2 + (l+m-n)^2\}$$

$$= \frac{4}{3}(l^2 + m^2 + n^2) = \frac{4}{3}$$

● **Ex. 80** *Let PM be the perpendicular from the point* $P(1, 2, 3)$ *to XY-plane. If OP makes an angle* θ *with the positive direction of the Z-axis and OM makes an angle* ϕ *with the positive direction of X-axis, where O is the origin, then find* θ *and* ϕ.

Sol. Here, P be (x, y, z) shown as,

then, $x = r\sin\theta \cdot \cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta \quad ...(i)$



$\Rightarrow 1 = r\sin\theta\cdot\cos\phi, 2 = r\sin\theta\sin\phi, 3 = r\cos\theta$

$\Rightarrow 1^2 + 2^2 + 3^2 = r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta$

$= r^2\sin^2\theta(\cos^2\phi + \sin^2\phi) + r^2\cos^2\theta$

$= r^2\sin^2\theta + r^2\cos^2\theta = r^2$

$\Rightarrow \quad r = \pm\sqrt{14}$

$\therefore$ From Eq. (i), we have

$$\sin\theta\cos\phi = +\frac{1}{\sqrt{14}},$$

$$\sin\theta\sin\phi = \frac{2}{\sqrt{14}}, \cos\theta = \frac{3}{\sqrt{14}}$$

(neglecting – ve sign as acute angles)

$$\therefore \quad \frac{\sin\theta\sin\phi}{\sin\theta\cos\phi} = \frac{2}{1}$$

and $$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{5}}{3}$$

$$\Rightarrow \quad \tan\phi = 2 \text{ and } \tan\theta = \frac{\sqrt{5}}{3}$$

$$\Rightarrow \quad \phi = \tan^{-1}2 \text{ and } \theta = \tan^{-1}\left(\frac{\sqrt{5}}{3}\right)$$

● ***Ex. 81*** *Find the distance of the point* $(1, 0, -3)$ *from the plane* $x - y - z = 9$ *measured parallel to the line,* $\frac{x-2}{2} = \frac{y+2}{3} = \frac{2-6}{-6}$.

Sol. Given plane is

$$x - y - z = 9 \quad \ldots(i)$$

Given line AB is $\frac{x-2}{2} = \frac{y+2}{3} = \frac{z-6}{-6}$ ...(ii)

Equation of line passing through $(1, 0, -3)$ and parallel to

$$\frac{x-2}{2} = \frac{y+2}{3} = \frac{z-6}{-6}$$

is $$\frac{x-1}{2} = \frac{y-0}{3} = \frac{z+3}{-6} = r \quad \ldots(iii)$$

Coordinate of any point on Eq. (iii) may be given as

$$P(2r + 1, 3r, -6r - 3)$$

If P is intersection of Eqs. (i) and (iii), then it must lie on Eq. (i).

$$(2r + 1) - (3r) - (-6r - 3) = 9$$

$$2r + 1 - 3r + 6r + 3 = 9$$

$$5r = 5$$

$\Rightarrow \quad r = 1$

$\therefore$ Coordinate of $P(3, 3, -9)$.

$\Rightarrow$ Distance between $(1, 0, -3)$ and $(3, 3, -9)$

$$= \sqrt{(3-1)^2 + (3-0)^2 + (-9+3)^2}$$

$$= \sqrt{4 + 9 + 36} = 7$$

● ***Ex. 82*** *Find the equation of the plane which passes through* $a_1x + b_1y + c_1z + d_1 = 0, a_2x + b_2y + c_2z + d_2 = 0$ *and which is parallel to the line* $\frac{x-a}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$.

Sol. Given, $a_1x + b_1y + c_1z + d_1 = 0$

$$a_2x + b_2y + c_2z + d_2 = 0 \quad \ldots(i)$$

and $$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad \ldots(ii)$$

Equation of plane through the intersection of plane (i) is given by

$$(a_1x + b_1y + c_1z + d_1) + \lambda(a_2x + b_2y + c_2z + d_2) = 0$$

or $$(a_1 + \lambda a_2)x + (b_1 + \lambda b_2)y + (c_1 + \lambda c_2)z + (d_1 + \lambda d_2) = 0 \quad \ldots(iii)$$

DR's of normal to Eq. (iii) are

$$(a_1 + \lambda a_2), (b_1 + \lambda b_2), (c_1 + \lambda c_2)$$

$\therefore$ Eq. (iii) is parallel to Eq. (ii).

$\Rightarrow$ Normal to plane (iii) should be perpendicular to line (ii).

$\therefore \quad (a_1 + \lambda a_2)l + (b_1 + \lambda b_2)m + (c_1 + \lambda c_2)n = 0$

$\Rightarrow \quad \lambda = -\frac{(a_1l + b_1m + c_1n)}{(a_2l + b_2m + c_2n)}$, putting in (iii), we get

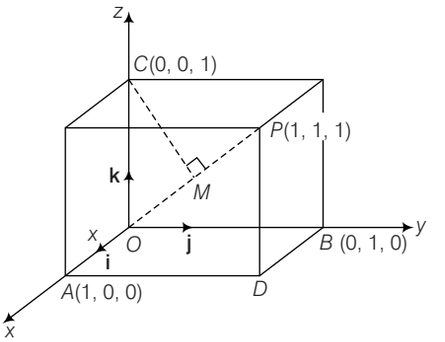
$$(a_1x + b_1y + c_1z + d_1)(a_2l + b_2m + c_2n) - (a_1l + b_1 + c_1n)(a_2x + b_2y + c_2z + d_2) = 0$$

Hence, the equation of required plane.

● ***Ex. 83*** *Find the perpendicular distance of a corner of a unit cube from a diagonal not passing through it.*

Sol. Let the edges OA, OB, OC of the unit cube be along OX, OY and OZ, respectively. Since, $OA = OB = OC = 1$ unit

$\therefore \quad \mathbf{OA} = \hat{\mathbf{i}}, \mathbf{OB} = \hat{\mathbf{j}}, \mathbf{OC} = \hat{\mathbf{k}}$



Let CM be perpendicular from the corner C on the diagonal OP. The vector equation of OP is

$$\mathbf{r} = \lambda(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$$

$\therefore$ $OM =$ Projection of **OC** on **OP**

$= \mathbf{OC} \cdot \mathbf{OP}$

$= \hat{\mathbf{k}} \dfrac{(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})}{\sqrt{3}} = \dfrac{1}{\sqrt{3}}$

Now, $OC^2 = OM^2 + CM^2$

$\Rightarrow$ $CM^2 = |OC|^2 - OM^2 = 1 - \dfrac{1}{3} = \dfrac{2}{3}$

$\Rightarrow$ $CM = \sqrt{\dfrac{2}{3}}$

● ***Ex. 84*** *If a variable plane forms a tetrahedron of constant volume $64k^3$ with the coordinate planes, then find the locus of the centroid of the tetrahedron.*

Sol. Let the variable plane intersects the coordinate axes at $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$. Then, the equation of the plane will be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \text{...(i)}$$

Let $P(\alpha, \beta, \gamma)$ be the centroid of tetrahedron $OABC$, then

$$\alpha = \frac{a}{4}, \beta = \frac{b}{4}, \gamma = \frac{c}{4}$$

or $a = 4\alpha, b = 4\beta, c = 4\gamma$

$\Rightarrow$ Volume of tetrahedron $= \dfrac{1}{3}$ (Area of $\Delta AOB) \cdot OC$

$\Rightarrow$ $64k^3 = \dfrac{1}{3}\left(\dfrac{1}{2} ab\right) c = \dfrac{abc}{6}$

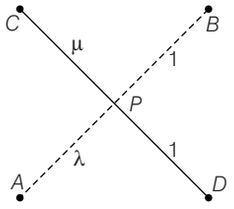
$\Rightarrow$ $64k^3 = \dfrac{(4\alpha)(4\beta)(4\gamma)}{6}$

$\Rightarrow$ $k^3 = \dfrac{\alpha\beta\gamma}{6}$

$\therefore$ Required locus of $P(\alpha, \beta, \gamma)$ is $xyz = 6k^3$.

● ***Ex. 85*** *Show that the line segments joining the points $(4, 7, 8), (-1, -2, 1)$ and (2, 3, 4) (1, 2, 5) intersect. Verify whether the four points concyclic.*

Sol. Here, $A(4, 7, 8)$, B $(-1, -2, 1)$, $C(2, 3, 4)$ and $D(1, 2, 5)$. If the lines AB and CD intersect at P, then let



$$\frac{AP}{PB} = \frac{\lambda}{1} \text{ and } \frac{CP}{PD} = \frac{\mu}{1}$$

Then, $P\left(\dfrac{-\lambda + 4}{\lambda + 1}, \dfrac{-2\lambda + 7}{\lambda + 1}, \dfrac{\lambda + 8}{\lambda + 1}\right)$

$$= \left(\frac{\mu + 2}{\mu + 1}, \frac{2\mu + 3}{\mu + 1}, \frac{5\mu + 4}{\mu + 1}\right)$$

$\Rightarrow$ $\dfrac{-\lambda + 4}{\lambda + 1} = \dfrac{\mu + 2}{\mu + 1}, \dfrac{-2\lambda + 7}{\lambda + 1} = \dfrac{2\mu + 3}{\mu + 1}$

$$\frac{\lambda + 8}{\lambda + 1} = \frac{5\mu + 4}{\mu + 1}$$

$\Rightarrow$ $\dfrac{-(\lambda + 1) + 5}{\lambda + 1} = \dfrac{(\mu + 1) + 1}{\mu + 1}$

$$\frac{-2(\lambda + 1) + 9}{\lambda + 1} = \frac{2(\mu + 1) + 1}{\mu + 1}$$

$$\frac{(\lambda + 1) + 7}{\lambda + 1} = \frac{5(\mu + 1) - 1}{\mu + 1}$$

$\Rightarrow$ $-1 + \dfrac{5}{\lambda + 1} = 1 + \dfrac{1}{\mu + 1}$

$$-2 + \frac{9}{\lambda + 1} = 2 + \frac{1}{\mu + 1};$$

$$1 + \frac{7}{\lambda + 1} = 5 - \frac{1}{(\mu + 1)}$$

Let $\dfrac{1}{\lambda + 1} = x$ and $\dfrac{1}{\mu + 1} = y$

$\Rightarrow$ $5x - y = 2; 9x - y = 4; 7x + y = 4$

On solving, $x = \dfrac{1}{2}, y = \dfrac{1}{2}$

$\Rightarrow$ $\lambda + 1 = 2, \mu + 1 = 2$

$\lambda = 1, \mu = 1$

Clearly, if $\lambda = 1$ and $\mu = 1$,

AB and CD bisects each other.

$\therefore$ $P = \left(\dfrac{3}{2}, \dfrac{5}{2}, \dfrac{9}{2}\right)$

Now, $AP = \sqrt{\left(4 - \dfrac{3}{2}\right)^2 + \left(7 - \dfrac{5}{2}\right)^2 + \left(8 - \dfrac{9}{2}\right)^2}$

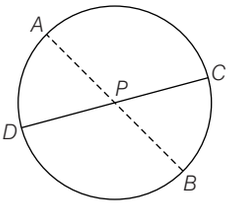
$$= \frac{\sqrt{155}}{2} = PB$$

Also, $CP = \sqrt{\left(2 - \dfrac{3}{2}\right)^2 + \left(3 - \dfrac{5}{2}\right)^2 + \left(4 - \dfrac{9}{2}\right)^2}$

$$= \frac{\sqrt{3}}{2} = PD$$

We know four points A, B, C and D are concylic, if

$$AP \cdot PB = PC \cdot PD$$



But here,

$$AP \cdot PB = \frac{155}{4} \text{ and } PC \cdot PD = \frac{3}{4}$$

$\therefore$ Points are non-concyclic.

● ***Ex. 86*** *If P be any point on the plane $lx + my + nz = p$ and Q be a point on the line OP such that $OP \cdot OQ = p^2$, show that the locus of the point Q is $p(lx + my + nz) = x^2 + y^2 + z^2$.*

Sol. Let $P(\alpha, \beta, \gamma)$ and $Q(x_1, y_1, z_1)$

DR's of OP are (α, β, γ) and DR's of OQ are (x_1, y_1, z_1).

$\because$ O, Q and P are collinear.

$$\therefore \quad \frac{\alpha}{x_1} = \frac{\beta}{y_1} = \frac{\gamma}{z_1} = k \quad \text{(say)} \quad ...(i)$$

Since, $P(\alpha, \beta, \gamma)$ lie on the plane

$$lx + my + nz = p,$$

$$l\alpha + m\beta + n\gamma = p$$

Since, $P(\alpha, \beta, \gamma)$ lie on the plane $lx + my + nz = p$,

$l\alpha + m\beta + n\gamma \pm p$

$$\Rightarrow \quad klx_1 + kmy_1 + knz_1 = p \quad \text{[using Eq. (i)]} \quad ...(ii)$$

Since, $OP \cdot OQ = p^2$

$$\therefore \quad \sqrt{\alpha^2 + \beta^2 + \gamma^2} \cdot \sqrt{x_1^2 + y_1^2 + z_1^2} = p^2$$

$$\sqrt{k^2x_1^2 + k^2y_1^2 + k^2z_1^2} \cdot \sqrt{x_1^2 + y_1^2 + z_1^2} = p^2$$

$$\Rightarrow \quad k(x_1^2 + y_1^2 + z_1^2) = p^2 \quad ...(iii)$$

From Eqs. (ii) and (iii), we get

$$\frac{lx_1 + my_1 + nz_1}{x_1^2 + y_1^2 + z_1^2} = \frac{1}{p}$$

$$\Rightarrow \quad p(lx_1 + my_1 + nz_1) = (x_1^2 + y_1^2 + z_1^2)$$

Hence, locus of Q

$$\Rightarrow \quad p(lx + my + nz) = x^2 + y^2 + z^2$$

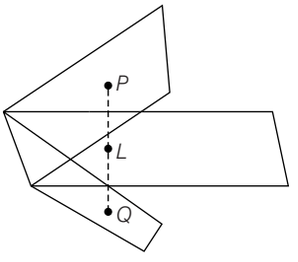
● ***Ex. 87*** *Find the reflection of the plane $ax + by + cz + d = 0$ in the plane $a'x + b'y + c'z + d' = 0$*

Sol. Given planes are

$$ax + by + cz + d = 0 \quad ...(i)$$

and $$a'x + b'y + c'z + d' = 0 \quad ...(ii)$$

Let $P(\alpha, \beta, \gamma)$ be an arbitrary point in the plane (i) and $Q(p, q, r)$ be the reflection of the point P in plane (ii). Locus of Q will be the required reflection of plane (i) in plane (ii), let L be the mid-point of PQ.



Then, $$L\left(\frac{p+\alpha}{2}, \frac{q+\beta}{2}, \frac{r+\gamma}{2}\right)$$

L lies in plane (ii), we get

$$\therefore \quad a'\left(\frac{p+\alpha}{2}\right) + b'\left(\frac{q+\beta}{2}\right) + c'\left(\frac{r+\gamma}{2}\right) + d' = 0$$

$$\Rightarrow \quad a'(p+\alpha) + b'(q+\beta) + c'(r+\gamma) + 2d' = 0 \quad ...(iii)$$

DR's of PQ are $\alpha - p, \beta - q, \gamma - r$.

Since, PQ perpendicular on plane Eq. (iii), we get

$$\frac{\alpha - p}{a'} = \frac{\beta - q}{b'} = \frac{\gamma - r}{c'} = k \quad \text{(say)}$$

$$\therefore \quad \alpha = p + a'k, \beta = q + b'k, \gamma = r + c'k$$

Putting the values of α, β and γ in Eq. (iii), we get

$$a'(2p + ka') + b'(2q + kb') + c(2r + kc') + 2d' = 0$$

$$\Rightarrow \quad 2(a'p + b'q + c'r + d') = -k(a'^2 + b'^2 + c'^2) \quad ...(iv)$$

Since, $P(\alpha, \beta, \gamma)$ lies on plane (i), we get

$$a\alpha + b\beta + c\gamma + d = 0$$

$$\Rightarrow \quad a(p + ka') + n(q + kb') + c(r + kc') + d = 0$$

$$\therefore \quad k = -\frac{(ap + bq + cr + d)}{(aa' + bb' + cc')}$$

Putting the value of k in Eq. (iv), we get

$$2(a'p + b'q + c'r + d') = \frac{(a'^2 + b'^2 + c'^2)(ap + bq + cr + d)}{aa' + bb' + cc'}$$

$\therefore$Locus of $Q(p, q, r)$.

i.e. equation reflection of plane (i) in plane (ii) is,

$$2(aa' + bb' + cc')(a'x + b'y + c'z + d') = (a'^2 + b'^2 + c'^2)(ax + by + cz + d)$$

● ***Ex. 88*** *A point P moves on a plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. A plane through P and perpendicular to OP meets the coordinate axes in A, B and C. If the planes through A, B and C parallel to the planes $x = 0$, $y = 0$ and $z = 0$ intersect in Q, then find the locus of Q.*

Sol. Given plane is $$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad ...(i)$$

Let $P(h, k, l)$ be the point on plane.

$$\therefore \quad \frac{h}{a} + \frac{k}{b} + \frac{1}{c} = 1 \quad ...(ii)$$

$$\Rightarrow \quad OP = \sqrt{h^2 + k^2 + l^2}$$

DC's of $OP = \left(\frac{h}{\sqrt{h^2+k^2+l^2}}, \frac{k}{\sqrt{h^2+k^2+l^2}}, \frac{l}{\sqrt{h^2+k^2+l^2}}\right)$

$\therefore$ Equation of the plane through P and normal to OP is,

$$\frac{hx}{\sqrt{h^2+k^2+l^2}} + \frac{ky}{\sqrt{h^2+k^2+l^2}} + \frac{lz}{\sqrt{h^2+k^2+l^2}} = \sqrt{h^2+k^2+l^2}$$

$$\Rightarrow \quad hx + ky + lz = h^2 + k^2 + l^2$$

$$\therefore \quad A \equiv \left(\frac{h^2+k^2+l^2}{h}, 0, 0\right)$$

$$B \equiv \left(0, \frac{h^2+k^2+l^2}{k}, 0\right)$$

$$C \equiv \left(0, 0, \frac{h^2 + k^2 + l^2}{l}\right)$$

Let $Q(\alpha, \beta, \gamma)$, then

$$\alpha = \frac{h^2 + k^2 + l^2}{h},$$

$$\beta = \frac{h^2 + k^2 + l^2}{k},$$

$$\gamma = \frac{h^2 + k^2 + l^2}{l} \quad ...(iii)$$

Now, $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{h^2 + k^2 + l^2}{(h^2 + k^2 + l^2)^2}$

$$= \frac{1}{h^2 + k^2 + l^2} \quad ...(iv)$$

From Eq. (iii), we get

$$h = \frac{h^2 + k^2 + l^2}{\alpha}$$

$$\Rightarrow \quad \frac{h}{a} = \frac{h^2 + k^2 + l^2}{a\alpha}$$

Similarly, $\frac{k}{b} = \frac{h^2 + k^2 + l^2}{b\beta}$

and $\frac{l}{c} = \frac{h^2 + k^2 + l^2}{c\gamma}$

$$\frac{h^2 + k^2 + l^2}{a\alpha} + \frac{h^2 + k^2 + l^2}{b\beta} + \frac{h^2 + k^2 + l^2}{c\gamma}$$

$$= \frac{h}{a} + \frac{k}{b} + \frac{1}{c} = 1 \quad \text{[from Eq. (ii)]}$$

$$\Rightarrow \frac{1}{a\alpha} + \frac{1}{b\beta} + \frac{1}{c\gamma} = \frac{1}{h^2 + k^2 + l^2}$$

$$= \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} \quad \text{[from Eq. (iv)]}$$

∴Required equation of locus is

$$\frac{1}{ax} + \frac{1}{by} + \frac{1}{cz} = \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}.$$

● ***Ex. 89*** *Prove that the shortest distance between any two opposite edges of a tetrahedron formed by the planes* $y + z = 0, x + z = 0, x + y = 0, x + y + z = \sqrt{3}a$ *is* $\sqrt{2}a$.

Sol. Here, planes

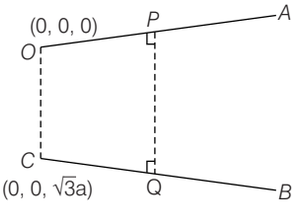
$$y + z = 0, z + x = 0, x + y = 0 \text{ meet at } O(0, 0, 0).$$

Let the tetrahedron be $OABC$.

Let the equation of one of the pair of opposite edges OA and BC be

$$y + z = 0, x + z = 0 \quad ...(i)$$

and $$x + y = 0, x + y + z = \sqrt{3}a \quad ...(ii)$$



Eqs. (i) and (ii) can be expressed in symmetrical form as

$$\frac{x - 0}{1} = \frac{y - 0}{1} = \frac{z - 0}{-1} \quad ...(iii)$$

and $$\frac{x - 0}{1} = \frac{y - 0}{-1} = \frac{z - \sqrt{3a}}{0} \quad ...(iv)$$

DR's of OA and BC are $(1, 1, -1)$ and $(-1, 1, 0)$.

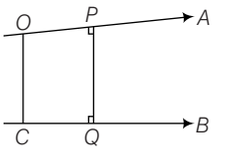
Let PQ be the shortest distance between OA and BC having direction cosine (l, m, n).

∴ PQ is perpendicular to both OA and BC.

$$l + m + n = 0 \quad ...(v)$$

$$l - m = 0 \quad ...(vi)$$

On solving Eqs. (v) and (vi), we get



$$\frac{l}{1} = \frac{m}{1} = \frac{n}{2} = k$$

Also, $$l^2 + m^2 + n^2 = 1$$

∴ $$k^2 + k^2 + 4k^2 = 1 \Rightarrow k = \frac{1}{\sqrt{6}}$$

∴ $$l = \frac{1}{\sqrt{6}} = m \text{ and } n = \frac{2}{\sqrt{6}}$$

Shortest distance between OA and BC,

i.e. PQ = Length of projection of OC and PQ

$$= |(x_2 - x_1)l + (y_2 - y_1)m + (z_2 - z_1)n|$$

$$= \left| 0 \cdot \frac{1}{\sqrt{6}} + 0 \cdot \frac{1}{\sqrt{6}} + \sqrt{3}a \cdot \frac{2}{\sqrt{6}} \right|$$

$$= \sqrt{2}a$$

Three Dimensional Coordinate System Exercise 1 :
Single Option Correct Type Questions

1. The xy-plane divides the line joining the points $(-1, 3, 4)\,(2, -5, 6)$.
(a) Internally in the ratio 2 : 3
(b) externally in the ratio 2 : 3
(c) internally in the ratio 3 : 2
(d) externally in the ratio 3 : 2

2. Ratio in which the zx-plane divides the join of (1, 2, 3) and (4, 2, 1).
(a) 1 : 1 internally (b) 1 : 1 externally
(c) 2 : 1 internally (d) 2 : 1 externally

3. If $P\,(3, 2, -4)$, $Q\,(5, 4, -6)$ and $R\,(9, 8, -10)$ are collinear, then R divides PQ in the ratio
(a) 3 : 2 internally (b) 3 : 2 externally
(c) 2 : 1 internally (d) 2 : 1 externally

4. $A\,(3, 2, 0)$, $B\,(5, 3, 2)$ and $C\,(-9, 6, -3)$ are the vertices of a triangle ABC. If the bisector of $\angle ABC$ meets BC at D, then coordinates of D are
(a) $\left(\frac{19}{8}, \frac{57}{16}, \frac{17}{16}\right)$ (b) $\left(-\frac{19}{8}, \frac{57}{16}, \frac{17}{16}\right)$
(c) $\left(\frac{19}{8}, -\frac{57}{16}, \frac{17}{16}\right)$ (d) None of these

5. A line passes through the points $(6, -7, -1)$ and $(2, -3, 1)$. The direction cosines of the line so directed that the angle made by it with the positive direction of x-axis is acute, are
(a) $\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}$ (b) $-\frac{2}{3}, \frac{2}{3}, \frac{1}{3}$
(c) $\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}$ (d) $\frac{2}{3}, \frac{2}{3}, \frac{1}{3}$

6. If P is a point in space such that OP is inclined to OX at 45° and OY to 60° then **OP** is inclined to OZ at
(a) 75°
(b) 60° and 120°
(c) 75° and 105°
(d) 255°

7. l_1, m_1, n_1 and l_2, m_2, n_2 are direction cosines of the two lines inclined to each other at an angle θ, then the direction cosines of the internal bisector of the angle between these lines are
(a) $\frac{l_1 + l_2}{2\sin\frac{\theta}{2}}, \frac{m_1 + m_2}{2\sin\frac{\theta}{2}}, \frac{n_1 + n_2}{2\sin\frac{\theta}{2}}$ (b) $\frac{l_1 + l_2}{2\cos\frac{\theta}{2}}, \frac{m_1 + m_2}{2\cos\frac{\theta}{2}}, \frac{n_1 + n_2}{2\cos\frac{\theta}{2}}$
(c) $\frac{l_1 - l_2}{2\sin\frac{\theta}{2}}, \frac{m_1 - m_2}{2\sin\frac{\theta}{2}}, \frac{n_1 - n_2}{2\sin\frac{\theta}{2}}$ (d) $\frac{l_1 - l_2}{2\cos\frac{\theta}{2}}, \frac{m_1 - m_2}{2\cos\frac{\theta}{2}}, \frac{n_1 - n_2}{2\cos\frac{\theta}{2}}$

8. The equation of the plane perpendicular to the line $\frac{x-1}{1}, \frac{y-2}{-1}, \frac{z+1}{2}$ and passing through the point (2, 3, 1), is
(a) $\mathbf{r}.(\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) = 1$ (b) $\mathbf{r}.(\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) = 1$
(c) $\mathbf{r}.(\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) = 7$ (d) None of these

9. The locus of a point which moves so that the difference of the squares of its distances from two given points is constant, is a
(a) straight line (b) plane
(c) sphere (d) None of these

10. The position vectors of points $\mathbf{a}$ and $\mathbf{b}$ are $\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and $3\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ respectively. The equation of a plane is $\mathbf{r}.(5\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 7\hat{\mathbf{k}}) + 9 = 0$. The points $\mathbf{a}$ and $\mathbf{b}$
(a) lie on the plane
(b) are on the same side of the plane
(c) are on the opposite side of the plane
(d) None of the above

11. The vector equation of the plane through the point $2\hat{\mathbf{i}} - \hat{\mathbf{j}} - 4\hat{\mathbf{k}}$ and parallel to the plane $\mathbf{r}.(4\hat{\mathbf{i}} - 12\hat{\mathbf{j}} - 3\hat{\mathbf{k}}) - 7 = 0$, is
(a) $\mathbf{r}.(4\hat{\mathbf{i}} - 12\hat{\mathbf{j}} - 3\hat{\mathbf{k}}) = 0$ (b) $\mathbf{r}.(4\hat{\mathbf{i}} - 12\hat{\mathbf{j}} - 3\hat{\mathbf{k}}) = 32$
(c) $\mathbf{r}.(4\hat{\mathbf{i}} - 12\hat{\mathbf{j}} - 3\hat{\mathbf{k}}) = 12$ (d) None of these

12. Let L_1 be the line $\mathbf{r}_1 = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}} + \lambda(\hat{\mathbf{i}} + 2\hat{\mathbf{k}})$ and let L_2 be the another line $\mathbf{r}_2 = 3\hat{\mathbf{i}} + \hat{\mathbf{j}} + \mu(\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}})$. Let π be the plane which contains the line L_1 and is parallel to L_2. The distance of the plane π from the origin is
(a) $\sqrt{\frac{2}{7}}$ (b) $\frac{1}{7}$
(c) $\sqrt{6}$ (d) None of these

13. For the line $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$, which one of the following is incorrect ?
(a) it lies in the plane $x - 2y + z = 0$
(b) it is same as line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$
(c) it passes through (2, 3, 5)
(d) it is parallel to the plane $x - 2y + z - 6 = 0$

14. The value of m for which straight line $3x - 2y + z + 3 = 0 = 4x - 3y + 4z + 1$ is parallel to the plane $2x - y + mz - 2 = 0$ is
(a) –2 (b) 8
(c) –18 (d) 11

15. The length of projection of the line segment joining the points (1, 0, −1) and (−1, 2, 2) on the plane $x + 3y - 5z = 6$, is equal to

(a) 2 (b) $\sqrt{\frac{271}{53}}$

(c) $\sqrt{\frac{472}{31}}$ (d) $\sqrt{\frac{474}{35}}$

16. The number of planes that are equidistant from four non-coplanar points is

(a) 3 (b) 4

(c) 7 (d) 9

17. In a three dimensional co-ordinate system, P, Q and R are images of a point $A(a, b, c)$ in the xy, yz and zx planes, respectively. If G is the centroid of triangle PQR, then area of triangle AOG is (O is the origin)

(a) 0 (b) $a^2 + b^2 + c^2$

(c) $\frac{2}{3}(a^2 + b^2 + c^2)$ (d) None of these

18. A plane passing through (1, 1, 1) cuts positive direction of coordinate axes at A, B and C, then the volume of tetrahedron $OABC$ satisfies

(a) $V \le \frac{9}{2}$ (b) $V \ge \frac{9}{2}$

(c) $V = \frac{9}{2}$ (d) None of these

19. If lines $x = y = z$ and $x = \frac{y}{2} = \frac{z}{3}$ and third line passing through (1, 1, 1) form a triangle of area $\sqrt{6}$ units, then point of intersection of third line with second line will be

(a) (1, 2, 3) (b) (2, 4, 6)

(c) $\left(\frac{4}{3}, \frac{8}{3}, \frac{12}{3}\right)$ (d) None of these

20. The point of intersection of the line passing through (0, 0, 1) and intersecting the lines $x + 2y + z = 1$, $-x + y - 2z = 2$ and $x + y = 2$, $x + z = 2$ with xy plane is

(a) $\left(\frac{5}{3}, -\frac{1}{3}, 0\right)$ (b) (1, 1, 0)

(c) $\left(\frac{2}{3}, -\frac{1}{3}, 0\right)$ (d) $\left(-\frac{5}{3}, \frac{1}{3}, 0\right)$

21. Two systems of rectangular axes have the same origin. If a plane cuts them at distance a, b, c and a', b', c' from the origin, then :

(a) $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2} = 0$

(b) $\frac{1}{a^2} - \frac{1}{b^2} - \frac{1}{c^2} + \frac{1}{a'^2} - \frac{1}{b'^2} - \frac{1}{c'^2} = 0$

(c) $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} - \frac{1}{a'^2} - \frac{1}{b'^2} - \frac{1}{c'^2} = 0$

(d) $\frac{1}{a^2} - \frac{1}{b^2} + \frac{1}{c^2} - \frac{1}{a'^2} + \frac{1}{b'^2} - \frac{1}{c'^2} = 0$

22. The line $\frac{x+6}{5} = \frac{y+10}{3} = \frac{z+14}{8}$ is the hypotenuse of an isosceles right angled triangle whose opposite vertex is (7, 2, 4). Then which of the following is not the side of the triangle ?

(a) $\frac{x-7}{2} = \frac{y-2}{-3} = \frac{z-4}{6}$

(b) $\frac{x-7}{3} = \frac{y-2}{6} = \frac{z-4}{2}$

(c) $\frac{x-7}{3} = \frac{y-2}{5} = \frac{z-4}{-1}$

(d) None of these

23. Consider the following 3 lines in space

$L_1 : \mathbf{r} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} + 4\mathbf{j} - \mathbf{k})$

$L_2 : \mathbf{r} = \mathbf{i} + \mathbf{j} - 3\mathbf{k} + \mu(4\mathbf{i} + 2\mathbf{j} + 4\mathbf{k})$

$L_3 : \mathbf{r} = 3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k} + t(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

Then, which one of the following pair(s) is/are in the same plane ?

(a) Only L_1L_2 (b) Only L_2L_3

(c) Only L_3L_1 (d) L_1L_2 and L_2L_3

24. Let $\mathbf{r} = \mathbf{a} + \lambda\mathbf{l}$ and $\mathbf{r} = \mathbf{b} + \mu\mathbf{m}$ be two lines in space, where $\mathbf{a} = 5\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = -\mathbf{i} + 7\mathbf{j} + 8\mathbf{k}$, $\mathbf{l} = -4\mathbf{i} + \mathbf{j} - \mathbf{k}$, and $\mathbf{m} = 2\mathbf{i} - 5\mathbf{j} - 7\mathbf{k}$, then the position vector of a point which lies on both of these lines, is

(a) $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

(b) $2\mathbf{i} + \mathbf{j} + \mathbf{k}$

(c) $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

(d) non-existent as the lines are skew

25. L_1 and L_2 are two lines whose vector equations are

$L_1 : \mathbf{r} = \lambda[(\cos\theta + \sqrt{3})\mathbf{i} + (\sqrt{2}\sin\theta)\mathbf{j} + (\cos\theta - \sqrt{3})\mathbf{k}]$

and $L_2 : \mathbf{r} = \mu(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})$

where, λ and μ are scalars and α is the acute angle between L_1 and L_2. If the angle α is independent of θ, then the value of α is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

26. The vector equations of two lines L_1 and L_2 are respectively,

$$\mathbf{r} = 17\mathbf{i} - 9\mathbf{j} + 9\mathbf{k} + \lambda(3\mathbf{i} + \mathbf{j} + 5\mathbf{k})$$

and $$\mathbf{r} = 15\mathbf{i} - 8\mathbf{j} - \mathbf{k} + \mu(4\mathbf{i} + 3\mathbf{j})$$

I. L_1 and L_2 are skew lines.

II. (11, −11, −1) is the point of intersection of L_1 and L_2.

III. (−11, 11, 1) is the point of intersection of L_1 and L_2.

IV. $\cos^{-1}\left(\frac{3}{\sqrt{35}}\right)$ is the acute angle between, L_1 and L_2.

Then, which of the following is true ?

(a) II and IV (b) I and IV

(c) Only IV (d) III and IV

27. Consider three vectors $\mathbf{p} = \mathbf{i} + \mathbf{j} + \mathbf{k}, \mathbf{q} = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ and $\mathbf{r} = \mathbf{i} + \mathbf{j} + 3\mathbf{k}$. If $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ denotes the position vector of three non-collinear points, then the equation of the plane containing these points is

(a) $2x - 3y + 1 = 0$ (b) $x - 3y + 2z = 0$

(c) $3x - y + z - 3 = 0$ (d) $3x - y - 2 = 0$

28. The intercept made by the plane $\mathbf{r}.\mathbf{n} = q$ on the x-axis is

(a) $\dfrac{q}{\mathbf{i}.\mathbf{n}}$ (b) $\dfrac{\mathbf{i}.\mathbf{n}}{q}$

(c) $(\mathbf{i}.\mathbf{n})\, q$ (d) $\dfrac{q}{|\mathbf{n}|}$

29. If the distance between the planes

$$8x + 12y - 14z = 2 \text{ and } 4x + 6y - 7z = 2$$

can be expressed in the form $\dfrac{1}{\sqrt{N}}$, where N is natural, then the value of $\dfrac{N(N+1)}{2}$ is

(a) 4950 (b) 5050

(c) 5150 (d) 5151

30. A plane passes through the points $P(4, 0, 0)$ and $Q(0, 0, 4)$ and is parallel to the Y-axis. The distance of the plane from the origin is

(a) 2 (b) 4

(c) $\sqrt{2}$ (d) $2\sqrt{2}$

31. If from the point $P(f, g, h)$ perpendiculars PL and PM be drawn to yz and zx-planes, then the equation to the plane OLM is

(a) $\dfrac{x}{f} + \dfrac{y}{g} - \dfrac{z}{h} = 0$ (b) $\dfrac{x}{f} + \dfrac{y}{g} + \dfrac{z}{h} = 0$

(c) $\dfrac{x}{f} - \dfrac{y}{g} + \dfrac{z}{h} = 0$ (d) $-\dfrac{x}{f} + \dfrac{y}{g} + \dfrac{z}{h} = 0$

32. The plane XOZ divides the join of $(1, -1, 5)$ and $(2, 3, 4)$ in the ratio $\lambda : 1$, then λ is

(a) -3 (b) $-\dfrac{1}{3}$

(c) 3 (d) $\dfrac{1}{3}$

33. A variable plane forms a tetrahedron of constant volume $64K^3$ with the coordinate planes and the origin, then locus of the centroid of the tetrahedron is

(a) $x^3 + y^3 + z^3 = 6k^3$ (b) $xyz = 6k^3$

(c) $x^2 + y^2 + z^2 = 4k^2$ (d) $x^{-2} + y^{-2} + z^{-2} = 4k^{-2}$

34. Let $ABCD$ be a tetrahedron such that the edges AB, AC and AD are mutually perpendicular. Let the area of ΔABC, ΔACD and ΔADB be 3, 4 and 5 sq units, respectively. Then, the area of the ΔBCD, is

(a) $5\sqrt{2}$ (b) 5

(c) $5/\sqrt{2}$ (d) $\dfrac{5}{2}$

35. Equation of the line which passes through the point with position vector (2, 1, 0) and perpendicular to the plane containing the vectors $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + \mathbf{k}$ is

(a) $\mathbf{r} = (2, 1, 0) + t(1, -1, 1)$

(b) $\mathbf{r} = (2, 1, 0) + t(-1, 1, 1)$

(c) $\mathbf{r} = (2, 1, 0) + t(1, 1, -1)$

(d) $\mathbf{r} = (2, 1, 0) + t(1, 1, 1)$

Where, t is a parameter.

36. Which of the following planes are parallel but not identical ?

$$P_1 : 4x - 2y + 6z = 3$$
$$P_2 : 4x - 2y - 2z = 6$$
$$P_3 : -6x + 3y - 9z = 5$$
$$P_4 : 2x - y - z = 3$$

(a) P_2 and P_3 (b) P_2 and P_4

(c) P_1 and P_3 (d) P_1 and P_4

37. A parallelopiped is formed by planes drawn through the points (1, 2, 3) and (9, 8, 5) parallel to the coordinate planes, then which of the following is not the length of an edge of this rectangular parallelopiped ?

(a) 2 (b) 4

(c) 6 (d) 8

38. vector equation of the plane $\mathbf{r} = \mathbf{i} - \mathbf{j} + \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k}) + \mu(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$ in the scalar dot product form is

(a) $\mathbf{r}.(5\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) = 7$

(b) $\mathbf{r}.(5\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) = 7$

(c) $\mathbf{r}.(5\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}) = 7$

(d) $\mathbf{r}.(5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = 7$

39. The vector equations of the two lines L_1 and L_2 are given by $L_1 : \mathbf{r} = (2\mathbf{i} + 9\mathbf{j} + 13\mathbf{k}) + \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$ and $L_2 : \mathbf{r} = (-3\mathbf{i} + 7\mathbf{j} + p\mathbf{k}) + \mu(-\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$.

Then, the lines L_1 and L_2 are

(a) skew lines for all $p \in R$

(b) intersecting for all $p \in R$ and the point of intersection is $(-1, 3, 4)$

(c) intersecting lines for $p = -2$

(d) intersecting for all real $p \in R$

40. Consider the plane $(x, y, z) = (0, 1, 1) + \lambda(1, -1, 1) + \mu(2, -1.0)$. The distance of this plane from the origin is

(a) $\dfrac{1}{3}$ (b) $\dfrac{\sqrt{3}}{2}$

(c) $\sqrt{\dfrac{3}{2}}$ (d) $\dfrac{2}{\sqrt{3}}$

41. The value of a for which the lines $\dfrac{x-2}{1} = \dfrac{y-9}{2} = \dfrac{z-13}{3}$ and $\dfrac{x-a}{-1} = \dfrac{y-7}{2} = \dfrac{z+2}{-3}$ intersect, is

(a) -5 (b) -2

(c) 5 (d) -3

42. For the line $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$, which one of the following is incorrect ?
(a) It lies in the plane $x-2y+z=0$.
(b) It is same as line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$.
(c) It passes through (2, 3, 5).
(d) It is parallel to the plane $x-2y+z-6=0$.

43. Given planes $P_1: cy+bz=x$;
$P_2: az+cx=y$
$P_3: bx+ay=z$
P_1, P_2 and P_3 pass through one line, if
(a) $a^2+b^2+c^2=ab+bc+ca$
(b) $a^2+b^2+c^2+2abc=1$
(c) $a^2+b^2+c^2=1$
(d) $a^2+b^2+c^2+2ab+2bc+2ca+2abc=1$

44. The lines $\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{-k}$ and $\frac{x-1}{k}=\frac{y-4}{2}=\frac{z-5}{1}$ are coplanar, if
(a) $k=0$ or -1 (b) $k=1$ or -1
(c) $k=0$ or -3 (d) $k=3$ or -3

45. The line $\frac{x-2}{3}=\frac{y+1}{2}=\frac{z-1}{-1}$ intersects the curve $xy=c^2$, in xy-plane, if c is equal to
(a) ± 1 (b) $\pm\frac{1}{3}$
(c) $\pm\sqrt{5}$ (d) None of these

46. The line which contains all points (x, y, z) which are of the form $(x, y, z)=(2,-2,5)+\lambda(1,-3,2)$ intersects the plane $2x-3y+4z=163$ at P and intersects the YZ-plane at Q. If the distance PQ is $a\sqrt{b}$, where $a, b\in N$ and $a>3$, then $(a+b)$ is equal to
(a) 23 (b) 95
(c) 27 (d) None of these

47. If the three planes $\mathbf{r}.\mathbf{n}_1=p_1, \mathbf{r}.\mathbf{n}_2=p_2$ and $\mathbf{r}.\mathbf{n}_3=p_3$ have a common line of intersection, then $p_1(\mathbf{n}_2\times\mathbf{n}_3)+p_2(\mathbf{n}_3\times\mathbf{n}_1)+p_3(\mathbf{n}_1\times\mathbf{n}_2)$ is equal to
(a) 1 (b) 2
(c) 0 (d) -1

48. The equation of the plane which passes through the line of intersection of the planes $\mathbf{r}.\mathbf{n}_1=q_1, \mathbf{r}.\mathbf{n}_2=q_2$ and is parallel to the line of intersection of the planes $\mathbf{r}.\mathbf{n}_3=q_3$ and $\mathbf{r}.\mathbf{n}_4=q_4$, is
(a) $[\mathbf{n}_2\ \mathbf{n}_3\ \mathbf{n}_4](\mathbf{r}.\mathbf{n}_1-q_1)=[\mathbf{n}_1\ \mathbf{n}_3\ \mathbf{n}_4](\mathbf{r}.\mathbf{n}_2-q_2)$
(b) $[\mathbf{n}_1\ \mathbf{n}_2\ \mathbf{n}_3](\mathbf{r}.\mathbf{n}_4-q_4)=[\mathbf{n}_4\ \mathbf{n}_3\ \mathbf{n}_1](\mathbf{r}.\mathbf{n}_2-q_2)$
(c) $[\mathbf{n}_4\ \mathbf{n}_3\ \mathbf{n}_1](\mathbf{r}.\mathbf{n}_4-q_4)=[\mathbf{n}_1\ \mathbf{n}_2\ \mathbf{n}_3](\mathbf{r}.\mathbf{n}_2-q_2)$
(d) None of the above

49. A straight line is given by $\mathbf{r}=(1+t)\,\mathbf{i}+3t\,\mathbf{j}+(1-t)\,\mathbf{k}$, where $t\in R$. If this line lies in the plane $x+y+cz=d$, then the value of $(c+d)$ is
(a) -1 (b) 1
(c) 7 (d) 9

50. The distance of the point $(-1,-5,-10)$ from the point of intersection of the line $\frac{x-2}{2}=\frac{y+1}{4}=\frac{z-2}{12}$ and the plane $x-y+z=5$ is
(a) $2\sqrt{11}$ (b) $\sqrt{126}$
(c) 13 (d) 14

51. $P(\mathbf{p})$ and $Q(\mathbf{q})$ are the position vector of two fixed points and $R(\mathbf{r})$ is the position vector of a variable point. If R moves such that $(\mathbf{r}-\mathbf{p})\times(\mathbf{r}-\mathbf{q})=0$, then the locus of R is
(a) a plane containing the origin O and parallel to two non-collinear vector **OP** and **OQ**.
(b) the surface of a sphere described on PQ as its diameter.
(c) a line passing through the points P and Q.
(d) a set of lines parallel to the line PQ.

52. The three vectors $\mathbf{i}+\mathbf{j}, \mathbf{j}+\mathbf{k}, \mathbf{k}+\mathbf{i}$ taken two at a time form three planes. The three unit vectors drawn perpendicular to these three planes form a parallelopiped of volume
(a) $\frac{1}{3}$ (b) 4
(c) $3\frac{\sqrt{3}}{4}$ (d) $\frac{4}{3\sqrt{3}}$

53. The orthogonal projection A' of the point A with position vector (1, 2, 3) on the plane $3x-y+4z=0$ is
(a) $(-1, 3, -1)$ (b) $\left(-\frac{1}{2}, \frac{5}{2}, 1\right)$
(c) $\left(\frac{1}{2}, -\frac{5}{2}, -1\right)$ (d) $(6., -7, -5)$

54. The equation of the line passing through (1, 1, 1) and perpendicular to the line of intersection of the planes $x+2y-4z=0$ and $2x-y+2z=0$ is
(a) $\frac{x-1}{5}=\frac{1-y}{1}=\frac{z-1}{2}$ (b) $\frac{x-1}{-5}=\frac{1-y}{1}=\frac{z-1}{2}$
(c) $\frac{x-1}{0}=\frac{1-y}{-10}=\frac{z-1}{-5}$ (d) $\frac{x-1}{-10}=\frac{y-1}{0}=\frac{z-1}{-5}$

55. A variable plane at a distance of 1 unit from the origin cuts the axes at A, B and C. If the centroid $D(x, y, z)$ of ΔABC satisfies the relation $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=K$, then the value of K is
(a) 3 (b) 1
(c) $\frac{1}{3}$ (d) 9

56. The angle between the lines AB and CD, where $A=(0,0,0)$, $B=(1,1,1)$, $C=(-1,-1,-1)$ and $D=(0,1,0)$ is given by

(a) $\cos\theta=\frac{1}{\sqrt{3}}$ (b) $\cos\theta=\frac{4}{3\sqrt{2}}$

(c) $\cos\theta=\frac{1}{\sqrt{5}}$ (d) $\cos\theta=\frac{1}{2\sqrt{2}}$

57. The shortest distance of a point $(1,2,-3)$ from a plane making intercepts 1, 2 and 3 units on position X, Y and Z-axes respectively, is

(a) 2 (b) 0

(c) $\frac{13}{12}$ (d) $\frac{12}{7}$

58. A tetrahedron has vertices $O(0,0,0)$, $A(1,2,1)$, $B(2,1,3)$ and $C(-1,1,2)$. Then the angle between the faces OAB and ABC will be

(a) $\cos^{-1}\left(\frac{19}{35}\right)$ (b) $\cos^{-1}\left(\frac{17}{31}\right)$

(c) 30° (d) 90°

59. The direction ratios of line I_1 passing through $P(1,3,4)$ and perpendicular to line $I_2\ \frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ (where, I_1 and I_2 are coplanar) is

(a) 14, 8, 1 (b) −14, 8, −1

(c) 14, −8, −1 (d) −14, −8, 1

60. Equation of the plane through three points A, B and C with position vectors $-6\mathbf{i}+3\mathbf{j}+2\mathbf{k}$, $3\mathbf{i}-2\mathbf{j}+4\mathbf{k}$ and $5\mathbf{i}+7\mathbf{j}+3\mathbf{k}$ is equal to

(a) $\mathbf{r}.(\mathbf{i}-\mathbf{j}+7\mathbf{k})+23=0$ (b) $\mathbf{r}.(\mathbf{i}+\mathbf{j}+7\mathbf{k})=23$

(c) $\mathbf{r}.(\mathbf{i}+\mathbf{j}-7\mathbf{k})+23=0$ (d) $\mathbf{r}.(\mathbf{i}-\mathbf{j}-7\mathbf{k})=23$

61. $OABC$ is a tetrahedron. The position vectors of A, B and C are $\mathbf{i}$, $\mathbf{i}+\mathbf{j}$ and $\mathbf{j}+\mathbf{k}$, respectively. O is origin. The height of the tetrahedron (taking plane ABC as base) is

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$

(c) $\frac{1}{2\sqrt{2}}$ (d) None of these

62. The plane $x-y-z=4$ is rotated through an angle 90° about its line of intersection with the plane $x+y+2z=4$. Then the equation of the plane in its new position is

(a) $x+y+4z=20$ (b) $x+5y+4z=20$

(c) $x+y-4z=20$ (d) $5x+y+4z=20$

63. Let A_{xy}, A_{yz}, A_{zx} be the area of the projection of a plane area A on the xy, yz, zx plane respectively Then $A^2=$

(a) $A_{xy}^2+A_{yz}^2+A_{zx}^2$ (b) $\sqrt{A_{xy}^2+A_{yz}^2+A_{zx}^2}$

(c) $A_{xy}+A_{yz}+A_{xz}$ (d) $\sqrt{A_{xy}+A_{yz}+A_{zx}}$

64. Through the point $P(h,k,l)$ a plane is drawn at right angles to OP to meet co-ordinate axes at A, B and C. If $OP=p$ then the area of the ΔABC is

(a) $\frac{p^5}{2hkl}$ (b) $\frac{p^5}{hkl}$

(c) $\frac{p^3}{2hkl}$ (d) $\frac{p^3}{hkl}$

65. The volume of the tetrahedron included between the plane $3x+4y-5z-60=0$ and the co-ordinate planes is

(a) 60 (b) 600

(c) 720 (d) 400

66. The angle between the lines whose direction cosines are given by the equations $l^2+m^2-n^2=0, l+m+n=0$ is

(a) $\cos^{-1}(2\sqrt{3})$ (b) $\cos^{-1}\sqrt{3}$

(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

67. The distance between the line $\mathbf{r}=2\hat{\mathbf{i}}-2\hat{\mathbf{j}}+3\hat{\mathbf{k}}+\lambda(\hat{\mathbf{i}}-\hat{\mathbf{j}}+4\hat{\mathbf{k}})$ and the plane $\mathbf{r}\cdot(\hat{\mathbf{i}}+5\hat{\mathbf{j}}+\hat{\mathbf{k}})=5$ is

(a) $\frac{10}{3\sqrt{3}}$ (b) $\frac{10}{3}$

(c) $\frac{10}{9}$ (d) $\frac{10}{\sqrt{3}}$

68. The Cartesian equation of the plane perpendicular to the line $\frac{x-1}{2}=\frac{y-3}{-1}=\frac{z-4}{2}$ and passing through the origin is

(a) $2x-y+2z-7=0$ (b) $2x+y+2z=0$

(c) $2x-y+2z=0$ (d) $2x-y-z=0$

69. Let $P(3,2,6)$ be a point in space and Q be a point on the line $\mathbf{r}=(\hat{\mathbf{i}}-\hat{\mathbf{j}}+2\hat{\mathbf{k}})+\mu(-3\hat{\mathbf{i}}+\hat{\mathbf{j}}+5\hat{\mathbf{k}})$. Then the value of μ for which the vector $\mathbf{PQ}$ is parallel to the plane $x-4y+3z=1$ is

(a) $\frac{1}{4}$ (b) $-\frac{1}{4}$

(c) $\frac{1}{8}$ (d) $-\frac{1}{8}$

70. A plane makes intercepts OA, OB and OC whose measurements are a, b and c on the OX, OY and OZ axes. The area of triangle ABC is

(a) $\frac{1}{2}(ab+bc+ca)$ (b) $\frac{1}{2}abc(a+b+c)$

(c) $\frac{1}{2}(a^2b^2+b^2c^2+c^2a^2)^{1/2}$ (d) $\frac{1}{2}(a+b+c)^2$

71. The radius of the circle in which the sphere $x^2+y^2+z^2+2x-2y-4z-19=0$ is cut by the plane $x+2y+2z+7=0$ is

(a) 2 (b) 3

(c) 4 (d) 1

72. Let $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\mathbf{b} = 2\hat{\mathbf{i}} - \hat{\mathbf{k}}$, then the point of intersection of the lines $\mathbf{r} \times \mathbf{a} = \mathbf{b} \times \mathbf{a}$ and $\mathbf{r} \times \mathbf{b} = \mathbf{a} \times \mathbf{b}$ is
(a) (3, −1, 1) (b) (3, 1, −1)
(c) (−3, 1, 1) (d) (−3, −1, −1)

73. The co-ordinates of the point P on the line $\mathbf{r} = (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) + \lambda(-\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}})$ which is nearest to the origin is
(a) $\left(\frac{2}{3}, \frac{4}{3}, \frac{2}{3}\right)$ (b) $\left(-\frac{2}{3}, -\frac{4}{3}, \frac{2}{3}\right)$
(c) $\left(\frac{2}{3}, \frac{4}{3}, -\frac{2}{3}\right)$ (d) None of these

74. The 3-dimensional vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ satisfying $\mathbf{v}_1 . \mathbf{v}_1 = 4, \mathbf{v}_1 . \mathbf{v}_2 = -2, \mathbf{v}_1 . \mathbf{v}_3 = 6, \mathbf{v}_2 . \mathbf{v}_2 = 2, \mathbf{v}_2 . \mathbf{v}_3 = -5, \mathbf{v}_3 . \mathbf{v}_3 = 29$, then $\mathbf{v}_3$ may be
(a) $-3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} \pm 4\hat{\mathbf{k}}$ (b) $3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} \pm 4\hat{\mathbf{k}}$
(c) $-2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} \pm 4\hat{\mathbf{k}}$ (d) $2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} \pm 4\hat{\mathbf{k}}$

75. The points $\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and $3\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ are equidistant from the plane $\mathbf{r} . (5\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 7\hat{\mathbf{k}}) + 9 = 0$, then they are
(a) on the same sides of the plane
(b) parallel of the plane
(c) on the opposite sides of the plane
(d) None of the above

76. A, B, C, D are four points in space. Then, $AC^2 + BD^2 + AD^2 + BC^2 \geq$
(a) $AB^2 + CD^2$ (b) $\frac{1}{AB^2} - \frac{1}{CD^2}$
(c) $\frac{1}{CD^2} - \frac{1}{AB^2}$ (d) None of these

77. If $|x_1| > |y_1| + |z_1|, |y_2| > |x_2| + |z_2|, |z_3| > |x_3| + |y_3|$, then $x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}} + z_1\hat{\mathbf{k}}$, $x_2\hat{\mathbf{i}} + y_2\hat{\mathbf{j}} + z_2\hat{\mathbf{k}}$ and $x_3\hat{\mathbf{i}} + y_3\hat{\mathbf{j}} + z_3\hat{\mathbf{k}}$ are
(a) perpendicular (b) collinear
(c) coplanar (d) non-coplanar

78. The position vector of the point of intersection of three planes $\mathbf{r} . \mathbf{n}_1 = q_1, \mathbf{r} . \mathbf{n}_2 = q_2, \mathbf{r} . \mathbf{n}_3 = q_3$, where $\mathbf{n}_1, \mathbf{n}_2$ and $\mathbf{n}_3$ are non-coplanar vectors, is
(a) $\frac{1}{[\mathbf{n}_1\ \mathbf{n}_2\ \mathbf{n}_3]}[q_3(\mathbf{n}_1 \times \mathbf{n}_2) + q_1(\mathbf{n}_2 \times \mathbf{n}_3) + q_2(\mathbf{n}_3 \times \mathbf{n}_1)]$
(b) $\frac{1}{[\mathbf{n}_1\ \mathbf{n}_2\ \mathbf{n}_3]}[q_1(\mathbf{n}_1 \times \mathbf{n}_2) + q_2(\mathbf{n}_2 \times \mathbf{n}_3) + q_3(\mathbf{n}_3 \times \mathbf{n}_1)]$
(c) $-\frac{1}{[\mathbf{n}_3\ \mathbf{n}_1\ \mathbf{n}_2]}[q_1(\mathbf{n}_1 \times \mathbf{n}_2) + q_2(\mathbf{n}_2 \times \mathbf{n}_3) + q_3(\mathbf{n}_3 \times \mathbf{n}_1)]$
(d) None of the above

79. A pentagon is formed by cutting a triangular corner from a rectangular piece of paper. The five sides of the pentagon have length 13, 19, 20, 25 and 31 not necessarily in that order. The area of the pentagon is
(a) 459 sq units (b) 600 sq units
(c) 680 sq units (d) 745 sq units

80. In a three dimensional coordinate system P, Q and R are images of a point $A(a, b, c)$ in the XY they YZ and the ZX planes respectively. If G is the centroid of triangle PQR, then area of triangle AOG is (O is the origin)
(a) 0 (b) $a^2 + b^2 + c^2$
(c) $\frac{2}{3}(a^2 + b^2 + c^2)$ (d) None of these

81. A plane $2x + 3y + 5z = 1$ has a point P which is at minimum distance from line joining $A(1, 0, -3)$, $B(1, -5, 7)$, then distance AP is equal to
(a) $3\sqrt{5}$ (b) $2\sqrt{5}$
(c) $4\sqrt{5}$ (d) None of these

82. The locus of a point which moves in such a way that its distance from the line $\frac{x}{1} = \frac{y}{1} = \frac{z}{-1}$ is twice the distance from the plane $x + y + z = 0$ is
(a) $x^2 + y^2 + z^2 - 5x - 3y - 3z = 0$
(b) $x^2 + y^2 + z^2 + 5x + 3y + 3z = 0$
(c) $x^2 + y^2 + z^2 - 5xy - 3yz - 3zx = 0$
(d) $x^2 + y^2 + z^2 + 5xy + 3yz + 3zx = 0$

83. A cube $C = \{(x, y, z) \mid 0 \leq x, y, z \leq 1\}$ is cut by a sharp knife along the plane $x = y$, $y = z$, $z = x$. If no piece is moved until all three cuts are made, the number of pieces is
(a) 6 (b) 7
(c) 8 (d) 27

84. A ray of light is sent through the point $P(1, 2, 3)$ and is reflected on the XY-plane. If the reflected ray passes through the point $Q(3, 2, 5)$, then the equation of the reflected ray is
(a) $\frac{x-3}{1} = \frac{y-2}{0} = \frac{z-5}{1}$ (b) $\frac{x-3}{1} = \frac{y-2}{0} = \frac{z-5}{-4}$
(c) $\frac{x-3}{1} = \frac{y-2}{0} = \frac{z-5}{4}$ (d) $\frac{x-1}{1} = \frac{y-2}{0} = \frac{z-3}{4}$

85. A plane cutting the axes in P, Q, R passes through $(\alpha - \beta, \beta - \gamma, \gamma - \alpha)$. If O is the origin, then locus of centre of sphere $OPQR$ is
(a) $\alpha x + \beta y + \gamma z = 4$
(b) $(\alpha - \beta)x + (\beta - \gamma)y + (\gamma - \alpha)z = 0$
(c) $(\alpha - \beta)yz + (\beta - \gamma)zx + (\gamma - \alpha)xy = 2xyz$
(d) $\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}\right)(x^2 + y^2 + z^2) = xyz$

86. The shortest distance between any two opposite edges of the tetrahedron formed by planes $x + y = 0$, $y + z = 0$, $z + x = 0$, $x + y + z = a$ is constant, equal to
(a) $2a$ (b) $\frac{2a}{\sqrt{6}}$
(c) $\frac{a}{\sqrt{6}}$ (d) $\frac{2a}{\sqrt{3}}$

87. The angle between the pair of planes represented by equation $2x^2 - 2y^2 + 4z^2 + 6xz + 2yz + 3xy = 0$ is

(a) $\cos^{-1}\left(\frac{1}{3}\right)$ (b) $\cos^{-1}\left(\frac{4}{21}\right)$

(c) $\cos^{-1}\left(\frac{4}{9}\right)$ (d) $\cos^{-1}\left(\frac{7}{\sqrt{84}}\right)$

88. Let (p, q, r) be a point on the plane $2x + 2y + z = 6$, then the least value of $p^2 + q^2 + r^2$ is equal to

(a) 4 (b) 5 (c) 6 (d) 8

89. The four lines drawing from the vertices of any tetrahedron to the centroid of the opposite faces meet in a point whose distance from each vertex is 'k' times the distance from each vertex to the opposite face, where k is

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) $\frac{3}{4}$ (d) $\frac{5}{4}$

90. The shortest distance from (1, 1, 1) to the line of intersection of the pair of planes $xy + yz + zx + y^2 = 0$ is

(a) $\sqrt{\frac{8}{3}}$ (b) $\frac{2}{\sqrt{3}}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\frac{2}{3}$

91. The shortest distance between the two lines $L_1 : x = k_1$; $y = k_2$ and $L_2 : x = k_3$; $y = k_4$ is equal to

(a) $\left|\sqrt{k_1^2 + k_2^2} - \sqrt{k_3^2 + k_4^2}\right|$ (b) $\sqrt{k_1k_3 + k_2k_4}$

(c) $\sqrt{(k_1 + k_3)^2 + (k_2 + k_4)^2}$ (d) $\sqrt{(k_1 - k_3)^2 + (k_2 - k_4)^2}$

92. $A = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$ and $B = \begin{bmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{bmatrix}$, where p_i, q_i, r_i are the cofactors of the elements l_i, m_i, n_i for $i = 1, 2, 3$. If $(l_1, m_1, n_1), (l_2, m_2, n_2)$ and (l_3, m_3, n_3) are the direction cosines of three mutually perpendicular lines, then $(p_1, q_1, r_1), (p_2, q_2, r_2)$ and (p_3, q_3, r_3) are

(a) the direction cosines of three mutually perpendicular lines

(b) the direction ratios of three mutually perpendicular lines which are not direction cosines

(c) the direction cosines of three lines which need not be perpendicular

(d) the direction ratios but not the direction cosines of three lines which need not be perpendicular

93. If $ABCD$ is a tetrahedron such that each ΔABC, ΔABD and ΔACD has a right angle at A. If $\text{ar}(\Delta ABC) = k_1$, $\text{ar}(\Delta ABD) = k_2$, $\text{ar}(\Delta BCD) = k_3$, then $\text{ar}(\Delta ACD)$ is

(a) $\sqrt{k_1^2 + k_2^2 + k_3^2}$ (b) $\sqrt{\frac{k_1k_2k_3}{k_1 + k_2 + k_3}}$

(c) $\sqrt{|k_1^2 + k_2^2 - k_3^2|}$ (d) $\sqrt{|k_2^2 - k_1^2 - k_3^2|}$

94. In a regular tetrahedron, if the distance between the mid-points of opposite edges is unity, its volume is

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$

(c) $\frac{1}{\sqrt{2}}$ (d) $\frac{1}{6\sqrt{2}}$

95. A variable plane makes intercepts on X, Y and Z-axes and it makes a tetrahedron of volume 64 cu. u. The locus of foot of perpendicular from origin on this plane is

(a) $(x^2 + y^2 + z^2)^3 = 384\ xyz$

(b) $xyz = 681$

(c) $(x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)^2 = 16$

(d) $xyz\ (x + y + z) = 81$

96. If P, Q, R, S are four coplanar points on the sides AB, BC, CD, DA of a skew quadrilateral, then $\frac{AP}{PB} \cdot \frac{BQ}{QC} \cdot \frac{CR}{RD} \cdot \frac{DS}{SA}$ equals

(a) 1 (b) -1 (c) 3 (d) -3

Three Dimensional Coordinate System Exercise 2 : More than One Correct Option Type Questions

97. Given the equations of the line $3x - y + z + 1 = 0$ and $5x + y + 3z = 0$. Then, which of the following is correct ?

(a) Symmetrical form of the equations of line is $\frac{x}{2} = \frac{y - \frac{1}{8}}{-1} = \frac{z + \frac{5}{8}}{1}$.

(b) Symmetrical form of the equations of line is $\frac{x + \frac{1}{8}}{1} = \frac{y - \frac{5}{8}}{-1} = \frac{z}{-2}$.

(c) Equation of the plane through (2, 1, 4) and perpendicular to the given lines is $2x - y + z - 7 = 0$.

(d) Equation of the plane through (2, 1, 4) and perpendicular to the given lines is $x + y - 2z + 5 = 0$.

98. Consider the family of planes $x + y + z = c$ where c is a parameter intersecting the coordinate axes at P, Q and R and α, β and γ are the angles made by each member of this family with positive x, y and z-axes. Which of the following interpretations hold good for this family?

(a) Each member of this family is equally inclined with coordinate axes.
(b) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 1$
(c) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 2$
(d) For $c = 3$ area of the ΔPQR is $3\sqrt{3}$ sq units.

99. Equation of the line through the point (1, 1, 1) and intersecting the lines $2x - y - z - 2 = 0 = x + y + z - 1$ and $x - y - z - 3 = 0 = 2x + 4y - z - 4$
(a) $x - 1 = 0, 7x + 17y - 3z - 134 = 0$
(b) $x - 1 = 0, 9x + 15y - 5z - 19 = 0$
(c) $x - 1 = 0, \frac{y-1}{1} = \frac{z-1}{3}$
(d) $x - 2y + 2z - 1 = 0, 9x + 15y - 5z - 19 = 0$

100. Through a point $P(h, k, l)$ a plane is drawn at right angles to OP to meet the coordinate axes in A, B and C. If $OP = p$, A_{xy} is area of projection of ΔABC on xy-plane, A_{yz} is area of projection of ΔABC on yz-plane, then
(a) $\Delta = \left|\frac{p^5}{hkl}\right|$ (b) $\Delta = \left|\frac{p^5}{2hkl}\right|$ (c) $\frac{A_{xy}}{A_{yz}} = \left|\frac{l}{h}\right|$ (d) $\frac{A_{xy}}{A_{yz}} = \left|\frac{h}{l}\right|$

101. Which of the following statements is/are correct?
(a) If $\mathbf{n}.\mathbf{a} = 0$, $\mathbf{n}.\mathbf{b} = 0$ and $\mathbf{n}.\mathbf{c} = 0$ for some non-zero vector $\mathbf{n}$, then $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$
(b) There exist a vector making angles 30° and 45° with x-axis and Y-axis.
(c) Locus of point for which $x = 3$ and $y = 4$ is a line parallel to the Z-axis whose distance from the Z-axis is 5
(d) The vertices of regular tetrahedron are O, A, B, C where 'O' is the origin. The vector $\mathbf{OA} + \mathbf{OB} + \mathbf{OC}$ is perpendicular to the plane ABC

102. Which of the following is/are correct about a tetrahedron ?
(a) Centroid of a tetrahedron lies on lines joining any vertex to the centroid opposite face
(b) Centroid of a tetrahedron lies on the lines joining the mid point of the opposite faces
(c) Distance of centroid from all the vertices are equal
(d) None of the above

103. A variable plane cutting coordinate axes in A, B, C is at a constant distance from the origin. Then the locus of centroid of the ΔABC is
(a) $x^{-2} + y^{-2} + z^{-2} = 16$ (b) $x^{-2} + y^{-2} + z^{-2} = 9$
(c) $\frac{1}{9}\left\{\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right\}^{-1} = 0$ (d) $X + Y = 0$

104. Equation of any plane containing the line $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$ is $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$, then pick correct alternatives
(a) $\frac{A}{a} = \frac{B}{b} = \frac{C}{c}$ is true for the line to be perpendicular to the plane
(b) $A(a + 3) + B(b - 1) + C(c - 2) = 0$
(c) $2aA + 3bB + 4cC = 0$
(d) $Aa + Bb + Cc = 0$

105. The line $\frac{x-2}{3} = \frac{y+1}{2} = \frac{z-1}{-1}$ intersects the curve $x^2 + y^2 = r^2, z = 0$ then
(a) Equation of the plane through (0, 0, 0) perpendicular to the given line is $3x + 2y - z = 0$
(b) $r = \sqrt{26}$ (c) $r = 6$
(d) $r = 7$

106. A vector equally inclined to the vectors $\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$ then the plane containing them is
(a) $\frac{\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}}{\sqrt{3}}$ (b) $\hat{\mathbf{j}} - \hat{\mathbf{k}}$ (c) $2\hat{\mathbf{i}}$ (d) $\hat{\mathbf{i}}$

107. Consider the plane through $(2, 3, -1)$ and at right angles to the vector $3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$ from the origin is
(a) The equation of the plane through the given point is $3x - 4y + 7z + 13 = 0$
(b) perpendicular distance of plane from origin $\frac{1}{\sqrt{74}}$
(c) perpendicular distance of plane from origin $\frac{13}{\sqrt{74}}$
(d) perpendicular distance of plane from origin $\frac{21}{\sqrt{74}}$

108. A plane passes through a fixed point (a, b, c) and cuts the axes in A, B, C. The locus of a point equidistant from origin, A, B and C must be
(a) $ayz + bzx + cxy = 2xyz$ (b) $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1$
(c) $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2$ (d) $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 3$

109. Let $\mathbf{A}$ be vector parallel to line of intersection of planes P_1 and P_2. Plane P_1 is parallel to the vectors $2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and $4\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$ and that P_2 is parallel to $\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $3\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$, then the angle between vector $\mathbf{A}$ and a given vector $2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ is
(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{6}$ (d) $\frac{3\pi}{4}$

110. Consider the lines $x = y = z$ and the line $2x + y + z - 1 = 0 = 3x + y + 2z - 2$, then
(a) the shortest distance between the two lines is $\frac{1}{\sqrt{2}}$
(b) the shortest distance between the two lines is $\sqrt{2}$
(c) plane containing 2nd line parallel to 1st line is $y - z + 1 = 0$
(d) the shortest distance between the two lines $\frac{\sqrt{3}}{2}$

111. If p_1, p_2, p_3 denote the perpendicular distances of the plane $2x - 3y + 4z + 2 = 0$ from the parallel planes.
(a) $p_1 + 8p_2 - p_3 = 0$ (b) $p_3 = 16p_2$
(c) $8p_2 = p_1$ (d) $p_1 + 2p_2 + 3p_3 = \sqrt{29}$

112. A line segment has length 63 and direction ratios are 3, −2,6. The components of the line vector are
(a) −27, 18, 54 (b) 27, −18, −54
(c) 27, −18, 54 (d) −27, 18, −54

113. The lines $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{-k}$ and $\frac{x-1}{k} = \frac{y-4}{2} = \frac{z-5}{1}$ are coplanar if
(a) $k = 0$ (b) $k = -1$
(c) $k = 2$ (d) $k = -3$

114. The points $A(4, 5, 10)$, $B(2, 3, 4)$ and $C(1, 2, -1)$ are three vertices of a parallelogram $ABCD$, then
(a) Vector equation of AB is $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + 3\mathbf{k})$
(b) Cartesian equation of BC is $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{5}$
(c) Coordinates of D are (3, 4, 5)
(d) $ABCD$ is a rectangle

115. The line $x = y = z$ meets the plane $x + y + z = 1$ at the point P and the sphere $x^2 + y^2 + z^2 = 1$ at the points R and S, then
(a) $PR + PS = 2$ (b) $PR \times PS = \frac{2}{3}$
(c) $PR = PS$ (d) $PR + PS = RS$

116. A rod of length 2 units whose one end is $(1, 0, -1)$ and other end touches the plane $x - 2y + 2z + 4 = 0$, then
(a) The rod sweeps the figure whose volume is π cubic units.
(b) The area of the region which the rod traces on the plane is 2π.
(c) The length of projection of the rod on the plane is $\sqrt{3}$ units.
(d) The centre of the region which the rod traces on the plane is $\left(\frac{2}{3}, \frac{2}{3}, \frac{-5}{3}\right)$.

117. Consider the planes $P_1 : 2x + y + z + 4 = 0$
$P_2 : y - z + 4 = 0$ and $P_3 : 3x + 2y + z + 8 = 0$
Let L_1, L_2, L_3 be the lines of intersection of the planes P_2 and P_3, P_3 and P_1, and P_1 and P_2 respectively. Then,
(a) at least two of the lines L_1, L_2 and L_3 are non-parallel
(b) at least two of the lines L_1, L_2 and L_3 are parallel
(c) the three planes intersect in a line
(d) the three planes form a triangular prism

118. The volume of a right triangular prism $ABCA_1B_1C_1$ is equal to 3. Find the coordinates of the vertex A_1, if the coordinates of the base vertices of the prism are $A(1, 0, 1)$, $B(2, 0, 0)$ and $C(0, 1, 0)$.
(a) $(-2, 0, 2)$ (b) $(0, -2, 0)$
(c) (0, 2, 0) (d) (2, 2, 2)

119. Let a plane pass through origin and is parallel to the line $\frac{x-1}{2} = \frac{y+3}{-1} = \frac{z+1}{-2}$ such that distance between plane and the line is $\frac{5}{3}$. Then, equation of the plane is ..... .
(a) $x - 2y + 2z = 0$ (b) $x - 2y - 2z = 0$
(c) $2x + 2y + z = 0$ (d) $x + y + z = 0$

120. $OABC$ is a regular tetrahedron of side unity, then
(a) the length of perpendicular from one vertex to opposite face is $\sqrt{2/3}$
(b) the perpendicular distance from mid-point of $\overline{OA}$ to the plane ABC is $1/\sqrt{6}$
(c) the angle between two skew edges is $\pi/2$
(d) the distance of centroid of the tetrahedron form any vertex is $\sqrt{3/8}$

121. If $OABC$ is a tetrahedron such that $OA^2 + BC^2 = OB^2 + CA^2 = OC^2 + AB^2$, then
(a) $OA \perp BC$ (b) $OB \perp AC$
(c) $OC \perp AB$ (d) $AB \perp AC$

122. If the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ intersects the line
$3\beta^2 x + 3(1 - 2\alpha)y + z = 3 = -\frac{1}{2}\{6\alpha^2 x + 3(1 - 2\beta)y + 2z\}$
then point $(\alpha, \beta, 1)$ lie on the plane
(a) $2x - y + z = 4$ (b) $x + y - z = 2$
(c) $x - 2y = 0$ (d) $2x - y = 0$

123. Let PM be the perpendicular from the point $P(1, 2, 3)$ to XY plane. If $\overline{OP}$ makes an angle θ with the positive direction of Z-axis and $\overline{OM}$ makes an angle ϕ with the positive direction of X-axis, where O is the origin and θ and ϕ are acute angles, then
(a) $\tan\theta = \frac{\sqrt{5}}{3}$ (b) $\sin\theta \sin\phi = \frac{2}{\sqrt{14}}$
(c) $\tan\phi = 2$ (d) $\cos\theta \cos\phi = \frac{1}{\sqrt{14}}$

124. A variable plane which remains at a constant distance P from the origin (0) cuts the coordinate axes in A, B, C
(a) Locus of centroid of tetrahedron $OABC$ is
$x^2y^2 + y^2z^2 + z^2x^2 = \frac{16}{p^2}x^2y^2z^2$
(b) Locus of centroid of tetrahedron $OABC$ is
$x^2y^2 + y^2z^2 + z^2x^2 = \frac{4}{p^2}x^2y^2z^2$
(c) Parametric equation of the centroid of the tetrahedron is of the form $\left(\frac{P}{4}\sec\alpha\sec\beta, \frac{P}{4}\sec\alpha \operatorname{cosec}\beta, \frac{P}{4}\operatorname{cosec}\alpha\right)$,
$\alpha, \beta \in (0, 2\pi) - \{\pi/2, \pi, 3\pi/2\}$
(d) None of the above

Three Dimensional Coordinate System Exercise 3 : Statement I and II Type Questions

■ **Directions** (Q. Nos. 125 to 138) For the following questions, choose the correct answers from the codes (a), (b), (c) and (d) defined as follows :

(a) Statement I is true, Statement II is also true; Statement II is the correct explanation of Statement I

(b) Statement I is true, Statement II is also true; Statement II is not the correct explanation of Statement I

(c) Statement I is true, Statement II is false

(d) Statement I is false, Statement II is true

125. **Statement I** Let $A(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})$ and $B(\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}})$ be two points, $P(2\hat{\mathbf{i}}+3\hat{\mathbf{j}}+\hat{\mathbf{k}})$ lies exterior to the sphere with AB as one of its diameters.

Statement II If A and B are any two points and P is a point in space such that $\mathbf{PA}.\mathbf{PB} > 0$, then the point P lies exterior to the sphere with AB as one of its diameters.

126. **Statement I** If $\mathbf{r} = x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}$, then equation $\mathbf{r}\times(2\hat{\mathbf{i}}-\hat{\mathbf{j}}+3\hat{\mathbf{k}}) = 3\hat{\mathbf{i}}+\hat{\mathbf{k}}$ represents a straight line.

Statement II If $\mathbf{r} = x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}$, then equation $\mathbf{r}\times(\hat{\mathbf{i}}+2\hat{\mathbf{j}}-3\hat{\mathbf{k}}) = 2\hat{\mathbf{i}}-\hat{\mathbf{j}}$ represents a straight line.

127. **Statement I** Let θ be the angle between the line $\frac{x-2}{2}=\frac{y-1}{-3}=\frac{z+2}{-2}$ and the plane $x+y-z=5$.

Then, $\theta = \sin^{-1}\left(\frac{1}{\sqrt{51}}\right)$

Statement II Angle between a straight line and a plane is the complement of angle between the line and normal to the plane.

128. **Statement I** A point on the straight line $2x+3y-4z=5$ and $3x-2y+4z=7$ can be determined by taking $x=k$ and then solving the two equations for y and z, where k is any real number.

Statement II If $c' \neq kc$, then the straight line $ax+by+cz+d=0$, $Kax+Kby+c'z+d'=0$, does not intersect the plane $z=\alpha$, where α is any real number.

129. Let the line L having equation $\frac{x-1}{2}=\frac{y-3}{5}=\frac{z-1}{3}$, intersects the plane P, having equation $x-y+z=5$ at the point A.

Statement I Equation of the line L' through the point A, lying in the plane P and having minimum inclination with line L is $8x+y-72-4=0=x-y+z-5$

Statement II Line L' must be the projection of the line L in the plane P.

130. Given lines $\frac{x-4}{2}=\frac{y+5}{4}=\frac{z-1}{-3}$ and $\frac{x-2}{1}=\frac{y+1}{3}=\frac{z}{2}$

Statement I The lines intersect.

Statement II They are not parallel.

131. Consider the lines $L_1 : \mathbf{r} = \mathbf{a}+\lambda\,\mathbf{b}$ and $L_2 : \mathbf{r} = \mathbf{b}+\mu\,\mathbf{a}$, where $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-collinear vectors.

Statement I L_1 and L_2 are coplanar and the plane containing these lines passes through origin.

Statement II $(a-b).(b\times a)=0$ and the plane containing L_1 and L_2 is $[\mathbf{r}\,\mathbf{a}\,\mathbf{b}]=0$ which passes through origin.

132. **Statement I** P is a point (a,b,c). Let A, B, C be the images of P in yz, zx and xy planes respectively, then equation of the plane passing through the points A, B and C is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Statement II The image of a point P in a plane is the foot of the perpendicular drawn from P on the plane.

133. **Statement I** The locus of a point which is equidistant from the points whose position vectors are $3\hat{\mathbf{i}}-2\hat{\mathbf{j}}+5\hat{\mathbf{k}}$ and $\hat{\mathbf{i}}+2\hat{\mathbf{j}}-\hat{\mathbf{k}}$ is $\mathbf{r}(\hat{\mathbf{i}}-2\hat{\mathbf{j}}+3\hat{\mathbf{k}}) = 8$.

Statement II The locus of a point which is equidistant from the points whose position vectors are $\mathbf{a}$ and $\mathbf{b}$ is $\left|\mathbf{r}-\frac{\mathbf{a}+\mathbf{b}}{2}\right|.(\mathbf{a}-\mathbf{b})=0$

134. **Statement I** If the vectors $\mathbf{a}$ and $\mathbf{c}$ are non-collinear then the lines $\mathbf{r} = 6\mathbf{a}-\mathbf{c}+\lambda(2\mathbf{c}-\mathbf{a})$ and $\mathbf{r} = \mathbf{a}-\mathbf{c}+\mu(\mathbf{a}+3\mathbf{c})$ are coplanar.

Statement II There exist λ and μ such that the two values of $\mathbf{r}$ in Statement I becomes same.

135. **Statement I** The lines $\frac{x-1}{1}=\frac{y}{-1}=\frac{z+1}{1}$ and $\frac{x-2}{1}=\frac{y+1}{2}=\frac{z}{3}$ are coplanar and equation of the plane containing them is $5x+2y-3z-8=0$

Statement II The line $\frac{x-2}{1}=\frac{y+1}{2}=\frac{z}{3}$ is perpendicular to the plane $3x+6y+9z-8=0$ and parallel to the plane $x+y-z=0$

136. The equation of two straight lines are $\frac{x-1}{2}=\frac{y+3}{1}=\frac{z-2}{-3}$ and $\frac{x-2}{1}=\frac{y-1}{-3}=\frac{z+3}{2}$.

Statement I The given lines are coplanar.

Statement II The equations $2x_1-y_1=1$, $x_1+3y_1=4$ and $3x_1+2y_1=5$ are consistent.

137. **Statement I** A plane passes through the point $A(2, 1, -3)$. If distance of this plane from origin is maximum, then its equation is $2x + y - 3z = 14$.

Statement II If the plane passing through the point $A(\mathbf{a})$ is at maximum distance from origin, then normal to the plane is vector $\mathbf{a}$.

138. **Statement I** At least two of the lines L_1, L_2 and L_3 are non-parallel.

Statement II The three planes do not have a common point.

Three Dimensional Coordinate System Exercise 4 : Passage Based Questions

Passage I

(Q. Nos. 139 to 142)

Let $A(1, 2, 3)$, $B(0, 0, 1)$ and $C(-1, 1, 1)$ are the vertices of ΔABC.

139. The equation of internal angle bisector through A to side BC is

(a) $\mathbf{r} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}} + \mu(3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$

(b) $\mathbf{r} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}} + \mu(3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$

(c) $\mathbf{r} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}} + \mu(3\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

(d) $\mathbf{r} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}} + \mu(3\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}})$

140. The equation of altitude through B to side AC is

(a) $\mathbf{r} = \mathbf{k} + t(7\hat{\mathbf{i}} - 10\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

(b) $\mathbf{r} = \mathbf{k} + t(-7\hat{\mathbf{i}} + 10\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

(c) $\mathbf{r} = \mathbf{k} + t(7\hat{\mathbf{i}} - 10\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$

(d) $\mathbf{r} = \mathbf{k} + t(7\hat{\mathbf{i}} + 10\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

141. The equation of median through C to side AB is

(a) $\mathbf{r} = -\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}} + p(3\hat{\mathbf{i}} - 2\hat{\mathbf{k}})$

(b) $\mathbf{r} = -\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}} + p(3\hat{\mathbf{i}} + 2\hat{\mathbf{k}})$

(c) $\mathbf{r} = -\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}} + p(-3\hat{\mathbf{i}} + 2\hat{\mathbf{k}})$

(d) $\mathbf{r} = -\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}} + p(3\hat{\mathbf{i}} + 2\hat{\mathbf{j}})$

142. The area of (ΔABC) is equal to

(a) $\frac{9}{2}$ (b) $\frac{\sqrt{17}}{2}$ (c) $\frac{17}{2}$ (d) $\frac{7}{2}$

Passage II

(Q. Nos. 143 to 144)

Consider a plane $x + y - z = 1$ and the point $A(1, 2, -3)$. A line L has the equation $x = 1 + 3r$, $y = 2 - r$, $z = 3 + 4r$

143. The coordinate of a point B of line L, such that AB is parallel to the plane, is

(a) $(10, -1, 15)$ (b) $(-5, 4, -5)$

(c) $(4, 1, 7)$ (d) $(-8, 5, -9)$

144. Equation of the plane containing the line L and the point A has the equation

(a) $x - 3y + 5 = 0$ (b) $x + 3y - 7 = 0$

(c) $3x - y - 1 = 0$ (d) $3x + y - 5 = 0$

Passage III

(Q. Nos. 145 to 148)

Consider a triangular pyramid $ABCD$ the position vector of whose angular points are $A(3, 0, 1)$, $B(-1, 4, 1)$, $C(5, 2, 3)$ and $D(0, -5, 4)$. Let G be the point of intersection of the medians of the ΔBCD.

145. The length of the vector **AG** is

(a) $\sqrt{17}$ (b) $\frac{\sqrt{51}}{3}$ (c) $\frac{\sqrt{51}}{9}$ (d) $\frac{\sqrt{59}}{4}$

146. Area of the ΔABC (in sq units) is

(a) 24 (b) $8\sqrt{6}$

(c) $4\sqrt{6}$ (d) None of these

147. The length of the perpendicular from the vertex D on the opposite face is

(a) $\frac{14}{\sqrt{6}}$ (b) $\frac{2}{\sqrt{6}}$

(c) $\frac{3}{\sqrt{6}}$ (d) None of these

148. Equation of the plane ABC is

(a) $x + y + 2z = 5$ (b) $x - y - 2z = 1$

(c) $2x + y - 2z = 4$ (d) $x + y - 2z = 1$

Passage IV

(Q. Nos. 149 to 151)

A line L_1 passing through a point with position vector $\mathbf{p} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and parallel $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, Another line L_2 passing through a point with position vector $= 2\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$ and parallel to $\mathbf{b} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$.

149. Equation of plane equidistant from line L_1 and L_2 is

(a) $\hat{\mathbf{r}} \cdot (\mathbf{i} - 7\mathbf{j} - 5\mathbf{k}) = 3$ (b) $\hat{\mathbf{r}} \cdot (\mathbf{i} + 7\mathbf{j} + 5\mathbf{k}) = 3$

(c) $\hat{\mathbf{r}} \cdot (\mathbf{i} - 7\mathbf{j} - 5\mathbf{k}) = 9$ (d) $\hat{\mathbf{r}} \cdot (\mathbf{i} + 7\mathbf{j} - 5\mathbf{k}) = 9$

150. Equation of a line passing through the point $(2, -3, 2)$ and equally inclined to the line L_1 and L_2 may be equal to

(a) $\frac{x-2}{2} = \frac{y-3}{-1} = \frac{z-2}{1}$ (b) $\frac{x-2}{-2} = y + 3 = z - 2$

(c) $\frac{x-2}{-4} = \frac{y+3}{3} = \frac{z-5}{2}$ (d) $\frac{x+2}{4} = \frac{y+3}{3} = \frac{z-2}{-5}$

151. The minimum distance of origin from the plane passing through the point with position vector $\mathbf{p}$ and perpendicular to the line L_2 is

(a) $\sqrt{14}$ (b) $\frac{7}{\sqrt{14}}$

(c) $\frac{11}{\sqrt{14}}$ (d) None of these

Passage V

(Q. Nos. 152 to 154)

For positive l, m and n, if the planes $x = ny + mz$, $y = lz + nx$, $z = mz + ly$ intersect in a straight line, then

152. l, m and n satisfy the equation

(a) $l^2 + m^2 + n^2 = 2$ (b) $l^2 + m^2 + n^2 + 2lmn = 1$

(c) $l^2 + m^2 + n^2 = 1$ (d) None of these

153. $\cos^{-1} l + \cos^{-1} m + \cos^{-1} n$ is equal to

(a) 90° (b) 50°

(c) 180° (d) None of these

154. The equation of the straight line is $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$, where the ordered traid (a, b, c) is

(a) $\sqrt{1-l^2}, \sqrt{1-m^2}, \sqrt{1-n^2}$

(b) l, m and n

(c) $\frac{1}{\sqrt{1-l^2}}, \frac{m}{\sqrt{1-m^2}}$ and $\frac{n}{\sqrt{1-n^2}}$

(d) None of the above

Passage VI

(Q. Nos. 155 to 157)

If $\mathbf{a} = 6\hat{\mathbf{i}} + 7\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$, $\mathbf{b} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$, $P(1, 2, 3)$

155. The position vector of L, the foot of the perpendicular from P on the line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ is

(a) $6\hat{\mathbf{i}} + 7\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$ (b) $3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$

(c) $3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 9\hat{\mathbf{k}}$ (d) $9\hat{\mathbf{i}} + 9\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$

156. The image of the point P in the line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ is

(a) (11, 12, 11) (b) (5, 2, −7)

(c) (5, 8, 15) (d) (17, 16, 7)

157. If A is the point with position vector $\mathbf{a}$ then area of the triangle ΔPLA is sq. units is equal to

(a) $3\sqrt{6}$ (b) $\frac{7\sqrt{17}}{2}$

(c) $\sqrt{17}$ (d) $\frac{7}{2}$

Passage VII

(Q. Nos. 158 to 160)

$A(-2, 2, 3)$ and $B(13, -3, 13)$ and L is a line through A.

158. A point P moves in the space such that $3PA = 2PB$, then the locus of P is

(a) $x^2 + y^2 + z^2 + 28x - 12y + 10z - 247 = 0$

(b) $x^2 + y^2 + z^2 - 28x + 12y + 10z - 247 = 0$

(c) $x^2 + y^2 + z^2 + 28x - 12y - 10z + 247 = 0$

(d) $x^2 + y^2 + z^2 - 28x + 12y - 10z + 247 = 0$

159. Coordinates of the point P which divides the join of A and B in the ratio 2 : 3 internally are

(a) $\left(\frac{33}{5}, -\frac{2}{5}, 9\right)$ (b) (4, 0, 7)

(c) $\left(\frac{32}{5}, -\frac{12}{5}, \frac{17}{5}\right)$ (d) (20, 0, 35)

160. Equation of a line L, perpendicular to the line AB is

(a) $\frac{x+2}{15} = \frac{y-2}{-5} = \frac{z-3}{10}$

(b) $\frac{x-2}{3} = \frac{y+2}{13} = \frac{z+3}{2}$

(c) $\frac{x+2}{3} = \frac{y-2}{13} = \frac{z-3}{2}$

(d) $\frac{x-2}{15} = \frac{y+2}{-5} = \frac{z+3}{10}$

Passage VIII

(Q. Nos. 161 to 163)

The vector equation of a plane is a relation satisfied by position vectors of all the points on the plane. If P is a plane and $\hat{\mathbf{n}}$ is a unit vector through origin which is perpendicular to the plane P then vector equation of the plane must be $\mathbf{r}.\hat{\mathbf{n}} = d$ where d represents perpendicular distance of plane p from origin.

161. If A is a point vector $\mathbf{a}$ then perpendicular distance of A from the plane $\mathbf{r}.\hat{\mathbf{n}} = d$ must be

(a) $|d + \mathbf{a}\hat{\mathbf{n}}|$ (b) $|d - \mathbf{a}\hat{\mathbf{n}}|$

(c) $|\mathbf{a}| - d|$ (d) $|d - \hat{\mathbf{a}}|$

162. If $\mathbf{b}$ be the foot of perpendicular from A to the plane $\mathbf{r}.\hat{\mathbf{n}} = d$ then $\mathbf{b}$ must be

(a) $\mathbf{a} + (d - \mathbf{a}.\hat{\mathbf{n}})\,\hat{\mathbf{n}}$ (b) $\mathbf{a} - (d - \mathbf{a}.\hat{\mathbf{n}})\,\hat{\mathbf{n}}$

(c) $\mathbf{a} + \mathbf{a}.\hat{\mathbf{n}}$ (d) $\mathbf{a} - \mathbf{a}.\hat{\mathbf{n}}$

163. The position vector of the image of the point $\mathbf{a}$ in the plane $\mathbf{r}.\hat{\mathbf{n}} = d$ must be $(d \neq 0)$

(a) $-\mathbf{a}.\hat{\mathbf{n}}$ (b) $\mathbf{a} - 2(d - \mathbf{a}.\hat{\mathbf{n}})\,\hat{\mathbf{n}}$

(c) $\mathbf{a} + 2(d - \mathbf{a}.\hat{\mathbf{n}})\,\hat{\mathbf{n}}$ (d) $\mathbf{a} + d(-\mathbf{a}.\hat{\mathbf{n}})$

Passage IX

(Q. Nos. 164 to 166)

A circle is the locus of a point in a plane such that its distance from a fixed point in the plane is constant. Anologously, a sphere is the locus of a point in space such that its distance from a fixed point in space is constant.

The fixed point is called the centre and the constant distance is called the radius of the circle/sphere.

In anology with the equation of the circle $|z-c|=a$, *the equation of a sphere of radius a is* $|\mathbf{r}-\mathbf{c}|=a$, *where* $\mathbf{c}$ *is the position vector of the centre and* $\mathbf{r}$ *is the position vector of any point on the surface of the sphere. In Cartesian system, the equation of the sphere, with centre at* $(-g,-f,-h)$ *is* $x^2+y^2+z^2+2gx+2fy+2hz+c=0$ *and its radius is* $\sqrt{f^2+g^2+h^2-c}$.

164. Radius of the sphere, with $(2,-3,4)$ and $(-5,6,-7)$ as extremities of a diameter, is

(a) $\sqrt{\frac{251}{2}}$ (b) $\sqrt{\frac{251}{3}}$

(c) $\sqrt{\frac{251}{4}}$ (d) $\sqrt{\frac{251}{5}}$

165. The centre of the sphere $(x-4)(x+4)+(y-3)(y+3)+z^2=0$ is

(a) (4, 3, 0) (b) (− 4, − 3, 0)

(c) (0, 0, 0) (d) None of these

166. Equation of the sphere having centre at $(3,6,-4)$ and touching the plane $\mathbf{r}\cdot(2\hat{\mathbf{i}}-2\hat{\mathbf{j}}-\hat{\mathbf{k}})=10$, is $(x-3)^2+(y-6)^2+(z+4)^2=k^2$, where k is equal to

(a) 3 (b) 4

(c) 6 (d) $\sqrt{17}$

Passage X

(Q. Nos. 167 to 168)

Let $A(2,3,5)$, $B(-1,3,2)$, $C(\lambda,5,\mu)$ *are the vertices of a triangle and its median through A (i.e.) AD is equally inclined to the coordinates axes.*

On the basis of the above information answer the following

167. The value of $2\lambda-\mu$ is equal to

(a) 13 (b) 4

(c) 3 (d) None of these

168. Projection of AB on BC is

(a) $\frac{8\sqrt{3}}{11}$ (b) $\frac{-8\sqrt{3}}{11}$

(c) − 48 (d) 48

Passage XI

(Q. Nos. 169 to 171)

The line of greatest slope on an inclined plane P_1 *is that line in the plane which is perpendicular to the line of intersection of plane* P_1 *and a horizontal plane* P_2.

169. Assuming the plane $4x-3y+7z=0$ to be horizontal, the direction cosines of the line of greatest slope in the plane $2x+y-5z=0$ are

(a) $\left(\frac{3}{\sqrt{11}},\frac{-1}{\sqrt{11}},\frac{1}{\sqrt{11}}\right)$ (b) $\left(\frac{3}{\sqrt{11}},\frac{1}{\sqrt{11}},\frac{-1}{\sqrt{11}}\right)$

(c) $\left(\frac{-3}{\sqrt{11}},\frac{1}{\sqrt{11}},\frac{1}{\sqrt{11}}\right)$ (d) $\left(\frac{1}{\sqrt{11}},\frac{3}{\sqrt{11}},\frac{-1}{\sqrt{11}}\right)$

170. The equation of a line of greatest slope can be

(a) $\frac{x}{3}=\frac{y}{1}=\frac{z}{-1}$ (b) $\frac{x}{3}=\frac{y}{-1}=\frac{z}{1}$

(c) $\frac{x}{-3}=\frac{y}{1}=\frac{z}{1}$ (d) $\frac{x}{1}=\frac{y}{3}=\frac{z}{-1}$

171. The coordinates of a point on the plane $2x+y-5z=0$ is $\sqrt{11}$ units away from the line of intersection of the given two planes are

(a) (3, 1, − 1) (b) (− 3, 1, 1)

(c) (3, − 1, 1) (d) (1, 3, − 1)

Passage XII

(Q. Nos. 172 to 174)

Given four points $A(2,1,0)$, $B(1,0,1)$, $C(3,0,1)$ *and* $D(0,0,2)$. *Point D lies on a line L orthogonal to the plane determined by the points A, B and C.*

172. The equation of the plane ABC is

(a) $x+y+z-3=0$ (b) $y+z-1=0$

(c) $x+z-1=0$ (d) $2y+z-1=0$

173. The equation of the line L is

(a) $\mathbf{r}=2\hat{\mathbf{k}}+\lambda(\hat{\mathbf{i}}+\hat{\mathbf{k}})$

(b) $\mathbf{r}=2\hat{\mathbf{k}}+\lambda(2\hat{\mathbf{j}}+\hat{\mathbf{k}})$

(c) $\mathbf{r}=2\hat{\mathbf{k}}+\lambda(\hat{\mathbf{j}}+\hat{\mathbf{k}})$

(d) None of the above

174. The perpendicular distance of D from the plane ABC is

(a) $\sqrt{2}$ (b) 1/2

(c) 2 (d) $1/\sqrt{2}$

Three Dimensional Coordinate System Exercise 5 : Matching Type Questions

175. Consider the following four pairs of line in Column I and match them with one or more entries in Column II.

Column I	Column II
(A) $L_1: x = 1 + t, y = t, z = 2 - 5t$ $L_2: r = (2, 1, -3) + \lambda(2, 2, -10)$	(p) non-coplanar lines
(B) $L_1: \frac{x-1}{2} = \frac{y-3}{2} = \frac{z-2}{-1}$ $L_2: \frac{x-2}{1} = \frac{y-6}{-1} = \frac{z+2}{-1}$	(q) lines lie in a unique plane
(C) $L_1: x = -6t, y = 1 + 9t, z = -3t$ $L_2: x = 1 + 2s, y = 4 - 3s, z = s$	(r) infinite planes containing both the lines
(D) $L_1: \frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ $L_2: \frac{x-3}{-4} = \frac{y-2}{-3} = \frac{z-1}{2}$	(s) lines are not intersecting at a unique point

176. $P(0, 3, -2)$, $Q(3, 7, -1)$ and $R(1, -3, -1)$ are 3 given points. Let L_1 be the line passing through P and Q and L_2 be the line through R and parallel to the vector $\mathbf{V} = \hat{\mathbf{i}} + \hat{\mathbf{k}}$.

Column I	Column II
(A) Perpendicular distance of P from L_1	(p) $7\sqrt{3}$
(B) Shortest distance between L_1 and L_2	(q) 2
(C) Area of the ΔPQR	(r) 6
(D) Distance from (0, 0, 0) to the plane PQR	(s) $\frac{19}{\sqrt{147}}$

177. Match the statements of Column I with values of Column II.

Column I	Column II
(A) If the line $\frac{x-1}{1} = \frac{y+1}{-2} = \frac{z+1}{\lambda}$ lies in the plane $3x - 2y + 5z = 0$, then λ is equal to	(p) $\sin^{-1}\sqrt{\frac{6}{25}}$
(B) If $(3, \lambda, \mu)$ is a point on the line $2x + y + z - 3 = 0 = x - 2y + z - 1$, then $\lambda + \mu$ is equal to	(q) $-\frac{7}{5}$
(C) The angle between the line $x = y = z$ and the plane $4x - 3y + 5z = 2$ is	(r) -3
(D) The angle between the planes $x + y + z = 0$ and $3x - 4y + 5z = 0$	(s) $\cos^{-1}\sqrt{\frac{8}{75}}$

178. Consider the lines given by $L_1 : x + 3y - 5 = 0$, $L_2 : 3x - ky - 1 = 0$ and $L_3 : 5x + 2y - 12 = 0$.

Match the statement of Column I with values of Column II.

Column I	Column II
(A) L_1, L_2 and L_3 are concurrent, if	(p) $k = -9, -\frac{6}{5}, 5$
(B) One of L_1, L_2 and L_3 is parallel to atleast one of the other two, if	(q) $k = -\frac{6}{5}, -9$
(C) L_1, L_2 and L_3 form a triangle, if	(r) $k = \frac{5}{6}$
(D) L_1, L_2 and L_3 do not form a triangle, if	(s) $k = 5$ (t) $k = 0$

179. A variable plane cuts the x, y and z-axes at the points, A, B and C, respectively such that the volume of the tetrahedron $OABC$ remain constant equal to 32 cu units and O is the origin of the coordinate system.

Column I	Column II
(A) The locus of the centroid of the tetrahedron is	(p) $xyz = 24$
(B) The locus of the point equidistant from O, A, B and C is	(q) $(x^2 + y^2 + z^2) = 192\, xyz$
(C) The locus of the foot of perpendicular from origin to the plane is	(r) $xyz = 3$
(D) If PA, PB and PC are mutually perpendicular, then the locus of P is	(s) $(x^2 + y^2 + z^2)^3 = 1536\, xyz$

180. Match the statements of Column I with values of Column II.

Column I	Column II
(A) The area of the triangle whose vertices are (0, 0, 0) (3, 4, 7) and (5, 2, 6) is	(p) 0
(B) The smallest radius of the sphere passing through (1, 0, 0), (0, 1, 0) and (0, 0, 1) is	(q) $\frac{70}{3}$
(C) The value(s) of λ for which the triangle with vertices $A(6., 10, 10)$ $B(1, 0, -5)$ and $C(6, -10, \lambda)$ will be a right angled triangle (right angled at A) is/are	(r) $\sqrt{\frac{2}{3}}$
(D) d is the perpendicular distance from (1, 3, 4) to $\frac{x-1}{-1} = \frac{y-1}{1} = \frac{z}{1}$, then value of $\frac{d}{2\sqrt{3}}$	(s) $\frac{3}{2}\sqrt{65}$

181. Match the statements of Column I with values of Column II. Consider the cube

	Column I		Column II
(A)	Angle between any two solid diagonal	(p)	$\cos^{-1}\frac{2}{\sqrt{6}}$
(B)	Angle between a solid diagonal and a plane	(q)	$\cos^{-1}\left(+\frac{1}{2}\right)$
(C)	Angle between plane diagonals of adjacent faces	(r)	$\cos^{-1}\frac{1}{3}$
(D)	The values of $\lvert\mathbf{a}\times\mathbf{b}\rvert$	(s)	$\frac{1}{2}$

Three Dimensional Coordinate System Exercise 6 : Single Integer Answer Type Questions

182. In a tetrahedron $OABC$, if $\mathbf{OA}=\hat{\mathbf{i}}$, $\mathbf{OB}=\hat{\mathbf{i}}+\hat{\mathbf{j}}$ and $\mathbf{OC}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}}$, if shortest distance between edges OA and BC is m, then $\sqrt{2}m$ is equal to ... (Where O is the origin)

183. A parallelopiped is formed by planes drawn through the points (2, 4, 5) and (5, 9, 7) parallel to the coordinate planes. The length of the diagonal of the parallelopiped is .....

184. If the perpendicular distance of the point (6, 5, 8) from the Y-axis is 5λ units, then λ is equal to .....

185. If the shortest distance between the lines $\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$ and $\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{2}$ is $\lambda\sqrt{30}$ units, then the value of λ is .....

186. If the planes $x-cy-bz=0$, $cx-y+az=0$ and $bx+ay-z=0$ pass through a line then the values of $a^2+b^2+c^2+2abc$ is .....

187. If xz-plane divide the join of point $(2,3,4)$ and $(1,-1,5)$ in the ratio $\lambda:1$, then the integer λ should be equal to

188. If the triangle ABC whose vertices are $A(-1,1,1)$, $B(1,-1,1)$ and $C(1,1,-1)$ is projected on xy-plane, then the area of the projected triangles is .....

189. The equation of a plane which bisects the line joining (1, 5, 7) and (−3, 1, −1) is $x+y+2z=\lambda$, then λ must be .....

190. The shortest distance between origin and a point on the space curve $x=2\sin t$, $y=2\cos t$, $z=3t$ is .....

191. The plane $2x-2y+z+12=0$ touches the surface $x^2+y^2+z^2-2x-4y+2z-3=0$ only at point $(-1,\lambda,-2)$. The value of λ must be

192. If the centroid of the tetrahedron $OABC$ where A, B, C are the points $(a,2,3)$, $(1,b,2)$ and $(2,1,c)$ be (1, 2, 3), then the point (a,b,c) is at distance $5\sqrt{\lambda}$ from origin, then λ must be equal to .....

193. If the circumcentre of the triangle whose vertices are $(3,2,-5)$, $(-3,8,-5)$ and $(-3,2,1)$ is $(-1,\lambda,-3)$ the integer λ must be equal to .....

194. If $\overline{P_1P_2}$ is perpendicular to $\overline{P_2P_3}$, then the value of k is, where $P_1(k,1,-1)$, $P_2(2k,0,2)$ and $P_3(2+2k,k,1)$ is .....

195. Let the equation of the plane containing line $x-y-z-4=0=x+y+2z-4$ and parallel to the line of intersection of the planes $2x+3y+z=1$ and $x+3y+2z=2$ be $x+Ay+Bz+C=0$. Then the values of $|A+B+C-4|$ is ..... .

196. Let $P(a,b,c)$ be any point on the plane $3x+2y+z=7$, then find the least value of $2(a^2+b^2+c^2)$.

197. The plane denoted by $P_1: 4x+7y+4z+81=0$ is rotated through a right angle about its line of intersection with the plane $P_2: 5x+3y+10z=25$. If the plane in its new position be denoted by P, and the distance of this plane from the origin is d, then the value of $\left[\frac{k}{2}\right]$ (where [.] represents greatest integer less than or equal to k) is .....

198. The distance of the point $P(-2,3,-4)$ from the line $\frac{x+2}{3}=\frac{2y+3}{4}=\frac{3z+4}{5}$ measured parallel to the plane $4x+12y-3z+1=0$ is d, then find the value of $(2d-8)$. is .....

199. The position vectors of the four angular points of a tetrahedron $OABC$ are (0, 0, 0), (0, 0, 2), (0, 4, 0) and (6, 0, 0), respectively. A point P inside the tetrahedron is at the same distance 'r' from the four plane faces of the tetrahedron. Then, the value of $9r$ is ..... .

200. Value of λ do the planes $x-y+z+1=0$, $\lambda x+3y+2z-3=0, 3x+\lambda y+z-2=0$ form a triangular prism must be

201. If the lattice point $P(x, y, z)$; $x, y, z>0$ and $x, y, z \in I$ with least value of z such that the 'P' lies on the planes $7x+6y+2z=272$ and $x-y+z=16$, then the value of $(x+y+z-42)$ is equal to

202. If the line $x=y=z$ intersect the line $x\sin A+y\sin B+z\sin C-2d^2=0$ $=x\sin 2A+y\sin 2B+z\sin C-d^2$, where A, B, C are the internal angles of a triangle and $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=k$, then the value of $64k$ is equal to

203. The number of real values of k for which the lines $\frac{x}{1}=\frac{y-1}{k}=\frac{z}{-1}$ and $\frac{x-k}{2k}=\frac{y-k}{3k-1}=\frac{z-2}{k}$ are coplanar, is

204. Let G_1, G_2 and G_3 be the centroids of the triangular faces OBC, OCA and OAB of a tetrahedron $OABC$. If V_1 denotes the volume of tetrahedron $OABC$ and V_2 that of the parallelepiped with OG_1, OG_2 and OG_3 as three concurrent edges, then the value of $4V_1/V_2$ is (where O is the origin)

205. A variable plane which remains at a constant distance p from the origin cuts the coordinate axes in A, B, C. The locus of the centroid of the tetrahedron $OABC$ is $x^2y^2+y^2z^2+z^2x^2=\frac{k}{p^2}x^2y^2z^2$, then $\sqrt[5]{2k}$ is

206. If $(l_1, m_1, n_1); (l_2, m_2, n_2)$ are D.C's of two lines, then $(l_1m_2-l_2m_1)^2+(m_1n_2-m_2n_1)^2$ $+(n_1l_2-n_2l_1)^2+(l_1l_2+m_1m_2+n_1n_2)^2=$

207. If the coordinates (x, y, z) of the point S which is equidistant from the points $O(0,0,0)$, $A(n^5,0,0)$ $B(0,n^4,0)$, $C(0,0,n)$ obey the relation $2(x+y+z)+1=0$. If $n \in Z$, then $|n|=$ _____ ($|\cdot|$ is the modulus function).

Three Dimensional Coordinate System Exercise 7 : Subjective Type Questions

208. Find the angle between the lines whose direction cosines has the relation $l+m+n=0$ and $2l^2+2m^2-n^2=0$.

209. Prove that the two lines whose direction cosines are connected by the two relations $al+bm+cn=0$ and $ul^2+vm^2+wn^2=0$ are perpendicular if

$a^2(v+w)+b^2(w+u)+c^2(u+v)=0$

and parallel if $\frac{a^2}{u}+\frac{b^2}{v}+\frac{c^2}{w}=0$.

210. Find the point on the line $\frac{x+2}{3}=\frac{y+1}{2}=\frac{z-3}{2}$ at a distance of $3\sqrt{2}$ from the point (1, 2, 3).

211. A line passes through $(2,-1,3)$ and is perpendicular to the lines $\mathbf{r}\cdot(\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}})+\lambda(2\hat{\mathbf{i}}-2\hat{\mathbf{j}}+\hat{\mathbf{k}})$ and $\mathbf{r}=(2\hat{\mathbf{i}}-\hat{\mathbf{j}}-3\hat{\mathbf{k}})+\mu(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\hat{\mathbf{k}})$ obtain its equation.

212. Find the equations of the two lines through the origin which intersect the line $\frac{x-3}{2}=\frac{y-3}{1}=\frac{z}{1}$ at angle of $\frac{\pi}{3}$ each.

213. Vertices B and C of ΔABC lie along the line $\frac{x+2}{2}=\frac{y-1}{1}=\frac{z-0}{4}$. Find the area of the triangle given that A has coordinates $(1,-1,2)$ and line segment BC has length 5.

214. A point P moves on the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ which is fixed. The plane through P perpendicular to OP meets the axes in A, B and C. The planes through A, B, C parallel to yz, zx, xy planes intersect in Q. Prove that if the axis bc rectangular, then the locus of Q is

$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}$

215. Prove that the distance of the point of intersection of the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ and the plane $x-y+z=5$ from the point $(-1,-5,-10)$ is 13.

216. Find the equation of the plane through the intersection of the planes $x+3y+6=0$ and $3x-y-4z=0$, whose perpendicular distance from the origin is unity.

217. Find the equation of the image of the plane $x-2y+2z-3=0$ in the plane $x+y+z-1=0$.

Three Dimensional Coordinate System Exercise 8 : Questions Asked in Previous Years Exam

(i) JEE Advanced & IIT JEE

218. Consider a pyramid $OPQRS$ located in the first octant $(x \geq 0, y \geq 0, z \geq 0)$ with O as origin and OP and OR along the X-axis and the Y-axis, respectively. The base $OPQR$ of the pyramid is a square with $OP = 3$. The point S is directly above the mid-point T of diagonal OQ such that $TS = 3$. Then,

[More than One Correct Type Question, 2016 Adv.]

(a) the acute angle between OQ and OS is $\frac{\pi}{3}$

(b) the equation of the plane containing the ΔOQS is $x - y = 0$

(c) the length of the perpendicular from P to the plane containing the ΔOQS is $\frac{3}{\sqrt{2}}$

(d) the perpendicular distance from O to the straight line containing RS is $\sqrt{\frac{15}{2}}$

219. Let P be the image of the point (3, 1, 7) with respect to the plane $x - y + z = 3$. Then, the equation of the plane passing through P and containing the straight line $\frac{x}{1} = \frac{y}{2} = \frac{z}{1}$ is

[Single Option Correct Type Question, 2016 Adv.]

(a) $x + y - 3z = 0$ (b) $3x + z = 0$
(c) $x - 4y + 7z = 0$ (d) $2x - y = 0$

220. From a point $P(\lambda, \lambda, \lambda)$, perpendiculars PQ and PR are drawn respectively on the lines $y = x, z = 1$ and $y = -x, z = -1$. If P is such that $\angle QPR$ is a right angle, then the possible value(s) of λ is (are)

[Single Option Correct Type Question, 2014 Adv.]

(a) $\sqrt{2}$ (b) 1 (c) -1 (d) $-\sqrt{2}$

221. Two lines $L_1 : x = 5, \frac{y}{3-\alpha} = \frac{z}{-2}$ and $L_2 : x = \alpha, \frac{y}{-1} = \frac{z}{2-\alpha}$ are coplanar. Then, α can take value(s)

[More than One Correct Type Question, 2013 Adv.]

(a) 1 (b) 2
(c) 3 (d) 4

222. A line l passing through the origin is perpendicular to the lines **[More than One Correct Type Question, 2013 Adv.]**

$l_1 : (3 + t)\hat{\mathbf{i}} + (-1 + 2t)\hat{\mathbf{j}} + (4 + 2t)\hat{\mathbf{k}}, -\infty < t < \infty$

$l_2 : (3 + 2s)\hat{\mathbf{i}} + (3 + 2s)\hat{\mathbf{j}} + (2 + s)\hat{\mathbf{k}}, -\infty < s < \infty$

Then, the coordinate(s) of the point(s) on l_2 at a distance of $\sqrt{17}$ from the point of intersection of l and l_1 is (are)

(a) $\left(\frac{7}{3}, \frac{7}{3}, \frac{5}{3}\right)$ (b) $(-1, -1, 0)$
(c) $(1, 1, 1)$ (d) $\left(\frac{7}{9}, \frac{7}{9}, \frac{8}{9}\right)$

223. Perpendicular are drawn from points on the line $\frac{x+2}{2} = \frac{y+1}{-1} = \frac{z}{3}$ to the plane $x + y + z = 3$. The feet of perpendiculars lie on the line

[Single Option Correct Type Question, 2013 Adv.]

(a) $\frac{x}{5} = \frac{y-1}{8} = \frac{z-2}{-13}$ (b) $\frac{x}{2} = \frac{y-1}{3} = \frac{z-2}{-5}$
(c) $\frac{x}{4} = \frac{y-1}{3} = \frac{z-2}{-7}$ (d) $\frac{x}{2} = \frac{y-1}{-7} = \frac{z-2}{5}$

224. Consider the lines $L_1 : \frac{x-1}{2} = \frac{y}{-1} = \frac{z+3}{1}$, $L_2 : \frac{x-4}{1} = \frac{y+3}{1} = \frac{z+3}{2}$ and the planes $P_1 : 7x + y + 2z = 3, P_2 : 3x + 5y - 6z = 4$. Let $ax + by + cz = d$ the equation of the plane passing through the point of intersection of lines L_1 and L_2 and perpendicular to planes P_1 and P_2.

Match List I with List II and select the correct answer using the code given below the lists.

[Single Option Correct Type Question, 2013 Adv.]

List I		List II	
P.	$a =$	1.	13
Q.	$b =$	2.	-3
R.	$c =$	3.	1
S.	$d =$	4.	-2

Codes

	P	Q	R	S		P	Q	R	S
(a)	3	2	4	1	(b)	1	3	4	2
(c)	3	2	1	4	(d)	2	4	1	3

225. If the straight lines $\frac{x-1}{2} = \frac{y+1}{k} = \frac{z}{2}$ and $\frac{x+1}{5} = \frac{y+1}{2} = \frac{z}{k}$ are coplanar, then the plane(s) containing these two lines is/are

[More than One Correct Type Question, IIT-JEE 2012]

(a) $y + 2z = -1$ (b) $y + z = -1$
(c) $y - z = -1$ (d) $y - 2z = -1$

226. If the distance between the plane $A\,x - 2y + z = d$ and the plane containing the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ is $\sqrt{6}$, then $|d|$ is equal to....

[Single Option Correct Type Question, IIT-JEE 2010]

Passage
(Q. Nos. 227-229)

Read the following passage and answer the questions.
Consider the lines

$$L_1 : \frac{x+1}{3} = \frac{y+2}{1} = \frac{z+1}{2}$$

$$L_2 : \frac{x-2}{1} = \frac{y+2}{2} = \frac{z-3}{3}$$

[Passage Type Question, IIT-JEE 2008]

227. The distance of the point (1, 1, 1) from the plane passing through the point $(-1, -2, -1)$ and whose normal is perpendicular to both the lines L_1 and L_2 , is
(a) $2/\sqrt{75}$ unit (b) $7/\sqrt{75}$ unit
(c) $13/\sqrt{75}$ units (d) $23/\sqrt{75}$ units

228. The shortest distance between L_1 and L_2 is
(a) 0 unit (b) $17/\sqrt{3}$ units
(c) $41/5\sqrt{3}$ units (d) $17/5\sqrt{3}$ units

229. The unit vector perpendicular to both L_1 and L_2 is
(a) $\frac{-\hat{\mathbf{i}} + 7\hat{\mathbf{j}} + 7\hat{\mathbf{k}}}{\sqrt{99}}$ (b) $\frac{-\hat{\mathbf{i}} - 7\hat{\mathbf{j}} + 5\hat{\mathbf{k}}}{5\sqrt{3}}$
(c) $\frac{-\hat{\mathbf{i}} + 7\hat{\mathbf{j}} + 5\hat{\mathbf{k}}}{5\sqrt{3}}$ (d) $\frac{7\hat{\mathbf{i}} - 7\hat{\mathbf{j}} - \hat{\mathbf{k}}}{\sqrt{99}}$

■ **Directions** (Q. Nos. 230-231) For the following questions, choose the correct answer from the codes (a), (b), (c) and (d) defined as follows.
(a) Statement I is true, Statement II is also true; Statement II is the correct explanation of Statement I
(b) Statement I is true, Statement II is also true; Statement II is not the correct explanation of Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

230. Consider three planes

$$P_1 : x - y + z = 1,\ P_2 : x + y - z = -1$$

and $P_3 : x - 3y + 3z = 2$

Let L_1, L_2, L_3 be the lines of intersection of the planes P_2 and P_3, P_3 and P_1, P_1 and P_2, respectively.

Statement I Atleast two of the lines L_1, L_2 and L_3 are non-parallel.

Statement II The three planes do not have a common point. **[Assertion and Reason Type Question, IIT-JEE 2008]**

231. Consider the planes

$$3x - 6y - 2z = 15 \text{ and } 2x + y - 2z = 5.$$

Statement I The parametric equations of the line of intersection of the given planes are $x = 3 + 14t$, $y = 1 + 2t$, $z = 15t$.

Statement II The vectors $14\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 15\hat{\mathbf{k}}$ is parallel to the line of intersection of the given planes.

[Assertion and Reason Type Question, IIT-JEE 2007]

232. Consider the following linear equations
$ax + by + cz = 0, bx + cy + az = 0, cx + ay + bz = 0$

[Matching Type Question, IIT-JEE 2007]

	Column I		Column II
A.	$a+b+c \neq 0$ and $a^2+b^2+c^2 = ab+bc+ca$	p.	The equations represent planes meeting only at a single point
B.	$a+b+c = 0$ and $a^2+b^2+c^2 \neq ab+bc+ca$	q.	The equations represent the line $x = y = z$
C.	$a+b+c \neq 0$ and $a^2+b^2+c^2 \neq ab+bc+ca$	r.	The equations represent identical planes
D.	$a+b+c = 0$ and $a^2+b^2+c^2 = ab+bc+ca$	r.	The equations represent the whole of the three-dimensional space

(ii) JEE Main and AIEEE

233. If the image of the point $P(1, -2, 3)$ in the plane $2x + 3y - 4z + 22 = 0$ measured parallel to the line $\frac{x}{1} = \frac{y}{4} = \frac{z}{5}$ is Q, then PQ is equal to **[2017 JEE Main]**
(a) $3\sqrt{5}$ (b) $2\sqrt{42}$
(c) $\sqrt{42}$ (d) $6\sqrt{5}$

234. The distance of the point (1, 3, –7) from the plane passing through the point $(1, -1, -1)$ having normal perpendicular to both the lines $\frac{x-1}{1} = \frac{y+2}{-2} = \frac{z-4}{3}$ and $\frac{x-2}{2} = \frac{y+1}{-1} = \frac{z+7}{-1}$, is **[2017 JEE Main]**
(a) $\frac{20}{\sqrt{74}}$ units (b) $\frac{10}{\sqrt{83}}$ units
(c) $\frac{5}{\sqrt{83}}$ units (d) $\frac{10}{\sqrt{74}}$ units

235. The distance of the point $(1, -5, 9)$ from the plane $x - y + z = 5$ measured along the line $x = y = z$ is **[2016 JEE Main]**
(a) $3\sqrt{10}$ (b) $10\sqrt{3}$
(c) $\frac{10}{\sqrt{3}}$ (d) $\frac{20}{3}$

236. If the line, $\frac{x-3}{2} = \frac{y+2}{-1} = \frac{z+4}{3}$ lies in the plane, $lx + my - z = 9$, then $l^2 + m^2$ is equal to **[2016 JEE Main]**
(a) 26 (b) 18 (c) 5 (d) 2

237. The distance of the point (1, 0, 2) from the point of intersection of the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x - y + z = 16$, is **[2015 JEE Main]**
(a) $2\sqrt{14}$ (b) 8
(c) $3\sqrt{21}$ (d) 13

238. The equation of the plane containing the line $2x-5y+z=3$, $x+y+4z=5$ and parallel to the plane $x+3y+6z=1$, is **[2015 JEE Main]**

(a) $2x+6y+12z=13$ (b) $x+3y+6z=-7$

(c) $x+3y+6z=7$ (d) $2x+6y+12z=-13$

239. The angle between the lines whose direction cosines satisfy the equations $l+m+n=0$ and $l^2=m^2+n^2$ is **[2014 JEE Main]**

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{6}$ (d) $\frac{\pi}{2}$

240. The image of the line $\frac{x-1}{3}=\frac{y-3}{1}=\frac{z-4}{-5}$ in the plane $2x-y+z+3=0$ is the line **[2014 JEE Main]**

(a) $\frac{x+3}{3}=\frac{y-5}{1}=\frac{z-2}{-5}$ (b) $\frac{x+3}{-3}=\frac{y-5}{-1}=\frac{z+2}{5}$

(c) $\frac{x-3}{3}=\frac{y+5}{1}=\frac{z-2}{-5}$ (d) $\frac{x-3}{-3}=\frac{y+5}{-1}=\frac{z-2}{5}$

241. Distance between two parallel planes $2x+y+2z=8$ and $4x+2y+4z+5=0$ is **[2013 JEE Main]**

(a) $\frac{3}{2}$ (b) $\frac{5}{2}$

(c) $\frac{7}{2}$ (d) $\frac{9}{2}$

242. If the lines

$$\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{-k} \text{ and } \frac{x-1}{k}=\frac{y-4}{2}=\frac{z-5}{1}$$

are coplanar, then k can have **[2013 JEE Main]**

(a) any value (b) exactly one value

(c) exactly two values (d) exactly three values

243. An equation of a plane parallel to the plane $x-2y+2z-5=0$ and at a unit distance from the origin is **[AIEEE 2012]**

(a) $x-2y+2z-3=0$ (b) $x-2y+2z+1=0$

(c) $x-2y+2z-1=0$ (d) $x-2y+2z+5=0$

244. If the line $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$ and $\frac{x-3}{1}=\frac{y-k}{2}=\frac{z}{1}$ intersect, then k is equal to **[AIEEE 2012]**

(a) -1 (b) $\frac{2}{9}$

(c) $\frac{9}{2}$ (d) 0

245. If the angle between the line $x=\frac{y-1}{2}=\frac{z-3}{\lambda}$ and the plane $x+2y+3z=4$ is $\cos^{-1}\left(\sqrt{\frac{5}{14}}\right)$, then λ equals **[AIEEE 2011]**

(a) $\frac{3}{2}$ (b) $\frac{2}{5}$

(c) $\frac{5}{3}$ (d) $\frac{2}{3}$

246. **Statement I** The point $A(1,0,7)$ is the mirror image of the point $B(1,6,3)$ in the line $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$.

Statement II The line $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$ bisects the line segment joining $A(1,0,7)$ and $B(1,6,3)$. **[AIEEE 2011]**

(a) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I

(b) Statement I is true, Statement II is false

(c) Statement I is false, Statement II is true

(d) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I

247. The length of the perpendicular drawn from the point $(3,-1,11)$ to the line $\frac{x}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ is **[AIEEE 2011]**

(a) $\sqrt{66}$ (b) $\sqrt{29}$

(c) $\sqrt{33}$ (d) $\sqrt{53}$

248. The distance of the point $(1,-5,9)$ from the plane $x-y+z=5$ measured along a straight line $x=y=z$, is **[AIEEE 2010]**

(a) $3\sqrt{5}$ (b) $10\sqrt{3}$

(c) $5\sqrt{3}$ (d) $3\sqrt{10}$

249. A line AB in three-dimensional space makes angles 45° and 120° with the positive X-axis and the positive Y-axis, respectively. If AB makes an acute angle θ with the positive Z-axis, then θ equals **[AIEEE 2010]**

(a) 30° (b) 45°

(c) 60° (d) 75°

250. **Statement I** The point $A(3, 1, 6)$ is the mirror image of the point $B(1, 3, 4)$ in the plane $x-y+z=5$.

Statement II The plane $x-y+z=5$ bisects the line segment joining $A(3, 1, 6)$ and $B(1, 3, 4)$. **[AIEEE 2010]**

(a) Statement I is true, Statement II is true; Statement II is the correct explanation of Statement I

(b) Statement I is true, Statement II is true; Statement II is not the correct explanation of Statement I

(c) Statement I is true, Statement II is false

(d) Statement I is false, Statement II is true

251. Let the line $\frac{x-2}{3}=\frac{y-1}{-5}=\frac{z+2}{2}$ lies in the plane $x+3y-\alpha z+\beta=0$. Then, (α,β) equals **[AIEEE 2009]**

(a) $(6,-17)$ (b) $(-6,7)$ (c) $(5,-15)$ (d) $(-5,15)$

252. The projections of a vector on the three coordinate axes are $6,-3,2$, respectively. The direction cosines of the vector are **[AIEEE 2009]**

(a) $6,-3,2$ (b) $\frac{6}{5},-\frac{3}{5},\frac{2}{5}$

(c) $\frac{6}{7},-\frac{3}{7},\frac{2}{7}$ (d) $-\frac{6}{7},-\frac{3}{7},\frac{2}{7}$

253. The line passing through the points $(5, 1\ a)$, and $(3,\ b,\ 1)$ crosses the YZ-plane at the point $\left(0,\ \frac{17}{2},\ \frac{-13}{2}\right)$. Then, **[AIEEE 2008]**

(a) $a = 8, b = 2$ (b) $a = 2, b = 8$

(c) $a = 4, b = 6$ (d) $a = 6, b = 4$

254. If the straight lines

$$\frac{x-1}{k} = \frac{y-2}{2} = \frac{z-3}{3} \text{ and } \frac{x-2}{3} = \frac{y-3}{k} = \frac{z-1}{2}$$

intersect at a point, then the integer k is equal to **[AIEEE 2008]**

(a) -2 (b) -5

(c) 5 (d) 2

255. Let L be the line of intersection of the planes $2x + 3y + z = 1$ and $x + 3y + 2z = 2$. If L makes an angle α with the positive X-axis, then $\cos\alpha$ equals **[AIEEE 2007]**

(a) $1/\sqrt{3}$ (b) $1/2$

(c) 1 (d) $1/\sqrt{2}$

256. If a line makes an angle of $\frac{\pi}{4}$ with the positive directions of each of X-axis and Y-axis, then the angle that the line makes with the positive direction of the Z-axis is **[AIEEE 2007]**

(a) $\pi/6$ (b) $\pi/3$

(c) $\pi/4$ (d) $\pi/2$

257. If $(2, 3, 5)$ is one end of a diameter of the sphere $x^2 + y^2 + z^2 - 6x - 12y - 2z + 20 = 0$, then the coordinates of the other end of the diameter are **[AIEEE 2007]**

(a) $(4, 9, -3)$ (b) $(4, -3, 3)$

(c) $(4, 3, 5)$ (d) $(4, 3, -3)$

258. The two lines $x = ay + b, z = cy + d$ and $x = a'y + b', z = c'y + d'$ are perpendicular to each other, if **[AIEEE 2006,2003]**

(a) $aa' + cc' = 1$ (b) $\frac{a}{a'} + \frac{c}{c'} = -1$

(c) $\frac{a}{a'} + \frac{c}{c'} = 1$ (d) $aa' + cc' = -1$

259. The image of the point $(-1, 3, 4)$ in the plane $x - 2y = 0$ is **[AIEEE 2006]**

(a) $(15, 11, 4)$ (b) $\left(-\frac{17}{3}, -\frac{19}{3}, 1\right)$

(c) $(8, 4, 4)$ (d) $\left(\frac{9}{5}, -\frac{13}{5}, 4\right)$

260. If the plane $2ax - 3ay + 4az + 6 = 0$ passes through the mid-point of the line joining the centres of the spheres $x^2 + y^2 + z^2 + 6x - 8y - 2z = 13$ and $x^2 + y^2 + z^2 - 10x + 4y - 2z = 8$, then a equals **[AIEEE 2005]**

(a) 2 (b) -2

(c) 1 (d) -1

261. If the angle θ between the line $\frac{x+1}{1} = \frac{y-1}{2} = \frac{z-2}{2}$ and the plane $2x - y + \sqrt{\lambda}z + 4 = 0$ is such that $\sin\theta = \frac{1}{3}$. The value of λ is **[AIEEE 2005]**

(a) $-\frac{4}{3}$ (b) $\frac{3}{4}$

(c) $-\frac{3}{5}$ (d) $\frac{5}{3}$

262. The angle between the lines $2x = 3y = -z$ and $6x = -y = -4z$ is **[AIEEE 2005]**

(a) $30°$ (b) $45°$

(c) $90°$ (d) $0°$

263. The plane $x + 2y - z = 4$ cuts the sphere $x^2 + y^2 + z^2 - x + z - 2 = 0$ in a circle of radius **[AIEEE 2005]**

(a) $\sqrt{2}$ (b) 2

(c) 1 (d) 3

Answers

Exercise for Session 1

1. 8 **2.** $\sqrt{3}\,|k|$ **4.** (1,2, 3), (3, 4, 5), (– 1, 6, – 7)
5. 5 **6.** (4, 5, 6) **7.** 90°
8. –1 **9.** 0 **11.** $\left(\frac{2}{3}, \frac{-1}{3}, \frac{2}{3}\right)$ or $\left(\frac{-2}{3}, \frac{1}{3}, \frac{-2}{3}\right)$
12. $\sqrt{14}$

Exercise for Session 2

1. $\mathbf{r} = 3\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}} + \lambda(2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 5\hat{\mathbf{k}})$
2. $\mathbf{r} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}} + \lambda(3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 5\hat{\mathbf{k}})$, $\frac{x-2}{3} = \frac{y+3}{4} = \frac{z-4}{-5}$
3. $\left(\frac{13}{5}, \frac{23}{5}, 0\right)$ **4.** $\cos^{-1}\left(\frac{19}{21}\right)$ **5.** (– 1, – 1, – 1)
6. $\frac{1}{\sqrt{3}}$ **7.** $\frac{41}{10}$
8. $\frac{x-1}{-1} = \frac{y}{2} = \frac{z-2}{-7}$ or $\frac{x-1}{1} = \frac{y}{-2} = \frac{z-2}{7}$
9. $\frac{x-1}{1} = \frac{y-2}{-2} = \frac{z+1}{1}$ **10.** (5, 8, 15)

Exercise for Session 3

1. $2x - y + 3z = 9$ **2.** $\pm\frac{1}{3}(2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}})$
3. $4x - 3y + 2z = 3$ **4.** $5x + 18y + 6z = 0$
5. $x - 5y - 2z + 6 = 0$, $3x - y + 4z - 2 = 0$
6. (– 3, 5, 2) **7.** $\sin^{-1}\left(\frac{15}{7\sqrt{11}}\right)$
8. $y + z = 2$ **9.** 13
10. $17x - 47y - 24z + 172 = 0$
11. $3x - y + 3z + 10 = 0$
12. $x - 2y + 2z = 0$ and $x - 2y + 2z - 6 = 0$
13. $25x + 17y + 62z = 238$ (acute angle bisector)
$x + 35y - 10z = 256$ (obtuse angle bisector)
14. $x - 8y + 4z = 7$
15. $2x + 2y + z = 9$

Exercise for Session 4

1. Centre (2, – 2, 0), Radius = $\sqrt{\frac{51}{2}}$
2. $x^2 + y^2 + z^2 - 4x + 4y - 4z + 9 = 0$, Centre (2, – 2, 2)
3. $x^2 + y^2 + z^2 - 2\sqrt{3}y - 1 = 0$
4. $9x^2 + 9y^2 + 9z^2 - 54x - 108y + 72z + 545 = 0$
5. $\lambda = \sqrt{3} \pm 3$
6. $2x^2 + 2y^2 + 2z^2 - 6x + 2y - 4z = 25$
7. (i) $\left(\frac{3}{2}, \frac{7}{2}, -2\right)$ (ii) $\frac{\sqrt{78}}{2}$ (iii) 5

Chapter Exercises

1. (b) **2.** (b) **3.** (b) **4.** (a) **5.** (a) **6.** (b)
7. (b) **8.** (b) **9.** (b) **10.** (c) **11.** (b) **12.** (a)
13. (c) **14.** (a) **15.** (d) **16.** (c) **17.** (a) **18.** (b)
19. (b) **20.** (a) **21.** (c) **22.** (c) **23.** (d) **24.** (a)
25. (a) **26.** (b) **27.** (d) **28.** (a) **29.** (d) **30.** (d)
31. (a) **32.** (d) **33.** (b) **34.** (a) **35.** (a) **36.** (c)
37. (b) **38.** (c) **39.** (c) **40.** (c) **41.** (d) **42.** (c)
43. (c) **44.** (c) **45.** (c) **46.** (a) **47.** (b) **48.** (a)
49. (d) **50.** (c) **51.** (c) **52.** (d) **53.** (b) **54.** (a)
55. (d) **56.** (b) **57.** (b) **58.** (a) **59.** (c) **60.** (a)
61. (b) **62.** (d) **63.** (a) **64.** (a) **65.** (b) **66.** (c)
67. (a) **68.** (c) **69.** (a) **70.** (c) **71.** (b) **72.** (b)
73. (a) **74.** (b) **75.** (c) **76.** (a) **77.** (d) **78.** (a)
79. (d) **80.** (a) **81.** (b) **82.** (c) **83.** (a) **84.** (c)
85. (c) **86.** (b) **87.** (c) **88.** (a) **89.** (c) **90.** (a)
91. (d) **92.** (a) **93.** (c) **94.** (a) **95.** (a) **96.** (a)
97. (b,d) **98.** (a,b,c) **99.** (b,c)
100. (b,e) **101.** (a,c,d) **102.** (a,b)
103. (b,c) **104.** (a,b) **105.** (a,b)
106. (c,d) **107.** (b,c) **108.** (a,c)
109. (b,d) **110.** (a,c) **111.** (a,b,c,d)
112. (c,d) **113.** (a,d) **114.** (a,b,c)
115. (a,b,d) **116.** (a,c,d) **117.** (b,c)
118. (b,d) **119.** (a,c) **120.** (a,b,c,d)
121. (a,b,c) **122.** (a,b,c) **123.** (a,b,c)
124. (a,b) **125.** (d) **126.** (d)
127. (a) **128.** (b) **129.** (b) **130.** (d) **131.** (a) **132.** (c)
133. (a) **134.** (a) **135.** (b) **136.** (a) **137.** (a) **138.** (d)
139. (d) **140.** (b) **141.** (b) **142.** (b) **143.** (d) **144.** (b)
145. (b) **146.** (c) **147.** (a) **148.** (d) **149.** (d) **150.** (b)
151. (c) **152.** (b) **153.** (c) **154.** (a) **155.** (c) **156.** (c)
157. (b) **158.** (a) **159.** (b) **160.** (c) **161.** (b) **162.** (a)
163. (c) **164.** (c) **165.** (c) **166.** (b) **167.** (b) **168.** (b)
169. (a) **170.** (b) **171.** (c) **172.** (b) **173.** (c) **174.** (d)
175. (A) $\to r$ (B) $\to q$, (C) $\to (q,s)$ (D) $\to (p,s)$
176. (A) $\to r$ (B) $\to q$, (C) $\to p$, (D) $\to s$
177. (A) $\to q$ (B) $\to r$, (C) $\to p$ (D) $\to s$
178. (A) $\to s$ (B) $\to q$, (C) $\to (r,t)$ (D) $\to (p,s)$
179. (A) $\to r$ (B) $\to q$, (C) $\to q$, (D) $\to (s)$
180. (A) $\to s$, (B) $\to r$, (C) $\to q$, (D) $\to (s)$
181. (A) $\to r$ (B) $\to p$, (C) $\to q$
182. (1) **183.** (7) **184.** (2) **185.** (3) **186.** (1) **187.** (3)
188. (2) **189.** (8) **190.** (2) **191.** (4) **192.** (3) **193.** (4)
194. (3) **195.** (7) **196.** (7) **197.** (7) **198.** (9) **199.** (6)
200. (4) **201.** (4) **202.** (4) **203.** (1) **204.** (9) **205.** (2)
206. (1) **207.** (1)
208. $\cos^{-1}\left(\frac{-1}{3}\right)$ **210.** (–2, – 1, 3) and $\left(\frac{56}{17}, \frac{43}{17}, \frac{111}{17}\right)$
211. $\mathbf{r} = (2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}) + \mu(2\hat{\mathbf{j}} + \hat{\mathbf{i}} - 2\hat{\mathbf{k}})$
212. $\frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$ and $\frac{x}{-1} = \frac{y}{1} = \frac{z}{-2}$
213. $\sqrt{\frac{1775}{28}}$ sq units **216.** $2x + y - 2z + 3 = 0$ and $x - 2y - 2z - 3 = 0$
217. $x - 8y + 4z - 7 = 0$
218. (b, c, d) **219.** (c) **220.** (c) **221.** (a,d) **222.** (b,d)
223. (d) **224.** (a) **225.** (b, c) **226.** (b) **227.** (c) **228.** (d)
229. (b) **230.** (d) **231.** (d)
232. (A) $\to$ (r); (B) $\to$ (q); (C) $\to$ (b); (D) $\to$ (s)
233. (b) **234.** (b) **235.** (b) **236.** (d) **237.** (d) **238.** (c)
239. (a) **240.** (a) **241.** (c) **242.** (c) **243.** (a) **244.** (c)
245. (d) **246.** (d) **247.** (d) **248.** (b) **249.** (c) **250.** (a)
251. (b) **252.** (c) **253.** (d) **254.** (b) **255.** (a) **256.** (d)
257. (a) **258.** (d) **259.** (d) **260.** (b) **261.** (d) **262.** (c)
263. (c)

Solutions

1. Suppose xy-plane divides the line joining the given points in the ratio $\lambda : 1$. The coordinate of the point of division are $\left(\frac{2\lambda-1}{\lambda+1}, \frac{-5\lambda+3}{\lambda+1}, \frac{6\lambda+4}{\lambda+1}\right)$.

This point lies on xy-plane.

$\therefore \quad \frac{6\lambda+4}{\lambda+1} = 0 \Rightarrow \lambda = -\frac{2}{3}$

Hence, xy-plane divides externally in the ratio $2 : 3$.

Aliter We know that the xy-plane divides the line segment joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio $-z_1 : z_2$.

Therefore, xy-plane divides the segment joining $(-1, 3, 4)$ and $(2, -5, 6)$ in the ratio $-4 : 6$ i.e. $2 : 3$ externally.

2. Suppose zx-plane divides the join of $(1, 2, 3)$ and $(4, 2, 1)$ in the ratio $\lambda : 1$. Then, the co-ordinates of the point of division are

$$\left(\frac{4\lambda+1}{\lambda+1}, \frac{2\lambda+2}{\lambda+1}, \frac{\lambda+3}{\lambda+1}\right)$$

This point lies on zx-plane

$\therefore \quad y$-coordinate $= 0 \Rightarrow \frac{2\lambda+2}{\lambda+1} = 0 \Rightarrow \lambda = -1$

Hence, zx-plane divides the join of $(1, 2, 3)$ and $(4, 2, 1)$ externally in the ratio $1 : 1$.

Aliter We know that the zx-plane divides the segment joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio $-y_1 : y_2$.

$\therefore zx$-plane divides the join of $(1, 2, 3)$ and $(4, 2, 1)$ in the ratio $-2 : 2$ i.e. $1 : 1$ externally.

3. Suppose R divides PQ in the ratio $\lambda : 1$. Then, the coordinates of R are

$$\left(\frac{5\lambda+3}{\lambda+1}, \frac{4\lambda+2}{\lambda+1}, \frac{-6\lambda-4}{\lambda+1}\right)$$

But, the coordinates of R are given as $(9, 8, -10)$.

$\therefore \quad \frac{5\lambda+3}{\lambda+1} = 9, \quad \frac{4\lambda+2}{\lambda+1} = 8$

and $\quad \frac{-6\lambda-4}{\lambda+1} = -10 \Rightarrow \lambda = -\frac{3}{2}$

Hence, R divides PQ externally in the ratio $3 : 2$.

4. D divides BC in the ratio $AB : AC$ i.e. $3 : 13$. Therefore, coordinates of D are

$$\left(\frac{3\times-9+13\times5}{3+13}, \frac{3\times6+13\times3}{3+13}, \frac{3\times-3+13\times2}{3+13}\right) \text{ or } \left(\frac{19}{8}, \frac{57}{16}, \frac{17}{16}\right).$$

5. Let l, m, n be the direction cosines of the given line. Then, as it makes an acute angle with x-axis. Therefore, $l > 0$. The line passes through $(6, -7, -1)$ and $(2, -3, 1)$. Therefore, its direction ratios are

$$6-2, -7+3, -1-1 \text{ or } 4, -4, -2 \text{ or } 2, -2, -1$$

Hence, direction cosines of the given line are $\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}$

6. We have, $\alpha = 45°$ and $\beta = 60°$

Suppose **OP** makes angle γ with OZ. Then,

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow \quad \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + \cos^2\gamma = 1$

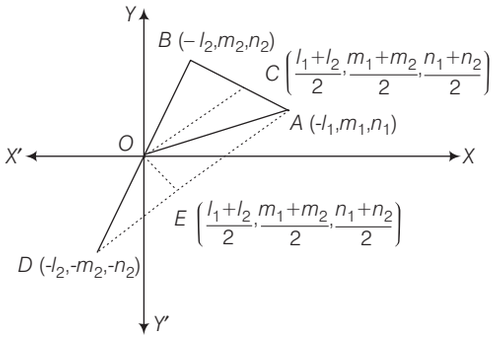
$\Rightarrow \quad \cos^2\gamma = \frac{1}{4} \Rightarrow \cos\gamma = \pm\frac{1}{2}$

$\Rightarrow \quad \gamma = 60°, 120°$

7. Let OA and OB be two lines with direction l_1, m_1, n_1 and l_2, m_2, n_2.

Let $OA = OB = 1$. Then, the coordinates of A and B are (l_1, m_1, n_1) and (l_2, m_2, n_2) respectively. Let OC be the bisector of $\angle AOB$. Then, C is the mid point of AB and so its coordinates are

$\left(\frac{l_1+l_2}{2}, \frac{m_1+m_2}{2}, \frac{n_1+n_2}{2}\right)$



$\therefore$ Direction ratios of OC are $\frac{l_1+l_2}{2}, \frac{m_1+m_2}{2}, \frac{n_1+n_2}{2}$

Now, $\quad OC = \sqrt{\left(\frac{l_1+l_2}{2}\right)^2 + \left(\frac{m_1+m_2}{2}\right) + \left(\frac{n_1+n_2}{2}\right)^2}$

$$OC = \frac{1}{2}\sqrt{(l_1^2+m_1^2+n_1^2) + (l_2^2+m_2^2+n_2^2) + 2(l_1l_2+m_1m_2+n_1n_2)}$$

$\Rightarrow \quad OC = \frac{1}{2}\sqrt{2+2\cos\theta}$ $[\because \cos\theta = l_1l_2 + m_1m_2 + n_1n_2]$

$\Rightarrow \quad OC = \frac{1}{2}\sqrt{2+2\cos\theta} = \cos\left(\frac{\theta}{2}\right)$

$\therefore$ Direction cosines of **OC** are

$$\frac{l_1+l_2}{2(OC)}, \frac{m_1+m_2}{2(OC)}, \frac{n_1+n_2}{2(OC)}$$

or, $\quad \frac{l_1+l_2}{2(\cos\frac{\theta}{2})}, \frac{m_1+m_2}{2(\cos\frac{\theta}{2})}, \frac{n_1+n_2}{(2\cos\frac{\theta}{2})}$

8. The given line is parallel to the vector $\mathbf{n} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$. The required plane passes through the point $(2, 3, 1)$ i.e., $2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and is perpendicular to the vector $\mathbf{n} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$. So, its equation is

$$\{\mathbf{r} - (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + \hat{\mathbf{k}})\} \cdot (\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) = 00000$$

$\Rightarrow \quad \mathbf{r} \cdot (\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) = 1$

9. Le the position vectors of the given points A and B be $\mathbf{a}$ and $\mathbf{b}$ respectively and that of the variable point P be $\mathbf{r}$. It is given that

$$PA^2 - PB^2 = k \qquad \text{(constant)}$$

$$\Rightarrow \quad |\mathbf{AP}|^2 - |\mathbf{BP}|^2 = k$$

$$\Rightarrow \quad |\mathbf{r}-\mathbf{a}|^2 - |\mathbf{r}-\mathbf{b}|^2 = k$$

$$\Rightarrow \quad \{|\mathbf{r}|^2 + |\mathbf{a}|^2 - 2\mathbf{r}\cdot\mathbf{a}\} - \{|\mathbf{r}|^2 + |\mathbf{b}|^2 - 2\mathbf{r}\cdot\mathbf{b}\} = k$$

$$\Rightarrow \quad 2\mathbf{r}\cdot(\mathbf{b}-\mathbf{a}) = k + |\mathbf{b}|^2 - |\mathbf{a}|^2$$

$$\Rightarrow \quad \mathbf{r}\cdot(\mathbf{b}-\mathbf{a}) = \lambda, \text{ where } \lambda = \frac{1}{2}\{k + |\mathbf{b}|^2 - |\mathbf{a}|^2\}$$

Clearly, it represents a plane.

10. The position vectors of two given points are $\mathbf{a} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and $\mathbf{b} = 3\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and the equation of the given plane is

$$\mathbf{r} = (5\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 7\hat{\mathbf{k}}) + 9 = 0$$

or $\quad \mathbf{r}.\mathbf{n} + d = 0$

We have $\quad \mathbf{a}.\mathbf{n} + d = (\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}})\cdot(5\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 7\hat{\mathbf{k}}) + 9$

$$= 5 - 2 - 21 + 9 < 0$$

and $\quad \mathbf{b}.\mathbf{n} + d = (3\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 3\hat{\mathbf{k}})\cdot(5\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 7\hat{\mathbf{k}}) + 9$

$$= 15 + 6 - 21 + 9 > 0$$

So, the points $\mathbf{a}$ and $\mathbf{b}$ are on the opposite sides of the plane.

11. The equation of a plane parallel to the plane

$$\mathbf{r}\cdot(4\hat{\mathbf{i}} - 12\hat{\mathbf{j}} - 3\hat{\mathbf{k}}) - 7 = 0 \text{ is,}$$

$$\mathbf{r}\cdot(4\hat{\mathbf{i}} - 12\hat{\mathbf{j}} - 3\hat{\mathbf{k}}) + \lambda = 0$$

This passes through $2\hat{\mathbf{i}} - \hat{\mathbf{j}} - 4\hat{\mathbf{k}}$

$$\therefore \quad (2\hat{\mathbf{i}} - \hat{\mathbf{j}} - 4\hat{\mathbf{k}})\cdot(4\hat{\mathbf{i}} - 12\hat{\mathbf{j}} - 3\hat{\mathbf{k}}) + \lambda = 0$$

$$\Rightarrow \quad 8 + 12 + 12 + \lambda = 0$$

$$\Rightarrow \quad \lambda = -32$$

So, the required plane is $\mathbf{r}\cdot(4\hat{\mathbf{i}} - 12\hat{\mathbf{j}} - 3\hat{\mathbf{k}}) - 32 = 0$

12. Equation of the plane containing L_1,

$$A(x-2) + B(y-1) + C(z+1) = 0$$

where $\quad A + 2C = 0;\ A + B - C = 0$

$$\Rightarrow \quad A = -2C,\ B = 3C,\ C = C$$

$\Rightarrow$ Plane is $-2(x-2) + 3(y-1) + z + 1 = 0$

or $\quad 2x - 3y - z - 2 = 0$

Hence, $\quad p = \left|\frac{-2}{\sqrt{14}}\right| = \sqrt{\frac{2}{7}}$

13. (1, 2, 3) satisfies the plane $x - 2y + z = 0$ and also

$$(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}})\cdot(\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}) = 0$$

Since the lines $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$ and

$\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ both satisfy $(0, 0, 0)$ and (1, 2, 3), both are same. Given line is obviously parallel to the plane $x - 2y + z = 6$

14. Vector $(3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}) \times (4\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}})$ is perpendicular to $2\hat{\mathbf{i}} - \hat{\mathbf{j}} + m\hat{\mathbf{k}}$

$$\Rightarrow \quad \begin{vmatrix} 3 & -2 & 1 \\ 4 & -3 & 4 \\ 2 & -1 & m \end{vmatrix} = 0 \ \Rightarrow \ m = -2$$

15. Let $A(1, 0, -1), B(-1, 2, 2)$

Direction ratios of segment AB are $< 2, -2, -3 >$.

$$\cos\theta = \frac{|2\times 1 + 3(-2) - 5(-3)|}{\sqrt{1+9+25}\sqrt{4+4+9}} = \frac{11}{\sqrt{17}\sqrt{35}} = \frac{11}{\sqrt{595}}$$

Length of projection $= (AB)\sin\theta$

$$= \sqrt{(2)^2 + (2)^2 + (3)^2} \times \sqrt{1 - \frac{121}{595}}$$

$$= \sqrt{17}\,\frac{\sqrt{474}}{\sqrt{17}\sqrt{35}} = \sqrt{\frac{474}{35}} \text{ units}$$

16. Let the point be A, B, C and D

The number of planes which have three points on one side and the fourth point on other side is 4. The number of planes which have two points on each side of the plane is 3.

$\Rightarrow$ Number of planes is 7.

17. Point A is $(a, b, c) \Rightarrow$ Points P, Q, R are $(a, b, -c)$, $(-a, b, c)$ and $(a, -b, c)$ respectively.

$\Rightarrow$ Centroid of triangle PQR is $\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right) \Rightarrow G \equiv \left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$

$\Rightarrow A, O, G$ are collinear $\Rightarrow$ area of triangle AOG is zero.

18. Let the equation of the plane be $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

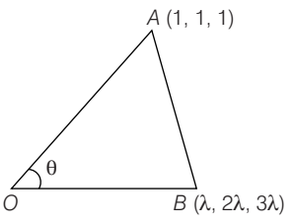
$$\Rightarrow \quad \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$$

$\Rightarrow$ Volume of tetrahedron $OABC = V = \frac{1}{6}(a\ b\ c)$

Now, $\quad (abc)^{1/3} \geq \frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} \geq 3$ (G.M $\geq$ H.M)

$$\Rightarrow \quad a\ b\ c \geq 27 \ \Rightarrow \ V \geq \frac{9}{2}$$

19.



Let any point of second line be $(\lambda, 2\lambda, 3\lambda)$

$$\cos\theta = \frac{6}{\sqrt{42}},\ \sin\theta = \frac{\sqrt{6}}{\sqrt{42}}$$

$$\Delta_{OAB} = \frac{1}{2}(OA)OB\sin\theta$$

$$= \frac{1}{2}\sqrt{3}\,\lambda\sqrt{14} \times \frac{\sqrt{6}}{\sqrt{42}} = \sqrt{6} \ \Rightarrow \ \lambda = 2$$

So, B is (2, 4, 6).

20. Equation of line $x + 2y + z - 1 + \lambda(-x + y - 2z - 2) = 0$ ...(i)

$$x + y - 2 + \mu(x + z - 2) = 0 \qquad \text{...(ii)}$$

(0, 0, 1) lies on it

$$\Rightarrow \quad \lambda = 0, \mu = -2$$

For point of intersection, $z = 0$ and solve (i) and (ii).

21. The planes are $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ and $\frac{x}{a'}+\frac{y}{b'}+\frac{z}{c'}=1$

Since the perpendicular distance of the origin on the planes is same, therefore

$$\left|\frac{-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}\right|=\left|\frac{-1}{\sqrt{\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}}}\right|$$

$$\Rightarrow \quad \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}-\frac{1}{a'^2}-\frac{1}{b'^2}-\frac{1}{c'^2}=0$$

22. Given one vertex $A(7, 2, 4)$ and line

$$\frac{x+6}{5}=\frac{y+10}{3}=\frac{z+14}{8}$$

General point on above line, $B\equiv(5\lambda-6, 3\lambda-10, 8\lambda-14)$

Direction ratios of line AB are $<5\lambda-13, 3\lambda-12, 8\lambda-18>$

Direction ratios of line BC are $<5, 3, 8>$

Since, angle between AB and BC is $\frac{\pi}{4}$.

$$\cos\frac{\pi}{4}=\frac{(5\lambda-3)5+3(3\lambda-12)+8(8\lambda-18)}{\sqrt{5^2+3^2+8^2}\cdot\sqrt{\begin{matrix}(5\lambda-13)^2\\+(3\lambda-12)^2\\+(8\lambda-18)^2\end{matrix}}}$$

Squaring and solving, we have $\lambda=3, 2$

Hence, equation of lines are $\frac{x-7}{2}=\frac{y-2}{-3}=\frac{z-4}{6}$

and $$\frac{x-7}{3}=\frac{y-2}{6}=\frac{z-4}{2}$$

23. L_1L_2 intersecting; L_2L_3 parallel; L_3L_1 skew.

24. $\lambda=\mu=1$ (point of intersection of two lines)

$\Rightarrow \mathbf{r}=\mathbf{a}+\mathbf{l}$ or $\mathbf{b}+\mathbf{m}$, i.e., $\mathbf{r}=\mathbf{i}+2\mathbf{j}+\mathbf{k}$

25. Both the lines pass through origin.

Line L_1 is parallel to the vector

$\mathbf{V}_1=(\cos\theta+\sqrt{3})i+(\sqrt{2}\sin\theta)j+(\cos\theta-\sqrt{3})k$ and L_2 is parallel to the vector

$$\mathbf{V}_2=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$$

$$\therefore \quad \cos\alpha=\frac{\mathbf{V}_1\cdot\mathbf{V}_2}{|\mathbf{V}_1||\mathbf{V}_2|}$$

$$=\frac{a(\cos\theta+\sqrt{3})+(b\sqrt{2})\sin\theta+c(\cos\theta-\sqrt{3})}{\sqrt{a^2+b^2+c^2}\sqrt{\begin{matrix}(\cos\theta+\sqrt{3})^2+2\sin^2\theta\\+(\cos\theta-\sqrt{3})^2\end{matrix}}}$$

$$=\frac{(a+c)\cos\theta+b\sqrt{2}\sin\theta+(a-c)\sqrt{3}}{\sqrt{a^2+b^2+c^2}\sqrt{2+6}}$$

In order that $\cos\alpha$ in independent of θ

$$a+c=0$$

and $$b=0$$

$$\therefore \quad \cos\alpha=\frac{2a\sqrt{3}}{a\sqrt{2}\cdot 2\sqrt{2}}=\frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \alpha=\frac{\pi}{6}$$

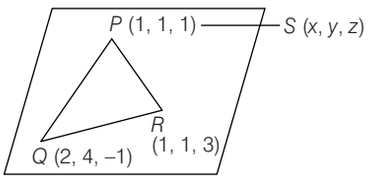
26. Given lines are skew lines and angle between them

$$=\cos^{-1}\left[\frac{12+3+0}{\sqrt{35}\sqrt{25}}\right]$$

$$=\cos^{-1}\frac{5}{\sqrt{35}}$$

27. Equation of plane

$$\begin{vmatrix}x-1 & y-1 & z-1\\ 1 & 3 & -2\\ 0 & 0 & 2\end{vmatrix}=0$$



$$\Rightarrow \quad 2(3x-3-y+1)=0$$

$$\Rightarrow \quad 3x-y=2$$

28. x-intercept $=\frac{q}{\hat{\mathbf{i}}\cdot\mathbf{n}}$

$$\therefore \quad x_1\mathbf{i}\cdot\mathbf{n}=q \Rightarrow x_1=\frac{q}{\mathbf{i}\cdot\mathbf{n}}$$

29. $P_1=4x+6y-7z-1=0$

$P_2=4x+6y-7z-2=0$

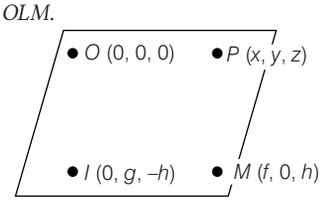
$$d=\frac{1}{\sqrt{16+36+49}}=\frac{1}{\sqrt{101}}$$

Hence, $\frac{101\times 102}{2}=5151$

30. x and z-intercept of the plane is 4 and it is parallel to y-axis, hence equation of the plane is $x+z=4$.

Its distance from $(0, 0, 0)$ is $2\sqrt{2}$.

31. Coordinate of $L(0, g, h)$ and $M(f, 0, h)$. Now, to find the equation of OLM.



$$\Rightarrow \quad \begin{vmatrix}x & y & z\\ 0 & g & h\\ f & 0 & h\end{vmatrix}=0$$

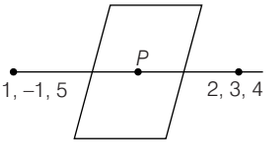
$$\Rightarrow (gh)x+(fh)y-(gf)z=0$$

or $$\frac{x}{f}+\frac{y}{g}-\frac{z}{h}=0$$

32. y-coordinate of P is zero.

$$\Rightarrow \quad =\frac{3\lambda+(-1)}{\lambda+1}$$

$$\Rightarrow \quad \lambda=\frac{1}{3}$$

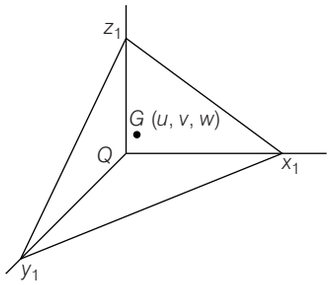


33. $\frac{x_1}{4} = u, \frac{y_1}{4} = v, \frac{z_1}{4} = w$

$$x_1 = 4u, y_1 = 4v, z_1 = 4w$$

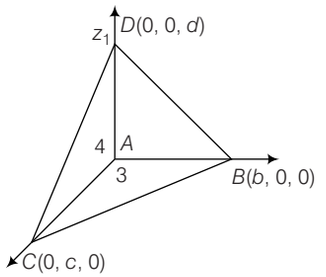
$$V = \frac{1}{6}\begin{vmatrix} 4u & 0 & 0 \\ 0 & 4v & 0 \\ 0 & 0 & 4w \end{vmatrix} = \left(\frac{64}{6}\right)uvw$$

$\therefore \quad 64\left(\frac{uvw}{6}\right) = 64k^3$



$xyz = 6k^3$

34.



Area of $\Delta BCD = \frac{1}{2}|\,\mathbf{BC} \times \mathbf{BD}\,|$

$$= \frac{1}{2}|\,(b\hat{\mathbf{i}} - c\,\hat{\mathbf{j}}) \times (b\hat{\mathbf{i}} - d\hat{\mathbf{k}})\,| = \frac{1}{2}|\,bd\,\hat{\mathbf{j}} + bc\,\hat{\mathbf{k}} + dc\,\hat{\mathbf{i}}\,|$$

$$= \frac{1}{2}\sqrt{b^2c^2 + c^2d^2 + d^2b^2} \qquad \ldots(i)$$

Now, $6 = bc, 8 = cd, 10 = bd$

$$b^2c^2 + c^2d^2 + d^2b^2 = 200$$

On substituting the value in Eq. (i), we get

$$A = \frac{1}{2}\sqrt{200} = 5\sqrt{2}$$

35. $\mathbf{r} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 0\hat{\mathbf{k}} + t(\hat{\mathbf{i}} + \hat{\mathbf{j}}) \times (\hat{\mathbf{j}} + \hat{\mathbf{k}})$

$$= (2, 1, 0) + t(\hat{\mathbf{k}} - \hat{\mathbf{j}} + \hat{\mathbf{i}}) = (2, 1, 0) + t(1, -1, 1)$$

36. Option (a), $-\frac{2}{3} = -\frac{2}{3} \neq \frac{2}{9}$

Option (b), $2 = 2 = 2 = 2$ identical

Option (c), $\frac{-2}{3} = \frac{-2}{3} = \frac{-2}{3} \neq \frac{3}{5}$

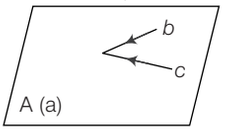
Option (d), $2 = 2 \neq -6$

37. $\left.\begin{matrix} x = 9;\ x = 1 \\ y = 8;\ y = 2 \\ z = 5;\ z = 3 \end{matrix}\right\}$

Edges of the cuboid are 8, 6 and 2.

38. Plane through a and parallel to two non-collinear vector

$(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{b} \times \mathbf{c}) = 0$ (takes dot with $\mathbf{b} \times \mathbf{c}$ both sides)



i.e., $(\mathbf{r} - (\hat{\mathbf{i}} - \hat{\mathbf{j}}) \cdot (5\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}) = 0$

$\Rightarrow \quad \mathbf{r} \cdot (5\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}) = 7$

39. Intersecting, if

$$\begin{vmatrix} 5 & 2 & 13 - p \\ 1 & 2 & 3 \\ -1 & 2 & -3 \end{vmatrix} = \begin{vmatrix} 5 & 2 & 13 - p \\ 0 & 2 & 0 \\ -1 & 2 & -3 \end{vmatrix}$$

$$-4(-15 + 13 - p) = 0$$

$$p = -2$$

Aliter

$$(\lambda + 2) = -(\mu + 3) \qquad \ldots(i)$$

$$2\lambda + p = 2\mu + 7 \qquad \ldots(ii)$$

$$3\lambda + 13 = p - 3\mu \qquad \ldots(iii)$$

From Eq. (i) $\mu = (-\lambda + 5)$

On putting in Eq. (ii), $2\lambda + 9 = -2(\lambda + 5) + 7$

$$\lambda = -3$$

Now, from Eq. (iii), $-9 + 13 = p + 6$

$$p = -2$$

40. $\mathbf{r} = \mathbf{a} + \gamma\mathbf{b} + \mu\mathbf{c}$

Taking dot with $\mathbf{b} \times \mathbf{c}$

$$[\mathbf{r}\ \mathbf{b}\ \mathbf{c}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \qquad [\text{where, } \mathbf{a} = (0, 1, 1)]$$

$\mathbf{b} = (1, -1, 1)$ and $\mathbf{c} = (2, -1, 0)$

$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \begin{vmatrix} 0 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & -1 & 0 \end{vmatrix} = 0 - (0 - 2) + 1(-1 + 2) = 3$$

and $[\mathbf{r}\ \mathbf{b}\ \mathbf{c}] = \begin{vmatrix} x & y & z \\ 1 & -1 & 1 \\ 2 & -1 & 0 \end{vmatrix} = x(0 + 1) - y(0 - 2) + z(-1 + 2)$

$$= x + 2y + z$$

Hence, equation of plane is $x + 2y + z = 3$

$\therefore \quad p = \left|\frac{-3}{\sqrt{6}}\right| = \sqrt{\frac{3}{2}}$

41. $\begin{vmatrix} 2 - a & 9 - 7 & 13 - (-2) \\ 1 & 2 & 3 \\ -1 & 2 & -3 \end{vmatrix} = 0$

$\Rightarrow \quad a = -3$

42. On (1, 2, 3) satisfies the plane $x - 2y + z = 0$ and also

$(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) \cdot (\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}) = 0$

Since, the lines $\frac{x - 1}{1} = \frac{y - 2}{2} = \frac{z - 3}{3}$

and $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ both satisfy $t + 1$ and $3t + 3$.

Hence, both are same.

Given line is obviously parallel to the plane $x - 2y + z = 6$

43. Infinite solution $\begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 0$

$\Rightarrow \quad a^2 + b^2 + c^2 = 1$

Note that 3 such planes can meet only at one point i.e. (0, 0, 0) or they may have the same line of intersection i.e., at infinite solution.

44. The given lines are coplanar, if

$$\begin{vmatrix} 2-1 & 3-4 & 4-5 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 1 & -1 & -1 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1-k \\ k & k+2 & 1+k \end{vmatrix} = 0$$

$\Rightarrow \quad 2(1 + k) - (k + 2)(1 - k) = 0$

if $\quad k^2 + 3k = 0 \Rightarrow k = 0$ or -3

45. Put $z = 0$ in the line given $x = 5$ and $y = 1$

$\Rightarrow \quad 5 \cdot 1 = c^2$

46. Equation of the line is $\dfrac{x-2}{1} = \dfrac{y+2}{-3} = \dfrac{z-5}{2} = \lambda$ ...(i)

Hence, any point on the line (i) can be taken as

$$x = \lambda + 2$$
$$y = -(3\lambda + 2)$$
$$z = (2\lambda + 5)$$

From some λ point lies on the plane

$$2x - 3y + 4z = 163 \quad \text{...(ii)}$$
$$2(\lambda + 2) + 3(3\lambda + 2) + 4(2\lambda + 5) = 163$$
$$19\lambda = 133$$

$\Rightarrow \quad \lambda = 7$

Hence, $\quad P \equiv (9, -23, 19)$

Also, Eq. (i) intersect YZ-plane i.e., $x = 0$

$$\lambda + 2 = 0$$

Hence, $\quad \lambda = -2$

$\therefore \quad Q(0, 4, 1)$

So, $\quad PQ = \sqrt{9^2 + 27^2 + 18^2}$

$$= 9\sqrt{1 + 3^2 + 2^2} = 9\sqrt{14}$$

$\Rightarrow \quad a = 9$ and $b = 14$

Hence, $a + b = 9 + 14 = 23$

47. Equation of the plane passing through the line of intersection of the first two planes is $\mathbf{r} \cdot (\mathbf{n}_1 + \lambda \mathbf{n}_2), = p_1 + \lambda p_2$. where λ is a parameter

Since, three planes have a common line of intersection the above equation should be identical with $\mathbf{r} \cdot \mathbf{n}_3 = p_3$ for some λ. That is for some λ,

$$\mathbf{n}_1 + \lambda \mathbf{n}_2 = k\mathbf{n}_3 \quad \text{...(i)}$$

and $\quad p_1 + \lambda p_2 = kp_3 \quad \text{...(ii)}$

From Eq. (i)

$$\mathbf{n}_1 \times \mathbf{n}_3 + \lambda \mathbf{n}_2 \times \mathbf{n}_3 = 0$$

and $\quad \mathbf{n}_1 \times \mathbf{n}_2 = k\mathbf{n}_3 \times \mathbf{n}_2 \quad \text{...(iii)}$

From Eq. (ii)

$$(p_1 + \lambda p_2)(\mathbf{n}_2 \times \mathbf{n}_3) = kp_3(\mathbf{n}_2 \times \mathbf{n}_3)$$
$$= p_3(\mathbf{n}_2 \times \mathbf{n}_1) \quad \text{...(iv)}$$
$$p_1(\mathbf{n}_2 \times \mathbf{n}_3) + p_2\lambda(\mathbf{n}_2 \times \mathbf{n}_3) + p_3(\mathbf{n}_1 + \mathbf{n}_2) = 0$$

$\Rightarrow \quad p_1(\mathbf{n}_2 \times \mathbf{n}_3) + p_2(\mathbf{n}_3 \times \mathbf{n}_1) + p_3(\mathbf{n}_1 \times \mathbf{n}_2) = 0$

[Using Eq. (iii)]

48. Equation of any plane through the intersection of $\mathbf{r} \cdot \mathbf{n}_1 = q_1$ and $\mathbf{r} \cdot \mathbf{n}_2 = q_2$ is of the form

$$\mathbf{r} \cdot \mathbf{n}_1 + \lambda \mathbf{r} \cdot \mathbf{n}_2 = q_1 + \lambda q_2 \quad \text{...(i)}$$

where λ is a parameter.

So, $\mathbf{n}_1 + \lambda \mathbf{n}_2$ is normal to the plane (i). Now, any plane parallel to the line of intersection of the planes $\mathbf{r} \cdot \mathbf{n}_3 = q_3$ and $\mathbf{r} \cdot \mathbf{n}_4 = q_4$ is of the form.

$\mathbf{r} \cdot (\mathbf{n}_3 + \mu \mathbf{n}_4) = q_3 + \mu q_4$, hence we must have

$\mathbf{n}_1 + \lambda \mathbf{n}_2 = k(\mathbf{n}_3 + \mu \mathbf{n}_4)$ for some k

$\Rightarrow \quad [\mathbf{n}_1 + \lambda \mathbf{n}_2] \cdot [\mathbf{n}_3 \times \mathbf{n}_4] = 0$

$\Rightarrow \quad [\mathbf{n}_1\ \mathbf{n}_3\ \mathbf{n}_4] + \lambda [\mathbf{n}_2\ \mathbf{n}_3\ \mathbf{n}_4] = 0$

$\Rightarrow \quad \lambda = -\dfrac{[\mathbf{n}_1\ \mathbf{n}_3\ \mathbf{n}_4]}{[\mathbf{n}_2\ \mathbf{n}_3\ \mathbf{n}_4]}$

On putting this value in Eq. (i), we have the equation of required plane as

$$\mathbf{r} \cdot \mathbf{n}_1 - q_1 = \frac{[\mathbf{n}_1\ \mathbf{n}_3\ \mathbf{n}_4]}{[\mathbf{n}_2\ \mathbf{n}_3\ \mathbf{n}_4]}(r \cdot \mathbf{n}_2 - q_2)$$

$\Rightarrow \quad [\mathbf{n}_2\ \mathbf{n}_3\ \mathbf{n}_4](\mathbf{r} \cdot \mathbf{n}_1 - q_1)$

$$= [\mathbf{n}_1\ \mathbf{n}_3\ \mathbf{n}_4](\mathbf{r} \cdot \mathbf{n}_2 - q_2)$$

49. Equation of line is

$$\mathbf{r} = \hat{\mathbf{i}} + 0\hat{\mathbf{j}} + \hat{\mathbf{k}} + t(\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}) \quad \text{...(i)}$$

Eq. (i) lies in $x + y + cz = d$

$\therefore \quad 1 + 0 + c = d$

$$1 + c = d$$

Also, $\quad 1 \cdot 1 + 1 \cdot 3 + c(-1) = 0$

$$c = 4$$

$\therefore \quad 1 + 4 = d \Rightarrow d = 5$

$\Rightarrow \quad (c + d) = 4 + 5 = 9$

50. Any point on $\dfrac{x-2}{2} = \dfrac{y+1}{4} = \dfrac{z-2}{12}$ can be

$(2r + 2, 4r - 1, 12r + 2)$

which lies on $x - y + z = 5$

$\therefore \quad (2r + 2) - (4r - 1) + 12r + 2 = 5$

$$r = 0$$

$\therefore$Point on the plane $\equiv (2, -1, 2)$

Distance between $(2, -1, 2)$ and $(-1, -5, -10)$

$$= \sqrt{(2+1)^2 + (-1+5)^2 + (2+10)^2}$$
$$= 13$$

51. $R(\mathbf{r})$ moves on $\mathbf{PQ}$.

$$\xleftarrow{\qquad R(\mathbf{r}) \qquad}$$
$$P(\mathbf{p}) \qquad\qquad Q(\mathbf{q})$$

52. $(\hat{\mathbf{i}} + \hat{\mathbf{j}}) \times (\hat{\mathbf{j}} + \hat{\mathbf{k}}) = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}} \Rightarrow$ Unit vector perpendicular as to the plane of $\hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\hat{\mathbf{j}} + \hat{\mathbf{k}}$ is $\dfrac{1}{\sqrt{3}}(\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}})$.

Similarly, other two unit vectors are

$\frac{1}{\sqrt{3}(\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}})}$ and $\frac{1}{\sqrt{3}}(-\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})$.

$$V=[\hat{\mathbf{n}}_1,\hat{\mathbf{n}}_2,\hat{\mathbf{n}}_3]=\frac{1}{3\sqrt{3}}\begin{vmatrix}1&-1&1\\1&1&-1\\-1&1&1\end{vmatrix}=\frac{4}{3\sqrt{3}}$$

Aliter

Let $\mathbf{a}=\hat{\mathbf{i}}+\hat{\mathbf{j}}$, $\mathbf{b}=\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $\mathbf{c}=\hat{\mathbf{k}}+\hat{\mathbf{i}}$.

Now, $[\mathbf{a}\times\mathbf{b},\mathbf{b}\times\mathbf{c},\mathbf{c}\times\mathbf{a}]=[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$

$$=\begin{vmatrix}1&1&0\\0&1&1\\1&0&1\end{vmatrix}^2=[1(1)-1(0-1)]^2=4$$

Hence, actual volume with unit vectors

$$=\frac{4}{|\mathbf{a}\times\mathbf{b}||\mathbf{b}\times\mathbf{c}||\mathbf{c}\times\mathbf{a}|}$$

Now, $|\mathbf{a}\times\mathbf{b}|=\sqrt{a^2b^2-(\mathbf{a}\cdot\mathbf{b})^2}=\sqrt{4-1}=\sqrt{3}$ etc.

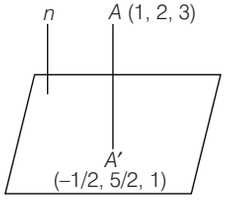
$$V_{\text{actual}}=\frac{4}{3\sqrt{3}}$$

53. $\mathbf{n}=3\hat{\mathbf{i}}-\hat{\mathbf{j}}+4\hat{\mathbf{k}}$

Line through A are parallel to $\mathbf{n}$ is

$$\mathbf{r}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}}+\lambda(3\hat{\mathbf{i}}-\hat{\mathbf{j}}+4\hat{\mathbf{k}})$$

$$=3\lambda+1,2-\lambda,3+4\lambda \qquad \ldots(i)$$



Hence, Eq. (i) must satisfy the plane

$$3x-y+4z=0$$

$$3(3\lambda+1)-(2-\lambda)+4(3+4\lambda)=0$$

$$26\lambda+13=0$$

$$\lambda=-\frac{1}{2}$$

Hence, A' is $\left(-\frac{1}{2},\frac{5}{2},1\right)$ which is the foot of the perpendicular from A on the given plane.

54. On solving $x+2y-4z=0$ and $2x-y+2z=0$, we get

$$\frac{x}{0}=\frac{y}{-10}=\frac{z}{-5}$$

One point P on line is $(0,-10t,-5t)$ and $Q\equiv(1,1,1)$

Direction ratio of $PQ\equiv(1,1+10t,1+5t)$

$$0-10-100t-5-25t=0$$

$$\Rightarrow \quad t=-\frac{3}{25}$$

$$\Rightarrow \quad P\equiv\left(0,\frac{6}{5},\frac{3}{5}\right)$$

Hence, required equation $\frac{x-1}{5}=\frac{1-y}{1}=\frac{z-1}{2}$.

55. Let $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ be the variable.

So that, $\left|\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}\right|=1$

Then, the coordinates of ΔABC are $A(a,0,0)$, $B(0,b,0)$ and $C(0,0,c)$.

The centroid of ΔABC is $\left(\frac{a}{3},\frac{b}{3},\frac{c}{3}\right)$

$$x=\frac{a}{3},\ y=\frac{b}{3},\ z=\frac{c}{3}$$

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=9$$

56. Direction ratios of AB are $1:1:1$.

Direction ratios of CD are $1:2:1$.

Angle between AB and CD,

$$\cos\theta=\frac{1\times1+1\times2+1\times1}{\sqrt{3}\sqrt{6}}=\frac{4}{3\sqrt{2}}$$

57. Equation of plane is $\frac{x}{1}+\frac{y}{2}+\frac{z}{3}=1$

$$\therefore \text{ Required distance } \left|\frac{\frac{1}{1}+\frac{2}{2}-\frac{3}{3}-1}{\sqrt{\left(\frac{1}{1}\right)^2+\left(\frac{1}{2}\right)^2+\left(\frac{1}{3}\right)^2}}\right|=0$$

58. Angle between the faces = Angle between the normals

$\mathbf{n}_1$ = Vector normal to face OAB

$$=\mathbf{OA}\times\mathbf{OB}=\begin{vmatrix}\hat{\mathbf{i}}&\hat{\mathbf{j}}&\hat{\mathbf{k}}\\1&2&1\\2&1&3\end{vmatrix}=5\hat{\mathbf{i}}-\hat{\mathbf{j}}-3\hat{\mathbf{k}}$$

$\mathbf{n}_2$ = Vector normal to face ABC

$$=\mathbf{AB}\times\mathbf{AC}=\begin{vmatrix}\hat{\mathbf{i}}&\hat{\mathbf{j}}&\hat{\mathbf{k}}\\1&-1&2\\-2&-1&1\end{vmatrix}$$

$$=\hat{\mathbf{i}}-5\hat{\mathbf{j}}-3\hat{\mathbf{k}}$$

Angle between faces $=\cos^{-1}\frac{\mathbf{n}_1\cdot\mathbf{n}_2}{[\mathbf{n}_1][\mathbf{n}_2]}=\cos^{-1}\left(\frac{19}{35}\right)$

59. $Q\equiv(1+2\lambda,2+3\lambda,3+4\lambda)$

Direction ratio of $PQ=2\lambda,3\lambda-1,4\lambda-1$

Now, $(2\lambda)2+(3\kappa-1)3+(4\lambda-1)4=0)$

$$29\lambda=7$$

$$\lambda=\frac{7}{29}$$

Direction ratio of line PQ is $(14,-8,-1)$.

60. Equation of the plane passing through three points A, B and C with position vector $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is

$$\mathbf{r}\cdot(\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a})=\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}$$

So that, if $\mathbf{a}, \mathbf{b}, \mathbf{c}$ represent the given vectors, then

$$\mathbf{a}\times\mathbf{b}+\mathbf{b}\times c+\mathbf{c}\times\mathbf{a}=\begin{vmatrix}\hat{\mathbf{i}}&\hat{\mathbf{j}}&\hat{\mathbf{k}}\\-6&3&2\\3&-2&4\end{vmatrix}+\begin{vmatrix}\hat{\mathbf{i}}&\hat{\mathbf{j}}&\hat{\mathbf{k}}\\3&-2&4\\5&7&3\end{vmatrix}+\begin{vmatrix}\hat{\mathbf{i}}&\hat{\mathbf{j}}&\hat{\mathbf{k}}\\5&7&3\\-6&3&2\end{vmatrix}$$

$$=-13\hat{\mathbf{i}}+13\hat{\mathbf{j}}-912\hat{\mathbf{k}}$$

and $\mathbf{a}\cdot\mathbf{a}\times\mathbf{c}=\begin{vmatrix}-6&3&2\\3&-2&4\\5&7&3\end{vmatrix}=299$

So, the required equation of the plane is
$\mathbf{r}.(-13\hat{\mathbf{i}}+13\hat{\mathbf{j}}-91\hat{\mathbf{k}})=\mathbf{299}$ or $\mathbf{r}\cdot(\hat{\mathbf{i}}-\hat{\mathbf{j}}+7\hat{\mathbf{k}})+23=0$

61. The volume of tetrahedron

$$=\frac{1}{6}(\mathbf{OA}\ \mathbf{OB}\ \mathbf{OC})=\frac{1}{6}\begin{vmatrix}1&0&0\\1&1&0\\0&1&1\end{vmatrix}=\frac{1}{6}\text{ units}$$

Area of the base $=\frac{1}{2}|(\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{i}})\times(\hat{\mathbf{j}}+\hat{\mathbf{k}}-\hat{\mathbf{i}})|$

$$=\frac{1}{2}|\hat{\mathbf{i}}+\hat{\mathbf{k}}|=\frac{1}{\sqrt{2}}$$

$$\text{Height}=\frac{3\times\text{Volume}}{\text{Area of base}}=\frac{3\sqrt{2}}{6}=\frac{1}{\sqrt{2}}$$

62. $x-y-z-4=0,\ x+y+2z-4=0$

Required plane is of the form

$(x-y-z-4)+\lambda(x+y+2z-4)=0$

Since, this plane is perpendicular to the plane $x-y-z-4=0$

$\therefore\quad 1+\lambda+(\lambda-1)(-1)+(2\lambda-1)(-1)=0,\ \lambda=\frac{3}{2}$

$\therefore$ Required plane $5x+y+4z=20$

63. Let l, m, n be the direction cosines of the normal to the plane on which lies the plane area A.

Then, $A_{xy}=$ projection of A on the xy-plane

$=A\cos\alpha$, where α is the angle between the plane and xy-plane.

$\therefore\qquad \cos\alpha=\frac{l.0+m.0+n.1}{1}$

Since, the normal to the xy-plane has direction cosines $(0, 0, 1)$

$\therefore\qquad A_{xy}=A_n$

Similarly, $\quad A_{yxz}=A_1$

$$A_{zx}=A_m$$

$\therefore A_{xy}^2+A_{yz}^2+A_{zx}^2=A^2$

64. Equation of the plane through $P(h, k, l)$ perpendicular to OP is

$xh+yk+zl=h^2+k^2+l^2=p^2$

where, $\qquad p^2=h^2+k^2+l^2$

$$\Rightarrow\qquad \frac{x}{\frac{p^2}{h}}+\frac{y}{\frac{p^2}{k}}+\frac{z}{\frac{p^2}{l}}=1$$

$$A_{xy}=\frac{1}{2}\cdot\frac{p^2}{h}\cdot\frac{p^2}{k},\ A_{yz}=\frac{1}{2}\cdot\frac{p^2}{k}\cdot\frac{p^2}{l},$$

$$A_{zx}=\frac{1}{2}\cdot\frac{p^2}{l}\cdot\frac{p^2}{h}$$

$$A=\sqrt{A_{xy}^2+A_{yz}^2+A_{zx}^2}$$

$$=\frac{p^4}{4}\sqrt{\frac{l^2+h^2+k^2}{h^2k^2l^2}}=\frac{p^4}{2}\sqrt{\frac{p^2}{h^2k^2l^2}}=\frac{p^5}{2hkl}$$

Hence $Ar(DABC)=\frac{p^5}{2hkl}$

65. Equation of the given plane can be written as

$$\frac{x}{20}+\frac{y}{15}+\frac{z}{-12}=1$$

which meets the co-ordinates axes in points $A(20, 0, 0)$, $B(0, 15, 0)$ and $C(0, 0, -12)$ and the co-ordinates of the origin are $(0, 0, 0)$.

$\therefore$ The volume of the tetrahedron $OABC$ is

$$\left|\frac{1}{6}\begin{vmatrix}20&0&0\\0&15&0\\0&0&-12\end{vmatrix}\right|=\left|\frac{1}{6}\times20\times15\times(-12)\right|=600$$

66. $l+m+n=0,\ l^2+m^2-n^2=0$

$$\Rightarrow\qquad l^2+m^2-(-l-m)^2=0$$

$$\Rightarrow\qquad 2lm=0 \text{ i.e., } l=0 \text{ or } m=0$$

If $l=0$, $m=-n$ and if $m=0$, $l=-n$

Since d.r.'s of the two lines are $0, 1, -1$ and $1, 0, -1$

$$\cos\theta=\frac{0\times1+1\times0+(-1)\times(-1)}{\sqrt{2}\sqrt{2}}=\frac{1}{2}$$

$$\Rightarrow\qquad \theta=\frac{\pi}{3}$$

67. $(\hat{\mathbf{i}}-\hat{\mathbf{j}}+4\hat{\mathbf{k}})\cdot(\hat{\mathbf{i}}+5\hat{\mathbf{j}}+\hat{\mathbf{k}})=0$

Therefore the line and the plane are parallel. A point on the line is $(2, -2, 3)$. Required distance is equal to distance of $(2, -2, 3)$ from the given plane $=\frac{|2+5(-2)+3-5|}{\sqrt{1^2+5^2+1^2}}=\frac{10}{\sqrt{27}}$

68. $\because$ Plane is perpendicular to the line

$\therefore$ Equation of plane is of the form $2x-y+2z+k=0$

$\because$ If passes through origin $\therefore\ k=0$

$$\therefore\qquad 2x-y+2z=0$$

69. $\mathbf{PQ}=\hat{\mathbf{i}}(-2-3\mu)+\hat{\mathbf{j}}(\mu-3)+\hat{\mathbf{k}}(5\mu-4)$

$\mathbf{PQ}$ is parallel to $x-4y+3z=1$

$\Rightarrow\ 1(-2-3\mu)-4(\mu-3)+3(5\mu-4)=0$

$$\Rightarrow\qquad \mu=\frac{1}{4}$$

70. Plane meets axes at $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.

Then area of $\Delta ABC=\frac{1}{2}|\mathbf{AB}\times\mathbf{AC}|$

$$=\frac{1}{2}|(-a\hat{i}+b\hat{j})\times(-a\hat{i}\times c\hat{k})|$$

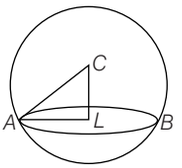
$$=\frac{1}{2}\sqrt{(a^2b^2+b^2c^2+c^2a^2}$$

71. Centre of the sphere is $(-1, 1, 2)$ and its radius

$$= \sqrt{1 + 1 + 4 + 19} = 5$$

CL, perpendicular distance of C from plane, is

$$\left| \frac{-1 + 2 + 4 + 7}{\sqrt{1 + 4 + 4}} \right| = 4$$



Now, $AL^2 = CA^2 - CL^2 = 25 - 16 = 9$

Hence, radius of the circle $= \sqrt{9} = 3$

72. Let $\mathbf{r} \times \mathbf{a} = \mathbf{b} \times \mathbf{a}$

$\Rightarrow$ $(\mathbf{r} - \mathbf{b}) \times \mathbf{a} = \mathbf{0}$

$\Rightarrow$ $\mathbf{r} = \mathbf{b} + t\mathbf{a}$

Similarly, other line $\mathbf{r} = \mathbf{a} + k\mathbf{b}$, where t and k are scalars.

Now $\mathbf{a} + k\mathbf{b} = \mathbf{b} + t\mathbf{a}$

$\Rightarrow$ $t = 1, k = 1$ (equation the coefficients of $\mathbf{a}$ and $\mathbf{b}$)

$\therefore$ $\mathbf{r} = \mathbf{a} + \mathbf{b} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{i}} - \hat{\mathbf{k}}$

$= 3\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$

i.e. $(3, 1, -1)$

73. Let the point P be (x, y, z), then the vector $x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ will lie on the line

$\Rightarrow$ $(x-1)\hat{\mathbf{i}} + (y-1)\hat{\mathbf{j}} + (z-1)\hat{\mathbf{k}}$

$= -\lambda\hat{\mathbf{i}} + \lambda\hat{\mathbf{j}} - \lambda\hat{\mathbf{k}}$

$\Rightarrow$ $x = 1 - \lambda, y = 1 + \lambda$ and $z = 1 - \lambda$

Now point P in nearest to the origin.

$\Rightarrow$ $D = (1-\lambda)^2 + (1+\lambda)^2 + (1-\lambda)^2$

$\Rightarrow$ $\frac{dD}{d\lambda} = -4(1-\lambda) + 2(1+\lambda) = 0$

$\Rightarrow$ $\lambda = \frac{1}{3}$

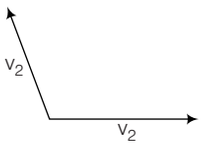
$\Rightarrow$ The point is $\left(\frac{2}{3}, \frac{4}{3}, \frac{2}{3}\right)$.

74. We have, $|\mathbf{v}_1| = 2, |\mathbf{v}_2| = \sqrt{2}$ and $|\mathbf{v}_3| = \sqrt{29}$

If θ is the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$, then

$2\sqrt{2}\cos\theta = -2$

$\Rightarrow$ $\cos\theta = -\frac{1}{\sqrt{2}} \Rightarrow \theta = 135°$



Let $\mathbf{v}_1 = 2\mathbf{i}$, $\mathbf{v}_2 = -\hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\mathbf{v}_3 = \alpha\hat{\mathbf{i}} + \beta\hat{\mathbf{j}} + \gamma\hat{\mathbf{k}}$

Since $\mathbf{v}_3 \cdot \mathbf{v}_1 = 6 = 2\alpha$

$\Rightarrow$ $\alpha = 3$

Also, $\mathbf{v}_3 \cdot \mathbf{v}_2 = -5 = -\alpha + \beta$

$\Rightarrow$ $\beta = -2$

and $\mathbf{v}_3 \cdot \mathbf{v}_2 = 29 = \alpha^2 + \beta^2 + \gamma^2$

$\Rightarrow$ $\gamma = \pm 4$

$\therefore$ $\mathbf{v}_3 = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} \pm 4\hat{\mathbf{k}}$

75. The given plane is $\mathbf{r} \cdot (5\mathbf{i} + 2\mathbf{j} - 7\mathbf{k}) = -9$

Length of the perpendicular from $\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ to it is

$$\frac{-9 - (\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}) \cdot (5\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 7\hat{\mathbf{k}})}{\sqrt{5 + 4 + 49}} = \frac{-9 - 5 + 2 + 21}{\sqrt{78}} = \frac{9}{\sqrt{78}}$$

Length of the perpendicular from $3\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$

$$\frac{-9 - (3\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) \cdot (5\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 7\hat{\mathbf{k}})}{\sqrt{78}} = \frac{-9 - 15 - 6 + 21}{\sqrt{78}} = -\frac{9}{\sqrt{78}}$$

Thus, the length of the two perpendiculars are equal in magnitude but opposite in sign. Hence, they are located on opposite side of the plane.

76. Let the position vector of A, B, C, D be $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ respectively.

Then, $AC^2 + BD^2 + AD^2 + BC^2$

$$\begin{aligned}
&= (\mathbf{c} - \mathbf{a}) \cdot (\mathbf{c} - \mathbf{a}) + (\mathbf{d} - \mathbf{b}) \cdot (\mathbf{d} - \mathbf{b}) \\
&\quad + (\mathbf{d} - \mathbf{a}) \cdot (\mathbf{d} - \mathbf{a}) + (\mathbf{c} - \mathbf{b}) \cdot (\mathbf{c} - \mathbf{b}) \\
&= |\mathbf{c}|^2 + |\mathbf{a}|^2 - 2\mathbf{a} \cdot \mathbf{c} + |\mathbf{d}|^2 + |\mathbf{b}|^2 \\
&\quad - 2\mathbf{d} \cdot \mathbf{b} + |\mathbf{d}|^2 + |\mathbf{a}|^2 - 2\mathbf{a} \cdot \mathbf{d} + |\mathbf{c}|^2 \\
&\quad + |\mathbf{b}|^2 - 2\mathbf{b} \cdot \mathbf{c} \\
&= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{c}|^2 + |\mathbf{d}|^2 \\
&\quad - 2\mathbf{c} \cdot \mathbf{d} + |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 + |\mathbf{d}|^2 \\
&\quad + 2\mathbf{a} \cdot \mathbf{b} + 2\mathbf{c} \cdot \mathbf{d} - 2\mathbf{a} \cdot \mathbf{c} - 2\mathbf{b} \cdot \mathbf{d} \\
&\quad - 2\mathbf{a} \cdot \mathbf{d} - 2\mathbf{d} \cdot \mathbf{c} \\
&= (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{d}) + (\mathbf{c} - \mathbf{d}) \cdot (\mathbf{c} - \mathbf{d}) + \\
&\quad + (\mathbf{a} + \mathbf{b} - \mathbf{c} - \mathbf{d}) \cdot (\mathbf{a} + \mathbf{b} - \mathbf{c} - \mathbf{d}) \\
&= AB^2 + CD^2 + (\mathbf{a} + \mathbf{b} - \mathbf{c} - \mathbf{d}) \cdot (\mathbf{a} + \mathbf{b} - \mathbf{c} - \mathbf{d}) \\
&\geq AB^2 + CD^2
\end{aligned}$$

77. If the given vectors are coplanar, then $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$

The set of equation

$$\begin{aligned}
x_1x + y_1y + z_1z &= 0 \\
x_2x + y_2y + z_2z &= 0 \\
x_3x + y_3y + z_3z &= 0
\end{aligned}$$

has a non-trivial solution.

Let the given set has a non-trivial solution x, y, z without loss of generality, we can assume that $x \geq y \geq z$.

For the given equation $x_1x + y_1y + z_1z = 0$, we have

$x_1x = -y_1y - z_1z$

$\Rightarrow$ $|x_1x| = |y_1y + z_1z| \leq |y_1y| + |z_1z|$

$\Rightarrow$ $|x_1x| \leq |y_1x| + |z_1x|$

$\Rightarrow$ $|x_1| < |y_1| + |z_1|$

Which is a contradiction to the given inequality.

$|x_1| > |y_1| + |z_1|$

Similarly, the other inequalities rule out the possibility of a non-trivial solution.

Therefore, the given equations have only a trivial solution.

So, the given vectors are non-coplanar.

78. The vectors $\mathbf{n}_1 \times \mathbf{n}_2$, $\mathbf{n}_2 \times \mathbf{n}_3$ and $\mathbf{n}_3 \times \mathbf{n}_1$ are non-coplanar vectors, so every vector can be written as

$$\mathbf{r} = a(\mathbf{n}_1 \times \mathbf{n}_2) + b(\mathbf{n}_2 \times \mathbf{n}_3) + c(\mathbf{n}_3 \times \mathbf{n}_1)$$

Substituting this value in $\mathbf{r} \cdot \mathbf{n}_1 = q_1, \mathbf{a}, \mathbf{b}$, we get

$$a(\mathbf{n}_1 \times \mathbf{n}_2) \cdot \mathbf{n}_1 + b(\mathbf{n}_2 \times \mathbf{n}_3) \cdot \mathbf{n}_1 + c(\mathbf{n}_3 \times \mathbf{n}_1) \cdot \mathbf{n}_1 = q_1$$

$$b(\mathbf{n}_1\ \mathbf{n}_2\ \mathbf{n}_3) = q_1 \ \Rightarrow\ b = \frac{q_1}{\mathbf{n}_1\ \mathbf{n}_2\ \mathbf{n}_3}$$

Since, the required point of intersection will have the position vector,

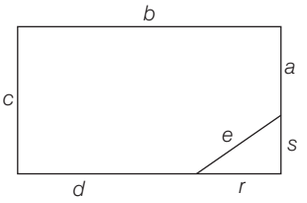
$$\mathbf{r} = \frac{1}{(\mathbf{n}_1\ \mathbf{n}_2\ \mathbf{n}_3)} \quad [q_3(\mathbf{n}_1 \times \mathbf{n}_2) + q_1(\mathbf{n}_2 \times \mathbf{n}_3) + q_2(\mathbf{n}_3 \times \mathbf{n}_1)]$$

79. Since $r^2 + s^2 = e^2$

$\Rightarrow e = 31$ or $e = 19$ is not possible.

Therefore, e equals 13, 20 or 25.

The possibility for triplet $\{r, s, e\}$ are {5, 12, 13}, {12, 16, 20}, {15, 20, 25}, {7, 24, 25}.



Since 16, 15 and 24 do not appear among any of pair wise differences of 13, 19, 20, 25, 31

$\Rightarrow a = 19, b = 25, c = 31, d = 20, e = 13$

Hence, required area = 745 sq units.

80. Point A is (a, b, c)

$\Rightarrow$ Points P, Q, R are $(a, b, -c), (-a, b, c)$ and $(a, -b, c)$ respectively.

$\Rightarrow$ Centroid of triangle PQR is $\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$

$\Rightarrow G \equiv \left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$

$\Rightarrow A, O, G$ are collinear $\Rightarrow$ Area of triangle AOG is zero.

81. Let line joining AB meet plane $2x + 3y + 5z = 1$ at P.

Let $$P = \left(\frac{\lambda + 1}{\lambda + 1}, \frac{-5\lambda}{\lambda + 1}, \frac{7\lambda - 3}{\lambda + 1}\right) \qquad \left[\frac{AP}{PB} = \lambda\right]$$

$$2.1 + 3\left(\frac{-5\lambda}{\lambda + 1}\right) + 5\left(\frac{7\lambda - 3}{\lambda + 1}\right) = 1$$

$$2(+1) - 15\lambda + 35\lambda - 15 = \lambda + 1$$

$$\Rightarrow \quad \lambda = \frac{2}{3}$$

$$\Rightarrow \quad P = (1, -2, 1)$$

$$\Rightarrow \quad AP = 2\sqrt{5}$$

82. $$Q = (r, r, -r)$$

PQ perpendicular $\frac{x}{1} = \frac{y}{1} = \frac{z}{-1}$

$$\therefore \quad (\alpha - r) \cdot 1 + (\beta - r) \cdot 1 + (\gamma + r)(-1) = 0$$

$$r = \frac{\alpha + \beta - \gamma}{3}$$

$$\therefore \quad PQ^2 = \frac{2}{3}\{\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta + \beta\gamma + \gamma\}$$

But $PQ = 2$

{perpendicular distance from $P(\alpha, \beta, \gamma)$ to plane $x + y + z = 0$}

$$\therefore \quad \frac{2}{3}(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta + \beta\gamma + \gamma\alpha) = 4\left\{\frac{\alpha + \beta + \gamma}{\sqrt{3}}\right\}^2$$

$$\therefore \quad \alpha^2 + \beta^2 + \gamma^2 + 5\alpha\beta + 3\beta\gamma + 3\gamma\alpha = 0$$

83. The cut $x = y$ separates the cube into points with $x < y$ and those with $x > y$.

$\therefore$ So, number of pieces equals to the number of ways of arrangements of x, y and z which is $3! = 6$.

Aliter Since in each coordinate there is inequality $x > y > z$.

So, number of pieces = number of ways of arranging $x, y, z = 6$

84. Here P and Q lie on the same side of XY plane

Image $P(1, 2, 3)$ on the XY plane is $P'(1, 2, -3)$

Reflected ray is $P'Q \Rightarrow \frac{x - 3}{2} = \frac{y - 2}{0} = \frac{z - 5}{8}$

$$\Rightarrow \quad \frac{x - 3}{1} = \frac{y - 2}{0} = \frac{z - 5}{4}$$

85. Let $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ be required plane.

Let the sphere be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

$d = 0$ if it passes through origin.

Also, $a = -2u, b = -2v, c = -2w$

and $$\frac{\alpha - \beta}{-2u} + \frac{\beta - \gamma}{-2v} + \frac{\gamma - \alpha}{-2w} = 1$$

Locus of centre $(-u, -v, -w)$ is $\Sigma(\alpha - \beta)yz = 2xyz$

86. On solving the given planes, the vertices are $O(0, 0, 0)$, $A(-a, a, a)$, $B(a, -a, a)$, $C(a, a, -a)$.

Consider the edges OA, BC whose equations are $\frac{x}{-1} = \frac{y}{1} = \frac{z}{1}$;

$$\frac{x - a}{0} = \frac{y + a}{1} = \frac{z - a}{-1}$$

Now, find S.D. between the lines.

87. The angle between the pair of planes is $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ is

$$\theta = \tan^{-1}\left(\frac{2\sqrt{f^2 + g^2 + h^2 - ab - bc - ca}}{a + b + c}\right)$$

88. $$2p + 2q + r = 6$$

$$\Rightarrow (2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}) \cdot (p\hat{\mathbf{i}} + q\hat{\mathbf{j}} + r\hat{\mathbf{k}}) = 6$$

$$(\mathbf{a} \cdot \mathbf{b})^2 \le |\mathbf{a}|^2 |\mathbf{b}|^2$$

$$\Rightarrow \quad 6^2 \le 9(p^2 + q^2 + r^2)$$

$$p^2 + q^2 + r^2 \le 4$$

89. Let $A(x_1, y_1, z_1), B(x_2, y_2, z_2), C(x_3, y_3, z_3), D(x_4, y_4, z_4)$ be the vertices of tetrahedron. If E is the centroid of face BCD and G is the centroid of $ABCD$ then $AG = 3/4(AE)$

$\therefore \quad K = 3/4.$

90. $y(x+y) + z(x+y) = 0$

$x + y = 0 \Rightarrow$ dr's are $\mathbf{b}_1 = (1, 1, 0)$

and $y + z = 0 \Rightarrow$ dr's are $\mathbf{b}_2 = (0, 1, 1)$

Now, $\mathbf{b}_1 \times \mathbf{b}_2 = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = (1, -1, 1)$

and $\mathbf{a}_1 - \mathbf{a}_1 = (1, 1, 1) - (0, 0, 0) = (1, 1, 1)$

$\therefore \quad SD = \dfrac{|(1, 1, 1) \times (1, -1, 1)|}{|(1, -1, 1)|} = \dfrac{|(-2, 0, 2)|}{|(1, -1, 1)|}$

$= \dfrac{\sqrt{4+0+4}}{\sqrt{1+1+1}} = \sqrt{\dfrac{8}{3}}$

91. $L_1 \| L_2$ Then required distance = distance between $(k_1, k_2, 0)$, $(k_3, k_4, 0)$

$= \sqrt{(k_1 - k_3)^2 + (k_2 - k_4)^2}$

92. Let $\mathbf{a} = l_1\hat{\mathbf{i}} + m_1\hat{\mathbf{j}} + n_1\hat{\mathbf{k}}$, $\mathbf{b} = l_2\hat{\mathbf{i}} + m_2\hat{\mathbf{j}} + n_2\hat{\mathbf{k}}$

and $\mathbf{c} = l_3\hat{\mathbf{i}} + m_3\hat{\mathbf{j}} + n_3\hat{\mathbf{k}}$

Given that **a, b, c** are three mutually perpendicular unit vectors.

Then, $p_1\hat{\mathbf{i}} + q_1\hat{\mathbf{j}} + r_1\hat{\mathbf{k}} = \mathbf{b} \times \mathbf{c} = \mathbf{a}$

($\because \mathbf{b} \times \mathbf{c}$ parallel to $\mathbf{a}$ and $\mathbf{b} \times \mathbf{c}$, $\mathbf{a}$ are unit vectors)

Similarly, $p_2\hat{\mathbf{i}} + q_2\hat{\mathbf{j}} + r_2\hat{\mathbf{k}} = \mathbf{c} \times \mathbf{a} = \mathbf{b}$

and $p_3\hat{\mathbf{i}} + q_3\hat{\mathbf{j}} + r_3\hat{\mathbf{k}} = \mathbf{a} \times \mathbf{b} = \mathbf{c}$

These vectors also mutually perpendicular unit vectors.

93. Let us suppose A be origin.

$\text{ar}(\Delta ABC)^2 + \text{ar}(\Delta ACD)^2 + \text{ar}(\Delta ABD)^2 = \text{ar}(\Delta BCD)^2$

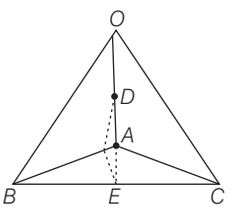
Hence the result follows.

94. $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be P.V. of $A, B, C, |\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = K$

$$\mathbf{OD} = \frac{\mathbf{a}}{2}\left|\ |\mathbf{OE}| = \frac{\mathbf{b}+\mathbf{c}}{2}\right|$$

Given, $|\mathbf{DE}| = 1 \Rightarrow \left|\dfrac{\mathbf{b}+\mathbf{c}-\mathbf{a}}{2}\right| = 1 \Rightarrow |\mathbf{b}+\mathbf{c}-\mathbf{a}| = 2$

$\Rightarrow \quad 3k^2 + 2k^2\left(\dfrac{1}{2} - \dfrac{1}{2} - \dfrac{1}{2}\right) = 4$



$\Rightarrow \quad K = \sqrt{2}$

$$\text{Volume} = \frac{1}{3}(\text{base area} \times \text{height}) = \frac{1}{3}\left(\frac{\sqrt{3}}{4}(k)^2 \times \sqrt{\frac{2}{3}}k\right) = \frac{1}{3}$$

95. The plane equation in the intercept forms is $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$.

Volume of tetrahedron $OABC$ is

$$v = \frac{abc}{6} = 64 \Rightarrow abc = 384$$

Foot of perpendicular from $(0, 0, 0)$ on this plane is

$$\frac{x}{1/a} = \frac{y}{1/b} = \frac{z}{1/c} = \frac{1}{\dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{1}{c^2}} = k$$

$\Rightarrow \quad x = \dfrac{k}{a}, y = \dfrac{k}{b}, z = \dfrac{k}{c}$

and $\dfrac{1}{k} = \dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{1}{c^2}$

$\Rightarrow \quad \dfrac{1}{k} = \dfrac{x^2 + y^2 + z^2}{k^2}$

$\Rightarrow \quad x^2 + y^2 + z^2 = k$

$\therefore (x^2 + y^2 + z^2)^3 = k^3 = abc \ xyz = 384 \ xyz$ is the required locus.

96. Let $A(x_1, y_1, z_1), B(x_2, y_2, z_2)$

$C(x_3, y_3, z_3), D(x_4, y_4, z_4)$

and the equation of the plane containing P, Q, R and S is $ax + by + cz + d = 0$ and $k_R = ax_r + by_r + cz_r + d$

$$\therefore \quad \frac{AP}{PB} \cdot \frac{BQ}{QC} \cdot \frac{CR}{RD} \cdot \frac{DS}{SA} = \frac{-K_1}{K_2} \cdot \frac{-K_2}{K_3} \cdot \frac{-K_3}{K_4} \cdot \frac{-K_4}{K_1} = 1$$

97. Let a, b and c be the DR's of the given line. Then,

we have $3a - b + c = 0$

$5a + b + 3c = 0$

On solving, we get $\dfrac{a}{1} = \dfrac{b}{1} = \dfrac{c}{-2}$

Again, suppose the given line intersect the plane $z = 0$ at $(x_1, y_1, 0)$, then $3x_1 - y_1 + 1 = 0$ and $5x_1 + y_1 = 0$

On solving, we get $x_1 = -\dfrac{1}{8}, y_1 = \dfrac{5}{8}$

Hence, the symmetrical form of the line is

$$\frac{x + \frac{1}{8}}{1} = \frac{y - \frac{5}{8}}{1} = \frac{z}{-2}$$

Equation of plane through $(2, 1, 4)$ is

$a(x-2) + b(y-1) + c(z-4) = 0$

when $a = 1, b = 1$ and $c = -21$

$x - 2 + y - 1 - 2(z - 4) = 0$

$x + y - 2z + 5 = 0$

98. Vector normal to the plane

$\mathbf{n} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$

$\mathbf{V}_x = \hat{\mathbf{i}}, \mathbf{V}_y = \hat{\mathbf{j}}, \mathbf{V}_z = \hat{\mathbf{k}}$

$$\cos(90° - \alpha) = \frac{\mathbf{V}_x \cdot \mathbf{n}}{|\mathbf{n}|}$$

$\Rightarrow \quad \sin\alpha = \dfrac{1}{\sqrt{3}}$

Similarly, $\sin\beta = \dfrac{1}{\sqrt{3}}$ and $\sin\gamma = \dfrac{1}{\sqrt{3}}$

Hence, $\sum \sin^2\alpha = 1$

and $\sum \cos^2\alpha = 2$

Also, plane is equally inclined with the coordinate axes.

Also, $A = \frac{1}{2}\sqrt{9^2 + 9^2 + 9^2} = \frac{9\sqrt{3}}{2}$

99. $2x - y - z - 2 + \lambda(x + y + z - 1) = 0$ satisfies (1, 1, 1)

$$2 - 1 - 1 - 2 + \lambda(3 - 1) = 0$$
$$\lambda = 1$$
$$x = 1 \quad \ldots(i)$$
$$x - y - z - 3 + \mu(2x + 4y - z - 4) = 0$$
$$-4 + \mu(1) = 0$$
$$\mu = 4$$
$$x - y - z - 3 + 4(2x + 4y - z - 4) = 0$$
$$9x + 15y - 5z - 19 = 0 \quad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$x - 1 = 0, 9x + 15y - 5z - 19 = 0$$

$a = 0$ and $c = 3b$

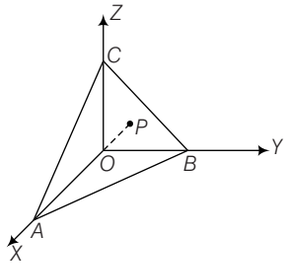
$$x - 1 = 0, \frac{y - 1}{1} = \frac{z - 1}{3}$$

100. $OP = \sqrt{h^2 + k^2 + l^2}$

Direction ratios of OP are $\left(\frac{h}{p}, \frac{k}{p}, \frac{l}{p}\right)$

Equation of plane is $\frac{hx}{p} + \frac{ky}{p} + \frac{lz}{p} = p$

$$A\left(\frac{p^2}{h}, 0, 0\right), B\left(0, \frac{p^2}{k}, 0\right), C\left(0, 0, \frac{p^2}{l}\right)$$



101. (a) Since, $\mathbf{n}\cdot\mathbf{a} = \mathbf{n}\cdot\mathbf{b} = \mathbf{n}\cdot\mathbf{c} = \mathbf{0}$

$\therefore$ **a**, **b** and **c** are coplanar

$\therefore$ $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = 0$

(b) $\cos^3 30° + \cos^2 45° + \cos^2\gamma = 1$

$\therefore$ $\sin^2 r = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$ which is not possible.

(c) Obvious

(d) $\mathbf{AB}\times\mathbf{BC}$ is perpendicular to the plane ABC.

$$\mathbf{AB}\times\mathbf{BC} = (\mathbf{OB} - \mathbf{OA})\times(\mathbf{OC} - \mathbf{OB})$$
$$= \mathbf{OB}\times\mathbf{OC} - \mathbf{OA}\times\mathbf{OC} + \mathbf{OA}\times\mathbf{OB}$$
$$= \mathbf{OA}\times\mathbf{OB} + \mathbf{OB}\times\mathbf{OC} + \mathbf{OA}\times\mathbf{OB}$$

i.e. $\mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{a}$ is perpendicular the plane ABC.

$(\mathbf{a} + \mathbf{b} + \mathbf{c})\times(\mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{a})$

$$= \{(\mathbf{a} + \mathbf{b} + \mathbf{c})\cdot\mathbf{c}\}\mathbf{b} - \{(\mathbf{a} + \mathbf{b} + \mathbf{c})\cdot\mathbf{a}\}\mathbf{b}$$
$$+ \{(\mathbf{a} + \mathbf{b} + \mathbf{c})\cdot\mathbf{c}\}\mathbf{b} - \{(\mathbf{a} + \mathbf{b} + \mathbf{c})\cdot\mathbf{b}\}\times\mathbf{c}$$
$$+ \{(\mathbf{a} + \mathbf{b} + \mathbf{c})\cdot\mathbf{a}\}\mathbf{c} - \{(\mathbf{a} + \mathbf{b} + \mathbf{c})\cdot\mathbf{a}\}\mathbf{a}$$
$$= (\mathbf{a} + \mathbf{b} + \mathbf{c})\cdot(\mathbf{b} - \mathbf{c})\mathbf{a} + (\mathbf{a} + \mathbf{b} + \mathbf{c})$$
$$\cdot(\mathbf{c} - \mathbf{a})\mathbf{b} + \{(\mathbf{a} + \mathbf{b} + \mathbf{c})\cdot(\mathbf{a} - \mathbf{b})\}\mathbf{c}$$
$$= (\mathbf{a}\cdot\mathbf{b} - \mathbf{a}\cdot\mathbf{c} + \mathbf{b}^2 - \mathbf{c}^2)\mathbf{a} + (\mathbf{b}\cdot\mathbf{c} - \mathbf{b}\cdot\mathbf{a}$$
$$+ (\mathbf{c}^2 - \mathbf{a}^2)\mathbf{b} + (\mathbf{a}^2 - \mathbf{b}^2 + \mathbf{c}\cdot\mathbf{a} - \mathbf{c} - \mathbf{b})\mathbf{c} = \mathbf{0}$$

Thus, the statement is true.

102. Let $A(\mathbf{a})$, $B(\mathbf{d})$, $C(\mathbf{c})$ and $D(\mathbf{d})$ be the vertices of a tetrahedron, then centroid of the tetrahedron is

$$\frac{\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}}{4}$$

centroid G_1 of the face BCD is $\frac{\mathbf{b} + \mathbf{c} + \mathbf{d}}{3}$

Now, centroid of the tetrahedron G_1 divides AG_1 in the ratio 3 : 1.

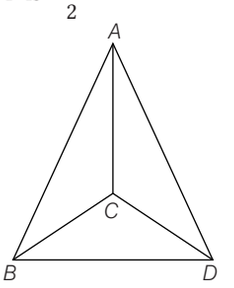
i.e. $\frac{3(\mathbf{b} + \mathbf{c} + \mathbf{d}) + \mathbf{a}}{3 + 1} = \frac{\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{a}}{4}$

$\therefore$ C lies on AC_1.

(b) The edges AB and CD. Let E be the mid point of AB and F be the mid point of CD

$\therefore$Positive vector of E is $\frac{\mathbf{a} + \mathbf{b}}{2}$

Positive vector of F is $\frac{\mathbf{c} + \mathbf{a}}{2}$



Mid-point of EF is $\frac{\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}}{4}$ which is the centroid of the tetrahedron $ABCD$.

103. Let the plane be $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$$\left|\frac{1}{\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2 + \left(\frac{1}{c}\right)^2}}\right| = 1 \quad \text{or} \quad \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = 1$$

The plane cuts the coordinates axes at $A(a, 0, 0)$, $B(0, b, 0)$, $c(0, 0, c)$. The centroid of ΔABC is

$$\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right) = (x, y, z) \quad \text{(let)}$$
$$x^{-2} + y^{-2} + z^{-2} = 9$$

or $$\frac{1}{9}\left\{\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right\}^{-1} = 0$$

104. When a line lies in a plane, then it is at right angles to the normal to the plane. Here, d.r's of the line are $< a, b, c >$ and altitude numbers of the plane are being taken as $< A, B, C >$. So, we must $aA + bB + cC = 0$.

105. For the given curve $z = 0$, therefore, the line and the curve meet where $\frac{x-2}{3} = \frac{y+1}{2} = \frac{0-1}{-1}$

i.e. where $\frac{x-2}{3} = 1, \frac{y+1}{z} = 1$ i.e., where $x = 5, y = 1$

So, the given line and the given curve meet in the point (5, 1, 0). Since, this point lies on the curve also, therefore, $5^2 + 1^2 = r^2$

$$\Rightarrow \quad r^2 = (\sqrt{26})^2$$

$$\Rightarrow \quad r = \pm\sqrt{26}$$

106. A vector coplanar with given vectors is $(1+\lambda)\hat{\mathbf{i}} + (\lambda-1)\hat{\mathbf{j}} + (1-\lambda)\hat{\mathbf{k}}$. Since it is equally inclined to the two given vectors

$$\frac{(1+\lambda)\hat{\mathbf{i}} + (\lambda-1)\hat{\mathbf{j}} + (1-\lambda)\hat{\mathbf{k}}}{|(1+\lambda)\hat{\mathbf{i}} + (\lambda-1)\hat{\mathbf{j}} + (1-\lambda)\hat{\mathbf{k}}|} \cdot \frac{(\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}})}{\sqrt{3}}$$

$$= \frac{(1+\lambda)\hat{\mathbf{i}} + (\lambda-1)\hat{\mathbf{j}} + (1-\lambda)\hat{\mathbf{k}}}{|(1+\lambda)\hat{\mathbf{i}} + (\lambda-1)\hat{\mathbf{j}} + (1-\lambda)\hat{\mathbf{k}}|} \cdot \frac{(\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}})}{\sqrt{3}}$$

$$\therefore \quad \lambda = 1$$

Required vector is $2\hat{\mathbf{i}}$ or $\hat{\mathbf{i}}$

107. The equation of the plane through (2, 3, −1) and perpendicular to the vector $3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$ is

$$3(x-2) + (-4)(y-3) + 7(z-(-1)) = 0$$

or
$$3x - 4y + 7z + 13 = 0$$

Distance of this plane from the origin

$$= \frac{|3\times 0 - 4\times 0 + 7\times 0 + 13|}{\sqrt{3^2 + (-4)^2 + 7^2}} = \frac{12}{\sqrt{74}}$$

108. Let A, B, C be $(\alpha, 0, 0), (0, \beta, 0)$ and $(0, 0, \gamma)$ then the plane ABC is $\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1$

Since it always passes through a, b, c

$$\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = 1 \qquad \ldots(i)$$

If p is (u, v, w) then $OP^2 = AP^2 = BP^2 = CP^2$

$$\Rightarrow \quad u^2 + v^2 + w^2 = (u-\alpha)^2 + v^2 + w^2$$

$$= \ldots \Rightarrow \alpha = \frac{u}{2}, \beta = \frac{v}{2}, \gamma = \frac{w}{2}$$

On putting α, β, γ in (i), we get

$$\frac{a}{u} + \frac{b}{v} + \frac{c}{w} = 2$$

$\Rightarrow$ Locus of (u, v, w) is $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2$

109. Normal of plane P_1 is

$$\mathbf{n}_1 = (2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) \times (4\hat{\mathbf{j}} - 3\hat{\mathbf{k}}) = -18\hat{\mathbf{i}}$$

Normal to plane P_2 is

$$\mathbf{n}_2 = (\hat{\mathbf{j}} - \hat{\mathbf{k}}) \times (3\hat{\mathbf{i}} + 3\hat{\mathbf{j}}) = 3\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$$

$\therefore$ $\mathbf{A}$ is parallel to $(\mathbf{n}_1 < \mathbf{n}_2) = \pm(-54\hat{\mathbf{j}} - 54\hat{\mathbf{k}})$

$\therefore$ Angle between $\mathbf{A}$ and $2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ is

$$\cos\theta = \pm\frac{(-54\hat{\mathbf{j}} + 54\hat{\mathbf{k}}) \cdot (2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}})}{54\sqrt{2.3}} = \pm\frac{1}{\sqrt{2}}$$

$$\theta = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}$$

110. Any plane through the second line is

$$2x + y + z - 1 + k(3x + y + 2z - 2) = 0$$

If this is parallel to the first line, then

$$(2 + 3\mathbf{k}) + (1 + \mathbf{k}) + (1 + 2\mathbf{k}) = 0$$

$$\Rightarrow \quad \mathbf{k} = -\frac{2}{3}$$

$\Rightarrow$ Plane is $2x + y + z - 1 - \frac{2}{3}(3x + y + 2z - 2) = 0$

or $y - z + 1 = 0$. The required SD must be distance of this plane from any point on the line $x = y = z$ say (1, 1, 1)

$$\Rightarrow \quad \text{SD} = \frac{1 - 1 + 1}{\sqrt{0^2 + 1^2 + (-1)^2}} = \frac{1}{\sqrt{2}}$$

111. $p_1 = \frac{4}{\sqrt{29}}, p_2 = \frac{1/2}{\sqrt{29}} = \frac{1}{2\sqrt{29}}, p_3 = \frac{8}{\sqrt{29}}$

For these values all the choices are easily verified.

112. Let the components of the line segment vector be a, b, c, then

$$a^2 + b^2 + c^2 = (63)^2 \qquad \ldots(i)$$

also
$$\frac{a}{3} = \frac{b}{-2} = \frac{c}{6} = \lambda \text{ (say) then}$$

$$a = 3\lambda, b = -2\lambda \quad \text{and} \quad c = 6\lambda$$

and from (i), we have

$$9\lambda^2 + 4\lambda^2 + 36\lambda^2 = (63)^2$$

$$\Rightarrow \quad 49\lambda^2 = (63)^2$$

$$\Rightarrow \quad \lambda = \pm\frac{63}{7} = \pm 9.$$

The required components are 27, −18, 54 or −27 , 18, −54

113. The given lines are coplanar if

$$0 = \begin{vmatrix} 2-1 & 3-4 & 4-5 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -1 & -1 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 1-k \\ k & k+2 & 1+k \end{vmatrix}$$

or if $2(1 + k) - (k + 2)(1 - k) = 0$

or if $k^2 + 3k = 0$ or if $k = 0, -3$.

114. Direction ratios of AB are $4-2, 5-3, 10-4$ or 1, 1, 3. So AB in parallel to the vector $\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and passes through B (2, 3, 4), the vector $2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$, its equation is $2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}} + \lambda(\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}})$

Similarly, BC passes through the points B (2, 3, 4) and its direction ratios are $2-1, 3-2, 4+1$ or 1, 1, 5.

So its cartesian equation is

$$\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{5}$$

Next, if D is (a, b, c), then since $ABCD$ is a parallelogram mid point of AC and BD is same. (diagonals of a parallelogram bisect each other)

$$\Rightarrow \quad \left(\frac{a+2}{2}, \frac{b+3}{2}, \frac{c+4}{2}\right) = \left(\frac{5}{2}, \frac{7}{2}, \frac{9}{2}\right)$$

$$\Rightarrow \quad (a, b, c) = (3, 4.5)$$

AB is not perpendicular to BC because

$$1 \times 1 + 1 \times 1 + 3 \times 5 \neq 0.$$

$ABCD$ is not a rectangle.

115. The coordinates of P where the line $x = y = z$ meets the plane $x + y + z = 1$ are $(1/3, 1/3, 1/3)$ and the co-ordinates of R and S where the line meets the sphere $x^2 + y^2 + z^2 = 1$ are $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ and $(-1/\sqrt{3}, -1/\sqrt{3}, -1/\sqrt{3})$

So that $\quad PR = \sqrt{3}\left|\left(\frac{1}{3} - \frac{1}{\sqrt{3}}\right)\right| = \sqrt{3}\left(\frac{1}{\sqrt{3}} - \frac{1}{3}\right)$

and $\quad PS = \sqrt{3}\left(\frac{1}{3} + \frac{1}{\sqrt{3}}\right)$

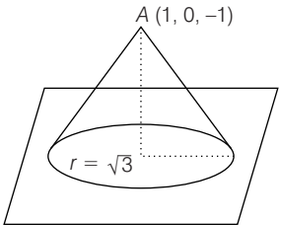
$$\Rightarrow \quad PR \cdot PS = 3\left(\frac{1}{3} - \frac{1}{9}\right) = \frac{2}{3}$$

$$PR + PS = 2$$

and $\quad RS = \sqrt{(1/\sqrt{3} + 1/\sqrt{3})^2 \times 3} = 2$

So that, $\quad PR + PS = RS$

116. The rod sweeps out the figure which is a cone.



The distance of point $A(1, 0, -1)$ from the plane is $\frac{|1 - 2 + 4|}{\sqrt{9}} = 1$ unit.

The slant height l of the cone is 2 units.

Then the radius of the base of the cone is

$$\sqrt{l^2 - 1} = \sqrt{4 - 1} = \sqrt{3}.$$

Hence, the volume of the cone is $\frac{\pi}{3}(\sqrt{3})^2 \cdot 1 = \pi$ cubic units.

Area of the circle on the plane which the rod traces is 3π.

Also, the centre of the circle is $Q(x, y, z)$.

Then $\quad \frac{x-1}{1} = \frac{y-0}{-2} = \frac{z+1}{2}$

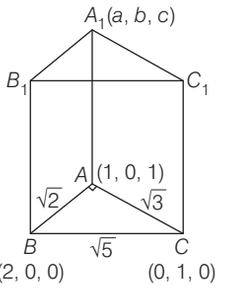
$$= \frac{-(1 - 0 - 2 + 4)}{l^2 + (-2)^2 + 2^2}$$

or $\quad Q(x, y, z) \equiv \left(\frac{2}{3}, \frac{2}{3}, \frac{-5}{3}\right).$

117. Observe that the lines L_1, L_2 and L_3 are parallel to the vector $(1, -1, -1)$.

Also, $\Delta = 0 = \Delta_1$ and $b_1c_2 - b_2c_1 \neq 0$

118. Volume = Area of base × height



$$3 = \frac{1}{2} \times \sqrt{2} \times \sqrt{3} \times h$$

$$h = \sqrt{6}$$

$$(A_1A)^2 = h^2 = 6$$

$$\mathbf{A_1A} \cdot \mathbf{AB} = 0$$

$$\mathbf{A_1A} \cdot \mathbf{AC} = 0$$

$$\mathbf{AA_1} \cdot \mathbf{BC} = 0$$

On solving, we get position vector of A_1 are $(0, -2, 0)$ or $(2, 2, 2)$.

119. Let the equation of plane be $lx + my + nz = 0$, where l, m, n be d.c's $\Rightarrow l^2 + m^2 + n^2 = 1 \rightarrow$ (i)

Given line $\frac{x-1}{2} = \frac{y+3}{-1} = \frac{z+1}{-2} \Rightarrow 2l - m - 2n = 0 \rightarrow$ (ii);

Also, $\quad \frac{l - 3m - n}{\sqrt{l^2 + m^2 + n^2}} = \frac{5}{3}$

$$\Rightarrow \quad l - 3m - n = \frac{5}{3} \rightarrow \text{(iii)}$$

Solving (i), (ii) and (iii), we get equation of plane as

$$x - 2y + 2z = 0 \quad \text{or} \quad 2x + 2y + z = 0.$$

120. (a) Height $= h = \sqrt{1 - \frac{1}{3}} = \sqrt{\frac{2}{3}}$

(b) Required distance $= \frac{1}{2}\left(\sqrt{\frac{2}{3}}\right) = \frac{1}{\sqrt{6}}$

(c) Angle $= \frac{\pi}{2}$

(d) Required distance $= \frac{3}{4}(h) = \frac{3}{4}\left(\sqrt{\frac{2}{3}}\right) = \sqrt{\frac{3}{8}}$

121. Let $OA = a$, $OB = b$, $OC = c$ then

$$a \cdot a + (b - c) \cdot (b - c) = b \cdot b + (c - a) \cdot (c - a)$$

$$\Rightarrow \quad -2b \cdot c = -2c \cdot a \Rightarrow (a \cdot b) \cdot c = 0$$

or $\quad BA \cdot OC = 0$

Hence, $AB \perp OC$ similarly $BC \perp OA$ and $CA \perp OB$.

122. Intersection of line with both the planes are the same

$$\Rightarrow \quad \frac{3}{3\beta^2 + 6(1 - 2\alpha) + 3} = \frac{-6}{6\alpha^2 + 6(1 - 2\beta) + 6}$$

$$\Rightarrow \quad 2(\beta - 1)^2 + 3(\alpha - 2)^2 = 0 \Rightarrow \alpha = 2, \beta = 1$$

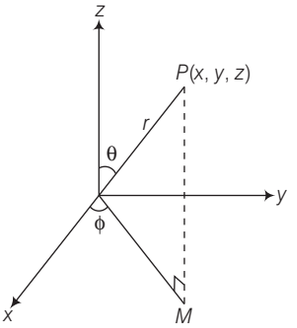
123. If P be (x, y, z) then from the figure,

$$x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$$

$$\Rightarrow \quad 1 = r\sin\theta\cos\phi, 2 = r\sin\theta\sin\phi, 3 = r\cos\theta$$

$\Rightarrow 1^2 + 2^2 + 3^2 = r^2 \Rightarrow r = \pm\sqrt{14}$

$\therefore \quad \sin\theta\cos\phi = \frac{1}{\sqrt{14}},\ \sin\theta\sin\phi = \frac{2}{\sqrt{14}},\ \cos\theta = \frac{3}{\sqrt{14}}$



(neglecting negative sign as θ and ϕ are acute)

$\therefore \qquad \frac{\sin\theta\sin\phi}{\sin\theta\cos\phi} = \frac{2}{1} \Rightarrow \tan\phi = 2$

Also, $\qquad \tan\theta = \frac{\sqrt{5}}{3}$

124. Let (l, m, n) be the direction cosines of the line perpendicular to the plane.

$\Rightarrow$ Equation of the plane $lx + my + nz = p$

$$\frac{x}{\left(\frac{p}{l}\right)} + \frac{y}{\left(\frac{p}{m}\right)} + \frac{z}{\left(\frac{p}{n}\right)} = 1$$

$A(p/l, 0, 0)$, $B(0, p/m, 0)$, $C(0, 0, p/n)$

Centroid of tetrahedron $OABC$ is

$$(x, y, z) = \left(\frac{p}{4l}, \frac{p}{4m}, \frac{p}{4n}\right)$$

Using, $\quad l^2 + m^2 + n^2 = 1$

$$x^2y^2 + y^2z^2 + z^2x^2 = \frac{16}{p^2}x^2y^2z^2$$

$$\Rightarrow \qquad \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{16}{p^2}$$

Put $\quad x = \frac{p}{4}\sec\alpha\sec\beta,\ y = \frac{p}{4}\sec\alpha\ \operatorname{cosec}\beta\ ,\ z = \frac{p}{4}\operatorname{cosec}\alpha$

$$\frac{1}{x} = \frac{4}{p}\cos\alpha\cos\beta,\ \frac{1}{y} = \frac{4}{p}\cos\alpha\sin\beta,\ \frac{1}{z} = \frac{4}{p}\sin\alpha$$

$$\left(\frac{1}{x}\right)^2 + \left(\frac{1}{y}\right)^2 + \left(\frac{1}{z}\right)^2 = \frac{16}{p^2}$$

$$[\cos^2\alpha\cos^2\beta + \cos^2\alpha\sin^2\beta + \sin^2\alpha]$$

$$= \frac{16}{p^2}[\cos^2\alpha + \sin^2\alpha] = \frac{16}{p^2}$$

125. Statement I $\mathbf{PA}\cdot\mathbf{PB} = 9 > 0$

$\therefore P$ is exterior to the sphere

Statement II is true (standard result)

126. **Statement II** $\mathbf{r}\times(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ x & y & z \\ 1 & 2 & -3 \end{vmatrix}$

$$\hat{\mathbf{i}}(-3y - 2z) - \hat{\mathbf{j}}(-3x - z) + \hat{\mathbf{k}}(2x - y)$$

$\therefore \quad -3y - 2z = 2, 3x + z = -1, 2x - y = 0$

i.e. $\quad -6x - 2z = 2, 3x + z = 1.$

$\therefore$ Straight line $2x - y = 0, 3x + z = -1$

Statement I

$$\mathbf{r}\times(2\mathbf{i} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ x & y & z \\ 2 & -1 & 3 \end{vmatrix}$$

$$= \hat{\mathbf{i}}(3y + z) - \hat{\mathbf{j}}(3x - 2z) + \hat{\mathbf{k}}(-x - 2y)$$

$\therefore \qquad 3y + z = 3, 3x - 2z = 0, -x - 2y = 1$

$$3x - 2(3 - 3y) = 0$$

$$3x + 6y = 6$$

$\Rightarrow \qquad x + 2y = 2$

Now, $\quad x + 2y = -1,\ x + 2y = 2$ are parallel planes.

$\therefore \mathbf{r}\times(2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}) = 3\hat{\mathbf{i}} + \hat{\mathbf{k}}$ is not a straight line.

127. $\sin\theta = \left|\frac{2 - 3 + 2}{\sqrt{4 + 9 + 4\sqrt{3}}}\right| = \frac{1}{\sqrt{51}}$

$\therefore$ Statement I is true, Statement II is true by definition.

128. **Statement I**

$$3y - 4z = 5 - 2k$$

$$-2y + 4z = 7 - 3k$$

$\therefore\ x = k,\ y = 12 - 5k,\ z = \frac{31 - 13k}{4}$ is a point on the line for all real value of k.

Statement I is true.

Statement II Direction ratios of the straight line are $< bc' - kbc,\ kac - ac',\ 0 >$ direction ratios of normal to be plane $< 0, 0, 1 >$

Now, $0\times(bc' - kbc) + 0\times(kac - ac') + 1\times 0 = 0$

$\therefore$ The straight line is parallel to the plane.

Statement II is the true but does not explain Statement I.

129. Equation of the plane, perpendicular to the plane P and containing line L is $8x + y - 7z = 4$.

130. L_1 and L_2 are obviously not parallel.

Consider the determinent

$$D = \begin{vmatrix} 2 & -4 & 1 \\ 2 & 4 & -3 \\ 1 & 3 & 2 \end{vmatrix}$$

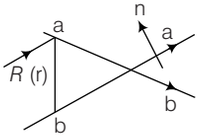
$$= 2(8 + 9) + 4(4 + 3) + 1(6 - 4)$$

$$= 34 + 28 + 2$$

$D \neq 0 \Rightarrow$ Skew

Hence, Statement I is false.

131. $\mathbf{n} = \mathbf{a}\times\mathbf{b}$. Equation of the plane



$$(\mathbf{r} - \mathbf{a})\cdot(\mathbf{a}\times\mathbf{b}) = 0$$

$$[\mathbf{r}\ \mathbf{a}\ \mathbf{b}] = 0$$

132. Statement II is not true because image of P in a plane is a point M such that PM is perpendicular to the plane and the mid-point of PM lies on the plane.

The point A, B, C are respectively $(-a, b, c), (a, -b, c)$ and $(a, b, -c)$

which lie on the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ and thus Statement I is true.

133. In Statement II, let $\mathbf{r}$ be the position vector of the point on the locus, then

$$|\mathbf{r}-\mathbf{a}|=|\mathbf{r}-\mathbf{b}| \Rightarrow (\mathbf{r}-\mathbf{a})^2=(\mathbf{r}-\mathbf{b})^2$$

$$\Rightarrow |\mathbf{r}|^2+|\mathbf{a}|^2-2\mathbf{r}\cdot\mathbf{a}=|\mathbf{r}|^2+|\mathbf{b}|^2-2\mathbf{r}\cdot\mathbf{b}$$

$$\Rightarrow 2\mathbf{r}\cdot(\mathbf{a}-\mathbf{b})+|\mathbf{b}|^2-|\mathbf{a}|^2=0$$

$$\Rightarrow 2\mathbf{r}\cdot(\mathbf{a}-\mathbf{b})+(\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}+\mathbf{a})=0$$

$$\Rightarrow \left(\mathbf{r}-\frac{\mathbf{a}+\mathbf{b}}{2}\right)\cdot(\mathbf{a}-\mathbf{b})=0$$

Showing that Statement II is true using it for Statement I.

we get the required locus as

$$\left[\mathbf{r}-\frac{3\hat{\mathbf{i}}-2\hat{\mathbf{j}}+5\hat{\mathbf{k}}+\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}}}{2}\right]\cdot(3\hat{\mathbf{i}}-2\hat{\mathbf{j}}+5\hat{\mathbf{k}}-(\hat{\mathbf{i}}+2\hat{\mathbf{j}}-\hat{\mathbf{k}}))=0$$

$$\Rightarrow [\mathbf{r}-(2\hat{\mathbf{i}}+2\hat{\mathbf{k}})\cdot(2\hat{\mathbf{i}}-4\hat{\mathbf{j}}+6\hat{\mathbf{k}})]=0$$

$$\Rightarrow \mathbf{r}\cdot(\hat{\mathbf{i}}-2\hat{\mathbf{j}}+3\hat{\mathbf{k}})=2\times1-0\times2+2\times3=8$$

and thus Statement I is also true.

134. Since $\mathbf{a}$ and $\mathbf{c}$ are non-collinear. Equating the coefficients of $\mathbf{a}$ and $\mathbf{c}$ in the two values of $\mathbf{r}$ we get.

$$6-\lambda=1+\mu, 2\lambda-1=3\mu-1 \Rightarrow \lambda=3, \mu=2$$

So there exist values for λ and μ such that the two values of $\mathbf{r}$ are same showing that the lines intersect and hence they are coplanar. Thus, statement I and statement II both are true and the first follows from the second.

135. Since, $\begin{vmatrix}1-2 & 0+1 & -1-0\\ 1 & -1 & 1\\ 1 & 2 & 3\end{vmatrix}=\begin{vmatrix}-1 & 1 & -1\\ 1 & -1 & 1\\ 1 & 2 & 3\end{vmatrix}=0$

The lines in Statement I are coplanar and equation of the plane containing them is

$$\begin{vmatrix}x-1 & y & z+1\\ 1 & -1 & 1\\ 1 & 2 & 3\end{vmatrix}=-(5x+2y-3z-8)=0$$

So Statement I is true.

Also, Statement II is true because $\frac{1}{3}=\frac{2}{6}=\frac{3}{9}$ and $1+2-3=0$

But does not lead to Statement I.

136. Any point on the first line is $(2x_1+1, x_1-3, -3x_1+2)$.

Any point on the second line is $(y_1+2, -3y_1+1, 2y_1-3)$

If two lines are coplanar, then $2x_1-y_1=1$, $x_1+3y_1=4$ and $3x_1+2y_1=5$ are consistent.

137. The direction cosines of segment OA are $\frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}$ and $\frac{-3}{\sqrt{14}}$.

$OA=\sqrt{14}$

This means OA will be normal to the plane and the equation of the plane is $2x+y-3z=14$.

138. If l, m, n denote the direction ratios of L_1 and $l+m-n=0$ and $l-3m+3n=0 \Rightarrow l=0, m=n$

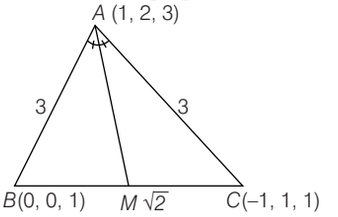
$\Rightarrow$ direction ratios of L_1 are 0, 1, 1 similarly for L_2 and L_3, we find that the direction ratios of both are 0, 1, 1 showing that L_1, L_2, L_3 are parallel, thus Statement I is False.

Statement II is True, because solving the given equation we get $x=0, y-z=-1$ and $y-z=\frac{-2}{3}$ which is not possible.

Solution (Q. Nos. 139-142)

139. Here, ΔABC is an isosceles with $AB=AC$

So, internal bisector of A is perpendicular to BC.



$\Delta AMB \cong \Delta AMC$ (RHS rule)

M is mid-point of BC.

So, $M \equiv \left(\frac{-1}{2}, \frac{-1}{2}, 1\right)$

$\therefore$Equation of internal bisector through A to side BC is

$$\mathbf{r}=(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}})+\mu\left(\frac{3}{2}\hat{\mathbf{i}}+\frac{3}{2}\hat{\mathbf{j}}+2\hat{\mathbf{k}}\right)$$

$$\Rightarrow \mathbf{r}=(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}})+\mu(3\hat{\mathbf{i}}+3\hat{\mathbf{j}}+4\hat{\mathbf{k}})$$

Aliter Equation of BC is $\mathbf{r}=\hat{\mathbf{k}}+\lambda(\hat{\mathbf{i}}-\hat{\mathbf{j}})$

Let position vector of M on BC be $\mathbf{r}$.

Now, $\mathbf{AM}$ = Position vector of M − Position vector of A

$$=(\lambda\hat{\mathbf{i}}-\lambda\hat{\mathbf{j}}+\hat{\mathbf{k}})-(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}})$$

$$=(\lambda-1)\hat{\mathbf{i}}-(\lambda+2)\hat{\mathbf{j}}-2\hat{\mathbf{k}}$$

Since, $\mathbf{AM}\cdot(\hat{\mathbf{i}}-\hat{\mathbf{j}})=0 \Rightarrow \lambda=\frac{-1}{2}$

Position vector of point, $M=\frac{-1}{2}\hat{\mathbf{i}}+\frac{1}{2}\hat{\mathbf{j}}+\hat{\mathbf{k}}$

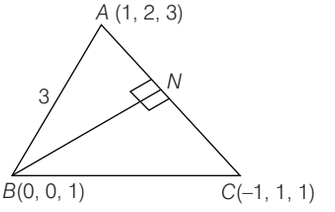
Equation of internal bisector through A to the side BC is

$$\mathbf{r}=(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}})+\mu\left(\frac{3}{2}\hat{\mathbf{i}}+\frac{3}{2}\hat{\mathbf{j}}+2\hat{\mathbf{k}}\right)$$

$$\Rightarrow \mathbf{r}=(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}})+\mu(3\hat{\mathbf{i}}+3\hat{\mathbf{j}}+4\hat{\mathbf{k}})$$

140. Now, equation of AC is

$$\mathbf{r}=(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}})+\lambda(2\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}})$$



Also, $\mathbf{BM}=(1+2\lambda)\hat{\mathbf{i}}+(2+\lambda)\hat{\mathbf{j}}+2(1+\lambda)\hat{\mathbf{k}}$

$$\mathbf{BM}\cdot(2\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}})=0$$

$\Rightarrow \quad 2(1+2\lambda)+(2+\lambda)+4(1+\lambda)=0$

$\Rightarrow \quad \lambda=\frac{-8}{9}$

Position vector of $N=\frac{-7\hat{\mathbf{i}}+10\hat{\mathbf{j}}+11\hat{\mathbf{k}}}{9}$

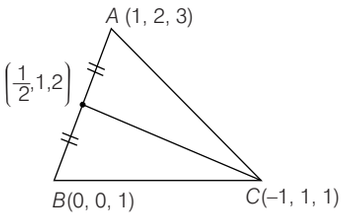
Equation of altitude through B to side AC is

$$\mathbf{r}=\hat{\mathbf{k}}+t\left(-\frac{7}{9}\hat{\mathbf{i}}+\frac{10}{9}\hat{\mathbf{j}}+\frac{11}{9}\hat{\mathbf{k}}-\hat{\mathbf{k}}\right)$$

$$\mathbf{r}=\hat{\mathbf{k}}+t(-7\hat{\mathbf{i}}+10\hat{\mathbf{j}}+2\hat{\mathbf{k}})$$

141. Clearly, mid-point L of AB is $\left(\frac{1}{2},1,2\right)$.

Equation of median through C to AB is



$$\mathbf{r}=(-\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})+p\left(\frac{3}{2}\hat{\mathbf{i}}+\hat{\mathbf{k}}\right)$$

$\Rightarrow \quad \mathbf{r}=(-\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})+p(3\hat{\mathbf{i}}+2\hat{\mathbf{k}})$

142. We have, $\cos A=\frac{3^2+3^2-(\sqrt{2})^2}{2(3)(3)}$

$\therefore \quad \cos A=\frac{16}{18}=\frac{8}{9}$

Now, area $(\Delta\ ABC)=\frac{1}{2}(3)(3)$

$$\sin A=\frac{9}{2}\sqrt{1-\frac{64}{81}}$$

$$=\frac{9}{2}\times\frac{\sqrt{17}}{9}=\frac{\sqrt{17}}{2}\text{ sq units}$$

Solution (Q. Nos. 143-144)

143. Line $\frac{x-1}{3}=\frac{y-2}{-1}=\frac{z-3}{4}=r$

Any point $B\equiv 3r+1, 2-r, 3+4r$ (on the line L)

$\mathbf{AB}=3r, -r, 4r+6$

Hence, $\mathbf{AB}$ is parallel to $x+y-z=1$.

Hence, $\quad 3r-r-4r-6=0$

$$2r=-6\ ;\quad r=-3$$

Hence, B is $(-8, 5, -9)$

144. Equation of plane containing the line L is

$$A(x-1)+B(y-2)+C(z-3)=0$$

where, $\quad 3A-B+4C=0 \qquad \ldots(ii)$

$\therefore$Eq. (i) also contains the point $A(1, 2, -1)$

Hence, $\quad C=0, 3A=B$

Equation of plane $\ x-1+3(y-2)=9$

$$x+3y-7=0$$

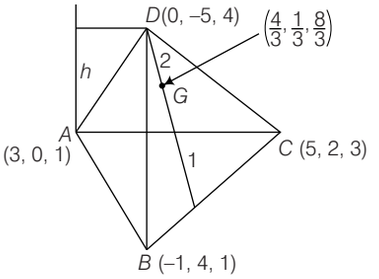
Solution (Q. Nos. 145-148)

145. $|\ \mathbf{AG}\ |^2=\left(\frac{5}{3}\right)^2+\frac{1}{9}+\left(\frac{5}{3}\right)^2=\frac{51}{9}$

$$|\ \mathbf{AG}\ |=\frac{\sqrt{51}}{3}$$

146. $\mathbf{AB}=-4\hat{\mathbf{i}}+4\hat{\mathbf{j}}+0\hat{\mathbf{k}}$

$\mathbf{AC}=2\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\hat{\mathbf{k}}$



$$\therefore \quad \mathbf{AB}\times\mathbf{AC}=-8\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & -1 & 0\\ 1 & 1 & 1\end{vmatrix}$$

$$=-8(-\hat{\mathbf{i}}-\hat{\mathbf{j}}+2\hat{\mathbf{k}})=8(\hat{\mathbf{i}}+\hat{\mathbf{j}}-2\hat{\mathbf{k}})=\mathbf{n}$$

$\therefore$Area of $\Delta ABC=\frac{1}{2}|\ \mathbf{AB}\times\mathbf{AC}\ |=4\sqrt{6}$

147. $h=|$ Projection of $\mathbf{AD}$ on $\mathbf{n}\ |$

$\mathbf{AD}=-3\hat{\mathbf{i}}-5\hat{\mathbf{j}}+3\hat{\mathbf{k}}$

$$=\left|\frac{\mathbf{AD}\cdot\mathbf{n}}{|\ \mathbf{n}\ |}\right|=\left|\frac{(-3\hat{\mathbf{i}}-5\hat{\mathbf{j}}+3\hat{\mathbf{k}})(\hat{\mathbf{i}}+\hat{\mathbf{j}}-2\hat{\mathbf{k}})}{\sqrt{6}}\right|$$

$$=\left|\frac{-3-5-6}{\sqrt{6}}\right|=\frac{14}{\sqrt{6}}$$

148. Equation of the plane ABC

$$A(x-3)+By+(z-1)=0$$

where, $\quad A=1, B=1, C=-2$

$\therefore \quad x-3+y-2z+2=0$

$$x+y-2z=1$$

Solution (Q. Nos. 149-151)

149. Line L_1 is parallel to $\mathbf{a}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}}$

Line L_2 is parallel to $\mathbf{b}=3\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}}$

Normal to the plane perpendicular to line L_1 and L_2 is

$$\mathbf{a}\times\mathbf{b}=(\hat{\mathbf{i}}+7\hat{\mathbf{j}}-5\hat{\mathbf{k}})$$

and plane passes through the point with positive vector

$$=\frac{3}{2}\hat{\mathbf{i}}+\frac{5}{2}\hat{\mathbf{j}}+2\hat{\mathbf{k}}$$

Equation of plane is $\mathbf{r}\cdot(\hat{\mathbf{i}}+7\hat{\mathbf{j}}-5\hat{\mathbf{k}})=9$

150. Angle bisector of vector $\mathbf{a}$ and $\mathbf{b}$ is,

$$\mathbf{r}_1=\frac{1}{\sqrt{14}}(2\hat{\mathbf{i}}-\hat{\mathbf{j}}-\hat{\mathbf{k}})$$

and $\quad \mathbf{r}_2=\frac{1}{\sqrt{14}}(4\hat{\mathbf{i}}+3\hat{\mathbf{j}}+5\hat{\mathbf{k}})$

Hence, the plane with either $(2, -1, -1)$ or $(4, 3, 5)$ as the direction ratio of normal and passing through $(2, -3, 2)$ is the required plane.

$\therefore$ Equation of line is $\frac{x-2}{2} = \frac{y+3}{-1} = \frac{3-2}{-1}$

and $\frac{x-2}{4} = \frac{y+3}{3} = \frac{3-2}{5}$

$\frac{x-2}{2} = y+3 = z-2$ or $\frac{x-2}{4} = \frac{y+3}{3} = \frac{z-2}{5}$

151. $\therefore$ Equation of required plane is

$\mathbf{r} \cdot (3\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) = (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$

$\Rightarrow \mathbf{r} \cdot (3\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) - 11 = 0$

$\therefore$ Required distance $= \left|\frac{11}{\sqrt{9+1+4}}\right| = \frac{11}{\sqrt{14}}$

■ **Solutions** (Q. Nos. 152-154)

The three plane intersect in a straight line. All three plane pass through origin (clearly).

$$\begin{vmatrix} 1 & -n & -m \\ n & -1 & l \\ m & l & -1 \end{vmatrix} = 1(1-l^2) + n(-n-lm) - m(nl+m)$$

$1 = l^2 + m^2 + n^2 + 2lmn$

Let $l = \cos\theta_1$, $m = \cos\theta_2$, $n = \cos\theta_3$ [since, $l, m, n \in (0, 1)$]

$\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 + 2\cos\theta_1\cos\theta_2\cos\theta_3 = 1$

$\cos^2\theta_1 + (2\cos\theta_2\cos\theta_3)\cos\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 - 1 = 0$

$$\cos\theta_1 = \frac{-2\cos\theta_2\cos\theta_3 \pm \sqrt{4\cos^2\theta_2\cos^2\theta_3 - 4\cos^2\theta_2 - 4\cos^2\theta_3 + 4}}{2}$$

$$= -\cos\theta_2\cos\theta_3 \pm \sqrt{1-\cos\theta_2}\sqrt{1-\cos^2\theta_3}$$

$\Rightarrow \cos\theta_1 = (\cos\theta_2\cos\theta_3 - \sin\theta_2\sin\theta_3)$

$\Rightarrow \cos\theta_1 = -\cos(\theta_2 + \theta_3)$

$\theta_1 + \theta_2 + \theta_3 = \pi$

$ny + mz = \frac{y - lz}{n}$

$n^2y + mnz = y - lz$

$(1 + mn)z = (1 - n^2)y$

$\frac{z}{y} = \frac{1-n^2}{1+mn}$

$\frac{x-ny}{m} = \frac{y-nx}{1}$

$lx - nly = my - mnx$

$(l + mn)x = (m + nl)y$

$\frac{y}{x} = \frac{l+mn}{m+nl}$

$\frac{y-lz}{n} = \frac{z-ly}{m}$

$my - mlz = nz - nly \Rightarrow (m + nl)y = (n + ml)z$

$\frac{z}{y} = \frac{m+nl}{n+ml} = \frac{\sin\theta_1\sin\theta_3}{\sin\theta_2\sin\theta_1} = \frac{\sin\theta_3}{\sin\theta_2}$

$\frac{x}{y} = \frac{m+nl}{l+mn} = \frac{\sin\theta_1}{\sin\theta_2}$

$\frac{x}{\sin\theta_1} = \frac{y}{\sin\theta_2} = \frac{z}{\sin\theta_3}$

$\frac{x}{\sqrt{1-l^2}} = \frac{y}{\sqrt{l-m^2}} = \frac{z}{\sqrt{1-n^2}}$

152. (b) **153.** (c) **154.** (a)

Solution (Q. Nos. 155-157)

155. Let the position vector of L be $\mathbf{a} + \lambda\mathbf{b}$

$= (6 + 3\lambda)\hat{\mathbf{i}} + (7 + 2\lambda)\hat{\mathbf{j}} + (7 - 2\lambda)\hat{\mathbf{k}}$

So, $\mathbf{PL} = (6 + 3\lambda)\hat{\mathbf{i}} + (7 + 2\lambda)\hat{\mathbf{j}} + (7 - 2\lambda)\hat{\mathbf{k}} - (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$

$= (5 + 3\lambda)\,\hat{\mathbf{i}} + (5 + 2\lambda)\,\hat{\mathbf{j}} + (4 - 2\lambda)\,\hat{\mathbf{k}}$

Since, $\mathbf{PL}$ is perpendicular to the given line which is parallel to

$\mathbf{b} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$

$\Rightarrow 3(5 + 3\lambda) + 2(5 + 2\lambda) - 2(4 - 2\lambda) = 0$

$\Rightarrow \lambda = -1$ and thus the position vector of L is $3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 9\hat{\mathbf{k}}$

156. Let the position vector of Q, the image of P in the given line be $x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}} + z_1\hat{\mathbf{k}}$, then L is the mid-point of PQ.

$\Rightarrow 3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 9\hat{\mathbf{k}} = \frac{\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}} + x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}} + z_1\hat{\mathbf{k}}}{2}$

$\Rightarrow \frac{x_1+1}{2} = 3,\ \frac{y_1+2}{2} = 5,\ \frac{z_1+3}{2} = 9$

$\Rightarrow x_1 = 5, y_1 = 8, z_1 = 15$

$\Rightarrow$ Image of P in the line is $(5, 8, 15)$

157. Area of the $\Delta PLA = \frac{1}{2}|\mathbf{PL}||\mathbf{AL}|$

$= \frac{1}{2}\left|2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}\right|\left|-3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}\right|$

$= \frac{1}{2}\sqrt{4+9+36}\ \sqrt{9+4+4} = \frac{7\sqrt{17}}{2}$ sq units.

Solution (Q. Nos. 158-160)

158. Let $P(x, y, z)$ be any point on the locus then $3PA = 2PB$

$\Rightarrow 9\,(PA)^2 = 4\,(PB)^2$

$\Rightarrow 9[(x+2)^2 + (y-2)^2 + (z-3)^2]$

$= 4[(x-13)^2 + (y+3)^2 + (z-13)^2]$

$\Rightarrow 5\,(x^2 + y^2 + z^2) + 140x - 60y + 50z - 1235 = 0$

$\Rightarrow x^2 + y^2 + z^2 + 28x - 12y + 10z - 247 = 0$

159. The required coordinates are

$\left(\frac{2\times13 + 3(-2)}{2+3}, \frac{2\times(-3) + 3(2)}{2+3}, \frac{2\times13 + 3\times3}{2+3}\right) = (4, 0, 7)$

160. Direction ratios of AB are $13 + 2, -3 - 2, 13 - 3$

i.e. $15, -5, 10$

Let the equation of the required line L be

$\frac{x+2}{l} = \frac{y-2}{m} = \frac{z-3}{n}$

then $15l - 5m + 10n = 0$

which satisfied by (c)

Solution (Q. Nos. 161-163)

161. Equation $\mathbf{r} = \mathbf{a} + t\hat{\mathbf{n}}$ is line passing through $\mathbf{a}$ and parallel to $\hat{\mathbf{n}}$. This will meet the plane $\mathbf{r} \cdot \hat{\mathbf{n}} = d$ at point for which

$$(\mathbf{a} + t\hat{\mathbf{n}}) \cdot \hat{\mathbf{n}} = d \Rightarrow t = d - \mathbf{a} \cdot \hat{\mathbf{n}}$$

Required distance $= \left|(\mathbf{a} + (d - \mathbf{a}.\hat{\mathbf{n}})\hat{\mathbf{n}} - \mathbf{a}\right| = \left|d - \mathbf{a}.\hat{\mathbf{n}}\right|$

162. Foot of the perpendicular from the point A to plane $\mathbf{r}.\hat{\mathbf{n}} = d$

$$= \mathbf{a} + (d - \mathbf{a}.\hat{\mathbf{n}})\,\hat{\mathbf{n}}$$

163. Let $\mathbf{b}$ be position vector of image of $\mathbf{a}$

$$\therefore \quad \frac{\mathbf{b} + \mathbf{a}}{2} = \mathbf{a} + (d - \mathbf{a}.\hat{\mathbf{n}})\,\hat{\mathbf{n}}$$

$$\mathbf{b} = \mathbf{a} + 2(d - \mathbf{a}.\hat{\mathbf{n}})\,\hat{\mathbf{n}}$$

Solution (Q. Nos. 164-166)

164. The centre of the sphere is at the mid-point of the extremities of a diameter $\Rightarrow$ the centre $\left(-\frac{3}{2}, \frac{3}{2}, -\frac{3}{2}\right)$

and hence the radius $= \sqrt{\left(\frac{7}{2}\right)^2 + \left(\frac{9}{2}\right)^2 + \left(\frac{11}{2}\right)^2}$

165. Equation of the circle can be written as

$$x^2 - 16x + y^2 - 9 + z^2 = 0$$

or $$x^2 + y^2 + z^2 = 25$$

166. Distance of the point $(3, 6, -4)$ from the given plane is equal to the radius of the sphere $\Rightarrow$ the radius of the sphere

$$= \left|\frac{(3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 4\hat{\mathbf{k}}) \cdot (2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - \hat{\mathbf{k}}) - 10}{\sqrt{4 + 4 + 1}}\right| = \left|\frac{6 - 12 + 4 - 10}{3}\right| = 4$$

Solution (Q. Nos. 167-168)

167. Mid-point of BC $= \left(\frac{\lambda - 1}{2}, \frac{\mu + 2}{2}\right)$

DR's $\quad AD = \left(\frac{\lambda - 5}{2}, 1, \frac{\mu - 8}{2}\right)$

AD is equally inclined to axes $\Rightarrow \lambda = 7, \mu = 10, 2\lambda - \mu = 4$

168. $A(2, 3, 5)\ B(-1, 2, 3)\ C(7, 5, 10)$

Projection of $AB = -3\hat{\mathbf{i}} - 3\hat{\mathbf{k}}$ on $BC = 8\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 8\hat{\mathbf{k}}$

$$\frac{AB \cdot BC}{|BC|} = \frac{-8\sqrt{3}}{11}$$

Solution (Q. Nos. 169-171)

169. Horizontal plane P_1 is of the form

$$\mathbf{r} \cdot \mathbf{n}_1 = 0, \qquad \text{where } \mathbf{n}_1 = (4, -3, 7)$$

Plane P_2 is of the form $\mathbf{r} \cdot \mathbf{n}_2 = 0$, where $\bar{n}_2 = (2, 1, -5)$

The vector $\mathbf{b}$ along the line of interaction

$$= \mathbf{n}_1 \times \mathbf{n}_2 = (4, 17, 5) = n_3 \text{ (say)}$$

Since the line of greatest slope is perpendicular to $\mathbf{n}_2$ and $\mathbf{n}_3$, the vector along the line of greatest slope

$$= \mathbf{n}_2 \times \mathbf{n}_3 = (3, -1, 1) = \mathbf{n}_4$$

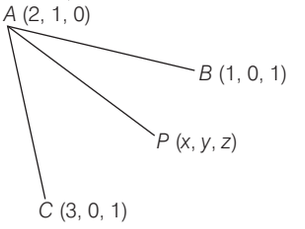
and $$\mathbf{n}_4 = \left(\frac{3}{\sqrt{11}}, \frac{-1}{\sqrt{11}}, \frac{1}{\sqrt{11}}\right)$$

170. Since $(0, 0, 0)$ is a point on both the planes, it is a point on the line of intersection and hence the equation of a line of greatest slope is

$$\frac{x}{\frac{3}{\sqrt{11}}} = \frac{y}{\frac{-1}{\sqrt{11}}} = \frac{z}{\frac{1}{\sqrt{11}}}$$

171. The point on the line at a distance $\sqrt{11}$ from the origin is the required point and it is $(3, -1, 1)$

Solution (Q. Nos. 172-174)



$$\begin{vmatrix} x-2 & y-1 & z \\ -1 & -1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 0$$

$$(x-2)[(-1) - (1)] - (y-1)[(-1) - 1] + z[1 + 1] = 0$$

$$2(y - 1) + 2z = 0$$

$$\Rightarrow \quad y + z - 1 = 0$$

The vector normal to the plane is $\mathbf{r} = 0\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$

The equation of the line through $(0, 0, 2)$ and parallel to $\mathbf{n}$ is

$$\mathbf{r} = 2\hat{\mathbf{k}} + \lambda(\hat{\mathbf{j}} + \hat{\mathbf{k}})$$

The perpendicular distance of $D(0, 0, 2)$ from plane.

172. (b) ***173.*** (c) ***174.*** (d)

175. (A) $L_1 : \frac{x-1}{1} = \frac{y-0}{1} = \frac{z-2}{-5}$; $\mathbf{V}_1 = \hat{\mathbf{i}} + \hat{\mathbf{j}} - 5\hat{\mathbf{k}}$;

$$L_2 : \frac{x-2}{2} = \frac{y-1}{2} = \frac{z+3}{-10}; \quad \mathbf{V}_2 = 2(\hat{\mathbf{i}} + \hat{\mathbf{j}} - 5\hat{\mathbf{k}})$$

Hence, lines are parallel and both contains the points $(1, 0, 2)$ and $(2, 1, -3)$ Coincident lines both L_1 and L_2 may lie in an infinite number of planes.

(B) $\left.\begin{matrix}\mathbf{V}_1 = 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}} \\ \mathbf{V}_2 = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}\end{matrix}\right] \Rightarrow$ Lines not parallel

Also, both intersect at $(3, 5, 1)$

Hence, lines are intersecting, hence they lie on a unique plane.

(C) $L_1 : \frac{x-0}{-6} = \frac{y-1}{9} = \frac{z-0}{-3} = t$

$$L_2 : \frac{x-1}{2} = \frac{y-4}{-3} = \frac{z-0}{1} = s$$

$\left.\begin{matrix} L_1 \text{ is parallel to } -6\hat{\mathbf{i}} + 9\hat{\mathbf{j}} - 3\hat{\mathbf{k}} \\ L_2 \text{ is parallel to } 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}} \end{matrix}\right]$

$\Rightarrow$ Lines parallel but not coincident.

Since, $(0, 1, 0)$ does not lie on L_2, not intersecting.

Hence L_1, L_2 lies in unique planes.

(D) Lines are skew can be verified.

176. $L_1 : \frac{x}{3} = \frac{y-3}{4} = \frac{z+2}{1}$ ...(i) (passing through P and Q)

$$L_2 : \frac{x-1}{1} = \frac{y-3}{0} = \frac{z+1}{1} \qquad \text{...(ii)}$$

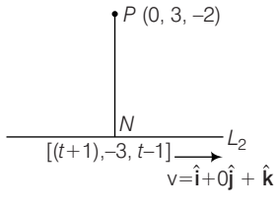
(passing through R and parallel to $\mathbf{v} = \hat{\mathbf{i}} + \hat{\mathbf{k}}$)

(A) Distance of $P(0, 3, -2)$ from L_2

$$\mathbf{PN} = (t+1)\,\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 2(t-1)\,\hat{\mathbf{k}}$$

Now, $\mathbf{PN} \cdot \mathbf{V} = 0$

$$[(t+1)\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + (t+1)\hat{\mathbf{k}}] \cdot (\hat{\mathbf{i}} + \hat{\mathbf{k}}) = 0$$

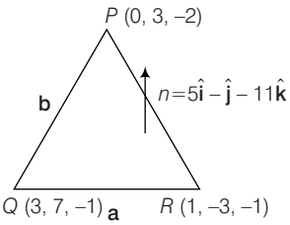


$$(t+1) + (t+1) = 0\ ;\ t = -1$$

Hence, $\mathbf{PN} = 6\hat{\mathbf{j}}$

$$|\mathbf{PN}| = |-6\hat{\mathbf{j}}| = 6$$

(B) Distance between L_1 and L_2



Equation of plane containing L_1 and parallel to L_2

$$Ax + B(y-3) + C(z+2) = 0$$

where $3A + 4B + C = 0$

And $A + B + C = 0$

$$A + C = 0$$

$C = \lambda, A = -\lambda, B = +\lambda/2$

$\therefore$ Equation of plane

$$-\lambda x + \frac{\lambda}{2}(y-3) + \lambda(z+2) = 0$$

$$2x - y + 3 - 2z - 4 = 0$$

$$2x - y - 2z = 1 \qquad \ldots(i)$$

Now, distance of the point $(1, -3, -1)$ lying on the line L_1 from the plane (i)

$$d = \left|\frac{2+3+2-1}{3}\right| = 2$$

(C) Area of ΔPQR

$$\mathbf{QR} = a = 2\hat{\mathbf{i}} + 10\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$$

$$\mathbf{QP} = b = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + \hat{\mathbf{k}}$$

$$\mathbf{a} \times \mathbf{b} = 2\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 5 & 0 \\ 3 & 4 & 1 \end{vmatrix}$$

$$= 2[\hat{\mathbf{i}}(5) - \hat{\mathbf{j}}(1) + \hat{\mathbf{k}}(4-15)] = 2[5\hat{\mathbf{i}} - \hat{\mathbf{j}} - 11\hat{\mathbf{k}}]$$

$$\frac{|\mathbf{a} \times \mathbf{b}|}{2} = \sqrt{25+1+121} = \sqrt{147} = \sqrt{3 \cdot 49} = 7\sqrt{3}$$

(D) Distance of $(0, 0, 0)$ from PQR

Equation of plane PQR is $(r - p) \cdot n$

$$= [x\hat{\mathbf{i}} + (y-3)\hat{\mathbf{j}} + (z+2)\hat{\mathbf{k}}] \cdot [5\hat{\mathbf{i}} - \hat{\mathbf{j}} - 11\hat{\mathbf{k}}]$$

$$= 5x - (y-3) - 11(z+2) = 0$$

$$= 5x - y - 11z - 19 = 0$$

Distance from $(0, 0, 0)$ of the plane

$$d = \left|\frac{19}{\sqrt{25+1+121}}\right| = \frac{19}{\sqrt{147}}$$

177. (A) $3 \cdot 1 - 2(-2) + 5(\lambda) = 0$

$$\Rightarrow \qquad \lambda = -\frac{7}{5}$$

(B) Point $(3, \lambda, \mu)$ lies on

$$2x + y + z - 3 = 0$$

$$= x - 2y + z - 1$$

$$3 \cdot 2 + \lambda + \mu - 3 = 0 \text{ and } 3 - 2\lambda + \mu - 1 = 0$$

$$\lambda + \mu + 3 = 0 \text{ and } 2\lambda - \mu - 2 = 0$$

So, $\lambda + \mu = -3$

(C) $\sin\theta = \dfrac{1 \cdot 4 + 1(-3) + 1 \cdot 5}{\sqrt{1^2+1^2+1^2}\,\sqrt{16+9+25}} = \dfrac{6}{\sqrt{3}\,\sqrt{50}}$

$$\theta = \sin^{-1}\sqrt{\frac{6}{25}}$$

(D) $\cos\theta = \dfrac{1 \cdot 3 + 1(-4) + 1 \cdot 5}{\sqrt{3}\,\sqrt{16+9+25}} = \dfrac{4}{\sqrt{3}\,\sqrt{50}}$

$$\theta = \cos^{-1}\sqrt{\frac{8}{75}}$$

178. $\begin{vmatrix} 1 & 3 & -5 \\ 3 & -k & -1 \\ 5 & 2 & -12 \end{vmatrix} = 1(12k+2) - 3(-36+5) - 5(6+5k)$

$$= 12k + 2 + 108 - 15 - 30 - 25k = 0$$

$k = 5$

L_1, L_2 and L_3 are concurrent for $k = 5$.

Slope of $L_1 = -\dfrac{1}{3}$, Slope of $L_2 = \dfrac{3}{k}$,

Slope of $L_3 = -\dfrac{5}{2}$

$$\frac{3}{k} = -\frac{1}{3} \Rightarrow k = -9$$

$$\frac{3}{k} = -\frac{5}{2} \Rightarrow k = -\frac{6}{5}$$

L_1, L_2 and L_3 form a triangle, if they are non-concurrent or any two out or three are not parallel.

$$k \neq -9, -\frac{6}{5}, 5$$

$k = \dfrac{5}{6}$ and 0 will be the values for which L_1, L_2 and L_3 form a triangle.

179. Given, $\dfrac{abc}{6} = 32$, where A, B and C are respectively, $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$.

(A) Centroid of tetrahedron $[\alpha, \beta, \gamma] = \left(\dfrac{a}{4}, \dfrac{b}{4}, \dfrac{c}{4}\right)$

$$a = 4\alpha,\ b = 4\beta,\ c = 4\gamma$$

$$64\alpha\beta\gamma = 32 \times 6$$

$$\alpha\beta\gamma = 3$$

(B) Equidistant point $(\alpha, \beta, \gamma) \equiv \left(\dfrac{a}{2}, \dfrac{b}{2}, \dfrac{c}{2}\right)$

$$a = 2\alpha,\ b = 2\beta,\ c = 2\gamma$$

$$8\alpha\beta\gamma = 32 \times 6$$

$$\alpha\beta\gamma = 24$$

(C) The equation of the plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$\therefore$ Foot of the perpendicular from the origin

$$\equiv (\alpha, \beta, \gamma) \equiv \left(\frac{1/a}{\sum 1/a^2}, \frac{1/b}{\sum 1/b^2}, \frac{1/c}{\sum 1/c^2} \right)$$

$$\frac{1}{a\alpha} = \frac{1}{b\beta} = \frac{1}{c\gamma} = t$$

where, $t = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \sum \frac{1}{a^2}$

or $t = (\alpha^2 + \beta^2 + \gamma^2)t^2$

$$t = \frac{1}{\alpha^2 + \beta^2 + \gamma^2}$$

and $\alpha = \frac{\alpha^2 + \beta^2 + \gamma^2}{\alpha},\ b = \frac{\alpha^2 + \beta^2 + \gamma^2}{\beta},$

$$c = \frac{\alpha^2 + \beta^2 + \gamma^2}{\gamma}$$

Now, $abc = 6 \times 32$

$(\alpha^2 + \beta^2 + \gamma^2) = 192\ \alpha\beta\gamma$

(D) Let P be (α, β, γ), then $PA \perp PB$

$\Rightarrow$ $a(\alpha - a) + \beta(\beta - b) + \gamma\gamma = 0$

$\Rightarrow$ $a\alpha + b\beta = \alpha^2 + \beta^2 + \gamma^2$

$PB \perp PC$

$\Rightarrow$ $a\alpha + b(\beta - b) + c(\gamma - c) = 0$

$\Rightarrow$ $b\beta + c\gamma = \alpha^2 + \beta^2 + \gamma^2$

$\therefore$ $\frac{a}{1/\alpha} = \frac{b}{1/\beta} = \frac{c}{1/\gamma}$

$$a = \frac{\alpha^2 + \beta^2 + \gamma^2}{2\alpha},\ b = \frac{\alpha^2 + \beta^2 + \gamma^2}{2\beta}$$

and $c = \frac{\alpha^2 + \beta^2 + \gamma^2}{2\gamma}$

$\therefore$ $abc = 6 \times 32$

$\Rightarrow$ $(\alpha^2 + \beta^2 + \gamma^2) = 192 \times 8\alpha\beta\gamma$

$$= 1536\alpha\beta\gamma$$

180. Let $O(0, 0, 0)$, $A(3, 4, 7)$ and $B(5, 2, 6)$ be the given point

Area of $\Delta OAB = \frac{1}{2} OA \cdot OB \sin(\angle AOB)$

Now, $OA = \sqrt{3^2 + 4^2 + 7^2} = \sqrt{74}$

$$OB = \sqrt{5^2 + 2^2 + 6^2} = \sqrt{65}$$

Also dc's of the line OA and OB are

$$= \frac{3}{\sqrt{74}}, \frac{4}{\sqrt{74}}, \frac{7}{\sqrt{74}} \text{ and } \frac{5}{\sqrt{65}}, \frac{2}{\sqrt{65}}, \frac{6}{\sqrt{65}}$$

$\therefore$ Required area $\frac{1}{2} \times \sqrt{74} \times \sqrt{65} \times \frac{3}{\sqrt{74}} = \frac{3}{2}\sqrt{65}$

(B) Let the required sphere be

$x^2 + y^2 + z^2 + 2ux + 2vy + wz + d = 0$ ...(1) substituting given points then we get $1 + 2u + d = 0$

$$1 + 2v + d = 0 \text{ and } 1 + 2w + d = 0$$

$\Rightarrow$ $u = v = w = \frac{1 + d}{2}$

If R be the radius of the sphere,

then $R^2 = u^2 + v^2 + w^2 - d$

covert above equation in terms of d differentiate, equate to zero solve for d.

(C) Let the given points be A, B and C respectively.

Then find AB, AC, BC and then apply $AB^2 + AC^2 = BC^2$ then solve for the λ.

(D) Any point on the line is $(1 - r, r + 1, r)$

The direction ratio of the line joining $(1, 3, 4)$ & $(1 - r, r + 1, r)$ is $-r, r - 2, r - 4$

$\therefore$ $(-1)(-r) + 1 \cdot (r - 2) + (r - 4) = 0$

$$r + r - 2 + r - 4 = 0$$

$$3r = 6 \Rightarrow r = 2$$

$\therefore$ Foot of the perpendicular is $(-1, 3, +2)$

$\therefore$ Distance $= \sqrt{(2)^2 + 0 + 4} = 2\sqrt{2}$

$\therefore$ $d = 2\sqrt{2}$

$$\frac{d}{2\sqrt{3}} = \frac{2\sqrt{2}}{2\sqrt{3}} = \frac{\sqrt{2}}{\sqrt{3}}$$

181. The solid diagonals may be taken as the lines join $(0, 0, 0)$, (a, a, a) and $(a, a, 0)$ and $(0, 0, a)$. The direction ratios will be $a, a, a,$; $a, a, -a$.

$\Rightarrow$ $\cos\theta = \frac{a^2 + a^2 - a^2}{\sqrt{3a^2} \times \sqrt{3a^2}} \frac{1}{3} \Rightarrow \theta = \cos^{-1}\frac{1}{3}$

Let us take the solid diagonal as the one joining $(0, 0, 0)$ (a, a, a) and plane diagonal as joining $(0, 0, 0)$ and $(a, a, 0)$. We easily get the angle as $\cos^{-1}\frac{2}{\sqrt{6}}$.

The third part is easily found as $\cos^{-1}\left(\frac{1}{2}\right)$

Hence, matching follows (A) $\rightarrow$ (r) ; (B) $\rightarrow$ (p) ; (C) $\rightarrow$ (q)

182. (i) Shortest distance

$$= \frac{|\mathbf{OB} \cdot \mathbf{OA} \times \mathbf{BC}|}{|\mathbf{OA} \times \mathbf{BC}|} = \frac{|(\hat{\mathbf{i}} + \hat{\mathbf{j}}) \cdot \hat{\mathbf{i}} \times (\hat{\mathbf{j}} \times \hat{\mathbf{k}})|}{\hat{\mathbf{i}} \times (\hat{\mathbf{j}} + \hat{\mathbf{k}})} = \frac{1}{\sqrt{2}}$$

$\Rightarrow$ $\sqrt{2}m = 1$

183. The length of the edges are given by $a = 5 - 2 = 3$, $b = 9 - 3 = 6$ and $c = 7 - 5 = 2$, so length of the diagonal

$$= \sqrt{a^2 + b^2 + c^2} = \sqrt{9 + 36 + 4} = 7 \text{ units}$$

184. Foot of perpendicular r from $(6, 5, 8)$ on y-axis is $(0, 5, 0)$.

Required distance

$$= \sqrt{(6 - 0)^2 + (5 - 5)^2 + (8 - 0)^2} = 10$$

$\Rightarrow$ $5\lambda = 10 \Rightarrow \lambda \Rightarrow \frac{10}{5} = 2$

185. Given lines are

$$\mathbf{r} = (3, 8, 3) + \lambda(-3, 1, 1)$$

and $\mathbf{r} = (-3, -7, 6) + \mu(-3, 2, 4)$

where, λ and μ are parameters

Shortest distance

$$= \frac{(-3-3, -7-8, 6-3)\cdot[(3,-1,1)\times(-3,2,4)]}{|(3,-1,1)\times(-3,2,4)|}$$

$$= \frac{(-6,-15,3)\cdot(-6,-15,3)}{\sqrt{36+225+9}}$$

$$= \sqrt{270} = 3\sqrt{30} \text{ units} = \lambda\sqrt{30}$$

$\therefore \quad \lambda = 3$

186. Given planes are

$$x - cy - bz = 0 \quad \text{...(i)}$$

$$cx - y + az = 0 \quad \text{...(ii)}$$

$$bx + acy - z = 0 \quad \text{...(iii)}$$

Equation of plane passing through the line of intersection of planes (i) and (ii) may be taken as

$$(x - cy - bz) + \lambda(cx - y + az) = 0 \quad \text{...(iv)}$$

Now, eliminating λ we get

$$a^2 + b^2 + c^2 + 2abc = 1$$

187. We must have $\frac{\lambda(-1) + 1\times 3}{\lambda + 1} = 0 \Rightarrow \lambda = 3$

$\therefore \quad \lambda = 3$

188. The coordinates of vertices of projected triangle will be $A'(-1, 1, 0)$, $B'(1, -1, 0)$, $C'(1, 1, 0)$

$$\text{area of triangle} = \frac{1}{2}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} \text{(Two dimension area formula)}$$

$= 2$ square units.

189. Plane must pass through

$$\left(\frac{1-3}{2}, \frac{5+1}{2}, \frac{7-1}{2}\right) \text{ or } (-1, 3, 3)$$

$\Rightarrow \quad -1 + 3 + 2\times 3 = \lambda \Rightarrow \lambda = 8$

190. $x^2 + y^2 + z^2 =$ square of distance from origin

$$4\sin^2 t + 4\cos^2 t + 9t^2 = 4 + 9t^2$$

which is shortest at $t = 0$

$\Rightarrow$ Shortest distance $= 2$

191. The point $(-1, \lambda, -2)$ must be lie on the plane

$$2x - 2y + z + 12 = 0$$

$$-2 - 2\lambda - 2 + 12 = 0$$

$$\lambda = 4$$

We can easily show that the distance of $(-1, 4, -2)$ from centre of the sphere $(1, 2, -1)$ is equal to its radius.

192. $1 = \frac{a+1+2+0}{4}$, $2 = \frac{2+b+1+0}{4}$,

$$3 = \frac{3+2+c+0}{4}$$

$\Rightarrow \quad a = 1, b = 5, c = 7$

$\Rightarrow$ Distance of centroid from origin is

$$\sqrt{1^2 + 25 + 49} = \sqrt{75} = 5\sqrt{3} \Rightarrow \lambda = 3$$

193. Equating the distances of circumcentre $(-1, \lambda, -3)$ from $(3, 2, -5)$ and $(-3, 8, -5)$ we get

$$2^2 + (\lambda + 2)^2 + (-3 + 5)^2 = (-1 + 3)^2 + (\lambda - 8)^2 + (-3 + 5)^2$$

$\Rightarrow \quad \lambda = 4$

Note : Verify

(i) $(-1, \lambda, -3)$ is at the same distance from third vertex.

(ii) $(-1, \lambda, -3)$ lies on the plane containing three points $(3, 2, -5)$; $(-3, 8, -5)$ and $(-3, 2, 1)$.

194. D.R's of $P_1P_2 = (k, -1, 3)$

D.R's of $P_2P_3 = (2, k, -1)$

$\because \quad P_1P_2 \perp P_2P_3$

$\therefore \quad k(2) - k - 3 = 0 : k = 3$

195. A plane containing line of intersection of the given planes is

$$x - y - z - 4 + \lambda(x + y + 2z - 4) = 0$$

i.e., $(\lambda + 1)x + (\lambda - 1)y + (2\lambda - 1)z - 4(\lambda + 1) = 0$

vector normal to it

$$V = (\lambda + 1)\hat{\mathbf{i}} + (\lambda - 1)\hat{\mathbf{j}} + (2\lambda - 1)\hat{\mathbf{k}}$$

Now the vector along the line of intersection of the planes

$$2x + 3y + z - 1 = 0$$

and $x + 3y + 2z - 2 = 0$ is given by

$$\mathbf{n} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 3 & 1 \\ 1 & 3 & 2 \end{vmatrix} = 3(\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}})$$

As $\mathbf{n}$ is parallel to the plane (i), therefore $\mathbf{n}\cdot\mathbf{V} = 0$

$$(\lambda + 1) - (\lambda - 1) + (2\lambda - 1) = 0$$

$$2 + 2\lambda - 1 = 0 \Rightarrow \lambda = \frac{-1}{2}$$

Hence, the required plane is $\frac{x}{2} - \frac{3y}{2} - 2z - 2 = 0$

$$x - 3y - 4z - 4 = 0$$

Hence $\quad |A + B + C - 4| = 7$

196. Clearly, minimum value of $a^2 + b^2 + c^2$

$$= \left(\frac{|3(0) + 2(0) + (0) - 7|}{\sqrt{(3)^2 + (2)^2 + (1)^2}}\right) = \frac{49}{14} = \frac{7}{2} \text{ units}$$

197.

$$4x + 7y + 4z + 81 = 0 \quad \text{...(i)}$$

$$5x + 3y + 10z = 25 \quad \text{...(ii)}$$

Equation of plane passing through their line of intersection is

$$(4x + 7y + 4z + 81) + \lambda(5x + 3y + 10z - 25) = 0$$

or $(4 + 5\lambda)x + (7 + 3\lambda)y + (4 + 10\lambda)z + 81 - 25\lambda = 0 \quad \text{...(iii)}$

plane (iii) $\perp$ to (i), so

$$4(4 + 5\lambda) + 7(7 + 3\lambda) + 4(4 + 10\lambda) = 0$$

$\therefore \quad \lambda = -1$

From (iii), equation of plane is

$$-x + 4y - 6z + 106 = 0 \quad \text{...(iv)}$$

Distance of (iv) from (0, 0, 0)

$$= \frac{106}{\sqrt{1 + 16 + 36}} = \frac{106}{\sqrt{53}}$$

198. Line through point $P(-2, 3, -4)$ and parallel to the given line

$$\frac{x+2}{3}=\frac{2y+3}{4}=\frac{3z+4}{5}$$

is $$\frac{x+2}{3}=\frac{y+\frac{3}{2}}{2}=\frac{z+\frac{4}{3}}{\frac{5}{3}}=\lambda$$

Any point on this line is $Q\left[3\lambda-2, 2\lambda-\frac{3}{2}, \frac{5}{3}\lambda-\frac{4}{3}\right]$.

Direction ratios of PQ are $\left[3\lambda, \frac{4\lambda-9}{2}, \frac{5\lambda+8}{3}\right]$

Now, PQ is parallel to the given plane $4x+12y-3z+1=0$

$\Rightarrow$ line is perpendicular to the normal to the plane

$$\Rightarrow \quad 4(3\lambda)+12\left(\frac{4\lambda-9}{2}\right)-3\left(\frac{5\lambda+8}{3}\right)=0$$

$$\Rightarrow \quad \lambda=2 \Rightarrow Q\left(4, \frac{5}{2}, 2\right)$$

$$\Rightarrow \quad PQ=\sqrt{(6)^2+\left(\frac{5}{2}-3\right)^2+(6)^2}=\frac{17}{2}$$

199. The given points are $O(0, 0, 0)$, $A(0, 0, 0)$, $B(0, 4, 0)$ and $C(6, 0, 0)$

Here, three faces of tetrahedron are xy, yz, zx plane.

Since point P is equidistance from zx , xy and yz planes, its coordinates are $P(r, r, r)$

Equation of plane ABC is

$$2x+3y+6z=12 \qquad \text{(from intercept form)}$$

P is also at distance r from plane ABC

$$\Rightarrow \quad \frac{|2r+3r+6r-12|}{\sqrt{4+9+36}}=r \Rightarrow |11r-12|=7r$$

$$\Rightarrow \quad 11r-12=\pm 7r \Rightarrow r=\frac{12}{18}, 3$$

$$\therefore \quad r=2/3 \qquad (\text{as } r<2)$$

200. The equation of the given planes can be written as

$$x-y+z+1=0$$
$$\lambda xz+3y+2z-3=0$$
$$3x+\lambda y+z-2=0$$

The rectangular array is

$$\begin{Vmatrix} 1 & -1 & 1 & 1 \\ \lambda & 3 & 2 & -3 \\ 3 & \lambda & 1 & -2 \end{Vmatrix}=0$$

$$\therefore \quad \Delta_4=\begin{vmatrix} 1 & -1 & 1 \\ \lambda & 3 & 2 \\ 3 & \lambda & 1 \end{vmatrix}$$

Applying $C_2 \to C_2+C_1$ and $C_3 \to C_3+C_1$, then

$$\Delta_4=\begin{vmatrix} 1 & \ldots 0 & \ldots 0 \\ \lambda & 3+\lambda & 2-\lambda \\ 3 & 3+\lambda & -2 \end{vmatrix}=(\lambda-4)(\lambda+3) \qquad \ldots (i)$$

Also, $$\Delta_1=\begin{vmatrix} -1 & 1 & 1 \\ 3 & 2 & -3 \\ \lambda & 1 & -2 \end{vmatrix}$$

Applying $C_2 \to C_2+C_1$ and $C_3 \to C_3+C_1$, then

$$\Delta_1=\begin{vmatrix} -1 & 0 & 0 \\ 3 & 5 & 0 \\ \lambda & \lambda+1 & \lambda-2 \end{vmatrix}=-5(\lambda-2) \qquad \ldots (ii)$$

$$\Delta_2=\begin{vmatrix} 1 & 1 & 1 \\ \lambda & 2 & -3 \\ 3 & 1 & -2 \end{vmatrix}$$

Applying $C_2 \to C_2-C_1$ and $C_3 \to C_3-C_1$, then

$$\Delta_2=\begin{vmatrix} 1 \ldots & 1 \ldots & 0 \\ \lambda & 2-\lambda & -3-\lambda \\ 3 & -2 & -5 \end{vmatrix}=3\lambda-16 \qquad \ldots (iii)$$

$$\Delta_3=\begin{vmatrix} 1 & -1 & 1 \\ \lambda & 3 & -3 \\ 3 & \lambda & -2 \end{vmatrix}$$

Applying $C_2 \to C_2+C_1$ and $C_3 \to C_3-C_1$, then

$$\Delta_3=\begin{vmatrix} 1 \ldots & 1 \ldots & 0 \\ \lambda & 3+\lambda & -3-\lambda \\ 3 & 3+\lambda & -5 \end{vmatrix}=(\lambda+3)(\lambda-2) \qquad \ldots (iv)$$

If the given planes form a triangular prism, then we know that $\Delta_4=0$ and none of Δ_1, Δ_2, Δ_3 is zero. Here from Eqs. (i), (ii), (iii) and (iv) we find that if $\lambda=4$, then $\Delta_4=0$ and none of Δ_1, Δ_2, Δ_3 is zero. Consequently for $\lambda=4$, then given planes form a triangular prism.

201. $7x+6y+2z=272$ and $x-y+z=16$

$$\Rightarrow \quad 5x+8y=240 \Rightarrow x=48-\frac{8}{5}y$$

Let $y=5\lambda$, $\lambda \in I \Rightarrow x=48-8\lambda$

and $z=16+y-x=13\lambda-32$

But $x>0, y>0$ and $z>0 \quad \Rightarrow \quad 48-8\lambda>0 \Rightarrow \lambda>\frac{48}{8}$

$$\Rightarrow \quad \lambda \le 5 \quad \text{and} \quad 13\lambda-32>0 \Rightarrow \quad \lambda>\frac{32}{13}$$

$$\Rightarrow \quad \lambda \ge 3$$

$$\therefore \quad \lambda \in [3, 5]$$

$$\therefore \quad Z_{\min}=39-32=7 \Rightarrow x=24, y=15$$

$$\therefore \quad x+y+z-42=4$$

202. The given two lines are intersect each other, then

$$\frac{a_1\alpha+b_1\beta+c_1\gamma+d_1}{a_1l+b_1m+c_1n}=\frac{a_2\alpha+b_2\beta+c_2\gamma+d_2}{a_2l+b_2m+c_2n}$$

$$\Rightarrow \quad \frac{-2d^2}{\sin A+\sin B+\sin C}=\frac{-d^2}{\sin 2A+\sin 2B+\sin 2C}$$

$$\Rightarrow \quad \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{1}{16}$$

203. $$[\mathbf{c}-\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=0 \quad \Rightarrow \quad \begin{vmatrix} k & k-1 & 2 \\ 1 & k & -1 \\ 2k & 3k-1 & k \end{vmatrix}=0$$

$$\Rightarrow \quad k^3-4k^2+8k-2=0$$

Here $$f'(k)=3k^2-8k+8>0 \ \forall \ k \in R$$

($\because$ Its discriminate is negative)

$\therefore$ The equation has only one real root.

204. Taking O as the origin, let the position vectors of A, B and C be $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ respectively. Then the position vectors of G_1, G_2 and G_3 are

$$\frac{\mathbf{b}+\mathbf{c}}{3}, \frac{\mathbf{c}+\mathbf{a}}{3} \text{ and } \frac{\mathbf{a}+\mathbf{b}}{3}$$

$$V_1 = \frac{1}{6}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \text{ and } V_2 = [\mathbf{OG}_1\ \mathbf{OG}_2\ \mathbf{OG}_3]$$

$$\Rightarrow \quad V_2 = \frac{1}{27}[\mathbf{b}+\mathbf{c}\ \mathbf{c}+\mathbf{a}\ \mathbf{a}+\mathbf{b}] = \frac{2}{27}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$

$$\Rightarrow \quad V_2 = \frac{2}{27} \times 6V_1 \Rightarrow 9V_2 = 4V_1$$

205. Let the equation of planes is $lx + my + nz = p$

$$\therefore A = \left(\frac{p}{l}, 0, 0\right) B = \left(0, \frac{p}{m}, 0\right) C = \left(0, 0, \frac{p}{n}\right) \text{ respectively}$$

$$\text{Centroid of } OABC = \left(\frac{p}{4l}, \frac{p}{4m}, \frac{p}{4n}\right) = (x_1, y_1, z_1) \qquad \text{(say)}$$

$$\because \quad l^2 + m^2 + n^2 = 1$$

$$\therefore \quad \frac{p^2}{16x_1^2} + \frac{p^2}{16y_1^2} + \frac{p^2}{16z_1^2} = 1$$

$$\Rightarrow \quad x_1^2y_1^2 + y_1^2z_1^2 + z_1^2x_1^2 = \frac{16}{p^2}x_1^2y_1^2z_1^2$$

$$\therefore \quad k = 16 \Rightarrow 2k = 32 \Rightarrow \sqrt[5]{2k} = 2$$

206. $l_1^2 + m_1^2 + n_1^2 = 1, l_2^2 + m_2^2 + n_2^2 = 1$

$$(l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2) - (l_1l_1 + m_1m_2 + n_1n_2)^2$$

$$= (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2$$

$$(l_1m_2 - l_2m_1)^2 + (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1l_2 + m_1m_2 + n_1n_2)^2 = 1$$

207. Coordinates of the point, $S = \left(\frac{n^5}{2}, \frac{n^4}{2}, \frac{n}{2}\right)$

$$\Rightarrow \quad 2 \times \left(\frac{n^5}{2} + \frac{n^4}{2} + \frac{n}{2}\right) = -1$$

$$\Rightarrow \quad n(n^4 + n^3 + 1) = -1$$

$n = -1$ is the only solution.

208. We have, $l + m + n = 0$ ...(i)

and $2l^2 + 2m^2 - n^2 = 0$ ...(ii)

Now, $2(l^2 + m^2) - n^2 = 0$

$$\Rightarrow \quad 2(1 - n^2) - n^2 = 0 \qquad [\because l^2 + m^2 + n^2 = 1]$$

$$\Rightarrow \quad 3n^2 = 2$$

$$\Rightarrow \quad n = \pm\sqrt{\frac{2}{3}}$$

Again, $2(l^2 + m^2) = n^2$

$$\Rightarrow \quad 2[(l + m)^2 - 2lm] = (-(l + m))^2$$

$$\Rightarrow \quad l = m$$

$$\therefore \quad l + m = \pm\sqrt{\frac{2}{3}} \Rightarrow 2l = \pm\sqrt{\frac{2}{3}}$$

$$\Rightarrow \quad l = \pm\frac{1}{\sqrt{6}} = m$$

$\therefore$ Direction cosines are

$$\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}\right), \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\sqrt{\frac{2}{3}}\right)$$

or $\left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}\right)$

and $\left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\sqrt{\frac{2}{3}}\right)$

The angle between in both the cases is $\cos^{-1}\left(\frac{-1}{3}\right)$.

209. Elimination n between the given relations, we get

$$ul^2 + vm^2 + w\left(\frac{al + bm}{-c}\right)^2 = 0$$

$$\Rightarrow \quad (c^2u + a^2w)\frac{l^2}{m^2} + 2abw.\frac{l}{m} + (b^2w + c^2v) = 0 \qquad \text{...(i)}$$

$$\therefore \quad \frac{l_1}{m_1} \cdot \frac{l_2}{m_2} = \text{product of roots} = \frac{b^2w + c^2v}{c^2u + a^2w}$$

or $\frac{l_1l_2}{b^2w + c^2v} = \frac{m_1m_2}{c^2u + a^2w} = \frac{n_1n_2}{a^2v + b^2u}$ (by symmetry)

If lines are perpendicular, then

$$l_1l_2 + m_1m_2 + n_1n_2 = 0$$

$$\Rightarrow \quad a^2(v + w) + b^2(w + u) + c^2(u + v) = 0$$

Again, if the lines be parallel, then their $d'c$ are equal so that the roots of Eq. (i) should be equal, i.e. discriminate = 0

$$\therefore \quad 4a^2b^2w^2 - 4(c^2u + a^2w)(b^2w + c^2v) = 0$$

$$\Rightarrow \quad a^2c^2vw + b^2c^2uw + c^4uv = 0$$

$$\Rightarrow \quad \frac{a^2}{u} + \frac{b^2}{v} + \frac{c^2}{w} = 0$$

210. The coordinates of any point on the line $\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2} = \lambda$ are given by

$(3\lambda - 2, 2\lambda - 1, 2\lambda + 3)$

The distance between the above point and $(1, 2, 3)$ is $3\sqrt{2}$.

$$\therefore \quad \sqrt{(3\lambda - 2 - 1)^2 + (2\lambda - 1 - 2)^2 + (2\lambda + 3 - 3)^2} = 3\sqrt{2}$$

$$\Rightarrow \quad \lambda = \frac{30}{17}, 0$$

$\therefore$ Required points are $(-2, -1, 3)$ and $\left(\frac{56}{17}, \frac{43}{17}, \frac{111}{17}\right)$

211. The required line is perpendicular to the lines which are parallel to vectors $\mathbf{b}_1 = 2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{b}_2 = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ respectively. So, it is parallel to the vector $\mathbf{b} = \mathbf{b}_1 \times \mathbf{b}_2$.

$$\text{Now, } \mathbf{b} = \mathbf{b}_1 \times \mathbf{b}_2 = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{vmatrix} = -6\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$$

Thus, the required line passes through the point $(2, -1, 3)$ and is parallel to the vector $\mathbf{b} = -6\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$.

So, its vector equation is

$$\mathbf{r} = (2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}) + \lambda(-6\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}})$$

or $\mathbf{r} = (2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}) + \mu(2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}})$,

where $\mu = -3\lambda$.

212. The coordinates of any point on the line $\frac{x-3}{2}=\frac{y-3}{1}=\frac{z}{1}$ are given by $\frac{x-3}{2}=\frac{y-3}{1}=\frac{z}{1}=\lambda$.

So, let the coordinates of A be $(2\lambda+3, \lambda+3, \lambda)$.

Let the line through $O(0, 0, 0)$ and making an angle $\frac{\pi}{3}$ with the given line be along OA. Then, its $d'r$ are proportional to

$$2\lambda+3-0, \lambda+3-0, \lambda-0$$

or $$2\lambda+3, \lambda+3, \lambda$$

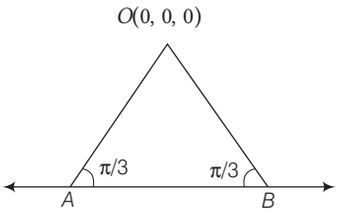
The direction ratios of the given line are proportional to 2, 1, 1.

It is given that the angle between the given line and the line along OA is $\frac{\pi}{3}$.

$$\therefore \quad \cos\frac{\pi}{3}=\frac{(2\lambda+3)\times 2+(\lambda+3)\times 1+\lambda\times 1}{\sqrt{(2\lambda+3)^2+(\lambda+3)^2+\lambda^2}\sqrt{2^2+1^2+1^2}}$$

$$=\frac{6\lambda+9}{\sqrt{6\lambda^2+18\lambda+18}\sqrt{6}}$$

$$\therefore \quad \lambda=-1, -2.$$



$$\frac{x-3}{2}=\frac{y-3}{1}=\frac{z}{1}$$

Putting these values of λ in the coordinates of A i.e. $(2\lambda+3, \lambda+3, \lambda)$, we find the coordinates of A and B i.e. $A(1, 2, -1)$ and $B(-1, 1, -2)$.

So, the equations of OA and OB are

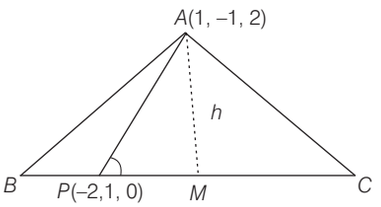
$$\frac{x-0}{1-0}=\frac{y-0}{2-0}=\frac{z-0}{-1-0}$$

$$\frac{x-0}{-1-0}=\frac{y-0}{1-0}=\frac{z-0}{-2-0}$$

or $$\frac{x}{1}=\frac{y}{2}=\frac{z}{-1}$$

and $$\frac{x}{-1}=\frac{y}{1}=\frac{z}{-2}$$

213. Clearly, height h of ΔABC is the length of perpendicular from $A(1, -1, 2)$ to the line $\frac{x+2}{2}=\frac{y-1}{1}=\frac{z-0}{4}$ which passes through $P(-2, 1, 0)$ and is parallel to $\mathbf{b}=2\hat{\mathbf{i}}+\hat{\mathbf{j}}+4\hat{\mathbf{k}}$.



$$\therefore \quad h=\frac{|\mathbf{PA}\times\mathbf{b}|}{|\mathbf{b}|}$$

Now, $$\mathbf{PA}=-3\hat{\mathbf{i}}+2\hat{\mathbf{j}}-2\hat{\mathbf{k}} \text{ and } \mathbf{b}=2\hat{\mathbf{i}}+\hat{\mathbf{j}}+4\hat{\mathbf{k}}$$

$$\therefore \quad \mathbf{PA}\times\mathbf{b}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ -3 & 2 & -2\\ 2 & 1 & 4\end{vmatrix}=10\hat{\mathbf{i}}+8-2\hat{\mathbf{k}}$$

$$\therefore \quad |\mathbf{PA}\times\mathbf{b}|=\sqrt{10^2+8^2+(-7)^2}=\sqrt{213}$$

and $$|\mathbf{b}|=\sqrt{2^2+1^2+4^2}=\sqrt{21}$$

$$\therefore \quad h=\frac{|\mathbf{PA}\times\mathbf{b}|}{|\mathbf{b}|}$$

$$=\frac{\sqrt{213}}{\sqrt{21}}=\sqrt{\frac{71}{7}}$$

It is given that the length of BC is 5 units.

$$\therefore \quad \text{Area of } \Delta ABC=\frac{1}{2}(BC\times h)$$

$$=\frac{1}{2}\times 5\times\sqrt{\frac{71}{7}}=\sqrt{\frac{1775}{28}} \text{ sq units.}$$

214. If the coordinates of the point P be (α, β, γ).

Then, $$\frac{\alpha}{a}+\frac{\beta}{\beta}+\frac{\gamma}{c}=1 \qquad \ldots(i)$$

Again d′c of OP are proportional to α, β, γ and hence these are also the d′r of the normal to the plane which is perpendicular to OP and since it passes through P, its equation is

$$\alpha(x-\alpha)+\beta(y-\beta)+\gamma(z-\gamma)=0$$

or $$\alpha x+\beta y+\gamma z=\alpha^2+\beta^2+\gamma^2 \qquad \ldots(ii)$$

It meets the axes in A, B, C and hence the coordinates of these points are $\left(\frac{\alpha^2+\beta^2+\gamma^2}{\alpha}, 0, 0\right)$ etc.

The equation of the plane through A and parallel to the YZ plane is $x=\frac{\alpha^2+\beta^2+\gamma^2}{\alpha}$.

Similarly the equations of other planes are

$$y=\frac{\alpha^2+\beta^2+\gamma^2}{\beta} \text{ and } z=\frac{\alpha^2+\beta^2+\gamma^2}{\gamma}.$$

The locus of their point of intersection is obtained by elimination α, β, γ between the three equations of the planes and relation (i)

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{\alpha^2+\beta^2+\gamma^2}{(\alpha^2+\beta^2+\gamma^2)^2}$$

$$=\frac{1}{\alpha^2+\beta^2+\gamma^2}$$

Again, $$\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}=\frac{\left(\frac{\alpha}{a}\right)+\left(\frac{\beta}{b}\right)+\left(\frac{\gamma}{c}\right)}{\alpha^2+\beta^2+\gamma^2}$$

$$=\frac{1}{\alpha^2+\beta^2+\gamma^2} \qquad \text{[from Eq. (i)]}$$

$$\therefore \quad \frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}$$

215. Any point on the line is $(3r + 2, 4r - 1, 12r + 2)$.

If it lies on the plane $x - y + z = 5$, then

$(3r + 2) - (4r - 1) + (12r + 2) = 5$

$\Rightarrow r = 0$

Hence, point of intersection is $(2, -1, 2)$.

Its distance from $(-1, -5, -10)$ is

$$\sqrt{(2+1)^2 + (-1+5)^2 + (2+10)^2}$$

$$= \sqrt{9 + 16 + 144} = \sqrt{169} = 13$$

216. Any plane through the intersection of given planes is $(x + 3y + 6 + \lambda(3x - y - 4z) = 0$

or $(1 + 3\lambda)x + (3 - \lambda)y - 4\lambda z + 6 = 0$ ...(i)

Its perpendicular distance from $(0, 0, 0)$ is 1.

$$\therefore \quad \left| \frac{6}{\sqrt{(1+3\lambda)^2 + (3-\lambda)^2 + (-4\lambda)^2}} \right| = 1$$

$\Rightarrow \lambda = \pm 1$

$\therefore$ Required planes are $2x + y - 2z + 3 = 0$ and $x - 2y - 2z - 3 = 0$.

217. The image of the plane

$$x - 2y + 2z - 3 = 0 \quad \text{...(i)}$$

in the plane $x + y + z - 1 = 0$ ...(ii)

passes through the line of intersection of the given planes.

Therefore, the equation of such a plane is

$$(x - 2y + 2z - 3) + t(x + y + z - 1) = 0$$

$$\Rightarrow (1 + t)x + (-2 + t)y + (2 + t)z - 3 - t = 0 \quad \text{...(iii)}$$

Now, plane (ii) makes the same angle with plane (i) and image plane (iii). Thus,

$$\frac{1 - 2 + 2}{3\sqrt{3}} = \pm \frac{1 + t - 2 + t + 2 + t}{\sqrt{3}\sqrt{(t+1)^2 + (t-2)^2 + (2+t)^2}}$$

$$\Rightarrow \quad t = 0, -\frac{2}{3}$$

For $t = 0$, we get plane (i); hence for image plane, $t = -\frac{2}{3}$

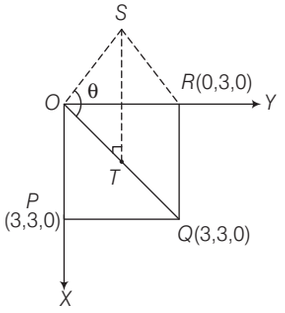
The equation of the image plane

$$3(x - 2y + 2z - 3) - 2(x + y + z - 1) = 0$$

$\Rightarrow x - 8y + 4z - 7 = 0$.

218. Given, square base $OP = OR = 3$

$\therefore \quad P(3, 0, 0),\ R = (0, 3, 0)$



Also, mid-point of OQ is $T\left(\frac{3}{2}, \frac{3}{2}, 0\right)$.

Since, S is directly above the mid-point T of diagonal OQ and $ST = 3$.

i.e. $S\left(\frac{3}{2}, \frac{3}{2}, 3\right)$

Here, DR's of OQ (3, 3, 0) and DR's of $OS\left(\frac{3}{2}, \frac{3}{2}, 3\right)$.

$$\therefore \quad \cos\theta = \frac{\frac{9}{2} + \frac{9}{2}}{\sqrt{9+9+0}\sqrt{\frac{9}{4} + \frac{9}{4} + 9}} = \frac{9}{\sqrt{18} \cdot \sqrt{\frac{27}{2}}} = \frac{1}{\sqrt{3}}$$

$\therefore$ Option (a) is incorrect.

Now, equation of the plane containing the ΔOQS is

$$\begin{vmatrix} x & y & z \\ 3 & 3 & 0 \\ 3/2 & 3/2 & 3 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x & y & z \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = 0$$

$$\Rightarrow \quad x(2 - 0) - y(2 - 0) + z(1 - 1) = 0$$

$$\Rightarrow \quad 2x - 2y = 0 \text{ or } x - y = 0$$

$\therefore$ Option (b) is correct.

Now, length of the perpendicular from $P(3, 0, 0)$ to the plane containing ΔOQS is

$$\frac{|3 - 0|}{\sqrt{1 + 1}} = \frac{3}{\sqrt{2}}$$

$\therefore$ Option (c) is correct.

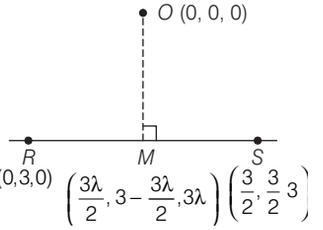
Here, equation of RS is

$$\frac{x - 0}{3/2} = \frac{y - 3}{-3/2} = \frac{z - 0}{3} = \lambda$$

$$\Rightarrow \quad x = \frac{3}{2}\lambda,\ y = -\frac{3}{2}\lambda + 3,\ z = 3\lambda$$

To find the distance from $O(0, 0, 0)$ to RS.

Let M be the foot of perpendicular.



$\because \quad \overline{OM} \perp \overline{RS} \Rightarrow \overline{OM} \cdot \overline{RS} = 0$

$$\Rightarrow \quad \frac{9\lambda}{4} - \frac{3}{2}\left(3 - \frac{3\lambda}{2}\right) + 3(3\lambda) = 0$$

$$\Rightarrow \quad \lambda = \frac{1}{3}$$

$$\therefore \quad M\left(\frac{1}{2}, \frac{5}{2}, 1\right)$$

$$\Rightarrow \quad OM = \sqrt{\frac{1}{4} + \frac{25}{4} + 1} = \sqrt{\frac{30}{4}} = \sqrt{\frac{15}{2}}$$

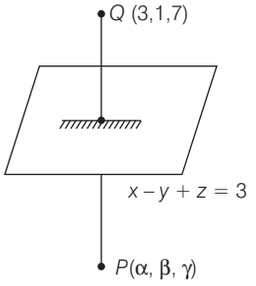
$\therefore$ Option (d) is correct.

219. Let image of $Q(3, 1, 7)$ w.r.t. $x - y + z = 3$ be $P(\alpha, \beta, \gamma)$.

$$\therefore \quad \frac{\alpha - 3}{1} = \frac{\beta - 1}{-1} = \frac{\gamma - 7}{1} = \frac{-2(3 - 1 + 7 - 3)}{1^2 + (-1)^2 + (1)^2}$$

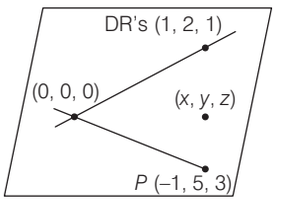
$$\Rightarrow \quad \alpha - 3 = 1 - \beta = \gamma - 7 = -4$$

$$\therefore \quad \alpha = -1, \beta = 5, \gamma = 3$$



Hence, the image of $Q(3, 1, 7)$ is $P(-1, 5, 3)$.

To find equation of plane passing through $P(-1, 5, 3)$ and containing $\frac{x}{1} = \frac{y}{2} = \frac{z}{1}$



$$\Rightarrow \quad \begin{vmatrix} x - 0 & y - 0 & z - 0 \\ 1 - 0 & 2 - 0 & 1 - 0 \\ -1 - 0 & 5 - 0 & 3 - 0 \end{vmatrix} = 0$$

$$\Rightarrow x(6 - 5) - y(3 + 1) + z(5 + 2) = 0$$

$$\therefore \quad x - 4y + 7z = 0$$

220. (i) Direction ratios of a line joining two points (x_1, y_1, z_1) and (x_2, y_2, z_2) are $x_2 - x_1, y_2 - y_1, z_2 - z_1$.

(ii) If the two lines with direction ratios $a_1, b_1, c_1; a_2, b_2, c_2$ are perpendicular, then $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$

Line L_1 is given by $y = x; z = 1$ can be expressed

$$L_1: \frac{x}{1} = \frac{y}{1} = \frac{z - 1}{0} = \alpha \qquad \text{[say]}$$

$$\Rightarrow \quad x = \alpha, y = \alpha, z = 1$$

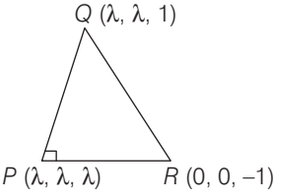
Let the coordinates of Q on L_1 be $(\alpha, \alpha, 1)$.

Line L_2 given by $y = -x, z = -1$ can be expressed as

$$L_2: \frac{x}{1} = \frac{y}{-1} = \frac{z + 1}{0} = \beta \qquad \text{[say]}$$

$$\Rightarrow \quad x = \beta, y = -\beta, z = -1$$

Let the coordinates of R on L_2 be $(\beta, -\beta, -1)$.



Direction ratios of PQ are $\lambda - \alpha, \lambda - \alpha, \lambda - 1$.

Now, $PQ \perp L_1$

$$\therefore \quad 1(\lambda - \alpha) + 1 \cdot (\lambda - \alpha) + 0 \cdot (\lambda - 1) = 0$$

$$\Rightarrow \quad \lambda = \alpha$$

Hence, $Q(\lambda, \lambda, 1)$

Direction ratios of PR are $\lambda - \beta, \lambda + \beta, \lambda + 1$.

Now, $PR \perp L_2$

$$\therefore \quad 1(\lambda - \beta) + (-1)(\lambda + \beta) + 0(\lambda + 1) = 0$$

$$\lambda - \beta - \lambda - \beta = 0 \Rightarrow \beta = 0$$

Hence, $R(0, 0, -1)$

Now, as $\angle QPR = 90°$

[as $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$, if two lines with DR's $a_1, b_1, c_1; a_2, b_2, c_2$ are perpendicular]

$$\therefore \quad (\lambda - \lambda)(\lambda - 0) + (\lambda - \lambda)(\lambda - 0) + (\lambda - 1)(\lambda + 1) = 0$$

$$\Rightarrow (\lambda - 1)(\lambda + 1) = 0 \Rightarrow \lambda = 1 \quad \text{or} \quad \lambda = -1$$

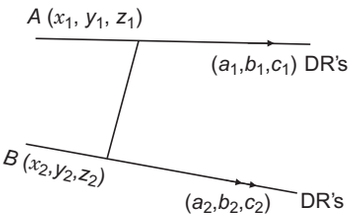
$\lambda = 1$, rejected as P and Q are different points.

$$\Rightarrow \quad \lambda = -1$$

221. If two straight lines are coplanar,

i.e. $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$

and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$ are coplanar



Then, $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$, (a_1, b_1, c_1) and (a_2, b_2, c_2) are coplanar,

i.e. $$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

Here, $$x = 5, \frac{y}{3 - \alpha} = \frac{z}{-2}$$

$$\Rightarrow \quad \frac{x - 5}{0} = \frac{y - 0}{-(\alpha - 3)} = \frac{z - 0}{-2} \qquad \ldots(i)$$

and $$x = \alpha, \frac{y}{-1} = \frac{z}{2 - \alpha}$$

$$\Rightarrow \quad \frac{x - \alpha}{0} = \frac{y - 0}{-1} = \frac{z - 0}{2 - \alpha} \qquad \ldots(ii)$$

$$\Rightarrow \quad \begin{vmatrix} 5 - \alpha & 0 & 0 \\ 0 & 3 - \alpha & -2 \\ 0 & -1 & 2 - \alpha \end{vmatrix} = 0$$

$$\Rightarrow \quad (5 - \alpha)[(3 - \alpha)(2 - \alpha) - 2] = 0$$

$$\Rightarrow \quad (5 - \alpha)[\alpha^2 - 5\alpha + 4] = 0$$

$$\Rightarrow \quad (5 - \alpha)(\alpha - 1)(\alpha - 4) = 0$$

$$\therefore \quad \alpha = 1, 4, 5$$

222. Equation of straight line is $l : \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

Since, l is perpendicular to l_1 and l_2.

So, its DR's are cross-product of l_1 and l_2.

Now, to find a point on l_2 whose distance is given, assume a point and find its distance to obtain point.

Let $l : \frac{x-0}{a} = \frac{y-0}{b} = \frac{z-0}{c}$

which is perpendicular to

$$l_1 : (3\hat{\mathbf{i}} - \hat{\mathbf{j}} + 4\hat{\mathbf{k}}) + t(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$$

$$l_2 : (3\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) + s(2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}})$$

$\therefore$ DR's of l is $\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & 2 \\ 2 & 2 & 1 \end{vmatrix} = -2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$

$$l : \frac{x}{-2} = \frac{y}{3} = \frac{z}{-2} = k_1, k_2$$

Now, $A(-2k_1, 3k_1, -2k_1)$ and $B(-2k_2, 3k_2, -2k_2)$.

Since, A lies on l_1.

$\therefore \quad (-2k_1)\hat{\mathbf{i}} + (3k_1)\hat{\mathbf{j}} - (2k_1)\hat{\mathbf{k}} = (3+t)\hat{\mathbf{i}} + (-1+2t)\hat{\mathbf{j}} + (4+2t)\hat{\mathbf{k}}$

$\Rightarrow \quad 3 + t = -2k_1, -1 + 2t = 3k_1, 4 + 2t = -2k_1$

$\therefore \quad k_1 = -1$

$\Rightarrow \quad A(2, -3, 2)$

Let any point on $l_2 (3 + 2s, 3 + 2s, 2 + s)$

$$\sqrt{(2-3-2s)^2 + (-3-3-2s)^2 + (2-2-s)^2} = \sqrt{17}$$

$$\Rightarrow \quad 9s^2 + 28s + 37 = 17$$

$$\Rightarrow \quad 9s^2 + 28s + 20 = 0$$

$$\Rightarrow \quad 9s^2 + 18s + 10s + 20 = 0$$

$$\Rightarrow \quad (9s + 10)(s + 2) = 0$$

$$\therefore \quad s = -2, \frac{-10}{9}.$$

Hence, $(-1, -1, 0)$ and $\left(\frac{7}{9}, \frac{7}{9}, \frac{8}{9}\right)$ are required points.

223. Any point on $\frac{x+2}{2} = \frac{y+1}{-1} = \frac{z}{3} = \lambda$

$\Rightarrow \quad x = 2\lambda - 2, y = -\lambda - 1, z = 3\lambda$

Let foot of perpendicular from $(2\lambda - 2, -\lambda - 1, 3\lambda)$ to $x + y + z = 3$ be (x_2, y_2, z_2).

$$\therefore \quad \frac{x_2 - (2\lambda - 2)}{1} = \frac{y_2 - (-\lambda - 1)}{1} = \frac{z_2 - (3\lambda)}{1} = -\frac{(2\lambda - 2 - \lambda - 1 + 3\lambda - 3)}{1 + 1 + 1}$$

$$\Rightarrow \quad x_2 - 2\lambda + 2 = y_2 + \lambda + 1 = z_2 - 3\lambda = 2 - \frac{4\lambda}{3}$$

$$\therefore \quad x_2 = \frac{2\lambda}{3}, y_2 = 1 - \frac{7\lambda}{3}, z_2 = 2 + \frac{5\lambda}{3}$$

$$\Rightarrow \quad \lambda = \frac{x_2 - 0}{2/3} = \frac{y_2 - 1}{-7/3} = \frac{z_2 - 2}{5/3}$$

Hence, foot of perpendicular lie on

$$\frac{x}{2/3} = \frac{y-1}{-7/3} = \frac{z-2}{5/3} \Rightarrow \frac{x}{2} = \frac{y-1}{-7} = \frac{z-2}{5}$$

224. $L_1 : \frac{x-1}{2} = \frac{y-0}{-1} = \frac{z-(-3)}{1}$

Normal of plane $P : \mathrm{n} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 7 & 1 & 2 \\ 3 & 5 & -6 \end{vmatrix}$

$$= \hat{\mathbf{i}}(-16) - \hat{\mathbf{j}}(-42 - 6) + \hat{\mathbf{k}}(32)$$

$$= -16\hat{\mathbf{i}} + 48\hat{\mathbf{j}} + 32\hat{\mathbf{k}}$$

DR's of normal $\mathrm{n} = \hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$

Point of intersection of L_1 and L_2.

$\Rightarrow \quad 2K_1 + 1 = K_2 + 4$

and $\quad -k_1 = k_2 - 3$

$\Rightarrow \quad k_1 = 2$ and $k_2 = 1$

$\therefore$ Point of intersection $(5, -2, -1)$

Now equation of plane,

$$1 \cdot (x - 5) - 3(y + 2) - 2(z + 1) = 0$$

$\Rightarrow \quad x - 3y - 2z - 13 = 0$

$\Rightarrow \quad x - 3y - 2z = 13$

$\therefore \quad a = 1, b = -3, c = -2, d = 13$

225. Since, $\frac{x-1}{2} = \frac{y+1}{K} = \frac{z}{2}$

and $\frac{x+1}{5} = \frac{y+1}{2} = \frac{z}{k}$ are coplanar.

$\Rightarrow \quad \begin{vmatrix} 2 & 0 & 0 \\ 2 & K & 2 \\ 5 & 2 & K \end{vmatrix} = 0$

$\Rightarrow \quad K^2 = 4 \Rightarrow K = \pm 2$

$\therefore \quad \mathbf{n}_1 = \mathbf{b}_1 \times \mathbf{d}_1 = 6\hat{\mathbf{j}} - 6\hat{\mathbf{k}}$, for $k = 2$

$\therefore \quad \mathbf{n}_2 = \mathbf{b}_2 \times \mathbf{d}_2 = 14\hat{\mathbf{j}} + 14\hat{\mathbf{k}}$, for $k = -2$

So, equation of planes are $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n}_1 = 0$

$\Rightarrow \quad y - z = -1$ and $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n}_2 = 0$

$\Rightarrow \quad y + z = -1$

226. Equation of the plane containing the lines

$$\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$$

and $\quad \frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$

is $a(x-2) + b(y-3) + c(z-4) = 0$ ...(i)

where, $\quad 3a + 4b + 5c = 0$ ...(ii)

$\quad 2a + 3b + 4c = 0$ ...(iii)

and $a(1-2) + b(2-3) + c(2-3) = 0$

i.e. $\quad a + b + c = 0$ ...(iv)

From Eqs. (ii) and (iii), $\frac{a}{1} = \frac{b}{-2} = \frac{c}{1}$, which satisfy Eq. (iv).

Plane through lines is $x - 2y + z = 0$.

Given plane is $Ax - 2y + z = d$ is $\sqrt{6}$.

$\therefore$ Planes must be parallel, so $A = 1$ and then

$$\frac{|d|}{\sqrt{6}} = \sqrt{6}$$

$$\Rightarrow \quad |d| = 6$$

227. The equation of the plane passing through the point $(-1,-2,-1)$ and whose normal is perpendicular to both the given lines L_1 and L_2 may be written as

$(x+1)+7(y+2)-5(z+1)=0 \Rightarrow x+7y-5z+10=0$

The distance of the point (1, 1, 1) from the plane

$$=\left|\frac{1+7-5+10}{\sqrt{1+49+25}}\right|=\frac{13}{\sqrt{75}} \text{ units}$$

228. The shortest distance between L_1 and L_2 is

$$\left|\frac{\{(2-(-1))\,\hat{\mathbf{i}}+(2-2)\,\hat{\mathbf{j}}+(3-(-1))\,\hat{\mathbf{k}}\}\cdot(-\hat{\mathbf{i}}-7\hat{\mathbf{j}}+5\hat{\mathbf{k}})}{5\sqrt{3}}\right|$$

$$=\left|\frac{(3\hat{\mathbf{i}}+4\hat{\mathbf{k}})\cdot(-\hat{\mathbf{i}}-7\hat{\mathbf{j}}+5\hat{\mathbf{k}})}{5\sqrt{3}}\right|$$

$$=\frac{17}{5\sqrt{3}} \text{ units}$$

229. The equations of given lines in vector form may be written as

$$L_1:\vec{\mathbf{r}}=(-\hat{\mathbf{i}}-2\hat{\mathbf{j}}-\hat{\mathbf{k}})+\lambda(3\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}})$$

and $L_2:\vec{\mathbf{r}}=(2\hat{\mathbf{i}}-2\hat{\mathbf{j}}+3\hat{\mathbf{k}})+\mu(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}})$

Since, the vector is perpendicular to both L_1 and L_2.

$$\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 3 & 1 & 2\\ 1 & 2 & 3\end{vmatrix}=-\hat{\mathbf{i}}-7\hat{\mathbf{j}}+5\hat{\mathbf{k}}$$

$\therefore$ Required unit vector

$$=\frac{(-\hat{\mathbf{i}}-7\hat{\mathbf{j}}+5\hat{\mathbf{k}})}{\sqrt{(-1)^2+(-7)^2+(5)^2}}$$

$$=\frac{1}{5\sqrt{3}}(-\hat{\mathbf{i}}-7\hat{\mathbf{j}}+5\hat{\mathbf{k}})$$

230. Given three planes are

$$P_1: x-y+z=1 \quad \text{...(i)}$$

$$P_2: x+y-z=-1 \quad \text{...(ii)}$$

and $$P_3: x-3y+3z=2 \quad \text{...(iii)}$$

On solving Eqs. (i) and (ii), we get

$$x=0,\ z=1+y$$

which does not satisfy Eq. (iii).

As $x-3y+3z=0-3y+3(1+y)=3\ (\neq 2)$

So, Statement II is true.

Next, since we know that direction ratios of line of intersection of planes $a_1x+b_1y+c_1z+d_1=0$

and $a_2x+b_2y+c_2z+d_2=0$ is

$$b_1c_2-b_2c_1,\ c_1a_2-a_1c_2,\ a_1b_2-a_2b_1$$

Using above result,

Direction ratios of lines L_1, L_2 and L_3 are

$$0, 2, 2\ ;\ 0, -4, -4\ ;\ 0, -2, -2$$

Since, all the three lines L_1, L_2 and L_3 are parallel pairwise.

Hence, Statement I is false.

231. Given planes are $3x-6y-2z=15$ and $2x+y-2z=5$.

For $z=0$, we get $x=3,\ y=-1$

Since, direction ratios of planes are

$$<3,-6,-2> \text{ and } <2,1,-2>$$

Then the DR's of line of intersection of planes is $<14,\ 2,\ 15>$ and line is

$$\frac{x-3}{14}=\frac{y+1}{2}=\frac{z-0}{15}=\lambda \quad \text{[say]}$$

$$\Rightarrow \quad x=14\lambda+3,\ y=2\lambda-1,\ z=15\lambda$$

Hence, Statement I is false.

But Statement II is true.

232. Let $\Delta=\begin{vmatrix}a & b & c\\ b & c & a\\ c & a & b\end{vmatrix}$

$$=-\frac{1}{2}(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]$$

(A) If $a+b+c\neq 0$ and $a^2+b^2+c^2=ab+bc+ca$

$\Rightarrow \Delta=0$ and $a=b=c\neq 0$

$\Rightarrow$ The equations represent identical planes.

(B) $a+b+c=0$ and $a^2+b^2+c^2\neq ab+bc+ca$

$\Rightarrow \Delta=0$

$\Rightarrow$ The equations have infinitely many solutions.

$ax+by=(a+b)z,\quad bx+cy=(b+c)z$

$\Rightarrow (b^2-ac)y=(b^2-ac)z \Rightarrow y=z$

$\Rightarrow ax+by+cy=0 \Rightarrow ax=ay \Rightarrow x=y=z$

(C) $a+b+c\neq 0$ and $a^2+b^2+c^2\neq ab+bc+ca$

$\Rightarrow \quad \Delta\neq 0$

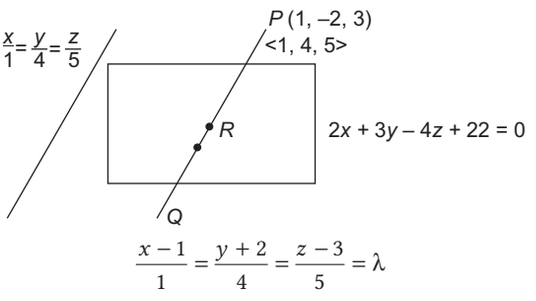
The equations represent planes meeting at only one point.

(D) $a+b+c=0$ and $a^2+b^2+c^2=ab+bc+ca$

$\Rightarrow a=b=c=0$

$\Rightarrow$ The equations represent whole of the three-dimensional space.

233. Any line parallel to $\frac{x}{1}=\frac{y}{4}=\frac{z}{5}$ and passing through $P(1,-2,3)$ is



$$\frac{x-1}{1}=\frac{y+2}{4}=\frac{z-3}{5}=\lambda \quad \text{(say)}$$

Any point on above line can be written as $(\lambda+1,\ 4\lambda-2,\ 5\lambda+3)$.

$\therefore$ Coordinates of R are $(\lambda+1,\ 4\lambda-2,\ 5\lambda+3)$.

Since, point R lies on the above plane.

$\therefore \quad 2(\lambda+1)+3(4\lambda-2)-4(5\lambda+3)+22=0 \quad \Rightarrow \quad \lambda=1$

So, point R is (2, 2, 8).

Now, $PR=\sqrt{(2-1)^2+(2+2)^2+(8-3)^2}=\sqrt{42}$

$\therefore \quad PQ=2PR=2\sqrt{42}$

234. Given, equations of lines are

$$\frac{x-1}{1}=\frac{y+2}{-2}=\frac{z-4}{3} \text{ and } \frac{x-2}{2}=\frac{y+1}{-1}=\frac{z+7}{-1}$$

Let $\mathbf{n}_1=\hat{\mathbf{i}}-2\hat{\mathbf{j}}+3\hat{\mathbf{k}}$ and $\mathbf{n}_2=2\hat{\mathbf{i}}-\hat{\mathbf{j}}-\hat{\mathbf{k}}$

$\therefore$ Any vector $\mathbf{n}$ perpendicular to both $\mathbf{n}_1$, $\mathbf{n}_2$ is given by

$$\mathbf{n}=\mathbf{n}_1\times\mathbf{n}_2$$

$$\Rightarrow \quad \mathbf{n}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & -2 & 3\\ 2 & -1 & -1\end{vmatrix}=5\hat{\mathbf{i}}+7\hat{\mathbf{j}}+3\hat{\mathbf{k}}$$

$\therefore$ Equation of a plane passing through $(1,-1,-1)$ and perpendicular to $\mathbf{n}$ is given by

$$5(x-1)+7(y+1)+3(z+1)=0$$

$$\Rightarrow \quad 5x+7y+3z+5=0$$

$$\therefore \text{Required distance}=\left|\frac{5+21-21+5}{\sqrt{5^2+7^2+3^2}}\right|=\frac{10}{\sqrt{83}} \text{ units}$$

235. Equation of line passing through $(1,-5,9)$ and parallel to $x=y=z$ is

$$\frac{x-1}{1}=\frac{y+5}{1}=\frac{z-9}{1}=\lambda \quad \text{(say)}$$

Thus, any point on this line is of the form $(\lambda+1, \lambda-5, \lambda+9)$.

Now, if $P(\lambda+1, \lambda-5, \lambda+9)$ is the point of intersection of line and plane, then

$$(\lambda+1)-(\lambda-5)+\lambda+9=5$$

$$\Rightarrow \quad \lambda+15=5$$

$$\Rightarrow \quad \lambda=-10$$

$\therefore$ Coordinates of point P are $(-9,-15,-1)$.

Hence, the required distance

$$=\sqrt{(1+9)^2+(-5+15)^2+(9+1)^2}$$

$$=\sqrt{10^2+10^2+10^2}=10\sqrt{3}$$

236. Since, the line $\frac{x-3}{2}=\frac{y+2}{-1}=\frac{z+4}{3}$ lies in the plane $lx+my-z=9$, therefore we have

$$2l-m-3=0$$

[$\because$ normal will be perpendicular to the line]

$$\Rightarrow \quad 2l-m=3 \quad \text{...(i)}$$

and $\quad 3l-2m+4=9$

[$\because$ point $(3,-2,-4)$ lies on the plane]

$$\Rightarrow \quad 3l-2m=5 \quad \text{...(ii)}$$

On solving Eqs. (i) and (ii), we get

$$l=1 \text{ and } m=-1$$

$$\therefore \quad l^2+m^2=2$$

237. Given equation of line is

$$\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}=\lambda \quad \text{(say) ...(i)}$$

and equation of plane is

$$x-y+z=16 \quad \text{...(ii)}$$

Any point on the line (i) is

$$(3\lambda+2, 4\lambda-1, 12\lambda+2)$$

Let this point be point of intersection of the line and plane.

$$\therefore \quad (3\lambda+2)-(4\lambda-1)+(12\lambda+2)=16$$

$$\Rightarrow \quad 11\lambda+5=16$$

$$\Rightarrow \quad 11\lambda=11$$

$$\Rightarrow \quad \lambda=1$$

$\therefore$ Point of intersection is $(5, 3, 14)$.

Now, distance between the points $(1, 0, 2)$ and $(5, 3, 14)$

$$=\sqrt{(5-1)^2+(3-0)^2+(14-2)^2}$$

$$=\sqrt{16+9+144}$$

$$=\sqrt{169}$$

$$=13$$

238. Let equation of plane containing the lines $2x-5y+z=3$ and $x+y+4z=5$ be

$$(2x-5y+z-3)+\lambda(x+y+4z-5)=0$$

$$\Rightarrow (2+\lambda)x+(\lambda-5)y+(4\lambda+1)z-3-5\lambda=0 \quad \text{...(i)}$$

This plane is parallel to the plane $x+3y+6z=1$.

$$\therefore \quad \frac{2+\lambda}{1}=\frac{\lambda-5}{3}=\frac{4\lambda+1}{6}$$

On taking first two equalities, we get

$$6+3\lambda=\lambda-5$$

$$\Rightarrow \quad 2\lambda=-11$$

$$\Rightarrow \quad \lambda=-\frac{11}{2}$$

On taking last two equalities, we get

$$6\lambda-30=3+12\lambda$$

$$\Rightarrow \quad -6\lambda=33$$

$$\Rightarrow \quad \lambda=-\frac{11}{2}$$

So, the equation of required plane is

$$\left(2-\frac{11}{2}\right)x+\left(\frac{-11}{2}-5\right)y+\left(-\frac{44}{2}+1\right)z-3+5\times\frac{11}{2}=0$$

$$\Rightarrow \quad -\frac{7}{2}x-\frac{21}{2}y-\frac{42}{2}z+\frac{49}{2}=0$$

$$\Rightarrow \quad x+3y+6z-7=0$$

239. Given, $l+m+n=0 \Rightarrow l=-(m+n)$

$$\Rightarrow \quad (m+n)^2=l^2$$

$$\Rightarrow \quad m^2+n^2+2mn=m^2+n^2 \quad [\because l^2=m^2+n^2, \text{ given}]$$

$$\Rightarrow \quad 2mn=0$$

Case I When $m=0$, then

$$l=-n$$

Hence, (l, m, n) is $(1, 0, -1)$.

Case II When $n=0$, then

$$l=-m$$

Hence, (l, m, n) is $(1, 0, -1)$.

$$\therefore \quad \cos\theta=\frac{1+0+0}{\sqrt{2}\times\sqrt{2}}=\frac{1}{2}$$

$$\Rightarrow \quad \theta=\frac{\pi}{3}$$

240. Plane and line are parallel to each other. Equation of normal to the plane through the point (1, 3, 4) is

$$\frac{x-1}{2}=\frac{y-3}{-1}=\frac{z-4}{1}=k \qquad \text{[Say]}$$

Any point in this normal is

$$(2k+1,-k+3,4+k).$$

$$\Rightarrow \quad \left(\frac{2k+1+1}{2},\frac{3-k+3}{2},\frac{4+k+4}{2}\right) \text{lies on plane.}$$

$$\Rightarrow \quad 2(k+1)-\left(\frac{6-k}{2}\right)+\left(\frac{8+k}{2}\right)+3=0$$

$$\Rightarrow \quad k=-2$$

Hence, point through which this image pass is

$$(2k+1,3-k,4+k)$$

i.e. $(2(-2)+1,3+2,4-2)=(-3,5,2)$

Hence, equation of image line is

$$\frac{x+3}{3}=\frac{y-5}{1}=\frac{z-2}{-5}$$

241. Given planes are

$$2x+y+2z-8=0$$

and $$2x+y+2z+\frac{5}{2}=0$$

Distance between two parallel planes

$$=\frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}=\left|\frac{-8-\frac{5}{2}}{\sqrt{2^2+1^2+2^2}}\right|$$

$$=\frac{\frac{21}{2}}{3}=\frac{7}{2}$$

242. The given line are

$$\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{-k} \qquad \ldots(i)$$

and $$\frac{x-1}{k}=\frac{y-k}{2}=\frac{z-5}{1} \qquad \ldots(ii)$$

Condition for two lines are coplanar.

$$\begin{vmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

wher,e (x_1, y_1, z_1) and $(x_2, y_2\ z_2)$ are any points on the lines (i) and (ii), respectively and $< l_1, m_1, n_1 >$ and $< l_2, m_2, n_2 >$ are direction cosines of lines (i) and (ii), respectively.

$$\therefore \quad \begin{vmatrix} 2-1 & 3-4 & 4-5 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix}=0$$

$$\Rightarrow \quad \begin{vmatrix} 1 & -1 & -1 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix}=0$$

$$\Rightarrow \quad 1(1+2k)+1(1+k^2)-(2-k)=0$$

$$\Rightarrow \quad k^2+2k+k=0$$

$$\Rightarrow \quad k^2+3k=0$$

$$\Rightarrow \quad k=0,-3$$

Note : If 0 appears in the denominator, then the correct way of representing the equation of straight line is

$$\frac{x-2}{1}=\frac{y-3}{1};\ z=4 \text{ and } x=l;\ \frac{y-4}{2}=\frac{z-5}{1}$$

243. **Given** A plane $P: x-2y+2z-5=0$

To find The equation of a plane parallel to given plane P and at a distance of 1 unit from origin. Equation of family of planes parallel to the given plane P is

$$Q: x-2y+2z+d=0$$

Also, perpendicular distance of Q from origin is 1 unit.

$$\Rightarrow \quad \left|\frac{0-2(0)+2(0)+d}{\sqrt{1^2+2^2+2^2}}\right|=1$$

$$\Rightarrow \quad \left|\frac{d}{3}\right|=1 \Rightarrow d=\pm 3$$

Hence, the required equation of the plane parallel to P and at unit distance from origin is

$$x-2y+2z\pm 3=0$$

Hence, out of the given equations, option (a) is the only correct option.

244. **Given** Two lines $L_1: \frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$

and $$L_2: \frac{x-3}{1}=\frac{y-k}{2}=\frac{z-0}{1}$$

To find The value of 'k' of the given lines L_1 and L_2 are intersecting each other.

Let $$L_1: \frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}=p$$

and $$L_2: \frac{x-3}{1}=\frac{y-k}{2}=\frac{z-0}{1}=q$$

$\Rightarrow$ Any point P on line L_1 is of type

$P(2p+1, 3p-1, 4p+1)$ and any point Q on line L_2 is of type $Q(q+3, 2q+k, q)$.

Since, L_1 and L_2 are intersecting each other, hence both points P and Q should coincide at the point of intersection, *i.e.,* corresponding coordinates of P and Q should be same.

$$2p+1=q+3,\ 3p-1=2q+k \text{ and } 4p+1=q$$

On solving $2p+1=q+3$ and $4p+1=q$, we get the values of p and q as

$$p=\frac{-3}{2} \quad \text{and} \quad q=-5$$

On substituting the values of p and q in the third equation $3p-1=2q+k$, we get

$$3\left(\frac{-3}{2}\right)-1=2(-5)+k$$

$$\Rightarrow \quad k=\frac{9}{2}$$

245. Angle between straight line $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$ and plene $\mathbf{r}\cdot\hat{\mathbf{n}}=\mathbf{d}$ is

$$\sin\theta=\frac{\mathbf{b}\cdot\hat{\mathbf{n}}}{|\mathbf{b}||\hat{\mathbf{n}}|}$$

$$\therefore \quad \sin\theta=\frac{(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\lambda\hat{\mathbf{k}})\cdot(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}})}{\sqrt{1+4+\lambda^2}\sqrt{1+4+9}}$$

$$= \frac{5+3\lambda}{\sqrt{\lambda^2+5}\ \sqrt{14}}$$

Given, $\cos\theta = \sqrt{\frac{5}{14}}$

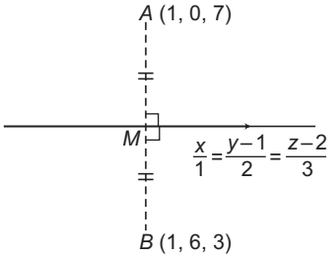
$\therefore \quad \sin\theta = \frac{3}{\sqrt{14}} \Rightarrow \frac{3}{\sqrt{14}} = \frac{5+3\lambda}{\sqrt{\lambda^2+5}\cdot\sqrt{14}}$

$\Rightarrow \quad 9(\lambda^2+5) = 9\lambda^2 + 30\lambda + 25$

$\Rightarrow \quad 9\lambda^2 + 45 = 9\lambda^2 + 30\lambda + 25$

$\Rightarrow \quad 30\lambda = 20 \Rightarrow \lambda = \frac{2}{3}$

246. Mid-point of AB is $M(1, 3, 5)$.



which lies on $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$

as $\frac{1}{1} = \frac{3-1}{2} = \frac{5-2}{3} \Rightarrow 1 = 1 = 1$

Hence, Statement II is true.

Also, directions ratios of AB is

$(1-1, 6-0, 3-7)$

i.e. $(0, 6, -4)$ ...(i)

and direction ratios of straight line is

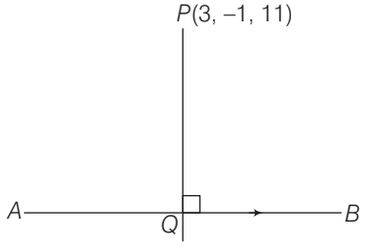
$(1, 2, 3)$ ...(ii)

The two lines are perpendicular, if

$0(1) + 6(2) - 4(3) = 12 - 12 = 0$

Hence, Statement I is true and statement II is a correct explanations of statement II.

247. Let the coordinates of Q be $(2\lambda, 3\lambda+2, 4\lambda+3)$ which is any point on the straight line AB.



$\therefore$ DR's of PQ is $(2\lambda-3, 3\lambda+3, 4\lambda-8)$

Also, perpendicular to straight line $AB\ \frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4} = \lambda$

having DR's $(2, 3, 4)$.

Thus, $2(2\lambda-3) + 3(3\lambda+3) + 4(4\lambda-8) = 0$

$\Rightarrow \quad 4\lambda - 6 + 9\lambda + 9 + 16\lambda - 32 = 0$

$\Rightarrow \quad 29\lambda - 29 = 0$

$\therefore \quad \lambda = 1$

Hence, coordinates of Q are $(2, 5, 7)$

$\therefore \quad |PQ| = \sqrt{(3-2)^2 + (-1-5)^2 + (7-11)^2}$

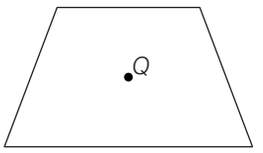
$= \sqrt{1+36+16} = \sqrt{53}$

248. Let Q be any point on the plane.

Then equation of PQ is

$$\frac{x-1}{1} = \frac{y+5}{1} = \frac{z-9}{1} = \lambda$$

where $P = (1, -5, 9)$



$\therefore \quad x = \lambda + 1, y = \lambda - 5, z = \lambda + 9$ lies on the plane

$x - y + z = 5$

$\Rightarrow \quad \lambda + 1 - \lambda + 5 + \lambda + 9 = 5$

$\therefore \quad \lambda = -10$

Hence, coordinate of Q is $Q(-9, -15, -1)$

$\therefore \quad |PQ| = \sqrt{(10)^2 + (10)^2 + (10)^2} = 10\sqrt{3}$

249. We know that, $\cos^2 45° + \cos^2 120° + \cos^2\theta = 1$

$\Rightarrow \quad \frac{1}{2} + \frac{1}{4} + \cos^2\theta = 1 \Rightarrow \cos^2\theta = \frac{1}{4}$

$\Rightarrow \quad \cos\theta = \pm\frac{1}{2} \Rightarrow \theta = 60° \text{ or } 120°$

250. The image of the point $(3, 1, 6)$ with respect to the plane $x - y + z = 5$ is

$$\frac{x-3}{1} = \frac{y-1}{-1} = \frac{z-6}{1} = \frac{-2(3-1+6-5)}{1+1+1}$$

$$\left[\because \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} = \frac{-2(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}\right]$$

$\Rightarrow \quad \frac{x-3}{1} = \frac{y-1}{-1} = \frac{z-6}{1} = -2$

$\Rightarrow \quad x = 3 - 2 = 1$

$y = 1 + 2 = 3$

and $z = 6 - 2 = 4$

which shows that Statement I is true.

We observe that the line segment joining the points $A(3, 1, 6)$ and $B(1, 3, 4)$ has direction ratios $2, -2, 2$ which are proportional to $1, -1, 1$. The direction ratios of the normal to the plane. Hence, Statements II is true. Thus, the Statements I and II are true and Statement II is correct explanation of Statement I.

251. Dr's of given line are $(3, -5, 2)$.

Dr's of normal to the plane $= (1, 3, -\alpha)$

$\therefore$ Line is perpendicular to the normal.

$\Rightarrow \quad 3(1) - 5(3) + 2(-\alpha) = 0$

$\Rightarrow \quad 3 - 15 - 2\alpha = 0$

$\Rightarrow \quad 2\alpha = -12 \Rightarrow \alpha = -6$

Also, point $(2, 1, -2)$ lies on the plane.

$\therefore \quad 2 + 3 + 6(-2) + \beta = 0 \Rightarrow \beta = 7$

$\Rightarrow \quad (\alpha, \beta) = (-6, 7)$

252. Projection of a vector on coordinate axes are

$$x_2 - x_1, y_2 - y_1, z_2 - z_1$$
$$x_2 - x_1 = 6,$$
$$y_2 - y_1 = -3,$$
$$z_2 - z_1 = 2$$

Now, $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

$$= \sqrt{36 + 9 + 4} = 7$$

So, the DC's of the vector are $\frac{6}{7}, -\frac{3}{7}, \frac{2}{7}$.

253. Equation of line passing through (5, 1, a) and (3, b, 1) is

$$\frac{x-3}{5-3} = \frac{y-b}{1-b} = \frac{z-1}{a-1} \quad \text{... (i)}$$

$$\left[\because \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}\right]$$

Point $\left(0, \frac{17}{2}, -\frac{13}{2}\right)$ satisfies Eq. (i), we get

$$-\frac{3}{2} = \frac{\frac{17}{2} - b}{1-b} = \frac{-\frac{13}{2} - 1}{a-1}$$

$$\Rightarrow \quad a - 1 = \frac{\left(-\frac{15}{2}\right)}{\left(-\frac{3}{2}\right)} = 5 \Rightarrow a = 6$$

Also, $\quad -3(1-b) = 2\left(\frac{17}{2} - b\right)$

$$\Rightarrow \quad 3b - 3 = 17 - 2b$$

$$\Rightarrow \quad 5b = 20 \Rightarrow b = 4$$

254. Given, $\quad \frac{x-1}{k} = \frac{y-2}{2} = \frac{z-3}{3}$ ... (i)

and $\quad \frac{x-2}{3} = \frac{y-3}{k} = \frac{z-1}{2}$ ... (ii)

Since, lines intersect at a point. Then, shortest distance between them is zero.

$$\therefore \quad \begin{vmatrix} k & 2 & 3 \\ 3 & k & 2 \\ 1 & 1 & -2 \end{vmatrix} = 0$$

$$\Rightarrow \; k(-2k - 2) - 2(-6 - 2) + 3(3 - k) = 0$$

$$\Rightarrow \quad -2k^2 - 5k + 25 = 0$$

$$\Rightarrow \quad 2k^2 + 5k - 25 = 0$$

$$\Rightarrow \; 2k^2 + 10k - 5k - 25 = 0$$

$$\Rightarrow \; 2k(k + 5) - 5(k + 5) = 0$$

$$\Rightarrow \quad k = \frac{5}{2}, -5$$

Hence, integer value of k is -5.

255. Let the drection cosines of line L be l, m and n. Since, the line intersect the given planes, then the normal to the planes are perpendicular to the line L.

$$\therefore \quad 2l + 3m + n = 0 \quad \text{...(i)}$$

and $\quad l + 3m + 2n = 0 \quad \text{...(ii)}$

From Eqs. (i) and (ii), we get

$$\frac{l}{3} = \frac{m}{-3} = \frac{n}{3} = k \quad \text{[say]}$$

We know that, $l^2 + m^2 + n^2 = 1$

$$\therefore \quad (3k)^2 + (-3k)^2 + (3k)^2 = 1$$

$$\Rightarrow \quad 27k^2 = 1 \;\Rightarrow\; k = \frac{1}{3\sqrt{3}}$$

$$\therefore \quad l = \frac{1}{\sqrt{3}} \;\Rightarrow\; \cos\alpha = \frac{1}{\sqrt{3}}$$

256. Since, a line makes an angle of $\frac{\pi}{4}$ with positive direction of each of X-axis and Y-axis, therefore

$$\alpha = \frac{\pi}{4}, \beta = \frac{\pi}{4}$$

We know that, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$$\Rightarrow \quad \cos^2\frac{\pi}{4} + \cos^2\frac{\pi}{4} + \cos^2\gamma = 1$$

$$\Rightarrow \quad \frac{1}{2} + \frac{1}{2} + \cos^2\gamma = 1$$

$$\Rightarrow \quad \cos^2\gamma = 0 \;\Rightarrow\; \gamma = 90°$$

257. Given, equation of sphere is

$$x^2 + y^2 + z^2 - 6x - 12y - 2z + 20 = 0$$

whose coordinates of centre are (3, 6, 1).

Since, one end of diameter are (2, 3, 5) and the other end of diameter be (α, β, γ),

then $\quad \frac{\alpha + 2}{2} = 3, \frac{\beta + 3}{2} = 6, \frac{\gamma + 5}{2} = 1$

$$\Rightarrow \quad \alpha = 4, \beta = 9$$

and $\quad \gamma = -3.$

Hence, the coordinates of other point are (4, 9, – 3).

258. Given equations of lines are

$$x = ay + b, z = cy + d$$

and $\quad x = a'y + b', z = c'y + d'$

These equations can be rewritten as

$$\frac{x-b}{a} = \frac{y-0}{1} = \frac{z-d}{c}$$

and $\quad \frac{x-b'}{a'} = \frac{y-0}{1} = \frac{z-d'}{c'}$

These lines will perpendicular, if $aa' + 1 + cc' = 0$

$$\because \quad l_1l_2 + m_1m_2 + n_1n_2 = 6$$

259. We know that, the image (x, y, z) of a point (x_1, y_1, z_1) in a plane $ax + by + cz + d = 0$ is given by

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \frac{-2(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}.$$

Thus, the image of point $(-1, 3, 4)$ in a plane $x - 2y = 0$ is given by

$$\frac{x+1}{1} = \frac{y-3}{-2} = \frac{z-4}{0}$$

$$= \frac{-2\,[1\times(-1)+(-2)\times 3+0\times 4]}{1+4}$$

$$\Rightarrow \quad \frac{x+1}{1}=\frac{y-3}{-2}=\frac{z-4}{0}=\frac{-2\,(-7)}{5}$$

$$\Rightarrow \quad x=\frac{14}{5}-1=\frac{9}{5},\ y=-\frac{28}{5}+3=-\frac{13}{5}$$

and $z=4$

Hence, the image of point $(-1, 3, 4)$ is $\left(\frac{9}{5}, -\frac{13}{5}, 4\right)$.

260. Centre of sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ is $(-u, -v, -w)$.

Given equation of first sphere is

$$x^2+y^2+z^2+6x-8y-2z=13 \qquad \ldots(i)$$

whose centre is $(-3, 4, 1)$

and equation of second sphere is

$$x^2+y^2+z^2-10x+4y-2z=8 \qquad \ldots(ii)$$

whose centre is $(5, -2, 1)$.

Mid-point of $(-3, 4, 1)$ and $(5, -2, 1)$ is $(1, 1, 1)$.

Since, the plane passes through $(1, 1, 1)$.

$\therefore \quad 2a-3a+4a+6=0$

$\Rightarrow \quad 3a=-6 \quad \Rightarrow a=-2$

261. Direction ratios of line normal are

$(a_1, b_1, c_1)=(1, 2, 2)$

and direction ratios of a plane are

$(a_2, b_2, c_2)=(2, -1, \sqrt{\lambda})$

Since, $\sin\theta=\dfrac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$

$$=\frac{1\times 2+2(-1)+2\times\sqrt{\lambda}}{\sqrt{(1)+(2)^2+(2)^2}\sqrt{(2)^2+(1)^2+(\sqrt{\lambda})^2}}$$

$$\Rightarrow \quad \frac{1}{3}=\frac{2\sqrt{\lambda}}{3\sqrt{5+\lambda}} \Rightarrow 5+\lambda=4\lambda$$

$$\Rightarrow \quad \lambda=\frac{5}{3}$$

262. If a_1, b_1, c_1 and a_2, b_2, c_2 are DR's of two lines, then the angle between them is given by

$$\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

The given equations can be rewritten as

$$\frac{x}{3}=\frac{y}{2}=\frac{z}{-6} \text{ and } \frac{x}{2}=\frac{y}{-12}=\frac{z}{-3}$$

$\therefore$ Angle between the lines is given by

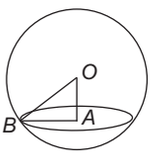
$$\cos\theta=\frac{6-24+18}{\sqrt{9+4+36}\sqrt{4+144+9}}$$

$$=\frac{0}{\sqrt{49}\sqrt{157}}=0$$

$$\Rightarrow \quad \theta=90^\circ$$

263. Since, the centre of sphere $x^2+y^2+z^2-x+z-2=0$ is $\left(\frac{1}{2}, 0, -\frac{1}{2}\right)$ and radius of sphere

$$=\sqrt{\frac{1}{4}+\frac{1}{4}+2}=\frac{\sqrt{10}}{2}$$



Distance of plane from centre of sphere

$$=\left|\frac{\frac{1}{2}+\frac{1}{2}-4}{\sqrt{1+4+1}}\right|=\frac{3}{\sqrt{6}}$$

So, radius of circle $=\sqrt{\frac{10}{4}-\frac{9}{6}}$

$$=\sqrt{\frac{30-18}{12}}=\sqrt{\frac{12}{12}}=1$$